AN APPLICATION OF A THEOREM OF G. ZWIRNER TO A CLASS OF NON-LINEAR ELLIPTIC SYSTEMS IN DIVERGENCE FORM

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Abstract. A theorem on the solutions of the problem $U'(w) = \gamma F(U(w), w), \quad U(w_1) = u_1, U(w_2) = u_2$ is applied for finding the functional solutions of the system of partial differential equations

$$\nabla \cdot (a(u, w)\nabla u) = 0, \quad u = u_1 \text{ on } \Gamma_1, \quad u = u_2 \text{ on } \Gamma_2, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$

$$\nabla \cdot (b(u, w)\nabla w) = 0, \quad w = w_1 \text{ on } \Gamma_1, \quad w = w_2 \text{ on } \Gamma_2, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_3.$$

The problem of existence and uniqueness of solutions is considered.

1. Introduction

The problem of finding the solutions of the ordinary differential equation

$$U'(w) = \gamma F(U(w), w)$$

which satisfy the two conditions

$$U(w_1) = u_1, \quad U(w_2) = u_2, \quad w_2 > w_1$$

was the object of several papers mainly of the Italian and Japanese school. We quote in particular [4], [8], [1], [7], [9], [6], [10]. In this paper we show that the theorem given by G. Zwirner in [9] on the existence and uniqueness for problem (1.1), (1.2) can be used to find a class of solutions, physically relevant, of the boundary value problem

$$\nabla \cdot (a(u, w)\nabla u) = 0 \quad \text{in } \Omega$$

$$u = u_1 \text{ on } \Gamma_1, \quad u = u_2 \text{ on } \Gamma_2, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$

$$\nabla \cdot (b(u, w)\nabla w) = 0 \quad \text{in } \Omega$$

$$w = w_1 \text{ on } \Gamma_1, \quad w = w_2 \text{ on } \Gamma_2, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_3, \quad w_2 > w_1$$

2010 Mathematics Subject Classification. 34L99, 35J66.

Key words and phrases. Existence and uniqueness, two-point problem for O.D.E., systems of P.D.E in divergence form.
where $\Omega$ is an open and bounded subset of $\mathbb{R}^N$ with boundary $\Gamma$ divided into three parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. $u_1$, $u_2$ are arbitrary constants, whereas $w_1$, $w_2$ are constants with the restriction $w_2 > w_1$.

When $N = 3$ the problem (1.3)-(1.6) has a simple physical interpretation. For, let $u(x)$, $x \in \Omega$ represent the temperature and $w(x)$ the concentration of a substance in a liquid at rest which occupies $\Omega$. Suppose that on $\Gamma_1$ and $\Gamma_2$ the temperature $u$ and the concentration $w$ are kept fixed at the two constant values $u_1$, $u_2$ and $w_1$, $w_2$ respectively, whereas $\Gamma_3$ is the part of the boundary of $\Omega$ which is thermally insulated and impermeable to the substance dissolved in the fluid. By the Fourier’s law we have for the density of heat flow $q = -a(u, w)\nabla u$ and for the density of molecular mass flow $J = -b(u, w)\nabla w$.

In absence of sources of heat and mass we have $\nabla \cdot q = 0$, $\nabla \cdot J = 0$ i.e. (1.3) and (1.5).

2. Existence and uniqueness of functional solutions

We assume that the boundary of $\Omega$ has a degree of regularity which makes solvable the mixed problem

\[ \Delta z = 0, \quad z = 0 \text{ on } \Gamma_1, \quad z = 1 \text{ on } \Gamma_2, \quad \frac{\partial z}{\partial n} = 0 \text{ on } \Gamma_3. \]  

We are interested in the functional solutions of problem (1.3)-(1.6) according to the following

Definition 2.1. A classical solution $(u(x), w(x))$ of problem (1.3)-(1.6) is termed functional if a function $U(t) \in C^1([w_1, w_2])$ exists such that $u(x) = U(w(x))$.

Example 2.2. Let us consider the special case of (1.3)-(1.6) in which

\[ a(u, w) = b(u, w), \quad a(u, w) \geq a_0 > 0. \]  

We claim that every classical solution $(u(x), w(x))$ of (1.3)-(1.6) is a functional solution with respect to the function

\[ U(t) = \alpha t + \beta, \quad \alpha = \frac{u_2 - u_1}{w_2 - w_1}, \quad \beta = \frac{u_1 w_2 - u_2 w_1}{w_2 - w_1}. \]

For, let $(u(x), w(x))$ be any solution of (1.3)-(1.6) and define $\zeta(x) = u(x) - (\alpha w(x) + \beta)$. We have

\[ \nabla \cdot (a(u, w)\nabla \zeta) = 0 \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Gamma_1, \quad \zeta = 0 \text{ on } \Gamma_2, \quad \frac{\partial \zeta}{\partial n} = 0 \text{ on } \Gamma_3. \]

Multiplying (2.3) by $\zeta$ and integrating by parts over $\Omega$ we have, in view of (2.2), $\zeta(x) = 0$. Hence $(u(x), w(x))$ is a functional solution since we have $u(x) = U(w(x))$. For other applications of the functional solutions of systems of partial differential equations in divergence form we refer to [2] and [3].

\footnote{The assumption $w_2 > w_1$, (or, more generally, $w_2 \neq w_1$) is essential to make problem (1.1), (1.2) meaningful. On the other hand, if we assume $w_1 = w_2 = w$ the problem (1.3)-(1.6) is immediately uncoupled. In fact, from (1.5) and (1.6) we have $w(x) = \bar{w}$, under the sole assumption $b(u, w) > 0$. Substituting this value of $w$ in (1.3), the problem (1.6) can be solved using the Kirchhoff transformation.}

\footnote{In certain situations the dependence of $a$ and $b$ on $u$, $w$ can be quite strong.}
Associated with the problem \[1.3\]-\[1.6\] we consider the two-point problem

\[\text{(2.4)}\]
\[U'(w) = \gamma \frac{b(U(w), w)}{a(U(w), w)},\]

\[\text{(2.5)}\]
\[U(w_1) = u_1, \quad U(w_2) = u_2, \quad w_2 > w_1.\]

To this problem we can apply the following theorem (see [9] for the proof).

**Theorem 2.3.** Let \(F(U, w)\) be measurable with respect to \(w\) and continuous with respect to \(U\) in the rectangle \(R = \{w_1 \leq w \leq w_2, \ u_1 \leq U \leq u_2\}, \ w_1 < w_2\). Assume that there exist two functions \(q(w), p(w) \in L^1(w_1, w_2)\) such that

\[p(w) \leq F(U, w) \leq q(w)\]

\[p(w) \geq 0, \quad \int_{w_1}^{w_2} p(t) dt > 0.\]

Then the problem

\[\text{(2.6)}\]
\[U'(w) = \gamma F(U(w), w), \quad U(w_1) = u_1, \quad U(w_2) = u_2,\]

in the unknown \(\gamma\) (a real number) and \(U(w)\), has at least one solution absolutely continuous in \([w_1, w_2]\). If \(F(U, w) \in C^k(R)\) then \(u(t) \in C^{k+1}([w_1, w_2])\). Moreover, if \(F(U, w)\) satisfies a Lipschitz condition in \(R\) with respect to \(U\) the solution of \(\text{(2.6)}\) is unique. \(^3\)

The link between the problem \((1.3)-(1.6)\) and the problem \((2.4), (2.5)\) is established in the theorems below using the following elementary

**Lemma 2.4.** Let \(w(x) \in C^0(\bar{\Omega})\) and

\[\min_{\bar{\Omega}} w(x) = w_1 \leq w(x) \leq w_2 = \max_{\bar{\Omega}} w(x).\]

Assume \(\mathcal{F}(t), \mathcal{G}(t) \in C^0([w_1, w_2])\), then, if

\[\text{(2.7)}\]
\[\mathcal{F}(w(x)) = \mathcal{G}(w(x)), \quad x \in \bar{\Omega},\]

we have, for all \(w \in [w_1, w_2]\),

\[\mathcal{F}(w) = \mathcal{G}(w).\]

**Proof.** Assume \(w^* \in [w_1, w_2]\). There exists \(x^* \in \bar{\Omega}\) such that \(w(x^*) = w^*\). Hence, by \(\text{(2.7)}\),

\[\text{(2.8)}\]
\[\mathcal{F}(w^*) = \mathcal{F}(w(x^*)) = \mathcal{G}(w(x^*)) = \mathcal{G}(w^*).\]

\[\square\]

\(^3\)Other criteria which guarantee the uniqueness of the solution can be found in [9].
Theorem 2.5. Let \( w_2 > w_1 \) and \( R = \{(u, w); \ u_1 \leq u \leq u_2, \ w_1 \leq w \leq w_2 \} \).

Assume \( b(u, w) \in C^0(\Omega) \) and

\[
\begin{align*}
(2.9) \quad a(u, w), \ b(u, w) & > 0 \quad \text{in} \quad R. \\
\end{align*}
\]

Let \( (u(x), w(x)) \) be a functional solution of the problem

\[
\begin{align*}
(2.10) \quad \nabla \cdot (a(u, w)\nabla u) &= 0 \quad \text{in} \quad \Omega \\
(2.11) \quad u = u_1 \quad \text{on} \quad \Gamma_1, \quad u = u_2 \quad \text{on} \quad \Gamma_2, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_3 \\
(2.12) \quad \nabla \cdot (b(u, w)\nabla w) &= 0 \quad \text{in} \quad \Omega \\
(2.13) \quad w = w_1 \quad \text{on} \quad \Gamma_1, \quad w = w_2 \quad \text{on} \quad \Gamma_2, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_3,
\end{align*}
\]

then the function \( U(w) \) entering in the definition of functional solution solves the two point-problem

\[
\begin{align*}
(2.14) \quad U'(w) &= \frac{b(U(w), w)}{a(U(w), w)} \\
(2.15) \quad U(w_1) = u_1, \quad U(w_2) = u_2, \quad w_2 > w_1.
\end{align*}
\]

Proof. Let \( (u(x), w(x)) \) be a functional solution of \( (2.10)-(2.13) \). By \( (2.12) \) the maximum principle \([5]\) implies

\[
\begin{align*}
(2.16) \quad w_1 \leq w(x) \leq w_2.
\end{align*}
\]

Moreover, by assumption \( u(x) = U(w(x)) \). Define

\[
\begin{align*}
(2.17) \quad \theta(w) &= \int_{u_1}^{w} a(U(t), t)U'(t)dt, \quad \psi(w) = \int_{u_1}^{w} b(U(t), t)dt
\end{align*}
\]

and

\[
\begin{align*}
(2.18) \quad \Theta(x) &= \theta(w(x)), \quad \Psi(x) = \psi(w(x)).
\end{align*}
\]

We have \( \nabla \Theta = a(u, w)\nabla u, \quad \nabla \Psi = b(u, w)\nabla w. \) On the other hand, \( (u(x), w(x)) \) solves \( (2.10)-(2.13) \), thus we have

\[
\begin{align*}
\Delta \Theta &= 0 \quad \text{in} \quad \Omega, \quad \Theta = 0 \quad \text{on} \quad \Gamma_1 \\
\Theta &= \theta(w_2) \quad \text{on} \quad \Gamma_2, \quad \frac{\partial \Theta}{\partial n} = 0 \quad \text{on} \quad \Gamma_3 \\
\Delta \Psi &= 0 \quad \text{in} \quad \Omega, \quad \Psi = 0 \quad \text{on} \quad \Gamma_1
\end{align*}
\]
\[ \Psi = \psi(w_2) \text{ on } \Gamma_2, \quad \frac{\partial \Theta}{\partial n} = 0 \text{ on } \Gamma_3. \]

By \( (2.9) \) we have \( \psi(w_2) \neq 0 \). Let \( z(x) \) be the solution of the problem \( (2.1) \). We obtain \( \Theta(x) = \theta(w_2)z(x) \) and \( \Psi(x) = \psi(w_2)z(x) \). Hence

\[ (2.19) \quad \Theta(x) = \gamma \Psi(x), \quad \gamma = \frac{\theta(w_2)}{\psi(w_2)}. \]

From \( (2.17), (2.18) \) and \( (2.19) \) we have

\[ (2.20) \int_{w_1}^{w(x)} a(U(t), t)U'(t)dt = \gamma \int_{w_1}^{w(x)} b(U(t), t)dt. \]

Applying Lemma 1.4 with

\[ \mathcal{F}(t) = \int_{w_1}^{t} a(U(\eta), \eta)U'(\eta)d\eta, \quad \mathcal{G}(t) = \int_{w_1}^{t} b(U(\eta), \eta)d\eta \]

by \( (2.20) \) we have

\[ \int_{w_1}^{w} a(U(t), t)U'(t)dt = \gamma \int_{w_1}^{w} b(U(t), t)dt. \]

Hence

\[ a(U(w), w)U'(w) = \gamma b(U(w), w) \]

and \( (2.14) \) holds. Moreover, also the boundary conditions \( (2.15) \) are verified. \( \square \)

Vice-versa we have

**Theorem 2.6.** Assume \( (2.4) \), then to every solution \( U(w) \) of class \( C^1([w_1, w_2]) \) of the problem

\[ (2.21) \quad U'(w) = \frac{b(U(w), w)}{a(U(w), w)}, \quad U(w_1) = u_1, \quad U(w_2) = u_2, \quad w_2 > w_1 \]

there corresponds a functional solution of the problem \( (2.11)-(2.13) \).

**Proof.** Let \( U(t) \) be a solution of \( (2.21) \) and consider the non-linear elliptic problem

\[ (2.22) \quad \nabla \cdot (b(U(w), w)\nabla w) = 0 \quad \text{in } \Omega \]

\[ (2.23) \quad w = w_1 \text{ on } \Gamma_1, \quad w = w_2 \text{ on } \Gamma_2, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_3. \]

There exists one and only one solution of \( (2.22), (2.23) \). For, let us define

\[ \psi(w) = \int_{w_1}^{w} b(U(t), t)dt. \]

By \( (2.9) \) \( \psi \) maps one-to-one \([w_1, w_2]\) onto \([0, \psi(w_2)]\). Hence, if we define \( \varphi(x) = \psi(w(x)) \), the problem \( (2.22), (2.23) \) can be restated as
\[ (2.24) \quad \Delta \varphi = 0 \quad \text{in} \quad \Omega, \quad \varphi = 0 \quad \text{on} \quad \Gamma_1 \]

\[ (2.25) \quad \varphi = \psi(w_2) \quad \text{on} \quad \Gamma_2, \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \Gamma_3. \]

By \((2.1)\) the solution of \((2.24)\) and \((2.25)\) exists and is unique and \(w(x) = \psi^{-1}(\varphi(x))\) gives the unique solution of \((2.22), (2.23)\). Define now

\[ u(x) = U(w(x)). \]

Thus \((2.22)\) can be written

\[ \nabla \cdot (b(u, w) \nabla w) = 0 \quad \text{in} \quad \Omega. \]

Setting \(w = w(x)\) in \((2.21)\) we obtain

\[ a(U(w(x)), w(x))U'(w(x)) = \gamma b(U(w(x)), w(x)) \]

and also

\[ a(U(w(x)), w(x))U'(w(x))\nabla w = \gamma b(U(w(x)), w(x))\nabla w \]

and, by \((2.22)\),

\[ \nabla \cdot (a(u, w) \nabla u) = 0 \quad \text{in} \quad \Omega. \]

On the other hand, the functions \((u(x), w(x))\) just defined satisfies also the boundary conditions \((2.11)\) and \((2.13)\). \(\square\)

This proof shows that the problem \((2.10)-(2.13)\) is solvable (i) if we can solve the linear problem \((2.24), (2.25)\), which in turn is immediately reducible to \((2.1)\) which contains the “geometric” part, (ii) a solution of problem \((2.14), (2.15)\) is known. This last solution contains the non-linear features of the original problem \((1.3)-(1.6)\) if we limit ourselves to consider functional solutions.

The uniqueness of the functional solutions of problem \((1.3)-(1.6)\) is also a consequence of the uniqueness for problem \((1.1), (1.2)\). In fact we have

**Theorem 2.7.** Let \((2.4)\) hold. If the problem

\[ (2.26) \quad U'(w) = \gamma \frac{b(U(w), w)}{a(U(w), w)}, \quad U(w_1) = u_1, \quad U(w_2) = u_2 \]

has a unique solution also the corresponding functional solution of

\[ (2.27) \quad \nabla \cdot (a(u, w) \nabla u) = 0 \quad \text{in} \quad \Omega \]

\[ (2.28) \quad u = u_1 \quad \text{on} \quad \Gamma_1, \quad u = u_2 \quad \text{on} \quad \Gamma_2, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_3 \]

\[ (2.29) \quad \nabla \cdot (b(u, w) \nabla w) = 0 \quad \text{in} \quad \Omega \]
is unique in the class of functional solutions.

Proof. Let, by contradiction, \((u^*, w^*)\), \((u^{**}, w^{**})\) be two functional solutions of problem (2.27)-(2.30). We have

\[ u^*(x) = U^*(w^*(x)), \quad u^{**}(x) = U^{**}(w^{**}(x)). \]

\(U^*(w)\) and \(U^{**}(w)\) are both solutions of the problem (2.26). Thus \(U^*(w) = U^{**}(w)\).

Let us define

\[ \psi^*(w) = \int_{w_1}^{w} b(U^*(t), t) \, dt, \quad \psi^{**}(w) = \int_{w_1}^{w} b(U^{**}(t), t) \, dt \]

and

\[ \Psi^*(x) = \psi^*(w^*(x)), \quad \Psi^{**}(x) = \psi^{**}(w^{**}(x)). \]

We have \(\psi^*(w_2) = \psi^{**}(w_2)\), therefore \(\Psi^*(x)\) and \(\Psi^{**}(x)\) are both solutions of the problem

\[ \Delta \varphi = 0 \quad \text{in} \quad \Omega, \quad \varphi = 0 \quad \text{on} \quad \Gamma_1, \]

\[ \varphi = \psi^*(w_2) \quad \text{on} \quad \Gamma_2, \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \Gamma_3 \]

which has a unique solution. Hence \(\Psi^*(x) = \Psi^{**}(x)\) and we have \(\psi^*(w) = \psi^{**}(w)\) by Lemma 1.3. This in turn implies

\[ w^*(x) = (\psi^*)^{-1}(\varphi(x)) = (\psi^{**})^{-1}(\varphi(x)) = w^{**}(x) \]

and

\[ u^*(x) = U^*(w^*(x)) = U^{**}(w^{**}(x)) = u^{**}(x). \]

We summarize our results in the following

**Theorem 2.8.** Let \(\frac{b(U, w)}{a(U, w)}\) be of class \(C^1\) in the rectangle \(R = \{w_1 \leq w \leq w_2, u_1 \leq U \leq u_2\}, w_1 < w_2\). Assume (2.9) and that there exist two functions \(q(w), p(w) \in L^1(w_1, w_2)\) such that

\[ 0 \leq p(w) \leq \frac{b(U, t)}{a(U, t)} \leq q(w), \quad \int_{w_1}^{w_2} p(t) \, dt > 0. \]

Then the problem (1.3)-(1.6) has at least one functional solution. Moreover, if \(\frac{b(U, w)}{a(U, w)}\) satisfies a Lipschitz condition in \(R\) with respect to \(U\) the solution of (1.3)-(1.6) is unique in the class of functional solutions.
Compliance with ethical standard

Conflict of interest. The author declares that he has no conflicts of interest.

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