1 Introduction

By Drinfeld [1], [2] and Jimbo [3], quantum group was invented by quantizing the universal enveloping algebras of Kac–Moody algebras. It was defined by deforming the defining relations for Chevalley generators. Later Drinfeld found another realization of quantum group for affine Kac–Moody algebras in [4]. Finally, Faddeev, Reshetikhin and Takhtatjan [5] defined quantum group by the $RLL = LLR$ relations, which deforms the matrix realization of finite dimensional simple Lie algebras. This was later generalized to the affine case by Reshetikhin and Semenov-Tian-Shansky [6]. Up to now, the relations among these three definitions are clarified as follows. Universal $R$ matrix by Drinfeld [2] gives the map from the third formulation to the first [5], [6]. In the original paper [4], the isomorphism from the first definition to the second is given. (See [7] for the explicit verification.) Later in the nontwisted affine case, the map in the opposite direction is obtained in [8], without the use of the Braid group, and in [9] by using the Braid group actions. The correspondence between the second and third definitions is found by Ding and Frenkel [10] for $g = \hat{gl}_n$. The purpose of this paper is to generalize the last result by utilizing the multiplicative formula of universal $R$ matrix. (See [11] for a similar calculation in the case $DY(\hat{sl}_2)$.) This formula in the nontwisted affine case is already obtained in [12] without the use of Braid group. In this paper, we follow the approach by Beck [9], [13] and Lusztig [14]. Their results easily give the multiplicative formula, and this formula immediately yields the desired correspondence. The results in this paper would be well known conceptually. We only hope that the explicit expressions here help to clarify the problem.
This paper is organized as follows. In section 2, we fix our notations. In section 3, we determine the defining relations of quantum group in the L operator formalism for \( g = A_{l}^{(1)}, B_{l}^{(1)}, C_{l}^{(1)}, D_{l}^{(2)}, A_{2l}^{(2)}, A_{2l-1}^{(2)}, D_{l+1}^{(2)} \), since we cannot find the proof in literature. Here we also consider the twisted case since the proof is essentially the same as the one in the nontwisted case. In section 4, after a review of the result by Beck, we write down the multiplicative formula of universal \( R \) matrix for nontwisted affine Kac–Moody algebras. In section 5, utilizing the result in section 3, the multiplicative formula and the result by Beck, we calculate the correspondence between the \( L \) operators and Drinfeld’s generators for \( g = A_{l}^{(1)}, B_{l}^{(1)}, C_{l}^{(1)}, D_{l}^{(1)} \). The work by B. Edwards, announced in [4], would immediately generalize the results here to the twisted case.

2 Notations

2.1

Let \( g \) be an affine Kac–Moody algebra over \( \mathbb{Q} \) and \((a_{ij})_{0 \leq i,j \leq l}\) its generalized Cartan matrix. Let further \((a_{ij})_{0 \leq i \leq l}\) and \((a_{ij}^{\vee})_{0 \leq i \leq l}\) be relatively prime positive integers such that \( \sum_{i}a_{ij}a_{ij} = 0 \) and \( \sum_{j}a_{ij}a_{ij} = 0 \), respectively. We set \( h^{\vee} = \sum_{i=0}^{l}a_{ij}^{\vee} \). We follow the enumeration of the vertices of the Dynkin diagram in [1] Chapter 4 except in the \( A_{2l}^{(2)} \) case, in which we enumerate them in reverse order. In our notation, \( a_{0} = 1 \) for any \( g \) and \( a_{0}^{\vee} = 2 \) for \( g = A_{2l}^{(2)} \), = 1 otherwise. We denote the finite type Kac–Moody algebra corresponding to the Cartan matrix \((a_{ij})_{0 \leq i,j \leq l}\) by \( \hat{g} \).

Let \( h^{*} \) be a \( \mathbb{Q} \) vector space with basis \( \alpha_{0}, \cdots, \alpha_{l}, \Lambda_{0} \). Let further \((d_{i})_{0 \leq i \leq l}\) be relatively prime positive integers such that \( d_{i}a_{ij} = d_{j}a_{ji} \). We introduce a symmetric bilinear form \(|\cdot,\cdot|: h^{*} \times h^{*} \rightarrow \mathbb{Q} \) by \(|\alpha_{i}|\alpha_{j}| = d_{i}a_{ij}, (\alpha_{i}|\Lambda_{0}) = d_{i}a_{i0} \) and \((\Lambda_{0}|\Lambda_{0}) = 0\). Set \( \delta = \sum_{i=0}^{l}a_{i}a_{i} \) and \( \alpha_{i}^{\vee} = \alpha_{i}/d_{i}, (0 \leq i \leq l) \). Define \( \omega_{i} \) and \( \omega_{i}^{\vee} \) in \( h^{*} \) by \(|\alpha_{i}|\omega_{i}| = \delta_{ij}, (1 \leq j \leq l) \), \(|\delta|\omega_{i}| = (\Lambda_{0}|\omega_{i}) = 0 \). We further define several \( \mathbb{Z} \) submodules of \( h^{*} \) by \( Q = \oplus_{i=0}^{l}Z\alpha_{i}, \Sigma = Q \oplus \mathbb{Z}\Lambda_{0}/d_{0}, \Gamma = \Sigma + p\mathbb{Z}\omega_{1} \) and \( Q = \oplus_{i=0}^{l}Z\alpha_{i} \). Here and hereafter \( p = 2 \) for \( g = A_{2l}^{(2)} \) and \( = 1 \) otherwise. Finally, as usual, we introduce an order relation on \( Q \) by \( \lambda \geq \mu \iff \lambda - \mu \in Q_{+} = \oplus_{i=0}^{l}Z_{\geq 0}\alpha_{i} \).

2.2

Let \( q \) be an indeterminate and set \( F = \mathbb{Q}(q) \). Following [1], [3], let \( U = U_{q}^{\Gamma}(g) \) be the \( F \) algebra generated by \( e_{i}, f_{i} \) \((0 \leq i \leq l)\) and \( k_{\lambda} \) \((\lambda \in \Gamma)\) with the relations,

\[
k_{\lambda}k_{\mu} = k_{\lambda+\mu}, \quad k_{0} = 1, \quad (2.1)
\]
where $k_i^{±1} = k_±^α_i$ and

$$q_i = q^{d_i}, \quad x_i^{(n)} = \frac{x_i}{[n]_i!}, \quad (x = e, f), \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i.$$  

This algebra has the following Hopf algebra structure,

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_λ) = k_λ \otimes k_λ, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_i k_i, \quad S(k_λ) = k_{-λ},$$

$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad \epsilon(k_λ) = 1. \quad (2.6)$$

We define several subalgebras of $U$ by

$$U^+ = \langle e_i | 0 \leq i \leq l \rangle, \quad U^- = \langle f_i | 0 \leq i \leq l \rangle, \quad U^0 = \langle k_λ | λ \in Γ \rangle, \quad U_{≥0} = U^0U^+, \quad U_{≤0} = U^-U^0.$$  

Then $U^± = \oplus_{μ∈Q} U^±_μ$ where

$$U^±_μ = \{ x ∈ U^± | k_λ x k_{-λ} = q^{(λ|μ)} x \ (∀ λ ∈ Γ) \} \quad (μ ∈ Q).$$

Moreover the following is known.

**Proposition 2.1** [2], [3], [4]

(1) The multiplication map $U^- \otimes U^0 \otimes U^+ → U$ induces an isomorphism of vector spaces.

(2) $\{ k_λ | λ ∈ Γ \}$ forms a basis of $U^0$.

(3) $\dim_E U^±_μ = P(μ)$ for $μ ∈ Q$, where $P(μ)$ is Kostant’s partition function for $g$.

Similarly we define $U = U^q (g), U' = U^q (g')$ and $\tilde{U} = U^q (\tilde{g})$. We identify $U$ (resp. $U'$) with the subalgebra of $\tilde{U}$ (resp. $U$). In particular, $U^± = U^± = U'^±$. Finally, for a $\tilde{U}$ module $M$ and $μ ∈ \tilde{P} = \oplus_{1 ≤ i ≤ q} \mathbb{Z}ω_i$, we define its weight space $M_μ$ by

$$M_μ = \{ v ∈ M | k_λ v = q^{(λ|μ)} v \ (∀ λ ∈ \tilde{Q}) \}.$$
2.3
Hereafter to the end of the next section, we consider $g = A_l^{(1)}$ ($l \geq 1$), $B_l^{(1)}$ ($l \geq 3$), $C_l^{(1)}$ ($l \geq 2$), $D_l^{(1)}$ ($l \geq 4$), $A_{2l-1}^{(2)}$ ($l \geq 1$), $D_{2l-1}^{(2)}$ ($l \geq 3$), $D_l^{(2)}$ ($l \geq 2$). Let $V(p taco_1)$ be the irreducible highest weight module with highest weight $p taco_1$ of $U$. Let further $v_i$ ($1 \leq i \leq \dim V(p taco_1)$) be a basis of this module consisting of weight vectors such that $i < j$ if $\eta_i > \eta_j$, where $\eta_i \in \hat{Q}$ is the weight of $v_i$ as a $\hat{U}$ module. $V(\omega_1) \oplus F$ in the case $g = D_l^{(2)}$, and $V(p taco_1)$ otherwise can be made into representations of $U'$, on which $k_{\pm \delta}$ acts as 1. We shall denote them by $(\rho, V)$ and set $N = \dim V$. In the case $g = D_l^{(2)}$, by $v_N$ we denote the basis vector of the trivial representation as a $\hat{U}$ module. The representations $(\rho, V)$ are the same as the ones used in [19] and, in the limit $q = 1$, reduce to the ones in [16, Chapters 7 and 8]. Let $\tau_a$ ($a \in F^\times$) be the automorphism of $U$ determined by $\tau_a(e_i) = a^{\delta_{ij}}e_i$, $\tau_a(f_i) = a^{-\delta_{ij}}f_i$ and $\tau_a(k_l) = k_l$. Put $\rho_a = \rho \circ \tau_a|_{U'}$. Set $M = l + 1$ and $a_i = q^{-2(i-1)}(1 \leq i \leq M)$ for $g = A_l^{(1)}$. In other cases, set $M = 2$ and $a_1 = 1$, $a_2 = \sigma q_0^{-h'/a_0}$. Here $\sigma = -1$ for $g = A_2^{(2)}$, $A_{2l-1}^{(2)}$ and 1 otherwise. Then in $(\rho_{a_1} \otimes \cdots \rho_{a_M}, V^\otimes M)$, there exists a trivial representation of $U'$. We denote its basis vector by $w$. In the case $M = 2$, we define a matrix $J = (J_{ij})_{1 \leq i, j \leq N}$ by $w = \sum_{i,j} J_{ij} v_i \otimes v_j$. See Appendix for the explicit expressions of $N$, $\eta_i$, $(\rho, V)$ and $w$. Below we identify $\text{End} V$ with $M_N(F)$ through the basis $v_1, \cdots, v_N$ normalized as in Appendix.

2.4
Let $\Theta$ be the quasi-universal $R$ matrix of $U$, $[20]$, $[21]$, $[14]$. This is an invertible element of $(\text{an appropriate completion of})$ $U \otimes U$ uniquely determined by

$$\Theta = \sum_{\mu \geq 0} \Theta_{\mu}, \quad \Theta_{\mu} \in U_\mu^+ \otimes U_{-\mu},$$

$$\Theta_0 = 1 \otimes 1, \quad \sigma \circ \Delta(x) \Theta = \Theta \Psi \circ \Delta(x) \quad (x \in U) \quad (2.7)$$

Here $\sigma(x \otimes y) = y \otimes x$, and $\Psi$ is the automorphism of $U \otimes U$ defined by

$$\Psi(e_i \otimes 1) = e_i \otimes k_i^{-1}, \quad \Psi(f_i \otimes 1) = f_i \otimes k_i,$$

$$\Psi(k_\lambda \otimes k_\mu) = k_\lambda \otimes k_\mu, \quad \Psi = \sigma \circ \Psi$$

This element further satisfies,

$$\Theta_{a_i} = -(q_i - q_i^{-1}) e_i \otimes f_i \quad (0 \leq i \leq l), \quad (2.8)$$

$$(1 \otimes \Delta) \Theta = \Theta_{13} \Psi_{13}(\Theta_{12}), \quad (\Delta \otimes 1) \Theta = \Theta_{13} \Psi_{13}(\Theta_{23}), \quad (2.9)$$

$$\Theta_{12} \Psi_{12}(\Theta_{13}) \Theta_{23} = \Theta_{23} \Psi_{23}(\Theta_{13}) \Theta_{12}, \quad (2.10)$$

where $\Theta_{13}$, for example, denotes $\sum_i a_i \otimes 1 \otimes b_i$ for $\Theta = \sum_i a_i \otimes b_i$ as usual.
For $\Theta^{-1}$, define $(\Theta^{-1})_{\mu}$ $(\mu \in Q_+)$ as for $\Theta$. Let us introduce $\hat{L}^\pm(z) \in (M_N(F) \otimes U^\pm)[[z^{\pm 1}]]$ and $\hat{R}(z) \in (M_N(F) \otimes M_N(F))[[z]]$ by

$$
\hat{L}^+(z) = \sum_{\mu \geq 0} (\rho \otimes 1)(\Theta_{\mu}) z^{(\lambda_{\mu})}, \quad \hat{L}^-(z) = \sum_{\mu \geq 0} (\rho \otimes 1) \left( ((\Theta_{21})^{-1})_{\mu} \right) z^{-(\lambda_{\mu})},
$$

$$
\hat{R}(z) = (1 \otimes \rho)\hat{L}^+(z).
$$

(2.11)

We further introduce $L$ operators and $R$ matrix by

$$
\hat{L}^+(z) = \hat{L}^+(z) \hat{T}^{-1}, \quad \hat{L}^-(z) = \hat{T} \hat{L}^-(z),
$$

$$
R(z) = \hat{R}(z) T^{-1} = \sum_{n \geq 0} \sum_{1 \leq i_1, i_2, j_1, j_2 \leq N} R[n]_{i_2 j_2} E_{i_1 i_2} \otimes E_{j_1 j_2} z^n,
$$

(2.12)

where

$$
\hat{T}^\pm = \sum_{1 \leq i \leq N} E_{ii} \otimes k^\pm_{\eta_i}, \quad T^\pm = \sum_{1 \leq i, j \leq N} q^\pm(\eta_i, -\eta_i) E_{ii} \otimes E_{jj},
$$

and $E_{ij}$ is a matrix unit.

Finally we state the simple properties of $R$ matrix, which we shall need in the next section. Let $(\hat{\rho}, Q^N)$ denote the representation of $g' = [g, g]$ obtained by taking the limit $q = 1$ of $\rho(\cdot) \in M_N(F)$. Let $r(z) \in (\hat{\rho}(g') \otimes \hat{\rho}(g'))[[z]]$ be the trigonometric classical $r$ matrix [21] of type $g$ normalized by the condition that the diagonal part of the $z^0$ component of $r(z)$ is $\sum_{1 \leq i, j \leq N} (\eta_i, -\eta_i) E_{ii} \otimes E_{jj} (=; r_0)$. Let further $A$ be the $Q$ subalgebra of $F$ consisting of the elements which have no pole at $q = 1$.

**Lemma 2.1**

1. $R[0]_{i_2 j_1} \neq 0$ only if $i_1 \leq i_2$ and $j_1 \geq j_2$

2. $\eta_i + \eta_j = \eta_{i_2} + \eta_{j_2}, \quad i_1 = i_2 \Leftrightarrow j_1 = j_2$ when $R[n]_{i_2 j_2} \neq 0$

3. $R(z) = 1 - (q - q^{-1}) (r(z) - r_0) \mod (q - q^{-1})^2 (M_N(A) \otimes_A M_N(A))[[z]]$

**Proof.** (3) Let $r_{\mu}$ $(\mu > 0)$ be the elements of $M_N(Q) \otimes Q M_N(Q)$ uniquely determined by the following conditions,

$$
\begin{align*}
\alpha_i &= d_i \hat{\rho}(e_i) \otimes \hat{\rho}(f_i) \quad (0 \leq i \leq l), \\
[1 \otimes \hat{\rho}(e_i), r_{\mu}] + [\hat{\rho}(e_i) \otimes 1, r_{\mu-\alpha_i}] &= 0 \quad (\mu \neq \alpha_i, 0 \leq i \leq l), \\
[\hat{\rho}(f_i) \otimes 1, r_{\mu}] + [1 \otimes \hat{\rho}(f_i), r_{\mu-\alpha_i}] &= 0 \quad (\mu \neq \alpha_i, 0 \leq i \leq l), \\
\sum_{k=1}^M (r_{\mu})_k M^{W_1} \otimes 1_{Q^N} &= 0 \quad (\mu > 0) \quad \text{for } g \neq A^{(2)}_{2l}, A^{(2)}_{2l-1}, \\
(tr \otimes 1)(r_{\mu}) &= 0 \quad (\mu > 0) \quad \text{for } g = A^{(2)}_{2l}, A^{(2)}_{2l-1},
\end{align*}
$$

(3)
where \( \dot{w} = w|_{q=1} \). Then \( r(z) = \sum_{\mu \geq 0} r_{\mu} z^{(\Delta_{\mu} + \mu)} \). Hence by considering the equations from (2.7), (2.8) and (2.9) in the limit proved by induction on \( h \mu \).

\[ \dot{\mu} \]

\belowwith{135x-231}{\mu} \]

Then \( g = A_{2l}^{(2)} \), \( A_{2l-1}^{(2)} \), we also use (3.14) below with \( l_{ij}[-n] = -L_{ij}^+[n]/(q - q^{-1}) \). (In the nontwisted case, it is also possible to use the multiplicative formula of \( \Theta \) below. See [3, Section 6] and [22 Section 1].)

3 \ L \ operator formalism

3.1

Let \( g, (\rho, V), N, \eta_i \) \( (1 \leq i \leq N) \), \( R(z) \), \( M \), \( a_i \) \( (1 \leq i \leq M) \) and \( w \) as before. Following [3, 8, 10, 23], let \( \mathcal{A} = \mathcal{A}(g) \) be the \( F \) algebra generated by \( L_{ij}^+[0] \) \( (1 \leq i \leq j \leq N) \), \( L_{ij}^+[1/n] \) \( (1 \leq j \leq i \leq N) \), \( L_{ij}^+[1/n] \) \( (1 \leq i, j, n \in \mathbb{Z}_0) \), \( C^{\pm 1} \) and \( D^{\pm 1} \) with the relations

\[
L_{ij}^+[0] L_{ij}^+[0] = C^{\pm 1} C^{\mp 1} = D^{\pm 1} D^{\mp 1} = 1 \quad (3.1)
\]

\[
L_{ij}^+[0] = 1 \quad \text{for} \quad i \text{ such that } \eta_i = 0 \quad (3.2)
\]

\[
C^{\pm 1} \text{ central} \quad (3.3)
\]

\[
DL_{ij}^+[1] D^{-1} = q^n L_{ij}^+[1] \quad (3.4)
\]

\[
R_{12}(Cz/w) L_{ij}^+[1] = L_{i2}^+[1] L_{ij}^+[1] R_{12}(C^{-1} z/w) \quad (3.5)
\]

\[
R_{12}(z/w) L_{ij}^+[1] = L_{ij}^+[1] R_{12}(z/w) \quad (3.6)
\]

\[
L_{ij}^+[1] (a_1 z) \cdots L_{ij}^+[1] (a_M z) w \otimes 1 = w \otimes 1 \quad (3.7)
\]

\[
(G \otimes 1) L_{ij}^+[1] (G^{-1} \otimes 1) = L_{ij}^+[1] (-z) \quad \text{for } g = D_{l+1}^{(2)} \quad (3.8)
\]

Here \( L_{ij}^+(z) \) is defined in terms of the generating functions \( L_{ij}^+[1/n] = \sum_{n \geq 0} L_{ij}^+[1/n] z^{\pm n} \) as

\[
L_{ij}^+(z) = \sum_{1 \leq i, j \leq N} E_{ij} \otimes L_{ij}^+[1/n] \in (M_N(F) \otimes \mathcal{A})[[z^{\pm 1}]],
\]

\( L_{ij}^+[1/n] \) denotes \( L_{ij}^+(z) \) whose matrix part acts on the \( i \) th space, and

\[
G = \text{diag}(1, \cdots, 1, -1) \in M_N(F) \quad \text{for } g = D_{l+1}^{(2)}.
\]

It is well known and easy to check that \( \mathcal{A} \) has the following Hopf algebra structure

\[
\Delta(L_{ij}^+(z)) = \sum_{k=1}^{N} L_{kj}^+(C_{2}^{-1} z) \otimes L_{ik}^+(z), \quad \Delta(L_{ij}^-(z)) = \sum_{k=1}^{N} L_{kj}^-(z) \otimes L_{ik}^-(C_{1}^{-1} z),
\]

\[
\Delta(C^{\pm 1}) = C^{\pm 1} \otimes C^{\pm 1}, \quad \Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1},
\]

\[
\epsilon(L_{ij}^+(z)) = \delta_{ij}, \quad \epsilon(C^{\pm 1}) = \epsilon(D^{\pm 1}) = 1,
\]

\[
S(L_{ij}^+(z)) = S(L_{ij}^+(Cz)^{-1}), \quad S(C^{\pm 1}) = C^{\mp 1}, \quad S(D^{\pm 1}) = D^{\mp 1},
\]

(3.9)
where \( C_1 = C \otimes 1, C_2 = 1 \otimes C \) and \(^t\) denotes the matrix transpose in the first space. It is also well known and easy to check that there is a surjective Hopf algebra homomorphism \( \phi : A \to U \) defined by

\[
L^\pm(z) \mapsto \hat{L}^\pm(z), \quad C_{\pm 1} \mapsto k_{\pm 4}, \quad D_{\pm 1} \mapsto k_{\pm A_0/d_0}.
\]

In the next subsection, we shall show the injectivity of \( \phi \) by the standard specialization argument at \( q = 1 \) [18, 22]. The only subtlety appears in the case \( A^{(2)}_{2l}, A^{(2)}_{2l-1} \).

### 3.2

Set

\[
\mathcal{L}^+(z) = L^+(z)T, \quad \mathcal{L}^-(z) = T^{-1}L^-(z), \quad T^\pm = \sum_{1 \leq i \leq N} E_{ii} \otimes L^\mp_{ii}[0],
\]

and define the components \( \mathcal{L}^\pm_{ij}[\mp n] \) as before. We further define subalgebras of \( A \) by

\[
A^\pm = \langle \mathcal{L}^\pm_{ij}[\mp n] | 1 \leq i, j \leq N, n \geq 0 \rangle, \quad A^0 = \langle L^+_i[0], C_{\pm 1}, D_{\pm 1} | 1 \leq i \leq N \rangle,
\]

and, for \( \mu \in Q \), we set

\[
A^\pm_{\mu} = \{ x \in A^\pm | L^+_i[0]xL^+_i[0] = q^{(n_i, \mu)}x \ (1 \leq i \leq N), \quad DxD^{-1} = q^{(n_\mu/d_0)}x \}. \tag{3.10}
\]

**Lemma 3.1**

1. \( L^\pm_{kk}[0]L^\mp_{ij}[\mp n] = q^{-(n_k, n_i - n_j)}L^\pm_{ij}[\mp n]L^\mp_{kk}[0] \).
2. \( A^0 \) is commutative.
3. \( \prod_{i=1}^N L^\pm_{ii}[0]^{n_i} = 1 \) if \( \sum_i n_i n_i = 0 \).
4. \( A = A^- A^0 A^+ \).

Proof. (3) follows from [22] and the \( z^0 \) components of [3, 9] and [3, 9].

**Lemma 3.2**

\( A^\pm = \oplus_{\mu \in Q,} A^\pm_{\mu} \) and \( \dim_F A^\pm_{\mu} \leq P(\mu) \).

Proof. This can be proven as in [22 Proposition 1.13]. Let \( \mathcal{N} \) be the \( F \) algebra generated by \( \mathcal{L}_{ij}[0] \) (1 \( \leq i < j \leq N \)) and \( \mathcal{L}_{ij}[-n] \) (1 \( \leq i, j \leq N, n > 0 \)) with the relations

\[
\mathcal{R}_{12}(z/w)T_{12}^{-1}\mathcal{L}_1(z)T_{12}\mathcal{L}_2(w) = \mathcal{L}_2(w)T_{12}\mathcal{L}_1(z)T_{12}^{-1}\mathcal{R}_{12}(z/w) \tag{3.11}
\]

\[
\mathcal{L}_1(a_1 z)T_{12}\mathcal{L}_2(a_2 z)T_{12}^{-1} \cdots T_{1,M}\mathcal{L}_M(a_M z)T_{1,M}^{-1} w \otimes 1 = w \otimes 1 \tag{3.12}
\]

\[
(G \otimes 1)\mathcal{L}(z)(G^{-1} \otimes 1) = \mathcal{L}^{-1}(z) \quad \text{for} \ g = D^{(2)}_{i+1} \tag{3.13}
\]

where \( T_{1:k} = T_{1} \cdots T_{k-1} \), and \( \mathcal{L}(z) \) are defined as \( L^+(z) \) with \( \mathcal{L}_{ii}[0] = 1 \). Next let \( \mathcal{N}_A \) be the \( A \) algebra generated by \( l_{ij}[0] \) (1 \( \leq i < j \leq N \)) and \( l_{ij}[-n] \).
Theorem 3.1

For $l(z)$ be defined as before with $l_{ii}[0] = 0$. Firstly substitute $\mathcal{L}(z) = 1 - (q - q^{-1}) l(z)$ into (3.13) and then divide (3.11) by $(q - q^{-1})^2$ and the rest by $q - q^{-1}$. We take the thus obtained equations as the defining relations of $\mathcal{N}_A$. In the case $g = A^{(2)}_{2l}, A^{(2)}_{2l-1}$, we impose another relation, 

$$\frac{\text{tr} l(a_{2} z) - \text{tr} l(a_{2}^{-1} z)}{q - q^{-1}} = \text{tr} \tilde{l}(z)^2 - \text{tr} l(z)^2 + (q - q^{-1}) \times \left( \text{tr} l(z)^2 \bar{l}(a_{2} z) - \text{tr} l(a_{2}^{-1} z) \bar{l}(z)^2 \right)$$

(3.14)

where $\bar{l}(z) = \sum_{n \geq 0} \sum_{ij} E_{ij} \otimes \frac{\eta}{\eta} [q_{ij}[0]]_{l_{N+1 - j} N+1 - i}[0] - n] z^n$ (see Appendix A.3 for the $u_i$) and $\text{tr}$ denotes the trace of the matrix part. Finally let $n^-$ be the subalgebra of $g$ generated by the $f_j$ in the notations of [16] and $U(n^-)$ its enveloping algebra. $U(n^-)$ can be defined by the generators $l_{ij}[0]$ ($1 \leq i < j \leq N$), $l_{ij}[-n]$ ($1 \leq i, j \leq N, n > 0$) and the relations

\[
\begin{align*}
[r_{12}(w), l_{1}(z)] + [l_{1}(z), l_{2}(w)] + [(r_{0})_{12}, l_{1}(z) - l_{2}(w)] &= 0, \\
l(z) (\tilde{J} \otimes 1) + (\tilde{J} \otimes 1)^t l(\sigma z) &= 0 \quad \text{for } g \neq A^{(1)}_i, \\
\text{tr} l(z) &= 0 \quad \text{for } g = A^{(1)}_{2l}, A^{(2)}_{2l-1}, \\
(G \otimes 1) l(z) (G^{-1} \otimes 1) &= l(-z) \quad \text{for } g = D^{(2)}_{l+1},
\end{align*}
\]  

(3.15)

where $\tilde{J} = J_{[q=1]}$. (See [16] Chapters 7 and 8.)

Set $\mathcal{N}_F = \mathcal{N}_A \otimes_A F$ and $\mathcal{N}_Q = \mathcal{N}_A \otimes_A Q$, where $A$ acts on $F$ naturally and on $Q$ via $q \to 1$. In the case $g = A^{(2)}_{2l}, A^{(2)}_{2l-1}$ (3.14) follows from (3.12) in $\mathcal{N}$. Thanks to Lemma 2.3 (3), in the limit $q = 1$ the relations of $\mathcal{N}_A$ reduce to (3.12), cf. [8, 23]. Therefore

$$\mathcal{N}_F \simeq \mathcal{N} \quad (l_{ij}[-n] \otimes 1 \mapsto -\mathcal{L}_{ij}[-n] / (q - q^{-1})),$$

$$\mathcal{N}_Q \simeq U(n^-) \quad (l_{ij}[-n] \otimes 1 \mapsto l_{ij}[-n]).$$

(3.16)

As easily checked, $\mathcal{N}_A$ is a $Q$-graded algebra by assigning $\eta_j - \eta_i - n\delta \in Q$ to $l_{ij}[-n]$ and each homogeneous subspace $\mathcal{N}_A[\mu]$ ($\mu \in Q$) is finitely generated over $A$. Moreover there is a surjective $F$ algebra homomorphism from $\mathcal{N}$ to $\mathcal{L}_{ij}[-n]$ which maps $\mathcal{N}_A[\mu]$ to $\mathcal{L}_{ij}^\mu[0]$. Therefore, thanks to (3.16) we obtain

$$\dim_F A_{\mu} \leq \dim_F (\mathcal{N}_A)_{\mu} \otimes_A F \leq \dim_Q (\mathcal{N}_A)_{\mu} \otimes_A Q = \mathcal{P}(\mu).$$

Theorem 3.1

For $g = A^{(1)}_{2l}, B^{(1)}_{l}, C^{(1)}_{l}, D^{(1)}_{l}, A^{(2)}_{2l}, A^{(2)}_{2l-1}, D^{(2)}_{l+1}, \phi : A(g) \to U_q^\mu(g)$ is a Hopf algebra isomorphism.

Proof. Clearly $\phi(A^0) = U^0$ and $\phi(A^{\pm}_{(\pm)\mu}) = U^{\pm}_{\pm\mu}$. Since $\Gamma = \oplus_{i=1}^l \mathbb{Z} \gamma_i \oplus \mathbb{Z} \delta \oplus \mathbb{Z} \Lambda_0 / d_0$, Proposition 2.1 (2), Eq. (3.1) and Lemma 2.3 (2) (3) show that $\phi|\mathcal{A}^0$ is
injective. Proposition 2.3 (3) and Lemma 3.2 prove that \( \phi|_{A^\pm} : A^\pm \to U^\pm \) are isomorphisms. Hence Proposition 2.1 (1) and Lemma 3.1 (4) yield the claim.

**Remark.** In the case \( g = A_1^{(1)} \), we can replace (3.7) in the definition of \( A(g) \) by the following well known conditions for quantum determinants,

\[
\sum_{\gamma \in S_{i+1}} (-q)^{l(\gamma)} L_{1, \gamma(1)}^\pm (a_1 z) \cdots L_{i+1, \gamma(i+1)}^\pm (a_{i+1} z) = 1
\]

where \( l(\gamma) \) denotes the length function of \( S_{i+1} \). For the above relation is contained in (3.7) and is sufficient to give Lemma 3.1 (3) and \( \text{tr} l(z) = 0 \) in (3.15).

### 4 Multiplicative formula of \( \Theta \)

#### 4.1

In this section, we shall consider nontwisted affine Kac–Moody algebra \( g \). In this subsection, for completeness and a later purpose, we review the result by Beck 14 on the isomorphism of \( U \) and Drinfeld’s realization \( D \) (see below). We slightly change the notations in order to do without the square root of the central element.

Let \( P^\vee = \bigoplus_{i=1}^l \mathbb{Z} \tilde{\alpha}_i^\vee \) be the coweight lattice of \( g \) and \( Q^\vee = \bigoplus_{i=1}^l \mathbb{Z} \alpha_i^\vee \) the coroot lattice. Let \( W \) and \( \Lambda \) be the Weyl groups of \( g \) and \( g \), respectively. Set \( \bar{W} = W \ltimes P^\vee \), where \( W \) acts on \( P^\vee \) naturally. \( \bar{W} \) is known to be isomorphic to \( \Pi \ltimes W \). Here \( \Pi \) is the subgroup of the Dynkin diagram automorphism isomorphic to \( P^\vee/Q^\vee \), and \( \pi \in \Pi \) acts on \( s_i \), a simple reflection with respect to \( \alpha_i \), as \( \pi s_i \pi^{-1} = s_{\pi(i)} \). \( \bar{W} \) acts on \( Q \). In particular, \( x \in \bar{P}^\vee \) acts on \( \alpha \in Q \) as \( x(\alpha) = \alpha - (\alpha|x) \delta \).

Let \( T_i \) \((0 \leq i \leq l) \) [14] and \( T_\pi \) \((\pi \in \Pi) \) be the automorphisms of \( U' \) defined by

\[
T_i(e_i) = -f_i k_i, \quad T_i(f_i) = -k_i^{-1} e_i, \quad T_i(k_\lambda) = k_{s_i(\lambda)} \\
T_i(e_i) = \text{ad}(e_i^{(-a_i)}), \quad T_i(f_i) = f_i \cdot \widetilde{\text{ad}}(f_i^{(-a_i)}) \quad (i \neq j) \\
T_\pi(e_i) = c_{\pi(i)}, \quad T_\pi(f_i) = f_{\pi(i)}, \quad T_\pi(k_\alpha) = k_{\alpha_{\pi(i)}}
\]

where \( \text{ad} \) (resp. \( \widetilde{\text{ad}} \)) is a left (resp. right) adjoint action defined by

\[
\text{ad}(x) \cdot y = \sum x^{(1)} y S(x^{(2)}), \quad y \cdot \widetilde{\text{ad}}(x) = \sum S(x^{(1)}) y x^{(2)}
\]

for \( x, y \in U \) and \( \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \). For \( \tilde{w} \in \bar{W} \), we say an expression \( \tilde{w} = \pi s_{i_1} \cdots s_{i_n} \) \((\pi \in \Pi) \) is reduced if \( s_{i_1} \cdots s_{i_n} \) is so in \( W \). For any reduced expression \( \tilde{w} = \pi s_{i_1} \cdots s_{i_n} \), we can set \( T_{\tilde{w}} = T_{\pi} T_{i_1} \cdots T_{i_n} \). Then \( T_{\tilde{w}} \) \((\tilde{w} \in \bar{W}) \) gives a Braid group action on \( U' \). Let \( \Omega \) be the \( \mathbb{Q} \) algebra anti–automorphism of \( U' \) defined by

\[
e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad k_\lambda \mapsto k_{-\lambda}, \quad q \mapsto q^{-1}.
\]
This satisfies $\Omega T_{\tilde{w}} = T_{\tilde{w}} \Omega$.

Fix $o(i) \in \{-1, 1\}$ ($1 \leq i \leq l$) such that $o(i) = -o(j)$ if $a_{ij} < 0$. Let us define $x_{i,k}^{(\pm)}$ and $h_{i,r} \in U$ ($1 \leq i \leq l, k \in \mathbb{Z}, r \in \mathbb{Z} - \{0\}$) by

$$x_{i,k}^{(+)} = o(i)T_{\omega_i^\vee}(e_i), \quad x_{i,k}^{(-)} = \Omega(x_{i,-k}^{(+)}) = \Omega(x_{i,-k}),$$

$$k_{i,r}^{\mp 1} \exp \left( \mp (q_i - q_i^{-1}) \sum_{r > 0} h_{i,\mp r} z^{\pm r} \right) = \sum_{r \geq 0} \phi_{i,\mp r}^{\pm}, \quad (4.2)$$

where

$$\phi_{i,0}^{-} = k_i, \quad \phi_{i,0}^{+} = (q_i - q_i^{-1})k_{i,0}, \quad (4.3)$$

Let $\mathcal{D} = \mathcal{D}(g)$ be the $F$-algebra generated by $x_{i,k}^{(\pm)}, h_{i,r}, k_{i}^{\pm 1}, C^{\pm 1}, D^{\pm 1}$ for $1 \leq i \leq l, k \in \mathbb{Z}, r \in \mathbb{Z} - \{0\}$) with the relations,

$$k_i^{\pm 1}k_{i,r}^{\mp 1} = C^{\pm 1}C^{\mp 1} = D^{\pm 1}D^{\mp 1} = 1, \quad \text{C}^{\pm 1} \text{central}, \quad (4.4)$$

$$[k_i, k_j] = [k_i, D] = [k_i, h_{j,r}] = 0, \quad (4.5)$$

$$[h_{i,r}, h_{j,s}] = \delta_{r+s,0} \left[ [ra_{i,s}], C^r - C^{-r} \right], \quad (4.6)$$

$$k_i x_{i,k}^{(\pm)} k_i^{-1} = q^{\pm(a_i, \alpha_j)} x_{i,k}, \quad (4.7)$$

$$Dx_{i,k}^{(\pm)} D^{-1} = q^k x_{i,k}^{(\pm)}, \quad Dh_{i,r} D^{-1} = q^r h_{i,r}, \quad (4.8)$$

$$[h_{i,r}, x_{j,k}^{(\pm)}] = \pm \left[ [ra_{i,k}], C^r - x_{j,r+k}^{(\pm)} \right], \quad (4.9)$$

$$\delta_{i,j} = \frac{\gamma_{i,j}}{q_i - q_i^{-1}}(C^{-n} \phi_{i,m+n} - C^{-m} \phi_{i,m+n}^{+}), \quad (4.10)$$

$$x_{i,m+1} x_{j,n} = q^{-(a_i, \alpha_j)} x_{i,m} x_{j,n} x_{i,m+1}, \quad (4.11)$$

$$x_{i,m} \cdots x_{i,m} x_{j,n} x_{i,m+1} \cdots x_{i,m+r} = 0, \quad (4.12)$$

$$\text{Sym}_{m_1, \ldots, m_r} \sum_{s=0}^{r} (-1)^s \frac{[r]!}{[s]! [r-s]!} x_{i,m_1}^{(\pm)} \cdots x_{i,m_{s+1}}^{(\pm)} x_{i,m_{s+2}}^{(\pm)} \cdots x_{i,m_r}^{(\pm)} = 0, \quad (i \neq j, r = 1 - a_{ij}), \quad (4.13)$$

where $\phi_{i,\mp r}^{\pm}$ ($r \geq 0$) are expressed in terms of $k_{i}^{\pm 1}$ and $h_{i,r}$'s by (4.2).

**Theorem 4.1** For nontwisted affine Kac–Moody algebra $g$, there exists an algebra isomorphism $\psi : \mathcal{D}(g) \rightarrow U_{q}^{Z}(g)$ determined by

$$x_{i,k}^{(\pm)} \mapsto x_{i,k}^{(\pm)}, \quad h_{i,r} \mapsto h_{i,r}, \quad k_{i}^{\pm 1} \mapsto k_{i}^{\pm 1}, \quad C^{\pm 1} \mapsto C^{\pm 1}, \quad D^{\pm 1} \mapsto k_{\pm a_{ij} / d_i}. \quad (4.14)$$
4.2

In this subsection, we shall write down the multiplicative formula of $\Theta$, following the approach by Beck [3], [4] and Lusztig [4]. Most of the necessary results have already been obtained by them.

Firstly we shall describe their results. Fix $x = \omega_{j_1} \cdots \omega_{j_l} \in \bar{Q} \subset W$ such that $(x|\alpha_i) > 0$ (1 ≤ i ≤ l). Let $s_{i_1} \cdots s_{i_k}$ be its reduced expression obtained by the concatenation of the reduced presentations of $\omega_{j_i}$'s and canceling the elements of $\Pi$. Let us define $\beta_k (k \in \mathbb{Z})$ by

\[ \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad \text{and} \quad \beta_{k+rn} = x^{-r}(\beta_k) (r \geq 0) \quad \text{and} \quad \beta_{k-rn} = x^r(-\beta_k) (r > 0) \]

where 1 ≤ k ≤ n. Then they are distinct and run over the whole set of positive real roots of $g$. Moreover $\beta_k$ is a positive (resp. negative) root of $\bar{g}$ mod $\mathbb{Z} \delta$ if $k > 0$ (resp. $k \leq 0$). We define $E_k \in U_{\beta_k}^+$ and $F_k \in U_{-\beta_k}^- (k \in \mathbb{Z})$ by

\[ E_{k+rn} = T_{x^{-r}}T_{i_1} \cdots T_{i_k}^{-1}(e_{i_k}) \quad (1 \leq k \leq n, r \geq 0) \]
\[ E_{k-rn} = \bar{E}_{k+rn} \quad (1 \leq k \leq n, r > 0) \]
\[ F_k = \Omega(E_k) \]

and set

\[ F^m = (F^{m_1}_1 F^{m_2}_2 \cdots) \prod_{i \leq j \leq l} h_{i,r}^{m_i} \cdots F_{0}^{m_0} \]
\[ (4.15) \]

Here $h_{i,r} \in U_{r \delta}^r (r > 0)$ is defined in [12], and $m = (m_k, m_{j,r})_{k \in \mathbb{Z}, r > 0, 1 \leq j \leq l}$ is a family of nonnegative integers such that all vanish except for a finite number of them. For $k \in \mathbb{Z}$ we let $\bar{k}$ denote the integer which is between 1 and $n$, and equal to $k$ mod $n$. Finally, let $(, ) : U_{\geq 0}^r \times U_{\leq 0}^r \to F$ be the Hopf pairing [2, 20] Proposition 2.1.1] determined by bilinearity and the following,

\[ (k, k_{\mu}) = q^{-(\lambda|\mu)}, \quad (e_{i}, k_{\lambda}) = (k_{\lambda}, f_{i}) = 0, \quad (e_{i}, f_{j}) = -\delta_{ij}/(q_i - q_i^{-1}), \]
\[ (x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2), \quad (x_1 x_2, y) = (x_2 \otimes x_1, \Delta(y)) \]
\[ (x, x_1, x_2) \in U_{\geq 0}, \quad (y, y_1, y_2) \in U_{\leq 0} \]

[13, Proposition 3] and [14, Proposition 40.2.4] give

Proposition 4.1 [13], [14]

1. $(E_m^m)$ and $(F_m^m)$ form a basis of $U^+$ and $U^-$, respectively.
2. $(E_m^m, F_n^m) = (\prod_{i \leq j \leq l} h_{i,r}^{m_i}, \prod_{i \leq j \leq l} h_{i,-r}^{m_i}) \times \prod_{k \in \mathbb{Z}} \delta_{m_k,n_k} (e_{i_k}^{m_k}, f_{i_k}^{m_k})$.

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Therefore we have only to show the following. For \( r \in \mathbb{Z}_{>0} \), let \( (C_{ij}(r))_{1 \leq i,j \leq l} \) be the inverse matrix of \( \left( \frac{|ra_{ij}|(q - q^{-1})}{r(q_j - q^{-1})} \right) \) and set \( \tilde{h}_{i,-r} = \sum_{j=1}^{l} C_{ij}(r)h_{j,-r} \).

**Lemma 4.1**

\[
\prod_{i,r} \tilde{h}_{i,r}^{m_{i,r}} \prod_{i,r} \hat{h}_{i,-r}^{n_{i,r}} = \prod_{i,r} \delta_{m_{i,r}, n_{i,r}, m_{i,r}!} \left( \frac{-1}{q - q^{-1}} \right)^{m_{i,r}}
\]

**Proof.** Let \( \langle \cdot, \cdot \rangle : U \times U \rightarrow F(q^{1/2}) \) be the Killing form introduced in [20]. This form is bilinear and has the following properties

\[
\langle \text{ad}(u) \cdot v_1, v_2 \rangle = \langle v_1, \text{ad}(u) \cdot v_2 \rangle \quad (u, v_1, v_2 \in U) \tag{4.17}
\]

\[
\langle xk_\lambda, yk_\mu \rangle = \langle x, y \rangle q^{-((\lambda)\mu)/2} \quad (x \in U^+, y \in U^-) \tag{4.18}
\]

\[
\{U^{\geq 0}U_{-\mu}, U^{\leq 0} \} = \{0\} \quad (\mu > 0) \tag{4.19}
\]

In particular, (4.18) implies that \( \langle x, yk_\lambda \rangle \) is independent of \( \lambda \in \Sigma \) if \( x \in U^+ \) and \( y \in U^{\leq 0} \). Since \( \tilde{h}_{i,-r} \in U_{-\hat{\delta}} \),

\[
\Delta(\tilde{h}_{i,-r}) = 1 \otimes \tilde{h}_{i,-r} + \tilde{h}_{i,-r} \otimes k_{-r\hat{\delta}} \mod \bigoplus_{r \hat{\delta} > \beta > 0} U_{-r(\hat{\delta} - \beta)} \otimes U_{-\beta}U^0
\]

Noting the above and (4.7), we obtain from (4.17) with \( u = \tilde{h}_{j,-s}, v_1 = \prod_{i,r} \tilde{h}_{i,r}^{m_{i,r}} \) and \( v_2 = \prod_{i,r} \hat{h}_{i,-r}^{n_{i,r}} \cdot k_\lambda, \)

\[
(q^{-((\lambda)\hat{\delta})} - 1)\{\prod_{i,r} \tilde{h}_{i,r}^{m_{i,r}}, \tilde{h}_{j,-s} \prod_{i,r} \hat{h}_{i,-r}^{n_{i,r}}\} + \frac{m_{j,s}}{q - q^{-1}}(\prod_{(i,r) \neq (j,s)} \tilde{h}_{i,r}^{m_{i,r}} \prod_{i,r} \hat{h}_{i,-r}^{n_{i,r}}) = \sum_{s \hat{\delta} > \beta \geq 0} q^{-((\lambda)\beta)x_\beta}
\]

Here \( x_\beta \)'s are the elements of \( F \) independent of \( \lambda \). Since the above equality holds for any \( \lambda \in \Sigma \), the inside of the curly bracket vanishes. From this, the lemma follows.

The following is essentially due to Beck and Lusztig, cf. [24], [12], [8], [25].

Set

\[
\Theta^0 = \exp\left( -(q - q^{-1}) \sum_{1 \leq i \leq l} C_{ij}(r)h_{i,r} \otimes h_{j,-r} \right),
\]

\[
\Theta^+ = \Theta_1 \Theta_2 \cdots, \quad \Theta^- = \cdots \Theta_{-1} \Theta_0, \tag{4.20}
\]

where

\[
\Theta_k = \sum_{m \geq 0} (-1)^m q_i^{-\frac{m(m-1)}{2}} (q_i - q_i^{-1})^m \cdot E_k^m \otimes F_k^m \quad (i = i_k).
\]
where \( a \) is diagonal, and such that \( L \) and \( T \) claim follows from Proposition 4.1 and Lemma 4.1.

Proof. (1) Since \( \Theta_\mu \) is the canonical element of \((\cdot,\cdot)_{U_n^+ \times U_{-\mu}}\), the property of \( \hat{\theta} \) and \( \mu \) as a root of \( \odot \) and \( \Psi \) in the last equations. Noting the property of \( g \) mod \( \mathbb{Z} \delta \) and the weight of \( h_{s,t} \) as the diagonal \( \mathbb{Z}_+ \). For \( \Theta^a (a = \pm, 0) \), define \((\Theta^a)_{\mu}^{\pm} \mu \) as for \( \Theta \). Let us introduce \( L^\pm, a(z) \in (M_N(F) \otimes \mathcal{U})[[z^{\pm 1}]] \) \( a = u, d, l \) by

\[
\hat{L}^+, a(z) = \hat{L}^+, a(z), \quad \hat{L}^+, d(z) = \hat{L}^+, d(z) T^{-1}, \quad \hat{L}^+, l(z) = \hat{T} \hat{L}^+, l(z) T^{-1},
\]

\[
\hat{L}^-, a(z) = \hat{T} \hat{L}^-, a(z) T^{-1}, \quad \hat{L}^-, d(z) = \hat{T} \hat{L}^-, d(z), \quad \hat{L}^-, l(z) = \hat{L}^-, l(z)
\]

where

\[
\hat{L}^+, a(z) = \sum_{\mu \geq 0} (\rho \otimes 1)(\Theta^a_{\mu})^z \frac{(L_{\mu}^+)_{0}}{z_0}, \quad \hat{L}^-, a(z) = \sum_{\mu \geq 0} (\rho \otimes 1) \left((\Theta^a_{\mu})^{-1}\right)_{-\mu} z^{-\frac{(L_{\mu}^-)_{0}}{z_0}},
\]

and \( u' = l'' = +, d' = d'' = 0 \) and \( l' = u'' = - \) in the last equations. Noting the property of \( \beta_k \) as a root of \( g \) mod \( \mathbb{Z} \delta \) and the weight of \( h_{s,t} \), we obtain

Lemma 5.1 If we identify \( A \) with \( \mathcal{U} \) by the map \( \phi \), \( L^\pm, a(z) = \hat{L}^\pm, a(z) \).

This, together with Proposition 4.2 and Theorem 4.1, gives the relation between the \( L \) operators and Drinfeld’s generators. Before we state the results, we prepare some notations. For \( 1 \leq i \leq l \), set

\[
e^+_{i}(z) = \sum_{m > 0} e^+_{i,m} z^m, \quad f^+_{i}(z) = \sum_{m > 0} e^+_{i,m} z^m, \quad \phi^+(z) = \sum_{m \geq 0} \phi^+_{i,m} z^m,
\]

\[
e^-_{i}(z) = \sum_{m \geq 0} e^-_{i,m} z^m, \quad f^-_{i}(z) = \sum_{m \geq 0} e^-_{i,m} z^m, \quad \phi^-(z) = \sum_{m \geq 0} \phi^-_{i,m} z^m.
\]

(5.2)
Let $\rho_\pm(x_i)$ ($x = e, f$) denote the expressions $x_i^\pm$ in Appendix A.2. Let further $\epsilon(i)$ ($1 \leq i \leq N$) denote $i$ for $\epsilon = +$ and $N + 1 - i$ for $\epsilon = -$. For $L^{\pm,a}(z)$ ($a = u, l$), let $\equiv$ denote the equality modulo $\mathbb{C}[F_{ij}]$, where the sum is taken over $i$, $j$ such that $\eta_i - \eta_j \neq 0, e\alpha_k \ (1 \leq k \leq l, \epsilon = +$ for $a = u, = -$ for $a = l$). Finally, define $b \in F^\times$ by $-a(l)b = q^{-l(l+1)}$ for $g = A_l^{(1)}$, $= x/y \times q_0^{-h/2}$ (see (A.3) for $x$ and $y$) for $g = B_l^{(1)}$ and $= q_0^{-h/2}$ for $g = C_l^{(1)}$, $D_l^{(1)}$. (is an inessential constant determined by the correspondence between the $e_j$ and $x_{i1}^\pm$).

By direct calculations, we obtain

**Theorem 5.1.** Let $g$ be $A_l^{(1)}$, $B_l^{(1)}$, $C_l^{(1)}$ or $D_l^{(1)}$. If we identify $D(g)$ with the subalgebra of $A(g)$ via $(\tau \circ \phi)^{-1} |_{\mathbb{C}[g]} \circ \psi$, the following holds.

1. $g = A_l^{(1)}$:  

$$\tau(L^{\pm,u}(z) - 1) = (q - q^{-1}) \sum_{1 \leq i \leq l} \rho(e_i) \otimes f_i^\pm (q^i C^{(1)} - \frac{1}{12} z)$$

$$\tau(L^{\pm,l}(z) - 1) = (q - q^{-1}) \sum_{1 \leq i \leq l} \rho(f_i) \otimes e_i^\pm (q^i C^{(1)} - \frac{1}{12} z)$$

$$L_{ii}^\pm(z)L_{i+1,i+1}^\pm(z)^{-1} = \phi_i^\pm (q^i z) \quad (1 \leq i \leq l)$$

2. $g = B_l^{(1)}$, $C_l^{(1)}$, $D_l^{(1)}$:  

$$\tau(L^{\pm,u}(z) - 1) = \sum_{1 \leq i \leq l} (q_i - q^{-1}_i) \rho(e_i) \otimes f_i^\pm (q^i c_i q_0^{-h/2} C^{(1)} - \frac{1}{12} z)$$

$$+(q_i - q^{-1}_i) \times \left\{ \sum_{e = \pm} \rho(e_i) \otimes f_i^\pm (q^i C^{(1)} - \frac{1}{12} z) \quad \text{for } g = B_l^{(1)} \right.$$  

$$\left. \rho(e_i) \otimes f_i^\pm (C^{(1)} - \frac{1}{12} z) \quad \text{for } g = C_l^{(1)}, D_l^{(1)} \right\}$$

$$\tau(L^{\pm,l}(z) - 1) = \sum_{1 \leq i \leq l} (q_i - q^{-1}_i) \rho(f_i) \otimes e_i^\pm (q^i c_i q_0^{-h/2} C^{(1)} - \frac{1}{12} z)$$

$$+(q_i - q^{-1}_i) \times \left\{ \sum_{e = \pm} \rho(f_i) \otimes e_i^\pm (q^i C^{(1)} - \frac{1}{12} z) \quad \text{for } g = B_l^{(1)} \right.$$  

$$\left. \rho(f_i) \otimes e_i^\pm (C^{(1)} - \frac{1}{12} z) \quad \text{for } g = C_l^{(1)}, D_l^{(1)} \right\}$$

$$L_{i(i)}^{\pm,d}(z)L_{i(i+1)+(i+1)}^{\pm,d}(z)^{-\epsilon} = \phi_i^\pm (q^i c_i q_0^{-h/2} z) \quad (1 \leq i \leq l - 1, \epsilon = \pm)$$

$$L_{\epsilon(i)}^{\pm,d}(z)L_{i(i+1)+(i+1)}^{\pm,d}(z)^{-\epsilon} = \phi_i^\pm (q^i z) \quad (\epsilon = \pm) \quad \text{for } g = B_l^{(1)}$$

$$L_{ii}^{\pm,d}(z)L_{i+1,i+1}^{\pm,d}(z)^{-1} = \phi_i^\pm (z) \quad \text{for } g = C_l^{(1)}$$

$$L_{i-1,i-1}^{\pm,d}(z)L_{i+1,i+1}^{\pm,d}(z)^{-1} = L_{ii}^{\pm,d}(z)L_{i+2,i+2}^{\pm,d}(z)^{-1} = \phi_i^\pm (z) \quad \text{for } g = D_l^{(1)}$$

**Remark.** In [23], another Hopf structure of quantum group found by Drinfeld is shown to be obtained by the twisting by $F = (\Theta_{21})^{-1}$ (in our notation). The
above correspondence between the $L$ operators and Drinfeld’s generators gives an intuitive explanation of its comultiplication formula as follows. After the twist, $\Theta^F = \Theta^0 \Theta^f \Psi(\Theta_{23}^f)$. Define $\hat{L}^{\pm,F}(z)$ for $\Theta^F$ as before. Then $\hat{L}^{+,F}(z) = \hat{L}^{+,d}(z) \hat{L}^{+,l}(z) \hat{L}^{-l}(k_{-i}z)^{-1}$ and $\hat{L}^{-,F}(z) = \hat{L}^{-,u}(k_{-i}z)^{-1} \hat{L}^{-,d}(z)$. These have the lower and upper triangular forms, respectively, and satisfy the comultiplication formula in $[3,4]$ for $\Delta^F(\cdot) = \mathcal{F}(\Delta(\cdot))^{-1}$. From this, the comultiplication formula for Drinfeld’s generators easily follows. The above argument is formal, but the justification would be possible.

A Appendix $U'$ module $(\rho, V)$ and $w$

A.1 $N$ and $\eta = (\eta_1, \cdots, \eta_N)$

We set $\epsilon_i = \rho \omega_i - (\alpha_1 + \cdots + \alpha_{i-1})$ $(1 \leq i \leq l)$. (Note that in the case $g = D^{(1)}_l$, $v_l$ and $v_{l+1}$ can be interchanged.)

(1) $A^{(1)}_l (l \geq 1): N = l + 1, \quad \eta = (\epsilon_1, \cdots, \epsilon_l, - \sum_{i=1}^l \epsilon_i)

(2) $B^{(1)}_l (l \geq 3), A^{(2)}_{2l} (l \geq 1): N = 2l + 1, \quad \eta = (\epsilon_1, \cdots, \epsilon_l, 0, -\epsilon_l, \cdots, -\epsilon_1)$

(3) $C^{(1)}_l (l \geq 2), A^{(2)}_{2l-1} (l \geq 3), D^{(1)}_l (l \geq 4): N = 2l, \quad \eta = (\epsilon_1, \cdots, \epsilon_l, -\epsilon_l, \cdots, -\epsilon_1)$

(4) $D^{(2)}_{l+1} (l \geq 2): N = 2l + 2, \quad \eta = (\epsilon_1, \cdots, \epsilon_l, 0, -\epsilon_l, \cdots, -\epsilon_1, 0)$

A.2 $(\rho, V)$

We give the matrix representations of $\epsilon_i$ and $f_i$ $(0 \leq i \leq l)$ with respect to the basis $v_1, \cdots, v_N$. $E_{ij}$ is a matrix unit and $f^i$ denotes the transpose of $f_i$, $x^B_i$ and $x^C_i$ $(x = e, f, 1 \leq i \leq l)$ below stand for the expressions $x_i$ given for $B^{(1)}_l$ and $C^{(1)}_l$, respectively.

(1) $A^{(1)}_l$:

(2) $B^{(1)}_l$, $C^{(1)}_l$, $D^{(1)}_l$:

(i) $1 \leq i < l$:

\[ e_i = e_i^+ + e_i^- \quad f_i = f_i^+ + f_i^- \quad e_i^+ = E_{i+1} = f_i^+ \quad e_i^- = -E_{N-i} = f_i^- \]

(ii) $i = l, 0$:

\[ B^{(1)}_l:
\]

\[ e_l = e_l^+ + e_l^- \quad f_l = f_l^+ + f_l^- \quad x^{-1}e_l^+ = E_{l+1} = y^{-1}f_l^+ \quad x^{-1}e_l^- = -E_{l+1} = y^{-1}f_l^- \quad e_0 = E_{N-1} = E_{N-2} = f_0 \]

\[ (A.1) \]

\[ (A.2) \]
\[ x, y \in A \text{ such that } xy = [2]_l \]  
(A.3)

\[ C_l^{(1)} : \quad e_l = E_{l+1} = t_l, \quad e_0 = E_{N+1} = t_0 \]  
(A.4)

\[ D_l^{(1)} : \quad e_l = E_{l-1+l+1} - E_{l+2} = t_l, \quad e_0 = E_{N-1} - E_{N+2} = t_0 \]  
(A.5)

(3) \[ A_{2l}^{(2)} : \quad x_i = M x_i^{2l} M^{-1} \quad (x = e, f, \quad 1 \leq i \leq l) \quad e_0 = E_{N+1} = t_0 \]

\[ M = \text{diag}(1 \cdots 1, -1, 1, \cdots, (-1)^l) \]  
(A.6)

(4) \[ A_{2l-1}^{(2)} : \quad x_i = M x_i^{2l-1} M^{-1} \quad (x = e, f, \quad 1 \leq i \leq l) \quad e_0 = E_{N-1} + E_{N+1} = t_0 \]

\[ M = \text{diag}(1 \cdots 1, -1, 1, \cdots, (-1)^{l-1}) \]  
(A.7)

(5) \[ D_{l+1}^{(2)} : \quad x_i = x_i^{2l} \quad (x = e, f, \quad 1 \leq i \leq l) \quad x^{-1} e_0 = E_{N+1} + E_{N-1} = y^{-1} t_0 \quad x, y \in A \text{ such that } xy = [2]_0 \]  
(A.8)

A.3 \quad w

(1) \quad g = A_{l}^{(1)} : \quad w = \sum_{\gamma \in \delta_{i+1}} (-1)^{l(\gamma)} v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(l+1)}.

\[ (l(\gamma) : \text{length function}) \]

(2) \quad g \neq A_{l}^{(1)} : \quad w = \sum_{1 \leq i, j \leq N} J_{ij} v_i \otimes v_j,

\[ J_{ij} = u_i \delta_{i+j,N+1} \text{ for } g \neq D_{l+1}^{(2)} = u_i \delta_{i+j,N} + u_N \delta_{i,N} \delta_{j,N} \text{ for } g = D_{l+1}^{(2)}. \]

List of \( (u_1, \ldots, u_N) \)

(i) \( B_l^{(1)} : \) \( (q^{-2(l-1)}, \ldots, q^{-3}, q^{-2}, q, q, q^2, \ldots, q^{2l-1}) \)

(ii) \( C_l^{(1)} : \) \( (-q^{-l}, \ldots, -q^{-2}, q^{-1}, q, q^2, \ldots, q^l) \)

(iii) \( D_l^{(1)} : \) \( (q^{-l-1}, \ldots, q^{-2}, 1, 1, q, \ldots, q^{l-1}) \)

(iv) \( A_{2l}^{(2)} : \) \( ((-1)^l q^{2(l-1)}, \ldots, q^{-3}, -q^{-1}, q, -q, q^3, \ldots, (-1)^l q^{2l-1}) \)

(v) \( A_{2l-1}^{(2)} : \) \( ((-1)^l q^{-l-1}, \ldots, q^{-2}, -q^{-1}, q, -q^2, \ldots, (-1)^l q^{l-1}) \)

(vi) \( D_{l+1}^{(2)} : \) \( (q^{-2(l-1)}, \ldots, q^{-3}, q^{-1}, q, q^2, \ldots, q^{2l-1}, -q) \)

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