Factorization in Exclusive and Seminclusive Decays and Effective Theories for Massless Particles

Ugo Aglietti, Guido Corbò

Dipartimento di Fisica, Università di Roma ‘La Sapienza’
INFN, Sezione di Roma I, P.le A. Moro 2, 00185 Roma, Italy.

Abstract

We prove rigorously factorization in the seminclusive decay $B \to D^{(*)} + jet$ using the large energy effective theory. It is also shown that this effective theory is unable to consistently describe completely exclusive processes, such as the decay $B \to D^{(*)} + \pi(\rho)$, and therefore also related properties such as factorization. This is due to an oversimplification of transverse momentum dynamics. We present a variant of the large energy effective theory, i.e. a new effective theory for massless particles, which properly takes into account transverse momentum dynamics and is therefore the natural framework to study exclusive non-leptonic decays. Finally, it is shown that the collinear instability of the large energy effective theory disappears when seminclusive observables are considered.
1 Introduction and Summary of the Results

The Large Energy Effective Theory (LEET) has been introduced in ref.\[1\] in connection with factorization of exclusive non-leptonic decay amplitudes of heavy mesons, i.e. in processes like

\[ B \rightarrow D^{(*)} + \pi(\rho). \]  \hspace{1cm} (1)

By factorization we mean that the decay (1) consists of two independent, i.e. non-interacting, subprocesses: the transition of a $B$ meson into a $D^{(*)}$ meson by the action of the weak $b \rightarrow c$ current (carrying a momentum $q$),

\[ B \rightarrow D^{(*)}, \]  \hspace{1cm} (2)

and the creation of a light meson by the action on the vacuum of the light $ud$ current (carrying a momentum $-q$, with $q^2 = m_\pi^2 (m_\rho^2)$),

\[ 0 \rightarrow \pi(\rho). \]  \hspace{1cm} (3)

The importance of factorization is that it allows the prediction of the rate of (1) in terms of the rates of two simpler (and well known) processes. To compute the non-leptonic rate under factorization assumption, we need only the form factors entering the semileptonic decay

\[ B \rightarrow D^{(*)} + e + \nu_e \]  \hspace{1cm} (4)

and the $\pi(\rho)$ decay constant, the former entering the experimentally well known leptonic decay

\[ \pi \rightarrow \mu + \nu_\mu \]  \hspace{1cm} (5)

(for experimental tests see [3]). Factorization essentially means that there is no interaction between the system composed of the $B$ and $D^{(*)}$ mesons and the system composed of the $\pi$ or the $\rho$. In other words factorization means that the exchange of hadrons between the two systems does not give a correction to the rate. In perturbation theory, the interaction between these two systems is represented by gluons connecting the valence quarks of the heavy mesons with the valence quarks of the light meson, and factorization means that these gluons have no dynamical effect. A formal condition for factorization to hold is that the one-loop anomalous dimension of the four
fermion operator $O$ inducing the decay (1) equals the sum of the one-loop anomalous dimensions of the current $J_h$ inducing the semileptonic decay (4) and of the current $J_l$ inducing the leptonic $\pi$-decay (5):

$$\gamma_O = \gamma_{J_h} + \gamma_{J_l}. \quad (6)$$

This condition is, in general, not satisfied [3].

Let us present the qualitative ideas about factorization in the decay (1). The aim of this paper is to give a theoretical support to these ideas, or to disprove them, by means of effective theories. We start from the picture of the process given by Bjorken [4]. The light quark and the light antiquark forming the $\pi$($\rho$) are created by the weak $ud$ current at the decay time in the same spatial point in a color singlet state (let us neglect for the moment the $ud$ color octet current, i.e. let us neglect hard gluon effects setting $C_+, C_- = 1$, see sec.(4)). This implies that there is no net color charge at the decay time. As the light quarks fly away, they separate from each other generating a growing color dipole. If the quarks are emitted with a very large energy $E$, color dipole growth will be slow because of time dilation. More specifically, if

$$\tau \sim 10^{-23} \text{sec} \quad (7)$$

is the time for the light pair to hadronize into a light meson at rest, the time $t$ for the formation of the meson in the decay (1) is

$$t = \gamma_L \tau \quad (8)$$

where $\gamma_L = E/\mu \gg 1$ is the Lorentz factor and $\mu$ is the pair mass. As a consequence, the color dipole is very small when the light pair is inside the heavy meson, so the interaction between the two systems is negligible in the infinite energy limit,

$$E \to \infty. \quad (9)$$

This picture suggests factorization of the non-leptonic decay amplitude (1). The argument is not rigorous in many respects. Criticism is related to general properties of quantum field theory dynamics. Particles cannot be localized inside a region smaller than their Compton wavelength

$$\lambda_C = 1/m. \quad (10)$$
This length is of the order of the confinement radius (or more) \( r_C \sim 1/\Lambda_{QCD} \sim 1 \text{ fm} \) for a light quark:

\[
\lambda_C \sim r_C. \tag{11}
\]

That means that the light quarks are created at a relative distance \( d \) of the order of the confinement radius,

\[
d \sim r_C. \tag{12}
\]

The increase of the emission energy \( E \) boosts the longitudinal momenta of the quarks and consequently shrinks the system in the longitudinal direction; it does not change the transverse momenta and the transverse dimension of the system:

\[
d_L \sim \frac{r_C}{\gamma}, \quad d_T \sim r_C. \tag{13}
\]

Therefore the color dipole strength is not zero at the decay time: it is large, of order \( g r_C \), where \( g \) is the quark color charge. A large color dipole means strong color interactions with the heavy system and an (expected) substantial violation of factorization.

The above argument can be rephrased in terms of the uncertainty principle. For the dipole field strength to be zero and factorization to hold, we require the transverse separation \( b \) between the light quarks, the so-called impact parameter, to be zero:

\[
b = 0. \tag{14}
\]

As a consequence, the average relative transverse momentum squared of the light pair is infinite:

\[
\langle q_T^2 \rangle = \infty. \tag{15}
\]

This implies that a bound state \( | \psi \rangle \), which is a state with a finite relative transverse momentum between the constituents,

\[
\langle q_T^2 \rangle \sim \Lambda_{QCD}^2, \tag{16}
\]

has zero overlap with the state \( | b = 0 \rangle \):

\[
\langle \psi | b = 0 \rangle = 0. \tag{17}
\]
To sum up: the probability of hadronization into a given meson state by a light quark pair generating a vanishing color dipole, is zero. The chain of arguments is the following:

\[ (\text{factorization}) \Rightarrow b = 0 \Rightarrow \langle q_T^2 \rangle = \infty \Rightarrow \langle \psi \mid b = 0 \rangle = 0. \quad (18) \]

Furthermore, it is well known that if one localizes a particle in a region smaller than \( \lambda_C \), additional particle-antiparticle pairs are created. The result is a mutiparticle state. The probability for a multiparticle state to hadronize into a single meson is very small.

In conclusion, we believe that factorization cannot be rigorously proved for the completely exclusive process (1). The point to keep in mind is that factorization holds as long as we consider states of the light quark system with small impact parameter

\[ b \sim 0. \quad (19) \]

We can change our viewpoint and consider the seminclusive process

\[ B \to D^{(s)} + \text{jet}. \quad (20) \]

The criticism above does not apply to this seminclusive process, in which the light quarks produce a jet of hadrons, instead of a single meson. Squeezing the color dipole (necessary for factorization!), as we have seen, induces pair creation, but this is not a problem for a jet, which is a multiparticle state. The large relative transverse momentum squared of the squeezed pair is not even a problem for the seminclusive decay. Assume that the light quarks are created at a very small transverse separation \( b \),

\[ b \ll r_C. \quad (21) \]

Because of the uncertainty principle, this implies a broad range of relative transverse momenta,

\[ \langle q_T^2 \rangle \sim \frac{1}{b^2} \gg \Lambda_{QCD}^2. \quad (22) \]

Partons with high energy, producing jets, can sustain large transverse momentum fluctuations. Their motion is specified by a space direction, so if we call \( \theta \) the deflection angle due to a transverse momentum transfer \( q_T \), we have:

\[ \theta \sim \frac{q_T}{E} \to 0 \quad (23) \]
in the ‘hard’ limit

\[ E \to \infty, \quad (24) \]
as long as we keep \( b > 0 \), i.e. small but not zero. Formally, we take the limits in the following order:

(i) we send \( E \to \infty \) keeping \( b \) constant but not zero;

(ii) we send \( b \to 0 \) to have complete color screening and factorization.

In words: the motion of the hard parton is not modified by transverse momentum fluctuations. These fluctuations affect only parton shower development and the hadronization process, i.e. the latter stages of jet dynamics. In the seminclusive process we get rid of these effects summing over all possible jet developments.

To summarize, we argue that factorization cannot be proved rigorously in the exclusive process \( (1) \) due to basic quantum field theory phenomena: these problems can be circumvented considering in place of the exclusive process the seminclusive one \( (20) \).

The rest of this paper is a formalization of the above ideas and considerations. We will see that the intuitive idea that a vanishing color dipole has no interactions, so its dynamics is factorized, can be rigorously formalized in gauge theories through the use of Wilson loops.

Let us discuss now the organization of the paper and outline the main results. In sec.4 we review the basic elements of the LEET. The physical content is very simple. This effective theory describes massless quarks with a very large energy \( E \) or, equivalently, quarks suffering soft interactions only. If \( k \) is the momentum exchanged by the quark, we require:

\[ | k_\mu | \ll E. \quad (25) \]

This means that the motion of the effective quark can be approximated by a straight line, with the velocity of light, in the (soft) collisions. In other words, we consider the action of the hard parton on soft quarks and gluons, but we neglect the reaction of the soft particles on the hard parton (recoil effect). This is very well seen looking at the propagator in configuration space in an arbitrary external gauge field,

\[ iS_F(x,0) = \int_0^\infty d\tau \ \delta^{(4)}(x - n\tau) \ P e^{i \int d^4y \ J_\mu(y) A^\mu(y)} \quad (26) \]
where \( J_\mu(y) \) is a color current associated with the classical motion of the quark:

\[
J_\mu(y) = g \int_0^\tau ds \ n_\mu \ \delta^{(4)}(y - ns).
\] (27)

The propagator has support (i.e. it is not zero) only on the points of the classical trajectory

\[
x_\mu = n_\mu \tau, \quad \tau \geq 0,
\] (28)

and the interaction produces only a phase factor in color space (Wilson P-line). The probability for the effective quark to reach a given point is not modified by the interaction. By this fact we mean that the reaction of the light particles on the effective quark is neglected. All this is fine. However, it has been shown in ref. [5] that the LEET suffers from consistency problems. We believe that the proof of the consistency of the LEET is a preliminary step for any phenomenological application, such as the justification of factorization. The problem originates from the fact that the LEET contains states with negative energies, because the dispersion relation is of the form

\[
\epsilon = k_z,
\] (29)
i.e. the modulus of the space momentum is missing (we have taken for simplicity a motion along the \( z \) axis). The negative energies cannot be removed with a Lorentz transformation as in the case of the HQET, and render the model unstable. The effective quark \( Q \) can decay into states with arbitrarily negative energies emitting collinear particles:

\[
Q \rightarrow Q + (\text{coll. part}).
\] (30)

Therefore one-particle states are unstable, with the consequence for example that:

\[
|Q\rangle_{\text{in}} \neq |Q\rangle_{\text{out}}.
\] (31)

In sec. 3 we review the collinear instability phenomenon and we prove that it disappears when semi-inclusive observables are considered. The idea is that of replacing the single particle states of the effective quark \( Q \) with cones centered along the line of motion of \( Q \) containing any particle.

In sec. 4 we discuss the application of the LEET to exclusive processes, and in particular to the factorization of non-leptonic amplitudes. As it is clear from the physical considerations done before, the LEET cannot describe
any bound state effect. It is clearly impossible for an effective quark and an
effective antiquark, each one moving with uniform rectilinear motion,
\begin{align*}
x^{(Q)}_\mu &= n\tau, \\
x^{(\bar{Q})}_\mu &= n\tau',
\end{align*}
(32)
to produce any bound state. Technically, we will see that all the correlation
functions describing exclusive dynamics turn out to be singular and trivial in
the \textit{LEET}. All this comes from neglecting transverse momentum dynamics
(just look at the dispersion relation (29)): the effective quarks are created
and remain at any time at a fixed transverse separation, \( b = 0 \) in the case of
the local \( ud \) current.

In sec.5 we simply reinterpret the results of the previous section. The
singularity of the correlators related to exclusive dynamics disappears once
seminclusive processes are considered, and their triviality does mean factorization
in this case. We also discuss the connection of the \textit{LEET} with the theory of color coherence, describing interjet activity in the perturbative
\textit{QCD} framework [6].

Having discovered that the \textit{LEET} is not the relevant theory for studying
exclusive processes like (1), we may ask whether it is possible to formulate
another effective theory for massless particles which does the job. In sec.6 we
introduce such a new effective theory which we call Modified Large Energy
Effective Theory (\textit{LEET}). The idea is that of including in the dispersion
relation (29) the first correction dependent on the transverse momentum, so
that the latter is replaced by:
\[ \epsilon = k_z + \frac{\vec{k}^2_T}{2E}, \]
(33)
where \( \vec{k}^2_T = k_x^2 + k_y^2 \). We will see that the effect of the transverse momentum
in the propagator in configuration space is a diffusive one. The propagator is
not concentrated only on the classical trajectory, as is the case for the \textit{LEET},
but it is diffused in a region which is growing with time. The probability for
the quark to remain into the classical trajectory decays as the inverse of the
time squared:
\[ P(\text{classical path; } t) \sim \frac{1}{t^2}. \]
(34)
The inclusion of transverse momentum makes the composite correlation functions non-singular and not trivial at the same time. In general, the dynamics of the LEET is much richer and complicated than that of the LEET, but it is still much simpler than in the original Dirac theory. The simplification essentially occurs because pair creation is absent, so the vacuum in the matter sector is trivial. The LEET is not related to the theory of Wilson loops, as it was the case of the LEET; as a consequence, for example, the propagator of the LEET cannot be computed in closed form in the interacting case, as it was for the LEET. The LEET bears some resemblance with the Non-Relativistic QCD (NRQCD) \[7\], which is an effective field theory with the non-relativistic energy-momentum relation:

\[
\epsilon = \frac{\vec{k}^2}{2m} \tag{35}
\]

where \(m\) is the heavy quark mass (compare eq.(33) with eq.(35)).

In general, we believe that the LEET is the effective theory for massless particles which is the right framework to study exclusive processes, and consequently also factorization of exclusive non-leptonic decays.

Sect.7 contains the conclusions of our investigation.

2 The Large Energy Effective Theory

Let us briefly review the basic elements of the LEET \[1, 5\]. It is derived decomposing the momentum \(P\) of a massless quark into a ‘classical part’ \(En\) and a fluctuation \(k\):

\[
P = En + k. \tag{36}
\]

\(n\) is a light-like vector \(n^2 = 0\), normalized by the condition \(v \cdot n = 1\), where \(v\) is a reference time-like vector, \(v^2 = 1\), \(v_0 > 0\). Therefore \(E\) is the classical quark energy in the rest frame of \(v\) and has to be considered large,

\[
| k_\mu | \ll E. \tag{37}
\]

The propagator is given by

\[
iS_F(En + iD) = \frac{i}{E\hat{n} + i\hat{D} + i\epsilon}
\]

\(\hat{n}\) is a normalizable one-dimensional representation of the Lorentz algebra. The propagator is valid at distances \(| x - y | \gg \frac{1}{\Lambda}\), where \(\Lambda\) is the strong coupling constant. The propagator is composed of an intermediate state propagator and a factor representing the difference of quark charge between the initial and final state.

In the next section, we will briefly review the basic elements of the LEET. It is derived decomposing the momentum \(P\) of a massless quark into a ‘classical part’ \(En\) and a fluctuation \(k\):

\[
P = En + k. \tag{36}
\]

\(n\) is a light-like vector \(n^2 = 0\), normalized by the condition \(v \cdot n = 1\), where \(v\) is a reference time-like vector, \(v^2 = 1\), \(v_0 > 0\). Therefore \(E\) is the classical quark energy in the rest frame of \(v\) and has to be considered large,

\[
| k_\mu | \ll E. \tag{37}
\]

The propagator is given by

\[
iS_F(En + iD) = \frac{i}{E\hat{n} + i\hat{D} + i\epsilon}
\]

\(\hat{n}\) is a normalizable one-dimensional representation of the Lorentz algebra. The propagator is valid at distances \(| x - y | \gg \frac{1}{\Lambda}\), where \(\Lambda\) is the strong coupling constant. The propagator is composed of an intermediate state propagator and a factor representing the difference of quark charge between the initial and final state.
\[
\begin{align*}
= & \left( E\hat{n} + i\hat{D} \right) \frac{i}{2E \hat{n} \cdot iD - D^2 + \sigma_{\mu\nu} g G^{\mu\nu} / 2 + i\epsilon} \\
= & \frac{\hat{n}}{2} \frac{i}{n \cdot iD + i\epsilon} + \frac{1}{2E} \frac{i}{2E \frac{\hat{n}}{2} \frac{i}{n \cdot iD + i\epsilon} \frac{i}{(-i)D^2} \frac{i}{n \cdot iD + i\epsilon}} \\
+ & \frac{ig \frac{i}{4E} \frac{i}{2E} \frac{i}{n \cdot iD + i\epsilon} \frac{i}{n \cdot iD + i\epsilon} \frac{i}{2E} \frac{i}{n \cdot iD + i\epsilon} + O \left( \frac{1}{E^2} \right). \quad (38)
\end{align*}
\]

where \( \hat{a} = \gamma_\mu a^\mu, \sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2, D_\mu = \partial_\mu - igA_\mu \) and \(-igG_{\mu\nu} = [D_\mu, D_\nu].\)

We made the replacement \( k \to iD \) to include magnetic interactions (we assume the usual convention that \( \psi(x) = \exp(-ik \cdot x) \)).

We treat \( 1/E \) corrections as perturbations to the leading term in eq.(38), so that the LEET propagator is

\[
i\hat{S}_F(k) = \frac{\hat{n}}{2} \frac{i}{n \cdot k + i\epsilon} \quad (39)
\]

The energy-momentum relation (the pole of the propagator) is

\[
\epsilon = \vec{u} \cdot \vec{k} \quad (40)
\]

where \( \vec{u} = \vec{n}/n_0 \) is the kinematical velocity, \( |\vec{u}| = 1.\)

The propagator in configuration space is the scalar density of a particle moving along a ray with the velocity of light:

\[
i\hat{S}_F^0(x) = \frac{\hat{n}}{2} \int_0^\infty d\tau \delta^{(4)}(x - n\tau) = \frac{\hat{n}}{2} \theta(t) \frac{\delta^{(3)}(\vec{x} - \vec{u}t)}{n_0} \quad (41)
\]

The interaction with the gauge field produces a P-line factor joining the origin with the point \( x \) along the light-like trajectory specified by \( n:\)

\[
i\hat{S}_F(x) = i\hat{S}_F^0(x) \exp \left[ i g \int_0^{t/n_0} dt' n_\mu A^\mu(nt') \right]. \quad (42)
\]

Note the factorization of both the spin and the color degrees of freedom. The propagator can be written in a completely covariant form, reminiscent of the Dyson formula for the S-matrix:

\[
i\hat{S}_F(x) = \frac{\hat{n}}{2} \int_0^\infty d\tau \delta^{(4)}(x - n\tau) \exp \left[ i \int d^4y J_\mu(y) A^\mu(y) \right] \quad (43)
\]
where $J_\mu$ is a classical current

$$J_\mu(y) = g \int_0^\tau ds \ n_\mu \delta^{(4)}(y - ns).$$

(44)

In momentum space, we have:

$$iS_F(k) = \frac{\hat{n}}{2} \int_0^\infty d\tau \ e^{i(k\cdot n + i\epsilon)\tau} \left[ 1 + ig \int \frac{d^4p}{(2\pi)^4} n_\mu A^\mu_{-}(p) \frac{e^{i(p\cdot n + i\epsilon)\tau} - 1}{i(p\cdot n + i\epsilon)} + \ldots \right]$$

(45)

The Feynman rule for the vertex is

$$V = i\gamma_\mu t^a.$$ (46)

The LEET Lagrangian, omitting the spin structure, is

$$\mathcal{L}(x) = \overline{Q}_n(x) \in \cdot DQ_n(x) + O(\frac{1}{E}).$$ (47)

The LEET describes massless particles suffering soft interactions only.

3 The Instability and Its Cure

Let us briefly review the instability phenomenon \[5\]. A simple example is the collision between an effective quark $Q$ with (residual) momentum $k$ and a full quark $q$ with momentum $p$. Energy and momentum conservation give

$$|\vec{p}| + \vec{u} \cdot \vec{k} = |\vec{p}'| + \vec{u} \cdot \vec{k}'$$

$$\vec{p} + \vec{k} = \vec{p}' + \vec{k}'$$

(48)

where $q$ is taken massless for simplicity. Let us assume that $p + k$ is a time-like vector, $(p + k)^2 > 0$. In the COM frame, $\vec{p} + \vec{k} = 0$, with $\vec{n}$ oriented along the +z axis, we have

$$p(1 - \cos \theta) = p'(1 - \cos \theta') = \mathcal{E}$$

(49)

\[1\] This formula is easily derived from eq.\[44\] making use of the eikonal identity

$$\frac{1}{k \cdot n + i\epsilon} - \frac{1}{(k + p) \cdot n + i\epsilon} = \frac{n \cdot p}{(n \cdot k + i\epsilon)((k + p) \cdot n + i\epsilon)}.$$
where $\theta$ is the angle between the quark 3-momentum and $\vec{n}$. The prime denotes final state quantities and

$$\mathcal{E} = \sqrt{\left(\sqrt{\varepsilon^2 + \|\varepsilon\|}\right)^2} \ll \mathcal{E}$$

(50)

is the total energy of the system in the LEET; it is expected to be of the order $\Lambda_{QCD}$ in QCD applications, while $E$ is the hard scale of the process. The instability originates because

$$p' = \frac{\mathcal{E}}{1 - \cos \theta'} \to \infty$$

(51)

when

$$\theta' \to 0.$$  

(52)

It is related to the emission of particles in the forward direction $\vec{n}$, the flight direction of $Q$.

Let us assume now a finite angular resolution $\delta > 0$ of the detectors. This implies that $Q$ cannot be distinguished from almost-collinear partons. We consider a cone of half-opening angle $\delta$ with the axis along $\vec{n}$. We have that $q$ is observed as an individual particle if it is emitted in the final state outside the cone,

$$\theta' > \delta.$$  

(53)

In this case the energy is bounded by

$$p' < \frac{\mathcal{E}}{1 - \cos \delta} < \infty,$$  

(54)

and cannot diverge anymore. Therefore, the divergence (51) does not occur for an observable parton. On the other hand, if the final parton is inside the cone,

$$\theta' < \delta,$$  

(55)

the finite angular resolution makes impossible to detect $q$ and $Q$ as separated particles and to measure their individual energies. A single particle (jet) is observed with the sum of the parton energies

$$p' + \epsilon' = \frac{\mathcal{E}}{1 - \cos \theta'} - \cos \theta' \frac{\mathcal{E}}{1 - \cos \theta'} = \mathcal{E} < \infty$$  

(56)
The individual energies are separately divergent but the sum is finite (small) by assumption (it equals the initial energy). The instability is therefore eliminated by the angular separation requirement. We stress that the cone has to be centered around the vector $\vec{n}$ and not around the residual momentum $\vec{k}$. In a general process, the instability is eliminated replacing the states of the effective quarks with cones centered around their line of flight.

### 3.1 Matching

The angular resolution $\delta$ is a parameter generated inside the effective theory by a consistency requirement, but its value is completely arbitrary. Let us see now the restrictions imposed on the values on $\delta$ by comparison with the full theory, the so-called matching. Before the expansion, the energy-momentum relation is

$$
\epsilon = \sqrt{P_Z^2 + P_T^2 - E} = \sqrt{(E + k_Z)^2 + k_T^2 - E} = |E + k_Z| - E + \frac{k_T^2}{2 |E + k_Z|} + \frac{k_T^4}{8 |E + k_Z|^3} + O(k_T^8) \quad (57)
$$

where in the last line a power expansion in $k_T$ has been done.

Going to the LEET implies two different kinds of approximations:

(i) Neglecting the modulus in eq.\((57)\). Keeping the modulus of the momentum would imply non-local operators in the effective theory, which are against the spirit of the effective theories themselves. Omitting the modulus we loose the positivity of the energy and the instability problem comes up. As we have seen, this problem is solved redefining the observables, i.e. considering seminclusive ones. This approximation is unavoidable.

(ii) Neglecting the transverse momentum $k_T$ in eq.\((57)\). The effect of this approximation is to mistreat bound state dynamics. As we will see, the correlators describing bound states, come out trivial and singular at the same time, as a consequence of this approximation. It is possible to remedy to this problem keeping transverse momentum terms in eq.\((57)\):

$$
\epsilon = k_Z + \frac{k_T^2}{2E} - \frac{k_Z k_T^2}{2E^2} + \frac{k_Z^2 k_T^2}{2E^3} - \frac{k_T^4}{8E^3} + \ldots \quad (58)
$$
Note, in particular, that keeping terms in the residual momentum to any finite order (i.e. $1/E$ corrections) does not stabilize the theory, as one could instead naively think.

The LEET dispersion relation is therefore:

$$\epsilon = k_Z.$$  \hfill (59)  

Let us take $k_T = 0$ in eq.(57). We see that the LEET is a good approximation as long as

$$|k_Z| \leq E$$  \hfill (60)  

The above condition for the final state momentum, $|k'_Z| \leq E$, implies

$$\frac{\cos \theta'}{1 - \cos \theta} \leq \frac{E}{\epsilon}. \hfill (61)$$

Comparing with eq.(54) we have, for small angles:

$$\delta \geq \delta_C = \sqrt{\frac{2\epsilon}{E}} \quad (\delta \ll 1) \hfill (62)$$

Eq.(62) is our final result. It says that the LEET is a good approximation of the full theory as long as $\delta$ is taken above a critical value $\delta_C$ below which fictitious energies come into the game.

The angular resolution $\delta$ we have introduced is similar in spirit to the parameter introduced to cancel collinear singularities in massless theories [8]. There are however differences. Collinear singularities induce a divergence in the cross section, while in our case cross sections are finite but involve states with infinite energies. Furthermore, in the process we considered, Rutherford scattering, there are no collinear singularities.

4 Exclusive Processes and Factorization

In this section we show that the LEET cannot be used consistently to describe exclusive processes such as the decay [1]. The starting point is the reduction formula, according to which any scattering matrix element $S_{fi}$ can be derived from the correlation functions of the theory [1],

$$G(x_1, x_2, \ldots, x_n) = \langle 0 | T\phi(x_1)\phi(x_2)\ldots\phi(x_n) | 0 \rangle \hfill (63)$$
where $\phi$ denotes generically a field, and $T$ is the Dyson time-ordering operator. This assumption implies that if a given process cannot be derived from any correlation function of the theory, the latter does not describe the process.

4.1 Spectroscopy

Let us begin by considering the simplest correlator, the propagator of a light meson such as a $\pi$ or a $\rho$, i.e. the 2-point function

$$C(x) = \langle 0 | TO(\vec{x}, t) O_\uparrow(\vec{0}, 0) | 0 \rangle$$

where

$$O(x) = \pi(x) \Gamma d(x).$$

and $\Gamma$ is a matrix in the Dirac algebra. We can set to zero the residual spatial momentum of the meson because we are interested in internal dynamics only, so we consider

$$C(t) = \int d^3 x \ C(t, \vec{x}).$$

The correlation $C(t)$ has the spectral decomposition [10] for $t > 0$

$$C(t) = \sum_n \frac{\langle 0 | O(0) | M_n(\vec{p} = 0) \rangle^2}{2m_n} e^{-im_n t} + (\text{multiparticle states})$$

where $M_n$ denotes all the meson states and the lightest state, depending on the spin-parity of the current, can be a $\pi$, a $\rho$, etc. [7] C(t) has the functional integral representation:

$$C(t) = \int d^3 x \ N \int [dA_\mu] e^{iS_{eff}[A_\mu]} \langle 0 | TO(\vec{x}, t) O_\uparrow(\vec{0}, 0) | 0 \rangle_A$$

where

$$N^{-1} = \int [dA_\mu] e^{iS_{eff}[A]}$$

The states have been normalized in a covariant way:

$$\langle B(p) | B(p') \rangle = (2\pi)^3 2E(p) \delta^{(3)} (\vec{p} - \vec{p}').$$
is a normalization factor, \( S_{\text{eff}}[A] = S_{YM}[A] - n_l/2 \log \det[i\hat{D}] \) is the effective gauge field action, and \( n_l \) is the number of light flavors. The \( T \)-ordered product with the subscript \( A \) means the expectation value in the fermionic vacuum in a background gauge field \( A_\mu \). The Wick contraction gives:

\[
\langle 0 | TO(\vec{x}, t)O^\dagger(\vec{0}, 0) | 0 \rangle_A = - Tr[ iS_F(x | 0; A_\mu) \tilde{\Gamma} iS_F(0 | x; A_\mu) \Gamma ]
\]

where \( \tilde{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0 \). Substituting the effective quark propagator, eq.(42), for \( iS_F(x | 0) \), and an analogous effective propagator for the antiquark,

\[
iS_F(0 | x) \to iS_{\text{eff}}(0 | x) = \frac{n}{2} \theta(t) \frac{\delta(3)(\vec{x} - \vec{u}t)}{n_0} P e^{i \int_x^0 A_\mu dx_\mu},
\]

we derive:

\[
C(t) =
\]

\[
= - \theta(t)^2 \int d^3x \frac{\delta(3)(\vec{x} - \vec{u}t)^2}{n_0^2} Tr \left[ \frac{\hat{n} \cdot \vec{n}}{2} \tilde{\Gamma} \frac{\hat{n} \cdot \vec{n}}{2} \Gamma \right] Tr \left[ P e^{ig \int_0^x A_\mu dx_\mu} IP e^{ig \int_0^x A_\mu dx_\mu} I \right]
\]

\[
= - \frac{\theta(t)}{n_0^2} Tr_{\text{spin}} \left[ \frac{\hat{n} \cdot \vec{n}}{2} \tilde{\Gamma} \frac{\hat{n} \cdot \vec{n}}{2} \Gamma \right] Tr_{\text{col}} \left[ I \right] \delta^{(3)}(\vec{x} = 0)
\]

where \( Tr_{\text{col}}[I] = N_C = 3 \). A few remarks are in order.

(i) There is an ultraviolet (UV) power divergence coming from the product of the two delta functions, which we have regularized with a sharp cut-off on the spatial momenta

\[
\delta^{(3)}_\Lambda(\vec{x}) = \int^\Lambda \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} = \frac{\Lambda^3}{6\pi^2} \delta_{\vec{x},0}
\]

This divergence originates from the fact that the LEET quarks are created in the same point by the (local) current and move along the same (classical) trajectory.

(ii) The P-lines of the quark and the antiquark are one the inverse of the other, so they give the unit operator in color space, independent of the gauge field \( A_\mu \). The functional integration over \( A_\mu \) is therefore trivial. This implies that there is not any color interaction, i.e. there are no gluons exchanged between the quarks: the theory is free.
The main point is that the correlator is constant in time: comparing with the spectral decomposition (67) we derive that all the meson masses vanish:

\[ m_n = 0 \quad \text{any } n. \quad (74) \]

Spectroscopy disappeared, but this is an infrared property which should be represented by an effective theory. As we will see in sec. 6, it is still possible to build up an effective theory for massless particles in which spectroscopy is not made trivial.

Furthermore, since the contribution of any state to the correlator is time-independent, there is no way to separate in principle the contribution of any selected meson (i.e. any exclusive state), such as a \( \pi \) or a \( \rho \). The simultaneous singular behaviour and, more important, trivial behaviour of the effective correlator is related to neglecting altogether transverse momentum dynamics (see sec. 6).

We conclude that the correlator (64) describes the propagation of a jet in a color singlet state, i.e. a state defined in a seminclusive way, in which it is not possible to separate any exclusive state.

### 4.2 Nonleptonic Decays

Let us consider now the non-leptonic decay (1). The rate can be computed, according to the standard reduction formula, from the values of the following 4-point correlation function

\[ C_4(z, y, x) = \langle 0 \mid T O(z) O_D(y) \mathcal{H}_W(t) O_B^\dagger(\xi) \mid t \rangle \quad (75) \]

taking the asymptotic limits

\[ t_x \to -\infty, \quad t_y \to +\infty, \quad t_z \to +\infty. \quad (76) \]

\( O_B = \overline{\eta} \Gamma b \) and \( O_D = \overline{\eta} \Gamma c \) are two interpolating fields for the \( B \) and \( D^{(*)} \) mesons respectively. \( \mathcal{H}_W \) is the effective non-leptonic weak hamiltonian, which may be written as

\[ \mathcal{H}_W(\xi) = \frac{G_F}{\sqrt{\xi}} \left[ \mathcal{C}_\infty \mathcal{O}_\infty(\xi) + \mathcal{C}_V \mathcal{O}_V(\xi) \right] \quad (77) \]
where $G_F$ is the Fermi constant and $O_1(x)$ and $O_8(x)$ are local 4-fermion operators which are the product of two singlet and octet currents in color space,

$$O_i(x) = \bar{d}(x) \gamma_{\mu L} \xi_i \ u(x) \ \bar{c}(x) \ \gamma_{\mu L} \xi_i \ b(x)$$

(78)

where $\gamma_L^i = \gamma^\mu (1 - \gamma_5), \ \xi_i = 1, t^a$ for $i = 1, 8$ respectively, and $C_1$ and $C_8$ are Wilson coefficients resumming hard gluon effects of the form $\alpha_s^n \log^k (m_W^2/m_b^2)$, $k \leq n$.[11]

Inserting the expression (77) for the weak hamiltonian in eq.(75), and making the Wick contractions as we did for the 2-point function, we derive:

$$\langle 0 \ | T \ O(z) \ O_D(y) \ H_{WY}(t) \ O_B^\dagger(z) \ | \eta \rangle_A$$

$$= \sum_{i=1,8} C_i \ Tr \left[ iS_c(y \ | \ 0; A_\mu) \ \gamma_L \xi_i \ iS_b(0 \ | \ x; A_\mu) \ \tilde{\Gamma}_B \ iS_s(x \ | \ y; A_\mu) \ \Gamma_D \right]$$

$$Tr \left[ iS_d(z \ | \ 0; A_\mu) \ \gamma_L^i \xi_i \ iS_u(0 \ | \ z; A_\mu) \ \Gamma \right],$$

(79)

where we have taken a valence strange quark ($q = s$) to avoid unimportant contractions. Each contribution involves the product of two separate fermionic traces. The term with $i = 8$ (i.e. the contribution of the octet operator $O_8$) does not contribute to the correlation because the trace of the light pair in color space involves a matrix $t^a$ and is therefore proportional to

$$Tr \left[ P \ e^{i \int_0^z A_\mu \ dx^\mu} \ t^a \ P \ e^{i \int_0^z A_\mu \ dx^\mu} \ I \right] = Tr \ t^a = 0.$$  (80)

The physical reason of this result is that a quark and an antiquark in the same spatial point in a color singlet state cannot emit a gluon and go into an octet state: the dipole field strength is zero.

The only contribution comes from the term with $i = 1$ (i.e. from $O_1$ only). The correlation (79) has a factor describing $ud$ dynamics similar to the 2-point function (70),

$$\langle 0 \ | T O(z) \ J_{\mu}^{u\rightarrow d}(0) \ | 0 \rangle = -Tr \left[ iS_d(z \ | \ 0) \ \gamma_L^\mu \ iS_u(0 \ | \ z) \ \Gamma \right]$$

$$= -\theta(t) \ n_0^2 \ Tr_{spin} \left[ \hat{n} \ \Gamma \ \hat{n} \ \gamma_L^\mu \right] \ Tr_{col} \left[ I \right] \delta^{(3)}(\vec{x} = 0)$$

(81)

where $J_{\mu}(x)^{u\rightarrow d} = \bar{d}(x) \gamma_L^\mu u(x)$. This implies similar considerations as those given for the 2-point function: the dynamics of the light pair is independent
of the gauge field, so the $u$ and $d$ quarks do not have any interaction with the heavy system (the $B$ and $D^{(*)}$ mesons) and do not have any reciprocal interactions either. Therefore, there is factorization in the sense that the $u-d$ system does not interact with the heavy system, but there is not any interaction either between the quarks giving rise to the $\pi$ or the $\rho$ meson. A given exclusive channel such as [ ] cannot be selected in principle from the correlation.

To summarize, we say that the state created by the $ud$ current acting on the vacuum is a jet, an indistinguishable ensemble of exclusive states (consisting of strictly collinear $u$ and $d$ quarks in lowest order). The LEET is not adequate to discuss issues such as factorization in exclusive channels.

### 4.3 Perturbative Expansion

If we compute the meson propagator (64) in the full theory (in ordinary 4-momentum space), we find in lowest order perturbation theory:

$$ C_F \sim \int \frac{1}{(xP + k)^2 + i\epsilon} \frac{1}{((1-x)P - k)^2 + i\epsilon} $$

$$ \sim \int \frac{1}{k_0 - k_z + k^2/(2xE) + i\epsilon} \frac{1}{k_0 - k_z - k^2/(2(1-x)E) - i\epsilon} $$

$$ \sim \int \frac{d^2k_T \, dk_+ \, dk_-}{(k_- + k_+ k_-(2xE) - k_T^2/(2xE) + i\epsilon)} $$

$$ \sim \frac{1}{(k_- - k_+ k_-(2(1-x)E) + k_T^2/(2(1-x)E) - i\epsilon)} $$

where we have taken an external momentum $P = E n$ with $n = (1; 0, 0, 1)$ and $E > 0$. The variable $x$ represents the quark momentum fraction in the infinite momentum frame ($0 \leq x \leq 1$) and $k_+ = k_0 + k_z$ and $k_- = k_0 - k_z$ are the usual light-cone variables and $k_T^2 = k_T^2$. The poles in the $k_-$-plane are

---

3We neglect the numerator structure of the amplitude because we are looking only at infrared singularities, which appear as zeros of the denominator.

4We assume that the quark momentum distribution in the meson $q(x)$ is not singularly peaked at the endpoints $x = 0, 1$, so that $xE$ and $(1-x)E$ can always be considered as large energies. A physical justification of this assumption comes from an expected Sudakov suppression of the elastic region.
located at

\[ k_- = \frac{k_T^2/(2xE) - i\epsilon}{1 + k_+//(2xE)}, \quad k_+ = \frac{-k_T^2/(2(1-x)E) + i\epsilon}{1 - k_+//(2(1-x)E)} \]  \quad (82)

Assuming a cutoff \( \Lambda \) on \( k_+ \) satisfying \( \Lambda \ll E \), the integral is approximated by

\[ C_F \sim \int d^2k_T \int dk_+ \int dk_- \frac{1}{k_- - k_T^2/(2xE) + i\epsilon} \frac{1}{k_+ + k_T^2/(2(1-x)E) - i\epsilon} \]  \quad (83)

There is a pinching of the poles in the \( k_- \)-plane for \( k_T = 0 \): in other words, the integration contour is trapped between two poles which coalesce in the limit \( k_T \to 0 \). The integral is logarithmically divergent \[ \text{with } \epsilon \] with \( \epsilon \):

\[ C_F \sim \int \frac{dk_T^2}{k_T^2 - i\epsilon} \sim \log \frac{1}{\epsilon}. \]  \quad (84)

The effective theory amplitude is obtained taking the limit \( E \to \infty \) in the integrand:

\[ C_E(r = 0) \sim \int d^2k_T \int dk_0dk_z \frac{1}{k_0 - k_z + i\epsilon} \frac{1}{k_0 - k_z - i\epsilon} \]  \quad (85)

where \( r \) is the meson residual momentum, which we set to zero.

The integrations over the transverse momentum and over \( k_+ \) give the cubic ultraviolet divergence found before. The integral over \( k_- \) involves instead a pinch singularity due to the infinitesimally close poles at \( k_- = \pm i\epsilon \). We note that pinching occurs in the whole transverse momentum space, while it occurs only for \( k_T \to 0 \) in the full theory. Integrating over \( k_- \) we pick up a \( 1/\epsilon \), i.e. a linearly divergent contribution:

\[ C_E \sim \frac{1}{\epsilon}. \]  \quad (86)

---

5The infrared logarithmic singularity originates because the integral (82) does not contain any scale \( (P^2 = 0) \), so it is of the form \( \int d^4k/(k^2)^2 \), as can be seen explicitly introducing a Feynman parameter.
Thus the infrared behaviour of the full theory is not reproduced by the LEET: a logarithmic infrared singularity in the full theory is in fact converted into a linear (i.e. much stronger) singularity in the effective theory.

The conclusion is that the LEET cannot be consistently applied to exclusive processes. Radiative corrections (powers in $\alpha_S$) and power suppressed corrections (powers in $1/E$) do not make the situation better. As a typical example of a radiative correction, let us consider the vertex correction of order $\alpha_S$ to the meson propagator, which gives an integral of the form

$$V \sim \int dk_+ dk_- d^2k_T dl_+ dl_- d^2l_T \frac{1}{k_+ + i\epsilon} \frac{1}{k_- - i\epsilon} \frac{1}{(k - l)^2} \frac{1}{l_+ + i\epsilon} \frac{1}{l_- - i\epsilon}$$

(87)

The diagram is affected by pinch singularities of higher order with respect to the tree diagram.

Corrections in $1/E$ are more singular than the lowest order term because they give rise to expressions of the form

$$\frac{1}{(k_+ + i\epsilon)^2} \frac{k_T^2}{2E} \frac{1}{k_- - i\epsilon'} \frac{1}{k_+ + i\epsilon} \frac{1}{(k_- - i\epsilon)^2} \frac{k_T^2}{2E'} \cdots$$

(88)

These integrands have pinch singularities of higher order with respect to the tree diagram. Therefore the ‘disease’ is not even cured by corrections in $1/E$, which actually made the situation worse.

### 4.4 Analogies with the HQET

There is a close analogy with the Heavy Quark Effective Theory (HQET) when the latter is applied to describe quarkonium states (for example the $b\bar{b}$-spectroscopy). The quarkonium propagator $Y$ (the analog of the light meson propagator in eq.64) involves a tree level contribution in the full theory

$$Y_F \sim \int d^4k \int \frac{1}{(P + k)^2 - M^2 + i\epsilon} \frac{1}{(-P + k)^2 - M^2 + i\epsilon}$$

$$\sim \int d^4k \int dk_0 \frac{1}{k_0 + k^2/(2M) + i\epsilon} \frac{1}{k_0 - k^2/(2M) - i\epsilon}$$

(89)

---

6As we have shown before, diagram [87] vanishes together with all the radiative corrections after making the trace over color, but here we look only at the singular structure of the integrand.
where we have taken an external momentum $2P$ with $P = (M, 0)$. In the complex $k_0$-plane there are four poles at

$$
k_0 = \pm \sqrt{k^2 + M^2} - M \mp i\epsilon, \quad k_0 = \pm \sqrt{k^2 + M^2} + M \mp i\epsilon,
$$

(90)

There is a pinching of the poles at $k_0 = \pm \sqrt{k^2 + M^2} \mp M \pm i\epsilon$ when $k = |\vec{k}| \rightarrow 0$. Integrating over $k_0$ we get

$$
Y_F \sim \int \frac{d^3k}{k^2/M + O(k^4) + i\epsilon},
$$

(91)

i.e. an infrared finite amplitude.\[1\]

The HQET amplitude is obtained taking the limit $M \rightarrow \infty$ in the integrand:

$$
Y_E(r^{\mu} = 0) \sim \int d^3k \int dk_0 \frac{1}{k_0 + i\epsilon} \frac{1}{k_0 - i\epsilon}
$$

(92)

In the $k_0$-plane there are poles at $k_0 = \pm i\epsilon$, so that there is a pinch singularity for every value of $\vec{k}$. The integral in $k_0$ is consequently divergent as $1/\epsilon$, contrary to the full theory amplitude which is finite for $\epsilon \rightarrow 0$. The infrared behaviour of the full theory is consequently not correctly reproduced by the HQET.

 Corrections of order $1/M$ involve higher order pinch singularities, because they generate integrands of the form

$$
\frac{1}{(k_0 + i\epsilon)^2} \frac{\vec{k}^2}{2M} \frac{1}{k_0 - i\epsilon}, \quad \frac{1}{k_0 + i\epsilon} \frac{1}{(k_0 - i\epsilon)^2} \frac{\vec{k}^2}{2M}, \ldots
$$

(93)

In the case of mesons composed of a heavy and a light quark, (such as for example a $B$-meson), the heavy quark can be consistently treated in the HQET because the light quark $q$ is described by the full theory and carries all the dynamics. The propagator of $q$ can be written in a background gauge field, because of covariance, as [12]

$$
iS_F(x \mid 0; A_\mu) = \sum_C f[C] \int_C \frac{\partial A_\mu}{\partial x^\mu} (94)$$

\[7\] In the massless case we found instead a logarithmic term related to the collinear singularity.
where $f[C]$ is a functional and its argument $C$ is any path connecting points 0 and $x$. Unlike the LEET, $iS_F$ is not concentrated on a single path (the classical one), but it involves many paths with the associated (many) P-lines. These properties of $iS_F$ make the $B$-meson propagator non-singular and non-trivial.

The meson propagator is given in the full theory in lowest order perturbation theory by

$$C_F \sim \int d^4k \frac{1}{k^2 + i\epsilon} \frac{1}{(P + k)^2 - M^2 + i\epsilon}$$

$$\sim \int d^3k dk_0 \frac{1}{k_0 - E_k + i\epsilon} \frac{1}{k_0 + E_k - i\epsilon} \frac{1}{k_0 + k^2/(2M) + i\epsilon} \tag{95}$$

where $E_k = |\vec{k}|$ since we have taken $q$ massless for simplicity. There are poles in the $k_0$-plane at

$$k_0 = \pm \sqrt{k^2 + M^2} - M \mp i\epsilon,$$

$$k_0 = \pm E_k \mp i\epsilon. \tag{96}$$

There is a pinching of the poles at $k_0 = \sqrt{k^2 + M^2} - M - i\epsilon$ and $k_0 = -E_k + i\epsilon$. The distance between these poles is

$$d = E_k + \sqrt{k^2 + M^2} - M \simeq k + O(k^2), \tag{97}$$

i.e. it is linear in $k$ for $k \ll M$.

The HQET amplitude is given by:

$$C_{HQET} \sim \int d^3k dk_0 \frac{1}{k_0 - E_k + i\epsilon} \frac{1}{k_0 + E_k - i\epsilon} \frac{1}{k_0 + i\epsilon}. \tag{98}$$

The poles of the heavy quark propagator are replaced by a single pole at $k_0 = -i\epsilon$, so there is a pinching between the latter pole and the one at $-E_k + i\epsilon$. The distance between them is

$$d = k, \tag{99}$$

so it goes to zero with $k$ as in the full theory. This implies that the infrared behaviour of the correlator is the same in the full theory and in the HQET.

The above analysis confirms the validity of the HQET approach to the description of heavy-light systems.
5 Factorization in Seminclusive Processes

In this section we show that the LEET does describe seminclusive processes and can be used to prove approximate factorization (in a sense specified below) in the process

\[ B \rightarrow D^{(*)} + \text{jet} \quad (100) \]

in the limit

\[ m_b - m_c \rightarrow \infty. \quad (101) \]

If we treat the jet as a massless system, its energy is given in the COM frame by

\[ E_{\text{jet}} = \frac{m_B^2 - m_D^2}{2m_B}. \quad (102) \]

The limit in which factorization holds is \( E_{\text{jet}} \rightarrow \infty \), which is implied by the limit (101), because \( m_B \rightarrow \infty \):

\[ E_{\text{jet}} > \frac{m_B - m_D}{2} \simeq \frac{m_b - m_c}{2}. \quad (103) \]

To prove factorization, we just have to reinterpret the results of sec.4. Replacing the up and down quarks with LEET quarks with the same velocity \( n \) in eq.(79), their color interactions with the heavy system disappear and factorization comes out. Actually, the problem is that we have a stronger property than factorization: the light quark and the light antiquark do not have any color interaction also with each other. However, in the seminclusive process the latter property does not cause any problem, because the \( u \) and \( d \) quarks have to be considered hard partons, i.e. short distance excitations, unrelated to exclusive dynamics.

To summarize: in the seminclusive case, the independence on the gauge field \( A_\mu \) of the light quark trace in eq.(79) does mean factorization.

In sec.4.3 we found that the correlators of the LEET are affected by strong pinch singularities in the perturbative expansion. In a seminclusive approach these singularities do not occur anymore because jets are defined integrating over energy and momentum intervals. Pinch singularities disappear from the correlations after integrating in the region of phase space

---

Note in particular that for factorization to hold it is not necessary to send the charm mass to infinity, but only to send the beauty mass to infinity (the latter acts as an energy reservoir).
specified by the jet definition. For the 2-point function, for example, eq.(85) is replaced by an integral of the form:

\[ \int dp_- f(p_-) \int_{-\infty}^{\infty} dk_- \frac{1}{k_- + p_- + i\epsilon} \frac{1}{k_- - i\epsilon} = 2\pi i \int dp_- f(p_-) \frac{1}{p_- + i\epsilon} = \text{(definite and finite)}. \quad (104) \]

where \( f(p_-) \) is a shape function, whose form depends on the specific definition of the jet.

To summarize, we proved in a non-perturbative way that a pair of hard quarks moving along the same classical trajectory do not have any color interaction. This implies, in particular, that they do not produce any radiation field. The suppression of the radiation field from a pair of hard partons in a color singlet state at small angular separation is a well known phenomenon in perturbative QCD. It is a particular case of the so called ‘color coherence’ \[6\]. The latter is an interference effect according to which a quark and an antiquark at an angular separation \( \delta \) in a color singlet state generate a radiation field restricted to a cone of width \( \delta \). Outside the cone, complete destructive interference takes place and no radiation is emitted. In this respect the decay (100) is analogous to the high-energy annihilation:

\[ e^+ e^- \to q + \bar{q} + \gamma \quad (105) \]

with the photon energy \( E_\gamma \) close to its kinematical endpoint,

\[ E_\gamma \sim \frac{\sqrt{s}}{2} \quad (COM \ frame). \quad (106) \]

The quark and the antiquark are in a color singlet state and are emitted at a small angle \( \delta \ll 1 \) with the \( \gamma \) recoiling in the opposite direction. According to color coherence, secondary soft partons are emitted at angles

\[ \theta < \delta \quad (107) \]

with respect to the quark and antiquark line of flight. In the limit

\[ \delta \to 0^+, \quad (108) \]
no radiation field is emitted. Another process analogous to the decay (100) is the radiative decay of the $Y$,

$$Y \rightarrow g + g + \gamma$$

(109)

close to the endpoint of the photon spectrum,

$$E_\gamma \sim m_b.$$  

(110)

The gluon pair is emitted in a color singlet state at a small angular separation $\delta \ll 1$. As in the decay (100), one can prove in a non-perturbative way that there are no color interactions of the gluon pair in the limit $\delta \rightarrow 0$ at leading order in $1/N_C$. The vanishing of color interactions at the endpoint (110) implies, in particular, that there is no typical Sudakov suppression of the differential rate $d\Gamma/dE_\gamma$ close to the endpoint as we have for example in the semileptonic $b \rightarrow u$ decay

$$b \rightarrow u + e + \nu_e.$$  

(111)

5.1 Corrections to Factorization

There are corrections to the factorization in the seminclusive process (100) related to the fact that the jet formed by the light $ud$ pair is not infinitely narrow. A quantitative discussion requires a detailed definition of a jet, such as for example the Sterman-Weinberg one, which is beyond the scope of this paper. We present only a qualitative discussion. Let us assume a jet angular width

$$2 \delta \ll 1.$$  

(112)

This implies that the light pair can be emitted with a relative angle up to $2 \delta$, so that the light-like vectors $n$ and $n'$ of the quark and the antiquark respectively, can be written up to first order as

$$n = (1; \delta, 0, 1), \quad n' = (1; -\delta, 0, 1).$$

(113)

We have taken the relative motion of the pair in the $x$ direction and $n^2 = n'^2 = \delta^2 \sim 0$.  

\footnote{We wish to thank G. Martinelli for having explained this point to us.}
5.1.1 Jet with Finite Angular Width

As usual, let us begin by considering the simplest case, a 3-point correlation function representing the creation of a pair of LEET quarks moving along $n$ and $n'$:

$$J_3(x, x') = \langle 0 \mid T B(x, x') L^\dagger(0) \mid 0 \rangle$$

(114)

where $L(y)$ is a local operator which annihilates a pair of effective quarks with velocities $n$ and $n'$:

$$L(y) = \overline{Q}_{n'}(y) \Gamma Q_n(y),$$

(115)

while $B(x, x')$ is a bilocal operator which, by covariance, may be written as

$$B(x, x') = \overline{Q}_{n'}(x') \Gamma P \exp \left[ ig \int_{x}^{x'} A_\mu dx^\mu \right] Q_n(x)$$

(116)

There is an ambiguity (of non-perturbative kind) in the choice of the path connecting point $x$ with $x'$. We consider small angles of emission of the quarks and we simply take as the path the segment joining $x$ with $x'$.

This correlation function gives a nonperturbative representation of a jet, so we may call it the jet correlator. The functional representation of the jet correlator is

$$\langle 0 \mid T B(x, x') L^\dagger(0) \mid 0 \rangle_A = -Tr \left[ i S_n(x \mid 0) \overline{\Gamma} i S_{n'}(0 \mid x') \Gamma P(x' \mid x) \right]$$

(117)

Inserting the expressions for the LEET propagators, we derive:

$$J_3(x, x') = -\theta(t) \theta(t') \frac{\delta^{(3)}(\vec{x} - \vec{u}t)}{n_0} \frac{\delta^{(3)}(\vec{x}' - \vec{u}'t')}{n'_0}$$

$$N \int [dA] e^{i S_{x,t} \mid [A]} Tr \left[ P(x \mid 0) P(0 \mid x') P(x' \mid x) \right]$$

(118)

where a more compact notation $P(y \mid x)$ has been assumed for a P-line joining $x$ with $y$.

From eq.(118) we see that the estimate of the jet correlator requires a non-perturbative computation of a Wilson loop on a triangular path, having as vertices the points

$$x = 0, \quad x = n\tau, \quad x' = n'\tau',$$

(119)

with some selected value for $\tau$ and $\tau'$.
5.1.2 Finite Jet Width in Non-Leptonic Decay

The seminclusive process \((100)\) is related to the following 5-point function

\[
J_5(z, z', y, x) = \langle 0 \mid T B(z, z') O_D(y) \mathcal{H}_W(t) O_B^\dagger(\xi) \mid t \rangle
\]

where the weak hamiltonian contains two \(\text{LEET}\) fields with different velocities \(n\) and \(n'\), so that the operators \(O_1\) and \(O_8\) in eq.(77) are of the form

\[
O_i(x) = \overline{Q}_n(x) \gamma_\mu \xi_i Q_{n'}(x) \gamma_\mu \xi_i b(x)
\]

We have:

\[
\langle 0 \mid T B(z, z') O_D(y) \mathcal{H}_W(t) O_B^\dagger(\xi) \mid t \rangle_A = \sum_{i=1,8} C_i \text{Tr} \left[ \begin{array}{c} i S_c(y \mid 0) \gamma_\mu \xi_i i S_b(0 \mid x) \hat{\Gamma}_B i S_s(x \mid y) \Gamma_D \end{array} \right]
\]

\[
\text{Tr} \left[ \begin{array}{c} i S_n(z \mid 0) \gamma_\mu \xi_i i S_n'(0 \mid z') \Gamma P(z' \mid z) \end{array} \right]
\]

\[
= \sum_{i=1,8} C_i \text{Tr} \left[ \begin{array}{c} i S_c(y \mid 0) \gamma_\mu \xi_i i S_b(0 \mid x) \hat{\Gamma}_B i S_s(x \mid y) \Gamma_D \end{array} \right]
\]

\[
\theta(t_z) \theta(t'_z) \frac{\delta^3(z - \hat{u} t_z)}{n_0} \frac{\delta^3(z' - \hat{u}' t'_z)}{n'_0} \text{Tr} \left[ \begin{array}{c} \frac{\hat{n}}{2} \gamma_\mu \hat{n}' \end{array} \right]
\]

\[
\text{Tr} \left[ \begin{array}{c} P(z \mid 0) \xi_i P(0 \mid z') P(z' \mid z) \end{array} \right].
\]

Unlike the previous case in which \(n' = n\), the P-lines do not cancel each other any more and there is a dependence on the gauge field: gauge dynamics is not trivial, factorization does not hold anymore and both the operators \(O_1\) and \(O_8\) give a non-vanishing contribution to the correlator. The corrections to factorization are related to matrix elements of the following form:

\[
\langle D^{(x)} \mid T_i J_{\mu, i}^{b+ rc}(0) \mid B \rangle
\]

where \(J_{\mu, i}^{b+ rc}(x) = \overline{\tau}(x) \gamma_\mu \xi_i b(x)\) and \(T_i\) is the following gauge invariant/covariant operator:

\[
T_i = \text{Tr} \left[ \begin{array}{c} P(z \mid 0) \xi_i P(0 \mid z') P(z' \mid z) \end{array} \right],
\]

i.e. it is the trace of a Wilson loop on the thin triangle considered in the previous section, with the insertion of \(\xi_i = 1, t_a\) at the vertex with the small angle \(2\delta \ll 1\).
According to this line of reasoning, we believe that factorization should be strongly violated in multiple jet production, i.e. in processes like

\[ B \rightarrow D^{(*)} + 2 \text{jets}. \]  \hspace{1cm} (125)

That is because, according to the picture of the process given in the introduction, in this process there is a large color dipole field of the light quarks, strongly interacting with the $B$ and $D^{(*)}$ mesons.

## 6 A New Effective Theory

We can remedy to the inadequacy of the \textit{LEET} to describe exclusive processes by including the leading kinetic correction (see eq.(58)) into the propagator:

\[ iS_F(k) = \frac{\hat{n}}{2} \left( \frac{i}{n \cdot k - k_T^2/2E + i\epsilon} \right). \]  \hspace{1cm} (126)

where we may take $n^\mu = (1; 0, 0, 1)$, $k_T^\mu = (0; k_T, 0)$ \hspace{1cm} (10). In the derivation of eq.(126) we followed an idea of ‘minimal correction’ of the \textit{LEET} pathologies; we have neglected for example the term $\hat{k}$ in the numerator of eq.(126), so that the spin structure is factorized.

We may call this new effective theory ‘Modified Large Energy Effective Theory’, \textit{LEET} for short. Note that, unlike the \textit{LEET} case, the hard scale $E$ is still present in the theory, i.e. it cannot be completely removed.

The problem of pinch singularities discussed in sec.4.3 is solved replacing \textit{LEET} propagators with \textit{LEET} propagators: the meson propagator in the \textit{LEET} has the form (83) so that pinch singularities occur only for $k_T = 0$ instead of in the whole transverse momentum space. The \textit{LEET} amplitude coincides with the full theory amplitude (82) for $k_+ \ll E$ and has the same infrared behaviour.

Eq.(126) still defines an effective theory, even though much more complicated than the \textit{LEET}: the propagator contains the 4-velocity $n$, is forward in time, so that antiparticles are removed and the vacuum is consequently trivial. This is just what we expect from an effective theory describing hard

\[ \text{We can give a Lorentz invariant representation of the transverse momentum } k_T \text{ with the Sudakov basis. If we define a second light-like vector } \eta \text{ such that } n \cdot \eta = 2 \text{ (in the usual frame, } \eta = (1; 0, 0, -1)), \text{ we have } n \cdot k_T = \eta \cdot k_T = 0 \text{ and } k_T^2 = -k_T^2 = k^2 - n \cdot k \eta \cdot k. \]
partons, in which particle-antiparticle pairs have a threshold energy of order $2E$ and cannot be excited with soft interactions. Furthermore, their virtual effect is expected to be small on the basis of the decoupling theorem [14].

The propagator is given as a function of time and spatial momentum by

$$iS_F(t, \vec{k}) = \frac{\hat{n}}{2} \theta(t) \exp \left[ -i k_Z t - i \frac{k_T^2}{2E} t \right]$$  \hspace{1cm} (127)

and in configuration space by

$$iS_F(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} iS_F(t, \vec{k}) = \frac{\hat{n}}{2} \theta(t) \delta(z - t) \frac{E}{2\pi it} e^{iEb^2/(2t)}$$

$$= iS_F^{LEET} (x) \frac{E}{2\pi it} e^{iEb^2/(2t)}$$ \hspace{1cm} (128)

where $b = |\vec{x}_T|$ is the impact parameter. The effect of the transverse momentum term is factorized and produces a diffusion in the impact parameter space: the factor in the last line of eq.(128) represents a gaussian process after analytic continuation $t_M = -it_E$. At large times, we have:

$$S(t, \vec{x}) \simeq \frac{\hat{n}}{2} \theta(t) \delta(z - t) \frac{E}{2\pi it}. \hspace{1cm} (129)$$

We see that there is a diffusion normal to the classical particle trajectory $z = t$ produced by transverse momentum fluctuations, which is instead absent in the $LEET$. The amplitude for the particle to remain into the classical trajectory decays like $1/t$, so the probability decays like $1/t^2$.

The lagrangian of the $LEET$, omitting the spin dependence, is

$$\mathcal{L}(\mathcal{Q}) = Q^\dagger(\mathcal{Q}) \left[ \bar{\mathcal{Q}} \cdot \mathcal{D} + \frac{D_T^E}{\mathcal{E}} \mathcal{Q} \right] Q(\mathcal{Q}).$$  \hspace{1cm} (130)

We believe that the $LEET$ is the correct effective theory for massless particles as long as exclusive processes are concerned.

Let us make a general observation. As we have shown in detail in sec.4, $1/E$ corrections cannot be considered perturbations in exclusive processes (i.e. local operator insertions), even though they are related to higher dimension operator: they must be kept in the unperturbed propagator. For
example, expanding the propagators in powers of $1/E$ in the meson propagator (64) is not a ‘legal’ operation. Here we have a counter-example of the general validity of Wilson’s Operator Product Expansion [15]: the kinetic operator, a higher dimensional operator, cannot be considered ‘irrelevant’ because of infrared effects.

It is hard to reach a conclusion about factorization in the exclusive decay on the basis of simple analytical computations with the LEET. That is because of diffusion in the impact parameter space, according to which light quark dynamics is represented by a superposition of Wilson loops on strips with a width of order $b$. A non-perturbative technique is needed: the LEET lagrangian can be discretized on an euclidean lattice and its dynamics can be computed numerically with lattice QCD.

We think that the LEET can describe also some perturbative QCD effects in which transverse momentum dynamics is important, and cannot be neglected altogether as in the LEET, such as for example initial state interactions in Drell-Yan processes [16]. We argue that the absence of antiparticle excitations is related to the analogous property of the vacuum state in the infinite momentum frame.

6.1 Analogies with Non Relativistic QCD

The LEET is the analog of Non Relativistic QCD (NRQCD) for massive quarks, which must be used in place of the HQET to describe quarkonium dynamics. The propagator of the NRQCD encodes the non-relativistic energy-momentum relation and is given by

$$iS_F(k) = \frac{1 + \gamma_0}{2} \frac{i}{k_0 - k^2/2M + i\epsilon}. \quad (131)$$

The basic approximations are:

(i) the spin fluctuation $\tilde{k}$ is neglected;

(ii) the antiparticle pole is eliminated, so that

$$k^2 \rightarrow -\tilde{k}^2. \quad (132)$$

The main point is that the term $\tilde{k}^2/(2M)$ is kept in the denominator, i.e. no expansion in $1/M$ is done as in the HQET. This means that a partial resummation of $1/M$ corrections is performed.
The problem of pinch singularities in quarkonium propagators discussed in sec.4.4 is eliminated replacing HQET propagators with NRQCD propagators [7]. The quarkonium propagator in NRQCD has the form:

\[ Y_{NRQCD} \sim \int d^3k \frac{1}{k_0 - \vec{k}^2/(2M) + i\epsilon} \frac{1}{k_0 + \vec{k}^2/(2M) - i\epsilon} \] (133)

The pinching of the poles occurs only in the point \( \vec{k} = 0 \) instead of in the whole 3-momentum space as with HQET propagators. Performing the integration over \( k_0 \) we derive:

\[ Y_{NRQCD} \sim \int d^3k \frac{1}{k^2/M - i\epsilon} \] (134)

which is infrared finite as the full theory correlator. Therefore the NRQCD propagator has the same infrared behaviour of the full theory propagator (eq.(89)).

7 Conclusions

The main conclusion of our analysis is that the Large Energy Effective Theory can be used to prove factorization in the semi inclusive process

\[ B \rightarrow D^{(*)} + jet \] (135)

in the limit

\[ m_b - m_c \rightarrow \infty. \] (136)

We tried our best to convince the reader that factorization in seminclusive decays can be proved pretty rigorously in a non-perturbative way with the theory of Wilson loops. The idea is that, in the limit of a very narrow jet, the Wilson loop collapses into a rectangle with two infinitesimal edges, which does not depend on the gauge field any more and can be either one or zero, depending on the operators acting at the vertices.

Factorization is exact in the limit of an infinitely narrow jet, and corrections related to a finite angular width \( \delta \ll 1 \) can be written in a gauge invariant way in terms of Wilson loops on triangular paths. The evaluation of these corrections requires a non-perturbative technique, such as for example
lattice QCD. Following the same line of reasoning, we argue that factorization should be strongly violated in multiple jet production, i.e. in processes like for example

\[ B \rightarrow D^{(*)} + 2 \text{jets}. \]  

(137)

On the other hand, we believe that factorization cannot be rigorously proved in exclusive non-leptonic decays, such as for example

\[ B \rightarrow D^{(*)} + \pi (\rho). \]  

(138)

That is because the theoretical tool, the Large Energy Effective Theory, is intrinsically incapable to describe bound state effects and exclusive hadron dynamics. This is due to neglecting transverse momentum dynamics. We have introduced a new effective theory for massless particles, the Modified Large Energy Effective Theory, which takes into account transverse momentum dynamics, and which is the right framework for studying exclusive processes. The latter theory is however more complicated so we did not succeed in deriving any conclusion about factorization on the basis of analytical computations with it.

Finally, we have also shown that the Large Energy Effective Theory is a consistent theory once seminclusive observables are considered instead of completely exclusive ones.

Acknowledgements

We wish to thank G. Martinelli and C. Sachrajda for discussions.

References

[1] M. J. Dugan and B. Grinstein, Phys. Lett. B 255 (1991) 583; the formalism is modelled in analogy with that one developed for the HQET in: H. Georgi, Phys. Lett. B 240 (1990) 447; for experimental tests of factorization in the effective theory framework see: C. Reader and N. Isgur, CEBAF-TH-91-23 preprint; T. Mannel, W. Roberts and Z. Ryzak, Phys. Rev. D 44 (1991) R18-R21.
[2] M. Bauer, B. Stech and M. Wirbel, Z Phys. C-Particles and Fields 34, 103-115 (1987); D. Bortoletto and S. Stone, Phys. Rev. D 65 n.24 (1990) 2951-2954.

[3] H. D. Politzer and M. B. Wise, Phys. Lett. B 257 (1991) 399-402.

[4] J. D. Bjorken, Nucl. Phys. B (Proc. Suppl.) 11 (1989) 325-341.

[5] U. Aglietti, Phys. Lett. B 292 (1992) 424-426.

[6] See for example: Y. L. Dokshitzer et al., Basics of Perturbative QCD, Edition Frontiers (1991), and references therein; see also: R. K. Ellis et al., QCD and Collider Physics, Cambridge University Press (1996).

[7] W. E. Caswell and G. P. Lepage, Phys. Lett. B 167 n.4 (1986) 437-442; G. P. Lepage and B. A. Thacker, Nucl. Phys. B (Proc. Suppl.) n.4 (1988) 199-203.

[8] G. Sterman and S. Weinberg, Phys. Rev. Lett. 39 (1972) 1436.

[9] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 1425 (1955).

[10] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 1442 (1955).

[11] M. K. Gaillard and B. W. Lee, Phys. Rev. Lett. 33 (1974) 108; G. Altarelli and M. Maiani, Phys. Lett. B 52 351; for the two-loop computation see: G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, Nucl. Phys. B 187 (1981) 461-513.

[12] A. M. Polyakov, Gauge Fields and Strings, Harwood Academic Press (1986).

[13] G. Altarelli et al., Nucl. Phys. B 208 (1982) 365-380.

[14] T. Appelquist and J. Carazzone, Phys. Rev. D 11, 2856 (1975).

[15] K. Wilson, Phys. Rev. 179, 1499 (1969).

[16] G. T. Bodwin, S. J. Brodsky and G. P. Lepage, Phys. Rev. Lett 47 n.25 (1981) 1799-1803.