A GROSHEV THEOREM FOR SMALL LINEAR FORMS

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1. Introduction

In this paper the absolute value or distance from the origin analogue of the classical Khintchine-Groshev theorem is established for a single linear form with a 'slowly decreasing' error function. To explain this in more detail, some notation is introduced. Throughout this paper, $m,n$ are positive integers i.e., $m,n \in \mathbb{N}$, $x = (x_1, \ldots, x_n)$ will denote a point or vector in $\mathbb{R}^n$, $q = (q_1, \ldots, q_n)$ will denote a non-zero vector in $\mathbb{Z}^n$ and

$$|x| := \max\{|x_1|, \ldots, |x_n|\} = \|x\|_{\infty}$$

will denote the height of the vector $x$. Let $\psi : \mathbb{N} \to (0, \infty)$ be a (non-zero) function which converges to 0 at $\infty$. The notion of a slowly decreasing function $\psi$ is defined in [3] as a function for which given $c \in (0,1)$, there exists a $K = K(c) > 1$ such that $\psi(ck) \leq K\psi(k)$. Of course since $\psi$ is decreasing, $\psi(k) \leq \psi(ck)$. For any set $X$, $|X|_n$ will denote the $n$-dimensional Lebesgue measure of $X$ (the suffix $n$ will usually be omitted; there should be no confusion with the height of a vector).

For convenience, the Khintchine-Groshev theorem will be stated for a measurable set $U \subseteq [0,1]^n$, where $n \in \mathbb{N}$. The distance of a real number $x$ from the integers $\mathbb{Z}$ is denoted by $\|x\| = \min\{|x-[x], |x-[x+1]|\}$, where $[x]$ is the integer part of $x$. Let $W(U; \psi) = W([0,1]^{m \times n}, U; \psi)$ be the set of $x \in U$ such that the system of inequalities in $m$ linear forms in $n$ variables given by

$$\|q_1x_{11} + \cdots + q_nx_{1n}\| < \psi(|q|)$$

$$\vdots$$

$$\|q_1x_{m1} + \cdots + q_nx_{mn}\| < \psi(|q|)$$

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has infinitely many solutions in \( q \in \mathbb{Z}^n \). The Khintchine-Groshev theorem states that the Lebesgue measure \( |W(\psi)| \) depends on the convergence of a sum involving \( \psi(k), k \in \mathbb{N} \), as follows.

**Theorem 1.** The set \( W(U; \psi), m, n \in \mathbb{N} \) has measure

\[
|W(U; \psi)| = \begin{cases} 
0 & \text{if } \sum_{k=1}^{\infty} k^{n-1} \psi^m(k) \text{ converges}, \\
|U| & \text{if } \sum_{k=1}^{\infty} k^{n-1} \psi^m(k) \text{ diverges and } \psi(k) \text{ is decreasing for } n = 1 \text{ or } 2.
\end{cases}
\]

We note that if we consider \( C\psi \) for some positive constant \( C \), the conclusion does not change, since the constant does not affect the convergence of the sum \( \sum_{k=1}^{\infty} k^{n-1} \psi^m(k) \).

Let the set of points \( x \in \mathbb{R}^n \) for which the linear form

\[
q \cdot x = q_1 x_1 + \cdots + q_n x_n
\]

satisfies

\[
|q \cdot x| = |q_1 x_1 + \cdots + q_n x_n| < \psi(|q|)
\]

for infinitely many \( q \in \mathbb{Z}^n \) be denoted by \( V(\mathbb{R}^n, \psi) \). The set \( V(\mathbb{R}^n, \psi) \) is the absolute value analogue of the Khintchine-Groshev theorem for a single linear form. It will be shown that the Lebesgue measure of \( V(\mathbb{R}^n, \psi) \) depends on the convergence or divergence of the sum

\[
\sum_{k=1}^{\infty} k^{n-2} \psi(k).
\]

For convenience, the subset

\[
V([0, 1]^n; \psi) = V(\mathbb{R}^n; \psi) \cap [0, 1]^n
= \{ x \in [0, 1]^n : |q \cdot x| < \psi(|q|) \text{ for infinitely many } q \}
\]

is considered and will be called simply \( V(\psi) \). The analogue of the Khintchine-Groshev theorem for \( V(\psi) \) is now stated. Since \( V(\psi) = \{0\} \) when \( n = 1 \), from now on take \( n \geq 2 \).

**Theorem 2.** For \( n \geq 2 \),

\[
|V(\psi)| = \begin{cases} 
0 & \text{if } \sum_{k=1}^{\infty} k^{n-2} \psi(k) \text{ converges}, \\
1 & \text{if } \sum_{k=1}^{\infty} k^{n-2} \psi(k) \text{ diverges and } \psi(k) \text{ is slowly decreasing.}
\end{cases}
\]
2. Proof of theorem

The set
\[ B_\delta(q) = \{ x \in [0, 1]^n : |q \cdot x| < \delta \} \]  
(3)
is a neighbourhood of the resonant set
\[ R(q) = \{ x \in [0, 1]^n : q \cdot x = 0 \} \]  
(4)and is a 'thickened' hyperplane or parallelepiped of volume
\[ |B_\delta(q)| \leq \frac{2\delta}{|q|}. \]  
(5)
The error arises when two or more coordinates of \( q \) are equal to the height of \( q \). This error, when it occurs, is negative and therefore has no consequence in any questions of convergence. It is readily verified the set \( V(\psi) \) can be expressed in the following 'limsup' form:
\[ V(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} B_{\psi(|q|)}(q) \subseteq \bigcup_{q=N}^{\infty} B_{\psi(|q|)}(q). \]  
(6)
The proof falls into two parts, depending on whether the sum (2) converges or diverges.

2.1. Convergence case. Suppose the sum (2) converges. Replacing \( \delta \) with \( \psi(|q|) \) and summing over all non zero \( q \), we have by (5),
\[ \sum_q |B_{\psi(|q|)}(q)| \leq \sum_q \frac{2\psi(|q|)}{|q|} = 2 \sum_{k=1}^{\infty} \sum_{|q|=k} \frac{\psi(|q|)}{|q|} \]
\[ = 2 \sum_{k=1}^{\infty} \frac{\psi(k)}{k} \sum_{|q|=k} 1 \]
\[ = 2^{n+1} n \sum_{k=1}^{\infty} \psi(k)k^{n-2} + O\left( \sum_{k=1}^{\infty} \psi(k)k^{n-3} \right). \]

A Borel-Cantelli Lemma argument is now applied to (6) to complete the proof of the convergence case. By (6), for each \( N = 1, 2, \ldots, \)
\[ |V(\psi)| \leq \sum_{k=N}^{\infty} \sum_{|q|=k} |B_{\psi(k)}(q)| \ll \sum_{k=N}^{\infty} \sum_{|q|=k} \frac{\psi(|q|)}{|q|} \]
\[ \ll \sum_{k=N}^{\infty} \psi(k)k^{n-2} \]
by (5). But the sum (2) converges, whence the tail tends to 0 as \( N \to \infty \), and so \( |V(\psi)| = 0. \)
It is readily verified that the argument used here extends to systems of $m$ linear forms so that if the sum $\sum_{k=1}^{\infty} k^{n-2}\psi(k)$ converges, the set of points in $\mathbb{R}^{m \times n}$ for which the inequalities

$$|q_1 x_1 + \cdots + q_n x_1| < \psi(|q|)$$

$$\vdots$$

$$|q_1 x_m + \cdots + q_n x_m| < \psi(|q|)$$

have infinitely many solutions in $q \in \mathbb{Z}^n$ is of measure zero.

2.2. **Divergence case.** Now assume that the sum (2) diverges. Since $\psi$ is slowly decreasing, there exists a constant $C > 1$ such that for all $k \in \mathbb{N}$,

$$\psi(k/n) \leq C \psi(k). \quad (7)$$

Define the set $F_j = \{x \in [0, 1]^n : x_j = 1\}$, which will be referred to as the $j$th face of the hypercube $[0, 1]^n$. For points in the set $V(\psi) \cap F_n$, the inequality (11) reduces to

$$|q_1 x_1 + q_2 x_2 + \cdots + q_{n-1} x_{n-1} + q_n| < \psi(|q|). \quad (8)$$

It is shown that $|V(\psi) \cap F_n| = 1$ by proving

$$W([0, 1]^{1 \times (n-1)}, [0, 1]^{n-1}; \psi/C) \times \{1\} \subseteq V(\psi) \cap F_n.$$ 

Then the sum

$$\sum_{k=1}^{\infty} K \psi(k) k^{n-2} = \infty$$

for any constant $K > 0$. By the $(n-1)$-dimensional Groshev theorem, the $(n-1)$ dimensional Lesbegue measure is given by

$$|W([0, 1]^{1 \times (n-1)}, [0, 1]^{n-1}; \psi/C)| = 1.$$ 

First we consider the case $j = n$. To discuss the validity of (8), let $\tilde{q} = (q_1, \ldots, q_{n-1})$ and define $\tilde{x}$ similarly. Note that $|q| \geq |\tilde{q}|$. Now let $\tilde{x} \in W([0, 1]^{1 \times (n-1)}, [0, 1]^{n-1}; \psi/C)$. Then by definition there exist infinitely many $\tilde{q} = (q_1, \ldots, q_{n-1}) \in \mathbb{Z}^{n-1}$ such that

$$||\tilde{q} \cdot \tilde{x}|| < \frac{1}{C} \psi(|\tilde{q}|),$$

that is, such that

$$|q_1 x_1 + \cdots + q_{n-1} x_{n-1} + q_n| < \frac{1}{C} \psi(|\tilde{q}|)$$

for infinitely many $\tilde{q} \in \mathbb{Z}^{n-1}, q_n \in \mathbb{Z}$. Now suppose $|\tilde{q}| = |q|$ for infinitely many $q$. Then since $C > 1$, it is clear that $(\tilde{x}, 1) \in V(\psi) \cap F_n$. Otherwise, suppose $|\tilde{q}| = |q|$ for only finitely many $q$. 

Then \(|q| > |\hat{q}|\) for all \(q\) with \(|q|\) sufficiently large. It follows that \(|q| = |q_n|\) for all \(q\) with \(|q|\) sufficiently large. Moreover there exists a \(q_n \in \mathbb{Z}\) such that,

\[
|q_1x_1 + \cdots + q_{n-1}x_{n-1} + q_n| < 1,
\]

i.e.,

\[
|q \cdot x| < 1,
\]

where \(x = (x_1, \ldots, x_{n-1}, x_n)\). Thus there exists \(j, 1 \leq j \leq n - 1\) such that

\[
|\hat{q}| \geq |q_j| > \frac{|q|}{n} = \frac{|q_n|}{n}.
\]

For if not, then for each \(j = 1, \ldots, n - 1\), \(|q_j| \leq |q|/n = |q_n|/n\) and so

\[
|\hat{q} \cdot \hat{x}| = |q_1x_1 + \cdots + q_{n-1}x_{n-1}| \leq \frac{(n-1)|q_n|}{n},
\]

whence

\[
|\hat{q} \cdot \hat{x} + q_n| \geq |q_n| - |\hat{q} \cdot \hat{x}| \geq |q_n| - \left(1 - \frac{1}{n}\right)|q_n| = \frac{|q_n|}{n} > 1.
\]

Therefore

\[
|q \cdot x| = |\hat{q} \cdot \hat{x} + q_n| < \frac{1}{C'}\psi(|\hat{q}|) \leq \frac{1}{C'}C\psi\left(\frac{|q|}{n}\right) < \psi(|q|)
\]

by (7) for infinitely many \(q\), so \(x \in V(\psi) \cap F_n\), and

\[
W([0,1]^{(n-1)}, [0,1]^{n-1}; \psi/C) \times \{1\} \subseteq V(\psi) \cap F_n.
\]

The argument for \(F_j\) is similar, \(1 \leq j < n\).

**Lemma 1.** Suppose \(t \in [0,1]\) and \(x \in V(\psi)\). Then \(tx \in V(\psi)\).

**Proof.** Clearly \(0 \in V(\psi)\), so the implication is true for \(t = 0\). Suppose \(t > 0\). By definition of \(x = (x_1, \ldots, x_n) \in [0,1]^n\), there exist infinitely many \(q\) such that

\[
|q_1x_1 + \cdots + q_nx_n| < \psi(|q|).
\]

Since \(\psi(|q|) > 0\),

\[
|q_1tx_1 + \cdots + q_ntx_n| = t|q_1x_1 + \cdots + q_nx_n| \leq t\psi(|q|) < \psi(|q|).
\]

Hence \(tx \in V(\psi)\) for all \(t \in [0,1]\). \(\Box\)

**Corollary 1.** If \(x \in F_j \cap V(\psi)\), then \(tx \in V(\psi)\).
For each \( j = 1, \ldots, n \) let \( P_j \) be the ‘pyramid’ with vertex at the origin and base \( F_j \), i.e.
\[
P_j = \{ x \in [0, 1]^n : |x| = x_j \} = \{ tx : t \in [0, 1], x \in F_j \}.
\]
Note that \( F_j \subset P_j \). Also for each \( U \subseteq [0, 1]^{n-1} \), write
\[
\hat{W}(U; \psi) = W([0, 1]^{n-1}, U; \psi)
\]
\[
= \{ y \in U : \|q \cdot y\| < \psi(|q|) \text{ for infinitely many } q \}.
\]
Then by Theorem 1,
\[
|\hat{W}(U; \psi)| = |U| \text{ since } (2) \text{ diverges.}
\]
Now
\[
|U| \geq |F_j \cap (V(\psi) \cap U)| \geq |\hat{W}(U; \psi/C)| = |U|.
\]
The characteristic function of the set \( S \) will be denoted \( \chi_S \).

Lemma 2. For each \( t \in [0, 1] \) and \( x \in [0, 1]^n \),
\[
\chi_{V(\psi) \cap P_j}(tx) \geq \chi_{V(\psi) \cap P_j}(x).
\]

Proof. If \( \chi_{V(\psi) \cap P_j}(x) = 1 \), then \( \chi_{V(\psi) \cap P_j}(tx) = 1 \) by Lemma 1. If \( \chi_{V(\psi) \cap P_j}(x) = 0 \), then \( \chi_{V(\psi) \cap P_j}(tx) = 0 \) or 1. \( \square \)

Lemma 3. For each \( j = 1, \ldots, n \),
\[
|V(\psi) \cap P_j| = 1/n.
\]

Proof. Take \( j = n \). Then
\[
|V(\psi) \cap P_n| = \int_0^1 \int_0^t \cdots \int_0^t \chi_{V(\psi) \cap P_n}(tx_1, \ldots, tx_{n-1}, t)dx_1 \ldots dx_{n-1}dt
\]
\[
\geq \int_0^t \left[ \int_0^t \cdots \int_0^t \chi_{V(\psi) \cap P_n}(x)dx_1 \ldots dx_{n-1} \right] dt
\]
Take \( U = [0, t]^{n-1} \). Then the inner multiple integral can be evaluated as
\[
\int_0^t \cdots \int_0^t \chi_{V(\psi) \cap P_n}(x)dx_1 \ldots dx_{n-1} = |V(\psi) \cap F_n \cap [0, t]^{n-1}|
\]
\[
= \hat{W}([0, t]^{n-1}, \psi)
\]
\[
= |[0, t]^{n-1}|
\]
\[
= t^{n-1}.
\]
Therefore
\[
|V(\psi) \cap P_n| \geq \int_0^1 t^{n-1}dt = 1/n.
\]
The cases for \( j \neq n \) are similar.
It follows that
\[ |V(\psi)| = \bigcup_{j=1}^{n} |V(\psi) \cap P_j| \geq \sum_{j=1}^{n} |V(\psi) \cap F_j| = n.1/n = 1, \]
whence \( |V(\psi)| = 1 \) as claimed. \( \square \)

A direct proof would be more satisfactory but there is a similar configuration of curved lines in the plane for which the analogous measure result does not hold (see [1]).

Unlike the case of convergence, the arguments in the divergence case do not extend to the case of more than one linear form. Now, in [2], H. Dickinson considered the system of linear forms in the case when \( \psi(k) = k^{-\tau} \) and showed that the Hausdorff dimension is given by
\[ \dim_H V = \begin{cases} (m-1)n + \frac{m}{\tau+1}, & \tau > (m/n) - 1 \\ mn, & 0 < \tau \leq (m/n) - 1 \end{cases} \]
The argument establishing the upper bound is essentially the same as the argument in the case for convergence. The argument for the lower bound uses ubiquity and there is a close connection between this and the divergence case of Khintchine’s theorem. However, Dickinson’s approach does not appear to be adaptable in a straightforward manner to more than one linear form in the divergence case.

Although less powerful, the methods used in this paper are concise and direct. They can also be used to provide a lower bound for the number of solutions from the formula for the classical case (see [4] or [5, Chap. 1, §7]) applied to the \((n-1)\)-dimensional faces of the unit cube. Then using the same argument as the divergence case to overcome the difficulty that arises when \( |q| = |q_n| \) discussed in §2.2, it can be verified that the number of solutions \( N(x, Q) \) satisfies
\[ N(x, Q) \gg \sum_{k=1}^{Q} k^{n-2} \psi(k). \]

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