**Abstract.** We prove that the moduli space of stable sheaves of rank 2 with a certain Chern classes on a smooth quadric $Q$ in $\mathbb{P}_3$, is isomorphic to $\mathbb{P}_3$. Using this identification, we give a new proof that a certain Brill-Noether locus on a non-hyperelliptic curve of genus 4, is isomorphic to the Donagi-Izadi cubic threefold.

1. Introduction

In [5], the geometry of the Brill-Noether loci $W^r$ of the moduli spaces $SU_C(2, K_C)$ of semi-stable vector bundles of rank 2 with canonical determinant over curves with small genus, was investigated. In [2], the explicit description of $W^1$ when $g(C) = 3$, was rediscovered, using the rational map from the moduli space $\overline{M}(1, 2)$ of stable sheaves of rank 2 with Chern classes $(1, 2)$ over $\mathbb{P}_2$ to $W^1$, defined by the restriction. In this article, we use the same method as in [2] to rediscover the geometry of $W^2$ stated in [5], using the rational restriction map

$$\Phi : \overline{M}(2) \dashrightarrow W^2,$$

sending $E$ to $E|_C$, where $\overline{M}(k)$ is the moduli space of semi-stable sheaves of rank 2 with the Chern classes $c_1 = (1, 1)$ and $c_2 = k$ over a smooth quadric surface $Q \subset \mathbb{P}_3$, containing $C$, and $W^2$ is a Brill-Noether locus of a non-hyperelliptic curve of genus 4. This rational map makes sense because $C$ is canonically embedded into $\mathbb{P}_3$ and there exists the unique quadric containing it. We will deal with the case when $Q$ is smooth.

First, we give the explicit description of $\overline{M}(2)$. We construct a morphism $\Psi : \overline{M}(2) \rightarrow \mathbb{P}_3$, sending $E$ to the point $p_E$ which is passed by the lines containing zeros of the sections of $E$. In fact, $\Psi$ can be proven to be an isomorphism.

Second, we study the jumping conics of $E \in \overline{M}(2)$. In fact, the isomorphism $\Psi$ can be redefined by sending $E$ to the set of jumping conics of $E$, which is a hyperplane in $\mathbb{P}_{3}^*$. 

Last, we investigate the restriction map $\Phi : \overline{M}(2) \dashrightarrow W^2$. We prove that this map is birational and given by the complete linear system $|I_C(3)|$. This will give us an easy proof that $W^2$ is isomorphic to the Donagi-Izadi cubic threefold.

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threefold. If we composite \( \Phi \) with the projection \( \mathcal{W}^2 \to \mathbb{P}_3^* \) at the unique point of \( \mathcal{W}^3 \), we have an isomorphism of \( \overline{M}(2) \) with \( \mathbb{P}_3^* \). We also describe this isomorphism in terms of sheaves in \( \overline{M}(2) \).

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2. Description of \( \overline{M}(2) \)

Let \( Q \) be a smooth quadric in \( \mathbb{P}_3 \) over \( \mathbb{C} \) and \( \overline{M}(k) \) be the moduli space of semi-stable sheaves of rank 2 with the Chern classes \( c_1 = \mathcal{O}_Q(1, 1) \) and \( c_2 = k \) on \( Q \) with respect to the ample line bundle \( H = \mathcal{O}_Q(1, 1) \). We know, by the Bogomolov theorem, that \( \overline{M}(k) = \emptyset \) for \( k \leq 0 \). For the case when \( k = 1 \), the following statement can be easily derived.

**Lemma 2.1.** \( \overline{M}(1) \) is the one-point space with the strictly semi-stable bundle,

\[
E_0 = \mathcal{O}_Q(1, 0) \oplus \mathcal{O}_Q(0, 1).
\]

**Proof.** Let \( E \in \overline{M}(1) \) be a stable bundle and then, from \( \chi(E) = 4 \) and the stability condition, we have

\[
0 \to \mathcal{O}_Q \to E \to I_p(1, 1) \to 0,
\]

where \( I_p \) is the ideal sheaf of a point \( p \in Q \). If we tensor the sequence with \( \mathcal{O}_Q(-1, 0) \), it can be shown that \( h^0(E(-1, 0)) = 1 \) which is a contradiction to the stability of \( E \). If \( E \) is strictly semi-stable, it should be fitted into the following sequence,

\[
0 \to \mathcal{O}_Q(a, 1 - a) \to E \to \mathcal{O}_Q(1 - a, a) \to 0,
\]

where \( a = 0 \) or \( 1 \). Since \( \text{Ext}^1(\mathcal{O}_Q(a, 1 - a), \mathcal{O}_Q(1 - a, a)) \) is trivial, so \( E \) must be the direct sum of \( \mathcal{O}_Q(1, 0) \) and \( \mathcal{O}_Q(0, 1) \). \( \square \)

Now let us deal with the case \( k = 2 \). Note that the stability and semi-stability conditions are equivalent. In particular, \( \overline{M}(2) \) is a projective space whose points correspond to isomorphism classes of stable sheaves on \( Q \) with given numerical invariants. From the standard computation, we can have \( \text{Ext}^2(E, E) = 0 \) for any \( E \in \overline{M}(2) \). Thus \( \overline{M}(2) \) is smooth.

Let \( E \) be a sheaf in \( \overline{M}(2) \). For a subsheaf \( \mathcal{O}_Q(a, b) \) of \( E \), we have \( a + b < 1 \) by the stability condition. Since \( \chi(E) = 3 \) and

\[
h^2(E) = h^0(E^*(-2, -2)) = h^0(E(-3, -3)) = 0,
\]

so we have \( h^0(E) > 0 \). Hence, \( E \) can be obtained from the following extension,

\[
0 \to \mathcal{O}_Q \to E \to I_Z(1, 1) \to 0, \tag{1}
\]

where \( Z \) is a zero-dimensional subscheme of \( Q \) with length 2. In particular, we have \( h^0(E) = 3 \) and \( h^1(E) = 0 \). Assume that there exists a line \( l \) on \( Q \) containing \( Z \) with the ideal sheaf \( \mathcal{O}_Q(-1, 0) \). If we tensor the sequence (1) with \( \mathcal{O}_Q(0, -1) \), it can be easily checked that \( h^0(E(0, -1)) = 0 \) contradicting
to the stability of $E$. Thus, the line $l$ containing $Z$ should intersect with $Q$ only at $Z$.

**Lemma 2.2.** The sheaves in the extension (1), are all stable if $Z$ is not contained in any line on $Q$.

**Proof.** From the condition on the numerical invariants of $E$, the only possibility of the sub-bundle $O_Q(a, b) \subset E$ is $O_Q$ or $O_Q(a, 1 - a)$ with $a = 0, 1$. The second case is impossible due to the condition on $Z$. □

Note that $P(Z) := P \text{Ext}^1(I_Z(1, 1), O_Q) \simeq P_1$. From the isomorphism

$$\text{Ext}^1(I_Z(1, 1), O_Q) \simeq O_{z_1} \oplus O_{z_2},$$

we can denote $(c_1, c_2)$ for the coordinates of $P(Z)$. As in the case of the projective plane [3], we can prove the following lemma.

**Lemma 2.3.** An extension $(c_1, c_2)$ gives a bundle if and only if all $c_i \neq 0$.

**Proof.** An easy application of [4]. □

**Lemma 2.4.** The set of non-bundles in $\overline{M}(2)$ is parametrized by $Q \subset P_3$.

**Proof.** Let $E$ be a non-bundle in $\overline{M}(2)$, then we have the following exact sequence,

$$0 \to E \to E^{**} \to O_Z \to 0,$$

where $Z$ is a zero-dimensional subscheme of $Q$. Since $E^{**}$ is semi-stable, we have $E^{**} \simeq E_0 = O_Q(1, 0) \oplus O_Q(0, 1)$ by [2.1]. In particular, the length of $Z$ is 1.

Now let $f : E_0 \to O_{p_E}$ be a surjection with the kernel $E$, where $p_E$ is a point in $Q$. We can denote $f$ by $(f_1, f_2)$, where $f_i : O_Q(i - 1, i) \to O_{p_E}$, i.e. the parametrization of such surjections is $P_1$ with the coordinates $(f_1, f_2)$. If $f_1 = 0$, then $\ker(f)$ is decomposed to $O_Q(1, 0) \oplus I_{p_E}(0, 1)$, which is not stable. Similarly, we have an unstable kernel when $f_2 = 0$. Let us assume that $f_i \neq 0$ for all $i$. Then we have the following diagram,

$$\begin{array}{ccccccc}
0 & \to & I_{p_E}(1, 0) & \to & E & \to & O_Q(0, 1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_Q(1, 0) & \to & E_0 & \to & O_Q(0, 1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_{p_E} & \to & O_{p_E} & \to & 0 \\
& & & & & & 0, \\
\end{array}$$
where $E$ is a non-trivial extension. It is easily checked that this non-trivial sheaf is stable. Since the dimension of $\text{Ext}^1(O_Q(0,1), I_{p_E}(1,0))$ is 1, the non-trivial extension $E$ is uniquely determined, i.e. we have only one non-bundle on $Q$ associated to a point $p_E \in Q$. Hence, we get the assertion. □

Now let $E$ be a stable bundle in $\overline{M}(2)$ and $s$ be a section of $E$. From $s$, we have an exact sequence of type (1) and so can consider a morphism

$$\psi_E : \mathbb{P} H^0(E) \simeq \mathbb{P}_2 \to \text{Gr}(1,3) \subset \mathbb{P}_5,$$

sending a section $s$ of $E$ to the line containing $Z$ in $\mathbb{P}_3$.

**Remark 2.5.** In the previous proof, if we choose a section of $E_0$ whose zero is $q \neq p_E$, then we have an exact sequence

$$0 \to O_Q \to E \to I_Z(1,1) \to 0,$$

where $Z = \{p_E, q\}$. In particular, for a non-bundle $E$, the image of $\varphi_E$ is the plane in $\text{Gr}(1,3)$ whose points correspond to lines passing through the singularity point $p_E$.

Let $E$ be a stable bundle in $\overline{M}(2)$ with the exact sequence (1).

**Lemma 2.6.** The determinant map

$$\lambda_E : \wedge^2 H^0(E) \to H^0(O_Q(1,1)),$$

is injective.

**Proof.** Let us assume that $\lambda_E$ is not injective. Since every element in $\wedge^2 H^0(E)$ is decomposable, there exist two sections $s_1$ and $s_2$ of $E$ such that $s_1 \wedge s_2$ is a non-trivial element in $\ker(\lambda_E)$. Then $s_1$ and $s_2$ generate a subsheaf $F$ of $E$ with $h^0(F) \geq 2$. Hence, $F$ is of the form $I_Z'(a,b)$, where $Z'$ is a 0-cycle on $Q$ and $a, b \geq 0$. From the stability condition of $E$, we have $F \simeq O_Q(a, 1 - a)$ with $a = 0$ or 1, which is not possible since $Z$ is not contained in any line on $Q$. □

Let $p_E$ be the point in $\mathbb{P}_3 \simeq \mathbb{P} H^0(O_Q(1,1))^*$ corresponding to the cokernel of $\lambda$ and consider the following exact sequence,

$$0 \to H^0(I_Z(1,1)) \to H^0(O_Q(1,1)) \to H^0(O_Z) \to 0.$$

In the previous lemma, $H^0(E)$ can be expressed as the direct sum of $H^0(O_Q)$ and $H^0(I_Z(1,1))$, which would give the following identification,

$$\wedge^2 H^0(E) \simeq [H^0(O_Q) \otimes H^0(I_Z(1,1))] \oplus [\wedge^2 H^0(I_Z(1,1))].$$

From this identification, clearly we have,

$$H^0(I_Z(1,1)) \subset \lambda_E(\wedge^2 H^0(E)).$$

In other words, the dual of the cokernel of $\lambda$ is contained in $H^0(O_Z)^*$. Note that $\mathbb{P} H^0(O_Z)^*$ is a line in $\mathbb{P}_3$ passing through $Z$. Thus the line passing through $Z$, also contains $p_E$. With the previous remark, we get the following lemma.
Lemma 2.7. $\psi_E$ is a linear embedding of $\mathbb{P}^0(E)$ into $\mathbb{P}_5$. Moreover, the image corresponds to the set of lines passing through one point $p_E$ in $\mathbb{P}_3$.

Now, let us define a map

$$\Psi : \mathcal{M}(2) \to \mathbb{P}_3,$$

sending $E$ to $p_E$, where $p_E$ is the unique point in $\mathbb{P}_3$, passed by the lines in the image of $\varphi_E$.

Proposition 2.8. $\Psi : \mathcal{M}(2) \to \mathbb{P}_3$ is an isomorphism.

Proof. Let $p$ be a point in $\mathbb{P}_3$. If $Z$ is a 0-cycle on $Q$ such that $p \in Z$ and the line passing through $Z$, is not contained in $Q$, then there exists $E \in \mathbb{P}(Z)$ for which $p_E = p$. Thus, $\Phi$ is surjective.

Moreover, assume that $p$ is not in $Q$. If we take the projection $\pi_p : \mathbb{P}_3 \to \mathbb{P}_2$ from $p$, then the restriction map $\pi_p : Q \to \mathbb{P}_2$ is a finite morphism of degree 2. Again, if we take the pull-back of $\Omega_{\mathbb{P}_2}(2)$ on $Q$, we get a stable vector bundle on $Q$ with the Chern classes $c_1 = \mathcal{O}_Q(1,1)$ and $c_2 = 2$ (this will be proved later in Lemma 4.3). Clearly, this defines an inverse map of $\Psi$. Hence, $\Psi$ is a birational morphism and it is an isomorphism over the stable vector bundles.

Since $\Phi$ is also an isomorphism over the non-bundles from Lemma 2.4, $\Phi$ is an isomorphism. $\square$

Remark 2.9. From the identification of $\mathcal{M}(2)$ with $\mathbb{P}_3$, we know that $\mathbb{P}(Z)$ is exactly the secant line of $Q$ in $\mathbb{P}_3$ passing through $Z$.

Remark 2.10. Alexander Kuznetsov pointed out that the moduli space $\mathcal{M}(2; 1, 1, 1)$ of stable sheaves on $\mathbb{P}_3$ of rank 3 with the Chern classes $(c_1, c_2, c_3) = (1, 1, 1)$ is isomorphic to $\mathbb{P}_3$, whose points correspond to the cokernels $E$ of $\mathcal{O}_{\mathbb{P}_3} \to T_{\mathbb{P}_3}(-1)$, where $T_{\mathbb{P}_3}$ is the tangent bundle of $\mathbb{P}_3$. Note that $h^0(\mathbb{P}_3, T_{\mathbb{P}_3}(-1)) = 4$. The restriction map from $\mathcal{M}(2; 1, 1, 1) \to \mathcal{M}(2)$ turns out to be an isomorphism.

3. Jumping Conics

Definition 3.1. Let $E$ be a stable sheaf in $\mathcal{M}(2)$ and $H$ be a hyperplane in $\mathbb{P}_3$. A conic $C_H = Q \cap H$ on $Q$, is called a jumping conic of $E$ if the splitting type of $E|_{C_H}$ is different from the generic splitting type.

Assume that $E$ is locally free. If $H$ is a hyperplane, which does not contain $p_E$, then we can choose a line $l$ through $p_E$, with the unique intersection point with $C_H$, say $q$. Let $Z = l \cap Q$ and then $E$ is fitted into the exact sequence (1), i.e. $E \in l = \mathbb{P}(Z)$. If we tensor the sequence with $\mathcal{O}_{C_H}$, then $E|_{C_H}$ lies in $\text{Ext}^1(\mathcal{O}_{C_H}(1 - q), \mathcal{O}_{C_H}(q)) = 0$, i.e.

$$E|_{C_H} \simeq \mathcal{O}_{C_H}(p) \oplus \mathcal{O}_{C_H}(1 - p).$$

(6)

If $H$ contains $p_E$, let us choose a line $l \subset H$ containing $p_E$. If we let $Z = l \cap Q$ again, we have $E \in l = \mathbb{P}(Z)$. But in this case, $E|_{C_H}$ lies in
Ext^1(\mathcal{O}_{C_H}, \mathcal{O}_{C_H}(1)) = 0, i.e.
(7) \quad E|_{C_H} \simeq \mathcal{O}_{C_H}(1) \oplus \mathcal{O}_{C_H}.

Similarly, when $E$ is not locally free, we can derive the same result. Hence, all jumping conics can be characterized by $h^0(E(-1)|_{C_H}) \neq 0$.

**Proposition 3.2.** For a stable bundle $E \in \overline{M}(2)$, the set of jumping conics $C_H$, form a hyperplane $H_E \subset \mathbb{P}_3$ corresponding to $p_E \in \mathbb{P}_3$. In particular, $\Psi$ can be also defined by sending $E$ to the set of jumping conics of $E$.

4. The Restriction Map

Let $C$ be a non-hyperelliptic curve of genus 4 and it is embedded into $\mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_3$ and there is the unique quadric surface $Q \subset \mathbb{P}_3$ containing $C$. Let $g_3^1$ and $h_3^1$ be the two trigonal line bundles in $\Theta \subset \text{Pic}^3(C)$ such that $g_3^1 \otimes h_3^1 = \mathcal{O}_C(K_C)$. $Q$ is smooth if and only if $|g_3^1| \neq |h_3^1|$. From now on, we assume that $Q$ is smooth. Let $SU_C(2, K_C)$ be the moduli space of semi-stable vector bundles of rank 2 with canonical determinant over $C$. Let $\mathcal{W}$ be the closure of the following set
(8) \quad \{ E \in SU_C(2, K_C) \mid h^0(C, E) \geq r + 1 \}.

Then we have the following inclusions [5] on the Brill-Noether loci,
(9) \quad SU_C(2, K_C) \supset W \supset W^1 \supset W^2 \supset W^3 \supset W^4 = \emptyset,

where $W = W^0$. Many geometric descriptions of these Brill-Noether loci have been investigated in [5].

Since $C$ is a divisor of $Q$ with the divisor class $(3, 3)$, we have the exact sequence
(10) \quad 0 \rightarrow \mathcal{O}_Q(-3, -3) \rightarrow \mathcal{O}_Q \rightarrow \mathcal{O}_C \rightarrow 0.

Twisting the sequence [10] with a stable bundle $E \in \overline{M}(2)$, we obtain that $h^0(C, E|_C) = h^0(Q, E) = 3$.

**Lemma 4.1.** For a stable bundle $E \in \overline{M}(2)$, its restriction to $C$, $E|_C$, is stable and so we have a rational map
$$\Phi : \overline{M}(2) \rightarrow \mathcal{W}^2$$

sending $E$ to $E|_C$.

**Proof.** Since the embedding of $C$ into $\mathbb{P}_3$ is canonical, the restriction of $\wedge^2 E \simeq \mathcal{O}_Q(1, 1)$ is $\mathcal{O}_C(K_C)$, i.e. $\det(E|_C) = \mathcal{O}_C(K_C)$. If we tensor the exact sequence [1] by $\mathcal{O}_C$, we have
$$0 \rightarrow \mathcal{O}_C(D) \rightarrow E|_C \rightarrow \mathcal{O}_C(K_C - D) \rightarrow 0,$$

where $D = Z \cap C$ as a scheme with $l(D) \leq l(Z) = 2$. Suppose that there exists a subbundle $\mathcal{O}_C(D') \subset E|_C$ with $\deg(D') = d' \geq 3$. If the natural composite $f : \mathcal{O}_C(D') \rightarrow \mathcal{O}_C(K_C - D)$ is zero, then $\mathcal{O}_C(D') \subset \mathcal{O}_C(D)$, which is not possible. Thus $f$ is not zero and must be injective so that $d'$ can be at most 6. Since $h^0(E|_C) = 3$ and $h^0(\mathcal{O}_C(K_C - D')) < 2$, so $H^0(\mathcal{O}_C(D'))$
Lemma 4.2. For a general $E$ in $W^2$, the determinant map

$$\lambda_E : \wedge^2 H^0(E) \rightarrow H^0(K_C),$$

is injective.

Proof. As in [lemma 2.6], let us assume that $s_1$ and $s_2$ are two sections of $E$ for which $s_1 \wedge s_2$ is a non-trivial element in $\ker(\lambda_E)$. It would generate a sub-bundle $L \subset E$ with $h^0(L) \geq 2$ and $\deg(L) \leq 2$, contrary to the fact that $C$ is non-hyperelliptic. □

Remark 4.5. Note that $g_3^1$ and $h_3^1$ are isomorphic to $O_Q(1,0)|_C$ and $O_Q(0,1)|_C$. In particular, $E_0|_C = g_3^1 \oplus h_3^1$ and $h^0(E_0) = h^0(E_0|_C) = 4$. As pointed in [5], $g_3^1 \oplus h_3^1$ can be considered as the unique point of $W^3$. 

is not trivial. Now we can assume that $D'$ is effective and the zeros of a section in $H^0(E|_C)$. Note that $H^0(E) \simeq H^0(E|_C)$ and so every section of $E|_C$ comes as the restriction of a section of $E$. Since the zero of a section of $E$ has only 2 points as its support, the degree of $D'$ must be less than 3. Hence, $E|_C$ is stable. □

Remark 4.5. Note that $g_3^1$ and $h_3^1$ are isomorphic to $O_Q(1,0)|_C$ and $O_Q(0,1)|_C$. In particular, $E_0|_C = g_3^1 \oplus h_3^1$ and $h^0(E_0) = h^0(E_0|_C) = 4$. As pointed in [5], $g_3^1 \oplus h_3^1$ can be considered as the unique point of $W^3$. 

Corollary 4.4. The restriction map $\Phi$ is birational.

Remark 4.5. Note that $g_3^1$ and $h_3^1$ are isomorphic to $O_Q(1,0)|_C$ and $O_Q(0,1)|_C$. In particular, $E_0|_C = g_3^1 \oplus h_3^1$ and $h^0(E_0) = h^0(E_0|_C) = 4$. As pointed in [5], $g_3^1 \oplus h_3^1$ can be considered as the unique point of $W^3$. 

Corollary 4.4. The restriction map $\Phi$ is birational.
Remark 4.6. Let $E$ be a non-bundle in $\overline{M}(2)$ and $p$ be its singularity point. If $p \notin C$, then $E|_C \simeq E_0|_C \in \mathcal{W}^3$. If $p \in C$, then $E|_C$ is not torsion-free. In particular, the indeterminacy locus of $\Phi$ is exactly $C \subset \mathbb{P}_3 \simeq \overline{M}(2)$.

Proposition 4.7. The map $\Phi$ is given by the complete linear system $|I_C(3)|$.

Proof. We know that $\Phi$ is an isomorphism on $\mathbb{P}_3 \setminus Q$ and sends $Q \setminus C$ to one point $E_0|_C$. Let $H$ be a general hyperplane in $\mathbb{P}_3$. The intersection of $H$ with $C$ is 6 points on $Q$, say $P_1, \cdots, P_6$ and these points lie on a conic $C_2 = H \cap Q$. The restriction of $\Phi$ to $H$ is not defined on $P_i$'s and maps the other points of $C_2$ to $E_0|_C$.

Let $S_6$ be the blow-up of $H$ at $P_i$'s and then we obtain a morphism $f_H$ from $S_6$ to $\mathcal{W}^2$.

\[
\begin{array}{c}
S_6 \\
\downarrow f_H \\
H \rightarrow \mathcal{W}^2
\end{array}
\]

The proper transform of $C_2$ in $S_6$ is a line $l_H$ and $f_H$ contracts $l_H$ to a point. Since $S_6$ is a cubic surface in $\mathbb{P}_3$, the degree of $f_H$ is 3 and so is the degree of $\Phi$.

Recall that the indeterminacy of $\Phi$ is $C$. Since $h^0(I_C(3)) = 5$ and $\mathcal{W}^2$ is a 3-fold which is not isomorphic to $\mathbb{P}_3$, so $\Phi$ must be given by the complete linear system $|I_C(3)|$. □

Remark 4.8. The image of $\mathbb{P}_3$ via $|I_C(3)|$ is known to be the Donagi-Izadi cubic threefold in $\mathbb{P}_4^* \{1\}$. This is singular with a nodal point $P = E_0|_C$.

Let $\pi_P$ be the projection from $\mathbb{P}_4^*$ at $P$ and then we have the following commutative diagram,

\[
\begin{array}{c}
\overline{M}(2) \xrightarrow{\Phi} \mathcal{W}^2 \\
\downarrow \Psi \quad \downarrow \pi_P \\
\mathbb{P}_3 \xrightarrow{\sim} \mathbb{P}_3^*,
\end{array}
\]

where the map $f_Q$ is defined as follows: Let $H'$ be a hyperplane in $\mathbb{P}_3^*$ and then it pulls back via $\pi_P$ to a hyperplane in $\mathbb{P}_3^*$ containing $P$, and again to $Q$ and a residual hyperplane $H \subset \mathbb{P}_3$ by $\Phi$.

Let $p_E$ be a point in $\mathbb{P}_3$ corresponding to $E \in \overline{M}(2)$ and $H'_E$ be its hyperplane in $\mathbb{P}_3^*$. As above, we can assign a residual hyperplane $H_E \subset \mathbb{P}_3$ and a conic $C_E = H_E \cap Q$ to $E$. Simply, $f_Q$ is a polar map given by

\[
[x_0, \cdots, x_3] \mapsto [\frac{\partial f}{\partial t_0}(x), \cdots, \frac{\partial f}{\partial t_3}(x)],
\]
where $f$ is the homogeneous polynomial of degree 2 defining $Q$. The hyperplane $H_E \subset \mathbb{P}_3$, corresponding to $f_Q(p_E)$, is given by the equation,

\begin{equation}
3 \sum_{i=0}^{3} \frac{\partial f}{\partial t_i}(p_E)t_i = 0,
\end{equation}

and, from the Euler formula, it is clear that $p_E \in Q$ is equivalent to $p_E \in H_E$. Assume that $p_E \notin Q$. Let us recall that $C_E = H_E \cap Q$ is the set of points $p \in Q$ for which $p_E$ is contained in the tangent plane of $Q$ at $p$. In particular, $E$ is fitted into an extension (1) where $Z$ is a point $p$ with multiplicity 2. In other words, there exists a section $s$ of $E$ whose zero is $p$ with multiplicity 2. We can have the same argument for the case when $p_E \in Q$.

**Proposition 4.9.** Let $E$ be a stable sheaf in $\overline{M}(2)$. The set of points $p \in Q$ which is the zero with multiplicity 2 for some section of $E$, forms a conic $C_E$ in $Q$. This defines an isomorphism

$$f_Q \circ \Psi : \overline{M}(2) \to \mathbb{P}^*_3.$$

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