Lower bounds on the quantum Fisher information based on the variance and various types of entropies

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We examine important properties of the difference between the variance and the quantum Fisher information over four, i.e., $(\Delta A)^2 - F_Q[\rho, A]/4$. We find that it is equal to a generalized variance defined in Petz [J. Phys. A 35, 929 (2002)] and Gibilisco, Hiai, and Petz [IEEE Trans. Inf. Theory 55, 439 (2009)]. We present an upper bound on this quantity that is proportional to the linear entropy. As expected, our relation shows that for states that are close to being pure, the quantum Fisher information over four is close to the variance. We also obtain the variance and the quantum Fisher information averaged over all Hermitian operators, and examine its relation to the von Neumann entropy. Apart from the usual quantum Fisher information, we also consider the Kubo-Mori-Bogoliubov quantum Fisher information.

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I. INTRODUCTION

Quantum metrology is a subfield of metrology that takes advantage of quantum phenomena to achieve a high precision in magnetometry, frequency measurements, and several other areas of interferometry [1–9]. There have been successful experiments with cold gases, trapped ions and photons to create quantum states useful for high precision metrology such as spin-squeezed states [10–18], Greenberger-Horne-Zeilinger (GHZ) states [19–27], symmetric Dicke states [28–33], and many-body singlet states [34]. Quantum metrology played a role even in the recent experiments with the squeezed-light-enhanced gravitational wave detector GEO 600 [35]. Experiments are being carried out achieving a larger and larger precision, reaching recently a 10-times improvement compared to the shot-noise limit, i.e., the best precision achievable by uncorrelated particle ensembles [36].

Partly due to the experimental successes, there has been a rapid theoretical development in quantum metrology. In particular, there has been a large effort to understand better the quantum Fisher information, which is a central notion in quantum metrology. It is connected to the task of estimating the phase $\theta$ for the unitary dynamics of a linear interferometer

$$U = \exp(-iA\theta),$$

assuming that we start from $\rho$ as the initial state, where $A$ is a Hermitian operator. A tight bound on the precision of the phase estimation is given by the Cramér-Rao bound as

$$(\Delta \theta)^2 \geq 1/F_Q[\rho, A],$$

where $F_Q[\rho, A]$ is the quantum Fisher information of the state [1–4, 7–9, 37, 38].

It has been found that the quantum Fisher information is strongly connected to quantum entanglement [39–42], which has been used to estimate multipartite entanglement by direct measurement of the sensitivity [32, 43, 44]. It has been investigated how various sets of quantum states, such as random bosonic states and states with a positive partial transpose perform metrologically [45, 46]. New approaches have been found to obtain the quantum Fisher information in systems in thermal equilibrium by measuring certain observables [47, 48]. Using the theory of quantum Fisher information, it has been examined how the precision scales with the size of a noisy quantum system [49, 50], which will help to identify cases when very high precision can be achieved with large systems [51]. Connected to these questions, new uncertainty relations have been derived with the quantum Fisher information, which improve the Heisenberg uncertainty [52]. Moreover, new relations between the quantum Fisher information and the entropy have been presented [53, 54].

Recently, a surprising property of the quantum Fisher information has been discovered: it is, up to a constant factor, the convex roof of the variance [55, 56]. This result connects the theory of the quantum Fisher information to entanglement measures, which are also defined by convex roofs [5]. The findings of Ref. [55, 56] were used to sharpen statements concerning the continuity of the quantum Fisher information [57]. As another consequence, the quantum Fisher information can efficiently be bounded from below based on few measurements [58]. Finally, the definition of the quantum Fisher information as a convex roof could be used to study the role of entanglement and quantum correlations in interferometry [59].

In this paper, we present a new approach to bound the quantum Fisher information with the variance and various entropies. In order to list our main results in...
we have to substitute \( \sigma_{\max}(\hat{J}^2) = N^2/4 \) into Eq. (7).

So far we considered bounds on \( V(\varrho, A) \) for a particular \( A \). We now arrived at computing the average of the quantity (3) for traceless Hermitian operators.

**Observation 3.** The average of \( V \) over traceless Hermitian matrices is given as

\[
V(\varrho) = \frac{2}{d^2-1} \left[ S_{\text{lin}}(\varrho) + H(\varrho) - 1 \right],
\]

where the averaging is over matrices \( A \) with a fixed norm \( \text{Tr}(A^2) = 2 \), \( d \) is the dimension of the system, and

\[
H(\varrho) = 2 \sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} = 1 + 2 \sum_{k \neq l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l}
\]

is the sum of the pairwise harmonic means of the eigenvalues of \( \varrho \). The quantity \( H(\varrho) \), in a certain sense, measures the purity of the quantum state, since for pure states \( H(\varrho) = 1 \), while for mixed states \( H(\varrho) > 1 \).

Our paper is organized as follows. In Sec. II, we present the basics of quantum metrology relevant to our paper, and we show that \( V(\varrho, A) \) is a generalized variance defined in Refs. [60, 61]. In Sec. III, we obtain upper bounds on the difference between the variance and the quantum Fisher information over four given in Eq. (3). In Sec. IV, we show some concrete examples for the application of our inequalities. In Sec. V, we calculate averages of the quantum Fisher information and \( V(\varrho, A) \) over all Hermitian operators, and relate them to the von Neumann entropy. Finally, we calculate similar averages for the Kubo-Mori-Bogoliubov quantum Fisher information.

**II. BASICS OF QUANTUM METROLOGY**

**A. The quantum Fisher information and the variance**

In this section, we review important properties of the quantum Fisher information and the variance. We will also stress the relations that connect these two quantities, which will motivate us to study \( V(\varrho, A) \) in the rest of the paper.
The quantum Fisher information can be computed as follows. Let us assume that a density matrix is given in its eigenbasis as
\[ \varrho = \sum_{k=1}^{d} \lambda_k |k\rangle \langle k|, \]  
where \(d\) is the dimension of the quantum system. Then, the quantum Fisher information is obtained as \([7, 8, 37, 38]\)
\[ F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |A_{kl}|^2, \]  
where \(A_{kl}\) is defined as \(A_{kl} = \langle k|A||l\rangle\). Equation (12) can be rewritten as \([5]\)
\[ F_Q[\varrho, A] = 4 \sum_{k,l} \lambda_k |A_{kl}|^2 - 8 \sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |A_{kl}|^2 \]
\[ = 4\langle A^2 \rangle - 8 \sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |A_{kl}|^2. \]  
The advantage of Eq. (13) is that the \(\langle A^2 \rangle\) appears in the formula, which makes it easy to compare the quantum Fisher information to the variance given as
\[ (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \]
\[ = \sum_{k,l} \lambda_k |A_{kl}|^2 - \left( \sum_k \lambda_k A_{kk} \right)^2. \]  
For any decomposition \(\{p_k, |\Psi_k\rangle\}\) of the density matrix \(\varrho\) we have \([55, 56]\)
\[ \frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2_{\Psi_k} \leq (\Delta A)^2_\varrho, \]  
where the upper and the lower bounds are both tight in the sense that there are decompositions that saturate the first inequality, and there are others that saturate the second one. Note that the latter statement could be generalized to covariance matrices \([62, 63]\).

These statements can also be expressed saying that the quantum Fisher information over four is the convex roof of the variance
\[ \frac{1}{4} F_Q[\varrho, A] = \inf_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\Psi_k}, \]  
while the variance is the concave roof of the itself
\[ (\Delta A)^2_\varrho = \sup_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\Psi_k}, \]  
where the infimum and the supremum are over all possible convex decompositions of \(\varrho\) of the type
\[ \varrho = \sum_k p_k |\Psi_k\rangle \langle \Psi_k|. \]  
where \(p_k\) are probabilities and \(|\Psi_k\rangle\) are pure states. Finally, we note that we can also interpret the relation of the quantum Fisher information and the variance as follows. We write the variance as \([55]\)
\[ (\Delta A)^2 = \sum_k p_k (\Delta A)^2_k + \sum_k p_k (\langle A \rangle - \langle A \rangle_k)^2. \]  
Equation (19) is valid for all decompositions of \(\varrho\) of the type (18). The first term on the right-hand side of Eq. (19) we can call "quantum" part, since it comes from the variance of the operator on pure quantum states \([55]\). The second term we can call the "classical" part, since it is just a classical variance formula for the subensemble expectation values \([64]\). In this picture, we can interpret the quantum Fisher information as the minimal "quantum" part of the variance, while \(V(\varrho, A)\) given in Eq. (3) is the maximum of the "classical" part. Hence, we can define \(V(\varrho, A)\) as a concave roof as
\[ V(\varrho, A) = \sup_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\langle A \rangle_{\Psi_k} - \langle A \rangle)^2, \]  
which can also be rewritten as
\[ V(\varrho, A) = \sup_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\langle A \rangle_{\Psi_k}^2 - \langle A \rangle^2). \]  

B. The difference between the variance and the quantum Fisher information over four

In this section, we discuss some important properties of \(V(\varrho, A)\). We also discuss that \(V(\varrho, A)\) equals a generalized variance given in Refs. \([60, 61]\).

The quantity (3) has the following important properties.

(i) \(V(\varrho, A) = 0\) for all pure states, and for all states for which \((\Delta A)^2 = 0\).

(ii) \(V(\varrho, A) = 0\) for all \(A\) if and only if \(\varrho\) is pure.

(iii) Since the variance is concave in the state and the quantum Fisher information is convex, it is also concave in the state

(iv) The quantity \(V(\varrho, A)\) is clearly independent from \(\text{Tr}(A)\), that is, \(V(\varrho, A) = V(\varrho, A + c\mathbb{1})\) for any \(c\).

(v) The relation \(V(U\varrho U^\dagger, A) = V(\varrho, U^\dagger AU)\) holds for any unitary \(U\).

(vi) For a bipartite system, for product states \(V(\varrho_A \otimes \varrho_B, \mathbb{1} \otimes A + B \otimes \mathbb{1}) = V(\varrho_A, A) + V(\varrho_B, B)\) holds. This can be proven noting that a similar relation holds for the variance and the quantum Fisher information.
where the lower and upper bounds, respectively, are given for simplicity. The corresponding mean is the arithmetic mean $f_{\text{mean}}$ of $f(x)$ is obtained as
\[
V(\varrho, A) = 2 \sum_{k \neq l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |A_{kl}|^2 - \left( \sum \lambda_i A_{ii} \right)^2. \tag{22}
\]

It is instructive to regroup the terms such that the double sum is only over indices that are not equal with each other. Hence, an alternative form of Eq. (22) becomes
\[
V(\varrho, A) = 2 \sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |A_{kl}|^2 + \sum \lambda_k A_{kk}^2 - \left( \sum \lambda_k A_{kk} \right)^2. \tag{23}
\]

One is tempted to think that the last two terms in Eq. (23) is the variance $\Delta A)^2$. Indeed, they are equal to the variance if $A$ is diagonal in the eigenbasis of $\varrho$. Otherwise, one can realize that the equality does not hold in the general case by comparing these terms to Eq. (14). It is instructive to connect $V(\varrho, A)$ to the family of generalized variances defined in Refs. [60, 61] as
\[
\text{var}_f(\varrho, A) = \sum_{ij} f(\lambda_i, \lambda_j) |A_{ij}|^2 - \left( \sum \lambda_i A_{ii} \right)^2, \tag{24}
\]
where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a matrix monotone function, and
\[
m_f(a, b) = bf(a/b) \tag{25}
\]
is a corresponding mean. With Eq. (24) we can define a large set of generalized variances. The $f(x)$ are boudned as
\[
f_{\text{min}}(x) \leq f(x) \leq f_{\text{max}}(x), \tag{26}
\]
where the lower and upper bounds, respectively, are given as
\[
f_{\text{min}}(x) = \frac{2x}{1+x}, \quad f_{\text{max}}(x) = \frac{1+x}{2}. \tag{27}
\]
The generalized variance (24) with $f(x) = f_{\text{max}}(x)$ is the usual variance
\[
\text{var}_f(\varrho, A) = \langle A^2 \rangle - \langle A \rangle^2, \tag{28}
\]
where in the subscript there is "max" rather than $f_{\text{max}}$ for simplicity. The corresponding mean is the arithmetic mean $m_{\text{max}}(a, b) = (a + b)/2$. Let us now consider the generalized variance (24) with $f_{\text{min}}(x)$. The corresponding mean is the harmonic mean $m_{\text{min}}(a, b) = 2ab/(a+b)$. Straightforward calculations show that
\[
\text{var}_f(\varrho, A) \equiv V(\varrho, A) \tag{29}
\]
holds [65]. Clearly, $\text{var}_f(\varrho, A)$ is the smallest of the generalized variances, while the usual variance, $\text{var}_f(\varrho, A)$, is the largest [60, 61]. For any generalized variance given in Eq. (24) we have
\[
(\Delta A)^2 - F_Q[\varrho, A] \leq \text{var}_f(\varrho, A) \leq (\Delta A)^2. \tag{30}
\]
Hence, if $F_Q[\varrho, A] = 0$ then all the generalized variances give the same value.

III. UPPER BOUND ON $V(\varrho)$ WITH THE PURITY

In this section, we prove Observations 1-2.

Proof of Observation 1. For the rank-2 case, $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda$, and all other eigenvalues are zero. Then, Eq. (22) becomes
\[
V(\varrho, A) = \lambda(1 - \lambda) \left[ (A_{11} - A_{22})^2 + 4|A_{12}|^2 \right]. \tag{31}
\]
Now, first we have to use the following relation
\[
\lambda(1 - \lambda) = \frac{1}{2} \left[ 1 - \text{Tr}(\varrho^2) \right], \tag{32}
\]
which can easily be proved with direct calculation. Second, we have to show that
\[
(A_{11} - A_{22})^2 + 4|A_{12}|^2 = (\omega_1 - \omega_2)^2, \tag{33}
\]
where $\omega_{1,2}$ are the eigenvalues of the matrix
\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{pmatrix}. \tag{34}
\]
This can be seen using the usual formula for the eigenvalues of a $2 \times 2$ Hermitian matrix given as
\[
\omega_{1,2} = \frac{(A_{11} + A_{22}) \pm \sqrt{D}}{2}, \tag{35}
\]
where
\[
D = (A_{11} + A_{22})^2 - 4(A_{11}A_{22} - |A_{12}|^2). \tag{36}
\]
Equations (35) and (36) yield Eq. (33). With these, we have proved the equality, (4), giving $V(\varrho, A)$ as a function of the purity and the eigenvalues.

So far, we found an upper bound on $V(\varrho, A)$ for states with rank at most two. Next, we will look for a bound for states with an arbitrary rank, in order to prove Observation 2.

Proof of Observation 2. Based on the definition of $V(\varrho, A)$ as a convex roof given in Eq. (20), we see that the relation (7) is true, if and only if
\[
\frac{1}{2} \text{S}_{\text{lin}} \left( \sum_k p_k |\psi_k\rangle \langle \psi_k| \right) [\sigma_{\text{max}}(A) - \sigma_{\text{min}}(A)]^2 \\
- \sum_k p_k (\langle A \psi_k \rangle - \langle A \rangle)^2 \tag{37}
\]
is non-negative for all possible choices for $p_k$ and $|\psi_k\rangle$. In order to verify that Eq. (37) cannot be negative, we need to minimize it over $p_k$ and $|\psi_k\rangle$. Let us consider now only a minimization over $\mathcal{M} = \{p_1, p_2, p_3, \ldots\}$ under the constraints $p_k \geq 0$, $\sum_k p_k = 1$. We consider a further constraint for the expectation value, $\langle A \rangle = \sum_k p_k \langle A \psi_k \rangle = A_0$, where $A_0$ is a constant. While we minimize over $p_k$, we keep the $|\psi_k\rangle$ fixed.

Next, we will determine the characteristics of the $\mathcal{M}$’s that minimize Eq. (37). The first term in Eq. (37) is
Eq. (7) only for the sider $H$ is maximal for the completely mixed state where we assumed that $\rho$’s are non-zero. (If we did not have the $\sum_k P_k \langle A \rangle_{\psi_k} = A_0$ constraint, then the extreme points would correspond to $\rho$’s with at most a single nonzero $P_k$.) Thus, we need to prove that Eq. (37) is non-negative for such cases. This can straightforwardly be done based on Eq. (4), which gives $V(\rho, A)$ for rank-2 states with the purity.

We have just proved that for any choice of $|\psi_k\rangle$, minimizing over $\rho$ will lead to a nonnegative value for Eq. (37). From this, the statement of the observation follows. ■

IV. EXAMPLES

Next, we examine how our lower bounds on the quantum Fisher information behave in some relevant situations.

A. Pure states

As a warm-up exercise, let us consider pure states. For a pure state $|\Psi\rangle$ the relations

$$
V(|\Psi\rangle, A) = 0,
$$

$$
S_{\text{lin}}(|\Psi\rangle) = 0,
$$

$$
H(|\Psi\rangle) = 1
$$

hold for any $A$. Clearly, pure states saturate Eq. (7).

B. Completely mixed state

The completely mixed state is defined as

$$
\rho_{\text{cm}} = \frac{1}{d},
$$

where $d$ is the dimension of the system. The state $\rho_{\text{cm}}$ is not useful for metrology since $F_Q[\rho, A] = 0$ for all $A$. In fact, $\rho_{\text{cm}}$ is the only quantum state that has this property. Hence, $V(\rho_{\text{cm}}, A)$ equals the variance of the state

$$
V(\rho_{\text{cm}}, A) = (\Delta A)^2_{\rho_{\text{cm}}} = \frac{1}{d} \text{Tr}(A^2),
$$

where we assumed that $A$ is traceless. The linear entropy is maximal for the completely mixed state

$$
S_{\text{lin}}(\rho_{\text{cm}}^2) = 1 - \frac{1}{d},
$$

Finally, $H$ defined in Eq. (10) is also maximal

$$
H(\rho_{\text{cm}}) = d.
$$

C. GHZ states

In this section, we consider states that live in the two-dimensional subspace

$$
\{ |000..00\rangle, |111..11\rangle \}. \tag{43}
$$

Such states are very relevant for experiments with trapped ions aiming to create GHZ states [25–27] defined as

$$
|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000..00\rangle + |111..11\rangle). \tag{44}
$$

For states of the type

$$
\rho_p = p \frac{P_{000..00} + P_{111..11}}{2} + (1 - p)|\text{GHZ}\rangle\langle\text{GHZ}| \tag{45}
$$

a relation giving the quantum Fisher information as a function of the density matrix

$$
F_Q[\rho, J_1] = 2 N^2 \left[ \text{Tr}(\rho^3) - 1 \right] \tag{46}
$$

holds [66]. Equation (46) has also been found in the context of relating the visibility to the metrological performance in ion-trap experiments in Ref. [67].

Let us apply now our theory to obtain a bound for any state living in the space (43). Such states satisfy Eq. (7) with $A = J_z$. For states of the two-dimensional subspace given in (43), the variance of $J_z$ is given as

$$
(\Delta J_z)^2 = \left( 1 - \langle P_{000..00} \rangle^2 - \langle P_{111..11} \rangle^2 \right) \frac{N^2}{2}. \tag{47}
$$

Using that Eq. (7) is saturated and from Eq. (47) we obtain

$$
\frac{F_Q[\rho, J_z]}{N^2} = 2 \left[ \text{Tr}(\rho^3) - \langle P_{000..00} \rangle^2 - \langle P_{111..11} \rangle^2 \right]. \tag{48}
$$

For noisy GHZ states of the form (45), the relation (48) reduces to Eq. (46).

We mention that another lower bound on the quantum Fisher information with the fidelity $F_{\text{GHZ}}$ is given by [58]

$$
\frac{F_Q[\rho, J_z]}{N^2} \geq \begin{cases} 
(1 - 2 F_{\text{GHZ}})^2, & \text{if } F_{\text{GHZ}} > 1/2, \\
0, & \text{if } F_{\text{GHZ}} \leq 1/2.
\end{cases} \tag{49}
$$

The bound (49) is valid for any quantum state, even for the ones that do not live in the two-dimensional subspace. The states within the two-dimensional subspace do not all saturate Eq. (49).

D. Ensemble of spin-$\frac{1}{2}$ particles

In this section, we apply our bound Eq. (7) to an ensemble of spin-$\frac{1}{2}$ particles.
It has been shown that for separable states in a linear interferometer, the quantum Fisher information is bounded as [39]

\[ F_Q[\varrho,J] \leq N. \]  
(50)

Any state that violates Eq. (50) is entangled. For general states the bound is

\[ F_Q[\varrho,J] \leq N^2, \]  
(51)

which is called the Heisenberg-limit.

It is an important question in metrology, what the conditions are for the Heisenberg scaling given by

\[ F_Q[\varrho,J] = O(N^2), \]  
(52)

where \( O \) is the usual Landau symbol. Rewriting Eq. (7) for \( A = J_z \), we arrive at

\[ (\Delta J_z)^2 - \frac{1}{2} F_Q[\varrho,J] \leq \left[ 1 - \text{Tr}(\varrho^2) \right] \frac{N^2}{2}. \]  
(53)

Let us now consider a family of states \( \varrho_N \) such that \( S_{\text{lin}}(\varrho_N^2) \leq s \), where \( s \) is some constant. Then, if

\[ (\Delta J_z)^2_{\varrho_N} \geq s \frac{N^2}{2}, \]  
(54)

then we have Heisenberg scaling. That is, it is sufficient that the variance scales as \( O(N^2) \) and the state is sufficiently pure.

**E. Systems in thermal equilibrium**

If our bounds are used in systems in thermal equilibrium then the purity \( \text{Tr}(\varrho^2) \) can straightforwardly be obtained from the temperature, using that the eigenvalues of the density matrix are given as

\[ \lambda_l \propto e^{-\frac{E_l}{kT}}, \]  
(55)

where \( E_k \) are the energy levels of the system, \( T \) is the temperature and \( k_B \) is the Boltzmann constant. Hence, using Observation 2, we can bound the quantum Fisher information from below if we know the variance and the temperature of the system. The method gives a useful bound if \( kT \lesssim E_1 - E_0 \).

**V. AVERAGING OVER OPERATORS**

In this section, we determine the averages over all operators for the variance, the quantum Fisher information and \( V(\varrho,A) \). This sheds new light on the relation between these quantities and entropies.

In Sec. II B it has been discussed that \( V(\varrho,A) = 0 \) for all \( A \) if and only if the state \( \varrho \) is pure. If we then average \( V(\varrho,A) \) over all observables \( A \), we obtain a quantity that is zero only for pure states. This quantity is concave in \( \varrho \), since \( V(\varrho,A) \) is also concave. Hence, it seems to be interesting to ask, how it is related to entropies.

**A. Averaging over the Hermitian matrices**

Next, we will discuss how to interpret the averaging over all traceless Hermitian matrices with a given norm. All such matrices can be obtained as a linear combination of the SU(\( d \)) generators as

\[ A_\vec{n} := \vec{A}^T \vec{n}, \]  
(56)

where \( \vec{A} = [A^{(1)}, A^{(2)}, A^{(3)}, \ldots]^T \) and \( \vec{n} \) is a unitvector with real elements, and \((\cdot)^T \) denotes matrix transpose. We consider the following normalization

\[ \text{Tr}(A^{(k)}A^{(l)}) = 2 \delta_{kl}. \]  
(57)

According to well-known results of linear algebra, the number of SU(\( d \)) generators is

\[ N_g = d^2 - 1. \]  
(58)

We now define the average over unit vectors as

\[ \mathcal{T} = \int f(\vec{n}) M(d\vec{n}). \]  
(59)

where \( M \) is a measure over unitvectors with the usual invariance properties such that \( f M(d\vec{n}) = 1 \), and \( f(\vec{n}) \) is some function depending on the unit vector \( \vec{n} \). Hence, we can average an expression over traceless Hermitian matrices with a given norm as

\[ \mathcal{T} = \int f(A_\vec{n}) M(d\vec{n}). \]  
(60)

Let us now calculate the average of the variance over the Hermitian matrices. For any operator \( A_\vec{n} \) one can obtain the variance as

\[ (\Delta A_{\vec{n}})^2 = \vec{n}^T C \vec{n}, \]  
(61)

where \( C \) is the covariance matrix defined as

\[ C_{mn} = \frac{1}{2} \left( \langle A^{(m)} A^{(n)} \rangle + \langle A^{(n)} A^{(m)} \rangle \right) - \langle A^{(m)} \rangle \langle A^{(n)} \rangle. \]  
(62)

Then, the average variance can be written as a sum of the variances of the generators \( A^{(k)} \), since

\[ \text{var}(\varrho) = \int M(d\vec{n}) \vec{n}^T C \vec{n} = \int M(d\vec{n}) \text{Tr}(C \vec{n} \vec{n}^T) \]

\[ = \frac{1}{N_g} \text{Tr}(C) = \frac{1}{N_g} \sum_{m=1}^{N_g} (\Delta A^{(m)})^2, \]  
(63)

where we used that

\[ \int M(d\vec{n}) \vec{n} \vec{n}^T = \frac{1}{N_g}. \]  
(64)

Based on Eq. (63), \( \text{var}(\varrho) \) is independent from the concrete choice of the \( A^{(k)} \).
Following the ideas above also for the quantum Fisher information, we present now relations for the averages of various quantities.

**Observation 4.** For $d \times d$ systems, the averages of the variance and the quantum Fisher information, respectively, are

$$\overline{\text{var}}(g) = \frac{2}{N_g} [S_{\text{lin}}(g) + d - 1], \quad (65a)$$

$$\overline{F}_Q[g] = \frac{8}{N_g} [d - H(\rho)]. \quad (65b)$$

The averages of the off-diagonal and diagonal elements of $A$, respectively, used later in calculations are

$$|A_{k\ell}|^2 = \frac{2}{N_g}, \quad (66a)$$

$$|A_{kl}|^2 = \frac{2}{N_g} \left( 1 - \frac{1}{d} \right), \quad (66b)$$

where $k \neq l$. The proof is given in Appendix A.

Simple numerical optimization shows a remarkable relation between $H(\rho)$ and the von Neumann entropy

$$S = -\text{Tr}(\rho \log \rho) = \sum_{k=1}^{d} \lambda_k \ln \lambda_k, \quad (67)$$

where $\ln(x)$ is the natural logarithm. In Fig. 2, we indicated the part of the $(H, \exp(S))$-space allowed for physical states. The exponential of the entropy defined as

$$\exp(S) = \prod_{k=1}^{d} \lambda_k^{-\lambda_k} \quad (68)$$

has attracted considerable attention [68, 69]. Fig. 2 also supports the that $H(\rho)$ is, essentially, a measure of purity that is related to the von Neumann entropy. We find the approximate relation

$$H(\rho) \sim \exp[S(\rho)]. \quad (69)$$

Equation (69) can be proved for states close to the completely mixed state based on an expansion of both sides around the point given by $\lambda_k^{(0)} = 1/d$ for $k = 1, 2, \ldots, d$. After we set $\lambda_d = 1 - \sum_{k=1}^{d-1} \lambda_k$, such an expansion involves $\lambda_k$ for $k = 1, 2, \ldots, d-1$. The left-hand side and the right-hand side of Eq. (69) are equal up to second order in the quantities ($\lambda_k - \lambda_k^{(0)}$).

Based on Eq. (65b), for the quantum Fisher information we obtain

$$\overline{F}_Q[g] \sim \frac{8}{N_g} \left\{ d - \exp[S(\rho)] \right\}. \quad (70)$$

Note that the relation Eq. (69) can be used to approximate the von Neumann entropy with a quantity that is easier to compute. Note also that our findings are very relevant to recent efforts to obtain inequalities between the quantum Fisher information and the von Neumann entropy [53, 54]. We will discuss this in Sec. V C in more detail.

**B. Averaging $V(\rho, A)$ over traceless Hermitian operators**

**Proof of Observation 3.** Finally, we arrived at computing the average of the quantity (3) for Hermitian operators. We have to use Observation 3 and that $\overline{V}(\rho) = \overline{\text{var}}(g) - \overline{F}_Q[g]/4$. The result is given in Eq. (9).

In Fig. 3, Eq. (9) and $\exp(S)$ are shown for random states of dimension $d = 3$ and $d = 10$. Now the correlation with $\exp(S)$ seems to be even more pronounced than in the case of the average quantum Fisher information. We find the approximate relation

$$\overline{V}(\rho) \sim \frac{2}{N_g} \left( 1 - \frac{1}{d^2} \right) \exp[S(\rho)]. \quad (71)$$

**C. The Kubo-Mori-Bogoliubov quantum Fisher information**

We now consider another form of the quantum Fisher information used frequently in mathematics [9], and calculate its average over traceless Hermitian operators. The Kubo-Mori-Bogoliubov quantum Fisher information $F_Q^\text{log}[\rho, A]$ can be expressed as

$$F_Q^\text{log}[\rho, A] = \sum_{k,l} |\lambda_k - \lambda_l| A_{kl}^2, \quad (72)$$
which is related to the relative entropy
\[ S(\rho|\sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma) \]  
(73)
via the following expression
\[ \frac{d^2}{d^2 \theta} S(\rho|e^{-i\theta} A e^{i\theta} |)_{\theta=0} = F^\log_Q[\rho, A]. \]  
(74)
[For a derivation of Eqs. (72) and (74), see Appendix B.]

Using Eq. (66a), the average of the quantum Fisher information over the SU(d) generators is given as
\[ F^\log_Q[\rho, A] = \frac{2}{N^g} \sum_{k,l} (\log(\lambda_k) - \log(\lambda_l)) (\lambda_k - \lambda_l). \]  
(75)
This can be rearranged as sum of a term containing the von Neumann entropy and another term with the logarithms of the eigenvalues as
\[ F^\log_Q[\rho] = -\frac{2}{N^g} \left( 2dS + \sum_k \log \lambda_k \right), \]  
(76)
where the von Neumann entropy \( S \) is given in Eq. (67).

Next, we ask what the minimal value of \( F^\log_Q[\rho] \) is for a given value of \( S \). This can be determined by minimizing Eq. (76) with some constraints. The constraints of \( \sum_{k=1}^d \lambda_k = 1 \) can be taken into account by minimizing
\[ f(\vec{\lambda}) = \frac{2}{N^g} \left( 2d \sum_{k=1}^d \lambda_k \log \lambda_k - 2 \sum_{k=1}^d \log \lambda_k \right), \]  
(77)
where \( \vec{\lambda} = \{\lambda_k\}_{k=1}^{d-1} \) and we set \( \lambda_d := 1 - \sum_{k=1}^{d-1} \lambda_k \). Considering the other constraint for the entropy, we arrive at
\[ g(\vec{\lambda}, \mu_1) = \frac{4}{N^g} \left[ -dS_0 - \sum_{k=1}^d \log \lambda_k \right. \\
- \mu \left. \left( -\sum_{k=1}^d \lambda_k \log \lambda_k - S_0 \right) \right], \]  
(78)
where \( \mu \) is a Lagrange multiplier and \( S_0 \) is a constant. The allowed region for \( \vec{\lambda} \) is determined by the conditions \( \lambda_k \geq 0 \) for \( k = 1, 2, ..., d - 1 \) and \( \sum_{k=1}^{d-1} \lambda_k \leq 1 \).

We are looking for the \( \vec{\lambda} \) that minimizes \( g \) for some \( \mu \). In principle, the minimum could be taken on the boundary of the allowed region for \( \vec{\lambda} \). However, \( \lambda_k \rightarrow +0 \) leads to \( g \rightarrow \infty \), hence a minimum cannot be obtained this way. The other possibility is that the minimum is taken at the \( \vec{\lambda} \) that fulfills
\[ \frac{\partial g}{\partial \lambda_k} = 0 \quad \text{for} \quad k = 1, 2, ..., d - 1, \]  
(79a)
\[ \frac{\partial g}{\partial \mu} = 0. \]  
(79b)
From the condition that the derivatives with respect to \( \lambda_k \) are zero, \( (79a) \), follows
\[ \frac{1}{\lambda_k} - \frac{1}{\lambda_d} - \mu (\log \lambda_k - \log \lambda_d) = 0 \]  
(80)
for \( k = 1, 2, ..., d - 1 \). For a given \( k \), \( \lambda_k = \lambda_d \) is clearly a solution. For some \( \mu \) values, there is a second solution. Then, the \( \vec{\lambda} \) satisfying \( (79a) \) has the following properties. A possibility is that all elements of \( \vec{\lambda} \) are equal to each other
\[ \lambda_k = (1 - \lambda_d)/(d - 1) \quad \text{for} \quad k = 1, 2, ..., d - 1 \]  
(81)
Another possibility is that some of the elements of \( \vec{\lambda} \) are equal to \( \lambda_d \). The other elements are different from \( \lambda_d \), and they are all equal to each other.

So far we looked for \( \vec{\lambda} \) for which the derivative of \( g \) is zero, which is only a necessary condition for obtaining a minimum of \( f(\vec{\lambda}) \) with the given constraints. Simple calculations show that the \( \vec{\lambda} \) given in Eq. (81) minimizes \( f(\vec{\lambda}) \) for a given value of the von Neumann entropy if 1/\( d \) \( \leq \lambda_d \leq 1 \). Such eigenvalues correspond to a quantum state that is a mixture of a pure state and white noise. The average quantum Fisher information \( F^\log_Q[\rho] \) as a function of \( \exp(S) \) corresponding to the eigenvalues given in Eq. (81) is plotted in Fig. 4.

The results of this section complement the results of Ref. [53], where they established a quantum version of the classical isoperimetric inequality relating the quantum Fisher information and the entropy power of a quantum state. They studied multi-mode continuous variable systems, where the averaging was carried out for the canonical operators \( x_k \) and \( p_k \). The quantum Fisher information was defined based on a second order derivative of the relative entropy, just as in our last example Eq. (74). In contrast, we considered systems of finite dimension, and averaged the quantum Fisher information over all Hermitian observables.
VI. CONCLUSIONS

We considered a generalized variance defined as the difference between the variance and the quantum Fisher information over four. We obtained lower bounds on it with the purity of the state. We also considered the generalized variance averaged over all Hermitian operators. We found that it is a weighted sum of the linear entropy and another simple term that is the sum of the pairwise harmonic means of the eigenvalues of the density matrix. We examined the relation of our quantity to the von Neumann entropy. We found also relations between the Kubo-Mori-Bogoliubov quantum Fisher information averaged over all Hermitian operators and the von Neumann entropy.

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Appendix A: Calculations for averages over the Hermitian operators

In this Appendix, we prove Observation 4. Let us evaluate the sum of variances over all generators in Eq. (63). Based on the well-known identities (e.g., see Ref. [70])

\[ \sum_{m=1}^{N_\xi} \langle (A^{(m)})^2 \rangle = 2 \left( d - \frac{1}{d} \right), \]
\[ \sum_{m=1}^{N_\xi} \langle A^{(m)} \rangle^2 = 2 \left[ \text{Tr}(\rho^2) - \frac{1}{d} \right], \]

we obtain for the average variance Eq. (65a). Note that here we used the normalization given in Eq. (57).

Let us obtain an equation for averages of the elements of A explicitly. Based on Eq. (A1a), averaging the second moment of A_\xi can be rewritten as

\[ \sum_{m=1}^{N_\xi} \langle (A^{(m)})^2 \rangle = 2 \left( d - \frac{1}{d} \right), \]
\[ \sum_{m=1}^{N_\xi} \langle A^{(m)} \rangle^2 = 2 \left[ \text{Tr}(\rho^2) - \frac{1}{d} \right], \]

where \( A_\xi \) is defined in Eq. (56). We took into account that the averages for diagonal elements are equal to each other, hence

\[ |A_{11}|^2 = |A_{kk}|^2 \]  \hspace{1cm} (A3)

holds for all \( k \). Similarly, the averages for off-diagonal elements are also equal to each other, and we obtain

\[ |A_{12}|^2 = |A_{kl}|^2 \]  \hspace{1cm} (A4)

for all \( k \neq l \).

After calculating the average of the variance for traceless Hermitian operators, we calculate an analogous quantity for the quantum Fisher information. The quantum Fisher information for a traceless Hermitian operator can be obtained as

\[ F_Q[\rho, A_\xi] = n^TF_n, \] \hspace{1cm} (A5)

where \( F \) is the Fisher matrix defined as [2]

\[ F_{mn} = F_Q[\rho, A^{(m)}, A^{(n)}] = \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} A^{(m)}_{kl} A^{(n)}_{kl}. \] \hspace{1cm} (A6)

Note that Eq. (A6) coincides with Eq. (12) if \( A^{(m)} = A^{(n)} \).

Next, we consider the quantum Fisher information averaged over the Hermitian operators [71]. The average has been calculated in Refs. [72]. Here, we present an alternative proof for completeness, as well as, for proving Eqs. (66a) and (66b).

Based on ideas similar to the ones used for the average variance given in Eq. (63), we obtain

\[ \mathcal{T}_Q[\rho] = \frac{1}{N_\xi} \sum_{m=1}^{d^2-1} F_Q[\rho, A^{(m)}]. \] \hspace{1cm} (A7)

Let us now evaluate the sum of the quantum Fisher information for all generators in Eq. (12). Averaging Eq. (12) can be rewritten as

\[ \mathcal{F}_Q[\rho] = \frac{1}{N_\xi} \sum_{m=1}^{d^2-1} F_Q[\rho, A^{(m)}] \]

Using the identity

\[ \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} = \frac{1}{N_\xi} \sum_{m=1}^{d^2-1} \frac{(\lambda_k + \lambda_l)^2}{\lambda_k + \lambda_l} - \frac{4\lambda_k \lambda_l}{\lambda_k + \lambda_l}, \] \hspace{1cm} (A9)

Eq. (A8) can be further rewritten with \( H(\rho) \) as

\[ \mathcal{F}_Q[\rho] = 4 \left( d - H(\rho) \right) |A_{12}|^2. \] \hspace{1cm} (A10)

Next, we determine \( |A_{12}|^2 \). For that we write down the average quantum Fisher information for pure states in two different ways. On the one hand, from Eq. (A10) we obtain for a pure state \( |\Psi\rangle \) the average quantum Fisher information as

\[ \mathcal{F}_Q[|\Psi\rangle] = 4 (d - 1) |A_{12}|^2. \] \hspace{1cm} (A11)
On the other hand, we know that for pure states the quantum Fisher information is four times the variance. Hence, using the formula (65a) for the the average variance, for pure states $|\Psi\rangle$ we arrive at
\[ F_Q[|\Psi\rangle] = 4 \text{var}(\theta) = \frac{8}{N_g}(d-1). \] (A12)

Comparing Eq. (A11) and Eq. (A12) we arrive at Eq. (66a) and obtain the average quantum Fisher information as Eq. (65b).

The quantum Fisher information $F_Q[\varrho, A^{(k)}]$ is convex in the state. Hence, the quantum Fisher information averaged over the Hermitian operators (65b) is also convex in the state. From Eq. (65b) it also follows that $H(\varrho)$ is concave in the state. Equation (65b) is maximal for pure states. Hence, the average quantum Fisher information, i.e., the metrological usefulness of the quantum state is the largest for pure states.

Finally, we prove the formula for the average for the diagonal elements of $A$ given in Eq. (66b). Based on Eq. (14) we know that for the completely mixed state, (39), for any $A$
\[ \langle A \rangle_{\varrho_{cm}} = \frac{1}{d} \sum_{k,l} |A_{kl}|^2 = \frac{1}{d} \text{Tr}(A^2) \] (A13)
holds. Due to the normalization of the basis matrices given in Eq. (57) we arrive at
\[ \langle A \rangle_{\tilde{n}} = \frac{2}{d} \] (A14)
for any $\tilde{n}$. From Eq. (66a), Eq. (A14) and Eq. (A2) we obtain Eq. (66b).

**Appendix B: The relation between the Kubo-Mori-Bogoliubov Fisher information and the relative entropy**

In Ref. [73], the generalized quantum Fisher information is defined as
\[ F_Q^{\log}[\varrho; A] = \sum_{ij} \frac{1}{m_f(\lambda_i, \lambda_j)} |A_{ij}|^2, \] (B1)
where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a matrix monotone function, and a corresponding mean $m_f$ is defined as in Eq. (25). The quantum Fisher information family, (B1), includes the usual quantum Fisher information for $f(\varrho) = (1 + x)/2$. It also includes the Kubo-Mori-Bogoliubov Fisher information with
\[ f(\varrho) = (x - 1)/ \log(x) \] (B2)
defined as
\[ F_Q^{\log}[\varrho; A] = \sum_{k,l} \frac{\log(\lambda_k) - \log(\lambda_l)}{\lambda_k - \lambda_l} |A_{kl}|^2. \] (B3)
The quantum Fisher information, (B3), corresponds to linear dynamics of the type
\[ \varrho_t = \varrho_0 + B \theta, \] (B4)
where $t$ is the parameter of the quantum state and $B$ is a traceless Hermitian matrix. However, in physics we typically consider a unitary dynamics of the type
\[ \varrho_\theta = e^{-iA^\theta} \varrho e^{iA^\theta}. \] (B5)

The Kubo-Mori-Bogoliubov Fisher information corresponding to such a dynamics can be obtained as
\[ F_Q^{\log}[\varrho; A] = F_Q^{\log}[\varrho; i[\varrho, A]], \] (B6)
which is identical to the formula given in Eq. (72).

Equation (74) can be proved as follows. In Ref. [74], it is shown that for small $\theta$
\[ F_Q^{\log}[\varrho; A] \approx \frac{2}{\theta^2} S(\varrho||\varrho_\theta) \] (B7)
holds. Now, note the trivial relation
\[ S(\varrho||\varrho_\theta) = 0 \] (B8)
for $\theta = 0$. Note also that
\[ \frac{d}{d\theta} S(\varrho||\varrho_\theta)|_{\theta=0} = 0, \] (B9)
since $S(\varrho||\varrho_\theta)$ is minimal for $\theta = 0$. Hence,
\[ S(\varrho||\varrho_\theta) \approx \frac{\theta^2}{2} \frac{d^2}{d\theta^2} S(\varrho||\varrho_\theta) + O(\theta^3). \] (B10)
From Eqs. (B7) and (B10) follows Eq. (74). On the relation between the Kubo-Mori-Bogoliubov Fisher information and the relative entropy, see also Refs. [60, 69].

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(b − a)/(log(b) − log(a)). The quantity varKL(A) is also 
called the Kubo-Mori-Bogoliubov variance [60, 61]. It 
is connected to the magnetic susceptibility κ = 
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