Recent rank-based attacks have reduced the security of Rainbow below the security requirements set out by NIST by speeding up repeated kernel finding operations using classical mathematics techniques. If quantum algorithms are applied to perform these repeated operations, the rank-based attacks may be more threatening and could dramatically lower the security level of Rainbow. In this paper, we propose a novel MinRank attack called Q-rMinRank attack, the first quantum approach to the key recovery attacks on Rainbow. By designing quantum circuits that can find the kernel, we achieved quadratic speedup for the MinRank attack to recover the private keys of Rainbow. We show that even the parameter set V of Rainbow does not meet the 128-bit security level, the minimum security requirement through our Q-rMinRank attack. It means that Rainbow is no longer secure in quantum computing environments.

**Keywords** Post Quantum Cryptography, Rainbow signature scheme, MinRank attack, Grover’s algorithm

1 Introduction

Due to recent advances in the development of quantum computers, such as Google’s 53-qubit quantum processor “Sycamore” [1] and IBM’s 127-qubit quantum processor “Eagle” [2], NIST (National Institute of Standards and Technology) estimates that quantum computers will be capable of breaking 2048-bit RSA as early as 2026. This is because 4096 qubits in a quantum computing environment using Shor's quantum algorithm [3] is enough to break 2048-bit RSA [4]. The threat that the current public key cryptographic algorithms such as RSA and ECDSA will be broken in a quantum computing environment has led NIST to conduct a PQC (Post Quantum Cryptography) standardization project. To be securely used in quantum computing environments, the PQC candidates must meet the security requirements set out by NIST, from the 128-bit security for Level I to the 256-bit security for Level V.

Among the PQC candidates for the NIST standardization project, MQ (Multivariate Quadratic)-based signature schemes are expected to be highly utilized in IoT devices with limited resources due to the advantages of short signature length as well as fast signature generation and verification speed [5]. So, Rainbow, a representative MQ-based signature scheme by Jintai Ding [6], had been considered one of the NIST PQC standardization finalists until recently. The security of Rainbow mainly relies on the difficulty of the MQ problem, the Extended Isomorphism of Polynomials (EIP) problem, and the MinRank problem. The existential unforgeability of Rainbow is based on the intractability of the MQ problem (solving multivariate systems of quadratic equations). The difficulty of recovering private keys from public keys of Rainbow depends on the difficulties of the EIP and the MinRank (finding a non-zero \(k\)-tuple such that the rank of a linear combination of some matrices is less than some small rank) problems. Thus, signature forgery attacks and key recovery attacks on Rainbow have attempted to solve these underlying problems.
1.1 Related Works

So far, there have been several attempts to forge the Rainbow signature by trying to solve the MQ problem using mathematical techniques such as XL [7] and Gröbner Basis algorithms [8], but they have not been successful.

Since Olivier Billet et al. proposed a MinRank attack algorithm to recover private keys from the public keys of the Rainbow in 2006 [9], key recovery attacks such as the MinRank attack have become a potential threat to Rainbow. In 2020, Bardet et al. reduced the complexity of the MinRank attacks on Rainbow to \( \frac{1}{2} \) of existing MinRank attacks in the parameter sets III and V of Rainbow [10]. However, the attack proposed in [10] did not even threaten the parameter sets submitted for the second round of the NIST PQC project.

In 2021, Ward Beullens proposed a rectangular MinRank attack [11]. He converted the public keys of Rainbow to polar forms and reduced the size of input matrices of the MinRank attack using the polar forms. Then, he recovered the private keys of Rainbow using the Support Minors Modeling algorithm introduced in [10]. This attack reduced the security level of the parameter sets I, III, and V of Rainbow to 127-bit, 177-bit, and 226-bit security levels, respectively. In 2022, Ward Beullens proposed two new key recovery attacks (simple attack and combined attack) against Rainbow, further reducing the security level of Rainbow [12]. He reduced the complexity of the MinRank attack by suggesting a method to guess the kernel with a high probability in a simple attack. They also reduced the attack complexity for the parameter sets III and V by combining a rectangular MinRank attack [11] with a simple attack. Moreover, he reduced the security level of the parameter sets I, III, and V of Rainbow to 69-bit, 160-bit, and 257-bit security levels in their simple attack and to 99-bit, 157-bit, 206-bit security levels in their combined attack, respectively.

Until now, only rank-based attacks using the classical mathematical method such as the Support Minors Modeling algorithm [10] have been proposed for the key recovery attacks against Rainbow. The parameter set I of Rainbow does not fall short of the minimum-security requirement, the 128-bit security level, against such key recovery attacks. However, the parameter sets III and V of Rainbow still satisfy the 128-bit (I) and the 196-bit (III) security level set by NIST even against the most threatening key recovery attack proposed by Ward Beullens [11,12]. Hence, the Rainbow team argued to NIST that they would replace the parameter set I of Rainbow with the parameter set III and the parameter set III with the parameter set V.

1.2 Our Contributions

The complexities of these rank-based attacks against Rainbow are dominated by the cost of finding a kernel because of its repetitive operations. The classical key recovery attacks have been improving attack complexity by finding faster mathematical techniques that can find the kernel.

We try to dramatically reduce attack complexity by utilizing quantum properties such as superposition to find the kernel. Based on this idea, we propose a novel quantum rectangular MinRank attack on Rainbow, called a Q-rMinRank attack, which is the first quantum key recovery attack on Rainbow. We reduced the attack complexity by finding a kernel with low complexity in the MinRank attack using Grover’s quantum search algorithm. To demonstrate the Q-rMinRank attack, we designed a Q-rMinRank-Grover algorithm that allows our Q-rMinRank attack to find the kernel faster. Our quantum circuit measures the kernel of a matrix with a high probability by using appropriate quantum gates. To further reduce the complexity of finding the kernel with a quantum algorithm, we reduced the size of the matrix by converting the public keys of Rainbow into polar form.

Then, we conducted a security analysis on the Q-rMinRank attack, to determine how secure Rainbow is against our Q-rMinRank attack by analyzing the overall complexity of the Q-rMinRank attack. According to the results of our security analysis, the complexity of the Q-rMinRank attack is \( \frac{1}{2} \sqrt{2^{\left(o_2 - o_1\right)} q_k \log q} \) when \( o_1 \) and \( o_2 \) are the numbers of oil variables in the first and second layers, respectively, and \( q \) is the size of the finite field used in Rainbow. Since the parameter set I of Rainbow has the same number of oil variables in the first and second layers of Rainbow at 32 each, the complexity of the Q-rMinRank attack becomes \( O(1) \). This result means that the kernel can be found with a high probability by running the Q-rMinRank oracle circuit one time only, and the private key of Rainbow using the parameter set I can be quickly recovered. Furthermore, our Q-rMinRank attack has complexities of \( 2^{94} \) for the parameter set III and \( 2^{112} \) for V. It means that all the parameter sets of Rainbow fall short of the 128-bit security level against our Q-rMinRank attack, as shown in Table 1.

We also implemented the quantum circuit designed for a toy example of Rainbow with quantum simulators Qiskit [13] and ProjectQ [14] to verify the feasibility of the Q-rMinRank attack. In addition, we analyzed the number of quantum resources required for the Q-rMinRank attack against third-round Rainbow. To perform the Q-rMinRank attack and recover the private keys of Rainbow, our quantum circuits require 661, 1832, and 2376 qubits for the parameter sets I, III, and V of Rainbow. Rainbow, which can only be broken with up to 2376 qubits, is no longer secure in quantum computing environments.
Table 1: A complexity comparison of the MinRank attack [9], the Rectangular MinRank attack [11], the new MinRank attacks [12], and our Q-rMinRank attack. Complexities below $2^{128}$ are written in bold.

| Parameter set ID       | Security complexity |
|------------------------|---------------------|
| I (128-bit security)   | $2^{144}$           |
|                        | $2^{127}$           |
|                        | $2^{69}$            |
|                        | $2^{99}$            |
|                        | $1$                 |
| III (192-bit security) | $2^{544}$           |
|                        | $2^{177}$           |
|                        | $2^{160}$           |
|                        | $2^{157}$           |
|                        | $2^{64}$            |
| V (256-bit security)   | $2^{768}$           |
|                        | $2^{226}$           |
|                        | $2^{257}$           |
|                        | $2^{206}$           |
|                        | $2^{112}$           |

The summary of the main contributions of this paper are as follows:

- We proposed the quantum rectangular MinRank attack (Q-rMinRank attack), the first quantum-based key recovery attack on Rainbow. And we also designed a Q-rMinRank-Grover oracle circuit, that finds the kernel with low complexity, for the Q-rMinRank attack.
- We evaluated the security level of Rainbow against the Q-rMinRank attack. According to our security analysis, even the parameter set V of Rainbow does not satisfy the 128-bit security level against our Q-rMinRank attack.
- We implemented the quantum circuit for the toy example of Rainbow using quantum simulators Qiskit and ProjectQ to show the feasibility of our attack. We also analyzed the required quantum resources to evaluate the quantum security of Rainbow.

1.3 Organization

The remaining paper is organized as follows. Section 2 provides preliminaries of the Rainbow signature scheme, MinRank attack, Grover’s algorithm [15], and the elementary quantum gates required. In Section 3 we present a Q-rMinRank attack. Finally, in Section 4 we evaluate the security level of the Rainbow scheme against the Q-rMinRank attack and analyze the quantum resources required for it.

2 Preliminaries

We introduce elementary quantum gates of quantum circuits, Rainbow signature scheme, and Grover’s quantum algorithm in this section.

2.1 Elementary Quantum Gates

2.1.1 NOT Gate

The quantum NOT gate is also called the X gate. The operation of the NOT gate in quantum computing, as shown in Figure 1, is the same as the operation of the NOT gate in classical computing.

\[ |x\rangle \xrightarrow{\text{NOT}} |\bar{x}\rangle = |x\rangle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\bar{x}\rangle \]

Figure 1: The NOT gate

2.1.2 CNOT Gate

The operation of the CNOT (Controlled-NOT) gate in quantum computing, as shown in Figure 2, is the same as an XOR operation in classical computing. When two qubits $x$ and $y$ are input to this CNOT gate, the CNOT gate results are $x$ for the input $x \oplus y$ for the input $y$. 

3
2.1.3 $C^N$-NOT Gate

• CCNOT Gate
The CCNOT (Controlled-Controlled-NOT) gate in Figure 3 is also called a Toffoli gate ($N = 2$). The Toffoli gate has three inputs, and the output is almost the same as the input, except that the third qubit is flipped only if the first and second qubits are all 1. That is, when three qubits $x$, $y$, and $z$ are input to this Toffoli gate, the result of this Toffoli gate is $x$ and $y$ for the $x$ and $y$, and $z \oplus xy$ for the input $z$.

$$
\begin{array}{c|c|c}
    x \quad y \quad z & x' \quad y' \quad z' \\
    \hline
    0 \quad 0 \quad 0 & 0 \quad 0 \quad 0 \\
    0 \quad 1 \quad 0 & 1 \quad 0 \quad 1 \\
    1 \quad 0 \quad 1 & 1 \quad 1 \quad 1 \\
    1 \quad 1 \quad 1 & 1 \quad 1 \quad 0 \\
\end{array}
$$

Figure 3: The CCNOT gate

• C$^N$-NOT Gate
The C$^N$-NOT gate in Figure 4 reverses qubit $z$ when $N$ control qubits $c_i$ are all 1. That is, when $N + 1$ qubits $c_0, c_1, \ldots, c_{n-1}$, and $z$ are input to this C$^N$-NOT gate, the result of this C$^N$-NOT gate is $c_0, c_1, \ldots, c_{n-1}$, and $z' = z \oplus c_0c_1\ldots c_{n-1}$.

$$
\begin{array}{c|c|c|c|c}
    c_0 \quad c_1 \quad \ldots \quad c_{n-2} \quad c_{n-1} \quad z & c_0' \quad c_1' \quad \ldots \quad c_{n-2}' \quad c_{n-1}' \quad z' \\
    \hline
    0 \quad 0 \quad \ldots \quad 0 \quad 0 & 0 \quad 0 \quad \ldots \quad 0 \quad 0 \\
    0 \quad 1 \quad \ldots \quad 0 \quad 1 & 1 \quad 0 \quad \ldots \quad 1 \quad 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 \quad 1 \quad \ldots \quad 0 \quad 1 & 1 \quad 0 \quad \ldots \quad 1 \quad 0 \\
    1 \quad 0 \quad \ldots \quad 1 \quad 0 & 0 \quad 1 \quad \ldots \quad 1 \quad 1 \\
    1 \quad 1 \quad \ldots \quad 1 \quad 1 & 1 \quad 1 \quad \ldots \quad 1 \quad 0 \\
\end{array}
$$

Figure 4: The C$^N$-NOT gate

2.2 Rainbow Signature Scheme

Rainbow is one of the Multivariate Quadratic based digital signature schemes, which is a candidate for the third round of the NIST Post Quantum Cryptography project. Rainbow based on the UOV (Unbalanced Oil-Vinegar) problem has the advantage of a relatively short signature and fast signing and verifying algorithm. In this section, we describe the key generation, signature generation, and verification method of the Rainbow signature scheme.

2.2.1 Parameters

Rainbow combines five NIST security categories into three, and the parameters used in Rainbow are shown in Table 2. The total number of variables used in Rainbow is $n = v_1 + o_1 + o_2$. 

\[ n = v_1 + o_1 + o_2. \]
Table 2: The Rainbow parameter sets with NIST security level categories

| Set ID | $q$ in $F_q$ | $v_1$ | $o_1$ | $o_2$ | NIST security level categories (N-category number) |
|--------|--------------|-------|-------|-------|-----------------------------------------------|
| I      | $2^4$        | 36    | 32    | 32    | N-I, N-II                                     |
| III    | $2^8$        | 68    | 32    | 48    | N-III, N-IV                                   |
| V      | $2^8$        | 96    | 36    | 64    | N-V                                           |

2.2.2 Key Generation

- **Private Key**
  The private key of the Rainbow consists of two affine maps $S : F^m \rightarrow F^m$ and $T : F^n \rightarrow F^n$, and central map $F : F^n \rightarrow F^n$. The central map $F$ consists of $m (m = n - v_1)$ multivariate equations $f^{(v_1+1)}$, ..., $f^{(n)}$. When $k \in v_1 + 1, ..., n$, $f^{(k)}$ is as follows.

$$f^{(k)}(x_1, ..., x_n) = \sum_{i,j \in V_l} \alpha_{ij}^{(k)} x_i x_j + \sum_{i \in V_l, j \in O_l} \beta_{ij}^{(k)} x_i x_j + \sum_{i \in V_l \cup O_l} \gamma_i^{(k)} x_i + \delta^{(k)}$$

(1)

- **Public Key**
  The public key of the Rainbow $P$ is the composition of the private keys $S$, $F$, and $T$.

$$P = S \circ F \circ T : F^n \rightarrow F^m$$

2.2.3 Signature Generation and Verification

- **Signature Generation**
  When the message to be signed is $d$, the hash function is $H : 0, 1 \rightarrow F^m$, and the signature is $Z \in F^n$, the signature generation process is as follows.

1. Compute the hash value $h = H(d) \in F^m$
2. Compute $x = S^{-1}(h) \in F^m$
3. Find $y$ that satisfies $F(y) = x$
4. Compute the signature $z = T^{-1}(y) \in F^n$

- **Signature Verification**
  Given message $d$ and signature $z$, the signature verification process is as follows.

1. Compute the hash value $h = H(d) \in F^m$
2. Compute the $h' = P(z) \in F^m$
3. If $h = h'$, the signature $z$ is verified
2.3 MinRank Attack

The MinRank problem is to find a linear combination \( Q = \sum_{i=1}^{m} \lambda_i Q_i \) with a rank smaller than some rank \( r \), given \( m \) \((n \times n)\) matrices \( Q_1, ..., Q_m \). The MinRank attack finds central map \( F \) of Rainbow by solving the MinRank problem. In the case of Rainbow, a linear combination of public keys with a rank of \( v_2 \), which is the number of vinegar variables of the second layer, corresponds to a linear combination of central maps of the first layer. By finding \( o_1 \) linear combinations, the number of oil variables in the first layer, the central maps of the first layer can be reconstructed, thereby finding the secret keys of Rainbow. Algorithm 1 shows the overall process of the MinRank attack on Rainbow.

Algorithm 1 The MinRank attack

**Input:** matrices \( P^{(v_1+1)}, ..., P^{(n)} \)

**Output:** Linear combination \( C = \sum_{i=v_1+1}^{n} c_i \cdot P^{(i)} \) of rank \( \leq v_2 \)

1: repeat
2: Choose randomly a vector \( \lambda \in \mathbb{F}_m \) and compute \( P = \sum_{i=v_1+1}^{n} \lambda_i \cdot P^{(i)} \)
3: if \( \text{Rank}(P) > 1 \) and \( \text{Rank}(P) < n \) then
4: Choose randomly a vector \( \gamma \) from \( \text{Ker}(P) \)
5: \( C \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)} \)
6: end if
7: until \( \text{Rank}(C) \leq v_2 \)
8: return \( C \)

The complexity of the MinRank attack is \( o_1 \cdot q^{v_1+1} \). In this attack, it takes \( q^{v_1} \) complexity to find a kernel of \( P \) (Ker\( (P) \) in Algorithm 1). The complexity of finding a kernel in the parameter sets of the third round Rainbow takes \( 2^{144} \) in the parameter set I, \( 2^{544} \) in the parameter set III, and \( 2^{768} \) in the parameter set V.

2.4 Grover’s Algorithm

An unstructured search problem is to find a solution \( x^* \) in a set \( x = x_1, x_2, ..., x_N \) such that \( f(x^*) = 1 \) when a boolean function \( f : x \rightarrow 0, 1 \) is given. The unstructured search problem takes \( O(N) \) complexity to solve with classical computers. Grover’s algorithm [15] proposed by Lov Grover in 1996 allows unstructured search problems to be solved in \( O(\sqrt{N}) \) complexity using quantum properties. Figure 6 shows the entire circuit of Grover’s algorithm (See Algorithm 2). The circuit of Grover’s algorithm consists of a NOT gate \((X)\), Hadamard gates \((H)\), oracle gates \((O_f^\pm)\), and diffusion operators \((D)\).

![Figure 6: The entire circuit of the Grover’s algorithm](image)

To increase the probability that a solution will be measured, Grover’s algorithm proceeds by repeatedly using the oracle gate \( O_f^\pm \) and Grover diffusion operator \( D \). The repetition time, \( r_z \), can be selected in two ways [16]. First, if the repetition time is \( \sqrt{\frac{N}{8}} \), the probability of measuring a solution exceeds \( \frac{2}{3} \) when operating the whole Grover’s algorithm more than 110 times. Secondly, the solution will be measured with a high probability when the repetition time is \( \frac{\pi}{4} \sqrt{N} \). In this paper, we repeat the Grover oracle and diffusion gate pair \( \frac{\pi}{4} \sqrt{N} \) times to measure the solution at once. The
Algorithm 2 The Grover’s algorithm

\textbf{Input:} black box function \( f \), \( n \)-qubit register \( x \), qubit \( b \)

\textbf{Output:} solution \( x^* \)

1: for \( i = 0 \) to \( n - 1 \) do
2: \( H(x_i) \)
3: end for
4: \( H(X(b)) \)
5: for \( i = 1 \) to \( r_t \) do
6: \( O_f(x,b) \)
7: \( D(x) \)
8: end for
9: solution \( x^* \leftarrow \text{Measure}(x) \)
10: return \( x^* \)

The oracle gate should be designed to find a solution to the problem that we want to solve. The oracle gate reverses the qubit \( |b\rangle \) only if the input \( x \) is the solution. Figure 7 shows how the oracle gate works.

The Grover diffusion operator increases the probability of \( x^* \) which is a solution among \( 2^n \) superpositioned qubit states and is designed as follows.

\[
D = H^\otimes n Z_0 H^\otimes n = H^\otimes n (2 |0^n\rangle \langle 0^n| - I) H^\otimes n = 2((H |0\rangle)^\otimes n ((H |0\rangle)^\otimes n)^\dagger) - H^\otimes n H^\otimes n
\]

\[
= 2 |+^n\rangle \langle +^n| - I = \sum_{x \in 0,1^n} (2\mu - \alpha_x) |x\rangle
\]

3 Quantum Rectangular MinRank Attack

In this section, we propose a quantum rectangular MinRank attack, called a Q-rMinRank attack, the first quantum approach for a key recovery attack on Rainbow. The Q-rMinRank attack consists of three main steps: \textit{preprocessing} step, \textit{quantum kernel extraction} step, and \textit{key recovery} step as shown in Figure 8. The \textit{preprocessing} step performs mathematical operations to convert the public keys of Rainbow into a single matrix in the Classical Computing Part. Then, the \textit{quantum kernel extraction} step finds a kernel of the converted matrix using Q-rMinRank-Grover algorithm that we designed in the Quantum Computing Part. Finally, the private keys are recovered during the \textit{key recovery} step by constructing a linear combination of the public keys using the kernel.

Figure 8: The conceptual diagram of our Q-rMinRank attack.
Our Q-rMinRank attack utilizes Grover’s quantum search algorithm to speed up the kernel search. Among all operations in the MinRank attack, the kernel search operation is the most complicated and time-consuming as it is repeated until the rank condition of the linear combination is satisfied. Since Grover’s quantum search algorithm is capable of parallel processing, the kernel search operation can be performed much faster in our Q-rMinRank attack. To this end, it is necessary to properly design a Grover oracle quantum circuit that could find a kernel of a matrix. Our Q-rMinRank attack achieved a quadratic speedup in the complexity of the classical MinRank attacks. The overall Q-rMinRank attack algorithm is shown in Algorithm 3.

**Algorithm 3** The Q-rMinRank attack

**Input:** matrices \( P^{(v_1+1)}, \ldots, P^{(n)} \)

**Output:** Linear combination \( C = \sum_{i=v_1+1}^{n} c_i \cdot P^{(i)} \) of rank \( \leq v_2 \)

1: repeat
2: Convert the public keys to polar form \( P'(e_i, x), e_i \): basis vector for \( \mathbb{F}_q^n \)
3: Choose randomly a vector \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \in \mathbb{F}_q^m \) and compute \( \bar{P} = \sum_{i=1}^{n} \lambda_i \cdot L_{e_i} \)
4: if \( \text{Rank}(\bar{P}) > 1 \) and \( \text{Rank}(P) < n \) then
5: \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \)
6: \( C \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)} \)
7: else
8: Go to 3
9: end if
10: until \( \text{Rank}(C) \leq v_2 \)
11: return \( C \)

### 3.1 The Preprocessing Step

Rainbow has \( m \times (n \times n) \) matrices \( P^{(v_1+1)}, \ldots, P^{(n)} \) as public keys. In the original MinRank attack, these public keys are converted into an \((n \times n)\) matrix \( P \) by choosing a random vector \( \Lambda \in \mathbb{F}_q^m \) and computing a \( \bar{P} \) as follows:

\[
P = \sum_{i=v_1+1}^{n} \lambda_i \cdot P^{(i)}.
\]

The complexity of finding a kernel decreases as the input matrix size is smaller. In order to reduce the complexity of finding the kernel, the \( m \times (n \times n) \) public keys of Rainbow are converted to the polar form \( P'(e, x) \) in our preprocessing step, as shown in Line 2 in Algorithm 3. The polar form \( [11] \) of multivariate quadratic polynomial \( p(x) \) is defined as

\[
p'(x, y) := p(x + y) - p(x) - p(y) + p(0).
\]

For a multivariate quadratic map \( P(x) = p_1(x), \ldots, p_m(x) \), its polar form is defined as

\[
P'(x, y) := p'_1(x, y), \ldots, p'_m(x, y).
\]

By using the polar form, each column at the same position in the \( m \times (n \times n) \) public key matrix is grouped into a single \((n \times m)\) matrix as shown in Figure 9. Eventually the \( m \times (n \times n) \) public key matrix is converted into \( n \times (n \times m) \) matrices. After that, the linear combination of the \( n \times (n \times m) \) converted matrices, called \( P \), is computed as the process in Line 3 of Algorithm 3. Then, the rank of \( P(\text{Rank}(P)) \) is checked, if \( \text{Rank}(P) = 0 \) or \( \text{Rank}(P) \geq n \), the process in Line 3 of Algorithm 3 is repeatedly executed until the condition \( (1 < \text{Rank}(P) < n) \) is satisfied. If \( 1 < \text{Rank}(P) < n \), the \( P \) is transferred to the quantum kernel extraction step, a quantum computing part, to find the kernel efficiently.

### 3.2 The Quantum Kernel Extraction Step

To find the kernel of the matrix \( P \), the quantum kernel extraction step performs a Q-rMinRank-Grover algorithm (See Algorithm 4) using \( m \cdot n_q \)-qubit registers \( x \), \( n_q \)-qubit register \( t \), \( n \cdot n_q \)-qubit registers \( e \), and qubit \( b \). The \( n_q (= \log_2 q) \)
Algorithm 4 The Q-rMinRank-Grover algorithm

**Input:** \((n \times m)\) matrix \(P\), \(m \cdot n_q\)-qubit registers \(x\), \(n_q\)-qubit register \(t\), \(n \cdot n_q\)-qubit registers \(e\), qubit \(b\)

**Output:** \((m \times 1)\) kernel vector \(\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}\)

1. for \(i = 0\) to \(m\) do
2. for \(j = 0\) to \(n_q\) do
3. \(H(x_{ij})\)
4. end for
5. end for
6. \(H(X(b))\)
7. for \(i = 1\) to \(r_t\) do
8. \(\text{Q-rMinRank}_\text{Oracle}(P, x, t, e, b)\)
9. \(D(x)\)
10. end for
11. kernel vector \(\Gamma \leftarrow \text{Measure}(x)\)
12. return \(\Gamma\)

is the number of qubits required to express a number on \(GF(q)\). The required qubit registers and their roles are shown in Table 3.

Hadamard gate is first applied to the all qubits in quantum registers \(x\) bringing the qubits in superposition states. The qubit \(b\) is also transformed to the superposition state using Hadamard gate. Then, to increase the probability of measuring the kernel, the pair of Q-rMinRank_Oracle() and diffusion circuit \(D()\) is repeatedly performed \(r_t\) times. 

\[
\left( r_t = \frac{\pi}{4} \sqrt{2^{o_2-o_1}n_q} \right)
\]

To represent a linear combination of the \(x_j\) and \(i\)-th row of \(P\) in the quantum register \(e_i\) (\(0 \leq i < n\), \(0 \leq j < m\)), Multiply() and Add() circuits are performed in Q-rMinRank_Oracle (see Algorithm 5) as follows: The \(x_j\) is multiplied by \(P_{i,j}\) and the result is stored in the quantum register \(t\) using the Multiply(). And then, the quantum register \(t\) is added to the quantum register \(e_i\) using the Add(). The dagger circuit of the multiplier, Multiply_dagger(), is performed to initialize the quantum register \(t\). The \(X\) gates are executed on the quantum register \(e_i\) so that the \((n \cdot n_q)\)-controlled-NOT gate reverses the qubit \(b\) only when all the \(e\) are 0. The above processes, called FrontOracle, are repeatedly executed \(n\) times. After the FrontOracle process, the \((n \cdot n_q)\)-controlled-NOT gate is taken using all the qubits in quantum register \(e\).

| Quantum registers | Number of qubits | Role of qubit registers |
|-------------------|------------------|-------------------------|
| \(x\)             | \(m \cdot n_q\)  | At the end of Q-rMinRank-Grover algorithm, the kernel is measured at qubit register \(x\). |
| \(t\)             | \(n_q\)          | The qubit register \(t\) is ancillary qubit register. The multiplication of the element of input matrix and qubit register \(x\) is stored in the qubit register \(t\). |
| \(e\)             | \(n \cdot n_q\)  | The sum of multiplications stored in qubit registers \(t\), which is the sum of multiplications of each row of input matrix and qubit register \(x\), is stored in qubit register \(e\). |
| \(b\)             | 1                | The qubit \(b\) is reversed when the qubit register \(e\) is all 0. |
Q-rMinRank attack: The first quantum approach for key recovery attacks on Rainbow

**Figure 10:** The Q-rMinRank oracle circuit for the Q-rMinRank attack.

**Figure 11:** The quantum multiplier over GF($2^n$)

**Figure 12:** The dagger circuit of the quantum multiplier over GF($2^n$)

**Figure 13:** The quantum adder over GF($2^n$)

as the control qubits to check whether all the $e$ are 0 and then to reverse the qubit $b$. The dagger circuit of the FrontOracle is applied to initialize the quantum registers $e$, at the end of the Q-rMinRank oracle circuit. These processes increase the probability that the kernel is measured. If the quantum registers $x$ are measured after the Q-rMinRank Oracle and diffusion circuit pair is repeated $\sqrt{2^{(q_2-q_1)n_q}}$ times, the kernel vector $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is measured with a high probability. After that, the kernel vector $\Gamma$ is transferred to the key recovery step. Table 4 shows the quantum circuits used in our Q-rMinRank Oracle.
Algorithm 5 The Q-rMinRank_Oracle circuit

Input: $(n \times m)$ matrix $P$, $m \cdot n_q$-qubit registers $x$, $n_q$-qubit register $t$, $n \cdot n_q$-qubit registers $e$, qubit $b$

\begin{align*}
1: & \text{for } i = 0 \text{ to } n - 1 \text{ do} \\
2: & \text{for } j = 0 \text{ to } m - 1 \text{ do} \\
3: & \text{Multiply}(x_j, P_{i,j}, t) \\
4: & \text{Add}(t, e_i) \\
5: & \text{Multiply_dagger}(x_j, P_{i,j}, t) \\
6: & \text{end for} \\
7: & \text{for } j = 0 \text{ to } n_q - 1 \text{ do} \\
8: & X(e_{ij}) \\
9: & \text{end for} \\
10: & \text{end for} \\
11: & (n \cdot n_q)-\text{controlled-NOT} (e_{00} \sim e_{(n-1)(n_q-1)}, b) \\
12: & \text{for } i = n - 1 \text{ to } 0 \text{ do} \\
13: & \text{for } j = n_q - 1 \text{ to } 0 \text{ do} \\
14: & X(e_{ij}) \\
15: & \text{end for} \\
16: & \text{end for} \\
17: & \text{Multiply}(x_j, P_{i,j}, t) \\
18: & \text{Add}(t, e_i) \\
19: & \text{Multiply_dagger}(x_j, P_{i,j}, t) \\
20: & \text{end for} \\
21: & \text{end for}
\end{align*}

Table 4: The quantum circuits used in our Q-rMinRank_Oracle

| Quantum circuits       | Operations of the quantum circuits                                      |
|------------------------|-------------------------------------------------------------------------|
| $H(\cdot)$             | Hadamard gate which brings the input qubit in superposition state       |
| $X(\cdot)$             | NOT gate which reverses the input qubit                                  |
| Multiply(\cdot)        | a quantum multiplier over $GF(q)$ [17]                                  |
| Add(\cdot)             | a quantum adder over $GF(q)$ [18]                                      |
| Multiply_dagger(\cdot) | a dagger circuit of Multiply(\cdot)                                    |
| $(n \cdot n_q)$-controlled-NOT(\cdot) | controlled-NOT gate using $(n \cdot n_q)$ qubits as the control qubits |

3.3 The Key Recovery Step

After the kernel vector $\Gamma$ is obtained in the quantum kernel extraction step, the key recovery step recovers the central map of the Rainbow by computing the linear combination $C$ of $m$ public keys with $\Gamma$ again as follows:

$$C \leftarrow \sum_{i=v_1+1}^{n} \gamma_i \cdot P^{(i)} \quad (7)$$

If the rank of linear combination $C$ is less than $v_2$ (the number of vinegar variables of the second layer), our Q-rMinRank attack outputs the linear combination $C$ and recovers the central map $F (\equiv C)$ in the key recovery step. Otherwise, the preprocessing and quantum kernel extraction steps are repeated until the rank of linear combination $C$ is less than or equal to $v_2$.

4 Complexity Analysis

In this section, we analyze the security of Rainbow based on our Q-rMinRank attack. First, we evaluate the security level of Rainbow by calculating the Q-rMinRank attack’s complexity for each parameter set of third-round Rainbow. Then, we analyze the quantum resources required for the Q-rMinRank attack.
4.1 Security Level Analysis to the Parameter Sets of Rainbow

The Rainbow signature scheme has three security levels with each parameter set shown in Table 2. Its relationship with the NIST security levels is shown in Table 5.

Table 5: Description of the NIST security categories and relation with the Rainbow security levels

| Security level | Description | Security complexity | Parameters of Rainbow |
|----------------|-------------|---------------------|-----------------------|
| N-I            | hard to break AES128 | $2^{128}$          | I (2^4, 36, 32, 32)   |
| N-II           | hard to break SHA256  | $2^{128}$          |                       |
| N-III          | hard to break AES192  | $2^{192}$          | III (2^8, 68, 32, 48) |
| N-IV           | hard to break SHA384  | $2^{192}$          |                       |
| N-V            | hard to break AES256  | $2^{256}$          | V (2^8, 96, 36, 64)   |

In the Q-rMinRank attack, the public keys of Rainbow are converted to $(n \times m)$ matrices using polar forms. This conversion method changes the instances of the MinRank problem [11]. Table 6 shows the differences between the MinRank attack and the Rectangular MinRank attack.

Table 6: Comparison of the new instances with the known instances of the MinRank problem

| Elements of MinRank problem | Known instances | New instances |
|-----------------------------|-----------------|---------------|
| Size of matrices           | $(n \times n)$  | $(n \times m)$|
| Number of matrices          | $o_2 + 1$       | $n - o_2 + 1$ |
| Rank of linear combination  | $m$             | $o_2$         |

Because instance $o_2 + 1$ is converted to $n - o_2 + 1$, $n - v_1 - o_1$ is converted to $n - o_2 (n = v_1 + o_1 + o_2)$. Therefore, $v_1$ is converted to $o_2 - o_1$ due to the conversion of the public keys. In this case, the kernel’s size is changed from $v_1$ to $o_2 - o_1$. The number of queries repeating the Q-rMinRank oracle and diffusion gate pair is $\frac{n}{r} \sqrt{2^r}$ according to [16] when the number of qubits to find is $x$. And thus, the number of queries in our Q-rMinRank attack with the kernel size $o_2 - o_1$ is $\frac{n}{r} \sqrt{2^{(o_2 - o_1)m_q}}$, where $n_q$ is the required qubit per one element of the kernel vector. In the case of parameter set I of Rainbow, where the number of oil variables in the first layer $o_1$ and the number of oil variables in the second layer $o_2$ is the same, the attack complexity becomes $O(1)$. Similarly, when the instances of parameter sets III and V are substituted, the attack complexity for each parameter set becomes $2^{64}$ and $2^{112}$.

Figure 14 shows the comparison of attack complexities among the MinRank attack [9], the rectangular MinRank attack [12], and our Q-rMinRank attack. The attack complexities of our attack on Rainbow specified in Figure 14 are calculated based on the complexity equation. All of the parameter sets of Rainbow fall short of the 128-bit security level, the minimum security requirement set out by NIST, against our Q-rMinRank attack. It means that Rainbow is no longer secure in quantum computing environments.

4.2 Quantum Resources

For quantum resource analysis, we estimate the number of qubits and and quantum gates required to perform our Q-rMinRank-Grover algorithm.

When the parameters of Rainbow are $(v_1, o_1, o_2)$, our Q-rMinRank-Grover algorithm for the Q-rMinRank attack requires total $(n + o_1 + o_2 + 1)n_q + 1$ qubits including $m = o_1 + o_2$ $n_q$-qubit registers $x$, an $n_q$-qubit register $t$, $n n_q$-qubit registers $e$, and a qubit $b$ as shown in Table 3.

The Q-rMinRank-Grover algorithm consists of Q-rMinRank oracle and diffusion circuits. The diffusion quantum circuit already proposed to increase the probability that the solution is measured requires $2n \cdot n_q$ NOT gates, $2n \cdot n_q + 2$ Hadamard gates, and an $(o_2 - o_1)n_q - 1)$-controlled-NOT gate. Our Q-rMinRank oracle quantum circuit executes Multiply($\cdot$) $2nm$ times, Multiply_dagger($\cdot$) $2nm$ times, and Add($\cdot$) $2nm$ times, and requires extra $2n \cdot n_q$ NOT gates and an $(n \cdot n_q)$-controlled-NOT gate. Multiply($\cdot$) and Multiply_dagger($\cdot$) require $n_q^2$ CNOT gates, and Add($\cdot$) requires $n_q$ CNOT gates. Therefore, our Q-rMinRank oracle quantum circuit requires total $2n \cdot n_q$ NOT gates, $4n((o_1 + o_2)n_q + 1)n_q$ CNOT gates, and an $(n \cdot n_q)$-controlled-NOT gates as shown in Table 4.
Figure 14: Comparison of security levels among the MinRank attack [9], the rectangular MinRank attack [11], the improved rectangular MinRank attack [12], and our Q-rMinRank attack.

Table 7: The number of quantum gates required for our Q-rMinRank oracle quantum circuit for the Q-rMinRank attack

| Gates Type               | Equation                              |
|-------------------------|---------------------------------------|
| NOT Gate                | $2n \cdot n_q$                        |
| CNOT Gate               | $4n((o_1 + o_2)n_q + 1)n_q$            |
| $(n \cdot n_q)$-controlled-NOT Gate | $1$                                    |

Finally, considering the number of repetitions of the Q-rMinRank oracle and diffusion pairs, there needs to be $\pi \cdot n \cdot n_q \sqrt{2(o_2 - o_1)n_q}$ NOT gates, $(o_2 - o_1)n_q + \frac{3}{8}\sqrt{2(o_2 - o_1)n_q}(n \cdot n_q + 1)$ Hadamard gates, $n\pi((o_1 + o_2)n_q + 1)\sqrt{2(o_2 - o_1)n_q}$ CNOT gates, $\frac{3}{8}\sqrt{2(o_2 - o_1)n_q}((o_2 - o_1)n_q - 1)$-controlled-NOT gates, and $\frac{3}{8}\sqrt{2(o_2 - o_1)n_q}(n \cdot n_q)$-controlled-NOT gates for our Q-rMinRank-Grover algorithm. Table 8 shows the number of qubits and quantum gates required for the Q-rMinRank attack considering the number of repetitions and variables for each parameter set of Rainbow.

Table 8: A comparison of quantum resources for our Q-rMinRank attack

| Parameter Set ID | The number of qubits | The kinds of quantum gates | The number of controlled qubits | The number of gates |
|------------------|----------------------|---------------------------|-------------------------------|--------------------|
| I                | 661                  | H                         | NOT                          | $3.78 \times 10^{24}$ |
|                  |                      | H                         | $(o_2 - o_1)n_q - 1$         | $1.93 \times 10^{27}$ |
|                  |                      | CNOT                      | $(o_2 - o_1)n_q - 1$         | $\frac{272}{2^{72}}$ |
| III              | 1832                 | H                         | NOT                          | $1.79 \times 10^{85}$ |
|                  |                      | H                         | $(o_2 - o_1)n_q - 1$         | $2.30 \times 10^{88}$ |
|                  |                      | CNOT                      | $(o_2 - o_1)n_q - 1$         | $\frac{272}{2^{72}}$ |
| V                | 2376                 | H                         | NOT                          | $1.23 \times 10^{119}$ |
|                  |                      | H                         | $(o_2 - o_1)n_q - 1$         | $1.97 \times 10^{121}$ |
|                  |                      | CNOT                      | $(o_2 - o_1)n_q - 1$         | $\frac{384}{2^{384}}$ |
4.3 Quantum Simulation Result

To show the feasibility of our Q-rMinRank attack, we implemented the quantum circuit of our Q-rMinRank-Grover algorithm to find a kernel of the following matrix over $GF(8)$ as a toy example of our model using quantum simulators Qiskit [13] and ProjectQ [14].

\[
\begin{bmatrix}
5 & 2 \\
7 & 1 \\
3 & 4 \\
\end{bmatrix}
\]  

(8)

ProjectQ, an open-source software effort for quantum computing, provides a function of drawing quantum circuits. Appendix 5 shows our Q-rMinRank oracle quantum circuit of the Q-rMinRank attack drawn using ProjectQ. Qiskit, another open-source framework for quantum computing, has the advantage of fast execution speed and supports multi-shot simulation that shows the frequency of measurement when executed several times. We measured the probability that our Q-rMinRank-Grover algorithm finds the kernel utilizing the multi-shot function of Qiskit. Figure 15 shows the output of implementing our Q-rMinRank-Grover algorithm circuit for the toy example in the Qiskit Aer simulator for 1000 shots.

![Figure 15: The histogram obtained by running a quantum circuit in Qiskit Aer simulator](image)

Figure 15 shows a histogram excluding qubit values measured less than 10 times in 1000 shots. Qubit values measured less than 10 times are not kernel values, and the sum of their probabilities is indicated in the bar named ‘NotKernel’. Other bars represent each of the probabilities when the kernels are accurately measured. For example, in our Q-rMinRank-Grover algorithm circuit, $x_0$, $x_1$, and $x_2$ are measured 120 times out of 1000 times, with kernel values of ‘001’, ‘101’, and ‘110’, respectively. Our Q-rMinRank-Grover algorithm finds the kernel with an accuracy of 91.4% probability.

5 Conclusion

In this paper, we first proposed a quantum rectangular MinRank (Q-rMinRank) attack that applied quantum algorithms to key recovery attacks against Rainbow, especially MinRank attacks. By designing a Q-rMinRank oracle circuit, the Q-rMinRank attack finds a kernel faster than existing MinRank attacks. Also, we evaluated the security level of Rainbow. The security levels of all parameter set I, III, and V of Rainbow are less than 128-bit security levels against our Q-rMinRank attack. Then, we estimated the quantum resources required for the Q-rMinRank attack. In order to recover the private keys of Rainbow using the Q-rMinRank attack, our quantum circuit requires 661, 1832, and 2376 qubits to the parameter sets I, III, and V of Rainbow. Finally, we demonstrated the reliability of the Q-rMinRank attack by implementing our quantum circuit using quantum simulators Qiskit and ProjectQ. Our Q-rMinRank-Grover algorithm finds the kernel with an accuracy of 91.4% probability.

If a roughly 2000-qubit quantum computer can be used, it can break the Rainbow by performing the Q-rMinRank attack algorithm. It means that the Rainbow can be broken even using half as many qubits as RSA-2048 bits break. Thus, we showed that Rainbow is no longer a PQC algorithm.
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Q-rMinRank-Grover algorithm circuit for the toy example of Section 4.3

Figure 16: Q-rMinRank-Grover algorithm quantum circuit for toy example of Section 4.3 when the Q-rMinRank oracle and diffusion circuits pairs repeat 1