MANIFOLDS WITH VANISHING CHERN CLASSES: HYPERELLIPITC MANIFOLDS, MANIFOLDS
ISOGNENOUS TO A TORUS PRODUCT, AND SOME QUESTIONS BY SEVERI/BALDASSARRI

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In memory of Mario Baldassarri (1920-1964).

Abstract. We first give a negative answer to a question posed by Severi in 1951, whether the Abelian Varieties are the only projective manifolds with vanishing Chern classes. We exhibit Hyperelliptic Manifolds which are not Abelian varieties (nor complex tori) and whose Chern classes are zero not only in integral homology, but also in the Chow ring.

We prove moreover that the Bagnera-de Franchis manifolds (quotients $T/G$ of a complex torus $T$ by the free action of a cyclic group $G$) have topologically trivial tangent bundle.

Motivated by a more general question addressed by Mario Baldassarri in 1956, we discuss the Hyperelliptic Manifolds, the Pseudo-Abelian Varieties introduced by Roth, and we introduce a new notion, of Manifolds Isogenous to a $k$-Torus Product: the latter have the last $k$ Chern classes trivial in rational cohomology and vanishing Chern numbers.

We show that in dimension 2 the latter class is the correct substitute for some incorrect assertions by Enriques, Dantoni, Roth and Baldassarri: these are the surfaces with $K_X$ nef and $c_2(X) = 0$.

We observe that a similar picture does not hold in higher dimension; and we discuss, as a class of solutions to Baldassarri’s question, (manifolds isogenous to) projective (respectively: Kähler) manifolds whose tangent (resp. cotangent bundle) has a trivial subbundle. The former ones are, in the case where $K_X$ is nef, the Pseudo-Abelian varieties of Roth, while the latter are not yet understood: we can however formulate some open questions and conjectures.

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Mario Baldassarri ’s article [Bald56] is dedicated to the attempt to characterize the smooth projective manifolds $X$ whose first $h$ canonical systems $K_0(X), \ldots, K_{h-1}(X)$ have degree zero. The canonical systems of a manifold, defined in a geometric way by Todd, Eger and later in a simpler way by Beniamino Segre [Seg52, Seg54], after proposals made by Severi, were shown in 1955 by Nakano [Nak55] to be the so called Chern classes of the cotangent bundle (see [At98] for an historical account and [Ful84] as a general reference): more precisely, up to sign, to the system $K_h(X)$ corresponds the Chern class $c_{n-h}(X)$, where $n$ is the dimension of $X$ (and we can consider the Chern classes either as elements of the Chow ring of $X$, or as integral cohomology classes).

Hence we shall formulate some questions raised by Severi, and by the work of Baldassarri, in terms of Chern classes.

Baldassarri’s paper begins citing a problem raised by Severi in [Sev51]: it asks whether Abelian varieties can be characterized as the projective manifolds whose Chern classes are all trivial.

Our first result is that Severi’s question has a negative answer, both if we assume that the Chern classes are zero in integral homology ($c_i(X) = 0 \in H^{2i}(X, \mathbb{Z}), \forall i$: here we show that a counterexample is given already in dimension 2 by the Hyperelliptic surfaces), or even if we make the stronger assumptions that the Chern classes are zero in the Chow ring of $X$: here the counterexamples start in dimension 3, since for hyperelliptic surfaces $c_1(X) \neq 0 \in \text{Pic}(X)$.

We have more precisely:

**Theorem 0.1.** The tangent bundle of a Bagnera-de Franchis manifold $X = T/G$ (X is the quotient of a complex torus $T$ by a cyclic group $G$) has degree zero in the Chow ring.
acting freely and containing no translations) is topologically trivial, equivalently all its integral Chern classes $c_i(X) = 0 \in H^*(X, \mathbb{Z})$.

There are Bagnera de Franchis manifolds $X = T/G$, which are not complex tori, such that all its Chern classes $c_i(X)$ are zero in the Chow ring of $X$.

There are some Hyperelliptic manifolds $X = T/G$ (these are the compact Kähler manifolds with trivial Chern classes in rational cohomology) such that not all their integral Chern classes $c_i(X) \in H^*(X, \mathbb{Z})$ are equal to zero.

To clarify the history of the problem, the characterization of the Manifolds with all Chern classes zero in rational (or real) cohomology was solved only in 1978, thanks to Yau’s celebrated theorem [Yau78] on the existence of Kähler-Einstein metrics on manifolds with $c_1(X) = 0 \in H^2(X, \mathbb{Q})$.

From this, as explained in Kobayashi’s book, page 116 of [Kob87], follows that the compact Kähler manifolds (cKM) with $c_1(X) = c_2(X) = 0 \in H^*(X, \mathbb{Q})$ are the Hyperelliptic manifolds, the quotients of a complex torus by the free action of a finite group $G$. Once one knows that we have a Kähler-Einstein metric, that the manifold is flat was proven by Apte [Ap55] in the 50’s.

Hence, after 1978, Severi’s question became a question concerning Hyperelliptic manifolds, and our above theorem shows that for these the situation is mixed: there are examples with Chern classes trivial also in integral cohomology, or even in the Chow ring of rational equivalence classes, but there are also examples with nontrivial Chern classes in integral cohomology. Hence our theorem raises the interesting problem of a complete classification of the (hyperelliptic) manifolds with $c_i(X) = 0 \in H^*(X, \mathbb{Z})$, $\forall i$.

Baldassarri’s paper [Bald56] deals instead with the manifolds such that the last Chern classes vanish in rational cohomology, namely $c_i(X) = 0 \in H^{2i}(X, \mathbb{Q})$ for $i \geq k + 1$: and he claimed that these manifolds are the Pseudo-Abelian varieties introduced by Roth [Roth54]. Todd’s review of [Bald56] pointed out a wrong intermediate result, admitting as counterexample the blow up $X$ of $\mathbb{P}^3$ with centre a smooth curve of genus 3, which is a regular manifold $X$ having $c_3 = 0$.

Todd’s counterexample could somehow be quickly dismissed as only pointing out the need to assume that $X$ is a minimal manifold, say with $K_X$ nef, as this assumption was already known to be necessary in dimension 2.

We show here however that the flaw is not simply a technical problem, the main claim by Baldassarri that the solutions are Roth’s Pseudo-Abelian varieties is incorrect also for minimal manifolds. And we show this first by introducing the concept of Manifolds isogenous to a $k$-torus product. This is a larger class than the class of Roth’s Pseudo-Abelian
varieties, and varieties in this class are solutions to Baldassarri’s problem, because of the

**Remark 0.1. (Isogeny principle):** If we have a finite unramified map \( f : Z \to X \), then \( c_i(Z) = 0 \in H^{2i}(Z, \mathbb{Q}) \) if and only if \( c_i(X) = 0 \in H^{2i}(X, \mathbb{Q}) \).

Defining *isogeny* between manifolds as the equivalence relation generated by the existence of such finite unramified maps, we see that the solutions to Baldassarri’s question consist of isogeny classes.

On the other hand Roth saw that the crucial flaw in Baldassarri’s argument was the attempt to show that manifolds with top Chern class equal to zero have positive irregularity. If one takes this as an assumption, and makes it stronger by requiring that there exists a holomorphic 1-form without zeros, then Baldassarri’s claim holds at least in dimension 3, as shown by [HS21b] (see also a similar result in Theorem 7.3).

What is historically interesting is to observe that the ‘original sin’ of [Bald56] was to try to extend to higher dimension some wrong results by Enriques, Dantoni and Roth (indeed the error of Enriques is also reproduced in the classification theorem of Castelnuovo and Enriques [CastEn15]).

The work by Baldassarri and Roth is indeed inspired by a paper by Dantoni [Dant43], devoted to the minimal surfaces \( X \) with \( c_2(X) = 0 \). Dantoni uses surface classification, but indeed especially an article by Enriques of 1905 [Enr05b], claiming that the non ruled surfaces with these properties are the hyperelliptic manifolds and the ‘elliptic’ surfaces. But ‘elliptic’ for Enriques here does not have the same standard meaning introduced later by Kodaira and others: Enriques requires the action of a fixed elliptic curve on \( X \), with all orbits of dimension 1.

Enriques and Dantoni in their classification omit to consider the case of quotients \( X = (E \times C)/G \) where the action of the finite group \( G \) is free, of product type, but \( G \) does not act on the elliptic curve \( E \) via translations, and moreover \( C \) is a curve of genus \( g \geq 2 \), such that the quotient \( C/G \) is an elliptic curve (see for instance [CatLi19] or [Bea78] for the special case \( p_g = 0 \), and in general [CB] or [Cat22]). Indeed, in this latter case the automorphism group of \( X \) has dimension zero.

Dantoni’s paper inspired Roth [Roth53], who defined the Pseudo-Abelian varieties as the manifolds admitting the action of a complex torus of positive dimension = \( k \) having all orbits of dimension \( k \), and such that \( k \) is maximal with this property.

But for instance, in [Roth53], Roth does not realize about the existence of Hyperelliptic threefolds with automorphism groups of dimension zero, and believes that these are only Pseudo-Abelian varieties with \( k \geq 1 \).
Our conclusion is simple: the ‘original sin’ is to consider only quotients $X = (T \times Y)/G$ where $T$ is a torus, $G$ acts via a product action, which is free and such that $G$ acts on $T$ via translations. Obviously under these assumptions $T$ acts on $X$ and the orbits have all the same dimension $k = \text{dim}(T)$!

We can repair things just by making the weaker requirement that the action of $G$ is free, and I define in this way the varieties isogenous to a k-Torus product as the free quotients $X = (T \times Y)/G$ where $T$ is a torus of dimension $k$.

For these quotient manifolds it follows from Remark 0.1 that the last $k$ Chern classes are zero in rational cohomology, and that the Chern numbers of $X$ are equal to zero. Whereas for Pseudo-Abelian varieties the last $k$ Chern classes are zero in integral cohomology, because for them we have a splitting of the tangent bundle $\Theta_X \cong \mathcal{O}_X^k \oplus \mathcal{F}$.

As already observed, the solutions to Baldassarri’s question consists of isogeny classes, and the special isogeny classes are here those which contain a product $T \times Y$, where $T$ is a $k$-dimensional torus. These are all the solutions in the case of dimension $n = 2$, and for the case $k = n$ of hyperelliptic manifolds.

Work of Chad Schoen [Schoen88] shows however that in dimension 3 there are simply connected manifolds, actually Calabi Yau manifolds $(K_X$ trivial in $\text{Pic}^0(X))$ with $c_3 = 0$: and we show here that the Schoen threefolds are not birationally covered by a family of isomorphic Abelian surfaces, see Proposition 9.1 and that they do not admit a fibration onto a surface with general fibres isomorphic to a fixed elliptic curve $E$, see Proposition 9.2. Hence that they are also birationally far away from being isogenous to a torus product.

Surely Baldassarri’s question inspires also a similar question of characterizing the minimal manifolds with vanishing Chern numbers; among these are the manifolds isogenous to a torus product, and these are all in dimension $n = 2$.

Interesting quite recent results by Hao-Schreieder [HS21a], applied to threefolds with $c_1c_2 = 0$, is an important step towards a solution in dimension three.

At any rate a solution to Baldassarri’s question is also provided by the manifolds $X$ which are isogenous to a partially framed or co-framed manifold $Z$: that is, $Z$ is either partially (tangentially) framed, meaning that the tangent bundle $\Theta_Z$ admits a trivial subbundle $\mathcal{O}_Z^k$, or $Z$ is partially co-framed (cotangentially framed), meaning by the cotangent bundle $\Omega_Z^1$ admits such a trivial subbundle $\mathcal{O}_Z^k$.

What we observe is that, thanks to the work of Fujiki [Fuj78] and Lieberman [Li78], the class of partially (tangentially) framed projective manifolds coincides, in the case where $K_X$ is nef, with the class of the Pseudo-Abelian varieties of Roth, see Theorem 7.1 which contains also more general results.
We also observe [Li78, AMN12] that the picture becomes more complicated if we enlarge our consideration to the wider realm of cKM (compact Kähler Manifolds).

Instead, the second class of partially co-framed manifolds is more mysterious and presents some intriguing questions (see the discussion following Proposition 7.2 and Theorem 7.3). In the projective case we do not have yet more examples than the Pseudo-Abelian varieties.

In fact, we ask two general questions such that a positive answer to both of them would imply that we have nothing more than the Pseudo-Abelian varieties.

1. Hyperelliptic manifolds and varieties

The French school of Appell, Humbert, Picard, Poincaré defined the Hyperelliptic Varieties as those smooth projective varieties whose universal covering is biholomorphic to \( \mathbb{C}^n \) (in particular the Abelian varieties are in this class). For \( n = 1 \) these are just the elliptic curves, whereas the Hyperelliptic varieties of dimension 2 were classified by Enriques and Severi [EnrSev09] and by Bagnara and De Franchis [BdF08]: both pairs were awarded the prestigious Bordin Prize for this achievement.

Kodaira [Kod66] showed instead that if we take the wider class of compact complex manifolds of dimension 2 whose universal covering is \( \mathbb{C}^2 \), then there are other non-algebraic and non-Kähler surfaces, called nowadays Kodaira surfaces (beware: these are not the so-called Kodaira fibred surfaces!).

Iitaka conjectured that if a compact Kähler Manifold \( X \) has universal covering biholomorphic to \( \mathbb{C}^n \), then necessarily \( X = T/G \) of a complex torus \( T \) by the free action of a finite group \( G \) (which we may assume to contain no translations).

The conjecture by Iitaka was proven in dimension 2 by Kodaira, and in dimension 3 by Campana and Zhang [CamZha05]. Whereas it was shown in [CHK13] that, if the abundance conjecture holds, then a projective smooth variety \( X \) with universal covering \( \mathbb{C}^n \) is a Hyperelliptic variety according to the following definition.

**Definition 1.1.** A Hyperelliptic Manifold \( X \) is defined to be a quotient \( X = T/G \) of a complex torus \( T \) by the free action of a finite group \( G \) which contains no translations.

We say that \( X \) is a Hyperelliptic Variety if moreover the torus \( T \) is projective, i.e., it is an Abelian variety \( A \), that is, \( A \) possesses an ample line bundle \( L \).

If the group \( G \) is a cyclic group \( \mathbb{Z}/m \), then such a quotient is called ([BCP15, Cat15]) a Bagnara-De Franchis manifold.

In dimension \( n = 2 \), a hyperelliptic manifold \( X \) is necessarily projective, and \( G \) is necessarily cyclic, whereas in dimension \( n \geq 3 \) the only
examples with $G$ non Abelian have $G = D_4$ and were classified in [UY76] and [CD20b] (for us $D_4$ is the dihedral group of order 8).

Indeed, (see for instance [CD20a]) every Hyperelliptic Manifold is a deformation of a Hyperelliptic Variety, so that a posteriori the two notions are related to each other, in particular the underlying differentiable manifolds are the same.

There are at least three important research directions concerning Hyperelliptic Varieties:

1. Establish Iitaka’s conjecture.
2. Understand and classify Hyperelliptic Manifolds.
3. Construct interesting manifolds as submanifolds (e.g., Hyper-surfaces) of Hyperelliptic Manifolds.

Question (1) is essentially a question about fundamental groups of compact Kähler Manifolds: since (cf. for instance [Cat15] Coroll. 82, page 356) any compact Kähler Manifold $X$ with contractible universal cover and with $\pi_1(X)$ Abelian is a complex torus. Hence the main point is to show that if a compact Kähler Manifold $X$ has universal covering biholomorphic to $\mathbb{C}^n$, then necessarily $\pi_1(X)$ has an Abelian subgroup of finite index.

More generally one can ask:

**Question 1.2.** Given a compact complex manifold with universal covering $\tilde{X} \cong \mathbb{C}^n$, is the fundamental group $\pi_1(X)$ a solvable group?

Question (2) above is instead essentially a difficult algebraic question: since if $X$ is a Hyperelliptic Manifold, and $\Gamma := \pi_1(X)$, then we have an exact sequence of groups

\[ \begin{align*}
0 & \to \Lambda \to \Gamma \to G \to 1, \\
\end{align*} \]

where $\Lambda = \pi_1(T) \cong \mathbb{Z}^{2n}$.

This leads (see [CC17]) to the following definition of an abstract torsion free even Euclidean crystallographic group.

**Definition 1.3.** (i) We say that a group $\Gamma$ is an abstract Euclidean crystallographic group if there exists an exact sequence of groups

\[ \begin{align*}
\star & \quad 0 \to \Lambda \to \Gamma \to G \to 1 \\
\end{align*} \]

such that

1. $G$ is a finite group
2. $\Lambda$ is free abelian (we shall denote its rank by $r$)
3. Inner conjugation $Ad : \Gamma \to Aut(\Lambda)$ has Kernel exactly $\Lambda$, hence $Ad$ induces an embedding, called Linear part,

\[ L : G \to GL(\Lambda) := Aut(\Lambda) \]

(\text{thus } L(g)(\lambda) = Ad(\gamma)(\lambda) = \gamma \lambda \gamma^{-1}, \ \forall \gamma \text{ a lift of } g)

(ii) A crystallographic group $\Gamma$ is said to be even if:

- (ii.1) $\Lambda$ is a free abelian group of even rank $r = 2n$
(ii.2) \(\Gamma\) is \(G\)-even, equivalently, there exists a Hodge decomposition
\[\Lambda \otimes \mathbb{Z} \mathbb{C} = H^{1,0} \oplus H^{1,0}\]
which is invariant for the \(G\)-action (i.e., \(H^{1,0}\) is a \(G\)-invariant subspace); this is equivalent to:

(ii.2 bis) considering the associated faithful representation \(G \to Aut(\Lambda)\), for each real representation \(\chi\) of \(G\), the \(\chi\)-isotypical component
\[M_\chi \subset \Lambda \otimes \mathbb{Z} \mathbb{R}\]
has even dimension (over \(\mathbb{C}\)).

(iii) \(\Gamma\) is said to be torsion-free if there are no elements of finite order inside \(\Gamma\).

(iv) An affine realization defined over a field \(K \supset \mathbb{Z}\) of an abstract Euclidean crystallographic group \(\Gamma\) is a homomorphism (necessarily injective)
\[\rho : \Gamma \to Aff(\Lambda \otimes \mathbb{Z} K)\]
such that
\[\rho(\gamma)(v) = v + \lambda, \quad \forall \gamma \text{ a lift of } g \in G\]
and
\[V_K \ni v \mapsto L(g)v + u_\gamma, \quad \text{for some } u_\gamma \in V_K.\]

(v) More generally we can say that an affine realization of \(\Gamma\) is obtained via a lattice \(\Lambda' \subset \Lambda \otimes \mathbb{Z} \mathbb{Q}\) if there exists a homomorphism \(\rho' : \Gamma \to Aff(\Lambda')\) such that \(\rho = \rho' \otimes \mathbb{Z} K\) (then necessarily \(\Lambda \subset \Lambda'\)).

Extending previous classical results of Bieberbach \([\text{Bieb11, Bieb12}]\), in \([\text{CC17}]\) was proven:

**Theorem 1.4.** Given an abstract Euclidean crystallographic group there is one and only one class of affine realization, for each field \(K \supset \mathbb{Z}\). There is moreover an effectively computable minimal number \(d \in \mathbb{N}\) such that the realization is obtained via \(\frac{1}{d}\Lambda\).

The above theorem of \([\text{CC17}]\) says in particular that conversely, given such a torsion free even Euclidean crystallographic group \(\Gamma\), there are Hyperelliptic Manifolds with \(\pi_1(X) \cong \Gamma\).

Moreover the Hyperelliptic Manifolds are the compact Kähler Manifolds which are \(K(\Gamma, 1)\)'s for abstract torsion-free even Euclidean crystallographic groups \(\Gamma\) (recall that a \(K(\Gamma, 1)\) is a space \(X\) with contractible universal covering and with \(\pi_1(X) \cong \Gamma\)).

Euclidean crystallographic groups were investigated by Bieberbach \([\text{Bieb11, Bieb12}]\) who proved that, in each dimension, there is a finite set of isomorphism classes (the proof uses Minkowski’s geometry of numbers, but is to our knowledge not effective and does not lead to a classification).
We end this section with an observation, on the automorphism group of Hyperelliptic Manifolds, which will be quite important in the sequel.

**Proposition 1.5.** Let \( X = T/G \) be a Hyperelliptic Manifold. Then its group of Automorphisms is the quotient \( \text{Aut}(X) = \text{Aut}(T)^G / G \), where

\[
\text{Aut}(T)^G := N_{\text{Aut}(T)}(G) \subset \text{Aut}(T)
\]

is the Normalizer of \( G \) in \( \text{Aut}(T) \).

In particular the connected component of the Identity \( \text{Aut}^0(X) \) is isomorphic to the subtorus

\[
T' \subset T, \quad T' = \{ x | g(x) = x, \forall g \in G \}.
\]

The group \( \text{Aut}^0(X) \) may then be trivial if \( n := \dim(X) \geq 3 \) and \( G \) is not cyclic. Equivalently, \( \text{Aut}^0(X) \) is trivial if and only if \( H^1(X, \mathcal{O}_X) = 0 \).

**Proof.** The subgroup \( \Lambda \) is a characteristic subgroup of \( \Gamma \), hence, for each \( \Phi \in \text{Aut}(X) \), \( \Phi \) lifts to an automorphism \( \phi \in \text{Aut}(T) \), which normalizes \( G \).

Then the linear part of \( \phi \) defines a homomorphism of \( \text{Aut}(X) \to GL(\Lambda) \), and, since the image is discrete, \( \text{Aut}^0(X) \) consists of translations. A translation \( z \mapsto z + b \) normalizes \( G \) if and only if \( G(b) = b \).

Finally, writing \( T = V/\Lambda \), the subspace \( V^G := \{ v | Gv = v \} \) is not trivial if \( G \) is cyclic, since \( G \) acts freely, but for \( n \geq 3 \) we have the case of \( G = D_4 \), where \( V^G = 0 \) (see [CD20b]).

Now, \( V^G = 0 \) if and only if \( (V^\vee)^G = 0 \), equivalently, \( H^1(X, \mathcal{O}_X) = 0 \). \( \square \)

2. The Picard Group of a Hyperelliptic Manifold

If \( X = T/G \) is a Hyperelliptic Manifold, we want to analyze the exponential exact sequence:

\[
0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \text{Pic}(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)
\]

which leads to:

\[
0 \to \text{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \to \text{Pic}(X) \to \text{NS}(X) \subset H^2(X, \mathbb{Z}).
\]

Since \( f : T \to X \) is finite, we have, by the Leray spectral sequence,

\[
(*) \quad H^i(X, \mathcal{O}_X) = H^i(T, \mathcal{O}_T)^G
\]

for \( i \geq 1 \).

We write \( T = V/\Lambda \), and \( V = V_1 \oplus V_2 \), in such a way that the linear part \( L \) of the action of \( G \) acts as the identity on \( V_1 \), while \( V_2 \) is a direct sum of nontrivial irreducible representations of \( G \). Follows then from Dolbeault’s theorem:

\[
H^1(X, \mathcal{O}_X) \cong \overline{V}_1^\vee, \quad H^2(X, \mathcal{O}_X) \cong \Lambda^2(\overline{V}_1^\vee)^G = \Lambda^2(\overline{V}_1^\vee) \oplus \Lambda^2(\overline{V}_2^\vee)^G.
\]
Write $\Lambda_i := \Lambda \cap V_i$. Since $V_1$ is defined over $\mathbb{Q}$ and contains $\Lambda_1$ as a lattice, we have an exact sequence
\[
0 \to \Lambda_1 \oplus \Lambda_2 \to \Lambda \to \Lambda^* \to 0,
\]
where $\Lambda^*$ is a finite group, we obtain
\[(00) 0 \to \text{Hom} (\Lambda, \mathbb{Z}) \to (\text{Hom} (\Lambda_1, \mathbb{Z}) \oplus \text{Hom} (\Lambda_2, \mathbb{Z})) \to \text{Ext}^1 (\Lambda^*, \mathbb{Z}) \to 0.
\]
Apply now the Grothendieck spectral sequence
\[
H^n (\Gamma, H^q (T, \mathcal{F})) \Rightarrow H^{n+q} (X, f_* \mathcal{F}^G),
\]
first to $\mathcal{F} = \mathbb{Z}$, then to $\mathcal{F} = \mathcal{O}_T^*$. 

In the first case $\mathcal{F} = \mathbb{Z}$ we get, since $H^1 (G, \mathbb{Z}) = \text{Hom} (G, \mathbb{Z}) = 0$, and from the spectral sequence diagram (here $\Lambda^\vee := \text{Hom} (\Lambda, \mathbb{Z})$)
\[
(2.1) \quad \begin{pmatrix}
H^2 (T, \mathbb{Z})^G & \cdots & \cdots & \cdots \\
\Lambda^\vee = H^1 (T, \mathbb{Z})^G & H^1 (G, \Lambda^\vee) & H^2 (G, \Lambda^\vee) & H^3 (G, \Lambda^\vee) \\
\mathbb{Z} & 0 & H^2 (G, \mathbb{Z}) & H^3 (G, \mathbb{Z})
\end{pmatrix}.
\]

that
\[
(**) 0 \to H^1 (X, \mathbb{Z}) = H^1 (\Gamma, \mathbb{Z}) = \text{Hom} (\Gamma, \mathbb{Z}) \cong \text{Ker} (\psi),
\]
where
\[
\psi : H^1 (T, \mathbb{Z})^G = \text{Hom} (\Lambda, \mathbb{Z})^G \to H^2 (G, \mathbb{Z}).
\]

Also, we have a filtration on $H^2 (X, \mathbb{Z})$ with graded pieces:
\[
H^2 (G, \mathbb{Z}) / \text{Im} (\psi), \quad \text{Ker} [\varphi : H^1 (G, \text{Hom} (\Lambda, \mathbb{Z})) \to H^3 (G, \mathbb{Z})],
\]
\[
\text{Ker} [\text{Ker} [H^2 (T, \mathbb{Z})^G \to H^2 (G, \text{Hom} (\Lambda, \mathbb{Z})] \to \text{Coker} (\varphi)].
\]

Observe now that, due to exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$, and since $H^i (G, \mathbb{C}) = 0$ for $i \geq 1$,
\[
H^2 (G, \mathbb{Z}) \cong H^1 (G, \mathbb{C}^*) \cong \text{Hom} (G, \mathbb{C}^*) \cong \text{Hom} (G^{ab}, \mathbb{C}^*),
\]
while $H^3 (G, \mathbb{Z}) \cong H^2 (G, \mathbb{C}^*)$, the group which classifies the central extensions
\[
1 \to \mathbb{C}^* \to G' \to G \to 1.
\]

In the second case ($\mathcal{F} = \mathcal{O}_T^*$) the Grothendieck spectral sequence yields the exact sequence
\[
0 \to H^1 (G, \mathbb{C}^*) = \text{Hom} (G, \mathbb{C}^*) \to H^1 (\Gamma, \mathcal{O}_T^*) = \text{Pic} (X) \to H^1 (\Lambda, \mathcal{O}_T^*) = \text{Pic} (T)^G \to H^2 (G, \mathbb{C}^*).
\]

This exact sequence is more geometrical, it is the standard sequence saying that $G$-linearized line bundles on $T$ map to $G$-invariant line bundle classes, and two linearizations differ by a character $\chi : G \to \mathbb{C}^*$. The sequence gives an obstruction, for a $G$-invariant line bundle class, to admitting a $G$-linearization, and the obstruction takes values in $H^2 (G, \mathbb{C}^*)$. This obstruction group is trivial for instance if $G$ is a cyclic group.
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The group $H^2(G, \mathbb{C}^*)$ is called the group of Schur multipliers, and the element that we obtain from the above sequence is the class of the Thetagroup of $L$: if a line bundle class $L$ on $T$ is $G$-invariant, Mumford, [Mum70] pages 221 and foll., defined the Thetagroup $\Theta(L)$ as the group of the isomorphisms of $L$ with $g^*(L)$, so that we have the exact sequence

$$1 \to \mathbb{C}^* \to \Theta(L) \to G \to 1.$$ 

This is a central extension, hence it is classified by an element in $H^2(G, \mathbb{C}^*)$ which measures the obstruction to splitting the above exact sequence (that is, to lifting the action of $G$ to $L$).

Example 2.1. Consider the canonical line bundle $K_X$ on $X$. Its pull back is the canonical line bundle $K_A$, which is a trivial line bundle $K_A \cong \mathcal{O}_A$. Both line bundles are $G$-linearized, but the corresponding linearizations are different. $G$ acts trivially on $H^0(A, \mathcal{O}_A)$, while it acts on $H^0(A, K_A)$ through the representation $\det(L(G))$. Hence the canonical line bundle $K_X$ of a Hyperelliptic manifold is trivial if and only if the representation $L : G \to GL(V)$ is unimodular (has determinant $= 1$).

From the previous discussion it is apparent that the main group to be investigated is then $H^1(\Lambda, \mathcal{O}_V^*)^G = \text{Pic}(T)^G$. We use here then the exact sequence for the Picard group of $T$ (derived from the exponential sequence):

$$(* * *) 0 \to \text{Pic}^0(T) \to \text{Pic}(T) \to \text{NS}(T) \to 0,$$

$$\text{NS}(T) = \text{Ker}[H^2(T, \mathbb{Z}) \to H^2(T, \mathcal{O}_T)].$$

This sequence is very explicit: by the Theorem of Appell-Humbert, $\text{NS}(T)$ is the space of Hermitian forms $H$ on $V$ whose imaginary part $E$ takes integral values on $\Lambda$.

Indeed, interpreting $\text{Pic}(T) = H^1(\Lambda, \mathcal{O}_V^*)$ we get the cocycles in Appell-Humbert normal form:

$$f_\lambda(z) = \rho(\lambda) \exp(\pi(H(z, \lambda) + \frac{1}{2}\pi H(\lambda, \lambda))),$$

where $\rho$ is a semicharacter for $E$, that is, $\rho : \Lambda \to \mathbb{C}^*$ satisfies

$$\rho(\lambda + \lambda') = \rho(\lambda)\rho(\lambda') \exp(\pi i E(\lambda, \lambda')).$$

In this interpretation $\text{Pic}^0(T) = T^* := \overline{\nu^*} / \text{Hom}(\Lambda, \mathbb{Z}) \cong \text{Hom}(\Lambda, \mathbb{R}) / \text{Hom}(\Lambda, \mathbb{Z})$, and we get a character $\chi : \Lambda \to \mathbb{C}^*$ by composing with $y \mapsto \exp(2\pi iy)$.

We take the exact sequence of $G$-invariants associated to $(** *)$:

$$(** *) 0 \to (T^*)^G = \text{Pic}^0(T)^G \to \text{Pic}(T)^G \to \text{NS}(T)^G \to H^1(G, T^*).$$

The last arrow measures the obstruction for an invariant class $H$ in $\text{NS}(T)$ to come from an invariant class in $\text{Pic}(T)$ and is induced by the usual arrow

$$L \mapsto g^*(L) \otimes L^{-1},$$

where $g^*$ is the pull back. 


applied to a line bundle $L$ with Chern class $H$.
Using the Appell-Humbert theorem it is easy to calculate $\text{NS}(T)^G$: these are the Hermitian forms $H$ as above which are $G$-invariant, hence $\text{NS}(T)^G = H^2(T, \mathbb{Z})^G \cap H^{1,1}(T)$.

As a final remark, since $(T^*) = \nabla^\vee / \text{Hom}(\Lambda, \mathbb{Z})$, taking $G$-invariants we obtain:

$$(\text{Pic})\, 0 \to (\nabla^\vee)^G / \text{Hom}(\Lambda, \mathbb{Z})^G \to (T^*)^G \to H^1(G, \text{Hom}(\Lambda, \mathbb{Z})) \to 0,$$

and

$$H^1(G, T^*) \cong H^2(G, \text{Hom}(\Lambda, \mathbb{Z})),$$

which brings us back to the first spectral sequence.

**Lemma 2.2.** If $X$ is a Bagnera-De Franchis Manifold, then

$$\psi : H^1(T, \mathbb{Z})^G = \text{Hom}(\Lambda, \mathbb{Z})^G \to H^2(G, \mathbb{Z})$$

is onto.

Hence in particular

$$(**)\, 0 \to H^1(X, \mathbb{Z}) \to H^1(T, \mathbb{Z})^G \to H^2(G, \mathbb{Z}) \to 0$$

is exact.

**Proof.** Abelianizing the exact sequence

$$0 \to \Lambda \to \Gamma \to G \to 1$$

we obtain

$$0 \to \Lambda / (\Lambda \cap [\Gamma, \Gamma]) \to \Gamma^{ab} \to G^{ab} \to 0.$$

In particular we have that the Kernel, $\Lambda / (\Lambda \cap [\Gamma, \Gamma])$, is a quotient of the space of coinvariants $\Lambda_G = \Lambda / ([\Lambda, \Gamma])$.

In the case where $G$ is cyclic, generated by the image of $\gamma$, then $[\Gamma, \Gamma]$ equals just $[\Lambda, \Gamma]$, since the brackets $[\gamma^i, \gamma^j]$ are trivial.

We have (in general) the exact sequence

$$0 \to N := \langle \text{Im}(I - L(g)) \rangle_{g \in G} \to \Lambda \to \Lambda_G \to 0,$$

Hence, setting $M^\vee := \text{Hom}(M, \mathbb{Z})$,

$$0 \to (\Lambda_G)^\vee \to \Lambda^\vee \to N^\vee$$

and we have

$$(\Lambda_G)^\vee = (\Lambda^\vee)^G \subset (\Lambda_1)^\vee,$$

the first equality is by definition and the second inclusion holds since we have

$$0 \to \Lambda^\vee \to (\Lambda_1)^\vee \oplus (\Lambda_2)^\vee \to \text{Ext}^1(\Lambda^*, \mathbb{Z}) \to 0,$$

hence $(\Lambda^\vee)^G \subset ((\Lambda_1)^\vee \oplus (\Lambda_2)^\vee)^G = (\Lambda_1)^\vee$.

In the BdF case, where $G$ is cyclic, starting from the exact sequence

$$0 \to \Lambda_G \to \Gamma^{ab} \to G^{ab} = G \to 0,$$
we make the following

**CLAIM:** we have the exact sequence

\[(ES) \quad 0 \to H^1(X, \mathbb{Z}) = (\Gamma^{ab})^\vee \to (\Lambda_G)^\vee = (\Lambda^\vee)^G = H^1(T, \mathbb{Z})^G \to \text{Ext}^1(G^{ab}, \mathbb{Z}) \to 0.\]

Once the above is shown we can conclude since

\[\text{Ext}^1(G^{ab}, \mathbb{Z}) \cong G^{ab} \cong \text{Hom}(G, \mathbb{C}^*) \cong H^2(G, \mathbb{Z}).\]

To show that the above sequence (ES) is exact, we need to show the Ext\(^1\)(\Gamma^{ab}, \mathbb{Z}) \cong \text{Ext}^1(\Lambda_G, \mathbb{Z})\), which in turn follows if we show that we have an isomorphism \(\text{Tors}(\Lambda_G) \cong \text{Tors}(\Gamma^{ab})\).

In order to show this, we go back to the description of Bagnera de Franchis manifolds, as done in [Cat15], Proposition 16 page 309:

\(X = T/G\), with \(T = (A_1 \times A_2)/\Lambda^*,\) where \(\Lambda^* \subset A_1 \times A_2\) is a finite subgroup, such that

1. \(\Lambda^*\) is the graph of an isomorphism between subgroups \(\mathcal{T}_1 \subset A_1, \mathcal{T}_2 \subset A_2,\)
2. \((\alpha_2 - \text{Id})\mathcal{T}_2 = 0,\)
3. \(G\) is generated by \(g\) such that \(g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2))\), such that the subgroup of order \(m\) generated by \(\beta_1\) intersects \(\mathcal{T}_1\) only in \(\{0\}\).
4. In particular \(X = (A_1 \times A_2)/(G \times \Lambda^*).\)

By property (2) follows that \((\text{Id} - L_g)(\Lambda^*) = 0,\) hence \((\text{Id} - L_g)(\Lambda) \subset \Lambda_1 \oplus \Lambda_2\) and indeed, if we define \(\Lambda'_2 \subset \Lambda_2 \otimes \mathbb{Q}\) via the property that \(\Lambda'_2/\Lambda_2 \cong \mathcal{T}_2,\) then \((\text{Id} - L_g)(\Lambda) = (\alpha_2 - \text{Id})\Lambda'_2 \subset \Lambda_2,\) since the vectors in the image have first coordinate equal to zero. Then we have an exact sequence

\[0 \to \Lambda_1 \oplus [\Lambda_2/(\alpha_2 - \text{Id})(\Lambda'_2)] \to \Lambda_G \to \Lambda^* \to 0.\]

We apply now Proposition 25, page 315 of [Cat15], stating that

\[\text{Alb}(X) = A_1/(\mathcal{T}_1 \oplus \langle \beta_1 \rangle),\]

hence if we write \(H_1(X, \mathbb{Z}) = \text{Tors}(H_1(X, \mathbb{Z})) \oplus H_1(X, \mathbb{Z})_{\text{free}},\)

\[H_1(X, \mathbb{Z})_{\text{free}}/\Lambda_1 = \Gamma^{ab}_{\text{free}}/\Lambda_1 \cong \Lambda^* \oplus (\mathbb{Z}/m).\]

Hence the torsion group of \(\Lambda_G\) contains \([\Lambda_2/(\alpha_2 - \text{Id})(\Lambda'_2)],\) which is therefore contained in the torsion group of \(\Gamma^{ab};\) the latter cannot however be larger since the quotient \(H_1(X, \mathbb{Z})/(\Lambda_1 \oplus [\Lambda_2/(\alpha_2 - \text{Id})(\Lambda'_2)]) \cong \Lambda^* \oplus (\mathbb{Z}/m) = H_1(X, \mathbb{Z})_{\text{free}}/\Lambda_1.\)

\[\square\]

**Remark 2.3.** In general the surjective map \(\Lambda_G \to \Lambda/(\Lambda \cap [\Gamma, \Gamma])\) is not injective: for instance, in the case of the Hyperelliptic threefold with \(G = D_4,\) it has a kernel \(\cong \mathbb{Z}/2.\)
Lemma 2.4. If $X$ is a Bagnera-De Franchis Manifold, then we have an exact sequence

\[ \ast \ast \] \[ 0 \to H^1(G, (\Lambda^\vee)) \to H^2(X, \mathbb{Z}) \to \ker[H^2(T, \mathbb{Z})^G \to H^2(G, (\Lambda^\vee))] \to 0. \]

In particular, the torsion group of $H^2(X, \mathbb{Z})$ is the group $H^1(G, (\Lambda^\vee))$, which is an $m$-torsion group.

Proof. The first assertion follows from the first spectral sequence, since $H^3(G, \mathbb{Z}) = H^2(G, \mathbb{C}^*) = 0$.

For the second assertion, we notice that the third term in the exact sequence is contained in $H^2(T, \mathbb{Z})$, hence it is torsion free.

Observe moreover that, $G$ being cyclic and generated by $g$, a cocycle in $H^1(G, (\Lambda^\vee))$ is fully determined by the element $f(g) \in \Lambda^\vee$, actually $f(g) \in \ker(1 + g + g^2 \cdots + g^{m-1})$.

Whereas the coboundaries are the elements inside $\text{Im}(1 - g^2)$. Hence $x := f(g)$ is cohomologous to $gx$, which is cohomologous to $g^2x$, hence $mx$ is cohomologous to zero as we wanted to show.

We summarize our results in the following Theorem, part (1') therein is due to Andreas Demleitner.

Theorem 2.5. Let $X = T/G$ be a hyperelliptic manifold. The following statements hold:

1. The sequence of $G$-linearized line bundles on $X$ is

\[ 0 \to H^1(G, \mathbb{C}^*) = \text{Hom}(G, \mathbb{C}^*) \to H^1(T, \mathcal{O}_T^\vee) = \text{Pic}(X) \to H^1(\Gamma, \mathcal{O}_V^\vee)^G = \text{Pic}(T)^G \to H^2(G, \mathbb{C}^*), \]

and a line bundle $L \in \text{Pic}(T)^G$ admits a linearization if and only if its class in $H^2(G, \mathbb{C}^*)$ is zero. Moreover, two linearizations on $L$ differ by a character $\chi: G \to \mathbb{C}^*$.

2. $H^1(X, \mathbb{Z})$ sits in an exact sequence

\[ 0 \to H^1(X, \mathbb{Z}) \to H^1(T, \mathbb{Z})^G \to \text{Im}(\psi) \to 0, \]

\[ \psi: H^1(T, \mathbb{Z})^G \to H^2(X, \mathbb{Z}). \]

If $X = T/G$, $T = (V_1 \oplus V_2)/\Lambda$ is a Bagnera-De Franchis manifold with group $G \cong \mathbb{Z}/m$, then statements (1) and (2) specialize to

(1') Every line bundle $L \in \text{Pic}(T)^G$ admits a $G$-linearization, and two linearizations on $L$ differ by an $m$-th root of unity. Moreover, if $\mathcal{L} \in \text{Pic}(X)$ pulls back to a line bundle $L \in \text{Pic}(T)^G$ with Appell-Humbert data $(H, \rho)$, a cocycle $[f] \in H^1(\Gamma, \mathcal{O}_V^\vee)$ corresponding to $\mathcal{L}$ is determined by

\[ f_\gamma(z) := \rho(\lambda)^{1/m} \exp(\frac{\pi}{m}H(z, \lambda) + \frac{\pi}{2m^2}H(\lambda, \lambda)), \]

where

- $\gamma$ is a lift of a generator $g$ of $G$, which acts on $V_1 \oplus V_2$ as $(z_1, z_2) \mapsto (z_1 + b_1, \alpha z_2 + b_2)$.
we can choose $f$ to give an element $[f]$ a Bagnera-De Franchis manifold and $L$.

It remains to check that the definition of $\lambda \in \Lambda$, $\rho(\lambda)^{1/m}$ is an $m$-th root of $\rho(\lambda)$ in $\mathbb{C}$.

(2′) The map $\psi$ is onto, in particular, we have exact sequences

$$0 \to H^1(X, \mathbb{Z}) \to H^1(T, \mathbb{Z})^G \to H^2(G, \mathbb{Z}) \to 0,$$

and

$$0 \to H^1(G, (L')) \to H^2(X, \mathbb{Z}) \to \ker[H^2(T, \mathbb{Z})^G \to H^2(G, (L'))] \to 0,$$

with $H^1(G, (L')) = \text{Tors}(H^2(X, \mathbb{Z}))$.

(3′) The first Chern class map $c_1$ applied to the exact sequence

$$0 \to H^1(G, \mathbb{C}^*) \to \text{Pic}(X) \to \text{Pic}(T)^G \to H^2(G, \mathbb{C}^*)$$

sends $H^1(G, \mathbb{C}^*)$ to $\text{Pic}^0(X)$, and sends $(T^*)^G = \text{Pic}^0(T)^G$ onto $H^1(G, \Lambda^c)$.

Proof. The last part of assertion (1′) is not proven yet. If $X = T/G$ is a Bagnera-De Franchis manifold and $L$ is a line bundle on $X$, we aim to give an element $[f] \in H^1(\Gamma, \mathcal{O}_T^*)$ corresponding to $L$. If $\lambda' \in \Lambda \subset \Gamma$, we can choose $f_{\lambda'}(z)$ to be in Appell-Humbert normal form,

$$f_{\lambda'}(z) = \rho(\lambda') \exp(\pi H(z, \lambda') + \frac{\pi}{2} H(\lambda', \lambda')), $$

as already noted. Since $G$ is cyclic, it remains to determine an element $f_{\lambda}(z)$, which satisfies the cocycle condition

$$f_{\lambda}(z) = f_{\gamma}((\gamma^{m-1}z) \cdot \cdots \cdot f_{\gamma}(z), \quad \lambda := \gamma^m = mb_1 \in \Lambda \cap V_1.$$

It remains to check that the definition of $f_{\gamma}$ in the statement of the Theorem satisfies this condition. We calculate

$$f_{\gamma}((\gamma^{m-1}z) \cdot \cdots \cdot f_{\gamma}(z) = \rho(\lambda) \exp(\frac{\pi}{m} H((\gamma^{m-1}+\gamma+1)z, \lambda) + \frac{\pi}{2m} H(\lambda, \lambda)).$$

Writing $z = (z_1, z_2), z_j \in V_j$, we obtain that

$$((\gamma^{m-1} + \gamma + 1)z = (mz_1 + \frac{m(m-1)}{2} b_1, b_2'), \text{ for some } b_2' \in V_2.$$

We note that, since $H$ is $G$-invariant, we obtain that $H(w_2, w_1) = 0$ for any $w_j \in V_j$. This implies, together with $\lambda = mb_1 \in V_1$, that

$$\frac{\pi}{m} H((\gamma^{m-1} + \gamma + 1)z, \lambda) = \pi H(z, \lambda) + \frac{\pi}{2m} H(b_1, \lambda),$$

and finally the desired

$$f_{\gamma}((\gamma^{m-1}z) \cdot \cdots \cdot f_{\gamma}(z) = f_{\lambda}(z).$$

For the other yet unproven assertion (3′), we use the exact sequence $(\text{Pic})$ stating that we have a surjection $(T^*)^G \to H^1(G, \Lambda^c)$. In particular, $H^1(G, \mathbb{C}^*)$ maps to zero in $(T^*)^G$, hence has trivial integral Chern class. Indeed $H^1(G, \mathbb{C}^*)$ maps to $H^2(G, \mathbb{Z})$ which maps to zero inside $H^2(X, \mathbb{Z})$. 
3. Tangent bundles of Bagnera de Franchis manifolds and counterexamples to the Severi conjecture

Theorem 3.1. The tangent bundle of a Bagnera de Franchis manifold $X = T/G$ is topologically trivial, equivalently all its integral Chern classes $c_i(X) = 0 \in H^*(X, \mathbb{Z})$.

Proof. It is sufficient to show that the Chern classes vanish, since then the classifying map to the infinite Grassmannian is homotopic to a constant, hence $\Theta_X$ is trivial.

Since $G$ is Abelian, if $T = V/\Lambda$, the vector space $V$ splits as a direct sum of character spaces $V = \bigoplus_{\chi \in G^*} V_{\chi}$, and $\Theta_X$ is a direct sum of line bundles $L_1, \ldots, L_n$ corresponding to some character $\chi_i \in G^* = H^1(G, \mathbb{C}^*)$.

We saw in (3') of Theorem 2.5 that $c_1(L_i) = 0$, hence

$$c(\Theta_X) = \prod_i (1 + c_1(L_i)) = 1,$$

and all Chern classes of $X$ vanish.

The previous theorem gives a counterexample of the topological version of Severi’s conjecture, but we can give a stronger counterexample, where all Chern classes are zero in the Chow ring and not only in the cohomology ring.

Theorem 3.2. There are Bagnera de Franchis manifolds $X = T/G$, which are not complex tori, such that all their Chern classes $c_i(X)$ are rationally equivalent to zero (zero in the Chow group).

Proof. Consider a product of three elliptic curves, $T = E_1 \times E_2 \times E_3$ and the affine action of $G = \mathbb{Z}/2$ on $T$ such that

$$g(a_1, a_2, a_3) = (a_1 + \eta, -a_2, -a_3),$$

where $\eta$ is a nontrivial point of order 2.

Then $\Theta_X \cong \mathcal{O}_X \oplus L \oplus L$, where $L$ is the nontrivial bundle of 2-torsion corresponding to the unique embedding $G \to \mathbb{C}^*$.

Since as we saw $H^1(G, \mathbb{C}^*)$ maps to zero in $H^2(X, \mathbb{Z})$, $L$ is an element of $\text{Pic}^0(X) \cong E_1/\langle \eta \rangle$, hence $L$ pulls back from the elliptic curve $\text{Alb}(X)$.

$L = \pi^*(\mathcal{L})$.

Then, since $2c_1(L) = 0$,

$$c(\Theta_X) = (1 + c_1(L))^2 = 1 + L^2 = 1 + \pi^*(\mathcal{L})^2 = 1$$

in the Chow ring of $X$, since $(\mathcal{L})^2$ in the Chow ring of $\text{Alb}(X)$. 

□
We shall later recall that any compact Kähler manifold $X$ with $c_i(X) = 0 \in H^i(X, \mathbb{Q})$ is a Hyperelliptic manifold. However

**Theorem 3.3.** There are Hyperelliptic manifolds $X = T/G$ such that not all their integral Chern classes $c_i(X) \in H^i(X, \mathbb{Z})$ are equal to zero.

**Proof.** Consider the product of two elliptic curves and an Abelian surface, $T = E_1 \times E_2 \times A_3$ and the affine action of $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ such that

$g_{12}(a_1, a_2, a_3) = (-a_1 + \eta_1, -a_2, a_3 + \eta_3),$

$g_{13}(a_1, a_2, a_3) = (-a_1, a_2 + \eta_2, -a_3 + \eta_3),$

$g_{23}(a_1, a_2, a_3) = (a_1 + \eta_1, -a_2 + \eta_2, -a_3),$

where $\eta_1, \eta_2, \eta_3$ are respective nontrivial points of order 2. Then $\Theta_X \cong L_1 \oplus L_2 \oplus L_3 \oplus L_3$, where the $L_i$ are nontrivial bundles of 2-torsion, corresponding to the three nontrivial characters $G \to \mathbb{C}^*$. In this case $H^1(T, \mathbb{Z})^G = 0$, hence Pic$^0(X)$ is trivial. Hence the three Chern classes $c_1(L_i)$ are the three nontrivial elements of $H^2(G, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ (as $\psi = 0$).

Hence

$c(\Theta_X) = (1 + c_1(L_1))(1 + c_1(L_2))(1 + c_1(L_3))^2,$

and $c_1(\Theta_X) = c_1(L_3) \neq 0 \in H^2(X, \mathbb{Z}).$

The above theorem raises the interesting question

**Question 3.4.** Is there an easy classification of the Hyperelliptic manifolds $X = T/G$ whose integral Chern classes $c_i(X) \in H^i(X, \mathbb{Z})$ are all equal to zero? Equivalently, of all compact Kähler manifolds whose tangent bundle is topologically trivial?

4. **Surfaces with second Chern class equal to zero, and a Chern class characterization of Hyperelliptic and Abelian surfaces**

The classification theorem by Castelnuovo and Enriques [CastEn15], was extended by Kodaira [Kod68], and a crucial result is the following (see [Bea78], Chapter 6, Theorem VI.13, Theorem 1.4 of [CatLi19], and especially the crucial Theorem of [Cat22]):

**Theorem 4.1.** Let $S$ be a compact smooth complex surface, minimal in the strong sense that $K_S$ is nef, and such that $\chi(S) = 0$, which implies that $K_S^2 = 0$ and the topological Euler number $e(S) = c_2(S) = 0$. Equivalently, assume that $K_S$ is nef, and that $e(S) = c_2(S) = 0$. Then either

1) $p_g(S) = 1$, hence $q(S) = 2$, and $S$ is a complex torus $A$ (a hyperelliptic surface of grade 1), or
2) \( p_g(S) = p, q(S) = p + 1 \) and \( S \) is isogenous to an elliptic product, i.e. \( S \) is the quotient \((C_1 \times C_2)/G\) of a product of curves of genera

\[
g_1 := g(C_1) = 1, g_2 := g(C_2) \geq 1,
\]

by a free action of a finite group of product type \((G \text{ acts faithfully on } C_1, C_2 \text{ and we take the diagonal action } g(x, y) := (gx, gy))\), and such that moreover if we denote by \( g'_j = g(C_j/G) \), then \( g'_1 + g'_2 = p + 1 \).

Case 2) bifurcates into two subcases:

(2.1,p) \( g'_1 = 1 \) (hence \( G \text{ acts on } C_1 \text{ by translations}, C_2/G \) has genus \( p \)), and we assume, for \( p = 1 \), that \( g_2 \geq 2 \); or

(2.0,p) \( g'_1 = 0 \) (hence \( C_1/G \cong \mathbb{P}^1 \)), \( C_2/G \) has genus \( p + 1 = q(S) \), and we assume, for \( p = 0 \), that \( g_2 \geq 2 \); here the image of Albanese map \( \alpha : S \to \text{Alb}(S) \) equals \( C_2/G \subset \text{Alb}(S) \).

(2.1,0) with \( g_2 = 1 \) is the case where \( S \) is a properly hyperelliptic (bien elliptic) surface (a hyperelliptic surface of grade \( \geq 2 \)):

\[
S = (E_1 \times C_2)/G, \text{ where } E_1, C_2 \text{ are elliptic curves, and } G \text{ acts via an action of product type, such that } G \text{ acts on } E_1 \text{ via translations, and faithfully on } C_2 \text{ with } C_2/G \cong \mathbb{P}^1.
\]

In this case all the fibres of the Albanese map are isomorphic to \( C_2 \), \( P_12(S) = 1 \) and \( S \) admits also an elliptic fibration \( \psi : S \to C_2/G \cong \mathbb{P}^1 \).

In the other cases (2.0,p), (2.1,p), for \( p \geq 1 \), (2.1,0) with \( g_2 \geq 2 \), \( S \) is isogenous to a higher genus elliptic product, this means that \( C_2 \) has genus \( g_2 \geq 2 \). Here \( S \) is properly elliptic and \( P_12(S) \geq 2 \).

The cases are distinguished mainly by the geometric genus \( p_g(S) = p \).

In the case of the torus and of the hyperelliptic surfaces \( K_S \) is numerically equivalent to zero, whereas in the other cases \( K_S \) is not numerically equivalent to zero.

Moreover, the three cases are also distinguished (notice that \( c_1(S) \) is the class of the divisor \(-K_S\)) by

- \( K_S = 0 \in \text{Pic}(S) \) for case 1) of a complex torus,
- \( c_1(S) = 0 \in H^2(S,\mathbb{Q}) \) but \( K_S \neq 0 \in \text{Pic}(S) \) in the case of properly hyperelliptic surfaces,
- \( c_1(S) \neq 0 \in H^2(S,\mathbb{Q}) \) in the other case where \( S \) is isogenous to a higher genus elliptic product.

Proof. The part which is not contained in the cited sources concerns how to distinguish the several cases.

Everything is straightforward, except the assertion that for hyperelliptic surfaces (case (2.1,0) with \( g_1 = 1 \)) the first integral Chern class is zero: but this is a special case of our result on Bagnera de Franchis manifolds, Theorem 3.1. □

The following can be readily checked:
Proposition 4.2. In case 1) $A$ acts transitively and freely on $A$, so $\text{Aut}^0(S)$ has dimension 2.

In case (2.1,p), $\text{Aut}^0(S)$ has dimension 1: $C_1 = E_1$ acts on $S$, transitively on the orbit closures, but with stabilizer $H \subset G \subset E_1$ for the classes of points $(x, y) \in E_1 \times C_2$ such that $Hy = y$.

Finally, in case (2.0,p), $\text{Aut}^0(S)$ has dimension 0: there is no action of $C_1$ on $S$, but the general fibres of the Albanese map $\alpha: S \to C_2/G$ are isomorphic to $C_1$ (a finite number shall only be isogenous to $C_1$).

With a weaker notion of minimality we have a counterexample to the previous Theorem 4.1, as we shall now see.

Proposition 4.3. If the smooth surface satisfies $c_1^2 = c_2 = 0$, that is $K^2_S = e(S) = 0$, then $S$ is minimal.

Proof. By the Noether’s formula, from $K^2_S = 0$ and $e(S) = c_2(S) = 0$ follows $\chi(O_S) = 0$.

And $\chi(O_S)$ is a birational invariant.

If $S$ is not minimal, then $S$ is the blow-up of a minimal surface $S'$ with $e(S') < 0$. By Castelnuovo’s theorem $S'$ is a ruled surface with $q(S') \geq 2$. But then $0 > \chi(O_{S'}) = \chi(O_S) = 0$, a contradiction.

□

Corollary 4.1. Consider the minimal surfaces $S$ with Chern numbers $c_1^2 = c_2 = 0$, that is with $K^2_S = e(S) = 0$.

If $K_S$ is nef, $S$ is isogenous to a product $Y \times A$, with a torus $A$ of dimension $\geq 1$.

If $K_S$ is not nef, then $S$ is a $\mathbb{P}^1$-bundle over an elliptic curve.

Proof. By Theorem 4.1 there remains only to consider the case where $K_S$ is not nef, hence $S$ is ruled. Therefore, the case of $S = \mathbb{P}^2$ yielding $e(S) = 3$, we have a $\mathbb{P}^1$-bundle over a curve $C$. In this case, since $0 = e(S) = 4(1 - g(C))$, we get that $C$ has genus $g(C) = 1$.

□

Remark 4.4. If we take an elliptic curve $B$, and a vector bundle $V$ of rank 2 on $B$, say $V = L \oplus M$, where $\text{deg}(L) = \text{deg}(M)$, then $\mathbb{P}(V)$ is minimal, but the group of automorphisms of $\mathbb{P}(V)$ consist of $\mathbb{C}^*$ for general choice of $L, M$ (see [Mar71], also Theorem 7.3 of [CatLiu21]). Hence the action is not transitive on the orbit closures, because the two sections of $\mathbb{P}(V)$ are left invariant by the automorphism group.

For completeness we show that:

Proposition 4.5. Surfaces in the class (2.0,0) do exist.

Proof. Let $G := (\mathbb{Z}/2)^3$, and make it first act on an elliptic curve $C_1$ as the group of transformations

$$z \mapsto \pm z + \eta, \ 2\eta = 0,$$
so that $G$ has generators $\eta_1, \eta_2, \epsilon$, where $\epsilon(z) = -z$.
To get a second action on $C_2$ such that $C_2/G =: E_2$ is an elliptic curve, we take $E_2$ to be an elliptic curve, $\mathcal{B} = \{x_1, x_2\}$ a branch set, so that
\[
\pi_1(E_2 \setminus \mathcal{B}) = \langle \alpha, \beta, \gamma_1, \gamma_2 | \gamma_1 \gamma_2 = [\alpha, \beta] \rangle,
\]
hence $H_1(E_2 \setminus \mathcal{B}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma_1$.
Define $\mu : H_1(E_2 \setminus \mathcal{B}) \to G$ by:
\[
\mu(\alpha) = \epsilon, \ \mu(\beta) = \eta_1, \ \mu(\gamma_1) = \eta_2 \Rightarrow \mu(\gamma_2) = \eta_2.
\]
We want to prove that the product action of $G$ on $C_1 \times C_2$ is free.
To this purpose we observe that $\eta_2$ has eight fixed points on $C_2$, and, since $C_2$ has genus 5, $E'_2 := C_2/\langle \eta_2 \rangle$ is an elliptic curve, so that $E'_2 \to E_2$ is étale, that is, $G/\langle \eta_2 \rangle$ acts freely on $E'_2$. The conclusion is that the only element acting on $C_2$ with fixed points is $\eta_2$; since $\eta_2$ acts freely on $C_1$, the product action is free.

We end this section observing that if $S$ is a minimal surface with $e(S) = 0$, then either $S$ is a $\mathbb{P}^1$ bundle over an elliptic curve, or $K_S$ is nef. In the latter case it must be $K_S^2 = 0$, since for $K_S^2 < 0$ $S$ is ruled, and $K_S$ is not nef, and for $K_S^2 > 0$ $S$ is of general type, and then $e(S) > 0$.
The following is then the correction of the theorem of Dantoni [Dant 43]; it is amended, since we show that if $S$ is not a torus the surface is elliptic in the weak sense that it has a rational map with fibres elliptic curves, and moreover, we see that Dantoni’s assertion that a torus of dimension at least 1 acts on $S$, is not true in general, as we saw in Proposition 4.2 and in the previous remark 4.3.

**Theorem 4.6.** A surface $S$ with $e(S) = 0$ is either the blow up of a $\mathbb{P}^1$-bundle over a curve of genus at least 2, or it is minimal, and then it is either a complex torus, or a hyperelliptic surface, or it is isogenous to a higher genus elliptic product, or it is a $\mathbb{P}^1$-bundle over an elliptic curve.
In the last three cases $S$ contains a 1-dimensional family of isomorphic elliptic curves whose union is dense in $S$.
If the canonical divisor is numerically trivial, then $S$ is either a complex torus, or a hyperelliptic surface.

5. A Chern classes characterization of Hyperelliptic Manifolds as Manifolds with vanishing Chern classes
A complex torus $T$ has holomorphically trivial tangent bundle and conversely a compact Kähler manifold $X$ with holomorphically trivial tangent (or cotangent) bundle is a complex torus.
Because, if $\Omega_X^1 \cong \mathcal{O}_X^n$, then $H^0(\Omega_X^1) \cong \mathbb{C}^n$, hence the Albanese Variety has dimension $n = \dim(X)$ and the Albanese map $\alpha : X \to Alb(X)$ is a finite unramified covering.
More generally, see for instance [AMN12, Prop. 8.1], a compact complex manifold $X$ with $\Theta_X \cong \mathcal{O}_X^n$ is a complex torus. In particular, all the Chern classes of $T$, $c_i(T) \in H^{2i}(T, \mathbb{Z})$ are trivial. If we take a hyperelliptic manifold $X = T/G$, then since $H^j(X, \mathbb{Q}) = H^j(T, \mathbb{Q})^G$, and $c_i(X)$ pulls back to $c_i(T)$, it follows that the rational Chern classes are trivial, equivalently, $c_i(X) = 0 \in H^{2i}(X, \mathbb{R})$. In particular all the Chern numbers of $X$ are equal to 0.

Already in the surface case the condition that all the Chern numbers are zero (this means $K^2_X = e(X) = 0$) is not sufficient, as we have seen, to imply that $X$ is a complex torus or a hyperelliptic surface (in the old notation a torus was also called a hyperelliptic surface, of grade, or rank, 1). But it implies (see Theorem 4.6) that the surface is either a torus or it is birationally (but not necessarily biregularly) covered by a 1-dimensional family of isomorphic elliptic curves.

5.1. **Baldassarri’s first problem.** Baldassarri [Bald56] was trying to extend Dantoni’s theorem to varieties of higher dimension, following a proposal made by Severi.

At that time, even if Baldassarri in his Ergebnisse book [Bald56book] (the book was rather influential, it was for instance translated in Russian by Manin) was exposing the new methods in the theory of Algebraic Varieties, the concept of canonical systems of all dimension was rather based on more geometric approaches, the latest of which was the rather clear approach by Beniamino Segre [Seg52], [Seg54] using the embedding covariants for the case of the diagonal $\Delta_X \subset X \times X$. The approach of Segre finally related the canonical systems of all intermediate dimensions [Roth56] to the theory of Chern classes, see [Nak55], and since, see [Ful84], the total Segre classes is the inverse of the total Chern class.

Baldassarri was of course aware of the theory of Chern classes (but respectful for Severi’s terminology) and we can reformulate the first basic questions he was approaching as follows:

**Question 5.1.** (I) How to characterize a variety with all the Chern classes of degree equal to zero? Equivalently, with all the Chern classes trivial in rational homology?

(II) How to characterize a variety with all the last $k$ Chern classes of degree zero?

The new differential theoretic methods turned out in the 50’s to be quite powerful, starting from Kodaira’s vanishing theorem [Kod53], which resolved the long standing question concerning the regularity of the adjoint, and later brought to the solution to other classical problems [Kod54a, Kod54b, Hir56].

---

1. An approach also followed later by Grothendieck.
In particular, concerning Question (I) above, Apte proved (see also [Kob87], page 116) that a compact Kähler-Einstein manifold such that \(c_1(X), c_2(X) = 0 \in H^*(X, \mathbb{R})\) is flat, that is, it is a hyperelliptic manifold.

In 1978 Yau (he obtained the Fields medal for this result) showed that a compact Kähler manifold with \(c_1(X) = 0 \in H^2(X, \mathbb{R})\) admits a Kähler-Einstein metric, that is a metric such that its Ricci form is identically zero. Hence the following theorem was proven:

**Theorem 5.2.** A compact Kähler manifold \(X\) such that \(c_1(X) = 0, c_2(X) = 0, \text{ in } H^*(X, \mathbb{R})\), is a hyperelliptic manifold.

The above result was amply generalized, to the case where we have a singular quotient \(T/G\), because the higher dimensional classification theory requires the study of singular varieties in order to obtain a good theory of minimal models.

The latest touch in this direction was obtained in a paper by Steven Lu-Behrouz Taji [Lu-Ta18], following important results by Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau on stable vector bundles on compact Kähler manifolds having trivial first two Chern classes.

But in this article we shall mainly deal with smooth varieties and manifolds, since the situation is already complicated enough even under this assumption.

We have shown in a previous section that, against Severi’s expectation, it is not true that a projective variety is an Abelian variety (respectively: a compact Kähler manifold \(X\) is a torus) if and only if all its integral Chern classes \(c_1(X), \ldots, c_n(X) = 0 \in H^*(X, \mathbb{Z})\).

Concerning Question II), and the forthcoming Questions III) and IV), we try to briefly explain in the next section a definition due to Roth [Roth54].

6. **Pseudo Abelian Varieties and Varieties Isogenous to a \(k\)-Torus-Product**

Leonard Roth [Roth54], [Roth55] defined the Pseudo-Abelian varieties as follows:

**Definition 6.1.** (Roth)
A projective variety \(X\) of dimension \(n\) is said to be Pseudo Abelian of order \(k\) if
i) \(Aut^0(X)\) contains a complex torus \(T\), of maximal dimension = \(k\), with the property that its orbits are all of dimension \(k\).

Recall in fact the following theorem by Fujiki and Lieberman [Fuj78], [Li78]
Theorem 6.1. If \( X \) is a compact Kähler manifold, then there is an exact sequence of groups
\[
1 \to L \to Aut^0(X) \to T_X \to i
\]
where \( L \) is a linear algebraic group, and \( T_X \) is a complex torus.
If \( X \) is not uniruled, then \( L = \{1\} \).

We explain now how to derive from Roth’s definition a simpler one.
We have an action \( T \times X \to X \), and we denote by \( T_x \) the orbit of \( x \), image of \( T \times \{x\} \).
Hence the orbits \( T_x \) of \( T \) give a variety \( V \) of dimension \( n - k \) in the Hilbert scheme \( \mathcal{H} \) (Douady space) of \( X \), and the restriction of the universal family to \( V \), \( \phi : U \to V \), yields \( U \) which maps isomorphically to \( X \). In fact, \( dim(U) = n = dim(X) \), and the identity of \( X \) factors as:
\[
X \to U \to V \times X \to X,
\]
where \( x \in X \mapsto (T_x, x) \).
Hence we may write: \( \phi : X \to V \), and since the action of \( T \),
\[
a : T \times X \to X
\]
commutes with the projection over \( V \), and is effective, then the general fibre of \( \phi \) is isomorphic to \( T \).
The other fibres are instead of the form \( T/G' \), where \( G' \) is a finite subgroup of \( T \). Since the general fibres are isomorphic, over an open set \( V' \) of \( V \) we have a holomorphic bundle (for instance, as a consequence of Kuranishi’s theorem).
Because of the action of \( T \) on the fibres the monodromy of the bundle centralizes the group of translations hence the monodromy transformations consist of translations, and we have a principal bundle.
If \( X \) is projective, then the monodromy is finite, hence we get a finite group \( G \subset T \).
The fibration is isotrivial, hence there exists a finite Galois base change \( f : Y \to V \) with group \( G \) such that the fibre product is birational to a product
\[
Y \times_V X \sim Y \times T.
\]
At each point of \( V \) the local monodromy \( G' \) is a subgroup of \( G \), hence the fibre product \( Y \times_V X \) yields an unramified covering of \( X \) (since \( G' \) yields an unramified covering of the corresponding fibre).
Hence the fibre product \( Y \times_V X \) is smooth, is birational to \( Y \times T \), and all the fibres are isomorphic to \( T \): therefore, \( Y \times_V X \cong Y \times T \), compatibly with the projection onto \( Y \).
This motivates an equivalent definition, and some related definitions:

Definition 6.2. (I) A complex manifold \( X \) of dimension \( n \) is said to be a \( k \)-Pseudo-Torus (or Pseudo-Torus of order \( k \)) if there is a torus \( T \) of dimension \( k \), a manifold \( Y \), and a finite Abelian group \( G \) acting
on $T$ faithfully via translations, acting faithfully on $Y$, such that the quotient of the product action is isomorphic to $X$:

$$X = (Y \times T)/G.$$ 

(II) If moreover $X$ is projective, we shall say that $X$ is a Pseudo-Abelian Variety of order $k$ in the strong sense.

(III) A compact Kähler Manifold $X$ is said to be Seifert fibred, cf. [Li78], if there is a finite abelian unramified covering $Z \to X$ with Abelian Galois group $G$ (hence $X = Z/G$) such that $Z$ is a principal bundle $Z \to Y$, with fibre a (positive dimensional) complex torus $T$, and where the action of $G$ commutes with the action of $T$ on $Z$.

(IV) A compact complex manifold $X$ is called ([AMN12]) a suspension over a complex torus $T = \mathbb{C}^k/\Lambda$ if there is a compact complex manifold $Y$ and a homomorphism $\rho : \Lambda \to Aut(Y)$ such that

$$X = (\mathbb{C}^k \times Y)/\Lambda, \lambda(z, y) := (z + \lambda, \rho(\lambda)(y)).$$

Remark 6.2. (a) The reason for definition (III) is that if $X$ is only a cKM, and it has an action of a complex torus $T$, with all the orbits tori of the same dimension, then the global monodromy is not necessarily finite. But the local monodromies around the points corresponding to multiple fibres are finite subgroups of $T$, and since we have a finite number of them, the local monodromies generate a finite subgroup $G$ of $T$. Associated to this subgroup there is an covering $Y \to V$ which is a principal $T$-bundle.

(b) In the case of definition (IV) of a suspension over a torus, $\mathbb{C}^k$ acts on $X$ via translations on the first summand, but in general there is no complex torus acting on $X$.

Indeed the $\mathbb{C}^k$-orbits are the projections $\pi(\mathbb{C}^k \times \{y\})$, which are isomorphic to $\mathbb{C}^k/\text{Stab}_y$, hence they do not need be compact.

The suspension over a torus is a Pseudo-torus if and only if the subgroup $\text{Im}(\rho) \subset Aut(Y)$ is finite. Equivalently, if and only if all orbits are tori (which amounts to $\text{Stab}_y$ being a subgroup of finite index in $\Lambda$ for all $y \in Y$): since then $\text{Ker}(\rho)$ has finite index.

In particular the suspension is Pseudo-Abelian if and only if it is a Pseudo-torus.

(c) Observe that for such a suspension we have a splitting of the tangent bundle

$$\Theta_X \cong \mathcal{O}_X^k \oplus \mathcal{F},$$

hence also of the cotangent bundle.

In [AMN12] Example 2.4, page 1005, it is observed that a suspension $X$ as above is Kähler if and only if $Y$ is Kähler and a finite index subgroup $\Lambda' \subset \Lambda$ maps to $Aut^0(Y)$. 

In this case, taking $G := \Lambda/\Lambda'$, there is a Galois covering $Z$ of $X$ with group $G$. If moreover $\text{Aut}^0(Y)$ is trivial, then $Z = T' \times Y$ and we have a pseudo-torus.

If $\text{Aut}^0(Y)$ is non-trivial, one cannot conclude that $Z$ is a principal torus bundle, since the action of $\Lambda'$ on $Y$ need not be properly discontinuous.

(d) If $X$ is Seifert fibred, then $X = Z/G$, and we have an exact sequence corresponding to the principal bundle $Z \to Y$:

$$0 \to \mathcal{O}_Z^k \to \Theta_Z \to F_Z \to 0.$$  

Since the action of $G$ commutes with the action of $T$, the exact sequence descends to $X$, and we have

$$0 \to \mathcal{O}_X^k \to \Theta_X \to F \to 0.$$  

As we shall soon see, the above notions of Seifert-fibred manifold, or of pseudo-torus, and of suspension over a torus are not general enough in order to deal with manifolds with Chern classes of degree zero, hence we give another definition (important for the case of projective varieties):

**Definition 6.3.** A complex manifold $X$ of dimension $n$ is said to be isogenous to a $k$-torus product (MITP of order $k$) if there is a torus $T$ of dimension $k$, a manifold $Y$, and a finite group $G$ acting freely on the product $Y \times T$ such that:

$$X = (Y \times T)/G.$$  

If $X$ is projective, we shall say that it is a Variety Isogenous to a $k$-Torus product.

The following are easy important properties of such Manifolds and Varieties Isogenous to a Torus product.

**Proposition 6.4.** Let the complex Manifold $X$ of dimension $n$ be Isogenous to a $k$-Torus product $Y \times T$ where $\text{dim}(T) = k > 0$.

Then the integral Chern classes $c_i(Y \times T) \in H^*(Y \times T, \mathbb{Z})$ vanish for $i \geq n - k + 1$.

And the rational Chern classes $c_i(X) \in H^*(X, \mathbb{Q})$ vanish for $i \geq n - k + 1$.

Moreover all the Chern numbers of $X$ vanish.

If moreover $X$ is (respectively is isogenous to) a pseudo-torus, or more generally the suspension over a torus, or is Seifert fibred, then all the integral Chern classes $c_i(X) \in H^*(X, \mathbb{Z})$ vanish for $i \geq n - k + 1$ (respectively the corresponding rational Chern classes vanish).

**Proof.** Since the tangent bundle of $Y \times T$ is the direct sum of the pull back of the tangent bundle of $Y$ with the pull back of the tangent bundle of $T$ which is trivial, the total Chern class of $Y \times T$ is the pull back of the total Chern class of $Y$, and this proves the first assertion. The second assertion follows since the Chern classes of $X$ pull back to the Chern classes of the product $Y \times T$, and $H^*(X, \mathbb{Q}) = H^*(Y \times T, \mathbb{Q})^G$. 

The third assertion follows since any isobaric polynomial of weight \( n \) in the Chern classes of \( X \) is a rational multiple of the same isobaric polynomial of weight \( n \) in the Chern classes of \( Y \), and we are assuming \( k > 0 \).

The last assertion follows since the tangent bundle of \( X \) either splits as \( \Theta_X = \mathcal{O}_X^k \oplus \mathcal{F} \), or has a bundle exact sequence

\[
0 \to \mathcal{O}_X^k \to \Theta_X \to \mathcal{F} \to 0.
\]

\[\Box\]

7. Partially framed and co-framed manifolds

We begin with a simple definition yielding complex manifolds with all the last \( k \) integral Chern classes equal to zero.

**Definition 7.1.** A complex manifold \( X \) of dimension \( n \) is said to be \( k \)-tangentially framed, or simply \( k \)-framed, if the holomorphic tangent bundle \( \Theta_X \) admits a trivial subbundle \( \cong \mathcal{O}_X^k \), and where \( k > 0 \) is maximal.

A complex manifold \( X \) of dimension \( n \) is said to be \( k \)-cotangentially framed, or simply \( k \)-co-framed, if the holomorphic cotangent bundle \( \Omega_X^1 \) admits a trivial subbundle \( \cong \mathcal{O}_X^k \), and where \( k > 0 \) is maximal.

In the rest of the section we shall use the results of \([\text{Li78}], [\text{Fuj78}], [\text{AMN12}]\), often for simplicity we might refer to the exposition given in the last paper.

**Proposition 7.2.** Assume that \( X \) is a \( k \)-coframed compact Kähler manifold and let \( W \subset h^0(\Omega_X^1) \) be the corresponding maximal vector subspace consisting entirely of nowhere vanishing holomorphic 1-forms (of dimension \( k \geq 1 \)). Then \( W \) determines an analytically integrable foliation with trivial normal bundle.

i) If moreover the subspace \( W \) corresponds to a quotient torus \( T' \) of \( \text{Alb}(X) = H^0(\Omega_X^1)^\vee/(H_1(X, \mathbb{Z})/\text{Tors}) \), then the foliation is algebraically integrable, consisting of the fibres of \( \Psi : X \to T' \), which is a differentiable fibre bundle.

ii) If we have a splitting \( \Omega_X^1 \cong \mathcal{O}_X^k \oplus \mathcal{F}^\vee \), then \( \Psi \) is a holomorphic fibre bundle with fibre \( Y \).

iii) If moreover \( Y \) has finite automorphism group, then \( X \) is a \( k \)-Pseudo-Torus product \( X = (Y \times T)/G \) with \( \dim(T) = k > 0 \).

**Proof.** A subspace \( W \) which is maximal with the property that all forms \( \omega \in W \setminus \{0\} \) are nowhere vanishing, has a basis \( \omega_1, \ldots, \omega_k \) such that the forms \( \omega_j \) are linearly independent at each point, hence they generate a trivial rank \( k \) subbundle of \( \Omega_X^1 \).

The associated foliation is analytically integrable, because \( X \) is a cKM, and holomorphic 1-forms are closed, hence the distribution induced by \( W \) is integrable, and spans a trivial subbundle of the cotangent bundle.
The foliation on $X$ is induced by a foliation on $Alb(X)$, corresponding to the annihilator of $W$. This foliation is algebraically integrable if, as we assume, it corresponds to the projection onto a quotient torus $T'$. The composed map $\Psi : X \to T'$ has fibres of dimension $k$, which are therefore union of leaves. Since the fibres are smooth, if they are not connected, we would get by the Stein factorization on unramified covering of $T'$, which is again a quotient of $Alb(X)$ by the universal property of the Albanese variety.
Hence we may assume that the fibres of $\Phi$ are connected, and we have a differentiable fibre bundle.
The splitting $\Omega_X^1 \cong \mathcal{O}_X^k \oplus \mathcal{F}$ guarantees that the Kodaira-Spencer map for the family is identically zero, hence by Kuranishi’s theorem we have a holomorphic bundle.
If this is a holomorphic bundle, with fibre $Y$, and $Aut(Y)$ is finite, there is an unramified covering $T \to T'$ with group $G$ such that the pull back is a product. Therefore $X = (T \times Y)/G$, where $G$ acts on $T$ via translations.

**Theorem 7.1.** Let $X$ be a compact Kähler complex Manifold of dimension $n$: then $X$ is the suspension over a torus $T$ with $dim(T) = k > 0$ if and only if there is a $k$-framing yielding a partial tangential splitting

$$\Theta_X \cong \mathcal{O}_X^k \oplus \mathcal{F}.$$ 

A $k$-framing of $X$ yields the structure of a Seifert fibration on $X$ in the case where $h^0(\Theta_X) = k$.
Moreover, a $k$-framed projective manifold $X$ is a suspension over an Abelian Variety and is indeed a Pseudo-Abelian variety if $K_X$ is nef.

**Proof.** We have already seen that for a suspension over a torus we have such a partial tangential splitting.
Conversely, recall that $H^0(\Theta_X)$ is the Lie algebra of the Lie group $Aut(X)$.
The Lie Algebra $H^0(\Theta_X) =: \mathcal{H}_X$ contains the Lie ideal $\mathcal{H}_X^1$ of the vector fields admitting zeros, and there is (see [AMN12], page 1002), a direct sum $\mathcal{H}_X = \mathcal{H}_X^1 \oplus \mathcal{A}$, where $\mathcal{A}$ is a maximal Abelian subalgebra generated by nowhere vanishing vector fields.
By Theorem 3.14 of [Li78], $\mathcal{H}_X^1$ is precisely the subspace of $\mathcal{H}_X$ yielding the zero flow on $Alb(X)$.
If $\mathcal{H}_X^1 = 0$ then trivial subbundle yields $k$ everywhere linearly independent vector fields, which, see [Fuj78], [Li78], and also Theorem 1.2 of [AMN12], generate the action of a $k$-dimensional complex torus $T$ with smooth orbits $T_x$, quotients of $T$ by a finite group $H_x$.
Hence $X$ is Seifert fibred, as we have discussed earlier, cf. Theorem 4.9 by Lieberman [Li78].
If we have a tangential splitting $\Theta_X \cong \mathcal{O}_X^k \oplus \mathcal{F}$, we get a corresponding cotangent splitting,

$$\Omega^1_X \cong \mathcal{O}_X^k \oplus \mathcal{F}^\vee.$$ 

It suffices, by the preceding proposition, to show that the coframing defines a subspace $W \subset H^0(\Omega^1_X)$ corresponding to a quotient torus $T'$ of $\text{Alb}(X)$.

Since every automorphism of $X$ yields an affine action on $\text{Alb}(X)$, the action of the torus $T$, which spans $W^\vee$, yields a subtorus $A$ of $\text{Alb}(X)$. Then we define $T' := \text{Alb}(X)/A$, and we get $\Psi : X \to T'$ which is is a holomorphic bundle, with parallel transport given by the action of $T$: hence $X$ is the suspension over a torus.

Finally, if $X$ is projective, $T'$ is an Abelian variety. We defer the reader to Theorem 0.3 of [AMN12] for the proof of the last assertions, see also [Li77].

\[\square\]

**Remark 7.2.** Theorem 0.3 of [AMN12] proves the following very interesting result: if $X$ is a compact Kähler manifold which is $k$-framed, then $X$ admits a small deformation which is a suspension over a $k$-dimensional torus, and which is a $k$-Pseudo-Torus if $Kod(X) \geq 0$.

If $X$ is projective and $k$-framed, as we saw, they show that $X$ is Pseudo-Abelian.

**Theorem 7.3.** Assume that $X$ is a $k$-coframed compact Kähler manifold, and that $h^0(\Omega^1_X) = k$: then the Albanese map $a_X$ is a differentiable fibre bundle.

If moreover $X$ is projective, $k = n - 1$, and $K_X$ is nef, then $X$ is a pseudo-Abelian variety.

**Proof.** The first assertion follows from i) of Proposition 7.2.

For the second, the fibres of $a_X$ are smooth curves of genus $g \geq 1$. Since the fibration induces a holomorphic map to the Teichmüller space $\mathcal{T}_g$, which is biholomorphic to a bounded domain, this map is constant and we have a holomorphic fibre bundle.

Since $X$ is projective, the monodromy is finite, so there exists a finite unramified map $A' \to A : +\text{Alb}(X)$ such that the pull back is a product, hence $X$ is an Abelian variety.

\[\square\]

**Example 7.3.** This example concerns $k$-coframed varieties.

(1) A smooth fibration onto an Abelian variety need not be isotrivial, and not even a holomorphic bundle.

Let in fact $A$ be an elliptic curve and assume that we have an embedding $j : A \to S$, where $S$ is a surface of general type, say with $K_S$ ample.

Let $\Gamma \subset A \times S$ be the graph of $j$, and take $X$ to be the blow up of $A \times S$ with centre the smooth curve $\Gamma$. 

\[\square\]
Then the differentiable fibre bundle $f : X \to A$ is not a holomorphic bundle, since $Aut(S)$ is finite, and the pairs $(S, p_1)$ and $(S, P_2)$ are not isomorphic for general $P_1, P_2 \in j(A)$.

(2) In this case $K_X$ is not nef. Else, one may ask whether the fibration is isotrivial. This has been shown by Kovacs in the case where the fibres have ample canonical divisor [Kov97].

We pose now the following general questions:

**Question 7.4.** (a) Assume that $X$ is a $k$-coframed projective manifold. Then there exists a coframing $V$ (a subbundle $V \cong \mathcal{O}_X^k$ of $\Omega^1_X$) such that i) of Proposition 7.2 holds: namely the subspace $\tilde{W} = H^0(V) \subset H^0(\Omega^1_X)$ defines a quotient Abelian variety $A'$.

(b) If $X$ is projective with $K_X$ nef and it admits a fibration $f : X \to A'$ onto an Abelian variety $A'$ with all fibres smooth, then $f$ is a holomorphic bundle hence $X$ is a pseudo-Abelian variety.

As already discussed, question (b) is motivated by the result of [Kov97], and it fits into a pattern of conjectures of classification theory.

**Remark 7.4.** Question (a) asks, in the case where $X$ is projective, whether the subspace $W \subset H^0(\Omega^1_X)$ corresponds to an Abelian subvariety of $Alb(X)$, equivalently, whether $\Lambda^{2k}(W \oplus \tilde{W})$ is a point defined over $\mathbb{Q}$ in the Grassmann manifold $Grass(2k, H^1(X, \mathbb{C})) \subset \mathbb{P}(\Lambda^{2k}(H^1(X, \mathbb{C}))).$

One may conjecture that this is true if $V$ is geometrically defined, that is, it is unique and invariant by all automorphisms in $Aut(\mathbb{Q})$. Here it is important that $X$ is projective, and one may reduce to the case where $X$ is defined over an algebraic extension of $\mathbb{Q}$.

I initially thought that question (a) has a positive answer, trying to use (see [Miya87]) the generic semipositivity of $\Omega^1_X$ for a non uniruled variety, and its Harder-Narasimhan filtration to define a geometrically unique coframing $V$. But Deligne spotted a trivial mistake.

8. **Mathematical and Historical comments on Baldassarri’s paper [Bald56] and the questions it suggests**

As already mentioned, the article by Baldassarri dealt with the question of characterizing all the projective varieties $X$ such that the rational Chern classes $c_i(X) \in H^*(X, \mathbb{Q})$ vanish for $n - k + 1 \leq i \leq n$, but not for $i = n - k$.

This question is still wide open, except for the case $k = n$, as we saw in Theorem 5.2.

Baldassarri’s question also suggests (see question (IV) below) to classify, if possible, the varieties (cKM) $X$ whose integral Chern classes $c_i(X) \in H^*(X, \mathbb{Z})$ vanish for $n - k + 1 \leq i \leq n$.

Again, the question is quite open, even for the case $k = n$, as we saw in section 3.
The paper [Bald56] by Baldassarri motivates indeed the following more general questions. But, in view of what we have seen in the case of surfaces, one might want to include a condition of minimality of $X$ in a strong sense, for instance that $K_X$ is nef. Or require only a birational isomorphism with a product (because a $\mathbb{P}^1$-bundle over an elliptic curve $T$ is only birational to $\mathbb{P}^1 \times T$), or that $X$ be only birational to $(Y \times T)/G$, where the action is only free in codimension 1 (as in [HS21a]).

**Question 8.1.** (II) Is a variety (respectively, a compact Kähler manifold) $X$ with $K_X$ nef and such that its rational Chern classes $c_i(X) \in H^*(X, \mathbb{R})$ vanish for $k + 1 \leq i \leq n$ isogenous to a $k$-Torus Product (respectively, isogenous to a $k$-framed or $k$-coframed manifold)?

(III) Same question for a variety (or compact Kähler manifold) $X$ with $K_X$ nef and with all the Chern numbers equal to zero.

(IV) What can we say about a variety (cKM) $X$ whose integral Chern classes $c_i(X) \in H^*(X, \mathbb{Z})$ vanish for $k + 1 \leq i \leq n$?

We shall see in the next sections that work of Chad Schoen [Schoen88] gives a negative answer to Questions (II) and (III).

Baldassarri was trying to extend the ‘results’ in dimension 2 and 3 by Dantoni and Roth, showing that his varieties are the Pseudo-Abelian varieties of order $k$.

Already for $k = n$ their answers are incorrect, as Theorem 5.2 shows: because a Pseudo-Abelian Variety of rank $k = n$ is an Abelian variety, and not a Hyperelliptic Variety.

**Remark 8.2.** a) Let me repeat that this assertion is wrong already for surfaces, because the amended version of the theorem of Dantoni (Theorem 4.6) shows that we have more varieties than the Pseudo-Abelian varieties. Indeed, in the case of surfaces, with the hypothesis that $S$ is minimal, we also have the surfaces which are Isogenous to a 1-Torus Product, or birational to it.

b) The assertion is also wrong in dim = 3 since here, as shown in Proposition 1.5 there are Hyperelliptic Threefolds whose group of Automorphisms is discrete, hence they are not Pseudo-Abelian. Indeed, if one looks at the paper by Roth [Roth53] on Hyperelliptic threefolds, one sees that Roth does not consider the case of Hyperelliptic Threefolds for which $Aut^0(X)$ consists only of the Identity.

c) In the same paper Roth calls, following Enriques, [Enr05a], [Enr05b], ‘the elliptic surfaces’ the surfaces such that $Aut^0(X)$ is an elliptic curve. Here the modern terminology, introduced by Kodaira, differs: an elliptic surface is a surface admitting a fibration $f : S \to C$ with fibres elliptic curves; in general it does not possess nontrivial automorphisms.

As we saw in Proposition 1.5, however, there are Hyperelliptic Threefolds and Varieties (hence for them $c_n(X) = 0$) which are regular ($H^1(O_X) = 0$).
On the other hand, Todd’s remark points out a crucial fact which Baldassarri wants to use, that an irregular variety possessing a holomorphic 1-form without zeros has necessarily \( c_n(X) = 0 \); to give an idea of the difficulty of the type of questions considered by Baldassarri, let us notice that for instance the investigation of varieties such that there is \( \omega \in H^0(\Omega^1_X) \) without zeros has been taken up (before our present investigations) in the last years by Schreieder and Hao ([Schre21] [HS21b]), and a classification has been achieved only in dimension 3.

What is more interesting is that, under this much stronger assumption of the existence of such a form \( \omega \), the results confirms, in a special case, the result claimed by Baldassarri (see also our Theorem 7.3).

**Theorem 8.3.** (Hao-Schreieder [HS21b])

Let \( X \) be a smooth projective threefold satisfying property

\[(A) : \text{admitting a holomorphic 1-form } \omega \in H^0(\Omega^1_X) \text{ without zeros.} \]

Then the minimal model program for \( X \) yields a birational morphism \( \sigma : X \to X_{\text{min}} \) blowing up smooth elliptic curves which are not contracted by the Albanese map, and such that

\[(2) \text{ there is a smooth morphism } \pi : X_{\text{min}} \to A \text{ to an Abelian variety of positive dimension;} \]

\[(3) \text{ If the Kodaira dimension is non negative, then a finite étale cover of } X_{\text{min}} \text{ is a product } A' \cong A' \times S', \text{ where } S' \text{ is smooth projective, and the composite map } A' \cong A' \times \{s'\} \to A \text{ is finite and étale.} \]

\[(4) \text{ If the Kodaira dimension is equal to } -\infty, \text{ then either} \]

\[(4a) X_{\text{min}} \text{ has a smooth Del Pezzo fibration over an elliptic curve, or} \]

\[(4b) X_{\text{min}} \text{ has a conic bundle fibration } f : X_{\text{min}} \to S \text{ over a smooth surface } S \text{ satisfying property } (A). \text{ Moreover, either } f \text{ is smooth, or } A \text{ is smooth and the degeneracy locus of } f \text{ is a finite union of elliptic curves which are étale over } A. \]

**Corollary 8.1.** If \( X \) is a smooth projective threefold satisfying property (A) of admitting a holomorphic 1-form \( \omega \in H^0(\Omega^1_X) \) without zeros, and the Kodaira dimension of \( X \) is non negative, then \( X_{\text{min}} \) is a Pseudo-Abelian variety in the sense of Roth.

**Proof.** We are in case (3), and obviously \( A' \) acts on the product \( A' \times S' \).

Since the map \( A' \to A \) is finite and unramified, it is a quotient map, with Galois group \( G \), a finite Abelian group of translations of \( A' \).

We have \( A' \times S' \to X_{\text{min}} \to A \), and since \( A' \times S' \to A' \) is a smooth fibration with fibre \( S' \), hence also \( X_{\text{min}} \to A \) is a smooth fibration with fibre \( S' \).

Since the pull back of \( X_{\text{min}} \to A \) to \( A' \) is isomorphic to the product \( A' \times S' \), we conclude that \( A' \) acts on \( X_{\text{min}} \) and \( X_{\text{min}} = (A' \times S')/G \), where \( G \) acts on \( A' \) by translations. \( \square \)

A few words concerning the \( k \)-co-framed manifolds \( X \), in view of Proposition 7.2 and Example 7.3.
The historical conclusion that we can draw is that Baldassarri’s paper was vitiated by the ‘original sin’ of trying to extend results which were not correct already in small dimension \( n = 2, 3 \).

Also, the author, like other algebraic geometers of his generation, realized that more modern methods were needed, be they coming from differential geometry (see for instance Theorem 5.2), or from homological algebra and sheaf theory or from topology (as the crucial concept of isogeny).

However, in spite of the tremendous substantial and technical progress which took place in the last 60 or more years, the problems considered by Baldassarri are still open and at the centre of forefront research.

9. Manifolds with vanishing Chern numbers

The title of this section is on purpose ambiguous: one may ask about Manifolds for which a certain Chern number vanishes, or all the way consider Manifolds for which all the Chern numbers are equal to zero.

We have given the example of Manifolds \( X \) Isogenous to a \( k \)-Torus Product as a prototype of manifolds with all the Chern numbers equal to zero.

We have also observed that, at least in dimension 2, all the surfaces with all the Chern numbers \( c_1^2 = c_2 = 0 \), or the minimal surfaces with \( c_2 = 0 \), in view of Theorem 4.6, are the manifolds of this type, if \( K_S \) is nef, or birational to one of this type if \( S \) is ruled.

Because if \( S \) is minimal and not elliptic ruled, there exists a Galois étale covering \( S' \to S \) such that \( S' \cong T \times Y \), where \( T \) is a complex torus with \( \dim(T) > 0 \).

More generally, we have the class of manifolds \( X \) isogenous to a partially cotangentially framed manifold \( X \).

If we go up to dimension 3, there are three Chern numbers, \( c_1^3, c_3, \) and \( c_1c_2 = \chi(\mathcal{O}_X) \).

A recent result by Hao and Schreieder goes in the direction of answering the question, \[ {\text{HS21}} \] in the special case where the Kodaira dimension is \( n - 1 \):

**Theorem 9.1. (Hao-Schreieder)** Let \( X \) be a minimal model with \( \dim(X) = n \) and Kodaira dimension \( n - 1 \). Then \( c_1^{n-2}c_2(X) = 0 \) if and only if \( X \) is birational to a quotient \( Z = (E \times Y)/G \), where

1. \( Z \) has canonical singularities;
2. \( E \) is an elliptic curve and \( Y \) is a normal projective variety with \( K_Y \) ample;
3. \( G \) acts diagonally, faithfully on each factor, and freely in codimension two on \( E \times Y \).
Now, the case where $X$ is a threefold of general type (in this case it can be $c_1c_2 = 0$, as shown by Ein and Lazarsfeld, [EL97]), is excluded if $X$ is minimal, since then $c_3^2 > 0$.

Hence the missing cases are the cases of Kodaira dimension 0 and 1. For Kodaira dimension 0, $c_1(X) = 0 \in H^2(X, \mathbb{Q})$, hence remains to see what happens for $c_3 = 0$. Some examples of simply connected Calabi-Yau threefolds with $c_3 = 0$ have been constructed by Chad Schoen [Schoen88] and other examples were later found by Volker Braun [Bra12] as hypersurfaces in a toric fourfold.

We want to discuss now the former examples by Schoen, and show some partial results which seem to indicate that they should not be birational to a quotient of a torus product.

These examples are constructed as small resolutions of fibre products $X = S_1 \times_{\mathbb{P}^1} S_2$ where $f_i : S_i \to \mathbb{P}^1$ is a rational elliptic surface with a section. Let $f : X \to \mathbb{P}^1$ be the fibre product of $f_1$ and $f_2$.

We are interested in the special case where the critical values of $f_1, f_2$ are different; then the fibre product $X$ is smooth, and all the fibres $F$ of $f$ are either a product of two elliptic curves, or the product of an elliptic curve with a degenerate fibre. Hence all the fibres $F$ have Euler number $e(F) = 0$, and $c_3(X) = e(X) = 0$.

The canonical divisor of $X$ is trivial, for instance we can take the $f_i = \frac{F_i}{G_i}$ to be given by a pencil of plane cubics with simple base points: then $X$ is the small resolution of a hypersurface of bidegree $(3, 3)$ $X' \subset \mathbb{P}^2 \times \mathbb{P}^2$,

$$X' = \{(x, y)|F_1(x)G_2(y) = F_2(y)G_1(x)\}$$

hence $X'$ and $X$ have trivial canonical divisor (see [Schoen88] page 181). Moreover, essentially by the hyperplane theorem of Lefschetz, $X$ is simply connected (see at any rate (2.1) of [Schoen88], page 181).

Thus $X$ is a Calabi-Yau threefold with $c_3(X) = 0$.

**Proposition 9.1.** The Schoen threefolds $X$ cannot be birationally covered by a 1-dimensional family of subvarieties which are isomorphic to a fixed Abelian surface $T$.

**Proof.** Assume that $X$ is birationally covered by a family $T \times C$ (where by the way $C = \mathbb{P}^1$ since $q(X) = 0$).

For a general $b \in C$, $T_b := T \times \{b\}$ cannot map to a fibre of $f : X \to \mathbb{P}^1$, since $f$ is the fibre product of $f_1$ and $f_2$ and we may assume that the fibrations $f_i$ do not have constant moduli.

Hence $T_b$, which is a subvariety of $X$, dominates $\mathbb{P}^1$ through the morphism $f$.

But a linear system of dimension one on an Abelian surface yields a morphism to $\mathbb{P}^1$ only if the system is not ample, that is, the fibre is a union of translates of an elliptic curve $E \subset T$. Varying $b$, the elliptic curve $E$ is fixed (since it corresponds to a subgroup of the first homology group of $T$), therefore all the fibres of $f$ contain an elliptic
curve isomorphic to $E$. This is a contradiction, as the general fibres of $f_1$ and $f_2$ are not even isogenous to a fixed elliptic curve $E$.

$\square$

**Proposition 9.2.** The Schoen threefolds $X$ do not admit a fibration $\psi : X \to S$ onto a surface $S$ such that the general fibre is isomorphic to a fixed elliptic curve $E$.

**Proof.** Since $X$ is simply connected, and the general fibre of $\psi$ is connected, it follows that $S$ is also simply connected.

Let $D$ be the divisorial part of the set of critical values of $\psi$, and let $D^*$ be its smooth locus: define

$$S^* := (S \setminus Sing(D)), \quad X^* = \psi^{-1}(S^*).$$

Then we have an exact sequence

$$\pi_1(E) \to \pi_1(X^*) \to \pi_1(S^*) \to 1,$$

and we observe that also $X^*, S^*$ are simply connected, since we have removed a subvariety of real codimension 4.

The image of $\pi_1(E) \to \pi_1(X^*)$ is the quotient of $\pi_1(E)$ by the local monodromies around the irreducible components $D_j$ of $D^*$.

To understand these local monodromies, take a general curve section $C$ of $S$. The inverse image of $C$ is an elliptic fibration $\Sigma \to C$ over $C$ such that all the smooth fibres are isomorphic to $E$.

The fibration is isotrivial, hence $\Sigma = (C' \times E)/G$, where $C' \to C$ is Galois with group $G$.

$G$ acts on $C' \times E$ via a product action. If there are fixpoints for the action of $G$ on $C'$, then, since the quotient is smooth, it follows that the isotropy subgroups act freely on $E$, hence by translations. So the local holomorphic monodromies are translations by torsion points, and the monodromy acts trivially on the first homology of $E$.

The conclusion is that the image of $\pi_1(E) \to \pi_1(X^*)$ is an infinite group, and this is a contradiction since $\pi_1(X^*)$ is trivial.

$\square$

The construction of Chad Schoen [Schoen88] applied to other elliptic fibrations leads to threefolds $X$ with Kodaira dimension 1, and $c_3(X) = 0$. We have not yet investigated whether we can achieve with this construction trivial Chern number $c_1(X)c_2(X) = 0$.

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