Partially quenched QCD and staggered fermions

Claude Bernard and Maarten Golterman

aDepartment of Physics, Washington University
St. Louis, MO 63130-4899, USA

We summarize results for partially quenched chiral perturbation theory and indicate an application to staggered fermion QCD in which the square root of the determinant is taken to reduce the number of flavors from four to two.

1. Partially quenched chiral perturbation theory

We summarize the results of a recent investigation of partially quenched theories [1]. This extends previous work on the fully quenched case [2,3] (see also refs. [4,5]). Partially quenched theories are theories in which not all fermions are quenched: only for \( k \) of the \( n \) fermions present in the theory will the determinant in the functional integral be replaced by 1. A lagrangian formulation of such a theory is obtained by considering QCD with \( n + k \) quarks, where the first \( n \) quarks are just normal quarks (denoted by \( q_i \), \( i = 1 \ldots n \)), whereas the other \( k \) quarks (denoted by \( \tilde{q}_j \), \( j = 1 \ldots k \)) will be given bosonic statistics. We will refer to such a theory as an \( SU(n|k) \) theory. The same method that we developed for studying ChPT for a completely quenched theory can also be applied in this case.

Our motivation for considering partially quenched theories is threefold:

First, one may learn more about the peculiar infrared behavior [2,3] by considering what happens when only part of the fermion content of a theory is quenched. In particular, if different fermion mass scales are present, one might ask how the infrared behavior depends on whether all or only some of the fermions with a common mass are quenched. Also, it is interesting to know what happens in the unquenched sector of the theory: is a theory with \( n \) fermions, out of which \( k \) are quenched, the same as an unquenched theory with just \( n - k \) fermions?

Second, partially quenched theories arise naturally in the description of simulations in which the valence quark masses are not chosen equal to the sea-quark masses. This is a not uncommon numerical technique which, for example, allows one to use Wilson valence quarks and staggered sea-quarks. One would like to have a chiral theory for such simulations.

A third motivation comes from staggered fermion QCD, which describes QCD with four flavors of quarks in the continuum limit. In order to use these fermions for simulations of QCD with only two flavors, a common trick is to take the square root of the fermion determinant, thereby effectively reducing the number of flavors which appear in virtual quark loops from four to two.

We can state three theorems about partially quenched theories:

I. In the subsector where all valence quarks are unquenched the \( SU(n|k) \) theory is completely equivalent to a normal, completely unquenched \( SU(n - k) \) theory.

II. The “super-\( \eta' \),” \( \Phi_0 \), defined as

\[
\Phi_0 = \frac{1}{\sqrt{n - k}} \left( \sum_{i=1}^{n} \bar{q}_i \gamma_5 q_i + \sum_{j=1}^{k} \bar{\tilde{q}}_j \gamma_5 \tilde{q}_j \right),
\]  

is equivalent to the \( \eta' \) constructed in the unquenched sector of the \( SU(n|k) \) theory, and is therefore, by I, equivalent to the \( SU(n - k) \) \( \eta' \). “Equivalent” here means that Green’s functions constructed from an arbitrary number of super-\( \eta' \) fields and unquenched quarks, will be equal to the corre-
responding Green’s functions with the super-$\eta'$ replaced by the $\eta'$ of the $SU(n-k)$ theory.

III. Quenched infrared divergences, coming from a double pole in the $\eta'$ propagator and associated with some quark mass of mass $m$, will arise if and only if the scale $m$ is fully quenched, i.e., if there is a pseudoquark of mass $m$ for every quark of mass $m$.

Theorems I and II can be proved by simple physical arguments, which rely on the cancellation between quarks and pseudoquarks (bosonic quarks) in virtual loops (Thrm. I) or valence lines (Thrm. II). Theorem III requires a detailed examination of the propagator in the neutral meson sector in partially quenched chiral perturbation theory. It is then possible to show the offending double poles can only arise when a mass scale is fully quenched.

2. Staggered fermions

In this section, we will consider the definition of two-flavor meson operators in the mass degenerate two-flavor theory obtained from the degenerate four-flavor theory in which the square root of the determinant is taken. Theorem I tells us that we can obtain the two-flavor unquenched theory in this way, and that no problems are to be expected from taking the square root. For nonsinglet mesons no tuning of the operators is required because one may use the same operators as in the four-flavor theory.

However, one expects that the definition of an operator for the $\eta'_{SU(2)}$ in the four-flavor theory will require tuning, even with degenerate quark masses. What we wish to show here is that nevertheless two ways exist for choosing a mass matrix and a meson operator which do not require tuning of the operator in order to define a pure $\eta'_{SU(2)}$ in the four-flavor theory. The first method consists of applying theorem II, whereas the second method makes use of a peculiarity of nonlocal staggered fermion mass terms.

For staggered fermion QCD, mass terms can be constructed which lead to the most general four flavor mass matrix $M$ in the continuum limit:

$$M = m + m_\mu \xi_\mu + \frac{1}{2} m_{\mu\nu}(- i \xi_\mu \xi_\nu) + m_5^5 i \xi_\mu \xi_5 + m_5^5 \xi_5. \quad (2)$$

The $4 \times 4$ $\xi$-matrices form a representation of the Clifford algebra $\xi_\mu \xi_\nu + \xi_\nu \xi_\mu = 2 \delta_{\mu\nu}$, and are identified with $SU(4)$ flavor generators in the continuum limit. We will denote the terms in eq. (2) with scalar (S), vector (V), tensor (T), axial vector (A) and pseudoscalar (P) respectively. They correspond to $0, \ldots, 4$ link operators in the staggered fermion action.

It can be shown that this form of the mass matrix is stable under renormalization, in the sense that the coefficients $m, m_\mu, \ldots$ will only receive multiplicative renormalizations, one for each tensor structure in eq. (2). Note that the mass matrix $M$ needs to be diagonalized in order to determine what the mass eigenstates are.

Let us first consider the simplest possible mass matrix, by choosing only the single site mass $m$ to be nonzero, corresponding to four degenerate flavors. In this case, the simplest operator for an $\eta'_{SU(4)}$ will be a four link operator, which in the continuum limit corresponds to the operator $\bar{\psi}\gamma_5\psi$, where $\psi$ is a continuum Dirac field with four flavor components.

In this basis, an $\eta'_{SU(2)}$ would be created by the continuum operator

$$\eta'^{\text{cont}}_{SU(2)} = \bar{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma_5 \psi. \quad (3)$$

Clearly, in order to construct a staggered operator with this continuum limit, we need an operator with flavor matrix of type $S$ to get a nonzero trace because the $\eta'_{SU(2)}$ flavor matrix in eq. (3) has a nonvanishing trace, and $V, T, A$ and $P$ are all traceless. In addition, we need an operator of the type $V, T, A$ or $P$, since the matrix contains two zero eigenvalues. The fact that these operators renormalize differently from $S$ leads to the need to tune their relative coefficient. We conclude that with a single site mass term no explicit $\eta'_{SU(2)}$ operator can be constructed in the four-flavor staggered theory without tuning. The
only way to avoid tuning in this case, is to compute the diagrams for the $\eta_{SU(4)}$ and adjust the relative coefficients of the straight-through and the two-hairpin diagrams, as implied by theorem II, and explained in detail in section 2 of ref. [1].

Actually, the special properties of the tensor operator make it possible to construct an $\eta_{SU(2)}$ without tuning in a different way. To discuss this, we will choose an explicit representation of the $\xi$-matrices:

$$\xi_i = \sigma_i \otimes \tau_1, \quad \xi_4 = \tau_2, \quad \xi_5 = \tau_3.$$  \tag{4}

In this case, it is necessary to choose a mass term of the tensor type. For definiteness we choose

$$M_0 = m(-\xi_1 \xi_2) = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & -m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix}.$$  \tag{5}

which corresponds again to four flavors with a degenerate mass $m$. The minus signs can be removed by a nonanomalous chiral transformation. The $\eta_{SU(4)}$ with this mass matrix is

$$\eta_{SU(4)}^{\text{cont}} \propto \bar{\psi}(-i\xi_1 \xi_2)\gamma_5 \psi = \bar{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma_5 \psi.$$  \tag{6}

Projecting to $SU(2)$, we get for the $\eta_{SU(2)}^{\text{cont}}$

$$\eta_{SU(2)}^{\text{cont}} \propto \bar{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma_5 \psi = \bar{\psi}(-i\xi_1 \xi_2 - i\xi_3 \xi_4)\gamma_5 \psi.$$  \tag{7}

Unlike the previous case, this $\eta_{SU(2)}^{\text{cont}}$ flavor matrix is now traceless, which allows us to write it as a sum of two tensor terms.

The $\eta_{SU(2)}^{\text{cont}}$ of eq. (7) is now constructed from two tensor operators rather than one scalar and one of some other type. Since all tensor operators get renormalized in the same way, no tuning is needed here. The price, however, is the use of a tensor mass term, which would make this approach awkward for standard simulations. Using the $\eta_{SU(4)}$, and readjusting the relative weight of the diagrams by hand, will be preferable in most cases. We note that such readjustment is standard practice in weak matrix element calculations with staggered fermions [1].

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