Abstract. \(q\)-vertex operators for quantum affine algebras have played important role in the theory of solvable lattice models and the quantum Knizhnik-Zamolodchikov equation. Explicit constructions of these vertex operators for most level one modules are known for classical types except for type \(C_n^{(1)}\), where the level \(-1/2\) have been constructed. In this paper we survey these results for the quantum affine algebras of types \(A_n^{(1)}, B_n^{(1)}, C_n^{(1)}\) and \(D_n^{(1)}\).

1. Introduction

Algebraic conformal field theory can be simply described by the beautiful properties of the Virasoro algebra, vertex operators of affine Lie algebras, and the Knizhnik-Zamolodchikov equation. The vertex operators or intertwining operators between certain representations of affine Lie algebras stand at the core of the foundation since they also provide solutions to the KZ equation. Most advances in this part of Lie theory around 1980s circled around this interesting subject. Vertex operators also helped the introduction of the vertex algebra by Borcherds [Bo] (cf. [FLM]). In 1985 Drinfeld [D1] and Jimbo [Jb] introduced the notion of quantum group, a \(q\)-deformation of enveloping algebras of Kac-Moody algebras. This has inspired torrential activity to quantize existed phenomena in Lie theory. Lusztig first \(q\)-deformed the integrable modules of Kac-Moody algebras [L]. Then the \(q\)-deformed level one modules of quantum affine algebras were constructed by vertex representations [FR]. Quantum KZ equations were defined by Frenkel-Reshetikhin using the \(q\)-vertex operators [FR], and their matrix coefficients provide the solutions for the \(q\)-KZ equations. The Kyoto school used the \(q\)-vertex operators in solvable lattice models (see [JM]). They proposed that the half of the space of states \(H\), on which Baxter's corner transfer matrix is acting, can be identified with highest weight representations of quantum affine algebras. Then the correlation functions and the form factors can be computed in terms of the \(q\)-deformed vertex operators. Moveover their integral formulas are immediately given by explicit bosonization of the \(q\)-vertex operators. Because of its applications in statistical mechanics models and \(q\)-conformal field theory, the program for realizing various \(q\)-vertex operators attracted the attention of several researchers. It started with the explicit bosonization of level one case for \(U_q(A_n^{(1)})\) first for \(n = 1\) [JMMN], then for general \(n\) by Koyama [Ko1]. Kang, Koyama, and the first author [JKK] generalized the bosonization to type \(D_n^{(1)}\). More recently we gave explicit bosonizations

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for other types of quantum affine algebras \([JM1, JM2, JK]\). Some lower rank cases of other modules are also considered in \([Ko2]\).

In this work we will review the techniques involved in the realization of q-vertex operators and provide a unified description for the untwisted quantum affine algebras of classical types. To minimize the length of the paper we do not include twisted quantum affine algebras which have been considered in \([JM2]\). However the basic ingredients are similar in all cases.

Currently two methods of deriving q-vertex operators are used: coproduct formula (cf. Theorem \([3.1]\)) and direct computation using commutation relations. We refer the reader to \([JK]\) for more detailed information about the direct computation approach. Due to lack of space we do not include the intertwining operators. Their bosonization agrees with our vertex operators in the special computable component. The rest of the components in \([DI]\) are for the Drinfeld comultiplication. These constructions are used to construct the q-vertex operators in the last section.

2. Notation and Preliminaries

Let \(A = (A_{ij}), i, j \in I = \{0, 1, \cdots, n\}\) be the generalized Cartan matrix of type \(A_n^{(1)}, B_n^{(1)}, C_n^{(1)},\) or \(D_n^{(1)}\).

Let \(\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(A)\) be the associated Kac-Moody algebra, and \(h_i (i \in I)\) and \(d\) be the generators \(\mathfrak{h}\) of the Cartan subalgebra \(\mathfrak{h}\). We define the affine root lattice of \(\hat{\mathfrak{g}}\) to be

\[
\hat{Q} = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_n,
\]

where \(\alpha_i \in \hat{\mathfrak{h}}^*\) such that \(<\alpha_i, h_j> = A_{ij}\) and \(<\alpha_i, d> = \delta_{i0}\). The affine weight lattice \(\hat{P}\) is defined to be

\[
\hat{P} = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta,
\]

where \(\Lambda_i(h_j) = \delta_{ij}, \Lambda_i(d) = 0,\) and \(\delta(h_j) = 0, \delta(d) = 1\) for \(j \in I\). The dual affine weight lattice is then defined as

\[
\hat{P}^* = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d.
\]

We will denote the corresponding weight (resp. root) lattice of the finite dimensional Lie algebra \(\mathfrak{g}\) by \(P\) (resp. \(Q\)).

The nondegenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\) on \(\hat{\mathfrak{h}}^*\) satisfies

\[
(2.1) \quad \langle \alpha_i | \alpha_j \rangle = d_i A_{ij}, \quad \langle \delta | \alpha_i \rangle = \langle \delta | \delta \rangle = 0 \quad \text{for all } i, j \in I,
\]

where \(d_i = 1\) except for \((d_0, \cdots, d_n) = (1, \cdots, 1, 1/2)\) in the case of \(B_n^{(1)}\), and \((d_0, \cdots, d_n) = (1, 1/2, \cdots, 1, 2, 1)\) in the case of \(C_n^{(1)}\).

Let \(q_i = q^{d_i} = q^{\hat{\alpha}_i | \alpha_i}, i \in I\). The quantum affine algebra \(U_q(\hat{\mathfrak{g}})\) is the associative algebra with 1 over \(\mathbb{C}(q^{1/2})\) generated by the elements \(e_i, f_i (i \in I)\) and \(q^h\).
simplicity we will denote $k$ where $e$ generated by the elements $x$.

For $h \in \hat{P}^\vee$, we have the following relations:

$$
q^h q^{h'} = q^{h+h'}, \quad q^0 = 1, \quad \text{for } h, h' \in \hat{P}^\vee,
$$

$$
q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in \hat{P}^\vee(i \in I),
$$

$$
e_i f_j - f_j e_i = \delta_{ij} t_i - t_i^{-1}, \quad \text{where } t_i = q_i^{h_i} = q^\hat{\alpha_i}h_i, \quad \text{and } i, j \in I,
$$

$$
\sum_{m+n=1-a_{ij}} (-1)^m e_i^{(m)} e_i^{(n)} = 0, \quad \text{and}
$$

$$
\sum_{m+n=1-a_{ij}} (-1)^m f_i^{(m)} f_i^{(n)} = 0 \quad \text{for } i \neq j,
$$

where $e_i^{(k)} = e_i^k/[k]!$, $f_i^{(k)} = f_i^k/[k]!$, $[m]! = \prod_{k=1}^m [k]$, and $[k] = q_i^k - q_i^{-k}$. For simplicity we will denote $[k] = [k]$ for $i = 1, \ldots, n - 1$. The derived subalgebra generated by $e_i$, $f_i$, $t_i$ ($i \in I$) is denoted by $U_q(\hat{g})$.

The algebra $U_q(\hat{g})$ has a Hopf algebra structure with comultiplication $\Delta$, counit $u^*$, and antipode $S$ defined by

$$
\Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in \hat{P}^\vee, \\
\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i, \\
\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \quad \text{for } i \in I, \\
u^*(q^h) = 1 \quad \text{for } h \in \hat{P}^\vee, \\
u^*(e_i) = u^*(f_i) = 0 \quad \text{for } i \in I, \\
S(q^h) = q^{-h} \quad \text{for } h \in \hat{P}^\vee, \\
S(e_i) = -t_i^{-1} e_i, \quad S(f_i) = -f_i t_i \quad \text{for } i \in I.
$$

Let $V, W$ be two $U_q(\hat{g})$-modules. The tensor product $V \otimes W$ is defined as the $U_q(\hat{g})$-module via the coproduct $\Delta$. The (restricted) dual $U_q(\hat{g})$-module $V^*$ is defined by

$$(x \cdot v^*)(u) = v^*(S(x) \cdot u)$$

for $x \in U_q(\hat{g})$, $u \in V$, and $v^* \in V^*$.

3. Drinfeld realization and level zero modules

We now recall Drinfeld’s realization of the quantum affine algebra $U_q(\hat{g})$ (and of $U_q'(\hat{g})$)(cf. [D2, B, J3]). Let $U$ be the associative algebra with 1 over $C(q^{1/2})$ generated by the elements $x_i^\pm(k)$, $a_i(l)$, $K_i^\pm 1$, $\gamma^{\pm 1/2}$, $q^{\pm d}$ ($i = 1, 2, \ldots, n, k \in \mathbb{Z}, l \in$
where $\mathbb{Z} \setminus \{0\}$ with the following defining relations:
\[
\begin{align*}
[\gamma^{\pm 1/2}, u] &= 0 \text{ for all } u \in U, \\
[a_i(k), a_j(l)] &= \delta_{k+i,0} \frac{[A_{ij}k]}{q_j-q_j^{-1}}, \\
[a_i(k), K_j^{\pm 1}] &= [q^{\pm d}, K_j^{\pm 1}] = 0, \\
q^d x_i^+(k)q^{-d} &= q^k x_i^+(k), \quad q^d a_i(l)q^{-d} = q^l a_i(l), \\
K_i x_j^-(k) K_i^{-1} &= q^{\pm (a_i|a_j)} x_j^-(k), \\
\end{align*}
\]
\[
[a_i(k), x_j^+(l)] = \pm \frac{[A_{ij}k]}{q_i-q_i^{-1}} \gamma^{|k|/2} x_j^+(k+l), \\
x_i^+(k+1) x_j^+(l) &= q^{\pm (a_i|a_j)} x_j^-(l) x_i^+(k) + q^{a_i(k) x_j^+(l) - x_j^+(l+1) x_i^+(k)}, \\
\end{align*}
\]
\[
\begin{align}
[x_i^+(k), x_j^-(l)] &= \frac{\delta_{ij}}{q_i-q_i^{-1}} \left( \gamma^{\frac{k+1}{2}} \psi_i(k+l) - \gamma^{\frac{1}{2}} \varphi_i(k+l) \right), 
\end{align}
\]
where $\psi_i(m)$ and $\varphi_i(-m)$ ($m \in \mathbb{Z}_{\geq 0}$) are defined by
\[
\begin{align*}
\sum_{m=0}^{\infty} \psi_i(m) z^{-m} &= K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(k) z^{-k} \right), \\
\sum_{m=0}^{\infty} \varphi_i(-m) z^{-m} &= K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(-k) z^{k} \right), 
\end{align*}
\]
and the Serre relations are:
\[
\begin{align}
\text{Sym}_{k_1, \ldots, k_m} \sum_{r=0}^{m=1-A_{ij}} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] x_i^+(k_1) \cdots x_i^+(k_r) x_j^-(l) \\
x_i^+(k_{r+1}) \cdots x_i^+(k_m) = 0 \quad \text{if } i \neq j, 
\end{align}
\]
We denote by $U'$ the subalgebra of $U$ generated by the elements $x_i^+(k), a_i(l), K_i^{\pm 1},$ $\gamma^{\pm 1/2}$ ($i = 1, 2, \ldots, n, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}$).

Based on the general result of [12] one can compute directly the isomorphism as given in [13]. The $\epsilon$-sequences correspond to reduced expressions of the longest element in the Weyl group for details see [13].

**Proposition 3.1.** [13] Let $i_1, i_2, \ldots, i_{h-1}$ be a sequence of indices given in the Table. Then the $C(q^{1/2})$-algebra isomorphism $\Psi : U_q(\widehat{\mathfrak{g}}) \to U$ is given by
\[
\begin{align*}
\epsilon_i &\mapsto x_i^+(0), \quad f_i \mapsto x_i^-(0), \quad t_i \mapsto K_i \text{ for } i = 1, \ldots, n, \\
e_0 &\mapsto \left[ x_{i_{h-1}}^+(0), \cdots, x_{i_2}^+(0), x_{i_1}^+(1) \right] q_{1}^{1} \cdots q_{h-2}^{1} \gamma K_{\theta}^{-1}, \\
f_0 &\mapsto \alpha(-q)^{-1} \gamma^{-1} K_{\theta} [x_{i_{h-1}}^+(0), \cdots, x_{i_2}^+(0), x_{i_1}^+(-1)] q_{1}^{1} \cdots q_{h-2}^{1}, \\
t_0 &\mapsto \gamma K_{\theta}^{-1}, \quad q^d \mapsto q^d, 
\end{align*}
\]
where $K_{\theta} = K_{i_1} \cdots K_{i_{h-1}}, h$ is the Coxeter number, $\epsilon = \sum_{i=1}^{h-2} \epsilon_i$. The constant $\alpha$ is 1 for simply laced types $A_{i}^{(1)}, D_{n}^{(1)}$, $\alpha = [2]_1$ for $C_{n}^{(1)}$, and $\alpha = [2]^{1-\delta_{1,1}}$ for $B_{n}^{(1)}$.

The restriction of $\Psi$ to $U_q'(\widehat{\mathfrak{g}})$ defines an isomorphism of $U_q'(\widehat{\mathfrak{g}})$ and $U'$.

Table: $\epsilon$-Sequences for simple Lie algebras
$\mathcal{g}$ | $\epsilon$-Sequence. $\epsilon = \sum \epsilon_i$ | $\epsilon$
---|---|---
$A_n$ | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n$ | $-n+1$
$B_n$ | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_{n-1} \alpha_0 \alpha_n \rightarrow \cdots \rightarrow \alpha_2$ | $-2n+4$
$C_n$ | $\alpha_1 \rightarrow 1/2 \rightarrow \cdots \rightarrow \alpha_n \rightarrow 1/2 \rightarrow \cdots \rightarrow \alpha_2 \rightarrow \alpha_1$ | $-n+1$
$D_n$ | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \cdots \rightarrow \alpha_2$ | $-2n+4$

Through the isomorphism of Proposition 3.2, the coproduct will be carried over to the Drinfeld realization. The following result is true for all type one quantum affine algebras.

**Theorem 3.1.** Let $k \in \mathbb{Z}_{\geq 0}$, $l \in \mathbb{N}$, and let $N^+_l$ (resp. $N^-_l$) be the subalgebra of $\mathbf{U}$ generated by the elements $x^+_i(-m_1) \cdots x^+_i(-m_s)$ (resp. $x^-_i(m_1) \cdots x^-_i(m_s)$) with $m_i \in \mathbb{Z}_{\geq 1}$. Then the comultiplication $\Delta$ of the algebra $\mathbf{U}$ has the following form:

$$
\Delta(x^+_i(k)) = x^+_i(k) \otimes \gamma^k + \gamma^{2k} K_i \otimes x^+_i(k) + \sum_{j=0}^{k-1} \gamma^{\frac{k-j}{2}} \psi(k-j) \otimes \gamma^{k-j} x^+_i(j) \pmod{N_+ \otimes N_-^2},
$$

$$
\Delta(x^-_i(-l)) = x^-_i(-l) \otimes \gamma^{-l} + K_i^{-1} \otimes x^-_i(-l) + \sum_{j=1}^{l-1} \gamma^{l-j} \varphi(-l+j) \otimes \gamma^{-l+j} x^-_i(-j) \pmod{N_- \otimes N_+^2},
$$

$$
\Delta(x^-_i(l)) = x^-_i(l) \otimes K_i + \gamma^l \otimes x^-_i(l) + \sum_{j=1}^{l-1} \gamma^{l-j} x^-_i(j) \otimes \gamma^{l-j} \psi(l-j) \pmod{N_\pm^2 \otimes N_\pm},
$$

$$
\Delta(x^-_i(-k)) = x^-_i(k) \otimes \gamma^{-2k} K_i^{-1} + \gamma^{-k} \otimes x^-_i(-k) + \sum_{j=0}^{k-1} \gamma^{2-k-j} x^-_i(-j) \otimes \gamma^{2-k-j} \varphi(i-j) \pmod{N_-^2 \otimes N_+},
$$

$$
\Delta(a_i(l)) = a_i(l) \otimes \gamma^l + \gamma^{\frac{l}{2}} \otimes a_i(l) \pmod{N_- \otimes N_+},
$$

$$
\Delta(a_i(-l)) = a_i(-l) \otimes \gamma^{-\frac{l}{2}} + \gamma^{-\frac{l}{2}} \otimes a_i(-l) \pmod{N_- \otimes N_+}.
$$

Moreover the same formulas are true for the derived subalgebra $\mathbf{U}'$.

Let $V$ be the finite dimensional representation described by the following representation graphs. Note that the graphs are crystal graphs, and they are also perfect crystals $\mathbf{KMN}$ except in the case of $C_n^{(1)}$. Let $e_i$ be the basis elements ($i \in I$), and $I$ is the index set. Let $E_{ij}$ be the matrix units. We can read the actions from the graphs.

The $U_q(\mathfrak{g})$-module structure on the affinization or evaluation module of $V$ $\mathbf{KMN}$ is given as follows. We equip the affinization $V_\gamma = V \otimes C(q^{1/2})[z, z^{-1}]$ with the following actions:

$$
e_i(v \otimes z^m) = e_i(v \otimes z^{m+\delta_i,0}), \quad f_i(v \otimes z^m) = f_i(v \otimes z^{m-\delta_i,0}),
$$

$$
t_i(v \otimes z^m) = t_i(v \otimes z^m), \quad q^d(v \otimes z^m) = q^d v \otimes z^m,
$$

for $i \in I$, $v \in V$. 

\[ \text{(3.4)} \]
The evaluation module $V_z$ is a level zero $U_q(\hat{\mathfrak{g}})$-module, i.e., $\gamma$ acts as identity ($= q^0$). Through the isomorphism $\Psi$ the evaluation module is also a $\mathfrak{U}$-module. The action of the Drinfeld generators are given by the following result.

Define $\tau = \tau(A)$ as follows:

$$\tau = \begin{cases} 
  n + 1 & A^{(1)}_n \\
  2n - 1 & B^{(1)}_n \\
  2n + 2 & C^{(1)}_n \\
  2n - 2 & D^{(1)}_n 
\end{cases}$$

We define the content of each vertex in the representation graph by

$$c(i) = \begin{cases} 
  i & \text{if } i < n \\
  \tau - i & \text{if } i \in \{\overline{1}, \ldots, \overline{n}\} 
\end{cases}$$

Here we adopt the convention that $\overline{i} = i$.  

Figure 1. Representation graphs
\textbf{Theorem 3.2.} [Kol, JKK, LM, IK] For \( j \in \{1, \ldots, n\} \), the Drinfeld generators act on the evaluation module \( V_z \) as follows.

\begin{align*}
  x^+_{\pm}(k) &= \sum_{\langle (i(j), e(j)) \rangle} (q_{i(j)} \pm q_{e(j)})^k E_{i(j), e(j)} \\
  x^-_{\pm}(k) &= \sum_{\langle (i(j), e(j)) \rangle} (q_{i(j)} \pm q_{e(j)})^k E_{i(j), e(j)} \\
  a_j(l) &= \sum_{\langle (i(j), e(j)) \rangle} \frac{l}{l} (q_{i(j)} \pm q_{e(j)})^k (q_{i(j)} \pm q_{e(j)})^k E_{i(j), e(j)} - q_{i(j)} E_{i(j), e(j)} \\
  x^+_n(k) &= \begin{cases} 
  (q^{n-1} z)^2[k]_n E_{n,0} + (q^{n-1} z)^k E_{0,\pi}, & B_n(1) \\
  (q^{n-1} z)^k E_{n,\pi}, & C_n(1) \\
  (q^{n-1} z)^k (E_{n-1,\pi} + E_{n,n-1}), & D_n(1) 
  \end{cases} \\
  x^-_n(k) &= \begin{cases} 
  (q^{n-1} z)^2[k]_n E_{n,0} + (q^{n-1} z)^k E_{0,\pi}, & B_n(1) \\
  (q^{n-1} z)^k E_{n,\pi}, & C_n(1) \\
  (q^{n-1} z)^k (E_{n-1,\pi} + E_{n-1,n}), & D_n(1) 
  \end{cases} \\
  a_n(l) &= \begin{cases} 
  \frac{1}{l} (q^{n-1} z)^2 E_{n,0} - q^{l-1} E_{n,0} + (q^{l-1} E_{\pi,\pi} - E_{0,\pi}), & B_n(1) \\
  \frac{1}{l} (q^{n-1} z)^k (E_{n,n-1} + E_{n-1,n}), & C_n(1) \\
  \frac{1}{l} (q^{n-1} z)^k (q^{l-1} E_{n-1,n} - q^{l-1} E_{0,\pi} + q^{l-1} E_{n,n}), & D_n(1) 
  \end{cases}
\end{align*}

where the summation runs through all possible pairs \( i(j) \rightarrow e(j) \) in the representation graphs connected by \( j \). \( j = 1, 2, \ldots, n-1 \) (but we allow \( j = n \) for type \( A_n(1) \)), \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z} \setminus \{0\} \).

\textbf{Theorem 3.3.} [Kol, JKK, LM, IK] The Drinfeld generators act on the dual evaluation module \( V^*_z = V^* \otimes C(q^{1/2})[z, z^{-1}] \) as follows.

\begin{align*}
  x^+_{\pm}(k) &= \begin{cases} 
  -(q^{-1}) \sum_{\langle (i(j), e(j)) \rangle} (q_{i(j)} - q_{e(j)})^k E_{i(j), e(j)} \\
  -(q^{-1}) \sum_{\langle (i(j), e(j)) \rangle} (q_{i(j)} - q_{e(j)})^k E_{i(j), e(j)} \\
  (q^{-1}) \sum_{\langle (i(j), e(j)) \rangle} \frac{l}{l} (q_{i(j)} - q_{e(j)})^k (q_{i(j)} - q_{e(j)})^k E_{i(j), e(j)} - q_{i(j)} E_{i(j), e(j)} \\
  (q^{-1}) ((q^{-n-1} z)^k E_{n,0} + (q^{n+1} z)^k q E_{0,\pi}), & B_n(1) \\
  (q^{-1}) (q^{-n-1} z)^k E_{n,\pi}, & C_n(1) \\
  (q^{-1}) (q^{-n-1} z)^k E_{n-1,\pi} + E_{n-1,n}, & D_n(1) \\
  (q^-) ((q^{n+1} z)^k q^{-1} E_{0,\pi} + (q^{-n} z)^k E_{n,0}), & B_n(1) \\
  (q^-) (q^{n+1} z)^k E_{n,\pi}, & C_n(1) \\
  (q^-) (q^{n+1} z)^k E_{n-1,\pi} + E_{n-1,n}, & D_n(1) \\
  \frac{1}{l} (q^{-n+1} z)^2 (E_{n,n} - E_{n,n} + (q^{-l} E_{\pi,\pi} - E_{0,\pi})), & B_n(1) \\
  \frac{1}{l} (q^{-n+1} z)^k (q^{-l} E_{\pi,\pi} - q^{l} E_{n,n}), & C_n(1) \\
  \frac{1}{l} (q^{-n+1} z)^k (q^{-l} E_{\pi,\pi} - q^{l} E_{n-1,n} - q^{l} E_{0,\pi}, & D_n(1)
  \end{cases}
\end{align*}

where the summation runs through all possible pairs \( i(j) \rightarrow e(j) \) in the representation graphs connected by \( j \). \( j = 1, 2, \ldots, n-1 \) (but we allow \( j = n \) for type \( A_n(1) \)), \( k \in \mathbb{Z} \) and \( l \in \mathbb{Z} \setminus \{0\} \).
4. Vertex representation of $U_q(\mathfrak{g})$

In this section we recall the level one realizations of $U_q(\mathfrak{g})$ for $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}$ as given in [JM] and $\hat{\mathfrak{g}} = B_n^{(1)}$ in [JKM] (cf. [FR] for $V(\Lambda_0)$). We also recall the level $-1/2$ realization of $U_q(C_n^{(1)})$ as given in [JKM]. The quantum affine algebra $U_q(\mathfrak{g})$ has level one irreducible representations: $V(\Lambda_i)$, $i = 0, 1, \cdots, n$ for $A_n^{(1)}$; $i = 0, 1, n$ for $B_n^{(1)}$ and $i = 0, 1, n - 1, n$ for $D_n^{(1)}$.

The weight lattice $P$ has two coset representatives $Q$ and $Q + \lambda_n$, where $\lambda_i$ denote the fundamental weights in $P$. We will express any level one fundamental weight $\lambda_i$ as a linear combination of the simple roots $\alpha_i$ and $\lambda_n$.

Let $\varepsilon: Q \times Q \rightarrow \mathbb{Z}_2 = \{ \pm 1 \}$ be a cocycle such that

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{\langle \alpha | \beta \rangle + \langle \alpha | \beta \rangle},$$

whose existence can be constructed directly on the simple roots. Then there corresponds a central extension of the group algebra $C(Q)$ associated with $\varepsilon$ such that

$$1 \rightarrow \mathbb{Z}_2 \rightarrow C\{Q\} \rightarrow C(Q) \rightarrow 1.$$ 

If we still use $e^\alpha$ to denote the generators of $C\{Q\}$, then we have

$$e^\alpha e^\beta = (-1)^{\langle \alpha | \beta \rangle + \langle \alpha | \beta \rangle} e^\beta e^\alpha,$$

for any $\alpha, \beta \in Q$.

Moreover we define the vector space $C\{P\} = C\{Q\} \oplus C\{Q\} e^\lambda_n$ as a $C\{Q\}$-module by formally adjoining the element $e^\lambda_n$.

For $\alpha \in P$ define the operator $\partial_\alpha$ on $C\{P\}$ by

$$\partial_\alpha e^\beta = (\alpha | \beta) e^\beta.$$

Let $U_q(\mathfrak{h})$ be the infinite dimensional Heisenberg algebra generated by $a_i(k)$ and the central element $\gamma$ ($k \in \mathbb{Z}\setminus\{0\}, i = 1, \cdots, n$) subject to the relations

$$[a_i(k), a_j(l)] = \delta_{k+l,0} \left[ A_{ij} k \right] \frac{q^k - q^{-k}}{q_j - q_j^{-1}}.$$

The algebra $U_q(\mathfrak{h})$ acts on the space of symmetric algebra $\text{Sym}(\mathfrak{h}^-)$ generated by $a_i(-k)$ ($k \in \mathbb{N}, i = 1, \cdots, n$) with $\gamma = q$ by the action:

$$a_i(-k) \mapsto \text{multiplication by } a_i(-k),$$

$$a_i(k) \mapsto \sum_j [A_{ij} k] \frac{q^k - q^{-k}}{q_j - q_j^{-1}} d \frac{d}{a_j(-k)}.$$

For the case of $B_n^{(1)}$ we need to consider an extra fermionic field. For $Z = \mathbb{Z} + s$ ($s = 1/2$, NS-case; $s = 0$, R-case), let $C_q^s$ be the $q$-deformed Clifford algebra generated by $\kappa(k)$, $k \in Z$ satisfying the relations:

$$(\kappa(k), \kappa(l)) = (q^k + q^{-k}) \delta_{k,-l},$$

where $k, l \in Z$ and $\{a, b\} = ab + ba$.

Let $\Lambda(C_q^{-})^s$ be the polynomial algebra generated by $\kappa(-k)$, $k \in \mathbb{Z}_{>0}$, and $\Lambda(C_q^{-})^0$ (resp. $\Lambda(C_q^{-})^1$) be the subalgebra generated by products of even (resp. odd) number of generators $\kappa(-k)$'s. Then

$$\Lambda(C_q^{-})^s = \Lambda(C_q^{-})^0 \oplus \Lambda(C_q^{-})^1.$$
The algebra $C_q^*$ acts on $\Lambda(C_q^{-})^*$ canonically by $(k \in \mathbb{Z}_{>0})$

$$\kappa(-k) \mapsto \text{multiplication by } \kappa(-k),$$
$$\kappa(k) \mapsto -\frac{d}{d \kappa(-k)}.$$  

We define that

$$V(\Lambda_i) = \text{Sym}(\mathfrak{h})^\sim \otimes (\mathbb{C}\{Q\}) e^{\lambda_i}, \quad A_n^{(1)}, D_n^{(1)}$$

$$V(\Lambda_0) = \text{Sym}(\mathfrak{h})^\sim \otimes (\mathbb{C}\{Q\}) \otimes \Lambda(C_q^{-})^{1/2} \otimes (\mathbb{C}\{Q\}) e^{\lambda_i} \otimes \Lambda(C_q^{-})^{1/2}, B_n^{(1)}$$

$$V(\Lambda_1) = \text{Sym}(\mathfrak{h})^\sim \otimes (\mathbb{C}\{Q\}) e^{\lambda_i} \otimes \Lambda(C_q^{-})^{1/2} \otimes (\mathbb{C}\{Q\}) \otimes \Lambda(C_q^{-})^{1/2}, B_n^{(1)}$$

$$V(\Lambda_n) = \text{Sym}(\mathfrak{h})^\sim \otimes (\mathbb{C}\{Q\}) e^{\lambda_i} \otimes \Lambda(C_q^{-})^0, B_n^{(1)}$$

where $i = 0, \cdots, n$ for $A_n^{(1)}$ and $i = 0, 1, n-1, n$ for $D_n^{(1)}$. $\mathbb{C}\{Q\}$ is the group algebra of the sublattice of long roots in the case of $B_n^{(1)}$. Here for convenience we denote $\lambda_0 = 0$.

**Proposition 4.1.** [FJ] [JM1] The space $V(\Lambda_i)$ of level one irreducible modules of the quantum affine Lie algebra $U_q(\mathfrak{g})$ is realized under the following action:

$$\gamma \mapsto q, \quad K_j \mapsto q^\alpha_{ij}, \quad a_j(k) \mapsto a_j(k) \quad (1 \leq j \leq n),$$

$$x_i^+(z) \mapsto X_i^+(z) = \exp(\sum_{k=1}^\infty \frac{a_i(-k)}{k} q^{\mp k/2} z^k) \exp(\sum_{k=1}^\infty \frac{a_i(k)}{k} q^{\mp k/2} z^{-k})$$

$$\times e^{\pm \alpha_i z \pm \alpha_j, i+1}, \quad 1 \leq i \leq n; \text{when } \gamma = B_n^{(1)}, 1 \leq i \leq n - 1$$

$$x_n^+(z) \mapsto X_n^+(z) = \exp(\sum_{k=1}^\infty \frac{a_n(-k)}{2k} q^{\mp k/2} z^k) \exp(\sum_{k=1}^\infty \frac{a_n(k)}{2k} q^{\mp k/2} z^{-k})$$

$$\times e^{\pm \alpha_n z \pm \alpha_n, n+1/2}(\pm \kappa(z)), \quad \text{for } \gamma = B_n^{(1)}$$

where $\kappa(z) = \sum_{m \in \mathbb{Z}} \kappa(m) z^{-m}$. The highest weight vectors are $|\Lambda_i\rangle = e^{\lambda_i}$.

We now recall the level $-1/2$ realization of $U_q(\mathfrak{c})^{(1)}$. There are four level $-1/2$ weights: $\mu_1 = -\frac{1}{2} \Lambda_0, \mu_2 = -2\Lambda_0 + \Lambda_1 - \frac{1}{2} \Lambda_0, \mu_3 = -\frac{1}{2} \Lambda_0 + \Lambda_n - \frac{1}{2} \Lambda_0, \mu_4 = -\frac{1}{2} \Lambda_0 + \Lambda_n - 1/2$.

Let $a_i(m)$ be the bosonic operators satisfying (4.4) with $\gamma = q^{-1/2}$. Let $b_i(m)$ be another set of bosonic operators satisfying the following defining relations.

$$[b_i(m), b_j(l)] = m \delta_{ij} \delta_{m+l,0},$$
$$[a_i(m), b_j(l)] = 0.$$  

We define the Fock space $\mathcal{F}_{\alpha, \beta}$ for $\alpha \in P + \frac{1}{2} \Lambda_n, \beta \in P$ by the defining relations

$$a_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0), \quad b_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0),$$

$$a_i(0)|\alpha, \beta\rangle = (\alpha_i|\alpha\rangle|\alpha, \beta\rangle, \quad b_i(0)|\alpha, \beta\rangle = (2\varepsilon_i|\beta\rangle|\alpha, \beta\rangle),$$

where $|\alpha, \beta\rangle$ is the vacuum vector, and $\varepsilon_i$ are the usual orthonormal vectors.

We set

$$\tilde{\mathcal{F}} = \bigoplus_{\alpha \in P + \frac{1}{2} \Lambda_n, \beta \in P} \mathcal{F}_{\alpha, \beta}.$$  

Let $e^{\alpha_i} = e^\alpha_i$ and $e^{b_i} (1 \leq i \leq n)$ be operators on $\tilde{\mathcal{F}}$ given by:

$$e^{\alpha_i}|\alpha, \beta\rangle = |\alpha + \varepsilon_i, \beta\rangle, \quad e^{b_i}|\alpha, \beta\rangle = |\alpha, \beta + \varepsilon_i\rangle.$$
Let $\alpha$ be the usual bosonic normal ordering defined by

$$a_i(m) a_j(l) := a_i(m) a_j(l) \quad (m \leq l), \quad a_j(l) a_i(m) \quad (m > l),$$

and similar normal products for the $b_i(m)$'s. Let $\partial = \partial_{q^{1/2}}$ be the $q$-difference operator:

$$\partial_{q^{1/2}}(f(z)) = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})z}.$$

We introduce the following operators.

$$Y_i^\pm(z) = \exp(\sum_{k=1}^{\infty} \frac{a_i(k)}{k} q^{\pm\frac{k}{4}} z^k) \exp(\sum_{k=1}^{\infty} \frac{a_i(-k)}{k} q^{\mp\frac{k}{4}} z^{-k}) e^{\pm a_i z^2} a_i(0),$$

$$Z_i^\pm(z) = \exp(\sum_{k=1}^{\infty} \frac{b_i(k)}{k} z^k) \exp(\sum_{k=1}^{\infty} \frac{b_i(-k)}{k} q^{-\frac{k}{4}} z^{-k}) e^{\pm b_i z^2} b_i(0).$$

We define the operators $x_i^\pm(m) \ (i = 1, \ldots, n, m \in \mathbb{Z})$ by the following generating functions. $X_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_i^\pm(m) z^{-m-1}$.

$$X_i^+(z) = Z_i^+(z) Z_{i+1}^+(y_i^+(z)), \quad i = 1, \ldots, n - 1$$

$$X_i^-(z) = Z_i^-(z) Z_{i+1}^-(y_i^-(z)), \quad i = 1, \ldots, n - 1$$

$$X_n^+(z) = \left( \frac{1}{q^{\frac{z}{2}} + q^{-\frac{z}{2}}} : Z_n^+(z) \partial^2 Z_n^+(z) - : \partial Z_n^+(q^{\frac{z}{2}} z) \partial Z_n^+(q^{-\frac{z}{2}} z) : \right) Y_n^+(z)$$

$$X_n^-(z) = \left( \frac{1}{q^{\frac{z}{2}} + q^{-\frac{z}{2}}} : Z_n^-(q^{\frac{z}{2}} z) Z_n^-(q^{-\frac{z}{2}} z) : Y_n^-(z).$$

**Remark.** Our $a_i(k)$ differs from that in [JKM], where we took $a_i(k)/|d_i|$ for $a_i(k)$.

Furthermore, we know that $\hat{F}$ contains the four irreducible highest weight modules ([JKM]). Let $F_{\alpha,\beta}^1$ be the subspace of $F_{\alpha,\beta}$ generated by $a_i(m)$ \ ($i = 1, \ldots, n \in \mathbb{Z}$, $m \in \mathbb{Z}$) Similarly, let $F_{\alpha,\beta}^2$ \ ($j = 1, \ldots, n$) be the subspace of $F_{\alpha,\beta}$ generated by $b_j(m) \ (m \in \mathbb{Z})$. We can define the following isomorphism by $|\alpha,\beta\rangle \otimes |\alpha',\beta'\rangle \rightarrow |\alpha + \alpha',\beta + \beta'\rangle$.

$$F_{\alpha,\beta}^1 \otimes F_{\alpha',\beta'}^1 \otimes \cdots \otimes F_{\alpha,\beta}^n \rightarrow F_{\alpha,\beta_1 + \cdots + \beta_n}.$$ Let $Q_j^-$ be the operator from $F_{\alpha,\beta}^j$ to $F_{\alpha,\beta - \varepsilon_j}^j$ defined by

$$Q_j^- = \frac{1}{2\pi i} \int Z_j^-(z) dz.$$

We set subspaces $F_i \ (i = 1, 2, 3, 4)$ of $\hat{F}$ as follows.

$$F_1 = \bigoplus_{\alpha \in Q} F_{\alpha,\alpha}, \quad F_2 = \bigoplus_{\alpha \in Q} F_{\alpha + \varepsilon_1,\alpha + \varepsilon_1}$$

$$F_3 = \bigoplus_{\alpha \in Q} F_{\alpha - \frac{1}{2} \lambda_n,\alpha}, \quad F_4 = \bigoplus_{\alpha \in Q + \varepsilon_n} F_{\alpha - \frac{1}{2} \lambda_n,\alpha},$$

where

$$F_{\alpha,\beta}^\prime = F_{\alpha,0}^1 \otimes \bigotimes_{j=1}^{n} \text{Ker}_{F_{\alpha,\beta}^{2,j}} Q_j^-,$$

for $\beta = l_1 \varepsilon_1 + \cdots + l_n \varepsilon_n$. Then we have the following proposition.
Proposition 4.1. (JKM) Each $\mathcal{F}_i$ ($i=1,2,3,4$) is an irreducible highest weight $U_q$-module isomorphic to $V(\mu_i)$, The highest weight vectors are given by $|\mu_1\rangle = |0,0\rangle$, $|\mu_2\rangle = b_1(-1)|\lambda_1,\lambda_1\rangle$, $|\mu_3\rangle = | -\frac{1}{2}\lambda_n,0\rangle$, $|\mu_4\rangle = | -\frac{1}{2}\lambda_n - \varepsilon_n,-\varepsilon_n\rangle$.

5. REALIZATION OF THE LEVEL ONE VERTEX OPERATORS

We first recall the notion of Frenkel-Reshetikhin vertex operators [FR]. Let $V$ be a finite dimensional representation of the derived quantum affine Lie algebra $U'_q(\hat{g})$ with the associated affinization space $V_z$ (cf. [R]). Let $V(\lambda)$ and $V(\mu)$ be two integrable highest weight representations of $U_q(\hat{g})$. The type I (resp. type II) vertex operator is the $U_q(\hat{g})$-intertwining operator $\Phi(z): V(\lambda) \rightarrow V(\mu) \otimes V_z$ (resp. $V(\lambda) \rightarrow V_z \otimes V(\mu)$), which equals to $\Phi(z)z^{\Delta_\mu - \Delta_\lambda}$ where $\Phi(z)$ is the $U'_q(\hat{g})$-intertwining operator. Here the tensor product $\otimes$ is completed in certain sense, and we will write $\otimes$ for $\otimes$ in the sequel. $\Delta_\mu$ is the conformal weight for $\mu$. For simplicity we will compute the intertwining operators $\Phi(z)$ for the derived subalgebra $U'_q(\hat{g})$.

The existence of vertex operators is given by the fundamental fact [FR] (cf. DJO) (true for both types, though stated for type I):

$$Hom_{U'_q(\hat{g})}(V(\lambda), V(\mu) \otimes V_z) \simeq \{ v \in V | \ wt(v) = \lambda - \mu \mod \delta \ \text{and} \ e_i^{(\mu,h_i)+1}v = 0 \ \text{for} \ i = 0, \ldots, n \},$$

(5.1)

where the isomorphism is defined as follows. We say a pair of weights $(\lambda, \mu)$ is admissible if they satisfy (5.3). For each admissible pair $(\lambda, \mu)$ there corresponds uniquely a special vector $v_1$ in the crystal graph such that $wt(v_1) = \lambda - \mu$. We then normalize the corresponding vertex operator as follows.

$$\Phi^\mu_\lambda(z)|\lambda\rangle = |\mu\rangle \otimes v_1 + \text{higher terms in } z$$

(5.2)

Type II vertex operators assume similar normalization.

For the evaluation module $V_z$, we define the components of the vertex operator $\Phi^\mu_\lambda: V(\lambda) \rightarrow V(\mu) \otimes V_z$ by

$$\Phi^\mu_\lambda(z)|u\rangle = \sum_{j \in J} \Phi^{\mu V}_\lambda(z)|u\rangle \otimes v_j,$$

(5.3)

for $|u\rangle \in V(\lambda)$, and $j$ runs through the index set $J$ for the basis of $V$ as in the Figure 1.

We also consider the intertwining operators of modules of the following form:

$$\Phi^\mu_\nu(z): V(\lambda) \otimes V_z \rightarrow V(\mu) \otimes \mathbb{C}[z,z^{-1}]$$

by means of the vertex operators with respect to the dual space $V^*_z$:

$$\Phi^\nu_\lambda(z)(|v\rangle \otimes v_i) = \Phi^\mu_{\lambda V}(z)|v\rangle$$

(5.4)

for $|v\rangle \in V(\lambda)$ and $i$ belongs to $J$.

For a node $j$ in the crystal graph we define the weight of $j$ to be $\sum_i (f_i(j) - e_i(j))\lambda_i$, where $f_i(j)$ is the number of $i$-arrows going out of $j$, and $e_i(j)$ is the number of $i$-arrows coming into $j$.

By the intertwining property it is easy to see the following determinant relations. For more explicit relations see [Ko1], [JM2], [JK], [JKK].
Proposition 5.1. The vertex operator \( \Phi(z) \) of type I with respect to \( V_z \) is determined by its component \( \Phi_T(z) \), where \( T \) is the last vertex in the representation graph. In particular, for each pair \( i \rightarrow j \) in the graph, we have

\[
\Phi_i(z) = [r_j]_{q_k}^{-1} \Phi_j(z),
\]

where \( r_j \) counts the number of \( k \)-arrows coming into \( j \). With respect to \( V_z^* \), the vertex operator \( \Phi^*(z) \) of type I is determined by \( \Phi_T^*(z) \), and for each pair \( i \rightarrow j \)

\[
\Phi_j^*(z) = [r_j]_{q_k}^{-1} \Phi_j(z),
\]

where \( r_j \) counts the number of \( k \)-arrows coming into \( j \).

Proposition 5.2. Let \( \Phi(z) \) be a type II vertex operator with respect to \( V_z : V(\lambda) \rightarrow V_z \otimes V(\mu) \). Then \( \Phi(z) \) is determined by the component \( \Phi_1(z) \). More precisely, for each with pair \( i \rightarrow j \) we have:

\[
\Phi_j(z) = [r'_i]_{q_k}^{-1} \Phi_i(z),
\]

where \( r'_i \) counts the number of \( k \)-arrows going out of \( i \). With respect to \( V_z^* \) the vertex operator \( \Phi^*(z) \) of type II is determined by its component \( \Phi_T^*(z) \), where \( T \) is the last vertex in the crystal graph. For each pair \( i \rightarrow j \) we have

\[
\Phi_j^*(z) = q_k^2 [r'_i]_{q_k}^{-1} \Phi_j(z),
\]

where \( r'_i \) counts the number of \( k \)-arrows going out of \( i \).

Proposition 5.3. Let \( U_q(\mathfrak{g}) \) be the quantum affine algebra of type \( A_n^{(1)}, B_n^{(1)} \), or \( D_n^{(1)} \). (a) Let \( \Phi(z) : V(\lambda) \rightarrow V_1(z) \otimes V_2(z) \) be a vertex operator of type I, where \( (\lambda, \mu) \) is an admissible pair of weights. Then we have for each \( j = 1, \ldots, n \) and \( k \in \mathbb{N} \)

\[
[\Phi_T(z), X_j^-(w)] = 0, \quad \text{for type } A, \text{ we let } T = n + 1
\]

\[
t_j \Phi_T(z) t_j^{-1} = q^j \Phi_T(z),
\]

\[
[a_j(k), \Phi_T(z)] = \delta_{j1} \frac{[k]}{k} q^{(r + 3/2)k} z^k \Phi_T(z),
\]

\[
[a_j(-k), \Phi_T(z)] = \delta_{j1} \frac{[k]}{k} q^{-(r + 1/2)k} z^{-k} \Phi_T(z).
\]

(b) If \( \Phi(z) \) is a vertex operator of type I associated with \( V_z^* \), then

\[
[\Phi_T^*(z), X_j^-(w)] = 0, \quad \text{for type } A, \text{ we let } T = n + 1
\]

\[
t_j \Phi_T^*(z) t_j^{-1} = q^j \Phi_T^*(z),
\]

\[
[a_j(k), \Phi_T^*(z)] = \delta_{j1} \frac{[k]}{k} q^{3k} z^k \Phi_T^*(z),
\]

\[
[a_j(-k), \Phi_T^*(z)] = \delta_{j1} \frac{[k]}{k} q^{-3k} z^{-k} \Phi_T^*(z).
\]
(c) If $\Phi(z)$ is a vertex operator of type II associated with $V_z$, then

$$[\Phi_1(z), X_j^-(w)] = 0,$$

$$t_j \Phi_1(z) t_j^{-1} = q^{-\delta_{j1}} \Phi_1(z),$$

$$[a_j(k), \Phi_1(z)] = -\delta_{j1} [k] q^{-k/2} z^k \Phi_1(z),$$

$$[a_j(-k), \Phi_1(z)] = -\delta_{j1} [k] q^{-3k/2} z^{-k} \Phi_1(z).$$

(d) If $\Phi^*(z)$ is a vertex operator of type II associated with $V^*_z$, then

$$[\Phi^*_1(z), X_j^-(w)] = 0,$$

$$t_j \Phi^*_1(z) t_j^{-1} = q^{-\delta_{j1}} \Phi^*_1(z),$$

$$[a_j(k), \Phi^*_1(z)] = -\delta_{j1} [k] \begin{cases} q^{2n+2} z^k \Phi^*_1(z), & \text{type } A^{(1)}_n \\ q^{-(n+1/2)} z^k \Phi^*_1(z), & \text{type } B^{(1)}_n \text{ or } D^{(1)}_n \end{cases},$$

$$[a_j(-k), \Phi^*_1(z)] = -\delta_{j1} [k] \begin{cases} q^{2n+2} z^{-k} \Phi^*_1(z), & \text{type } A^{(1)}_n \\ q^{(n-1)} z^{-k} \Phi^*_1(z), & \text{type } B^{(1)}_n \text{ or } D^{(1)}_n \end{cases}.$$
Theorem 5.2. The 1-components of the type II vertex operator $\Phi^V_{\lambda}(z)$ with respect to $V_z : V(\lambda) \rightarrow V_z \otimes V(\mu)$ can be realized explicitly as follows:

$$\Phi^V_1(z) = \exp\left(-\sum \frac{[k]}{k} q^{\tau/2} a_{\tau}(-k) z^k\right) \exp\left(-\sum \frac{[k]}{k} q^{(\tau-3/2)k} a_{\tau}(k) z^{-k}\right)$$

$$e^{-\lambda_1(qz)-\partial_{\lambda_1}+(\lambda_1|\lambda_1+\bar{\psi})(-1)-\partial_{\lambda_n}+(\lambda_n|\lambda_n+\bar{\psi})\mu},$$

where $b^\mu_k$ is a constant.

The 1-component of type I vertex operator $\Phi^*_1(z)$ associated with $V^*_z$ is given by

$$\Phi^*_1(z) = \exp\left(-\sum \frac{[k]}{k} q^{-(\tau-1/2)k} a_{\tau}(-k) z^k\right) \exp\left(-\sum \frac{[k]}{k} q^{(\tau-3/2)k} a_{\tau}(k) z^{-k}\right)$$

$$e^{-\lambda_1(q^{-2n+2}z)-\partial_{\lambda_n}+(\lambda_1|\lambda_1+\bar{\psi})(-1)-\partial_{\lambda_n}+(\lambda_n|\lambda_n+\bar{\psi})\mu},$$

for type $B_n^{(1)}$ and $D_n^{(1)}$, and where $b^\mu_k$ is a constant.

$$\Phi^*_{n+1}(z) = \exp\left(-\sum \frac{[k]}{k} q^{2n+2k} a_{n+1}(-k) z^k\right) \exp\left(-\sum \frac{[k]}{k} q^{-2n+2k} a_{n+1}(k) z^{-k}\right)$$

$$e^{-\lambda_1(q^{n+2}z)-\partial_{\lambda_n}+(\lambda_1|\lambda_1+\bar{\psi})(-1)-\partial_{\lambda_n}+(\lambda_n|\lambda_n+\bar{\psi})\mu},$$

for type $A_n^{(1)}$ and where $b^\mu_k$ is a constant.

For type $C_n^{(1)}$ level $-1/2$ the vertex operators are determined by the following result.

Theorem 5.3. For the type one vertex operators associated to the admissible pair of weights $(\lambda, \mu) = (\mu_1, \mu_2), (\mu_2, \mu_1), (\mu_3, \mu_4), \text{ and } (\mu_4, \mu_3)$ respectively, one has

$$\Phi^{V}_{\mu_1}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{(n+1/4-\delta_{\mu_1})k} a_{\tau}(-k) z^k\right)$$

$$\exp\left(\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{(n+3/4-\delta_{\mu_1})k} a_{\tau}(k) z^{-k}\right)$$

$$e^{-\lambda_1(q^{n+1/2})z-\lambda_1(0)+1-(\lambda_1|\mu)\partial Z^+(q^{n+1/2})c_{ij},}$$

where $c_{ij}$ are constants for the four cases $(\mu_1, \mu_2)$ with $c_{12} = 1$.

Moveover the type two vertex operators are given by

$$\Phi^{V}_{\lambda}(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{-k/4} a_{\tau}(-k) z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{-3k/4} a_{\tau}(k) z^{-k}\right) c'_{ij}$$

$$e^{-\lambda_1(q^{-1/2})z-\lambda_1(0)+1-(\lambda_1|\mu)\partial Z^+(q^{-(1/2)})c'_{ij},}$$

where $c'_{ij}$ are constants for the four cases $(\mu_1, \mu_2)$ with $c'_{12} = 1$.

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