Dynamics of Fermat potentials in non-perturbative gravitational lensing

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We present a framework, based on the null-surface formulation of general relativity, for discussing the dynamics of Fermat potentials for gravitational lensing in a generic situation without approximations of any kind. Additionally, we derive two lens equations: one for the case of thick compact lenses and the other one for lensing by gravitational waves. These equations in principle generalize the astrophysical scheme for lensing by removing the thin-lens approximation while retaining the weak fields.

I. INTRODUCTION

In the astrophysical approach to lensing \cite{1,2}, background light sources are considered to lie on a plane at a distance \( D_s \) from the observer, and the deflector or lens is described by a mass distribution on a plane at distance \( D_l \) from the observer. Light rays from the source in the source plane travel along straight null geodesic paths on a flat or cosmological background until they reach the lens plane, at which point they suffer a sharp bend in direction, subsequently traveling again on a straight line towards the observer. The direction at which a light ray reaches the observer determines the angular location of the image on the observer’s celestial sphere. The celestial sphere can be identified with the lens plane (which is sometimes referred to as the image plane). A fundamental element of gravitational lensing is a lens equation of the form:

\[
\eta = \frac{D_s}{D_l} \xi - D_{ls} \alpha(\xi) \tag{1}
\]

where \( \eta \) is a position vector on the source plane representing the source’s location, \( \xi \) is a position vector on the lens plane representing the location where the light ray emitted by the source pierces the lens plane, \( D_{ls} \) is the distance between the lens plane and the source plane and \( \alpha \) is the bending angle that the light ray suffers at the lens plane, which is determined by the distribution of mass on the lens plane and is thus a function on the lens plane. The bending angle \( \alpha \) can be obtained from a deflection potential \( \psi(\xi) \) via

\[
\alpha = \frac{\partial \psi}{\partial \xi} \tag{2}
\]

and the deflection potential \( \psi \) in turn is obtained from the mass distribution \( \Sigma(\xi) \) by means of a 2-dimensional Poisson equation

\[
\Delta_\xi \psi(\xi) = \Sigma(\xi), \tag{3}
\]

where \( \Delta_\xi \) is the 2-dimensional Laplacian on the lens plane.

A very elegant scheme that leads to the lens equation is based on Fermat’s principle, according to which light takes the path of least travel time between any two fixed points. In order to use this principle, a set of paths between the fixed source at \( \eta \) and the observer are considered, all of which consist of two straight segments, but which are allowed to pierce the lens plane at arbitrary points \( \xi \). The value of \( \xi \) thus becomes a label for the path. The travel time along such paths is thus a function of \( \xi \), which must be minimized with respect to \( \xi \). The travel time along such paths is a Fermat potential \( \phi(\xi, \eta) \), in general depending on both the source point as well as the point on the lens plane. The Fermat potential has the form

\[
\phi(\xi, \eta) = \frac{1}{2} \frac{D_l D_s}{D_{ls}} \left| \frac{\eta}{D_s} - \frac{\xi}{D_l} \right|^2 - \psi(\xi), \tag{4}
\]

where the first term represents the geometric length of the broken path with respect to the length of a straight path between the source and the observer, and the second term represents the gravitational time delay suffered by the
lightray as a consequence of the presence of mass at the lens plane. The minimization of the travel time with respect to the path requires

\[ \frac{\partial \phi(\xi, \eta)}{\partial \xi} = 0. \]  

(5)

This equation provides the lens map \( \xi \rightarrow \eta \) from the lens plane to the source plane, i.e., Eq (1). The Fermat potential is affected by the presence of a mass distribution \( \Sigma(\xi) \) on the lens plane essentially because of Eq. (3). As a consequence of Eq. (3), one can see that the Fermat potential also satisfies a Poisson-type equation

\[ \Delta \xi \phi(\xi, \eta) = -S(\xi), \]  

(6)

where \( S(\xi) \) differs from \( \Sigma(\xi) \) by an additive constant. This is the dynamical equation for the Fermat potential in the astrophysical approach to lensing. It encodes the information of the curvature of the spacetime due to the presence of mass on the lens plane, and it thus represents the role of the Einstein equations in lensing. One can see that the Fermat-potential formulation of lensing is fully tailored to the approximations in use in the astrophysical approach to lensing, namely: thin lenses, weak fields and small angles.

We are interested in a generalization of the Fermat-potential approach to lensing for a generic situation with no approximations. The first step to this goal is to construct a Fermat potential for a generic situation, such that a lens mapping from the observer’s celestial sphere to the lightsource locations results from a variational principle applied to it. We refer to this phase of the project as the kinematics of the Fermat potential. The second step is to find a “field equation” for the Fermat potential, namely: an equation that relates the Fermat potential to the sources of gravitational field. We regard this as the dynamics of the Fermat potential.

In a previous work [3] we have described a setting for an approach to the kinematics of Fermat potentials using both a generalization of a Fermat potential and a generalized Fermat principle consistent with a generic situation, which we briefly summarize in the following.

The aim of the scheme is to construct a function on spacetime that, by extremization, leads to an expression of the observer’s past lightcone. The observer’s past lightcone can be parametrized in terms of the celestial sphere (the null directions at the apex of the lightcone) and an affine length along the null rays that rule the lightcone. Such a parametrization leads directly to a lens mapping from the celestial sphere out to lightsource, and a time-of-arrival approximation. The first step to this goal is to construct a Fermat potential for a generic situation, such that a lens mapping from the observer’s past lightcone can be constructed. Since lightcones are level surfaces of solutions to the eikonal equation, such solutions are our starting point.

Consider a lensing spacetime with coordinates \( x^a \) and metric \( g_{ab} \). Suppose we have a (sufficiently generic) 2-parameter family of solutions \( Z(x^a, \zeta, \tilde{\zeta}) \) of the eikonal equation \( g^{ab} Z_a Z_b = 0 \) (with \( \bar{\partial} = \partial/\partial x^a \)), such that the surfaces of constant value of \( Z(x^a, \zeta, \tilde{\zeta}) \) foliate the spacetime for every fixed value of \( (\zeta, \tilde{\zeta}) \). Define

\[ G(x_0^a, x^a, \zeta, \tilde{\zeta}) \equiv Z(x^a, \zeta, \tilde{\zeta}) - Z(x_0^a, \zeta, \tilde{\zeta}) \]  

(7)

Then, for every fixed value of \( (\zeta, \tilde{\zeta}) \), the surface defined by the points \( x^a \) that satisfy

\[ G(x_0^a, x^a, \zeta, \tilde{\zeta}) = 0 \]  

(8)

is null and contains the point \( x_0^a \), at which the vector \( \ell^a \equiv g^{ab} G_{,b} \) sweeps out the sphere of null directions as \( (\zeta, \tilde{\zeta}) \) is allowed to vary. The parameters \( (\zeta, \tilde{\zeta}) \) are thus coordinates on the observer’s celestial sphere, and can be taken as complex stereographic coordinates. Solving (8) for \( x^0 \equiv t \) we have

\[ t = T(x_0^a, x^i, \zeta, \tilde{\zeta}), \]  

(9)

namely, the value of the coordinate time of the point of intersection of the family of null surfaces \( G = 0 \) with the worldline \( (x^i, t) \) of a static source at the spatial position \( x^i \), \( i = 1, 2, 3 \). (This can be reformulated to include a moving source.) Varying \( (\zeta, \tilde{\zeta}) \), different null surfaces in the family \( G = 0 \) are considered, all of which contain the observer \( x_0^a \) and intersect the worldline of a potential source at \( x^i \) at time \( T(x_0^a, x^i, \zeta, \tilde{\zeta}) \). This time cannot be interpreted in terms of curves joining the points \( x_0^a \) and \( (x^i, \tilde{T}) \) because there is an infinity of curves lying on the surface \( G = 0 \) joining the two points for which the coordinate time of the end point is \( t \): there is no way to pick a preferred curve on the null surface. In fact, among all possible curves joining the two points on a given null surface \( G = 0 \), there may even not be one that is \( geodesic \), and there are certainly many that become spacelike at least one point. Thus, even though \( T(x_0^a, x^i, \zeta, \tilde{\zeta}) \) does represent a time function in a sense, we have no reason to identify it with the time of arrival of a light signal. However, \( T(x_0^a, x^i, \zeta, \tilde{\zeta}) \) is the time at which a wavefront of the \( (\zeta, \tilde{\zeta}) \)–null surface passes by the source at \( x^i \), so we can think that instead of looking at travel times along null paths we are in effect looking at travel times.
of wavefronts from the observer to the source. Both points of view are equivalent in the end, because ultimately the values of \((\zeta, \bar{\zeta})\) for which the intersection time \(T(x^0_a, x^i, \zeta, \bar{\zeta})\) is stationary are null surfaces for which there exists a null geodesic that joins \(x^0_a\) with \((x^i, T)\).

Our recipe is: to extremize the function \(T(x^0_a, x^i, \zeta, \bar{\zeta})\) with respect to variations of \((\zeta, \bar{\zeta})\) keeping \(x^i\) fixed, and interpret the extremum \(T_0 = T(x^0_a, x^i, \zeta_0, \bar{\zeta}_0)\) as the time of arrival of a light signal from \(x^0_a\) to \(x^i\), giving an image in the direction \((\zeta_0, \bar{\zeta}_0)\). These choices of \((\zeta, \bar{\zeta})\) lead to null geodesics connecting the observer with the source. It follows that the function \(T(x^0_a, x^i, \zeta, \bar{\zeta})\) can be viewed as a generalized Fermat potential. On the other hand, since \(T(x^0_a, x^i, \zeta, \bar{\zeta})\), in general will be given implicitly via \((8)\), then extremizing \(T(x^0_a, x^i, \zeta, \bar{\zeta})\) is equivalent to imposing the conditions

\[
G_{,\zeta} |_{x^a} = G_{,\bar{\zeta}} |_{x^a} = 0
\]

which are, in fact, the conditions for the envelope of the level surfaces of \(G\). This results from implicit differentiation \(\partial/\partial \zeta |_{x^i}\) of \((8)\) where \(t\) is assumed to be given by \((\bar{\zeta})\). One obtains

\[
G_{,\zeta} |_{x^a} + G_{,t} \ T_{,\zeta} |_{x^i} = 0.
\]

Since \(G_{,t}\) must be non-zero (otherwise it could not define \(T\) implicitly), then \(T_{,\zeta} |_{x^i} = 0 \Leftrightarrow G_{,\zeta} |_{x^a} = 0\). Likewise with \(T_{,\bar{\zeta}} |_{x^i}\). We refer to \(G(x^0_a, x^i, \zeta, \bar{\zeta})\) as an “implicit” generalized Fermat potential, making a slight abuse of terminology.

In fact, \(G = 0\) plus the envelope condition for \(G\) as a joint set of equations produces for us two things. 1.) A lens equation: an assignment of a value of \((\zeta, \bar{\zeta})\) on the celestial sphere for each source location \(x^i\), through light signals. 2.) A time of arrival of the light signal. Thus, the envelope conditions \((10)\) on a level surface of \(G\) embody our generalized Fermat’s principle: a version in terms of surfaces, rather than paths.

Up to this point, we have assumed that the Fermat potential \(G\) is somehow known, and satisfies the eikonal equation. Thus we have only considered the kinematics of such implicit Fermat potentials: we have not considered how a Fermat potential is affected by the presence of matter in the spacetime (the dynamics). The matter (the deflector) is, of course, hidden away in the metric \(g_{ab}\) that determines the eikonal equation, so the eikonal constitutes part of the dynamics, but the field equations for the metric itself are the Einstein equations \(R_{ab} = \frac{1}{2}g_{ab} R = T_{ab}\). In principle, one could first solve for the metric, and then use the metric in the eikonal to obtain the Fermat potential \(G\). There is no general solution to the Einstein equations in terms of data or matter sources, because of their nonlinearity, therefore this route does not have the potential to provide us with a one-step dynamical equation for the Fermat potential driven directly by the matter source, which would be analogous to Eq. \((9)\).

Notwithstanding, there is an approach to general relativity which has the potential to directly lead to dynamical equations for implicit Fermat potentials \(12\). We describe the resulting dynamics of implicit Fermat potentials through such an approach in Section 11. An application of the dynamics to the case of weak fields is worked out in Section 11. The main results of the application to weak fields are two lens equations: one for the case of thick compact objects and the other one for the case of lensing by gravitational waves. Whether these equations can be of use in astrophysical situations remains to be seen, but our point is that they can be written down in principle.

**II. DYNAMICS OF THE IMPLICIT FERMAT POTENTIAL**

The formulation of general relativity via null surfaces \(13\) provides a setting for discussing the dynamics of implicit Fermat potentials in the sense of the previous Section. This formulation is described in complete detail in \(13\). Here we provide a (necessarily brief) summary of the main aspects of the formulation, hoping that the unfamiliar reader is able to complement this Section with \(13\).

In the null-surface approach, the aim is to rewrite the Einstein equations in terms of variables that are to be thought of as more fundamental than the metric \(g_{ab}\), which can then be generated from them. The fundamental objects of the formulation are two functions of six variables, \(Z(x^a, \zeta, \bar{\zeta})\) and \(\Omega(x^a, \zeta, \bar{\zeta})\). The variables \(x^a\) represent points in a four-dimensional spacetime with a Lorentzian metric to be determined. The variables \((\zeta, \bar{\zeta})\) represent a sphere fiber at every point \(x^a\); eventually \((\zeta, \bar{\zeta})\) acquires the meaning of the sphere of null directions at each spacetime point. The first function, \(Z\), whose level surfaces are null for all values of \((\zeta, \bar{\zeta})\) encodes the conformal structure of the spacetime. That is to say that if the metric of the spacetime is known, then \(Z(x^a, \zeta, \bar{\zeta})\) satisfies the eikonal equation \(g^{ab}Z_{,a}Z_{,b} = 0\) for all values of \((\zeta, \bar{\zeta})\). The second function, \(\Omega\), represents the scale (or conformal) factor of the spacetime with conformal structure given by \(Z\) (all conformally related spacetimes have the same conformal structure; the scale factor breaks the conformal invariance). From the two variables, \(Z\) and \(\Omega\), a Lorentzian metric can easily be constructed. Field equations, equivalent to the Einstein equations can be imposed on them.
These field equations for $Z$ and $\Omega$ are in general quite complicated. However, for the weak field applications that we will treat they become quite simple and are easily reduced to quadratures. We will first present the full equations before giving the linearization.

It is perhaps easiest if the equations are expressed in two steps. We first introduce several auxiliary variables obtained from derivatives of $Z(x^a, \zeta, \bar{\zeta})$: we define four variables $\theta^i(x^a, \zeta, \bar{\zeta})$ which can be considered as coordinates given intrinsically by the families of null surfaces by

$$
\begin{align*}
\theta^0 & \equiv Z \equiv u, \\
\theta^+ & \equiv \partial Z \equiv \omega, \\
\theta^- & \equiv \bar{\partial} Z \equiv \bar{\omega}, \\
\theta^1 & \equiv \partial \bar{\partial} Z \equiv R.
\end{align*}
$$

Using these relations we can go back and forth between $x^a$ and $\theta^i$. Further, it is assumed that these relations can be inverted locally so that we have

$$x^a = x^a(\theta^i, \zeta, \bar{\zeta}).$$

In addition to $\theta^i$, we define another fundamental variable (and its complex conjugate) by

$$\Lambda(\theta^i, \zeta, \bar{\zeta}) = \partial Z^2,$$

where the $x^a$ on the right-hand side have been replaced by the $\theta^i$ via Eq. (13).

For the second step, the Einstein equations are written in terms of $\Lambda(\theta^i, \zeta, \bar{\zeta})$ and $\Omega(\theta^i, \zeta, \bar{\zeta})$ as

$$
\begin{align*}
\Omega_{11} &= Q(\Lambda)\Omega + T\Omega^3, \\
\hat{\partial}\Omega &= \frac{1}{2}W(\Lambda)\Omega, \\
M(\Lambda) &= 0,
\end{align*}
$$

where we have used the notation $,i = \partial / \partial \theta^i$ and in particular

$$\Omega_{11} = \frac{\partial^2 \Omega}{\partial R^2}.$$  

The quantity $T$ encodes the information in the stress-energy tensor $T_{ab}$, while $(Q, W, M)$ are explicit functions of $\Lambda$ and its derivatives:

$$
\begin{align*}
Q & \equiv \frac{1}{4q} \left( \Lambda_{11,11} \Lambda_{11} + \frac{3}{2q}(q_{11})^2 - q_{11} \right), \\
M & \equiv \hat{\partial}(\Lambda_{11}) - 2\Lambda_{1} - \left( W + \hat{\partial} \ln q \right)\Lambda_{1} , \\
W & \equiv \left( \Lambda_{1} + \frac{1}{2}\Lambda_{11,1} \Lambda_{11} \right) + \frac{1}{2}\Lambda_{1} \Lambda_{1} - \frac{1}{4}\Lambda_{11} \hat{\partial}(\Lambda_{1}) - \frac{1}{2}\hat{\partial} \ln q - \frac{1}{4}\Lambda_{1} \hat{\partial} \ln q \right) \left( 1 - \frac{1}{4}\Lambda_{11} \Lambda_{11} \right)^{-1},
\end{align*}
$$

with

$$q \equiv 1 - \Lambda_{11} \Lambda_{11}.$$  

The operator $\hat{\partial}$ is the usual $\partial$ with $x^a$ held constant; but operating on functions $f(\theta^i, \zeta, \bar{\zeta})$ it takes the form

$$\hat{\partial}f(\theta^i, \zeta, \bar{\zeta}) = \partial f|_{\theta^0} + \theta^+ f_{,0} + \Lambda f_{,1+} + \theta^1 f_{,1-} + \left( \Lambda_{1} \frac{\partial + J}{1 - \Lambda_{11} \Lambda_{11}} \right) f_{,1}$$

with

$$J = -2\theta^+ + \theta^- \Lambda_{1} + \Lambda_{11} \frac{\theta^0}{\Lambda_{11}} + \theta^1 \Lambda_{1} + \Lambda_{11} \Lambda_{1}.$$  

For the purposes of applying the $\hat{\partial}$ operator, $\Lambda$ and $\Omega$ have spin weights 2 and 0.
Eq. (15a) is a direct translation of the Einstein equation, $G_{ab} = T_{ab}$, while Eqs. (15b) and (15c), referred to as the metricity conditions, are the requirements that a unique metric can be constructed from $Z$ and $\Omega$.

The field equations couple $\Lambda$ and $\Omega$ in a complicated non-linear manner that makes finding exact solutions an almost impossible task. Nevertheless, as we will shortly see, the linearization becomes quite simple. In principle, the procedure for finding the Fermat potential would be to solve Eqs. (15) for $\Lambda = \Lambda(\theta^a, \zeta, \bar{\zeta}) = \Lambda(Z, \partial Z, \partial\bar{Z}, \zeta, \bar{\zeta})$ and then return to Eq. (14), written in the form

$$\bar{\sigma}^2 Z = \Lambda(Z, \partial Z, \partial\bar{Z}, \partial\bar{Z}, \zeta, \bar{\zeta})$$

(21)

so that it becomes a dynamical equation for $Z(x^a, \zeta, \bar{\zeta})$ with a known right-hand side. Actually, since $\Lambda$ is a complex function, (21) is a pair of over-determined second-order PDEs for $Z$ in the two variables $(\zeta, \bar{\zeta})$. From general considerations [1], the solution space is four-dimensional; the solutions have four constants of integration, $x^a$, that become the spacetime coordinates. Eq. (21) is our dynamical equation for the Fermat potential. In this regard, we point out that Eq. (15c) is precisely the vanishing of the generalized Wunschmann invariant that is central to the problem of the classification of this type of overdetermined systems of PDE’s [9,10].

Some effort has been devoted to the issue of rewriting the field equations in terms of $Z$ and $\Omega$ by eliminating $\Lambda$ via (21) and (23), with partial success [11]. To do this exactly appears to be a formidable task. Nevertheless, in the case of weak fields, the linearization of the field equations allows us to combine the content of the field equations (15) and (21) and (12), with partial success [11]. To do this exactly appears to be a formidable task. Nevertheless, as we will shortly see, the linearization becomes quite simple. In principle, the application of our program to weak fields.

### III. Lens Equations in the Case of Weak Fields

The field equations (15) have input of two types. In the first place, there is the matter and stresses, encoded into the symbol $T$. Additionally, there is also gravitational radiation at the boundaries of the spacetime that must be considered free input. This is because the field equations are equivalent to the characteristic problem for the Einstein equations, in which case the gravitational radiation incoming into the spacetime must be specified [11], usually in the form of the complex shear $\sigma(u, \zeta, \bar{\zeta})$ of a null congruence of the Bondi type evaluated at the boundary of the spacetime, $\mathcal{J}^-$. This is a complex function of three arguments, the local Bondi coordinates on $\mathcal{J}^-$. We concentrate on the case in which all input sources or data are small, namely, there exists a smallness parameter $\epsilon$ by which the magnitude of the source and data can be measured, and such that its powers can be neglected. This will lead to a linearized version of the dynamics of the Fermat potentials discussed in the previous section. However, within this regime there are two distinct main cases of interest. One is the case where the spacetime is empty of matter and the gravitational radiation is small, that is: $T = 0$ and $|\sigma| \sim \epsilon$. We refer to this case as the gravitational-radiation case. The other case is such that the matter source is weak but finite, and the gravitational radiation is negligible, namely: $|T| \sim \epsilon$ and $|\sigma| \sim \epsilon^k$ with $k \geq 2$. This case will be referred to as the near-Newtonian case.

#### A. The gravitational-radiation case

We assume that we have a purely radiative spacetime, in which the stress-energy tensor vanishes and the gravitational radiation is small ($T = 0$ and $|\sigma| \sim \epsilon$). The spacetime is then flat up to a small perturbation. In flat spacetime, null planes in all possible directions constitute a 2-parameter family of solutions to the eikonal equation, leading to $Z_{\text{flat}}(x^a, \zeta, \bar{\zeta}) = x^a \ell_a$ where

$$\ell^a = \frac{1}{\sqrt{2(1 + \zeta \bar{\zeta})}}(1 + \zeta \bar{\zeta}, \zeta + \bar{\zeta}, -i(\zeta - \bar{\zeta}), \zeta \bar{\zeta} - 1)$$

(22)

is a future directed null vector pointing in the $(\zeta, \bar{\zeta})$-direction. We may expect to have then

$$Z(x^a, \zeta, \bar{\zeta}) = x^a \ell_a + \bar{Z}(x^a, \zeta, \bar{\zeta})$$

(23)

with $|\bar{Z}| \sim \epsilon$. Since $\bar{\sigma}^2 \ell^a = 0$, then the variable $\Lambda$ in the field equations (15) is of first order in $\epsilon$, and products of $\Lambda$ and its derivatives can be neglected. Therefore the field equations (15) can be linearized with respect to $\Lambda$ around $\Lambda = 0$. They become
\[ \Omega_{11} = 0, \]  
(24a)  
\[ \hat{\partial} \Omega = \frac{1}{2} \left( \Lambda_{++} + \frac{1}{2} \hat{\partial}(\Lambda_{11}) \right) \Omega, \]  
(24b)  
\[ 2\Lambda_{1-} - \hat{\partial}(\Lambda_{11}) = 0. \]  
(24c)  

By Eq. (24a) with boundary conditions consistent with an asymptotically flat spacetime (in which \( \Omega \rightarrow 1 \) along null directions), we must have \( \Omega = 1 \). Thus (24b) becomes

\[ 2\Lambda_{1+} + \hat{\partial}(\Lambda_{11}) = 0 \]  
(25)

Equations (24c) and (23) can be manipulated through algebra, differentiation and integration (with boundary conditions consistent with an asymptotically flat spacetime), to yield

\[ \bar{\partial}^2 \Lambda = \bar{\partial}^2 \sigma(\theta^0, \zeta, \tilde{\zeta}) + \bar{\partial}^2 \sigma(\theta^0, \zeta, \tilde{\zeta}). \]  
(26)

Using (12) and (21), this equation can be rewritten as an equation for \( Z \):

\[ \bar{\partial}^2 \sigma Z = \bar{\partial}^2 \sigma(x^a \ell_a, \zeta, \tilde{\zeta}) + \bar{\partial}^2 \sigma(x^a \ell_a, \zeta, \tilde{\zeta}). \]  
(27)

Notice that in (27) the operator \( \bar{\partial} \) acts at fixed \( x^a \), so that all three arguments in \( \sigma(x^a \ell_a, \zeta, \tilde{\zeta}) \) contribute to the eth-derivative. In order to obtain this equation from (20), \( \sigma(\theta^0, \zeta, \tilde{\zeta}) \) is approximated by \( \sigma(x^a \ell_a, \zeta, \tilde{\zeta}) \), an approximation that is justified because \( \sigma \) is a first-order quantity, therefore first-order terms in its arguments can be neglected. Since \( Z \) leads to the Fermat potential \( G \) in a trivial manner, we consider this the dynamical equation for the implicit Fermat potential in the presence of weak gravitational radiation. The operator \( \bar{\partial} \bar{\partial} \) is the double Laplacian on the sphere \( (\zeta, \tilde{\zeta}) \), whereas the right-hand side acts as a known source. This is the analog of Eq. (3) for lensing by gravitational waves without the approximations of thin lenses or small angles.

There exists a simple Green function [12] for the double Laplacian \( \bar{\partial}^2 \bar{\partial}^2 \), given by

\[ F(\zeta, \zeta', \zeta', \tilde{\zeta}) = \frac{1}{4\pi} \ell'_a \ell^a \ln(\ell'_a \ell^a), \]  
(28)

which can be used to express the general solution to (27) in closed form in terms of the source:

\[ Z(x^a, \zeta, \tilde{\zeta}) = x^a \ell_a + \frac{1}{4\pi} \int_{S^2} (\bar{\partial}^2 \sigma(x^a \ell'_a, \zeta', \tilde{\zeta}') + \bar{\partial}^2 \sigma(x^a \ell'_a, \zeta', \tilde{\zeta}')) \ell'_a \ell^a \ln(\ell'_a \ell^a) dS^2'. \]  
(29)

This expression for \( Z \) can now be used to construct the implicit Fermat potential \( G = Z(x^a, \zeta, \tilde{\zeta}) - Z(x^a_0, \zeta, \tilde{\zeta}) \). The two envelope conditions \( \partial G = \overline{\partial G} = 0 \) then take the form

\[ 0 = (x^a - x^a_0) \partial \ell_a + \frac{1}{4\pi} \int_{S^2} \left[ (\bar{\partial}^2 [\sigma(x^a \ell'_a, \zeta', \tilde{\zeta}') - \sigma(x^a_0 \ell'_a, \zeta', \tilde{\zeta}')] + \bar{\partial}^2 [\sigma(x^a \ell'_a, \zeta', \tilde{\zeta}') - \sigma(x^a_0 \ell'_a, \zeta', \tilde{\zeta}')] \right] \partial [\ell'_a \ell^a \ln(\ell'_a \ell^a)] dS^2'. \]  
(30)

By our discussion in Section 1, Equations (30) and \( Z(x^a, \zeta, \tilde{\zeta}) - Z(x^a_0, \zeta, \tilde{\zeta}) = 0 \) combined are equivalent to the lens mapping and time delay for the case of lensing by weak gravitational waves without the approximations of thin lenses or small angles. Since there is no lens plane in this case, the lens equation is interpreted as a mapping from the observer’s celestial sphere to the source’s spatial location. The lens mapping is understood in the following sense. We assume a spin-2 function \( \sigma(u, \zeta, \tilde{\zeta}) \) is given, perhaps constructed with some model in mind. This function represents the gravitational radiation that reaches the boundary of an asymptotically flat vacuum spacetime. We can specify our coordinates by setting the origin on the observer’s worldline, so \( x^a_0 = (\tau, 0, 0, 0) \). The point \( x^a = (t, x, y, z) \) that satisfies both equations \( Z(x^a, \zeta, \tilde{\zeta}) - Z(x^a_0, \zeta, \tilde{\zeta}) = 0 \) and (30) then lies on the observer’s lightcone, and is reached by a null geodesic in the direction \( (\zeta, \tilde{\zeta}) \) at the observer. These are three equations for the four variables \( x^a \), so one of them will be free. As usual, we assume that this free coordinate represents a distance to the source, or can be obtained from a measured distance to the source. The remaining two spatial coordinates then specify the angular location of the source, and the time coordinate represents the travel time of the light signal from the source to the observer.
It may not be a simple matter to solve for the two angular coordinates and time from $Z(x^a, \zeta, \tilde{\zeta}) - Z(x_0^a, \zeta, \tilde{\zeta}) = 0$ and (30), in order to express the lens mapping in closed form even if a function $\sigma$ is given explicitly. In fact, it is not even clear that the lens mapping can be expressed in closed form. But the point that we wish to make is that in principle, the lens mapping is there. Work on an explicit example of this lens mapping is in progress and will be reported elsewhere.

What would be the use of lensing by gravitational radiation? Lensing by extremely compact objects provides a means to study the mass structure of the deflector. Likewise, we can think of using lensing by gravitational waves to infer information about the gravitational waves themselves. In principle, lensing events are natural detectors of gravitational waves. In practice, however, these lensing events are expected to lie below the observational limit, and be overshadowed by the magnitude of lensing effects by compact objects.

B. The near-Newtonian case

We turn our attention now to the opposite case, namely: there are sources of weak strength but there is little or no incoming gravitational radiation. This would include, for instance, the case of static spacetimes with weak (isolated) matter sources, or any matter sources that are static in first approximation. We thus have, by assumption, $|T| \sim \epsilon$ and $|\sigma| \sim \epsilon^k$ with $k \geq 2$. The source $T$ is a function of six variables obtained from the stress energy tensor $T_{ab}$ by

$$\tilde{T}(\theta^i, \zeta, \tilde{\zeta}) \equiv T_{ab}(x^a)\ell^a \ell^b,$$  \hspace{1cm} \text{(31)}

where

$$x^a = (2\theta^0 + \theta^1)\ell^a + \theta^0 \delta \ell^a + \theta^- \delta \ell^a - \theta^+ \delta \ell^a,$$  \hspace{1cm} \text{(32)}

where $\ell^a$ is given by (22). In this case, the Einstein equation (15a) splits into two hierarchical equations, one for the zeroth-order term $\Omega_0$ and the other one for the first-order term $\Omega$:

$$\begin{align*}
(\Omega_0)_{11} &= 0, \\ \Omega_{11} &= T \Omega_0. 
\end{align*}$$ \hspace{1cm} \text{(33a)} \hspace{1cm} \text{(33b)}

It has been shown that in the asymptotically flat regime, one can take $\Omega_0 = 1$ consistently with Eq. (15b) up to first order. Thus we take $\Omega_0 = 1$ and the field equations (15) become

$$\begin{align*}
\Omega_{11} &= T, \\ \tilde{\partial} \Omega &= \frac{1}{2} \left( \Lambda_{++} + \frac{1}{2} \tilde{\partial}(\Lambda_{11}) \right), \\ 2\Lambda_{--} - \tilde{\partial}(\Lambda_{11}) &= 0.
\end{align*}$$ \hspace{1cm} \text{(34a)} \hspace{1cm} \text{(34b)} \hspace{1cm} \text{(34c)}

where we have used the fact that also in this case the spacetime can be represented as a small perturbation off flat space, in which case $\Lambda$ is of first-order in $\epsilon$. By virtue of Eq. (34a), the conformal factor $\Omega$ can be obtained from the source by quadratures:

$$\Omega = 1 - \int_{\theta^0}^{\infty} \int_{\theta^1}^{\infty} T \, d\theta^{11} \, d\theta^{1\nu}$$  \hspace{1cm} \text{(35)}

where the upper limit of the integral has been chosen at $\theta^1 = \infty$, the null boundary of the asymptotically flat spacetime. Thus $\Omega$ can be considered as a given source of Eq. (34a). Equations (34b) and (34c) can be manipulated through algebra, differentiation and integration, to yield

$$\tilde{\partial}^2 \Lambda = \int_{\theta^0}^{\infty} \left( 3\tilde{\partial}^2 \int_{\theta^1}^{\infty} (\tilde{\partial} \Omega)_{-1} d\theta^{11} + 3\tilde{\partial} \int_{\theta^1}^{\infty} (\tilde{\partial} \Omega)_{++} d\theta^{11} - 2\tilde{\partial} \Omega (2\Omega - \tilde{\partial} \Omega) \right) d\theta^0. $$ \hspace{1cm} \text{(36)}

The upper limit $\theta^0 = \infty$ represents the point $i^0$ at the null boundary of an asymptotically flat spacetime (timelike infinity).

Using (12) and (21), Eq. (34a) can be rewritten as an equation for $Z$:

$$\tilde{\partial}^2 \Omega^2 Z = \int_{\theta^0}^{\infty} \left( 3\tilde{\partial}^2 \int_{\theta^1}^{\infty} (\tilde{\partial} \Omega)_{-1} d\theta^{11} + 3\tilde{\partial} \int_{\theta^1}^{\infty} (\tilde{\partial} \Omega)_{++} d\theta^{11} - 2\tilde{\partial} \Omega (2\Omega - \tilde{\partial} \Omega) \right) d\theta^0. $$ \hspace{1cm} \text{(37)}
This is a Poisson-type equation, since the operator in the left-hand side is the double Laplacian on the sphere \((\zeta, \tilde{\zeta})\), and the right-hand side is a given source, by virtue of \([33]\). Again, since \(Z\) leads to the Fermat potential \(G\) in a trivial manner, we consider this the dynamical equation for the Fermat potential. This equation thus plays a role analogous to \([3]\) and generalizes Eq. \([3]\) by removing the approximations of thin lenses and small angles.

We can now proceed to write down the lens mapping explicitly. We start by writing down the solution of \([33]\) in terms of the source in the right-hand side, by means of the Green function \(F(\zeta, \tilde{\zeta}, \zeta', \tilde{\zeta}')\) given by Eq. \([28]\):

\[
Z(x^a, \zeta, \tilde{\zeta}) = x^a \ell_a + \frac{1}{4\pi} \int_{S^2} S(x^a, \zeta', \tilde{\zeta}') \ell_a^b \ell^c_s \ln(\ell_a^a \ell^c_s) dS^{2'},
\]

with

\[
S(x^a, \zeta, \tilde{\zeta}) = \int_{x^a \ell_a}^{\infty} \left( 3\Delta^2 \int_{x^a \ell_a}^{\infty} (\bar{\Omega})_+ d0^{1'} + 3\Delta^2 \int_{x^a \ell_a}^{\infty} (\bar{\Omega})_+ d0^{1'} - 2\bar{\Omega}(2\Omega - \bar{\Omega}) \right) d0^{1'}.
\]

We can now construct the Fermat potential \(G = Z(x^a, \zeta, \tilde{\zeta}) - Z(x_0^a, \zeta, \tilde{\zeta})\) and write down the two envelope conditions \(\partial G = \bar{\Omega} = 0\), which then form the take the form

\[
0 = (x^a - x_0^a) \partial_0 \ell_a + \frac{1}{4\pi} \int_{S^2} S(x^a, \zeta', \tilde{\zeta}') \partial_0 [\ell_a^b \ell^c_s \ln(\ell_a^a \ell^c_s)] dS^{2'}.
\]

This equation, with \(Z(x^a, \zeta, \tilde{\zeta}) - Z(x_0^a, \zeta, \tilde{\zeta}) = 0\), is a generalization of the astrophysical lens equation and time delay that removes the thin-lens and small-angle approximations. As in the previous subsection, there is no lens plane, and the mapping takes points on the observer’s celestial sphere into the source’s spatial locations. To be more specific, we can choose the coordinates so that the observer lies at the origin \(x_0^a = (\tau, 0, 0, 0)\). The stress-energy tensor \(T_{ab}\) of a given matter configuration on a flat background is assumed to be given, perhaps with some model in mind. The quantity \(T = T_{ab} \hat{\ell}^a \hat{\ell}^b\) is constructed, and is used as the source of the quadrature for the quantity \(\Omega\) via Eq. \([33]\). The quantity \(\Omega\) thus obtained is used to obtain \(S\) according to \([33]\). The combined equations \(Z(x^a, \zeta, \tilde{\zeta}) - Z(x_0^a, \zeta, \tilde{\zeta}) = 0\) and \([40]\) provide three equations for the four variables \(x^a\). The values of \(x^a\) that satisfy the three equations lie on the observer’s lightcone. One of the four variables \(x^a\) remains free, and is assumed to be given in terms of some observed physical distance. The remaining three variables represent the angular location of the source and the time of arrival of the light signal.

It may be complicated to write down the lens mapping in closed form, even if \(T_{ab}\) is given explicitly. However, one can envision a physical situation that is close to a thin lens, in which case we would expect this scheme to return a kind of “post-thin” lens equation. In other words, it is worth considering the possibility of obtaining corrections to the thin-lens approach by means of this scheme. The effect of the thickness of the lens has only rarely been considered in the literature \([13, 17]\).

IV. REMARKS AND OUTLOOK

By setting up a framework within the null–surface formulation to discuss the dynamics of Fermat potentials for gravitational lensing in full generality, we have achieved two main goals, as argued in the following.

In the first place, we have provided two lens equations that generalize the astrophysical approach to lensing in two ways: 1.) both lens equations remove the approximations of thin lenses and small angles; and 2.) the lens equations include the case of lensing by gravitational waves. Even though the exceeding complexity of the case of non-vanishing matter sources – Eq. \([11]\) – puts its applicability into question, the point of our work is to show that, in principle, the thin-lens approximation can be removed in a meaningful manner. On the other hand, the obvious simplicity of the case of vanishing matter sources – Eq. \([43]\) – opens a door to the phenomenon of lensing by gravitational waves, which has occasionally been treated before, within the framework of the thin-lens approximation \([15]\) and in a statistical framework \([19]\). In principle, both cases could be combined in a straightforward manner, simply by adding their respective \(Z\) functions, therefore the spectrum of many possible applications to weak fields is opened in this work.

One might question the necessity of the use of any type of Fermat’s principle when one moves away from the thin-lens approximation on the basis that the direct integration of the null geodesics on a linearized perturbation off flat space would clearly yield a lens map. In \([10]\), for instance, the equations for the null geodesics of a linearized perturbation off flat space are written down explicitly and integrated to obtain the bending angle of a moving thin lens, whereas in \([4]\) the direct integration of the null geodesics of a perturbation off an isotropic cosmology is used to derive the cosmological thin-lens equation without the use of broken paths. Still, so far as we are aware, very rarely if ever has the direct integration of the null geodesics led to a scheme in any way comparable with the thin-lens
scheme in practicality. Ideally, one would like to have a correction to the thin-lens scheme that represents the effects of the width of the lens and an estimate of the regime in which such corrections cannot be neglected. One can easily argue that the approach that we present here may have the potential to make a contribution towards such a goal and deserves to be pursued. The usefulness of our approach must first be tested in particular cases where results are already available. In this respect, the thin-lens scheme has been easily reproduced in our approach, including the case of moving lens planes. It would be of great interest to be able to relate our lens equations to existing results involving lensing by gravitational waves, such as, among others. The reader should be aware that, because of the statistical nature of the problem considered in, our equation cannot be compared directly with results in that article.

One should keep in mind, as well, that the treatment in this work is limited in at least two senses. We have only considered weak fields as perturbations off flat space. Therefore, the application to isotropic cosmologies is not covered here. There are reasons to think that the scheme developed here can be adapted to isotropic cosmologies with little effort. Work on this line is in progress and will be reported elsewhere. Secondly, we have only considered a coordinate-based treatment of lensing, as opposed to an optical-distance treatment. By virtue of this we have stayed away from a great technical and practical difficulty in observational lensing, which is the use of a meaningful observable distance to the source. We have bypassed this difficulty simply by using abstract coordinates. The use of an observable coordinate distance to the source would greatly complicate the lens equations obtained here in form, but it would not affect their underlying content.

In second place, we have here demonstrated that the null-surface formulation of general relativity has the potential for physical applications of interest, in spite of its apparent level of abstraction.

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