A contact manifold \((M, \xi)\) is a \((2n + 1)\)-dimensional manifold \(M\) equipped with a smooth maximally nonintegrable hyperplane field \(\xi \subset TM\), i.e., locally \(\xi = \ker \alpha\), where \(\alpha\) is a 1-form which satisfies \(\alpha \wedge (d\alpha)^n \neq 0\). Since \(d\alpha\) is a nondegenerate 2-form when restricted to \(\xi\), contact geometry is customarily viewed as the odd-dimensional sibling of symplectic geometry. Although

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contact geometry in dimensions $\geq 5$ is still in an incipient state, contact structures in dimension 3 are much better understood, largely due to the fact that symplectic geometry in two dimensions is just the study of area. The goal of this article is to explain some of the recent developments in 3-dimensional contact geometry, with an emphasis on methods from 3-dimensional topology. Basic references include [Ac, El, Et, Ge]. The article [Kz] is similar in spirit to ours.

Three-dimensional contact geometry lies at the interface between 3- and 4-manifold geometries, and has been an essential part of the flurry in low-dimensional geometry and topology over the last 20 years. In dimension 3, it relates to foliation theory and knot theory; in dimension 4, there are rich interactions with symplectic geometry. In both dimensions, there are relations with gauge theories such as Seiberg-Witten theory and Heegaard Floer homology.

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1. Introduction

From now on we will restrict our attention to contact structures on 3-manifolds. We will implicitly assume that our contact structures $\xi$ on $M$ satisfy the following:

1. $\xi$ is oriented, and hence given as the kernel of a global 1-form $\alpha$.
2. $\alpha \wedge d\alpha > 0$, i.e., the contact structure is positive.

Such contact structures are often said to be cooriented.

HW 1. Show that if $\xi$ is a smooth oriented 2-plane field, then $\xi$ can be written as the kernel of a global 1-form $\alpha$.

1.1. First examples.

Example 1: $(\mathbb{R}^3, \xi_0)$, where $\mathbb{R}^3$ has coordinates $(x, y, z)$, and $\xi_0$ is given by $\alpha_0 = dz - ydx$. Then $\xi_0 = \ker \alpha_0 = \mathbb{R}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$. According to the standard “propeller picture” (see Figure 1), all the straight lines parallel to the $y$-axis are everywhere tangent to $\xi_0$, and the 2-planes rotate in unison along these straight lines.

Example 2: $(T^3, \xi_n)$. Here $T^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$, with coordinates $(x, y, z)$, and $n \in \mathbb{Z}^+$. Then $\xi_n$ is given by $\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$. We have

$$\xi_n = \mathbb{R}\left\{\frac{\partial}{\partial z}, \cos(2\pi nz)\frac{\partial}{\partial x} - \sin(2\pi nz)\frac{\partial}{\partial y}\right\}.$$

This time, the circles $x = y = const$ (parallel to the $z$-axis) are everywhere tangent to $\xi_n$, and the contact structure makes $n$ full twists along such circles.
HW 2. Verify that \((\mathbb{R}^3, \xi_0)\) and \((T^3, \xi_n)\) are indeed contact manifolds.

The significance of Example 1 is the following:

**Theorem 1.1** (Pfaff). Every contact 3-manifold \((M, \xi)\) locally looks like \((\mathbb{R}^3, \xi_0)\), i.e., for all \(p \in M\) there is an open set \(U \supset p\) such that \((U, \xi) \simeq (\mathbb{R}^3, \xi_0)\).

Note that an isomorphism in the contact category (usually called a contactomorphism) is a diffeomorphism \(\phi : (M_1, \xi_1) \rightarrow (M_2, \xi_2)\) which maps \(\phi_*\xi_1 = \xi_2\). Pfaff’s theorem says that there are no local invariants in contact geometry.

**Remark.** A contactomorphism usually does not preserve the contact 1-form.

HW 3. Prove Pfaff’s theorem in dimension 3. Then generalize it to higher dimensions.

**Example 3:** \((S^3, \xi)\), the standard contact structure on \(S^3\). Consider \(B^4 = \{|z_1|^2 + |z_2|^2 \leq 1\} \subset \mathbb{C}^2\). Then take \(S^3 = \partial B^4\). The contact structure \(\xi\) is defined as follows: for all \(p \in S^3\), \(\xi_p\) is the unique complex line \(\subset T_p S^3\) (the unique 2-plane invariant under the complex structure \(J\)).

HW 4. Write down a contact 1-form \(\alpha\) for \((S^3, \xi)\) and verify that \(\alpha \wedge d\alpha > 0\).

1.2. **Legendrian knots.** Given a contact manifold \((M, \xi)\), a curve \(L \subset M\) is Legendrian if \(L\) is everywhere tangent to \(\xi\), i.e., \(\dot{L}(p) \in \xi_p\) at every point \(p \in L\). In this section we describe the invariants that can be assigned to a Legendrian knot (= embedded closed curve) \(L\). For a more thorough discussion, see the survey article [Et3].

**Twisting number/Thurston-Bennequin invariant:** Our first invariant is the relative Thurston-Bennequin invariant \(t(L, \mathcal{F})\), also known as the twisting number, where \(\mathcal{F}\) is some fixed framing for \(L\). Although \(t(L, \mathcal{F})\) is an invariant of the unoriented knot \(L\), for convenience pick one orientation of \(L\). \(L\) has a natural framing called the normal framing, induced from \(\xi\) by taking \(v_p \in \xi_p\).
so that \((v_p, \hat{L}(p))\) form an oriented basis for \(\xi_p\). We then define \(t(L, F)\) to be the integer difference in the number of twists between the normal framing and \(F\). By convention, left twists are negative. Now, the framing \(F\) that we choose is often dictated by the topology. For example, if \([L] = 0 \in H_1(M; \mathbb{Z})\) (which is the case when \(M = S^3\)), then there is a compact surface \(\Sigma \subset M\) with \(\partial \Sigma = L\), i.e., a Seifert surface. Now \(\Sigma\) induces a framing \(F_\Sigma\), which is the normal framing to the 2-plane field \(T\Sigma\) along \(L\), and the Thurston-Bennequin invariant \(tb(L)\) is given by:

\[ tb(L) = t(L, F_\Sigma). \]

**HW 5.** Show that \(tb(L)\) does not depend on the choice of Seifert surface \(\Sigma\).

In Example 2, if \(L = \{x = y = \text{const}\}\), then a convenient framing \(F\) is induced from tori \(x = \text{const}\) (or equivalently from \(y = \text{const}\)). We have \(t(L, F) = -n\).

**Rotation number:** Given an oriented Legendrian knot \(L\) in \(S^3\), we define the rotation number \(r(L)\) as follows: Choose a Seifert surface \(\Sigma\) and trivialize \(\xi|_\Sigma\). Then \(r(L)\) is the winding number of \(\dot{L}\) along \(L\) with respect to the trivialization.

**HW 6.** Show that \(r(L)\) does not depend on the choice of trivialization or Seifert surface.

**Front projection:** We now consider Legendrian knots in the standard contact \((\mathbb{R}^3, \xi_0)\) given by \(dz - ydx = 0\). Consider the front projection \(\pi : \mathbb{R}^3 \to \mathbb{R}^2\), where \((x, y, z) \mapsto (x, z)\). Generic Legendrian knots \(L\) (the genericity can be achieved by applying a small contact isotopy) can be projected to closed curves in \(\mathbb{R}^2\) with cusps and ordinary double points but no vertical tangencies. Conversely, such a closed curve in \(\mathbb{R}^2\) can be lifted to a Legendrian knot in \(\mathbb{R}^3\) by setting \(y\) to be the slope of the curve at \((x, z)\). (Observe that if \(dz - ydx = 0\), then \(\frac{dz}{dx} = y\).) The Thurston-Bennequin invariant and rotation number of a Legendrian knot \(L\) can be computed in the front projection using the following formula:

\[
\begin{align*}
\text{tb}(L) & = -\frac{1}{2} (\#\text{cusps}) + \#\text{positive crossings} \\
& \quad - \#\text{negative crossings}, \\
\text{r}(L) & = \frac{1}{2} (\#\text{downward cusps} - \#\text{upward cusps})
\end{align*}
\]

**HW 7.** Prove the above formulas for \(\text{tb}\) and \(r\) in the front projection.

**Stabilization:** Given an oriented Legendrian knot \(L\), its positive stabilization (resp. negative stabilization) \(S_+(L)\) (resp. \(S_-(L)\)) is an operation that decreases \(\text{tb}\) by adding a zigzag in the front projection as in Figure 2.

We have \(\text{tb}(S_\pm(L)) = \text{tb}(L) - 1\) and \(r(S_\pm(L)) = r(L) \pm 1\).

**HW 8.** Prove that the stabilization operation is well-defined (independent of the location where the zigzag is added).

The following theorem of Eliashberg-Fraser [EF] enumerates all the Legendrian unknots:
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**Theorem 1.2** (Eliashberg-Fraser). Legendrian unknots in the standard contact $\mathbb{R}^3$ (or $S^3$) are completely determined by $tb$ and $r$.

In fact, all the Legendrian unknots are stabilizations $S^k_+ S^k_-(L_0)$ of the unique maximal $tb$ Legendrian unknot $L_0$ with $tb(L_0) = -1$ and $r(L_0) = 0$, given on the left-hand side of Figure 3. The right-hand picture is $S^2_+ S^1_-(L_0)$.

**Figure 2.** Positive and negative stabilizations.

**Figure 3.** Legendrian unknots in the front projection.

For an oriented Legendrian knot in $\mathbb{R}^3$ or $S^3$, the topological knot type, the Thurston-Bennequin invariant, and the rotation number are called the **classical invariants**. Although Legendrian unknots are completely determined by their classical invariants according to Theorem 1.2, Legendrian knots in general are not completely classified by the classical invariants. One way of distinguishing two Legendrian knots with the same classical invariants is through **contact homology**. (See [ChE1G] for more details.)

1.3. **Tight vs. overtwisted.** In the 1970’s, Lutz [Lu] and Martinet [Ma] proved the following:

**Theorem 1.3** (Lutz, Martinet). Let $M$ be a closed oriented 3-manifold, $Dist(M)$ be the set of smooth 2-plane field distributions on $M$, and $Cont(M)$ be the set of smooth contact 2-plane field distributions on $M$. Then

$$\pi_0(Cont(M)) \to \pi_0(Dist(M))$$
is surjective.

Strategy of Proof.

(1) Start with a 2-plane field $\xi$. Take a fine enough triangulation of $M$ so that on each 3-simplex $\Delta$, $\xi$ is close to a linear foliation by planes.

(2) It is easy to homotop $\xi$ near the 2-skeleton so it becomes contact. Now we have an extension problem to the interior of each 3-simplex.

(3) Insert a Lutz tube. A Lutz tube is a contact structure on $S^1 \times D^2$ (with cylindrical coordinates $(z, r, \theta)$, where $D^2 = \{(r, \theta) | r \leq 1\}$) given by the 1-form
\[
\alpha = \cos(2\pi r)dz + r \sin(2\pi r)d\theta.
\]

□

HW 9. Think about how to use a Lutz tube (“perform a Lutz twist”) to finish the construction. Keep in mind that the homotopy class of the 2-plane field needs to be preserved.

Having introduced Lutz twists, we can now write down more contact structures on $\mathbb{R}^3$:

Example 1$_R$: $(\mathbb{R}^3, \zeta_R)$, where $\mathbb{R}^3$ has cylindrical coordinates $(r, \theta, z)$, $R$ is a positive real number, and $\zeta_R$ is given by $\alpha_R = \cos f_R(r)dz + r \sin f_R(r)d\theta$. Here $f_R(r)$ is a function with positive derivative satisfying $f_R(r) = r$ near $r = 0$ and $\lim_{r \to +\infty} f_R(r) = R$.

HW 10. Show that $(\mathbb{R}^3, \xi_0) \simeq (\mathbb{R}^3, \zeta_R)$ for all $R \leq \pi$.

However, we have the following key result of Bennequin [Be]:

Theorem 1.4 (Bennequin). $(\mathbb{R}^3, \xi_0) \not\simeq (\mathbb{R}^3, \zeta_R)$ if $R > \pi$.

The distinguishing feature is the existence of an overtwisted (OT) disk, i.e., an embedded disk $D \subset (M, \xi)$ such that $\xi_p = T_pD$ at all $p \in \partial D$. A typical OT disk looks like $\{pt\} \times D^2$ in the Lutz tube $S^1 \times D^2$ described above (also see Figure 4). While it is not hard to see that $(\mathbb{R}^3, \zeta_R)$ has OT disks if $R > \pi$, what Bennequin proved was that $(\mathbb{R}^3, \xi_0)$ contains no OT disks. It turns out that the existence of an OT disk is equivalent to the existence of a Legendrian unknot $L$ with $tb(L) = 0$.

HW 11 (Hard). Try to prove that $(\mathbb{R}^3, \xi_0)$ has no overtwisted disks.

It is not an exaggeration to say that modern contact geometry has its beginnings in Bennequin’s theorem. There is a dichotomy in the world of contact structures, those that contain OT disks (called overtwisted contact structures) and those that do not (called tight contact structures). In view of Theorem 1.4, every contact structure is locally tight, and therefore the question of overtwistedness is a global one.

The following is an important inequality for knots in tight contact manifolds.

Theorem 1.5 (Bennequin inequality). Let $L$ be nullhomologous Legendrian knot in a tight $(M, \xi)$. If $\Sigma$ is a Seifert surface for $L$ with Euler characteristic $\chi(\Sigma)$, then
\[
tb(L) \pm r(L) \leq -\chi(\Sigma).
\]
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Circle of tangencies

Figure 4. An overtwisted disk $D$. (Precisely speaking, the disk should end at the circle of tangencies.) The straight lines represent the singular (characteristic) foliation that $\xi \cap TD$ traces on $D$, and the circle is the set of points where $\xi = TD$. There is also an elliptic tangency at the center.

1.4. Classification of contact structures. When discussing the classification of contact structures, it is important to keep in mind the following theorem:

Theorem 1.6 (Gray). Let $\xi_t$, $t \in [0, 1]$, be a 1-parameter family of contact structures on a closed manifold $M$. Then there is a 1-parameter family of diffeomorphisms $\varphi_t$ such that $\varphi_0 = \text{id}$ and $\varphi_t^* \xi_t = \xi_0$.

In other words, a homotopy of contact structures gives rise to a contact isotopy.

The overtwisted classification (on closed 3-manifolds) was shown by Eliashberg [El3] to be essentially the same as the homotopy classification of 2-plane fields. (The result is quite striking, especially when contrasted with the tight classification on $T^3$ below.)

Theorem 1.7 (Eliashberg). Let $M$ be a closed oriented 3-manifold, and $\text{Cont}^{OT}(M) \subset \text{Dist}(M)$ be the overtwisted 2-plane field distributions. Then

$$\pi_0(\text{Cont}^{OT}(M)) \simeq \pi_0(\text{Dist}(M)).$$

On the other hand, tight contact structures tend to reflect the underlying topology of the manifold, and are more difficult to understand. The goal of this article is to introduce techniques which enable us to better understand tight contact structures. In the meantime, we list a couple of examples:

1. $S^3$. Eliashberg [El2] proved that there is a unique tight contact structure up to isotopy. It is the one given in Example 3.

2. $T^3$. Giroux [Gi2] and Kanda [Ka] independently proved that (a) every tight contact structure is isomorphic to some $\xi_n$ and (b) $(T^3, \xi_m) \not\simeq (T^3, \xi_n)$ if $m \neq n$.

HW 12. Try to prove that $(T^3, \xi_m) \not\simeq (T^3, \xi_n)$ if $m \neq n$.

In Section 4 we will give a classification of tight contact structures for the lens spaces $L(p, q)$. 

1.5. **A criterion for tightness.** A contact structure \((M, \xi)\) is *symplectically fillable* if there exists a compact symplectic 4-manifold \((X, \omega)\) such that \(\partial X = M\) and \(\omega|_\xi > 0\). \((X, \omega)\) is said to be a *symplectic filling* of \((M, \xi)\). (Technically speaking, what we are calling “symplectically fillable” is usually called “weakly symplectically fillable”, but since we have no need of such taxonomy in this article, we will stick to “symplectically fillable” or even just “fillable”. For more information, refer to [EH].)

**HW 13.** Show that \((S^3, \xi)\) in Example 3 is symplectically fillable.

**HW 14.** Show \((T^3, \xi_n)\) in Example 2 is symplectically fillable. (Hint: first modify \(\alpha_n \mapsto dz + t\alpha_n\) with \(t\) small.)

A powerful general method for producing tight contact structures is the following theorem of Gromov and Eliashberg [El1, Gr]:

**Theorem 1.8** (Gromov-Eliashberg). A symplectically fillable contact structure is tight.

It immediately follows from the symplectic filling theorem that the standard \((S^3, \xi)\) from Example 3 and the contact structures \((T^3, \xi_n)\) from Example 2 are tight.

Symplectic filling is a 4-dimensional way of checking whether \((M, \xi)\) is tight. We will discuss other methods (including a purely 3-dimensional one) of proving tightness in Section 5.

1.6. **Relationship with foliation theory.** Foliations are the other type of locally homogeneous 2-plane field distributions. The following table is a brief list of analogous objects from both worlds (note that the analogies are not precise):

| Foliations         | Contact Structures       |
|--------------------|--------------------------|
| \(\alpha \wedge d\alpha = 0\) integrable | \(\alpha \wedge d\alpha > 0\) nonintegrable |
| \(\alpha = dz\) Frobenius                  | \(\alpha = dz - ydx\) Pfaff       |
| Reeb components    | Overtwisted disks         |
| Taut               | Tight                     |

A (rank 2) *foliation* \(\xi\) is an integrable 2-plane field distribution, i.e., locally given as the kernel of a 1-form \(\alpha\) with \(\alpha \wedge d\alpha = 0\). According to Frobenius’ theorem, \(\xi\) can locally be written as the kernel of \(\alpha = dz\). The world of foliations also breaks up into the topologically significant *taut* foliations (i.e., foliations for which there is a closed transversal curve through each leaf), and the foliations with *generalized Reeb components*, which exist on every 3-manifold. A generalized Reeb component is a compact submanifold \(N \subset M\) whose boundary \(\partial N\) is a union of torus leaves, and such that there are no transversal arcs which begin and end on \(\partial N\). The primary example of a generalized Reeb component is a *Reeb component*, i.e., a foliation of the solid torus \(S^1 \times D^2\) whose boundary \(S^1 \times S^1\) is a leaf and whose interior is foliated by planes as in Figure 5.
The following is a key theorem which allows us to transfer information from foliation theory to contact geometry.

**Theorem 1.9** (Eliashberg-Thurston). Let $M$ be a closed, oriented 3-manifold $\neq S^1 \times S^2$. Then every taut foliation admits a $C^0$-small perturbation into a tight contact structure.

For a thorough treatment of the relationship with foliation theory, see [ET]. In Section 2.3 we will discuss one aspect, namely the relationship with Gabai’s *sutured manifold* theory.

2. Convex surfaces

In this section, we investigate embedded surfaces $\Sigma$ in the contact manifold $(M, \xi)$. The principal notion is that of *convexity*. For the time being, $\xi$ may be tight or overtwisted.

2.1. Characteristic foliations. Before discussing convexity, we first examine how $\xi$ traces a singular line field on an embedded surface $\Sigma$.

**Definition 2.1.** The characteristic foliation $\Sigma_\xi$ is the singular foliation induced on $\Sigma$ from $\xi$, where $\Sigma_\xi(p) = \xi_p \cap T_p \Sigma$. The singular points (or tangencies) are points $p \in \Sigma$ where $\xi_p = T_p \Sigma$.

**Lemma 2.2.** A $C^\infty$-generic characteristic foliation $\Sigma_\xi$ is of Morse-Smale type, i.e., satisfies the following:

1. the singularities and closed orbits are dynamically hyperbolic, i.e, hyperbolic in the dynamical systems sense,
2. there are no saddle-saddle connections, and
3. every point $p \in \Sigma$ limits to some isolated singularity or closed orbit in forward time and likewise in backward time.

The proof of Lemma 2.2 uses the fact that a $C^\infty$-small perturbation of $\xi$ is still contact. We choose the perturbation of $\xi$ to be compactly supported near $\Sigma$, and hence the isotopy in Gray’s theorem is compactly supported near $\Sigma$. Therefore, generic properties of 1-forms (in particular the Morse-Smale condition) are satisfied.
HW 15. Show that if $\alpha$ is a contact 1-form and $\beta$ is any 1-form, then $\alpha + t\beta$ is contact for sufficiently small $t$.

There are two types of dynamically hyperbolic singularities: elliptic and hyperbolic (not in the dynamical systems sense). Choose coordinates $(x, y)$ on $\Sigma$ and let the origin be the singular point. If we write $\alpha = dz + f \, dx + g \, dy$, then $X = g \frac{\partial}{\partial x} - f \frac{\partial}{\partial y}$ is a vector field for the characteristic foliation near the origin. If the determinant of the matrix

$$
\begin{pmatrix}
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}
\end{pmatrix}
$$

is positive (resp. negative), then the singular point is elliptic (resp. hyperbolic). An example of an elliptic singularity is $\alpha = dz + (xdy - ydx)$, and an example of a hyperbolic singularity is $\alpha = dz + (2xdy + ydx)$.

Next we discuss signs. Assume $\Sigma$ and $\xi$ are both oriented. Then a singular point $p$ is positive (resp. negative) if $T_p\Sigma$ and $\xi_p$ have the same orientation (resp. opposite orientations).

Claim. The characteristic foliation $\Sigma_\xi$ is oriented.

We use the convention that positive elliptic points are sources and negative elliptic points are sinks. If $p$ is a nonsingular point of a leaf $L$ of the characteristic foliation, then we choose $v \in T_pL$ so that $(v, n)$ is an oriented basis for $T_p\Sigma$. Here $n \in T_p\Sigma$ is an oriented normal vector to $\xi_p$.

Examples of characteristic foliations:

1. Consider $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset (\mathbb{R}^3, \zeta_{\pi/2})$. Then $S^2$ will have two singular points, the positive elliptic point $(0, 0, 1)$ and the negative elliptic point $(0, 0, -1)$, and the leaves spiral downward from $(0, 0, 1)$ to $(0, 0, -1)$.

2. An example of an overtwisted disk $D$ is one which has a positive elliptic point at the center and radial leaves emanating from the center, such that $\partial D$ is a circle of singularities. Often in the literature one sees overtwisted disks whose boundary is transverse to $\xi$ and whose leaves emanating from the center spiral towards the limit cycle $\partial D$. (Strictly speaking, such a $D$ with a limit cycle is not an OT disk according to our definition, but can easily be modified to fit our definition.)

The importance of the characteristic foliation $\Sigma_\xi$ comes from the following proposition:

Proposition 2.3. Let $\xi_0$ and $\xi_1$ be two contact structures which induce the same characteristic foliation on $\Sigma$. Then there is an isotopy $\varphi_t$, $t \in [0, 1]$, rel $\Sigma$, with $\varphi_0 = \text{id}$ and $(\varphi_1)_*\xi_0 = \xi_1$.

2.2. Convexity. The notion of a convex surface, introduced by Giroux in [Gir] and extended to the case of a compact surface with Legendrian boundary by Kanda in [Ka], is the key ingredient in the cut-and-paste theory of contact structures.

Definition 2.4. A properly embedded oriented surface $\Sigma$ is convex if there exists a contact vector field $v \pitchfork \Sigma$. Here, a contact vector field is a vector field whose corresponding flow preserves the
contact structure $\xi$. In this article we assume that our convex surfaces are either closed or compact with Legendrian boundary.

If $\Sigma = \Sigma \times \{0\}$ is convex, then there is an invariant neighborhood $\Sigma \times [-\varepsilon, \varepsilon] \subset M$. We usually assume that $v$ agrees with the normal orientation to $\Sigma$.

Properties of convex surfaces:

1. A $C^\infty$-generic closed embedded surface $\Sigma$ is convex. This is because an embedded surface $\Sigma$ with a Morse-Smale characteristic foliation is convex. (The same is almost true for compact surfaces with Legendrian boundary, but more care is needed along the boundary.)
2. To a convex surface $\Sigma$ we may associate a multicurve (i.e., a properly embedded (smooth) 1-manifold, possibly disconnected and possibly with boundary)

$$\Gamma_\Sigma = \{x \in \Sigma | v(x) \in \xi_x\},$$

called the dividing set. It satisfies the following:

(a) $\Gamma_\Sigma \cap \Sigma_\xi$.
(b) The isotopy class of $\Gamma_\Sigma$ does not depend on the choice of $v$.
(c) $\Sigma \setminus \Gamma_\Sigma = R_+ (\Gamma_\Sigma) \sqcup R_- (\Gamma_\Sigma)$, where $R_+ (\Gamma_\Sigma) \subset \Sigma$ (resp. $R_- (\Gamma_\Sigma)$) is the set of points $x$ where the normal orientation to $\Sigma$ given by $v(x)$ agrees with (resp. is opposite to) the normal orientation to $\xi_x$.

Remark. We may think of $\Gamma_\Sigma$ as the set of points where $\xi \perp \Sigma$, where $\perp$ is measured with respect to $v$.

Write $\#\Gamma_\Sigma$ for the number of connected components of $\Gamma_\Sigma$.

The usefulness of the dividing set $\Gamma_\Sigma$ comes from the following:

**Theorem 2.5** (Giroux’s Flexibility Theorem). Assume $\Sigma$ is convex with characteristic foliation $\Sigma_\xi$, contact vector field $v$, and dividing set $\Gamma_\Sigma$. Let $F$ be another singular foliation on $\Sigma$ which is adapted to $\Gamma_\Sigma$ (i.e., there is a contact structure $\xi'$ in a neighborhood of $\Sigma$ such that $\Sigma_{\xi'} = F$ and $\Gamma_\Sigma$ is also a dividing set for $\xi'$). Then there is an isotopy $\varphi_t$, $t \in [0, 1]$, of $\Sigma$ in $(M, \xi)$ such that:
(1) $\varphi_0 = id$ and $\varphi_t|_{\Gamma_{\Sigma}} = id$ for all $t$.
(2) $\varphi_t(\Sigma) \pitchfork v$ for all $t$.
(3) $\varphi_1(\Sigma)$ has characteristic foliation $\mathcal{F}$.

In essence, $\Gamma_{\Sigma}$ encodes ALL of the essential contact-topological information in a neighborhood of $\Sigma$. Therefore, having discussed characteristic foliations in Section 2.1 we may proceed to discard them and simply remember the dividing set.

**HW 16. Prove Giroux Flexibility.**

**Examples on $T^2$:** There are two common characteristic foliations on $T^2$.

1. *Nonsingular Morse-Smale.* This is when the characteristic foliation is nonsingular and has exactly $2n$ closed orbits, $n$ of which are sources (repelling periodic orbits) and the other $n$ are sinks (attracting periodic orbits). $\Gamma_{T^2}$ consists of $2n$ closed curves parallel to the closed orbits. Each dividing curve lies inbetween two periodic orbits.
2. *Standard form.* An example is $x = const$ inside $(T^3, \xi_n)$. The torus is fibered by closed Legendrian fibers, called *ruling curves*, and the singular set consists of $2n$ closed curves, called *Legendrian divides*. The $2n$ curves of $\Gamma_{T^2}$ lie between the Legendrian divides.

**HW 17. Find an explicit example of a $T^2$ inside a contact manifold with nonsingular Morse-Smale characteristic foliation.**

![Figure 7](image_url)

**Figure 7.** The left-hand side is a torus with nonsingular Morse-Smale characteristic foliation. The right-hand side is a torus in standard form. Here the sides are identified and the top and bottom are identified.

What Giroux Flexibility tells us is that it is easy to switch between the two types of characteristic foliations – nonsingular Morse-Smale and standard form. The following corollary of Giroux Flexibility is a crucial ingredient in the cut-and-paste theory of contact structures.

**Corollary 2.6 (Legendrian Realization Principle, abbreviated LeRP).** Let $\Sigma$ be a convex surface and $C$ be a multicurve on $\Sigma$. Assume $C \pitchfork \Gamma_{\Sigma}$ and $C$ is nonisolating, i.e., each connected component of $\Sigma \setminus C$ nontrivially intersects $\Gamma_{\Sigma}$. Then there is an isotopy (as in the Giroux Flexibility Theorem) such that $\varphi_1(C)$ is Legendrian.
HW 18. Try to prove LeRP, assuming Giroux Flexibility.

Remark. \( C \) may have extraneous intersections with \( \Gamma_\Sigma \), i.e., the actual number of intersections \( \#(C \cap \Gamma_\Sigma) \) is allowed to be larger than the geometric intersection number.

Fact: If \( C \) is a Legendrian curve on the convex surface \( \Sigma \), then the twisting number \( t(C, \Sigma) \) relative to the framing from \( \Sigma \) is \(-\frac{1}{2}\#(C \cap \Gamma_\Sigma)\). Here \( \#(\cdot) \) represents cardinality, not geometric intersection.

Now we present the criterion (see [Gi3]) for determining when a convex surface has a tight neighborhood.

**Proposition 2.7** (Giroux’s Criterion). A convex surface \( \Sigma \neq S^2 \) has a tight neighborhood if and only if \( \Gamma_\Sigma \) has no homotopically trivial dividing curves. If \( \Sigma = S^2 \), then there is a tight neighborhood if and only if \( \#\Gamma_\Sigma = 1 \).

HW 19. Prove that if \( \Gamma_\Sigma \) has a homotopically trivial dividing curve, then there exists an overtwisted disk in a neighborhood of \( \Sigma \), provided we are not in the situation where \( \Sigma = S^2 \) and \( \#\Gamma_\Sigma = 1 \). (Hint: use LeRP, together with a trick when \( \Gamma_\Sigma \) has no other components besides the homotopically trivial curve.)

The “only if” direction in Giroux’s Criterion follows from HW 19. The “if” direction follows from constructing an explicit model inside a tight 3-ball or gluing (for the latter, see [Co1]).

Suppose that \( (M, \xi) \) is tight. If \( \Sigma = S^2 \) is a convex surface in \( (M, \xi) \), then \( \Gamma_\Sigma \) is unique up to isotopy, consisting on one (homotopically trivial) circle. If \( \Sigma = T^2 \) is convex, then it consists of \( 2n \) parallel, homotopically essential curves. Therefore \( \Gamma_{T^2} \) is determined by \( \#\Gamma_{T^2} \) and the slope, once a trivialization \( T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2 \) is fixed.

2.3. **Convex decomposition theory.** The reader may have already noticed certain similarities between convex surfaces and the theory of sutured manifolds due to Gabai [Ga].

**Definition 2.8.** A sutured manifold \( (M, \Gamma) \) consists of the following data:

1. \( M \) is a compact, oriented, irreducible 3-manifold; each component of \( M \) has nonempty boundary,
2. \( \Gamma \) is a multicurve on \( \partial M \) which has nonempty intersection with each component of \( \partial M \), and
3. \( \Gamma \) divides \( \partial M \) into positive and negative regions, whose sign changes every time \( \Gamma \) is crossed. We write \( \partial M \setminus \Gamma = R_+(\Gamma) \cup R_-(\Gamma) \).

Here, a 3-manifold \( M \) is irreducible if every embedded 2-sphere \( S^2 \) bounds a 3-ball \( B^3 \).

Note that our definition of a sutured manifold, chosen to simplify the exposition in this paper, is slightly different from that of Gabai [Ga].

**Definition 2.9.** Let \( S \) be a compact oriented surface with connected components \( S_1, \ldots, S_n \). The **Thurston norm** of \( S \) is:

\[
x(S) = \sum_{i \text{ such that } \chi(S_i) < 0} |\chi(S_i)|.
\]
**Definition 2.10.** A sutured manifold \((M, \Gamma)\) is taut if \(R_{\pm}(\Gamma)\) are incompressible in \(M\) and minimize the Thurston norm in \(H_2(M, \Gamma)\). Here, a surface \(S \subset M\) is incompressible if for every embedded disk \(D \subset M\) with \(D \cap S = \partial D\), there is a disk \(D' \subset S\) such that \(\partial D = \partial D'\).

Roughly speaking, \((M, \Gamma)\) is taut if \(R_{\pm}(\Gamma)\) attain the minimum genus amongst all the embedded representatives in the relative homology class \(H_2(M, \Gamma)\).

We have the following theorem which gives the equivalence between tightness and tautness in the case of a manifold with boundary (see [HKM1]):

**Theorem 2.11** (Kazez-Matić-Honda). Let \((M, \Gamma)\) be a sutured manifold. Then the following are equivalent:

1. \((M, \Gamma)\) is taut.
2. \((M, \Gamma)\) carries a taut foliation.
3. \((M, \Gamma)\) carries a universally tight contact structure.
4. \((M, \Gamma)\) carries a tight contact structure.

A contact structure \(\xi\) on \(M\) is carried by \((M, \Gamma)\) if \(\partial M\) is a convex surface for \(\xi\) with dividing set \(\Gamma\). A transversely oriented foliation \(\xi\) on \(M\) is carried by \((M, \Gamma)\) if there exists a thickening of \(\Gamma\) to a union \(\gamma \subset \partial M\) of annuli, so that \(\partial M \setminus \gamma\) is a union of leaves of \(\xi\), \(\xi\) is transverse to \(\gamma\), and the orientations of \(R_{\pm}(\Gamma)\) and \(\xi\) agree. (Strictly speaking, in this case \(M\) is a manifold with corners.) A tight contact structure is universally tight if it remains tight when pulled back to the universal cover of \(M\).

In the rest of this section, we explain how sutured manifold decompositions have an analog in the contact world, namely the theory of convex decompositions. Using it we outline the proof of \((1) \Rightarrow(4)\).

**Definition 2.12.** Let \(S\) be an oriented, properly embedded surface in \((M, \Gamma)\) which intersects \(\Gamma\) transversely. Then a sutured manifold splitting \((M, \Gamma) \leadsto (M', \Gamma')\) is given as follows (see Figure 8 for an illustration): Define \(M' = M \setminus S\), and let \(S_+\) (resp. \(S_-\)) be the copy of \(S\) on \(\partial M'\) where the orientation inherited from \(S\) and the outward normal agree (are opposite). Then set \(R_{\pm}(\Gamma') = (R_{\pm}(\Gamma)\setminus S) \cup S_{\pm}\). The new suture \(\Gamma'\) forms the boundary between the regions \(R_+(\Gamma')\) and \(R_-(\Gamma')\).

![Figure 8](image-url)
A sutured manifold \((M, \Gamma)\) is decomposable, if there is a sequence of sutured manifold splittings:
\[
(M, \Gamma) \xrightarrow{S_1} (M_1, \Gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \Gamma_n) = \sqcup (B^3, S^1).
\]

Gabai, in [Ga], proved the following theorem:

**Theorem 2.13** (Gabai).

1. *(Decomposition)* If \((M, \Gamma)\) is taut, then it is decomposable.
2. *(Reconstruction)* Given a sutured manifold decomposition, we can backtrack and construct a taut foliation which is carried by \((M, \Gamma)\).

Now, in the contact category, we choose a dividing set \(\Gamma_S\) so that every component of \(\Gamma_S\) is \(\partial\)-parallel, i.e., cuts off a half-disk of \(S\) which does not intersect any other component of \(\Gamma_S\). Such a dividing set \(\Gamma_S\) is also called \(\partial\)-parallel.

If there is an invariant contact structure defined in a neighborhood of \(\partial M\) with dividing set \(\Gamma = \Gamma_{\partial M}\), then by an application of LeRP, we may take \(\partial S\) to be Legendrian. (There are some exceptional cases, but we will not worry about them here.) Extend the contact structure to be an invariant contact structure in a neighborhood of \(S\) with \(\partial\)-parallel dividing set \(\Gamma_S\). Now, if we cut \(M\) along \(S\), we obtain a manifold with corners. To smooth the corners, we apply edge-rounding. This is given in Figures 9 and 10. Figure 9 gives the surface \(S\) before rounding, and Figure 10 after rounding. Notice that we may think of \(S\) as a lid of a jar, and the edge-rounding operation as twisting to close the jar.

HW 20. Explain why edge-rounding works as in Figures 9 and 10.
Observe that the dividing set in Figure 10 is isotopic to the sutures in Figure 8. Therefore, given a sutured manifold splitting \((M, \Gamma) \xrightarrow{S} (M', \Gamma')\), with a \(\partial\)-parallel dividing set \(\Gamma_S\). Using the decomposition theorem of Gabai, if \((M, \Gamma)\) is taut, then there exists a convex decomposition:

\[
(M, \Gamma) \xrightarrow{(S_1, \Gamma_{S_1})} (M_1, \Gamma_1) \xrightarrow{(S_2, \Gamma_{S_2})} \ldots \xrightarrow{(S_n, \Gamma_{S_n})} (M_n, \Gamma_n) = \sqcup (B^3, S^1).
\]

We now work backwards, starting with the following theorem of Eliashberg [El2]:

**Theorem 2.14** (Eliashberg). Fix a characteristic foliation \(\mathcal{F}\) adapted to \(\Gamma_{\partial B^3} = S^1\). Then there is a unique tight contact structure on \(B^3\) up to isotopy relative to \(\partial B^3\).

The following gluing theorem of Colin [Co1] allows us to inductively build a universally tight contact structure carried by \((M, \Gamma)\).

**Theorem 2.15** (Colin). Let \(\Sigma\) be an incompressible surface with \(\partial \Sigma \neq \emptyset\). If \(\Gamma_{\Sigma}\) is \(\partial\)-parallel and \((M \setminus \Sigma, \xi|_{M\setminus\Sigma})\) is universally tight, then \((M, \xi)\) is also universally tight.

This theorem and other similar theorems will be discussed in Section 5.

Theorem 2.11 is a refinement, in the case of manifolds with boundary, of the following theorem:

**Theorem 2.16** (Gabai-Eliashberg-Thurston). Let \(M\) be an oriented, closed, irreducible 3-manifold with \(H_2(M; \mathbb{Z}) \neq 0\). Then \(M\) carries a universally tight contact structure.

The Gabai-Eliashberg-Thurston theorem was originally proved in two parts: Gabai [Ga] proved that such an \(M\) carries a taut foliation, and Eliashberg-Thurston [ET] proved that the taut foliation can be perturbed into a universally tight contact structure. There is also an alternate, purely 3-dimensional method for proving this theorem [HKM2, HKM3, HKM4].

### 3. Bypasses

In this section, we introduce the other chief ingredient in the cut-and-paste theory of tight contact structures: the *bypass*. As a surface is isotoped inside the ambient tight contact manifold \((M, \xi)\), the dividing set changes in discrete units, and the fundamental unit of change is effected by the bypass. Bypasses would be quite useless if they were difficult to find. For the cases we examine in Section 4, namely solid tori, \(T^2 \times I\), and lens spaces, they can be found relatively easily by examining the next step in the Haken hierarchy. This will be explained in Section 3.2. For more information on bypasses, refer to [H1].

#### 3.1. Definition and examples.

**Definition 3.1.** Let \(\Sigma\) be a convex surface and \(\alpha\) be a Legendrian arc in \(\Sigma\) which intersects \(\Gamma_{\Sigma}\) in three points \(p_1, p_2, p_3\), where \(p_1\) and \(p_3\) are endpoints of \(\alpha\). A bypass half-disk is a convex half-disk \(D\) with Legendrian boundary, where \(D \cap \Sigma = \alpha\) and \(tb(\partial D) = -1\). \(\alpha\) is called the arc of attachment of the bypass, and \(D\) is said to be a bypass along \(\alpha\) or \(\Sigma\).
Remark. Most bypasses do not come for free. Finding a bypass is equivalent to raising the twisting number (or Thurston-Bennequin invariant) by 1. Although it is easy to lower the twisting number by attaching “zigzags” in a front projection, raising the twisting number is usually a nontrivial operation.

Lemma 3.2 (Bypass Attachment Lemma). Let $D$ be a bypass for $\Sigma$. If $\Sigma$ is isotoped across $D$, then we obtain a new convex surface $\Sigma'$ whose dividing set is obtained from $\Gamma_\Sigma$ via the move in Figure 12.

Example: $T^2$. Let us enumerate the possible bypass attachments – see Figure 13. (a) is the case where $\#\Gamma_{T^2} = 2n > 2$, and the bypass reduced $\#\Gamma$ by two, while keeping the slope fixed. (b) is the case where $\#\Gamma_{T^2} = 2$, and the slope is modified. In addition, there also are trivial and disallowed moves, which are moves locally given in Figure 14. It turns out that the trivial move
always exists inside a tight contact manifold, whereas the disallowed move can never exist inside a tight contact manifold.

**HW 21.** *Is there a bypass attachment which increases $\#\Gamma$?*

**Intrinsic interpretation:** Observe that, in case (b), the bypass move is equivalent to performing a *positive Dehn twist* along a particular curve. We can therefore reformulate this bypass move and give an *intrinsic interpretation* in terms of the Farey tessellation of the hyperbolic unit disk $H$ (Figure 15). The set of vertices of the Farey tessellation is $\mathbb{Q} \cup \{\infty\}$ on $\partial H$. (More precisely, fix a
fractional linear transformation $f$ from the upper half-plane model of hyperbolic space to the unit disk model $H$. Then the set of vertices is the image of $\mathbb{Q} \cup \{\infty\}$ under $f$. There is a unique edge between $\frac{p}{q}$ and $\frac{p'}{q'}$ if and only if the corresponding shortest integer vectors form an integral basis for $\mathbb{Z}^2$. (The edge is usually taken to be a geodesic in $H$.)

![Figure 15. The Farey tessellation. The spacing between vertices are not drawn to scale.](image)

**Proposition 3.3.** Let $s = \text{slope}(\Gamma_{T^2})$. If a bypass is attached along a closed Legendrian curve of slope $s'$, then the resulting slope $s''$ is obtained as follows: Let $(s', s) \subset \partial H$ be the counterclockwise interval from $s'$ to $s$. Then $s''$ is the point on $(s', s)$ which is closest to $s'$ and has an edge to $s$.

See Figure 16 for an illustration.

**HW 22.** Prove Proposition 3.3.

![Figure 16. Intrinsic interpretation of the bypass attachment.](image)
3.2. **Finding bypasses.** We now explain how to find bypasses. Let $M$ be a closed manifold and $\Sigma \subset M$ be a closed surface. In order to find a bypass along $\Sigma$, we consider $M \setminus \Sigma$. Let $S \subset M \setminus \Sigma$ be an incompressible surface with nonempty boundary, for example the next cutting surface in the Haken hierarchy. Under mild conditions on $\partial S$, we can take $S$ to be a convex surface with nonempty Legendrian boundary.

**Lemma 3.4.** Suppose that $\Gamma_S$ has a $\partial$-parallel component and either $S \neq D^2$ or else if $S = D^2$ then $tb(\partial S) < -1$. Then there exists a bypass along $\partial S$ and hence along $\Sigma$.

**Proof.** Draw an arc $\delta' \subset S$ so that $\delta'$ cuts off a half-disk with only the $\partial$-parallel arc $\delta$ on it. The condition on $S$ is needed to ensure that we can use LeRP to find a Legendrian arc $\delta''$. The half-disk cut off by $\delta''$ (and containing a copy of $\delta$) is the bypass for $\Sigma$. \[\Box\]

**Corollary 3.5.** Let $S = D^2$ be a convex disk with Legendrian boundary so that $tb(\partial S) < -1$. Then there exists a bypass along $\partial S$.

Corollary 3.5 follows from Lemma 3.4 by observing that all components of $\Gamma_{D^2}$ cut off half-disks of $D^2$ and that a $\partial$-parallel component is simply an outermost arc of $\Gamma_{D^2}$.

**Remark.** Corollary 3.5 does not work when $tb(\partial D) = -1$.

Similarly, we can prove the following:

**Corollary 3.6 (Imbalance Principle).** Let $S = S^1 \times [0, 1]$ be a convex annulus. If $t(S^1 \times \{1\}, F_S) < t(S^1 \times \{0\}, F_S)$, then there is a $\partial$-parallel arc and hence a bypass along $S^1 \times \{1\}$. Here $F_S$ is the framing induced from the surface $S$.

Figure 17 gives an example of a convex annulus with $t(S^1 \times \{1\}, F_S) < t(S^1 \times \{0\}, F_S)$. There is necessarily a bypass along $S^1 \times \{1\}$.

![Figure 17](image)
4. Classification of tight contact structures on lens spaces

As an illustration of the technology introduced in the previous two sections, we give a complete classification of tight contact structures on the lens spaces $L(p, q)$. This classification was obtained independently by Giroux [Gi2] and Honda [H1]; partial results had been obtained previously by Etnyre [Et2]. In this article, we follow the method of [H1].

4.1. The standard neighborhood of a Legendrian curve. Consider a (closed) Legendrian curve $L$ with $t(L, F) = -n < 0$, $n \in \mathbb{Z}^+$. (Pick some framing $F$ for which the twisting number is negative.) Then a standard neighborhood $S^1 \times D^2 = \mathbb{R}/\mathbb{Z} \times \{x^2 + y^2 \leq \varepsilon\}$ (with coordinates $z, x, y$) of the Legendrian curve $L = S^1 \times \{(0, 0)\}$ is given by

$$\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy,$$

and satisfies the following:

1. $T^2 = \partial(S^1 \times D^2)$ is convex.
2. $\#\Gamma_{T^2} = 2$.
3. $\text{slope}(\Gamma_{T^2}) = -\frac{1}{n}$, if the meridian has zero slope and the longitude given by $x = y = \text{const}$ has slope $\infty$.

The following is due to Kanda [Ka] and Makar-Limanov [ML1].

**Proposition 4.1** (Kanda, Makar-Limanov). Given a solid torus $S^1 \times D^2$ and boundary conditions (1), (2), (3), there exists a unique tight contact structure on $S^1 \times D^2$ up to isotopy rel boundary, provided we have fixed a characteristic foliation $F$ adapted to $\Gamma_{\partial(S^1 \times T^2)}$.

**Remark.** The precise characteristic foliation is irrelevant in view of Giroux Flexibility.

**Proof.**

1. Let $L \subset T^2$ be a curve which bounds the meridian $D$. Using LeRP, realize it as a Legendrian curve with $tb(L) = -1$.
2. Using the genericity of convex surfaces, realize the surface $D$ with $\partial D = L$ as a convex surface with Legendrian boundary. Since $tb(L) = -1$, there is only one possibility for $\Gamma_D$, up to isotopy.
3. Next, using Giroux Flexibility, fix some characteristic foliation on $D$ adapted to $\Gamma_D$. Note that any two tight contact structures on $S^1 \times D$ with boundary condition $F$ can be isotoped to agree on $T^2 \cup D$.
4. The rest is a 3-ball $B^3$. Use Eliashberg’s uniqueness theorem for tight contact structures on $B^3$. \qed

**HW 23.** Try to prove Eliashberg’s theorem, using convex surfaces.
4.2. **Lens spaces.** Let $p > q > 0$ be relatively prime integers. The lens space $L(p, q)$ is obtained by gluing $V_1 = S^1 \times D^2$ and $V_2 = S^1 \times D^2$ together via $A : \partial V_2 \to \partial V_1$, where $A = \begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix} \in -SL(2, \mathbb{Z})$. Here we are making an oriented identification $\partial V_i \simeq \mathbb{R}^2 / \mathbb{Z}^2$, where the meridian of $V_i$ is mapped to $\pm (1, 0)$, and some chosen longitude is mapped to $\pm (0, 1)$.

**Continued fractions:** Let $-\frac{p}{q}$ have a continued fraction expansion

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots - \frac{1}{r_k}}}$$

where $r_i \leq -2$.

**Example:** $-\frac{14}{5} = -3 - \frac{1}{-5}$. We write $-\frac{14}{5} \leftrightarrow (-3, -5)$.

**Theorem 4.2** (Giroux, Honda). On $L(p, q)$, there are exactly $|r_0 + 1| |r_1 + 1| \cdots |r_k + 1|$ tight contact structures up to isotopy. They are all holomorphically fillable.

A surgery presentation for $L(p, q)$ is given as follows:

![Figure 18](image)

**Legendrian surgery:** Given a Legendrian knot $K = K_0$ or link $L = \sqcup_{i=1}^k K_i$ in a contact manifold $(M, \xi)$, we can perform a surgery along the $K_i$ with coefficient $tb(K_i) - 1$. At the 4-dimensional level, if $M = S^3$, then we start with a Stein domain $B^4$ with $\partial B^4 = S^3$, and attach 2-handles in a way which makes the resulting 4-manifold $X^4$ a Stein domain (and in particular symplectic). The resulting contact 3-manifold $(M', \xi')$ with $\partial X = M'$ is said to be holomorphically fillable. Similarly, if $(M, \xi)$ is symplectically fillable, then $(M', \xi')$ obtained by Legendrian surgery is also symplectically fillable. The Stein construction was done by Eliashberg in [EH] and the symplectic construction by Weinstein [We].

Suppose $K_i$ is a Legendrian unknot with $tb(K_i) = r_i + 1$ and $r(K_i) = \text{one of } r_i + 2, r_i + 4, \ldots, -(r_i + 2)$. There are precisely $|r_i + 1|$ choices for the rotation number $r(K_i)$. (In fact, these are all the Legendrian unknots with $tb(K_i) = r_i + 1$ by Theorem 1.2)

**HW 24.** Show that the $|r_0 + 1||r_1 + 1| \cdots |r_k + 1|$ holomorphically fillable contact structures are distinct.
Therefore, we have the lower bound:

\[ \# \text{Tight}(L(p, q)) \geq |(r_0 + 1)(r_1 + 1)\ldots(r_k + 1)|. \]

Here \( \text{Tight}(M) \) refers to the set of isotopy classes of tight contact structures on \( M \). In order to prove Theorem 4.2 it remains to show the reverse inequality.

4.3. Solid tori. We now consider tight contact structures on the solid torus \( S^1 \times D^2 \) with the following conditions on the boundary \( T = S^1 \times D^2 \):

1. \( \# \Gamma_T = 2 \).
2. \( \text{slope}(\Gamma_T) = -\frac{p}{q} \), where \( -\infty < -\frac{p}{q} \leq -1 \). (After performing Dehn twists, we can normalize the slope as such.)
3. The fixed characteristic foliation \( \mathcal{F} \) is adapted to \( \Gamma_T \).

**Theorem 4.3.** There are exactly \( |(r_0 + 1)(r_1 + 1)\ldots(r_{k-1} + 1)r_k| \) tight contact structures on \( S^1 \times D^2 \) with this boundary condition.

**Step 1:** In this step we factor \( S^1 \times D^2 \) into a union of \( T^2 \times I \) layers and a standard neighborhood of a Legendrian curve isotopic to the core curve of \( S^1 \times D^2 \). Assume \( -\frac{p}{q} < -1 \), since \( -\frac{p}{q} = -1 \) has already been treated.

Let \( D \) be a meridional disk with \( \partial D \) Legendrian and \( \text{tb}(\partial D) = -p < -1 \). Then by Lemma 3.5 there is at least one bypass along \( \partial D \). Attach the bypass to \( T \) from the interior and apply the Bypass Attachment Lemma. We obtain a convex torus \( T' \) isotopic to \( T \), such that \( T \) and \( T' \) cobound a \( T^2 \times I \). Denote slope(\( \Gamma_{T'} \)) = \( -\frac{p'}{q'} \).

**HW 25.** If \( -\frac{p}{q} \leftrightarrow (r_0, r_1, \ldots, r_{k-1}, r_k) \), then \( -\frac{p'}{q'} \leftrightarrow (r_0, r_1, \ldots, r_{k-1}, r_k + 1) \).

We successively peel off \( T^2 \times I \) layers according to the Farey tessellation. The sequence of slopes is given by the continued fraction expansion, or, equivalently, by the shortest sequence of counterclockwise arcs in the Farey tessellation from \( -\frac{p}{q} \) to \( -1 \). Once slope \(-1\) is reached, \( S^1 \times D^2 \) with boundary slope \(-1\) is the standard neighborhood of a Legendrian core curve with twisting number \(-1\) (with respect to the fibration induced from the \( S^1 \)-fibers \( S^1 \times \{pt\} \)).

**Step 2:** (Analysis of each \( T^2 \times I \) layer)

**Fact:** Consider \( T^2 \times [0, 1] \) with convex boundary conditions \( \# \Gamma_0 = \# \Gamma_1 = 2 \), \( s_0 = \infty \), and \( s_1 = 0 \). Here we write \( \Gamma_i = \Gamma_{T^2 \times \{i\}} \) and \( s_i = \text{slope}(\Gamma_i) \). (More invariantly, the shortest integers corresponding \( s_0, s_1 \) form an integral basis for \( \mathbb{Z}^2 \).) Then there are exactly two tight contact structures (up to isotopy rel boundary) which are minimally twisting, i.e., every convex torus \( T' \) isotopic to \( T^2 \times \{i\} \) has slope(\( \Gamma_{T'} \)) in the interval \((0, +\infty)\). They are distinguished by the Poincaré duals of the relative half-Euler class, which are computed to be \( \pm((1, 0) - (0, 1)) \in H_1(T^2 \times [0, 1]; \mathbb{Z}) \). We call these \( T^2 \times [0, 1] \) layers basic slices.

The proof of the fact will be omitted, but one of the key elements in the proof is the following lemma:
HW 26. Prove, using the Imbalance Principle, that for any tight contact structure on \( T^2 \times [0, 1] \) with boundary slopes \( s_0 \neq s_1 \) and any rational slope \( s \) in the interval \((s_1, s_0)\), there exists a convex surface \( T' \subset T^2 \times [0, 1] \), which is parallel to \( T^2 \times \{pt\} \) and has slope \( s \). Here, if \( s_0 < s_1 \), \((s_1, s_0)\) means \((s_1, +\infty) \cup (-\infty, s_0)\).

Step 3: (Shuffling) Consider the example of the solid torus where \(-\frac{p}{q} = -\frac{14}{5}\). We have the following factorization:

\[
\begin{align*}
-\frac{14}{5} & \leftrightarrow (-3, -5) \\
-\frac{3}{5} & \leftrightarrow (-3, -4) \\
-\frac{2}{5} & \leftrightarrow (-3, -3) \\
-\frac{1}{5} & \leftrightarrow (-3, -2) \\
-2 & \leftrightarrow (-3, -1) = (-2) \\
-1 & \leftrightarrow (-1)
\end{align*}
\]

We group the basic slices into continued fraction blocks. Each block consists of all the slopes whose continued fraction representations are of the same length. In the example, we have two blocks: slope \(-\frac{14}{5}\) to \(-2\), and slope \(-2\) to \(-1\). All the relative half-Euler classes of the basic slices in the first block are \(\pm(-1, 3)\); for the second block, they are \(\pm(0, 1)\). Therefore, a naive upper bound for the number of tight contact structures would be 2 to the power \#(basic slices).

A closer inspection however reveals that we may shuffle basic slices which are in the same continued fraction block. More precisely, if \(T^2 \times [0, 2]\) admits a factoring into basic slices \(T^2 \times [0, 1]\) and \(T^2 \times [1, 2]\) with relative half-Euler classes \((a, b)\) and \(- (a, b)\), then it also admits a factoring into basic slices where the relative half-Euler classes are \(-(a, b)\) and \((a, b)\), i.e., the order is reversed.

Shuffling is (more or less) equivalent to the following proposition:

Lemma 4.4. Let \(L\) be a Legendrian knot. Then \(S_+S_-(L) = S_-S_+(L)\).

HW 27. Prove Lemma 4.4 (Observe that the ambient contact manifold is irrelevant and that the commutation can be done in a standard tubular neighborhood of \(L\)).

Returning to the example at hand, the first continued fraction block has at most \(|-5| = 4 + 1\) tight contact structures (distinguished by the relative half-Euler class), and the second has at most \(|-3 + 1| = 2\) tight contact structures. We compute \#Tight \(\leq 2 \cdot 5\).

In general, for the solid torus with slope \(-\frac{p}{q} \leftrightarrow (r_0, r_1, \ldots, r_k)\) we have:

\[
\text{(2)} \quad \#\text{Tight} \leq |(r_0 + 1)(r_1 + 1) \ldots (r_{k-1} + 1)r_k|.
\]

4.4. Completion of the proof of Theorems 4.2 and 4.3 We prove the following, which instantaneously completes the proof of both theorems.

\[
\text{(3)} \quad \#\text{Tight}(L(p, q)) \leq |(r_0 + 1)(r_1 + 1) \ldots (r_k + 1)|.
\]

Recall that on \( \partial V_1 \), the meridian of \( V_2 \) has slope \(-\frac{p}{q} \leftrightarrow (r_0, r_1, \ldots, r_{k-1}, r_k)\). First, take a Legendrian curve \( \gamma \) isotopic to the core curve of \( V_2 \) with largest twisting number. (Such a Legendrian curve exists, since any closed curve admits a \( C^0 \)-small approximation by a Legendrian curve;
the upper bound exists by the Thurston-Bennequin inequality.) We may assume $V_2$ is the standard neighborhood of $\gamma$; the tight contact structure on $V_2$ is then unique up to isotopy. Next, $	ext{slope}(\Gamma_{V_2}) = -\frac{p'}{q'} \leftrightarrow (r_0, \ldots, r_{k-1}, r_k + 1)$, and we have already computed the upper bound for $\#\text{Tight}(V_2)$ to be $|(r_0 + 1)\ldots(r_{k-1} + 1)(r_k + 1)|$ by Equation 2. This completes the proof of Equation 3 and hence of Theorems 4.2 and 4.3.

**Open Question.** Give a complete classification of tight contact structures on $T^2 \times [0,1]$ when $\#\Gamma_{T^2 \times \{i\}} > 2$, $i = 0, 1$. (Contrary to what is claimed in [H1], the general answer is not yet known.)

5. **GLUING**

There are three general methods for proving tightness:

1. symplectic filling,
2. gauge theory (in particular Heegaard Floer homology), and
3. gluing (state traversal).

Symplectic filling was already discussed in Section 1.5. We briefly explain the relationship between contact structures and the *Heegaard Floer homology* of Ozsvath and Szabo [OSZ1, OSZ2]. To an oriented closed 3-manifold $M$ one can assign a Heegaard Floer homology group $\tilde{HF}(M)$, constructed out of the Heegaard decomposition of $M$. In [OSZ3], Ozsvath and Szabo assigned a class $c(\xi) \in \tilde{HF}(-M)$ to every contact structure $(M, \xi)$ (tight or overtwisted). This was done via the work of Giroux [Gi4] in which it was shown that every contact structure $(M, \xi)$ corresponds to an equivalence class of open book decompositions of $M$ (and hence an equivalence class of fibered knots). Lisca and Stipsicz [LS3] showed that large families of contact structures are tight (but not fillable) by showing that their Heegaard Floer homology class is nonzero. The Heegaard Floer homology approach appears to be very promising at the time of the writing of this article.

In this section we focus on the last technique, namely gluing. Many of the key ideas in gluing were introduced by Colin [Co1, Co2] and Makar-Limanov [ML2], and subsequently enhanced by Honda [H2] who combined them with the bypass technology.

Let us start by asking the following question:

**Question 5.1.** Let $\Sigma$ be a convex surface in $(M, \xi)$. If $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ is tight, then is $(M, \xi)$ tight?

**Answer:** This is usually not true. Our goal is to understand to what extent it is true.

**HW 28.** Give an example of an overtwisted $T^2 \times [0,1]$ which is tight when restricted to $T^2 \times [0, \frac{1}{2}]$ and to $T^2 \times [\frac{1}{2}, 1]$.

5.1. **Basic examples with trivial state transitions.**

**Example A:** (Colin [Co2], Makar-Limanov [ML2]) Suppose $\Sigma = S^2$. If $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ is tight, then $(M, \xi)$ is tight.
Proof. Recall that there is only one possibility for $\Gamma_{S^2}$ inside a tight contact manifold. We argue by contradiction. Suppose there is an OT disk $D \subset M$. A priori, the OT disk $D$ can intersect $\Sigma$ in a very complicated manner. We obtain a contradiction as follows:

1. Isotop $\Sigma$ to $\Sigma'$ so that $\Sigma' \cap D = \emptyset$.
2. Discretize the isotopy $\Sigma_0 = \Sigma \to \Sigma_1 \to \cdots \to \Sigma_n = \Sigma'$, so that each step is obtained by attaching a bypass.
3. If $(M \setminus \Sigma_i, \xi|_{M \setminus \Sigma_i})$ is tight, then $\Gamma_{\Sigma_i} = \Gamma_{\Sigma_{i+1}} = S^1$ and the bypass must be trivial. Hence, $(M \setminus \Sigma_i, \xi|_{M \setminus \Sigma_i}) \simeq (M \setminus \Sigma_{i+1}, \xi|_{M \setminus \Sigma_{i+1}})$.

We have proved inductively that $(M \setminus \Sigma', \xi|_{M \setminus \Sigma'})$ is tight, a contradiction. $\Box$

More generally, one can prove:

Theorem 5.2 (Colin [Co2]). If $M = M_1 \# M_2$, then $\text{Tight}(M) \simeq \text{Tight}(M_1) \times \text{Tight}(M_2)$.

HW 29. Classify tight contact structures on $S^1 \times S^2$.

Example B: (Colin [Co1]) If $\Sigma = D^2$ and $\Gamma_{\Sigma}$ is $\partial$-parallel, then $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ tight $\Rightarrow (M, \xi)$ tight.

Example C: (Colin [Co1]) Let $\Sigma$ be an incompressible surface with $\partial \Sigma \neq \emptyset$. If $\Gamma_{\Sigma}$ is $\partial$-parallel and $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ is universally tight, then $(M, \xi)$ is universally tight. (This is Theorem 2.15 above.)

Question 5.3. In Example C, does $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ tight imply $(M, \xi)$ tight? In other words, can universal tightness be avoided?

All of the above examples can be characterized by the fact that the state transitions are trivial. However, to create more interesting examples, we need to “traverse all states”.

5.2. More complicated example.

Example D: (Honda [H2]) Let $H$ be a handlebody of genus $g$ and $D_1, \ldots, D_g$ be compressing disks so that $H \setminus (D_1 \cup \cdots \cup D_g) = B^3$. Fix $\Gamma_{\partial H}$ (and a compatible characteristic foliation). Note that we need $tb(D_i) \leq -1$, since otherwise we can find an OT disk using LeRP.

Let $C$ be the configuration space, i.e., the set of all possible $C = (\Gamma_{D_1}, \ldots, \Gamma_{D_g})$, where each $\Gamma_{D_i}$ has no closed curves. The cardinality of $C$ is finite. If we cut $H$ along $\Sigma = D_1 \cup \cdots \cup D_g$, then we obtain a 3-ball with corners. Given a configuration $C$, we can round the corners, as previously explained in Section 2.3. Now, if $\Gamma_{\partial (H \setminus \Sigma)} = S^1$ after rounding, then $C$ is said to be potentially allowable.

State transitions: The smallest unit of isotopy (in the contact world) is a bypass attachment. Therefore we examine the effect of one bypass attachment onto $D_i$. First we need to ascertain whether a candidate bypass exists.
Criterion for existence of state transition: The candidate bypass exists if and only if attaching the bypass from the interior of $B^3 = H \setminus \Sigma$ does not increase $\#\Gamma_{\partial B^3}$.

We construct a graph $\Gamma$ with $C$ as the vertices. We assign an edge from $(\Gamma_{D_1}, \ldots, \Gamma_{D_i}, \ldots, \Gamma_{D_g})$ to $(\Gamma_{D_1'}, \ldots, \Gamma_{D_i'}, \ldots, \Gamma_{D_g'})$ if there is a state transition $D_i \rightarrow D_i'$ given by a single bypass move. Note that the bypass may be from either side of $D_i$. Then we have:

**Theorem 5.4.** $\text{Tight}(H, \Gamma_{\partial H})$ is in 1-1 correspondence with the connected components of $\Gamma$, all of whose vertices $C$ are potentially allowable.

HW 30. Explain why $\text{Tight}(H, \Gamma_{\partial H})$ is finite.

**Remark.** Since $C$ is a finite graph, in theory we can compute $\text{Tight}(H, \Gamma_{\partial H})$ for any handlebody $H$ with a fixed boundary $\Gamma_{\partial H}$. Tanya Cofer, a (former) graduate student at the University of Georgia, has programmed this for $g = 1$, and the experiment agrees with the theoretical number from Theorem 4.2, in case $\#\Gamma_{\partial H} = 2$ and the slope is $-\frac{p}{q}$ with $p \leq 10$.

HW 31. Using the state transition technique, analyze tight contact structures on $S^1 \times D^2$, where $\Gamma_{T^2}, T^2 = \partial(S^1 \times D^2)$, satisfies the following:

1. $\#\Gamma_{T^2} = 2$ and slope($\Gamma_{T^2}$) = $-2$.
2. $\#\Gamma_{T^2} = 2$ and slope($\Gamma_{T^2}$) = $-3$.
3. $\#\Gamma_{T^2} = 4$ and slope($\Gamma_{T^2}$) = $\infty$.

Here the slope of the meridian is 0 and the slope of some preferred longitude is $\infty$.

5.3. Tightness and fillability. We present two examples which show that the world of tight contact structures is larger than the world of symplectically fillable contact structures.

**Example E:** (Honda [H2]) We present a tight handlebody $H$ of genus 4 which becomes OT after a Legendrian surgery. Since Legendrian surgery preserves fillability, the tight handlebody cannot be embedded inside any closed fillable contact 3-manifold.

We take the union $H = M_1 \cup M_2$, where $M_1 = S^1 \times D^2$ is the standard tubular neighborhood of a Legendrian curve and $M_2$ is an $I$-invariant neighborhood of a convex disk $S$ with 4 holes. Here $\partial S = \gamma - \cup_{i=1}^4 \gamma_i$ and $\Gamma_S$ consists of 4 arcs, one each from $\gamma_i$ to $\gamma_{i+1}$ ($i \mod 4$). The gluing is presented in Figure 19 where $T^2 = \partial(S^1 \times D^2)$ is drawn so that $\Gamma_{T^2}$ has slope $\infty$, the $\gamma_i$ have slope 0, and the meridian of $M_1$ has slope 1.

A Legendrian surgery along the core curve of $M_1$ yields a new meridional slope of 0 along $T^2$, and hence allows $S$ to be completed to an OT disk. Using the state transition method, one can prove that the contact structure is tight.

HW 32. Verify the tightness.

**Example F:** (Etnyre-Honda [EH]) Consider the torus bundle $M = (T^2 \times [0, 1]) / \sim$, where $(x, 1) \sim (Ax, 0)$, $T^2 = R^2/Z^2$, and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $T^2 \times [0, 1]$ be a basic slice with boundary slopes
$s_0 = \infty$ and $s_1 = 0$. The glued-up contact structure $\xi$ is proved to be tight using state traversal. However, $\xi$ is not symplectically fillable by the following contradiction argument:

1. $M$ is a Seifert fibered space over $S^2$ with Seifert invariants $\left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$.
2. There exists a Legendrian surgery taking $(M, \xi)$ to $(M', \xi')$, where $M'$ is a Seifert fibered space over $S^2$ with invariants $\left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$. Since Legendrian surgery preserves fillability, if $\xi$ is fillable, then $\xi'$ is also fillable.
3. A theorem of Lisca [Li], proved using Seiberg-Witten theory, states that there are no fillable contact structures on $M'$.

**Remark.** Example F was the first example of a tight contact structure which is not fillable. Since then, numerous other examples have been discovered by Lisca and Stipsicz [LS1, LS2, LS3].

**Open Question.** Elucidate the difference between the world of tight contact structures and the world of fillable contact structures.

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