Research Article

Convergence Theorems for the Variational Inequality Problems and Split Feasibility Problems in Hilbert Spaces

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In this paper, we establish an iterative algorithm by combining Yamada’s hybrid steepest descent method and Wang’s algorithm for finding the common solutions of variational inequality problems and split feasibility problems. The strong convergence of the sequence generated by our suggested iterative algorithm to such a common solution is proved in the setting of Hilbert spaces under some suitable assumptions imposed on the parameters. Moreover, we propose iterative algorithms for finding the common solutions of variational inequality problems and multiple-sets split feasibility problems. Finally, we also give numerical examples for illustrating our algorithms.

1. Introduction

In 2005, Censor et al. [1] introduced the multiple-sets split feasibility problem (MSSFP), which is formulated as follows:

\[
\text{find } x \in \bigcap_{i=1}^{N} C_i \text{ such that } Ax \in \bigcap_{j=1}^{M} Q_j,
\]

where \( C_i \) (\( i = 1, 2, \ldots, N \)) and \( Q_j \) (\( j = 1, 2, \ldots, M \)) are nonempty closed convex subsets of Hilbert spaces \( H_1 \) and \( H_2 \), respectively, and \( A : H_1 \rightarrow H_2 \) is a bounded linear mapping. Denote by \( \Omega \) the set of solutions of MSSFP (1). Many iterative algorithms have been developed to solve the MSSFP (see [1–3]). Moreover, it arises in many fields in the real world, such as inverse problem of intensity-modulated radiation therapy, image reconstruction, and signal processing (see [1, 4, 5] and the references therein).

When \( N = M = 1 \), the MSSFP is known as the split feasibility problem (SFP); it was first introduced by Censor and Elfving [5], which is formulated as follows:

\[
\text{find } x \in C \text{ such that } Ax \in Q.
\]

Denote by \( \Gamma \) the set of solutions of SFP (2). Assume that the SFP is consistent (i.e., (2) has a solution). It is well known that \( x \in C \) solves (2) if and only if it solves the fixed point equation

\[
x = Tx,
\]

\[
T = P_C(I - \gamma A^*(I - P_Q)A), \quad x \in C,
\]

where \( \gamma \) is a positive constant, \( A^* \) is the adjoint operator of \( A \), and \( P_C \) and \( P_Q \) are the metric projections of \( H_1 \) and \( H_2 \) onto \( C \) and \( Q \), respectively (for more details, see [6]).

The variational inequality problem (VIP) was introduced by Stampacchia [7], which is finding a point

\[
x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in C,
\]

where \( C \) is a nonempty closed convex subset of a Hilbert space \( H \) and \( F : C \rightarrow H \) is a mapping. The ideas of the VIP are being applied in many fields including mechanics, nonlinear programming, game theory, and economic equilibrium (see [8–12]).
In [13], we see that \( x \in C \) solves (4) if and only if it solves the fixed point equation
\[
x = Sx, \\
S = P_C(I - \mu F), \quad x \in C.
\]
(5)

Moreover, it is well known that if \( F \) is \( k \)-Lipschitz continuous and \( \eta \)-strongly monotone, then VIP (4) has a unique solution (see, e.g., [14]).

Since SFP and VIP include some special cases (see [15, 16]), indeed, convex linear inverse problem and split equality problem are special cases of SFP, and zero point problem and minimization problem are special cases of VIP. Jung [17] studied the common solution of variational inequality problem and split feasibility problem: find a point
\[
x^* \in \Gamma \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0, \quad \text{for all } x \in \Gamma,
\]
(6)

where \( \Gamma \) is the solution set of SFP (2) and \( F: H_1 \rightarrow H_1 \) is an \( \eta \)-strongly monotone and \( k \)-Lipschitz continuous mapping. After that, for solving problem (6), Buong [2] considered the following algorithms, which were proposed in [14, 18], respectively:
\[
x_{n+1} = (I - t_n \mu F)Tx_n, \quad n \geq 0,
\]
(7)
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) (I - t_n \mu F)Tx_n, \quad n \geq 0,
\]
(8)

where \( T = P_C(I - \gamma A^* (I - P_Q)A) \), and under the following conditions:

(C1) \( t_n \in (0, 1) \), \( t_n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \sum_{n=1}^{\infty} t_n = \infty \).

(C2) \( 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1 \).

Moreover, Buong [2] considered the sequence \( \{x_n\} \) that is generated by the following algorithm, which is weakly convergent to a solution of MSSFP (1):
\[
x_{n+1} = P_1(I - \gamma A^* (I - P_2)A)x_n,
\]
(9)

where \( P_1 = P_{C_1} \ldots P_{C_N} \) and \( P_2 = P_{Q_1} \ldots P_{Q_M} \) or \( P_1 = \sum_{i=1}^{N} \alpha_i P_{C_i} \) and \( P_2 = \sum_{j=1}^{M} \beta_j P_{Q_j} \) in which \( \alpha_i \) and \( \beta_j \), for \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \), are positive real numbers such that \( \sum_{i=1}^{N} \alpha_i = \sum_{j=1}^{M} \beta_j = 1 \).

Motivated by the aforementioned works, we establish an iterative algorithm by combining algorithms (7) and (8) for finding the solution of problem (6) and prove the strong convergence of the sequence generated by our iterative algorithm to the solution of problem (6) in the setting of Hilbert spaces. Moreover, we propose iterative algorithms for solving the common solutions of variational inequality problems and multiple-sets split feasibility problems. Finally, we also give numerical examples for illustrating our algorithms.

2. Preliminaries

In order to solve our results, we now recall the following definitions and preliminary results that will be used in the sequel. Throughout this section, let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \).

Definition 1. A mapping \( T: H \rightarrow H \) is called

(i) \( k \)-Lipschitz continuous, if \( \| Tx - Ty \| \leq k \| x - y \| \) for all \( x, y \in H \), where \( k \) is a positive number.

(ii) Nonexpansive, if (i) holds with \( k = 1 \).

(iii) \( \eta \)-strongly monotone, if \( \eta \| x - y \| \leq \langle Tx - Ty, x - y \rangle \) for all \( x, y \in H \), where \( \eta \) is a positive number.

(iv) Firmly nonexpansive, if \( \| Tx - Ty \|^2 \leq \langle Tx - Ty, x - y \rangle \) for all \( x, y \in H \).

(v) \( \alpha \)-Averaged, if \( T = (1 - \alpha)I + \alpha N \) for some fixed \( \alpha \in (0, 1) \) and a nonexpansive mapping \( N \).

In [5], we know that the metric projection \( P_C: H \rightarrow C \) is firmly nonexpansive and \( (1/2) \)-averaged.

We collect some basic properties of averaged mappings in the following results.

Lemma 1 (see [16]). We have

(i) The composite of finitely many averaged mappings is averaged. In particular, if \( T_i \) is \( \alpha_i \)-averaged, where \( \alpha_i \in (0, 1) \) for \( i = 1, 2 \), then the composite \( T_1T_2 \) is \( \alpha \)-averaged, where \( \alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2 \).

(ii) If the mappings \( \{T_i\}_{i=1}^{N} \) are averaged and have a common fixed point, then
\[
\text{Fix}(T_1, T_2, \ldots, T_N) = \bigcap_{i=1}^{N} \text{Fix}(T_i).
\]

Proposition 1 (see [19]). Let \( D \) be a nonempty subset of \( H \), \( m \geq 2 \) be an integer, and \( \psi: (0, 1)^m \rightarrow (0, 1) \) be defined by
\[
\phi(a_1, \ldots, a_m) = \frac{1}{1 - (1/\sum_{i=1}^{m} (a_i/1 - a_i))}
\]
(11)

For every \( i \in \{1, \ldots, m\} \), let \( \alpha_i \in (0, 1) \) and \( T_i: D \rightarrow D \) be \( \alpha_i \)-averaged. Then, \( T = T_1 \ldots T_m \) is \( \alpha \)-averaged, where \( \alpha = \phi(a_1, \ldots, a_m) \).

The following properties of the nonexpansive mappings are very convenient and helpful to use.

Lemma 2 (see [20]). Assume that \( H_1 \) and \( H_2 \) are Hilbert spaces. Let \( A: H_1 \rightarrow H_2 \) be a linear bounded mapping such that \( A \neq 0 \) and let \( T: H_2 \rightarrow H_2 \) be a nonexpansive mapping. Then, for \( 0 \leq \gamma < \| A \|^2, \quad I - \gamma A^* (I - T)A = \gamma\| A \|^2 \)-averaged.

Proposition 2 (see [19]). Let \( C \) be a nonempty subset of \( H \), and let \( \{T_i\}_{i=1}^{N} \) be a finite family of nonexpansive mappings from \( C \) to \( H \). Assume that \( \{\alpha_i\}_{i=1}^{N} \subset (0, 1) \) and \( \{\delta_i\}_{i=1}^{N} \subset (0, 1) \) such that \( \sum_{i=1}^{N} \delta_i = 1 \). Suppose that, for every \( i \in I \), \( T_i \) is \( \alpha_i \)-averaged; then, \( T = \sum_{i=1}^{N} \delta_i T_i \) is \( \alpha \)-averaged, where \( \alpha = \sum_{i=1}^{N} \delta_i \alpha_i \).

The following results play a crucial role in the next section.

Lemma 3 (see [14]). Let \( t \) be a real number in \( (0, 1] \). Let \( F: H \rightarrow H \) be an \( \eta \)-strongly monotone and \( k \)-Lipschitz continuous mapping.
There is a problem in the document where the content is not clearly legible. However, I can provide a general understanding based on the visible parts and surrounding context. It seems to deal with advanced mathematical concepts such as monotone mappings, variational inequalities, and iterative algorithms. The document is discussing the convergence of sequences defined by iterative algorithms and comparing them with other methods. There is a focus on the strong convergence of these sequences to unique solutions of variational inequalities. The context suggests it is part of a research paper or a detailed mathematical exposition.
\[ y_n = (1 - \alpha_n)x_n + \alpha_n(I - t_n\mu F)P_1(I - \gamma A(I - P_2)A)x_n, \]
\[ x_{n+1} = (I - t_n\mu F)P_1(I - \gamma A(I - P_2)A)x_{n}, \quad \forall n \geq 1, \]
(20)

with one of the following cases:

(A1) \( P_1 = P_{C_1} \cup \ldots \cup P_{C_N} \) and \( P_2 = P_{Q_1} \cup \ldots \cup P_{Q_M} \)

(A2) \( P_1 = \sum_{i=1}^{N} \delta_i P_{C_i} \) and \( P_2 = \sum_{j=1}^{M} \zeta_j P_{Q_j} \)

(A3) \( P_1 = P_{C_1} \cup \ldots \cup P_{C_N} \) and \( P_2 = \sum_{j=1}^{M} \zeta_j P_{Q_j} \)

(A4) \( P_1 = \sum_{i=1}^{N} \delta_i P_{C_i} \) and \( P_2 = P_{Q_1} \cup \ldots \cup P_{Q_M} \)

converges to the element \( x^* \) in the solution set of (19).

**Proof.** Let \( T = P_1(I - \gamma A^*(I - P_2)A) \). We will show that \( T \) is averaged.

In the case of (A1), \( P_1 = P_{C_1} \cup \ldots \cup P_{C_N} \) and \( P_2 = P_{Q_1} \cup \ldots \cup P_{Q_M} \). Since \( P_{C_i} \) is \((1/2)\)-averaged for all \( i = 1, \ldots, N \), by Proposition 1, we get that \( P_1 \) is \( \lambda_1 \)-averaged, where \( \lambda_1 = N/(N + 1) \). Similarly, we have that \( P_2 \) is also averaged and so \( P_2 \) is nonexpansive. By Lemma 2, we deduce that \( I - \gamma A^*(I - P_2)A \) is \( \lambda_2 \)-averaged, where \( \lambda_2 = \gamma \|A\|^2 \).

\[ \text{Fix}(T) = \text{Fix}(P_1) \cap \text{Fix}(I - \gamma A^*(I - P_2)A) = \text{Fix}(P_1) \cap A^{-1}\text{Fix}(P_2) \]
\[ = \bigcap_{i=1}^{N} C_i \cap A^{-1} \left( \bigcap_{j=1}^{M} Q_j \right) = \Omega. \]

Then, iterative algorithm (20) can be rewritten as follows:

\[ x_{n+1} = (I - t_n\mu F)\bar{T}x_n, \]
(22)

where \( \bar{T} = (1 - \alpha_n)I + \alpha_n(I - t_n\mu F)T \) and \( T = (1 - \lambda)I + \lambda S \). Since \((1 - \lambda)I + \lambda S \) and \( I - t_n\mu F \) are nonexpansive, then \((I - t_n\mu F)T \) is nonexpansive. Thus, the strong convergence of (20) to the element \( x^* \) in the solution set of (19) follows by Theorem 2.

**Theorem 6.** Let \( [C_i]_{i=1}^{N}, \{Q_j\}_{j=1}^{M}, \{t_n\}, \{\delta_n\}, \) and \( \{\zeta_n\} \) be as in Theorem 5. Then, as \( n \rightarrow \infty \), the sequence \( \{x_n\} \), defined by

\[ \begin{align*}
  y_n &= (1 - \alpha_n)x_n + \alpha_n(I - t_n\mu F)P_1(I - \gamma A(I - P_2)A)x_n, \\
  x_{n+1} &= (1 - \beta_n)x_n + \beta_n(I - t_n\mu F)P_1(I - \gamma A(I - P_2)A)x_n, \\
\end{align*} \]
(23)

with one of the cases (A1)–(A4), converges strongly to an element in the solution set of (19).

**Proof.** In the proof of Theorem 5, one can rewrite iterative algorithm (23) as follows:

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(I - t_n\mu F)\bar{T}x_n, \]
(24)

where \( \bar{T} = (1 - \beta_n)I + \beta_n(I - t_n\mu F)T \) and \( T = (1 - \lambda)I + \lambda S \). Since \((I - t_n\mu F)T \) is nonexpansive, the strong convergence of (23) to the element \( x^* \) in the solution set of (19) follows by Theorem 1.

**4. Numerical Example**

In this section, we present the numerical example comparing algorithm (8) which is given by Buong [2] and algorithm (15) (new method) to solve the following test problem in [2]: find an element \( x^* \in \Omega \) such that
\[ \phi(x^*) = \min_{x \in \Omega} \phi(x), \]

\[ \Omega = C_i \cap A^{-1}Q_j \neq \emptyset, \]

where \( \phi \) is a convex function, having a strongly monotone and Lipschitz continuous derivative \( \phi'(x) \) on the Euclidian space \( E^n \), \( C = \cap_{i=1}^{N} C_i \) and \( Q = \cap_{j=1}^{M} Q_j \) where

\[ C_i = \left\{ x \in E^n : \sum_{k=1}^{n} a_{ik} x_k \leq b_i \right\}, \quad (26) \]

\[ a_{ik}, b_i \in (-\infty, +\infty), \text{ for } 1 \leq k \leq n \text{ and } 1 \leq i \leq N, \]

\[ Q_j = \left\{ y \in E^m : \sum_{l=1}^{m} \left( y_l - a_{lj} \right)^2 \leq R_j^2 \right\}, \quad R_j > 0, \quad (27) \]

\[ \phi(x) = (1-a)\|x\|^2/2 \text{ for some fixed } a \in (0,1), \text{ and} \]

**Example 1.** We consider test problem (25), where \( N = M = 1, n = m = 2, \phi(x) = (1-a)\|x\|^2/2 \text{ for some fixed } a \in (0,1), \text{ and} \]

**Table 2:** Computational results for Example 2 with different methods.

| Initial point | A1   | A2   | A3   | A4   |
|--------------|------|------|------|------|
| (-2,1)\(^T\) | n    | 28577| 24264| 28577| 24264|
|              | s    | 1.491225| 1.355074| 1.534414| 1.282528|
| (1,3)\(^T\)  | n    | 33407| 31438| 33407| 31438|
|              | s    | 1.746868| 1.693069| 1.816897| 1.690618|

**Figure 1:** The convergence behavior of \( E_n \) with the initial point \((-2,1)^T\).

**Figure 2:** The convergence behavior of \( E_n \) with the initial point \((1,3)^T\).
So, we have that $F = \varphi' = (1 - a)I$ is a $k$-Lipschitz continuous and $\eta$-strongly monotone mapping with $k = \eta = (1 - a)$. For each algorithm, we set $a_i = (1/i, -1), b_i = 0$, for all $i = 1, \ldots, N$, and $a_j = (1/j, 0), R_j = 1$, for all $j = 1, \ldots, M$. Taking $a = 0.5, \gamma = 0.3$, the stopping criterion is defined by $E_n = \|x_{n+1} - x_n\| < \varepsilon$ where $\varepsilon = 10^{-4}$ and $10^{-6}$. The numerical results are listed in Table 1 with different initial points $x^i$, where $n$ is the number of iterations and $s$ is the CPU time in seconds. In Figures 1 and 2, we present the graphs illustrating the number of iterations for both methods using the stopping criterion defined as above with the different initial points shown in Table 1.

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$ (28)
Remark 1. From the numerical analysis of our results in Table 1 and Figures 1 and 2, we get that algorithm (15) (new method) has less number of iterations and faster convergence than algorithm (8) (Buong method).

Example 2. In this example, we consider algorithm (23) for solving test problem (25), where $N = 5$ and $M = 4$. Let $\{C_j\}_{j=1}^{N}$, $\{Q_j\}_{j=1}^{M}$, $\varphi$, $a$, and $A$ be as in Example 1. In the numerical experiment, we take the stopping criterion $E_2 < 10^{-4}$. The numerical results are listed in Table 2 with different cases of $P_1$ and $P_2$. In Figures 3 and 4, we present the graphs illustrating the number of iterations for all cases of $P_1$ and $P_2$ using the stopping criterion as above with the different initial points appeared in Table 2. Moreover, Table 3 shows the effect of different choices of $\gamma$.

Remark 2. We observe from the numerical analysis of Table 2 that algorithm (23) has the fastest convergence when $P_1$ and $P_2$ satisfy (A4) and the slowest convergence when $P_1$ and $P_2$ satisfy (A3). Moreover, we require less iteration steps and CPU times for convergence when $\gamma$ is chosen very small and close to zero.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest.

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