Supremum of the function $S_1(t)$ on short intervals

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Abstract

We prove a lower bound on the supremum of the function $S_1(T)$ on short intervals, defined by the integration of the argument of the Riemann zeta-function. The same type of result on the supremum of $S(T)$ have already been obtained by Karatsuba and Korolev. Our result is based on the idea of the paper of Karatsuba and Korolev. Also, we show an improved Omega-result for a lower bound.

1 Introduction

We consider the argument of the Riemann zeta function $\zeta(s)$, where $s = \sigma + ti$ is a complex variable, on the critical line $\sigma = \frac{1}{2}$.

We introduce the functions $S(t)$ and $S_1(t)$. When $T$ is not the ordinate of any zero of $\zeta(s)$, we define

$$S(T) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + Ti \right).$$

This is obtained by continuous variation along the straight lines connecting $2$, $2 + Ti$, and $\frac{1}{2} + Ti$, starting with the value zero. When $T$ is the ordinate of some zero of $\zeta(s)$, we define

$$S(T) = \frac{1}{2} \{S(T + 0) + S(T - 0)\}.$$

Next, we define $S_1(T)$ by

$$S_1(T) = \int_0^T S(t)dt + C,$$

where $C$ is the constant defined by

$$C = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma)| d\sigma.$$
It is a classical result of von Mangoldt (cf. chapter 9 of Titchmarsh [14]) that there exists a number $T_0 > 0$ such that for $T > T_0$ we have

$$S(T) = O(\log T).$$

Also, Littlewood [9] proved that there exists a number $T_0 > 0$ such that for $T > T_0$ we have

$$S_1(T) = O(\log T).$$

Further, Littlewood proved that under the Riemann Hypothesis we have

$$S(T) = O\left(\frac{\log T}{\log \log T}\right)$$

and

$$S_1(T) = O\left(\frac{\log T}{(\log \log T)^2}\right).$$

There exist some known results for $S(t)$ on short intervals. In 1946, Selberg [12] proved the inequalities

$$\sup_{T \leq t \leq 2T} (\pm S(t)) \geq \frac{A (\log T)^{1/3}}{(\log \log T)^{1/3}},$$

where $A$ is a positive absolute constant. Also, a similar result for $S_1(t)$ is

$$S_1(t) = \Omega_\pm \left(\frac{\log t^{1/3}}{(\log \log t)^{1/3}}\right).$$ (1)

Also, Tsang [15] proved for $S_1(t)$ that

$$S_1(t) = \begin{cases} 
\Omega_+ \left(\frac{\log t^{1/3}}{(\log \log t)^{1/3}}\right), & \text{unconditionally,} \\
\Omega_- \left(\frac{\log t^{1/3}}{(\log \log t)^{1/3}}\right), & \text{unconditionally,} \\
\Omega\pm \left(\frac{\log t^{1/3}}{(\log \log t)^{1/3}}\right), & \text{assuming R.H.}
\end{cases}$$ (2)

In 1977, Montgomery [10] established the following result under the assumption of the Riemann hypothesis: In the interval $(T^{1/2}, T)$, there exist points $t_0$ and $t_1$ such that

$$(-1)^j S(t_j) \geq \frac{1}{20} \left(\frac{\log T}{\log \log T}\right)^{1/2}, \quad j = 0, 1.$$

In 1986, Tsang [15] improved the methods of [12] to obtain the following inequalities strengthening the above results of Selberg and Montgomery:

$$\sup_{T \leq t \leq 2T} (\pm S(t)) \geq A \left(\frac{\log T}{\log \log T}\right)^a,$$
where $A > 0$ is an absolute constant and the value of $a$ is equal to $\frac{1}{2}$ if the Riemann hypothesis is true and equal to $\frac{1}{3}$ otherwise.

In 2005, Karatsuba and Korolev [6] established the following result: Let $0 < \epsilon < \frac{1}{10\pi}$, $T \geq T_0(\epsilon) > 0$, and $H = T^\frac{2}{\log T}$. Then

$$\sup_{T-H \leq t \leq T+2H} (\pm S(t)) \geq \frac{\epsilon^2}{1000} \left(\frac{\log T}{\log \log T}\right)^\frac{1}{2}.$$ 

Our result in the present paper is obtained by applying the method of proving the above result to the function $S_1(t)$.

**Theorem 1.**

Let $0 < \epsilon < \frac{1}{10\pi}$, $T \geq T_0(\epsilon) > 0$, and $H = T^\frac{2}{\log T}$. Then

$$\sup_{T-H \leq t \leq T+2H} (\pm S_1(t)) \geq \frac{\epsilon}{4000\pi} \left(\frac{\log T}{\log \log T}\right)^\frac{1}{2}.$$ 

This can be proven similarly to the above result of Karatsuba and Korolev [6]. So in this paper, we describe just the outline of the proof of Theorem 1. However, lemmas to apply for the proof of Theorem 1 are different from those in [6]. There are five lemmas to apply, four lemmas among them are different. Therefore, we describe the details of the proofs of those lemmas, which are Lemma 1, Lemma 2, Lemma 3 and Lemma 4. The basic ideas of the proofs of Lemmas 1, 2, 3 and 4 are based on the proof of Theorem 2, Lemma 2, Lemma 4 and Lemma 3, respectively, of Chapter 3 in Karatsuba and Korolev [6].

**Theorem 2.**

$$S_1(t) = \Omega_{\pm} \left(\frac{(\log t)^{\frac{1}{2}}}{(\log \log t)^{\frac{1}{2}}}\right).$$

Theorem 2 can be seen immediately from Theorem 1. This is an improvement of Selberg’s result (1). Moreover, for $\Omega_+$, Theorem 2 is also an improvement of Tsang’s result (2).

There are functions $S_2(t)$, $S_3(t)$, · · · defined by

$$S_m(t) = \int_0^t S_{m-1}(u)du + C_m$$

for $m \geq 2$, where constants $C_m$ depend on $m$. It seems that we cannot apply the method in Karatsuba and Korolev [6] for $S_2(t)$, $S_3(t)$, etc. because $S_2(t)$, etc. do not have the expression like

$$S_1(t) = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log |\zeta(\sigma + ti)|d\sigma + O(1)$$

for $S_1(t)$ in p. 274 of Selberg [13]. This expression is essential in the proof of Lemma 1. The basic idea of the method in Karatsuba and Korolev [6] relies on Lemma 1. Therefore, the method in this paper cannot be applied to $S_2(t)$, etc.

Therefore, some new idea or the expression like (3) will be necessary to obtain the result similar to our Theorem 1 for functions $S_2(t)$, etc.
2 Some lemmas

Here we introduce the following notations. Let $2 \leq x \leq t^2$. We set

$$\sigma_{x,t} = \frac{1}{2} + 2 \max \left( \left| \beta - \frac{1}{2} \right|, \frac{1}{\log x} \right),$$

where $\beta$ ranges over the real parts of the zeros $\rho = \beta + \gamma i$ of the Riemann zeta function that satisfy the condition

$$|\gamma - t| \leq \frac{x}{3} |\beta - \frac{1}{2}| \log x.$$

Also, we set

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Using these notations, we state the following lemmas.

**Lemma 1.**

Let $f(z)$ be a function taking real values on the real line, analytic on the strip $|\Im z| \leq 1$, and satisfying the inequality $|f(z)| \leq c(|z| + 1)^{-(1+\alpha)}$, $c > 0$, $\alpha > 0$, on this strip. Then, the formula

$$\int_{-\infty}^{\infty} f(u)S_1(t + u)du = \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^2(\log n)^2} \Re \left( \frac{1}{n^{it}} \hat{f}(\log n) \right) - C \hat{f}(0)$$

$$+ 2 \left\{ \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_{0}^{\beta-\sigma} \Re f(\gamma - t - xi)dxd\sigma \\
- \int_{\frac{1}{2}}^{1} \int_{0}^{1-\sigma} \Re f(-t - xi)dxd\sigma \right\},$$

where $\hat{f}(x)$ is given by the formula

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{-ixu}du,$$

holds for any $t$, where the summation in the last sum is taken over all complex zeros $\rho = \beta + \gamma i$ of $\zeta(s)$ to the right of the critical line, and where

$$C = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma)|d\sigma.$$

**Lemma 2.**

For any sufficiently large positive values of $H$, $t$, and $\tau$ with $\tau < \log t$ and $H < t$,

$$\int_{-\frac{1}{2}H\tau}^{\frac{1}{2}H\tau} \left( \frac{\sin u}{u} \right)^2 S_1 \left( t + \frac{2u}{\tau} \right) du = W(t) + R(t) + O \left( \frac{\log t}{\tau H} \right) + O(1),$$

where $S_1(x)$ is a certain function and $W(t)$, $R(t)$ are certain integrals.

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where
\[ W(t) = \sum_{p \leq e^t} \frac{\cos(t \log p)}{p^{\frac{3}{2}} \log p} \left( 1 - \frac{\log p}{\tau} \right), \]
\[ R(t) = \tau \sum_{\beta > \frac{2}{3}} \int_{\frac{\beta}{2}}^{\beta} \int_0^{\beta - \sigma} \Re \left( \frac{\sin \frac{\tau}{2} (\gamma - t - xi)}{\tau (\gamma - t - xi)} \right)^2 \, dx \, d\sigma. \]

Lemma 3. @
Let \( \epsilon \) with \( 0 < \epsilon < \frac{1}{1000} \) be fixed. Let \( T \geq T_0(\epsilon) > 0, H = T^{\frac{32}{15}} + \epsilon \) and \( k \) be an integer such that \( k \geq k_0(\epsilon) > 1 \), let \( m = 2k + 1 \), \( \tau = 2 \log \log H \), and \( m \tau < \frac{1}{10} \epsilon \log T \). Then the function \( R(t) \) defined by Lemma 2 satisfies the inequality
\[ \int_T^{T+H} |R(t)|^m \, dt < H \left\{ 25^m + (\log T)^3 \left( \frac{50 \tau m^2}{e^3 \log T} \right)^m \right\}. \]

Lemma 4. @
Let \( T \geq T_0 > 0, e^2 < H < T, 2 < \tau < \log H \), and \( k \) be an integer such that \( k \geq k_0 > 1 \) and \( (2k \log k)^2 < e^{k \tau} \). Then
\[ \int_T^{T+H} W(t)^{2k} \, dt > \left( \frac{1}{5 \sqrt{10} e} \cdot \frac{k^2}{\log k} \right)^{2k} \left( H - e^{3k \tau} \right), \quad (4) \]
\[ \left| \int_T^{T+H} W(t)^{2k+1} \, dt \right| < e^{3k \tau + \frac{3}{2} \tau}. \quad (5) \]

This lemma is Lemma 3 of Chapter 3 in Karatsuba and Korolev [6]. But in Karatsuba and Korolev [6], the function \( W(t) \) is defined by
\[ W(t) = - \sum_{p \leq e^t} \frac{\sin(t \log p)}{p^{\frac{3}{2}}} \left( 1 - \frac{\log p}{\tau} \right), \]
which are different from the definition in this paper.

The following lemma is given in Karatsuba and Korolev [6].

Lemma 5. @
Let \( H > 0 \) and \( M > 0 \), let \( k \geq 1 \) be an integer, and let \( W(t), R(t) \) be real functions which satisfy the conditions
1) \( \int_T^{T+H} |W(t)|^{2k} \, dt \geq HM^{2k}, \)
2) \( \left| \int_T^{T+H} W(t)^{2k+1} \, dt \right| \leq \frac{1}{2} HM^{2k+1}, \)
3) \( \int_T^{T+H} |R(t)|^{2k+1} \, dt < H \left( \frac{M}{2} \right)^{2k+1}. \)
Then
\[
\max_{T \leq t \leq T + H} \pm(W(t) + R(t)) \geq \frac{1}{8} M.
\]

This lemma is Lemma 1 of Chapter 3 in Karatsuba and Korolev [6].

### 3 Proof of Lemma 1

This proof is an analogue of the proof of Theorem 2 of Chapter 3 in Karatsuba and Korolev [6].

**Proof.** Put \(\frac{1}{2} \leq \sigma \leq \frac{3}{2}\). We set \(\psi(z) = f((\sigma - z)i - t)\) and take \(X \geq 2(|t| + 10)\) such that the distance from the ordinate of any zero of \(\zeta(s)\) to \(X\) is not less than \(c \log X^{-1}\), where \(c\) is a positive absolute constant.

Let \(\Gamma\) be the boundary of the rectangle with the vertices \(\sigma \pm Xi, \frac{3}{2} \pm Xi\), and let a horizontal cut be drawn from the line \(\Re s = \sigma\) inside this rectangle to each zero \(\rho = \beta + \gamma i\) and also to the point \(z = 1\). Then the functions \(\log \zeta(z)\) and \(\psi(z)\) are analytic inside \(\Gamma\).

By the residue theorem, the following equality holds:
\[
0 = \int_{\Gamma} \psi(z) \log \zeta(z) dz = \left( \int_{\frac{3}{2}+Xi}^{\frac{3}{2}-Xi} - \int_{\sigma+Xi}^{\sigma-Xi} + \int_{\sigma-Xi}^{\sigma+Xi} \right) \psi(z) \log \zeta(z) dz
\]
\[
= I_1 - I_2 - I_3 + I_4,
\]
say. Then, we have
\[
I_1 = i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti} \log n} \hat{f}(\log n) + O \left( \frac{1}{X^\alpha} \right)
\]
since for \(\alpha = \frac{3}{2} - \sigma\)
\[
\int_{-\infty}^{\infty} \psi \left( \frac{3}{2} + ui \right) \log \zeta \left( \frac{3}{2} + ui \right) du = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti} \log n} \int_{-\infty}^{\infty} \frac{1}{nu} f(u - t - \alpha i) \, du
\]
\[
= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti} \log n} \hat{f}(\log n).
\]

Also,
\[
I_2 = O \left( \frac{(\log X)^2}{X^{(1+\alpha)}} \right),
\]
\[
I_4 = O \left( \frac{(\log X)^2}{X^{(1+\alpha)}} \right).
\]
as in p. 461 of Karatsuba and Korolev [6].

We denote by $L$ the cut going from the point $\sigma + \gamma i$ to the each points $\beta + \gamma i$, and denote by $I(L)$ the integral over the banks of this cut. Then,

$$I(L) = \int_L \psi(z) \log \zeta(z) dz = 2\pi i \sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx$$

as in p. 462 of Karatsuba and Korolev [6].

If $L$ is the cut going to the point $z = 1$, then

$$I(L) = -2\pi i \int_0^{1-\sigma} f(-t - xi) dx.$$ 

Hence, we have

$$I_3 = \int_{\sigma - X i}^{\sigma + XI} \psi(z) \log \zeta(z) dz$$

$$ - 2\pi i \left( \sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - \int_0^{1-\sigma} f(-t - xi) dx \right).$$

When $X$ tends to infinity, we obtain

$$\lim_{X \to \infty} \int_{\sigma - X i}^{\sigma + XI} \psi(z) \log \zeta(z) dz = i \int_{-\infty}^{\infty} f(u) \log \zeta(\sigma + (t + u)i) du$$

$$= i \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti}} \cdot \hat{f}(\log n)$$

$$+ 2\pi i \left( \sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - \int_0^{1-\sigma} f(-t - xi) dx \right).$$

Dividing by $i$, we get for $\sigma \geq \frac{1}{2}$

$$\int_{-\infty}^{\infty} f(u) \log \zeta(\sigma + (t + u)i) du = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma+ti}} \cdot \hat{f}(\log n)$$

$$+ 2\pi \left( \sum_{\beta > \sigma} \int_0^{\beta - \sigma} f(\gamma - t - xi) dx - K(\sigma) \int_0^{1-\sigma} f(-t - xi) dx \right),$$

where

$$K(\sigma) = \begin{cases} 
1 & \text{for } \frac{1}{2} \leq \sigma \leq 1, \\
0 & \text{for } \sigma > 1.
\end{cases}$$
Here, taking the real part and applying (3) and integrating in \( \sigma \) over the interval \([\frac{1}{2}, \frac{3}{2}]\), we have

\[
\int_{-\infty}^{\infty} \int_{\frac{1}{2}}^{\frac{3}{2}} f(u) \log |\zeta(\sigma + (t + u)i)|d\sigma du = \pi \int_{-\infty}^{\infty} S_1(t + u) f(u)du + \pi \int_{-\infty}^{\infty} f(u)Cdu
\]

\[
= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^2 (\log n)^2} \Re \left( \frac{1}{n^s} \hat{f}(\log n) \right) + 2\pi \left( \int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\beta > \sigma} \int_{0}^{\beta - \sigma} \Re f(\gamma - t - xi)d\sigma - \int_{\frac{1}{2}}^{\frac{3}{2}} \int_{0}^{1-\sigma} \Re f(-t - xi)d\sigma \right).
\]

Therefore

\[
\int_{-\infty}^{\infty} S_1(t + u) f(u)du = \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^2 (\log n)^2} \Re \left( \frac{1}{n^s} \hat{f}(\log n) \right) - C\hat{f}(0)
\]

\[
+ 2\pi \left( \int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\beta > \sigma} \int_{0}^{\beta - \sigma} \Re f(\gamma - t - xi)d\sigma - \int_{\frac{1}{2}}^{\frac{3}{2}} \int_{0}^{1-\sigma} \Re f(-t - xi)d\sigma \right).
\]

\[
= \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^2 (\log n)^2} \Re \left( \frac{1}{n^s} \hat{f}(\log n) \right) - C\hat{f}(0)
\]

\[
+ 2\pi \left( \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta - \sigma} \Re f(\gamma - t - xi)d\sigma - \int_{\frac{1}{2}}^{\frac{3}{2}} \int_{0}^{1-\sigma} \Re f(-t - xi)d\sigma \right).
\]

\[
\square
\]

4 Proof of Lemma \( \mathcal{L}_2 \)

This proof is an analogue of the proof of Lemma 2 of Chapter 3 in Karatsuba and Korolev [1].

Proof. Put \( f(z) = \left( \frac{\sin \frac{\pi z}{\tau}}{\frac{\pi z}{\tau}} \right)^2 \). By

\[
\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi x^2 u} f(u)du = \frac{2\pi}{\tau} \max \left( 0, 1 - \left| \frac{x}{\tau} \right| \right),
\]

we get

\[
\hat{f}(\log n) = \begin{cases} \frac{2\pi}{\tau} \left( 1 - \frac{\log n}{\tau} \right) & (1 \leq n \leq e^\tau) \\ 0 & (n > e^\tau) \end{cases}.
\]
Then, we have

\[
\int_{-\infty}^{\infty} \left( \sin \frac{\pi}{2} u \right)^2 S_1(t + u) du = \frac{1}{\pi} \sum_{n \leq e^t} \frac{\Lambda(n)}{n^2 (\log n)^2} \frac{2\pi}{\tau} \left( 1 - \frac{\log n}{\tau} \right) \cos(t \log n) + \frac{2}{\tau} \left( \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_{\frac{1}{2}}^{\beta - \sigma} \Re \left( \sin \frac{\pi}{2} \frac{\gamma - t - \xi i}{\tau} \right)^2 d\xi d\sigma \right) - \int_{\frac{1}{2}}^{1} \int_{0}^{1 - \sigma} \Re \left( \sin \frac{\pi}{2} \frac{\gamma - t - \xi i}{\tau} \right)^2 d\xi d\sigma
\]

by Lemma 1. Since for \( 0 \leq \xi \leq 1 - \sigma \)

\[
2 \left| \sin \frac{\pi}{2} \left( t + \xi i \right) \right|^2 < \frac{1}{5\tau}
\]

as in p. 473 of Karatsuba and Korolev [6], we have

\[
\left| 2 \int_{\frac{1}{2}}^{1} \int_{0}^{1 - \sigma} \Re \left( \sin \frac{\pi}{2} \frac{\gamma - t - \xi i}{\tau} \right)^2 d\xi d\sigma \right| = O \left( \frac{1}{\tau} \right)
\]

In the first term of the right-hand side in (6), we single out the terms corresponding to the \( n = p \) in the sum and estimate the remainder terms. Then, we have

\[
\sum_{2 \leq k \leq p \leq e^t} \frac{\Lambda(p^k)}{p^2 (\log p^k)^2} \cdot \frac{2}{\tau} \left( 1 - \frac{\log p^k}{\tau} \right) < \sum_{2 \leq k \leq p \leq e^t} \frac{\log p^k}{p^2 (\log p^k)^2} \cdot \frac{2}{\tau} \ll \frac{1}{\tau}
\]

Hence

\[
\int_{-\infty}^{\infty} \left( \sin \frac{\pi}{2} u \right)^2 S_1(t + u) du = \frac{2}{\tau} \sum_{p \leq e^t} \frac{\cos(t \log p)}{p^2 \log p} \left( 1 - \frac{\log p}{\tau} \right) - C \cdot \frac{2\pi}{\tau} + \sum_{\beta > \frac{1}{2}} \int_{\frac{1}{2}}^{\beta} \int_{0}^{1 - \sigma} \Re \left( \sin \frac{\pi}{2} \frac{\gamma - t - \xi i}{\tau} \right)^2 d\xi d\sigma
\]

\[
+ O \left( \frac{1}{\tau} \right)
\]

(7)

Put \( v = \frac{\pi}{2} u \). Then the left-hand side of the above is equal to

\[
\left( \int_{-\frac{1}{2} H \tau}^{\frac{1}{2} H \tau} + \int_{-\infty}^{-\frac{1}{2} H \tau} + \int_{\frac{1}{2} H \tau}^{\infty} \right) \left( \sin v \right)^2 v S_1 \left( t + \frac{2v}{\tau} \right) \frac{2}{\tau} dv.
\]

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Since $S_1(t) = O(\log t)$, we have

$$\left| \left( \int_{-\infty}^{\frac{t+H}{2}} + \int_{\frac{t+H}{2}}^{\infty} \right) \left( \frac{\sin v}{v} \right)^2 S_1 \left( t + \frac{2v}{\tau} \right) dv \right| \ll \frac{1}{\tau} \int_{H}^{\infty} \log(t + v') \frac{1}{v'^2} dv'$$

$$\ll \frac{1}{\tau} \left\{ \int_{H}^{t} \frac{\log t}{v'^2} dv' + \int_{t}^{\infty} \frac{\log v'}{v'^2} dv' \right\}$$

$$\ll \frac{1}{\tau} \left( \frac{\log t}{H} + \frac{\log t}{t} \right) \ll \frac{\log t}{\tau H}.$$ 

Inserting these estimates into \(7\) and dividing by $\frac{2}{\tau}$ the both sides, we obtain the result.

\[\square\]

5 Proof of Lemma 3

This proof is an analogue of the proof of Lemma 4 of Chapter 3 in Karatsuba and Korolev [6].

Proof. We put

$$L_k = \int_{T}^{T+H} |R(t)|^{2k+1} dt$$

and note the inequality

$$\left| \Re \left( \frac{\sin(x - yi)}{x - yi} \right)^2 \right| < \frac{8ye^{2y}}{1 + x^2 + y^2}$$

for any $x, y \in \mathbb{R}, y \geq 0$ similarly to pp. 476 – 477 of Karatsuba and Korolev [6]. Then,

$$|R(t)| \leq \tau \sum_{\gamma \geq \frac{1}{4}} \left| \int_{\frac{1}{2}}^{\beta} \int_{0}^{\beta - \sigma} \Re \left( \frac{\sin \frac{\pi}{2}(\gamma - t - \xi i)}{\frac{\pi}{2}(\gamma - t - \xi i)} \right)^2 d\xi d\sigma \right|$$

$$\leq \tau \sum_{\gamma \geq \frac{1}{4}} \left| \int_{\frac{1}{2}}^{\beta} \int_{0}^{\beta - \sigma} \frac{8 \cdot \frac{\pi}{2} e^{\pi \xi}}{1 + \left( \frac{\pi}{2}(\gamma - t) \right)^2 + \left( \frac{\pi}{2} \right)^2} d\xi d\sigma \right|$$

$$< 4\tau^2 \sum_{\gamma \geq \frac{1}{4}} \left( \frac{\pi}{2} \right)^3 \frac{\xi e^{\pi(\beta - \frac{1}{2})}}{1 + \left( \frac{\pi}{2}(\gamma - t) \right)^2 + \left( \frac{\pi}{2} (\beta - \frac{1}{2}) \right)^2} d\xi d\sigma$$

$$= 8 \sum_{\gamma \geq \frac{1}{4}} \left( \frac{\pi}{2} \right)^3 \frac{e^{\pi(\beta - \frac{1}{2})}}{(\frac{\pi}{2})^2 + (\gamma - t)^2 + (\beta - \frac{1}{2})^2}.$$
We split the last sum into two sums. The first sum $\Sigma_1$ is the sum of the terms satisfying $|\gamma - t| > (\log T)^2$, and the second sum $\Sigma_2$ is the sum of the other terms.

Here, we denote by $\theta_t$ the largest difference of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + \gamma i$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| \leq (\log T)^2$. Also, we denote by $\theta'_t$ the supremum of the form $\beta - \frac{1}{2}$ for zeros $\rho = \beta + \gamma i$ in the rectangle $\frac{1}{2} < \beta \leq 1$, $|\gamma - t| > (\log T)^2$.

As in p. 478 of Karatsuba and Korolev [6], we apply the estimation related to $\sigma_{x,t}$ and the result $N(t + 1) - N(t) < 18 \log t$ which is obtained by the Riemann-von Mangoldt formula and $|S(t)| < 8 \log t$ for $t \geq t_0 > 0$. Then we take $x = (\log T)^\frac{\epsilon}{4}$, and we have

$$
\Sigma_1 < \left( \beta - \frac{1}{2} \right) \sum_{|\gamma - t| > (\log T)^2} \frac{2e_\tau}{(\gamma - t)^2} < \frac{2}{3} \theta'_t \log T \sum_{|\gamma - t| > (\log T)^2} \frac{1}{n^2} \sum_{n < |\gamma - t| \leq n + 1} 1
$$

and

$$
\Sigma_2 < 8\theta^3 e^{\tau \theta} \sum_{|\gamma - t| \leq (\log T)^2} \frac{1}{(\frac{2}{\tau})^2 + (\gamma - t)^2} \left( \frac{\theta'_t}{\theta_t} \right)^2 \frac{1}{(\beta - \frac{1}{2})^2}
$$

$$
< 8\theta^3 e^{\tau \theta} \sum_{\rho} \frac{1}{(\sigma_{x,t} - \beta)^2 + (\gamma - t)^2} < 8\theta^3 e^{\tau \theta} \frac{13}{5} \frac{1}{\sigma_{x,t} - \frac{1}{2}} \log T
$$

$$
\leq 8\theta^3 e^{\tau \theta} \frac{13}{5} \frac{5\tau}{39} \log T = 8\theta^3 e^{\tau \theta} \cdot \frac{\tau}{3} \log T.
$$

From the definitions of $\theta_t$ and $\theta'_t$, we get $\theta_t < \frac{1}{2}$ and $\theta'_t < \frac{1}{2}$. Hence, we have

$$
|R(t)| < 25 \left( \theta'_t + \frac{7}{2} \theta^2 e^{\tau \theta_t} \log T \right) < 25 \left( 1 + \frac{7}{2} \theta^2 e^{\tau \theta_t} \log T \right).
$$

Hence

$$
L_k < \left( \frac{25}{2} \right)^m \int_T^{T + H} \left( 1 + \frac{7}{2} \theta^2 e^{\tau \theta_t} \log T \right)^m dt.
$$

This integrand is the same as that in p. 479 of Karatsuba and Korolev [6]. Hence the estimation of the last integral is the same as in pp. 480 – 481 of Karatsuba and Korolev [6]. Along that way, we have

$$
L_k < 25^m H \left\{ 1 + \frac{24}{5} \cdot \frac{1}{m} (\log T)^3 2m! \left( \frac{7}{2} \tau \log T \right)^m \left( \frac{1}{10} \log T \right)^{-2m} \right\}
$$

$$
< 25^m H \left\{ 1 + (\log T)^3 \left( \frac{2m^2 \tau}{e^3 \log T} \right)^m \right\}
$$

$$
< H \left( 25^m + (\log T)^3 \left( \frac{50m^2 \tau}{e^3 \log T} \right)^m \right).
$$

□
6 Proof of Lemma 4

This proof is an analogue of the proof of Lemma 3 of Chapter 3 in Karatsuba and Korolev [6].

Proof. As in pp. 474 – 475 of Karatsuba and Korolev [6], we can write

$$\int_T^{T+H} W(t)^2 \, dt = I_k = \left(\frac{2k}{k}\right) \frac{H}{2^{2k}} \Sigma + \theta e^{3k\tau},$$

where

$$\Sigma = \sum_{\substack{p_1, \ldots, p_k = q_1, \ldots, q_k \\ p_1, \ldots, p_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2, \quad f(p) = \frac{1}{p^2 \log p} \left(1 - \frac{\log p}{\tau}\right).$$

Then,

$$\Sigma \geq k! \sum_{\substack{p_1, \ldots, p_k \text{ are distinct} \\ p_1, \ldots, p_k \leq e^\tau}} f(p_1)^2 \cdots f(p_k)^2 \geq k! \sum_{p_1 \leq e^\tau} f(p_1)^2 \sum_{p_2 \leq e^\tau, p_1 \neq p_2} f(p_2)^2 \cdots \sum_{p_k \leq e^\tau, p_1, \ldots, p_{k-1} \neq p_k} f(p_k)^2.$$

Since $\frac{1}{np} f(p)^2 < 0$, $f(p)^2$ is monotonically decreasing function for $p \geq 2$. Also, since $(k - 1)$th prime does not exceed $2k \log k$, the inner sum of the above inequality is greater than the same sum over $2k \log k < p_k < e^{4\tau}$. Hence the inner sum over $p_k$ is greater than

$$\left(\frac{1}{5}\right)^2 \sum_{2k \log k < p \leq e^{4\tau}} \frac{1}{p (\log p)^2}.$$

For $(2k \log k)^2 \leq e^{4\tau}$, since

$$\sum_{U < p \leq U/2} \frac{1}{p (\log p)^2} \geq \frac{1}{4 (\log U)^2} \sum_{U < p \leq U/2} \frac{1}{p} = \frac{1}{4 (\log U)^2} \left(\log \log U^2 - \log \log U + o(1)\right) > \frac{1}{8 (\log U)^2},$$

the sum over $p_k$ is greater than $\frac{1}{10} \left(\frac{1}{\log k}\right)^2$. Also, the same lower bound holds for the sums over $p_1, p_2, \ldots, p_{k-1}$. Therefore, we see

$$\Sigma \geq k! \left(\frac{1}{250 (\log k)^2}\right)^k \geq \sqrt{2\pi k} \left(\frac{1}{5 \sqrt{10 e} \log k}\right)^{2k}.$$
So,

\[ I_k > H \left( \frac{1}{5\sqrt{10e}} \cdot \frac{k^{\frac{1}{2}}}{\log k} \right)^{2k} - e^{3k^\tau}. \]

This is the first part of Lemma 4. The second part is proved similarly to [6]. □

7 Outline of the proof of the Theorem

As described in section 1, our result can be proven similarly to Theorem 5 of Chapter 3 in Karatsuba and Korolev [6]. Therefore, we describe the outline of the proof.

Outline of the proof. Put \( \tau = 2 \log \log H \). Consider the right-hand side of the inequality in the statement of Lemma 3. We see easily that

\[ \frac{50\tau m^2}{e^3 \log T} < \frac{500k^2}{e^3} \cdot \frac{\log \log T}{\log T} \leq \frac{k^{\frac{1}{2}}}{\log k} \cdot \frac{500k^{\frac{1}{2}}}{e^3} \cdot \frac{\log \log T}{\log T} = \frac{k^{\frac{1}{2}}}{\log k} \cdot \delta, \]

say.

Here, putting \( k = \left\lfloor \frac{\tau}{1000} \left( \frac{\log T}{\log \log T} \right)^\frac{1}{2} \right\rfloor \), we have \( \delta < \frac{1}{60}, (2k \log k)^2 < e^{4\tau} \) and \( e^{3k\tau} < H^{\frac{1}{2}} \). Hence, we can apply Lemma 3 and Lemma 4. Then we have

\[
\int_T^{T+H} W(t)^{2k} \, dt > H M^{2k},
\]

\[
\left| \int_T^{T+H} W(t)^{2k+1} \, dt \right| < \frac{1}{2} H M^{2k+1},
\]

\[
\int_T^{T+H} |R(t)|^{2k+1} \, dt < H \left( \frac{M}{2} \right)^{2k+1},
\]

with \( M = \frac{k^{\frac{1}{2}}}{30 \log k} \). Thus, we see that \( W(t) \) and \( R(t) \) satisfy the conditions of Lemma 5 with \( M = \frac{k^{\frac{1}{2}}}{30 \log k} \). Hence there are two points \( t_0 \) and \( t_1 \) such that

\[ W(t_0) + R(t_0) \geq \frac{M}{8}, \quad W(t_1) + R(t_1) \leq -\frac{M}{8} \]

in the interval \( T \leq t \leq T + H \). By Lemma 2 we have

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left( \frac{\sin u}{u} \right)^2 S_1 \left( t + \frac{2u}{\tau} \right) \, du \geq \frac{M}{8} + O \left( \frac{\log t_0}{\tau H} \right),
\]

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left( \frac{\sin u}{u} \right)^2 S_1 \left( t + \frac{2u}{\tau} \right) \, du \leq -\frac{M}{8} + O \left( \frac{\log t_1}{\tau H} \right).
\]
Here, putting

\[
M_0 = \sup_{T-H \leq t \leq T+T} S_1(t), \quad M_1 = \inf_{T-H \leq t \leq T+2T} S_1(t),
\]

we have

\[
\int_{-\gamma}^{\gamma} \left( \frac{\sin u}{u} \right)^2 S_1 \left( t_0 + \frac{2u}{\tau} \right) \, du < M_0 \int_{-\infty}^{\infty} \left( \frac{\sin u}{u} \right)^2 = \frac{\pi}{2} M_0 \quad (M_0 > 0),
\]

\[
\int_{-\gamma}^{\gamma} \left( \frac{\sin u}{u} \right)^2 S_1 \left( t_1 + \frac{2u}{\tau} \right) \, du > M_1 \int_{-\infty}^{\infty} \left( \frac{\sin u}{u} \right)^2 = \frac{\pi}{2} M_1 \quad (M_1 < 0).
\]

Therefore, we obtain for \( r = 0, 1 \)

\[
(-1)^r M_r > \frac{2}{\pi} \cdot \frac{M}{8} + O \left( \frac{\log t}{\tau H} \right) > \frac{1}{4\pi} \cdot \frac{k^{\frac{1}{2}}}{30 \log k} > \frac{\epsilon}{4000\pi} \left( \frac{\log T}{(\log \log T)^{\frac{1}{2}}} \right).
\]

Thus, we obtain the result.

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