POSITIVE SOLUTION BRANCHES OF TWO-SPECIES
COMPETITION MODEL IN OPEN
ADVECTIVE ENVIRONMENTS

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Abstract. The effect of competition is an important topic in spatial ecology. This paper deals with a general two-species competition system in open advective and inhomogeneous environments. At first, the critical values on the interspecific competition coefficients are established, which determine the stability of semi-trivial steady states. Secondly, by analyzing the nonexistence of coexistence steady states and using the theory of monotone dynamical system, we find that the competitive exclusion principle holds if one of the interspecific competition coefficients is large and the other is in a certain range. Thirdly, in terms of these critical values, the structure and direction of bifurcating branches of positive equilibria arising from two semi-trivial steady states are given by means of the bifurcation theory and stability analysis. Finally, we show that multiple coexistence occurs under certain regimes.

1. Introduction. Competition between species is a fundamental ecological process. In the past few years, various special cases and variants have been qualitatively investigated in the following model

\[
\begin{aligned}
  u_t &= d_1 u_{xx} - q_1 u_x + u(r_1(x) - u - bv), & x \in (0, L), & t > 0, \\
  v_t &= d_2 v_{xx} - q_2 v_x + v(r_2(x) - cu - v), & x \in (0, L), & t > 0, \\
  d_1 u_x(0, t) - q_1 u(0, t) &= d_2 v_x(0, t) - q_2 v(0, t) = 0, & t > 0, \\
  d_1 u_L(L, t) - q_1 u(L, t) &= -\gamma q_1 u(L, t), & t > 0, \\
  d_2 v_L(L, t) - q_2 v(L, t) &= -\gamma q_2 v(L, t), & t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & x \in (0, L), \\
  v(x, 0) &= v_0(x) \geq 0, & x \in (0, L),
\end{aligned}
\]

(1.1)

which can describe the competition between two species, whose population densities at time \( t > 0 \) and location \( x \in [0, L] \) are presented by \( u(x, t) \) and \( v(x, t) \) respectively.

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Here, $d_1, d_2 > 0$ are random diffusion rates of species $u$ and $v$, respectively; $q_1, q_2$ denote the advective rates of two species; $r_1(x) > 0$ and $r_2(x) > 0$ which stand for the local carrying capacity or intrinsic growth rates of species $u$ and $v$, respectively, depend nontrivially on the spatial variable (spatially inhomogeneous environment); $b, c > 0$ measure the interspecific competition abilities, while the intraspecific competition coefficients are normalized by 1. The no-flux boundary conditions of two species at the upstream end $x = 0$ mean that no individuals can move in or out the left-hand boundary; the boundary conditions at the downstream end $x = L$ are flexible including Dirichlet, Neumann and Robin type conditions as special cases, where $\gamma \geq 0$ measures the loss rate of individuals at the downstream end (see [18] for a detailed derivation). Clearly, when $\gamma = 0$, we obtain the no-flux condition again; when $\gamma = 1$, the free-flow condition is obtained, which is also called the Danckwerts condition (see, e.g., [1, 13]) and when $\gamma \to \infty$, we get the hostile condition.

System (1.1) with $q_1 = q_2 = 0$, i.e., two-species competition model in non-advection environments has been studied extensively; see for instance [5, 6, 7, 12] and the references therein. A recently remarkable work is the study of He and Ni [7], where they provided a complete classification on all possible long-time dynamical behaviors under the basic assumption that $\{(b, c) : b, c > 0, bc \leq 1\}$.

However, many species may also take directed movements actively or passively besides random movements in some environments, such as rivers, water columns, the guts and so on. In these situations, the advection terms are considered in (1.1). Generally speaking, the global dynamic behavior of system (1.1) is comparatively complex and remains open now. It is worth mentioning that some special cases of system (1.1) have been investigated in two scenarios: (i) closed advective environments, where individuals cannot pass through the boundary, which are modelled by no-flux conditions at both ends ($\gamma = 0$ in (1.1)); (ii) open advective environments, in which there is always a net loss of individuals at the downstream end ($\gamma \neq 0$ in (1.1)).

In the case of closed advective environments, some works such as [15, 23, 28] are on the homogeneous versions of system (1.1), i.e., $r_1$ and $r_2$ are constants; and some are on the inhomogeneous environments, such as [10, 16, 27, 29]. Particularly, under the assumption that $\frac{dr_1}{dx} = \frac{dr_2}{dx} = k$ and $\{(b, c) : b, c > 0, bc \leq 1\}$, Zhou and Xiao [29] presented a complete classification on all possible long-time dynamical behaviors under the theory of monotone dynamical system.

In the case of open advective environments, (1.1) has been studied by several authors when interspecific competition is symmetric, i.e., $b = c = 1$; see, e.g., [13, 14, 17, 24, 31] for spatially homogeneous case and [16, 26, 30] for spatially inhomogeneous case. In particular, by considering the general boundary conditions at the downstream end and assuming that $r_1(x) = r_2(x) = r(x)$ is decreasing in the spatial variable, Lou et al.[16] explored the joint impact of movement strategy and environmental heterogeneity on the outcome of competition. Cantrell et al.[2] studied the long term behavior of a competitive system where one species uses an ideal free strategy and the other uses a fickian-type diffusion in a spatially inhomogeneous environment. Under the Danckwerts boundary conditions, Wang et al. [25] investigated the dynamic behavior of general model (1.1) with $d_1 = d_2, q_1 = 0$ in homogeneous environments.

The global dynamics of the general two-species competition system (1.1) is far from being completely understood. In this paper, we investigate system (1.1) with
Danckwerts boundary conditions, i.e.,

\[
\begin{align*}
  u_t &= d_1 u_{xx} - q_1 u_x + u(r_1(x) - u - bv), & x \in (0, L), & t > 0, \\
  v_t &= d_2 v_{xx} - q_2 v_x + v(r_2(x) - cu - v), & x \in (0, L), & t > 0, \\
  d_1 u_x(0, t) - q_1 u(0, t) &= u_x(L, t) = 0, & t > 0, \\
  d_2 v_x(0, t) - q_2 v(0, t) &= v_x(L, t) = 0, & t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & v(x, 0) &= v_0(x) \geq 0, & x \in [0, L].
\end{align*}
\]

(1.2)

At the downstream end \( x = L \), it is assumed that water flow will cause a hundred percent loss. That is, two species are in open advective inhomogeneous environments.

To study the dynamics of model (1.2), we mainly investigate the following steady-state system

\[
\begin{align*}
  d_1 u_{xx} - q_1 u_x + u(r_1(x) - u - bv) &= 0, & x \in (0, L), \\
  d_2 v_{xx} - q_2 v_x + v(r_2(x) - cu - v) &= 0, & x \in (0, L), \\
  d_1 u_x(0) - q_1 u(0) &= u_x(L) = 0, \\
  d_2 v_x(0) - q_2 v(0) &= v_x(L) = 0.
\end{align*}
\]

(1.3)

At first, let us introduce some notations. Define \( \lambda_1(d, q, h(x)) \) to be the principal eigenvalue of

\[
\begin{align*}
  d\phi_{xx} - q\phi_x + h(x)\phi &= \lambda\phi, & x \in (0, L), \\
  d\phi_x(0) - q\phi(0) &= \phi_x(L) = 0.
\end{align*}
\]

(1.4)

It is well known that for \( d > 0, L > 0 \) and \( h(x) \in C([0, L]) \), there exists a unique critical value \( q^* \in (0, 2\sqrt{d\max_{x \in [0, L]} h(x)}) \) such that

\[
\begin{align*}
  \lambda_1(d, q, h(x)) &> 0, & 0 \leq q < q^*, \\
  \lambda_1(d, q, h(x)) &= 0, & q = q^*, \\
  \lambda_1(d, q, h(x)) &< 0, & q > q^*.
\end{align*}
\]

(see, e.g., [13] or [14]).

Consider the following problem

\[
\begin{align*}
  dw_{xx} -qw_x + w(r(x) - w) &= 0, & x \in (0, L), \\
  dw_x(0) - qw(0) &= w_x(L) = 0.
\end{align*}
\]

(1.5)

As we know, (1.5) admits a unique positive solution \( w = \theta(d, q, r(x)) \) if and only if \( 0 \leq q < q^* \) (see, e.g., Corollary 4.4 in [13] or Theorem 2.1(b) in [17]). Here \( q^* \) is uniquely determined by \( \lambda_1(d, q^*, r(x)) = 0 \).

Therefore, (1.3) has two semi-trivial solutions \((u^*, 0)\) and \((0, v^*)\) when \( 0 \leq q_1 < q_1^* \) and \( 0 \leq q_2 < q_2^* \), where \( u^* := \theta(d_1, q_1, r_1(x)), v^* := \theta(d_2, q_2, r_2(x)) \), and \( q_1^*, q_2^* \) are uniquely determined by \( \lambda_1(d_1, q_1^*, r_1(x)) = 0 \) and \( \lambda_1(d_2, q_2^*, r_2(x)) = 0 \), respectively.

We are interested in the interplay between the interspecific competition coefficients \( b, c \) on the global dynamics of system (1.2). It follows from similar arguments as in Lemma 5.4 of [3] that system (1.2) induces a strongly monotone dynamical system in the sense that if \((u_1(x, t), v_1(x, t)), (u_2(x, t), v_2(x, t))\) are two solutions of (1.2) with \( u_1(x, 0) \geq u_2(x, 0), v_1(x, 0) \leq v_2(x, 0) \) and \((u_1(x, 0), v_1(x, 0)) \neq (u_2(x, 0), v_2(x, 0))\) on \([0, L]\), then \( u_1(x, t) > u_2(x, t), v_1(x, t) > v_2(x, t)\) for all \( 0 \leq x \leq L \) and \( t > 0 \). We will adopt stability analysis, bifurcation theory and monotone dynamical system theory to study the competition exclusion, positive solution branches and multiple coexistence of system (1.3).
Our first main result shows that the competition exclusion of two species may occur if one of the interspecific competition coefficients, i.e., $b$ or $c$ is large. To state the main results, we define

$$b_0 = \sup_{\varphi \in H^1(0,L), \varphi \neq 0} \frac{-q_1 e^{\frac{2 q_1 L}{\varphi}} \varphi^2(L) - d_1 \int_0^L e^{\frac{2 q_1 x}{\varphi}} \varphi^2 dx + \int_0^L r_1(x) e^{\frac{2 q_1 x}{\varphi}} \varphi^2 dx}{\int_0^L v \varphi e^{\frac{2 q_1 x}{\varphi}} \varphi^2 dx},$$

$$c_0 = \sup_{\psi \in H^1(0,L), \psi \neq 0} \frac{-q_2 e^{\frac{2 q_2 L}{\psi}} \psi^2(L) - d_2 \int_0^L e^{\frac{2 q_2 x}{\psi}} \psi^2 dx + \int_0^L r_2(x) e^{\frac{2 q_2 x}{\psi}} \psi^2 dx}{\int_0^L u \psi e^{\frac{2 q_2 x}{\psi}} \psi^2 dx}.$$  \hspace{1cm} (1.6)

\textbf{Theorem 1.1.} Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$.

(i) There exists $\tilde{b} > b_0$ such that $(0,v^*)$ is globally asymptotically stable when $b > \tilde{b}$ and $0 < c < c_0$;

(ii) there exists $\tilde{c} > c_0$ such that $(u^*,0)$ is globally asymptotically stable when $0 < b < b_0$ and $c > \tilde{c}$.

As we will see in Lemmas 3.1 and 3.2, positive solution bifurcations from semi-trivial solution branches occur when $b = b_0$ and $c = c_0$, respectively. For saving spaces, we mainly analyze the bifurcation from semi-trivial steady state $(0,v^*)$ by taking $b$ as bifurcation parameter. Using the global bifurcation theory, we obtain the following results.

\textbf{Theorem 1.2.} Suppose $c > 0$, $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. Let $B^+$ be a continuum containing the branch of positive solutions $(u,v,b)$ to (1.3) bifurcating from the semi-trivial solution branch $\Gamma_v := \{(0,v,b) : b > 0\}$ at $(0,v^*,b_0)$ and excluding the negative ones. The following assertions hold:

(i) If $c = c_0$, then $B^+$ meets the other semi-trivial solution branch $\Gamma_u := \{(u^*,0,b) : b > 0\}$ at $(u^*,0,b^*)$.

(ii) If $0 < c < c_0$, then for $(u,v,b) \in B^+$, we have $(u,v) \to (u^*,\hat{v}^*)$ as $b \to 0^+$.

(iii) If $c > c_0$, then for $(u,v,b) \in B^+$, we have $v \to 0$ as $b \to \infty$.

Here $b^*$ is given by (4.8) and $\hat{v}^*$ is the unique positive solution of

$$\begin{cases}
d_2 \hat{v}_{xx} - q_2 \hat{v}_x + \hat{v}(r_2(x) - c \hat{v} - \hat{v}) = 0, & x \in (0,L), \\
0 \hat{v}(0) - q_2 \hat{v}(L) = \hat{v}_x(L) = 0.
\end{cases} \hspace{1cm} (1.8)$$

The definition of $B^+$ in Theorem 1.2 is given in Section 4.3. We illustrate the track of this continuum of positive steady states in Figures 1.1–1.3, where $S^+$ accounts for $\{(\|u\|_\infty,b) : (u,v,b) \in B^+\}$ and $c^*$ is given by (4.4) which determines the direction of $B^+$ (also $S^+$) according to the parameter $b$ in a small neighbourhood of $b_0$ (see Remark 4.3).

It follows from Theorem 1.2 that the coexistence phenomena occur.

\textbf{Proposition 1.3.} Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. System (1.3) has at least a positive solution if one of the following three conditions holds:

(i) $c = c_0$ and $\min\{b_0, b^*\} < b < \max\{b_0, b^*\}$;

(ii) $0 < c < c_0$ and $0 < b < b_0$;

(iii) $c > c_0$ and $b > b_0$.

Specially, when $b, c$ are in a certain range, such as in Figure 1.1(i)(iv), Figure 1.2(i) and Figure 1.3(i), the structure of continuum $B^+$ implies that (1.2) admits multiple positive steady states.
Figure 1.1. The schematic diagrams of the curves $S^+ = \{(\|u\|_\infty, b) : (u, v, b) \in B^+\}$ with $B^+$ defined in Section 4.3. Here $b_0 < b^*$ in (i)-(ii), and $b_0 > b^*$ in (iii)-(iv).

Figure 1.2. The schematic diagrams of the curves $S^+ = \{(\|u\|_\infty, b) : (u, v, b) \in B^+\}$ with $B^+$ is defined in Section 4.3.

Figure 1.3. The schematic diagrams of the curves $S^+ = \{(\|u\|_\infty, b) : (u, v, b) \in B^+\}$ with $B^+$ defined in Section 4.3.

Proposition 1.4. Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. System (1.3) admits at least two positive solutions if one of the following conditions holds:

(i) $c = c_0 < c^*$ and $b_0 - \varepsilon < b < b_0 < b^*$ for small $\varepsilon$;
Lemma 2.3. Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. System (1.3) admits at least two positive solutions if one of the following conditions holds:

(i) $b = b_0 < b^*$ and $c_0 - \varepsilon < c < c_0 < c^*$ for small $\varepsilon$;
(ii) $c^* < c < c_0$ and $b_0 < b < b_0 + \varepsilon$ for small $\varepsilon$;
(iii) $c_0 < c < c^*$ and $b_0 - \varepsilon < b < b_0$ for small $\varepsilon$,

where $b^*$ and $c^*$ are defined by (4.8) and (4.4), respectively.

By the symmetry of $b$ and $c$, we can obtain the following multiple coexistence results.

Remark 1.5. Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. System (1.3) admits at least two positive solutions if one of the following conditions holds:

(i) $b = b_0 < b^*$ and $c_0 - \varepsilon < c < c_0 < c^*$ for small $\varepsilon$;
(ii) $b = b_0 > b^*$ and $c^* < c < c_0 + \varepsilon$ for small $\varepsilon$;
(iii) $b^* < b < b_0$ and $c_0 < c < c_0 + \varepsilon$ for small $\varepsilon$;
(iv) $b < b_0 < b^*$ and $c_0 - \varepsilon < c < c_0$ for small $\varepsilon$.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results which are useful in later analysis. The goal of Section 3 is to establish the stability of two semi-trivial solutions and the nonexistence of coexistence solutions of (1.3), and to prove Theorem 1.1 by means of the theory of monotone dynamical system. The structure and stability of bifurcating solutions are investigated in Section 4 and Theorem 1.2 is proven. Some brief discussions and numerical examples are given in Section 5, which implies the existence of multiple steady states numerically.

2. Preliminaries. In this section, we present some well-known lemmas including results about the related linear eigenvalue problem, properties of the steady state to single-species model, and the monotone dynamical system theory.

Lemma 2.1. ([3], [8], [19]) Suppose $d, q$ are constants with $d > 0$, $q \geq 0$, and $h(x) \in C([0, L])$. Then (1.4) admits a principal eigenvalue

$$
\lambda_1(d, q, h(x)) = \sup_{\phi \in H^1(0, L), \phi \neq 0} \frac{\int_0^L e^{-\frac{\lambda}{2} x}(-d \phi'^2 + h(x)\phi^2)dx - q\phi^2(0)}{\int_0^L e^{-\frac{\lambda}{2} x}\phi^2dx},
$$

which corresponds to a positive eigenfunction $\phi_1$. Moreover,

(i) $h_1(x) \geq h_2(x)$ implies $\lambda_1(d, q, h_1(x)) \geq \lambda_1(d, q, h_2(x))$, and the equality holds only if $h_1(x) \equiv h_2(x)$;
(ii) $h_n(x) \to h(x)$ in $L^\infty(0, L)$ implies $\lambda_1(d, q, h_n(x)) \to \lambda_1(d, q, h(x))$.

Lemma 2.2. For $0 \leq q < q^*$, the unique positive solution $\theta = \theta(d, q, r(x))$ of single population model (1.5) has the following properties:

(i) $0 < \theta \leq \bar{r}$ on $[0, L]$, where $\bar{r} = \max_{x \in [0, L]} r(x)$;
(ii) $\lambda_1(d, q, r(x) - 2\theta) < 0 < \lambda_1(d, q, r(x))$.

Proof. By the super-sub solution method and the uniqueness of $\theta$, we can get that $0 < \theta \leq \bar{r}$ on $[0, L]$. Since $r(x) - 2\theta < r(x) - \theta < r(x)$ and $\lambda_1(d, q, r(x) - \theta) = 0$, we get that $\lambda_1(d, q, r(x) - 2\theta) < 0 < \lambda_1(d, q, r(x))$ by Lemma 2.1.

Lemma 2.3. ([9], [21]) For the monotone dynamical system (1.2),

(i) if there is no coexistence state, then one of the semi-trivial equilibria is unstable and the other one is the global attractor;
By using variational formulation, we obtain (1.6). The proof is completed.

Using similar arguments as Corollary 2.10 in [11], we know that the stability of \( \mu \) and asymptotically stable if
\[
0 < b < b_0 \quad \text{and} \quad \lim_{b \to \infty} \lambda_1(d_1, q_1, r_1(x) - b v^*) < 0.
\]

3. Competition exclusion. As mentioned before, system (1.2) can be cast into a strongly monotone dynamical system (see Lemma 5.4 of [3]). In this section, we analyze the stability of two semi-trivial solutions and the nonexistence of the coexistence solutions to (1.3) by taking the interspecific competition coefficients \( b \) and \( c \) as parameters. It is well known that the local stability is determined by linear stability.

We first establish the linear stability of the steady state \((0, v^*)\).

**Lemma 3.1.** Suppose \( 0 \leq q_1 < q_1^* \) and \( 0 \leq q_2 < q_2^* \). Then the semi-trivial steady state \((0, v^*)\) of (1.2) is unstable when \( 0 < b < b_0 \), and it is asymptotically stable when \( b > b_0 \), where \( b_0 \) is defined by (1.6).

**Proof.** Let \( \mu_1(b) \) be the principal eigenvalue of the following problem
\[
\begin{cases}
    d_1 \varphi_{xx} - q_1 \varphi_x + (r_1(x) - b v^*) \varphi = \mu \varphi, & x \in (0, L), \\
    d_1 \varphi_x(0) - q_1 \varphi(0) = \varphi_x(L) = 0.
\end{cases}
\]  
(3.1)

Using similar arguments as Corollary 2.10 in [11], we know that the stability of \((0, v^*)\) is determined by the sign of \( \mu_1(b) \), namely, \((0, v^*)\) is unstable if \( \mu_1(b) > 0 \) and asymptotically stable if \( \mu_1(b) < 0 \).

Notice that \( \mu_1(b) = \lambda_1(d_1, q_1, r_1(x) - b v^*) \) and \( \mu_1(0) = \lambda_1(d_1, q_1, r_1(x)) > 0 \) by Lemma 2.2(ii). On the other hand, \( \lim_{b \to \infty} \mu_1(b) = \lim_{b \to \infty} \lambda_1(d_1, q_1, r_1(x) - b v^*) < \lambda_1(d_1, q_1, 0) \), and \( \lambda_1(d_1, q_1, 0) \leq 0 \) by some direct computation or by Lemma 2.1. Therefore, \( \mu_1(b) < 0 \) for sufficiently large \( b \). Moreover, it is easy to see that \( \mu_1(b) \) is continuous and monotone decreasing with respect to \( b \) by Lemma 2.1. Hence, there exists a unique \( b_0 > 0 \) such that
\[
\mu_1(b) > 0 \quad \text{if} \quad 0 < b < b_0, \quad \mu_1(b) = 0 \quad \text{if} \quad b = b_0, \quad \text{and} \quad \mu_1(b) < 0 \quad \text{if} \quad b > b_0.
\]
That is, \((0, v^*)\) is unstable when \( 0 < b < b_0 \) and asymptotically stable when \( b > b_0 \).

To calculate the value of \( b_0 \), we rewrite (3.1) (with \( b = b_0 \) and \( \mu = 0 \)) as
\[
\begin{cases}
    d_1(\varphi_0)_{xx} - q_1(\varphi_0)x + r_1(x)\varphi_0 = b_0 v^* \varphi_0, & x \in (0, L), \\
    d_1(\varphi_0)(0) - q_1 \varphi_0(0) = (\varphi_0)_x(L) = 0,
\end{cases}
\]  
(3.2)

where \( \varphi_0 > 0 \) is the corresponding eigenfunction of \( \mu_1(b_0) = 0 \), which is uniquely determined by the normalization \( \max_{x \in [0, L]} \varphi_0(x) = 1 \).

Let \( \tilde{\varphi}_0 = \varphi_0 e^{-\frac{q_1}{b_0} x} \). Then \( \tilde{\varphi}_0 \) satisfies
\[
\begin{cases}
    (d_1 e^{\frac{q_1}{b_0} x}(\tilde{\varphi}_0)_x) + r_1(x)e^{\frac{q_1}{b_0} x} \tilde{\varphi}_0 = b_0 v^* e^{\frac{q_1}{b_0} x} \tilde{\varphi}_0, & x \in (0, L), \\
    (\tilde{\varphi}_0)_x(0) = d_1(\tilde{\varphi}_0)(L) + q_1 \tilde{\varphi}_0(L) = 0.
\end{cases}
\]

By using variational formulation, we obtain (1.6). The proof is completed. \( \square \)

Similarly, we can establish the local stability of \((u^*, 0)\) as follows.
Lemma 3.2. Suppose $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. Then the semi-trivial steady state $(u^*, 0)$ of (1.2) is unstable when $0 < c < c_0$, and it is asymptotically stable when $c > c_0$, where $c_0$ is defined by (1.7).

Remark 3.3. In fact, by similar arguments as in Lemma 3.1, we can deduce that $c_0$ is uniquely determined by

$$
\begin{align*}
\frac{d^2(\Psi_0)_{xx} - q_2(\Psi_0)_{x} + r_2(x)\Psi_0 = c_0^* u^* \Psi_0, }{x \in (0, L),}
\frac{d^2(\Psi_0)_{x}(0) - q_2(\Psi_0)(0) = (\Psi_0)_{x}(L) = 0,}
\end{align*}
$$

(3.3)

where $\Psi_0$ is a positive function, which is uniquely determined by the normalization $\max_{x \in [0, L]} \Psi_0(x) = 1$.

In order to analyze the nonexistence of coexistence steady states, we obtain a priori estimate.

Lemma 3.4. Suppose that $(u, v)$ is a nonnegative solution of (1.3) with $u \neq 0$, $v \neq 0$ for $q_1, q_2 \geq 0$. Then $0 < u < \tau_1$, $0 < v < \tau_2$ on $[0, L]$, and $q_1 < q_1^*$, $q_2 < q_2^*$, where $\tau_i = \max_{x \in [0, L]} r_i(x), i = 1, 2$. Moreover, we also have $u < u^*$, $v < v^*$ on $[0, L]$.

Proof. Because $(u, v)$ is a nonnegative solution of (1.3), $u$ satisfies

$$
\begin{align*}
\frac{d^1 u_{xx} - q_1 u_x + u(r_1(x) - u - bv) = 0, }{x \in (0, L),}
\frac{d^1 u_x(0) - q_1 u(0) = u_x(L) = 0.}
\end{align*}
$$

Let $\tilde{u} = \tau_1 - u$. Then $\tilde{u}$ satisfies

$$
\begin{align*}
-(d^1 \tilde{u}_{xx} - q_1 \tilde{u}_x) + (2\tau_1 - r_1(x))\tilde{u} = \tilde{u}^2 + bwv + \tau_1(r_1 - r_1(x)) \geq 0, \quad x \in (0, L),
-d^1 \tilde{u}_x(0) + q_1 \tilde{u}(0) = q_1 \tau_1 \geq 0,
\tilde{u}_x(L) = 0.
\end{align*}
$$

Hence, $\tilde{u} > 0$ on $[0, L]$ by the strong maximum principle and Hopf Lemma, that is, $u < \tau_1$ on $[0, L]$. Similarly, $v < \tau_2$ on $[0, L]$ by the equation for $v$.

Next, we sort out the equation (1.3). Then

$$
\begin{align*}
-(d^1 v_{xx} + q_2 v_x + v(cu + v) = r_2(x)v \geq 0, \quad x \in (0, L),
-d^1 v_x(0) - q_1 v(0) = v_x(L) = 0, 
\end{align*}
$$

By the strong maximum principle and Hopf Lemma, we have $u > 0$ for $x \in [0, L]$. Similarly, $v > 0$ for $x \in [0, L]$. Hence, $0 < u < \tau_1$ and $0 < v < \tau_2$ for $x \in [0, L]$. Since $u > 0$ and $v > 0$, we know that $\lambda_1(d_1, q_1, r_1(x)) > \lambda_1(d_1, q_1, r_1(x) - u - bv) = 0$ by the definition of $q^*_1$, we get that $q_1 < q^*_1$. Similarly, $q_2 < q^*_2$.

Finally, we consider the transformation $\vartheta = u^* - u$ to obtain that $\vartheta$ satisfies

$$
\begin{align*}
\frac{d^1 \vartheta_{xx} - q_1 \vartheta_x + \vartheta(r_1(x) - 2u^*) = -(bwv + \vartheta^2) \leq 0, }{x \in (0, L),}
\frac{d^1 \vartheta_x(0) - q_1 \vartheta(0) = \vartheta_x(L) = 0.}
\end{align*}
$$

Clearly, $\vartheta \geq 0$ on $[0, L]$ based on $\lambda_1(d_1, q_1, r_1(x) - 2u^*) < 0$ from Lemma 2.2(ii). Moreover, $\vartheta > 0$ on $[0, L]$ by the strong maximum principle and Hopf Lemma. That is, $u < u^*$ on $[0, L]$. The conclusion for $v$ can be established similarly. The proof is finished. \qed
Theorem 3.5. There exists a constant $\bar{b} > 0$ such that for all $b > \bar{b}$ and $0 < c \leq c_0$, system (1.2) has no coexistence steady states.

Proof. Arguing indirectly, we suppose that there is a sequence of coexistence steady states of system (1.2) denoted by $\{ (u_n, v_n) \}_{n=1}^{\infty}$ with $u_n > 0$, $v_n > 0$ on $[0, L]$, corresponding to the parameters $0 < c_n \leq c_0$ and $b_n \to \infty$. Without loss of generality, we can assume that $c_n \to \hat{c}$ with $0 \leq \hat{c} \leq c_0$ by choosing a subsequence. Then $(u_n, v_n)$ satisfies

$$
\begin{cases}
  d_1(u_n)_{xx} - q_1(u_n)_x + u_n(r_1(x) - u_n - b_n v_n) = 0, & x \in (0, L), \\
  d_2(v_n)_{xx} - q_2(v_n)_x + v_n(r_2(x) - c_n u_n - v_n) = 0, & x \in (0, L), \\
  d_1(u_n)(0) - q_1 u_n(0) = (u_n)_x(L) = 0, \\
  d_2(v_n)(0) - q_2 v_n(0) = (v_n)_x(L) = 0.
\end{cases}
$$

We integrate the first equation of (3.4) over $(0, L)$ and apply corresponding boundary conditions to obtain

$$
\int_0^L u_n(r_1(x) - u_n - b_n v_n) \, dx - q_1 u_n(L) = 0,
$$

which implies that

$$
\int_0^L u_n v_n \, dx = \frac{1}{b_n} \left[ \int_0^L (r_1(x) u_n - u_n^2) \, dx - q_1 u_n(L) \right].
$$

Noting that $u_n$ is bounded (see Lemma 3.4) and $b_n \to \infty$, we have that $\int_0^L u_n v_n \, dx \to 0$ and $u_n v_n \to 0$ in $L^p(0, L)$ for all $p \geq 1$ as $n \to \infty$. Integrating the equation for $v_n$ of (3.4) over $(0, x)$ and applying $d_2(v_n)_x(0) - q_2 v_n(0) = 0$, we have

$$
(v_n)_x = \frac{1}{d_2} \left[ q_2 v_n - \int_0^x v_n(r_2(x) - c_n u_n - v_n) \, dx \right].
$$

Because $u_n$, $v_n$ and $c_n$ are bounded, we obtain that $(v_n)_x$ is bounded. Thus, from the second equation of (3.4), we see that $\{-d_2(v_n)_{xx}\}$ is bounded in $L^p(0, L)$. By $L^p$ estimates and the Sobolev embedding theorem, we may assume that $v_n \to \hat{v} \geq 0$ in $C^1([0, L])$ by passing to a subsequence if necessary. Letting $n \to \infty$ in the equation of $v_n$, we obtain that $\hat{v}$ satisfies

$$
\begin{cases}
  d_2 \hat{v}_{xx} - q_2 \hat{v}_x + \hat{v}(r_2(x) - \hat{v}) = 0, & x \in (0, L), \\
  d_2 \hat{v}_x(0) - q_2 \hat{v}(0) = \hat{v}_x(L) = 0.
\end{cases}
$$

Thus, $\hat{v} = v^*$ or $\hat{v} = 0$. If $\hat{v} = v^*$, then $b_n \to \infty$ implies that $u_n(r_1(x) - u_n - b_n v_n) < 0$ for $n$ large enough. Clearly, this immediately leads to a contradiction with equality (3.5). Therefore, $\hat{v} = 0$.

Let $V_n = \frac{v_n}{\| v_n \|_\infty}$. Then $(u_n, V_n)$ satisfies

$$
\begin{cases}
  d_1(u_n)_{xx} - q_1(u_n)_x + u_n(r_1(x) - u_n - b_n \| V_n \|_\infty V_n) = 0, & x \in (0, L), \\
  d_2(V_n)_{xx} - q_2(V_n)_x + V_n(r_2(x) - c_n u_n - \| v_n \|_\infty V_n) = 0, & x \in (0, L), \\
  d_1(u_n)(0) - q_1 u_n(0) = (u_n)_x(L) = 0, \\
  d_2(V_n)(0) - q_2 V_n(0) = (V_n)_x(L) = 0.
\end{cases}
$$

Hence, it suffices to consider two cases:

(i) $b_n \| v_n \|_\infty \to \infty$ as $n \to \infty$;
(ii) $b_n \| v_n \|_\infty \to D$ as $n \to \infty$, with $D \geq 0$. 
Suppose case (i) holds. Because \( \|v_n\|_\infty \to 0 \) and \( u_n, V_n, c_n \) are bounded in the second equation of (3.6), by passing to a subsequence if necessary we may assume that \( V_n \to V \geq 0 \) in \( C^1([0, L]) \). It follows from the strong maximum principle that \( \hat{V} > 0 \) on \([0, L]\). Integrating the first equation of (3.6) over \((0, L)\), we have
\[
\int_0^L u_n(r_1(x) - u_n - b_n\|v_n\|_\infty V_n)dx - q_1 u_n(L) = 0.
\]

Since \( \hat{V} > 0 \) and \( b_n\|v_n\|_\infty \to \infty \), one can immediately obtain a contradiction from the above equation for large \( n \).

So case (ii) holds, that is, \( b_n\|v_n\|_\infty \) is bounded. Combining with the fact that \( u_n, V_n \) are bounded, by \( L^p \) estimates and the Sobolev embedding theorem for the first equation of (3.6), we may assume \( u_n \to \hat{u} \) in \( C^1([0, L]) \). Then \( (u_n, V_n) \to (\hat{u}, \hat{V}) \) with \( 0 \leq \hat{u} \leq \tau_1, \hat{V} > 0 \) on \([0, L]\), and
\[
\begin{align*}
\begin{cases}
d_1 \hat{u}_{xx} - q_1 \hat{u}_x + \hat{u}(r_1(x) - \hat{u} - D\hat{V}) = 0, & x \in (0, L), \\
d_2 \hat{V}_{xx} - q_2 \hat{V}_x + \hat{V}(r_2(x) - \hat{c}\hat{u}) = 0, & x \in (0, L), \\
d_1 \hat{u}_x(0) - q_1 \hat{u}(0) = \hat{u}_x(L) = 0, \\
d_2 \hat{V}_x(0) - q_2 \hat{V}(0) = \hat{V}_x(L) = 0.
\end{cases}
\end{align*}
\]

(3.7)

If \( \hat{c} < c_0 \), then \( r_2(x) - \hat{c}\hat{u} > r_2(x) - c_0\hat{u} \geq \hat{r}_2(x) - c_0\hat{u}^* \), by Lemma 3.4, and consequently \( \lambda_1(d_2, q_2, r_2(x) - c_0\hat{u}^*) < \lambda_1(d_2, q_2, r_2(x) - c_0\hat{u}) = 0 \) based on Lemma 2.1 and \( \hat{V} > 0 \) on \([0, L]\). However, it follows from Remark 3.3 that \( \lambda_1(d_2, q_2, r_2(x) - c_0\hat{u}^*) = 0 \). This contradiction implies that \( \hat{c} < c_0 \) can not happen. Then we have \( \hat{c} = c_0 \). In this case, we claim that \( \hat{u} \equiv \hat{u}^* \). In fact, if \( \hat{u} \leq \hat{u}^* \) and \( \hat{u}^* \neq \hat{u} \), then
\[
0 = \lambda_1(d_2, q_2, r_2(x) - \hat{c}\hat{u}) = \lambda_1(d_2, q_2, r_2(x) - c_0\hat{u}) > \lambda_1(d_2, q_2, r_2(x) - c_0\hat{u}^*) = 0,
\]
a contradiction. Hence, \( \hat{c} = c_0 \) and \( \hat{u} \equiv \hat{u}^* \). Replacing \( \hat{u} \) by \( \hat{u}^* \) in the first equation of (3.7), we obtain
\[
\begin{align*}
\begin{cases}
d_1 \hat{u}_{xx}^* - q_1 \hat{u}_x^* + \hat{u}^*(r_1(x) - \hat{u}^*) = D\hat{V}\hat{u}^*, & x \in (0, L), \\
d_1 \hat{u}_x^*(0) - q_1 \hat{u}^*(0) = \hat{u}_x^*(L) = 0.
\end{cases}
\end{align*}
\]

Since \( \hat{u}^* = \theta(d_1, q_1, r_1(x)) > 0 \) and \( \hat{V} > 0 \) on \([0, L]\), it is easy to see that \( D = 0 \) from the definition of \( \theta \). Hence, \( b_n\|v_n\|_\infty \to 0 \).

Denote \( \varphi_n = u_n - \hat{u}^* \). Then \( \varphi_n < 0 \) and \( \varphi_n \) satisfies
\[
\begin{align*}
\begin{cases}
d_1(\varphi_n)_{xx} - q_1(\varphi_n)_x + \varphi_n(r_1(x) - 2\hat{u}^* - \varphi_n) = b_n\|v_n\|_\infty V_n\varphi_n + b_n\|v_n\|_\infty u^*V_n, & x \in (0, L), \\
d_1(\varphi_n)_x(0) - q_1\varphi_n(0) = (\varphi_n)_x(L) = 0.
\end{cases}
\end{align*}
\]

By integrating the above equation over \((0, x)\), similarly as before, we obtain that \( (\varphi_n)_x \) is bounded. Furthermore, \( \{-d_1(\varphi_n)_{xx}\} \) is bounded in \( L^p(0, L) \). By \( L^p \) estimates and the Sobolev embedding theorem, we may assume the existence of a subsequence (if necessary), which satisfies \( \varphi_n \to 0 \) in \( C^1([0, L]) \), and there exists \( C_1 > 0 \) such that
\[
\|\varphi_n\|_{2,p} \leq C_1(b_n\|v_n\|_\infty\|V_n\varphi_n\|_p + b_n\|v_n\|_\infty\|u^*V_n\|_p).
\]
Easily, we conclude that \( \xi_n = \frac{\tilde{\varphi}_n}{\|\varphi_n\|_\infty} \) is uniformly bounded. Now, we write system (3.6) in terms of \( \xi_n \) and \( V_n \) as

\[
\begin{align*}
  d_1(\xi_n)_{xx} - q_1(\xi_n)x + \xi_n(r_1(x) - 2u^* - \varphi_n) &= f_1, & x \in (0, L), \\
  d_2(V_n)_{xx} - q_2(V_n)x + V_n(r_2(x) - c_0u^*) &= f_2, & x \in (0, L), \\
  d_1(\xi_n)_x(0) - q_1\xi_n(0) &= (\xi_n)_x(L) = 0, \\
  d_2(V_n)_x(0) - q_2V_n(0) &= (V_n)_x(L) = 0,
\end{align*}
\]

(3.8)

where

\[
\begin{align*}
  f_1 &= \varphi_nV_n + u^*V_n, \\
  f_2 &= (c_n - c_0)u^*V_n + b_n\|\varphi_n\|_\infty V_n + \|\varphi_n\|_\infty V_n^2.
\end{align*}
\]

Since \( \varphi_n, \xi_n, V_n, u^* \) are bounded, by \( L^p \) estimates and the Sobolev embedding theorem for the first equation of (3.8), we may assume the existence of a subsequence (if necessary), which satisfies \( \xi_n \to \xi \) in \( C^1([0, L]) \). Passing to the limit for the equation of \( \xi_n \) of (3.8), we have

\[
\begin{align*}
  d_1\hat{\xi}_{xx} - q_1\hat{\xi}_x + \hat{\xi}(r_1(x) - 2u^*) &= u^*\hat{V}, & x \in (0, L), \\
  d_1\hat{\xi}_x(0) - q_1\hat{\xi}(0) &= \hat{\xi}(L) = 0.
\end{align*}
\]

Due to the fact that \( \lambda_1(d_1, q_1, r_1(x) - 2u^*) < 0 \) by Lemma 2.2(ii) and \( u^*, \hat{V} > 0 \) on \([0, L]\), we obtain \( \hat{\xi} < 0 \) by the maximum principle, that is \( \xi_n < 0 \) for \( n \) large enough.

Multiplying the second equation of (3.8) by \( e^{-\frac{\xi_n}{2}\hat{V}} \) and the second equation of (3.7) (in the case of \( \hat{\varphi} = c_0 \) and \( \hat{u} \equiv u^* \)) by \( e^{-\frac{\xi_n}{2}V_n} \), integrating over \((0, L)\) and subtracting, we have

\[
(c_n-c_0)\int_0^Lu^*V_ne^{-\frac{\xi_n}{2}\hat{V}}dx = -\|\varphi_n\|_\infty(b_n\int_0^L e^{-\frac{\xi_n}{2}V_n}\hat{V}dx + \int_0^L e^{-\frac{\xi_n}{2}V_n^2}\hat{V}dx).
\]

In this equality, the left-hand side is nonpositive since \( 0 < c_n \leq c_0 \) and for \( n \) large enough the right-hand side is strictly positive since \( \xi_n < 0 \), a contradiction. Hence, system (1.2) has no coexistence steady states when \( 0 < c \leq c_0 \) and \( b \to \infty \). We complete the proof.

By using similar arguments as Theorem 3.5, we have the following nonexistence result.

**Theorem 3.6.** There exists a constant \( \bar{c} > 0 \) such that for all \( c > \bar{c} \) and \( 0 < b \leq b_0 \), system (1.2) has no coexistence steady states.

**Proof of Theorem 1.1.** Since the proofs are similar, we only prove the conclusion (i). It follows from Lemmas 3.1 and 3.2 that \( (0, v^*) \) is stable when \( b > \bar{b} > b_0 \) and \( (u^*, 0) \) is unstable when \( 0 < c < c_0 \). Combining with the nonexistence result in Theorem 3.5 and the monotone dynamical system theorem (see Lemma 2.3), we get that \( (0, v^*) \) is globally asymptotically stable when \( b > \bar{b} > b_0 \) and \( 0 < c < c_0 \).

4. **Positive solution branches of system (1.3).** As mentioned before, for \( 0 \leq q_1 < q_1^* \) and \( 0 \leq q_2 < q_2^* \) fixed, (1.2) has two semi-trivial steady states \((u^*, 0)\) and \((0, v^*)\), and their stabilities may change when the parameters change (e.g., \( b \) crossing \( b_0 \) for \((0, v^*)\) and \( c \) crossing \( c_0 \) for \((u^*, 0))\). In this section, we analyze the structure and the stability of bifurcating solutions from two semi-trivial steady states by taking \( b \) or \( c \) as the bifurcation parameter.
4.1. Local bifurcating solutions from \((0, v^*)\). We take \(b\) as the bifurcation parameter to construct a positive solution branch that bifurcates from the semi-trivial solution \((0, v^*)\). By Lemma 3.1, bifurcation of positive solutions from \((0, v^*)\) can only occur when \(b = b_0\).

Let
\[
X_i = \{u \in W^{2,p}(0, L) : d_i u_x(0) - q_i u(0) = u_x(L) = 0\}, i = 1, 2,
\]
\[
X = X_1 \times X_2,
\]
\[
Y = L^p(0, L) \times L^p(0, L),
\]
where \(p > 1\). Then \(X \hookrightarrow C^1([0, L]) \times C^1([0, L])\). Define \(T : X \times \mathbb{R}^+ \to Y\) by
\[
T(u, v, b) = \begin{pmatrix}
d_1 u_{xx} - q_1 u_x + u(r_1(x) - u - b v) \\
d_2 v_{xx} - q_2 v_x + v(r_2(x) - c u - v)
\end{pmatrix}.
\]
Then \(T(u, v, b)\) is a continuously differential mapping and the positive solutions of \(T(u, v, b) = 0\) correspond to the positive solutions of (1.3).

**Theorem 4.1.** Suppose \(c > 0\), \(0 \leq q_1 < q_1^*\) and \(0 \leq q_2 < q_2^*\). Then \((0, v^*, b_0)\) is a bifurcation point of \(T(u, v, b) = 0\) with respect to \{\((0, v^*, b) : b > b_0\}\). There is a smooth curve of nonconstant solutions \{\((u(\epsilon), v(\epsilon), b(\epsilon)) : \epsilon \in (-\epsilon_0, \epsilon_0)\}\} to \(T(u, v, b) = 0\) for some \(\epsilon_0 > 0\) small enough, satisfying \(u(\epsilon) = \epsilon(\varphi_0 + \tilde{u}(\epsilon)), v(\epsilon) = v^* + \epsilon(\psi_0 + \tilde{v}(\epsilon))\), \(b(\epsilon) = b_0 + \tilde{b}(\epsilon)\), and \(\tilde{u}(0) = \tilde{v}(0) = 0\), where \(\varphi_0 > 0\) and \(\psi_0 < 0\) satisfies
\[
\begin{aligned}
d_2(\psi_0)_{xx} - q_2(\psi_0)_x + (r_2(x) - 2v^*)\psi_0 &= cv^*\varphi_0, \quad x \in (0, L), \\
d_2(\psi_0)_x(0) - q_2\psi_0(0) &= (\psi_0)_x(L) = 0.
\end{aligned}
\]

**Proof.** The Fréchet derivative of \(T(u, v, b_0)\) with respect to \((u, v)\) at \((0, v^*)\) is denoted by \(D_{(u,v)}T(0, v^*, b_0)\). Then
\[
D_{(u,v)}T(0, v^*, b_0)(\varphi, \psi)^T = \begin{pmatrix}
d_1\varphi_{xx} - q_1\varphi_x + (r_1(x) - b_0 v^*)\varphi \\
d_2\psi_{xx} - q_2\psi_x + (r_2(x) - 2v^*)\psi - cv^*\varphi
\end{pmatrix}.
\]
Clearly, \(D_{(u,v)}T(0, v^*, b_0)\) is a Fredholm operator. In order to apply the standard bifurcation theorem from a simple eigenvalue in [4] and [20], we first calculate the null space of \(D_{(u,v)}T(0, v^*, b_0)\). Let \(D_{(u,v)}T(0, v^*, b_0)(\varphi, \psi)^T = 0\) with \((\varphi, \psi) \neq (0, 0)\). Then
\[
\begin{aligned}
d_1\varphi_{xx} - q_1\varphi_x + (r_1(x) - b_0 v^*)\varphi &= 0, \quad x \in (0, L), \\
d_2\psi_{xx} - q_2\psi_x + (r_2(x) - 2v^*)\psi - cv^*\varphi &= 0, \quad x \in (0, L), \\
d_1\varphi_x(0) - q_1\varphi(0) &= \varphi_x(L) = 0, \\
d_2\psi_x(0) - q_2\psi(0) &= \psi_x(L) = 0.
\end{aligned}
\]
If \(\varphi = 0\), then \(\psi\) satisfies
\[
\begin{aligned}
d_2\psi_{xx} - q_2\psi_x + (r_2(x) - 2v^*)\psi &= 0, \quad x \in (0, L), \\
d_2\psi_x(0) - q_2\psi(0) &= \psi_x(L) = 0.
\end{aligned}
\]
Since \(\lambda_1(d_2, q_2, r_2(x) - 2v^*) < 0\) by Lemma 2.2(ii), we get that \(\psi = 0\), that is, \((\varphi, \psi) \equiv (0, 0)\), a contradiction. Hence, \(\varphi \neq 0\), which implies that \(\varphi\) is the corresponding principal eigenfunction of \(\lambda_1(d_1, q_1, r_1(x) - b_0 v^*) = 0\) and satisfies (3.2). Thus, we can take \(\varphi = \varphi_0 > 0\). Then \(\psi\) satisfies
\[
\begin{aligned}
d_2\psi_{xx} - q_2\psi_x + (r_2(x) - 2v^*)\psi &= cv^*\varphi_0, \quad x \in (0, L), \\
d_2\psi_x(0) - q_2\psi(0) &= \psi_x(L) = 0.
\end{aligned}
\]
We can derive \( \psi = \psi_0 = (d_2 \frac{d^2}{dx^2} - q_2 \frac{d}{dx} + r_2(x) - 2v^*)^{-1}(cv^*\varphi_0) < 0 \) based on Lemma 2.2(ii), that is, \( \psi_0 \) satisfies (4.1). Therefore, the kernel \( \ker(D_{(u,v)}T(0,v^*,b_0)) = \text{span}\{\langle \varphi_0, \psi_0 \rangle \} \) and

\[
\dim \ker(D_{(u,v)}T(0,v^*,b_0)) = 1.
\]

Next, we determine the range of the operator \( D_{(u,v)}T(0,v^*,b_0) \) which is denoted by \( \ker(D_{(u,v)}T(0,v^*,b_0)) \). Suppose \( (f,g)^T \in \ker(D_{(u,v)}T(0,v^*,b_0)) \). Then there exists \( (\varphi, \psi) \in X \) such that \( D_{(u,v)}T(0,v^*,b_0)(\varphi, \psi)^T = (f,g)^T \). Direct computation leads to

\[
\begin{cases}
  d_1 \varphi_{xx} - q_1 \varphi_x + (r_1(x) - b_0 v^*) \varphi = f, & x \in (0, L), \\
  d_2 \psi_{xx} - q_2 \psi_x + (r_2(x) - 2v^*) \psi - cv^* \varphi = g, & x \in (0, L), \\
  \varphi_x(0) - q_1 \varphi(0) = \varphi_x(L) = 0, \\
  \psi_x(0) - q_2 \psi(0) = \psi_x(L) = 0.
\end{cases}
\] (4.2)

Multiplying the first equation of (4.2) by \( e^{-\frac{q_1}{4}x} \) and the equation of (3.2) by \( e^{-\frac{q_2}{4}x} \phi \), integrating over \((0, L)\) and subtracting, we obtain

\[
\int_0^L f e^{-\frac{q_1}{4}x} \varphi_0 dx = 0.
\]

Hence, the range of \( D_{(u,v)}T(0,v^*,b_0) \) is

\[
\ker(D_{(u,v)}T(0,v^*,b_0)) = \{(f,g)^T \in Y^T : \int_0^L f e^{-\frac{q_1}{4}x} \varphi_0 dx = 0\},
\]

and

\[
\text{codim} \ker(D_{(u,v)}T(0,v^*,b_0)) = 1.
\]

At last,

\[
D_{b(u,v)}T(0,v^*,b_0)(\varphi_0, \psi_0)^T = D_{(u,v)}T(0,v^*,b_0)(\varphi_0, \psi_0)^T |_{b=b_0}
\]

\[
= \begin{pmatrix} -v^* \varphi_0 \\ 0 \end{pmatrix} \notin \ker(D_{(u,v)}T(0,v^*,b_0)).
\]

Let

\[
Z = \{(f,g) \in X : \int_0^L f e^{-\frac{q_1}{4}x} \varphi_0 dx = 0\}.
\]

It is easy to see that \( Z \bigoplus \text{span}\{\langle \varphi_0, \psi_0 \rangle \} = X \). By the application of the standard bifurcation theorem from a simple eigenvalue, there exists a \( \epsilon_0 > 0 \) and \( C^1 \) curve \( (u(\epsilon), v(\epsilon), b(\epsilon)) : (-\epsilon_0, \epsilon_0) \to Z \times \mathbb{R}^+ \) such that \( u(0) = 0, v(0) = v^*, b(0) = b_0 \) and

\[
(u(\epsilon), v(\epsilon), b(\epsilon)) = (c(\varphi_0 + \tilde{u}(\epsilon)), v^* + c(\psi_0 + \tilde{v}(\epsilon)), b_0 + \tilde{b}(\epsilon)),
\]

where \( \tilde{u}(0) = \tilde{v}(0) = \tilde{b}(0) = 0 \), which concludes the proof.

The bifurcating branch \( \Gamma := \{(u(\epsilon), v(\epsilon), b(\epsilon)) : 0 < \epsilon < \epsilon_0\} \) is exactly the positive solution branch of the steady-state system (1.3). Next, we need to discuss the stability of bifurcating solutions and the direction of the bifurcating branch.

Let \( \zeta = \frac{v_0}{c} \). Then \( \zeta < 0 \) and

\[
\begin{cases}
  d_2 \zeta_{xx} - q_2 \zeta_x + (r_2(x) - 2v^*)\zeta = v^* \varphi_0, & x \in (0, L), \\
  d_2 \zeta_x(0) - q_2 \zeta(0) = \zeta_x(L) = 0.
\end{cases}
\] (4.3)
Proposition 4.2. Set
\[ c^* = -\frac{\int_0^L e^{-\frac{2\pi x}{\lambda^*}} \phi_3^2 dx}{b_0 \int_0^L e^{-\frac{2\pi x}{\lambda^*}} \phi_0^2 \zeta^2 dx}, \]  
(4.4)
where \( \phi_0 \) and \( \zeta \) are defined in (3.2) and (4.3), respectively. The bifurcating solutions given by Theorem 4.1 are unstable for \( c > c^* \), and locally asymptotically stable for \( c < c^* \).

Proof. Substituting
\[ (u(\epsilon), v(\epsilon), b(\epsilon)) = (\epsilon(\phi_0 + \bar{u}(\epsilon)), v^* + \epsilon(\psi_0 + \bar{v}(\epsilon)), b_0 + \bar{b}(\epsilon)) \]
into the first equation of (1.3) and dividing by \( \epsilon \), we obtain
\[ d_1(\phi_0 + \bar{u}(\epsilon))_{xx} - q_1(\phi_0 + \bar{u}(\epsilon))_x + (\phi_0 + \bar{u}(\epsilon))[r_1(x) - \epsilon(\phi_0 + \bar{u}(\epsilon)) - (b_0 + \bar{b}(\epsilon))(v^* + \epsilon(\psi_0 + \bar{v}(\epsilon)))] = 0. \]  
(4.5)
For simplicity of notation, we denote \( \frac{\partial \bar{u}}{\partial \epsilon} = \dot{\bar{u}} \), etc. Differentiating (4.5) with respect to \( \epsilon \) and setting \( \epsilon = 0 \), we get
\[ d_1(\dot{\bar{u}}(0))_{xx} - q_1(\dot{\bar{u}}(0))_x + \dot{\bar{u}}(0)(r_1(x) - b_0 v^*) + \varphi_0(-\varphi_0 - \dot{\bar{b}}(0)v^* - b_0 \psi_0) = 0, \]
since \( \dot{\bar{u}}(0) = \dot{\bar{v}}(0) = \bar{b}(0) = 0 \). Replacing \( \psi_0 \) by \( c\zeta \), multiplying this equation by \( e^{-\frac{2\pi x}{\lambda^*}} \varphi_0 \) and integrating over \((0, L)\), we have
\[ \dot{\bar{b}}(0) \int_0^L e^{-\frac{2\pi x}{\lambda^*}} \varphi_0^2 v^* dx = -cb_0 \int_0^L e^{-\frac{2\pi x}{\lambda^*}} \varphi_0^2 \zeta^2 dx - \int_0^L e^{-\frac{2\pi x}{\lambda^*}} \varphi_0^2 dx \]
\[ = (-b_0 \int_0^L e^{-\frac{2\pi x}{\lambda^*}} \varphi_0^2 \zeta^2 dx)(c - c^*). \]
Hence, \( \dot{\bar{b}}(0) > 0 \) when \( c > c^* \), and \( \dot{\bar{b}}(0) < 0 \) when \( c < c^* \).

We have already known that 0 is an \( i \)-simple eigenvalue (see [22]) of \( D_{(u,v)}T(0, v^*, b_0) \). Then by Lemma 13.7 in [22], there exist functions
\[ b \rightarrow (\gamma(b), (z_1(b), z_2(b))), \]
\[ \epsilon \rightarrow (\tau(\epsilon), (y_1(\epsilon), y_2(\epsilon))) \]
defined on neighborhoods of \( b_0 \) and 0, respectively, into \( \mathbb{R} \times X \), such that
\[ (\gamma(b_0), (z_1(b_0), z_2(b_0))) = (0, (\varphi_0, \psi_0)) = (\tau(0), (y_1(0), y_2(0))), \]
\[ (z_1(b), z_2(b)) - (\varphi_0, \psi_0) \in Z, (y_1(\epsilon), y_2(\epsilon)) - (\varphi_0, \psi_0) \in Z, \]
and on these neighborhoods,
\[ D_{(u,v)}T(0, v^*, b)(z_1(b), z_2(b))^T = \gamma(b)(z_1(b), z_2(b))^T, \]
\[ D_{(u,v)}T(u(\epsilon), v(\epsilon), b(\epsilon))(y_1(\epsilon), y_2(\epsilon))^T = \tau(\epsilon)(y_1(\epsilon), y_2(\epsilon))^T. \]
Namely, the sign of \( \tau(\epsilon) \) will determine whether the bifurcating solutions are stable or unstable. Moreover, we know
\[ \lim_{\epsilon \rightarrow 0} \frac{\dot{\bar{b}}(0)\gamma'(b_0)}{\tau(\epsilon)} = \lim_{\epsilon \rightarrow 0} \frac{\dot{\bar{b}}(0)\gamma'(b_0)}{\tau(\epsilon)} = -1. \]
Hence, \( \text{sign}(\tau(\epsilon)) = \text{sign}(-\dot{\bar{b}}(0)\gamma'(b_0)). \)

Now, we need to show that 0 is the eigenvalue of the operator \( D_{(u,v)}T(0, v^*, b_0) \) with the largest real part. To this end, suppose that \( \lambda_0 \) is an eigenvalue of the
operator $D_{(u,v)}T(0,v^*, b_0)$ and $(\varphi, \psi)$ is the corresponding eigenfunction. Then we have $D_{(u,v)}T(0,v^*, b_0)(\varphi, \psi)^T = \lambda_0(\varphi, \psi)^T$, that is,

\[
\begin{cases}
    d_1\varphi_{xx} - q_1\varphi_x + (r_1(x) - b_0v^*)\varphi = \lambda_0\varphi, & x \in (0, L), \\
    d_2\psi_{xx} - q_2\psi_x + (r_2(x) - 2v^*)\psi - cv^*\varphi = \lambda_0\psi, & x \in (0, L), \\
    d_1\varphi_x(0) - q_1\varphi(0) = \varphi_x(L) = 0, \\
    d_2\psi_x(0) - q_2\psi(0) = \psi_x(L) = 0.
\end{cases}
\]  

(4.6)

It is easy to see that the eigenvalues of (4.6) consist of the eigenvalues of the following two operators:

\[\mathcal{L}_1 = d_1\frac{d^2}{dx^2} - q_1\frac{d}{dx} + r_1(x) - b_0v^* \quad \text{and} \quad \mathcal{L}_2 = d_2\frac{d^2}{dx^2} - q_2\frac{d}{dx} + r_2(x) - 2v^* .\]

Hence, $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \leq \max\{\lambda_1(\mathcal{L}_1), \lambda_1(\mathcal{L}_2)\}$. Note that $\lambda_1(\mathcal{L}_1) = \lambda_1(d_1, q_1, r_1(x) - b_0v^*) = 0$ from (3.2) and $\lambda_1(\mathcal{L}_2) = \lambda_1(d_2, q_2, r_2(x) - 2v^*) < 0$ by Lemma 2.2(ii).

One can conclude that $0$ is the largest eigenvalue of the operator $D_{(u,v)}T(0,v^*, b_0)$.

Due to $D_{(u,v)}T(0,v^*, b)(z_1(b), z_2(b))^T = \gamma(b)(z_1(b), z_2(b))^T$, we have

\[
\begin{cases}
    d_1(z_1)_{xx} - q_1(z_1)_{x} + (r_1(x) - bv^*)z_1 = \gamma(b)z_1, & x \in (0, L), \\
    d_2(z_2)_{xx} - q_2(z_2)_{x} + (r_2(x) - 2v^*)z_2 - cv^*z_1 = \gamma(b)z_2, & x \in (0, L), \\
    d_1(z_1)(0) - q_1z_1(0) = (z_1)_x(L) = 0, \\
    d_2(z_2)(0) - q_2z_2(0) = (z_2)_x(L) = 0.
\end{cases}
\]

Noting that $z_1(b) \neq 0$ when $b$ is close to $b_0$, we conclude that

\[\gamma(b) = \lambda_1(d_1, q_1, r_1(x) - bv^*)\]

on the neighborhood of $b_0$. In view of Lemma 2.1, one can deduce that $\gamma'(b_0) < 0$. Hence, if $c > c^*$ and $\epsilon > 0$, then $\tau(\epsilon) > 0$ which implies that the bifurcating solutions are unstable; if $c < c^*$ and $\epsilon > 0$, then $\tau(\epsilon) < 0$ and the bifurcating solutions are stable. This completes the proof.

\[\Box\]

**Remark 4.3.** From the proof of Proposition 4.2, we can deduce that, for $(u, v, b) \in \Gamma$,

\[
\frac{du}{db} \bigg|_{c=0} = \frac{d}{d\epsilon} \frac{db}{d\epsilon} \bigg|_{\epsilon=0} = \frac{\varphi_0}{\tilde{b}(0)} > 0
\]

when $c > c^*$, which means that the bifurcating solution $\Gamma$ from $(0, v^*, b_0)$ upwards in $b$ to $\infty$ in the neighbourhood of $b_0$ if $c > c^*$.

Similarly, when $c < c^*$, we get $\frac{du}{db} \bigg|_{c=0} < 0$, which means that the bifurcating solution $\Gamma$ from $(0, v^*, b_0)$ downwards in $b$ to $0^+$ in the neighbourhood of $b_0$.

### 4.2. Local bifurcating solutions from $(u^*, 0)$.

We take $c$ as the bifurcation parameter to construct a positive solution branch that bifurcates from the semi-trivial solution $(u^*, 0)$. By Lemma 3.2, bifurcation of positive steady states from $(u^*, 0)$ can only occur when $c = c_0$. We define $X$ and $Y$ as before, and the mapping $F : X \times \mathbb{R}^+ \to Y$ as

\[F(u, v, c) = \left( d_1u_{xx} - q_1u_x + u(r_1(x) - u - bv) \\ d_2v_{xx} - q_2v_x + v(r_2(x) - cu - v) \right), \]

Then $F(u, v, c)$ is a continuously differential mapping and the positive solutions of $F(u, v, c) = 0$ correspond to the positive solutions of (1.3).

By similar arguments as subsection 4.1, we have following conclusions.
Theorem 4.4. Suppose $b > 0$, $0 \leq q_1 < q_1^*$ and $0 \leq q_2 < q_2^*$. Then $(u^*, 0, c_0)$ is a bifurcation point of $F(u, v, c) = 0$ with respect to $\{u^*, 0, c : c > 0\}$. There is a smooth curve of nonconstant solutions $\{(u(s), v(s), c(s)) : s \in (-s_0, s_0)\}$ to $F(u, v, c) = 0$ for some $s_0 > 0$ small enough, satisfying $u(s) = u^* + s(\Phi_0 + \tilde{u}(s))$, $v(s) = s(\Psi_0 + \tilde{v}(s))$, $c(s) = c_0 + c(s)$, and $\tilde{u}(0) = \tilde{v}(0) = 0$, where $\Psi > 0$ satisfies $(3.3)$ and $\Phi_0 < 0$ satisfies
\[
\begin{align*}
&\left\{d_3(\Phi_0)_{xx} - q_1(\Phi_0)_x + (r_1(x) - 2u^*)\Phi_0 = bu^*\Phi_0, \quad x \in (0, L), \right. \\
&\left.\left\{d_3(\Phi_0)_x(0) - q_1\Phi_0(0) = (\Phi_0)_x(L) = 0. \right.
\end{align*}
\]

Let $\eta = \frac{\Phi_0}{b}$. Then $\eta < 0$ and satisfies
\[
\begin{align*}
&\left\{d_3\eta_{xx} - q_1\eta_x + (r_1(x) - 2u^*)\eta = u^*\Psi_0, \quad x \in (0, L), \right. \\
&\left.\left\{d_1\eta_x(0) - q_1\eta(0) = \eta_x(L) = 0. \right.
\end{align*}
\]
(4.7)

Proposition 4.5. Set
\[
b^* = -\frac{\int_0^L e^{-\frac{q_2}{2}x}\Psi_0^2dx}{c_0\int_0^L e^{-\frac{q_2}{2}x}\Psi_0^2dx},
\]
where $\Psi_0$ and $\eta$ are defined in $(3.3)$ and $(4.7)$, respectively. The bifurcating solutions given by Theorem 4.4 are unstable for $b > b^*$, and locally asymptotically stable for $b < b^*$.

Remark 4.6. The bifurcating solution from $(u^*, 0, c_0)$ upwards in $c$ to $\infty$ in the neighbourhood of $c_0$ when $b > b^*$, and downwards in $c$ to $0^+$ in the neighbourhood of $c_0$ when $b < b^*$.

4.3. Global bifurcation. Now, we extend the local bifurcating branch $\Gamma$ to the global one by the global bifurcation results for Fredholm operators (see Theorem 4.3–4.4 in [20]).

Note that $\Gamma$ is the positive solution branch and $D_{(u, v)}T(u, v, b)$ is a Fredholm operator. First, by Theorem 4.3 in [20], we can obtain a connected component $\mathcal{B}$ of the set $\{(u, v, b) \in X \times \mathbb{R}^+ : T(u, v, b) = 0, (u, v) \neq (0, 0)\}$. Moreover, either $\mathcal{B}$ is not compact in $X \times \mathbb{R}^+$ or $\mathcal{B}$ contains a point $(0, v^*, b)$ with $b \neq b_0$. Then $\Gamma \subset \mathcal{B}$. Let
\[
P = \{(u, v) \in C^1([0, L]) \times C^1([0, L]) : u > 0, v > 0, x \in [0, L]\}.
\]
Then $\mathcal{B} \cap (P \times \mathbb{R}^+) \neq \emptyset$.

Let $\mathcal{B}^* = \mathcal{B} \cap (P \times \mathbb{R}^+)$. Then $\mathcal{B}^*$ consists of the local positive solution branch $\Gamma$ near the bifurcation point $(0, v^*, b_0)$. That is, $\mathcal{B}^* \subset P \times \mathbb{R}^+$ in a small neighborhood of $(0, v^*, b_0)$.

Let $\mathcal{B}^+$ be the connected component of $\mathcal{B} \setminus \{(u(\epsilon), v(\epsilon), b(\epsilon)) : -\epsilon_0 < \epsilon < 0\}$. Then $\mathcal{B}^* \subset \mathcal{B}^+$. It follows from Theorem 4.4 in [20] that $\mathcal{B}^+$ satisfies one of the following alternatives:

(i) it is not compact in $X \times \mathbb{R}^+$;
(ii) it contains a point $(0, v^*, b)$ with $b \neq b_0$;
(iii) it contains a point $(u, v^* + v, b)$ where $(u, v) \neq (0, 0)$ and $(u, v) \in Z$.

Lemma 4.7. For all $b > 0$, $(0, 0, b) \notin \mathcal{B}^+$.

Proof. We argue by contradiction. Assume that there exists a $(0, 0, b) \in \mathcal{B}^+$. Then there exists a sequence $\{(u_n, v_n, b_n)\} \subset \mathcal{B}^+$ which converges to $(0, 0, b)$ as $n \to \infty$. 


We have that $u_n$ satisfies
\[
\begin{aligned}
&d_1(u_n)_{xx} - q_1(u_n)_{x} + u_n(r_1(x) - u_n - b_n v_n) = 0, \quad x \in (0, L), \\
&d_1(u_n)(0) - q_1 u_n(0) = (u_n)_x(L) = 0.
\end{aligned}
\]
It immediately implies
\[
0 = \lambda_1(d_1, q_1, r_1(x) - u_n - b_n v_n) \to \lambda_1(d_1, q_1, r_1(x)), \quad \text{as } n \to \infty.
\]
However, $\lambda_1(d_1, q_1, r_1(x)) > 0$ by Lemma 2.2(ii), a contradiction. This contradiction finishes the proof. \hfill \Box

**Lemma 4.8.** Suppose $(u, v, b) \in \mathcal{B}^+$ and $\min_{x \in [0, L]} u = 0$. Then $(u, v) = (0, v^*)$.

**Proof.** The function $u(x)$ satisfies
\[
\begin{aligned}
&d_1 u_{xx} - q_1 u_x + u(r_1(x) - u - bv) = 0, \quad x \in (0, L), \\
&d_1 u_x(0) - q_1 u(0) = u_x(L) = 0.
\end{aligned}
\]
Since $\min_{x \in [0, L]} u = 0$, we have $u \equiv 0$ on $[0, L]$ by the strong maximum principle.

Then, the function $v(x)$ satisfies
\[
\begin{aligned}
&d_2 v_{xx} - q_2 v_x + v(r_2(x) - v) = 0, \quad x \in (0, L), \\
&d_2 v_x(0) - q_2 v(0) = v_x(L) = 0.
\end{aligned}
\]
We obtain $v = v^*$ or $v = 0$. If $v = 0$, then $(0, 0, b) \in \mathcal{B}^+$. Clearly, this immediately leads to a contradiction with Lemma 4.7. Thus, the proof is finished. \hfill \Box

**Proposition 4.9.** The connected component $\mathcal{B}^+$ is not compact in $\mathcal{X} \times \mathbb{R}^+$.

**Proof.** According to the above results, we need to point out that case (ii) and case (iii) can not occur.

Suppose (ii) holds. Then we can find a sequence of points $\{(u_n, v_n, b_n)\} \subset \mathcal{B}^+ \cap (\mathcal{P} \times \mathbb{R}^+)$ with $u_n > 0$, $v_n > 0$ on $[0, L]$, which converges to $(0, v^*, b)$ as $n \to \infty$. Let $U_n = \frac{u_n}{\|u_n\|_\infty}$. Then $U_n$ satisfies
\[
\begin{aligned}
&d_1(U_n)_{xx} - q_1(U_n)_x + U_n(r_1(x) - u_n - b_n v_n) = 0, \quad x \in (0, L), \\
&d_1(U_n)(0) - q_1 U_n(0) = (U_n)_x(L) = 0.
\end{aligned}
\]
By integrating the above equation over $(0, x)$, similarly as before, we obtain that $(U_n)_x$ is bounded in $L^p(0, L)$. Hence, $\{-d_1(U_n)_{xx}\}$ is bounded in $L^p(0, L)$. We can deduce that there exists a convergent subsequence of $U_n$, which we still denote by $U_n$ for the sake of convenience such that $U_n \to \hat{U} \geq 0$ in $C^1([0, L])$ by $L^p$ estimates and the Sobolev embedding theorem. It follows from the strong maximum principle that $\hat{U} > 0$ on $[0, L]$. Passing to the limit for the equation of $U_n$, we have
\[
\begin{aligned}
&d_1 \hat{U}_{xx} - q_1 \hat{U}_x + \hat{U}(r_1(x) - b v^*) = 0, \quad x \in (0, L), \\
&d_1 \hat{U}_x(0) - q_1 \hat{U}(0) = \hat{U}_x(L) = 0.
\end{aligned}
\]
One easily deduces $b = b_0$ based on $\hat{U} > 0$ on $[0, L]$ and (3.2), a contradiction.

Suppose (iii) holds, i.e., there is $(u, v^* + v, b) \in \mathcal{B}^+$ with $(u, v) \neq (0, 0)$ and $(u, v) \in Z$. Then
\[
\int_0^L e^{-\frac{u}{4x^2}} u \varphi_0 dx = 0,
\]
which implies that $\min_{x \in [0, L]} u \leq 0$. In the other hand, taking $(u_1, v^* + v_1, b_1) \in \Gamma$, we get that $\min_{x \in [0, L]} u_1 > 0$. It follows from the intermediate value theorem that there exists $(u_2, v^* + v_2, b_2) \in \mathcal{B}^+$ such that $\min_{x \in [0, L]} u_2 = 0$. By Lemma 4.8, we know that $(u_2, v^* + v_2) = (0, v^*)$. Since $\mathcal{B}^+ \setminus \{(0, v^*, b_0)\}$ is connected, $b_2 \neq b_0$. That is $(u_2, v^* + v_2, b_2) = (0, v^*, b_2) \in \mathcal{B}^+$ with $b_2 \neq b_0$, which is the option (ii) that has been ruled out above. Therefore we complete the proof. 

Lemma 4.10. Suppose $\{(u_n, v_n, b_n)\} \subset \mathcal{B}^+$, $v_n \to 0$ in $W^{2,p}(0, L)$ and $b_n$ is bounded. Then $u_n \to u^*$ in $W^{2,p}(0, L)$ and $c = c_0$.

Proof. Without loss of generality, we suppose $b_n \to \hat{b}$ by choosing a subsequence. Due to $v_n \to 0$, we use a transformation $V_n = \frac{u_n}{\|v_n\|_\infty}$. Then $(u_n, V_n)$ satisfies

$$
\begin{align*}
&d_1(u_n)_{xx} - q_1(u_n)_x + u_n r_1(x) - u_n - b_n v_n = 0, \quad x \in (0, L), \\
&d_2(V_n)_{xx} - q_2(V_n)_x + V_n r_2(x) - c u_n - v_n = 0, \quad x \in (0, L), \\
&d_1(u_n)(0) - q_1 u_n(0) = (u_n)_x(L) = 0, \\
&d_2(V_n)(0) - q_2 V_n(0) = (V_n)_x(L) = 0.
\end{align*}
$$

So by integrating the equations for $u_n$ and $V_n$ over $(0, x)$, similarly as before, we obtain that $(u_n)_x$ and $(V_n)_x$ are bounded. Clearly, $\{-d_1(u_n)_{xx}\}$ and $\{-d_2(V_n)_{xx}\}$ are bounded in $L^p(0, L)$. By $L^p$ estimates and the Sobolev embedding theorem, we can choose a subsequence (if necessary), which satisfies $(u_n, V_n) \to (\hat{u}, \hat{V})$ with $0 \leq \hat{u} \leq \tau_1$ and $\hat{V} \geq r_0 \neq 0$ in $C^1([0, L]) \times C^1([0, L])$ as $n \to \infty$. It follows from the strong maximum principle that $\hat{V} > 0$ on $[0, L]$. Passing to the limit for the function $u_n$, we obtain

$$
\begin{align*}
&d_1 \hat{u}_{xx} - q_1 \hat{u}_x + \hat{u}(r_1(x) - \hat{u}) = 0, \quad x \in (0, L), \\
&d_1 \hat{u}_x(0) - q_1 \hat{u}(0) = \hat{u}_x(L) = 0.
\end{align*}
$$

One finds that $\hat{u} = 0$ or $\hat{u} = u^*$. If $\hat{u} = 0$, then $(u_n, v_n, b_n) \to (0, 0, \hat{b})$, which is a contradiction with Lemma 4.7. Hence, $\hat{u} = u^*$. Passing to the limit for the function $V_n$, we see

$$
\begin{align*}
&d_2 \hat{V}_{xx} - q_2 \hat{V}_x + \hat{V}(r_2(x) - c u^*) = 0, \quad x \in (0, L), \\
&d_2 \hat{V}_x(0) - q_2 \hat{V}(0) = \hat{V}_x(L) = 0.
\end{align*}
$$

This combines with $\lambda_1(d_2, q_2, r_2(x) - c_0 u^*) = 0$ by Lemma 3.2 and $\hat{V} > 0$ on $[0, L]$ to imply $c = c_0$. The proof is finished. 

Proposition 4.11. Suppose $\{(u_n, v_n, b_n)\} \subset \mathcal{B}^+$ and $v_n \to 0$ for $c = c_0$. Then $u_n \to u^*$ and $b_n \to \hat{b}^*$.

Proof. Theorem 3.5 implies that $b_n$ is bounded when $c = c_0$ and $\{(u_n, v_n, b_n)\} \subset \mathcal{B}^+$. Without loss of generality, there exists a $\hat{b} \geq 0$ such that $b_n \to \hat{b}$ by choosing a subsequence. Consider $V_n = \frac{v_n}{\|v_n\|_\infty}$. This case can be treated similarly as the Lemma 4.10 to obtain $(u_n, V_n) \to (u^*, \hat{V})$ in $C^1([0, L]) \times C^1([0, L])$ as $n \to \infty$. Passing to the limit for the function $V_n$, we obtain

$$
\begin{align*}
&d_2 \hat{V}_{xx} - q_2 \hat{V}_x + \hat{V}(r_2(x) - c_0 u^*) = 0, \quad x \in (0, L), \\
&d_2 \hat{V}_x(0) - q_2 \hat{V}(0) = \hat{V}_x(L) = 0.
\end{align*}
$$

(4.9)
Consider \( \varphi_n = u_n - u^* \). Then \( V_n \) satisfies

\[
\begin{aligned}
\begin{cases}
 d_2(V_n)_{xx} - q_2(V_n)_x + V_n(r_2(x) - c_0u^*) = v_nV_n + c_0\varphi_nV_n, & x \in (0, L), \\
 d_2(V_n)(0) - q_2V_n(0) = (V_n)_x(L) = 0.
\end{cases}
\end{aligned}
\]

Multiplying the equation of (4.10) by \( e^{-\frac{\gamma_2}{2}\varphi} \), the equation of (4.9) by \( e^{-\frac{\gamma_2}{2}V} \), integrating over \((0, L)\) and then subtracting the resulting equations, we get

\[
\int_0^L e^{-\frac{\gamma_2}{2}\varphi}v_nV_n\dot{V}dx + c_0 \int_0^L e^{-\frac{\gamma_2}{2}\varphi}V_n\dot{V}dx = 0.
\]

Next, we rewrite the equation of \( u_n \) in terms of \( \varphi_n \) as

\[
\begin{aligned}
\begin{cases}
 d_1(\varphi_n)_{xx} - q_1(\varphi_n)_x + \varphi_n(r_1(x) - 2u^* - \varphi_n) = b_nv_nu^* + b_nv_n\varphi_n, & x \in (0, L), \\
 d_1(\varphi_n)(0) - q_1\varphi_n(0) = (\varphi_n)_x(L) = 0.
\end{cases}
\end{aligned}
\]

By \( L^p \) estimates, we conclude that there exists a \( C_2 > 0 \) such that

\[
\| \varphi_n \|_{2,p} \leq C_2(b_n\| u_n u^* \|_p + b_n\| v_n \varphi_n \|_p).
\]

Clearly, we can conclude that \( w_n = \frac{\varphi_n}{\| \varphi_n \|_{\infty}} \) is uniformly bounded. Then \( w_n \) satisfies

\[
\begin{aligned}
\begin{cases}
 d_1(w_n)_{xx} - q_1(w_n)_x + w_n(r_1(x) - 2u^*) = \varphi_nw_n + b_nV_nu^* + b_nv_nw_n, & x \in (0, L), \\
 d_1(w_n)(0) - q_1w_n(0) = (w_n)_x(L) = 0.
\end{cases}
\end{aligned}
\]

So by integrating the above equation for \( w_n \) over \((0, x)\), similarly as before, we obtain that \( (w_n)_x \) is bounded. Clearly, \( \{-d_1(w_n)_{xx}\} \) is bounded in \( L^q(0, L) \). By \( L^p \) estimates and the Sobolev embedding theorem, we can choose a subsequence (if necessary), which satisfies \( w_n \to \hat{w} \) in \( C^1([0, L]) \) as \( n \to \infty \). Passing to the limit for the equation \( w_n \), we get

\[
\begin{aligned}
\begin{cases}
 d_1\hat{w}_x - q_1\hat{w} + \hat{w}(r_1(x) - 2u^*) = b\hat{U}u^*, & x \in (0, L), \\
 d_1\hat{w}(0) - q_1\hat{w}(0) = \hat{w}_x(L) = 0.
\end{cases}
\end{aligned}
\]

Let \( \varpi = \frac{\hat{w}}{b} \). Then \( \varpi \) satisfies

\[
\begin{aligned}
\begin{cases}
 d_1\varpi_x - q_1\varpi + \varpi(r_1(x) - 2u^*) = \hat{V}u^*, & x \in (0, L), \\
 d_1\varpi(0) - q_1\varpi(0) = \varpi_x(L) = 0.
\end{cases}
\end{aligned}
\]

Finally, dividing equality (4.11) by \( \| v_n \|_\infty \) and taking limit, we get

\[
\int_0^L e^{-\frac{\gamma_2}{2}\varpi}dx + c_0 \int_0^L e^{-\frac{\gamma_2}{2}\varpi}\varpi dx = 0.
\]

So

\[
\hat{b} = -\frac{\int_0^L e^{-\frac{\gamma_2}{2}\varphi}dx}{c_0 \int_0^L e^{-\frac{\gamma_2}{2}\varpi}\varpi dx}.
\]

Comparing the equations for \( \hat{V} \) and \( \varpi \) with the equations for \( \Psi_0 \) and \( \eta \) (see (4.9), (4.12), (3.3) and (4.7)), respectively, we get that \( \hat{b} = b^* \). The proof is finished. \( \square \)

**Proposition 4.12.** Suppose \( \{(u_n, v_n, b_n)\} \subset B^+ \) and \( b_n \to 0^+ \). Then \( c < c_0, \ u_n \to u^* \) and \( v_n \to \hat{v}^* \), where \( \hat{v}^* \) is the unique positive solution of (1.8).
Proposition 4.13. We obtain that \((u_n)_x\) and \((v_n)_x\) are bounded by integrating the equations for \(u_n\) and \(v_n\) over \((0, x)\), similarly as before. Clearly, we derive that \(\{ -d_1(u_n)_{xx}\} \) and \(\{ -d_2(v_n)_{xx}\} \) are bounded in \(L^p(0, L)\). By \(L^p\) estimates and the Sobolev embedding theorem, we can choose a subsequence (if necessary), which satisfies \((u_n, v_n) \to (\hat{u}, \hat{v})\) in \(C^1([0, L]) \times C^1([0, L])\) as \(n \to \infty\). Passing to the limit, we have

\[
\begin{align*}
d_1\hat{u}_{xx} - q_1\hat{u}_x + \hat{u}(r_1(x) - \hat{u}) &= 0, \quad x \in (0, L), \\
d_2\hat{v}_{xx} - q_2\hat{v}_x + \hat{v}(r_2(x) - c\hat{u} - \hat{v}) &= 0, \quad x \in (0, L), \\
d_1\hat{u}_x(0) - q_1\hat{u}(0) &= \hat{u}_x(L) = 0, \\
d_2\hat{v}_x(0) - q_2\hat{v}(0) &= \hat{v}_x(L) = 0.
\end{align*}
\]

From the equation and the boundary condition of \(\hat{u}\), we get \(\hat{u} = u^*\) or \(\hat{u} = 0\). If \(\hat{u} = 0\), i.e., \(u_n \to 0\), we consider \(U_n = \frac{u_n}{\|u_n\|}\). Similarly as Proposition 4.9, we have \(U_n \to \hat{U} \geq_1 \not= 0\) in \(C^1([0, L])\). Passing to the limit for the equation of \(U_n\), we have

\[
\begin{align*}
d_1\hat{U}_{xx} - q_1\hat{U}_x + r_1(x)\hat{U} &= 0, \quad x \in (0, L), \\
d_1\hat{U}_x(0) - q_1\hat{U}(0) &= \hat{U}_x(L) = 0.
\end{align*}
\]

One easily deduces \(\hat{U} \equiv 0\) based on \(\lambda_1(d_1, q_1, r_1(x)) > 0\) from Lemma 2.2(ii), which contradicts \(\|\hat{U}\|_{\infty} = 1\). Hence, \(\hat{u} = u^*\). Observe that the equation of (1.8) has a unique positive solution \(\hat{v}^*\) if and only if \(c < c_0\). If \(\hat{v} = 0\), then we have \(c = c_0\) by Lemma 4.10 and \(b_n \to b^*\) by Proposition 4.11, which is a contradiction with \(b_n \to 0\). Therefore, \(\hat{v} = \hat{v}^*\) and \(c < c_0\). The proof is finished.

**Proposition 4.13.** Suppose \(\{u_n, v_n, b_n\} \subset B^+\) and \(b_n \to \infty\). Then \(c > c_0\) and \(v_n \to 0\).

**Proof.** Theorem 3.5 implies that when \(b_n \to \infty\) and \(\{u_n, v_n, b_n\} \subset B^+\), we have \(c > c_0\). The function \(u_n\) satisfies

\[
\begin{align*}
d_1(u_n)_{xx} - q_1(u_n)_x + u_n(r_1(x) - u_n - b_nv_n) &= 0, \quad x \in (0, L), \\
d_1(u_n)_x(0) - q_1u_n(0) &= (u_n)_x(L) = 0.
\end{align*}
\]

By integrating above equation over \((0, L)\) similarly to Theorem 3.5, we obtain that \(\int_0^L u_nv_n\,dx \to 0\) as \(n \to \infty\) and \(u_nv_n \to 0\) in \(L^p(0, L)\) for all \(p \geq 1\).

The function \(v_n\) satisfies

\[
\begin{align*}
d_2(v_n)_{xx} - q_2(v_n)_x + v_n(r_2(x) - cu_n - v_n) &= 0, \quad x \in (0, L), \\
d_2(v_n)_x(0) - q_2v_n(0) &= (v_n)_x(L) = 0.
\end{align*}
\]

Moreover, we get that \((v_n)_x\) is bounded by integrating the equation for \(v_n\) over \((0, x)\). Clearly, \(\{ -d_2(v_n)_{xx}\} \) is bounded in \(L^p(0, L)\). By \(L^p\) estimates and the Sobolev embedding theorem, we can choose a subsequence (if necessary), which satisfies \(v_n \to \hat{v}\) in \(C^1([0, L])\) as \(n \to \infty\). Taking the limit for the above equation, we get

\[
\begin{align*}
d_2\hat{v}_{xx} + \hat{v}(r_2(x) - \hat{v}) &= 0, \quad x \in (0, L), \\
d_2\hat{v}_x(0) - q_2\hat{v}(0) &= \hat{v}_x(L) = 0,
\end{align*}
\]

which implies \(\hat{v} = 0\) or \(\hat{v} = v^*\). In addition, by integrating the equation for \(u_n\) over \((0, L)\), we obtain

\[
\int_0^L u_n(r_1(x) - u_n - b_nv_n)\,dx = q_1u_n(L).
\]
If \( \dot{v} = v^* \), the right-hand side is nonnegative, but the left-hand side is strictly negative due to \( b_n \to \infty \), a contradiction. Thus, \( \dot{v} = 0 \). The proof is finished. \( \square \)

How about the convergence of \( u_n \)? The following result shows that there are more than one limit for \( \{u_n\} \).

**Proposition 4.14.** Suppose \( \{u_n, v_n, b_n\} \subset B^+ \) and \( b_n \to \infty \). Then \( b_n v_n \) is uniformly bounded. Let \( V_n = \frac{u_n}{\|v_n\|_{\infty}} \) and \( D \in \mathbb{R} \) be a limit point of \( b_n \|v_n\|_{\infty} \), then \( D > 0 \) and there exists a subsequence of \( (u_n, V_n) \) (if necessary) converging to \((\hat{u}, \hat{V})\) in \( X \), which satisfies

\[
\begin{aligned}
d_1 \hat{u}_{xx} - q_1 \hat{u}_x + \hat{u}(r_1(x) - \hat{u} - D\hat{V}) &= 0, \quad x \in (0, L), \\
d_2 \hat{V}_{xx} - q_2 \hat{V}_x + \hat{V}(r_2(x) - c\hat{u}) &= 0, \quad x \in (0, L), \\
d_1 \hat{u}_x(0) - q_1 \hat{u}(0) &= \hat{u}_x(L) = 0, \\
d_2 \hat{V}_x(0) - q_2 \hat{V}(0) &= \hat{V}_x(L) = 0. \\
\end{aligned}
\]

(4.13)

**Proof.** The proof of this proposition is partly similar to that of Theorem 3.5 including some notations. Notice that \( (u_n, V_n) \) satisfies

\[
\begin{aligned}
d_1 (u_n)_{xx} - q_1 (u_n)_x + u_n(r_1(x) - u_n - b_n \|v_n\|_{\infty} V_n) &= 0, \quad x \in (0, L), \\
d_2 (V_n)_{xx} - q_2 (V_n)_x + V_n(r_2(x) - cu_n - \|v_n\|_{\infty} V_n) &= 0, \quad x \in (0, L), \\
d_1 (u_n)_x(0) - q_1 u_n(0) &= (u_n)_x(L) = 0, \\
d_2 (V_n)_x(0) - q_2 V_n(0) &= (V_n)_x(L) = 0. \\
\end{aligned}
\]

(4.14)

It follows from \( L^p \) estimates and the Sobolev embedding theorem for the equation of \( V_n \) that \( V_n \to \hat{V} \) in \( W^{2,p}(0, L) \) by passing to a subsequence if necessary. Suppose \( b_n \|v_n\|_{\infty} \to \infty \). Integrating the first equation of (4.14) over \( (0, L) \), we get that 
\[
\int_0^L u_n V_n dx \to 0 \quad \text{and} \quad u_n V_n \to 0 \quad \text{in} \ L^p(0, L) \quad \text{for all} \ p \geq 1 \quad \text{as} \ n \to \infty.
\]

Then \( \hat{V} \) satisfies

\[
\begin{aligned}
d_2 \hat{V}_{xx} - q_2 \hat{V}_x + r_2(x)\hat{V} &= 0, \quad x \in (0, L), \\
d_2 \hat{V}_x(0) - q_2 \hat{V}(0) &= \hat{V}_x(L) = 0. \\
\end{aligned}
\]

Since \( \lambda_1(d_2, r_2(x)) > 0 \) based on Lemma 2.2(ii), we get \( \hat{V} \equiv 0 \), which contradicts \( \|\hat{V}\|_{\infty} = 1 \).

Assume \( \lim_{n \to \infty} b_n \|v_n\|_{\infty} = D \) by passing to a subsequence if necessary. Then \( D \in [0, \infty) \) and there exists a subsequence of \( (u_n, V_n) \) (if necessary), which converges to \((\hat{u}, \hat{V})\) in \( X \), and satisfies (4.13). If \( D = 0 \), then \( \hat{u} = u^* \) or \( \hat{u} = 0 \). We claim \( \hat{u} = u^* \).

Otherwise, \( \hat{u} = 0 \) implies \((\hat{u}, \hat{V}) = (0, 0) \) from (4.13), which is a contradiction with \( \|\hat{V}\|_{\infty} = 1 \). Hence, combining with the fact that \( \lambda_1(d_2, r_2(x) - c_0 u^*) = 0 \) from Remark 3.3, we obtain \( c = c_0 \). This contradicts the conclusion of Proposition 4.13. Therefore \( D > 0 \). \( \square \)

**Theorem 4.15.** For the connected component \( B^+ \), the following assertions hold:

(i) \( c = c_0 \) if and only if the case \( v \to 0 \) holds and \( b \) is bounded. In this case, \( \overline{B^+} = B^+ \cup \{(u^*, 0, b^*)\} \), in other words, \( (0, v^*, b_0) \) connects with the equilibrium \((u^*, 0, b)\) only at \( b = b^* \). That is, system (1.3) has at least a positive solution when \( c = c_0 \), \( b \) is between \( b_0 \) and \( b^* \);

(ii) \( c < c_0 \) if and only if the case \( b \to 0^+ \) holds. In this case, \( \overline{B^+} = B^+ \cup \{(u^*, \hat{v}^*, 0)\} \) where \( \hat{v}^* \) satisfies (1.8). That is, system (1.3) has at least a stable positive solution when \( c < c_0 \) and \( b < b_0 \);
Dependent on parameters, approach. May change signs for different parameters, which will be illustrated by a numerical analysis.

Proof. Clearly, case (i) holds by Lemma 4.10 and Proposition 4.11; case (ii) and case (iii) hold by Proposition 4.12, Proposition 4.13, Lemma 3.1, Lemma 3.2 and Lemma 2.3.

Remark 4.16. Theorem 1.2 and Proposition 1.3 are directly obtained by Theorem 1.2 and Remark 4.3.

5. Discussion and numerical simulation. We investigate a general two-species competition system (1.2) in open advective inhomogeneous environments, where the movement strategies, growth rates and competition abilities of two species are allowed to be different. We mainly focus on the effect of competition abilities on the dynamical behaviors of such a model. Similar questions are addressed for the model with weak competition case \(b = c = 1\) and identical competition case \(b = c = 0\) in [7, 29] and the references therein. Especially, He and Ni [7] provided a complete classification on the global dynamics of system (1.2) with \(q_1 = q_2 = 0\); Zhou and Xiao [29] obtain a complete classification on all possible long-time dynamical behaviors for system (1.2) under the condition that the movement strategies of two competitors are proportional, i.e., \(\frac{d\varepsilon}{dt} = \frac{2\varepsilon}{q_1} = k\). In this paper, we get rid of these assumptions and present a relatively complete classification on the dynamical behavior of system (1.2) in \(b - c\) plane.

From Theorem 1.1, we know that when the competitive impact of species \(u\) allows invasion by species \(v\) (i.e., \(0 < b < b_0\)), \((u^*, 0)\) is globally asymptotically stable if species \(u\) has strong competition ability \((c > \tilde{c} > c_0)\); Likewise, for \(0 < c < c_0\), \((0, c^*)\) is globally asymptotically stable if \(b > \tilde{b} > b_0\). In a word, the competitive exclusion principle holds if one of the interspecific competition abilities is sufficiently large and the other is in a certain range.

Theorem 1.2 yields information about two things by the analysis on the structure of bifurcating branches of positive steady states arising from semi-trivial steady states.

(i) System (1.3) has at least a positive solution if the interspecific competition abilities of two species are both small \((0 < b < b_0, 0 < c < c_0)\) or large \((b > b_0, c > c_0)\), or \(c = c_0, \min\{b_0, b^*\} < b < \max\{b_0, b^*\}\) (see Proposition 1.3). It is worth pointing out that the coexistence results in the first two cases, i.e., both \(b\) and \(c\) are small or large, can also be obtained by the stability analysis of two semi-trivial steady states (see Lemmas 3.1 and 3.2) and the monotone dynamical system theory (see Lemma 2.3). However, the result in the third case \((c = c_0\) and \(\min\{b_0, b^*\} < b < \max\{b_0, b^*\}\) cannot be deduced by the monotone dynamical system theory.

(ii) System (1.3) admits multiple positive solutions in four cases: \(1\) \(c = c_0 < c^*, b_0 - \varepsilon < b < b_0 < b^*\), \(2\) \(c = c_0 > c^*, b^* < b_0 < b < b_0 + \varepsilon\), \(3\) \(c^* < c < c_0, b_0 < b < b_0 + \varepsilon\), and \(4\) \(c_0 < c < c^*, b_0 - \varepsilon < b < b_0\), where \(\varepsilon > 0\) is small.

Unfortunately, we can’t determine the signs of \(b_0 - b^*\) and \(c_0 - c^*\). In fact, they may change signs for different parameters, which will be illustrated by a numerical approach.

It follows from the definitions of \(b_0\), \(b^*\), \(c_0\) and \(c^*\) that these critical values are dependent on parameters \(d_1, d_2, q_1, q_2, r_1\) and \(r_2\). To numerically compute them, we
fix

\[ q_1 = 0.1, q_2 = 0.1, r_1 = 1, r_2 = 0.9. \]  \hspace{1cm} (5.1)

In Figure 5.1(i), by taking \( d_2 = 0.01, 0.05, 0.2, 0.6, 1 \) respectively, we calculate the values of \( b_0 - b^* \) vs. \( d_1 \in (0, 1) \). In Figure 5.1(ii), we take \( d_1 = 0.01, 0.05, 0.2, 0.6, 1 \) respectively to calculate the values of \( b_0 - b^* \) vs. \( d_2 \in (0, 1) \). From Figure 5.1, it is easy to see that \( b_0 - b^* > 0 \) when both \( d_1 \) and \( d_2 \) are small, and \( b_0 - b^* \) maybe negative when \( d_1 \) or \( d_2 \) is relatively large. Due to the symmetry of \( b \) and \( c \), \( c_0 - c^* \) has the similar properties as \( b_0 - b^* \) and we omit the simulations of \( c_0 - c^* \) here. The information implies that multiple coexistence can occur.

**Figure 5.1.** The graphs of \( b_0 - b^* \) vs. \( d_1 \) in (i) and vs. \( d_2 \) in (ii) with other parameters fixed as (5.1).

It is worth noting that \( b_0 = b^* \) \( (c_0 = c^* \) symmetrically) always holds when we take \( q_1 = q_2 = 0 \) in our simulations (see, e.g., Figure 5.2). This phenomenon strongly suggests that there is no multiple coexistence for system (1.2) with \( q_1 = q_2 = 0 \), which coincides with the results in [7].

**Figure 5.2.** The graphs of \( b_0 - b^* \) vs. \( d_1 \) in (i) and vs. \( d_2 \) in (ii) with \( q_1 = q_2 = 0 \) and other parameters fixed as (5.1).

Finally, for the sake of arguments, assume that \( c^* < c_0 \) and \( b^* < b_0 \). We give a schematic overview of our results in terms of the \( b - c \) plane (see Figure 5.3). In region (I) of Figure 5.3, there is a stable coexistence of system (1.3), and two semi-trivial equilibria are both unstable, while in region (III) there is one unstable
coexistence of system (1.3), and two semi-trivial equilibria are both stable. In regions (II) and (IV), one of the semi-trivial steady states (i.e., \((0, v^*)\) or \((u^*, 0)\)) is globally asymptotically stable via competitive overmatch. In the region (V), multiple coexistence states of system (1.3) occur.

![Diagram](image)

**Figure 5.3.** Schematic diagram of the global dynamics on system (1.3) in \(b - c\) plane.

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