MODULAR CATEGORIES AND ORBIFOLD MODELS II

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INTRODUCTION

This paper is a continuation of the paper [Ki]. Its main goal is to study so-called orbifold models in conformal field theory. In mathematical language, this question can be reformulated as follows: given a vertex operator algebra \( V \) and a finite group of automorphisms \( G \), describe the category \( C \) of modules over the fixed-point algebra \( V^G \). In what follows, we assume that both \( V \) and \( V^G \) are rational VOA’s and that categories of module over \( V, V^G \) are modular tensor categories.

It is well known that \( C \) cannot be determined from the category of \( V \)-modules and the group \( G \) only. However, in the holomorphic case (when \( V \) has a unique simple module, \( V \) itself) it was suggested in [DVVV], [DPR] and proved in [Ki] (for \( \omega = 1 \)) that if we additionally assume that \( C \) is modular, then \( C \) is equivalent to the category of representations of a twisted Drinfeld double \( D^\omega(G) \). (This result had also been announced — without full proof — in [Mu]). Thus, in the holomorphic case the extra data we need to completely determine \( C \) is the cocycle \( \omega \in H^3(G, \mathbb{C}^\times) \).

In this paper, we present some partial results in non-holomorphic case. In particular, we show that the category \( C \) is completely determined by the category of “twisted” \( V \)-modules and the action of the group \( G \) on this category. Our work is based on the results of [KO], [Ki], where it was shown that under certain assumptions on \( V, V^G \), this problem can be reformulated in the language of tensor categories. We give this reformulation below, and in the remainder of the paper only use the language of braided tensor categories. Vertex operator algebras will not appear in the paper at all.

Almost all results of this paper had been announced in [Mu]; however, [Mu] does not contain proofs of many important results, referring the reader to a manuscript in preparation [Mu2]. Thus, we feel that the present paper may help the readers to fill this gap. Also, our methods are somewhat different from those of [Mu].

Finally, it should be noted that some of the results of the current paper had been proved in the language of VOA’s in the paper [DY]; see Remark 4.3 for details. It also seems that there is a close relation between he current paper and recent paper by Yamagami [Y]. We plan to study this relation in detail in future publications.

1. FORMULATION OF THE PROBLEM

In this section, we list the main conventions used in this paper and formulate the main problem in the language of tensor categories.

Throughout the paper, we keep the same notation as in [KO], [Ki]. In particular,

- \( C \) is a semisimple rigid balanced braided tensor category over \( \mathbb{C} \) (later we will assume that \( C \) is modular),

The author was supported in part by NSF grant DMS9970473.
• $A$ is a rigid commutative associative algebra in $\mathcal{C}$ and $\theta_A = \text{id}$ (where $\theta$ is the universal twist in the category).

• $G$ a finite group acting faithfully by automorphisms $\pi_g, g \in G$ on $A$ such that $A^G = 1$

As in [KO], we use two categories of $A$-modules, $\text{Rep} A, \text{Rep}^0 A$ and functors $F: \mathcal{C} \to \text{Rep} A, G: \text{Rep} A \to \mathcal{C}$.

In such a situation, we will say that $\mathcal{C}$ is a “$G$-orbifold of $\text{Rep}^0 A$” (this terminology is motivated by physics, as mentioned in the introduction).

1.1. Problem. Reconstruct $\mathcal{C}$ from the category $\text{Rep}^0 A$ and the group $G$ and probably some extra data.

The paper [Ki] gave a proof of the result suggested in [DVVV], namely, the answer to this question in the case when $\text{Rep}^0 A \simeq \text{Vec}$ is the category of finite-dimensional vector spaces (so-called holomorphic case). Here we will try to consider the general case.

Before giving the answer, let us introduce some notation.

1.2. Definition. $\mathcal{C}_1 \subset \mathcal{C}$ is the full subcategory generated (as an abelian category) by simple objects $V_i$ such that $\langle V_i, X \rangle \neq 0$ for some $X \in \text{Rep}^0 A$.

Then we have the following diagram:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \text{Rep} A \\
\downarrow G & & \downarrow G \\
\mathcal{C}_1 & \xrightarrow{F} & \text{Rep}^0 A
\end{array}
$$

Note that at the moment it is not clear why for every $V \in \mathcal{C}_1$, $F(V) \in \text{Rep}^0 A$. This will be proved later (see Corollary 4.6).

So, our goal can be formulated as follows: we need to reconstruct this diagram from its lower right corner. However, we need more data. The first piece of data is that in this situation, we have an “action” of $G$ on the category $\text{Rep}^0 A$.

As in [Ki, Theorem 4.7], for $X \in \text{Rep} A, h \in G$ we denote by $X^h$ the object of $\text{Rep} A$ which coincides with $X$ as an object of $\mathcal{C}$ but has the action of $A$ twisted by $\pi_h$. It follows from the definition that $(X^h)^g = X^{hg}$. Thus, this construction defines an “action of $G$ by functors on $\text{Rep} A$”. To be precise, we have the following lemma.

1.3. Lemma. For $g \in G$ let $\Pi_g$ be a functor $\text{Rep} A \to \text{Rep} A$ defined by

$$
\Pi_g(X) = X^{g^{-1}},
$$

and for $f \in \text{Hom}_A(X, Y)$, $\Pi_g(f) = f$ considered as a morphism $X^{g^{-1}} \to Y^{g^{-1}}$.

Then we have canonical functor isomorphisms $\Pi_g \Pi_h = \Pi_{gh}$ and $\Pi_g(X \otimes_A Y) = \Pi_g(X) \otimes_A \Pi_g(Y)$.

In particular, this shows that $G$ acts on the set of isomorphism classes of objects in $\text{Rep} A$ by

$$
g[X] = [X^{g^{-1}}].
$$
Note that we use $X^g^{-1}$ rather than $X^g$ in order to get the left action; the formula $[X] \mapsto [X^g]$ would give right action.

The following construction allows us to pass from $\text{Rep} A$ on which $G$ acts by functors to new category on which $G$ acts by automorphisms of objects:

### 1.4. Definition

Let $(\text{Rep} A)^G$ be the category with

$$\text{Obj}(\text{Rep} A)^G = \{(X, \varphi_X)\}$$

where $X$ is an object of $\text{Rep} A$ and $\varphi_X$ is a collection of $C$-morphisms $\varphi_X(g): X \to X, g \in G$ such that the following diagram is commutative:

$$\begin{array}{ccc}
A \otimes X & \xrightarrow{\mu} & X \\
\pi_g \otimes \varphi_X(g) & \downarrow & \varphi_X(g) \\
A \otimes X & \xrightarrow{\mu} & X \\
\end{array}$$

and

$$(1.5) \quad \varphi_X(g_1) \varphi_X(g_2) = \varphi_X(g_1 g_2), \quad \varphi(1) = \text{id}.$$ 

Morphisms in $(\text{Rep} A)^G$ are defined to be morphisms in $\text{Rep} A$ which commute with $\varphi(g)$.

Finally, we denote by $(\text{Rep}^0 A)^G$ the full subcategory in $(\text{Rep} A)^G$ with objects $(X, \varphi_X)$ such that $X \in \text{Rep}^0 A$.

Condition (1.4) is equivalent to saying that $\varphi_X(g)$ is a morphism of $A$-modules $X \to X^g$, or, more generally, for every $h$, $\varphi_X(g)$ is a morphism of $A$-modules $X^h \to X^{gh}$.

Note that (1.5) requires that $\varphi(g)$ define an action of $G$ by $C$-automorphisms of $X$. One might think that this is too restrictive and we need to allow projective action. However, as we explain later (see Theorem 3.5), it is not necessary.

Then the main result of this paper can be formulated as follows.

### 1.5. Theorem

One has natural equivalences of braided tensor categories

$$\mathcal{C} \simeq (\text{Rep} A)^G,$$

$$\mathcal{C}_1 \simeq (\text{Rep}^0 A)^G.$$ 

Proof of this theorem and further results will be given in Section 4, Section 5.

Comparing this with the original question of recovering the diagram (1.1) from its lower right corner, i.e. $\text{Rep}^0 A$, we see that so far we have succeeded in reconstructing along the horizontal lines: we can reconstruct $\mathcal{C}_1$ from $\text{Rep}^0 A$, and $\mathcal{C}$ from $\text{Rep} A$ (and the action of $G$ by automorphisms on $\text{Rep} A$).

### 1.6. Remark

It should be noted that the bottom line of the diagram (1.1) is “modularization” in the sense of [3].

Reconstruction of $\mathcal{C}$ from $\mathcal{C}_1$ presents serious difficulties, some of which will be discussed in Section 6, and at the moment seems to be out of reach in non-holomorphic case.
2. Model example

Let $M$ be a finite group, and $N \subset M$ a normal subgroup. Let $G = M/N$, $\mathcal{C} = \text{Rep } M$. As in [KO], let $A = \mathcal{F}(M/N)$ be the algebra of functions on $M/N = G$; we denote by $\delta_x, x \in G$ the standard basis of delta-functions in $A$. Then $A$ is a $\mathcal{C}$-algebra, and $\text{Rep } A = \text{Rep}^0 A$. Moreover, we have an action of $G$ by automorphisms on $A$:

$$\pi_g \delta_x = \delta_{xg^{-1}}, \quad x, g \in G.$$  

Thus we are in the situation discussed in Section 1, but with the diagram (1.1) degenerating to a single line:

$$\mathcal{C} \xrightarrow{F} \text{Rep } A.$$

In this situation, $\text{Rep } A = \text{Rep } N$, and the functors $F$ and $G$ are the restriction and induction functors, respectively (see [KO]), so the previous diagram becomes

$$\text{Rep } M \xrightarrow{\text{Res}} \text{Rep } N.$$

Note that in particular, by Lemma 1.3 we have an action of $G$ by functor automorphisms on $\text{Rep } N$. This action looks especially simple if $M = G \ltimes N$ is a semidirect product: in this case, $G$ acts on $N$ by conjugation, and the action on $\text{Rep } N$ is nothing by twisting the action of $N$ by this conjugation.

2.1. Theorem. One has canonical equivalence of categories

$$\mathcal{C} \simeq (\text{Rep } N)^G.$$

Proof. It follows from the definition that objects of $(\text{Rep } N)^G = (\text{Rep } A)^G$ are complex vector spaces $V$ with the following extra structure:

- action of $M$
- $G$-grading: $V = \bigoplus_{g \in G} V_g$ such that $mV_g \subset V_{gm}, m \in M$
- action of $G$ which commutes with the action of $M$ and satisfies $\varphi(g)V_x \subset V_{xg^{-1}}$

Define now the functors $(\text{Rep } N)^G \to \text{Rep } M$ and $\text{Rep } M \to (\text{Rep } N)^G$ as follows: for $V \in (\text{Rep } N)^G$,

$$V \mapsto V^G = \text{collections of vectors } \{v_x\}_{x \in G}$$

such that $v_x \in V_x, \varphi(g)v_x = v_{xg^{-1}}$

and for $W \in \text{Rep } M$,

$$W \mapsto W \otimes \mathcal{F}(G)$$

which is considered as an object of $(\text{Rep } N)^G$ with the grading given by $(W \otimes \mathcal{F}(G))_x = W \otimes C\delta_x, x \in G$, action of $M$ given by $m(w \otimes \delta_x) = mw \otimes \delta_{mx}$ and action of $G$ by $g(w \otimes \delta_x) = w \otimes \delta_{xg^{-1}}$.

It is easy to see that these functors are well-defined and inverse to each other. \hfill \square

In this example, however, $\mathcal{C}$ is symmetric, so $\text{Rep } A = \text{Rep}^0 A$. Let us consider an example of a non-symmetric tensor category.
Let $M$ be a finite group, $N \subset M$ — a normal subgroup, and $D(M) = \mathbb{C}[M] \rtimes \mathcal{F}(M)$ the Drinfeld double of $M$. Then we have following commutative diagram of Hopf algebras:

$$
\begin{array}{ccc}
D(M) & \xrightarrow{\mathbb{C}[N] \rtimes \mathcal{F}(M)} & \mathbb{C}[N] \rtimes \mathcal{F}(M) \\
\downarrow & & \downarrow \\
\mathbb{C}[M] \rtimes \mathcal{F}(N) & \xleftarrow{\mathbb{C}[M] \rtimes \mathcal{F}(N)} & D(N)
\end{array}
$$

where the vertical arrows are restriction maps $\mathcal{F}(M) \to \mathcal{F}(N)$ and horizontal arrows are inclusions $\mathbb{C}[N] \subset \mathbb{C}[M]$.

As before, let $A = \mathcal{F}(G) = \mathcal{F}(M/N)$, which we consider as a module over $D(M)$ by taking the grading to be identically 1. One easily sees that this is a commutative associative algebra in the category $\mathcal{C} = \text{Rep } D(M)$. As before, we have action of $G$ by automorphisms on $A$.

2.2. Lemma. In the situation above, one has canonical equivalences of categories

$$
\begin{align*}
\text{Rep } A &= \text{Rep}(\mathbb{C}[N] \rtimes \mathcal{F}(M)), \\
\text{Rep }^0 A &= \text{Rep } D(N), \\
\mathcal{C}_1 &= \text{Rep}(\mathbb{C}[M] \rtimes \mathcal{F}(N)).
\end{align*}
$$

Proof. The proof is quite parallel to the calculations in [Ki, Section 3].

Thus, in this case the diagram [1.1] is obtained by taking representations of the terms in diagram [2.4]:

$$
\begin{array}{ccc}
\text{Rep } D(M) & \xrightarrow{F} & \text{Rep } \mathbb{C}[N] \rtimes \mathcal{F}(M) \\
\cup & & \cup \\
\text{Rep } \mathbb{C}[M] \rtimes \mathcal{F}(N) & \xleftarrow{F} & \text{Rep } D(N)
\end{array}
$$

3. Category $(\text{Rep } A)^G$

In this section, we study some properties of the category $(\text{Rep } A)^G$ introduced in Definition 1.4. We start by giving examples of objects in this category.

3.1. Example. (1) $X = A, \varphi(g) = \pi_g$. From now on, we will denote this object of $(\text{Rep } A)^G$ by just $A$.

(2) Let $X \in \text{Rep } A$. Let

$$
\text{Ind } X = \bigoplus_{g \in G} X^g.
$$

Then $\text{Ind } X$ has a natural action of $G$ by permutation of summands, which shows that $\text{Ind } X$ is an object of $(\text{Rep } A)^G$.

Note that if $X \in (\text{Rep } A)^G$ and $V$ is a finite-dimensional representation of $G$, then $V \boxtimes X \in \text{Rep } A$ has a natural action of $G$ and thus a structure of an object in $(\text{Rep } A)^G$ (see [K], Section 1] for definition of $\boxtimes$). In other words, $(\text{Rep } A)^G$ is
a module category over \( \text{Rep}_G \) of finite-dimensional complex representations of \( G \). In particular, applying this to \( X = A \), we see that

\[
V \mapsto V \boxtimes A
\]

identifies \( \text{Rep}_G \) with a subcategory in \( (\text{Rep}^0 A)^G \).

### 3.2. Proposition
(\( \text{Rep}_A \))^\( G \) has a natural structure of a rigid tensor category. \( (\text{Rep}^0 A)^G \) has a natural structure of a balanced rigid braided tensor category, with the braiding inherited from \( C \).

**Proof.** Define \( (X, \varphi_X) \otimes (Y, \varphi_Y) = (X \otimes_A Y, \varphi_X \otimes \varphi_Y) \) and \( (X, \varphi_X)^* = (X^*, (\varphi_X(g^{-1}))^*) \), where for \( f \in \text{Hom}_{\text{Rep}_A}(X, Y) \) we denote by \( f^* \in \text{Hom}_{\text{Rep}_A}(Y^*, X^*) \) the adjoint morphism. It is straightforward to check that so defined tensor product and dual object satisfy all required properties. \( \square \)

Note that the definition above also shows that the forgetful functor \( (\text{Rep}_A)^G \rightarrow \text{Rep}_A \) is a tensor functor.

### 3.3. Remark
We can not define a braiding on \( (\text{Rep}_A)^G \) by just using the braiding in \( C \); it will not be a morphism of \( A \)-modules (for this reason, \( \text{Rep}_A \) in general is not a braided category). However, we will show later (see Theorem 5.5) that \( (\text{Rep}_A)^G \) does have a braided structure.

It is possible to give an explicit description of \( (\text{Rep}_A)^G \) as an abelian category. Namely, let \( S, S^0 \) be the sets of isomorphism classes of simple objects in \( \text{Rep}_A \) and \( \text{Rep}^0 A \) respectively. Let us fix for every \( s \in S \) a representative \( X_s \). Formula (1.3) defines an action of \( G \) on \( S \). Let \( \mathcal{O} \subset S \) be a \( G \)-orbit. Let \( \mathcal{F}[\mathcal{O}] \) be the algebra of complex-valued functions on \( \mathcal{O} \) and \( \mathcal{F}^s[\mathcal{O}] \) the group of non-vanishing functions on \( \mathcal{O} \); both \( \mathcal{F}[\mathcal{O}], \mathcal{F}^s[\mathcal{O}] \) are naturally \( G \)-modules. Then, as described in [DY], we have a natural projective action of \( G \times \mathcal{F}[\mathcal{O}] \) by \( A \)-morphisms on \( X_\mathcal{O} = \bigoplus_{s \in \mathcal{O}} X_s \). Namely, the action of \( \mathcal{F}[\mathcal{O}] \) is given by \( \delta_s |_{X_{g}} = \text{id}, \delta_s |_{X_{g'}} = 0 \) for \( s \neq s' \). To define action of \( G \), note that for every \( s \in \mathcal{O}, g \in G \) there exists a unique up to a constant \( A \)-isomorphism

\[
\varphi_s(g) : X_{g^{-1}}^g \xrightarrow{\sim} X_{g(s)}. \tag{3.3}
\]

Let us fix a choice of \( \varphi_s(g) \) and define \( \varphi_X(g) = \bigoplus_{s \in S} \varphi_s(g) \). This gives a projective action of \( G \) on \( X_\mathcal{O} \). Equivalently, we can say that \( \mathcal{O} \) defines a cohomology class \( [\alpha] \in H^2(G, \mathcal{F}^s[\mathcal{O}]) \) and we have a natural action of the twisted algebra \( A_\alpha(G, \mathcal{O}) \) by \( A \)-morphisms on \( X_\mathcal{O} \). We refer the reader to [DY] for details.

In particular, if we choose \( s \in \mathcal{O} \) and denote

\[
G_s = \{ g \in G \mid gs = s \} = \{ g \in G \mid X_{g}^g \simeq X_{s} \},
\]

then we have a projective action of \( G_s \) on \( X_s \), or a true action of the twisted group algebra \( C^{\alpha_s}[G_s] \).

### 3.4. Remark
Please note that we consider \( \mathcal{F}[\mathcal{O}] \) as a left \( G \)-module, whereas [DY] consider it as a right \( G \)-module. Thus, formulas of [DY] should be suitably modified for our setup (which is actually more standard one). This modification is straightforward enough and we leave it to the reader.
3.5. **Theorem.** As an abelian category,

$$(\text{Rep } A)^G \simeq \bigoplus_{\mathcal{O}} \text{Rep } A_{\alpha^{-1}}(G, \mathcal{O}),$$

where $\mathcal{O}$ runs through a set of orbits in $S$. The equivalence is given by

$$V \mapsto V \boxtimes_{F[\mathcal{O}]} X_{\mathcal{O}} = \bigoplus_{s \in \mathcal{O}} V_s \boxtimes X_s, \quad V \in \text{Rep } A_{\alpha^{-1}}(G, \mathcal{O})$$

$$X \mapsto V = \bigoplus_{s \in S} \text{Hom}_A(X_s, X), \quad X \in (\text{Rep } A)^G,$$

where $X_{\mathcal{O}} = \bigoplus_{s \in \mathcal{O}} X_s$, $V = \bigoplus_{s \in \mathcal{O}} V_s$ is the decomposition given by the action of $F[\mathcal{O}]$, and the action of $G$ on $X$ and $V$ is related by $\varphi_X(g) = \bigoplus \varphi_{V_s}(g) \otimes \varphi_s(g)$. Here $\varphi_s(g)$ is as defined in (3.3) and $\varphi_{V_s}(g) : V_s \to V_{gs}$ is the action of $G$ on $V$.

**Proof.** We need to check that the functors defined in the theorem are well-defined and inverse to each other (up to a functorial isomorphism), which is straightforward from the definition. The only step worth mentioning is that both $\varphi_{V_s}$ and $\varphi_s$ are projective representations of $G$, with cocycles $\alpha^{-1}$ and $\alpha$ respectively; thus, the tensor product $\varphi_X(g) = \bigoplus \varphi_{V_s}(g) \otimes \varphi_s(g)$ is a true representation of $G$. \(\square\)

This theorem shows that the structure of $(\text{Rep } A)^G$ as an abelian category is completely determined by the set $S$ with the action of $G$ and the cohomology class $[\alpha_s] = \bigoplus_{\mathcal{O}} [\alpha_{\mathcal{O}}] \in H^2(G, F^*(S))$.

Using the equivalence of categories

$$\text{Rep } C^{\alpha^{-1}}[G_s] \simeq \text{Rep } A_{\alpha^{-1}}(G, \mathcal{O})$$

(see [DY], Theorem 3.5]), the statement of the theorem can be rewritten in simpler but less invariant way: there exists an equivalence of abelian categories

$$(\text{Rep } A)^G \simeq \bigoplus_s \text{Rep } C^{\alpha^{-1}}[G_s],$$

given by

$$(3.4) \quad X \mapsto \bigoplus_s \text{Hom}_A(X_s, X)$$

In both formulas, $s$ runs through a set of representatives of the orbits.

Combining this theorem with semisimplicity of $C^{\alpha}[G_s]$ (see [Kar]), we get the following corollary.

3.6. **Corollary.**

1. $(\text{Rep } A)^G$, $(\text{Rep } A)^G$ are semisimple.
2. Simple objects in $(\text{Rep } A)^G$ (respectively, $(\text{Rep } A)^G$) are

$$(3.5) \quad X_{\lambda, \mathcal{O}} = V_{\lambda} \boxtimes_{F[\mathcal{O}]} X_{\mathcal{O}} = \bigoplus_{s \in \mathcal{O}} (V_{\lambda})_s \boxtimes X_s,$$

where $\mathcal{O}$ is an orbit of the action of $G$ on the set $S$ of simple objects in $\text{Rep } A$ (respectively, the set $S^0$ of simple objects in $\text{Rep } A$), and $V_\lambda$ is an irreducible representation of $A_{\alpha^{-1}}(G, \mathcal{O})$.

3.7. **Example.** Assume that $\text{Rep } A \simeq \text{Vec}$, i.e. the only simple object in $\text{Rep } A$ is $A$. Then $V \mapsto V \boxtimes A$ defines an equivalence of tensor categories $\text{Rep } G \simeq (\text{Rep } A)^G$. This is the so-called holomorphic case; it was studied in [K].
3.8. Lemma. Let $s \in S$ and let $\mathcal{O}$ be the orbit of $s$. Let $\text{Ind } X_s$ be as in Example [3.1]. Then, as an object of $(\text{Rep } A)^G$,

$$\text{Ind } X_s \simeq \bigoplus_{\lambda} V_\lambda^* \boxtimes X_{\lambda, \mathcal{O}},$$

where $\lambda$ runs over the set of irreducible representations of $\mathbb{C}[\alpha^{-1}\mathcal{O}]$. In this formula, the multiplicity spaces $V_\lambda^* \in \text{Rep } \mathbb{C}[\alpha^{-1}\mathcal{O}]$ are considered as vector spaces, with no extra structure.

Proof. It is immediate from the definitions that $\text{Ind } X_s \simeq \bigoplus_{s' \in \mathcal{O}} \mathbb{C}[G_{s'}] \boxtimes X_{s'}$. Now the statement of the theorem follows from the formula (3.4) for the equivalence of categories $(\text{Rep } A)^G \simeq \text{Rep } \mathbb{C}[\alpha^{-1}\mathcal{O}]$ and identity $\mathbb{C}[\mathcal{O}] \simeq V_\lambda^* \otimes V_\lambda$, where $\lambda$ runs over the set of irreducible representations of $\mathbb{C}[\alpha^{-1}\mathcal{O}]$. The latter formula holds for any finite-dimensional semisimple associative algebra. Note that if $V_\lambda$ is a representation of $\mathbb{C}[\alpha^{-1}\mathcal{O}]$, then $V_\lambda^*$ is naturally a representation of $\mathbb{C}[\alpha^{\mathcal{O}}]$. \hfill \Box

4. Untwisted sector

Our first goal is to describe the “untwisted sector” of $\mathcal{C}$, i.e. $\mathcal{C}_1$. Part of it has been done in [Ki] where we showed that the subcategory in $\mathcal{C}$ generated by summands of $G(A)$ is equivalent to $\text{Rep } G$.

In holomorphic case, this is the complete description of the untwisted sector. In non-holomorphic case, more work is needed.

Define a functor $\Phi: (\text{Rep } A)^G \to \mathcal{C}$ by

$$\Phi(X) = X^G,$$

where we use the action of $G$ by $\mathcal{C}$-automorphisms of $X$ defined by $\varphi_X$. The definition of “invariants” $X^G$ is straightforward; interested reader can find the details in [Ki].

For example, it is easy to see that one has canonical isomorphisms

\begin{align}
\Phi(A) &= 1, \\
\Phi(\text{Ind } X) &= G(X), \quad X \in \text{Rep } A.
\end{align}

Note that $X^G$ is canonically a $\mathcal{C}$-sub-object in $X$; moreover, it is easy to write a $\mathcal{C}$-morphism $\text{Sym}_X: X \to X$ which is a projector on $X^g$: $\text{Sym}^2 = \text{Sym}$, $\text{Im } \text{Sym} = X^G$. It is given by

$$\text{Sym} = \frac{1}{|G|} \sum_{g \in G} \varphi_X(g).$$

4.1. Theorem. (1) $\Phi$ is an equivalence of tensor categories $(\text{Rep } A)^G \simeq \mathcal{C}$

(2) Restriction of $\Phi$ to $(\text{Rep }^0 A)^G \subset (\text{Rep } A)^G$ is an equivalence of braided tensor categories $(\text{Rep }^0 A)^G \simeq \mathcal{C}_1$, where $\mathcal{C}_1$ is the full subcategory in $\mathcal{C}$ generated (as an abelian category) by simple objects in $\mathcal{C}$ which appear in decomposition of $G(X)$, $X \in \text{Rep }^0 A$.

Proof. The proof repeats, with suitable changes, the proof of Theorem 2.11 in [Ki], with $\text{Rep } G$ replaced by $(\text{Rep } A)^G$. We sketch below those steps which are not completely identical.
First, note that $\Phi(A) = 1$ and $\langle \Phi(X_{\lambda,\mathcal{O}}), 1 \rangle = 0$ if $X_{\lambda,\mathcal{O}}$ is a simple object in $(\text{Rep} \, A)^G$ which is not isomorphic to $A$. Indeed, the first identity is obvious; the second follows from the fact that if $X_{\lambda,\mathcal{O}} \neq A$, then $\langle G(X), 1 \rangle_{C} = \langle X, A \rangle_{\text{Rep} \, A} = 0$.

Next, define the functorial morphism $J: \Phi(X) \otimes \Phi(Y) \to \Phi(X \otimes_A Y)$ as the following composition:

\begin{equation}
X^G \otimes Y^G \hookrightarrow (X \otimes Y)^G \to (X \otimes_A Y)^G
\end{equation}

(the second morphism is induced by the canonical projection $X \otimes Y \to X \otimes_A Y$, see [KO]).

Now we can repeat the same steps as in [Ki] to show that $\Phi$ is compatible with associativity, commutativity (for $X, Y \in (\text{Rep}^0 A)^G$) and unit isomorphisms, and with duality. Then we show that the subcategory $D \subset (\text{Rep} \, A)^G$ generated (as an abelian category) by simple objects $X_{\lambda,\mathcal{O}} \in (\text{Rep} \, A)^G$ such that $\Phi(X_{\lambda,\mathcal{O}}) \neq 0$ is closed under tensor product and duality. By results of [Ki], $D$ contains $\text{Rep} \, G \subset (\text{Rep}^0 A)^G$.

4.2. Lemma. Let $D$ be a full subcategory in $(\text{Rep} \, A)^G$ which is closed under duality, tensor product, and taking sub-objects and contains $\text{Rep} \, G$. Then $D$ is generated by $X_{\lambda,\mathcal{O}}$ where $\mathcal{O}$ runs over some set of orbits in $S$ and $\lambda$ runs over the set of all irreducible representations of $A_\alpha(G, \mathcal{O})$.

Assume that $\mathcal{O}$ is an orbit from the complement to the set of orbits mentioned in the lemma, i.e. $\Phi(X_{\lambda,\mathcal{O}}) = 0$ for all $\lambda$. Choose $s \in \mathcal{O}$ and consider $\text{Ind} X_s$ as in Example [4.1.4]. On one hand, by [4.2], $\Phi((X_s)) = X_s$. On the other hand, it follows from Lemma [3.8] that $\Phi((X_s)) = 0$. This contradiction shows that $D = (\text{Rep} \, A)^G$, and thus, $\Phi(X_{\lambda,\mathcal{O}}) \neq 0$ for any simple $X_{\lambda,\mathcal{O}}$.

Now the same arguments as in [Ki] show that $\Phi$ is a tensor functor which is an isomorphism on morphisms. Thus, $\Phi$ is an equivalence $(\text{Rep} \, A)^G \simeq C'$ where $C'$ is a subcategory in $C$ generated by $\Phi(X_{\lambda,\mathcal{O}})$, each of which is a simple object in $C$. In other words, $C'$ is the essential image of $\Phi$.

To show that $C' = C$, let $L$ be a simple object in $C$. Then $L \subset G(X)$ for some $X \in \text{Rep} \, A$. Consider $\text{Ind} X = \bigoplus X^g$. Then $L \subset G(X) = \Phi(\text{Ind} X)$ (see [4.2]), which shows that $L \in C'$.

Replacing $\text{Rep} \, A$ by $\text{Rep}^0 A$ we get part (2) of the theorem. □

4.3. Remark. This theorem contains as a special case the main result of [DY], which in our notation reads as follows: $C_1$ is equivalent to $(\text{Rep}^0 A)^G$ as an abelian category. Note, however, [DY] does not discuss the tensor structure of $C_1$.

4.4. Theorem. Under the equivalence $(\text{Rep} \, A)^G \simeq C$ described in Theorem 4.1, the functors $F: C \to \text{Rep} \, A, G: \text{Rep} \, A \to C$ are given by

$F((X, \varphi_X)) = X,$

$G(X) = \text{Ind} X.$

Proof. For $G$, it easily follows from [4.2]. For $F$, note that by definition $F(X) = A \otimes X^G$. Define morphisms of $A$-modules $A \otimes X^G \to X, A \otimes X^G \to X$ by

$A \otimes X^G \hookrightarrow A \otimes X \xrightarrow{\varphi} X$

$X \xrightarrow{i_A \otimes \text{id}} A \otimes A \otimes X \xrightarrow{\text{id} \otimes \mu} A \otimes X \to A \otimes X^G$
where \( X^G \hookrightarrow X \), \( X \to X^G \) are the canonical projection and embedding and \( i_A : 1 \to A \otimes A \) is the rigidity morphism (see [KO, Definition 1.9]).

We leave it to the reader to check that these two morphisms are inverse to each other and thus, define isomorphism of \( A \)-modules \( X \cong A \otimes X^G \). □

4.5. Corollary. \( A \) is a “transparent”, or “central”, object in \( C_1 \): for any \( X \in C_1 \), \( \tilde{R}_X A \tilde{R}_A X = id_{A \otimes X} \).

Indeed, it suffices to note that \( \text{Rep} G \subset (\text{Rep}^0 A)^G \) is “central” in \( (\text{Rep}^0 A)^G \): for every \( X \in (\text{Rep}^0 A)^G, Y \in \text{Rep} G, \tilde{R}_{XY} \tilde{R}_{YX} = id \), which is obvious from the definitions. In fact, it can be shown (see [Mu]) that \( (\text{Rep}^0 A)^G \) is exactly the centralizer in \( \text{Rep} G \) of \( \text{Rep} G \), or, equivalently, \( C_1 \) is the centralizer in \( C \) of \( \text{Rep} G \).

4.6. Corollary. For every \( V \in C_1 \), \( F(V) \in \text{Rep}^0 A \).

4.7. Remark. The results of this section are almost identical to the results in [Mu] if we note that the subcategory \( \text{Rep} G \subset (\text{Rep}^0 A)^G \) is the same subcategory which is denoted by \( S \) in [Mu], and our \( \text{Rep} A \) is the same as \( C \times S \). The only difference is that [Mu] uses unitarity of \( C \) in a non-trivial way.

5. Twisted sector

From now on, let us assume that \( C \) is modular. Let us describe all of \( C \), not just the untwisted sector. Recall the definition of \( g \)-twisted \( A \)-module (see [Ki, Definition 4.1]).

5.1. Theorem. Every simple \( X \in \text{Rep} A \) is \( g \)-twisted for some \( g \in G \).

Proof. The proof is parallel to the proof of Theorem 4.3 in [Ki], with the following changes. Let \( H \) be the following formal linear combination of objects in \( \text{Rep}^0 A \):

\[
H = \bigoplus \frac{\dim_A X_s}{D^2} X_s,
\]

where \( s \) runs over the set \( S^0 \) of simple objects in \( \text{Rep}^0 A \), and \( D = \sqrt{\sum_s (\dim_A X_s)^2} \) (the normalization is chosen so that \( \dim_A H = 1 \)). Then we have the following lemma:

5.2. Lemma. If \( X_i \) is a simple object in \( \text{Rep} A \), then

\[
\frac{1}{\dim A} \quad X_i \quad \bigoplus \quad H \quad = \delta_{i0} \text{id}_{X_i}
\]

where \( i = 0 \) is the index of the unit object: \( X_0 = A \).

Proof of the lemma. First, by results of [KO, Lemma 5.4], the left hand side is zero if \( X_i \notin \text{Rep}^0 A \). If \( X_i \in \text{Rep}^0 A \), then the result follows from the identity \( (s^A)^2 = \delta_{i0} \) (cf., for example, [BK, Corollary 3.1.11]). □
This lemma immediately implies the following result (cf. [K1, Lemma 4.4])

5.3. Corollary.

\[
\frac{1}{(\dim A)^2} \begin{array}{c}
\text{X*} \\
\text{X} \\
\text{H}
\end{array} = \frac{1}{\dim X} \begin{array}{c}
\text{X*} \\
\text{X} \\
\text{X*}
\end{array}
\]

Now we can repeat the same steps as in the proof of Theorem 4.3 in [K1], replacing as needed $A$ by $H$ and using Corollary 4.5 instead of $\tilde{\mathcal{R}}^2_{AA} = \text{id}$. □

Let $\text{Rep}_g A$ be the full subcategory in $\text{Rep} A$ consisting of $g$-twisted $A$-modules; then $\text{Rep}_1 A$ is the same category which we had previously denoted $\text{Rep}^0 A$. It follows from Theorem 5.1 that

\begin{equation}
\text{Rep} A = \bigoplus_{g \in G} \text{Rep}_g A.
\end{equation}

As in the holomorphic case ([K1, Theorem 4.7]), this grading has some natural properties:

5.4. Theorem. (1) If $X \in \text{Rep}_g A$ then $X^h \in \text{Rep}_{h^{-1}gh} A$.

(2) If $X_1 \in \text{Rep}_g A, X_2 \in \text{Rep}_{g_2} A$ then $X_1 \otimes_A X_2 \in \text{Rep}_{g_1 g_2} A$.

(3) If $X \in \text{Rep}_g A$, then $X^* \in \text{Rep}_{g^{-1}} A$.

The proof of this theorem is completely parallel to the proof of corresponding parts of [K1, Theorem 4.7] and is omitted.

Note that unlike holomorphic case, it is not true in general that there is only one simple module in each of $\text{Rep}_g A$, and it is not true that every simple module is invertible.

We can now use decomposition (5.2) to construct a braiding on $(\text{Rep} A)^G$. For every $X \in \text{Rep} A$, let $\delta_g : X \to X$ be the projection on the $g$-twisted sector. i.e. $\delta_g = \text{id}$ for $X \in \text{Rep}_g A$ and $\delta_g = 0$ for $X \in \text{Rep}_h A, h \neq g$.

5.5. Theorem. For $X, Y \in (\text{Rep} A)^G$, define functorial morphism $\sigma_{X,Y} \in \text{Hom}_C(X \otimes Y, Y \otimes X)$ by the following composition

\begin{equation}
X \otimes Y \sum \delta_g \otimes \varphi_Y(g) X \otimes Y \to Y \otimes X
\end{equation}

(the second morphism is the usual commutativity isomorphism in $C$). Then $\sigma$ descends to a $(\text{Rep} A)^G$-morphism $X \otimes_A Y \to Y \otimes_A X$ and defines a structure of a braided tensor category on $(\text{Rep} A)^G$.

Proof. Let us start by showing that $\sigma$ descends to a $C$-morphism $X \otimes_A Y \to Y \otimes_A X$. To do so, recall (see [KO]) that $X \otimes_A Y$ is defined as $X \otimes Y / I$, where $I = \text{Im}(\mu_1 - \mu_2)$. Thus we need to show that $\sigma I \subset I$.

Without loss of generality, we can assume that $X \in \text{Rep}_g A$. In this case, $\sigma = \text{id} \otimes \varphi_Y(g)$ and we can rewrite the composition $\sigma \circ (\mu_1 - \mu_2)$ as shown in Figure [□], where the notation $f_1 \equiv f_2$ stands for $\text{Im}(f_1 - f_2) \subset I$. Thus, $\sigma(\mu_1 - \mu_2) \equiv 0$, or, equivalently, $\sigma(I) \subset I$. □
The remaining parts of the theorem (i.e., that $\sigma$ is a morphism of $A$-modules, commutes with the action of $G$ and that it satisfies the hexagon axioms) are easily shown by direct calculation which we omit.

\[
\phi(g) - \phi(g) = \phi(g) - \phi(g) \equiv 0
\]

Figure 1. Proof of Theorem 5.5

5.6. Remark. It should be noted that $\sum \delta_g \otimes g$ is the $R$-matrix for the Drinfeld double $D(G)$; thus, (5.3) combines the action of Drinfeld double on objects $X \in (\text{Rep } A)^G$ with the commutativity isomorphism in $\mathcal{C}$. For example, in the holomorphic case $\text{(Rep } A = \text{Vec})$, as described in [K1], $(\text{Rep } A)^G \simeq \mathcal{C}$ is just the category of representations of (twisted) Drinfeld double $D^\omega(G)$, and the commutativity isomorphism (5.3) is the usual commutativity isomorphism in $\text{Rep } D^\omega(G)$.

5.7. Remark. The decomposition (5.2) and Theorem 5.5 together are equivalent to the statement that $\text{Rep } A$ is a “$G$-crossed braided category” as defined in [Tu], [Mu].

5.8. Proposition. The functor $\Phi: (\text{Rep } A)^G \rightarrow \mathcal{C}$ identifies commutativity isomorphism $\sigma$ in $(\text{Rep } A)^G$ defined by (5.3) with the commutativity isomorphism in $\mathcal{C}$.

Proof. Trivial: on $Y^G$, $\varphi_Y(g) = \text{id}$, so on $X^G \otimes_A Y^G$,
\[
\sum \delta_g \otimes \varphi_Y(g) = (\sum \delta_g) \otimes \text{id} = \text{id}.
\]

5.9. Corollary. $(\text{Rep } A)^G$ is equivalent to $\mathcal{C}$ as a braided tensor category.
This result shows that the category $C$ can be recovered from the category $\text{Rep} A$ and the action of $G$ on $\text{Rep} A$.

It would be desirable to describe $\text{Rep} A$ in terms of $\text{Rep}^0 A$ with some extra structure. However, at the moment we do not have such a description. We do have some partial result, though.

5.10. **Theorem.** If $C$ is modular, then for all $g \in G$, $\text{Rep}_g A \neq 0$.

**Proof.** Let $s$ be the $s$-matrix for $C \simeq (\text{Rep} A)^G$. Let $X_i$ be a simple object in $(\text{Rep} A)^G$ and $V_\lambda$ — a simple object in $\text{Rep} G \subset (\text{Rep} A)^G$. Then it follows from the definition of $s$ and explicit formula (5.3) for the commutativity isomorphism in $(\text{Rep} A)^G$ that

$$s_{V_\lambda, X_i} = \frac{1}{D_C} \sum_g (\dim_A(X_i)_g) \text{tr}_{V_\lambda} g$$

where $X = \bigoplus_g (X_i)_g$ is the decomposition of $X_i$ as an $A$-module defined by (5.2).

Now let $H = \{ g \in G \mid \text{Rep}_g A \neq 0 \}$. It immediately follows from Theorem 5.4 that $H$ is a normal subgroup in $G$. Assume that $H \neq G$. Then there exists a non-zero formal linear combination $V = \bigoplus c_\lambda V_\lambda$ of irreducible representations of $G$ such that $\text{tr}_V g = 0$ for all $g \in H$. Using (5.4), we get

$$s_{V, X_i} = \sum c_\lambda s_{V_\lambda, X_i} = 0$$

for all $X_i$, and thus the matrix $s$ is singular, which contradicts the definition of a modular category. \qed

6. **Conclusion**

Returning to the original question of recovering the diagram (1.1) from its lower right corner, i.e. $\text{Rep}^0 A$, we see that so far we have succeeded in reconstructing along the horizontal lines: we can reconstruct $C_1$ from $\text{Rep}^0 A$, and $C$ from $\text{Rep} A$ (and the action of $G$ by automorphisms on $\text{Rep} A$).

However, we have failed to do the reconstruction along the vertical lines: we cannot recover $C$ from $C_1$, or $\text{Rep} A$ from $\text{Rep}^0 A$. The following example gives some insight in the difficulty of the problem:

6.1. **Example.** Let $C$ be the semisimple subquotient of the category of representations of $U_q(\mathfrak{sl}_2)$ at root of unity, $q = e^{\pi i/4}$, as in [KO]. Assume that $k = l - 2$ is divisible by 4: $k = 4m$. Then, as discussed in [KO], there is a unique structure of a commutative associative algebra on $A = 1 + \delta, \delta = V_k$ (this corresponds to $D_{2m}$ in the ADE classification of “subgroups” in $U_q(\mathfrak{sl}_2)$). Define an action of the group $\mathbb{Z}_2$ on $A$ by $\sigma|_1 = 1, \sigma|_\delta = -1$, where $\sigma$ is the non-trivial element of $\mathbb{Z}_2$. Then one easily sees that $\sigma$ is an automorphism of $A$ as a $C$-algebra, and $A^\sigma = 1$, so this is an example of an $\mathbb{Z}_2$-orbifold. In this case, it follows from the results of [KO] that $C_1$ is the subcategory in $C$ consisting of the modules with even highest weight (in physical terminology, integer spin). So in this case, recovering $C$ from $C_1$ is essentially equivalent to recovering the category of $U_q(\mathfrak{sl}_2)$-modules from the subcategory of integer-spin modules. The classical analog of this problem would be recovering the category of representations of $\text{SL}(2, \mathbb{C})$ from the category of representations of $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2 = \text{PSL}(2, \mathbb{C})$. Even in the classical case, we do not know any such construction.
This example shows that a model question for recovering $\mathcal{C}$ from $\mathcal{C}_1$ is reconstructing the category of representations of a group $M$ knowing the normal subgroup $G \subset M$ (in the example above, $G = \mathbb{Z}_2$) and the category $\text{Rep} \, N$, $N = M/G$. This question is in a sense dual to the model example discussed in Section 2, where the roles of $G$ and $N$ were reversed. It turns out that the dual question is much harder. Of course, in principle this model question is solvable: any finite group can be reconstructed from its category of representations, so we recover $N$ from $\text{Rep} \, N$, then use the results of Section 2. Unfortunately, this recipe is very indirect; even more importantly, it completely fails in the cases when $\mathcal{C}_1$ is not a Tannakian category, so it is not a category of representations of a group.

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