Parabolic Coordinates and the Hydrogen Atom in Spaces $H_3$ and $S_3$

V.M. Red’kov and E.M. Ovsiyuk

Institute of Physics, NAS of Belarus
Mozyr State Pedagogical University

The Coulomb problem for Schrödinger equation is examined, in spaces of constant curvature, Lobachevsky $H_3$ and Riemann $S_3$ models, on the base of generalized parabolic coordinates. In contrast to the hyperbolic case, in spherical space $S_3$ such parabolic coordinates turn to be complex-valued, with additional constraint on them. The technique of the use of such real and complex coordinates in two space models within the method of separation of variables in Schrödinger equation with Kepler potential is developed in detail; the energy spectra and corresponding wave functions for bound states have been constructed in explicit form for both spaces; connections with Runge-Lenz operators in both curved space models are described.

I. INTRODUCTION

Quantum mechanics had been started with the theory of the hydrogen atom, so considering the Quantum mechanics in Riemannian spaces it is first natural step to turn to just this system. A common quantum-mechanical hydrogen atom model is based materially on the assumption of the Euclidean character of the physical 3-space geometry. In this context, natural questions arise: what in such a model description is determined by this assumption, and which changes will be entailed by allowing for other spatial geometries: for instance, Lobachevsky’s $H_3$, Riemann’s $S_3$, or de Sitter geometry. The question is of fundamental significance, even beyond its possible experimental testing.

Firstly, the hydrogen atom in 3-dimensional space of constant positive curvature $S_3$ was considered by Schrödinger [1]. He had been studied the so-called factorization method in quantum mechanics; in particular, application of this techniques to discrete part of the energy spectrum for hydrogen atom had been elaborated. An idea was to modify the basic atom system so as to cover all the energy spectrum including the region $E > 0$ as well. However, mere placing of the atom system inside a finite box in order to make the whole energy spectrum discrete did not seem attractive, so Schrödinger had placed the atom into the curved background of the Riemann space

*Electronic address: redkov@dragon.bas-net.by, e.ovsiyuk@mail.ru
model \( S_3 \). Due its compactness, the spherical Riemann model may simulate the effect of the finite box – see Schrödinger [1] and Stevenson [2].

In spherical coordinates of \( S^3 \)

\[
dl^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 d\phi^2)
\]

the Schrödinger Hamiltonian in dimensionless units has the form

\[
H = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} - \frac{e}{\tan \chi} ;
\]

\( \rho \) is a curvature radius, a unite for length; \( M \) is a mass of the electron; \( h^2/M\rho^2 \) is taken as a unit for energy; \( e = \frac{\alpha}{\rho} \frac{h^2}{M\rho^2} \) stands for a Coulomb interaction constant; the sign at \( e/\tan \chi \) corresponds to the attracting Coulomb force. The energy spectrum is entire discrete and given by

\[
\epsilon_n = -\frac{e^2}{2n^2} + \frac{1}{2} (n^2 - 1) , \quad n = 1, 2, 3, ...
\]

Hydrogen atom in the Lobachevsky space \( H_3 \) was considered firstly by Infeld and Shild [3]

\[
dl^2 = d\chi^2 + \cosh^2 \chi (d\theta^2 + \sin^2 d\phi^2) ,
\]

\[
H = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} - \frac{e}{\tanh \chi} ;
\]

Energy spectrum contains a discrete and continuous parts. The number of discrete levels is finite, they are specified by

\[
-\frac{e^2}{2} \leq \epsilon \leq \left( \frac{1}{2} - e \right) , \quad \epsilon_n = -\frac{e^2}{2n^2} - \frac{1}{2} (n^2 - 1) , \quad n = 1, 2, 3, ..., N .
\]

In the region \( \epsilon \geq \left( \frac{1}{2} - e \right) \) the energy spectrum is continuous.

Thus, the models of the hydrogen atom in Euclid, Riemann, and Lobachevsky spaces significantly differ from each other, which is the result of differences in three spatial geometries: \( E^3, H^3, S^3 \). To present time, we see a plenty of investigations on this matter:

Higgs [4], Leemon [5], Kurochkin – Otchik [6], Bogush – Kurochkin – Otchik [7], Parker [8], [9], Ringwood – Devreese [10], Kobayshi [11], Bessis – Bessis – Shamseddine [12], Grinberg – Maranon – Vucetich [13], Bogush – Otchik – Red’kov [14], Bessis – Bessis – Shamseddine [15], [16], [17], Chondming – Dianyan [18], Xu – Xu [19], Melnikov – Shikin [20], Shamseddine [21], Otchik – Red’kov [22], Barut – Inomata – Junker [23], Bessis – Bessis – Roux [24], Bogush – Otchik – Red’kov [25], Gorbatsievich – Pribe [26], Groshe [27], Barut – Inomata – Junker [28], Katayama [29], Chernikov [30], Mardoyan – Sisakyan [31], Granovskii – Zhedanov – Lutsenko [32], Kozlov – Harin [33], Vintsikii – Marfoyan – Pogosyan – Sisakyan – Strizh [34], Shamseddine [35], Bogush –
II. COORDINATES IN SPACE H₃ AND S₃

In Euclidean 3-dimension space $E_3$ there exist 11 coordinate systems [48], allowing for the complete separation of variables in the Helmholtz equation

$$
\left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} + \lambda \right) \Phi(x^1, x^2, x^3) = 0 ;
$$

$g^{\alpha\beta}(x)$ stands for the metric tensor of space $E_3$ specified for curvilinear coordinates $(x^1, x^2, x^3)$. Solution of the same problem for spaces of constant positive and negative curvature, Riemannian $S_3$ and Lobachevsky $H_3$ models was given by Olevsky in [49]. It was established that there exist 34 such coordinate systems for hyperbolic space $H_3$, whereas in the case of spherical model $S_3$ the number of those systems is only 6. Extension of the analysis to the complex sphere was given by Kalnins and Miller in [51]. Also see [53–55, 55, 56].

The above asymmetry between $H_3$ and $S_3$ may be seen as even more strange if one calls the known relations of these models through the analytical continuation

$$
H_3 \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 = +1 ,
$$

$$
S_3 \quad y_0^2 + y_1^2 + y_2^2 + y_3^2 = +1 ;
$$

the curvature radius $R$ is taken as a unit for length.

The asymmetry of the models $H_3$ and $S_3$ with respect to coordinate systems finds its logical corollary when turning to the study of the quantum mechanical model for a hydrogen atom on the background of a curved space. In particular, an additional degeneracy like in the case of flat space was observed, which presumes existence of a hidden symmetry in the (curved space) problem. In [4–7], the symmetry operators accounting for such additional degeneracy in Kepler problem on curved space background ware found for both model $H_3$ and $S_3$, and an analog of the conventional Runge–Lenz vector in the flat space was constructed.

Connection between the Runge–Lenz operator $\vec{A}$ in the quantum Kepler problem and parabolic coordinates in Euclidean space is well known: by solving the Schrödinger equation in these coordinates the eigenfunctions of the third component $A_3$ arise [13]. Analogous situation exists in the hyperbolic space $H_3$ but not in the spherical $S_3$. In the Lobachewskyan space, among 34 coordinates established by Olevsky [49] one may select one special case of parabolic system of coordinates.

Kurochkin – Otchik [36], Otchik [37], Nersessian – Pogosyan [38], Red’kov [39], Bogush – Kurochkin – Otchik [40], Kurochkin – Otchik – Shoukavy [41], Kurochkin – Shoukavy [42], Bogush – Otchik – Red’kov [43], Bessis – Bessis [44], Iwai [45], Cohen – Powers [46], Ovsiyuk [47].
in $H_3$, in which the Schrödinger equation allows the separation of variables and the wave functions arisen turn out to be eigenfunctions of the operator $B = A_3 + L^2$. Among six coordinate systems mentioned in [4] an analog of parabolic coordinates is not encountered.

If one looks at 34 and 6 systems in $H_3$ and $S_3$ respectively, one can note that all six ones from $S_3$ have their counterparts in $H_3$. The main purpose of the present paper consists is the search of some counterparts of remaining $34 - 6 = 28$ systems. It turns out that such 28 systems in $S_3$ can be constructed, but they should be complex-valued; to preserve real nature of the geometrical space one must impose additional restrictions including complex conjugation.

In particular, the complex analog for parabolic coordinates in space of the positive curvature $S_3$ can be introduced and used in studying the quantum mechanical Kepler problem in this space (this possibility was partly studied before in [14], [22], [43]).

Let us start with the following fact: from the the metrics in Lobachevsky space (note $\chi \in [0, +\infty)$)

$$dl^2 = d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta \, d\phi^2)$$

by means of the change $\chi \rightarrow i\chi$, $\sinh \chi \rightarrow i \sin \chi$ one can obtain the corresponding metrics of the Riemannian space (note that $\chi \in [0, \pi]$)

$$dl^2 = -\left[d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta \, d\phi^2) \right].$$

This simple observation on $H_3 - S_3$ connection leads us to interesting consequences. Indeed, let us compare, for instance, wave functions and spectra for hydrogen atom in spaces of negative and constant curvature

$$\begin{align*}
\overline{H}_3, & \quad \Psi_{nlm}(\chi, \theta, \phi) = NS(\chi)Y_{lm}(\theta, \phi), \\
S(\chi) & = \sinh^l \chi \exp \left[\left(n - l - 1 - \frac{e}{n}\right) \chi\right] \\
& \times F\left(\frac{e}{n} + l + 1, l - n + 1, 2l + 2; 1 - e^{-2\chi}\right), \\
\epsilon_n & = -\frac{e^2}{2n^2} - \frac{1}{2}(n^2 - 1);
\end{align*}$$

$$\begin{align*}
\overline{S}_3, & \quad \Psi_{nlm}(\chi, \theta, \phi) = KS(\chi)Y_{lm}(\theta, \phi), \\
S(\chi) & = \sin^l \chi \exp \left[\left(i(n - l - 1) - \frac{e}{n}\right) \chi\right] \\
& \times F\left(-ie^n + l + 1, l - n + 1, 2l + 2; 1 - e^{-2\chi}\right), \\
\epsilon_n & = -\frac{e^2}{2n^2} + \frac{1}{2}(n^2 - 1), \quad e = \frac{\alpha}{R}/\hbar^2, \quad \frac{M \hbar^2}{R^2}.
\end{align*}$$
quantity \( (M\hbar^2/R^2) \) provides us with natural unit for energy, \( e \) is a dimensionless parameter characterizing intensity of the Coulomb interaction. One may readily note that these two solutions turn into each other at the following formal replacement

\[ \chi \rightarrow i\chi, \ e \rightarrow -i\ e, \ \epsilon \rightarrow -\epsilon. \]  

(6)

This example indicates that the relation between \( H_3 \) and \( S_3 \) reflected by substitution \( \chi \rightarrow i\chi \) is meaningful. In the context of the described above situation with coordinate systems in \( H_3 \) and \( S_3 \), let us make use of this correspondence \( (\chi \rightarrow i\chi) \) as follows.

Let in the Lobachevsky space \( H_3 \) be chosen a coordinate system \((\rho_1, \rho_2, \rho_3)\) (one of those 34 found by Olevsky), then as a first step one has to establish connection of such a system with spherical one:

\[ \rho_k = f_k(\chi, \theta, \phi), \]  

(7)

and a second step is to introduce a corresponding coordinate system in the space \( S_3 \) through the formal change \( \chi \rightarrow i\chi \):

\[ \rho_k = f_k(i\chi, \theta, \phi). \]  

(8)

With help of this prescription one can determine 34 coordinate systems in space \( S_3 \) in comparison to six ones given in [49]. It turns out that 28 new (added) coordinate systems are complex-valued and therefore additional restrictions should be imposed which involve complex conjugation. All these extra coordinate systems permit the full separation of variables in the Helmholtz equation on the sphere \( S_3 \).

Below, only one example of such coordinates, analog of the parabolic ones in space \( H_3 \), will be examined in detail and applied to the study of the quantum-mechanical Kepler problem on the sphere \( S_3 \).

III. PARABOLIC COORDINATES IN SPACE MODELS \( S_3, H_3 \)

In [49], the following coordinate system (the case XXV) in Lobachevsky space had been given

\[ dl^2 = \frac{(\rho_1 - \rho_2)}{4(\rho_1 - a)(\rho_1 - b)^2} d\rho_1^2 + \frac{(\rho_2 - \rho_1)}{4(\rho_2 - a)(\rho_2 - b)^2} d\rho_2^2 - (\rho_1 - a)(\rho_2 - a) d\rho_3^2, \]  

(9)
where \((\rho_1, \rho_2, \rho_3)\) are connected with the four quasi-Cartesian coordinates \((x_0, x_1, x_2, x_3)\)

\[-(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, \quad x_0 > +1\]

by the formulas

\[
\frac{x_2}{x_1} = \tan \left[(a - b) \rho_3\right], \quad b < \rho_1 < a < \rho_2,
\]

\[
\frac{x_1^2 + x_2^2}{\rho_i - a} + \frac{x_3^2 - x_0^2}{\rho_i - b} + \frac{(x_3 - x_0)^2}{(\rho_i - b)^2} = 0 \quad (i = 1, 2).
\]  

(10)

With the notation

\[
x_1^2 + x_2^2 = \sigma^2, \quad x_3 - x_0 = U, \quad x_3 + x_0 = V,
\]

and \(a = +1, b = 0\), eq. (10) gives

\[
\frac{\sigma^2}{\rho_1 - 1} + \frac{UV}{\rho_1} + \frac{U^2}{\rho_1^2} = 0, \quad \frac{\sigma^2}{\rho_2 - 1} + \frac{UV}{\rho_2} + \frac{U^2}{\rho_2^2} = 0.
\]

Combining two last equations results in

\[
(\frac{\rho_1}{\rho_1 - 1} - \frac{\rho_2}{\rho_2 - 1}) \sigma^2 + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) U^2 = 0,
\]

\[
(\frac{\rho_1^2}{\rho_1 - 1} - \frac{\rho_2^2}{\rho_2 - 1}) \sigma^2 + (\rho_1 - \rho_2) UV = 0.
\]

From this, taking into account \(\sigma^2 + UV = -1\), we arrive at

\[
\frac{U}{V} = \frac{\rho_1 \rho_2}{\rho_1 \rho_2 - \rho_1 - \rho_2}, \quad UV = \rho_1 \rho_2 - \rho_1 - \rho_2,
\]

and further

\[
U^2 = \rho_1 \rho_2, \quad V = U \frac{\rho_1 \rho_2 - \rho_1 - \rho_2}{\rho_1 \rho_2}.
\]

Thus, for \(U, V, \sigma\) we have found (the Lobachevsky space is realized on the branch \(x_0 > +1\), so that \((x_3 - x_0) \leq 0\))

\[
U = x_3 - x_0 = -\sqrt{\rho_1 \rho_2}, \quad V = x_3 + x_0 = \frac{\rho_1 + \rho_2 - \rho_1 \rho_2}{\sqrt{\rho_1 \rho_2}}, \quad \sigma = \sqrt{-1 - UV} = \sqrt{-(1 - \rho_1)(1 - \rho_2)}.
\]

Explicit formulas relating \(\rho_1, \rho_2, \rho_3\) with Cartesian coordinates \((x_0, x_1)\) look as

\[
x_1 = \sqrt{-(1 - \rho_1)(1 - \rho_2)} \cos \rho_3,
\]

\[
x_2 = \sqrt{-(1 - \rho_1)(1 - \rho_2)} \sin \rho_3,
\]

\[
x_3 = \frac{\rho_1 + \rho_2 - 2 \rho_1 \rho_2}{2 \sqrt{\rho_1 \rho_2}}, \quad x_0 = \frac{\rho_1 + \rho_2}{2 \sqrt{\rho_1 \rho_2}}; \tag{11}
\]
and the inverse formulas are
\[
\rho_1 = \frac{x_0 - x_3}{x_0 + x}, \quad \rho_2 = \frac{x_0 - x_3}{x_0 - x}, \\
\rho_3 = \arctan \frac{x_2}{x_1}, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. 
\tag{12}
\]

Now, instead of the introduced \(\rho_1, \rho_2, \rho_3\) one can define other coordinates which behave simply in the limit \(R \to \infty\) (the curvature vanishes). Such a limiting procedure for spherical coordinates of the hyperbolic space \(H_3\) with metrics
\[
dl^2 = \rho^2 \left[ d\chi^2 + \sinh^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right]
\]
going over into spherical ones of the flat space \(E_3\)
\[
\lim_{\rho \to \infty} (\rho \chi) = r, \quad \lim_{\rho \to \infty} (\rho \sinh \chi) = r. \tag{13}
\]
Eliminating \(x_0\) as follows
\[
q_1 = \frac{x_1}{x_0} = \frac{x_1}{\sqrt{1 + x^2}}, \quad q_2 = \tanh \chi \, n_1, \quad n_1 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]
we note that when \(R \to \infty\) the coordinates \(q_1\) reduce to
\[
\lim_{\rho \to \infty} (\rho q_1) = \lim_{\rho \to \infty} (\rho \tanh \chi \, n_1) = r \, n_1. \tag{14}
\]
So, to have coordinates with known and understandable behavior in the limit \(R \to \infty\) we define new coordinates \(t_1, t_2, \phi\)
\[
t_1 = 1 - \rho_1 = \frac{q_3 + q}{1 + q}, \quad t_2 = 1 - \rho_2 = \frac{q_3 - q}{1 - q}, \quad \phi = \rho_3 = \arctan \frac{q_2}{q_1}; \tag{15}
\]
in the limit of the flat space they provide us with the known parabolic coordinates \((\xi, \eta, \phi)\)
\[
\lim_{\rho \to \infty} (\rho t_1) = z + r = \xi, \quad \lim_{\rho \to \infty} (\rho t_2) = z - r = -\eta. \tag{16}
\]
The metrics \((\ref{eq:12})\) in coordinates \((t_1, t_2, \phi)\) takes the form
\[
dl^2 = \frac{t_1 - t_2}{4t_1(1-t_2)^2} \, dt_1^2 + \frac{t_2 - t_1}{4t_2(1-t_2)^2} \, dt_2^2 - t_1 t_2 \, d\phi^2, \\
0 \leq t_1 \leq 1, \quad t_2 \leq 0, \quad 0 \leq \phi \leq 2\pi. \tag{17}
\]
Now, with the help of the rules \((\ref{eq:7})\) and \((\ref{eq:8})\), one can define corresponding parabolic coordinates \(t_1, t_2\) on the sphere \(S_3\). To this end, coordinates \((t_1, t_2)\) in \(H_3\) must be expressed in terms of spherical ones \((\chi, \theta)\)
\[
t_1 = (1 + \cos \theta) \frac{\tanh \chi}{1 + \tanh \chi}, \quad t_2 = (1 - \cos \theta) \frac{-\tanh \chi}{1 - \tanh \chi}. \tag{18}
\]
from whence we get defining relations for corresponding coordinates in $S_3$

$$t_1 = (1 + \cos \theta) \frac{i \tan \chi}{1 + i \tan \chi}, \quad t_2 = (1 - \cos \theta) \frac{-i \tan \chi}{1 - i \tan \chi}.$$ (19)

Take special notice that $(t_1$ and $t_2$) in (19) are complex-valued expressed through two real $(\chi, \theta)$.

The inverse formulas are readily found

$$1 + \cos \theta = t_1(1 + \frac{1}{iq}), \quad 1 - \cos \theta = t_1(1 - \frac{1}{iq}),$$

$$\cos \theta = \frac{t_1 + t_2 - 2t_1t_2}{t_1 - t_2}, \quad iq = \frac{t_1 - t_2}{2 - t_1 - t_2}.$$ (20)

So defined parametrization of $S_3$ by coordinates $t_1, t_2$ can be additionally detailed by the formulas

$$t_1 = (1 + \cos \theta) \varphi(\chi), \quad t_2 = (1 - \cos \theta) \varphi^*(\chi),$$

$$\varphi(\chi) = \sin \chi \exp [i(\frac{\pi}{2} - \chi)].$$ (21)

From (21) one can derive the relationship between $t_1$ and $t_2$

$$\frac{t_1}{t_2} = \frac{t_1^*}{t_2} = -\frac{t_1(1 - t_2)}{t_2(1 - t_1)}, \quad t_1t_2 = (t_1t_2)^*,$$ (22)

which are equivalent to

$$t_1^* = -t_1 \frac{1 - t_2}{1 - t_1}, \quad t_2^* = -t_2 \frac{1 - t_1}{1 - t_2},$$ (23)

or

$$(1 - t_2) = (1 - t_1) \left( -\frac{t_1}{t_1^*} \right), \quad (1 - t_1) = (1 - t_2) \left( -\frac{t_2}{t_2^*} \right).$$ (24)

Its existence may evidently be referred to the real nature of the space $S_3$. In particular, one consequence is: if $t_1 \to 1 - 0$ then $t_2 \to 1 + 0$, and so on.

In the following, so defined coordinates $(t_1, t_2, \phi)$ are called parabolic coordinates on the sphere $S_3$. In the limit of the flat space, they reduce to the ordinary parabolic coordinates

$$\lim_{\rho \to \infty} (-i\rho t_1) = \xi, \quad \lim_{\rho \to \infty} (-i\rho t_2) = -\eta.$$ (25)

Now, we transform the metrics of the space $S_3$ in spherical coordinates

$$dl^2 = -d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2),$$

to complex parabolic $t_1, t_2, \phi$. As a first step, with the help of

$$\sin^2 \theta = t_1 t_2 \frac{1 + q^2}{q^2}, \quad \sin^2 \chi = \frac{q^2}{1 + q^2},$$
we obtain

\[ \sin^2 \chi \sin^2 \theta \, d\phi^2 = t_1 \, t_2 \, d\phi^2 . \]  

(26)

As a second step, we have

\[(d\theta)^2 = \frac{1}{\sin^2 \theta} \, (d \cos \theta)^2 = \frac{1}{t_1 t_2} \, \frac{q^2}{1 + q^2} \, [d(\frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2})]^2 , \]

and further

\[ \sin^2 \chi \, (d\theta)^2 = \frac{q^4}{(1 + q^2)^2} \, \frac{4}{t_1 t_2(t_1 - t_2)^4} \, [t_2(t_2 - 1) \, dt_1 - t_1(t_1 - 1) \, dt_2]^2 , \]

or

\[ \sin^2 \chi \, (d\theta)^2 = \frac{1}{4(1-t_1)^2(1-t_2)^2 t_1 t_2} \, [t_2 \, (t_2 - 1) \, dt_1 - t_1 \, (t_1 - 1) \, dt_2]^2 . \]  

(27)

Taking into account relations

\[ \frac{i \, d\chi}{\cos^2 \chi} = dt_1 \, \frac{2(1-t_2)}{(2-t_1-t_2)^2} - dt_2 \, \frac{2(1-t_1)}{(2-t_1-t_2)^2} \]

and

\[ \cos^2 \chi = \frac{1}{1 + \tan^2 \chi} = \frac{(2-t_1-t_2)^2}{4(1-t_1)(1-t_2)} , \]

we get

\[ (d\chi)^2 = -\frac{[ (1-t_2)dt_1 - (1-t_1)dt_2 ]^2}{4(1-t_1)^2(1-t_2)^2} . \]  

(28)

Therefore, for the metrics in parabolic coordinates in \( S_3 \) we have arrived at the form

\[ dl^2 = \frac{t_2 - t_1}{4t_1(1-t_1)^2} \, dt_1^2 + \frac{t_1 - t_2}{4t_2(1-t_2)^2} \, dt_2^2 + t_1 t_2 \, d\phi^2 . \]  

(29)

Formally, this formula differs from its counterpart in the space \( H_3 \) only by presence of \((-1)\) in the expression for \( dl^2 \).

The more clarity in the complex coordinates may be achieved if one determines them in terms of Cartesian coordinates

\[ i \, y_1 = \sqrt{-t_1 t_2} \, \cos \phi , \quad i \, y_2 = \sqrt{-t_1 t_2} \, \sin \phi , \]

\[ i \, y_3 = \frac{t_1 + t_2 - 2t_1 t_2}{2\sqrt{(1-t_1)(1-t_2)}} , \quad y_0 = \frac{2 - t_1 - t_2}{2\sqrt{(1-t_1)(1-t_2)}}, \]

they evidently obey an identity (below the notation \( y = \sqrt{y_1^2 + y_2^2 + y_3^2} \) is used)

\[ y_1^2 + y_2^2 + y_3^2 + y_0^2 = (y + iy_0)(y - iy_0) = 1 . \]
Inverse to (30) formulas are

\[ t_1 = (y + y_3)(y + iy_0) = a e^{i\alpha}, \]
\[ t_2 = (y - y_3)(y - iy_0) = b e^{-i\alpha}, \]
\[ t_1 t_2 = y_1^2 + y_2^2, \quad \tan \phi = \frac{y_2}{y_1}. \]  

(31)

Two complex coordinates are detailed by

\[ a = y + y_3, \quad b = y - y_3, \quad ab = y_1^2 + y_2^2, \]
\[ \cos \alpha = y, \quad \sin \alpha = y_0; \]  

(32)

from this it follows

\[ \cos \alpha = y = \frac{a + b}{2}, \quad \frac{a - b}{2} = y_3. \]

The domain for variables \((a, b)\) can be illustrated by the Fig. 1.

Fig. 1. The domain for \((a, b)\).

Note that associated points in the sphere \((y_0, y_k)\) and \((-y_0, -y_k)\) are parameterized according to

\[ (+y_0, +y_k) \implies (t_1, t_2, \phi), \]
\[ (-y_0, -y_k) \implies (t_2, t_1, \phi + \pi), \]  

(33)

or

\[ (+y_0, +y_k) \implies (a, b, \phi), \]
\[ (-y_0, -y_k) \implies (b, a, \phi + \pi). \]  

(34)

Let us describe peculiarities of parametrization of \(S_3\) by the variables \((a, b, \alpha, \phi)\). The first that is a closed line

\[ y_1 = 0, \quad y_2 = 0, \quad y_3 + y_0^2 = 1; \]  

(35)
in this case, \( ab = 0 \), and to this line there corresponds only a part of the boundary in Fig. 1, \( a = 0 \) and \( b = 0 \) (and the coordinate \( \phi \) is "dumb")

\[
y_3 > 0, \ t_1 = 2y_3(y_3 + iy_0), \quad t_2 = 0, \quad \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];
\]

\[
y_3 < 0, \ t_1 = 0, \ t_2 = -2y_3(y_3 - iy_0), \quad \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].
\]

Now, let us consider another closed line

\[
y_3 = 0, \ y_0 = 0, \ y_1^2 + y_2^2 = 1;
\]

characterized by

\[
a = b = \sqrt{y_1^2 + y_2^2} = 1, \quad e^{\pm i\alpha} = 1 \pm i0, \quad a + b = 2.
\]

To this line there correspond a single point \((1, 1)\) on the boundary (see Fig. 1), and the coordinate \( \phi \) is not "dump" now

\[
y_1 = \cos \phi, \quad y_2 = \sin \phi.
\]

There exists one other peculiar region to which there corresponds the line \( a + b = 2 \) on the boundary (see Fig. 1). Indeed, let

\[
y_0 = 0, \quad y_1^2 + y_2^2 + y_3^2 = 1,
\]

this sphere is parameterized in accordance with

\[
\sin \alpha = 0, \quad \cos \alpha = 1 \quad \implies \quad a + b = 2, \quad \phi \in [0, 2\pi],
\]

\[
y_1 = +\sqrt{a(2 - a)} \cos \phi, \quad y_2 = +\sqrt{a(2 - a)} \sin \phi, \quad y_3 = a - 1.
\]

Here one can introduce the variable \( a - 1 = \cos \theta \), then

\[
y_1 = +\sin \theta \cos \phi, \quad y_2 = +\sin \theta \sin \phi, \quad y_3 = \cos \theta.
\]

IV. SEPARATION OF VARIABLES IN PARABOLIC COORDINATES, THE MODELS \( H_3 \) AND \( S_3 \)

Now let us turn to a Coulomb problem. General expression for Schrödinger Hamiltonian

\[
H = -\frac{1}{2} \frac{\partial}{\sqrt{g} \partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} - \frac{e}{\sqrt{g}} \frac{t_1 - t_2}{4(1 - t_1)(1 - t_2)},
\]

gives
\( S_3 \)
\[
H = 2 \frac{1-t_1}{t_1-t_2} \frac{\partial}{\partial t_1} t_1(1-t_1) \frac{\partial}{\partial t_1} + 2 \frac{1-t_2}{t_2-t_1} \frac{\partial}{\partial t_2} t_2(1-t_2) \frac{\partial}{\partial t_2} - \frac{1}{2t_1t_2} \frac{\partial^2}{\partial \phi^2} - ie \frac{2-t_1-t_2}{t_1-t_2} .
\] (43)

Transition to the Hamiltonian in the model \( H_3 \) is achieved by performing two formal replacements:
\( e \implies ie \), and \( H \implies -H \). Thus we obtain
\( H_3 \)
\[
H = -2 \frac{1-t_1}{t_1-t_2} \frac{\partial}{\partial t_1} t_1(1-t_1) \frac{\partial}{\partial t_1} - 2 \frac{1-t_2}{t_2-t_1} \frac{\partial}{\partial t_2} t_2(1-t_2) \frac{\partial}{\partial t_2} + \frac{1}{2t_1t_2} \frac{\partial^2}{\partial \phi^2} - e \frac{2-t_1-t_2}{t_1-t_2} .
\] (44)

Now, let us separate the variables in the Schrödinger equation in coordinates \((t_1, t_2, \phi)\) – first let us specify the case of the model \( S_3 \)
\[
\Psi(t_1, t_2, \phi) = f_1(t_1) f_2(t_2) e^{im\phi} ;
\] (45)

from \( H \Psi = e \Psi \) it follows
\[
f_2 \frac{2(1-t_1)}{t_1-t_2} \frac{d}{dt_1} t_1(1-t_1) \frac{d}{dt_1} f_1 + f_1 \frac{2(1-t_2)}{t_2-t_1} \frac{d}{dt_2} t_2(1-t_2) \frac{d}{dt_2} f_2 + \frac{m^2}{2t_1t_2} f_1 f_2 - ie \frac{2-t_1-t_2}{t_1-t_2} f_1 f_2 = e f_1 f_2 .
\] (46)

and then multiplying (46) by \((t_1-t_2)/2 \ f_1 f_2 \), one derives
\[
\frac{1}{f_1} (1-t_1) \frac{d}{dt_1} t_1(1-t_1) \frac{d}{dt_1} f_1 - \frac{m^2}{4t_1} + ie \frac{2-t_1-t_2}{2} f_1 f_2 = 0 .
\] (47)

where two separation constants \( k_1 \) and \( k_2 \) obey an additional constraint
\[
k_1 - k_2 = -ie .
\] (48)

Thus, the problem in \( S_3 \) consists in solving the system
\( S_3 \)
\[
(1-t_1) \frac{d}{dt_1} t_1(1-t_1) \frac{d}{dt_1} f_1 + (\frac{ie - \epsilon}{2} t_1 - \frac{m^2}{4t_1} + k_1) f_1 = 0 ,
\]
\[
(1-t_2) \frac{d}{dt_2} t_2(1-t_2) \frac{d}{dt_2} f_2 + (\frac{-ie - \epsilon}{2} t_2 - \frac{m^2}{4t_1} + k_2) f_2 = 0 .
\] (49)
In the model $H_3$ we obtain (separation constants obey the identity $k_1 - k_2 = e$)

\[ H_3 \]

\[
(1-t_1) \frac{d}{dt_1} t_1(1-t_1) \frac{d}{dt_1} f_1 + \left( \frac{-e + \epsilon}{2} t_1 - \frac{m^2}{4t_1} + k_1 \right) f_1 = 0 ,
\]

\[
(1-t_2) \frac{d}{dt_2} t_2(1-t_2) \frac{d}{dt_2} f_2 + \left( \frac{e + \epsilon}{2} t_2 - \frac{m^2}{4t_1} + k_2 \right) f_2 = 0 .
\]

Solutions of (49) and (50) are searched in the form

\[
f_1 = t_1^{q_1} (1-t_1)^{b_1} S_1(t_1) , \quad f_2 = t_2^{q_2} (1-t_2)^{b_2} S_2(t_2) .
\]

It suffice to consider the case (49); transition from $S_3$ to $H_3$ is realized through the change

\[
S_3 \Rightarrow H_3 , \quad e \Rightarrow -\epsilon , \quad e \Rightarrow +ie .
\]

The first equation in (49) gives (to obtain analogous result for $f_2$, it suffices to change the index 1 by 2 and the parameter $e$ by $-e$)

\[
t_1(1-t_1) S_1'' + S_1' \left[ 2a(1-t_1)2bt_1 + (1-2t_1) \right] - \left[ a_1(a_1 - 1) \left( \frac{1}{t_1} - 1 \right) - 2a_1b_1 + b_1(b_1 - 1) \left( \frac{1}{t_1} - 1 \right) + a_1 \left( \frac{1}{t_1} - 2 \right) - b_1 \left( 2 - \frac{1}{1-t_1} \right) + \frac{ie - \epsilon}{2} \left( \frac{1}{1-t_1} - 1 \right) - \frac{m^2}{4} \left( \frac{1}{t_1} + \frac{1}{1-t_1} \right) + k_1 \frac{1}{1-t_1} \right] S_1(t_1) = 0 .
\]

Terms proportional to $t_1^{-1}$ and $(1-t_1)^{-1}$ can be eliminated by imposing additional restriction:

\[ a_1^2 - \frac{m^2}{4} = 0 , \quad b_1^2 + \frac{ie - \epsilon}{2} - \frac{m^2}{4} + k_1 = 0 ; \]

which results in

\[
t_1(1-t_1) S_1'' + S_1' \left[ (2a_1 + 1) - (2a_1 + 2b_1 + 2)t_1 \right] - \left[ a_1(a_1 + 1) + 2a_1b_1 + b_1(b_1 + 1) + \frac{ie - \epsilon}{2} \right] S_1 = 0 .
\]

This means that $S(t_1)$ is expressed in terms of hypergeometric functions $S_1(t_1) = F(\alpha_1, \beta_1, \gamma_1; t_1)$ with parameters obeying to

\[
\gamma_1 = 2a_1 + 1 , \quad \alpha_1 + \beta_1 + 1 = 2a_1 + 2b_1 + 2 , \quad \alpha_1 \beta_1 = a_1(a_1 + 1) + 2a_1b_1 + b_1(b_1 + 1) + \frac{ie - \epsilon}{2} ,
\]

from whence it follows

\[ \alpha_1 = a_1 + b_1 + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{e - ie}{2}} , \quad \beta_1 = a_1 + b_1 + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{e - ie}{2}} , \quad \gamma_1 = 2a_1 + 1 .
\]
Let us summarize:

for $S_3$

$$f_1 = t_1^{a_3} (1 - t_1^{b_3}) S_1, \quad f_2 = t_2^{a_3} (1 - t_2^{b_2}) S_2,$$

$$S_1 = F(\alpha, \beta, \gamma_1; t_1), \quad S_2 = F(\alpha, \beta, \gamma_2; t_2),$$

$$S_3 = F(\alpha, \beta, \gamma_1; t_1),$$

$$S_4 = F(\alpha, \beta, \gamma_2; t_2).$$

(55)

and for $H_3$

$$f_1 = t_1^{a_3} (1 - t_1^{b_3}) S_1, \quad f_2 = t_2^{a_3} (1 - t_2^{b_2}) S_2,$$

$$S_1 = F(\alpha, \beta, \gamma_1; t_1), \quad S_2 = F(\alpha, \beta, \gamma_2; t_2),$$

$$S_3 = F(\alpha, \beta, \gamma_1; t_1),$$

$$S_4 = F(\alpha, \beta, \gamma_2; t_2).$$

(56)

V. THE HYDROGEN ATOM IN $H_3$, BOUND STATES

The task consists in separating from all solutions found in Section IV those which describe possible bound states. To treat this problem we will need some additional details in parametrization of the model $H_3$ by coordinates $t_1, t_2, \phi$ (see (11))

$$t_1 = \frac{x_3 + x}{x_0 + x} = \frac{q_3 + q}{1 + q}, \quad 0 \leq t_1 < 1,$$

$$t_2 = \frac{x_3 - x}{x_0 - x} = \frac{q_3 - q}{1 - q},$$

$$-\infty \leq t_2 \leq 0,$$

$$\tan \phi = \frac{x_2}{x_1} = \frac{q_2}{q_1}, \quad \phi \in [0, 2\pi].$$

(58)
Besides, we will use relations
\[
q_1 = \frac{\sqrt{-t_1 t_2} \sqrt{(1 - t_1) (1 - t_2)}}{2 - t_1 - t_2} \cos \phi ,
\]
\[
q_2 = \frac{\sqrt{-t_1 t_2} \sqrt{(1 - t_1) (1 - t_2)}}{2 - t_1 - t_2} \sin \phi ,
\]
\[
q_3 = \frac{t_1 + t_2 - 2t_1 t_2}{2 - t_1 - t_2} .
\]

Note that the origin is characterized by
\[
t_1 = 0, \ t_2 = 0 \iff q_1 = 0, \ q_2 = 0, \ q_3 = 0 ;
\]
and the whole boundary for the domain \(G(t_1, t_2)\) is detailed as follows (see Fig. 2)
\[
(a) \quad t_1 = 0, \ t_2 \in (-\infty, 0] \implies \quad q_1 = 0, \ q_2 = 0, \ q_3 = -q = \frac{t_2}{2 - t_2} ,
\]
\[
t_2 \to -\infty, \ q_3 = -q \to -1 ;
\]
\[
(b) \quad t_1 \to +1, \ t_2 \in (-\infty, 0] \implies \quad q_1 = 0, \ q_2 = 0, \ q_3 \to \frac{1 - t_2}{1 - t_2} = +1 ,
\]
so the boundary (b) parameterizes one single point;
\[
(c) \quad t_1 \in [0, +1), \ t_2 = 0 \implies \quad q_1 = 0, \ q_2 = 0, \ q_3 = +q = \frac{t_1}{2 - t_1} ;
\]
\[
(d) \quad t_1 \in [0, +1), \ t_2 \to -\infty \implies \quad q_1 = \sqrt{t_1(1 - t_1)} \cos \phi ,
\]
\[
q_2 = \sqrt{t_1(1 - t_1)} \sin \phi , \quad q_3 = 2t_1 - 1 .
\]

It is ellipsoid \(4(q_1^2 + q_2^2) + q_3^2 = 1\) passing through two points
\[
t_1 \to 1, \ q_1 \to 0, \ q_2 \to 0, \ q_3 \to +1 ,
\]
\[
t_1 \to 0, \ q_1 \to 0, \ q_2 \to 0, \ q_3 \to -1 . \quad (59)
\]

The structure of the boundary may be illustrated by Fig. 2. Some clarity can be added with the help of the inverse formulas (let \(q_i = q n_i\)):
\[
t_1 = \frac{q(n_3 + 1)}{1 + q}, \quad t_2 = \frac{q(n_3 - 1)}{1 - q}, \quad \tan \phi = \frac{n_2}{n_1} ;
\]
from whence it follows
\[
n_3 = +1, \quad t_1 = 0, \quad t_2 = \frac{2q}{q - 1} ,
\]
\[
n_3 = -1, \quad t_1 = \frac{2q}{q + 1}, \quad t_2 = 0 .
\]
In particular, at \( n_3 \neq \pm 1 \) the boundary \( q = (1 - \Delta) \), \( \Delta \to +0 \) is parameterized according to

\[
t_1 \to \frac{(1 + n_3)}{2}, \quad t_2 \to -\frac{(1 - n_3)}{\Delta}, \quad \tan \phi = \frac{n_2}{n_1}.
\]

Now we are ready to construct the bound states solutions. In the first place, note that to have solutions vanishing in the origin \( q_i = 0 \), we must take \( a_1 \) and \( a_2 \) positive

\[
a_1 = +\frac{|m|}{2}, \quad a_2 = +\frac{|m|}{2}. \tag{60}
\]

To have solution single-valued and continuous in the region \( q_3 \to +1 \), \( q_1 = 0 \), \( q_2 = 0 \) (the boundary (b) in the Fig. 2), we must take positive \( b_1 \) and negative \( b_2 \):

\[
b_1 = +\sqrt{-\epsilon + \epsilon + \frac{m^2}{4} - k_1} > 0, \tag{61}
\]

\[
b_2 = -\sqrt{-\epsilon - \epsilon + \frac{m^2}{4} - k_1} < 0;
\]

besides one should check that the total negative power \((a_2 + b_2)\) compensates a positive power \( n_2 \) of the quantum number of the main term of a polynomial at infinity

\[
a_2 + b_2 + n_2 < 0. \tag{62}
\]

Additionally we assume positiveness of two expressions under the square roots in (61).

To obtain a polynomial in the variable \( t_1 \), we require

\[
\beta_1 = -n_1, \quad n_1 = 0, 1, 2, 3, \ldots \tag{63}
\]

In turn, the same in variable \( t_2 \) can be reached by

\[
\beta_2 = -n_2, \quad n_2 = 0, 1, 2, 3, \ldots \tag{64}
\]

Equations (63) and (64) will give (let \( N_1 = 2n_1 + |m| +1 \) and \( N_2 = 2n_2 + |m| +1 \))

\[
\sqrt{2(\epsilon - \epsilon) + m^2 - 4k_1} = \sqrt{1 + 2(+\epsilon - \epsilon)} - N_1, \tag{60}
\]

\[
\sqrt{2(-\epsilon - \epsilon) + m^2 - 4k_2} = N_2 - \sqrt{1 + 2(-\epsilon - \epsilon)}.
\]
Squaring both ones, after simple manipulation we get

\[
\left[ \sqrt{1 + 2(+e - \epsilon)} - N_1 \right]^2 = \left[ N_2 - \sqrt{1 + 2(-e - \epsilon)} \right]^2,
\]

so that

\[
\sqrt{1 + 2(+e - \epsilon)} - N_1 = N_2 - \sqrt{1 + 2(-e - \epsilon)}
\]
or

\[
\sqrt{1 + 2(e - \epsilon)} + \sqrt{1 + 2(-e - \epsilon)} = 2k, \quad k = \frac{N_1 + N_2}{2} = n_1 + n_2 + |m| + 1.
\]

Squaring the above equation, we arrive at a quadratic equation (let \( x = 1 - 2\epsilon \))

\[
x^2 - 4e^2 = (2k^2 - x)^2,
\]

with solution

\[
\epsilon = -\frac{e^2}{2k^2} - \frac{k^2 - 1}{2}, \quad k = n_1 + n_2 + |m| + 1 = 1, 2, 3, ... \quad (65)
\]

The energy levels belong to the interval

\[
-\frac{e^2}{2} \leq \epsilon \leq \left( \frac{1}{2} - e \right), \quad k < \sqrt{e} \quad (66)
\]

It is a matter of simple calculations to find expressions for \( k_1, k_2 \) and involved parameters. To this end, first let us derive simple expressions for roots

\[
\begin{align*}
+ \sqrt{\frac{1}{4} + \frac{e - \epsilon}{2}} &= +\frac{1}{2} (k + \frac{e}{k}) , \\
+ \sqrt{\frac{1}{4} + \frac{-e - \epsilon}{2}} &= +\frac{1}{2} (k - \frac{e}{k}) ;
\end{align*}
\]

then

\[
a_1 + b_1 + \frac{1}{2} = -n_1 + \frac{1}{2} (k + \frac{e}{k}) , \quad a_2 + b_2 + \frac{1}{2} = -n_2 + \frac{1}{2} (k - \frac{e}{k}) . \quad (68)
\]

It is easily checked (62):

\[
a_2 + b_2 + n_2 < 0 \quad \implies \quad \left( -n_2 - \frac{1}{2} + \frac{1}{2} (k - \frac{e}{k}) \right) + n_2 = -\frac{1}{2} + \frac{1}{2} (k - \frac{e}{k}) < 0 ; \quad (69)
\]

which holds if (see (66))

\[
k - \frac{e}{k} < 0 \quad \iff \quad k < \sqrt{e}. \quad (70)
\]
Now, expressions for $\alpha_1, \alpha_2$ are

$$
\alpha_1 = -n_1 + (k + \frac{e}{k}) = n_2 + |m| + 1 + \frac{e}{k},
$$

$$
\alpha_2 = -n_2 + (k - \frac{e}{k}) - N_1 = n_1 + |m| + 1 - \frac{e}{k}.
$$

And finally, for $b_1, b_2$

$$
b_1 = +\frac{1}{2} \sqrt{2(e - \epsilon) + m^2 - 4k_1} = \frac{1}{2} \left[ +\sqrt{1 + 2(e - \epsilon) - N_1} \right]
$$

$$
\beta_1 = \frac{|m|}{2} + \frac{1}{2} b_1 - \sqrt{\frac{1}{4} + \frac{-\epsilon + \epsilon}{2}}
$$

$$
\beta_2 = \frac{|m|}{2} + \frac{1}{2} b_2 - \sqrt{\frac{1}{4} + \frac{-\epsilon - \epsilon}{2}}
$$

from whence it follows that

$$
b_1 + b_2 = 0 . \quad (71)
$$

Additionally, expressions for $\beta_1, \beta_2$ can be verified

$$
\beta_1 = \frac{|m|}{2} + \frac{1}{2} + \frac{1}{2} \left[ (k + \frac{e}{k}) - N_1 \right] - \frac{1}{2} \left( k + \frac{e}{k} \right) = \frac{|m|}{2} + \frac{1}{2} - N_1 = -n_1 ,
$$

$$
\beta_2 = \frac{|m|}{2} + \frac{1}{2} + \frac{1}{2} \left[ (k - \frac{e}{k}) - N_2 \right] - \frac{1}{2} \left( k - \frac{e}{k} \right) = \frac{|m|}{2} + \frac{1}{2} - N_2 = -n_2 . \quad (72)
$$

Let us find expressions for separation constants $k_1, k_2$

$$
-2\epsilon + 2e + m^2 - 4k_1 = \left( n_2 - n_1 \right) + \frac{e}{k} \right]^2 ,
$$

$$
-2\epsilon - 2e + m^2 - 4k_2 = \left( n_2 - n_1 \right) + \frac{e}{k} \right]^2 ;
$$

from whence it follows

$$
4k_1 = (k + \frac{e}{k})^2 - \left( n_2 - n_1 \right) + \frac{e}{k} \right]^2 + m^2 - 1 ,
$$

$$
4k_2 = (k - \frac{e}{k})^2 - \left( n_2 - n_1 \right) + \frac{e}{k} \right]^2 + m^2 - 1 . \quad (73)
$$

In particular, one obtains $k_1 - k_2 = -ie$. 

VI. THE HYDROGEN ATOM IN SPACE $S_3$, BOUND STATES IN COMPLEX PARABOLIC COORDINATES

The task consists in separating from all solutions found in Section 4 those which describe possible bound states. We will need some details in parametrization of the model $H_3$

$$i \ y_1 = \sqrt{-t_1 t_2} \ \cos \phi , \quad i \ y_2 = \sqrt{-t_1 t_2} \ \sin \phi ,$$

$$i \ y_3 = \frac{t_1 + t_2 - 2 t_1 t_2}{2 \sqrt{(1 - t_1)(1 - t_2)}}, \quad y_0 = \frac{2 - t_1 - t_2}{2 \sqrt{(1 - t_1)(1 - t_2)}},$$

$$t_1 = (y + y_3) \ (y + iy_0) = a \ e^{i \alpha}, \quad t_2 = (y - y_3)(y - iy_0) = b \ e^{-i \alpha},$$

$$\tan \phi = \frac{y_2}{y_1}, \quad t_1 t_2 = y_1^2 + y_2^2. \quad (74)$$

Let us consider peculiarities of the parametrization with $(t_1, t_2; \phi) - (a, b, \alpha; \phi)$. The origin is described by

$$t_1 = 0, \ t_2 = 0 \ (a = 0, \ b = 0) \implies y_1 = 0, \ y_0 = +1 .$$

The closed line $y_0^2 + y_3^2 = 1$ consists of two parts:

$$t_1 = 0, \ t_2 \neq 0 \ (a = 0, \ b \neq 0) \implies iy_3 = \frac{t_2}{2 \sqrt{1 - t_2}}, \ y_0 = \frac{2 - t_2}{2 \sqrt{1 - t_2}}; \quad (75)$$

and

$$t_1 \neq 0, \ t_2 = 0, \ (a \neq 0, \ b = 0) \implies iy_3 = \frac{t_1}{2 \sqrt{1 - t_1}}, \ y_0 = \frac{2 - t_1}{2 \sqrt{1 - t_1}}; \quad (76)$$

here the coordinate $\phi$ is a "dump" one. For another closed line

$$y_3 = 0, \quad y_0 = 0, \quad y_1^2 + y_2^2 = 1 ;$$

we have description

$$a = b = \sqrt{y_1^2 + y_2^2} = 1, \quad e^{\pm i \alpha} = 1 \pm i \ 0 ,$$

$$a + b = 2, \quad t_1 = e^{i \alpha} \to 1, \quad t_2 = e^{-i \alpha} \to 1 ,$$

$$t_1 t_2 \to 1, \quad y_1 = \cos \phi, \quad y_2 = \sin \phi ,$$

$$iy_3 = \frac{t_1 + t_2 - 2}{2 \sqrt{2 - t_1 - t_2}} = \frac{i}{2} \ \sqrt{2 - t_1 - t_2} = \frac{i}{\sqrt{2}} \ \sqrt{1 - \cos \alpha} = 0 ,$$

$$y_0 = \frac{2 - t_1 - t_2}{2 \sqrt{2 - t_1 - t_2}} = \frac{1}{2} \ \sqrt{2 - t_1 - t_2} = \frac{1}{\sqrt{2}} \ \sqrt{1 - \cos \alpha} = 0 . \quad (78)$$

This means that to the line (77) there corresponds one single point $(1, 1)$ on the boundary (see Fig. 1), and here the coordinate $\phi$ is not "dump". There exists one other peculiar region to which the boundary $a + b = 2$ (see Fig. 1) is referred

$$y_0 = 0, \quad y_1^2 + y_2^2 + y_3^2 = 1 , \quad (79)$$
at this

\[
\sin \alpha = 0, \cos \alpha = 1, \quad t_1 = a, \quad t_2 = b \implies
\]

\[
t_1 + t_2 = a + b = 2, \quad \phi \in [0, 2\pi],
\]

\[
y_1 = +\sqrt{a(2-a)} \cos \phi,
\]

\[
y_2 = +\sqrt{a(2-a)} \sin \phi, \quad y_3 = a - 1. \tag{80}
\]

Finally, we must remember on special additional restriction to which obey two complex coordinates

\[
(1 - t_2) = (1 - t_1) \left( -\frac{t_1}{t_1^*} \right), \quad (1 - t_1) = (1 - t_2) \left( -\frac{t_2}{t_2^*} \right). \tag{81}
\]

In particular, these means that if \( t_1 \to 1 \pm 0 \) then \( t_2 \to 1 \mp 0 \), and inversely.

Now let us separate solutions for bound states. To have functions vanishing on the axis \( y_1 = 0, y_2 = 0 \), we must take positive \( a_1 \) and \( a_2 \):

\[
a_1 = +\left\lvert \frac{m}{2} \right\rvert, \quad a_2 = +\left\lvert \frac{m}{2} \right\rvert. \tag{82}
\]

Let us consider behavior of the functions on the line \( y_1^2 + y_2^2 = 1 \) – see \( \text{(78)} \). Because here \( t_1 \to 1, t_2 \to 1 \), to have continuous and finite solutions we must impose restrictions

\[
\Re b_1 > 0, \quad b_1 = +\sqrt{\frac{\epsilon - i\epsilon}{2} + \frac{m^2}{4} - k_1},
\]

\[
\Re b_2 > 0, \quad b_2 = +\sqrt{\frac{\epsilon + i\epsilon}{2} + \frac{m^2}{4} - k_2}. \tag{83}
\]

In general, instead it is enough to require only

\[
\Re (b_1 + b_2) \geq 0. \tag{84}
\]

Assuming

\[
\Re \left( +\sqrt{\frac{1}{4} + \frac{\epsilon - i\epsilon}{2}} \right) > 0, \quad \Re \left( +\sqrt{\frac{1}{4} + \frac{\epsilon + i\epsilon}{2}} \right) > 0,
\]

let us reduce hypergeometric functions into polynomials

\[
\beta_1 = -n_1, \quad n_1 = 0, 1, 2, \ldots;
\]

\[
\beta_2 = -n_2, \quad n_2 = 0, 1, 2, \ldots; \tag{85}
\]
thereby it is supposed that there exist such values for energy and complex \( k_1, k_2 \) at which imaginary parts of two square roots in \( \beta_1 \) and \( \beta_2 \) will cancel out each other. Eqs. (85) give

\[
\beta_1 = \left| \frac{m}{2} \right| + \sqrt{\frac{1}{2} \left( \epsilon - ie \right) + \frac{m^2}{4} - k_1} + \frac{1}{2} \sqrt{\frac{1}{2} \left( \epsilon - ie \right)} = -n_1 ,
\]

\[
\beta_2 = \left| \frac{m}{2} \right| + \sqrt{\frac{1}{2} \left( \epsilon + ie \right) + \frac{m^2}{4} - k_2} + \frac{1}{2} \sqrt{\frac{1}{2} \left( \epsilon + ie \right)} = -n_2 ,
\] (86)

or differently

\[
\left| m \right| + 1 + 2n_1 + \sqrt{2(\epsilon - ie) + m^2 - 4k_1} = \sqrt{1 + 2(\epsilon - ie)} ,
\]

\[
\left| m \right| + 1 + 2n_2 + \sqrt{2(\epsilon + ie) + m^2 - 4k_2} = \sqrt{1 + 2(\epsilon + ie)} .
\] (87)

Let

\[ N_1 = 2n_1 + \left| m \right| + 1 , \quad N_2 = 2n_2 + \left| m \right| + 1 , \]

then eqs. (87) take the form

\[
\sqrt{2\epsilon - 2ie + m^2 - 4k_1} = \sqrt{1 + 2(\epsilon - ie)} - N_1 ,
\]

\[
\sqrt{2\epsilon + 2ie + m^2 - 4k_2} = \sqrt{1 + 2(\epsilon + ie)} - N_2 .
\] (88)

Squaring both equations, after simple manipulation we get (remembering on \( k_1 - k_2 = -ie \))

\[
\left[ \sqrt{1 + 2(\epsilon - ie)} - N_1 \right]^2 = \left[ \sqrt{1 + 2(\epsilon + ie)} - N_2 \right]^2 ;
\]

from this two equations follow

\[
\sqrt{1 + 2(\epsilon - ie)} - N_1 = \pm \sqrt{1 + 2(\epsilon + ie)} - N_2 ,
\]

\[
\sqrt{1 + 2(\epsilon - ie)} - N_1 = \mp \sqrt{1 + 2(\epsilon + ie)} - N_2 .
\]

They give respectively

\[
\sqrt{1 + 2(\epsilon - ie)} - \sqrt{1 + 2(\epsilon + ie)} = N_1 - N_2 = 2n ,
\]

\[
\sqrt{1 + 2(\epsilon - ie)} + \sqrt{1 + 2(\epsilon + ie)} = N_1 + N_2 = 2k .
\] (89)

Note, that the first of them cannot be valid, further we will consider only second equation in (89) – it represent a correct rule for energy quantization. Squaring it we obtain

\[
(1 + 2\epsilon) + \sqrt{(1 + 2\epsilon)^2 + 4\epsilon^2} = 2k^2 .
\] (90)
Its solution is (let $2\epsilon + 1 = x$)

$$\epsilon = -\frac{\epsilon^2}{2k^2} + \frac{k^2 - 1}{2}, \quad k = n_1 + n_2 + |m| + 1.$$ (91)

With the used of the formula for energy levels

$$2\epsilon + 1 = -\frac{\epsilon^2}{k^2} + k^2,$$

one can derives simple expressions for all involved parameters. First, one finds

$$+ \sqrt{\frac{1}{4} + \frac{\epsilon - ie}{2}} = + \frac{1}{2} \sqrt{k^2 - \frac{\epsilon^2}{k^2} - ie} = + \frac{1}{2} (k - i\frac{\epsilon}{k}) ,$$

$$+ \sqrt{\frac{1}{4} + \frac{\epsilon + ie}{2}} = + \frac{1}{2} \sqrt{k^2 - \frac{\epsilon^2}{k^2} + ie} = + \frac{1}{2} (k + i\frac{\epsilon}{k}) .$$

Now expressions for

$$a_1 + b_1 + \frac{1}{2} = -n_1 + \frac{1}{2} (k - i\frac{\epsilon}{k}) ,$$

$$a_2 + b_2 + \frac{1}{2} = -n_2 + \frac{1}{2} (k - i\frac{\epsilon}{k}) ,$$

and then $\alpha_1, \alpha_2$:

$$\alpha_1 = -n_1 + (k - i\frac{\epsilon}{k}) = n_2 + |m| + 1 - i\frac{\epsilon}{k} ,$$

$$\alpha_2 = -n_2 + (k + i\frac{\epsilon}{k}) = n_1 + |m| + 1 + i\frac{\epsilon}{k} .$$

For $b_1, b_2$ we easily produce

$$b_1 = + \frac{1}{2} \sqrt{2\epsilon - 2ie + m^2 - 4k_1} = \frac{1}{2} \left[ + \sqrt{1 + 2(\epsilon - ie)} - N_1 \right] = \frac{1}{2} \left[ + (n_2 - n_1) - i\frac{\epsilon}{k} \right] ,$$

$$b_2 = + \frac{1}{2} \sqrt{2\epsilon + 2ie + m^2 - 4k_2} = \frac{1}{2} \left[ + \sqrt{1 + 2(\epsilon + ie)} - N_2 \right] = \frac{1}{2} \left[ - (n_2 - n_1) + i\frac{\epsilon}{k} \right] .$$

It should be noted that an identity holds

$$b_1 + b_2 = 0 ;$$ (92)

the latter is enough to ensure finiteness of the solutions at the region $t_1 \to 1$, $t_2 \to 1$.

It was emphasized above that it has sense to examine continuity properties in accordance with (see Fig. 1)

$$t_1 = t_2 = a + b = 2 \implies (1 - t_1) = -(1 - t_2) ;$$

from whence it follows that if $(1 - t_1) = 0$, then $(1 - t_2) = 0$. In other words, we have no ground to expect continuity of the following types

$$\Psi(t_1 \to 1, t_2) , \quad \text{or} \quad \Psi(t_1, t_2 \to 1) .$$
Additionally, one can check expressions for $\beta_1, \beta_2$

$$\beta_1 = \frac{|m|}{2} + \frac{1}{2} + b_1 - \sqrt{\frac{1}{4} + \frac{\epsilon - ie}{2}} =$$

$$\frac{|m|}{2} + \frac{1}{2} + \frac{1}{2} \left[ (k - i\frac{e}{k}) - N_1 \right] - \frac{1}{2} (k - i\frac{e}{k}) = \frac{|m| + 1 - N_1}{2} = -n_1,$$

$$\beta_2 = \frac{|m|}{2} + \frac{1}{2} + b_2 - \sqrt{\frac{1}{4} + \frac{\epsilon + ie}{2}} =$$

$$\frac{|m|}{2} + \frac{1}{2} + \frac{1}{2} \left[ (k + i\frac{e}{k}) - N_2 \right] - \frac{1}{2} (k + i\frac{e}{k}) = \frac{|m| + 1 - N_2}{2} = -n_2.$$  

Finally, it is the matter of simple calculations to specify $k_1$ and $k_2$:

$$2\epsilon - 2ie + m^2 - 4k_1 = \left[ + (n_2 - n_1) - i\frac{e}{k} \right]^2,$$

$$2\epsilon + 2ie + m^2 - 4k_2 = \left[ -(n_2 - n_1) + i\frac{e}{k} \right]^2,$$

that is

$$4k_1 = (k - i\frac{e}{k})^2 - \left[ (n_2 - n_1) - i\frac{e}{k} \right]^2 + m^2 - 1,$$

$$4k_2 = (k + i\frac{e}{k})^2 - \left[ (n_2 - n_1) - i\frac{e}{k} \right]^2 + m^2 - 1.$$  

In particular, the identity $k_1 - k_2 = -ie$ holds.

**VII. THE RUNGE–LENZ VECTOR AND PARABOLIC COORDINATES**

At separating the variables in Schrödinger equation two constants were introduced $k_1, k_2$; the problem is to find an operator that is diagonalized on wave functions (31) with eigenvalues $(k_1 + k_2)$

$$\hat{B} \ f_1 \ f_2 \ e^{im\phi} = (k_1 + k_2) \ f_1 \ f_2 \ e^{im\phi}.$$  

(93)

For the operator $\hat{B}$ one can obtain the following representation

$$\hat{B} = -(1 - t_1) \frac{\partial}{\partial t_1} \ t_1 (1 - t_1) \ \frac{\partial}{\partial t_1} - t_1 \ \frac{(-H + ie)}{2} - \frac{1}{4t_1} \ \frac{\partial^2}{\partial \phi^2},$$

$$- (1 - t_2) \ \frac{\partial}{\partial t_2} \ t_2 (1 - t_2) \ \frac{\partial}{\partial t_2} - t_2 \ \frac{(-H - ie)}{2} - \frac{1}{4t_2} \ \frac{\partial^2}{\partial \phi^2},$$  

(94)

or after substituting the expression for $H$

$$\hat{B} = -ie \ \frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2} + \frac{2t_2 (1 - t_1)(1 - 2t_1)}{t_1 - t_2} \ \frac{\partial}{\partial t_1} + \frac{2t_1 (1 - t_2)(1 - 2t_2)}{t_2 - t_1} \ \frac{\partial}{\partial t_2}$$

$$+ \frac{2t_1 t_2 (1 - t_1)^2}{t_1 - t_2} \ \frac{\partial^2}{\partial t_1^2} + \frac{2t_2 t_1 (1 - t_2)^2}{t_2 - t_1} \ \frac{\partial^2}{\partial t_2^2} - \frac{t_1 + t_2}{2t_1 t_2} \ \frac{\partial^2}{\partial \phi^2};$$  

(95)
note the identity

$$-\i e \frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2} = -\i e \cos \theta = -\i e \frac{q_3}{q}.$$ 

Now we turn to establishing connection between $\hat{B}$ and Runge – Lenz vector. It is convenient to solve the task in the same time both in space $H_3$ and $S_3$.

In $H_3$ and $S_3$ the quantum mechanical Runge-Lenz operator is constructed from momentum and orbital momentum by the formula [4–7]

$$\vec{A} = \frac{e}{q} \vec{q} + \frac{1}{2} ([\vec{L} \vec{P}] - [\vec{P} \vec{L}]) ,$$

(96)

where

$$P_i = -i(\delta_{ij} + q_i q_j) \frac{\partial}{\partial q_j}, \ \vec{L} = [\vec{q} \vec{P}] ,$$

(97)

upper sign corresponds to the model $H_3$, lower corresponds to $S_3$ model; operators $\vec{L}$ and $\vec{P}$ are measured in units $\hbar$ and $\hbar/\rho$ respectively.

In correspondence with symmetry of space models, the components of $\vec{P}$, $\vec{L}$ obey the commutation relations of Lie algebras so(3.1) and so(4):

$$[L_a, L_b] = i \epsilon_{abc} L_c , \ [L_a, P_b] = i \epsilon_{abc} P_c , \ [P_a, P_b] = \pm i \epsilon_{abc} L_c .$$

(98)

Since in the above expression for $\hat{B}$ specific term $-\i e q_3/q$ is present, (in the model $H_3$ we see the term $e q_3/q$), it is natural to look for certain relationship between $\hat{B}$ and $A_3$.

Rather long calculation give the following result

$$\text{in } H_3, \quad \hat{B} = (A + \vec{L}^2) , \quad \text{in } S_3, \quad i \hat{B} = (A + i \vec{L}^2) .$$

(99)

VIII. DISCUSSION

In should be emphasized that the possibility to employ complex-valued coordinates in space of positive constant curvature $S_3$ can be used in other coordinate systems as well – Olevsky’s results provide us with $34-6 = 28$ such special cases. For instance, a complex analogue in spherical space $S_3$ for horospherical coordinates of Lobachevsky space was introduced in [57, 58], ant it was used to examine Shapiro’s plane wave solutions of the Schrödinger equation in spaces $S_3$ by analogy with $H_3$ model.

Such a possibility evidently will extend the class of integrable problems in these spaces – see [53]. Also, complex coordinates in 3D-spaces of constant curvature can be of interest in the context of the theory of the Lorentz group $SO(3,1)$ – see in [59, 60].
IX. ACKNOWLEDGEMENT

Authors are grateful to participants of the scientific seminar of Laboratory of theoretical physics of Institute of physics, National academy of sciences of Belarus, for discussion and advices.

[1] E. Schrödinger. A method of determining quantum-mechanical eigenvalues and eigenfunctions. Proc. Roy. Irish. Soc. A. 46. 9–16 (1940).
[2] A.F. Stevenson. A note on the "Kepler problem" in a spherical space, and the factorization method of solving eigenvalue problems. Phys. Rev. 59. 842–843 (1941).
[3] L. Infeld, A. Schild. A note on the Kepler problem in a space of constant negative curvature. Phys. Rev. 67. No 3/4. 121–122 (1945).
[4] P.W. Higgs. Dynamical symmetries in a spherical geometry. I. J. Phys. A. 12. No 3. 309–323 (1979).
[5] H.I. Leemon. Dynamical symmetries in a spherical geometry. II. J. Phys. A. 12. No 14. 489–501. (1979).
[6] Yu.A. Kurochkin, V.S. Otchik. Analogue of the Runge-Lenz vector and energy spectrum for Kepler problem in 3-dimensional sphere. Doklady Akad. Nauk BSSR. 23. No 11. 987–990 (1979).
[7] A.A. Bogush, Kurochkin Yu.A., Otchik V.S. On quantum-mechanical Kepler problem in Lobachevsky space. Doklady Akad. Nauk BSSR. 24. No 1. 19–22 (1979).
[8] L. Parker. One-electron atom in curved space-time. Phys. Rev. lett. 44. No 23. 1559–1562 (1980).
[9] L. Parker. The atom as a probe of curved space-time. Gen. Relat. and Grav. 13. No 4. 307–311 (1981).
[10] G.A. Ringwood, J.T.Devreese. The hydrogen atom: Quantum mechanics on the quotient of a conformally flat manifold. J. Math. Phys. 21. 1390–1392 (1980).
[11] K. Kobayashi. A derivation of the Pauli-Lenz vector and its variants. J. Phys. A. 13. No 2. 425–430 (1980).
[12] N. Bessis, G. Bessis. R. Shamseddine. Atomic fine-structure in a space of constant curvature. J. Phys. A. 15. No 10. 3131–3144 (1982).
[13] H. Grinberg, J. Marañon, H. Vucetich. The hydrogen atom as a projection of an homogeneous space. Z. Phys. C. 20. 147–149 (1983).
[14] A.A. Bogush, V.S. Otchik., V.M. Red'kov. Separation of variables in Schrödinger equation and normed wave functions for the Kepler problem in tree-dimensional spaces of constant curvature. Proceedings of the National Academy of Sciences of Belarus. Ser. fiz.-mat. 3. 56–62 (1983).
[15] N. Bessis, G. Bessis, R. Shamseddine. Space-curvature effects in atomic fine- and hyperfine-structure calculations. Phys. Rev. A. 29. No 5. 2375–2388 (1984).
[16] N. Bessis, G. Bessis, D. Roux. Atomic fine-structure calculations in a space of constant negative curvature. Phys. Rev. A. 30. No 2.1094–1097 (1984).
[17] N. Bessis, G. Bessis. Atomic fine and hyper-fine structure calculations in a space of constant curvature.
Lectures Notes in Physics. 212. 143–153 (1984).

[18] C.M. Xu, D.Y. Xu. Dirac equation and energy levels of hydrogen-like atoms in Robertson – Walker metrics. Nuovo Cim. B. 83, No 2. 162–172 (1984).

[19] C.M. Xu, D.Y. Xu. Dirac equation and energy-levels of hydrogen like atoms in Robertson – Walker metrics. Nuovo Cim. B. 3, No 2. 162–172 (1984).

[20] V.N. Melnikov, G.N. Shikin. Hydrogen-like atom in gravitational field of the universe. Izvestiz Vuzov. Fizika. 1. 55–59 (1985).

[21] R. Shamseddine. Structure fine et hyperfine atomique dans un espace à courbure constante. J. Phys. A. 19, No 5. 717–724 (1986).

[22] V.S. Otchik, V.M. Red’kov. Quantum mechanical Kepler problem in spaces of constant curvature. Preprint 298, Institute of Physics, NANB. Minsk (1986).

[23] A.O. Barut, A. Inomata and G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. J. Phys. A: Math. Gen. 20, No 18. 6271–6280 (1987).

[24] N. Bessis, G. Bessis, D. Roux Space-curvature effects in the interaction between atom and external fields: Zeeman and Stark effects in a space of constant positive curvature. Phys. Rev. A. 33, No 1. 324–336 (1988).

[25] A.A. Bogush, V.S. Otchik, V.M. Red’kov. Complex parabolic coordinates and hydrogen atom on the sphere. Minsk (1988) 40 pages. Deposited in VINITI 12.04.88, 2722 - B88.

[26] A.K. Gorbatsievich, A. Priebe. On the hydrogen atom in Kerr space time. Acta Phys. Polon. B. 20, No 11. 901–909 (1989).

[27] C. Groshe. The path integral for the Kepler problem on the pseudosphere. Ann. Phys. (N.Y.). 204. 208–222 (1990).

[28] A.O. Barut, A. Inomata and G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. II. Hyperbolic space curvature. J. Phys. A: Math. Gen. 23, No 7. 1179–1190 (1990).

[29] N. Katayama. A note on the Kepler problem in a space of constant curvature. Nuovo Cim. B. 105, No 1. 113–119 (1990).

[30] N.A. Chernikov. The Kepler problem in the Lobachevsky space and its solution. Acta Phys. Polonica. B. 23. 115–122 (1992).

[31] L.G. Mardoyan, A.N. Sisakyan. The hydrogen-atom in curved space – orthogonality of the radial wave-functions with respect to the orbital angular momentum. Soviet J. Nuclear Physics-USSR. 55, No 9. 1366–1367 (1992).

[32] Ya.I. Granovskii, A.S. Zhedanov, I.M. Lutsenko. Quadric algebras and dynamics in curved space. I. An oscillator. Theor. Math. Phys. 91. 474–480 (1992); Quadric algebras and dynamics in curved space. II. The Kepler problem. Theor. Math. Phys. 91. 604–612 (1992).

[33] V.V. Kozlov, A.O. Harin. Kepler’s problem in constant curvature spaces. Celest. Mech. and Dynam. Astron. 54. 393–399 (1992).
[34] S.I. Vinitskii, L.G. Marfoyan, G.S. Pogosyan, A.N. Sisakyan, T.A. Strizh. Hydrogen-atom in curved space – expansion in free solutions on a 3-dimensional sphere. Physics of Atomic Nuclei. 56, No 3. 321–327 (1993).

[35] R. Shamseddine. On the resolution of the wave equations of electron in a space of constant curvature. Can. J. Phys. 75. 805–811 (1997).

[36] A.A. Bogush, Yu.A. Kurochkin, V.S.Otchik. Algebra of conserved operators for the Kepler-Coulomb problem in the spaces of constant curvature. Yad. Fiz. 61, No 10. 1889–1892 (1998).

[37] V.S. Otchik. On the connection between spherical and parabolic bases in the quantum mechanical Kepler problem in Lobachevsky space. Proc. of the National Acad. of Science of Belarus. Phys. Math. ser. 4. 67–72 (1999).

[38] A. Nersessian, G. Pogosyan. Relation of the oscillator and Coulomb systems on spheres and pseudospheres. Phys. Rev. A. 63, No 2. 020103(R) (2001).

[39] V.M. Red’kov. On WKB-quantization in Lobachevski and Riemann 3-spaces. Nonlinear phenomena in complex systems. 6, No 2. 654–668 (2003).

[40] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik. Coulomb scattering in the Lobachevsky space. Nonlinear Phenomena in Complex Systems. 6. 894–897 (2003).

[41] Yu.A. Kurochkin, V.S. Otchik, Dz.V. Shoukavy. MIC-Kepler scattering problem in the three-dimensional Lobachevsky space. Non-Euclidean Geometry in Modern Physics: Proc. of the International Conference BGL-5 (Bolyai - Gauss - Lobachevsky). 10-13 Oct 2006, Minsk, Belarus. 116–121 (2006).

[42] Yu. Kurochkin, Dz. Shoukavy. Regge trajectories of the Coulomb potential in the space of constant negative curvature J. Math. Phys. 47, No 2. 022103 (2006).

[43] A.A. Bogush, V.C. Otchik, V.M. Red’kov. The Runge-Lenz vector for quantum Kepler problem in the space of positive constant curvature and complex parabolic coordinates. Proc. of 5th International Conference Bolyai-Gauss-Lobachevsky: Non-Euclidean Geometry In Modern Physics (BGL-5). 10-13 Oct 2006, Minsk, Belarus. 135–144 (2006); arxiv:hep-th/0612178

[44] N. Bessis, G. Bessis. Electronic wave functions in a space of constant curvature. J. Phys. A. 12, No 11. 1991–1997 (1979).

[45] T. Iwai. Quantization of the conformal Kepler problem and its application to the hydrogen-atom. J. Math. Phys. 23, No 6. 1093–1099 (1982).

[46] J.M. Cohen, R.T. Powers. The general relativistic hydrogen-atom. Comm. Mat. Phys. 86, No 1. 69–86 (1982).

[47] E.M. Ovsiyuk. Quantum Kepler problem for spin 1/2 particle in spaces on constant curvature. I. Pauli theory. NPCS. 14, No 1. 14–26 (2011).

[48] L.P. Eisenhart. Separable systems in Euclidean space Phys. Phys. 45, 427 (1934).

[49] M.N. Olevsky. Three-orthogonal coordinate systems in spaces of constant curvature, in which equation \( \Delta_2 U + \lambda U = 0 \) permits the full separation of variables, Matem. Sbornik. 27, 379 – 426 (1950).
[50] Herranz J., Ballesteros A., Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications 2 010 [22 pages] (2006).

[51] Kalnins E., Miller W. Lie Theory and the Wave Equation in Space-Time. 2. The Group SO(4, C). SIAM J. Math. Anal. 9, No 1. 12–33 (1978).

[52] Pogosyan G.S.; Yakhno A. Lie Algebra contractions and separation of variables. Three-dimensional sphere. Physics of Atomic Nuclei. 72, No. 5. 836–844 (2009).

[53] Herranz J., Ballesteros A. Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature, SIGMA. 2, 010, [22 pages] (2006).

[54] Grosche C.; Pogosyan G.S.; Sissakian A.N. Path Integral discussion for Smorodinsky–Winternitz potentials: II. Two and three dimensional sphere. Fortschritte der Physik. 43(6). 523–563 (1995).

[55] Grosche C.; Pogosyan G.S.; Sissakian A.N. Path Integral Approach to Superintegrable Potentials. Two- Dimensional Hyperboloid. Phys. Part. Nucl. 1996. 27(3). 244–278; Path Integral discussion for Superintegrable Potentials: IV. Three Dimensional Pseudosphere. Phys. Part. Nucl. 28. 486–519 (1997).

[56] Pogosyan G.S.; Yakhno A. Lie Algebra Contractions and Separation of Variables. Three-Dimensional Sphere. Physics of Atomic Nuclei. 72. No. 5. 836–844 (2009).

[57] E.M. Ovsiyuk, N.G. Tokarevskay, V.M. Red’kov. Analogue of the plane waves in spherical Riemann space and complex horospherical coordinates. Reports of National Academy of Sciences of Belarus. Ser. phys.-mat. 3 84–89 (2009).

[58] E.M. Bychkovskaya, N.G. Tokarevskaya, V.M. Red’kov Shapiro’s plane waves in spaces of constant curvature and separation of variables in real and complex coordinates. Nonlinear Phenomena in Complex Systems. 12 No 1. P. 1–15 (2009).

[59] A.A. Bogush, V.M. Red’kov On Unique parametrization of the linear group GL(4,C) and its subgroups by using the Dirac algebra basis. Nonlinear Phenomena in Complex Systems. 11, No 1. P. 1–24 (2008).

[60] V.M. Red’kov, A.A. Bogush, N.G. Tokarevskaya On Parametrization of the Linear GL(4,C) and Unitary SU(4) Groups in Terms of Dirac Matrices. SIGMA. 4, 021. – 46 pages (2008).