Spherically symmetric false vacuum: no-go theorems and global structure

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We enumerate all possible types of spacetime causal structures that can appear in static, spherically symmetric configurations of a self-gravitating, real, nonlinear, minimally coupled scalar field \( \phi \) in general relativity, with an arbitrary potential \( V(\phi) \), not necessarily positive-definite. It is shown that a variable scalar field adds nothing to the list of possible structures with a constant \( \phi \) field, namely, Minkowski (or AdS), Schwarzschild, de Sitter and Schwarzschild — de Sitter. It follows, in particular, that, whatever is \( V(\phi) \), this theory does not admit regular black holes with flat or AdS asymptotics. It is concluded that the only possible globally regular, asymptotically flat solutions are solitons with a regular center, without horizons and with at least partly negative potentials \( V(\phi) \). Extension of the results to more general field models is discussed.

1. Introduction

The attractive idea of replacing the black hole (BH) singularities by nonsingular vacuum cores traces back to the papers of the 60s by Gliner \( \text{[1]} \) and Bardeen \( \text{[2]} \) but remains in the scope of modern studies. Possible manifestations of regular BHs vary from fundamental particles to largest astrophysical objects and created universes — for a review see Ref. \( \text{[3]} \).

Regular BHs with phenomenological sources have been discussed in, e.g., Refs. \( \text{[2, 4–7]} \). A class of nonlinear electrodynamics Lagrangians leading to regular BHs with magnetic field sources has been found in \( \text{[3]} \).

A natural question is whether or not a regular BH can be obtained as a false vacuum configuration with a nonlinear scalar field in general relativity, i.e., from the equations of motion due to the action

\[
S = \int d^4x \sqrt{-g} \left[ R + (\partial \phi)^2 - 2V(\phi) \right]
\]  

(1)

where \( R \) is the scalar curvature, \( (\partial \phi)^2 = g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \) and \( V(\phi) \) is a potential. This action, with many particular forms of \( V(\phi) \), has been vastly used to describe the vacuum (sometimes interpreted as a variable cosmological “constant”) in inflationary cosmology, for the description of growing vacuum bubbles, etc.

An attempt to construct a regular false vacuum BH was made in Ref. \( \text{[3]} \), with a potential having two slightly different minima, \( V(\phi_1) > V(\phi_2) = 0 \), the Schwarzschild metric and \( \phi = \phi_2 \) outside the horizon, the de Sitter metric and \( \phi = \phi_1 \) inside the horizon. It was claimed that a reasonable matching of the solutions was possible on the horizon despite a finite jump of \( \phi \). Gal’tsov and Lemos \( \text{[10]} \) showed that the piecewise solution of Ref. \( \text{[3]} \) cannot be described in terms of distributions and requires a singular matter source on the horizon. They proved \( \text{[10]} \) that asymptotically flat regular BH solutions are absent in theory \( \text{[1]} \) with any nonnegative potential \( V(\phi) \) (the no-go theorem). For the region outside the horizon, the only asymptotically flat BH solution is Schwarzschild, as follows from the well-known no-hair theorems (see Ref. \( \text{[1]} \) for a recent review).

Less is known when the asymptotic flatness and/or \( V \geq 0 \) assumptions are abandoned. Meanwhile, negative potential energy densities, in particular, the cosmological constant \( V = \Lambda < 0 \) giving rise to the anti–de Sitter (AdS) solution, do not lead to catastrophes (if restricted below), are often treated in various contexts and readily appear from quantum effects like vacuum polarization. Systems with an AdS rather than flat asymptotic cause great interest in connection with the AdS/CFT correspondence \( \text{[12]} \). BHs with less symmetric asymptotics were also considered \( \text{[13]} \).

We will study the possible global behavior of static, spherically symmetric solutions in theory \( \text{[1]} \) with arbitrary \( V(\phi) \) and arbitrary asymptotics. It happens that the field equations leave a very narrow spectrum of opportunities. According to Theorem 2 to be proved here, the set of causal structures is the same as known for constant \( \phi \): Minkowski (or AdS), Schwarzschild, de Sitter, and Schwarzschild — de Sitter (not to be confused with the de Sitter — Schwarzschild structure containing a de Sitter core, discussed in \( \text{[3]} \)).

A conclusion much stronger than in Ref. \( \text{[10]} \), namely, the absence of regular BH solutions for any \( V(\phi) \) and any asymptotic, then simply follows as a corollary.

The only possible singularity-free solutions are either de Sitter-like, with a single “cosmological” horizon (such an example was recently described by Hosotani \( \text{[14]} \)), or solutions without horizons, including asymptotically flat ones. The latter are, however, impossible if \( V \geq 0 \), as follows from a simple theorem proved in the manner of the no-hair theorems.

The conclusions obtained here can be more or less easily extended to other field models, as is pointed out in the last section.
In what follows, all statements apply to static, spherically symmetric configurations, and all relevant functions are assumed to be sufficiently smooth, unless otherwise indicated. The symbol □ marks the end of a proof.

2. Field equations

The field equations due to (1) are

\[ \nabla^\alpha \nabla_\alpha \varphi + V = 0, \]  
\[ R^\mu_{\nu} - \frac{1}{2} g^\mu_{\rho} R + T^\nu_{\mu} = 0, \]  
where \( V \) is the potential, \( R^\mu_{\nu} \) is the Ricci tensor, \( T^\nu_{\mu} \) is the energy-momentum tensor of the field:

\[ T^\nu_{\mu} = \varphi,_{\mu} \varphi^\nu - \frac{1}{4} \delta^\nu_{\mu} (\partial \varphi)^2 + \frac{1}{2} \delta^\nu_{\mu} V(\varphi). \]  

For a static, spherically symmetric configuration, the metric can be written in the form

\[ ds^2 = A(\rho) dt^2 - \frac{d\rho^2}{A(\rho)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  
and \( \varphi = \varphi(\rho) \) (We choose the coordinate gauge \( g_{tt} g_{\rho \rho} = -1 \) and suppose that large radii \( r \) correspond to large \( \rho \).) Eq. (2) and certain combinations of Eqs. (3) lead to

\[ (Ar^2 \varphi)' = r^2 V(\varphi); \]  
\[ (Ar')^2 = -r^2 V; \]  
\[ 2r' \rho - r^2 \varphi^2 = 2; \]  
\[ A(r^2)' - r^2 A' = 2; \]  
\[ A'r^2 + Ar^2 - 1 = \frac{1}{2} Ar^2 \varphi^2 - r^2 V, \]  
where the prime denotes \( d/d\rho \). Only three of these five equations are independent: the scalar equation (6) follows from the Einstein equations, while Eq. (10) is a first integral of the others. Given a potential \( V(\varphi) \), this is a determined set of equations for the unknowns \( r, A, \varphi \).

Eq. (3) can be integrated giving

\[ \left( \frac{A}{r^2} \right)' = -\frac{2(\rho - \rho_0)}{r^4}, \]  
where \( \rho_0 \) is an integration constant.

3. No-go theorems and other observations

Our interest will be in the generic global behavior of the solutions and the existence of BHs and globally regular configurations, in particular, regular BHs.

Let us first make sure that (unless the potential \( V \) is singular at some \( \rho \)) the full range of the \( \rho \) coordinate covers all values of \( r \), from the center \( (\rho = \rho_c, r(\rho_c) = 0) \), regular or singular, to infinity. To do that, we will rule out such nonsingular configurations as wormholes, horns and flux tubes.

By definition, a (traversable, Lorentzian) wormhole with the metric (8) has two asymptotics at which \( r \to \infty \), hence the function \( r(\rho) \) has at least one regular minimum. A horn is a region of space where, as \( \rho \) tends to some value \( \rho^* \), \( r(\rho) \neq \text{const} \) and \( g_{tt} = A \) have finite limits while the length integral \( l = \int d\rho/A \) diverges. In other words, a horn is an infinitely long 3-dimensional “tube” of finite radius, where the clock rate remains finite. Such “horned particles” were, in particular, discussed as possible remnants of black hole evaporation (1). Lastly, a flux tube is a configuration with \( r = \text{const} \).

**Theorem 1.** The field equations due to (1) do not admit (i) solutions where the function \( r(\rho) \) has a regular minimum, (ii) solutions describing a horn, and (iii) flux-tube solutions with \( \varphi \neq \text{const} \).

**Proof.** Since \( r(\rho) \geq 0 \) by its geometric meaning, Eq. (8) gives \( r'' \leq 0 \), which rules out regular minima. The same equation leads to \( \varphi = \text{const} \) as soon as \( r = \text{const} \). Thus items (i) and (iii) have been proved.

Suppose now that there is a horn. Then, by definition, \( A \) has a finite limit whereas \( l \to \infty \) as \( \rho \to \rho^* \). This is only possible if \( \rho^* = \pm \infty \). Under these circumstances, the left-hand side of Eq. (11) vanishes at the “horn end”, \( \rho \to \rho^* = \pm \infty \), whereas its right-hand side tends to infinity. This contradiction proves item (ii). □

Due to the local nature of the proof, Theorem 1 rules out wormholes or horns with any large \( r \) behavior — flat, de Sitter or any other. Moreover, since \( r' > 0 \) at large \( \rho \), the function \( r(\rho) \) is monotonic in the whole range.

Let us now address to the causal structure of the solutions, determined by the disposition of static \( (A > 0) \) and nonstatic \( (A < 0) \) regions of spacetime (also labeled R and T regions, respectively). This relationship is unambiguous in the sense that a particular disposition of regions leads to a certain Penrose-Carter diagram (7, 8). The latter may be further complicated by identification of isometric surfaces, if any, and by branching that leads to structures like Riemann surfaces (13, 18).

Horizons that separate the regions are regular spheres of nonzero radius, corresponding to zeros of the function \( A(\rho) \). Such zeros, if any, are regular points of Eqs. (6)–(10) due to our choice of the coordinates. Moreover, near a horizon, \( \rho \) varies (up to a positive constant factor) like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric (1). Therefore one can jointly consider regions on both sides of a horizon.

A horizon is simple or multiple according to whether the zero of \( A(\rho) \) is simple or multiple. A simple or, in general, odd-order horizon separates a static region from a nonstatic one (as, e.g., the Schwarzschild horizon). Even-order horizons separate regions of the same
nature (as the double horizon in the extreme Reissner-Nordström metric).

The following theorem severely restricts the set of possible structures.

**Theorem 2.** Consider solutions of the theory \([1]\) with the metric \([3]\) and \(\varphi = \varphi(\rho)\). Let there be a static region \(a < \rho < b \leq \infty\). Then:

(i) all horizons are simple;

(ii) no horizons exist at \(\rho < a\) and at \(\rho > b\).

**Proof.** Let \(\rho = h\) be a horizon: \(A(h) = 0\). It follows from Eq. \([3]\) that \(A''(h) = -2/r_h^2 < 0\). Therefore \(h\) cannot be a zero of \(A(\rho)\) of order higher than two. Consider the function \(B(\rho) = A/r^2\). A horizon is also a zero of \(B(\rho)\) of the same multiplicity as that of \(A\). If it is double, \(A'(h) = B'(h) = 0\), then \(h = \rho_0\), so that \(B' > 0\) at \(\rho < h\) and \(B' < 0\) at \(\rho > h\). Thus \(B < 0\) for all \(\rho \neq h\), and the spacetime has no static region (curve 1 in Fig. 1). So item (i) is proved.

Consider now the boundary \(\rho = a\) of the static region. If \(r(a) \neq 0\), then it is a horizon, \(A(a) = B(a) = 0\). (One cannot have \(B(a) = \infty\) since by \([1]\) \(|B'(a)| < \infty\).) By item (i), the horizon is simple, and \(B'(a) > 0\), therefore in \([1]\) we have \(a < \rho_0\) whence it follows that \(B' > 0\) everywhere to the left of \(\rho = a\): \(B(\rho)\) is an increasing function and cannot return to zero, ruling out horizons at \(\rho < a\) (see curves 2 and 3 in Fig. 1).

If \(b < \infty\), horizons at \(\rho > b\) are ruled out in a similar manner. \(\Box\)

According to Theorem 2, the list of possible global structures is the same as the one for constant \(\varphi\):

- [TR]: Schwarzschild (curve 3 in Fig. 1),
- [R]: Minkowski or AdS (curve 5),
- [TRT]: Schwarzschild – de Sitter (curve 2), and
- [TT], [T]: spacetimes without static regions (curve 1 and still below).

The R and T letters in brackets show the sequence of static and nonstatic regions, ordered from center to infinity. The center is generically singular. The only possible nonsingular solutions have either Minkowski/AdS or de Sitter structures, and, in particular, solitonlike asymptotically flat solutions are not excluded.

**Corollary.** The theory \([3]\) does not admit static, spherically symmetric, regular BHs.

Indeed, such a BH, with any large \(r\) behavior, must have static regions at small and large \(r\), separated by at least two simple or one double horizon (in the above notation, the structure must be [RTR] or [RR] or more complex). This is impossible according to Theorem 2.

The above two theorems did not use any assumptions on the asymptotic behavior of the solutions or the shape and even sign of the potential. Let us now mention some more specific but important results. One of them is the well-known no-hair theorem, first proved by Bekenstein \([14]\) for \(V(\varphi)\) without local maxima and later extended to any \(V \geq 0\) and some more general Lagrangians (see, e.g., Ref. \([3]\) for proofs and references):

**Theorem 3.** Suppose \(V \geq 0\). Then the only asymptotically flat BH solution to Eqs. \([4]\) in the range \((h, \infty)\) (where \(\rho = h\) is the event horizon) comprises the Schwarzschild metric, \(\varphi = \text{const}\) and \(V \equiv 0\).

Another restriction \([20]\) concerns the properties of globally regular configurations and can be called the generalized Rosen theorem (G. Rosen \([21]\) studied similar restrictions for nonlinear fields in flat spacetime):

**Theorem 4.** An asymptotically flat solution with a positive mass \(M\) and a regular center is impossible with \(V(\varphi) \geq 0\).

Here is a proof slightly different from the one given in Ref. \([20]\). Integrate Eq. \([7]\) from the center \((\rho = \rho_c)\) to infinity:

\[
A' r^2 \bigg|_{\rho_c}^\infty = -2 \int_{\rho_c}^\infty r^2 V \, d\rho. \tag{12}
\]

In an asymptotically flat metric, \(A(\rho)\) behaves at large \(\rho\) as \(1 - 2M/\rho^2\), where \(M\) is the Schwarzschild mass in geometric units, and \(r = \rho + O(1)\), therefore the upper limit of \(A' r^2\) equals \(2M\). At a regular center \(r = 0\) and, as is easily verified, \(A' = 0\), so the lower limit is zero. Consequently,

\[
M = -\int_{\rho_c}^\infty r^2 V \, d\rho. \tag{13}
\]

Thus \(M > 0\) requires that \(V(\varphi)\) should be at least partly negative. \(\Box\)
This simple theorem, proved previously for particle-like solutions without horizons, equally applies to regular BHs, if any. Hence, independently of Theorem 2, it also rules out regular BHs with $V \geq 0$. Theorem 2 tells us more: even negative potentials do not create such BHs.

4. Generalizations

One can notice that Theorem 1 actually rests on the validity of the null energy condition which, with the metric (1), reads $T^\rho_\nu - T^\nu_\rho \geq 0$. This, via the appropriate Einstein equation, leads to $r'' \leq 0$. In turn, Theorem 2 rests on Eq. (5) following from

$$T^\rho_\nu = T^0_\theta.$$  (14)

Thus both theorems hold for all kinds of matter whose energy-momentum tensors satisfy these two conditions.

Consider, for instance, the following action, more general than (1):

$$S = \int d^4x \sqrt{-g} \left[ R + F(I, \varphi) \right]$$  (15)

where $I = (\partial \varphi)^2$ and $F(I, \varphi)$ is an arbitrary function. The scalar field energy-momentum tensor is

$$T^\rho_\nu = \frac{\partial F}{\partial I} \varphi^\nu \varphi^\rho - \frac{1}{2} \delta^\rho_\nu F(\varphi).$$  (16)

In the static, spherically symmetric case, Eq. (14) holds automatically due to $\varphi = \varphi(\rho)$, while the null energy condition holds as long as $\partial F/\partial I \geq 0$, which actually means that the kinetic energy is nonnegative. Under this condition, both Theorems 1 and 2 are valid for the theory (5). Otherwise Theorem 2 alone holds; it still correctly describes the $\rho$ dependence of $A$ and consequently the possible horizons disposition, but $r(\rho)$ is not necessarily monotonic, so that wormholes and horns are not forbidden.

An extension of the present results to higher dimensions, with coordinate spheres $S^{D-2}$ instead of $S^2$, is straightforward. Other extensions, which need investigation, concern theories connected with (1) and (15) by $\varphi$-dependent conformal transformations, such as theories with nonminimally coupled scalar fields (e.g., scalar-tensor theories) and nonlinear gravity (e.g., with the Lagrangian function $f(R)$). I hope to consider them in future papers.

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References

[1] E. Gliner, Sov. Phys. JETP 22, 378 (1966).
[2] J. Bardeen, in: GR5 Proceedings, 1968.
[3] I. Dymnikova, “Variable cosmological constant — geometry and physics”. [gr-qc/0010010]
[4] I. Dymnikova, Gen. Rel. Grav. 24, 235 (1992).
[5] M. Mars, M.M. Martin-Prats and J.M.M. Senovilla, Clas. Qu. Grav. 13, L51 (1996).
[6] A. Borde, Phys. Rev. D 55, 7615 (1997).
[7] W.F. Kao, “Regular charged black holes with a charged de Sitter core”, hep-th/0009049.
[8] K.A. Bronnikov, Phys. Rev. D 63, 044005 (2001).
[9] R.G. Daghigh, J.I. Kapusta and Y. Hosotani, “False vacuum black holes and universes”, gr-qc/0008006.
[10] D.V. Gal’tsov and J.P.S. Lemos, “No-go theorem for false vacuum black holes”, gr-qc/0008076.
[11] J.D. Bekenstein, “Black holes: classical properties, thermodynamics, and heuristic quantization”, gr-qc/9808033.
[12] J. Maldacena, Adv, Theor. Math. Phys. 2, 231 (1998).
[13] K.C.K. Chan, J.H. Horne and R.B. Mann, Nucl. Phys. B 447, 441 (1995).
[14] Y. Hosotani, “Solitons in the false vacuum”, gr-qc/0104006.
[15] K.A. Bronnikov, G. Clément, C.P. Constantinidis and J.C. Fabris, Phys. Lett. A243, 121 (1998), gr-qc/9801050, Grav.& Cosmol. 4, 128 (1998), gr-qc/9804064.
[16] T. Banks, A. Dabholkar, M.R. Douglas and M. O’Loughlin, Phys. Rev. D 45, 3607 (1992); T. Banks and M. O'Loughlin, Phys. Rev. D 47, 540 (1993).
[17] M. Walker, J. Math. Phys. 11, 2280 (1970).
[18] K.A. Bronnikov, Izv. vuzov SSSR, Fiz., 1979, No.6, p. 32.
[19] J.D. Bekenstein, Phys. Rev. D 5, 1239 (1972); ibid., 2403.
[20] K.A.Bronnikov and G.N.Shikin, “Self-gravitating particle models with classical fields and their stability”. Series “Izdat Nauki i Tekhniki” (“Results of Science and Engineering”), Subseries “Classical Field Theory and Gravitation Theory”, v. 2, p. 4, VINITI, Moscow 1991 (in Russian).
[21] G. Rosen, J. Math. Phys. 7, 2066 (1966).