A classification of Poisson homogeneous spaces of complex reductive Poisson-Lie groups

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Abstract

Let $G$ be a complex reductive connected algebraic group equipped with the Sklyanin bracket. A classification of Poisson homogeneous $G$-spaces with connected isotropy subgroups is given. This result is based on Drinfeld’s correspondence between Poisson homogeneous $G$-spaces and Lagrangian subalgebras in the double $D(\mathfrak{g})$ (here $\mathfrak{g} = \text{Lie } G$). A geometric interpretation of some of Poisson homogeneous $G$-spaces is also proposed.

Let $G$ be a Poisson-Lie group, $\mathfrak{g} = \text{Lie } G$, let $D(\mathfrak{g})$ be the double corresponding to the Lie bialgebra $\mathfrak{g}$. We say that a subalgebra $\mathfrak{l} \subset D(\mathfrak{g})$ is called Lagrangian if it is a maximal isotropic subspace with respect to the natural scalar product in $D(\mathfrak{g})$. It follows from [4] that there is a one-to-one correspondence between Poisson homogeneous $G$-spaces (up to isomorphism) with connected stabilizers and Lagrangian subalgebras $\mathfrak{l} \subset D(\mathfrak{g})$ such that $\mathfrak{l} \cap \mathfrak{g}$ is a Lie algebra of a certain closed subgroup in $G$ (up to $G$-conjugacy).

Now let $G$ be a connected complex reductive algebraic group equipped with the Sklyanin bracket. By $\langle \cdot, \cdot \rangle$ denote any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that its restriction on a compact real form of $[\mathfrak{g}, \mathfrak{g}]$ is positive definite. Then $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$, and the natural scalar product in $D(\mathfrak{g})$ is given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle,$$

(1)

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where \( x_1, x_2, y_1, y_2 \in \mathfrak{g} \) (see Section 1).

In this paper we obtain a description of orbits of the diagonal \( G \)-action on the set of all Lagrangian subalgebras in \( \mathfrak{g} \times \mathfrak{g} \) (see Theorem 2.1) and specify the orbits of Lagrangian subalgebras \( I \subset \mathfrak{g} \times \mathfrak{g} \) such that the subalgebra \( I \cap \mathfrak{g}_{\text{diag}} \subset \mathfrak{g}_{\text{diag}} \simeq \mathfrak{g} \) corresponds to a certain closed subgroup in \( G \) (see Theorem 2.2; here by \( \mathfrak{g}_{\text{diag}} \) we denote the diagonal image of \( \mathfrak{g} \) in \( \mathfrak{g} \times \mathfrak{g} \)). Thus we get a classification of all Poisson homogeneous \( G \)-spaces with connected stabilizers.

Note that the description of \( G \)-orbits on the set of Lagrangian subalgebras \( I \subset \mathfrak{g} \times \mathfrak{g} \) such that \( I \cap \mathfrak{g}_{\text{diag}} = 0 \) was obtained in [1]; this result is related to a classification of the solutions of the classical Yang-Baxter equation. A classification of structures of a Poisson homogeneous space on \( G/H \), where \( H \) is a Cartan subgroup, was independently obtained by Jiang-Hua Lu; this structures are closely related to the solutions of the classical dynamical Yang-Baxter equation (see [8]).

This paper is organized as follows. In Section 1 we recall the definition of the Sklyanin bracket on \( G \). In Section 2 we formulate classification theorems. Section 3 presents methods of the proof of Theorem 2.1. In Section 4 we propose a geometric interpretation of some of Poisson homogeneous \( G \)-spaces, i.e., we construct a Poisson manifold \( X \) with a Poisson \( G \)-action such that \( G \)-orbits on \( X \) are Poisson homogeneous \( G \)-spaces, and different orbits are not isomorphic (note that in the case when the Poisson bracket on \( G \) is zero, an analogue of \( X \) is \( \mathfrak{g}^* \) with the Kirillov’s bracket and the coadjoint action of \( G \)).

Note that in this paper we only formulate the main results and give a brief description of methods of proofs. The complete proofs will be presented elsewhere.

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## 1 Poisson structure on \( G \)

Let us recall the definition of the Poisson structure on \( G \). Fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). Let \( \mathbf{R} \) be the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \), \( \mathbf{R}_+ \) the set of positive roots with respect to a certain system of simple roots \( \Gamma \subset \mathbf{R} \).
Set
\[ n_+ = \bigoplus_{\alpha \in R_+} g_{\alpha}, \quad n_- = \bigoplus_{\alpha \in R_+} g_{-\alpha}, \]
\[ b_+ = h \oplus n_+, \quad b_- = h \oplus n_. \]

Consider \( r = \frac{1}{2} t_0 + t_1 \) (here the tensor \( t = t_0 + t_1 + t_2 \in g \otimes g \) corresponds to the bilinear form \( \langle \cdot, \cdot \rangle \), \( t_0 \in h \otimes h, t_1 \in n_+ \otimes n_, t_2 \in n_- \otimes n_+ \). We have \( r = r_{sym} + r_{alt} \), where \( r_{sym} \) is symmetric and \( r_{alt} \) is skew-symmetric. Let \( r^{\mu\nu} \) be the components of the tensor \( r \) in some basis \( \{ e_\mu \} \subset g \). Denote by \( \partial_\mu \) (respectively by \( \partial'_\mu \)) the right-invariant (respectively left-invariant) vector field corresponding to \( e_\mu \). Since \( r \) satisfies the classical Yang-Baxter equation and \( r_{sym} \) is \( g \)-invariant (see [4, §4]), we see that the Sklyanin's formula
\[ \{ \phi, \psi \} = r^{\mu\nu} (\partial'_\mu \phi \cdot \partial'_\nu \psi - \partial_\mu \phi \cdot \partial_\nu \psi) \]
(here \( \phi, \psi \) are regular functions on \( G \)) defines the structure of Poisson-Lie group on \( G \). The structure of a Lie bialgebra on \( g = \text{Lie} G \) is defined by the Manin triple \( (g \times g, g_{\text{diag}}, m) \), where \( g \times g \) equipped with the scalar product (1),
\[ m = \{(x, y) \in b_- \times b_+ \mid x_h + y_h = 0\}, \]
\( x_h \) (respectively \( y_h \)) is the image of \( x \) (respectively of \( y \)) in \( h \) (see [4, §3, Example 3.2]). In particular, the double \( D(g) \) is equal to \( g \times g \) equipped with the scalar product (1).

2 Classification theorems

Fix a Cartan subalgebra \( h \subset g \). Let \( R \) be the root system of \( g \) with respect to \( h \).

Let \( P, P' \subset R \) be parabolic subsets (see [2, Ch.6, §1.7]). Set
\[ p = h \oplus (\bigoplus_{a \in P} g_a), \quad p' = h \oplus (\bigoplus_{a \in P'} g_a). \]

Then \( p \) and \( p' \) are parabolic subalgebras in \( g \). Set \( A = P \cap (-P), \quad A' = P' \cap (-P') \). Let \( a \) and \( a' \) be the semisimple subalgebras in \( g \) generated by \( A \) and \( A' \) respectively. Let \( h = a \cap h, \quad h' = a' \cap h \). Note that \( \hat{h} \) (respectively \( \hat{h}' \)) is the linear span of the coroots \( \alpha^\vee \in h \) such that \( \alpha \in A \) (respectively \( \alpha \in A' \)).
Let $\sigma : A \to A'$ be an isomorphism of the root systems such that $\sigma$ preserves the scalar product. Set
\[ U = \{ \alpha \in A' \mid \sigma^{-k}(\alpha) \in A' \ \forall \ k \in \mathbb{N} \}. \]
Since the sets $A, A' \subset R$ are finite, and $\sigma : A \to A'$ is a bijection, we have
\[ U = \{ \alpha \in A \mid \sigma^k(\alpha) \in A \ \forall \ k \in \mathbb{N} \} = \{ \alpha \in A \cap A' \mid \sigma^l(\alpha) \in A \cap A' \ \forall \ l \in \mathbb{Z} \}. \]
It is easy to prove that
\[ u = h \oplus \bigoplus_{\alpha \in U} g_{\alpha} \]
is a Levi subalgebra in $g$ (i.e., a reductive Levi subalgebra of a certain parabolic subalgebra in $g$), and $U$ is the root system of $u$. We consider only the case when $\sigma$ preserves a certain system of simple roots in $U$.

Let $\xi : a \to a'$ be an isomorphism such that $\xi(g_\alpha) = g_{\sigma(\alpha)}$ for all $\alpha \in A$; then $\xi(h) = h'$, and $\xi$ preserves $(\cdot, \cdot)$. Let the linear map $\sigma^\vee : h \to h'$ be given by $\sigma^\vee(\alpha^\vee) = \sigma(\alpha)^\vee$, where $\alpha \in A$; then $\xi(x) = \sigma^\vee(x)$ for all $x \in h$. Note that
\[ [u, u]^\xi = \{ x \in [u, u] \mid \xi(x) = x \} \]
is a reductive Lie algebra, and $h^\xi = [u, u]^\xi \cap h$ is a Cartan subalgebra in $[u, u]^\xi$ (see [1, Ch.4, §4.2]).

Consider a nilpotent element $x \in [u, u]^\xi$ (we say that an element $x \in g$ is called nilpotent if $x \in [g, g]$ and $ad x$ is nilpotent). Let $h \in h^\xi$ be the characteristic of the nilpotent element $x$ (see [3, Ch.6, §2.1]; recall that one can reconstruct $x$ by $h$ uniquely up to conjugation). Let the isomorphism $\theta : a \to a'$ be given by $\theta = \xi \cdot \exp(ad x)$.

Let $3$ (respectively $3'$) be the orthogonal complement to $h$ (respectively $h'$) in $h$. Note that the natural maps $3 \to p/[p, p]$ and $3' \to p'/[p', p']$ are isomorphisms. Consider $3 \times 3'$ equipped with the scalar product [(4)]. Let $l_0 \subset 3 \times 3'$ be a Lagrangian subspace. Consider
\[ l = \{(x, y) \in g \times g \mid x \in p, y \in p', \theta(x_a) = y_{a'}, (x_3, y_{3'}) \in l_0 \}; \]
here $x_a$ is the image of $x$ in $a$, $x_3$ is the image of $x$ in $3 = p/[p, p]$, $y_{a'}$ is the image of $y$ in $a'$, $y_{3'}$ is the image of $y$ in $3' = p'/[p', p']$. Then $l$ is a Lagrangian subalgebra in $g \times g$. By $L(P, P', \sigma, \xi, h, l_0)$ denote the class of $G$-conjugacy of $l$. 

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Theorem 2.1  1) Any $G$-orbit on the set of all Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ is of the form $L(P, P', \sigma, \xi, h, l_0)$.

2) $L(P, P', \sigma, \xi, h, l_0) = L(\tilde{P}, \tilde{P}', \tilde{\sigma}, \tilde{\xi}, \tilde{h}, \tilde{l}_0)$ iff $(P, P', \xi, h, l_0)$ and $(\tilde{P}, \tilde{P}', \tilde{\xi}, \tilde{h}, \tilde{l}_0)$ are $N(h)$-conjugate (here by $N(h)$ we denote the normalizer of $h$ in $G$).

Notes.  1) Let $W$ be the Weyl group of the root system $\mathcal{R}$. If $(P, P', \xi, h, l_0)$ and $(\tilde{P}, \tilde{P}', \tilde{\xi}, \tilde{h}, \tilde{l}_0)$ are $N(h)$-conjugate, then $(P, P', \sigma, h)$ and $(\tilde{P}, \tilde{P}', \tilde{\sigma}, \tilde{h})$ are $W$-conjugate.

2) Every class of $G$-conjugacy of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ depends on the discrete parameters $(P, P', \sigma, h)$ and the continuous parameters $(\xi, l_0)$. Fix $(P, P', \sigma)$ and denote by $\Xi$ (respectively by $\Lambda$) the space of parameters $\xi$ (respectively $l_0$) such that $\xi$ (respectively $l_0$) corresponds to $(P, P', \sigma)$. Let $\langle A \rangle$ be the linear span of $A$, $n = \dim \mathfrak{z}$. It can be proved that

$$\dim \Xi = \dim \{ \alpha \in \langle A \rangle \mid \sigma(\alpha) = \alpha \},$$

$$\dim \Lambda = \frac{n(n-1)}{2}$$

(note that $\Lambda$ is the Lagrangian Grassmann manifold for $\mathfrak{z} \times \mathfrak{z}'$).

We shall say that a class of $G$-conjugacy $L(P, P', \sigma, \xi, h, l_0)$ is called integrable (respectively algebraic integrable) if the subalgebra $l \cap \mathfrak{g}_{\text{diag}} \subset \mathfrak{g}_{\text{diag}} \simeq \mathfrak{g}$ corresponds to a closed (respectively closed by Zariski) subgroup in $G$ for a certain (and then for every) Lagrangian subalgebra $l \in L(P, P', \sigma, \xi, h, l_0)$.

Theorem 2.2 gives a test of the integrability and the algebraic integrability of $L(P, P', \sigma, \xi, h, l_0)$.

Let $H \subset G$ be the connected subgroup such that $\text{Lie } H = \mathfrak{h} \subset \mathfrak{g}$.

Theorem 2.2 A class of $G$-conjugacy $L(P, P', \sigma, \xi, h, l_0)$ is integrable (respectively algebraic integrable) iff the subspace

$$V = \{ x \in \mathfrak{h} \mid (x_3, x_{\tilde{3}'}) \in l_0, \ \sigma^{\tilde{\gamma}}(x_\tilde{h}) = x_{\tilde{3}'}, \ \sigma^{\tilde{\gamma}}(x_\tilde{h}) \in \mathfrak{h} \} \subset \mathfrak{h}$$

(3) is the Lie algebra of a closed (respectively closed by Zariski) subgroup in $H$.

Note. It follows from the Theorem 2.2 that the (algebraic) integrability of a class of $G$-conjugacy $L(P, P', \sigma, \xi, h, l_0)$ depends only on $\sigma$ and $l_0$ (and is independent of $\xi$ and $h$).
Now we recall a well-known method to verify that a subspace $V \subset \mathfrak{h}$ is the Lie algebra of a closed (respectively closed by Zariski) subgroup in $H$.

Consider the lattice

$$\mathcal{H} = \text{Ker}(\exp: \mathfrak{h} \to H) \subset \mathfrak{h}.$$ 

**Proposition 2.3** (see [9, Ch.3, §2, Theorem 5]) A subspace $V \subset \mathfrak{h}$ corresponds to a closed by Zariski subgroup in $H$ iff $V$ is defined over $\mathbb{Q}$ with respect to the lattice $\mathcal{H}$, i.e., $V = \mathcal{V} \otimes \mathbb{C}$ for a certain sublattice $\mathcal{V} \subset \mathcal{H}$.

Let $t = \mathcal{H} \otimes \mathbb{R} \subset \mathfrak{h}$.

**Proposition 2.4** A subspace $V \subset \mathfrak{h}$ corresponds to a closed subgroup in $H$ iff $V \cap t$ is defined over $\mathbb{Q}$ with respect to the lattice $\mathcal{H}$, i.e., $V \cap t = V \otimes \mathbb{R}$ for a certain sublattice $V \subset \mathcal{H}$.

### 3 Methods of the proof of Theorem 2.1

Now we present a way to prove Theorem 2.1.

Let $p, p' \subset \mathfrak{g}$ be parabolic subalgebras. We have $p/p^\perp = \mathfrak{a} \oplus \mathfrak{z}$, where $\mathfrak{a}$ is semisimple, and $\mathfrak{z}$ is abelian; the same holds for $p'$. Let $\theta: \mathfrak{a} \to \mathfrak{a}'$ be an isomorphism such that $\theta$ preserves $\langle \cdot, \cdot \rangle$. We shall say that a triple $(p, p', \theta)$ is called *admissible*. By $T(\mathfrak{g})$ denote the set of all admissible triples.

Consider $(p, p', \theta) \in T(\mathfrak{g})$. Let $l_0 \subset \mathfrak{z} \times \mathfrak{z}'$ be a Lagrangian subspace with respect to the bilinear form $(\mathbb{1})$. We say that a quadruple $(p, p', \theta, l_0)$ is called *admissible*. Suppose $(p, p', \theta, l_0)$ is an admissible quadruple; then set

$$l(p, p', \theta, l_0) := \{(x, y) \in p \times p' \mid \theta(x_\mathfrak{a}) = y_\mathfrak{a}', (x_\mathfrak{z}, y_\mathfrak{z}') \in l_0\} \subset \mathfrak{g} \times \mathfrak{g},$$

where $x_\mathfrak{a}$ is the image of $x$ in $\mathfrak{a}$, $x_\mathfrak{z}$ is the image of $x$ in $\mathfrak{z}$, $y_\mathfrak{a}'$ is the image of $y$ in $\mathfrak{a}'$, $y_\mathfrak{z}'$ is the image of $y$ in $\mathfrak{z}'$. It is not hard to prove the following proposition.

**Proposition 3.1** 1) $l(p, p', \theta, l_0)$ is a Lagrangian subalgebra.

2) The correspondence $(p, p', \theta, l_0) \mapsto l(p, p', \theta, l_0)$ is a $G$-equivariant bijection between the set of all Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ and the set of all admissible quadruples $(p, p', \theta, l_0)$.

3) Lagrangian subalgebras $l(p, p', \theta, l_0)$ and $l(p, p', \theta, \tilde{l}_0)$ are $G$-conjugate iff $l_0 = \tilde{l}_0$. 

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Thus a classification of Lagrangian subalgebras is reduced to a classification of admissible triples up to $G$-conjugacy. It can be shown that the theory of admissible triples is quite similar to the theory of automorphisms of complex semisimple Lie algebras. In fact, there exists a natural notion of a semisimple admissible triple; we can define a notion of an invariant subalgebra for an admissible triple; for any semisimple admissible triple there exists an invariant Cartan subalgebra; it is possible, using invariant Cartan subalgebras, to give a complete description of semisimple admissible triples up to $G$-conjugacy; for any admissible triple there exists an analogue of the Jordan decomposition, etc. The realization of this program leads us to Theorem 2.1.

4 A geometric interpretation

In this section we give a geometric interpretation of some of Poisson homogeneous $G$-spaces.

By $\bar{G}$ denote the group of all automorphisms $g : \mathfrak{g} \to \mathfrak{g}$ such that the following conditions hold: (1) $g$ preserves the scalar product $\langle \cdot, \cdot \rangle$; (2) $g$ is equal to the identity mapping on the center of $\mathfrak{g}$. Suppose $g \in \bar{G}$ and set

$$I_g = \{(x, y) \mid x = g(y)\} \subset \mathfrak{g} \times \mathfrak{g}.$$ 

Then $I_g$ is a Lagrangian subalgebra. Note that the Lagrangian subalgebras $I_g$ form the classes of $G$-conjugacy $L(P, P', \sigma, \xi, h, l_0)$ such that $P = P' = R$ and $l_0$ is the image of the center of $\mathfrak{g}$ under the diagonal mapping $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$.

Let us give a geometric interpretation of Poisson homogeneous $G$-spaces corresponding to Lagrangian subalgebras of the form $I_g$. Note that the connected component of the center of $G$ acts trivially on the subalgebras $I_g$; therefore it is enough to consider the case when $G$ is semisimple, i.e., $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In the following part of this section we consider the case $G = \text{Int} \mathfrak{g}$.

Let $\phi, \psi$ be regular functions on $\bar{G}$. Consider

$$\{\phi, \psi\} = -r_{\mu
u}^{\text{alt}} \cdot (\partial'_\mu \phi - \partial_\mu \phi) \cdot (\partial'_\nu \psi - \partial_\nu \psi) + r_{\text{sym}}^{\mu\nu} \cdot (\partial'_\mu \phi - \partial_\mu \phi) \cdot (\partial'_\nu \psi + \partial_\nu \psi), \quad (2)$$

where $r$, $\partial_\mu$ and $\partial'_\mu$ are defined in Section 1. By $X$ we denote the manifold $\bar{G}$ equipped with the bracket (2).
Theorem 4.1 The bracket (2) is a Poisson bracket, the action of $G$ on $X$ by conjugations is Poisson, and the orbits of this action are Poisson homogeneous $G$-spaces such that the Lagrangian subalgebra $l_g$ corresponds to a point $g \in X$.

Note. The bracket (2) is a special case of the bracket from [10, Theorem 3.1], when $J_1 = -J_2$ (using the notation from [10]). See also [7].

Theorem 4.1 can be proved by using the following general result (see Theorem 4.2). Suppose $G$ is an arbitrary Poisson-Lie group. We say that a double of $G$ is a Lie group $D$ such that the following conditions hold:

1. $\text{Lie } D = D(g)$;
2. The natural scalar product in $D(g)$ is invariant with respect to the adjoint action of $D$ (then $D$ becomes a Poisson-Lie group by means of the canonical element $r \in g \otimes g^* \subset D(g) \otimes D(g)$, see [4, §13]);
3. $G$ is a closed Poisson-Lie subgroup in $D$.

Theorem 4.2 Let $G$ be a Poisson-Lie group, $g = \text{Lie } G$. Let $D$ be a double of $G$. Consider the action of $G$ on the Poisson manifold $D/G$ by left translations. Suppose $w \in D$ and denote by $x$ the image of $w$ in $D/G$; then $X = G \cdot x$ is a Poisson homogeneous $G$-space, and the Lagrangian subalgebra $l_x := w \cdot g \cdot w^{-1} \subset D(g)$ corresponds to the pair $(X,x)$.

In our case take $D = \tilde{G} \times \tilde{G}$; then Theorem 4.1 follows from Theorem 4.2.

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