Cosmology and Static Spherically Symmetric solutions in
$D$–dimensional Scalar Tensor Theories: Some Novel Features

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Abstract

We consider scalar tensor theories in $D$–dimensional spacetime, $D \geq 4$. They consist of metric and a non minimally coupled scalar field, with its non minimal coupling characterised by a function. The probes couple minimally to the metric only. We obtain vacuum solutions - both cosmological and static spherically symmetric ones - and study their properties. We find that, as seen by the probes, there is no singularity in the cosmological solutions for a class of functions which obey certain constraints. It turns out that for the same class of functions, there are static spherically symmetric solutions which exhibit novel properties: e.g. near the “horizon”, the gravitational force as seen by the probe becomes repulsive.

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I. INTRODUCTION

Scalar tensor theories of gravity, also referred to as generalised Brans-Dicke theories, are a natural generalisation of Einstein’s general theory of relativity [1, 2, 3]. Such theories appear naturally in various contexts, for example in Kaluza-Klein theories, low energy effective actions of string theory, and five dimensional brane world theories [3, 4, 5, 6, 7].

In these theories the gravity sector consists of the graviton and a non minimally coupled scalar field $\phi$, the non minimal coupling being characterised by a function of $\phi$. There may also be a potential for $\phi$. The matter sector consists of probes and/or matter fields of various types which, in general, couple to the metric and to $\phi$. For the sake of simplicity, in the following, we will not consider matter fields and, moreover, assume that probes couple minimally to the metric only.

In the following, we consider scalar tensor theories in $D-$dimensional spacetime, $D \geq 4$, consisting of the metric $g_{\mu\nu}$, a non minimally coupled scalar field $\phi$ with its non minimal coupling characterised by a function $\psi(\phi)$, and probes coupled minimally to the metric $g_{\mu\nu}$ only. The probes will follow the geodesics of $g_{\mu\nu}$ which we, therefore, refer to as the physical metric.

In this paper we obtain vacuum solutions of the scalar tensor theories in $D-$dimensional spacetime - both cosmological and static spherically symmetric ones - and study their properties. These solutions generalise those in [8] for the $D = 4$ case, and are obtained as follows. We first transform the gravity sector to “Einstein frame” where the action for the Einstein metric $g^{\ast}_{\mu\nu}$ is the standard Einstein-Hilbert action and the canonically normalised scalar field $\phi$ is minimally coupled to $g^{\ast}_{\mu\nu}$. The corresponding equations of motion can be solved rather easily. We then transform the solutions back to “physical frame” to obtain the physical metric $g_{\mu\nu}$ and study their properties.

Note that: (i) In Einstein frame, the probes generically couple to the scalar field also and thus experience a corresponding force besides the gravitational force due to $g_{\mu\nu}$. Therefore, the probes do not follow the geodesics of $g_{\mu\nu}$. Clearly, if the scalar field forces are also taken into account then the resulting motion of the probes in the Einstein frame will be same as that in the physical frame - where the probes couple minimally to the physical metric $g_{\mu\nu}$, and follow its geodesics [3, 4, 9]. (ii) If a potential for $\phi$ and/or matter fields, even if coupled minimally to $g_{\mu\nu}$, are present then the resulting equations of motions are in general difficult.
to solve either in the Einstein frame or in the physical frame. See, however, [10, 11, 12].

Hence, in this paper, we consider only vacuum case where the potential for \( \phi \) and the matter fields are absent.

We find that, as measured by the probes which couple minimally to \( g_{\mu\nu} \), there is no singularity in the cosmological solutions for a class of functions \( \psi(\phi) \) which obey certain constraints. It turns out that for the same class of functions \( \psi(\phi) \), there are static spherically symmetric solutions which exhibit interesting properties: e.g. near the “horizon”, the gravitational force as seen by the probe becomes repulsive. These features are likely to have novel implications for realistic cases, as discussed at the end of the paper.

The plan of the paper is as follows. In section II we give a brief outline of scalar tensor theories. In section III we obtain cosmological solutions and study their properties. In section IV we obtain static spherically symmetric solutions and study their properties. In section V we summarise our results and conclude with a discussion of their relevance to realistic cases, thereby also pointing out various issues that need to be studied further.

II. SCALAR TENSOR THEORY: GENERAL CONSIDERATIONS

In this paper we consider the following action

\[
S_{total} = S + S_{probe}(g_{\mu\nu})
\]

in \( D \)-dimensional spacetime with \( D \geq 4 \). The scalar-tensor part \( S \) of the total action is given by

\[
S = -\frac{1}{16\pi} \int d^Dx \sqrt{-g} \, e^{(\frac{D-2}{2})\psi} \left( R - \frac{A}{2} (\nabla \phi)^2 + e^\psi V(\phi) \right)
\]

where \( \psi \) is a function of scalar field \( \phi \), \( A = 1 - \frac{(D-1)(D-2)}{2} \psi_\phi^2 \), and \( \psi_\phi = \frac{d\psi}{d\phi} \). The probe action \( S_{probe} \) is, for example, that of a point particle of mass \( m_0 \) which couples to the metric \( g_{\mu\nu} \) only and is given by

\[
S_{probe} = -m_0 \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}.
\]

The spacetime would be governed by the equations in the scalar tensor sector and the probe will test this background without distorting it. The probe follows the geodesics of \( g_{\mu\nu} \). So, \( g_{\mu\nu} \) is the physical metric. For a realistic scenario, one must include matter fields coupled to
$g_{\mu\nu}$, and possibly to $\phi$ also. However, with their inclusion, the resulting equations of motion are difficult to solve explicitly and the required analysis becomes very involved. Hence, in the present paper, we will not include the matter fields.

Writing the action given in (2) in terms of the Einstein frame metric $g_{\ast\mu\nu}$ given by

$$g_{\ast\mu\nu} = e^{\psi(\phi)} g_{\mu\nu},$$

the action $S$ becomes

$$S_{\ast} = -\frac{1}{16\pi} \int d^D x \sqrt{-g_{\ast}} \left( R_{\ast} - \frac{1}{2} (\nabla_{\ast}\phi)^2 + V(\phi) \right) + S_{\text{probe}}(e^{-\psi} g_{\ast\mu\nu}).$$

Written in terms of $g_{\ast\mu\nu}$, the action for the graviton and scalar is in the canonical form. Hence, $g_{\ast\mu\nu}$ is the Einstein frame metric and $\phi$ is the canonically normalised scalar field. The probe however couples to $\phi$ also through the function $\psi(\phi)$. Indeed, this is our definition of $\psi(\phi)$. This is why we get the rather unusual coefficients for the scalar field terms in the action $S$ given in (2). It is straightforward to write the action $S$ equivalently in the generalised Brans-Dicke (BD) form [1, 2, 3]. Then, the corresponding BD function $\omega_{BD}$ is given by

$$\omega_{BD} = \frac{2}{(D-2)^2 \psi^2_{\phi}} - \frac{(D-1)}{(D-2)}.$$

The equations of motion for $g_{\ast\mu\nu}$ and $\phi$ in the Einstein frame are given by

$$2R_{\ast\mu\nu} - \nabla_{\ast\mu}\phi \nabla_{\ast\nu}\phi + \frac{2}{D-2} V(\phi) = 0$$

$$\nabla^2_{\phi} + \frac{\partial V}{\partial \phi} = 0.$$

The Ricci scalar $R$ in the physical frame is related to the Ricci scalar $R_{\ast}$ in the Einstein frame by

$$R = e^{\psi} \left( R_{\ast} + (D-1)(\psi_{\phi} \nabla^2_{\ast}\phi + \psi_{\phi\phi} (\nabla_{\ast}\phi)^2) - \frac{(D-1)(D-2)}{4} \psi^2_{\phi} (\nabla_{\ast}\phi)^2 \right).$$

Other curvature invariants in the physical frame can also be similarly related to those in the Einstein frame. Note that if $V = 0$ then $R$ is given by

$$R = e^{\psi} R_{\ast} \left( 1 + 2(D-1)\psi_{\phi\phi} - \frac{(D-1)(D-2)}{2} \psi^2_{\phi} \right).$$

We will solve the equations of motion in the Einstein frame. Without matter included, solving the differential equations of motion in this frame is much simpler. For getting the
physical quantities as seen by the probe we will transform back to the physical frame. With
matter included however, solving the equations of motion is difficult in any frame. In this
paper, therefore, we consider only the vacuum solutions with matter absent and study their
physical properties as seen by the probe. For some solutions with $V(\phi)$ and/or matter
present, see [10, 11, 12].

This procedure of obtaining solutions in one frame, here Einstein frame, and then obtain-
ing the physical quantities by transforming back to the physical frame using $g_{\mu \nu} = e^{-\psi}g_{*\mu \nu}$
(see equation (4)) is certainly valid as long as the factor $e^{-\psi}$ is not zero or infinity. If this
factor vanishes (or diverges) at a point then the validity of this procedure is not automatic;
often, the equivalence between the different frames will be destroyed at this point. One then
has to study the implications of the transformed solutions in the physical frame, and check
whether they satisfy the equations of motion at this point also. 1 In fact, in the case of a
conformally coupled scalar field [13], the transformed solutions imply distributional sources
at such a point in the physical frame, as shown by a beautiful analysis in [14]. We will
comment more on this issue below.

III. COSMOLOGICAL SOLUTIONS

Consider the FRW metric for flat universe in the physical frame. It is given by

$$ds^2 \equiv g_{\mu \nu} \, dx^\mu \, dx^\nu = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2_{D-2} \right), \quad (11)$$

where $t$ and $a$ are the physical time and the scale factor as seen by the probe. The corre-
sponding metric in the Einstein frame is given by

$$ds^2_* = g_{*\mu \nu} \, dx^\mu \, dx^\nu = -dt^2_* + a_*^2(t_*) \left( dr^2 + r^2 d\Omega^2_{D-2} \right). \quad (12)$$

Equation (11) relates the cosmic times and the scale factors in the two frames as follows:

$$\frac{dt}{dt_*} = e^{-\psi/2}, \quad a = a_* e^{-\psi/2}. \quad (13)$$

1 We thank the referee for stressing the importance of this point which we had overlooked earlier.
So, $t$ is a strictly increasing function of $t_*$ if $e^{-\psi/2} > 0$ strictly.

Now putting in the specific forms of the Ricci tensor components the equations of motion (7), (8) take the following form.

\[
\ddot{\phi} + (D - 1) \frac{\dot{a}_* \dot{\phi}}{a_*} - \frac{\partial V}{\partial \phi} = 0 \tag{14}
\]

\[
\frac{\dot{a}_*}{a_*} + (D - 2) \frac{\dot{a}_*^2}{a_*^2} + \frac{1}{D - 2} V(\phi) = 0 \tag{15}
\]

\[
2(D - 1)(D - 2) \frac{\dot{a}_*^2}{a_*^2} - \dot{\phi}^2 + 2V(\phi) = 0 \tag{16}
\]

where $(\dot{\cdot}) \equiv \frac{d(\cdot)}{dt}$. The above equations, with $V(\phi) \neq 0$, can be solved by the ‘method of prepotentials’ [15]. For a given $V(\phi)$ find the prepotential $W(\phi)$ which satisfies the nonlinear differential equation

\[
2(D - 2)^2 W_\phi^2 - (D - 1)(D - 2) W^2 = V
\]

where $W_\phi \equiv \frac{dW}{d\phi}$. Then $\phi(t_*)$ and $a(t_*)$ can be obtained from the equations

\[
\dot{\phi} = -2(D - 2) W_\phi \quad, \quad \frac{\dot{a}}{a} = W.
\]

The solutions thus obtained can be shown to satisfy the equations (14) - (16). The difficulty in solving these equations is now essentially transferred to solving for the prepotential $W$ for a given $V$. Although solutions can be obtained easily by this method for a class of potentials $V$, we consider in this paper $V = 0$ case only. Our analysis can be extended to $V \neq 0$ cases also.

So, let $V = 0$ in the following. Solving the equations of motion, we get

\[
e^\phi = e^{\phi_0} \left( \frac{t_*}{t_{*0}} \right)^m \tag{17}
\]

\[
a_* = a_{*0} \left( \frac{t_*}{t_{*0}} \right)^n \tag{18}
\]

where $\epsilon = \pm 1$, the range of $t_*$ is $0 \leq t_* \leq \infty$, $m$ and $n$ are positive constants, and $a_* = a_{*0}$, $\phi = \phi_0$ at some initial time $t_{*0} > 0$. The solution to equations (14) - (16) gives

\[
(m, n) = \left( \frac{1}{(D - 1)}, \sqrt{\frac{2(D - 2)}{D - 1}} \right) \tag{19}
\]

The Ricci scalar $R$ then becomes

\[
R = -\left( \frac{D - 2}{D - 1} \right) \frac{1}{t_*^2} \tag{20}
\]
The physical Ricci scalar $R$ is given by (10). Note that as $t_\ast \to 0$, $R_\ast \to \infty$.

To understand the behaviour of the physical quantities $t$, $a$, $R$, and their dependence on the function $\psi(\phi)$, first consider

$$\psi(\phi) = k\phi$$

where $k$ is a constant which can be assumed to be positive without loss of generality. Using equations (13), and defining $K \equiv 1 - \frac{\epsilon km}{2}$, we get

$$a = a_0 \left(\frac{t_\ast}{t_{\ast 0}}\right)^{n-\epsilon km/2}$$

$$t - t_0 = \frac{B}{K} \left(\frac{t^K_{\ast}}{t^K_{\ast 0}} - 1\right) \quad \text{if} \quad K \neq 0
= B \ln \left(\frac{t_\ast}{t_{\ast 0}}\right) \quad \text{if} \quad K = 0$$

$$R = C t^{-2K}_\ast$$

where $a_0$, $B$, and $C$ are some constants whose explicit forms are not needed and $t_0$ is the value of the physical time $t$ when $t_\ast = t_{\ast 0}$.

The evolution of $t$, $a$, and $R$ as $t_\ast$ varies from 0 to $\infty$ can be easily seen from the above equations. The values of these quantities vary monotonically for $0 < t_\ast < \infty$. Asymptotically, as $t_\ast \to 0$ or $\infty$, they tend to 0, $\pm \infty$, or to a finite value depending on whether $K$ is positive, zero, or negative i.e whether $\epsilon km$ is less than, equal to, or greater than 2. These asymptotic behaviours, which can be obtained from the above equations, are summarised conveniently in the Tables I and II given below.

| $\epsilon km$ | $t$   | $a$    | $|R|$      |
|--------------|-------|--------|-----------|
| $< 2$        | finite| 0 or $\infty$ | $\infty$ |
| $= 2$        | $-\infty$ | $\infty$ | finite    |
| $> 2$        | $-\infty$ | $\infty$ | 0         |

TABLE I: Asymptotic behaviour of $t$, $a$, and $|R|$ as $t_\ast \to 0$.

For $\epsilon km < 2$, $a = 0$ or $\infty$ if $(2n - \epsilon km)$ is positive or negative.

| $\epsilon km$ | $t$   | $a$    | $|R|$      |
|--------------|-------|--------|-----------|
| $< 2$        | $\infty$ | $\infty$ or 0 | 0         |
| $= 2$        | $\infty$ | 0       | finite    |
| $> 2$        | finite | 0       | $\infty$  |
TABLE II: Asymptotic behaviour of $t, a,$ and $|R|$ as $t_* \to \infty$.

For $\epsilon km < 2$, $a = \infty$ or 0 if $(2n - \epsilon km)$ is positive or negative.

The behaviour of $t, a,$ and $R$ as $t_*$ varies from 0 to $\infty$ can be read off from the above Tables. Generically $\epsilon km \neq 2$. If $\epsilon km < 2$ then the time $t$ evolves to $\infty$ in future with no curvature singularity, whereas in the past there is a curvature singularity at a finite time $t$ beyond which the physical time $t$ cannot be extended. The corresponding asymptotic values of the scale factor $a$ depends on the sign of $(2n - \epsilon km)$ and are given in the Tables I and II.

For $2n - \epsilon km > 0$, $a \to 0$ and this is the usual Big Bang singularity encountered in general relativity. On the other hand, if $\epsilon km > 2$ then the past evolution is singularity free but the future evolution terminates in a singularity at a finite time where the scale factor vanishes - the Big Crunch. For $\epsilon km = 2$, which is non generic, the curvature scalar $R$ remains finite throughout the past and the future.

We now consider the behaviour of the physical quantities $t, a,$ and $R$ for a general function $\psi(\phi)$. We will assume that all the derivatives of $\psi$ with respect to $\phi$ are finite. Namely,

$$\psi^{(n)}(\phi) \equiv \frac{d^n \psi}{d\phi^n} = \text{finite} \quad \forall n \geq 1. \tag{25}$$

This will ensure that the curvature scalar or other curvature invariants will not diverge because of the divergences in $\psi^{(n)}$ for some $n$. The qualitative features of the evolution of $t, a,$ and $R$ can be obtained easily using equations (13), (17), (18), and the Tables I and II.

It is then clear that these quantities will evolve smoothly as a function of $t_*$ for $0 < t_* < \infty$, except perhaps in the asymptotic limit when $t_* = 0$ or $\infty$ and $|\phi| \to \infty$. Therefore, we now analyse the asymptotic behaviour of the solutions in this limit. Consider the class of functions $\psi(\phi)$ where

$$\psi(\phi) = -\lambda |\phi| \quad \text{as} \quad |\phi| \to \infty$$

and $\lambda$ is a positive constant. From the asymptotic behaviour of the solutions given in the Tables I and II, it can be seen by a straightforward analysis that if $\lambda m \geq 2$ then the physical time $t$ can be continued from $-\infty$ to $\infty$, the scale factor $a$ will remain non zero, and the curvature scalar will remain finite throughout the evolution. Therefore, we assume that the function $\psi(\phi)$ satisfies the constraint (25) and obeys the asymptotic condition above with $\lambda m \geq 2$; i.e. that

$$\psi(\phi) = -\lambda |\phi| \quad \text{as} \quad |\phi| \to \infty, \quad \lambda \geq \sqrt{\frac{2(D - 1)}{D - 2}}. \tag{26}$$
A wide class of functions $\psi(\phi)$ exists satisfying the above properties. A simple example is $\psi(\phi) = -\lambda \sqrt{\phi^2 + c^2}$ with $\lambda$ satisfying the condition given in equation (26). Note also that, in the language of generalised Brans-Dicke theories, this constraint on $\lambda$ and $\psi(\phi)$ translates into the following constraint on the BD function $\omega_{BD}$:

$$\omega_{BD} \leq -\frac{D}{D-1} < -1$$

which can be obtained from equation (26).

In obtaining the above results, it was implicitly assumed that $e^{-\psi} > 0$ strictly. Otherwise, $t$ will not be a strictly increasing function of $t_*$. It can now be seen that this assumption is satisfied. This is because it follows that any function $\psi(\phi)$ which satisfies the constraints given in equations (25) and (26) will have a finite maximum, namely $\psi \leq \psi_{max} < \infty$. This then implies that $e^{-\psi} > 0$ strictly.

The fact that the physical Ricci scalar remains finite and the physical cosmic time can be extended into the past and future without bound is indeed a remarkable result and indicates that the Big bang singularity may be absent in the limited context discussed here. The absence of singularity further requires all curvature invariants in the physical frame to be finite. This can indeed be shown to be true following the methods of [8] for general $D-$dimensions for functions $\psi(\phi)$ satisfying the constraints given above.

IV. THE SCHWARZSCHILD SOLUTION

We now consider static spherically symmetric solutions with $V(\phi) = 0$. As before it is easier to solve the equations of motion in the Einstein frame without matter incorporated. For solutions with $V \neq 0$ and/or with matter included, see [10,11]. Let the metric in the Einstein frame be given by

$$ds^2 = -f dt^2 + \frac{dr^2}{g} + h^2 d\Omega^2_{D-2}$$

where $f, g,$ and $h$ are functions of $r$. The equations of motion are

$$\frac{2f''}{f'} - \frac{f'}{f} + \frac{g'}{g} + 2(D-2)\frac{h'}{h} = 0$$

$$\frac{h'g'}{hg} - \frac{h'f'}{hf} + 2\frac{h''}{h} + \frac{\phi'^2}{D-2} = 0$$

$$2fghh'' + f'ghh' + f'ghh' - 2(D-3)f(1 - gh'^2) = 0$$

$$2\phi'' + \left(\frac{f'}{f} + \frac{g'}{g} + 2(D-2)\frac{h'}{h}\right)\phi' = 0$$

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where \( (\cdot)' \equiv \frac{d}{dr}(\cdot) \). Also, \( R_* = \frac{1}{2} g \phi'^2 \) and the physical Ricci scalar \( R \) is given by equation (10). Now the equations of motion can be solved by taking the ansatz

\[
\begin{align*}
    f &= Z^a \\
    g &= Z^b \\
    h^2 &= r^2 Z^q \\
    e^\phi &= e^{\phi_0} Z^p
\end{align*}
\]

where \( Z = 1 - \left( \frac{r_0}{r} \right)^{D-3} \) (37)

and \( \phi_0 \) and \( r_0 \) are constants. With this ansatz, \( R_* = \frac{p^2}{2} Z^2 Z^{b-2} \) and the physical Ricci scalar \( R \) is given by

\[
R = \frac{p^2}{2} e^\psi Z^2 Z^{b-2} \left( 1 + 2(D - 1)\psi_{\phi\phi} - \frac{(D - 1)(D - 2)}{2} \psi_\phi^2 \right).
\] (38)

Solving the equations of motion (29) - (32) gives

\[
\begin{align*}
    a &= 1 - (D - 3)q \\
    b &= 1 - q \\
    2p^2 &= (D - 2)(2 - (D - 3)q) q.
\end{align*}
\]

Note that if \( p = q = 0 \) then \( \phi = \phi_0 \) and \( a = b = 1 \), which is the standard Schwarzschild solution in \( D \)-dimensional spacetime with its horizon at \( r = r_0 \).

From equation (41), it follows that \( 0 \leq q \leq \frac{2}{D-3} \) since \( p^2 \geq 0 \) and \( D \geq 4 \). For a given \( p \) there are two solutions for \( q \), one on either side of \( \frac{1}{D-3} \). We choose the branch where \( 0 \leq q \leq \frac{1}{D-3} \) so that when \( p = 0 \) one obtains \( q = 0 \) only, \( i.e. \) the standard \( D \)-dimensional Schwarzschild solution only with horizon located at \( r = r_0 \). The ranges of \( q, a \) and \( b \) are thus

\[
0 \leq q \leq \frac{1}{D-3}, \quad 1 \geq a \geq 0, \quad 1 \geq b \geq \frac{D - 4}{D - 3}
\]
as can be obtained from equations (39) and (40).

Consider the physical Ricci scalar \( R \). It diverges at \( r = r_0 \) for \( p \neq 0 \). It also diverges at \( r = 0 \) about which we will comment later. The divergence at \( r_0 \) is absent for the standard Schwarzschild case where \( p = 0 \). The singular behaviour of the curvature scalar at \( r = r_0 \), when \( p \neq 0 \), is due to the piece \( e^\psi Z^{b-2} \). So in order that there is no singularity at \( r_0 \) we must have

\[
\lim_{|\phi| \to \infty} e^\psi Z^{b-2} = \text{finite}
\] (42)
and $\psi_\phi$, $\psi_{\phi\phi}$ must also be finite.

We wish to ensure that the Ricci scalar $R$ and other curvature invariants do not diverge because of the divergences in the derivatives of $\psi$ with respect to $\phi$. So here too we impose (25) and consider the class of functions $\psi(\phi)$ where $\psi(\phi) = -\lambda|\phi|$ as $|\phi| \to \infty$. Consider now the behaviour of $R$ near $r_0$ where $|\phi| \to \infty$. So, the above requirement implies

$$\lim_{|\phi| \to \infty} e^\psi Z^{b-2} = e^{-|\phi|(\lambda \frac{2+b}{|p|})}.$$ (43)

This is finite if $\lambda \geq \frac{2+b}{|p|}$, that is if

$$\lambda \geq \frac{1 + q}{\sqrt{(\frac{D-2}{2})(2 - (D-3)q)q}}.$$ (44)

The right hand side of (44) minimizes for $q = \frac{1}{D-2}$. Putting this minimum value in (44), we get

$$\lambda \geq \sqrt{\frac{2(D-1)}{D-2}}.$$ (45)

This is the same constraint on $\lambda$ as obtained from the cosmological case. Thus there is a solution with non zero $q$, i.e. with a non trivial scalar field, and for which the physical Ricci scalar $R$ does not diverge at $r = r_0$ whenever $\psi(\phi)$ satisfies the constraint (26). The parameter $q$ will lie in the range $0 < q_- \leq q \leq \text{min} \left(q_+, \frac{1}{D-3}\right)$ where $q_-$ and $q_+$ are the values of $q$ saturating the inequality in (44). Also, following the methods of [8] for general $D-$dimensions, all curvature invariants in the physical frame can be shown to be finite at $r_0$ for functions $\psi(\phi)$ satisfying constraints given in equations (25) and (26).

Let $\lambda = \sqrt{\frac{2(D-1)}{D-2}}$. Then, the solution with no divergence at $r_0$ is given by either $q = 0$ or $q = \frac{1}{D-2}$. The former one is the standard Schwarzschild solution. Consider the later one and the corresponding metric. The solutions give

$$a = q = \frac{1}{D-2}, \quad b = \frac{D - 3}{D-2}, \quad p^2 = \frac{\lambda^2}{4}.$$ 

Substitute these values into the physical line element, assuming that the function $\psi(\phi)$ satisfies the constraint (26) with $\lambda = \sqrt{\frac{2(D-1)}{D-2}}$. Then $\psi(\phi) = -\lambda|\phi|$ near $r_0$ and we get

$$ds^2 = \frac{1}{Z} \left(-dt^2 + r^2 d\Omega^2_{D-2}\right) + \frac{dr^2}{Z^2}.$$ (46)
Let $r^{D-3} \equiv r_0^{D-3}(1 + \rho)$. Then $\rho \simeq 0$ near $r = r_0$ and the physical line element near $r_0$ becomes

$$ds^2 \simeq \frac{1}{\rho} \left( -dt^2 + r_0^2 d\Omega^2_{D-2} \right) + \frac{r_0^2}{(D-3)^2} \frac{d\rho^2}{\rho^2}$$

which describes a $D-$dimensional anti de Sitter spacetime of radius $\frac{r_0}{D-3}$. This also shows that the curvature invariants are not diverging near $r_0$. For a detailed analysis of such metrics, see [16].

**Radial Geodesic Motion of a Massive Probe**

Another interesting feature of the solution is the gravitational force in the physical frame as seen by the probe. It is attractive for $r \to \infty$ but becomes repulsive for $r \to r_{0+}$. To show this let us look at the $tt$ component of the physical metric $g_{\mu\nu}$. Now, $g_{tt} = -e^{-\psi} Z^a$.

So, in the limit $r \to \infty$,

$$-g_{tt} = 1 - \left( \frac{r_0}{r} \right)^{D-3} + \mathcal{O}\left( \left( \frac{r_0}{r} \right)^2 \right) < 1 . \quad (47)$$

$e^{-\psi}$ is non negative and never vanishes since $\psi \leq \psi_{max} < \infty$. $Z > 0$ for $r > r_0$ and

$$\lim_{r \to r_0} (-g_{tt}) = \lim_{r \to r_0} Z^{a-\lambda |p|} \geq \lim_{r \to r_0} Z^{a+b-2} \to \infty . \quad (48)$$

So as $r$ decreases from $\infty$ to $r_{0+}$, the above mentioned factors ensure that $(-g_{tt})$ decreases from 1 to some minimum value at $r_{min} > r_0$, and then diverges to infinity as $r \to r_0$ always remaining positive and non vanishing in this range. The slope of the curve of $(-g_{tt})$ with respect to $r$ gives the nature of the gravitational force. In the standard Schwarzschild solution, this is always attractive. But the particular $r-$dependence here shows that for scalar tensor theories the force is attractive for $r > r_{min}$ and becomes repulsive for $r < r_{min}$.

The repulsive force can be seen explicitly by studying the geodesic motion of a radially incoming test particle with non zero rest mass. For a metric given by

$$ds^2 = -g_0 dt^2 + g_1 dr^2 + g_2 d\Omega^2_{D-2},$$

where $g_0$, $g_1$, and $g_2$ are functions of $r$ only, the radial geodesic equation becomes

$$r_{pp} + \frac{g_1 r_p^2}{2g_1} + \frac{g_0'}{2g_1 g_0^2} = 0 \quad (50)$$

$$t_p = \frac{1}{g_0} \quad (51)$$
where \( (\cdot)' \equiv \frac{d(\cdot)}{dr}\) and \((\cdot)_p \equiv \frac{d(\cdot)}{dp}\). Equation (50) can be integrated twice to get

\[
\int dt = \int dr \sqrt{\frac{g_1}{g_0(1 + E g_0)}}
\]

where \(E = -1 + v^2\), corresponding to releasing the test particle at \(r = \infty\) with an inward velocity \(v\) (in units where velocity of light = 1). Since the test particle has non-zero rest mass, its velocity \(v < 1\) and, hence, \(E < 0\).

In our case \(g_0 = -g_{tt}\) which diverges to \(\infty\) at \(r_0\). Therefore, the denominator in (52) vanishes at some \(r_t\), where \(1 + E g_0(r_t) = 0\) and \(r_0 < r_t < r_{\text{min}}\), indicating that \(r_t\) is the turning point. Equation (52) also shows that a test particle starting from \(r = r_{\text{initial}} < \infty\) reaches the turning point \(r_t\) at a finite physical time. Analysis of equation (50) then shows that the test particle travels outwards after reaching \(r_t\). It is clear that such a turning point exists irrespective of the value of \(v\) \((< 1)\) or, equivalently, the initial energy of the test particle. This shows that massive test particles feel a repulsive gravitational force as they approach \(r_{0+}\). Contrast this with the Schwarzschild black hole where \(g_0 = 1 - \left(\frac{m}{r}\right)^{D-3} \leq 1\): the factor \(1 + E g_0\) never vanishes and, hence, there is no turning point. In Einstein frame, where the action is given by (5) and the test particle couples to the scalar field \(\phi\) also, this repulsion can be thought of as arising due to the scalar field force.

### A Few Remarks

We now make a few remarks about the properties of the static spherically symmetric solutions obtained above. For \(q = 0\) they are the standard \(D\)-dimensional Schwarzschild solutions with horizon located at \(r_0\). Consider \(q \neq 0\).

(i) In the Einstein frame, the curvature invariants diverge at \(r_0\). Hence, the above solutions do not describe black holes. In the physical frame also, they do not describe black holes because if \(\psi(\phi)\) is unconstrained then the curvature invariants diverge at \(r_0\), whereas if they are constrained then the physical frame component \(g_{tt}\) does not vanish for any \(r\) where \(r_0 \leq r \leq \infty\). Thus, clearly, the parameter \(q\) is not an extra ‘black hole hair’ which is forbidden by no hair theorems [17].

(ii) The above solutions are valid in the region \(r > r_0\) where all fields are regular. Solutions in the region \(r < r_0\) can also be obtained straightforwardly - either from the above solutions with suitable modifications or otherwise. However, in all these solutions, the scalar field \(\phi\) will diverge as \(r \to r_0\) in both the regions \(r < r_0\) and \(r > r_0\). Hence, these solutions are likely to imply distributional sources at \(r = r_0\) as in [14], in which case they can not be said
to satisfy the equations of motion (29) - (32) at $r = r_0$.

(iii) There is a singularity $r = 0$ in the region $r < r_0$ and it is not removed even when $\psi$ satisfies the constraints (25) and (26).

(iv) The physical metric $g_{\mu\nu} = e^{-\psi}g^{\ast}_{\mu\nu}$. For the functions $\psi$ satisfying the constraint (26), the conformal factor $e^{-\psi}$ diverges at $r = r_0$. Therefore, the equivalence of solutions in the Einstein and the physical frame is likely to be destroyed at $r = r_0$. See also the remark (ii) above.

(v) In the physical frame, the curvature invariants are all finite at $r = r_0$ when the function $\psi(\phi)$ satisfies the constraints (25) and (26). It will be interesting to extend, if possible, the present solutions across $r_0$. The present coordinates are unlikely to be useful for such an extension and one has to find a coordinate chart that can cover appropriately the region around $r_0$. Note that when $\lambda = \sqrt{\frac{2(D-1)}{D-2}}$, one may effectively continue across $r_0$ by lifting the solution to one higher dimension, the details of which can be found in [16]. But it is not clear to us whether such an extension and interpretation as in [16] is possible for $\lambda > \sqrt{\frac{2(D-1)}{D-2}}$ also.

These aspects concerning $r \leq r_0$ are interesting and their resolution is an important issue. However, we will be concerned here only with $r > r_0$. Also, conservatively, we take the present solutions for $q \neq 0$ to be valid only in the region $r_0 < r \leq \infty$ since this suffices for our purposes here. As we will argue in section V, for realistic cases with matter fields included the solutions presented here and their features will remain unchanged for $r > r_0$, whereas they will be modified for $r \leq r_0$ the modifications being dependent on the details of matter fields.

V. DISCUSSION AND CONCLUSIONS

We considered $D$-dimensional scalar tensor theories, characterised by a function $\psi(\phi)$, and studied the vacuum solutions where matter fields are absent. The test particles are assumed to couple minimally, and only, to the metric in the physical frame. Therefore, they follow the geodesics of the physical frame metric and will probe the properties of the corresponding spacetime backgrounds. Of course, the motion of the probes will be invariant in any frame, e.g. Einstein frame, but it will not be along the geodesics of the corresponding metric since, generically, the probe will couple to the scalar field also and feel a force due to
We obtained vacuum cosmological and static spherically symmetric solutions and found that they exhibit interesting features for a class of theories where the function $\psi(\phi)$ satisfies the constraints (25) and (26). For cosmological solutions, the Ricci scalar remains finite and the time continues indefinitely into the past and the future. Other curvature invariants can also be shown to remain finite. So these cosmological solutions are free of singularities.

For the static spherically symmetric case we obtained solutions where the scalar field varies non trivially. If the scalar field is constant then these solutions reduce to the standard Schwarzschild ones, with horizon at $r = r_0$. Otherwise, the solutions have a new singularity at $r_0$ where the Ricci scalar diverges. But the divergence at $r_0$ is absent when $\psi(\phi)$ satisfies the constraints (25) and (26). Other curvature invariants at $r_0$ can also be shown to remain finite then. Also, a radially infalling probe feels a repulsive gravitational force as it approaches $r_0$, reaches a turning point $r_t > r_0$, and then travels outwards. However, $r = 0$ remains singular, as in the Schwarzschild solution, and will likely be seen by any conformally coupled probe, e.g. electromagnetic fields (photons) in $D = 4$ spacetime. Also, a proper extension of the solutions for $q \neq 0$ across $r_0$, satisfying the equations of motion (29) - (32) for all $r$, is not clear to us.

The above features are interesting. But, the most crucial issue is whether such solutions can arise in realistic cases with matter fields present. For conformally coupled matter, such as electromagnetic fields in $D = 4$ spacetime with action of the form $\int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$, the equations of motion for the metric, scalar, and matter fields always admit the constant scalar solution. Quite possibly then the asymptotic end points of the dynamics in these cases will be indistinguishable from that in the standard Einstein theory with no scalar present.

But, generically, the equations of motion in other cases will not admit the constant scalar as a solution. Then, it is possible that the solutions found here might describe the asymptotic end points of the dynamics. For example, see [12] for the $D = 4$ cosmological case with matter included. A similar analysis, with similar conclusions, is likely to be valid for $D > 4$ also.

Consider a ‘star’ made up of, say, a perfect fluid which couples minimally to the physical frame metric. Constant scalar is indeed a solution outside the star in the vacuum but, generically, not inside. Also, generically, the first derivative of the scalar will be non zero at the boundary. Its continuity across the boundary of the star would then imply that
the vacuum solution needs to be described by a solution with non-trivial scalar, namely a solution of the type presented here with \( q \neq 0 \) in the region \( r > r_0 \). The interior solution will depend on the matter content and its distribution, and will be different from that given here.

For the class of theories considered here, where \( \psi(\phi) \) satisfies the constraints (25) and (26), the gravitational force in the physical frame likely becomes repulsive when the radius of a collapsing star approaches \( r_0 \). Quite plausibly, this force will halt the collapse, stabilising the star radius at a value near, but greater than, \( r_0 \). If this is the case then the extension of the present solutions, with \( q \neq 0 \), across \( r_0 \) will be rendered unnecessary and the presence or absence of singularities in the interior will be dictated by the matter content and its distribution. The question of whether this is what actually happens in a collapse in the class of scalar tensor theories given here can only be answered by solving the relevant equations of motion. But equations are very complicated and solving them is beyond the scope of present paper. For some solutions of ‘stars’ in the scalar tensor theories, see [11]. The scalar functions considered in these works, however, do not satisfy the constraints (26).

Another important question is whether scalar tensor theories of the type considered here can arise naturally. Scalar tensor theories do appear naturally in various contexts e.g. in Kaluza-Klein (KK), string, and brane world theories [4, 6, 7]. Is the corresponding parameter \( \lambda \geq \sqrt{\frac{2(D-1)}{D-2}} \); equivalently is \( \omega_{BD} \leq -\frac{D}{D-1} < -1 \)?

It turns out that for KK theories \( \omega_{BD} > -1 \) [4]. For low energy string theory \( \omega_{BD} = -1 \) [4, 5]; however, see [18]. In string theory, if one considers the \( D = 10 \) spacetime probed by the \( D0 \)-brane probes then it turns out that the corresponding \( \omega_{BD} = -\frac{10}{9} \) [19] which is just on the margin. In brane world theories of Randall-Sundrum I type, having an extra fifth dimension of unit-interval topology, the radion acts as the scalar in the effective four dimensional theory on a brane located in the fifth dimension. For the negative tension brane located at one end of the interval, it turns out that \( \omega_{BD} = -\frac{3}{2} + \epsilon \) with \( \epsilon \) positive and very close to zero [6]. This value is \( < -\frac{D}{D-1} \) where \( D = 4 \) now. However, the implications are not completely clear to us since in the brane world scenario, gravity propagates in the fifth dimension and, moreover, the value of \( \omega_{BD} \) on a brane depends on its location in the fifth dimension. For example, \( \omega_{BD} = \infty \) on the positive tension brane, located at the opposite end of the interval.

Another issue that needs to be studied is the following: if \( D > 4 \) and the observed
four dimensional spacetime is to be part of the $D$–dimensional spacetime in the scalar tensor theories studied here then it is important to consider anisotropic cases also since $(D – 4)$ directions are likely to be compact or, in any case, have different dynamics from the observable four dimensional spacetime.

Conservatively, we had taken here the static spherically symmetric solutions for $q \neq 0$ to be valid only in the region $r_0 < r \leq \infty$. This was sufficient for our purposes here. Nevertheless it is interesting and important in its own right to study their extension across $r_0$, perhaps as in [10], which satisfies the equations of motion for all $r$, including $r_0$ and without any distributional sources as in [14].

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