On the class of caustic on the moduli space of odd spin curves.

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Abstract

We introduce a locus on the moduli space of odd spin curves that we call the caustic. The caustic parametrizes spin curves with spinors having one pole and a triple zero. The codimension of this locus is one, thus it is a divisor. We express the class of this divisor in the rational Picard group via the set of standard generators.

1 Introduction.

Consider a smooth projective curve $C$ with a spin line bundle $L \to C$ such that $h^0(C, L) = 1$. Let $p \in C$ be a zero of the corresponding spinor (i.e. the holomorphic section of $L$). Let $f : C \to PH^0(C, L(p))^\vee$ be the map that maps a point on $C$ to the hyperplane in $H^0(C, L(p))$ consisting of sections that vanish at this point. Riemann-Roch formula implies that $h^0(C, L(p)) = 2$, thus $PH^0(C, L(p))^\vee \simeq \mathbb{P}^1$ and we can identify $f$ with a meromorphic function on $C$. Generically $f$ has $4g - 2$ simple critical points (and one of them is $p$). Our goal is to study the divisor class of the locus consisting of spin curves $(C, L)$ such that $f$ has less than $4g - 2$ critical points for some choice of $p$. In particular we derive a formula expressing this class in terms of the standard generators of the rational Picard group of the moduli space of spin curves.

Let us introduce the necessary notation. Denote the moduli space of smooth odd spin curves by $S_g^-$ and its compactification by $\overline{S}_g^-$. Consider the locus

$$V = \{(C, L) \in S_g^- \mid \exists p \in C : L(p) \simeq O_C(3x_1 + x_2 + \cdots + x_{g-2})\}.$$

The closure of $V$ in $\overline{S}_g^-$ consists of two irreducible components $Z_g$ and $C_g$. The first one, $Z_g$, parametrizes the spinors with multiple zeros. The class of $Z_g$ in $\text{Pic}(\overline{S}_g^-) \otimes \mathbb{Q}$ was computed by G. Farkas in [8]. Following Arnold we call the second one caustic. We have the following

**Theorem.** Let $g \geq 5$ and $\lambda, \alpha_j, j = 0, \ldots, g-1, \beta_0$ be the standard generators of the rational Picard group $\text{Pic}(\overline{S}_g^-) \otimes \mathbb{Q}$. The following formula holds:

$$[C_g] = (4g^2 + 76g - 68) \lambda - \sum_{j=1}^{g-1} (g-j)(8g+12j-8)\alpha_j - \frac{1}{8}(3g^2 + 49g - 54)\alpha_0 - \frac{1}{2}(41g-42)\beta_0.$$  

For the definition of generators of the Picard group see the next section.

Let us outline the idea of the proof. Consider the variety $\overline{S}_{g,2} \subset \overline{M}_{g,2} \times_{\overline{M}_g} \overline{S}_g^-$ which is the closure of the locus parametrizing quadruples $(C, L, p, q)$, where $p$ is a zero of a spinor and $q$

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is a critical point for $C \to PH^0(C, L(p))'$ and $q \neq p$. The forgetful map $\phi : \mathcal{F}_{g,2} \to \mathcal{F}_g$ is a
$(g-1)(4g-3)$-sheeted branched cover away from the locus where $h^0(C, L) > 1$. We will always assume that $p$ is the first marked point and $q$ is the second one.

Note that holomorphic sections of $L(p)$ are in a natural correspondence with square roots of abelian differentials with zeros of even order and possibly a pole of order 2 at $p$. We introduce a line bundle $\mathcal{T} \to \mathcal{F}_{g,2}$ whose fiber over a generic $(C, L, p, q)$ is spanned by differentials with a pole of order 2 at $p$ and a zero of order 4 at $q$ and all zeros of even order. Note that such a differential is unique up to a constant for a generic $C$. Thus this defines $\mathcal{T}$ over an open subset of $\mathcal{F}_{g,2}$, and we will see that $\mathcal{T}$ can be extended to the whole $\mathcal{F}_{g,2}$.

Denote by $\mathcal{L}_q \to \mathcal{F}_{g,2}$ the pullback from $\mathcal{M}_{g,2}$ of the tautological line bundle whose fibers are the cotangent lines to the second marked point. We have a natural homomorphism $\mathcal{T} \to \mathcal{L}_q^{\otimes 5}$, which is defined by $\omega \mapsto \text{Res}_{z=0} \frac{\omega}{z}$, where $\omega \in \mathcal{T}(C, L, p, q)$ and $z$ is a local coordinate on $C$ with $z(q) = 0$. The projection of its degeneration locus under the forgetful map $\phi$ consists of $\mathcal{C}_g$ and some boundary components. This allows us to express the class of $\mathcal{C}_g$ in terms of $\det \phi_* \mathcal{L}_q, \det \phi_* \mathcal{T}$ and the boundary divisors, and if we find expressions for classes of $\phi_* \mathcal{L}_q$ and $\phi_* \mathcal{T}$ and coefficients of boundary divisors then we are done.

To express $\det \phi_* \mathcal{T}$ in terms of the standard generators we consider the homomorphism $\mathcal{T} \to \mathcal{L}_{p}^{\otimes (1-1)}$, where $\mathcal{L}_p \to \mathcal{F}_{g,2}$ is the pullback of the tautological line bundle whose fibers are the cotangent lines to the first marked point. This homomorphism is defined similarly to the previous one by $\omega \mapsto \text{Res}_{z=0} \frac{\omega}{z}$. The projection of its degeneration locus consists of $\mathcal{C}_g, \mathcal{Z}_g$ and some boundary components. The class of $\det \phi_* \mathcal{L}_q$ can be computed by a simple arguments the explicit formula for $[\mathcal{Z}_g]$. Therefore the formula for the class of $\det \phi_* \mathcal{T}$ follows from the formula for $\mathcal{Z}_g$ ([8], see also [2]) and the computation of the coefficients of the boundary divisors.

To compute the class of $\det \phi_* \mathcal{L}_q$ we use theta functions. Namely we construct a holomorphic section of the line bundle $\lambda \otimes \mathcal{T} \otimes \mathcal{L}_p \otimes \mathcal{L}_q$ and compute its divisor explicitly. Since the class of $\det \phi_* (\mathcal{T} \otimes \mathcal{L}_p)$ is known from the above mentioned arguments, we get obtain the formula for the class of $\det \phi_* \mathcal{L}_q$.

The paper is organized as follows. In Section 2 we remind basic facts about the moduli space of spin curves and its Picard group. In Section 3 we analyze the asymptotic behaviour of the linear system $[L + p]$, define $\mathcal{T}$ and homomorphisms from it to $\mathcal{L}_{p}^{\otimes (1-1)}$ and $\mathcal{L}_q^{\otimes 5}$, and compute the degeneration loci of these homomorphisms. Finally, in Section 4 we construct the section mentioned above using theta functions, compute its divisor and drive the formula for $[\mathcal{C}_g]$.

## 2 Moduli space of odd spin curves

Let $\mathcal{M}_g$, $g \geq 3$, be the moduli space of smooth genus $g$ algebraic curves. Let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. The boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ consists of $\lceil \frac{g}{2} \rceil + 1$ irreducible divisors $\Delta_0, \ldots, \Delta_{\lceil \frac{g}{2} \rceil}$ where $\Delta_0$ is the closure of the locus of irreducible curves with one node and $\Delta_j$ for $j \geq 1$ is the closure of the locus of reducible one-nodal curves such that two irreducible components are of genera $j$ and $g-j$.

The moduli space $\mathcal{S}_g$ of smooth odd spin curves is a degree $2g-1(2g-1)$ cover of $\mathcal{M}_g$. The cover is extended to a branched cover of $\overline{\mathcal{M}}_g$ by the Cornalba compactification $\overline{\mathcal{S}}_g$ of $\mathcal{S}_g$ ramified over $\Delta_0$.

**Cornalba compactification.** A nodal curve $C$ is called quasi-stable if it satisfies two conditions:

1) Any rational component $E$ of $C$ intersects $\overline{C \setminus E}$ at two or more points;
2) Any two rational components $E_1, E_2$ of $C$ such that $\# E_i \cap \overline{C \setminus E_i} = 2$ are disjoint.

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Rational component $E$ of $C$ intersecting $\overline{C \setminus E}$ at exactly two points is called **exceptional**.

Following [3] we define a **spin curve** as a triple $(C, \eta, \beta)$ consisting of a quasi-stable curve $C$, a line bundle $\eta$ of degree $g - 1$ on it and a homomorphism $\beta : \eta \otimes^2 \to \omega_C$ with the following properties:

1) $\eta$ is of degree one on every exceptional component of $C$;
2) $\beta$ is not zero on every non-exceptional component of $C$.

The parity of the spin curve $(C, \eta, \beta)$ is the parity of $\dim H^0(C, \eta)$. The parity is invariant under continuous deformations (see [13] or [1]).

An isomorphism between $(C, \eta, \beta)$ and $(C', \eta', \beta')$ is an isomorphism $\sigma : C \to C'$ such that $\sigma^* \eta'$ and $\eta$ are isomorphic and the following diagram

$$
\begin{array}{ccc}
\eta'^2 & \phi \circ \phi^{-1} & \sigma^*(\eta')^2 \\
\downarrow \beta & & \downarrow \sigma^*\beta'
\end{array}
$$

is commutative, where $\phi$ is an isomorphism between $\eta$ and $\sigma^* \eta'$. The moduli space $\overline{S}_g$ consists of all equivalence classes of odd spin curves under such isomorphisms.

Introduce the projection $\rho : \overline{S}_g \to \overline{M}_g$ which maps (an equivalence class of) a triple $(C, \eta, \beta)$ to (an equivalence class of) a curve $\tilde{C}$, where $\tilde{C}$ is obtained from $C$ by contracting all exceptional components to points.

**Rational Picard group of $\overline{S}_g$**. The boundary $\overline{S}_g \setminus S_g$ is the union of irreducible divisors $A_0, \ldots, A_{g-1}, B_0$ such that $\rho(A_0) = \rho(B_0) = \Delta_0$ and $\rho(A_{g-j}) = \rho(A_j) = \Delta_j$ for $j = 1, \ldots, \lfloor \frac{g}{2} \rfloor$.

Description of $A_j$ for $j \neq 0$. Note that there are no spin curves $(C, \eta, \beta)$ with a reducible one-nodal base curve $C$, since the relative dualizing sheaf $\omega_C$ on a reducible curve with one node being restricted to each component must be of odd degree (see [3, 8, p.5] for more details).

Let $(C, \eta, \beta)$ be a spin curve such that $C = C_- \cup E \cup C_+$ where $C_-$ and $C_+$ are smooth curves of genus $j$ and $g - j$ respectively and $E$ is an exceptional component. The divisor $A_j$ parametrizes the closure of the locus of such curves with the property that $\eta$ restricted to $C_-$ is odd.

Description of $A_0$ and $B_0$. Contrary to the case $j \neq 0$, a spin curve $(C, \eta, \beta)$ for which $\rho(C, \eta, \beta)$ is an irreducible one-nodal curve, does not necessarily have exceptional components. Let $A_0$ parametrize the closure of the locus of spin curves with one-nodal irreducible underlying curve and $B_0$ parametrize the closure of the locus of reducible spin curves (with an exceptional component) that are mapped to $\Delta_0$ by $\rho$.

Recall that $\rho$ has simple branching along $B_0$ and is unramified on $\overline{S}_g \setminus B_0$.

Denote by $\alpha_j$ the class of $A_j$ and by $\beta_0$ the class of $B_0$ in the rational Picard group $\text{Pic}(\overline{S}_g) \otimes \mathbb{Q}$ respectively. Let $\lambda$ be the pullback of the Hodge class on $\overline{M}_g$ under $\rho$. The Picard group is generated by these classes:

$$
\text{Pic}(\overline{S}_g) \otimes \mathbb{Q} = \text{span}_\mathbb{Q}(\lambda, \alpha_0, \ldots, \alpha_{g-1}, \beta_0). 
$$

(2.1)

## 3 Asymptotics of $|L + p|$.

Let us first of all define the bundle $T \to \overline{S}_{g,2}$. Consider the universal curve $\pi_2 : \overline{\mathcal{C}S}_g \to \overline{S}_{g,2}$, blown up along singularities (we will see below that it has $A_2$ type singularities). We have two sections $\tilde{p}, \tilde{q} : \overline{S}_{g,2} \to \overline{\mathcal{C}S}_g$ associated with marked points. Given a point $(C, L, p, q) \in \overline{S}_{g,2}$ such that $C$ is smooth and $h^0(C, L) = 1$ we associate with it points $x_1, \ldots, x_{g-1} \in \pi_{g-1}^{-1}(C, L, p, q)$ such that $x_1 + x_2 + \cdots + x_{g-1}$ is the unique divisor in the linear system $|L + p - q|$. Let $V \subset \overline{\mathcal{C}S}_g$.
be the minimal closed locus such that \( V \cap \pi_2^{-1}(C, L, p, q) = \{ x_1, \ldots, x_{g-1} \} \) for any \((C, L, p, q)\) satisfying the conditions above. Consider the divisor \( D = 2[\tilde{p}(\overline{S}_{g,2})] - 2[q(\overline{S}_{g,2})] - 2[V] \). The bundle \( \mathcal{T} \) is defined as the pushforward \((\pi_2)_*(\omega_C(D))\). The sheaf \( \omega_C(D) \) can be identified with the sheaf of sections of \( \omega_C\) having a double pole at \( \tilde{p}(\overline{S}_{g,2}) \) and zeros of order at least 2 at \( V \) and at least 4 at \( q(\overline{S}_{g,2}) \) respectively. Thus generic fibers of \( \mathcal{T} \) are naturally identified with spaces of meromorphic differentials having a double pole at \( p \), zero of order 4 at \( q \) and all other zeros of even order.

Now we are going to describe the local behavior of the linear system \(|L + p|\). Let \( \pi : \overline{S}_g \rightarrow \overline{S}_g \) be the universal spin curve. Consider a simply connected neighborhood of the origin \( U \subset \mathbb{C} \) and an inclusion \( U \hookrightarrow \overline{S}_g \) with image transversal to the boundary and intersecting exactly one of the boundary components at 0 \( \in U \). Let \( \mathcal{E} \rightarrow U \) be the pullback of \( \pi \). Then the fiber of \( \mathcal{E} \) over 0 \( \in U \) is a one-nodal irreducible curve or a curve with two nodes and one exceptional component. In the former case there exists a neighborhood in \( \mathcal{E} \) of the nodal point of the central fiber isomorphic to \( \{(x, y, \varepsilon) \in U^3 \mid xy = \varepsilon \} \), and in the latter case a neighborhood of the exceptional component isomorphic to the blow up of \( \{(x, y, \varepsilon) \in U^3 \mid xy = \varepsilon^2 \} \) at the origin. Functions \( x \) and \( y \) provide local coordinates on each fiber of \( \mathcal{E} \) which we call plumbing coordinates.

Consider the family \( \tilde{\mathcal{E}} = \phi^*\mathcal{E} \rightarrow \phi^{-1}(U) = \tilde{U} \). This family is not necessary smooth. We will see below that this can indeed be the case, but \( \tilde{\mathcal{E}} \) can have only \( A_2 \) singularities. The family \( \tilde{\mathcal{E}} \) is smooth if restricted to \( \tilde{U} \setminus \{0\} \tilde{\mathcal{E}} \), and its fiber over \( t \) is a quadriple \((C_t, L_t, p_t, q_t)\), where \((C_t, L_t)\) is a smooth spin curve, \( p_t \) is a zero of the corresponding spinor and \( q_t \) is a double point of the one-dimensional linear system \(|L_t + p_t|\). We assume that \( h^0(C_t, L_t) = 1 \) for all \( t \in \tilde{U} \setminus \{0\} \) and \( \mathcal{E} \cap (\mathbb{S}_g \cup \mathcal{S}_g) = \emptyset \). Let \( \omega : \tilde{U} \rightarrow \mathcal{T}|_{\tilde{U}} \) be a non-vanishing section (recall that \( U \) is simply connected, thus such a section exists). The section \( \omega \) gives the rise of holomorphic family of meromorphic differentials \( \omega_t \) on \( C_t \), such that \( \omega_t \) is non-zero restricted to some irreducible component of \( C_t \). We will use this notations throughout this section.

### 3.1 Asymptotics near \( A_j \) for \( j > 0 \).

Suppose that \((C_0, L_0) \in A_j\), in this case \( C_0 = C_- \cup E \cup C_+ \), where \( C_- \) is a genus \( j \) curve and \( E \) is the exceptional component. Denote \( L_0|_{C_\sigma} \) by \( L_\sigma \) for \( \sigma \in \{-, +\} \). Recall that \( L_- \) is odd. We claim that \( p_0 \) and \( q_0 \) do not belong to the exceptional component. It is true for \( p_0 \) because zeros of spinors of \( L_+ \) do not converge to points on \( E \). Thus we have to verify that \( q_0 \) does not belong to the exceptional component. For convenience let us call points lying on an exceptional component exceptional points. We have two cases:

1) Suppose that \( p_0 \in C_- \). Then it belongs to the unique divisor in the linear system \(|L_-|\). Note that the linear system \(|L_- + p_0| \) generically contains \( 4j - 2 \) divisors with double points and \( 4j - 3 \) of this points can be limits of \( q_t \), since \( q_t \) cannot converge to \( p_0 \), because \( p_0 \) is a simple critical point. The nodal point on \( C_1 \) can be chosen generically, thus we can assume that all this double points do not lie on the exceptional component.

Suppose now that \( q_0 \notin C_- \). If \( q_0 \in C_+ \) then it must belong to the linear system \(|L_+ + kp_+|\), where \( p_+ \) is the nodal point. Choose the minimal \( k \), we have \( k \geq 2 \). If \( k > 2 \) then each one-dimensional subsystem of \(|L_+ + kp_+| \) has more than \( 4(g - j) \) double points which contradicts with the condition that there are only \( 4g - 3 \) ways to choose \( q_t \). Thus \( k = 2 \), and there are \( 4(g - j) \) double points on \( C_+ \{ \text{node} \} \). Therefore \( q_0 \notin E \).

2) Suppose that \( p_0 \in C_+ \). Then it belongs to the unique divisor of the linear system \(|L_+ + p_+|\). Note that \( p_+ \) and \( p_0 \) are double points of \(|L_+ + p_+ + p_0|\). Thus \(|L_+ + p_+ + p_0| \) has \( 4(g - j) - 1 \) non-exceptional double points one of which is \( p_0 \). But \( q_t \) cannot converge to \( p_0 \) therefore we
have $4(g-j) - 2$ possible non-exceptional limits on $C_+$ for $q_0$. The arguments similar to above ones imply that there are $4j - 1$ possible limits for $q_t$ on $C_-$ which are non-exceptional double points of $|L_- + 2p_-|$. Since there are $4g - 3$ ways of choosing $q_t$ for fixed $t$ we see that the limit of $q_t$ must be non-exceptional.

To finalize our analysis note that $\phi: \tilde{U} \to U$ is of degree one in this case.

### 3.2 Asymptotics near $A_0$.

In this case $C_0 = C/_{p_-\sim p_+}$ for some smooth genus $g - 1$ curve $C$. Denote by $L$ the pullback of $L_0$ to $C$. We suppose that $p_-$ and $p_+$ are chosen generically. The point $p_0$ belong to the unique divisor of $|L + \frac{1}{2}(p_- + p_+)|$, in particular is non-exceptional. The linear system $|L + \frac{1}{2}(p_- + p_+) + p_0|$ has $4g - 4$ double points with $p_0$ being one of them, thus $q_0$ can be exceptional. Since there are only 2 double points which belong to the node in the limit, $E$ may have only $A_2$ singularity.

To examine exceptional $q_0$ let us consider two families $a_t, b_t \in C_t$ of points such that $a_0$ and $b_0$ do not belong to the same divisor in $|L + \frac{1}{2}(p_- + p_+)|$ and $a_0, b_0 \in C \setminus \{p_0, p_-, p_+\}$. We have a family of maps $|L_t + p_t|: C_t \to PH^0(C_t, L_t(p_t))$. Introduce a coordinate on $PH^0(C_t, L_t(p_t))$ in such a way that $(a_t, b_t, p_t)$ goes to $(0, 1, \infty)$ and consider a map

$$F: \tilde{E} \to \mathbb{P}^1 \times \tilde{U},$$

which commutes with the projection to $\tilde{U}$ and restricted to $C_t$ is equal to $|L_t + p_t|: C_t \to PH^0(C_t, L_t(p_t))$. There exists a neighborhood $U$ of $p_-$ in $\tilde{E}$ such that $F|_U$ is a double cover. The branching divisor of $F|_U$ corresponds to the family $q_t$ with exceptional $q_0$. Suppose that $\tilde{E}$ is smooth, then $\tilde{U} \to U$ is one-to-one and the branching locus of $F$ is not smooth, because we have two way to choose $q_t$ to $q_0$. This is impossible since the branching divisor of double cover is always smooth, therefore $\tilde{E}$ has indeed the $A_2$ singularity, and $\phi: \tilde{U} \to U$ is a double cover branched at the origin. The family $\tilde{E}$ is locally isomorphic to $\{(x, y, \varepsilon) \in \mathbb{C}^3 \mid xy = \varepsilon^2\}$ with $F(x, y, \varepsilon) = (x - y, \varepsilon)$.

### 3.3 Asymptotics near $B_0$.

In this case $C_0 = C \cup E$, where $E$ is rational, $C$ is smooth of genus $g - 1$ and $C \cap E = \{p_-, p_+\}$. We assume that $p_-$ and $p_+$ are generic. Denote by $L$ the pullback of $L_0$ to $C$ and recall that $L$ is an odd spin bundle. We have four cases:

1) The points $p_0$ and $q_0$ are not exceptional. Then $p_0$ is a zero of a spinor corresponding to $L$ and $q_0$ belongs to the linear system $|L + p_0|$. This system has $4g - 7$ double points.

2) Only $q_0$ is exceptional. Arguments similar to above imply that $E$ has $A_2$ singularities at $p_-$ or $p_+$, and we have two double points of $|L_t + p_t|$ converging to $p_-$ and two to $p_+$.

3) Only $p_0$ is exceptional. Then $q_0$ is a double point of $|L + p_- + p_+|$. There are $4g - 4$ such points.

4) Both $p_0$ and $q_0$ are exceptional.

In cases 1), 3) and 4) $E$ is smooth and $\phi: \tilde{U} \to U$ is of degree one.

### 3.4 Homomorphisms.

Now we are going to describe homomorphisms from $\mathcal{T}$ to $L_p^{\otimes(-1)}$ and $L_q^{\otimes5}$ mentioned in the introduction. Recall that $L_p \to S_{g,2}$ and $L_q \to S_{g,2}$ are tautological bundles corresponding
to the first and the second marked points respectively. To formulate our result we need the following divisors in $\overline{\mathcal{M}}_{g,2}$:

$$A^\mu_j = \{(C_\mu \cup E \cup C_+, L, p, q) \in \phi^{-1}(A_j) \mid p \in C_\mu, q \in C_\mu\}, \quad j = 1, \ldots, g - 1, \; \mu \in \{-, +\},$$

$$A^s_0 = \{(C/p_{\sim} + L, p, q) \in \phi^{-1}(A_0) \mid q \in C \setminus \{p_\sim \sim p_+\}\},$$

$$A^0_0 = \{(C/p_{\sim} + L, p, q) \in \phi^{-1}(A_0) \mid q \text{ is exceptional}\}.$$

$$B^{s,s}_0 = \{(C \cup E, L, p, q) \in \phi^{-1}(B_0) \mid q \in C \setminus \{p_\sim \sim p_+\}\},$$

$$B^{s,e}_0 = \{(C \cup E, L, p, q) \in \phi^{-1}(B_0) \mid \text{only } q \text{ is exceptional}\}.$$

$$B^{e,e}_0 = \{(C \cup E, L, p, q) \in \phi^{-1}(B_0) \mid \text{only } p \text{ is exceptional}\},$$

$$B^{e,e}_0 = \{(C \cup E, L, p, q) \in \phi^{-1}(B_0) \mid \text{both } q \text{ and } p \text{ are exceptional}\}.$$

On $\overline{\mathcal{M}}_{g,2}$ we have the divisor corresponding to the closure of the locus of curves of the form $(C \cup \mathbb{P}^1, p, q)$, where $C$ is a smooth genus $g$ curve, and $p$ and $q$ lie on the rational component. Denote by $X_g$ its pullback to $\overline{\mathcal{M}}_{g,2}$.

Let $\tilde{Z}_g$ be the divisor on $\overline{\mathcal{M}}_{g,2}$ parametrizing points $(C, L, p, q)$ such that $(C, L) \in Z_g$ and $q$ corresponds to the double zero of the spinor. Note that $X_g \subset \tilde{Z}_g$.

Let $\tilde{C}_g$ be the divisor on $\overline{\mathcal{M}}_{g,2}$ parametrizing points $(C, L, p, q)$ such that $(C, L) \in C_g$ and $q$ corresponds to the point of multiplicity 3.

**Lemma 3.1.** Fix $1 \leq j \leq g - 1$. Let $\xi \in \tilde{E}$.

(-) Suppose that $(C_0, L_0, p_0, q_0) \in A_j^-$. Then $t^{-1} \omega_l(\xi)$ converges to a non-zero differential on $C_+$ with a double pole at $p_+ \in C_+$ and even zeros as $\xi \in E$ converges to $\xi_+ \in C_+ \subset E$ in transverse direction. If $\xi \to C_-$ then $\omega_l(\xi)$ converges to a non-zero differential with even zeros, a double pole at $p_0$.

(+) Suppose that $(C_0, L_0, p_0, q_0) \in A_j^+$. Then $t^{-1} \omega_l(\xi)$ converges to a non-zero differential with a pole of order 4 at $p_+ \in C_+$ and even zeros as $\xi \in E$ converges to $\xi_+ \in C_+ \subset E$. If $\xi \to \xi_- \in C_-$ then $\omega_l(\xi)$ converges to a non-zero differential with even zeros, a double pole at $p_0$ and a double zero at $p_-$.

(--) Suppose that $(C_0, L_0, p_0, q_0) \in A_j^{+-}$. Then $t^{-1} \omega_l(\xi)$ converges to a non-zero differential with a pole of order 4 at $p_- \in C_-$ and even zeros as $\xi \in E$ converges to $\xi_- \in C_- \subset E$. If $\xi \to \xi_+ \in C_+$ then $\omega_l(\xi)$ converges to a non-zero differential with even zeros, a double pole at $p_0$ and a double zero at $p_+$.

(++) Suppose that $(C_0, L_0, p_0, q_0) \in A_j^{++}$. Then $t^{-1} \omega_l(\xi)$ converges to a non-zero differential with double poles at $p_+ \in C_+$ and $p_0$ and even zeros as $\xi \in E$ converges to $\xi_+ \in C_+ \subset E$. If $\xi \to \xi_- \in C_-$ then $\omega_l(\xi)$ converges to a non-zero holomorphic differential with even zeros.

**Proof.** Consider the expansion of $\omega_l$ in plumbing coordinates:

$$\omega_l = \sum_k c_k(t)x^k \frac{dx}{x} = -\sum_k c_k(t)x^k \frac{dy}{y} \quad (3.1)$$

To prove the asymptotics in the case (--) note that for some $l$ the differential $t^l \omega_l$ converges to a non-zero meromorphic differential on $C_-$ regular at $p_-$. This is true because we know its zeros: they correspond to the divisor of the linear system $|L_- + p_0|$ passing through $q_0$. Then the expansion above implies that if $\omega_l$ vanishes with the order $l$ along $C_-$ then it vanishes with the order at least $l + 1$ along $C_+$. In fact the order is equal to $l + 1$, which can be deduced from the order of a zero of $\omega_0|C_+$ at the node which we know. Since $\omega_0$ is non-zero, then $l = 0$. 

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The case (–+). Choose \( l \) such that \( t^l \omega_t \) converges to some non-zero differential on \( C_+ \). From the analysis in Subsection 3.1 we know that the limit differential has a pole of order 4 at \( p_+ \). From this observation and the expansion (3.1) it follows that the divisor of \( \omega \) is equal to \( l[C_+] + (l - 3)[C_-] \) and \( c_2(t) = O(t) \), \( c_1(t) = O(t^2) \), thus \( l = 3 \) and \( \lim_{\xi \to \xi_-} \omega_t(\xi) \) has double zero at \( p_- \).

The case (++) is similar to the previous one.

The case (+++) Suppose that \( t^l \omega_t \) converges to a non-zero differential of \( C_+ \). Since it must have a double pole at \( p_+ \), using the expression (3.1) we get that the divisor of \( \omega \) is equal to \( (l - 1)[C_-] + l[C_+] \). This completes the proof of (++).

\[ \square \]

**Lemma 3.2.** Suppose that \((C_0, L_0) \in A_0\).

1. If \((C_0, L_0, p_0, q_0) \in A_0^e\) then \( \omega_0 \) is a non-zero differential on \( C \) with even zeros, simple poles at \( p_- \), \( p_+ \) and a double pole at \( p_0 \).

2. If \((C_0, L_0, p_0, q_0) \in B_0^e\) then \( \omega_0 \) is a meromorphic differential on \( C \) with double pole at \( p_0 \) and other zeros of even order.

**Proof.** The proof follows easily from the analysis in Subsection 3.2.

\[ \square \]

**Lemma 3.3.** Suppose that \((C_0, L_0) \in B_0\).

1. If \((C_0, L_0, p_0, q_0) \in B_0^e\) then \( \omega_0 \) is a meromorphic differential on \( C \) with simple zeros at \( p_- \) and \( p_+ \) and a double pole at \( p_0 \).

2. If \((C_0, L_0, p_0, q_0) \in B_0^e\) then \( \omega_0 \) is a zero of order 2 at \( p_0 \).

3. Choose \( \xi \in \xi_- \), thus \( \omega_0(\xi) \) has double zero at \( p_- \).

4. Choose \( \xi \in \xi_- \), thus \( \omega_0(\xi) \) has double zero at \( p_- \).

5. Choose \( \xi \in \xi_- \), thus \( \omega_0(\xi) \) has double zero at \( p_- \).

6. Choose \( \xi \in \xi_- \), thus \( \omega_0(\xi) \) has double zero at \( p_- \).

7. Choose \( \xi \in \xi_- \), thus \( \omega_0(\xi) \) has double zero at \( p_- \).

**Proof.** The proof follows directly from the analysis in Subsection 3.3.

\[ \square \]

**Lemma 3.4.** There exists a non-trivial homomorphism \( h_p : T(-B_0^e) \to L_p^\otimes(-1) \) such that

\[
\text{div } h_p = 2\hat{\mathcal{Z}}_g + 3\mathcal{X}_g + \sum_{j=1}^{g-1} A_j^{++} + B_0^e.
\]

(3.2)

Here we understood \( h_p \) as a holomorphic section of \( T(-B_0^e) \otimes L_p \).

**Proof.** Take a point \((C, L, p, q) \in \Sigma_{g,2}\) such that \( \phi(C, L, p, q) \) does not belong to the boundary or \( \mathcal{Z}_g \) and \( h^0(C, L) = 1 \). Introduce a local coordinate \( z \) on \( C \) with \( z(p) = 0 \). For \( \omega \in T_{|(C, L, p, q)} \) define \( h_p(\omega) = \text{Res}_{z=0} z \omega \). It is clear that \( h_p \) is independent of \( z \) and does not vanish away from the boundary \( \Sigma_{g,2} \) and \( \hat{\mathcal{Z}}_g \).

Let \( \mathcal{U} \subset \Sigma_{g,2} \) be a small neighborhood of a generic point \((C, L, p, q) \in \hat{\mathcal{Z}}_g \) and let \( v : \mathcal{U} \to T_{|\mathcal{U}} \) be a non-vanishing section. Let us analyze the case when \((C, L, p, q) \in \hat{\mathcal{Z}}_g \times \mathcal{X}_g \) first. We can find a neighborhood \( \mathcal{V} \subset \mathcal{C}\hat{\mathcal{S}}_g \) of \( p \in \pi_2^{-1}(\mathcal{U}) \) and functions \( z, t : \mathcal{V} \to \mathbb{C} \) such that

\[
v = \frac{(z^2 - t)^2}{z^2} dz,
\]

and \( \text{div } t = \pi_2^{-1}(\hat{\mathcal{Z}}_g) \cap \mathcal{V} \). Therefore we can extend \( h_p \) to \( \hat{\mathcal{Z}}_g \times \mathcal{X}_g \) in such a way that it has a zero of order 2 on along an open dense subset of \( \hat{\mathcal{Z}}_g \times \mathcal{X}_g \).
Suppose now that \((C, L, p, q) \in \mathcal{X}_g\). Then we can find a neighborhood \(\mathcal{V}\) of \(p\) and two functions \(z, t\) on \(\mathcal{V}\) such that
\[
v = \frac{(tz-t)^4(tz+at)^2}{(tz)^2} \quad \text{for some} \quad a \neq 0.
\]
Thus we can extend \(h_p\) to \(\mathcal{X}_g\) in such a way that it has a zero of order 2 on along an open dense subset of \(\mathcal{X}_g\). Since \(\mathcal{X}_g \subset \tilde{\mathcal{X}}_g\), this explains the first two terms in the right-hand side of (3.2).

Let us now analyze what happens near the boundary of \(\mathcal{X}_g\). Suppose that the fiber \((C, L, p, q)\) belongs to \(\partial \mathcal{X}_g\). Then \(h_p\) extends to \((C, L, p, q)\) by Lemma 3.3 and is non-zero there.

Let \((C, L, p_0, q_0) \in B_g^{e,s}\) be a generic point. Consider the local expansion of \(\omega_t\) in plumbing coordinates \(xy = t^2\):
\[
\omega_t = \frac{a(t)}{(x-t)^2} + \sum_k c_k(t)x^k \quad \text{for some} \quad b \neq 0.
\]
We have \(a(t) = t^{-1}(1+O(t))\), since \(a(0) \neq 0\) (this inequality follows from Lemma 3.3), we see that \(h_p\) can be extended to \(B_g^{e,s}\) as a non-trivial homomorphism between \(\mathcal{T}(-B_g^{e,s})\) and \(L_p^{e,s}\).

Suppose now that \((C, L, p_0, q_0) \in B_g^{e,e}\) is generic. Consider once again the expansion (3.3). We know that \(x(q_t) = t(b+O(t))\) for some \(b \neq 0\) and \(\omega_t\) has a zero of order 4 at \(q_t\). Substituting \(x(q_t)\) in (3.3) and using the fact that \(\omega_t\) has a zero of order 4 at \(q_t\) we get that the limit \(t^{-4}a(t)\) exists and is non-zero. Thus using (3.3) once again we see that \(h_p\) can be extended to \(B_g^{e,e}\) and has a zero of order 1 along it.

If \((C, L, p_0, q_0)\) belongs to \(\phi^{-1}(A_0)\), then Lemma 3.2 immediately implies that \(h_p\) can be extended to \((C, L, p_0, q_0)\) and does not vanish there. Finally, the term \(\sum_{j=1}^{g-1} A^{j+}_{e,e}\) appears in (3.2) as a consequence of Lemma 3.1.

\[\square\]

**Lemma 3.5.** There exists a non-trivial homomorphism \(h_q : \mathcal{T} \rightarrow L_q^{e,s}\) such that
\[
\text{div} \ h_q = 5\mathcal{X}_g + 2\tilde{\mathcal{C}}_g + \sum_{j=1}^{g-1} (3A^{j+}_{e,e} + 3A^{j-}_{e,e} + A^{j+}_{e,e}) + 2A^e_0 + 3B^{e,s}_0 + B^{e,e}_0.
\]

**Proof.** Let \((C, L, p, q) \in \mathcal{S}_g\) be such that \(\phi(C, L, p, q)\) does not belong to the boundary of \(\mathcal{S}_g\) or \(\mathcal{X}_g \cup \tilde{\mathcal{C}}_g\). Introduce a local coordinate \(z\) on \(C\) with \(z(q) = 0\). For \(\omega \in \mathcal{T}|_{(C, L, p, q)}\) define \(h_p(\omega) = \text{Res}_{z=0} z^{-5}\omega\). It is clear that \(h_q\) is independent of \(z\) and does not vanish away from the boundary \(\mathcal{S}_g \setminus \mathcal{S}_g \setminus \mathcal{X}_g \cup \tilde{\mathcal{C}}_g\).

Let \(\mathcal{U} \subset \mathcal{S}_g\) be a small neighborhood of a generic point \((C, L, p, q) \in \mathcal{X}_g \cup \tilde{\mathcal{C}}_g\) and let \(v : \mathcal{U} \rightarrow \mathcal{T}|_{\mathcal{U}}\) be a non-vanishing section. Let us analyze the case when \((C, L, p, q) \in \tilde{\mathcal{C}}_g\) first. We can find a neighborhood \(\mathcal{V} \subset \mathcal{C}\) of \(p \in \pi_2^{-1}(\mathcal{U})\) and functions \(z, t : \mathcal{V} \rightarrow \mathbb{C}\) such that
\[
v = (z - t)^4(z + at)^2 \quad \text{for some} \quad a \neq 0.
\]
for some $a \neq 0$, and $\div t = \pi_2^{-1}((\tilde{C}_g) \cap \mathcal{V})$. Therefore we can extend $h_q$ to an open subset of $\tilde{C}_g$ such that it has a zero of order 2 there. Suppose now that $(C, L, p, q) \in \mathcal{X}_g$. Then we can find a neighborhood $\mathcal{V}$ of $p$ and $z, t$ such that

$$v = \frac{(tz - t)^4(tz + at)^2}{(tz)^2} \ t dz = t^5 \frac{(z - 1)^4(z + a)^2}{z^2} \ dz$$

for some $a \neq 0$. Thus we can extend $h_q$ to an open subset of $\mathcal{X}_g$ such that it has a zero of order 5 there. This explains the appearance of the first two terms in the right-hand side of (3.2).

We know that

$$\text{Consider a smooth curve } C \text{ of genus } g \text{ with a fixed Torelli marking } \rho \text{ (i.e. a symplectic basis in } H_1(C, \mathbb{Z}) \text{). Let } \theta[\alpha] \text{ be a theta function with an odd characteristic } \alpha \text{ on the Jacobian of } C. \text{ The prime form is defined as }$$

$$E(x, y) = \frac{\theta[\alpha](\mathcal{A}(x - y))}{\zeta_\alpha(x)\zeta_\alpha(y)},$$

where $\mathcal{A} : \text{Div}_0(C) \to \text{Jac}(C)$ is the Abel map and $\zeta_\alpha(x) = (d_x \theta[\alpha](\mathcal{A}(x - y)))|_{x=y}^{1/2}$ is the spinor corresponding to $\alpha$. One can easily check that $E$ is a holomorphic section of the line

$$\begin{align*}
\text{4 Theta functions construction.}
\end{align*}$$
bundle $\pi_i^1L_0 \otimes \pi_i^2L_0 \otimes \mathcal{O}_{C \times C}(\delta^s(\div \theta)) \to C \times C$, where $\pi_i$ is a projection to the $i$-th factor, $\delta(x, y) = A(x - y)$ and $L_0 \to C$ is a spin bundle with zero characteristic. It does not depend on the choice of $\alpha$ but does depend on the choice of the Torelli marking.

Let now $(C, L, p, q)$ represents a point in $\overline{S}_{g,2}$ with $C$ smooth, and consider a meromorphic differential $\omega$ on $C$ representing a point in the fiber $\mathcal{T}|_{(C, L, p, q)}$. The following proposition provides a formula for $\omega$ in terms of the theta function.

**Proposition 4.1.** There exists a constant $c \in \mathbb{C}$ such that

$$
\left( \frac{d\theta[y][A(x - p + q - y)]_{y=q}}{E(x, p)} \right)^2 = c \omega(x),
$$

where $\eta$ is the theta characteristic corresponding to $L$. If $h^0(C, L) > 1$ then the left-hand side is equal to zero.

**Proof.** Suppose that $h^0(C, L) > 1$. By modular properties of the theta function $\theta[y](A(p - x + q - y))$ is equal to $\theta(A(L_0 - L + x + p - q - y))$ up to a non-zero factor, but $\dim |L| \geq 2$, though $L - x - q$ can be represented by an effective divisor. Therefore $\theta[y](A(p - x + q - y))$ by the Riemann theorem.

Suppose that $h^0(C, L) = 1$. Note that the left-hand side behaves as a differential in $x$ when all other variables are fixed. Both the left-hand side and the right-hand side of (4.1) have a double pole at $p$ and no other poles. Therefore it is enough to check that $\theta[y](A(x-p+q-y)) = 0$ if $\omega(x) = 0$. In the latter case $L + p - x - q$ is represented by an effective divisor and $\theta[y](A(x - p + q - y)) = 0$ by the Riemann theorem. \[\square\]

Let $\mathcal{Y}_g \subset \overline{S}_{g,2}$ be the divisor parametrizing points $(C, L, p, q) \in S_{g,2}$ with $h^0(C, L) > 1$. Let us study the modular properties of

$$
\varsigma(x; C, \eta, p, q) = \left( \frac{d\theta[y][A(x - p + q - y)]_{y=q}}{E(x, p)} \right)^2.
$$

**Proposition 4.2.** $\varsigma$ is a non-trivial holomorphic section of the line bundle $\mathcal{L}_y \otimes \mathcal{L}_q^2 \otimes \mathcal{L} \otimes \lambda(-2\mathcal{Y}_g) \to \overline{S}_{g,2}$.

**Proof.** The transformation law of the theta function (see [12]) implies that $\varsigma$ is indeed a section of the above line bundle over a dense open subset of $\overline{S}_{g,2}$. A simple consideration based on asymptotics of the theta function under degenerations of the curve (see [2] Section 3, Propositions 3.1, 3.2, 3.3) shows that $\varsigma$ can be extended to the whole $\overline{S}_{g,2}$. \[\square\]

**Lemma 4.1.** The section $\varsigma$ has the following divisor:

$$
\text{div } \varsigma = 2\overline{Z}_g + 3X_g + \sum_{j=1}^{g-1}(2A_{j+}^{-} + A_{j+}^{+}) + B_0^s + B_0^e + B_1^s + B_1^e + \frac{1}{4}A_0^s + \frac{1}{2}A_0^e.
$$

**Proof.** It is clear that $\varsigma$ does not vanish outside the divisors appearing in the right-hand side of (4.3). Consider a point $(C_0, L_0, p_0, q_0) \in \mathcal{Y}_g$ such that $h^0(C_0, L_0) = 1$. Take a neighborhood $\mathcal{V}$ of $q$ in the universal curve and a function $z : \mathcal{V} \to C$ such that $\text{div } z \cdot \phi^{-1}(C, L, p, q) = q$ for each $(C, L, p, q) \in \pi(\mathcal{V})$. Let $t$ be a function on $\pi(\mathcal{V})$, which gives a coordinate in a transversal direction to $\mathcal{X}_g$. Then there exists a trivialization $u$ of $\mathcal{T}|_{\pi(\mathcal{V})}$ such that

$$
u \circ \varsigma(C, L, p, q) = t^5((z(q)/t - 1)dz(q)/t)^2dz(p)/t.$$
Thus $\varsigma$ has a zero of order 5 along $X_g$. Similar arguments show that $\varsigma$ has a zero of order 2 along $\tilde{Z}_g \setminus X_g$. Since $X_g$ is an irreducible component of $\tilde{Z}_g$ this explain the first two terms in the expression for $\text{div} \, \varsigma$. The other ones follow directly from asymptotics of the theta function under degenerations of the curve.

Lemma 4.2. The following relation holds in $\text{Pic}(\mathcal{S}_g) \otimes \mathbb{Q}$:

$$[\det \phi_* L_p] \equiv (4g - 3) \left( \frac{3}{2} \lambda - \frac{1}{8} \alpha_0 + \frac{3}{2} |Z_g| \right). \quad (4.4)$$

Proof. Let $F \to \mathcal{S}_g$ be the pushforward of the universal spin bundle from the universal spin curve. Over $\mathcal{S}_g \setminus Z_g$ we have the following exact sequence:

$$0 \to \oplus_{j=1}^{4g-3} F^{\otimes 2} \to \oplus_{j=1}^{4g-3} E_g \to (\phi_* L_p) \to 0,$$

where the second arrow is the natural inclusion and the third one is the restriction map. From this sequence by simple local computations we compute that $2[F] + \frac{1}{4g-3} [\det \phi_* L_p] + \frac{3}{2} |Z_g| = \lambda$. We know that $2F = \frac{1}{8} \alpha_0 - \frac{1}{2} \lambda$ holds in $\text{Pic}(\mathcal{S}_g) \otimes \mathbb{Q}$ (see [8]). This completes the proof.

Let us recall the formula for the class of $Z_g$ in $\text{Pic}(\mathcal{S}_g) \otimes \mathbb{Q}$.

Lemma 4.3 (G. Farkas, [8]). For $g \geq 3$ the following relation holds in $\text{Pic}(\mathcal{S}_g) \otimes \mathbb{Q}$

$$[Z_g] = (g + 8) \lambda - 2 \sum_{j=1}^{g-1} (g - j) \alpha_j - \frac{g+2}{4} \alpha_0 - 2 \beta_0. \quad (4.5)$$

Remark 4.1. Using Lemma 4.2 one can in fact prove (4.3). To exclude $[\det \phi_* L_p]$ from the left-hand side of (4.4) it is enough to notice that there exists a homomorphism $F^{\otimes 2(g-1)(4g-3)} \to [\det \phi_* L_p]$ whose degenerating locus consists of $Z_g$ and the of boundary divisors.

Now we are ready to prove our main result.

Theorem. Fix a number $g \geq 5$. Let $\alpha_j, \ j = 0, \ldots, g - 1, \beta_0, \lambda$ be standard generators in $\text{Pic}(\mathcal{S}_g) \otimes \mathbb{Q}$. Then the following formula holds:

$$[C_g] \equiv \left( 4g^2 + \frac{303}{4} g - 68 \right) \lambda - \frac{1}{8} (3g^2 + 59g - 54) \alpha_0 - \frac{1}{2} (37g - 58) \beta_0 - \sum_{j=1}^{g-1} (g - j) \left( 8g + 12j - \frac{563}{2} \right) \alpha_j.$$

Proof. Lemma 4.4 and the analysis of the behaviour of $|L + p|$ near the boundary (see Subsections 5.1, 5.2, 5.3) we see that

$$[\det \phi_* T] + [\det \phi_* L_p] \equiv -(2g - 1) |Z_g| - \sum_{j=1}^{g-1} (g - j)(4g - j - 2) \alpha_j + (4g - 5) \beta_0.$$

Substituting the expression for $[\det \phi_* L_p]$ from Lemma 4.2 we get

$$[\det \phi_* T] \equiv \frac{3}{2} (4g - 3) \lambda - \left( 8g - \frac{11}{2} \right) |Z_g| - \sum_{j=1}^{g-1} (g - j)(4g - j - 2) \alpha_j + \frac{1}{8} (4g - 3) \alpha_0 + (4g - 5) \beta_0.$$
Lemma 4.1 now implies

$$2[\det \phi, L_q] = -(g-1)(4g-3)\lambda + (4g-2)[Z_g] + \sum_{j=1}^{g-1} 4(g-j)(2g-1)\alpha_j + \frac{1}{4}(g-1)(6g-7)\alpha_0 + (2g-2)\beta_0.$$ 

Finally using Lemma 3.5 and the above relations we get

$$[C_g] \equiv (4g^2 + 76g - 68)\lambda - \sum_{j=1}^{g-1} (g-j)(8g + 12j - 8)\alpha_j - \frac{1}{8}(3g^2 + 49g - 54)\alpha_0 - \frac{1}{2}(41g - 42)\beta_0.$$ 

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References

[1] M. Atiyah, *Riemann surfaces and spin structures*, Ann. Scient. Ec. Norm. Sup. 4, 47–62 (1971).

[2] M. Basok, *Tau Function and Moduli of Spin Curves*, International Mathematics Research Notices, (2015).

[3] M. Cornalba, *Moduli of curves and theta-characteristics*, Lectures on Riemann surfaces (Trieste), 560–589 (1987).

[4] R. Donagi, *The Schottky problem*, Theory of Moduli Lecture Notes in Mathematics Volume 1337, 84–137 (1988).

[5] A. Eskin, M. Kontsevich, A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmuller geodesic flow*, Publications mathematiques de l’IHES, Volume 120, Issue 1 , 207–333 (2014).

[6] G. Farkas, *The birational type of the moduli space of even spin curves*, Advances in Mathematics 223, 433–443 (2010).

[7] G. Farkas, *Theta characteristics and their moduli*, arXiv:1201.2557

[8] G. Farkas, A. Verra, *The geometry of the moduli space of odd spin curves*, arXiv:1004.0278

[9] A. Kokotov, D. Korotkin, *Tau-functions on spaces of Abelian differentials and higher genus generalization of Ray-Singer formula*, J. Diff. Geom. 82, 35–100 (2009).

[10] M. Kontsevich, A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Inventiones Mathematicae, 153, 631–678 (2003).

[11] D. Korotkin, P. Zograf, *Tau function and moduli of differentials*, Math. Res. Lett. 18, no.3, 447–458 (2011).

[12] D. Mumford, *Tata lectures on theta I*, Birkhauser (2007).
[13] D. Mumford, *Theta-characteristics of an algebraic curve*, Ann. Scient. Ec. Norm. Sup. 2, 181–191 (1971).

[14] Montserrat Teixidor i Bigas, *The divisor of curves with a vanishing theta-null*, Compositio Mathematica 66.1, 15–22 (1988).

[15] A. Yamada, *Precise variational formulas for abelian differentials*, Kodai Math. J. 3(1), 114–143 (1980).