MARGINALLY TRAPPED SURFACES IN $\mathbb{L}^4$ AND THREE WEIERSTRASS REPRESENTATIONS

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Abstract. We construct new integrable systems to present Weierstrass type representations for spacelike surfaces whose mean curvature vector $H$ satisfies the null condition $\langle H, H \rangle = 0$ in the four dimensional Lorentz-Minkowski space $\mathbb{L}^4$. Our new Weierstrass presentations extend simultaneously classical Weierstrass representations (of the first kind and the second kind) for maximal surfaces in $\mathbb{L}^3$ and minimal surfaces in $\mathbb{R}^3$. We solve a linear partial differential equation to construct explicit examples of marginally trapped surfaces with nowhere vanishing mean curvature vector.

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1. MOTIVATION AND MAIN RESULTS

A spacelike space in the four dimensional Lorentz-Minkowski space $\mathbb{L}^4$ is marginally trapped if its mean curvature vector $H$ verifies the null condition:

$$\langle H, H \rangle = 0.$$ 

We briefly review known conformal representations [2, 4, 6] for marginally trapped surfaces.

The Aledo-Gálvez-Mira representation [2, Theorem 9] generates marginally trapped surfaces of Bryant type in $\mathbb{L}^4$. Their conformal representation yields marginally trapped surfaces satisfying two geometric conditions that they have flat normal bundle and are locally isometric to some minimal surfaces in Euclidean space $\mathbb{R}^4$ or some maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$. The Aledo-Gálvez-Mira representation extends simultaneously the Bryant representation [3, Theorem A] for surfaces with mean curvature one in hyperbolic space $\mathbb{H}^3 \subset \mathbb{L}^4$ and the representation [1, Theorem 7.1] for spacelike surfaces with mean curvature one in de-Sitter space $\mathbb{S}_1^3 \subset \mathbb{L}^4$.

Recently, Dussan, Filho, Simões [4, Section 5] presented an interesting method to construct explicit examples of marginally trapped surfaces sitting in $\mathbb{H}^3 \subset \mathbb{L}^4$. They solved a Riccati partial differential equation for the triple of a holomorphic function and two $\mathbb{C}$-valued functions. The conformal representation due to Liu [5, Theorem 4.2] for general marginally trapped surfaces in $\mathbb{L}^4$ requires to solve a system of partial differential equations for the triple of a holomorphic function and two $\mathbb{C}$-valued functions. Our work is strongly motivated by the Liu representation.

We sketch the main results of this paper, the first one in a series of papers on the marginally trapped surfaces. We present the three Weierstrass type representations (Theorem 7, Corollary 8, Corollary 9) for marginally trapped surfaces in the four dimensional Lorentz-Minkowski space $\mathbb{L}^4$. Our representations require to solve the linear partial differential equations.

Inspired by the Liu integrable system (5 Theorem 4.2] and Lemma 5 for the triple of a holomorphic function and two $\mathbb{C}$-valued functions, we construct integrable systems (Definition 2.1).
and Definition 2.2 for the triple of a holomorphic function and two \( \mathbb{R} \)-valued functions. The fundamental features of our integrable systems are investigated in Section 2 and Section 5.

The key geometric idea hidden in our algebraic reduction of the Liu integrable system is revealed in our Weierstrass type representation (in Corollary \( \text{Corollary 8} \)) for marginally trapped surfaces in \( L^4 \). Our conformal representation naturally extends the Weierstrass type representation of the second kind (due to O. Kobayashi [5, Corollary 1.3]) for maximal surfaces in \( \mathbb{L}^3 \). (See Remark \( \text{Remark 4.1} \)).

Let \( h(u + iv) \) be an arbitrary prescribed nowhere vanishing holomorphic function. Our conformal representation in Corollary \( \text{Corollary 8} \) indicates that the pair \( (\mathcal{M}(u,v), \mathcal{N}(u,v)) \) of the \( \mathbb{R} \)-valued functions, which solves the linear partial differential equation

\[
\mathcal{M}_{uu} + \mathcal{M}_{vv} = \text{Re} \; h \left( \mathcal{N}_{uu} + \mathcal{N}_{vv} \right),
\]
yields the marginally trapped surface in \( L^4 \). (The holomorphic function \( g := \frac{1}{h} \) can be viewed as a complexified null Gauss map of the marginally trapped surface. See Remark \( \text{Remark 4.1} \)). Our representation in Corollary \( \text{Corollary 8} \) can be viewed a natural generalization of classical representations of the first kind for maximal surfaces in \( \mathbb{L}^3 \) and minimal surfaces in \( \mathbb{R}^3 \). (See Remark \( \text{Remark 4.2} \)).

Solving the above linear partial differential equation yields explicit examples of marginally trapped surfaces in \( L^4 \). In particular, in Section 6 we construct two families of marginally trapped surfaces with nowhere vanishing mean curvature vector. Both families contain a catenoid cousin with mean curvature one in the three dimensional hyperbolic space \( \mathbb{H}^3 \subset L^4 \).

2. Two Equivalente Integrable Systems

Throughout this section, \( \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \) denotes a simply connected domain with the complex coordinate \( z \).

**Definition 2.1 (Weierstrass data of the first kind).** We call the triple \( (g, \mathcal{P}, \mathcal{Q}) \) the Weierstrass data of the first kind if a function \( g : \Omega \to \mathbb{C} - \{0\} \) and two \( C^2 \) functions \( \mathcal{P}, \mathcal{Q} : \Omega \to \mathbb{R} \) satisfy the three conditions

\[
\begin{align*}
g_\tau &= 0, \\
\mathcal{P}_\tau &= |g|^2 \mathcal{Q}_\tau, \\
\mathcal{P}_z - |g|^2 \mathcal{Q}_z &\neq 0.
\end{align*}
\]

**Definition 2.2 (Weierstrass data of the second kind).** We call the triple \( (h, \mathcal{M}, \mathcal{N}) \) the Weierstrass data of the second kind if a function \( h : \Omega \to \mathbb{C} - \{0\} \) and two \( C^2 \) functions \( \mathcal{M}, \mathcal{N} : \Omega \to \mathbb{R} \) satisfy the three conditions

\[
\begin{align*}
h_\tau &= 0, \\
\mathcal{M}_\tau &= (\text{Re} \; h) \mathcal{N}_\tau, \\
\mathcal{M}_z &= (\text{Re} \; h) \mathcal{N}_z \neq 0.
\end{align*}
\]

**Theorem 1 (Equivalence transformation from the first data to the second data).** Let \( (g, \mathcal{P}, \mathcal{Q}) \) be a Weierstrass data of the first kind. We define \( h := \frac{1}{g} \) and \( \mathcal{N} := 2\mathcal{P} \). Then, there exists a \( C^2 \) function \( \mathcal{M}_\lambda : \Omega \to \mathbb{R} \) such that

\[
\mathcal{M}_z = \frac{1}{g} \mathcal{P}_z + g \mathcal{Q}_z.
\]

Moreover, the triple \( (h, \mathcal{M}, \mathcal{N}) \) becomes a Weierstrass data of the second kind such that

\[
\mathcal{M}_z - (\text{Re} \; h) \mathcal{N}_z = -\frac{1}{g} \left( \mathcal{P}_z - |g|^2 \mathcal{Q}_z \right).
\]

**Proof.** The function \( g : \Omega \to \mathbb{C} - \{0\} \) and two \( C^2 \) functions \( \mathcal{P}, \mathcal{Q} : \Omega \to \mathbb{R} \) satisfy

\[
\begin{align*}
g_\tau &= 0, & \quad \mathcal{P}_\tau &= |g|^2 \mathcal{Q}_\tau, & \quad \mathcal{P}_z - |g|^2 \mathcal{Q}_z &\neq 0.
\end{align*}
\]

It is immediate that \( h_\tau = \left( \frac{1}{g} \right)_\tau = 0 \). We introduce the function \( \mathcal{E} : \Omega \to \mathbb{R} \) by

\[
\mathcal{E} := \frac{1}{g} \mathcal{P}_z + g \mathcal{Q}_z.
\]

We have

\[
\mathcal{E}_\tau = \frac{1}{g} \mathcal{P}_\tau + g \mathcal{Q}_\tau = \left( \frac{1}{g} + \frac{1}{2} \right) \mathcal{P}_\tau = 2 (\text{Re} \; h) \mathcal{P}_\tau \in \mathbb{R}.
\]
Since the domain $\Omega$ is simply connected, by Poincaré’s Lemma, this implies the existence of an $\mathbb{R}$-valued function $\mathcal{M}_\lambda$ defined on $\Omega$ such that

$$\mathcal{M}_z = \mathcal{E}.$$ 

We have

$$\mathcal{M}_z = \mathcal{E} = 2 (\text{Re } h) \mathcal{P}_z = (\text{Re } h) \mathcal{N}_z$$

and

$$\mathcal{M}_z - (\text{Re } h) \mathcal{N}_z = \mathcal{E} = \frac{1}{2} \left( \frac{1}{g} + \frac{1}{\bar{g}} \right) 2 \mathcal{P}_z = \frac{1}{g} \mathcal{P}_z + g \mathcal{Q}_z = \left( \frac{1}{g} + \frac{1}{\bar{g}} \right) \mathcal{P}_z = - \frac{1}{g} \left( \mathcal{P}_z - |g|^2 \mathcal{Q}_z \right) \neq 0.$$

We conclude that the triple $(h, M, N)$ is a Weierstrass data of the second kind:

$$h_z = 0, \quad M_z = (\text{Re } h) \mathcal{N}_z, \quad M_z - (\text{Re } h) \mathcal{N}_z \neq 0.$$

□

**Theorem 2** (Equivalence transformation from the second data to the first data). Let $(h, M, N)$ be a Weierstrass data of the second kind. We define $g := \frac{1}{h}$ and $P := \frac{1}{g^2} N$. Then, there exists a $C^2$ function $Q_\lambda : \Omega \rightarrow \mathbb{R}$ such that

$$Q_z = h \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z.$$

Moreover, the triple $(g, P, Q)$ becomes a Weierstrass data of the first kind such that

$$\frac{1}{g} \left( \mathcal{P}_z - |g|^2 \mathcal{Q}_z \right) = - (\mathcal{M}_z - (\text{Re } h) \mathcal{N}_z).$$

**Proof.** The function $h : \Omega \rightarrow \mathbb{C} - \{0\}$ and two $C^2$ functions $\mathcal{M}_z, \mathcal{N}_z : \Omega \rightarrow \mathbb{R}$ satisfy

$$h_z = 0, \quad \mathcal{M}_z = \frac{h + \bar{h}}{2} \mathcal{N}_z, \quad \mathcal{M}_z - \frac{h + \bar{h}}{2} \mathcal{N}_z \neq 0.$$

It is immediate that $g_z = \left( \frac{1}{h} \right)_z = 0$. We introduce the function $\mathcal{E} : \Omega \rightarrow \mathbb{R}$ by

$$\mathcal{E} := h \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z.$$

We have

$$\mathcal{E} = h \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z = \left( \frac{h + \bar{h}}{2} - \frac{h^2}{2} \right) \mathcal{N}_z = \frac{|h|^2}{2} \mathcal{N}_z \in \mathbb{R}.$$ 

Since the domain $\Omega$ is simply connected, by Poincaré’s Lemma, this implies the existence of an $\mathbb{R}$-valued function $Q_\lambda$ defined on $\Omega$ such that

$$Q_z = \mathcal{E}.$$

We have

$$Q_z = \mathcal{E} = \frac{|h|^2}{2} \mathcal{N}_z = \frac{1}{|g|^2} \mathcal{P}_z$$

and

$$\frac{1}{|g|^2} \mathcal{P}_z - Q_z = \frac{|h|^2}{2} \mathcal{N}_z - \left( \frac{h}{2} \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z \right) = h \left( (\text{Re } h) \mathcal{N}_z - \mathcal{M}_z \right) \neq 0.$$

We conclude that the triple $(g, P, Q)$ is a Weierstrass data of the first kind:

$$g_z = 0, \quad P_z = |g|^2 \mathcal{Q}_z, \quad P_z - |g|^2 \mathcal{Q}_z \neq 0.$$

□

**Remark 2.1.** It is straightforward to check that the transformation in Theorem 2 is the reverse transformation in Theorem 1 using the integrability conditions.

We present three fundamental deformations of Weierstrass data of the first kind and the second kind.
Proposition 3 (Parabolic Deformations). Let the triple \((g, \mathcal{P}, Q)\) be a Weierstrass data of the first kind. Let \(\lambda \in \mathbb{R}\) be a parameter constant such that \(g(z) \neq -\frac{1}{\lambda} z\) for all \(z \in \Omega\). We set \(g_{\lambda} := \frac{1}{g + i\lambda} g\) and \(\mathcal{P}_{\lambda} := \mathcal{P}\). Then, there exists a \(C^2\) function \(Q_{\lambda} : \Omega \to \mathbb{R}\) satisfying the integrability condition

\[
(Q_{\lambda})_z = \left(\frac{1}{g} + i\lambda\right) (gQ_z - i\lambda P_z).
\]

Moreover, the triple \((g_{\lambda}, \mathcal{P}_{\lambda}, Q_{\lambda})\) becomes a Weierstrass data of the first kind.

First Proof of Proposition 3 We observe that both \(g_{\lambda}\) and \(g = g_0\) vanishes nowhere. We have

\[
\frac{1}{g_{\lambda}} = \frac{1}{g} + i\lambda \quad \text{and} \quad \frac{1}{g_{\lambda}} = \frac{1}{g} - i\lambda.
\]

We introduce the function \(E : \Omega \to \mathbb{R}\) by

\[
E := \left(\frac{1}{g} + i\lambda\right) (gQ_z - i\lambda P_z) = \frac{1}{g_{\lambda}} (gQ_z - i\lambda P_z).
\]

Note that both \(g_{\lambda}\) and \(g = g_0\) are holomorphic and that \(Q_{\lambda} = \frac{1}{|g|} \mathcal{P}_{\lambda}\). We have

\[
E_{\lambda} = \frac{g}{g_{\lambda}} Q_{\lambda} + \frac{1}{g_{\lambda}} (-i\lambda) \mathcal{P}_{\lambda} = \frac{g}{g_{\lambda}} \frac{\mathcal{P}_{\lambda}}{|g|^2} + \frac{1}{g_{\lambda}} (-i\lambda) \mathcal{P}_{\lambda} = \frac{1}{|g_{\lambda}|^2} \mathcal{P}_{\lambda} = \frac{1}{|g_{\lambda}|^2} \mathcal{P}_{\lambda}.
\]

Since the domain \(\Omega\) is simply connected, by Poincaré’s Lemma, the observation

\[
E_{\lambda} = \frac{1}{|g_{\lambda}|} \mathcal{P}_{\lambda} \in \mathbb{R}
\]

implies the existence of an \(\mathbb{R}\)-valued function \(Q_{\lambda}\) defined on \(\Omega\) such that

\[
(Q_{\lambda})_{z\lambda} = E.
\]

It follows that

\[
(Q_{\lambda})_{z\lambda} = E = \frac{1}{|g_{\lambda}|} \mathcal{P}_{\lambda} = \frac{1}{|g_{\lambda}|} (\mathcal{P}_{\lambda})_{z\lambda}.
\]

Finally, we have the equality

\[
\frac{1}{g_{\lambda}} (\mathcal{P}_{\lambda})_z - g_{\lambda} (Q_{\lambda})_z = \frac{1}{g_{\lambda}} (\mathcal{P})_z - g_{\lambda} E = \frac{1}{g_{\lambda}} (\mathcal{P})_z - g_{\lambda} \left(\frac{1}{g_{\lambda}} (gQ_z - i\lambda P_z)\right) = \left(\frac{1}{g_{\lambda}} + i\lambda\right) P_z - g Q_z = \frac{1}{g} P_z - g Q_z,
\]

This implies the last condition

\[
(\mathcal{P}_{\lambda})_z - \frac{|g_{\lambda}|^2}{g} (Q_{\lambda})_z = \frac{3}{g} \left(\frac{g}{g_{\lambda}} - |g|^2 Q_z\right) \neq 0.
\]

□

Though the first proof of Proposition 3 is self-contained, it does not explain the method how to discover the integrability condition in Proposition 3.

\[
(Q_{\lambda})_z = \left(\frac{1}{g} + i\lambda\right) (gQ_z - i\lambda P_z).
\]

We use the equivalence of the Weierstrass data of the first kind and the Weierstrass data of the second kind to discover the integrability condition.

Second Proof of Proposition 3 We set \(h = \frac{1}{g}\) and \(N = 2\mathcal{P}\). According to Theorem 1, we can find a \(C^2\) function \(M : \Omega \to \mathbb{R}\) such that

\[
M_z = \frac{1}{g} P_z + g Q_z.
\]

Moreover, the triple \((h, M, N)\) is a Weierstrass data of the second kind:

\[
h_{\lambda} = 0, \quad M_{z\lambda} = (\text{Re} \, h) N_{z\lambda}, \quad M_z - (\text{Re} \, h) N_z \neq 0.
\]
It follows from $\lambda \in \mathbb{R}$ that $\text{Re}(h + i\lambda) = \text{Re} h$. We find immediately that the triple

$$(h_\lambda, M_\lambda, N_\lambda) = (h + i\lambda, M, N)$$

is also a Weierstrass data of the second kind:

$$(h_\lambda)_{12} = 0, \quad (M_\lambda)_{2} = (\text{Re} h_\lambda) (N_\lambda)_{22}, \quad (M_\lambda)_{2} - (\text{Re} h_\lambda) (N_\lambda)_{22} \neq 0.$$

Theorem 2 guarantees the existence of a $C^2$ function $Q_\lambda : \Omega \to \mathbb{R}$ such that

$$(Q_\lambda)_2 = h_\lambda (M_\lambda)_2 - \frac{h_\lambda^2}{2} (N_\lambda)_2.$$ 

It follows from the definitions $g_\lambda = \frac{g}{1 + i\lambda g}$, $h = \frac{1}{g}$, and $h_\lambda = h + i\lambda$ that

$$\frac{1}{g_\lambda} = \frac{1}{g} + i\lambda = h + i\lambda = h_\lambda.$$

We set

$$P_\lambda := P = \frac{1}{2} \lambda = \frac{1}{2} N_\lambda.$$

Theorem 2 shows that the triple $(g_\lambda, P_\lambda, Q_\lambda)$ is a Weierstrass data of the second kind. It now remains to deduce the integrability condition:

$$(Q_\lambda)_2 = h_\lambda (M_\lambda)_2 - \frac{h_\lambda^2}{2} (N_\lambda)_2 = \frac{1}{g_\lambda} M_\lambda - \frac{1}{g_\lambda^2} P_\lambda$$

$$= \frac{1}{g_\lambda} \left( \frac{1}{g} P_\lambda + g Q_\lambda - \frac{1}{g_\lambda} P_\lambda \right) = \frac{1}{g_\lambda} \left( g Q_\lambda + \left( \frac{1}{g} - \frac{1}{g_\lambda} \right) P_\lambda \right)$$

$$= \left( \frac{1}{g} + i\lambda \right) (gQ_\lambda - i\lambda P_\lambda).$$

□

Proposition 4 (Elliptic Deformations). Let the triple $(h, M, N)$ be a Weierstrass data of the second kind. Given a parameter $\tau \in \mathbb{R}$, we set $h_\tau := e^{-i\tau} h$ and $N_\tau := N$. Then, there exists a $C^2$ function $M_\tau : \Omega \to \mathbb{R}$ satisfying the integrability condition

$$(M_\tau)_2 = e^{i\tau} M_\lambda - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_\lambda.$$

Moreover, the triple $(h_\tau, M_\tau, N_\tau)$ becomes a Weierstrass data of the second kind.

First Proof of Proposition 4. Since $h = h_0$ vanishes nowhere, $h_\tau = e^{-i\tau} h$ also vanishes nowhere. We introduce the function $E : \Omega \to \mathbb{R}$ by

$$E := e^{i\tau} M_\lambda - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_\lambda.$$

Note that $h$ is holomorphic and that $M_\tau = \frac{h + h_\tau}{2} N_\tau$. We have

$$E_\tau = e^{i\tau} M_\tau - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_\tau = \left( e^{i\tau} \frac{h + h_\tau}{2} - \frac{e^{i\tau} - e^{-i\tau}}{2} h \right) N_\tau$$

$$= \left( \frac{e^{-i\tau} h + e^{i\tau} h_\tau}{2} \right) N_\tau = (\text{Re} h_\tau) N_\tau.$$

Since the domain $\Omega$ is simply connected, by Poincaré’s Lemma, the observation

$$E_\tau = (\text{Re} h_\tau) N_\tau \in \mathbb{R}$$

implies the existence of an $\mathbb{R}$-valued function $M_\lambda$ defined on $\Omega$ such that

$$(M_\lambda)_2 = E.$$

It follows that

$$(M_\lambda)_2 = E_\tau = (\text{Re} h_\tau) N_\tau = (\text{Re} h_\tau) (N_\lambda)_2.$$
Finally, we deduce the last condition
\[
(M_\lambda)_z - (\text{Re } h_\nu) (N_\lambda)_z = E - \left( \frac{e^{-i\tau} h + e^{i\tau} \overline{h}}{2} \right) N_z
= e^{i\tau} M_z - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_z - \left( \frac{e^{-i\tau} h + e^{i\tau} \overline{h}}{2} \right) N_z
= e^{i\tau} (M_z - (\text{Re } h) N_z) \neq 0.
\]

Though the first proof of Proposition 4 is self-contained, it does not explain how to discover the integrability condition in Proposition 4:
\[
(M_{\tau})_z = e^{i\tau} M_z - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_z.
\]

We use the equivalence of the Weierstrass data of the first kind and the Weierstrass data of the second kind to discover the integrability condition.

**Second Proof of Proposition 4** We set \( g = \frac{1}{h} \) and \( P = \frac{1}{2} N \). According to Theorem 2, we can find a \( C^2 \) function \( Q : \Omega \to \mathbb{R} \) such that
\[
Q_z = h M_z - \frac{h^2}{2} N_z.
\]
Moreover, the triple \((g, P, Q)\) is a Weierstrass data of the second kind:
\[
g_{\nu} = 0, \quad P_{\nu} = |g|^2 Q_{\nu}, \quad P_z = |g|^2 Q_z \neq 0.
\]
We observe that \( |e^{i\tau} g| = |g| \). We find immediately that the triple \((g_{\tau}, P_{\tau}, Q_{\tau}) = (e^{i\tau} g, P, Q)\) is also a Weierstrass data of the first kind:
\[
(g_{\tau})_{\nu} = 0, \quad (P_{\tau})_{\nu} = |g_{\tau}|^2 (Q_{\tau})_{\nu}, \quad (P_{\tau})_z = |g_{\tau}|^2 (Q_{\tau})_z \neq 0.
\]

Theorem 1 guarantees the existence of a \( C^2 \) function \( M_{\tau} : \Omega \to \mathbb{R} \) such that
\[
(M_{\tau})_z = \frac{1}{g_{\tau}} (P_{\tau})_z + g_{\tau} (Q_{\tau})_z.
\]
It follows from the definitions \( h_{\tau} = e^{-i\tau} h, g = \frac{1}{h}, \) and \( g_{\tau} = e^{i\tau} g \) that
\[
h_{\tau} = \frac{1}{g_{\tau}},
\]
We set
\[
N_{\tau} := N = 2P = 2P_\lambda.
\]

Theorem 1 shows that the triple \((h_{\tau}), M_{\tau}, N_{\tau})\) is a Weierstrass data of the second kind. It now remains to deduce the integrability condition:
\[
(M_{\tau})_z = \frac{1}{g_{\tau}} (P_{\tau})_z + g_{\tau} (Q_{\tau})_z = e^{-i\tau} h N_{\tau} - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_z
= e^{-i\tau} h \frac{N_{\tau}}{2} + \frac{e^{i\tau} - e^{-i\tau}}{2} h N_z.
\]

**Proposition 5 (Hyperbolic Deformations in terms of the Weierstrass data of the first kind).** Let the triple \((g, P, Q)\) be a Weierstrass data of the first kind. Let \( \eta \in \mathbb{R} \) be a parameter. Then, the triple \((g_\eta, P_\eta, Q_\eta) := (e^{i\eta} g, e^{i\eta} P, e^{i\eta} Q)\) becomes a Weierstrass data of the first kind.
Proof. The assumption says that the triple \((g, \mathcal{P}, \mathcal{Q})\) satisfy
\[
g_{\tau} = 0, \quad \mathcal{P}_{\tau} = |g|^2 \mathcal{Q}_{\tau}, \quad \mathcal{P}_z - |g|^2 \mathcal{Q}_z \neq 0.
\]
Hence, the triple \((g_\tau, \mathcal{P}_\tau, \mathcal{Q}_\tau)\) satisfy
\[
\begin{align*}
(\mathcal{P}_\tau)_z &= (e^\eta g)_{\tau} = 0, \\
(\mathcal{P}_\tau)_z &= (e^\eta \mathcal{P})_{\tau} = e^\eta |g|^2 \mathcal{Q}_{\tau}, \\
(\mathcal{P}_\tau)_z - |g_\tau|^2 (\mathcal{Q}_\tau)_z &= e^\eta \left( \mathcal{P}_z - |g|^2 \mathcal{Q}_z \right) \neq 0.
\end{align*}
\]
\(\square\)

3. Reduction of Liu integrable system

The purpose of this section is to explain one way to discover our integrable systems for marginally trapped surfaces in \(L^4\). An algebraic reduction of the following version of the Liu integrable system [3] for the triple of a holomorphic function and two \(\mathbb{C}\)-valued functions yields to our integrable systems for the Weierstrass triple of a holomorphic function and two \(\mathbb{R}\)-valued functions. We emphasize that the hidden geometric idea of our algebraic reduction is implicitly contained in our new Weierstrass representations (Theorem 2 Corollary 2 and Corollary 2) for marginally trapped surfaces.

Lemma 6 (Liu’s integrable system for marginally trapped surfaces in \(L^4\) [6]). Let \(\Omega\) be an open domain in \(\mathbb{C}\) with the complex coordinate \(z\). We assume that a spacelike surface \(\Sigma\) in Lorentz-Minkowski space \(L^4\) is parameterized by a \(C^2\) conformal patch \(X: \Omega \rightarrow L^4\) of the form
\[
X_z = \begin{bmatrix}
(x_1)_z \\
(x_2)_z \\
(x_3)_z \\
(x_4)_z
\end{bmatrix} = \Psi \begin{bmatrix}
f_1 + f_2 \\
-i(f_1 - f_2) \\
1 - f_1 f_2 \\
1 + f_1 f_2
\end{bmatrix},
\]
where \(\Psi: \Omega \rightarrow \mathbb{C} - \{0\}\) and \(f_1, f_2: \Omega \rightarrow \mathbb{C}\). Then, the following three integrability conditions hold for all \(z \in \Omega\):
\[
\begin{align*}
(1) \quad & \Psi_{\tau} = \bar{\Psi}_{\tau}, \\
(2) \quad & (\Psi f_1 f_2)_{\tau} = (\bar{\Psi} f_1 f_2)_{\tau}, \\
(3) \quad & (\Psi f_1)_{\tau} = (\Psi f_2)_{\tau} \iff (\bar{\Psi} f_1)_{\tau} = (\bar{\Psi} f_2)_{\tau}.
\end{align*}
\]
If \(\Sigma\) is marginally trapped, then the following holomorphism condition holds for each point \(z \in \Omega\):
\[
f_1(z) = 0 \quad \text{or} \quad f_2(z) = 0.
\]

Proof. It follows from [6] Theorem 4.1 and Theorem 4.2. For the reader’s convenience, we sketch a self-contained proof. We first observe that
\[
\begin{align*}
\Psi &= \frac{(x_3)_z + (x_4)_z}{2}, \\
\Psi f_1 f_2 &= \frac{-(x_3)_z + (x_4)_z}{2}, \\
\Psi f_1 &= \frac{(x_1)_z + i(x_2)_z}{2}, \\
\Psi f_2 &= \frac{(x_1)_z - i(x_2)_z}{2}.
\end{align*}
\]
(These relations reveal how to prescribe the three functions \(\Psi, f_1, f_2\) in terms of the four component functions of \(X_z\).) Since \(x_1 = x_\tau, x_2 = x_\bar{\tau}, x_3 = x_\tau, x_4 = x_\bar{\tau}\), we have
\[
\begin{align*}
\Psi_{\tau} &= \left( \frac{(x_3)_\tau + (x_4)_\tau}{2} \right)_{\tau} = \left( \frac{(x_3)_z + (x_4)_z}{2} \right) = \bar{\Psi}_{\tau} \\
(\Psi f_1 f_2)_{\tau} &= \left( \frac{-(x_3)_\tau + (x_4)_\tau}{2} \right)_{\tau} = \left( \frac{-(x_3)_z + (x_4)_z}{2} \right) = (\bar{\Psi} f_1 f_2)_{\tau}
\end{align*}
\]
and
\[
\Psi f_1 = \frac{(x_1)_\tau + i(x_2)_\tau}{2} = \left( \frac{(x_1)_z + i(x_2)_z}{2} \right)_{\tau} = \bar{\Psi} f_1
\]
\[
\Psi f_2 = \frac{(x_1)_\tau - i(x_2)_\tau}{2} = \left( \frac{(x_1)_z - i(x_2)_z}{2} \right)_{\tau} = \bar{\Psi} f_2.
\]
We also have
\[
\left( \Psi f_2 \right)_\Sigma = \left( \frac{(x_1)_z - i(x_2)_z}{2} \right)_\Sigma = \left( \frac{(x_1)_z + i(x_2)_z}{2} \right)_z = \left( \Psi f_1 \right)_\Sigma.
\]

Now, we assume that the spacelike surface \( \Sigma \) with the mean curvature vector \( H \) is marginally trapped. Since \( X \) is a conformal patch, we find that the null condition
\[
\langle H, H \rangle = 0
\]
is equivalent to
\[
\langle X, X \rangle = 0.
\]

We differentiate both sides of
\[
X_z = \Psi F_0, \quad \text{where } F_0 := \begin{bmatrix} f_1 + f_2 \\ -i \left( f_1 - f_2 \right) \\ 1 - f_1 f_2 \\ 1 + f_1 f_2 \end{bmatrix},
\]
with respect to \( \Sigma \) to obtain the decomposition
\[
X_{\Sigma} = \Psi_{\Sigma} F_0 + \Psi (f_1)_{\Sigma} F_2 + \Psi (f_2)_{\Sigma} F_1,
\]
where we define
\[
F_2 := \begin{bmatrix} 1 \\ -i \\ -f_2 \\ f_2 \end{bmatrix} \quad \text{and} \quad F_1 := \begin{bmatrix} 1 \\ i \\ -f_1 \\ f_1 \end{bmatrix}.
\]

A straightforward computation shows that
\[
\begin{cases}
(F_1, F_2) = 2, \\
(F_1, F_j) = 0 \text{ for } (i, j) \neq (1, 2), (2, 1).
\end{cases}
\]

It follows that
\[
0 = \langle X, X_{\Sigma} \rangle = 2 \Psi^2 (f_1)_{\Sigma} (f_2)_{\Sigma} (F_1, F_2) = 4 \Psi^2 (f_1)_{\Sigma} (f_2)_{\Sigma}
\]
We recall the assumption \( \Psi(z) \neq 0 \) for all \( z \in \Omega \). We conclude that \( (f_1)_{\Sigma} = 0 \) or \( (f_2)_{\Sigma} = 0 \) at each point \( z \in \Omega \). This completes the proof. \( \square \)

It remains to explain how to use the Liu integrable system for the marginally trapped surface

1. \( h_{\Sigma} = 0 \),
2. \( \mathcal{M}_{\Sigma} = (\text{Re } h) \mathcal{N}_{\Sigma} \),

which were essential in our second Weierstrass representation in Corollary[8]. We keep the notations in Lemma[8] and consider the case \( (f_2)_{\Sigma} \equiv 0 \). We have
\[
\Psi f_1 = \frac{(x_1)_z + i(x_2)_z}{2} \quad \text{and} \quad \Psi f_2 = \frac{(x_1)_z - i(x_2)_z}{2}.
\]

We make an additional assumption that the function \( \Psi = \frac{(x_2)_z + (x_1)_z}{2} \) never vanish on \( \Omega \). (This assumption will be not required in our second Weierstrass representation in Corollary[8].) We define the three functions \( \mathcal{M}, \mathcal{N} : \Omega \to \mathbb{R} \) and \( h : \Omega \to \mathbb{C} \) by
\[
\begin{aligned}
\mathcal{M} := x_1, \quad \mathcal{N} := x_3 + x_4, \quad h := f_2 \frac{\Psi f_2}{\Psi} = \frac{(x_1)_z - i(x_2)_z}{(x_3)_z + (x_4)_z} = \frac{(x_1)_z - i(x_2)_z}{N_z}.
\end{aligned}
\]
We note that $h = f_2$ is holomorphic in $\Omega$ and that $N = \overline{\nabla}$. We observe

$$2\Psi f_2 = (x_1)_z - i(x_2)_z = \mathcal{N}_z h,$$
$$-i(x_2)_z = -(x_1)_z + ((x_1)_z - i(x_2)_z) = -\mathcal{M}_z + \mathcal{N}_z h,$$
$$2\Psi f_1 = (x_1)_z + i(x_2)_z = \mathcal{M}_z - (-\mathcal{M}_z + \mathcal{N}_z h) = 2\mathcal{M}_z - \mathcal{N}_z h.$$

We conclude that

$$0 = (\Psi f_1)_z - (\Psi f_2)_z = \left(\mathcal{M}_z - \frac{1}{2}\mathcal{N}_z h\right)_z - \left(\frac{1}{2}\mathcal{N}_z h\right)_z$$
$$= \left(\mathcal{M}_z - \frac{1}{2}\mathcal{N}_z h - \frac{1}{2}\mathcal{N}_z h\right)_z - \left(\frac{1}{2}\mathcal{N}_z h + \frac{1}{2}\mathcal{N}_z h\right)_z$$
$$= \mathcal{M}_z_2 - (\text{Re } h) \mathcal{N}_z_2 - \text{Re } (h_\overline{\mathcal{N}_z})$$
$$= \mathcal{M}_z_2 - (\text{Re } h) \mathcal{N}_z_2.$$

This completes the desired reduction.

**Remark 3.1.** The geometric idea to construct our algebraic reduction of the Liu integrable system is revealed our Weierstrass representation (Corollary 1.3) and Remark 3.1 for marginally trapped surfaces in $L^4$, which generalizes Weierstrass representation of the second kind (due to O. Kobayashi [1] Corollary 1.3) for maximal surfaces in $L^3$.

### 4. Three Weierstrass representations

The 4-dimensional Lorentz-Minkowski space $L^4$ is the real vector space $\mathbb{R}^4$ equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where $x_1, x_2, x_3, x_4$ denotes the canonical coordinates in $\mathbb{R}^4$. The standard complexification of $\langle \cdot, \cdot \rangle$ induces the symmetric bilinear form on $\mathbb{C}^4$.

The purpose of this section is to present various conformal representation for marginally trapped surfaces in $L^4$. A spacelike surface in Lorentz-Minkowski space $L^4$ is called a marginally trapped surface if its mean curvature vector $H$ satisfies the null condition:

$$\langle H, H \rangle = 0.$$

**Theorem 7 (First Weierstrass representation for marginally trapped surfaces in $L^4$).** Let $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ be a simply connected domain with the complex coordinate $z$. Let the triple $(g, P, Q)$ be a Weierstrass data of the first kind. The holomorphic function $g : \Omega \to \mathbb{C} - \{0\}$ and two $C^2$ functions $P, Q : \Omega \to \mathbb{R}$ satisfy the equation

$$P_z = |g|^2 Q_z,$$

and the condition

$$P_z \neq |g|^2 Q_z.$$

Then, there exists a conformal parameterization $X = X_{(g, P, Q)} : \Omega \to \mathbb{L}^4$ of the marginally trapped surface $\Sigma = X(\Omega)$ in Lorentz-Minkowski space $\mathbb{L}^4$ satisfying the following two equalities:

$$X_z = \left[\begin{array}{c} (x_1)_z \\ (x_2)_z \\ (x_3)_z \\ (x_4)_z \end{array} \right] = P_z \left[\begin{array}{c} \frac{1}{g} \\ -i g^{-1} \\ 1 \\ 0 \end{array} \right] + Q_z \left[\begin{array}{c} g \\ -i g \\ -1 \\ 1 \end{array} \right]$$

and

$$X_z = Q_z \left[\begin{array}{c} 2 \text{Re } g \\ 2 \text{Im } g \\ -1 + |g|^2 \\ 1 + |g|^2 \end{array} \right].$$

In particular, we have $x_3(z) = P(z) - Q(z)$ and $x_4(z) = P(z) + Q(z)$, up to additive constants. The Gauss map $N$ of the marginally trapped surface $\Sigma$ defined by

$$N := \left[\begin{array}{c} 2 \text{Re } g \\ 2 \text{Im } g \\ -1 + |g|^2 \\ 1 + |g|^2 \end{array} \right].$$
satisfies the orthogonality conditions \( \langle X_z, N \rangle = 0 \) and \( \langle N, N \rangle = 0 \). (In particular, \( N \) is null.) The conformal metric induced by the patch \( X(z) \) is

\[
d s_{\gamma}^2 = \frac{4}{|g|^2} | \mathcal{P}_z - |g|^2 Q_z |^2 |dz|^2.
\]

**Proof.** The vector \( N \) is null:

\[
\langle N, N \rangle = (2 \Re g)^2 + (2 \Im g)^2 = (1 + |g|^2)^2 - (1 + |g|^2)^2 = 0.
\]

We introduce the mappings \( \mathcal{E}, G_1, G_2 \) defined on the domain \( \Omega \):

\[
G_1 = \begin{bmatrix} \frac{1}{g} \\ g \\ 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -ig \\ -1 \\ 1 \end{bmatrix}, \quad \mathcal{E} = \mathcal{P}_z G_1 + Q_z G_2 \in \mathbb{C}^4.
\]

We note that \( Q_z \in \mathbb{R} \) and \( \mathcal{N} \in \mathbb{R}^4 \). We use the holomorphicity of \( g \) and the assumption \( \mathcal{P}_z = g \overline{\mathcal{Q}_z} \) to find that the mapping \( \mathcal{E}_z \) is \( \mathbb{R}^4 \)-valued. Indeed, we have

\[
\mathcal{E}_z = \mathcal{P}_z \mathcal{E} + Q_z \mathcal{N} = \mathbb{R}^4.
\]

Since the domain \( \Omega \) is simply connected, by Poincaré’s Lemma, the observation \( \mathcal{E}_z \in \mathbb{R}^4 \) implies the existence of an \( \mathbb{R}^4 \)-valued mapping \( X \) defined on \( \Omega \) such that

\[
\begin{bmatrix} (x_1)_z \\ (x_2)_z \\ (x_3)_z \\ (x_4)_z \end{bmatrix} = X_z = \mathcal{E} = \mathcal{P}_z G_1 + Q_z G_2 = \mathcal{P}_z \begin{bmatrix} \frac{1}{g} \\ g \\ 1 \end{bmatrix} + Q_z \begin{bmatrix} g \\ -ig \\ 1 \end{bmatrix}.
\]

It follows from the equalities \((x_3)_z = \mathcal{P}_z - Q_z \) and \((x_4)_z = \mathcal{P}_z + Q_z \) that

\[
x_3 = \mathcal{P} - Q \quad \text{and} \quad x_4 = \mathcal{P} + Q,
\]

up to additive constants. The conformality of the mapping \( z \mapsto X(z) \) follows from \( \langle X_z, X_z \rangle = 0 \). Indeed, we have

\[
\langle X_z, X_z \rangle = \mathcal{P}_z^2 \langle G_1, G_1 \rangle + 2 \mathcal{P}_z Q_z \langle G_1, G_2 \rangle + Q_z \langle G_2, G_2 \rangle = 0.
\]

It is straightforward to check the identities

\[
\langle G_1, G_1 \rangle = \frac{2}{|g|^2}, \quad \langle G_1, G_2 \rangle = -2, \quad \langle G_2, G_2 \rangle = -2.
\]

Since \( X \) is \( \mathbb{R}^4 \)-valued, we have \( X_z = \overline{X}_z \). To find the conformal metric \( ds_{\gamma}^2 = \Lambda(z) |dz|^2 \) induced by the patch \( X \), we compute the conformal factor \( \Lambda = 2 \langle X_z, X_z \rangle \). We have

\[
\frac{1}{2} \Lambda(z) = \langle X_z, X_z \rangle = \langle X_z, \overline{X}_z \rangle = \langle \mathcal{P}_z G_1 + Q_z G_2, \mathcal{P}_z G_1 + \overline{Q_z G_2} \rangle = \mathcal{P}_z^2 |G_1|^2 + |Q_z|^2 |G_2|^2 + \mathcal{P}_z Q_z \overline{G_1} \langle G_2, \overline{G_2} \rangle + \mathcal{P}_z \overline{Q_z} \langle G_1, \overline{G_2} \rangle = \mathcal{P}_z^2 \left( \frac{2}{|g|^2} \right) + |Q_z|^2 \left( 2 |g|^2 \right) + \mathcal{P}_z Q_z (-2) + \mathcal{P}_z \overline{Q_z} (-2) = 2 \left| \mathcal{P}_z \frac{1}{g} - Q_z \overline{\mathcal{P}_z} \right|^2 = \frac{2}{|g|^2} \left| \mathcal{P}_z - |g|^2 Q_z \right|^2 > 0.
\]
It follows that
\[ ds^2 = \lambda(z) \left| dz \right|^2 = \frac{4 |g|}{\left| \rho - |g|^2 Q_z \right|^2} \left| dz \right|^2 \]

We recall that \( X_{\tau} = \mathcal{E}_{\tau} = Q_{\tau} \mathcal{N} \). Since the vector \( \mathcal{N} \) is null, we find that the mean curvature vector \( \mathbf{H} \) is also null:
\[ \mathbf{H} := \triangle_{ds^2} X = \triangle_{\lambda(z) \left| dz \right|^2} X = \frac{4}{\lambda(z)} X_{\tau} = \frac{4}{\lambda(z)} Q_{\tau} \mathcal{N} \]

It remains to verify that \( \mathcal{N} \) is the Gauss map in the sense that \( \langle X_{\tau}, \mathcal{N} \rangle = 0 \). We observe that
\[
\langle G_1, \mathcal{N} \rangle = \begin{pmatrix} \frac{1}{g} \\ g \\ 1 \end{pmatrix} \begin{pmatrix} 2 \Re g \\ 2 \Im g \\ -1 + |g|^2 \end{pmatrix} = 2 \frac{\Re g + i \Im g}{g} \left( -1 + |g|^2 \right) - \left( 1 + |g|^2 \right) = 0,
\]
and that
\[
\langle G_2, \mathcal{N} \rangle = \begin{pmatrix} g \\ -i g \\ -1 \end{pmatrix} \begin{pmatrix} 2 \Re g \\ 2 \Im g \\ -1 + |g|^2 \end{pmatrix} = 2 g \left( \Re g - i \Im g \right) - \left( -1 + |g|^2 \right) - \left( 1 + |g|^2 \right) = 0.
\]

It is immediate that
\[
\langle X_{\tau}, \mathcal{N} \rangle = \langle \mathcal{P}_z G_1 + Q_z G_2, \mathcal{N} \rangle = \mathcal{P}_z \langle G_1, \mathcal{N} \rangle + Q_z \langle G_2, \mathcal{N} \rangle = 0.
\]

\[ \square \]

**Corollary 8 (Second Weierstrass representation for marginally trapped surfaces in \( \mathbb{L}^4 \)).** Let \( \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \) be a simply connected domain with the complex coordinate \( z \). Let the triple \((h, \mathcal{M}, \mathcal{N})\) be a Weierstrass data of the second kind. The holomorphic function \( h : \Omega \to \mathbb{C} - \{0\} \) and two \( C^2 \) functions \( \mathcal{M}, \mathcal{N} : \Omega \to \mathbb{R} \) satisfy the equation
\[ \mathcal{M}_{\tau} = (\Re h) \mathcal{N}_{\tau}, \]
and the condition
\[ \mathcal{M}_z \neq (\Re h) \mathcal{N}_z. \]

Then, there exists a conformal parameterization \( X = X_{(h, \mathcal{M}, \mathcal{N})} : \Omega \to \mathbb{L}^4 \) of the marginally trapped surface \( \Sigma = X(\Omega) \) in Lorentz-Minkowski space \( \mathbb{L}^4 \) satisfying
\[
X_{\tau} = X_{(h, \mathcal{M}, \mathcal{N})} = \mathcal{M}_z \begin{pmatrix} 1 \\ -i \\ -h \\ h \end{pmatrix} + \mathcal{N}_z \begin{pmatrix} 0 \\ ih \\ \frac{1}{2} (1 + h^2) \\ \frac{1}{2} (1 - h^2) \end{pmatrix}.
\]

In particular, we have \( x_1(z) = \mathcal{M}(z) \) and \( x_4(z) = \mathcal{N}(z) \), up to additive constants. The conformal metric induced by the patch \( X(z) \) is
\[ ds^2 = 4 \left| \mathcal{M}_z - (\Re h) \mathcal{N}_z \right|^2 |dz|^2. \]

**Proof.** Using the conditions \( h_{\tau} = 0 \) and \( \mathcal{M}_{\tau} = \frac{h + \mathcal{M}}{2} \mathcal{N}_{\tau} \), we deduce
\[
\frac{\partial}{\partial z} \left( h \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z \right) = h \left( \frac{h + \mathcal{M}}{2} \mathcal{N}_{\tau} \right) - \frac{h^2}{2} \mathcal{N}_{\tau} = \frac{|h|^2}{2} \mathcal{N}_{\tau} \in \mathbb{R}.
\]

Since the domain \( \Omega \) is simply connected, by Poincaré’s Lemma, this implies the existence of an \( \mathbb{R} \)-valued mapping \( Q \) defined on \( \Omega \) satisfying the equalities
\[ Q_z = h \mathcal{M}_z - \frac{h^2}{2} \mathcal{N}_z \quad \text{and} \quad Q_{\tau} = \frac{|h|^2}{2} \mathcal{N}_{\tau}. \]
We define \( P := \frac{1}{4} \mathcal{N} \) and \( g := \frac{1}{h} \). Since \( h : \Omega \to \mathbb{C} - \{0\} \) is holomorphic, \( g : \Omega \to \mathbb{C} - \{0\} \) is also holomorphic. We deduce remaining conditions in Theorem [7]

\[
\mathcal{P}_{\ast\ast} = \frac{1}{2} \mathcal{N}_{\ast\ast} = \frac{1}{|h|^2} \mathcal{Q}_{\ast\ast} = |g|^2 \mathcal{Q}_{\ast\ast},
\]

and

\[
\mathcal{P}_{\ast} - |g|^2 \mathcal{Q}_{\ast} = \frac{1}{2} \mathcal{N}_{\ast} - \frac{1}{h} \left( h \mathcal{M}_{\ast} - \frac{h^2}{2} \mathcal{N}_{\ast} \right) = -\frac{1}{h} \left( \mathcal{M}_{\ast} - (\text{Re } h) \mathcal{N}_{\ast} \right) \neq 0.
\]

Taking the triple \((g, P, Q)\) in Theorem [7] yields the existence of \( X = X_{[g, P, Q]} : \Omega \to \mathbb{L}^4 \) of the marginally trapped surface \( \Sigma = X(\Omega) \) in Lorentz-Minkowski space \( \mathbb{L}^4 \) satisfying

\[
\begin{bmatrix}
(x_1)_{\ast} \\
(x_2)_{\ast} \\
(x_3)_{\ast} \\
(x_4)_{\ast}
\end{bmatrix} = X_{\ast} = \mathcal{P}_{\ast} + \mathcal{Q}_{\ast} = \mathcal{M}_{\ast} + \mathcal{N}_{\ast} = \begin{bmatrix}
\frac{1}{2} & 0 & g \\
1 & -i & -ig \\
1 & -1 & 1 \\
1 & 0 & \frac{1}{2} (1 + h^2)
\end{bmatrix}.
\]

It follows from the equalities \((x_1)_{\ast} = \mathcal{M}_{\ast} \) and \((x_3)_{\ast} + (x_4)_{\ast} = \mathcal{N}_{\ast} \) that \( x_1(z) = \mathcal{M}(z) \) and \( x_3(z) + x_4(z) = \mathcal{N}(z) \), up to additive constants. We recall that \( g = \frac{1}{h} \) and \( \mathcal{P}_{\ast} - |g|^2 \mathcal{Q}_{\ast} = -\frac{1}{h} \left( \mathcal{M}_{\ast} - (\text{Re } h) \mathcal{N}_{\ast} \right) \). We use Theorem [7] to compute the conformal metric induced by the patch \( X(z) \):

\[
d s_{\ast\ast}^2 = \frac{4}{|g|^2} |\mathcal{P}_{\ast} - |g|^2 \mathcal{Q}_{\ast}|^2 |d z|^2 = 4 |\mathcal{M}_{\ast} - (\text{Re } h) \mathcal{N}_{\ast}|^2 |d z|^2.
\]

\[\square\]

**Remark 4.1.**

(1) We assume that the two triples \((h, \mathcal{M}, \mathcal{N})\) and \((g, P, Q)\) satisfy the relations

\[
g = \frac{1}{h}, \quad P = \frac{1}{2} \mathcal{N}, \quad Q = h \mathcal{M} - \frac{h^2}{2} \mathcal{N}.
\]

The proof of Corollary [8] shows the claim that \( X_{(h, \mathcal{M}, \mathcal{N})} = X_{[g, P, Q]} \) up to an additive vector constant. In other words, the Weierstrass representation induced by the triple \((g, P, Q)\) in Theorem [7] and the Weierstrass representation induced by the triple \((h, \mathcal{M}, \mathcal{N})\) in Corollary [8] yields the same marginally trapped surface in \( \mathbb{L}^4 \), up to translations.

(2) In the particular case when \( \mathcal{M}_{\ast} \equiv 0 \) (or equivalently, \( \mathcal{M} \) is constant) and \( \mathcal{N}_{\ast} \) is holomorphic (or equivalently, \( \mathcal{N} \) is harmonic), the representation in Corollary [8] for marginally trapped surface in \( \mathbb{L}^4 \) reduces to

\[
X_{\ast} dz = \begin{bmatrix}
(x_1)_{\ast} \\
(x_2)_{\ast} \\
(x_3)_{\ast} \\
(x_4)_{\ast}
\end{bmatrix} dz = \mathcal{N}_{\ast} = \begin{bmatrix}
0 \\
\frac{ih}{2} (1 + h^2) \\
\frac{ih}{2} (1 - h^2) \\
1
\end{bmatrix} dz,
\]

which recovers the Weierstrass representation of the second kind (due to O. Kobayashi [8] Corollary 1.3) for maximal surfaces in \( \mathbb{L}^4 = \mathbb{L}^4 \cap \{ x_4 = 0 \} \) equipped with the Lorentzian metric \( dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \). The induced conformal metric by the patch \( X \) is

\[
d s_{\ast\ast}^2 = 4 |\mathcal{M}_{\ast} - (\text{Re } h) \mathcal{N}_{\ast}|^2 |d z|^2 = 4 (\text{Re } h)^2 |\mathcal{N}_{\ast}|^2 |d z|^2.
\]

Minimal surfaces in Euclidean space \( \mathbb{R}^3 = \mathbb{L}^4 \cap \{ x_4 = 0 \} \) and maximal surfaces in Lorentz-Minkowski space \( \mathbb{L}^4 = \mathbb{L}^4 \cap \{ x_3 = 0 \} \) are examples of spacelike surfaces with null mean curvature vector in \( \mathbb{L}^4 \). We present a geometric representation for marginally trapped surfaces, which contains the classical Weierstrass representations for minimal surfaces and maximal surfaces simultaneously. The idea is to use Theorem [8] to prescribe the two height functions and the complexified Gauss map for the marginally trapped surface in \( \mathbb{L}^4 \).
**Corollary 9 (Third Weierstrass representation for marginally trapped surfaces in \( \mathbb{L}^4 \)).** Let \( \Omega \subset \mathbb{R}^2 \cong \mathbb{C} \) be a simply connected domain with the complex coordinate \( z \). Given a holomorphic function \( g : \Omega \to \mathbb{C} \setminus \{0\} \), we consider two \( C^2 \) functions \( A, B : \Omega \to \mathbb{R} \) satisfying the equation

\[
A_z = \frac{-1 + |g|^2}{1 + |g|^2} B_z,
\]

and the condition

\[
A_z \neq \frac{-1 + |g|^2}{1 + |g|^2} B_z.
\]

Then, there exists a conformal parameterization \( X = X_{\{g, A, B\}} : \Omega \to \mathbb{L}^4 \) of the marginally trapped surface \( \Sigma = X(\Omega) \) in Lorentz-Minkowski space \( \mathbb{L}^4 \) satisfying

\[
X_z = \begin{bmatrix}
(x_1)_z \\
(x_2)_z \\
(x_3)_z \\
(x_4)_z
\end{bmatrix} = A_z \begin{bmatrix}
\frac{i}{2} \left( \frac{\bar{g} - g}{|g|^2} \right) \\
\frac{1}{2} \\
1 \\
0
\end{bmatrix} + B_z \begin{bmatrix}
\frac{1}{2} \left( \frac{\bar{g} + g}{|g|^2} \right) \\
0 \\
0 \\
1
\end{bmatrix}.
\]

We have \( x_3(z) = A(z) \) and \( x_4(z) = B(z) \), up to additive constants. The Gauss map \( N \) of the marginally trapped surface \( \Sigma \) defined by

\[
N := \begin{bmatrix}
2 \Re g \\
2 \Im g \\
-1 + |g|^2 \\
1 + |g|^2
\end{bmatrix}
\]
satisfies the orthogonality condition \( \langle X_z, N \rangle = 0 \). The conformal metric induced by the patch \( X(z) \) is

\[
ds_{x}^2 = \left( |g| + \frac{1}{|g|^2} \right)^2 \left| A_z - \frac{-1 + |g|^2}{1 + |g|^2} B_z \right|^2 |dz|^2.
\]

**Proof.** We take \( (P, Q) = (\frac{A + B}{2}, -\frac{A + B}{2}) \). We obtain

\[
P_{z\bar{z}} - |g|^2 Q_{z\bar{z}} = \frac{1}{2} \left( \left( 1 + |g|^2 \right) A_{z\bar{z}} - \left( -1 + |g|^2 \right) B_{z\bar{z}} \right) = 0,
\]

and

\[
P_z - |g|^2 Q_z = \frac{1}{2} \left( \left( 1 + |g|^2 \right) A_z - \left( -1 + |g|^2 \right) B_z \right) \neq 0.
\]

Taking the triple \( (g, P, Q) = (g, \frac{A + B}{2}, -\frac{A + B}{2}) \) in Theorem 7 yields the existence of \( X_{\{g, P, Q\}} : \Omega \to \mathbb{L}^4 \) of the marginally trapped surface \( \Sigma = X_{\{g, P, Q\}}(\Omega) \) in \( \mathbb{L}^4 \) satisfying

\[
\frac{\partial}{\partial z} X_{\{g, P, Q\}} = P_z \begin{bmatrix}
\frac{i}{2} \\
1 \\
1 \\
1
\end{bmatrix} + Q_z \begin{bmatrix}
g \\
-ig \\
1 \\
1
\end{bmatrix}.
\]

Setting \( X_{\{g, A, B\}}(z) := X_{\{g, P, Q\}}(z) \) and using \( (P, Q) = (\frac{A + B}{2}, -\frac{A + B}{2}) \), we have

\[
\begin{bmatrix}
(x_1)_z \\
(x_2)_z \\
(x_3)_z \\
(x_4)_z
\end{bmatrix} = \frac{\partial}{\partial z} X_{\{g, A, B\}} = \frac{\partial}{\partial z} X_{\{g, P, Q\}} = A_z \begin{bmatrix}
\frac{i}{2} \left( \frac{\bar{g} - g}{|g|^2} \right) \\
\frac{1}{2} \left( \frac{\bar{g} + g}{|g|^2} \right) \\
1 \\
0
\end{bmatrix} + B_z \begin{bmatrix}
\frac{1}{2} \left( \frac{\bar{g} + g}{|g|^2} \right) \\
1 \\
0 \\
1
\end{bmatrix}.
\]

It follows from the equalities \( (x_3)_z = A_z \) and \( (x_4)_z = B_z \) that

\[
x_3(z) = A(z) \quad \text{and} \quad x_4(z) = B(z),
\]
up to additive constants. It follow from Theorem\(7\) that
\[
d_{\mathbb{H}_n}^2 = \frac{4}{|g|^2} \left| P_z - |g|^2 Q_z \right|^2 |dz|^2 = \left( |g| + \frac{1}{|g|} \right)^2 \left| A_z - \frac{-1 + |g|^2}{1 + |g|^2} B_z \right|^2 |dz|^2.
\]

Remark 4.2.

(1) Corollary\(9\) shows the method how to prescribe two height functions \(A = x_i, B = x_i,\) and the complexified Gauss map \(g\) of the marginally trapped surface in \(L^4.\)

(2) The proof of Corollary\(9\) indicates that \(X_{(g, A, B)} = X_{(g, P, Q)},\) up to an additive vector constant. In other words, the Weierstrass representation induced by the triple \((g, A, B)\) in Theorem\(9\) and the Weierstrass representation induced by the triple \((g, P, Q)\) in Corollary\(9\) yields the same marginally trapped surface in \(L^4,\) up to translations.

(3) In the particular case when \(A_z\) is holomorphic and \(B_z \equiv 0,\) the representation in Corollary\(9\) for marginally trapped surface in \(L^4\) reduces to
\[
X_z dz = \begin{bmatrix} (x_1)_z \cr (x_2)_z \cr (x_3)_z \cr (x_4)_z \end{bmatrix} dz = A_z \begin{bmatrix} \frac{1}{2} \left( \frac{1}{g} - g \right) \cr \frac{1}{2} \left( \frac{1}{g} + g \right) \cr 1 \cr 0 \end{bmatrix} dz,
\]
which recovers the Weierstrass representation for minimal surfaces in Euclidean space \(\mathbb{R}^3 = L^4 \cap \{x_4 = 0\}.\) The holomorphic differential \(A_z dz\) is the height differential in the \(x_3\)-coordinate. The induced conformal metric by the patch \(X\) is
\[
d_{\mathbb{H}_n}^2 = \left( |g| + \frac{1}{|g|} \right)^2 \left| A_z - \frac{-1 + |g|^2}{1 + |g|^2} B_z \right|^2 |dz|^2 = \left( |g| + \frac{1}{|g|} \right)^2 |A_z|^2 |dz|^2.
\]

(4) In the particular case when \(A_z \equiv 0\) and \(B_z\) is holomorphic, the representation in Corollary\(9\) for marginally trapped surface in \(L^4\) reduces to
\[
X_z dz = \begin{bmatrix} (x_1)_z \cr (x_2)_z \cr (x_3)_z \cr (x_4)_z \end{bmatrix} dz = B_z \begin{bmatrix} \frac{1}{2} \left( \frac{1}{g} - g \right) \cr \frac{1}{2} \left( \frac{1}{g} + g \right) \cr 0 \cr 1 \end{bmatrix} dz,
\]
which recovers the classical Weierstrass representation for maximal surfaces in Lorentz-Minkowski space \(L^4 = L^4 \cap \{x_3 = 0\}\) equipped with the Lorentzian metric \(dx_1^2 + dx_2^2 - dx_4^2.\) The holomorphic differential \(B_z dz\) is the height differential in the \(x_4\)-coordinate. The induced conformal metric by the patch \(X\) is
\[
d_{\mathbb{H}_n}^2 = \left( 1 + \frac{|g|^2}{|g|} \right)^2 \left| A_z - \frac{-1 + |g|^2}{1 + |g|^2} B_z \right|^2 |dz|^2 = \left( 1 + \frac{|g|}{|g|} \right)^2 \left| B_z \right|^2 |dz|^2.
\]

5. Geometric Interpretations of Parameters

The purpose of this section is to reveal geometric meanings of parameters in the elliptic, parabolic, hyperbolic deformations of our integral systems (introduced in Section\(2\)) in terms of the geometry of marginally trapped surfaces in Lorentz-Minkowski space \(L^4.\)

Proposition 10 (Geometric meaning of the parameter \(\lambda\) in the parabolic deformation in Proposition\(9\)). Let the triple \((g, P, Q)\) be a Weierstrass data of the first kind. Let \(\lambda \in \mathbb{R}\) be a parameter constant such that \(g(z) \neq \frac{1}{\lambda^*} \) for all \(z \in \Omega.\) We set \(g_\lambda := \frac{g}{1 + ig\lambda} \) and \(P_\lambda := P.\) Then, the following statements hold:

(1) There exists a \(C^2\) function \(Q_\lambda : \Omega \to \mathbb{R}\) satisfying the integrability condition
\[
(Q_\lambda)_z = \left( \frac{1}{g} + i\lambda \right) (gQ_z - i\lambda P_z).
\]
Moreover, the triple \((g_\lambda, P_\lambda, Q_\lambda)\) becomes a Weierstrass data of the first kind.

**Theorem 2.** The marginally trapped surface \(\Sigma_\lambda = X_{\{g_\lambda, P_\lambda, Q_\lambda\}}(\Omega)\) is congruent to the marginally trapped surface \(\Sigma = \Sigma_0 = X_{\{g, P, Q\}}(\Omega)\) in \(\mathbb{L}^4\). More concretely, we have
\[
X_{\{g_\lambda, P_\lambda, Q_\lambda\}} = L^{\text{parabolic}}_\lambda \left( X_{\{g, P, Q\}}(z) \right),
\]
up to a translation in \(\mathbb{L}^4\). Here, the parabolic rotation \(L^{\text{parabolic}}_\lambda\) in \(\mathbb{L}^4\) denotes the linear transformation represented by the matrix
\[
L^{\text{parabolic}}_\lambda = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\lambda & -\lambda \\
0 & \lambda & 1 - \frac{\lambda^2}{2} & -\frac{\lambda^2}{2} \\
0 & -\lambda & \frac{\lambda^2}{2} & 1 + \frac{\lambda^2}{2}
\end{bmatrix}.
\]

The item (2) indicates that the parabolic rotation \(L^{\text{parabolic}}_\lambda\) of the marginally trapped surface in \(\mathbb{L}^4\) induces the parabolic deformation in Proposition 3.

**Proof.** We need the contents of the second proof of Proposition 3 and the proof of Corollary 8. The deformation in the item (1) is proved in Proposition 3. First, the second proof of Proposition 3 reveals that the Weierstrass triple \((g_\lambda, P_\lambda, Q_\lambda)\) of the first kind corresponds to the Weierstrass triple of the second kind
\[
(h_\lambda, M_\lambda, N_\lambda) = (h + i\lambda, M, N).
\]
Second, the proof of Corollary 8 (and Remark 4.1) guarantees the existence of a conformal parameterization \(X_\lambda : \Omega \to \mathbb{L}^4\) of the marginally trapped surface in Lorentz-Minkowski space \(\mathbb{L}^4\):
\[
X_\lambda := X_{\{g_\lambda, P_\lambda, Q_\lambda\}} = X_{\{h_\lambda, M_\lambda, N_\lambda\}} = X_{(h + i\lambda, M, N)},
\]
up to an additive vector constant. Corollary 8 shows that
\[
(X_0)_z = M_z \begin{bmatrix}
1 \\
-ih \\
\frac{1}{2} (1 - h^2)
\end{bmatrix} + N_z \begin{bmatrix}
0 \\
i (h + i\lambda) \\
\frac{1}{2} (1 - (h + i\lambda)^2)
\end{bmatrix},
\]
and that
\[
(X_\lambda)_z = M_z \begin{bmatrix}
1 \\
-ih \\
\frac{1}{2} (1 - h^2)
\end{bmatrix} + N_z \begin{bmatrix}
0 \\
i (h + i\lambda) \\
\frac{1}{2} (1 + (h + i\lambda)^2)
\end{bmatrix}.
\]
However, a straightforward computation shows that
\[
M_z \begin{bmatrix}
1 \\
-ih \\
\frac{1}{2} (1 - h^2)
\end{bmatrix} + N_z \begin{bmatrix}
0 \\
i (h + i\lambda) \\
\frac{1}{2} (1 + (h + i\lambda)^2)
\end{bmatrix} = L^{\text{parabolic}}_\lambda \left( M_z \begin{bmatrix}
1 \\
-ih \\
\frac{1}{2} (1 - h^2)
\end{bmatrix} + N_z \begin{bmatrix}
0 \\
i (h + i\lambda) \\
\frac{1}{2} (1 + h^2)
\end{bmatrix} \right).
\]
Integrating the differential equation
\[
(X_\lambda)_z = L^{\text{parabolic}}_\lambda \left( (X_0)_z \right) = \left( L^{\text{parabolic}}_\lambda X_0 \right)_z,
\]
yields that, up to an additive vector constant,
\[
X_\lambda(z) = L^{\text{parabolic}}_\lambda \left( X_0(z) \right).
\]

**Proposition 11.** Geometric meaning of the parameter \(\tau\) in the elliptic deformation in Proposition 4. Let the triple \((h, M, N)\) be a Weierstrass data of the second kind. Given a parameter \(\tau \in \mathbb{R}\), we set \(h_\tau := e^{-\tau h}\) and \(N_\tau := N\). Then, the following statements hold:
(1) Then, there exists a \( C^2 \) function \( \mathcal{M}_\tau : \Omega \to \mathbb{R} \) satisfying the integrability condition
\[
(\mathcal{M}_\tau)_z = e^{i\tau} \mathcal{M}_z - \frac{e^{i\tau} - e^{-i\tau}}{2} h N_z.
\]
Moreover, the triple \((h_\ell, \mathcal{M}_\tau, N_\tau)\) becomes a Weierstrass data of the second kind.

(2) The marginally trapped surface \( \Sigma_\tau = X_{(h_\ell, \mathcal{M}_\tau, N_\tau)}(\Omega) \) is congruent to the marginally trapped surface \( \Sigma = \Sigma_0 = X_{(h, \mathcal{M}, N)}(\Omega) \) in \( \mathbb{L}^4 \). More concretely, we have
\[
X_{(h_\ell, \mathcal{M}_\tau, N_\tau)} = \mathcal{L}_\tau^{\text{elliptic}} \left( X_{(h, \mathcal{M}, N)}(z) \right),
\]
up to a translation in \( \mathbb{L}^4 \). Here, the elliptic rotation \( \mathcal{L}_\tau^{\text{elliptic}} \) in \( \mathbb{L}^4 \) denotes the linear transformation represented by the matrix
\[
\mathcal{L}_\tau^{\text{elliptic}} = \begin{bmatrix}
\cos \tau & -\sin \tau & 0 & 0 \\
\sin \tau & \cos \tau & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
The item (2) indicates that the elliptic rotation \( \mathcal{L}_\tau^{\text{elliptic}} \) of the marginally trapped surface in \( \mathbb{L}^4 \) induces the elliptic deformation in Proposition 8.

Proof. First, the second proof of Proposition 8 reveals that the Weierstrass triple \((h_\ell, \mathcal{M}_\tau, N_\tau)\) of the first kind corresponds to the Weierstrass triple of the second kind
\[
(g_\ell, \mathcal{P}_\tau, \mathcal{Q}_\tau) = (e^{i\tau} g, \mathcal{P}, \mathcal{Q}).
\]
Second, the proof of Corollary 8 (and Remark 4.1) guarantees the existence of a conformal parameterization \( X_\tau : \Omega \to \mathbb{L}^4 \) of the marginally trapped surface in Lorentz-Minkowski space \( \mathbb{L}^4 \):
\[
X_\tau := X_{(h_\ell, \mathcal{M}_\tau, N_\tau)} = X_{[g_\ell, \mathcal{P}_\tau, \mathcal{Q}_\tau]} = X_{[e^{i\tau} g, \mathcal{P}, \mathcal{Q}]}
\]
up to an additive vector constant. Corollary 8 shows that
\[
(X_0)_z = \mathcal{P}_z \begin{bmatrix}
\frac{1}{g} \\
\frac{1}{g}
\end{bmatrix} + \mathcal{Q}_z \begin{bmatrix}
g \\
-i g
\end{bmatrix}
\]
and that
\[
(X_\tau)_z = \mathcal{P}_z \begin{bmatrix}
e^{i\tau}g \\
e^{i\tau}g
\end{bmatrix} + \mathcal{Q}_z \begin{bmatrix}
e^{i\tau}g \\
-e^{i\tau}g
\end{bmatrix} = \mathcal{L}_\tau^{\text{elliptic}} ((X_0)_z) = \left( \mathcal{L}_\tau^{\text{elliptic}} X_0 \right)_z.
\]
Integrating this differential equation yields that, up to an additive vector constant,
\[
(X_\tau)(z) = \mathcal{L}_\tau^{\text{elliptic}} X_0(z).
\]

\]
Corollary 8 implies that Lorentz-Minkowski space $L^4$ admits the existence of a conformal parameterization $X\eta$ of the marginally trapped surface in $L^4$.

Proof. The deformation in the item (1) is proved in Proposition 5. Corollary 8 guarantees the existence of a conformal parameterization $X\eta: \Omega \to L^4$ of the marginally trapped surface in Lorentz-Minkowski space $L^4$:

$$X\eta := X_{\{g_\eta, \mathcal{P}_\eta, \mathcal{Q}_\eta\}} = X_{\{e^{\eta q}, e^{\eta p}, e^{-\eta q}\}}.$$  

Corollary 8 implies that

$$(X_0)_z = \mathcal{P}_z \begin{bmatrix} \frac{1}{g} \\ \frac{-i}{g} \\ 1 \\ 1 \end{bmatrix} + \mathcal{Q}_z \begin{bmatrix} g \\ -ig \\ -1 \\ 1 \end{bmatrix}$$

and that

$$(X\eta)_z = e^{\eta q} \mathcal{P}_z \begin{bmatrix} \frac{1}{e^g} \\ \frac{-i e^g}{e^g} \\ 1 \\ 1 \end{bmatrix} + e^{-\eta q} \mathcal{Q}_z \begin{bmatrix} e^g \\ -ie^g \\ -1 \\ 1 \end{bmatrix} = L^{\text{hyperbolic}}_{\eta}(X_0)_z = (L^{\text{hyperbolic}}_{\eta} X_0)_z.$$

Integrating this differential equation yields that, up to an additive vector constant,

$$X\eta(z) = L^{\text{hyperbolic}}_{\eta}(X_0(z)).$$

☐

6. EXAMPLES OF MARGINA LLY TRAPPED SURFACES

We use our integrable systems and Weierstrass representations to construct explicit examples of the marginally trapped surface in $L^4$ with nowhere vanishing mean curvature vector. We adopt the complex coordinate $z = u + iv$ with $u, v \in \mathbb{R}$. To use Corollary 8 we need to prepare the Weierstrass triple $(h, \mathcal{M}, \mathcal{N})$ of the second kind, which solves the system

$$\begin{cases} h_{\overline{z}} = 0, \\ \mathcal{M}_{\overline{z}} = (\text{Re} h) N_{\overline{z}}, \\ \mathcal{M}_z = (\text{Re} h) N_z \neq 0. \end{cases}$$

We take the normalization with the nowhere vanishing holomorphic function

$$h(z) = e^{iz} = e^{-v} (\cos u + i \sin u) \neq 0.$$  

Then, we need to find a pair $(\mathcal{M}(u, v), \mathcal{N}(u, v))$ of $\mathbb{R}$-valued functions solving the Poisson equation of the form

$$\mathcal{M}_{uu} + \mathcal{M}_{vv} = e^{-v} \cos u (N_{uu} + N_{vv}).$$

We observe that this linear equation admits the following two independent solutions:

$$\begin{align*}
(\mathcal{M}_1(u, v), \mathcal{N}_1(u, v)) &= (\sinh u \sin u, e^v \cosh u), \\
(\mathcal{M}_2(u, v), \mathcal{N}_2(u, v)) &= (\cos u \cos v, e^v \sin v).
\end{align*}$$

Using these two solutions and the linear structure of the above Poisson equation, we construct families of marginally trapped surfaces with nowhere vanishing mean curvature vector.
Example 6.1 (A family of marginally trapped surfaces in $\mathbb{L}^4$ connecting from a CMC-1 surface in $\mathbb{H}^3$ to a CMC-1 surface in $\mathbb{S}^3$). We construct a one parameter family $\{\Sigma_\theta\}_{\theta \in [0, \pi]}$ of marginally trapped surfaces with nowhere vanishing mean curvature vector field. Moreover, the surface $\Sigma_\theta$ lies in the hypersurface in $\mathbb{L}^4$:

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = - \cos (2\theta).$$

Given $\theta \in \left[0, \frac{\pi}{2}\right]$, we choose an open domain $\Omega_\theta \subset \mathbb{R}^2$ satisfying the condition

$$\cos \theta \cosh u + \sin \theta \cos v \neq 0,$$

for all $(u, v) \in \Omega_\theta$. For instance, if $\theta \in \left(0, \frac{\pi}{2}\right)$ we take

$$\Omega_\theta = \mathbb{R}^2.$$

For $\theta = \frac{\pi}{2}$, we take

$$\Omega_\frac{\pi}{2} = \left\{ (u, v) \in \mathbb{R}^2 \mid u > 0, \ v \in (-\pi, \pi) \right\};$$

For $\theta = \frac{\pi}{4}$, we take

$$\Omega_\frac{\pi}{4} = \left\{ (u, v) \in \mathbb{R}^2 \mid u \in \mathbb{R}, \ v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}.$$

Given $\theta \in \left[0, \frac{\pi}{2}\right]$, we define the triple $(h_\theta, M_\theta, N_\theta)$ by

$$h_\theta(z) = e^{\xi z} = e^{-\nu} (\cos u + i \sin u) \neq 0,$$

$$M_\theta(z) = \cos \theta \sinh u \sin u - \sin \theta \cos u \cos v,$$

$$N_\theta(z) = e^{\nu} (\cos \theta \cosh u + \beta \sin v).$$

The triple $(h, M, N) = (h_\theta, M_\theta, N_\theta)$ is a Weierstrass data of the second kind:

$$h_{\pm} = 0,$$

$$4N_{\pm} = e^{\nu} (2 \cos \theta \cosh u \sin v) ,$$

$$4M_{\pm} = \cos u (2 \cos \theta \cosh u \sin v) = (\text{Re} \ h) (4N_{\pm}),$$

$$M_z = (\text{Re} \ h) N_z = \frac{i}{2} e^{-iu} (\cos \theta \cosh u \sin v \cos v).$$

Applying Corollary 8 to the triple $(h_\theta, M_\theta, N_\theta)$, we can find a conformal patch $X_\theta := X_{(h_\theta, M_\theta, N_\theta)} : \Omega_\theta \to \mathbb{L}^4$ of the marginally trapped surface $\Sigma_\theta$ in $\mathbb{L}^4$:

$$X_\theta(u, v) = \cos \theta X_0(u, v) + \sin \theta X_{\pm}(u, v),$$

where we introduce

$$X_0(u, v) = \begin{bmatrix} \sinh u \sin u \\ \sinh u \cos u \\ \cosh u \sin v \\ \cosh u \cosh v \end{bmatrix} \quad \text{and} \quad X_{\pm}(u, v) = \begin{bmatrix} -\cos u \cos v \\ \sin u \cos v \\ \cosh v \sin v \\ \sinh v \sin v \end{bmatrix}.$$

The metric of the spacelike surface $\Sigma_\theta$ induced by the patch $X_\theta$ is

$$\Lambda_\theta(z) |dz|^2 = (\cos \theta \cosh u + \sin \theta \cos v)^2 \left( du^2 + dv^2 \right).$$

We claim that the mean curvature vector $H$ vanishes nowhere on the surface $\Sigma_\theta$:

$$H = \Delta_x_\theta(z) |dz|^2 X_\theta = \frac{1}{\Lambda_\theta(z)} \left( (X_\theta)_{uu} + (X_\theta)_{vv} \right).$$

Recall that $\cos \theta \cosh u + \sin \theta \cos v \neq 0$ for all $(u, v) \in \Omega_\theta$. The null vector

$$\left( X_\theta \right)_{uu} + \left( X_\theta \right)_{vv} = 2 (\cos \theta \cosh u + \sin \theta \cos v) \begin{bmatrix} \cos u \\ -\sin u \\ \sinh v \\ \cosh v \end{bmatrix}$$

vanishes nowhere because it is not possible that $\cos u$ and $-\sin u$ can be zero simultaneously. It now remains to show that the the spacelike surface $\Sigma_\theta$, defined by the patch

$$X_\theta = \cos \theta X_0 + \sin \theta X_{\pm},$$

is marginally trapped.
lies in the hypersurface \( x_1^2 + x_2^2 + x_3^2 - x_4^2 = -\cos(2\theta) \) in \( \mathbb{L}^4 \). We have

\[
\langle X_0, X_0 \rangle = \begin{pmatrix}
\sinh u \sin u \\
\sinh u \cos u \\
\cosh u \sinh v \\
\cosh u \cosh v \\
\end{pmatrix}, \quad \begin{pmatrix}
\sinh u \sin u \\
\sinh u \cos u \\
\cosh u \sinh v \\
\cosh u \cosh v \\
\end{pmatrix} = -1,
\]

\[
\langle X_0, X_\pi \rangle = \begin{pmatrix}
\sinh u \sin u \\
\sinh u \cos u \\
\cosh u \sinh v \\
\cosh u \cosh v \\
\end{pmatrix}, \quad \begin{pmatrix}
-\cos u \cos v \\
\sin u \cos v \\
\cosh v \sin v \\
\sinh v \sin v \\
\end{pmatrix} = 0,
\]

\[
\langle X_\pi, X_\pi \rangle = \begin{pmatrix}
-\cos u \cos v \\
\sin u \cos v \\
\cosh v \sin v \\
\sinh v \sin v \\
\end{pmatrix}, \quad \begin{pmatrix}
-\cos u \cos v \\
\sin u \cos v \\
\cosh v \sin v \\
\sinh v \sin v \\
\end{pmatrix} = 1.
\]

We conclude that

\[
\langle X_\theta, X_\theta \rangle = \langle \cos \theta X_0 + \sin \theta X_\pi, \cos \theta X_0 + \sin \theta X_\pi \rangle = -\cos^2 \theta + \sin^2 \theta = -\cos(2\theta).
\]

**Remark 6.1.**

(1) Taking \( \theta = 0 \) in Example 6.1 yields a marginally trapped surface \( \Sigma_0 \subset \mathbb{L}^4 \) parameterized by

\[
X_0(u, v) = \begin{pmatrix}
\sinh u \sin u \\
\sinh u \cos u \\
\cosh u \sinh v \\
\cosh u \cosh v \\
\end{pmatrix}, \quad (u, v) \in \mathbb{R}^2.
\]

The surface \( \Sigma_0 \) is a Bryant surface [3], which lies in the hyperboloid model of the three dimensional hyperbolic space \( \mathbb{H}^3 \subset \mathbb{L}^4 \) given by the hypersurface

\[
x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1,
\]

and has mean curvature one in \( \mathbb{H}^3 \). The surface \( \Sigma_0 \) is called a Bryant surface [3] in \( \mathbb{H}^3 \) in the sense that it is locally isometric to a catenoid, which is a rotational surface with zero mean curvature in \( \mathbb{R}^3 \). The conformal metric on the surface \( \Sigma_0 \) induced by the patch \( X_0(u, v) \) is

\[
cosh^2 u \left( du^2 + dv^2 \right),
\]

which is the conformal metric induced by the patch \( X_{\text{cat}}(u, v) \) of the catenoid \( \Sigma_{\text{cat}} \) in the three dimensional Euclidean space \( \mathbb{R}^3 = L^4 \cap \{x_4 = 0\} \):

\[
X_{\text{cat}}(u, v) = \begin{pmatrix}
\cosh u \sin v \\
\cosh u \cos v \\
u \\
0
\end{pmatrix}, \quad (u, v) \in \mathbb{R}^2.
\]

(2) Taking \( \theta = \frac{\pi}{2} \) in Example 6.1 yields a marginally trapped surface \( \Sigma_\pi \subset \mathbb{L}^4 \) parameterized by

\[
X_\pi(u, v) = \begin{pmatrix}
-\cos u \cos v \\
\sin u \cos v \\
\cosh v \sin v \\
\sinh v \sin v \\
\end{pmatrix}, \quad (u, v) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).
\]

The surface \( \Sigma_\pi \) is lies in the model of the three dimensional de-Sitter space \( S^3 \subset \mathbb{L}^4 \) given by the hypersurface

\[
x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1,
\]

and has mean curvature one in \( S^3 \). The surface \( \Sigma_\pi \) can be viewed as a hyperbolic catenoid cousin in \( S^3 \). The conformal metric on the surface \( \Sigma_\pi \) induced by the patch \( X_\pi(u, v) \) is

\[
cos^2 v \left( du^2 + dv^2 \right),
\]

\[\]
which is the conformal metric induced by the patch $X_{\text{heat}}(u, v)$ of the hyperbolic catenoid $\Sigma_{\text{heat}}$ in the three dimensional Lorentz-Minkowski space $\mathbb{L}^3 = \mathbb{L}^4 \cap \{ x_1 = 0 \}$:

$$X_{\text{heat}}(u, v) = \begin{bmatrix} 0 \\
\cos v \sinh u \\
\cos v \cosh u \end{bmatrix}, \quad (u, v) \in \mathbb{R} \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).$$

(3) Taking $\theta = \frac{\pi}{2}$ in Example 6.1 yields a marginally trapped surface $\Sigma_{\|} \subset \mathbb{L}^4$ parameterized by

$$X_{\|}(u, v) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sinh u \sin v \\
\sinh u \cos v \\
\cosh u \sinh v \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -\cos v \sin u \\
\sin v \cos u \\
\sinh v \sin v \end{bmatrix}, \quad (u, v) \in \mathbb{R} \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).$$

The surface $\Sigma_{\|}$ is null in the sense that it lies in the light cone $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$.

**Example 6.2** (Two parameter deformations of a catenoid cousin in $\mathbb{H}^3$ to marginally trapped surfaces in $\mathbb{L}^4$). We take $\Omega = \mathbb{R}^2 \equiv \mathbb{C}$ and the complex coordinate. We produce a two parameter family of marginally trapped surfaces with nowhere vanishing mean curvature vector field. Given a pair $(\alpha, \beta)$ of real constants such that $\alpha^2 + \beta^2 < 1$, we define the triple $(h(\alpha, \beta), M(\alpha, \beta), N(\alpha, \beta))$ by

$$h(\alpha, \beta)(z) = e^{i\alpha} = e^{-v} (\cos u + i \sin u) \neq 0,$$

$$M(\alpha, \beta)(z) = \alpha u + \beta v + \sinh u \sin u,$$

$$N(\alpha, \beta)(z) = e^v \cosh u.$$

The triple $(h, M, N) = (h(\alpha, \beta), M(\alpha, \beta), N(\alpha, \beta))$ is a Weierstrass data of the second kind:

$$h_{\|} = 0,$$

$$4M_{\|} = 2 \cos u \cosh u = (e^{-v} \cos u) (2e^v \cosh u) = (\Re h) (4N_{\|}),$$

$$M_{\|} - (\Re h) N_{\|} = (\alpha + \cosh u \sin u) + i (-\beta + \cosh u \cos u) \neq 0.$$

The proof of the last statement is given as follows. Assume to the contrary that there exists a pair $(u, v)$ of real numbers such that $M_{\|} - (\Re h) N_{\|} = 0$, or equivalently, that

$$\cosh u \sin u = -\alpha \quad \text{and} \quad \cosh u \cos u = \beta.$$

Combining the assumption $a^2 + b^2 < 1$ and these two conditions yields

$$1 > \alpha^2 + \beta^2 = (\cosh u \sin u)^2 + (\cosh u \cos u)^2 = \cosh^2 u \geq 1,$$

which is a contradiction. Applying Corollary to the triple $(h, M, N)$, we can find a conformal patch $X_{(\alpha, \beta)} := X_{(h, M, N)} : \mathbb{R}^2 \to \mathbb{L}^4$ of the marginally trapped surface $\Sigma_{(\alpha, \beta)}$ in $\mathbb{L}^4$:

$$X_{(\alpha, \beta)}(u, v) = \alpha \begin{bmatrix} u \\
-e^{-v} \sin u \\
e^{-v} \sin u \end{bmatrix} + \beta \begin{bmatrix} -u \\
-\cosh u \cos v \\
\cosh u \sin v \end{bmatrix} + \begin{bmatrix} \sinh u \sin u \\
\sinh u \cos u \\
\cosh u \sin v \end{bmatrix}.$$

The metric of the spacelike surface $\Sigma_{(\alpha, \beta)}$ induced by the patch $X_{(\alpha, \beta)}$ is

$$\Lambda_{(\alpha, \beta)}(z)|dz|^2 = \left( (\alpha + \cosh u \sin u)^2 + (-\beta + \cosh u \cos u)^2 \right) \left( du^2 + dv^2 \right).$$

Finally, we claim that the mean curvature vector $H$ vanishes nowhere on the surface $\Sigma_{(\alpha, \beta)}$:

$$H = \Delta \Lambda_{(\alpha, \beta)}(z)|dz|^2 X_{(\alpha, \beta)} = \frac{1}{\Lambda_{(\alpha, \beta)}(z)} \left( (X_{(\alpha, \beta)})_{uu} + (X_{(\alpha, \beta)})_{vv} \right).$$

However, the null vector

$$(X_{(\alpha, \beta)})_{uu} + (X_{(\alpha, \beta)})_{vv} = 2 \cosh u \begin{bmatrix} \cos u \\
-\sin u \\
\sinh v \cosh v \end{bmatrix}$$

vanishes nowhere because it is not possible that $\cos u$ and $-\sin u$ can be zero simultaneously.
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