Certain new integral formulas involving the generalized $k$-Bessel function

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Abstract
In this present paper, we investigate generalized integration formulas containing the generalized $k$-Bessel function $W_{k,v}^c(z)$ based on the well known Oberhettinger formula [12] and obtain the results in term of Wright-type function. Also, we establish certain special cases of our main result.

Keywords: Gamma function, $k$-gamma function, Generalized hypergeometric function, Oberhettinger formula, Wright function, Generalized $k$-Bessel function

1 Introduction
We begin with the generalized hypergeometric function $pF_q(z)$ is defined in [6] as:

$$pF_q(z) = pF_q\left[\begin{array}{c}
(\alpha_1), (\alpha_2), \cdots (\alpha_p) \\
(\beta_1), (\beta_2), \cdots (\beta_q)
\end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $\alpha_i, \beta_j \in \mathbb{C}; i = 1, 2, \cdots , p$, $j = 1, 2, \cdots , q$ and $b_j \neq 0, -1, -2, \cdots$ and $(z)_n$ is the Pochhammer symbols. The familiar gamma function is defined as by the following formula:

$$\Gamma(\mu) = \int_0^{\infty} t^{\mu-1} e^{-t} dt, \mu \in \mathbb{C},$$

$$\Gamma(z + 1) = z\Gamma(z), z \in \mathbb{C},$$

and beta function is defined as:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$
The Wright type hypergeometric function is defined (see [16]-[18]) by the following series as:

\[ p \Psi_q(z) = \frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} \left[ \frac{(\alpha_1, \ldots, \alpha_p)}{1} : z \right] \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \left[ (\beta_1, \ldots, \beta_q) : z \right]. \tag{1.5} \]

where \( \beta_j \) and \( \alpha_i \) are real positive numbers such that

\[ 1 + \sum_{s=1}^{q} \beta_s - \sum_{r=1}^{p} \alpha_r > 0. \]

The generalized hypergeometric function \( pF_q(z) \) is a special case of \( p \Psi_q(z) \) for \( A_i = B_j = 1 \), where \( i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, q \):

\[ \frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} \left[ (\alpha_1, \ldots, \alpha_p) : z \right] \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \left[ (\beta_1, \ldots, \beta_q) : z \right]. \tag{1.6} \]

The generalized \( k \)-Bessel function defined in [11] as:

\[ W_{\nu,k}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(nk + \nu + k)n!} \left( \frac{z}{2} \right)^{2n+\frac{1}{k}}, \tag{1.7} \]

where \( k > 0, \nu > -1 \), and \( c \in \mathbb{R} \) and \( \Gamma_k(z) \) is the \( k \)-gamma function defined in [5] as:

\[ \Gamma_k(z) = \int_{0}^{\infty} e^{-\frac{t^k}{k}} dt, z \in \mathbb{C}. \tag{1.8} \]

By inspection the following relation holds:

\[ \Gamma_k(z + k) = z \Gamma_k(z) \tag{1.9} \]

\[ \Gamma_k(z) = k \Gamma_k \left( \frac{z}{k} \right). \tag{1.10} \]

The Pochhammer \( k \)-symbols can be defined as:

\[ (x)_{n,k} = x(x+k) \cdots (x + (n-1)k), n \neq 0, n \in \mathbb{N}, (x)_0,k = 1. \]

The relation between Pochhammer \( k \)-symbols and \( k \)-gamma function is defined as:

\[ (x)_{n,k} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)}. \]

If \( k \rightarrow 1 \) and \( c = 1 \), then the generalized \( k \)-Bessel function defined in (2.12) reduces to the well known classical Bessel function \( J_p \), defined in [7]. For further detail about \( k \)-Bessel function and its properties (see [8]-[10]).

In this paper, we define a class of integral formulas which containing the generalized \( k \)-Bessel function as defined in (1.7). Also, we investigate some special cases as the corollaries. For this continuation of our study, we recall the following result of Oberhettinger [12].

\[ \int_{0}^{\infty} z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} dz = 2\beta b^{-\beta} \left( \frac{b}{2} \right) \Gamma(2\alpha) \Gamma(\beta - \alpha) \Gamma(1 + \alpha + \beta), \]

\[ 0 < \Re(\alpha) < \Re(\beta). \tag{1.11} \]

For various other investigation containing special function, the reader may refer to the recent work of researchers (see [3], [4], [13], [14], [15]).
2 Main Result

In this section, we establish two generalized integral formulas containing $k$-Bessel function defined (1.7), which represented in terms of Wright-type function defined in (1.5) by inserting with the suitable argument defined in (1.11).

**Theorem 2.1.** For $\alpha, \beta, v, c \in \mathbb{C}$ with $\Re(\beta + \frac{v}{k}) > \Re(\alpha) > 0$, $\Re(v/k) > -1$ and $z > 0$, then the following result holds:

$$
\begin{align*}
\int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2a} \right)^{-\beta} W_{i/\varepsilon} \left( \frac{y}{z + a + \sqrt{z^2 + 2a} \varepsilon} \right) dz \\
= \frac{\Gamma(2\alpha) x^{\beta - 1}}{2^{\frac{\alpha - 1}{\varepsilon}} k^\frac{\beta}{2}} \times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(k(n + v + k))n!} \left( \frac{y}{2(\varepsilon + z + a + \sqrt{z^2 + 2a})} \right)^{2n + \frac{\beta}{2} + \frac{v}{2k}} \cdot \left( \frac{\varepsilon^2}{4\alpha^2} \right) \cdot (2.12)
\end{align*}
$$

**Proof.** Let $\mathcal{L}_1$ be the left hand side of (2.1) and applying (1.7) to the integrand of (2.12), we have

$$
\begin{align*}
\mathcal{L}_1 &= \int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2a} \right)^{-\beta} \\
&\times \sum_{n=0}^{\infty} \Gamma(k(n + v + k))n! \left( \frac{y}{2(\varepsilon + z + a + \sqrt{z^2 + 2a})} \right)^{2n + \frac{\beta}{2} + \frac{v}{2k}} dz.
\end{align*}
$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.1, we have

$$
\begin{align*}
\mathcal{L}_1 &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(k(n + v + k))n!} \left( \frac{y}{2} \right)^{2n + \frac{\beta}{2}} \\
&\times \int_0^\infty z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2b^2} \right)^{-(\beta + \frac{v}{2} + 2n)} dz \cdot (2.13)
\end{align*}
$$

By considering the assumption given in theorem 2.1, since $\Re(\beta + \frac{v}{k}) > \Re(\alpha) > 0$, $k > 0$ and applying (1.11) to (2.13), we obtain

$$
\begin{align*}
\mathcal{L}_1 &= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(k(n + v + k))n!} \left( \frac{y}{2} \right)^{2n + \frac{\beta}{2}} 2(\beta + \frac{v}{k} + 2n)b^{-(\beta + \frac{v}{2} + 2n)} \left( \frac{b}{2} \right)^\alpha \\
&\times \frac{\Gamma(2\alpha) \Gamma(\beta + \frac{v}{k} + 2n - \alpha)}{\Gamma(1 + \beta + \frac{v}{k} + \alpha + 2n)}.
\end{align*}
$$

Applying (1.7) and (1.3), we get

$$
\begin{align*}
\mathcal{L}_1 &= \frac{\Gamma(2\alpha) b^{\beta - 1}}{2^{\frac{\alpha - 1}{\varepsilon}} k^{\beta/2}} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\frac{\beta}{2} + 1 + n)n!} \left( \frac{y^{2n}}{4^n k^n b^{2n}} \right) \\
&\times \frac{\Gamma(\beta + \frac{v}{k} + 2n + 1) \Gamma(\beta + \frac{v}{k} + 2n - \alpha)}{\Gamma(\beta + \frac{v}{k} + 2n + 1) \Gamma(\alpha + \beta + \frac{v}{k} + 2n)}.
\end{align*}
$$

which upon using (1.5), we get the required result.
Theorem 2.2. For $\alpha, \beta, \nu, c \in \mathbb{C}$ with $\Re(\beta + \frac{\nu}{k}) > \Re(\alpha + \frac{\nu}{k}) > 0$, $\Re(\nu/k) > -1$ and $z > 0$, then the following result holds:

$$\int_0^\infty z^{\alpha-1} \left(z + b + \sqrt{z^2 + 2bc}\right)^{-\beta} W_{\nu}^k \left(\frac{yz}{z + b + \sqrt{z^2 + 2bc}}\right) dz$$

$$= \frac{(y)^\frac{1}{2} \Gamma(\beta - \alpha) h^{a-\nu}}{2^{\frac{1}{2} + \frac{\nu}{k}} z^{\frac{1}{2} + \frac{\nu}{k}}} \times 2 \Psi_3 \left[\begin{array}{c} (\beta + \frac{\nu}{k} + 1; 2), (2\beta + 2\frac{\nu}{k}; 4); \\ (\beta + \frac{\nu}{k}, 2), (\frac{\nu}{k} + 1, 1), (\alpha + \beta + \frac{\nu}{k} + 1, 4) \end{array} \right] - \frac{c^2}{16} \right]. \quad (2.14)$$

Proof. Let $\mathcal{L}_2$ be the left hand side of (2.13) and applying (1.7) to the integrand of (2.14), we have

$$\mathcal{L}_2 = \int_0^\infty z^{\alpha-1} \left(z + a + \sqrt{z^2 + 2ac}\right)^{-\beta} \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(k(nk + \nu + k)n!)} \left(\frac{yz}{2(z + b + \sqrt{z^2 + 2bc}}\right)^{2n+\frac{\nu}{k}} dz$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.13, we have

$$\mathcal{L}_2 = \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(k(nk + \nu + k)n!)} \left(\frac{y}{2}\right)^{2n+\frac{\nu}{k}} \times \int_0^\infty z^{\alpha+\frac{\nu}{k}+2n-1} \left(z + b + \sqrt{z^2 + 2bc}\right)^{-(\beta + \frac{\nu}{k} + 2n)} dz. \quad (2.15)$$

By considering the assumption given in theorem 2.13, since $\Re(\beta + \frac{\nu}{k}) > \Re(\alpha + \frac{\nu}{k}) > 0, k > 0$ and applying (1.11) to (2.15), we obtain

$$\mathcal{L}_2 = \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(k(nk + \nu + k)n!)} \left(\frac{y}{2}\right)^{2n+\frac{\nu}{k}} 2(\beta + \frac{\nu}{k} + 2n) h^{a-\nu} \frac{1}{2^{\alpha+\frac{\nu}{k}+2n}} \times \frac{\Gamma(2\alpha + 2\frac{\nu}{k} + 4n) \Gamma(\beta - \alpha)}{\Gamma(1 + \beta + \frac{\nu}{k} + \alpha + 4n)}.$$ 

Applying (1.10) and (1.3), we get

$$\mathcal{L}_2 = \frac{(y)^\frac{1}{2} \Gamma(2\alpha) h^{a-\nu}}{2^{\frac{1}{2} + \frac{\nu}{k}} z^{\frac{1}{2} + \frac{\nu}{k}}} \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(\frac{\nu}{k} + 1 + n)n!} \left(\frac{y^{2n}}{16^kh^n}\right) \frac{\Gamma(2\alpha + 2\frac{\nu}{k} + 4n) \Gamma(\beta - \alpha)}{\Gamma(1 + \beta + \frac{\nu}{k} + \alpha + 4n)}$$

which upon using (1.5), we get the required result.

3 Special Cases

In this section, we present the generalized form of classical and modified Bessel functions which are the special cases of $k$-Bessel function defined (1.7). Also, we prove two corollaries which are the special cases of obtained theorems in Section 2.
Case 1. If we set $c = 1$ in (1.7), then we get another definition of $k$-Bessel function. We call it the classical $k$-Bessel function

$$J^0_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^n}{\Gamma(n) (\frac{k}{2})^n}$$

(3.16)

Case 2. If we set $c = -1$ in (1.7), then we get another definition of $k$-Bessel function. We call it the modified $k$-Bessel function

$$J^0_k(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n}}{\Gamma(n) (\frac{k}{2})^{2n}}$$

(3.17)

Corollary 3.1. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

$$\int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} J^0_{\alpha-\beta} \left( \frac{y}{z + a + \sqrt{z^2 + 2az}} \right) dz$$

$$= \left( \frac{\Gamma(2\alpha)\alpha^{\alpha-\beta}z}{2^{\alpha-1}k^\beta} \right) \times 2^\Psi_3 \left[ \begin{array}{c} (\beta + \frac{\alpha}{2}, \beta + \frac{\alpha}{2},) \\ (\beta + \frac{\alpha}{2}, \frac{\alpha}{2} + 1, 1, ) \end{array} \right]$$

Corollary 3.2. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

$$\int_0^\infty z^{\alpha-1} \left( z + a + \sqrt{z^2 + 2az} \right)^{-\beta} I^0_{\alpha-\beta} \left( \frac{y}{z + a + \sqrt{z^2 + 2az}} \right) dz$$

$$= \left( \frac{\Gamma(2\alpha)\alpha^{\alpha-\beta}z}{2^{\alpha-1}k^\beta} \right) \times 2^\Psi_3 \left[ \begin{array}{c} (\beta + \frac{\alpha}{2}, \beta + \frac{\alpha}{2},) \\ (\beta + \frac{\alpha}{2}, \frac{\alpha}{2} + 1, 1, ) \end{array} \right]$$

Corollary 3.3. Assume that the conditions of Theorem 2.13 are satisfied. Then the following integral formula holds:

$$\int_0^\infty z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} J^0_{\alpha-\beta} \left( \frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz$$

$$= \left( \frac{\Gamma(\beta - \alpha)\beta^{\alpha-\beta}z}{2^{\alpha-1}k^\beta} \right) \times 2^\Psi_3 \left[ \begin{array}{c} (\beta + \frac{\alpha}{2}, 2\beta + \frac{\alpha}{2}, 4) \\ (\beta + \frac{\alpha}{2}, \frac{\alpha}{2} + 1, 1, ) \end{array} \right]$$

Corollary 3.4. Assume that the conditions of Theorem 2.13 are satisfied. Then the following integral formula holds:

$$\int_0^\infty z^{\alpha-1} \left( z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} I^0_{\alpha-\beta} \left( \frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz$$

$$= \left( \frac{\Gamma(\beta - \alpha)\beta^{\alpha-\beta}z}{2^{\alpha-1}k^\beta} \right) \times 2^\Psi_3 \left[ \begin{array}{c} (\beta + \frac{\alpha}{2}, 2\beta + \frac{\alpha}{2}, 4) \\ (\beta + \frac{\alpha}{2}, \frac{\alpha}{2} + 1, 1, ) \end{array} \right]$$
Remark 3.1. In this paper, we introduced two integral representation for \( k \)-Bessel function. If letting \( k = 1 \) and \( c = \pm 1 \) respectively, then we obtained the results of classical Bessel function \( J_v(z) \) and modified Bessel function \( I_v(z) \).

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