Solving Linear Equations with Separable Problem
Data over Directed Networks

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Abstract—This paper deals with linear algebraic equations where the global coefficient matrix and constant vector are given respectively, by the summation of the coefficient matrices and constant vectors of the individual agents. Our approach is based on reformulating the original problem as an unconstrained optimization. Based on this exact reformulation, we first provide a gradient-based, centralized algorithm which serves as a reference for the ensuing design of distributed algorithms. We propose two sets of exponentially stable continuous-time distributed algorithms that do not require the individual agent matrices to be invertible, and are based on estimating non-distributed terms in the centralized algorithm using dynamic average consensus. The first algorithm works for time-varying weight-balanced directed networks, and the second algorithm works for general directed networks for which the communication graphs might not be balanced. Numerical simulations illustrate our results.

Index Terms—Linear algebraic equations; Distributed algorithms; Directed graphs.

I. INTRODUCTION

The importance of solving linear algebraic equations is paramount. They appear frequently in core mathematics as well as in applications, in physics and engineering. Nonlinear systems can often be well understood by their linear approximation. Due to the recent development of large-scale networks coupled with parallel processing power and fast communication capabilities, there is a growing effort aimed at developing distributed algorithms to solve systems of linear equations. Distributed algorithms preserve the privacy of the agents, are robust against single point of failures, and scale well with the network size. Keeping these considerations in mind, this paper is a contribution to the growing body of distributed algorithms to solve linear algebraic equations.

Literature Review: Justifying the ubiquity of linear equations, there is a vast and expanding literature to solve them efficiently, cf. [1]–[5] and references therein. However, most of the works consider the information structure where each agent knows some rows of the coefficient matrix and the constant vector. In those cases, the collective problem has a solution if and only if the individual equations are solvable. Instead, our approach here uses dynamic average consensus [8], [9] to estimate certain non-distributed terms in a gradient-based algorithm for the reformulated optimization problem. We also draw inspiration from [10], [11] on distributed optimization to extend our treatment to deal with unbalanced networks. However, unlike the aforementioned works where the desired solution is not an equilibrium of the dynamics, requiring a diminishing time-varying stepsize-like parameter to ensure convergence, here we make sure that any solution of the linear equation is indeed an equilibrium of the proposed dynamics. This enables us to employ Lyapunov stability analysis to establish convergence and offers a framework to study robustness against disturbances and errors. Our work [12] requires bidirectional 2-hop communication. In contrast, the distributed algorithms here require information exchange only with immediate neighbors and work for arbitrary directed graphs.

Statement of Contributions: We consider linear algebraic equations where the coefficient matrices and constant vector for the overall problem are given, respectively, by the summation of the individual agents’ coefficient matrices and constant vectors. Our starting point is the exact reformulation of this problem as a constrained optimization problem. Using the observation that the optimal value of this optimization is zero, the updates and prove stability. [5] uses element-wise average consensus for the coefficient matrix as well as the constant vector, which does not scale with either the problem dimension or the network size, and is not desirable from a privacy standpoint. [6] also exploits the positive definite property of the individual matrices and requires the agents to know the state as well as the matrices of the neighbors. [7] proposes a distributed algorithm without any positive definiteness condition, but agents are allowed to converge to different solutions. Our approach here uses dynamic average consensus [8], [9] to estimate certain non-distributed terms in a gradient-based algorithm for the reformulated optimization problem. We also draw inspiration from [10], [11] on distributed optimization to extend our treatment to deal with unbalanced networks. However, unlike the aforementioned works where the desired solution is not an equilibrium of the dynamics, requiring a diminishing time-varying stepsize-like parameter to ensure convergence, here we make sure that any solution of the linear equation is indeed an equilibrium of the proposed dynamics. This enables us to employ Lyapunov stability analysis to establish convergence and offers a framework to study robustness against disturbances and errors. Our work [12] requires bidirectional 2-hop communication. In contrast, the distributed algorithms here require information exchange only with immediate neighbors and work for arbitrary directed graphs.

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we reformulate it as optimization of an unconstrained function, and propose a centralized algorithm which works for weight-balanced networks and serves as a reference for the design of distributed algorithms. Using dynamic average consensus, we then propose a distributed algorithm that does not require the agent matrices to be positive definite, works for time-varying weight-balanced networks and is guaranteed to converge to a solution of the original problem exponentially fast. Building on the insights gained in establishing these results, we propose a distributed algorithm that is not limited to weight-balanced networks and is also guaranteed to converge to a solution of the linear equation exponentially fast.

II. PRELIMINARIES

Here we review basic notions from graph theory [10, 13], [14] and dynamic average consensus [8, 9].

Graph Theory: Let \( G = (V, E, A) \) be a weighted directed graph (or digraph), with \( V \) as the set of vertices (or nodes) and \( E \subseteq V \times V \) as the set of edges: \( (v_i, v_j) \in E \) iff there is an edge from node \( v_i \) to node \( v_j \). With \(|V| = n\), the adjacency matrix \( A \in \mathbb{R}^{n \times n} \) of \( G \) is such that \( A_{ij} > 0 \) if \((v_i, v_j) \in E\) and \( A_{ij} = 0\), otherwise. A directed path is an ordered sequence of vertices such that any pair of consecutive vertices is an edge. A digraph is strongly connected if there is a directed path between any two distinct vertices. The out- and in-degree of a node are, resp., the number of outgoing edges from and incoming edges to it. The weighted out-degree and weighted in-degree of a node \( v_i \) are \( d^{\text{out}}(v_i) = \sum_{j=1}^{n} A_{ij} \) and \( d^{\text{in}}(v_i) = \sum_{j=1}^{n} A_{ji} \), resp. The out-degree matrix \( D^{\text{out}} \in \mathbb{R}^{n \times n} \) and in-degree matrix \( D^{\text{in}} \in \mathbb{R}^{n \times n} \) are diagonal matrices defined as \( D^{\text{out}} = d^{\text{out}}(v_j) \) and \( D^{\text{in}} = d^{\text{in}}(v_i) \), resp. A graph is weight-balanced if \( D^{\text{out}} = D^{\text{in}} \). The Laplacian \( L \in \mathbb{R}^{n \times n} \) is \( L = D^{\text{in}} - A \). All eigenvalues of \( L \) have nonnegative real parts, 0 is simple with left eigenvector \( \vec{v} \) iff \( G \) is strongly connected, and \( L \vec{1} = 0 \) iff \( G \) is weight-balanced if \( L + L^{\top} \) is positive semidefinite, cf. [13, Theorem 1.37]. If \( G \) is strongly connected, it follows from [14, Lemma 3] that there exists a positive right eigenvector \( \vec{x} \in \mathbb{R}^{n} \) associated to 0.

Dynamic Average Consensus: Consider a group of \( n \in \mathbb{Z}_{>1} \) agents communicating over a weight-balanced digraph \( G \) with Laplacian \( L \). Each agent \( i \in \{1, \ldots, n\} \) has a state \( x_i \in \mathbb{R} \) and an input \( z_i \in \mathbb{R} \). The dynamic average consensus algorithm aims at making all the agents track the average \( \frac{1}{n} \sum_{i=1}^{n} x_i \) asymptotically. Here we present the algorithm following [8], where it was introduced for undirected graphs. Consider

\[
\dot{x} = -L x + \vec{z}.
\]

If \( \sum_{i=1}^{n} x_i(0) = \sum_{i=1}^{n} z_i(0) \) and the input \( z \) is bounded, then \( x_i(t) \rightarrow \frac{1}{n} \sum_{i=1}^{n} z_i(t) \) as \( t \rightarrow \infty \) for \( i \in \{1, \ldots, n\} \), cf. [8].

III. PROBLEM FORMULATION

Consider a group of \( n \) agents interacting over a digraph that seek to solve in a distributed way the linear algebraic equation

\[
\begin{pmatrix}
A \\
b
\end{pmatrix} x = \begin{pmatrix}
b_i
\end{pmatrix},
\]

where \( x \in \mathbb{R}^m \) is the unknown solution vector, and \( A_i \in \mathbb{R}^{m \times m} \) and \( b_i \in \mathbb{R}^m \) are the coefficient matrix and constant vector corresponding to agent \( i \in \{1, \ldots, n\} \). We assume that (1) has at least one solution. The formulation (1) includes, as a particular case, scenarios where each agent \( i \) knows only some rows of the coefficient matrix \( A \) and constant vector \( b \).

Our approach consists of first formulating (1) as a system involving \( n \) unknown solution vectors, one per agent, and then reformulating it as a convex optimization problem. Based on this reformulation, we propose two sets of (out-)distributed algorithms (where each agent only needs information from its out-neighbors) to find the solutions of (1). We start by endowing each agent with its own version \( x_i \in \mathbb{R}^m \) of \( x \).

Then (1) can be equivalently written as

\[
\sum_{i=1}^{n} A_i x_i = \sum_{i=1}^{n} b_i,
\]

\[
x_i = x_j \quad \forall i, j.
\]

Equation (2b) ensures that \( x_i = x \) for all the agents. Clearly the set of equations (2) and the original problem (1) are equivalent. Next we formulate (2) as a convex optimization problem. Consider the quadratic function \( f : \mathbb{R}^{mn} \rightarrow \mathbb{R} \)

\[
f(x) = \left( \sum_{i=1}^{n} (A_i x_i - b_i) \right) \top \left( \sum_{i=1}^{n} (A_i x_i - b_i) \right),
\]

which is convex and attains its minimum over the solution set of (2a). For convenience, we use \( L = L \otimes I \) and \( f(x) = (A x - b) \top (A x - b) \), where \( I \) denotes the block-diagonal matrix obtained after putting the matrices \( \{A_i\}_{i=1}^{n} \) along the principal diagonal, and \( b = [b_1; \ldots; b_n] \in \mathbb{R}^{mn} \). If \( G \) is strongly connected, the solutions of (2) are the same as the optimizers of

\[
\min_{x} f(x)
\]

s.t.

\[
L \top x = 0.
\]

Remark 1. (Distributed algorithmic solutions to optimization problem): The problem (3) can be solved over an undirected graph by reformulating it using the techniques in [15] and employing the saddle-point dynamics, cf. [16, 17]. These dynamics involve terms of the form \( L \top \) and, to be implemented over a digraph, would need information from in- as well as out-neighbors and hence are not suitable for our setup. Works that deal with distributed optimization under consensus constraints over digraphs, see e.g. [10, 18] and references therein, require the objective function to be separable, and therefore are not applicable either here.

IV. DISTRIBUTED ALGORITHMS OVER WEIGHT-BALANCED NETWORKS

We present distributed algorithms to solve (1) over weight-balanced networks.

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A. Centralized Algorithm

We first introduce a centralized algorithm using the fact that the objective function vanishes at the optimizers of (1). Let

$$\min_x \frac{1}{2} \alpha x^T (L + L^T) x + \beta f(x),$$

(4)

where \(\alpha, \beta > 0\). Clearly, (3) and (4) have the same set of solutions if \(G\) is strongly connected and weight-balanced. Since problem (4) is unconstrained, one can use gradient descent to find its optimizers. However, the gradient

$$-\alpha(L + L^T)x - \beta A^T \mathbb{1} \mathbb{1}^T (Ax - b)$$

(5)

of the objective function in (4) involves terms with \(L^T\), whose computation would require information from in-neighbors. Instead, we consider the following gradient-based dynamics

$$\dot{x} = -\alpha L x - \beta A^T \mathbb{1} \mathbb{1}^T (Ax - b).$$

(5)

Whenever convenient, we refer to (5) as \(\psi_{grad}\). Note that the first term in the dynamics (5) is distributed, meaning that each agent can implement it with information from its out-neighbors. The second term, however, requires collective information from all the agents because of the summation across the network. Nevertheless, this algorithm serves as the basis for our distributed algorithm design in the next section.

The next result formally characterizes the equivalence between the equilibria of (3) and the solutions of (1).

**Lemma IV.1.** (Equivalence between (3) and (1)): Let \(G\) be a strongly connected and weight-balanced digraph. Then for all \(\alpha, \beta \in \mathbb{R}_{>0}\), \(x^*\) is an equilibrium of (3) if and only if \(x^* = 1 \otimes x^\ast\), where \(x^\ast \in \mathbb{R}^m\) solves (1).

**Proof:** The implication from right to left is immediate. To prove the implication in the other direction, let \(\bar{x} \in \mathbb{R}^n\) be a solution of (1) and consider \(\bar{x} = 1 \otimes \bar{x}\). Since \(x^\ast\) and \(\bar{x}\) are equilibria of (3),

$$\alpha L (x^\ast - \bar{x}) + \beta A^T \mathbb{1} \mathbb{1}^T A (x^\ast - \bar{x}) = 0.$$

(6)

Let \(Q_{11} = \frac{1}{2} \alpha (L + L^T) + \beta A^T \mathbb{1} \mathbb{1}^T A\). Then (6) implies

$$(x^\ast - \bar{x})^T Q_{11} (x^\ast - \bar{x}) = 0.$$

Since \(G\) is weight-balanced, \((L + L^T) \succeq 0\). This along with the fact that \(A^T \mathbb{1} \mathbb{1}^T A \succeq 0\) implies \(L^T (x^\ast - \bar{x}) = 0\) and \(\mathbb{1}^T A (x^\ast - \bar{x}) = 0\). Therefore, \(x^\ast = 1 \otimes x^\ast\), for some \(x^\ast \in \mathbb{R}^m\) which satisfies \(Ax^\ast = A\bar{x}\), as claimed.

The next result characterizes the convergence of (5).

**Proposition IV.2.** (Exponential stability of (5)): Let \(G\) be a strongly connected and weight-balanced digraph. Then for all \(\alpha, \beta \in \mathbb{R}_{>0}\), any trajectory of (5) converges exponentially to a point of the form \(x^\ast = 1 \otimes x^\ast\), where \(x^\ast \in \mathbb{R}^m\) solves (1).

**Proof:** Consider a vector \(w \in \mathbb{R}^m\) in the null space of \(Q_{11}\). Using the same line of arguments as in the proof of Lemma IV.1, this implies that \(L^T w = 0\) and \(\mathbb{1}^T A w = 0\). Therefore, along (5),

$$\dot{x}^T w = -(\alpha x^T L^T + \beta(A x - b)^T \mathbb{1} \mathbb{1}^T A) w = 0.$$

This means that the dynamics (5) are orthogonal to the null space of \(Q_{11}\) and hence the component of \(x\) in the null space of \(Q_{11}\), say \(x_{null}\), remains constant. Given the initial condition \(x(0)\), consider the particular equilibrium \(x^\ast\) of (5) satisfying \(x_{null}^\ast = x_{null}(0)\). Since different equilibria differ only in their null space component, \(x^\ast\) defined this way is unique. Consider the Lyapunov function candidate \(V : \mathbb{R}^{mn} \to \mathbb{R}\)

$$V(x) = \frac{1}{2}(x - x^\ast)^T (x - x^\ast).$$

The Lie derivative of \(V\) along the dynamics (5) is given by

$$\mathcal{L}_{\psi_{grad}} V = -(x - x^\ast)^T (\alpha L x + \beta A^T \mathbb{1} \mathbb{1}^T (Ax - b))$$

$$= -(x - x^\ast)^T Q_{11} (x - x^\ast) \leq -2\alpha_2 (Q_{11}) V.$$

The last inequality follows from applying the Courant-Fischer theorem [19, Theorem 4.2.11] together with the fact that \((x - x^\ast)^T w = 0\) as \(x_{null}\) is constant. Using the monotonicity theorem [19, Corollary 4.3.3], we further have

$$\mathcal{L}_{\psi_{grad}} V \leq -2 \min \left\{\frac{1}{2} \alpha_2 (L + L^T), \beta_2 (A^T \mathbb{1} \mathbb{1}^T A)\right\} V.$$

Hence, the dynamics (5) is exponentially stable with a rate depending on \(\alpha, \beta, L\) and \(\{|A_i|\}_{i=1}^n\).

B. Distributed Algorithm

We present a distributed algorithm to find a solution of (1), which is based on the centralized algorithm (5) and involves employing dynamic average consensus (cf. Section II) to estimate the aggregate \(\mathbb{1}^T (Ax - b)\). Formally,

$$\dot{x} = -\alpha L x - n \beta A^T y,$$

(7a)

$$\dot{y} = -\alpha A L x - n \beta A A^T y - \gamma L y,$$

(7b)

with design parameter \(\gamma > 0\). Here, each agent \(i \in \{1, \ldots, n\}\) updates \(y_i \in \mathbb{R}\) which estimates the average mismatch \(\frac{1}{n} \mathbb{1}^T (Ax - b)\). The dynamics (7) is distributed as each agent just needs to know its state and that of its out-neighbors. Whenever convenient, we refer to it as \(\psi_{grad}\). The following result characterizes the equilibria of (7) and shows that the total deviation from the average mismatch is conserved.

**Lemma IV.3.** (Equilibria of (7) and invariance of total deviation): Let \(G\) be a strongly connected and weight-balanced digraph. Then, if \((x^\ast, 0)\) is an equilibrium of (7) then \(x^\ast = 1 \otimes x^\ast\), where \(x^\ast \in \mathbb{R}^m\). Moreover, for all \(\alpha, \beta, \gamma \in \mathbb{R}_{>0}\), \(\mathbb{1}^T (y - Ax)\) remains constant along the evolution of (7).

**Proof:** Let \((x^\ast, 0)\) be an equilibrium of (7). From (7a), it follows that \(L x = 0\), and hence \(x^\ast = 1 \otimes x^\ast\) for some \(x^\ast \in \mathbb{R}^m\), establishing the first statement. Now, consider the derivative \(\mathbb{1}^T \dot{y} - A x = -\gamma \mathbb{1}^T L y = 0\). Hence, \(\mathbb{1}^T (y - Ax)\) is conserved along the evolution of (7).

**Remark 2.** (Distributed initialization of the \(\psi_{grad}\) algorithm): From Lemma IV.3, we observe that in order for a trajectory of (7) to converge to an equilibrium of the form \((x^\ast, y^\ast) = (1 \otimes x^\ast, 0)\), where \(x^\ast \in \mathbb{R}^m\) solves (1), its initial condition must satisfy \(\mathbb{1}^T y(0) = \mathbb{1}^T (Ax(0) - b)\). This could be
implemented in a distributed way if each agent $i \in \{1, \ldots, n\}$ chooses its initial states satisfying $y_i(0) = A_i x_i(0) - b_i$. One trivial selection, for example, is $x(0) = 0$ and $y(0) = -b$.

The next result characterizes the convergence of (7).

**Theorem IV.4.** (Exponential stability of (7) over balanced networks: Let $G$ be a strongly connected and weight-balanced digraph and assume null($A$) $\subseteq$ null($A_i$), for all $i \in \{1, \ldots, n\}$. Let $\alpha, \beta \in \mathbb{R}_{>0}$ and define

$$\tilde{\gamma} = \max \left\{ \frac{2}{\lambda_2(L + L^T)} \lambda_{\text{max}} \left( Q_{12} \right), 0 \right\},$$

where $Q_{11} = \frac{1}{2} \alpha (L + L^T) + \beta A A^T$ and $Q_{12} = \frac{1}{2} (n \beta A^T + \beta A^T A A^T + \beta A A^T L^T)$. Then, for all $\gamma \in (\tilde{\gamma}, \infty)$, any trajectory of (7) with initial condition satisfying $1^T y(0) = 1^T (A x(0) - b)$ converges exponentially to $(x^*, 0)$, where $x^* = 1 \otimes x^*$ and $x^*$ $\in \mathbb{R}^m$ solves (1).

**Proof:** Define the error variable

$$e = y - \frac{1}{n} 1 1^T (A x - b),$$

measuring the difference between the agents’ estimates and the actual value of average mismatch. Note that

$$\dot{e} = -\alpha \Pi L x - n \beta \Pi A A^T y - \gamma L y,$$

where $\Pi = I - \frac{1}{n} 1 1^T$. Rewriting (7) in terms of $x$ and $e$, we get

$$\dot{x} = -\alpha L x - x - \beta A 1 1^T (A x - b) - n \beta A e,$$

(9a)

$$\dot{e} = -\alpha \Pi L x - x - \beta \Pi A A^T 1 1^T (A x - b) - n \beta \Pi A A^T e - \gamma L e.$$  

(9b)

From the proof of Proposition IV.2, we know that if $w \in \mathbb{R}^{mn}$ is in the null space of $Q_{11}$, then $w^T \dot{w} = 0$ and $1^T A w = 0$. Therefore, $w = 1 \otimes w$, where $w \in \mathbb{R}^m$ belongs to $w \in \text{null}(A)$. By hypothesis, $A_i w = 0$ for all $i \in \{1, \ldots, n\}$. Moreover, from (9a), $x^T w = 0$, and the $x$ component of the equilibrium $(x^*, y^*)$ of (7) satisfies $x_{\text{null}} = x(0)_{\text{null}}$ and is unique. With the initialization of the statement, it follows from Lemma IV.3 that $y^* = 1 \otimes \frac{1}{n} 1^T (A x^* - b)$. Substituting this value of $y^*$ in (7) and following the proof of Lemma IV.1, one can establish that the corresponding equilibrium is of the form $(1 \otimes x^*, 0)$, where $x^* \in \mathbb{R}^m$ is a solution of (1). Consider the Lyapunov function candidate $V_2 : \mathbb{R}^{2mn} \to \mathbb{R}$

$$V_2(x, e) = \frac{1}{2} (x - x^*)^T (x - x^*) + \frac{1}{2} e^T e.$$

The Lie derivative of $V_2$ along (9) is given by

$$\mathcal{L}_{\dot{\psi}_{\text{sys}}} V_2 = -e^T (\alpha L x + \beta A 1 1^T (A x - b)) - n \beta (x - x^*)^T A e - \gamma L e,$$

which is negative definite.

**Remark 3.** (Lower bound on $\gamma$): The lower bound $\tilde{\gamma}$ in Theorem IV.4 is conservative in general. In fact, the algorithm may converge even if this condition is not satisfied, something that we have observed in simulation. Note also that although $\alpha$ and $\beta$ are free parameters, they should still be carefully chosen as $\tilde{\gamma}$ depends on them.

The result above can be extended to time-varying networks. In case $G(t)$ is time-varying, the algorithm in (7) reads as

$$\dot{x} = -\alpha L(t) x - n \beta A A^T y,$$

(10a)

$$\dot{y} = -\alpha A L(t) x - n \beta A A^T y - \gamma L(t) y.$$  

(10b)

The next result formally characterizes the convergence of (10). Its proof is similar to that of Theorem IV.4 and hence omitted.

**Theorem IV.5.** (Exponential stability of (10) over time-varying balanced networks: Let $\{G(t)\}_{t=0}^\infty$ be a sequence of strongly connected and weight-balanced digraphs with uniformly bounded edge weights (i.e., there exists $a \in (0, \infty)$ such that $A_{ij}(t) < a$ for all $(i, j)$ and $t \geq 0$), and assume null(A) $\subseteq$ null(A_i), for all $i \in \{1, \ldots, n\}$. Let $\alpha, \beta \in \mathbb{R}_{>0}$ and define $\gamma(t)$ as

$$\gamma(t) = \frac{2}{\lambda_2(L(t) + L^T)} \lambda_{\text{max}} \left( Q_{12}(t) \right),$$

where $Q_{11}(t) = \frac{1}{2} \alpha (L(t) + L^T) + \beta A A^T$ and $Q_{12}(t) = \frac{1}{2} (n \beta A^T + \beta A A^T L^T)$. Then, we have the fact that due to the mentioned initialization, $1^T e = 0$ from Lemma IV.3. Since $x_{\text{null}}$ is constant, $(x - x^*)^T w = 0$ and from the Courant-Fischer theorem [1] Theorem 4.2.11, (x - x^*)^T Q_{11} (x - x^*) \leq -\lambda_2(Q_{11}) (x - x^*)^T (x - x^*)$.

Also, since $1^T e = 0$ and $G$ is weight-balanced, it again follows from the Courant-Fischer theorem that

$$-e^T Q_{22} e \leq -\frac{1}{2} \gamma L_2(L(t) + L^T) e - n \beta A A^T e.$$  

Therefore, we can upper bound the Lie derivative as

$$\mathcal{L}_{\dot{\psi}_{\text{sys}}} V_2 \leq -e^T Q_{22} e \leq -\frac{1}{2} \gamma L_2(Q_{11}) I - n \beta A A^T - \frac{1}{2} \lambda_2(Q_{11}) Q_{12} Q_{22} > 0.$$  

Hence, $\gamma(t) > 0$ if $\gamma > \tilde{\gamma}$, and $\mathcal{L}_{\dot{\psi}_{\text{sys}}} V_2 \leq -2 \lambda_{\text{min}}(Q) V_2$.

The null space condition in Theorem IV.4 makes sure that $x_{\text{null}}$ remains invariant along the evolution of (7) and all the agents approach the solution of (1) closest to $x(0)$. This condition is automatically satisfied if the matrix $A$ is full rank, or in other words, equation (1) has a unique solution. We believe (and simulations also suggest) that if this condition is not satisfied, the $x$ component of the dynamics still converges to a solution of (1).

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Then for all $\gamma \in (\bar{\gamma}, \infty)$, where $\bar{\gamma} = \sup_{t \geq 0} \gamma(t)$, any trajectory of (10) with initial conditions $1^T y(0) = 1^T (A x(0) - b)$ converges exponentially to $(x^*, 0)$, where $x^* = 1 \otimes x^*$ and $x^* \in \mathbb{R}^m$ solves (11).

V. DISTRIBUTED ALGORITHM OVER UNBALANCED NETWORKS

In this section, we extend our approach to solve problem (11) over graphs that are not necessarily balanced. In those scenarios, since $L \neq 0$, the one-to-one correspondence between the desired equilibria of (3) or (7) and the solutions of (11) does not hold anymore. To overcome this, we propose

\[
\begin{align*}
\dot{x} &= -\alpha L V x - \beta A^T y, \quad \text{(11a)} \\
\dot{y} &= -\alpha A L V x - \beta A A^T y - \gamma L V y, \quad \text{(11b)} \\
\dot{v} &= -L V, \quad \text{(11c)}
\end{align*}
\]

where $V = \text{diag}(\bar{v})$, $\bar{v} = 1 \otimes \bar{v}$, and $\bar{v}$ is a positive right eigenvector with eigenvalue 0 of $L$. Exponential stability of (11) can be established by interpreting $L \text{-diag}(\bar{v})$ as the Laplacian of a weight-balanced graph and then following the same steps as in the proof of Theorem [V.4] but we omit it here for reasons of space. Although (11) is distributed, it assumes that agents have a priori knowledge of the corresponding entries of $\bar{v}$ which might be limiting in practice. To deal with this limitation, we propose an algorithm that does not require such knowledge by augmenting (11) with an additional dynamics converging to $\bar{v}$,

\[
\begin{align*}
\dot{x} &= -\alpha L V x - \beta A^T y, \quad \text{(12a)} \\
\dot{y} &= -\alpha A L V x - \beta A A^T y - \gamma L V y, \quad \text{(12b)} \\
\dot{v} &= -L V, \quad \text{(12c)}
\end{align*}
\]

where $V = \text{diag}(\bar{v})$. Whenever convenient, we refer to dynamics (12) as $\psi_{\text{dist}}$. Note that, unlike all the dynamics discussed so far, $\psi_{\text{dist}}$ is nonlinear.

Remark 4. (Distributed nature of (12)): The dynamics (12) is out-distributed, but requires each agent $i \in \{1, \ldots, n\}$ to have knowledge of its in-degree because $L = D^m - A$ and the graph is not weight-balanced. If we use instead the out-Laplacian $L = D^\text{out} - A$, then one could still define an equivalent algorithm for (11) with $L \text{V}$ replaced by $V L$, but (12c) would look like $\dot{v} = -L^T v$, which would require state information from in-neighbors too.

The next result characterizes the convergence of (12).

Theorem V.1. (Exponential stability of (12) over unbalanced networks): Let $G$ be a strongly connected digraph and assume null($A$) $\subseteq$ null($A_1$), for all $i \in \{1, \ldots, n\}$. Let $\alpha, \beta \in \mathbb{R}_{>0}$ and define

\[
\bar{\gamma} = \max \left\{ \frac{2}{\lambda_3(L^+ V + V L^+)} \lambda_{\text{max}}(Q_{11}^2 Q_{12}^2 - \beta A A^T), 0 \right\},
\]

where $Q_{11} = \frac{1}{2}(\alpha L V + V L^+) + \beta A^T I I^T A$, $Q_{12} = \frac{1}{2}(n \beta A^T + \alpha V L^+ A^T + \beta A^T I I^T A)$, $\bar{v}$ is the positive eigenvector with eigenvalue 0 of $L$ satisfying $1^T \bar{v} = 1$, and $V = \text{diag}(\bar{v})$. Then, for all $\gamma \in (\bar{\gamma}, \infty)$, any trajectory of (12) with initial condition satisfying $1^T y(0) = 1^T (A x(0) - b)$ and $v(0) = \frac{1}{n} 1$, converges exponentially to $(x^*, 0, v)$, where $x^* = 1 \otimes x^*$ and $x^* \in \mathbb{R}^m$ solves (11), and $\bar{v} = 1 \otimes \bar{v}$.

Proof: From [11, Proposition 2.2], we have that $v(t) > 0$ for all $t \geq 0$. Also, since $1^T L = 0$, $1^T v$ is conserved along the evolution of (12c). Hence $v(t) \to \bar{v}$ exponentially fast with a rate determined by the non-zero eigenvalue of $L$ with the smallest real part. Let us interpret the dynamics (12a), (12b) as the dynamics (11) with some disturbance $d(t)$ defined by

\[
d = \left[ \begin{array}{c} d^\alpha \\ -\alpha L(V - \bar{V}) x \\ -\alpha A L(V - \bar{V}) x - \gamma L(V - \bar{V}) y \end{array} \right],
\]

which goes to 0 as $t \to \infty$. Consider a vector $w \in \text{null}(Q_{11})$. Then as in the proof of Theorem [V.4], $w \in \text{null}(A_i)$ and by hypothesis, $A_i w = 0$ for all $i \in \{1, \ldots, n\}$. Since $1^T L = 0$, therefore, $w^T d^\alpha = 0$ and we still have $w^T x = 0$, and the $x$ component of the equilibrium $(x^*, y^*, v^*)$ of (12) satisfies $x^*_\text{null} = x(0)_{\text{null}}$ and is unique. With the initialization of the statement and following the same steps as in the proof of Lemma [V.3], one can establish that $y^* = 1 \otimes \bar{v} = 1^T (A x^* - b)$. Substituting this value of $y^*$ in (12b) and following the proof of Lemma [V.1], one can establish that the corresponding equilibrium is of the form $(1 \otimes x^*, 0, \bar{v})$, where $x^* \in \mathbb{R}^m$ is a solution of (11). Consider now the Lyapunov function candidate $V_3 : \mathbb{R}^{3mn} \to \mathbb{R}$

\[
V_3(x, e, v) = V_2(x, e) + \frac{\delta}{2} (v - \bar{v})^T P(v - \bar{v}),
\]

where $\delta > 0$, $P = \bar{V}^{-1}$, $e$ is defined as in (8), and $V_2$ is the same function as in the proof of Theorem [V.4]. The Lie derivative of $V_3$ along (12) is given by

\[
\begin{align*}
\mathcal{L}_{\psi_{\text{dist}}} V_3 &= - \left[ (x - x^*)^T Q_{11}^2 Q_{12}^2 \left( \frac{1}{n} x \right) e \\ + (x - x^*)^T d^\alpha \right] \\
&\quad + e^T d^\alpha - (v - \bar{v})^T (L^T P + P L)(v - \bar{v}),
\end{align*}
\]

where $d^\alpha = -\alpha \Pi A L(V - \bar{V}) x - \gamma L(V - \bar{V}) e$, and $Q_{22} = \frac{1}{2}(L V + V L^+) + n \beta A A^T$. Interestingly, $L \text{V}$ can be interpreted as the Laplacian of a weight-balanced graph and as a result, $L V + V L^+ \succeq 0$ implying that $L^T P + P L \succeq 0$. Once again, following Lemma [V.1], one can establish that with the initialization of the statement, $1^T e = 0$ and therefore using the Courant-Fischer theorem [19, Theorem 4.2.11] together with the fact that $(x - x^*)^T w = 0$ due to invariance of $x_{\text{null}}$, we can upper bound the Lie derivative as

\[
\begin{align*}
\mathcal{L}_{\psi_{\text{dist}}} V_3 &\leq - \left[ (x - x^*)^T \left( \frac{\lambda_2(Q_{11})}{Q_{12}} \right) I_I^T Q_{12}^2 \right] \left[ (x - x^*)^T e \right] \\
&\quad + \alpha \Pi A L \|v - \bar{v}\| (\|x - x^*\| + \|x^*\|) \\
&\quad + \gamma \|e\| \Pi A L \|v - \bar{v}\| (\|x - x^*\| + \|x^*\|) \\
&\quad + \gamma \|e\| \Pi A L \|v - \bar{v}\| \|x - x^*\|,
\end{align*}
\]

\[\pi_2(\gamma, \infty) \quad \Box\]

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hypotheses of [21, Theorem 4.10] for exponential stability. This along with the fact that (12a)-(12b) and (11) are essentially the same. For algorithm (12), exponential convergence of algorithms (5) and (7)

\[ z^*(t) = \frac{1}{\sqrt{\lambda_{\min}(Q)}} z(0) \] 

for weight-balanced graphs, and (11) for unbalanced graphs evolves exponentially fast. Future work will involve formally characterizing the convergence when the null space condition is not satisfied, and explore the design of distributed algorithms for finding least-square solutions when exact ones do not exist, extension to cases where the problem data is time-varying, and the communication graph is unbalanced and time-varying.

VII. CONCLUSIONS AND FUTURE WORK

We have presented continuous-time algorithms to solve linear algebraic equations whose problem data is represented as the summation of the data of individual agents. The proposed algorithms are distributed over general directed networks, do not require the individual agent matrices to be positive definite, and are guaranteed to converge to a solution of the linear equation exponentially fast. Future work will involve formally characterizing the convergence when the null space condition is not satisfied, and explore the design of distributed algorithms for finding least-square solutions when exact ones do not exist, extension to cases where the problem data is time-varying, and the communication graph is unbalanced and time-varying.

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