HYPERBOLIC TOPOLOGICAL INVARIANTS AND THE BLACK HOLE GEOMETRY

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Abstract. We discuss the isometry group structure of three-dimensional black holes and Chern–Simons invariants. Aspects of the holographic principle relevant to black hole geometry are analyzed.

1. Introduction

The discovery of black hole solutions in three–dimensional gravity offers a promising new area for the analysis of interesting and difficult problems that were posed in the four-dimensional case. We begin here with the discussion of some geometrical aspects of the non–rotating, three–dimensional black hole [1]. It is known that a static Lorentzian metric is a solution of the three–dimensional vacuum Einstein equation with negative cosmological constant $\Lambda$, i.e. $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$, $R = 6\Lambda = -6\sigma^{-2}$. The sectional curvature $k$ is constant and negative, namely $k = -\sigma^{-2}$. The metric describes a space–time which is locally isometric to the anti–de Sitter space. The Euclidean section is obtained by the Wick rotation $t \rightarrow i\tau$, and reads

$$ds^2 = \left( r^2\sigma^{-2} - M \right) d\tau^2 + \left( r^2\sigma^{-2} - M \right)^{-1} dr^2 + r^2 d\varphi^2 , \quad (1.1)$$

where the coordinates $(t, r, \varphi)$ have been used (8G=1 is assumed, so that the mass $M$ is dimensionless), and $\sigma$ is a dimensional constant. The metric (1.1) has a horizon radius given by $r_+ = \sigma M^{1/2}$. With a change of coordinates, $(\tau, r, \varphi) \rightarrow (y, x_1, x_2)$, of the form

$$y = \frac{r_+}{r} \exp \left( \frac{r_+}{\sigma} \varphi \right), \quad x_1 + ix_2 = \frac{1}{r} \left( r^2 - r_+^2 \right)^{1/2} \exp \left( i \frac{r_+}{\sigma^2 \tau} + \frac{r_+}{\sigma} \varphi \right) , \quad (1.2)$$

the metric becomes

$$ds^2 = \frac{\sigma^2}{y^2} \left( dy^2 + dx_1^2 + dx_2^2 \right) . \quad (1.3)$$

As a consequence, the metric describes a manifold homeomorphic to the real hyperbolic space $\mathbb{H}^3$. The group of isometries of $\mathbb{H}^3$ is $SL(2, \mathbb{C})$. 

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Let us now consider a discrete subgroup $\Gamma \subset PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C})/\{\pm Id\}$ ($Id$ is the identity element), which acts discontinuously at the point $z$ belonging to the extended complex plane $\mathbb{C} \cup \{\infty\}$. We recall that a transformation $\gamma \neq Id$, $\gamma \in \Gamma$, with $\gamma z = (az + b)(cz + d)^{-1}$, $ad - bc = 1$, $a, b, c, d \in \mathbb{C}$, is called elliptic if $(\text{Tr} \gamma)^2 = (a + d)^2$ satisfies $0 \leq (\text{Tr} \gamma)^2 < 4$, hyperbolic if $(\text{Tr} \gamma)^2 > 4$, parabolic if $(\text{Tr} \gamma)^2 = 4$, and loxodromic if $(\text{Tr} \gamma)^2 \in \mathbb{C}\{0, 4\}$.

The periodicity of the angular coordinate $\varphi$ allows to describe the black hole manifold as the quotient $\Gamma \backslash \mathbb{H}^3$, $\Gamma$ being a discrete group of isometries possessing a primitive element $\gamma_h \in \Gamma$, defined by the identification $\gamma_h(y, w) = (\exp(2\pi r_+ \sigma^{-1})y, \exp(2\pi r_+ \sigma^{-1}w)) \sim (y, w)$. Therefore, the matrix

$$
\gamma_h = \begin{pmatrix}
ed^\frac{\varphi}{\sigma} & 0 \\
0 & e^{-\frac{r_\pm}{\sigma}}
\end{pmatrix},
$$

(1.4)

corresponds to an hyperbolic element $(\text{Tr} \gamma_h > 2)$ consisting in a pure dilatation. Furthermore, since in Euclidean coordinates $\tau$ becomes an angular variable, with period $\beta$, one is led also to the identification $\gamma_e(y, w) = (y, \exp(i\beta r_+ \sigma^{-2})w) \sim (y, w)$. This identification is generated by an elliptic element in the group $\Gamma$, namely

$$
\gamma_e = \begin{pmatrix}
ed^{i\beta r_+} & 0 \\
0 & e^{-i\beta r_+}
\end{pmatrix},
$$

(1.5)

as long as $\text{Tr} \gamma_e < 2$, and a conical singularity will be present. If $\beta r_+ \sigma^{-2} = 2\pi$, then $\gamma_e = Id$ and the conical singularity is absent. As a result, the period is determined to be $\beta_H = 2\pi \sigma^2 (r_+)^{-1}$ and this is interpreted as the inverse of the Hawking temperature. The $M = 0$ line element has the form $ds_0^2 = r^2 \sigma^{-2}dr^2 + \sigma^2 r^2 d\sigma^2$. It is clear that $\tau$ can be identified with any period $\beta$ (in particular with $\beta = \infty$), and that $\varphi$ has period $2\pi$.

Through the coordinate change: $r = y^{-1} \sigma^2$, $\tau = x_1$, $\varphi = x_2 \sigma^{-1}$, one gets the metric of the hyperbolic space: $ds_0^2 = \sigma^2 y^{-2}(dx_1^2 + dx_2^2 + dy^2)$, and the identification $\gamma_p(w, y) = (w + \beta + 2i\pi \sigma y) \simeq (w, y)$. This identification is generated by elements of $\Gamma$ of the form

$$
\gamma_{p_1} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \gamma_{p_2} = \begin{pmatrix} 1 & 2i\pi \sigma \\ 0 & 1 \end{pmatrix},
$$

(1.6)

which are parabolic. Therefore the on-shell three-dimensional black hole can be regarded as a strictly hyperbolic, non-compact manifold $\Gamma \backslash \mathbb{H}^3$ (see Sect. 2).

The group $SL(2, \mathbb{C})$ being the universal covering of the Lorentz group, for a Euclidean space there are one-forms which can be combined to build a single $SL(2, \mathbb{C})$ connection $A^\mu_\nu$. It has been shown [2] that the Einstein action can be reduced to the Chern–Simons action associated with the connection $A^\mu_\nu$. $SL(2, \mathbb{C})$ gauge transformations in the Chern–Simons formulation are equivalent to diffeomorphisms and local Lorentz transformations in the metric formulation of the black hole [3]. In Sect. 3 we will work with the
Chern–Simons formulation, which follows from the relationship (non-trivial, in fact, since it has a nonvanishing holonomy) between flat connections and geometric structures related to one–forms. Some aspects of the holographic principle for hyperbolic geometry will be considered in Sect. 4.

2. The Arithmetic Geometry of $\Gamma = SL(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \text{Id}\}$

Here we shall summarize the geometry and local isometry associated with a simple three–dimensional complex Lie group. We shall consider that the discrete subgroup $\Gamma$ acts discontinuously at the points of the extended complex plane. The isometric circle of a transformation $\gamma \in G = SL(2, \mathbb{C})/\{\pm \text{Id}\}$ for which $\infty$ is not a fixed point is defined to be the circle $I(\gamma) = \{z : |\gamma z| = 1\}$, or $I(\gamma) = \{z : |z + d/c| = |c|^{-1}\}, \ c \neq 0$. A transformation $\gamma$ carries its isometric circle $I(\gamma)$ into $I(\gamma^{-1})$. For $\Gamma \subset G$, one can choose a subgroup $\Gamma$ of the form $SL(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \text{Id}\}$, where $\mathbb{Z}$ is the ring of integers. The element $\gamma \in \Gamma$ will be identified with $-\gamma$. The group $\Gamma$ has, within a conjugation, one maximal parabolic subgroup $\Gamma_{\infty}$. The fundamental domain related to $\Gamma$ has one parabolic vertex, and can be taken to be of the form: $F(\Gamma) = \{(y, w) : x_1^2 + x_2^2 + y^2 > 1, \ -1/2 < x_2 < x_1 < 1/2\}$. The Selberg trace formula can be constructed as an expansion in eigenfunctions of the automorphic Laplacian. Since the discrete group $\Gamma$ has a cusp at $\infty$, each element of the stabilizer $\Gamma_{\infty}$ is a translation. The conjugacy class $\{\gamma\}_G$, $\gamma \in \Gamma_{\infty}$, with $\gamma$ different from the identity, and the centralizers related to this class of $\gamma$, have been calculated in Refs. 4, 5. Let $\lambda_j$ be the isolated eigenvalues of the self–adjoint extension of the Laplace operator and let us introduce a suitable analytic function $h(r)$ and $r_j^2 = \lambda_j - 1$. For one parabolic vertex let us introduce a subdomain $F_y$ of the fundamental region $F(\Gamma)$ by means of $F_y = \{z \in F(\Gamma), \ z = \{y, x\} | y \leq \mathcal{Y}\}$, where $\mathcal{Y}$ is a sufficiently large positive number.

**Lemma 1** (Ref. [4], Eq. (13)). Suppose $h(r)$ to be an even analytic function in the strip $|3r| < 1 + \varepsilon \ (\varepsilon > 0)$, and $h(r) = \mathcal{O}(1 + |r|^2)^{-2}$. Then the following formula holds:

$$
\sum_j h(r_j) = \lim_{\mathcal{Y} \to \infty} \left\{ \int_{F_y} \sum_{\{\gamma\}_G} k(u(z, \gamma z)) \ d\mu(z) \right\},
$$

$$
-\frac{1}{2\pi} \int_{R_+} h(r) \left( \int_{F_y} |E(z, 1 + ir)|^2 \ d\mu(z) dr \right),
$$

where $d\mu(z) = y^{-3} dy dx_1 dx_2$ is the invariant measure on $\mathbb{H}^3$, $k(z, z') = k(u(z, z'))$ is the kernel of the invariant operator, $u(z, z') = |z - z'|^2 / yy'$, and $E(z, s)$ is the Eisenstein–Maass series associated with one cusp.

The final result (see for detail Ref. [5]) could be considered as an addition to this Lemma 1.
Theorem 2  For the special discrete group $SL(2,\mathbb{Z} + i\mathbb{Z})/\{\pm Id\}$ and $h(r)$ satisfying the conditions of Lemma 1, the Selberg’s trace formula holds

$$\sum_{\gamma} h(r_{\gamma}) - \sum_{\gamma, \gamma \neq Id, \gamma \text{--non--parabolic}} \int k(u(z, \gamma z)) \, d\mu(z) = \text{vol}(F) \int_{\mathbb{R}_+} \frac{r^2}{2\pi^2} h(r) \, dr$$

$$+ \frac{1}{4\pi} \int_{\mathbb{R}} h(r) \left( \frac{d}{ds} \log S(s)|_{s=1+ir} - \psi(1 + ir/2) \right) + \frac{h(0)}{4} (1 - S(1)) + Cg(0).$$

(2.2)

Here $\text{vol}(F)$ is the (finite) volume of the fundamental domain $F$ with respect to the measure $d\mu$, the function $S(s)$ (in the general case it is the $S$-matrix) is given by a generalized Dirichlet series, convergent for $\Re s > 1$. The functions $E(z, s)$ and $S(s)$ can be analytically extended on the whole complex $s$–plane, where they satisfy the functional equations $S(s)S(1 - s) = Id$, $S(s) = S(s)$, $E(z, s) = S(s)E(z, 1 - s)$. $\psi(s)$ is the logarithmic derivative of the Euler $\Gamma$–function, $C$ a computable constant and $g(u)$ denotes the Fourier transform of $h(r)$.

2.1. The functional determinant. In general, the determinant of an elliptic differential operator requires a regularization. It is convenient to introduce the operator $\Sigma_{\Gamma}(\delta) = \Sigma_{\Gamma} + \delta^2 - 1$, with $\delta$ a suitable parameter. One of the most widely used regularization is the zeta–function regularization (see Refs. [6,7,8]). Thus one has $\log\det\Sigma_{\Gamma}(\delta) = -(d/ds)\zeta(s|\Sigma_{\Gamma}(\delta))|_{s=0}$. In standard cases, the zeta function at $s = 0$ is well defined and one gets a finite result. The meromorphic structure of the analytically continued zeta function is related to the asymptotic properties of the heat–kernel trace. For the rank one symmetric space $\Gamma \backslash \mathbb{H}^3$ the trace of the operator $\exp[-(t\Sigma_{\Gamma}(\delta))]$ could be computed by using Theorem 2 (Eq. (2.2)) with the choice $h(r) = \exp[-t(r^2 + \delta^2)]$ (we use units in which the curvature of $\mathbb{H}^3$ is equal to $-1$). Thus we have $g(u) = (4\pi t)^{-1/2} \exp(-t\delta^2 - u^2(4t)^{-1})$, $g(0) = (4\pi t)^{-1/2} \exp(-t\delta^2)$, and $h(0) = \exp(-t\delta^2)$.

We shall consider additive terms of the zeta function associated with the identity and parabolic elements of the group $\Gamma$ only (the heat kernel and the zeta–function analysis for co–compact discrete group $\Gamma$ has been done, for example, in Refs. [6,7]).

$$\text{Tr} \left( \exp(-t(\Sigma_{\Gamma}(\delta) - \delta^2)) \right) = \frac{\text{vol}(F) + 4\pi t C}{(4\pi t)^{3/2}} + \frac{1}{4\pi} \int_{\mathbb{R}} \psi(1 + ir/2) e^{-tr^2} \, dr.$$

(2.3)

Making use of the trace formula, we compute the functional determinant of the Laplace–type operator on $\Gamma \backslash \mathbb{H}^3$. The zeta function for $\Re s$ sufficiently large, can be rewritten in the form

$$\zeta(s|\Sigma_{\Gamma}(\delta)) = \sum_{\sigma} \rho_{\sigma} \left( \lambda_{\sigma} + \delta^2 - 1 \right)^{-s} = \sum_{\gamma} \left( \lambda_{\gamma} + \delta^2 - 1 \right)^{-s}$$
where the sum over $j$ runs over the discrete spectrum, $\{\lambda_j\} \in \text{Spec}(\mathcal{L}_\Gamma)$. For the continuous spectrum, $\rho_\lambda$ is proportional to the logarithmic derivative of the $S$–matrix $S(s)$.

One has $(d/ds)\zeta(s|\mathcal{L}_\Gamma(\delta)) = -\sum_\sigma \rho_\sigma (\lambda_\sigma + \delta^2 - 1)^{-s} \log(\lambda_\sigma + \delta^2 - 1)$, and

$$
\frac{d}{ds} \left( \frac{1}{2\delta} \frac{d}{d\delta} \frac{d}{ds} \zeta(s|\mathcal{L}_\Gamma(\delta)) \right) = 2\delta \sum_\sigma \rho_\sigma (\lambda_\sigma + \delta^2 - 1)^{-s-2} + O(s). \tag{2.5}
$$

A standard Tauberian argument gives a Weyl’s estimate for large $\sigma$, namely [5]: $(\lambda_\sigma + \delta^2 - 1)^{-1} \simeq \sigma^{-2/3}$. As a consequence, in the limit $s \to 0$, the right hand side of Eq. (2.5) is finite. This works for two– and three–dimensional cases. In higher dimensions it is necessary to take further derivatives with respect to $\delta$ [9]. The inclusion of the contribution related to the hyperbolic elements is almost straightforward and can be found, for example, in Refs. [6,7]. It is additive and reads simply $\log Z_\Gamma(1 + \delta)$, like before $Z_\Gamma(s)$ is the Selberg zeta function. Summarizing, the final result is:

**Theorem 3** (Ref. [5], Theorem 2). The following identity holds

$$
\det \mathcal{L}_\Gamma(\delta) = \frac{2}{(\pi\delta)^{1/2}\Gamma(\delta/2)} \exp \left( -\frac{\text{vol}(F)\delta^3}{6\pi} + C\delta \right) Z_\Gamma(1 + \delta). \tag{2.6}
$$

### 3. Topological Invariants of the Hyperbolic Geometry

The Chern–Simons partition function may be expressed through the asymptotics which lead to a series of $C^\infty$– invariants associated with triplets $\{X; F; \xi\}$, with $X$ a smooth homology three–sphere, $F$ a homology class of framings of $X$, and $\xi$ an acyclic conjugacy class of orthogonal representations of the fundamental group $\pi_1(X)$ [10]. In addition, the cohomology $H(X; \text{Ad}\xi)$ of $X$ with respect to the local system related to $\text{Ad}\xi$ vanishes.

We turn into Chern–Simons invariants related to real hyperbolic spaces. Let $\mathfrak{M} = G/K$ be an irreducible rank one symmetric space of non–compact type, where $G$ is a connected non–compact simple split rank one Lie group with finite centre, and $K \subset G$ a maximal compact subgroup [11]. Let $\Gamma \subset G$ be a discrete, co–compact free subgroup. Then $X = \Gamma \backslash \mathfrak{M}$ is a compact Riemannian manifold with fundamental group $\Gamma$, i.e. $X$ is a compact locally symmetric space. For a real hyperbolic manifold, we have $G = SO(1, n) \ (n \in \mathbb{Z}^+_\infty), \ K = SO(n), \text{ and } X = \Gamma \backslash \mathbb{H}^n$. Given a finite–dimensional unitary representation $\chi$ of $\Gamma$, there is the corresponding vector bundle $V_\chi \to X$ over $X$, given by $V_\chi = \Gamma \backslash (\mathfrak{M} \otimes F_\chi)$, where $F_\chi$ (the fibre of $V_\chi$) is the representation space of $\chi$ and where $\Gamma$ acts on $\mathfrak{M} \otimes F_\chi$ by the rule $\gamma \cdot (m, f) = (\gamma \cdot m, \chi(\gamma)f)$ for $(\gamma, m, f) \in (\Gamma \otimes \mathfrak{M} \otimes F_\chi)$. Let $\mathcal{L}_\Gamma$ be the Laplace–Beltrami operator of $X$ acting on smooth sections of $V_\chi$; we obtain $\mathcal{L}_\Gamma$ by projecting the Laplace–Beltrami operator of $\mathfrak{M}$ (which is $G$–invariant
and thus $\Gamma$–invariant) to $X$. For any representation $\chi : \Gamma \to U(n)$ one can construct a vector bundle $E_\chi$ over a certain 4-manifold $Y$ with boundary $\partial Y = X$ which is an extension of a flat vector bundle $E_X$ over $X$. Let $A_X$ be any extension of a flat connection $A_X$ corresponding to $\chi$.

The index theorem of Atiyah–Patodi–Singer for the twisted Dirac operator $D_{A_X}$ [12, 13, 14] gives here

$$\text{Index} \left( D_{A_X} \right) = \int_Y \text{ch}(\overline{E}_\chi) \hat{\Delta}(Y) - \frac{1}{2} (\eta(0, \mathcal{D}_X) + h(0, \mathcal{D}_X)), \quad (3.1)$$

where $\text{ch}(\overline{E}_\chi)$ and $\hat{\Delta}(Y)$ are the Chern character and $\hat{\Delta}$–genus respectively, $\hat{\Delta} = 1 - p_1(Y)/24$, $p_1(Y)$ is the 1–st Pontryagin class, $\eta(0, \mathcal{D}_X)$ is the holomorphic eta function, $h(0, \mathcal{D}_X)$ is the dimension of the space of harmonic spinors on $X$ ($h(0, \mathcal{D}_X) = \dim\ker\mathcal{D}_X = \text{multiplicity of the } 0\text{–eigenvalue of } \mathcal{D}_X$ acting on $X$); and $\mathcal{D}_X$ is a Dirac operator on $X$ acting on spinors with coefficients in $\chi$. The Chern–Simons invariants of $X$ can be derived from Eq. (3.1). Indeed we have:

$$CS(A_X) \equiv \frac{1}{2} (\dim\chi(0, \mathcal{D}) - \eta(0, \mathcal{D}_X)), \mod(\mathbb{Z}/2). \quad (3.2)$$

A remarkable formula relating $\eta(s, \mathcal{D})$, to the closed geodesics on an hyperbolic manifold of $(4n - 1)$–dimension has been derived in [15, 16]. More explicitly the following function can be defined, initially for $s^2 \gg 0$, by the formula

$$\log Z_{\Gamma}(s) \overset{\text{def}}{=} \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} (-1)^q \frac{L(\gamma, \mathcal{D})}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)}, \quad (3.3)$$

where $\mathcal{E}_1(\Gamma)$ is the set of those conjugacy classes $[\gamma]$ for which $X_\gamma$ has the property that the Euclidean de Rham factor of $\tilde{X}_\gamma$ is 1-dimensional ($\tilde{X}$ is a simply connected cover of $X$ which is a symmetric space of noncompact type), the number $q$ is half the dimension of the fibre of the centre bundle $\mathcal{C}(TX)$ over $X_\gamma$, $\ell(\gamma)$ is the length of the closed geodesic $c_\gamma$ (with multiplicity $m(\gamma)$) in the free homotopy class corresponding to $[\gamma]$, $P_h(\gamma)$ is the restriction of the linear Poincaré map $P(\gamma)$ at $(c_\gamma, \tilde{c}_\gamma) \in TX$ to the directions normal to the geodesic flow and $L(\gamma, \mathcal{D})$ is the Lefschetz number (see Ref. 16).

### 3.1. The twisted Dirac operator

Let now $\chi : \Gamma \to U(F)$ be a unitary representation of $\Gamma$ on $F$. The Hermitian vector bundle $F = \tilde{X} \times_F F$ over $X$ inherits a flat connection from the trivial connection on $\tilde{X} \times F$. We specialize to the case of locally homogeneous Dirac operators $\mathcal{D} : C^\infty(X, E) \to C^\infty(X, E)$ in order to construct a generalized operator $O_\chi$, acting on spinors with coefficients in $\chi$. If $\mathcal{D} : C^\infty(X, V) \to C^\infty(X, V)$ is a differential operator acting on the sections of the vector bundle $V$, then $\mathcal{D}$ extends canonically to a differential operator $\mathcal{D}_\chi : C^\infty(X, V \otimes F) \to C^\infty(X, V \otimes F)$, uniquely characterized by the property that $\mathcal{D}_\chi$ is locally isomorphic to
$D \otimes \ldots \otimes D$ (dim $F$ times) \cite{16}. One can repeat the arguments to construct a twisted zeta function $Z^\chi_T(s)$. It follows that \cite{17, 18, 19}: $Z^\chi_T(0) = Z^\chi_T(0)^{\dim \chi} \exp (-2i\pi CS(A_\chi))$, and eventually the Chern–Simons functional takes the form

$$CS(A_\chi) \equiv \frac{1}{2i\pi} \log \left( \frac{Z_T(0)^{\dim \chi}}{Z^\chi_T(0)} \right), \quad \text{mod}(\mathbb{Z}/2). \quad (3.4)$$

### 3.2. Hyperbolic manifolds with cusps

We have used the formula for the relation between the eta invariant of the signature operator and the Selberg zeta function of odd type, which is defined by the geometric data for a compact hyperbolic manifold. The corresponding formula between the analytic torsion and the Ruelle zeta function for odd dimensional hyperbolic manifold has been proven in Ref. \cite{20}. In addition, the symmetry that is similar to the Lefschetz fixed point theorem has been used to reduce the combination of the Selberg zeta functions to the Ruelle zeta function. Later, these results were generalized to the case of compact locally symmetric spaces of higher ranks \cite{16, 21}, and they were extended to the case of non–compact locally symmetric spaces with finite volumes \cite{22}. In the extension of these results there are two main difficulties: the fact that the heat kernel of the Laplace operator is not a trace class operator, and the correct analysis of the weighted orbital integral term which appears in the Selberg trace formula for a non–compact locally symmetric space. These difficulties has been overcome in Ref. \cite{22} with the help of relative spectral invariants \cite{23} and the Fourier transform of the weighted orbital integral for the $\mathbb{R}$–rank one cases \cite{24}, respectively.

### 3.3. Structure on symmetric spaces

Let us explain the results more precisely. Consider $G$, a noncompact semisimple Lie group with finite center $Z_G$, $K$ its maximal compact subgroup, and $G/K$ to be a noncompact symmetric space of $\mathbb{R}$–rank one. Let $\Gamma$ a torsion free discrete subgroup of $G$ with the volume of $\Gamma \backslash G$ finite. Taking into account the fixed Iwasawa decomposition $G = KAN$, consider a $\Gamma$–cuspidal minimal parabolic subgroup $P$ of $G$ with the Langlands decomposition $P = BAN$, being $B$ the centralizer of $A$ in $K$. Let us define the Dirac operator $\mathcal{D}$, assuming a spin structure for $X = \Gamma \backslash (\text{Spin}(2n + 1, 1)/\text{Spin}(2n + 1))$. The spin bundle $E_{\tau_s}$ is the locally homogeneous vector bundle defined by the spin representation $\tau_s$ of the maximal compact group $\text{Spin}(2n + 1)$. One can decompose the space of sections of $E_{\tau_s}$ into two subspaces, which are given by the half spin representations $\sigma_{\pm}$ of $\text{Spin}(2n) \subset \text{Spin}(2n + 1)$.

The Selberg trace formula could be used to prove a relation of the eta invariant and the Selberg zeta function of odd type. Let us consider a family of functions $K_\ell$ over $G = \text{Spin}(2n + 1, 1)$, which is given by taking the local trace for the integral kernel $\exp(-t\mathcal{D}^2)$ (or $\mathcal{D} \exp(-t\mathcal{D}^2)$). The Selberg trace formula applied to the scalar kernel $K_\ell$ has the form \cite{22}:
\[ \sum_{\sigma=\sigma_p} \sum_{\lambda_k \in \sigma_p} \hat{K}_t (\sigma, i\lambda_k) - \frac{d(\sigma_\pm)}{4\pi} \int_{\mathbb{R}} \text{Tr} (S_T (\sigma_\pm, -i\lambda) \\
\times (d/ds)S_T (\sigma_\pm, s)|_{s=i\lambda} \pi_T (\sigma_\pm, i\lambda)(\mathcal{K}_t)) \, d\lambda = I_T (\mathcal{K}_t) + H_T (\mathcal{K}_t) + U_T (\mathcal{K}_t), \]

where \( \sigma_p \in \text{Spec} \mathcal{D} \), \( d(\sigma_\pm) \) is the degree of the half spin representation of \( \text{Spin}(2n) \), \( S_T (\sigma_\pm, i\lambda) \) is the scattering matrix and \( I_T (\mathcal{K}_t) \), \( H_T (\mathcal{K}_t) \) and \( U_T (\mathcal{K}_t) \) are identity, hyperbolic and unipotent orbital integrals respectively. The analysis of the unipotent orbital integral \( U_T (\mathcal{K}_t) \) gives the following result [22]: all the unipotent terms are vanishing in the Selberg trace formula applied to the odd kernel function. This means that we can get the same formula as the result given in Ref. [15] in the relation of the eta invariant and the Selberg zeta function of odd type for the compact hyperbolic manifolds without any additional terms, which might exist due to the cusps.

**Note.** Observe [22] that in the functional equation of the eta invariant and the Selberg zeta function of odd type, a term \( \text{tr}_s (S_T (\sigma_\pm, s)(d/dz)S_T (\sigma_\pm, z)) \), given by the scattering matrix \( S_T (\sigma_\pm, z) \) appears, where the supertrace \( \text{tr}_s \) is the trace taken over the subspace related to the representation space of \( \sigma_\pm \) with weight \( \pm 1 \). This is an odd type formula, for the functional equation of Selberg zeta functions, which has been proved in Ref. [25].

### 4. Results on the Holographic Principle

According to the holographic principle, there exist strong ties between certain field theories on a manifold (“bulk space”) and on its boundary (at infinity). A few mathematically exact results relevant to that program are the following. The class of Euclidean AdS\(_3\) spaces which we have considered here are quotients of the real hyperbolic space \( \mathbb{H}^3 \) by a Schottky group. The boundaries of these spaces can be compact oriented surfaces with conformal structure (compact complex algebraic curves).

In Ref. [26], a principle associated with the Euclidean AdS\(_2\) holography has been established. The bulk space is there a modular curve, which is the global quotient of the hyperbolic plane \( \mathbb{H}^2 \) by a finite index group, \( \Gamma \), of \( G = PSL(2, \mathbb{Z}) \). The boundary at infinity is then \( \mathbb{P}^1(\mathbb{R}) \). Let \( \mathbb{P} \) be a coset space \( \mathbb{P} = G/\Gamma \). Then, the modular curve \( X_G := \Gamma \backslash \mathbb{H}^2 \) can be regarded as the quotient \( C \Gamma \mathbb{H} = \mathbb{H} \backslash (\mathbb{H}^2 \times \mathbb{P}) \); its non–commutative boundary (in the sense of Connes [27]) as the \( C^* \)– algebra \( C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma \) [28, 29, 26]. The results which have been regarded as manifestations of the holography principle are [26]:

1. There is a correspondence between the eigenfunctions of the transfer operator \( L_s \) and the eigenfunctions of the Laplacian (Maas wave forms). This correspondence is established in Refs. [30,31,32].
2. The cohomology classes in \( H_1(X_G, \text{cusps}, \mathbb{R}) \) can be regarded as elements in the cyclic cohomology of the algebra \( C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma \).
Cohomology classes of certain geodesics in the bulk space correspond to projectors in the algebra of observables on the boundary space.

(3) An explicit correspondence exists between a certain class of fields in the bulk space (Mellin transforms of modular forms of weight two) and the class of fields on the boundary.

In the three-dimensional case we can interpret the statement in Eq. (2.6) as an instance of a kind of holographic correspondence, if we regard $\det \mathcal{L}_\Gamma$ as a partition function in the bulk space and the right hand side as a theory related to the boundary at infinity.

Other constructions associated with the symmetric space can be considered for convex–cocompact groups. In fact, let $\partial X$ be a geodesic boundary of the symmetric space $X$ of a real, rank one, semisimple Lie group $G$. If $\Gamma \subset G$ is a discrete torsion–free subgroup, then a $\Gamma$–equivalent decomposition, $\partial X = \Omega \cup \Lambda$, can be constructed, where $\Lambda$ is the limit set of $\Gamma$. The subgroup $\Gamma$ is called convex–cocompact if $\Gamma \backslash X \cup \Omega$ is a compact manifold with boundary [33]. The orbit space, $X = \Gamma \backslash H^n$, may be viewed as the interior of a compact manifold with boundary, namely the Klein manifold for $\Gamma$ [34], $X = (\Gamma \backslash H^n) \cup (\Gamma \backslash \Omega(\Gamma))$, so that the boundary at infinity is given by $\partial_{\infty} X = \partial X = \Gamma \backslash \Omega(\Gamma)$.

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