The graph with spectrum $14^1 2^{40} (-4)^{10} (-6)^9$

Aart Blokhuis · Andries E. Brouwer ·
Willem H. Haemers

Received: 5 April 2011 / Revised: 19 May 2011 / Accepted: 27 May 2011 /
Published online: 14 June 2011
© The Author(s) 2011. This article is published with open access at Springerlink.com

Abstract We show that there is a unique graph with spectrum as in the title. It is a subgraph of the McLaughlin graph. The proof uses a strong form of the eigenvalue interlacing theorem to reduce the problem to one about root lattices.

Keywords Graph spectrum · Strongly regular graph · Root lattice

Mathematics Subject Classification (2000) 05C50 · 05E30 · 05C62

1 The graph $\Delta$

It was shown in [5] that there is a unique graph $Z$ with spectrum $30^1 2^{90} (-10)^{21}$ (with multiplicities written as exponents), namely the collinearity graph of the unique generalized quadrangle with parameters $GQ(3, 9)$. It is strongly regular with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$. Its automorphism group is $U_4(3) \cong PGO_{6}^{*}$ (3) (of order $2^{10} \cdot 3^6 \cdot 5 \cdot 7$), where the $*$ denotes that the form may be multiplied by a constant.

It was shown in [1] that there is a unique graph $Y$ with spectrum $20^1 2^{60} (-7)^{20}$. It is strongly regular with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$, and is the second subconstituent of $Z$, the subgraph induced on the set of vertices at distance 2 from a fixed
vertex $a$ of $Z$. Its automorphism group is $3^4 : ((2 \times S_6) \cdot 2)$ acting rank 3, the point stabilizer in $\text{Aut}(Z)$. One construction of $Y$ is found by taking $1^4 / \{1\}$ (where $1$ denotes the all-1 vector) inside $F_3^6$, where two cosets are adjacent when they differ by a weight-3 vector.

Let $\Delta$ be the second subconstituent of $Y$, the subgraph induced on the set of vertices at distance 2 from a fixed vertex $b$ of $Y$. Then $\Delta$ has spectrum $14^1 2^{40} (-4)^{10} (-6)^9$ (apply Theorem 5.1 of [5]) and automorphism group $(2^2 \times S_6) \cdot 2$, the stabilizer of the unordered pair $\{a, b\}$ in $\text{Aut}(Z)$, twice as large as the point stabilizer of $\text{Aut}(Y)$. The above description of $Y$ leads to a description of $\Delta$ as the graph on the cosets in $F_3^6$ with coordinates (up to permutation) either $000012 + \{1\}$ or $001122 + \{1\}$, where two cosets are adjacent when they differ by a weight 3 vector.

In this note we show that the graph $\Delta$ is determined by its spectrum.

This is an interesting case. The uniqueness proof is elegant and quite different from the methods found in the literature (cf. [3,4]).

2 Interlacing

An important tool is the following lemma on interlacing eigenvalues ([6], Theorem 2.1 (i), (ii); see also [2], Theorem 3.3.1).

**Lemma 2.1** Let $\Gamma$ be a graph on $n$ vertices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let $\{X_1, \ldots, X_m\}$ be a partition of the vertex set of $\Gamma$ into nonempty parts. Let $r_{ij}$ be the average number of neighbours in $X_j$ of a vertex in $X_i$. Then the matrix $R = (r_{ij})$ has real eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, which satisfy

(i) (interlacing) $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$;

(ii) if $\mu_i = \lambda_i$, or $\mu_i = \lambda_{n-m+i}$ for some $i \in \{1, \ldots, m\}$, then $R$ has a $\mu_i$-eigenvector $v = (v_1, \ldots, v_m)^\top$, such that the vector $w \in \mathbb{R}^n$ whose entries are equal to $v_j$ for all vertices in $X_j$ ($j = 1, \ldots, m$) is a $\mu_i$-eigenvector of $\Gamma$.

For example if $m = 1$ it follows that the average valency $\bar{k}$ of $\Gamma$ is at most equal to $\lambda_1$, and equality implies that the all-1 vector is a $\lambda_1$-eigenvector of $\Gamma$. Since $n\bar{k} = \sum \lambda_i^2$ it follows that $\Gamma$ is regular of valency $\lambda_1$ if $n\lambda_1 = \sum \lambda_i^2$.

3 Graphs cospectral to $\Delta$

Let $\Gamma$ be a graph with the same spectrum $14^1 2^{40} (-4)^{10} (-6)^9$ as $\Delta$.

We shall write $x \sim y$ ($x \not\sim y$) when $x$ is a (non)neighbour of $y$ in $\Gamma$, and denote the number of common neighbours of $x$ and $y$ by $\lambda(x, y)$ ($\mu(x, y)$).

(i) By Lemma 2.1 we know that $\Gamma$ is regular of valency 14. Moreover $\Gamma$ is connected, because the multiplicity of the eigenvalue 14 equals 1.

If $\Gamma$ has adjacency matrix $A$, then $(A - 2I)(A + 4I)(A + 6I) = 72J$ so that $(A^3)_{xx} = 8$, and it follows that each vertex is in four triangles.

(ii) For a vertex $x$, let $T_x$ be a set of eight neighbours of $x$ such that $\{x\} \cup T_x$ contains the four triangles on $x$. Let $S_x$ be the set of the remaining six neighbours of $x$, and let $N_x$ be the set of 45 nonneighbours of $x$. The matrix of average row sums of $A$, partitioned according to $\{\{x\}, T_x, S_x, N_x\}$ is
the row sum is nonzero, contradiction. It follows that the representation is injective.

If \( y \) is a column vector of squared norm 2, with entries either 2 1 18 042 \( 0 \), then \( y \) is a root of the form \( \sum_{\lambda} a \lambda \) with \( a \in \mathbb{Z} \), and since there can be at most two more nonzero entries, the row sum is nonzero, contradiction. It follows that the representation is injective.

\( B \) has 1, 3 neighbours in \( S \), and hence 3 neighbours in \( S \), so that \( a = 6 - \mu(x, z) \). In particular, \( \mu(x, z) = 3 \) implies that \( z \) has no neighbours in \( S \).

(iii) The rank 10 matrix \( B = 4J - (A - 2I)(A + 6I) \) is positive semi-definite and hence can be written \( B = N^\top N \) for a 10 \times 60 matrix \( N \).

Let \( \bar{x} \) be column \( x \) of \( N \). Then \( x \mapsto \bar{x} \) is a representation of \( \Gamma \) in Euclidean 10-space, with

\[
(x, y) = \begin{cases}
  2 & \text{if } x = y \\
  -\mu(x, y) & \text{if } x \sim y \\
  4 - \mu(x, y) & \text{if } x \not\sim y
\end{cases}
\]

It follows that for nonadjacent vertices \( x, y \) one has \( 2 \leq \mu(x, y) \leq 6 \).

If \( \{x, y, z\} \) is a triangle, then \( \bar{x} + \bar{y} + \bar{z} = 0 \) (since this sum has squared norm 0).

The matrix \( B \) satisfies \( JB = 0 \) and \( AB = -4B \) and \( B^2 = 12B \) so that the rows of \( B \) are integral vectors with sum 0 and squared norm 24.

Row \( x \) of \( B \) has a 2 at the \( x \)-position, and a -1 at the 8 positions \( z \in T_x \) (with \( \lambda(x, z) = 1 \)). If \( \bar{x} = \bar{y} \), so that rows \( x \) and \( y \) of \( B \) are identical, then \( \mu(x, y) = 2 \) and we see two 2’s and at least fourteen -1’s in each row, and since there can be at most two more nonzero entries, the row sum is nonzero, contradiction. It follows that the representation is injective.

If \( (\bar{x}, \bar{y}) = -2 \), then \( \bar{y} = -\bar{x} \). Given \( x \), this happens for at most one \( y \). It follows that a row of \( B \) has entries either \( 2^1 1^8 0^2 (-1)^8 (-2)^1 \) or \( 2^1 1^9 0^3 9 \) (-1) 11 (with multiplicities written as exponents).

(iv) Let us call a triangle a line. If \( \mu(x, y) = 3 \) then each of the six edges connecting \( x \) and \( y \) with their common neighbours are in a line. Now there are 24 lines not on \( x \) meeting \( T_x \), and each \( y \) with \( \mu(x, y) = 3 \) determines three such lines, so if there are 9 such points \( y \) then some line is seen twice. We find a line \( \{y, y', z\} \) with \( x \sim z \). Now \( 0 = (\bar{x}, \bar{y}) + (\bar{x}, \bar{y'}) + (\bar{x}, \bar{z}) = 1 + 1 + (-1) = 1 \), contradiction. It follows that no row of \( B \) has pattern \( 2^1 1^9 0^3 9 (-1) 11 \).

(v) A set of roots (vectors of squared norm 2) with integral inner products spans a root lattice ([2], §3.10), so \( \Lambda = \{\bar{x} \mid x \in V\Gamma\} \) is a 10-dimensional root lattice, orthogonal direct sum of summands of the form \( A_n (n \geq 1) \), \( D_n (n \geq 4) \), \( E_6 \), \( E_7 \), or \( E_8 \).

(vi) The roots of the orthogonal direct sum of root lattices are the roots of the summands, so that an orthogonal direct sum decomposition of \( \Lambda \) gives a partition of \( V\Gamma \) such that \( (\bar{y}, \bar{z}) = 0 \) if \( y, z \) are vertices from different parts. It follows that the three vertices of a triangle belong to the same part.

Consider the graph \( T \) with vertex set \( V\Gamma \) where two vertices \( x, y \) are adjacent when \( (\bar{x}, \bar{y}) = -1 \), i.e., when \( xy \) is an edge in a triangle of \( \Gamma \). Given \( x \), consider the five subsets \( S_i = \{u \in V\Gamma \mid (\bar{x}, \bar{u}) = i\} \) for \( i = 2, 1, 0, -1, -2 \). We have \( |S_2| = |S_{-2}| = 1, |S_{-1}| = |S_1| = 8, |S_0| = 42 \). The graph \( T \) is regular of valency 8. In \( T \), any vertex \( y \in S_{-1} \) has 1 neighbour \( x \), 1 neighbour in \( S_{-1} \), 3 neighbours in \( S_1 \), and hence 3 neighbours in \( S_0 \). A vertex \( z \in S_0 \) has 0 or 2 \( \Gamma \)-neighbours in \( S_{-1} \), so at most 2 \( T \)-neighbours. We see that the connected component of \( T \) containing \( x \) has at least \( 1 + 8 + 8 + 1 + (8 \cdot 3)/2 = 30 \) vertices.
It follows that either the root lattice $\Lambda$ is indecomposable, i.e., is $A_{10}$ or $D_{10}$, or has precisely two summands. Since $A_n$ has $n(n+1)$ roots, and $D_n$ has $2n(n-1)$ roots, the possibilities in the latter case are $A_5 + A_5$, $A_5 + D_5$, $D_5 + D_5$.

(vii) Suppose $\Lambda$ has a direct summand $D_5$. The root system $D_5$ has 40 roots, and 30 occur as images of vertices in the corresponding connected component $C$ of $T$. Let $\Phi$ be the graph on the 40 roots of $D_5$, adjacent when they have inner product $-1$, and consider $C$ a subset of the vertex set of $\Phi$. Let $D$ be the set of 10 roots not in $C$. The graph $\Phi$ is regular of valency 12. The valency inside $C$ is 8, so each vertex in $C$ has 4 neighbours in $D$. This gives 120 edges meeting $D$, so there are no internal edges in $D$ and no two roots of $D$ have inner product $-1$. Both $\Phi$ and $C$ are closed under $u \rightarrow -u$, so also $D$ is, and no two roots of $D$ have inner product 1. Consequently, $D$ has only inner products 2, 0, $-2$ and consists of five mutually orthogonal pairs of opposite roots. But $D_5$ does not contain 5 mutually orthogonal roots. Contradiction.

(viii) Consider the graph $\Pi$ with as vertices the 30 pairs $\pm \bar{x}$, adjacent when they have non-zero inner product. Then $\Pi$ has valency 8 and $\lambda = 4$. Using a Weetman argument (cf. [7]) we see that a connected component of $\Pi$ has fewer than 30 vertices. It will follow that $\Lambda \simeq A_5 + A_5$.

As follows. For geodesics $x_0 \sim x_1 \sim x_2 \sim \ldots$ we find lower bounds $n_i$ for the number of common neighbours of $x_i$ and $x_{i+1}$ at distance $i$ from $x_0$. We can take $n_1 = 2$ since two nonadjacent vertices in a 4-regular graph on 8 vertices must have at least 2 common neighbours. We can take $n_2 = 3$ since the set of common neighbours of $x_2$ and $x_0$ has valency at least $n_1 = 2$, and hence size at least 3 (and an 8-vertex graph of degree 4 cannot have a cut set of size 2). Now the local graph at $x_3$ has at least 4 vertices at distance 2 from $x_0$, and hence cannot have any at distance 4 from $x_0$ and a connected component of $\Pi$ has diameter at most 3 and size at most $1 + 8 + (8 \cdot 3)/3 + (8 \cdot 2)/4 = 21$, as desired.

(ix) Thus far, we identified the 60 vertices of $\Gamma$ with the 60 roots of $A_5 + A_5$, and can recognize the triangles of $\Gamma$. It remains to find the edges of $\Gamma$ that are not in a triangle.

Let $C$ and $D$ be the two sets of vertices belonging to the two systems $A_5$. Given $x \in C$, the 12 vertices $y \in C$ with $(\bar{x}, \bar{y}) = 0$ have common $T$-neighbours with $x$, so are nonadjacent to $x$ in $\Gamma$. That determines the induced subgraph on $C$ and on $D$, and we have to find the edges between $C$ and $D$.

Suppose $x \in C$. If $\bar{y} = -\bar{x}$, then $\mu(x,y) = 6$, and the six common neighbours of $x$ and $y$ live in $D$, and form all neighbours of $x$ in $D$. If $u$ is a common neighbour of $x$ and $y$, and $\bar{v} = -\bar{u}$, then also $v$ is a common neighbour of $x$ and $y$. This means that for the edges across we can identify pairs of opposite roots, and have a geometry with 15 points and 15 lines, where each point is on 3 lines and each line has 3 points. The points can be identified with the pairs from a 6-set. Then subgraph on the set of points is $T(6)$. The lines consist of three mutually disjoint pairs. This is the unique generalized quadrangle of order 2.

This proves that $\Gamma$ is uniquely determined by its spectrum, and hence must be isomorphic to $\Delta$.

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Brouwer A.E., Haemers W.H.: Structure and uniqueness of the $(81, 20, 1, 6)$ strongly regular graph. Discrete Math. 106/107, 77–82 (1992).
The graph with spectrum $14^1 2^{40} (-4)^{10} (-6)^9$

2. Brouwer A.E., Cohen A.M., Neumaier A.: Distance-Regular Graphs. Springer, Heidelberg (1989).
3. van Dam E.R., Haemers W.H.: Which graphs are determined by their spectrum? Linear Algebra Appl. 373, 241–272 (2003).
4. van Dam E.R., Haemers W.H.: Developments on spectral characterizations of graphs. Discrete Math. 309, 576–586 (2009).
5. Cameron P.J., Goethals J.-M., Seidel J.J.: Strongly regular graphs having strongly regular subconstituents. J. Algebra 55, 257–280 (1978).
6. Haemers W.H.: Interlacing eigenvalues and graphs. Linear Algebra Appl. 226–228, 593–616 (1995).
7. Weetman G.: Diameter bounds for graph extensions. J. Lond. Math. Soc. 50, 209–221 (1994).