A New Number Theory
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Abstract
The article proposes a mathematical method that permits to treat numbers with more than 2 dimensions. The sums and the products must be done by two distinct definitions. Despite this little limitation you have the same algebra of the standard complex numbers. The limitations occur only when you have to really compute the sums and the products between numbers, not in the symbolic algebra.

The product and the sum are commutative.

Keywords: Quaternions; Number Theory; Operator theory; Algebra; Tensor methods

Definitions
Let us consider the 3d space that can be represented by the tern \( \vec{u}, \vec{v}, \vec{w} \) as shown in the Figure 1
The point P can be written as: \( P = (x, y, z) = \vec{u} \cdot x + \vec{v} \cdot y + \vec{w} \cdot z \)
But also, using the polar notation the point P can be written as \( P = r \cdot e^{i \alpha} \beta \)
where \( r = \sqrt{x^2 + y^2 + z^2} \)

The operator \( e^{i \beta} \) raises the lying vectors in the plane \( \{ \vec{u}, \vec{v} \} \) of \( \beta \) radians along \( \vec{w} \);

The operator \( e^{j \alpha} \) rotates the lying vectors in the plane \( \{ \vec{u}, \vec{v} \} \) of \( \alpha \) radians.

Definition of the sum:
1) \( P_1 + P_2 = (u_1 + u_2) \vec{u} \cdot (v_1 + v_2) \vec{v} \cdot (w_1 + w_2) \vec{w} \cdot \) \[ Cartesian notation \]

Definition of the product:
2) \( P_1 \cdot P_2 = (u_1 \cdot e^{i \alpha_1} \beta_1) \cdot (v_1 \cdot e^{i \alpha_1} \beta_1) = (u_1 \cdot v_1 \cdot e^{i \alpha_1} \beta_1) \cdot (v_1 \cdot \beta_1) \) \[ Polar notation \]

The Number
\( s = U \cdot x + \vec{v} \cdot y + \vec{w} \cdot z \equiv R e^{i \alpha + k \beta} \)
Behaves as a complex number with three dimensions provided that:

To do the sums it must always be used the definition (1) \[ Cartesian notation \]. To do the products it must always be used the definition (2) \[ polar notation \]. The definition can be extended to 4 dimensions.
The relations between \( P = (x, y, z) \) and \( (r, \alpha, \beta) \) are given by the following formulas:
\( z = r \cdot \sin(\beta) \)
\( y = r \cdot \cos(\beta) \sin(\alpha) \)
\( x = r \cdot \cos(\beta) \cos(\alpha) \)

For \( \alpha = 0 \), the vector is lying in the plane \( \{ \vec{u}, \vec{w} \} \), and the polar notation coincides with a vector in the vertical rotation. The y values for \( \alpha = 0 \) are 0 by default. The transition from one format to another is always possible, because the tern \( (x, y, z) \) always and uniquely identifies the tern \( (r, \alpha, \beta) \) through the formulas (3). The methods of symbolic computation are identical to those of the standard complex numbers.

The operations, of calculation, must always take into account the two rules (1) and (2) above for the sums and products. To assess the calculator expressions you can use the Reverse Polish Notation (RPN).

Calculus
The question is: does the calculus work?
Fixing:
\( ds = u \cdot dx + v \cdot dy + w \cdot dz \)

The definition above, can’t coincides at the infinitesimal to the differential of
\( ds = d(e^{i \alpha + k \beta}) \)
The calculus works using the formulas (3)
\( dz = dr \cdot \sin(\beta) + r \cdot \cos(\beta) \cdot d\beta \)

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Practically the polar notation is useful only to define the concept of the commutative product, necessary for the symbolic operations and for the real calculation of the product between the numbers.

As a last consideration, we can also define $s \cdot s = r^2 \cdot e^{(\alpha+\beta)\pi/2}$

and in general: $s^t = r^t \cdot e^{(\alpha+\beta)t\pi/2}, \ t \in \mathbb{R}$

4 Dimensions

Let us consider $s$ as a 4 dimension number as defined above:

$s = u \cdot x + v \cdot y + w \cdot z + q \cdot t = r \cdot e^{i\alpha+\beta+\gamma+i\lambda}$

Is it still true the analysis above? The answer is yes, but we have to introduce the operator $e^{i\lambda}$ which raises the vectors in the cube $\{ u, v, w \}$ of $\lambda$ radians along $q$.

The unity vector $q$ is orthogonal to $u$, $v$, and $w$, but it can’t be represented graphically; the formulas between the two representations, one Cartesian, the other polar, can be detected from an idea given by the Figure 2.

The formulas:

$$r = \sqrt{x^2 + y^2 + z^2 + t^2} \quad \alpha \in [0, 2\pi]$$

$$r' = \sqrt{x'^2 + y'^2 + z'^2} = r \cdot \cos(\gamma) \quad \beta \in [0, 2\pi]$$

$$t = r \cdot \sin(\gamma)$$

$$z = r' \cdot \sin(\beta)$$

$$y = r' \cdot \cos(\beta) \cdot \sin(\alpha)$$

$$x = r' \cdot \cos(\beta) \cdot \cos(\alpha)$$

Because $r'$ can’t be negative, it is clear that $\gamma$ must be reduced to $\gamma = [\pi/2, \pi/2]$.

When $|r| = \pi/2$, $x = y = x = 0$, i.e. a pure $q$ vector.

Examples: Calculating the volume of a sphere

We need a little core of the new algebra, see below.

Here a couple of routines (written in Visual Basic) to estimates the volume of a sphere with the algebra above. The more $Kloop$ is high, the more accurate is the estimation.

Sub SphereA (R0 as Double, Kloop As Integer)

```
Dim S As Complex3d
Dim dAlfa As Double, dBeta As Double, dr As Double
Dim I As Long, J As Long, K As Long, KLoop As Long
Dim Vol As Double, dv As Double

dAlfa=2 * Pi / KLoop
'dLet consider half a sphere
dBeta=Pi / 2 / KLoop
dr=R0 / KLoop

Vol=0
S=Init_Vector(0, 0, 0)
For K=1 To K Loop
For I=1 To K Loop
For J=1 To K Loop
'Strange differential
dv=S.R ^ 2 * dAlfa * Cos(S.Beta) * dBeta * dr
Vol=Vol + dv
S=Rotation_3d(S, dAlfa, 0)
Next J
S=Rotation_3d(S, 0, dBeta)
Next I
S.Beta=0
S=Rotation_3d(S, 0, 0, dr)
Next K

MsgBox "Sphere: estimated Vol: " + Format(2 * Vol) + vbCrLf + "Exact:" + Format(4 / 3 * Pi * R0 ^ 3)
End Sub
```

Sub SphereB (R0 As Double, Kloop As Integer)

```
Dim S As Complex3d
Dim dAlfa As Double, dBeta As Double, dr As Double
Dim I As Long, J As Long, K As Long
Dim Vol As Double, dv As Complex3d

dAlfa=2 * Pi / KLoop
dBeta=Pi / 2 / KLoop
dr=R0 / KLoop

Vol=0
S=Init_Vector(0, 0, 0)
For K=1 To K Loop
For I=1 To K Loop
For J=1 To K Loop
'Strange differential
dv=S.R ^ 2 * dAlfa * Cos(S.Beta) * dBeta * dr
Vol=Vol + dv
S=Rotation_3d(S, dAlfa, 0)
Next J
S=Rotation_3d(S, 0, dBeta)
Next I
S.Beta=0
S=Rotation_3d(S, 0, 0, dr)
Next K

MsgBox "Sphere: estimated Vol: " + Format(2 * Vol) + vbCrLf + "Exact:" + Format(4 / 3 * Pi * R0 ^ 3)
End Sub
```
dv=Differentiate_Vector_3d(S, dAlfa, dBeta, dr)
dv=Project_A_on_B_3d(dv, S)
Vol=Vol + dv.X * dv.Y * dv.Z
S=Rotation_3d(S, dAlfa, 0)
Next J
S=Rotation_3d(S, 0, dBeta)
Next I
S.Beta=0
S=Rotation_3d(S, 0, 0, dr)
DoEvents
Next K
'Smsg
MsgBox 'Sphere: estimated Vol: " + Format(2 * Vol) + vbCrLf + "Exact:
" + Format(4 / 3 * Pi * R0 ^ 3)
End Sub

Here the code written in Visual Basic that define the sum, the product's, and s^n...
Option Explicit
'-----------------------------------
' CORE 3d ALGEBRA
' V2.4d * OPTIMIZED
'-----------------------------------
Public Const Pi=3.14159265358979
'AVOID THE USE OF SMALL NUMBER IN SIMULATION (OR
'VERY BIG NUMBERS)
'THE PRECISION IS LIMITED, THE MANTISSA HAVE 15 DIGIT
'
Public Const MaxDigit=12, AsZero=10 ^ -12
'
'We can round the results of calculus or not
Private Const Round_Results=True
'The definition of the Complex3d type
Type Complex3d
X As Double
Y As Double
Z As Double
R As Double
Alfa As Double
Beta As Double
End Type
'The initialization number as in an Algebriac notation
Function Init_Algebric(X As Double, Y As Double, Z As Double) As Complex3d
Dim R As Complex3d
R.X=X
R.Y=Y
R.Z=Z
Calc_Vector_Notation R
Init_Algebric=R
End Function
'The initialization number as in a Vector notation
Function Init_Vector(R As Double, Alfa As Double, Beta As Double) As Complex3d
Dim S As Complex3d
S.R=R
S.Alfa=Alfa
S.Beta=Beta
To_Algebric_Notation S
Init_Vector=S
End Function
'The Sum A+B
Function Sum_3d (A As Complex3d, B As Complex3d) As Complex3d
Dim R As Complex3d
R.X=A.X + B.X
R.Y=A.Y + B.Y
R.Z=A.Z + B.Z
Calc_Vector_Notation R
Sum_3d=R
End Function
'The Difference A-B
Function Diff_3d (A As Complex3d, B As Complex3d) As Complex3d
Dim R As Complex3d
R.X=A.X - B.X
R.Y=A.Y - B.Y
R.Z=A.Z - B.Z
Calc_Vector_Notation R
Diff_3d=R
End Function
'The Product A*B
Function Mul_3d (A As Complex3d, B As Complex3d) As Complex3d
Dim R As Complex3d
R.X=A.X * B.X
R.Y=A.Y * B.Y
R.Z=A.Z * B.Z
Calc_Vector_Notation R
Diff_3d=R
End Function
'The Product A*B
Function Mul_3d (A As Complex3d, B As Complex3d) As Complex3d
Dim R As Complex3d, X As Double
R.R=A.R * B.R
R.Alfa=Modulus (A.Alfa + B.Alfa, 2 * Pi)
R.Beta=Modulus (A.Beta + B.Beta, 2 * Pi)
To_Algebric_Notation R
Mul_3d=R
End Function
'The Division A/B
Function Div_3d (A As Complex3d, B As Complex3d) As Complex3d
Dim R As Complex3d
'R=A.R / B.R
R.Alfa=Modulus(A.Alfa - B.Alfa, 2 * Pi)
R.Beta=Modulus(A.Beta - B.Beta, 2 * Pi)
To_Algebric_Notation R
Div_3d=R
End Function
'The Division A/B
Function Inverse_3d(S As Complex3d) As Complex3d
Dim R As Complex3d, X As Double
R.R=1 / S.R
R.Alfa=-S.Alfa
R.Beta=-S.Beta
To_Algebric_Notation R
Inverse_3d=R
End Function
'S^X; X Real
Function S_elev_X_3d(S As Complex3d, X As Double) As Complex3d
Dim R As Complex3d
R.R=S.R ^ X
If Abs(X) <= 1 Then
R.Alfa=S.Alfa * X
R.Beta=S.Beta * X
Else
R.Alfa=Modulus(S.Alfa * X, 2 * Pi)
R.Beta=Modulus(S.Beta * X, 2 * Pi)
End If
To_Algebric_Notation R
S_elev_X_3d=R
End Function
'S^X; X Real
Function Sqr_3d(S As Complex3d) As Complex3d
Dim R As Complex3d
R.R=Sqr(S.R)
R.Alfa=S.Alfa / 2
R.Beta=S.Beta / 2
To_Algebric_Notation R
Sqr_3d=R
End Function
'Rotation and Elongation
Function Rotation_3d(S As Complex3d, dAlfa As Double, dBeta As Double, Optional dr As Double=0) As Complex3d
Dim R As Complex3d
R=R
If Near0(R.R)=0 And Near0(dr)=0 Then
Rotation_3d=R
Exit Function
End If
R.R=R.R + dr
R.Alfa=Modulus(S.Alfa + dAlfa, 2 * Pi)
R.Beta=Modulus(S.Beta + dBeta, 2 * Pi)
To_Algebric_Notation R
Rotation_3d=R
End Function
'Creates ds from a vector S and dAlfa, dBeta and dr
Function Differentiate_Vector_3d(S As Complex3d, dAlfa As Double, dBeta As Double, dr As Double) As Complex3d
Dim dx As Double, dy As Double, dz As Double, ds As Complex3d
dz=dr * Sin(S.Beta) + S.R * Cos(S.Beta) * dBeta
dy=dr * Cos(S.Beta) * Sin(S.Alfa) - S.R * Sin(S.Beta) * Sin(S.Alfa) * dBeta + S.R * Cos(S.Beta) * Cos(S.Alfa) * dAlfa
dx=dr * Cos(S.Beta) * Cos(S.Alfa) - S.R * Sin(S.Beta) * Cos(S.Alfa) * dBeta + S.R * Cos(S.Beta) * Sin(S.Alfa) * dAlfa
ds=Init_Algebric(dx, dy, dz)
Differentiate_Vector_3d=ds
End Function
'Internal product
Function A_V_B_3d(A As Complex3d, B As Complex3d) As Double
A_V_B_3d=A.X * B.X + A.Y * B.Y + A.Z * B.Z
End Function
'External product
Function A_X_B_3d(A As Complex3d, B As Complex3d) As Complex3d
Dim X As Double, Y As Double, Z As Double, R As Complex3d
\[X = A \cdot Y \cdot B \cdot Z - A \cdot Z \cdot B \cdot Y \]
\[Y = A \cdot Z \cdot B \cdot X - A \cdot X \cdot B \cdot Z \]
\[Z = A \cdot X \cdot B \cdot Y - A \cdot Y \cdot B \cdot X \]

`End Function`

`R = Init_Algebric(X, Y, Z)`

`A_X_B_3d = R`

`Function Versor_3d(S As Complex3d) As Complex3d`

`Dim R As Complex3d, R0 As Double`

`R = S`

`R0 = R.R`

`R.R = 1`

`Versor_3d = R`

`Exit Function`

`Set_To_Zero: R.X = 0 R.Y = 0 R.Z = 0 R.R = 0 R.Alpha = 0 R.Beta = 0 Versor_3d = R End Function`

`Function Project_A_on_B_3d(A As Complex3d, B As Complex3d) As Complex3d`

`Dim Wx As Complex3d, Wy As Complex3d, Wz As Complex3d, R As Complex3d, R0 As Double`

`Dim X As Double, Y As Double, Z As Double`

`Dim BVx As Double, BVy As Double, BVz As Double`

`If Near0(B.R) = 0 Then GoTo Set_To_Zero If Near0(A.R) = 0 Then GoTo Set_To_Zero`

`Wx = Versor_3d(B)`

`Wx.X = Wx.X / R0 Wx.Y = Wx.Y / R0 Wx.Z = Wx.Z / R0 Check_Algebric_Zero R Versor_3d = R Exit Function Set_To_Zero: R.X = 0 R.Y = 0 R.Z = 0 R.R = 0 R.Alpha = 0 R.Beta = 0 Project_A_on_B_3d = R End Function`
THE TRASFORMATION FROM ALGEBRIC TO POLAR NOTATION

Private Sub Calc_Vector_Notation(S As Complex3d)
    Dim SinBeta As Double, CosBeta As Double, SinAlfa As Double, CosAlfa As Double
    Check_Algebric_Zero S
    'Calc R
    S.R=Sqr(S.X ^ 2 + S.Y ^ 2 + S.Z ^ 2)
    If Near0(S.R)=0 Then GoTo Set_To_Zero
    'Solve Beta....
    SinBeta=S.Z / S.R
    CosBeta=Sqr(S.X ^ 2 + S.Y ^ 2) / S.R
    'CosBeta is always >=0
    'SinBeta can be <=0
    If Round(CosBeta, MaxDigit)=0 Then
        If Round(SinBeta, MaxDigit)=0 Then GoTo Set_To_Zero
        S.Beta=Pi / 2 * Sgn(S.Z)
    Else
        S.Beta=ArcSin(SinBeta)
    End If
    'Solve Alfa....
    If Round(CosBeta, MaxDigit) <> 0 Then
        SinAlfa=S.Y / S.R / CosBeta
        CosAlfa=S.X / S.R / CosBeta
        'CosAlfa can be <=0 ...
        If Round(CosAlfa, MaxDigit)=0 Then
            If Round(SinAlfa, MaxDigit)=0 Then S.Alfa=0
        Else
            S.Y=S.R * CosBeta * Sin(S.Alfa)
            S.X=S.R * CosBeta * Cos(S.Alfa)
        End If
    Check_Algebric_Zero S
    If Near0(S.Alfa)=0 And S.X<0 Then S.Alfa=Pi
    If Near0(S.Z)=0 And S.Beta<>0 Then S.Beta=0
    Exit Sub
Set_To_Zero:
    S.X=0
    S.Y=0
    S.Z=0
    S.R=0
    S.Alfa=0
    S.Beta=0
End Sub

THE TRASFORMATION FROM VECTOR TO ALGEBRIC

Private Sub To_Algebric_Notation(S As Complex3d)
    Dim CosBeta As Double
    If Near0(S.R)=0 Then GoTo Set_To_Zero
    'Solve X,Y,Z
    S.Z=S.R * Sin(S.Beta)
    CosBeta=Cos(S.Beta)
    If Near0(CosBeta)=0 Then
        S.Y=0
        S.X=0
    Else
        S.Y=S.R * CosBeta * Sin(S.Alfa)
        S.X=S.R * CosBeta * Cos(S.Alfa)
    End If
    Check_Algebric_Zero S
    If Near0(S.Alfa)=0 And S.X<0 Then S.Alfa=Pi
    If Near0(S.Z)=0 And S.Beta<>0 Then S.Beta=0
    Exit Sub
Set_To_Zero:
    S.X=0
    S.Y=0
    S.Z=0
    S.R=0
    S.Alfa=0
    S.Beta=0
End Sub

Private Sub Check_Algebric_Zero(S As Complex3d)
S.Z=Near0(S.Z)
S.Y=Near0(S.Y)
S.X=Near0(S.X)
End Sub

Function Modulus(X As Double, Y As Double) As Double
Dim Resto As Double
Resto=X / Y - Fix(X / Y)
If Round(Resto * Y, MaxDigit)=0 Then
Modulus=0
Else
Modulus=Resto * Y
End If
End Function

Function ArcSin(X As Double) As Double
If Round(Abs(X), MaxDigit)=1 Then
ArcSin=Pi / 2 * Sgn(X)
Exit Function
End If
ArcSin=Atn(X / Sqr(1 - X ^ 2))
End Function

Function ArcCos(X As Double) As Double
If Round(Abs(X), MaxDigit)=1 Then
If X > 0 Then
ArcCos=0
End If
ArcCos=Atn(-X / Sqr(1 - X ^ 2)) + 2 * Atn(1)
End Function

Function Near0(X As Double) As Double
Dim R As Double
R=X
If Round_Results Then R=Round(R, MaxDigit)
Near0=R
If Abs(R) <= AsZero Then
Near0=0
End If
End Function

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