A Gas-Kinetic Stability Analysis of Self-Gravitating and Collisional Particulate Disks with Application to Saturn’s Rings

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Abstract

Linear theory is used to determine the stability of the self-gravitating, rapidly (and nonuniformly) rotating, two-dimensional, and collisional particulate disk against small-amplitude gravity perturbations. A gas-kinetic theory approach is used by exploring the combined system of the Boltzmann and the Poisson equations. The effects of physical collisions between particles are taken into account by using in the Boltzmann kinetic equation a Krook model integral of collisions modified to allow collisions to be inelastic. It is shown that as a direct result of the classical Jeans instability and a secular dissipative-type instability of small-amplitude gravity disturbances (e.g. those produced by a spontaneous perturbation and/or a companion system) the disk is subdivided into numerous irregular ringlets, with size and spacing of the order of $4\pi \rho \approx 2\pi h$, where $\rho \approx c_r/\kappa$ is the mean epicyclic radius, $c_r$ is the radial dispersion of random velocities of particles, $\kappa$ is the local epicyclic frequency, and $h \approx 2\rho$ is the typical thickness of the system. The present research is aimed above all at explaining the origin of various structures in highly flattened, rapidly rotating systems of mutually gravitating particles. In particular, it is suggested that forthcoming Cassini spacecraft high-resolution images may reveal this kind of hyperfine $\sim 2\pi h \lesssim 100$ m structure in the main rings A, B, and C of the Saturnian ring system.

1 Introduction

Self-gravitating disk systems are of great interest in astrophysics because of their widespread appearance, e.g. disks in spiral galaxies, pancakes and accretion disks around massive objects, low mass X-ray binaries, the protoplanetary clouds, and, finally, the main rings of Saturn. Such systems are highly dynamic and are subject to various instabilities of small-amplitude gravity perturbations. This is because the evolution of these disks is primarily driven by angular momentum redistribution. This might take place through global mechanisms like (1) nonaxisymmetric instabilities caused by self-gravity or (2) instabilities caused by simultaneous action of self-gravity and effective viscous coupling between neighbouring disk annuli.

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2The strongly flattened, disk-shaped form of all these objects is due to their rapid rotation. Thus, equilibrium is established in a simple manner in such disks, i.e. it is governed mainly by the balance between the centrifugal and gravitational forces.
Because of the long-range nature of the gravitational forces between particles, a self-gravitating system exhibits collective modes of motions — modes in which the particles in large regions move coherently or in unison. In turn, collective-type oscillations have been studied in a system of electrically charged particles in a plasma (Alexandrov et al., 1984; Krall and Trivelpiece, 1986; Swanson, 1989). The similarity between the gravitational and Coulomb interactions is well known and already explored, e.g. Bertin (1980), Fridman and Polyachenko (1984), and Binney and Tremaine (1987). Therefore, in this paper we employ certain developed mathematical formalisms from plasma physics.

The dynamics of flattened gravitating systems has now been studied quite thoroughly. This research has aimed to explain the origin of various observed structures: spiral and ring formations in flat galaxies, the thin ringlets around Saturn, etc. One of the main trends has therefore been to analyze the perturbation dynamics in such systems, in both linear and nonlinear regimes (Fridman and Polyachenko, 1984; Binney and Tremaine, 1987).

In the current research, the linear stability theory of small oscillations (and their stability) of a disk of mutually gravitating particles is reexamined by using the method of gas-kinetic theory which incorporates interparticle collisions through a Krook (or the so-called $\tau$-) approximation (Lifshitz and Pitaevskii, 1981; Krall and Trivelpiece, 1986, p. 315). This representation gives new insight into the gravitating disk stability. In particular, we will investigate in detail the important limit of strong and frequent interparticle collisions. The results of the analysis are applied to Saturn’s rings, composed of rock and ice: we predict the small-scale, hyperfine structure of order 100 m to be observed in the main rings A, B, and C.

Saturnian ring system populated primarily by centimeter- to a several meter-sized mutually gravitating and physically colliding particles (Zebker et al., 1985). A particle size distribution function exhibits approximately inverse-cubic power-law behavior. In addition, Saturn has extensive but much more tenuous rings containing mainly micrometer-sized particles (Cuzzi et al., 1984; Lissauer and Cuzzi, 1985). Since collisions tend to dissipate the highly ordered motions involved in wave propagation, collisions will lead to wave dissipation. Except for rings of very small collision frequency, impacts are certain to dominate over individual gravitational encounters. Some of the properties of inelastic physical collisions in Saturn’s main rings are discussed in the article.

In fact, it should be made clear right from the start that the suggestion of hyperfine $\lesssim 100$ m structure in Saturn’s rings due to the combined effects of gravity and interparticle collisions is not an entirely new idea. Salo (1992, 1995), Willerding (1992), Richardson (1994), Osterbart and Willerding (1995), and Sterzik et al. (1995) have already predicted such a structure in Saturn’s rings by \( N \)-body computer simulations.

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Saturn is, of course, most famous for its spectacular set of rings. Attempts to find a plausible naturalistic explanation of the origin of Saturn’s rings began about 250 years ago but have not yet been quantitatively successful, making this one of the oldest unsolved problems in modern science. Many of famous mechanics and mathematics investigators, starting with Laplace and Maxwell, studied their structure, composition, and stability.
Our significant contribution is just a kinetic theory derivation of results obtained before in simulations by Salo, Osterbart, Willerding, Richardson, and Sterzik et al. (see also Griv (1998)) or through other analytic approximations (Fridman and Polyachenko, 1984, Vol. 2; Schmit and Tscharnuter, 1995, 1999).

In turn, the fairly recent Voyager missions have shown that Saturn’s main rings (the brightest A and B rings, and the semitransparent inner C ring) are divided into a huge number of irregular concentric rings (Smith et al., 1982); see Cuzzi et al. (1984), Lissauer and Cuzzi (1985), Esposito (1986, 1993), Cuzzi (1989), and Nicholson and Dones (1991) as reviews. The Voyager 2 spacecraft close-up view of Saturn’s rings shows that even the so-called gaps demonstrate a complicate structure — Cassini’s division, for example, contains perhaps 100 ringlets (e.g. Flynn and Cuzzi, 1989). It was found that the main rings exhibit large irregular variations in optical depth that are not associated with any resonances with known satellites. Actually, a new class of objects in the solar system was discovered. Moreover, the planetary rings turned out to be a necessary element and consistent phenomenon in the satellite systems of all giant planets. Because low-velocity collisions of ice particles will always involve some dissipation of acoustic energy, some source of energy must be available to keep them from total collapse to a featureless monolayer.

The best analysis of the Voyager 2 photopolarimeter PPS data and imaging science data for structure was done by Showalter and Nicholson (1990) and Horne and Cuzzi (1996), respectively. The Voyager 2 PPS experiment obtained the highest resolution of any ring observation of Saturn, profiling the variation of optical depth in radial steps of about 100 m. However, below a few kilometers scale, the PPS data is too noisy to extract information about irregular structure: the finest “structure” observed by PPS is well fit by models of statistical noise combined with stochastic variations resulting from large particles or clumps of particles (Showalter and Nicholson, 1990). So, such hyperfine ∼ 100 m structure might exist, but the Voyager 2 spacecraft could not see it. Imaging science data were sensitive only down to 10 – 20 km scales. Some ring regions only exhibit well-defined characteristics scales larger than that, while others do show power on scales all the way down to the 10 – 20 km lower limit (Horne and Cuzzi, 1996). Kilometer-scale and larger irregular structure is primarily confined to Saturn’s high optical depth B ring, where particles collide frequently. It is clear, however, that despite the lack of evidence for the types of ∼ 100 m ring structure that we are trying to explain, neither is there evidence that such structure does not exist; we simply do not have observations with resolution comparable to the scale height of the main rings.

As was stated by Wisdom and Tremaine (1988), the presence of the irregular structure in the Saturnian ring system is surprising because the timescale on which such irregularities in the distribution of particles should be removed by viscous diffusion is much shorter than the age of the solar system.

In summary, the Voyager space missions have provided accurate data regarding the dynamics of planetary rings. The Voyager 2 flyby of Saturn has revealed that the Saturnian disk shows a complex irregular density structure ranging from a few
kilometers down to the several hundred meters’ resolution of the spacecraft’s camera (Lane et al., 1982; Smith et al., 1982; Cuzzi et al., 1984; Esposito, 1986, 1993; Sicardy and Brahic, 1990). Most irregular microstructures are likely much younger than the solar system and new rings are created by some unknown mechanisms (Esposito, 1986, 1993). For instance, Esposito (1986, 1993) advocates the hypothesis of a larger role for catastrophic events in the Saturnian rings: new rings are episodically created by the destruction of small moons near the planet. Rings arise from singular events like the destruction of a ringmoon or comet, and a ring’s physical nature is the result of a competition between fragmentation and accretion in the planet’s Roche zone. These processes often involve occasional major events. Ring history is disorderly. Dones (1991) even proposed a recent cometary origin for Saturn’s rings. Accordingly, an origin for Saturn’s rings in tidal disruption by a comet of the scale of Chiron, which passed within Saturn’s Roche radius, is explored. Shan and Goertz (1991) suggested that the electromagnetically induced radial transport of angular momentum associated with radial transport of charged submicron-size dust particles may explain the features in the Saturnian B ring. This mechanism induces an instability which produces, over geological times, significant radial structuring of the ring. The radial scale of this irregular structure extends to very small sizes, down to the resolution limits. As was stated by Esposito (1986): “every time we improve our resolution we see more structure.”

On a small scale the irregular rings have been observed to be undergoing variation and oscillations with time and ring longitude (Smith et al., 1982). Actually, it was found that the individual rings of Saturn are in various states of oscillations. The latter indicates that probably such features are wave phenomena, and different instabilities of gravity perturbations may play important roles in ring’s dynamics. In this regard, the wealth of ring data from the Voyager spacecraft already motivated studies of wave propagation in planetary rings (see Goldreich and Tremaine (1982), Shu et al. (1983), Borderies (1989), Araki (1991), and Nicholson and Dones (1991) as reviews of the problem). This includes externally driven satellite resonances which clear gaps by perturbing ring particles at certain radial distances (Holberg et al., 1982; Thiessenhusen et al., 1995; Horn et al., 1996), bending waves in Saturn’s rings (Gresh et al., 1986; Rosen and Lissauer, 1988), spiral Lin-Shu density waves of the type invoked to explain the spiral structure of disk galaxies, a nonlinear density wave theory, viscous damping, and a great number of moonlets orbiting inside the optically thick parts of Saturn’s rings (Colwell, 1994; Spahn et al., 1994). Franklin et al. (1982) presented an evidence that two previously unidentified, yet conspicuous gaps in Saturn’s rings lie at distances Gaps such as these can be produced in a ring of large bodies or small

4 The rings of Saturn consist of thousands of smaller ringlets. In this paper, however, we do not consider few truly isolated ringlets with adjacent empty gaps, located in the low-density C-ring and the Cassini Division resembling those of Uranus (Tyler et al., 1983; Porco and Nicholson, 1987; Porco, 1990). Many of these narrow ringlets (with typical widths of a few tens of kilometers) with extremely sharp edges are found in the isolated resonance locations of different satellites, e.g. Prometheus 2 : 1 inner Lindblad resonance. An adequate theoretical explanation for these isolated narrow ringlets is still missing (Hanninen and Salo, 1995; Goldreich et al., 1995).
uncharged particles by a nonaxisymmetric gravitational field (both of the above can be associated with the $l = m = 3$ harmonic), a fact that is relevant to models of planetary interiors. The F ring is highly stirred by shepherds, and embedded moonlets are suggested on the basis of charged particle absorption as well as both azimuthal and radial structure (Cuzzi and Burns, 1988). Through a numerical modeling, Kolvoord and Burns (1992) demonstrated that the modest, out-of-plane satellite-induced vertical and horizontal distortions of three narrow bands generate a structure akin to the “braided” F ring. In the later case, both the structure and the origin of rings can be explained by a multitude of small moons, still unseen, orbiting nearby or among the rings (Esposito, 1992).

It is important that the Voyager’s stellar occultation data revealed some indirect evidence for structuring in the densest central parts of opaque Saturn’s B ring down to the 100 m length scale (Showalter and Nicholson, 1990). One cannot exclude the existence of such kinds of small-scale structure in other, low optical depth regions of the system.

Although direct observations on the small scales of interest (< 100 m or so) in Saturn’s rings are nonexistent, there is indirect evidence for structure on this scale in the A ring. The reflectivity of the A ring varies strongly with longitude in Earth-based and Voyager observations (Cuzzi et al., 1984; Franklin et al., 1987; Dones et al., 1993). The full amplitude of this variation peaks at 30 – 40% in the mid-A ring and smaller toward the inner and outer ring edges. By using the International Planetary Patrol network, Thompson et al. (1981) detected that the azimuthal brightness variations for the brighter portion of ring A increased as the ring tilt decreased from $B = 26$ deg to less than 16 deg, reaching the order of + or 20%detectable for ring B or the outer portion of ring A. The variation is generally known as the “quadrupole azimuthal asymmetry,” because the effect is not symmetric about the ring ansae. This asymmetry has been known for several decades. A simple semiquantitative explanation for the quadrupole asymmetry in terms of numerous unresolved density “wakes” caused by gravitational interactions of the particles has been presented by Colombo et al. (1976) and Franklin and Colombo (1978). The presence of wakes causes the effective area covered by particles, hence brightness, of the ring to vary at different longitudes (Franklin et al., 1987). This explanation, which requires some degree of self-gravitation between nearby orbiting bodies, accounts both for the presence of the azimuthal brightness variations in Saturn’s ring A and for their absence in ring B. A bias in the particle distribution and corresponding photometric effects are thereby produced the latter corresponding very closely to the variations observed in ring A. Their absence in ring B is primarily a consequence of the higher optical thickness and decreasing importance of self-gravitation in that ring.

5A systematic, uniform search of Voyager 2 photopolarimeter system data set for 216 significant features of Saturn’s rings with the spatial resolution in the radial direction in the ring plane better than 100 m was described by Esposito et al. (1987). Also, Brophy and Rosen (1992) conducted a parallel examination of Voyager radio and photopolarimeter occultation observations of the Saturn A ring’s satellite-excited density waves.
What caused the stratification of the main Saturn’s rings? Various theories have been advanced to explain the creation of the small-scale structure in Saturn’s rings. For instance, Lin and Bodenheimer (1981), Łukkary (1981), and Ward (1981) explained this microstructure by diffusional instabilities (or negative diffusion instabilities). According to Goldreich and Tremaine (1978, 1982), Cuzzi et al. (1981), Lissauer (1989), and Goldreich et al. (1995), the fine radial ring structure can be associated with resonant forcing by external moons. Shu et al. (1983) developed the theory of forced bending waves which cause small-scale vertical corrugations of the local ring plane. As a matter of fact, a few dozen spiral density wave trains that are resonantly forced in the plane of the system under study by external moons have been detected (Smith et al., 1982; Shu et al., 1985; Esposito, 1986, 1993). Lissauer (1985) has found features within Saturn’s rings that might be produced by vertical resonances of external moons. However, it is clear that only a small part of the structure of Saturn’s rings is determined by plane and vertical resonances with satellite orbits. Also, it has been suggested that a part of the irregular structure may be due to embedded moonlets orbiting within the ring system (Lissauer et al., 1981; Colwell, 1994; Spahn et al., 1994). By N-body simulations, Osterbart and Willerding (1995) also concluded that the most promising explanation for the ringlet structure of the B ring of Saturn is the assumption that a great number of moonlets within the ring system can trigger trailing density wakes of the type studied by Julian and Toomre (1966). As a matter of fact, a small moon, Saturn’s eighteenth satellite has been discovered embedded within the Encke gap of the ring system (Showalter, 1991). But no more satellite has yet been discovered within the main rings. In turn, Durisen et al. (1989) and Durisen et al. (1996) suggested that some of features of Saturn’s rings — radial optical depth structures near the inner edges of Saturn’s A and B rings, including the edges themselves — can be produced or maintained by “ballistic transport,” that is, radial transport of mass and angular momentum due to exchanges of ejecta from meteoroid impacts on ring particles. But it is clear, these observations and theories do not imply that most of the small-scale structure in the rings results from external or embedded moonlets, diffusional instabilities, ballistic transport, and other mechanisms may also be proposed.

In regard to the existence of the irregular structure in the Saturnian ring system, it seems likely that a universal mechanism that will generate all types of the structure does not exist: there should be several possible mechanisms. Moreover, different regions of Saturn’s rings may prefer different types of instabilities of small-amplitude gravity perturbations for ring generation.

In the following we argue that small-scale, hyperfine \( \sim 2\pi h < \sim 100 \text{ m} \) structures could be primarily produced by the classical gravitational Jeans-type instability in low optical depth regions and a secular dissipative-type instability in high optical depth regions in the system under study\(^6\). Here \( h \) is the typical thickness of the system. Thus, we propose that the numerous ringlets in the Saturnian ring system are the manifestation of tightly

\(^6\)Generally, the term “Jeans instabilities” identifies nonresonant instabilities associated with growing accumulations of mass (cf. electrostatic bunching instabilities or a fire-hose instability of a plasma).
wound spiral and/or radial density waves in the disk, which remain quasi-stationary in a frame of reference rotating around the center of the system at a proper speed. In this regard, the idea that the radial gravitational instability of a gas-dust disk may have played a vital role in the formation process of the planetary system seems to have been first suggested by Ginzburg et al. (1972), Polyachenko and Fridman (1972) (see also Fridman and Polyachenko (1984, Vol. 2, p. 261)).

As mentioned above, in this paper we will give a self-consistent asymptotic solution to the kinetic equations of particle dynamics for a thin, rapidly rotating disk of mutually gravitating identical particles in Keplerian rotation around a central gravitating mass, taking into account the effects of interparticle physical collisions. This work is of general interest for the theory of particulate disks. Even though some aspects of the results were previously known, the authors feel that the new technique provides additional physical insight into the process involved. The linear stability analysis presented in this paper does make predictions about the morphology of ring structure that could be compared to future Cassini measurements — tightly wound spiral and/or radial wavelike structures with size and spacing of the order $2\pi h$. This is a first step towards a general theory of planetary ring dynamics.

2 Basic equations

Under the reasonable assumption that the inclusion of motions normal to the plane makes little difference to the evolution of the rapidly rotating system of particles, let us consider dynamics of an infinitesimally thin self-gravitating disk (Griv, 1996; Griv and Peter, 1996). This is a valid approximation if one considers perturbations with a radial wavelength that is much greater than $h/2 \approx c^2/\sqrt{4\pi G\Sigma}$, where $c$ is the mean dispersion of random velocities, $G$ is Newton’s gravitational constant, and $\Sigma$ is the volume mass density in the mid-plane (Shu, 1970; Ginzburg et al., 1972; Griv and Yuan, 1997). Note that it has been shown by $N$-body simulations of disk-shaped galaxies of stars that the inclusion of motions normal to the plane makes little difference to the evolution of the rapidly rotating thin disk (Hohl, 1978). (Common dynamical processes act in the stellar disks of flat galaxies and in Saturn’s ring system of particles; Tremaine (1989)). This justifies the two-dimensional treatment of the main part of a rapidly rotating particulate disk. From now on, as a simplification, we assume the disk is two-dimensional and, therefore we consider the special case of waves propagating in the equatorial plane of the system under study.\footnote{While the theoretical model is two dimensional, actual self-gravitating disks have finite transverse dimension. When the disk is “strongly magnetized” (rapidly rotating), motion along the “magnetic field” (along the vector of the angular rotation) may be neglected so that the motion of the particles is two dimensional. In this case the two-dimensional model is appropriate provided the waves generated are also nearly two dimensional; in general, in the rapidly rotating disks the coupling between waves propagating in the plane and along the axis of rotation is expected to be small. We suggest that waves in the plane are coupled with waves propagating in the normal to the plane direction at the position of the so-called vertical resonances , and here the self-excitation of spontaneous kinetically-unstable}
The collision motion of an ensemble of identical particles in the plane, in the frame of reference rotating with angular velocity $\Omega$, can be described by the Boltzmann equation (Lin and Shu, 1966; Lin et al., 1969; Griv and Peter, 1996)

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \left( \Omega + \frac{v_\varphi}{r} \right) \frac{\partial f}{\partial \varphi} + \left( 2\Omega v_\varphi + \frac{v_\varphi^2}{r} - \frac{\partial \Phi_1}{\partial r} \right) \frac{\partial f}{\partial v_r}$$

$$- \left( \frac{\kappa^2}{2\Omega} v_r + \frac{v_r v_\varphi}{r} + \frac{1}{r} \frac{\partial \Phi_1}{\partial \varphi} \right) \frac{\partial f}{\partial v_\varphi} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}},$$

(1)

where the total azimuthal velocity of the particles is represented as a sum of $v_\varphi$ and the circular velocity $r\Omega$, $\Omega(r)$ is the angular velocity at the distance $r$ from the planet, and $v_r$ and $v_\varphi$ are the minor residual (random) velocities in the radial and azimuthal directions, respectively, $|v_r|$ and $|v_\varphi| \ll r\Omega$. Applying the linear theory of disk stability, in the kinetic equation (1) the total gravitational potential $\Phi(\vec{r}, t)$ is divided into a smooth basic part $\Phi_0(\vec{r})$ satisfying the equilibrium condition

$$\Omega^2 r = \frac{\partial \Phi_0}{\partial r},$$

and a small fluctuating part $\Phi_1(\vec{r}, t)$ with $|\Phi_1/\Phi_0| \ll 1$ for all $\vec{r}$ and $t$.

The left-hand side of the Boltzmann equation (1) represents the total time rate of change of the distribution function $f$ along a particle trajectory in $(\vec{r}, \vec{v})$ space as defined by Langrange’s system of characteristic equations:

$$d\vec{r}/dt = \vec{v} \quad \text{and} \quad d\vec{v}/dt = -\nabla \Phi/\partial \vec{r}.$$

In Eq. (1), $(\partial f/\partial t)_{\text{coll}}$ is the so-called collision integral which takes into account effects due to the discrete-point nature of the gravitational charges, or collision effects (including diffusion in space and velocity), and defines the change of the distribution function $f(\vec{r}, \vec{v}, t)$ arising from ordinary interparticle collisions (in a plasma this term represents the change of $f$ arising from collisions with particles at distances shorter than a Debye length). The Boltzmann form for the collision integral is based on an assumption that the duration of a collision is much less than the time between collisions — instantaneous collisions are considered. We assume that there are only binary physical collisions, and momentum is conserved in collisions. In addition there is no correlation in motion between the colliding species, that is, Boltzmann’s hypothesis of molecular chaos is adopted. The left-hand side in Eq. (1) is the total rate of change of the phase space density following the motion. The Boltzmann equation then says this density changes because of the collisions, following the phase space trajectories.

In plasma physics, Lifshitz and Pitaevskii (1981, p. 115) have discussed phenomena in which interparticle collisions are unimportant, and such a plasma is said to be bending waves is expected (Griv et al., 1997a). Clearly, such effects can only be studied by using a three-dimensional model.
collisionless (and in the lowest-order approximation of the theory one can neglect the collision integral in the kinetic equation). It was shown that a necessary condition is that $\nu_c \ll |\omega|$, where $\omega$ is the frequency of excited oscillations: then the collision operator in the kinetic equation above is small in comparison with $\partial f/\partial t$. Lifshitz and Pitaevskii (1981) have pointed out that collisions may be neglected also if the particle mean free path is large compared with the wavelength of collective oscillations. Then the collision integral in Eq. (1) is small in comparison with the term $\vec{v} \cdot (\partial f/\partial \vec{r})$.

Equation (1) resembles the Boltzmann equation for a collisional plasma in a nonuniform magnetic field, thus the techniques of plasma theory may be applied. In plasma physics, methods for investigating oscillations and the stability of a collisional system have been developed using either the exact Boltzmann integral formulation or an approximate collisional term in the form of Krook (or Bhatnagar et al.) model. Reviews of plasma kinetic theory, taking into account collisions between particles, are given by Rukhadze and Silin (1969), Mikhailovskii (1974), and Alexandrov et al. (1984).

In general, the Boltzmann equation is nearly intractable because of the complicated collision integral. The collisional term can be approximated in various ways, the simplest is to assume that it vanishes in the collisionless model. In this work we use the simple kinetic model when the exact, but complicated, Boltzmann integral $(\partial f/\partial t)_{\text{coll}}$ is replaced by an approximate, phenomenological term in the form of the Krook model (Shu and Stewart, 1985). The Krook integral is called a model because it cannot be derived from the exact Boltzmann integral but can only be constructed by general physical reasoning, i.e. the need to satisfy the laws of conservation of particle number, momentum, and energy.

The simple Krook integral in the case of a two-dimensional disk of identical particles has the form

$$\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = -\nu_c (f - f_0),$$

where $f$ is the actual distribution function of particles and $f_0$ is the steady-state equilibrium distribution function (Shu and Stewart, 1985). In Eq. (2), $\nu_c$ plays the role of the velocity-independent collision frequency; $\nu_c = n \langle sv \rangle$, where $n$ is the number density of particles, $s$ is the effective radius of a particle, and $\langle ... \rangle$ denotes the average over particles of all random velocities $v$ in a Maxwellian distribution. Results following from Eq. (2) depend on $\nu_c$, and one must seek other, external means to obtain the value of $\nu_c$. In order to conserve particles in this model, we demand $\int (f - f_0) d\vec{v} dr = 0$, since $(f - f_0)$ will eventually decay away. On the other hand, it will relax the distribution towards equilibrium with an increase in entropy. The distribution function $f_0$ is necessarily a Maxwellian-like. In the case of an infinitesimally thin, differentially rotating ($d\Omega/dr \neq 0$) disk with collisions, the function $f_0$ is given by

$$f_0 = \frac{\sigma_0(r)}{2\pi c_r(r)c_\phi(r)} \exp \left\{ -\frac{v_r^2}{2c_r^2(r)} - \frac{v_\phi^2}{2c_\phi^2(r)} \right\},$$

with $c_r$ and $c_\phi$ being the averaged dispersion of radial and azimuthal random velocities of particles, respectively, in general $c_r \neq c_\phi$, and $\sigma_0(r)$ being the equilibrium surface
mass density (Griv, 1996; Griv and Peter, 1996). In the case of an almost collisionless system, the function $f_0$ can be constructed (satisfying the unperturbed part of the kinetic equation) with the use of the constants $I_1, I_2, \ldots$, of the equilibrium particle motion $f_0 = F_0(I_1, I_2, \ldots)$, where $F_0$ is, generally speaking, an arbitrary function. In a system with frequent collisions, the postcollisional velocities distributed isotropically, $c_r = c_\varphi$.

The Krook collision integral (2) can be interpreted as the following. As a result of scattering, particles are lost at the rate $\nu_c f$ (an "absorption" process) and re-emitted at the rate $\nu_c f_0$ with Maxwellian distribution of the postcollisional velocities at the mean surface density. The greatest defects of the Krook collision integral is that it cannot be derived from the Boltzmann integral but can only be constructed by general physical reasoning and is that in the case of small-angle (gravitational or Coulomb) collisions the "diffusion" coefficient $\nu_c$ does not fall of with increasing velocity, as do those given by the Fokker-Planck equation (Rosenbluth et al., 1957; Rukhadze and Silin, 1969). However, by replacing the exact Boltzmann collision integral in the kinetic equation by an approximate term in the form of Krook model the problem of the stability can be solved with simple methods similar to those used for the collisionless case: by applying the usual procedure in particle path integration (Mikhailovskii and Pogutse, 1966; Mikhailovskii, 1974, Vol. 2; Griv and Chiueh, 1997).

The right-hand side of Eq. (1) as given by Eq. (2) is a gross approximation to the collision integrals. It represents a relaxation term to a steady state distribution, $f_0$. The solution will be assumed to depend analytically upon $\nu_c$.

It has to be noted that at least in the case of small-angle collisions the Krook model cannot be used in the case of perturbations of a particulate disk with too small, $k^2 \rho^2 \gg 1$, and too large, $k^2 \rho^2 \ll 1$, wavelengths. Pitaevskii (1963), Rukhadze and Silin (1969), and Griv et al. (1997b) already explained the problem. Here $k$ is the wavenumber, $\rho \approx c/\kappa$ is the mean epicyclic radius (the Larmor radius in a plasma), and

$$\kappa = 2\Omega \left(1 + \frac{r \ d\Omega}{2\Omega \ dr}\right)^{1/2}$$

is the ordinary epicyclic frequency. Also, the collisional term (2) is applicable only in the case of purely elastic collisions, e.g. in the case of gravitational encounters. Let us modify Eq. (2) to allow the effects of inelastic (physical) collisions. Following Shu and Stewart (1985), it is supposed here that each collision reduces the magnitude of the component of the relative velocity along the line of centers of particles by a factor $\epsilon < 1$, where $\epsilon$ is the coefficient of restitution averaged over all encounters, and is defined from the equation

$$3c_f^2 = (2 + \epsilon^2)c^2,$$

where $c_f$ is the velocity dispersion after collision and $c$ is the velocity dispersion before collision. A very popular model of the particles of Saturn’s rings is a smooth ice sphere,
whose restitution coefficient $\epsilon$ is quite high, $0.5 < \epsilon < 1$, $\epsilon$ is decreased as the collision velocity increases, and impact velocities are only a few millimetres per second (Goldreich and Tremaine, 1982; Bridges et al., 1984). Momentum is conserved in inelastic collisions, so that in the equation of motion the viscosity can be determined by the effective intercollision time $\propto 1/\nu_c$. Thus, $c_I < c$, and in Eq. (2) one can replace $f_0$ by a Maxwellian-like distribution with the postcollisional velocities.

One should keep in mind that without inelastic collisions a planetary ring cannot come into a state of equilibrium and will be “heated up.” Without dissipation the velocity dispersion of random velocities of colliding particles in a rapidly rotating system grows without limit and any structures are transient structures only. The coefficient of restitution $\epsilon$ in inelastic collisions can be a function of the (thermal) impact velocity $v$, and this fact should be taken into account by modelling in great detail the equilibrium distribution (Goldreich and Tremaine, 1978; Shu and Stewart, 1985). Planetary rings are gravitationally and collisionally dominated systems, and a thermal quasi-equilibrium in these systems is achieved via gravitational (due to gravitational instabilities) “heating” and viscous (due to velocity shear) heating and collisional cooling (damping of the impact velocities due to dissipative collisions). We discuss this problem in brief in Sections 4.1 and 4.2.

Shu and Stewart (1985) have estimated the velocity-independent frequency of inelastic collisions, using a similar relaxation-time approximation to the collision term:

$$\nu_c \approx \frac{8}{\pi} \mu \tau,$$

(5)

where $\mu \approx \sqrt{4\pi G \Sigma} \approx \Omega$ is the frequency of vertical epicyclic oscillations and $\tau$ is the normal optical depth. In the case of spherical particles with a power-law distribution of sizes (and masses), $\tau$ should be replaced by the quantity $\tau_e$ which is smaller than the normal optical depth by the factor 3 or 4 (Shu and Stewart, 1985).

As one can see, the collision integral (2) will vanish when the equilibrium Maxwellian distribution is substituted, $f = f_0$. It is assumed in Eq. (2) that there is no systematic, mean movement of particles excepting for the circular rotation. The collision integral of the form (2) does not take into account detailed mechanisms of the inelastic interaction such as spin degrees of freedom, the particle size distribution, and the finite size of the particles (Araki and Tremaine, 1986). Such effects can, of course, be included in the analysis if necessary. Thus, it is not entirely clear to what extent the results obtained from our study are relevant to the structure of Saturn’s rings. Nonetheless, we apply our results to the standard uniform-size hard sphere model (Goldreich and Tremaine, 1978) for simplicity.

It may be shown by integrating over velocities that the collisional term in such a form conserves the number of particles and the momentum only on the average over a cycle.
Random kinetic energy is always dissipated on the average over finite times by the inelastic part of the collisional process (this is because $c_f^2 < c^2$ is always true). The collision term of the form (2) forces the distribution function to relax in position space upon each collision to the uniform isotropic Maxwellian distribution of postcollisional velocities at the mean density.

Of course, the question that remains is whether the $f_0$ used in the Krook collisional integral is appropriate to investigate the effects of frequent and strong collisions; it could probably be strongly modified by frequent collisions. For instance, Wisdom and Tremaine (1988) show that, as optical depth increases, significant stresses are communicated through the finite size of closely packed particles. In a future investigation, several results of the kinetic theory of oscillations of a collisional particulate disk obtained with the aid of the exact Boltzmann integral will be presented (Griv, Gedalin, and Yuan, 1999, in preparation). This will show clearly which of the results obtained with the Krook model collision integral are qualitatively correct.

Perturbations in the gravitational field cause perturbations to the particle distribution function. In the linear approximation, one can therefore write $f(\vec{r}, \vec{v}, t) = f_0(\vec{r}, \vec{v}) + f_1(\vec{r}, \vec{v}, t)$, where $|f_1| \ll f_0$ and $f_1$ is a function rapidly oscillating in space and time. If an initial perturbation grows, the system is called unstable. The function $f_0$ describes the differentially rotating “background” against which small perturbations develop. Initially, the disk is in an equilibrium, $\partial f_0 / \partial t = 0$.

The linearized kinetic equation (1) for the perturbed distribution function $f_1(\vec{r}, \vec{v}, t)$ becomes

$$\frac{df_1}{dt} = \frac{1}{r} \frac{\partial \Phi_1}{\partial \varphi} \frac{\partial f_0}{\partial v_r} + \frac{1}{r} \frac{\partial \Phi_1}{\partial \varphi} \frac{\partial f_0}{\partial v_\varphi} - \nu_c f_1,$$

where $d/dt$ is taken along the unperturbed orbits of particles in the local rotating frame. In turn, the linearized kinetic equation (1) for the unperturbed distribution function $f_0(\vec{r}, \vec{v})$ takes the form

$$v_r \frac{\partial f_0}{\partial r} + \left( 2 \Omega v_\varphi + \frac{v_r^2}{r} \right) \frac{\partial f_0}{\partial v_r} - \left( \frac{\kappa^2}{2 \Omega} v_r + \frac{\nu_c v_\varphi}{r} \right) \frac{\partial f_0}{\partial v_\varphi} = 0.$$

Initially the disk is in equilibrium, $\partial f_0 / \partial t = 0$.

Generally, the wave is not plane. However, if we assume that the solution is nearly a plane wave, the expression for the field may be written in the form

$$N = \delta N(r)e^{iA(r)}$$

The more complicated phenomenological Bhatnagar-Gross-Krook integral has the form

$$\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = -\nu_c \left( f - \frac{\sigma}{\sigma_0} f_0 \right),$$

where $\sigma$ is the local surface density (Bhatnagar et al., 1954). In contrast to the simpler case of Krook’s collision integral, the collision term in such a form instantaneously conserves the number of particles, the momentum, and the particle’s energy (Griv and Chiueh, 1997).

It was a question posed by the referee of the paper: whether a simple gas-kinetic theory approach breaks down in the regime of the high optical depth.
(we omit the $\mathbb{R}$; it is understood that the real part of all expressions is taken). In homogeneous media $A(r) = k_r r$. If the medium is sufficiently slowly varying, the simple plane wave solution above represents a good starting point.

If geometrical optics (or the standard WKBJ method in quantum mechanics) is applicable, the amplitude $\delta N$ is, generally speaking, a function of the coordinates and time, and the phase $A$, which is called the eikonal, is a large quantity. Following the WKBJ method, the perturbations will be taken to be of the form

$$f_1, \Phi_1 \sim \sum_{m=-\infty}^{\infty} \delta f_m(r), \delta \Phi_m(r), \exp\{-i\omega t + im\varphi + ik_r r\},$$

$|k_r| r \gg 1$, which corresponds to the fact that in each small region of space (and each small interval of time) the wave can be considered as plane. Evidently $f_1$ and $\Phi_1$ are periodic functions of $\varphi$, and hence $m$ must be an integer. Consideration will be limited to the region between the turning points in a disk (the transparency region). Here $\omega$ is the frequency of excited waves, $m$ is the positive azimuthal mode number, which gives the number of spiral arms (for axisymmetric perturbations $m = 0$), and $k_r$ and $k_\varphi \equiv m/r$ are the radial and azimuthal components of the wavevector $\vec{k}$. In the framework of the linear theory we are interesting, we can select one of the Fourier harmonics: $\delta f, \delta \Phi \exp (ik_r r + im\varphi - \omega t)$. In the local approximation of the WKBJ method (with no derivatives of $\delta N$ and neglecting $d^2 A/dr^2$, e.g. Swanson (1989, p. 13)), $\delta f$ and $\delta \Phi$ are constants. In addition, perturbations with a wavelength $\lambda = 2\pi/k_r$ such that $h \lesssim \lambda \ll R$ are investigated, where $R$ is the radial size of of a system. Then, the disk may be regarded as infinitesimally thin.

In the case of the two-dimensional disk we are interested in, the Poisson equation is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \sigma \delta(z),$$

where $\sigma(r)$ is the surface mass density, and $\delta(z)$ is the Dirac delta-function. The improved asymptotic, $k_r^2 \gg k_\varphi^2$, Lin-Shu type solution of this equation may be written in the form (Lau and Bertin, 1978; Lin and Lau, 1979; Bertin, 1980)

$$\sigma_1(r) = -\text{sign}(k) \frac{k \Phi_1(r)}{2\pi G} \left\{1 - \frac{i}{k_r r} \frac{d \ln r}{d \ln r} \left[r^{1/2} \delta \Phi\right]\right\},$$

where $\sigma_1$ is the small perturbation of the equilibrium surface density. The wave vector $\vec{k}$ in the two-dimensional system is given by $k^2 = k_r^2 + k_\varphi^2$ and the angle between the direction of the wave front and the tangent to the circular orbit of the particle (the pitch angle) is defined by

$$\tan \psi = \frac{k_\varphi}{k_r} = \frac{m}{r k_r}.$$

Solution (7) which determines the perturbed surface mass density required to support the perturbed gravitational potential was found in the asymptotic Lin-Shu type (Lin
and Shu, 1966; Shu, 1970) approximation of moderately tightly wound perturbations
\[
\tan^2 \psi \ll 1
\]
which has traditionally been used in all such investigations (see Bertin and Mark (1978), Lin and Lau (1979), and Bertin (1980) for a discussion). The improved solution (7) of the Poisson equation, which includes effects of finite inclination of spiral arms \((k_\phi \neq 0)\), is accurate to two orders in the familiar Lin-Shu asymptotic approximation.

In order to find the perturbed distribution \(f_1\), it is convenient to integrate Eq. (6) along the unperturbed trajectories of particles. Since this is difficult to do exactly, epicyclic orbits and series expansions are used instead. Defining the epicycle phase \(\phi_0\) at \(t = 0\) by \([v_r, v_\phi] = [v_\perp \cos \phi_0, (\kappa/2\Omega)v_\perp \sin \phi_0]\), where \(\phi_0\) and \(v_\perp\) are constants of integration, the solution for the ordinary Lindblad elliptic-epicyclic trajectories in an unperturbed central force field of Saturn can be expressed as

\[
\begin{align*}
    r &= r_0 - \frac{v_\perp}{\kappa} \left[ \sin(\phi_0 - \kappa t) - \sin \phi_0 \right], \\
    \varphi &= \Omega t + \frac{2\Omega}{\kappa} \frac{v_\perp}{v_\perp} \left[ \cos(\phi_0 - \kappa t) - \cos \phi_0 \right],
\end{align*}
\]

where \(r_0\) is the radius of the chosen circular orbit in the \((r, \varphi)\) plane. The equations above describe the small departure of the actual radius \(r(t)\) from \(r_0\), which is chosen so that the constant of areas for the circular orbit \(r_0^2 \dot{\varphi}_0\) (and \(r_0 \dot{\varphi}_0^2 = (\partial \Phi_0/\partial r)_0\)) is equal to the angular momentum integral \(r^2 \dot{\varphi} = \text{const}\). In Eqs. (8), \(v_\perp / r_0 \kappa \sim \rho / r_0 \ll 1\), and \(\rho \simeq v_\perp / \kappa\) is the epicyclic radius (Grivnev, 1988; Griv and Peter, 1996). The zeroth order approximation is simply a circle on which the particle moves with angular velocity \(\Omega = V/r\) and rotational velocity \(V(r) = (r \partial \Phi_0 / \partial r)_0^{1/2}\). The first-order Lindblad epicyclic theory superposes on this regular rotational motion harmonic oscillations both in the radial and azimuthal directions with a characteristic frequency \(\kappa\) called the epicyclic frequency, Eqs. (8) (cf. a simple cyclotron gyration of a charged particle of a plasma). This describes a circular orbit of a guiding center. In the Saturnian ring disk \(\kappa \approx \mu \approx \Omega\).

The closure condition for the orbit up to first order in Lindblad’s epicyclic theory may be written as

\[
\frac{\kappa}{\Omega} = \frac{n}{s},
\]

where \(n, s\) are positive integers. Obviously, the above condition determines only two types of fields in which all orbits are closed: the limit of Keplerian rotation, \(n = s = 1\), and the limit of rigid rotation, \(n = 2s = 2\). In the former limit, after one complete revolution along the circular orbit and one complete revolution along the epicycle, the particle will occupy its original position.

11 The postepicyclic orbits take into account the second-order effects of the orbital eccentricity and describe additional directional motions of particles, i.e., oscillations with combined epicyclic frequencies, the small displacement of the center of the epicycle along the field gradient, and the small non-oscillatory drifting motions both along the circular orbit and along the epicycle (Griv, 1996; Griv and Peter, 1996; Griv et al., 1999a).
From Eqs. (8) it easy to find \( u_r \) and \( u_\varphi \), the components of the particle’s random velocity relative to the planetary center. In the lowest order, the solutions are

\[ u_r = v_\perp \cos(\phi_0 - \kappa t), \quad u_\varphi = \frac{2\Omega}{\kappa} v_\perp \sin(\phi_0 - \kappa t). \]

To integrate Eq. (6) over \( t \), we need to determine the components \( v_r \) and \( v_\varphi \) of the particle’s velocity at each point relative to the local standard of rest. Since the circular velocity in this rotating system, is \( -r_1 r_0 (d\Omega/dr) \), where \( r_1 = r - r_0 \) (Spitzer and Schwarzschild, 1953; Binney and Tremaine, 1987, p. 120; Griv and Peter, 1996), we have

\[ v_r = u_r; \quad v_\varphi = u_\varphi + r_0 \frac{v_\perp d\Omega}{\kappa} \sin(\phi_0 - \kappa t) \approx \frac{\kappa}{2\Omega} v_\perp \sin(\phi_0 - \kappa t). \]  \tag{9}

Additionally, to perform the integral in Eq. (6) an expression for the equilibrium distribution function \( f_0 \), Eq. (3), is needed. For an infinitesimally thin disk with rare collisions between particles, \( \nu_c \ll \Omega \), we choose the Schwarzschild distribution function (the anisotropic Maxwellian distribution function) satisfying the unperturbed part of the kinetic equation (Shu, 1970)

\[ f_0 = \frac{2\Omega(r_0)}{\kappa(r_0)} \frac{\sigma_0(r_0)}{2\pi c^2(r_0)} \exp \left\{ -\frac{v_\perp^2}{2c_\perp^2(r_0)} \right\}. \]

The function \( f_0 \) is a function of the epicyclic constants of motion \( E = v_\perp^2/2 \) and \( r_0 \), and the quantity \( r_0 \) represents approximately the \( r \)-coordinate of the particle guiding center. For an infinitesimally thin disk with frequent collisions between particles, \( \nu_c \gg \Omega \), we choose the isotropic Maxwellian distribution with the postcollisional velocities

\[ f_0 = \frac{\sigma_0(r_0)}{2\pi c^2(r_0)} \exp \left\{ -\frac{v_\perp^2}{2c_\perp^2(r_0)} \right\}. \]  \tag{10}

In the first equation of the system (10) the fact was used that the radial and azimuthal dispersions in Eq. (3) are not independent but, according to Lindblad’s theory of epicyclic orbits (Eqs. [8] and [9]), are related in the rotating frame through

\[ c_r = (2\Omega/\kappa)c_\varphi. \]

Of course, this relation is valid only when \( \nu_c \ll \kappa \) or \( \Omega \). In the opposite limit of frequent collisions (the hydrodynamical limit), when \( \nu_c \gg \kappa \) or \( \Omega \),

\[ c_r = c_\varphi. \]

Hence, in the case of rapidly rotating disk with rare interparticle collisions the partial derivatives in Eq. (6) transform as follows (see also Shu (1970)):

\[ \frac{\partial}{\partial v_r} = v_r \frac{\partial}{\partial E}; \quad \frac{\partial}{\partial v_\varphi} \approx \left( \frac{2\Omega}{\kappa} \right)^2 v_\varphi \frac{\partial}{\partial E} + \frac{2\Omega}{\kappa^2} \frac{\partial}{\partial r}. \]
Using above equations, the solution of the kinetic equation (6) in the considered case of a spatially inhomogeneous, \( \partial f_0 / \partial r \neq 0 \), almost collisionless disk may be written in the form (Griv, 1996; Griv and Peter, 1996; Griv and Chiueh, 1997)

\[
f_1 = e^{-\nu_c t} \int_{-\infty}^{t'} dt' e^{\nu_c t'} \left( \overline{v}_\perp \frac{\partial \Phi_1}{\partial r} \frac{\partial f_0}{\partial \overline{v}} + \frac{2\Omega}{\kappa^2 r_0} \frac{1}{\nu_c} \frac{\partial \Phi_1}{\partial r} \frac{\partial f_0}{\partial r} \right). \tag{11}
\]

### 3 The generalized dispersion relation

Now it is possible to integrate Eq. (11) along the unperturbed trajectories (8), \( \overline{R}(\overline{r}, \overline{v}, t) \), that end at \( \overline{R}(\overline{r}, \overline{v}, t) \) when \( t' \to t \), using the equilibrium distribution functions and relation

\[
J_{l-1}(\chi) + J_{l+1}(\chi) = \frac{2l}{\chi} J_l(\chi),
\]

where \( J_l(\chi) \) is the Bessel function of the first kind of the order \( l \). The method of integration has been described, e.g. by Krall and Trivelpiece (1987, p. 395) and Swanson (1989, p. 142). The following expression may be easily obtained:

\[
f_1 = -\Phi_1(r_0) \left[ \kappa \frac{\partial f_0}{\partial \overline{v}} \sum_{l=\infty}^{\infty} \frac{J_l^2(k_s v_\perp / \kappa)}{\omega_s - l\kappa + i\nu_c} + \frac{2\Omega m}{\kappa^2 r_0} \sum_{l=\infty}^{\infty} \frac{J_l^2(k_s v_\perp / \kappa)}{\omega_s - l\kappa + i\nu_c} \right], \tag{12}
\]

where

\[
k_s = k \left\{ 1 + \left[ (2\Omega/\kappa)^2 - 1 \right] \sin^2 \psi \right\}^{1/2}
\]

is the effective wavenumber and \( \omega_s = \omega - m\Omega \) is the Doppler-shifted frequency of excited waves as seen by a particle in a rotating frame. The addition of the Krook term is equivalent to the change

\[
\omega_s \to \omega_s \left( 1 + i \frac{\nu_c}{\omega_s} \right) \quad \text{and} \quad \left| \frac{\nu_c}{\omega_s} \right| \ll 1.
\]

Clearly, if \( |\nu_c/\omega_s| \) is small enough and \( |\omega_s| \lesssim \Omega \) we can ignore these collisions. Then, the replacement \( \omega_s \to \omega_s + i\nu_c \) would merely introduce some collisional damping with the modes.

We assume that the perturbations vanish as \( t' \to -\infty \) and take some values at \( t \), so we may neglect the effects of the initial conditions. The method of integration has also been described by Griv (1996), Griv and Peter (1996), and Griv and Chiueh (1997). It is convenient to write the eigenfrequency in a form of the sum of the real part \( \Re \omega_s \) and the imaginary part \( i\Im \omega_s \). In accordance with the definition of perturbations \( f_1, \Phi_1, \) and \( \sigma_1 \), the existence of solutions with \( \Im \omega_s \) greater than zero implies instability; the solutions with \( \Im \omega_s = 0 \) describe long-lived natural oscillations. The solutions with \( (\Re \omega_s)^2 < 0 \) describe the aperiodic instabilities. Integrating Eq. (12) over velocity space

\[
\int_{-\infty}^{\infty} dv_r \int_{-\infty}^{\infty} f_1 dv_\varphi = 2\pi \frac{\kappa}{2\Omega} \int_{-\infty}^{\infty} f_1 v_\perp dv_\perp \equiv \sigma_1
\]
and equating the result to the “in-phase” perturbed surface density given by the asymptotic solution of the Poisson equation (7) \( \sigma_1 = -|k|\Phi_1/2\pi G \), the generalized Lin-Shu type dispersion relation \( \omega_s = \omega_s(k_s) \), is obtained

\[
\frac{k^2 c_s^2}{2\pi G\sigma_0 |k|} = -\kappa \sum_{l=-\infty}^{\infty} \frac{l e^{-\nu_c I_l(x)}}{\omega_s - l\kappa + i\nu_c} + 2\Omega \frac{m p^2}{r_0 L} \sum_{l=-\infty}^{\infty} \frac{e^{-\nu_c I_l(x)}}{\omega_s - l\kappa + i\nu_c},
\]

where \( x = k_s^2 c_s^2 / \kappa^2 \approx k_s^2 \rho^2 \), \( I_l(x) \) is the modified Bessel function of the order \( l \), and

\[
|L| \approx \left| \frac{\partial \ln \left( 2\Omega \sigma_0 / \kappa c_s^2 \right)}{\partial r} \right|^{-1}
\]

is the radial scale of a spatial inhomogeneity. In the local version of the WKBJ method, we are using, \(|k_r|^{-1} < |L| < r\). In the derivation of Eq. (13), the following formula has been used

\[
\int_0^\infty e^{-r^2} J_l(\alpha r) J_l(\beta r) = \frac{1}{2\pi^2} \exp \left( -\frac{\alpha^2 + \beta^2}{4r^2} \right) I_l \left( \frac{\alpha\beta}{2r^2} \right).
\]

Lynden-Bell and Kalnajs (1972) first derived the generalized Lin-Shu type dispersion relation (13) for the case of non-axisymmetric Jeans-type perturbations by neglecting the effect of collisions, \( \nu_c = 0 \), and inhomogeneity, \( L \to \infty \) (Eq. [A11] in their paper).

We pay attention mainly to the long-wavelength oscillations, \( x \equiv k_s^2 c_s^2 / \kappa^2 \leq 1 \), the case of epicyclic radius small compared with wavelength (but, of course, in order to be appropriate for WKBJ wave we consider the perturbations with \(|k| r \gg 1\). In this limit, one can use the following asymptotic expansion of the modified Bessel functions \( I_0(x) \approx 1 + x^2/4 \), \( I_1(x) \approx x/2 \), and \( \exp(-x) \approx 1 - x + x^2/2 \). The short-wavelength perturbations, \( k_s^2 \rho^2 \gg 1 \), are not as dangerous in the problem of disk stability as oscillations with \( k_s^2 \rho^2 \leq 1 \), since they lead only to very small-scale perturbations of the density with the radial-scale \( \lambda_r \ll 2\pi \rho \). For the parameters of Saturn’s rings the velocity dispersion of the largest ring particles is \( \lesssim 0.5 \text{ cm s}^{-1} \) and the mean epicyclic radius is \( \rho \lesssim 10 \text{ m} \), thus, \( \rho \) is of the order the largest particle size (Esposito, 1986, 1993). It makes little sense to speak of collective effects on scales smaller than the finite-sized particles. The asymptotic expansion of the Bessel functions \( I_l(x) \) in the short-wavelength limit, \( x = k_s^2 c_s^2 / \kappa^2 \approx k_s^2 \rho^2 \gg 1 \), the case of epicyclic radius that is large compared with wavelength:

\[
I_l(x) \approx \frac{e^x}{\sqrt{2\pi x}} \left[ 1 + O \left( \frac{1}{x} \right) \right],
\]

while in a more rigorous approximation \( I_l(x) \) is a monotonically decreasing function of \( l \) for a fixed \( x \gg 1 \).

In Eq. (13), the functions \( \Lambda_l(x) = e^{-x} I_l(x) \) appear commonly in a theoretical treatment of Maxwellian plasmas in a magnetic field. It is instructive to note: (i)
0 ≤ \Lambda_l(x) ≤ 1; (ii) \Lambda_0(x) decreases monotonically from \Lambda_0(0) = 1; and (ii) \Lambda_l(x) for \( l \neq 0 \) starts from \Lambda_l(0) = 0, reaches a maximum, and then decreases.

The dispersion relation (13) describes the ordered behavior of a medium near its metaequilibrium state and generalizes the standard Lin-Shu dispersion relation (Lin and Shu, 1966; Lin et al., 1969; Shu, 1970) for nonaxisymmetric perturbations, \( \sin \psi \neq 0 \), to the case where physical collisions occur in an inhomogeneous disk. The effects of tangential gravitational forces (pitch angle dependent effects, \( \psi \neq 0 \)) for the collisionless model of a galactic disk have previously been analysed by Lau and Bertin (1978) and Lin and Lau (1979) in the framework of the hydrodynamical approach and Bertin and Mark (1978), Bertin (1980), Morozov (1980, 1981), Griv (1996), and Griv and Peter (1996) in stellar dynamics. (As mentioned above, in galaxies and Saturn’s rings \( c_r/r_0 \Omega \ll 1 \), and therefore the systems may be treated as an almost cold gas. Thus with the exceptions of resonant regions a kinetic description yields results no different from those obtained hydrodynamically.) In sharp contrast to the original Lin-Shu dispersion relation (Lin and Shu, 1966; Lin et al., 1969; Shu, 1970), Eq. (13) is valid for relatively open spiral waves (\( k_r \gtrsim k_{\varphi} \)), and in addition describes the effects of interparticle collisions in a spatially inhomogeneous medium.

The dispersion relation (13) is complicated: the basic dispersion relation above is highly nonlinear in the frequency \( \omega^* \). To see the physical meaning of solutions of Eq. (13), one does not need the exact solutions. Rather, in order to deal with the most interesting oscillation types, let us consider various limiting cases of perturbations described by some simplified variations of Eq. (13). For instance, similar to the plasma physics method, it is sufficient to consider only the principal part of the disk between the inner \( l = -1 \) and outer \( l = 1 \) Lindblad resonances (Shu, 1970; Morozov, 1980; Griv et al., 1999a); the treatment of the Lindblad resonances as well as the corotation resonance is beyond the scope of the present paper (Griv, 1996). In addition, we consider the case of weakly inhomogeneous medium, when the second term on the right-hand side in Eq. (13) is only a small correction.

4 The Jeans-type and dissipative-type oscillations

Two different cases may be considered in Eq. (13): (a) weak \( \omega^*_s \gg \nu_c^2 \) and rare \( \nu_c^2 \ll \Omega^2 \), and (b) strong \( \omega^*_s \ll \nu_c^2 \) and frequent \( \nu_c^2 \gg \Omega^2 \) collisions. When \( \omega^*_s \gg \nu_c^2 \) and \( \nu_c^2 \ll \Omega^2 \), the collisions cause only small corrections of the perturbed distribution function and in the zero-order approximation of the theory all dissipative effects may be ignored. This is just the opposite of the procedure in ordinary gas dynamics, where collisions are the dominant effect. This approach is valid for high “temperatures” (the random velocity
temperatures) and low densities, when the mean potential between neighboring particles is small compared with the thermal energy. Collisions are to be relatively unimportant and hence the form of the collision integral can be grossly approximated. In the opposite limiting case of strong and frequent collisions, the corrections are large and dissipative effects play the main role in a system dynamics.

4.1 The Jeans oscillations — weak and rare collisions ($\omega^2_\ast \gg \nu^2_\ast$ and $\nu^2_\ast \ll \Omega^2$)

Such low optical depth regions can be found in the C ring at distances $r < 92\,000$ km from Saturn’s center, where the average optical depth $\tau < 0.1$, in the inner portions of the densest B ring at distances $r < 100\,000$ km, where $\tau = 0.5 - 0.8$, and in the A ring (including the Encke gap) at distances $r > 122\,000$ km, where $\tau \leq 0.5$ (Esposito, 1986, Fig. 2 in his paper).

The equation (13) can be represented in the simplest form ($|l| \leq 1$)

$$ (\omega_\ast + i\nu_\ast)^3 - (\omega_\ast + i\nu_\ast)\omega^2_J + \omega^2_{gr}\kappa^2 = 0, \quad (14) $$

where the square of the Jeans frequency $\omega_J$ is

$$ \omega^2_J \approx \kappa^2 - 2\pi G\sigma_0|k|F(x). \quad (15) $$

In Eq. (15), $F(x) = (2\kappa^2/k^2c^2_r)\exp(-x)I_1(x)$ is the so-called “reduction factor,” and $F(x) \to 1$ in a dynamically cold system ($c_r = 0$) and decreases with increasing $x$ (increasing the velocity dispersion) in a dynamically hot disk ($c_r > 0$). The reduction factor takes into account the fact that the wave field affects only weakly the particles with high random velocities. On the left-hand side in Eq. (14), $|\omega_\ast|$ and $|\omega_J| \gg \nu_\ast$.

Also in Eq. (14),

$$ \omega_{gr} = 2\Omega e^{-x}I_0(x) \frac{2\pi G\sigma_0|k| m \rho^2}{k^2c^2_r} \frac{r_0L}{r_0} $$

is the frequency of the so-called gradient oscillations.

In general, Eq. (14) describes two ordinary Jeans branches of oscillations — the most important long-wavelength branch, $x = k^2c^2_r/\kappa^2 \lesssim 1$, and the short-wavelength one, $x > 1$ — and a new gradient branch of oscillations modified by collisions (Griv and Chiueh, 1997). The Jeans instability occurs when $\omega^2_J < 0$. In the current subsection, we study the physics of this instability and the condition of instability taking into account the additional effects of rare and weak collisions and inhomogeneity. Note that in plasma physics an instability of the Jeans type is known as the negative-mass instability of a relativistic charged particle ring or the diocotron instability of a nonrelativistic ring that caused azimuthal clumping of beams in synchrotrons, betatrons, and mirror machines (Davidson, 1992).

Equation (14) is cubic in $\omega_\ast + i\nu_\ast$, but a useful limit is $\kappa^2 \gtrsim \omega^2_J \gg \omega^2_{gr}$. Finally, analyzing the simplified dispersion relation (14), it is useful to distinguish between the cases of axisymmetric ($m = 0$) and nonaxisymmetric ($m \neq 0$) perturbations.
From relation (14) in the frequency range
\[ |\omega_*^2| \sim |\omega_J^3| \gg |\omega_{gr}| \kappa^2 \quad \text{and} \quad |\omega_*| \gg \nu_c \]
the dispersion law for the Jeans branch of oscillations is
\[ \omega_{*1,2} \approx \pm p|\omega_J| - i\nu_c, \]
where \( p = 1 \) for Jeans-stable perturbations with \( \omega_J^2 > 0 \) and \( p = i \) for Jeans-unstable perturbations with \( \omega_J^2 < 0 \). In Eq. (17), the term involving \( \omega_{gr} \) is the small correction, and in general \( \omega_J^2 \sim \kappa^2/2 \). To repeat ourselves, in this subsection only oscillations in a disk with rare collisions are considered. From Eq. (17) one concludes that both Jeans-stable \( (\omega_J^2 > 0) \) and Jeans-unstable \( (\omega_J^2 < 0) \) perturbations will weakly decay as a result of rare collisions. Accordingly, a spatial inhomogeneity will not influence the stability condition of Jeans modes. Because oscillations in the range (16) are being considered here, the collisional correction \( \sim \nu_c \) and the inhomogeneity correction \( \sim \omega_{gr} \sim L^{-1} \) to the Jeans frequency \( |\omega_J| \lesssim \Omega \) are small, \( \sim \nu_c \ll \Omega \).

In the weakly inhomogeneous disk considered in the paper, from Eq. (17) it follows that the Jeans-unstable perturbations grow in an oscillatory way, \( R\omega_{*1,2} \neq 0 \). The gradient of macroscopic parameters \( \Omega, \sigma_0, c_r, \) and azimuthal mode number \( m \) determine the small real part of such Jeans-unstable modes, \( |R\omega_{*1,2}/3\omega_{*1,2}| \ll 1 \).

Thus, it is found that weak and rare collisions between particles lead to the damping of Jeans modes in a particulate disk. The effect is not great: the time necessary for the perturbation amplitude to fall to 1/e of its initial value is about the collision time, \( \nu_c^{-1} \). This is much longer than, for instance, the characteristic time of a single revolution in Saturn’s rings, \( \sim \Omega^{-1} \). The Jeans instabilities are fastest in the weakly collisional regime, in the sense that their growth rates are slowed down for increasing interparticle collisions.

It follows from Eq. (17) that the collisional effects do not depend on the wavenumber \( k \). The latter contradicts our recent results obtained with the exact Landau integral of collisions (Griv et al., 1997b): in fact, the collision frequency \( \nu_c \) used here should be replaced in the case of small-angle collisions by the effective collision frequency \( \nu_{eff} \approx \nu_c k^2 \rho^2 \), which describes properly the more rapid collisional smoothing of the small-scale inhomogeneities of the particle distribution function, \( k^2 \rho^2 \to \infty \), and \( \nu_{eff} \to 0 \) for long-wavelength perturbations, \( k^2 \rho^2 \to 0 \). Therefore, interparticle collisions are poorly represented by an approximate method presented here. The results obtained in this subsection indicate only a tendency of Jeans perturbations to be damped in a collisional system, and the damping rate given by Eq. (17) is correct only to the order of magnitude.

The Jeans-type perturbations can be stabilized by the random velocity dispersion. Indeed, if one recalls that such unstable perturbations are possible only when \( \omega_{*1,2} = \omega_J^2 < 0 \), then by using the condition \( \omega_{*1,2}^2 \geq 0 \) for all possible \( k \), a local stability criterion
against arbitrary Jeans-type perturbations can be written in the form (Morozov, 1980, 1981; Griv, 1996; Griv and Peter, 1996)

\[ c_r \geq c_T \left( 1 + \left[ \frac{(2\Omega/\kappa)^2 - 1}{\sin^2 \psi} \right] \right)^{1/2} \approx c_T \left[ 1 + 3\sin^2 \psi \right]^{1/2}, \quad (18) \]

where \( c_T = 3.36G\sigma_0/\kappa \) is the well-known Toomre’s (1964) critical velocity dispersion to suppress the instability of only axisymmetric (radial) gravity perturbations, and the fact is used that in Saturn’s rings \( 2\Omega/\kappa \approx 2 \). The stability criterion thus obtained represents generalization of Toomre’s (1964) criterion to the case when additionally nonaxisymmetric (spiral) gravity perturbations (\( \psi \neq 0 \)) are taken into account. The parameter \( \left\{ 1 + \left[ (2\Omega/\kappa)^2 - 1 \right] \sin^2 \psi \right\}^{1/2} \) is an additional stability parameter that depends on both the pitch angle \( \psi \) and the amount of differential rotation in the disk \( d\Omega/dr \) (cf. the parameter \( J \) introduced by Lau and Bertin (1978), Lin and Lau (1979), and Bertin (1980)).

As one can see from Eq. (18), the modified critical velocity dispersion \( c_{\text{crit}} \) grows with \( \psi \). Consequently, in order to suppress the most “dangerous,” in the sense of the loss of gravitational stability, very open nonaxisymmetric perturbations with \( \psi > 45^\circ \), \( c_{\text{crit}} \) should obey the following generalized local stability criterion:

\[ c_r \geq c_{\text{crit}} = \frac{2\Omega}{\kappa} c_T. \]

In Saturn’s rings \( 2\Omega/\kappa \approx 2 \). A relationship exists between Eq. (18) and what Toomre (Toomre, 1977, 1981; Binney and Tremaine, 1987, p. 375) called “swing amplification.”

It is clear from criterion (18) that stability of nonaxisymmetric Jeans perturbations, \( m \) or \( \psi \neq 0 \), in particular very open perturbations with \( \psi \rightarrow 90^\circ \), in a differentially rotating disk \( (2\Omega/\kappa > 1) \) requires a larger velocity dispersion than Toomre’s critical value \( c_T \). It is only for arbitrary perturbations in a rigidly rotating disk \( (2\Omega/\kappa = 1) \) and/or for axisymmetric perturbations in a differentially rotating disk \( c_{\text{crit}} = c_T \). Thus, nonaxisymmetric Jeans-type disturbances in a nonuniformly rotating system are more difficult to suppress than the axisymmetric ones, in general agreement with the work by Goldreich and Lynden-Bell (1965) and Julian and Toomre (1966). The result is quite

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13To obtain Eq. (18), one first finds the critical wavenumber \( k_{\text{crit}} \approx \frac{\kappa}{2\Omega}(1/\rho) \) from the relation \( \partial \omega_j^2/\partial k = 0 \), corresponding to the minimum on the dispersion curve (15). Then this \( k_{\text{crit}} \) is substituted into the dispersion relation and from the condition \( \omega_j^2 \geq 0 \) the critical velocity dispersion is found. This critical velocity dispersion will stabilize arbitrary but not only axisymmetric Jeans perturbations.

14As was pointed out to us by a second anonymous referee of the paper, Lau and Bertin (1978) and Lin and Lau (1979) suggested of \( c_r < (2\Omega/\kappa)c_T \) as a criterion for appreciable swing amplification rather than as a criterion for local nonaxisymmetric instability of gravity perturbations. The point is that there exists ambiguity in the interpretation of “swing” as a transient temporal amplification of single wavelets, or as a steady amplification of propagating waves that reflect and form standing patterns (global normal modes). Most workers in the field feel more comfortable with nonaxisymmetric stability criteria that derive from global normal-mode calculations than local (shearing-sheet) analyses. Because of collisional damping, the former is inappropriate for Saturn’s rings.
obvious: spiral perturbations, in contrast with radial ones, are subject to the influence of the differential character of the rotational motion. An expression for the critical velocity dispersion that is similar to formula (18) was first obtained by Lau and Bertin (1978) and Lin and Lau (1979, p. 130) in the framework of the simple hydrodynamical model and Morozov (1980, 1981) in more complicated stellar dynamical model. The free kinetic energy associated with the differential rotation of the system under study is only one possible source for the growth of the energy of these spiral Jeans-type perturbations, and appears to be released when angular momentum is transferred outward. According to Polyachenko (1989) and Polyachenko and Polyachenko (1997), the marginal stability condition for Jeans perturbations of an arbitrary degree of axial asymmetry has been available since 1965 (Goldreich and Lynden-Bell, 1965), though in a slightly masked form. See Polyachenko and Polyachenko (1997) for a detailed discussion of the problem.

As one can see from Eq. (18), the modified critical velocity dispersion \( c_r \) grows with \((2\Omega/\kappa - 1) \propto |d\Omega/dr|\). This finding can be regarded as evidence of the fierce spiral Jeans-type instability of disks with a strong degree of differential rotation: the shear, \( 2\Omega/\kappa > 1 \), gives rise to a destabilization effect.

Strictly speaking, the above expression for the critical \( c_r \) cannot be used when the pitch angle is large, since the asymptotic expansion is valid only for \( \tan^2 \psi \ll 1 \) (Eq. [7]). It indicates only a tendency with increasing \( \psi \), and in the case of very open spirals with \( \psi > 45^\circ \) a special more accurate analysis, e.g. Polyachenko and Polyachenko (1997), is necessary. Hence, the above expression for the critical velocity dispersion is clearly only approximate. Note only that according to \( N \)-body simulations made by Griv et al. (1999b) the pitch angle of the most unstable Jeans-type perturbations \( \psi \approx 35^\circ \), thus \( \tan^2 \psi = k_\psi^2/k_r^2 \ll 1 \) and the asymptotic Lin-Shu type approximation of moderately tightly wound perturbations used throughout the theory in the present paper does not fail.

The velocity dispersion in a particulate disk is conveniently expressed in terms of Toomre’s (1964) parameter \( Q \), which measures the ratio of actual radial velocity dispersion to the minimum required to suppress the instability of axisymmetric perturbations:

\[
Q = \frac{c_r}{c_T} = \frac{c_r \kappa}{3.36G\sigma_0}.
\] (19)

It follows from Eqs. (18) and (19) that \( Q \approx 2\Omega/\kappa \approx 2 \) is sufficient to suppress the instability of arbitrary gravity perturbations in the disk with Keplerian-like rotation, including the most unstable open ones. Thus, in the differentially rotating disk with rare collisions the value of \( Q \) has to be maintained under about 2 if nonaxisymmetric Jeans-type instabilities are to be developed. The latter result has been obtained in stellar dynamics both analytically (Morozov, 1980, 1981; Griv and Peter, 1996) and by \( N \)-body simulations (Sellwood and Carlberg, 1984; Griv, 1998). Interestingly, Bottema (1993) has found that \( Q \) between 2 and 2.5 over a large range of galactic disks.

Some indirect estimates have probably indicated the value of \( Q \approx 2 \) in the B ring of the Saturnian ring system (Lane et al., 1982, p. 543). So, other Saturn’s rings are likely to have the same or even smaller \( Q \)-values, and therefore they may be Jeans-unstable to
spiral Jeans-type perturbations. According to the above, the characteristic scale of this instability is \( \lambda_{\text{crit}} = 2\pi/k_{\text{crit}} \approx 4\pi\rho \), and only wavelengths close to \( \lambda_{\text{crit}} \) are unstable.

Notice also that the critical value of \( Q \approx 2 \) predicted in our analysis is close to Salo (1992, 1995), Willerding (1992), Osterbart and Willerding (1995), Sterzik et al. (1995), and Griv et al. (1999b) numerical results.

Summarizing, collective motion connected with the Jeans-unstable mode is excited in the plane of a disk of mutually gravitating particles when the random velocity dispersion is not sufficiently large, \( c_r < (2\Omega/\kappa) c_T \), in other words, if the effective Toomre’s \( Q \)-value is \( Q < 2\Omega/\kappa \sim 2 \). The characteristic local scale of this instability is about \( \lambda_{\text{crit}} = 4\pi c_r/\kappa \).

The principal new contribution of our investigation already presented by Griv (1996) and Griv and Yuan (1996) is that in the low optical depth portion of Jeans-stable, differentially rotating, and spatially inhomogeneous particulate disk with \( \tau \approx \nu c/\Omega \ll 1 \) there remains however a Landau-type oscillating microinstability, which is due to a resonant interaction of particles drifting at the phase velocity of the nonaxisymmetric Jeans-stable wave with the wave field at corotation, i.e. Cherenkov radiation effect in plasmas. This new kinetic-type, \( \Re \omega_{1,2} \neq 0 \), \( \Im \omega_{1,2} > 0 \), and \( |\Im \omega_{1,2}/\Re \omega_{1,2}| \ll 1 \), instability of a particulate system (or “a strong resonant version of the diocotron instability”) is fundamentally different from the ordinary aperiodic Jeans (“diocotron”) instability just discussed above. The density waves propagating in the plane of a system are effectively produced at resonant wave-particle interaction with the growth rate of the mode of maximum instability \( \Im \omega_{1,2} \sim 0.1\Omega \). The typical radial wavelength of oscillatory unstable waves \( \sim 2\pi\rho \sim h \). Rare interparticle collisions cannot suppress this resonant self-excitation of spiral density waves in Saturn’s rings. The free kinetic energy associated with the differential rotation is only one possible source for the growth of the average wave energy. Such kinetic-type instabilities distinguish themselves in that only a rather small group of so-called resonant particles takes part in their generation. For this reason the energy capacity of the kinetic instabilities, as a rule, is considerably less than the energy capacity of the hydrodynamic Jeans-type instabilities; the latter are generated by almost all the particles of the phase space. To stress, the kinetic Landau-type instability of the particulate medium may be excited in the gravitating, differentially rotating, and spatially inhomogeneous disk provided that Jeans instabilities are already completely suppressed by the combined effect of rotation and thermal motions, Eq. (18).

It follows from Eq. (17), the wavelength of maximum Jeans instability \( \lambda_{\text{crit}} = 2\pi/k_{\text{crit}} \) is given by

\[
\lambda_{\text{crit}} \approx (2\Omega/\kappa)2\pi\rho \approx 4\pi c_r/\kappa, \tag{20}
\]

where \( c_r \) is expressed by Eq. (18). For the parameters of Saturn’s rings \( \lambda_{\text{crit}} \sim 100 \)

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15 As has been pointed out to us by the referee of the paper, R. H. Durisen, estimating the velocity dispersion in different Saturn’s ring regions with a steep particle-size distribution is not a simple problem: the dispersion velocities are only “measured” through analysis of density waves damping. Also, there are real differences in the “opacity” of different ring regions, e.g. Cuzzi and Estrada (1998). In this regard, we note the related idea of DEB’s (Dynamic Ephemeral Bodies) in Saturn’s rings (Davis et al., 1984; Longaretti, 1989).
m, and such scales of a few hundred meters correspond to the hyperfine ringlet structure of Saturn’s rings probably already discovered by Voyager 2 in low optical depth regions. This dynamical model may help to explain the observed quadrupole azimuthal brightness asymmetry of the A ring (see the Introduction).

Recently Griv (1998) and Griv et al. (1999b) studied the almost collisionless self-gravitating particulate disk using the numerical method of local simulations (or N-body simulations in a Hill’s approximation). It was shown that the local stability criterion obtained from the computer models is in general agreement with the theoretical prediction as outlined in the present paper. Moreover, it has been shown that, as a direct result of the Jeans instability of nonaxisymmetric perturbations, the low optical depth of such a system is subdivided into numerous irregular ringlets, with size and spacing of the order $\lambda_{\text{crit}}$.

According to Eq. (17), the Jeans-unstable perturbations ($\omega_2^2 < 0$) grow almost aperiodically, so the arbitrary perturbation therefore does not propagate, but stands still and grows. To orders of magnitude the growth rate of this fierce instability

$$\Im \omega_{1,2} \sim \sqrt{2\pi G\sigma_0 (k_2^2/|k|)} \exp(-k_2^2 \rho^2) \lesssim \Omega,$$

where $k_2^2 \rho^2 \lesssim 1$. This means that the instability under investigation will develop rapidly on a dynamical time scale $\sim 1/\Omega$. Inevitably, the velocity dispersion of particles would be expected to increase in the field of unstable waves with an amplitude increasing with time at the expense of the regular circular velocity (Griv et al., 1994). That is, if at the beginning the dispersion remains below the critical value $c_{\text{crit}} \approx 2c_T$ (or the critical $Q_{\text{crit}} \approx 2$, respectively) in Saturn’s rings, owing to the gravitational collapses it will increase until this value is reached, i.e. the situation is stabilized. The Jeans instability grows on a dynamical time-scale and heats the disk with rare interparticle collisions until $Q \approx Q_{\text{crit}}$ (Morozov, 1980, 1981; Griv et al., 1999b). With the disappearance of the Jeans instabilities the increase of velocity dispersion will stop. The theory of such collective wave-particle interaction was initially developed in a theory of plasmas, which is the quasi-linear theory (Alexandrov et al., 1984, p. 408; Krall and Trivelpiece, 1986, p. 520). From the plasma theory, it follows that the characteristic time in which the square of total spread of random velocities approximately doubles, is equal to $t_\text{d} \sim 1/\Im \omega_{1,2}$. Therefore, in the case of the instability $t_\text{d}$ is about only the time of a single revolution of a system $\sim 1/\Omega$. The increase of velocity dispersion takes place because the particles gain additional oscillatory energy in the gravitational field of unstable density waves (Griv et al., 1994). This nonresonant process of the heating of the medium by a rise in the amplitude of the unstable waves growing almost aperiodically recalls the case of nonresonant quasi-linear relaxation in plasmas, which can effectively heat the medium even without interparticle collisions (Alexandrov et al., 1984; Krall and Trivelpiece, 1986). Of course, this is an apparent heating and there is no change in entropy (Griv et al., 1994). As a result of this increase of the velocity dispersion up to $c_r \approx 2c_T$ (or $Q \approx 2$, respectively), the Jeans instability will be switched off. Note that such fast
growth in the velocity dispersion has been observed in N-body simulations of Jeans-unstable stellar disks (Sellwood and Carlberg, 1984; Grivnev 1985; Griv and Chiueh, 1998). Apparently, Toomre (1964) and Kulsrud (1972) were the first who discussed an enhancement the rate of relaxation of particulate systems toward thermal equilibrium (or quasi-equilibrium) by collective effects. That is, relaxation may occur by collective collisions between one particle and many others which are collected together by some coherent process such as a wave (see Kulsrud (1972) for an explanation).

Thus, in the nonlinear regime, the particles in low optical depth regions of Saturn’s rings (and the stars in galaxies) can continue developing Jeans-unstable condensations only if some effective mechanism of “cooling” exists. It seems that just this approach is assumed in the modern version of the long-lived spiral structure in numerical models of a stellar-gaseous galactic disk — suggesting the dissipation in the gas and accretion, and formation of new dynamically cold stars (Sellwood and Carlberg, 1984; Griv and Chiueh, 1998). Obviously, in Saturn’s rings such cooling mechanism is actually operating: inelastic/dissipative interparticle collisions reduce the magnitude of the relative velocity of particles. By local N-body simulations, Salo (1992, 1995), Richardson (1994), and Osterbart and Willerding (1995) already investigated the role of the Jeans instability type mechanism in long-term sculpting of the structure of Saturn’s rings by including both gravitational interactions and dissipative impacts between particles. It seems likely that this cooling mechanism might operate effectively in the low-optical depth regions of the Saturnian ring system, and might play an important role in the development of a hyperfine $\sim 4\pi\rho$ long-lived ring structure therein.

In turn, in the another frequency range

$$|\omega_3| \approx |\omega_{gr}| \ll |\omega_s| \quad \text{and} \quad |\omega_s| \gg \nu_c,$$

the dispersion relation (14) has also another root equal to

$$\omega_{3s} \approx \omega_{gr} \frac{\kappa^2}{\omega_3^2} - i\nu_c.$$

This root describes the gradient, $L^{-1} \neq 0$, branch of oscillations modified by collisions. Apparently, the gradient perturbations are stable and are independent of the stability of Jeans modes. These low-frequency, $|\omega_{3s}| \ll \Omega$, gradient waves are weakly damped ($\Im \omega_{3s} < 0$) so that in the plane of a collisional disk weakly damped small-amplitude waves can propagate.

4.2 The dissipative oscillations — strong and frequent collisions ($\omega^2 \ll \nu_c^2$ and $\nu_c^2 \gg \Omega^2$)

Such high optical depth regions, in which one cannot treat collisions as a small perturbation, can be found in the central portions of the B ring at distances $100,000 < r < 122,000$ km from Saturn’s center, where $\tau > 1$ (Esposito, 1986, Fig. 2). This is just the procedure in ordinary gas dynamics, where collisions are the dominant effect and the mean
potential between neighboring particles is large compared with the thermal energy. This is the usual hydrodynamic approximation accepted by Lynden-Bell and Pringle (1974), Morozov et al. (1985), Willerding (1992), and Schmit and Tscharnuter (1995, 1999) for perturbations with small epicyclic radius, \( k_* \rho \leq 1 \) (and \( \nu_c^2 \gg \kappa^2 \)).

In the limit when the frequency of oscillations is smaller than the collision frequency and the collisional frequency is greater than the rotational (epicyclic) frequency, the effect of the system’s rotation is negligible. In a plasma, this limit corresponds to the isotropic plasma without an external magnetic field. In such a nonrotating system, in the lowest approximation of the theory the particle dynamics and wave propagation properties in a system correspond to an unmagnetized plasma and free-streaming particle orbits

\[
\vec{r}' = \vec{r} + \vec{v}(t' - t) \quad \text{and} \quad \vec{v}' = \vec{v},
\]

where \( \vec{r} \) is the radius-vector of a particle and \( \vec{v} \) is the particle’s velocity. This is physically obvious since a typical particle should follow at least one epicycle between collisions. Only in this limit can one speak of the epicycle rotation. The equilibrium axially symmetric distribution function is the Maxwellian with the “temperature” \( c^2 \) and the surface density \( \sigma_0 \):

\[
f_0(|\vec{v}|) = \frac{\sigma_0}{2\pi c^2} \exp\left( -\frac{v^2}{2c^2} \right).
\]

The formal transition to the limit \( \kappa \to 0 \) in Eq. (13) is nontrivial and connected with the problem of the asymptotic representation of the Bessel functions of high order at large arguments. For \( \kappa \to 0 \), the arguments of the Bessel functions in Eq. (13) become large. Then, all terms with \( |l| < l_{\text{max}} \approx k_* v_\perp / \kappa \) contribute to the same order, whereas the terms with \( |l| > l_{\text{max}} \) are exponentially small. Therefore, the summation in Eq. (13) must be extended up to \( |l| = l_{\text{max}} \). Such a solution has been attempted in plasma physics, however, the calculation was tedious. A more well-defined approach is to integrate the basic Boltzmann equation (1) over the free-streaming orbits given above when the collision term becomes the principal one, and obtain a dispersion relation in this “rotationless” case. Then, the resulting Boltzmann equation for the perturbed distribution function of particles (in a rotating frame we are using),

\[
\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial \Phi_1}{\partial \vec{r}} - \frac{\partial \Phi_1}{\partial \vec{v}} \cdot \frac{\partial f_0}{\partial \vec{v}} = -\nu_c f_1,
\]

can be solved by successive approximations, neglecting the influence of the first two terms on the left-hand side \( \propto \partial / \partial t \) and \( \partial / \partial \vec{r} \) on the distribution of particles in the zero-order approximation (cf. Rukhadze and Silin (1969) and Alexandrov et al. (1984, pp. 66 and 101)).\(^{16}\) As a result in the zero-order approximation, one gets

\[
f_1 = i \frac{\Phi_1 \nu_c}{\nu_c} \frac{\partial f_0}{\partial \vec{v}}.
\]

\(^{16}\)The above is analogous to the efficient Chapman-Enskog method of solution of the Boltzmann equation for a monatomic gas (Lifshitz and Pitaevskii, 1981).
The perturbed distribution \( f_1 \) given by Eq. (21) can be used in the continuity equation to determine perturbation of the surface density. The continuity equation for a small density perturbation \( \sigma_1(\vec{r}, t) \) in a disk is

\[
\sigma_1 = - \int_{-\infty}^{t} \frac{dT}{\partial \vec{r}} \int_{-\infty}^{\infty} \vec{v} f_1 d\vec{v},
\]

(22)

where \( |\sigma_1/\sigma_0| \ll 1 \) and \( \sigma_1 = 0 \) as \( t \to -\infty \). The fact that the density of the disk can vary only as a result of diffusion of particles in the disk, i.e. any external (or nondiffusion) fluxes of matter are absent, has taken into account in the equation of continuity. Utilizing in Eq. (22) the above equation for the \( f_1 \), we thus arrive at the following formula for the perturbed surface density in a spatially homogeneous disk:

\[
\sigma_1 = \Phi_1 k^2 i \nu c \int_{-\infty}^{\infty} \vec{v} \cdot \frac{\partial f_0}{\partial \vec{v}} d\vec{v} \omega - \vec{k} \cdot \vec{v} - \omega \nu - i \epsilon,
\]

(23)

where the positive infinitesimal \( \epsilon \) was introduced, which serves to assure the adiabatic turning on the perturbation (Lifshitz and Pitaevskii, 1981; Alexandrov et al., 1984). Following Lifshitz and Pitaevskii (1981, p. 121), we take the \( r \)-axis along \( \vec{k} \). Then

\[
\sigma_1 = \Phi_1 k^2 \int_{-\infty}^{\infty} v_r \frac{\partial f_0(v_r)}{\partial v_r} dv_r \omega \left( 1 + k_r v_r - \omega + i \epsilon \right),
\]

(24)

where we used the distribution function only with respect to \( v_r \):

\[
f_0(v_r) = \int_{-\infty}^{\infty} f_0 dv.
\]

In Eq. (24), by considering the most interesting high-frequencies perturbations when \( \omega^2 \gg k_r^2 v_r^2 \), the integral may be evaluated by using the binomial theorem:

\[
\int_{-\infty}^{\infty} v_r \frac{\partial f_0(v_r)}{\partial v_r} \frac{dv_r}{\omega - k_r v_r} = \int_{-\infty}^{\infty} v_r \frac{\partial f_0(v_r)}{\partial v_r} \frac{dv_r}{\omega} \left( 1 + \frac{k_r v_r}{\omega} + \frac{k_r^2 v_r^2}{\omega^2} + \cdots \right).
\]

The integrals of the terms even in \( \omega \) are zero; the remainder give

\[
\sigma_1 \approx \frac{\Phi_1 k^2 \sigma_0}{i \nu c \omega} \left( 1 + \frac{3 k_r^2 v_r^2}{\omega^2} \right),
\]

(25)

where \( k_r^2 v_r^2/\omega^2 \ll 1 \).

A comparison of Eqs. (7) and (25) leads to the sought dispersion relation in the fast wave range where the phase velocity exceeds the thermal velocities \( \omega/k_r > c_r \),

\[
1 + \frac{2 \pi G \sigma_0 |k_r|}{i \nu c \omega} \left( 1 + \frac{3 k_r^2 c_r^2}{\omega^2} \right) = 0,
\]

(26)

27
which finally yields

\[ \omega = \frac{i2\pi G\sigma_0|k_r|}{\nu_c} \left[ 1 - 3\nu_c^2 \left( \frac{c_r}{2\pi G\sigma_0} \right)^2 \right]. \]  

(27)

We have took into account that \( \omega \approx i2\pi G\sigma_0|k_r|/\nu_c \) and substituted this value for \( \omega \) in a small term in Eq. (26). Due to condition \( \omega^2/k_r^2 \gg c_r^2 \), the second term on the right-hand side in Eq. (27) is a small correction.

Equation (27) gives almost the same condition for secular dissipative-type instability as in the usual hydrodynamic description adopted by Lynden-Bell and Pringle (1974), Fridman and Polyachenko (1984, Vol. 2), Morozov et al. (1985), and others: a disk with frequent collisions is always aperiodically unstable for all wavelengths. Frequent binary collisions thus remove the rotational stabilization in a flat system. This fundamental result was found first by Lynden-Bell and Pringle (1974). Contrary to Lynden-Bell and Pringle, Fridman and Polyachenko, Morozov et al., and others, the result (27) obtained in the present paper through the use of the kinetic approximation states that the given instability in a particulate system develops for all modes, but not only for modes with wavelengths longer than a critical one. To orders of magnitude the growth rate of the instability in Saturn’s rings \( 3\omega = (0.3 - 0.5)\Omega \). This is because in the B ring \( 2\pi G\sigma_0|k_r| \sim \Omega^2 \) and \( \nu_c/\Omega = 2 - 3 \). Thus, the instability will develop rapidly on a time scale of \( 3 - 5 \) rotations. Interestingly, the growth rate of secular-unstable perturbations decreases with increasing \( \nu_c \) and \( c_r^2 \). This is physically obvious: as a result of interparticle collisions and thermal pressure, the organized motion tends to be lost. Similar to the case of the classical Jeans instability considered in the previous subsection, it may be shown that the inclusion of spatial inhomogeneity leads to a small real part of the wavefrequency \( \omega \). Like the Jeans instability, the secular dissipative-type instability presumably heats the disk with frequent interparticle collisions. According to the equation above, the latter leads to a decrease in the growth rate of the oscillation amplitude. Eventually, as a result of such “heating” the secular instability will be switched off. It seems likely that, in the nonlinear regime, the particles in Saturn’s rings can continue developing dissipative-unstable condensations: inelastic physical collisions reduce the magnitude of the relative velocity of particles, and, thus, effectively cool the system.

There are similarities between the aforementioned almost aperiodic instability in a particulate disk with frequent collisions and the secular dissipative-type instability in a rigidly rotating gaseous viscous sheet that was first discovered by Lynden-Bell and Pringle (1974). They claimed this instability to be analogous to the well-known viscous mechanism that converts Maclaurin spheroids to Jacobi ellipsoids. (In fact, Lynden-Bell and Pringle derived the dispersion relation for a rigidly rotating sheet.) Fridman and Polyachenko (1984, Vol. 2, p. 239), Morozov et al. (1985), and Willerding (1992) improved the Lynden-Bell and Pringle calculation by taking into account the effect of nonuniform rotation in a two-dimensional self-gravitating system. Fridman and Polyachenko showed that the perturbed, sliced state is energetically preferable. Fridman and Polyachenko, and Morozov et al. explained the cause of the instability:
it results from perturbations which have negative energy in the dissipative medium. Thus, this instability might arise merely from the dissipation of the energy of regular circular rotation into ever larger amounts of heat, i.e. the energy of medium’s regular motion transforms into random motions of particles. According to Morozov et al. (1985) and Eq. (27) the introduction of differential rotation leaves the result of Lynden-Bell and Pringle unchanged. Similar studies of the secular dissipative-type ring instability in the framework of the simple hydrodynamic model were made also by Schmit and Tscharnuter (1995, 1999). To emphasize, the instability under consideration, which has an essential dependence on the self-gravitation of the disk matter, will remain even in a rigidly rotating disk. Gorkavyi et al. (1990) attempt to apply the secular dissipative-type instability to the distance law of planets. By a particle N-body simulations, Sterzik et al. (1995) probably confirmed predictions regarding wavelength and growth time of the instability. Sterzik et al. especially stressed that any dissipative mechanism (e.g. convectively or magnetically or shear-generated turbulence, inelastic particle collisions) can cause the instability, as long as it reflects a hydrodynamic shear viscosity.

Equation (27) only indicates that the growth rate is a maximum for short radial wavelength, \( \lambda_r = 2\pi/k_r \to 0 \), but cannot determine the region of maximally unstable wavelengths. This actually means that the characteristic scale of stratification of the disk to which the secular instability leads is unknown. However, we have restricted ourselves to the consideration of relatively long-wavelengths modes with \( \lambda_r \gg h/2 \).

We conclude that, as in the case of rare and weak collisions, the instability growth rate is maximum for the wavelengths \( \lambda_r \sim (2 - 4)\pi h \), that is, the growth rate is a maximum for perturbations of the order the Jeans-Toomre wavelength, \( \sim \lambda_J = c^2/G\sigma_0 \). Here, in accordance with the results of the previous subsection, we took into account that in marginally Jeans-stable disks \( c \approx (2\Omega/\kappa)c_T \). In the hydrodynamical model of a galactic gaseous disk, Morozov et al. (1985), and in the hydrodynamical model of Saturn’s ring disk, Schmit and Tscharnuter (1995, 1999) already pointed to the most unstable perturbations \( \sim 4\lambda_J \) in marginally Jeans-stable parts of a self-gravitating system with frequent interparticle collisions.

To emphasize, the instability has an essential dependence on the self-gravitation of the disk matter, and it has nothing to do with a “purely viscous” mechanism suggested by Lin and Bodenheimer (1981), Lukkari (1981), and Ward (1981) to describe the irregular ringlets in regions of high optical depth around Saturn by taking into account the effects of viscous forces, but neglecting self-gravity.\(^{18}\)

\(^{17}\)Small-scale perturbations with \( \lambda_r < h \) are stable (Fridman and Polyachenko, 1984).

\(^{18}\)Araki and Tremaine (1986), Wisdom and Tremaine (1986), Brophy and Esposito (1989), and Richardson (1994) have proved that the “purely viscous” type of instability would not actually occur in Saturn’s rings.
5 Conclusions

Collective-type oscillations (and their stability) of highly flattened self-gravitating systems have been investigated both experimentally by gravitational $N$-body simulations and theoretically for more than three decades. The linear theory of such oscillations is well known. In this paper the linear kinetic theory of oscillations and their stability of the highly flattened, rapidly (and nonuniformly) rotating disk of mutually gravitating particles is reexamined and extended. The simultaneous effect of self-gravity and collisions between particles is taken into account. The stability analysis of small gravity perturbations is based on the Boltzmann kinetic equation with the Krook model collision integral, modified in accord with Shu and Stewart (1985) to allow for the effects of inelastic collisions between particles. Interest in the kinetic theory is due to the efforts to solve the problem of small-scale irregular radial structure in Saturn’s rings probably already revealed by the Voyager mission.

It is found that in the two limiting cases of weak and rare and strong and frequent physical inelastic collisions the particulate disk may be unstable with respect to the Jeans-type perturbations and to the secular dissipative-type perturbations, respectively. These instabilities of small-amplitude gravity perturbations (e.g. those produced by a spontaneous perturbation and/or a companion system) do not depend on the behavior of the particle distribution function in the neighborhood of a particular speed, but the determining factors of the instabilities are the macroscopic parameters of a self-gravitating system — the mean mass density, the angular velocity of regular rotation, the dispersion of random velocities of particles, and the frequency of interparticle collisions. Generally, the growth rate of these almost aperiodic ($|\Re \omega / \Im \omega | \ll 1$) nonresonant instabilities is $\lambda_{\text{crit}} \approx 4\pi \rho \approx 2\pi h$. For the parameters of Saturn’s rings $\lambda_{\text{crit}} \sim 100$ m.

We suggest that the instabilities we are investigating may be considered as an effective generating mechanism for hyperfine structure of the order of one Jeans length, i.e. $\sim 4\pi \rho \approx 2\pi h$ in the main C, A, and B rings of the Saturnian ring system. Because modern observations indicate that the ring thickness ranges from 1 m or less in the C ring to $1 - 5$ m in the B ring and $5 - 30$ m in the A ring (Esposito, 1986, 1993), more accurate estimations give the critical value of wavelength $\lambda_{\text{crit}}$ in the C ring $\lambda_{\text{crit}} \approx 7$ m, in the B ring $\lambda_{\text{crit}} = 7 - 30$ m, and in the A ring $\lambda_{\text{crit}} = 30 - 200$ m.

In addition, we can expect the formation of small-scale spiral arm and/or radial fragments as precursors of small “moonlets” inside the Roche limit in the Saturnian ring system and protoplanetary disks of particles. The latter is an important step towards an understanding of a main question of protoplanetary disk evolution as well as the evolutionary processes in galactic disks: what kind of evolutionary processes leads to the formation of moons, planets, and stars in different astrophysical disk system?

As has been pointed out in the Introduction, by Schmit and Tscharnuter (1995) and
Griv (1996, 1998), structures on the 100 m scale fall below the noise of Voyager’s stellar occultation data. More precise Cassini spacecraft observations should help to settle this question almost 6 years from now, but at present the paper would have to be regarded as a prediction of structure in ring systems based on theoretical modeling rather than as an explanation for observed features.

Further work is in progress to extend the calculations presented in the article. In addition, it would be desirable to have experimental tests (N-body experiments) of the theory.

Acknowledgements. The authors wish to thank numerous colleagues for many discussions, especially Kai-Wing Chen, Tzihong Chiueh, Alexei M. Fridman, Peter Goldreich, Edward Liverts, Muzañar N. Maksumov, William Peter, Shlomi Pistinner, Frank H. Shu, and Raphael Steinitz. We are grateful to the first referee of the paper, R. H. Durisen, for very useful comments and constructive criticism that improved our understanding of the problems under consideration. This research was sponsored in part by the Ministry of Science and the Science Foundation founded by the Academy of Sciences and Humanities in Israel, and the Academia Sinica in Taiwan. One of us (E. G.) was financially supported by the Israeli Ministry of Immigrant Absorption in the framework of the program “Giladi.”

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