Finiteness in $N=1$ SYM Theories

C. Lucchesi*

*Institut de Physique, Université de Neuchâtel
1 rue Breguet, CH – 2000 Neuchâtel (Switzerland)

I present a criterion for all-order finiteness in $N=1$ SYM theories. Three applications are given; they yield all-order finite $N=1$ SYM models with global symmetries of the superpotentials.

1. INTRODUCTION

The aim of this paper is to present applications of the criterion for all-order finiteness in $N=1$ SYM of [1]. All-order finiteness is here meant in the sense of exact vanishing of the perturbative $\beta$-functions.

The all-order finiteness criterion is an exact result, with hypotheses operating exclusively at the one-loop level. It is based on the structure of the supercurrent anomaly multiplet, which relates the conformal anomalies to the axial ones. The axial anomalies being non-renormalized, they are given by their one-loop values. Vanishing of the latter is guaranteed by the hypothesis that the one-loop gauge $\beta$-function, as well as the one-loop anomalous dimensions, vanish (these two conditions are known to yield one-loop finiteness [2]). A further hypothesis on the unicity of the solution to the conditions of vanishing one-loop Yukawa $\beta$-functions comes from imposing, as a consistency requirement, that reduction of the couplings be verified. Therefore, the all-order finiteness result is the one-loop result supplemented by a consistency requirement for higher orders [hyp. (iv) below]. We shall first review one-loop finiteness (section 2). In section 3 we state the all-order result. Section 4 presents three applications. For related approaches to all-order finiteness in $N=1$ SYM, see [3, 4, 5].

2. ONE-LOOP FINITENESS

Consider an $N=1$ super-Yang-Mills theory with simple gauge group (gauge coupling $g$) and superpotential $W(\phi^i) = \frac{1}{6} m_{ij} \phi^i \phi^j + \frac{1}{5} \lambda_{ijk} \phi^i \phi^j \phi^k$. The matter fields $\phi^i$ transform under the irrep. $R_i$. The one-loop gauge $\beta$-function is given by

$$\beta^{(1)}_g = \frac{g^3}{16\pi^2} \left[ \sum_i T(R_i) - 3 C_2(G) \right].$$

(2.1)

The one-loop Yukawa $\beta$-functions

$$\beta^{(1)}_{ij} = \lambda_{ijk} \gamma^{(1)}_k + \lambda_{ikl} \gamma^{(1)}_j + \lambda_{jkl}\gamma^{(1)}_i$$

(2.2)

are combinations of the one-loop anomalous dimensions

$$\gamma^{(1)}_j = \frac{1}{32\pi^2} \left[ \lambda^{ikl} \lambda_{jkl} - 2 g^2 C_2(R_i) \delta_i^j \right].$$

(2.3)

Necessary and sufficient conditions for one- (and two-) loop finiteness [1] result from demanding that (2.1), resp. (2.3), vanish, i.e.,

$$\sum_i T(R_i) = 3 C_2(G),$$

(2.4)

$$\lambda^{ikl} \lambda_{jkl} = 2 g^2 C_2(R_i) \delta^i_j.$$ 

(2.5)

3. ALL-ORDER FINITENESS

At all orders, finiteness is guaranteed by the following criterion [1]: if (i) the gauge anomaly vanishes, (ii) $\beta^{(1)}_g = 0$, (iii) the conditions $\gamma^{(1)}_j = 0$ possess solutions of the form $\lambda_{ijk} = \rho_{ijk} g$, $\rho_{ijk} \in \mathbb{C}$, and (iv) these solutions of $\gamma^{(1)}_j = 0$ are isolated and non-degenerate, solutions of $\beta^{(2)}_{ij} = 0$, then each of these solutions corresponds to an $N=1$ SYM model with one independent coupling constant (e.g., the gauge coupling $g$) which does not run, i.e., $\beta_g = 0$ at all orders.

For the proof, see [1] and the original literature [2, 3]. In order to obtain a SYM model with

By “isolated”, we mean that the zeroes cannot be multiple, whereas by “non-degenerate” we forbid parametric families.
one isolated and non-degenerate solution \((i.e., a unique solution for that model)\), one generally needs to restrict the superpotential by imposing global, chiral or discrete, symmetries. It turns out that one solution of \(\gamma_j^{(1)} = 0\) which is isolated and non-degenerate when regarded as a solution of \(\beta_{jk}^{(1)} = 0\) corresponds, if it exists, to a given global chiral symmetry of the superpotential. Such chiral symmetries are denoted by
\[ \delta_a \phi_i^\dagger = i e_{a}^\dagger \phi_i^\dagger, \quad \delta_a \phi_i = -i e_a^\dagger \phi_i, \quad \delta_a e_i = e_a, \] (3.1)
where \(e_a = e_i^\dagger\) are the Hermitian charges. Symmetry of the superpotential is then guaranteed provided
\[ \lambda_{ij} e_k + \lambda_{kji} e_i^\dagger + \lambda_{kij} e_j^\dagger = 0. \] (3.2)

### 4. APPLICATIONS

#### 4.1. SU(6)\(_\text{gauge}\) \(N=1\) SYM finite model

The first application \([1]\) is based on an \(SU(6)\) gauge model which is known to be one-loop finite \([3]\). It contains matter supermultiplets in the \(6, \bar{6}, \mathbf{15}\) and \(21\) with multiplicities \((8, 16, 1, 1)\), and has been chosen for purely illustrative purposes. The superpotential writes (repr. indices suppressed)
\[ W = \lambda_{(ab)} \mathbf{6} \mathbf{6}^{\dagger} \mathbf{21} + \lambda_{[ij]} \mathbf{6} \bar{\mathbf{6}} \mathbf{15} + \lambda_3 \mathbf{15} \mathbf{15}^\dagger, \] (4.1)
where \(\lambda_{(ab)}\), \(\lambda_{[ij]}\) and \(\lambda_3\) are the Yukawa couplings, \(i, j = 1..8, a, b = 1..16\). The \(\gamma^{(1)}\) conditions (2.3) yield
\[ \lambda^{(ac)}\lambda_{(cb)} = \alpha g^2 \delta_a^c, \quad \lambda^{(ac)}\lambda^{(ca)} = 16 \alpha g^2, \quad 4 \lambda_{[ik]} \lambda^{[kj]} = 7 \alpha g^2 \delta_i^k, \quad 2 \lambda_{(jk)} \lambda^{|k|j} = 9 |\lambda_i|^2 = 28 \alpha g^2, \] (4.2)
with \(\alpha\) some constant. The \(\lambda_{(ab)}\)'s and \(\lambda_{[ij]}\)'s obeying \((1.2)\) form parametric families of zeroes, and \(\lambda_3\) is a degenerate (double) zero. These solutions are hence neither isolated nor non-degenerate solutions of \(\beta^{(1)}_{jk} = 0\), and hypothesis (iv) of the finiteness criterion is not satisfied.

Let us then pick one of the above solutions, denote it by \((l_{(ab)}, k_{[ij]}),\) and replace it into the superpotential. The latter, restricted superpotential, is invariant under chiral transformations of the type \((1.1)\), i.e.,
\[ \delta_{(l)} \mathbf{6}^\dagger = i e_{(l)}^\dagger \mathbf{6}^\dagger, \quad \delta_{(k)} \mathbf{6}_i = i e_{(k)}^\dagger \mathbf{6}_i, \] (4.3)
provided relations of the form \((1.2)\) among generators and Yukawa couplings
\[ l_{(ac)} e_{(l)}^\dagger + l_{(bc)} e_{(l)}^\dagger = 0, \quad k_{ji} e_{(k)}^\dagger + k_{ij} e_{(k)}^\dagger = 0, \] (4.4)
hold. Considering now the Yukawa couplings as metrics, we infer that the \(e_{(l)}\)'s generate the group preserving \(16 \times 16\) bilinear symmetric metrics, \(i.e.,\) the orthogonal group \(O(16)\), whereas the \(e_{(k)}\)'s generate the group preserving \(8 \times 8\) bilinear antisymmetric metrics, \(i.e.,\) the symplectic group \(Sp(8)\). Other symmetries (see \([1]\)) make \(\lambda_3\) an isolated zero. We have thus obtained an all-order finite \(N=1\) SYM model with symmetry group \(SU(6)_{\text{gauge}} \times \text{O}(16) \times Sp(8)\) global.

#### 4.2. SU(5)\(_\text{gauge}\) \(N=1\) SYM finite model

We consider now a more phenomenologically-minded application of all-order finiteness (see \([3]\) and references therein). Among the \(SU(5)\) one-loop finite models of \([3]\), only one allows for three fermion generations and adjoint matter. It is built out of the chiral supermultiplets \(5, \bar{5}, 10, \bar{10}\) and \(24\) with multiplicities \((4, 7, 3, 0, 1)\). The superpotential (repr. indices suppressed) is
\[ W = \frac{2}{5} 10 \times 10, \bar{10}_a + \frac{2}{5} 10 \times 10, \bar{H}_a \bar{H}_b + \frac{2}{3} \times 5, \bar{10}_a 5, \bar{10}_b + \frac{2}{3} \times 5, \bar{10}_a \bar{H}_a \bar{H}_b + \frac{2}{3} \times 5, \bar{10}_a \bar{H}_a \bar{A}_a + \frac{2}{3} \times 24^3, \] (4.5)
where \(i, j, k = 1..3\) and \(a, b = 1..4\). The \(10\)'s and \(5\)’s are the usual three generations, and the \(24\) contains the scalar superfield. The \((5 + 5)\) Higgses are denoted by \(H_a, \bar{H}_a\). The \(\gamma^{(1)}\) conditions (2.3) can be computed to be
\[ 4 g_{i j a} g^{j b} + \frac{24}{5} f_{a c} f^{b c} + \frac{4}{5} q_{i a} q^{j b} = \frac{24}{5} g_{i b}^2, \]
\[ 3 g_{i j a} g^{j b} + \frac{24}{5} f_{a c} f^{c b} + \frac{24}{5} h_{i a} h^{j b} = \frac{24}{5} g_{i b}^2, \]
\[ 4 g_{k i a} g^{k j a} + \frac{24}{5} h_{i a} h^{j a} + \frac{4}{5} g_{i k a} g^{k j a} = \frac{24}{5} g_{i b}^2, \]
(4.6)
\[ 2 g_{k i a} g^{k j a} + 3 g_{i k a} g^{k j a} + \frac{24}{5} q_{k a} q^{j a} = \frac{24}{5} g_{i b}^2, \]
\[ f_{a b} f^{a b} + \frac{24}{5} p p^* + h_{i a} h^{i a} = 10 g^2. \]
Finiteness at all orders is achieved by imposing on the superpotential a \(Z_7 \times \bar{Z}_3\) discrete symmetry with \(Z_7\)-charges \((1;2;4;1;2;5;3;6;0)\) and \(Z_3\)-charges \((1;2;0;0;0;0;1;2;0;0)\) [for the ten fields \((10; 5; H_a)\)], plus a multiplicative \(Q\)-parity (see \([3]\)). These symmetries enforce \(g_{i i}^2 = \frac{8}{9} g^2, \quad \bar{g}_{i i}^2 = \)
\[
\frac{\partial}{\partial \theta^a} f_{4i} = \theta^a \quad p^2 = \frac{12}{5} \theta^a, \quad \text{for } i = 1, 2, 3, \text{ and all other Yukawa couplings vanish.}
\]

4.3. SYM finite models with global U(1)’s

Finiteness can be attained by imposing that the superpotential be invariant under a product of global U(1) groups. One specific example has been successfully worked out [1]. Starting from some one-loop finite N=1 SYM model extracted from the list of [2], one constructs one explicit solution to the conditions of vanishing one-loop \(\beta\)-functions. The latter solution is then replaced into the superpotential, hence yielding a restricted superpotential \(\hat{W}\). The next task is to find a minimal set of global U(1) symmetries, say \(n\) such groups, that constrain the original superpotential to be just \(\hat{W}\). The \(n\) associated Abelian charges carried by the matter superfields have to be chosen accordingly. If this can be done and yields isolated and non-degenerate solutions to \(\beta^{(1)}_{ijk} = 0\), what one ends up with is an all-order finite \(N=1\) SYM model with symmetry group \(G_{\text{gauge}} \times U(1)^n_{\text{global}}\). It is not clear whether such an approach can be used in general, or if it applies to a definite subset of all lower-orders finite models.

5. CONCLUSIONS

In the process of constructing all-order finite SYM theories, one first reduces the number of independent Yukawa couplings by means of global symmetries. Then one checks if the solution of \(\gamma^j_{(1)} = 0\) considered as a solution of \(\beta^{(1)}_{ijk} = 0\) is isolated and non-degenerate. If not, the process has to be restarted, imposing an enlarged global symmetry of the superpotential. The process stops successfully if unicity of the solution of \(\beta^{(1)}_{ijk} = 0\) is attained.

There can be different, arbitrarily multiple or degenerate solutions to \(\gamma^j_{(1)} = 0\). Each of them \textit{may} yield a finite SYM model with global symmetries, assuming that such symmetries exist. If more than one finite model can be constructed for a given unconstrained \(N=1\) SYM theory, then each of these models corresponds to the original theory with an additional global symmetry specific to that model. One may hope that the global symmetries which are necessary for finiteness turn out to be physically relevant and to carry predictive power. Of course, one can also check for finiteness after imposing global symmetries that are motivated by phenomenology (as, e.g., family symmetry).

Applications of the finiteness criterion to SGUTS with SU(6) and SU(5) gauge groups, and specified matter contents, have been presented. The latter models are shown to be all-order finite provided one imposes the global Lie-type, resp. discrete, symmetries \([O(16) \times Sp(8)]_{\text{global}}\) and \(Z_7 \times Z_3\). A more generic example has been outlined, for which all-order finiteness is achieved by imposing a product of global U(1) symmetries with suitable charges.

Acknowledgements: The author thanks O. Piguet for discussions and a careful reading of the manuscript.

REFERENCES

1. C. Lucchesi, O. Piguet and K. Sibold, Helv. Phys. Acta 61, (1988) 321; Phys. Lett. B201 (1988) 241; C. Lucchesi. Proc. of SUSY 95, Paris, June 15-19, 1995, [hep-th/9510073].
2. A.J. Parkes and P.C. West, Phys. Lett. B138 (1984) 99; Nucl. Phys. B256 (1985) 340; P. West, Phys. Lett. B137 (1984) 371; D.R.T. Jones and A.J. Parkes, Phys. Lett. B160 (1985) 267; D.R.T. Jones and L. Mezincescu, Phys. Lett. B136 (1984) 242; B138 (1984) 293; A.J. Parkes, Phys. Lett. B156 (1985) 73.
3. A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, Nucl. Phys. B281, (1972) 87; D.I. Kazakov, Phys. Lett. B179 (1986) 952.
4. X.D. Jiang and X.J. Zhou, Phys. Rev. D 42 (1990) 2109.
5. M.J. Strassler, RU-95-86, [hep-th/9602021], and references therein.
6. C. Lucchesi and G. Zoupanos, Proc. of STU-Dualities Workshop, CERN, Nov 27 - Dec 1, 1995, [hep-ph/9604216].
7. O. Piguet and K. Sibold, Int. J. Mod. Phys. A1 (1986) 913; Phys. Lett. B177 (1986) 373.
8. S. Hamidi, J. Patera and J.H. Schwarz, Phys. Lett. B141 (1984) 349.
9. C. Lucchesi and O. Piguet, unpubl. notes.