ORTHOGONAL GEODESIC CHORDS, BRAKE ORBITS AND HOMOCLINIC ORBITS IN RIEMANIAN MANIFOLDS

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ABSTRACT. The study of solutions with fixed energy of certain classes of Lagrangian (or Hamiltonian) systems is reduced, via the classical Maupertuis–Jacobi variational principle, to the study of geodesics in Riemannian manifolds. We are interested in investigating the problem of existence of brake orbits and homoclinic orbits, in which case the Maupertuis–Jacobi principle produces a Riemannian manifold with boundary and with metric degenerating in a non trivial way on the boundary. In this paper we use the classical Maupertuis–Jacobi principle to show how to remove the degeneration of the metric on the boundary, and we prove in full generality how the brake orbit and the homoclinic orbit multiplicity problem can be reduced to the study of multiplicity of orthogonal geodesic chords in a manifold with regular and strongly concave boundary.

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1. INTRODUCTION

The study of periodic and homoclinic orbits of Lagrangian and Hamiltonian systems is an extremely active research field in classical and modern mathematics, having a huge number of applications in physical sciences. One of the peculiarities of the problem is that, although already very popular among classical analysts and geometers, it has never been out of fashion, and it has been studied along the time with techniques of an increasing level of sophistication. Indeed, the study of solutions of Hamiltonian systems has motivated...
many recent developments of several mathematical theories, including Calculus of Variations, Symplectic Geometry and Morse Theory, among others, and the vast literature on the topic witnesses the leading role of the subject in modern mathematics.

The central interest of the present paper is to study solutions of an autonomous Lagrangian (or Hamiltonian) system, having prescribed energy, in a manifold $M$ that belong to two special classes of solutions: the homoclinic orbits and the brake orbits. Homoclinic orbits are solutions $x : \mathbb{R} \to M$ of the system for which the limits $\lim_{t \to \pm \infty} x(t)$ and $\lim_{t \to \pm \infty} \dot{x}(t)$ exist and are equal, and $\lim_{t \to \pm \infty} \dot{x}(t) = 0$. Such limits must then be a critical point of the potential function of the system. Brake orbits are a special class of periodic solutions that have an oscillating character, i.e., periodic solutions $x : \mathbb{R} \to M$ having period $2T$, with $x(T + t) = x(T - t)$ and $\dot{x}(T + t) = -\dot{x}(T - t)$ for all $t \in \mathbb{R}$. Clearly, $\dot{x}(kT) = 0$ for all $k \in \mathbb{Z}$.

By a classical variational principle, known as the Maupertuis–Jacobi principle, solutions of autonomous Lagrangian or Hamiltonian systems having a fixed value of the energy correspond to geodesics relatively to a Riemannian metric, called the Jacobi metric. When dealing with homoclinic orbits issuing from a critical point of the potential function, or with brake orbits, then the classical formulation of the Maupertuis–Jacobi principle fails, due to the fact that such solutions pass through a region where the Jacobi metric degenerates in a non-trivial way. An accurate analysis of the geodesic behavior near such degeneracies, that occur on the boundary of the level set of the potential function, has lead many authors to obtain existence results by perturbation techniques. More specifically, following an original idea by Seifert [11], some authors (see [2]) have been able to perform a geometrical construction consisting in attaching a smooth, convex and sufficiently small collar (see Figure 1) to the degenerate region, in such a way that the geodesics in the resulting manifold could be counted by standard techniques in convex Riemannian geometry ([3, 9]). Then, a limit argument was used to obtain existence results for geodesics in the original degenerate metric by letting the size of the collar go to zero. The same idea cannot be used if one wants to obtain multiplicity results, due to the fact that such limit procedure does not guarantee that possibly distinct geodesics in the perturbed metric converge to geometrically distinct geodesics in the original Jacobi metric, unless one poses ad hoc "non resonance"
assumptions (see [7]). Here, by geometrically distinct, we mean geodesics having different images; the non resonance assumptions mentioned above guarantees that it is avoided the situation in which distinct geodesics in the perturbed metric tend to the same periodic geodesic travelled a different number of times.

The starting point of this paper is the idea that, if one wants to preserve the number of distinct geodesics, then one has to perform a geometrical construction that avoids limits procedure. Such construction would obviously be based on a careful investigation of the geodesic behavior near the boundary of the level set of the potential function. Working in this direction has lead to the quite remarkable observation that the boundary of a non critical level set of the potential function, or of a small ball around a non degenerate maximum point of the potential, are near certain hypersurfaces that are strongly concave relatively to the Jacobi metric, and that have the property that orthogonal geodesic chords arriving on one of these hypersurfaces can be uniquely extended to geodesic chords up to the degenerate boundary. The presence of concave hypersurfaces near the degenerate boundary can be interpreted as an indication that Seifert’s technique of gluing a convex collar would be somewhat unnatural in order to study the multiplicity problem in full generality.

The main results of this paper are contained in Theorem 5.9 relating the brake orbits problem to the orthogonal geodesic chords problem, and Theorem 5.19 that deals with the homoclinics problem.

The issue of concavity, as opposed to the convexity property used in the classical literature, is the key point to develop a multiplicity theory for brake orbits and homoclinic orbits under purely topological assumptions on the underlying manifolds. These multiplicity results constitute the topic of two forthcoming papers by the authors ([5, 6]).

2. GEODESICS AND CONCAVITY

Let \((M, g)\) be a smooth (i.e., of class \(C^2\)) Riemannian manifold with \(\dim(M) = m \geq 2\), let dist denote the distance function on \(M\) induced by \(g\); the symbol \(\nabla\) will denote the covariant derivative of the Levi-Civita connection of \(g\), as well as the gradient differential operator for smooth maps on \(M\). The Hessian \(H^f(g)\) of a smooth map \(f : M \to \mathbb{R}\) at a point \(q \in M\) is the symmetric bilinear form \(H^f(q)(v, w) = g((\nabla_v \nabla f)(q), w)\) for all \(v, w \in T_q M\); equivalently, \(H^f(q)(v, v) = \frac{d^2}{dt^2}k_0 f(\gamma(s))\), where \(\gamma : [-\varepsilon, \varepsilon] \to M\) is the unique (affinely parameterized) geodesic in \(M\) with \(\gamma(0) = q\) and \(\dot{\gamma}(0) = v\). We will denote by \(\|\cdot\|\) the covariant derivative along a curve, in such a way that \(\frac{d^2}{dt^2}k_0 = 0\) is the equation of the geodesics. A basic reference on the background material for Riemannian geometry is [4].

Let \(\Omega \subset M\) be an open subset; \(\overline{\Omega} = \Omega \cup \partial \Omega\) will denote its closure. There are several notions of convexity and concavity in Riemannian geometry, extending the usual ones for subsets of the Euclidean space \(\mathbb{R}^m\). In this paper we will use a somewhat concavity assumption for compact subsets of \(M\), that we will refer as "strong concavity" below, and which is stable by \(C^2\)-small perturbations of the boundary. Let us first recall the following:

**Definition 2.1.** \(\overline{\Omega}\) is said to be convex if every geodesic \(\gamma : [a, b] \to \overline{\Omega}\) whose endpoints \(\gamma(a)\) and \(\gamma(b)\) are in \(\Omega\) has image entirely contained in \(\Omega\). Likewise, \(\overline{\Omega}\) is said to be concave if its complement \(M \setminus \overline{\Omega}\) is convex.

If \(\partial \Omega\) is a smooth embedded submanifold of \(M\), let \(\mathbb{I}_n(x) : T_x(\partial \Omega) \times T_x(\partial \Omega) \to \mathbb{R}\) denote the second fundamental form of \(\partial \Omega\) in the normal direction \(n \in T_x(\partial \Omega)^\perp\). Recall that \(\mathbb{I}_n(x)\) is a symmetric bilinear form on \(T_x(\partial \Omega)\) defined by:

\[
\mathbb{I}_n(x)(v, w) = g(\nabla_v W, n), \quad v, w \in T_x(\partial \Omega),
\]
where $W$ is any local extension of $w$ to a smooth vector field along $\partial \Omega$.

**Remark 2.2.** Assume that it is given a smooth function $\phi : M \to \mathbb{R}$ with the property that $\Omega = \phi^{-1}([-\infty, 0])$ and $\partial \Omega = \phi^{-1}(0)$, with $d\phi \neq 0$ on $\partial \Omega$. The following equality between the Hessian $H^\phi$ and the second fundamental form $^2\mathbb{II}$ of $\partial \Omega$ holds:

\[
H^\phi(x)(v, v) = -^2\mathbb{II}_{\phi(x)}(x)(v, v), \quad x \in \partial \Omega, \ v \in T_x(\partial \Omega);
\]

Namely, if $x \in \partial \Omega$, $v \in T_x(\partial \Omega)$ and $V$ is a local extension around $x$ of $v$ to a vector field which is tangent to $\partial \Omega$, then $v(g(\nabla \phi, V)) = 0$ on $\partial \Omega$, and thus:

\[
H^\phi(x)(v, v) = v(g(\nabla \phi, V)) - g(\nabla \phi, \nabla_v V) = -^2\mathbb{II}_{\phi(x)}(x)(v, v).
\]

Note that the second fundamental form is defined intrinsically, while there is general no natural choice for a function $\phi$ describing the boundary of $\Omega$ as above.

**Definition 2.3.** We will say that that $\overline{\Omega}$ is strongly concave if $^2\mathbb{II}_n(x)$ is positive definite for all $x \in \partial \Omega$ and all inward pointing normal direction $n$.

**Remark 2.4.** Strong concavity is evidently a $C^2$-open condition. It should also be emphasized that if $\overline{\Omega}$ is strongly concave, then for any smooth map $\phi : M \to \mathbb{R}$ as in Remark 2.2, then for all $q \in \partial \Omega$, the Hessian $H^\phi(q)$ is negative definite on $T_q(\partial \Omega)$. From this observation, it follows immediately that geodesics starting tangentially to $\partial \Omega$ move inside $\Omega$.

The main objects of our study are geodesics in $M$ having image in $\overline{\Omega}$ and with endpoints orthogonal to $\partial \Omega$. We distinguish a special class of such geodesics, called "weak", whose relevance will not be emphasized in the present paper, but it will be used in a substantial way in the proof of the multiplicity results in [5,6].

**Definition 2.5.** A geodesic $\gamma : [a, b] \to M$ is called a geodesic chord in $\overline{\Omega}$ if $\gamma([-\infty, 0]) \subset \Omega$ and $\gamma(a), \gamma(b) \in \partial \Omega$; by a weak geodesic chord we will mean a geodesic $\gamma : [a, b] \to M$ with image in $\overline{\Omega}$ and endpoints $\gamma(a), \gamma(b) \in \partial \Omega$. A (weak) geodesic chord is called orthogonal if $\dot{\gamma}(a^-) \in (T_{\gamma(a)}\partial \Omega)^\perp$ and $\dot{\gamma}(b^-) \in (T_{\gamma(b)}\partial \Omega)^\perp$, where $\dot{\gamma}(\cdot^-)$ denote the lateral derivatives (see Figure 4). An orthogonal geodesic chord in $\overline{\Omega}$ whose endpoints belong to distinct connected components of $\partial \Omega$ will be called a crossing orthogonal geodesic chord in $\overline{\Omega}$.

For shortness, we will write OGC for “orthogonal geodesic chord” and WOGC for “weak orthogonal geodesic chord”.

For the proof of the multiplicity results in [5,6], we will use a geometrical construction that will work in a situation where one can exclude a priori the existence in $\overline{\Omega}$ of (crossing) weak orthogonal geodesic chords in $\partial \Omega$. We will now show that one does not lose generality in assuming that there are no such WOGC’s in $\overline{\Omega}$ by proving the following:

**Proposition 2.6.** Let $\Omega \subset M$ be an open set whose boundary $\partial \Omega$ is smooth and compact and with $\overline{\Omega}$ strongly concave. Assume that there are only a finite number of (crossing) orthogonal geodesic chords in $\overline{\Omega}$. Then, there exists an open subset $\Omega' \subset \Omega$ with the following properties:

1. $\overline{\Omega'}$ is diffeomorphic to $\overline{\Omega}$ and it has smooth boundary;
2. $\overline{\Omega'}$ is strongly concave;
3. For example one can choose $\phi$ such that $|\phi(q)| = \text{dist}(q, \partial \Omega)$ for all $q$ in a (closed) neighborhood of $\partial \Omega$.
4. Observe that, with our definition of $\phi$, then $\nabla \phi$ is a normal vector to $\partial \Omega$ pointing outwards from $\Omega$. 

Figure 2. A weak orthogonal geodesic chord (WOGC) in $\Omega'$ (above), and a crossing OGC (below).

(3) the number of (crossing) OGC’s in $\Omega'$ is less than or equal to the number of (crossing) OGC’s in $\Omega$;

(4) every (crossing) WOGC in $\Omega'$ is a (crossing) OGC in $\Omega'$.

Proof. The desired set $\Omega'$ will be taken of the form:

$$\Omega' = \phi^{-1}([-\infty, -\delta]),$$

with $\delta > 0$ small, and with $\phi$ a smooth map as in Remark 2.2 such that $|\phi(q)| = \text{dist}(q, \partial \Omega)$ for $q$ near $\partial \Omega$. Observe that if $\delta$ is small enough, then by continuity $d\phi \neq 0$ on $\phi^{-1}([-\delta, 0])$, which implies that $\partial \Omega'$ is smooth and that $\Omega'$ is diffeomorphic to $\Omega$, as we see using the integral curves of $\nabla \phi$. Since strong concavity is an open condition in the $C^2$-topology, if $\delta > 0$ is small enough then $\Omega'$ is strongly concave, proving (2).

Moreover, $\delta$ must be chosen small enough so that the exponential map gives a diffeomorphism from an open neighborhood of the zero section of the normal bundle of $\partial \Omega$ to the set $\phi^{-1}([-2\delta, 2\delta])$; the existence of such $\delta$ is guaranteed by our compactness assumption on $\partial \Omega$. Since $\phi(q) = -\text{dist}(q, \partial \Omega)$ near $\partial \Omega$, then every (crossing) geodesic in $\Omega'$ that arrives orthogonally at $\partial \Omega'$ can be smoothly extended to a (crossing) geodesic in $\Omega$ that arrives orthogonally at $\partial \Omega$; observe that any such extended geodesic only touches $\partial \Omega$ at the endpoints, i.e., it is a (crossing) OGC in $\Omega$. This proves part (3).

We claim that there exists $\delta > 0$ arbitrarily small such that every (crossing) WOGC is a (crossing) OGC in $\phi^{-1}([-\infty, -\delta])$. Assume on the contrary that there exists a sequence $\delta_n > 0$ with $\delta_n \to 0$ as $n \to \infty$, a sequence $0 < s_n < 1$ and a sequence of (crossing) geodesics $\gamma_n : [0, 1] \to \Omega$ with $\phi(\gamma_n(0)) = \phi(\gamma_n(s_n)) = \phi(\gamma_n(1)) = -\delta_n$, $\dot{\gamma}_n(0)$ and $\dot{\gamma}_n(1)$ orthogonal to $\phi^{-1}(-\delta_n)$ and $\phi(\gamma_n(s)) \leq -\delta_n$ for all $s \in [0, 1]$ and all $n \in \mathbb{N}$. As we have observed, for $n$ large each geodesic $\gamma_n$ can be smoothly extended to a (crossing) OGC in $\Omega$, and clearly all such extensions cannot make a finite set of geometrically distinct (crossing) OGC’s in $\Omega$. Namely, each $\gamma_n$ is tangent to the surface $\phi^{-1}(-\delta_n)$, and to no other surface of the form $\phi^{-1}(-\delta)$ with $\delta < \delta_n$. This says that the extensions of the $\gamma_n$ are all geometrically distinct, which contradicts the fact that there is only a finite number of (crossing) OGC’s in $\Omega'$ and proves part (4). \qed
3. Brake and Homoclinic Orbits of Hamiltonian Systems

Let $p = (p_i), q = (q^i)$ be coordinates on $\mathbb{R}^{2m}$, and let us consider a natural Hamiltonian function $H \in C^2(\mathbb{R}^{2m}, \mathbb{R})$, i.e., a function of the form

$$
H(p, q) = \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(q)p_ip_j + V(q),
$$

where $V \in C^2(\mathbb{R}^m, \mathbb{R})$ and $A(q) = (a^{ij}(q))$ is a positive definite quadratic form on $\mathbb{R}^m$:

$$
\sum_{i,j=1}^{m} a^{ij}(q)p_ip_j \geq \nu(q)|q|^2
$$

for some continuous function $\nu : \mathbb{R}^m \to \mathbb{R}^+$ and for all $(p, q) \in \mathbb{R}^{2m}$.

The corresponding Hamiltonian system is:

$$
\begin{align*}
\dot{p} &= -\frac{\partial H}{\partial q} \\
\dot{q} &= \frac{\partial H}{\partial p}
\end{align*}
$$

where the dot denotes differentiation with respect to time.

For all $q \in \mathbb{R}^m$, denote by $\mathcal{L}(q) : \mathbb{R}^m \to \mathbb{R}^m$ the linear isomorphism whose matrix with respect to the canonical basis is $(a_{ij}(q))$, the inverse of $(a^{ij}(q))$; it is easily seen that, if $(p, q)$ is a solution of class $C^1$ of (3.2), then $q$ is actually a map of class $C^2$ and

$$
p = \mathcal{L}(q)\dot{q}.
$$

With a slight abuse of language, we will say that a $C^2$-map $q : I \to \mathbb{R}^m$ is a solution of (3.2) if $(p, q)$ is a solution of (3.2) where $p$ is given by (3.3). Since the system (3.2) is autonomous, i.e., time independent, then the function $H$ is constant along each solution, and it represents the total energy of the solution of the dynamical system. There exists a large amount of literature concerning the study of periodic solutions of autonomous Hamiltonian systems having energy $H$ prescribed (see for instance [8] and the references therein).

We will be concerned with a special kind of periodic solutions of (3.2), called brake orbits. A brake orbit for the system (3.2) is a non constant periodic solution $p : t \mapsto (p(t), q(t)) \in \mathbb{R}^{2m}$ of class $C^2$ with the property that $p(0) = p(T) = 0$ for some $T > 0$. Since $H$ is even in the variable $p$, a brake orbit $(p, q)$ is $2T$-periodic, with $p$ odd and $q$ even about $t = 0$ and about $t = T$. Clearly, if $E$ is the energy of a brake orbit $(p, q)$, then $V(q(0)) = V(q(T)) = E$.

The link between solutions of brake orbits and orthogonal geodesic chords is obtained in Theorem 5.9 (used in [6] to obtain the multiplicity result for brake orbits). Its proof is based on a well known variational principle, that relates solutions of (3.2) having prescribed energy $E$ with curves in the open subset $\Omega_E \subset \mathbb{R}^m$:

$$
\Omega_E = V^{-1}(]-\infty, E[) = \{ x \in \mathbb{R}^m : V(x) < E \}
$$

endowed with the Jacobi metric (see Proposition 4.1):

$$
g_E(x) = (E - V(x)) \cdot \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \, dx^i \, dx^j.
$$

Let us now consider the problem of homoclinics on a Riemannian manifold $(M, g)$. 

Assume that we are given a map $V \in C^2(M, \mathbb{R})$; the corresponding second order Hamiltonian system is the equation:

$$\frac{D}{dt} \dot{q} + \nabla V(q) = 0. \quad (3.6)$$

Note that if $M = \mathbb{R}^m$ and $g$ is the Riemannian metric

$$g = \frac{1}{2} \sum_{i,j=1}^m a_{ij}(x) \, dx^i \, dx^j, \quad (3.7)$$

where the coefficients $a_{ij}$ are as above, then equation (3.6) is equivalent to (3.2), in the sense that $x$ is a solution of (3.6) if and only if the pair $q = x$ and $p = \mathcal{L}(x)\dot{x}$ is a solution of (3.2).

Let $x_0 \in M$ be a critical point of $V$, i.e., such that $\nabla V(x_0) = 0$. We recall that a homoclinic orbit for the system (3.6) emanating from $x_0$ is a solution $q \in C^2(\mathbb{R}, M)$ of (3.6) such that:

$$\lim_{t \to -\infty} q(t) = \lim_{t \to +\infty} q(t) = x_0, \quad (3.8)$$

$$\lim_{t \to -\infty} \dot{q}(t) = \lim_{t \to +\infty} \dot{q}(t) = 0. \quad (3.9)$$

To the authors’ knowledge, the only result available in the literature on multiplicity of homoclinics in the autonomous case is due to Ambrosetti and Coti–Zelati [11], to Rabinowitz [10] and to Tanaka [12]. A quite general multiplicity result for homoclinics, generalizing those in [11] and in [12], will be given in [5] using the result of Theorem 5.19.

It should also be mentioned that very likely all the results in this paper can be extended to the case of Hamiltonian functions $H$ more general than (3.1). As observed by Weinstein in [13], Hamiltonians that are positively homogeneous in the momenta lead to Finsler metrics rather than Riemannian metrics.

4. The Maupertuis Principle

Throughout this section, $(M, g)$ will denote a Riemannian manifold of class $C^2$; all our constructions will be made in suitable (relatively) compact subsets of $M$, and for this reason it will not be restrictive to assume, as we will, that $(M, g)$ is complete.

4.1. The variational framework. The symbol $H^1([a, b], \mathbb{R}^m)$ will denote the Sobolev space of all absolutely continuous function $f : [a, b] \to \mathbb{R}^m$ whose weak derivative is square integrable. Similarly, $H^1([a, b], M)$ will denote the infinite dimensional Hilbert manifold consisting of all absolutely continuous curves $x : [a, b] \to M$ such that $\varphi \circ x|_{[c,d]} \in H^1([c, d], \mathbb{R}^m)$ for all chart $\varphi : U \subset M \to \mathbb{R}^m$ of $M$ such that $x([c, d]) \subset U$. By $H^1_{bc}(]a, b], \mathbb{R}^m)$ we will denote the vector space of all continuous maps $f : [a, b] \to \mathbb{R}^m$ such that $f|_{[c,d]} \in H^1([c, d], \mathbb{R}^m)$ for all $[c, d] \subset ]a, b]$; the set $H^1_{bc}(]a, b], M)$ is defined similarly. The Hilbert space norm of $H^1([a, b], \mathbb{R}^m)$ will be denoted by $\| \cdot \|_{a,b}$; for the purposes of this paper it will not be necessary to make the choice among equivalent norms of $H^1([a, b], \mathbb{R}^m)$.

4.2. The Maupertuis–Jacobi principle for brake orbits. Let $V \in C^2(M, \mathbb{R})$ and let $E \in \mathbb{R}$. Consider the sublevel $\Omega_E$ of $V$ in $\{3.4\}$ and the Maupertuis integral $f_{a,b} : H^1([a, b], \Omega_E) \to \mathbb{R}$, which is the geodesic action functional relative to the metric $g_E$ $\{3.5\}$, given by:

$$f_{a,b}(x) = \frac{1}{2} \int_a^b (E - V(x)) \, g(\dot{x}, \dot{x}) \, dt, \quad (4.1)$$
where $g$ is the Riemannian metric $\Omega^m$. Observe that the metric $g_E$ degenerates on $\partial \Omega_E$.

The functional $f_{a,b}$ is smooth, and its differential is readily computed as:

$$
(4.2) \quad df_{a,b}(x)W = \int_a^b \left( E - V(x) \right) g(\dot{x}, \frac{D}{dt}W) \, dt - \frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) g(\nabla V(x), W) \, dt,
$$

where $W \in H^1([a, b], \mathbb{R}^m)$. The corresponding Euler–Lagrange equation of the critical points of $f_{a,b}$ is

$$
(4.3) \quad \left( E - V(x(s)) \right) \frac{D}{dt} \dot{x}(s) - g(\nabla V(x(s)), \dot{x}(s)) = 0, 
$$

for all $s \in [a, b]$.

Solutions of the Hamiltonian system having fixed energy $E$ and critical points of the functional $f_{a,b}$ of (4.1) are related by the following variational principle, known in the literature as the Maupertuis–Jacobi principle:

**Proposition 4.1.** Assume that $E$ is a regular value of the function $V$.

Let $x \in C^0([a, b], \mathbb{R}^m) \cap H^1([a, b], \mathbb{R}^m)$ be a non constant curve such that

$$
(4.4) \quad \int_a^b \left( E - V(x) \right) g(\dot{x}, \frac{D}{dt}W) \, dt - \frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) g(\nabla V(x), W) \, dt = 0
$$

for all $W \in C_0^\infty([a, b], \mathbb{R}^m)$, and such that:

$$
(4.5) \quad V(x(s)) < E, \quad \text{for all } s \in [a, b];
$$

and

$$
(4.6) \quad V(x(a)), V(x(b)) \leq E.
$$

Then, $x \in H^1([a, b], \mathbb{R}^m)$, and if $V(x(a)) = V(x(b)) = E$, it is $x(a) \neq x(b)$. Moreover, in the above situation, there exist positive constants $c_x$ and $T$ and a $C^1$-diffeomorphism $\sigma : [0, T] \rightarrow [a, b]$ such that:

$$
(4.7) \quad \left( E - V(x) \right) g(\dot{x}, \dot{x}) \equiv c_x \quad \text{on } [a, b],
$$

and, setting $q = x \circ \sigma : [0, T] \rightarrow \mathbb{R}^m$, and $p(s) = L(q(s))\dot{q}(s)$, the pair $(q, p) : [0, T] \rightarrow \mathbb{R}^{2m}$ is a solution of (4.2) having energy $E$ with $q(0) = x(a)$, $q(T) = x(b)$. If $V(x(a)) = V(x(b)) = E$ then $q$ can be extended to a $2T$-periodic brake orbit of (4.2).

**Proof.** A proof when $L$ is the identity map $\text{id}$ can be found for instance in [2]. For convenience of the reader we give here a sketch of the proof in the general case.

Since $x$ satisfies (4.4), standard regularization arguments show that $x$ is of class $C^2$ on $[a, b]$, while integration by parts gives (4.3) $\forall s \in [a, b]$. Equation (4.5) follows contracting both sides of (4.3) with $\dot{x}$ using $g$. Now set

$$
(4.8) \quad t(s) = \frac{1}{2} \int_a^s \frac{c_x}{E - V(x(\tau))} \, d\tau.
$$

A simple estimate shows that $T = t(b) < +\infty$. Indeed, setting

$$
C = \sup \{ g(\nabla V(x), \nabla V(x))^{1/2} : x \in \overline{\Omega_E} \},
$$

and using (4.2), one has

$$
\left| \frac{d}{ds} \left( \frac{1}{E - V(x(s))} \right) \right| \leq \frac{C g(\dot{x}, \dot{x})^{1/2}}{(E - V(x))^2} = \frac{C \sqrt{c_x}}{(E - V(x))^{5/2}}.
$$
Therefore, standard estimates for ordinary differential equations gives the existence of a constant $D_\epsilon$ such that
\[
\frac{1}{E - V(x(s))} \leq D_\epsilon \left( \frac{1}{(s-a)^{2/3}} + \frac{1}{(b-s)^{2/3}} \right), \quad \forall s \in [a,b],
\]
proving that $t(b) < +\infty$ and that $x \in H^1([a,b], \mathbb{R}^m)$.

Now, denote by $\sigma : [0, T] \to [a, b]$ the inverse map of $\alpha_3$, and set $q(t) = x(\sigma(t))$. Since $\sigma'(t) = 2(c_\epsilon)^{-1}(E - V(x(\sigma(t))))$, a straightforward computation shows that $\frac{D}{dt} \dot{q} = -\nabla V(q)$ and $\frac{1}{2}g(\dot{q}, \dot{q}) + V(q) = E$. Therefore, the pair $(q, \xi(q)\dot{q} : [0, T] \to \mathbb{R}^{2m}$ is a solution of (3.3) with energy $E$.

Moreover $q(0) = x(a)$ and $q(T) = x(b)$, and by the uniqueness of the Cauchy problem, if $V(x(a)) = V(x(b)) = E$ it must be $q(0) \neq q(T)$, and $q$ can be extended to a periodic brake orbit. \(\square\)

\subsection{The Maupertuis–Jacobi Principle near a nondegenerate maximum of the potential energy} The above formulation of the Maupertuis–Jacobi principle is not suited to study homoclinic orbits issuing from a critical point of the potential function $V$. Our next goal is to establish an extension of the principle that will be applied in this situation.

\begin{proposition}
Let $(M, g)$ be a Riemannian manifold, $V \in C^2(M, \mathbb{R})$, let $x_0 \in M$ be a nondegenerate maximum of $V$, and set $E = V(x_0)$. Assume that $x$ is a curve in the set $C^0([a,b], \overline{\Omega_E}) \cap H^1_{loc}([a,b], \overline{\Omega_E})$ such that:
\begin{equation}
\int_a^b (E - V(x)) g(\dot{x}, \frac{D}{dt}W) \, dt - \frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) g(\nabla V(x), W) \, dt = 0
\end{equation}
for all $W \in C_0^\infty([a,b], \mathbb{R}^m)$, and such that
\begin{align}
V(x(s)) < E, & \text{ for } s \in [a,b]; \\
x(b) = x_0.
\end{align}
Then, there exists a $C^1$-diffeomorphism $\sigma : [0, +\infty[ \to [a,b]$ such that the curve $q = x \circ \sigma$ is a solution of (3.6) satisfying $q(0) = x(a)$ and $\lim_{t \to +\infty} q(t) = x_0$, $\lim_{t \to +\infty} \dot{q}(t) = 0$.
\end{proposition}

\begin{proof}
Choose $g \in ]0, \text{dist}(x(a), x_0)]$ and define $\alpha_1 \in [a,b]$ as the first instant $s$ at which $\text{dist}(x(s), x_0) = g$. By (4.9), the restriction $x|_{[a,\alpha_1]}$ is a geodesic relatively to the metric $g_E$, since $x([a,\alpha_1])$ is contained in a region where $E - V$ is positive. Denote by $c_x$ the constant value of $(E - V(x))g(\dot{x}, \dot{x})$; for all $s \in [a, \alpha_1]$ set:
\[
t(s) = \frac{1}{2} \int_a^s \frac{c_x}{E - V(x(\tau))} \, d\tau
\]
and denote by $\sigma : [0, t(\alpha_1)] \to [a, \alpha_1]$ the inverse function of $s \mapsto t(s)$. Then, a straightforward calculations shows that the map $q = x \circ \sigma$ is a solution of the equation (3.6) with $\frac{1}{2}g(\dot{q}, \dot{q}) + V(q) \equiv E$ on $[0, s(\alpha_1)]$.

Let us choose $\alpha_2 \in ]\alpha_1, b]$ be the first instant $s$ at which $\text{dist}(x(s), x_0) = \frac{g}{2}$; we can repeat the construction above obtaining a solution $q_*$ of (3.6) defined on an interval $[0, t(\alpha_2)]$. The key observation here is that, in fact, such a function $q_*$ is an extension of $q$, and therefore it satisfies the same conservation law $\frac{1}{2}g(\dot{q}_*, \dot{q}_*) + V(q_*) \equiv E$ on $[0, t(\alpha_2)]$. An iteration of this construction produces a sequence $a < \alpha_1 < \alpha_2 < \ldots < b$
such that \( \text{dist}(x(\alpha_k), x_0) = \frac{1}{\nu(k)} \), maps of class \( C^1 \), \( t : [a, L] \to [0, T] \), its inverse \( \sigma : [0, T] \to [a, L] \), where:

\[
T = \frac{1}{2} \int_a^L \frac{c_x}{E - V(x(\tau))} \, d\tau \in [0, +\infty], \quad L = \lim_{k \to \infty} \alpha_k \in [a, b],
\]

and a curve of class \( C^2 \), \( q = x \circ \sigma : [0, T] \to \Omega_E \), that satisfies (3.6), and with

\[
(4.12) \quad \frac{1}{2} g(\dot{q}, \dot{q}) + V(q) \equiv E
\]
on \([0, T]\); in particular, \( g(\dot{q}, \dot{q}) \) is bounded.

Let us prove that \( T = +\infty \) and that \( \lim_{t \to +\infty} q(t) = x_0 \). We know that, by construction, \( \lim_{k \to \infty} t(\alpha_k) = T \) and \( \lim_{k \to \infty} q(t(\alpha_k)) = x_0 \); suppose by absurd that there exists \( \bar{\nu} > 0 \), and a sequence \( \beta_k \) such that \( \lim_{k \to \infty} \beta_k = L \) and \( \text{dist}(q(t(\beta_k)), x_0) \geq \bar{\nu} \) for all \( k \). Since \( x_0 \) is an isolated maximum point, we can assume \( \bar{\nu} \) small enough so that

\[
(4.13) \quad \inf_{\frac{1}{2} \bar{\nu} \leq \text{dist}(q, x_0) \leq \bar{\nu}} (E - V(Q)) \equiv \bar{\epsilon} > 0.
\]

Up to subsequences, we can obviously assume that \( \beta_k \in ]\alpha_k, \alpha_{k+1}] \) for all \( k \); for \( k \) sufficiently large, there exists \( \gamma_k \in ]\alpha_k, \beta_k[ \) which is the first instant \( t \in ]\alpha_k, \beta_k[ \) at which \( \text{dist}(q(s(t)), x_0) = \frac{\bar{\nu}}{2} \). Since \( g(\dot{q}, \dot{q}) \) is bounded, there exists \( \nu > 0 \) such that

\[
(4.14) \quad t(\gamma_k) - t(\alpha_k) \geq \nu, \quad \text{for all } k;
\]

from (4.13) and (4.14) we get:

\[
(4.15) \quad \int_0^{t(\alpha_{N+1})} (E - V(q(\tau))) \, d\tau \geq \sum_{k=1}^N \int_{t(\alpha_k)}^{t(\gamma_k)} (E - V(q(\tau))) \, d\tau \geq \sum_{k=1}^N \bar{\epsilon} \nu = N \bar{\epsilon} \nu \to +\infty
\]
as \( N \to \infty \). On the other hand, for all \( s \in ]a, L[ \),

\[
\int_0^{t(s)} (E - V(q(\tau))) \, d\tau = \frac{1}{2} \int_a^s c_x \, d\theta = \frac{1}{2} \frac{b - a}{c_x},
\]

which is obviously inconsistent with (4.15), and therefore proves that \( \lim_{t \to T^-} q(t) = x_0 \). Moreover, the conservation law (4.12) implies that \( \lim_{t \to T^-} \dot{q}(t) = 0 \).

Finally, the local uniqueness of the solution of an initial value problem implies immediately that \( T \) cannot be finite; for, the only solution \( q \) of (3.6) satisfying \( q(T) = x_0 \) and \( \dot{q}(T) = 0 \) is the constant \( q \equiv x_0 \).

\[\square\]

5. Orthogonal Geodesic Chords and the Maupertuis Integral.

In this section we will prove the main result of the paper, showing how to reduce the brake orbit and the homoclinics multiplicity problem to a multiplicity result for orthogonal geodesic chords.

We will begin with the study of the Jacobi metric near the level surface \( V^{-1}(E) \), with \( E \) regular value of \( V \).
5.1. **The Jacobi distance near a regular value of the potential.** Let \( g \) be a Riemannian metric, \( g_E = (E - V(x))g \), \( \Omega_E \) as in (3.4); assume \( \nabla V(x) \neq 0 \) for all \( x \in V^{-1}(E) \) and that \( \Omega_E \) is compact.

**Lemma 5.1.** For all \( Q \in \Omega_E \), the infimum:

\[
d_E(Q) := \inf \left\{ \int_0^1 ((E - V(x))g(\dot{x}, \dot{x}))^{1/2} \, dt : x \in H^1([0, 1], \Omega_E), \ x(0) = Q, \ x(1) \in \partial \Omega \right\}
\]

is attained on at least one curve \( \gamma_Q \in H^1([0, 1], \Omega_E) \) such that \( (E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q) \) is constant, \( \gamma_Q([0, 1]) \subset \Omega \), and \( \gamma_Q \) is a \( C^2 \) curve on \( [0, 1] \). Moreover, such a curve satisfies assumption (4.4) of Proposition 4.1 on the interval \([a, b] = [0, 1]\).

**Proof.** For all \( k \in \mathbb{N} \) sufficiently large, set \( \Omega_k = V^{-1}(]-\infty, E - \frac{1}{k}[^1) \subset \Omega_E \), and consider the problem of minimization of the \( g_E \)-length functional:

\[
L_E(x) = \int_0^1 \left( (E - V(x))g(\dot{x}, \dot{x}) \right)^{1/2} \, ds,
\]

in the space \( \mathfrak{G}_k \) consisting of curves \( x \in H^1([0, 1], \Omega_k) \) with \( x(0) = Q \) and \( x(1) \in \partial \Omega_k \).

It is not hard to prove, by standard arguments, that for all \( \Omega_k \neq \emptyset \), the above problem has a solution \( \gamma_k \) which is a \( g_E \)-geodesic, and with \( \gamma_k([0, 1]) \subset \Omega_k \).

Set \( q_k = \gamma_k(1) \in \partial \Omega_k \) and \( l_k = L_E(\gamma_k) \). Since \( q_k \) approaches \( \partial \Omega \) as \( k \to \infty \), arguing by contradiction we get:

\[
\liminf_{k \to \infty} l_k \geq d_E(Q).
\]

Now, if by absurd it was:

\[
\liminf_{k \to \infty} l_k > d_E(Q),
\]

then we could find a curve \( x \in H^1([0, 1], \Omega) \) with \( x(0) = Q \), \( x(1) \in \partial \Omega \), and with \( L_E(x) < \liminf_{k \to \infty} l_k \). Then, a suitable reparameterization of \( x \) would yield a curve \( y \in \mathfrak{G}_k \) with \( L_E(y) < l_k \), which contradicts the minimality of \( l_k \) and proves that

\[
\liminf_{k \to \infty} l_k = d_E(Q).
\]

Now, arguing as in the proof of Proposition 4.1, we see that the sequence:

\[
\int_0^1 \frac{dt}{E - V(\dot{\gamma}_k(t))}
\]

is bounded. Now, \( \int_0^1 (E - V(\gamma_k))g(\dot{\gamma}_k, \dot{\gamma}_k) \, d\tau = l_k^2 \equiv (E - V(\gamma_k))g(\dot{\gamma}_k, \dot{\gamma}_k) \) is bounded, which implies \( \int_0^1 g(\dot{\gamma}_k, \dot{\gamma}_k) \, d\tau \) bounded, namely the sequence \( \gamma_k \) is bounded in \( H^1([0, 1], \Omega_E) \). Up to subsequences, we have a curve \( \gamma_Q \in H^1([0, 1], \Omega_E) \) which is an \( H^1 \)-weak limit of the \( \gamma_k \)'s; in particular, \( \gamma_k \) is uniformly convergent to \( \gamma_Q \).

We claim that such a curve \( \gamma_Q \) satisfies the required properties. First, \( \gamma_Q([0, 1]) \subset \Omega_E \). Otherwise, if \( b < 1 \) is the first instant where \( \gamma_Q(b) \in \partial \Omega_E \), by (5.1) and the conservation law of the energy for \( \gamma_k \) one should have

\[
(b - 1)l_k^2 = \int_b^1 (E - V(\gamma_k))g(\dot{\gamma}_k, \dot{\gamma}_k) \, d\tau \to 0,
\]

in contradiction with \( Q \notin \partial \Omega_E \). Then \( \gamma_Q \) satisfies (4.4) in \([0, 1]\) since it is a \( H^1 \)-weak limit of \( \gamma_k \), which is a sequence of \( g_E \)-geodesics.

Clearly, \( \gamma_Q \) is of class \( C^2 \) on \([0, 1]\), because the convergence on each interval \([0, b]\) is indeed smooth for all \( b < 1 \).
Finally, since $L_E(z) \leq \liminf_{k \to \infty} l_k$, from (5.1) it follows that $L_E(\gamma_Q) = d_E(Q)$, and this concludes the proof. \hfill \Box

Remark 5.2. It is immediate to see that, $\gamma_Q$ is a minimizer as in Lemma 5.1 if and only if it is a minimizer for the functional

\begin{equation}
(5.3) \quad f_{0,1}(x) = \frac{1}{2} \int_0^1 (E - V(x))g(\dot{x}, \dot{x}) \, dt
\end{equation}

in the space of curves

\begin{equation}
(5.4) \quad X_Q = \{ x \in H^1([0,1], \Omega_E) : x(0) = Q, x([0,1]) \subset \Omega_E, x(1) \in \partial \Omega_E \}.
\end{equation}

Then, by Lemma 5.1, $f_{0,1}$ has at least one minimizer on $X_Q$.

Using a simple argument, we also have:

Lemma 5.3. The map $d_E : \Omega_E \to [0, +\infty]$ defined in the statement of Lemma 5.1 is continuous, and it admits a continuous extension to $\Omega_E$ by setting $d_E = 0$ on $\partial \Omega_E$. \hfill \Box

Now we shall study the map

\begin{equation}
(5.5) \quad \psi(y) = \frac{1}{2} d_E^2(y),
\end{equation}

proving that it is $C^2$ and satisfies a convex condition when $y$ is nearby $\partial \Omega_E$.

Proposition 5.4. If $Q$ is sufficiently close to $\partial \Omega_E$ then the minimizer of the functional (5.3) in the space $X_Q$ is unique.

Proof. Let $z = z(t, 0, Q)$ the solution of the Cauchy problem

\begin{equation}
(5.6) \quad \begin{cases}
\dot{z}(t) = J \cdot D^2 H(z(t)) \\
z(0) = (0, Q), \quad Q \in \partial \Omega_E,
\end{cases}
\end{equation}

where $H$ is the Hamiltonian function (5.1), and $J$ is the matrix

\begin{equation}
J = \begin{pmatrix}
0 & -I_m \\
I_m & 0
\end{pmatrix}
\end{equation}

and $I_m$ is the $m \times m$ identity matrix. Since $V$ and $a_{ij}$ are $C^2$, $z = (p, q)$ is of class $C^1$ with respect to $(t, Q)$, therefore $\dot{z} = \dot{z}(t, Q)$ is of class $C^1$ with respect to $(t, Q)$ so $\dot{q} = \dot{q}(t, Q)$ is of class $C^1$. Since $\dot{q} = \dot{q}(0, Q) = 0$, in a neighborhood of a fixed point $Q_0 \in \partial \Omega_E$ it is

\begin{equation}
(5.7) \quad \dot{q}(t, Q) = t \dot{q}(0, Q_0) + \varphi(t, Q) = -t \nabla V(Q_0) + \varphi(t, Q)
\end{equation}

where $\varphi$ is of class $C^1$ and $d \varphi(0, Q_0) = 0$. Moreover

\begin{equation}
(5.8) \quad q(t, Q) = Q - \frac{t^2}{2} \nabla V(Q_0) + \varphi(t, Q)
\end{equation}

where $\varphi_0(t, Q) = \int_0^t \varphi(s, Q) \, ds$. Then, if $\{y_1, \ldots, y_{m-1}\}$ is a coordinate system of $V^{-1}(E)$ in a neighborhood of $Q_0$, by (5.8) we deduce that, setting $\tau = t^2$, the set $\{y_1, \ldots, y_{m-1}, \tau\}$ is a local coordinate system on the manifold with boundary $\partial \Omega_E$ and $(\tau, Q) \mapsto q(\tau, Q)$ defines a local chart.

Then, due to the compactness of $\partial \Omega_E$, and denoted by $\text{dist}(\cdot, \cdot)$ the distance induced by $g$, there exists $\bar{\rho} > 0$ having the following property:

\begin{equation}
(5.9) \quad \forall y \in \Omega_E \text{ with } \text{dist}(y, \partial \Omega_E) \leq \bar{\rho} \text{ there exists a unique solution } (p_y, q_y) \text{ of (3.2)}
\end{equation}

with energy $E$, and a unique $t_y > 0$ such that $q_y(0) \in \partial \Omega_E, q_y(t_y) = y$. 

Then, by Proposition 5.1, \( \forall y \in \Omega_E \) with \( \text{dist}(y, \partial \Omega_E) \leq \bar{\rho} \) there exists a unique minimizer \( \gamma_y \) for \( f_{0,1} \) on \( X_y \).

**Remark 5.5.** Note that \( q_y(t) = q(t, Q_y) \) where \( Q_y \) is implicitly defined by \( q(t_y, Q_y) = y \). By the variable change used in Proposition 5.1, it turns out that

\[
(5.10) \quad q(t, Q_y) = \gamma_y(1 - \sigma), \quad \text{where} \quad t(\sigma) = \psi(y) \int_0^\sigma \frac{1}{E - V(\gamma_y(\tau))} \, d\tau.
\]

In particular, since \( \sigma = \sigma(t) \) is the inverse of \( t(\sigma) \) we have

\[
(5.11) \quad \psi(y)q(t, Q_y) = -(E - V(y))\dot{\gamma}_y(0).
\]

Note also that \( t_y = \sqrt{\tau_y} \) is of class \( C^1 \) when \( \tau_y > 0 \) since \( (\tau, Q) \) is a local coordinate system.

In the following result we are assuming \( \bar{\Omega}_E \subset \mathbb{R}^m \).

**Proposition 5.6.** Let \( \bar{\rho} \) satisfy property 5.5. Whenever \( 0 < \text{dist}(y, \partial \Omega_E) \leq \bar{\rho} \), \( \psi \) is differentiable at \( y \) and

\[
(5.12) \quad d\psi(y)[\xi] = -(E - V(y))g(\gamma_y(0), \xi) \quad \forall \xi \in \mathbb{R}^m.
\]

**Proof.** Given the local nature of the result, it will not be restrictive to assume that \( M \) is topologically embedded as an open subset of \( \mathbb{R}^m \). Consider

\[
v_\xi(s) = (1 - 2s)^+ \xi,
\]

where \( (\cdot)^+ \) denotes the positive part. For \( \epsilon \) sufficiently small (with respect to \( \xi \)) the curve \( \gamma_y(s) + \epsilon v_\xi(s) \) belongs to \( X_{y+\xi} \) (see 5.4). Then, by the definition of \( \psi \) as minimum value,

\[
\psi(y + \epsilon \xi) \leq f_{0,1}(\gamma_y + \epsilon v_\xi)
\]

and therefore

\[
\psi(y + \epsilon \xi) - \psi(y) \leq f_{0,1}(\gamma_y + \epsilon v_\xi) - f_{0,1}(\gamma_y).
\]

Now

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( f_{0,1}(\gamma_y + \epsilon v_\xi) - f_{0,1}(\gamma_y) \right) = \int_0^1 (E - V(\gamma_y))g(\dot{\gamma}_y, \frac{D}{\epsilon} v_\xi) - \frac{1}{2}g(\nabla V(\gamma_y), v_\xi)g(\dot{\gamma}_y, \dot{\gamma}_y) \, ds
\]

uniformly as \( |\xi| \leq 1 \). Moreover, since \( v_\xi = 0 \) in the interval \([\frac{1}{2}, 1]\), using the differential equation satisfied by \( \gamma_y \) and integrating by parts gives

\[
\int_0^1 (E - V(\gamma_y))g(\dot{\gamma}_y, \frac{D}{\epsilon} v_\xi) - \frac{1}{2}g(\nabla V(\gamma_y), v_\xi)g(\dot{\gamma}_y, \dot{\gamma}_y) \, ds = -(E - V(\gamma_y(0)))g(\dot{\gamma}_y(0), v_\xi(0)) = -(E - V(y))g(\dot{\gamma}_y(0), \xi).
\]

Therefore, uniformly as \( |\xi| \leq 1 \),

\[
(5.13) \quad \lim_{\epsilon \to 0^+} \sup \frac{1}{\epsilon} \left( \psi(y + \epsilon v_\xi) - \psi(y) \right) + (E - V(y))g(\dot{\gamma}_y(0), \xi) \leq 0.
\]

Moreover, since \( \psi(y + \epsilon \xi) = f_{0,1}(\gamma_y + \epsilon \xi) \) and \( \psi(y) \leq f_{0,1}(\gamma_y + \epsilon \xi) - \epsilon v_\xi \) one has

\[
(5.14) \quad \psi(y + \epsilon \xi) - \psi(y) \geq f_{0,1}(\gamma_y + \epsilon \xi) - f_{0,1}(\gamma_y + \epsilon \xi) - \epsilon v_\xi = \epsilon(f'_{0,1}(\gamma_y + \epsilon \xi), v_\xi) - \frac{\epsilon^2}{2}f''_{0,1}(\gamma_y + \epsilon \xi - \partial_\xi \epsilon v_\xi)[v_\xi, v_\xi],
\]
for some $\partial_\xi \in [0,1]$. Here $\langle \cdot, \gamma \rangle_1$ denotes the standard scalar product in $H^1$ and $f'$, $f''$ are respectively gradient and Hessian with respect to $\langle \cdot, \gamma \rangle_1$.

Now, it is $\gamma_{y+\varepsilon\xi}(0) = y + \varepsilon \xi$ and $y \notin V^{-1}(E)$. Moreover, by the uniqueness of the minimizer it is not difficult to prove that, $\forall \delta > 0 \exists \varepsilon(\delta) > 0$ such that

$$\text{dist}(\gamma_{y+\varepsilon\xi}(s), \gamma_y(s)) \leq \delta \quad \text{for any } \varepsilon \in [0, \varepsilon(\delta)], \quad |\xi| \leq 1, \ s \in [0,1].$$

Then, since $\gamma_y$ is uniformly far from $V^{-1}(E)$ on the interval $[0, \frac{1}{2}]$, the same holds for $\gamma_{y+\varepsilon\xi}$ whenever $\varepsilon$ is small and $|\xi| \leq 1$. Thus, recalling the definition of $d_E$ in Lemma 5.1 the conservation law satisfied by the minimizer $\gamma_{y+\varepsilon\xi}$ is

$$(E - V(\gamma_{y+\varepsilon\xi}))g(\dot{\gamma}_{y+\varepsilon\xi}, \dot{\gamma}_{y+\varepsilon\xi}) = d_E^2(y + \varepsilon \xi).$$

This implies the existence of a constant $C > 0$ such that

$$\int_0^{1/2} g(\dot{\gamma}_{y+\varepsilon\xi}, \dot{\gamma}_{y+\varepsilon\xi}) \, ds \leq C$$

for any $\varepsilon$ small and $|\xi| \leq 1$.

Therefore $\langle f''_0,1(\gamma_{y+\varepsilon\xi} - \partial_\varepsilon\varepsilon v_\xi + v_\xi) \rangle_1$ is uniformly bounded with respect to $\varepsilon$ small and $|\xi| \leq 1$, due to $v_\xi = 0$ on $[\frac{1}{2}, 1]$, and by (5.14) we get

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle f''_0,1(\gamma_{y+\varepsilon\xi} - \partial_\varepsilon\varepsilon v_\xi + v_\xi) \rangle_1 = \lim_{\varepsilon \to 0} \langle f'_0,1(\gamma_{y+\varepsilon\xi} + v_\xi) \rangle_1$$

uniformly as $|\xi| \leq 1$.

Now, using the differential equation (4.3) satisfied by $\gamma_{y+\varepsilon\xi}$ and integrating by parts one obtains

$$\langle f'_0,1(\gamma_{y+\varepsilon\xi} + v_\xi) \rangle_1 = -(E - V(y + \varepsilon \xi))g(\dot{\gamma}_{y+\varepsilon\xi}(0), \xi),$$

while by (5.11) and the continuity of $\dot{q}(t_y, Q_y)$ and $\psi(y)$ we have

$$\lim_{\varepsilon \to 0} (E - V(y + \varepsilon \xi))\dot{\gamma}_{y+\varepsilon\xi}(0) = (E - V(y))\dot{\gamma}_y(0)$$

uniformly as $|\xi| \leq 1$. Therefore, by (5.14)–(5.16) it is

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \psi(y + \varepsilon \xi) - \psi(y) + (E - V(y))g(\dot{\gamma}_y(0), \xi) \right) \geq 0$$

uniformly as $|\xi| \leq 1$. Finally, combining (5.15) and (5.17) one has (5.12). \hfill \square

**Remark 5.7.** By (5.11) we deduce that $(E - V(y))\dot{\gamma}_y(0)$ is continuous, therefore by (5.12), $\psi$ is of class $C^1$. Again by (5.11) and the $C^1$–regularity of $\dot{q}_y(t_y, Q_y)$ we deduce that $(E - V(y))\dot{\gamma}_y(0)$ is of class $C^1$ whenever $y \notin V^{-1}(E)$, and by (5.12) it turns out that $\psi$ is of class $C^2$.

In the following proposition we will show that $\psi$ satisfies a strongly convex assumption nearby $V^{-1}(E)$.

**Proposition 5.8.** There exists $\tilde{\rho} \leq \rho$ with the property that, for any $y \in \Omega_E$ such that $0 < \text{dist}(y, V^{-1}(E)) \leq \tilde{\rho}$ the Hessian (with respect to the Jacobi metric $g_E$) of $\Psi$ at $y$ satisfies

$$H^\psi(y)[v,v] > 0 \quad \forall v : d\psi(y)[v] = 0, \ v \neq 0.$$
Proof. Recall that

\[ H^y(v)[v, v] = \frac{\partial^2}{\partial s^2} (\psi(\eta(s)))|_{s=0}, \]

where \( \eta(s) \) is a geodesic with respect to the Jacobi metric \( g_E \), namely a solution of the differential equation (5.3) satisfying the initial data conditions

\[ \eta(0) = y, \quad \dot{\eta}(0) = \xi. \]

Now, by (5.11) and (5.12)

\[ d\psi(\eta(s))[\dot{\eta}(s)] = -(E - V(\eta(s)))g(\dot{\eta}(s)(0), \dot{\eta}(s)) = \psi(\eta(s))g(\dot{\eta}(t_{\eta(s)}, \eta(s)), \dot{\eta}(s)). \]

Since \( \lim_{t \to 0} Q_{\eta(s)} = Q_y \), using (5.7) we can write

\[ \dot{\eta}(t, Q_{\eta(s)}) = -t
V(y) + \varphi(t, Q_{\eta(s)}) \]

as \( d\varphi(0, Q_y) = 0 \), and

\[ \frac{\partial^2}{\partial s^2} (\psi(\eta(s))) = \psi(\eta(s)) (g(\dot{\eta}(t_{\eta(s)}, Q_{\eta(s)}), \dot{\eta}(s)))^2 + \psi(\eta(s))g(\dot{\eta}(t_{\eta(s)}, Q_{\eta(s)}), \frac{D}{D s} \dot{\eta}(s)) + \psi(\eta(s))g(\dot{\eta}(t_{\eta(s)}, \eta(s)) - dt_{\eta(s)}[\dot{\eta}(s)] \nabla V(y) + \frac{\partial \varphi}{\partial t}(t, Q_{\eta(s)}) dt_{\eta(s)}[\dot{\eta}(s)] + \frac{\partial \varphi}{\partial Q} \frac{\partial Q}{\partial \dot{\eta}} [\dot{\eta}(s)], \dot{\eta}(s)). \]

Since \( \eta(s) \) satisfies (5.3) and \( d\varphi(0, Q_y) = 0 \), it suffices to show that for any \( y \) sufficiently close to \( \partial \Omega \),

\[ \psi(y) (g(\dot{\eta}(t, Q_y), v))^2 + \psi(y) dt_y[v] g(-\nabla V(y), v) + \frac{\psi(y)}{E - V(y)} \left( g(\nabla V(y), v) g(\dot{\eta}(t, Q_y), v) - \frac{1}{2} g(\dot{\eta}(t, Q_y), \nabla V(y)) g(v, v) \right) > 0 \]

for any \( v \) such that \( d\psi(y)[v] = 0 \). This means that \( g(\dot{\eta}(t, Q_y), v) = 0 \) so it will suffice to show

\[ (5.19) \sup_{|v|=1} |dt_y[v]| g(\nabla V(y), \nabla V(y))^{1/2} - \frac{1}{2(E - V(y))} g(\dot{\eta}(t, Q_y), \nabla V(y)) > 0 \]

for any \( y \) close to \( V^{-1}(E) \).

Since \( \eta(t, Q_y) = y \) we get

\[ dt_y[v] \dot{\eta}(t_y, Q_y) + \frac{\partial \varphi}{\partial Q} \frac{\partial Q_y}{\partial \dot{\eta}} [v] = v. \]

Moreover, \( \frac{\partial \varphi}{\partial Q}(t_y, Q_y) \) goes to the identity map as \( y \) tends to \( \partial \Omega \), while \( \frac{\partial Q_y}{\partial \dot{\eta}} [v] \) tends to \( v \) uniformly as \( |v| \leq 1 \), since \( (0, Q) \) is a coordinate system for \( V^{-1}(E) \). Then, as \( y \to V^{-1}(E), dt_y[v] \dot{\eta}(t_y, Q_y) \to 0 \) uniformly in \( v \).

Note that \( \frac{1}{2} g(\dot{\eta}, \dot{\eta}) = E - V(\eta) \), therefore

\[ (5.20) \quad g(\dot{\eta}(t, Q_y), \dot{\eta}(t, Q_y)) = 2(E - V(y)) \]

so

\[ (5.21) \quad \lim_{y \to \partial \Omega} \sqrt{E - V(y)} |dt_y[v]| = 0 \]

uniformly in \( |v| \leq 1 \).
Finally, by (5.7) we have
\[
\lim_{y \to V^{-1}(E)} g \left( \frac{\dot{q}(t, Q)}{\sqrt{g(\dot{q}(t, Q), \dot{q}(t, Q))}} \frac{\nabla V(y)}{\sqrt{g(\nabla V(y), \nabla V(y))}} \right) = -1
\]
therefore by (5.20)
\[
(5.22) \quad \lim_{y \to V^{-1}(E)} -\frac{g(\dot{q}(t, Q), \nabla V(y))}{\sqrt{E - V(y)}} > 0
\]
and combining (5.21) with (5.22) one obtains (5.19) and the proof is complete. \[\square\]

By Proposition 5.6, Remark 5.7 and Proposition 5.8 one immediately obtains the following proposition, which is the main result of the section:

**Theorem 5.9.** Let \( E \) be a regular value for \( V(x) \), and let \( d_E : \Omega \to [0, +\infty) \) be the map defined in the statement of Lemma 5.1, and assume that \( \Omega_E \) is compact. There exists a positive number \( \delta^* \) such that, setting:
\[
\Omega^* = \{ x \in \Omega_E : d_E(x) > \delta^* \},
\]
the following statements hold:

1. \( \partial \Omega^* \) is of class \( C^2 \);
2. \( \Omega^* \) is homeomorphic to \( \overline{\Omega}_E \);
3. \( \overline{\Omega}_* \) is strongly concave relatively to the Jacobi metric \( g_E \);
4. if \( x : [0, 1] \to \overline{\Omega}_* \) is an orthogonal geodesic chord in \( \overline{\Omega}_* \) relatively to the Jacobi metric \( g_E \), then there exists \( [\alpha, \beta] \supset [0, 1] \) and a unique extension \( \hat{x} : [\alpha, \beta] \to \overline{\Omega} \) of \( x \) with \( \hat{x} \in H^1([\alpha, \beta], \overline{\Omega}) \) satisfying:
   - assumption (4.4) of Proposition 4.1 on the interval \([\alpha, \beta] \);
   - \( \hat{x}(s) \in d_E^{-1}([-\delta^*, 0]) \) for all \( s \in [\alpha, 0[ \cup ]1, \beta[ \);
   - \( V(\hat{x}(\alpha)) = V(\hat{x}(\beta)) = E \).

**Remark 5.10.** Theorem 5.9 tells us that the study of multiple brake orbits can be reduced to the study of multiple orthogonal geodesic chords in a Riemannian manifold with regular and strongly concave boundary.

**5.2. The Jacobi distance near a nondegenerate maximum point of the potential.** Let us now assume that \( x_0 \in M \) is a nondegenerate maximum point of \( V \), with \( V(x_0) = E \), and let us make the following assumptions:

- \( V^{-1}([-\infty, E]) \) is compact;
- \( V^{-1}(E) \setminus \{ x_0 \} \) is a regular embedded hypersurface of \( M \).

We will show how to get rid of the singularity of the Jacobi metric at \( x_0 \), while the singularity on \( V^{-1}(E) \setminus \{ x_0 \} \) can be removed as in the case of brake orbits, using Theorem 5.9.

First, we need a preparatory result. Let \( \delta > 0 \) be fixed in such a way that the set:
\[
\{ p \in M : V(p) > E - \delta \}
\]
has precisely two connected components; let \( \Omega_\delta \) denote the connected component of the point \( x_0 \).
Lemma 5.11. Let $Q \in \Omega_\delta \setminus \{x_0\}$ be fixed; then, the infimum:

$$
(5.23) \quad d_E(Q) := \inf \left\{ \left[ \int_0^1 (E - V(x)) g(\dot{x}, \dot{x}) \, dt \right]^{1/2} : x \in C^0([0, 1], \mathcal{M}) \cap H^1_{\text{loc}}([0, 1], \mathcal{M}), \ x(0) = Q, \ x(1) = x_0 \right\}
$$

is attained on some curve $\gamma_Q$ with the property $(E - V(\gamma_Q))g(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant and $\gamma_Q([0, 1]) \subset \overline{\mathcal{M}} \setminus \{x_0\}$. Moreover

$$
(5.24) \quad \lim_{Q \to x_0} d_E(Q) = 0,
$$

$$
(5.25) \quad \lim_{Q \to x_0} \left[ \sup_{s \in [0, 1]} \text{dist}(\gamma_Q(s), x_0) \right] = 0,
$$

$$
(5.26) \quad \text{In particular, for } Q \text{ sufficiently close to } x_0,
$$

$$
(5.27) \quad \gamma_Q([0, 1]) \subset \Omega_\delta,
$$

so it is of class $C^2$ and satisfies assumption $\mathcal{A}$ of Proposition 4.2 on the interval $[a, b] = [0, 1]$.

Proof. Let $x_n \in C^0([0, 1], \overline{\mathcal{M}}) \cap H^1([0, 1], \overline{\mathcal{M}})$ be a minimizing sequence for the length functional $\int_0^1 ((E - V(x)) g(\dot{x}, \dot{x}))^{1/2} \, dt$, leaving $(E - V(x)) g(\dot{x}, \dot{x})$ constant. Choose $\rho > 0$ such that $\text{dist}(Q, x_0) > \rho$ and, for all $n \in \mathbb{N}$, define $\alpha_n \in [0, 1]$ to be the first instant $s$ such that $\text{dist}(x(s), x_0) = \rho$.

The sequence $\alpha_n$ stays away from 0 and 1, because for all interval $I \subset x_n^{-1}(\frac{\rho}{2}, \rho)$ the integral $\int_I g(\dot{x}_n, \dot{x}_n) \, ds$ is bounded. We can therefore find a subsequence $\alpha_{n_k}$ converging to $\alpha_1 \in [0, 1]$. Furthermore, since $\int_0^{\alpha_1} g(\dot{x}_n, \dot{x}_n) \, ds$ is bounded, taking a subsequence $x_n$ we can assume that $x_n$ is $H^1$-weakly and uniformly convergent to some $x_1 \in H^1([0, \alpha_1], \Omega_\delta)$; then, $\text{dist}(x(\alpha_1), x_0) = \rho$. Repeating the construction, we can find $\alpha_2 \in [\alpha_1, 1]$ and a subsequence $x_n$ of $x_n$ which is $H^1$-weakly and uniformly convergent to a curve $x_2 \in H^1([0, \alpha_2], \Omega_\delta)$ with $\text{dist}(x(\alpha_2), x_0) = \frac{\rho}{2}$ and $x_2|_{[0, \alpha_1]} = x_1$. Iteration of this construction yields a weak-$H^1$ limit of $x_n$, which is a curve $x \in H^1_{\text{loc}}([0, \alpha_1], \overline{\mathcal{M}})$, where $\tilde{\alpha} = \lim_{k \to \infty} \alpha_k$, and $\text{dist}(x(\tilde{\alpha}), x_0) = \frac{\rho}{2}$.

Now, for all $k \geq 1$:

$$
\int_0^{\alpha_k} ((E - V(x)) g(\dot{x}, \dot{x}))^{1/2} \, ds \leq \liminf_{n \to \infty} \int_0^{\alpha_k} ((E - V(x_n)) g(\dot{x}_n, \dot{x}_n))^{1/2} \, ds \\
\leq \liminf_{n \to \infty} \int_0^1 ((E - V(x_n)) g(\dot{x}_n, \dot{x}_n))^{1/2} \, ds = d_E(Q),
$$

hence:

$$
\int_0^{\tilde{\alpha}} ((E - V(x)) g(\dot{x}, \dot{x}))^{1/2} \, ds = \lim_{k \to \infty} \int_0^{\alpha_k} ((E - V(x)) g(\dot{x}, \dot{x}))^{1/2} \, ds \leq d_E(Q),
$$

and we can assume, as usual, $(E - V(x)) g(\dot{x}, \dot{x})$ constant (and positive since $Q \neq x_0$). The curve $x$ can be extended continuously to $\overline{\mathcal{M}}$ by setting $x(\overline{\mathcal{M}}) = x_0$. Indeed, if by contradiction there exists a sequence $\beta_n < \alpha_n < \overline{\mathcal{M}}$ such that $\lim_{k \to \infty} \beta_k = \overline{\mathcal{M}}$ and a positive number $\overline{\mathcal{M}}$ such that $\text{dist}(x(\beta_k), x_0) \geq \overline{\mathcal{M}}$, there exist $\beta_k$ such that $\text{dist}(x(\beta_k), x_0) = \overline{\mathcal{M}}$. 


and \( \text{dist}(x(s),x_0) \geq \frac{\nu}{2}, \forall s \in [\beta_k^1,\beta_k]. \) But \( E - V(x(s)) \) is far from zero in \([\beta_k^1,\beta_k]\) therefore \( g(\dot{x},\dot{x}) \leq K \in \mathbb{R}^+ \) on \([\beta_k^1,\beta_k]\) for some \( K, \) and then

\[
\frac{\nu}{2} \leq \text{dist}(x(\beta_k^1),x(\beta_k)) \leq \int_{\beta_k^1}^{\beta_k} g(\dot{x},\dot{x})\,dt \leq K(\beta_k - \beta_k^1) \to 0
\]

which is a contradiction.

Clearly, up to reparameterizations on \( x \) we can assume \( \bar{\alpha} = 1 \) and \( x([0,1]) \subset \overline{\Omega_\delta} \setminus \{x_0\}. \)

Taking \( \gamma_Q = x \) we have the existence of a minimizer satisfying the conservation law \((E - V(\gamma_Q))g(\dot{\gamma}_Q,\dot{\gamma}_Q)\) constant.

Now, taking a chord \( C_Q \) joining \( Q \) and \( x_0 \) we have that \( l(C_Q) \to 0 \) as \( Q \to x_0, \) and since \( d_E(Q) \leq l(C_Q) \) we obtain (5.24).

Moreover, if by contradiction (5.25) does not hold for any \( Q \) sufficiently close to \( x_0, \) there exists \( s_Q \) such that

\[
\text{dist}(\gamma_Q(s_Q),x_0) \geq \frac{\nu}{2} > 0.
\]

Let \( t_Q > s_Q \) such that \( \text{dist}(\gamma_Q(t_Q),x_0) = \frac{\nu}{2} \) and \( \text{dist}(\gamma_Q(s),x_0) \geq \frac{\nu}{2} \forall s \in [s_Q,t_Q]. \)

Since \( g(\gamma_Q) \) is bounded in \([s_Q,t_Q]\) it must be \( t_Q - s_Q \) far from zero as \( Q \to x_0. \) But also \( E - V(\gamma_Q) \) and \( g(\dot{\gamma}_Q,\dot{\gamma}_Q) \) are far from zero in \([s_Q,t_Q]\) so we deduce that

\[
\int_{s_Q}^{t_Q} \left( \int_0^1 (E - V(x))g(\dot{\gamma}_Q,\dot{\gamma}_Q)\,dt \right)^{1/2}
\]

which is in contradiction with (5.24).

Note that (5.25) immediately implies (5.24) and since \( \gamma_Q \) is a minimizer satisfying \((E - V(\gamma_Q))g(\dot{\gamma}_Q,\dot{\gamma}_Q)\) constant, we immediately see that (4.4) is satisfied in the interval \([0,1]. \)

As for Lemma 5.11 a simple argument shows

**Lemma 5.12.** The map \( d_E : \Omega_\delta \to [0, +\infty) \) defined in the statement of Lemma 5.11 is continuous.

For any \( y \) sufficiently close to \( x_0, \) let \( q_y \) be the reparameterization of \( \gamma_y \) given by Proposition 4.7. We have

\[
\begin{aligned}
\frac{D}{dt}q_y + \nabla Y(q_y) &= 0 \\
q_y(0) &= y \\
limit_{t \to +\infty} q_y(t) &= x_0 \\
limit_{t \to +\infty} \dot{q}_y(t) &= 0.
\end{aligned}
\]

The following estimate holds

**Proposition 5.13.** Let \( q_y \) be as above. Then there exists \( \bar{\rho} \) and a constant \( \alpha > 0 \) such that

\[
\text{dist}(q_y(t),x_0) \leq \text{dist}(y,x_0)e^{-\alpha t}
\]

for any \( y \) such that \( \text{dist}(y,x_0) \leq \bar{\rho}. \)

To obtain the above result we need the following maximum principle in \( \mathbb{R}. \)

**Lemma 5.14.** Let \( \varphi : [0, +\infty) \to \mathbb{R} \) be a \( C^2 \) map with \( \lim_{t \to +\infty} \varphi(t) = 0. \) Let \( \nu > 0 \) such that \( \varphi''(t) \geq \nu \varphi(t), \forall t \geq 0. \) Then \( \varphi \leq \varphi(0)e^{-\sqrt{\nu}t}. \)
Proof. Consider the map $\psi = \varphi - \varphi_0$ where $\varphi_0(t) = \varphi(0)e^{-\sqrt{r}t}$. Clearly $\psi(0) = \lim_{t \to +\infty} \psi(t) = 0$ and so $\psi$ has a global maximum at some $\hat{t} \in [0, +\infty]$. If $\hat{t} > 0$ then $\psi(\hat{t}) \leq \frac{1}{\hat{t}} \psi''(\hat{t}) \leq 0$. □

Remark 5.15. Clearly, an analogous result as in the above Lemma 5.14 holds, reversing all inequalities.

Proof of Proposition 5.16. Let $q$ be a solution of (5.28) (with $q(0) = y$), and let $\varphi(t) = \frac{1}{2}\dist(q(t), x_0)^2$. By (5.29) we can choose $\tilde{\rho}$ sufficiently small so that

$$\dist(q(t), x_0) < \rho_0, \quad \text{for any } t \geq 0,$$

where $\rho_0$ is chosen so that the function $\overline{\vartheta}(z) = \frac{1}{2}\dist(z, x_0)^2$, in the open ball $B(x_0, \rho_0)$ of center $x_0$ and radius $\rho_0$, is of class $C^2$, strictly convex and, called $x_z$ the unique minimal geodesic with respect to $g$ such that $x_z(0) = x_0, x_z(1) = z$ (see [4]), one has

$$\nabla \overline{\vartheta}(z) = \dot{x}_z(1).$$

Now $\varphi'(t) = g(\nabla \overline{\vartheta}(q(t)), \dot{q}(t))$ and

$$\varphi''(t) = H^q(q(t))|\dot{q}(t), \dot{q}(t)| + g(\nabla \overline{\vartheta}(q(t)), \frac{\partial}{\partial q} \overline{\vartheta}(t)) \geq g(\nabla \overline{\vartheta}(q(t)), \nabla V(q(t))).$$

Now, take $z$ in $B(x_0, \rho_0)$, consider the minimal geodesic $x_z$ as above, and define the map

$$\rho(s) := g(\nabla \overline{\vartheta}(x_z(s)), -\nabla V(x_z(s))).$$

By the choice of $x_z$ it is $\nabla \overline{\vartheta}(x_z(s)) = s \dot{x}_z(s)$, so

$$\dot{\rho}(s) = g(\dot{x}_z(s), -\nabla V(x_z(s))) - s H^V(x_z(s))|\dot{x}_z(s), \dot{x}_z(s)| \geq g(\dot{x}_z(s), -\nabla V(x_z(s))) + s v g(\dot{x}_z(s), \dot{x}_z(s))$$

for a suitable choice of $\nu$ ($x_0$ is a nondegenerate maximum point). Since $\rho(0) = 0$ then

$$g(\nabla \overline{\vartheta}(z), -\nabla V(z)) = \varphi(1) = \int_0^1 \dot{\rho}(s) \, ds \geq$$

$$\int_0^1 g(\dot{x}_z(s), -\nabla V(x_z(s))) + s v g(\dot{x}_z(s), \dot{x}_z(s)) \, ds =$$

$$-V(x_z(s))_{s=0}^1 + \nu \dist(z, x_0)^2 \int_0^1 s \, ds =$$

$$(E - V(z)) + \frac{\nu}{2} \dist(z, x_0)^2 \geq \frac{\nu}{2} \dist(z, x_0)^2,$$

where $V(x_0) = E$ has also been used. Therefore $\varphi''(t) \geq \frac{\nu}{2} \dist(q(t), x_0)^2 = \nu q(t)$, and by Lemma 5.14

$$\dist(q(t), x_0)^2 \leq \dist(q(0), x_0^0) e^{-\sqrt{\nu}t},$$

and (5.29) follows taking the square root of both members above. □

The regularity of the distance function from $x_0$ with respect to the Jacobi metric is based on the following proposition.

Proposition 5.16. For any $y$ close to $x_0$ there exists a unique $q_y$ satisfying (5.28). Moreover, the map

$$q \mapsto \dot{q}_y(0)$$

(5.30)
is of class \( C^1 \) and its differential satisfies \( d\bar{q}(0)[v] = \dot{\xi}(0) \), where \( \xi(t) \) is the unique solution of

\[
\begin{cases}
\frac{D^2}{dt^2} \xi(t) + R(\dot{q}_y, \xi(t))\dot{q}_y + L^V(q_y)\xi(t) = 0 \\
\xi(0) = 0 \\
\lim_{t \to +\infty} \xi(t) = \lim_{t \to +\infty} \dot{\xi}(t) = 0
\end{cases}
\]

(5.31)

where \( \frac{D^2}{dt^2} \xi \) is the second covariant derivative and \( R(\cdot, \cdot) \) the Riemann tensor with respect to \( g \), and \( L^V(x)[v] \in T_xM \) is the vector defined through \( g(L^V(x)[v], w) = H^V(x)[v, w] \) for all \( w \in T_xM \).

**Proof.** Consider the ball \( B(x_0, \rho) \), with \( \rho > 0 \) small, and the spaces

\( X_2 = \{ q \in C^2(\mathbb{R}^+, B(x_0, \rho)) : \lim_{t \to +\infty} q(t) = x_0, \lim_{t \to +\infty} \dot{q}(t) = \lim_{t \to +\infty} \ddot{q}(t) = 0 \} \)

with the norm (we can assume to work in a local chart)

(5.32) \( \|q_2 - q_1\| := \sup_{t \in \mathbb{R}^+} |q_2(t) - q_1(t)| + \sup_{t \in \mathbb{R}^+} |\dot{q}_2(t) - \dot{q}_1(t)| + \sup_{t \in \mathbb{R}^+} |\ddot{q}_2(t) - \ddot{q}_1(t)| \)

and

\( X_0 = \{ q \in C^0(\mathbb{R}^+, \mathbb{R}^m) : \lim_{t \to +\infty} |q(t)| = 0 \} \)

with the norm

\( \|q_2 - q_1\| := \sup_{t \in \mathbb{R}^+} |q_2(t) - q_1(t)|, \)

that are clearly Banach spaces. Now, consider the open set

\( A_2 = \{ q \in X_2 : \sup_{t \in \mathbb{R}^+} \text{dist}(q(t), x_0) < \rho \} \subset X_2 \)

and the map

\( F : A_2 \times B(x_0, \rho) \to X_0 \times \mathbb{R}^m \)

given by

\( F(q, y) = (\frac{D}{dt} \dot{q} + \nabla V(q), q(0) - y). \)

Thanks to the behaviour at infinity, we can use the same standard arguments exploited in finite intervals to prove that \( F \) is differentiable and (see [4])

\( dF(q, y)[\xi, v] = (\frac{D^2}{dt^2} \xi + R(\dot{q}, \xi)\dot{q} + L^V(q)[\xi], \xi(0) - v). \)

Moreover, thank again to the behaviour at infinity, it is a straight check to verify that \( dF(q, y) \) is continuous (recall that \( g \) and \( V \) are of class \( C^2 \)).

Now consider \( \frac{\partial F}{\partial q}(x_0, 0)[\xi] = (\ddot{\xi} + L^V(0)[\xi], \xi(0)) \) where \( x_0 \) denotes the constant curve with image \( x_0 \). We claim that

(5.33) \( \frac{\partial F}{\partial q}(x_0, 0) : X_2 \to X_0 \times \mathbb{R}^m \)

is an isomorphism.

Recalling the definition of \( L^V \), and since \( H^V(0) \) is symmetric and negative definite, using a base consisting of eigenvectors for \( H^V(0) \), it is sufficient to show that for any
function $h \in C^0(\mathbb{R}^+, \mathbb{R})$ such that $\lim_{t \to +\infty} h(t) = 0$ and for any $\theta \in \mathbb{R}$, the solution of
\begin{align*}
\begin{cases}
\ddot{x} - \alpha^2 x = h \\
x(0) = \theta \\
\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \dot{x}(t) = 0
\end{cases}
\end{align*}
(5.34)
exists and is unique (where $x : \mathbb{R}^+ \to \mathbb{R}$).

The general solution of the differential equation above is
\begin{align*}
x(t) &= \left( a + \frac{1}{2\alpha} \int_0^t h(s)e^{-\alpha s} \, ds \right) e^{\alpha t} + \left( b - \frac{1}{2\alpha} \int_0^t h(s) e^{\alpha s} \, ds \right) e^{-\alpha t}.
\end{align*}

Since $\lim_{t \to +\infty} h(t) = 0$ it is $\lim_{t \to +\infty} x(t) = 0$ only if we choose
\begin{align*}
a = -\frac{1}{2\alpha} \int_0^{+\infty} h(s) e^{-\alpha s} \, ds.
\end{align*}

With such a choice indeed $\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \dot{x}(t) = 0$, while $x(0) = \theta$ for
\begin{align*}
b = \theta - a = \theta + \frac{1}{2\alpha} \int_0^{+\infty} h(s) e^{-\alpha s} \, ds,
\end{align*}
proving that the solution of (5.34) exists and is unique, and therefore the map defined in (5.33) is an isomorphism.

Then, by the Implicit Function Theorem and Proposition 5.13 we have the uniqueness of $q_y$ for any $y$ close to $x_0$ and its $C^1$–differentiability in $X_2$. In particular the map (5.30) is of class $C^2$. Denoting by $\xi$ the differential $dq_y[v]$, and differentiating the expression $F(q_y, y) \equiv 0$, in particular we obtain that $\xi$ solves (5.31). Since, we have already seen, the solution exists and is unique for $y = x_0$, Proposition 5.13 ensures that this remains true for $y$ close to $x_0$ also.

Finally, $C^1$–regularity of $q_y$ with respect to the norm (5.32) immediately implies that
\begin{align*}
d\check{q}_y(0)[v] = \check{\dot{q}}(0),
\end{align*}
where $\check{\xi}$ is the solution of (5.31), and then $d\check{q}_y[v](t) = \check{\xi}(t)$. \hfill \qed

Now set
\begin{align*}
\psi(y) = \frac{1}{2} d_E(y)^2
\end{align*}
(5.35)
where $l$ is the map defined in (5.23) of Lemma 5.11. Thanks to the above proposition we can repeat the proof of Proposition 5.16 to get its counterpart in the case of a nondegenerate maximum point.

**Proposition 5.17.** There exists $\bar{\rho} > 0$ such that for any $y$ with $\text{dist}(y, x_0) \leq \bar{\rho}$ the map $\psi$ defined in (5.35) is of class $C^2$ and its differential is given by
\begin{align*}
d\psi(y)[v] = -(E - V(y)) g(\check{q}_y(0), v) = -\psi(y) g(\check{q}_y(0), v).
\end{align*}
(5.36)
Proposition 5.18. There exists \( \tilde{\rho} \leq \rho \) such that for any \( y \) with \( \text{dist}(y, x_0) \leq \tilde{\rho} \) it is
\[
H^\psi(y)[v, v] > 0, \quad \forall v : d\psi(y)[v] = 0.
\]

Proof. We need to evaluate
\[
\frac{\partial^2}{\partial s^2} \left( \psi(\eta(s)) \right)_{s=0},
\]
where \( \eta(s) \) is the geodesic with respect to the Jacobi metric \( g_E \) such that \( \eta(0) = y, \eta(0) = v \), where \( d\psi(y)[v] = 0 \). We also recall that \( \eta(s) \) satisfies equation (5.28). By (5.36) and exploiting (5.30), one gets
\[
H^\psi(y)[v, v] = \psi(y)g(\dot{\eta}(0), v)^2 - \psi(y)g(\dot{d}\eta(0)[v], v) - \frac{\psi(y)}{E - V(y)} \left( -\frac{1}{2} g(\dot{v}, v)g(\dot{\eta}(0), \nabla V(y)) + g(\nabla V(y), v)g(\dot{\eta}(0), v) \right).
\]

Since \( g(\dot{\eta}(0), v) = d\psi(y)[v] = 0 \), it suffices to show the existence of \( \nu_0 > 0 \) such that
\[
\inf_{|v|=1} g\left( d\dot{\eta}(0)[v], v \right) + \frac{g(\dot{\eta}(0), \nabla V(y))}{2(E - V(y))} \geq \nu_0
\]
for any \( y \) close sufficiently to \( x_0 \). Let us consider the map \( \mu(t) = g(\dot{\eta}(t), \nabla V(q_y(t))) \).
By (5.28) it is
\[
\mu(t) - \mu(0) = \int_0^t \mu'(\tau) \, d\tau = \int_0^t \left[ g(-\nabla V(q_y), \nabla V(q_y)) + H^V(y)(q_y)[\dot{q}_y, \dot{q}_y] \right] \, d\tau
\]
then, by Proposition 5.13, and nondegeneracy of the maximum point \( x_0 \), we see that there exists \( \nu > 0 \) such that
\[
\mu(t) - \mu(0) \leq -\nu \int_0^t |\dot{q}_y|^2 \, d\tau = -\nu \int_0^t (E - V(q_y(\tau))) \, d\tau,
\]
and since \( \lim_{t \to +\infty} \mu(t) = 0 \) we have
\[
g(\dot{q}_y(0), \nabla V(y)) = \mu(0) \geq \nu \int_0^{+\infty} (E - V(q_y(\tau))) \, d\tau.
\]
Now, consider the map \( \kappa(t) = E - V(q_y(t)) \): it is
\[
\kappa''(t) = -H^V(q_y)[\dot{q}_y, \dot{q}_y] + g(\nabla V(q_y), \nabla V(q_y)).
\]
Again, by nondegeneracy of \( x_0 \) as maximum point and Proposition 5.13 there exists \( A > 0 \) such that
\[
g(\nabla V(q_y(t)), \nabla V(q_y(t))) \leq A(E - V(q_y(t))).
\]
while the conservation law of the energy for $q_y$ gives $\frac{1}{2}g(\dot{q}_y, q_y) = E - V(q_y)$. Then there exists $B > 0$ such that $\kappa''(t) \leq B\kappa(t)$ for $t \geq 0$, and by Remark 5.15

$$E - V(q_y(t)) \geq (E - V(y))e^{-\sqrt{B}t}.$$ 

Then

$$g(\dot{q}_y(0), \nabla V(y)) \geq \nu(E - V(y)) \int_0^{+\infty} e^{-\sqrt{B}\tau} d\tau.$$ 

Finally, by Proposition 5.16, $d\dot{q}_y(0) \to d\dot{q}_{x_0}(0)$ while $d\dot{q}_{x_0}(0)[v] = \dot{\xi}_0(0)$ where $\xi_0(t)$ is the unique solution of

$$\begin{cases}
\dot{\xi}_0 + LV(x_0)[\xi_0] = 0 \\
\xi_0(0) = v \\
\lim_{t \to +\infty} \xi_0(t) = \lim_{t \to +\infty} \dot{\xi}_0(t) = 0.
\end{cases}$$

But, denoting by $e_i$ a basis of eigenvectors for $LV(x_0)$ and by $\lambda_i < 0$ the corresponding eigenvalues we have

$$\xi_0(t) = \sum_{i=1}^{m} v_ie^{-\sqrt{\lambda_i}t}e_i.$$ 

Since $d\dot{q}_{x_0}(0)[v] = \dot{\xi}_0(0)$ and $-H^V(x_0)$ is positive definite, there exists $\mu_0 > 0$ such that

$$g(d\phi_0(0)[v], v) \geq \mu_0 g(v, v),$$

and (5.37) is completely proved. □

Finally, we give the result needed to prove our multiplicity result for homoclinics in [5]. To this aim, take $y \in \{x : V(x) < E\}$ and consider

(5.38) $$d(y) = \text{dist}_E(y, V^{-1}(E))$$

where $\text{dist}_E$ is the distance with respect to the Jacobi metric. Combining the results of Theorem 5.9, Lemma 5.11, Propositions 5.17, 5.18 and using the function (5.38) gives us the following:

**Theorem 5.19.** Assume that:

- (a) $V^{-1}([-\infty, E]) \cup \{x_0\}$ is homeomorphic to an open ball of $\mathbb{R}^m$;
- (b) $dV(x) \neq 0$ for all $x \in V^{-1}(E) \setminus \{x_0\}$;

 moreover, let $d$ be as in (5.38). Then, there exists a positive number $\delta_*$ such that, setting

$$\Omega_* = \{x \in \mathbb{R}^M : d(x) > \delta_*\}$$

and denoting by $D_0$ the connected component of $\partial \Omega_*$ close to $x_0$ and by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{x_0\}$, the following results hold:

1. $\partial \Omega_*$ is of class $C^2$;
2. $\Omega_*$ is homomorphic to an annulus;
3. $\Omega_*$ is strongly concave with respect to the Jacobi metric $g_E$;
4. if $x : [0, 1] \to \overline{\Omega_*}$ is an orthogonal geodesic chord in $\overline{\Omega_*}$ relatively to the Jacobi metric $g_E$ such that $x(0) \in D_0$ and $x(1) \in D_1$, then there exists $[\alpha, \beta] \supset [0, 1]$ and a unique extension $\widehat{x} : [\alpha, \beta] \to \overline{\Omega_*}$, $x \in C^0 \cap H_{loc}^{1,2}([\alpha, \beta], \overline{\Omega_*})$ satisfying
   - $\widehat{x}$ is a geodesic with respect to the Jacobi metric;
   - $\widehat{x}(s) \in d^{-1}([-\delta_*, 0])$ for all $s \in [0, \overline{1}]$;
   - $\widehat{x}(\alpha) = x_0, \widehat{x}(\beta) \in V^{-1}(E) \setminus \{x_0\}$. 

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