First Post-Minkowskian approach to turbulent gravity

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We compute the metric fluctuations induced by a turbulent energy-matter tensor within the first order Post-Minkowskian approximation. It is found that the turbulent energy cascade can in principle interfere with the process of black hole formation, leading to a potentially strong coupling between these two highly nonlinear phenomena. It is further found that a power-law turbulent energy spectrum $E(k) \sim k^{-n}$ generates metric fluctuations scaling like $x^{n-2}$, where $x$ is the four-dimensional spacelike distance from an arbitrary origin in Minkowski spacetime, highlighting the onset of metric singularities whenever $n < 2$. Finally, the effect of metric fluctuations on the geodesic motion of test particles is also discussed as a potential technique to extract information on the spectral characteristics of fluctuating spacetime.

I. INTRODUCTION

The key informing principle of general relativity stipulates that matter/energy and spacetime co-evolve through a self-consistent loop, whereby, to say it with Wheeler, “spacetime tells matter how to move; matter tells spacetime how to curve” \cite{1}. Despite its logical simplicity, the mathematical formulation of the above statement (Einstein equations, EEs for short) faces with a daunting complexity barrier, mostly on account of the strong non-linearity of the matter-spacetime interactions. Even for the “simple” case of matter at rest, the exact solutions of the EEs are restricted to very few precious instances, usually characterized by highly idealized geometries with very special symmetry properties (often too special), which impair a general understanding of the problem \cite{2}.

Evaluating the gravitational field generated by matter in motion clearly adds another layer of mathematical complexity, particularly in the case where such motion is not regular but turbulent instead. For instance, it is not known what kind of spacetime metric results from a given fluctuating energy-momentum tensor (energy density, pressure, velocity field). Likewise, we do not know the fate of turbulent flows in the presence of gravity: do the associated scales (gravitational and turbulent) compete or cooperate among them? Does gravity always dominate in the end, erasing, perhaps beyond some threshold, all fluid scales, or do the latter leave an appreciable long-standing signature on the gravitational field despite its dominance? In other words: can turbulence play an appreciable role on the gravitational collapse process (or in a cosmological context) of a turbulent fluid?

The relevance of these questions for modern astrophysics and cosmology cannot be overstated \cite{3, 11}, and this work represents a preliminary attempt to gain semi-quantitative insights into the above matters.

More specifically, we proceed within a Post-Minkowskian (PM) framework, i.e. starting from a flat space situation (zero gravity, and Minkowskian fluid dynamics) and adding corrections to the first order in the gravitational constant $G$, eventually to be continued with high-order iterative corrections. This is a standard approach in the study of the two-body problem in general relativity and seems to offer a promising avenue also for the case of fluid-driven gravitational field. In the following, we shall present a “warm-up” investigation along these lines, highlighting on the various difficulties which stand on the way of a quantitative understanding of the turbulence-gravity coupling.

II. THE TURBULENT ENERGY CASCADE AND ITS INTERFERENCE WITH METRIC LENGTH SCALES: DIMENSIONAL ESTIMATES

We begin by considering a gravity-free (flat space) turbulent fluid, whose velocity fluctuations at statistical steady-state, obey the following generic power-law statistics:

$$u(L) = u(L_0)(L/L_0)^\alpha,$$

where $L$ is a generic, running, length scale and $L_0$ is the typical size of the fluid, related to the typical velocity size $u(L_0)$, $\alpha$ is a scaling exponent in the range $0 \leq \alpha \leq 1$, with $\alpha = 0$ corresponding to white uncorrelated noise (total randomness), while $\alpha = 1$ denotes a smooth, differentiable field. In the following we shall refer to $\alpha$ as to the velocity roughness exponent.

Starting from a mother eddy of size $L_0$, the nonlinear cascade generates eddies of progressively smaller size, till
the smallest active length is reached, below which nonlinearity is no longer capable of sustaining coherent motion against dissipation. This happens at the Kolmogorov (or dissipative) length, which is given by the following expression [12,13]

\[ L_d = \frac{L_0}{Rey^{1/(1+\alpha)}} \]  \hspace{1cm} (2.2)

where \( Rey = \frac{U_0 L_0}{\nu} \) denotes the Reynolds number of a turbulent fluid with kinematical viscosity \( \nu \) [13]. The situation is schematically represented in Fig. 1. Let \( L_m = \nu/U_0 \) (such that \( Rey = L_0/L_m \)) be a microscale length fixed by the ratio kinematic of the kinematic viscosity \( \nu \) and the macroscopic velocity \( U_0 \) of a fluid of macroscale \( L_0 \). A simple rearrangement leads to the following compact expression:

\[ L_d = L_0^p L_m^{1-p}, \]  \hspace{1cm} (2.3)

where the scaling exponent \( p \) relates to the roughness via \( p = \alpha/(1+\alpha) \); for example, \( 0 \leq p \leq 1/2 \), assuming \( \alpha \in [0,1] \).

Since the kinematic viscosity shows surprisingly small variations across disparate states of matter [12], we keep \( \alpha \in [0,1] \).

To express this condition in a dimensional form let us divide both sides, for example, by the mass of the Sun, \( M_{\text{sun}} = 2 \cdot 10^{30} \text{ kg} \), \( GM_{\text{sun}}/c^2 = L_{\text{sun}} \sim 10^7 \text{ m} \) and introduce the gravitational length to replace \( c^2/G = L_g^2 \rho \). We find the relation

\[ m = \frac{M_0}{M_{\text{sun}}} > L_g^p L_m^{1-a_p} L_{\text{sun}}^{-1} \equiv m^*, \]  \hspace{1cm} (2.7)

where

\[ a_p = \frac{2}{3} \frac{p}{1 - \frac{p}{3}}. \]  \hspace{1cm} (2.8)

Here \( m^* \) represents the critical mass above which the dissipative length falls below the Schwarzschild scale.

Eq. (2.7) can also we written as

\[ m^* = \frac{L_s^*}{L_{\text{sun}}}, \quad L^* = L_g^{a_p} L_m^{1-a_p}, \]  \hspace{1cm} (2.9)

where the numerator defines an effective length, \( L^* \), interpolating between the microscale \( L_m \) and the gravitational macroscale \( L_g \). Clearly, \( L^* \) is an increasing function of \( p \), going from \( L_m \) at \( p = 0 \) (random fluid) to \( L_g^{2/3} L_m^{3/5} \) for \( p = 1/2 \) (smooth fluid).

The expression (2.7) is the main result of this section. Summarizing, for any value of \( p \), we have the following...
length scales related to the density of the fluid
\[ m_\star = 10^{-3} L_g^{\alpha_p} L_m^{1-\alpha_p}, \quad L_d = L_0^{\alpha} L_m^{1-\alpha}, \]
\[ L_g = \frac{c}{\sqrt{G \rho^{3/2}}} = \frac{3.7 \cdot 10^{13}}{\rho^{3/2}}, \]
\[ L_m = \frac{10^{-12}}{\rho^{3/7}}, \]
\[ L_0 = \left( \frac{M_0}{\rho} \right)^{1/3} = \left( \frac{m_\star M_{\odot}}{\rho} \right)^{1/3} = 10 m_\star^{1/3} L_g^{2/3}, \]
\[ L_* = \frac{G M_0}{c^2} = L_{\odot} m_\star = 10^9 m_\star. \quad (2.10) \]

Despite their simplicity, the above expressions invite a number of informative remarks. In the following we analyze three distinguished scenarios of decreasing roughness, namely:

1. Fully random fluid \((\alpha = 0, p = 0, a_p = 0);\)
2. Three-dimensional incompressible fluid \((\alpha = 1/3, p = 1/4, a_p = 2/11);\)
3. Two-dimensional incompressible fluid \((\alpha = 1, p = 1/2, a_p = 2/5).\)

### A. Fully random fluid

In this case we have \(\alpha = 0\), hence \(p = 0, a_p = 0\), and a \(|k|^{-1}\) spectrum. The expression (2.7) reduces to
\[ m^* = L_m L_{\odot}^{-1} \sim 10^{-3} L_m, \quad (2.11) \]
and \(L_d = L_m\). The above relation shows \(L_m = L_d = L_*\) varies from \(10^{-12} m\) to \(10^{-14} m\), while \(L_0\) from \(10^0 m\) to \(10^{-2} m\). Moreover the critical mass \(m^*\) ranges from \(10^{-16} \div 10^{-18}\) solar masses, indicating that pretty small black-holes can potentially interfere with a turbulent cascade at Reynolds in the order of \(10^{3}\). Since full randomness is less realistic than correlated turbulence, it is of interest to directly inspect the turbulent cases.

### B. Three-dimensional turbulence

As mentioned above, turbulence is a subtly correlated form of chaos far from pure randomness. As a result, it shows high sensitivity to spatial dimensionality. In 3d, energy is dissipated even in the (singular) limit of zero viscosity, through the nonlinear energy cascade from \(L_0\) down to \(L_d = L_0/\text{Rey}^{3/4}\), \(\text{Rey}\) being the Reynolds number. This leads to a roughness exponent \(\alpha = 1/3\) and a \(|k|^{-5/3}\) power spectrum. Hence, we have \(\alpha = 1/3, p = 1/4, a_p = 2/11\). The expression (2.7) now gives
\[ m^* = L_g^{2/11} L_m^{9/11} L_{\odot}^{-1}. \quad (2.12) \]
As an explicit example in Fig. (2) we show \(m^*, L_0\) and \(L_m\) as a function of \(\rho\).

![FIG. 2: 3d Turbulence](image-url)

The leading factor \(L_g\) is now active, but still largely suppressed by the small \(2/11\) exponent, yet providing a boost of about five orders of magnitude with respect to the case of a fully random fluid. The range of Reynolds numbers (\(\text{Rey} = L_0/L_m\)) is more less the same as for the fully random case.

### C. Two-dimensional turbulence

In two spatial dimensions, energy is conserved while enstrophy (vorticity squared) is dissipated, which implies a direct (large to small) enstrophy cascade and an inverse energy cascade (from small to large) [12].

The result is a smooth flow field with \(\alpha = 1\), corresponding to \(p = 1/2, a_p = 2/5\) and a much steeper energy spectrum \(|k|^{-3}\). The expression (2.7) now gives
\[ m^* = L_g^{2/5} L_m^{3/5} L_{\odot}^{-1}. \quad (2.13) \]

From the above relations it is apparent that putative black holes (BHs) are another five orders of magnitude more massive than in the 3d case. In fact, they range from \(10^{-5} \div 10^{-9}\) solar masses, namely \(10^{25} \div 10^{21}\) kg, significantly more massive than in the previous cases.

It is worth noting that all turbulent cascades above involve pretty large values of the Reynolds number \(\text{Rey} = L_0/L_m\) around \(10^{20}\) (as a matter of reference, the Reynolds number for a standard airline is about \(10^8\)).

So far we have established that turbulent flows are capable of potentially strong interactions with growing BHs, since they can reach down to the background curvature scale (i.e., the mass in the case of a Schwarzschild
black hole). The key question, however, is how to describe such strong coupling in quantitative terms. As discussed in the Introduction, a fully-fledged answer to this question must necessarily rely upon the non-perturbative solution of Einstein’s equations, driven by a turbulent matter tensor.

A few heuristic arguments can be brought up, without undertaking such a demanding task head-on. To this regard, let us recall that the standard fate of a dissipative eddy of size $L_d$ is to de-cohere into a “spray” of droplets too small to sustain collective motion: that’s where hydrodynamic bows away to a microscopic description. Under strong coupling conditions it is plausible to assume that instead of being turned into heat, dissipative eddies would rather feed the black hole (Primordial Black hole, PBH, could be more appropriate in a cosmological context) growth in a sort of preferential way as compared to eddies of larger size. This speculation is grounded into the principle of locality of turbulence in reciprocal (Fourier) space, according to which eddies interact strongly only with eddies of comparable size. This is known to hold for fluid turbulence, but does by no means imply that the same principle applies to turbulence-gravity interactions as well.

Indeed, recent results, based on the numerical and analytic solution of the EEs, show that black hole (BH) horizons can themselves turn turbulent and in a way which is highly reminiscent of Kolmogorov 3d turbulence [12]. Such results suggest that the aforementioned principle of locality may indeed hold true for BH-turbulence interactions as well. It appears therefore reasonable to speculate that dissipative eddies might function as “catalyzers” of gravitational collapse should be liable to numerical verification.

III. TURBULENCE-DRIVEN GRAVITY: FIRST ORDER POST-MINKOWSKIAN APPROACH

So far, we have presented statistical steady-state considerations based on dimensional analysis, an approach which proves exceedingly insightful in the theory of (gravity-free) turbulence. In this section, we endeavor to sketch a quantitative analysis of the EEs under the stochastic drive of a turbulence matter-energy tensor. To this purpose, we work in a Post-Minkowskian (PM) context which implies a weak gravitational field (first order in the gravitational constant $G$, treated as a place-holder in the perturbative expansion) but is not restricted to small velocities, limiting our considerations to the first-order (1PM, or $O(G^1)$) approximation level.

Before plunging into the 1PM formalism, we wish to mention that the effects of stochastic fluctuations of the energy-matter tensors on the gravitational metric have been considered before in the context of stochastic gravity [17]. In the approach of Ref. [17] metric fluctuations are treated by means of a generalized Langevin formalism [18], whereas in the present work we adopt a strategy inspired by a merger between 1PM and turbulence modeling techniques [19].

Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (with $\eta_{\mu\nu} = \text{diag}[1,1,1,1]$) denote a 1PM perturbation of the flat space, with inverse $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and such that

$$\Box_x h^{\mu\nu} = -16\pi G S^{\mu\nu} + O(\partial^2 h h + h^2),$$  \hspace{1cm} (3.1)

where $\Box_x = \eta^{\mu\nu} \partial_{\mu} \partial_{\nu}$ and

$$S^{\mu\nu}(x) = T^{\mu\nu}(x) - \frac{1}{2}T(x)g^{\mu\nu}(x),$$

$$T(x) = g_{\alpha\beta}(x)T^{\alpha\beta}(x).$$  \hspace{1cm} (3.2)

In this case all indices are raised/lowered by using the flat metric $g_{\mu\nu}$, as standard. Following Ref. [20], let us introduce the Green function $G(x - y)$ of the flat-space D’Alembert operator, such that

$$\Box_x G(x - y) = -4\pi \delta^{(4)}(x - y).$$  \hspace{1cm} (3.3)

This implies

$$h^{\mu\nu}(x) = 4G \int d^4 y G(x - y) S^{\mu\nu}(y) + O(G^2),$$  \hspace{1cm} (3.4)

and it is fully determined as soon as the source $S^{\mu\nu}(y)$ is specified. Let us assume $T^{\mu\nu}$ to represent a perfect fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu}, \quad T = -\rho + 3p,$$  \hspace{1cm} (3.5)

so that

$$S^{\mu\nu} = (\rho + p) u^\mu u^\nu - \frac{1}{2}(p - \rho)g^{\mu\nu}$$

$$= \rho \left( u^\mu u^\nu + \frac{1}{2}g^{\mu\nu} \right) + p \left( u^\mu u^\nu - \frac{1}{2}g^{\mu\nu} \right).$$  \hspace{1cm} (3.6)

Furthermore, when studying turbulent motions, characterized by the coexistence of slowly-varying and rapidly-varying excitations, it is convenient to split the fluid components into an averaged and a fluctuating component $X(x) = X_0(x) + \delta X(x)$, e.g.,

$$\rho(x) = \rho_0(x) + \delta \rho(x),$$

$$p(x) = p_0(x) + \delta p(x),$$

$$u^\mu(x) = u_0^\mu(x) + \delta u^\mu(x),$$  \hspace{1cm} (3.7)

\footnote{At the 1PM order, retarded and time-symmetric propagators are equivalent [24].}
where \( X_0(x) \), is a slowly varying quantity whereas \( \Delta X(x) \) is a rapidly varying component. In the above "slow (rapid)" implies scales longer (shorter) than the typical averaging length, namely the heterogeneity scale of the fluid (infinity in the case of homogeneous turbulence).

Within the 1PM approximation, the source \( S^{\mu\nu} \), being prefactored by \( G \), can be treated as a zeroth-order quantity i.e., with \( \theta^{\mu\nu} = \eta^{\mu\nu} \) in Eqs.

As a result, we are left with the following source terms in the equation (3.3):

\[
S^{\mu\nu} = S_0^{\mu\nu} + S_1^{\mu\nu} + S_2^{\mu\nu} + S_3^{\mu\nu},
\]

where

\[
S_0^{\mu\nu} = \left( \rho_0 + p_0 \right) u_0^\mu u_0^\nu - \frac{1}{2} \left( p_0 - \rho_0 \right) \eta^{\mu\nu},
\]

\[
S_1^{\mu\nu} = \left( \rho_0 + p_0 \right) \left( u_0^\mu \delta u^\nu + u_0^\nu \delta u^\mu \right) + \left( \delta \rho + \delta p \right) u_0^\mu u_0^\nu - \frac{1}{2} \left( \delta \rho - \delta p \right) \eta^{\mu\nu},
\]

\[
S_2^{\mu\nu} = \left( \rho_0 + p_0 \right) R_{\rho u, u}^{\mu \nu} + 2 \eta^{\mu \nu} R_{p, u}^{\rho} + 2 \eta^{\mu \nu} R_{p, u}^{\rho},
\]

\[
S_3^{\mu\nu} = R_{\rho u, u}^{\mu \nu} + R_{p, u}^{\rho},
\]

In the above, we have introduced the "correlators" 

\[
R_{\rho u, u}^{\mu \nu} = \delta \rho \delta u^\mu, \quad R_{\rho, u}^{\nu} = \delta \delta \rho u^\nu, \quad R_{p, u}^{\rho} = \delta \rho u^\rho \delta u^\nu,
\]

where \( X_{(ab)} = \frac{1}{2} \left( X_{ab} + X_{ba} \right) \) denotes symmetrization.

Notice that \( R_{\rho u, u}^{\mu \nu} \) (or more precisely its averaged version) is a direct analogue of the Reynolds stress tensor in Kolmogorov turbulence, while \( R_{\rho, u}^{\nu} \) and \( R_{p, u}^{\rho} \) reflect compressibility effects (in case the pressure is a linear function of energy they are basically the same).

Next, we move to Fourier space, where the inverse box operator becomes:

\[
\hat{G}(k) = -\frac{1}{k^2}, \quad k^2 = k \cdot k = \eta_{ab} k^a k^b,
\]

with \( \frac{1}{k^2} \) denoting a Principal Value (PV) kernel. Consequently,

\[
h^{\mu\nu}(x) = -16\pi G \int \frac{d^4 k}{(2\pi)^4} \frac{\hat{S}_{\rho u}(k) e^{i k \cdot x}}{k^2} = -16\pi G \sum_{r=0}^{3} \int \frac{d^4 k}{(2\pi)^4} \frac{\hat{S}_{r}(k) e^{i k \cdot x}}{k^2},
\]

where

\[
\hat{S}_{\rho u}(k) = \int d^4 x e^{-i k \cdot x} S_{\rho u}(x).
\]

We can then consider the parts of \( h^{\mu\nu} \) sourced by the various components of \( S^{\mu\nu} \),

\[
h_r^{\mu\nu}(x) = -16\pi G \int \frac{d^4 k}{(2\pi)^4} \frac{\hat{S}_{r}(k) e^{i k \cdot x}}{k^2}, \quad r = 0, \ldots, 3.
\]

To make further analytical progress, we need to make physically reasonable assumptions on the source terms \( \hat{S}_{\rho u}(k), r = 0, 1, 2, 3 \). For the sake of concreteness, let us start by discussing the case \( \hat{S}_{\rho u}(k) \) (other source terms can be added later), under the simplifying hypothesis that \( u_0^\mu \) is a constant field,

\[
\hat{S}_{\rho u}(k) = \hat{\rho}_0(k) H_+^{\mu \nu} + \hat{\rho}_0(k) H_-^{\mu \nu},
\]

where

\[
H_+^{\mu \nu} = u_0^\mu u_0^\nu \pm \frac{1}{2} \eta^{\mu \nu}, \quad H_-^{\mu \nu} = u_0^\mu u_0^\nu \pm \frac{1}{2} \eta^{\mu \nu},
\]

and

\[
\hat{\rho}_0(k) = \mathcal{E} k^n, \quad \hat{\rho}_0(k) = \mathcal{P} k^m,
\]

with \( \mathcal{E} \) and \( \mathcal{P} \) constants, and scaling exponents \( n, m \) both negative.

Finally, let us introduce the notation

\[
J_q(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{k^q e^{i k \cdot x}}{k^2} = \int \frac{d^4 k}{(2\pi)^4} \frac{k^q e^{i k \cdot x}}{k^2} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left( -\omega^2 + k^2 q/2 \right).
\]

For negative values of \( q \) this integral is divergent along the lightcones \( \omega = \pm |k| \) and should be then regularized (see Appendix A).

We find

\[
\hat{S}_{\rho u}(k) = H_+^{\mu \nu} \mathcal{E} k^n + H_-^{\mu \nu} \mathcal{P} k^m,
\]

and

\[
h_0^{\mu\nu}(x) = -16\pi G H_+^{\mu \nu} \mathcal{E} \int \frac{d^4 k}{(2\pi)^4} \frac{k^{n-2} e^{i k \cdot x}}{k^2} - 16\pi G H_-^{\mu \nu} \mathcal{P} \int \frac{d^4 k}{(2\pi)^4} \frac{k^{m-2} e^{i k \cdot x}}{k^2} = -16\pi G [H_+^{\mu \nu} \mathcal{E} J_{n-2}(x) + H_-^{\mu \nu} \mathcal{P} J_{m-2}(x)].
\]

A direct computation shows that, after a Wick rotation and suitable regularization techniques (see Appendix A), we obtain:

\[
J_q(x) \rightarrow J_q^{\text{Wick}}(x) = \frac{C_q}{x^{1+\gamma}}, \quad C_q = \frac{2^q \Gamma(1+\frac{1}{2})}{\pi^2 \Gamma(-1+\frac{1}{2})^q}.
\]
We note that $C_q$ is well defined. In the range $q \in (-2, 0)$, $C_q$, with values $C_q \in [-\frac{1}{3}, 0]$ (lim$_{q \to -2} C_q = -\frac{1}{4\pi}$ and lim$_{q \to 0} C_q = 0$). This implies

$$h^\mu_0(x) = -16\pi G \left( H^\mu_+ \mathcal{E} \frac{C_{n-2}}{x^{n+2}} + H^\mu_- P \frac{C_{m-2}}{x^{m+2}} \right),$$

(3.23)

where $x^2 = -t^2 + x^2$ is assumed to be positive, i.e. space-like (details in Appendix A). Note that the result (3.23) carries a coordinate-dependent information (however, it depends on the choice of the origin of the coordinate system, which we implicitly placed at $x = 0$).

This simple example shows the onset of a metric singularity (along the lightcone) for $n > -2$ (and same for $m$). In particular, 2d turbulence, $n = -3$, yields a smooth metric, while 3d turbulence, $n = 5/3$ leads to a mildly singular one, with the same exponent as turbulent velocity fluctuations, but opposite sign, i.e. $-1/3$. Next, let us examine the $S_{1\mu}^\nu$ term,

$$\tilde{S}_{1\mu}^\nu(k) = \left. u_0^\mu \right\{ \frac{dl}{(2\pi)^4} \left[ \tilde{\rho}(k) + \tilde{\rho}(k') \right] \delta u^\nu(k - k') + \frac{dl}{(2\pi)^4} \left[ \tilde{\rho}(k) + \tilde{\rho}(k') \right] \delta u^\nu(k - k') + H^\mu_+ \delta \rho(k) + H^\mu_- \delta \rho(k) \right\}. \quad (3.25)$$

Using Eq. (3.15), we find

$$\tilde{S}_{1\mu}^\nu(k) = \left. u_0^\mu \right\{ \frac{dl}{(2\pi)^4} \left[ \tilde{\rho}(k) + \tilde{\rho}(k') \right] \delta u^\nu(k - k') + \frac{dl}{(2\pi)^4} \left[ \tilde{\rho}(k) + \tilde{\rho}(k') \right] \delta u^\nu(k - k') + H^\mu_+ \delta \rho(k) + H^\mu_- \delta \rho(k) \right\}, \quad (3.26)$$

Again, to proceed further analytically we need (physically reasonable) assumptions on $\delta u^\nu(k)$, $\delta \rho(k)$ and $\delta \rho(k)$. For instance, the simplest case is:

$$\tilde{\delta} u^\nu(k) = (2\pi)^4 C_{\delta u}^\nu \delta(k), \quad (3.27)$$

with $C_{\delta u}^\nu$ a constant vector.

This implies

$$\tilde{S}_{1\mu}^\nu(k) = \left. u_0^\mu \left( \frac{dl}{(2\pi)^4} \right) \left[ \tilde{\rho}(k) + \tilde{\rho}(k') \right] \delta u^\nu(k - k') + H^\mu_+ \delta \rho(k) + H^\mu_- \delta \rho(k) \right\}. \quad (3.28)$$

Assuming a power-law for $\tilde{\rho}(k)$ and $\tilde{\rho}(k')$ (e.g., $\tilde{\rho}(k) \sim C_{\delta\rho} k^p$), we are redirected exactly to the same mathematical treatment as for the previous case of $S_{1\mu}^\nu$.

Let us now examine the case of $S_{2\mu}^\nu$, and limit our considerations to the first term,

$$S_{2\mu}^\nu(x) = (\rho_0 + p_0) R^\mu_\nu, \quad (3.29)$$

since all the others can be treated similarly. Passing to the Fourier space we find

$$\tilde{S}_{2\mu}^\nu(k) = \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} (\tilde{\rho}(k_1) + \tilde{\rho}(k_1)) \delta u^\nu(k_2) \times \delta u^\mu(k - k_1 - k_2), \quad (3.30)$$

Using Eq. (3.18), the above expression becomes

$$\tilde{S}_{2\mu}^\nu(k) = \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} (\tilde{\rho}(k_1) + \tilde{\rho}(k_1)) \delta u^\nu(k_2) \times \delta u^\mu(k - k_1 - k_2), \quad (3.31)$$

and again to proceed further we need a physically reasonable expression for $\delta u^\mu(k)$. In the simple case (3.27) we obtain

$$\tilde{S}_{2\mu}^\nu(k) = C_{\delta u}^\nu C_{\delta u}^\mu \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} (\tilde{\rho}(k_1) + \tilde{\rho}(k_1)) \delta u^\nu(k_2) \times \delta u^\mu(k - k_1 - k_2) \times C_{\delta u}^\nu C_{\delta u}^\mu (\tilde{\rho}(k_1) + \tilde{\rho}(k_1)). \quad (3.32)$$

Correspondingly

$$\frac{h_{2\mu\nu}(x)}{16\pi G} = -C_{\delta u}^\mu C_{\delta u}^\nu \int \frac{dk}{(2\pi)^4} \left( \tilde{\rho}(k) + \tilde{\rho}(k') \right) e^{ikx}, \quad (3.33)$$

Extending these considerations to the other terms entering $S_{2\mu}^\nu$ or to the remaining component of the fluid source $S_{3\mu}^\nu$ is performed along the same lines as above. Namely, it is conceptually straightforward, albeit a bit more involved from a mathematical standpoint.

**A. Timelike geodesics and particle scattering in fluctuating spacetime**

Armored with the above formalism, we next proceed to study the timelike geodesics (the orbits of massive particles with mass $m$) of the turbulence–perturbed metric $g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)$, with unit tangent vector

$$\frac{dp_\alpha}{ds} = \frac{d\tau}{ds}, \quad (3.34)$$

such that

$$\frac{dp_\alpha}{ds} = \frac{1}{2} \partial_\alpha h_{\mu\nu}(x) p^\mu p^\nu, \quad (3.35)$$
where $\sigma = \frac{\tau}{m}$ and $\tau$ the proper time parameter. The situation is schematically depicted in Fig. 3

It is also straightforward to derive the variation of the particle’s 4-momentum when scattered by the (spacetime metric generated) fluid, namely

$$\Delta p_\alpha = \frac{1}{2} \int_{-\infty}^{\infty} d\sigma \partial_\alpha h_{\mu\nu}(x) \bigg|_{x=x(\sigma)}^{} p^\mu p^\nu. \quad (3.36)$$

Working at the first order in $G$, since $h_{\mu\nu}$ is already $O(G)$, all the other ingredients entering the right-hand-side of Eq. (3.36) can be replaced by their zeroth-order (free motion in a flat spacetime) approximations, i.e., constant momenta ($p^\alpha = p^\alpha_0 =$constant) and straight (incoming) world lines:

$$x^\alpha(\sigma) = x^\alpha(0) + p^\alpha_0 \sigma \equiv b^\alpha + p^\alpha_0 \sigma, \quad (3.37)$$

where we have denoted $x(0) = b$. As a result:

$$\Delta p_\alpha = \frac{1}{2} p_0^\mu p_0^\nu \int_{-\infty}^{\infty} d\sigma \partial_\alpha h_{\mu\nu}(x) \bigg|_{x=b^\mu + p^\alpha_0} = -4G(2\pi)^2 p_0^\mu p_0^\nu \int \frac{dk^i}{(2\pi)^2} k_\alpha \delta(k \cdot p_-) e^{ik \cdot b}, \quad (3.38)$$

with $\delta(k \cdot p_-)$ arising from the integration over $\sigma$.

Here, we can take $\hat{S}_{\mu\nu}(k) = \hat{S}_{\mu\nu}^0(k)$ as given, for example, by Eq. (3.20).

$$\hat{S}_{\mu\nu}(k) = H_{+\mu\nu} E k^n + H_{-\mu\nu} \mathcal{P} k^m. \quad (3.39)$$

Let us introduce the notation

$$H_{\pm\mu\nu} p^\mu p^\nu = L_\pm = \text{const}. \quad (3.40)$$

From Eq. (3.42), we obtain:

$$\Delta p_\alpha = -4G(2\pi)^2 \int \frac{dk_0 dk^3 k}{m \gamma} [L_+ E k^{n-2} + L_- \mathcal{P} k^{m-2}] k_\alpha e^{ik \cdot b} \delta(k_0 - \sqrt{\gamma^2 - 1} k_y), \quad (3.43)$$

with $k_0 = \sqrt{\gamma^2 - 1} k_y$ the on-shell condition and $d^3k = dk_x dk_y dk_z$.

Therefore $k^2 = k_x^2 + \left(\frac{k_y}{\gamma}\right)^2 + k_z^2 \equiv k^2_1 + k^2_2$ on-shell. Finally

$$\Delta p_\alpha = -4G \int \frac{d^3k}{m \gamma(2\pi)^2} [L_+ E \int d^3k k^{n-2} k_\alpha e^{ik \cdot b} + L_- \mathcal{P} \int d^3k k^{m-2} k_\alpha e^{ik \cdot b}], \quad (3.44)$$

On symmetry grounds, we have:

$$\Delta p_z = 0, \quad (3.45)$$

and, because of the on-shell condition, we obtain

$$\Delta p_0 = \sqrt{\gamma^2 - 1} \Delta p_y. \quad (3.46)$$
For \( n < 1 \), the above rescaled expression cannot be used. Working directly with the un-rescaled expression, we have

\[
Y_{-1}(b) = \pi \gamma \int dk_x \frac{e^{ik_x b}}{k_x}, \tag{3.55}
\]

implying the following Dirac-delta result

\[
\frac{d}{db} Y_{-1}(b) = 2i\pi^2 \gamma \delta(b). \tag{3.56}
\]

Other values of \( n \) are also of interest; for \( n = -3 \), we compute:

\[
Y_{-3}(b) = \sqrt{\pi} \frac{\Gamma(\frac{4}{3})}{\Gamma(2)} b^{2/3} \int_{-\infty}^{\infty} du u^{-5/3} e^{iu} = \frac{1}{2} \pi^2 b^2 i \int_{-\infty}^{\infty} du u^{\sin u / u^3} = -\frac{\pi^2}{4} \gamma b^2 i, \tag{3.57}
\]

where the last integral diverges at \( u = 0 \) and it has been evaluated by using the Partie Finie. For \( n = -5/3 \) we find instead:

\[
Y_{-5/3}(b) = \sqrt{\pi} \frac{\Gamma(\frac{4}{3})}{\Gamma(3/4)} b^{5/3} \int_{-\infty}^{\infty} du u^{-5/3} e^{iu} = \sqrt{\pi} \frac{\Gamma(\frac{4}{3})}{\Gamma(2)} b^{5/3} + 3\gamma^{-3/2} \frac{\Gamma(\frac{5}{6})}{\Gamma(4/3) \Gamma(4/3)} i, \tag{3.58}
\]

where

\[
\Gamma \left( \frac{4}{3} \right) \Gamma \left( \frac{2}{3} \right) = \frac{2}{9} \pi \sqrt{3}, \quad \Gamma \left( \frac{5}{6} \right) \approx 1.1288. \tag{3.59}
\]

Therefore

\[
Y_{-5/3}(b) = 27\pi^{1/2} \frac{\Gamma(\frac{5}{6})}{\Gamma(2)} b^{5/3} i \approx 15.5941 \gamma b^{5/3} i. \tag{3.60}
\]

Summarizing, in the case \( n = -1 \) we end up with a \( \Delta p_x \propto \delta(b) \), namely a particle scattered by a fluid with an energy spectrum of the type \( \sim k^{-1} \) experiences an instantaneous variation of its (initially constant) linear momentum. In the case of a energy spectrum of the type \( \sim k^n \), we find a corresponding power law for the variation of the linear momentum, \( \Delta p_x \propto b^{-n-1} \). This may permit to discriminate between various equations of state of the fluid (including a fluid undergoing a turbulent behavior) by examining the variation of linear momentum for a particle scattered by the fluid itself.
B. Dissipative effects: viscosity and heat conduction

As discussed in Sec. II, the Kolmogorov cascade is terminated by dissipative effects; as a result it is of interest to extend our analysis to the case of a viscous fluid with heat conduction, as discussed in Ex. 22.7 of Ref. [21].

To this purpose, we increment Eq. (3.61) with additional dissipative components:

\[ T_{\text{visc}} = -\zeta \Theta(u) P(u) \mu^\nu - 2\eta \sigma(u) \mu^\nu , \]
\[ T_{\text{visc}} = -3\zeta \Theta(u) , \]
\[ T_{\text{heat}}^{\mu\nu} = 2u^{(\mu}q^{\nu)} , \]
\[ T_{\text{heat}} = 0 , \]

(3.61)

where \( P(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \) projects orthogonally to \( u \), \( \sigma(u)^{\alpha\beta} = \text{STF}_{\alpha\beta}[P(u)_{\mu}^\nu P(u)^\mu_\beta \nabla_\mu u_\alpha] \) is the symmetric and tracefree (STF) shear tensor of the fluid and \( \Theta(u) = \nabla_\alpha u^\alpha \) is the expansion scalar. Moreover, \( \zeta \geq 0 \) denotes the coefficient of bulk viscosity, \( \eta > 0 \) the coefficient of dynamical viscosity and

\[ q^\alpha = -\kappa P(u)^{\alpha\beta}[T_\beta + Ta(u)_\beta] , \]

(3.62)

is the Eckart law of conduction heat (later developed by C. Cattaneo [23] with \( \kappa \) the (constant) coefficient of thermal conductivity and \( a(u)_\alpha = \nabla_\alpha a_\alpha \) the acceleration of the fluid world lines.

It proves expedient to introduce the dissipative tensors \( S^{\mu\nu}_{\text{visc}} = T^{\mu\nu}_{\text{visc}} - \frac{1}{3} T_{\text{visc}} g^{\mu\nu} \) and \( S^{\mu\nu}_{\text{heat}} = T^{\mu\nu}_{\text{heat}} \) and consider their representation in the Fourier space. Assuming again \( u^\mu = u_0^\mu = \text{constant} \) in a Minkowski (flat) spacetime referred to Cartesian coordinates \( \nabla_\alpha a_\beta = 0 \), hence \( T^{\mu\nu}_{\text{visc}} = 0 \) (i.e., a viscosity contribution appears at higher orders of the PM procedure), we obtain:

\[ S^{\mu\nu}_{\text{heat}} = u_0^{\mu} q^{\nu} + u_0^{\nu} q^{\mu} = -\kappa(u_0^{\mu} P(u)_0^{\nu}) T_\beta + u_0^{\nu} P(u_0) T_\beta = -\kappa(u_0^{\mu} T_\nu + u_0^{\nu} T_\mu + 2u_0^{\mu} u_0^{\nu} T_\sigma) , \]

(3.63)

where \( T_\mu = T^\mu \) denotes the temperature gradient.

Moving to Fourier space:

\[ \tilde{S}^{\mu\nu}_{\text{heat}}(k) = -\kappa[2u_0^{(\mu} \tau^{\nu)}(k) + 2u_0^{\mu} u_0^{\nu} \tau_\sigma(k)] , \]

(3.64)

where

\[ \tau^{\nu}(k) = \int d^4x e^{-ik.x} T^{\nu} = ik^{\nu} \hat{T}(k) , \]

(3.65)

and

\[ \hat{T}(k) = \int d^4x e^{-ik.x} T(x) \]

(3.66)

is the Fourier transform of the temperature.

Finally, we find

\[ \tilde{S}^{\mu\nu}_{\text{heat}}(k) = -ik \hat{T}(k)[2u_0^{(\mu} k^{\nu)} + 2u_0^{\mu} u_0^{\nu} k_\sigma k_\sigma] . \]

(3.67)

To proceed further, consistently with the present analysis, we consider a power-law spectrum

\[ \hat{T}(k) = T_0 k^\ell , \]

(3.68)

for some power exponent \( \ell \).

This takes us back to the same basic integrals encountered in the non-dissipative treatment, without adding any further layer of mathematical complexity, at least at the 1PM level of approximation.

Future work is however needed to investigate the consequences of releasing the main simplifying assumptions, such as \( u_\mu = u_0^\mu = \text{constant} \). Work along these lines is currently in progress.

C. Nonlocal effects and link to fractional calculus

Finally, we observe that the integral (3.19) is conducive to fractional calculus, and particularly to a fractional version of the D’Alembert operator. As is well known, fractional derivatives describe nonlocal effects in space and time, which is consistent with the nature of turbulence [24, 21]. Indeed, coherent structures display a finite lifetime, meaning that by the time their effects are felt in a given spacetime location, they have already or moved elsewhere in the fluid, or possibly dissipated away, whence the memory effect in both space and time.

It is very plausible to speculate that such form of dynamic memory should be enhanced by coupling to the gravitational degrees of freedom, since the latter are driven by curvature, itself a source of nonlocality even in non-relativistic physics (think of the Poisson equation in electrostatics).

Clearly, the 1PM framework falls short of capturing the interaction of turbulence with black-holes, but, as mentioned above, it is plausible to speculate that the emergence of nonlocal effects would only be strengthened in the presence of strongly curved spacetimes.

IV. CONCLUDING REMARKS

We have inspected the perturbative effects of fluid turbulence on the gravitational metric and vice versa. Based on purely statistical steady-state scaling arguments, we have first studied the qualitative viability of gravitational interference on the turbulent energy cascade. Next, we have performed a detailed dynamic analysis within the simplified framework of first-order Post-Minkowskian (1PM) gravity. Despite not being analytically solvable and far from the strong-curvature conditions characterizing black-hole physics, the 1PM analysis strongly hints at the onset of turbulence-driven non-local effects on spacetime evolution. In fact, it permits us to pin down the real-space and time scaling exponents of the perturbed metric as a function of the spectral exponents of turbulence.

Although firm conclusions are necessarily hinging on a more quantitative non-perturbative analysis, most likely
by numerical means, it is plausible to speculate that such turbulence-driven nonlocal effects would only be accruable in the presence of strongly curved spacetimes.

Appendix A: Evaluating the integral \( J_q(x) \)

We start from the integral

\[
J_q(x) = \int \frac{d^4k}{(2\pi)^4} k^q e^{ik \cdot x}, \tag{A1}
\]

where \( x = (t, \mathbf{x}) \) is a spacetime vector, \( x^2 = -t^2 + \mathbf{x}^2 \),

\[ k = -\omega dt + K_i d\omega^i, \quad K = \sqrt{\delta_{ij} K_i K_j}, \tag{A2} \]

(having denoted \( k_0 = -\omega, k_i = K_i \) for convenience) and

\[ k^q = (-\omega^2 + K^2)^{q/2}. \tag{A3} \]

For negative values of \( q \) this integral is divergent along the lines \( \omega = \pm K \) and should be regularized, see below. Let us first reduce this integral as follows. Using spherical coordinates in the 3-space of \( \mathbf{K} \), we find

\[
d^4k = d\omega d^3K = d\omega K^2 dK \sin \theta d\theta d\phi \tag{A4}
\]

and

\[
J_q(x) = \int \frac{d^3K}{(2\pi)^3} e^{iK \cdot x^3} A(t; K, q). \tag{A5}
\]

having defined

\[ A(t; K, q) = \int \frac{d\omega}{2\pi} e^{-i\omega t} (-\omega^2 + K^2)^{q/2}. \tag{A6} \]

To compute the integral (A5) let us assume that \( \mathbf{K} \) is aligned with the \( z \) axis of the spherical coordinates so that

\[ K_i x^i = K |\mathbf{x}| \cos \theta, \tag{A7} \]

and (after trivial integration over \( \phi \))

\[ J_q(x) = \int_0^\infty \frac{K^2 dK}{(2\pi)^2} A(t; K, q) \int_0^\pi \sin \theta d\theta e^{iK |\mathbf{x}| \cos \theta}. \tag{A8} \]

The integral in \( \theta \) can be easily performed

\[
\int_0^\pi \sin \theta d\theta e^{iK |\mathbf{x}| \cos \theta} = \frac{2 \sin(K |\mathbf{x}|)}{K |\mathbf{x}|}, \tag{A9} \]

and therefore

\[
J_q(x) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^\infty K dKA(t; K, q) \sin(K |\mathbf{x}|). \tag{A10} \]

Under assumptions \(-2 < q < 0 \) (q real), \( t \neq 0, |\mathbf{x}| \to |\mathbf{x}| - i\epsilon \) (with \( \epsilon > 0 \), later sent to 0) we find

\[
J_q(x) = \frac{2^q}{\pi^2 x^{4+q}} \Gamma(1 + \frac{q}{2}) \Gamma(-1 + \frac{q}{2}) = iC_q x^{4+q}, \tag{A11} \]

with \( C_q \) given in Eq. (3.22) above and repeated below for convenience,

\[ C_q = \frac{2^q}{\pi^2 \Gamma(1 + \frac{q}{2}) \Gamma(-1 + \frac{q}{2})}. \tag{A12} \]

The Minkowskian (singular) integral leads to a complex quantity, meaning that a Wick-rotation is required to turn it into a real, hence to a physically acceptable metric.

To show this in detail, we repeat the previous computation in the Euclidean case, by using Eqs. 3 and 5 of Ref. [27], conveniently rewritten here as

\[
I_{m,0,d} = \int d^d k (k^2)^m = (-1)^m \pi^{d/2} \Gamma(m + 1) \delta_{m+d/2,0}, \tag{A13} \]

when \( d = 4 \). The result is:

\[
J_q^{\text{Wick}}(x) = \int \frac{d^4k}{(2\pi)^4} k^q e^{ik \cdot x} = \sum_{n=0}^\infty \frac{i^n}{n!(2\pi)^2} \int d^d k (k^2)^{q/2} (2k \cdot x)^n = \sum_{n=0}^\infty \frac{i^{n+q}(q/2)!n!}{n!(2\pi)^2} (x^2)^n \delta_{m+d+n,2+0} = \frac{C_q}{x^{q+4}}, \tag{A14} \]

where \( C_q \) is given in Eq. (A12) and \( x = x_{\text{Euclidean}} \).

We see indeed that \( J_q^{\text{Wick}}(x) \), Eq. (A14), differs from its Minkowskian analogue, Eq. (A11), by the imaginary unit, consistently with the Wick rotation performed to regularize it (see also Ref. [27] for the convergence conditions controlling the validity of this result).

However, the spacetime distance in Eq. (A14) is Euclidean, and extending the result to the Minkowskian case raises a question whenever \( x^2 < 0 \) (timelike distance).

A possible solution is to reinstate positive-definiteness is to introduce the absolute value,

\[ x_{\text{Euclidean}}^2 = t^2 + \mathbf{x}^2 = |t^2 + \mathbf{x}^2| \rightarrow |(it)^2 + \mathbf{x}^2| = |-t^2 + \mathbf{x}^2|; \tag{A15} \]

the latter leaves the Euclidean distance unaffected but plays the desired goal of removing ambiguities in the Minkowskian case.

However, at this stage the above fix and the subsequent discussion is purely empirical, indicating that further detailed investigation is required to fully unveil the nature of the turbulence-driven metric singularities in the Minkowskian case.
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