Perfect single error-correcting codes in the Johnson scheme

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Abstract—Delsarte conjectured in 1973 that there are no nontrivial perfect codes in the Johnson scheme. Etzion and Schwartz recently showed that perfect codes must be k-regular for large k, and used this to show that there are no perfect codes correcting single errors in J(n, w) for n ≤ 50,000. In this paper we show that there are no perfect single error-correcting codes for n ≤ 2^{250}.

I. INTRODUCTION

The Johnson graph J(n, w) has vertices corresponding to V^n_n, the w-subsets of the set N = {1, 2, ..., n}, with two vertices adjacent if their intersection has size w − 1.

The distance between two w-sets is half the size of their symmetric difference. The e-sphere of a point, the set of all w-sets within distance e, has cardinality

Φ_e(n, w) = \sum_{i=0}^{e} \binom{w}{i} \binom{n-w}{i}.

A code C ⊂ J(n, w) is called e-perfect if the e-spheres of all the codewords of C form a partition of V^n_n. Delsarte [2] conjectured that no nontrivial perfect codes exist in J(n, w).

Etzion and Schwartz [3] introduced the concept of k-regular codes. In this paper we use their results to improve the lower bound on the size of a 1-perfect code. The method of proof will be to look at the factors of Φ_1(w, a). We show that Φ_1(w, a) is squarefree, and for each prime p_i | Φ_1(w, a), there is an integer α_i such that p_i^{α_i} must be close to n − w. Then we will show that the α_i’s are distinct and pairwise coprime, and the sum of their reciprocals is close to two. A computer search for perfect powers in short intervals then shows that no such codes exist with n < 2^{250}.

For the rest of this paper we will deal with the case e = 1, and write n = 2w + a. This may be done without loss of generality, since the complement of an e-perfect code in J(n, w) is e-perfect in J(n, n − w). Also, to simplify the statements of theorems, we will assume throughout the paper that C is a nontrivial 1-perfect code in J(n, w).

II. REGULARITY OF 1-PERFECT CODES

In this section we summarize the results of Etzion and Schwartz [3] that we will need. Let A be a k-subset of N = {1, 2, ..., n}. For all 0 ≤ i ≤ k, define

C_A(i) = \{c ∈ C : |c ∩ A| = i\},

and for each I ⊆ A, define

C_A(I) = \{c ∈ C : c ∩ A = I\}.

Theorem 2: C is k-regular if:

1) There exist numbers α(0), α(1), ..., α(k) such that for any k-set A in N, C_A(i) = α(i), for i = 0, 1, ..., k.
2) For any k-set A in N, there exist numbers β_A(0), β_A(1), ..., β_A(k) such that if I ⊆ A, then C_A(I) = β_A(|I|).

Proof: Theorem 13 in [3], which is a strengthening of a theorem of Roos [7], gives C is k-regular if:

1) There exist numbers α(0), α(1), ..., α(k) such that for any k-set A in N, C_A(i) = α(i), for i = 0, 1, ..., k.
2) For any k-set A in N, there exist numbers β_A(0), β_A(1), ..., β_A(k) such that if I ⊆ A, then C_A(I) = β_A(|I|).

Etzion and Schwarz give a necessary condition for a code to be regular:

Theorem 1: If C is k-regular, then

Φ_1(w, a) = 1 + w(w + a) \left(\frac{2w + a - i}{w + a}\right)

for i = 0, ..., k.

They then show that 1-perfect codes must be highly regular.

Theorem 2: C is k-regular if the polynomial

σ_1(w, a, m) = m^2 − (2w + a + 1)m + w(w + a) + 1

has no integer roots for 1 ≤ m ≤ k.

Let

L(w, a) = \frac{2w + a + 1 - \sqrt{(a + 1)^2 + 4(w - 1)}}{2}.

The smallest root of L is L(w, a), so

Theorem 3: C is k-regular for any k < L(w, a).

This means that we can rule out 1-perfect codes by showing that there is some i with 0 ≤ i ≤ L(w, a) such that Φ_1 is not satisfied. L(w, a) is an increasing function of a, so

Lemma 1: L(w, a) ≥ L(w, 0) > w − \sqrt{w}.

Lemma 2: We have

0 < a < w/2.

Proof: Theorem 13 in [3], which is a strengthening of a theorem of Roos [7], gives a < w − 3. If a = 0 then C is a trivial code.

If a ≥ w/2, then

L(w, a) > L \left(\frac{w + 7}{2}\right) = w = 2,

so C is (w − 2)-regular. C is also (w − 1)-regular, since

σ_1(w, a, w − 1) = a − (w − 3) ≠ 0

for a < w − 3.

Since C corrects single errors, any two codewords are at least distance 3 apart in J(n, w). Let A be a (w − 1)-set contained in some codeword c_1. Remove any element of A and add one not in c_1 to get a new (w − 1)-set A'. Since C is (w − 1)-regular, there is a codeword c_2 containing A', but c_1 and c_2 have distance 2 in J(n, w), a contradiction.

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III. Divisors of $\Phi_1(w, a)$

We will derive necessary conditions for 1-perfect codes by looking at possible prime divisors of $\Phi_1(w, a)$. One tool will be:

**Lemma 3**: (Kummer) Let $p$ be a prime. The number of times $p$ appears in the factorization of $\binom{a}{b}$ equals the number of carries when adding $b$ to $a - b$ in base $p$.

Theorem 3 and Lemmas 1 and 2 imply

**Corollary 1**: If $p$ is a prime with $p|\Phi_1(w, a)$, then there are at least $k$ carries when adding $w + a$ to $j = w - i$ for $j = \lceil \sqrt{w} \rceil + 1, \lfloor \sqrt{w} \rfloor + 2, \ldots, w$.

Let

$$ w + a = (r_m, r_{m-1}, \ldots, r_1, r_0)_p $$

(3)

be the base $p$ representation of $w + a$, with $r_m \geq 1$. Let $l = \lceil m/2 \rceil$.

**Lemma 4**: $r_i = p - 1$ for $i = l + 1, l + 2, \ldots, m$.

**Proof**: For any $i$ with $\lfloor \sqrt{w} \rfloor + 1 \leq p^i \leq w$, adding $p^i$ to $w + a$ must have a carry by Corollary 1, so the lemma follows for $i = l + 1, \ldots, m - 1$. To complete the proof, we need to show that $w \geq p^m$. We have

$$ w + a \geq p^m + (p - 1)p^{m-1} \geq \frac{3}{2} p^m. $$

Since $a < w/2$ by Lemma 2, this implies $w > p^m$.

**Theorem 4**: $\Phi_1(w, a)$ must be squarefree.

**Proof**: Adding $p^m$ to $w + a$ has only one carry, so by Corollary 1 only one power of $p$ divides $\Phi_1(w, a)$.

**Theorem 5**: For any prime $p$ dividing $\Phi_1(w, a)$, let $\alpha = m + 1 = \lceil \log_p (w + a) \rceil + 1$. Then

$$ p^\alpha - \lfloor \sqrt{w} \rfloor - 1 \leq w + a < p^\alpha $$

(4)

**Proof**: We have $w + a < p^\alpha$ from 3. By Lemma 4 we must have $r_i = p - 1$ for $i = l + 1, l + 2, \ldots, m$. Let

$$(t_1, t_1 - 1, \ldots, t_0)_p$$

be the base $p$ representation of $\lfloor \sqrt{w} \rfloor$. The left inequality of 3 is equivalent to

$$ p^\alpha - 1 - (w + a) = (p - 1 - r_i, \ldots, p - 1 - r_0)_p \leq (t_1, t_1 - 1, \ldots, t_0)_p = \lfloor \sqrt{w} \rfloor. $$

If this is not satisfied, let $i$ be the largest integer such that $p - 1 - r_i > t_i$. The number $(t_1, t_1 - 1, \ldots, t_{i+1}, t_i + 1, 0, \ldots, 0)_p$ is greater than $\lfloor \sqrt{w} \rfloor$ and has no carries when added to $w + a$ in base $p$, which contradicts Corollary 1.

Thus we have that $p^\alpha$ is in a short interval around $w + a$.

We will use this result in the following form:

**Corollary 2**: For a prime $p$ dividing $\Phi_1(w, a)$, we have

$$ 0 < \log_{w+a} p - \frac{1}{\alpha} < \frac{3}{2} \left( \frac{1}{w + a} + \frac{4}{w + a} \right). $$

(5)

**Proof**: From 3, we have

$$ p^\alpha > w + a \geq p^\alpha \left( 1 - \frac{1}{\sqrt{w + a}} - \frac{2}{w + a} \right) $$

using $\lceil \sqrt{w} \rceil + 1 < \sqrt{w + a} + 2$. Taking the log base $w + a$, we have

$$ \alpha \log_{w+a} p > 1 > \alpha \log_{w+a} p + \log_{w+a} \left( \frac{1}{w + a} - \frac{2}{w + a} \right) $$

Using the bound $-\log(1 - x) < x + x^2$ for $x < 1/2$ gives the corollary.

IV. POWERS IN SHORT INTERVALS

**Theorem 5** shows that for a 1-perfect code to exist, several prime powers must be close to $w + a$. Having a large number of prime powers in a short interval seems unlikely. Loxton [6] showed (a gap in the proof was later fixed by Bernstein [1]) that the number of perfect powers in $[w, w + \sqrt{w}]$ is at most

$$ \exp(40 \sqrt{\log \log w \log \log \log w}). $$

Loxton conjectured that the number of perfect powers in such an interval is bounded by a constant, but a proof seems very far off.

For the rest of this paper, take

$$ p_1 p_2 \ldots p_r = \Phi_1(w, a) = 1 + w(w + a). $$

(6)

Taking the log of 6 gives

$$ \sum_{i=1}^{r} \log_{w+a} p_i = \log_{w+a} \left( w(w + a) + 1 \right), $$

so

$$ 0 < \sum_{i=1}^{r} \log_{w+a} p_i - (1 + \log_{w+a} w) $$

$$ = \log_{w+a} \left( 1 + \frac{1}{w(w + a)} \right) $$

$$ \leq \frac{1}{w(w + a)}. $$

(7)

**Theorem 6**: If

$$ \left| \sum_{i=1}^{r} \frac{1}{\alpha_i} - (1 + \log_{w+a} w) \right| < \frac{4}{\sqrt{w + a}}. $$

**Proof**: If $\sum_{i=1}^{r} \frac{1}{\alpha_i} - (1 + \log_{w+a} w) \geq 0$, then the theorem follows immediately from 7 and Corollary 2. Otherwise, summing 5 we have

$$ 0 < (1 + \log_{w+a} w) - \sum_{i=1}^{r} \frac{1}{\alpha_i} $$

$$ < \sum_{i=1}^{r} \log_{w+a} p_i - \sum_{i=1}^{r} \frac{1}{\alpha_i} $$

$$ < \sum_{i=1}^{r} \frac{1}{\alpha_i} \left( \frac{1}{\sqrt{w + a}} + \frac{4}{w + a} \right) $$

$$ \leq \frac{2}{\sqrt{w + a}}. $$

Clearly the constant 4 in Theorem 6 can be strengthened, but this will be enough for our purposes.

For $0 < a < w/2$, we have $w + a < 3w/2$, so

$$ 1 - \log_{w+a} 3/2 < \log_{w+a} a < 1 $$

using $\sqrt{3} < \sqrt{w + a} < 2$.
and Theorem 5 says that we have an Egyptian fraction representing a number close to 2. Etzion and Schwartz showed that there are no 1-perfect codes with $n \leq 50000$, and so
\[
\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots + \frac{1}{\alpha_r} \in [1.934, 2.026]. \quad (8)
\]

**Lemma 5:** The $\alpha_i$’s are distinct and pairwise coprime.

**Proof:** We cannot have $\alpha_i = \alpha_j = 1$, since then $p_i, p_j > (w+a)$ implies $p_ip_j > 1+w(w+a) = \Phi_1(w, a)$, contradicting (6).

Suppose we have $\alpha_i, \alpha_j$ with $\gcd(\alpha_i, \alpha_j) = g > 1$. Then by Theorem 5, $p_i^{\alpha_i}$ and $p_j^{\alpha_j}$ are two $g^{th}$ powers in an interval around $w + a$, of length $\sqrt{w + a}$, which is impossible. 

For an integer $k$, let $p^-(k)$ denote the smallest prime factor of $k$.

**Corollary 3:** Some $\alpha_i$ has $p^-(\alpha_i) \geq 7$.

**Proof:** If there are more than four $\alpha$’s, clearly one of them must have a prime factor bigger than 5. For four $\alpha$’s, the set $\{1, 2, 3, 5\}$ has sum of reciprocals 2.033, which by (6) is too big, and an easy computation finds that any set of powers of these numbers has a sum of reciprocals that is too small. The largest is $\{1, 2, 3, 25\}$, with sum 1.8733.

Let $\gamma(n)$ denote the largest squarefree divisor of $n$. The abc conjecture asserts that, for any $\epsilon > 0$ there are only finitely many integers $a, b$ and $c$ such that $a + b = c$ and

\[
\max\{a, b, c\} \leq C_\epsilon \gamma(abc)^{1+\epsilon}.
\]

See [4] for information and references about the abc conjecture.

For any choice of $\alpha$’s satisfying (6), Masser-Oesterlé’s abc conjecture implies there are only a finite number of solutions. For example, take $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 7$. Let $a = p_3^{\alpha_3}, b = p_7^{\alpha_4}$ and $c$ be their difference, which is at most

\[
\max\{p_3^{\alpha_3}, p_7^{\alpha_4}\} \quad \text{by Theorem 5}
\]

\[
\max\{a, b, c\} = w + a \leq C_\epsilon p_3 p_4 c < (w + a)^{1+\epsilon}(1/3 + 1/7 + 1/2) < (w + a)^{0.98}
\]

for all but finitely many $w$’s.

**V. A New Lower Bound for n**

While we cannot show that there are no perfect codes, Theorem 5 gives us an efficient way to search for possible codes, by searching for powers in short intervals.

To show a bound of $2^C$ for $n$, we need to check for primes $a, b \geq 2$ and integers $3 \leq p, q < C$ with

\[
0 < a^p - b^q < \sqrt{a^p}.
\]

It suffices to consider prime values of $p$ and $q$, since any $k$th power is also a $p^-(k)$th power. It is possible to run through the possibilities efficiently. Let $\{p_1 = 3, p_2 = 5, \ldots, p_k\}$ be the odd primes up to $C$. The following procedure will find all pairs $i, j$ and integers $b_i, b_j$ for which $b_i^{p_i}$ and $b_j^{p_j}$ are close:

1. Start with $b_1 = b_2 = \cdots = b_k = 2$. Compute powers $c_i = b_i^{p_i}$ for $i = 1, 2, \ldots, k$.
2. Let $c_i$ be the smallest power, and $c_j$ the second smallest. Compare them to see if they are close enough.

3) Increment the base $b_i$, recompute $c_i$, and continue.

4) Stop when all powers are larger than $2^C$.

If two powers less than $2^C$ are in a short interval, they will eventually be the two smallest powers in the list, and will be found. A heap (see, for example, [5]) is an efficient data structure to maintain the powers in, requiring only one comparison to find the two smallest powers, and $\leq \log_{2^C}$ steps to reorder the heap after changing $c_i$.

Note that the above algorithm looks for any integers $b_i$ and $b_j$ with powers in a short interval, not just primes. Only considering primes would reduce the number of comparisons, but complicate the rule for stopping the bases $b_i$.

In five hours on a 2.6 GHz Opteron, an implementation of this algorithm eliminated everything up to $2^{10^6}$. It found 60 powers higher than squares in short intervals, most of which involved a cube and fifth power. By Corollary 3 we may discount these. The only higher powers are given in Table 1.

| $p_i^{\alpha_i}$ | $p_j^{\beta_j}$ | difference |
|------------------|-----------------|------------|
| $2^{10}$         | $5^5$           | 3          |
| $13^3$           | $3^7$           | 10         |
| $3251^{13}$      | $32^7$          | 838863     |
| $33^1$           | $3493^3$        | 178820     |
| $1965781^3$      | $498^7$         | 1539250669 |

No set of four $\alpha$’s have a sum of reciprocals in this interval, and the only sets of five that do are $\{1, 2, 3, 7, k\}$, where $k \in [41, 71]$ with $\gcd(k, 2 \cdot 3 \cdot 7) = 1$. Any set of six $\alpha$’s clearly have two $\alpha$’s with a factor $\geq 7$, so we have

**Corollary 4:** At least two $\alpha$’s have $p^-(\alpha_i) \geq 7$.

Therefore we may do a search as above, but starting with $p_1 = 7$ instead of 3. The search work is proportional to $2^{C/p_1}$, so this greatly reduces the search time. A search for seventh and higher powers up to $2^{250}$ in a short interval took four hours and found none, so

**Theorem 8:** There are no 1-perfect codes in $J(n, w)$ for all $n < 2^{250}$.

**Acknowledgments.** The author would like to thank the anonymous referee, who suggested changes which greatly improved the presentation of this paper, and pointed out Lemma 2.

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