Hom-quantum groups: I. Quasi-triangular Hom-bialgebras

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Abstract
We introduce a twisted generalization of quantum groups, called quasi-triangular Hom-bialgebras. They are non-associative and non-coassociative analogues of Drinfel’d’s quasi-triangular bialgebras, in which the non-(co)associativity is controlled by a twisting map. A family of quasi-triangular Hom-bialgebras can be constructed from any quasi-triangular bialgebra, such as Drinfel’d’s quantum enveloping algebras. Each quasi-triangular Hom-bialgebra comes with a solution of the quantum Hom–Yang–Baxter equation, which is a non-associative version of the quantum Yang–Baxter equation. Solutions of the Hom–Yang–Baxter equation can be obtained from modules of suitable quasi-triangular Hom-bialgebras. The results here are related to q-deformations of Lie algebras and have potential applications to invariants of knots and quandle cohomology.

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1. Introduction
This paper is part of an on-going effort [53, 56, 57] to study twisted generalizations of the various Yang–Baxter equations (YBEs) and related algebraic structures. A twisted generalization of the YBE [7, 8, 50], called the Hom–Yang–Baxter equation (HYBE), and its relationships to the braid relations and braid group representations [4, 5] were studied in [53, 56]. Twisted versions of the classical YBE [46, 47] and Drinfel’d’s Lie bialgebras [11, 12] were studied in [57]. From a physical viewpoint, the YBE is closely related to invariants of knots. The author is currently investigating the relationship between the HYBE, invariants of knots and quandle cohomology.

The purpose of this paper is to study a twisted generalization of quantum groups and the quantum Yang–Baxter equation (QYBE). The quantum groups being generalized in this paper are Drinfel’d’s quasi-triangular bialgebras [12]. Our generalized quantum groups,
called quasi-triangular Hom-bialgebras, are in general non-associative, non-coassociative, non-commutative and non-cocommutative. We also refer to these objects colloquially as Hom-quantum groups. As we describe below, suitable quasi-triangular Hom-bialgebras give rise to solutions of the HYBE.

Let us first recall the definition of a quasi-triangular bialgebra and its relationships to the various YBEs. A quasi-triangular bialgebra \((A, R)\) [12] consists of a unital bialgebra \(A\) and an invertible element \(R \in A^{\otimes 2}\) such that the following three conditions are satisfied:

\[
\Delta^\text{op}(x)R = R\Delta(x),
\]

\[
(\Delta \otimes \text{Id})(R) = R_{13}R_{23}, \quad (\text{Id} \otimes \Delta)(R) = R_{13}R_{12}.
\]

Here \(\Delta^\text{op} = \tau \circ \Delta\) with \(\tau : A^{\otimes 2} \to A^{\otimes 2}\) the twist isomorphism, \(R_{12} = R \otimes 1\), \(R_{23} = 1 \otimes R\) and \(R_{13} = (\tau \otimes \text{Id})R_{23}\). The invertible 2-tensor \(R\), called the quasi-triangular structure, satisfies the QYBE

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

Examples of quasi-triangular bialgebras include Drinfel’d’s quantum enveloping algebra \(U_h(g)\) [12] of a semi-simple Lie algebra or a Kac–Moody algebra \(g\) [24], the anyonic quantum groups [35] and the quantum line [34], among many others. The QYBE and quasi-triangular bialgebras are motivated by work on the quantum inverse scattering method [14–18, 46, 47] and exactly solved models in statistical mechanics [7, 8, 50]. Comprehensive expositions on quasi-triangular bialgebras can be found in [10, 13, 25, 36].

Quasi-triangular bialgebras and the QYBE are related to the YBE as follows. Consider a module \(V\) over a quasi-triangular bialgebra \((A, R)\) and the operator \(B : V^{\otimes 2} \to V^{\otimes 2}\) defined as

\[
B(v \otimes w) = \tau(R(v \otimes w)).
\]

As a consequence of the QYBE (1.0.2), the operator \(B\) satisfies the YBE [7, 8, 50]

\[
(B \otimes \text{Id})(\text{Id} \otimes B)(B \otimes \text{Id}) = (\text{Id} \otimes B)(B \otimes \text{Id})(\text{Id} \otimes B).
\]

So one quasi-triangular bialgebra \((A, R)\) gives rise to many solutions of the YBE via its modules. For this reason, the quasi-triangular structure \(R\) is also known as the universal \(R\)-matrix. The reader may consult [42] for discussion of different versions of the YBE and of their uses in physics.

Our generalizations of quantum groups and the QYBE (as well as the HYBE) are all motivated by Hom–Lie algebras and other Hom-type algebras. Roughly speaking, a Hom-type structure arises when one strategically replaces the identity map in the defining axioms of a classical structure by a general twisting map \(\alpha\). A classical structure should be a particular example of a Hom-type structure in which the twisting map is the identity map.

From a physical viewpoint, Hom-type structures are important because they are related to vertex operator algebras (VOA) and string theory. In particular, a Hom–Lie algebra \((L, [-,-], \alpha)\) has an anti-symmetric bracket \([-,-] : L^{\otimes 2} \to L\) that satisfies the Hom–Jacobi identity

\[
[[x, y], \alpha(z)] + [[z, x], \alpha(y)] + [[y, z], \alpha(x)] = 0.
\]

Hom–Lie algebras were introduced in [20] to describe the structures on some \(q\)-deformations of the Witt and the Virasoro algebras, both of which are important in VOA and string theory. Earlier precursors of Hom–Lie algebras can be found in [2, 21, 33]. Hom–Lie algebras are also closely related to deformed vector fields [3, 20, 29–31, 43, 45] and number theory [28].

One can similarly define a Hom-associative algebra [37], which satisfies the Hom-associativity axiom

\[
(xy)\alpha(z) = \alpha(x)(yz);
\]
see definition 2.2. So a Hom-associative algebra is not associative, but the non-associativity is controlled by the twisting map \( \alpha \). One obtains a Hom–Lie algebra from a Hom-associative algebra via the commutator bracket \([37, \text{proposition 1.7}].\) Conversely, the enveloping Hom-associative algebra of a Hom–Lie algebra is constructed in \([51]\) and is studied further in \([54]\).

The reader may consult \([37–40]\) and \([51–60]\) for other Hom-type structures.

We now describe the main results of this paper concerning Hom-type generalizations of quasi-triangular bialgebras and the QYBE. Precise definitions, statements of results and proofs are given in later sections.

Following the patterns of Hom–Lie and Hom-associative algebras, one can define Hom-bialgebras, which are non-associative and non-coassociative generalizations of bialgebras in which the non-(co)associativity is controlled by the twisting map \( \alpha \). In section 2, we introduce quasi-triangular Hom-bialgebras (definition 2.7), generalizing Drinfel’d’s quasi-triangular bialgebras by strategically replacing the identity map by a twisting map \( \alpha \) in the defining axioms. We show that a quasi-triangular Hom-bialgebra comes equipped with a solution \( R \) of the quantum Hom–Yang–Baxter equation (QHYBE) (theorem 2.10), which is a non-associative analogue of the QYBE. In fact, due to the non-associative nature of a Hom-bialgebra, there are two different versions of the QHYBE ((2.10.1) and (2.10.2)), both of which hold in a quasi-triangular Hom-bialgebra.

In section 3, we give two general procedures by which quasi-triangular Hom-bialgebras can be constructed. First we show that every quasi-triangular bialgebra \( A \) can be twisted into a family of quasi-triangular Hom-bialgebras \( A_\alpha \), where \( \alpha \) runs through the bialgebra endomorphisms on \( A \). Here the twisting procedure is applied to the (co)multiplication in \( A \) (theorem 3.1). On the other hand, if \( A \) is a quasi-triangular Hom-bialgebra with a surjective twisting map \( \alpha \), then we obtain a sequence of quasi-triangular Hom-bialgebras by replacing \( R \in A \otimes^\otimes \) with \((\alpha^r \otimes \alpha^l)(R)\) for \( n \geq 1 \) (theorem 3.3). These twisting procedures yield lots of examples of quasi-triangular Hom-bialgebras. As illustrations, we apply these twisting procedures to Majid’s anyon-generating quantum groups \([35]\) (example 3.5), the quasi-triangular group bialgebra \( kG \) (example 3.6) and function bialgebra \( k(G) \) (example 3.7) for finite Abelian groups \( G \), and Drinfel’d’s quantum enveloping algebras \( U_h(g) \) (examples 3.8 and 3.9) for semi-simple Lie algebras \( g \). Since \( U_h(g) \) is non-commutative and non-cocommutative, the quasi-triangular Hom-bialgebras obtained by twisting \( U_h(g) \) are simultaneously non-(co)associative and non-(co)commutative.

The relationship between the QYBE (1.0.2) and the YBE (1.0.3) described above is generalized to the Hom-type setting in section 4. We show that a suitably defined module over a quasi-triangular Hom-bialgebra with \( \alpha \)-invariant \( R \) has a canonical solution of the HYBE given by \( B = \tau \circ R \) (theorem 4.5). This generalizes the solution of the YBE associated with a module over a quasi-triangular bialgebra, as discussed above.

We illustrate the concept of modules over a quasi-triangular Hom-bialgebra with \( \alpha \)-invariant \( R \) in section 5. In particular, we consider the quasi-triangular Hom-bialgebra \( U_h(sl_2)_\alpha \) (example 3.9) and a sequence of modules \( \tilde{V}_n \) over it (proposition 5.2). Here \( \tilde{V}_n \) is topologically free of rank \( n + 1 \) over \( C[[\hbar]] \). We also write down explicitly the matrix representing the canonical solution of the HYBE associated with \( \tilde{V}_1 \) (proposition 5.3).

2. Quasi-triangular Hom-bialgebras and the QHYBEs

In this section, we first recall the definition of a Hom-bialgebra (definition 2.2). Then we define quasi-triangular Hom-bialgebras (definition 2.7) and establish the QHYBEs (theorem 2.10). Several characterizations of the axioms of a quasi-triangular Hom-bialgebra are given at the...
end of this section (theorems 2.13 and 2.14). Concrete examples of quasi-triangular Hombialgebras are given in the following section.

2.1. Conventions and notations

We work over a fixed commutative ring $k$ of characteristic 0. Modules, tensor products and linear maps are all taken over $k$. If $V$ and $W$ are $k$-modules, then $\tau : V \otimes W \to W \otimes V$ denotes the twist isomorphism, $\tau(v \otimes w) = w \otimes v$. For a map $\phi : V \to W$ and $v \in V$, we sometimes write $\phi(v)$ as $\langle \phi, v \rangle$. If $k$ is a field and $V$ is a $k$-vector space, then the linear dual of $V$ is $V^* = \text{Hom}(V,k)$. From now on, whenever the linear dual $V^*$ is in sight, we tacitly assume that $k$ is a characteristic 0 field.

Given a bilinear map $\mu : V^\otimes 2 \to V$ and elements $x, y \in V$, we often write $\mu(x, y)$ as $xy$ and put in parentheses for longer products. For a map $\Delta_1 : V \to V^\otimes 2$, we use Sweedler's notation for comultiplication:

\[ \Delta_1(x) = \sum (x) x_1 \otimes x_2. \]

To simplify the typography in computations, we often omit the summation sign $\sum (x)$.

**Definition 2.2.**

(i) A Hom-associative algebra $[37]$ $(A, \mu, \alpha)$ consists of a $k$-module $A$, a bilinear map $\mu : A^\otimes 2 \to A$ (the multiplication) and a linear self-map $\alpha : A \to A$ such that the Hom-associativity condition

\[ \mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha) \]

is satisfied.

(ii) A Hom-coassociative coalgebra $[38, 40]$ $(C, \Delta, \alpha)$ consists of a $k$-module $C$, a linear map $\Delta : C \to C^\otimes 2$ (the comultiplication) and a linear self-map $\alpha : C \to C$ such that the Hom-coassociativity condition

\[ (\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta \]

is satisfied.

(iii) A Hom-bialgebra $[38, 54]$ is a quadruple $(A, \mu, \Delta, \alpha)$ in which $(A, \mu, \alpha)$ is a Hom-associative algebra, $(A, \Delta, \alpha)$ is a Hom-coassociative coalgebra and the following compatibility condition holds:

\[ \Delta \circ \mu = \mu^\otimes 2 \circ (\text{Id} \otimes \tau \otimes \text{Id}) \circ \Delta^\otimes 2. \quad (2.2.1) \]

A Hom-(co)associative (co)algebra is **multiplicative** if $\alpha$ is (co)multiplicative with respect to the (co)multiplication. A Hom-bialgebra is multiplicative if its Hom-associative algebra and Hom-coassociative coalgebra are both multiplicative. The map $\alpha$ is called the **twisting map**.

The compatibility condition (2.2.1) can be restated as

\[ \Delta(xy) = \sum (xy) x_1 y_1 \otimes x_2 y_2. \]

In a Hom-bialgebra, the (co)multiplication is not (co)associative, but the non-(co)associativity is controlled by the twisting map $\alpha$. In particular, a bialgebra is a multiplicative Hom-bialgebra when equipped with $\alpha = \text{Id}$. More generally, any bialgebra can be twisted into a multiplicative Hom-bialgebra via a bialgebra morphism, as explained in the example below.
Example 2.3.
(i) If \((A, \mu)\) is an associative algebra and \(\alpha : A \to A\) is an algebra morphism, then
\[ A_\alpha = (A, \mu_\alpha = \alpha \mu, \alpha) \]
is a multiplicative Hom-associative algebra \([52]\). Indeed, the Hom-associativity condition in \(A_\alpha\) is equal to \(\alpha^2\) applied to the associativity condition of \(\mu\).
(ii) Dually, if \((C, \Delta)\) is a coassociative coalgebra and \(\alpha : C \to C\) is a coalgebra morphism, then
\[ C_\alpha = (C, \Delta_\alpha = \Delta \alpha, \alpha) \]
is a multiplicative Hom-coassociative coalgebra.
(iii) Combining the previous two cases, if \((A, \mu, \Delta)\) is a bialgebra and \(\alpha : A \to A\) is a bialgebra morphism, then
\[ A_\alpha = (A, \mu_\alpha = \alpha \mu, \Delta_\alpha = \Delta \alpha, \alpha) \]
is a multiplicative Hom-bialgebra. The compatibility condition \((2.2.1)\) for \(\Delta_\alpha\) and \(\mu_\alpha\) is straightforward to check.

It is clear that the axioms of a Hom-coassociative coalgebra are dual to those of a Hom-associative algebra. The following examples have to do with this duality.

Example 2.4.
(i) Let \((C, \Delta, \alpha)\) be a Hom-coassociative coalgebra and \(C^*\) be the linear dual of \(C\). Then we have a Hom-associative algebra \((C^*, \Delta^*, \alpha^*)\), where
\[ \langle \Delta^*(\phi, \psi), x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle \quad \text{and} \quad \alpha^*(\phi) = \phi \circ \alpha \] \hspace{1cm} (2.4.1)
for all \(\phi, \psi \in C^*\) and \(x \in C\). This is checked in exactly the same way as for (co)associative algebras \([1, 2.1]\), as was done in \([40, \text{corollary 4.12}]\).
(ii) Likewise, suppose that \((A, \mu, \alpha)\) is a finite-dimensional Hom-associative algebra. Then \((A^*, \mu^*, \alpha^*)\) is a Hom-coassociative coalgebra, where
\[ \langle \mu^*(\phi), x \otimes y \rangle = \langle \phi, \mu(x, y) \rangle \quad \text{and} \quad \alpha^*(\phi) = \phi \circ \alpha \] \hspace{1cm} (2.4.2)
for all \(\phi \in A^*\) and \(x, y \in A\) \([40, \text{corollary 4.12}]\). In what follows, whenever \(\mu^* : A^* \to A^* \otimes A^*\) is in sight, we tacitly assume that \(A\) is finite dimensional.
(iii) Combining the previous two examples, suppose that \((A, \mu, \Delta, \alpha)\) is a finite-dimensional Hom-bialgebra. Then so is \((A^*, \Delta^*, \mu^*, \alpha^*)\), where \(\Delta^*, \mu^*\) and \(\alpha^*\) are defined as in \((2.4.1)\) and \((2.4.2)\).

To generalize quantum groups to the Hom-type setting, we need a suitably weakened notion of a multiplicative identity for Hom-associative algebras.

Definition 2.5.
(i) Let \((A, \mu, \alpha)\) be a Hom-associative algebra. A weak unit \([19]\) of \(A\) is an element \(c \in A\) such that
\[ \alpha(x) = cx = xc \]
for all \(x \in A\). In this case, we call \((A, \mu, \alpha, c)\) a weakly unital Hom-associative algebra.
(ii) Let \((A, \mu, \alpha, c)\) be a weakly unital Hom-associative algebra and \(R \in A^{\otimes 2}\). Define the following elements in \(A^{\otimes 3}\):
\[ R_{12} = R \otimes c, \quad R_{23} = c \otimes R, \quad R_{13} = (\tau \otimes Id)(R_{23}). \] \hspace{1cm} (2.5.1)
The elements in (2.5.1) are our Hom-type generalizations of the elements involved in the QYBE (1.0.2) and in the definition of a quasi-triangular bialgebra.

Example 2.6 ([19] Example 2.2). If \((A, \mu, 1)\) is a unital associative algebra, then the multiplicative Hom-associative algebra \(A_\alpha = (A, \mu_\alpha, \alpha)\) (example 2.3) has a weak unit \(c = 1\). In fact, we have
\[
\mu_\alpha(1, x) = \alpha(\mu(1, x)) = \alpha(x) = \alpha(\mu(x, 1)) = \mu_\alpha(x, 1).
\]

So \((A, \mu_\alpha, \alpha, 1)\) is a weakly unital multiplicative Hom-associative algebra.

We are now ready for the main definition of this paper, which gives a non-associative and non-coassociative version of a quasi-triangular bialgebra.

Definition 2.7. A quasi-triangular Hom-bialgebra is a tuple \((A, \mu, \Delta, \alpha, c, R)\) in which

(i) \((A, \mu, \Delta, \alpha)\) is a multiplicative Hom-bialgebra,

(ii) \(c\) is a weak unit of \((A, \mu, \alpha)\)

(iii) \(R \in A^{\otimes 2}\) satisfies the following three axioms:

\[
(\Delta \otimes \alpha)(R) = R_{13}R_{23}, \quad (2.7.1a)
\]

\[
(\alpha \otimes \Delta)(R) = R_{13}R_{12}, \quad (2.7.1b)
\]

\[
[(\tau \circ \Delta)(x)]R = R\Delta(x) \quad (2.7.1c)
\]

for all \(x \in A\). The elements \(R_{12}, R_{13}\) and \(R_{23}\) are defined in (2.5.1).

Example 2.8. A quasi-triangular bialgebra \((A, R)\) (1.0.1) in the sense of Drinfel’d [12] becomes a quasi-triangular Hom-bialgebra when equipped with \(\alpha = \text{Id}\) and \(c = 1\). In a quasi-triangular bialgebra, it is usually assumed that the quasi-triangular structure \(R\) is invertible. In that case, axiom (2.7.1c) can be restated as
\[
(\tau \circ \Delta)(x) = R\Delta(x)R^{-1}.
\]

However, in this paper, even when we refer to a quasi-triangular bialgebra (e.g., in theorem 3.1 and corollary 3.4), we will not have to use its counit or the invertibility of its quasi-triangular structure. See also remark 3.2.

Some remarks are in order.

Remark 2.9.

(i) The multiplications in (2.7.1a) are computed in each tensor factor using the multiplication \(\mu\) in \(A\). They all make sense because there is no iterated multiplication.

(ii) Since a weak unit \(c\) (definition 2.5) is not actually a multiplicative identity, it does not make much sense to talk about multiplicative inverse in a weakly unital Hom-associative algebra. This is the reason for not insisting on \(R\) (in definition 2.7) being invertible and for stating (2.7.1c) without using \(R^{-1}\).

(iii) Write \(R = \sum s_i \otimes t_i \in A^{\otimes 2}\). From definitions (2.5.1) of the \(R_{ij}\) and the requirement that \(c\) be a weak unit, the three axioms in (2.7.1a)–(2.7.1c) can be restated as

\[
(\Delta \otimes \alpha)(R) = \sum \alpha(s_i) \otimes \alpha(s_j) \otimes t_it_j, \quad (2.9.1a)
\]

\[
(\alpha \otimes \Delta)(R) = \sum s_is_j \otimes \alpha(t_i) \otimes \alpha(t_j), \quad (2.9.1b)
\]

\[
\sum s_is_j \otimes x_1t_i = \sum s_it_j \otimes x_1x_2, \quad (2.9.1c)
\]

where \(\Delta(x) = \sum x_1 \otimes x_2\).
A major reason for introducing quasi-triangular bialgebras \((A, R)\) [12] is that the quasi-triangular structure \(R\) satisfies the QYBE (1.0.2). As we mentioned in the introduction, the fact that \(R\) is a solution of the QYBE leads to the solutions of the YBE (1.0.3) in the representations of \(A\). We will generalize this relationship between the QYBE and the YBE to the Hom-type setting in section 4. As a first step, we now show that the element \(R\) in a quasi-triangular Hom-bialgebra satisfies two non-associative versions of the QYBE.

**Theorem 2.10.** Let \((A, \mu, \Delta, \alpha, c, R)\) be a quasi-triangular Hom-bialgebra. Then \(R\) satisfies the QHYBEs

\[
(R_{12}R_{13})R_{23} = R_{23}(R_{13}R_{12}) \quad \text{(2.10.1)}
\]

and

\[
R_{12}(R_{13}R_{23}) = (R_{23}R_{13})R_{12}. \quad \text{(2.10.2)}
\]

**Proof.** First consider (2.10.1). Recall that \(R_{23} = c \otimes R\), and we have

\[
R_{12}R_{13} = (\text{Id} \otimes \tau)(R_{13}R_{12}). \quad \text{(2.10.3)}
\]

We compute as follows:

\[
(R_{12}R_{13})R_{23} = [(\text{Id} \otimes \tau)(R_{13}R_{12})]R_{23} \quad \text{by (2.10.3)}
\]

\[
= [(\text{Id} \otimes \tau)(\alpha \otimes \Delta)(R)]R_{23} \quad \text{by (2.7.1b)}
\]

\[
= [(\alpha \otimes (\tau \circ \Delta))(R)](c \otimes R) \quad \text{by (2.7.1c)}
\]

\[
= (c \otimes R)[(\alpha \otimes \Delta)(R)] \quad \text{by (2.7.1c)}
\]

\[
= R_{23}(R_{13}R_{12}) \quad \text{by (2.7.1b)}.
\]

This proves that \(R\) satisfies the QHYBE (2.10.1).

The other QHYBE (2.10.2) is proved by a similar computation. Since \(R_{12} = R \otimes c\) and

\[
(\tau \otimes \text{Id})(R_{13}R_{23}) = R_{23}R_{13}, \quad \text{(2.10.5)}
\]

we have

\[
R_{12}(R_{13}R_{23}) = (R \otimes c)[(\Delta \otimes \alpha)(R)] \quad \text{by (2.7.1a)}
\]

\[
= [(\tau \circ \Delta) \otimes \alpha)(R)](R \otimes c) \quad \text{by (2.7.1c)}
\]

\[
= [(\tau \otimes \text{Id}) \circ (\Delta \otimes \alpha)(R)]R_{12}
\]

\[
= [(\tau \otimes \text{Id})(R_{13}R_{23})]R_{12} \quad \text{by (2.7.1a)}
\]

\[
= (R_{23}R_{13})R_{12} \quad \text{by (2.10.5)}.
\]

This finishes the proof. \(\square\)

**Remark 2.11.** Let us make it clear that the two QHYBEs (2.10.1) and (2.10.2) are indeed different in general. Writing \(R = \sum s_i \otimes t_i\), the left-hand side of (2.10.1) is

\[
(R_{12}R_{13})R_{23} = (s-js_i \otimes t_i c \otimes c t_i)(c \otimes s_k \otimes t_k)
\]

\[
= (s-js_i \otimes \alpha(t_i) \otimes \alpha(t_i))(c \otimes s_k \otimes t_k)
\]

\[
= \alpha(s, s_i) \otimes \alpha(t_i)s_k \otimes \alpha(t_i)t_k
\]

\[
= \alpha(s)\alpha(s_i) \otimes \alpha(t_i)s_k \otimes \alpha(t_i)t_k. \quad \text{(2.11.1)}
\]
Likewise, the left-hand side of (2.10.2) is
\[
R_{12}(R_{13}R_{23}) = (s_j \otimes t_j \otimes c)(s_c \otimes c_{sk} \otimes t_{tk}) \\
= (s_j \otimes t_j \otimes c)(\alpha(s_j) \otimes \alpha(s_k) \otimes t_{tk}) \\
= s_j\alpha(s_j) \otimes t_j\alpha(s_k) \otimes \alpha(t_{tk}) \\
= s_j\alpha(s_j) \otimes t_j\alpha(s_k) \otimes \alpha(t_{ti})\alpha(t_k).
\tag{2.11.2}
\]

One observes that the last lines in (2.11.1) and in (2.11.2) are different. A similar computation for \(R_{23}(R_{13}R_{12})\) and \((R_{23}R_{13})R_{12}\) shows that they are different as well.

**Remark 2.12.** On the other hand, suppose that \(R\) is \(\alpha\)-invariant, i.e.
\[
\alpha^{\otimes 2}(R) = \sum \alpha(s_j) \otimes \alpha(t_k) = R.
\]

Then one can see from (2.11.1) and (2.11.2) (and a similar computation for \(R_{23}(R_{13}R_{12})\) and \((R_{23}R_{13})R_{12}\)) that the two QHYBEs (2.10.1) and (2.10.2) coincide.

We conclude this section with some alternative characterizations of the axioms (2.7.1a) and (2.7.1b) of a quasi-triangular Hom-bialgebra. They generalize an observation in [12, page 812 (5)] that characterizes two of the axioms of a quasi-triangle bialgebra in terms of a certain map being an algebra morphism and a coalgebra anti-morphism. Let \((A, \mu, \Delta, \alpha, c)\) be a Hom-bialgebra with a weak unit \(c\), \(R \in A^{\otimes 2}\) be an arbitrary element and \(A^*\) be the linear dual of \(A\). Define four linear maps \(\lambda_1, \lambda'_1, \lambda_2, \lambda'_2: A^* \to A\) by setting
\[
\lambda_1(\phi) = (\phi \otimes \alpha, R), \quad \lambda'_1(\phi) = (\alpha(\phi) \otimes Id, R), \\
\lambda_2(\phi) = (\alpha \otimes \phi, R), \quad \lambda'_2(\phi) = (Id \otimes \alpha(\phi), R)
\tag{2.12.1}
\]
for \(\phi \in A^*\).

In the following characterizations of axioms (2.7.1a) and (2.7.1b), we use the operations \(\Delta^*: A^* \otimes A^* \to A^*\) and \(\mu^*: A^* \to A^* \otimes A^*\) in (2.4.1) and (2.4.2).

**Theorem 2.13.** Let \((A, \mu, \Delta, \alpha, c)\) be a Hom-bialgebra with a weak unit \(c\) and \(R \in A^{\otimes 2}\) be an arbitrary element. With the notations in (2.12.1), the following statements are equivalent.

(i) Axiom (2.7.1a) holds, i.e., \((\Delta \otimes \alpha)(R) = R_{13}R_{23}\).

(ii) The diagram
\[
\begin{array}{ccc}
A^* \otimes A^* & \xrightarrow{\lambda'_1 \otimes \lambda'_2} & A \otimes A \\
\Delta^* \downarrow & & \downarrow \mu \\
A^* & \xrightarrow{\lambda_1} & A
\end{array}
\tag{2.13.1}
\]
is commutative.

If \(A\) is finite dimensional, then the two statements above are also equivalent to the commutativity of the diagram
\[
\begin{array}{ccc}
A^* & \xrightarrow{\lambda'_2} & A \\
\mu^* \downarrow & & \downarrow \Delta \\
A^* \otimes A^* & \xrightarrow{\lambda_2 \otimes \lambda_2} & A \otimes A.
\end{array}
\tag{2.13.2}
\]
Proof. First we show the equivalence between axiom (2.7.1a) and the commutativity of the square (2.13.1). Write \( R = \sum s_i \otimes t_i \). Axiom (2.7.1a) holds if and only if
\[
\langle \phi \otimes \psi \otimes \text{Id}, (\Delta \otimes \alpha)(R) \rangle = \langle \phi \otimes \psi \otimes \text{Id}, R_{13}R_{23} \rangle \quad (2.13.3)
\]
for all \( \phi, \psi \in A^* \). The left-hand side of (2.13.3) is
\[
\langle \phi \otimes \psi \otimes \text{Id}, (\Delta \otimes \alpha)(R) \rangle = \langle \phi \otimes \psi, \Delta(s_i)\alpha(t_i) \rangle
= \langle \Delta^*(\phi, \psi), s_i\alpha(t_i) \rangle
= \langle \Delta^*(\phi, \psi) \otimes \alpha, s_i \otimes t_i \rangle
= \lambda_1(\Delta^*(\phi, \psi)).
\]
On the other hand, we have
\[
R_{13}R_{23} = \sum \alpha(s_i) \otimes \alpha(s_j) \otimes t_it_j.
\]
So the right-hand side of (2.13.3) is
\[
\langle \phi \otimes \psi \otimes \text{Id}, R_{13}R_{23} \rangle = \langle \phi \otimes \psi \otimes \text{Id}, \alpha(s_i) \otimes \alpha(s_j) \otimes t_it_j \rangle
= \langle \phi, \alpha(s_i) \rangle \langle \psi, \alpha(s_j) \rangle t_it_j
= \langle (\alpha^*(\phi), s_i t_i), (\alpha^*(\psi), s_j t_j) \rangle
= \langle (\alpha^*(\phi) \otimes \text{Id}, s_i \otimes t_i), (\alpha^*(\psi) \otimes \text{Id}, s_j \otimes t_j) \rangle
= \mu(\lambda'_1(\phi), \lambda'_1(\psi)).
\]
Therefore, condition (2.13.3) holds for all \( \phi, \psi \in A^* \) if and only if the square (2.13.1) is commutative.

Next we show the equivalence between axiom (2.7.1a) and the commutativity of the square (2.13.2) when \( A \) is finite dimensional. The finite dimensionality of \( A \) ensures that \( \mu^* \) is well defined. Axiom (2.7.1a) holds if and only if
\[
\langle \text{Id} \otimes \text{Id} \otimes \phi, (\Delta \otimes \alpha)(R) \rangle = \langle \text{Id} \otimes \text{Id} \otimes \phi, R_{13}R_{23} \rangle \quad (2.13.6)
\]
for all \( \phi \in A^* \). The left-hand side of (2.13.6) is
\[
\langle \text{Id} \otimes \text{Id} \otimes \phi, (\Delta \otimes \alpha)(R) \rangle = \Delta(s_i)\langle \phi, \alpha(t_i) \rangle
= \Delta(s_i, \alpha^*(\phi), \alpha(t_i))
= \Delta(\text{Id} \otimes \alpha^*(\phi), s_i \otimes t_i)
= \Delta(\lambda'_2(\phi)).
\]
Below we write \( \mu^*(\phi) = \sum \phi_1 \otimes \phi_2 \). The right-hand side of (2.13.6) is
\[
\langle \text{Id} \otimes \text{Id} \otimes \phi, R_{13}R_{23} \rangle = \langle \text{Id} \otimes \text{Id} \otimes \phi, \alpha(s_i) \otimes \alpha(s_j) \otimes t_it_j \rangle
= \alpha(s_i) \otimes \alpha(s_j) \langle \mu^*(\phi), t_i \otimes t_j \rangle
= \alpha(s_i) \langle \phi_1, t_i \rangle \otimes \alpha(s_j) \langle \phi_2, t_j \rangle
= \langle \alpha \otimes \phi_1, s_i \otimes t_i \rangle \otimes \langle \alpha \otimes \phi_2, s_j \otimes t_j \rangle
= \lambda'^{\otimes 2}_2(\mu^*(\phi)).
\]
Therefore, condition (2.13.6) holds for all \( \phi \in A^* \) if and only if the square (2.13.2) is commutative.

The following result is the analogue of theorem 2.13 for axiom (2.7.1b).

Theorem 2.14. Let \((A, \mu, \Delta, \alpha, c)\) be a Hom-bialgebra with a weak unit \( c \) and \( R \in A^{\otimes 2} \) be an arbitrary element. With the notations in (2.12.1), the following statements are equivalent.
(i) Axiom (2.7.1b) holds, i.e., \((\alpha \otimes \Delta)(R) = R_{13}R_{12}\).

(ii) The diagram

\[
\begin{array}{ccc}
A^* \otimes A^* & \xrightarrow{\lambda_2' \otimes \lambda_2} & A \otimes A \\
\Delta^* & \downarrow & \mu^* \\
A^* & \xrightarrow{\lambda_2} & A
\end{array}
\]

is commutative, where \(\mu^* = \mu \circ \tau\).

If \(A\) is finite dimensional, then the two statements above are also equivalent to the commutativity of the diagram

\[
\begin{array}{ccc}
A^* & \xrightarrow{\lambda_1} & A \\
\mu^{*op} & \downarrow & \Delta \\
A^* \otimes A^* & \xrightarrow{\lambda_1 \otimes \lambda_1} & A \otimes A
\end{array}
\]

where \(\mu^{op} = \tau \circ \mu^*\).

**Proof.** This proof is completely analogous to that of theorem 2.13, so we give only a sketch. Axiom (2.7.1b) is equivalent to the equality

\[
(\text{Id} \otimes \phi \otimes \psi, (\alpha \otimes \Delta)(R)) = (\text{Id} \otimes \phi \otimes \psi, R_{13}R_{12})
\]

for all \(\phi, \psi \in A^*\). Some calculation shows that this equality is in turn equivalent to the commutativity of the square (2.14.1). To show the equivalence between (2.7.1b) and the commutativity of the square (2.14.2), one uses \(\phi \otimes \text{Id} \otimes \text{Id}\) instead of \(\text{Id} \otimes \phi \otimes \psi\). \(\square\)

In the special case \(\alpha = \text{Id}\), we have \(\lambda_1 = \lambda_1'\) and \(\lambda_2 = \lambda_2'\). So in this case, the commutative diagrams (2.13.1), (2.13.2), (2.14.1) and (2.14.2) mean, respectively, that \(\lambda_1\) is an algebra morphism, \(\lambda_2\) is a coalgebra morphism, \(\lambda_2\) is an algebra anti-morphism and \(\lambda_1\) is a coalgebra anti-morphism.

### 3. Examples of quasi-triangular Hom-bialgebras

Before we discuss the relationships between the QHYBEs ((2.10.1) and (2.10.2)) and the Hom version of the YBE, in this section we describe several classes of quasi-triangular Hom-bialgebras (examples 3.5–3.9).

We begin with some general twisting procedures by which quasi-triangular Hom-bialgebras can be constructed (theorem 3.1 and corollary 3.4). The first twisting procedure, applied to the (co)multiplication, produces a family of quasi-triangular Hom-bialgebras from any given quasi-triangular bialgebra. Recall the definitions of a quasi-triangular Hom-bialgebra (definition 2.7) and of a quasi-triangular bialgebra (in the paragraph containing (1.0.2)).

**Theorem 3.1.** Let \((A, \mu, \Delta, R)\) be a quasi-triangular bialgebra and \(\alpha : A \to A\) be a bialgebra morphism (not-necessarily preserving 1). Then

\[
A_\alpha = (A, \mu_\alpha = \alpha \mu, \Delta_\alpha = \Delta \alpha, \alpha, 1, R)
\]

is a quasi-triangular Hom-bialgebra.
Proof. As noted in examples 2.3 and 2.6, \((A, \mu, \Delta, \alpha, 1)\) is a multiplicative Hom-bialgebra with a weak unit \(c = 1\). It remains to verify the three axioms (2.7.1a)–(2.7.1c) for \(A_u\).

For (2.7.1a) note that, since \(R = \sum s_i \otimes t_i\) is a quasi-triangular structure (1.0.1), we have
\[
(\Delta \otimes \text{Id})(R) = R_{13}R_{23} = \sum s_i \otimes s_j \otimes t_j,
\]
where \(t_j t_j = \mu(t_i, t_j)\). Also, we have \(\Delta_a = \alpha \otimes \Delta\) because \(\alpha\) is a bialgebra morphism. Therefore, we have
\[
(\Delta_a \otimes \alpha)(R) = ((\alpha \otimes \Delta) \otimes \alpha)(R) = \alpha \otimes (\Delta \otimes \text{Id})(R) = \alpha \otimes \alpha \otimes (s_i \otimes s_j \otimes t_i t_j).
\]
This proves (2.7.1a) (in the alternative formulation (2.9.1a)) for \(A_u\). Similarly, for (2.7.1b) we use
\[
(\text{Id} \otimes \Delta)(R) = R_{13}R_{12} = \sum s_j s_i \otimes t_i \otimes t_j.
\]
This gives
\[
(\alpha \otimes \Delta_a)(R) = (\alpha \otimes (\alpha \otimes \Delta))(R) = \alpha \otimes \alpha \otimes (s_j s_i \otimes t_i \otimes t_j) = \mu_a(s_j, s_i) \otimes \alpha(t_i) \otimes \alpha(t_j).
\]
This proves (2.7.1b) (in the alternative formulation (2.9.1b)) for \(A_u\).

For (2.7.1c) note that we have \(((\tau \circ \Delta)(y))R = R\Delta(y)(1.0.1)\) for \(y \in A\), i.e.
\[
\sum y_2 s_i \otimes y_1 t_i = \sum s_j y_1 \otimes t_j y_2,
\]
where \(\Delta(y) = \sum y_1 \otimes y_2\). We use it below when \(y = \alpha(x)\) for \(x \in A\). We have
\[
\mu_a((\tau \circ \Delta_a)(x), R) = \mu_a(\alpha(x)_2 \otimes \alpha(x)_1, s_j \otimes t_i) = \alpha(\alpha(x)_2 s_i) \otimes \alpha(\alpha(x)_1 t_i) = \alpha(s_i \alpha(x)_1) \otimes \alpha(t_i \alpha(x)_2) = \mu_a(R, \Delta_a(x)).
\]
This proves (2.7.1c) for \(A_u\). \(\square\)

Remark 3.2. In the proof of theorem 3.1, we did not use the invertibility of \(R\). So theorem 3.1 is still true even if \(R\) is not invertible. The same goes for corollary 3.4 below.

The second twisting procedure, applied to the element \(R\), produces a family of quasi-triangular Hom-bialgebras from any given quasi-triangular Hom-bialgebra with a surjective twisting map.

Theorem 3.3. Let \((A, \mu, \Delta, \alpha, c, R)\) be a quasi-triangular Hom-bialgebra with \(c\) surjective and \(n\) be a positive integer. Then
\[
A^{(n)} = (A, \mu, \Delta, \alpha, c, R^n)
\]
is also a quasi-triangular Hom-bialgebra, where \(R^n = (\alpha^n \otimes \alpha^n)(R)\).
Proof. By induction it suffices to prove the case \( n = 1 \). We need to check the three axioms (2.7.1a)--(2.7.1c) for
\[
R^\alpha = (\alpha \otimes \alpha)(R) = \sum \alpha(s_i) \otimes \alpha(t_i),
\]
where \( R = \sum s_i \otimes t_i \). Using the assumption that \( \alpha \) is (co)multiplicative and that \( c \) is a weak unit, we compute as follows:
\[
(\Delta \otimes \alpha)(R^\alpha) = (\Delta \otimes \alpha)(\alpha \otimes \alpha)(R)
= \alpha^{\otimes 3}((\Delta \otimes \alpha)(R))
= \alpha^{\otimes 3}(\alpha(s_i) \otimes \alpha(s_j) \otimes t_i t_j) \quad \text{by (2.9.1a)}
= \alpha^2(s_i) \otimes \alpha^2(s_j) \otimes \alpha(t_i t_j)
= \alpha(s_i) c \otimes c \alpha(s_j) \otimes \alpha(t_i) \alpha(t_j)
= (\alpha(s_i) \otimes c \otimes \alpha(t_i))(c \otimes \alpha(s_j) \otimes \alpha(t_j))
= R_1^\alpha R_2^\alpha.
\]
This proves (2.7.1a) for \( R^\alpha \). Axiom (2.7.1b) for \( R^\alpha \) is proved similarly. Note that we have not used the surjectivity assumption of \( \alpha \) so far.

To prove (2.7.1c) for \( R^\alpha \), pick an element \( x \in A \). Since \( \alpha \) is assumed to be surjective, we have \( x = \alpha(y) \) for some (not-necessarily unique) element \( y \in A \). By the comultiplicativity of \( \alpha \), we have
\[
\Delta(x) = \sum x_1 \otimes x_2 = \sum \alpha(y_1) \otimes \alpha(y_2) = \sum \alpha(y_1) \otimes \alpha(y_2) = \alpha^{\otimes 2}(\Delta(y)).
\]
Using the multiplicativity of \( \alpha \), we compute as follows:
\[
[[\tau \circ \Delta](x)]R^\alpha = \alpha(y_2)\alpha(s_i) \otimes \alpha(y_1)\alpha(t_i)
= \alpha^{\otimes 2}((\tau \circ \Delta)(y))R
= \alpha^{\otimes 2}(R \Delta(y)) \quad \text{by (2.7.1c)}
= \alpha(s_i) \alpha(y_1) \otimes \alpha(t_i) \alpha(y_2)
= \alpha(s_i) x_1 \otimes \alpha(t_i) x_2
= R^\alpha \Delta(x).
\]
This proves (2.7.1c) for \( R^\alpha \).

The following result is an immediate consequence of theorems 3.1 and 3.3.

**Corollary 3.4.** Let \((A, \mu, \Delta, R)\) be a quasi-triangular bialgebra, \( \alpha : A \to A \) be a surjective bialgebra morphism (not-necessarily preserving 1) and \( n \) be a positive integer. Then
\[
A^{(n)}_\alpha = (A, \mu_\alpha, \Delta_\alpha, \alpha, 1, R^\alpha)
\]
is a quasi-triangular Hom-bialgebra, where \( \mu_\alpha = \alpha \circ \mu, \Delta_\alpha = \Delta \circ \alpha \) and \( R^\alpha = (\alpha^n \otimes \alpha^n)(R) \).

Since a quasi-triangular Hom-bialgebra is a new kind of algebraic structure, it is helpful to see some very concrete examples. We now give a series of examples of quasi-triangular Hom-bialgebras using theorem 3.1 and corollary 3.4. In each example below, we start with a well-known quantum group and produce a concrete family of quasi-triangular Hom-bialgebras.

**Example 3.5** (Anyon-generating Hom-quantum groups). Consider the group bialgebra \( CZ/n \) over the complex numbers \( C \) generated by the multiplicative cyclic group \( \mathbb{Z}/n \) with a generator \( g \) and the relation \( g^n = 1 \). Its comultiplication is determined by
\[
\Delta(g) = g \otimes g.
\]
which is cocommutative. It becomes a quasi-triangular bialgebra when equipped with the non-trivial quasi-triangular structure

$$R = \frac{1}{n} \sum_{p,q=0}^{n-1} \exp(-2\pi ipq/n) g^p \otimes g^q.$$  

The quasi-triangular bialgebra \((\mathbb{C}Z/n, R)\) is called the anyon-generating quantum group ([35] and [36, example 2.1.6]) and is denoted by \(Z/n\). We can obtain bialgebra morphisms on \(\mathbb{C}Z/n\) by extending the group morphisms

$$\alpha_k : \mathbb{Z}/n \to \mathbb{Z}/n, \quad \alpha_k(g) = g^k$$

for \(k \in \{1, \ldots, n-1\}\). Moreover, \(\alpha_k\) is surjective if and only if \(k\) and \(n\) are relatively prime.

By theorem 3.1, for each \(k \in \{1, \ldots, n-1\}\), we have a quasi-triangular Hom-bialgebra

\[(Z_{/n})_{\alpha_k} = (\mathbb{C}Z/n, \mu_{\alpha_k}, \Delta_{\alpha_k}, \alpha_k, 1, R)\]

with twisted (co)multiplication. Here \(\mu_{\alpha_k} = \alpha_k \circ \mu\) (with \(\mu\) the multiplication in \(\mathbb{C}Z/n\)) and \(\Delta_{\alpha_k}\) is determined by

$$\Delta_{\alpha_k}(g) = g^k \otimes g^k.$$  

Suppose that \(k\) and \(n\) are relatively prime, so \(\alpha_k\) is surjective. By corollary 3.4, for each integer \(t \geq 1\) we have a quasi-triangular Hom-bialgebra

\[(Z_{/n})^{(t)}_{\alpha_k} = (\mathbb{C}Z/n, \mu_{\alpha_k}, \Delta_{\alpha_k}, \alpha_k, 1, R^t),\]

where the twisted quasi-triangular structure is

$$R^t = (\alpha_k^t \otimes \alpha_k^t)(R) = \frac{1}{n} \sum_{p,q=0}^{n-1} \exp(-2\pi ipq/n) g^{tpk} \otimes g^{tpk}.$$  

**Example 3.6** (Hom-quantum group bialgebras). This is a generalization of the previous example. Let \(k\) be a characteristic 0 field, \(G\) be a finite Abelian group, written multiplicatively with identity \(e\), and \(kG\) be its group bialgebra [1, page 58, example 2.4]. Its comultiplication is determined by (3.5.1) for \(g \in G\). Since \(G\) is commutative, \(kG\) is both commutative and cocommutative.

Identify \(kG \otimes kG\) with \(kH\), where \(H = G \times G\). A (not-necessarily invertible) quasi-triangular structure on the group bialgebra \(kG\) is equivalent to a function \(R : G \times G \to k\) such that

$$\sum_{x,y=u} R(u, x)R(w, y) = \delta_{u,w} R(u, v) \quad \text{and} \quad \sum_{x,y=v} R(x, v)R(y, w) = \delta_{v,w} R(u, v)$$

for all \(u, v, w \in G\) [36, example 2.1.17], where \(\delta_{u,w}\) denotes the Kronecker delta. In fact, writing

$$R = \sum_{u, v \in G} R(u, v)u \otimes v,$$

the two conditions in (3.6.1) are equivalent to the axioms \((\Delta \otimes Id)(R) = R_{13}R_{23}\) and \((Id \otimes \Delta)(R) = R_{13}R_{12}\) (1.0.1), respectively. The condition \((\tau \circ \Delta(x)(R) = R\Delta(x)\) is automatic because \(kG\) is both commutative and cocommutative. Fix such a quasi-triangular structure \(R\) for the rest of this example.

If \(\alpha : G \to G\) is any group morphism, then it extends naturally to a bialgebra morphism on \(kG\), where

$$\alpha \left( \sum_u c_u u \right) = \sum_u c_u \alpha(u).$$
By theorem 3.1 (and remark 3.2), we have a quasi-triangular Hom-bialgebra
\[ kG_\alpha = (kG, \mu_\alpha = \alpha \circ \mu, \Delta_\alpha = \Delta \circ \alpha, \alpha, e, R) \]
where \( \mu \) and \( \Delta \) are the multiplication and the comultiplication in \( kG \), respectively.

Suppose, in addition, that \( \alpha : G \to G \) is a group automorphism. Then by corollary 3.4 (and remark 3.2), we have a quasi-triangular Hom-bialgebra
\[ kG_\alpha^n = (kG, \mu_\alpha^n, \Delta_\alpha^n, \alpha, e, R_\alpha^n) \]
for each \( n \geq 1 \). We can make the twisted quasi-triangular structure \( R_\alpha^n = (\alpha^n \otimes \alpha^n)(R) \) more explicit as follows. Thinking of \( R \) as a function \( \mathbb{R}^{G \times G} \to k \), we have
\[ \alpha^{\otimes 2}(R) = \sum_{u, v} R(u, v) \alpha(u) \otimes \alpha(v) = \sum_{u, v} R(\alpha^{-1}(u), \alpha^{-1}(v)) u \otimes v, \]
since \( \alpha \) is invertible. Therefore, \( R_\alpha^n \) is equivalent to the function \( G \times G \to k \) given by
\[ R_\alpha^n(u, v) = R((\alpha^{-1})^n(u), (\alpha^{-1})^n(v)) \quad (3.6.2) \]
for \( u, v \in G \).

Example 3.7 (Hom-quantum function bialgebras). This example is closely related to the previous example. Let \( G \) be a finite Abelian group, \( k \) be a characteristic 0 field and \( k(G) \) be the bialgebra of functions \( G \to k \) [1, page 58, example 2.4]. Its multiplication \( \mu \) is defined pointwise, i.e.,
\[ \langle \mu(\phi, \psi), u \rangle = \langle \phi, u \rangle \langle \psi, u \rangle, \]
for \( u \in G \) and \( \phi, \psi \in k(G) \). Its comultiplication \( \Delta \) is dual to the multiplication in \( G \), i.e.,
\[ \langle \Delta(\phi), (u, v) \rangle = \langle \phi, uv \rangle \]
for \( \phi \in k(G) \) and \( u, v \in G \). Since \( G \) is commutative, \( k(G) \) is both commutative and cocommutative.

A (not-necessarily invertible) quasi-triangular structure \( R \) on the function bialgebra \( k(G) \) is equivalent to a function \( R : G \times G \to k \) such that
\[ R(\mu u w, ) = R(u, w)R(v, w) \quad \text{and} \quad R(u, vw) = R(u, w)R(u, v) \quad (3.7.1) \]
for all \( u, v, w \in G \) [36, example 2.1.18]. In fact, the two conditions in (3.7.1) are equivalent to the axioms \((\Delta \otimes Id)(R) = R_{12}R_{23} \) and \((Id \otimes \Delta)(R) = R_{13}R_{12} \) (1.0.1), respectively. The condition \((\xi \circ \Delta(\xi))(R) = R\Delta(\xi)\) is automatic because \( k(G) \) is both commutative and cocommutative. Fix such a quasi-triangular structure \( R \) for the rest of this example.

If \( \alpha : G \to G \) is a group morphism, then it extends naturally to a bialgebra morphism \( \alpha^* : k(G) \to k(G) \) given by
\[ \alpha^*(\phi) = \phi \circ \alpha. \]
By theorem 3.1 (and remark 3.2), we have a quasi-triangular Hom-bialgebra
\[ k(G)^{\alpha^*} = (k(G), \mu_{\alpha^*} = \alpha^* \circ \mu, \Delta_{\alpha^*} = \Delta \circ \alpha^*, \alpha^*, e, R). \]
The twisted multiplication and comultiplication are given by
\[ \langle \mu_{\alpha^*}(\phi, \psi), u \rangle = \langle \phi, \alpha(u) \rangle \langle \psi, \alpha(u) \rangle \quad \text{and} \quad \langle \Delta_{\alpha^*}(\phi), (u, v) \rangle = \langle \phi, \alpha(\alpha(v)) \rangle. \]
If the group morphism \( \alpha : G \to G \) is invertible, then so is the induced bialgebra morphism \( \alpha^* : k(G) \to k(G) \). By corollary 3.4 (and remark 3.2), we have a quasi-triangular Hom-bialgebra
\[ k(G)^{\alpha^*} = (k(G), \mu_{\alpha^*}, \Delta_{\alpha^*}, \alpha^*, e, R_{\alpha^*}) \]
for each $n \geq 1$. As a function $G \times G \to k$, the twisted quasi-triangular structure $R^{\alpha}$ is given by

$$R^{\alpha}(u, v) = \{(\alpha^*)^n \otimes (\alpha^*)^n)(R)(u, v) = R(\alpha^u, \alpha^v)\) for $u, v \in G$. This is the function (3.6.2) with $\alpha$ and $\alpha^{-1}$ interchanged.

**Example 3.8** (Hom-quantum enveloping algebras). Let us first recall Drinfel’d’s quantum enveloping algebra $U_q(\mathfrak{g})$ [12, section 13], using the notations in [36, section 3.3]. Another exposition of $U_q(\mathfrak{g})$ is given in [25, XVII]. We refer the reader to [22, 23] for discussion of semi-simple Lie algebras and to [6, 9, 41] for basics of topological algebras over the power series algebra $C[[t]]$.

Let $\mathfrak{g}$ be a finite-dimensional complex semi-simple Lie algebra, $A = (a_{ij})_{1 \leq i, j \leq n}$ be its Cartan matrix, $(\beta_j)_{1 \leq j \leq n}$ be a system of positive simple roots and $d_i = (\beta_i, \beta_i)/2$ be its root lengths, where $(\cdot, \cdot)$ is the inverse of the Killing form. Define the $q$-symbols:

$$q_i = e^{d_{i/2}}, \quad [m]_q^m = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad \left[ \begin{array}{c} m \\ r \end{array} \right]_q = \frac{[m]_q!}{[r]_q! [m-r]_q!}, \quad (3.8.1)$$

where

$$[r]_q! = [r]_q [r-1]_q \cdots [1]_q \quad \text{and} \quad [0]_q! = 1.$$

The quantum enveloping algebra $U_q(\mathfrak{g})$ is defined as the topological $C[[t]]$-algebra that is topologically generated by the set $\{H_i, X_{\pm i}\}_{1 \leq i \leq n}$ of $3n$ generators with the following relations:

$$[H_i, H_j] = 0, \quad [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j},$$

$$[X_{\pm i}, X_{\mp j}] = \delta_{ij} q_i^{H_0} - q_i^{-H_0}/q_i - q_i^{-1}, (3.8.2)$$

and if $i \neq j$,

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} X_{\pm i}^{1-a_{ij}-k} X_{\pm j}^k X_{\pm i}^k = 0. \quad (3.8.3)$$

The comultiplication $\Delta$ in $U_q(\mathfrak{g})$ is determined by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta(X_{\pm i}) = X_{\pm i} \otimes q_i^{H_0/2} + q_i^{-H_0/2} \otimes X_{\pm i}$$

for $1 \leq i \leq n$. The bialgebra $U_q(\mathfrak{g})$ becomes a quasi-triangular bialgebra when equipped with the quasi-triangular structure

$$R = \sum_{a \in \mathbb{N}} \left\{ \exp \left( \frac{1}{2} t_0 + \frac{1}{4} (H_0 - 1 - 1 \otimes H_0) \right) \right\} P_a, \quad (3.8.4)$$

where $\mathbb{N}$ is the set of non-negative integers,

$$H_a = \sum_{i=1}^n a_i H_i$$

for $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, and

$$t_0 = \sum_{i,j} (DA)^{-1}_{ij} H_j \otimes H_j$$

with $D = \text{diag}(d_1, \ldots, d_n)$ (i.e., the diagonal matrix with $d_i$ as its $i$th diagonal entry). The symbol $P_a$ denotes a certain polynomial in the variables $u_i = X_{+i} \otimes 1$ and $v_i = 1 \otimes X_{-i}$.
that is homogeneous of degree \( a_t \) in \( u_i \) and \( v_i \), and \( P_b = 1 \otimes 1 \). More information about the quasi-triangular structure \( R \) (3.8.5) can be found in [26, 32, 36, 44].

We can obtain bialgebra morphisms on \( U_h(g) \) as follows. Let \( e = (c_1, \ldots, c_n) \in \mathbb{C}^n \) be any \( n \)-tuple of complex numbers. Define

\[
H_e = \sum_{i=1}^n c_i H_i
\]

and the \( \mathbb{C}[[h]] \)-linear map \( \alpha_e : U_h(g) \to U_h(g) \) by

\[
\alpha_e(u) = e^{hH_e} u e^{-hH_e}
\]

(3.8.6) for \( u \in U_h(g) \). Then \( \alpha_e \) is a consequence of the second relation in (3.8.2). Therefore, the map \( \alpha_e \) preserves relations (3.8.2) and (3.8.4) implies that \( \alpha_e^{\otimes 2} \circ \Delta \) and \( \Delta \circ \alpha_e \) coincide when applied to \( H_j \). Also, we have

\[
\alpha_e(X_{\pm j}) = \gamma_j^{\pm 1} X_{\pm j},
\]

(3.8.8) where

\[
\gamma_j = \exp \left( \sum_{i=1}^n c_i a_{ij} \right),
\]

by the second relation in (3.8.2). More precisely, we are using [25, page 408 (2.5)]

\[
e^{hH_j} X_{\pm j} e^{-hH_j} = e^{\pm c_{ij} X_{\pm j}},
\]

which is a consequence of the second relation in (3.8.2). Since \( \alpha_e \) fixes

\[
q_j^{\pm H_i/2} = e^{\pm h\delta_{ij} H_i/4},
\]

we infer from (3.8.4) that

\[
\alpha_e^{\otimes 2}(\Delta(X_{\pm j})) = \gamma_j^{\pm 1} \Delta(X_{\pm j}) = \Delta(\alpha_e(X_{\pm j})).
\]

Therefore, the map \( \alpha_e \) is a bialgebra automorphism on \( U_h(g) \). Alternatively, one can use (3.8.7) and (3.8.8) as the definition of the map \( \alpha_e \) (on the generators). Then one checks directly that \( \alpha_e \) preserves relations (3.8.2) and (3.8.3) and is compatible with comultiplication (3.8.4).

By theorem 3.1 and corollary 3.4, for each \( n \)-tuple \( e \in \mathbb{C}^n \) and each integer \( t \geq 0 \), we have a quasi-triangular Hom-bialgebra

\[
U_h(g)_{(t)}^a = (U_h(g), \mu_a, \Delta_a, \alpha, 1, R^a)
\]

with \( \alpha = \alpha_e \) (3.8.6). The twisted operations are given by

\[
\mu_a(u, v) = e^{h \mu_a(u, v)} e^{-h \mu_a},
\]

\[
\Delta_a(H_j) = \Delta(H_j),
\]

\[
\Delta_a(X_{\pm j}) = \gamma_j^{\pm 1} \Delta(X_{\pm j}) = e^{\pm \sum_{i=1}^n c_{ij} \Delta(X_{\pm i})}.
\]

The twisted quasi-triangular structure \( R^a \) is \( (\alpha' \otimes \alpha')(R) \), where \( \alpha'^0 = Id \). To make it more explicit, note that

\[
\alpha(H_a) = H_a \quad \text{and} \quad \alpha^{\otimes 2}(t_0) = t_0
\]

because each \( H_j \) is fixed by \( \alpha \) (3.8.7). So the entire exponential term in \( R \) (3.8.5) is fixed by \( \alpha^{\otimes 2} \). For the polynomial \( P_a, \) let us write it as \( P_a(u_1, \ldots, u_n, v_1, \ldots, v_n) \). Since

\[
\alpha^{\otimes 2}(u_j) = \alpha(X_{\pm j}) \otimes 1 = \gamma_j(X_{\pm j} \otimes 1) = \gamma_j u_j
\]
and similarly
\[ \alpha^{\otimes 2}(v_j) = \gamma_j^{-1} v_j, \]
we have
\[ (\alpha' \otimes \alpha') P_a(u_1, \ldots, u_n, v_1, \ldots, v_n) = P_a(\gamma'_1 u_1, \ldots, \gamma'_n u_n, \gamma^{-1}_1 v_1, \ldots, \gamma^{-1}_n v_n). \]
Therefore, we have
\[ R^\vee' = \sum_{a \in \mathbb{N}} \left\{ \exp \left[ \frac{1}{2} t_0 + \frac{1}{4} (H_a \otimes 1 - 1 \otimes H_a) \right] \right\} P_a(\gamma'_1 u_1, \ldots, \gamma'_n u_n, \gamma^{-1}_1 v_1, \ldots, \gamma^{-1}_n v_n), \]
where
\[ \gamma'_j = \exp \left( \pm t \sum_{i=1}^s c_i a_{ij} \right). \]

**Example 3.9** (Hom-quantum enveloping algebra of \( \mathfrak{sl}_2 \)). Let us examine the special case \( \mathfrak{g} = \mathfrak{sl}_2 \) of the previous example. The bialgebra \( U_h(\mathfrak{sl}_2) \) was first studied in [27, 48], and its quasi-triangular structure \((3.9.1)\) was given in [12, page 816]. Using the notations of the previous example with \( \mathfrak{g} = \mathfrak{sl}_2 \), we have \( n = 1, a_{11} = 2 \) and \( d_1 = 1 \), and \( U_h(\mathfrak{sl}_2) \) is the topological \( \mathbb{C}[[\hbar]] \)-algebra generated by \( \{H, X_\pm \} \) with relations \((3.8.2)\), where
\[ q_1 = q = e^{\hbar/2}. \]
Relations \((3.8.3)\) are empty for \( \mathfrak{g} = \mathfrak{sl}_2 \). The quasi-triangular structure is
\[ R = \sum_{a \geq 0} \frac{(q - q^{-1})^a}{[a]_q!} \exp \left[ \frac{\hbar}{4} [H \otimes H + a(H \otimes 1 - 1 \otimes H)] \right] (X_+^a \otimes X_-^a). \] \((3.9.1)\)
As in the previous example, given any complex number \( c \), we have a bialgebra automorphism \( \alpha = \alpha_c : U_h(\mathfrak{sl}_2) \rightarrow U_h(\mathfrak{sl}_2) \) defined as
\[ \alpha(u) = e^{i c \hbar} u e^{-i c \hbar} \] \((3.9.2)\)
for \( u \in U_h(\mathfrak{sl}_2) \).

By theorem 3.1, we have a quasi-triangular Hom-bialgebra
\[ U_h(\mathfrak{sl}_2)_c = (U_h(\mathfrak{sl}_2), \mu_c, \Delta_c, \alpha, 1, R). \]
Moreover, \( R \) \((3.9.1)\) is \( \alpha \)-invariant, i.e. \( \alpha^{\otimes 2}(R) = R \). This is an immediate consequence of
\[ \alpha(H) = H \quad \text{and} \quad \alpha(X_\pm) = e^{\pm 2c} X_\pm, \]
which are special cases of \((3.8.7)\) and \((3.8.8)\). Quasi-triangular Hom-bialgebras with \( \alpha \)-invariant \( R \) play a major role in theorem 4.5 below. We will revisit this example in section 5.

**4. Solutions of the HYBE from quasi-triangular Hom-bialgebras**

In this section, we extend the relationship between the QYBE \((1.0.2)\) and the YBE \((1.0.3)\), as discussed in the introduction, to the Hom-type setting. Let us first recall the following generalization of the YBE.

**Definition 4.1** ([54, 57]). Let \( V \) be a \( k \)-module, \( \alpha : V \rightarrow V \) be a linear map and \( B : V^{\otimes 2} \rightarrow V^{\otimes 2} \) be a bilinear map that commutes with \( \alpha^{\otimes 2} \). We say that \( B \) is a solution of the HYBE for \((V, \alpha)\) if it satisfies
\[ (\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha). \] \((4.1.1)\)
The YBE \((1.0.3)\) is the special case of the HYBE \((4.1.1)\) in which \( \alpha = Id. \)
As in the classical case, solutions of the HYBE are closely related to the braid relations and braid group representations [4, 5]. Indeed, suppose that $B : V^\otimes 2 \rightarrow V^\otimes 2$ is a solution of the HYBE for $(V, \alpha)$. Then for $n \geq 3$ and $1 \leq i \leq n - 1$, the operators

$$B_i = \alpha^\otimes(i-1) \otimes B \otimes \alpha^\otimes(n-i-1) : V^\otimes n \rightarrow V^\otimes n$$

satisfy the braid relations

$$B_iB_j = B_jB_i \quad \text{if} \quad |i - j| > 1 \quad \text{and} \quad B_iB_{i+1}B_i = B_{i+1}B_iB_{i+1}.$$

In particular, if $\alpha$ and $B$ are both invertible, then so are the operators $B_i$. In this case, there is a corresponding representation of the braid group on $V^\otimes n$ [53, theorem 1.4]. Many examples of the solutions of the HYBE can be found in [53, 56].

We will show that every quasi-triangular Hom-bialgebra in which $R$ is fixed by $\alpha^\otimes 2$ gives rise to many solutions of the HYBE via its modules. To make this precise, we need a suitable notion of modules over a Hom-associative algebra.

**Definition 4.2.**

(i) A Hom-module is a pair $(V, \alpha)$ consisting of a $k$-module $V$ and a linear map $\alpha$.

(ii) Let $(A, \mu, \alpha_A)$ be a Hom-associative algebra (definition 2.2). By an $A$-module we mean a Hom-module $(M, \alpha_M)$ together with a linear map $\lambda : A \otimes M \rightarrow M$ such that

$$\lambda(ax) = \alpha_A(a)(bx) \quad \text{and} \quad \alpha_M(ax) = \alpha_A(a)\alpha_M(x) \quad (4.2.1)$$

for $a, b \in A$ and $x \in M$, where $\lambda(a, x)$ is abbreviated to $ax$.

Note that a slightly different notion of a module over a Hom-associative algebra was defined in [38]. The difference with definition 4.2 is that in [38], the second condition in (4.2.1) is not required.

**Example 4.3.**

(i) A multiplicative Hom-associative algebra $(A, \mu, \alpha)$ is an $A$-module with the structure map $\lambda = \mu$. In this case, axioms (4.2.1) are exactly the Hom-associativity and the multiplicativity of $\alpha$ in $A$.

(ii) Let $(A, \mu)$ be an associative algebra, $M$ be an $A$-module in the usual sense with the structure map $\lambda, \alpha_A : A \rightarrow M$ be an algebra morphism and $\alpha_M : M \rightarrow M$ be a linear map. Suppose that

$$\alpha_M \circ \lambda = \lambda \circ (\alpha_A \otimes \alpha_M).$$

This is the case, for example, if $\alpha_A = Id$ and $\alpha_M$ is an $A$-module morphism. Define the twisted action

$$\lambda_M = \alpha_M \circ \lambda.$$

Then it is easy to check that $(M, \alpha_M)$ becomes a module over the multiplicative Hom-associative algebra $A_\alpha = (A, \mu_\alpha = \alpha_A \circ \mu, \alpha_A)$ (example 2.3) with the structure map $\lambda_M$.

**Definition 4.4.** In a quasi-triangular Hom-bialgebra (definition 2.7), we say that the element $R$ is $\alpha$-invariant if $\alpha^\otimes 2(R) = R$.

Some examples of $\alpha$-invariant $R$ were given in example 3.9. When $R$ is $\alpha$-invariant, the two versions of the QHYBE ((2.10.1) and (2.10.2)) coincide, as was noted in remark 2.12.

We can now describe the relationship between quasi-triangular Hom-bialgebras and the HYBE (4.1.1).
Theorem 4.5. Let \((A, \mu, \Delta, \alpha, \epsilon, R)\) be a quasi-triangular Hom-bialgebra in which \(R\) is \(\alpha\)-invariant and \((M, \alpha_M)\) be an \(A\)-module. Then the operator
\[
B = \tau \circ R : M^{\otimes 2} \to M^{\otimes 2}
\]
is a solution of the HYBE (4.1.1) for \((M, \alpha_M)\).

**Proof.** To simplify the typography, we will omit the subscripts in \(\alpha_A\) and \(\alpha_M\). Write \(R = \sum s_i \otimes t_i\). Then the \(\alpha\)-invariance of \(R\) means that
\[
\sum s_i \otimes t_i = \sum \alpha(s_i) \otimes \alpha(t_i). \quad (4.5.1)
\]
Recall that \(\tau\) denotes the twist isomorphism. So the map \(B = \tau \circ R\) is given by
\[
B(v \otimes w) = \sum t_i w \otimes s_i v
\]
for \(v, w \in M\). That \(B\) commutes with \(\alpha^{\otimes 2}\) follows from the \(\alpha\)-invariance of \(R\) and the second axiom in (4.2.1).

It remains to check that \(B = \tau \circ R\) satisfies the HYBE (4.1.1). Note that the \(\alpha\)-invariance of \(R\) (4.5.1) and computation (2.11.1) imply that the QHYBE (2.10.1) now takes the form
\[
\sum s_j s_i \otimes t_j t_i \otimes t_k t_0 = (R_{13} R_{12}) R_{23}
\]
\[
= R_{23} (R_{13} R_{12})
\]
\[
= \sum s_j s_i \otimes t_i t_j \otimes t_k t_0. \quad (4.5.2)
\]
Let \(x\) denote a typical generator \(u \otimes v \otimes w \in M^{\otimes 3}\). Using the \(\alpha\)-invariance of \(R\) (4.5.1) and the module axioms (4.2.1), a direct computation gives
\[
(B \otimes \alpha)(\alpha \otimes B)(B \otimes \alpha)(x) = t_k (t_i \alpha(w)) \otimes s_j \alpha(t_i v) \otimes \alpha(s_j(s_i u))
\]
\[
= t_k (t_i \alpha(w)) \otimes s_j (\alpha(t_i) \alpha(v)) \otimes \alpha(s_j(\alpha(s_i \alpha(u)))
\]
\[
= \alpha(t_i) (t_i \alpha(w)) \otimes \alpha(t_j) (s_j \alpha(v)) \otimes \alpha(s_j(s_i \alpha(u))
\]
\[
= \alpha(t_i) (t_i \alpha(w)) \otimes t_j (\alpha(s_j \alpha(v)) \otimes \alpha(s_j(s_i \alpha(u))
\]
\[
= \alpha(t_i (t_i w)) \otimes t_j (\alpha(s_j v) \otimes s_j(\alpha(s_i \alpha(u))
\]
\[
= (\alpha \otimes B)(B \otimes \alpha)(\alpha \otimes B)(x).
\]

This proves that \(B = \tau \circ R\) is a solution of the HYBE for \((M, \alpha_M)\). \(\square\)

**Corollary 4.6.** Let \((A, \mu, \Delta, R)\) be a quasi-triangular bialgebra, \(M\) be an \(A\)-module with the structure map \(\lambda, \alpha_A : A \to A\) be a bialgebra morphism such that \(\alpha^{\otimes 2}_A(R) = R\) and \(\alpha_M : M \to M\) be a linear map such that
\[
\alpha_M \circ \lambda = \lambda \circ (\alpha_A \otimes \alpha_M).
\]
Then the operator \(B_\alpha : M^{\otimes 2} \to M^{\otimes 2}\) defined by
\[
B_\alpha(v \otimes w) = \sum \alpha_M(\lambda(t_i, w)) \otimes \alpha_M(\lambda(s_i, v)) \quad (4.6.1)
\]
for \(v, w \in M\), where \(R = \sum s_i \otimes t_i\), is a solution of the HYBE (4.1.1) for \((M, \alpha_M)\).
Proof. By theorem 3.1, \(A_A = (A, \mu_A, \Delta_A, \alpha, 1, R)\) is a quasi-triangular Hom-bialgebra and \((M, \alpha_M)\) is an \(A_A\)-module (example 4.3) with the structure map \(\lambda_A = \alpha_M \circ \lambda\). Therefore, theorem 4.5 implies that there is a solution of the HYBE for \((M, \alpha_M)\) of the form
\[
\begin{align*}
B_a(v \otimes w) &= \sum \lambda_A(t_i, w) \otimes \lambda_A(s_i, v) \\
&= \sum \alpha_M(\lambda(t_i, w)) \otimes \alpha_M(\lambda(s_i, v)),
\end{align*}
\]
as was to be shown. \(\square\)

The following result is the special case of the previous corollary with \(\alpha_A = Id\).

**Corollary 4.7.** Let \((A, \mu, \Delta, R)\) be a quasi-triangular bialgebra, \(M\) be an \(A\)-module and \(\alpha_M\) be an \(A\)-module morphism. Then the operator \(B_a : M^\otimes 2 \to M^\otimes 2\) defined in (4.6.1) is a solution of the HYBE (4.1.1) for \((M, \alpha_M)\).

5. Modules over \(U_h(sl_2)\)

In this section, we illustrate the results of the previous section with certain modules over the quasi-triangular Hom-bialgebra \(U_h(sl_2)_\alpha\), which was discussed in example 3.9. We use the same notations as in examples 3.8 and 3.9. In particular, \(U_h(sl_2)\) is the topological \(\mathbb{C}[\hbar]]\)-algebra generated by \([H, X_\pm]\) with relations (3.8.2), where \(q = e^{\hbar/2}\), and its comultiplication is defined as in (3.8.4). It becomes a quasi-triangular Hom-bialgebra when equipped with the quasi-triangular structure \(R\) (3.9.1).

Fix a complex number \(c\), and let \(\alpha : U_h(sl_2) \to U_h(sl_2)\) be the bialgebra automorphism defined by
\[
\alpha(H) = H \quad \text{and} \quad \alpha(X_\pm) = \gamma^{\pm 1} X_\pm,
\]
where
\[
\gamma = e^{\sqrt{c}}.
\]
Equivalently, \(\alpha\) is the inner automorphism (3.9.2) induced by \(e^{cH}\). Then \(U_h(sl_2)_\alpha\) is the quasi-triangular Hom-bialgebra obtained from \(U_h(sl_2)\) by twisting its (co)multiplication along \(\alpha\) (theorem 3.1). Moreover, the element \(R\) (3.9.1) is \(\alpha\)-invariant, i.e. \((\alpha \otimes \alpha)(R) = R\).

Fix a non-negative integer \(n\). Let \(\overline{V}_n\) be the free \(\mathbb{C}[\hbar]]\)-module with a basis \([v_i]_{0 \leq i \leq n}\). Then \(\overline{V}_n\) becomes a (topological) \(U_h(sl_2)\)-module via the map \(\rho : U_h(sl_2) \otimes \overline{V}_n \to \overline{V}_n\) determined by
\[
\begin{align*}
\rho(X_+, v_i) &= [n + 1 - i] v_{i-1}, \\
\rho(X_-, v_i) &= [i + 1] v_{i+1}, \\
\rho(H, v_i) &= (n - 2i) v_i
\end{align*}
\]
for \(0 \leq i \leq n\). In (5.0.1), we set
\[
v_{-1} = 0 = v_{n+1},
\]
and \([m]_q\) is defined as in (3.8.1). See, for example, [25, XVII.4] and [12, 27, 48]. We will apply corollary 4.6 to the \(U_h(sl_2)\)-module \(\overline{V}_n\).

Consider the \(\mathbb{C}[\hbar]]\)-linear automorphism \(\alpha : \overline{V}_n \to \overline{V}_n\) defined by
\[
\alpha(v_i) = \gamma^{-i} v_i
\]
for \(0 \leq i \leq n\).

**Lemma 5.1.** We have
\[
\alpha \circ \rho = \rho \circ (\alpha \otimes \alpha)
\]
as maps \(U_h(sl_2) \otimes \overline{V}_n \to \overline{V}_n\).
Consider the action of the first term in Proposition 5.3. With respect to the basis $m$ for explicitly for the case

It suffices to check (5.1.1) on the elements

Proof. Let us first compute the action of $R_{i}$

Moreover, by lemma 5.1 and corollary 4.6, there is a solution of the HYBE for $(\tilde{V}_{n}, \alpha)$ of the form (4.6.1):

where $R$ (3.9.1) acts on $\tilde{V}_{n}^{\otimes 2}$ via the original $U_{h}(sl_{2})$-module structure $\rho$. Let us write down $B_{a}$ explicitly for the case $\tilde{V}_{1}$.

Proposition 5.3. With respect to the basis $\{v_{0} \otimes v_{0}, v_{0} \otimes v_{1}, v_{1} \otimes v_{0}, v_{1} \otimes v_{1}\}$ of $\tilde{V}_{1}^{\otimes 2}$, the solution

of the HYBE (4.1.1) for $(\tilde{V}_{1}, \alpha)$ is given by the matrix

where $q = e^{h/2}$ and $\gamma = e^{2\alpha}$.

Proof. Let us first compute the action of $R$ (3.9.1) on $\tilde{V}_{1}^{\otimes 2}$. It follows from definition (5.0.1) that, when $n = 1$, both $X_{+}$ and $X_{-}$ act trivially on $\tilde{V}_{1}$ for $a \geq 2$, and

for $i = 0, 1$. Thus, only the first two terms in $R$ (3.9.1) (corresponding to $a = 0, 1$) can act non-trivially on $\tilde{V}_{1}^{\otimes 2}$. Since $q = e^{h/2}$, these two terms are

Consider the action of the first term in $R'$. It follows from (5.3.2) that

for $m \geq 0$. Thus, we have



Proof. It suffices to check (5.1.1) on the elements $X_{+} \otimes v_{i}$ and $H \otimes v_{i}$. When applied to $H \otimes v_{i}$, both sides of (5.1.1) are equal to $(n - 2i)\gamma^{-i}v_{i}$. On the other hand, we have

A similar computation shows that both sides of (5.1.1), when applied to $X_{-} \otimes v_{i}$, are equal to $[i + 1]q\gamma^{-i}v_{i+1}$.

\[\boxed{\text{Proposition 5.2. The map } \rho_{a} = \alpha \circ \rho (5.1.1) \text{ gives } (\tilde{V}_{n}, \alpha) \text{ the structure of a } U_{h}(sl_{2})_{a}-\text{module.}\} \]
For the second term in $R'$, note that $X_+ \otimes X_-$ only acts non-trivially on $v_1 \otimes v_0$ among the four basis elements $\{v_i \otimes v_j\}_{0 \leq i,j \leq 1}$. We have

\[(X_+ \otimes X_-)(v_1 \otimes v_0) = v_0 \otimes v_1 \] (5.3.5)

and

\[q^{1/2}(H \otimes H \otimes 1 - 1 \otimes H \otimes H + H \otimes 1 - 1 \otimes H): v_0 \otimes v_1 \mapsto q^{1/2}v_0 \otimes v_1. \] (5.3.6)

The result now follows from (5.0.2), (5.2.1) and (5.3.3)–(5.3.6).

Regard $\gamma = e^{2c}$ as a parameter, where $c$ runs through the complex numbers. The operators $B_\alpha$ (5.3.1) thus form a one-parameter family of deformations of

\[B = q^{-\frac{1}{4}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \]

which is a solution of the YBE (1.0.3) for $\tilde{V}_1$.

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