On limit theory for Lévy semi-stationary processes

Andreas Basse-O’Connor∗  Claudio Heinrich‡  Mark Podolskij‡

October 17, 2016

Abstract

In this paper we present some limit theorems for power variation of Lévy semi-stationary processes in the setting of infill asymptotics. Lévy semi-stationary processes, which are a one-dimensional analogue of ambit fields, are moving average type processes with a multiplicative random component, which is usually referred to as volatility or intermittency. From the mathematical point of view this work extends the asymptotic theory investigated in [14], where the authors derived the limit theory for $k$th order increments of stationary increments Lévy driven moving averages. The asymptotic results turn out to heavily depend on the interplay between the given order of the increments, the considered power $p > 0$, the Blumenthal–Getoor index $\beta \in (0, 2)$ of the driving pure jump Lévy process $L$ and the behaviour of the kernel function $g$ at 0 determined by the power $\alpha$. In this paper we will study the first order asymptotic theory for Lévy semi-stationary processes with a random volatility/intermittency component and present some statistical applications of the probabilistic results.

Key words: Power variation, Lévy semi-stationary processes, limit theorems, stable convergence, high frequency data.

AMS 2010 subject classifications. Primary 60F05, 60F15, 60G22; secondary 60G48, 60H05.

1 Introduction and main results

Over the last ten years there has been a growing interest in the theory of ambit fields. Ambit fields is a class of spatio-temporal stochastic processes that has been originally introduced by Barndorff-Nielsen and Schmiegel in a series of papers [11, 12, 13] in the context of turbulence modelling, but which has found manifold applications in mathematical finance and biology among other sciences; see e.g. [3, 9].

Ambit processes describe the dynamics in a stochastically developing field, for instance a turbulent wind field, along curves embedded in such a field. A key characteristic of the

*Department of Mathematics, Aarhus University, E-mail: basse@math.au.dk.
†Department of Mathematics, Aarhus University, E-mail: claudio.heinrich@math.au.dk
‡Department of Mathematics, Aarhus University, E-mail: mpodolskij@math.au.dk.
modelling framework is that beyond the most basic kind of random noise it also specifically incorporates additional, often drastically changing, inputs referred to as volatility or intermittency. In terms of mathematical formulae an ambit field is specified via

\[
X_t(x) = \mu + \int_{A_t(x)} g(t, s, x, \xi) \sigma_s(\xi) \, L(ds, d\xi) + \int_{D_t(x)} q(t, s, x, \xi) a_s(\xi) \, ds \, d\xi, \tag{1.1}
\]

where \(t\) denotes time while \(x\) gives the position in space. Further, \(A_t(x)\) and \(D_t(x)\) are Borel measurable subsets of \(\mathbb{R} \times \mathbb{R}^d\), \(g\) and \(q\) are deterministic weight functions, \(\sigma\) represents the volatility or intermittency field, \(a\) is a drift field and \(L\) denotes an independently scattered infinitely divisible random measure on \(\mathbb{R} \times \mathbb{R}^d\) (see e.g. [36] for details). In the literature, the sets \(A_t(x)\) and \(D_t(x)\) are usually referred to as ambit sets. In the framework of turbulence modelling the stochastic field \((X_t(x))_{t \geq 0, x \in \mathbb{R}^3}\) describes the velocity of a turbulent flow at time \(t\) and position \(x\), while the ambit sets \(A_t(x), D_t(x)\) are typically bounded.

In this paper we will consider a purely temporal analogue of ambit fields (without drift) \((X_t)_{t \in \mathbb{R}}\), defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})\), which is given as

\[
X_t = \int_{-\infty}^t \{g(t - s) - g_0(-s)\} \sigma_s \, dL_s, \tag{1.2}
\]

and is usually referred to as a Lévy semi-stationary process. Here \(L = (L_t)_{t \in \mathbb{R}}\) is a symmetric Lévy process on \(\mathbb{R}\) with respect to \((\mathcal{F}_t)_{t \in \mathbb{R}}\) with \(L_0 = 0\) and without a Gaussian component. That is, for all \(u \in \mathbb{R}\), the process \((L_{t+u} - L_u)_{t \geq 0}\) is a symmetric Lévy process on \(\mathbb{R}_+\) with respect to \((\mathcal{F}_{t+u})_{t \geq 0}\). Moreover, \((\sigma_t)_{t \in \mathbb{R}}\) is a càdlàg process adapted to \((\mathcal{F}_t)_{t \in \mathbb{R}}\), and \(g\) and \(g_0\) are deterministic functions from \(\mathbb{R}\) into \(\mathbb{R}\) vanishing on \((-\infty, 0)\). The name Lévy semi-stationary process refers to the fact that the process \((X_t)_{t \in \mathbb{R}}\) is stationary whenever \(g_0 = 0\) and \((\sigma_t)_{t \in \mathbb{R}}\) is stationary and independent of \((L_t)_{t \in \mathbb{R}}\). It is assumed throughout this paper that \(g, g_0, \sigma\) and \(L\) are such that the process \((X_t)\) is well-defined, which will in particular be satisfied under the conditions stated in Remark 3.3 below. We are interested in the asymptotic behaviour of power variation of the process \(X\). More precisely, let us consider the \(k\)th order increments \(\Delta^n_{i,k} X\) of \(X, k \in \mathbb{N}\), that are defined by

\[
\Delta^n_{i,k} X := \sum_{j=0}^{k} (-1)^j {k \choose j} X_{i-j/n}, \quad i \geq k.
\]

For instance, we have that \(\Delta^0_{i,1} X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\) and \(\Delta^1_{i,2} X = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}\). The main functional of interest is the power variation processes computed on the basis of \(k\)th order increments:

\[
V(p; k)^n_t := \sum_{i=k}^{\lfloor nt \rfloor} |\Delta^n_{i,k} X|^p, \quad p > 0. \tag{1.3}
\]

At this stage we remark that power variation of stochastic processes has been a very active research area in the last decade. We refer e.g. to [8, 28, 29, 35] for limit theory for power variations of Itô semimartingales, to [11, 7, 20, 27, 34] for the asymptotic results
in the framework of fractional Brownian motion and related processes, and to \[19\] \[41\] for investigations of power variation of the Rosenblatt process. More specifically, power variation of Brownian semi-stationary processes, which is the model \[12\] driven by a Brownian motion, has been studied in \[5\] \[6\] \[25\]. Under proper normalisation the authors have shown convergence in probability for the statistic \(V(p; k)^{p}_t\) and proved its asymptotic mixed normality.

However, when the driving motion in \[12\] is a pure jump Lévy process, the asymptotic theory is very different from the Brownian case. To see this, let us recall the first order asymptotic results investigated in \[14\], who studied power variation of a class of Lévy semi-stationary processes with \(\sigma = 1\) and \(t = 1\). Throughout the paper we will need the notion of Blumenthal–Getoor index of \(L\), which is defined via

\[
\beta := \inf \left\{ r \geq 0 : \int_{-1}^{1} |x|^r \nu(dx) < \infty \right\} \in [0, 2],
\]

where \(\nu\) denotes the Lévy measure of \(L\). It is well-known that \(\sum_{s \in [0, 1]} |\Delta L_s|^p\) is finite when \(p > \beta\), while it is infinite for \(p < \beta\). Here \(\Delta L_s = L_s - L_{s-}\) where \(L_{s-} = \lim_{u \uparrow s, u < s} L_u\).

The paper \[14\] imposes the following set of assumptions on \(g, g_0\) and \(\nu\):

**Assumption (A):** The function \(g: \mathbb{R} \to \mathbb{R}\) satisfies

\[
g(t) \sim c_0 t^\alpha \quad \text{as } t \downarrow 0 \quad \text{for some } \alpha > 0 \text{ and } c_0 \neq 0,
\]

where \(g(t) \sim f(t)\) as \(t \downarrow 0\) means that \(\lim_{t \downarrow 0} g(t)/f(t) = 1\). For some \(\theta \in (0, 2]\), \(\limsup_{t \to \infty} \nu(x : |x| \geq t)^t \theta < \infty\) and \(g - g_0\) is a bounded function in \(L^\theta(\mathbb{R}_+)\). Furthermore, \(g\) is \(k\)-times continuously differentiable on \((0, \infty)\) and there exists a \(\delta > 0\) such that \(|g^{(k)}(t)| \leq Ct^{\alpha-k}\) for all \(t \in (0, \delta)\), and such that both \(|g'|\) and \(|g^{(k)}|\) are in \(L^\theta((\delta, \infty))\) and are decreasing on \((\delta, \infty)\).

**Assumption (A-log):** In addition to (A) suppose that \(\int_{\delta}^{\infty} |g^{(k)}(s)|^\theta \log(1/|g^{(k)}(s)|) ds < \infty\).

Assumption (A) ensures, in particular, that the process \(X\) with \(\sigma = 1\) is well-defined, cf. \[14\]. When \(L\) is a \(\beta\)-stable Lévy process, we can and will always choose \(\theta = \beta\) in assumption (A). Before we proceed with the main statement of \[14\], we need some more notation. Let \(h_k: \mathbb{R} \to \mathbb{R}\) be given by

\[
h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x - j)^\alpha_+, \quad x \in \mathbb{R},
\]

where \(y_+ = \max\{y, 0\}\) for all \(y \in \mathbb{R}\). Let \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) and \((T_m)_{m \geq 1}\) be a sequence of \(\mathcal{F}\)-stopping times that exhausts the jumps of \((L_t)_{t \geq 0}\). That is, \((T_m(\omega) : m \geq 1) \cap \mathbb{R}_+ = \{t \geq 0 : \Delta L_t(\omega) \neq 0\}\) and \(T_m(\omega) \neq T_n(\omega)\) for all \(m \neq n\) with \(T_m(\omega) < \infty\). Let \((U_m)_{m \geq 1}\) be independent and uniform \([0, 1]\)-distributed random variables, defined on an extension \((\Omega', \mathcal{F}', \mathbb{P}')\) of the original probability space, which are independent of \(\mathcal{F}\).

The following first order limit theory for the power variation \(V(p; k)^{p}_t\) has been proved in \[14\] for \(\sigma \equiv 1\). We refer to \[11\] \[37\] for the definition of \(\mathcal{F}\)-stable convergence in law which will be denoted \(\mathcal{L}_{\mathcal{F}}^{p} \rightarrow\). Moreover, \(\mathbb{P} \rightarrow\) will denote convergence in probability.
Theorem 1.1 (First order asymptotics [13]). Suppose that $X = (X_t)_{t \geq 0}$ is a stochastic process defined by (1.2) with $\sigma \equiv 1$. Suppose (A) is satisfied and assume that the Blumenthal–Getoor index satisfies $\beta < 2$. Set $V(p; k) := V(p; k_1^n)$. We have the following three cases:

(i) Suppose that (A-log) holds if $\theta = 1$. If $\alpha < k - 1/p$ and $p > \beta$ then the $F$-stable convergence holds as $n \to \infty$

$$n^{\alpha p} V(p; k)^n \xrightarrow{L^s} \sum_{m: T_m \in [0, 1]} |\Delta L_{T_m}|^p V_m \quad \text{where} \quad V_m = \sum_{t=0}^{\infty} |h_k(l + U_m)|^p.$$

(ii) Suppose that $L$ is a symmetric $\beta$-stable Lévy process with scale parameter $\gamma > 0$. If $\alpha < k - 1/\beta$ and $p < \beta$ then it holds

$$n^{-1 + p(\alpha + 1/\beta)} V(p; k)^n \xrightarrow{P} m_p$$

where $m_p = |c_0|^p \gamma^p (\int_\mathbb{R} |h_k(x)|^\beta dx)^p/\beta \mathbb{E}[|Z|^p]$ and $Z$ is a symmetric $\beta$-stable random variable with scale parameter 1.

(iii) Suppose that $p \geq 1$. If $p = \theta$ suppose in addition that (A-log) holds. For all $\alpha > k - 1/(\beta \lor p)$ we have that

$$n^{-1 + pk} V(p; k)^n \xrightarrow{P} \int_0^1 |F_u|^p \, du$$

where $(F_u)_{u \in \mathbb{R}}$ is a measurable process satisfying

$$F_u = \int_{-\infty}^u g^{(k)}(u - s) \, dL_s \quad \text{a.s. for all } u \in \mathbb{R} \quad \text{and} \quad \int_0^1 |F_u|^p \, du < \infty \quad \text{a.s.}$$

The aim of this work is twofold. Firstly, we extend Theorem 1.1 to Lévy semi-stationary processes with a non-trivial volatility process $\sigma$. Such extensions are important in applications, say in the framework of turbulence, since the volatility $\sigma$ are often of key importance. Secondly, we show that the convergence in all the three cases are functional in the Skorokhod topology or in the uniform norm, see Theorem 1.2 below. As we will see later, first order asymptotic theory for Lévy semi-stationary processes can be used to draw inference on the parameters $\alpha$, $\beta$ and on certain volatility functionals in the context of high frequency observations, see Section 2. Furthermore, this type of limit theory is an intermediate step towards asymptotic results for general ambit fields of the form (1.1). We remark that, in contrast to the Brownian setting, the extension of Theorem 1.1 to Lévy semi-stationary processes is a more complex issue. This is due to the fact that it is harder to estimate various norms of $X$ and related processes when the driving process $L$ is a Lévy process. Our estimates on $X$ rely heavily on decoupling techniques and isometries for stochastic integral mappings presented in the book of Kwapien and Wojczyinski [32], see Section 3 for more details. To state our main result we introduce the following assumptions.
Assumptions: Throughout this paper we suppose that (A) holds and let \((H_s)_{s \in \mathbb{R}}\) denote the stochastic process \(H_s = g^{(k)}(-s)\sigma_s \mathbb{1}_{(-\infty, -\delta)}(s), s \in \mathbb{R}\), where the constant \(\delta\) is defined in assumption (A). We now state two additional integrability assumptions on process \(H\) to be used in Theorem 1.2 below.

Assumption (B1): Suppose there exists \(\rho > 0\) with \(\rho \leq 1 \wedge \theta\) and \(\beta' > \beta\) and \(\beta' \geq p\) such that
\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}} \left( |H_s|^\rho \vee |H_s|^\beta' \right) ds \right)^{\frac{1}{2}} \right] < \infty. \tag{1.6}
\]
For \(\theta = 1\) suppose in addition that we may choose \(\rho < 1\) in (1.6).

Assumption (B2): Suppose that
\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}} |H_s|^\beta ds \right) \right] < \infty.
\]

Assumption (B2) will only be used in case \(L\) is a symmetric \(\beta\)-stable Lévy process, where we have \(\theta = \beta\). In this case we note that (B1) is a stronger assumption than (B2).

Let in the following \(D(\mathbb{R}_+; \mathbb{R})\) denote the Skorokhod space of càdlàg functions from \(\mathbb{R}_+\) into \(\mathbb{R}\), equipped with the Skorokhod \(M_1\)-topology. For a detailed introduction and basic properties of this space we refer to [44, Chapter 11.5]. We denote by \(\mathcal{L}_{M_1-}\) the \(\mathcal{F}\)-stable convergence of càdlàg stochastic processes, regarded as random variables taking values in the Polish space \(D(\mathbb{R}_+; \mathbb{R})\). By \(\xrightarrow{\text{u.c.p.}}\) we denote uniform convergence on compact sets in probability. That is, \((Y^n_t)_{t \geq 0}\) \(\xrightarrow{\text{u.c.p.}}\) \((Y_t)_{t \geq 0}\) as \(n \to \infty\) means that
\[
P \left( \sup_{t \in [0,N]} |Y^n_t - Y_t| > \varepsilon \right) \to 0
\]
for all \(N \in \mathbb{N}\) and all \(\varepsilon > 0\). Below, let \((T_m)_{m \geq 1}\) and \((U_m)_{m \geq 1}\) be defined as before Theorem 1.1 and the constant \(m_p\) be defined as in Theorem 1.1(ii).

The following extension of Theorem 1.1, to include a non-trivial \(\sigma\) process and functional convergence, is the main result of this paper.

**Theorem 1.2.** Let \(X = (X_t)_{t \geq 0}\) be a stochastic process defined by (1.2). Assume that the Blumenthal–Getoor index satisfies \(\beta < 2\).

(i) Suppose that (B1) holds and that \(\alpha < k - 1/p\), \(p > \beta\) and \(p \geq 1\). Then, as \(n \to \infty\), the functional \(\mathcal{F}\)-stable convergence holds
\[
n^\alpha p V (p; k)_t \xrightarrow{\mathcal{L}_{M_1-}\} \left| c_0 \right|^p \sum_{m: T_m \in [0,t]} |\Delta L_{T_m} \sigma_{T_m}|^p V_m \quad \text{where} \quad V_m = \sum_{l=0}^{\infty} \left| h_k(l + U_m) \right|^p.
\]

(ii) Suppose that \(L\) is a symmetric \(\beta\)-stable Lévy process with \(\beta \in (0,2)\) and scale parameter \(\gamma > 0\). Suppose that (B2) holds and that \(\alpha < k - 1/\beta\) and \(p < \beta\). Then as \(n \to \infty\)
\[
n^{-1+p(\alpha+1/\beta)} V (p; k)_t \xrightarrow{\text{u.c.p.}} m_p \int_0^t |\sigma|_p^p ds.
\]
(iii) Suppose that (B1) holds, $\theta > 1$, $\alpha > k - 1/(\beta \vee p)$ and $p \geq 1$. Then as $n \to \infty$

$$n^{-1+pk} V(p; k)_t^n \overset{u.c.p.}{\to} \int_0^t |F_u|^p du,$$

where $(F_u)_{u \in \mathbb{R}}$ is a measurable process satisfying

$$F_u = \int_{-\infty}^u g^{(k)}(u-s) \sigma_s^- dL_s \text{ a.s. for all } u \in \mathbb{R} \quad \text{and} \quad \int_0^t |F_u|^p du < \infty \text{ a.s.}$$

Remark 1.3. Under the integrability assumption (B1), Theorem 1.2 covers all possible choices of $\alpha > 0, \beta \in (0, 2)$ and $p \geq 1$ except the critical cases where $p = \beta$, $\alpha = k - 1/p$ or $\alpha = k - 1/\beta$. The two critical cases $\alpha = k - 1/p, p > \beta$ and $\alpha = k - 1/\beta, p < \beta$ have been discussed in [15] in the case $\sigma \equiv 1$.

Remark 1.4. In his original work [42], Skorokhod introduced 4 different topologies on $D(\mathbb{R}_+; \mathbb{R})$, commonly referred to as $J_1, J_2, M_1$ and $M_2$ topology, $J_1$ being by far the most popular one. It can be shown that the functional stable convergence in Theorem 1.2(i) does not hold with respect to the $J_1$-topology. Neither does it hold with respect to $J_2$, whereas $M_2$-convergence is a direct consequence of $M_1$-convergence, since $M_2$ is weaker than $M_1$.

This paper is structured as follows. Section 2 is devoted to various statistical applications of our limit theory. In Section 3 we discuss properties of Lévy integrals of predictable processes and recall essential estimates from [32] for those integrals. All proofs are demonstrated in Section 4.

2 Some statistical applications

We start this section by giving an interpretation to the parameters $\alpha > 0$ and $\beta \in (0, 2)$. Let us consider the linear fractional stable motion defined by

$$Y_t := c_0 \int_{\mathbb{R}} \{(t-s)^\alpha_+ - (-s)^\alpha_+ \} dL_s,$$

where the constant $c_0$ has been introduced in assumption (A). It is well known that the process $(Y_t)_{t \geq 0}$ is well defined whenever $H = \alpha + 1/\beta \in (1/2, 1)$. Furthermore, the process $(Y_t)_{t \geq 0}$ has stationary symmetric $\beta$-stable increments, Hölder continuous paths of all orders smaller than $\alpha$ and self-similarity index $H$, i.e.

$$(Y_{at})_{t \geq 0} \overset{d}{=} (a^H Y_t)_{t \geq 0} \quad \text{for any } a \in \mathbb{R}_+.$$ 

We refer to e.g. [16] for more details. As it has been discussed in [14, 15] in the setting $\sigma = 1$, the small scale behaviour of the process $X$ is well approximated by the corresponding behaviour of the linear fractional stable motion $Y$. In other words, when the intermittency process $\sigma$ is smooth, we have that

$$X_{t+\Delta} - X_t \approx \sigma_t (Y_{t+\Delta} - Y_t)$$
for small $\Delta > 0$. Thus, intuitively speaking, the properties of $Y$ (Hölder smoothness, self-similarity) transfer to the process $X$ on small scales.

Having understood the role of the parameters $\alpha > 0$ and $H = \alpha + 1/\beta \in (1/2, 1)$ from the modelling perspective, it is obviously important to investigate estimation methods for these parameters. We note that the conditions $\alpha > 0$ and $H \in (1/2, 1)$ imply the restrictions $\beta \in (1, 2)$ and $\alpha < 1 - 1/\max\{p, \beta\}$. Hence, the regime of Theorem 1.2 (iii) is never applicable.

We start with a direct estimation procedure, which identifies the convergence rates in Theorem 1.2 (i)-(ii). We apply these convergence results only for $t = 1$ and $k = 1$. For $p \in [\underline{p}, \overline{p}]$ with $\underline{p} \in (0, 1)$ and $\overline{p} > 2$, we introduce the statistic

$$S_{\alpha, \beta}(n, p) := -\frac{\log V(p)^n}{\log n} \quad \text{with} \quad V(p)^n = V(p; 1)^n.$$ 

When the underlying Lévy motion $L$ is symmetric $\beta$-stable and the assumptions of Theorems 1.2 (i)-(ii) are satisfied, we obtain that

$$S_{\alpha, \beta}(n, p) \xrightarrow{P} S_{\alpha, \beta}(p) := \begin{cases} \alpha p : & \alpha < 1 - 1/p \text{ and } p > \beta \\ pH - 1 : & \alpha < 1 - 1/\beta \text{ and } p < \beta \end{cases} \quad (2.1)$$

Indeed, the result of Theorem 1.2 (i) shows that

$$\frac{\alpha p \log n + \log V(p)^n}{\log n} \xrightarrow{\mathcal{L}} 0 \quad \Rightarrow \quad \frac{\alpha p \log n + \log V(p)^n}{\log n} \xrightarrow{P} 0.$$ 

This explains the first line in (2.1). Similarly, Theorem 1.2 (ii) implies the second convergence result of (2.1). At this stage we remark that the limit $S_{\alpha, \beta} : [\underline{p}, \overline{p}] \setminus \{\beta\} \to \mathbb{R}$ is a piecewise linear function with two different slopes. It can be continuously extended to the function $S_{\alpha, \beta} : [\underline{p}, \overline{p}] \to \mathbb{R}$, whose definition can be further extended to include all values

$$(\alpha, \beta) \in J := \{(\alpha, \beta) \in \mathbb{R}^2 : \beta \in [1, 2], \alpha \in [0, 1 - 1/\beta]\}.$$ 

Let $(\alpha_0, \beta_0) \in J^o$, where $J^o$ is the set of all inner points of $J$, denote the true parameter of the model (1.2). Now, it is natural to consider the $L^2$-distance between the observed scale function $S_{\alpha_0, \beta_0}(n, p)$ and the theoretical limit $S_{\alpha, \beta}(p)$:

$$(\hat{\alpha}_n, \hat{\beta}_n) \in \arg\min_{(\alpha, \beta) \in J} \|S_{\alpha_0, \beta_0}(n, p) - S_{\alpha, \beta}\|_{L^2([\underline{p}, \overline{p}])} \quad (2.2)$$

with $S_{\alpha_0, \beta_0}(n) := S_{\alpha_0, \beta_0}(n, \cdot)$. This approach is somewhat similar to the estimation method proposed in [26]. We notice that, for a finite $n$, the minimum of the $L^2([\underline{p}, \overline{p}])$-distance at (2.2) is not necessarily obtained at a unique point, and we take an arbitrary measurable minimiser $(\hat{\alpha}_n, \hat{\beta}_n)$. Our next result shows consistency of the estimator $(\hat{\alpha}_n, \hat{\beta}_n)$.

**Corollary 2.1.** Let $(\alpha_0, \beta_0) \in J^o$ and let $L$ be a symmetric $\beta$-stable Lévy motion. Assume that the conditions of Theorem 1.2 (i) (resp. Theorem 1.2 (ii)) hold when $\alpha \in (0, 1 - 1/p)$ and $p > \beta$ (resp. $\alpha \in (0, 1 - 1/\beta)$ and $p < \beta$). Then we obtain convergence in probability

$$(\hat{\alpha}_n, \hat{\beta}_n) \xrightarrow{P} (\alpha_0, \beta_0).$$
Proof. Set \( r_0 = (\alpha_0, \beta_0) \) and \( \hat{r}_n = (\hat{\alpha}_n, \hat{\beta}_n) \). We first show the convergence
\[
\|S_{r_0}(n) - S_{r_0}\|_{L^2([p, \overline{p}])} \overset{p}{\longrightarrow} 0.
\] (2.3)

From (2.1) we deduce that \( S_{r_0}(n, p) \overset{p}{\longrightarrow} S_{r_0}(p) \) for all \( p \in [p, \overline{p}] \setminus \{\beta\} \). Furthermore, for any \( p \in [p, \overline{p}] \), it holds that
\[
(V(p))^{1/p} \leq (V(\overline{p}))^{1/p} \leq (V(p))^{1/p}.
\]
Hence, we deduce the inequality
\[
\left| \frac{\log V(p)^n}{\log n} \right| \leq \max \left\{ \frac{p}{\overline{p}} \left| \frac{\log V(\overline{p})^n}{\log n} \right|, \frac{\overline{p}}{p} \left| \frac{\log V(p)^n}{\log n} \right| \right\}.
\]
Since \( |\log V(p)^n/\log n| \overset{p}{\longrightarrow} pH - 1 \) and \( |\log V(\overline{p})^n/\log n| \overset{p}{\longrightarrow} op \), because \( p < 1 < \beta \) and \( \overline{p} > 2 > \beta \), we readily deduce the convergence at (2.3) by dominated convergence theorem.

Now, we note that the mapping \( G : J \rightarrow G(J) \subset L^2([p, \overline{p}]), r \mapsto S_r \), is a homeomorphism. Thus, it suffices to prove that \( \|S_{\hat{r}_n} - S_{r_0}\|_{L^2([p, \overline{p}])} \overset{p}{\longrightarrow} 0 \) to conclude \( \hat{r}_n \overset{p}{\longrightarrow} r_0 \). To show the latter we observe that
\[
\|S_{\hat{r}_n} - S_{r_0}\|_{L^2([p, \overline{p}])} \leq \|S_{r_0}(n) - S_{r_0}\|_{L^2([p, \overline{p}])} + \|S_{r_0}(n) - S_{\hat{r}_n}\|_{L^2([p, \overline{p}])}
\]
\[
= \|S_{r_0}(n) - S_{r_0}\|_{L^2([p, \overline{p}])} + \min_{r \in J} \|S_{r_0}(n) - S_r\|_{L^2([p, \overline{p}])}
\]
\[
\leq 2\|S_{r_0}(n) - S_{r_0}\|_{L^2([p, \overline{p}])} \overset{p}{\longrightarrow} 0.
\]
This completes the proof of Corollary 2.1. \( \square \)

In practice the integral in (2.2) needs to be discretised. We further remark that the estimator \( S_{\alpha, \beta}(n, p) \) has the rate of convergence \( \log n \) due to the bias \( V(p)/\log n \), where \( V(p) \) denotes the limit of \( V(p)^n \).

As for the estimation of the self-similarity parameter \( H = \alpha + 1/\beta \in (1/2, 1) \), there is an alternative estimator based on a ratio statistic. Recalling that \( \beta \in (1, 2) \), we deduce for any \( p \in (0, 1] \)
\[
R(n, p) := \frac{\sum_{i=2}^{n-2} |X_{i/n} - X_{i-2/n}|^p}{\sum_{i=1}^{n-1} |X_{i/n} - X_{i-1/n}|^p} \overset{p}{\longrightarrow} 2^{pH}
\]
by a direct application of Theorem 1.2 (ii). Thus, we immediately conclude that
\[
\hat{H}_n := \frac{\log R(n, p)}{p \log 2} \overset{p}{\longrightarrow} H.
\]
This type of idea is rather standard in the framework of a fractional Brownian motion with Hurst parameter \( H \). It has been also applied for Brownian semi-stationary processes in [5, 6]. Theorem 1.2 (i) in [14], which has been shown in the setting \( \sigma = 1 \), suggests that the statistic \( \hat{H}_n \) has convergence rate \( n^{1 - 1/(1-\alpha)\beta} \) whenever \( p \in (0, 1/2] \). Furthermore, the rate of convergence can be improved to \( \sqrt{n} \) via using \( k \)th order increments with \( k \geq 2 \) (cf. [14, Theorem 1.2 (ii)]). However, we dispense with the precise proof of these statements for non-constant intermittency process \( \sigma \).
Remark 2.2. We remark that the linear fractional stable motion \((Y_t)_{t \geq 0}\) is well defined for \(H = \alpha + 1/\beta \in (0, 1)\) and \(\alpha \in (-1/\beta, 1 - 1/\beta)\). In this case the process \(Y\) has unbounded paths whenever \(\alpha < 0\). Since in this framework there is no a priori lower bound on the parameter \(\beta\), it is hard to apply Theorem 1.2 (ii) to estimate the parameter \(H\), because it requires the condition \(p < \beta\). An elegant solution of this problem has been found in a recent work [22] in the context of a linear fractional stable motion. It turns out that in this setting the asymptotic result of Theorem 1.2 (ii) remains valid for powers \(p \in (-1, 0)\). Hence, it holds that
\[
\hat{H}_n \xrightarrow{p} H \quad \text{for } p \in (-1, 0)
\]
when the underlying process is a linear fractional stable motion. However, proving this result for a general Lévy semi-stationary process is a much more delicate issue.

Another important object for applications in turbulence modelling is the intermittency process \(\sigma\). First of all, we remark that the process \(\sigma\) in the general model (1.2) is statistically not identifiable. This is easily seen, because multiplication of \(\sigma\) by a constant cannot be distinguished from the multiplication of, say, Lévy process \(L\) by the same constant. However, it is very well possible to estimate the relative intermittency, which is defined as
\[
RI(p) := \frac{\int_0^t |\sigma_s|^p \, ds}{\int_0^1 |\sigma_s|^p \, ds}, \quad t \in (0, 1),
\]
for \(p \in (0, 1]\). The relative intermittency, which has been introduced in [10] for \(p = 2\) in the context of Brownian semi-stationary processes, describes the relative amplitude of the velocity process on an interval \([0, 1]\). Applying the convergence result of Theorem 1.2 (ii) for \(p \in (0, 1]\), the relative intermittency can be consistently estimated via
\[
RI(n, p) := \frac{V(p)^n}{V(p)^n_1} \xrightarrow{p} RI(p).
\]
Again we suspect that the associated convergence rate is \(n^{1-1/(1-\alpha)/\beta}\) whenever \(p \in (0, 1/2]\) as suggested by [14, Theorem 1.2 (i)].

3 Preliminaries: Estimates on Lévy integrals

To prove the various limit theorems we need very sharp estimates of the \(p\)th moments of the increments of process \(X\) defined in (1.2). In fact, we need such estimates for several different processes related to \(X\) obtained by different truncations. Below we explain some intuition behind the techniques we use to estimate the \(p\)th moments of \(X\). Recall that if \(B\) is a Brownian motion and \(F\) is predictable process then we have the following estimate:

For any \(q > 0\) there exists a finite constant \(C\), only depending on \(p\), such that
\[
E \left[ \left| \int_0^t F_s \, dB_s \right|^q \right] \leq C E \left[ \left( \int_0^t F_s^2 \, ds \right)^{q/2} \right] = C E \left[ \|F\|^q_{L^2([0,t])} \right], \quad (3.1)
\]
which follows by the Burkholder-Davis-Gundy inequality, see [31, Theorem 3.28]. The estimate (3.1) is crucial for proving limit theorems when the driving process is a Brownian
motion, see e.g. [5]. But the situation becomes more complicated when the Brownian motion \( B \) is replaced by a Lévy process \( L \) as considered in the present paper. For integrals with respect to general Lévy processes \( L \) we cannot estimate the sample paths \( s \mapsto F_s(\omega), \omega \in \Omega \), in the \( L^2([0,t]) \)-norm. We need to consider other functionals, which depend on the Lévy measure \( \nu \) of \( L \). When \( F : \mathbb{R}_+ \to \mathbb{R} \) is a deterministic function and \( L \) is a Lévy process, such estimates go back to Rajput and Rosiński [36, Theorem 3.3]. Their results imply the existence of a constant \( C > 0 \) such that
\[
\mathbb{E}[\left| \int_0^t F_s dL_s \right|^q] \leq C \| F \|_{L,q}^q,
\]
where \( \| \cdot \|_{L,q} \) is a certain functional to be defined below (when \( L \) is symmetric and without Gaussian component). The decoupling approach used in Kwapién and Woyczyński [32] provides an extension of the results to general predictable \( F \), see Lemmas 3.1 and 3.2 below. These results can be thought of as extensions of (3.1) to integrals with respect to Lévy processes. Before stating the results precisely, we need the following notation.

Let \( L = (L_t)_{t \in \mathbb{R}} \) be a symmetric Lévy process on the real line with \( L_0 = 0 \), Lévy measure \( \nu \) and without a Gaussian component. For a predictable process \( (F_t)_{t \in \mathbb{R}} \) and for \( q = 0 \) or \( q \geq 1 \) we define
\[
\Phi_{q,L}(F) := \int_{\mathbb{R}^2} \phi_q(F_s u) \, ds \nu(du),
\]
where
\[
\phi_q(x) := |x|^q 1_{\{|x|>1\}} + x^2 1_{\{|x|\leq 1\}}.
\]
A predictable process \( F = (F_t)_{t \in \mathbb{R}} \) is integrable with respect to \( (L_t)_{t \in \mathbb{R}} \) in the sense of [32] if and only if \( \Phi_{0,L}(F) < \infty \) almost surely (cf. [32, Theorem 9.1.1]). The linear space of predictable processes satisfying \( \Phi_{q,L}(F) < \infty \) will be denoted by \( L^q(dL) \). In order to estimate the \( p \)-moments of stochastic integrals we introduce for all \( q \geq 1 \)
\[
\|F\|_{q,L} := \inf\{\lambda \geq 0 : \Phi_{q,L}(F/\lambda) \leq 1\}, \quad F \in L^q(dL). \tag{3.2}
\]
The following two results from Chapter 9.5 in [32] will play a key role for our proofs.

**Lemma 3.1** ([32], Equation (9.5.3)). For all \( q \geq 1 \) there is a constant \( C \), depending only on \( q \), such that we obtain for all \( F \in L^q(dL) \)
\[
\mathbb{E}\left[ \left| \int_\mathbb{R} F_s \, dL_s \right|^q \right] \leq C \mathbb{E}[\| F \|_{q,L}^q]. \tag{3.3}
\]

The above lemma follows by [32, Equation (9.5.3)] and the comments following it. Actually, [32, Equation (9.5.3)] only treats the case where the stochastic integral in (3.3) is over a finite time interval, say \( \int_0^t F_s \, dL_s \). However, the definition of the stochastic integral and the estimates of the integral in [32, Chapters 8–9] extend to the case of \( \int_\mathbb{R} F_s \, dL_s \) in a natural way. For example, the set function \( m((s,t]) = L_t - L_s \) for \( s < t \), extends only to a \( \sigma \)-finite stochastic measure defined on the \( \delta \)-ring of bounded Borel subsets of \( \mathbb{R} \) (cf.
guarantees the existence of the process $X_0$ that $\Phi$ then, an application of the mean value theorem combined with assumption (3.4) yields deduce from (A) and simple calculations the estimate

$\mathbb{E}[\|Z\|^\beta]^{1/q} \leq \|Z\|_{\beta,\infty} \leq \mathbb{E}[\|Z\|^\beta]^{1/\beta}$.

In the literature, $\| \cdot \|_{\beta,\infty}$ is often referred to as the weak $L^\beta$-norm. However, $\| \cdot \|_{\beta,\infty}$ generally fails to satisfy the triangle inequality.

**Remark 3.3.** Suppose that (A) is satisfied and define the two processes $F^{(1)}$ and $F^{(2)}$ by $F_s^{(1)} = (g(s) - g_0(s))\sigma_s$ and $F_s^{(2)} = g'(s)\sigma_s$ for $s < 0$. Then the process $X$ given by $\mathbf{1.2}$ is well-defined if there exists a $\beta' > \beta$ such that

$$\int_{-\infty}^{-\delta} (|F_s^{(i)}|^\beta \mathbb{1}_{\{|F_s^{(i)}| \leq 1\}} + |F_s^{(i)}|^{\beta'} \mathbb{1}_{\{|F_s^{(i)}| > 1\}}) \, ds < \infty$$

almost surely for $i = 1, 2$. To show the above we argue as follows: For any $\beta' \in (\beta, 2]$ we deduce from (A) and simple calculations the estimate

$$\int_{\mathbb{R}} (|ux|^2 \wedge 1) \nu(dx) \leq C(|u|^\beta \mathbb{1}_{\{|u| \leq 1\}} + |u|^{\beta'} \mathbb{1}_{\{|u| > 1\}}), \quad u \in \mathbb{R}.$$  

Then, an application of the mean value theorem combined with assumption (3.4) yields that $\Phi_{0,L}(H^{(t)}) < \infty$ almost surely for all $t > 0$, where $H^{(t)} = (g(t-s) - g_0(s))\sigma_s$. This guarantees the existence of the process $X$ due to $\mathbf{3.2}$ Theorem 9.1.1.

In our proofs we will need the following properties of the functional $\| \cdot \|_{L,q}$ defined in $\mathbf{3.2}$.

**Remark 3.4.** The functional $\| \cdot \|_{L,q}$ satisfies the following three properties:

(i) Homogeneity: For all $\lambda \in \mathbb{R}$, $F \in L^q(dL)$, $\|\lambda F\|_{q,L} = |\lambda|\|F\|_{q,L}$.

(ii) Triangle inequality (up to a constant): There exists a constant $C > 0$ such that for all $m \in \mathbb{N}$ and $F^1, \ldots, F^m \in L^q(dL)$ we have

$$\|F^1 + \cdots + F^m\|_{q,L} \leq C(\|F^1\|_{q,L} + \cdots + \|F^m\|_{q,L}).$$  

(3.6)
(iii) Upper bound: For all \( F \in \mathbf{L}^q(dL) \) we have
\[
\|F\|_{q,L} \leq \Phi_{q,L}^{1/2}(F) \vee \Phi_{q,L}^{1/4}(F).
\] (3.7)

Property (iii) follows directly from the definition of \( \| \cdot \|_{L,q} \) in (3.2). To show property (ii) it is sufficient to derive (3.6) for \( F^1, \ldots, F^m \in \mathbf{L}^q_{nr}(dL) \), where \( \mathbf{L}^q_{nr}(dL) \) denotes the subspace of nonrandom processes in \( \mathbf{L}^q(dL) \). We will show that there is a norm \( \| \cdot \|'_{q,L} \) on \( \mathbf{L}^q_{nr}(dL) \) and \( c > 0 \) and \( C > 0 \) such that
\[
c \|F\|'_{q,L} \leq \|F\|_{q,L} \leq C \|F\|'_{q,L},
\]
for all \( F \in \mathbf{L}^q_{nr}(dL) \), which then implies (3.6). To this end, let
\[
\tilde{\phi}_q(x) := \left(2/q|x|^q + 1 - 2/q\right) \mathbf{1}_{\{|x| > 1\}} + x^2 \mathbf{1}_{\{|x| \leq 1\}}.
\]
Clearly, there exist \( c, C > 0 \) such that
\[
c \tilde{\phi}_q(x) \leq \phi_q(x) \leq C \tilde{\phi}_q(x) \quad \text{for all } x \in \mathbb{R}.
\]
Moreover, since the function \( \tilde{\phi}_q \) is convex, the functional
\[
\|F\|_{q,L} = \inf \left\{ \lambda \geq 0 : \int_{\mathbb{R}^2} \tilde{\phi}_q(F_s u/\lambda) \, ds \, \nu(du) \leq 1 \right\}
\]
is a norm on \( \mathbf{L}^q_{nr}(dL) \), called the Luxemburg norm (cf. [33, Chapter 1]). Using convexity of \( \tilde{\phi}_q \) it follows by straightforward calculations that \( c \|F\|'_{q,L} \leq \|F\|_{q,L} \leq C \|F\|'_{q,L} \) for all \( F \in \mathbf{L}^q_{nr}(dL) \). This implies (3.6). Finally, property (iii) follows by the fact that \( \phi_q(\lambda x) \leq (\lambda^2 + \lambda^q) \phi_q(x) \) for all \( \lambda \geq 0 \).

We conclude this subsection with a remark on the situation when the integrator is a non-symmetric Lévy process \( (\tilde{L}_t)_{t \in \mathbb{R}} \) with \( \tilde{L}_0 = 0 \), Lévy measure \( \tilde{\nu} \), shift parameter \( \eta \), without a Gaussian part, and the truncation function \( \tau : x \mapsto \mathbf{1}_{\{|x| < 1\}} + \text{sign}(x) \mathbf{1}_{\{|x| \geq 1\}} \). That is, for all \( \theta \in \mathbb{R} \),
\[
\mathbb{E}[e^{i\theta \tilde{L}_t}] = \exp \left( i\theta \eta + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta \tau(x)) \, \tilde{\nu}(dx) \right).
\]
For a predictable process \( (F_t)_{t \in \mathbb{R}} \) define
\[
\Psi_{0,\tilde{L}}(F) = \left| \int_{\mathbb{R}^2} \tau(u F_s) - \tau(u) F_s \, ds \, \tilde{\nu}(du) + \eta F_s \right|.
\]
Then, the condition
\[
\Phi_{0,\tilde{L}}(F) + \Psi_{0,\tilde{L}}(F) < \infty \quad \text{almost surely}
\]
(3.8)
is sufficient for the integral \( \int_{\mathbb{R}} F_s \, d\tilde{L}_s \) to exist, and we write \( F \in \mathbf{L}^0(\tilde{dL}) \). Indeed, this is a consequence of [32] Theorem 9.1.1 and pp. 217–218 combined with the estimate [36] Lemma 2.8].
4 Proofs

In this section we present the proofs of our main results. Let us first briefly comment on some of the techniques used in the proofs of Theorem 1.2.

The proof of Theorem 1.2 (i) is divided into two parts. First we show the theorem under the assumption that \( L \) is a compound Poisson process with jumps bounded away from zero in absolute value by some \( a > 0 \). In this situation, (B1) ensures that the integral in (1.2) can be defined \( \omega \) by \( \omega \), and the limit of \( V(p; k)_n \) can be derived by similar means as in [14]. Thereafter, we argue that the contribution of the jumps of \( L \) with absolute value \( \leq a \) to the power variation becomes negligible as \( a \to 0 \).

The proof of Theorem 1.2 (ii) relies on Bernstein’s blocking technique combined with Theorem 1.1 (ii) and several approximation steps. A key step in the proof of Theorem 1.2 (ii) is an application of a suitable stochastic Fubini result. For this purpose we present and prove a stochastic Fubini theorem for Lévy integrals with predictable integrands that is applicable under our assumptions.

Throughout the proofs we denote all positive constants that do not depend on \( n \) or \( \omega \) by \( C \), eventhough they may change from line to line. Similarly, we will denote by \( K \) any positive random variable that does not depend on \( n \), but may change from line to line.

For a random variable \( Y \) and \( q > 0 \) we denote \( \|Y\|_q = \mathbb{E}[|Y|^q]^{1/q} \). We frequently use the notation
\[
g_{i,n}(s) = \sum_{j=0}^{k} (-1)^j {k \choose j} g((i - j)/n - s),
\]
which allows us to express the \( k \)-th order increments of \( X \) as
\[
\Delta_{i,k}^{n}X = \int_{-\infty}^{i/n} g_{i,n}(s) \sigma_\omega \, dL_s.
\]
Recalling that \( |g^{(k)}(s)| \leq Ct^{\alpha-k} \) for all \( s \in (0, \delta) \) and \( g^{(k)} \) is decreasing on \((\delta, \infty)\) by assumption (A), Taylor expansion leads to the following important estimates.

Lemma 4.1. Suppose that assumption (A) is satisfied. It holds that
\[
|g_{i,n}(s)| \leq C(i/n - s)^\alpha \quad \text{for } s \in [(i - k)/n, i/n],
\]
\[
|g_{i,n}(s)| \leq Cn^{-k}((i - k)/n - s)^{\alpha-k} \quad \text{for } s \in (i/n - \delta, (i - k)/n), \text{ and}
\]
\[
|g_{i,n}(s)| \leq Cn^{-k}(\mathbb{1}_{[(i - k)/n - \delta, i/n - \delta]}(s) + g^{(k)}((i - k)/n - s)\mathbb{1}_{(-\infty, (i - k)/n - \delta)}(s)) \quad \text{for } s \in (-\infty, i/n - \delta].
\]

Proof. The first inequality follows directly from condition (1.4) of (A). The second inequality is a straightforward consequence of Taylor expansion of order \( k \) and the condition \( |g^{(k)}(t)| \leq Kt^{\alpha-k} \) for \( t \in (0, \delta) \). The third inequality follows again through Taylor expansion and the fact that the function \( g^{(k)} \) is decreasing on \((\delta, \infty)\).

Remark 4.2. Throughout the proofs we will generally assume that the process \( \sigma \) is uniformly bounded on \([-\delta, \infty)\). That is, there exists a deterministic constant \( C > 0 \) such
that $|\sigma_s| < C$ for all $s \geq -\delta$. This does not restrict the generality of our results, since we can apply the following localisation argument. Let $(S_m)_{m \geq 1}$ be a sequence of $(\mathcal{F}_t)_{t \geq -\delta}$-stopping times with $S_m \uparrow \infty$, such that $|\sigma_s|\mathbb{1}_{\{S_m > -\delta\}}$ is bounded for all $s \in [-\delta, S_m]$. Let the process $\sigma^{(m)}$ be defined by

$$\sigma_s^{(m)} = \sigma_s \mathbb{1}_{\{s < S_m\}} + \sigma_{S_m} - \mathbb{1}_{\{s \geq S_m > -\delta\}}.$$  

The process $\sigma^{(m)}$ is again càdlàg and adapted. We define the process $(X_t^{(m)})_{t \geq 0}$ by replacing $\sigma$ in the definition of $X$ by $\sigma^{(m)}$. We note that $(X_t^{(m)})$ is well-defined since $(X_t)$ is well-defined and that assumption (B1) and (B2) hold for $(X_t^{(m)})$ if they hold for $(X_t)$. It holds that $\Delta_{i,k}^{n}X^{(m)}\mathbb{1}_{\{S_m > t\}} = \Delta_{i,k}^{n}X\mathbb{1}_{\{S_m > t\}}$ almost surely. It is therefore sufficient to show that Theorem 1.2 holds for the processes $X^{(m)}$. Then the theorem follows for the process $X$ by letting $m \to \infty$.

### 4.1 Proof of Theorem 1.2 (i)

For the proof of Theorem 1.2 (i) we follow the strategy from [14] Thm. 1.1 (i). We assume first that $L$ is a compound Poisson process with jumps bounded in absolute value away from zero by some $a > 0$. Later on, we argue that the small jumps of $L$ are asymptotically negligible. Recall that in order to show functional $\mathcal{F}$-stable convergence on $\mathbb{D}(\mathbb{R}^+; \mathbb{R})$ it is sufficient to show $\mathcal{F}$-stable convergence on $\mathbb{D}([0, t_\infty]; \mathbb{R})$, for arbitrary but fixed $t_\infty > 0$ (cf. [14] Chapter 3.3]). Throughout this subsection we will therefore fix a $t_\infty > 0$, and denote by $\mathbb{D}$ the space $\mathbb{D}([0, t_\infty]; \mathbb{R})$ equipped with the Skorokhod $M_1$-topology, and by $\overset{\mathcal{L}M_1-s}{\longrightarrow}$ the $\mathcal{F}$-stable convergence of $\mathbb{D}$-valued processes.

#### 4.1.1 Compound Poisson Case

Suppose that $(L_t)_{t \in \mathbb{R}}$ is a symmetric compound Poisson process with Lévy measure $\nu$, satisfying $\nu([-a,a]) = 0$ for some $a > 0$. Let $0 < T_1 < T_2 < \ldots$ denote the jump times of $(L_t)_{t \geq 0}$ in increasing order. For $\varepsilon > 0$ we define

$$\Omega_{\varepsilon} = \{ \omega \in \Omega : \text{for all } m \text{ with } T_m(\omega) \in [0, t_\infty) \text{ we have } |T_m(\omega) - T_{m-1}(\omega)| > \varepsilon$$

and $\Delta L_s(\omega) = 0$ for all $s \in [-\varepsilon, 0]$}.\]

We note that $\Omega_{\varepsilon} \uparrow \Omega$, as $\varepsilon \downarrow 0$. Letting

$$M_{i,n,\varepsilon} := \int_{i/n-\varepsilon}^{i/n} g_{i,n}(s) \sigma_s \ dL_s, \quad \text{and} \quad R_{i,n,\varepsilon} := \int_{i/n}^{s} g_{i,n}(s) \sigma_s \ dL_s,$$

we have the decomposition $\Delta_{i,k}^{n}X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon}$. It turns out that $M_{i,n,\varepsilon}$ is the asymptotically dominating term, whereas $R_{i,n,\varepsilon}$ is negligible as $n \to \infty$.

We show that, on $\Omega_{\varepsilon}$,

$$n^{-p} \sum_{i=k}^{[nt]} |M_{i,n,\varepsilon}|^p \overset{\mathcal{L}M_1-s}{\longrightarrow} Z_t, \quad \text{where} \quad Z_t = \sum_{m : T_m(0,t]} |\Delta L_{T_m} \sigma_{T_m}|^p V_m, \quad (4.1)$$

...
where \((V_m)_{m \geq 1}\) are defined in Theorem \[\text{I.2}\] (i). Denote by \(i_m\) the random index such that \(T_m \in ((i_m - 1)/n, i_m/n]\). Then, we have on \(\Omega_{\varepsilon}\)

\[
n^{op} \sum_{i=1}^{\lfloor nt \rfloor} |M_{i,n,\varepsilon}|^p = V_t^{n,\varepsilon},
\]

where \(V_t^{n,\varepsilon} = n^{op} \sum_{m: T_m \in (0, \lfloor nt \rfloor/n]} |\Delta L_{T_m} \sigma_{T_m-}|^p \left( \sum_{l=0}^{s} |g_{i_m+l,n}(T_m)|^p \right)\).

Here the random index \(v_t^m\) is defined as

\[
v_t^m = v_t^m(\varepsilon, n) = \begin{cases} \lfloor \varepsilon n \rfloor \land (\lceil nt \rceil - i_m) & \text{if } T_m - (\lfloor \varepsilon n \rfloor + i_m)/n > -\varepsilon, \\ \lfloor \varepsilon n \rfloor - 1 \land (\lceil nt \rceil - i_m) & \text{if } T_m - (\lfloor \varepsilon n \rfloor + i_m)/n \leq -\varepsilon. \end{cases}
\]

Additionally, we set \(v_t^m = \infty\) if \(T_m > \lfloor nt \rfloor/n\). We remark that \(v_t^m\) attains the value \([nt] - i_m\) only if \(T_m \in (t - \varepsilon, t]\), which is the case for at most one \(m\). For the proof of (4.1) we first show stable convergence of the finite dimensional distributions of \(V^{n,\varepsilon}\). Thereafter, we show that the sequence \((V^{n,\varepsilon})_{n \geq 1}\) is tight and deduce the functional convergence \(V^{n,\varepsilon} \xrightarrow{L^1} Z\).

**Lemma 4.3.** For \(r \geq 1\) and \(0 \leq t_1 < \cdots < t_r \leq t_{\infty}\) we obtain on \(\Omega_{\varepsilon}\) the \(\mathcal{F}\)-stable convergence

\[(V_{t_1}^{n,\varepsilon}, \ldots, V_{t_r}^{n,\varepsilon}) \xrightarrow{L^{\varepsilon}} (Z_{t_1}, \ldots, Z_{t_r})\]

as \(n \to \infty\).

**Proof.** Let \((U_i)_{i \geq 1}\) be i.i.d. \(U([0,1])\)-distributed random variables, defined on an extension \((\Omega', \mathcal{F}', \mathbb{P}')\) of the original probability space, independent of \(\mathcal{F}\). By arguing as in \[\text{I.4}\] Section 5.1, we deduce for any \(d \geq 1\) the \(\mathcal{F}\)-stable convergence

\[
\{n^{op} g_{i_m+l,n}(T_m)\}_{l,m \leq d} \xrightarrow{L^{\varepsilon}} \{h_k(l + U_m)\}_{l,m \leq d}
\]

as \(n \to \infty\), where \(h_k\) is defined in \[\text{I.5}\]. Defining

\[
V_{t_1}^{n,\varepsilon,d} := n^{op} \sum_{m \leq d: T_m \in (0, \lfloor nt \rfloor/n]} |\Delta L_{T_m} \sigma_{T_m-}|^p \left( \sum_{l=0}^{d} |g_{i_m+l,n}(T_m)|^p \right)
\]

and

\[
Z_t^d := \sum_{m \leq d: T_m \in (0,t]} |\Delta L_{T_m} \sigma_{T_m-}|^p \left( \sum_{l=0}^{d} |h_k(l + U_m)|^p \right),
\]

the continuous mapping theorem for stable convergence yields

\[(V_{t_1}^{n,\varepsilon,d}, \ldots, V_{t_r}^{n,\varepsilon,d}) \xrightarrow{L^{\varepsilon}} (Z_{t_1}^d, \ldots, Z_{t_r}^d), \quad \text{for } n \to \infty,
\]

for all \(d \geq 1\). It follows by Lemma \[\text{I.4}\] for all \(l\) with \(k \leq l < [n\delta]\) that

\[n^{op}|g_{i_m+l,n}(T_m)|^p \leq C|l - k|^{(a-k)p},\]
where we recall that $(\alpha - k)p < -1$. Consequently, we find a random variable $K > 0$ such that for all $t \in [0, t_\infty]$

\[ |V_{t}^{n,\varepsilon,d} - V_{t}^{n,\varepsilon}| \leq K \left( \sum_{m > d:T_m \in [0, t_\infty]} |\Delta L_{m}\sigma_{m}|^p + \sum_{m:T_m \in [0, t_\infty]} \sum_{l = n^m \wedge d}^\infty |l - k|^{(\alpha - k)p} \right). \]

By definition, the random index $v_{m}^m = v_{m}^m(n, \omega)$ satisfies $\lim \inf_{n \to \infty} v_{m}^m(n, \omega) = \infty$ for all $\omega$ with $T_m(\omega) \neq t$. Consequently, we obtain that $\lim \sup_{n \to \infty} |V_{t}^{n,\varepsilon,d} - V_{t}^{n,\varepsilon}| \to 0$ almost surely as $d \to \infty$. It follows that on $\Omega_\varepsilon$

\[ \lim \sup_{n \to \infty} \left\{ \sup_{t \in \{t_1, \ldots, t_r\}} |V_{t}^{n,\varepsilon} - V_{t}^{n,\varepsilon,d}| \right\} \to 0, \quad \text{almost surely, as } d \to \infty. \tag{4.3} \]

By monotone convergence theorem we obtain that $\sup_{t \in [0, t_\infty]} |Z_{t}^d - Z_t| \to 0$ as $d \to \infty$. Together with (4.2) and (4.3), this implies the statement of the lemma by a standard approximation argument, see for example [17, Thm 3.2].

Recall that the stable convergence $V_{t}^{n,\varepsilon} \xrightarrow{L_{M_1}} Z$ is equivalent to the convergence of $(V_{t}^{n,\varepsilon},X) \xrightarrow{F} (Z,X)$ for all $F$-measurable random variables $X$, cf. [30, Prop. 5.33]. Consequently, Lemma 4.3 and the following lemma together with Prokhorov’s theorem imply (4.1), where we recall that $D([0, t_\infty])$ equipped with the $M_1$ topology is a Polish space.

**Lemma 4.4.** The sequence of $\mathbb{D}$-valued processes $(V_{t}^{n,\varepsilon})_{n \geq 1}$ is tight.

**Proof.** It is sufficient to show that the conditions of [44, Theorem 12.12.3] are satisfied. Condition (i) is satisfied, since the family of real valued random variables $(V_{t_\infty}^{n,\varepsilon})_{n \geq 1}$ is tight by Lemma 4.3. Condition (ii) is satisfied, since the oscillating function $w_s$ introduced in [44, chapter 12, (5.1)] satisfies $w_s(V_{t_\infty}^{n,\varepsilon}, \theta) = 0$ for all $\theta > 0$ and all $n$, since $V_{t_\infty}^{n,\varepsilon}$ is increasing. This concludes the proof of (4.1). Next we show that

\[ n^{\alpha p} \sum_{i = k}^{[nt_\infty]} |R_{i,n,\varepsilon}|^p \xrightarrow{\mathbb{P}} 0. \tag{4.4} \]

Recalling that $\alpha < k - 1/p$, it is sufficient to show that

\[ \sup_{n \in \mathbb{N}, i \in \{k, \ldots, [nt_\infty]\}} n^k |R_{i,n,\varepsilon}| < \infty, \quad \text{almost surely.} \]

It follows from Lemma 4.1 that

\[ n^k |g_{i,n}(s)\sigma_{\pm}| \leq C(I_{[-\delta,t_\infty]}(s) + |g^{(k)}(-s)\sigma_{\pm} - I_{(-\infty,-\delta)}(s)|) := \psi_s. \]

Let $\tilde{L} = (\tilde{L}_t)_{t \in \mathbb{R}}$ denote the total variation process defined as $\tilde{L}_0 = 0$ and $\tilde{L}_t - \tilde{L}_u$ is the total variation of $v \mapsto L_v$ on $(u, t]$ for all $u < t$. Since $L$ is a compound Poisson process,
the process $\tilde{L}$ is well-defined, finite and we deduce from Theorem 21.9 that $\tilde{L}$ is a Lévy process with Lévy measure $\tilde{\nu} = 2\nu|_{\mathbb{R}_+}$ and shift parameter $\eta$ with respect to the truncation function $\tau: x \mapsto 1_{\{|x|<\bar{a}\}} + \text{sign}(x)1_{\{|x|\geq \bar{a}\}}$ given by $\eta = \int_\mathbb{R} \tau(x)\tilde{\nu}(dx)$. Next we use the following estimate:

$$n^k|R_{i,n,\varepsilon}| \leq \int_{(-\infty,\frac{1}{n}-\varepsilon]} n^k|g_{i,n}(s)|\sigma_s-|d\tilde{L}_s| \leq \int_\mathbb{R} \psi_s d\tilde{L}_s.$$ 

The right-hand side is finite almost surely due to the following Lemma 4.5 and the proof of (4.4) is complete.

**Lemma 4.5.** Let $L$ be a symmetric compound Poisson process with Lévy measure $\nu$ satisfying $\nu([-a,a]) = 0$ for some $a \in (0,1]$ and let $\tilde{L}$ and $\psi$ be given as above. Suppose, in addition, that (B1) is satisfied. Then the stochastic integral $\int_\mathbb{R} \psi_s d\tilde{L}_s$ exists and is finite almost surely.

**Proof.** To show that the stochastic integral $\int_\mathbb{R} \psi_s d\tilde{L}_s$ is well-defined it is enough to prove that $\Phi_{0,L}(\psi) + \Psi_{0,\tilde{L}}(\psi) < \infty$ almost surely (see (3.17) of Section 3). For some $\beta' > \beta$ we have from (B1) that

$$\int_\mathbb{R} |\psi_s|^\beta \mathbb{1}_{\{|\psi_s|\leq 1\}} + |\psi_s|^\beta' \mathbb{1}_{\{|\psi_s|> 1\}} \, ds < \infty, \quad \text{a.s.}$$

This implies that $\Phi_{0,L}(\psi) < \infty$ almost surely (cf. (3.15)). Next we note that

$$\Psi_{0,L}(\psi) = \left| \int_{\mathbb{R}^2} \tau(x\psi_s) - \tau(u)\psi_s \, ds \tilde{\nu}(dx) + \eta \psi_s \right| = \left| \int_{\mathbb{R}^2} \tau(x\psi_s) \, ds \tilde{\nu}(dx) \right|,$n

where the second equality follows by definition of $\eta$ above. Hence, to show that $\Psi_{0,L}(\psi) < \infty$ almost surely, it suffices according to (B1) to derive the following estimate. There exists a constant $C > 0$ such that for all $u \in \mathbb{R}$

$$\int_\mathbb{R} |\tau(u)\psi_s| \tilde{\nu}(dx) \leq C(\|u\|^\rho \mathbb{1}_{\{|u|\leq 1\}} + \mathbb{1}_{\{|u|> 1\}}).$$

(4.5)

where $\rho$ is as in assumption (B1). By the definitions of $\tau$ and $\tilde{\nu}$ we have that

$$\int_\mathbb{R} |\tau(u)\psi_s| \tilde{\nu}(dx) = |u| \int_{\{|x|\leq |u|^{-1}\}} |x| \nu(dx) + \nu(x \in \mathbb{R} : |xu| > 1).$$

(4.6)

We recall that $\limsup_{t \to \infty} \nu([t,\infty))^{\theta} < \infty$. Since $\nu$ is finite, there exists $C_0 > 0$ such that $\nu([t,\infty)) \leq C_0/t^\theta$ for all $t \geq a$. Consequently, we obtain for all $t \geq a$ and $f(u) = \mathbb{1}_{[t,\infty)}(u)$

$$\int_a^\infty f(x) \nu(dx) \leq \frac{C_0}{\theta} \int_a^\infty f(x)x^{-\theta-1} \, dx.$$
For all \( u \)

For the second term on the right-hand side of (4.6) we use the following estimate

\[
|u| \int_{\{|x| \leq |u|^{-1}\}} |x| \nu(dx) \leq (C_0/\theta) \mathbb{1}_{\{|u| \leq a^{-1}\}} |u| \int_a^{1-a^{-1}} |x|^{-\theta} dx
\]

\[
\leq C \mathbb{1}_{\{|u| \leq a^{-1}\}} \begin{cases} |u|^\theta & \theta < 1 \\ |u| (\log(1/|u|) + \log(1/a)) & \theta = 1 \\ |u| & \theta > 1. \end{cases}
\]

For the second term on the right-hand side of (4.6) we use the following estimate

\[
\nu\left( x \in \mathbb{R} : |xu| > 1 \right) \leq C (\mathbb{1}_{\{|u| > 1\}} + |u|^{-\theta} \mathbb{1}_{\{|u| \leq 1\}}) = C (\mathbb{1}_{\{|u| > 1\}} + |u|^\theta \mathbb{1}_{\{|u| \leq 1\}})
\]

for all \( u \in \mathbb{R} \), which completes the proof of (4.5) and hence of the lemma.

Recalling the decomposition \( \Delta_{\varepsilon n} \cdot X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon} \) we obtain by Minkowski’s inequality

\[
\sup_{t \in [0,t_{\infty}]} \left( n^{\alpha p} V(p; k)^{n} \right) \frac{1}{p} - \left( n^{\alpha p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^{p} \right) \frac{1}{p} \leq \left( n^{\alpha p} \sum_{i=k}^{n} |R_{i,n,\varepsilon}|^{p} \right) \frac{1}{p}.
\]

Therefore, by virtue of (4.1) and (4.3), we conclude that

\[
n^{\alpha p} V(p; k)^{n} \mathcal{L}_{\varepsilon n} \xrightarrow{t_{\infty}} Z_t \quad \text{on } \Omega_{\varepsilon}.
\]

By letting \( \varepsilon \to 0 \) we conclude that Theorem 1.2 (i) holds, when \( L \) is a compound Poisson process with jumps bounded away from 0.

### 4.1.2 Decomposition into big and small jumps

In this section we extend the proof of Theorem 1.2 (i) to general symmetric Lévy processes \( (L_t)_{t \in \mathbb{R}} \). We need the following preliminary result.

**Lemma 4.6.** Let \( q \geq 1 \) and \( a \in (0,1] \). The function

\[
\xi(y) = \int_{-a}^{a} |y|^2 \mathbb{1}_{\{|yx| \leq 1\}} + |y|^q \mathbb{1}_{\{|yx| > 1\}} \nu(dx)
\]

satisfies \( |\xi(y)| \leq C (|y|^2 \mathbb{1}_{\{|y| \leq 1\}} + |y|^\beta^q \mathbb{1}_{\{|y| > 1\}}) \) for any \( \beta' > \beta \), where \( C \) does not depend on \( a \).

**Proof.** Use the decomposition \( \xi = \xi_1 + \xi_2 \) with

\[
\xi_1(y) = \int_{-a}^{a} |y|^2 \mathbb{1}_{\{|yx| \leq 1\}} \nu(dx), \quad \text{and} \quad \xi_2(y) = \int_{-a}^{a} |y|^q \mathbb{1}_{\{|yx| > 1\}} \nu(dx).
\]

We obtain

\[
\xi_1(y) \mathbb{1}_{\{|y| \leq 1\}} \leq |y|^2 \int_{-1}^{1} x^2 \nu(dx) \mathbb{1}_{\{|y| \leq 1\}},
\]
and \( \xi_1(y) \mathbb{I}_{\{|y|>1\}} \leq C|y|^\beta q \mathbb{I}_{\{|y|>1\}} \) follows from (3.5), showing that \( \xi_1 \) satisfies the estimate given in the lemma. For \( q > \beta \) we obtain
\[
\xi_2(y) = 2|y|^q \mathbb{I}_{\{|y|>1\}} \int_{1/|y|}^a |x|^q \nu(dx) \leq C|y|^q \mathbb{I}_{\{|y|>1\}}.
\]
If \( q \leq \beta \) we have for any \( \beta' > \beta \)
\[
\xi_2(y) \leq 2|y|^\beta q \mathbb{I}_{\{|y|>1\}} \int_{1/|y|}^a |x|^\beta q \nu(dx) \leq C|y|^\beta q \mathbb{I}_{\{|y|>1\}},
\]
which completes the proof. \( \Box \)

Now, given a general symmetric Lévy process \((L_t)_{t \in \mathbb{R}}\), consider for \( a > 0 \) the compound Poisson process \((L^a_t)_{t \in \mathbb{R}}\) defined by
\[
L^a_t - L^a_s = \sum_{s < u \leq t} \Delta L_u \mathbb{I}_{\{|\Delta L_u| > a\}}, \quad L^a_0 = 0.
\]
Moreover, let \((L^a_t)_{t \in \mathbb{R}}\) denote the Lévy process \((L_t - L^a_t)_{t \in \mathbb{R}}\). The key result of this section is the following approximation lemma. Intuitively, the lemma shows that replacing \((L_t)_{t \in \mathbb{R}}\) by \((L^a_t)_{t \in \mathbb{R}}\) in the definition of \(X\) has a negligible effect for \(a \to 0\).

**Lemma 4.7.** It holds that
\[
\lim_{a \to 0} \lim_{n \to \infty} \sup_{t \in [0,1]} \left| n^{op} \sum_{i=1}^{\lfloor nt \rfloor} \int_{-\infty}^{i/n} g_{i,n}(s) \sigma_{s-} \, dL^a_s \right|^p_1 = 0.
\]

*Proof.* We make the decomposition
\[
\int_{-\infty}^{i/n} g_{i,n}(s) \sigma_{s-} \, dL^a_s = A_{i,n} + B_{i,n},
\]
where
\[
A_{i,n} = \int_{-\delta}^{i/n} g_{i,n}(s) \sigma_{s-} \, dL^a_s \quad \text{and} \quad B_{i,n} = \int_{-\infty}^{-\delta} g_{i,n}(s) \sigma_{s-} \, dL^a_s.
\]
Lemma 3.1 shows that
\[
\left| n^{op} \sum_{i=k}^{\lfloor nt \rfloor} |A_{i,n}|^p \right|_1 = n^{-1} \sum_{i=k}^{\lfloor nt \rfloor} \left| \int_{-\delta}^{i/n} n^{a+1/p} g_{i,n}(s) \sigma_{s-} \, dL^a_s \right|^p_1 \leq Cn^{-1} \sum_{i=k}^{\lfloor nt \rfloor} \mathbb{E} \left[ \left\| F^i_{n,\delta} \right\|_{p,L^a}^p \right],
\]
where the process \((F^i_{t,\delta})_{t \in \mathbb{R}}\) is defined as \( F^i_{t,\delta} = n^{a+1/p} g_{i,n}(t) \mathbb{I}_{(-\delta,i/n]}(t) \sigma_t \). Since the random variable \( \sup_{t \in [-\delta,\infty)} |\sigma_t| \) is uniformly bounded (see Remark 4.2), we obtain by
We obtain by Lemma 3.1 that

\[ E\left[\|F^{i,n}\|^p_{p, L^a}\right] \leq C\|n^{\alpha+1/p}g_{i,n}\|_{p, L^a}^p \]

\[ \leq C\Phi_{p, L^a}\left(n^{\alpha+1/p}g_{k,n}\right)^{p/2} \vee \Phi_{p, L^a}\left(n^{\alpha+1/p}g_{k,n}\right) \]

\[ \leq C\left(\int_{|x|\leq a}|x|^p + x^2\nu(dx)\right)^{p/2} \vee \left(\int_{|x|\leq a}|x|^p + x^2\nu(dx)\right), \]

for all \( n \in \mathbb{N} \) and \( i \in \{k, \ldots, [nt_\infty]\} \). Since \( p > \beta \) by assumption, we conclude that

\[ \limsup_{n \to \infty} \left\| n^{\alpha p} \sum_{i=k}^{[nt_\infty]} |A_{i,n}|^p \right\|_1 \to 0, \quad \text{as } a \to 0. \quad (4.7) \]

Next, we show that for all \( a > 0 \)

\[ \limsup_{n \to \infty} \left\| n^{\alpha p} \sum_{i=k}^{[nt_\infty]} |B_{i,n}|^p \right\|_1 = 0. \quad (4.8) \]

Introducing the processes \((Y^{i,n}_t)_{t \in \mathbb{R}}\) and \((Y_t)_{t \in \mathbb{R}}\) defined as

\[ Y^{i,n}_t = n^{\alpha+1/p}g_{i,n}(t)\sigma_t \mathbb{1}_{(-\infty,-\delta]}(t), \quad \text{and} \]

\[ Y_t = |g^{(k)}(-t)\sigma_t \mathbb{1}_{(-\infty,-\delta]}(t)|, \]

we obtain by Lemma 3.1 that

\[ \left\| n^{\alpha p} \sum_{i=k}^{[nt_\infty]} |B_{i,n}|^p \right\|_1 \leq C n^{-1} \sum_{i=k}^{[nt_\infty]} E\left[\|Y^{i,n}\|^p_{p, L^a}\right]. \]

Moreover, recalling that \(|g^{(k)}|\) is decreasing on \((\delta, \infty)\), an application of Lemma 4.1 shows that

\[ E\left[\|Y^{i,n}\|^p_{p, L^a}\right] \leq n^{p(\alpha+1/p-k)}E\left[\|Y\|^p_{p, L^a}\right], \]

for all \( i \in \{k, \ldots, n\} \). Since \( \alpha + 1/p - k < 0 \), equation (4.8) follows if \( E\left[\|Y\|^p_{p, L^a}\right] < \infty \).

Applying the estimate (3.7) shows that this is satisfied if \( E\left[\Phi_{p, L^a}(Y)\right] < \infty \), which is a consequence of (B1) and Lemma 4.6 where we used that \( p > \beta \). Now, the result follows from (4.7) and (4.8).

We can complete the proof of Theorem 1.2 (i) by combining Lemma 4.7 with the results of Section 4.1.1. To this end, let

\[ X_t^{\infty} := \int_{-\infty}^t (g(t-s) - g_0(-s))\sigma_s^- \, dL_s^{\infty}, \quad X_t^{\infty} := \int_{-\infty}^t (g(t-s) - g_0(-s))\sigma_s^- \, dL_s^{\infty}, \]

and introduce the stopping times

\[ T_m^{\infty} := \begin{cases} T_m & \text{if } |\Delta L_{T_m}| > a, \\ \infty & \text{else}. \end{cases} \]
The results of Subsection 4.1.1 show that
\[ n^{\alpha_p} V(X_{>a}, p; k)^n_t \xrightarrow{\mathcal{L} M_1-s} Z_{>a}^n_t := \sum_{m:T_m^n \in [0,t]} |\Delta L_{T_m^n} \sigma_{T_m^n} - \sigma V_m |^{p} \]
for all \( a > 0 \), where \( V(X_{>a}, p; k)^n_t \) denotes the power variation of the process \( X_{>a} \). Making the decomposition
\[ (n^{\alpha_p} V(p; k)^n_t)^{1/p} = (n^{\alpha_p} V(X_{>a}, p; k)^n_t)^{1/p} + \left( (n^{\alpha_p} V(p; k)^n_t)^{1/p} - (n^{\alpha_p} V(X_{>a}, p; k)^n_t)^{1/p} \right) \]
we have by Minkowski’s inequality
\[ \lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}\left( \sup_{t \in [0,\infty]} |U^n_{t,a}^n > \varepsilon | \right) \leq \lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}(n^{\alpha_p} V(X_{>a}, p; k)^n_t > \varepsilon^p) = 0, \]
for all \( \varepsilon > 0 \), where we applied Lemma 4.7. Since \( U^n_{t,a}^n \xrightarrow{\mathcal{L} M_1-s} Z_{>a}^n t \) as \( n \to \infty \), and \( \sup_{t \in [0,\infty]} |Z_{>a}^n - Z_t| \to 0 \) almost surely, as \( a \to 0 \), Theorem 1.2 (i) follows from [17 Thm 3.2].

**Remark 4.8.** A popular strategy for extending laws of large numbers from a class of processes with constant volatility to a more general model, which includes a stochastic volatility factor, is Bernstein’s blocking technique. This technique has for example been applied for Brownian semi-stationary processes and Itô semimartingales; see e.g. [5, 8]. Also, we will apply this technique in the next subsection to prove Theorem 1.2 (ii). However, in the framework of Theorem 1.2 (i) this approach is not applicable. A crucial step for the blocking technique is showing that the asymptotics of the power variation does not change if we replace the increments \( \Delta^n_{i,k} X \) by \( \sigma_{i-k} \Delta^n_{i,k} G \), where the process \( G \) is defined as
\[ G_t = \int_{-\infty}^t \{g(t - s) - g_0(-s)\} dL_s, \]
see e.g. Section 4.2 below. It can be shown that this is generally not true in the framework of Theorem 1.2 (i). For instance, when \( L \) is a compound Poisson process with Lévy measure \( \nu = \delta_{\{1\}} + \delta_{\{-1\}} \) and \( \sigma = L \), this approximation fails.

### 4.2 Proof of Theorem 1.2 (ii)

Since \( t \mapsto V(p; k)^n_t \) is increasing and the limiting function is continuous, uniform convergence on compact sets in probability follows if we show
\[ n^{-1+p(\alpha+1/\beta)} V(p; k)^n_t \xrightarrow{p} m_p \int_0^t |\sigma_s|^p ds \]
for a fixed \( t > 0 \), which we will do in the following.

As already mentioned, we will use Bernstein’s blocking technique to prove Theorem 1.2 (ii). A crucial step in the proof is to show that the asymptotic behavior of the
power variation does not change if we replace $\Delta_{i,k}^n X$ in (1.3) by $\sigma_{(i-k)/n} \Delta_{i,k}^n G$, where the process $(G_t)_{t \geq 0}$ is defined as in Remark 4.8. Note that assumption (A) ensures that $G$ is well-defined. Thereafter, we apply the following blocking technique. We divide the interval $[0, t]$ into subblocks of size $1/l$ and freeze $\sigma$ at the beginning of each block. The limiting power variation for the resulting process can then be derived by applying part (ii) of Theorem 1.1 on every block. The proof of Theorem 1.2 (ii) is then completed by letting $l \to \infty$.

The following lemma plays an important role when replacing $\Delta_{i,k}^n X$ in (1.3) by $\sigma_{(i-k)/n} \Delta_{i,k}^n G$. Here and in the following we denote by $v_{\sigma}$ the modulus of continuity of $\sigma$ defined as

$$v_{\sigma}(s, \eta) = \sup\{|\sigma_s - \sigma_r| : r \in [s - \eta, s + \eta]\}.$$

**Lemma 4.9.** Let $(\sigma_t)_{t \in \mathbb{R}}$ be a process with càdlàg or càglàd sample paths that is uniformly bounded on $[-\delta, \infty)$. For any $\alpha, q \in (0, \infty)$ and any (deterministic) sequence $(a_n)_{n \geq 1}$ with $a_n \to 0$ we have

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left( \frac{1}{n} \sum_{i=k}^{[nt]} \|v_{\sigma}(i/n, \varepsilon + a_n)\|_q^\alpha \right) = 0.$$

**Proof.** Since $v_{\sigma}$ is bounded and $x \mapsto x^\alpha$ is locally Lipschitz for $\alpha > 1$, we may assume w.l.o.g. that $\alpha \leq 1$ and $q \geq 1$. For $\kappa > 0$ we use the decomposition $\sigma = \sigma_{<\kappa} + \sigma_{\geq \kappa}$, where

$$\sigma_{<\kappa} = \sum_{-\delta < u \leq s} \Delta \sigma_u 1\{|\Delta \sigma_u| \geq \kappa\},$$

and $\sigma_{\geq \kappa} = \sigma - \sigma_{<\kappa}$. Even though $\sigma$ is uniformly bounded on $[-\delta, \infty)$, $\sigma_{<\kappa}$ and $\sigma_{\geq \kappa}$ might not be. For this reason we introduce the sets

$$\Omega_m := \{\omega : |\sigma_{<\kappa}(\omega)| + |\sigma_{\geq \kappa}(\omega)| \leq m \text{ for all } s \in [-\delta, t + \delta],$$

and $\sigma_{\geq \kappa}(\omega)$ has less than $m$ jumps in $[-\delta, t + \delta]\}$. Note that $\Omega_m \uparrow \Omega$, as $m \to \infty$. By triangular inequality we have $v_{\sigma}(s, \eta) \leq v_{\sigma_{<\kappa}}(s, \eta) 1_{\Omega_m} + v_{\sigma_{\geq \kappa}}(s, \eta) 1_{\Omega_m} + C1_{\Omega_m}$ for all $s \in [0, t], \eta < \delta$ and $m \geq 1$. Since $P(\Omega_m^c) \to 0$ as $m \to \infty$, we can choose $m$ sufficiently large such that

$$\frac{1}{n} \sum_{i=k}^{[nt]} \|v_{\sigma}(i/n, \varepsilon + a_n)\|_q^\alpha \leq \frac{1}{n} \sum_{i=k}^{[nt]} \|v_{\sigma_{<\kappa}}(i/n, \varepsilon + a_n) 1_{\Omega_m}\|_q^\alpha + \frac{1}{n} \sum_{i=k}^{[nt]} \|v_{\sigma_{\geq \kappa}}(i/n, \varepsilon + a_n) 1_{\Omega_m}\|_q^\alpha + \kappa, \quad (4.9)$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$. We show that

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \left( \frac{1}{n} \sum_{i=k}^{[nt]} \|v_{\sigma_{<\kappa}}(i/n, \varepsilon + a_n) 1_{\Omega_m}\|_q^\alpha \right) \leq 2\kappa^\alpha. \quad (4.10)$$
In order to do so, we assume the existence of sequences \((\varepsilon_l), (n_l), (i_l)\) with \(\varepsilon_l \to 0, n_l \to \infty\) and \(i_l \in \{1, \ldots, \lceil n_l \rceil\}\) such that
\[
\|v_{\sigma^{<\kappa}}(i_l/n_l, \varepsilon_l + a_{n_l})1_{\Omega_m}\|_q^\alpha > 2\kappa^\alpha
\]  
for all \(l\), and derive a contradiction. Since \((i_l/n_l)_{l \geq 1}\) is a bounded sequence we may assume that \(i_l/n_l\) converges to some \(s_0 \in [0, \ell]\) by considering a suitable subsequence \((l_k)_{k \geq 1}\). For all \(\omega \in \Omega_m\) it holds that \(\lim_{\gamma \to 0} v_{\sigma^{<\kappa}}(s_0, \gamma) = |\Delta \sigma_{s_0}^{<\kappa}| \leq \kappa\). Therefore, by the dominated convergence theorem, we can find a \(\gamma > 0\) such that \(\|v_{\sigma^{<\kappa}}(s_0, \gamma)1_{\Omega_m}\|_q^\alpha \leq 2\kappa^\alpha\). This is a contradiction to (4.11), since for sufficiently large \(l\), we have \([i_l/n_l - \varepsilon_l - |a_{n_l}|, i_l/n_l + \varepsilon_l + |a_{n_l}|] \subset [s_0 - \gamma, s_0 + \gamma]\). This completes the proof of (4.10).

Next, we show that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left(\frac{1}{n} \sum_{i=k}^{\lceil n \rceil} \|v_{\sigma^{\geq \kappa}}(i/n, \varepsilon + a_n)1_{\Omega_m}\|_q^\alpha \right) = 0.
\]  
Recalling that \(q/\alpha \geq 1\), an application of Jensen’s inequality yields
\[
\frac{1}{n} \sum_{i=k}^{\lceil n \rceil} \|v_{\sigma^{\geq \kappa}}(i/n, \varepsilon + a_n)1_{\Omega_m}\|_q^\alpha \leq \left(\frac{q/\alpha - 1}{n} \sum_{i=k}^{\lceil n \rceil} \|v_{\sigma^{\geq \kappa}}(i/n, \varepsilon + a_n)1_{\Omega_m}\|_q^q\right)^{\alpha/q} = \left(\frac{q/\alpha - 1}{n} \sum_{i=k}^{\lceil n \rceil} (v_{\sigma^{\geq \kappa}}(i/n, \varepsilon + a_n)1_{\Omega_m})^q\right)^{\alpha/q},
\]  
for all \(n \in \mathbb{N}, \varepsilon > 0\). Now, (4.12) follows from the estimate
\[
\frac{1}{n} \sum_{i=k}^{\lceil n \rceil} (v_{\sigma^{\geq \kappa}}(i/n, \varepsilon + a_n)1_{\Omega_m})^q \leq \sup_{s \in [-\delta, t + \delta]} |\Delta \sigma_s^{\geq \kappa}|^q N1_{\Omega_m}2(\varepsilon + a_n) \leq Cn^{q+1}(\varepsilon + a_n),
\]  
for all \(n \in \mathbb{N}\). Here \(N = N(\omega)\) denotes the number of jumps of \(\sigma^{\geq \kappa}\) in \([-\delta, t + \delta]\). Using (4.10) and (4.12), the lemma now follows from (4.9) by letting \(\kappa \to 0\).

The proof of (1.2) (ii) heavily relies on the estimate given in Lemma 3.2. This lemma assumes the role that Itô’s isometry typically plays for the blocking technique when the involved stochastic integral is driven by a Brownian motion. In order to apply Lemma 3.2 the following estimates will be crucial.

**Lemma 4.10.** Suppose that assumption (B2) holds, and assume that \(\alpha + 1/\beta < k\). For \(\varepsilon > 0\) with \(\varepsilon \leq \delta\) there is a constant \(C > 0\) such that
\[
\mathbb{E}\left[\int_{n^{1/\beta}}^n |g_{i,n}(s)\sigma_{s-}^{\beta}|^\beta ds\right] + \int_{n^{1/\beta}}^n |g_{i,n}(s)|^{\beta} ds \leq Cn^{-\alpha\beta - 1}, \quad \text{and}
\]
\[
\mathbb{E}\left[\int_{-\infty}^{n^{1/\beta}} |g_{i,n}(s)\sigma_{s-}^{\beta}|^\beta ds\right] + \int_{-\infty}^{n^{1/\beta}} |g_{i,n}(s)|^{\beta} ds \leq Cn^{-k\beta},
\]  
for all \(i \in \{k, \ldots, n\} \).
Proof. By Lemma 4.1 we have that
\[
|g_{i,n}(s)|^\beta \mathbb{I}_{[i/n-\varepsilon,i/n]}(s) \\
\leq C((i/n - s)^{\alpha\beta} \mathbb{I}_{(i-k)/n,i/n]}(s) + n^{-k\beta}((i-k)/n-s)^{(\alpha-k)\beta} \mathbb{I}_{[i/n-\varepsilon,(i-k)/n]}(s)).
\]

Recalling that \(\sigma\) is bounded on \([-\delta, \infty)\), the first inequality follows by calculating the integral of the right hand side. The second inequality is a direct consequence of Lemma 4.1 and assumptions (A) and (B2).

A crucial step in the proof of Theorem 1.2 (ii) is showing that
\[
n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} \|\Delta_{i,k}^n X - \sigma_{(i-k)/n} \Delta_{i,k}^n G\|_p^p \to 0, \tag{4.13}
\]
as \(n \to \infty\), where the process \(G\) is defined in Remark 4.8. We fix some \(\varepsilon > 0\) and make the decomposition
\[
\Delta_{i,k}^n X - \sigma_{(i-k)/n} \Delta_{i,k}^n G = A_i^{n,\varepsilon} + B_i^{n,\varepsilon} + C_i^{n,\varepsilon},
\]
where
\[
A_i^{n,\varepsilon} = \int_{i/n-\varepsilon}^{i/n} g_{i,n}(s)(\sigma_{s-} - \sigma_{i/n-\varepsilon}) \, dL_s,
\]
\[
B_i^{n,\varepsilon} = (\sigma_{i/n-\varepsilon} - \sigma_{(i-k)/n}) \int_{i/n-\varepsilon}^{i/n} g_{i,n}(s) \, dL_s,
\]
\[
C_i^{n,\varepsilon} = \int_{-\infty}^{i/n-\varepsilon} g_{i,n}(s)\sigma_{s-} dL_s - \sigma_{(i-k)/n} \int_{-\infty}^{i/n-\varepsilon} g_{i,n}(s) \, dL_s.
\]

We deduce (4.13) by showing that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left( n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} \|A_i^{n,\varepsilon}\|_p^p \right) = 0,
\]
and the same for \(B_i^{n,\varepsilon}\) and \(C_i^{n,\varepsilon}\), respectively. For \(A_i^{n,\varepsilon}\) we obtain by Lemma 3.2
\[
n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} \|A_i^{n,\varepsilon}\|_p^p \leq C n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} \|E \left[ \int_{i/n-\varepsilon}^{i/n} |g_{i,n}(s)(\sigma_{s-} - \sigma_{i/n-\varepsilon})|^\beta \, ds \right] \|^{p/\beta} \]
\[
\leq C n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} \|v_\sigma(i/n, \varepsilon + 1/n)\|_{p/\beta}^p \left( \int_{i/n-\varepsilon}^{i/n} |g_{i,n}(s)|^\beta \, ds \right)^{p/\beta}.
\]
By Lemma 4.9 and Lemma 4.10 we conclude that
\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \left( n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{nt} \| A_i^{n,\epsilon} \|^p \right) = 0. \tag{4.14}
\]

For \( B_i^{n,\epsilon} \) we apply Hölder’s inequality with \( p' \) and \( q' \) satisfying \( 1/p' + 1/q' = 1 \) and \( pq' < \beta \), which is possible due to our assumption \( p < \beta \). This yields
\[
n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{nt} \left\| B_i^{n,\epsilon} \right\|^p \leq C n^{-1} \sum_{i=k}^{nt} \left\| v_\sigma \left( i/n, \epsilon + k/n \right) \right\|^p.
\]

Here we have used that, as a consequence of Lemma 3.2 and Lemma 4.10 whenever \( pq' < \beta \) there exists a \( C > 0 \) such that \( \| n^{\alpha+1/\beta} \int_{i/n-\epsilon}^{i/n} g_{i,n}(s) \, dL_s \|_{pq'} < C \) for all \( n \in \mathbb{N}, i \in \{k, ..., nt\} \). Thus, by Lemma 4.9
\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \left( n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{nt} \| B_i^{n,\epsilon} \|^p \right) = 0. \tag{4.15}
\]

Moreover, by Lemma 3.2 and Lemma 4.10 it follows that for all \( \epsilon > 0 \)
\[
\lim_{n \to \infty} \left( n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{nt} \| C_i^{n,\epsilon} \|^p \right) \leq C \limsup_{n \to \infty} (n^{p(\alpha+1/\beta-k)}) = 0,
\]

which together with (4.14) and (4.15) completes the proof of (4.13).

By Minkowski’s inequality for \( p \geq 1 \) and subadditivity for \( p < 1 \), it is now sufficient to show that
\[
n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{nt} \left| \sigma_{i-1/n} \Delta_{i,k}^n G \right|^p \stackrel{p}{\to} m_p \int_0^t |\sigma_s|^p \, ds, \tag{4.16}
\]
in order to prove Theorem 1.2 (ii).

Intuitively, replacing \( |\Delta_{i,k}^n X| \) by \( |\sigma_{i-1/n} \Delta_{i,k}^n G| \) corresponds to freezing the process \((\sigma_t)_{t \in \mathbb{R}}\) over blocks of length \( 1/n \). For the prove of (4.16) we freeze \( \sigma \) now over small blocks with block size \( 1/l \) that does not depend on \( n \). This will allow us to apply Theorem
\[ I(i) \] (ii) on every block. Thereafter, \ref{4.16} follows by letting \( l \to \infty \). For \( l > 0 \) we decompose
\[
n^{-1+p(\alpha+1/\beta)} \sum_{i=k}^{[nt]} |\sigma(i-k)/n \Delta_{i,k}^n G|^p - m_p \int_0^t |\sigma_s|^p ds
\]
\[
= n^{-1+p(\alpha+1/\beta)} \left( \sum_{i=k}^{[nt]} |\Delta_{i,k}^n G|^p \left( |\sigma(i-k)/n|^p - |\sigma(j_{i,n,i-1})/l|^p \right) \right) + \left( n^{-1+p(\alpha+1/\beta)} \sum_{j=1}^{[tl]+1} |\sigma(j-1)/l|^p \left( \sum_{i \in I(j)} |\Delta_{i,k}^n G|^p - m_p l^{-1} \right) \right) + \left( m_p l^{-1} \sum_{j=1}^{[tl]} |\sigma(j-1)/l|^p - m_p \int_0^t |\sigma_s|^p ds \right)
\]
\[
= D_{n,l} + E_{n,l} + F_l.
\]
Here, \( j_{l,i} \) denotes the index \( j \in \{1, \ldots, [tl] + 1 \} \) such that \((i-k)/n \in ((j-1)/l, j/l]\) and \( I_l(j) \) is the set of indices \( i \) such that \((i-k)/n \in ((j-1)/l, j/l]\). We show that
\[
\lim_{l \to \infty} \limsup_{n \to \infty} \mathbb{P}(|D_{n,l} + E_{n,l} + F_l| > \varepsilon) = 0
\]
for any \( \varepsilon > 0 \). Note that \( F_l \overset{a.s.}{\to} 0 \) as \( l \to \infty \), since the Lebesgue integral of any càdlàg function exists. For every \( l \in \mathbb{N} \) we have \( \limsup_{n \to \infty} \mathbb{P}(|E_{n,l}| > \varepsilon) = 0 \) by Theorem \ref{1.1} (ii).

For \( \lim_{l \to \infty} \limsup_{n \to \infty} \mathbb{P}(|D_{n,l}| > \varepsilon) = 0 \) we argue as follows. Choose some \( p' > 1 \) such that \( pp' < \beta \) and let \( q' \) be such that \( 1/p' + 1/q' = 1 \). We find
\[
\|D_{n,l}\|_1 = \left\| n^{-1+p(\alpha+1/\beta)} \left( \sum_{i=k}^{[nt]} |\Delta_{i,k}^n G|^p \left( |\sigma(i-k)/n|^p - |\sigma(j_{i,n,i-1})/l|^p \right) \right) \right\|_1
\]
\[
\leq n^{-1} \sum_{i=k}^{[nt]} \left\| n^{\alpha+1/\beta} \Delta_{i,k}^n G \right\|_{p''} \left\| |\sigma(i-k)/n|^p - |\sigma(j_{i,n,i-1})/l|^p \right\|_{q'}
\]
\[
\leq \left( n^{-1} \sum_{i=k}^{[nt]} \left\| n^{\alpha+1/\beta} \Delta_{i,k}^n G \right\|_{pp''}^{p/2} \right)^{1/2} \left( n^{-1} \sum_{i=k}^{[nt]} \left\| |\sigma(i-k)/n|^p - |\sigma(j_{i,n,i-1})/l|^p \right\|_{q''}^{2} \right)^{1/2}
\]

The first factor is bounded by Lemmas \ref{3.2} and \ref{4.10}. For the second factor we can apply Lemma \ref{4.9} since the process \((|\sigma_t|^p)_{t \in \mathbb{R}}\) is càdlàg and bounded on \([−\delta, \infty)\), and conclude that \( \lim_{l \to \infty} \limsup_{n \to \infty} \|D_{n,l}\|_1 = 0 \). This completes the proof of \ref{4.16}, and hence of Theorem \ref{1.2} (ii).

\[ \square \]

4.3 Proof of Theorem \ref{1.2} (iii)

For the proof of Theorem \ref{1.2} (iii) we show that under the conditions of the theorem the process \( X \) admits a modification with \( k \)-times differentiable sample paths with \( k \)-th derivative \( F \) as defined in the theorem. Then the result follows by an application of the following stochastic Fubini theorem. We remark that a Fubini theorem for Lévy integrals of deterministic integrands was derived by similar means in [2, Theorem 3.1].
Lemma 4.11. Let \( f : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R} \) be a random field that is measurable with respect to the product \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \otimes \Pi \), where \( \Pi \) denotes the \((\mathcal{F}_t)_{t \in \mathbb{R}}\)-predictable \( \sigma \)-algebra on \( \mathbb{R} \times \Omega \). That is, \( \Pi \) is the \( \sigma \)-algebra generated by all sets \( A \times (s, t] \), where \( s < t \) and \( A \in \mathcal{F}_s \). Let \((L_t)_{t \in \mathbb{R}}\) be a symmetric Lévy process that has finite first moment. Assume that we have

\[
\mathbb{E} \left[ \int_{\mathbb{R}} \| f(u, \cdot) \|_{1,L} \, du \right] < \infty. \tag{4.17}
\]

Then, we obtain

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(u, s) \, du \right) \, dL_s = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(u, s) \, dL_s \right) \, du \quad \text{almost surely},
\]

and all the integrals are well-defined.

The proof of this result relies on the following lemma.

Lemma 4.12. Let \( L \) be a Lévy process that has \( q \)-th moment for some \( q \geq 1 \). There exists a \( C > 0 \) such that the following holds. For all random fields \( f : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R} \) that are measurable with respect to the product \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \otimes \Pi \), where \( \Pi \) denotes the \((\mathcal{F}_t)_{t \in \mathbb{R}}\)-predictable \( \sigma \)-algebra on \( \mathbb{R} \times \Omega \), we have that

\[
\left\| \int_{\mathbb{R}} f(u, \cdot) \, du \right\|_{q,L} \leq C \int_{\mathbb{R}} \| f(u, \cdot) \|_{q,L} \, du \quad \text{almost surely}. \tag{4.18}
\]

Proof. Let us first remark that the stochastic process \((\| f(u, \cdot) \|_{q,L})_{u \in \mathbb{R}}\) indeed admits a measurable modification by the following argument. For any \( g \in \mathbf{L}^q(dL) \), the mapping \( \Gamma_g : \mathbb{R} \to \mathbb{R} \), \( u \mapsto \mathbb{E}[\| f(u, \cdot) - g \|_{q,L} \wedge 1] \) is measurable by product measurability of \( f \). Therefore, the mapping \( \mathbb{R} \to \mathbf{L}^q(dL) \), \( u \mapsto f(u, \cdot) \) is measurable, where we equip \( \mathbf{L}^q(dL) \) with the (semi)metric induced by \( \mathbb{E}[(\| \cdot \|_{q,L} \wedge 1)] \). Since the mapping \( \mathbf{L}^q(dL) \to \mathbf{L}^0(\Omega) \), \( g \mapsto \| g \|_{q,L} \) is continuous, the existence of a measurable modification of \((\| f(u, \cdot) \|_{q,L})_{u \in \mathbb{R}}\) follows from [21, Theorem 3].

The proof relies on the monotone class theorem, cf. [24, Theorem 6.1.3]. We fix a compact set \( K \subset \mathbb{R} \) and denote by \( \mathbf{L}^0_K \) the linear space of all random fields \( f : K \times K \times \Omega \to \mathbb{R} \) that are measurable with respect to \( \mathcal{B}(K) \otimes \Pi \). Denote by \( \mathcal{R}_K \) the set of all random fields in \( \mathbf{L}^0_K \) of the form

\[
f(u, s) = \sum_{j=1}^{l} F^j \mathbb{1}_{A_j}(u),
\]

where \( l \in \mathbb{N} \), and \( F^1, \ldots, F^l \in \mathbf{L}^q(dL) \) are bounded, and \( A_1, \ldots, A_l \) are disjoint sets in \( \mathcal{B}(K) \). Then, (4.18) holds for all \( f \in \mathcal{R}_K \) with \( C \) as in (3.6). Let

\[
\mathcal{H}_K := \left\{ f \in \mathbf{L}^0_K : \text{There is a sequence } (f_n)_{n \in \mathbb{N}} \subset \mathcal{R}_K \text{ such that} \right\}
\]

\[
\int_{K} \| f(u, \cdot) - f_n(u, \cdot) \|_{q,L} \, du \to 0, \text{ a.s., and}
\]

\[
\left\| \int_{K} | f(u, \cdot) - f_n(u, \cdot) | \, du \right\|_{q,L} \to 0, \text{ a.s.}
\].
Note that $\mathcal{H}_K$ is a linear space containing $\mathcal{R}_K$ that is closed under bounded monotone convergence. Therefore, the monotone class theorem yields that $\mathcal{H}_K$ contains all bounded functions in $L^0_K$. Recalling the equivalence of $\| \cdot \|_{q,L}$ and the (random) norm $\| \cdot \|_{q,L}'$, which has been introduced in Section 3, it is easy to see that all $f \in \mathcal{H}_K$ satisfy (4.18), with the constant $C$ not depending on $K$. Now, the lemma follows for general $f$ by an approximation argument.

With this result at hand we can now prove Lemma 4.11.

**Proof of Lemma 4.11.** Let us first remark that the stochastic process $u \mapsto \int_{\mathbb{R}} f(u,s) \, dL_s$ admits a measurable modification by measurability of the map $\mathbb{R} \to L^1(dL)$, $u \mapsto f(u,\cdot)$ and continuity of the integral mapping $L^1(dL) \to L^1(\Omega)$, $f(u,\cdot) \mapsto \int_{\mathbb{R}} f(u,s) \, dL_s$, which follows from Lemma 3.1. Then, the existence of a measurable modification follows from [21, Theorem 3].

The result obviously holds for $f$ of the form

$$f(u,s,\omega) = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}(u) \mathbb{1}_{B_i}(s,\omega), \quad (4.19)$$

where $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{B}(\mathbb{R})$ and $B_i \in \Pi$ for $i = 1, \ldots, k$. By an application of the monotone class theorem we see that for every $\mathcal{B}(\mathbb{R}) \otimes \Pi$ measurable $f$ satisfying (4.17) we can find a sequence $f_n$ of the form (4.19) such that $\mathbb{E}[\int_{\mathbb{R}} \| f_n(u,s) - f(u,s) \|_{L,1} \, du] \to 0$. Letting

$$Y := \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(u,s) \, dL_s \right) \, du, \quad \text{and} \quad Y_n := \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f_n(u,s) \, dL_s \right) \, du \right],$$

we have by (4.17) and Lemma 3.1

$$\mathbb{E}[|Y|] \leq C \int_{\mathbb{R}} \mathbb{E}[[f(u,\cdot)]_{1,L}] \, du < \infty.$$ 

Thus, we deduce that

$$\mathbb{E} [|Y - Y_n|] \leq C \mathbb{E} \left[ \int_{\mathbb{R}} \| f(u,\cdot) - f_n(u,\cdot) \|_{1,L} \, du \right] \to 0, \quad \text{as } n \to \infty.$$ 

By applying Lemma 4.12 we obtain that

$$Z := \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(u,s) \, du \right) \, dL_s \quad \text{and} \quad Z_n = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f_n(u,s) \, du \right) \, dL_s$$

are in $L^1(\Omega)$, and that $\mathbb{E}[|Z_n - Z|] \to 0$, as $n \to \infty$. Now the result follows since $Y_n = Z_n$ almost surely.

In order to apply Fubini’s theorem in the proof of Theorem 1.2 (iii) we need the following lemma.
Lemma 4.13. Suppose that assumption (B1) holds and let \( \alpha > k - 1/(\beta \vee 1) \). For any \( t > 0 \) the random field \( f_t(u,s) := g^k(u-s)\sigma_s \mathbb{1}_{[0,t]}(u) \mathbb{1}_{(-\infty,u]}(s) \) satisfies the conditions of Lemma [4.11].

Proof. The measurability with respect to \( \mathcal{B}(\mathbb{R}) \otimes \Pi \) is obvious. By (3.7) for the function \( f_t \) to satisfy (4.17) it is sufficient to show that

\[
\int_0^t \mathbb{E} [\Phi_{1,L}(f_t(u,\cdot))] \, du = \int_0^t \mathbb{E} [\Phi_{1,L}(f_t(u,\cdot) \mathbb{1}_{(-\infty,-\delta])}] \, du + \int_0^t \mathbb{E} [\Phi_{1,L}(f_t(u,\cdot) \mathbb{1}_{(-\delta,t]})] \, du
\]

\[
:= I_1 + I_2 < \infty.
\]

We first show that \( I_1 < \infty \). Since \( |g^k| \) is decreasing on \( (\delta, \infty) \), we have for all \( u \in [0,t] \)

\[
\Phi_{1,L}(f_t(u,\cdot) \mathbb{1}_{(-\infty,-\delta])}) \leq \Phi_{1,L}(f_t(0,\cdot) \mathbb{1}_{(-\infty,-\delta])})
\]

\[
= \int_{-\infty}^{-\delta} \int_{\mathbb{R}} |f_t(0,s)x| \mathbb{1}_{\{|f_t(0,s)x| \leq 1\}} \nu(dx) \, ds
\]

\[
+ \int_{-\infty}^{-\delta} \int_{\mathbb{R}} |f_t(0,s)x| \mathbb{1}_{\{|f_t(0,s)x| > 1\}} \nu(dx) \, ds
\]

\[
= J_1 + J_2.
\]

By (3.5) we obtain for any \( \beta' > \beta \)

\[
J_1 \leq C \int_{-\infty}^{-\delta} |f_t(0,s)|^\theta \mathbb{1}_{\{|f_t(0,s)| \leq 1\}} + |f_t(0,s)|^{\beta'} \mathbb{1}_{\{|f_t(0,s)| > 1\}} \, ds. \tag{4.20}
\]

For \( \beta' \) close enough to \( \beta \), the right hand side of (4.20) is in \( L^1(\Omega) \) by assumption (B1). For \( J_2 \) we observe for all \( \beta' > \beta \) with \( \beta' \geq 1 \) that

\[
J_2 \leq \int_{-\infty}^{-\delta} \int_{-1}^1 |f_t(0,s)x|^{\beta'} \mathbb{1}_{\{|f_t(0,s)x| > 1\}} \nu(dx) \, ds + 2 \int_{-\infty}^{-\delta} |f_t(0,s)| \, ds \int_1^\infty |x| \, \nu(dx)
\]

\[
\leq C \int_{-\infty}^{-\delta} |f_t(0,s)|^{\beta'} \mathbb{1}_{\{|f_t(0,s)| > 1\}} + |f_t(0,s)| \, ds, \tag{4.21}
\]

which is in \( L^1(\Omega) \) by assumption (B1) for sufficiently small \( \beta' > \beta \). Thus, we conclude that \( I_1 < \infty \).

Now we show that \( I_2 < \infty \). By boundedness of \( \sigma \) on the interval \([-\delta, t] \), it holds for all \( u \in [0,t] \) that

\[
\Phi_{1,L}(f_t(u,\cdot) \mathbb{1}_{(-\delta,t]})
\]

\[
\leq C \int_{-\delta}^t \int_{\mathbb{R}} |g^k(t-s)x| \mathbb{1}_{\{|g^k(t-s)x| \leq 1\}} + |g^k(t-s)x|^2 \mathbb{1}_{\{|g^k(t-s)x| \leq 1\}} \nu(dx) \, ds.
\]

We recall that \( |g^k(s)| \leq Cs^{\alpha-k} \) for \( s \in (0,\delta] \). Thus, choosing \( \beta' > \beta \) such that \( (\alpha-k)\beta' > -1 \), we have by (3.5)

\[
\int_{-\delta}^t \int_{\mathbb{R}} |g^k(t-s)x|^2 \mathbb{1}_{\{|g^k(t-s)x| \leq 1\}} \nu(dx) \, ds \leq C + \int_{-\delta}^t |g^k(t-s)|^{\beta'} \, ds < \infty.
\]
Moreover, by arguing as in [4.21]
\[
\int_{-\delta}^{t} \int_{\mathbb{R}} |g^{(k)}(t-s)x| \mathbb{I}_{\{ |g^{(k)}(t-s)x| > 1 \}} \nu(dx) \, ds
\]
\[
\leq C \int_{-\delta}^{t} |g^{(k)}(t-s)|^{\beta} \mathbb{I}_{\{ |g^{(k)}(t-s)| > 1 \}} + |g^{(k)}(t-s)| \, ds < \infty,
\]
where we used that \( \alpha - k > -1 \). Therefore, we conclude that \( I_2 < \infty \), which completes the proof. \( \square \)

In the proof of Theorem 1.2 (iii) we will again separate the large jumps and small jumps of the Lévy process \( L \). For the small jump part the result given in Lemma 4.13 can be strengthened as follows. Note that for this result it is in fact necessary to cut off the large jumps of \( L \), since \( L \) might not have \( p \)-th moment.

**Lemma 4.14.** Let \( t > 0 \) be fixed, and let \( p \geq 1 \). Suppose, (B1) is satisfied and suppose that \( \alpha > k - 1/(\beta \lor p) \). For any \( a \in (0,1) \), the random field \( f_t(u,s) = g^{(k)}(u-s)\sigma_{s-} \mathbb{I}_{[0,t]}(u) \mathbb{I}_{(-\infty,u)}(s) \) satisfies
\[
\int_{0}^{t} \mathbb{E}[||f_t(u,\cdot)||^p_{p,L \leq a}] \, du < \infty,
\]
where process \( (L^a_s)_{s \in \mathbb{R}} \) is defined in Lemma 4.7.

**Proof.** We decompose
\[
\int_{0}^{t} \mathbb{E}[||f_t(u,\cdot)||^p_{p,L \leq a}] \, du \leq C \int_{0}^{t} \mathbb{E}[||f_t^{(1)}(u,\cdot)||^p_{p,L \leq a}] \, du + C \int_{0}^{t} \mathbb{E}[||f_t^{(2)}(u,\cdot)||^p_{p,L \leq a}] \, du
\]
\[
= I_1 + I_2,
\]
where
\[
f_t^{(1)}(u,s) = g^{(k)}(u-s)\sigma_{s-} \mathbb{I}_{[0,t]}(u) \mathbb{I}_{(-\delta,u)}(s), \quad f_t^{(2)}(u,s) = g^{(k)}(u-s)\sigma_{s-} \mathbb{I}_{[0,t]}(u) \mathbb{I}_{(-\infty,-\delta)}(s).
\]
For \( I_1 \) we use that \( \sigma \) is bounded on \([-\delta, \infty)\). Thus, denoting \( e_t(u,s) = g^{(k)}(u-s)\mathbb{I}_{[0,t]}(u) \mathbb{I}_{(-\delta,u)}(s) \), we have that
\[
I_1 \leq C \int_{0}^{t} ||e_t(u,\cdot)||^p_{p,L \leq a} \, du
\]
\[
\leq C \int_{0}^{t} \Phi_{p,L \leq a}(e_t(u,\cdot)) + \Phi_{p,L \leq a}^p(e_t(u,\cdot)) \, du
\]
\[
\leq C t \left( \Phi_{p,L \leq a}(e_t(t,\cdot)) + \Phi_{p,L \leq a}^p(e_t(t,\cdot)) \right),
\]
where the second inequality follows from [3.7], the third inequality holds since \( |e_t(u,s)| \leq |e_t(t,s+t-u)| \), and since \( \Phi_{p,L \leq a}(f) \) is invariant under shifting the argument of the function \( f \). We obtain by virtue of Lemma 4.6
\[
\Phi_{p,L \leq a}(e_t(t,\cdot)) \leq C \int_{-\delta}^{t} |g^{(k)}(t-s)|^2 \mathbb{I}_{\{ |g^{(k)}(t-s)| \leq 1 \}} + |g^{(k)}(t-s)|^{2p} \mathbb{I}_{\{ |g^{(k)}(t-s)| > 1 \}} \, ds,
\]
where we choose $\beta' > \beta$ such that $(\alpha - k)(\beta' \lor p) > -1$. Now, recalling that $|g^{(k)}(t)| \leq Ct^{\alpha - k}$ for all $t \in (0, \delta)$, we see that

$$
\int_{-\delta}^{\delta} \Phi^{p,L}_{\epsilon}(t, s) \, ds < \infty.
$$

Consequently, $\Phi^{p,L}_{\epsilon}(t, s, s)$ is finite which implies $I_1 < \infty$. 

For $I_2$ we obtain that $\mathbb{E}[[f_t(\sigma, \cdot, \cdot)^p]_{p,L \leq a}] \leq \mathbb{E}[[f_t(\sigma, \cdot, \cdot)^p]_{p,L \leq a}]$ for all $u \in [0, \delta]$, because $|g^{(k)}| \leq \frac{1}{b}$ is decreasing on $(\delta, \infty)$. By (3.7) it holds that

$$
\mathbb{E}[[f_t(\sigma, \cdot, \cdot)^p]_{p,L \leq a}] \leq \mathbb{E} \left[ \Phi^{p,L}_{\epsilon}(f_t(\sigma, \cdot, \cdot)) \right] + \mathbb{E} \left[ \Phi^{p,L}_{\epsilon}(\frac{1}{2}f_t(\sigma, \cdot, \cdot)) \right],
$$

which is finite by Lemma 4.1 and assumption (B1).

With these preliminaries at hand, we can finally prove Theorem 1.2 (iii). As we remarked at the beginning of Subsection 4.2, it is sufficient to show convergence in probability for a fixed $t > 0$ in order to obtain uniform convergence on compacts in probability. Therefore, the theorem is an immediate consequence of the following lemma and Lemma 4.3 in [14].

**Lemma 4.15.** Under the conditions of Theorem 1.2 (iii), there is a process $(Z_t)_{t \geq 0}$ that satisfies almost surely $V(Z, p; k)^a_t = V(X, p; k)^a_t$ for all $n \in \mathbb{N}$ and $t \geq 0$, and has the following properties. The sample paths of $Z$ are with probability 1 $k$-times differentiable and it holds for Lebesgue almost all $t \geq 0$ that

$$
\frac{\partial^k Z_t}{(\partial t)^k} = \int_{-\delta}^{\delta} g^{(k)}(t - s) \sigma_s \, dL_s.
$$

Moreover, for any $t_0 > 0$ it holds with probability 1 that

$$
\frac{\partial^k Z_t}{(\partial t)^k} \in L^p([0, t_0]).
$$

**Proof.** For ease of notation we consider the case $k = 1$. The general case follows by similar arguments. The strategy of the proof is the following. We let $a \in (0, 1]$ and define the processes $(F_u^a)_{u \in \mathbb{R}}$ and $(F_u^{a^+})_{u \in \mathbb{R}}$ by

$$
F_u^a = \int_{-\delta}^{u} g'(u - s) \sigma_s \, dL_s^a, \quad F_u^{a^+} = \sum_{s \in (-\delta, u)} g'(u - s) \sigma_s \Delta L_s^a \mathbb{1}_{\{|\Delta L_s^a| > a\}},
$$

where the process $(L^a_t)_{t \in \mathbb{R}}$ has been introduced in Section 4.1.2. We show that both processes $F_u^a$ and $F_u^{a^+}$ are well-defined and that they both admit a modification with sample paths in $L^p([0, t])$. Then, we define the process

$$
Z_t := \int_{0}^{t} (F_u^a + F_u^{a^+}) \, du,
$$

...
and show that it satisfies the properties given in the lemma.

We begin by analysing $F_u^{\leq a}$. The process $f_t(u, s) = g'(u - s)\sigma_s - \mathbb{1}_{[0, t]}(u)\mathbb{1}_{(-\infty, u)}(s)$ is integrable in $s$ with respect to $L^{\leq a}$ for Lebesgue almost all $u$, according to Lemma 4.14. As we argued in Lemma 4.11, $(F_u^{\leq a})_{u \geq 0}$ admits a modification with measurable sample paths. Moreover, applying Lemmas 3.1 and 4.14 we obtain $F^{\leq a} \in L^p([0, t])$, almost surely, since

$$\mathbb{E}\left[ \int_0^t |F_u^{\leq a}|^p \, du \right] \leq C \int_0^t \mathbb{E}\left[ |f_t(u, -)|^{p, L^{\leq a}} \right] \, du < \infty.$$  

For the process $F_u^{>a}$ we make the decomposition

$$F_u^{>a} = F_u^{>a, \leq -\delta} + F_u^{>a, >-\delta},$$

where

$$F_u^{>a, \leq -\delta} = \sum_{s \in (-\infty, -\delta)} g'(u - s)\sigma_s - \Delta L_s \mathbb{1}_{\{|\Delta L_s| > a\}} + \sum_{s \in (-\delta, u)} g'(u - s)\sigma_s - \Delta L_s \mathbb{1}_{\{|\Delta L_s| > a\}}.$$

We argue first that $F^{>a, \leq -\delta}$ is well-defined and in $L^p([0, t])$ almost surely. Applying Lemma 4.15 we obtain that

$$\sum_{s \in (-\infty, -\delta]} |g'(u - s)\sigma_s - \Delta L_s| \mathbb{1}_{\{|\Delta L_s| > a\}} < \infty$$

almost surely. Since $g'$ is decreasing on $[\delta, \infty)$, we conclude that the sum in the definition of $F^{>a, \leq -\delta}$ is absolutely convergent and bounded by the left hand side of (4.22), which does not depend on $u$. It follows that $F^{>a, \leq -\delta}$ is well-defined and in $L^p([0, t])$ almost surely.

For $F^{>a, >-\delta}$ we use that $L$ has only finitely many jumps of size $> a$ on $[-\delta, t]$. Therefore, $F^{>a, >-\delta}$ is well-defined and we find a positive random variable $K < \infty$ such that

$$\int_0^t |F^{>a, >-\delta}|^p \, du = \int_0^t \left| \sum_{s \in (-\delta, u)} g'(u - s)\sigma_s - \Delta L_s \mathbb{1}_{\{|\Delta L_s| > a\}} \right|^p \, du$$

$$\leq K \int_0^t \sum_{s \in (-\delta, u)} \left| g'(u - s)\sigma_s - \Delta L_s \mathbb{1}_{\{|\Delta L_s| > a\}} \right|^p \, du$$

$$\leq K \sum_{s \in (-\delta, t)} \left| \sigma_s - \Delta L_s \mathbb{1}_{\{|\Delta L_s| > a\}} \right|^p \int_0^t |g'(u - s)|^p \, du$$

$$< \infty.$$

Here, the last inequality follows by $|g'(s)| \leq C s^{a-1}$ for $s \in (0, \delta)$ and $(\alpha - 1)p > -1$.

Define the process $(Z_t)_{t \geq 0}$ by

$$Z_t := \int_0^t (F_u^{\leq a} + F_u^{>a}) \, du.$$

All that remains to show is that $V(X, p; 1)^n = V(Z, p; 1)^n$ for all $n \in \mathbb{N}$ and all $t > 0$ with probability 1. For any $t > 0$ it holds with probability 1 that

$$X_t - X_0 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f_t(u, s) \, du \right) \, dL_s = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f_t(u, s) \, dL_s \right) \, du = Z_t,$$
where we have applied Lemmas 4.11 and 4.13. Consequently, it holds that \( \mathbb{P}[X_t = Z_t + X_0 \text{ for all } t \in \mathbb{Q}_+] = 1 \) which implies \( V(X, p; 1)^n_t = V(Z, p; 1)^n_t \) for all \( n \in \mathbb{N} \) and all \( t > 0 \) almost surely.

References

[1] D.J. Aldous and G.K. Eagleson (1978). On mixing and stability of limit theorems. *Ann. Probab.* 6(2), 325–331.

[2] O.E. Barndorff-Nielsen and A. Basse-O’Connor (2011). Quasi Ornstein–Uhlenbeck processes. *Bernoulli* 17(3), 916–941.

[3] O.E. Barndorff-Nielsen, F.E. Benth and A. Veraart (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli* 19(3), 803–845.

[4] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2009). Power variation for Gaussian processes with stationary increments. *Stochastic Process. Appl.* 119(6), 1845–1865.

[5] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2011): Multipower variation for Brownian semistationary processes. *Bernoulli* 17(4), 1159–1194.

[6] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2013): Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In *Prokhorov and Contemporary Probability Theory: In Honor of Yuri V. Prokhorov*, eds. A.N. Shiryaev, S.R.S. Varadhan and E.L. Presman. Springer.

[7] O.E. Barndorff-Nielsen, J.M. Corcuera, M. Podolskij and J.H.C. Woerner (2009). Bipower variation for Gaussian processes with stationary increments. *J. Appl. Probab.* 46(1), 132–150.

[8] O.E. Barndorff-Nielsen, S.E. Graversen, J. Jacod, M. Podolskij and N. Shephard (2005). A central limit theorem for realised power and bipower variations of continuous semimartingales. In: Kabanov, Yu., Liptser, R., Stoyanov, J. (eds.), *From Stochastic Calculus to Mathematical Finance. Festschrift in Honour of A.N. Shiryaev*, 33–68, Springer, Heidelberg.

[9] O.E. Barndorff-Nielsen, E.B.V. Jensen, K.Y. Jónsdóttir and J. Schmiegel (2007): Spatio-temporal modelling - with a view to biological growth. In B. Finkenstdt, L. Held and V. Isham: *Statistical Methods for Spatio-Temporal Systems*. London: Chapman and Hall/CRC, 47–75.

[10] O.E. Barndorff-Nielsen, M. Pakkanen and J. Schmiegel (2014): Assessing relative volatility/intermittency/energy dissipation. *Electronic Journal of Statistics* 8, 1996–2021.
[11] O.E. Barndorff-Nielsen and J. Schmiegel (2007): Ambit processes; with applications to turbulence and cancer growth. In F.E. Benth, Nunno, G.D., Linstrøm, T., Øksendal, B. and Zhang, T. (Eds.): Stochastic Analysis and Applications: The Abel Symposium 2005. Heidelberg: Springer. Pp. 93–124.

[12] O.E. Barndorff-Nielsen and J. Schmiegel (2008): Time change, volatility and turbulence. In A. Sarychev, A. Shiryaev, M. Guerra and M.d.R. Grossinho (Eds.): Proceedings of the Workshop on Mathematical Control Theory and Finance. Lisbon 2007. Berlin: Springer. Pp. 29–53.

[13] O.E. Barndorff-Nielsen and J. Schmiegel (2009): Brownian semistationary processes and volatility/intermittency. In: Albrecher, H., Runggaldier, W., Schachermayer, W. (Eds.): Advanced Financial Modelling, 1–26, Germany: Walter de Gruyter.

[14] A. Basse-O’Connor, R. Lachièze-Rey and M. Podolskij (2015). Limit theorems for stationary increments Lévy driven moving averages. [arXiv:1506.06679 [math.PR]].

[15] A. Basse-O’Connor and M. Podolskij (2015). On critical cases in limit theory for stationary increments Lévy driven moving averages. Preprint. CREATEs Research Paper 2015–57. Available at http://econ.au.dk/fileadmin/site_files/filer_oekonomi/Working_Papers/CREATEs/2015/rp15_57.

[16] A. Benassi, S. Cohen and J. Istas (2004). On roughness indices for fractional fields. Bernoulli 10(2), 357–373.

[17] P. Billingsley (1999). Convergence of Probability Measures. Second Ed. Wiley Series in Probability and Statistics: Probability and Statistics.

[18] S. Cambanis, C.D. Hardin, Jr., and A. Weron (1987). Ergodic properties of stationary stable processes. Stochastic Process. Appl. 24(1), 1–18.

[19] A. Chronopoulou, C.A. Tudor and F.G. Viens (2009). Variations and Hurst index estimation for a Rosenblatt process using longer filters. Electron. J. Stat. 3, 1393–1435.

[20] J.-F. Coeurjolly (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. Stat. Inference Stoch. Process. 4(2), 199–227.

[21] D.L. Cohn (1972). Measurable choice of limit points and the existence of separable and measurable processes. Z. Wahrsch. Verw. Gebiete 22(2), 161–165.

[22] T.T.N. Dang and J. Istas (2015): Estimation of the Hurst and the stability indices of a H-self-similar stable process. Working paper. Available at http://arxiv.org/abs/1506.05593.

[23] S. Delattre and J. Jacod (1997). A central limit theorem for normalized functions of the increments of a diffusion process, in the presence of round-off errors. Bernoulli 3(1), 1–28.

[24] R. Durrett (2010). Probability: Theory and Examples. Cambridge university press.
[25] K. Gärtner and M. Podolskij (2014): On non-standard limits of Brownian semi-stationary processes. *Stochastic Processes and Their Applications* 125(2), 653–677.

[26] D. Grahovac, N.N. Leonenko and M.S. Taqqu (2015): Scaling properties of the empirical structure function of linear fractional stable motion and estimation of its parameters. *J. Stat. Phys.* 158(1), 105–119.

[27] L. Guyon and J. Leon (1989): Convergence en loi des $H$-variations d’un processus gaussien stationnaire sur $\mathbb{R}$. *Ann. Inst. H. Poincaré Probab. Statist.* 25(3), 265–282.

[28] J. Jacod (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Process. Appl.* 118(4), 517–559.

[29] J. Jacod and P. Protter (2012). *Discretization of Processes*. Springer, Berlin.

[30] J. Jacod and A. Shiryaev (2013). *Limit Theorems for Stochastic Processes*. Springer Science & Business Media.

[31] I. Karatzas and S.E. Shreve (1992). *Brownian Motion and Stochastic Calculus* (second edition). Graduate Texts in Mathematics 113, Springer-Verlag, New York.

[32] S. Kwapién and W. A. Woyczyński (1992). *Random Series and Stochastic Integrals: Single and Multiple*. Probability and Its Applications, Birkhäuser Boston.

[33] J. Musielak (1983). *Orlicz Spaces and Modular Spaces. Lecture notes in mathematics* 1034, Springer: New York Berlin Heidelberg.

[34] I. Nourdin and A. Réveillac (2009). Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. *Ann. Probab.* 37(6), 2200–2230.

[35] M. Podolskij and M. Vetter (2010). Understanding limit theorems for semimartingales: a short survey. *Stat. Neerl.* 64(3), 329–351.

[36] B. Rajput and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields* 82(3), 451–487.

[37] A. Rényi (1963). On stable sequences of events. *Sankhyā Ser. A* 25, 293–302.

[38] G. Samorodnitsky and M.S. Taqqu (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapmann and Hall, New York.

[39] K. Sato (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics 68, Cambridge University Press, Cambridge.

[40] K. Takashima (1989). Sample path properties of ergodic self-similar processes. *Osaka J. Math.* 26(1), 159–189.

[41] C.A. Tudor and F.G. Viens (2009). Variations and estimators for self-similarity parameters via Malliavin calculus. *Ann. Probab.* 37(6), 2093–2134.
[42] A.V. Skorokhod (1956). Limit theorems for stochastic processes. *Theory Probab. Appl.* 1, 261–190.

[43] J.W. Tukey (1938). On the distribution of the fractional part of a statistical variable. Rec. Math. [Mat. Sbornik] N.S., 4(46):3, 561–562.

[44] W. Whitt (2002). *Stochastic-Process Limits.* Springer Verlag New York Berlin Heidelberg.