DYNAMIC BEHAVIOR OF THE ROOTS OF THE TAYLOR POLYNOMIALS OF THE RIEMANN XI FUNCTION WITH GROWING DEGREE

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ABSTRACT. We establish a uniform approximation result for the Taylor polynomials of the xi function of Riemann which is valid in the entire complex plane as the degree grows. In particular, we identify a domain growing with the degree of the polynomials on which they converge to Riemann’s xi function. Using this approximation we obtain an estimate of the number of “spurious zeros” of the Taylor polynomial which are outside of the critical strip, which leads to a Riemann - von Mangoldt type of formula for the number of zeros of the Taylor polynomials within the critical strip. Super-exponential convergence of Hurwitz zeros of the Taylor polynomials to bounded zeros of the xi function are established along the way, and finally we explain how our approximation techniques can be extended to a collection of analytic L-functions.

1. INTRODUCTION

Consider Riemann’s ζ-function defined by

\[ \xi(z) = \frac{1}{2} \pi^{-z/2} \Gamma \left( \frac{z}{2} \right) z(z - 1) \zeta(z) \]

where \( \zeta(z) \) is the Riemann ζ-function. The pre-factors of the ζ-function in the above definition absorb the poles and trivial zeros of the ζ-function so that \( \xi \) is an entire function whose only zeros are the nontrivial zeros of \( \zeta(z) \), i.e. those lying in the critical strip \( 0 < \text{Re} \ z < 1 \). As a consequence, the functional equation for the \( \xi \)-function is much simplified

\[ \xi(z) = \xi(1 - z). \]

The infamous Riemann Hypothesis is equivalent to the statement that the only zeros of \( \xi(z) \) lie on the critical line \( \text{Re} \ z = 1/2 \). There is a vast body of literature concerning the properties of the \( \zeta \)-function and the Riemann Hypothesis, and we cannot do any justice to summarizing those works here. We refer the reader to the classical works [6, 14] at the tip of that iceberg.

In Riemann’s 1859 paper [12] he considered the quantity

\[ N(T) = \{ z \in \mathbb{C} \mid \zeta(z) = 0, \ \text{Re}(z) \in (0, 1), \ \text{Im}(z) \in (0, T) \}, \]

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and proposed that \( N(T) \approx T/(2\pi) \log(T/(2\pi)) - T/(2\pi) \). This was subsequently proved by von Mangoldt with an explicit bound on the remainder:

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]

In this paper we study the zeros of the Taylor polynomial approximations of Riemann’s \( \xi \)-function. We establish a version of the Riemann-von Mangoldt formula for these zeros, by using a new uniform asymptotic description of the Taylor polynomials when the degree is large. The techniques used here are general and can be applied to a broad class of functions. In the last section of this paper we extend the analysis of Taylor polynomials to a larger collection of analytic L-functions.

Studying the zeros of Taylor approximates to given functions goes back at least to the 1920s, and probably earlier. In [13], Szegő considered the distribution of zeros of \( p_n(z) = \sum_{k=0}^{n} z^k/k! \), the partial sums of the exponential series. He showed that the zeros of the rescaled function \( p_n(nz) \) converge as \( n \to \infty \) to a curve \( D_\infty \), now called the Szegő curve, which is a branch of the level curve \( \{ z : |ze^{1-z}| = 1 \} \) and computed the asymptotic distribution of zeros along the Szegő curve. Subsequent work in this direction [1, 3, 10] has provided detailed results bounding the distance of the zeros of \( p_n(nz) \) from \( D_\infty \). Similar results on the zeros of the partial sums of \( \cos, \sin \), and other exponential functions have been derived in [2, 4]. An extension to partial sums of analytic functions defined by exponential integrals appears in [15], which also contains some further historical discussion and references.

The starting point for our analysis of the Taylor polynomials of the \( \xi \)-function is the recent work of [8] in which the authors utilize basic facts of complex analysis to represent the partial sums, \( p_n \), of the exponential series as Cauchy integrals over certain contours in the complex plane. Steepest descent analysis of the (rescaled) Taylor polynomials and properties of Cauchy integrals lead to a uniform asymptotic description of the polynomials as \( n \to \infty \) in the entire complex plane. In doing so, [8] re-derives many of Szegő’s classic results on the zeros of the partial sums. Additionally, the method naturally accommodates the presence of critical points in the asymptotic analysis which complicate the approximation theory in the more classical works mentioned previously.

Recall the Cauchy integral representation of the \( n^{th} \) Taylor polynomial approximating a given function \( F(z) \) (which we assume to be entire to avoid fretting about domain issues):

\[
T_{n-1}(F; z) = F(z) \left[ \chi_S(z) - \frac{z^n}{F'(z)} \int_{\partial S} \frac{F(s)}{s^n \cdot s - z} \right],
\]

where \( S \) is taken to be a simply connected open set whose boundary \( \partial S \) is either a finite union of smooth arcs forming a simple closed (obviously rectifiable) curve, or a reasonable extension (which will be described as needed below), and \( \chi_S(z) \) is the characteristic function of \( S \). Basic results concerning Taylor approximation are obtained from this representation by taking \( S \) to be a disc of fixed size and then estimating the \( n \)-dependence of the last term on the right hand side of (1.4). More interestingly, the integral’s dependence on \( n \) can be
estimated precisely, using the steepest descent method for integrals, provided the function $F$ is so nice as to permit the application of the steepest descent method.

A portion of this paper is dedicated to showing very explicitly that this is so for the function $\xi$ defined above. However, it is useful to describe the general conditions, as cryptic as they might appear to be: one requires that for $n$ sufficiently large there should be a number of “stationary phase points”, and that the original contour of integration can be deformed to a contour of controllable arc length which passes through one or more of these stationary phase points while otherwise remaining in regions where the integrand is exponentially smaller than its behavior near one (or more) of these critical points. The simple case of $F(z) = e^z$ is useful to clarify the above discussion (see [8] for more information, including a brief discussion of the various contributions to this example). Evaluating (1.4) by steepest descent methods, it is convenient to introduce a rescaling map $z \mapsto \lambda(n)z$ which renormalizes the stationary phase points, which typically grow with $n$, to remain $O(1)$ as $n \to \infty$. In the case $F(z) = e^z$, there is a single stationary phase point $z_0 = n := \lambda(n)$ and (1.4) becomes for any $\delta > 0$,

\begin{equation}
T_{n-1}(e^z; nz) = e^{nz} \left[ \chi_S(nz) - \frac{(ze^{1-z})^n}{\sqrt{2\pi n}} \frac{1}{1 - z} \left( 1 + O\left(\frac{1}{n}\right) \right) \right], \quad |z - 1| > \delta.
\end{equation}

Formula (1.5) demonstrates that the Taylor polynomials approximate $e^z$ on sets that grow with $n$. We can characterize the largest such set, $\Omega(e^z)$, as the closure of the connected

\[\]
component of $|ze^{1-z}| < 1$ containing $z = 0$:

$$\Omega(e^z) = \{ z : |ze^{1-z}| < 1 \text{ and } |\text{Re } z| < 1 \}.$$  

The boundary $D_\infty = \partial\Omega(e^z)$ is the Szegő curve mentioned previously. For $z$ away from the Szegő curve, the asymptotic formula (1.5) clearly cannot vanish. Szegő showed that:

1) every accumulation point of the zeros $\{z_{k,n}\}_{k=1}^n$ of $T_{n-1}(e^z; nz)$ must lie on $D_\infty$;
2) Every point on $D_\infty$ is an accumulation point of $\{z_{k,n}\}_{k=1}^n$. It was later shown, [3], that $\text{dist}(z_{k,n}, D_\infty) = O\left(\log n \right)$ for each zero $z_{k,n}$ of $T_{n-1}(e^z; nz)$ which is uniformly bounded away from the stationary point at $z = 1$ (for $z_{k,n}$ near 1 the rate of convergence to $D_\infty$ slows to $O\left(n^{-1/2}\right)$). It’s also possible to improve on the Szegő curve; one can consider the curve

$$D_n^{(1)} = \left\{ z : \frac{|ze^{1-z}|}{\sqrt{2\pi n}} = 1 \text{ and } |\text{Re } z| < 1 \right\};$$

it was shown in [4] that for any $\delta > 0$, $\text{dist}(z_{k,n}, D_n^{(1)}) = O\left(n^{-2}\right)$ for each $z_{k,n}$ such that $|z_{k,n} - 1| > \delta$. The curve $D_n^{(1)}$ is only the first in a countable family of improved Szegő curves $D_n^{(j)}$; the further improved Szegő curves result from keeping $j$ terms from the complete asymptotic series which in (1.5) is represented simply by $(1 + O\left(n^{-1}\right))$. In Figure 1 we plot the Szegő curve and its (first) improvement for $e^z$ along with the roots of $T_{n-1}(e^z; nz)$ for $n = 201$. The plot was produced using the software package Mathematica [16].

![Figure 1](image1.png)

**Figure 2.** Each dot represents a zero of the (rescaled) Taylor polynomial $T_n(\cosh(z); (n + 1)z)$ of degree $n = 200$. The Szegő curve (dashed line); and an improved Szegő curve (solid line) are also given. Here, the zeros in the imaginary interval $[-ie^{-1}, ie^{-1}]$ are the Hurwitz zeros of $T_n(\cosh(z); z)$.

The situation for functions which have zeros is somewhat modified. Suppose that $s$ is a root of order $k$ of a function $f$ analytic at $s$. Then given any sufficiently small neighborhood $\mathcal{N}$ of the root $s$, the Taylor polynomials $T_n(f; z)$ converge (uniformly) to $f$ in $\mathcal{N}$ and so by Hurwitz’s theorem (cf. [5]) $T_n(f; z)$ will have exactly $k$ zeros in $n$ for all $n \geq n_0(\mathcal{N})$. This
imposes a natural dichotomy on the zeros of the Taylor polynomials: those which converge to the zeros of \( f \) we label, ‘Hurwitz zeros’; those which do not converge to zeros of \( f \) we label ‘spurious zeros’, and these accumulate on the analogue of the Szegő curve for the function \( f \). To illustrate this dichotomy see Figures 2 and 3 where the zeros of rescaled Taylor polynomials of \( \cosh(z) \) and \( \xi(z + 1/2) \) are given together with their Szegő curves.

In both Figure 2 and Figure 3 the zeros of the functions \( \cosh((n+1)z) \) and \( \xi(\lambda(n)z + 1/2) \) do not appear, because they agree with the computed zeros of the Taylor polynomials to well beyond the plotting resolution. In Table 1 the 24 roots of \( T_n(\cosh(z); (n + 1)z) \) with \( n = 200 \) which lie on the imaginary axis in Figure 2 are compared to the first 24 zeros of \( \cosh((n + 1)z) \). The convergence rate is striking. These numerical calculations required very high precision calculations using [16]. In Section 5 below, we will show that the rate at which any fixed Hurwitz zeros converges to a fixed root of the function \( \xi \) is super-exponential. We believe that this is true for a large class of entire functions \( f \), of which, as Table 1 suggests, \( \cosh \) is certainly a member.

1.1. Taylor polynomials of \( \xi \). In the remainder of the paper we will be interested in the Taylor (Maclaurin) polynomials of the function

\[
(1.8) \quad f(z) = \xi(1/2 + z).
\]

The function \( f \) is entire and possesses the symmetries \( f(z^*) = f(z^*) \) and \( f(-z) = f(z) \), the latter of which follows from (1.2). The Taylor polynomials \( T_n \) inherit the symmetries of \( f \): \( T_n(f; z^*) = T_n(f; z) = T_n(f; -z) \), so that for any \( n \in \mathbb{N}, \) i) \( T_{2n+1}(f; z) = T_{2n}(f; -z) \); and ii) zeros of \( T_n \), excepting purely real or imaginary roots, come in quartets. In what follows we will omit the dependence of the Taylor polynomials upon \( f \) and write simply \( T_{2n}(z) \) for \( T_{2n}(f; z) \).
The exponential decay of $|\Gamma(z)|$ along vertical lines—the other factors in (1.1) being polynomially bounded in $\text{Im}(z)$—allows us to deform the set $\mathcal{S}$ in (1.4) to an infinite vertical strip. For any number $\lambda > 0$, let

$$\mathcal{S}_\lambda = \{ z \in \mathbb{C} : |\text{Re } z| < \lambda \}. \quad (1.9)$$

Anticipating the introduction of a scaling parameter $\lambda = \lambda(n)$, and letting $\chi = \chi_{\mathcal{S}_1}$ be the characteristic function of $\mathcal{S}_1$, we have

$$T_{2n-2}(\lambda z) = f(\lambda z) \left[ \chi(z) - \frac{e^{n\phi(z)}}{\sqrt{n}} h(z) \right] \quad (1.10)$$

where we have defined

$$e^{n\phi(z)} := \frac{2^n f(\lambda)}{f(\lambda z)}, \quad (1.11)$$

$$h(z) := \frac{\sqrt{n}}{2\pi i} \int_{\partial \mathcal{S}_1} e^{-n\phi(s)} \frac{ds}{s - z}. \quad (1.12)$$

2. Preliminaries

The methods of Korobov and Vinogradov produce the following zero free region (c.f. [14, §6.19]) of $\zeta$ extending inside the critical strip: for any choice of $A > 0$, $\zeta(s)$ has no zeros for $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ with $|t|$ large and $\sigma > 1 - \frac{A}{(\log t)^{2/3} (\log \log t)^{1/3}}$ and we have the bounds

$$|\zeta(s)| = O \left( (\log t)^{2/3} (1 + |t|^{100(1-\sigma)^3/2}) \right), \quad 1/2 \leq \sigma \leq 1, \quad (2.1)$$

$$\frac{\zeta'(s)}{\zeta(s)} = O \left( (\log t)^{2/3} (\log \log t)^{1/3} \right), \quad \frac{1}{\zeta(s)} = O \left( (\log t)^{2/3} (\log \log t)^{1/3} \right).$$

the best bounds of this type are those of Ford [7].

It follows that our rescaled function $f(\lambda z)$ is zero free in the domain

$$\mathcal{F}_\lambda = \left\{ z = x + iy \in \mathbb{C}^+ : x \geq \frac{1}{2\lambda} - \frac{A}{\lambda (\log \lambda y)^{2/3} (\log \log \lambda y)^{1/3}} \right\}. \quad (2.2)$$

Outside the critical strip we have the more elementary bound from [7]

**Lemma 2.1.** Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ and $\sigma > 1$, then

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \frac{1}{\sigma - 1}. \quad \text{Proof.}$$

For $\sigma > 1$ we have $|\zeta'(s)/\zeta(s)| \leq -\zeta'(\sigma)/\zeta(\sigma)$ and

$$-\zeta'(\sigma) = \sum_{m=2}^{\infty} \log m m^\sigma = \sum_{m=2}^{\infty} \left[ \sum_{n=1}^{m-1} \log \left( \frac{n+1}{n} \right) \right] m^{-\sigma} = \sum_{n=1}^{\infty} \left[ \sum_{m \geq n+1} m^{-\sigma} \right] \log \left( \frac{n+1}{n} \right).$$
The result follows from bounding the interior sum by the integral \( \int_{n}^{\infty} m^{-\sigma} \, dm \) and recalling that for \( x > 0 \), \( \log(1 + x) < x \):

\[
-\zeta'(\sigma) \leq \sum_{n=1}^{\infty} \left( \frac{n^{1-\sigma}}{\sigma - 1} \right) \frac{1}{n} = \frac{\zeta(\sigma)}{\sigma - 1}.
\]

The above bounds on the logarithmic derivative, both near the strips edge and outside it, give a bound on the argument of \( \zeta(s) \) at the edge of the critical strip.

**Lemma 2.2.** There exist \( t_0 > 0 \) such that for all \( t > t_0 \) we have

\[
\arg \zeta(1 + it) \leq \frac{2}{3} \log \log t + O(\log \log \log t).
\]

**Proof.** Since \( \zeta(2) > 0 \) and \( \Re \zeta(2 + i\tau) \geq 1 - \sum_{n=1}^{\infty} n^{-2} > 0 \) for all \( \tau \geq 0 \), \( \Re \zeta \) is strictly positive on the vertical line from \( s = 2 \) to \( s = 2 + it \). It follows that \( |\arg \zeta(2 + it)| \leq \pi \).

Using (2.1) and Lemma 2.1 for all sufficiently large \( t \) there exist a constant \( A > 0 \), such that for any \( q \in (0, 1) \) we have

\[
|\arg \zeta(1 + it) - \arg \zeta(2 + it)| \leq \int_{1+q}^{2} \frac{d\sigma}{\sigma - 1} + Aq(\log t)^{2/3}(\log \log t)^{1/3}
= \log \frac{1}{q} + Aq(\log t)^{2/3}(\log \log t)^{1/3}.
\]

The minimizer of this last expression, as a function of \( q \), is \( q_0 = A^{-1}(\log t)^{-2/3}(\log \log t)^{-1/3} \). Computing the minimum completes the proof. \( \square \)

2.1. **The phase** \( \phi_{\lambda}(z) \). The phase, implicitly defined by (1.11),

\[
\phi_{\lambda}(z) = 2 \log z + \frac{1}{n} \log \frac{f(\lambda)}{f(\lambda z)}
\]

is analytic in any region in which \( f(\lambda z) = \xi(1/2 + \lambda z) \) is zero free. In particular \( \phi_{\lambda} \) is well defined along the contour of integration \( |\Re z| = 1 \). Moreover, the choice of branch can be chosen such that \( \phi_{\lambda}(z) \) is positive real for \( z \in (1, \infty) \) and satisfies the symmetry \( \phi_{\lambda}(z) = \phi_{\lambda}(-z) \).

The following formula for \( \phi_{\lambda} \) is well suited for a large \( \lambda \) expansion. For any fixed \( c > 0 \), if \( |z| > c \) and \( \lambda \gg 1 \) we have

\[
\phi_{\lambda}(z) = 2 \log z + \left( \frac{\lambda}{2n} \log \frac{\lambda}{2\pi} \right) (1 - z) - \frac{\lambda}{2n} [1 - z + z \log z]
- \frac{1}{n} \log \zeta \left( \frac{\lambda z + 1}{2} \right) + \frac{1}{n} r(z; \lambda)
\]
where the remainder \( r(z; \lambda) \) is given by

\[
r(z; \lambda) = \log \frac{\Gamma(\lambda/2 + 1/4)}{\Gamma(\lambda z/2 + 1/4)} - \left( \frac{\lambda}{2} \log \frac{\lambda}{2} \right) (1 - z) + \frac{\lambda}{2} [1 - z + z \log z]
\]

\[+ \log \left[ \frac{(\lambda^2 - 1/4)\zeta(\lambda + 1/2)}{(\lambda^2 z^2 - 1/4)} \right].
\]

(2.5)

This remainder term is bounded provided that \( z \) stays away from its obvious singularities. More precisely, let \( c > 0 \) be fixed, then using Stirling’s expansion of \( \log \Gamma(s) \), one may verify that

\[
r(z; \lambda) = O(1) \quad \text{Re} z \geq \frac{1}{2\lambda} \quad \text{and} \quad |z - \frac{1}{2\lambda}| > c.
\]

(2.6)

The explicit \( \zeta \) term in (2.4) becomes meaningful only near the critical strip; elsewhere, it is comparable to the remainder \( r \). One can similarly compute the \( z \)-derivative of the phase:

\[
\partial_z \phi_\lambda(z) = \frac{2}{z} - \frac{\lambda}{2n} \log \frac{\lambda}{2\pi} - \frac{\lambda}{2n} \log z - \frac{\lambda \zeta'(\lambda z + 1/2)}{n \zeta(\lambda z + 1/2)} + \frac{1}{n} \partial_z r(z; \lambda).
\]

(2.7)

The representation (1.10) places the essential \( n \)-dependence of the Taylor polynomials in the phase \( \phi_\lambda \) defined by (1.11) which appears in the exponential term of the integral (1.12). As the following lemma shows, for large \( n \) the phase has two stationary points outside the critical strip, and these points’ magnitudes increase with \( n \). We choose the scaling parameter \( \lambda = \lambda(n) \) according to Lemma 2.3 below precisely so that these stationary points lie at \( z = \pm 1 \) in the rescaled plane\(^\dagger\). This completes the definition of \( T_{2n-2}(\lambda z) \) so that the representation (1.10) is now well defined.

**Lemma 2.3.** For all sufficiently large \( n \) there is a unique choice of \( \lambda = \lambda(n) \), with \( \lambda > 1/2 \) (i.e. right of the shifted critical strip) satisfying \( \partial_z \phi_\lambda(z) \bigg|_{z=1} = 2 - (\lambda/n)\partial_\lambda \log f(\lambda) = 0 \). This choice of \( \lambda \) satisfies the relation

\[
2 - \frac{\lambda}{2n} \log \left( \frac{\lambda}{2\pi} \right) = O \left( \frac{1}{n} \right),
\]

and asymptotically

\[
\lambda = \lambda(n) = \frac{4n}{W(2n/\pi)} \left[ 1 + O \left( n^{-1} \right) \right].
\]

Here \( W(z) \) is the branch of the inverse function to \( We^W = z \) which is real and increasing for \( z \in (-e^{-1}, \infty) \) sometimes called the Lambert-W function\(^\ddagger\).

Moreover, for this choice of \( \lambda \) the critical point at \( z = 1 \) is simple and

\[
\phi''_\lambda(1) = -2 + O \left( \frac{1}{\log n} \right).
\]

\(^\dagger\)By symmetry the stationary points must be opposites.

\(^\ddagger\)For more information on \( W(z) \) see §4.13 of [11]
Proof. As $f$ is entire, $\partial_\lambda \log f(\lambda)$ is bounded for any finite $\lambda$ outside the open critical strip as $f(\lambda)$ is zero free in this region. It follows that any root $\lambda(n)$ of $2 - (\lambda/n) \partial_\lambda \log f(\lambda)$ outside the strip must grow without bound as $n \to \infty$.

Let $\epsilon = 1/n,$

$$G(\epsilon, \lambda) = 2 - \epsilon \lambda \partial_\lambda \log f(\lambda), \quad \lambda(\epsilon, \nu) = \frac{4}{\epsilon W(2/(\epsilon \pi))} \left[ 1 + \epsilon \nu \right],$$

and let $\tilde{G}(\epsilon, \nu) = \epsilon^{-1} G(\epsilon, \lambda(\epsilon, \nu))$. As $\phi'_\lambda(1) = 0$ is equivalent to $G(\epsilon, \lambda) = 0$, the theorem is proved if we can show that $\tilde{G}(\epsilon, \nu) = 0$ implicitly defines a unique function $\nu(\epsilon)$ which is bounded for $\epsilon$ near 0. Using (2.7) we have

$$G(\epsilon, \lambda) = 2 - \frac{\epsilon \lambda}{2} \log \left( \frac{\lambda}{2\pi} \right) - \epsilon R(\lambda)$$

where $R$ is given by

$$R(\lambda) := \frac{\lambda}{2} \left[ \psi \left( \frac{\lambda}{2} + \frac{1}{4} \right) - \log \frac{\lambda}{2} + \frac{4}{\lambda} \left( 1 + \frac{1}{4\lambda^2 - 1} \right) + \frac{2\zeta'((\lambda + 1)/2)}{\zeta((\lambda + 1)/2)} \right]$$

Here $\psi$ denotes the digamma function, the logarithmic derivative of $\Gamma$. For $\lambda$ large and $|\arg \lambda| < \pi$ Stirling’s series gives $\psi(\lambda/2 + 1/4) - \log(\lambda/2) = 1/(2\lambda) + O(\lambda^{-2})$. So as $\lambda \to \infty$ the leading order terms in $R$ cancel and $R(\lambda) = 7/(2\lambda) + O(\lambda^{-1})$. Inserting this fact into $G(n^{-1}, \lambda) = 0$ shows that (2.8) is the correct asymptotic model.

The defining relation $We^W = 2/(\epsilon \pi)$ for $W = W(2/(\epsilon \pi))$ implies, by taking logarithms, that $W^{-1} \log(2/(\epsilon \pi W)) = 1$. After some simplification we have

$$\tilde{G}(\epsilon, \nu) = -2\nu - \frac{2\nu}{W(2/(\epsilon \pi))}(1 + \epsilon \nu) \frac{\log(1 + \epsilon \nu)}{\epsilon \nu} - R(\lambda(\epsilon, \nu)).$$

Using the fact that $W(2n/\pi) = O(\log n)$ and computing the derivative of $R$ one may verify that $\tilde{G}(0, 0) = 0$ and $\tilde{G}_\nu(0, 0) = -2$. Thus, we can apply the implicit function theorem to conclude that a bounded (locally in $\epsilon$) solution $\nu = \nu(\epsilon)$ exists in a neighborhood of $\epsilon = 0$. □

Lemma 2.3 has the following useful and immediate corollary:

**Corollary 2.4.** For $\lambda = \lambda(n)$ as given in Lemma 2.3 the asymptotic expansion of the phase becomes

$$(2.10) \quad \phi(\lambda) = 2(\log z + 1 - z) - \frac{\lambda}{2n}(1 - z + z \log z) - \frac{1}{n} \log \zeta \left( \lambda z + \frac{1}{2} \right) + \frac{1}{n} \tilde{r}(z; \lambda),$$

where

$$\frac{1}{n} \tilde{r}(z; \lambda) = \frac{1}{n} r(z; \lambda) + \left( \frac{\lambda}{2n} \log \frac{\lambda}{2\pi} - 2 \right) (1 - z)$$

satisfies the same boundedness conditions (2.6) as the original $r(z; \lambda)$. 

We complete this section by showing that \( \partial_z \phi_\lambda \) has no other bounded zeros outside the critical strip.

**Lemma 2.5.** Let \( \lambda = \lambda(n) \) be as given by Lemma 2.3 and fix \( R > \epsilon > 0 \). Then for any \( z \) such that

\[
(2.11) \quad z \in \mathcal{F}_\lambda \quad \text{and} \quad \epsilon \leq |z| \leq R
\]

we have

\[
(2.12) \quad \partial_z \phi_\lambda(z) = 2(z^{-1} - 1) + O_R \left( \left( \frac{\log n}{\log \log n} \right)^{1/3} \right).
\]

Additionally, given a fixed \( \rho \in (0, 1) \), if \( |z - 1| > \rho \), then there exist \( n_0 = n_0(R, \rho) > 0 \) such that for all \( n > n_0 \) we have

\[
(2.13) \quad |\partial_z \phi_\lambda(z)| \geq \rho.
\]

**Proof.** Differentiating (2.10) one has

\[
(2.14) \quad \partial_z \phi_\lambda(z) = 2(z^{-1} - 1) - \frac{\lambda}{n} \frac{\zeta'((\lambda z + 1/2)}{\zeta(\lambda z + 1/2)} - \frac{\lambda}{2n} \log z + \frac{1}{n} \partial_z \tilde{r}(z; \lambda)
\]

Then for any \( z \) as described in (2.11) we use (2.1) to bound the \( \zeta'/\zeta \) term in the expression above and note that Lemma 2.3 implies that \( \lambda/n = O \left( (\log n)^{-1} \right) \) to arrive at (2.12). The last statement follows from the fact that for \( |z - 1| > \rho \), \( 2|z^{-1} - 1| \geq 2\rho/(1 + \rho) = \rho + \rho(1-\rho)/(1+\rho) \). Then using (2.12) it is clear that we may choose \( n_0(R, \rho) \) such that (2.13) is satisfied whenever \( n > n_0 \).

3. **Uniform approximation of \( T_{2n}(z) \) in the plane**

In this section we construct in a piecewise fashion a uniform approximation of the function \( h(z) \) (defined by (1.12)). Inserting this approximation into the representation of the Taylor polynomials \( T_{2n}(\lambda z) \) in (1.10) immediately yields a uniform asymptotic representation of the rescaled Taylor polynomials in the plane; this is the result of our Theorem 3.1 below.

Lemma 2.3 implies that the contour integral (1.12) defining \( h(z) \) has two regular stationary points at \( z = \pm 1 \) and is otherwise non-stationary. Specifically,

\[
(3.1) \quad w^2 = \phi_\lambda(z) = \frac{\phi''_\lambda(1)}{2} (z - 1)^2 [1 + O((z - 1))]
\]

defines a map \( w = w(z) \) which, when restricted to any sufficiently small neighborhood \( B_{1,\delta} \) of \( z = 1 \) (or \( B_{-1,\delta} \) of \( z = -1 \)), is an invertible conformal map onto a bounded neighborhood of \( w = 0 \). We choose the branch such that \( w \) maps \( \partial S_1 \) locally to a nearly horizontal contour in the \( w \)-plane oriented left-to-right:

\[
(3.2) \quad w(z) = -i \sqrt{\frac{-\phi''_\lambda(1)}{2}} (z - 1) [1 + O((z - 1))] \quad z \in B_{1,\delta},
\]
and enforce symmetry by demanding that $w(z) = w(-z)$ for $z \in B_{-1,\delta}$. The estimate on $\phi''_\lambda(1)$ in Lemma 2.3 implies that $\sqrt{-\phi''_\lambda(1)/2} = 1 + O(1/\log n)$ so that $w = w(z)$ is asymptotically isometric for $z$ near 1 and $n \gg 1$. We fix the neighborhoods $B_{\pm 1,\delta}$ by requiring that $B_{\pm 1,\delta}$ are, for any sufficiently small $\delta > 0$, the two pre-images of the disk of radius $\delta$ in the $w$-plane:

\begin{equation}
B_{\pm 1,\delta} = \{ w \in \mathbb{C} : |w| < \delta \}
\end{equation}

and we let $B_\delta = B_{1,\delta} \cup B_{-1,\delta}$.

For $z$ bounded away from $\pm 1$ a standard stationary phase calculation gives

\begin{equation}
h(z) = h_0(z) \left[ 1 + O(n^{-1}) \right], \quad h_0(z) = \frac{1}{\sqrt{2\pi|\phi''_\lambda(1)|}} \frac{2}{1 - z^2}.
\end{equation}

As $z \to \pm 1$ this approximation breaks down as the pole of the integrand in (1.12) at $s = z$ approaches the stationary points. At these points a more careful analysis is required which we give below; we prove the following theorem.

**Theorem 3.1.** Let $\lambda = \lambda(n)$ be as described in Lemma 2.3, $\chi(z)$ the characteristic function of the set $|\text{Re } z| < 1$, and $h_0(z)$, defined by (3.4), the leading order stationary phase approximation of $h(z)$. Then as $n \to \infty$ the Taylor polynomials described by (1.10) admit the asymptotic expansion

\begin{equation}
T_{2n-2}(\lambda z) = T_{2n-1}(\lambda z) = \begin{cases} 
 f(\lambda z) \left[ \chi(z) - \frac{e^{n\phi_\lambda(z)}}{\sqrt{n}} h_0(z) (1 + \mathcal{E}(z)) \right] & z \in \mathbb{C} \setminus B_\delta \\
 f(\lambda z) \left[ \frac{1}{2} \text{erfc}(i\sqrt{n}w(z)) - \frac{e^{n\phi_\lambda(z)}}{\sqrt{n}} \mathcal{E}(z) \right] & z \in B_\delta.
\end{cases}
\end{equation}

where the residual error function $\mathcal{E}(z)$ is bounded, analytic in $\mathbb{C} \setminus ((\partial S_1 \setminus B_\delta) \cup \partial B_\delta)$, and satisfies

\begin{equation}
\mathcal{E}(z) = \begin{cases} 
 O(n^{-1}) & z \in B_\delta^c \\
 h_0(z) + \frac{1}{2i\sqrt{\pi w(z)}} + O(n^{-1}) & z \in B_\delta.
\end{cases}
\end{equation}

**Corollary 3.2.** Let $\lambda = \lambda(n)$ be as described in Lemma 2.3 and $e^{n\phi_\lambda(z)}$ be as defined in (1.11). Define

\begin{align*}
\Omega &= \left\{ z \in \mathbb{C} : |\text{Re } z| < 1 \text{ and } |e^{\phi_\lambda(z)}| < 1 \right\}, \\
\Omega_- &= \left\{ z \in \mathbb{C} : |\text{Re } z| > 1 \text{ and } |e^{\phi_\lambda(z)}| < 1 \right\}, \\
\Omega_+ &= \left\{ z \in \mathbb{C} : |e^{\phi_\lambda(z)}| > 1 \right\}.
\end{align*}
Then the relative error satisfies
\[
\lim_{n \to \infty} \left| \frac{T_{2n}(\lambda z)}{\xi(1/2 + \lambda z)} - 1 \right| = 0 \quad z \in \Omega,
\]
\[
\lim_{n \to \infty} \left| \frac{T_{2n}(\lambda z)}{\xi(1/2 + \lambda z)} \right| = \begin{cases} 
0 & z \in \Omega_- \\
\infty & z \in \Omega_+ 
\end{cases}
\]

Let us begin to develop the tools to prove Theorem 3.1. For \( z \in B_{\delta} \), we define the function \( k : B_{\delta} \setminus \partial S_1 \to \mathbb{C} \) by
\[
k(z) = \sqrt{n} \hat{k}(\sqrt{n} w(z)), \quad \hat{k}(s) = \frac{1}{2\pi i} \int_{\gamma} e^{-t^2} \frac{dt}{t - s}
\]
where \( \gamma \) is the left-to-right oriented contour passing through the origin formed by extending the scaled image \( \sqrt{n} w(\partial S_1 \cap B_{1, \delta}) \) horizontally to infinity in both directions. We will show that this function well approximates \( h(z) \) in \( B_{\delta} \). For our purposes, the essential fact is that \( k(z) \) is analytic in \( B_{\delta} \setminus \partial S_1 \) and satisfies the same jump relation on \( \partial S_1 \) as the function \( h(z) \) which we are attempting to approximate:
\[
h_+(z) - h_-(z) = \sqrt{n} e^{-n\phi_\delta(z)} = \sqrt{n} e^{-nw(z)^2} = k_+(z) - k_-(z) \quad z \in \partial S_1 \cap B_{\delta}.
\]

The integral defining \( \hat{k} \) can be explicitly evaluated: integrating by parts one easily shows that \( \hat{k} \) satisfies \( \hat{k} + 2sk = \frac{i}{\sqrt{\pi}} \); using (3.7) and the residue calculus one sees that \( \hat{k}_\pm(0) = \pm 1/2 \).

Solving the differential equation for \( \hat{k} \) yields, upon composition with \( \sqrt{n} w(z) \):
\[
k(z) = \sqrt{n} e^{-n\phi_\delta(z)} \left[ \chi(z) - \frac{1}{2} \text{erfc}(i\sqrt{n} w(z)) \right].
\]

Using the known asymptotic behavior [11, eq. 7.12.1] of the complementary error function
\[
e^{-s^2} \text{erfc}(s) \sim \begin{cases}
\frac{1}{\sqrt{\pi s}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(1/2 + m)}{\Gamma(1/2)} s^{-2m} & |\arg(s)| < 3\pi/4 \\
2e^{-s^2} + \frac{1}{\sqrt{\pi s}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(1/2 + m)}{\Gamma(1/2)} s^{-2m} & |\arg(-s)| < 3\pi/4,
\end{cases}
\]

it follows that uniformly in the \( s \)-plane
\[
\hat{k}(s) = -\frac{1}{2i\sqrt{\pi s}} \left[ 1 + O(s^{-2}) \right] \quad s \to \infty.
\]

Putting together the steepest descent approximation (3.4), valid in \( B_{\delta}^c \), and our local model \( k \) we define the residual error function
\[
\mathcal{E}(z) = \begin{cases} 
h(z) - h_0(z) & z \in B_{\delta}^c \\
h(z) - k(z) & z \in B_{\delta} \end{cases}
\]
Orienting the contour \( \partial B_{\delta} \) counterclockwise we have the following lemma.
Lemma 3.3. The residual $\mathcal{E}(z)$, defined by (3.12), is analytic in $\mathbb{C} \setminus \Gamma_{\mathcal{E}}$, $\Gamma_{\mathcal{E}} = (\partial S_1 \setminus B_\delta) \cup \partial B_\delta$, and given by

$$\mathcal{E}(z) = \frac{1}{2\pi i} \int_{\Gamma_{\mathcal{E}}} \frac{v_{\mathcal{E}}(w)}{w - z} \, dw,$$

$$v_{\mathcal{E}}(z) = \begin{cases} h_0(z) - k(z) & z \in \partial B_\delta \\ \sqrt{n} e^{-n\phi_\lambda(z)} & z \in \gamma \setminus B_\delta. \end{cases}$$

Moreover, there exist $n_0, \delta_0 > 0$ such that for any $n \geq n_0$ and $\delta \leq \delta_0$

$$\mathcal{E}(z) = \begin{cases} h_0(z) + \frac{1}{2\sqrt{\pi} w(z)} & z \in B_\delta \\ O(n^{-1}) & z \in B_\delta^c \end{cases}$$

uniformly for $z$ in each set.

Proof. From (1.12) and (3.4) we see that $\mathcal{E}(z)$ is analytic in $B_\delta^c$ except along $\partial S_1$ where it inherits the jump discontinuity of $h$ and that it vanishes as $z \to \infty$. Inside $B_\delta$, (3.8) implies that $\mathcal{E}$ is continuous across $\partial S_1$, and hence is analytic. The jump $v_{\mathcal{E}}$ and Cauchy integral representation of $\mathcal{E}(z)$ in (3.13) follow immediately.

The bounds in (3.14) follow from two observations: first, the jump $v_{\mathcal{E}}$ is exponentially small on $\partial S_1 \setminus B_\delta$, specifically $v_{\mathcal{E}}(z) = O(e^{-cn})$ for $z \in \partial S_1 \setminus B_\delta$ where $c = \min_{y \in [\delta, \infty)} \Re \phi_\lambda(1+iy) > 0$; and secondly, on the disk boundary $\partial B_\delta$ we have

$$v_{\mathcal{E}}(z) = \left[ h_0(z) + \frac{1}{2\sqrt{\pi} w(z)} \right] - \left[ k(z) + \frac{1}{2\sqrt{\pi} w(z)} \right].$$

Using (3.4) and (3.2) it’s easy to see that the first bracketed term has vanishing residues at $z = \pm 1$; it therefore extends to a bounded analytic function for $z \in B_\delta$. The second bracketed term is not analytic in $B_\delta$, but using (3.7) and (3.11) it admits a Laurent expansion on $\partial B_\delta$ which is uniformly $O(n^{-1})$. Thus, the Cauchy transform of the first bracketed term can be explicitly evaluated for any $z \in \mathbb{C} \setminus \partial B_\delta$ by the Cauchy integral formula; using the boundedness of the Cauchy projection operators the Cauchy transform of the second bracketed term above is everywhere $O(n^{-1})$. The expansion (3.14) follows immediately.

Proof of Theorem 3.1. Equation (3.12) and Lemma 3.3 yield an asymptotic expansion of $h(z)$ in $B_\delta$ and $B_\delta^c$. Plugging this result into (1.10) gives (3.5) which completes the proof.

4. Counting the zeros of the Taylor polynomials

Here and in what follows, $\lambda = \lambda(n)$ is as described by Lemma 2.3, so that in particular $\lambda$ satisfies $4n - \lambda \log \left( \frac{1}{2\pi} \right) = O(1)$. It follows from (1.10) that any zero $z_{k,n}$ of $T_{2n-2}(\lambda z)$ satisfies

$$\mathcal{G}(z_{k,n}) = \frac{\log n}{2n} + \frac{2k\pi i}{n}, \quad \mathcal{G}(z) := \phi_\lambda(z) + \frac{1}{n} \log h(z),$$

$$G(z_{k,n}) = \frac{\log n}{2n} + \frac{2k\pi i}{n}, \quad G(z) := \frac{\phi_\lambda(z)}{n} + \frac{1}{n} \log h(z),$$
Figure 3. Zeros of $T_{202}(\lambda z)$, the 202nd degree Taylor polynomial of $\xi(z + 1/2)$ in the rescaled plane, computed using [16]. The scaling parameter $\lambda$ is given by Lemma 2.3. As $n \to \infty$, spurious zeros approach the level curve $D^{(0)}_n$ (dashed line); for finite $n$, the improved curve $D^{(1)}_n$ (solid line) more accurately approximates zeros. A particular fraction lie inside the curve $D^{(0)}_n$. These Hurwitz zeros converge to shifted and scaled nontrivial roots of the $\zeta$ function. The difference between the 11 numerically computed zeros of $T_{202}(\lambda z)$ on the positive critical line and the (rescaled) first 11 nontrivial zeros of $\zeta$ are given in Table 2.

This can happen in one of two ways, either: a) $z_{k,n}$ is a Hurwitz zero converging to a zero of $f(\lambda z)$—and thus lies inside the rescaled critical strip; or b) $z_{k,n}$ is a spurious zero, and does not approach a root of $f(\lambda z)$; in both cases $z_{k,n}$ must approach the level curve

$$D^{(0)}_n = \{ z \in \mathbb{C} : \text{Re} \phi_\lambda(z) = 0, \text{ and } |z| < 1 \}$$

which is nearly† the Szegő curve for $f$. Although not necessary for this paper, this set can be shown to consist of a collection of disjoint components collapsing upon Hurwitz zeros (and the corresponding zeros of the function $f(\lambda z)$) together with an additional large component attracting those zeros that are spurious.

Let

$$Z_n = \left\{ z_{k,n} : T_{2n-2}(\lambda z_{k,n}) = 0 \text{ and } \text{Re } z_{k,n} > \frac{1}{2\lambda} \right\}$$

†we do not call $D^{(0)}_n$ the actual Szegő curve because $\phi_\lambda$ still has weak $n$ dependence. Strictly speaking, the Szegő curve should be defined as $D_\infty = \lim_{n \to \infty} D^{(0)}_n$. 
denote the set of (spurious) zeros of $T_{2n-2}(\lambda z)$ outside the rescaled critical strip. The results in this section culminate in the following theorem:

**Theorem 4.1.** Let $T_{2n-2}(\lambda z)$ be the rescaled Taylor polynomial of degree $2n - 2$ defined by (1.10) and Lemma 2.3. Then as $n \to \infty$

$$|Z_n| = n - \frac{\lambda \mathcal{Y}}{2\pi} \log \left( \frac{\lambda \mathcal{Y}}{2\pi} \right) + \frac{\lambda \mathcal{Y}}{2\pi} - \frac{1}{4\pi \mathcal{Y}} \log \left( \frac{\lambda \mathcal{Y}}{2\pi} \right) + O(\log \log \lambda \mathcal{Y}).$$

Here $\mathcal{Y}$, defined in Lemma 4.5 below, is the imaginary part of a point on $D_n^{(1)}$—a further improvement to the curve $D_n^{(0)}$—at the edge of the critical strip.

As $T_{2n-2}$ has exactly $2n - 2$ zeros this has the immediate and obvious corollary:

**Corollary 4.2.** As $n \to \infty$, the Taylor polynomial $T_{2n-2}(\lambda z)$ has

$$\frac{\lambda \mathcal{Y}}{2\pi} \log \left( \frac{\lambda \mathcal{Y}}{2\pi} \right) - \frac{\lambda \mathcal{Y}}{2\pi} + \frac{1}{4\pi \mathcal{Y}} \log \left( \frac{\lambda \mathcal{Y}}{2\pi} \right) + O(\log \log \lambda \mathcal{Y})$$

zeros in the rescaled critical strip.

**Remark 4.3.** Well known estimates on the behavior of $\zeta$ within the critical strip show that the level set $\text{Re } \mathcal{G} = \frac{\log n}{2n}$, on which all zeros of $T_{2n-2}(\lambda z)$ must live, remains within a rectangle whose height is bounded by $\mathcal{Y} + O\left(\log \frac{n}{n}\right)$. Corollary 4.2 is therefore consistent with the Riemann-von Mangoldt formula (1.3) using $T = \lambda \mathcal{Y}$. The precision of the error bound for the zeros of the Taylor polynomials suggests that there are a growing number of spurious zeros within the rescaled critical strip.

Theorem 4.1 is proved below using the asymptotic representation in Theorem 3.1. We first count those zeros bounded away from the stationary points $z = \pm 1$ by constructing a set of approximate zeros $\alpha_{k,n}$ and then demonstrating that each of these is in one to one correspondence with an actual zero $z_{k,n}$ of the Taylor polynomial in the zero free region $F_\lambda$. We then count the zeros near each of the stationary points using a Rouche theorem type argument. Finally, note that the four-fold symmetry $T_{2n}(z) = T_{2n}(-z) = T_{2n}(z^*) = T_{2n}(-z^*)$ implies that it is sufficient to study only those zeros in the closed positive quadrant: $\text{Re } z, \text{Im } z \geq 0$.

**4.1. Number of zeros outside the critical strip, away from the stationary points.**

Let

$$\mathcal{U} = \left\{ z \in \mathbb{C}^+ : \frac{1}{2\lambda} \leq \text{Re } z \leq 1 \and z \notin B_{1,\delta} \right\}.$$  

(4.4)

denote the vertical strip in $\mathbb{C}^+$ between the critical strip and the stationary point at $z = 1$ with a small neighborhood of $z = 1$ deleted. Both $f(\lambda z)$ and $h(z)$ are analytic and zero free in $\mathcal{U}$, so $\phi_\lambda(z)$ and $\log h(z)$ are each well defined (we choose the branches real valued for $z \in \mathcal{U} \cap \mathbb{R}$).
As a first step toward Theorem 4.1 we want to estimate the number of zeros of \( T_{2n-2}(\lambda z) \) in \( U \). We could approximate the zeros \( z_{k,n} \) by points along \( D_{\infty} \), but for our purposes it will be more convenient to work with

\[
D_{n}^{(1)} = \{ z \in C : \text{Re} G_1(z) = \log \frac{n}{2n}, \text{ and } |z| < 1 \},
\]

which is the (first) correction to the level curve \( D_n^{(0)} \) that better attracts the spurious zeros, analogous to the improved Szegő curve (1.7), which comes from keeping the first term in the asymptotic series for \( h \). We define the \textit{approximate zeros}, \( \alpha_{k,n} \), as roots of the equation.

\[
G_1(\alpha_{k,n}) = \log \frac{n}{2n} + \frac{2k\pi i}{n},
\]

and denote by \( A_n \) the set of approximate zeros of \( T_{2n-2}(\lambda z) \) which lie in \( U \):

\[
A_n = \left\{ \alpha_{k,n} \in U : G_1(\alpha_{k,n}) = \log \frac{n}{2n} + \frac{2k\pi i}{n} \right\}.
\]

We begin by describing the shape of the improved level curve \( D_{n}^{(1)} \) along which our approximate zeros accumulate in the region

\[
F_{1,\lambda} = \{ z \in F_{\lambda} \setminus B_{1,\delta} : \text{Re} z \in [0, 1] \}
\]

\textbf{Lemma 4.4.} Let \( z = x + iy \). Fix \( A > 0 \) defining the zero free region \( F_{\lambda} \). Then there exist \( n_0 > 0 \) such that for any \( n > n_0 \) the level curve \( \text{Re} G_1 = \log \frac{n}{2n} \) implicitly defines a single smooth non-intersecting curve \( y = Y(x) \) for \( z \in F_{1,\lambda} \) as defined above. Near the edge of the critical strip, that is for,

\[
-\frac{A}{\lambda(\log \lambda)^{2/3}(\log \log \lambda)^{1/3}} < x - \frac{1}{2\lambda} < \frac{A}{\lambda},
\]

the curve \( y = Y(x) \) satisfies

\[
Y(x) = \frac{8n}{\pi \lambda} W \left( \frac{\pi \lambda}{8n} e^{-1 + x + (\log n)/(4n)} \right) + O \left( \frac{\log \log n}{n} \right),
\]

where \( W \) is the Lambert-W function.

\textbf{Proof.} Both \( \phi_\lambda \) and \( h_0 \) are analytic in \( F_{1,\lambda} \). Lemma 2.5 bounds \( |\partial_z \phi_\lambda| \) below uniformly in \( n \), and \( h_0 \) has a bounded derivative in \( F_{1,\lambda} \). It follows that for all sufficiently large \( n \), \( \partial_z G_1(z) \neq 0 \) for all \( z \in F_{1,\lambda} \) and thus the level set \( \text{Re} G_1 = \log \frac{n}{2n} \) must consist of a collection of smooth nonintersecting arcs in \( F_{1,\lambda} \) with no finite endpoint in \( F_{1,\lambda} \).

As \( \lim_{z \to 0} \text{Re} \phi_\lambda(z) = -\infty \), \( \phi_\lambda(1) = 0 \), and \( \partial_z \phi_\lambda \) has no zeros on \((0, 1)\), \( \text{Re} \phi_\lambda < 0 \) for \( z \in (0, 1) \). Thus, for any \( x_0 \in (0, 1) \), for all sufficiently large \( n \), \( \text{Re} G_1 < 0 \) for all \( x \in (0, x_0) \).
So no branch of the level curve may leave $\mathcal{F}_{1,\lambda}$ through the real axis. Away from $z = 0$ we use (2.10) to write

$$
\text{Re} \mathcal{G}_1(z) = 2(\log |z| + 1 - x) - \frac{\lambda}{2n}(1 - x + x \log |z| - y \arg(z))
$$

$$
- \frac{1}{n} \log \zeta(\lambda z + 1/2)| + \frac{1}{n} \text{Re} \tilde{r}(z; \lambda) + \frac{1}{n} \log |h_0(z)|.
$$

From this expansion we observe that: (1) the level curves are bounded above since $\text{Re} \mathcal{G}_1(z)$ grows without bound as $y \to \infty$ with $x$ bounded; (2) for any $y_0 > 0$, if $z = 1 + iy$, with $y > y_0$, then for all $n$ large enough $\text{Re} \mathcal{G}_1(z) > c(y_0) > 0$. So all branches of the level curve $\text{Re} \mathcal{G}_1 = \frac{\log n}{2n}$ in $\mathcal{F}_{1,\lambda}$ must enter $\mathcal{F}_{1,\lambda}$ through its left edge and leave by entering $B_{1,\delta}$.

Since all branches of the level set are bounded away from the origin and infinity, for any $z = x + iy$ along the level set $\text{Re} \phi(z) = \log n/(2n)$ with $\text{Re} z = x < A/\lambda$ we have:

$$
\log |z| = \log y + O(\lambda^{-2}), \quad \arg(z) = \frac{\pi}{2} - \frac{x}{y} + O(\lambda^{-3}).
$$

Inserting these into (4.8) one has

$$
\text{Re} \phi(z) = g(x, y) + O\left(\frac{\log n}{n}\right), \quad g(x, y) = 2(\log y + 1 - x) - \frac{\lambda}{2n}(1 - \frac{\pi}{2}y),
$$

where we’ve used (2.1) to bound $\log \zeta(\lambda z + 1/2)$. For each $0 < x < A/\lambda$ there is a single solution $y$ of $g(x, y) = (\log n)/(2n)$. It follows that there is only a single branch of the level curve $\text{Re} \mathcal{G}_1 = \frac{\log n}{2n}$ in $\mathcal{F}_{1,\lambda}$. One may then solve $g(x, y) = (\log n)/(2n)$ for $y$ using the Lambert-$W$ function, which gives the leading term of $y = Y(x)$ for $x \in \mathcal{F}_{1,\lambda}$ with $\text{Re} x < A/\lambda$. The error bound is immediate.

**Lemma 4.5.** As $n \to \infty$, the number of approximate zeros in $U$ satisfies

$$
|\mathcal{A}_n| = \frac{n}{2} - \frac{\lambda Y}{4\pi} \log \left(\frac{\lambda Y}{2\pi}\right) + \frac{\lambda Y}{4\pi} - \frac{1}{8\pi Y} \log \left(\frac{\lambda Y}{2\pi}\right) - \frac{n\delta^2}{2\pi} + O(\log \log \lambda Y).
$$

Here, $Y = Y \left(\frac{1}{2\pi}\right)$, with $Y(x)$ as described by Lemma 4.4, is the imaginary part of $z$ where the level curve $\text{Re} \mathcal{G}_1 = \frac{\log n}{2n}$ meets the edge of the critical strip, and $\delta$ is the radius of the image-disk $w(B_{1,\delta})$ in the $w$-plane.

**Proof.** Moving along the level curve $\text{Re} \mathcal{G}_1 = \frac{\log n}{2n}$ from $\text{Re} z = 1$ towards the critical strip, $\text{Im} \mathcal{G}_1$ is strictly increasing, so denoting by $z_0$ the point at which $\text{Re} \mathcal{G}_1 = \frac{\log n}{2n}$ intersects the boundary of $B_{1,\delta}$, and noting that $\log h_0$ is bounded outside $B_{1,\delta}$, the number of approximate zeros in $U$ is given by

$$
\frac{n}{2\pi} \left[\text{Im} \mathcal{G}_1 \left(\frac{1}{2\lambda} + iY\right) - \text{Im} \mathcal{G}_1 (z_0)\right] = \frac{n}{2\pi} \left[\text{Im} \phi(z) \left(\frac{1}{2\lambda} + iY\right) - \text{Im} \phi(z_0)\right] + O(1).
$$

Recall that the set $B_{1,\delta}$ is chosen such that the image $w(B_{1,\delta})$ under the map $w = w(z)$ defined by (3.1)-(-3.2) is a disk of radius $\delta$. The condition that $\text{Re} \phi(z_0) = \text{Re} w^2 =
\(\delta^2 \cos(2 \arg w) = (\log n)/2n\) gives
\[
\text{Im} \phi_\lambda(z_0) = \delta^2 \sin(2 \arg w) = \delta^2 - \mathcal{O}(\log n)/(n\delta^2).
\]
Similarly, using Lemma 2.2, (2.10) and (4.9) we have
\[
\text{Im} \phi_\lambda\left(\frac{1}{2\lambda} + i\Upsilon\right) = 2 \left[\pi/2 - \Upsilon - \frac{1}{2\lambda\Upsilon}\right] + \frac{\lambda}{2n} [\Upsilon - \Upsilon \log \Upsilon] + \mathcal{O}\left(\frac{\log \log \lambda\Upsilon}{n}\right)
\]
where we have used the estimate \(\lambda = \mathcal{O}(n/\log n)\) implied by Lemma 2.3 to drop lower order terms and in the last equality we’ve used the first asymptotic statement in Lemma 2.3 to simplify. The result follows immediately.

**Lemma 4.6.** Fix \(A > 0\) to define a zero free region \(\mathcal{F}_\lambda\) as in (2.2). Let \(T_{2n-2}(\lambda z)\) be the rescaled Taylor polynomial of degree \(2n - 2\) defined by (1.10) and Lemma 2.3; let \(z_{k,n}\) and \(\alpha_{k,n}\) denote actual and approximate zeros of \(T_{2n-2}(\lambda z)\) defined by (4.1) and (4.6) respectively. Then for all sufficiently large \(n\), each approximate zero \(\alpha_{k,n} \in \mathcal{F}_\lambda\) corresponds to a distinct zero \(z_{k,n}\) of \(T_{2n-2}(\lambda z)\). Moreover,
\[
|z_{k,n} - \alpha_{k,n}| = \mathcal{O}(n^{-2}).
\]

**Proof.** Fix \(n\). Clearly for \(k \neq \ell\), any solutions \(z_{k,n}\) and \(z_{\ell,n}\) of (4.1) in the zero free region of \(f\) are distinct as each corresponds to a distinct value of the single valued function \(\text{Im}(\phi_\lambda(z) - n^{-1} \log h(z))\). Using (4.6), the root condition (4.1) can be rewritten in the form
\[
\mathcal{G}_k(z_{k,n}, n^{-1}) = 0
\]
where \(\mathcal{G}_k(z, \epsilon) = \phi_\lambda(z) - \phi_\lambda(\alpha_{k,n}) + \epsilon [\log h(z) - \log h_0(\alpha_{k,n})]\).

Now, \(\mathcal{G}_k(\alpha_{k,n}, 0) = 0\) and Lemma 2.5 guarantees that for all sufficiently large \(n\) (independent of \(k\)), \(|\partial_{z} \mathcal{G}_k(\alpha_{k,n}, 0)| = |\partial_{z} \phi_\lambda(\alpha_{k,n})|\geq \rho > 0\) for all sufficiently large \(n\). Invoking the implicit function theorem, there exist a unique solution \(z_{k,n}\) of (4.1) for all sufficiently large \(n\) in a neighborhood of \(\alpha_{k,n}\). Expanding, we have (again uniformly in \(k\))
\[
z_{k,n} = \alpha_{k,n} - \frac{1}{n} \frac{\log h(\alpha_{k,n}) - \log h_0(\alpha_{k,n})}{\partial_{z} \phi_\lambda(\alpha_{k,n})} + \mathcal{O}\left(\frac{1}{n^2}\right).
\]
Recalling (3.4), we observe that \(\log h(\alpha_{k,n}) - \log h_0(\alpha_{k,n}) = \mathcal{O}(n^{-1})\) which completes the result. \(\square\)

### 4.2. Number of zeros near the stationary points.
Near the stationary points \(z = \pm 1\) the zeros of \(T_{2n}(\lambda z)\) are not spaced uniformly along the level curve \(D_{1}^{1}\). Theorem 3.1 suggest that the zeros of \(T_{2n-2}(\lambda z)\) should be well approximated by the zeros of \(g(in^{1/2}w(z))\).

Recall that \(B_\delta\) is chosen such that the scaling map \(v(z) = in^{1/2}w(z)\) maps \(B_\delta\) to a disk of radius \(n^{1/2}\delta\) centered at the origin in the \(v\)-plane, i.e., \(v(B_1) = B(0, n^{1/2}\delta)\). The zeros of \(\text{erfc}(v)\) are well known and come in conjugate pairs \([11, §7.13(ii)]\). Enumerating the zeros of
erfc(v) in $\mathbb{C}^+$ by $v_k = \mu_k + i\nu_k$, according to increasing absolute value, the large modulus zeros of erfc(v) are asymptotically given, for $k \gg 1$, by

$$v_k := \mu_k + i\nu_k$$

$$\mu_k = -\varsigma + \frac{1}{4} \tau \varsigma^{-1} - \frac{1}{16} (1 - \tau + \frac{1}{2} \tau^2) \varsigma^{-3} + \ldots$$

$$\varsigma = \sqrt{(k - 1/8)\pi}$$

$$\nu_k = \varsigma + \frac{1}{4} \tau \varsigma^{-1} + \frac{1}{16} (1 - \tau + \frac{1}{2} \tau^2) \varsigma^{-3} + \ldots$$

$$\tau = \log \left(2\varsigma \sqrt{2\pi}\right)$$

from which it follows that

$$|v_k|^2 = 2\pi(k - 1/8) + O\left(k^{-1}\log(k)^2\right).$$

To count the number of zeros of $T_{2n-2}(\lambda z)$ in $B_{\pm 1,\delta}$ we first introduce the integer valued functions

$$K^-(n, \delta) := \left\lfloor \frac{n\delta^2}{2\pi} - \frac{3}{8} \right\rfloor$$

and

$$K^+(n, \delta) := \left\lceil \frac{n\delta^2}{2\pi} - \frac{3}{8} \right\rceil,$$

where $[x]$ and $\lceil x \rceil$ are the floor and ceiling functions respectively.

**Lemma 4.7.** There exist $\delta_0 > 0$ such that for any fixed $\delta$, $0 < \delta < \delta_0$, there exist $n_0(\delta_0, \delta)$ such that for any $n > n_0$ the Taylor polynomial $T_{2n-2}(\lambda z)$ has either $2K^-(n, \delta)$ or $2K^+(n, \delta)$ zeros in $B_{\pm 1,\delta}$.

**Proof.** Due to even symmetry of $T_{2n-2}(\lambda z)$ we consider only $B_{1,\delta}$. For simplicity, temporarily let $v(z) = in^{1/2}w(z)$ and write $g(v) = \frac{1}{2}e^v \text{erfc}(v)$. As $f(\lambda z)$ is zero free in $B_{1,\delta}$, the representation (3.5) implies that $T_{2n-2}(\lambda z)$ has the same number of zeros in $B_{1,\delta}$ as the function $g(v(z)) + n^{-1/2}\mathcal{E}(z)$.

For a fixed choice of $\delta > 0$ define the radii

$$R^-_\delta = \sqrt{2\pi \left(K^-(n, \delta) + \frac{3}{8}\right)}, \quad R^+\delta = \sqrt{2\pi \left(K^+(n, \delta) + \frac{3}{8}\right)}.$$

The proof follows from Rouche’s theorem. From (4.11) it follows that there are exactly $2K^-(n, \delta)$ zeros of $g(v)$ in $\mathbb{D}(0, R^-_\delta)$ and $2K^+(n, \delta)$ zeros of $g(v)$ in $\mathbb{D}(0, R^+_\delta)$ for any sufficiently large $n$. Lemma 3.3 guarantees for all $z \in v^{-1}(\mathbb{D}(0, R^+_\delta))$ that $|\mathcal{E}(z)| < C_0$ for some fixed positive constant $C_0 > 0$ independent of $\delta$. As we will show below, there also exist a constant $C_1 > 0$ such that $|g(v)| > C_1/R^+_\delta$ on the circles of radii $R^+_\delta$. By choosing $\delta$ such that $R^+\delta / n^{1/2} < \delta_0 = C_1/C_0$, Rouche’s theorem implies that $T_{2n-2}(\lambda z)$ has $2K^-(n, \delta)$ zeros in $v^{-1}(\mathbb{D}(0, R^-_\delta))$ and $2K^+(n, \delta)$ zeros in $v^{-1}(\mathbb{D}(0, R^+_\delta))$. The result then follows from observing that $v^{-1}(\mathbb{D}(0, R^-_\delta)) \subseteq B_{1,\delta} \subseteq v^{-1}(\mathbb{D}(0, R^+_\delta))$, so we have determined the number

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1We are slightly abusing notation here, in (3.12) $\mathcal{E}(z)$ is piecewise defined inside and outside $B_\delta$. What we mean here is the analytic extension of $\mathcal{E}(z)$ from inside $B_{1,\delta}$ to a set containing $v^{-1}(\mathbb{D}(0, R^+_\delta))$ which can be see to exist simply by deforming the contour $\Gamma_\mathcal{E}$ in (3.13).
of zeros $T_{2n-2}(\lambda z)$ in $B_{1,\delta}$ to within two, depending on the location of the extra pair of zeros in $v^{-1}(\mathbb{D}(0, R_\delta^+))$.

It remains to show that $|g(v)| > C_1/R_\delta^+$ for $|v| = R_\delta^+$. For any $\delta > 0$, $R_\delta^+ \sim \delta n^{1/2}$ so for all sufficiently large $n$ the asymptotic series for $g(v)$ given in (3.10) can be applied. Away from the rays $\arg(v) = \pm 3\pi/4$ the desired bound is immediate; when one needs the second expansion in (3.10) the exponential term is either beyond all orders small or dominant away from these rays—in either case the previous bound holds. To bound the behavior of $g(v)$ on the disk boundary near the lines $\pm \arg(v) = 3\pi/4$ first observe that $g(\overline{v}) = \overline{g(v)}$ so it is sufficient to only consider $\arg(v)$ near $3\pi/4$. Write $v = R_\delta^{-} e^{i(3\pi/4 + \alpha)}$, the case when $|v| = R_\delta^+$ can be treated identically. Letting $K^- = K^-(n, \delta)$, the first terms in the asymptotic expansion gives

$$g(v) = e^{(R_\delta^-)^2 \sin 2\alpha} \left[ \cos \left( (2\pi K^- + 3\pi/4) \cos 2\alpha \right) - i \sin \left( (2\pi K^- + 3\pi/4) \cos 2\alpha \right) \right]$$

$$+ \frac{1}{2\sqrt{\pi} R_\delta^-} \left[ \cos(3\pi/4 + \alpha) - i \sin(3\pi/4 + \alpha) \right] + O \left( (R_\delta^-)^2 \right).$$

For $2|\alpha| \leq \arccos \left( \frac{2\pi K^-}{2\pi K^- + 3\pi/4} \right)$, the imaginary parts of the first two terms are both negative and so the sum has a larger (in absolute value) imaginary part than either term separately, hence

$$|g(v)| \geq |\text{Im} g(v)| \geq \frac{C}{R_\delta^-} \text{ for } 2|\alpha| \leq \arccos \left( \frac{2\pi K^-}{2\pi K^- + 3\pi/4} \right).$$

On the other hand for $\arccos \left( \frac{2\pi K^-}{2\pi K^- + 3\pi/4} \right) < 2|\alpha| \leq \pi/50$,

$$|e^{i^2}| = e^{(R_\delta^-)^2 \sin 2\alpha} \geq e^{\pm 2\pi(K^- + 3/8)\sqrt{1 - \left( \frac{2\pi K^-}{2\pi K^- + 3\pi/4} \right)^2}} = e^{\pm \sqrt{\pi} \sqrt{K^-}} \left[ 1 + O \left( \frac{1}{\sqrt{K^-}} \right) \right],$$

where the upper (lower) inequality and signs are taken if $\alpha$ is positive (negative). Now, since $K^- \sim n\delta^2$, the exponential is already beyond all orders separated in scale from the algebraic terms in the expansion and we see that $|g(v)| \leq C/R_\delta^-$ for $|\alpha| < \pi/100$. \hfill \Box

Combining the results of Sections 4.1-4.2 we can now prove Theorem 4.1, the main result of Section 4.

**Proof of Theorem 4.1.** Since $\mathcal{G} - \mathcal{G}_1 = \frac{1}{n} \log \frac{h}{h_0}$, and $\frac{1}{n} \log \frac{h}{h_0}$ as well as its derivative are uniformly $O\left( n^{-2} \right)$ in $\mathcal{U}$, there is a single level curve of $\text{Re} \mathcal{G} = \frac{\log n}{2n}$ in $\mathcal{U}$, on which all zeros of $T_{2n-2}(\lambda z)$ in $\mathcal{U}$ must live. As one traverses this level curve from $\partial B_{1,\delta}$ to the point where it intersects the vertical line $\{ z : \text{Re} z = \frac{b}{2n} \}$, $\text{Im} \mathcal{G}$ is strictly monotone increasing. Each root of $T_{2n-2}(\lambda z)$ in $\mathcal{U}$ must satisfy (4.1) for some integer $k$. The monotonicity of both $\text{Im} \mathcal{G}$ and $\text{Im} \mathcal{G}_1$ along the associated level curves, and the fact that $|\text{Im} \mathcal{G} - \text{Im} \mathcal{G}_1| \leq cn^{-2}$ uniformly in $\mathcal{U}$, imply that the only roots of $T_{2n-2}(\lambda z)$ within $\mathcal{U}$ are those identified in Lemma 4.6.
Table 2. Tabulated here are the differences between the 11 numerical calculated of the zeros $z_{k,n}$ of $T_{202}(\lambda z)$ on the critical line depicted in Figure 3, and the first 11 (rescaled) zeros of the $\xi(s+1/2)$ function, denoted here as $s_k$.

We observe that: 1) the boundedness of the derivative $\partial_s \phi_\lambda$ near the edge of the critical strip (and in any compact subset of $\mathcal{U}$) along with the smoothness of the level curve $\text{Re} \ G_1 = \log n / 2n$ implies that there exist a constant $c > 0$ such that zeros $|z_{k,n} - z_{j,n}| > c/n$; and 2) Lemma 4.6 guarantees that $|z_{k,n} - \alpha_{k,n}| = O\left(n^{-2}\right)$.

Now the left-most approximate zero in $\mathcal{U}$ corresponds to a root of $T_{2n-2}(\lambda z)$ that may or may not lie within $\mathcal{U}$. Likewise, the approximate zero $\alpha_{k,n}$ within the critical strip that is closest to the vertical line $\{\text{Re} z = \frac{1}{2\lambda}\}$ corresponds to a root of $T_{2n-2}(\lambda z)$ that may or may not lie within $\mathcal{U}$. Similar considerations for those approximate roots near $\partial B_{1,\delta}$ show that, again, there could be up to 2 additional roots (or 2 fewer roots) of $T_{2n-2}(\lambda z)$ near $\partial B_{1,\delta}$, because of boundary effects. So we have shown that

$$|\mathcal{Z}_n| = |\{\mathcal{Z}_n \cap \mathcal{U}\} - |\mathcal{A}_n| | \leq 4.$$  

Combining the this observation with Lemmas 4.5 and 4.6 we get an expression for the number of true zeros $z_{k,n}$ outside the critical strip (by left-right symmetry we multiply $|\mathcal{A}_n|$ by 2) which are bounded away from the stationary points $z = \pm 1$. Lemma 4.7 gives an exact count of the number of zeros in $B_{\pm 1,\delta}$ (of which half of each are in $\mathcal{F}_\lambda$). Summing these contributions we find that

$$|\mathcal{Z}_n| = n - \frac{\lambda \gamma}{2\pi} \log\left(\frac{\lambda \gamma}{2\pi}\right) + \frac{\lambda \gamma}{2\pi} - \frac{1}{4\pi \gamma} \log\left(\frac{\lambda \gamma}{2\pi}\right) - \frac{n\delta^2}{\pi} + 2K^-(n, \delta) + O\left(\log\log \lambda \gamma\right).$$

The result then follows from observing that

$$\left|2K^-(n, \delta) - \frac{n\delta^2}{\pi}\right| = \left|2\left[\frac{n\delta^2}{2\pi} - \frac{3}{8}\right] - \frac{n\delta^2}{\pi}\right| \leq 2.$$  

□
5. Convergence rates of true zeros

In this section we turn our attention to those zeros of $T_{2n-2}(\lambda z)$ which converge to roots of $\xi(z+1/2)$ as $n \to \infty$. Recall that Hurwitz theorem guarantees that near any root $s$ of order $m$ of $f(z)$ in the unscaled plane, there will be exactly $m$ zeros of $T_n(z)$ for all sufficiently large $n$, and that these will converge to $s$ as $n \to \infty$. These are the ‘Hurwitz zeros’ of $T_n(z)$.

In Figure 3 there are 11 zeros of $T_{202}(\lambda z)$ on the positive critical line below the level curve $D^{(0)}_n$. The absolute error between these numerically computed Hurwitz zeros of the Taylor polynomial zeros and the first 11 nontrivial roots of the $\xi(\lambda z + 1/2)$ function are given in Table 2. The agreement is surprising good, particularly considering that the scaling factor $\lambda \approx 133$ for $T_{202}(\lambda z)$.

Suppose that $s$ is an order $m$ zero† of $f(s) = \xi(s + 1/2)$, and suppose that $\lambda z = s + \mu$ is a Hurwitz zero of the Taylor polynomial $T_{2n-2}(\lambda z)$. As $\text{Re } s < \frac{1}{2} < \lambda$ our representation (1.10) of $T_{2n}(\lambda z)$ gives

\[
T_{2n-2}(s + \mu) = f(s + \mu) + \frac{(s + \mu)^{2n} f(\lambda)}{\lambda^{2n} \sqrt{n}} h(s/\lambda + \mu/\lambda).
\]

Expanding around $s$ gives

\[
T_{2n-2}(s + \mu) = \frac{f^{(m)}(s)}{m!} \mu^m + \frac{s^{2n} f(\lambda)}{\lambda^{2n} \sqrt{n}} h(s/\lambda) + R(s, \mu)
\]

where $R$ is the explicit remainder

\[
R(s, \mu) = \left[ f(s + \mu) - \frac{f^{(m)}(s)}{m!} \mu^m \right] + \frac{s^{2n} f(\lambda)}{\lambda^{2n} \sqrt{n}} \left[ h \left( \frac{s + \mu}{\lambda} \right) - h \left( \frac{s}{\lambda} \right) + \left( \frac{\mu}{s} \right) (1 - \frac{\mu}{s})^2 - 1 \right] h \left( \frac{s + \mu}{\lambda} \right).
\]

Consider how the factor $\frac{s^{2n} f(\lambda)}{\lambda^{2n} \sqrt{n}}$ behaves as $n \to \infty$. Using Stirling’s series for $\Gamma$, we find

\[
f(\lambda) = \left( \frac{\lambda}{2\pi e} \right)^{\lambda/2} \left( \frac{\lambda}{2\pi} \right)^{7/4} 2\sqrt{2\pi} \left[ 1 - \frac{1}{48\lambda} + O(\lambda^{-2}) \right]
\]

where we note that $\zeta(\lambda + 1/2) - 1$ is beyond all orders small and so makes no contribution to the asymptotic expansion. This can be further simplified using Lemma 2.3; setting $u = \frac{2n}{\pi}$, and recalling the defining relation for $W = W(u)$, i.e., $We^W = u$ we have

\[
\frac{\lambda}{2\pi} = \frac{u}{W} \left[ 1 + O \left( n^{-1} \right) \right] = e^W \left[ 1 + O \left( n^{-1} \right) \right]
\]

which allows us to write

\[
f(\lambda) \sim \left( \frac{\lambda}{2\pi e} \right)^{\lambda/2} \left( \frac{\lambda}{2\pi} \right)^{7/4} \sim e^{2n(1-W^{-1} + \frac{7}{8n} W)}.
\]

†It is widely believed, but unproven, that all zeros of $\xi$ are simple
so finally we have the asymptotic estimate

\[(5.7) \quad \frac{s^{2n}f(\lambda)}{\lambda^{2n}\sqrt{n}} \sim \exp \left[ -2n \left( \frac{\log |\lambda|}{s} - 1 + W^{-1} - \frac{7}{8n} W + \frac{1}{4n} \log n \right) \right] \]

which is uniformly exponentially small provided that \(|s| < \frac{\lambda}{e}\).

**Theorem 5.1.** Let \(s\) be a fixed zero of \(\xi(s+1/2)\) of order \(m\). Then there exist an \(N_0 = N_0(s)\) such that for all \(n \geq N_0\) the Taylor polynomial \(T_{2n-2}(\lambda z)\) defined by (1.10) and Lemma 2.3 has exactly \(m\) zeros \(\{z_{k,n}^H\}_{k=1}^m\) converging to \(s/\lambda\). Moreover, these zeros converge at a super-exponential rate:

\[(5.8) \quad \max_{1 \leq k \leq m} |\lambda z_{k,n}^H - s| = O \left( \exp \left[ -\frac{2n}{m} \left( \frac{\log |\lambda|}{s} - 1 + W^{-1} - \frac{7}{8n} W + \frac{1}{4n} \log n \right) \right] \right). \]

**Proof.** The first half of the theorem is just a restatement of Hurwitz’s theorem in the case of Taylor polynomials. It remains to establish our superexponential bound on the rate of convergence. Let \(s\) be a fixed root of order \(m\) of \(f(s)\), write \(\lambda z = s + \mu\) and define the function

\[g(\mu) := T_{2n-2}(s + \mu) - \frac{f^{(m)}(s)}{m!} \mu^m = \frac{s^{2n}f(\lambda)}{\lambda^{2n}\sqrt{n}} h(s/\lambda) + R(s, \mu).\]

Let \(\rho(n) = \left| \frac{s^{2n}f(\lambda)}{\lambda^{2n}\sqrt{n}} \right|^{1/m}\). It follows from (1.12) and (3.4) that \(h(\lambda z)\) is analytic and bounded for all \(z\) in the critical strip. Then using (5.3), Taylor’s remainder theorem and (5.7), on the circle \(|\mu| = A \rho(n)\) we have

\[|f^{(m)}(s)/m! - \mu^m| \leq \frac{|f^{(m)}(s)| A^m}{m!} \rho(n)^m \left| h(z) \right|_{L^\infty(S_{1/(2n)})} + O \left(n \rho(n)^{m+1}\right) \]

Taking \(A\) and \(N\) sufficiently large we use Rouche’s theorem to conclude that \(T_{2n-2}(s + \mu)\) has exactly \(m\) zeros inside the circle or radius \(A \rho(n)\). \(\square\)

**Remark 5.2.** Though we have only considered fixed roots \(s\) in the above Theorem which do not scale with parameter \(\lambda(n)\), these may also exhibit super-exponential convergence under certain assumptions. To consider growing roots, in the proof above one must include the asymptotic behavior of \(s\) and \(f^{(m)}(s)\). Essentially one must know that \(s\) does not grow faster than \(\lambda\), that as \(|s|\) grows the order \(m\) of the roots is bounded, and that the first non-zero derivative \(f^{(m)}(s)\) is not too close to zero. Skipping the other details, the proof goes through as before where one considers the new radial scaling factor:

\[\tilde{\rho}(n) = \frac{s^{2n}f(\lambda)m!}{|f^{(m)}(s)| \lambda^{2n}\sqrt{n}}^{1/m}\]

then in light of (5.7) the condition for super-exponential convergence amounts to knowing that the quantity

\[r_{n,s} = \log |\lambda| - \log |s| - \frac{1}{n} \log \frac{m!}{f^{(m)}(s)} \gg 1.\]
and exponential convergence is maintained as long as $r_{n,s}$ is positive and bounded away from zero.

6. Extensions to a class of L-functions

In this section we briefly explain how the results can be extended to a class of functions that includes many functions of interest in the theory of numbers, referred to as analytic L-functions.

Suppose that a function $L$ is defined via a Dirichlet series,
\[ L(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}, \]
with the $a_n$'s being real. The following conditions are sufficient to extend the analysis described above to a collection of these analytic $L$ functions.

(A) The function $L$ extends to an analytic function in $\mathbb{C}$.

(B) The function $L$ satisfies a functional equation of the form
\[ \Lambda(z) := N^{z/2} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(z + \mu_j) \prod_{k=1}^{K} \Gamma_{\mathbb{C}}(z + \eta_j) L(z) = \Lambda(1 - z) , \]
where, following [9], $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$.

(C) The function $L$ satisfies a polynomial bound in $\Im(z)$ for $|z| \to \infty$: for $y$ sufficiently large, $|L(x + iy)| \leq C|y|^a$ for some positive constants $C$ and $a$.

(D) The analogue of the last two estimates in (2.1) for the function $L$ holds true.

Remark 6.1. The last two estimates in (2.1) for the function $\zeta$, or their analogues for $L$ (condition (D)), are more than what is needed for the asymptotic analysis of the Taylor approximants. In order to establish the uniform asymptotic description of the Taylor approximants, it is sufficient to represent the Taylor approximant as an integral over two vertical lines, and then have enough analytical control on the phase function $\phi_\lambda$ in order to apply the steepest descent method. This is guaranteed by conditions (A), (B), and (C) alone, and we present those results below. The additional detailed control near the edge of the critical strip was used to confine the “spurious zeros” of $T_{2n-2}(\xi; \lambda z)$, using our asymptotic analysis, establishing Theorem 4.1 and its Corollary. One can easily state results analogous to Theorem 4.1 and its Corollary. However, since this relies upon estimates which are known only in special cases (to the best of our knowledge), we will refrain from stating these conditional results. Rather, in this section we will only state the extension of Theorem 3.1 to a general class of analytic L-functions whose existence is already established.

Remark 6.2. Of course, we could also relax some of the above conditions, and there are in principle no additional obstacles if we permit complex Dirichlet coefficients $a_n$ (which
can lead to a functional equation of the form $\Lambda(z) = \overline{\Lambda}(1 - z)$, where $\overline{\Lambda}$ is an L-function dual to $\Lambda$, nor if we permit the existence of a pole at $z = 1$ (and hence at $z = 0$). But the additional complication is perhaps worth consideration only in specific examples.

Under assumptions (A), (B), and (C), we can define the function $F(z) = \Lambda(1/2 + z)$, and then express the rescaled Taylor polynomial of degree $2n - 2$ for the function $F(z)$ via a contour integral over two vertical lines, as in (1.10):

\begin{equation}
T_{2n-2}(F; \lambda z) = F(\lambda z) \left[ \chi(z) - \frac{e^{n\phi_\lambda(z)}}{\sqrt{n}} \mathcal{H}(z) \right],
\end{equation}

\begin{equation}
e^{n\phi_\lambda(z)} := \frac{z^{2n} F(\lambda)}{F(\lambda z)}, \quad \mathcal{H}(z) := \frac{\sqrt{n}}{2\pi i} \int_{\partial S_1} e^{-n\phi_\lambda(s)} \frac{ds}{s - z}.
\end{equation}

Moreover, the phase function $\phi_\lambda$ can be expressed in manner analogous to (2.4):

\begin{equation}
\phi_\lambda(z) = 2 \log z + \frac{\lambda}{n} \left[ \log \left( \frac{2K}{N^{1/2}} \right) - \left( \frac{J}{2} + K \right) \log \frac{\lambda}{2\pi} \right] (z - 1) +
\end{equation}

\begin{equation}
- \frac{\lambda}{n} \left[ \frac{J}{2} + K \right] (1 - z + z \log z) - \frac{1}{n} \log L \left( \frac{1}{2} + \lambda z \right) + \frac{1}{n} r(z; \lambda)
\end{equation}

where the function $r(z; \lambda)$, as well as its derivative in $z$, is bounded. We may then compute $\phi'_\lambda$:

\begin{equation}
\phi'_\lambda(z) = 2 + \frac{\lambda}{n} \left[ \log \left( \frac{2K}{N^{1/2}} \right) - \left( \frac{J}{2} + K \right) \log \frac{\lambda}{2\pi} \right] +
\end{equation}

\begin{equation}
- \frac{\lambda}{n} \left[ \frac{J}{2} + K \right] \log z - \frac{\lambda}{n} \frac{L'}{L} \left( \frac{1}{2} + \lambda z \right) + \frac{1}{n} r'(z; \lambda)
\end{equation}

Following the arguments of Section 2, one may verify the analogue of Lemma 2.3, which sets the stage for the application of the steepest descent method.

**Lemma 6.3.** Suppose that the function $L$ satisfies assumptions (A), (B), and (C) above. Then, for all sufficiently large $n$ there is a unique choice of $\lambda = \lambda(n)$, with $\lambda > 1/2$ (i.e. right of the shifted critical strip) satisfying $\partial_z \phi_\lambda(z) \big|_{z=1} = 2 - (\lambda/n) \partial_\lambda \log F(\lambda) = 0$. This choice of $\lambda$ satisfies the relation

\begin{equation}
2 + \frac{\lambda}{n} \left[ \log \left( \frac{2K}{N^{1/2}} \right) - \left( \frac{J}{2} + K \right) \log \frac{\lambda}{2\pi} \right] = \mathcal{O} \left( \frac{1}{n} \right),
\end{equation}

and asymptotically

\[\lambda = \lambda(n) = \frac{4n}{J + 2K} \left[ W \left( \frac{2n}{\pi(J + 2K)} \left( \frac{N}{4K} \right)^{\frac{1}{J+2K}} \right) \right]^{-1} \left[ 1 + \mathcal{O} \left( n^{-1} \right) \right].\]
Here $W(z)$ is the branch of the inverse function to $We^W = z$ which is real and increasing for $z \in (-e^{-1}, \infty)$ sometimes called the Lambert-$W$ function.

Moreover, for this choice of $\lambda$, the critical point at $z = 1$ is simple and

$$\phi''_\lambda(1) = -2 + O\left(\frac{1}{\log n}\right).$$

In addition, for any $\sigma > 1$, there exists $N(\sigma)$ sufficiently large so that for all $n > N(\sigma)$, the only critical point of $\phi_\lambda$ in $\{z : \Re(z) > \frac{\sigma}{2\pi}\}$ is $z = 1$.

The analog of Theorem 3.1, establishing the uniform asymptotic behavior of the Taylor approximants $T_{2n-2}(F;\lambda z)$, is also clear, under these assumptions. The computations are a bit more involved because of the flurry of Gamma functions, but are otherwise straightforward.

The analogue of the analytic transformation $w(z)$, defined in the beginning of Section 3, is defined via

$$w^2 = \phi_\lambda(z) = \phi''_\lambda(1) \frac{(z-1)^2}{2} [1 + O(z-1)].$$

This function obeys all of the properties described in the beginning of Section 3. In particular, when restricted to any sufficiently small neighborhood $B_{1,\delta}$ of $z = 1$ (or $B_{-1,\delta}$ of $z = -1$), it is an invertible conformal map onto a bounded neighborhood of $w = 0$, and the branch is chosen so that $w$ maps the vertical line $\partial S_1$ locally to a nearly horizontal contour in the $w$-plane oriented left-to-right. In addition, we may use symmetry so that $w(z) = w(-z)$ for $z \in B_{-1,\delta}$. Moreover, the estimate on $\phi''_\lambda(1)$ in Lemma 6.3 implies that $w = w(z)$ is nearly isometric for $z$ near 1 and $n \gg 1$. We again fix the neighborhoods $B_{\pm 1,\delta}$ by requiring that $B_{\pm 1,\delta}$ are, for any sufficiently small $\delta > 0$, the two pre-images of the disk of radius $\delta$ in the $w$-plane:

$$w(B_{\pm 1,\delta}) = \{w \in \mathbb{C} : |w| < \delta\}$$

and we let $B_\delta = B_{1,\delta} \cup B_{-1,\delta}$.

**Theorem 6.4.** Suppose the function $L$ satisfies assumptions (A), (B), and (C) above. Let $\lambda = \lambda(n)$ be as described in Lemma 6.3, $\chi(z)$ the characteristic function of the set $|\Re z| < 1$, and $H_0(z)$, defined via

$$H_0(z) = \frac{1}{\sqrt{2\pi|\phi''_\lambda(1)|}} \frac{2}{1 - z^2}.$$
be the leading order stationary phase approximation of $H(z)$. Then as $n \to \infty$ the Taylor polynomials described by (6.2) admit the asymptotic expansion

$$T_{2n-2}(F; \lambda z) = T_{2n-1}(F; \lambda z) = \begin{cases} F(\lambda z) \left[ \chi(z) - \frac{e^{n\phi_{\lambda}(z)}}{\sqrt{n}} (H_0(z) + \mathcal{E}(z)) \right] & z \in \mathbb{C}\setminus B_\delta \\ F(\lambda z) \left[ \frac{1}{2} \text{erfc}(i\sqrt{n}w(z)) - \frac{e^{n\phi_{\lambda}(z)}}{\sqrt{n}} \mathcal{E}(z) \right] & z \in B_\delta. \end{cases}$$

(6.11)

where the residual error function $\mathcal{E}(z)$ is bounded, analytic in $\mathbb{C}\setminus ((\partial S_1 \setminus B_\delta) \cup \partial B_\delta)$, and satisfies

$$\mathcal{E}(z) = \begin{cases} \mathcal{O}(n^{-1}) & z \in B_\delta^c \\ h_0(z) + \frac{1}{2i\sqrt{\pi}w(z)} + \mathcal{O}(n^{-1}) & z \in B_\delta. \end{cases}$$

(6.12)

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