A systolic inequality for compact quotients of Carnot groups with Popp’s volume

Kenshiro Tashiro

Abstract

In this paper, we give a systolic inequality for quotient spaces of Carnot groups $\Gamma \backslash G$ with Popp’s volume. Namely we show the existence of a positive constant $C > 0$ such that the systole of $\Gamma \backslash G$ is less than $C \operatorname{vol}(\Gamma \backslash G)^\frac{1}{Q}$, where $Q$ is the Hausdorff dimension. Moreover the constant depends only on the dimension of grading of the Lie algebra $g = \bigoplus V_i$.

1 Introduction

A sub-Riemannian manifold is a triple $(M, E, g)$, where $M$ is a smooth manifold, $E$ is a sub-bundle of the tangent bundle $TM$, and $g$ is a metric on $E$. If $E = TM$, then it is an ordinary Riemannian manifold. Geometry and analysis on sub-Riemannian manifold has been actively studied in relation to differential operator theory, geometric group theory, control theory and optimal transport theory. In this paper, we study a systolic inequality on compact quotient spaces of Carnot groups.

We recall systolic inequalities on Riemannian manifolds. On a Riemannian manifold $(M, g)$, the systole $\operatorname{sys}(M, g)$ is defined by

$$\operatorname{sys}(M, g) = \inf \{ \text{length}(c) \mid c : \text{non-contractible closed curve} \}.$$ 

If $M$ is closed, then the minimum exists. The systolic inequality asserts that for a large class of closed Riemannian manifold $M$, there is a constant $C > 0$ such that

$$\operatorname{sys}(M, g) \leq C \cdot \operatorname{vol}(M, g)^\frac{1}{d},$$

where $d = \dim M$ and $\operatorname{vol}(M, g) = \int_M d\mu_g$ is the total volume with respect to the natural volume form $d\mu_g$. A constant $C$ may depend on the topological type such as the dimension or the genus of the surface.

As an example, let us consider a flat torus $\Gamma \backslash E^d$, where $\Gamma$ is a lattice in the $d$-dimensional Euclidean space $E^d$. Then the systole $\operatorname{sys}(\Gamma \backslash E^d)$ is equal to $\min\{ ||\gamma|| \mid \gamma \in \Gamma \subset E^d \}$, and the total volume is equal to that of the fundamental domain of $\Gamma$.

The systolic inequality on flat tori is obtained in the following procedure, for example please see Section 1 of [6]. Let $p : E^d \to \Gamma \backslash E^d$ be the covering map.
Trivially there is a unique positive number \( R_0 > 0 \) such that the volume of the \( R_0 \)-ball in \( \mathbb{E}^d \) is equal to that of the flat torus \( \Gamma \backslash \mathbb{E}^d \) i.e. \( \text{vol}(B(R_0)) = \text{vol}(\Gamma \backslash \mathbb{E}^d) \). Then the restriction of the covering map \( p \) to the \( R_0 \)-ball is not injective. This implies that there is a non contractible closed curve in \( \Gamma \backslash \mathbb{E}^d \) such that its length is less than or equal to \( 2R_0 \). Hence the systole of the flat manifold \( \Gamma \backslash \mathbb{E}^d \) is bounded above by

\[
\text{sys}(\Gamma \backslash \mathbb{E}^d) \leq 2R_0 = 2\omega_d^{-\frac{1}{d}} \text{vol}(\Gamma \backslash \mathbb{E}^d)^{\frac{1}{d}},
\]

where \( \omega_d \) is the volume of the unit ball in the Euclidean space \( \mathbb{E}^d \).

Such systolic inequalities are proved for surfaces such as non-flat dimensional torus (its proof is not in the literature, but mentioned in [11]), projective space [11], and higher genus ones [4, 8, 5]. For higher dimensional Riemannian manifolds, Gromov showed the existence of the systolic constant \( C_d \) for essential manifolds [5]. Here a manifold is said to be essential if the fundamental class \([M] \in H_d(M)\) does not vanish via the natural homomorphism to the homology of the Eilenberg–Maclean space \( H_d(K(\pi_1 M, 1)) \). This class contains closed surfaces, closed aspherical manifolds and lens spaces.

We give a systolic inequality on compact quotient spaces of Carnot groups.

Let \((G, V_1, \langle \cdot, \cdot \rangle_1)\) be a Carnot group, and \( g = \bigoplus V_i \) its grading of the Lie algebra (please see Definition 2.1). We call a discrete subgroup \( \Gamma \) of a simply connected nilpotent Lie group \( G \) lattice if it is cocompact and discrete. Since \( \Gamma \) acts isometrically on \( G \) from the left, we can define the quotient sub-Riemannian metric on \( \Gamma \backslash G \). The systole on the quotient space \( \Gamma \backslash G \) is defined from this quotient metric. The volume is given by the integral of Popp’s volume form (Definition 2.2). We skip the description of its definition and properties to Section 2. Set \( d_i = \dim V_i \) and \( Q = \sum_{i=1}^{k} id_i \). Notice that \( Q \) is equal to the Hausdorff dimension of the Carnot group [9]. The main theorem of this paper is the following.

**Theorem 1.1.** There is a positive constant \( C = C(d_1, \ldots, d_k) > 0 \) such that for any Carnot group \((G, V_1, \langle \cdot, \cdot \rangle_1)\) with the grading \( \dim V_i = d_i \), and any lattice \( \Gamma < G \),

\[
\text{sys}(\Gamma \backslash G) \leq C \cdot \text{vol}(\Gamma \backslash G)^{\frac{1}{d}},
\]

An example of Carnot group is the Euclidean space \( \mathbb{E}^d \). Thus Theorem 1.1 is a generalization of the systolic inequality on flat tori.

**Remark 1.1.**

- We can show Theorem 1.1 for any \( \Gamma \) which acts isometrically, freely and cocompactly on \( G \).

- We can write a constant \( C \) by using the Hausdorff dimension \( Q \) since \( d_1, \ldots, d_k \) can be controlled by \( Q \).

The idea of the proof is similar to the flat tori case. Namely, let \( B_{CC}(R) \) be the ball in \((G, V_1, \langle \cdot, \cdot \rangle_1)\) of radius \( R \) centered at \( e \). It is well known that the volume of the \( R \)-ball satisfies

\[
\text{vol}(B_{CC}(R)) = \text{vol}(B_{CC}(1)) R^Q,
\]
please see [1]. On the other hand, there is a positive number $R_0 > 0$ such that $\text{vol}(B_{CC}(R_0)) = \text{vol}(\Gamma \setminus G)$. It implies that the systole of $\Gamma \setminus G$ is less than or equal to $2R_0$, and

$$\text{sys}(\Gamma \setminus G) \leq 2R_0 = 2\text{vol}(B_{CC}(1))^\frac{1}{d_1}\text{vol}(\Gamma \setminus G)^\frac{1}{d_1}.$$ 

The volume of the unit ball in Carnot group is more complicated than the Euclidean one. In [7], Hassannezhad–Kokarev estimated the volume of the unit ball of corank 1 Carnot groups, and gave an upper bound of the volume which depends on the dimension. It seems hard to apply their method to general Carnot groups. We indirectly compute a lower bound of the volume to obtain the following proposition, which implies the main theorem.

**Proposition 1.1** (Proposition 3.2). There exists a constant $D = D(d_1, \ldots, d_k) > 0$ such that for any $k$-step Carnot group $(G, V, \langle \cdot, \cdot \rangle_1)$ with $\dim V_i = d_i$, the volume of the unit ball $\text{vol}(B_{CC}(1))$ is greater than $D$.

Roughly speaking, the main idea is to construct the balls $B^{d_i}(R_i) \subset V_i$ $(i = 1, \ldots, k)$ such that

1. $R_1, \ldots, R_k$ depends on $d_1, \ldots, d_k$, and
2. their product $B^{d_1}(R_1) \times \cdots \times B^{d_k}(R_k)$ inscribes the unit ball $B_{CC}(1)$ via the identification $\exp : g \simeq G$.

The Ball–Box theorem asserts a similar statement. It claims the existence of a direct product of boxes which inscribes a ball. However its construction is abstract, so we cannot apply it to estimate the volume of the ball. In other words, our work can be regarded as a quantitative version of (a part of) the Ball–Box theorem.

**Acknowledgement**

The author would appreciate to Professor Takumi Yokota for many helpful comments. This research is supported by JSPS KAKENHI grant number 20J13261.

2 Carnot groups

Let $G$ be a simply connected nilpotent Lie group, and $g$ its Lie algebra. The nilpotent Lie algebra $g$ is called graded if $g$ has a direct sum decomposition $g = \bigoplus_{i=1}^{k} V_i$ such that $[V_i, V_j] = V_{i+j}$. We will identify the Lie algebra $g$ to the tangent space $T_eG$, and call a left invariant vector field $v \in g$ is horizontal vector if it is in $V_1$.

Let $\langle \cdot, \cdot \rangle_1$ be an inner product on $V_1$. Then we can define a sub-Riemannian structure on $G$ as follows. For all $g \in G$, we define a fiberwise inner product...
〈·,·〉g on $L_gV_1$ by $〈L_gv_1, L_gv_2〉_g = 〈v_1, v_2〉$ for all horizontal vector $v_1, v_2 \in V_1$. Then the sub-bundle $E = \bigsqcup_{g \in G} L_gV_1$ and the inner metric $〈·,·〉_g$ defines a left invariant sub-Riemannian metric.

**Definition 2.1** (Carnot group). A triple $(G,V_1,〈·,·〉)$ is called a Carnot group.

We call a curve $c : [a,b] \rightarrow G$ is horizontal if the derivative $\dot{c}(t)$ is in $L_{c(t)}V_1$ for a.e. $t \in [a,b]$. The length of a horizontal curve is given by the integral $l(c) = \int_a^b ||\dot{c}(t)||_{c(t)} dt$, and the distance of two points in the sub-Riemannian manifold $(G,V_1,〈·,·〉)$ is defined by the infimum of the length of curves joining these points. We call it the *Carnot–Carathéodory distance* and denote by $d_{CC}$. We also denote by $B_{CC}(R)$ the ball in $G$ of radius $R > 0$ centered at the identity element $e$.

**Dilation** For a positive number $t > 0$, define the Lie algebra isomorphism $\delta_t : g \rightarrow g$ by

$$
\delta_t \left( \sum_{j=1}^k X_j \right) = \sum_{j=1}^k t^j X_j.
$$

This isomorphism $\delta_t$ is called the *dilation*.

Let $\exp : g \rightarrow G$ be the exponential map. Since the group $G$ is simply connected and nilpotent, the exponential map is a diffeomorphism. So we will identify group elements in $G$ to vectors in $g$ via the exponential map. Moreover we can regard the dilation $\delta_t$ as the Lie group automorphism.

The origin of the name dilation comes from the following property. Let $c$ be a length minimizing curve joining two points $x,y \in G$. Since the derivative of the curve $c$ is in left translation of $V_1$ a.e., the length of the curve $\delta_t \circ c$ is equal to $t \cdot length(c)$. It implies that $d_{CC}(\delta_t(x),\delta_t(y)) = td_{CC}(x,y)$. Moreover, it also implies that the volume of the ball of radius $R$ satisfies

$$
vol(B_{CC}(R)) = vol(B_{CC}(1))R^Q,
$$

with respect to a Haar volume.

**Popp’s volume** Popp’s volume is introduced by Montgomery in his book [10] as a generalization of a Riemannian canonical volume form $d\mu_g = \sqrt{\det(g)}dx_1 \wedge \cdots \wedge dx_d$. The qualitative properties of Popp’s volume has been actively studied. By the result of Barilari–Rizzi [3], Popp’s volume is characterized as an (local) isometric invariant volume on equiregular sub-Riemannian manifolds. Agrachev–Barilari–Boscain exhibited a condition for Popp’s volume being a scalar multiple of the spherical Hausdorff volume in [2]. The qualitative study of Popp’s volume on a Carnot group is not interesting since it is a Haar volume. Our interest is in the quantitative properties of Popp’s volume. As far as our knowledge, there is no literature on its quantitative study.
Popp’s volume is constructed as follows. For \( i \geq 2 \), we inductively define the linear map \( \phi_i : V_i^\otimes i \to V_i \) by
\[
\phi_2(X_1, X_2) = [X_1, X_2], \quad \text{and} \\
\phi_i(X_1, \ldots, X_i) = [X_1, \phi_{i-1}(X_2, \ldots, X_i)] \quad (i \geq 3).
\]
We will shortly write
\[
[X_1, \ldots, X_i] := \phi_i(X_1, \ldots, X_i).
\]
Recall that on the tensor product space \( V_i^\otimes i \), we can define the canonical inner product by
\[
\langle u_1 \otimes \cdots \otimes u_i, v_1 \otimes \cdots \otimes v_i \rangle_{\otimes i} = \prod_{j=1}^i \langle u_j, v_j \rangle_1.
\]
From the inner product space \( (V_i^\otimes i, \langle \cdot, \cdot \rangle_{\otimes i}) \), we set an inner product on \( V_i \) by using the following lemma.

**Lemma 2.1** (Lemma 20.3 in [1]). Let \( E \) be an inner product space, \( V \) a vector field, and \( f : E \to V \) a surjective linear map. Then \( f \) induces an inner product on \( V \) such that the norm of \( v \in V \) is
\[
\|v\|_V = \min \{\|u\|_E \mid f(u) = v\}.
\]

Applying the above lemma to the surjective linear map \( \phi_i \), we obtain the inner product \( \langle \cdot, \cdot \rangle_i \) on each \( V_i \) for \( i = 2, \ldots, k \), and on their direct sum \( g = \bigoplus V_i \). We denote by \( \langle \cdot, \cdot \rangle_g \) the induced inner product on \( g \).

**Definition 2.2.** Popp’s volume on a Carnot group \( (G, V_1, \langle \cdot, \cdot \rangle_1) \) is the left invariant volume form induced from the inner product \( \langle \cdot, \cdot \rangle_g \) on \( g \simeq T_e G \).

**The Baker–Campbell–Hausdorff formula** Roughly speaking, the Baker–Campbell–Hausdorff formula (BCH formula) describes the solution for \( Z \) to the equation \( Z = \log(\exp(X) \exp(Y)) = X \cdot Y \). It links the Lie group product on \( G \) and the Lie bracket on \( g \) by using combinatorial coefficients. In this paper, we will use the BCH formula in the following three form. Set \( \mathbb{I}_j^p = \{1, \ldots, j\}^p \) for positive integers \( j, p \in \mathbb{N} \).

- For each \( (i_1, \ldots, i_p) \in \mathbb{I}_2^k \), there is a constant \( \alpha_{(i_1, \ldots, i_p)} \) such that for any vector \( X_{i_1}, \ldots, X_{2k} \) with \( X_i, X_{i+k} \in V_i \) (\( i = 1, \ldots, k \)),
\[
\left( \sum_{i=1}^k X_i \right) \cdot \left( \sum_{i=1}^k X_{i+k} \right) = \sum_{i=1}^{2k} X_i + \sum_{p \geq 2} \sum_{(i_1, \ldots, i_p) \in \mathbb{I}_2^p} \alpha_{(i_1, \ldots, i_p)} [X_{i_1}, \ldots, X_{i_p}],
\]

\( (2) \)

\[
= \sum_{i=1}^{2k} X_i + \sum_{p \geq 2} \sum_{(i_1, \ldots, i_p) \in \mathbb{I}_2^p} \alpha_{(i_1, \ldots, i_p)} [X_{i_1}, \ldots, X_{i_p}].
\]

\( (3) \)
For each \((n_1, \ldots, n_p) \in \mathbb{Z}_N^p\), there is a constant \(\beta(n_1, \ldots, n_p)\) such that for any vector \(X_1, \ldots, X_N \in g\),
\[
X_1 \cdots X_N = \sum_{n=1}^{N} X_n + \sum_{p \geq 2} \sum_{(n_1, \ldots, n_p) \in \mathbb{Z}_N^p} \beta(n_1, \ldots, n_p)[X_{n_1}, \ldots, X_{n_p}].
\]

For each \((i_1, \ldots, i_p) \in \mathbb{Z}_j^p\), there is a constant \(\gamma(i_1, \ldots, i_p)\) such that for any vector \(X_1, \ldots, X_j \in g\),
\[
[X_1, \cdots, X_j] = [X_1, \cdots, X_j] + \sum_{p \geq j+1} \sum_{(i_1, \ldots, i_p) \in \mathbb{Z}_j^p} \gamma(i_1, \ldots, i_p)[X_{i_1}, \cdots, X_{i_p}].
\]

Here we write the commutator of the group by \([x, y]_c = xyx^{-1}y^{-1}\), and define the map \(\psi_n : G^n \to G\) by
\[
\psi_2(x_1, x_2) = [x_1, x_2]_c \quad \text{and} \quad \psi_n(x_1, \ldots, x_n) = [x_1, \psi_{n-1}(x_2, \ldots, x_n)]_c \quad (i \geq 3).
\]

We shortly wrote
\([x_1, \ldots, x_n]_c := \psi_n(x_1, \ldots, x_n)\).

Notice that the constants \(\alpha, \beta\) and \(\gamma\) depend only on the choice of indices.

## 3 Proof of the main theorem

### 3.1 2-step case

We start from 2-step Carnot groups. Denote by \(B^{d_i}(R) \subset V_i\) the ball centered at 0 of radius \(R\) in the inner metric \(\langle \cdot, \cdot \rangle_i\). We simply write the distance from the identity element \(e\) to \(X \in g \simeq G\) by \(d_{CC}(X)\).

**Proposition 3.1.** There exists a positive constants \(\epsilon_1, \epsilon_2 > 0\) such that for any 2-step Carnot group \((G, V_1, \langle \cdot, \cdot \rangle_1)\) with \(d_i = \dim V_i\),
\[
B^{d_1}(\epsilon_1) \times B^{d_2}(\epsilon_2) \subset B_{CC}(1).
\]

In particular, the volume of the unit ball is greater than \(\epsilon_1^{d_1} \epsilon_2^{d_2} \omega_{d_1} \omega_{d_2}\).

Throughout the paper, the following construction of the group element and the vector is important.

**Definition 3.1.** For a set of vectors \(\{X_{ni}\}_{n=1, \ldots, N, i=1, \ldots, j}\), define the group element \(y(\{X_{ni}\})\) in \(G\) by
\[
y(\{X_{ni}\}) = \prod_{n=1}^{N} [X_{n1}, \cdots, X_{nj}]_c,
\]
and define the vector $Y(\{X_{ni}\})$ in $\mathfrak{g}$ by

$$Y(\{X_{ni}\}) = \sum_{n=1}^{N} [X_{n1}, \ldots, X_{nj}].$$

**Remark 3.1.** For a given set of horizontal vectors $\{X_{ni}\}$, the group element $y(\{X_{ni}\})$ coincides with $\exp(Y(\{X_{ni}\}))$ if and only if $G$ is abelian or 2-step.

**Proof of Proposition 3.1.** Let $Z$ be a given vector in $V_2$, and put $\nu = \|Z\|_2$. By the definition of the norm $\|\cdot\|_2$, there is a set of horizontal vectors $\{X_{ni}\}_{n=1}^{d_1}$ such that

$$Z = \sum_{n=1}^{d_1} [X_{n1}, X_{n2}], \quad \text{and}$$

$$\nu = \sqrt{\sum_{n,m=1}^{d_1} \langle X_{n1}, X_{m1} \rangle \langle X_{n2}, X_{m2} \rangle} = \sqrt{\sum_{n=1}^{d_1} \|X_{n1}\|_1^4 \|X_{n2}\|_1^4}. \quad (9)$$

The second equality of (9) holds by letting $\{X_{n1} \otimes X_{n2}\}_{n=1}^{d_1}$ be orthogonal. From the bi-linearity of the Lie bracket, we can assume

$$\|X_{n1}\|_1 = \|X_{n2}\|_1 \quad \text{for all } n = 1, \ldots, d_1. \quad (10)$$

As we see in Remark 3.1, $Z = Y(\{X_{ni}\}) = y(\{X_{ni}\})$. Hence by the left invariance of the distance and the triangle inequality,

$$d_{CC}(Z) \leq \sum_{n=1}^{d_1} d_{CC}(X_{n1}) + d_{CC}(X_{n2}) + d_{CC}(-X_{n1}) + d_{CC}(-X_{n2})$$

$$= 2 \sum_{n=1}^{d_1} \|X_{n1}\|_1 + \|X_{n2}\|_1.$$  

By the assumption (10),

$$2 \sum_{n=1}^{d_1} \|X_{n1}\|_1 + \|X_{n2}\|_1 = 4 \sum_{n=1}^{d_1} \|X_{n1}\|_1,$$

and

$$\nu = \sqrt{\sum_{n=1}^{d_1} \|X_{n1}\|_1^4 \|X_{n2}\|_1^4} = \sqrt{\sum_{n=1}^{d_1} \|X_{n1}\|_1^4}.$$

Combined with the above three inequalities, we obtain an upper bound of $d_{CC}(Z)$ by

$$d_{CC}(Z) \leq 4 \sum_{n=1}^{d_1} \|X_{n1}\|_1 \leq 4d_1^2 \sqrt{\sum_{n=1}^{d_1} \|X_{n1}\|_1^4} \leq 4d_1^2 \sqrt{\nu}. \quad (11)$$
Set $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \frac{1}{16}$. If $U_i \in V_i$ satisfies $\|U_i\|_i \leq \epsilon_i$ for $i = 1, 2$, then the inequality (11) yields

$$d_{CC}(U_1 + U_2) \leq d_{CC}(U_1) + d_{CC}(U_2) \leq 1.$$  

By using the volume of the unit ball in the $d_i$-dimensional Euclidean space $\omega_{d_i}$, we have

$$Vol(B_{CC}(1)) \geq Vol(B_{d_1}(\epsilon_1) \times B_{d_2}(\epsilon_2)) = \frac{1}{2^{d_1+4d_2}} \frac{1}{\omega_1^{d_1} \omega_2^{d_2}}.$$  

### 3.2 Higher step cases

For a $k$-step Carnot group $G$, a similar procedure can be applied.

**Proposition 3.2.** There exists a positive constants $\epsilon_1, \ldots, \epsilon_k > 0$ such that for any $k$-step Carnot group $(G, V_1, \langle \cdot, \cdot \rangle_1)$ with $d_i = \dim V_i$,

$$\prod_{i=1}^{k} B_{d_i}(\epsilon_i) \subset B_{CC}(1).$$

In particular, the volume of the unit ball is greater than $\prod_{i=1}^{k} \epsilon_i^{d_i} \omega_{d_i}$.

The difficulty compared from the 2-step case is the non-equality between $y(\{X_{ni}\})$ and $Y(\{X_{ni}\})$. We will compute its difference by using the BCH formula (7).

Let $Z_j$ be a given vector in $V_j$, and put $\nu_j = \|Z_j\|_j$. By the definition of the norm $\| \cdot \|_j$, there is a set of horizontal vectors $\{X_{ni}\}_{n=1,\ldots,N,j=1,\ldots,j}$ such that

$$Z_j = \sum_{n=1}^{d_j} [X_{n1}, \ldots, X_{nj}],$$

$$\nu_j = \sqrt{\sum_{n,m=1}^{d_j} \langle X_{n1}, X_{m1} \rangle \cdots \langle X_{nj}, Y_{nj} \rangle} = \sqrt{\sum_{n=1}^{d_j} \|X_{n1}\|^2_1 \cdots \|X_{nj}\|^2_j},$$

$$\|X_{n1}\|_1 = \cdots = \|X_{nj}\|_1 \text{ for all } n = 1, \ldots, N.$$  

**Definition 3.2.** We say that a set of horizontal vectors $\{X_{ni}\}$ is adjusted to $Z_j$ if the three conditions (12), (13) and (14) holds.

Let us consider the group element $y(\{X_{ni}\})$ for a set of horizontal vectors adjusted to $Z_j$. Define the map $P_l : g \to V_l$ to be the linear projection. Then by the BCH formula (7), $P_l(y(\{X_{ni}\})) = 0$ for $l \leq j - 1$, $P_j(y(\{X_{ni}\})) = Y(\{X_{ni}\}) = Z_j$, and $P_i(y(\{X_{ni}\}))$ does not vanish for $i \geq l + 1$. We label the error term as follows.
Lemma 3.1. From the bi-linearity of the Lie bracket and the subadditivity of the norm, we have
\[
\|Z_P, Z_Q\|_{p+q} \leq 2^{p+q} \|Z_P\|_p \|Z_Q\|_q,
\]
where \(p \land q = \min\{p, q\}\).

**Proof.** By the anti-symmetry of the Lie bracket, we can assume \(p \leq q\). Let \(\{X^{(p)}\}_{m} \) be a set of horizontal vectors adjusted to \(Z_P\) (resp. \(Z_Q\). From the bi-linearity of the Lie bracket and the subadditivity of the norm \(||\cdot||_{p+q}\), we have
\[
\|Z_P, Z_Q\|_{p+q} = \left\| \sum_{n=1}^{d_1^p} \sum_{m=1}^{d_1^q} X^{(p)}_{n1}, \ldots, X^{(p)}_{np}, \sum_{m=1}^{d_1^q} X^{(q)}_{m1}, \ldots, X^{(q)}_{mq} \right\|_{p+q}
\leq \sum_{n=1}^{d_1^p} \sum_{m=1}^{d_1^q} \left\| \left[ X^{(p)}_{n1}, \ldots, X^{(p)}_{np}, X^{(q)}_{m1}, \ldots, X^{(q)}_{mq} \right] \right\|_{p+q}.
\]
By applying the Jacobi identity \([X, [Y, Z]] = [X, [Y, Z]] - [Y, [X, Z]]\) several times, we can rewrite
\[
\sum_{n=1}^{d_1^p} \sum_{m=1}^{d_1^q} \left\| \epsilon_{\sigma} X^{(p)}_{n\sigma(1)}, \ldots, X^{(p)}_{n\sigma(2)}, \ldots, X^{(q)}_{m1}, \ldots, X^{(q)}_{mq} \right\|_{p+q},
\]
where \(S\) is a subset of the symmetric group \(S_p\) such that \(#S = 2^p\), and \(\epsilon_{\sigma} \in \{\pm 1\}\). Then we can proceed as
\[
\sum_{n=1}^{d_1^p} \sum_{m=1}^{d_1^q} \left\| \sum_{\sigma \in S} \epsilon_{\sigma} X^{(p)}_{n\sigma(1)}, \ldots, X^{(p)}_{n\sigma(2)}, \ldots, X^{(q)}_{m1}, \ldots, X^{(q)}_{mq} \right\|_{p+q}
\leq \sum_{n=1}^{d_1^p} \sum_{m=1}^{d_1^q} 2^p \prod_{i=1}^{p} \|X^{(p)}_{ni}\|_1 \prod_{j=1}^{q} \|X^{(q)}_{mj}\|_1
= 2^p \left( \prod_{n=1}^{d_1^p} \|X^{(p)}_{ni}\|_1 \right) \left( \sum_{m=1}^{d_1^q} \prod_{j=1}^{q} \|X^{(q)}_{mj}\|_1 \right)
= 2^p \|Z_P\|_p \|Z_Q\|_q.
\]

\[
\square
\]
Next we will control an upper bound of the $l$-error vector $A_l$ in the norm $\| \cdot \|_l$.

**Lemma 3.2.** For each $j = 1, \ldots, k$, there is a positive constant $\theta_j = \theta_j(d_1, j, k)$ such that for any vector $Z_j \in V_j$ ($v_j = \| Z_j \|_j$) and any set of horizontal vectors $\{ X_{ni} \}$ adjusted to $Z_j$,

$$\| A_l(\{ X_{ni} \}) \|_l \leq \theta_j \gamma_j^{\frac{m}{m_j}}.$$  

**Proof.** For each $n = 1, \ldots, d_1$, let $U_n = y(\{ X_{ni} \}_{i=1}^{k})$. The BCH formula (7) gives

$$U_n = [X_{n1}, \ldots, X_{nj}] + \sum_{m \geq j+1} \sum_{(i_1, \ldots, i_m) \in \mathcal{I}_m} \gamma(i_1, \ldots, i_m) [X_{n1}, \ldots, X_{nm}].$$  

By using the Baker–Campbell–Hausdorff formula (5), the product of the elements $U_n$ is written by

$$\prod_{n=1}^{d_1} U_n = Z_j + \sum_{q \geq 2} \sum_{(n_1, \ldots, n_q) \in \mathcal{I}_q^{d_1}} \beta(n_1, \ldots, n_q) [U_{n1}, \ldots, U_{nq}].$$

Notice that the element $\prod_{n=1}^{d_1} U_n$ coincides with $Z_j + \sum_{l=1}^{k} A_l = y(\{ X_{ni} \})$. Thus we have an explicit form of $A_l$ by

$$A_l = \sum_{q \geq 2} \sum_{(n_1, \ldots, n_q) \in \mathcal{I}_q^{d_1}} \sum_{m_1 + \cdots + m_q = l} \beta(n_1, \ldots, n_q) [P_{m_1}(U_{n1}), \ldots, P_{m_q}(U_{nq})].$$

From (15), we can compute $\| P_m(U_n) \|_m$ by

$$\| P_m(U_n) \|_m \leq \begin{cases} 0 & m = 1, \ldots, j - 1, \\ \| X_{n1} \|_j^{\frac{m}{m_j}} & m = j, \\ \sqrt{\sum_{(i_1, \ldots, i_m) \in \mathcal{I}_m^{d_1}} \gamma(i_1, \ldots, i_m)} \| X_{n1} \|_1^{m_j} & m = j + 1, \ldots, k. \end{cases}$$

In any cases, $\| P_m(U_n) \|_m$ is less than or equal to $\tilde{\gamma} \nu_j^{\frac{m}{m_j}}$, where

$$\tilde{\gamma} = \max \{ 1, \sqrt{\sum_{(i_1, \ldots, i_m) \in \mathcal{I}_m^{d_1}} \gamma(i_1, \ldots, i_m)} \}.$$  

By the subadditivity of the norm $\| \cdot \|_m$.
and Lemma 3.1, we have
\[
\|A_l\| \leq \sum_{q \geq 2} \sum_{(n_1, \ldots, n_q) \in \mathbb{N}^q_{d_1^q}} \|P_{m_1}(U_{n_1})\| \cdots \|P_{m_q}(U_{n_q})\|_{m_q}
\]
\[
\leq \sum_{q \geq 2} \sum_{(n_1, \ldots, n_q) \in \mathbb{N}^q_{d_1^q}} 2^q \beta_{(n_1, \ldots, n_q)} \left( \frac{l + q}{q} \right) \tilde{\gamma}^q \nu_j^l
\]
\[
\leq \sum_{q \geq 2} 2^q d_1^q \tilde{\beta} \left( \frac{l + q}{q} \right) \tilde{\gamma}^q \nu_j^l
\]
\[
\leq (d_1^q \tilde{\beta})^{k+1+2l} \tilde{\gamma}^k \nu_j^k
\]
where \( \tilde{\beta} = \max \{|\beta_{n_1, \ldots, n_q}| \mid (n_1, \ldots, n_q) \in \mathbb{N}^q_{d_1^q}, q = 2, \ldots, k\} \). Hence we have obtained a constant \( \theta_j \) by \( \theta_j = (d_1^q \tilde{\beta})^{k+1+2l} \tilde{\gamma}^k \nu_j^k \).

The norm \( \| \cdot \|_j \) controls the Carnot–Carathéodory distance \( d_{CC} \) in the following sense.

**Lemma 3.3.** For any \( Z_j \in V_j \) with \( \nu_j = \|Z_j\|_j \) and any set of horizontal vector \( \{X_{ni}\} \) adjusted to \( Z_j \), there is an upper bound of the Carnot-Carathéodory distance from \( e \) to \( y(\{X_{ni}\}) \) by
\[
d_{CC}(y(\{X_{ni}\})) \leq 2^{j+1} d_1^{2j+1} \nu_j^l.
\]
In particular, when \( j = k \), we obtain
\[
d_{CC}(Z_k) \leq 2^{k+1} d_1^{2k+1} \nu_k^l.
\]

**Proof.** By using the triangle inequality and (14), we obtain an upper bound of the distance from \( e \) to \( y(\{X_{ni}\}) \) by
\[
d_{CC}(y(\{X_{ni}\})) \leq \sum_{n=1}^{d_1^j} \sum_{i=1}^{j-1} 2^i \|X_{ni}\|_1 + 2^{j-1} \|X_{nj}\|_1 \leq 2^{j+1} \sum_{n=1}^{d_1^j} \|X_{ni}\|_1.
\]
By the Jensen’s inequality, we obtain
\[
2^{j+1} \sum_{n=1}^{d_1^j} \|X_{ni}\|_1 \leq 2^{j+1} d_1^{2j+1} \sqrt{\sum_{n=1}^{d_1^j} \|X_{ni}\|_1^2}.
\]
Again by (13), we have
\[
2^{j+1} d_1^{2j+1} \sqrt{\sum_{n=1}^{N} \|X_{ni}\|_1^2} = 2^{j+1} d_1^{2j+1} \nu_j^l.
\]

\[\square\]
We have considered a set of vectors adjusted to a vector $Z_j \in V_j$. We will introduce a similar notion adjusted to a vector $Z$ in the whole Lie algebra $\mathfrak{g}$.

Let $Z = \sum_{j=1}^k Z_j$ be a vector in $\mathfrak{g} = \bigoplus_{j=1}^k V_j$ ($Z_j \in V_j$). We will inductively define a set of vectors $\{X_{ni}^{(j)}\}_{n=1}^{\infty}$, $a_{i,j}^{(j)}$ for $j = 1, \ldots, k$ as follows. For $Z_1 \in V_1$, define a set of vector $\{X_{ni}^{(1)}\} = \{Z_1, 0, \ldots, 0\}$. Let $\{X_{ni}^{(2)}\}$ be a set of vectors adjusted to $Z_2$. Then there is vectors $B_l^{(2)}(\{X_{ni}^{(1)}\}, \{X_{ni}^{(2)}\}) \in V_l$ for $l > 2$ such that

$$y(\{X_{ni}^{(1)}\})y(\{X_{ni}^{(2)}\}) = Z_1 + Z_2 + \sum_{l=3}^{k} B_l^{(2)}(\{X_{ni}^{(1)}\}, \{X_{ni}^{(2)}\}).$$

Next let $\{X_{ni}^{(3)}\}$ be a set of vectors adjusted to $Z_3 - B_3^{(2)}$. Then again there are vectors $B_l^{(3)}(\{X_{ni}^{(1)}\}, \{X_{ni}^{(2)}\}, \{X_{ni}^{(3)}\}) \in V_l$ for $l > 3$ such that

$$\prod_{j=1}^{3} y(\{X_{ni}^{(j)}\}) = \sum_{l=1}^{3} Z_l + \sum_{l=4}^{k} B_l^{(3)}(\{X_{ni}^{(1)}\}, \{X_{ni}^{(2)}\}, \{X_{ni}^{(3)}\}).$$

In this way, we can inductively define a set of vectors $\{X_{ni}^{(j)}\}$ and error vectors $B_l^{(j)}$. We will summarize this argument in the following definition.

**Definition 3.4.** For a vector $\sum_{j=1}^k Z_j \in \bigoplus V_j$, (non-unique) sets of horizontal vectors $\left(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(k)}\}\right)$ and vectors $B_l^{(j)}(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(j)}\}) \in V_l$ are inductively defined by

1. $\{X_{ni}^{(1)}\} = \{Z_1, 0, \ldots, 0\}$,
2. $B_l^{(1)}(\{X_{ni}^{(1)}\}) = 0$ for $l = 1, \ldots, k$,
3. $\{X_{ni}^{(j+1)}\}$ is a set of horizontal vectors adjusted to $Z_{j+1} - B_{j+1}^{(j)}(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(j)}\})$,
4. $B_l^{(j)}(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(j)}\}) = P_l\left(\prod_{m=1}^{j} y(\{X_{ni}^{(m)}\})\right)$ for $l = j + 1, \ldots, k$.

We call $\left(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(k)}\}\right)$ sets of horizontal vectors adjusted to $Z$, and call $B_l^{(j)}(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(j)}\})$ an $(l, j)$-error vector.

We will simply write $B_l^{(j)}$. Since $\mathfrak{g}$ is $k$-step, the error vectors $B_l^{(k)}$ vanishes and

$$\prod_{j=1}^{k} y(\{X_{ni}^{(j)}\}) = \exp\left(\sum_{j=1}^{k} Z_j\right).$$

**Remark 3.2.** Since the choice of a set of horizontal vectors adjusted to $Z_j \in V_j$ is not unique, the choice error vectors are not unique.
Error vectors $B_l^{(j)}$’s can be controlled as follows.

**Lemma 3.4.** For $Z = \sum_{j=1}^{k} Z_j \in \mathfrak{g}$, let $\nu_j = \|Z_j\|$ and $\left(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(k)}\}\right)$ sets of horizontal vectors adjusted to $Z$. For $j = 1, \ldots, k-1$ and $l = j+1, \ldots, k$, there are polynomials $Q_{lj}(\beta_1, \ldots, \beta_j)$ such that $\|B_l^{(j)}\|_l \leq Q(\beta_1, \ldots, \beta_j)$. Moreover, their coefficients and the number of terms depend only on the dimensions $d_1, \ldots, d_k$.

**Proof.** We prove by inductions on $j$. When $j = 1$, then $B_l^{(1)} = 0$ for all $l = 2, \ldots, k$, so the lemma trivially follows.

Assume the lemma holds for $j > 1$. From the definition of the $l$-error vector $A_l := A_l(\{X_{ni}^{(j+1)}\})$ and the $(l, j)$-error vector $B_l^{(j)} := B_l^{(j)}(\{X_{ni}^{(1)}\}, \ldots, \{X_{ni}^{(j)}\})$, we have

$$y(\{X_{ni}^{(j+1)}\}) = \exp \left( Z_{j+1} + \sum_{l=j+2}^{k} A_l \right),$$

and

$$\prod_{l=1}^{j} y(\{X_{ni}^{(l)}\}) = \exp \left( \sum_{l=1}^{j} Z_l + \sum_{l=j+1}^{k} B_l^{(j)} \right).$$

In particular, we can see that the $(l, j+1)$-error vector $B_l^{(j+1)}$ can be written by the $l$-error vector $A_l$ and $(l, j)$-error vector $B_l^{(j)}$ by the following equality:

$$\sum_{l=1}^{j+1} \frac{Z_l + \sum_{l=j+2}^{k} B_l^{(j+1)}}{\prod_{l=1}^{j} y(\{X_{ni}^{(l)}\})} = \left( \sum_{l=1}^{j} Z_l + \sum_{l=j+1}^{k} B_l^{(j)} \right) \left( Z_{j+1} + \sum_{l=j+2}^{k} A_l \right).$$

More precisely, let $\{S_p\}_{p=1, \ldots, 2k}$ be the finite set of vectors defined by

$$S_p = \begin{cases} Z_p & (p = 1, \ldots, j), \\ B_p^{(j)} & (p = j+1, \ldots, k), \\ Z_{j+1} & (p = k+1, \ldots, k+j), \\ A_p & (p = k+j+2, \ldots, 2k). \end{cases}$$

By the BCH formula, the $(l, j+1)$-error vector $B_l^{(j+1)}$ can be written by

$$B_l^{(j+1)} = P_l \left( \sum_{q=2}^{k} \sum_{(p_1, \ldots, p_q) \in \mathbb{I}_q^{2k}} \alpha_{(p_1, \ldots, p_q)} [S_{p_1}, \ldots, S_{p_q}] \right).$$
Hence we obtain a polynomial $Q_{l+1}(\nu_1, \ldots, \nu_{j+1})$ from the induction hypothesis, Lemma 3.1 and 3.2.

Now we have finished the preparation to prove the main proposition 3.2.

**Proof of Proposition 3.2.** We will prove by induction on $k$. We have already shown this proposition for 2-step Carnot group with $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \frac{1}{16}$ in Proposition 3.1.

Assume that the assertion is true for $k-1$ with $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_{k-1}$, and that $\|Z_j\| \leq t^j \tilde{\epsilon}_j$ ($t > 0, j = 1, \ldots, k - 1$). By the triangle inequality and the induction hypothesis, we have

$$d_{CC}(Z) = d_{CC}(\prod_{j=1}^{k} g(\{X_{n_i}^{(j)}\}))$$

$$\leq d_{CC} \left( \sum_{j=1}^{k-1} Z_j + B_k^{(k-1)} \right) + d_{CC} \left( g(\{X_{n_i}^{(k)}\}) \right)$$

$$\leq t + d_{CC} \left( g(\{X_{n_i}^{(k)}\}) \right).$$

By Lemma 3.4 there is a polynomial $Q_{kk-1}$ such that

$$\|B_k^{(k-1)}\|_k \leq Q_{kk-1}(t \tilde{\epsilon}_1, \ldots, t^{k-1} \tilde{\epsilon}_{k-1}).$$

Since the set of horizontal vectors $\{X_{n_i}^{(k)}\}$ is adjusted to $U_k - B_k^{(k-1)}$, Lemma 3.3 yields

$$d_{CC} \left( g(\{X_{n_i}^{(k)}\}) \right) \leq 2^{k+1} d_{1}^{2k-1} \|U_k - B_k^{(k-1)}\|_k^{\frac{1}{k}}$$

$$\leq 2^{k+1} d_{1}^{2k-1} \left( \epsilon_k + Q_{kk-1}(t \tilde{\epsilon}_1, \ldots, t^{k-1} \tilde{\epsilon}_{k-1}) \right)^{\frac{1}{k}}.$$

It is easily checked that $d_{CC}(\exp(Z))$ is less than 1 if we choose sufficiently small $t$ and $\epsilon_k$, which can be controlled by the dimension $d_i$'s. 

**References**

[1] Agrachev A., Barilari D., Boscain U.: A Comprehensive Introduction to Sub-Riemannian Geometry.

[2] Agrachev A., Barilari D., Boscain U.: On the Hausdorff volume in sub-Riemannian geometry. *Calc. Var. and PDE’s*, 43(3-4):355–388 (2012).

[3] Barilari D., Rizzi L.: A formula for Popp’s volume in sub-Riemannian geometry, *Anal. Geom. Metr. Spaces*, 2013, 1, 42–57.

[4] Bavard C.: Inégalité isosystolique pour la bouteille de Klein. *Math. Ann.* 274(3): 439–441 (1986).
[5] M. Gromov, Filling Riemannian manifolds, *J. Differential Geom.* **18**(1), 1–147 (1983).

[6] Gromov M.: Systoles and intersystolic inequalities, *Actes de la Table Ronde de Geometrie Differentielle en Vhonneur de Marcel Berger* (Arthur L. Besse, ed.), Seminaires and Congres of the SMF, No. 1, 1996.

[7] Hassanabzad, A., Kokarev, G.; Sub-Laplacian eigenvalue bounds on sub-Riemannian manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **XVI**(4), 1049–1092 (2016).

[8] Hebda J.: Some lower bounds for the area of surfaces, *Invent. Math.* **65**(3), 485–490 (1981/82).

[9] J. Mitchell, On Carnot Carathéodory metrics, *J. Diff. Geom.*, 21, 35-45 (1985).

[10] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications, 2000, *American Mathematical Society* (Mathematical Surveys and Monographs Vol. **91**).

[11] Pu P. M.: Some inequalities in certain nonorientable Riemannian manifolds, *Pacific J. Math.*, **2**, 55-71, (1952).