GENERALIZED LATTICE POINT VISIBILITY IN $\mathbb{N}^k$

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Abstract. A lattice point $(r, s) \in \mathbb{N}^2$ is said to be visible from the origin if no other integer lattice point lies on the line segment joining the origin and $(r, s)$. It is a well-known result that the proportion of lattice points visible from the origin is given by $\frac{1}{\zeta(2)}$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ denotes the Riemann zeta function. Goins, Harris, Kubik and Mbirika, generalized the notion of lattice point visibility by saying that for a fixed $b \in \mathbb{N}$, a lattice point $(r, s) \in \mathbb{N}^2$ is $b$-visible from the origin if no other lattice point lies on the graph of a function $f(x) = mx^b$, for some $m \in \mathbb{Q}$, between the origin and $(r, s)$. In their analysis they establish that for a fixed $b \in \mathbb{N}$, the proportion of $b$-visible lattice points is $\frac{1}{\zeta(b+1)}$, which generalizes the result in the classical lattice point visibility setting. In this short note we give an $n$-dimensional notion of $b$-visibility that recovers the one presented by Goins et. al. in 2-dimensions, and the classical notion in $n$-dimensions. We prove that for a fixed $b = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ the proportion of $b$-visible lattice points is given by $\frac{1}{\zeta(\sum_{i=1}^{n} b_i)}$.

Moreover, we propose a $b$-visibility notion for vectors $b \in \mathbb{Q}_{>0}^n$, and we show that by imposing weak conditions on those vectors one obtains that the density of $b = (b_1 a_1, b_2 a_2, \ldots, b_n a_n) \in \mathbb{Q}_{>0}^n$-visible points is $\frac{1}{\zeta(\sum_{i=1}^{n} b_i)}$. Finally, we give a notion of visibility for vectors $b \in (\mathbb{Q}^*)^n$, compatible with the previous notion, that recovers the results of Harris and Omar for $b \in \mathbb{Q}^*$ in 2-dimensions; and show that the proportion of $b$-visible points in this case only depends on the negative entries of $b$.

1. Introduction

In classical lattice point visibility, a point $(r, s)$ in the integer lattice $\mathbb{Z}^2$ is said to be visible (from the origin) if the line segment joining the origin $(0, 0)$ and the point $(r, s)$ contains no other integer lattice points. One well-known result is that the proportion of visible lattice points in $\mathbb{Z}^2$ is given by $1/\zeta(2)$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ denotes the Riemann zeta function [1]. In fact, a similar argument establishes that for $k \geq 2$ the proportion of visible lattice points in $\mathbb{Z}^k$ (analogously defined) is given by $1/\zeta(k)$ [3].

In 2017, Goins, Harris, Kubik and Mbirika generalized the classical definition of lattice point visibility by fixing $b \in \mathbb{N}$ and considering curves of the form $f(x) = ax^b$ with $a \in \mathbb{Q}$, see [4]. In this new setting, a lattice point $(r, s) \in \mathbb{N}^2$ is said to be $b$-visible if it lies on the graph of a curve of the form $f(x) = ax^b$ with $a \in \mathbb{Q}$ and there are no other integer lattice points lying on this curve between $(0, 0)$ and $(r, s)$. Hence, setting $b = 1$ recovers the classical definition of lattice point visibility. In the $b$-visibility setting, Goins et. al. established that the proportion of $b$-visible lattice points in $\mathbb{N}^2$ is given by $\frac{1}{\zeta(b+1)}$. Harris and Omar expanded this work to power functions with rational exponents by establishing that the proportion of
\( (b/a) \)-visible lattice points in \( \mathbb{N}_a^2 \) (the set nonnegative integers that are \( a \)th powers) is given by \( \frac{1}{\zeta(b+1)} \), and the proportion of \((-b/a)\)-visible lattice points in \( \mathbb{N}_a^2 \) is given by \( \frac{1}{\zeta(b)} \).

In this work we extend the notion of visibility to lattice points in \( \mathbb{N}^k \).

First we do it for the classical notion and the one developed my Goins et al, by fixing \( b = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k \), and saying that a lattice point \( \mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k \) is \( b \)-visible if there does not exists a positive real number \( 0 < t < 1 \) such that \((n_1t^{b_1}, n_2t^{b_2}, \ldots, n_kt^{b_k}) \) is a lattice point in \( \mathbb{N}^k \) (Definition 2,3). Using this definition, we prove the following result regarding the proportion of \( b \)-visible points in \( \mathbb{N}^k \).

**Theorem 1.** Fix \( k \in \mathbb{N} \) and \( b = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k \), with \( b \) satisfying the condition that \( \gcd(b_1, b_2, \ldots, b_k) = 1 \). Then the proportion of points \( \mathbf{n} \in \mathbb{N}^k \) that are \( b \)-visible is \( \frac{1}{\zeta(\sum_{i=1}^k b_i)} \).

For an \( k \in \mathbb{N} \), note that setting \( b = (1,1,\ldots,1) \in \mathbb{N}^k \) in Theorem 1 recovers the proportionality result in classical lattice point visibility.

To state our next result, we consider \( a_1, a_2, \ldots, a_k \in \mathbb{N} \) and define \( \alpha := \text{lcm}(a_1, \ldots, a_k) \). Also for each \( 1 \leq i \leq k \) we let \( \mathbb{N}_a \) denote the set of integers of the form \( \ell n_i \) with \( \ell \in \mathbb{N} \), and we say that \( \gcd(b) = 1 \) if there is an integer linear combination of the entries of \( b \) equal to 1.

**Theorem 2.** Fix \( k \in \mathbb{N} \) and \( b = (b_1, b_2, \ldots, b_k) \in \mathbb{Q}_{>0}^k \), with \( b \) satisfying the conditions that \( a_1, a_2, \ldots, a_k \in \mathbb{N} \) and \( \gcd(a_1, a_2, \ldots, a_k) = 1 \). Then the proportion of points in \( \mathbb{N}_{a_{1}} \times \cdots \times \mathbb{N}_{a_{n}} \) that are \( b \)-visible is \( \frac{1}{\zeta(\sum_{i=1}^k b_i)} \).

**Theorem 3.** Let \( b = (b_1, \ldots, b_k) \in (\mathbb{Q}^*)^k \) be such that its negative entries are indexed by the set \( J \subseteq [k] \), with \( b \) satisfying the conditions that \( a_1, a_2, \ldots, a_k \in \mathbb{N} \) and \( \gcd(b_1, b_2, \ldots, b_k) = 1 \). Then the proportion of points in \( \mathbb{N}_{a_{1}} \times \cdots \times \mathbb{N}_{a_{n}} \) that are \( b \)-visible is \( \frac{1}{\zeta(\sum_{j \in J} |b_j|)} \).

Note that setting \( k = 2 \) and \( b = (1, b) \in \mathbb{Q}^2 \) in Theorems 2 and 3 recovers the analogous results in the \( b \)-visibility setting as considered by Harris and Omar [3], thereby extending them to \( n \)-dimensions.

We organize this work as follows: Section 2 is separated into two parts. The first considers \( b \in \mathbb{N}^k \) and provides the definition of \( b \)-visibility along with a number-theoretic characterization of what it means for a lattice point to be \( b \)-visible. The second part extends this definition and provides an analogous number-theoretic result in the case where \( b \in \mathbb{Q}^k \). Section 3 contains the results on the proportion of lattice points that are \( b \)-visible, work which establishes our main results (Theorem 1,3).

## 2. Background

Goins et. al. gave the following definition in [4]: Fix \( b \in \mathbb{N} \), then a lattice point \((r, s) \in \mathbb{N}^2 \) is said to be \( b \)-visible if it lies on the graph of a curve of the form \( f(x) = ax^b \) with \( a \in \mathbb{Q} \) and there does not exist \((r', s') \in \mathbb{N}^2 \) on \( f \) with \( 0 < r' < r \). This definition of \( b \)-visibility is dependent on a power function \( f(x) \) on which the lattice point \((r, s) \) lies. However, we start by presenting an equivalent definition which is a parametrized version of the \( b \)-visibility and bypasses the need for such a function \( f \).
Definition 2.1. Fix $b \in \mathbb{N}$. A lattice point $(r, s) \in \mathbb{N}^2$ is said to be $b$-visible if there does not exist a real number $0 < t < 1$ such that $(rt, st) \in \mathbb{N}^2$.

Definition 2.1 will be the key in defining $b$-visibility to $\mathbb{N}^k$, for $b \in \mathbb{Q}^k$. In order to stay consistent with known literature on lattice point visibility, we begin next section by stating a general definition of $b$-visibility which depends on lattice points lying on the graph of certain real-valued functions. We then show that this definition of $b$-visibility is in fact independent of what function the point lies on. This leads naturally to a definition of lattice point $b$-visibility relying solely on a number-theoretic description of the lattice point $n \in \mathbb{N}^k$. We separate the remainder of this section into three cases: $b \in \mathbb{N}^k$, $b \in \mathbb{Q}^k_{>0}$, and the case where $b \in \mathbb{Q}^k$.

2.1. On $b$-visible lattice points, with $b \in \mathbb{N}^k$. In this section we consider $b \in \mathbb{N}^k$ and begin by presenting a definition of $b$-visibility, which depends on a lattice point lying on a certain real-valued function.

Definition 2.2. Fix $b = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$ and define

$$\mathcal{F}(b) := \{ f : \mathbb{R} \to \mathbb{R}^k \mid f(t) = (m_1 t^{b_1}, m_2 t^{b_2}, \ldots, m_k t^{b_k}) \text{ where } m_1, m_2, \ldots, m_k \in \mathbb{N} \}.$$ 

If $n = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ and there exists $f \in \mathcal{F}(b)$ such that

1. $f(t) = n$ for some $t \in \mathbb{R}_{>0}$ and
2. there does not exist $0 < t' < t$ such that $f(t') \in \mathbb{N}^k$

then $n$ is said to be $b$-visible with respect to $f$. If Condition (1) is satisfied, but (2) is not, then we say that $n$ is $b$-invisible with respect to $f$.

In order to illustrate Definition 2.2 we present the following example.

Example 1. The point $(4, 16, 40, 128)$ lies on the graph of $f(t) = (4t^2, 16t^4, 40t^3, 128t^7)$ since $f(1) = (4, 16, 40, 128)$, but it is not $(2, 4, 3, 7)$-visible, because

$$f \left( \frac{1}{2} \right) = \left( 4 \left( \frac{1}{2} \right)^2, 16 \left( \frac{1}{2} \right)^4, 40 \left( \frac{1}{2} \right)^3, 128 \left( \frac{1}{2} \right)^7 \right) = (1, 1, 5, 1) \in \mathbb{N}^4.$$

However, $(1, 1, 5, 1)$ is $(2, 4, 3, 7)$-visible since it lies on the curve $g(t) = (t^2, t^4, 5t^3, t^7)$ as $g(1) = (1, 1, 5, 1)$ and there does not exist $0 < t' < 1$ such that $g(t') \in \mathbb{N}^4$.

We remark that the classical notion of visibility in $\mathbb{N}^k$ is the particular case of taking $b = (1, \ldots, 1) \in \mathbb{N}^k$ in Definition 2.2 while $b$-visibility corresponds to taking $b = (1, b)$. For a fixed $b \in \mathbb{N}^k$ we now show that the $b$-visibility of the lattice point $n \in \mathbb{N}^k$ is independent of the choice of function $f$ satisfying Definition 2.2. This is the content of the following result.

Lemma 1. Fix $b = (b_1, \ldots, b_k)$, $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and suppose that $n$ is $b$-visible with respect to $f \in \mathcal{F}(b)$. If $g \in \mathcal{F}(b)$ and $g(r) = n$ for some $r \in \mathbb{R}_{>0}$, then $n$ is $b$-visible with respect to $g$.

Proof. By hypothesis $f(s) = (m_1 s^{b_1}, \ldots, m_k s^{b_k}) = n$ for some $s \in \mathbb{R}_{>0}$ and $m_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Likewise, $g(r) = (u_1 r^{b_1}, \ldots, u_k r^{b_k}) = n$ for some $r \in \mathbb{R}_{>0}$ and $u_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Thus $(m_1 s^{b_1}, \ldots, m_k s^{b_k}) = (u_1 t^{b_1}, \ldots, u_k t^{b_k})$, which implies $m_i (s/r)^{b_i} = u_i$ for each $1 \leq i \leq k$. If there exists $0 < t < r$ such that $g(t) = (v_1, \ldots, v_k) \in \mathbb{N}^k$ then for each $1 \leq i \leq k$ we have

$$v_i = u_i t^{b_i} = m_i (s/r)^{b_i} t^{b_i} = m_i (st/r)^{b_i}.$$
Since $t < r$ then $st/r < s$, and thus $f(st/r) \in \mathbb{N}^k$, which contradicts the fact that $n$ is $b$-visible with respect to $f$. Hence, such a $t$ can not exists and we conclude that $n$ is $b$-visible with respect to $g$. □

Note that for any $b = (b_1, b_2, \ldots, b_k)$ and $n = (n_1, n_2, \ldots, n_k)$ in $\mathbb{N}^k$, $n$ lies on the curve $f(t) = (n_1 t^{b_1}, n_2 t^{b_2}, \ldots, n_k t^{b_k}) \in F(b)$ since $f(1) = n$. Thus, by Lemma\ref{lemma:visibility} the $b$-visibility of $n$ can be reduced to determining the existence of $0 < t < 1$ for which $f(t) \in \mathbb{N}^k$. Hence, from now on we refer to the $b$-visibility of a lattice point $n$ with no reference to a specific function and we use the following definition of $b$-visibility, whenever $b \in \mathbb{N}^k$.

**Definition 2.3.** Fix $b = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$. Then the lattice point $n = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ is said to be $b$-visible if there does not exist a positive real number $0 < t < 1$ such that $(n_1 t^{b_1}, n_2 t^{b_2}, \ldots, n_k t^{b_k}) \in \mathbb{N}^k$.

We would like to have a number theoretic characterization of the points that are $b$-visible according to the above definition; and one would expect it to be almost identical to the characterization given by Goins et. al. However, a new obstacle arises in this context:

For the former notions of visibility, if a point $n = (n_1, n_2) \in \mathbb{N}^2$ was not $b$-visible then there had to exist a rational number $t < 1$ such that $(tn_1, tn_2) \in \mathbb{N}^2$ and $tn_1 = \frac{n_1}{n_2}(tn_2)^b$. This does not need to be the case now. Take for instance the point $(2, 4) \in \mathbb{N}^2$. The only point with integer coordinates $(2t^2, 4t^4)$ for $0 < t < 1$ is $(1, 1)$ which is obtained by making $t = \frac{1}{\sqrt{2}}$. Hence, in accordance with our definition, $(2, 4)$ is not $(2, 4)$-visible, but this can only be noticed if $t$ is allowed to be an irrational number.

This can be avoided if $b$ satisfies $\gcd(b) = 1$ where $\gcd(b)$ is the greatest common divisor of the entries of $b$. Indeed, in that case there must exist integers satisfying $m_1 b_1 + m_2 b_2 + \cdots + m_k b_k = 1$. So, by hypothesis one has that $(n_1 t^{b_1}, n_2 t^{b_2}, \ldots, n_k t^{b_k}) \in \mathbb{N}^k$, and then:

$$(n_1 t^{b_1})^{m_1} \cdots (n_k t^{b_k})^{m_k} = (n_1^{m_1}) \cdots (n_k^{m_k}) t^{m_1 b_1 + m_2 b_2 + \cdots + m_k b_k} = (n_1^{m_1}) \cdots (n_k^{m_k}) t \in \mathbb{Q}.$$  

Thus $t \in \mathbb{Q}$, and we only need to worry about the case $\gcd(b) > 1$.

**Lemma 2.** A point $n \in \mathbb{N}^k$ is $b$-visible if and only if it is $(\frac{1}{\gcd(b)} b)$-visible.

**Proof.** Suppose that $n$ is not $b$-visible. Then there exists $0 < t < 1$ with $(n_1 t^{b_1}, n_2 t^{b_2}, \ldots, n_k t^{b_k}) \in \mathbb{N}^k$. As a result $t^{\gcd(b)} \in \mathbb{Q}$ by an argument analogous to the one above. But then:

$$(n_1 t^{\gcd(b)})^{\frac{b_1}{\gcd(b)}}, n_2 t^{\gcd(b)} \frac{b_2}{\gcd(b)} \cdots, n_k t^{\gcd(b)} \frac{b_k}{\gcd(b)} \in \mathbb{N}^k$$

Where $t^{\gcd(b)} < 1$ because $t < 1$. So $n$ is not $(\frac{1}{\gcd(b)} b)$-visible.

For the converse direction, if $(n_1 t)\frac{b_1}{\gcd(b)}, n_2 (t)\frac{b_2}{\gcd(b)}, \ldots, n_k (t)\frac{b_k}{\gcd(b)} \in \mathbb{N}^k$, for $0 < t < 1$, then taking $t' = t^{\frac{1}{\gcd(b)}}$ evinces that $n$ is not $b$ visible either. □

In light of the foregoing, we will only study $b$-visibility for $\gcd(b) = 1$.

**Theorem 4.** Fix $b = (b_1, \ldots, b_k) \in \mathbb{N}^k$ satisfying $\gcd(b) = 1$. Then the lattice point $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ is $b$-visible if and only if there does not exist a prime $p$, such that $p^{b_i} | n_i$ for all $i = 1, 2, \ldots, k$. 

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Proof. Suppose that \( \mathbf{n} \) is \( \mathbf{b} \)-visible. If there exists a prime \( p \) satisfying \( p^{b_i} | n_i \) for all \( i = 1, \ldots, k \), then the point \( \mathbf{n}' = (n_1(\frac{1}{p})^{b_1}, n_2(\frac{1}{p})^{b_2}, \ldots, n_k(\frac{1}{p})^{b_k}) \) is an integer lattice point. Since \( \frac{1}{p} < 1 \), it follows that \( \mathbf{n} \) is not \( \mathbf{b} \)-visible, which is a contradiction.

On the other hand let us suppose that \( \mathbf{n} \) is not \( \mathbf{b} \)-visible. Then there exists \( 0 < t < 1 \) such that \( (n_1 t^{b_1}, n_2 t^{b_2}, \ldots, n_k t^{b_k}) \in \mathbb{N}^k \). Since \( \gcd(\mathbf{b}) = 1 \) and \( t \in \mathbb{Q} \), we can take \( a \) and \( c \) with \( \gcd(a, c) = 1 \), and let \( t = \frac{a}{c} \). Then \( c^{b_i} | n_i \) for all \( i = 1, \ldots, k \). Thus any prime factor \( p \) of \( c \) yields the desired result.

\[ \square \]

**Definition 2.4.** Let \( \mathbf{b}, \mathbf{n} \in \mathbb{N}^k \). We say that \( \mathbf{n} \) is \( \mathbf{b} \)-relatively prime if there does not exist a prime \( p \) such that \( p^{b_i} | n_i \) for all \( i = 1, \ldots, k \).

In light of Definition 2.4 we can restate Theorem 4 as follows.

**Corollary 1.** Fix \( \mathbf{b} \in \mathbb{N}^k \) satisfying \( \gcd(\mathbf{b}) = 1 \). Then the lattice point \( \mathbf{n} \in \mathbb{N}^k \) is \( \mathbf{b} \)-visible if and only if \( \mathbf{n} \) is \( \mathbf{b} \)-relatively prime.

Note that Theorem 4 and Corollary 1 present number-theoretic characterizations of \( \mathbf{b} \)-visible lattice points whenever \( \mathbf{b} \in \mathbb{N}^k \).

2.2. **On \( \mathbf{b} \)-visible lattice points, with \( \mathbf{b} \in \mathbb{Q}^k \).** We now extend the definition of \( \mathbf{b} \)-visibility to the case where \( \mathbf{b} \in \mathbb{Q}^k \), and present number-theoretic results analogous to Theorem 4 and Corollary 1. These results generalize the \( \mathbf{b} \)-visibility proportionality results for \( \mathbf{b} \in \mathbb{Q}^k \) obtained by Harris and Omar in [6] to lattice points in \( \mathbb{N}^k \).

In what follows, we begin by considering \( \mathbf{b} \in \mathbb{Q}_{>0}^k \), i.e. \( \mathbf{b} \) with all positive rational entries, and later we consider the case where \( \mathbf{b} \) has some negative entries.

**Definition 2.5.** Fix \( \mathbf{b} = (\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}) \in \mathbb{Q}_{>0}^k \) and suppose that \( \mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k \) lies on the curve

\[
\mathbf{f}(t) = (m_1 t^{\frac{b_1}{a_1}}, m_2 t^{\frac{b_2}{a_2}}, \ldots, m_k t^{\frac{b_k}{a_k}})
\]

for some \( \mathbf{m} = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k \). Then the point \( \mathbf{n} \) is said to be \( \mathbf{b} \)-visible (with respect to \( \mathbf{f} \)) if there does not exist another point in \( \mathbb{N}^k \) on the graph of \( \mathbf{f}(t) \) lying between the origin and \( \mathbf{n} \). If \( \mathbf{n} \) is not \( \mathbf{b} \)-visible, then we say \( \mathbf{n} \) is \( \mathbf{b} \)-invisible (with respect to \( \mathbf{f} \)).

As before, we note that for any \( \mathbf{b} = (\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}) \in \mathbb{Q}_{>0}^k \) the lattice point

\[
\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k
\]

lies on the curve \( \mathbf{f}(t) = (n_1 t^{\frac{b_1}{a_1}}, n_2 t^{\frac{b_2}{a_2}}, \ldots, n_k t^{\frac{b_k}{a_k}}) \) since \( \mathbf{f}(1) = \mathbf{n} \). Thus, for \( \mathbf{b} \in \mathbb{Q}_{>0}^k \), the \( \mathbf{b} \)-visibility of \( \mathbf{n} \) can again be reduced to determining the existence of a real number \( 0 < t < 1 \) for which \( \mathbf{f}(t) \in \mathbb{N}^k \). Moreover, note that the statement and proof of Lemma 1 hold when \( \mathbf{b} \in \mathbb{Q}_{>0}^k \). Thus, the \( \mathbf{b} \)-visibility of \( \mathbf{n} \in \mathbb{N}^k \) is independent of the function \( \mathbf{f} \) when \( \mathbf{b} \in \mathbb{Q}_{>0}^k \).

For \( a_1, a_2, \ldots, a_k \in \mathbb{N} \) we let \( \alpha := \text{lcm}(a_1, \ldots, a_k) \) and for each \( 1 \leq i \leq k \) we let \( \mathbb{N}^k_{\frac{\alpha}{a_i}} \) denote the set of integers of the form \( \ell \frac{\alpha}{a_i} \) with \( \ell \in \mathbb{N} \). We now state and prove the following technical result.

**Lemma 3.** Let \( \mathbf{b} = (\frac{b_1}{a_1}, \ldots, \frac{b_k}{a_k}) \in \mathbb{Q}_{>0}^k \). Then the \( \mathbf{b} \)-visibility of \( \mathbf{n} \in \mathbb{N}^k \) with respect to the family of functions \( \mathbf{f}(t) = (m_1 t^{\frac{b_1}{a_1}}, \ldots, m_k t^{\frac{b_k}{a_k}}) \) with \( (m_1, \ldots, m_k) \in \mathbb{N}^k \) is equivalent to the \( \mathbf{b} \)-visibility of \( \mathbf{n} \in \mathbb{N}^k \) with respect to the family of functions \( \mathbf{g}(t) = (m_1 t^{\frac{b_1}{a_1}}, \ldots, m_k t^{\frac{b_k}{a_k}}) \) with \( (m_1, \ldots, m_k) \in \mathbb{N}_{\frac{\alpha}{a_1}}^k \times \ldots \times \mathbb{N}_{\frac{\alpha}{a_k}}^k \).
Proof. Let \( n \in \mathbb{N}_{a_1} \times \ldots \times \mathbb{N}_{a_k} \), and suppose that \( n \) is not \( b \)-visible with respect to \( f(t) = (m_1 t^{b_1}, \ldots, m_k t^{b_k}) \) with \((m_1, \ldots, m_k) \in \mathbb{N}^k\). Then there exist \( t', t'' \in \mathbb{R} \) with \( t'' < t' \), and \( f(t'') \in \mathbb{N}_{a_1} \times \ldots \times \mathbb{N}_{a_k} \), \( f(t') = n \). But then

\[
\left( m_1(t')^{\frac{b_1}{a_1}} \left( \frac{t''}{t'} \right)^{\frac{b_1}{a_1}}, \ldots, m_k(t')^{\frac{b_k}{a_k}} \left( \frac{t''}{t'} \right)^{\frac{b_k}{a_k}} \right) = \left( n_1 \left( \frac{t''}{t'} \right)^{\frac{b_1}{a_1}}, \ldots, n_k \left( \frac{t''}{t'} \right)^{\frac{b_k}{a_k}} \right).
\]

And recall that \( n \in \mathbb{N}_{a_1} \times \ldots \times \mathbb{N}_{a_k} \), so the right side of the last equation is equal to \( g(t'') \) for a function \( g(t) \) belonging to the second family of functions, and because \( t'' < t' \) we have \( \frac{t''}{t'} < 1 \), so it is also not visible with respect to \( g(t) \).

The other direction is clear because the second family of functions \( g(t) \) is included in the family of functions \( f(t) \).

Note that Lemma 3 allows us to restrict our \( b \)-visibility study to lattice points \( n \in \mathbb{N}_{a_1} \times \ldots \times \mathbb{N}_{a_k} \). In order to present an analogous result to Theorem 4 we need the following technical result.

**Lemma 4.** Let \( b, c \in \mathbb{N} \) with \( \gcd(b, c) = 1 \). If \( t \in \mathbb{Q} \) and \( t^\frac{b}{c} \in \mathbb{Q} \), then \( t^\frac{1}{c} \in \mathbb{Q} \).

**Proof.** Let \( t = \frac{r}{s} \), we can take \( r \) and \( s \) so that \( \gcd(r, s) = 1 \). Suppose by contradiction that \( t^\frac{b}{c} \notin \mathbb{Q} \). Then, necessarily \( r \) or \( s \) is not a perfect \( c \)-th power. Without loss of generality, assume \( r \) is not a perfect \( c \)-th power. Then \( r^b \) is also not a perfect \( c \)-th power since \( \gcd(b, c) = 1 \) by hypothesis. Hence \( r^\frac{b}{c} \notin \mathbb{Q} \). There are two cases for \( s \), either it is a perfect \( c \)-th power, or it is not.

In the first case \( s^\frac{1}{c} \in \mathbb{Q} \), so \( s^\frac{b}{c} \in \mathbb{Q} \), thus \( \frac{r^b}{s^\frac{b}{c}} = t^\frac{b}{c} \notin \mathbb{Q} \) contradicting the hypothesis. In the second case, both \( r^b \) and \( s^b \) are not perfect \( c \)-th powers (again because \( \gcd(b, c) = 1 \)). Let \( \alpha := r^b \), \( \beta := s^b \), it follows that \( \gcd(\alpha, \beta) = 1 \). If \( \frac{\alpha^\frac{1}{c}}{\beta^\frac{1}{c}} \in \mathbb{Q} \), then \((\frac{\alpha}{\beta})^\frac{b}{c} \in \mathbb{Q} \), hence \((\alpha \cdot \beta^{-1})^\frac{b}{c} \in \mathbb{Q} \), but we chose \( \alpha \) and \( \beta \) with \( \gcd(\alpha, \beta) = 1 \), thus \( \alpha \cdot \beta^{-1} \) can not be a perfect \( c \)-th power, a contradiction. \( \square \)

**Remark 1.** The proof of Lemma 4 also shows that if \((\frac{\alpha}{\beta})^\frac{b}{c} \in \mathbb{Q} \), for some \( \alpha, \beta \in \mathbb{N} \) with \( \gcd(\alpha, \beta) = 1 \), then both \( \alpha \) and \( \beta \) must be perfect \( c \)-th powers.

We are now ready to state our next result. Here we use \( \gcd(b) = 1 \) to mean that there is an integer linear combination of the entries of \( b \) that equals 1.

**Theorem 5.** Fix \( b = (\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}) \in \mathbb{Q}_{>0}^k \) with \( \gcd(b) = 1 \). Then the lattice point \( n = (\ell_1 \frac{a_1}{\alpha}, \ell_2 \frac{a_2}{\alpha}, \ldots, \ell_k \frac{a_k}{\alpha}) \in \mathbb{N}_{\frac{a_1}{\alpha}} \times \ldots \times \mathbb{N}_{\frac{a_k}{\alpha}} \) is \( b \)-visible if and only if \((\ell_1, \ell_2, \ldots, \ell_k)\) is \((b_1, b_2, \ldots, b_k)\)-visible.

**Proof.** Suppose that \( n \) is \( b \)-visible and recall \( \alpha := \text{lcm}(a_1, \ldots, a_k) \). If \((\ell_1, \ell_2, \ldots, \ell_k)\) is not \((b_1, b_2, \ldots, b_k)\)-visible, then there exists a prime \( p \) satisfying \( p^{b_i} | \ell_i \) for all \( i = 1, \ldots, k \) and
$(\ell_1 \cdot (\frac{1}{p})^{b_1}, \ldots, \ell_k \cdot (\frac{1}{p})^{b_k})$ is an integer lattice point. Hence, the point
\[
\left( \ell_1^{\frac{a_1}{a_1}}, \left( \frac{1}{p^{a_1}} \right)^{\frac{b_1}{a_1}}, \ldots, \ell_k^{\frac{a_k}{a_k}}, \left( \frac{1}{p^{a_k}} \right)^{\frac{b_k}{a_k}} \right) = \left( \left( \frac{\ell_1}{p^{b_1}} \right)^{\frac{1}{a_1}}, \ldots, \left( \frac{\ell_k}{p^{b_k}} \right)^{\frac{1}{a_k}} \right)
\]
belongs to the set $\mathbb{N}_{\frac{a_1}{a_1}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a_k}}$. Recall that being $b$-visible does not depend on the choice of $(m_1, \ldots, m_k)$. This together with the fact that $\frac{1}{p^{a_i}} < 1$, implies that $n$ is $b$-invisible, which is a contradiction.

Now suppose that $n$ is not $b$-visible. Then there exists $t < 1$ such that $(\ell_1^{\frac{a_1}{a_1}} t^{\frac{b_1}{a_1}}, \ldots, \ell_k^{\frac{a_k}{a_k}} t^{\frac{b_k}{a_k}}) \in \mathbb{N}_{\frac{a_1}{a_1}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a_k}}$. In particular, we have that $t^{\frac{b_i}{a_i}} \in \mathbb{Q}$ for all $i = 1, \ldots, k$, and since $\gcd(b) = 1$ it follows that $t \in \mathbb{Q}$. By Lemma 4, we have that $t^{\frac{b_i}{a_i}} \in \mathbb{Q}$ for $i = 1, \ldots, k$, and by Remark 1 $t = \frac{s}{d}$ for some $r, s \in \mathbb{N}$, where $r$ and $s$ are perfect $a_i$-th powers for all $i = 1, \ldots, k$.

Hence, $s = \prod_{j=1}^{d} p_j^{c_j}$ for some prime numbers $p_1, \ldots, p_{\ell}$, and integers $c_1, \ldots, c_{\ell}$ satisfying $a_i | c_j$ for all $1 \leq j \leq d$. Thus, for each $1 \leq j \leq d$, there exist $c'_j$ such that $c_j = a_i \cdot c'_j$. Thus, for all $i = 1, \ldots, k$, we have
\[
\left( \frac{1}{s} \right)^{\frac{b_i}{a_i}} = \left( \frac{1}{\prod_{j=1}^{d} p_j^{c_j}} \right)^{\frac{1}{a_i}} = \left( \frac{1}{\prod_{j=1}^{d} p_j^{c'_j}} \right)^{\frac{1}{a_i}}.
\]

Thus, $\ell_i^{\frac{a_i}{a_i}} \cdot t^{\frac{b_i}{a_i}} \in \mathbb{N}$ implies that
\[
\ell_i^{\frac{a_i}{a_i}} \cdot \left( \frac{1}{s} \right)^{\frac{b_i}{a_i}} = \left( \frac{\ell_i}{\prod_{j=1}^{d} p_j^{c'_j}} \right)^{\frac{1}{a_i}} \in \mathbb{N}.
\]

In particular, for any $j = 1, \ldots, d$, we have that $p_j^{b_j} | \ell_i$ for all $i = 1, \ldots, k$. Therefore, $(\ell_1, \ell_2, \ldots, \ell_k)$ is not $(b_1, b_2, \ldots, b_k)$-visible, as desired.

Finally, we give a definition for $b$-visibility allowing for negative rational exponents. Let $b \in (\mathbb{Q}^*)^k$, that is, a $k$-tuple whose entries might be positive or negative rationals, but not 0. This will allow us to generalize the work of Harris and Omar [6]. From now on, whenever we write a rational as $\frac{b_i}{a_i} \cdot \alpha$ is assumed to be positive.

**Definition 2.6 (b-visibility for $b \in (\mathbb{Q}^*)^k$).** Let $b = (b_1, \ldots, b_k) \in (\mathbb{Q}^*)^k$, and let $\alpha := \text{lcm}(a_1, \ldots, a_k)$. Then an integer lattice point $n = (m_1, \ldots, m_k) \in \mathbb{N}_{\frac{a_1}{a_1}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a_k}}$ is $b$-visible if the following conditions hold:

1. $n$ lies on the graph of $f(t) = (m_1 t^{\frac{b_1}{a_1}}, \ldots, m_k t^{\frac{b_k}{a_k}})$ for some $m = (m_1, m_2, \ldots, m_k) \in \mathbb{N}_{\frac{a_1}{a_1}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a_k}}$.
2. There does not exist $t' \in \mathbb{R}$ such that $0 < t' < t$, and $(m_1(t')^{\frac{b_1}{a_1}}, \ldots, m_k(t')^{\frac{b_k}{a_k}})$ is a point in $\mathbb{N}_{\frac{a_1}{a_1}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a_k}}$.

If condition (1) is satisfied, but condition (2) is not, then we say that the point $n$ is $b$-invisible.
Remark 2. Being $b$-visible does not depend on the choice of $(m_1, \ldots, m_k)$, just as in Definition 2.2.

Theorem 6. Let $b = (\frac{b_1}{a_1}, \ldots, \frac{b_k}{a_k}) \in (\mathbb{Q}^*)^k$ be such that its negative entries are indexed by the set $J \subseteq [k]$, with $b$ satisfying $\gcd(b) = 1$, then the lattice point $n = (\ell_1 \frac{a_1}{a_i}, \ell_2 \frac{a_2}{a_i}, \ldots, \ell_k \frac{a_k}{a_i}) \in \mathbb{N}_{a_1} \times \cdots \times \mathbb{N}_{a_k}$ is $b$-visible if and only if there does not exist a prime $p$ such that $p^{b_j} | \ell_j$ for all $j \in J$.

Proof. Suppose that there exists a prime $p$ with $p^{b_j} | \ell_j$ for all $j \in J$. It follows that the $h$ entry of the point

$$\left(\ell_1 \frac{a_1}{a_h} \cdot (p^a)^{\frac{b_1}{a_h}}, \ell_2 \frac{a_2}{a_h} \cdot (p^a)^{\frac{b_2}{a_h}}, \ldots, \ell_k \frac{a_k}{a_h} \cdot (p^a)^{\frac{b_k}{a_h}}\right)$$

is $\ell_h \frac{a_h}{a_h} (p^a)^{\frac{b_h}{a_h}}$ if $h \notin J$ and $(\frac{\ell_h}{p^{b_h} a_h})^{a_h}$ if $h \in J$. Also the point in (1) belongs to the set $\mathbb{N}_{a_1} \times \cdots \times \mathbb{N}_{a_k}$ because all of the coordinates $h$ for which $h \notin J$ are products of integers, and the coordinates $h$ for which $h \in J$ are integer powers of integers, by assumption. Thus, for $p^a > 1$, $n$ is not $b$-visible.

The other direction is analogous to the argument presented in Theorem 5, restricted to the entries $h$ for which $h \in J$. \hfill \Box

Remark 3. Observe that the last theorem bears a close resemblance to Theorem 4, and it as if we had restricted the visibility of $n$ to the visibility of those entries of $n$ which are in $J$.

3. Proportion of $b$-visible lattice points

Given the definition and results in the previous section, we now present our main results. The arguments we make in the proof of Theorem 1 are a generalization of the arguments made by Pinsky in [10].

Theorem 1. Let $b \in \mathbb{N}^k$, such that $b = (b_1, \ldots, b_k)$ satisfies $\gcd(b) = 1$. Then the proportion of points in $\mathbb{N}^k$ that are $b$-visible is

$$\frac{1}{\zeta \left(\sum_{i=1}^{k} b_i\right)}$$

Proof. Let $N \in \mathbb{N}$ and set $[N] := \{1, \ldots, N\}$. We want to compute the proportion of points $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ which are $b$-visible. To this end, for each $i = 1, \ldots, k$, pick $n_i$ integers from the set $[N]$ with uniform probability. This results in a $k$-tuple $n \in [N]^k$. The associated probability space is $([N]^k, P_N)$, where $P_N$ is the uniform measure.

Fix $b = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$ and let $p$ be prime. If $C_{p,N}$ denotes the event that the prime $p$ satisfies $p^{b_i} | n_i$ for all $i = 1, \ldots, k$, then its probability is given by

$$P_N(C_{p,N}) = \frac{1}{N^k} \prod_{i=1}^{k} \left\lfloor \frac{N}{p^{b_i}} \right\rfloor$$
for \( p < N \) since there are \( \left\lfloor \frac{N}{p^i} \right\rfloor \) numbers divisible by \( p^i \) in \([N]\). We now establish that the events \( C_{p,N} \) are independent.

**Claim.** Let \( R = \prod_{i=1}^{r} p_i^{b_i} \), where \( p_i \) corresponds to the \( i \)-th prime (\( p_1 = 2, p_2 = 3, \ldots \)). If \( R | N \), then the events \( C_{p_i,N} \) are independent for \( i = 1, \ldots, r \).

**Proof of claim.** If \( p_i \) for \( i = 1, \ldots, r \) are distinct primes, and \( m \in [N] \) is arbitrary, then \( m \) can only be divisible by \( p_i^{b_i} \) for all \( i = 1, \ldots, r \) if \( m = \left( \prod_{j=1}^{r} p_j^{b_j} \right) l \), for some \( l \in \mathbb{N} \). Note that there must be \( \left\lfloor \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \right\rfloor \) such numbers \( m \in [N] \), and the events that \( \prod_{j=1}^{r} p_j^{b_j} | n_j \), and \( \prod_{j=1}^{r} p_j^{b_j} | n_k \) are independent for all \( j \neq k \). Hence

\[
\begin{align*}
P_N \left( \bigcap_{i=1}^{r} C_{p_i,N} \right) &= \frac{1}{N^k} \prod_{i=1}^{k} \left[ \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \right] \\
&= \prod_{j=1}^{k} \left[ \frac{1}{p_j^{b_j}} \right] \\
&= \prod_{j=1}^{r} P_N(C_{p_j,N})
\end{align*}
\]

Moreover, by hypothesis, \( R | N \), thus \( \left\lfloor \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \right\rfloor = \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \). The same argument using \( p_1, \ldots, p_k \), establishes that there are \( \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \) numbers in \([N]\) divisible by \( p_1^{b_1}, \ldots, p_k^{b_k} \). Hence

\[
\begin{align*}
P_N \left( \bigcap_{i=1}^{r} C_{p_i,N} \right) &= \frac{1}{N^k} \prod_{i=1}^{k} \left[ \frac{N}{\prod_{j=1}^{r} p_j^{b_j}} \right] \\
&= \frac{r}{\prod_{j=1}^{r} p_j^{b_j}} \\
&= \prod_{j=1}^{r} P_N(C_{p_j,N})
\end{align*}
\]

as we wanted to show.

We now return to the main proof. Clearly, for an arbitrary \( r \in \mathbb{N} \) and a fixed prime \( p_i \)

\[
\bigcap_{i=1}^{\infty} (C_{p_i,N})^c = \left( \bigcap_{i=1}^{r} (C_{p_i,N})^c \right) \setminus \left( \bigcup_{i=r}^{\infty} C_{p_i,N} \right)
\]

Since \( P_N \) is the uniform measure, it always takes finite values. Hence, by the relation in (7), and using the subadditivity property we obtain

\[
\begin{align*}
P_N \left( \bigcap_{j=1}^{r} (C_{p_j,N})^c \right) &= P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) + P_N \left( \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \cap \left( \bigcup_{i=r}^{\infty} C_{p_i,N} \right) \right)
\end{align*}
\]
rewriting \((8)\) yields

\[
P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) = P_N \left( \bigcap_{i=1}^{r} (C_{p_i,N})^c \right) - P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \cap \bigcap_{i=1}^{r} (C_{p_i,N})^c \right)
\]

and lastly

\[
P_N \left( \bigcap_{i=1}^{r} (C_{p_i,N})^c \right) - P_N \left( \bigcup_{i=1}^{\infty} C_{p_i,N} \right) \leq P_N \left( \bigcup_{i=1}^{\infty} (C_{p_i,N})^c \right) \leq \sum_{i=1}^{\infty} P_N(C_{p_i,N}).
\]

Furthermore, by \([2]\) we have that

\[
\sum_{i=1}^{\infty} P_N(C_{p_i,N}) = \sum_{i=1}^{\infty} \left( \frac{1}{N^k} \prod_{j=1}^{k} \left\lfloor \frac{N}{p_j} \right\rfloor \right) \leq \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^{k} p_j}.
\]

If we let \(R = \prod_{i=1}^{r} p_i^{b_i}\), and \(N\) is such that \(R \mid N\), we have by the previous claim that the events \(C_{p_i,N}\) are independent for \(i = 1, \ldots, r\). This implies that the events \((C_{p_i,N})^c\) are also independent. Then,

\[
P_N \left( \bigcap_{i=1}^{r} (C_{p_i,N})^c \right) = \prod_{i=1}^{r} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_j} \right).
\]

By \([11]\), if \(R \mid N\) we have that

\[
\prod_{i=1}^{r} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_j} \right) - \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^{k} p_j} \leq P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \leq \prod_{i=1}^{r} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_j} \right).
\]

Recall that \(P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right)\) is the probability that the \(k\)-tuple \(n\) is \(b\)-visible. Thus,

\[
N^k P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right),
\]

is the number of lattice points \(n\) that are \(b\)-visible and belong to \([N]^k\).

Now let \(N\) be arbitrary \((R \nmid N\) may happen\). There exist \(k_1, k_2 \in \mathbb{N}\) satisfying the inequality \(Rk_1 < N < Rk_2\), and \(k_1 \geq k_1', k_2 \leq k_2'\) for any other pair \(k_1', k_2' \in \mathbb{N}\) that satisfies the inequality. For that pair \(k_1, k_2\) we have by election that \(Rk_2 - N < R\) and \(N - Rk_1 < R\). Hence, \(N - Rk_2 < N + R\).

Moreover, because there are more \(b\)-visible points in \([N + 1]^k\) than in \([N]^k\), the expression \(N^k P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right)\) is increasing in \(N\). In particular if \(N_1 := Rk_1, N_2 := Rk_2\), then we have:

\[
N_1^k P_{N_1} \left( \bigcap_{i=1}^{\infty} (C_{p_i,N_1})^c \right) \leq N^k P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \leq N_2^k P_{N_2} \left( \bigcap_{i=1}^{\infty} (C_{p_i,N_2})^c \right)
\]
and hence

\( \frac{N_1}{N} k \cdot P_{N_1} \left( \bigcap_{i=1}^{\infty} (C_{p_i,N_1})^c \right) \leq P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \leq \frac{N_2}{N} k \cdot P_{N_2} \left( \bigcap_{i=1}^{\infty} (C_{p_i,N_2})^c \right). \)

Since \( N_1 = Rk_1 \) and \( N_2 = Rk_2 \), then \( N_1 \leq N \leq N_2 \) holds because of the choice of \( k_1 \) and \( k_2 \). It is immediate from the definition of \( N_1 \) and \( N_2 \) that \( R|N_1 \) and \( R|N_2 \). As a consequence of (13) it follows that (14) becomes

\( \frac{N_1}{N} k \left( \prod_{i=1}^{r} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_i^{b_j}} \right) - \sum_{i=r}^{\infty} \frac{1}{\prod_{j=1}^{k} p_i^{b_j}} \right) \leq P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \leq \frac{N_2}{N} k \left( \prod_{i=1}^{r} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_i^{b_j}} \right) \right). \)

Using Theorem 4, finding the proportion of \( b \)-visible points \( N^k \) is equivalent to finding

\( P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) \)

as \( N \) goes to infinity, which is equivalent to letting \( N \to \infty \) in the last inequality of (15), and then letting \( r \to \infty \). Doing so yields

\[ \lim_{N \to \infty} P_N \left( \bigcap_{i=1}^{\infty} (C_{p_i,N})^c \right) = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{\prod_{j=1}^{k} p_i^{b_j}} \right) = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{p_i^{\sum_{j=1}^{k} b_j}} \right) = \frac{1}{\zeta \left( \sum_{i=1}^{k} b_i \right)} \].

We conclude by noticing that Theorem 1 recovers the results obtained in \( b \)-visibility, by taking \( b = (1, b) \), and the generalization of the classical lattice point visibility in \( n \)-dimensions by taking \( b = 1 = (1, 1, \ldots, 1) \in \mathbb{N}^n \).

We now generalize the results obtained by Harris and Omar [6] regarding \( b \)-visibility allowing rational exponents.

**Theorem 2.** Fix \( b = \left( \frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k} \right) \in \mathbb{Q}_{>0}^k \), with \( b \) satisfying \( \gcd(b) = 1 \). Then the proportion of points in \( \mathbb{N}_{\frac{a_1}{a}} \times \ldots \times \mathbb{N}_{\frac{a_k}{a}} \) that are \( b \)-visible is \( \frac{1}{\zeta \left( \sum_{i=1}^{k} b_i \right)} \).

**Proof.** Define \( [N_{\frac{a_i}{a}}] := \left\{ \frac{a_i}{a}, 2 \frac{a_i}{a}, \ldots, \left\lfloor N \frac{a_i}{a} \right\rfloor \right\} \) for \( i = 1, \ldots, k \). Let \( n_{1, \frac{a_1}{a}}, \ldots, n_{k, \frac{a_k}{a}} \) be picked independently with uniform probability in \( [N_{\frac{a_1}{a}}], \ldots, [N_{\frac{a_k}{a}}] \), respectively.

Fix a prime \( p \). Then the probability that \( p^{b_i} | n_{i, \frac{a_i}{a}} \), is thus given by

\[ \frac{1}{\left\lfloor N_{\frac{a_i}{a}} \right\rfloor} \cdot \left\lfloor N_{\frac{a_i}{a}} \right\rfloor^{1/p^{b_i}} \].

By mutual independence, the probability \( P_{p,N} \) that \( p^{b_i} | n_{i, \frac{a_i}{a}} \) for all \( i = 1, \ldots, k \) is given by

\[ \prod_{i=1}^{k} \frac{1}{\left\lfloor N_{\frac{a_i}{a}} \right\rfloor} \cdot \left\lfloor N_{\frac{a_i}{a}} \right\rfloor^{1/p^{b_i}} \].
And since $P_{p,N} \to \prod_{i=1}^{k} \frac{1}{p^{b_i}} = \frac{1}{p^{\sum_i b_i}}$ as $N \to \infty$, it is a consequence of Theorem 5 that

$$
\lim_{N \to \infty} \prod_{p \text{ prime}} (1 - P_{p,N}) = \prod_{p \text{ prime}} (1 - \frac{1}{p^{\sum_i b_i}}) = \frac{1}{\zeta(\sum_i b_i)}.
$$


We remark that in [6], Harris and Omar considered the case when $b = (1, b/a)$ with $b/a \in \mathbb{Q}$. When $b$ was a negative integer, they thought of a point $(r, s) \in \mathbb{N}^2$ as being visible (or invisible) from a point at infinity, i.e. located at $(\infty, 0)$. From this they established that the proportion of visible lattice points in $\mathbb{N}_a \times \mathbb{N}$, where $\mathbb{N}_a = \{1^a, 2^a, 3^a, \ldots\}$, was given by $1/\zeta(b)$. However, their study considered the number-theoretic approach where the point $(r, s)$ was $(b/a)$-visible provided there was no prime $p$ such that $p^b | s$. Taking this number-theoretic approach we now establish our final result.

**Theorem 3.** Let $b = (\frac{b_1}{a_1}, \ldots, \frac{b_k}{a_k}) \in (\mathbb{Q}^*)^k$ be such that its negative entries are indexed by the set $J \subseteq [k]$, with $b$ satisfying the conditions that $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $\gcd(\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}) = 1$. Then the proportion of points in $\mathbb{N}_{\frac{b_1}{a_1}} \times \cdots \times \mathbb{N}_{\frac{b_k}{a_k}}$ that are $b$-visible is $\frac{1}{\zeta(\sum_{j \in J} |b_j|)}$.

**Proof.** The proof of this theorem is analogous to the proof of the previous theorem, and is a consequence of Theorem 6 and Remark 3.

**Remark 4.** Recall that in [6], Harris and Omar obtained that the density of $b$-visible points for $b = \frac{b}{a}$, was $\frac{1}{\zeta(b)}$ if $b$ is negative, whereas the density is $\frac{1}{\zeta(b+1)}$ if $b$ is positive. Our last theorem explains the reason for the difference between these densities: when the vector $b$ contains both positive and negative entries, only the negative entries matter.

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