Online Learning Demands in Max-min Fairness

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Abstract

We describe mechanisms for the allocation of a scarce resource among multiple users in a way that is efficient, fair, and strategy-proof, but when users do not know their resource requirements. The mechanism is repeated for multiple rounds and a user’s requirements can change on each round. At the end of each round, users provide feedback about the allocation they received, enabling the mechanism to learn user preferences over time. Such situations are common in the shared usage of a compute cluster among many users in an organisation, where all teams may not precisely know the amount of resources needed to execute their jobs. By understating their requirements, users will receive less than they need and consequently not achieve their goals. By overstating them, they may siphon away precious resources that could be useful to others in the organisation. We formalise this task of online learning in fair division via notions of efficiency, fairness, and strategy-proofness applicable to this setting, and study this problem under three types of feedback: when the users’ observations are deterministic, when they are stochastic and follow a parametric model, and when they are stochastic and nonparametric. We derive mechanisms inspired by the classical max-min fairness procedure that achieve these requisites, and quantify the extent to which they are achieved via asymptotic rates. We corroborate these insights with an experimental evaluation on synthetic problems and a web-serving task.

Keywords: Fair division, Mechanism design, Strategy-proofness, Fairness, Online learning

1. Introduction

The fair division of a finite resource among a set of rational agents (users) is a well-studied problem in game theory (Procaccia, 2013). In this paper, we study a formalism for fair division, when users have certain resource demands; a user’s utility for the amount of resources they receive increases up to this demand, but does not increase thereafter. Such use cases arise when sharing computational resources among different applications within a computer (lin), or when sharing among different human users in shared-computing platforms (lbl; psc; Boutin et al., 2014; Verma et al., 2015). Each user has an entitlement (fair share) to a resource; however, some users’ demands will be smaller than their entitlement, while some users’ demands might be larger. Hence, allocating the resource simply in proportion to their entitlements will result in unused resources allocated to the former agents that could have been allocated to the latter agents with an unmet demand. Users may request some amount of resources, which may not necessarily be equal their demand, from a mechanism responsible for allocating this resource among the users. Fair division is the design of mechanisms
for allocating a scarce resource in a way that is *efficient*, meaning that no resources are left unused when there is an unmet demand, *fair*, meaning that an agent participating in the mechanism is at least as happy as when she can only use her fair share, and *strategy-proof*, meaning that agents are incentivised to report their true demands when requesting resources.

Max-min Fairness (MMF) is one of the most popular mechanisms for fair division that satisfies the above desiderata. Since being first introduced in the networking literature (Demers et al., 1989), it has been used in a plethora of applications such as scheduling data-centre jobs (Chen et al., 2018; Ghodsi et al., 2013), load balancing (Nace and Pióro, 2008), fair queueing in the Linux OS (lin) and packet processing (Ghodsi et al., 2012), sharing wireless (Huang and Bensaou, 2001) and data-centre (Shieh et al., 2011) networks, and many more (Hahne, 1991; Li et al., 2015; Liu et al., 2013). Moreover, MMF and its variants have been implemented in popular open source platforms such as Hadoop (had), Spark (Zaharia et al., 2010), and Mesos (Hindman et al., 2011).

To the best of our knowledge, all mechanisms for fair division in the literature, including MMF, assume that users know their demands. However, in many practical applications, users typically care about achieving a certain desired level of performance. While their demands are determined by this performance level, users often have difficulty in translating their performance requirements to resource requirements since real world systems can be complex and hard to model (Venkataraman et al., 2016). In this work, we propose shifting the burden of estimating these demands from the user to the mechanism, and doing so in a manner that satisfies efficiency, fairness, and strategy-proofness. As an example, consider an organisation where a compute cluster is shared by users who are serving live web traffic. Each user wishes to meet a certain service level objective (SLO), such as a given threshold on the fraction of queries completed within a specified time limit. Applying a mechanism such as MMF and expecting an efficient allocation requires that all users precisely know their demands (the amount of resources needed to meet their SLOs). If a user understates this demand, she risks not meeting her own SLOs. If she instead overstates her demand, she may take away precious resources from other users who could have used those resources to achieve their SLOs, resulting in an inefficient allocation. Prior applied research suggests that while modern data centres operate well below capacity (usually 30-50%), most users are unable to meet their SLOs (Delimitrou and Kozyrakis, 2013, 2014; Rzadca et al., 2020).

In this work, we design a multi-round mechanism for such instances when agents may not know their demands to satisfy their performance objectives. At the beginning of each round, the agents report the load of the traffic they need to serve in that round, then the mechanism assigns an allocation to each user based on past information while accounting for the load. At the end of the round, agents provide feedback on the allocation (e.g. the extent to which their SLOs were achieved). Satisfying strategy-proofness and fairness is more challenging in this setting. Since an agent reports feedback on each round, it provides her more opportunity to manipulate outcomes than typical settings for fair allocation where she reports a single demand. Additionally, in order to find an efficient allocation, the mechanism needs to estimate the demands of all agents; when doing so it risks violating the fairness criterion, especially for agents whose demands are less than their entitlements.

More generally, fairness is an important topic that has garnered attention in recent times in the machine learning community. Fairness can be construed in many ways, and this paper studies a concrete instantiation that arises in resource allocation. While this topic has been studied in the game theory literature fairly extensively, our paper focuses on the learning problem when users do not
know their resource requirements. As we will see shortly, in our setting, fairness and efficiency can be conflicting: an exactly fair allocation can result in worse outcomes to all agents; however, by considering a weaker notions of fairness which hold probabilistically and/or asymptotically, we can achieve outcomes that are beneficial to everyone. We believe that many of these ideas can be applied in various other online learning settings where similar fairness constraints arise.

This manuscript is organised as follows. In Sections 2.1 and 2.2, we briefly review fair division and describe the MMF algorithm. In Sections 2.3 and 2.4, we formalise online learning in fair division and define notions of efficiency, fairness, and strategy-proofness which are applicable in this setting. We also propose three feedback models motivated by practical use cases. In Section 3, we describe our mechanisms and present our theoretical results, quantifying how fast they learn via asymptotic rates. In Section 4, we evaluate the proposed methods empirically in synthetic experiments and a web-serving task. All proofs are given in the Appendix.

Related Work

In addition to the many practical applications described above, the fair allocation of resources has inspired a line of theoretical work. The include mechanisms for dynamically changing demands (Cole et al., 2013; Freeman et al., 2018; Tang et al., 2014), for allocating multiple resource types (Ghodsi et al., 2011; Gutman and Nisan, 2012; Li and Xue, 2013; Parkes et al., 2015), and when there is a stream of resources (Aleksandrov and Walsh, 2017). None of these works consider the problem of learning agents’ demands when they are unknown.

There is a long line of work in the intersection of online learning and mechanism design (Amin et al., 2013; Athey and Segal, 2013; Babaioff et al., 2014; Mansour et al., 2015; Nazerzadeh et al., 2008). The majority of them focus on auction-like settings, and assume that agents know their preferences—such as their value for items in an auction—and the goal of the mechanism is to elicit those preferences truthfully. Some work has studied instances where the agents do not know their preferences, but can learn them via repeated participations in a mechanism. Some examples include Weed et al. (2016), where an agent learns to bid in a repeated Vickrey auction, and Liu et al. (2019), where agents on one side of a matching market learn their preferences for alternatives on the other side. In contrast, here, learning happens on the side of the mechanism, imposing minimal burden on agents who may not be very sophisticated. Therefore, the onus is on the mechanism to ensure that all agents sufficiently explore all allocations, while ensuring that they are incentivised to report their feedback truthfully, so that the mechanism can learn these preferences. This is similar to Kandasamy et al. (2020) who study VCG mechanism design with bandit feedback where a mechanism chooses outcomes and prices for the users; the users in turn provide feedback about the outcomes which the mechanism uses when determining future outcomes and prices. The novelty in our work relative to existing literature is our focus on combining online learning with fair allocation.

2. Problem Setup

In this section, we will first review fair allocation and describe MMF, one of the most common methods for fair allocation. We will then describe the learning problem.
2.1 Fair Division

There are \( n \) agents sharing a resource of size 1. Agent \( i \) has an entitlement \( e_i \) to the resource, where \( e_i > 0 \) and \( \sum_{i=1}^{n} e_i = 1 \). At any given instant, let \( d_i^* \) denote the true demand for user \( i \), where \( d_i^* \geq 0 \) for all users \( i \). Continuing with the example from Section 1, this resource could be a compute cluster shared by \( n \) users in an organisation. The entitlements are set by the management depending on whether a user’s workload consists of time-sensitive live traffic or offline job processing, and \( d_i^* \) is the amount of the resource user \( i \) needs to achieve her SLOs. In some use cases, \( e_i \) may represent the contribution of each agent to a federated system, such as in universities where it could be set based on the contribution of each research group to purchase a cluster.

In a mechanism for fair allocation, each agent reports their demand \( d_i \) (not necessarily truthfully) to the mechanism; the mechanism returns an allocation vector \( a \in \mathbb{R}_+^n \), where \( a_i \) is the amount of the resource allocated to agent \( i \). Here, \( \sum_{i=1}^{n} a_i \leq 1 \). Let \( u_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) denote agent \( i \)'s utility function where \( u_i(a) \) is the value agent \( i \) derives if the mechanism allocates an amount \( a \) of the resource to her. She wishes to receive resources up to her demand, but has no value for resources beyond her demand; i.e., her utility \( u_i \) is strictly increasing up to \( d_i \), but \( u_i(a) = u_i(d_i) \) for all \( a > d_i \).

The literature on fair division typically considers three desiderata for a mechanism: (i) efficiency, (ii) fairness, (iii) strategy-proofness. Efficiency means that there are no unused resources when there is an unmet demand. To define it formally, let \( \ell^{\text{ur}}, \ell^{\text{ex}}, \ell^{\text{ud}} \) be as defined below for given \( d, a \in \mathbb{R}_+^n \):

\[
\ell^{\text{ur}}(a) = 1 - \sum_{i=1}^{n} a_i, \quad \ell^{\text{ex}}(d, a) = \sum_{i=1}^{n} (a_i - d_i)^+, \quad \ell^{\text{ud}}(d, a) = \sum_{i=1}^{n} (d_i - a_i)^+.
\]

Here, \( y^+ = \max(y, 0) \). If \( d^* \in \mathbb{R}_+^n \) is the vector of true demands and \( a \in \mathbb{R}_+^n \) is a vector of allocations output by a mechanism, then \( \ell^{\text{ur}}(a) \) is the amount of unallocated resources in this instance, \( \ell^{\text{ex}}(d^*, a) \) is the sum of over-allocated resources (allocated over a user’s demand) and \( \ell^{\text{ud}}(d^*, a) \) is the sum of unmet demands (allocated under a user’s demand). Then, \( \ell^{\text{ur}}(a) + \ell^{\text{ex}}(d^*, a) \) is the total amount of resources that are not being used, and \( \ell(d^*, a) = \min(\ell^{\text{ur}}(a) + \ell^{\text{ex}}(d^*, a), \ell^{\text{ud}}(d^*, a)) \) is the amount of resources that are not being used but could have been used to improve the utility of some agent. A mechanism is efficient if, for all \( d^* \), and when all agents report their true demands, \( \ell(d^*, a) = 0 \). An efficient mechanism is Pareto-optimal, in that one user’s utility can be increased only by decreasing the utility of another. Next, a mechanism is fair (also known as sharing incentive or individual rationality) if the utility a truthful user derives from an allocation is at least as much as if she had been allocated her entitlement; i.e., for all \( i, a, d, u_i(a_i) \geq u_i(e_i) \); recall that the allocation an agent receives depends on the demands reported by the other agents. Finally, a mechanism is strategy-proof if no agent benefits by misreporting their demands. That is, consider any agent \( i \) and fix the demands reported by all other agents. Let \( a^* \) be the allocation returned by the mechanism when agent \( i \) reports her true demand \( d_i^* \), and \( a \) be the allocation vector when she reports any other demand \( d \). Strategy-proofness means that for any agent \( i \) and for all reported demands from other agents, \( u_i(a_i^*) \geq u_i(a_i) \).

Observe that the above formalism does not assume that agents’ utilities are comparable, i.e., we make no intertemporal comparisons of utility (Hammond, 1990). The utilities are used solely to specify an agent’s preferences over different allocations. Therefore, notions such as utilitarian welfare which accumulates the utilities of all agents are not meaningful in this setting.
Algorithm 1 MMF

Require: entitlements \( \{e_i\}_{i=1}^n \), reported demands \( \{d_i\}_{i=1}^n \).
1: \( r \leftarrow 1 \), \( e \leftarrow 1 \), \( S \leftarrow \{1, \ldots, n\} \), \( a \leftarrow 0_n \).
2: for \( j \) in ascending order of \( \frac{d_j}{e} \) do
3: if \( d_j < \frac{r}{e} \) then
4: \( a_j \leftarrow d_j \).
5: \( S \leftarrow S \setminus \{j\} \), \( r \leftarrow r - d_i \), \( e \leftarrow e - e_i \).
6: else # Allocate proportionally to all remaining agents and exit.
7: \( a_k \leftarrow r \cdot \frac{e_k}{e} \) for all \( k \in S \).
8: Break.
9: return \( a \)

2.2 Max-min Fairness

Algorithm 1 describes max-min fairness, a popular mechanism for fair division. First, it allocates the demands to users whose demands are small relative to their entitlement; for all other agents whose demands cannot be satisfied simultaneously, it allocates in proportion to their entitlements. The following theorem shows that MMF satisfies the above desiderata; its proof is given in Appendix A, where we have also established other useful properties of MMF.

Theorem 1 MMF (Algorithm 1) is efficient, fair, and strategy-proof.

As an example, consider 4 users with equal entitlements and true demands \( \{0.1, 0.28, 0.4, 0.5\} \). MMF returns the allocation \( \{0.1, 0.28, 0.31, 0.31\} \). Instead, had we allocated the resources equally according to their entitlements, i.e. \( \{0.25, 0.25, 0.25, 0.25\} \), agent 2 will not have met her demand, agents 3 and 4 will not have received as much, while agent 1 will have been sitting on 0.15 of the resource—this is fair and trivially strategy-proof, but inefficient.

2.3 Online Estimation of Demands

We begin our formalism for the online learning version of the fair division problem with a description of the environment. We consider a multi-round setting, where, on round \( t \), agent \( i \) faces a load \( w_{it} \) with (unknown) demand \( d_{it}^* \). In order to be able to effectively learn, we need a form of feedback for each agent which informs us of the agent’s utility, and additionally be able to relate the loads from different time steps. To this end, we define an (unknown) payoff function \( f_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) for each agent \( i \), which will characterise the agent’s utility function. When user \( i \) receives an allocation \( a_{it} \) on round \( t \), she observes a reward \( X_{it} \). We will consider different feedback models where these rewards can be deterministic, in which case \( X_{it} = f_i(a_{it}/w_{it}) \), or they can be stochastic, in which case \( X_{it} \) is drawn from a \( \sigma_{it} \) sub-Gaussian distribution with mean \( f_i(a_{it}/w_{it}) \).

The payoff \( f_i \) is a non-decreasing function of the amount of resources allocated per unit load. The agent wishes to achieve a certain (known) threshold payoff \( \alpha_i \) and hence, her true demand at round \( t \) is given by \( d_{it}^* = w_{it} f_i^{-1}(\alpha_i) \). This implies that each agent’s true demand increases proportionally with the load, with the demand per unit load, or unit demand for short, being \( f_i^{-1}(\alpha_i) =: \eta_i^* \). We will assume that \( \eta_i^* \in [0, \eta_{\text{max}}] \), where \( \eta_{\text{max}} \) is known.
The agent’s utility \( u_i : [0, \eta_{\text{max}}] \to \mathbb{R}_+ \), as illustrated in the figure to the right, is also a function of the amount of resources allocated per unit load which increases \textit{strictly} up to \( \eta_i^* \), and does not increase beyond \( \eta_i^* \). In the figure, we have shown \( f_i(a) = u_i(a) \) for \( a < \eta_i^* \), although this is not necessary—we only require that \( u_i \) be increasing up to \( \eta_i^* \) and remain flat thereafter. Additionally, we will assume that \( u_i \) is Lipschitz continuous with Lipschitz constant \( L_i^u \).

We will make some mild assumptions so as to avoid degenerate cases in our analysis. First, we will assume that both \( \{w_{it}\}_{i,t} \) and \( \{\sigma_{it}\}_{i,t} \) are fixed sequences. Second, for all \( i, t, w_{it} \in (w, \overline{w}] \) and \( \sigma_{it} \in (\underline{\sigma}, \overline{\sigma}] \) for some \( w, \sigma > 0 \) and \( \overline{\sigma}, \underline{\sigma} < \infty; w, \overline{\sigma}, \underline{\sigma} \) need not be known. We comment more on the bounded-from-below conditions on \( w_{it}, \sigma_{it} \) in our proofs. Third, we will assume \( \overline{w} \eta_{\text{max}} \leq 1 \), which states that at the very least each agent should be able to meet their demands if they have the entire resource to themselves; this assumption can also be relaxed, and we will comment further in our proofs. The following example helps us motivate the above formalism.

**Example 1 (Web serving)** Continuing with the example from Section 1, say that the SLO of each team is to ensure that a given threshold, say 0.95, of the queries are completed on time on average. The load \( w_{it} \) is the number of queries agent \( i \) receives in round \( t \). Her reward \( X_{it} \in [0, 1] \) is the fraction of queries completed on time. The probability each query will succeed increases with the amount of resources allocated per query; specifically, the success of each query is a Bernoulli event with probability \( f_i(a_{it}/w_{it}) \). Therefore, \( f_i(a_{it}/w_{it}) = \mathbb{E}[X_{it}] \) denotes the expected fraction of queries completed on time. The utility of the agent \( u_i = \min(f_i, 0.95) \) increases with this expected fraction but is capped at 0.95. Hence, her demand is \( d_{it}^* = w_{it}f_i^{-1}(0.95) \). Finally, since \( X_{it} \) is a sum of \( w_{it} \) bounded random variables, it is \( \sigma_{it} = 1/(2\sqrt{w_{it}}) \) sub-Gaussian. In web services, it is common to set SLOs via such thresholds since the amount of resources needed to complete all queries on time could be possibly infinite. Moreover, since the quality of the overall service is usually bottlenecked by external factors (Mogul and Wilkes, 2019), there is little value to exceeding such a threshold.

In an online mechanism for fair allocation with unknown demands, each agent states their threshold \( \alpha_i \) ahead of time. At the beginning of each round, each agent reports their load \( w_{it} \) to the mechanism, then the mechanism returns an allocation vector, and at the end of the round each agent reports their reward \( X_{it} \) back to the mechanism. The mechanism should use this feedback to estimate demands in an online fashion and quickly converge to an efficient ellocation in a manner that is fair and strategy-proof. As we will see, achieving these desiderata exactly is quite challenging in our setting and hence we will define asymptotic variants to make the problem tractable.

**Efficiency:** We define the \textit{loss} \( L_T \) to be the sum of resources left on the table over \( T \) rounds:

\[
L_T := \sum_{t=1}^{T} \ell(d_{it}^*, a_{it}), \quad \text{where } \ell(d_{it}^*, a_{it}) = \min(\ell_{\text{ur}}(a_{it}) + \ell_{\text{or}}(d_{it}^*, a_{it}), \ell_{\text{end}}(d_{it}^*, a_{it})).
\]  

(2)

Recall the definitions of \( \ell_{\text{ur}}, \ell_{\text{or}}, \ell_{\text{end}} \) from (1). A mechanism is \textit{asymptotically efficient} if, when all agents are reporting truthfully, \( L_T \in o(T) \). We will say that a mechanism is \textit{probably asymptotically efficient} if this holds with probability at least \( 1 - \delta \), where \( \delta \in (0, 1) \) is pre-specified.
**Fairness:** Let $U_{iT}, U^{\alpha_i}_{iT}$, respectively be the sum of an agent’s utilities when she participates in the mechanism truthfully for $T$ rounds, and when she has her entitlement to herself. Precisely,

$$U_{iT} = \sum_{t=1}^{T} u_i (a_{it}/w_{it}) , \quad U^{\alpha_i}_{iT} = \sum_{t=1}^{T} u_i (\alpha_i/w_{it}) ,$$

(3)

A mechanism is asymptotically fair if, $U^{\alpha_i}_{iT} - U_{iT} \in o(T)$. Similarly, a mechanism is probably asymptotically fair if $U^{\alpha_i}_{iT} - U_{iT} \in o(T)$ with probability at least $1 - \delta$, where $\delta \in (0, 1)$ is pre-specified. To understand why we consider an asymptotic version of fairness, observe that achieving an efficient allocation requires that the mechanism accurately estimates the demands. This is especially the case for agents whose demand is lower than their entitlement since the excess resources can be allocated to other users who might need them. However, in doing so, it invariably risks allocating less than the demand and consequently violating fairness. Asymptotic fairness means that these violations will vanish over time.

**Strategy-proofness:** We will introduce a variety of strategy-proofness definitions in this paper. To define them, let $U_{iT}$ be as defined in (3). Let $\pi$ be an arbitrary (non-truthful) policy that an agent may follow and let $U^{\pi}_{iT}$ be the sum of utilities when following this policy. A mechanism is strategy-proof if $U^{\pi}_{iT} - U_{iT} \leq 0$ regardless of the behaviour of the other agents. A mechanism is probably strategy-proof if the same holds with probability at least $1 - \delta$, where $\delta \in (0, 1)$ is given. A mechanism is asymptotically strategy-proof if $U^{\pi}_{iT} - U_{iT} \in o(T)$ regardless of the behaviour of the other agents. It is probably asymptotically strategy-proof if this holds with probability at least $1 - \delta$. A mechanism is probably asymptotically Bayes-Nash incentive-compatible if $U^{\pi}_{iT} - U_{iT} \in o(T)$ with probability at least $1 - \delta$ when all other agents are being truthful. In the above definition, an agent may adopt a non-truthful policy $\pi$ by misreporting her threshold $\alpha_i$ at the beginning, or by misreporting her load $w_{it}$ or the reward $X_{it}$ on any round $t$. For example, with the intention of getting more resources, she could inflate the mechanism’s estimate of her demand by overstating her threshold or load, or understating her reward. An agent could also be strategic over multiple rounds by adaptively using the information she gained when reporting her loads and rewards.

We focus on the above notions of incentive-compatibility since achieving dominant-strategy incentive-compatibility can be challenging in multi-round mechanisms (Babaioff et al., 2013, 2014; Kandasamy et al., 2020). Typically, authors circumvent this difficulty by adopting Bayesian notions which assume that agent values are drawn from known prior beliefs. However, such Bayesian assumptions may not be realistic, and even if it were, the prior beliefs may not be known in practice (Schummer, 2004). In this work, we rely on definitions of incentive-compatibility which hold with high probability and/or asymptotically in order to make the problem tractable. If a fair allocation mechanism is asymptotically strategy-proof, the maximum utility an agent may gain by not being truthful vanishes over time. In many use cases, it is reasonable to assume that agents would be truthful if the benefit of deviating is negligible. Prior work has similarly explored different concepts of approximate incentive-compatibility in various mechanism design problems when achieving dominant truthfulness is not possible (Daskalakis et al., 2006; Feder et al., 2007; Kojima and Manea, 2010; Lipton et al., 2003; Roberts and Postlewaite, 1976). Moreover, Kandasamy et al. (2020) and Nazerzadeh et al. (2008) study asymptotic strategy-proofness when learning in auctions.

Finally, we note that in some use cases, we may wish to learn without any strategy-proofness constraints while still satisfying fairness. For instance, in Example 1, the SLOs may be set by the management, while an organisation’s central monitoring system might be able to directly observe
the load and rewards. Therefore, we will also consider mechanisms for online learning without any strategy-proofness guarantees. In particular, we find that when we relax the strategy-proofness requirements, the rates for efficiency and fairness improve.

2.4 Feedback Models

All that is left to do to complete the problem setup is to specify the feedback model, i.e., a form for the payoffs \( \{ f_i \} \) and the (probabilistic) model for the rewards \( X_{it} \). In this paper, we will consider the following three models.

1. **Deterministic Feedback:** Our first model is the simplest of the three: all agents deterministically observe the payoff for their allocation, i.e., \( X_{it} = f_i(a_{it}/w_{it}) \) on all rounds.

2. **Stochastic Feedback with Parametric Payoffs:** Here, the rewards are stochastic, where \( X_{it} \) has expectation \( f_i(a_{it}/w_{it}) \) and is \( \sigma_{it} \) sub-Gaussian. For all users, \( f_i \) has parametric form \( f_i(a) = f_{\theta_i^*}(a) = \mu(a\theta_i^*) \), where \( \theta_i^* > 0 \) is an unknown user-specific parameter and \( \mu : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a known increasing function.

3. **Stochastic Feedback with Nonparametric Payoffs:** Here, \( X_{it} \) has expectation \( f_i(a_{it}/w_{it}) \) and is \( \sigma_{it} \) sub-Gaussian. Moreover, for all users \( i \), \( f_i \) is \((L/\eta_{\max})\)–Lipschitz continuous, where \( L \) is known. That is,

\[
\forall a_1, a_2 \in [0, \eta_{\max}], \quad |f(a_1) - f(a_2)| \leq \frac{L}{\eta_{\max}} |a_1 - a_2|.
\]

There are no additional assumptions required for the first model. For the second model, we chose a parametric family of the above form since it is a straightforward way to model increasing functions, as necessitated by our problem setup. In Example 1, an appropriate choice for \( \mu \) could be \( \mu(x) = \tanh(x) \) or \( \mu(x) = 1 - (1 + x)^{-1} \), which are bounded and increasing. Moreover, since they are concave, they can model situations which exhibit a diminishing returns property when more resources are allocated, as is commonly the case in practice (Venkataraman et al., 2016). In the second model, we will assume that we know a lower bound on the parameter \( \theta_i^* \) and the derivative of \( \mu \). Such regularity conditions are common in parametric models in the online learning literature (Chaudhuri et al., 2015; Filippi et al., 2010; Li et al., 2017). This is stated formally in Assumption 1.

**Assumption 1** There exists known \( \theta_{\min} > 0 \) such that \( \theta_i^* \geq \theta_{\min} \) for all \( i \). Moreover,

\[
\kappa_{\mu} \overset{\Delta}{=} \inf_{x \in [0, \eta_{\max}]} \frac{d\mu(x)}{dx} \geq 0.
\]

In the third model, we will additionally require that for each user, the payoff increases sharply at her demand, which we formalise by defining the near-threshold gradient (NTG) of a non-decreasing function at a given point.

**Definition 1 (Near-threshold Gradient)** The near-threshold gradient of a non-decreasing function \( f : \mathbb{R} \rightarrow \mathbb{R} \) at \( \eta \in \mathbb{R} \) with \( f(\eta) = \alpha \) is defined as,

\[
\text{NTG}(f, \eta) \overset{\Delta}{=} \sup \left\{ G \geq 0; \ \exists \epsilon > 0 \text{ such that, } \forall a \in (\eta - \epsilon, \eta + \epsilon), |f(x) - \alpha| \geq \frac{G}{\eta_{\max}} |a - \eta| \right\}.
\]
Figure 1: The payoff curve, data (×’s) and confidence intervals (shaded region) for two different scenarios. Left: A payoff curve with positive NTG where it is possible to accurately estimate the unit demand η_i when we get more data. Right: A payoff curve where the NTG is 0; here, we may not be able to tightly upperbound η_i since the lower confidence bound for f_i will necessarily be smaller than α_i for some x > η_i.

Moreover, for G ≥ NTG(f, η), define ε_G as follows:

\[ ε_G \triangleq \sup \left\{ \epsilon \geq 0; \forall a \in (\eta - \epsilon, \eta + \epsilon), |f(x) - \alpha| \geq \frac{G}{\eta_{\text{max}}} |a - \eta| \right\}. \]

If f is differentiable at η, then NTG(f, η) = f'(η). Assumption 2 below states that the NTG should be positive for all payoffs at their respective demands. To understand why such an assumption is necessary, assume that f_i increases up to the demand but is flat thereafter. Then, any lower confidence bound for the payoff at any a > η_i will necessarily be smaller than α_i and therefore, we will not be able to estimate this demand with any confidence. We have illustrated this in Figure 1.

Assumption 2 There exists G_0 > 0 such that, for all users i, NTG(f_i, α_i) ≥ G_0.

In this paper, we study the above models in order to delineate what can be achieved under various assumptions and under various strategy-proofness constraints (see Table 1). In real world settings, the deterministic model can be unsuitable since feedback can be noisy. Empirically, we find that the nonparametric model most meaningfully reflects practical settings, and consequently the corresponding algorithms outperform its parametric counterparts. However, the proof intuitions are simpler to understand in the deterministic and parametric models.

This completes the formulation of online learning in fair allocation. Next, we present our mechanisms.

3. Mechanisms & Theoretical Results

Our mechanisms are outlined in Algorithms 2 and 3 where we have hidden the round index subscript (e.g. t) for simplicity. In order to present a general framework for all models, we have abstracted out some of components of the mechanism. The most important of these is the definition of a user class USER-CLASS whose definition depends on the feedback model. Its main purpose is to estimate the unit demands of each user from past data. USER-CLASS is instantiated for each user in the system.

Algorithm 2 outlines a framework for online learning in this environment, which, generally speaking, provides stronger strategy-proofness guarantees. It operates over a sequence of brackets, indexed
As we will see, being conservative comes at a loss in efficiency, reflected via worse rates for the loss. 

Algorithm 2 MMF-LEARN-SSP

Require: entitlements \( \{e_i\}_i \), definitions for the function \( r' \), the function EXPLORE-PHASE, and the user class USER-CLASS.

1: \( \text{UC}_i \leftarrow \text{USER-CLASS.INITIALISE}(\cdot) \) for all users \( i \). \hspace{1cm} \# Instantiate USER-CLASS for all users
2: for \( q = 1, 2, \ldots \) do
3: \( \text{EXPLORE-PHASE(\{UC\}_i) \hspace{1cm} \# Execute exploration phase} \)
4: \( \hat{\eta}_i \leftarrow \text{UC}_i.\text{GET-UD-UB}() \) for each user \( i \). \hspace{1cm} \# Compute upper bound for unit demand
5: for \( r = 1, \ldots, r'(q) \) do,
6: \( \{w_i\}_i \leftarrow \text{obtain loads from agents} \).
7: \( a \leftarrow \text{MMF}(\{e_i\}_i, \{w_i \times \hat{\eta}_i\}_i) \). \hspace{1cm} \# MMF from Algorithm 1
8: Allocate according to \( a \).

Algorithm 3 MMF-LEARN-WSP

 Require: entitlements \( \{e_i\}_i \), user class definition USER-CLASS.

1: \( \text{UC}_i \leftarrow \text{USER-CLASS.INITIALISE}(\cdot) \) for all users \( i \). \hspace{1cm} \# Instantiate for all users
2: Allocate \( e_i \) to each user and collect rewards and SGCs \( \{(X_i, \sigma_i)\}_i \) from all users. \hspace{1cm} \# Round 1
3: \( \text{UC}_i.\text{RECORD-FEEDBACK}(a_i/w_i, X_i, \sigma_i) \) for each user \( i \).
4: for \( t = 2, 3, \ldots \) do,
5: \( \eta_i \leftarrow \text{UC}_i.\text{GET-UD-REC}() \) for each user \( i \). \hspace{1cm} \# Obtain recommendation for unit demand
6: \( \{w_i\}_i \leftarrow \text{obtain loads from agents} \).
7: \( a \leftarrow \text{MMF}(\{e_i\}_i, \{w_i \times \eta_i\}_i) \). \hspace{1cm} \# MMF from Algorithm 1
8: Allocate according to \( a \) and collect rewards and SGCs \( \{(X_i, \sigma_i)\}_i \) from all users.
9: \( \text{UC}_i.\text{RECORD-FEEDBACK}(a_i/w_i, X_i, \sigma_i) \) for each user \( i \).

By \( q \). Each bracket begins with an exploration phase, during which the mechanism tries different allocations for each user, collects the observed rewards, and reports them to the USER-CLASS. Then, on round \( t \), it computes upper (confidence) bounds \( \{\hat{\eta}_{it}\}_{i \in [n]} \) for the unit loads \( \{\eta_{it}^*\}_{i \in [n]} \) of all agents, based on data collected only from previous exploration phases; for this, it uses the GET-UD-UB method of USER-CLASS. This is then followed by \( r'(q) \) rounds during which the mechanism chooses the allocation by invoking MMF as a subroutine; when doing so, it sets the demand of agent \( i \) to be \( w_{it} \times \hat{\eta}_{it} \). The number of exploration phases remains fixed while \( r'(q) \) increases with the bracket index \( q \). In addition to the entitlements and the user class definition, the algorithm accepts two function definitions as inputs which depend on the feedback model: a function \( r' \) which specifies the length of the second phase for each bracket and a function EXPLORE-PHASE which implements the exploration phase for the model.

By using an upper confidence bound as the reported demand, the mechanism errs on the side of caution. Choosing these allocations conservatively\(^1\) is necessary to ensure both strategy-proofness and fairness. Intuitively, if the mechanism promises not to underestimate a user’s demands, there is less incentive for the user to inflate their resource requirements by not being truthful. Similarly, we also show that fairness is less likely to be violated if we do not underestimate user demands in MMF. As we will see, being conservative comes at a loss in efficiency, reflected via worse rates for the loss.

\(^1\) This can be contrasted with optimistic methods in the bandit literature which may take actions optimistically.
Algorithm 3 outlines a framework for online learning which, generally speaking, provides weaker strategy-proofness guarantees, but stronger efficiency and fairness guarantees than Algorithm 2. In round 1, it allocates in proportion to each agent’s entitlements. In each subsequent round $t$, it computes recommendations $\{\eta_{it}\}_{i \in [n]}$ for the unit demands using data from previous rounds; for this, it invokes the GET-UD-REC method of USER-CLASS. It then determines the allocations via MMF by setting the demand of agent $i$ to be $w_{it} \times \eta_{it}$. Finally it collects and records the rewards and sub-gaussian constants (SGCs) from all agents.

Table 1 summarises the main theoretical results in this paper. For each of the three feedback models, we have presented the rates for asymptotic efficiency, fairness, and strategy-proofness when using either Algorithm 2 or 3 along with model-specific definitions for USER-CLASS, $r'$, and EXPLORE-PHASE which we will describe shortly.

| # | Efficiency $L_T \in \tilde{O}(?)$ | Fairness $U_{iT}^e - U_{iT} \in \tilde{O}(?)$ | Strategy-proofness $U_{iT}^p - U_{iT} \in \tilde{O}(?)$ |
|---|---|---|---|
| 1. Deterministic Feedback | Algorithms 2 & 4 Theorem 2 | $n^{3/2} T^{1/2}$ | $n^{1/2} T^{1/2}$ | $\leq 0$ |
| Algorithms 2 & 5 Theorem 3 | $n \log(T)$ | $n \log(T)$ | 1 |
| Algorithms 3 & 5 Theorem 4 | $n$ | 1 | – |
| 2. Stochastic Feedback with Parametric Payoffs (bounds hold w/p $\geq 1 - \delta$) | Algorithms 2 & 6 Theorem 5 | $n T^{2/3}$ | $\leq 0$ | $\leq 0$ |
| Algorithms 3 & 6 Theorem 6 | $n T^{1/2}$ | $\leq 0$ | $n T^{1/2}$, BNIC |
| 3. Stochastic Feedback with Nonparametric Payoffs (bounds hold w/p $\geq 1 - \delta$) | Algorithms 2 & 7-10 Theorem 7 | $G^{-3/2} n^{4/3} T^{2/3}$ | $n^{1/3} T^{2/3}$ | $n^{-2/3} T^{2/3}$ |
| Algorithms 3 & 7-10 Theorem 8 | $G^{-3/2} n T^{1/2}$ | $G^{-3/2} T^{1/2}$ | – |

Table 1: Summary of asymptotic rates for efficiency, fairness, and strategy-proofness for the three feedback models when using different mechanisms, where we have hidden any logarithmic dependence on $T$ when there are polynomial terms. BNIC indicates that the mechanism is (asymptotically) Bayes-Nash incentive-compatible. When the bounds hold non-asymptotically, we write ‘$\leq 0$’, e.g. $U_{iT}^e - U_{iT} \leq 0$ in both mechanisms for the second model. In the third model, the bounds hold for any $G > G_0$ (Definition 1).
In the next three subsections, we present the algorithms for the three feedback models in detail. When describing \textit{USER-CLASS}, we will use OOP-style pseudocode. Therefore, \textit{method} refers to a function that can be called using an instance of this class, while \textit{private-method} refers to a function that can only be called by other methods of the instant. An \textit{attribute} refers to a variable belonging to the instant that can be accessed by all methods. When referring to an attribute or method from within a class, we will use the prefix \textit{SELF}. Finally, \textit{function} refers to function that can be defined and called without a class. For brevity, we will assume that all input arguments and variables available to the mechanism are also available to the model-specific functions and classes (e.g. $t, q, \eta_{\text{max}}$). Within each user class, we will hide the user index subscript (e.g. $i$). Moreover, the attributes within a user class are round-indexed versions which will be updated at the end of each round. We will hide the round index subscript as we have done in Algorithms 2 and 3; however our discussion and proofs will make the round index explicit.

### 3.1 Deterministic Feedback

With deterministic feedback, we will present mechanisms for learning under three different strategy-proofness constraints: non-asymptotic strategy-proofness, asymptotic strategy-proofnes, and without strategy-proofness. As we will see, the rate for the loss (efficiency) will improve as this strategy-proofness constraint is weakened.

#### 3.1.1 Learning with non-asymptotic strategy-proofness

In this setting, we will use Algorithm 2 along with the user class \textit{USER-CLASS} and functions $r'$, \textit{EXPLORE-PHASE} as given in Algorithm 4. In the user class for user $i$, we maintain an attribute $\hat{\eta}_{it}$ (recall that we have suppressed user and time indices) which is an upper bound on the user’s unit demand $\eta_{it}^*$. Each exploration consists of $n$ rounds, where each agent receives one round each; in the $q^\text{th}$ phase, it uses $\eta_{\text{max}} k/2^h$ as the unit demand for the agent on this round, where $h = \lceil \log_2 (q + 1) \rceil$, and $k = 2q - 2^h + 1$. That is, on round $t$ for such an agent, the allocation is $w_{it} \times \eta_{\text{max}} k/2^h$, where $w_{it}$ is the load of the agent for that round. If the received feedback $X_i$ was larger than the threshold $\alpha_i$, we update $\hat{\eta}_{it}$ to the minimum of the previous value and $\eta_{\text{max}} k/2^h$ (line 15).

After the $\hat{\eta}_{it}$ values are updated for each agent in an exploration phase, we use this value for the remainder of the bracket in line 7 of Algorithm 2. Finally, in this setting, we use $r'(q) = q$. We have the following theorem for this mechanism.

**Theorem 2** Under the deterministic feedback model, Algorithm 2, when using the definitions in Algorithm 4, satisfies the following. Under truthful reporting from all agents, it is asymptotically efficient with

$$L_T \leq 10n^{3/2}T^{1/2} \in \mathcal{O}(n^{3/2}T^{1/2}).$$

It is asymptotically fair with,

$$U^{e_i}_T - U^{u_i}_T \leq 2\sqrt{2} \eta_{\text{max}} n^{1/2}T^{1/2} \in \mathcal{O}(n^{1/2}T^{1/2}).$$

for all agents $i$. Finally, it is (non-asymptotically) strategy-proof, i.e. for all agents $i$ and all policies $\pi$, $U^{\pi_i}_T - U^{u_i}_T \leq 0$ for all $T \geq 1$.

While the above algorithm is asymptotically efficient, the rates are fairly slow for deterministic observations, with the loss growing at rate $\mathcal{O}(T^{1/2})$. This is not surprising, since we collect feedback
Algorithm 4 Definitions for the Deterministic Feedback Model with Strategy-Proofness

Function $r'(q)$
1: return $nq$

Function EXPLORE-PHASE\left(\{UC_i\}_i\right)
2: for $i = 1, \ldots, n$ do
3:    $w_i \leftarrow$ obtain load from user $i$.
4:    $h \leftarrow \lceil \log_2(q + 1) \rceil, \quad k \leftarrow 2q - 2^h + 1$.
5:    $\eta_i \leftarrow \eta_{\text{max}}k/2^h$.
6:    Allocate $\eta_i \times w_i$ to user $i$ only and obtain feedback $X_i$.
7:    UC_i.RECORD-FEEDBACK($\eta_i, X_i$).

Class USER-CLASS
8: attributes $\hat{\eta}$
9: method INITIALISE()
10:   SELF.$\hat{\eta} \leftarrow \eta_{\text{max}}$.
11: method GET-UD-UB()
12:   return SELF.$\hat{\eta}$
13: method RECORD-FEEDBACK($\overline{\alpha}, X_i$) \# SGCs are not relevant for deterministic feedback
14:   if $X_i > \alpha_i$ then
15:      SELF.$\hat{\eta} \leftarrow \min$(SELF.$\hat{\eta}, \overline{\alpha}$)

only during the exploration phase, where we essentially perform grid search for each agent separately. This is due to the stringent strategy-proofness requirement—achieving a (non-asymptotically) strategy-proof solution requires that the allocations used to estimate an agent’s demand do not depend on previous reports of any agent. In what follows, we will demonstrate that the rate for the loss can be considerably improved to have either logarithmic or no dependence on $T$ by considering weaker strategy-proofness criteria.

3.1.2 LEARNING WITH ASYMPTOTIC STRATEGY-PROOFNESS

In this section, we will study learning under deterministic feedback while satisfying asymptotic strategy-proofness. The user class USER-CLASS and the functions $r'$ and EXPLORE-PHASE for this model are given in Algorithm 5. We will describe these components individually.

The exploration phase in Algorithm 2 performs binary search for the $\eta_i^*$ value for each agent separately. It maintains upper and lower bounds $(\hat{\eta}_{it}, \tilde{\eta}_{it})$ for each agent initialised to $(0, \eta_{\text{max}})$. Each exploration phase consists of $2n$ rounds, where it performs two steps of binary search per agent. On a round $t$ for agent $i$ within this exploration phase, the mechanism tries $\eta_{it} = (\hat{\eta}_{it} + \tilde{\eta}_{it})/2$ for agent $i$. All other agents $j$ do not receive an allocation during this round, i.e. $\eta_{jt} = 0$. If $X_{it} < \alpha_i$, it updates the lower bound for the next round to $\eta_{it}$ and does not change the upper bound, and vice
Algorithm 5 Definitions for the Deterministic Feedback Model

Function $r'(q)$
1: return $\lfloor e^q \rfloor$

Function EXPLORE-PHASE($\{UC_i\}_i$)
2: for $j = 1, \ldots, 2n$ do
3: $i \leftarrow j \mod n$
4: $w_i \leftarrow$ obtain load from user $i$.
5: $\eta_i \leftarrow UC_i.GET-UD-REC-FOR-UB()$.
6: Allocate $\eta_i \times w_i$ to user $i$ only and obtain feedback $X_i$.
7: $UC_i.RECORD-FEEDBACK(\eta_i, X_i)$

Class USER-CLASS
8: attributes $\hat{\eta}, \tilde{\eta}$
9: method INITIALISE() # Upper and lower bounds on unit demand
10: SELF.$\tilde{\eta}$ $\leftarrow 0$, SELF.$\hat{\eta}$ $\leftarrow \eta_{\text{max}}$.
11: method GET-UD-UB() # Main interface for Algorithm 2
12: return SELF.$\tilde{\eta}$
13: method GET-UD-REC() # Main interface for Algorithm 3
14: return $\frac{1}{2} (\text{SELF.}\tilde{\eta} + \text{SELF.}\hat{\eta})$
15: method RECORD-FEEDBACK($\pi, X_i$) # SGCs are not relevant for deterministic feedback
16: if $X_i < \alpha_i$ then
17: SELF.$\tilde{\eta}$ $\leftarrow \max(\text{SELF.}\tilde{\eta}, \pi)$
18: else
19: SELF.$\hat{\eta}$ $\leftarrow \min(\text{SELF.}\hat{\eta}, \pi)$

versa if $X_{it} \geq \alpha_i$. After the two rounds for the agent, the $\hat{\theta}_{it}$ value computed as above is used for the remainder of the bracket in line 7 of Algorithm 2. Finally, for this model we use $r'(q) = \lfloor e^q \rfloor$.

We have the following theorem for this mechanism, whose proofs for the loss $L_T$ and fairness use the simple fact that binary search converges at an exponential rate.

Theorem 3 Under the deterministic feedback model, Algorithm 2, when using the definitions in Algorithm 5, satisfies the following. Under truthful reporting from all agents, it is asymptotically efficient with

$$L_T \leq 2n \log(eT) + 67\overline{w}\eta_{\text{max}}n^{2.39} \in \mathcal{O}(n \log T).$$

for all $T \geq 2n + 3$. It is asymptotically fair with

$$U_{iT}^{\alpha_i} - U_{iT} \leq 2nL_i^{\alpha_i}\eta_{\text{max}} \log(eT) \in \mathcal{O}(n \log T).$$
for all agents $i$. It is asymptotically strategy-proof with

$$U_{iT}^\pi - U_{iT} \leq L_i^n \eta_{\text{max}} \in \mathcal{O}(1).$$

for all policies $\pi$ and all $T \geq 1$.

When compared to Theorem 2, the rate for efficiency and fairness have improved significantly from $\mathcal{O}(T^{1/2})$ to $\mathcal{O}(\log T)$. However, the strategy-proofness guarantee is slightly weak: the maximum gain in the sum of a user’s utilities in $T$ rounds when deviating from the truth, is at most a constant.

### 3.1.3 Learning without Strategy-proofness Constraints

We will use Algorithm 3 with the same USER-CLASS as in Algorithm 5. Similar to above, the algorithm maintains upper and lower bounds $(\tilde{\eta}_{it}, \tilde{\eta}_{it})$ for each agent initialised to $(0, \eta_{\text{max}})$. In line 5 of Algorithm 3, it sets the recommendation to $\eta_{it} = (\tilde{\eta}_{it} + \tilde{\eta}_{it})/2$. When it receives feedback, it updates the upper and lower bounds as follows.

$$\begin{align*}
(\tilde{\eta}_{it+1}, \tilde{\eta}_{it+1}) &\left\{ \begin{array}{ll}
(\max(a_{it}/w_{it}, \tilde{\eta}_{it}), \tilde{\eta}_{it}) & \text{if } X_{it} < \alpha_i, \\
(\tilde{\eta}_{it}, \min(a_{it}/w_{it}, \tilde{\eta}_{it})) & \text{if } X_{it} \geq \alpha_i.
\end{array} \right.
\end{align*}$$

Here, $a \in \mathbb{R}^n$ is the allocation vector chosen by MMF in line 7 when we use $\{\eta_{it}, w_{it}\}_{i=1}^n$ as the reported demands. Despite some similarities to the strategy-proof case in computing $\eta_i$, there are some important differences. Crucially, as $\eta_i$ is only used to set the demands for MMF, there is no guarantee that the agent will be able to experience this allocation. Consequently, we might not be able to estimate the demands of some agents. For instance, if an agent’s demands $d_{it}^* = w_{it}\eta_i^*$ are large on all rounds, then the agent might never be able to experience this allocation due to contention for the limited resource. Therefore, we cannot estimate $\eta_i^*$ accurately as $X_{it} < \alpha_i$ on all rounds.

Fortunately, we can still effectively learn in this environment provided we estimate the demands of those agents whose demands are, loosely speaking, small. We have the following theorem.

**Theorem 4** Under the deterministic feedback model, Algorithm 3, when using the definitions in Algorithm 5, satisfies the following. Under truthful reporting from all agents,

$$L_T \leq 1 + 2n\bar{w}\eta_{\text{max}} \in \mathcal{O}(n).$$

Therefore, it is asymptotically efficient. Moreover, it is asymptotically fair. Precisely, for any user $i$, and for all $T \geq 1$,

$$U_{iT}^\pi - U_{iT} \leq L_i^n \eta_{\text{max}} \in \mathcal{O}(1).$$

When compared to Algorithm 2, Algorithm 3 has better asymptotic bounds for efficiency and fairness, and in particular, it does not grow with $T$. On the flip side, Algorithm 3 is not strategy-proof.

### 3.2 Stochastic Feedback with Parametric Payoffs

In this section, we will study learning user demands under asymptotic strategy-proofness and Bayes-Nash incentive-compatibility. Algorithm 6 defines USER-CLASS and the functions $r'$ and EXPLORE-PHASE for this model which we will use in both cases.
### Algorithm 6: Definitions for the Stochastic Feedback Model with Parametric Payoffs

#### Function $r'(q)$
1. **return** $\left\lfloor \frac{5q^{1/2}}{6} \right\rfloor$

#### Function EXPLORATION-PHASE($\{UC_i\}_i$)
2. $\{w_i\}_i \leftarrow$ obtain loads from agents.
3. Allocate $e_i$ to each user $i$ and obtain feedback $X_i$.
4. UC$_i$ RECORD-FEEDBACK($e_i/w_i, X_i, \sigma_i$) for each user $i$.

#### Class USER-CLASS
5. **attributes** $D, \hat{\eta}$  # Past data ($D$) and upper bound on unit demand ($\text{udub}$)
6. **method** INITIALISE($)$
7. SELF.$\hat{\eta} \leftarrow \eta_{\text{max}}$.
8. **method** GET-UD-UB($)$  
9. **return** SELF.$\hat{\eta}$  # Main interface for Algorithm 2
10. **method** GET-UD-REC($)$  
11. **return** SELF.$\hat{\eta}$  # Main interface for Algorithm 3
12. **method** RECORD-FEEDBACK($\bar{\pi}, X_i, \sigma_i$)
13. Add $(\bar{\pi}_i, X_i, \sigma_i)$ to SELF.$D$.
14. SELF.$\hat{\eta} \leftarrow$ Compute upper bound as described in (6) and (8).

We begin by describing a procedure to compute confidence intervals for the unit demands $\{\eta^*_i\}_i$ from past data. Let $\{(a_{is}, X_{is})\}_{s \in D_t}$ be a dataset of allocation–reward pairs for user $i$ where $D_t \subset \{1, \ldots, t-1\}$ is a subset of the first $t-1$ time indices. Then define,

$$
\theta_{it} = \arg\min_{\theta \geq \theta_{\text{min}}} \left| \sum_{s \in D_t} \frac{a_{is}}{w_{is}\sigma_{is}^2} \left( \mu \left( \frac{a_{is}\theta}{w_{is}} \right) - X_{is} \right) \right|, \quad A_{it}^2 := \sum_{s \in D_t} \frac{a_{is}^2}{w_{is}^2\sigma_{is}^2}, \quad (6)
$$

Here, $\theta_{it}$ can be interpreted as an estimate for $\theta_i^*$ using the data in round indices $D_t$. Recall, from Section 2, that for the stochastic models we wish our results to hold with a target success probability of at least $1 - \delta$, where $\delta$ is an input to the mechanism. We now define $\beta_t$ as follows,

$$
\beta_t = \frac{5}{\kappa_\mu} \sqrt{\log(n\pi^2 |D_t|^2/6\delta)} \in \mathcal{O}(\log(n|D_t|/\delta)) \quad (7)
$$

We then define $(\bar{\theta}_{it}, \hat{\theta}_{it}), (\bar{\eta}_{it}, \hat{\eta}_{it})$ as shown below, which are confidence intervals for $\theta_i^*$ and $\eta_i^*$ respectively (Lemma 21). We have:

$$
(\bar{\theta}_{it}, \hat{\theta}_{it}) = \left( \min\left\{ \theta_{\text{min}}, \theta_{it} - \frac{\beta_t}{A_{it}} \right\}, \theta_{it} + \frac{\beta_t}{A_{it}} \right), \quad (\bar{\eta}_{it}, \hat{\eta}_{it}) = \left( \frac{\mu^{-1}(\alpha_i)}{\theta_{it}}, \frac{\mu^{-1}(\alpha_i)}{\theta_{it}} \right). \quad (8)
$$
3.2.1 Learning with Asymptotic Strategy-Proofness

The exploration phase in Algorithm 2 consists of just a single round where the mechanism allocates in proportion to the entitlements, while the latter phase in bracket $q$ is executed for $r'(q) = \lceil 5q^{1/2}/6 \rceil$ rounds. The upper confidence bound $\hat{\eta}_t$ for $\eta^*_t$ in line 3 is set to be $\hat{\eta}_t$ as computed in (8), where we set $D_t$ to be the time indices corresponding to the exploration phase; observe that in Algorithm 2 and Algorithm 6, the dataset $D_{it}$ is updated in the RECORD-FEEDBACK method which is called in EXPLORE-PHASE. The following theorem states the properties of Algorithm 2 under this model.

**Theorem 5** Assume that the rewards follow the parametric feedback model outlined in Section 2.4 and that it satisfies Assumption 1. Then Algorithm 2, when using the definitions in Algorithm 5, satisfies the following statements with probability greater than $1 - \delta$. Under truthful reporting from all agents, it is asymptotically efficient with,

$$L_T \leq 3T^{2/3} + 3Cn\beta_T T^{2/3} \in \mathcal{O}\left(nT^{2/3}\sqrt{\log(nT/\delta)}\right),$$

for all $T > 0$. Here, $C = \sqrt{2}\max_i \frac{T^{2/3}}{\theta_{\min}(\text{min}_i e_i)}$. Moreover, $U_{iT}^\pi \leq U_{iT}$ for all users $i$ and for all $T \geq 1$. Finally, for any user $i$ and for any policy $\pi$, $U_{iT}^\pi - U_{iT} \leq 0$ for all $T \geq 1$. Therefore, Algorithm 2 is probably fair and probably strategy-proof.

3.2.2 Learning with Asymptotic Bayes-Nash Incentive-Compatibility

In Algorithm 3, on round $t$ we set the recommendation $\eta_t$ in line 5 to be $\hat{\eta}_t$ from (8), where we set $D_t = \{1, \ldots, t-1\}$. To state our theorem, we first define the following constants.

$$C_1 = \frac{\theta_{\min}}{2\sigma w^2}, \quad C_2 = \frac{1}{\sqrt{\log(1 + C_1^2)}}, \quad C_3 = \left(\frac{\bar{\sigma} w}{\sigma w \text{min}_i e_i}\right)^2. \quad (9)$$

We have the following theorem.

**Theorem 6** Assume that the rewards follow the parametric feedback model outlined in Section 2.4 and that it satisfies Assumption 1. Then Algorithm 2, when using the definitions in Algorithm 5, satisfies the following statements with probability greater than $1 - \delta$. Under truthful reporting from all agents, it is asymptotically efficient with,

$$L_T \leq 1 + C_2n\beta_T T^{1/2}\sqrt{\log(C_3 T)} \in \mathcal{O}\left(nT^{1/2}\sqrt{\log(nT/\delta)} \log(T)\right),$$

for all $T \geq 0$. Moreover, $U_{iT}^\pi \leq U_{iT}$ for all users $i$ and for all $T \geq 1$; i.e. it is probably fair. Finally, assume all agents except $i$ are truthful. Then, for all $\pi$, and for all $T > 0$,

$$U_{iT}^\pi - U_{iT} \leq \frac{L_i C_2}{w} (n-1)\beta_T T^{1/2}\sqrt{\log(C_3 T)} \in \mathcal{O}\left(nT^{1/2}\sqrt{\log(nT/\delta)} \log(T)\right).$$

That is, Algorithm 3 is probably asymptotically Bayes-Nash incentive-compatible.

In the parametric model, Algorithm 3 is asymptotically efficient with rate $\tilde{O}(T^{1/2})$, which is better than the $\tilde{O}(T^{2/3})$ rate of Algorithm 2. Moreover, unlike in Section 3.1, Algorithm 3 also satisfies an asymptotic Bayes-Nash incentive-compatibility condition on an agent’s truthful behaviour. While this is weaker than asymptotic strategy-proofness, it describes an approximate Nash equilibrium: an agent does not stand to gain much by deviating from truthfully reporting their threshold, loads, and feedback, if all other agents are being truthful.
3.3 Stochastic Feedback with Nonparametric Payoffs

In the nonparametric setting, we will use a branch-and-bound method for estimating the unit demands. For this, we will define an infinite binary tree for each user, with each node in the tree corresponding to a sub-interval of $[0, \eta_{\text{max}}]$. As we collect data, we will expand the nodes in this tree, and assign each point to multiple nodes of the expanded tree. We will then use the data assigned to these nodes to construct upper and lower confidence intervals for the values of the payoff in said interval.

We will index the nodes of our binary tree by a tuple of integers $(h, k)$ where $h \geq 0$ is the height of the node, and $k$, which lies between 1 and $2^h$ denotes its position among all nodes at height $h$. The root node is $(0, 1)$. The left child of node $(h, k)$ is $(h + 1, 2k - 1)$ and its right child is $(h + 1, 2k)$; i.e. the children are at the next height and occupy the position in between the children of the nodes immediately to the left and right of the parent. It follows that the parent of a node $(h, k)$ is $(h - 1, \lceil k/2 \rceil)$. Each node $(h, k)$ is associated with an interval $I_{hk} \subset [0, \eta_{\text{max}}]$, defined as follows:

$$I_{hk} = \left[ \frac{\eta_{\text{max}}(k - 1)}{2^h}, \frac{\eta_{\text{max}}k}{2^h} \right] \text{ if } 1 \leq k < 2^h, \quad I_{hk} = \left[ \frac{\eta_{\text{max}}(2^h - 1)}{2^h}, \eta_{\text{max}} \right] \text{ if } k = 2^h. \quad (10)$$

For example, the interval corresponding to the root node $(0, 1) = [0, \eta_{\text{max}})$; its children are $(1, 1)$ and $(1, 2)$ with corresponding intervals $[0, 1/2)$ and $[1/2, 1]$ respectively. We see that the nodes at a given height partition $[0, \eta_{\text{max}}]$, i.e. for all $h$, $\bigcup_{k=1}^{2^h} I_{hk} = [0, \eta_{\text{max}}]$ and for all $h$ and $k_1 < k_2$, $I_{hk_1} \cup I_{hk_2} = \varnothing$.

We will now describe the USER-CLASS, outlined in Algorithms 7-10. Recall that we have dropped the user and time subscripts in the algorithm. For what follows, we define the following quantities.

$$t^1 = 2^{\lfloor \log_2(t) \rfloor}, \quad \beta_t = \sqrt{(4 + 2 \log(2)) \log \left( \frac{n^2 \tau^3}{6 \delta} \right)}, \quad \tau_{ht} = \frac{\beta_t^2}{L^2} 4^h. \quad (11)$$

USER-CLASS will maintain an infinite binary tree for each user as described above. It will incrementally expand nodes in the tree as it collects data, with the expanded nodes reflecting the data it has received. Let $D_t^i \subset \{1, \ldots, t - 1\}$ denote a subset of the round indices in the first $t - 1$ rounds; we will use feedback from agent $i$ in rounds $D_t^i$ to learn her demand. Let $I_{t_i}$ denote the sub-tree of expanded nodes for each user $i$ at the beginning of round $t$. Consider a round $t$, at the end of which user $i$ receives feedback; this could be all rounds in Algorithm 3 or only during the exploration rounds in Algorithm 2. When the USER-CLASS receives a data point $(\bar{a}_{it}, X_{it}, \sigma_{it})$ in RECORD-FEEDBACK (line 24), it assigns that data point to the nodes along a path $P_{it}$, where

$$\bar{a}_{it} \in I_{hk}, \text{ for all } (h, k) \in P_{it}. \quad (12)$$

We will find it useful to view $P_{it}$ as a set which contains nodes. Accordingly, if user $i$ did not receive feedback during a round $t$, we let $P_{it} = \varnothing$. We now define $W_{it}(h, k)$ to be the sum of squared inverse sub-Gaussian constants assigned to node $(h, k)$ and $\bar{f}_{it}(h, k)$ to be the sample mean of the data assigned to $(h, k)$ weighted by the sub-Gaussian constants. We have:

$$W_{it}(h, k) = \sum_{s \in D_t^i} \frac{1}{\sigma_{is}^2} \mathbbm{1} \left( (h, k) \in P_{is} \right), \quad \bar{f}_{it}(h, k) = \frac{1}{W_{it}(h, k)} \sum_{s \in D_t^i} X_{it} \frac{X_{it}}{\sigma_{is}^2} \mathbbm{1} \left( (h, k) \in P_{is} \right) \quad (13)$$
Algorithm 7  Definitions for the Stochastic Feedback Model with Nonparametric Payoffs – Part I

Function \( r'(q) \)
1: \return \left[ 5nq^{1/2}/6 \right] 

Function EXPLORE-PHASE(\{UC_i\})
2: \textbf{for } i = 1, \ldots, n \textbf{ do}
3: \hspace{1em} w_i \leftarrow \text{obtain load from user } i.
4: \hspace{1em} \eta_i \leftarrow UC_i.GET-UD-REC-FOR-UB().
5: \hspace{1em} Allocate \( \eta_i \times w_i \) to user \( i \) only and obtain feedback \( X_i \) and SGC \( \sigma_i \).
6: \hspace{1em} UCI.RECORD-FEEDBACK(\eta_i, X_i, \sigma_i)

Class USER-CLASS
7: \attributes \( \mathcal{T}, W, \tilde{f}, \hat{f}, \tilde{B}, \hat{B}, \tilde{\sigma}, \hat{\sigma} \)
# \( \mathcal{T} \) stores the expanded tree, \( W \) stores
# the sum of inverse squared SGCs. \( \tilde{f}, \hat{f}, \tilde{B}, \hat{B} \) are used to compute the lower/
# upper bounds. \((\tilde{\sigma}, \hat{\sigma})\) are used in computing an upper confidence bound for \( \eta_i^* \).

8: \method INITIALISE()
9: \hspace{1em} SELF.\mathcal{T} \leftarrow \{(0,1)\}.
10: \hspace{1em} SELF.\tilde{B}(0,1) \leftarrow 0, \text{ SELF.}\hat{B}(0,1) \leftarrow 1.
11: \hspace{1em} SELF.EXPAND-NODE((0,1)). \hspace{1em} #Line 42

12: \method GET-UD-UB()
13: \hspace{1em} \return \eta_{\text{max}} \hat{k}/2 \hat{h}. \hspace{1em} #Right-side boundary of \( I_{hk} \). \((\tilde{h}, \hat{k})\) is updated in Line 33

14: \method GET-UD-REC()
15: \hspace{1em} \textbf{if } t = t^* \textbf{ then}
16: \hspace{2em} SELF.REFRESH-BOUNDS-IN-TREE() \hspace{1em} #Line 91
17: \hspace{2em} (h, k) \leftarrow (0, 1) \hspace{1em} #IS-A-LEAF returns true if \( (h, k) \) is a leaf of \( \mathcal{T} \)
18: \hspace{2em} \textbf{while not IS-A-LEAF(SELF.} \mathcal{T}, (h, k)) \textbf{ and SELF.} W(h, k) \geq \tau_{ht} \textbf{ do}
19: \hspace{3em} \textbf{if SELF.GET-B-VAL} (h+1, 2k-1) > SELF.GET-B-VAL(h+1, 2k) \textbf{ then}
20: \hspace{4em} (h, k) \leftarrow (h + 1, 2k - 1)
21: \hspace{2em} \textbf{else}
22: \hspace{3em} (h, k) \leftarrow (h + 1, 2k)
23: \hspace{2em} \textbf{return} \hspace{1em} An arbitrary point in \( I_{hk} \).

In Algorithm 7–10, \( W_{it}(h, k), \tilde{f}_{it}(h, k) \) are updated in the ASSIGN-TO-NODE method in line 53. In RECORD-FEEDBACK, when we traverse along the path \( P_{it} \) (12) assigning the node to each point in that path, we stop either when we reach a leaf node in \( \mathcal{T}_{it} \), or if there is insufficient data at the current node (line 26). The latter criterion is determined by \( W_{it}(h, k) < \tau_{ht} \), where \( \tau_{ht} \) is as defined in (11). If the last node in \( P_{it} \) was a leaf node with sufficient data, i.e. if \( W_{it}(h, k) \geq \tau_{ht} \), it expands that node and adds its children to the tree.

USER-CLASS maintains quantities \( \tilde{B}_{it}(h, k), \hat{B}_{it}(h, k) \) for each node in the tree, which can be interpreted as a lower (upper) bound on the infimum (supremum) of \( f_i \) in the interval \( I_{hk} \). To describe
Algorithm 8  Definitions for the Stochastic Feedback Model with Nonparametric Payoffs – Part II

24: method RECORD-FEEDBACK(\(\pi, X_i, \sigma_i\))
25: \((h, k) \leftarrow (0, 1)\)
26: while \((h, k) \in \text{SELF.T} \text{ and } \text{SELF.W}(h, k) \geq \tau_{ht}\) do #While loop computes \(P_t\)
27: \(\text{SELF.ASSIGN-TO-NODE}((h, k), \pi, X_i, \sigma_i)\) #Line 53
28: if \(\pi < \frac{1}{2}(\ell_{hk} + r_{hk})\), then \((h, k) \leftarrow (h + 1, 2k - 1)\), else \((h, k) \leftarrow (h + 1, 2k)\)
29: \(\text{SELF.UPDATE-BOUNDS-ON-PATH-TO-ROOT}(h, k)\) #Line 62
30: if IS-A-LEAF\((h, k)\) and SELF.W\((h, k) \geq \tau_{ht}\) then
31: \(\text{SELF.EXPAND-NODE}((h, k))\) #Line 42
32: method GET-UD-REC-FOR-UB( ) #Used by EXPLORE-PHASE (line 1)
33: \((\hat{h}, \hat{k}) \leftarrow \text{SELF.UB-TRAVERSE}( )\) #Line 35
34: return An arbitrary point in \(I_{\hat{h}\hat{k}}\).
35: private-method UB-TRAVERSE( ) #IS-A-LEAF returns true if \((h, k)\) is a leaf of \(T\)
36: \((h, k) \leftarrow (0, 1)\)
37: while not IS-A-LEAF\((\text{SELF.T}, (h, k))\) and SELF.W\((h, k) \geq \tau_{ht}\) do
38: if SELF.B\((h + 1, 2k) \geq \alpha\), then \((h, k) \leftarrow (h + 1, 2k - 1)\), else \((h, k) \leftarrow (h + 1, 2k - 1)\)
39: return \((h, k)\)
40: private-method GET-B-VAL\((h, k)\)
41: return SELF.min\((\text{SELF.B}(h, k) - \alpha, \alpha - \text{SELF.B}(h, k))\)
42: private-method EXPAND-NODE\((h, k)\) \(\text{SELF.T} \leftarrow \text{SELF.T} \cup \{(h + 1, 2k - 1), (h + 1, 2k)\}\).
43: \((\ell, u) \leftarrow \text{SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE}((h + 1, 2k - 1))\) #Line 83
44: SELF.B\((h + 1, 2k - 1) \leftarrow \ell, \text{SELF.B}(h + 1, 2k - 1) \leftarrow u.
45: SELF.B\((h + 1, 2k) \leftarrow \ell, \text{SELF.B}(h + 1, 2k) \leftarrow u.
46: return SELF.\(\tilde{\eta}\)
47: private-method UPDATE-BOUNDS-FOR-NODES-AT-SAME-DEPTH((h, k))
48: for \(k' = k + 1, \ldots, 2^h\) do
49: if SELF.\(\tilde{B}(h, k') < \text{SELF.\(\tilde{B}(h, k)\), then SELF.\(\tilde{B}(h, k') \leftarrow \text{SELF.\(\tilde{B}(h, k)\), else break}
50: for \(k' = k - 1, \ldots, 1\) do
51: if SELF.\(\tilde{B}(h, k') > \text{SELF.\(\tilde{B}(h, k)\), then SELF.\(\tilde{B}(h, k') \leftarrow \text{SELF.\(\tilde{B}(h, k)\), else break}

this bound, we first define upper and lower confidence bounds \(\tilde{f}_t, \tilde{f}_t\) for each node \((h, k)\) using only the data assigned to the node:

\[
\tilde{f}_t(h, k) = \begin{cases}
\tilde{f}_t(h, k) - \beta_t W_t(h, k)^{-1/2} - L \cdot 2^{-h} & \text{if } W_t(h, k) > 0, \\
-\infty & \text{if } W_t(h, k) = 0,
\end{cases}
\]

\[
\tilde{f}_t(h, k) = \begin{cases}
\tilde{f}_t(h, k) + \beta_t W_t(h, k)^{-1/2} + L \cdot 2^{-h} & \text{if } W_t(h, k) > 0, \\
\infty & \text{if } W_t(h, k) = 0,
\end{cases}
\]
Algorithm 9  Definitions for the Stochastic Feedback Model with Nonparametric Payoffs – Part III

53: private-method assign-to-node((h, k), a, X_i, \sigma_i)
54:  if SELF.W(h, k) = 0 then
55:    SELF.f(h, k) \leftarrow X_i
56:  else
57:    SELF.f(h, k) \leftarrow \frac{SELF.W(h, k) + X_i}{\sigma_i^2}
58:  SELF.W(h, k) \leftarrow SELF.W(h, k) + \sigma_i^2
59:  SELF.f(h, k) \leftarrow SELF.f(h, k) - \beta_t^{1/2} SELF.W(h, k)^{-1/2} - L \cdot 2^{-h}
60:  SELF.f(h, k) \leftarrow SELF.f(h, k) + \beta_t^{1/2} SELF.W(h, k)^{-1/2} + L \cdot 2^{-h}

62: private-method update-bounds-on-path-to-root((h, k))
63:  if IS-A-LEAF(SELF.T, (h, k)) then
64:    (\ell, u) \leftarrow SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE((h + 1, 2k - 1)) \# Line 83
65:    SELF.B(h, k) \leftarrow \max(SELF.f(h, k), SELF.B(h, k), \ell)
66:    SELF.B(h, k) \leftarrow \min(SELF.f(h, k), SELF.B(h, k), u)
67:    SELF.UPDATE-BOUNDS-FOR-NODES-AT-SAME-DEPTH((h, k)) \# Line 48
68:    (h, k) \leftarrow (h - 1, [((k + 1)/2)]) \# Set (h, k) to its parent
69:  while h \neq -1 do \# Stop when you reach (0,1)
70:    SELF.B(h, k) \leftarrow \max(SELF.B(h, k), SELF.B(h + 1, 2k - 1))
71:    SELF.B(h, k) \leftarrow \min(SELF.f(h, k), SELF.B(h, k), SELF.B(h + 1, 2k - 1))
72:    SELF.UPDATE-BOUNDS-FOR-NODES-AT-SAME-DEPTH((h, k)) \# Line 48
73:    (h, k) \leftarrow (h - 1, [((k + 1)/2)]) \# Set (h, k) to its parent

74: method get-conf-interval(\pi)
75:  (h, k) \leftarrow (0, 1)
76:  b \leftarrow 0, \quad \hat{b} \leftarrow 1
77:  while (h, k) \in SELF.T do
78:    \hat{b} \leftarrow \max(\hat{b}, SELF.B(h, k)), \quad \hat{b} \leftarrow \min(\hat{b}, SELF.B(h, k))
79:    if \pi < \frac{1}{2}(\ell_{hh} + r_{hh}), \quad \text{then} \quad (h, k) \leftarrow (h + 1, 2k - 1), \quad \text{else} \quad (h, k) \leftarrow (h + 1, 2k)
80:    (l, u) \leftarrow SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE(h, k) \# Line 83
81:    \hat{b} \leftarrow \max(\hat{b}, l), \quad \hat{b} \leftarrow \min(\hat{b}, u)
82:    return (\hat{b}, \hat{b})

Here, \beta_t, t^i are as defined in (11). Above, \beta_t^{1/2} W_{it}(h, k)^{-1/2} accounts for the stochasticity in the observed rewards, while \frac{L}{2^h} accounts for the variation in the function value in the interval I_{hh}. As we will show in our proofs, with high probability, f_i(a) \in (\tilde{f}_i(h, k), \hat{f}_i(h, k)) for all a \in I_{hh}.

While \tilde{f}_i(h, k), \hat{f}_i(h, k) provide us a preliminary confidence interval on the function values, this can be refined by considering the bounds of its children and accounting for monotonicity of the function.
Algorithm 10: Definitions for the Stochastic Feedback Model with Nonparametric Payoffs – Part IV

```plaintext
83: private-method GET-BOUNDS-FOR-UNEXPANDED-NODE((h, k))
84: if h > h_{max}(SELF.T) then \# h_{max}(T) = \max\{h; (h, k) has been expanded in T\}
85: return (0, 1)
86: else
87: ℓ ← \widehat{B}(h, k') where k' is the largest k'' < k such that (h, k'') has been expanded.
88: u ← \widehat{B}(h, k') where k' is the smallest k'' > k such that (h, k'') has been expanded.
89: (\ell', u') ← SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE(h + 1, 2k - 1) \# Recurse
90: return (max(ℓ, ℓ'), min(u, u'))
91: private-method REFRESH-BOUNDS-IN-TREE()
92: for (h, k) ∈ SELF.T do \# Update \(\tilde{f}, \tilde{f}\) values of all nodes with new \(t^i\) value
93: SELF.\tilde{f}(h, k) ← SELF.\tilde{f}(h, k) - β_{ℓt} SELF.W(h, k)^{-1/2} - L \cdot 2^{-h}
94: SELF.f(h, k) ← SELF.\tilde{f}(h, k) + β_{ℓt} SELF.W(h, k)^{-1/2} + L \cdot 2^{-h}
95: for h = h_{max}(SELF.T), \ldots , 0 do \# h_{max}(T) = \max\{h; (h, k) has been expanded in T\}
96: \(b_{max}^{\prime} \leftarrow 0\) \# Set \(\widehat{B}\) and ensure it is non-decreasing right to left
97: for k in increasing order among expanded nodes (h, k) at height h do
98: if IS-A-LEAF((h, k), SELF.T) then
99: (l, u) ← SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE(h + 1, 2k - 1)
100: \(b \leftarrow \max\{SELF.\tilde{f}(h, k), SELF.\widehat{B}(h, k), l\}\)
101: else
102: \(b \leftarrow \max\{SELF.\tilde{f}(h, k), SELF.\widehat{B}(h, k), SELF.\widehat{B}(h + 1, 2k - 1)\}\)
103: SELF.\widehat{B}(h, k) ← \max(b_{max}, b)
104: \(\hat{b}_{max} \leftarrow b\)
105: \(\hat{b}_{min} \leftarrow 1\) \# Set \(\widehat{B}\) and ensure it is non-increasing left to right
106: for k in decreasing order among expanded nodes (h, k) at height h do
107: if IS-A-LEAF((h, k), SELF.T) then
108: (l, u) ← SELF.GET-BOUNDS-FOR-UNEXPANDED-NODE(h + 1, 2k - 1)
109: \(b \leftarrow \min\{SELF.\tilde{f}(h, k), SELF.\widehat{B}(h, k), u\}\)
110: else
111: \(b \leftarrow \min\{SELF.\tilde{f}(h, k), SELF.\widehat{B}(h, k), SELF.\widehat{B}(h + 1, 2k)\}\)
112: SELF.\widehat{B}(h, k) ← \min(\hat{b}_{min}, b)
113: \(\hat{b}_{min} \leftarrow SELF.\widehat{B}(h, k)\)
```

The actual lower bounds \(\widehat{B}_{it}\) for the function are computed as follows,

\[
\widehat{B}_{it}(h, k) = \begin{cases} 
0 & \text{if } (h, k') \notin \mathcal{T}_{it} \text{ for all } k' \leq k, \\
\max\left(\tilde{f}_{it}(h, k), \widehat{B}_{it-1}(h, k), \widehat{B}_{it}(h + 1, 2k - 1)\right), & \text{if } (h, k) \in \mathcal{T}_{it}, \\
\widehat{B}_{it}(h, k') & \text{otherwise. Here, } k' = \max\{k'' < k; (h, k'') \in \mathcal{T}_{it}\}. 
\end{cases}
\]  

In the first case above, if a node has not been expanded, nor have any nodes to its left at the same height, we set \(\widehat{B}_{it}(h, k) = 0\). In the second case, if a node has been expanded, we set \(\widehat{B}_{it}(h, k)\) to the maximum of its \(\tilde{f}_{it}(h, k)\) value, the bound \(\widehat{B}_{it-1}(h, k)\) from the previous round, and the bound
\( \tilde{B}_{it}(h + 1, 2k - 1) \) of its left child. The third case applies to nodes which have not been expanded, but if some node at the same height to its left has been expanded; in such cases, we set it to \( \tilde{B}_{it}(h, k') \) where \((h, k')\) is the right-most expanded node to the left of \((h, k)\). Observe that (15) defines \( \tilde{B}_{it} \) values for all nodes \((h, k)\) in the tree: for all nodes where there is no other expanded node at its height, \( \tilde{B}_{it}(h, k) = 0 \) by the first line; for the deepest expanded nodes, \( \tilde{B}_{it}(h, k) \) is given by the second line; then, the \( \tilde{B}_{it}(h, k) \) values for the unexpanded nodes at the same height can be computed using the first or the third lines depending on whether any nodes to the left of the node have been expanded or not; we can then proceed to the previous height and compute \( \tilde{B}_{it} \) in a similar fashion, starting with those nodes that have been expanded.

The upper bounds \( \tilde{B}_{it} \) are computed in an analogous fashion:

\[
\tilde{B}_{it}(h, k) = \begin{cases} 
1 & \text{if } (h, k') \not\in T_{it} \text{ for all } k' \geq k, \\
\min \left( \tilde{f}_{it}(h, k), \tilde{B}_{it-1}(h, k), \tilde{B}_{it}(h + 1, 2k) \right), & \text{if } (h, k) \in T_{it}, \\
\tilde{B}_{it}(h, k'), & \text{otherwise. Here, } k' = \min\{k'' > k; (h, k'') \in T_{it}\}.
\end{cases}
\]

(16)

In the first case, if a node has not been expanded, nor have any nodes to its right at the same height, we set \( \tilde{B}_{it}(h, k) = 1 \). In the second case, if a node has been expanded, we set \( \tilde{B}_{it}(h, k) \) to the minimum of its \( \tilde{f}_{it}(h, k) \) value, the bound \( \tilde{B}_{it-1}(h, k) \) from the previous round, and the bound \( \tilde{B}_{it}(h + 1, 2k) \) of its right child. The third case applies to nodes which have not been expanded, but if some node at the same height to its right has been expanded, in such cases, we set it to \( \tilde{B}_{it}(h, k') \) where \((h, k')\) is the left-most expanded node to the right of \((h, k)\). Similar to above, (16) defines \( \tilde{B}_{it} \) for all nodes in the tree. This completes the description of the confidence intervals for the payoffs. In our analysis, we show that if \( \tilde{f}_{it}, \tilde{f}_{it} \) trap the function, so do \( \tilde{B}_{it}, \tilde{B}_{it} \). Precisely,

\[
f_{it}(a) \in (\tilde{f}_{it}(h, k), \tilde{f}_{it}(h, k)) \forall a \in I_{hk}, \text{ for all } t \text{ and } (h, k)
\Longrightarrow f_{it}(a) \in (\tilde{B}_{it}(h, k), \tilde{B}_{it}(h, k)) \forall a \in I_{hk}, \text{ for all } t \text{ and } (h, k).
\]

USER-CLASS computes and updates the confidence intervals in two different places as shown above. First, whenever a new data point is received, it is assigned to nodes along a chosen path \( P_{it} \) changing the \( \tilde{f}_{it}(h, k) \) values in (13) and (14). Therefore, the \( \tilde{B}, \tilde{B} \) values need to be updated, not only for those nodes in \( P_{it} \), but possibly also for neighbouring nodes at the same height, due to the third case in (15) and (16). This is effected via the UPDATE-BOUNDS-ON-PATH-TO-ROOT method (line 62) invoked in line 29. Second, whenever \( t = t^\dagger \), the value of \( \beta_{it} \) used in (14) changes, requiring that we update the confidence intervals throughout the entire tree. This is effected via the REFRESH-BOUNDS-IN-TREE method (line 91). For Algorithm 3, we have shown its invocation explicitly in line 16 in the GET-UD-REC method which is called for each user by Algorithm 3 in each round. For Algorithm 2, for brevity, we have not made this invocation explicit. An implementation of Algorithm 2 could, say, call REFRESH-BOUNDS-IN-TREE at the beginning of each round by checking if \( t = t^\dagger \).

We will now describe the mechanisms for learning in this model under asymptotic strategy-proofness and without any strategy-proofness constraints.
3.3.1 Learning with asymptotic strategy-proofness

We will use Algorithm 2 in this setting along with the definitions in Algorithm 7–10. We let \( r'(q) = \lfloor 5nq^{1/2}/6 \rfloor \). Each exploration phase consists of \( n \) rounds, one per user. In the round corresponding to user \( i \), we call the GET-UD-REC-FOR-UB method (line 32) of USER-CLASS. This method, invokes UB-TRAVERSE (line 35) which traverses a path of nodes in the tree which either contain \( \eta^*_i \) or is to the right of a node containing \( \eta^*_i \). Concretely, starting from the root node it hops to the root child if the right child’s lower confidence bound \( \tilde{B}_{it} \) is larger than \( \alpha_t \) and to the left child otherwise; intuitively, if it has enough evidence that \( \eta^*_i \) is not in the right child, it chooses the left child. We proceed this way, and stop either when we reach a leaf of \( T_{it} \) or if there are not sufficient data points assigned to the node, quantified by the condition \( W_{it}(h, k) < \tau_{it} \). We update \((\hat{h}_{it}, \hat{k}_{it})\) to be the value returned by UB-TRAVERSE and return an arbitrary point in \( I_{hk} \).

The \((\hat{h}_{it}, \hat{k}_{it})\) node, which is updated once every bracket during the exploration phase round for user \( i \), is used in calculating the value to be used as the reported demand in MMF. Precisely, when Algorithm 2 calls GET-UD-UB, we return \( \eta_{\text{max}} \hat{k}_{it}/2\hat{h}_{it} \) which is the right-most point of the interval \( I_{\hat{h}_{it}, \hat{k}_{it}} \). Lemma 32 in Appendix E shows that the point obtained in this manner is an upper confidence bound on \( \eta^*_i \). It is worth observing that \( \eta_{\text{max}} k/2^h \notin I_{hk} \), unless \( k = 2^h \) (see (10)).

The following theorem outlines the main properties of Algorithm 2 in the nonparametric setting.

**Theorem 7** Assume that the rewards follow the nonparametric feedback model outlined in Section 2.4 and that it satisfies Assumption 2. Let \( G \in (0, G_0] \) be given and let \( \epsilon \in G \) be as defined in Definition 1. Then Algorithm 2, when using the definitions in Algorithms 7–10, satisfies the following with probability greater than \( 1 - \delta \). Under truthful reporting from all agents, it is asymptotically efficient with

\[
L_T \leq 3n^{1/3}T^{2/3} + C_1 \frac{L^{1/2}n^3 \sigma}{G^{1/2}} \beta_2 n^{4/3}T^{2/3} +
C_2 n^{5/3}T^{1/3} \left( \frac{L \sigma^2 \eta_{\text{max}}^2}{G^3 \epsilon_G^3} \beta_2^2 + \frac{\eta_{\text{max}}^2 \sigma^2}{G^2 \epsilon_G^2} \beta_2 \frac{L^2 \eta_{\text{max}}^2}{G^2} + \frac{4L \eta_{\text{max}}}{G \epsilon_G} + 1 \right)
\in O \left( \frac{\log(nT/\delta)}{G^{3/2} \epsilon_G^3} n^{5/3}T^{1/3} + \frac{\sqrt{\log(nT/\delta)}}{G^{5/2}} n^{4/3}T^{2/3} \right)
\]

Here, \( C_1, C_2 \) are global constants. Moreover, it is asymptotically fair with

\[
U^{\epsilon_i}_{IT} - U_iT \leq 3L_i^u \eta_{\text{max}} n^{1/3}T^{2/3} \in O(n^{1/3}T^{2/3}).
\]

Finally, it is asymptotically strategy-proof with

\[
U^{\epsilon_i}_{IT} - U_iT \leq 3L_i^u \eta_{\text{max}} n^{-2/3}T^{2/3} \in O(n^{-2/3}T^{2/3}),
\]

for all policies \( \pi \) and all \( T \geq 1 \).

The \( n^{2/3}T^{2/3} \) rate in the dominant term for the loss is similar to the stochastic parametric model. However, there is also a fairly strong dependence on the near-threshold gradient: in addition to the \( G^{-3/2} \) dependence on the leading term, there is also a \( G^{-3/2} \epsilon_G^{-3} \) dependence on a lower order term. The
The main reason for this strong dependence is that we need to translate confidence intervals on the payoff obtained via the rewards to a confidence interval on the unit demand. As we outlined in Figure 1, this translation can be difficult if $G_0$ is small.

The results for fairness and strategy-proofness are also weaker than the parametric model, with the guarantees here holding only asymptotically. An interesting observation here is that the asymptotic rate for strategy-proofness has an $n^{-2/3}$ dependence on the number of agents. This says that there is less opportunity for an agent to manipulate the outcomes when there are many agents. Next, we will look at learning in the nonparametric model without any strategy-proofness constraints.

### 3.3.2 Learning without Strategy-proofness Constraints

We will use Algorithm 3 along with the definitions in Algorithm 7–10. The get-ud-rec method is given in line 14 of Algorithm 7–10. To describe this method, first define,

$$B_{it}(h, k) = \min \left( \hat{B}_{it}(h, k) - \alpha_i, \alpha_i - \tilde{B}_{it}(h, k) \right).$$

If $B_{it}(h, k)$ is small, this is either because the upper bound $\hat{B}_{it}(h, k)$ is close to or smaller than the threshold $\alpha_i$, or if the lower bound $\tilde{B}_{it}(h, k)$ is close to or larger than the $\alpha_i$. Intuitively, if $B_{it}(h, k)$ is small we are more confident that $\eta^*_i \notin I_{hk}$. When we call the get-ud-rec method (line 14), it traverses a path along this tree, where at each node, it chooses the child with the highest $B_{it}$ value; at each step, it refines its recommendation of $\eta^*_i$ in this manner. It stops either when it has reached a leaf of $\mathcal{T}_{it}$ or if there is not sufficient data assigned to the node, such that any finer estimate will not be meaningful. It then returns an arbitrary point in the interval corresponding to the last node.

The following theorem outlines the main theoretical results for the nonparametric model when using the above procedure along in Algorithm 3.

**Theorem 8** Assume that the rewards follow the nonparametric feedback model outlined in Section 2.4 and that it satisfies Assumption 2. Let $G \in (0, G_0]$ be given and let $\epsilon_G$ be as defined in Definition 1. Then Algorithm 3, when using the definitions in Algorithms 7–10 satisfies the following with probability greater than $1 - \delta$. Under truthful reporting from all agents, it is asymptotically efficient with

$$L_T \leq C_1n \left( \frac{L^{1/2} \eta_{\text{max}} \sigma}{G^{3/2}} \beta_{2T} T^{1/2} + \frac{L \eta_{\text{max}} \sigma^2}{G^3 \epsilon_G^2} \beta_{2T}^2 + \frac{\eta_{\text{max}} \sigma^2}{G^2 \epsilon_G^2} \beta_{2T}^2 + C_3 \right)$$

$$\in \mathcal{O} \left( \frac{n \log (nT/\delta)}{G^{3/2} \epsilon_G^2} + \sqrt{\frac{\log (nT/\delta)}{G^{3/2}}} T^{1/2} \right)$$

Moreover, it is asymptotically fair with

$$U_{it}^{e_i} - U_{it} \leq C_2 L_i^u \left( \frac{L^{1/2} \eta_{\text{max}} \sigma}{G^{3/2}} \beta_{2T} T^{1/2} + \frac{L \eta_{\text{max}} \sigma^2}{G^3 \epsilon_G^2} \beta_{2T}^2 + \frac{\eta_{\text{max}} \sigma^2}{G^2 \epsilon_G^2} \beta_{2T}^2 + C_3 \right)$$

$$\in \mathcal{O} \left( \frac{\log (nT/\delta)}{G^{3/2} \epsilon_G^2} + \sqrt{\frac{\log (nT/\delta)}{G^{3/2}}} T^{1/2} \right)$$

Above, $C_1, C_2$ are global constants and $C_3 = L^2 / G^2 + L \eta_{\text{max}} / G + 1$. 
As was the case for the previous models, we find that the rate for the efficiency and fairness are better when the strategy-proofness constraints are relaxed, improving from $\tilde{O}(T^{2/3})$ to $\tilde{O}(T^{1/2})$. The dependence on the near threshold gradient in the dominant term is similar to Theorem 7, although now the $G^{-3}\epsilon_G^{-3}$ is coupled with a $n \log(T)$ whereas in Theorem 5, it was $n^{5/3}T^{1/3}$.

This completes the description and results for the nonparametric model. It is worth emphasising that while the procedure is seemingly long, it is computationally very efficient in practice as most of the steps are relatively simple. Moreover, the height of the tree does not grow too rapidly since we wait for $W_{it}(h, k)$ to grow larger than $\tau_{ht}$ which increases exponentially in $h$ (11). In fact, the above method is being deployed in a real-time scheduling system, where sub-second response times are necessary. The method was designed with this important practical consideration of being able to determine the allocations fast. In Section 4, we have presented some empirical results on run time.

In Figure 2, we have illustrated the expansion of the tree and the construction of the confidence intervals in Algorithm 3. We show two different users whose demands are small and large relative to their entitlement. For the first user, the tree is expanded deep around $\eta^*_i$ enabling us to accurately estimate this user’s demand. In the latter case, the demand is large; due to resource contention and fairness constraints, we are not able to allocate many resources to this user and accurately estimate her demand. We have also illustrated confidence intervals for the payoff, computed via the GET-CONF-INTERVAL method (line 74). Observe that the confidence intervals are monotonic. For instance, in the bottom figure, one would expect the confidence intervals in the $(0.75, 1)$ interval to be large due to the lack of data. However, we are able to use monotonicity to clip the lower confidence bound using data from allocations in the range $(0, 0.75)$. Monotonicity of the confidence intervals is necessary for the correctness of our algorithms—see Remark 3.

### 3.4 Proof Sketches & Discussion

We conclude this section with a brief discussion on high-level proof techniques and relations to previous work in the online learning and bandit literature.

To control the $T$-period loss, we bound the instantaneous loss $\ell(d^*_t, a_t)$ using a two-way argument. First, if there are any unallocated resources $\ell^{ur} > 0$, we bound $\ell$ by the unmet demand $\ell^{ud}$; this is usually the easier case since it captures instances when there are more resources available than the sum of demands. For example, when we use an upper bound for $\eta^*_i$ in Algorithm 2, we show that when $\ell^{ur} > 0$, we also have $\ell^{ud} = 0$ as all agents have met their demands for that round. Even in Algorithm 3 when we do not use an upper bound, this term can be shown to be small. However, when $\ell^{ur}(d^*_t) = 0$, this means that there is a scarcity of resources. This is the harder case to analyse since the mechanism can risk an inefficient allocation by over-allocating for some agents. Generally speaking, this term will vanish if our estimate of the demands converge to the true demands, for which we use the properties of our allocation scheme. However, as noted before, the demands of all agents cannot be estimated accurately in Algorithm 3 due to resource contention and fairness constraints. For instance, in Figure 2, we cannot allocate enough resources for the agent in the bottom figure to accurately estimate her demand. Therefore, this requires a more careful analysis which argues that we only need to estimate the demands of agents whose demands are small in order to achieve an efficient allocation. This argument needs to be made carefully as we need to account for the changing demands at each round.
Figure 2: An illustration of the nonparametric tree-based estimator for two users whose demands \( \eta^*_i \) are smaller and larger than their entitlement in the top and bottom figures respectively. Here, \( e_i \) and \( \eta^*_i \) denote the entitlement and unit demand. The blue curve is the payoff \( f_i \) and the \( \times \)'s are the data collected, i.e. allocation-reward pairs. The shaded region represents the confidence interval for the payoff, which is computed using the GET-CONF-INTERVAL method in line 74 of Algorithms 7-10. In this simulation, we had four users with equal entitlement, and \( \alpha_i = 0.9, w_{it} = 1 \) for all \( i, t \).
Our proofs for fairness and strategy-proofness rely on several useful properties of MMF, which we prove in Appendix A. For example, in Algorithm 2, since we use upper bounds on the agents’ demands in line 7, it guarantees that an agent cannot gain by inflating the mechanism’s estimate of her demand. One interesting observation here is that the fairness guarantees for the parametric model hold non-asymptotically—albeit probabilistically—whereas the guarantees for the deterministic model hold only asymptotically. This is a consequence of the parametric assumption, as it allows us to estimate an agent’s payoff by simply estimating her parameter $\theta_i$.

While our algorithms bear some superficial similarities to optimistic bandit methods which use upper confidence bounds on the reward (Auer, 2003), there are no immediate connections as we do not maximise any function of the rewards. In fact, it can be argued that an optimistic strategy for resource allocation would allocate resources assuming that satisfying each user’s demand was as easy as possible given past data. Consequently, it would allocate resources based on a lower (confidence) bound of the demand. In contrast, our use of the upper bounds stems from the strategy-proofness and fairness requirements. That said, the construction of the confidence intervals for the parametric and nonparametric models borrows ideas from prior work in the bandit literature using generalised linear models (Dani et al., 2008; Filippi et al., 2010; Rusmevichientong and Tsitsiklis, 2010), and nonparametric models (Bubeck et al., 2010; Gheshlaghi Azar et al., 2014; Grill et al., 2015; Jones et al., 1993; Sen et al., 2018, 2019; Shang et al., 2018).

It is worth highlighting some of the differences in the nonparametric setting when compared to the above nonparametric bandit optimisation work. First, bandit analyses are concerned with minimising the cumulative regret which compares the optimal payoff value to the queried payoff value. However, here, our loss is given in terms of the allocations (see (2)) and not the payoff value achieved by the allocations. This requires us to translate payoff values we have observed to an estimate on the demand, which can be difficult. Second, in typical bandit settings, we may query the function at any point we wish. However, in our setting, we may not be able to do so due to contention on limited resources. For example, in our algorithm, we may not receive feedback for the chosen recommendation; rather, the allocation is chosen by MMF based on the recommendation which may be smaller than the recommendation itself. In the design of the algorithm, we need to ensure that recommendations are carefully chosen so that MMF does not repeatedly choose the same allocation for the user; ensuring that the confidence intervals are monotonic is critical for doing so. Moreover, as we stated before, the analysis needs to handle the fact that we may never be able to estimate a user’s demand due to this phenomenon. Additionally, we also need to account for the discrepancy between the recommended point and the evaluated point when recording feedback. In addition to these two main differences, there are a number of other differences that arise due to the differences in the problem set up. Since our goal here is to estimate the demands, the criterion for traversing a tree to select a recommendation (lines 32, 35 and lines 14, 40 for Algorithms 2 and 3 respectively) is different from prior tree-based bandit work (Bubeck et al., 2010; Gheshlaghi Azar et al., 2014). Moreover, the computation of the lower/upper confidence bounds in (15), (16) are markedly different from the usual way they are computed in the optimisation literature. Navigating these challenges requires new design and analysis techniques.

2. For example, in optimisation, finding the optimal value is typically is easier than finding the optimal point.
4. Experiments

In this section, we present our experimental evaluation on synthetic experiments and a prediction-serving task. We compare the following classes of methods in this evaluation.

- Entitlement based allocation: on all rounds we allocate in proportion to entitlements.
- The three methods in Section 3.1 using deterministic feedback, i.e. Algorithms 2 & 4, Algorithms 2 & 5, and 3 & 5.
- The two methods in Section 3.2 using parametric feedback, i.e. Algorithms 2 & 6 and Algorithms 3 & 6, when using \( \mu(x) = \tanh(x) \).
- The same parametric models when using an algebraic function \( 1 - (1 + x)^{-1} \) for \( \mu(x) \).
- The two methods in Section 3.3 using nonparametric feedback, i.e. Algorithms 2 & 7-10 and Algorithms 3 & 7-10.

Synthetic Experiments

Our synthetic experimental set up simulates the web-serving scenario outlined in Example 1. We have agents for whom the unit load is between \( 10^{-4} \) and \( 10^{-6} \) and on each round, the load is chosen uniformly randomly in \([5000, 15000]\). We perform three experiments, with 5, 10, and 15 agents respectively, and where the rewards are drawn from payoff functions \( f_i \) which have the parametric form in Section 3.2. For the first two synthetic experiments, we use \( \mu(x) = \tanh(x) \) and \( \mu(x) = 1 - (1 + x)^{-1} \) with the parameter \( \theta_i^* \) set based on the unit load for each agent’s model. In the third synthetic experiment, we use \( \mu(x) = 1/(1 + e^{-(x-b)}) \) (logistic function) where \( b \) is chosen so that it is \( 0.6 \times \) the unit demand for each user; observe that while the first two experiments conform to the parametric model, the third does not. For Det-SP and Det-NSP, we directly use the payoff \( f_i(a) = \mu(\theta_i^*a) \) as the feedback, whereas for the stochastic methods we use stochastic feedback.

The results are given in Figures 3(a)-3(c). As expected, assigning proportional to entitlements on each round performs poorly and has linear loss. As indicated in our analysis, Algorithm 3 does better than Algorithm 2 for the same feedback model. While the parametric models outperform entitlement-based allocation in all experiments, it can suffer when the model is misspecified. In contrast, the nonparametric models perform well across all the experiments as it does not make strong assumptions about the payoffs. Finally, the deterministic methods do better than the stochastic methods since they observe the payoff without noise.

In Figure 3(c), while some of the methods based on Algorithm 3 perform worse than simply allocating in proportion to the entitlements, we see that the loss grows sublinearly. The large loss in the initial rounds is due to the large exploration phases when there are many agents. It is also worth observing that in Figure 3(a), the parametric method using a \( \tanh \) function for \( \mu \) performs worse than the nonparametric method, even though the true payoff is a \( \tanh \) function. This could be due to the fact that the confidence intervals may be somewhat conservative (see below).

We wish to highlight that the tree-based procedure for the nonparametric model is computationally efficient in practice. For example, in Algorithm 3, in the second synthetic experiment with 10 users, the average time taken to obtain a recommendation for a user was \( \sim 0.0011 \)s after 100 rounds (i.e. 100 data points in the tree), \( \sim 0.0037 \)s after 1000 rounds, and \( \sim 0.0384 \)s after 10000 rounds.

We do not wish to highlight that the tree-based procedure for the nonparametric model is computationally efficient in practice. For example, in Algorithm 3, in the second synthetic experiment with 10 users, the average time taken to obtain a recommendation for a user was \( \sim 0.0011 \)s after 100 rounds (i.e. 100 data points in the tree), \( \sim 0.0037 \)s after 1000 rounds, and \( \sim 0.0384 \)s after 10000 rounds.
Kandasamy, Sela, Gonzalez, Jordan, Stoica

Figure 3: Figures (a)–(c): Results on the synthetic experiments. The rewards are drawn from models where the payoffs are tanh functions, algebraic functions of the form $1 - (1 + x)^{-1}$, or logistic functions, as indicated in the title. Figure (d): Results on the prediction-serving task. Figure (e): Legend for figures (a)–(d). In all figures, we plot the loss on the y-axis (lower is better). All figures were averaged over 5 runs and the shaded region (not visible in most curves) indicates one standard error.
procedure becomes more expensive in later rounds since the tree is expanded as we collect more data. However, since we expand a node only after the sum of inverse variances exceeds $\tau_t$—which grows at rate $4^h$ (see (11))—it does not expand very fast.

A Prediction-serving task

We evaluate our approach on latency-sensitive prediction serving (Crankshaw et al., 2018), which is used in applications such as Amazon Alexa. Here, each user deploys a queued serving system that takes query inputs and returns prediction responses. In this setting, although the application owner knows how to quantify strict performance requirements from the application, the appropriate resource allocation is far less clear a priori, due to the complexity of the system.

In our experiment we consider five users sharing 100 virtual machines, with equal entitlement to this resource. Each user specifies a 100ms, 0.95 percentile latency SLO target for their application response time, meaning that they allow at most 5% of queries in incoming traffic to have response time greater than 100ms. At the end of each round, users provide feedback on the fraction of queries that were completed on time. For the arrival traffic of three of the users we use the Waikato network dataset (McGregor et al., 2000), with different time-of-day and day-of-week regions for the different users. For one user, we use data from the Twitter streaming API (twi), and for the one user, we use the Wikipedia traffic data (wik). We use a query-level execution simulator from (Crankshaw et al., 2018), which uses power-of-2-choices load balancing to mimic real deployment conditions.

The results are shown in Figure 3(d); we obtained ground truth by exhaustively profiling the performance of each user for all values for the number of virtual machines, which enables us to numerically compute the unit demands. As expected, all methods outperform allocating resources in proportion to the entitlements, with the nonparametric model performing the best. The parametric models perform poorly in this experiment, which could be attributed to a mismatch between the model and the problem.

Some Implementation Details

For the parametric methods, we solved for $\theta_t$ in (6), by computing $\hat{\theta}_t^{ML}$ as outlined in D, and then clipping it at $\theta_{\min}$. However, in practice we found $\hat{\theta}_t^{ML} > \theta_{\min}$ in almost all cases. We solved for $\theta_t$, using the Newton-Raphson method.

In all our experiments, we used $\delta = 10^{-3}$. We found the theoretical value for $\beta_t$ (7) as described in Section 3.2 for the parametric model to be too conservative; therefore, we divided it by 5. This value was tuned using a hold-out set of synthetic experiments that were not included in Figure 3. In the bandit literature, it is common to tune upper confidence bounds in a similar manner (Filippi et al., 2010; Kandasamy et al., 2015; Srinivas et al., 2010).

5. Conclusion

We studied mechanisms for a multi-round fair allocation problem when agents do not know their resource requirements, but can provide feedback on an allocation assigned to them. We proposed three feedback models for this problem, and described mechanisms for each model that achieved varying degrees of strategy-proofness. In all cases, we provided upper bounds on the asymptotic rate
for efficiency, fairness, and strategy-proofness, and observed that as we relaxed the strategy-proofness constraints, the rates for efficiency and fairness improved. These insights are backed up by empirical evaluations on range of synthetic and real benchmarks.

One avenue for future work is to explore hardness results for this problem. In particular, while the $\tilde{O}(T^{1/2})$ rates for the loss are not surprising for the stochastic models when using Algorithm 3, it is worth exploring lower bounds for asymptotic fairness and strategy-proofness, and more interestingly how fairness and strategy-proofness constraints affect the rates for the loss.

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Appendix

This Appendix is organised as follows. In Appendix A, we prove Theorem 1. We also establish some properties about MMF which will be useful in subsequent proofs. In Appendix B, we will state and prove some intermediate results that will be useful throughout our analyses of the learning problem in all three models. Appendix C analyses the deterministic feedback model, Appendix D analyses the stochastic parametric model, and Appendix E analyses the nonparametric model.

Appendix A. Properties of Max-min Fairness and Proof of Theorem 1

In this section we state and prove some properties about Max-min fairness (MMF) that will be used in our proofs. Recall that MMF is outlined in Algorithm 1. We will let $r$ and $e$ be the variables in Algorithm 1, which are initialised in line 1 and then updated in line 5.

**Property 1** If agent $i$ reports $d_i < e_i$, her allocation is $d_i$.

**Proof.** In Algorithm 1, $d_j < e_j$ for all agents $j$ before user $i$ in the sorted order, therefore, $r/e$ is increasing until it reaches agent $i$. Since $r/e = 1$ at the beginning, we have $r/e \geq 1$ whenever we reach the if condition in line 3 for all users up to $i$. Therefore, $d_j/e_j < r/e$ for all agents $j$ until $i$. Hence, the for loop does not break, and $i$ is allocated $d_i$ in line 4. \hfill \Box

**Property 2** If user $i$ reports $d_i \geq e_i$, her allocation is at least $e_i$.

**Proof.** We will first show, by way of induction, that in Algorithm 1, $r \geq e$ each time we visit line 3. For the first user, this is true since $r = e = 1$. Now assume that it is true when the if condition is satisfied for a user $j$. Then, $d_j/e_j < r/e$. We therefore have,

$$\frac{r - d_j}{e - e_j} > \frac{r - r e_j/e}{e - e_j} = \frac{r}{e} \cdot \frac{1 - e_j/e}{1 - e_j/e} \geq 1.$$  \hfill (18)

Therefore, the statement is true when we visit line 3 the next time.
Now, in Algorithm 1, if a user was allocated her resource in line 4, then \( a_i = d_i \geq e_i \). Therefore, say she was allocated in line 7. When the condition in line 3 is violated for some user, \( r/e \geq 1 \) by the above argument in (18). Therefore, for all users \( j \) who are assigned in 7, \( a_j = e_j^{t_e} \geq e_j \). □

**Property 3** A user’s allocation is never more than her reported demand.

**Proof.** Let user \( j \)'s demand be \( d_j \) and consider any user \( i \). If she is allocated in line 4, then she is allocated \( d_i \). If she is allocated in line 7, then she is allocated \( r \frac{d_i}{e} \) which is at most \( d_i \) by the if condition in line 3 and the fact that agents are sorted in ascending order of \( d_j/e_j \).

**Property 4** Suppose we have a demand vector \( d \) and allocation vector \( a \) returned by MMF, where, for user \( i \), \( a_i < d_i \). Keeping all other reported demands constant, user \( i \)'s allocation would have been the same for all of her reports \( d_i' \geq a_i \).

**Proof.** If \( a_i < d_i \), then she was allocated \( a_i = re_i/e \) at line 7 in Algorithm 1. Therefore, for any bid greater than or equal to \( a_i \), while her ranking may have changed, she would still have been allocated the same amount in line 7, as the if condition in line 3 is not satisfied.

**Property 5** Fix the reported demands of all agents except \( i \). Let the allocations of agent \( i \) when reporting \( d_i', d_i'' \) be \( a_i \) and \( a_i' \) respectively. If \( d_i < d_i'' \), then \( a_i' \leq a_i' \) with equality holding only when the agent is allocated in line 7 when reporting both \( d_i \) and \( d_i'' \).

**Proof.** First consider the case where the agent was allocated in line 7 when reporting \( d_i \). Then, \( a_i \leq d_i \). Using the same argument used in the proof of Property 4, we have that her allocation would have been the same for all demands larger than \( d_i \), including, in particular \( d_i'' \). Therefore, \( a_i = a_i' \).

Second, consider the case where she was allocated in line 4 when reporting \( d_i \). If she would have been allocated in line 4, had she reported \( d_i'' \) instead of \( d_i \), then \( a_i = d_i < d_i'' = a_i'' \). Suppose instead, that she would have been allocated in line 7 when reporting \( d_i'' \). If \( r', e' \) are the values of \( r, e \) in Algorithm 1 when the if condition is first not satisfied, she would have received \( r'e_i/e' \) when reporting \( d_i'' \). We will show, by way of contradiction, that \( d_i < r'e_i/e' \) which proves the property.

To show the contradiction, assume instead \( d_i \geq r'e_i/e' \). Then, when \( i \) reports \( d_i \), the if condition is violated when either user \( i \) or an agent before user \( i \) in the ordering reaches the condition. Then, she will have been allocated at line 7 which contradicts the premise of the case, which is that \( i \) was allocated in line 4 when reporting \( d_i \).

**Property 6** Fix the reported demands of all agents except \( i \). Let the utilities of agent \( i \) when reporting \( d_i', d_i^* \) be \( \tilde{u}_i \) and \( \tilde{u}_i^* \) respectively. If \( d_i < d_i^* \), then \( \tilde{u}_i \leq \tilde{u}_i^* \).

**Proof.** This follows from monotonicity of the utility and Property 5, i.e. the fact that an agent’s allocation cannot decrease when she increases her bid, when all other bids are unchanged.

**Property 7** Fix the reported demands of all agents except \( i \). Let the utilities of agent \( i \) when reporting \( d_i, d_i^* \) be \( \tilde{u}_i \) and \( \tilde{u}_i^* \) respectively. If \( d_i > d_i^* \), then \( \tilde{u}_i = \tilde{u}_i^* \).

**Proof.** Assume the allocations were \( a_i^*, a_i \) when the user reports \( d_i^*, d_i \) respectively. By Property 3, \( a_i^* \leq d_i^* \). First assume that the agent was allocated \( a_i^* < d_i^* \). By Property 4, we have \( a_i = a_i^* \) which
implies ť_i = ť^*_i. Now say the agent was allocated a_i^* = d_i^*, and it changes to a_i when she switches to d_i. By Property 5, the allocation by MMF does not decrease when an agent increases her demand. Therefore, a_i ≥ a_i^*. The claim follows from the fact that u_i(a) = u_i(a_i^*) for all a ≥ a_i^*.

We can now use the above properties to prove Theorem 1.

**Proof of Theorem 1. Efficiency:** Assume all users report truthfully. Recall that a mechanism is efficient if ℓ(d^*, a) = min(ℓ^ur(a) + ℓ^or(d^*, a), ℓ^ad(d^*, a)) = 0 where ℓ^ur, ℓ^or, ℓ^ad are as defined in (1). Since MMF never allocates a user more than her reported demand (Property 3), ℓ^or(d^*, a) = 0. Therefore, if ℓ^ur(a) = 0, then ℓ(d^*, a) = 0. If ℓ^ur(a) > 0, this means Algorithm 1 never entered line 7 and therefore a_i = d_i^* for all users i. Hence, ℓ(d^*, a) ≤ ℓ^ad(d^*, a) = 0.

**Fairness:** Assume user i reports truthfully. If d_i^* < e_i, then by Property 1, u_i(e_i) = u_i(d_i^*) = u_i(a_i). If d_i^* ≥ e_i, then by Property 2 and the fact that u_i is non-decreasing, u_i(e_i) ≤ u_i(a_i).

**Strategy-proofness:** This follows from Properties 6 and 7.

**Appendix B. Some Intermediate Results**

In this section, we will state some intermediate lemmas that will be useful in the proofs of Theorems 3–8. We begin with some notation.

**Instantaneous allocations and losses:** We will let \( \bar{a}_{it} = a_{it}/w_{it} \) be the allocation per unit demand for agent i at round t. Let \( \ell_t = \ell(d_i^*, a_t) \) be the loss at round t and \( \ell^ur_t = \ell^ur(a_t), \ell^or_t = \ell^or(d_i^*, a_t), \) and \( \ell^ad_t = \ell^ad(d_i^*, a_t) \) be the unallocated resources, over allocated resources, and unmet demand respectively at round t (1). Recall, \( \ell_t = \min(\ell^ur_t, \ell^or_t, \ell^ad_t) \).

**Upper/lower (confidence) bounds on the demands:** We will use \( \tilde{\eta}_{it} \) and \( \bar{\eta}_{it} \) to denote upper and lower (confidence) bounds on agent i’s unit demands \( \eta_i^* \) at round t, i.e. they satisfy \( \bar{\eta}_{it} \leq \eta_i^* \leq \tilde{\eta}_{it} \) (with high probability). Moreover, unless otherwise specified, \( \bar{d}_{it} = w_{it}\tilde{\eta}_{it} \) and \( \bar{\bar{d}}_{it} = w_{it}\bar{\eta}_{it} \) will denote upper and lower (confidence) bounds on agent i’s demand \( d_i^* \) at round t.

**Exploration phase \& bracket indices in Algorithm 2:** We will use \( \mathcal{E} \) to denote the round indices belonging to the exploration phase in Algorithm 2. We will use \( q_t \) to denote the bracket index round \( t \) belongs to and \( T_q \) to denote the number of rounds completed by \( q \) brackets. Then,

\[
T_{q-1} < t \leq T_q. \tag{19}
\]

Finally, it is worth recalling the definition of a sub-Gaussian distribution. A random variable \( X \) is \( \sigma \) sub-Gaussian if,

\[
\text{for all } \lambda > 0, \quad \mathbb{E} \left[ e^{\lambda (X - \mathbb{E}X)} \right] \leq \exp \left( \frac{\lambda^2 \sigma^2}{2} \right). \tag{20}
\]

**B.1 Bounds on the Loss**

The following lemma will be useful in bounding the loss in round \( t \) for Algorithm 2.
**Lemma 9** Suppose on round $t$, a multi-round mechanism chose its allocations via MMF by using an upper bound $\tilde{d}_{it}$ for $d_{it}^*$ for all $i \in \{1, \ldots, n\}$ as the reported demand, i.e. $\tilde{d}_{it} \geq d_{it}^*$. Moreover, let $\tilde{d}_{it}$ be a lower bound for $d_{it}^*$ for all $i$, i.e. $\tilde{d}_{it} \leq d_{it}^*$. Then,

$$\ell_t \leq \sum_{i=1}^{n} (\tilde{d}_{it} - d_{it}^*) \leq \sum_{i=1}^{n} (d_{it} - \tilde{d}_{it}).$$

**Proof.** The second inequality follows from the first since $d_{it}^* \geq \tilde{d}_{it}$. To prove the first inequality, first let $\ell_t^{ur} > 0$. This means, in Algorithm 1, all agents were allocated in line 4 and therefore $a_{it} = \tilde{d}_{it} > d_{it}^*$ for all $i$. Hence $\ell_t \leq \ell_t^{ad} = 0$. The statement is true since $\tilde{d}_{it} \geq d_{it}^*$.

Now let $\ell_t^{ur} = 0$. We bound $\ell_t \leq \ell_t^{or} = \sum_i (a_{it} - d_{it}^*)^+$. Since the mechanism reports $\tilde{d}_{it}$ as the demand to MMF, and since MMF does not allocate more than the reported demand (Property 3), we have $a_{it} \leq \tilde{d}_{it}$. □

The following lemma will be useful in bounding the loss in round $t$ for Algorithm 3.

**Lemma 10** Suppose on round $t \geq 2$ of Algorithm 3, we chose the allocations via MMF by using a reported demand $d_{it}$ for all $i \in \{1, \ldots, n\}$. Then,

$$L_T \leq 1 + \sum_{i=1}^{n} \sum_{t=2}^{T} (a_{it} - d_{it}^*)^+ + \sum_{i=1}^{n} \sum_{t=2}^{T} 1(a_{it} = d_{it})(d_{it}^* - a_{it})^+.$$  

**Proof.** First consider the loss at round $t$, where, recall $\ell_t = \min(\ell_t^{ur}, \ell_t^{or}, \ell_t^{ad})$. If $\ell_t^{ur} > 0$, we will bound $\ell_t \leq \ell_t^{ad}$ and if $\ell_t^{ur} = 0$, we will bound $\ell_t \leq \ell_t^{or}$. Intuitively, if the resource is scarce we will show that we have not been inefficient by over-allocating to users, and if there are excess resources, we will show that we have not under-allocated to anyone. This leads us to,

$$L_T = \sum_{t=1}^{T} \ell_t \leq 1 + \sum_{t \geq 2, \ell_t^{ur} = 0}^{T} \ell_t^{or} + \sum_{t \geq 2, \ell_t^{ur} > 0}^{T} \ell_t^{ad}$$

$$= 1 + \sum_{t=2, \ell_t^{ur} = 0}^{T} \sum_{i=1}^{n} (a_{it} - d_{it}^*)^+ + \sum_{t \geq 2, \ell_t^{ur} > 0}^{T} \sum_{i=1}^{n} (d_{it}^* - a_{it})^+$$

$$\leq 1 + \sum_{i=1}^{n} \sum_{t=2}^{T} (a_{it} - d_{it}^*)^+ + \sum_{i=1}^{n} \sum_{t=2, a_{it} = d_{it}}^{T} (d_{it}^* - a_{it})^+.$$  

Here, the second step uses the definitions for $\ell_t^{ur}, \ell_t^{ad}$. The last step uses two relaxations. First, we remove the constraint $\ell_t^{ur} = 0$ in the first summation. In the second summation, we use the fact that if there are unallocated resources, it can only be when MMF allocates to all users their requested demand; therefore, $\ell_t^{ur} > 0$ implies $a_{it} = d_{it}$ for all $i$. □

**B.2 Bounds on Fairness**

The following two lemmas will be useful in the proofs of our fairness results for Algorithm 2. Recall from Section 2.3, user $i$’s utility $u_i$ is $L_i^3$-Lipschitz continuous.
Lemma 11 Suppose on round \( t \), the allocations \( a_t \) are set via MMF, and that for agent \( i \) we used a reported demand \( d_{it} = \eta_{it} w_{it} \). Say she received an allocation \( a_{it} \). Let \( \hat{\eta}_{it} \) be such that \( \hat{\eta}_{it} \geq \max(\eta^*_i, \eta_{it}) \). Then, regardless of the behaviour of the other agents, we have for agent \( i \),

\[
u_i(e_i/w_{it}) - u_i(a_{it}/w_{it}) \leq L^u_i(\hat{\eta}_{it} - \eta_{it}).
\]

**Proof.** Denote \( \hat{d}_{it} = \hat{\eta}_{it} w_{it} \) and \( d^*_{it} = \eta^*_i w_{it} \). First observe that if \( d_{it} \geq e_i \), by Property 2, we have \( u_i(e_i/w_{it}) - u_i(a_{it}/w_{it}) \leq 0 \) and the statement is true since the RHS is positive. Now say \( d_{it} < e_i \).

By Property 1, \( a_{it} = d_{it} \) and therefore,

\[
u_i(e_i/w_{it}) - u_i(a_{it}/w_{it}) \leq u_i(e_i/w_{it}) - u_i(d^*_{it}/w_{it}) + u_i(d^*_{it}/w_{it}) - u_i(d_{it}/w_{it}) \leq L^u_i(d^*_{it}/w_{it} - d_{it}/w_{it})^+ \leq L^u_i(\hat{\eta}_{it} - \eta_{it}).
\]

Above, we have used \( u_i(e_i/w_{it}) - u_i(d^*_{it}/w_{it}) \leq 0 \) since \( \eta^*_i \) maximises \( u_i \) and the fact that \( u_i \) is \( L^u_i \) Lipschitz and increasing. The last step uses the fact that \( \hat{\eta}_{it} \geq \eta^*_i \). \( \square \)

Lemma 12 Suppose we use upper bounds \( \{\hat{\eta}_{it}\}_{i=1}^n \) on the unit demand in line 7 of Algorithm 2 on round \( t \) and chose the allocations via MMF. Assume agent \( i \) was truthful. Let \( q_T \) be as defined in (19) and let \( r \) denote the number of rounds per exploration phase. Then, regardless of the behaviour of the other agents, we have for agent \( i \),

\[
U^e_{it} - U_{iT} \leq rL^u_i \sum_{t=1}^T \eta_{it} q_T.
\]

**Proof.** We decompose the sum of utilities into \( t \in E \) and \( t \notin E \) and apply Lemma 11 to obtain,

\[
U^e_{it} - U_{iT} = \sum_{t \in E} (u_i(e_i/w_{it}) - u_i(a_{it}/w_{it})) + \sum_{t \notin E} (u_i(e_i/w_{it}) - u_i(a_{it}/w_{it})) \\
\leq \sum_{t \in E} (u_i(e_i/w_{it}) - u_i(0)) + \sum_{t \notin E} (u_i(\eta^*_i) - u_i(0)) \leq rL^u_i \sum_{t=1}^T \eta_{it} q_T.
\]

Above, \( u_i(e_i/w_{it}) - u_i(a_{it}/w_{it}) \leq 0 \) for \( t \notin E \) by applying Lemma 11 with \( \eta_{it} \) and \( \hat{\eta}_{it} \) both set to the value \( \eta^*_i \) in line 7 of Algorithm 2 (which is an upper bound on \( \eta^*_i \)). The third step uses that \( \eta^*_i \) maximises \( u_i \). The last step uses Lipschitzness of agent \( i \)'s utility, the fact that \( \eta_{it} \leq \eta_{max} \), and that in \( T \) rounds there will have been \( r q_T \) exploration rounds. \( \square \)

The following lemma will be useful in the proofs of our fairness results for Algorithm 3.

Lemma 13 Suppose on round \( t \) of Algorithm 3, the allocations \( a_t \) are set via MMF, and that for agent \( i \) we used a reported demand \( d_{it} = \eta_{it} w_{it} \). Then, regardless of the behaviour of the other agents, we have for agent \( i \),

\[
U^e_{it} - U_{iT} \leq \sum_{t=1}^T \mathbb{1}(\overline{\eta}_{it} = \eta_{it} \wedge \overline{\eta}_{it} < \eta^*_i) \cdot (u_i(\eta^*_i) - u_i(\eta_{it}))
\]
Proof. Observe that \( u_i(e_i/w_{it}) < u_i(a_{it}/w_{it}) \) only when \( a_{it} < \min(e_i, d^*_it) \); if \( a_{it} \geq e_i \), then the agent’s utility will be larger than when just using her entitlement since \( u_i \) is non-decreasing. Further, \( u_i(\eta_i^*) = u_i(\eta) \) for all \( \eta \geq \eta_i^* \). Accounting for this, we can bound \( U_{iT} - U_{iT}^\pi \) as shown below.

\[
U_{iT}^\pi - U_{iT} = \sum_{t=1}^{T} \left( u_i \left( \frac{e_i}{w_{it}} \right) - u_i \left( \frac{a_{it}}{w_{it}} \right) \right) \\
\leq \sum_{t:a_{it} \leq \min(e_i, d^*_it)} \left( u_i \left( \frac{e_i}{w_{it}} \right) - u_i \left( \frac{d^*_it}{w_{it}} \right) \right) + \left( u_i \left( \frac{d^*_it}{w_{it}} \right) - u_i \left( \frac{\eta_{it}w_{it}}{w_{it}} \right) \right) \\
\leq \sum_{t:a_{it} \leq \min(e_i, d^*_it)} (u_i(\eta_i^*) - u_i(\eta_{it})) \\
\leq \sum_{t=1}^{T} \mathbb{1}(\eta_{it} = \eta_{it} \land \eta_{it} < \eta_i^*) \cdot (u_i(\eta_i^*) - u_i(\eta_{it}))
\]

Above, the second step adds and subtracts \( u_i(\eta_i^*) = u_i(d^*_it/w_{it}) \) while also observing \( a_{it} = d^*_it = \eta_{it}w_{it} \) by Property 1. The third step uses the fact that \( u_i(d^*_it/w_{it}) = u_i(\eta_i^*) \geq u_i(a) \) for all \( a \). In the fourth step, we have used Properties 1, 2, and 3 to conclude that \( \eta_{it} = \tilde{\eta}_{it} \) when \( \tilde{\eta}_{it} \leq e_i \). □

### B.3 Strategy-proofness

Our next lemma will be useful in establishing strategy-proofness for Algorithm 2. For this, consider any agent \( i \) and fix the behaviour of all other agents. Let \( U_{iT}, U_{iT}^\pi \) respectively denote the sum of utilities when \( i \) participates truthfully and when she is following any other (non-truthful) policy \( \pi \). Let \( \{a_{it}\}_t, \{a_{it}^\pi\}_t \) respectively denote the sequence of allocations for agent \( i \) when she is adopting these strategies. Let \( \{\tilde{a}_{it}\}_t \) denote the allocations when agent \( i \) follows \( \pi \) from rounds 1 to \( t - 1 \), but her allocation for round \( t \) is based on her true unit demand; in Algorithm 2, this means we use \( \tilde{\eta}_{it} = \eta_i^* \) in line 7. Note that in general, \( \tilde{\eta}_{it} \) depends on previous allocations when following \( \pi \), since it will have also affected the allocations and consequently the estimates of the unit demands for the other agents, which in turn will affect the allocation \( \tilde{a}_{it} \) chosen by MMF at round \( t \) for agent \( i \).

Our next Lemma is useful for establishing strategy-proofness results for Algorithm 2. Recall that in the exploration phase, the allocations for one agent are not affected by reports from other agents.

**Lemma 14** Consider Algorithm 2 and assume that allocations for any agent in the exploration phase are chosen independent of the reports by other agents. Suppose \( \tilde{\eta}_{i} \) in line 7 is an upper bound on \( \eta_i^* \). Then, for all policies \( \pi \) and all \( T \geq 1 \),

\[
U_{iT}^\pi - U_{iT} \leq \sum_{t=1}^{T} \mathbb{1}(t \in E) \left( u_i(a_{it}^\pi/w_{it}) - u_i(a_{it}/w_{it}) \right).
\]

**Proof.** We first decompose \( U_{iT}^\pi - U_{iT} \) as follows,

\[
U_{iT}^\pi - U_{iT} = \sum_{t \in E} \left( u_i(a_{it}^\pi/w_{it}) - u_i(a_{it}/w_{it}) \right) + \sum_{t \in \bar{E}} \left( u_i(a_{it}^\pi/w_{it}) - u_i(a_{it}/w_{it}) \right) \\
\leq \sum_{t \in E} \left( u_i(a_{it}^\pi/w_{it}) - u_i(a_{it}/w_{it}) \right) + \sum_{t \in \bar{E}} \left( u_i(\tilde{a}_{it}/w_{it}) - u_i(a_{it}/w_{it}) \right)
\]

37
Here, the last step uses Properties 6 and 7 to conclude that an agent’s utility is maximised when reporting her true demand. Agent $i$’s allocation when using her true demand is $a_{it}$ and therefore $u_i(a_{it}/w_{it}) \geq u_i(a_{it}/w_{it})$.

We now argue that each term in the second sum is 0 which will establish the claim. Recall that Algorithm 2 only uses values in the exploration phase to determine allocations, and moreover the allocations in the exploration phase for agent $j \neq i$ are chosen independent of the reports by the agent $i$. Therefore, the value used for $\tilde{\eta}_j$ in line (7) is the same for all agents $j \neq i$ when computing $\tilde{a}_{it}$ and $a_{it}$. By Property 7, we have $u_i(\tilde{a}_{it}/w_{it}) - u_i(a_{it}/w_{it}) = 0$ since $\tilde{\eta}_i$ is an upper bound on $\eta_i^*$.

□

B.4 Bounding the number of Exploration Phases in Algorithm 2

Lemmas 15, 16, and 17 will help us lower and upper bound the number of brackets $q_T$ (19) after $T$ rounds in Algorithm 2.

**Lemma 15** Suppose we execute Algorithm 2 with $n$ exploration phase rounds in each bracket followed by $r'(q) = nq$ rounds for the latter phase. Then, $\frac{1}{\sqrt{3}}n^{-1/2}T^{1/2} \leq q_T \leq 2\sqrt{2n^{-1/2}T^{1/2}}$.

**Proof.** The claim can be easily verified for $T \leq 2n$ so that $q_T = 1$. Therefore, let $T \geq 2n + 1$ and for brevity, write $q = q_T$. We have, $T_{q-1} < T \leq T_q$, where, $T_m = mn + \sum_{t=1}^m t$. Letting $S_m = \sum_{t=1}^m t^d$ and bounding the sum of an increasing function by an integral we have,

$$\int_0^m t^d \, dt < S_m < \int_1^{m+1} t^d \, dt \implies \frac{m^{d+1}}{d+1} < S_m < \frac{(m+1)^{d+1} - 1}{d+1}. \quad (21)$$

(We will use (21) again below in Lemma 17 with different $d$ values.) Setting $d = 1$ leads to the following bounds on $T$,

$$T \leq T_q \leq qn + \sum_{s=1}^q ns \leq qn + \frac{n}{2}((q+1)^2 - 1) \leq nq^2 + \frac{n}{2}(2q)^2 \leq 3nq^2, \quad (22)$$

$$T \geq T_{q-1} \geq n(q-1) + \sum_{t=1}^{q-1} nt \geq \frac{n}{2}(q-1)^2 \geq \frac{n}{2} \left(\frac{q}{2}\right)^2 = \frac{n}{8}q^2. \quad (23)$$

In (22), we have used the upper bound in (21) with $m = q$, and the facts $q \leq q^2$, $q + 1 \leq 2q$. In (23), we have used the lower bound in (21) with $m = q - 1$, and that $q - 1 \geq q/2$ when $q \geq 2$ which is true when $T \geq 2n + 1$. We therefore have, $q \leq 2\sqrt{2n^{-1/2}T}$ and $q \geq \frac{1}{\sqrt{3}}n^{-1/2}T^{1/2}$. □

**Lemma 16** Suppose we execute Algorithm 2 with $2n$ exploration phase rounds in each bracket followed by $r'(q) = \lceil e^q \rceil$ rounds for the latter phase. Then,

$$\log \left(\frac{T}{2n + e}\right) \leq q_T \leq 1 + \log (T).$$

**Proof.** The statement can be verified easily for $T \leq 2n + 2$ since $q_T = 1$. Therefore let $T \geq 2n + 3$. For brevity, write $q = q_T$. Using the notation in (19), we first observe $T_{q-1} < T \leq T_q$, where
We conclude this Section with some technical results. The first, given in Lemma 19, is a concentration of the following result from de la Pena et al. (2004). This leads to the following bounds on $T$.

$$T \leq T_q \leq 2nq + \sum_{t=1}^{q} e^t \leq 2nq + e(e^q - 1) \leq (2n + e)e^q,$$

$$T \geq T_{q-1} \geq 2n(q - 1) + \sum_{t=1}^{q-1} (e^t - 1) \geq (2n - 1)(q - 1) + e^{q-1} - 1 \geq e^{q-1}.$$

In the last step, we have used the fact that $(q - 1)(2n - 1) - 1 \geq 0$ when $q \geq 2$ which is true when $T \geq 2n + 3$. Inverting the above inequalities yields the claim. □

**Lemma 17** Suppose we execute Algorithm 2 with a exploration phase rounds in each bracket followed by $r'(q) = \lfloor 5aq/6 \rfloor$ rounds for the latter phase. Then, $\frac{a}{2}a^{-2/3}T^{2/3} \leq qT \leq 3a^{-2/3}T^{2/3}$.

**Proof.** The proof will follow along similar lines to the proof of Lemma 15. The claim can be easily verified for $T \leq 2a$ so that $qT = 1$. Therefore, let $T \geq 2a + 1$. For brevity, write $q = q_T$, $c = 5/6$, $d = 1/2$. We have, $T_{q-1} < T \leq T_q$, where, $T_m = am + \sum_{t=1}^{m} [ct^d]$. Using (21), we have the following bounds on $T$.

$$T \leq T_q \leq aq + ac \sum_{t=1}^{q} t^d \leq aq + \frac{ac}{d+1}((q+1)^{d+1} - 1)$$

(24)

$$\leq aq^{d+1} + \frac{ac}{d+1}(2q)^{d+1} \leq c_1aq^{3/2},$$

$$T \geq T_{q-1} \geq a(q - 1) + \sum_{t=1}^{q-1} (act^d - 1) \geq (a - 1)(q - 1) + \frac{ac}{d+1}(q - 1)^{d+1}$$

(25)

$$\geq \frac{ac}{d+1} \left( \frac{q}{2} \right)^{d+1} = c_2aq^{3/2}.$$

Here $c_1 = 1 + 2^{3/2}c/3$ and $c_2 = c/(3\sqrt{2})$. In (24), we have used the upper bound in (21) with $m = q$, and the facts $q \leq q^{3/2}$, $q + 1 \leq 2q$. In (25), we have used the lower bound in (21) with $m = q - 1$, and that $q - 1 \geq q/2$ when $q \geq 2$ which is true when $T \geq 2a + 1$. We therefore have, $q \leq c_2^{-2/3}a^{-2/3}T^{2/3}$ and $q \geq c_1^{-2/3}a^{-2/3}T^{2/3}$. Substituting $c = 5/6$ yields the desired result. □

**B.5 Technical Lemmas**

We conclude this Section with some technical results. The first, given in Lemma 19, is a concentration result that will help us establish confidence intervals in the second and third models. It is a corollary of the following result from de la Pena et al. (2004).

**Lemma 18 (de la Pena et al. (2004), Corollary 2.2)** Let $A, B$ be random variables such that $A \geq 0$ a.s. and $\mathbb{E} \left[ e^{\nu B - \frac{\nu^2 A^2}{2}} \right] \leq 1$ for all $\nu \in \mathbb{R}$. Then, $\forall c \geq 2$, $\mathbb{P} \left( |B| > cA\sqrt{2 + \log(2)} \right) \leq e^{-c^2/2}$. 

39
Lemma 19 Let \( \{ F_s \}_{s \geq 0} \) be a filtration, and \( \{ \gamma_s \}_{s \geq 1}, \{ \sigma_s \}_{s \geq 1} \) be predictable processes such that \( \gamma_s \in \mathbb{R} \) and \( \sigma_s > 0 \). Let \( \{ z_s \}_{s \geq 1} \) be a real-valued martingale difference sequence adapted to \( \{ F_s \}_{s \geq 0} \). Assume that \( z_s \) is conditionally \( \sigma_s \) sub-Gaussian, i.e. \( \forall \lambda \geq 0, \mathbb{E} \left[ e^{\lambda z_s} | F_{s-1} \right] \leq \exp \left( \frac{\lambda^2 \gamma_s^2}{2} \right) \). Then, for \( t \geq 1 \) and \( \delta \leq 1/e \), the following holds with probability greater than \( 1 - \delta \).

\[
\left| \sum_{s=1}^{t} \frac{\gamma_s z_s}{\sigma_s^2} \right| \leq \sqrt{(4 + 2 \log(2)) \log \left( \frac{1}{\delta} \right) \sum_{s=1}^{t} \frac{\gamma_s^2}{\sigma_s^2}}.
\]

Proof. We will apply Lemma 18 with \( A^2 \leftarrow \sum_{s=1}^{t} \frac{\gamma_s^2}{\sigma_s^2}, B \leftarrow \sum_{s=1}^{t} \frac{\gamma_s z_s}{\sigma_s^2} \). We first need to verify that the condition for the Lemma holds. Let \( \nu \in \mathbb{R} \) be given. Writing, \( Q_s = \frac{\nu \gamma_s z_s}{\sigma_s^2} - \frac{\nu^2 \gamma_s^2}{2 \sigma_s^2} \), we have

\[
\nu B - \frac{\nu^2 A^2}{2} = \sum_{s=1}^{t} Q_s.
\]

Then, by the sub-Gaussian property,

\[
\mathbb{E} \left[ e^{\nu Q_s} | F_{s-1} \right] = \exp \left( -\frac{\nu^2 \gamma_s^2}{2 \sigma_s^2} \right) \mathbb{E} \left[ \exp \left( \frac{\nu \gamma_s z_s}{\sigma_s^2} \right) | F_{s-1} \right] \leq 1.
\]

Here, we have used the fact that \( \gamma_s \) and \( \sigma_s \) are \( F_{s-1} \) measurable. This leads us to,

\[
\mathbb{E} \left[ e^{\sum_{s=1}^{t} Q_s} \right] = \mathbb{E} \left[ e^{\sum_{s=1}^{t} Q_s} \mathbb{E} \left[ e^{\nu Q_s} | F_{s-1} \right] \right] \leq \mathbb{E} \left[ e^{\sum_{s=1}^{t} Q_s} \right] \leq \ldots \leq 1.
\]

The claim follows by applying Lemma 18 with \( c = \sqrt{2 \log(1/\delta)} \). The \( \delta < 1/e \) condition arises from the \( c \geq 2 \) condition in Lemma 18.

Finally, we will also need the following inequality.

Lemma 20 Let \( c > 0 \). Then, for all \( x \in [0, c] \), \( x \leq \frac{c}{\log(1+c)} \log(1 + x) \).

Proof. Write \( f(x) = \frac{c}{\log(1+c)} \log(1 + x) - x \). We need to show \( f(x) \geq 0 \) for all \( x \in [0, c] \). This follows by observing that \( f \) is concave and that \( f(0) = f(c) = 0 \).

Appendix C. Proofs of Results in Section 3.1

In this section, we present our proofs for the deterministic feedback model.

C.1 Proof of Theorem 2

Efficiency: We will first decompose the \( T \)-period loss as follows,

\[
L_T = \sum_{t \in E} \ell_t + \sum_{t \not\in E} \ell_t \leq nq_T + \sum_{t \not\in E} \ell_t \leq 2\sqrt{2n} T^{1/2} + \sum_{t \not\in E} \sum_{i=1}^{n} (\tilde{d}_{it} - \tilde{d}_{it}).
\]

(26)

Above, we have used the fact that in \( T \) rounds there will have been \( nq_T \) exploration rounds and then used Lemma 15 to bound \( q_T \). To bound the second sum, we have used Lemma 9. Here, \( \tilde{d}_{it} = w_{it} \tilde{\eta}_t \) is an upper bound on the unit demand, computed at the end of the exploration phase.
in the RECORD-FEEDBACK method in line 13. Similarly, we will define \( \tilde{d}_{it} = w_{it}\tilde{\eta}_{it} \) to be a lower bound on the unit demand which is computed as follows. It is initialised to \( \tilde{\eta}_{it} = 0 \) at the beginning. At the end of an exploration phase round \( t \) for user \( i \) it is updated to \( \tilde{\eta}_{it} = \min(\eta_{\text{max}}k/2^h, \tilde{\eta}_{it-1}) \) if \( X_i \) was less than or equal to \( \alpha_t \). Here, \( h, k \) are as defined in EXPLORE-PHASE of Algorithm 4.

Consider any \( t \notin E \) and let \( q_t \) be as defined in (19). In the exploration phases up to the \( t^{\text{th}} \) round, we will have evaluated unit demands for all agents at least at values \( \{\eta_{\text{max}}k/2^h; h = 1, \ldots, \lfloor \log_2(q_t + 1) \rfloor \}, k = 1, \ldots, 2^h \). Therefore, assuming all agents were truthful, we will have constrained \( \eta_{it}^* \) to within the interval \([\tilde{\eta}_{lt}, \tilde{\eta}_{lt}]\) of width at most \( \eta_{\text{max}}/2^{\lfloor \log_2(q_t + 1) \rfloor} \). We have:

\[
\tilde{\eta}_{lt} - \tilde{\eta}_{lt} \leq \frac{\eta_{\text{max}}}{2^{\lfloor \log_2(q_t + 1) \rfloor}} \leq \frac{\eta_{\text{max}}}{2^{\log_2(q_t + 1) - 1}} \leq \frac{2\eta_{\text{max}}}{q_t + 1} \leq \frac{2\eta_{\text{max}}\sqrt{3n}}{t^{1/2}}.
\]

Here, the last step uses Lemma 15. We can now use (26) to bound the loss as follows,

\[
L_T \leq 2\sqrt{2}n^{1/2}T^{1/2} + \sum_{t \notin E} \sum_{i=1}^{n} w_{it}(\tilde{\eta}_{lt} - \tilde{\eta}_{lt}) \leq 2\sqrt{2}n^{1/2}T^{1/2} + 2n\eta_{\text{max}}\tilde{\eta}_{lt} \sum_{t=1}^{T} \frac{1}{t^{1/2}} \leq 2\sqrt{2}n^{1/2}T^{1/2} + 2\sqrt{3}\eta_{\text{max}}\sqrt{n} \left(2\sqrt{T} \right) \leq 10n^{3/2}\sqrt{T}.
\]

Here, the second step uses the conclusion from the previous display. The third step bounds \( \sum_{t=1}^{T} 1/\sqrt{t} \leq 2T^{1/2} \). The last step uses \( \eta_{\text{max}} \leq 1 \) by our assumptions, \( n^{1/2} \leq n^{3/2} \), and that \( 2\sqrt{2} + 4\sqrt{3} \leq 10 \).

**Fairness**: This follows by applying Lemma 12 with \( r = n \) and the upper bound for \( q_T \) in Lemma 15.

**Strategy-proofness**: This follows from Lemma 14 and noting that when \( t \in E \), the allocations do not depend on the strategy adopted by any of the users. Therefore, \( a_{it}^T = a_{it} \).

**C.2 Proof of Theorem 3**

**Efficiency**: We will first decompose the \( T \)-period loss as follows similar to (26),

\[
L_T = \sum_{t \in E} \ell_t + \sum_{t \notin E} \ell_t \leq 2nq_T + \sum_{t \notin E} \ell_t \leq 2n \log(eT) + \sum_{t \notin E} \sum_{i=1}^{n} (\tilde{d}_{it} - \tilde{d}_{it}). \tag{27}
\]

Above, we have used the fact that in \( T \) rounds there will have been \( 2nq_T \) exploration rounds and then used Lemma 16 to bound \( q_T \). To bound the second sum, we have used Lemma 9. Here, \( \tilde{d}_{it} = w_{it}\tilde{\eta}_{it} \) and \( \tilde{d}_{it} = w_{it}\tilde{\eta}_{it} \) are upper and lower bounds on the demands, where \( \tilde{\eta}_{lt}, \tilde{\eta}_{lt} \) are the upper and lower bounds on the unit demand, computed at the end of the exploration phase in the RECORD-FEEDBACK method in line 15. Since \( \tilde{\eta}_{\text{max}} \leq 1 \), the binary search procedure always halves the width of the current upper and lower bounds. We therefore have, for \( t \notin E \),

\[
(\tilde{d}_{it} - \tilde{d}_{it}) = w_{it}(\tilde{\eta}_{lt} - \tilde{\eta}_{lt}) \leq \frac{\tilde{\eta}_{\text{max}}}{2^{2q_t - 1}} \leq \frac{2\tilde{\eta}_{\text{max}}}{q_t^{\log(4)\log(4)\eta_{\text{max}}}} \leq \frac{2\log(4)\log(4)\eta_{\text{max}}}{t^{1/2}}.
\]

The second step observes that in \( t \) rounds, there will have been \( q_t \) brackets and therefore \( 2q_t \) exploration rounds for agent \( i \). The width \( \tilde{\eta}_{lt} - \tilde{\eta}_{lt} \) is \( \eta_{\text{max}} \) at round \( t = 1 \) and is halved after each
iteration of binary search. We then use the lower bound in Lemma 16 which is positive when $T \geq 2n + 3$, which is the case when $t \notin E$. The last inequality uses the fact that $a^b = b^a$ for $a, b > 0$. By summing over all agents and time steps, we have

$$\sum_t \sum_i \left( d_{it} - \hat{d}_{it} \right) \leq 2 \cdot 5^{\log(4)/\eta_{\max}} \log(4) - 1 \leq 67 \eta_{\max} n^{2.39}. $$

The claim on asymptotic efficiency follows by combining this with (27).

**Fairness:** This follows by applying Lemma 12 with $r = 2n$ and the upper bound for $q_T$ in Lemma 16.

**Strategy-proofness:** Denote $\pi_{it} = a_{it}/w_{it}$ and for a policy $\pi, \pi_{it}^* = a_{it}^*/w_{it}$. Let $\tilde{\eta}_{it}, \tilde{\eta}_{it}$ be the upper and lower bounds on the unit demand, as computed at the end of the exploration phase in the RECORD-FEEDBACK method in line 15. Then, using Lemma 14 we obtain,

$$U_{iT}^\pi - U_{iT} \leq \sum_{t \in E} (u_i(\pi_{it}^*) - u_i(\pi_{it})) = \sum_{t \in E, a_{it} > 0} (u_i(\pi_{it}) - u_i(\eta_{it}^*) + u_i(\eta_{it}^*) - u_i(\pi_{it}))
\leq \sum_{t \in E, a_{it} > 0} (u_i(\eta_{it}^*) - u_i(\pi_{it})) \leq L_i^u \sum_{t \in E, a_{it} > 0} (\beta_{it} - \beta_{it})^+
= \frac{L_i^u}{2} \sum_{t \in E, a_{it} > 0} (\beta_{it} - \beta_{it}) \leq L_i^u \eta_{\max} / 2 \sum_{t=1}^T \frac{1}{2^t-1} = L_i^u \eta_{\max}.$$

Here, in the second step we have added and subtracted $u_i(\eta_{it}^*)$ and moreover restricted the summation to rounds where user $i$ receives a non-zero allocation; recall that in EXPLORE-PHASE of Algorithm 5, each user receives two rounds of allocations. In the third step we have used the fact that $\eta_{it}^*$ maximises $u_i$. In the fourth step we have used the Lipschitz property of $u_i$. The fifth step uses the fact that at a round $t$ in the exploration phase for agent $i$, $\beta_{it} = (\beta_{it} + \beta_{it})/2$ and that $\eta_{it}^* \leq \beta_{it}$. The proof is completed by observing that the exploration phase performs binary search for $\eta_{it}^*$. Therefore, $\beta_{it} - \beta_{it}$ is initially equal to $\eta_{\max}$ and then halved at each round.

\[\square\]

**C.3 Proof of Theorem 4**

**Efficiency:** We will use Lemma 10 to bound the $T$-period loss. We will bound the first sum in the RHS of Lemma 10 by bounding $\sum_{t=2}^T (a_{it} - d_{it}^*)$ for user $i$. Recall that we denote $\alpha_{it} = a_{it}/w_{it}$. Let $s_1, s_2, \ldots$, denote the round indices after the second round in which, $a_{it} > d_{it}^*$. We have:

$$\alpha_{is_{r+1}} - \eta_{it}^* \leq \frac{1}{2} \left( \beta_{is_{r+1}} + \beta_{is_{r+1}} \right) - \eta_{it}^* = \frac{1}{2} (\alpha_{is_{r}} - \eta_{it}^*) \leq \ldots \leq \frac{1}{2^r} (\alpha_{is_1} - \eta_{it}^*) \leq \frac{\eta_{\max}}{2^r+1}. $$

Above, the first step uses the fact that the allocations are chosen by MMF, by setting the demand of user $i$ to be $\eta_{it} \cdot w_{it} = \frac{1}{2} (\beta_{it} + \beta_{it}) / w_{it}$ and the fact that MMF does not allocate more than the reported demand (Property 3); therefore, $\alpha_{is_{r+1}} \leq \eta_{is_{r+1}} = (\beta_{is_{r+1}} + \beta_{is_{r+1}})/2$. The second step uses that since we did not over-allocate between rounds $s_r$ and $s_{r+1}$, the upper bound $\beta_{is_{r+1}}$ at round $s_{r+1}$ is the unit allocation $\tilde{\alpha}_{is_{r}}$ at round $s_{r}$; additionally, $\beta_{is_{r+1}} \leq \eta_{is_{r+1}}$ for all $t$. By repeatedly applying this argument, we arrive at $(\alpha_{is_1} - \eta_{it}^*)/2^r$. The last step follows from the observation that when we first over-allocate, we have $\alpha_{is_1} - \eta_{it}^* \leq \frac{1}{2} \eta_{\max}$. Therefore,

$$\sum_{t=2}^T (a_{it} - d_{it}^*) \leq \sum_{r=1}^{\infty} w_{is_r} (\alpha_{is_r} - \eta_{it}^*) \leq \bar{w} \sum_{r=1}^{\infty} (\alpha_{is_r} - \eta_{it}^*) \leq \bar{w} \eta_{max}. $$

(28)
By summing over all agents we can bound the first sum by \( n \bar{w} \eta_{\text{max}} \).

Now let us turn to the second summation in Lemma 10. We will consider a user \( i \) and bound \( \sum_{t=2}^{T} \sum_{a_{it}=d_{it}} (d_{it}^a - a_{it})^+ \). Letting \( s_1, s_2, \ldots \) denote the round indices where \( a_{it} = d_{it} \), we have

\[
\hat{\eta}_{is_{r+1}} - \eta_{is_{r+1}} \leq \hat{\eta}_{is_{r+1}} - \hat{\eta}_{is_{r+1}} \leq \frac{1}{2} (\hat{\eta}_{is_r} - \eta_{is_r}) \leq \cdots \leq \frac{\eta_{\text{max}}}{2^{r+1}}. \tag{29}
\]

The first step above uses that both \( \hat{\eta}_{it}, \hat{\eta}_{it} \) are respectively non-increasing and non-decreasing with \( t \) and that \( s_{r+1} \geq s_r + 1 \). The second step uses the fact that since \( a_{is_r} = d_{is_r} = \eta_{is_r} \) in round \( s_r \), the gap between the upper and lower bounds would have halved at the next round. Via a similar argument to (28) and summing over all agents, we can bound the second sum in Lemma 10 by \( n \bar{w} \eta_{\text{max}} \). Therefore, \( L_T \leq 1 + 2n \bar{w} \eta_{\text{max}} \).

**Fairness:** We will use Lemma 13 to bound \( U_{iT}^w - U_{iT} \). Letting \( s_1, s_2, \ldots \), denote the round indices where \( \bar{\eta}_{it} = \eta_{it} \) and \( \overline{a}_{it} < \eta_{it}^* \), we have,

\[
U_{iT}^w - U_{iT} \leq \sum_{t=1}^{T} 1(\bar{\eta}_{it} = \eta_{it} \land \overline{a}_{it} < \eta_{it}^*) \cdot (u_i(\eta_{it}^*) - u_i(\eta_{it})) = \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} (u_i(\eta_{it}^*) - u_i(\eta_{is_r})) \leq L_i^w \sum_{r=1}^{\infty} (\hat{\eta}_{is_r} - \eta_{is_r}) \leq L_i^w \eta_{\text{max}}.
\]

Here, the second we have used that \( u_i \) is Lipschitz and non-decreasing. The third step simply observes \( \eta_{it}^* \leq \hat{\eta}_{it} \) and \( \eta_{it} = \frac{1}{2}(\hat{\eta}_{it} + \hat{\eta}_{it}) \) for all \( t \). The last step uses a similar argument to (29). \( \square \)

**Remark 1** Our proofs used the fact that \( w_{it} \geq 0 \) when arguing that \( \hat{\eta}_{it} - \eta_{it} \) is halved at the end of each exploration round in Appendix C.2. Had an agent experienced no load during one of her exploration rounds, this would not have been true. This assumption can be avoided by more careful book-keeping where the mechanism checks if an agent has non-zero traffic and if not, delaying that exploration round to a future round.

### Appendix D. Proofs of Results in Section 3.2

This section presents our analysis for the stochastic parametric model. In Appendix D.1, we will first show that the confidence intervals in (8) traps the true demands with probability larger than \( 1 - \delta \) in all rounds. In Appendices D.2 and D.3, we will establish our results for Algorithms 2 and 3 respectively assuming that this is true.

#### D.1 Confidence Intervals for \( \eta_{it}^* \)

The following lemma shows that the probability that the confidence intervals in (8) do not capture the true parameter and demands is bounded by \( \delta \). The lemma applies to both Algorithms 2 and 3.

**Lemma 21** Assume the feedback model described in Section 3.2 and let \( (\hat{\theta}_{it}, \hat{\theta}_{it}) \) and \( (\hat{\eta}_{it}, \hat{\eta}_{it}) \) be as defined in (8). Then, with probability greater than \( 1 - \delta \), for all \( i \in \{1, \ldots, n\} \) and all \( t \geq 0 \),

\[
\theta_{it}^* \in (\hat{\theta}_{it}, \hat{\theta}_{it}), \quad \eta_{it}^* \in (\hat{\eta}_{it}, \hat{\eta}_{it}).
\]
Proof. Consider any user $i$. First, we will show that the sum of deviations of $X_{it}$ from the true payoff scale with $A_{it}$. We will use Lemma 19 with $z_{s} \leftarrow X_{is} - f_{i}(\overline{a}_{is})$, $\gamma_{s} \leftarrow a_{is}/w_{is}$, $\sigma_{s} \leftarrow \sigma_{is}$, and let $F_{s}$ be the sigma-field generated by the data from all users up to round $s$. Accordingly, $\{\gamma_{s}\}$ is predictable since the recommendations $\{\hat{\eta}_{jt} \times w_{is}\}_{j=1}^{n}$ for all users are chosen based on their past data and then the allocations are chosen based on these recommendations via MMF; i.e., $\gamma_{s}$ is $F_{s-1}$-measurable. $\sigma_{s}$ is predictable by our assumptions. Finally, $z_{s}$ is $F_{s}$-measurable and $\mathbb{E}[z_{s} | F_{s-1}] = 0$.

Now define $B_{it}$ as shown below and recall the definition of $A_{it}$ from (6).

$$B_{it} = \sum_{s=1}^{t-1} \frac{a_{is}}{w_{is}\sigma_{is}^{2}} (X_{is} - f_{i}(\overline{a}_{is})), \quad A_{it}^{2} := \sum_{s \in D_{t}} \frac{a_{is}^{2}}{w_{is}^{2}\sigma_{is}^{2}}.$$  

By applying Lemma 18 for a given $\delta' \in (0, 1/e)$, the following holds with probability at least $1 - \delta'$:

$$|B_{it}| \leq A_{it}\sqrt{(4 + 2 \log(2)) \log(1/\delta')}.$$  

(30)

Next, we will bound the estimation error $|\theta_{it} - \theta_{i}^{*}|$ (see (6)) for the parameter $\theta_{i}^{*}$ in terms of $B_{it}$. This part of the analysis is based on prior work on generalised linear models (Filippi et al., 2010).

We first define,

$$g_{it}(\theta) = \sum_{s=1}^{t-1} \frac{a_{is}}{w_{is}\sigma_{is}^{2}} \mu\left(\frac{a_{is}}{w_{is}} \theta \right).$$

By an application of the mean value theorem we have,

$$|\theta_{it} - \theta_{i}^{*}| \leq \frac{1}{\hat{g}_{it}(\theta_{i}^{*})}|g_{it}(\theta_{it}) - g_{it}(\theta_{i}^{*})| \leq \frac{1}{\kappa^{2}_{it} A_{it}^{2}} \hat{g}_{it}(\theta_{i}^{*}) |g_{it}(\theta_{it}) - g_{it}(\theta_{i}^{*})|$$

Here $\theta_{it}'$ is between $\theta_{it}$ and $\theta_{i}^{*}$. We have upper bounded its integral using Assumption 1 to obtain

$$\hat{g}_{it}(\theta_{it}') = \sum_{s=1}^{t-1} \frac{a_{is}^{2}}{w_{is}^{2}\sigma_{is}^{2}} \mu\left(\frac{a_{is}}{w_{is}} \theta_{it}' \right) \geq \kappa^{2}_{it} A_{it}^{2}.$$  

Next, let $\theta_{it}^{\text{PL}}$ be the unique $\theta \in \mathbb{R}$ satisfying $g_{it}(\theta) = \sum_{s=1}^{t-1} \frac{a_{is}}{w_{is}\sigma_{is}^{2}} X_{is}$. Readers familiar with the literature on generalised linear models will recognise $\theta_{it}^{\text{PL}}$ as the maximum quasi-likelihood estimator (Chen et al., 1999). We can therefore rewrite $\theta_{it}$ in (6) as $\theta_{it} = \arg\min \theta>\theta_{\text{min}} |g_{it}(\theta) - g_{it}(\theta_{it}^{\text{PL}})|$. We therefore have,

$$|\theta_{it} - \theta_{i}^{*}| \leq \frac{1}{\kappa^{2}_{it} A_{it}^{2}} |g_{it}(\theta_{it}) - g_{it}(\theta_{i}^{*})| \leq \frac{1}{\kappa^{2}_{it} A_{it}^{2}} (|g_{it}(\theta_{it}) - g_{it}(\theta_{it}^{\text{PL}})| + |g_{it}(\theta_{it}^{\text{PL}}) - g_{it}(\theta_{i}^{*})|)$$

$$\leq \frac{2}{\kappa^{2}_{it} A_{it}^{2}} |g_{it}(\theta_{it}^{\text{PL}}) - g_{it}(\theta_{i}^{*})| = \frac{2}{\kappa^{2}_{it} A_{it}^{2}} |B_{it}|.$$  

(31)

The first step uses the conclusion from the previous display. The third step uses the fact that $\theta_{it}$ minimises $|g_{it}(\theta) - g_{it}(\theta_{it}^{\text{PL}})|$; therefore, $|g_{it}(\theta_{it}) - g_{it}(\theta_{it}^{\text{PL}})| \leq |g_{it}(\theta_{i}^{*}) - g_{it}(\theta_{it}^{\text{PL}})|$. The last step uses the definition of $B_{it}$.

Finally, we will apply (30) with $\delta' \leftarrow 6\delta/(\pi^{2}nD_{t}^{2})$ which is less than $1/e$ if $n \geq 2$. By (31), a union bound, and observing $\sum_{t=2}^{T} 1 = \pi^{2}/6$, we have, with probability greater than $1 - \delta$,

$$\text{for all } i, t, \quad |\theta_{it} - \theta_{i}^{*}| \leq \frac{2}{\kappa^{2}_{it} A_{it}^{2}} \sqrt{4 + 2 \log(2) \log(1/\delta')} A_{it} \leq \frac{\beta_{i}}{A_{it}}.$$  

This establishes the result for the confidence intervals for $\theta_{i}^{*}$. The result for the confidence intervals for $\eta_{i}^{*}$ follows from the fact that $\eta_{i}^{*} = \mu^{-1}(\alpha)/\theta_{i}^{*}$ is a decreasing function of $\theta_{i}^{*}$. □
D.2 Proof of Theorem 5

Throughout this proof we will assume that for all $i, n, d_{it}^* \in (\tilde{d}_{it}, \hat{d}_{it})$. By Lemma 21, this is true with probability at least $1 - \delta$.

Efficiency: We will first decompose the loss as follows,

$$L_T \leq \sum_{t \in E} \ell_t + \sum_{t \notin E} \ell_t \leq q_T + \sum_{t \notin E} \ell_t \leq 3T^{2/3} + \sum_{i=1}^n (\hat{d}_{it} - \tilde{d}_{it}).$$

We have used the fact that in $T$ rounds there will have been $q_T$ exploration rounds and then used Lemma 17 with $a = 1$ to upper bound $q_T$. To bound the second sum, we have used Lemma 9. Here, $\tilde{d}_{it} = w_{it}\hat{\eta}_{it}$ and $\tilde{d}_{it} = w_{it}\hat{\eta}_{it}$ are upper and lower confidence bounds on the instantaneous demands, where $\hat{\eta}_{it}, \tilde{\eta}_{it}$ are the upper and lower confidence bounds on the unit demand as given in (8). We can bound this further via,

$$I_{it} - \tilde{I}_{it} = w_{it}(\hat{\eta}_{it} - \tilde{\eta}_{it}) = w_{it}\mu^{-1}(\alpha) \left( \frac{1}{\tilde{\theta}_{it}} - \frac{1}{\hat{\theta}_{it}} \right) = \frac{w_{it}\mu^{-1}(\alpha)}{\tilde{\theta}_{it}\hat{\theta}_{it}} \left( \hat{\theta}_{it} - \tilde{\theta}_{it} \right)$$

Here, the first three steps substitutes the expressions for $\tilde{\theta}_{it}, \hat{\theta}_{it}, \tilde{\eta}_{it}, \hat{\eta}_{it}$ from (6). The fourth step uses $\tilde{\theta}_{it} \geq \theta^*_i$, $\hat{\theta}_{it} \geq \theta_{\min}$, and that $\hat{\theta}_{it} - \tilde{\theta}_{it} = 2\beta_i A_{it}$. The fifth step uses $\eta^*_i = \mu^{-1}(\alpha_i) / \theta^*_i$ and an upper bound for $A_{it}$ that we will show below. The last step simply uses the definition of $C$ given in the theorem. To obtain the upper bound on $A_{it}$, we use Lemma 17 and the fact that at all exploration rounds, agent $i$ receives an allocation $e_i$. We have:

$$A_{it}^2 = \sum_{s \in D_i} \frac{a_{is}^2}{w_{is}^2 \sigma^2} \geq \frac{e_i^2}{w^2 \sigma^2} q_t \geq \frac{e_i^2}{2w^2 \sigma^2} t^{2/3}.$$

This leads us the following bound on $L_T$,

$$L_T \leq 3T^{2/3} + 2C\beta_T \sum_{i=1}^n \sum_{t \notin E} t^{-1/3} \leq 3T^{2/3} + 3Cn\beta_T T^{2/3}.$$

Fairness: This follows as a consequence of Lemma 11. Precisely,

$$U_{iT}^{e_i} - U_{iT} = \sum_{t \notin E} (u_i(e_i/w_{it}) - u_i(a_{it}/w_{it})) \leq 0.$$

In the first step, we use the fact that during the exploration phase rounds $t \in E$, each agent gets her entitlement $a_{it} = e_i$. During the other rounds, we invoke MMF using an upper bound on agent $i$’s demand. In Lemma 11 with set both $\tilde{\eta}_{it}$ and $\eta_{it}$ to this upper bound, and consequently, the RHS is 0.

Strategy-proofness: This follows from Lemma 14 and the following two observations: first, under the event specified in Lemma 21, $\tilde{\eta}_{it}$ in line (7) of Algorithm 2 is an upper bound on $\eta^*_i$; second, the allocations in the exploration phase do not depend on reports of any agents from previous rounds, therefore $a_{it}^* = a_{it}$.

□
D.3 Proof of Theorem 6

We will begin with three intermediate lemmas. The first will help us establish the result for asymptotic efficiency, the second will help us with asymptotic BNIC, and the third is a technical lemma that will help us in controlling both loss terms. In all three lemmas, we will let \( \mathcal{E} \) denote the event that for all \( i, n \) \( d_{it}^* \in (\hat{d}_{it}, \check{d}_{it}) \). By Lemma 21, \( \mathbb{P}(\mathcal{E}) \geq 1 - \delta \).

**Lemma 22** Let \( C_1 \) be as defined in (9). The following bound holds on \( L_T \) under event \( \mathcal{E} \).

\[
L_T \leq 1 + \frac{\beta_T}{C_1} \sum_{t=1}^{T} \sum_{i=1}^{n} \min \left( C_1, \frac{a_{it}}{w_{it} \sigma_{it} A_{it}} \right).
\]

**Proof.** We will divide this proof into three steps. In the first two steps, we will consider a single round \( t \geq 2 \). Recall the notation from the beginning of Appendix B.

**Step 1:** First, we will argue that \( \ell_t \leq \min(1, \ell_t^{or}) \). Observe that if \( \ell_t^{ur} = 0 \), then \( a_t = \hat{d}_{it} \geq d_{it}^* \) and therefore \( \ell_t \leq \ell_t^{or} = 0 \) and the claim is true. When \( \ell_t^{ur} = 0 \), then we can bound \( \ell_t \leq \ell_t^{ur} + \ell_t^{pr} = \ell_t^{or} \). The above claim follows by observing that \( \ell_t \leq 1 \) trivially since the total amount of the resource is 1.

**Step 2:** Next, consider any user \( i \). We will argue that \( \max(a_{it} - d_{it}^*, 0) \leq 2\beta_T \frac{\sigma_{it}}{\sigma_{min} A_{it}} \). Let \( \{a_{jt}^{*}\}_{j=1}^{n} \) denote the allocations returned by MMF if we were to invoke with the true demands \( \{d_{jt}^*\}_{j=1}^{n} \) for round \( t \). Recall that \( a_{it}^* \leq d_{it}^* \).

First assume \( a_{it}^* < d_{it}^* \). Then, by Property 4, increasing \( i \)'s reported demand to \( \hat{d}_{it}( > d_{it}^* ) \) while keeping the demands of all other agents at \( \{d_{jt}^*\}_{j \neq i} \) does not increase \( i \)'s allocation. Moreover, increasing the demands of all other agents cannot increase \( i \)'s allocation and hence \( a_{it} \leq a_{it}^* < d_{it}^* \); therefore, the statement is true since the RHS is positive.

Now assume \( a_{it}^* = d_{it}^* \). If \( a_{it} \leq a_{it}^* = d_{it}^* = \check{d}_{it} \), the statement is trivially true. Therefore, let \( a_{it} > a_{it}^* = d_{it}^* = \check{d}_{it} \). As MMF does not allocate more than the reported demand, we also have \( a_{it} < \hat{d}_{it} \). This results in the following bound.

\[
\max(a_{it} - d_{it}^*, 0) \leq \hat{d}_{it} - \check{d}_{it} \leq \frac{\mu^{-1}(\alpha) w_{it}}{\theta_{it} \theta_{it}} (\theta_{it} - \check{d}_{it}) \leq \frac{\sigma_{it}}{\sigma_{min} A_{it}} \leq \frac{a_{it}}{\theta_{min}} A_{it}.
\]

**Step 3:** We now combine the results of the previous two steps to bound \( L_T \) as follows.

\[
L_T = \sum_{t=1}^{T} \ell_t \leq 1 + \sum_{t=2}^{T} \ell_t \leq 1 + \sum_{t=2}^{T} \min(1, \ell_t^{or}) = 1 + \sum_{t=2}^{T} \min \left( \frac{n}{\ell_t}, \max(a_{it} - d_{it}^*, 0) \right) \leq 1 + \frac{\beta_T}{C_1} \sum_{t=1}^{T} \sum_{i=1}^{n} \min \left( C_1, \frac{a_{it}}{\sigma_{it} w_{it} A_{it}} \right)
\]

\[
\leq 1 + \frac{\beta_T}{C_1} \sum_{i=1}^{n} \sum_{t=2}^{T} \min \left( C_1, \frac{a_{it}}{\sigma_{it} w_{it} A_{it}} \right)
\]

(32)
Here, the third step applies the results from step 1. The fifth step simply uses \( \min(a, \sum_i b_i) \leq \sum_i \min(a, b_i) \) when \( b_i \geq 0 \) for all \( i \). The sixth step applies the result from step 2. The eighth step uses the fact that \( \beta_t \geq 1 \) for all possible choices of \( n, \delta, |D_t| \).

**Lemma 23** Consider any agent \( i \in \{1, \ldots, n\} \) and assume all other agents are submitting truthfully. Let \( U_{IT}^\pi, U_{IT} \) be as defined in Theorem 6. The following bound holds on \( U_{IT}^\pi - U_{IT} \) under event \( \mathcal{E} \).

\[
U_{IT}^\pi - U_{IT} \leq \frac{L_u \beta T}{C_1 w} \sum_{j \neq i} \sum_{t=2}^T \min \left( C_1, \frac{a_{jt}}{w_j \sigma_j A_{jt}} \right).
\]

**Proof.** We will divide this proof into three steps.

**Step 1:** We will decompose \( U_{IT}^\pi - U_{IT} \) as follows. Let \( u_{it}^\pi \) denote the utility at time \( t \) when agent \( i \) is following \( \pi \). Let \( u_{it}^\dagger \) denote the utility at time \( t \) when agent \( i \) is follows \( \pi \) until round \( t - 1 \) and then on round \( t \) we invoke MMF (line 7, Algorithm 3) with the true demand \( d_{it}^* = \eta_j w_{it} \) for agent \( i \), and the upper bounds \( \hat{d}_{jt} = \hat{\eta}_{jt} w_{jt} \) for all other agents \( j \). Let \( u_{it}^\star \) denote the utility of agent \( i \) when we invoke MMF with the true demands for all agents. We then have,

\[
U_{IT}^\pi - U_{IT} = \sum_{t=1}^n (u_{it}^\pi - u_{it}) = \sum_{t=1}^n (u_{it}^\pi - u_{it}^\dagger + u_{it}^\dagger - u_{it}^\star + u_{it}^\star - u_{it}) \leq \sum_{t=1}^n (u_{it}^\star - u_{it}) .
\]

The last step is obtained via two observations. First, \( u_{it}^\pi - u_{it}^\dagger \leq 0 \) since, by Properties 6 and 7, reporting true demands is a weakly dominant strategy for agent \( i \). Second, \( u_{it}^\dagger - u_{it}^\star \leq 0 \) since the reported demands used for all other agents \( j \neq i \) when invoking MMF, is an upper bound on their true demands under \( \mathcal{E} \) as they are being truthful. When comparing the reported demands in \( u_{it}^\dagger \) and \( u_{it}^\star \), we see that agent \( i \)'s reported demand stays the same but the reported demands of other agents decrease, which cannot decrease the allocation and consequently the utility of agent \( i \).

Next, using the Lipschitz properties of \( u_i \), we can write \( u_{it}^\star - u_{it} = u_i(a_{it}^*/w_{it}) - u_i(a_{it}/w_{it}) \leq L_u \max(a_{it}^*/w_{it} - a_{it}/w_{it}, 0) \leq L_u \max(a_{it}^* - a_{it}, 0) \). Moreover, since increasing all user’s demands can only increase the sum of allocations, we have \( \sum_{j=1}^n a_{jt}^* \leq \sum_{j=1}^n a_{jt} \). Therefore, \( a_{it}^* - a_{it} \leq \sum_{j \neq i}(a_{jt} - a_{jt}^*) \) and hence \( \max(a_{it}^* - a_{it}, 0) \leq \sum_{j \neq i} \max(a_{jt} - a_{jt}^*, 0) \). This leads us to the following bound.

\[
U_{IT}^\pi - U_{IT} \leq \frac{L_u}{w} \sum_{t=1}^T \max(a_{it}^* - a_{it}, 0) \leq \frac{L_u}{w} \sum_{t=1}^T \sum_{j \neq i} \max(a_{jt} - a_{jt}^*, 0).
\]

**Step 2:** Consider any user \( j \neq i \). Here, we will argue that \( \max(a_{jt} - a_{jt}^*, 0) \leq \frac{2\beta_t}{\theta_{\min}} \frac{a_{jt}}{A_{jt}} \), where, recall \( \{d_{kt}^n\}_{k=1}^n \) is the allocation returned by MMF if we use the true demands \( \{d_{kt}^n\}_{k=1}^n \) for all agents \( k \). First observe that if \( a_{jt} \leq a_{jt}^* \), the statement is trivially true as the RHS is positive.

Next, if \( a_{jt} > a_{jt}^* \), we argue that \( a_{jt}^* = d_{jt}^* \); we will prove this via its contrapositive. Observe that \( a_{jt}^* \leq d_{jt}^* \) by properties of MMF. Let us assume that \( a_{jt}^* < d_{jt}^* \). Then, by Property 4, increasing \( j \)'s reported demand to \( \hat{d}_{jt}(> d_{jt}^*) \) while keeping the reported demands of all other agents at \( \{d_{kt}^*\}_{k \neq j} \)}
does not increase her allocation. Next, increasing the demands of all other agents cannot increase \( j \)'s allocation and hence \( a_{jt} \leq a_{jt}^* \). This proves the above statement.

By substituting \( d_{jt}^* \) for \( a_{jt}^* \), we obtain the following bound.

\[
\max(a_{jt} - a_{jt}^*, 0) \leq \tilde{d}_{jt} - \tilde{d}_{jt} \leq \frac{w_i \mu^{-1}(\alpha)}{\theta_{jt}\theta_{jt}}(\tilde{\theta}_{jt} - \tilde{\theta}_{jt}) \leq \frac{2w_{jt}^2 a_{jt} t}{\theta_{min} A_{jt}} \leq \frac{2w_{jt}^2 a_{jt}}{\theta_{min} A_{jt}}.
\]

Above, we have used the facts \( a_{jt} \leq \tilde{d}_{jt}, a_{jt}^* \geq \tilde{d}_{jt} \), and the expressions for \( \tilde{d}_{jt}, \tilde{\theta}_{jt}, \tilde{\theta}_{jt} \).

**Step 3:** Combining the results of the two previous steps, we obtain the following bound,

\[
U_{iT} - U_{iT - 1} \leq \frac{L_i^u}{w_i} \sum_{t=1}^{n} \min_{j \neq i} \left( 1, \frac{a_{jt}}{\min_{j \neq i} C_{jt}} \right) \leq \frac{L_i^u}{C_i w_i} \sum_{j \neq i} \sum_{t=2}^{T} \min \left( C_1, \frac{a_{jt}}{w_j \sigma_{jt} A_{jt}} \right).
\]

Here, the first step observes that \( \max(a_{jt} - a_{jt}^*, 0) \leq 1 \) and the last step is obtained by repeating the calculations in (32).

**Lemma 24** Let \( c > 0 \). Then, \( \sum_{t=1}^{T} \min \left( c, \frac{a_{jt}^2}{w_j^2 \sigma_{jt}^2 A_{jt}^2} \right) \leq \frac{c}{\log(1 + c)} \log(C_3 T) \), where \( C_3 \) is as given in (9).

**Proof.** We first simplify \( A_{iT}^2 \) as follows.

\[
A_{iT}^2 = A_{i,T-1}^2 + \frac{a_{jt}^2}{w_i^2 \sigma_{jt}^2} = A_{i,T-1}^2 \left( 1 + \frac{a_{jt}^2}{A_{i,T-1}^2 \sigma_{jt}^2} \right) = \cdots = A_{i2}^2 \prod_{s=1}^{T-1} \left( 1 + \frac{a_{jt}^2}{A_{i2}^2 w_i^2 \sigma_{jt}^2} \right)^{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2}.
\]

Now, observing that \( A_{iT}^2 = \sum_{t=1}^{T-1} \frac{a_{jt}^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2} \leq \frac{T-1}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2} \) and that \( A_{i2}^2 = \frac{a_{jt}^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2} \geq \frac{c_i^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2} \), we obtain,

\[
\sum_{t=1}^{T} \log \left( 1 + \frac{a_{jt}^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2 A_{jt}^2} \right) = \log \left( \frac{A_{iT}^2}{A_{i2}^2} \right) \leq \log(C_3 T).
\]

By applying Lemma 20, we obtain

\[
\sum_{t=1}^{T} \min \left( c, \frac{a_{jt}^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2 A_{jt}^2} \right) \leq \frac{c}{\log(1 + c)} \sum_{t=1}^{T} \log \left( 1 + \frac{a_{jt}^2}{\sigma_{jt}^2 \sigma_{jt}^2 \sigma_{jt}^2 A_{jt}^2} \right) \leq \frac{c}{\log(1 + c)} \log(C_3 T).
\]

We are now ready to prove the theorem.

**Proof of Theorem 6.** In this proof, denote \( Q_i = \sum_{t=2}^{T} \min \left( C_1, \frac{a_{jt}^2}{w_j \sigma_{jt} A_{jt}} \right) \), and recall the definitions of \( C_1, C_2, C_3 \) from (9). By an application of the Cauchy-Schwarz inequality and Lemma 24 we obtain,

\[
Q_i^2 \leq (T - 1) \sum_{t=2}^{T} \min \left( C_1^2, \frac{a_{jt}^2}{w_j \sigma_{jt} A_{jt}} \right) \leq (T - 1) C_1^2 C_2^2 \log(C_3 T).
\]
**Efficiency:** The claim for asymptotic efficiency follows by applying Lemma 22.

\[
L_T \leq 1 + \frac{\beta_T}{C_1} \sum_{i=1}^{n} Q_i \leq 1 + C_2 n \beta_T \sqrt{(T - 1) \log(C_3 T)}.
\]

**Fairness:** Similar to the argument in Appendix D.2, this follows as a direct consequence of Lemma 11. On round 1, each agent gets her entitlement. On each subsequent round, we invoke MMF using an upper bound on the agent’s demand.

**Strategy-proofness:** The claim for asymptotic Bayes-Nash incentive compatibility follows by applying Lemma 23.

\[
U_{iT} - U_{iT} \leq \frac{L^u \beta_T}{C_1 w} \sum_{i=1}^{n} Q_i \leq \frac{L^u C_2}{w} (n - 1) \beta_T \sqrt{(T - 1) \log(C_3 T)}.
\]

□

**Remark 2** In our proofs, if \( \sigma_{it} = 0 \), many of the above quantities become undefined, resulting in a degeneracy. However, if at any instant \( \sigma_{it} = 0 \), it means we will have observed \( f_{\theta^*_i} \) at \( a_{it}/w_{it} \) exactly, and will know \( \theta^*_i \). Since, \( f_{\theta^*_i} \) is completely determined by \( \theta^*_i \), we can compute the agent’s true unit demand and use it from thereon. We also require \( w_{it} > 0 \) for the strategy-proof case for a similar reason as in Remark 1. This can be avoided by more careful book-keeping which postpones an agent’s exploration round if \( w_{it} > 0 \).

**Appendix E. Proofs of Results in Section 3.3**

In this section, we analyse the stochastic model with nonparametric payoffs. We will first define some quantities and notation that will be used in our proofs.

Let \( \ell_{hk} = (k - 1)2^{-h} \eta_{\max} \) and \( r_{hk} = k2^{-h} \eta_{\max} \) denote the left and right points of the interval \( I_{hk} \) (10). Then, for a user \( i \), let \( \Delta_i(h, k) \) be,

\[
\Delta_i(h, k) = \begin{cases} 
  f_i(\ell_{hk}) - \alpha_i, & \text{if } \ell_{hk} > \alpha_i, \\
  \alpha_i - f_i(r_{hk}), & \text{if } r_{hk} < \alpha_i, \\
  0, & \text{otherwise.}
\end{cases}
\]

(33)

For an interval \((h, k)\) with \( \Delta_i(h, k) > 0 \), we define

\[
u_{it}(h, k) = \frac{4 \beta_i^2}{(\Delta_i(h, k) - L 2^{-h})^2}.
\]

(34)

In our analysis, we will consider a desirable event \( \mathcal{E} \). We will first show that \( \mathbb{P}(\mathcal{E}) > 1 - \delta \) and then show that our bounds will hold when \( \mathcal{E} \) is true, thus proving our theorems. In order to define \( \mathcal{E} \), we
first define $E_{it}(h, k)$ below. Recall that $f_{it}, \tilde{f}_{it}$ are defined in \(14\).

$$E_{it}(h, k) = \begin{cases} f_{it}(h, k) > \alpha_i \land \tilde{f}_{it}(h, k) < \alpha_i, & \text{if } \eta^*_i \in I_{hk}, \\ \tilde{f}_{it}(h, k) > \alpha_i, & \text{if } I_{hk} \subset (\eta^*_i, \eta_{\text{max}}], f_i(hk) - \alpha_i > \frac{L}{2h}, \\ f_{it}(h, k) < \alpha_i, & \text{if } I_{hk} \subset [0, \eta^*_i), \alpha_i - f_i(hk) > \frac{L}{2h}, \\ \text{True} & \text{otherwise.} \end{cases}$$

We then define $E_{it}, E_i, E$ as shown below.

$$E_{it} = \bigcap_{h=0}^{\infty} \bigcap_{k=1}^{2^h} E_{it}(h, k), \quad E_i = \bigcap_{t=1}^{\infty} E_{it}, \quad E = \bigcap_{i=1}^{n} E_i. \quad (36)$$

Recall that $P_{it}$ is the path chosen by RECORD-FEEDBACK for user $i$ in round $t$. We will let $(H^it_i, K^it_i)$ denote the last node in $P_{it}$. It is worth observing that $P_{it}$ is also the path chosen by UB-TRAVERSE when we are in the exploration phase in Algorithm 2, or by GET-UD-REC when $\eta_{it} = \pi_{it}$ in Algorithm 3. Next, we define $N_{it}(h, k), N'_{it}(h, k)$ as follows.

$$N_{it}(h, k) = \sum_{s=1}^{t-1} \mathbb{1}((h, k) \in P_{ts}), \quad N'_{it}(h, k) = \sum_{s=1}^{t-1} \mathbb{1}((H_{it}, K_{it}) = (h, k)). \quad (37)$$

Here, $N_{it}(h, k)$ is the number of points assigned to $(h, k)$ while $N'_{it}(h, k)$ only counts the point if it was the last node in the path chosen by RECORD-FEEDBACK. The following relations should be straightforward to verify.

$$N'_{it}(h, k) \leq N_{it}(h, k), \quad \sum_{(h, k) \in T_{it}} N'_{it}(h, k) = t - 1,$$

$$\sigma^2 W_{it}(h, k) \leq N_{it}(h, k) \leq \bar{\pi}^2 W_{it}(h, k).$$

We will find it useful to define $g_{it}(h, k)$ as shown below, which is similar to $B_{it}$, but is defined using the $f_{it}, \tilde{f}_{it}$ quantities.

$$g_{it}(h, k) = \min (f_{it}(h, k) - \alpha_i, \alpha_i - \tilde{f}_{it}(h, k)) \quad (38)$$

Recall that the intervals $\{I_{hk}\}_{k=1}^{2^h}$ at each height partitions $[0, \eta_{\text{max}}]$. Hence, at each height $h$, there is a unique interval $k^*_h$ which contains the unique demand:

$$\text{for all } h, \quad \eta^*_i \in I_{hk^*_h}. \quad (39)$$

We will refer to this sequence $(0, 1), (1, k^*_1), (2, k^*_2), \ldots$ as the threshold nodes.
Observe that we use $\beta_t^{\dagger}$ in the expressions for the confidence intervals (14), where $t^{\dagger}$ is as given in (11). The following statements are straightforward to verify.

$$t \leq t^{\dagger} \leq 2t, \quad \beta_t^{\dagger} \leq \beta_{2t} \in O\left(\sqrt{\log(nt/\delta)}\right).$$ (40)

Finally, in this proof, when we say that a node $(h, k)$ is an ancestor of $(h', k')$, we mean that $(h, k)$ could be $(h', k')$, or its parent, or its parent’s parent, etc. We say that $(h', k')$ is a descendant of $(h, k)$ if $(h, k)$ is an ancestor of $(h', k')$.

We can now proceed to our analysis, which will be organised as follows. In Appendix E.1, we will bound $\mathbb{P}(E^c)$ for both Algorithms 2 and 3. In Appendices E.2 and E.3, we will establish some intermediate lemmas and definitions that will be used in both algorithms. Then, in Appendices E.4 and E.5, we will prove our main results for Algorithms 2 and 3 respectively.

**E.1 Bounding $\mathbb{P}(E^c)$**

In this section, we will prove the following result which applies to both, Algorithms 2 and 3.

**Lemma 25** Let $E$ be as defined in (36). Then $\mathbb{P}(E^c) \leq \delta$.

The first step of the proof of this lemma considers nodes $(h, k)$ which satisfy the first case in (35).

We will first show the following result.

**Lemma 26** Consider user $i$ and let $(h, k)$ be such that $\eta^*_i \in I_{hk}$. Then,

$$\forall t \geq 2, \quad \mathbb{P}\left(\hat{\eta}_i(h, k) < \alpha_i \lor \hat{\eta}_i(h, k) > \alpha_i\right) \leq 6\delta \frac{\beta_t}{\sqrt{2\pi t^3}}.$$

**Proof.** First observe that if $W_{it}(h, k) = 0$, the statement is true by the definition of $\hat{\eta}_i(h, k)$, as the event inside $\mathbb{P}()$ holds with probability 0. Therefore, assume $W_{it}(h, k) > 0$ going forward. We first observe:

$$\hat{\eta}_i(h, k) < \alpha_i \iff \hat{\eta}_i(h, k) + \frac{\beta_t}{\sqrt{W_{it}(h, k)}} + \frac{L}{2\pi} < \alpha_i,$$

$$\iff \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} X_{is} + \beta_t^{\dagger} \sqrt{W_{it}(h, k)} < \left(\alpha_i - \frac{L}{2\pi}\right) W_{it}(h, k),$$

$$\iff \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is})) + \beta_t^{\dagger} \sqrt{W_{it}(h, k)} <$$

$$< \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} \left(\alpha_i - \frac{L}{2\pi} - f_i(\bar{a}_{is})\right),$$

$$\iff \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is})) < -\beta_t^{\dagger} \sqrt{\sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} \left(\alpha_i - \frac{L}{2\pi} - f_i(\bar{a}_{is})\right)}.$$
Here, the last step simply uses condition (4) to conclude, \( \bar{a}_{i,s} \in I_{hk} \implies |\eta_i^* - \bar{a}_{i,s}| \leq \frac{\eta_{\max}}{2^t} \implies \alpha_i - f_i(\bar{a}_{i,s}) < \frac{L}{2^t}. \) By a similar argument, we can show,

\[
\tilde{f}_{it}(h, k) > \alpha_i \iff \sum_{s=1}^{t-1} \frac{\mathbb{1}(\langle h, k \rangle \in P_{it})}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{i,s})) > \beta_{t^*} \sqrt{\sum_{s=1}^{t-1} \frac{\mathbb{1}(\langle h, k \rangle \in P_{is})}{\sigma_{is}^2}}.
\]

We will now apply Lemma 19 with \( \gamma_s \leftarrow \mathbb{1}(\langle h, k \rangle \in P_{is}), \sigma_s \leftarrow \sigma_{is}, z_s \leftarrow (X_{is} - f_i(\bar{a}_{i,s})). \) We will let \( \mathcal{F}_s \) be the sigma-field generated by the data from all users up to round \( s. \) Accordingly, \( \{\gamma_s\} \) is predictable since the recommendations \( \{\eta_{js} \times w_{js}\}_{j=1}^n \) for all users are chosen based on their past data, the allocations are chosen based on these recommendations via MMF, and finally \( P_{is} \) depends on this allocation and \( i \)'s past data in the tree; i.e. \( \gamma_s \) is \( \mathcal{F}_{s-1} \)-measurable. Moreover, \( \sigma_s \) is predictable by our assumptions. Finally, \( \mathbb{E}[z_s|\mathcal{F}_{s-1}] = 0. \) By combining the two previous displays, we have,

\[
\mathbb{P}(\tilde{f}_{it}(h, k) < \alpha_i \lor \tilde{f}_{it}(h, k) > \alpha_i) \\
\leq \mathbb{P} \left( \left| \sum_{s=1}^{t-1} \frac{\mathbb{1}(\langle h, k \rangle \in P_{it})}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{i,s})) \right| > \beta_{t^*} \sqrt{\sum_{s=1}^{t-1} \frac{\mathbb{1}(\langle h, k \rangle \in P_{is})}{\sigma_{is}^2}} \right) \\
\leq \frac{6\delta}{n^2 \pi t^{2^3}} \leq \frac{6\delta}{n^2 \pi t^{2^3}}.
\]

The first step uses Lemma 19 and the last step follows from the observation \( t^* \geq t. \) \( \square \)

Next, we consider nodes \( (h, k) \) which satisfy the second case in (35), for which we have the following result.

**Lemma 27** Consider user \( i \) and let \( (h, k) \) be such that \( I_{hk} \subset (\eta_i^*, \eta_{\max}) \) and \( \Delta_i(h, k) = f_i(\ell_{hk}) - \alpha_i > L/2^h. \) Let \( u_{it}(h, k) \) be as defined in (34). Then,

\[
\forall t \geq 2, \quad \mathbb{P}(\tilde{f}_{it}(h, k) > \alpha_i \land W_{it}(h, k) \geq u_{it}(h, k)) \leq \frac{6\delta}{n^2 \pi t^{2^3}}.
\]

**Proof.** Consider any \( t. \) First, the condition on \( W_{it}(h, k) \) implies the following.

\[
W_{it}(h, k) > \frac{4\beta_t^2}{(\Delta_i(h, k) - L2^{-h})^2} \implies f_i(\ell_{hk}) - \alpha_i - \frac{L}{2^h} > \frac{2\beta_t}{\sqrt{W_{it}(h, k)}} \\
\implies \forall a \in I_{hk}, \alpha_i - \frac{L}{2^h} - f_i(a) < -\frac{2\beta_t}{\sqrt{W_{it}(h, k)}}. \tag{41}
\]

We now observe:

\[
W_{it}(h, k) \geq u_{it}(h, k) \land \tilde{f}_{it}(h, k) < \alpha_i \\
\iff W_{it}(h, k) \geq u_{it}(h, k) \land \tilde{f}_{it}(h, k) - \frac{\beta_{t^*}}{\sqrt{W_{it}(h, k)}} < L/2^h + \alpha_i \\
\iff W_{it}(h, k) \geq u_{it}(h, k) \land \sum_{s=1}^{t-1} \frac{\mathbb{1}(\langle h, k \rangle \in P_{it})}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{i,s})) - \beta_{t^*} \sqrt{W_{it}(h, k)}
\]

52
We first note that in

We are now ready to prove Lemma 25.

\[ \frac{1}{\sigma_{is}^2} \sum_{s=1}^{t-1} \mathbb{1}((h, k) \in P_{it}) \left( \alpha_i + \frac{L}{2h} - f_i(\bar{a}_{is}) \right) , \]

\[ \Rightarrow \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is})) < -\beta_{it} \sqrt{\sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is}))} . \]

\[ \Rightarrow \left| \sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is})) \right| > \beta_{it} \sqrt{\sum_{s=1}^{t-1} \frac{1}{\sigma_{is}^2} (X_{is} - f_i(\bar{a}_{is}))} . \]

In the third step, we have used (41) along with fact that when \((h, k) \in P_{it}\), then \(a_{it} \in I_{hk}\) as in each step in the while loop of RECORD-FEEDBACK, we choose the child which contains \(a_{it}\). In the last step we have considered the absolute value of the LHS and used the fact that \(t^\dagger \geq t\). The claim follows by applying Lemma 19 with the same \(\gamma_s, \sigma_s, \epsilon_s\), and \(\mathcal{F}_s\) as we did in the proof of Lemma 26. \(\square\)

Next, we consider nodes \((h, k)\) which satisfy the third case in (35). The proof of the following lemma follows along similar lines to that of Lemma 27.

**Lemma 28** Consider user \(i\) and let \((h, k)\) be such that \(I_{hk} \subset [0, \eta^*_h)\) and \(\Delta_i(h, k) = \alpha_i - f_i(r_{hk}) > L/2^h\). Let \(u_{it}(h, k)\) be as defined in (34). Then,

\[ \forall t \geq 2, \quad \mathbb{P} \left( f_{it}(h, k) < \alpha_i \land W_{it}(h, k) \geq u_{it}(h, k) \right) \leq \frac{6\delta}{n\pi^2t^3} . \]

We are now ready to prove Lemma 25.

**Proof of Lemma 25.** Recall the definitions in (35) and (36). By the union bound, we first write,

\[ \mathbb{P}(\mathcal{E}_i^\epsilon) \leq \sum_{t=1}^{\infty} \sum_{(h,k)} \mathbb{P}(\mathcal{E}_{it}^\epsilon(h, k)) . \]

We first note that in \(t\) rounds, at most \(t\) nodes will have been expanded. For any node that has not been expanded, \(\mathbb{P}(\mathcal{E}_{it}^\epsilon(h, k)) = 0\); this is because, for the first case in (35), we have \(\tilde{f}_{it}(h, k) = \infty\) and \(\tilde{f}_{it}(h, k) = -\infty\) by definition (14); moreover, \(W_{it}(h, k) = 0\) for unexpanded nodes, and therefore the second and third cases do not occur. Therefore, there are only \(t\) non-zero terms in the inner summation above.

Next, note that any node \((h, k)\) for which \(\mathbb{P}(\mathcal{E}_{it}^\epsilon(h, k))\) is non-zero, satisfies \(\mathbb{P}(\mathcal{E}_{it}^\epsilon(h, k)) \leq 6\delta/(n\pi^2t^3)\) by Lemmas 26, 27, and 28. Therefore,

\[ \mathbb{P}(\mathcal{E}_i^\epsilon) \leq \sum_{t=1}^{\infty} \sum_{(h,k) \in T_{it}} \frac{6\delta}{n\pi^2t^3} \leq \sum_{t=1}^{\infty} \frac{6\delta}{n\pi^2t^2} \leq \frac{\delta}{n} . \]

The last step above uses the identity \(\sum t^{-2} = \pi^2/6\). The claim follows by a final application of the union bound over the \(n\) users \(\mathbb{P}(\mathcal{E}_i^\epsilon) \leq \sum_{i=1}^{n} \mathbb{P}(\mathcal{E}_i^\epsilon) \leq \delta\). \(\square\)
E.2 Some Intermediate Results

In this section, we will prove some technical lemmas that will be used in the proofs of both Theorems 7 and 8. The first shows that both $\tilde{B}_{it}(h, k), \widehat{B}_{it}(h, k)$ are non-decreasing with $k$ for a given $h$.

**Lemma 29** Let $h \geq 0$ and $k_1, k_1 \in \mathbb{N}$ such that $0 \leq k_1 < k_2 \leq 2^h$. Then, the following hold

\[
\tilde{B}_{it}(h, k_1) \leq \tilde{B}_{it}(h, k_2), \quad \widehat{B}_{it}(h, k_1) \leq \widehat{B}_{it}(h, k_2).
\]

**Proof.** We will prove the first result. The proof of the second result follows analogously. Recall that we update the bounds at two different places in Algorithms 7–10. First, the REFRESH-BOUNDS-IN-TREE method (line 91) recomputes the lower confidence bounds $\tilde{f}_{it}(h, k)$ and $\tilde{B}_{it}(h, k)$ for all expanded nodes $(h, k) \in T_{it}$. This is invoked by GET-UD-REC (line 14) when $t = t^i$, i.e. at the beginning of rounds 1, 2, 4, 8, etc. Second, the UPDATE-BOUNDS-ON-PATH-TO-ROOT (line 62) method, which is invoked when we add a new data point in RECORD-FEEDBACK (line 24), recomputes $\tilde{f}_{it}$ for the nodes the data point was assigned to, and updates $\tilde{B}_{it}$ for nodes whose values may have been affected. We will show that in the first case, the refresh operation ensures that $\tilde{B}_{it}$ is non-decreasing, and in the second case, the updates preserve monotonicity.

First, consider the REFRESH-BOUNDS-IN-TREE method. The $\tilde{b}_{\text{max}}$ variable keeps track of the maximum $\tilde{B}_{it}(h, k)$ value as we update the expanded nodes $(h, k)$ in increasing order of $k$ at height $h$. When we reach a node $(h, k')$, we ensure $\tilde{B}_{it}(h, k') \geq \tilde{b}_{\text{max}}$ which ensures monotonicity.

Second, consider the UPDATE-BOUNDS-ON-PATH-TO-ROOT method. Assume that node $k$ at height $h$ is updated at round $t$, and that $\tilde{B}_{i,t-1}$ is monotonic at all heights $h$ at round $t - 1$. Observe that the update ensures that $\tilde{B}_{it}(h, k) \geq \tilde{B}_{i,t-1}(h, k)$ and that the values $\tilde{B}_{i,k-1}(h, k')$ for $k' < k$ do not change from round $t - 1$ to round $t$. Therefore, since monotonicity is preserved at round $t - 1$, we have $\tilde{B}_{it}(h, k') \leq \tilde{B}_{it}(h, k)$ for all $k' \leq k$. Moreover, the UPDATE-BOUNDS-FOR-NODES-AT-SAME-DEPTH method updates $\tilde{B}_{it}(h, k') \leftarrow \max(\tilde{B}_{it}(h, k'), \tilde{B}_{it}(h, k))$ for $k' > k$. Both these updates ensure monotonicity at the end of round $t$. □

Our second technical result in this section expresses $B_{it}(h, k)$ as a function of $g_{it}(h, k)$ and the $B_{it}$ values of its children.

**Lemma 30** Let $g_{it}$ be as defined in (38) and $B_{it}$ be as defined in (17). Then, for $(h, k) \in T_{it}$,

\[
B_{it}(h, k) = \min \left( g_{it}(h, k), B_{i,t-1}(h, k), \max (B_{it}(h+1, 2k-1), B_{it}(h+1, 2k)) \right).
\]

**Proof.** First, recall the expressions for $\tilde{B}_{it}, \widehat{B}_{it}$ in (15), (16), when $(h, k) \in T_{it}$:

\[
\begin{align*}
\tilde{B}_{it}(h, k) &= \max (\tilde{f}_{it}(h, k), \tilde{B}_{i,t-1}(h, k), \tilde{B}_{it}(h+1, 2k-1)), \\
\widehat{B}_{it}(h, k) &= \min (\tilde{f}_{it}(h, k), \widehat{B}_{i,t-1}(h, k), \widehat{B}_{it}(h+1, 2k)).
\end{align*}
\]  \hspace{1cm} (42)

We now expand $B_{it}(h, k)$ as follows.

\[
B_{it}(h, k) = \min \left( \tilde{B}_{it}(h, k) - \alpha_i, \alpha_i - \tilde{B}_{it}(h, k) \right)
\]
We will first prove that $\alpha_i - \max(\tilde{f}_{it}(h,k), \tilde{B}_{it,t-1}(h,k), \tilde{B}_{it}(h+1,2k)) < \alpha_i$.

Consider any user $f_i$ such that $h > h_i$ be the deepest expanded threshold node at round $t$. Let $t$ be given. We will show, via induction, that $	ilde{B}_{it}(h,k) > \alpha_i$ for all threshold nodes $(h,k)$, under $E$, the lower and upper bounds $\tilde{B}_{it}, \tilde{B}'_{it}$ trap the threshold value $\alpha_i$.

**Lemma 31** Consider any user $i$ and let $h \geq 0$. Let $k_{h,i}^j$ be as defined above. Under $E$, for all $t \geq 1$ and $h \geq 0$, we have

$$\tilde{B}_{it}(h,k_{h,i}^j) < \alpha_i, \quad \tilde{B}'_{it}(h,k_{h,i}^j) > \alpha_i, \quad B_{it}(h,k_{h,i}^j) > 0,$$

**Proof.** We will first prove that $\tilde{B}_{it}(h,k_{h,i}^j) < \alpha_i$. Define,

$$\tilde{B}'_{it}(h,k) = \max\left(\tilde{f}_{it}(h,k), \min\left(\tilde{B}_{it}(h+1,2k-1), \tilde{B}_{it}(h+1,2k)\right)\right).$$

Let $t \geq 1$ be given. We will show, via induction, that $\tilde{B}'_{it}(h,k_{h,i}^j) < \alpha_i$ for all $h \geq 0$. Let $(h_t,k_{h_t}^j)$ be the deepest expanded threshold node at round $t$. As the base case, we have that $\tilde{B}'_{it}(h,k_{h,i}^j) < \alpha_i$ for all $h > h_t$ since $\tilde{B}'_{it}(\ell,m) = 0 < \alpha_i$ for any unexpanded node $(\ell,m)$. Now, assume that $\tilde{B}'_{it}(h+1,k_{h+1,i}^j) < \alpha_i$ for some $h$. We therefore have,

$$\tilde{B}'_{it}(h,k_{h,i}^j) \geq \max\left(\tilde{f}_{it}(h,k_{h,i}^j), \tilde{B}'_{it}(h+1,k_{h+1,i}^j)\right) < \alpha_i. \quad (43)$$

Here, the first step simply uses the definition for $\tilde{B}'_{it}(h,k)$ from the previous display, observing that $(h+1,k_{h+1,i}^j)$ is either $(h+1,2k_{h,i}^j-1)$ or $(h+1,2k_{h,i}^j)$. In the second step, we have used $\tilde{f}_{it}(h,k_{h,i}^j) < \alpha_i$ under $E$ (36), and that $\tilde{B}'_{it}(h+1,k_{h+1,i}^j) < \alpha_i$ by the inductive assumption.
Now, observe that,
\[
\bar{B}_{it}(h, k^i_h) = \max \left( \bar{f}_{it}(h, k^i_h), \bar{B}_{i,t-1}(h, k^i_h), \bar{B}_{i,t}(h + 1, 2k^i_h - 1) \right) \\
= \max \left( \bar{f}_{it}(h, k^i_h), \bar{B}_{i,t-1}(h, k^i_h), \min(\bar{B}_{i,t}(h + 1, 2k^i_h - 1), \bar{B}_{i,t}(h + 1, 2k^i_h)) \right) \\
= \max \left( \bar{B}'_{it}(h, k^i_h), \bar{B}_{i,t-1}(h, k^i_h) \right)
\]

Here, the first step is simply the definition for $\bar{B}_{it}$ (15), and the second step uses monotonicity of $\bar{B}_{it}(h, \cdot)$ (Lemma 29). We can now prove the claim via induction over the rounds $t$. As the base case, $\bar{B}_{i1}(h, k) = 0 < \alpha_i$ for all nodes $(h, k)$ at round $t = 1$; therefore, it is also true for all $(h, k^i_h)$. Now, assume $\bar{B}_{i,t-1}(h, k^i_h) < \alpha_i$ as the inductive hypothesis. We then have, by (43),
\[
\bar{B}_{it}(h, k^i_h) = \max(\bar{B}'_{it}(h, k^i_h), \bar{B}_{i,t-1}(h, k^i_h)) < \alpha_i.
\]

The proof of the second result follows along similar lines and the third result follows from the first two, as $B_{it}(h, k) = \min(\bar{B}_{it}(h, k) - \alpha_i, \alpha_i - \bar{B}_{it}(h, k)) > 0$. \(\square\)

E.3 Some Intermediate Definitions

In this section, we will define a few constructions that we will use in both proofs.

**Definitions** $h_G, k^i_1, k^i_2, \eta^i_1, \eta^i_2, r^i_1, r^i_2$: Let $G_0$ be as given in Assumption 2. Let $G \in (0, G_0]$ be given, and $\epsilon_G$ be as defined in Definition 1. We define $h_G = \min\{h; \eta_{\text{max}}2^{-h_G} \leq G\epsilon_G/(4L)\}$. It is straightforward to verify,
\[
\frac{4L\eta_{\text{max}}}{G\epsilon_G} \leq 2^{h_G} < \frac{8L\eta_{\text{max}}}{G\epsilon_G}.
\]

Next, for user $i$, we will consider two nodes $(h^i_G, k^i_1), (h^i_G, k^i_2)$ at height $h^i_G$ of the tree such that the following hold. Let $I_{h^i_G,k^i_1} = [\ell^i_1, r^i_1)$, $I_{h^i_G,k^i_2} = [\ell^i_2, r^i_2)$ be the corresponding intervals. We have:
\[
\eta^i_1 - \epsilon_G < r^i_1 < \eta^i_2 - \epsilon_G/2, \quad \eta^i_1 + \epsilon_G/2 < \ell^i_1 < \eta^i_2 + \epsilon_G \leq r^i_2.
\]

We can find such $k^i_1, k^i_2$ by our definition of $h^i_G$ (44).

**Definitions** $I_h, J_h$: Let $I_h$ be the nodes $(h, k)$ at height $h$ which satisfy the following conditions:
\[
\Delta_i(h, k) \leq \frac{L}{2^h}, \quad \exists a \in I_h, \quad a > \eta^*_i, \quad \forall a \in I_h, \quad a < \ell^*_i.
\]

We will let $J_h$ be the nodes $(h, k)$ at height $h$ such that, $(h, k) \notin I_h, \quad I_{hk} \cap (\eta^*_i, \ell^*_i) \neq \emptyset$, and whose parent is in $I_{h-1}$. Next, we will bound the sizes of $I_h$ and $J_h$. For any $a \in I_{hk}$ where $(h, k) \in I_h$,
\[
a - \eta^*_i \leq (a - \ell_{hk}) + (\ell_{hk} - \eta^*_i) \leq \frac{\eta_{\text{max}}}{2^h} + \frac{\eta_{\text{max}}}{G}(f_i(\ell_{hk}) - \alpha_i) \leq \frac{\eta_{\text{max}}}{2^h} + \frac{2L\eta_{\text{max}}}{G2^h} \Delta = \text{width}_h.
\]

Here, we have used the NTG condition and that the maximum width of any $I_{hk}$ is $\eta_{\text{max}}2^{-h}$. The size of $I_h$ is bounded by the number of intervals of size $\eta_{\text{max}}2^{-h}$ in an interval of size $\text{width}_h$ and the
leftmost interval which contains \( \eta^*_i \). Using the fact that \( L \geq G \) yields the following bound in (46) for \(|I_h|\). Moreover, since the parent of \( J_h \) is in \( I_{h-1} \), we can also bound \( J_h \). We have:

\[
|I_h| \leq 1 + \frac{\text{width}_h}{\eta_{\max} 2^{-h}} \leq 2 + \frac{2L}{G} \leq \frac{4L}{G}, \quad |J_h| \leq 2|I_{h-1}| \leq \frac{8L}{G}.
\] (46)

Next, for any \( a \in I_{hk} \), where \((h, k) \in I_h \), we can bound \( f_i(a) - \alpha_i \),

\[
f_i(a) - \alpha_i = f_i(a) - f_i(\ell_{hk}) + f_i(\ell_{hk}) - \alpha_i \leq 2L2^{-h} + \Delta_i(h, k) \leq 4L2^{-h}.
\] (47)

Here, we have used (4) to conclude \(|f_i(a) - f_i(\ell_{hk})| \leq 2L2^{-h}\). Similarly, since the parents of nodes in \( J_h \) are in \( I_{h-1} \), we have for all \( a \in I_{hk} \), where \((h, k) \in J_h \), \( f_i(a) - \alpha_i \leq 8L2^{-h}\).

**Definitions** \( I'_{h}, J'_{h} \): Similar to above, we let \( I'_{h} \) be the nodes \((h, k)\) at height \( h \) which satisfy the following three conditions,

\[
\Delta_i(h, k) \leq \frac{2L}{2^n}, \quad \exists a \in I_{hk} \quad a < \eta^*_i, \quad \forall a \in I_{hk} \quad a > r^i_{\downarrow}.
\]

We will let \( J'_{h} \) be the nodes \((h, k)\) at height \( h \) such that \((h, k) \notin I_h, I_{hk} \cap (r^i_{\uparrow}, \eta^*_i) \neq \emptyset\), and whose parent is in \( I'_{h-1} \). By following the same argument to (46), we can show \(|I'_{h}| \leq \frac{8L}{2^n}\), and \(|J'_{h}| \leq 2|I'_{h-1}| \leq \frac{8L}{2^n}\). Moreover, by following a similar argument to (47), we can show that for any \( a \in I_{hk} \), where \((h, k) \in I'_{h}, \alpha_i - f_i(a) \leq 4L2^{-h}\). Similarly, since the parents of nodes in \( J'_{h} \) are in \( I'_{h-1} \), we have for all \( a \in I_{hk} \), where \((h, k) \in J'_{h}, \alpha_i - f_i \leq 8L2^{-h}\).

**E.4 Proof of Theorem 7**

In this section, we will prove Theorem 7. Recall that in Algorithm 2, we collect feedback only during the exploration phases, and moreover, that each user receives just one non-zero allocation during each exploration phase. We then use the value returned by GET-UB-UB (line 12, Algorithm 7–10) as the reported demand for the latter phase in each bracket. In the remainder of this section, we will denote this value in the \( q^a \) bracket for user \( i \) by \( \hat{\eta}_{iq} \). Additionally, we will let \( t^i_q \) denote the round index in the exploration phase of the \( q^a \) bracket in which user \( i \) received a non-zero allocation.

Our first result shows that \( \hat{\eta}_{iq} \) is an upper bound on \( \eta^*_i \) under \( \mathcal{E} \).

**Lemma 32** Consider any user \( i \) and bracket \( q > 0 \). Let \( \hat{\eta}_{iq} \) denote the point returned by GET-UB-UB. Under \( \mathcal{E} \), we have \( \hat{\eta}_{iq} \geq \eta^*_i \).

**Proof.** Recall that GET-UB-UB invokes UB-TRAVERSE to obtain a node \((h, k)\), and then returns \( \eta_{\max} k/2^h \). If \((h, k)\) is a threshold node, i.e. \( \eta^*_i \in I_{hk} \), then the statement is trivially true as \( \eta_{\max} k/2^h \) is the right-most point of \( I_{hk} \). (It is worth observing that \( \eta_{\max} k/2^h \notin I_{hk} \), unless \( k = 2^h \), see (10).)

If \( \eta^*_i \notin I_{hk} \), we will show that at some node threshold node \((\ell, k^\ell_i)\), UB-TRAVERSE chose the right child \((\ell + 1, 2k^\ell_i)\) instead of the left child \((\ell + 1, 2k^\ell_i - 1)\), and moreover that the left child was the threshold node at height \( \ell + 1 \), i.e. \((\ell + 1, k^\ell_{\ell + 1}) = (\ell + 1, 2k^\ell_i - 1)\). Therefore, GET-UB-UB returns a point to the right of \( \eta^*_i \), hence proving the lemma.

To show the above claim, observe that \((0, 1)\) is a threshold node and \((h, k)\) is not. We will let \((\ell, k^\ell_i)\) be the last threshold node in the path chosen by UB-TRAVERSE. Next, assume, by way
of contradiction, that the right child was the threshold node. Under \( \mathcal{E} \), by Lemma 31, we have \( \tilde{B}_{i,t_q}^{(\ell + 1, 2k_i^\ell)} < \alpha_i \), which means that in line 38 we will have chosen the right node. This is a contradiction since \((\ell, k_i^\ell)\) was the last threshold node in the path. Therefore, the left child was the threshold node. Finally, since \((\ell, k_i^\ell)\) was the last threshold node, it means that in line 38, we chose the right child.

\[ \Box \]

The next step in proving Theorem 7 is to prove the following lemma.

**Lemma 33** Consider any user \( i \) and let \( Q > 0 \). Assume \( f_i \) satisfies Assumption 2. Let \( G \in (0, G_0] \) and let \( \epsilon_G \) be as given in Definition 1. Let \( t_{i_q}^i \) denote the round index during which user \( i \) receives a non-zero allocation in the exploration phase. Then, under \( \mathcal{E} \),

\[
\tilde{L}_{Q_i}^i \triangleq \sum_{q=1}^{Q} \hat{\eta}_{iq} - \eta_i^*
\]

\[
\leq C' \frac{L^{1/2} \sigma_{\max}^3}{G^{3/2}} \beta_{2t_q^i}^2 Q^{1/2} + 586 \frac{L \sigma_{\max}^2 \hat{\eta}_{iq}^3}{G^3 \epsilon_G^3} \beta_2^{2t_q^i} + \frac{64 \eta_{\max}^2 \sigma_i^2}{G^2 \epsilon_G^2} \beta_2^{2t_q^i} + 160 \frac{L^2 \eta_{\max}}{G^2} \beta_2^{2t_q^i} + 16 \frac{L \eta_{\max}}{G \epsilon_G} + 1.
\]

Here, \( C' \) is a global constant.

Since we collect feedback for each user only once per bracket, \( \hat{\eta}_{iq} \) is an upper confidence bound constructed using \( q \) observations. Therefore, the LHS of Lemma 33 can be interpreted as the loss term for the following online learning task that occurs over \( Q \) rounds: on each round, a learner may evaluate any point on the interval \([0, \eta_{\max}]\); at the end of each round, the learner needs to output an upper confidence bound for \( \eta_i^* \); her loss at round \( q \) is the difference between this upper bound and the true unit demand \( \eta_i^* \).

With the above interpretation, we will find it convenient to express some of the quantities we have seen before differently. First consider, \( N_{i,t_q^i+1}(h,k) \) which is the number of times \((h,k)\) was in the path chosen by \textsc{Record-feedback} in the first \( t_{i_q}^i \) rounds (37). By observing that the user will have received allocations in the rounds \( \{ t_{i_s}^i \}_{s=1}^q \) in the first \( q \) brackets, we can write

\[
N_{i,t_q^i+1}(h,k) = \sum_{s=1}^{q} \mathbb{1} ((h,k) \in P_{i_s}) = \sum_{s=1}^{q} \mathbb{1} ((h,k) \in P_{i_{t_q^i}}).
\]

Similarly, we have

\[
N_{i,t_q^i+1}(h,k) = \sum_{s=1}^{q} \mathbb{1} \left( (H_{i_s}^i, K_{i_s}^i) = (h,k) \right), \quad W_{i,t_q^i+1}(h,k) = \sum_{s=1}^{q} \frac{1}{\sigma_{i_s}^2} \mathbb{1} \left( (h,k) \in P_{i_{t_q^i}} \right).
\]

Additionally, since there is no resource contention when a user is allocated during the exploration phase, we may obtain feedback for any allocation we wish. Therefore, the path \( P_{i_{t_q^i}} \) chosen by \textsc{Record-feedback} will be the same as the path chosen by \textsc{UB-traverse}. The following lemma bounds \( N_{i,t_q^i+1}(h,k) \) for nodes \((h,k)\) that do not contain \( \alpha_i \).
Lemma 34 Consider user $i$ and let $(h, k)$ be such that $I_{hk} \subset (n_{i}^{+}, n_{\text{max}})$ and $\Delta_{i}(h, k) = f_{i}(\ell_{hk}) - \alpha_{i} > L/2^{h}$. Under $E$, for all $t \geq 1$,

$$N_{i,t_{h}^{+}+1}(h, k) \leq \sigma^{2} \max \left( \tau_{t_{h}^{+}}(h, k), u_{it_{h}^{+}}(h, k) \right) + 1 = \sigma^{2} \max \left( \frac{\beta_{t_{h}^{+}}^{2} 4^{h}}{L^{2}}, \frac{4 \beta_{t_{h}^{+}}^{2} (\Delta_{i}(h, k) - L 2^{-h})^{2}}{} \right) + 1.$$  

Proof. The statement is clearly true for unexpanded nodes, so we will show this for $(h, k) \in T_{it_{h}^{+}}$. First, we will decompose $N_{i,t_{h}^{+}+1}(h, k)$ as follows.

$$N_{i,t_{h}^{+}+1}(h, k) = \sum_{s=1}^{q} \mathbb{1} \left( (h, k) \in P_{it_{h}^{+}} \land N_{it_{h}^{+}}(h, k) \leq \sigma^{2} \tau_{t_{h}^{+}} \right) + \sum_{s=1}^{q} \mathbb{1} \left( (h, k) \in P_{it_{h}^{+}} \land N_{it_{h}^{+}}(h, k) > \sigma^{2} \tau_{t_{h}^{+}} \right) \leq [\sigma^{2} \tau_{t_{h}^{+}}] + \sum_{s=\lceil \sigma^{2} \tau_{t_{h}^{+}} \rceil}^{q} \mathbb{1} \left( (h, k) \in P_{it_{h}^{+}} \land N_{it_{h}^{+}}(h, k) > \sigma^{2} \tau_{t_{h}^{+}} \right)$$ (48)

In the second step, we have bound the first summation by observing that $N_{it_{h}^{+}}(h, k)$ values are constant from bracket $s$ to $s+1$ unless $(h, k) \in P_{it_{h}^{+}}$ in which case they increase by one. Therefore, at most $[\sigma^{2} \tau_{t_{h}^{+}}]$ such terms can be non-zero. For the second sum, we have used the fact that $N_{it_{h}^{+}}(h, k)$ cannot be larger than $\sigma^{2} \tau_{t_{h}^{+}}$ in the first $[\sigma^{2} \tau_{t_{h}^{+}}]$ rounds.

Now, consider bracket $s$. We will show, by way of contradiction, that the following cannot hold simultaneously.

$$N_{it_{h}^{+}}(h, k) > \sigma^{2} \max(u_{it_{h}^{+}}(h, k), \tau_{t_{h}^{+}}), \quad (h, k) \in P_{it_{h}^{+}}.$$  

Recall the definition of the threshold nodes $\{(h', k_{h'})\}_{h' \geq 0}$ from the beginning of Appendix E. Observe that $(0, 1), (h, k) \in P_{it_{h}^{+}}$, but $(0, 1)$ is a threshold node, while $(h, k)$ is not. Let $(\ell, k_{\ell}^{+})$ be the last threshold node in $P_{it_{h}^{+}}$ with children $(\ell + 1, k_{\ell+1}^{+})$ and $(\ell + 1, k_{\ell+1}^{+} + 1)$; here, the left child is the threshold node for the same reason outlined in the proof of Lemma 32, while the right child is an ancestor of $(h, k)$. Since $N_{it_{h}^{+}}(h, k) > \sigma^{2} \tau_{t_{h}^{+}}$, we have $W_{i,t_{h}^{+}}(h, k) > \tau_{t_{h}^{+}}$, and therefore, $W_{i,t_{h}^{+}}(h''', k''') > \tau_{t_{h}^{+}}$ for all of $(h, k)$’s ancestors $(h''', k''')$. This observation, along with the fact that $(h, k)$ is not a threshold node implies that UB-TRAVERSE will have reached node $(\ell, k_{\ell}^{+})$ and then proceeded one more step to choose the right child $(\ell + 1, k_{\ell+1}^{+} + 1)$. Therefore, $\alpha_{i} \geq \overline{B}_{it_{h}^{+}}(\ell + 1, k_{\ell+1}^{+} + 1)$, since the if condition (line 38, Algorithm 7–10) in UB-TRAVERSE chose the right node. However, $N_{it_{h}^{+}}(h, k) > \sigma^{2} u_{it_{h}^{+}}(h, k)$ and the fact that $(h, k)$ is a descendant of $(\ell + 1, k_{\ell+1}^{+} + 1)$ implies that $W_{i,t_{h}^{+}}(\ell + 1, k_{\ell+1}^{+} + 1) > W_{i,t_{h}^{+}}(h, k) \geq \tau_{t_{h}^{+}} > \tau_{t_{h}^{+}}$, and therefore, by the definition of the event $E$ (36), we have $\overline{B}_{it_{h}^{+}}(\ell + 1, k_{\ell+1}^{+} + 1) > \alpha_{i}$. This is a contradiction.

To complete the proof, we will relax (48) further to obtain,
We will bound

\[ \hat{\sigma}^2 \tau_{ht}^2 \wedge N_{it_q^i}(h, k) \leq \sigma^2 u_{it_q^i}(h, k) \]

\[ \leq \left[ \hat{\sigma}^2 \tau_{ht}^2 \right] + \sum_{s=\left[ \sigma^2 \tau_{ht} \right]}^t \mathbb{1} \left( (h, k) \in P_{it_q^i} \wedge N_{it_q^i}(h, k) > \sigma^2 \tau_{ht}^2 \wedge N_{it_q^i}(h, k) \leq \sigma^2 u_{it_q^i}(h, k) \right) \]

Here, the first sum in the first step vanishes by the above contradiction. To bound the remaining sum, observe that if \( u_{it_q^i}(h, k) < \tau_{ht}^2 \), each term in the sum is 0 and we have \( N_{i,t^i_q+1}(h, k) \leq \left[ \hat{\sigma}^2 \tau_{ht} \right] \).

If \( u_{it_q^i}(h, k) > \tau_{ht}^2 \), we can use that the sum starts at \( \left[ \hat{\sigma}^2 \tau_{ht} \right] \) and a similar reasoning as we did in (48), to show that there are at most \( \left[ \hat{\sigma}^2 u_{it_q^i}(h, k) \right] - \left[ \hat{\sigma}^2 \tau_{ht} \right] \) non-zero terms in this summation. Therefore, \( N_{i,t^i_q+1}(h, k) \leq \left[ \hat{\sigma}^2 u_{it_q^i}(h, k) \right] \).

We can now bound \( \hat{L}_Q \).

**Proof of Lemma 33:** Assume that \( \mathcal{E} \) holds, so that all claims hold with probability at least \( 1 - \delta \). Recall the definitions of the quantities \( h_G, k^i_\tau, \ell^i_\tau, r^i_\tau \) from Appendix E.3. We can bound \( \hat{L}_Q^i \) as follows,

\[
\hat{L}_Q^i = \sum_{q=1}^{Q} (\hat{\eta}_{i,q} - \eta^*_i) \leq \eta_{\max} \left( L_{Q}^{i1} + L_{Q}^{i2} + L_{Q}^{i3} + \frac{1}{G} \hat{L}_Q^{i4} \right) \quad \text{where,} \quad (49)
\]

\[
L_{Q}^{i1} = \sum_{q=1}^{Q} \mathbb{1}(H_{i,q}^i < h_G), \quad L_{Q}^{i2} = \sum_{q=1}^{Q} \mathbb{1}(H_{i,q}^i \geq h_G \wedge \hat{\eta}_{i,q} \in [\ell^i_\tau, r^i_\tau]),
\]

\[
L_{Q}^{i3} = \sum_{q=1}^{Q} \mathbb{1}(H_{i,q}^i \geq h_G \wedge \hat{\eta}_{i,q} \in [\ell^i_\tau, r^i_\tau]),
\]

\[
L_{Q}^{i4} = \sum_{q=1}^{Q} \mathbb{1}(H_{i,q}^i \geq h_G \wedge \hat{\eta}_{i,q} \in (\eta^*_i, \ell^i_\tau)) \cdot (f_i(\hat{\eta}_{i,q}) - \alpha_i),
\]

Here, \( L_{Q}^{i1} \) bounds the number of rounds in which \( H_{i,q}^i < h_G \); therefore, when bounding the remaining terms we can focus on the rounds where \( H_{i,q}^i \geq h_G \). \( L_{Q}^{i2} \) considers evaluations greater than or equal to \( \ell^i_\tau \) and \( L_{Q}^{i3} \) considers evaluations in \([\ell^i_\tau, r^i_\tau] \), i.e. rounds \( q \) where \( (h_G, k^i_\tau) \in P_{it_q^i} \). In both cases, we simply bound \( \hat{\eta}_{i,q} - \eta^*_i \) \( \leq \eta_{\max} \). Finally, \( L_{Q}^{i4} \) accounts for the rounds when \( \eta_{i,q} \in (\eta^*_i, \ell^i_\tau) \). Here, we have used the fact that when \( a \in (\eta^*_i, \ell^i_\tau) \subset (\eta^*_i - \epsilon_G, \eta^*_i + \epsilon_G) \), by Assumption 2, we have \( |a - \eta^*_i| \leq (\eta_{\max}/G)|f_i(a) - \alpha_i| \). We will now bound each of the above terms individually. For brevity, we will denote \( \beta_{it_q^i} = \beta \), where, recall \( \beta_t \) is as defined in (11), and \( \beta_{it_q^i} \) is the value used in the exploration phase round for user \( i \) in bracket \( Q \).

**Bounding \( L_{Q}^{i1} \):** We will bound \( L_{Q}^{i1} \) by summing up the \( N_{i,t+1}(h, k) \) values for all nodes up to height \( h_G - 1 \). When assigning points to nodes in RECORD-FEEDBACK, recall that we always proceed to the child node of \( (h, k) \) if \( W_{it}(h, k) > \tau_{ht} \), in which case it is not counted in \( N_{it_q^i}(h, k) \). Therefore,
We will now apply Lemma 33. Using the above conclusion and the fact that \(G < L\), bounding \(N_{i,t_Q} (h, k) \leq 1 + \tilde{\sigma}^2 \tau_{hQ}\). This leads us to the following bound,

\[
L_{iQ}^1 = \sum_{h=0}^{h_G-1} \sum_{k=1}^{2^h} \sum_{h'=0}^{h_G-1} N_{i,t_Q} (h, k) \leq \sum_{h=0}^{h_G-1} 2^h \left(1 + \frac{\sigma^2 \beta^2}{L^2} 4^h\right) \leq 2^{h_G} + \frac{\sigma^2 \beta^2}{L^2} \sum_{h=0}^{h_G-1} 8^h
\]

\[
\leq 2^{h_G} + \frac{\sigma^2 \beta^2}{L^2} 8 h_G \leq 8 L \eta_{\max} \frac{\sigma^2 \beta^2}{G \epsilon_G} + \frac{512 L \sigma^2 \eta_{\max}^2 \beta^2}{7 G^3 \epsilon_G^2}.
\]

(50)

Here, the fourth step uses the fact that \(\sum_{h=0}^{m} 8^h = (8^{m+1} - 1)/7\), and the last step uses (44).

Bounding \(L_{iQ}^2\): First observe that we can write \(L_{iQ}^2 = \sum_{k=1}^{2^h} 1 (k > k^i) N_{i,t_Q} (h_G, k)\). This follows from our definition of \(h_G, k^i, k^l\), and two observations. First, we can write \(\tilde{\eta}_{iQ} = \eta_{\max} k/2^h\) for some \((h, k)\), in which case, the allocation returned by GET-UD-REC-FOR-UB in the \(q^\text{th}\) round will have been in \(I_{hq}\). Second, \(N_{iQ} (h, k)\) counts all evaluations at node \(h\) and its children. Moreover, we can use the NTG condition and the definition of \(h_G\) to conclude

\[
\forall k > k^i, \quad \Delta_i (h_G, k) = f_i (\ell_h) - \alpha_i >= f_i (\ell_h) - f_i (\eta^*_i + \epsilon_G) + f_i (\eta^*_i + \epsilon_G) - f_i (\eta^*_i - \epsilon_G) - f_i (\eta^*_i - \epsilon_G) \geq \frac{G \epsilon_G}{\eta_{\max}}.
\]

We will now apply Lemma 33. Using the above conclusion and the fact that \(\frac{G \epsilon_G}{\eta_{\max}} < \frac{L}{2 \epsilon_G} < \frac{G \epsilon_G}{4 \eta_{\max}}\) from (44), we have,

\[
N_{i,t_Q} (h_G, k) \leq \sigma^2 \max \left(\frac{\beta^2}{L^2} 4^{h_G}, \frac{4 \beta^2}{\left(\Delta_i (h_G, k) - L / 2^h\right)^2}\right) + 1
\]

\[
\leq \sigma^2 \beta^2 \max \left(\frac{64 \eta_{\max}^2}{G^2 \epsilon_G^2}, \frac{64 \eta_{\max}^2}{9 G^2 \epsilon_G^2}\right) + 1 \leq \frac{64 \eta_{\max}^2 \sigma^2 \beta^2}{G^2 \epsilon_G^2} + 1.
\]

Finally, by applying (44) once again, we have,

\[
L_{iQ}^2 = \sum_{k=1}^{2^h} 1 (k > k^i) N_{i,t_Q} (h_G, k) \leq 2^h \left(\frac{64 \eta_{\max}^2 \sigma^2 \beta^2}{G^2 \epsilon_G^2} + 1\right)
\]

\[
\leq \frac{512 L \eta_{\max}^3 \sigma^2}{G^3 \epsilon_G^2} \beta^2 + \frac{8 L \eta_{\max}^3 \sigma^2}{G \epsilon_G}.
\]

(51)

Bounding \(L_{iQ}^3\): Observe that we can write \(L_{iQ}^3 = N_{i,t_Q} (h_G, k^l)\). By the NTG condition and the definition of \(h_G\) (44) we have,

\[
\Delta_i (h_G, k^l) = f_i (\ell^l) - \alpha_i \geq \frac{G}{\eta_{\max}} (\ell^l - \eta^*_l) > \frac{G \epsilon_G}{2 \eta_{\max}}.
\]

Additionally, by (44), we have \(\frac{L}{2 \epsilon_G} < \frac{G \epsilon_G}{4 \eta_{\max}}\). Applying Lemma 33 once again, we get,

\[
L_{iQ}^3 = N_{i,t_Q} (h_G, k^l) \leq \frac{64 \eta_{\max}^2 \sigma^2}{G^2 \epsilon_G^2} \beta^2 + 1.
\]

(52)
Bounding $L_{Q}^{i_1}$: Recall the definitions of $I_{h}, J_{h}$ from Appendix E.3. Let $H \geq h_G$ be a positive integer whose value will be determined shortly. We will define three subsets of nodes $N_1, N_2, N_3$ in our infinite tree. Recall, from the beginning of Appendix E, that a descendant of a node $(h,k)$ could be $(h,k)$ or its children, its children’s children, etc. Let $N_1$ denote the descendants of $I_{h}$, let $N_2 = \bigcup_{h=h_G}^{H-1} I_{h}$, and let $N_3$ denote the descendants of $\bigcup_{h=h_G}^{H} J_{h}$. We can now see that $L_{Q}^{i_1}$ can be bound as follows.

$$L_{Q}^{i_1} \leq L_1 + L_2 + L_3, \quad L_i = \sum_{q=1}^{Q} \mathbb{1}((H_{t_q}^i, K_{t_q}^i) \in N_i \land H_{t_q}^i \geq h_G) \cdot (f_i(\eta_{iq}) - \alpha_i).$$

Recall from (37), that $(H_{t_q}^i, K_{t_q}^i)$ are the last nodes in the path $P_{i,t_q}$ chosen by RECORD-FEEDBACK in round $t_q$ of bracket $q$, and moreover, in that bracket $q$, we used $\hat{\eta}_{iq}$ as the upper bound in the latter phase of Algorithm 2. Therefore, when $\hat{\eta}_{iq} \in (\eta^*_i, \ell^*_i)$ and $H_{t_q}^i \geq h_G$, we also have $(H_{t_q}^i, K_{t_q}^i) \in N_1 \cup N_2 \cup N_3$. We will now bound $L_1, L_2,$ and $L_3$.

By (47), we have $f_i(a) - \alpha_i < 4L2^{-H}$ when $a \in I_{hk}$, for any $(h,k) \in N_1$. This leads to the following straightforward bound for $L_1$,

$$L_1 = \sum_{s=1}^{Q} \mathbb{1}((H_{t_q}^i, K_{t_q}^i) \in N_1) \cdot (f_i(\hat{\eta}_{iq}) - \alpha_i) \leq \frac{4L}{2^H}T. \quad (53)$$

Next, we bound $L_2$ as shown below.

$$L_2 = \sum_{h=h_G}^{H-1} \sum_{(h,k) \in I_{h}} \frac{4L}{2^h} N_{i,t_q+1}(h,k) \leq \sum_{h=h_G}^{H-1} \frac{4L}{2^h} \left( \frac{4L}{2^h} \right) \left(1 + \frac{4h^2 \beta^2}{L^2} \right) \leq \frac{16L^2}{G} \sum_{h=h_G}^{H-1} 2^{-h} + \frac{16 \beta^2 \sigma^2}{G} \sum_{h=h_G}^{H-1} 2^{h} \leq \frac{32L^2}{G} + \frac{16 \beta^2 \sigma^2}{G} 2^H. \quad (54)$$

In the first step we have used $N_{i,t_q+1}(h,k)$ (instead of $N_{i,t_q+1}(h,k)$), since $L_2$ only counts allocations where $(H_{t_q}^i, K_{t_q}^i)$ were in $\bigcup_{h=h_G}^{H-1} I_{h}$. We have also used (47) to bound $f_i(\hat{\eta}_{iq}) - \alpha_i$. In the second step, by the same reasoning used in the bound for $L_{Q}^{i_1}$, we have $N_{i,t_q+1}(h,k) \leq 1 + \beta^2 \tau_{h,t_q}$ for any node $(h,k)$; moreover, we have used (46) to bound the number of nodes in $I_{h}$. The remaining steps are obtained by algebraic manipulations.

Finally, we bound $L_3$ as follows.

$$L_3 = \sum_{h=h_G}^{H} \sum_{(h,k) \in J_{h}} \frac{8L}{2^h} N_{i,t_q+1}(h,k) \leq \sum_{h=h_G}^{H} \frac{8L}{2^h} \left( \frac{4L}{2^h} \right) \left(1 + \frac{4h^2 \beta^2}{L^2} \right) \leq \frac{64L^2}{G} \sum_{h=h_G}^{H} 2^{-h} + \frac{256 \beta^2 \sigma^2}{G} \sum_{h=h_G}^{H} 2^{h} \leq \frac{128L^2}{G} + \frac{512 \beta^2 \sigma^2}{G} 2^H. \quad (55)$$

Above, in the first step we have used $N_{i,t_q+1}(h,k)$ (instead of $N_{i,t_q+1}(h,k)$), since $L_3$ counts allocations where $(H_{t_q}^i, K_{t_q}^i)$ belonged to the descendants of $\bigcup_{h=h_G}^{H} J_{h}$; additionally, we have used (47)
and the fact that parents of nodes in $\mathcal{J}_h$ are in $\mathcal{I}_{h-1}$ to bound $f_i(a_{it}) - \alpha_i$. In the second step, first we have used (46) to bound the number of nodes in $\mathcal{J}_h$; to bound the number of evaluations in each such node, we have applied Lemma 33 along with the fact that $\Delta_i(h, k) > 2L2^{-h}$ for nodes in $\mathcal{J}_h$
by their definition; therefore,

$$N_{i,t_Q + 1}(h, k) \leq 1 + \sigma^2 \max \left( \frac{\beta^2}{L^2} 4^h, \frac{4 \beta^2}{\left( \Delta_i(h, k) - 2^{-h} \right)^2} \right) \leq 1 + \frac{4 \sigma^2 \beta^2}{L^2} 4^h.$$

The remaining steps in (55) are obtained via algebraic manipulations. Combining (53), (54), and (55) results in the following bound for $L_{t_Q}^{i}$.

$$L_{t_Q}^{i} \leq 4L \frac{2^H}{G} Q + \frac{528 \sigma^2 \beta^2}{G} 2^H + \frac{160L^2}{G} \leq C' \frac{L^{1/2} \beta T^{1/2}}{G^{1/2}} + \frac{160L^2}{G} \tag{56}$$

Here $C'$ is a global constant. The last step is obtained by choosing $H$ such that $2^H \approx \frac{\sqrt{Gt_Q}}{\delta \beta}$.

The lemma now follows by combining (49), (50), (51), (52), and (56). Moreover, from (40) we have $\beta = \beta_{t_Q}^i \leq \beta_{2t_Q}^i$.

We are now ready to prove Theorem 7.

**Proof of Theorem 7.** Assume that $E$ holds, so that all claims hold with probability at least $1 - \delta$.

**Efficiency:** We will decompose and bound the loss as shown below:

$$L_T = \sum_{t \in E} \ell_t + \sum_{t \notin E} \ell_t \leq nq_T + \sum_{t \notin E} \ell_t \leq nq_T + \sum_{t \in E} \sum_{i=1}^n (\tilde{a}_{it} - d_{it}^*) \tag{57}$$

$$\leq nq_T + \overline{w} \sum_{i=1}^n \sum_{t \notin E} (\tilde{\eta}_{it} - \eta_i^*) \leq nq_T + \overline{w} \sum_{i=1}^n \sum_{q=1}^{q_T} r'(q) (\tilde{\eta}_{iq} - \eta_i^*) \leq nq_T + \overline{w} \sum_{i=1}^n \sum_{q=1}^{q_T} \tilde{l}_{iq}^i \leq nq_T + \frac{5\overline{w}}{6} nq_T^{1/2} \sum_{i=1}^n \tilde{l}_{i}^i \leq nq_T + \frac{5\overline{w}}{6} n^2 \left( \frac{CL^{1/2} \beta_{t_Q}^i}{G^{1/2}} q_T + 586 \frac{L^2 \sigma^2 \eta_{i_{max}}^2}{G^{3/2} \epsilon_G^2} \beta_{2T}^i q_T^{1/2} + \frac{64 \eta_{i_{max}}^2 \sigma^2}{G^2 \epsilon_G^2} \beta_{2T}^i q_T^{1/2} \right)$$

Here, in the third step we have used Lemma 9 to bound the instantaneous losses in the second sum. In the fourth step, we have observed $\tilde{a}_{it} - d_{it}^* = w_{it}(\tilde{\eta}_{it} - \eta_i^*) \leq \overline{w}(\tilde{\eta}_{it} - \eta_i^*)$. For the fifth step, we have upper expressed the second sum in terms of $\tilde{\eta}_{iq}$, by summing over the brackets—note that there may be more terms in the summation as we consider all rounds in the $q_T$th bracket. In the sixth step, we have observed that $r'$ is an increasing function, and in the seventh step, we have used that $r'(q) = \lfloor 5nq/6 \rfloor$ (line 1, Algorithm 7–10), and observed that the term inside the summation is $\tilde{L}_{t_Q}^i$ from Lemma 33. The last step applies the bound in Lemma 33 while also observing that $t_Q^i \leq T$ and that $\beta_t$ is increasing in $t$. The bound on the loss follows by an application of Lemma 17 with $a = n$ to upper bound $q_T$. 

63
Fairness: This follows by applying Lemma 12 with \( r = n \) and the upper bound for \( q_T \) in Lemma 17.

Strategy-proofness: For a policy \( \pi \), denote \( \bar{\pi}_i^\tau = a_i^\tau / w_i^\tau \). Using Lemma 14 we obtain,
\[
U_i^\tau - U_i^T \leq \sum_{t \in E} (u_i(\bar{\pi}_i^\tau) - u_i(\bar{\pi}_i^\tau)) \leq L_i^u \sum_{t \in E, \eta_t > 0} (\bar{\pi}_i^\tau - \bar{\pi}_i^\tau)^+ \leq L_i^u \eta_{\text{max}} q_T \leq 3L_i^u \eta_{\text{max}} n^{-2/3} T^{2/3}
\]

Here, we have observed that we allocate to each user only once in each exploration phase. The last step uses Lemma 17.

\[\Box\]

E.5 Proof of Theorem 8

In this section, we will prove Theorem 8. We will begin with two lemmas to bound the number of allocations away from the demand for each user \( i \) assuming that \( \mathcal{E} \) holds (36). The first of these, bounds the number of allocations larger than the demand.

Lemma 35 Consider user \( i \) and let \( (h, k) \) be such that \( I_{hk} \subset (\eta_i^*, \eta_{\text{max}}) \) and \( \Delta_i(h, k) = f_i(\ell_{hk}) - \alpha_i > L/2^h \). Under \( \mathcal{E} \), for all \( t \geq 1 \),
\[
N_{i,t+1}(h, k) \leq \sigma_2^2 \cdot \min(\tau_{ht}, u_{ht}(h, k)) + 1 = \sigma_2^2 \cdot \min\left(\frac{\beta_2^2}{L^2} 4^h, \frac{4\beta_2^2}{(\Delta_i(h, k) - L2^{-h})^2}\right) + 1
\]

Proof. The statement is clearly true for unexpanded nodes, so let we will show this for \( (h, k) \in T_{it} \).

First, we will decompose \( N_{i,t+1}(h, k) = \sum_{s=1}^t 1 \) \((h, k) \in P_{is}\) as follows,
\[
N_{i,t+1}(h, k) = \sum_{s=1}^t 1 \ ((h, k) \in P_{is} \wedge N_{is}(h, k) \leq \sigma_2^2 \tau_{ht})
+ \sum_{s=1}^t 1 \ ((h, k) \in P_{is} \wedge N_{is}(h, k) > \sigma_2^2 \tau_{ht})
\leq \lfloor \sigma_2^2 \tau_{ht} \rfloor + \sum_{s=\lfloor \sigma_2^2 \tau_{ht} \rfloor}^t 1 \ ((h, k) \in P_{is} \wedge N_{is}(h, k) > \sigma_2^2 \tau_{ht})
\]

In the second step, we have bound the first summation by observing that \( N_{is}(h, k) \) values are constant from round \( s \) to \( s + 1 \) unless \((h, k) \in P_{is} \) in which case they increase by one. Therefore, at most \( \lfloor \sigma_2^2 \tau_{ht} \rfloor \) such terms can be non-zero. For the second sum, we have used the fact that \( N_{is}(h, k) \) cannot be larger than \( \sigma_2^2 \tau_{ht} \) in the first \( \lfloor \sigma_2^2 \tau_{ht} \rfloor \) rounds. To bound the second summation, we will consider round \( s \) and show, by way of contradiction, that the following cannot hold simultaneously,
\[
N_{is}(h, k) > u_{is}(h, k) \sigma_2^2, \quad N_{is}(h, k) > \sigma_2^2 \tau_{ht}, \quad \eta_{is} \geq \ell_{hk}.
\]

Recall that \( \eta_{is} \) is the recommended unit demand at round \( s \). First observe that under \( \mathcal{E} \),
\[
N_{is}(h, k) > u_{is}(h, k) \sigma_2^2 \implies W_{is}(h, k) > u_{is}(h, k) \implies B_{is}(h, k) \geq f_{is}(h, k) \geq \alpha_i,
\implies \bar{B}_{is}(h, k') \geq \alpha_i, \quad \text{for all } k' \geq k,
\implies B_{is}(h, k') = \min(\bar{B}_{is}(h, k') - \alpha_i, \alpha_i - \bar{B}_{is}(h, k')) \leq 0, \quad \text{for all } k' \geq k,
\]
\]
\]

64
The second step uses the conditions for \( \mathcal{E} (35) \) and the definition for \( \tilde{B}_{it}(h, k) \) for expanded nodes (42). The third step uses Lemma 29, and the last step simply uses the definition of \( B_{is}(17) \). The conclusion in (60) says that all of the nodes to the right of \((h, k)\) at height \(h\) will have negative \( B_{is}(h, k) \) value if \( N_{is}(h, k) > u_{is}(h, k) \).

Note that if user \(i\) received an allocation \(a_{is} = \pi_{is}w_{is}\) at round \(s\), it could be because the reported demand was \(d_{is} = \eta_{is}w_{is} = \pi_{is}w_{is}\), or because it was \(d_{is} = \eta_{is}w_{is} > \pi_{is}w_{is}\), but received less due to resource contention. Let \((h, k')\) be the node at height \(h\) that contained the recommendations from GET-UD-REC (line 14, Algorithm 7–10) which was used as the reported demand for MMF (line 5, Algorithm 3), i.e. \( \eta_{is} \in I_{hk} \). Let \((\ell, k^i_{\ell+1})\) be the common ancestor of \((h, k^i_h), (h, k), \) and \((h, k')\) whose children are \((\ell + 1, k^i_{\ell+1} + 1)\), \((\ell + 1, k^i_{\ell+1} + 1)\); here, \((\ell + 1, k^i_{\ell+1} + 1)\) is the left child of \((\ell, k^i_{\ell})\) and an ancestor of \((h, k^i_h)\), while \((\ell + 1, k^i_{\ell+1} + 1)\) is the right child; this is the case since \(I_{hk} \subset \eta_{max}\).

Since \(k^i_h < k \leq k'\), there are three possible cases here, all of which lead to a contradiction of the statement in (59).

1. \((\ell + 1, k^i_{\ell+1} + 1)\) is a common ancestor of \((h, k^i_h)\) and \((h, k)\), while \((\ell + 1, k^i_{\ell+1} + 1)\) is an ancestor of \((h, k')\): By (60), all of \((\ell + 1, k^i_{\ell+1} + 1)\)’s descendants at height \(h\) will have negative \(B_{is}\) value. By Lemma 30, all of their parents at height \(h - 1\) will also have negative \(B_{is}\) value. Continuing this argument, we have that \(B_{is}(h + 1, k^i_{\ell+1} + 1) < 0\). Since \(W_{is}(h, k) > \tau_{ht}\), we also have \(W_{is}(h', k'') > \tau_{ht}\) for all of \((h, k)\)’s ancestors \((h', k'')\), including in particular, \((\ell, k^i_{\ell+1})\) and its ancestors. Moreover, since \(\eta_{is}\) was the chosen recommendation, the while loop in GET-UD-REC will have reached \((\ell, k^i_{\ell+1})\) and have proceeded one more step to select \((\ell + 1, k^i_{\ell+1} + 1)\). Since we always choose the child with the higher \(B_{is}\) value at each stage, \(B_{is}(\ell + 1, k^i_{\ell+1}) \leq B_{is}(\ell + 1, k^i_{\ell+1} + 1) < 0\). However, by Lemma 31, \(B_{is}(\ell + 1, k^i_{\ell+1}) > 0\), which is a contradiction.

2. \((\ell + 1, k^i_{\ell+1} + 1)\) is an ancestor of \((h, k)\) with \(k = k'\): Since \(W_{is}(h, k) > \tau_{ht}\), we have that the while loop in GET-UD-REC chose each node on the path from \((0, 1)\) to \((h, k)\). Since the \(B_{is}\) value never decreases along a chosen path,

\[
B_{is}(\ell + 1, k^i_{\ell+1}) \leq B_{is}(\ell + 1, k^i_{\ell+1} + 1) \leq \cdots \leq B_{is}(h, k) < 0.
\]

Once again, the contradiction follows from the fact that \(B_{is}(\ell + 1, k^i_{\ell+1}) > 0\) by Lemma 31.

3. \((\ell + 1, k^i_{\ell+1} + 1)\) is a common ancestor ancestor of \((h, k)\) and \((h, k')\) with \(k < k'\): Let \((p, q)\) be the common ancestor of \((h, k)\) and \((h, k')\), with the left child \((p + 1, 2q - 1)\) leading to \((h, k)\) and the right child \((p + 1, 2q)\) leading to \((h, k')\). Since \(W_{is}(h, k) > \tau_{ht}\) it means that GET-UD-REC chose each node on the path from \((0, 1)\) to \((p, q)\) and proceeded to choose \((p + 1, 2q)\). Since, by Lemma 31, \(B_{is}(\ell, k^i_{\ell}) \geq 0\) and since the \(B_{is}\) value never decreases along a chosen path, \(B_{is}(p + 1, 2q) < 0\). However, by (60) and a reasoning similar to the first case above, \(B_{is}(p + 1, 2q) < 0\), which is a contradiction.

To complete the proof, note that MMF does not allocate more than the requested demand for a user \(i\); therefore, \((h, k) \in P_{is} \implies \pi_{is} \in P_{is} \implies \eta_{is} \geq \ell_{hk}\). Using (58), we write,

\[
N_{i,t+1}(h, k) \leq [\sigma^2 \tau_{ht}] + \sum_{s=\lfloor \sigma^2 \tau_{ht} \rfloor}^{t} \mathbb{1} ((h, k) \in P_{is} \land N_{is}(h, k) > \sigma^2 \tau_{ht} \land N_{is}(h, k) \leq \sigma^2 u_{is}(h, k)) +
\]
Therefore, \( N \) values since users may not have received large allocations (see for example, the bottom figure in Figure 2). Consider user \( i \) and let \( \alpha_i \) be defined in (34). Our second main lemma in this section bounds \( N_{it} \) for allocations lower than \( \alpha_i \). Its proof follows along similar lines to the proof of Lemma 35 in several places; as such, we will frequently refer to calculations from above.

**Remark 3** It is worth pointing out why monotonicity of the confidence intervals is necessary for the correctness of the algorithm. If they were not, the \( B_{it} \) values may be large for large allocation values since users may not have received large allocations (see for example, the bottom figure in Figure 2); consequently, the \( \tilde{B}_{it} \) values will be large and the \( \bar{B}_{it} \) values will be small. Hence, the \( B_{it} = \min(\alpha_i - \bar{B}_{it}, \tilde{B}_{it} - \alpha_i) \) values will also be large causing GET-UD-REC to recommend a large allocation. This could lead to pathological situations where the GET-UD-REC keeps recommending large allocations, but a smaller allocation is repeatedly chosen due to contention on limited resources. If the confidence intervals are monotonic, then \( B_{it} \) will be small for large allocations even if they have not been evaluated. This ensures that GET-UD-REC does not recommend large allocations. In particular, if the lower confidence bound is larger than \( \alpha_i \) for any allocation, monotonicity ensures that \( B_{it} \) is negative for any larger allocation. The proof of Lemma 35 captures this intuition.

Our second lemma bounds the number of allocations smaller than the demand whenever MMF allocates an amount of resources equal to the demand. For this, we first define \( W_{it}^{\text{rec}}(h, k), N_{it}^{\text{rec}}(h, k) \) below, which are variants of \( W_{it}(h, k), N_{it}(h, k) \), but only consider rounds when the allocation was equal to the recommendation. We have:

\[
W_{it}^{\text{rec}}(h, k) = \sum_{s=1}^{t-1} \frac{1}{\sigma^2_{it}} \mathbb{1}(\eta_{it} = \bar{u}_{it} \land (h, k) \in P_{ts}),
\]

\[
N_{it}^{\text{rec}}(h, k) = \sum_{s=1}^{t-1} \mathbb{1}(\eta_{it} = \bar{u}_{it} \land (h, k) \in P_{ts}). \tag{61}
\]

Our second main lemma in this section bounds \( N_{it}^{\text{rec}}(h, k) \) for allocations lower than \( \eta_{it}^* \). Its proof follows along similar lines to the proof of Lemma 35 in several places; as such, we will frequently refer to calculations from above.

**Lemma 36** Consider user \( i \) and let \( (h, k) \) be such that \( I_{hk} \subseteq [0, \eta_{it}^*] \) and \( \Delta_i(h, k) = \alpha_i - f_i(r_{hk}) > L/2^h \). Under \( E \), let \( u_{it}(h, k) \) be as defined in (34). Then, for all \( t \geq 1 \),

\[
N_{it}^{\text{rec}}(h, k) \leq \sigma^2 \max(\tau_{ht}, u_{it}(h, k)) + 1 = \sigma^2 \max\left(\frac{\beta^2_{it}}{L^2}, \frac{4\beta^2_{it}}{(\Delta_i(h, k) - L2^{-h})^2}\right) + 1.
\]
Proof. The statement is true for unexpanded nodes, so, as before we will consider \((h, k) \in T_{it}\). By following the same reasoning as in (58), we have,

\[
N_{i,t+1}^{rec}(h, k) \leq \lceil \sigma^2 \tau_{ht} \rceil + \sum_{s=\lceil \sigma^2 \tau_{ht} \rceil}^{t} 1(\eta_{is} = \bar{a}_{is} \land (h, k) \in P_{is} \land N_{is}^{rec}(h, k) > \sigma^2 \tau_{ht}). \tag{62}
\]

To bound the summation in the RHS above, consider any round \(s \leq t\). We will show, by way of contradiction, that the following statements cannot hold simultaneously.

\[
N_{is}^{rec}(h, k) > u_{is}(h, k)\sigma^2, \quad N_{is}^{rec}(h, k) > \sigma^2 \tau_{ht}, \quad (h, k) \in P_{is}, \quad \bar{a}_{is} = \eta_{is}.
\]

Let \((\ell, k^0_{\ell})\) be the last threshold node on the path from \((0, 1)\) to \((h, k)\), with children \((\ell + 1, k^1_{\ell+1})\) and \((\ell + 1, k^2_{\ell+1})\); here, the left child \((\ell + 1, k^1_{\ell+1})\) is an ancestor of \((h, k)\) and \((\ell + 1, k^2_{\ell+1})\) is the threshold node at height \(\ell + 1\) since \(I_{hkh} \subset [0, \eta^*_{s}]\). Since \(W_{is}(h, k) > W_{is}^{rec}(h, k) > \tau_{hs}\), we also have \(W_{is}(h'', k'') > \tau_{ht}\) for all of \((h, k)\)’s ancestors \((h'', k'')\), including, in particular, \((\ell, k^1_{\ell})\) and its ancestors. Moreover, since \(\eta_{is}\) was the chosen recommendation, the while loop in GET-UD-REC will have reached \((\ell, k^1_{\ell})\) and have proceeded one more step to choose \((\ell + 1, k^2_{\ell+1})\) since the while condition is satisfied. By Lemma 30, and the fact that GET-UD-REC (line 14, Algorithm 7–10) chooses the child with the larger \(B_{it}\) value at each node, we have that the \(B_{it}\) values are non-decreasing along a chosen path. This leads us to the following conclusion:

\[
B_{it}(\ell + 1, k^0_{\ell+1}) \leq B_{it}(\ell + 1, k^2_{\ell+1}) \leq B_{it}(h, k) < 0.
\]

Here, the last inequality uses the definition of the event \(E_{it}\) (35), and that \(N_{it}(h, k) \geq u_{it}(h, k)\sigma^2 \implies W_{it}(h, k) \geq u_{it}(h, k)\). However, by Lemma 31, \(B_{it}(\ell + 1, k^2_{\ell+1}) > 0\), which is a contradiction.

Finally, using (62), we obtain the following bound.

\[
N_{i,t+1}^{rec}(h, k) \leq \lceil \sigma^2 \tau_{ht} \rceil + \sum_{s=\lceil \sigma^2 \tau_{ht} \rceil}^{t} 1(\bar{a}_{is} = \eta_{is}, (h, k) \in P_{is}, N_{is}^{rec}(h, k) > \sigma^2 \tau_{ht}, N_{is}^{rec}(h, k) \leq \sigma^2 u_{it}(h, k)) + \sum_{s=\lceil \sigma^2 \tau_{ht} \rceil}^{t} 1(\bar{a}_{is} = \eta_{is}, (h, k) \in P_{is}, N_{is}^{rec}(h, k) > \sigma^2 \tau_{ht}, N_{is}^{rec}(h, k) > \sigma^2 u_{it}(h, k)) \leq \lceil \sigma^2 \tau_{ht} \rceil + \sum_{s=\lceil \sigma^2 \tau_{ht} \rceil}^{t} 1(\bar{a}_{it} = \eta_{it}, (h, k) \in P_{is}, N_{is}^{rec}(h, k) > \sigma^2 \tau_{ht}, N_{is}^{rec}(h, k) \leq \sigma^2 u_{it}(h, k))
\]

The proof is completed by the same line of reasoning as at the end of the proof of Lemma 35 to show \(N_{i,t+1}^{rec}(h, k) \leq \max(\lceil \sigma^2 \tau_{ht} \rceil, \lceil \sigma^2 u_{it}(h, k) \rceil)\).

We are now ready to prove the Theorem. Some of the calculations used in this proof will be similar to those used in the proof of Lemma 33, where we bound \(\hat{L}_{iQ}\). However, for the sake of clarity, and to keep this proof self-contained, we will repeat those calculations.
**Proof of Theorem 8.** Efficiency: Recall the bound on the loss from Lemma 10. By observing 
\[(a_{it} - d_{it}^*)^+ = w_{it}(\bar{a}_{it} - \eta_{it}^*)^+ \leq \bar{w}(\bar{a}_{it} - \eta_{it})^+\], and similarly, 
\[(d_{it}^+ - a_{it})^+ \leq \bar{w}(\eta_{it}^* - \bar{a}_{it})^+\], we first relax this bound as follows.

\[
L_T \leq 1 + \bar{w} \sum_{i=1}^{n} \sum_{t=2}^{T} (\bar{a}_{it} - \eta_{it}^*)^+ + \bar{w} \sum_{i=1}^{n} \sum_{t=2}^{T} (\eta_{it}^* - \bar{a}_{it})^+ \tag{63}
\]

We can now bound the individual summations above as shown below. Recall the definitions of the quantities \(h_G, k_t^+, l_t^+, r_t^+, \ell_t^+\) from Appendix E.3. We have:

\[
\sum_{t=2}^{T} (\bar{a}_{it} - \eta_{it}^*)^+ \leq \eta_{\text{max}} \sum_{t=2}^{T} 1(\bar{a}_{it} \in [r_t^+, \eta_{\text{max}}]) + \eta_{\text{max}} \sum_{t=2}^{T} 1(\bar{a}_{it} \in I_{h_G k_t^+})
\]

\[
+ \sum_{t=2}^{T} 1(\bar{a}_{it} \in (\eta_{it}^*, \ell_t^+)) \cdot (1 - \bar{a}_{it} - \eta_{it}^*). \tag{64}
\]

\[
\sum_{t=2}^{T} (\eta_{it}^* - \bar{a}_{it})^+ \leq \eta_{\text{max}} \sum_{t=2}^{T} 1(\bar{a}_{it} \in [0, \ell_t^+]) + \eta_{\text{max}} \sum_{t=2}^{T} 1(\bar{a}_{it} \in I_{h_G k_t^+})
\]

\[
+ \sum_{t=2}^{T} 1(\eta_{it} = \bar{a}_{it} \land \bar{a}_{it} \in (r_t^+, \eta_{it}^*)) \cdot (\eta_{it}^* - \bar{a}_{it}). \tag{65}
\]

Here, (64) follows from the fact that \((1 - \bar{a}_{it} - \eta_{it}^*)^+\) is positive only when \(\bar{a}_{it} \in (\eta_{it}^*, \eta_{\text{max}}) = (\eta_{it}^*, \ell_t^+) \cup \ell_t^+ \cup r_t^+, \eta_{\text{max}}\). Moreover, we have bound \((\bar{a}_{it} - \eta_{it}^*)^+ \leq \eta_{\text{max}}\) when \(\bar{a}_{it} \in (\eta_{it}^*, \ell_t^+) \cup (\ell_t^+, r_t^+)\). We obtain (65) via analogous reasoning on the interval \([0, \eta_{it}^*]\). Combining this with (63), we have the following bound.

\[
L_T \leq 1 + \bar{w} \eta_{\text{max}} \sum_{i=1}^{n} \left( L_T^{11} + L_T^{12} + L_T^{13} + L_T^{14} + \frac{1}{G} L_T^{15} + \frac{1}{G} L_T^{16} \right), \tag{66}
\]

where,

\[
L_T^{11} = \sum_{t=1}^{T} 1(H_t^i < h_G), \quad L_T^{12} = \sum_{t=2}^{T} 1(\bar{a}_{it} \in [0, \ell_t^+] \cup [r_t^+, \eta_{\text{max}}] \land H_t^i \geq h_G),
\]

\[
L_T^{13} = \sum_{t=2}^{T} 1(\bar{a}_{it} \in I_{h_G k_t^+} \land H_t^i \geq h_G), \quad L_T^{14} = \sum_{t=2}^{T} 1(\bar{a}_{it} \in I_{h_G k_t^+} \land H_t^i \geq h_G),
\]

\[
L_T^{15} = \sum_{t=2}^{T} 1(\bar{a}_{it} \in (\eta_{it}^*, \ell_t^+) \land H_t^i \geq h_G) \cdot (f_i(\eta_{it}) - \alpha_i), \quad L_T^{16} = \sum_{t=2}^{T} 1(\eta_{it} = \bar{a}_{it} \land \bar{a}_{it} \in (r_t^+, \eta_{it}^*) \land H_t^i \geq h_G) \cdot (\alpha_i - f_i(\alpha_i)).
\]

In (66), \(L_T^{11}\) bounds the number of rounds in which \(H_t^i < h_G\); therefore, when bounding each of the terms in (64) and (65), we can focus on the rounds where \(H_t^i \geq h_G\). \(L_T^{12}\) accounts for the first term in the RHS of (64) and the first term of (65). \(L_T^{13}\) accounts for the second term in the RHS of (64), while \(L_T^{14}\) accounts for the second term of (65). Finally, \(L_T^{15}\) accounts for the
third term in the RHS of (64), while $L_{T}^{10}$ accounts for the third term of (65). Here, we have used the fact that when $a \in (r_{i}^{\prime}, \eta_{i}^{\prime}) \cup (\eta_{i}^{\prime}, \ell_{i}^{\prime}) \subset (\eta_{i}^{\ast} - \epsilon G, \eta_{i}^{\ast} + \epsilon G)$, we have, by Assumption 2, $|a - \eta_{i}^{\ast}| \leq (G/\eta_{\max})|f_{i}(a) - \alpha_{i}|$. We will now bound each of the above terms individually.

Bounding $L_{T}^{12}$. We will bound this by summing up the $N_{it}(h, k)$ values for all nodes up to height $h_{G} - 1$. When assigning points to nodes in RECORD-FEEDBACK, recall that we always proceed to the child node if $W_{it}(h, k) > \tau_{ht}$, in which case it is not counted in $N_{it}(h, k)$. Therefore, $N_{it}(h, k) \leq 1 + \sigma^{2}\tau_{ht}$. This leads us to the following bound,

$$L_{T}^{12} = \sum_{h=0}^{h_{G} - 1} \sum_{k = 1}^{2^{h}} N_{i,T+1}(h, k) \leq \sum_{h=0}^{h_{G} - 1} \sum_{k = 1}^{2^{h}} 2^{h} \left( 1 + \sigma^{2} \frac{2^{2}}{L^{2}} \right) \leq 2^{h_{G}} + \frac{\sigma^{2} \beta^{2}_{T+1}}{L^{2}} \sum_{h=0}^{h_{G} - 1} 8^{h} \leq \frac{8 L \eta_{\max}}{G \epsilon G} + \frac{512 L \sigma^{2} \eta_{\max}^{3} / \beta^{2}_{T+1}}{G^{3} \epsilon G}. \quad (67)$$

Here, the fourth step uses the fact that $\sum_{h=0}^{m} b^{h} = (b^{m+1} - 1)/7$, and the last step uses (44).

Bounding $L_{T}^{12}$: First observe that we can write $L_{T}^{12} = \sum_{k=1}^{2^{h_{G}}} 1(k < k_{i}^{\prime} \lor k > k_{i}^{\prime}) N_{i,T+1}(h, k)$; this follows from our definition of $h_{G}, k_{i}^{\prime}, k_{i}^{\prime}$, and the fact that $N_{it}(h, k)$ counts all evaluations at node $h$ and its children. Additionally, we have the following relations by the NTG condition and the definition of $h_{G}$ above,

$$\forall k > k_{i}^{\prime}, \quad \Delta_{i}(h_{G}, k) = f_{i}(\ell_{hk}) - \alpha_{i} = f_{i}(\ell_{hk}) - f_{i}(\eta_{i}^{\ast} + \epsilon G) + f_{i}(\eta_{i}^{\ast} + \epsilon G) - f_{i}(\eta_{i}^{\ast}) \geq f_{i}(\eta_{i}^{\ast} + \epsilon G) - f_{i}(\eta_{i}^{\ast}) \geq \frac{G \epsilon G}{\eta_{\max}},$$

$$\forall k < k_{i}^{\prime}, \quad \Delta_{i}(h_{G}, k) = \alpha_{i} - f_{i}(\eta_{hk}) \geq f_{i}(\eta_{i}^{\ast}) - f_{i}(\eta_{i}^{\ast} - \epsilon G) \geq \frac{G \epsilon G}{\eta_{\max}}.$$

Moreover, from (44) we have $\frac{L}{2^{h_{G}}} < \frac{G \eta_{\max}^{2}}{4 \epsilon G}$, and $\frac{\sigma^{2}}{L} < \frac{\eta_{\max}^{2}}{G \epsilon G}$. Applying these conclusions to Lemmas 35 and 36, we have for all $k < k_{i}^{\prime}$ or $k > k_{i}^{\prime}$,

$$N_{i,T+1}(h, k) \leq \sigma^{2} \max \left( \frac{\beta^{2}_{T+1}}{L^{2}} 4^{h_{G}}, \frac{4 \beta^{2}_{T+1}}{L^{2}} \left( \Delta_{i}(h_{G}, k) - L2^{-h_{G}} \right)^{2} \right) + 1 \leq \sigma^{2} \beta^{2}_{T+1} \max \left( \frac{64 \eta_{\max}^{2}}{G^{2} \epsilon G}, \frac{64 \eta_{\max}^{2}}{9 G^{2} \epsilon G} \right) + 1 \leq \frac{64 \eta_{\max}^{2} \sigma^{2} \beta^{2}_{T+1}}{G^{2} \epsilon G} + 1.$$

Finally, by applying (44) once again, we have,

$$L_{T}^{12} = \sum_{k=1}^{2^{h_{G}}} 1(k < k_{i}^{\prime} \lor k > k_{i}^{\prime}) N_{i,T+1}(h, k) \leq 2^{h_{G}} \left( \frac{64 \eta_{\max}^{2} \sigma^{2} \beta^{2}_{T+1}}{G^{2} \epsilon G} + 1 \right) \leq \frac{512 L \eta_{\max}^{3} \sigma^{2} \beta^{2}_{T+1}}{G^{3} \epsilon G} + 8 \frac{L \eta_{\max}}{G \epsilon G} . \quad (68)$$

Bounding $L_{T}^{13}$. Observe that we can write $L_{T}^{13} = N_{i,T+1}(h, k_{i}^{\prime})$. By the NTG condition and the definition of $h_{G}$ (44) we have,

$$\Delta_{i}(h_{G}, k_{i}^{\prime}) = \alpha_{i} - f_{i}(\eta_{i}^{\prime}) \geq \frac{G}{\eta_{\max}} (\eta_{i}^{\ast} - r_{i}^{\prime}) > \frac{G \epsilon G}{2 \eta_{\max}}.$$
Additionally, by (44), we have $\frac{L_T}{2G} < \frac{G \epsilon_G}{4 \eta_{\max}}$. Applying Lemma 36, we get,

$$L_T^{13} = N_{i,T+1}(h_G, k_i^j) \leq \frac{64 \eta_{\max} \pi^2}{G^2 \epsilon_G^2} \beta_{T+1}^2 + 1. \quad (69)$$

Bounding $L_T^{14}$: Following a similar argument as $L_T^{13}$ and via an application of Lemma 35, we have

$$L_T^{14} = N_{i,T+1}(h_G, k_i^j) \leq \frac{64 \eta_{\max} \pi^2}{G^2 \epsilon_G^2} \beta_{T+1}^2 + 1. \quad (70)$$

Bounding $L_T^{15}$: Recall the definitions of $I_h, \mathcal{J}_h$ from Appendix E.3. Let $H \geq h_G$ be a positive integer whose value will be determined shortly. We will define three subsets of nodes $N_1, N_2, N_3$ in our infinite tree. Let $N_1$ denote the descendants of $I_H$; let $N_2 = \bigcup_{h=h_G}^{H-1} I_h$; and let $N_3$ denote the descendants of $\bigcup_{h=h_G}^{H} \mathcal{J}_h$. We can now see that $L_T^{15}$ can be bound as follows.

$$L_T^{15} \leq L_1 + L_2 + L_3, \quad L_i = \sum_{t=2}^{T} 1((H_i^t, K_i^t) \in N_i \land H_i^t \geq h_G) \cdot (f_i(\pi_{it}) - \alpha_i).$$

Recall from (37), that $(H_i^t, K_i^t)$ are the last nodes in the path $P_{it}$ chosen by RECORD-FEEDBACK. The above bound follows from the fact that for any allocation satisfying, $a_{it} \in (\eta_i^*, \ell_i^*) \land H_i^t \geq h_G$, the last node $(H_i^t, K_i^t)$ should be in $N_1 \cup N_2 \cup N_3$.

By (47), we have $f_i(a) - \alpha_i < 4L2^{-H}$ when $a \in I_{hk}$, for any $(h,k) \in N_1$. This leads to the following straightforward bound for $L_1$,

$$L_1 = \sum_{t=2}^{T} 1((H_i^t, K_i^t) \in N_1 \land H_i^t \geq h_G) \cdot (f_i(\pi_{it}) - \alpha_i) \leq \frac{4L}{2^H} T. \quad (71)$$

Next, we bound $L_2$ as shown below.

$$L_2 = \sum_{h=h_G}^{H-1} \sum_{(h,k) \in I_h} \frac{4L}{2^h} N_{i,T+1}(h,k) \leq \sum_{h=h_G}^{H-1} \frac{4L}{2^h} \frac{4L}{2^H} \left(1 + \frac{4^h \pi^2 \beta_{T+1}^2}{L^2} \right)$$

$$\leq \frac{16L^2}{G} \sum_{h=h_G}^{H-1} 2^{-h} + \frac{16^2 \pi^2 \beta_{T+1}^2}{G} \sum_{h=h_G}^{H-1} 2^h \leq \frac{32L^2}{G} + \frac{16 \pi^2 \beta_{T+1}^2}{G} 2^H. \quad (72)$$

In the first step we have used $N_{i,T+1}(h,k)$, since $L_2$ only counts allocations where $(H_i^t, K_i^t)$ were in $\bigcup_{h=h_G}^{H-1} I_h$; additionally, we have used (47) to bound $f_i(a_{it}) - \alpha_i$. In the second step, by the same reasoning used in the bound for $L_T^{11}$, we have $N_{i,T+1}(h,k) \leq 1 + \pi^2 \tau h_T$ for any node $(h,k)$; moreover, we have used (46) to bound the number of nodes in $I_h$. The remaining steps are obtained by algebraic manipulations. Finally, we bound $L_3$ as follows.

$$L_3 = \sum_{h=h_G}^{H} \sum_{(h,k) \in \mathcal{J}_h} \frac{8L}{2^h} N_{i,T+1}(h,k) \leq \sum_{h=h_G}^{H} \frac{8L}{2^h} \frac{8L}{2^H} \left(1 + \frac{4^h \pi^2 \beta_{T+1}^2}{L^2} 4^h \right)$$

$$\leq \frac{64L^2}{G} \sum_{h=h_G}^{H} 2^{-h} + \frac{256 \pi^2 \beta_{T+1}^2}{G} \sum_{h=h_G}^{H} 2^h \leq \frac{128L^2}{G} + \frac{512 \pi^2 \beta_{T+1}^2}{G} 2^H. \quad (73)$$
Above, in the first step we have used \( N_{it}(h, k) \) (instead of \( N_{it}'(h, k) \)), since \( \mathcal{L}_3 \) counts allocations where \((H_t^i, K_t^i)\) belonged to the descendants of \( \bigcup_{h=h_G}^{H-1} \mathcal{J}_h \); additionally, we have used (47) and the fact that parents of nodes in \( \mathcal{J}_h \) are in \( \mathcal{I}_{h-1} \) to bound \( f_i(\alpha_{it}) - \alpha_i \). In the second step, first we have used (46) to bound the number of nodes in \( \mathcal{J}_h \); to bound the number of evaluations in each such node, we have applied Lemma 35 along with the fact that \( \Delta_i(h, k) > 2L2^{-h} \) for nodes in \( \mathcal{J}_h \) by their definition; therefore,

\[
N_{i,T+1}(h, k) \leq 1 + \sigma^2 \max \left( \frac{\beta^2}{L^2} 4^h, \frac{4\beta^2}{L^2} \left( \Delta_i(h, k) - L2^{-h} \right) \right) \leq 1 + 4\sigma^2 \frac{\beta^2}{L^2} 4^h.
\]

The remaining steps in (73) are obtained via algebraic manipulations. Combining (71), (72), and (73) results in the following bound for \( L_T^{i5} \).

\[
L_T^{i5} \leq \frac{4L}{2^H} T + \frac{528\sigma^2 \beta^2}{G} 2^H + \frac{160L^2}{G} \leq C' \frac{L^{1/2} \sigma \beta T^{1/2}}{G^{1/2}} + \frac{160L^2}{G}.
\]

(74)

Here \( C' \) is a global constant. The last step is obtained by choosing \( H \) such that \( 2^H \approx \frac{G}{\sigma \beta T^{1/2}} \).

Bounding \( L_T^{i6} \): Our method for obtaining this bound follows along similar lines to the bound of \( L_T^{i5} \), but using the \( \mathcal{I}_h' \), \( \mathcal{J}_h' \) sets as defined in Appendix E.3. Therefore, we will outline the argument highlighting only the important differences.

As before, let \( H \geq h_G \) be a positive integer whose value will be determined shortly. Next, let \( \mathcal{N}_1 \) denote the descendants of \( \mathcal{I}_H' \); let \( \mathcal{N}_2 = \bigcup_{h=h_G}^{H-1} \mathcal{I}_h' \); and let \( \mathcal{N}_3 \) denote the descendants of \( \bigcup_{h=h_G}^{H-1} \mathcal{J}_h' \). It follows that \( L_T^{i6} \) can be bound as follows.

\[
L_T^{i6} \leq \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \quad \mathcal{L}_i = \sum_{t=2}^{T} \mathbb{1}(\eta_{it} = \tilde{\sigma}_{it} \land (H_t^i, K_t^i) \in \mathcal{N}_i \land H_t^i \geq h_G) \cdot (f_i(\tilde{\sigma}_{it}) - \alpha_i).
\]

We will now bound \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \). As in (71), we can bound \( \mathcal{L}_1 \leq \frac{4\beta}{2^H} T \). Denoting \( N_{it}^{rec'}(h, k) = \sum_{s=1}^{t-1} \mathbb{1}(\eta_{is} = \tilde{\alpha}_{is}, (H_{is}, K_{is}) = (h, k)) \), we bound \( \mathcal{L}_2 \) as shown below.

\[
\mathcal{L}_2 = \frac{4L}{2^H} \sum_{h=h_G}^{H-1} \sum_{(h,k) \in \mathcal{I}_h} N_{i,T+1}^{rec'}(h, k) \leq \frac{4L}{2^H} \sum_{h=h_G}^{H-1} \sum_{(h,k) \in \mathcal{I}_h} N_{i,T+1}^{rec}(h, k) \leq \frac{32L^2}{G} + \frac{16\sigma^2 \beta^2}{G} 2^H.
\]

The first step simply uses \( N_{it}^{rec'}(h, k) \leq N_{it}^{rec}(h, k) \), while the last step follows from the same calculations as in (72). Finally, we have the following bound for \( \mathcal{L}_3 \).

\[
\mathcal{L}_3 = \frac{8L}{2^{H-1}} \sum_{h=h_G}^{H} N_{i,T+1}^{rec}(h, k) \leq \frac{8L}{2^H} \sum_{h=h_G}^{H} \frac{8L}{2^H} \left( 1 + \frac{4\beta^2}{L^2} 4^h \right) \leq \frac{128L^2}{G} + \frac{512\sigma^2 \beta^2}{G} 2^H.
\]
Above, in the first step we have used the bounds on \( \alpha_i - f_i \) for points in \( \mathcal{J}'_h \) given at the end of Appendix E.3. To bound \( N_{it}^{ec}(h, k) \), we have applied Lemma 36 along with the fact that \( \Delta_i(h, k) > 2L2^{-h} \) for nodes in \( \mathcal{J}'_h \) by their definition; therefore,

\[
N_{it}^{ec}(h, k) \leq 1 + \sigma^2 \max \left( \frac{\beta^2}{L^2} 4^h, \frac{4\beta^2}{(\Delta_i(h, k) - L2^{-h})^2} \right) \leq 1 + \frac{4\sigma^2 \beta^2}{L^2} 4^h.
\]

The remainder of the calculations for bounding \( \mathcal{L}_3 \), are similar to (73). The expressions for the bounds for \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \) are identical to the bounds for \( L_{it}^{35} \), and hence we obtain the following bound for \( L_{it}^{46} \), which is the same as the RHS of (74). Here \( C' \) is a global constant.

\[
L_{it}^{46} \leq C' \frac{L_{i/2}^{1/2} \sigma_{T/2}^{1/2} G_{T/2}^{1/2}}{G} + \frac{160L^2}{G} \quad (75)
\]

The bound on the loss is obtained by combining (66), (67), (68), (69), (70), (74), and (75) and observing that \( \beta_{T/2} \leq \beta_{2T} \approx \sqrt{\log(nT/\delta)} \) (40).

**Fairness:** We will bound \( U_{it}^{e_i} - U_{iT} \) for our asymptotic fairness result in the following manner.

\[
U_{it}^{e_i} - U_{iT} \leq \sum_{t=1}^{T} \mathbb{1}(\eta_{it} = \overline{\sigma}_{it} \wedge \overline{\sigma}_{it} < \eta_i^*) \cdot (u_i(\eta_i^*) - u_i(\eta_{it}))
\]

\[
\leq \sum_{t=2}^{T} \mathbb{1}(\overline{\sigma}_{it} \in [0, \ell_i^t) \wedge \overline{\sigma}_{it} = \eta_{it}) L_{IT}^u \eta_{it} + \sum_{t=2}^{T} \mathbb{1}(\overline{\sigma}_{it} \in I_{hGk_i^t} \wedge \overline{\sigma}_{it} = \eta_{it}) L_{iT}^u \eta_{max}
\]

\[
+ \sum_{t=2}^{T} \mathbb{1}(\eta_{it} = \overline{\sigma}_{it} \wedge \overline{\sigma}_{it} \in [r_i^t, \eta_i^*)) \cdot L_{iT}^u (\eta_i^* - \overline{\sigma}_{it}).
\]

\[
\leq L_{IT}^u \eta_{max} (L_{IT}^{11} + L_{IT}^{12} + L_{IT}^{14}) + \frac{L_{IT}^u \eta_{max}}{G} L_{IT}^{46}.
\]

Here, the first step uses the bound for \( U_{it}^{e_i} - U_{iT} \) in Lemma 13. The second step decomposes the allocations in \([0, \eta_i^*]\) into the intervals \([0, \ell_i^t), [\ell_i^t, r_i^t), [r_i^t, \eta_i^*)\) and \([\eta_i^*, \eta_i^*]_{\mathcal{L}^*}\) and used \( L_{IT}^u \)-Lipschitzness of the utility functions. Moreover, for the first two summations, we have used the fact that \( u_i(\eta_i^*) - u_i(\eta_{it}) \leq L_{IT}^u (\eta_i^* - \eta_{it})^+ \leq L_{IT}^u \eta_{max} \). The last step is obtained by comparing the expressions for \( L_{IT}^{11}, L_{IT}^{12}, L_{IT}^{14}, L_{IT}^{46} \) in (65) and (66). The claim follows from the bounds in (67), (68), (70), and (75). □

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