DYNAMICS ON NETWORKS OF MANIFOLDS

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ABSTRACT. We propose a precise definition of a continuous time dynamical system made up of interacting open subsystems. The interconnections of subsystems are coded by directed graphs. We prove that the appropriate maps of graphs called graph fibrations give rise to maps of the appropriate dynamical systems. Consequently surjective graph fibrations give rise to invariant subsystems and injective graph fibrations give rise to projections of dynamical systems.

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1. Introduction

Given a dynamical system, one is initially interested in studying invariant subsystems; this includes equilibria, periodic orbits, etc. Moreover, constructing projections onto smaller systems as well as conjugacies and semi-conjugacies with simpler systems are generally useful for understanding the qualitative properties of dynamical systems. In this paper, we study the combinatorial properties of dynamical systems defined on networks to present a unified theoretical framework for all of the above objects. In this paper we give a precise definition of a continuous time dynamical system made up of interacting open subsystems. We then exploit the combinatorial aspect of networks to produce maps of dynamical systems out of appropriate maps of graphs called graph fibrations (q.v. Definition 3.1). We show that surjective graph fibrations give rise to invariant subsystems and injective graph fibrations give rise to projections of dynamical systems.

Our results shed new light on the existence of invariant subsystems in coupled cell networks proved by Golubitsky, Pivato, Stewart and Török [8, 3], since we impose no groupoid symmetries on our systems. In this respect our work is close in spirit to the approach to dynamics on networks advocated by Field [6]; in contrast to Field we find it convenient to use the language of category theory. We also find it useful to borrow the notions of open systems and their interconnection from engineering (see, for example [2, 11, 7]) and the definition of a graph fibration from computer science [9] (see [10] for a history of the notion and alternative terminologies).

In more detail we construct a category of networks of continuous time systems. A network in our sense consists of

- a finite directed graph $G$ with a set of nodes $G_0$,
- a phase space function $\mathcal{P}$ that assigns to each node of the graph an appropriate phase space (which we take to be a manifold),
a family of open systems \( \{ w_a \}_{a \in G_0} \) (one for each node \( a \) of the graph \( G \)) consistent in an appropriate way with the structure of the graph, and

- an interconnection map \( J \) that turns these open systems into a vector field on the product \( \prod_{a \in G_0} P(a) \) of the phase spaces of the nodes.

Our main result, Theorem 3.10, shows that graph fibrations compatible with phase space functions give rise to maps of dynamical systems.

Putting aside all of the technicalities, however, the main idea of the construction of this paper is: given a map between any two graphs, we might like to induce a map on the associated dynamical systems via pullbacks. We observe that in order for this pullback to be well-defined, the map of graphs is required to have certain properties, and these are exactly the properties that make the graph map a \emph{fibration}, as we will show below. The perhaps surprising conclusion made below is that not only are the definition conditions for a fibration necessary, they are also sufficient — any graph fibration gives rise to a map of dynamical systems, in a sense that can be made precise.

The paper is organized as follows. We start by recalling the definition of a directed multigraph, define the notion of a network of manifolds and the total space of the network. We then recall open systems and their interconnections. We show how a network of manifolds naturally leads to spaces of open systems that can be interconnected. One can think of these spaces as forming an infinite dimensional vector bundle over the set of nodes of the appropriate graph. We then prove our main result, Theorem 3.10: graph fibrations give rise to maps of dynamical systems.

2. Definitions and constructions

Graphs and manifolds. Throughout the paper \emph{graphs} are finite directed multigraphs, possibly with loops. More precisely, we use the following definition:

2.1. Definition. A graph \( G \) consists of two finite sets \( G_1 \) (of arrows, or edges), \( G_0 \) (of nodes, or vertices) and two maps \( s, t : G_1 \to G_0 \) (source, target):

\[
G = \{ s, t : G_1 \to G_0 \}.
\]

We write \( G = \{ G_1 \rightrightarrows G_0 \} \).

The set \( G_1 \) may be empty, i.e., we may have \( G = \{ \emptyset \rightrightarrows G_0 \} \), making \( G \) a disjoint collection of vertices.

2.2. Definition. A map of graphs \( \varphi : A \to B \) from a graph \( A \) to a graph \( B \) is a pair of maps \( \varphi_1 : A_1 \to B_1 \), \( \varphi_0 : A_0 \to B_0 \) taking edges of \( A \) to edges of \( B \), nodes of \( A \) to nodes of \( B \) so that for any edge \( \gamma \) of \( A \) we have

\[
\varphi_0(s(\gamma)) = s(\varphi_1(\gamma)) \quad \text{and} \quad \varphi_0(t(\gamma)) = t(\varphi_1(\gamma)).
\]

We often omit the indices 0 and 1 and write \( \varphi(\gamma) \) for \( \varphi_1(\gamma) \) and \( \varphi(a) \) for \( \varphi_0(a) \).

2.3. Remark. The collection of (finite directed multi-)graphs and maps of graphs form a category \textbf{Graph}.

In order to construct networks from graphs we need to have a consistent way of assigning manifolds to nodes of our graphs. We formalize this idea by making the collection of graphs with manifolds assigned to vertices into a category \textbf{Graph/Man}.

2.4. Definition (Category of networks of manifolds \textbf{Graph/Man}). A network of manifolds is a pair \((G, P)\) where \( G \) is a (finite directed multi-)graph and \( P : G_0 \to \text{Man} \) is a function that assigns to each node \( a \) of \( G \) a manifold \( P(a) \). We think of \( P \) as an assignment of phase spaces to the nodes of the graph \( G \), and for this reason we refer to \( P \) as a \emph{phase space function}.

Networks of manifolds form a category \textbf{Graph/Man}. Its objects are are pairs \((G, P)\) as above. A morphism \( \varphi \) from \((G, P)\) to \((G', P')\) is a map of graphs \( \varphi : G \to G' \) with

\[
P' \circ \varphi = P.
\]

2.5. Notation. Given a category \( \mathcal{C} \) we denote the opposite category by \( \mathcal{C}^{\text{op}} \), i.e. the category with all of the same objects and all of the arrows reversed. We adhere to the convention that a \emph{contravariant functor} from a category \( \mathcal{C} \) to a category \( \mathcal{D} \) is a covariant functor

\[
F : \mathcal{C}^{\text{op}} \to \mathcal{D}.
\]
Then for any morphism \( c \rightarrow c' \) of \( \mathcal{C} \) we have \( F(c) \xleftarrow{F(y)} F(c') \) in \( \mathcal{D} \).

Next we recall the notion of a \textit{product} in a category \( \mathcal{C} \). We do this to draw a distinction between categorical and Cartesian products of finite families of manifolds which we will exploit throughout the paper.

2.6. \textbf{Definition.} A \textit{product} of a family \( \{c_s\}_{s \in S} \) of objects in a category \( \mathcal{C} \) indexed by a set \( S \) is an object \( \prod_{s \in S} c_s \) of \( \mathcal{C} \) together with a family of morphisms \( \{\pi_s : \prod_{s \in S} c_s \rightarrow c_s\}_{s \in S} \) with the following universal property: given an object \( c' \) of \( \mathcal{C} \) and a family of morphisms \( \{f_s : c' \rightarrow c_s\}_{s \in S} \) there is a unique morphism \( f : c' \rightarrow \prod_{s \in S} c_s \) with

\[
\pi_s \circ f = f_s \quad \text{for all } s \in S.
\]

2.7. \textbf{Remark.} If a product exists then it is unique up to a unique isomorphism [1].

2.8 (Categorical products in the category Man of manifolds).

Observe that in the category \textbf{Man} of (finite dimensional paracompact smooth) manifolds a product \( \prod_{s \in S} M_s \) exists for any family \( \{M_s\}_{s \in S} \) of manifolds indexed by a finite set. It can be constructed by ordering the elements of \( S \) and taking the corresponding Cartesian product. That is, products of manifolds indexed by a set \( S \) can be constructed by writing \( S = \{s_1, \ldots, s_n\} \) and setting \( \prod_{s \in S} M_s = \prod_{i=1}^n M_{s_i} \), where the right hand side is the Cartesian product. (This is the more common mathematical definition of “product”.)

However, for our purposes it is better to have a construction of the product that does not involve a choice of ordering of the indexing set in question. This may be done as follows. Given a family \( \{M_s\}_{s \in S} \) of manifolds, denote by \( \bigsqcup_{s \in S} M_s \) their disjoint union.\(^1\) Now define

\[
\prod_{s \in S} M_s := \{x : S \rightarrow \bigsqcup_{s \in S} M_s \mid x(s) \in M_s \text{ for all } s \in S\}.
\]

The projection maps \( \pi_s : \bigsqcup_{s' \in S} M_{s'} \rightarrow M_s \) are defined by

\[
\pi_s(x) = x(s).
\]

We denote \( x(s) \in M_s \) by \( x_s \) and think of it as \( s^{th} \) “coordinate” of an element \( x \in \bigsqcup_{s \in S} M_s \). Equivalently we may think of elements of the categorical product \( \prod_{s \in S} M_s \) as unordered tuples \( (x_s)_{s \in S} \) with \( x_s \in M_s \).

Our discussion of categorical versus Cartesian products may seem fuzzy, but it will be useful to maps between total phase of networks of manifolds from maps of networks in Proposition 2.12 below.

2.9. \textbf{Definition} (total phase space of a network \((G, \mathcal{P})\)). For a pair \((G, \mathcal{P})\) consisting of a graph \( G \) and an assignment \( \mathcal{P} : G_0 \rightarrow \text{Man} \), that is, for an object \((G, \mathcal{P})\) of \textbf{Graph/Man} we set

\[
\mathbb{P}G \equiv \mathbb{P}(G, \mathcal{P}) := \bigsqcup_{a \in G_0} \mathcal{P}(a),
\]

the categorical product of manifolds attached to the nodes of the graph \( G \) by the \textit{phase space function} \( \mathcal{P} \) and call the resulting manifold \( \mathbb{P}G \) the \textit{total phase space} of the network \((G, \mathcal{P})\).

2.10. \textbf{Example.} Consider the graph

\[
G = \begin{tikzpicture}
  \node (a) at (0,0) \{a\};
  \node (b) at (1,0) \{b\};
  \draw (a) edge[loop above] node[above] {$\beta$} (a);
  \draw (a) edge[loop below] node[below] {$\alpha$} (a);
\end{tikzpicture}
\]

Define \( \mathcal{P} : G_0 \rightarrow \text{Man} \) by \( \mathcal{P}(a) = S^2 \) (the two sphere) and \( \mathcal{P}(b) = S^3 \). Then

\[
\mathbb{P}(G, \mathcal{P}) = S^2 \times S^3.
\]

2.11. \textbf{Notation.} If \( G = \{\emptyset \Rightarrow \{a\}\} \) is a graph with one node \( a \) and no arrows, we write \( G = \{a\} \). Then for any phase space function \( \mathcal{P} : G_0 = \{a\} \rightarrow \text{Man} \) we abbreviate \( \mathbb{P}(\emptyset \Rightarrow \{a\}), \mathcal{P} : \{a\} \rightarrow \text{Man} \) as \( \mathbb{P}a \).

\(^1\) It may be defined by \( \bigsqcup_{s \in S} M_s = \bigsqcup_{s \in S} M_s \times \{s\} \).
2.12. **Proposition.** The assignment
\[(G, \mathcal{P}) \mapsto \mathbb{P} G := \bigcap_{a \in G_0} \mathcal{P}(a)\]
of phase spaces to networks extends to a contravariant functor
\[\mathbb{P} : (\text{Graph}/\text{Man})^{\text{op}} \to \text{Man}.\]

**Proof.** Suppose \(\varphi : (G, \mathcal{P}) \to (G', \mathcal{P}')\) is a morphism in \(\text{Graph}/\text{Man}\). That is, suppose \(\varphi : G \to G'\) is a map of graphs with \(\mathcal{P}' \circ \varphi = \mathcal{P}\). We need to define a map of manifolds
\[\mathbb{P}\varphi : \mathbb{P} G' \to \mathbb{P} G.\]
Since by definition \(\mathbb{P} G\) is the product \(\prod_{a \in G_0} \mathcal{P}(a)\), the universal property of products implies that in order to define \(\mathbb{P}\varphi\) it is enough to define a family of maps
\[\{ (\mathbb{P}\varphi)_a : \mathbb{P} G' \to \mathcal{P}(a) \}_{a \in G_0}.\]
For any node \(a'\) of \(G'\) we have the canonical projection
\[\pi'_{a'} : \mathbb{P} G' \to \mathcal{P}(a').\]
We therefore define
\[(\mathbb{P}\varphi)_a := \pi'_{\varphi(a)} : \mathbb{P} G' \to \mathcal{P}'(\varphi(a)) = \mathcal{P}(a)\]
for all \(a \in G_0\). By the universal property of \(\mathbb{P} G = \prod_{a \in G_0} \mathcal{P}(a)\) this defines the desired map \(\mathbb{P}\varphi : \mathbb{P} G' \to \mathbb{P} G\).

The universal property of products also implies that the map \(\mathbb{P}\) on morphisms of \(\text{Graph}/\text{Man}\) as defined above is actually a functor. That is,
\[\mathbb{P}(\psi \circ \varphi) = \mathbb{P}\varphi \circ \mathbb{P}\psi\]
for any pair \((\psi, \varphi)\) of composable morphisms in \(\text{Graph}/\text{Man}\). \(\square\)

2.13. **Example.** Suppose \(G\) is a graph with two nodes \(a, b\) and no edges, \(G'\) is a graph with one node \(\{c\}\) and no edges, \(\mathcal{P}'(c)\) is a manifold \(M\), \(\varphi : G \to G'\) is the only possible map of graphs (it sends both nodes to \(c\)). \(\mathcal{P} : G_0 \to \text{Man}\) is given by \(\mathcal{P}(a) = M = \mathcal{P}(b)\) (so that \(\mathcal{P}' \circ \varphi = \mathcal{P}\)). Then \(\mathbb{P} G' \cong M\),
\[\mathbb{P} G = \{(x_a, x_b) \mid x_a \in \mathcal{P}(a), x_b \in \mathcal{P}(b)\} \cong M \times M\]
and \(\mathbb{P}\varphi : M \to M \times M\) is the unique map with \((\mathbb{P}\varphi(x))_a = x\) and \((\mathbb{P}\varphi(x))_b = x\) for all \(x \in \mathbb{P} G'\). Thus \(\mathbb{P}\varphi : M \to M \times M\) is the diagonal map \(x \mapsto (x, x)\).

2.14. **Example.** Let \((G, \mathcal{P}), (G', \mathcal{P}')\) be as in Example 2.13 above and \(\psi : (G', \mathcal{P}') \to (G, \mathcal{P})\) be the map that sends the node \(c\) to \(a\). Then \(\mathbb{P}\psi : \mathbb{P} G \to \mathbb{P} G'\) is the map that sends \((x_a, x_b)\) to \(x_a\).

2.15. **Remark.** If \((G, \mathcal{P})\) is a graph with a phase function, that is, an object of \(\text{Graph}/\text{Man}\), and \(\varphi : H \to G\) a map of graphs then \(\mathcal{P} \circ \varphi : H \to \text{Man}\) is a phase function and \(\varphi : (H, \mathcal{P} \circ \varphi) \to (G, \mathcal{P})\) is a morphism in \(\text{Graph}/\text{Man}\). We then have a map of manifolds
\[\mathbb{P}\varphi : \mathbb{P}(H, \mathcal{P} \circ \varphi) \to \mathbb{P}(G, \mathcal{P}).\]

Similarly, a commutative diagram
\[
\begin{array}{ccc}
K & \xrightarrow{j} & H \\
\downarrow{\psi} & & \downarrow{\varphi} \\
G & \xrightarrow{\varphi} & \end{array}
\]
of maps of graphs and a phase space function \(\mathcal{P} : G \to \text{Man}\) give rise to the commutative diagram of maps of manifolds
\[
\begin{array}{ccc}
\mathbb{P}(K, \mathcal{P} \circ \psi) & \xleftarrow{Pj} & \mathbb{P}(H, \mathcal{P} \circ \varphi) \\
\downarrow{P\psi} & & \downarrow{P\varphi} \\
\mathbb{P}(G, \mathcal{P}) & & .
\end{array}
\]
Embeddings and submersions from maps of graphs. As we said in the introduction, the main goal of this paper is to construct maps of dynamical systems from graph fibrations. In Proposition 2.12 we showed that a map of networks $\varphi: (G, P) \to (G', P')$ defines a map of manifolds $\mathbb{P}\varphi: \mathbb{P}(G', P') \to \mathbb{P}(G, P)$. In this subsection we prove that:

1. If the map of graphs $\varphi: G \to G'$ is injective on nodes, then $\mathbb{P}\varphi$ is a surjective submersion.
2. If the map of graphs $\varphi: G \to G'$ is surjective on nodes, then $\mathbb{P}\varphi$ is an embedding.

[Recall that a smooth map between two manifolds is a submersion if its differential is onto at every point. A smooth map between two manifolds is an embedding if it is 1-1, its differential is 1-1 everywhere and it is a homeomorphism onto its image.] Combined with Theorem 3.10 below, this shows that surjective fibrations of networks of manifolds give rise to invariant dynamical subsystems and injective fibrations give rise to projections of dynamical systems.

2.16. Lemma. Suppose $\varphi: (G, P) \to (G', P')$ is a map of networks of manifolds such that the map on nodes, $\varphi_0: G_0 \to G'_0$, is surjective. Then $\mathbb{P}\varphi: \mathbb{P}G' \to \mathbb{P}G$ is an embedding whose image is the “polydiagonal”

$$\Delta_{\varphi} = \{ x \in \mathbb{P}G \mid x_a = x_b \text{ whenever } \varphi(a) = \varphi(b) \}.$$  

Proof. Assume first for simplicity that $G'$ has only one vertex $*$ and $P'(*)=M$. Then for any vertex $a$ of $G$ we have

$$\mathcal{P}(a) = \mathcal{P}'(\varphi(a)) = \mathcal{P}'(*) = M,$$

$\mathbb{P}G' = M$ and $\mathbb{P}G = M \times \cdots \times M$ ($|G_0|$ copies), where as before $G_0$ is the set of vertices of the graph $G$. In this case the proof of Proposition 2.12 shows that the map $\mathbb{P}\varphi: M \to M^{G_0}$ is of the form

$$\mathbb{P}\varphi(x) = (x, \ldots, x)$$

for all $x \in M$.

In general

$$\mathbb{P}\varphi: \mathbb{P}G' = \prod_{a' \in G'_0} \mathcal{P}'(a') \to \prod_{a \in G_0} \left( \prod_{a' \in \varphi^{-1}(a')} \mathcal{P}(a) \right) = \mathbb{P}G$$

is the product of maps of the form

$$\mathcal{P}'(a') \to \prod_{a \in \varphi^{-1}(a')} \mathcal{P}(a), \quad x \mapsto (x, \ldots, x).$$  

$\square$

2.17. Lemma. Suppose $\varphi: (G, P) \to (G', P')$ is a map of networks of manifolds such that the map $\varphi_0: G_0 \to G'_0$ on nodes is injective. Then $\mathbb{P}\varphi: \mathbb{P}G' \to \mathbb{P}G$ is a surjective submersion.

Proof. Since $\varphi: G \to G'$ is injective, the set of nodes $G_0'$ of $G'$ can be partitioned as the disjoint union of the image $\varphi(G_0)$, which is a copy of $G_0$, and the complement. Hence

$$\mathbb{P}G' \simeq \prod_{a \in G_0} \mathcal{P}(\varphi(a)) \times \bigcap_{a' \notin \varphi(G_0)} \mathcal{P}'(a') \simeq \mathbb{P}G \times \bigcap_{a' \notin \varphi(G_0)} \mathcal{P}'(a').$$

With respect to this identification of $\mathbb{P}G'$ with $\mathbb{P}G \times \bigcap_{a' \notin \varphi(G_0)} \mathcal{P}'(a')$ the map $\mathbb{P}\varphi: \mathbb{P}G' \to \mathbb{P}G$ is the projection

$$\mathbb{P}G \times \bigcap_{a' \notin \varphi(G_0)} \mathcal{P}'(a') \to \mathbb{P}G,$$

which is a surjective submersion.  

$\square$
Open systems and their interconnections. Having set up a consistent way of assigning phase spaces to graphs, we now take up continuous time dynamical systems. We start by recalling a definition of an open (control) systems, which is essentially due to Brockett [2].

2.18. Definition. A continuous time control system (or an open system) on a manifold $M$ is a surjective submersion $p: Q \to M$ from some manifold $Q$ together with a smooth map $F: Q \to TM$ so that

$$F(q) \in T_{p(q)}M$$

for all $q \in Q$. That is, the following diagram commutes. Here $\pi: TM \to M$ is the canonical projection.

2.19. Given a manifold $U$ of “control variables” we may consider control systems of the form

$$(2.20) \quad F: M \times U \to TM.$$ 

Here the submersion $p: M \times U \to M$ is given by

$$p(x,u) = x.$$ 

The collection of all such control systems forms a vector space that we denote by $\text{Control}(M \times U \to M)$:

$$\text{Control}(M \times U \to M) := \{F: M \times U \to TM \mid F(x,u) \in T_x M\}.$$ 

Now suppose we are given a finite family $\{F_i: M_i \times U_i \to TM_i\}_{j=1}^N$ of control systems and we want to somehow interconnect them to obtain a closed system $\mathcal{J}(F_1, \ldots, F_N)$ that is, a vector field, on the product $\prod_i M_i$. What additional data do we need to define the interconnection map

$$\mathcal{J}: \prod_i \text{Control}(M_i \times U_i \to M_i) \to \Gamma(T(\prod_i M_i))?$$

The answer is given by the following proposition:

2.21. Proposition. Given a family $\{p_j: M_j \times U_j \to M_j\}_{j=1}^N$ of projections on the first factor and a family of smooth maps $\{s_j: \prod M_i \to M_j \times U_j\}$ so that the diagrams

$$\begin{array}{ccc}
M_j \times U_j & \xrightarrow{p_1} & M_j \\
\downarrow s_j & \downarrow & \downarrow \\
\prod M_i & \xrightarrow{pr_j} & M_j
\end{array}$$

commute for each index $j$, there is an interconnection map $\mathcal{J}$ making the diagrams

$$\begin{array}{ccc}
\prod_i \text{Control}(M_i \times U_i \to M_i) & \xrightarrow{\mathcal{J}} & \Gamma(T(\prod_i M_i)) \\
\downarrow & \downarrow & \downarrow \\
\text{Control}(M_j \times U_j \to M_j) & \xrightarrow{\mathcal{J}_j} & \text{Control}(\prod_i M_i \xrightarrow{pr_j} M_j)
\end{array}$$

commute for each $j$. The components $\mathcal{J}_j$ of the interconnection map $\mathcal{J}$ are defined by $\mathcal{J}_j(F_j) := F_j \circ s_j$ for all $j$.

Proof. The space of vector fields $\Gamma(T(\prod_i M_i))$ on the product $\prod_i M_i$ is the product of vector spaces $\text{Control}(\prod_i M_i \to M_j)$:

$$\Gamma(T(\prod_i M_i)) = \prod_j \text{Control}(\prod_i M_i \xrightarrow{pr_j} M_j).$$

In other words a vector field $X$ on the product $\prod_i M_i$ is a tuple $X = (X_1, \ldots, X_N)$, where

$$X_j := D(pr_j) \circ X.$$

$(D(pr_j): T\prod_i M_i \to TM_j)$ denotes the differential of the canonical projection $pr_j: \prod M_i \to M_j$.) Each component $X_j: \prod_i M_i \to TM_i$ is a control system.
To define a map from a vector space into a product of vector spaces it is enough to define a map into each of the factors. We have canonical projections

\[ \pi_j: \bigcap_i \text{Control}(M_i \times U_i \to M_i) \to \text{Control}(M_j \times U_j \to M_j), \quad j = 1, \ldots, N. \]

Consequently to define the interconnection map \( \mathcal{I} \) it is enough to define the maps

\[ \mathcal{I}_j: \text{Control}(M_j \times U_j \to M_j) \to \text{Control}(\bigcap_i M_i \xrightarrow{pr_j} M_j), \]

for each index \( j \). We therefore define the maps \( \mathcal{I}_j: \text{Control}(M_j \times U_j \to M_j) \to \text{Control}(\bigcap_i M_i \xrightarrow{pr_j} M_j), \)

1 \( \leq j \leq N \), by

\[ \mathcal{I}_j(F_j) := F_j \circ s_j. \]

\[ \blacksquare \]

2.22. **Remark.** It will be useful for us to remember that the canonical projections

\[ \varpi_j: \Gamma(T \bigcap_i M_i)) \to \text{Control}(\bigcap_i M_i \to M_j) \]

are given by

\[ \varpi_j(X) = D(pr_j) \circ X, \]

where \( D(pr_j): T \bigcap_i M_i \to TM_j \) are the differentials of the canonical projections \( pr_j: \bigcap_i M_i \to M_j \).

**Interconnections and graphs.** We next explain how finite directed graphs whose nodes are decorated with phase spaces, that is, networks of manifolds in the sense of Definition 2.4 give rise to interconnection maps. To do this precisely it is useful to have a notion of an *input trees* of a directed graph. Given a graph, an input tree \( I(a) \) of a vertex \( a \) is — roughly — the vertex itself and all of the arrows leading into it. We want to think of this as a graph in its own right, as follows.

2.23. **Definition (Input tree).** Given a vertex \( a \) of a graph \( G \) we define the *input tree* \( I(a) \) to be a graph with the set of vertices \( I(a)_0 \) given by

\[ I(a)_0 := \{ a \} \sqcup t^{-1}(a); \]

where, as before, the set \( t^{-1}(a) \) is the set of arrows in \( G \) with target \( a \). The set of edges \( I(a)_1 \) of the input tree is the set of pairs

\[ I(a)_1 := \{(a, \gamma) \mid \gamma \in G_1, \ t(\gamma) = a\}, \]

and the source and target maps \( I(a)_1 \rightharpoonup I(a)_0 \) are defined by

\[ s(a, \gamma) = \gamma \quad \text{and} \quad t(a, \gamma) = a. \]

In pictures,

\[ (a, \gamma) \quad (a) \]

2.24. **Example.** Consider the graph

\[ G = \quad \beta \quad \alpha \]

as in Example 2.10. Then the input tree \( I(a) \) is the graph with one node \( a \) and no edges: \( I(a) = \bullet a \). The input tree \( I(b) \) has three nodes and two edges:
Notice that our definition of input tree “pulls apart” multiple edges coming from a common vertex.

2.25. **Remark.** For each node $a$ of a graph $G$ we have a natural map of graphs

$$\xi = \xi_a : I(a) \to G, \quad \xi_a(a, \gamma) = \gamma,$$

which need not be injective.

2.26. Let $G$ be a graph with a phase space function $\mathcal{P} : G \to \text{Man}$ and let $a$ be a node of $G$. We then have a graph $\{a\}$ with one node and no arrows. Denote the inclusion of $\{a\}$ in $G$ by $\iota_a$ and the inclusion into its input tree $I(a)$ by $j_a$. Then the diagram of maps of graphs

\[
\begin{array}{ccc}
\{a\} & \xrightarrow{j_a} & I(a) \\
\iota_a & \downarrow & \downarrow \xi \\
G & & G
\end{array}
\]

commutes. By Remark 2.15 we have a commuting diagram of maps of manifolds

\[
\begin{array}{ccc}
\mathbb{P}\{a\} & \xrightarrow{\mathbb{P}j_a} & \mathbb{P}I(a) \\
\mathbb{P}\iota_a & \downarrow & \downarrow \mathbb{P}\xi \\
\mathbb{P}G & & \mathbb{P}a
\end{array}
\]

2.27. Let us now examine more closely the map $\mathbb{P}j_a : \mathbb{P}I(a) \to \mathbb{P}a$ in 2.26 above. Since the set of nodes $I(a)_0$ of the input tree $I(a)$ is the disjoint union

$$I(a)_0 = \{a\} \sqcup t^{-1}(a),$$

and since $\xi_a(\gamma) = s(\gamma)$ for any $\gamma \in t^{-1}(a) \subset I(a)_0$, we have

$$\mathbb{P}I(a) = \mathcal{P}(a) \times \bigcap_{\gamma \in t^{-1}(a)} \mathcal{P}(s(\gamma)).$$

Since $j_a : \{a\} \to I(a)_0 = \{a\} \sqcup t^{-1}(a)$ is the inclusion,

$$\mathbb{P}j_a : \mathbb{P}I(a) \to \mathbb{P}a$$

is the projection

$$\mathcal{P}(a) \times \bigcap_{\gamma \in t^{-1}(a)} \mathcal{P}(s(\gamma)) \to \mathbb{P}a.$$

Similarly

$$\mathbb{P}\iota_a : \mathbb{P}G \to \mathbb{P}a$$

is the projection

$$\bigcap_{b \in G_0} \mathcal{P}(b) \to \mathcal{P}(a).$$

Putting 2.26 and 2.27 together we get
2.28. **Proposition.** Given a graph $G$ with a phase space function $\mathcal{P}: G_0 \to \text{Man}$, that is, a network $(G, \mathcal{P})$ of manifolds, we have commutative diagrams of maps of manifolds

$$
\mathbb{P}I(a) = \mathcal{P}(a) \times \bigcap_{\gamma \in \hat{E}^{-1}(a)} \mathcal{P}(s(\gamma)) \xrightarrow{\mathbb{P}_a} \mathcal{P}(a)
$$

for each node $a$ of the graph $G$.

2.29. **Example.** Suppose $G = a \xrightarrow{b} b$ is a graph as in Example 2.10 and suppose $\mathcal{P}: G_0 \to \text{Man}$ is a phase space function. Then

$$\mathbb{P}I(b) \simeq \mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b),$$

$\mathbb{P}j_b$ is the projection $\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \to \mathcal{P}(b)$, and

$$\text{Control}(\mathbb{P}I(b) \to \mathbb{P}b) = \text{Control}(\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \to \mathcal{P}(b)).$$

On the other hand $\mathbb{P}I(a) = \mathcal{P}(a)$, $\mathbb{P}j_a: \mathcal{P}(a) \to \mathcal{P}(a)$ is the identity map and

$$\text{Control}(\mathbb{P}I(a) \to \mathbb{P}a) = \Gamma(T\mathcal{P}(a)),$$

the space of vector fields on the manifold $\mathcal{P}(a)$.

2.30. **Notation.** Given a network $(G, \mathcal{P})$ of manifolds we have a product of vector spaces

$$\text{Ctrl}(G, \mathcal{P}) := \bigsqcap_{a \in G_0} \text{Control}(\mathbb{P}I(a) \to \mathbb{P}a).$$

The elements of $\text{Ctrl}(G, \mathcal{P})$ are unordered tuples of $(w_a)_{a \in G_0}$ of control systems (q.v. 2.8). We may think of them as sections of the vector bundle $\bigsqcap_{a \in G_0} \text{Control}(\mathbb{P}I(a) \to \mathbb{P}a) \to G_0$ over the vertices of $G$.

It is easy to see that Propositions 2.21 and 2.28 give us

2.31. **Theorem.** Given a network $(G, \mathcal{P})$ of manifolds, there exists a natural interconnection map

$$\mathcal{I}: \bigsqcap_{a \in G_0} \text{Control}(\mathbb{P}I(a) \to \mathbb{P}a) \to \Gamma(T\mathcal{P}G))$$

with

$$\varpi_a \circ \mathcal{I}((w_b)_{b \in G_0}) = w_a \circ \mathbb{P}j_a$$

for all nodes $a \in G_0$. Here $\varpi_a: \Gamma(T\mathcal{P}G) \to \text{Control}(\mathbb{P}G_0 \xrightarrow{\mathbb{P}a} \mathbb{P}a)$ are the projection maps; $\varpi_a = D(\mathbb{P}t_a)$ (q.v. Remark 2.22).

2.32. **Example.** Consider the graph $G$ as in Examples 2.10 and 2.32 with a phase space function $\mathcal{P}: G_0 \to \text{Man}$. Then the vector field

$$X = \mathcal{I}(w_a, w_b): \mathcal{P}(a) \times \mathcal{P}(b) \to T\mathcal{P}(a) \times T\mathcal{P}(b)$$

is of the form

$$X(x, y) = (w_a(x), w_b(x, x, y)) \quad \text{for all } (x, y) \in \mathcal{P}(a) \times \mathcal{P}(b).$$

2.33. **Example.** Consider the graph

$$G = \begin{array}{ccc}
\text{a} & \circlearrowright & \text{b} \\
 & \searrow & \\
\text{b} & \nearrow & \text{c}
\end{array}$$
and let $P: G_0 \to \text{Man}$ be a phase space function. Then

$$(\mathcal{S}(w_a, w_b, w_c))(x, y, z) = (w_a(x), w_b(x, x, y), w_c(y, z))$$

for all $(w_a, w_b, w_c) \in \mathcal{Ctrl}(G, P)$ and all $(x, y, z) \in P(a) \times P(b) \times P(c)$.

3. Maps of dynamical systems from graph fibrations

Following Boldi and Vigna [9] (see also [10]) we single out a class of maps of graphs called graph fibrations.

3.1. Definition. A map $\varphi: G \to G'$ of directed graphs is a graph fibration if for any vertex $a$ of $G$ and any edge $e'$ of $G'$ ending at $\varphi(a)$ there is a unique edge $e$ of $G$ ending at $a$ with $\varphi(e) = e'$.

3.2. Example. The map of graphs

sending the edge $\gamma$ to $\gamma'$ and the edge $\delta$ to $\delta'$ is a graph fibration.

3.3. Given any maps $\varphi: G \to G'$ of graphs and a node $a$ of $G$ there is an induced map of input trees

$$\varphi_a: I(a) \to I(\varphi(a)).$$

On edges of $I(a)$ the map is defined by

$$\varphi(a, \gamma) := (\varphi(a), \varphi(\gamma))$$

(cf. Definition 2.23). Moreover the diagram of graphs

commutes (the map $\xi_a: I(a) \to G$ from an input tree to the original graph is defined in Remark 2.25).

3.4. Lemma. If $\varphi: G \to G'$ is a graph fibration then the induced maps

$$\varphi_a: I(a) \to I(\varphi(a))$$

of input trees defined above are isomorphisms for all nodes $a$ of $G$.

Proof. Given an edge $(\varphi(a), \gamma')$ of $I(\varphi(a))$ there is a unique edge $\gamma$ of $G$ with $\varphi(\gamma) = \gamma'$ and $t(\gamma) = a$ and consequently $\varphi_a(a, \gamma) = (\varphi(a), \gamma')$. It follows that $\varphi_a$ is bijective on vertices and edges.

Recall that a map from a network $(G, P)$ to a network $(G', P')$ is a map of graphs $\varphi: G \to G'$ with the property that

$$P' \circ \varphi = P.$$

3.5. Definition (Fibration of networks of manifolds). A map of networks $\varphi: (G, P) \to (G', P')$ of manifolds is a fibration if $\varphi: G \to G'$ is a graph fibration.

The theorem 3.6 below is our reason for singling out fibrations of networks.

3.6. Theorem. A fibration $\varphi: (G, P) \to (G', P')$ of networks naturally induces a linear map

$$\varphi^*: \mathcal{Ctrl}(G', P') \to \mathcal{Ctrl}(G, P).$$
Proof. Since
\[
\mathcal{Ctrl}(G, \mathcal{P}) = \bigcap_{a \in G_0} \text{Control}(\mathcal{P}I(a) \to \mathcal{P} a)
\]
is a product of vector spaces, the map \(\varphi^*\) is uniquely determined by maps from \(\mathcal{Ctrl}(G', \mathcal{P}')\) to the factors \(\text{Control}(\mathcal{P}I(c) \to \mathcal{P} c) \to \text{Control}(\mathcal{P}I(b) \to \mathcal{P} b)\)
for all \(b \in G'_0\). Hence in order to define the map \(\varphi^*\) it is enough to define maps of vector spaces
\[
\varphi^*_a : \text{Control}(\mathcal{P}I(\varphi(a)) \to \mathcal{P} \varphi(a)) \to \text{Control}(\mathcal{P}I(a) \to \mathcal{P} a)
\]
for all nodes \(a\) of the graph \(G\). By 3.3 the diagram
\[
\begin{array}{ccc}
I(a) \ar[r]^\varphi \ar[d]_{\xi_a} & I(\varphi(a)) \ar[d]_{\xi_{\varphi(a)}} & \\
G \ar[r]^\varphi \ar[d]^p & G' \ar[r]_p & \text{Man} \ar[d]^\varphi
\end{array}
\]
commutes for each \(a \in G_0\). Let
\[
\varphi|_{\{a\}} : \{a\} \to \{\varphi(a)\}
\]
denote the restriction of \(\varphi : G \to G'\) to the subgraph \(\{a\} \hookrightarrow G\). It is easy to see that the diagrams
\[
\begin{array}{ccc}
I(a) \ar[r]^\varphi \ar[d]_{j_a} & I(\varphi(a)) \ar[d]_{j_{\varphi(a)}} & \\
\{a\} \ar[r]_{\varphi|_{\{a\}}} & \{\varphi(a)\} \ar[u]_{\varphi|_{\{a\}}}
\end{array}
\]
commutes as well. By Lemma 3.4 the map \(\varphi_a^*\) is an isomorphism of graphs. Hence
\[
\text{P} \varphi_a : \mathcal{P}I(a) \to \mathcal{P}I(\varphi(a))
\]
is an isomorphism of manifolds. Define
\[
\varphi^*_a : \text{Control}(\mathcal{P}I(\varphi(a)) \to \mathcal{P} \varphi(a)) \to \text{Control}(\mathcal{P}I(a) \to \mathcal{P} a)
\]
by
\[
\varphi^*_a(F) = D\mathcal{P}(\varphi|_{\{a\}}) \circ F \circ (\mathcal{P}\varphi_a)^{-1}
\]
for all \(F \in \text{Control}(\mathcal{P}I(\varphi(a))).\) By the universal property of products this gives us the desired map \(\varphi^*\). Moreover the diagrams
\[
\begin{array}{ccc}
\mathcal{Ctrl}(G', \mathcal{P}') \ar[r]^\varphi & \mathcal{Ctrl}(G, \mathcal{P}) \ar[d]_{\pi_{\varphi(a)}} & \\
\text{Control}(\mathcal{P}I(\varphi(a)) \to \mathcal{P} \varphi(a)) \ar[d]_{\varphi_a^*} & & \text{Control}(\mathcal{P}I(a) \to \mathcal{P} a)
\end{array}
\]
commute for all \(a \in G_0\). \(\square\)

3.9. Example. We write down an example of the map \(\varphi^*\) constructed in Theorem 3.6. Consider the graph fibration \(\varphi : G \to G'\), or
as in Example 3.2. Let $\mathcal{P}': G'_0 \to \text{Man}$ be a phase space function. Then
\[ \mathcal{C}t\mathcal{r}(G', \mathcal{P}') = \{(w'_a: \mathcal{P}(a) \to T\mathcal{P}(a), w'_b: \mathcal{P}(a) \times \mathcal{P}(b) \to T\mathcal{P}(b), w'_c: \mathcal{P}(c) \to T\mathcal{P}(c))\}; \]
\[ \mathcal{C}t\mathcal{r}(G, \mathcal{P}' \circ \varphi) = \{(w'_a: \mathcal{P}'(a) \to \mathcal{P}'(a), w'_b: \mathcal{P}'(a) \to \mathcal{P}'(b), w'_c: \mathcal{P}'(a) \times \mathcal{P}'(b) \to T\mathcal{P}'(b))\} \]
and
\[ \varphi^*(w'_a, w'_b, w'_c) = (w'_a, w'_a, w'_b). \]

We are now in the position to state and prove the main result of the paper.

3.10. **Theorem.** Let $\varphi: (G, \mathcal{P}) \to (G', \mathcal{P}')$ be a fibration of networks of manifolds. Then the pullback map $\varphi^*: \mathcal{C}t\mathcal{r}(G', \mathcal{P}') \to \mathcal{C}t\mathcal{r}(G, \mathcal{P})$ constructed in Theorem 3.6 is compatible with the interconnection maps
\[ \mathcal{I}' : \mathcal{C}t\mathcal{r}(G', \mathcal{P}') \to \Gamma(T\mathcal{P}'G') \quad \text{and} \quad \mathcal{I}: (\mathcal{C}t\mathcal{r}(G, \mathcal{P})) \to \Gamma(T\mathcal{P}G). \]

Namely for any collection $w' \in \mathcal{C}t\mathcal{r}(G', \mathcal{P}')$ of open systems on the network $(G', \mathcal{P}')$ the diagram
\[ \begin{array}{ccc}
T\mathcal{P}'G' & \xrightarrow{D\mathcal{P}\varphi} & T\mathcal{P}G \\
\mathcal{I}'(w') \downarrow & & \downarrow \mathcal{I}(\varphi^*w') \\
\mathcal{P}G' & \xrightarrow{\mathcal{P}\varphi} & \mathcal{P}G
\end{array} \]
(3.11)
commutes. Consequently
\[ \mathcal{P}\varphi : (\mathcal{P}(G', \mathcal{P}'), \mathcal{I}'(w')) \to (\mathcal{P}(G, \mathcal{P}), \mathcal{I}(\varphi^*w')) \]
is a map of dynamical systems.

**Proof.** Recall that the manifold $\mathcal{P}G$ is the product $\prod_{a \in G_0} \mathcal{P}a$. Hence the tangent bundle bundle $T\mathcal{P}G$ is the product $\prod_{a \in G_0} T\mathcal{P}a$. The canonical projections $T\mathcal{P}G \to T\mathcal{P}a$ are differential of the maps $\mathcal{P}\iota_a: \mathcal{P}G \to \mathcal{P}a$ where, as before, $\iota_a : \{a\} \hookrightarrow G$ is the canonical inclusion of graphs. Hence by the universal property of products, two maps into $T\mathcal{P}G$ are equal if and only if all their components are equal. Therefore, in order to prove that (3.11) commutes it is enough to show that
\[ D\mathcal{P}\iota_a \circ \mathcal{I}(\varphi^*w') \circ \mathcal{P}\varphi = D\mathcal{P}\iota_a \circ D\mathcal{P}\varphi \circ \mathcal{I}'(w') \]
for all nodes $a \in G_0$. By definition of the restriction $\varphi|_{\{a\}}$ of $\varphi: G \to G'$ to $\{a\} \hookrightarrow G$, the diagram
\[ \begin{array}{ccc}
\{a\} & \xrightarrow{\varphi|_{\{a\}}} & \{\varphi(a)\} \\
\iota_a \downarrow & & \downarrow \iota_{\varphi(a)} \\
G & \xrightarrow{\varphi} & \mathcal{P}G
\end{array} \]
(3.12)
commutes. By the definition of the pullback map \( \varphi^* \) and the interconnection maps \( \mathcal{I}, \mathcal{I}' \) the diagram

\[
\begin{array}{ccc}
\mathcal{I}(\varphi^* w') & \xrightarrow{\varphi^* \mathcal{I}(w')} & \mathcal{I}'(w') \\
\downarrow \mathcal{I}(w') & & \downarrow \mathcal{I}'(w') \\
\mathcal{P}_I(a) & \xrightarrow{\mathcal{P}_I(\varphi)} & \mathcal{P}_I(\varphi(a))
\end{array}
\]

(3.13)

commutes as well. We now compute:

\[
\begin{aligned}
D \mathcal{P}_t a \circ \mathcal{I}(\varphi^* w') \circ \mathcal{P}_\varphi &= (\mathcal{I}(\varphi^* w'))_a \circ \mathcal{P}_\varphi \\
&= D \mathcal{P}(\varphi|_a) \circ \mathcal{I}'(w')_{\varphi(a)} \\
&= D \mathcal{P}(\varphi|_a) \circ D \mathcal{P}_t \varphi(a) \circ \mathcal{I}'(w') \\
&= D \mathcal{P}(\varphi \circ t_a) \circ \mathcal{I}'(w') \\
&= D \mathcal{(t_a)} \circ D \mathcal{P} \varphi \circ \mathcal{I}'(w').
\end{aligned}
\]

And we are done. \( \square \)

3.14. Example. Consider the graph fibration

\[
\begin{array}{ccc}
a_1 & \xrightarrow{\gamma} & b \\
\downarrow \delta & & \downarrow \delta' \\
a_2 & \xrightarrow{\gamma'} & b & \xrightarrow{\delta'} & c
\end{array}
\]

as in Examples 3.2 and 3.9. Let \( \mathcal{P}' : G'_0 \to \text{Man} \) be a phase space function and let \( \mathcal{P} = \mathcal{P}' \circ \varphi \). Then

\[
\begin{aligned}
\mathcal{P}_G' &= \mathcal{P}'(a) \times \mathcal{P}'(b) \times \mathcal{P}'(c), \\
\mathcal{P}_G &= \mathcal{P}'(a) \times \mathcal{P}'(a) \times \mathcal{P}'(b), \\
\mathcal{P}_\varphi(x, y, z) &= (x, x, y),
\end{aligned}
\]

and

\[
D \mathcal{P}_\varphi(p, q, r) = (p, p, q).
\]

For any \( w' = (w'_a, w'_b, w'_c) \in \text{Ctrl}(G', \mathcal{P}') \),

\[
(\mathcal{I}'(w'))(x, y, z) = (w'_a(x), w'_b(x, y), w'_c(y, z)),
\]

and

\[
(\mathcal{I}(\varphi^* w'))(x_1, x_2, y) = (w'_a(x_1), w'_b(x_2), w'_b(x_1, x_2, y))
\]

and

\[
(\mathcal{I}(\varphi^* w') \circ \mathcal{P}_\varphi)(x, y, z) = (w'_a(x), w'_a(x), w'_b(x, x, y))
\]

while

\[
(D \mathcal{P}_\varphi \circ \mathcal{I}'(w'))(x, y, z) = D \mathcal{P}_\varphi(w'_a(x), w'_b(x, y, w'_c(y, z)) = (w'_a(x), w'_a(x), w'_b(x, x, y)).
\]

Hence

\[
(\mathcal{I}(\varphi^* w') \circ \mathcal{P}_\varphi) = (D \mathcal{P}_\varphi \circ \mathcal{I}'(w'))
\]

as expected.

3.15. Example. In the Example 3.14 above the map \( \varphi : G \to G' \) is neither injective nor surjective. It can, of course, be factored as a surjection \( \psi : G \to G'' \):
followed by an injection \( \iota: G'' \to G \):

\[
\begin{align*}
\gamma' \\
\gamma
\end{align*}
\]

The map \( \mathbb{P}\psi: \mathbb{P}G'' \to \mathbb{P}G \) is easily seen to be given by

\[
\mathbb{P}\psi(x, y) = (x, x, y).
\]

It is an embedding, as it should (q.v. Lemma 2.16). The map \( \mathbb{P}\iota: \mathbb{P}G' \to \mathbb{P}G'' \) is given by

\[
\mathbb{P}\iota(x, y, z) = (x, y).
\]

It is a submersion (q.v. Lemma 2.17). Since \( \mathbb{P} \) is a contravariant functor,

\[
\mathbb{P}\varphi = \mathbb{P}(\iota \circ \psi) = \mathbb{P}\psi \circ \mathbb{P}\iota.
\]

We leave it to the reader to check that for any \( w' = (w'_a, w'_b, w'_c) \in \text{Ctrl}(G', \mathbb{P}') \), the map \( \mathbb{P}\iota \) does project the integral curves of the vector field \( \mathcal{I}(w') \) to the integral curves of the vector field \( \mathcal{I}(\varphi^* w') \) on \( \mathbb{P}G'' \). Furthermore, \( \mathbb{P}\psi \) embeds the dynamical system \( (\mathbb{P}G'', \mathcal{I}(\varphi^* w')) \) into the dynamical system \( (\mathbb{P}G, \mathcal{I}(\varphi^* w')) \).

### 3.16. Example.

Consider the injective graph fibration \( \iota: G \to G' \):

Choose phase space functions \( \mathcal{P}, \mathcal{P}' \) so that \( i: (G, \mathcal{P}) \to (G', \mathcal{P}') \) is a map of networks. By Theorem 3.10, for any collection \( w' = (w'_a, w'_b, w'_c) \in \text{Ctrl}(G', \mathbb{P}') \) of open systems on the network \( (G', \mathbb{P}') \) the dynamics in the subsystem \( \mathcal{I}(\varphi^* w') \) drives the entire system \( (\mathbb{P}G', \mathcal{I}(\varphi^* w')) \). This is intuitively clear from the graph (3.17) since there are no “feedbacks” from vertices 4, \ldots, 10 back into 1, 2, 3.

### Concluding remarks.

Theorem 3.10 allows us to define a category \( \text{DSN} \) of continuous time dynamical systems on networks. The objects of this category are triples

\[
(G, \mathcal{P}: G_0 \to \text{Man}, w \in \text{Ctrl}(G, \mathcal{P})),
\]

where as before \( G \) is a finite directed graph, \( \mathcal{P} \) is a phase space function and \( w = (w_a)_{a \in G_0} \) is a tuple of control systems associated with the input trees of the graph \( G \) and the function \( \mathcal{P} \).

A morphism from \( (G', \mathcal{P}', w') \) to \( (G, \mathcal{P}, w) \) is a graph fibration \( \varphi: G \to G' \) (yes, the direction of the map is correct) with \( \mathcal{P}' \circ \varphi = \mathcal{P} \) and \( \varphi^* w' = w \). Alternatively we may think of a map from \( (G', \mathcal{P}', w') \) to \( (G, \mathcal{P}, w) \) as a fibration of networks of manifolds \( \varphi: (G, \mathcal{P}) \to (G', \mathcal{P}') \) with \( \varphi^* w' = w \).

Theorem 3.10 allows us to define a functor from \( \text{DSN} \) to the category \( \text{DS} \) of continuous time dynamical systems on manifolds. The objects \( \text{DS} \) are pairs \( (M, X) \), where \( M \) is a manifold and \( X \) is a vector field on \( M \). A morphism from \( (M, X) \) to \( (N, Y) \) in this category is a smooth map \( f: M \to N \) between two manifolds with \( Df \circ X = Y \circ f \). The functor

\[
\text{DSN} \to \text{DS}
\]
is defined by
\[
\left((G', \mathcal{P}', w') \xrightarrow{\varphi} (G, \mathcal{P}, w)\right) \mapsto \left(\left(\mathbb{P}G', \mathcal{I}(w')\right) \xrightarrow{\mathbb{P}\varphi} (\mathbb{P}G, \mathcal{I}(w))\right).
\]

Given a dynamical system on a network \((G, \mathcal{P}, w)\) we can forget the dynamics. This defines a functor
\[\text{DSN} \to (\text{Man}/\text{Graph})_{\text{fib}}\]
from the category of dynamical systems on networks to a subcategory of the category of networks of manifolds whose maps are fibrations of networks (hence the subscript \(\text{fib}\)). Composing the functor above with the functor \(\text{Man}/\text{Graph} \to \text{Graph}^{\text{op}}\) forgets all the information except for the graph. These two functors from \(\text{DSN}\) to \(\text{DN}\) and to \(\text{Graph}^{\text{op}}\), respectively, allow us to interpret continuous time dynamical systems on networks both as dynamical systems and as graphs.

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