Finite-time consensus of leader-following non-linear multi-agent systems via event-triggered impulsive control

Tao Chen | Shiguo Peng | Zhenhua Zhang

School of Automation, Guangdong University of Technology, Guangzhou, China

Correspondence
Shiguo Peng, School of Automation, Guangdong University of Technology, Guangzhou 510006, China. Email: sgpeng@gdut.edu.cn

Funding information
National Natural Science Foundation of China, Grant/Award Numbers: 61973092, 61374081; National Natural Science Foundation of China-Guangdong Joint Fund, Grant/Award Number: 2019A1515012104

Abstract
This work is concerned with the finite-time consensus of leader-following non-linear multi-agent systems by means of distributed event-triggered impulsive control. The finite-time consensus protocol is first put forward based on event-triggered impulsive strategies, where the impulsive instants are determined by the proposed event-triggered condition. For the event-triggered condition, it not only determines the impulsive instants but also effects the update time of the finite-time control. Moreover, compared with the existing finite-time controllers, the controller designed in this work does not contain any sign function, thereby overcoming the chattering phenomenon. In addition, without finding Zeno behaviour, the feasibility of the proposed control protocol is demonstrated. Lastly, simulations are employed to demonstrate the effectiveness of the proposed control schemes.

1 | INTRODUCTION

With advance of artificial intelligence (AI) and computer technology at a furious space, lots of multi-agent technologies could be seen everywhere in our daily life [1–4]. Consensus problem on multi-agent systems (MASs) has been a popular object among them and has been widely applied into many fields including swarm-based computing [3], formation control of advanced unmanned aerial vehicles (UAVs)[6], group consensus of robots [7], and so on. Thus, a great deal of researches have been published about consensus problem. In the meantime, various kinds of control methods served consensus problem have been vigorously developed, which consist of intermittent control [8, 9], feedback control [10, 11], sliding mode control [12], and so on [13–17].

As far as we know, impulsive control as a class of discontinuous-time control has attracted a considerable amount of attention owing to the real applications in many areas [18–22]. Compared with the continuous-time control, impulsive control has a number of merits, which can reduce the amount of data transmission and ameliorate robustness of the system. Moreover, the cost of control can be reduced because of smaller control gain. In [18], the MASs with stochastically switching topologies have been investigated by means of impulsive control. In [21], by using quantized relative state measurements, the authors designed a impulsive controller to deal with the fixed-time quantized consensus problem of leader-following non-linear multi-agent systems (LNMASs), which cut down the operational costs of the systems. Besides, the impulsive pinning control scheme was proposed to exponential consensus problem of stochastic LNMASs with time-varying delays in [22]. In a word, impulsive control can deal with the problem that can not be solved by continuous control.

However, in the existing results, most designed impulsive controllers are based on the time-triggered mechanism and are periodic [23–25]. Inevitably, in order to guarantee the consensus rate of the controlled system, the time-triggered impulsive frequency may be designed so high that it leads to unnecessary consumption of energy resources [26, 27]. Hence, for the sake of saving resources and achieving the expected performance, an appropriate candidate is the event-triggered impulsive control method. With regard to the event-triggered impulsive control approach, it has more merits than other time-triggered control approach. This method not only significantly reduces the update number of control but also effectively improves the utilization of the limited bandwidth resource. Therefore, many scholars have been interested in resorting to the event-triggered impulsive control approach to deal with MASs [28–32]. With the help of a distributed event-triggered impulsive control protocol, the authors achieved the consensus of leader-following
MASs in [29]. It should be noted that the impulsive instants relied on the event triggering instants. In [31], the memristive neural networks with time-varying delays realized the quasi-synchronization via state feedback and event-triggered impulsive strategies, which the update time of the state feedback control input and working time of impulsive controller are determined by the event-triggered instants. In [32], by means of the event-based impulsive control method, the exponential stabilization of continuous-time systems has been investigated. Also, the proposed strategy has been applied to the synchronization of memristive neural networks.

It is worth mentioning that most of the recent studies relating to event-triggered consensus focus only on the asymptotic convergence, which means that the convergence can be achieved within infinite time. In practical applications, however, the equipments need to complete all planned tasks in the finite time. Besides, the service life of equipment is limited. Therefore, the finite-time control is of great significance to accelerate convergence rate and demonstrate better robustness. Taking into account these advantages, many researches about the finite-time control have been established in recent years [33–37]. Based on the distributed event-triggered control, the authors in [35] verified that linear MASs can realize the finite-time consensus, which the consensus speed can be further adjusted. In [37], the distributed finite-time event-driven strategy was put forward for MASs with single-integrator model. The setting time depending on the initial state and the event-triggering threshold could be estimated. To the best of our knowledge, few works address the distributed finite-time event-driven strategy was put forward for the distributed event-triggered impulsive control, the authors in [35] demonstrated the effectiveness of analytical results in Section4. Finally, the conclusions are drawn in Section5.

Notations: Hopefully, \( \mathbb{R} \) stands for the real number set, \( \mathbb{R}^N \) denotes the set of N-dimensional Euclidean space, and \( \mathbb{R}_+ \) represents the set of nonnegative real numbers. The set of positive integers are denoted by \( \mathbb{N}_+ \). \( 1 = (1, 1, \ldots, 1)^T \) denotes an N-dimensional column vector with all elements being 1, and \( I_N \) stands for the N-dimensional unit matrix. \( \|x\| = (x^T x)^{1/2} \) is the Euclidean norm of \( x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N \), where the superscript \( T \) stands for transpose. \( |x| \) is the absolute value of \( x \). A \( P \) \( (P > 0) \) denotes a negative (positive) definite matrix. \( \lambda_2(P), \lambda_N(P) \), and \( P^T \) denotes the minimum non-zero eigenvalue, the maximum eigenvalue and the transpose of matrix \( P \), respectively. diag\{\cdots\} represents a diagonal matrix. \( \emptyset \) stands for the empty set. For \( \chi : \mathbb{R} \rightarrow \mathbb{R} \), denotes \( \chi(\ell^-) = \lim_{\tau \rightarrow 0^+} \chi(\ell + \tau) \) and \( \chi(\ell^+) = \lim_{\tau \rightarrow 0^-} \chi(\ell + \tau) \), and the upper-right Dini derivative \( D^+ \chi(\ell) = \limsup_{\tau \rightarrow 0^+} \frac{\chi(\ell + \tau) - \chi(\ell)}{\tau} \).

2 | PROBLEM FORMULATION AND PRELIMINARIES

2.1 Algebraic graph theory

Briefly, we will introduce the basic algebraic graph theory in this subsection. As a tool, graph theory plays an important role in information exchange among agents in the multi-agent systems. In this work, an undirected graph \( G = (\mathcal{V}, \mathcal{E}, A) \) with \( N \) agents is considered by us, where the vertex set \( \mathcal{V} = \{0, 1, \ldots, N\} \) represents the corresponding agent and \( \mathcal{G} \subseteq \mathcal{V} \times \mathcal{V} \) stands for the set of edges. Besides, \( A = [a_{ij}]_{N \times N} \) denotes the weighted adjacency matrix of undirected graph \( \mathcal{G} \), which its elements can be defined as: \( a_{ij} = \begin{cases} >0, & (i, j) \in \mathcal{G}; \\ 0, & \text{otherwise}. \end{cases} \)

Let \( D = \text{diag}[d_1, d_2, \ldots, d_N] \) denote the degree matrix of \( \mathcal{G} \).
with \( d_i = \sum_{j=1, j\neq i}^N a_{ij} \). Then, the Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{N \times N} \) of graph \( G \) is defined as \( L = D - A \). Moreover, let \( B = \text{diag}(a_{10}, a_{20}, \ldots, a_{N0}) \) stand for the leader adjacency matrix used for describing whether the followers have an exchange of information with the leader in graph \( \bar{G} \) consisting of \( N + 1 \) agents. The elements \( a_{0i} > 0 \) of \( B \) represent the leader connecting to the \( i \)-th agent, otherwise, \( a_{0i} = 0 \). Let \( H = L + B = [b_{ij}]_{N \times N} \).

### 2.2 Some lemmas

In this subsection, we list some lemmas, which are useful for the theoretical analysis in this article.

**Lemma 1 ([40])**. For any \( x_i \in \mathbb{R}, i = 1, 2, \ldots, N, \) and \( 0 < \rho \leq 1 \), then the following inequalities hold:

\[
\left( \sum_{i=1}^N |x_i| \right)^\rho \leq \sum_{i=1}^N |x_i|^\rho \leq N^{1-\rho} \left( \sum_{i=1}^N |x_i| \right)^\rho.
\]

**Lemma 2 ([41])**. Owing to a connected undirected graph \( \mathcal{G} \), the properties of the Laplacian matrix \( L \) of graph \( \mathcal{G} \) are presented in the following words: \( x^T L x = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (x_i - x_j)^2 \), for any \( x = (x_1, x_2, \ldots, x_N)^T \). The \( L \) is positive semi-definite matrix which has \( N \) positive real eigenvalues. Suppose that the eigenvalues of \( L \) are stood for \( 0, \lambda_2, \ldots, \lambda_N \) satisfying \( 0 < \lambda_2 \leq \cdots \leq \lambda_N \). Therefore, its minimum non-zero eigenvalue \( \lambda_2 > 0 \). Further more, \( x^T L x \geq \lambda_2 x^T x \) when \( \forall i, x = 0 \).

**Remark 1.** As we know, both adjacency matrix \( A \) and Laplacian matrix \( L \) of the connected undirected graph \( \mathcal{G} \) are symmetric. In this work, we consider the graph \( \mathcal{G} \) consisting of the followers and a leader, which the followers are undirected connection with each other and the leader at least has one directed path to the followers. Namely, the graph \( \mathcal{G} \) is a directed graph. So, according to the above-related graph theory and previous results [33], the matrix \( H \) of graph \( \mathcal{G} \) is positive definite and symmetric.

**Lemma 3 ([42]).** Let \( V : \mathbb{R} \to \mathbb{R}_+ \) be a positive definite and Lipschitz continuous function, if there exist two constants \( c > 0 \) and \( 0 < \alpha < 1 \) so that the Dini derivative of \( V(t) \) satisfies

\[
D^+ V(t) \leq -cV^\alpha(t), \forall t \geq t_0, \quad V(t_0) \geq 0.
\]

Then \( V(t) \) satisfies

\[
V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0) - c(1-\alpha)(t-t_0), \quad t_0 \leq t \leq \tau,
\]

and \( V(t) \equiv 0, \forall t \geq \tau \), with the settling time \( \tau \) given by

\[
\tau = t_0 + \frac{V^{1-\alpha}(t_0)}{c(1-\alpha)}.
\]

**Lemma 4 ([43]).** Assume that there exist three constants \( c > 0, 0 < \alpha < 1, 0 < \delta_k < 1 \), which \( k = \{1, 2, \ldots, m\} \) is a finite positive integer set and \( m \) is a positive integer, and a continuous, non-negative function \( V(t) \) such that the Dini derivative of \( V(t) \) satisfies the conditions as follows:

\[
\begin{align*}
D^+ V(t) & \leq -cV^\alpha(t), \quad t \neq t_k, \\
V(t_k^+) & \leq \delta_k V(t_k^-), \quad t = t_k.
\end{align*}
\]

Then the following inequality holds:

\[
V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0) - c(1-\alpha)(t-t_0), \quad t_0 \leq t \leq \tau,
\]

where \( \tau \) is a constant which represents the settling time.

**Lemma 5 ([44]).** Suppose a symmetric matrix \( P = [p_{ij}] \) where \( p_{11} \) is a symmetric matrix, then the following statements are equivalent:

1. \( P < 0 \),
2. \( P_{11} < 0, P_{22} - P_{12}^T P_{11}^{-1} P_{12} < 0 \),
3. \( P_{22} < 0, P_{11} - P_{12} P_{22}^{-1} P_{12}^T < 0 \).

### 2.3 Problem formulation

Suppose that the LNMSs consist of \( N \) followers and a leader, the communication topology is a connected graph which will be introduced later. Firstly, the dynamics of the follower agents can be described as follows

\[
\dot{x}_i(t) = f(x_i(t), t) + u_i(t),
\]

where \( x_i(t) \in \mathbb{R} \) is the state of agent \( i, i = 1, \ldots, N \), \( f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) is the continuous non-linear function, \( u_i(t) \in \mathbb{R} \) is the control input.

Then, consider the dynamics of the leader is defined as

\[
\dot{x}_0(t) = g(x_0(t), t),
\]

where \( x_0(t) \in \mathbb{R} \) is the state of leader \( 0, g : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) is the continuous non-linear function.

**Remark 2.** Note that a great deal of physical systems have the characteristic of non-linearity in nature. Compared with the existing results in [29] and [35], the models we studied include the non-linear part which is more suitable to the practical application.

Throughout the whole article, the following assumption and definition are put forward before starting the main results.

**Assumption 1.** For any \( x_i, x_j \in \mathbb{R} \), there exists a known and positive constant \( \varepsilon \) such that

\[
|f(x_i, t_j, t)| \leq \varepsilon|x_i - x_j|.
\]
Definition 1. The NMASs (1) and (2) are said to achieve leader-following finite-time consensus, if there exists a constant $\tau > 0$ which depends on any initial state $x_i(0)$ and $x_0(0)$ such that

$$\lim_{t \to \tau} |x_i(t) - x_0(t)| = 0, \forall i \in N.$$ 

Then, $|x_i(t) - x_0(t)| \equiv 0$ when $t \geq \tau$. Here, $\tau$ is called the settling time.

3 | MAIN RESULTS

In this section, the event-triggered impulsive control method is considered to investigate the finite-time consensus of LNMASs (1).

In order to achieve consensus of the LNMASs in finite time, we propose the following control protocol for agent $i$:

$$u_i(t) = -\rho_1 \left[ \sum_{j=0}^{N} a_{ij} \left( x_j(t_k) - x_j(t_{k+1}) \right) \right] + \rho_2 \left[ \sum_{j=0}^{N} a_{ij} \left( x_j(t_k) - x_j(t_{k+1}) \right) \right] + \sum_{j=1}^{\infty} \mu \left( \sum_{j=0}^{N} a_{ij} \left( x_j(t) - x_j(t') \right) \right) \delta(t - t'),$$

where $\rho_1, \rho_2$ denote the positive constants; $\nu < \omega$. $\mu \in R$ stands for the impulsive strength which satisfies $0 < \mu < 1, t \in N_+$. The sequence of event triggering instants $\{t_k\}$ is defined as $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots$, which are decided by the subsequent events, and $\delta(t)$ represents the Dirac function.

Remark 3. Different from the control protocol in [29], [34] and [37], both finite-time controller and event-triggered impulsive control are considered in this work. Inspired by [31], we design an event-triggered impulsive control to achieve the leader-following finite-time consensus in this work. It is noteworthy that the update time of the finite-time control and the working time of the impulsive controller are determined by the event-triggered instants. The impulsive control only works at $t_k$, and the impulsive instants are relied on the triggered condition. Therefore, the event-triggered time sequence is also impulsive instant sequence. Based on the designed control scheme, the distributed event-triggered scheme of control loop is shown as Figure 1.

Remark 4. Compared with the controller in [34] and [37], the controller (3) in this work does not contain the sign function. So far as anyone can tell, the sign function always gives rise to a chattering phenomenon which has an effect on systems and control signals. In the existing results on studying the finite-time problems, sign function is an essential part of the controller. In our study, the finite-time consensus will be investigated by designing a controller without sign function.

Inspired by [[29], Remark 3], similarly, we give a chronological order to amalgamate all the time sequence $\{t_k\} \cap N_+$, $i \in \mathbb{N}$. Let the set of time sequence all agents $\{t_k\} = \{t_k : i \in \mathbb{N}, n \in N_+\}$, which satisfy $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, $\lim_{k \to \infty} t_k = +\infty$. The sequence is shown as in Figure 2.

We can obtain $\Phi_i(k) = 1$ if the event of agent $i$ is triggered at $t_k$, then there exists an appropriate $k$ such that $\{t_k \mid \Phi_i(k) = 1, k \in N_+\} = \{t_k\}$; otherwise, $\{t_k \mid \Phi_i(k) = 0, k \in N_+\} \cap \{t_k\} = \emptyset$. 

FIGURE 1 The distributed event-triggering block of the finite-time consensus via the event-triggered impulsive control

FIGURE 2 Distributed event-triggered time sequence of all agents in the system
Therefore, the dynamics of follower agents can be further obtained from (1) and (3)

\[ \dot{x}_i(t) = f(x_i(t), t) - g_i(k)\rho_1 \left[ \sum_{j=0}^{N} a_{ij}(x_i(t) - x_j(t_k)) \right] + \rho_2 \left[ \sum_{j=0}^{N} a_{ij}(x_i(t_k) - x_j(t_k)) \right] + g_i(k) \delta(t - t_k), \]

where the state of agent \( i \) is changed from \( x_i(t_k) \) to \( x_i(t_{k+1}) \) when \( \tilde{g}_i(k) = 1 \).

Then, we define the tracking error as

\[ \Delta x_i(t) = x_i(t) - x_0(t), \quad i = 1, 2, \ldots, N, \]

and thus

\[ \Delta x(t) = (\Delta x_1(t), \Delta x_2(t), \ldots, \Delta x_N(t))^T. \]

For developing our event-triggered strategy, we consider the combined measurement function to be defined as

\[ q_i(t) = \sum_{j=0}^{N} a_{ij}(x_i(t) - x_j(t)). \]

Obviously, one can be obtained that \( q(t) = (q_1(t), q_2(t), \ldots, q_N(t))^T = H\Delta x(t) \in \mathbb{R}^N \).

We assume that at least one agent’s controller is triggered at time \( t_k \). So, motivated by [29], Remark 3], the event-triggered matrix \( \Lambda = \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_N) \) is used to express the triggered state of each agent in the system.

Therefore, the control law can be further described as

\[ u(t) = u_x(t) + u_0(t) \]

\[ = -\rho_1 \Delta x_0^2(t_k) - \rho_2 \Delta x_0(t_k) + \Lambda \sum_{j=1}^{\infty} [\mu q_j(t)] \delta(t - t_j), \]

where \( \Delta x_0(t_k) = (\Delta x_1(t_k), \Delta x_2(t_k), \ldots, \Delta x_N(t_k))^T \), \( u_x(t) = (u_1(t), u_2(t), \ldots, u_N(t))^T \), \( u_0(t) = (u_1(t), u_2(t), \ldots, u_N(t))^T \), \( u_x(t_k) = -\rho_1 \Delta x_0^2(t_k) - \rho_2 \Delta x_0(t_k) \) and \( u_0(t_k) = \Lambda \sum_{j=1}^{\infty} [\mu q_j(t)] \delta(t - t_j) \).

Further, \( \nu(t) \) can be rewritten as

\[ u(t) = u_x(t) + u_0(t) \]

\[ = -[\nu(t) + q_1^2(t_k)] + \Lambda \sum_{j=1}^{\infty} [\mu q_j(t)] \delta(t - t_j), \]

where \( \nu(t) = \rho_1 \Delta x_0^2(t_k) + \rho_2 \Delta x_0(t_k) - \rho_2 q_1^2(t_k) - \rho_2 q_2(t_k) \).

Remark 5. As we know, traditional measurement error functions in [29] and [36] are defined as \( \nu(t) = x_i(t) - x_i(t_k) \), where \( t_k \) is the event time of agent \( i \). In this work, the construction of the new measurement error function \( \nu(t) \) is decided by the control input \( u_x(t) \) and the combined measurement function \( q_i(t) \). Hence, the constructed measurement error function can simplify the process of the following theoretical proof.

Combining (1) with (2), and substituting (9) into (1), the track error dynamics (5) of LNMSs with event-triggered impulsive control and finite-time control can be depicted as the following form:

\[ \begin{cases} \dot{x}(t) = F(x(t), t) - \nu(t) + \rho_1 q_1^2(t_k) - \rho_2 q_2(t_k), & t \in [t_k, t_{k+1}] \setminus \mathbb{T}, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = \mu q_1(t_k), & t = t_k, \end{cases} \]

where \( F(x(t), t) = (f_i(x, t), f_2(x, t), \ldots, f_N(x, t))^T \), \( f_i(x, t) = f_i(x(t), t) - f_i(x_0(t), t) \). Without loss of generality, assuming that \( x_i(t) \) is right-hand continuous at \( t = t_k \), for instance, \( x_i(t_k) = x_i(t_k^+) \), \( x_i(t_k^-) = x_0(t_k^-) \), \( x_i(t_k^-) = x_i(t_k^+) \), \( x_i(t) = x_i(t_k^+) - x_i(t_k^-) \).

Next, we present a theoretical result to guarantee that the LNMSs (1) and (2) can achieve consensus within finite time by means of the control protocol (3), where the triggering instant \( t_k \) is decided by Algorithm 1.

**Theorem 1.** Under all the aforementioned assumptions and the control law (3), if there exist constants \( \rho_1 > 0, \rho_2 > 0, \varepsilon > 0, \theta > 0 \) satisfying

\[ \varepsilon \leq \frac{1}{2} \rho_2 \Lambda^2(H), \]

\[ b_i(t) < 0, \quad t \in [t_k, t_{k+1}], \]

\[ [-\theta H (I_N + \mu \Lambda H)^T] < 0, \]

\[ 0 < \theta < 1, \]

where \( \Lambda^2(H) \) is the minimum non-zero eigenvalue of matrix \( H \), \( b_i(t) = \nu_i(t) - \frac{1}{2} \rho_2 q_2(t_i) \) denotes the event-triggered function, and the triggered
Algorithm 1 Finite-time consensus Algorithm of LNMASs via event-triggered impulsive control strategy

Require:
2: Initialize all parameters, such as control protocol parameters \( \rho_1, \rho_2, \nu, \omega \), and so on.
3: Input the last triggering times \( t'_i \), state of agents \( x_i(t) \), \( i = 0, 1, \ldots, N \), and so on.

Ensure:
5: for \( t = t_0 \) to \( t_{\text{end}} \) do
6: Compute the threshold \( h_i(t) \).
7: for \( i = 0, 1, \ldots, N \) do
8: Compute the measurement error \( e_i(t) \) based on formula (6).
9: Compute the threshold \( h_i(t) \).
10: if \( h_i(t) \geq 0 \) then
11: The event has occurred, and the event instant has been recorded as \( t_{k+1} \).
12: Update the state \( x_i(t) \) of agent \( i \) at instant \( t_{k+1} \), where the finite-time control of agent \( i \) is also updated at the triggering instant \( t_{k+1} \).
13: else
14: Update the state \( x_i(t) \) of agent \( i \) at instant \( t \) which belongs to interval \( [t'_i, t_{k+1}] \), and the finite-time control is updated in the last triggering instant \( t'_i \).
15: end if
16: end for

An impulsive and finite-time control instant \( t_k \) is determined by the following event-triggered condition:

\[
\tau_k = \inf \left\{ t \in (t_k, +\infty) | h_i(t) \geq 0 \right\}. \quad (15)
\]

then, the LNMs with the controller (3) can realize the finite-time consensus in the settling time \( \tau \), and the settling time \( \tau \) bounded as follows:

\[
\tau = \frac{V^{\frac{\nu + \omega}{2w}}(0)}{2 - \rho_1 \lambda_H^{\frac{\nu + \omega}{2w}} \left( 1 - \frac{\nu}{\omega} \right)}. \quad (16)
\]

Proof. Define the Lyapunov candidate function as

\[
V(t) = \frac{1}{2} \tilde{x}^T(t)H\tilde{x}(t). \quad (17)
\]

From Lemma 2, it can be obtained that \( \sum_{i=1}^{N} q_i(t) = (H^T\tilde{x}(t)) \times (H^T\tilde{x}(t)) \geq \lambda_H(H^T\tilde{x}(t)) \tilde{x}(t) \) and \( \sum_{i=1}^{N} q_i(t) = (H^T\tilde{x}(t)) \times (H^T\tilde{x}(t)) \geq \lambda_H(H) \tilde{x}(t) \) for all \( t \in \mathbb{R} \). Therefore, one has \( q_i(t) \leq \|q_i(t)\| \leq \sqrt{2\lambda_H(H)}V(0), \) where \( \lambda_H(H) \) is the maximum eigenvalue of matrix \( H \).

For \( t \in [t_k, t_{k+1}) \), \( k \in \mathbb{N}_+ \), calculating \( D^+ V(t) \) with respect to the trajectories of (10), it yields that

\[
D^+ V(t) = \tilde{x}^T(t)H\tilde{x}(t) = \tilde{x}^T(t)(F(\tilde{x}(t), t) - e(t) - \rho_1 q_i(t) - \rho_2 q_i(t)) = \tilde{x}^T(t)H(F(\tilde{x}(t), t) - \tilde{x}^T(t)H e(t) - \rho_1 \tilde{x}^T(t)H q_i(t) - \rho_2 \tilde{x}^T(t)H q_i(t)). \quad (18)
\]

Based on the Assumption 1, one can obtain that

\[
\tilde{x}^T(t)H F(\tilde{x}(t), t) \leq \varepsilon \tilde{x}^T(t)H \tilde{x}(t). \quad (19)
\]

Then, according to Lemma 1 and condition (11) (12), we have

\[
D^+ V(t) \leq \sum_{i=1}^{N} q_i(t) \|e_i(t)\| - \frac{1}{2} \rho_2 \|q_i(t)\| + \varepsilon \tilde{x}^T(t)H \tilde{x}(t) - \rho_1 \sum_{i=1}^{N} q_i^2(t) \leq \rho_1 \left( \sum_{i=1}^{N} q_i^2(t) \right)^{\frac{\nu + \omega}{2w}} + \varepsilon \tilde{x}^T(t)H \tilde{x}(t) - \rho_2 \sum_{i=1}^{N} q_i^2(t) \leq \rho_1 \left( 2\lambda_2(H) \right)^{\frac{\nu + \omega}{2w}} \tilde{x}^T(t)H \tilde{x}(t) + 2\varepsilon V(t) - \rho_2 \lambda_2(H) V(t) \leq \rho_1 \left( 2\lambda_2(H) \right)^{\frac{\nu + \omega}{2w}} \tilde{x}^T(t)H \tilde{x}(t). \quad (20)
\]

By Lemma 3, it is obvious that

\[
V^{1-\frac{\nu + \omega}{2w}}(t) \leq V^{1-\frac{\nu + \omega}{2w}}(t_0) - \rho_1 \left( 2\lambda_2(H) \right)^{\frac{\nu + \omega}{2w}} \times \left( 1 - \frac{\nu + \omega}{2w} \right) (t - t_0). \quad (21)
\]

On the other hand, when \( t = t_k, k \in \mathbb{N}_+ \), according to Lemma 5 (10) and conditions (13) and (14),

\[
V(t_k) = \frac{1}{2} [ (I_N + \mu H \tilde{x}(t_k)^T) H (I_N + \mu H \tilde{x}(t_k)^T) ] \tilde{x} \leq \frac{1}{2} \theta V(t_k). \quad (22)
\]

From Lemma 4, one obtains

\[
V^{1-\frac{\nu + \omega}{2w}}(t) \leq V^{1-\frac{\nu + \omega}{2w}}(t_0) - \rho_1 \left( 2\lambda_2(H) \right)^{\frac{\nu + \omega}{2w}} \times \left( 1 - \frac{\nu + \omega}{2w} \right) (t - t_0), t_0 \leq t \leq \tau. \quad (23)
\]
Evidently, the error system can be found to tend to zero in the finite time, and the settling time is estimated as follows:

\[
\tau = \frac{\omega \gamma}{V \frac{\omega}{\gamma} (0)} \frac{V}{2 \rho_1 \lambda^2(H)} \frac{\gamma}{\omega}. \tag{24}
\]

As consequence, based on the above discussions, the LNMASs (1) and (2) can realize consensus within finite time \(\tau\) by means of event-triggered impulsive control. This completes the proof.

In order to get rid of the Zeno behaviour, the following Theorem is verified that the inter-execution time intervals \(t'_{k+1} - t'_k\) has a lower bound. For this bound, it is a strictly positive constant.

**Theorem 2.** Consider the tracking error dynamics of LNMAS (10) under the control rule (3), the impulsive instant \(t_k\) is triggered by the condition (15). For any initial condition, there exists a strictly positive constant \(t\) which is the lower bound of inter-execution time intervals \(t'_{k+1} - t'_k\) as follows. Namely, the system (10) is free of the Zeno behaviour.

\[
t = \frac{\rho_2}{2(\kappa_1 + \kappa_2)^2 + \rho_2(\kappa_1 + \kappa_2)}, \tag{25}
\]

where \(\kappa_1 = \rho_1 \frac{\gamma}{\omega} (\lambda_N \frac{\gamma}{V(0)} - \rho_2), \quad \kappa_2 = ||H|| (\rho_1 \lambda_N \frac{\gamma}{V(0)} + \rho_2 + N \zeta).\)

**Proof.** For \(t \in [t_k, t_{k+1})\), let

\[
\Psi_i(t) = \frac{|v_i(t)|}{|q_i(t)|}. \tag{26}
\]

According to (6), (8), (10) and the event-triggered condition (15), it can be obtained that

\[
D^+ \Psi_i(t) = \frac{[v_i(t)] |q_i(t)| - |v_i(t)| |\dot{q}_i(t)|}{|q_i(t)|^2}. \tag{27}
\]

Based on (8), substituting the first equation of (10) into (27), we can further get that

\[
D^+ \Psi_i(t) \leq \frac{|v_i(t)| |q_i(t)| - |v_i(t)| |\dot{q}_i(t)|}{|q_i(t)|^2} + \frac{|e_i(t)| |\dot{q}_i(t)|}{|q_i(t)|^2} \leq (\kappa_1 + \Psi_i(t))\left[|v_i(t)| - |v_i(t)| |\dot{q}_i(t)|\right] \leq (\kappa_1 + \Psi_i(t)) \leq (\kappa_1 + \Psi_i(t)) (\kappa_2 + \Psi_i(t)) \leq (\kappa_1 + \kappa_2 + \Psi_i(t))^2. \tag{28}
\]

Let \(\dot{\Gamma}(t) = (x_1 + x_2 + \Gamma(t))^2, \quad \text{and} \quad \Psi_i(t) \text{satisfies the bound} \quad \Psi_i(t) \leq \Gamma(t, \Gamma_0), \quad \text{where} \quad \Gamma(t, \Gamma_0) \text{is the solution of} \quad \dot{\Gamma}(t) \text{with the initial value being} \quad \Gamma(0, \Gamma_0) = \Gamma_0.\)

By computing, we can further obtain that

\[
\Gamma(t, 0) = \frac{t(\kappa_1 + \kappa_2)^2}{1 - t(\kappa_1 + \kappa_2)^2} > 0. \tag{29}
\]

Based on the above discussions, the lower bound \(\tau\) of inter-execution time interval is verified to exist and it satisfies \(t'_{k+1} - t'_k \geq \tau > 0.\) Therefore, the Zeno behaviour is excluded and the proof is completed.

When we remove \(f(x_i(t), t)\) and \(g(x_i(t), t)\), the systems become the leader-following linear multi-agent systems. The track error dynamics (10) can be reduced to the following form as

\[
\begin{align*}
\dot{\tilde{x}}(t) &= -e(t) - \rho_1 q(t) - \rho_2 q(t), \\
\dot{t} &\in [t_k, t_{k+1}), \quad k \in \mathbb{N}_+, \\
\Delta \tilde{x}(t_k) &= \tilde{x}(t_k) - \tilde{x}(t_k^-) = \mu \Lambda q(t_k^-), \\
\dot{t} &= t_k.
\end{align*} \tag{30}
\]

Then, we can easily obtain the following corollary.

**Corollary 1.** Under the control law (3), if there exist constants \(\rho_1 > 0, \rho_2 > 0, \theta > 0\) satisfying

\[
b_i(t) < 0, \quad t \in [t_k, t_{k+1}), \tag{31}
\]

\[
-\delta \langle H_N + \mu \Lambda H \rangle^T H < 0, \tag{32}
\]

\[
0 < \theta < 1, \tag{33}
\]

where \(\Lambda_2(H)\) is the minimum non-zero eigenvalue of matrix \(H\), \(b_i(t) = |e_i(t)| - \frac{\rho_1}{\rho_2} |q_i(t)|\) denotes the event-triggered function, and the triggered impulsive and finite-time control instant \(t_k\) is determined by the following event-triggered condition:

\[
t_{k+1} = \inf\{t \in (t_k, +\infty) | b_i(t) \geq \theta\}. \tag{34}
\]

then, the tracking error systems (30) with the controller (3) can realize the finite-time consensus in the settling time \(\tau\), and the settling time \(\tau\) bounded
Figure 3: The communication topology $\mathcal{G}$ of a multi-agent system as follows:

$$\tau = \frac{V^{\frac{\omega}{2\omega}}(0)}{2^\frac{\omega}{2\omega} \rho_1 \lambda_2(H)^{\frac{\omega}{2\omega}}(1 - \frac{\nu}{\omega})}.$$  \hspace{1cm} (35)

**Proof.** Similar proof method with that of Theorem 1, so we omit it. Therefore, the leader-following linear MASs can realize finite-time consensus by means of event-triggered impulsive control. This completes the proof. \hfill \Box

4 | NUMERICAL SIMULATION

In this section, we will provide one example to validate the effectiveness of the proposed control schemes.

It is assumed that the LNMAss consist of four followers and a leader, and the communication topology $\mathcal{G}$ is shown in Figure 3.

From the Figure 3, the Laplacian matrix $\mathcal{L}$ and the leader adjacency matrix $\mathcal{B}$ are obtained as follows:

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

thus,

$$\mathcal{H} = \mathcal{L} + \mathcal{B} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$  \hspace{1cm}

Considering the LNMAss with 4 follower agents and a leader agent are described by:

$$\begin{align*}
\dot{x}_i(t) &= f(x_i(t), t) + u_i(t), \quad i = 1, 2, 3, 4, \\
\dot{x}_0(t) &= g(x_0(t), t),
\end{align*}$$  \hspace{1cm} (36)

where non-linear characteristics are $f(x_i(t), t) = 0.5x_i(t) + 0.6\cos(x_i(t)), g(x_0(t), t) = 0.5x_0(t) + 1.5\sin(x_0(t))$, and $u_i(t)$ is in the form of (3).

Set the initial states of the followers as $x(0) = (1.5, 0.8, 0.1 - 2)^T$. The initial state of the leader is chosen as $x_0(0) = 0.3$. Figure 4 shows that the agents and the leader do not reach an agreement in the absence of the controller.

According to (3), the parameters of the controller are chosen as $\rho_1 = 1, \rho_2 = 2, \frac{\omega}{2\omega} = 3/5$. By simple calculation, it can be obtained that $\mu = 0.5$, which satisfies the condition (13) of Theorem 1. Besides, the condition (11) in Theorem 1 implies that $\epsilon = 0.5$ should be satisfied. The numerical results of the system under controller (3) are depicted in Figures 5–8. According to (16), the settling time can be calculated that $\tau = 1.42$.

In Figures 5 and 6 under protocol (3), we can see that the setting time is approximate $t = 0.2$, which proves the effectiveness and feasibility of Theorem 1. According to the event-triggered...
FIGURE 6 Tracking error trajectories of the agents under the event-triggered impulsive control.

FIGURE 7 Event triggering time instants and time of every agent in the MASs.

FIGURE 8 The finite-time strategy \( u_\omega \) with event-triggered control.

FIGURE 9 The state trajectories of system (1) only via finite-time strategy with \( \rho_1 = 12, \rho_2 = 4 \) and \( \omega = 9/11 \).

FIGURE 10 The state trajectories of system (1) only via event-triggered impulsive control with \( \mu = 0.8 \).

condition, the event-triggering instants of every agent in the MASs are illustrated in Figure 7.

To further illustrate the superiority of our strategy, we give two comparison simulations. Firstly, in [34], being short of event-triggered impulsive control, the state trajectories of LNMSAs only with finite-time strategy is depicted in Figure 9. Comparing with Figure 9, it can be obviously found that the LNMSAs controlled only by finite-time strategy need the bigger control gains to achieve consensus, which could result in the higher control cost.

On the other hand, we simulate Figure 10 based on [29]. Comparing with the Figure 10, it shows the state trajectories of LNMSAs only with event-triggered impulsive control, while the systems under our approach achieve consensus with a faster speed.

In a word, based on comparing the above results, the LNMSAs controlled by (3) can achieve consensus within the
finite time. Besides, the control cost can be cut down because of small control gains.

5 | CONCLUSION

This work studied the finite-time consensus of LNMA s by means of the distributed event-triggered impulsive controller. The designed controller does not include the sign function, which eliminates the chattering phenomenon. Several sufficient conditions are derived by the event-triggered condition, which determines the impulsive instants and update time of finite-time control. Also, the event-triggered rules have been demonstrated to perform well and can get rid of the Zeno behaviour. Numerical examples have been presented to show the effectiveness of the main results.

As we know, the distributed event-triggered impulsive mechanism is a popular research. Meantime, the stochastic discrete-time network is also a hotspot. How to realize finite-time consensus of stochastic discrete-time network via distributed event-triggered impulsive scheme is our future research interest.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China under Grants 61374081 and 61973092, and the Guangdong Basic and Applied Basic Research Foundation under Grant 2019A1515012104.

ORCID

Tao Chen  https://orcid.org/0000-0001-5320-9930

REFERENCES

1. Yin, X., Dong, Y., Hu, S.: Adaptive periodic event-triggered consensus for multi-agent systems subject to input saturation. Int. J. Control 89(4), 1–21 (2015)
2. He, W., et al.: Leader-following consensus of nonlinear multiagent systems with stochastic sampling. IEEE Trans. Cybern. 47(2), 327–338 (2017)
3. Su, S., Lin, Z.: Distributed consensus control of multi-agent systems with higher order agent dynamics and dynamically changing directed interaction topologies. IEEE Trans. Autom. Control 61(2), 515–519 (2016)
4. Tan, X., et al.: Leader-following mean square consensus of stochastic multi-agent systems with input delay via event-triggered control. IET Control Theory Appl. 12(2), 299–309 (2018)
5. Meng, Z., Lin, Z., Ren, W.: Leader-follower swarm tracking for networked lagrange systems. Syst. Control Lett. 61(1), 117–126 (2012)
6. Wang, X., Yadav, V., Balakrishnan, S.N.: Cooperative uav formation flying with obstacle/collision avoidance. IEEE Trans. Control Syst. Technol. 15(4), 672–679 (2007)
7. Liu, J., Jin, Z.: Distributed impulsive group consensus in second-order multi-agent systems under directed topology. Int. J. Control 88(5), 910–919 (2015)
8. Zhang, G., Shen, Y.: Exponential synchronization of delayed memristor-based chaotic neural networks via periodically intermittent control. Neural Netw. 55, 1–10 (2014)
9. Zhang, G., Shen, Y.: Exponential stabilization of memristor-based chaotic neural networks with time-varying delays via intermittent control. IEEE Trans. Neural Netw. Learn. Syst. 26(7), 1431–1441 (2015)
10. Wang, L., Shen, Y.: Finite-time stabilizability and instability stablility of delayed memristive neural networks with nonlinear discontinuous controller. IEEE Trans. Neural Netw. Learn. Syst. 26(11), 2914–2924 (2017)
11. Wu, A., Zeng, Z.: ‘Global mittag-leffler stabilization of fractional-order memristive neural networks’. IEEE Trans. Neural Netw. Learn. Syst. 28(1), 206–217 (2017)
12. Liu, X., et al.: H∞ stochastic synchronization for master–slave semi-markovian switching system via sliding mode control. Complexity 21(6), 430–441 (2016)
13. He, C., et al.: The adaptive distributed observer approach to the cooperative output regulation of linear multi-agent systems. Automatica 75, 299–305 (2017)
14. Sun, Q., et al.: A multiagent-based consensus algorithm for distributed coordinated control of distributed generators in the energy internet. IEEE Trans. Smart Grid 6(6), 3006–3019 (2015)
15. Qian, Y., et al.: Second-order consensus of multi-agent systems with nonlinear dynamics via impulsive control. Neurocomputing 125(3), 142–147 (2014)
16. Cheng, Y., et al.: Distributed finite-time attitude regulation for multiple rigid spacecraft via bounded control. Inf. Sci. 328, 144–157 (2016)
17. Han, T., et al.: Three-dimensional containment control for multiple unmanned aerial vehicles. J. Frankl. Inst. 353(13), 2925–2942 (2016)
18. Zhang, H., et al.: Impulsive consensus of multi-agent systems with stochastically switching topologies. Nonlinear Anal-Hybrid Syst. 26, 212–224 (2017)
19. Gao, Z.M., He, Y., Wu, M.: Improved stability criteria for the neural networks with time-varying delay via new augmented Lyapunov–Krasovskii functional. Appl. Math. Comput. 349, 258–269 (2019)
20. Li, X., Song, S.: Stabilization of delay systems: Delay-dependent impulsive control. IEEE Trans. Autom. Control 62(1), 406–411 (2017)
21. Xu, Z., Li, C., Han, Y.: Leader-following fixed-time quantized consensus of multi-agent systems via impulsive control. J. Frankl. Inst. 356(1), 441–456 (2019)
22. Ren, H., et al.: Exponential consensus of nonlinear stochastic multi-agent systems with rous and rons via impulsive pinning control. IET Control Theory Appl 11(2), 225–236 (2016)
23. Li, X., Ho, D.W.C., Cao, J.: ‘Finite-time stability and settling-time estimation of nonlinear impulsive systems’. Automatica 99, 361–368 (2019)
24. Lu, X., Zhang, X., Liu, Q.: Finite-time synchronization of nonlinear complex dynamical networks on time scales via pinning impulsive control. Neurocomputing 275, 2014–2110 (2018)
25. Li, H., et al.: Fixed-time stability and stabilization of impulsive dynamical systems. J. Frankl. Inst. 354(18), 8626–8644 (2017)
26. Ren, H., Deng, F., Peng, Y.: Finite time synchronization of markovian jumping stochastic complex dynamical systems with mix delays via hybrid control strategy. Neurocomputing 272, 683–693 (2018)
27. Xu, Z., Li, C., Han, Y.: Leader-following fixed-time quantized consensus of multi-agent systems via impulsive control. J. Frankl. Inst. 356(1), 441–456 (2019)
28. Xu, W., Ho, D.W.C.: Clustered event-triggered consensus analysis: An impulsive framework. IEEE Trans. Ind. Electron. 63(11), 7133–7143 (2016)
29. Tan, X., Cao, J., Li, X.: Consensus of leader-following multiagent systems: A distributed event-triggered impulsive control strategy. IEEE Trans Cybern 49(3), 792–801 (2019)
30. Ni, J., et al.: Consensus of second-order multi-agent systems via event-triggered impulsive control. In: 2017 Chinese Automation Congress (CAC), pp. 4346–4350, 2017
31. Zhou, Y., Zeng, Z.: Event-triggered impulsive control on quasi-synchronization of memristive neural networks with time-varying delays. Neural Netw. 110, 55–65 (2019)
32. Zhu, W., et al.: Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks. IEEE Trans. Neural Netw. Learn. Syst. 20(8), 3599–3609 (2018)
33. Liu, J., et al.: Distributed event-triggered fixed-time consensus for leader-follower multiagent systems with nonlinear dynamics and uncertain disturbances. Int. J. Robust Nonlinear Control 28(11), 3543–3559 (2018)
34. Dong, Y., Xian, J.: Finite-time event-triggered consensus for nonlinear multi-agent networks under directed network topology. IET Control Theory Appl 11(15), 2458–2464 (2017)
35. Cao, Z., et al.: Finite-time consensus of linear multi-agent system via distributed event-triggered strategy. J. Frankl. Inst. 355(3), 1338–1350 (2018)
36. Zhang, H., et al.: Finite-time distributed event-triggered consensus control for multi-agent systems. Inf. Sci. 339, 132–142 (2016)
37. Hu, B., Guan, Z.H., Fu, M.: Distributed event-driven control for finite-time consensus. Automatica 103, 88–95 (2019)
38. Yang, X., et al.: Fixed-time synchronization of complex networks with impulsive effects via nonchattering control. IEEE Trans. Autom. Control 62(11), 5511–5521 (2017)
39. Zhang, W., Yang, X., Li, C.: Fixed-time stochastic synchronization of complex networks via continuous control. IEEE Trans Cybern 49(8), 3099–3104 (2018)
40. Li, S., Du, H., Lin, X.: Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. Automatica 47(8), 1706–1712 (2011)
41. Olfati-Saber, R., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control 49(9), 1520–1533 (2004)

42. Bhat, S.P., Bernstein, D.S.: Finite-time stability of continuous autonomous systems. SIAM J. Control Optim. 38(3), 751–766 (2000)
43. Ren, H., Deng, F., Peng, Y.: Finite time synchronization of markovian jumping stochastic complex dynamical systems with mix delays via hybrid control strategy. Neurocomputing 272, 683–693 (2018)
44. Boyd, S., et al.: History of linear matrix inequalities in control theory. In: Proceedings of 1994 American Control Conference, pp. 31–34, ACC (1994)

How to cite this article: Chen T, Peng S, Zhang Z. Finite-time consensus of leader-following non-linear multi-agent systems via event-triggered impulsive control. IET Control Theory Appl. 2021;15:926–936. https://doi.org/10.1049/cth2.12092