DECOMPOSING ELEMENTS OF A RIGHT SELF-INJECTIVE RING

FEROZ SIDDIQUE AND ASHISH K. SRIVASTAVA

Abstract. It was proved independently by both Wolfson [An ideal theoretic characterization of the ring of all linear transformations, Amer. J. Math. 75 (1953), 358-386] and Zelinsky [Every Linear Transformation is Sum of Non-singular Ones, Proc. Amer. Math. Soc. 5 (1954), 627-630] that every linear transformation of a vector space \( V \) over a division ring \( D \) is the sum of two invertible linear transformations except when \( V \) is one-dimensional over \( \mathbb{Z}_2 \).

This was extended by Khurana and Srivastava [Right self-injective rings in which each element is sum of two units, J. Algebra and its Appl., Vol. 6, No. 2 (2007), 281-286] who proved that every element of a right self-injective ring \( R \) is the sum of two units if and only if \( R \) has no factor ring isomorphic to \( \mathbb{Z}_2 \).

In this paper we prove that if \( R \) is a right self-injective ring, then for each element \( a \in R \) there exists a unit \( u \in R \) such that both \( a + u \) and \( a - u \) are units if and only if \( R \) has no factor ring isomorphic to \( \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \).

All our rings are associative with identity element 1. Following Vámos \[19\], an element \( x \) in a ring \( R \) is called a \( k \)-good element if \( x \) can be expressed as the sum of \( k \) units in \( R \). A ring \( R \) is called a \( k \)-good ring if each element of \( R \) is a \( k \)-good element. Many authors including Chen \[2\], Ehrlich \[4\], Henriksen \[10\], Fisher-Snider \[5\], Khurana - Srivastava \([13, 14]\), Raphael \[17\], Vámos \([19, 20]\), Wiegand \[20\] and Wang - Zhou \[21\] have studied rings generated additively by their unit elements, in particular, 2-good rings. We refer the readers to \[18\] for a survey of rings generated by units.

In \[16\] a ring \( R \) is said to be a twin-good ring if for each \( x \in R \) there exists a unit \( u \in R \) such that both \( x + u \) and \( x - u \) are units in \( R \). Clearly every twin-good ring is 2-good. However, there are numerous examples of 2-good rings which are not twin-good. For example, \( \mathbb{Z}_3 \) is 2-good but not twin-good. We denote by \( J(R) \), the Jacobson radical of ring \( R \) and by \( U(R) \), the group of units of \( R \).

The following observations were noted in \[16\]. Their proofs are straightforward.

Lemma 1. If \( D \) is a division ring such that \( |D| \geq 4 \), then \( D \) is twin-good.

Lemma 2. For a ring \( R \), we have the following:

(i) If \( R \) is twin-good then for any proper ideal \( I \) of \( R \), the factor ring \( R/I \) is also twin-good.

(ii) If a factor ring \( R/I \) is twin-good and \( I \subseteq J(R) \), then \( R \) is twin-good. Thus, in particular, it follows that a ring \( R \) is twin-good if and only if \( R/J(R) \) is twin-good.

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(iii) If $R$ is a direct product of rings $R_i$ where each $R_i$ is a twin-good ring, then $R$ is also a twin-good ring.

1. Main Results

A ring $R$ is called right self-injective if each right $R$-homomorphism from any right ideal of $R$ to $R$ can be extended to an endomorphism of $R$. As the ring of linear transformations is a right self-injective ring, the result of Wolfson and Zelinsky attracted quite a bit of attention toward understanding which right self-injective rings are 2-good.

**Theorem 3.** (Vámos [19]) A right self-injective ring $R$ is 2-good if $R$ has no non-zero corner ring that is Boolean.

Khurana and Srivastava [13] extended the result of Wolfson and Zelinsky to the class of right self-injective rings and proved the following

**Theorem 4.** (Khurana, Srivastava [13]) A right self-injective ring $R$ is 2-good if and only if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$.

We will prove an analogue of this result for twin-good rings. But, first we have some definitions and useful lemmas.

We say that an $n \times n$ matrix $A$ over a ring $R$ admits a diagonal reduction if there exist invertible matrices $P, Q \in M_n(R)$ such that $PAQ$ is a diagonal matrix. Following Ara et. al. [1], a ring $R$ is called an elementary divisor ring if every square matrix over $R$ admits a diagonal reduction. This definition is less stringent than the one proposed by Kaplansky in [11]. The class of elementary divisor rings includes unit-regular rings and von Neumann regular right self-injective rings (see [1], [9]).

If $R$ is an elementary divisor ring, then clearly the matrix ring $M_n(R)$ is 2-good for each $n \geq 2$. In the case of twin-good rings, we have the following

**Lemma 5.** Let $R$ be an elementary divisor ring. Then the matrix ring $M_n(R)$ is twin-good for each $n \geq 3$.

**Proof.** Let $n \in N$ such that $n \geq 3$. Let $M$ be any arbitrary element of $M_n(R)$. Then there exist invertible matrices $E, F \in M_n(R)$ such that $EMF$ is a diagonal matrix. Set $A = EMF$. Then $A \in M_n(R)$ is a diagonal matrix. Suppose

$$A = \begin{bmatrix}
    d_1 & 0 & 0 & \cdots & 0 & 0 \\
    0 & d_2 & 0 & \cdots & 0 & 0 \\
    0 & 0 & d_3 & \cdots & 0 & 0 \\
    0 & 0 & 0 & d_4 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & d_{n-1} & 0 \\
    0 & 0 & 0 & \cdots & 0 & d_n
\end{bmatrix}.$$  

We consider the first $(n-1)$ columns of the first row of $A$ and call it $P$. Thus $P$ is a $1 \times (n-1)$ matrix given by $P = \left[ \begin{array}{cccc} d_1 & 0 & 0 & \cdots & 0 \end{array} \right]$. Similarly we consider the last $(n-1)$ rows of the last column of $A$ and call it $Q$. Thus $Q$ is a $(n-1) \times 1$
matrix given by \( Q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ d_n \end{bmatrix} \). Now we consider the lower left \((n-1) \times (n-1)\) block in \( A \) and call it \( B \). Thus \( B = \begin{bmatrix} 0 & d_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & d_4 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & d_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n d_1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \).

Let \( T = QP + I_{(n-1)} \). Then \( T = \begin{bmatrix} O_1 & 1 \\ T & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_n d_1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in M_{n-1}(R) \).

Now we create an \( n \times n \) matrix \( U = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & d_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & d_3 & \cdots & 0 & 0 \\ 0 & 0 & 1 & d_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n d_1 & 0 & 0 & \cdots & 1 & d_n \end{bmatrix} \in M_n(R) \), where \( a_{i,j} = 1, a_{n-1,2} = -d_n d_1, a_{n,1} = 1, \) and \( a_{i,j} = 0 \) elsewhere.

Clearly \( U \) is a unit in \( M_n(R) \) whose inverse is given by \( U^{-1} = [a_{i,j}] \), where \( a_{i,i+1} = 1, a_{n-1,2} = -d_n d_1, a_{n,1} = 1, \) and \( a_{i,j} = 0 \) elsewhere.

Now we consider the matrices \( A + U, A - U \) in \( M_n(R) \) which are of the form

\[
\begin{bmatrix}
  d_1 & 0 & 0 & \cdots & 0 & 1 \\
  1 & d_2 & 0 & \cdots & 0 & 0 \\
  0 & 1 & d_3 & \cdots & 0 & 0 \\
  0 & 0 & 1 & d_4 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  d_n d_1 & 0 & 0 & \cdots & 1 & d_n \\
\end{bmatrix}
\begin{bmatrix}
  d_1 & 0 & 0 & \cdots & 0 & -1 \\
 -1 & d_2 & 0 & \cdots & 0 & 0 \\
 0 & -1 & d_3 & \cdots & 0 & 0 \\
 0 & 0 & -1 & d_4 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 -d_n d_1 & 0 & 0 & \cdots & -1 & d_n \\
\end{bmatrix}
\]

respectively.

It can easily be checked that \( A + U \) and \( A - U \) are invertible matrices. Thus we have shown that there exists an invertible matrix \( U \in M_n(R) \) such that both \( A + U \) and \( A - U \) are invertible matrices. Clearly \( E^{-1} UF^{-1} \) is also invertible in \( M_n(R) \) such that both \( E^{-1} AF^{-1} + E^{-1} UF^{-1} \) and \( E^{-1} AF^{-1} - E^{-1} UF^{-1} \) are invertible. Thus it follows that \( M \) is twin-good. Hence the matrix ring \( M_n(R) \) is twin-good for each \( n \geq 3 \). \( \square \)
It follows from the result of Wolfson and Zelinsky that any proper matrix ring \( \mathbb{M}_n(D) \) is 2-good where \( D \) is a division ring and \( n \geq 2 \). For twin-good rings, we have the following.

**Lemma 6.** If \( R \) is an abelian regular ring, then \( \mathbb{M}_2(R) \) is twin-good.

**Proof.** Let \( A \) be an arbitrary element of \( \mathbb{M}_2(R) \). As \( R \) is an elementary divisor ring, there exist invertible matrices \( P, Q \in \mathbb{M}_2(R) \) such that

\[
P AQ = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}
\]

for some \( a, b \in R \). Since \( R \) is abelian regular, there exist \( u, v \in U(R) \) and central idempotents \( e_1, e_2 \in R \) such that \( a = e_1 u, b = e_2 v \).

Then we can write \( PAQ = UE \) where

\[
U = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}
\]

and

\[
E = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}.
\]

Clearly \( U \) is a unit in \( \mathbb{M}_2(R) \) and \( E \) is an idempotent in \( \mathbb{M}_2(R) \). We consider \( V \in \mathbb{M}_2(R) \) of the form

\[
V = \begin{bmatrix} 0 & -1 \\ -1 & e_2 \end{bmatrix}.
\]

Clearly the matrix \( V \) is a unit with inverse

\[
V^{-1} = \begin{bmatrix} e_2 & -1 \\ -1 & 0 \end{bmatrix}.
\]

Now we have

\[
E - V = \begin{bmatrix} e_1 & 1 \\ 1 & 2e_2 \end{bmatrix}.
\]

Clearly \( E - V \) is a unit with its inverse given by

\[
(E - V)^{-1} = \begin{bmatrix} 4e_1e_2 - 2e_2 & 1 - 2e_1e_2 \\ 1 - 2e_1e_2 & 2e_1e_2 - e_1 \end{bmatrix}.
\]

We have

\[
E + V = \begin{bmatrix} e_1 & -1 \\ -1 & 0 \end{bmatrix}.
\]

Clearly \( E + V \) is a unit with its inverse given by

\[
(E + V)^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & -e_1 \end{bmatrix}.
\]

Thus we have obtained a unit \( V \) such that both \( E - V \) and \( E + V \) are units. Clearly, then \( UV, UE - UV, UE + UV \) are units in \( \mathbb{M}_2(R) \). Thus \( PAQ - UV \) and \( PAQ + UV \) are units. Therefore \( PAQ \) is twin-good and consequently, multiplying by \( P^{-1} \) in left and \( Q^{-1} \) in right, we conclude that \( A \) is twin good. This shows that \( \mathbb{M}_2(R) \) is also twin-good. \( \square \)

**Corollary 7.** If \( R \) is an abelian regular ring, then the matrix ring \( \mathbb{M}_n(R) \) is twin-good for each \( n \geq 2 \).

In particular, if \( D \) is a division ring, then the matrix ring \( \mathbb{M}_n(D) \) is twin-good for each \( n \geq 2 \).
Proof. It is straightforward from Lemma 5 and Lemma 6. □

Remark 8. As a consequence of the above corollary, it follows that a semilocal ring $R$ is twin-good if and only if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Now we are ready to prove our main theorem.

Theorem 9. A right self-injective ring $R$ is twin good if and only if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. Let $R$ be a right self-injective ring such that $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. We know that $R/J(R)$ is a von Neumann regular right self-injective ring. From the type theory of von Neumann regular right self-injective rings it follows that $R/J(R) \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$ where $R_1$ is of type $I_f$, $R_2$ is of type $I_{\infty}$, $R_3$ is of type $II_f$, $R_4$ is of type $II_{\infty}$, and $R_5$ is of type $III$ (see [6, Theorem 10.22]). Taking $T = R_2 \times R_3 \times R_5$, we may write $R/J(R) \cong R_1 \times R_3 \times T$, where $T$ is purely infinite. We have $T_f \cong nT_f$ for all positive integers $n$ by [6, Theorem 10.16]. In particular, for $n = 3$, this yields $T \cong M_3(T)$. Since $T$ is an elementary divisor ring, by Lemma 5 we conclude that $M_3(T)$ is twin-good and consequently $T$ is twin-good.

Next we consider $R_1$. We know that $R_1 \cong \prod M_{n_i}(S_i)$ where each $S_i$ is an abelian regular self-injective ring (see [6, Theorem 10.24]). Since each $S_i$ is an elementary divisor ring, we know $M_{n_i}(S_i)$ is twin good whenever $n_i \geq 3$. If $n_i = 2$, then by Lemma 5 we have that $M_{n_i}(S_i)$ is twin-good.

Consider $n_i = 1$. Then we wish to prove that $S_i$ is twin-good. This was shown in [16] but we present the proof here for the sake of completeness. Assume to the contrary that $S_i$ is not twin-good. Then there exists an element $x \in S_i$ such that, for any $u \in U(S_i)$, either $x + u \notin U(S_i)$ or $x - u \notin U(S_i)$. Consider the set $S = \{I : I$ is an ideal of $S_i$ such that $\overline{x} + \overline{u} \notin U(S_i/I)$ or $\overline{x} - \overline{u} \notin U(S_i/I)$, for each $u \in U(S_i)\}$. Clearly, $S$ is a non-empty set. It may be shown that $S$ is an inductive set and hence, by Zorn’s lemma, $S$ has a maximal element, say $M$. Clearly then $S_i/M$ is indecomposable as a ring and therefore it has no nontrivial central idempotent. Since $S_i/M$ is an abelian regular ring, this yields that $S_i/M$ has no nontrivial idempotent. Hence, $S_i/M$ is a division ring. Therefore, by Lemma 1 it follows that $S_i/M \cong \mathbb{Z}_2$ or $S_i/M \cong \mathbb{Z}_3$. This yields a contradiction to our assumption. Hence, $S_i$ is twin-good.

We now consider $R_3$. Since $R_3$ is of type $II_f$, we can write $R_3 \cong n(e_n R_3)$ for each $n \in \mathbb{N}$ where $e_n$ is an idempotent in $R$ (see [6, Proposition 10.28]). In particular, for $n = 3$ we have $R_3 \cong M_3(e_3 R_3 e_3)$. As $e_3 R_3 e_3$ is an elementary divisor ring, it follows that $M_3(e_3 R_3 e_3)$ is twin good by Lemma 5.

Thus $R/J(R)$, being a direct product of twin-good rings, is twin good. Hence, by Lemma 2, $R$ is twin good.

The converse is obvious. □

As a consequence, we have the following

Corollary 10. For any linear transformation $T$ on a right vector space $V$ over a division ring $D$, there exists an invertible linear transformation $S$ on $V$ such that both $T - S$ and $T + S$ are invertible, except when $V$ is one-dimensional over $\mathbb{Z}_2$ or $\mathbb{Z}_3$. 


Consider the following three conditions on a module $M$:

**C1:** Every submodule of $M$.

**C2:** Every submodule of $M$ if quasi-injective.

**C3:** If this hold only for $A$-module $M$.

Now we may adapt the techniques of [13] and generalize our main result to the endomorphism rings of several classes of modules. Recall that a module $M$ is said to be $N$-injective if for every submodule $N_1$ of the module $N$, all homomorphisms $N_1 	o M$ can be extended to homomorphisms $N 	o M$. A right $R$-module $M$ is injective if $M$ is $N$-injective for every $N \in \text{Mod-}R$. A module $M$ is said to be quasi-injective if $M$ is $M$-injective.

Consider the following three conditions on a module $M$:

**C1:** Every submodule of $M$ is essential in a direct summand of $M$.

**C2:** Every submodule of $M$ isomorphic to a direct summand of $M$ is itself a direct summand of $M$.

**C3:** If $N_1$ and $N_2$ are direct summands of $M$ with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also a direct summand of $M$.

A module $M$ is called a **continuous module** if it satisfies conditions C1 and C2. A module $M$ is called **$\pi$-injective** (or **quasi-continuous**) if it satisfies conditions C1 and C3.

A right $R$-module $M$ is said to satisfy the exchange property if for every right $R$-module $A$ and any two direct sum decompositions $A = M' \oplus N = \oplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $B_i$ of $A_i$ such that $A = M' \oplus \left( \oplus_{i \in I} B_i \right)$. If this hold only for $|I| < \infty$, then $M$ is said to satisfy the finite exchange property. A ring $R$ is called an **exchange ring** if $R_R$ satisfies the (finite) exchange property.

Now, we have the following for endomorphism ring of a quasi-continuous module

**Corollary 12.** Let $S$ be any ring, $M$ be a quasi-continuous right $S$-module with finite exchange property and $R = \text{End}(M_S)$. If no factor ring of $R$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$, then $R$ is twin-good.

**Proof.** This proof is almost identical to the proof of [13] Theorem 3] but we write it here for the sake of completeness. Let $\Delta = \{ f \in R : \ker f \subset M \}$. Then $\Delta$ is an ideal of $R$ and $\Delta \subseteq J(R)$. By ([15], Cor. 3.13), $R = R/\Delta \cong R_1 \times R_2$, where $R_1$ is von Neumann regular right self-injective and $R_2$ is an exchange ring with no non-zero nilpotent element. We have already shown in Theorem 9 that $R_1$ is twin-good. Since, $R_2$ has no non-zero nilpotent element, each idempotent in $R_2$ is central. Now we proceed to show that $R_2$ is also twin-good. Assume to the contrary that there exists an element $a \in R_2$ which is not twin-good. Then as in the proof of Theorem 9 we find an ideal $I$ of $R_2$ such that $x = a + I \in R_2/I$ is not twin-good in $R_2/I$ and $R_2/I$ has no central idempotent. This implies that $R_2/I$ is an exchange ring without any non-trivial idempotent, and hence it must be local. If $S = R_2/I$ then $x + J(S)$ is not twin-good in $S/J(S)$, which is a division ring.
Therefore, $S/J(S) \cong \mathbb{Z}_2$, or $\mathbb{Z}_3$, a contradiction. Hence, every element of $R_2$ is twin-good. Therefore, every element of $R$ is twin-good and hence $R$ is twin-good. This completes the proof. □

**Corollary 13.** The endomorphism ring $R = \text{End}(M_S)$ of a continuous module $M_S$ is twin-good if $R$ has no factor isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

**Proof.** It follows from the above corollary in view of the fact that a continuous module is quasi-continuous and also has exchange property. □

A module $M$ is called cotorsion if every short exact sequence $0 \rightarrow M \rightarrow E \rightarrow F \rightarrow 0$ with $F$ flat, splits. It is known due to Guil Asensio and Herzog that if $M$ is a flat cotorsion right $R$-module and $S = \text{End}(M_R)$, then $S/J(S)$ is a von Neumann regular right self-injective ring (see [7]). As a consequence, we have the following

**Corollary 14.** The endomorphism ring $R = \text{End}(M_S)$ of a flat cotorsion (in particular, pure injective) module $M_S$ is twin-good if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Consider the following conditions on a module $N$;

(D1): For every submodule $A$ of $N$, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq A$ and $N_2 \cap A$ is small in $N$.

(D2): If $A$ is a submodule of $N$ such that $N/A$ is isomorphic to a direct summand of $N$, then $A$ is a direct summand of $N$.

A right $R$-module $N$ is called a discrete module if $N$ satisfies the conditions D1 and D2. It is well known that every discrete module is a Harada module.

**Corollary 15.** The endomorphism ring $R = \text{End}(M_S)$ of a Harada module $M_S$ is twin-good if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

**Proof.** It is known due to Kasch [12] that $R/J(R)$ is a direct product of right full linear rings and hence the corollary follows from Theorem 9. □

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Department of Mathematics and Computer Science, St. Louis University, St. Louis, MO-63103, USA
E-mail address: fsiddiq2@slu.edu

Department of Mathematics and Computer Science, St. Louis University, St. Louis, MO-63103, USA
E-mail address: asrivas3@slu.edu