Multiplicative relations among singular moduli
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1. Introduction

We consider some Diophantine problems of mixed modular-multiplicative type associated with the Zilber-Pink conjecture (ZP; see [4, 17, 25] and §2). Our results rely on the “modular Ax-Schanuel” theorem recently established by us [16].

Recall that a singular modulus is a complex number which is the \( j \)-invariant of an elliptic curve with complex multiplication; equivalently it is a number of the form \( \sigma = j(\tau) \) where \( j : \mathbb{H} \to \mathbb{C} \) is the elliptic modular function, \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) is the complex upper-half plane, and \( \tau \in \mathbb{H} \) is a quadratic point (i.e. \([\mathbb{Q}(\tau) : \mathbb{Q}] = 2\)).

1.1. Definition. An \( n \)-tuple \( (\sigma_1, \ldots, \sigma_n) \) of distinct singular moduli will be called a singular-dependent \( n \)-tuple if the set \( \{ \sigma_1, \ldots, \sigma_n \} \) is multiplicatively dependent (i.e. \( \prod \sigma_i^{a_i} = 1 \) for some integers \( a_i \) not all zero), but no proper subset is multiplicatively dependent.

1.2. Theorem. Let \( n \geq 1 \). There exist only finitely many singular-dependent \( n \)-tuples.

The independence of proper subsets is clearly needed to avoid trivialities. The result is ineffective. Some examples (including a singular-dependent 5-tuple) can be found among the rational singular moduli (listed in [20, A.4]; see 6.3). Bilu–Masser–Zannier [3] show that there are no singular moduli with \( \sigma_1 \sigma_2 = 1 \). This result is generalised by Bilu–Luca–Pizarro-Madariaga [2] to classify all solutions of \( \sigma_1 \sigma_2 \in \mathbb{Q}^\times \). Habegger [7] shows that only finitely many singular moduli are algebraic units.

In addition to the “modular Ax-Schanuel”, we make use of isogeny estimates and some other arithmetic ingredients, gathered in §6, and we require the following result showing that distinct rational “translates” of the \( j \)-function are multiplicatively independent modulo constants. To formulate it, recall that, for \( g_1, g_2 \in \text{GL}_2^+\mathbb{Q} \), the functions \( j(g_1z), j(g_2z) \) are identically equal iff \( [g_1] = [g_2] \) in \( \text{PSL}_2(\mathbb{Z})\backslash \text{PGL}_2^+\mathbb{Q} \); functions \( f_1, \ldots, f_k : \mathbb{H} \to \mathbb{C} \) will be called multiplicatively independent modulo constants if there is no relation \( \prod_{i=1}^k f_i^{n_i} = c \) where \( n_i \) are integers, not all zero, and \( c \in \mathbb{C} \).

1.3. Theorem. Let \( g_1, \ldots, g_k \in \text{GL}_2^+\mathbb{Q} \). If the functions \( j(g_1z), \ldots, j(g_kz) \) are pairwise distinct then they are multiplicatively independent modulo constants.

Theorem 1.3 is not predicted by ZP, nor would it follow from “Ax-Schanuel” for \( \text{exp} \) and \( j \) (see §3). But in view of Theorem 1.3, Theorem 1.2 is implied by ZP. The Zilber-Pink setting is introduced in §2. After the proofs of 1.3 and 1.2 in §4 and §6, we discuss further problems connected with the same setting in §7 and §8.
2. The Zilber-Pink setting

We identify varieties and subvarieties with their sets of complex points (thus $Y(1)(\mathbb{C}) = \mathbb{C}$ and $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$) and are assumed irreducible over $\mathbb{C}$.

For $m, n \in \mathbb{N} = \{0, 1, 2, \ldots \}$ set

$$X = X_{m,n} = Y(1)^m \times \mathbb{G}_m^n.$$

2.1. Definition.

1. A weakly special subvariety of $Y(1)^m = X_{m,0} = \mathbb{C}^m$ is a subvariety of the following form. There is a “partition” $m_0, \ldots, m_k$ of $\{1, \ldots, m\}$, in which $m_0$ only is permitted to be 0, but $k = 0$ is permitted such that $M = M_0 \times M_1 \times \ldots \times M_k$ where $M_0$ is a point in $\mathbb{C}^{m_0}$ (here $\mathbb{C}^m_i$ refers to the cartesian product of the coordinates contained in $m_i$, which is a subset of $\{1, \ldots, m\}$) and, for $i = 1, \ldots, k$, $M_i \subset \mathbb{C}^{m_i}$ is a modular curve.

2. A special point of $\mathbb{C}^m$ is a weakly special subvariety $M$ of dimension zero (so $n_0 = \{1, \ldots, n\}$ and $M = M_0$) such that each coordinate of $M$ is a singular modulus.

3. A special subvariety of $\mathbb{C}^m$ is a weakly special subvariety such that $m_0 = \emptyset$ or $M_0 \in \mathbb{C}^{m_0}$ is a special point. It is strongly special if $m_0 = \emptyset$.

4. A weakly special subvariety of $\mathbb{G}_m^n = X_{0,n} = (\mathbb{C}^\times)^n$ is a coset of a subtorus, i.e. a subvariety defined by a finite system of equations $\prod x_i^{a_{ij}} = \xi_j, j = 1, \ldots, k$ where, for each $j$, $a_{ij} \in \mathbb{Z}$ are not all zero, $\xi_j \in \mathbb{C}^\times$ and the lattice generated by the exponent vectors $(a_{1j}, \ldots, a_{nj}), j = 1, \ldots, k$ is primitive.

5. A special point of $\mathbb{G}_m^n$ is a torsion point.

6. A special subvariety of $\mathbb{G}_m^n$ is a weakly special subvariety such that each $\xi_j$ is a root of unity; i.e. it is a coset of a subtorus by a torsion point.

7. A weakly special subvariety of $X$ is a product $M \times T$ where $M, T$ are weakly special subvarieties of $Y(1)^m, \mathbb{G}_m^n$, respectively, and likewise for a special point of $X$ and special subvariety of $X$.

2.2. Definition. Let $W \subset X$. A subvariety $A \subset W$ is called an atypical component (of $W$ in $X$) if there is a special subvariety $T \subset X$ such that $A \subset W \cap T$ and

$$\dim A > \dim W + \dim T - \dim X.$$

The atypical set of $W$ (in $X$) is the union of all atypical components (of $W$ in $X$), and is denoted $\text{Atyp}(W, X)$, or $\text{Atyp}(W)$ if $X$ is implicit from the context.

2.3. Conjecture (Zilber-Pink for $X$). Let $W \subset X$. Then $\text{Atyp}(W)$ is a finite union of atypical components; equivalently, there are only finitely many maximal atypical components.

The full Zilber-Pink conjecture is the same statement about an arbitrary mixed Shimura variety (with its special subvarieties), and an algebraic subvariety $W \subset X$. See [24].
2.4. Definition. Let $A \subset X$ be a subvariety. We denote by $\langle A \rangle$ the smallest special subvariety containing $A$ (which exists as it is just the intersection of all special subvarieties containing $A$), and define the defect of $A$ by

$$\delta(A) = \dim \langle A \rangle - \dim A.$$  

Thus $A \subset W$ is atypical if $\delta(A) < \dim X - \dim W$, and $W$ itself is atypical if $\langle W \rangle \neq X$.

Now in Conjecture 2.3 we look only for maximal atypical components, and we do not care if a larger atypical component contains a smaller but more atypical (i.e. smaller defect) one. But in fact the conjecture (taken over all special subvarieties of $X$) implies a formally stronger version (see [9], Proposition 2.4).

2.5. Definition. A subvariety $W \subset V$ is called optimal for $V$ if there is no strictly larger subvariety $W \subset W' \subset V$ with $\delta(W') \leq \delta(W)$.

2.6. Conjecture. Let $V \subset X$. Then $V$ has only finitely many optimal subvarieties.

For a particular $V$ and $X$, finding (or establishing the finiteness of) all optimal subvarieties could be more difficult than finding (or establishing the finiteness of) all maximal atypical subvarieties.

Now (as in [9]) we can repeat the same pattern of definitions with weakly special subvarieties instead of special ones. The smallest weakly special subvariety containing $W$ we denote $\langle W \rangle_{\text{geo}}$, and we define the geodesic defect to be

$$\delta_{\text{geo}}(W) = \dim \langle W \rangle_{\text{geo}} - \dim W$$

A subvariety $W \subset V$ is called geodesic-optimal if there is no strictly larger subvariety $W' \subset V$ with $\delta_{\text{geo}}(W') \leq \delta_{\text{geo}}(W)$. (This property is termed “cd-maximal” in the multiplicative setting in [18]). The following fact was established for modular, multiplicative and abelian varieties separately in [9].

2.7. Proposition. Let $V \subset X_{m,n}$. An optimal component of $V$ is geodesic-optimal.

Proof. It is easy to adapt the proof of [9, Proposition 4.3] to show that $X_{m,n}$ has the “defect condition”, and then the above follows from the formal properties of weakly special and special subvarieties, as in [9, Proposition 4.5]. □

Now we consider

$$V = V_n = \{(x_1, \ldots, x_n, t_1, \ldots, t_n) : s_i = t_i, i = 1, \ldots, n \} \subset X_n = X_{n,n}.$$  

We see that if a tuple $(\sigma_1, \ldots, \sigma_n)$ of singular moduli satisfies a non-trivial multiplicative relation then the point

$$\Sigma = (\sigma_1, \ldots, \sigma_n, \sigma_1, \ldots, \sigma_n) \in V$$

lies in the intersection of $V$ with a special subvariety of $X$ of codimension $n + 1$. So such a point is an atypical component of $V_n$.  

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3. Mixed Ax-Schanuel

We now take again

\[ X = X_{m,n} = Y(1)^m \times \mathbb{G}_m^n, \quad U = U_{m,n} = \mathbb{H}^m \times \mathbb{C}^n, \quad \text{and} \quad \pi : U \to X \]

given by

\[ \pi(z_1, \ldots, z_m, u_1, \ldots, u_n) = (j(z_1), \ldots, j(z_m), \exp(u_1), \ldots, \exp(u_n)). \]

3.1. Definition.

1. An algebraic subvariety of \( U \) will mean a complex-analytically irreducible component of \( Y \cap U \) where \( Y \subset \mathbb{C}^m \times \mathbb{C}^n \) is an algebraic subvariety.

2. A weakly special subvariety of \( U \) is an irreducible component of \( \pi^{-1}(W) \) where \( W \) is a weakly special subvariety of \( X \). Likewise for special subvariety of \( U \).

The following result leads to the analogue of the “Weak Complex Ax” (WCA; [9, Conjecture 5.10]) in this mixed modular-multiplicative setting. It is deduced from the same statement in the two extreme special cases: WCA for \( Y(1)^n \), which is a consequence of the full modular Ax-Schanuel result established in [16], and WCA for \( \mathbb{G}_m^n \), which is a consequence of Ax-Schanuel [1].

Note that we could avoid talking about “algebraic subvarieties of \( U \)” by taking \( Y \) to be an algebraic subvariety of \( \mathbb{C}^m \times \mathbb{C}^n \) and \( A \) to be a complex-analytically irreducible component of \( Y \cap \pi^{-1}(V) \).

3.2. Theorem. Let \( V \subset X \) and \( W \subset U \) be algebraic subvarieties and \( A \subset W \cap \pi^{-1}(V) \) a complex-analytically irreducible component. Then

\[ \dim A = \dim V + \dim W - \dim X \]

unless \( A \) is contained in a proper weakly special subvariety of \( U \).

Proof. We suppose that \( A \) is not contained in a proper weakly special subvariety of \( U \), and prove the dimension statement. We may suppose that \( A \) is Zariski-dense in \( W \) and that \( \pi(A) \) is Zariski-dense in \( V \).

Let \( A_0, W_0, V_0 \) be the images of \( A, W, V \) under the projection to \( \mathbb{C}^n \) (for \( A, W \)) and \( \mathbb{G}_m^n \) (for \( V \)). Then \( A_0 \subset W_0 \cap \exp^{-1}(V_0) \), and \( A_0 \) is not contained in a proper weakly special subvariety of \( \mathbb{C}^n \), otherwise \( A \) would be contained in a proper weakly special subvariety of \( U \). So by Ax-Schanuel ([1]; see also [22]) we have

\[ \dim A_0 \leq \dim W_0 + \dim V_0 - \dim \mathbb{C}^n. \]

Now we look at fibres in \( \mathbb{H}^m \) and \( \mathbb{C}^m \). We let \( A_u, W_u \subset \mathbb{H}^m, V_t \subset \mathbb{C}^m \) be the fibres (of \( A, W, V \) respectively) over \( u = (u_1, \ldots, u_n) \in A_0, \ u \in W_0, \ t = (t_1, \ldots, t_n) \in V_0 \), respectively.
Now $A_0$ must be Zariski-dense in $W_0$, else $A$ could not be Zariski-dense in $W$, and similarly $\exp(A_0)$ must be Zariski-dense in $V_0$. The projection $W \to W_0$ has a generic fibre dimension away from a locus $W' \subset W$ of lower dimension, which does not contain $A$. So a generic fibre over $A_0$ outside the image of $W'$ is generic for $A_0$ as well as $W_0$, and so is the corresponding fibre over $V_0$.

For $u \in A_0$, if $A_u$ is not contained in a proper weakly special subvariety of $\mathbb{H}^m$, then by [16] we have,

$$\dim A_u \leq \dim W_u + \dim V_u - \dim \mathbb{H}^m.$$ 

If this holds generically, adding up the two last displays gives us the statement we want.

So we consider what happens when this fails generically. If the $A_u$ were contained in a fixed proper weakly special, than $A$ would be, which we have precluded. So the fibres must belong to a “moving family” of proper weakly specials. As elements of $GL^+_2(\mathbb{Q})$ can’t vary analytically, the only possibility is that some coordinates are constant on the fibres (though not constant on $A$).

Say these coordinates are $z_1, \ldots, z_h$, and for $1 \leq \ell \leq k$ we let $A_\ell, W_\ell$ and $V_\ell$ be the image of $A, W, V$ under projection to $\mathbb{H}^\ell \times \mathbb{C}^n$ (or $\mathbb{C}^\ell \times G_m^\ell$ for $V_\ell$) where these are the coordinates corresponding to $z_1, \ldots, z_\ell$ (or their images under $j$, for $V$).

Now prove inductively that the dimension inequality holds at “level” $\ell$, and once it holds at level $k$ we are done. We assume that, for some $0 \leq h < k$:

(A) $A_h$ is Zariski-dense in $W_h$ and $\pi_h(A_h)$ is Zariski-dense in $V_h$, and

(B) $\dim A_h \leq \dim W_h + \dim V_h - (n + h)$.

We know that these both hold for $h = 0$, and that (A) holds for all $h$.

Now $z_{h-1}$ is constant on the fibres, so $\dim A_{h+1} = \dim A_h$. To show (B) we need only show that either $\dim W_{h+1} > \dim W_h$ or $\dim V_{h+1} > \dim V_h$.

Suppose that $\dim W_{h+1} = \dim W_h$. This means that, as functions on $W$, $z_{h+1}$ is algebraic over $z_1, \ldots, z_h, u_1, \ldots, u_n$. But, as $W$ is not contained in a proper weakly special subvariety, $z_{h+1}$ is not constant on $W$ nor does it satisfy any relation $z_{h+1} = g z_i$ where $1 \leq i \leq h$ and $g \in GL^+_2(\mathbb{Q})$. But then, by the “Ax-Lindemann” result of [14] for the $j$-function, $j(z_{h+1})$ is algebraically independent of $j(z_1), \ldots, j(z_i), \exp(u_1), \ldots, \exp(u_n)$ as functions on $W$. Hence by the Zariski density these functions are independent as functions on $A_{h+1}$, and hence, by the Zariski-density of $\pi_{h+1}(A_{h+1})$ in $V_{h+1}$, we must have that $\dim V_{h+1} = \dim V + 1$. \qed

From this statement one may deduce, as explained in [15, above 5.7], the analogue of [9, Conjecture 5.10] (for $j$ itself this follows from [16]).

3.3. Theorem. Let $U' \subset U$ be a weakly special subvariety, and put $X' = \pi(U')$. Let $V \subset X'$ and $W \subset U'$ be subvarieties, and $A$ a component of $W \cap \pi^{-1}(V)$. Then

$$\dim A = \dim V + \dim W - \dim X'$$

unless $A$ is contained in a proper weakly special subvariety of $U'$. \qed
It is shown in [9] that Theorem 3.2 is equivalent by arguments using only the formal properties of the collection of weakly special subvarieties to the following version. We need the following definition from [9].

3.4. Definition. Fix a subvariety $V \subset X$.

1. A component with respect to $V$ is a complex analytically irreducible component of $W \cap \pi^{-1}(V)$ for some algebraic subvariety $W \subset U$.
2. If $A$ is a component w.r.t. $V$ we define its defect to be $\partial(A) = \dim \text{Zcl}(A) - \dim A$, where $\text{Zcl}(A)$ denotes the Zariski closure of $A$.
3. A component $A$ w.r.t. $V$ is called optimal for $V$ if there is no structly larger component $B$ w.r.t. $V$ with $\partial(B) \leq \partial(A)$.
4. A component $A$ w.r.t. $V$ is called geodesic if it is a component of $W \cap \pi^{-1}(V)$ for some weakly special subvariety $W$.

3.5. Proposition. Let $V \subset X$. An optimal component with respect to $V$ is geodesic.

Proof. The same as the proof that ‘Formulation A’ implies ‘Formulation B’ in [9]. (The proof of the reverse implication is also the same as given there.) □

4. Proof of Theorem 1.3

We start by recalling some background on trees and lattices associated to $\text{GL}_2^+(\mathbb{Q})$. Let $T_{\mathbb{Q}} = \text{PSL}_2(\mathbb{Z}) \backslash \text{PGL}_2^+(\mathbb{Q})$, where we assume their images are distinct. For a prime number $p$, $T_{\mathbb{Q}}$ maps into $T_p = \text{PSL}_2(\mathbb{Z}_p) \backslash \text{PGL}_2(\mathbb{Q}_p)$, and embeds into the product of the $T_p$ over all $p$.

Now $T_{\mathbb{Q}}$ may be identified with the space of $\mathbb{Z}$-lattices in $\mathbb{Q}^2$ up to scaling, by sending $g$ to the lattice spanned by $e_1 g, e_2 g$, where $e_1 = (1, 0), e_2 = (0, 1)$. Likewise, $T_p$ may be identified with the space of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ up to scale. Moreover, $T_p$ may be given the structure of a connected $(p + 1)$-regular tree by saying that two lattices $L, L'$ are adjacent if one can scale $L'$ to be inside $L$ with index $p$. There is a natural right action of $\text{PGL}_2(\mathbb{Q}_p)$ on $T_p$: it acts on $\mathbb{Q}_p^2$ in the natural way and thus on the lattices in it.

Since $T_p$ is a tree there is a unique shortest path between any two nodes, and any path between those nodes traverses that path.

Our proof will study curves isogenous to the curve $E_0$ whose $j$-invariant is 0. These curves have CM by $\mathbb{Z}[\zeta]$, where $\zeta = \exp(2\pi i/3)$. A point $z \in \mathbb{H}$ with $j(z) = 0$ corresponds to the elliptic curve $E_0$ together with a basis $v_1, v_2$ for its integral homology $H_1(E_0, \mathbb{Z})$. For any sub-lattice $L \subset H_1(E_0, \mathbb{Q})$ we can define an elliptic curve $E_L$ isogenous to $E_0$ which only depends on $L$ up to scale. To do this, scale $L$ until it contains $H_1(E_0, \mathbb{Z})$ and the quotient is cyclic. We can identify $Q_L = L/H_1(E_0, \mathbb{Z})$ with a subgroup of the torsion group of $E_0$ and take the quotient. Define $T'_{\mathbb{Q}}$ to be the space of lattices in $H_1(E_0, \mathbb{Q})$, up to scaling, and correspondingly $T'_p$ the space of $\mathbb{Z}_p$-lattices in $H_1(E_0, \mathbb{Q}_p)$, up to scaling.
Now suppose that $E_L$ is isomorphic to $E_0$. This implies that the quotient $Q_L$ is the same as that of the kernel of an endomorphism $x$ of $E_0$. If we identify $H_1(E,\mathbb{Z})$ with $\mathbb{Z}[\zeta]$, then the kernel of multiplication by $x$ is $(x^{-1})/\mathbb{Z}[\zeta]$, where $(m)$ denotes the fractional ideal generated by $m$. These correspond to elements of the fractional ideal group of $\mathbb{Z}_p[\zeta]$ (providing the endomorphisms giving the kernels) quotiented out by $\mathbb{Q}_p^\times$ (scaling). Explicitly we find the following.

1. If $p \equiv 1 \mod 3$ then $(p)$ has two distinct primes above it, whose product is $(p)$. Then $\mathbb{Z}_p[\zeta] = \mathbb{Z}_p \oplus \mathbb{Z}_p$ with ideal group $\mathbb{Z}^2$, which we quotient by the diagonal $\mathbb{Z}$. These nodes give a line in the tree: each such node being adjacent to two other such nodes for which the edges correspond to the two primes over $(p)$.

2. If $p \equiv 2 \mod 3$ then $\mathbb{Z}_p[\zeta] = \mathbb{Z}_{p^2}$, with ideal group $\mathbb{Z}$ which we quotient by $\mathbb{Z}$. Thus in this case there is just one node coming from curves isomorphic to $E_0$.

3. If $p = 3$ we get a ramified extension of $\mathbb{Z}_3$, which still has ideal group $\mathbb{Z}$ (generated by powers of the uniformiser) but now we quotient by $2\mathbb{Z}$ since 3 has valuation 2. We thus have two nodes coming from curves isomorphic to $E_0$, which are adjacent in the tree.

Note that in every case there is at least one node $N'$ of $T'_p$ adjacent to $H_1(E_0,\mathbb{Z})$ such that any lattice $L$ for which the shortest path from $H_1(E_0,\mathbb{Z})$ to $L$ goes through $N'$ is not isomorphic to $E_0$.

4.1. Proposition. Let $g_1, \ldots, g_k \in \text{GL}_2^+(\mathbb{Q})$ and suppose that the functions $j(g_i z)$ are distinct. Then there exists $z \in \mathbb{H}$ such that $j(g_i z) = 0$ for exactly one $i$.

Proof. First suppose that there exists a prime number $p$ such that the images of the $g_i$ in $T'_p$ are distinct. Without loss of generality we may assume that $g_1, g_2$ have images $u_1, u_2$ in $T_p$ whose distance is at least as large as that between the images of any distinct $g_i, g_k$. This implies there is a unique node $N$ adjacent to $u_1$ such that the shortest path from $g_1$ to any other $g_i$ goes through $N$. We may further suppose without loss of generality that $g_1 = 1$.

Fixing a basis $v_1, v_2$ for $H_1(E_0,\mathbb{Z})$ gives a map from $T_p$ to $T'_p$, sending $\mathbb{Z}^2$ to $H_1(E_0,\mathbb{Z})$. By choosing $v_1, v_2$ appropriately we can send $N$ to $N'$. It follows that the $z$ with $j(z) = 0$ corresponding to this choice has $j(g_i z) \neq 0$ for all $i > 1$.

Now we give the proof without the simplifying assumption. While no single $p$ may separate all the $g_i$, finitely many $p$ do. Let $S = \{g_1, \ldots, g_k\}$. Consider the image of $S$ in $T_2$ and pick two nodes with maximal distance among images of pairs from $S$. Let $u_2$ be one of these “extremal” nodes, and let $S_2$ be the subset of $S$ whose image in $T_2$ is $u_2$.

Now consider the image of $S_2$ in $T_3$, choose an extremal node $u_3$ and let $S_3$ be the subset of $S_2$ whose image in $T_3$ is $u_3$. After finitely many steps we arrive at a set $S_p$ with only a single element. We may assume this element is $g_1$ and that $g_1 = 1$.

For each prime $q \leq p$ we let $N_q$ be the unique node adjacent to $u_q$ through which all paths from $u_q$ to other images $S_r$ go, where $r$ is the prime preceding $q$ (or $r = 0$ for $p = 2$).
Choose a basis $v_1, v_2$ of $H_1(E_0, \mathbb{Z})$ such that the induced map from $T_q$ to $T'_q$ takes $N_q$ to $N'_q$ for all $q \leq p$. The fact that this is possible amounts to the fact that $\text{SL}_2(\mathbb{Z})$ subjects onto $\text{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for every $n$.

The claim now is that, for each $i > 1$, $j(g_iz) \neq 0$. To see this, let $q < p$ be the largest prime such that $g_i \in S_q$, and $q' \leq p$ the next prime after $q$. The above argument in the tree $T'_q$ shows that $g_iz$ does not represent $E_0$. This proves the claim and the proposition follows.

4.2. Proof of Theorem 1.3. Theorem 1.3 follows directly from Proposition 4.1.

5. Arithmetic estimates

The proof of Theorem 1.2, and further results considered in the sequel, use some basic arithmetic estimates which are gathered here. Several of them were used for similar purposes in [8]. The absolute logarithmic Weil height of a non-zero algebraic number $\alpha$ is denoted $h(\alpha)$; the absolute Weil height is $H(\alpha) = \exp h(\alpha)$.

Constants $c_1, c_2, \ldots$ here and in the sequel are positive and absolute (though not necessarily effective!), and have only the indicated dependencies (e.g. $c_3(\epsilon)$ is a constant depending on $\epsilon$).

Weak Lehmer inequality

A lower bound for the height by any fixed negative power of the degree suffices for our purposes. Loher has proved (see [11]): if $[K : \mathbb{Q}] = d \geq 2$ and $0 \neq \alpha \in K$ is not a root of unity then

$$h(\alpha) \geq \frac{1}{37} d^{-2} (\log d)^{-1}.$$ (5.1)

Singular moduli

For a singular modulus $\sigma$, we denote by $R_\sigma$ the associated quadratic order and $D_\sigma = D(R_\sigma)$ its discriminant. Habegger [7, Lemma 1] shows that

$$h(\sigma) \geq c_1 \log |D_\sigma| - c_2,$$ (5.2)

based on results of Colmez and Nakkajima-Taguchi.

Now no singular modulus is a root of unity (we thank Gareth Jones for pointing this out: a singular modulus has a Galois conjugate which is real, but $\pm 1$ are not singular moduli by inspecting the list of rational singular moduli e.g. in [20, A.4]. Finiteness follows ineffectively from my AO paper and it also follows from Habegger’s result [7] that only finitely many are algebraic units; effectively it follows from Paulin [12], who also gives a different proof elsewhere).
This together with Kronecker’s theorem imply, for a non-zero singular modulus \( \sigma \),

\[(5.3) \quad h(\sigma) > c_4.\]

In the other direction ([8], Lemma 4.3), for all \( \epsilon > 0 \),

\[(5.4) \quad h(\sigma) \leq c_3(\epsilon)|D_\sigma|^{\epsilon}.\]

Finally, we note that if \( \tau \) is a pre-image of a singular modulus \( \sigma \) in the classical fundamental domain for the \( \text{SL}_2(\mathbb{Z}) \) action then (see [14, 5.7])

\[(5.5) \quad H(\tau) \leq 2D_\sigma.\]

Class numbers of imaginary quadratic fields

The class number of an imaginary quadratic order \( R \) will be denoted \( \text{Cl}(R) \). Recall that, for a singular modulus \( \sigma \), \([\mathbb{Q}(\sigma) : \mathbb{Q}] = \text{Cl}(R_\sigma)\). By Landau-Siegel, for every \( \epsilon > 0 \),

\[(5.6) \quad \text{Cl}(R) \geq c_4(\epsilon)|D(R)|^{\frac{1}{2}-\epsilon}.\]

In the other direction,

\[(5.7) \quad \text{Cl}(R) \leq c_5(\epsilon)|D(R)|^{\frac{1}{2}+\epsilon}\]

with \( c_5(\epsilon) \) explicit (see e.g. Paulin [12], Prop. 2.2 for a precise statement).

Faltings height of an elliptic curve

Let \( E \) be an elliptic curve defined over a number field. Let \( h_F(E) \) denote the semi-stable Faltings height of \( E \), and \( j_E \) its \( j \)-invariant. Then ([21, 2.1]; see also [6])

\[(5.8) \quad |h(j_E) - \frac{1}{12}h_F(E)| \leq c \log \max\{2, h(j_E)\}\]

with an absolute constant \( c \).

Further, if \( E_1, E_2 \) are elliptic curves defined over a number field with a cyclic isogeny of order \( N \) between them (i.e. \( \Phi_N(j_{E_1}, j_{E_2}) = 0 \)) then ([19, 2.1.4]; see also [8, Proof of lemma 4.2])

\[(5.9) \quad |h_F(E_1) - h_F(E_2)| \leq \frac{1}{2} \log N.\]

Isogeny estimate

Let \( K \) be a number field with \( d = \max\{2, [K : \mathbb{Q}]\} \). Let \( E, E' \) be elliptic curves defined over \( K \), with \( h_F(E) \) and \( h_F(E') \) their semi-stable Faltings heights. Pellarin, [13, Theorem 2]) proves the following.
If $E, E'$ are isogenous then there exists an isogeny $E \to E'$ of degree $N$ satisfying
\begin{equation}
N \leq 10^{78} d^4 \max\{1, \log d\}^2 \max\{1, h_F(E)\}^2.
\end{equation}

**Estimate for the height of a multiplicative dependence**

The following result, due to Yu (see [11]), allows us to get control of the height of a multiplicative relation on our singular moduli in terms of their height. It is thus a kind of “multiplicative isogeny estimate”.

Let \( \alpha_1, \ldots, \alpha_n \) be multiplicatively dependent non-zero elements of a number field \( K \) of degree \( d \geq 2 \). Suppose that any proper subset of the \( \alpha_i \) is multiplicatively independent. Then there exist rational integers \( b_1, \ldots, b_n \) with \( \alpha_1 b_1 \cdots \alpha_n b_n = 1 \) and
\begin{equation}
|b_i| \leq c_7(n) d^n \log d h(\alpha_1) \cdots h(\alpha_n) / h(\alpha_i), \quad i = 1, \ldots, n.
\end{equation}

6. Proof of Theorem 1.2

Fix \( n \) and suppose Conjecture 1.4 holds for modular curves in \( \mathbb{C}^n \). Let \( X = Y(1)^n \times \mathbb{G}_m^n \),
\[ V = \{(x_1, \ldots, x_n, t_1, \ldots, t_n) \in X : t_i = x_i, i = 1, \ldots, n\}. \]
So \( \dim V = \text{codim} V = n \) and a singular-dependent \( n \)-tuple \( (x_1, \ldots, x_n) \) gives rise to an atypical point \( (x_1, \ldots, x_n, x_1, \ldots, x_n) \in V \).

**Lemma.** A singular-dependent \( n \)-tuple may not be contained in an atypical component of \( V \) of positive dimension.

**Proof.** A singular-dependent tuple can never be contained in a special subvariety of \( X \) defined by two (independent) multiplicative conditions, for between them we could eliminate one coordinate, contradicting the minimality.

Now a special subvariety of the form \( M \times \mathbb{G}_m^n \), where \( M \) is a special subvariety of \( Y(1)^n \) can never intersect \( V \) atypically; neither can one of the form \( Y(1)^n \times T \) where \( T \) is a special subvariety of \( \mathbb{G}_m^n \).

Let us consider a special subvariety of the form \( M \times T \) where \( T \) is defined by one multiplicative condition. The intersection of \( M \times T \) with \( V \) consists of those \( n \)-tuples of \( M \) which belong to \( T \). This would typically have dimension \( \dim M - 1 \), and so to be atypical we must have \( M \cap \mathbb{G}_m^n \subset T \). Now Theorem 1.3 implies that \( M \) has two identically equal coordinates, but then cannot contain a singular-dependent tuple. \( \square \)

6.2. **Proof of Theorem 1.2.** If \( \sigma = j(\tau) \) is a singular modulus, so that \( \tau \in \mathbb{H} \) is quadratic over \( \mathbb{Q} \), we define its **complexity** \( \Delta(\sigma) \) to be the absolute value of the discriminant of \( \tau \) i.e. \( \Delta(\sigma) = |D_\sigma| = |b^2 - 4ac| \) where \( ax^2 + bx + c \in \mathbb{Z}[x] \) with \( (a, b, c) = 1 \) has \( \tau \) as a root. For a tuple \( (\sigma_1, \ldots, \sigma_n) \) of singular moduli we define the complexity of \( \sigma \) to be \( \Delta(\sigma) = \max(\Delta(\sigma_1), \ldots, \Delta(\sigma_n)) \).
Now suppose that $V$ contains a point corresponding to a singular-dependent $n$-tuple of sufficiently large complexity, $\Delta$. By Landau-Siegel (5.6) with $\epsilon = 1/4$, such a tuple has, for sufficiently large (though ineffective) $\Delta$, at least $c_5 \Delta^{1/4}$ conjugates over $\mathbb{Q}$. Each is a singular-dependent $n$-tuple, and they give rise to distinct points in $V$.

Let $F_j$ be the standard fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and $F_{\text{exp}}$ the standard fundamental domain for the action of $2\pi i \mathbb{Z}$ (by translation) on $\mathbb{C}$.

We now consider the sets

$$X = \{(z, u, r, s) \in F^n_j \times F^n_{\text{exp}} \times \mathbb{R}^n \times \mathbb{R} : j(z) = \exp(u), r \cdot u = 2\pi i s\}$$

so that $(j(z), \exp(u)) \in V$ for $(z, u, r, s) \in X$ and

$$Z = \{(z, r, s) \in F^n_j \times \mathbb{R}^n \times \mathbb{R} : (z, u, r, s) \in X\}.$$

Then $Z$ is a definable set in the o-minimal structure $\mathbb{R}_{\text{an}}^{\text{exp}}$.

A singular-dependent $n$-tuple $\sigma \in V$ has a pre-image

$$\tau = (z_1, \ldots, z_n, u_1, \ldots, u_n) \in F^n_j \times F^n_{\text{exp}},$$

and this gives rise to a point in $Z$, where the coordinates in $\mathbb{R}^{n+1}$ register the multiplicative dependence of the tuple, as follows. The $F_j$ coordinates are the $z_i$, so they are quadratic points, and as recalled in (5.5) their absolute height is bounded by $2\Delta(\sigma_i)$. The point in $\mathbb{R}^{n+1}$ has integer coordinates $(b_1, \ldots, b_n, b)$, not all zero, such that

$$\sum_{i=1}^n b_i u_i = 2\pi i b.$$

By (5.11), the $b_i$ in a multiplicative relation among the $\sigma_i$ may be taken to be bounded in size by $c_7(n)\Delta^n$ and since the imaginary parts of the $u_i$ are bounded by $2\pi i$, we find that the height of $(z_1, \ldots, z_n, b_1, \ldots, b_n, b)$ is bounded by $c_9(n)\Delta^n$.

In view of the Galois lower bound, a singular-dependent $n$-tuple of complexity $\Delta$ gives rise to at least

$$T \frac{1}{\Delta^n} \text{ quadratic points on } Z \text{ with absolute height at most } T = c_{10}(n)\Delta^n.$$

For sufficiently large $\Delta$, the Counting Theorem (see [14, 3.2]) applied to quadratic points on $Z$ (considered in real coordinates) implies that it contains a semi-algebraic set of positive dimension. This implies (by the arguments used in [8, 9]) that there is a complex algebraic $Y \subset U$ which intersects $Z$ in a positive-dimensional component $A$ which is atypical in dimension and contains singular-dependent $n$-tuples.

By the mixed Ax-Schanuel of §3 this implies that there is a positive-dimensional weakly special subvariety $W$ containing $Y$ containing a component $B$ with $A \subset B$ and $\partial(B) \leq \partial(A)$. Moreover, it contains the special subvarieties that contain (some of) the singular-dependent points, so $W$ is a special subvariety of positive dimension containing singular-dependent points of $V$, which we have seen is impossible.

So $\Delta$ is bounded, giving the finiteness. $\square$
6.3. Example. An example of a singular-dependent 5-tuple is (see [20, A.4]):

\((-2^{15}3^{3}5^{3}11^{3}, -2^{15}, 2^{3}3^{3}11^{3}, 2^{6}3^{3}, 2^{15}9^{15}3^{3}).\)

One also has a 3-tuple \((-2^{15}, -2^{15}3^{3}, 2^{6}3^{3})\) and 4-tuple \((2^{4}3^{3}5^{3}, -2^{15}3^{15}3^{3}, -3^{3}5^{3}, 2^{6}5^{3}).\)

7. On the atypical set of \(V_n\)

The atypical set of \(V_n\) is the union of its proper optimal components (\(V_n\) itself is always optimal but never atypical). Since optimal components are geodesic-optimal (2.7), we will investigate the possibilities for these.

We observe that any geodesic-optimal components which dominate every coordinate can only come from an optimal strongly special subvariety. The finiteness of these, even if we cannot identify them, is guaranteed by \(o\)-minimality.

7.1. Definition. Complex numbers \(x, y\) will be called isogenous if \(\Phi_N(x, y) = 0\) for some \(N \geq 1\). I.e., if the elliptic curves with \(j\)-invariants \(x\) and \(y\) are isogenous.

7.2. Geodesic-optimal components of dimension \(n\)

As already observed, \(V_n\) is not atypical since it dominates both \(Y(1)^n\) and \(G_m^n\). In other words, the defect of \(V_n\) is equal to its codimension.

7.3. Geodesic-optimal components of dimension \(n - 1\)

Let \(T \subset X\) be a geodesic subvariety of co-dimension 2. Can \(T \cap V\) have dimension \(n - 1\)? There are two equations defining \(T\), each being one of four possible types: a single modular relation, a constant modular coordinate, a single multiplicative relation, a constant multiplicative coordinate.

Now if both equations are of modular (respectively multiplicative) type we never get an atypical component, because \(V\) dominates \(Y(1)^n\) (respectively \(G_m^n\)). The same is true for any \(T\) which is defined purely by modular (respectively multiplicative) relations.

So we consider \(T\) defined by one condition of each type. Let us call \(T_1\) the projection of \(T\) to the \(Y(1)^n\) factor, which is a geodesic subvariety of codimension 1, and \(T_2\) its projection to \(G_m^n\). We get an atypical component of dimension \(n - 1\) if either \(T_1 \cap G_m^n\) is contained in \(T_2\), or if \(T_2\) is contained in \(T_1\) (i.e. when both are considered in the same copy of \((C^\times)^n\)).

If the modular condition is a modular relation (rather than a constant coordinate) then the first is excluded by Conjecture 1.4 for \(n = 2\), which we have affirmed above, unless it is of the form \(x_i = x_j\). If the multiplicative relation is not a fixed coordinate, the other inclusion is also impossible unless it is of the form \(t_i = t_j\).

So we are reduced to considering constant coordinate conditions on both sides. This obviously leads to a component of dimension \(n - 1\) if the conditions coincide: \(x_i = \xi = t_i\). However such a component can only be atypical (i.e. arise from the intersection of \(V_n\) with a special subvariety of codimension (at most) 2 if \(\xi\) is both a singular modulus and a root of unity. But this never occurs, as remarked in §5.
This establishes ZP for $V_1$, which is the curve defined by $x_1 = t_1$ in $\mathbb{C} \times \mathbb{C}^\times$. And it shows that $V_2$ has no atypical subvarieties of positive dimension apart from the “diagonal” $x_1 = x_2$.

7.4. Proposition. ZP holds for $V_2$.

**Proof.** In view of the fact that the only atypical component of positive dimension is the “diagonal”, which has defect zero, we are reduced to showing that $V_2$ has only finitely many optimal points, i.e. points which are atypical but not contained in the “diagonal”. A point $(x_1, x_2, x_1, x_2) \in V_2$ is atypical if it lies on a special subvariety of codimension 3. There are then two cases: we have two independent modular conditions and one multiplicative, or two multiplicative and one modular relation.

The former case is exactly the question of singular-dependent 2 tuples, whose finiteness we have already established. The latter leads to the question of two (unequal) roots of unity which satisfy a modular relation. This is established in the following proposition, by a similar argument to that used in 5.2; and with this the proof is complete. □

We may observe that the optimal points of $V_2$ satisfy 3 special relations (never 4), so have “defect” 1.

7.5. Definition. A pair of distinct roots of unity is called a *modular pair* if they satisfy a modular relation.

7.6. Proposition. There exist only finitely many modular pairs.

**Proof.** Let $(\zeta_1, \zeta_2)$ be such a point, where the order of $\zeta_i$ is $M_i$ and $\Phi_L(\zeta_1, \zeta_2) = 0$. The point is that the order of the root of unity, and their bounded height, leads to a bound on the degree of the modular relation. Specifically, by (5.8), the semi-stable Faltings height of the corresponding elliptic curves $E_1, E_2$ with $j$-invariants $\zeta_1, \zeta_2$ are bounded, and so by the isogeny estimate (5.10) there is a modular relation $\Phi_N(\zeta_1, \zeta_2) = 0$ with $N \leq c_{11} \max\{M_1, M_2\}^5$. Thus such a point leads to a rational point on a suitable definable set whose height is bounded by a polynomial in the orders of the two roots, and if it is of sufficiently large complexity it forces the existence of a higher dimensional atypical intersection containing such points. But the only atypical set of dimension 1 is given by $x_1 = x_2, t_1 = t_2$. □

As modular relations always subsist between two numbers, there is no notion of “modular-multiplicative $n$-tuples” analogous to singular-dependent tuples. However, an immediate consequence of the above is that, for any $n$, there exists only finitely many $n$-tuples of distinct roots of unity which are pairwise isogenous (and none for sufficiently large $n$).
7.7. Geodesic-optimal components of dimension $n - 2$

These arise from intersecting $V_n$ with a geodesic subvariety $T$ of codimension (at least) 3. We must have at least 1 relation of each type, and if they are all of “non-constant” type (no fixed coordinates) then we get finiteness by $o$-minimality.

If there is one constant condition, this immediately gives a second such condition of the other type, and then any additional non-constant condition (i.e. not forcing any further constant coordinates) will give a component of dimension $n - 2$. However, no such component can be atypical.

Consider the case of 3 constant conditions. First the case of two fixed modular coordinates. This will give rise to an atypical intersection if the two fixed values are multiplicatively related. Next the case of two fixed multiplicative coordinates. This will give rise to an atypical component if the two fixed values are isogenous. The finiteness of such components follows from ZP for $V_2$, and they all have defect 2.

We conclude:

7.8. Proposition. For $n \geq 1$, $V_n$ has only finitely many maximal atypical components of dimension $n - 2$.

But for $n = 3$ we can in fact exclude “strongly atypical” altogether. Such a component has one of two shapes:

1. two modular relations and one multiplicative relation. This would be atypical if the resulting modular curve satisfied the multiplicative relation, which is impossible by our affirmation of conjecture 1.4 for $n = 3$.
2. two multiplicative relations and one modular relation. This gives a “multiplicative curve”, which can be parameterised as $(\zeta_1 t^{a_1}, \zeta_2 t^{a_2}, \zeta_3 t^{a_3})$, where $\zeta_i$ are roots of unity and $a_i$ integers. As the $\Phi_N, N \geq 2$ are symmetric, two coordinates cannot satisfy a modular equation unless $a_i = a_j$ (so that $N = 1$ and $\Phi_1 = X - Y$) and $\zeta_i = \zeta_j$.

7.9. Proposition. The positive dimensional atypical components of $V_3$ and their defects may be described as follows:

1. The intersection of $V_3$ with $x_i = x_j, i \neq j$ is a copy of $V_2$ contained in $X_2$ (hence of defect 2) and has some atypical points in it, which have defect 1. It contains also the subvariety with $x_1 = x_2 = x_3$, which has defect 0.
2. A singular-dependent 2-tuple $\sigma = (\sigma_1, \sigma_2)$ give rise to an atypical component $A_\sigma$ of dimension 1 and defect 2. [There may exist singular moduli which belong to two distinct such pairs $\sigma, \sigma'$. Then we get a point $(A_\sigma \cap A_{\sigma'})$ of defect 1.]
3. A modular pair $\zeta = (\zeta_1, \zeta_2)$ gives rise to an atypical component $B_\zeta$ of dimension 1 and defect 2. [There may exist roots of unity belonging to two distinct modular pairs $\zeta, \zeta'$. Then we get a point $(B_\zeta \cap B_{\zeta'})$ of defect 1.]

In particular, there are no positive dimensional “strongly atypical” components.
Thus ZP for $V_3$ depends on the finiteness of its atypical points off all the above positive dimensional atypical components. This leads to some Diophantine questions which would establish ZP for $V_3$, which we study in the next section.

7.10. Remark. Note that $V_n$ contains families of atypical weakly special subvarieties defined by imposing conditions of the form $x_i = x_j$ (and $t_i = t_j$) or $x_k = t_k = c_k \in \mathbb{C}^\times$ for various choices of $(i, j), i \neq j, k$. If $m$ such conditions are imposed, the resulting weakly special subvariety has dimension $2n - m$ and intersects $V_n$ in a component of dimension $n - m$, so has defect $n - m$.

7.11. Conjecture. The atypical geodesic components described in 7.10 give all geodesic optimal subvarieties of $V_n$ for any $n$; in particular there are no “strongly optimal” components.

8. Optimal points in $V_3$

The optimal points in $(x_1, x_2, x_3, x_1, x_2, x_3) \in V_3$ fall into two classes. Those which are atypical in satisfying at least 4 special conditions, but are not contained in atypical component of higher dimension. And those which are “more atypical”, satisfying 5 special conditions (it is not possible to have 6: only a triple of singular moduli which were also roots of unity could achieve this), though lying in an atypical set of larger dimension but larger defect. Those lying on diagonals $x_i = x_j, i \neq j$ are easy to describe, we consider here those that don’t.

Let us first consider points satisfying 5 special conditions. These also fall into two types: 3 modular, 2 multiplicative, or the other way around. If there are 3 modular conditions then each $x_i$ is a singular modulus. The two multiplicative conditions mean either than one $x_j$ is torsion, and the other two multiplicatively related, or the three are pairwise multiplicatively related. The former is impossible. Now only finitely many pairs of singular moduli have a multiplicative relation, so $x_1, x_2$ comes from a finite set, and $x_3$ comes also from a finite set. If there are three multiplicative relations then each $x_i$ is torsion. Only finitely many pairs of (distinct) roots of unity satisfy isogenies, and we get finiteness (there are no “isogenies” involving three points!). All these points have defect 1.

Now we consider points $(x_1, x_2, x_3, x_1, x_2, x_3) \in V_3$, away from positive dimensional atypical subvarieties, satisfying 4 special conditions. The “generic” situation involves no singular moduli or roots of unity.

8.1. Problem. Prove that there exist only finitely many triples $x_1, x_2, x_3$ of distinct non-zero algebraic numbers, which are not roots of unity and not singular moduli, such that they are pairwise isogenous, and also pairwise multiplicatively dependent.

The various arithmetic estimates seem insufficient to get a lower degree bound in terms of the “complexity”: the degrees of the two isogenies and the heights of the two multiplicative relations. This seems to be problem of a similar nature to that encountered in [8] dealing with curves which are not “asymmetric”.

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There are three “non-generic” variations of which we can resolve two. The modular relations may take the form that one coordinate is singular, the other two isogenous. Up to permutations we may assume the singular coordinate is $x_1$.

8.2. Proposition. There exist only finitely many triples $x_1, x_2, x_3$ of distinct non-zero algebraic numbers such that $x_3$ is a root of unity, $x_1, x_2$ are multiplicatively dependent, and the three points are pairwise isogenous.

Proof. Define the complexity $\Delta$ of such a triple to be the maximum of: the order $M$ of the root of unity $x_3$ and the minimum degrees of isogenies $N_1, N_2$ between $x_3$ and $x_1, x_2$, respectively. By (5.8), the stable Faltings height of an elliptic curves whose $j$-invariant is a root of unity is absolutely bounded. Now by (5.9), $h(x_j) \ll (1 + \log \max\{N_j\})$, $j = 1, 2$, so by (5.10) the degrees $d_j = [\mathbb{Q}(x_3, x_j) : \mathbb{Q}] \gg N_j^{1/5}$. By (5.11) and (5.1) (to get a lower bound for $h(x_i)$) the height of a multiplicative relation between $x_1, x_2$ is bounded by some $c_{12}\Delta^{c_{13}}$. And $[\mathbb{Q}(x_3) : \mathbb{Q}] \gg M$.

Thus, a triple of complexity $\Delta$ gives rise to “many” (i.e. at least $c_{14}\Delta^{c_{15}}$) quadratic points on a certain definable set, and so all but finitely many such points lie on atypical components of positive dimension.

But no such triples lie on positive dimensional atypical components: By 7.9, such components have either two singular coordinates or two modular coordinates, so the conditions on our triples would then force all $x_i$ to be singular, which is impossible (as then $x_3$ cannot be torsion) or all torsion, which leads to the same impossible requirement for $x_1$. $\square$

Symmetrically, we may have that the multiplicative relations take the form that one coordinate is a root of unity, the other two being multiplicatively dependent. We seem unable to establish finiteness here, so we pose it as a problem.

8.3. Problem. Prove that there exist only finitely many triples $x_1, x_2, x_3$ of distinct non-zero algebraic numbers such that $x_1$ is singular, $x_2, x_3$ are isogenous, and the three are pairwise multiplicatively dependent.

Finally, we have the following.

8.4. Proposition. There exist only finitely many triples $x_1, x_2, x_3$ of distinct non-zero algebraic numbers such that

1. $x_1$ is a singular modulus, $x_2, x_3$ are isogenous, and
2. $x_3$ is a root of unity, $x_1, x_2$ are multiplicatively dependent.

Proof. Let $D$ be the discriminant of $x_1$ (see §5), and $M$ the (minimal) order of $x_3$. Take $N$ minimal with $\Phi_N(x_2, x_3) = 0$, and $B$ minimal for a non-trivial multiplicative relation $x_1^{b_1}x_2^{b_2} = 1$ with $B = \max\{b_1, b_2\}$. Set $\Delta = \max\{|D|, M, N\}$ to be the complexity of the tuple $(x_1, x_2, x_3)$.

Let $E_\xi$ be the elliptic curve with $j$-invariant $\xi$. As in the proof of 8.2, $h_F(E_{x_2})$ is bounded by some absolute $c_{16}$. Then, as earlier, $N \leq c_{17}([\mathbb{Q}(x_2, x_3) : \mathbb{Q}])^5$. Also $M \ll [\mathbb{Q}(x_3) : \mathbb{Q}]$, and $|D| \ll [\mathbb{Q}(x_1) : \mathbb{Q}]^4$ by (5.6).
Arguing as in [8], the height inequalities (5.8, 5.9) imply that \( h(x_2) \) is bounded above by \( c_{18}(1 + \log N) \). By the Weak Lehmer estimate (5.1) it is bounded below by \( c_{19}d^{-3} \). Corresponding estimates for \( h(x_1) \) are provided by (5.4) and (5.3). Therefore (5.11) ensures that

\[
B \leq c_{20}d^3D.
\]

The rest of the proof is the same as the proof of 8.2. \( \square \)

Thus 8.1 and 8.3 imply (and are implied by) ZP for \( V_3 \).

If one takes two complex numbers and three conditions, then either two “modular” conditions or two “multiplicative” special conditions will force the points to be special, and one can prove finiteness. However one can consider two complex numbers satisfying a special condition of each of three different types.

8.5. Problems. Prove that there are only finitely many pairs of distinct non-zero algebraic numbers \( x_1, x_2 \) in each situation.

1. \( x_1, x_2 \) are isogenous, and multiplicatively dependent, and are also isogenous for some other Shimura curve.
2. \( x_1, x_2 \) are isogenous, and multiplicatively dependent, and the points with these \( x \)-coordinates are dependent in some specific elliptic curve.
3. As in the previous problems, but with more or different conditions: say the points are isogenous/dependent for 10 pairwise incommensurable Shimura curves.

Finally we state a conjecture on the height of “typical” intersections of mixed multiplicative-modular type under which 8.1 and 8.3 are affirmed. This is along the lines of a conjecture of Habegger [6], itself an analogue of the “Bounded Height Conjecture” for \((\mathbb{C}^*)^n\) formulated by Bombieri-Masser-Zannier [4] and proved by Habegger [5].

8.6. Definition. A modular-dependent pair is a point \((x, y) \in (\mathbb{C}^*)^2\) such that there exists integers \( N, a, b, c \) with \( N, c \geq 1 \) and \( \gcd(a, b) = 1 \) such that

\[
\Phi_N(x, y) = 0, \quad (x^ay^b)^c = 1.
\]

The complexity \( \Delta(x, y) \) of such a pair is the minimum of \( \max(N, |a|, |b|, c) \) over all \( N, a, b, c \) for which the above equations hold for \( x, y \).

8.7. Conjecture. For \( \epsilon > 0 \) we have \( h(x), h(y) \leq c_{\epsilon}\Delta(x, y)^{\epsilon} \) for all modular-dependent pairs \((x, y)\).

8.8. Proposition. Assume Conjecture 8.7. Then finiteness holds in 8.1 and 8.3.

Proof. Let \((x, y)\) be a modular-dependent pair with complexity \( \Delta = \Delta(x, y) \). We may assume that neither \( x \) nor \( y \) are roots of unity. Constants denoted \( C \) are absolute but may vary at each occurrence.

Let \( E_x, E_y \) be elliptic curves with \( j \)-invariants \( x, y \) and semistable Faltings heights \( h_F(x) = h_F(E_x) \) and \( h_F(y) = h_F(E_y) \) respectively. Then \( E_x, E_y \) may both be defined over \( \mathbb{Q}(x, y) \), and we set \( d = [\mathbb{Q}(x, y) : \mathbb{Q}] \).
By Pellarin’s isogeny estimate (5.10), \( N \leq C d^4 \max(1, \log d)^2 \max(1, h_F(x))^2 \). Now \( h_F(x) \) and \( h(x) \) differ by at most \( C \log \max(2, h(x)) \). So

\[
N \leq C d^4 \max(1, \log d)^2 (1 + h(x) + C \log \max(2, h(x))).
\]

We have \( d^4 \max(1, \log d)^2 \leq d^6 \), and under our conjecture (with \( \epsilon = 1/10 \)) we have

\[
N \leq C d^6 \Delta^{1/10}.
\]

By a Weak Lehmer inequality (5.1) \( h(x) \geq C d^{-3} \), \( h(y) \geq C d^{-3} \). Since neither \( x, y \) is a root of unity, we find (5.11) that there exists a non-trivial multiplicative relation \( x^\alpha y^\beta = 1 \) with

\[
|\alpha| \leq C d^3 h(y) \leq C d^3 \Delta^{1/10}, \quad |\beta| \leq C d^3 h(x) \leq C d^3 \Delta^{1/10}.
\]

Again since \( x, y \) are not roots of unity, we have that \( (\alpha, \beta) \) is a multiple of \( (ca, cb) \). So we find that

\[
|a|, |b|, c \leq C d^3 \Delta^{1/10}.
\]

Now \( \Delta = \max(N, |a|, |b|, |c|) \) and so combining the various inequalities we find

\[
\Delta \leq d^7.
\]

Now points \( x_1, x_2, x_3 \) as in Problem 8.1 give rise to rational points on some suitable definable set of height at most \( \max(\Delta(x_1, x_2), \Delta(x_2, x_3), \Delta(x_1, x_3)) \). This lower estimate for the degree is then suitable to complete a finiteness proof for isolated points of this form by point-counting and o-minimality as in the proofs of 1.2, 8.2, and 8.4. The argument for 8.3 is similar. \( \Box \)

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