AN EQUIVALENCE BETWEEN DESINGULARIZED AND RENORMALIZED VALUES OF MULTIPLE ZETA FUNCTIONS AT NEGATIVE INTEGERS

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ABSTRACT. It is known that the special values of multiple zeta functions at non-positive arguments are indeterminate in most cases due to the occurrences of infinitely many singularities. In order to give a suitable rigorous meaning of the special values there, Furusho, Komori, Matsumoto and Tsumura introduced the desingularized values by the desingularization method to resolve all singularities. While, Ebrahimi-Fard, Manchon and Singer introduced the renormalized values to keep the “shuffle” relation by the renormalization procedure à la Connes and Kreimer. In this paper, we reveal an equivalence, that is, an explicit interrelationship between these two values. As a corollary, we also obtain an explicit formula to describe renormalized values in terms of Bernoulli numbers.

CONTENTS

0. Introduction 1
1. Desingularizations 3
1.1. The desingularization method and desingularized MZF s 4
1.2. Desingularized values 5
2. Renormalizations 6
2.1. Algebraic frameworks 6
2.2. An explicit formula for the reduced coproduct \( \tilde{\Delta}_0 \) 8
2.3. The algebraic Birkhoff decomposition and renormalized values 11
3. Main results 14
3.1. Recurrence formulas among renormalized values 14
3.2. An equivalence between desingularized values and renormalized ones 19
References 21

0. INTRODUCTION

In 1776, Euler ([11]) considered a certain power series, the so-called double zeta values, and showed several relations among them. More than 200 years later than Euler, the multiple zeta value (MZV for short) which is more general series

\[
\zeta(k_1, \ldots, k_n) := \sum_{0 < m_1 < \cdots < m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}
\]

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converging for \( k_1, \ldots, k_n \in \mathbb{N} \) and \( k_n > 1 \), appeared in \cite{10} written by Ecalle again, in 1981. In 1990s, these values also came to be focused by Hoffman (\cite{15}) and Zagier (\cite{20}). The MZV admits an iterated integral expression, which enables us to regard it as a period of a certain motive. (\cite{7}, \cite{13} and \cite{19}). MZVs are related to mathematical physics in \cite{3} and \cite{4}. They are explained in \cite{22}.

MZVs are regarded as special values at positive integer points of the multiple zeta-function (MZF for short), the series

\[
\zeta(s_1, \ldots, s_n) := \sum_{0 < m_1 < \cdots < m_n} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}
\]

which converges absolutely in the region

\[
\{ (s_1, \ldots, s_n) \in \mathbb{C}^n \mid \Re(s_{n-k+1} + \cdots + s_n) > k \ (1 \leq k \leq n) \}. \tag{0.1}
\]

In the early 2000s, Zhao (\cite{21}) and Akiyama, Egami and Tanigawa (\cite{11}) independently showed that MZF can be meromorphically continued to \( \mathbb{C}^n \). Especially, in \cite{11}, the set of all singularities of the function \( \zeta(s_1, \ldots, s_n) \) is determined as

\[
\begin{align*}
  s_n &= 1, \\
  s_{n-1} + s_n &= 2, 1, 0, -2, -4, \ldots, \\
  s_{n-k+1} + \cdots + s_n &= k - r \ (3 \leq k \leq n, \ r \in \mathbb{N}_0).
\end{align*} \tag{0.2}
\]

Because almost all of integer points with non-positive arguments are located in the above singularities, the special values of MZF there are indeterminate in all cases except for \( \zeta(-k) \) at \( k \in \mathbb{N}_0 \), and \( \zeta(-k_1, -k_2) \) at \( k_1, k_2 \in \mathbb{N}_0 \) with \( k_1 + k_2 \) odd. Actually, giving a nice definition of \( \zeta(-k_1, \ldots, -k_n) \) for \( k_1, \ldots, k_n \in \mathbb{N}_0 \) is one of our most fundamental problems.

In order to resolve all infinitely many singularities of MZF, the desingularization method was introduced by Furusho, Komori, Matsumoto and Tsumura in \cite{12}. By applying this method to \( \zeta(s_1, \ldots, s_n) \), they constructed the desingularized MZF \( \zeta_{\text{FKMT}}(s_1, \ldots, s_n) \) which is entire on the whole space \( \mathbb{C}^n \) and they also showed its basic properties. The desingularized value

\[
\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n) \in \mathbb{C} \tag{0.3}
\]

is given as the special value of \( \zeta_{\text{FKMT}}(s_1, \ldots, s_n) \) at \( (s_1, \ldots, s_n) = (-k_1, \ldots, -k_n) \) for \( k_1, \ldots, k_n \in \mathbb{N}_0 \) (see Definition\( 1.4\)). In \cite{12}, its generating function given by

\[
Z_{\text{FKMT}}(t_1, \ldots, t_n) := \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(-t_1)^{k_1} \cdots (-t_n)^{k_n}}{k_1! \cdots k_n!} \zeta_{\text{FKMT}}(-k_1, \ldots, -k_n) \tag{0.4}
\]

in \( \mathbb{C}[[t_1, \ldots, t_n]] \) was calculated and the desingularized values were described in terms of the Bernoulli numbers. (See Proposition\( 1.5\)).

In contrast, Connes and Kreimer (\cite{10}) started a Hopf algebraic approach to the renormalization procedure in the perturbative quantum field theory. A fundamental tool in their work is the algebraic Birkhoff decomposition (Theorem\( 2.6\)). By applying this decomposition to a certain Hopf algebra parameterizing regularized MZVs, Guo and Zhang (\cite{14}) gave the renormalized values which satisfy the harmonic relations.

\footnote{It is denoted by \( \zeta_n^{\text{deq}}((s_j); (1)) \) in \cite{12}.}
introduced the different renormalized values which obey harmonic(-like) relations by using different Hopf algebras. Meanwhile, Ebrahimi-Fard, Manchon and Singer ([8]) also introduced another type of the renormalized values (cf. Definition 2.8) satisfying the “shuffle relations” (see Proposition 2.10 for precise), which in this paper we denote as

\[ \zeta_{\text{EMS}}(-k_1,\ldots,-k_n) \in \mathbb{C} \]

for \( k_1,\ldots,k_n \in \mathbb{N}_0 \), and which we consider with its generating function given by

\[ Z_{\text{EMS}}(t_1,\ldots,t_n) := \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(-t_1)^{k_1} \cdots (-t_n)^{k_n}}{k_1! \cdots k_n!} \zeta_{\text{EMS}}(-k_1,\ldots,-k_n) \in \mathbb{C}[[t_1,\ldots,t_n]]. \]

Our main theorem in this paper is an equivalence between the desingularized values (0.3) and the renormalized values (0.5):

**Theorem 3.5.** For \( n \in \mathbb{N} \), we have

\[ Z_{\text{EMS}}(t_1,\ldots,t_n) = \prod_{i=1}^{n} \frac{1 - e^{-t_i - \cdots - t_n}}{t_i + \cdots + t_n} \cdot Z_{\text{FKMT}}(-t_1,\ldots,-t_n). \]

As a consequence of this theorem, the renormalized values can be given as linear combinations of the desingularized values and vice versa (cf. Examples 3.7 and 3.8). By combining the above equivalence with the explicit formula (cf. Proposition 1.5) of the desingularized values shown in [12], we obtain the following explicit formula of the renormalized values.

**Corollary 3.9.** For \( k_1,\ldots,k_n \in \mathbb{N}_0 \), we have

\[ \zeta_{\text{EMS}}(-k_1,\ldots,-k_n) = (-1)^{k_1+\cdots+k_n} \sum_{\nu_1+\cdots+\nu_n = k_i, 1 \leq i \leq n} \prod_{i=1}^{n} \frac{k_i!}{\nu_i!} \prod_{j=1}^{n} \frac{B_{\nu_i_1+\cdots+\nu_i_1+1}}{\nu_i_1!} \cdot \ldots \cdot \nu_i_1 \nu_i_1 + \cdots + \nu_i_1 + 1. \]

Here \( B_n \) is the Bernoulli number in (1.1).

The plan of our paper goes as follows. In section 1 we recall the desingularization method, desingularized MZF and the desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [12]. In section 2 we review an algebraic framework on Hopf algebra in [8], and we prove an explicit formula of the reduced coproduct \( \tilde{\Delta}_0 \) (Proposition 2.5) which is required to prove the recurrence formula of renormalized values in [8] in section 3. We also review the algebraic Birkhoff decomposition and renormalized values in [8]. In section 3 by showing a recurrence formula (Proposition 3.3) we prove the above main results, that is, an equivalence between desingularized values and renormalized values (Theorem 3.5) and an explicit formula of renormalized values (Corollary 3.9).

1. Desingularizations

In this section, we review the desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [12]. In §1.1 we recall the desingularization
method and desingularized MZF, and explain some remarkable properties of this function. In §1.2, we review the desingularized values and their generating function.

1.1. The desingularization method and desingularized MZF

In this subsection, we review the desingularization method, the desingularized MZF. We also recall the basic properties of the desingularized MZF.

The desingularization method is a method to resolve all singularities of MZF. We recall the generating function

\[ \tilde{\mathcal{H}}_n(t_1, \ldots, t_n; c) \in \mathbb{C}\llbracket t_1, \ldots, t_n \rrbracket \]

which is defined by

\[ \tilde{\mathcal{H}}_n(t_1, \ldots, t_n; c) := \prod_{j=1}^{n} \left( \frac{1}{ \exp\left( \sum_{k=j}^{n} t_k \right) - 1 } - \frac{c}{\exp\left( c \sum_{k=j}^{n} t_k \right) - 1 } \right) \]

for \( c \in \mathbb{R} \). Here \( B_m (m \geq 0) \) is the Bernoulli number which is defined by

\[ x e^x - 1 = \sum_{m \geq 0} B_m \frac{x^m}{m!}. \]

We note that \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6} \).

**Definition 1.1** ([12] Definition 3.1). For non-integral complex numbers \( s_1, \ldots, s_n \), the desingularized MZF \( \zeta_{\text{FKMT}}(s_1, \ldots, s_n) \) is defined by

\[ \zeta_{\text{FKMT}}(s_1, \ldots, s_n) := \lim_{c \to 1} \frac{1}{(1-c)^n} \prod_{k=1}^{n} \frac{1}{(e^{2 \pi i s_k} - 1) \Gamma(s_k)} \int_{C^n} \tilde{\mathcal{H}}_n(t_1, \ldots, t_n; c) \prod_{k=1}^{n} t_k^{s_k-1} dt_k. \]

Here \( C \) is the path consisting of the positive real axis (top side), a circle around the origin of radius \( \varepsilon \) (sufficiently small), and the positive real axis (bottom side).

One of the remarkable properties of the desingularized MZF is that it is an entire function, i.e., the equation (1.2) is well-defined as an analytic function by the following proposition.

**Proposition 1.2** ([12] Theorem 3.4). The equation \( \zeta_{\text{FKMT}}(s_1, \ldots, s_n) \) can be analytically continued to \( \mathbb{C}^n \) as an entire function in \( (s_1, \ldots, s_n) \in \mathbb{C}^n \) by the following integral expression:

\[ \zeta_{\text{FKMT}}(s_1, \ldots, s_n) = \prod_{k=1}^{n} \frac{1}{(e^{2 \pi i s_k} - 1) \Gamma(s_k)} \times \int_{C^n} \prod_{j=1}^{n} \lim_{c \to 1 \in \mathbb{R}\setminus\{1\}} \frac{1}{1-c} \left( \frac{1}{\exp\left( \sum_{k=j}^{n} t_k \right) - 1 } - \frac{c}{\exp\left( c \sum_{k=j}^{n} t_k \right) - 1 } \right) \prod_{k=1}^{n} t_k^{s_k-1} dt_k. \]

\(^2\)It is denoted by \( \tilde{\mathcal{H}}_n((t_j); (1); c) \) in [12].
We explain another remarkable properties of the desingularized MZF. For indeterminates $u_j$ and $v_j$ ($1 \leq j \leq n$), we set
\[
\mathcal{G}((u_j),(v_j)) := \prod_{j=1}^{n} (1 - (u_j v_j + \cdots + u_n v_n)(v_j^{-1} - v_j^{-1}))
\]
with the convention $v_0^{-1} := 0$, and we define the set of integers $\{a_{l,m}\}$ by
\[
\mathcal{G}((u_j),(v_j)) = \sum_{l=(l_j)\in\mathbb{N}_0^n, m=(m_j)\in\mathbb{Z}^n, \sum_{j=1}^{n} m_j=0} a_{l,m} \prod_{j=1}^{n} u_j^j v_j^{m_j}.
\]

Another remarkable properties of the desingularized MZF is that the function is given by a finite ‘linear’ combination of MZF's.

**Proposition 1.3** ([12] Theorem 3.8). For $s_1, \ldots, s_n \in \mathbb{C}$, we have the following equality between meromorphic functions of the complex variables $(s_1, \ldots, s_n)$.

\[
\zeta_{\text{FKMT}}(s_1, \ldots, s_n) = \sum_{l=(l_j)\in\mathbb{N}_0^n, m=(m_j)\in\mathbb{Z}^n, \sum_{j=1}^{n} m_j=0} a_{l,m} \left(\prod_{j=1}^{n} (s_j)^{l_j}\right) \zeta(s_1 + m_1, \ldots, s_n + m_n).
\]

Here, $(s)_k$ is the Pochhammer symbol, that is, for $k \in \mathbb{N}$ and $s \in \mathbb{C}$ $(s)_0 := 1$ and $(s)_k := s(s+1) \cdots (s+k-1)$.

1.2. Desingularized values. We review the desingularized values and its explicit formula (Proposition [1.4]), and then we give a recurrence formula of the desingularized values (Corollary [1.6]).

The desingularized value is given as the special value at the integer points with non-positive arguments of an entire function:

**Definition 1.4.** For $k_1, \ldots, k_n \in \mathbb{N}_0$, the desingularized value $\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n) \in \mathbb{C}$ is defined to be the special value of desingularized MZF $\zeta_{\text{FKMT}}(s_1, \ldots, s_n)$ at $(s_1, \ldots, s_n) = (-k_1, \ldots, -k_n)$.

The generating function $Z_{\text{FKMT}}(t_1, \ldots, t_n)$ of $\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n)$ in the equation ([0.4]) is explicitly calculated as follows.

**Proposition 1.5** ([12] Theorem 3.7). We have
\[
Z_{\text{FKMT}}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{(1 - t_i - \cdots - t_n)(e^{t_i+\cdots+t_n} - 1)}{(e^{t_i+\cdots+t_n} - 1)^2}.
\]

In terms of $\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n)$ for $k_1, \ldots, k_n \in \mathbb{N}_0$, the above equation is reformulated to
\[
\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n) = (-1)^{k_1+\cdots+k_n} \sum_{\nu_1+\cdots+\nu_n=k_1} \prod_{1 \leq i \leq n} \frac{k_i!}{\nu_i!} B_{\nu_i+\cdots+\nu_n+1}.
\]

By the above proposition we have the following recurrence formula:
Corollary 1.6.

\[(1.4) \quad Z_{\text{FKMT}}(t_1, \ldots, t_n) = Z_{\text{FKMT}}(t_2, \ldots, t_n) \cdot Z_{\text{FKMT}}(t_1 + \cdots + t_n) \quad (n \in \mathbb{N}).\]

In terms of \(\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n)\), the equation (1.4) is reformulated to

\[(1.5) \quad \zeta_{\text{FKMT}}(-k_1, \ldots, -k_n) = \sum_{i_2 + j_2 = k_2} \prod_{a=2}^{n} \binom{k_a}{i_a} \zeta_{\text{FKMT}}(-i_2, \ldots, -i_n)\zeta_{\text{FKMT}}(-k_1-j_2-\cdots-j_n)\]

for \(k_1, \ldots, k_n \in \mathbb{N}_0\). Here we use \(\binom{k_a}{i_a} := \frac{k_a!}{i_a!(k_a-i_a)!}\).

In §3 we will show that the same formula as (1.5) holds for the renormalized value \(\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n)\) in the equation (3.8).

2. Renormalizations

In this section, we recall the renormalization procedure to define renormalized values which is introduced by Ebrahimi-Fard, Manchon and Singer. In §2.1, we start by recalling their framework of a Hopf algebra generated by words and in §2.2 we show an explicit formula in Proposition 2.5 to calculate the reduced coproduct \(\Delta_0\). This proposition is essential to show the recurrence formula of \(\zeta_{\text{FKMT}}(-k_1, \ldots, -k_n)\) in §3. In §2.3 we explain the algebraic Birkhoff decomposition à la Connes and Kreimer which is required to define renormalized values.

2.1. Algebraic frameworks. We follow the conventions of [8]. Let \(X_0 := \{j, d, y\}\) be the set of three elements \(j, d\) and \(y\). Let \(W_0\) be the associative monoid, with the empty word 1 as a unit, generated by \(X_0\) with the rule \(jd = dj = 1\). Any element \(w \in W_0\) can be uniquely represented by

\[w = j^{k_1} y \cdots j^{k_n}\]

for \(k_1, \ldots, k_n \in \mathbb{Z}\). An element of \(W_0\) is called a word. Put \(Y_0 := W_0 y \cup \{1\}\) and we call an element of \(Y_0\) admissible. We denote the \(\mathbb{Q}\)-linear space \(A_0\) generated by \(W_0\) by \(A_0 := \langle W_0 \rangle_{\mathbb{Q}}\). The linear space \(A_0\) is naturally equipped with a structure of a non-commutative algebra. We equip this \(A_0\) with a new product \(\uplus_0 : A_0 \otimes A_0 \to A_0\) which is a \(\mathbb{Q}\)-linear map recursively defined by

\[1 \uplus_0 w := w \uplus_0 1 := w \quad (w \in W_0),\]

\[yu \uplus_0 v := u \uplus_0 yv := y(u \uplus_0 v) \quad (u, v \in W_0),\]

\[ju \uplus_0 jv := j(u \uplus_0 jv) + j(j(u \uplus_0 v)) \quad (u, v \in W_0),\]

\[du \uplus_0 dv := d(u \uplus_0 dv) - u \uplus_0 d^2 v \quad (u, v \in W_0).\]

Then \((A_0, \uplus_0)\) forms a unitary, nonassociative, noncommutative \(\mathbb{Q}\)-algebra. We define

\[T := \langle \{j^{k_1} y \cdots j^{k_{n-1}} y j^{k_n} \in W_0 \mid k_n \neq 0, n \in \mathbb{N}\} \rangle_{\mathbb{Q}},\]

that is, to be the linear subspace of \(A_0\) linearly generated by words ending in \(d\) or \(j\) and

\[L := \langle j^k \{d(u \uplus_0 v) - du \uplus_0 v - u \uplus_0 dv\} \mid k \in \mathbb{Z}, u, v \in W_0 y \rangle_{(A_0, \uplus_0)},\]
that is, to be the two-sided ideal of \((\mathcal{A}_0, \sqcup \sqcup_0)\) algebraically generated by the above elements. The subspace \(\mathcal{T}\) forms a two-sided ideal of \(\mathcal{A}_0\) by [8] Lemma 3.4. We define the quotient algebra

\[
\mathcal{B}'_0 := \mathcal{A}_0/(\mathcal{T} + \mathcal{L}).
\]

We consider the map

\[
\zeta^\mu_t : \mathcal{B}'_0 \to \mathbb{Q}[t]
\]

by \(\zeta^\mu_t(1) := 1\) and for \(k_1, \ldots, k_n \in \mathbb{Z}\),

\[
\zeta^\mu_t(j^{k_1}y \cdots j^{k_1}y) := \text{Li}_{k_1, \ldots, k_n}(t).
\]

Here \(\text{Li}_{k_1, \ldots, k_n}(t)\) is the multiple polylogarithm defined by

\[
\text{Li}_{k_1, \ldots, k_n}(t) := \sum_{0 < m_1 < \cdots < m_n} \frac{t^{m_n}}{m_1 \cdots m_{k_n}}.
\]

**Lemma 2.1.** The map \(\zeta^\mu_t\) is well-defined and forms an algebra homomorphism.

The first half of the claim of Lemma 2.1 is proved in the same way to proof of [8] Proposition 3.5 and the latter half of the claim of Lemma 2.1 is proved in [8] Lemma 3.6.

**Remark 2.2.** The restriction of the shuffle product \(\sqcup \sqcup_0\) to admissible words at positive arguments corresponds the usual shuffle product \(\sqcup\) as is proved in [8] Lemma 3.7. Let \(\mathcal{C} := \mathbb{Q} \oplus \mathbb{Q}(j, y)\) and \(\mathcal{D} := \mathbb{Q} \oplus \mathbb{Q}(x_0, x_1)\). Then two algebras \((\mathcal{C}, \sqcup \sqcup_0)\) and \((\mathcal{D}, \sqcup)\) become isomorphic under the linear map \(\Phi : (\mathcal{D}, \sqcup) \to (\mathcal{C}, \sqcup \sqcup_0)\) by \(\Phi(1) := 1\) and for \(k_1, \ldots, k_n \in \mathbb{N}\) with \(k_1 > 1\),

\[
\Phi(j^{k_1-1}x_0 \cdots j^{k_n-1}x_1) := j^{k_1-1}y \cdots j^{k_n-1}y.
\]

Let \(L := \{d, y\}\) be the set of two elements \(d\) and \(y\). Let \(L^*\) be the free monoid of \(L\) with empty word \(1\) as a unit. This \(L^*\) forms a submonoid of \(W_0\). Put \(Y := L^*y \cup \{1\} \subset Y_0\). So all elements of \(Y\) are admissible. The *weight* \(\text{wt}(w)\) of a word \(w \in L^*\) means the number of letters appearing in \(w\) and the *depth* \(\text{dp}(w)\) of a word \(w \in L^*\) is given by the number of \(y\) appearing in \(w\). We denote the free unitary, associative, noncommutative \(\mathbb{Q}\)-algebra of \(L\) by \(\mathbb{Q}(L)\). Then \((\mathbb{Q}(L), \sqcup \sqcup_0)\) forms a unitary, nonassociative, noncommutative \(\mathbb{Q}\)-subalgebra of \(\mathcal{A}_0\). The algebra \(\mathbb{Q}(L)\) also forms a counital, cocommutative coalgebra. (See [8] §3.3.5.) We define

\[
\mathcal{T}_- := \langle \{wd \mid w \in L^*\} \rangle_{\mathbb{Q}} (= \mathcal{T} \cap \mathbb{Q}(L)),
\]

that is, to be the linear subspace of \(\mathbb{Q}(L)\) linearly generated by words ending in \(d\) and

\[
\mathcal{L}_- := \langle d^k(\mathcal{T}_-(\sqcup \sqcup_0 v) - du \sqcup \sqcup_0 v - u \sqcup \sqcup_0 dv) \mid k \in \mathbb{N}_0, u, v \in L^* \rangle_{\mathbb{Q}(L), \sqcup \sqcup_0},
\]

that is, to be the two-sided ideal of \((\mathbb{Q}(L), \sqcup \sqcup_0)\) algebraically generated by the above elements. We consider the Q-linear subspace

\[
\mathcal{S}_- := \mathcal{T}_- + \mathcal{L}_-
\]

of \(\mathbb{Q}(L)\) generated by \(\mathcal{L}_-\) and \(\mathcal{T}_-\). This \(\mathcal{S}_-\) also forms a two-sided ideal as our previous \(\mathcal{T} + \mathcal{L}\). We put the quotient

\[
\mathcal{H}_0 := \mathbb{Q}(L)/\mathcal{S}_-.
\]
Actually $\mathcal{H}_0$ forms a connected, filtered, commutative and cocommutative Hopf algebra (cf. [8] §3.3.6), whose product is equal to $\omega_0$ and whose coproduct is given by

$$\Delta_0(w) := \sum_{S \subseteq \{n\}} w_S \otimes w_{\overline{S}},$$

for $w \in Y \setminus \{1\}(\subset \mathcal{H}_0)$. In the summation, $S$ may be empty. we put $n := \text{wt}(w)$, $[n] := \{1, \ldots, n\}$ and $\overline{S} := [n] \setminus S$. For $w := x_1 \cdots x_n$ ($x_i \in L^*$, $i = 1, \ldots, n$) and $S := \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$, we define $w_S := x_{i_1} \cdots x_{i_k}$. We call the set $S$ admissible if both $w_S, w_{\overline{S}} \in Y$. See [8] §3.3.8 for combinatorial method using polygons to compute $\Delta_0(w)$. We define $\mathbb{Q}$-linear map $\tilde{\Delta}_0 : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ by

$$(2.2) \quad \tilde{\Delta}_0(w) := \Delta_0(w) - 1 \otimes w - w \otimes 1 \quad (w \in Y),$$

and we call $\tilde{\Delta}_0$ the reduced product.

2.2. An explicit formula for the reduced coproduct $\tilde{\Delta}_0$. We show an explicit formula (Proposition 2.3) to calculate the reduced coproduct $\tilde{\Delta}_0$ in this subsection. This proposition is important to prove the recurrence formula of $\zeta_{\mathbb{C}^n}(-k_1, \ldots, -k_n)$ in §3.

We consider the bilinear map $f : \mathbb{Q}(L) \times \mathbb{Q}(L)^{\otimes 2} \to \mathbb{Q}(L)^{\otimes 2}$ defined by

$$f(1, w \otimes w') := w \otimes w',$$

$$f(d, w \otimes w') := dw \otimes w' + w \otimes dw',$$

$$f(y, w \otimes w') := yw \otimes w' + w \otimes yw',$$

and inductively

$$f(x_0, w \otimes w') := f(x, f(x_0, w \otimes w')),$$

for $w, w' \in \mathbb{Q}(L)$, $x_0 \in L$ and $x \in L^*$. Then the following lemma holds:

**Lemma 2.3.** There is a map $\overline{f} : \mathbb{Q}(L) \times \mathcal{H}_0^{\otimes 2} \to \mathcal{H}_0^{\otimes 2}$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{Q}(L) \otimes \mathbb{Q}(L) & \xrightarrow{f(x, \cdot)} & \mathbb{Q}(L) \otimes \mathbb{Q}(L) \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{H}_0 \otimes \mathcal{H}_0 & \xrightarrow{\overline{f}(x, \cdot)} & \mathcal{H}_0 \otimes \mathcal{H}_0
\end{array}
$$

where $x \in \mathbb{Q}(L)$ and $\pi : \mathbb{Q}(L)^{\otimes 2} \to \mathcal{H}_0^{\otimes 2}$ is a natural projection.

**Proof.** It is sufficient to prove $f(x, \text{ker} \pi) \subset \text{ker} \pi$ for $x \in L^*$. Here $\text{ker} \pi = \mathbb{Q}(L) \otimes S_- + S_- \otimes \mathbb{Q}(L)$. We show this by induction on $\text{wt}(x)$. Let $x_0 = d$ or $y$ and put $v \in S_-$. If $v \in \mathcal{T}_-$, it is clear that $x_0 v \in \mathcal{T}_- \subset S_-$. If $v \in \mathcal{L}_-$, for $x_0 = d$ it is easy to see that $dv \in \mathcal{L}_- \subset S_-$ by the definition of $\mathcal{L}_-$. Because $\mathcal{L}_-$ is a two-sided ideal of $(\mathbb{Q}(L), \mathcal{U}_0)$, we have $y \mathcal{U}_0 v \in \mathcal{L}_-$ for $x_0 = y$. By the definition of $\mathcal{U}_0$, we get

$$y \mathcal{U}_0 v = y(1 \mathcal{U}_0 v) = yv \in \mathcal{L}_- \subset S_-.$$

Because $S_-$ is $\mathcal{L}_- + \mathcal{T}_-$, for $v \in S_-$ and $x_0 = d$ or $y$, we have $x_0 v \in S_-$. 
Let $w \in L^*$ and $v \in S_-$. Then $x_0v \in S_-$, so we have
\[
\pi \left( f(x_0, w \otimes v) \right) = \pi(x_0w \otimes v + w \otimes x_0v) \\
= \pi(x_0w \otimes v) + \pi(w \otimes x_0v) \\
= 0.
\]

Let $w \in L^*$ and $v \in S_-$. For $x \in L^*$, we get
\[
\pi(f(xx_0, w \otimes v)) = \pi(f(x, f(x_0, w \otimes v))) \\
= \pi(f(x, x_0w \otimes v + w \otimes x_0v)) \\
= \pi(f(x, x_0w \otimes v)) + \pi(f(x, w \otimes x_0v)) \\
= 0,
\]
by our induction assumption. This also applies to the case when $w \in S_-$ and $v \in L^*$, so the claim holds.
\[
\square
\]

For $x \in L^*$ and $w, w' \in Y$, we simply denote $\overline{f}(x, w \otimes w')$ by $x \bullet (w \otimes w')$ and we define
\[
w \otimes_{\text{sym}} w' := w \otimes w' + w' \otimes w \in H_0 \otimes H_0.
\]

Then, the following equations hold in $H_0 \otimes H_0$:
\[
d^n \bullet (w \otimes_{\text{sym}} w') = \sum_{i+j=n} \binom{n}{i} d^i w \otimes_{\text{sym}} d^j w',
\]
\[
(d^n y) \bullet (w \otimes_{\text{sym}} w') = \sum_{i+j=n} \binom{n}{i} \sum_{\{u,v\} = \{d^i y, d^j v\}} uw \otimes_{\text{sym}} vw',
\]
for $n \in \mathbb{N}$, $w, w' \in Y$. These equations can be proved inductively on $n \in \mathbb{N}$.

**Proposition 2.4.** For $w \in Y \setminus \{1\}$,
\[
\tilde{\Delta}_0(dw) = d \bullet \tilde{\Delta}_0(w),
\]
\[
\tilde{\Delta}_0(yw) = y \bullet \tilde{\Delta}_0(w) + y \otimes_{\text{sym}} w.
\]
Proposition 2.5. Let □ proved in the same way. We use \( d N \) (2.7) (2.8) \( \tilde{\Delta} \). Because we have

**Proof.** Let \( w \) be in \( Y \setminus \{1\} \). By the definition of \( \Delta_0 \) and the equation (2.2), we have

\[
\tilde{\Delta}_0(dw) = \Delta_0(dw) - 1 \otimes_{\text{sym}} dw
\]

\[
= \sum_{S \subset [n+1], S:\text{admissible}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\text{sym}} dw
\]

\[
= \sum_{1 \in S \subset [n+1], S:\text{admissible}} (dw)_S \otimes (dw)_{\overline{S}} + \sum_{1 \notin S \subset [n+1], S:\text{admissible}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\text{sym}} dw
\]

\[
= \sum_{S \subset [n], S:\text{admissible}} d \cdot w_S \otimes w_{\overline{S}} + \sum_{S \subset [n], S:\text{admissible}} w_S \otimes d \cdot w_{\overline{S}} - (d \otimes_{\text{sym}} w + 1 \otimes_{\text{sym}} dw)
\]

\[
= d \cdot \left( \sum_{S \subset [n], S:\text{admissible}} w_S \otimes w_{\overline{S}} - 1 \otimes_{\text{sym}} w \right)
\]

\[
= d \cdot \tilde{\Delta}_0(w).
\]

We use \( d \otimes_{\text{sym}} w = 0 \) in \( \mathcal{H}_0 \otimes \mathcal{H}_0 \) at the fourth equality. The equation (2.6) can be proved in the same way. \( \square \)

**Proposition 2.5.** Let \( w_m := d^m y \) for \( m \in \mathbb{N}_0 \). Then for \( n \in \mathbb{N}_{\geq 2} \) and \( k_1, \ldots, k_n \in \mathbb{N}_0 \), we have

(2.7)

\[
\tilde{\Delta}_0(w_{k_1} \cdots w_{k_n}) = \sum_{i_1+j_1=k_1} \binom{k_1}{i_1} d^i y \otimes_{\text{sym}} d^{i_1} w_{k_2} \cdots w_{k_n}
\]

\[
+ \sum_{p=2}^{n-1} \sum_{i_1+j_1=k_1} \prod_{a=1}^{p} \binom{k_a}{i_a} \sum_{u_q, v_q \in \{d^a y, d^{i_1} y\}} \sum_{1 \leq q \leq p-1} (u_1 \cdots u_{p-1} d^p y \otimes_{\text{sym}} v_1 \cdots v_{p-1} d^p w_{k_{p+1}} \cdots w_{k_n}).
\]

Here \( \{u_q, v_q\} = \{d^a y, d^{i_1} y\} \) means \( (u_q, v_q) = (d^a y, d^{i_1} y) \) or \( (d^a y, d^i y) \).

**Proof.** Because we have

(2.8)

\[
\tilde{\Delta}_0(d^a y w) = d^a \bullet \left( y \otimes_{\text{sym}} w + y \otimes_{\text{sym}} \tilde{\Delta}_0(w) \right) \quad (a \in \mathbb{N}_0)
\]

by Proposition (2.4), we compute

\[
\tilde{\Delta}_0(w_{k_1} w_{k_2} \cdots w_{k_n})
\]

\[
= d^{k_1} \bullet (y \otimes_{\text{sym}} w_{k_2} \cdots w_{k_n}) + (d^{k_1} y) \bullet \tilde{\Delta}_0(w_{k_2} \cdots w_{k_n})
\]

\[
= d^{k_1} \bullet (y \otimes_{\text{sym}} w_{k_2} \cdots w_{k_n}) + (d^{k_1} y d^{k_2}) \bullet (y \otimes_{\text{sym}} w_{k_3} \cdots w_{k_n})
\]

\[
+ (d^{k_1} y d^{k_2} y) \bullet \tilde{\Delta}_0(w_{k_3} \cdots w_{k_n}).
\]
By using the equation (2.8) repeatedly, we get

\[
\sum_{p=1}^{n-1} (d^{k_1} y \cdots d^{k_{p-1}} y) \bullet \left( \sum_{i_p+j_p=k_p} \binom{k_p}{i_p} d^{i_p} y \otimes_{\text{sym}} d^{j_p} w_{k_{p+1}} \cdots w_{k_n} \right). 
\]

Because \( \tilde{\Delta}_0 (d^a y) = 0 \) \((a \in \mathbb{N}_0)\) by the definition of \( \tilde{\Delta}_0 \), the second term vanishes. Therefore by (2.3), we get

\[
\tilde{\Delta}_0 (w_{k_1} w_{k_2} \cdots w_{k_n}) 
\]

And by using (2.4) repeatedly, we have

\[
= \sum_{i_1+j_1=k_1} \binom{k_1}{i_1} d^{i_1} y \otimes_{\text{sym}} d^{j_1} w_{k_2} \cdots w_{k_n} 
\]

\[
+ \sum_{p=2}^{n-1} \sum_{i_p+j_p=k_p} \prod_{a=1}^{p} \binom{k_a}{i_a} \sum_{1 \leq q \leq p-1} \{ u_q, v_q \} = \{ d^{q_1}, d^{q_2} \} \left( u_1 \cdots u_{p-1} d^{q_p} y \otimes_{\text{sym}} v_1 \cdots v_{p-1} d^{q_p} w_{k_{p+1}} \cdots w_{k_n} \right). 
\]

\[\square\]

2.3. The algebraic Birkhoff decomposition and renormalized values. We explain the algebraic Birkhoff decomposition. This decomposition is a fundamental tool in a work of Connes and Kreimer [6] on their Hopf algebraic approach to renormalization of perturbative quantum field theory. This decomposition is necessary to define renormalized values.

Based on [17], we recall the algebraic Birkhoff decomposition. We denote the product and the unit of \( \mathbb{Q} \)-algebra \( \mathcal{A} \) by \( m_\mathcal{A} \) and \( u_\mathcal{A} \). For a Hopf algebra \( \mathcal{H} \) over \( \mathbb{Q} \), we mean \( \Delta_\mathcal{H}, \varepsilon_\mathcal{H} \) and \( S_\mathcal{H} \) to be its coproduct, its counit and its antipode respectively. In this paper, we often use Sweedler’s notation:

\[
\tilde{\Delta}_0 (w) := \sum_{(w)} u' \otimes u''.
\]

Let \( \mathcal{H} \) be a Hopf algebra over \( \mathbb{Q} \), \( \mathcal{A} \) be a \( \mathbb{Q} \)-algebra and \( \mathcal{L} (\mathcal{H}, \mathcal{A}) \) be the set of \( \mathbb{Q} \)-linear maps from \( \mathcal{H} \) to \( \mathcal{A} \). We define the convolution \( \phi \ast \psi \in \mathcal{L} (\mathcal{H}, \mathcal{A}) \) by

\[
\phi \ast \psi := m_\mathcal{A} \circ (\phi \otimes \psi) \circ \Delta_\mathcal{H}
\]

for \( \mathbb{Q} \)-linear maps \( \phi \) and \( \psi \in \mathcal{L} (\mathcal{H}, \mathcal{A}) \). Let \( \mathcal{H} \) be a Hopf algebra over \( \mathbb{Q} \) and \( \mathcal{A} \) be a \( \mathbb{Q} \)-algebra. The subset

\[
G (\mathcal{H}, \mathcal{A}) := \{ \phi \in \mathcal{L} (\mathcal{H}, \mathcal{A}) \mid \phi(1) = 1_\mathcal{A} \}
\]

endowed with the above convolution product \( \ast \) forms a group. The unit is given by a map \( e = u_\mathcal{A} \circ \varepsilon_\mathcal{H} \).
Let \( \mathcal{H} \) be a connected filtered Hopf algebra over \( \mathbb{Q} \), that is, \( \mathcal{H} \) has a filtration of \( \mathbb{Q} \)-linear subspace:

\[
\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \bigcup_{n \in \mathbb{N}_0} \mathcal{H}^n = \mathcal{H}
\]

with \( \mathcal{H}^0 = \mathbb{Q} \) and with the conditions: \( \mathcal{H}^m \mathcal{H}^n \subset \mathcal{H}^{m+n} \) and \( S_\mathcal{H}(\mathcal{H}^n) \subset \mathcal{H}^n \) and \( \Delta_\mathcal{H}(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q \) for \( m, n \in \mathbb{N}_0 \).

Let \( \mathcal{A} := \mathbb{Q}[\frac{1}{z}, z] := \mathbb{Q}[[z]][\frac{1}{z}] \) be the algebra consisting of all Laurent series. And we decompose it as \( \mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+ \) where \( \mathcal{A}_- := \frac{1}{z} \mathbb{Q}[\frac{1}{z}] \) and \( \mathcal{A}_+ := \mathbb{Q}[[z]] \). We define a projection \( \pi : \mathcal{A} \rightarrow \mathcal{A}_- \) by

\[
\pi \left( \sum_{n=-k}^{\infty} a_n z^n \right) := \sum_{n=-k}^{-1} a_n z^n,
\]

with \( a_n \in \mathbb{Q} \) and \( k \in \mathbb{Z} \). Here we use the convention the sum over empty set is zero.

The following theorem is the fundamental tool of Connes and Kreimer (\cite{6}) in the renormalization procedure of perturbative quantum field theory.

**Theorem 2.6** (\cite{6}, \cite{8}, \cite{17}: algebraic Birkhoff decomposition). For \( \phi \in G(\mathcal{H}, \mathcal{A}) \), there are unique linear maps \( \phi_+ : \mathcal{H} \rightarrow \mathcal{A}_+ \) and \( \phi_- : \mathcal{H} \rightarrow \mathbb{Q} \oplus \mathcal{A}_- \) with \( \phi_-(1) = 1 \in \mathbb{Q} \) such that

\[
\phi = \phi_-^{-1} \circ \phi_+.
\]

Moreover the maps \( \phi_- \) and \( \phi_+ \) are algebra homomorphisms if \( \phi \) is an algebra homomorphism.

We define the \( \mathbb{Q} \)-linear map \( \phi : \mathcal{H}_0 \rightarrow \mathcal{A} \) by \( \phi(1) := 1 \) and for \( k_1, \ldots, k_n \in \mathbb{N}_0 \),

\[
(2.10) \quad \phi(d^{k_1} y \cdots d^{k_n} y) := \phi(d^{k_1} y \cdots d^{k_n} y)(z) := \partial_z^{k_1} (x \partial_z^{k_2}) \cdots (x \partial_z^{k_n}) (x(z))
\]

where \( x := x(z) := \frac{e^z - 1}{e^z - 1} \in \mathcal{A} \) and \( \partial_z \) is the derivative by \( z \).

**Proposition 2.7** (\cite{8} §4.2). The \( \mathbb{Q} \)-linear map \( \phi : \mathcal{H}_0 \rightarrow \mathcal{A} \) is well-defined and forms algebra homomorphism. Moreover, the following diagram is commutative:

\[
\begin{array}{ccc}
(\mathcal{H}_0, \cup_0) & \xrightarrow{\zeta^\mu} & (\mathbb{Q}[\mu], \cdot) \\
\phi \downarrow & & \downarrow_{t \rightarrow e^t} \\
(\mathcal{A}, \cdot) & & \\
\end{array}
\]

where \( \zeta^\mu \) is the map in (2.1).

Because the map \( \phi \) is algebraic by the above proposition, we obtain the algebraic map:

\[
(2.11) \quad \phi_+ : \mathcal{H}_0 \rightarrow \mathcal{A}_+
\]

which is an algebra homomorphism by Theorem 2.6.

**Definition 2.8** (\cite{8} §4.2). The renormalized value\(^3\) \( \zeta_{\text{ren}}(-k_1, \ldots, -k_n) \) is defined by

\[
(2.12) \quad \zeta_{\text{ren}}(-k_1, \ldots, -k_n) := \lim_{z \rightarrow 0} \phi_+ (d^{k_n} y \cdots d^{k_1} y)(z)
\]

\(^3\)If we follow the notations of \cite{8}, it should be denoted by \( \zeta_+(-k_n, \ldots, -k_1) \).
for \( k_1, \ldots, k_n \in \mathbb{N}_0 \).

It is remarkable that the renormalized values coincide with special values of the meromorphic continuation of MZFs at non-positive arguments which do not locate at their singularities.

**Proposition 2.9** ([8] Theorem 4.3). For \( k_1 \in \mathbb{N}_0 \), we have
\[
\zeta_{\text{EMS}}(-k_1) = \zeta(-k_1)
\]
and for \( k_1, k_2 \in \mathbb{N}_0 \) with \( k_1 + k_2 \) odd, we have
\[
\zeta_{\text{EMS}}(-k_1, -k_2) = \zeta(-k_1, -k_2).
\]

We remind that, as is showed in the set (0.2), \( \zeta(s_1, \cdots, s_n) \) is always irregular at \((s_1, \cdots, s_n) = (-k_1, \cdots, -k_n) \in \mathbb{Z}_0^n \) for \( n \geq 3 \).

Another remarkable property of the renormalized values is that a certain shuffle relation hold for them. Because \( \text{EMS} \) is the product of \( \mathcal{H}_0 \) and \( \phi_+ : \mathcal{H}_0 \to \mathbb{Q}[[z]] \) is a unital algebra homomorphism by Theorem 2.6 we obtain the following proposition:

**Proposition 2.10** ([8] §4.2: shuffle relation). For \( w, w' \in Y \), we have
\[
\phi_+(w \text{\ EMS} \ w') = \phi_+(w)\phi_+(w').
\]

Here are examples in lower depth:

**Examples 2.11.** For \( a, b, c \in \mathbb{N}_0 \), we have
\[
\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b) = \begin{cases} 
\sum_{k=0}^{a} (-1)^k \binom{a}{k} \zeta_{\text{EMS}}(-b - k, -a + k) & \text{if } b \geq 1, \\
\zeta_{\text{EMS}}(-a, 0) & \text{if } b = 0,
\end{cases}
\]
\[
\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b, -c) = \begin{cases} 
\sum_{k=0}^{c} (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b - k, -c - k - a + k) & \text{if } c \geq 1, \\
\sum_{k=0}^{b} (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b - k, -a + k, 0) & \text{if } b \geq 1, c = 0, \\
\zeta_{\text{EMS}}(-a, 0, 0) & \text{if } b = c = 0.
\end{cases}
\]

For our comparison, we remind below the usual shuffle relation for positive arguments. For \( a, b \in \mathbb{N}_{>1} \),
\[
\zeta(a) \cdot \zeta(b) = \sum_{k=0}^{a-1} \binom{b - 1 + k}{k} \zeta(a - k, b + k) + \sum_{k=0}^{b-1} \binom{a - 1 + k}{k} \zeta(b - k, a + k),
\]
and for \( a, c \in \mathbb{N}_{>1} \) and \( b \in \mathbb{N} \),
\[
\zeta(a) \cdot \zeta(b, c) = \sum_{k=0}^{a-1} \sum_{i=0}^{a-k-1} \binom{c - 1 + k}{k} \binom{b - 1 + i}{i} \zeta(a - k - i, b + i, c + k) + \sum_{k=0}^{a-1} \sum_{j=0}^{b-1} \binom{c - 1 + k}{k} \binom{a - k - 1 + j}{j} \zeta(b - j, a - k + j, c + k) + \sum_{k=0}^{c-1} \binom{a - 1 + k}{k} \zeta(b, c - k, a + k).
\]
3. Main results

In this section, we prove a recurrence formula among renormalized values of MZF in Proposition 3.3. Moreover, by showing that the renormalized value \( \zeta_{\text{EMS}}(\mathbf{k}, \cdots, -k_n) \) satisfies the recurrence formula similar to the one (1.5) for \( \zeta_{\text{EMS}}(-k_1, \ldots, -k_n) \), we prove an equivalence between the desingularized values and the renormalized values in Theorem 3.5. As a corollary of Theorem 3.5, we obtain an explicit formula of renormalized values (Corollary 3.9).

3.1. Recurrence formulas among renormalized values. The goal of this subsection is to prove Proposition 3.3 which is on recurrence formula among renormalized values.

We start with the following key lemma of [8] which is a method to compute recursively the image of \( \phi_+ \) (the equation (2.11)).

Lemma 3.1 ([8] Corollary 4.4). For \( w \in Y \) with \( dp(w) > 1 \), we have

\[
\phi_+(w) = \frac{1}{2^{dp(w)} - 1} \sum_{(w)} \phi_+(w') \phi_+(w'').
\]

Here we use Sweedler's notation (2.7).

Proposition 3.2. For \( n \in \mathbb{N}_{\geq 2} \) and \( k_1, \ldots, k_n \in \mathbb{N}_0 \), we have

\[
\zeta_{\text{EMS}}(-k_1, \ldots, -k_n) = \sum_{i_n + j_n = k_n} \left( \frac{k_n}{i_n} \right) \zeta_{\text{EMS}}(-i_n) \zeta_{\text{EMS}}(-k_1, \ldots, -k_n - j_n)
\]

\[
+ \sum_{p=2}^{n-1} \sum_{i_n + j_n = k_n} \prod_{a=p}^{n} \left( \frac{k_a}{i_a} \right) \prod_{i_p + j_p = k_p} \zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n-1} - i_n) \zeta_{\text{EMS}}(-k_1, \ldots, -k_{p-1} - j_p \circ_p \cdots \circ_{n-1} - j_n).
\]

Proof. By Proposition 2.5 and Lemma 3.1, for \( n \in \mathbb{N}_{\geq 2} \) and \( k_1, \ldots, k_n \in \mathbb{N}_0 \) we get

\[
\phi_+(w_{k_n} \cdots w_{k_1}) = \sum_{i_n + j_n = k_n} \left( \frac{k_n}{i_n} \right) \phi_+(d^{i_n} y) \phi_+(d^{i_n} w_{k_n-1} \cdots w_{k_1})
\]

\[
+ \sum_{p=2}^{n-1} \sum_{i_n + j_n = k_n} \prod_{a=p}^{n} \left( \frac{k_a}{i_a} \right) \sum_{u_q, v_q \in \{d^{i_q}, d^{j_q} y\}} \phi_+(u_n \cdots u_{p+1} d^{j_q} y) \phi_+(v_n \cdots v_{p+1} d^{i_q} w_{k_{p-1}} \cdots w_{k_1}).
\]
because $dp(w) = n$. For $p \leq q \leq n - 1$, we define

$$
(\circ_q, \circ_p) := \begin{cases} 
(+, +) & \text{if } (u_{q+1}, v_{q+1}) = (d^{q+1}, d^{q+1}y), \\
(\gamma, +) & \text{if } (u_{q+1}, v_{q+1}) = (d^{q+1}y, d^{q+1}y).
\end{cases}
$$

Then by the definition of $\zeta EMS(-k_1, \ldots, -k_n)$, the equation (3.1) holds. We prove (3.2) and (3.3) by induction on $n$.

We define the following generating functions in $\mathbb{C}[[x]]$ for $n \in \mathbb{N}_2$ and $k_1, \ldots, k_n \in \mathbb{N}_0$:

$$
\mathfrak{h} := \mathfrak{h}(x) := \sum_{k_1=0}^{\infty} \frac{(-x)^k_1}{k_1!} \zeta EMS(-k_1),
$$

$$
\mathfrak{h}_{k_1, \ldots, k_{n-1}}(x) := \sum_{k_n=0}^{\infty} \frac{(-x)^{k_n}}{k_n!} \zeta EMS(-k_1, \ldots, -k_n),
$$

$$
\mathfrak{h}_{k_1, \ldots, k_n}(x) := \mathfrak{h}^{k_1} \mathfrak{h}_{k_1, \ldots, k_{n-1}}(x).
$$

The equation (3.1) looks complicated. But it can be simplified to the following recurrence formula (3.2).

**Proposition 3.3.** For $n \in \mathbb{N}_2$ and $k_1, \ldots, k_n \in \mathbb{N}_0$, we have

$$
\zeta EMS(-k_1, \ldots, -k_n) = \sum_{i_n + j_n = k_n} \binom{k_n}{i_n} \zeta EMS(-i_n) \zeta EMS(-k_1, \ldots, -k_{n-1} - j_n),
$$

and

$$
\mathfrak{h}_{k_1, \ldots, k_{n-1}}(x) = (-1)^{k_1 + \ldots + k_{n-1}} \left( \mathfrak{h} \partial_x^{k_{n-1}} \right) \cdots \left( \mathfrak{h} \partial_x^{k_1} \right) \mathfrak{h}.
$$

**Proof.** We prove (3.2) and (3.3) by induction on $n \in \mathbb{N}_2$. Let $n = 2$. Then by the equation (3.1) of Proposition 3.2, the equation (3.2) clearly holds. And by the equation (3.3) for $n = 2$, we have

$$
\mathfrak{h}_{k_1}(x) = \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \zeta EMS(-k_1, -k_2)
$$

$$
= \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \sum_{i_2 + j_2 = k_2} \binom{k_2}{i_2} \zeta EMS(-i_2) \zeta EMS(-k_1 - j_2)
$$

$$
= \left\{ \sum_{i_2=0}^{\infty} \frac{(-x)^{i_2}}{i_2!} \zeta EMS(-i_2) \right\} \left\{ \sum_{j_2=0}^{\infty} \frac{(-x)^{j_2}}{j_2!} \zeta EMS(-k_1 - j_2) \right\}
$$

$$
= \mathfrak{h} \left\{ (-1)^{k_1} \partial_x^{k_1} (\mathfrak{h}) \right\}
$$

$$
= (-1)^{k_1} \left( \mathfrak{h} \partial_x^{k_1} \right) (\mathfrak{h}).
$$

Let $n = n_0 \geq 3$. We assume that (3.2) and (3.3) hold for $2 \leq n \leq n_0 - 1$. Firstly, we prove the equation (3.2). By Lemma 3.4 which will be proved later, the second
Lemma 3.4. Let \( n_0 \geq 3 \). We assume that (3.3) holds for \( n = 1 \) with \( 2 \leq l \leq n_0 - 1 \). Let \( 2 \leq p \leq n_0 - 1 \) and \( \phi_i \in \{+, \cdot\} \) for \( p \leq i \leq n_0 - 1 \). Then we have

\[
\sum_{i_{p}+j_{p}=k_{p}} \prod_{a=p}^{n_{0}} \left( k_{a} \right) \zeta_{\text{EM}} (-i_{p} \circ \cdots \circ_{a_{n_{0}-1} - i_{n_{0}}}) \zeta_{\text{EM}} (-k_{1} \cdots - k_{p-1} - j_{p} \circ \cdots \circ \cdot \cdot_{a_{n_{0}-1} - j_{n_{0}}})
\]

\[
= \sum_{i_{n_{0}}+j_{n_{0}}=k_{n_{0}}} \left( k_{n_{0}} \right) \zeta_{\text{EM}} (-i_{n_{0}}) \zeta_{\text{EM}} (-k_{1} \cdots - k_{n_{0}-1} - j_{n_{0}}).
\]

Here \( \phi_i \) is chosen to be with \( \{\phi_i, \phi_1\} = \{+, \cdot\} \) for \( p \leq i \leq n_0 - 1 \).
Proof. We get
\[
\sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} \text{(RHS of (3.5))} = (-1)^{k_{n_0}+1} h \partial_x^{k_{n_0}+1} \left( h_{k_1, \ldots, k_{n_0-2}} (x) \right)
\]
in the same way to the computations of \( h_{k_1} (x) \) in (3.4). By our induction hypothesis on (3.3), for \( n_0 \) we obtain
\[
(3.6) \quad = (-1)^{k_1+\cdots+k_{n_0}-1} \left( h \partial_x^{k_{n_0}-1} \right) \cdots \left( h \partial_x^{k_1} \right) (h).
\]
On the other hand, we have
\[
\sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} \text{(LHS of (3.5))}
= \sum_{i_p+j_p=k_p} \frac{n_0-1}{a_p} \left( k_a \right) \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EOB}} (-i_p \circ_p \cdots \circ_{n_0-1} - i_{n_0}) \right\}
\times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EOB}} (-k_1, \ldots, -k_{p-1} - j_p \circ_p \cdots \circ_{n_0-1} - j_{n_0}) \right\}.
\]
We also consider the following two cases:

Case i) : When \((\circ_{n_0-1} \circ_{n_0-1}) = (+, +)\), we compute
\[
\sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} \text{(LHS of (3.5))}
= \sum_{i_p+j_p=k_p} \frac{n_0-1}{a_p} \left( k_a \right) \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EOB}} (-i_p \circ_p \cdots \circ_{n_0-2} - i_{n_0-1} - i_{n_0}) \right\}
\times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EOB}} (-k_1, \ldots, -k_{p-1} - j_p \circ_p \cdots \circ_{n_0-2} - j_{n_0-1} - j_{n_0}) \right\}.
\]

Put \( m := \left\{ \begin{array}{ll} p - 1 & \text{when } \circ_i \text{ is } + \text{ for all } i, \\
\max \{ l \mid p \leq l \leq n_0 - 2, \circ_l = + \} & \text{otherwise.} \end{array} \right. \)

Then we have
\[
= \sum_{i_p+j_p=k_p} \frac{n_0-1}{a_p} \left( k_a \right) \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EOB}} (-i_p \circ_p \cdots \circ_{n_0-2} - i_{n_0-1} - i_{n_0}) \right\}
\times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EOB}} (-k_1, \ldots, -k_{p-1} - j_p \circ_p \cdots \circ_{m-1} - j_{m}) \right\}.
\]
Here $S := \begin{cases} k_{p-1} + j_p + \cdots + j_{n_0-1} & \text{when } \circ_i \text{ is } + \text{ for all } i, \\
_{m+1} + \cdots + j_{n_0-1} & \text{otherwise.} \end{cases}$

\begin{align*}
= \sum_{i_p+j_p = k_p} (-1)^S \prod_{a=p}^{n_0-1} \frac{k_a}{i_a} h_{i_p \circ_p \cdots \circ_{n_0-2} i_{n_0-1}}(x) \cdot h_{k_1, \ldots, k_{p-1} + j_p \circ_p \cdots \circ_{n_0-2} j_{n_0-1}}(x).
\end{align*}

Here we use the definitions of $h_{k_1, \ldots, k_{n-1}}(x)$ and $h_{k_1, \ldots, k_{n}}(x)$. And by using our induction hypothesis on (3.3), we have

\begin{align*}
= \sum_{i_p+j_p = k_p} \prod_{a=p}^{n_0-1} \frac{k_a}{i_a} \left\{ (-1)^{\sum_{i_p}^{n_0-1}} \left( h_{i_p \circ_p \cdots \circ_{n_0-2} i_{n_0-1}}(x) \cdot \left( h_{i_p \circ_p \cdots \circ_{n_0-2} i_{n_0-2}}(x) \cdots (h_{\delta_x^p \circ_x^p}(h)) \right) \right) \right\}
\end{align*}

Here we put $\delta_i := \begin{cases} 0 & \text{if } \circ_i = +, \\
1 & \text{if } \circ_i = \circ, \text{ for } p \leq i \leq n_0 - 2. \end{cases}$

\begin{align*}
= (-1)^S h \sum_{i_p+j_p = k_p} \prod_{a=p}^{n_0-1} \frac{k_a}{i_a} \left\{ \delta_x^{i_{n_0-1}} \left( h_{i_{n_0-2} \circ_x^{i_{n_0-2}}} \cdots (h_{\delta_x^p \circ_x^p})(h) \right) \right\}
\end{align*}

We use Leibniz rule in last equality. By using this rule repeatedly, we get

\begin{align*}
= (-1)^S h \sum_{i_p+j_p = k_p} \prod_{a=p}^{n_0-2} \frac{k_a}{i_a} \left\{ \delta_x^{i_{n_0-2}} \left( h_{i_{n_0-3} \circ_x^{i_{n_0-3}}} \cdots (h_{\delta_x^p \circ_x^p})(h) \right) \right\}
\end{align*}

We use Leibniz rule in last equality. By using this rule repeatedly, we get

\begin{align*}
= (-1)^S h \sum_{i_p+j_p = k_p} \prod_{a=p}^{n_0-2} \frac{k_a}{i_a} \left\{ h_{i_p \circ_p \cdots \circ_{n_0-2} i_{n_0-2}}(x) \cdots (h_{\delta_x^p \circ_x^p})(h) \right\}.
\end{align*}

This is equal to (3.3).

Case ii) : When $(\circ_{n_0-1}, \circ_{n_0-1}) = (+, \circ)$, it can be proved in the same way to Case i).
3.2. An equivalence between desingularized values and renormalized ones.

We reveal a close relationship among desingularized values and renormalized ones in Theorem 3.5. As a consequence, we get an explicit formula of renormalized values in terms of Bernoulli numbers in Corollary 3.9.

Our main theorem of this paper is the following explicit relationship between the generating function \( Z_{FKMT}(t_1, \ldots, t_n) \) of the desingularized values \( \zeta_{FKMT}(-k_1, \ldots, -k_n) \) in (0.4) and the generating function \( Z_{EMS}(t_1, \ldots, t_n) \) of the renormalized values \( \zeta_{EMS}(-k_1, \ldots, -k_n) \) in (0.6).

**Theorem 3.5.** For \( n \in \mathbb{N} \), we have

\[
(3.7) \quad Z_{EMS}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{1 - e^{-t_i} - \cdots - e^{-t_n}}{t_i + \cdots + t_n} \cdot Z_{FKMT}(-t_1, \ldots, -t_n).
\]

**Proof.** By Proposition 3.3 and Lemma 3.4 we get

\[
(3.8) \quad \zeta_{EMS}(-k_1, \ldots, -k_n) = \sum_{i_2 + j_2 = k_2, a=2}^{n} \prod_{i_n+j_n=k_n} \zeta_{EMS}(-i_2, \ldots, -i_n) \zeta_{EMS}(-k_1 - j_2 - \cdots - j_n).
\]

Here, we use Lemma 3.4 for \( p = 2 \) and for all \( i_q = j_q \), \( (2 \leq q \leq n) \). It is remarkable that the same recurrence formula holds for \( \zeta_{FKMT}(-k_1, \ldots, -k_n) \) of [15]. Thus, we get

\[
(3.9) \quad Z_{EMS}(t_1, \ldots, t_n) = Z_{EMS}(t_2, \ldots, t_n) \cdot Z_{EMS}(t_1 + \cdots + t_n) \quad (n \in \mathbb{N}).
\]

Now from [18] Theorem 4.3, \( \zeta_{EMS}(-k_1) = \zeta(-k_1) \) at \( k_1 \in \mathbb{N}_0 \), so we can write \( Z_{EMS}(x) \) by

\[
Z_{EMS}(x) = 1 + x - e^x \frac{x}{e^x - 1}.
\]

We get the following equation by \( Z_{EMS}(x) \) and \( Z_{FKMT}(x) \):

\[
(3.10) \quad Z_{EMS}(x) = \frac{1 - e^{-x}}{x} Z_{FKMT}(-x).
\]

By using (1.4), (3.9) and (3.10), we get (3.7). \( \square \)

By Theorem 3.5, we find that desingularized values and renormalized ones are equivalent. Namely, the renormalized values can be given as linear combinations of the desingularized ones.

**Examples 3.6.** The desingularized values and the renormalized values are equal at the origin:

\[
\zeta_{FKMT}(0, \ldots, 0) = \zeta_{EMS}(0, \ldots, 0) = B_n^n = \left( -\frac{1}{2} \right)^n
\]
Examples 3.7. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

\[
\zeta_{EMS}(-k_1) = \sum_{\nu_{01} + \nu_{11} = k_1} \binom{k_1}{\nu_{01}} \frac{(-1)^{\nu_{11}}}{\nu_{01} + 1} \zeta_{FKMT}(-\nu_{11}),
\]
\[
\zeta_{EMS}(-k_1, -k_2) = \sum_{\nu_{01} + \nu_{11} = k_1, \nu_{02} + \nu_{12} + \nu_{22} = k_2} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \nu_{12}} \frac{1}{\nu_{02} + 1} \frac{(-1)^{\nu_{11} + \nu_{22}}}{\nu_{01} + \nu_{12} + 1} \zeta_{FKMT}(-\nu_{11}, -\nu_{22}),
\]
\[
\zeta_{EMS}(-k_1, -k_2, -k_3) = \sum_{\nu_{01} + \nu_{11} = k_1, \nu_{02} + \nu_{12} + \nu_{22} = k_2, \nu_{03} + \nu_{13} + \nu_{23} + \nu_{33} = k_3} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \nu_{12}} \binom{k_3}{\nu_{03} \nu_{13} \nu_{23}} \\
\times \frac{1}{\nu_{03} + 1} \frac{1}{\nu_{01} + \nu_{13} + 1} \frac{1}{\nu_{01} + \nu_{12} + \nu_{23} + 1} \frac{(-1)^{\nu_{11} + \nu_{12} + \nu_{23}}}{\nu_{01} + 1} \zeta_{FKMT}(-\nu_{11}, -\nu_{22}, -\nu_{33}).
\]

Here $(k_{02} \nu_{12}) := \frac{k_{01}!}{\nu_{02}! \nu_{12}! (k_2 - \nu_{02} - \nu_{12})!}$ and $(k_{03} \nu_{13} \nu_{23}) := \frac{k_{01}!}{\nu_{03}! \nu_{13}! \nu_{23}! (k_3 - \nu_{03} - \nu_{13} - \nu_{23})!}$.

On the other hand, desingularized values can be also given as linear combinations of product of renormalized ones and Bernoulli numbers $B_n$:

Examples 3.8. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

\[
\zeta_{FKMT}(-k_1) = (-1)^{k_1} \sum_{\nu_{01} + \nu_{11} = k_1} \binom{k_1}{\nu_{01}} B_{\nu_{01}} \zeta_{EMS}(-\nu_{11}),
\]
\[
\zeta_{FKMT}(-k_1, -k_2) = (-1)^{k_1 + k_2} \sum_{\nu_{01} + \nu_{11} = k_1, \nu_{02} + \nu_{12} + \nu_{22} = k_2} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \nu_{12}} B_{\nu_{02}} B_{\nu_{01} + \nu_{12}} \zeta_{EMS}(-\nu_{11}, -\nu_{22}),
\]
\[
\zeta_{FKMT}(-k_1, -k_2, -k_3) = (-1)^{k_1 + k_2 + k_3} \sum_{\nu_{01} + \nu_{11} = k_1, \nu_{02} + \nu_{12} + \nu_{22} = k_2, \nu_{03} + \nu_{13} + \nu_{23} + \nu_{33} = k_3} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \nu_{12}} \binom{k_3}{\nu_{03} \nu_{13} \nu_{23}} \\
\times B_{\nu_{02}} B_{\nu_{01} + \nu_{13}} B_{\nu_{01} + \nu_{12} + \nu_{23}} \zeta_{EMS}(-\nu_{11}, -\nu_{22}, -\nu_{33}).
\]

By combining Proposition 1.5 and Theorem 3.5, we obtain the following corollary.

Corollary 3.9. For $n \in \mathbb{N}$, we have

\[
Z_{EMS}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{(t_i + \cdots + t_n) - (t_i + \cdots + t_n - 1)}{(t_i + \cdots + t_n)(t_i + \cdots + t_n - 1)}.
\]

The above equation is equivalent to the equation (1.4). Therefore the renormalized values are described explicitly in terms of Bernoulli numbers:
Examples 3.10. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\zeta_{EMS}(-k_1) = \frac{(-1)^{k_1}}{k_1 + 1} B_{k_1 + 1},$$

$$\zeta_{EMS}(-k_1, -k_2) = (-1)^{k_1+k_2} \sum_{\nu_1+\nu_2 = k_2} \binom{k_2}{\nu_2} B_{\nu_2 + 1} B_{k_1 + \nu_1 + \nu_2 + 1},$$

$$\zeta_{EMS}(-k_1, -k_2, -k_3) = (-1)^{k_1+k_2+k_3} \sum_{\nu_1+\nu_2+\nu_3 = k_3} \binom{k_2}{\nu_1} \binom{k_3}{\nu_2+\nu_3} \frac{B_{\nu_1+\nu_2+\nu_3+1}}{\nu_1+\nu_2+\nu_3+1} \frac{B_{k_1+\nu_1+\nu_2+\nu_3+1}}{\nu_1+\nu_2+\nu_3+1}.$$

As is explained in our introduction, other types of renormalized values were investigated in several places in the literature ([9], [14], [18] etc). However, their explicit relationships with the desingularized values $\zeta_{KMT}(-k_1, \cdots, -k_n)$ do not seem to be shown so far, actually which was posed as a question in [12] Question 4.8. It would be great if our equivalence (Theorem 3.3) could also lead a direction to settle their question.

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