Abstract

The question of spontaneous apparition of singularity in the 3D incompressible Euler equations is one of the most important and challenging open problems in mathematical fluid mechanics. In this survey article we review some of recent approaches to the problem. We first review Kato's classical local well-posedness result in the Sobolev space and derive the celebrated Beale-Kato-Majda criterion for finite time blow-up. Then, we discuss recent refinements of the criterion as well as geometric type of theorems on the sufficiency condition for the regularity of solutions. After that we review results excluding some of the scenarios leading to finite time singularities. We also survey studies of various simplified model problems. A dichotomy type of result between the finite time blow-up and the global in time regular dynamics is presented, and a spectral dynamics approach to study local in time behaviors of the enstrophy is also reviewed. Finally, progresses on the problem of optimal regularity for solutions to have conserved quantities are presented.
1 Introduction

The motion of homogeneous incompressible ideal fluid in a domain $\Omega \subset \mathbb{R}^n$ is described by the following system of Euler equations.

\[
\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\nabla p, \quad (x, t) \in \Omega \times (0, \infty) \\
\text{div } v &= 0, \quad (x, t) \in \Omega \times (0, \infty) \\
v(x, 0) &= v_0(x), \quad x \in \Omega
\end{aligned}
\]
where \( v = (v^1, v^2, \cdots, v^n) \), \( v^j = v^j(x, t) \), \( j = 1, 2, \cdots, n \), is the velocity of the fluid flows, \( p = p(x, t) \) is the scalar pressure, and \( v_0(x) \) is a given initial velocity field satisfying \( \text{div} \ v_0 = 0 \). Here we use the standard notion of vector calculus, denoting

\[
\nabla p = \left( \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \cdots, \frac{\partial p}{\partial x_n} \right), \quad (v \cdot \nabla) v^j = \sum_{k=1}^{n} v^k \frac{\partial v^j}{\partial x_k}, \quad \text{div} \ v = \sum_{k=1}^{n} \frac{\partial v^k}{\partial x_k}.
\]

The first equation of (E) follows from the balance of momentum for each portion of fluid, while the second equation can be derived from the conservation of mass of fluid during its motion, combined with the homogeneity (constant density) assumption on the fluid. The system (E) is first derived by L. Euler in 1755 ([77]). Unless otherwise stated, we are concerned on the Cauchy problem of the system (E) on \( \Omega = \mathbb{R}^n \), but many of the results presented here are obviously valid also for \( \Omega = \mathbb{R}^n / \mathbb{Z}^n \) (periodic domain), and even for the bounded domain with the smooth boundary with the boundary condition \( v \cdot \nu = 0 \), where \( \nu \) is the outward unit normal vector. We also suppose \( n = 2 \) or \( 3 \) throughout this paper. In this article our aim to survey recent results on the mathematical aspects the 3D Euler equations closely related to the problem of spontaneous apparition of singularity starting from a classical solutions having finite energy. If we add the dissipation term \( \mu \Delta v = \mu \sum_{j=1}^{n} \frac{\partial^2 v_j}{\partial x_j^2} \), where \( \mu > 0 \) is the viscosity coefficient, to the right hand side of the first equation of (E), then we have the Navier-Stokes equations, the regularity/singularity question of which is one of the seven millennium problems in mathematics. In this article we do not treat the Navier-Stokes equations. For details of mathematical studies on the Navier-Stokes equations see e.g. [144, 57, 112, 84, 107, 116, 109]. We also omit other important topics such as existence and uniqueness questions of the weak solutions of the 2D Euler equations, and the related vortex patch problems, vortex sheet problems, and so on. These are well treated in the other papers and monographs([116, 37, 45, 112, 133, 135, 153, 154, 148, 139]) and the references therein. For the survey related the stability question please see for example [79] and references therein. For the results on the regularity of the Euler equations with uniformly rotating external force we refer [2], while for the numerical studies on the blow-up problem of the Euler equations there are many articles including [101, 102, 94, 7, 80, 11, 89, 90, 91, 127]. For various mathematical and physical aspects of the Euler equations there are many excellent books, review articles including [11, 8, 45, 47, 49, 79, 86, 115, 116, 118, 29, 152].
Obviously, the references are not complete mainly due to author’s ignorance.

1.1 Basic properties

In the study of the Euler equations the notion of vorticity, $\omega = \text{curl } v$, plays important roles. We can reformulate the Euler system in terms of the vorticity fields only as follows. We first consider the 3D case. Let us first rewrite the first equation of (E) as

$$\frac{\partial v}{\partial t} - v \times \text{curl } v = -\nabla (p + \frac{1}{2}|v|^2).$$  \hfill (1.1)

Then, taking curl of (1.1), and using elementary vector identities, we obtain the following vorticity formulation:

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = \omega \cdot \nabla v,$$  \hfill (1.2)

$$\text{div } v = 0, \quad \text{curl } v = \omega,$$  \hfill (1.3)

$$\omega(x, 0) = \omega_0(x).$$  \hfill (1.4)

The linear elliptic system (1.3) for $v$ can be solved explicitly in terms of $\omega$, assuming $\omega$ decays sufficiently fast near spatial infinity, to provides us with the Biot-Savart law,

$$v(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \omega(y, t)}{|x - y|^3} dy.$$  \hfill (1.5)

Substituting this $v$ into (1.2), we obtain an integro-differential system for $\omega$. The term in the right hand side of (1.2) is called the vortex stretching term, and is regarded as the main source of difficulties in the mathematical theory of the 3D Euler equations. Let us introduce the deformation matrix

$$S(x, t) = (S_{ij}(x, t))_{i,j=1}^3$$

defined as the symmetric part of the velocity gradient matrix,

$$S_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right).$$

From the Biot-Savart law in (1.5) we can explicitly compute

$$S(x, t) = \frac{3}{8\pi p.v.} \int_{\mathbb{R}^3} \frac{[(y \times \omega(x + y, t)) \otimes y + y \otimes (y \times \omega(x + y, t))]}{|y|^5} dy.$$  \hfill (1.6)
(see e.g. [116] for the details on the computation). The kernel in the convolution integral of (1.6) defines a singular integral operator of the Calderon-Zygmund type (see e.g. [137, 138] for more details). Since the vortex stretching term can be written as \((\omega \cdot \nabla)v = S\omega\), we see that the singular integral operator and related harmonic analysis results could have important roles to study the Euler equations.

In the two dimensional case we take the vorticity as the scalar, \(\omega = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}\), and the evolution equation of \(\omega\) becomes

\[
\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = 0,
\]

where the velocity is represented in terms of the vorticity by the 2D Biot-Savart law,

\[
v(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-y_2 + x_2, y_1 - x_1)}{|x - y|^2} \omega(y, t) dy.
\]

Observe that there is no vortex stretching term in (1.7), which makes the proof of global regularity in 2D Euler equations easily accessible. In many studies of the Euler equations it is convenient to introduce the notion of ‘particle trajectory mapping’, \(X(\cdot, t)\) defined by

\[
\frac{\partial X(a, t)}{\partial t} = v(X(a, t), t), \quad X(a, 0) = a, \quad a \in \Omega.
\]

The mapping \(X(\cdot, t)\) transforms from the location of the initial fluid particle to the location at time \(t\), and the parameter \(a\) is called the Lagrangian particle marker. If we denote the Jacobian of the transformation, \(\det(\nabla_a X(a, t)) = J(a, t)\), then we can show easily (see e.g. [116] for the proof) that

\[
\frac{\partial J}{\partial t} = (\text{div} v) J,
\]

which implies that the velocity field \(v\) satisfies the incompressibility, \(\text{div} v = 0\) if and only if the mapping \(X(\cdot, t)\) is volume preserving. At this moment we note that, although the Euler equations are originally derived by applying the physical principles of mass conservation and the momentum balance, we could also derive them by applying the least action principle to the action defined by

\[
\mathcal{I}(A) = \frac{1}{2} \int_{t_1}^{t_2} \int_{\Omega} \left| \frac{\partial X(x, t)}{\partial t} \right|^2 dx dt.
\]
Here, $X(\cdot, t) : \Omega \to \Omega \subset \mathbb{R}^n$ is a parameterized family of volume preserving diffeomorphism. This variational approach to the Euler equations implies that we can view solutions of the Euler equations as a geodesic curve in the $L^2(\Omega)$ metric on the infinite dimensional manifold of volume preserving diffeomorphisms (see e.g. [1, 8, 75] and references therein for more details on the geometric approaches to the Euler equations).

The 3D Euler equations have many conserved quantities. We list some important ones below.

(i) Energy,

$$E(t) = \frac{1}{2} \int_\Omega |v(x, t)|^2 dx.$$ 

(ii) Helicity,

$$H(t) = \int_\Omega v(x, t) \cdot \omega(x, t) dx.$$ 

(iii) Circulation,

$$\Gamma_{C(t)} = \oint_{C(t)} v \cdot dl,$$

where $C(t) = \{X(a, t) | a \in C\}$ is a curve moving along with the fluid.

(iv) Impulse,

$$I(t) = \frac{1}{2} \int_\Omega x \times \omega dx.$$ 

(v) Moment of Impulse,

$$M(t) = \frac{1}{3} \int_\Omega x \times (x \times \omega) dx.$$ 

The proof of conservations of the above quantities for the classical solutions can be done without difficulty using elementary vector calculus (for details see e.g. [116, 118]). The helicity, in particular, represents the degree of knottedness of the vortex lines in the fluid, where the vortex lines are the integral curves of the vorticity fields. In [1] there are detailed discussions on this aspects and other topological implications of the helicity conservation. For
the 2D Euler equations there is no analogue of helicity, while the circulation conservation is replaced by the vorticity flux integral,

$$\int_{D(t)} \omega(x,t) dx,$$

where $D(t) = \{ X(a,t) | a \in D \subset \Omega \}$ is a planar region moving along the fluid in $\Omega$. The impulse and the moment of impulse integrals in the 2E Euler equations are replace by

$$\frac{1}{2} \int_{\Omega} (x_2, -x_1) \omega dx \quad \text{and} \quad -\frac{1}{3} \int_{\Omega} |x|^2 \omega dx \quad \text{respectively.}$$

In the 2D Euler equations we have extra conserved quantities; namely for any continuous function $f$ the integral

$$\int_{\Omega} f(\omega(x,t)) dx$$

is conserved. There are also many known explicit solutions to the Euler equations, for which we just refer [108, 116]. In the remained part of this subsection we introduce some notations to be used later for 3D Euler equations. Given velocity $v(x,t)$, and pressure $p(x,t)$, we set the $3 \times 3$ matrices,

$$V_{ij} = \frac{\partial v_j}{\partial x_i}, \quad S_{ij} = \frac{V_{ij} + V_{ji}}{2}, \quad A_{ij} = \frac{V_{ij} - V_{ji}}{2}, \quad P_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j},$$

with $i, j = 1, 2, 3$. We have the decomposition $V = (V_{ij}) = S + A$, where the symmetric part $S = (S_{ij})$ represents the deformation tensor of the fluid introduced above, while the antisymmetric part $A = (A_{ij})$ is related to the vorticity $\omega$ by the formula,

$$A_{ij} = \frac{1}{2} \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k, \quad \omega_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_{jk}, \quad (1.10)$$

where $\varepsilon_{ijk}$ is the skewsymmetric tensor with the normalization $\varepsilon_{123} = 1$. Note that $P = (P_{ij})$ is the hessian of the pressure. We also frequently use the notation for the vorticity direction field,

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}.$$
defined whenever \( \omega(x, t) \neq 0 \). Computing partial derivatives \( \partial / \partial x_k \) of the first equation of (E), we obtain the matrix equation

\[
\frac{DV}{Dt} = -V^2 - P, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)v.
\]  

(1.11)

Taking symmetric part of this, we obtain

\[
\frac{DS}{Dt} = -S^2 - A^2 - P,
\]

from which, using the formula (1.10), we have

\[
\frac{DS_{ij}}{Dt} = -\sum_{k=1}^{3} S_{ik} S_{kj} + \frac{1}{4} (|\omega|^2 \delta_{ij} - \omega_i \omega_j) - P_{ij},
\]

(1.12)

where \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) if \( i \neq j \). The antisymmetric part of (1.11), on the other hand, is

\[
\frac{DA}{Dt} = -SA - AS,
\]

which, using the formula (1.10) again, we obtain easily

\[
\frac{D\omega}{Dt} = S\omega,
\]

(1.13)

which is the vorticity evolution equation (1.2). Taking dot product (1.13) with \( \omega \), we immediately have

\[
\frac{D|\omega|}{Dt} = \alpha|\omega|,
\]

(1.14)

where we set

\[
\alpha(x, t) = \begin{cases} 
\sum_{i,j=1}^{3} \xi_i(x, t) S_{ij}(x, t) \xi_j(x, t) & \text{if } \omega(x, t) \neq 0 \\
0 & \text{if } \omega(x, t) = 0.
\end{cases}
\]
1.2 Preliminaries

Here we introduce some notations and function spaces to be used in the later sections. Given $p \in [1, \infty]$, the Lebesgue space $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, is the Banach space defined by the norm

$$\|f\|_{L^p} := \begin{cases} \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \text{ess. sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty. \end{cases}$$

For $j = 1, \cdots, n$ the Riesz transform $R_j$ of $f$ is given by

$$R_j(f)(x) = \frac{\Gamma(n+1)}{\pi^{n+1}} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy$$

whenever the right hand side makes sense. The Hardy space $H^1(\mathbb{R}^n)$ is defined by

$$f \in H^1(\mathbb{R}^n) \text{ if and only if } \|f\|_{H^1} := \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j f\|_{L^1} < \infty.$$ 

The space $BMO(\mathbb{R}^n)$ denotes the space of functions of bounded mean oscillations, defined by

$$f \in BMO(\mathbb{R}^n) \text{ if and only if } \|f\|_{BMO} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{\text{Vol}(Q)} \int_{Q} |f - f_Q| \, dx < \infty,$$

where $f_Q = \frac{1}{\text{Vol}(Q)} \int_{Q} f \, dx$. For more details on the Hardy space and BMO we refer [137, 138]. Let us set the multi-index $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in (\mathbb{Z} \cup \{0\})^n$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then, $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where $D_j = \partial / \partial x_j$, $j = 1, 2, \cdots, n$. Given $k \in \mathbb{Z}$ and $p \in [1, \infty)$ the Sobolev space, $W^{k,p}(\mathbb{R}^n)$ is the Banach space of functions consisting of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{W^{k,p}} := \left( \int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

where the derivatives are in the sense of distributions. For $p = \infty$ we replace the $L^p(\mathbb{R}^n)$ norm by the $L^\infty(\mathbb{R}^n)$ norm. In particular, we denote $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$. In order to handle the functions having fractional derivatives of
order \( s \in \mathbb{R} \), we use the Bessel potential space \( L^s_p(\mathbb{R}^n) \) defined by the Banach spaces norm,

\[
\| f \|_{L^s_p} := \| (1 - \Delta)^{\frac{s}{2}} f \|_{L^p},
\]

where \((1 - \Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)(\xi)]\). Here \( \mathcal{F}(\cdot) \) and \( \mathcal{F}^{-1}(\cdot) \) denoting the Fourier transform and its inverse, defined by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx,
\]

and

\[
\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \, d\xi,
\]

whenever the integrals make sense. Next we introduce the Besov spaces. We follow \[145\] (see also \[141, 109, 45, 130\]). Let \( \mathcal{S} \) be the Schwartz class of rapidly decreasing functions. We consider \( \varphi \in \mathcal{S} \) satisfying \( \text{Supp } \hat{\varphi} \subset \{ \xi \in \mathbb{R}^n | \frac{1}{2} \leq |\xi| \leq 2 \} \), and \( \hat{\varphi}(\xi) > 0 \) if \( \frac{1}{2} < |\xi| < 2 \). Setting \( \hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi) \) (In other words, \( \varphi_j(x) = 2^{jn} \varphi(2^jx) \)), we can adjust the normalization constant in front of \( \hat{\varphi} \) so that

\[
\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\]

Let \( s \in \mathbb{R} \), \( p, q \in [0, \infty] \). Given \( f \in \mathcal{S}' \), we denote \( \Delta_j f = \varphi_j * f \). Then the homogeneous Besov semi-norm \( \| f \|_{\dot{B}^s_{p,q}} \) is defined by

\[
\| f \|_{\dot{B}^s_{p,q}} = \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \varphi_j * f \|_{L^p}^q \right)^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\
\sup_{j \in \mathbb{Z}} (2^j s \| \varphi_j * f \|_{L^p}) & \text{if } q = \infty.
\end{array} \right.
\]

For \((s, p, q) \in [0, \infty) \times [1, \infty] \times [1, \infty]\) the homogeneous Besov space \( \dot{B}^s_{p,q} \) is a quasi-normed space with the quasi-norm given by \( \| \cdot \|_{\dot{B}^s_{p,q}} \). For \( s > 0 \) we define the inhomogeneous Besov space norm \( \| f \|_{B^s_{p,q}} \) of \( f \in \mathcal{S}' \) as \( \| f \|_{B^s_{p,q}} = \| f \|_{L^p} + \| f \|_{\dot{B}^s_{p,q}} \). Similarly, for \((s, p, q) \in [0, \infty) \times [1, \infty] \times [1, \infty]\), the homogeneous
Triebel-Lizorkin semi-norm \( \| f \|_{\dot{F}_{p,q}^s} \) is defined by

\[
\| f \|_{\dot{F}_{p,q}^s} = \begin{cases} 
\left\| \left( \sum_{j \in \mathbb{Z}} 2^{jsq} |\varphi_j * f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p} & \text{if } q \in [1, \infty) \\
\left\| \sup_{j \in \mathbb{Z}} (2^{js} |\varphi_j * f(\cdot)|) \right\|_{L^p} & \text{if } q = \infty
\end{cases}
\]

The homogeneous Triebel-Lizorkin space \( \dot{F}_{p,q}^s \) is a quasi-normed space with the quasi-norm given by \( \| \cdot \|_{\dot{F}_{p,q}^s} \). For \( s > 0 \), \((p,q) \in [1, \infty) \times [1, \infty)\) we define the inhomogeneous Triebel-Lizorkin space norm by

\[
\| f \|_{F_{p,q}^s} = \| f \|_{L^p} + \| f \|_{\dot{F}_{p,q}^s}.
\]

The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, \( \| \cdot \|_{F_{p,q}^s} \). We observe that \( \dot{B}_{p,p}^s(\mathbb{R}^n) = F_{p,p}^s(\mathbb{R}^n) \). The Triebel-Lizorkin space is a generalization of many classical function spaces. Indeed, the followings are well established (see e.g. [145])

\[
F_{0,2}^0(\mathbb{R}^n) = \dot{F}_{p,2}^0(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad (1 < p < \infty).
\]

\[
\dot{F}_{1,2}^0(\mathbb{R}^n) = H^1(\mathbb{R}^n) \quad \text{and} \quad \dot{F}_{\infty,2}^0 = BMO(\mathbb{R}^n).
\]

\[
F_{p,2}^s(\mathbb{R}^n) = L^{s,p}(\mathbb{R}^n).
\]

We also note sequence of continuous embeddings for the spaces close to \( L^\infty(\mathbb{R}^n) \)(([145], [95])).

\[
\dot{B}_{p,1}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,1}^0(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^n).
\]

(1.15)

Given \( 0 < s < 1 \), \( 1 \leq p \leq \infty \), \( 1 \leq q \leq \infty \), we introduce another function spaces \( \dot{F}_{p,q}^s \) defined by the seminorm,

\[
\| f \|_{\dot{F}_{p,q}^s} = \begin{cases} 
\left\| \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(x-y)|^q}{|y|^{n+sq}} \, dy \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n, dx)} & \text{if } 1 \leq p \leq \infty, 1 \leq q < \infty \\
\left\| \esssup_{|y| > 0} \frac{|f(x) - f(x-y)|}{|y|^s} \right\|_{L^p(\mathbb{R}^n, dx)} & \text{if } 1 \leq p \leq \infty, q = \infty
\end{cases}
\]

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On the other hand, the space $\dot{B}^s_{p,q}$ is defined by the seminorm,

$$
\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} 
\left( \int_{\mathbb{R}^n} \frac{\|f(\cdot) - f(\cdot - y)\|_{L_p}}{|y|^{n+sq}} dy \right)^{1/q} & \text{if } 1 \leq p \leq \infty, 1 \leq q < \infty \\
\text{ess sup} \frac{\|f(\cdot) - f(\cdot - y)\|_{L_p}}{|y|^s} & \text{if } 1 \leq p \leq \infty, q = \infty
\end{cases}
$$

Observe that, in particular, $\dot{F}^s_{\infty,\infty} = \dot{B}^s_{\infty,\infty} = C^s$, which is the usual Hölder seminormed space for $s \in \mathbb{R}_+ \mathbb{Z}$. We also note that if $q = \infty$, $\dot{B}^s_{p,\infty} = \dot{N}^s_p$, which is the Nikolskii space.

The inhomogeneous version of those spaces, $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ are defined by their norms,

$$
\|f\|_{\dot{F}^s_{p,q}} = \|f\|_{L_p} + \|f\|_{\dot{F}^s_{p,q}}, \quad \|f\|_{\dot{B}^s_{p,q}} = \|f\|_{L_p} + \|f\|_{\dot{B}^s_{p,q}},
$$

respectively. We note that for $0 < s < 1$, $2 \leq p < \infty$, $q = 2$, $\dot{F}^s_{2,2} \cong L_2^n(\mathbb{R}^n)$, introduced above (see pp. 163, [137]). If $\frac{n}{\min\{p,q\}} < s < 1$, $n < p < \infty$ and $n < q \leq \infty$, then $\dot{F}^s_{p,q}$ coincides with the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ defined above (see pp. 101, [145]). On the other hand, for wider range of parameters, $0 < s < 1$, $0 < p \leq \infty$, $0 < q \leq \infty$, $\dot{B}^s_{p,q}$ coincides with the Besov space $B^s_{p,q}(\mathbb{R}^n)$ defined above.

2 Local well-posedness and blow-up criteria

2.1 Kato’s local existence and the BKM criterion

We review briefly the key elements in the classical local existence proof of solutions in the Sobolev space $H^m(\mathbb{R}^n)$, $m > n/2 + 1$, essentially obtained by Kato in [97] (see also [116]). After that we derive the celebrated Beale, Kato and Majda’s criterion on finite time blow-up of the local solution in $H^m(\mathbb{R}^n)$, $m > n/2 + 1$ in [4]. Taking derivatives $D^\alpha$ on the first equation of (E) and then taking $L^2$ inner product it with $D^\alpha v$, and summing over the
multi-indices $\alpha$ with $|\alpha| \leq m$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|_{H^m}^2 = - \sum_{|\alpha| \leq m} (D^\alpha (v \cdot \nabla)v - (v \cdot \nabla) D^\alpha v, D^\alpha v)_{L^2}
- \sum_{|\alpha| \leq m} (v \cdot \nabla) D^\alpha v, D^\alpha v)_{L^2}
- \sum_{|\alpha| \leq m} (D^\alpha \nabla p, D^\alpha v)_{L^2}
= I + II + III.
\]
Integrating by part, we obtain
\[III = \sum_{|\alpha| \leq m} (D^\alpha p, D^\alpha \text{div} v)_{L^2} = 0.\]
Integrating by part again, and using the fact $\text{div} v = 0$, we have
\[II = -\frac{1}{2} \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} (v \cdot \nabla)|D^\alpha v|^2 dx = \frac{1}{2} \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} \text{div} v|D^\alpha v|^2 dx = 0.\]
We now use the so called *commutator type of estimate* ([104]),
\[\sum_{|\alpha| \leq m} \|D^\alpha (fg) - f D^\alpha g\|_{L^2} \leq C(\|\nabla f\|_{L^\infty} \|g\|_{H^{m-1}} + \|f\|_{H^m} \|g\|_{L^\infty}),\]
and obtain
\[I \leq \sum_{|\alpha| \leq m} \|D^\alpha (v \cdot \nabla)v - (v \cdot \nabla) D^\alpha v\|_{L^2}\|v\|_{H^m} \leq C\|\nabla v\|_{L^\infty} \|v\|_{H^m}^2.\]
Summarizing the above estimates, I,II,III, we have
\[\frac{d}{dt} \|v\|_{H^m}^2 \leq C\|\nabla v\|_{L^\infty} \|v\|_{H^m}^2. \tag{2.1}\]
Further estimate, using the *Sobolev inequality*, $\|\nabla v\|_{L^\infty} \leq C\|v\|_{H^m}$ for $m > n/2 + 1$, gives
\[\frac{d}{dt} \|v\|_{H^m}^2 \leq C\|v\|_{H^m}^3.\]
Thanks to Gronwall’s lemma we have the local in time uniform estimate
\[\|v(t)\|_{H^m} \leq \frac{\|v_0\|_{H^m}}{1 - Ct\|v_0\|_{H^m}} \leq 2\|v_0\|_{H^m} \tag{2.2}\]
for all $t \in [0, T]$, where $T = \frac{1}{2C\|v_0\|_{H^m}}$. Using this estimate we can also deduce the estimate

$$\sup_{0 \leq t \leq T} \left\| \frac{\partial v}{\partial t} \right\|_{H^{m-1}} \leq C(\|v_0\|_{H^m}) \tag{2.3}$$

directly from (E). The estimates (2.2) and (2.3) are the two key a priori estimates for the construction of the local solutions. For actual elaboration of the proof we approximate the Euler system by mollification, Galerkin projection, or iteration of successive linear systems, and construct a sequence of smooth approximate solutions to (E), say \(\{v_k(\cdot, t)\}_{k \in \mathbb{N}}\) corresponding to the initial data \(\{v_{0,k}\}_{k \in \mathbb{N}}\) respectively with \(v_k \to v_0\) in \(H^m(\mathbb{R}^n)\). The estimates for the approximate solution sequence provides us with the uniform estimates of \(\{v_k\}\) in \(L^\infty([0, T]; H^m(\mathbb{R}^n)) \cap Lip([0, T]; H^{m-1}(\mathbb{R}^n))\). Then, applying the standard Aubin-Nitche compactness lemma, we can pass to the limit \(k \to \infty\) in the equations for the approximate solutions, and can show that the limit \(v = v_\infty\) is a solution of the (E) in \(L^\infty([0, T]; H^m(\mathbb{R}^n))\). By further argument we can actually show that the limit \(v\) belongs to \(C([0, T]; H^m(\mathbb{R}^n)) \cap AC([0, T]; H^{m-1}(\mathbb{R}^n))\), where \(AC([0, T]; X)\) denotes the space of \(X\) valued absolutely continuous functions on \([0, T]\). The general scheme of such existence proof is standard, and is described in detail in [113] in the general type of hyperbolic conservation laws. The approximation of the Euler system by mollification was done for the construction of local solution of the Euler (and the Navier-Stokes) system in [116].

Regarding the question of finite time blow-up of the local classical solution in \(H^m(\mathbb{R}^n)\), \(m > n/2 + 1\), constructed above, the celebrated Beale-Kato-Majda theorem (called the BKM criterion) states that

$$\limsup_{t \to T_*} \|v(t)\|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(s)\|_{L^\infty} ds = \infty. \tag{2.4}$$

We outline the proof of this theorem below (for more details see [4, 116]). We first recall the Beale-Kato-Majda’s version of the logarithmic Sobolev inequality,

$$\|\nabla v\|_{L^\infty} \leq C\|\omega\|_{L^\infty}(1 + \log(1 + \|v\|_{H^m})) + C\|\omega\|_{L^2} \tag{2.5}$$

for \(m > n/2 + 1\). Now suppose \(\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt := M(T_*) < \infty\). Taking \(L^2\) inner product the first equation of (E) with \(\omega\), then after integration by part
we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \omega \|^2_{L^2} = (\omega \cdot \nabla v, \omega)_{L^2} \leq \| \omega \|_{L^\infty} \| \nabla v \|_{L^2} \| \omega \|_{L^2} = \| \omega \|_{L^\infty} \| \omega \|^2_{L^2},
\]
where we used the identity \( \| \nabla v \|_{L^2} = \| \omega \|_{L^2} \). Applying the Gronwall lemma, we obtain
\[
\| \omega(t) \|_{L^2} \leq \| \omega_0 \|_{L^2} \exp \left( \int_0^{T_*} \| \omega(s) \|_{L^\infty} ds \right) = \| \omega_0 \|_{L^2} \exp[M(T_*)].
\] (2.6)
for all \( t \in [0, T_*] \). Substituting (2.6) into (2.5), and combining this with (2.1), we have
\[
\frac{d}{dt} \| v \|^2_{H^m} \leq C \left[ 1 + \| \omega \|^2_{L^\infty} \right] \| v \|^2_{H^m}
\]
Applying the Gronwall lemma we deduce
\[
\| v(t) \|_{H^m} \leq \| v_0 \|_{H^m} \exp \left( C_1 \exp \left( C_2 \int_0^{T_*} \| \omega(\tau) \|_{L^\infty} d\tau \right) \right)
\] (2.7)
for all \( t \in [0, T_*] \) and for some constants \( C_1 \) and \( C_2 \) depending on \( M(T_*) \). The inequality (2.7) provides the with the necessity part of (2.4). The sufficiency part is an easy consequence of the Sobolev inequality,
\[
\int_0^{T_*} \| \omega(s) \|_{L^\infty} ds \leq T_* \sup_{0 \leq t \leq T_*} \| \nabla v(t) \|_{L^\infty} \leq CT_* \sup_{0 \leq t \leq T_*} \| v(t) \|_{H^m}
\]
for \( m > n/2 + 1 \). There are many other results of local well-posedness in various function spaces(see [14, 15, 17, 20, 43, 45, 96, 98, 99, 111, 142, 143, 147, 148, 153]). For the local existence proved in terms of a geometric formulation see [75]. For the BKM criterion for solutions in the Hölder space see [3]. Immediately after the BKM result appeared, Ponce derive similar criterion in terms of the deformation tensor([128]). Recently, Constantin proved local well-posedness and a blow-up criterion in terms of the active vector formulation([51]).

### 2.2 Refinements of the BKM criterion

The first refinement of the BKM criterion was done by Kozono and Taniuchi in [105], where they proved...
Theorem 2.1 Let \( s > n/p + 1 \). A solution \( v \) of the Euler equations belonging to \( C([0, T_*); W^{s,p}(\mathbb{R}^n) \cap C^1([0, T_*); W^{s-2,p}(\mathbb{R}^n)) \) blows up at \( T_* \) in \( W^{s,p}(\mathbb{R}^n) \), namely

\[
\limsup_{t \nearrow T_*} \|v(t)\|_{W^{s,p}} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega\|_{BMO} \, dt = \infty.
\]

The proof is based on the following version of the logarithmic Sobolev inequality for \( f \in W^{s,p}(\mathbb{R}^n), \ s > n/p, \ 1 < p < \infty, \)

\[
\|f\|_{L^\infty} \leq C (1 + \|f\|_{BMO}(1 + \log^+ \|f\|_{W^{s,p}})).
\]

(see [105] for details of the proof). We recall now the embedding relations (1.15). Further refinement of the above theorem is the following (see [14][20]).

Theorem 2.2 (i) (super-critical case) Let \( s > n/p + 1, \ p \in (1, \infty), \ q \in [1, \infty] \). Then, the local in time solution \( v \in C([0, T_*); B^{s}_{p,q}(\mathbb{R}^n)) \) blows up at \( T_* \) in \( B^{s}_{p,q}(\mathbb{R}^n) \), namely

\[
\limsup_{t \nearrow T_*} \|v(t)\|_{B^{s}_{p,q}} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{B^{0}_{\infty,\infty}} \, dt = \infty.
\]

(ii) (critical case) Let \( p \in (1, \infty) \). Then, the local in time solution \( v \in C([0, T_*); B^{n/p+1}_{p,1}(\mathbb{R}^n)) \) blows up at \( T_* \) in \( B^{n/p+1}_{p,1}(\mathbb{R}^n) \), namely

\[
\limsup_{t \nearrow T_*} \|v(t)\|_{B^{n/p+1}_{p,1}} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{B^{0}_{\infty,1}} \, dt = \infty.
\]

The proof of (i) is based on the following version of the logarithmic Sobolev inequality for \( f \in B^{s}_{p,q}(\mathbb{R}^n) \) with \( s > n/p \) with \( p \in (1, \infty), \ q \in [1, \infty] \).

\[
\|f\|_{L^\infty} \leq C(1 + \|f\|_{B^{0}_{\infty,\infty}}(\log^+ \|f\|_{B^{s}_{p,q}} + 1))
\]

In [106] Kozono, Ogawa and Taniuchi obtained similar results to (i) above independently.

In all of the above criteria, including the BKM theorem, we need to control all of the three components of the vorticity vector to obtain regularity. The following theorem proved in [22] states that actually we only need to control two components of the vorticity in the slightly stronger norm than the \( L^\infty \) norm (recall again the embedding (1.15)).
Theorem 2.3 Let $m > 5/2$. Suppose $v \in C([0, T_1); H^m(\mathbb{R}^3))$ is the local classical solution of (E) for some $T_1 > 0$, corresponding to the initial data $v_0 \in H^m(\mathbb{R}^3)$, and $\omega = \text{curl} \ v$ is its vorticity. We decompose $\omega = \tilde{\omega} + \omega_3 e_3$, where $\tilde{\omega} = \omega_1 e_1 + \omega_2 e_2$, and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. Then,

$$\limsup_{t \to T^*} \|v(t)\|_{H^m} = \infty \text{ if and only if } \int_0^{T^*} \|\tilde{\omega}(t)\|_{H^m}^2 \, dt = \infty.$$ 

Note that $\tilde{\omega}$ could be the projected component of $\omega$ onto any plane in $\mathbb{R}^3$.

For the solution $v = (v^1, v^2, 0)$ of the Euler equations on the $x_1 - x_2$ plane, the vorticity is $\omega = \omega_3 e_3$ with $\omega_3 = \partial_{x_1} v^2 - \partial_{x_2} v^1$, and $\tilde{\omega} \equiv 0$. Hence, as a trivial application of the above theorem we reproduce the well-known global time regularity for the 2D Euler equations.

Next we present recent results on the blow up criterion in terms of hessian of the pressure. As in the introduction we use $P = (P_{ij})$, $S = (S_{ij})$ and $\xi$ to denote the hessian of the pressure, the deformation tensor and the vorticity direction field respectively, introduced in section 1. We also introduce the notations

$$\frac{S\xi}{|S\xi|} = \zeta, \quad \zeta \cdot P\xi = \mu.$$ 

The following is proved in [30].

Theorem 2.4 If the solution $v(x, t)$ of the 3D Euler system with $v_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$, blows up at $T^*$, namely $\limsup_{t \to T^*} \|v(t)\|_{H^m} = \infty$, then necessarily,

$$\int_0^{T^*} \exp \left( \int_0^t \|\mu(s)\|_{L^\infty} \, ds \right) \, dt = \infty.$$ 

Similar criterion in terms of the hessian of pressure, but with different detailed geometric configuration from the above theorem is obtained by Gibbon, Holm, Kerr and Roulstone in [87]. Below we denote $\xi_p = \xi \times P\xi$.

Theorem 2.5 Let $m \geq 3$ and $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be a periodic box. Then, there exists a global solution of the Euler equations $v \in C([0, \infty); H^m(\mathbb{T}^3)) \cap C^1([0, \infty); H^{m-1}(\mathbb{T}^3))$ if

$$\int_0^T \|\xi_p(t)\|_{L^\infty} \, dt < \infty, \quad \forall t \in (0, T)$$

excepting the case where $\xi$ becomes collinear with the eigenvalues of $P$ at $T$. 


Next, we consider the axisymmetric solution of the Euler equations, which means velocity field \( v(r, x_3, t) \), solving the Euler equations, and having the representation

\[
v(r, x_3, t) = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3
\]

in the cylindrical coordinate system, where

\[
e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( -\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_3 = (0, 0, 1), \quad r = \sqrt{x_1^2 + x_2^2}.
\]

In this case also the question of finite time blow-up of solution is wide open (see e.g. [89, 90, 11] for studies in such case). The vorticity \( \omega = \text{curl} \ v \) is computed as

\[
\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,
\]

where

\[
\omega^r = -\partial_{x_3}v^\theta, \quad \omega^\theta = \partial_{x_3}v^r - \partial_r v^3, \quad \omega^3 = \frac{1}{r} \partial_r (rv^\theta).
\]

We denote

\[
\tilde{v} = v^r e_r + v^3 e_3, \quad \tilde{\omega} = \omega^r e_r + \omega^3 e_3.
\]

Hence, \( \omega = \tilde{\omega} + \tilde{\omega}_\theta \), where \( \tilde{\omega}_\theta = \omega^\theta e_\theta \). The Euler equations for the axisymmetric solution are

\[
\begin{aligned}
\frac{\partial v^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^r &= -\frac{\partial p}{\partial r}, \\
\frac{\partial v^\theta}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^\theta &= -\frac{v^r v^\theta}{r}, \\
\frac{\partial v^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^3 &= -\frac{\partial p}{\partial x_3}, \\
\text{div } \tilde{v} &= 0, \\
v(r, x_3, 0) &= v_0(r, x_3),
\end{aligned}
\]

where \( \tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3} \). In the axisymmetric Euler equations the vorticity formulation becomes

\[
\begin{aligned}
\frac{\partial \omega^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) \omega^r &= \omega^r (\tilde{\omega} \cdot \tilde{\nabla}) v^r, \\
\frac{\partial \omega^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) \omega^3 &= \omega^3 (\tilde{\omega} \cdot \tilde{\nabla}) v^3, \\
\left[ \frac{\partial}{\partial t} + \tilde{v} \cdot \tilde{\nabla} \right] \left( \frac{\omega^\theta}{r} \right) &= (\tilde{\omega} \cdot \tilde{\nabla}) \left( \frac{v^\theta}{r} \right), \\
\text{div } \tilde{v} &= 0, \quad \text{curl } \tilde{v} = \tilde{\omega}^\theta.
\end{aligned}
\]
In the case of axisymmetry we only need to control just one component of vorticity (the angular component) to get the regularity of solution. The following theorem is proved in [40].

**Theorem 2.6** Let \( v \in C([0,T_*]; H^m(\mathbb{R}^3)) \), \( m > 5/2 \), be the local classical axisymmetric solution of \((E)\), corresponding to an axisymmetric initial data \( v_0 \in H^m(\mathbb{R}^3) \). Then, the solution blows up in \( H^m(\mathbb{R}^3) \) at \( T_* \) if and only if for all \( (\gamma,p) \in (0,1) \times [1,\infty) \) we have

\[
\int_0^{T_*} \| \omega_\theta(t) \|_{L^\infty} dt + \int_0^{T_*} \exp \left[ \int_0^t \left\{ \| \omega_\theta(s) \|_{L^\infty} \left( 1 + \log^+ \| \omega_\theta(s) \|_{C^\gamma} \| \omega_\theta(s) \|_{L^p} \right) \right\} ds \right] dt = \infty. \tag{2.8}
\]

We observe that although we need to control only \( \omega_\theta \) to get the regularity, the its norm, which is in \( C^\gamma \), is higher than the \( L^\infty \) norm used in the BKM criterion. If we use the ‘critical’ Besov space \( \tilde{B}^0_{\infty,1}(\mathbb{R}^3) \) we can derive slightly sharper criterion than Theorem 2.6 as follows (see [22] for the proof).

**Theorem 2.7** Let \( v \in C([0,T_*]; H^m(\mathbb{R}^3)) \) be the local classical axisymmetric solution of \((E)\), corresponding to an axisymmetric initial data \( v_0 \in H^m(\mathbb{R}^3) \). Then,

\[
\limsup_{t \to T_*} \| v(t) \|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \| \omega_\theta(t) \|_{\tilde{B}^0_{\infty,1}} dt = \infty. \tag{2.9}
\]

We observe that contrary to (2.8) we do not need to control the high regularity norm, the \( C^\gamma \) norm of vorticity in (2.9). We can also have the regularity of the axisymmetric Euler equation by controlling only one component of the velocity, the swirl velocity \( v^\theta \) as in the follows proved in [38].

**Theorem 2.8** Let \( v \in C([0,T_*]; H^m(\mathbb{R}^3)) \), \( m > 5/2 \), be the local classical axisymmetric solution of \((E)\), corresponding to an axisymmetric initial data \( v_0 \in H^m(\mathbb{R}^3) \). Then, the solution blows up in \( H^m(\mathbb{R}^3) \) at \( T_* \) if and only if

\[
\int_0^{T_*} \left( \| \bar{\nabla} v^\theta \|_{L^\infty} + \| \frac{\partial v^\theta}{\partial r} \|_{L^\infty} + \| \frac{1}{r} \frac{\partial v^\theta}{\partial x_3} \|_{L^\infty} \right) dt = \infty.
\]
2.3 Constantin-Fefferman-Majda’s and other related results

In order to study the regularity problem of the 3D Navier-Stokes equations, Constantin and Fefferman investigated the geometric structure of the integral kernel in the vortex stretching term more carefully, and discovered the phenomena of ‘depletion effect’ hidden in the integration ([55], see also [48] for detailed exposition related to this fact). Later similar geometric structure of the vortex stretching term was studied extensively also in the blow-up problem of the 3D Euler equations by Constantin, Fefferman and Majda ([56]). Here we first present their results in detail, and results in [25], where the BKM criterion and the Constantin-Fefferman-Majda’s criterion are interpolated in some sense. Besides those results presented in this subsection we also mention that there are other interesting geometric approaches to the Euler equations such as the quaternion formulation by Gibbon ([85, 86, 87]).

We begin with a definition in [56]. Given a set \( W \subseteq \mathbb{R}^3 \) and \( r > 0 \) we use the notation \( B_r(W) = \{ y \in B_r(x) ; x \in W \} \).

**Definition 2.1** A set \( W_0 \subseteq \mathbb{R}^3 \) is called smoothly directed if there exists \( \rho > 0 \) and \( r, 0 < r \leq \rho/2 \) such that the following three conditions are satisfied.

(i) For every \( a \in W_0^* = \{ q \in W_0 ; |\omega_0(q)| \neq 0 \} \), and all \( t \in [0, T) \), the vorticity direction field \( \xi(\cdot, t) \) has a Lipshitz extension (denoted by the same letter) to the Euclidean ball of radius \( 4\rho \) centered at \( X(a,t) \) and

\[
M = \lim_{t \to T} \sup_{a \in W_0^*} \int_0^t \| \nabla \xi(\cdot, t) \|_{L^\infty(B_{4\rho}(X(a,t)))} dt < \infty.
\]

(ii) The inequality

\[
\sup_{B_{3r}(W_t)} |\omega(x,t)| \leq m \sup_{B_r(W_t)} |\omega(x,t)|
\]

holds for all \( t \in [0, T) \) with \( m \geq 0 \) constant.

(iii) The inequality

\[
\sup_{B_{4\rho}(W_t)} |v(x,t)| \leq U
\]

holds for all \( t \in [0, T) \).

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The assumption (i) means that the direction of vorticity is well behaved in a neighborhood of a bunch of trajectories. The assumption (ii) states that this neighborhood is large enough to capture the local intensification of $\omega$. Under these assumptions the following theorem is proved in [56].

**Theorem 2.9** Assume $W_0$ is smoothly directed. Then there exists $\tau > 0$ and $\Gamma$ such that

$$\sup_{B_r(W_t)} |\omega(x, t)| \leq \Gamma \sup_{B_\rho(W_{t_0})} |\omega(x, t_0)|$$

holds for any $0 \leq t_0 < T$ and $0 \leq t - t_0 \leq \tau$.

They also introduced the notion of regularly directed set, closely related to the geometric structure of the kernel defining vortex stretching term.

**Definition 2.2** We say that a set $W_0$ is regularly directed if there exists $\rho > 0$ such that

$$\sup_{aW^0_0} \int_0^T K_\rho(X(a, t))dt < \infty$$

where

$$K_\rho(x) = \int_{|y| \leq \rho} |D(\hat{y}, \xi(x+y), \xi(x))||\omega(x+y)| \frac{dy}{|y|^3}$$

and

$$D(\hat{y}, \xi(x+y), \xi(x)) = (\hat{y} \cdot \xi(x)) \text{Det}(\hat{y}, \xi(x+y), \xi(x)).$$

Under the above assumption on the regularly directed sets the following is proved also in [56].

**Theorem 2.10** Assume $W_0$ is regularly directed. Then there exists a constant $\Gamma$ such that

$$\sup_{a \in W_0} |\omega(X(a, t), t)| \leq \Gamma \sup_{a \in W_0} |\omega_0(a)|$$

holds for all $t \in [0, T]$.

The original studies by Constantin and Fefferman in [55] about the Navier-Stokes equations, which motivated the above theorems, are concerned mainly about the regularity of solutions in terms of the vorticity direction fields $\xi$. We recall, on the other hand, that the BKM type of criterion controls the
magnitude of vorticity to obtain regularity. Incorporation of both the direction and the magnitude of vorticity to obtain regularity for the 3D Navier-Stokes equations was first initiated by Beirão da Veiga and Berselli in [6], and developed further by Beirão da Veiga in [5], and finally refined in an ‘optimal’ form in [35] (see also [39] for a localized version). We now present the Euler equation version of the result in [35].

Below we use the notion of particle trajectory \( X(a, t) \), which is defined by the classical solution \( v(x, t) \) of (E). Let us denote \( \Omega_0 = \{ x \in \mathbb{R}^3 | \omega_0(x) \neq 0 \} \), \( \Omega_t = X(\Omega_0, t) \).

We note that the direction field of the vorticity, \( \xi(x, t) = \omega(x, t)/|\omega(x, t)| \), is well-defined if \( x \in \Omega_t \) for \( v_0 \in C^1(\mathbb{R}^3) \) with \( \Omega_0 \neq \emptyset \). The following is the main theorem proved in [25].

**Theorem 2.11** Let \( v(x, t) \) be the local classical solution to (E) with initial data \( v_0 \in H^m(\mathbb{R}^3) \), \( m > 5/2 \), and \( \omega(x, t) = \text{curl} \, v(x, t) \). We assume \( \Omega_0 \neq \emptyset \). Then, the solution can be continued up to \( T + \delta \) as the classical solution, namely \( v(t) \in C([0, T + \delta]; H^m(\mathbb{R}^3)) \) for some \( \delta > 0 \), if there exists \( p, p', q, q', s, r_1, r_2, r_3 \) satisfying the following conditions,

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad (2.10)
\]

and

\[
\frac{1}{r_1} + \frac{p'}{r_2} \left( 1 - \frac{sq'}{3} \right) + \frac{1}{r_3} \left\{ 1 - p' \left( 1 - \frac{sq'}{3} \right) \right\} = 1 \quad (2.11)
\]

with

\[
0 < s < 1, \quad 1 \leq \frac{3}{sq'} < p \leq \infty, \quad 1 \leq q \leq \infty, \quad (2.12)
\]

and

\[
r_1 \in [1, \infty], \quad r_2 \in \left[ p' \left( 1 - \frac{sq'}{3} \right), \infty \right], \quad r_3 \in \left[ 1 - p' \left( 1 - \frac{sq'}{3} \right), \infty \right] \quad (2.13)
\]

such that for direction field \( \xi(x, t) \), and the magnitude of vorticity \( |\omega(x, t)| \) the followings hold:

\[
\int_0^T \|\xi(t)\|_{F_{\infty, q}(\Omega_t)}^{r_1} dt < \infty, \quad (2.14)
\]

and

\[
\int_0^T \|\omega(t)\|_{L^{p'}(\Omega_t)}^{r_2} dt + \int_0^T \|\omega(t)\|_{L^{q'}(\Omega_t)}^{r_3} dt < \infty. \quad (2.15)
\]
In order to get insight implied by the above theorem let us consider the special case of $p = \infty, q = 1$. In this case the conditions (2.14)-(2.15) are satisfied if

$$\xi(x, t) \in L^{r_1}(0, T; C^s(\mathbb{R}^3)),$$

(2.16)

$$\omega(x, t) \in L^{r_2}(0, T; L^\infty(\mathbb{R}^3)) \cap L^{r_3}(0, T; L^\infty(\mathbb{R}^3)).$$

(2.17)

with

$$\frac{1}{r_1} + \frac{1}{r_2} \left(1 - \frac{s}{3}\right) + \frac{s}{3r_3} = 1.$$  

(2.18)

Let us formally pass $s \to 0$ in (2.16) and (2.18), and choose $r_1 = \infty$ and $r_2 = r_3 = 1$, then we find that the conditions (2.16)-(2.17) reduce to the BKM condition, since the condition $\xi(x, t) \in L^\infty(0, T; C^0(\mathbb{R}^3)) \cong L^\infty((0, T) \times \mathbb{R}^3)$ is obviously satisfied due to the fact that $|\xi(x, t)| \equiv 1$.

The other case of interest is $q' = 3/s$, where (2.14)+(2.15) are satisfied if

$$\xi(x, t) \in L^{r_1}(0, T; \hat{F}_s^{\infty}(\mathbb{R}^3)), \quad |\omega(x, t)| \in L^{r_2}(0, T; \hat{L}^{s}(\mathbb{R}^3)).$$

(2.19)

with $1/r_1 + 1/r_2 = 1$. The condition (2.19) shows explicitly the mutual compensation between the regularity of the direction field and the integrability of the vorticity magnitude in order to control regularity/singularity of solutions of the Euler equations.

Next we review the result of non-blow-up conditions due to Deng, Hou and Yu[71, 72]. We consider a time $t$ and a vortex line segment $L_t$ such that the maximum of vorticity over the whole domain is comparable to the maximum of vorticity on over $L_t$, namely

$$\Omega(t) := \sup_{x \in \mathbb{R}^3} |\omega(x, t)| \sim \max_{x \in L_t} |\omega(x, t)|.$$  

We denote $L(t) := \text{arc length of } L_t$; $\xi$, $n$ and $\kappa$ are the unit tangential and the unit normal vectors to $L_t$ and the curvature of $L_t$ respectively. We also use the notations,

$$U_\xi(t) := \max_{x, y \in L_t} |(v \cdot \xi)(x, t) - (v \cdot \xi)(y, t)|,$$

$$U_n(t) := \max_{x \in L_t} |(v \cdot n)(x, t)|,$$

$$M(t) := \max_{x \in L_t} |(\nabla \cdot \xi)(x, t)|,$$

$$K(t) := \max_{x \in L_t} \kappa(x, t).$$
We denote by $X(A, s, t)$ the image by the trajectory map at time $t > s$ of fluid particles at $A$ at time $s$. Then, the following is proved in [72].

**Theorem 2.12** Assume that there is a family of vortex line segment $L_t$ and $T_0 \in [0, T^*)$, such that $X(L_{t_1}, t_1, t_2) \supseteq L_{t_2}$ for all $T_0 < t_1 < t_2 < T^*$. Also assume that $\Omega(t)$ is monotonically increasing and $\max_{x \in L_t} |\omega(x, t)| \geq c_0 \Omega(t)$ for some $c_0$ when $t$ is sufficiently close to $T^*$. Furthermore, we assume there are constants $C_U, C_0, c_L$ such that

1. $[U \xi(t) + U_n(t)K(t)L(t)] \leq C_U(T^* - t)^{-A}$ for some constant $A \in (0, 1)$,
2. $M(t)L(t), K(t)L(t) \leq C_0$,
3. $L(t) \geq c_L(T^* - t)^B$ for some constant $B \in (0, 1)$.

Then there will be no blow-up in the 3D incompressible Euler flow up to time $T^*$, as long as $B < 1 - A$.

In the endpoint case of $B = 1 - A$ they deduced the following theorem([71]).

**Theorem 2.13** Under the same assumption as in Theorem 2.10, there will be no blow-up in the Euler system up to time $T^*$ in the case $B = 1 - A$, as long as the following condition is satisfied:

$$R^3K < y_1 \left( R^{A-1}(1 - A)^{1-A}/(2 - A)^{2-A} \right),$$

where $R = e^{C_0}/c_0, K := \frac{C_Uc_0}{c_L(1-A)}$, and $y_1(m)$ denotes the smallest positive $y$ such that $m = y/(1 + y)^{2-A}$.

We refer [71, 72] for discussions on the various connections of Theorem 2.10 and Theorem 2.11 with numerical computations.

### 3 Blow-up scenarios

#### 3.1 Vortex sheet collapse

We recall that a vortex line is an integral curve of the vorticity, and a vortex tube is a tubular neighborhood in $\mathbb{R}^3$ foliated by vortex lines. Numerical simulations(see e.g. [46]) show that vortex tubes grow and thinner(stretching),
and folds before singularity happens. We review here the result by Cordoba and Fefferman(66) excluding a type of vortex tube collapse.

Let \( Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3 \) be a closed rectangular box, and let \( T > 0 \) be given. A regular tube is a relatively open set \( \Omega_t \subset Q \) parameterized by time \( t \in [0, T) \), having the form \( \Omega_t = \{(x_1, x_2, x_3) \in Q : \theta(x_1, x_2, x_3, t) < 0\} \) with \( \theta \in C^1(Q \times [0, T)) \), and satisfying the following properties:

\[
|\nabla_{x_1, x_2} \theta| \neq 0 \quad \text{for} \quad (x_1, x_2, x_3, t) \in Q \times [0, T), \theta(x_1, x_2, x_3, t) = 0;
\]

\[
\Omega_t(x_3) := \{(x_1, x_2) \in I_1 \times I_2 : (x_1, x_2, x_3) \in \Omega_t\} \text{ is non-empty,}
\]

for all \( x_3 \in I_3, t \in [0, T) \);

\[
\text{closure}(\Omega_t(x_3)) \subset \text{interior}(I_1 \times I_2)
\]

for all \( x_3 \in I_3, t \in [0, T) \).

Let \( u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \) be a \( C^1 \) velocity field defined on \( Q \times [0, T) \). We say that the regular tube \( \Omega_t \) moves with the velocity field \( u \), if we have

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla_x \right) \theta = 0 \quad \text{whenever} \quad (x, t) \in Q \times [0, T), \theta(x, t) = 0.
\]

By the Helmholtz theorem we know that a vortex tube arising from a 3D Euler solution moves with the fluid velocity. The following theorem proved by Cordoba and Fefferman(66) says for the 3D Euler equations that a vortex tube cannot reach zero thickness in finite time, unless it bends and twists so violently that no part of it forms a regular tube.

**Theorem 3.1** Let \( \Omega_t \subset Q(t \in [0, T]) \) be a regular tube that moves with \( C^1 \), divergence free velocity field \( u(x, t) \).

\[
\text{If} \quad \int_0^T \sup_{x \in Q} |u(x, t)| dt < \infty, \quad \text{then} \quad \liminf_{t \to T^-} \text{Vol}(\Omega_t) > 0.
\]

### 3.2 Squirt singularity

The theorem of excluding the regular vortex tube collapse was generalized by Cordoba, Fefferman and de la Lave(70), which we review here. We first recall their definition of squirt singularities. Let \( \Omega \subset \mathbb{R}^n \) be an open set. We denote \( X_t(a) = X(a, t) \), which is a particle trajectory generated by a \( C^1 \)
vector field $u : \Omega \times [0, T) \to \mathbb{R}^n$ such that $\text{div} \ u = 0$. We also set $X_{t,s}(a)$ as the position at time $t$ of the trajectory which at time $t = s$ is $a$. We have obvious relations,

$$X_t(a) = X_{t,0}(a), \quad X_{t,s} = X_t \circ X_s^{-1}, \quad X_{t,s} \circ X_{s,s_1} = X_{t,s_1}.$$ 

For $S \subset \Omega$, we denote by

$$X_{t,s}^\Omega S = \{ x \in \Omega \mid x = X_t(a), \ a \in S, \ X_s(a) \in \Omega, \ 0 \leq s \leq t \}.$$ 

In other words, $X_{t,s}^\Omega S$ is the evolution of the set $S$, starting at time $a$, after we eliminate the trajectories which step out of $\Omega$ at some time. By the incompressibility condition on $u$, we have that $\text{Vol}(X_{t,s}^\Omega S)$ is independent of $t$, and the function $t \mapsto \text{Vol}(X_{t,s}^\Omega S)$ is nonincreasing.

**Definition 3.1** Let $\Omega_-, \Omega_+$ be open and bounded sets. $\overline{\Omega_-} \subset \Omega_+$. Therefore, $\text{dist} (\Omega_-, \mathbb{R}^n - \Omega_+) \geq r > 0$. We say that $u$ experiences a squirt singularity in $\Omega_-$, at time $T > 0$, when for every $0 \leq s < T$, we can find a set $S_s \subset \Omega_+$ such that

(i) $S_s \cap \Omega_-$ has positive measure, $0 \leq s < T$,

(ii) $\lim_{t \to T-} \text{Vol}(X_{t,s}^\Omega S_s) = 0$.

The physical intuition behind the above definition is that there is a region of positive volume so that all the fluid occupying it gets ejected from a slightly bigger region in finite time. Besides the vortex tube collapse singularity introduced in the previous subsection the potato chip singularity and the saddle collapse singularity, which will be defined below, are also special examples of the squirt singularity, connected with real fluid mechanics phenomena.

**Definition 3.2 (potato chip singularity)** We say that $u$ experiences a potato chip singularity when we can find continuous functions

$$f_\pm : \mathbb{R}^{n-1} \times [0, T) \to \mathbb{R}$$

such that

$$f_+(x_1, \cdots, x_{n-1}, t) \geq f_-(x_1, \cdots, x_{n-1}, t), \ t \in [0, T], x_1, \cdots, x_{n-1} \in B_{2r}(\Pi x^0),
$$

$$f_+(x_1, \cdots, x_{n-1}, 0) \geq f_-(x_1, \cdots, x_{n-1}, 0), \ x_1, \cdots, x_{n-1} \in B_r(\Pi x^0),
$$

$$\lim_{t \to T_-} [f_+(x_1, \cdots, x_{n-1}, t) - f_-(x_1, \cdots, x_{n-1}, t)] = 0 \forall x_1, \cdots, x_{n-1} \in B_{2r}(\Pi x^0)$$

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and such that the surfaces
\[ \Sigma_{\pm,t} = \{ x_n = f_{\pm}(x_1, \cdots, x_{n-1}, t) \} \subset \Omega \]
are transformed into each other by the flow
\[ X(\Sigma_{\pm,0}, t) \supset \Sigma_{\pm,t}. \]

In the above \( \Pi \) is projection on the first \( n-1 \) coordinates.

Previously to [70] potato chip singularities were considered in the 2D and 3D flows by Córdoba and Fefferman ([69], [67] respectively) in the name of ‘sharp front’. In particular the exclusion of sharp front in the 2D quasi-geostrophic equation is proved in [69]. The following notion of saddle collapse singularity is relevant only for 2D flows.

**Definition 3.3 (saddle collapse singularity)** We consider foliation of a neighborhood of the origin (with coordinates \( x_1, x_2 \)) whose leaves are given by equations of the form
\[ \rho := (y_1(\beta(t) + y_2) \cdot (y_1(\delta(t) + y_2) = \text{cons} \]
and \( (y_1, y_2) = F_t(x_1, x_2) \), where \( \beta, \delta : [0, T) \to \mathbb{R}^+ \) are \( C^1 \) foliations, \( F \) is a \( C^2 \) function of \( x,t \), for a fixed \( t \), and \( F_t \) is an orientation preserving diffeomorphism. We say that the foliation experiences a saddle collapse when
\[ \liminf_{t \to T} \beta(t) + \delta(t) = 0. \]

If the leaves of the foliation are transported by a vector field \( u \), we say that the vector field \( u \) experiences a saddle collapse.

The exclusion of saddle point singularity in the 2D quasi-geostrophic equation (see Section 4.3 below) was proved by Córdoba in [65]. The following ‘unified’ theorem is proved in [70].

**Theorem 3.2** If \( u \) has a squirt singularity at \( T \), then \( \int_s^T \sup_x |u(x,t)|dt = \infty \) for all \( s \in (0,T) \). Moreover, if \( u \) has a potato chip singularity, then
\[ \int_s^T \sup_x |\Pi u(x,t)|dt = \infty. \]
3.3 Self-similar blow-up

In this subsection we review the scenario of self-similar singularity studied in [32]. We first observe that the Euler system (E) has scaling property that if \((v, p)\) is a solution of the system (E), then for any \(\lambda > 0\) and \(\alpha \in \mathbb{R}\) the functions

\[
v^{\lambda,\alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1}t), \quad p^{\lambda,\alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1}t)
\]

are also solutions of (E) with the initial data \(v^{\lambda,\alpha}_0(x) = \lambda^\alpha v_0(\lambda x)\). In view of the scaling properties in (3.1), the self-similar blowing up solution \(v(x, t)\) of (E), if it exists, should be of the form,

\[
v(x, t) = \frac{1}{(T_* - t)^{\alpha/\alpha+1}} V \left( \frac{x}{(T_* - t)^{1/\alpha+1}} \right)
\]

for \(\alpha \neq -1\) and \(t\) sufficiently close to \(T_*\). If we assume that initial vorticity \(\omega_0\) has compact support, then the nonexistence of self-similar blow-up of the form given by (3.2) is rather immediate from the well-known formula, \(\omega(X(a, t), t) = \nabla_a X(a, t) \omega_0(a)\). We want to generalize this to a nontrivial case. Substituting (3.2) into (E), we find that \(V\) should be a solution of the system

\[
(SE) \left\{ \begin{array}{l}
\frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (x \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\
\text{div} \ V = 0
\end{array} \right. 
\]

for some scalar function \(P\), which could be regarded as the Euler version of the Leray equations introduced in [110]. The question of existence of nontrivial solution to (SE) is equivalent to the that of existence of nontrivial self-similar finite time blowing up solution to the Euler system of the form (3.2). Similar question for the 3D Navier-Stokes equations was raised by Leray in [110], and answered negatively by Necas, Ruzicka and Sverak [122], the result of which was refined later by Tsai in [146] (see also [119] for a generalization). Combining the energy conservation with a simple scaling argument, the author of this article showed that if there exists a nontrivial self-similar finite time blowing up solution, then its helicity should be zero (18). Mainly due to lack of the laplacian term in the right hand side of the first equations of (SE), we cannot expect the maximum principle,
which was crucial in the works in \cite{122} and \cite{146} for the 3D Navier-Stokes equations. Using a completely different argument from those previous ones, in \cite{32} it is proved that there cannot be self-similar blowing up solution to (E) of the form (3.2), if the vorticity decays sufficiently fast near infinity. Given a smooth velocity field \(v(x, t)\), the particle trajectory mapping \(a \mapsto X(a, t)\) The inverse \(A(x, t) := X^{-1}(x, t)\) is called the back to label map, which satisfies \(A(X(a, t), t) = a\), and \(X(A(x, t), t) = x\). The existence of the back-to-label map \(A(\cdot, t)\) for our smooth velocity \(v(x, t)\) for \(t \in (0, T^*)\), is guaranteed if we assume a uniform decay of \(v(x, t)\) near infinity, independent of the decay rate (see \cite{51}). The following is proved in \cite{32}.

**Theorem 3.3** There exists no finite time blowing up self-similar solution \(v(x, t)\) to the 3D Euler equations of the form (3.2) for \(t \in (0, T^*)\) with \(\alpha \neq -1\), if \(v\) and \(V\) satisfy the following conditions:

(i) For all \(t \in (0, T^*)\) the particle trajectory mapping \(X(\cdot, t)\) generated by the classical solution \(v \in C([0, T^*); C^1(\mathbb{R}^3; \mathbb{R}^3))\) is a \(C^1\) diffeomorphism from \(\mathbb{R}^3\) onto itself.

(ii) The vorticity satisfies \(\Omega = \text{curl } V \neq 0\), and there exists \(p_1 > 0\) such that \(\Omega \in L^p(\mathbb{R}^3)\) for all \(p \in (0, p_1)\).

We note that the condition (ii) is satisfied, for example, if \(\Omega \in L^1_{\text{loc}}(\mathbb{R}^3)\) and there exist constants \(R, K\) and \(\varepsilon_1, \varepsilon_2 > 0\) such that \(|\Omega(x)| \leq Ke^{-\varepsilon_1|x|^2}\) for \(|x| > R\), then we have \(\Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3)\) for all \(p \in (0, 1)\). Indeed, for all \(p \in (0, 1)\), we have

\[
\int_{\mathbb{R}^3} |\Omega(x)|^p dx = \int_{|x| \leq R} |\Omega(x)|^p dx + \int_{|x| > R} |\Omega(x)|^p dx \\
\leq |B_R|^{1-p} \left(\int_{|x| \leq R} |\Omega(x)| dx\right)^p + K^p \int_{\mathbb{R}^3} e^{-p\varepsilon_1|x|^2} dx < \infty,
\]

where \(|B_R|\) is the volume of the ball \(B_R\) of radius \(R\).

In the zero vorticity case \(\Omega = 0\), from \(\text{div } V = 0\) and \(\text{curl } V = 0\), we have \(V = \nabla h\), where \(h(x)\) is a harmonic function in \(\mathbb{R}^3\). Hence, we have an easy example of self-similar blow-up,

\[
v(x, t) = \frac{1}{(T^* - t)^{\frac{1}{\alpha + 1}}} \nabla h \left(\frac{x}{(T^* - t)^{\frac{1}{\alpha + 1}}}\right),
\]
in $\mathbb{R}^3$, which is also the case for the 3D Navier-Stokes with $\alpha = 1$. We do not consider this case in the theorem.

The above theorem is actually a corollary of the following more general theorem.

**Theorem 3.4** Let $v \in C([0, T); C^1(\mathbb{R}^3; \mathbb{R}^3))$ be a classical solution to the 3D Euler equations generating the particle trajectory mapping $X(\cdot, t)$ which is a $C^1$ diffeomorphism from $\mathbb{R}^3$ onto itself for all $t \in (0, T)$. Suppose we have representation of the vorticity of the solution, by

$$\omega(x, t) = \Psi(t)\Omega(\Phi(t)x) \quad \forall t \in [0, T) \quad (3.3)$$

where $\Psi(\cdot) \in C([0, T); (0, \infty))$, $\Phi(\cdot) \in C([0, T); \mathbb{R}^{3\times3})$ with $\det(\Phi(t)) \neq 0$ on $[0, T)$; $\Omega = \text{curl} V$ for some $V$, and there exists $p_1 > 0$ such that $|\Omega|$ belongs to $L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ on $[0, T)$, or $\Omega = 0$.

For the detailed proof of Theorem 3.3 and 3.4 we refer [32].

### 3.4 Asymptotic self-similar blow-up

In this subsection we consider the possibility of more refined scenario of self-similar singularity than in the previous subsection, called the asymptotic self-similar singularity. This means that the local in time smooth solution evolves into a self-similar profile as the possible singularity time is approached. The similar notion was considered previously by Giga and Kohn in their study of semilinear heat equation([88]). Their sense of convergence of solution to the self-similar profile is the pointwise sense with a time difference weight to make it scaling invariant, and cannot apply directly to the case of Euler system. It is found in [33] that if we make the sense of convergence strong enough, then we can apply the notion of asymptotic self-similar singularity to the Euler and the Navier-Stokes equations. The following theorem is proved in [33].

**Theorem 3.5** Let $v \in C([0, T); B^{\frac{4}{p+1}}_{p,1}(\mathbb{R}^3))$ be a classical solution to the 3D Euler equations. Suppose there exist $p_1 > 0$, $\alpha > -1$, $\bar{V} \in C^1(\mathbb{R}^3)$ with
\[
\lim_{R \to \infty} \sup_{|x| = R} |\bar{V}(x)| = 0 \text{ such that } \bar{\Omega} = \text{curl } \bar{V} \in L^q(\mathbb{R}^3) \text{ for all } q \in (0, p_1), \]
and the following convergence holds true:
\[
\lim_{t \to T} (T - t)^{\frac{2}{p+1}} \left\| v(\cdot, t) - \frac{1}{(T - t)^{\frac{1}{p+1}}} \bar{V} \left( \frac{\cdot}{(T - t)^{\frac{1}{p+1}}} \right) \right\|_{L^1} = 0,
\]
and
\[
\sup_{t \in (0, T)} (T - t) \left\| \omega(\cdot, t) - \frac{1}{T - t} \bar{\Omega} \left( \frac{\cdot}{(T - t)^{\frac{1}{p+1}}} \right) \right\|_{B_{\infty, 1}^0} < \infty.
\]

Then, \( \bar{\Omega} = 0 \), and \( v(x, t) \) can be extended to a solution of the 3D Euler system in \([0, T + \delta] \times \mathbb{R}^3\), and belongs to \( C([0, T + \delta]; B_{p,1}^{\frac{4}{p+1}}(\mathbb{R}^3)) \) for some \( \delta > 0 \).

We note that the above theorem still does not exclude the possibility that the sense of vorticity convergence to the asymptotically self-similar singularity is weaker than \( L^\infty \) sense. Namely, a self-similar vorticity profile could be approached from a local classical solution in the pointwise sense in space, or in the \( L^p(\mathbb{R}^3) \) sense for some \( p \) with \( 1 \leq p < \infty \). In [33] we also proved nonexistence of asymptotic self-similar solution to the 3D Navier-Stokes equations with appropriate change of functional setting (see also [93] for related results).

The proof of the above theorem follows without difficulty from the following blow-up rate estimate ([33]), which is interesting in itself.

**Theorem 3.6** Let \( p \in [1, \infty) \) and \( v \in C([0, T); B_{p,1}^{\frac{4}{p+1}}(\mathbb{R}^3)) \) be a classical solution to the 3D Euler equations. There exists an absolute constant \( \eta > 0 \) such that if
\[
\inf_{0 \leq t < T} (T - t) \| \omega(t) \|_{B_{\infty, 1}^2} < \eta,
\]
then \( v(x, t) \) can be extended to a solution of the 3D Euler system in \([0, T + \delta] \times \mathbb{R}^3\), and belongs to \( C([0, T + \delta]; B_{p,1}^{\frac{4}{p+1}}(\mathbb{R}^3)) \) for some \( \delta > 0 \).

We note that the proof of the local existence for \( v_0 \in B_{p,1}^{\frac{4}{p+1}}(\mathbb{R}^3) \) is done in [14, 20] (see also [147]). The above theorem implies that if \( T_* \) is the first time of singularity, then we have the lower estimate of the blow-up rate,
\[
\| \omega(t) \|_{B_{\infty, 1}^2} \geq \frac{C}{T_* - t} \quad \forall t \in [0, T_*)
\]
\[31\]
for an absolute constant $C$. The estimate (3.5) was actually derived previously by a different argument in [18]. We observe that (3.5) is consistent with both of the BKM criterion([4]) and the Kerr’s numerical calculation in [101] respectively.

The above continuation principle for a local solutions in $B^{\frac{5}{2}+1}_{p,1}(\mathbb{R}^3)$ has obvious applications to the solutions belonging to more conventional function spaces, due to the embeddings,

$$H^m(\mathbb{R}^3) \hookrightarrow C^1,\gamma(\mathbb{R}^3) \hookrightarrow B^{\frac{5}{2}+1}_{p,1}(\mathbb{R}^3)$$

for $m > 5/2$ and $\gamma = m - 5/2$. For example the local solution $v \in C([0,T); H^m(\mathbb{R}^3))$ can be continued to be $v \in C([0,T+\delta); H^m(\mathbb{R}^3))$ for some $\delta$, if (5.4) is satisfied. Regarding other implication of the above theorem on the self-similar blowing up solution to the 3D Euler equations, we have the following corollary(see [33] for the proof).

\textbf{Corollary 3.1} Let $v \in C([0,T_*); B^{\frac{5}{2}+1}_{p,1}(\mathbb{R}^3))$ be a classical solution to the 3D Euler equations. There exists $\eta > 0$ such that if we have representation for the velocity by (3.2), and $\bar{\Omega} = \text{curl} \bar{V}$ satisfies $\|\bar{\Omega}\|_{B^{\frac{5}{2}+1}_{p,1}} < \eta$, then $\bar{\Omega} = 0$, and $v(x,t)$ can be extended to a solution of the 3D Euler system in $[0,T_* + \delta] \times \mathbb{R}^3$, and belongs to $C([0,T_* + \delta]; B^{\frac{5}{2}+1}_{p,1}(\mathbb{R}^3))$ for some $\delta > 0$.

\section{Model problems}

Since the blow-up problem of the 3D Euler equations looks beyond the capability of current analysis, people proposed simplified model equations to get insight on the original problem. In this section we review some of them. Besides those results presented in the following subsections there are also studies on the other model problems. In [73] Dinaburg, Posvyanskii and Sinai analyzed a quasi-linear approximation of the infinite system of ODE arising when we write the Euler equation in a Fourier mode. Fridlander and Pavlović([80]) considered a vector model, and Katz and Pavlović([100]) studied dyadic model, both of which are resulted from the representation of the Euler equations in the wave number space. Okamoto and Ohkitani proposed model equations in [126], and a ‘dual’ system to the Euler equations was considered in [21].
4.1 Distortions of the Euler equations

Taking trace of the matrix equation (1.11) for $V$, we obtain $\Delta p = -trV^2$, and hence the hessian of the pressure is given by

$$P_{ij} = -\partial_i \partial_j (\Delta)^{-1} trV^2 = -R_i R_j trV^2,$$

where $R_j$ denotes the Riesz transform (see Section 1). Hence we can rewrite the Euler equations as

$$\frac{DV}{Dt} = -V^2 - R[trV^2], \quad R[\cdot] = (R_i R_j [\cdot])$$

(4.1)

In [47] Constantin studied a distorted version of the above system,

$$\frac{\partial V}{\partial t} = -V^2 - R[trV^2], \quad R[\cdot] = (R_i R_j [\cdot]),$$

(4.2)

where the convection term of the original Euler equations is deleted, and showed that a solution indeed blows up in finite time. Note that the incompressibility condition, $trV = 0$, is respected in the system (4.2). Thus we find that the convection term should have significant role in the study of the blow-up problem of the original Euler equations.

On the other hand, in [113] Liu and Tadmor studied another distorted version of (4.1), called the restricted Euler equations,

$$\frac{DV}{Dt} = -V^2 + \frac{1}{n} (trV^2) I.$$  (4.3)

We observe that in (4.3) the convection term is kept, but the non-local operator $R_i R_j (\cdot)$ is changed into a local one $-1/n \delta_{ij}$, where the numerical factor $-1/n$ is to keep the incompressibility condition. Analyzing the dynamics of eigenvalues of the matrix $V$, they showed that the system (4.3) actually blows up in finite time ([113]).

4.2 The Constantin-Lax-Majda equation

In 1985 Constantin, Lax and Majda constructed a one dimensional model of the vorticity formulation of the 3D Euler equations, which preserves the feature of nonlocality in vortex stretching term. Remarkably enough this model equation has an explicit solution for general initial data([58]). In this
subsection we briefly review their result. We first observe from section 1 that vorticity formulation of the Euler equations is \( \frac{D\omega}{Dt} = S\omega \), where \( S = \mathcal{P}(\omega) \) defines a singular integral operator of the Calderon-Zygmund type on \( \omega \). Let us replace \( \omega(x, t) \Rightarrow \theta(x, t), \frac{D\omega}{Dt} \Rightarrow \frac{\partial \theta}{\partial t}, \mathcal{P}(\cdot) \Rightarrow H(\cdot) \), where \( \theta(x, t) \) is a scalar function on \( \mathbb{R} \times \mathbb{R}_+ \), and \( H(\cdot) \) is the Hilbert transform defined by

\[
Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.
\]

Then we obtain, the following 1D scalar equation from the 3D Euler equation,

\[
(CL(\check{M})) : \frac{\partial \theta}{\partial t} = \theta H\theta.
\]

This model preserve the feature of nonlocality of the Euler system (E), in contrast to the more traditional one dimensional model, the inviscid Burgers equation. We recall the identities for the Hilbert transform:

\[
H(Hf) = -f, \quad H(fHg + gHf) = (Hf)(Hg) - fg,
\]

which imply \( H(\theta H\theta) = \frac{1}{2}[(H\theta)^2 - \theta^2] \). Applying \( H \) on both sides of the first equation of (CLM), and using the formula (4.4), we obtain

\[
(CL(\check{M}))^* : (H\theta)_t + \frac{1}{2}(\theta^2 - (\theta)^2) = 0.
\]

We introduce the complex valued function,

\[
z(x, t) = H\theta(x, t) + i\theta(x, t).
\]

Then, (CLM) and (CLM)* are the imaginary and the real parts of the complex Riccati equation,

\[
z_t(x, t) = \frac{1}{2}z^2(x, t)
\]

The explicit solution to the complex equation is

\[
z(x, t) = \frac{z_0}{1 - \frac{1}{2}t z_0(x)}.
\]

Taking the imaginary part, we obtain

\[
\theta(x, t) = \frac{4\theta_0(x)}{(2 - tH\theta_0(x))^2 + t^2\theta_0^2(x)}.
\]
The finite time blow-up occurs if and only if
\[ Z = \{ x \mid \theta_0(x) = 0 \text{ and } H\theta_0(x) > 0 \} \neq \emptyset. \]

In [134] Schochet find that even if we add viscosity term to (CLM) there is a finite time blow-up. See also [131, 132] for the studies of other variations of (CLM).

4.3 The 2D quasi-geostrophic equation and its 1D model

The 2D quasi-geostrophic equation (QG) models the dynamics of the mixture of cold and hot air and the fronts between them.

\[
\begin{align*}
\theta_t + (u \cdot \nabla)\theta &= 0, \\
v &= -\nabla^\perp(-\Delta)^{\frac{1}{2}}\theta, \\
\theta(x, 0) &= \theta_0(x),
\end{align*}
\]

where \( \nabla^\perp = (-\partial_2, \partial_1) \). Here, \( \theta(x, t) \) represents the temperature of the air at \((x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \). Besides its direct physical significance (QG) has another important feature that it has very similar structure to the 3D Euler equations. Indeed, taking \( \nabla^\perp \) to (QG), we obtain

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \nabla^\perp \theta = (\nabla^\perp \theta \cdot \nabla)v,
\]

where

\[
v(x, t) = \int_{\mathbb{R}^2} \frac{\nabla^\perp \theta(y, t)}{|x - y|} dy.
\]

This is exactly the vorticity formulation of 3D Euler equation if we identify

\[ \nabla^\perp \theta \iff \omega \]

After first observation and pioneering analysis of these feature by Constantin, Majda and Tabak[59] there have been so many research papers devoted to the study of this equation (also the equation with the viscosity term, \(-(-\Delta)^{\alpha}\theta, \alpha > 0\), added)[13, 53, 60, 61, 65, 64, 62, 63, 69, 70, 63, 149, 150, 151, 16, 25, 26, 42, 124, 73, 103, 10]). We briefly review some of them here concentrating on the inviscid equation (QG).
The local existence can be proved easily by standard method. The BKM type of blow-up criterion proved by Constantin, Majda and Tabak in \cite{59} is

$$\limsup_{t \to T_*} \| \theta(t) \|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \| \nabla^+ \theta(s) \|_{L^\infty} ds = \infty.$$  

(4.5)

This criterion has been refined, using the Triebel-Lizorkin spaces \cite{16}. The question of finite time singularity/global regularity is still open. Similarly to the Euler equations case we also have the following geometric type of condition for the regularity. We define the direction field $\xi = \nabla^\perp \theta / |\nabla^\perp \theta|$ whenever $|\nabla^\perp \theta(x,t)| \neq 0$.

**Definition 4.1** We say that a set $\Omega_0$ is smoothly directed if there exists $\rho > 0$ such that

$$\sup_{q \in \Omega_0^*} \int_0^T |v(X(q,t),t)|^2 dt < \infty,$$

$$\sup_{q \in \Omega_0^*} \int_0^T \| \nabla \xi(\cdot, t) \|_{L^\infty(B_\rho(X(q,t)))}^2 dt < \infty,$$

where $B_\rho(X)$ is the ball of radius $\rho$ centered at $X$ and

$$\Omega_0^* = \{ q \in \Omega_0 ; |\nabla \theta_0(q)| \neq 0 \}.$$

We denote $\Sigma_T(\Omega_0) = \{ (x,t) | x \in X(\Omega_0,t), 0 \leq t \leq T \}$. Then, the following theorem is proved(\cite{59}).

**Theorem 4.1** Assume that $\Omega_0$ is smoothly directed. Then

$$\sup_{(x,t) \in \Sigma_T(\Omega_0)} |\nabla \theta(x,t)| < \infty,$$

and no singularity occurs in $\Sigma_T(\Omega_0)$.

Next we present an ‘interpolated’ result between the criterion (4.5) and Theorem 4.1, obtained in \cite{25}. Let us denote bellow,

$$D_0 = \{ x \in \mathbb{R}^2 | ||\nabla^\perp \theta_0(x)|| \neq 0 \}, \quad D_t = X(D_0, t).$$

The following theorem(\cite{25}) could be also considered as the (QG) version of Theorem 2.9.
Theorem 4.2 Let $\theta(x,t)$ be the local classical solution to $(QG)$ with initial data $\theta_0 \in H^m(\mathbb{R}^2)$, $m > 3/2$, for which $D_0 \neq \emptyset$. Let $\xi(x,t) = \nabla^\perp \theta(x,t)/|\nabla^\perp \theta(x,t)|$ be the direction field defined for $x \in D_t$. Then, the solution can be continued up to $T < \infty$ as the classical solution, namely $\theta(t) \in C([0,T]; H^m(\mathbb{R}^2))$, if there exist parameters $p, p', q, q', s, r_1, r_2, r_3$ satisfying the following conditions,

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

(4.6)

and

$$\frac{1}{r_1} + \frac{1}{r_2} \left(1 - \frac{sq'}{2}\right) + \frac{1}{r_3} \left(1 - p' \left(1 - \frac{sq'}{2}\right)\right) = 1$$

(4.7)

with

$$0 < s < 1, \quad 1 \leq \frac{2}{sq'} < p \leq \infty, \quad 1 \leq q \leq \infty,$$

(4.8)

and

$$r_1 \in [1, \infty], \quad r_2 \in \left[p' \left(1 - \frac{sq'}{2}\right), \infty\right], \quad r_3 \in \left[1 - p' \left(1 - \frac{sq'}{2}\right), \infty\right]$$

(4.9)

such that the followings hold:

$$\int_0^T \|\xi(t)\|_{F_{\infty,q}^r(D_t)}^r dt < \infty,$$

(4.10)

and

$$\int_0^T \|\nabla^\perp \theta(t)\|_{L_{p'}^r(D_t)}^r dt + \int_0^T \|\nabla^\perp \theta(t)\|_{L_{q'}^r(D_t)}^r dt < \infty.$$

(4.11)

In order to compare this theorem with the Constantin-Majda-Tabak criterion (4.5), let us consider the case of $p = \infty, q = 1$. In this case the conditions (4.10)-(4.11) are satisfied if

$$\xi(x,t) \in L^{r_1}(0,T; C^s(\mathbb{R}^2)),$$

(4.12)

$$|\nabla^\perp \theta(x,t)| \in L^{r_2}(0,T; L^\infty(\mathbb{R}^2)) \cap L^{r_3}(0,T; L^\infty(\mathbb{R}^2)).$$

(4.13)

with

$$\frac{1}{r_1} + \frac{1}{r_2} \left(1 - \frac{s}{2}\right) + \frac{s}{2r_3} = 1.$$
If we formally passing $s \to 0$, and choosing $r_1 = \infty$, $r_2 = r_3 = 1$, we find that the conditions \((4.12)-(4.13)\) are satisfied if the Constantin-Majda-Tabak condition in \((4.5)\) holds, since the condition

$$\xi(x, t) \in L^\infty(0, T; C^0(\mathbb{R}^2)) \cong L^\infty((0, T) \times \mathbb{R}^2)$$

is automatically satisfied. The other is the case $q' = 2/s$, where \((4.10)-(4.11)\) are satisfied if

$$\xi(x, t) \in L^{r_1}(0, T; \dot{F}_\infty^{s, \frac{2}{2}}(\mathbb{R}^2)), \quad |\nabla^\perp \theta(x, t)| \in L^{r_2}(0, T; L^{\frac{2}{s}}(\mathbb{R}^2)) \quad (4.14)$$

with $1/r_1 + 1/r_2 = 1$, which shows mutual compensation of the regularity of the direction field $\xi(x, t)$ and the integrability of the magnitude of gradient $|\nabla^\perp \theta(x, t)|$ to obtain smoothness of $\theta(x, t)$.

There had been a conjectured scenario of singularity in (QG) in the form of hyperbolic saddle collapse of level curves of $\theta(x, t)$ (see Definition 3.3). This was excluded by Córdoba in 1998 ([65], see also Section 3.2 of this article). Another scenario of singularity, the sharp front singularity, which is a two dimensional version of potato chip singularity (see Definition 3.2 with $n = 2$) was excluded by Córdoba and Fefferman in [69] under the assumption of suitable velocity control (see Section 3.2).

We can also consider the possibility of self-similar singularity for (QG). We first note that (QG) has the scaling property that if $\theta$ is a solution of the system, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$\theta^{\lambda, \alpha}(x, t) = \lambda^\alpha \theta(\lambda x, \lambda^{\alpha+1} t) \quad (4.15)$$

are also solutions of (QG) with the initial data $\theta_0^{\lambda, \alpha}(x) = \lambda^\alpha \theta_0(\lambda x)$. Hence, the self-similar blowing up solution should be of the form,

$$\theta(x, t) = \frac{1}{(T_\ast - t)^{\frac{\alpha+1}{\alpha+1}}} \Theta \left( \frac{x}{(T_\ast - t)^{\frac{1}{\alpha+1}}} \right) \quad (4.16)$$

for $t$ sufficiently close $T_\ast$ and $\alpha \neq -1$. The following theorem is proved in [32].

**Theorem 4.3** Let $v$ generates a particle trajectory, which is a $C^1$ diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_\ast)$. There exists no nontrivial solution $\theta$ to the system (QG) of the form \((4.10)\), if there exists $p_1, p_2 \in (0, \infty)$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$. 

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We note that the integrability condition on the self-similar representation function $\Theta$ in the above theorem is ‘milder’ than the case of the exclusion of self-similar Euler equations in Theorem 3.3, in the sense that the decay condition is of $\Theta$ (not $\nabla^\perp \Theta$) near infinity is weaker than that of $\Omega = \text{curl} \, V$.

In the remained part of this subsection we discuss a 1D model of the 2D quasi-geostrophic equation studied in [36] (see [121] for related results). The construction of the one dimensional model can be done similarly to the Constantin-Lax-Majda equation introduced in section 4.2. We first note that

$$v = -R^\perp \theta = (-R^\perp \theta, R^\perp \theta),$$

where $R_j, j = 1, 2,$ is the two dimensional Riesz transform (see Section 1).

We can rewrite the dynamical equation of (QG) as

$$\theta_t + \text{div} [(R^\perp \theta) \theta] = 0,$$

since $\text{div}(R^\perp \theta) = 0$. To construct the one dimensional model we replace:

$$R^\perp (\cdot) \Rightarrow H(\cdot), \quad \text{div}(\cdot) \Rightarrow \partial_x$$

to obtain

$$\theta_t + (H(\theta)\theta)_x = 0.$$

Defining the complex valued function $z(x, t) = H\theta(x, t) + i\theta(x, t)$, and following Constantin-Lax-Majda([58]), we find that our equation is the imaginary part of

$$z_t + z z_x = 0,$$

which is complex Burgers’ equation. The characteristics method does not work here. Even in that case we can show that the finite time blow-up occurs for the generic initial data as follows.

**Theorem 4.4** Given a periodic non-constant initial data $\theta_0 \in C^1([\pi, \pi])$ such that $\int_{-\pi}^{\pi} \theta_0(x) \, dx = 0$, there is no $C^1([-\pi, \pi] \times [0, \infty))$ periodic solution to the model equation.

For the proof we refer [36]. Here we give a brief outline of the construction of an explicit blowing up solution. We begin with the complex Burgers equation:

$$z_t + z z_x = 0, \quad z = u + i\theta$$
with \( u(x,t) \equiv H\theta(x,t) \). Expanding it to real and imaginary parts, we obtain the system:

\[
\begin{aligned}
\left\{ \begin{array}{l}
    u_t + uu_x - \theta\theta_x = 0, \\
    \theta_t + u\theta_x + \theta u_x = 0
  \end{array} \right.
\]

In order to perform the hodograph transform we consider \( x(u,\theta) \) and \( t(u,\theta) \). We have,

\[
\begin{aligned}
    u_x &= Jt_\theta, \quad \theta_x = -Jt_u, \\
    u_t &= -Jx_\theta, \quad \theta_t = Jx_u,
\end{aligned}
\]

where \( J = (x_u t_\theta - x_\theta t_u)^{-1} \). By direct substitution we obtain,

\[
\begin{aligned}
\left\{ \begin{array}{l}
    -x_\theta + ut_\theta + \theta t_u = 0, \\
    x_u - ut_u + \theta t_\theta = 0
  \end{array} \right.
\]

as far as \( J^{-1} \neq 0 \). This system can be written more compactly in the form:

\[
\begin{aligned}
    -(x-tu)_\theta + (t\theta)_u &= 0, \\
    (x-tu)_u + (t\theta)_\theta &= 0,
\end{aligned}
\]

which leads to the following Cauchy-Riemann system,

\[
\begin{aligned}
    \xi_u = \eta_\theta, \quad \xi_\theta = -\eta_u,
\end{aligned}
\]

where we set \( \eta(u,\theta) := x(u,\theta) - t(u,\theta)u, \quad \xi(u,\theta) := t(u,\theta)\theta. \) Hence, \( f(z) = \xi(u,\theta) + i\eta(u,\theta) \) with \( z = u + i\theta \) is an analytic function. Choosing \( f(z) = \log z \), we find,

\[
t\theta = \log \sqrt{u^2 + \theta^2}, \quad x - tu = \arctan \frac{\theta}{u}, \tag{4.17}
\]

which corresponds to the initial data, \( z(x,0) = \cos x + i\sin x \). The relation (4.17) defines implicitly the real and imaginary parts \((u(x,t),\theta(x,t))\) of the solution. Removing \( \theta \) from the system, we obtain

\[
tu \tan(x-tu) = \log \left| \frac{u}{\cos(x-tu)} \right|,
\]

which defines \( u(x,t) \) implicitly. By elementary computations we find both \( u_x \) and \( \theta_x \) blow up at \( t = e^{-1} \).
4.4 The 2D Boussinesq system and Moffat’s problem

The 2D Boussinesq system for the incompressible fluid flows in $\mathbb{R}^2$ is

\[
\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\nabla p + \nu \Delta v + \theta e_2, \\
\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta &= \kappa \Delta \theta, \\
\text{div } v &= 0, \\
v(x,0) &= v_0(x), \quad \theta(x,0) = \theta_0(x),
\end{aligned}
\]

where $v = (v_1, v_2)$, $v_j = v_j(x,t)$, $j = 1, 2$, $(x,t) \in \mathbb{R}^2 \times (0, \infty)$, is the velocity vector field, $p = p(x,t)$ is the scalar pressure, $\theta(x,t)$ is the scalar temperature, $\nu \geq 0$ is the viscosity, and $\kappa \geq 0$ is the thermal diffusivity, and $e_2 = (0, 1)$.

The Boussinesq system has important roles in the atmospheric sciences (see e.g. [117]). The global in time regularity of $(B)_{\nu, \kappa}$ with $\nu > 0$ and $\kappa > 0$ is well-known (see e.g. [13]). On the other hand, the regularity/singularity questions of the fully inviscid case of $(B)_{0, 0}$ is an outstanding open problem in the mathematical fluid mechanics. It is well-known that inviscid 2D Boussinesq system has exactly same structure to the axisymmetric 3D Euler system off the axis of symmetry (see e.g. [115] for this observation). This is why the inviscid 2D Boussinesq system can be considered as a model equation of the 3D Euler system. The problem of the finite time blow-up for the fully inviscid Boussinesq system is an outstanding open problem. The BKM type of blow-up criterion, however, can be obtained without difficulty (see [38, 43, 74, 140] for various forms of blow-up criteria for the Boussinesq system). We first consider the partially viscous cases, i.e. either the zero diffusivity case, $\kappa = 0$ and $\nu > 0$, or the zero viscosity case, $\kappa > 0$ and $\nu = 0$. Even the regularity problem for partial viscosity cases has been open recently. Actually, in an article appeared in 2001, M. K. Moffatt raised a question of finite time singularity in the case $\kappa = 0, \nu > 0$ and its possible development in the limit $\kappa \to 0$ as one of the 21th century problems (see the Problem no. 3 in [120]). For this problem Cordoba, Fefferman and De La LLave ([70]) proved that special type of singularities, called ‘squirt singularities’, is absent. In [27] the author considered the both of two partial viscosity cases, and prove the global in time regularity for both of the cases. Furthermore it is proved that as diffusivity(viscosity) goes to zero the solutions of $(B)_{\nu, \kappa}$ converge strongly to those of zero diffusivity(viscosity) equations [27]. In particular the Problem no. 3 in [120] is solved. More precise statements of these results are stated.
in Theorem 1.1 and Theorem 1.2 below.

**Theorem 4.5**  Let \( \nu > 0 \) be fixed, and \( \text{div} \, v_0 = 0 \). Let \( m > 2 \) be an integer, and \( (v_0, \theta_0) \in H^m(\mathbb{R}^2) \). Then, there exists unique solution \((v, \theta)\) with \( \theta \in C([0, \infty); H^m(\mathbb{R}^2)) \) and \( v \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^2)) \) of the system \((B)_{\nu,0}\). Moreover, for each \( s < m \), the solutions \((v, \theta)\) of \((B)_{\nu,\kappa}\) converge to the corresponding solutions of \((B)_{\nu,0}\) in \( C([0, T]; H^s(\mathbb{R}^2)) \) as \( \kappa \to 0 \).

We note that Hou and Li also obtained the existence part of the above theorem independently in [92]. The following theorem is concerned with zero viscosity problem with fixed positive diffusivity.

**Theorem 4.6**  Let \( \kappa > 0 \) be fixed, and \( \text{div} \, v_0 = 0 \). Let \( m > 2 \) be an integer. Let \( m > 2 \) be an integer, and \( (v_0, \theta_0) \in H^m(\mathbb{R}^2) \). Then, there exists unique solutions \((v, \theta)\) with \( \theta \in C([0, \infty); H^m(\mathbb{R}^2)) \) and \( v \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^2)) \) of the system \((B)_{0,\kappa}\). Moreover, for each \( s < m \), the solutions \((v, \theta)\) of \((B)_{\nu,\kappa}\) converge to the corresponding solutions of \((B)_{0,\kappa}\) in \( C([0, T]; H^s(\mathbb{R}^2)) \) as \( \nu \to 0 \).

The proof of the above two theorems in [27] crucially uses the Brezis-Wainger inequality in [9, 76]. Below we consider the fully inviscid Boussinesq system, and show that there is no self-similar singularities under milder decay condition near infinity than the case of 3D Euler system. The inviscid Boussinesq system \((B) = (B)_{0,0}\) has scaling property that if \((v, \theta, p)\) is a solution of the system \((B)\), then for any \( \lambda > 0 \) and \( \alpha \in \mathbb{R} \) the functions

\[
\begin{align*}
v^{\lambda, \alpha}(x, t) &= \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \\
\theta^{\lambda, \alpha}(x, t) &= \lambda^{2\alpha+1} \theta(\lambda x, \lambda^{\alpha+1} t), \\
p^{\lambda, \alpha}(x, t) &= \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)
\end{align*}
\]

are also solutions of \((B)\) with the initial data

\[
\begin{align*}
v_0^{\lambda, \alpha}(x) &= \lambda^\alpha v_0(\lambda x), \\
\theta_0^{\lambda, \alpha}(x) &= \lambda^{2\alpha+1} \theta_0(\lambda x).
\end{align*}
\]

In view of the scaling properties in (4.18), the self-similar blowing-up solution \((v(x, t), \theta(x, t))\) of \((B)\) should of the form,

\[
\begin{align*}
v(x, t) &= \frac{1}{(T_* - t)^{\alpha + 1}} V \left( \frac{x}{(T_* - t)^{\alpha + 1}} \right), \\
\theta(x, t) &= \frac{1}{(T_* - t)^{2\alpha + 1}} \Theta \left( \frac{x}{(T_* - t)^{\frac{1}{\alpha + 1}}} \right),
\end{align*}
\]

where \( T_* \) is the blow-up time.
where $\alpha \neq -1$. We have the following nonexistence result of such type of solution (see [32]).

**Theorem 4.7** Let $v$ generates a particle trajectory, which is a $C^1$ diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_*)$. There exists no nontrivial solution $(v, \theta)$ of the system (B) of the form (4.20)-(4.21), if there exists $p_1, p_2 \in (0, \infty)$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$, and $V \in H^m(\mathbb{R}^2)$, $m > 2$.

Recalling the fact that the system (B) has the similar form as the axisymmetric 3D Euler system, we can also deduce the nonexistence of self-similar blowing up solution to the axisymmetric 3D Euler equations of the form (3.2), if $\Theta = rV^\theta$ satisfies the condition of Theorem 4.7, and curl $V = \Omega \in H^m(\mathbb{R}^3)$, $m > 5/2$, where $r = \sqrt{x_1^2 + x_2^2}$, and $V^\theta$ is the angular component of $V$. Note that in this case we do not need to assume strong decay of $\Omega$ as in Theorem 3.3. See [32] for more details.

### 4.5 Deformations of the Euler equations

Let us consider the following system considered in [34].

\[
(P_1) \begin{cases}
    \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla q + (1 + \varepsilon)\|\nabla u(t)\|_{L^\infty} u, \\
    \text{div } u = 0, \\
    u(x, 0) = u_0(x),
\end{cases}
\]

where $u = (u_1, \cdots, u_n)$, $u_j = u_j(x, t)$, $j = 1, \cdots, n$, is the unknown vector field $q = q(x, t)$ is the scalar, and $u_0$ is the given initial vector field satisfying $\text{div } u_0 = 0$. The constant $\varepsilon > 0$ is fixed. Below denote curl $u = \omega$ for ‘vorticity’ associated the ‘velocity’ $u$. We first note that the system of $(P_1)$ has the similar nonlocal structure to the Euler system (E), which is implicit in the pressure term combined with the divergence free condition. Moreover it has the same scaling properties as the original Euler system in (E). Namely, if $u(x, t), q(x, t)$ is a pair of solutions to $(P_1)$ with initial data $u_0(x)$, then for any $\alpha \in \mathbb{R}$

\[ u^\lambda(x, t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t), \quad q^\lambda(x, t) = \lambda^{2\alpha} q(\lambda x, \lambda^{\alpha+1} t) \]

is also a pair of solutions to $(P_1)$ with initial data $u_0^\lambda(x) = \lambda^\alpha u_0(x)$. As will be seen below, we can have the local well-posedness in the Sobolev space,
$H^m(\mathbb{R}^n), m > n/2 + 2$, as well as the BKM type of blow-up criterion for $(P_1)$, similarly to the Euler system (E). Furthermore, we can prove actual finite time blow-up for smooth initial data if $\omega_0 \neq 0$. This is rather surprising in the viewpoint that people working on the Euler system often have speculation that the divergence free condition might have the role of ‘desingularization’, and might make the singularity disappear. Obviously this is not the case for the system $(P_1)$. Furthermore, there is a canonical functional relation between the solution of $(P_1)$ and that of the Euler system (E); hence the word ‘deformation’. Using this relation we can translate the blow-up condition of the Euler system in terms of the solution of $(P_1)$. The precise contents of the above results on $(P_1)$ are stated in the following theorem.

**Theorem 4.8** Given $u_0 \in H^m(\mathbb{R}^n)$ with $\text{div} u_0 = 0$, where $m > \frac{n}{2} + 2$, the following statements hold true for $(P_1)$.

(i) There exists a local in time unique solution $u(t) \in C([0,T] : H^m(\mathbb{R}^n))$ with $T = T(\|u_0\|_{H^m})$.

(ii) The solution $u(x,t)$ blows-up at $t = t_*$, namely

$$\limsup_{t \to t_*} \|u(t)\|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|\omega(t)\|_{L^\infty} dt = \infty,$$

where $\omega = \text{curl} u$. Moreover, if the solution $u(x,t)$ blows up at $t_*$, then necessarily,

$$\int_0^{t_*} \exp \left[ (2 + \varepsilon) \int_0^\tau \|\nabla u(s)\|_{L^\infty} ds \right] d\tau = \infty$$

for $n = 3$, while

$$\int_0^{t_*} \exp \left[ (1 + \varepsilon) \int_0^\tau \|\nabla u(s)\|_{L^\infty} ds \right] d\tau = \infty$$

for $n = 2$.

(iii) If $\|\omega_0\|_{L^\infty} \neq 0$, then there exists time $t_* \leq \frac{1}{\varepsilon \|\omega_0\|_{L^\infty}}$ such that solution $u(x,t)$ of $(P_1)$ actually blows up at $t_*$. Moreover, at such $t_*$ we have

$$\int_0^{t_*} \exp \left[ (1 + \varepsilon) \int_0^\tau \|\nabla u(s)\|_{L^\infty} ds \right] d\tau = \infty.$$
(iv) The functional relation between the solution \( u(x,t) \) of \((P_1)\) and the solution \( v(x,t) \) of the Euler system \((E)\) is given by

\[
u(x,t) = \varphi'(t) v(x, \varphi(t)),\]

where

\[
\varphi(t) = \lambda \int_0^t \exp \left[ (1 + \varepsilon) \int_0^\tau \| \nabla u(s) \|_{L^\infty} \, ds \right] \, d\tau.
\]

(The relation between the two initial datum is \( u_0(x) = \lambda v_0(x) \).)

(v) The solution \( v(x,t) \) of the Euler system \((E)\) blows up at \( T_* < \infty \) if and only if for \( t_* := \varphi^{-1}(T_*) < \frac{1}{\varepsilon \| \omega_0 \|_{L^\infty}} \), both of the followings hold true.

\[
\int_0^{t_*} \exp \left[ (1 + \varepsilon) \int_0^\tau \| \nabla u(s) \|_{L^\infty} \, ds \right] \, d\tau < \infty,
\]

and

\[
\int_0^{t_*} \exp \left[ (2 + \varepsilon) \int_0^\tau \| \nabla u(s) \|_{L^\infty} \, ds \right] \, d\tau = \infty.
\]

For the proof we refer \[34\]. In the above theorem the result (iii) combined with (v) shows indirectly that there is no finite time blow-up in 2D Euler equations, consistent with the well-known result. Following the argument on p. 542 of \[18\], the following fact can be verified without difficulty:

We set

\[
 a(t) = \exp \left( \int_0^t (1 + \varepsilon) \| \nabla u(s) \|_{L^\infty} \, ds \right). \tag{4.22}
\]

Then, the solution \((u,q)\) of \((P_1)\) is given by

\[
 u(x,t) = a(t) U(x,t), \quad q(x,t) = a(t) P(x,t),
\]

where \((U,P)\) is a solution of the following system,

\[
 (aE) \begin{cases} 
 \frac{\partial U}{\partial t} + a(t)(U \cdot \nabla)U = -\nabla P, \\
 \text{div } U = 0, \\
 U(x,0) = U_0(x)
\end{cases}
\]

The system \((aE)\) was studied in \[18\], when \( a(t) \) is a prescribed function of \( t \), in which case the proof of local existence of \((aE)\) in \[18\] is exactly same as

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the case of \((E)\). In the current case, however, we need an extra proof of local existence, as is done in the next section, since the function \(a(t)\) defined by \((1.22)\) depends on the solution \(u(x,t)\) itself. As an application of Theorem 4.8 we can prove the following lower estimate of the possible blow-up time (see \[34\] for the detailed proof).

**Theorem 4.9** Let \(p \in (1, \infty)\) be fixed. Let \(v(t)\) be the local classical solution of the 3D Euler equations with initial data \(v_0 \in H^m(\mathbb{R}^3), m > 7/2\). If \(T_\ast\) is the first blow-up time, then

\[
T_\ast - t \geq \frac{1}{C_0 \|\omega(t)\|_{\dot{B}_{p,1}^3}}, \quad \forall t \in (0, T_\ast)
\]

where \(C_0\) is the absolute constant in \((Q_2)\).

In \[18\] the following form of lower estimate for the blow-up rate is derived.

\[
T_\ast - t \geq \frac{1}{\tilde{C}_0 \|\omega(t)\|_{\dot{B}_{\infty,1}^0}},
\]

where \(\tilde{C}_0\) is another absolute constant (see also the remarks after Theorem 3.6). Although there is (continous) embedding relation, \(\dot{B}_{p,1}^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,1}^0(\mathbb{R}^3)\) for \(p \in [1, \infty]\) (see Section 1), it is difficult to compare the two estimates \((4.23)\) and \((4.24)\) and decide which one is sharper, since the precise evaluation of the optimal constants \(C_0, \tilde{C}_0\) in those inequalities could be very difficult problem.

Next, given \(\varepsilon \geq 0\), we consider the following problem.

\[
P_2 \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla q - (1 + \varepsilon)\|\nabla u(t)\|_{L^\infty}u, \\
\text{div } u = 0, \\
u(x,0) = u_0(x),
\end{array} \right.
\]

Although the system of \((P_2)\) has also the same nonlocal structure and the scaling properties as the Euler system and \((P_1)\), we have the result of the global regularity stated in the following theorem (see \[34\] for the proof).
Theorem 4.10 Given $u_0 \in H^m(\mathbb{R}^n)$ with $\text{div} u_0 = 0$, where $m > \frac{n}{2} + 2$, then the solution $u(x,t)$ of $(P_2)$ belongs to $C([0, \infty) : H^m(\mathbb{R}^n))$. Moreover, we have the following decay estimate for the vorticity,

$$\|\omega(t)\|_{L^\infty} \leq \frac{\|\omega_0\|_{L^\infty}}{1 + \epsilon\|\omega_0\|_{L^\infty} t} \quad \forall t \in [0, \infty).$$

We also note that solution of the system $(P_2)$ has also similar functional relation with that of the Euler system as given in (iv) of Theorem 5.8 as will be clear in the proof of Theorem 1.1 in the next section.

Next, given $\epsilon > 0$, we consider the following perturbed systems of (E).

$$(E)^{\pm}_\epsilon \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla q \pm \epsilon \|\nabla u\|_{L^\infty}^{1+\epsilon} u, \\ \text{div} u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

If we set $\epsilon = 0$ in the above, then the system $(E)^0_\pm$ becomes $(E)$. For $\epsilon > 0$ we have finite time blow-up for the system $(E)^+_\epsilon$ with certain initial data, while we have the global regularity for $(E)^-_\epsilon$ with all solenoidal initial data in $H^m(\mathbb{R}^3)$, $m > 5/2$. More precisely we have the following theorem(see [31] for the proof).

Theorem 4.11 (i) Given $\epsilon > 0$, suppose $u_0 = u^\epsilon_0 \in H^m(\mathbb{R}^3)$ with $\text{div} u_0 = 0$ satisfies $\|\omega_0\|_{L^\infty} > (2/\epsilon)^{1/\epsilon}$, then there exists $T_*$ such that the solution $u(x,t)$ to $(E)^+_\epsilon$ blows up at $T_*$, namely

$$\limsup_{t \nearrow T_*} \|u(t)\|_{H^m} = \infty.$$  

(ii) Given $\epsilon > 0$ and $u_0 \in H^m(\mathbb{R}^3)$ with $\text{div} u_0 = 0$, there exists unique global in time classical solution $u(t) \in C([0, \infty) ; H^m(\mathbb{R}^3))$ to $(E)^-_\epsilon$. Moreover, we have the global in time vorticity estimate for the solution of $(E)^-_\epsilon$,

$$\|\omega(t)\|_{L^\infty} \leq \max \left\{ \|\omega_0\|_{L^\infty}, \left(\frac{1}{\epsilon}\right)^{1/\epsilon} \right\} \quad \forall t \geq 0.$$
The following theorem relates the finite time blow-up/global regularity of the Euler system with those of the system $(E)_\pm^\varepsilon$.

**Theorem 4.12** Given $\varepsilon > 0$, let $u_\pm^\varepsilon$ denote the solutions of $(E)_\pm^\varepsilon$ respectively with the same initial data $u_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$. We define

$$\varphi_\pm^\varepsilon(t, u_0) := \int_0^t \exp \left[ \pm \varepsilon \int_0^\tau \|\nabla u_\pm^\varepsilon(s)\|_L^{1+\varepsilon} ds \right] d\tau.$$ 

(i) If $\varphi_-(\infty, u_0) = \infty$, then the solution of the Euler system with initial data $u_0$ is regular globally in time.

(ii) Let $t_*$ be the first blow-up time for a solution $u_+^\varepsilon$ of $(E)_+^\varepsilon$ with initial data $u_0$ such that

$$\int_0^{t_*} \|\omega_+^\varepsilon(t)\|_{L^\infty} dt = \infty, \quad \text{where} \quad \omega_+^\varepsilon = \text{curl} \ u_+^\varepsilon.$$

If $\varphi_+(t_*, u_0) < \infty$, then the solution of the Euler system blows up at the finite time $T_* = \varphi_+(t_*, u_0)$.

We refer [31] for the proof of the above theorem.

5 **Dichotomy: singularity or global regular dynamics?**

In this section we review results in [28]. Below $S$, $P$ and $\xi(x,t)$ are the deformation tensor, the Hessian of the pressure and the vorticity direction field, associated with the flow, $v$, respectively as introduced in section 1. Let $\{ (\lambda_k, \eta^k) \}_{k=1}^3$ be the eigenvalue and the normalized eigenvectors of $S$. We set $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and

$$|\lambda| = \left( \sum_{k=1}^3 \lambda_k^2 \right)^{\frac{1}{2}}, \quad \rho_k = \eta^k \cdot P \eta^k \quad \text{for } k = 1, 2, 3.$$

We also denote

$$\eta^k(x, 0) = \eta_0^k(x), \quad \lambda_k(x, 0) = \lambda_{k,0}(x), \quad \lambda(x, 0) = \lambda_0(x), \quad \rho_k(x, 0) = \rho_{k,0}(x)$$
for the quantities at $t = 0$. Let $\omega(x,t) \neq 0$. At such point $(x,t)$ we define
the scalar fields
\[
\alpha = \xi \cdot S\xi, \quad \rho = \xi \cdot P\xi.
\]
At the points where $\omega(x,t) = 0$ we define $\alpha_0(x) = \alpha(x,0)$, $\rho_0(x) = \rho(x,0)$. Below we denote $f(X(a,t),t)' = \frac{Df}{Dt}(X(a,t),t)$ for simplicity.

Now, suppose that there is no blow-up of the solution on $[0,T_*]$, and the inequality
\[
\alpha(X(a,t),t)|\omega(X(a,t),t)| \geq \varepsilon|\omega(X(a,t),t)|^2
\]
(5.1) persists on $[0,T_*]$. We will see that this leads to a contradiction. Combining (5.1) with (1.14), we have
\[
|\omega|' \geq \varepsilon|\omega|^2.
\]
Hence, by Gronwall’s lemma, we obtain
\[
|\omega(X(a,t),t)| \geq \frac{|\omega_0(a)|}{1 - \varepsilon|\omega_0(a)|t},
\]
which implies that
\[
\limsup_{t \to T_*} |\omega(X(a,t),t)| = \infty.
\]
Thus we are lead to the following lemma.

**Lemma 5.1** Suppose $\alpha_0(a) > 0$, and there exists $\varepsilon > 0$ such that
\[
\alpha_0(a)|\omega_0(a)| \geq \varepsilon|\omega_0(a)|^2. \quad (5.2)
\]
Let us set
\[
T_* = \frac{1}{\varepsilon\alpha_0(a)}. \quad (5.3)
\]
Then, either the vorticity blows up no later than $T_*$, or there exists $t \in (0,T_*)$ such that
\[
\alpha(X(a,t),t)|\omega(X(a,t),t)| < \varepsilon|\omega(X(a,t),t)|^2. \quad (5.4)
\]
From this lemma we can derive the following:

**Theorem 5.1 (vortex dynamics)** Let $\nu_0 \in H^m(\Omega)$, $m > 5/2$, be given. We define
\[
\Phi_1(a,t) = \frac{\alpha(X(a,t),t)}{|\omega(X(a,t),t)|}
\]
and
\[ \Sigma_1(t) = \{ a \in \Omega \mid \alpha(X(a,t),t) > 0 \} \]
associated with the classical solution \( v(x,t) \). Suppose \( a \in \Sigma_1(0) \) and \( \omega_0(a) \neq 0 \). Then one of the following holds true.

(i) (finite time singularity) The solution of the Euler equations blows-up in finite time along the trajectory \( \{X(a,t)\} \).

(ii) (regular dynamics) On of the following holds true:

(a) (finite time extinction of \( \alpha \)) There exists \( t_1 \in (0,\infty) \) such that \( \alpha(X(a,t_1),t_1) = 0 \).

(b) (long time behavior of \( \Phi_1 \)) There exists an infinite sequence \( \{t_j\}_{j=1}^\infty \) with \( t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty \) as \( j \to \infty \) such that for all \( j = 1, 2, \cdots \) we have \( \Phi_1(a,0) > \Phi_1(a,t_1) > \cdots > \Phi_1(a,t_j) > \Phi_1(a,t_{j+1}) > 0 \) and \( \Phi_1(a,t) \geq \Phi_1(a,t_j) > 0 \) for all \( t \in [0,t_j] \).

As an illustration of proofs for the Theorem 5.2 and 5.3 below, we give outline of the proof of the above theorem. Let us first observe that the formula
\[
|\omega(X(a,t),t)| = \exp \left[ \int_0^t \alpha(X(a,s),s)ds \right] |\omega_0(a)|,
\]
which is obtained from [11.14] immediately shows that \( \omega(X(a,t),t) \neq 0 \) if and only if \( \omega_0(a) \neq 0 \) for the particle trajectory \( \{X(a,t)\} \) of the classical solution \( v(x,t) \) of the Euler equations. Choosing \( \varepsilon = \alpha_0(a)/|\omega_0(a)| \) in Lemma 4.1, we see that either the vorticity blows up no later than \( T_* = 1/\alpha_0(a) \), or there exists \( t_1 \in (0,T_*) \) such that
\[
\Phi_1(a,t_1) = \frac{\alpha(X(a,t_1),t_1)}{|\omega(X(a,t_1),t_1)|} < \frac{\alpha_0(a)}{|\omega_0(a)|} = \Phi_1(a,0).
\]
Under the hypothesis that (i) and (ii)-(a) do not hold true, we may assume \( a \in \Sigma_1(t_1) \) and repeat the above argument to find \( t_2 > t_1 \) such that \( \Phi_1(a,t_2) < \Phi_1(a,t_1) \), and also \( a \in \Sigma_1(t_2) \). Iterating the argument, we find a monotone increasing sequence \( \{t_j\}_{j=1}^{\infty} \) such that \( \Phi_1(a,t_j) > \Phi_1(a,t_{j+1}) \) for all \( j = 1, 2, 3, \cdots \). In particular we can choose each \( t_j \) so that \( \Phi_1(a,t) \geq \Phi_1(a,t_j) \) for all \( t \in (t_{j-1},t_j) \). If \( t_j \to t_\infty < \infty \) as \( j \to \infty \), then we can proceed further to have \( t_* > t_\infty \) such that \( \Phi_1(a,t_\infty) > \Phi_1(a,t_*) \). Hence, we may set \( t_\infty = \infty \),
which finishes the proof.

The above argument can be extended to prove the following theorems.

**Theorem 5.2 (dynamics of \( \alpha \))** Let \( v_0 \in H^m(\Omega), \ m > 5/2, \) be given. In case \( \alpha(X(a,t), t) \neq 0 \) we define

\[
\Phi_2(a,t) = \frac{|\xi \times S\xi|^2(X(a,t),t) - \rho(X(a,t),t)}{\alpha^2(X(a,t),t)},
\]

and

\[
\Sigma_2^+(t) = \{ a \in \Omega \mid \alpha(X(a,t),t) > 0, \Phi_2(X(a,t),t) > 1 \},
\]

\[
\Sigma_2^-(t) = \{ a \in \Omega \mid \alpha(X(a,t),t) < 0, \Phi_2(X(a,t),t) < 1 \},
\]

associated with \( v(x,t) \). Suppose \( a \in \Sigma_2^+(0) \cup \Sigma_2^-(0) \). Then one of the following holds true.

(i) (finite time singularity) The solution of the Euler equations blows-up in finite time along the trajectory \( \{X(a,t)\} \).

(ii) (regular dynamics) One of the following holds true:

(a) (finite time extinction of \( \alpha \)) There exists \( t_1 \in (0, \infty) \) such that \( \alpha(X(a,t_1),t_1) = 0 \).

(b) (long time behaviors of \( \Phi_2 \)) Either there exists \( T_1 \in (0, \infty) \) such that \( \Phi_2(a,T_1) = 1 \), or there exists an infinite sequence \( \{t_j\}_{j=1}^\infty \) with \( t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty \) as \( j \to \infty \) such that one of the followings hold:

(b.1) In the case \( a \in \Sigma_2^+(0) \), for all \( j = 1, 2, \cdots \) we have

\[
\Phi_2(a,0) > \Phi_2(a,t_1) > \cdots > \Phi_2(a,t_j) > \Phi_2(a,t_{j+1}) > 1 \quad \text{and} \quad \Phi_2(a,t) \geq \Phi_2(a,t_j) > 1 \quad \text{for all} \ t \in [0,t_j].
\]

(b.2) In the case \( a \in \Sigma_2^-(0) \), for all \( j = 1, 2, \cdots \) we have

\[
\Phi_2(a,0) < \Phi_2(a,t_1) < \cdots < \Phi_2(a,t_j) < \Phi_2(a,t_{j+1}) < 1 \quad \text{and} \quad \Phi_2(a,t) \leq \Phi_2(a,t_j) < 1 \quad \text{for all} \ t \in [0,t_j].
\]

**Theorem 5.3 (spectral dynamics)** Let \( v_0 \in H^m(\Omega), \ m > 5/2, \) be given. In case \( \lambda(X(a,t), t) \neq 0 \) we define

\[
\Phi_3(a,t) = \frac{\sum_{k=1}^3 \left[ -\lambda_k^3 + \frac{1}{4} |\eta_k \times \omega|^2 \lambda_k - \rho_k \lambda_k \right] (X(a,t),t)}{|\lambda(X(a,t),t)|^3},
\]
\[ \Sigma_3(t) = \{ a \in \Omega \mid \lambda(X(a,t),t) \neq 0, \Phi_3(X(a,t),t) > 0 \} \]

associated with \( v(x,t) \). Suppose \( a \in \Sigma_3(0) \). Then one of the following holds true:

(i) (finite time singularity) The solution of the Euler equations blows-up in finite time along the trajectory \( \{ X(a,t) \} \).

(ii) (regular dynamics) One of the followings hold true:

(a) (finite time extinction of \( \lambda \)) There exists \( t_1 \in (0, \infty) \) such that \( \lambda(X(a,t_1),t_1) = 0 \).

(b) (long time behavior of \( \Phi_3 \)) Either there exists \( T_1 \in (0, \infty) \) such that \( \Phi_3(a,T_1) = 0 \), or there exists an infinite sequence \( \{ t_j \}_{j=1}^\infty \) with \( t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty \) as \( j \to \infty \) such that for all \( j = 1, 2, \cdots \) we have \( \Phi_3(a,0) > \Phi_3(a,t_1) > \cdots > \Phi_3(a,t_j) > \Phi_3(a,t_{j+1}) > 0 \) and \( \Phi_3(a,t) \geq \Phi_3(a,t_j) > 0 \) for all \( t \in [0,t_j] \).

For the details of the proof of Theorem 5.2 and Theorem 5.3 we refer \[28\].

Here we present a refinement of Theorem 2.1 of \[30\], which is proved in \[28\].

**Theorem 5.4** Let \( v_0 \in H^m(\Omega) \), \( m > 5/2 \), be given. For such \( v_0 \) let us define a set \( \Sigma \subset \Omega \) by

\[ \Sigma = \{ a \in \Omega \mid \alpha_0(a) > 0, \omega_0(a) \neq 0, \exists \varepsilon \in (0,1) \ \text{such that} \ \rho_0(a) + 2\alpha_0^2(a) - |\xi_0 \times S_0\xi_0|^2(a) \leq (1 - \varepsilon)^2 \alpha_0^2(a) \}. \]

Let us set

\[ T_* = \frac{1}{\varepsilon \alpha_0(a)}. \quad (5.5) \]

Then, either the solution blows up no later than \( T_* \), or there exists \( t \in (0,T_*) \) such that

\[ \rho(X(a,t),t) + 2\alpha^2(X(a,t),t) - |\xi \times S\xi|^2(X(a,t),t) > (1 - \varepsilon)^2 \alpha^2(X(a,t),t). \quad (5.6) \]
We note that if we ignore the term $|\xi_0 \times S_0 \xi_0|^2(a)$, then we have the condition,

$$\rho_0(a) + \alpha_0^2(a) \leq (-2\varepsilon + \varepsilon^2)\alpha_0^2(a) < 0,$$

since $\varepsilon \in (0, 1)$. Thus $\Sigma \subset S$, where $S$ is the set defined in Theorem 2.1 of [30]. One can verify without difficulty that $\Sigma = \emptyset$ for the 2D Euler flows. Regarding the question if $\Sigma \neq \emptyset$ or not for 3D Euler flows, we have the following proposition (see [30] for more details).

**Proposition 5.1** Let us consider the system the domain $\Omega = [0, 2\pi]^3$ with the periodic boundary condition. In $\Omega$ we consider the Taylor-Green vortex flow defined by

$$u(x_1, x_2, x_3) = (\sin x_1 \cos x_2 \cos x_3, -\cos x_1 \sin x_2 \cos x_3, 0). \quad (5.7)$$

Then, the set

$$S_0 = \left\{ \left(0, \frac{\pi}{4}, \frac{7\pi}{4}\right), \left(0, \frac{7\pi}{4}, \frac{\pi}{4}\right) \right\}$$

is included in $\Sigma$ of Theorem 4.4. Moreover, for $x \in S_0$ we have the explicit values of $\alpha$ and $\rho$,

$$\alpha(x) = \frac{1}{2}, \quad \rho(x) = -\frac{1}{2}.$$ 

We recall that the Taylor-Green vortex has been the first candidate proposed for a finite time singularity for the 3D Euler equations, and there have been many numerical calculations of solution of (E) with the initial data given by it (see e.g. [7]).

### 6 Spectral dynamics approach

Spectral dynamics approach in the fluid mechanics was initiated by Liu and Tadmor([13]). They analyzed the restricted Euler system (4.3) in terms of (pointwise) dynamics of the eigenvalues of the velocity gradient matrix $V$. More specifically, multiplying left and right eigenvectors of $V$ to (4.3), they derived

$$\frac{D\lambda_j}{Dt} = -\lambda_j^2 + \frac{1}{n} \sum_{k=1}^{n} \lambda_k^2, \quad j = 1, 2, \cdots, n,$$

where $\lambda_j, j = 1, 2, \cdots, n$ are eigenvalues $V$, which are not necessarily real values. In this model system they proved finite time blow-up for suitable
initial data. In this section we review the results in [23], where the full Euler system is concerned. Moreover, we are working on the dynamics of eigenvalues of the deformation tensor $S$(hence real valued), not the velocity gradient matrix. We note that there were also application of the spectral dynamics of the deformation tensor in the study of regularity problem of the Navier-Stokes equations by Neustupa and Penel([123]). In this section for simplicity we consider the 3D Euler system (E) in the periodic domain, $\Omega = T^3(=\mathbb{R}^3/\mathbb{Z}^3)$. Below we denote $\lambda_1, \lambda_2, \lambda_3$ for the eigenvalues of the deformation tensor $S = (S_{ij})$ for the velocity fields of the 3D Euler system. We will first establish the following formula,

$$\frac{d}{dt} \int_{T^3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)dx = -4 \int_{T^3} \lambda_1 \lambda_2 \lambda_3 dx,$$

(6.1)

which has important implications(Theorem 6.1- Theorem 6.3 below). Indeed, using (1.12), we can compute

$$\frac{1}{2} \frac{d}{dt} \int_{T^3} S_{ij}S_{ij}dx = \int_{T^3} S_{ij} \frac{DS_{ij}}{dt}dx$$

$$= - \int_{T^3} S_{ik}S_{kj}S_{ij}dx - \frac{1}{4} \int_{T^3} \omega_i S_{ij}\omega_j + \frac{1}{4} \int_{T^3} |\omega|^2 S_{ij}dx + \int_{T^3} P_{ij}S_{ij}dx$$

$$= - \int_{T^3} S_{ik}S_{kj}S_{ij}dx - \frac{1}{8} \frac{d}{dt} \int_{T^3} |\omega|^2dx,$$

where we used the summation convention for the repeated indices, and used the $L^2$-version of the vorticity equation,

$$\frac{1}{2} \frac{d}{dt} \int_{T^3} |\omega|^2dx = \int_{T^3} \omega_i S_{ij}\omega_j, dx$$

(6.2)

which is immediate from (1.13). We note

$$\int_{T^3} |\omega|^2dx = \int_{T^3} |\nabla v|^2dx = \int_{T^3} V_{ij}V_{ij}dx = \int_{T^3} (S_{ij} + A_{ij})(S_{ij} + A_{ij})dx$$

$$= \int_{T^3} S_{ij}S_{ij}dx + \int_{T^3} A_{ij}A_{ij}dx = \int_{T^3} S_{ij}S_{ij}dx + \frac{1}{2} \int_{T^3} |\omega|^2dx.$$ 

Hence,

$$\int_{\mathbb{R}^3} S_{ij}S_{ij}dx = \frac{1}{2} \int_{T^3} |\omega|^2dx$$
Substituting this into (6.2), we obtain that
\[
\frac{d}{dt} \int_{T^3} S_{ij} S_{ij} dx = -\frac{4}{3} \int_{T^3} S_{ik} S_{kj} S_{ij} dx,
\]
which, in terms of the spectrum of \( S \), can be written as
\[
\frac{d}{dt} \int_{T^3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) dx = -\frac{4}{3} \int_{T^3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) dx. \tag{6.3}
\]
We observe from the divergence free condition, \( 0 = \text{div} \, v = \text{Tr} S = \lambda_1 + \lambda_2 + \lambda_3 \),
\[
0 = (\lambda_1 + \lambda_2 + \lambda_3)^3
= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + 3\lambda_1^2(\lambda_2 + \lambda_3) + 3\lambda_2^2(\lambda_1 + \lambda_3) + 3\lambda_3^2(\lambda_1 + \lambda_2) + 6\lambda_1\lambda_2\lambda_3
= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + 6\lambda_1\lambda_2\lambda_3.
\]
Hence, \( \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3 \). Substituting this into (6.3), we completes the proof of (6.1).

Using the formula (6.1), we can first prove the following new a priori estimate for the \( L^2 \) norm of vorticity for the 3D incompressible Euler equations(see [23] for the proof). We denote \( \mathbb{H}_m^\sigma = \{ v \in [H^m(T^3)]^3 \mid \text{div} \, v = 0 \} \).

**Theorem 6.1** Let \( v(t) \in C([0, T); \mathbb{H}_m^\sigma) \), \( m > 5/2 \) be the local classical solution of the 3D Euler equations with initial data \( v_0 \in \mathbb{H}_m^\sigma \) with \( \omega_0 \neq 0 \). Let \( \lambda_1(x, t) \geq \lambda_2(x, t) \geq \lambda_3(x, t) \) are the eigenvalues of the deformation tensor \( S_{ij}(v) = \frac{1}{2} (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \). We denote \( \lambda_2^+(x, t) = \max\{\lambda_2(x, t), 0\} \), and \( \lambda_2^-(x, t) = \min\{\lambda_2(x, t), 0\} \). Then, the following estimates hold.
\[
\exp \left[ \int_0^t \left( \frac{1}{2} \inf_{x \in T^3} \lambda_2^+(x, t) - \sup_{x \in T^3} |\lambda_2^-(x, t)| \right) dt \right] \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}}
\leq \exp \left[ \int_0^t \left( \sup_{x \in T^3} \lambda_2^+(x, t) - \frac{1}{2} \inf_{x \in T^3} |\lambda_2^-(x, t)| \right) dt \right]
\]
for all \( t \in (0, T) \).
The above estimate says, for example, that if we have the following compatibility conditions,
\[
\sup_{x \in \mathbb{T}^3} \lambda_2^+(x, t) \simeq \inf_{x \in \Omega} |\lambda_2^-(x, t)| \simeq g(t)
\]
for some time interval \([0, T]\), then
\[
\|\omega(t)\|_{L^2} \lesssim O\left(\exp\left[ C \int_0^t g(s)ds \right]\right) \quad \forall t \in [0, T]
\]
for some constant \(C\). On the other hand, we note the following connection of the above result to the previous one. From the equation
\[
\frac{D|\omega|}{Dt} = \alpha|\omega|, \quad \alpha(x, t) = \frac{\omega \cdot S\omega}{|\omega|^2}
\]
we immediately have
\[
\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} \exp\left(\int_0^t \sup_{x \in \mathbb{T}^3} \alpha(x, s)ds\right)
\]
\[
\leq \|\omega_0\|_{L^2} \exp\left(\int_0^t \sup_{x \in \mathbb{T}^3} \lambda_1(x, s)d\tau\right),
\]
where we used the fact \(\lambda_3 \leq \alpha \leq \lambda_1\), the well-known estimate for the Rayleigh quotient. We note that \(\lambda_2^+(x, t) > 0\) implies we have stretching of infinitesimal fluid volume in two directions and compression in the other one direction (planar stretching) at \((x, t)\), while \(|\lambda_2^-(x, t)| > 0\) implies stretching in one direction and compressions in two directions (linear stretching). The above estimate says that the dominance competition between planar stretching and linear stretching is an important mechanism controlling the growth/decay in time of the \(L^2\) norm of vorticity.

In order to state our next theorem we introduce some definitions. Given a differentiable vector field \(f = (f_1, f_2, f_3)\) on \(\mathbb{T}^3\), we denote by the scalar field \(\lambda_i(f), i=1,2,3\), the eigenvalues of the deformation tensor associated with \(f\). Below we always assume the ordering, \(\lambda_1(f) \geq \lambda_2(f) \geq \lambda_3(f)\). We also fix \(m > 5/2\) below. We recall that if \(f \in \mathbb{H}_m^m\), then \(\lambda_1(f) + \lambda_2(f) + \lambda_3(f) = 0\), which is another representation of \(\text{div } f = 0\).

Let us begin with introduction of admissible classes \(\mathcal{A}_\pm\) defined by
\[
\mathcal{A}_+ = \{f \in \mathbb{H}_m^m(\mathbb{T}^3) \mid \inf_{x \in \mathbb{T}^3} \lambda_2(f)(x) > 0\},
\]
Physically $A_+$ consists of solenoidal vector fields with planar stretching everywhere, while $A_-$ consists of everywhere linear stretching vector fields. Although they do not represent real physical flows, they might be useful in the study of searching initial data leading to finite time singularity for the 3D Euler equations. Given $v_0 \in \mathbb{H}^m_{\sigma}$, let $T_*(v_0)$ be the maximal time of unique existence of solution in $\mathbb{H}^m_{\sigma}$ for the system (E). Let $S_t : \mathbb{H}^m_{\sigma} \to \mathbb{H}^m_{\sigma}$ be the solution operator, mapping from initial data to the solution $v(t)$. Given $f \in A_+$, we define the first zero touching time of $\lambda_2(f)$ as

$$T(f) = \inf\{ t \in (0, T_*(v_0)) | \exists x \in T^3 \text{ such that } \lambda_2(S_t f)(x) < 0 \}.$$

Similarly for $f \in A_-$, we define

$$T(f) = \inf\{ t \in (0, T_*(v_0)) | \exists x \in T^3 \text{ such that } \lambda_2(S_t f)(x) > 0 \}.$$

The following theorem is actually an immediate corollary of Theorem 6.1, combined with the above definition of $A_\pm$ and $T(f)$. We just observe that for $v_0 \in A_+$ (resp. $A_-$) we have $\lambda_2^- = 0, \lambda_2^+ = \lambda_2$ (resp. $\lambda_2^+ = 0, \lambda_2^- = \lambda_2$) on $\Omega \times (0, T_*(v_0))$.

**Theorem 6.2** Let $v_0 \in A_\pm$ be given. We set $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ as the eigenvalues of the deformation tensor associated with $v(x,t) = (S_t v_0)(x)$ defined $t \in (0, T(v_0))$. Then, for all $t \in (0, T(v_0))$ we have the following estimates:

(i) If $v_0 \in A_+$, then

$$\exp\left( \frac{1}{2} \int_0^t \inf_{x \in T^3} |\lambda_2(x,s)| ds \right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left( \int_0^t \sup_{x \in T^3} |\lambda_2(x,s)| ds \right).$$

(ii) If $v_0 \in A_-$, then

$$\exp\left( -\int_0^t \sup_{x \in T^3} |\lambda_2(x,s)| ds \right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left( -\frac{1}{2} \int_0^t \inf_{x \in T^3} |\lambda_2(x,s)| ds \right).$$

(see [23] for the proof) If we have the compatibility conditions,

$$\inf_{x \in T^3} |\lambda_2(x,t)| \simeq \sup_{x \in T^3} |\lambda_2(x,t)| \simeq g(t) \quad \forall t \in (0, T(v_0)),$$

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which is the case for sufficiently small box $\mathbb{T}^3$, then we have

$$\frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \approx \begin{cases} \exp \left( \int_0^t g(s) ds \right) & \text{if } v_0 \in \mathcal{A}_+ \\ \exp \left( - \int_0^t g(s) ds \right) & \text{if } v_0 \in \mathcal{A}_- \end{cases}$$

for $t \in (0, T(v_0))$. In particular, if we could find $v_0 \in \mathcal{A}_+$ such that

$$\inf_{x \in \mathbb{T}^3} |\lambda_2(x,t)| \gtrsim O \left( \frac{1}{t_* - t} \right)$$

for time interval near $t_*$, then such data would lead to singularity at $t_*$. 

As another application of the formula (6.1) we have some decay in time estimates for some ratio of eigenvalues (see [23] for the proof).

**Theorem 6.3** Let $v_0 \in \mathcal{A}_+^\pm$ be given, and we set $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ as in Theorem 3.1. We define

$$\varepsilon(x,t) = \frac{|\lambda_2(x,t)|}{\lambda(x,t)} \quad \forall (x,t) \in \mathbb{T}^3 \times (0, T(v_0)),$$

where we set

$$\lambda(x,t) = \begin{cases} \lambda_1(x,t) & \text{if } v_0 \in \mathcal{A}_+ \\ -\lambda_3(x,t) & \text{if } v_0 \in \mathcal{A}_- \end{cases}.$$

Then, there exists a constant $C = C(v_0)$ such that

$$\inf_{(x,s) \in \mathbb{T}^3 \times (0,t)} \varepsilon(x,s) < \frac{C}{\sqrt{t}} \quad \forall t \in (0, T(v_0)).$$

Regarding the problem of searching finite time blowing up solution, the proof of the above theorem suggests the following:
Given $\delta > 0$, let us suppose we could find $v_0 \in \mathcal{A}_+$ such that for the associated solution $v(x,t) = (S_tv_0)(x)$ the estimate

$$\inf_{(x,s) \in \mathbb{T}^3 \times (0,t)} \varepsilon(x,s) \gtrsim O \left( \frac{1}{t_2^{\frac{1}{2} + \delta}} \right), \quad (6.4)$$

holds true, for sufficiently large time $t$. Then such $v_0$ will lead to the finite time singularity. In order to check the behavior (6.4) for a given solution we need a sharper and/or localized version of the equation (6.1) for the dynamics of eigenvalues of the deformation tensor.
7 Conservation laws for singular solutions

For the smooth solutions of the Euler equations there are many conserved quantities as described in Section 1 of this article. One of the most important conserved quantities is the total kinetic energy. For nonsmooth(weak) solutions it is not at all obvious that we still have energy conservation. Thus, there comes very interesting question of how much smoothness we need to assume for the solution to have energy conservation property. Regarding this question L. Onsager conjectured that a Hölder continuous weak solution with the Hölder exponent 1/3 preserve the energy, and this is sharp([125]). Considering Kolmogorov’s scaling argument on the energy correlation in the homogeneous turbulence the exponent 1/3 is natural. A sufficiency part of this conjecture is proved in a positive direction by an ingenious argument due to Constantin-E-Titi[54], using a special Besov type of space norm, $\dot{B}^{s}_{3,\infty}$ with $s > 1/3$ (more precisely, the Nikolskii space norm) for the velocity. See also [12] for related results in the magnetohydrodynamics. Remarkably enough Shnirelman[136] later constructed an example of weak solution of 3D Euler equations, which does not preserve energy. The problem of finding optimal regularity condition for a weak solution to have conservation property can also be considered for the helicity. Since the helicity is closely related to the topological invariants, e.g. the knottedness of vortex tubes, the non-conservation of helicity is directly related to the spontaneous apparition of singularity from local smooth solutions, which is the main theme of this article. In [19] the author of this article obtained a sufficient regularity condition for the helicity conservation, using the function space $\dot{B}^{s}_{p,\infty}$, $s > 1/3$, for the vorticity. These results on the energy and the helicity are recently refined in [24], using the Triebel-Lizorkin type of spaces, $\dot{F}^{s}_{p,q}$, and the Besov spaces $\dot{B}^{s}_{p,q}$ (see Section 1 for the definitions) with similar values for $s, p$, but allowing full range of values for $q \in [1, \infty]$.

By a weak solution of (E) in $\mathbb{R}^n \times (0, T)$ with initial data $v_0$ we mean a
vector field \( v \in C([0, T); L^2_{loc}(\mathbb{R}^n)) \) satisfying the integral identity:

\[
\begin{align*}
- \int_0^T \int_{\mathbb{R}^n} v(x, t) \cdot \frac{\partial \phi(x, t)}{\partial t} dx dt &- \int_{\mathbb{R}^n} v_0(x) \cdot \phi(x, 0) dx \\
- \int_0^T \int_{\mathbb{R}^n} v(x, t) \otimes v(x, t) : \nabla \phi(x, t) dx dt &- \int_0^T \int_{\mathbb{R}^n} \text{div} \phi(x, t) p(x, t) dx dt = 0, \\
\int_0^T \int_{\mathbb{R}^n} v(x, t) \cdot \nabla \psi(x, t) dx dt &= 0
\end{align*}
\]

(7.1)

for every vector test function \( \phi = (\phi_1, \cdots, \phi_n) \in C_0^\infty(\mathbb{R}^n \times [0, T)) \), and for every scalar test function \( \psi \in C_0^\infty(\mathbb{R}^n \times [0, T)) \). Here we used the notation \( (u \otimes v)_{ij} = u_i v_j \), and \( A : B = \sum_{i,j=1}^n A_{ij} B_{ij} \) for \( n \times n \) matrices \( A \) and \( B \).

The following is proved in [24].

**Theorem 7.1** Let \( s > 1/3 \) and \( q \in [2, \infty] \) be given. Suppose \( v \) is a weak solution of the \( n \)-dimensional Euler equations with \( v \in C([0, T]; L^2(\mathbb{R}^n)) \cap L^3(0, T; \hat{X}^s_{p,q}(\mathbb{R}^n)) \). Then, the energy is preserved in time, namely

\[
\int_{\mathbb{R}^n} |v(x, t)|^2 dx = \int_{\mathbb{R}^n} |v_0(x)|^2 dx
\]

(7.3)

for all \( t \in [0, T) \).

When we restrict \( q = \infty \), the above theorem reduce to the one in [54]. On the other hand, the results for Triebel-Lizorkin type of space are completely new.

**Theorem 7.2** Let \( s > 1/3 \), \( q \in [2, \infty] \), and \( r_1 \in [2, \infty], r_2 \in [1, \infty] \) be given, satisfying \( 2/r_1 + 1/r_2 = 1 \). Suppose \( v \) is a weak solution of the 3-D Euler equations with \( v \in C([0, T]; L^2(\mathbb{R}^3)) \cap L^{r_1}(0, T; \hat{X}^s_{p,q}(\mathbb{R}^3)) \) and \( \omega \in L^{r_2}(0, T; \hat{X}^s_{\frac{1}{p},q}(\mathbb{R}^3)) \), where the curl operation is in the sense of distribution.

Then, the helicity is preserved in time, namely

\[
\int_{\mathbb{R}^3} v(x, t) \cdot \omega(x, t) dx = \int_{\mathbb{R}^3} v_0(x) \cdot \omega_0(x) dx
\]

(7.4)
for all \( t \in [0, T) \).

Similarly to the case of Theorem 7.1, when we restrict \( q = \infty \), the above theorem reduce to the one in [19]. The results for the case of the Triebel-Lizorkin type of space, however, is new in [24].

As an application of the above theorem we have the following estimate from below of the vorticity by a constant depending on the initial data for the weak solutions of the 3D Euler equations. We estimate the helicity,

\[
\int_{\mathbb{R}^3} v(x, t) \cdot \omega(x, t) dx \leq \|v(\cdot, t)\|_{L^3} \|\omega(\cdot, t)\|_{L^{\frac{3}{2}}}
\]

\[
\leq C \|\nabla v(\cdot, t)\|_{L^{\frac{3}{2}}} \|\omega(\cdot, t)\|_{L^{\frac{3}{2}}} \leq C \|\omega(\cdot, t)\|_{L^{\frac{3}{2}}},
\]

where we used the Sobolev inequality and the Calderon-Zygmund inequality.

Combining this estimate with (7.4), we obtain the following:

**Corollary 7.1** Suppose \( v \) is a weak solution of the 3D Euler equations satisfying the conditions of Theorem 7.2. Then, we have the following estimate:

\[
\|\omega(\cdot, t)\|_{L^{\frac{3}{2}}}^2 \geq C H_0, \quad \forall t \in [0, T)
\]

where \( H_0 = \int_{\mathbb{R}^3} v_0(x) \cdot \omega_0(x) dx \) is the initial helicity, and \( C \) is an absolute constant.

Next we are concerned on the \( L^p \)-norm conservation for the weak solutions of (QG). Let \( p \in [2, \infty) \). By a weak solution of \((QG)\) in \( D \times (0, T) \) with initial data \( v_0 \) we mean a scalar field \( \theta \in C((0, T); L^p(\mathbb{R}^2) \cap L^{p-1}(\mathbb{R}^2)) \) satisfying the integral identity:

\[
- \int_0^T \int_{\mathbb{R}^2} \theta(x, t) \left[ \frac{\partial}{\partial t} + v \cdot \nabla \right] \phi(x, t) dx dt - \int_{\mathbb{R}^2} \theta_0(x) \phi(x, 0) dx = 0 \quad (7.5)
\]

\[
v(x, t) = -\nabla^\perp \int_{\mathbb{R}^2} \frac{\theta(y, t)}{|x - y|} dy \quad (7.6)
\]

for every test function \( \phi \in C_0^\infty(\mathbb{R}^2 \times [0, T)) \), where \( \nabla^\perp \) in (7.6) is in the sense of distribution. We note that contrary to the case of 3D Euler equations there is a global existence result for the weak solutions of (QG) for \( p = 2 \) due to Resnick([129]). The following is proved in [24].
Theorem 7.3 Let $s > 1/3$, $p \in [2, \infty)$, $q \in [1, \infty]$, and $r_1 \in [p, \infty], r_2 \in [1, \infty]$ be given, satisfying $p/r_1 + 1/r_2 = 1$. Suppose $\theta$ is a weak solution of (QG) with $\theta \in C([0, T]; L^p(\mathbb{R}^2) \cap L^{p-1}(\mathbb{R}^2)) \cap L^{r_1}(0, T; X_{s+p+1,q}^{s}(\mathbb{R}^2))$ and $v \in L^2(0, T; \dot{X}_{s+1,q}^{s}(\mathbb{R}^2))$. Then, the $L^p$ norm of $\theta(\cdot, t)$ is preserved, namely
\[
\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}
\] (7.7)
for all $t \in [0, T]$.

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