Mass sum rule of hadrons in the QCD instanton vacuum

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We briefly review the key aspect of the QCD instanton vacuum in relation to the quantum breaking of conformal symmetry and the trace anomaly. We use Ji’s invariant mass decomposition of the energy momentum tensor together with the trace anomaly, to discuss the mass budget of the nucleon and pion in the QCD instanton vacuum. A measure of the gluon condensate in the nucleon, is a measure of the compressibility of the QCD instanton vacuum as a dilute topological liquid.

I. INTRODUCTION

A remarkable feature of QCD is that in the chiral limit it is a scale free theory. Yet, all hadrons are massive, composing most of the visible mass in the universe. The typical hadronic scale is 1 fm, but where does it come from? The answer appears to be from a subtle quantum effect referred to as dimensional transmutation, and related to the quantum breaking of the conformal symmetry of QCD. This mechanism is non-perturbative. On the lattice, the lattice cutoff along with the running coupling combine to generate this scale. In the continuum, to achieve this mechanism requires a non-perturbative description of the vacuum state and its excitations.

The QCD vacuum as a topological liquid of instantons and anti-instantons, offers by far the most compelling non-perturbative description that is analytically tractable in the continuum, thanks to its QCD semi-classical origin and diluteness [1–3]. It is not the only description. Other candidates based on center vortices and monopoles to cite a few [4], are also suggested and may as well be present in addition to the instantons. However, the latters appear to trigger the dual breaking of conformal and chiral symmetry breaking, and dominate the vacuum state and its low-lying hadronic excitations. Center vortices maybe important for the disordering of the large Wilson loops and confinement, a mechanism likely at work in the orbitally excited hadrons as they Reggeize.

The spontaneous breaking of chiral symmetry rather than confinement drives the formation of the low-lying and stable hadrons such as the nucleon and pion. In the QCD instanton vacuum conformal symmetry is broken by the density of instantons: their continuous quantum rate of tunneling in the vacuum. The breaking of chiral symmetry follows simultaneously from the delocalization of the light quark zero modes, by leapfrogging the instantons and anti-instantons much like electrons leapfrogging atoms in a metal. Detailed

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numerical simulations of light hadronic correlators in the QCD instanton vacuum [5] show remarkable agreement with direct lattice measurements [6], and a wealth of correlators extracted from data [5]. The universal conductance fluctuations in the zero mode region of the Dirac spectrum, predicted by random matrix theory [7] and confirmed by lattice simulations [8], show unequivocally the topological character of the origin of mass.

In this note, we briefly review the salient aspects of the QCD instanton vacuum in relation to the quantum breaking of conformal symmetry in section II. We then discuss the role of the trace anomaly in the nucleon and pion mass in section III. The quark and gluon composition of the hadronic mass using Ji’s decomposition [9] is discussed in section IV. In section V, we show that the gluon condensate in the nucleon is tied to the QCD vacuum compressibility, a measure of the diluteness of the QCD instanton vacuum as a topological liquid. Our conclusions are in section VI.

II. QCD INSTANTON VACUUM

As we noted above, the chief aspect of the QCD vacuum (meaning quenched throughout) is its quantum breaking of conformal symmetry with the emergence of all light hadronic scales. The nature of the gauge fields at the origin of this breaking were mysterious and the subject of considerable debates and speculations for many decades, till stunning pictures were developed by Leinweber and his collaborators using cooling and/or projection techniques [10, 11]. Out of the fog of millions of gauge fluctuations, cooling has revealed a stunning vacuum landscape composed of inhomogeneous and topologically active gauge fields as shown in Fig. 1. Remarkably, the key features of this vacuum were predicted long ago by Shuryak [12]

\[ n_{I+I} \equiv \frac{1}{R^4} \approx \frac{1}{\text{fm}^4}, \quad \bar{\rho} R \approx \frac{1}{3} \]

for the instanton plus anti-instanton density and size, respectively. In other words, the hadronic scale \( R = 1 \text{fm} \) emerges as the mean quantum tunneling rate of the topological charge in the QCD vacuum. The dimensionfull parameters (1) combine in the dimensionless packing parameter \( \kappa \equiv \pi^2 \bar{\rho}^4 n_{I+I} \approx 0.1 \), a measure of the diluteness of the instanton-anti-instanton ensemble in the QCD vacuum. Fortunately, the smallness of \( \kappa \) is what will allow us to do reliable analytical calculations. In the cooled landscape shown in Fig. 1, most hadronic correlations are left unchanged with those before the cooling takes place [2] (and references therein).

The size distribution of the instantons and anti-instantons density (their tunneling rate per size) in the QCD vacuum is well captured semi-empirically by

\[ \frac{dn(\rho)}{d\rho} \sim \frac{1}{\rho^5} \left( \rho \Lambda_{QCD} \right)^{b} e^{-\#\rho^2/R^2} \]
with $b = 11N_c/3 - 2N_f/3$ (one loop). The small size distribution follows from the conformal nature of the instanton moduli and perturbation theory. The large size distribution is non-perturbative. A variational analysis of the QCD instanton vacuum including binary interactions [1], shows that the large size instantons are cutoff self-consistently by their density as in (2). Lattice parametrization of the same distribution suggests that the cutoff is due to the onset of confinement with $R \approx 1 \text{ fm} \rightarrow l_s \approx 0.2 \text{ fm}$ (the string length) [5, 13]. Here, we favor the former cutoff as it preserves the strictures of the renormalization group invariance following the quantum breaking of conformal symmetry, with a single scale overall in the chiral limit ($R$ sets the scale for both the gluon and chiral condensates).

FIG. 1. Instantons (yellow) and anti-instantons (blue) configurations in the cooled Yang-Mills vacuum [10]. They constitute the primordial gluon epoxy at the origin of the hadronic mass. See text.
A. Quantum conformal symmetry breaking and the trace anomaly

The quantum breaking of conformal symmetry is best captured through the trace of the energy momentum tensor. Indeed, consider its symmetric form

\[ T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{1+3}}{\delta g^{\mu\nu}} = F^a_{\mu\lambda} F^a_{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^2 + \frac{1}{4} \overrightarrow{\psi} \gamma^\mu \overleftarrow{\gamma}^\nu \psi \]

with \( \overrightarrow{\gamma} = \overrightarrow{\gamma} - \overleftarrow{\gamma} \) and \([\ ]_+\) denotes symmetrization. It is conserved \( \partial_\mu T^{\mu\nu} = 0 \), with an anomalous trace

\[ T^\mu_\mu = \beta(g^2) F^a_\mu F^a_\mu + m \overline{\psi} \psi \]

with the Gell-Mann-Low beta-function (2 loops)

\[ \beta(g^2) = -\frac{b g^4}{8 \pi^2} - \frac{\bar{b} g^6}{2(8 \pi^2)^2} + \mathcal{O}(g^8) \]

Throughout, we use the rescaling \( gF \to F \) for all operators in the instanton and anti-instanton gauge fields. In the QCD instanton vacuum, the gluon operator \( F^2/(32 \pi^2) \to (N_+ + N_-)/V \) counts the number of instantons plus anti-instantons per 4-volume \( V \). In the canonical ensemble with zero theta-angle, it is fixed by the instanton density with \( N_+/V = \bar{N}/2V \). Therefore we have

\[ \langle T^\mu_\mu \rangle \approx -b \left( \frac{\bar{N}}{V} \right) + m \langle \overline{\psi} \psi \rangle \approx -b \left( \frac{\bar{N}}{V} \right) \left( 1 + \mathcal{O}(mR) \right) \approx -10 \text{ fm}^{-4} \]

setting the scale of all hadrons. The current mass \( m \approx 8 \text{ MeV} \) in (6) is fixed at the soft renormalization point \( \bar{\rho} \approx 0.3 \text{ fm} \), about twice the commonly used value at the hard renormalization scale. The scale of the spontaneous breaking of chiral symmetry is also fixed by the finite instanton density, but its contribution to the vacuum scalar density is small since \( mR \approx (8 \text{ MeV})(1 \text{ fm}) \approx 1/25 \).

Since the vacuum is Lorentz symmetric, (6) amounts to a negative vacuum energy density \( B = \langle T^\mu_\mu \rangle /4 \approx -(250 \text{ MeV})^4 \), with no strict confinement at work. Note that \( B \approx -b \langle F^2 \rangle \), where the important overall negative sign inherited from the scale anomaly is ultimately a quantum magnetic effect (sign of the beta function). The gluon condensate \( \langle F^2 \rangle \) is always positive in Euclidean signature. This is usually referred to as the gluon epoxy (a term coined by the late Gerry Brown).
Some of the quantum scale fluctuations in QCD are captured in the QCD instanton vacuum using the grand-canonical description instead of the canonical one. In the former, the instanton number $N = N_+ + N_-$ is allowed to fluctuate with the measure $[3, 14, 15]$

$$P(N) = e^{\frac{N}{4}} \left( \frac{N}{N} \right)^{\frac{N}{4}}$$

(7)

which is stronger than Poisson ($b/4 \to 1$), to reproduce the vacuum compressibility

$$\frac{\langle (N - \bar{N})^2 \rangle_P}{\bar{N}} = \frac{4}{\bar{b}}$$

(8)

expected from QCD low-energy theorems [16].

### B. Spontaneous breaking of chiral symmetry and conductance fluctuations

The quantum breaking of conformal symmetry and the generation of the $R = 1$ fm and a gluon condensate, is a direct measure of the instanton tunneling rate or topological density in the vacuum. In the quenched approximation, it is solely a property of the gluon fields. This is a necessary but not a sufficient mechanism for hadronic mass generation. The sufficient mechanism, which relies on these topological fields, is the delocalization of the quark zero modes and the ensuing spontaneous breaking of chiral symmetry. The result is a vacuum chiral condensate and the emergence of a quark mass, both of which are fixed by the same $R = 1$ fm scale at the origin of the gluon condensate. This topological mechanism for mass generation leaves behind a fingerprint: universal conductance-like fluctuations in the quark spectrum [7]. This is a lesser known fundamental phenomenon, that we now briefly discuss.

Decades ago, Banks and Casher observed that the spontaneous breaking of chiral symmetry with a finite chiral condensate $\langle \bar{\psi}\psi \rangle$, is associated to a huge accumulation of the quark zero modes near the zero point of the virtual quark spectrum as illustrated in Fig. 2 (left), and captured by the relation [17]

$$\langle \bar{\psi}\psi \rangle = -\pi \nu(0) \equiv -\sigma_C$$

(9)

The quark density of states is

$$\nu(\lambda) = \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \left\langle \sum_n \delta(\lambda - \lambda_n[A]) \right\rangle_A \equiv \frac{1}{V} \frac{1}{\Delta \lambda}$$

(10)
with the virtual quark eigenstates solution to

\[(iD[A] + im)q_n[A] = (\lambda_n[A] + im)q_n[A]\]

for a given gauge configuration. A finite \(\nu(0)\) means that \(\Delta \lambda \approx R^3/V\), as opposed to \(\Delta \lambda \approx 1/4\sqrt{V}\) in the continuum. The quark spectrum is extremely dense near \(\lambda = 0\), as the disordering turn the quark zero modes to quasi-zero modes.

FIG. 2. Left: sketch of the virtual Dirac spectrum for the disordered quark zero modes. Right: the universal spectral or conductance fluctuations (dotted line) predicted by random matrix theory (11) [7], and measured in lattice generated gauge configurations (solid line) [8]. See text.

Remarkably, the Banks-Casher relation (9) resembles the Kubo formula for the DC conductivity in metals with \(\sigma_C \leftrightarrow -\langle \bar{\psi}\psi \rangle\) and the zero-virtuality point \(F = \lambda = 0\) playing the role of the Fermi-surface. The QCD vacuum turns metallic when chiral symmetry is spontaneously broken. The same holds for the QCD instanton vacuum. In the unquenched case, this statement is extremely non-trivial. A too dilute or too dense topologically active vacuum would result into either a neutral gas of instanton-anti-instanton molecules, or a crystal arrangement of instantons-anti-instantons, instead of a liquid, with no spontaneous breaking of chiral symmetry [1–3]. We are thankfull that nature has served us a dilute liquid!

In fact there are infinitely many Banks-Casher-like formula which capture the fluctuations of the conductance \(\sigma_C\) in the mesoscopic limit, with the connected moments \(\sigma^n_C = \langle (\bar{\psi}\psi)^n \rangle_C\). These conductance fluctuations are universal, and follow from chiral random matrix theory, with the result for the mesoscopic spectral density at zero virtuality given by the master formula [7]
\[ \nu_s(z = N\lambda) = \frac{z}{2} \left( J_{Nf}^2(z) - J_{Nf+1}(z)J_{Nf-1}(z) \right) \]  

(11)

with \( N, V \to \infty \) but fixed \( z = N\lambda \) and \( N/V \). These predicted conductance fluctuations were later measured in numerically generated lattice gauge configurations as shown in Fig. 2 [8] (right). The dotted line is the result (11) and the solid line is the lattice measurement. This is one of the signature for the topological origin of the hadronic mass scale, as it centers on the quark zero mode zone. The zero modes only occur in the presence of topological gauge fields such as the instantons and anti-instantons [18] (or their close cousins the instanton-dyons [19, 20]), a consequence of the Atiyah-Singer index theorem (or the Atiyah-Patodi-Singer for their cousins).

### III. MASS IDENTITY

We now focus on the anomalous trace of the QCD energy momentum tensor, and its relation to the hadron mass. It is worth stressing that the ensuing relation to the mass is just a bulk identity and not a mass decomposition. Having said that, the trace couples to a scalar dilaton which sources a sigma (2-pion) meson and/or a \( 0^{++} \) scalar glueball field. QCD perturbative arguments suggest that this trace may be accessible in the photo-production of charmonium at threshold [21, 22] (and references therein), although the coupling to the \( 2^{++} \) tensor glueball may still be very active in the threshold region [23]. Recall that the Reggeized form of the \( 2^{++} \) exchange transmutes to the Pomeron, and is dominant asymptotically.

#### A. Nucleon

For a nucleon state \( |P\rangle \) with the standard normalization \( \langle P|P' \rangle = 2E_P(2\pi)^3\delta^3(P-P') \), one has

\[ \langle P|T^\mu\nu|P \rangle = 2P^\mu P^\nu . \]  

(12)

with the trace in any frame (one-loop)

\[ \langle P|T^\mu_\mu|P \rangle = \langle P| \left( - \frac{b}{32\pi^2} F^2 + m\bar{\psi}\psi \right) |P \rangle = 2M_N^2 . \]  

(13)

with \( g^2F^2 \to F^2 \) for the strong instanton and anti-instanton gauge fields. It is renormalization group invariant. The identity (13) shows that the nucleon mass is the change of
the conformal anomaly or gluon field in a nucleon state. However, the formation of the state occurs only if chiral symmetry is spontaneously broken as we noted earlier. (13) is a QCD identity that is satisfied in the QCD instanton vacuum as we now show.

In the rest frame, the gluon contribution in (13) follows from the normalized and connected 3-point function asymptotically

\[
\frac{\langle P|F^2|P \rangle}{\langle P|P \rangle} = \lim_{T \to \infty} \frac{\langle J_P^I(T)F^2J_P(-T) \rangle_C}{\langle J_P^I(T)J_P(-T) \rangle} \tag{14}
\]

with \( J_P \) a pertinent nucleon source. In the canonical description of the QCD instanton vacuum \( F^2/(32\pi^2) \to \bar{N}/V \) is a number. It factors out in the 3-point correlator in (14) (numerator), and the connected correlator vanishes.

A non-vanishing contribution to the connected 3-point correlator follows from the grand-canonical description, where \( N \) is allowed to fluctuate as we noted in (7). With this in mind, it is straightforward to see that (14) is dominated by the variance

\[
\frac{V}{32\pi^2} \frac{\langle P|F^2|P \rangle}{\langle P|P \rangle} \approx \langle (N - \bar{N})^2 \rangle_P \frac{\partial}{\partial \bar{N}} \log \left( \lim_{T \to \infty} \frac{\langle J_P^I(T)J_P(-T) \rangle}{\langle J_P^I(T)J_P(-T) \rangle} \right) \tag{15}
\]

with the higher moments suppressed by \( 1/b^2 \sim 1/N_c^2 \). The result (15) was noted in [14] (see Eq. 5.8) using a fermionization method, and in [15] (see Eqs. 91,93) using a bosonization method, each of the QCD instanton vacuum in the \( 1/N_c \) approximation. The expectation value in the first bracket is carried using the distribution (7). (15) illustrates how the nucleon scoops the epoxy from the QCD instanton vacuum.

All dimensions in the QCD instanton vacuum are fixed by the density \( \bar{N}/V = 1/R^4 \) and the current quark masses. The nucleon mass is the sum of the chirally symmetric (invariant mass) plus the symmetry breaking contribution (pion-nucleon sigma term),

\[
M_N = M_{\text{inv}} + \sigma_{\pi N} = C \left( \frac{\bar{N}}{V} \right)^{\frac{1}{4}} + \bar{C} m \left( 1 + O(mR) \right) \tag{16}
\]

with [24–26]

\[
\sigma_{\pi N} = \frac{\langle P|m\bar{\psi}\psi|P \rangle}{\langle P|P \rangle} \approx 50 \text{ MeV} \tag{17}
\]

evaluated at the soft renormalization scale \( \bar{\rho} = 0.6 \text{ GeV} \), which is the appropriate scale for hadronic spectroscopy. The right-most relation in (16) follows from the QCD instanton vacuum. As a result, the anomalous contribution in the QCD instanton vacuum is
\[
\frac{V}{2T} \frac{-b}{32\pi^2} \frac{\langle P|F^2|P \rangle}{\langle P|P \rangle} = 4 \frac{\partial M_N}{\partial \log N} = M_{\text{inv}}
\]

which is seen to satisfy the sum rule

\[
\frac{\langle P|T_{\mu}^\mu|P \rangle}{2M_N} = M_{\text{inv}} + \frac{\langle P|m\bar{\psi}\psi|P \rangle}{2M_N} = M_N
\]

B. Pion

The preceding arguments apply also to the pion, with one major difference,

\[
m_\pi = C \sqrt{m} \left( \frac{\bar{N}}{V} \right)^{1/8} (1 + O(mR))
\]

since it is a Goldstone mode. The \(O(mR)\) corrections are small in the QCD instanton vacuum. It follows that

\[
\frac{V}{2T} \frac{-b}{32\pi^2} \frac{\langle \pi|F^2|\pi \rangle}{\langle \pi|\pi \rangle} = 4 \frac{\partial m_\pi}{\partial \log \bar{N}} = \frac{1}{2} m_\pi
\]

which was first observed in [27], with the sum rule

\[
\frac{\langle \pi|T_{\mu}^\mu|\pi \rangle}{2m_\pi} = \frac{1}{2} m_\pi + \frac{\langle \pi|m\bar{\psi}\psi|\pi \rangle}{2m_\pi} = m_\pi
\]

satisfied, as expected. The pion sigma-term follows from chiral reduction or the Feynman-Hellmann theorem

\[
\frac{\langle \pi|m\bar{\psi}\psi|\pi \rangle}{2m_\pi} = \frac{\partial E_\pi}{\partial \log m} = \frac{1}{2} m_\pi
\]
IV. JI MASS SUM RULE

The trace identity (12) reflects on the general fact that all hadron masses in QCD are tied to the quantum breaking of conformal symmetry as we noted earlier, and should be enforced by any non-perturbative quantum description, whether numerical such as the lattice or analytical such as the QCD instanton vacuum. However, it does not specifically budget this mass breaking in terms of the hadron constituents. In a strongly interacting theory, this issue may be elusive, especially with a soft renormalization scale, as the gluons are strongly untwined with the light quarks. This is more so in the unquenched and screened formulation.

This notwithstanding, a specific and physically motivated proposal to budget the mass, was put forth by Ji in [9, 27], and since revisited by many [28–30] (and references therein). The ensuing mass composition involves the sum of partonic contributions, some of which may be measurable using DIS experiments. The proposal relies on an invariant decomposition of the energy momentum tensor which we now detail.

The energy-momentum tensor (3) can be decomposed as the sum of a traceless and tracefull part [27, 31]

$$ T^{\mu \nu} = \tilde{T}^{\mu \nu} + \hat{T}^{\mu \nu} + g^{\mu \nu} \frac{1}{4} T_{\alpha}^{\alpha} , $$

(24)

where the traceless part reads

$$ \tilde{T}^{\mu \nu} = \left( - F^{a \mu \tau} F_{a \nu}^{\tau \nu} + \frac{1}{4} g^{\mu \nu} F^2 \right) + \frac{1}{4} \psi [D^\mu, D^\nu] \psi - g^{\mu \nu} \frac{1}{4} m \bar{\psi} \psi , $$

(25)

and the tracefull part is

$$ \hat{T}^{\mu \nu} = g^{\mu \nu} \frac{1}{4} \left( \frac{\beta (g^2)}{4 g^4} F^2 + m \bar{\psi} \psi \approx - \frac{b}{32 \pi^2} F^2 + m \bar{\psi} \psi \right) $$

(26)

We note that this decomposition is commensurate with the analysis of the nucleon energy momentum tensor in holographic QCD, through dual gravitons in bulk [23]. (Holography is a good example of a strong coupling description of a gauge theory via its gravity dual, where the partonic structure is elusive).

The tracefull and traceless part of the energy momentum tensor (24-26) correspond to the spin-2 and spin-0 representations of the Lorentz group, and do not mix under renormalization by symmetry. Their renormalization at the instanton size scale $\bar{\rho} \approx 0.3$ fm is subsumed throughout. On the lattice, this soft renormalization scale is best achieved using a cooling procedure where only the UV quantum and non-singular fluctuations are
subtracted (our instantons are classical fields in singular gauge!). Note that our renormalization scale is softer than the one used in currently fine lattices with $1/\mu \approx 0.1 \text{fm (MS scheme)}$ [32]. This difference will be further discussed below.

With this in mind, the matrix elements of the split energy-momentum tensor are constrained by Lorentz symmetry

$$\langle P | \bar{T}^{\mu\nu} | P \rangle = 2 \left( P^\mu P^\nu - \frac{1}{4} g^{\mu\nu} M_N^2 \right)$$

$$\langle P | \bar{T}^{\mu\nu} | P \rangle = \frac{1}{2} g^{\mu\nu} M_N^2$$

(27)

The corresponding Hamiltonian in Minkowski signature, follows from the 00-component of (24-26) modulo BRST exact and gauge dependent contributions,

$$H_G = \int d^3 x \, \bar{T}^{00}_G = \int d^3 x \frac{1}{2} (E^2 + B^2)$$

$$H_Q = \int d^3 x \, \bar{T}^{00}_Q = \int d^3 x \left( \frac{1}{2} \bar{\psi} \gamma \cdot i \bar{\nabla} \psi + \frac{3}{4} m \bar{\psi} \psi \right)$$

$$H_A = \int d^3 x \, \bar{T}^{00}_A = \int d^3 x \left( \frac{1}{4} \beta \left( g^2 \right) F^2 + m \bar{\psi} \psi \approx - \frac{b}{32 \pi^2} F^2 + m \bar{\psi} \psi \right)$$

$$H_m = \int d^3 x \, \bar{T}^{00}_G = \int d^3 x \, m \bar{\psi} \psi$$

(28)

where the time $t = 0$ is subsumed. The mass term can be rearranged so that (28) reads

$$H_G = \int d^3 x \, \bar{T}^{00}_G = \int d^3 x \frac{1}{2} (E^2 + B^2)$$

$$H_Q = \int d^3 x \, \bar{T}^{00}_Q = \int d^3 x \left( \frac{1}{2} \bar{\psi} \gamma \cdot i \bar{\nabla} \psi \right)$$

$$H_A = \int d^3 x \, \bar{T}^{00}_A = \int d^3 x \left( \frac{1}{4} \beta \left( g^2 \right) F^2 \approx - \frac{b}{32 \pi^2} F^2 \right)$$

$$H_m = \int d^3 x \, \bar{T}^{00}_G = \int d^3 x \, m \bar{\psi} \psi$$

(29)

The nucleon mass budget is then

$$M_N = \frac{\langle P | H_G + H_Q + H_A + H_m | P \rangle}{\langle P | P \rangle} \equiv M_N^G + M_N^Q + M_N^A + M_N^m$$

(30)

which shows that the combination

$$M_{\text{inv}} = M_N^G + M_N^Q + M_N^A$$

(31)
is chirally symmetric and equal to the invariant mass in (16).

In Euclidean signature, whether on the lattice or using the QCD instanton vacuum, (30) can be evaluated by trading $T^{00} \rightarrow T^{44}$ and $t = 0 \rightarrow i0$. In the dilute QCD instanton vacuum, the gluonic operator in (28-30) is the sum of multi-instanton contributions of the form

$$T^{44}_G[A] = \sum_{I=1}^{N_\pm} T^{44}_G[A_I(\xi_I)] + \sum_{I \neq J=1}^{N_\pm} T^{44}_G[A_I(\xi_I), A_J(\xi_J)] + \ldots = \sum_{I \neq J=1}^{N_\pm} T^{44}_G[A_I(\xi_I), A_J(\xi_J)] + \ldots$$

(32)

Since the first one-instanton contribution in (32) is composed of self-dual fields it vanishes. So we are left with only the two and higher multi-instanton contributions. When averaged over a measure of independent instantons, the remaining terms in (32) are suppressed by the diluteness factor $\kappa \approx 0.1$. As a result, the contribution of $M^N_G$ is parametrically small in comparison to $M^N_Q$ or $4M^N_{\pi}$, i.e. $M^N_G/M^N_Q \approx \kappa \approx 0.1$. The contributions $M^N_{Q,m}$ are solely given in terms of the fermionic zero modes (modulo the instanton gauge fields in the long derivative).

With this in mind, the breakdown in the mass budget (30) for the nucleon yields the estimates

$$\frac{M^N_Q}{M_N} \approx \frac{3}{4} \frac{1}{1 + \kappa} \left(1 - \frac{\sigma_{\pi N}}{M_N}\right) \approx 64\%$$
$$\frac{M^N_G}{M_N} \approx \frac{3}{4} \frac{\kappa}{1 + \kappa} \left(1 - \frac{\sigma_{\pi N}}{M_N}\right) \approx 7\%$$
$$\frac{M^N_A}{M_N} = \frac{1}{4} \left(1 - \frac{\sigma_{\pi N}}{M_N}\right) \approx 24\%$$
$$\frac{M^N_{\pi}}{M_M} = \frac{\sigma_{\pi N}}{M_N} \approx 5\%$$

(33)

with the empirical pion-nucleon sigma term (17). The anomalous contribution is scale and scheme independent at one-loop order. The mass contribution is also renormalization group invariant. (33) shows that in the QCD instanton vacuum, about 70% of the nucleon mass stems from the valence quarks (hopping zero modes), 25% from the gluon condensate or epoxy (displaced vacuum instanton field), and 7% from emerging valence gluons. The nucleon is composed mostly of quark constituents hopping and dragging the gluon epoxy.

The gluon epoxy in the nucleon is the quantum anomalous energy in the nucleon discussed recently in [33].

We note that the budgeting of the nucleon mass in (33) differs from the one reported on the lattice in [32], with a noticeably larger valence gluon fraction in the lattice nucleon.
In our analysis, this can only beaccommodated by a stronger instanton packing fraction of $\kappa \approx 0.5$ instead of 0.1, which is unlikely. (Note that larger values of $\kappa$ that include close instanton-anti-instanton pairs, not responsible for the breaking of chiral symmetry, were reported when analyzing certain correlations at zero cooling time [34]). The harder renormalization scale $\mu = 2$ GeV used in the reported lattice results, is the likely source of the valence and perturbative gluon enhancement reported in the lattice nucleon. Quantum evolution will enhance $M_G^N$ at the expense of $M_Q^N$, which in (33) would amount to effectively dressing $\kappa \approx 0.1 \rightarrow 0.5$ at $\mu = 2$ GeV.

Finally, a similar mass decomposition holds for the pion at the same soft renormalization scale of $\bar{\rho} = 0.6$ GeV, with the estimates

$$
\frac{M_Q^\pi}{m_\pi} \approx \frac{3}{8} \frac{1}{1 + \kappa} \approx 34% \\
\frac{M_G^\pi}{m_\pi} \approx \frac{3}{8} \frac{\kappa}{1 + \kappa} \approx 3% \\
\frac{M_A^\pi}{m_\pi} = \frac{1}{8} \approx 13% \\
\frac{M_m^\pi}{m_\pi} = \frac{1}{2} \approx 50% 
$$

(34)

About 85% of the pion mass stems from the valence quarks (hopping zero modes), 13% from the gluon condensate or epoxy (displaced vacuum instanton field), and 3% from emerging valence gluons. Needless to say that all mass contributions in the pion vanish smoothly in the chiral limit. Again, quantum evolution will enhance $M_G^\pi$ at the expense of $M_Q^\pi$, with effectively dressing $\kappa \approx 0.1 \rightarrow 0.5$ at $\mu = 2$ GeV.

V. MEASURING THE QCD VACUUM COMPRESSIBILITY

While the present discussion has focused on some key aspects of the QCD vacuum and the hadronic mass sum rule, it is worth noting that the results (15-21) can be recast in the following form

$$
\frac{\langle P \mid F^2(0) \mid P \rangle}{(4\pi(m_N - \sigma_{\pi N}/2))^2} \approx -\sigma_{F^2} 
$$

(35)

with the QCD vacuum compressibility

$$
\sigma_{F^2} = \frac{1}{32\pi^2} \int d^4x \frac{\langle F^2(x) F^2(0) \rangle_C}{\langle F^2(0) \rangle} 
$$

(36)
A measure of the gluon condensate or epoxy inside the proton (left hand-side) is a measure of the QCD vacuum compressibility $\sigma_{F^2}$ (right hand-side), modulo the pion-nucleon sigma term which is small. Since (35) is a nucleon connected matrix element, it is natural that it probes the fluctuations of $F^2$. While in the vacuum state the gluon condensate is positive, (36) shows that it is negative in the nucleon state. The nucleon state carries less epoxy.

The cooled Yang-Mills vacuum in Fig. 1 is composed of interacting topological charges. The vacuum compressibility $\sigma_{F^2}$ captures the squared variance of their interactions: $\sigma_{F^2} = 1$ for a non-interacting gas phase, $\sigma_{F^2} < 1$ for an interacting liquid phase, and $\sigma_{F^2} \ll 1$ for a strongly interacting crystal phase. QCD low-energy theorems suggest $\sigma_{F^2} \approx 4/b \approx 4/11$ (one-loop and quenched) [16], so the QCD instanton vacuum appears to be a dilute quantum topological liquid. A measure of $\sigma_{F^2}$ is a measure of a fundamental and universal parameter of the QCD vacuum.

VI. CONCLUSIONS

The QCD instanton vacuum is populated with topological tunneling configurations, with each costing zero energy. The way a light quark can propagate coherently through this maze of tunneling configurations is through its zero mode, scattering and hopping from an instanton to an anti-instanton and so on. The scattering through the instanton flips chirality, an amazing effect caused by a non-perturbative vector interaction (a perturbative gluon interaction preserves chirality). The hopping generates a very dense band in the virtual quark spectrum, reminiscent of the conduction band in conductors. As a result, chiral symmetry is spontaneously broken, a chiral condensate is formed and a running constituent quark mass emerges.

The QCD instanton vacuum breaks simultaneously conformal symmetry, with a large and negative vacuum energy density, or equivalently a large and positive gluon condensate (gluon epoxy). A hadronic excitation in this vacuum, whether a quark, a meson or a baryon costs energy or mass. A useful and physical way to budget this mass is Ji’s mass decomposition of the energy momentum tensor [9, 27]. In the QCD instanton vacuum, we find that the hadronic masses are largely due to the contribution of the valence quarks as they hop and drag the gluon epoxy.

Finally, a measure of the gluon condensate or epoxy in the nucleon, is a measure of the compressibility of the QCD instanton vacuum as a topological liquid. The diluteness of this liquid is central in our non-perturbative understanding of the emergence of mass in QCD using analytical methods. This gluonic content of the proton may be accessible through threshold electromagnetic production of heavy quarkonia [22], and perhaps diffractive cluster production in hadron-hadron collisions [35] at current and future facilities.

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