Analytical Analysis and Numerical Solution of Two Flavours Skyrmion*

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Abstract

Two flavours Skyrmion will be analyzed analytically, in case of static and rotational
Skyrme equations. Numerical solution of a nonlinear scalar field equation, i.e. the
Skyrme equation, will be worked with finite difference method. This article is a more
comprehensive version of SU(2) Skyrme Model for Hadron which have been published
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1 Introduction

Nonlinear physics is a general phenomenon in physics. It includes very long range from
e.g. particle, nuclear, condensed matter, fluids, plasmas, biophysics (for example: nonlinear
diffusion-reaction, DNA), to cosmology.

Soliton is defined as classical solution of nonlinear wave equation which has properties:
finite total energy, localized, nondispersive, stable with its profile of energy density distribution
is like pulse centered in finite space [1].

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Skyrmion is a topological soliton in three dimensions. Following, we will discuss soliton as baryon model, especially Skyrme model for hadron which is formulated in extended nonlinear Sigma model \[1\], \[2\].

2 Topological Solitons

Suppose that \( F_\phi = 0 \) is a differential equation or a system of differential equation involving a field or a set of fields denoted by \( \phi \) which are functions of \( D \) space coordinates \( \vec{x} \) and a time coordinate \( t \). We shall assume that \( F_\phi = 0 \) is the equation of motion which results from some Lagrangian field theory. We also assume that the system has a corresponding "energy" \( \varepsilon_\phi \) and "energy density" \( E_\phi(\vec{x}, t) \) \[3\]

\[ \varepsilon_\phi = \int d^D x \ E_\phi(\vec{x}, t) \] (1)

where for all allowed field configurations \( \phi \), \( E_\phi \) is greater than or equal to zero. If \( E_\phi \) is zero for all \( \vec{x} \), we call \( \phi \) a ground state or a vacuum solution, and denote it by \( \phi_{\text{vac}} \). The vacuum solution need not be unique.

Now suppose that \( \phi = \phi_{cl} \) is non-vacuum solution to \( F_\phi = 0 \). Following Coleman, we shall call \( \phi_{cl}(\vec{x}, t) \) a soliton solution if the following properties are satisfied \[3\]:

1. \( \varepsilon_{\phi=\phi_{cl}} \) is finite.
2. \( E_{\phi=\phi_{cl}}(\vec{x}, t) \) is nonsingular (finite) for all value of \( \vec{x} \) and \( t \), and localized for all times \( t \). A solution is localized at any time \( t \) if there is a bounded region of space defined by \( E_{\phi=\phi_{cl}}(\vec{x}, t) \geq \delta \), \( \delta \) being any arbitrary number, fulfilling

\[ 0 < \delta < \max_{\vec{x}} E_{\phi=\phi_{cl}}(\vec{x}, t). \] (2)

We say that the solution is localized for all times \( t \) if the bounded region can be chosen independent of \( t \).

3. \( \phi_{cl}(\vec{x}, t) \) is nonsingular.
4. \( \phi_{cl}(\vec{x}, t) \) is nondissipative.

Concerning (4), a solution \( \phi_{cl}(\vec{x}, t) \) is considered dissipative if

\[ \lim_{t \to \infty} \max_{\vec{x}} E_{\phi=\phi_{cl}}(\vec{x}, t) = 0. \] (3)

In our discussions, we will be concerned primarily with "static" solutions for which we denote \( \phi_{cl}(\vec{x}, t) \) by \( \Phi_{cl}(\vec{x}, t) \).

If we exclude the vacuum, then static, nonsingular, localized solutions are automatically nondissipative. Thus for static solitons, we only require (1), (2) and (3).

We will, however, be interested in static solutions which satisfy yet another requirements; namely that they be
Variational modes $\delta \phi$ which leave the energy unchanged are said to be "zero" frequency modes" or "zero modes". Thus, for zero modes, $\varepsilon_{\phi_{cl} + \delta \phi} \geq \varepsilon_{\phi_{cl}}$. Zero modes are generally associated with various symmetries of the soliton. Excluding the zero modes, a classically stable solution is a local minimum of the energy.

Although the question of stability is ultimately a dynamical one, topology often plays an important role. We shall now elaborate on this point. Let $Q$ be the set of all finite energy and nonsingular field configurations $\phi$ at some fixed time $t$. It is thus the configuration space of the system. A subset $Q_1$ of $Q$ is said to be (path-)connected if any field $\phi_1$ of $Q_1$ can be continuously deformed to any other field $\phi'_1$ in $Q_1$.

Two subsets $Q_1$ and $Q_2$ of $Q$ are said to be disconnected if any field $\phi_1$ of $Q_1$ cannot be continuously deformed to any field $\phi_2$ in $Q_2$. We shall consider the case where $Q$ has $N > 1$ disconnected components $Q_n$, each $Q_n$ being connected. $Q$ is the union of all these disconnected components,

$$Q = \bigcup_{n=1}^{N} Q_n. \quad (5)$$

Let $\phi(\vec{x})$ and $\phi'(\vec{x})$ be two fields in $Q$. $\phi(\vec{x})$ and $\phi'(\vec{x})$ are homotopic to each other, and write $\phi \sim \phi'$, if there exists a sequence of fields $\phi^{(\tau)}(\vec{x})$, $0 \leq \tau \leq 1$, which is continuous in $\tau$ and $\vec{x}$, with $\phi^{(0)}(\vec{x}) = \phi(\vec{x})$ and $\phi^{(1)}(\vec{x}) = \phi'(\vec{x})$. With this definition, all fields within one component $Q_n$ are "homotopic" to each other, while a field $\phi^{(n)}$ belonging to $Q_n$ is not homotopic to a field $\phi^{(n')}$ belonging to $Q_{n'}$ when $n' \neq n$. [3]

We can treat all our examples assuming that $n$ is countable. We make this assumption: What is the physical significance of this classification? Consider the initial conditions $(\phi^{(n)}, d\phi^{(n)}/dt)$ at time $t = 0$ for the equations of motion. (These equations for simplicity are assumed to be second order in time). Assume that $\phi^{(n)} \in Q_n$. After a lapse of time $T$, suppose that $\phi^{(n)}$ becomes $\phi^{(n')}$. $\phi^{(n)}$ and $d\phi^{(n)}/dt$ becomes $d\phi^{(n')}/dt$. Since time evolution is assumed to be a continuous operation, it follows that $\phi^{(n)}$ is homotopic to $\phi^{(n')}$. Hence, $\phi^{(n')} \in Q_{n'}$. In other words, the value of $n$ associated with the field $\phi^{(n)}$ is a constant of the motion. Hence, if we define $Q_0$ to be the sector which contains the vacuum solution $\phi_{\text{vac}}$, then the configurations $\phi^{(n)} \in Q_n$, $n \neq 0$, cannot be time evolved to $\phi_{\text{vac}}$ or in fact, to any other $\phi^{(m)} \in Q_m$ with $m \neq n$. For such reasons, the sectors $Q_{n \neq 0}$ of field configurations are said to be topological stable.

Soliton configurations which fulfill the five properties listed previously and which are in topologically stable sectors are called topological solitons [3].

3 Two Flavours Skyrmion as Hadron

In particle physics, flavour is a quantum number of elementary particles. In quantum chromodynamics (QCD), flavour is a global symmetry. In the electroweak theory, on the other hand, this symmetry is broken, and flavour changing processes exist [3].

If there are two or more particles which have identical interactions, then they may be interchanged without affecting the physics. Any (complex) linear combination of these two
particles give the same physics, as long as they are orthogonal or perpendicular to each other. In other words, the theory possesses symmetry transformations such as \( M \begin{pmatrix} u \\ d \end{pmatrix} \), where \( u \) and \( d \) are the two fields, and \( M \) is any 2 \( \times \) 2 unitary matrix with a unit determinant. Such matrices form a Lie group called \( SU(2) \), special unitary group. This is an example of flavour symmetry. The term "flavour" was first coined for use in the quark model of hadrons in 1968 [4].

In order to understand the charge radius of nucleon which have size roughly 1 Fermi, Tony Hilton Royle Skyrme in 1962 proposed idea that strongly interacting particles (hadrons) were locally concentrated static solution of extended nonlinear Sigma (Chiral) model. In (3+1)-dimensions of space-time, we will observe Skyrme model which describes hadrons as solitons (Skyrmions) from nonlinear Sigma (chiral) model field theory in internal symmetrical group of SU(2): two flavours Skyrmion.

Skyrme’s idea is unifying bosons and fermions in a common framework which provide a fundamental fields model consisted of only pion. The nucleon was obtained, as a certain classical configuration of the pion fields. Skyrmion is a topological soliton object, i.e. solution to the classical field equation with localized energy density. Various atoms should correspond to vortices of different connectivities in some underlying liquid.

Following, some ideas which are proposed by Skyrme in relations with baryon model [1, 8, 12]:

(1) Meson field can take its value in \( S_3 \) manifold. As the result of this assumption, Skyrme discovers conserved quantity i.e. topological charge or winding number and he interprets it as baryon number.

(2) Solution of Skyrme field equation in spherical coordinate takes form:

\[
F_a(r, \theta, \phi) = g(r)n_a(\theta, \phi) \tag{6}
\]

where \( g(r) \) is profile function which has spherical symmetric character and \( n_a \) (\( a = 1, 2, 3 \)) is component of unit vector, \( \hat{n} \). \( F_a(r, \theta, \phi) \) is called Skyrme ansatz or hedgehog.

(3) Skyrme ansatz describes stable extended particle with unit topological charge.

(4) Skyrme ansatz with unit topological charge can be quantized as fermion, so it is possible to identify state of isotopic spin, \( I \), and total spin, \( J \), with nucleon doublet in case of \( I = J = \frac{1}{2} \), and with \( \Delta \) resonance in case of \( I = J = \frac{3}{2} \).

4  \( SU(2) \) Skyrme Model

\( SU(2) \) Skyrme model is very simple model, because it only consists of two flavours. This model is described by \( U = U(x^\mu) \) function which has \( SU(2) \) group valued of (3+1)D space-time coordinates [1 12 14].

The dynamics is determined by action

\[
S = \int d^4x \mathcal{L}, \tag{7}
\]
where
\[ \mathcal{L} = \text{Tr} \left[ -\frac{F^2}{16} L_\mu L^\mu + \frac{1}{32a^2} [L_\mu, L_\nu] [L^\mu, L^\nu] + \frac{F^2}{16} M_\pi^2 (U^{-1} + U - 2I) \right] \] (8)
is the related Lagrangian density and
\[ L_\mu = U^{-1} \partial_\mu U \] (9)
are left chiral currents, where \( F \approx 123 \text{ MeV} \) is pion decay constant and \( a \) is dimensionless constant. The first term in equation (8) is the SU(2) chiral model Lagrangian density, the second term is Skyrme term for stabilizing solitonic solution. The last term describes the mass term where \( M_\pi \) is pion (meson) mass.

Euler-Lagrange equation of SU(2) Skyrme model is derived from least action principle
\[ \delta S = 0. \] (10)
If we take variation of action to equation (7), we obtain
\[ \delta S = \int d^4x \delta \mathcal{L}. \] (11)
Substitute equation (8) into (11), and use least action principle (10) [1]
\[ \partial_\mu \left( L^\mu - \frac{1}{a^2 F^2} [L_\nu, [L^\mu, L^\nu]] \right) + \frac{1}{2} M_\pi^2 (U - U^{-1}) = 0. \] (12)
In general case (nonstatic), energy of SU(2) Skyrme model is
\[ E = \int d^3x \, T^{00}, \] (13)
where \( T^{00} \) is energy-momentum tensor. Explicitly,
\[
E = \int d^3x \, \text{Tr} \left[ -\frac{F^2}{16} L_a L_a - \frac{1}{32a^2} [L_a, L_c] [L_a, L_c] \\
+ \frac{F^2}{16} L_0 L_0 + \frac{1}{16a^2} [L_0, L_a] [L_0, L_a] - \frac{F^2}{16} M_\pi^2 (U^{-1} + U - 2I) - \frac{F^2}{8} L_0 L_0 \\
- \frac{1}{8a^2} [L_a, L_0] [L_a, L_0] \right] \\
= E_{\text{static}} + E_{\text{rotation}}
\] (14)
where
\[ E_{\text{static}} = -\int d^3x \, \text{Tr} \left[ \frac{F^2}{16} L_a^2 + \frac{1}{32a^2} [L_a, L_c]^2 + \frac{F^2}{16} M_\pi^2 (U^{-1} + U - 2I) \right] \] (15)
and
\[ E_{\text{rotation}} = -\int d^3x \, \text{Tr} \left[ \frac{F^2}{16} L_0^2 + \frac{1}{16a^2} [L_0, L_a]^2 \right]. \] (16)
Solitonic properties of SU(2) Skyrme model for static energy in equation (15) is studied by scaling the spatial coordinates

\[ x \rightarrow 2\bar{x}/aF \]

and express energy in \( F/4a \), i.e. by taking

\[ (F/4a) = (1/12\pi^2). \]

In this unit, equation (15) becomes

\[
E_{\text{static}} = \frac{1}{12\pi^2} \int d^3x \left( -\frac{1}{2} \right) \text{Tr} \left[ L_a^2 + \frac{1}{8} ([L_a, L_c]^2 + m_{\pi}^2 (U^{-1} + U - 2I)) \right]
\]

where

\[ m_{\pi} = 2M_{\pi}/aF. \] (18)

Euler-Lagrange equation (12) in static case becomes

\[
\partial_a \left( L_a \left( L_a - \frac{1}{4} [L_c, [L_a, L_c]] \right) - \frac{m_{\pi}^2}{2} (U - U^{-1}) \right) = 0.
\] (19)

5 Scale Stability

Let us look at scale transformation below

\[ x \rightarrow \lambda x. \] (20)

We find that \( L_a \) currents, by scale transformation (20), transform into

\[
L_a(x) \rightarrow U^{-1}(\lambda x) \frac{\partial U(\lambda x)}{\partial x^a} = \lambda L_a(\lambda x).
\] (21)

The effect of scale transformation to static energy (17), by ignoring pion mass term (because pion mass is small), is

\[
E[\lambda]_{\text{static}} = \frac{1}{\lambda} E_{\sigma} + \lambda E_{\text{Sky}}
\]

where \( E_{\sigma} \) is static chiral energy term and \( E_{\text{Sky}} \) is Skyrme energy term.

From equation (22), we obtain

\[
\frac{dE[\lambda]}{d\lambda} \bigg|_{\lambda=1} = \left( -\frac{1}{\lambda^2} E_{\sigma} + E_{\text{Sky}} \right) \bigg|_{\lambda=1} = -E_{\sigma} + E_{\text{Sky}},
\] (23)

and

\[
\frac{d^2E[\lambda]}{d\lambda^2} \bigg|_{\lambda=1} = \frac{2}{\lambda^3} E_{\sigma} \bigg|_{\lambda=1} = 2E_{\sigma}.
\] (24)

The requirement for extremum condition is

\[
\frac{dE[\lambda]}{d\lambda} = 0.
\] (25)
We apply extremum condition, (25), to equation (23), we obtain

$$E = E_{\text{Sky}},$$

which it shows

$$E \geq 0.$$  

So that equation (24) fulfills condition

$$d^2 E [\lambda] \over dx^2 > 0.$$  

Equation (28) is minimum stable condition which implies that static energy (22) is stable against scale perturbation.

6 Topological Charge

Static energy of SU(2) Skyrme model can be expressed, by ignoring pion mass, as

$$E_{\text{static}} = \frac{1}{12\pi^2} \int d^3 x \left( -\frac{1}{2} \right) \text{Tr} \left[ \left( L_a \pm \frac{1}{4} \epsilon_{abc} [L_b, L_c] \right)^2 \right] + \frac{1}{24\pi^2} \int d^3 x \epsilon_{abc} \text{Tr} \left[ [L_a, L_b, L_c] \right]$$  

where $\epsilon_{abc}$ is Levi-Civita symbol, $\epsilon_{abc} = \delta_{123}^{abc}$. At energy lower bound

$$E_{\text{static}} \geq B,$$

where

$$B = -\frac{1}{24\pi^2} \int d^3 x \epsilon_{abc} \text{Tr} \left( [L_a, L_b, L_c] \right).$$

B integral, (31), is independent of space-metric tensor, i.e. topological quantity which is known as topological charge of SU(2) Skyrme model.

7 Static Skyrme Equation in Spherical Coordinate

In spherical coordinate, $(r, \theta, \phi)$, static Skyrme equation has the following form [1]

$$0 = \partial_r \left( L_r - \frac{1}{4} \left\{ \frac{1}{r^2} [L_\theta, [L_r, L_\theta]] + \frac{1}{r^2 \sin^2 \theta} [L_\phi, [L_r, L_\phi]] \right\} \right) + \partial_\theta \left( \frac{1}{r^2} L_\theta - \frac{1}{4} \left\{ \frac{1}{r^2} [L_r, [L_\theta, L_r]] + \frac{1}{r^4 \sin^2 \theta} [L_\phi, [L_\theta, L_r]] \right\} \right) + \partial_\phi \left( \frac{1}{r^2 \sin^2 \theta} L_\phi - \frac{1}{4} \left\{ \frac{1}{r^2} L_r, [L_\phi, L_r] \right\} \right)$$  

Its solution takes form

$$U(r) = \exp(i F_a(r, \theta, \phi) \sigma_a),$$

where

$$F_a(r, \theta, \phi) = g(r)n_a(\theta, \phi)$$

is Skyrme ansatz, $g(r)$ is profile function, $n_a$, $a = 1, 2, 3$ is component of unit vector, $\hat{n}$, in the internal space of SU(2), and $\sigma_a$ is Pauli matrix.
8 Skyrmion Static Energy and Its Solution

As a result of scaling spatial coordinates, by ignoring pion mass, static energy of SU(2) Skyrme model can be stated as [1]

\[ E_{\text{static}} = \frac{1}{12\pi^2} \left( 4\pi \right) \int r^2 dr \left[ \left( \frac{dg}{dr} \right)^2 + \frac{2}{r^2} \sin^2 g \left( 1 + \left( \frac{dg}{dr} \right)^2 \right) + \frac{1}{r^4} \sin^4 g \right]. \] (35)

From equation (35), by using least action principle

\[ \delta g E_{\text{static}} = 0, \] (36)
we can derive Euler-Lagrange equation for profile function [1]

\[ \frac{d^2 g}{dr^2} \left[ 1 + \frac{2}{r^2} \sin^2 g \right] + \left( \frac{dg}{dr} \right)^2 \left[ \frac{1}{r^2} \sin 2g \right] + \left( \frac{dg}{dr} \right) \left( \frac{2}{r} \right) - \frac{1}{r^2} \sin 2g - \frac{1}{r^4} \sin^2 g \sin 2g = 0. \] (37)

Equation (37) is second order of nonlinear differential equation.

The solution of equation (37) will be worked numerically with finite difference method. In order that static energy (35) has finite value at \( r = 0 \) and \( r = \infty \), then profile function \( g(r) \) must fulfills boundary conditions

\[ g(0) = \pi, \quad g(\infty) = 0. \] (38)

9 Quantized Rotational Energy

Skyrmion quantization is worked by involving time dependent of \( U(r) \) as

\[ U(r) \rightarrow U(r, t) = A(t)U(r)A(t)^\dagger, \] (39)
where

\[ A(t) \in SU(2)_{\text{internal}}, \quad AA^\dagger = A^\dagger A = I \] (40)

\( A \) is time dependent unitary matrix.

By using equation (39), rotational energy can be stated as [1, 12]

\[ E_{\text{rotation}} = - \left( \frac{\pi F^2}{6} \int_0^\infty dr r^2 \sin^2 g + \frac{2\pi}{3a^2} \int_0^\infty dr r^2 \sin^2 g \left[ \left( \frac{dg}{dr} \right)^2 + \frac{1}{r^2} \sin^2 g \right] \right) \left( R^{-1} \frac{\partial R}{\partial t} \right)^2 \]

\[ = \frac{1}{2} I \left( \text{Tr} \ \Omega^2 \right) \] (41)

where

\[ \text{Tr} \ \Omega^2 = -\text{Tr} \left( R^{-1} \frac{\partial R}{\partial t} \right)^2 \] (42)

with \( \Omega \) is angular velocity matrix of Skyrmion, and

\[ I = 2 \left( \frac{\pi F^2}{6} \int_0^\infty dr r^2 \sin^2 g + \frac{2\pi}{3a^2} \int_0^\infty dr r^2 \sin^2 g \left[ \left( \frac{dg}{dr} \right)^2 + \frac{1}{r^2} \sin^2 g \right] \right) \] (43)

is Skyrmion moment of inertia.
10 Finite Difference Method

Assume that \( g(x) \) is a continuous function of one variable. The value of this function is given only for discrete value, and equidistant \( x \):

\[
g_k \equiv g(x_k),
\]

where \( x_k \equiv x_0 + k\Delta x \). Let us define quantity

\[
\Delta g_k \equiv g_{k+1} - g_k
\]

which is called forward difference at point \( x_k \). Apply (45) for higher order of forward difference, we get

\[
\Delta^2 g_k \equiv \Delta g_{k+1} - \Delta g_k = g_{k+2} - 2g_{k+1} + g_k;
\]

\[
\Delta^3 g_k \equiv \Delta^2 g_{k+1} - \Delta^2 g_k = g_{k+3} - 3g_{k+2} + 3g_{k+1} - g_k
\]

and so on. General form of forward difference is:

\[
\Delta^r g_k \equiv \sum_{i=0}^{r} \left( -1 \right)^i \binom{r}{i} g_{k+r-i}.
\]

Backward difference is defined as

\[
\nabla g_k \equiv g_k - g_{k-1}
\]

and higher order of backward difference is

\[
\nabla^2 g_k \equiv \nabla g_k - \nabla g_{k-1} = g_k - 2g_{k-1} + g_{k-2}
\]

so on. General form of backward difference is

\[
\nabla^r g_k \equiv \sum_{i=0}^{r} \left( -1 \right)^i \binom{r}{i} g_{k-r+i}.
\]

Following, it is defined central difference which is symmetric with \( x_k \):

\[
\delta g_k \equiv g_{k+1/2} - g_{k-1/2}
\]

and higher order of central difference is

\[
\delta^2 g_k = g_{k+1} - 2g_k + g_{k-1}
\]

so on. General form of central difference is

\[
\delta^r g_k \equiv \sum_{i=0}^{r} \left( -1 \right)^i \binom{r}{i} g_{k-r/2+i}.
\]

If central difference is observed, \( k+1/2 \) index is out of the value which is served by discrete value. In order to obtain the integer value, mean value operator, \( \mu \), is used to operate with \( g(x) \) function as central mean

\[
\mu g = \frac{1}{2} \left[ g_{k+1/2} + g_{k-1/2} \right]
\]
and higher order of central mean is
\[ \mu^2 g_k \equiv \frac{1}{2} \left[ \mu g_{k+1/2} + \mu g_{k-1/2} \right] \] (56)
so on. **Central mean of central difference** is defined as
\[ \mu \delta g_k \equiv \frac{1}{2} \left[ \delta g_{k+1/2} + \delta g_{k-1/2} \right] = \frac{1}{2} \left[ g_{k+1} - g_{k-1} \right]. \] (57)

### 11 Numerical Solution of Skyrme Equation

Skyrme equation is stated in finite difference form, i.e. by substituting equation (49), (53), into Skyrme equation (37) then we obtain
\[ \frac{\delta^2 g_k}{\Delta r^2} \left[ 1 + \frac{2}{r^2} \sin^2 g_k \right] + \left( \frac{\nabla g_k}{\Delta r} \right)^2 \left[ \frac{1}{r^2} \sin 2g_k \right] + \left( \frac{\nabla g_k}{\Delta r} \right) \left( \frac{2}{r} \right) - \frac{1}{r^2} \sin 2g_k - \frac{1}{r^4} \sin^2 g_k \sin 2g_k = 0. \] (58)

By reposision of equation (58), we get
\[ \delta^2 g_k = -\left( \frac{\nabla g_k}{\Delta r} \right)^2 \left[ \frac{1}{r^2} \sin 2g_k \right] - \left( \frac{\nabla g_k}{\Delta r} \right) \left( \frac{2}{r} \right) \frac{2}{r^4} \sin^2 g_k \sin 2g_k \Delta r^2 \]
\[ = -\left( \frac{\nabla g_k}{\Delta r} \right)^2 \sin 2g_k - \left( \frac{\nabla g_k}{\Delta r} \right) \left( \Delta r \right) r + \sin 2g_k \Delta r^2 + \frac{1}{r^2} \sin^2 g_k \sin 2g_k \Delta r^2 \] \[ = \frac{r^2 + 2 \sin^2 g_k}{r^2 + 2 \sin^2 g_k} \] \[ g_{k+1} = -\left( \frac{\nabla g_k}{\Delta r} \right)^2 \sin 2g_k - \left( \frac{\nabla g_k}{\Delta r} \right) \left( \Delta r \right) r + \sin 2g_k \Delta r^2 + \frac{1}{r^2} \sin^2 g_k \sin 2g_k \Delta r^2 \]
\[ = \frac{r^2 + 2 \sin^2 g_k}{r^2 + 2 \sin^2 g_k} \] \[ = g_{k-1} + 2g_k. \] (60)
The problems are initial values which are needed to be defined, i.e.
\[ g(r = 0), \quad g(r = dr). \] (61)

Boundary conditions, which are given by Skyrme, are
\[ g(r = 0) = \pi, \quad g(r \to \infty) = 0. \] (62)

Apply the boundary conditions (62) for initial values (61). It is suitable for \( g(r = 0) \), and \( g(r = dr) \) must be chosen, so that it is suitable for \( g(r \to \infty) = 0 \). This method is known as **shooting method** (13). The plotting result, as shown in Figure 1: Profile function \( \phi(r) \), can be compared with trial and error method, i.e.
\[ g(x) = 4 \arctan e^{-r}. \] (63)

In order to calculate static energy (35), the following integral is used
Figure 1: Profile function $\phi(r)$

$$E(g) = \frac{2\pi f_\pi}{e} \int_0^\infty dr \left[ (r^2 g'^2 + 2 \sin^2 g) + \sin^2 g \left( 2g'^2 + \frac{\sin^2 g}{r^2} \right) \right]$$

(64)

where $g' = dg/dr$. Equation (64), is solved numerically with trapezoid method and it gives:

$$23.2154 \, \pi f_\pi / e.$$

Let us calculate numerically $E_{\text{static}}$ (65), using numerical value of $g(r)$

$$E_{\text{static}} = \frac{1}{12\pi^2} (4\pi) \left( \frac{23.2}{2} \right).$$

(65)

In the form of $F$ and $a$, using definition $(1/12\pi^2) = (F/4a)$, we obtain

$$E_{\text{static}} = \frac{\pi F}{a} \left( \frac{23.2}{2} \right).$$

(66)

The numerical value of $g(r)$ can be used to calculate

(1) nucleon ($m_N$) and delta static masses ($m_\Delta$);

(2) Skyrmion moment of inertia, $I$.

We get benefit from numerical value of profile function, $g(r)$, to calculate Skyrmion moment of inertia. It gives

$$I = \frac{1}{4} \left( \frac{447}{F a^2} \right).$$

(67)

In quantum mechanics, angular momentum $J$ is quantized as

$$J^2 = j(j + 1)\hbar^2$$

(68)
where \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \)

\[
h = \frac{\hbar}{2\pi},
\]

\( \hbar \) is Planck constant. Here, we use using natural unit, \( \hbar = 1 = c \).

Rotational energy is

\[
E_{\text{rotation}} = \frac{J^2}{2I} = \frac{j(j + 1)}{2I}.
\] (69)

### 12 Hadron Mass

Skyrmion is solution of Skyrme equation which has finite energy. Based on Einstein energy-mass formula, Skyrmion mass is

\[
m = \frac{E_{\text{static}} + E_{\text{rotation}}}{c^2} = M_{\text{static}} + M_{\text{rotation}}.
\] (70)

From equation (69) and (70), we obtain

\[
m = M_{\text{static}} + \frac{j(j + 1)}{2I}.
\] (71)

Wess-Zumino quantization condition [3], requires:

\[
j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots
\]

It means that Skyrmion is fermion.

In case of nucleon with spin \( \frac{1}{2} \), \( j = \frac{1}{2} \), equation (71) gives

\[
m_N = M_{\text{static}} + \frac{\frac{1}{2}(\frac{3}{2} + 1)}{2I}.
\] (72)

By substituting the values of \( F = 123 \) MeV and \( a = 4.95 \) into (66) and (67), give

\[
M_{\text{static}} \cong 905.08 \text{ MeV}
\] (73)

and

\[
I \cong 7.49 \times 10^{-3}.
\] (74)

Substitute (73) and (74) into (72) gives the result

\[
m_N \cong 955.15 \text{ MeV}.
\] (75)

In case of delta particle with spin \( \frac{3}{2} \), \( j = \frac{3}{2} \), equation (71) gives

\[
m_\Delta = M_{\text{static}} + \frac{\frac{3}{2}(\frac{5}{2} + 1)}{2I}.
\] (76)

By substituting the values of \( M_{\text{static}} \) and \( I \) above, it gives

\[
m_\Delta \cong 1155.41 \text{ MeV}.
\] (77)
13 Discussions

The calculation of nucleon energy-mass gives the value about 955.15 MeV. The experimental value of nucleon energy-mass is about 939 MeV. The different value of nucleon energy-mass based on calculation and experiment is about 16.15 MeV. The calculation of delta energy-mass gives the value about 1155.41 MeV. The experimental value of delta energy-mass is about 1232 MeV [3]. The different value of delta energy-mass based on calculation and experiment is about 76.59 MeV.

The different values among calculation and experiment are becaused of SU(2) Skyrme model is very simple model, i.e. it consists of two flavours, \((u,d)\) or \((u,s)\) or \((d,s)\). It is necessary keeping in mind related with these different values [15]:

1. Meson field Lagrangian only includes pseudoscalar field. Other low mass meson (vector) should be included.
2. In nature, there are three flavour families than two "light" flavour families which must be included in more realistic formula.
3. The effects of chiral and flavour symmetry breaking aren’t calculated yet.
4. \(N_C\) (number of colour) correction of nucleon mass isn’t included yet.

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[3] A.P. Balachandran, G. Marmo, B.S. Skagerstam dan A. Stern, Classical Topology and Quantum States, World Scientific, 1991.

[4] Wikipedia Encyclopedia, http://en.wikipedia.org/wiki/Flavour.

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15 Appendix: Flow Chart of Skyrme Profile Function Calculation and Its Integration

clear;
clf;
okmainloop=0;
N=1;
dr=.001;
$\%$dr=.01;
rf=50;
%rf=8;
rf=50;
rf=80;

nnl=length([0 dr:dr:rf rf+dr]);
g(nnl)=0;
n0=1;
n=2;
%

v=1+.0041
v0=v;
g(n0)=pi;%+0.01;
g(n)=g(n0)-v*dr;%0.02;

if okmainloop
% Looping utama program ------------
for r=[dr:dr:rf]
    n=n0+1;
    n1=n+1;
    
    gg=g(n);
    bg=g(n)-g(n0);
    bg2=bg*bg;
    sin2g=sin(gg)*sin(gg);
    sing2=sin(2*gg);
    r2=r*r;
    dr2=dr*dr;
    
    % x=eFr (catto)
    g(n1)=(-2*r*bg*dr-sing2*(4*bg2-4*sin2g*dr2/r2-dr2))/(r2+8*sin2g)+2*g(n)-g(n0);
    
    n0=n0+1;
    %if (g(n1)<=0) break; end;
end
% akhir looping utama----------------
save simpan.mat
else
    load simpan.mat
end

rr=[0 dr:dr:rf rf+dr];
nn=n1;
gg=g(1:nn);
% subplot(2,1,1); subplot(1,1,1); plot(rr(1:nn-1),gg(1:nn-1),'b-');

rr=[0 dr:dr:rf rf+dr]; %rre=pi*exp(-rr); rre=4*atan(exp(-rr));

hold on; plot(rr(1:nn-1),rre(1:nn-1),'r-'); grid on

%%%%%% negatif change to zero %g=g.*(g>0);

% Statical calculation
IIEUc=0; IIaUc=0; IIbUc=0; IIcUc=0; IIrUc=0;

jumlah=0; dgdr2=v*v; % from initial value gg=g(1); sin2g=sin(gg)*sin(gg);

for n=[1:nn1-1]
    dg=g(n+1)-g(n); %dg1=g(n+2)-g(n);
    if n==1
dgdr2=(dg*dg)/(dr*dr);
    else
        bg=g(n)-g(n-1);
        Dg=(g(n+1)-g(n-1))*0.5;
        dgdr2=(dg*dg)/(dr*dr);
        %dgdr2=(dg*bg)/(dr*dr);
        %dgdr2=(Dg*Dg)/(dr*dr);
    end
    gg=g(n);
    r=n*dr;
end
\[ \sin 2g = \sin(g g) \times \sin(g g) \]

\[ \text{jumlah} = \text{jumlah} + dr \times gg; \]
\[ r2 = r \times r; \]
\[ \text{IIEUc} = \text{IIEUc} + dr \times \left( r2 \times dgdr2 + \sin 2g \times \left( 2 + 8 \times dgdr2 + 4 \times \sin 2g / (r \times r) \right) \right); \]
\[ \% \text{ catto} \]
\[ \text{IIaUc} = \text{IIaUc} + dr \times r2 \times \sin 2g \times \left( 1 + 4 \times (dgdr2 + \sin 2g / (r \times r)) \right); \]
\[ \% \text{ catto} \]
\[ \text{IIbUc} = \text{IIbUc} + dr \times r2 \times (1 - \cos(g g)) \times \left( 1 + dgdr2 + 2 \times \sin 2g / (r \times r) \right); \]
\[ \text{IIICUc} = \text{IIICUc} + dr \times r2 \times (1 - \cos(g g)); \]
\[ \text{IIrUc} = \text{IIrUc} + dr \times r2 \times \sin 2g \times (bg / dr); \]
\[ \text{if } (g(n+1) < 0) \text{ break; end} \]
\[ \text{IEUc} = \text{IEUc} \times \pi \times 4 / 8; \]
\[ \text{IaUc} = \text{IaUc} \times \pi \times 2 / 3; \]
\[ \text{IbUc} = \text{IbUc} \times \pi \times 1 / 2; \]
\[ \text{ICUc} = \text{ICUc} \times \pi \times 1 / 2; \]
\[ \text{IrUc} = \text{IrUc} \times 2 / \pi; \]
\[ \text{save II.mat II* dr v0} \]