Three-Point Functions
of
Quarter BPS Operators in $\mathcal{N}=4$ SYM

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Abstract

In a recent paper \cite{19}, $\frac{1}{4}$-BPS chiral primaries were constructed in
the fully interacting four dimensional $\mathcal{N}=4$ Super-Yang-Mills theory
with gauge group $SU(N)$. These operators are annihilated by four super-
charges, and at order $g^2$ have protected scaling dimension and normal-
ization. Here, we compute three-point functions involving these $\frac{1}{4}$-BPS
operators along with $\frac{1}{2}$-BPS operators. The combinatorics of the prob-
lem is rather involved, and we consider the following special cases: (1)
correlators $\langle O_1 O_2 O_{\text{BPS}} \rangle$ of two $\frac{1}{2}$-BPS primaries with an arbitrary chi-
ral primary; (2) certain classes of $\langle O_4 O_4 O_{\text{BPS}} \rangle$ and $\langle O_4 O_4 O_4 \rangle$ three-point
functions; (3) three-point functions involving the $\Delta \leq 7$ operators found
in \cite{19}; (4) $\langle O_2 O_2 O_4 \rangle$ correlators with the special $O_4$ made of single and
double trace operators only. The analysis in cases (1)-(3) is valid for gen-
eral $N$, while (4) is a large $N$ approximation. Order $g^2$ corrections to all
three-point functions considered in this paper are found to vanish.

In the AdS/CFT correspondence, $\frac{1}{4}$-BPS chiral primaries are dual to
threshold bound states of elementary supergravity excitations. We present
a supergravity discussion of two- and three-point correlators involving
these bound states.

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1 Introduction

The AdS/CFT correspondence \[1, 2, 3\] provides a powerful tool for deriving dynamical information in \( \mathcal{N}=4 \) superconformal YM theory outside the regime of weak coupling perturbation theory. Comparison of weak and strong coupling behaviors has given rise to a number of surprising new conjectures \[4\], \[5\], \[6\], \[7\], \[8\], \[9\] that 2- and 3-point functions, as well as certain “extremal” n-point correlators of \( \frac{1}{2} \)-BPS operators are “unrenormalized”, \[10\], i.e. when properly defined, their form is independent of the gauge coupling \( g \). \( \mathcal{N}=2 \) superspace methods have since further confirmed the validity of these results \[11\], \[12\], \[13\]. Extensions to “near-extremal correlators” have unveiled a hierarchical pattern of \( g \)-dependence that further generalizes the non-renormalization conjectures in exciting ways \[14\].

The AdS/CFT correspondence relies on a direct duality between the \( \frac{1}{2} \)-BPS operators and the canonical fields and states of Type IIB supergravity on \( \text{AdS}_5 \times S^5 \), since they both belong to the shortest multiplets of the superconformal group \( SU(2,2|4) \), \[1\], \[2\], \[3\], \[15\]. However, in addition the \( \frac{1}{2} \)-BPS operators, also \( \frac{1}{4} \) - and \( \frac{1}{8} \)-BPS operators enjoy certain non-renormalization properties, such as the fact that their scaling dimension is fixed entirely by their internal quantum numbers \[16\], \[17\], \[18\], \[19\]. The \( \frac{1}{2} \) - and \( \frac{1}{4} \)-BPS operators are dual on the AdS side to threshold bound states of elementary supergravity excitations, typically consisting of at least two and three supergravity states respectively, which have not been explored from the AdS point of view.

Through the study of four point functions of \( \frac{1}{2} \)-BPS operators, certain couplings, such as those of two \( \frac{1}{2} \)-BPS operators and one \( \frac{1}{4} \)-BPS operator have been analyzed in SYM theory. In weak-coupling perturbation theory this was done in \[9\], while arguments based on \( \mathcal{N}=2 \) superfield methods were presented in \[20\]. Furthermore, using the \( \mathcal{N}=4 \) superfield approach, a general study of 3-point functions and their non-renormalization properties was initiated in \[21\]. \( \mathcal{N}=4 \) superfield methods, however, require on-shell superfields whose use in the study of off-shell correlators is not fully understood.

In a previous paper \[19\], a construction was presented for \( \frac{1}{4} \)-BPS chiral primaries in the fully interacting \( \mathcal{N}=4 \) SYM theory. In general, these \([p,q,p]\) operators are linear combinations of all local, polynomial, gauge invariant, scalar composite operators with the correct \( SU(4) \) labels. Besides the double trace operators from the classification of \[17\], \[18\], single trace and other multiple trace operators made of the same scalar fields also have to be taken into account. The coefficients with which they all combine into operators with a well defined scaling dimension are quite involved. However, in the large \( N \) limit, \( \frac{1}{4} \)-BPS primaries of a special form (those made of the single trace operator and the double trace operator from the classification of \[18\]) become surprisingly simple. The \( \frac{1}{4} \)-BPS chiral primaries, like the \( \frac{1}{2} \)-BPS operators extensively studied in the literature, also have protected two-point functions, at least at order \( g^2 \) \[19\].

Presently, we investigate the (non-) renormalization properties of three-point
correlators involving $\frac{1}{2}$-BPS operators along with $\frac{1}{4}$-BPS operators. Given the elaborate combinatorics of the problem, we concentrate on the following special cases. First, we discuss several group theoretic simplifications of the combinatorial factors multiplying the Feynman graphs that contribute to three-point functions of chiral primaries. Based on $SU(4)$ group theory and conformal invariance only, we argue that certain classes of such correlators are protected at order $g^2$, for all $N$. In particular, this allows us to compute $O(g^2)$ corrections to correlators of the form $\langle O_2^a O_2^b O_{\text{BPS}} \rangle$, where $O_2^a$ are two $\frac{1}{2}$-BPS operators, and $O_{\text{BPS}}$ is an arbitrary ($\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$)-BPS chiral primary. Next, we look at the three-point functions $\langle O_2^a O_4^b O_4^c \rangle$ and $\langle O_4^a O_4^b O_4^c \rangle$, also for general $N$, where $O_2^a$ are the $\Delta \leq 7 \frac{1}{2}$-BPS primaries found in [19]. Then, we carry out a large $N$ analysis of $\langle O_2^a O_2^b O_{\text{BPS}} \rangle$ correlators involving the special $\frac{1}{2}$-BPS operators (mixtures of single and double trace scalar composite operators), for arbitrary $\Delta$. Also in the large $N$ limit, we identify the corresponding objects in the supergravity description, and compute the correlators on the AdS side of the correspondence.

Finally, we make some speculations. Based on the broad range of special cases studied in this paper and in [19], we conjecture that two- and three-point functions of $\frac{1}{2}$- and $\frac{1}{4}$-operators receive no quantum corrections, for arbitrary $N$. Additionally, a set of group theoretic considerations of this paper extends straightforwardly from three-point functions to extremal correlators. Therefore, we suggest that extremal correlators involving $\frac{1}{2}$- and $\frac{1}{4}$-operators are protected as well.

2 The operators

We begin setting the stage for computing three-point functions, by describing the operators we will deal with. The construction of gauge invariant scalar composite operators was explained in [19], so here we will briefly review the main points of that discussion, as well as some well established facts.

Four dimensional $\mathcal{N}=4$ SYM is a superconformal theory, and has a global $SU(2,2|4)$ superconformal symmetry group. Operators in the theory fall into multiplets of $SU(2,2|4)$ [10, 13], and chiral primary operators are classified by its maximal bosonic subgroup [13], which includes the $R$-symmetry group $SU(4) \sim SO(6)$. The most widely studied operators in the theory are the $\frac{1}{2}$-BPS primaries, i.e. chiral primary operators annihilated by 8 out of 16 Poincaré supercharges. These are totally symmetric rank $q$ tensors of the flavor $SO(6)$, with highest weight operators of the form $\text{tr} (\phi^q)^2$, minus $SO(6)$ traces. The $SU(4)$ labels of these representations are $[0, q, 0]$, and the $SO(6)$ Young tableau corresponding to totally symmetric rank $q$ tensor is $\young(\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\·
1/4-BPS chiral primaries belong to \([p, q, p]\) representations with \(p \geq 2\). In the
\(SO(6)\) notation, the highest weight state of \([p, q, p]\) corresponds to the
\[
\begin{array}{ccc}
\text{1} & \cdots & \text{1} \\
\text{2} & \cdots & \text{p+q}
\end{array}
\]

Young tableau. In the free theory, 1/4-BPS primaries with the highest \(SU(4)\) weights
are of the form \(\text{tr} (\phi^1)^{p+q} \text{tr} (\phi^2)^p\) (modulo \((\phi^1, \phi^2)\) antisymmetrizations,
and subtraction of the \(SO(6)\) traces). However, there are many other ways
to partition a given Young tableau. Each partition may result in a different
operator after we take the \(SU(N)\) traces. To be more explicit, consider the
simplest example of \([p, q, p]\) scalar composite operators, namely \([2, 0, 2]\). The
ways to partition the Young tableaux corresponding to this representation are
\[
\begin{array}{c}
\young(1,1,2), \\
\young(1,1,2), \\
\young(1,1,2), \\
\young(1,1,2), \\
\young(1,1,2), \\
\young(1,1,2)
\end{array}
\]

where each continuous group of boxes corresponds to a single \(SU(N)\) trace.
After taking traces, the last three partitions vanish identically (since \(\text{tr} \phi^I = 0\)); and the “4=2+2” partitions turn out to give the same operator.
For a general \([p, q, p]\) representation the arguments are similar, although they become
progressively more tedious as \(2p + q\) gets larger. Scalar composite operators
with \(2p + q \leq 7\) are listed in Appendix A.

Sometimes, the way we take the \(SU(N)\) traces to obtain a \([p, q, p]\) scalar
composite operator is not important, and the only relevant information is what
fields are used, and that the operator is actually gauge invariant. In such cases,
we shall use the notation \([...]\) to denote gauge invariant combinations of the
fields in brackets. For example, operators corresponding to the highest weight
state of representations \([p, q, p]\) will be written as \([(\phi^1)^{p+q} (\phi^2)^p] - SO(6)\) traces,
etc.

In the interacting theory, none of the operators obtained by simply partitioning
Young tableaux are eigenstates of the dilatation operator (or pure) \[19\]. Instead, they are mixtures of operators with different scaling dimensions.
Proper 1/4-BPS primaries are certain linear combinations of these mixtures. To
find the correct linear combinations, one can look at two-point functions, and
The 1/4-BPS primaries are identified as the operators which receive no \(\mathcal{O}(g^2)\)
corrections to two-point functions among themselves or with other scalar composite operators. The protected scaling dimension of a \([p, q, p]\) chiral primary is
\(\Delta = 2p + q\).

The classical (Euclidean theory) Lagrangian can be written as \[10\]
\[
L = \text{tr} \left\{ \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \tilde{\lambda} \gamma^\mu D_\mu \lambda + \overline{D_\mu z_j} D^\mu z_j + \frac{i}{2} \tilde{\psi}^j \gamma^\mu D_\mu \psi^j \right\}

+ \sqrt{2} g f^{abc} \left( \tilde{\lambda}_a z_b^c L \psi^j_c - \tilde{\psi}^j_a z_b^c R z^j_b \lambda_c \right) - \frac{1}{2} N^2 f^{abc} f^{ade} \epsilon_{ijk} \epsilon_{ilm} z^j_d z^k_m z^l_c z^m_c
\]

They are 1/4-BPS for \(p = 0\); and vanish for \(p = 1\) after we take the \(SU(N)\) traces.

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where $L$ and $R$ are chirality projectors. Separate coupling constants $g$ and $Y$ are used to distinguish the terms coming from the gauge and superpotential sectors. The terms in the Lagrangian (3) proportional to $g$ and $g^2$ come from the $D$-terms; while the ones appearing with $Y$ and $Y^2$, from the $F$-terms.

The theory defined by (3) has a global symmetry group $SU(3) \times U(1)$. When $Y = g \sqrt{2}$, SUSY is enhanced from $\mathcal{N} = 1$ to $\mathcal{N} = 4$; and the $R$-symmetry group becomes $SU(4)$, although its manifestly realized part is still $SU(3) \times U(1)$.

Elementary fields in the Lagrangian (3) have good quantum numbers under this subgroup of $SU(4)$. The scalars $z_i$ and $\bar{z}_i$ are related to the fields $\phi^I, I = 1, ..., 6$ of a manifestly $SO(6)$ invariant formulation by a projection $\phi_a = \frac{1}{\sqrt{2}}(z_a + \bar{z}_a)$, $\phi_{a+3} = \frac{1}{\sqrt{2}}(z_a - \bar{z}_a), a = 1, 2, 3$. Order $g^2$ calculations are simpler in terms of the fields $\{\lambda, z_i, \bar{z}_i, \psi_i, \bar{\psi}_i | i = 1, 2, 3\}$, rather than $\{\phi^I, \lambda^j | I = 1, ..., 6, j = 1, ..., 4\}$, so we will use the Lagrangian (3) throughout this paper.

### 3 Contributing diagrams

Next, we to sort out the diagrams which contribute to correlators of these operators at order $g^0$ and $g^2$.

The two-point functions of the scalar composites discussed in (3) have the schematic form

\[
\langle [z_1^{(p+q)} z_2^{p}] (x) [\bar{z}_1^{(p+q)} \bar{z}_2^{p}] (y) \rangle
\]

where [...] are some gauge invariant combinations. The free field part of (3) is given by a power of the free correlator $[G(x, y)]^{(2p+q)}$, times a combinatoric factor. From the Lagrangian (3) we can read off the structures for the leading correction to the propagator, and the four-scalar blocks. These are shown in Figure 1, where they are categorized according to their gauge group (color) index structure (we use the same notation as in (3)). Gauge fixing and ghost terms in the Lagrangian do not contribute at this order (as we are only looking at operators made up of scalars).

Three-point functions to be considered in this paper are of the form

\[
\langle [z^{k+1}] (x) [z^{k+m}] (y) [z^l z^m] (w) \rangle
\]

The free result is just the product of appropriate powers of free correlators $[G(x, y)]^{k} [G(x, w)]^{l} [G(y, w)]^{m}$. The same structures that contribute to the two-point functions at order $g^2$ (see Figure 1), also contribute to the three-point functions (3). Apart from these, there are new building blocks, shown in Figure 2. They have the same index structure, but are now functions of three space-time coordinates rather than two.

Notice that the $F$-term corrections proportional to $\tilde{B}(x, y)$ in Figure 1, and the last graph (proportional to $\tilde{C}(x; y, w)$) in Figure 2, are antisymmetric in $i$ and $j$, hence they are absent when the scalars in the four legs have pairwise the
Figure 1: Structures contributing to two-point functions of scalars at order $g^2$ through four-scalar blocks and the propagator. Thick lines correspond to exchanges of the gauge boson, and of the auxiliary fields $F_i$ and $D$ (in the $\mathcal{N}=1$ formulation; after integrating out $F_i$ and $D$, the $zz\bar{z}\bar{z}$ vertex). The scalar propagator remains diagonal in both color and flavor indices at order $g^2$.

Figure 2: Building blocks for $g^2$ corrections to three-point functions. The three-points are $x$ (with two legs attached) and $y$ and $w$ (with a single leg each).

same flavor. For the same reason, these corrections are also absent when the operator at point $x$ is symmetric in all of its flavor indices. In particular, this is the case when the operator at $x$ is $\frac{1}{2}$-BPS.
4 Restrictions from $\mathcal{N}=4$ SUSY and gauge invariance

The form of quantum corrections to two and three-point functions is known \[5\]. Space-time coordinate dependence of the Feynman diagrams contributing to these correlators at order $\mathcal{O}(g^2)$ is constrained, since all exchanged fields are massless. We know the parametric form of the functions $A(x,y), B(x,y), C(x; y, w)$; and $C'(x; y, w), C''(x; y, w)$, and $C(x; y, w)$, without having to perform integrals explicitly. Functions which depend on two space-time points, are of the form $A(x_1, x_2) = a\log x_{12}^2 + b$ with $x_{ij} = x_i - x_j$; three-point contributions look like $C(x_1; x_2, x_3) = a'\log x_{12}^2 x_{13}^2 + a''\log x_{23}^2 + b'$ (making use of the $x_2 \leftrightarrow x_3$ symmetry of these building blocks).

$\mathcal{N}=4$ SUSY tells us more. From non-renormalization of two and three-point functions of operators in the stress tensor multiplet, one can see \[5\] that $B(x, y) = -2A(x, y)$, and $C'(0; x, y) + C(0; x, y) = -C(0; x, y)$; the authors of \[5\] chose to combine these and call it just $C'$. The coefficients $a', a''$ and $b'$ are determined\footnote{One way to see this is to consider the protected correlators of [0,2,0] scalar composite operators ($\text{tr} z_1 z_2(x) \text{tr} z_1 z_2(y)$), and $\langle [z^2]() \langle \bar{z}^2(y) [\bar{z} \bar{z}] () \rangle$ and $\langle [\bar{z} \bar{z}]() [\bar{z} \bar{z}]() [\bar{z} \bar{z}] () \rangle$.} in terms of $a$ and $b$:

$$A(x, 0) = -\frac{1}{2} B(x, 0) = a \log x^2 \mu^2 + b$$

$$C(0; x, y) = a \log \frac{x^2 y^2 \mu^2}{(x - y)^2} + b$$

Therefore, the net contribution to the three-point function \[5\] of the $\mathcal{O}(g^2)$ diagrams involving a gauge boson exchange (the ones proportional to $A$, $B$, and $C$), is

$$\langle [z^2]() [\bar{z}^2()] [z^m \bar{z}^n]() \rangle_{(A+B+C)}$$

$$= a (c_g^{12} \log x_{12}^2 \mu^2 + c_g^{13} \log x_{13}^2 \mu^2 + c_g^{23} \log x_{23}^2 \mu^2) + b$$

\footnote{This follows from $C(x; y, w) + C(y; x, w) + C(w; x, y) + A(x, y) + A(y, w) + A(x, w) = 0.$} (7)

where and $c_g^{ij}$ and $c_g^{123}$ are some combinatorial coefficients.

Now we use gauge invariance of the theory. On the one hand, we observe that the coefficients $a$ and $b$ are gauge dependent \footnote{Hence, the D-term diagrams proportional to $A$, $B$, and $C$ all cancel; their net contribution to the three-point functions \[5\] is zero.}:

$$A(x, 0) = \frac{1}{2} \pi^2 g^2 \xi \left[ \log x^2 \mu^2 + \log 4\pi - \gamma \right] + (\xi\text{-independent})$$

where $\xi$ is the gauge fixing parameter. On the other hand, a correlator of gauge invariant operators can not depend on $\xi$. Therefore, the combinatorial coefficients multiplying $a$ and $b$ in equation \(5\) must vanish, $c_g^{ij} = c_g^{123} = 0$. Hence, the $D$-term diagrams proportional to $A$, $B$, and $C$ all cancel; their net contribution to the three-point functions \[5\] is zero.
So just like in the case of two-point functions, we only have to consider the $F$-term graphs. They are proportional to $\tilde{B}$ and $\tilde{C}$, the only gauge independent diagrams around ($C' = -(C + \tilde{C})$ and $C'' = C - \tilde{C}$ do not have to be treated separately as they are linear combinations of the other ones).

In the $O(g^2)$ calculations of correlators of $1/4$-BPS operators \[5\] and \[7\], there were no other contributions to three-point functions except for those proportional to $A$ and $B$. Thus, gauge invariance together with $\mathcal{N}=4$ SUSY (which is needed to relate $C$ and $B$ to $A$) guarantees that the correlators of \[5\] and \[7\] receive no order $g^2$ corrections.

5 Position dependence of $\tilde{B}$ and $\tilde{C}$

Having shown that $D$-term corrections to three-point functions \[6\] are absent, it remains to consider the $F$-term interactions. In this Section we derive a relation between functions $\tilde{B}$ and $\tilde{C}$, which will play a key role in the analysis of three-point functions of $1/4$-BPS chiral primaries, see Section 7.1.1.

Space-time position dependence of $\tilde{B}$ and $\tilde{C}$ (shown Figures 1 and 2) is parametrically determined to be $\tilde{B}(x,0) = \tilde{a}\log(x^2\mu^2) + \tilde{b}$ and $\tilde{C}(0;x,y) = \tilde{a}'\log((x-y)^2\mu^2) + \tilde{c}$; furthermore, the leading divergent behavior can be read off from the integrals unambiguously and so from the limit $\tilde{C}(0;x,y \to x)$ we infer $\tilde{a}' = \frac{1}{2}\tilde{a}$.

To evaluate the remaining coefficients $\tilde{a}$, $\tilde{a}''$, and $\tilde{b}$, we use differential regularization \[23\] or a simpler equivalent prescription: replace $1/x^2 \to 1/(x^2 + \epsilon^2)$ for scalar propagators inside integrals. With this prescription

$$\tilde{B}(x,0) = -\frac{1}{4}Y^2 \int \frac{(d^4z)\left[4\pi^2 x^2\right]^2}{\left[4\pi^2((z-x)^2 + \epsilon^2)\right]^2 \left[4\pi^2(z^2 + \epsilon^2)\right]^2}$$

$$= -Y^2 \frac{1}{32\pi^2} \log(x^2/\epsilon^2) - 1$$

is the regularized two-point function, while the three-point function becomes

$$\tilde{C}(x; y, 0) = \frac{1}{4}Y^2 \int \frac{(d^4z)\left[4\pi^2 x^2\right] \left[4\pi^2(x-y)^2\right]}{\left[4\pi^2((z-x)^2 + \epsilon^2)\right]^2 \left[4\pi^2((z-y)^2 + \epsilon^2)\right] \left[4\pi^2(z^2 + \epsilon^2)\right]}$$

$$= -Y^2 \frac{1}{64\pi^2} \log \frac{x^2(x-y)^2}{y^2\epsilon^2}$$

(The numerators inside the integrals come about because of the powers of free scalar propagator in the definitions of $\tilde{B}$ and $\tilde{C}$, see Figures 1 and 2.) Therefore,

$$\tilde{C}(x; y, 0) + \tilde{C}(y; x, 0) - \tilde{B}(x, y) = -Y^2 \times \frac{1}{32\pi^2}$$

\[11\]

\[5\] This is the fastest way to calculate the integral for $\tilde{C}$, but one can obtain the same results using dimensional regularization.
is a nonzero constant (for $N=4$ SUSY, $Y^2 = 2g^2$).

The value of the constant $-Y^2/32\pi^2$ in equation (11) does not depend on the regulator $\epsilon$. Also note that with the “point splitting regularization” one would get the incorrect result of vanishing constant in (11).

6 Structure of the three-point functions

With the results of Section 4 at hand, we can write down the form of a general three-point function of scalar composite operators (5) to order $g^2$:

$$
\langle [z^k] (x) [\bar{z}^l z^m] (y) \rangle = G(x, y)^k G(x, w)^l G(w, y)^m
$$

$$
\times \left( \alpha_{\text{free}} + \beta_{xy} B(x, y) + \beta_{zw} B(x, w) + \beta_{yw} B(y, w)
+ \gamma_x \tilde{C}(x; y, w) + \gamma_y \tilde{C}(y; x, w) + \gamma_w \tilde{C}(w; x, y)
+ O(g^4) \right)
$$

(12)

where $\alpha_{\text{free}}, \beta$-s and $\gamma$-s are some combinatorial coefficients. Using the expressions (9) and (10) from Section 5, we can determine the $O(g^2)$ position dependence of (12) completely — if we know these combinatorial coefficients. Together with conformal invariance, and the SU(4) symmetry properties of the operators in (12), we can often go a long way to figuring out which of the combinatorial coefficients must vanish, without doing any actual calculations.

6.1 Space-time coordinate dependence

Like in the case of two-point functions, conformal invariance restricts the position dependence of three-point correlators of pure operators (i.e. ones which have a well defined scaling dimension). Consider three (gauge invariant Lorentz scalar) operators $O_1, O_2,$ and $O_3,$ of dimensions $\Delta_i = k_i + \delta_i,$ inserted at corresponding points $x_i.$ Let $k_i$ be integers and $\delta_i$ the order $g^2$ corrections to the scaling dimensions (which may or may not be zero). The three-point function $\langle O_1 O_2 O_3 \rangle$ is completely determined up to a multiplicative constant $C_{123} = C^0_{123} + C^1_{123}$ (where again $C^0_{123}$ is the free field result and $C^1_{123} \sim g^2$),

$$
\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{13}^{\Delta_2 + \Delta_3 - \Delta_1} x_{23}^{\Delta_1 + \Delta_3 - \Delta_2}}
$$

$$
= \langle O_1 O_2 O_3 \rangle_{\text{free}} \left( 1 + \frac{C^1_{123}}{C^0_{123}} \right)
$$

$$
- \delta_1 \log \frac{x_{12}^2 x_{13}^2}{x_{23}^2 \epsilon^2} - \delta_2 \log \frac{x_{21}^2 x_{23}^2}{x_{13}^2 \epsilon^2} - \delta_3 \log \frac{x_{31}^2 x_{32}^2}{x_{12}^2 \epsilon^2}
$$

$$
+ O(g^4) \right)
$$

(13)
with $x_{ij} = x_i - x_j$ as usual.

Suppose that all three operators have protected scaling dimensions, $\delta_i = 0$. Then (13) reduces to

$$\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \langle O_1O_2O_3 \rangle_{\text{free}} \left( 1 + C_{123}^1/C_{123}^0 \right)$$

and no logs arise. In this case, the combinatorial factors in (12) satisfy

$$\tilde{\gamma}_x = - (\tilde{\beta}_{xy} + \tilde{\beta}_{xw})$$

$$\tilde{\gamma}_y = - (\tilde{\beta}_{xy} + \tilde{\beta}_{yw})$$

$$\tilde{\gamma}_w = - (\tilde{\beta}_{xw} + \tilde{\beta}_{yw})$$

and we need to calculate only three coefficients (the $\tilde{\beta}$-s, for example), to find all the $O(g^2)$ corrections to this correlator. In fact, the only allowed correction is the constant $\tilde{\beta} = \tilde{\beta}_{xy} + \tilde{\beta}_{xw} + \tilde{\beta}_{yw}$ times $(-Y^2/32\pi^2)$.

To show that a three-point function of BPS operators is protected, we have to demonstrate that $\tilde{\beta} = 0$.

### 6.2 Group theory simplifications

There are several simplifications which set some of the combinatorial coefficients in (12) to zero. These considerations are based on the underlying $SU(4) \sim SO(6)$ symmetry of the theory only, and are applicable for general $N$. We will leave aside the trivial case when the correlator is forced to vanish by group theory, and assume that the Born level coefficient in (12) $\alpha_{\text{free}} \neq 0$.

The simplest BPS operators are $\frac{1}{2}$-BPS chiral primaries, gauge invariant scalar composites in $[0, q, 0]$ representations of $SO(6)$. These are totally symmetric tensors of $SO(6)$, so if for example the operator $O_x$ is $\frac{1}{2}$-BPS, the coefficients $\tilde{\beta}_{xy} = \tilde{\beta}_{xw} = \tilde{\gamma}_x = 0$ since the diagrams they multiply are antisymmetric in the flavor indices of $O_x$. Similarly, if both $O_x$ and $O_y$ are $\frac{1}{2}$-BPS and $O_w$ is any BPS operator, we have $\tilde{\beta}_{xy} = \tilde{\beta}_{xw} = \tilde{\gamma}_x = \tilde{\beta}_{yw} = \tilde{\gamma}_y = 0$, and hence $\tilde{\beta} = \tilde{\beta}_{xy} + \tilde{\beta}_{xw} + \tilde{\beta}_{yw} = 0$; there are no $O(g^2)$ corrections in this case. In particular, this reproduces the result of [5] when all three $O_{x,y,w}$ are $\frac{1}{2}$-BPS chiral primaries.

Now suppose that $O_w$ is $\frac{1}{2}$-BPS, and furthermore we can “partition the correlator into two flavors,” i.e. choose the operators such that $O_w = [z_1^m z_2^n]$ while $O_x = [z_1^k z_2^l]$ and $O_y = [z_1^{(k-m)} z_2^{(l-n)}]$. Consider a diagram proportional to $C(x; y, w)$, see Figure 3. The sum of all such diagrams is symmetric in the

---

6 If we choose $O_w$ as such a $\frac{1}{2}$-BPS operator, we would not be able to conclude that $\tilde{\gamma}_w = 0$ just from the symmetries of $\tilde{\gamma}_w$; the fourth diagram of Figure 2, is not antisymmetric in flavor indices at the vertex where the operator is made of both $z$ and $\bar{z}$-s. However, using equation (13) we find $\tilde{\gamma}_w = \tilde{\beta}_{xw} + \tilde{\beta}_{yw} = 0$ since $\tilde{\beta}_{xw} = \tilde{\beta}_{yw} = 0$ when all three operators have protected scaling dimensions.
Figure 3: $F$-term contributions to $\langle \mathcal{O}_{\text{BPS}}(x) \mathcal{O}'_{\text{BPS}}(y) \mathcal{O}_{\frac{1}{2}}(w) \rangle_{g^2}$ in the case when the correlator can be “partitioned into two flavors;” (a) proportional to $\tilde{B}(x, y)$; (b) proportional to $\tilde{C}(x; y, w)$; (c) proportional to $\tilde{C}(y; x, w)$.

Figure 4: Order $g^2$ corrections to correlators of the form (18): (a) and (b) includes a gauge boson exchange; (c) and (d) $F$-terms. Self energy contributions (not shown) also include a gauge boson exchange.

flavors at $y$ (since they are the same), and symmetric in flavors at $w$ (since $\mathcal{O}_w$ is $\frac{1}{2}$-BPS). But it must be antisymmetric in the flavors $z_1$ and $z_2$ leaving the interaction vertex, so all such diagrams cancel, and so $\tilde{\gamma}_x = 0$. In the same fashion, we conclude that $\tilde{\gamma}_y = 0$ as well, and together with $\tilde{\beta}_{yw} = \tilde{\beta}_{xw} = 0$ (as $\mathcal{O}_w$ is $\frac{1}{2}$-BPS), we find that $\beta = 0$ when $\mathcal{O}_x$ and $\mathcal{O}_y$ are any operators with protected scaling dimensions.

There is another type of three-point functions of BPS chiral primaries which receive no $O(g^2)$ corrections by similar considerations. Consider a correlator such that $\mathcal{O}_x$ is made of $z_1$ and $z_2$; $\mathcal{O}_y$ made of $\bar{z}_1$ and $\bar{z}_3$; and $\mathcal{O}_w$, made of $\bar{z}_2$ and $z_3$, i.e. a correlator of the form

$$\langle (z_1^n z_2^m)(x) [\bar{z}_1^m \bar{z}_3^k](y) [z_3^k \bar{z}_2^n](w) \rangle$$

This correlator is “partitioned into three disjoint flavors.” Order $g^2$ contributions to this three-point function are shown in Figure 4. There are no corrections proportional to any $\tilde{B}$-s since all lines within any rainbow carry the same flavor, so immediately $\tilde{\beta}_{xy} = \tilde{\beta}_{yw} = \tilde{\beta}_{xw} = 0$ and hence there are no $O(g^2)$ corrections here, as well.

Finally, extremal three-point functions can be analyzed in a simple way. Here, the scaling dimension of one of the operators is equal to the sum of
Figure 5: Order $g^2$ corrections to extremal correlators: (a) and (b) within a single peddle; (c) and (d) between the two peddles. Self energy contributions (not shown) and diagrams (a) and (c) are gauge dependent, while (b) and (d) diagrams arise from the $F$-terms.

scaling dimensions of the other two. Suppose that $\Delta_x + \Delta_y = \Delta_w$ in (12). At Born level, there are no $G(x,w)$ propagators, and so there are no corrections proportional to $B(x,y), \tilde{C}(x; y, w)$, or $\tilde{C}(y; x, w)$, see Figure 5. Together with the constraints (13-17), this determines $\beta = 0$ when the three (Lorentz scalar) operators inserted $x$, $y$, and $w$ are arbitrary operators with protected scaling dimensions.

Another remark about extremal correlators is in order. As it is easy to see, one of the above group theory simplifications generalizes straightforwardly to extremal correlators of chiral primaries. Namely, if all operators except for one are $1/2$-BPS (and the remaining one is an arbitrary chiral primary), extremal correlators receive no order $g^2$ corrections.

7 Three-point functions of BPS operators

We are now ready to discuss correlators of three BPS chiral primaries. The simplest correlators $\langle O_1 O_2 O_3 \rangle$ (where each $O_i$ stands for a $1/2$-BPS operator), were considered by [5], who found that three-point functions of $1/2$-BPS operators do not get corrected at order $g^2$, for any $N$. These are a special case of correlators of the form $\langle O_2 O_3 O_{BPS} \rangle$, which we discussed in Section 5.2 (here $O_{BPS}$ is an arbitrary BPS operator). These three-point functions receive no $O(g^2)$ corrections by group theory reasoning.

7.1 Correlators $\langle O_2 O_3 O_{BPS} \rangle$

Not all three-point functions of chiral primaries can be simplified using the results of Section 5.2, so occasionally we will have to actually compute some of the combinatorial coefficients. In this Section we will look at correlators of two $1/2$-BPS operators with one $1/2$-BPS operator.

7 In general, $(n + 1)$-point functions $\langle O_0(x_0) O_1(x_1)... O_n(x_n) \rangle$ are called extremal if one of the scaling dimensions is the sum of all the others, $\Delta_0 = \Delta_1 + ... + \Delta_n$. 

11
7.1.1 \( \langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y)(\text{tr } z z_t)(w) \rangle \)

The simplest \( \langle O_{\underline{4}} O_{\underline{4}} O_{\underline{2}} \rangle \) three-point functions are of the form

\[
\langle O(x) \bar{O}'(y)(\text{tr } X^2)(w) \rangle
\]

where the \( \frac{1}{2} \)-BPS primary \( \text{tr } X^2 \) is a scalar composite operator in the \([0,2,0]\) of \( SU(4) \). Group theory restricts the quantum numbers of operators which can have nontrivial tree point functions. Tensoring \([p,q,p] \otimes [0,2,0]\) using Young diagrams of \( SO(6) \) gives

\[
\begin{array}{c}
\begin{array}{c}
P \quad q \\
Q \\
Q \\
\end{array}
\end{array}
\otimes
\begin{array}{c}
\begin{array}{c}
P \quad q \\
Q \\
Q \\
\end{array}
\end{array}
\]

where in the first row there are no contractions (i.e. \( SO(6) \) traces), only symmetrizations and antisymmetrizations; in the second row, one contraction; and in the third row, two contractions; the “...” stands for tableaux with more than two rows.

In terms of Dynkin labels, equation \( \text{(20)} \) reads

\[
[p,q,p] \otimes [0,2,0] = [p,q+2,p] \oplus [p+1,q,p+1] \oplus [p+2,q-2,p+2]
\oplus [p,q,p] \oplus [p+1,q-2,p+1] \oplus [p-1,q+2,p-1]
\oplus [p,q-2,p] \oplus [p-1,q,p-1] \oplus [p-2,q+2,p-2]
\oplus ...
\]

Now, the “...” stands for representations with \([r,s,r+2k]\) Dynkin labels with \( k \neq 0 \). Thus the only three-point functions of the form \( \text{(19)} \) which can possibly have a nonzero value are the extremal correlators

\[
\langle O_{[p,q,p]}(x) \bar{O}_{[p,q-2,p]}(y)(\text{tr } z z_t^2)(w) \rangle
\]

\[\text{(22)}\]

\[
\langle O_{[p,q,p]}(x) \bar{O}_{[p-2,q+2,p-2]}(y)(\text{tr } z z_t^2)(w) \rangle
\]

\[\text{(23)}\]

\[
\langle O_{[p,q,p]}(x) \bar{O}_{[p-1,q,p-1]}(y)(\text{tr } z z_t^2)(w) \rangle
\]

\[\text{(24)}\]

which correspond to those diagrams in \( \text{(20)} \) with zero or maximal number of contractions; and non-extremal correlators

\[
\langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y)(\text{tr } z t z_t)(w) \rangle
\]

\[\text{(25)}\]

\[
\langle O_{[p,q,p]}(x) \bar{O}_{[p-1,q+2,p-1]}(y)(\text{tr } z z_{t1})(w) \rangle
\]

\[\text{(26)}\]

where \( t \) is a diagonal \( SU(3) \) generator. All other correlators of the form \( \text{(19)} \) either vanish because the tensor product of irreps \([p,q,p]\) and \([0,2,0]\) does not contain \([r,s,t]\), or are related to the ones in \( \text{(22)-(26)} \).
Extremal three-point functions were discussed in Section 6.2, and were found to be protected at order $g^2$. The only correlators of the form $\langle O \bar{O} (\text{tr } z t z) \rangle$ we need to consider are those given by (25) and (26). However, the three-point functions of Figure 6(c), must in fact vanish: $\text{tr } z_2 z_1 = \frac{1}{2} z_2^a z_1^a$ is diagonal in color indices, and hence the combinatorial factors for the Born graph of $\langle O_{[p,q,p]}(x) \bar{O}_{[p-1,q+2,p-1]}(y) (\text{tr } z_2 z_1)(w) \rangle$ are proportional to the ones for the two-point function $\langle O_{[p,q,p]} \bar{O}_{[p-1,q+2,p-1]} \rangle = 0$. The same thing happens at order $g^2$, etc. So correlators (26), although allowed by (21), are in fact forbidden by a combination of $SU(N)$ and $SU(4)$ group theory.

Correlators $\langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y)(\text{tr } z t z)(w) \rangle$ are the only ones that remain to be considered. The contributing Born level diagrams are shown in Figure 6(a,b), and the $O(g^2)$ graphs appear in Figure 7 (corrections to the scalar propagator are not shown, but are also present). Repeating the arguments of [5] from the $\frac{1}{2}$-BPS calculations, we see that the combinatorial structure of this three-point function $\langle O \bar{O} (\text{tr } z t z) \rangle$ is the same as that of the two-point function $\langle O \bar{O} \rangle$. At Born level, we find that

$$\langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y)(\text{tr } z t z)(w) \rangle|_{\text{free}} = \frac{1}{2}[(p+q)t_{11} + pt_{22}] \frac{G(x,w)G(y,w)}{G(x,y)} \langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y) \rangle|_{\text{free}}$$

At order $g^2$, the contributions proportional to $\tilde{B}$ and $\tilde{C}$ (diagrams (a1) and (b1) in Figure 6) have the same index structure, which in turn is identical to that of the two-point functions $\langle O_{[p,q,p]}(x) \bar{O}_{[p,q,p]}(y) \rangle$. Because $\text{tr } z_1 z_1$ is diagonal in color indices, its only effect on the combinatorics is to distinguish the pair of indices which go to $O_w$ rather than stretch directly between $O_x$ and $O_y$.

There is a curious relation between the functions $\tilde{B}(x,y)$ and $\tilde{C}(x;y,w)$, 

---

9 Explicitly, in Section 6.2 we saw that the $O(g^2)$ part of this three-point function vanishes.
which can be graphically expressed as:

\[
\begin{align*}
\langle O_{1} \rangle &= \text{constant} \\
\langle O_{1}(x)O_{2}(y) \rangle &= \frac{C_{12}}{(x-y)^{2\Delta}} \\
\langle O_{1}(x)O_{2}(y) \rangle &= \frac{\tilde{C}_{12} G(x,w)G(y,w)}{(x-y)^{2\Delta}} \frac{G(y,x)}{G(x,y)}
\end{align*}
\]

in \( d = 4 \). In other words, coordinate dependence (modulo the ratio of free scalar correlators) is the same, and the difference is just a constant factor. Assume now that \( O_{1} \) is constructed of only \( z_{i} \)-s; and \( O_{2} \), \( \tilde{z}_{j} \)-s. Then, the only contributions to the two-point functions \( \langle O_{1}(x)O_{2}(y) \rangle \) are proportional to \( \tilde{B}(x,y) \); similarly, the correlators \( \langle O_{1}(x)O_{2}(y) \rangle \) are proportional to \( \tilde{B}(x,y) + \tilde{C}(x;w,y) + \tilde{C}(y;w,x) \). Index structure of these building blocks is the same (as discussed after equation \( 27 \)), so

\[
\frac{\langle O_{1}(x)O_{2}(y) \rangle}{\langle O_{1}(x)O_{2}(y) \rangle}_{\text{free}} = \tilde{B}(x,y)
\]

\[
\frac{\langle O_{1}(x)O_{2}(y) \rangle}{\langle O_{1}(x)O_{2}(y) \rangle}\text{tr}X^{2}(w)}{\langle O_{1}(x)O_{2}(y) \rangle}\text{tr}X^{2}(w)}_{\text{free}} = \frac{1}{2}\zeta_{12} \left[ \tilde{C}(x;w,y) + \tilde{C}(y;w,x) + \tilde{B}(x,y) \right]
\]

with the same \( \zeta_{12} \). As was discussed in Section \( 4 \), \( \tilde{B} \) and \( \tilde{C} \) have the form

\[
\tilde{B}(x,y) = \tilde{a} \log \frac{(x-y)^{2}}{(x-w)^{2}} + \tilde{b}, \quad \tilde{C}(x;w,y) = \tilde{a} \log \frac{(x-w)^{2}}{(x-y)^{2}} - \tilde{a} \log \frac{(w-y)^{2}}{(x-w)^{2}} + \tilde{b}'.
\]

10 The fact that \( \tilde{C}(x;w,y) + \tilde{C}(y;w,x) - \tilde{B}(x,y) \) is just a constant was established in Section \( 5 \) by an explicit calculation. The value of this constant was also found there.

11 In particular, if \( \Delta = \Delta_{0} \) is not corrected, then neither are \( \langle O_{1}(x)O_{2}(y) \rangle \) and \( \langle O_{1}(x)O_{2}(y) \rangle \text{tr}X^{2}(w) \), at least at one loop.
By comparing (30) and (29), we see that expression (32) must have the same coordinate dependence as (31). This restricts \( \hat{a}' = \hat{a}'' = \frac{1}{2} \hat{a} \), which reproduces the “winking cat” identity (28).

Finally, we can relate \( \langle O_1(x) O_2(y) \text{ tr } X^2(w) \rangle \) to \( \langle O_1(x) O_2(y) \rangle \) by a Ward identity. As shown in [24], the ratio

\[
\frac{\langle O(x) O(y) T_{\mu\nu}(0) \rangle}{\langle O(x) O(y) \rangle} = \frac{2 \Delta}{3 \pi^2} \frac{t_{\mu\nu}(\gamma)(x-y)^4}{x^4 y^4}
\]

depends on the scaling dimension \( \Delta \) of the operator \( O \) (here, \( \gamma = \frac{x}{y} - \frac{y}{x} \) and \( t_{\mu\nu}(\gamma) = \frac{2 \Delta}{3 \pi^2} - \frac{1}{4} \eta_{\mu\nu} \)). Since the energy momentum tensor \( T_{\mu\nu} \) is in the same \( N = 4 \) multiplet with \( \text{tr } X^2 \), there is also nothing peculiar about the fact that \( \tilde{C}_{12}/C_{12} \) can in general receive \( O(g^2) \) correction. This ratio also depends on \( \Delta \).

### 7.1.2 General \( \langle O_{\frac{1}{2}} O_{\frac{3}{2}} O_{\frac{1}{2}} \rangle \) correlators

Three-point functions of two \( \frac{1}{2} \)-BPS operator and one \( \frac{3}{2} \)-BPS operator are similar to the ones described in Section 7.1.1. It suffices to consider a single three-point function (such that the Clebsch-Gordon coefficient\(^\text{12}\) for these three vectors in the given irreps of \( SU(4) \) is nonzero) for each set of three representations. Without loss of generality, we can choose a \([p, q, p] \) scalar composite \( O(x) \) to be made of only \( z \)-s; a \([r, s, r] \) scalar composite \( O'(y) \) to be made of only \( \bar{z} \)-s; and a \([0, k, 0] \) scalar composite \( \text{tr } X^{\alpha_1 + \alpha_2} \) at \( w \) of the form

\[
t_{i_1 \ldots i_{\alpha_1}; j_1 \ldots j_{\alpha_2}} \text{str } z_{i_1} \ldots z_{i_{\alpha_1}} \bar{z}_{j_1} \ldots \bar{z}_{j_{\alpha_2}}
\]

where \( \alpha_1 + \alpha_2 = k \), and \( t_{i_1 \ldots i_{\alpha_1}; j_1 \ldots j_{\alpha_2}} \) is the appropriate irreducible \( SU(3) \) tensor (like in [3]). The correlators we are after are

\[
\langle O(x) O'(y) (\text{tr } X^{\alpha_1 + \alpha_2})(w) \rangle
\]

Position dependence of (35) is

\[
[G(x, y)^{2(p+r)+(2r+s)-k} G(x, w)^{k+(2p+r)-(2r+s)} G(w, y)^{k-(2p+r)+(2r+s)}]^{1/2}
\]

at Born level. The contributing free diagrams are similar to the ones shown in Figure 4 and \( O(g^2) \) diagrams, to those of Figure 3 but now there can be a different number of lines stretching between \( x \) and \( w \) and between \( w \) and \( y \). Apart from the factor (36), the general \( \langle O_{\frac{1}{2}} O_{\frac{3}{2}} O_{\frac{1}{2}} \rangle \) correlator (33) is given by

\[
\alpha_{\text{free}} + \bar{\alpha}_{xy} \bar{B}(x, y) + \gamma_x \bar{C}(x; y, w) + \gamma_y \bar{C}(y; x, w) + O(g^4). \tag{37}
\]

\(^{12}\) By Wigner-Eckart theorem, for any three representations we only need to calculate one (nonvanishing) correlator of any representatives from these irreps.
According to the discussion of Section 6, the remaining combinatorial coefficients vanish, \( \beta_{xw} = \beta_{yw} = \gamma_w = 0 \). Moreover, \( \tilde{\beta}_{xy} = \tilde{\gamma}_w = -\tilde{\beta}_{xy} \) as follows from equations (15-17), so the \( O(g^2) \) corrections in (37) add up to

\[
\alpha_{\text{free}} - \tilde{\beta}_{xy} \left( \tilde{C}(x;y,w) + \tilde{C}(y;x,w) - \tilde{B}(x,y) \right) + O(g^4)
\]

and we only need to verify that \( \tilde{\beta}_{xy} = 0 \).

The simplifications we can use to deduce that \( \tilde{\beta}_{xy} = 0 \) without doing calculations, are discussed in Section 6.2. Extremal three-point functions are always easy to identify, and with the BPS primaries in representations \([p,q,p]\), \([r,s,r]\), and \([0,k,0]\), the restrictions on the scaling dimension translate into

\[
2r + s = 2p + q + k, \quad 2p + q = 2r + s + k, \quad \text{or} \quad 2p + q + 2r + s = k,
\]

depending on which scaling dimension is the sum of the other two.

The “three flavor partition” boils down to being able to choose a single flavor (at Born level) for the lines between the two \( \frac{1}{4} \)-BPS operators, when the third operator is \( \frac{1}{2} \)-BPS. This is possible whenever

\[
2r + s \leq k + q \quad \text{and} \quad 2p + q \leq k + s.
\]

Alternatively, suppose we can choose the \( \frac{1}{4} \)-BPS operator \( O_w \) to be made of only \( \bar{z}_1 \)-s and \( z_2 \)-s; and the \( \frac{1}{2} \)-BPS operators as \( O_x \) of \( z_1 \)-s and \( \bar{z}_2 \)-s, \( O_y \) of \( \bar{z}_1 \)-s and \( z_2 \)-s. This is the “two flavor partition” of Section 6.2. Flavors can be chosen this way if

\[
k \leq q + s.
\]

In all three cases \((39), (40), \) and \((41)\), there are no \( O(g^2) \) corrections, as established in Section 6.2 using only \( SU(4) \) group theory and conformal invariance arguments. However, there are allowed three-point functions of the form \( \langle O_{\frac{1}{4}} O_{\frac{1}{4}} O_{\frac{1}{2}} \rangle \) where we cannot choose irrep representatives in such a nice way.

Throughout the rest of this Section, we will concentrate on the \( \frac{1}{4} \)-BPS operators with dimensions 7 and smaller, constructed in \([19]\). In particular, we will consider scalar composite operators in \( SU(4) \) representations of the form \([p,q,p]\), with \( 2p + q \leq 7 \). These are \([2,0,2],[2,1,2],[2,2,2],[3,1,3] \), and \([2,3,2] \). We will take \( \frac{1}{2} \)-BPS (single trace) operators as whichever ones are allowed by group theory. Of the triple products of the form \([p,q,p] \otimes [r,s,r] \otimes [0,k,0] \) containing the singlet, most satisfy at least one of the simplifying constraints \((29),(40),(41)\). The exceptions are \([2,0,2] \otimes [2,0,2] \otimes [0,2,0] \) and \([3,1,3] \otimes [3,1,3] \otimes [0,4,0] \).

---

13 The expression multiplying \( \tilde{\beta}_{xy} \) in (38) is a nonzero, renormalization scale independent constant. In Section 3, its value was computed to be \(- \frac{1}{32} \frac{\pi^2}{2} \).

14 We omit the tedious details here. In order to find the allowed triple products, we used the method of BZ triangles, see Appendix B. Then we just went through the list and checked if any of the conditions \((39),(40),(41)\) apply.
Correlators of the form \( \langle \mathcal{O}_{[p,q,r]}(x) \mathcal{O}_{[p,q,r]}(y) \operatorname{tr} X^2(w) \rangle \) were considered in Section 7.1.1, so the only three-point function we actually have to calculate is \( \langle \mathcal{O}_{[3,1,3]} \mathcal{O}_{[3,1,3]} \operatorname{tr} X^{2+2} \rangle \). Explicitly, we can take

\[
\mathcal{O}_x = \sum_{j=1}^{4} C^j_x \mathcal{O}_j, \quad \mathcal{O}_y = \sum_{j=1}^{4} C^j_y \mathcal{O}_j \quad \text{with} \quad \mathcal{O}_j \sim [z_1^2 z_2^2 z_3],
\]

\[
\mathcal{O}_w \sim [z_1^2 z_2^2] - \text{SO}(6) \text{ traces},
\]

to be the scalar composite operators\(^{15}\) in the [3,1,3] of \( SU(4) \). With this choice of flavors, the free combinatorial factor for this three-point function is

\[
\alpha_{\text{free}} = \frac{(N^2 - 1)(N^2 - 2)}{41472 N^2} (189540 C^2_x C^2_y - 4860 C^2_y C^1_x N - \\
4860 C^2_y C^2_y N - 131220 C^2_x C^2_y N - 131220 C^2_y C^1_x N + \\
360 C^1_y N^2 + 13500 C^1_x C^1_x N^2 - 79380 C^2_y C^2_y N^2 - \\
22680 C^2_x C^2_y N^2 - 22680 C^2_x C^3_y N^2 + 5184 C^2_y C^3_y N^2 + \\
13500 C^1_x C^4_y N^2 - 30780 C^4_x C^1_y N^2 + 2700 C^2_x C^1_y N^3 - \\
270 C^2_x C^1_y N^3 + 2700 C^2_x C^2_y N^3 + 43740 C^3_y C^1_y N^3 - \\
270 C^3_y C^1_y N^3 - 9720 C^4_y C^3_y N^3 + 43740 C^3_y C^1_y N^3 - \\
9720 C^3_y C^3_y N^3 - 115 C^3_y C^1_y N^4 - 2760 C^4_y C^1_y N^4 + \\
13500 C^2_x C^2_y N^4 + 4410 C^3_x C^2_y N^4 + 4410 C^2_y C^3_y N^4 - \\
1332 C^4_y C^3_y N^4 - 2760 C^1_y C^4_y N^4 + 13680 C^2_y C^1_y N^4 - \\
450 C^2_x C^1_y N^5 + 240 C^3_x C^1_y N^5 - 450 C^1_x C^2_y N^5 - \\
4500 C^4_y C^2_y N^5 + 240 C^3_y C^3_y N^5 + 2340 C^4_y C^3_y N^5 - \\
450 C^3_y C^1_y N^5 + 2340 C^3_y C^1_y N^5 - 15 C^4_y C^4_y N^6 - \\
990 C^2_y C^2_y N^6 - 126 C^3_y C^3_y N^6 - 1980 C^4_y C^4_y N^6)
\]

so \( \langle \mathcal{O}_x \mathcal{O}_y \mathcal{O}_w \rangle \neq 0 \) in general (or when \( \mathcal{O}_x \) and \( \mathcal{O}_y \) are \( \frac{1}{4} \)-BPS, in particular).

We have also explicitly calculated\(^{16}\) the \( \mathcal{O}(g^2) \) combinatorial factor in \( [13] \):

\[
\tilde{\beta}_{xy} = \frac{(N^2 - 1)(N^2 - 4)}{13824} (-10800 C^2_x C^1_y + 4320 C^3_x C^1_y - \\
10800 C^1_x C^2_y - 259200 C^2_x C^2_y + 4320 C^1_y C^3_y + \\
103680 C^4_x C^3_y - 259200 C^2_x C^4_y + 103680 C^3_x C^4_y - \\
2025 C^1_y C^1_y N - 27000 C^2_x C^1_y N - 32400 C^2_x C^2_y N + \\
38880 C^3_x C^2_y N + 38880 C^2_x C^3_y N - 25920 C^3_x C^3_y N - \)

\(^{15}\) \( \mathcal{O}_1, \ldots, 4 \) were studied in \([14]\), and the results are summarized here in Appendix \(A\).

\(^{16}\) This calculation was done using \textit{Mathematica} and took about 200 hours.
27000\text{C}_x^4\text{C}_y^4\text{N} - 129600\text{C}_x^2\text{C}_y^4\text{N} - 600\text{C}_x^2\text{C}_y^1\text{N}^2 + 2940\text{C}_x^3\text{C}_y^1\text{N}^2 - 600\text{C}_x^1\text{C}_y^2\text{N}^2 + 50400\text{C}_x^2\text{C}_y^2\text{N}^2 - 2940\text{C}_x^1\text{C}_y^3\text{N}^2 - 7200\text{C}_x^2\text{C}_y^3\text{N}^2 + 50400\text{C}_x^2\text{C}_y^4\text{N}^2 - 7200\text{C}_x^3\text{C}_y^1\text{N}^2 + 175\text{C}_x^1\text{C}_y^3\text{N}^3 + 5400\text{C}_x^4\text{C}_y^1\text{N}^3 + 7200\text{C}_x^2\text{C}_y^2\text{N}^3 - 7920\text{C}_x^2\text{C}_y^3\text{N}^3 - 7200\text{C}_x^3\text{C}_y^1\text{N}^3 + 3888\text{C}_x^3\text{C}_y^3\text{N}^3 + 5400\text{C}_x^4\text{C}_y^4\text{N}^3 + 28800\text{C}_x^2\text{C}_y^4\text{N}^3 + 600\text{C}_x^2\text{C}_y^2\text{N}^4 - 780\text{C}_x^3\text{C}_y^3\text{N}^4 + 600\text{C}_x^4\text{C}_y^4\text{N}^4 - 780\text{C}_x^3\text{C}_y^1\text{N}^4 - 2880\text{C}_x^4\text{C}_y^3\text{N}^4 - 2880\text{C}_x^3\text{C}_y^4\text{N}^4 + 50\text{C}_x^4\text{C}_y^4\text{N}^5 + 2880\text{C}_x^2\text{C}_y^5\text{N}^5)

If we choose the coefficients \(C_x^{\alpha}, C_y^{\beta}, C_y^{\gamma}\) independently from the set \(\{-\frac{12N}{N-2}, 1, -\frac{5N}{N-2}, 0\}, \left\{-\frac{4N}{N-2}, -\frac{4N}{N-2}, \frac{10N}{N-2}, 1\right\}\), we recover \(\beta_{xy} = 0\). This corresponds to taking \(O_x\) and \(O_y\) as the \(\frac{1}{4}\)-BPS chiral primaries found in \(\text{Appendix C}\), so there are no \(O(\beta)\) corrections in this case either.

### 7.2 Three-point functions of \(\frac{1}{4}\)-BPS operators

When all three operators are \(\frac{1}{4}\)-BPS, the arguments get progressively more tedious. We will chose \(2l + k \leq 2p + q \leq 2r + s\). The simplifications discussed in Section 6.2 applicable to correlators \(\langle O_{[p,q,r]}\rangle\langle O_{[r,s,l]}\rangle\langle O_{[l,k,l]}\rangle\) are: the extremality condition

\[2r + s = 2p + q + 2l + k,\]

and the “partition into three disjoint flavors” condition

\[2r + s \leq 2l + k + q,\]
\[2r + s \leq 2p + q + k,\]
\[2p + q \leq 2l + k + s,\]

which have to be satisfied simultaneously (there are three more, but they are satisfied trivially since we took \(2l + k \leq 2p + q \leq 2r + s\)). For example, \([14,13]\) are true when all the \(\frac{1}{4}\)-BPS operators are in the \(84 = [2,0,2]\) of \(SU(4)\); for example, we can take\[17\]

\[\mathcal{Y}(x) = \left\{\text{tr} \ z_1^2\right\}\left\{\text{tr} \ z_2^2\right\} - \left\{\text{tr} \ z_1 z_2\right\}\left\{\text{tr} \ z_1 z_2\right\} + \frac{1}{N} \left\{\text{tr} \ z_1, z_2\right\}^2,\]

\[\mathcal{Y}(y) = \left\{\text{tr} \ z_1^2\right\}\left\{\text{tr} \ z_3^2\right\} - \left\{\text{tr} \ z_1 z_3\right\}\left\{\text{tr} \ z_1 z_3\right\} + \frac{1}{N} \left\{\text{tr} \ z_1, z_3\right\}^2,\]

\[\mathcal{Y}(w) = \left\{\text{tr} \ z_3^2\right\}\left\{\text{tr} \ z_2^2\right\} - \left\{\text{tr} \ z_3 z_2\right\}\left\{\text{tr} \ z_3 z_2\right\} + \frac{1}{N} \left\{\text{tr} \ z_3, z_2\right\}^2.\]

\[\text{[17]}\] Which just says that the number of scalars exchanged between each pair of \(O\)-s is no larger than the length of the first column in the corresponding Young tableaux.

\[\text{[18]}\] As shown in Appendix 3, such operators are in fact in the \(84\) of \(SU(4)\).
The Born amplitude

\[ \langle \mathcal{Y}(x) \mathcal{Y}(y) \mathcal{Y}(w) \rangle_{\text{free}} \propto (N^2 - 1)(N^2 - 4)(2N^2 - 15) \]  

(53)
does not vanish\(^{16}\) for \( N > 2 \), so we can’t blame the lack of corrections on group theory, and since the correlator \( \langle \mathcal{Y}(x) \mathcal{Y}(y) \mathcal{Y}(w) \rangle \) is of the form \((58)\), it receives no radiative corrections at order \( g^2 \).

Of the allowed \( \langle \mathcal{O}^\dagger(x) \mathcal{O}^\dagger(y) \mathcal{O}^\dagger(w) \rangle \) three-point functions where each \( \mathcal{O}^\dagger \) is a scalar composite in a \([p, q, p]\) of \( SU(4) \) with \( 2p + q \leq 7 \), ten more satisfy \((53)\) or \((47-49)\). For the remaining five correlators

\[
\begin{align*}
\langle \mathcal{O}_{[2,0,2]}(x) \mathcal{O}_{[2,0,2]}(y) \mathcal{O}_{[2,2,2]}(w) \rangle \\
\langle \mathcal{O}_{[2,0,2]}(x) \mathcal{O}_{[2,1,2]}(y) \mathcal{O}_{[2,3,2]}(w) \rangle \\
\langle \mathcal{O}_{[2,0,2]}(x) \mathcal{O}_{[2,1,2]}(y) \mathcal{O}_{[3,1,3]}(w) \rangle \\
\langle \mathcal{O}_{[2,0,2]}(x) \mathcal{O}_{[3,1,3]}(y) \mathcal{O}_{[2,3,2]}(w) \rangle \\
\langle \mathcal{O}_{[2,0,2]}(x) \mathcal{O}_{[3,1,3]}(y) \mathcal{O}_{[3,1,3]}(w) \rangle
\end{align*}
\]

(54)
we have to verify that there are no contributions proportional to any of the functions \( B(x, y) \), \( B(x, w) \), or \( B(y, w) \). In fact, with \( 2l + k \leq 2p + q \leq 2r + s \), we automatically have

\[ 2l + k \leq 2r + s + q \quad \text{and} \quad 2p + q \leq 2r + s + k \]  

(55)
so we can always choose the operators as

\[
\begin{align*}
\mathcal{O}_{[l,k,t]}(x) & \sim [z_1^{a} z_2^{b} z_3^c] \\
\mathcal{O}_{[p,a,r]}(y) & \sim [z_1^{e} z_2^{f} z_3^{g}] \\
\mathcal{O}_{[r,s,r]}(w) & \sim [z_1^{h} z_2^{i} z_3^{j}]
\end{align*}
\]

(56-58)
where \( c = \frac{1}{2}[(2l + k) + (2p + q) - (2r + s)] \leq l + k, p + q \); and integers \( a, b, c, d \) partition \( r + s = a + c, r = b + d \). Then \( \beta_{xy} = 0 \) since the operators exchanged between \( \mathcal{O}_x \) and \( \mathcal{O}_y \) all have the same flavor, and we only need to calculate \( \beta_{zw} \) and \( \beta_{yw} \). Details of these calculations are given in Appendix \([\text{II}]\) and here we just quote the result: as in the cases considered so far, \( \beta_{xy} = \beta_{zw} = \beta_{yw} = 0 \), and none of the three-point functions \((54)\) receive any \( \mathcal{O}(g^2) \) corrections.

8 \( \langle \mathcal{O}^\dagger \mathcal{O}^\dagger \mathcal{O}^\dagger \rangle \) correlators in the large \( N \) limit

Like the two-point functions studied in \([\text{II}]\), \( \langle \mathcal{O}_x \mathcal{O}_y \mathcal{O}_w \rangle \) calculations get progressively more cumbersome as the representations of the \( \mathcal{O} \)-s become larger. In this Section we will calculate correlators of two \( \frac{1}{2} \)-BPS operators with one \( \frac{1}{2} \)-BPS operator, in the large \( N \) limit. The situation when all three operators are \( \frac{1}{4} \)-BPS is even less tractable, and we avoid it here.

\(^{19}\) \( N > 1 \) in general since the gauge group is \( SU(N) \), while for \( N = 2 \) the single and double trace operators are proportional, and there are no \( \frac{1}{4} \)-BPS operators in the 84 of \( SU(4) \).

\(^{20}\) We used the method of BZ triangles (see Appendix \([\text{D}]\)) to find the allowed triple products.
8.1 Large $N$ operators

We will use the $\frac{1}{4}$-BPS operators found in [19]. Schematically, the special double and single trace operators can be written as

$$O_{[p,q,p]} \sim \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & q & \cdots \\ p & \cdots \end{pmatrix}, \quad K_{[p,q,p]} \sim \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & q & \cdots \\ p & \cdots \end{pmatrix}$$ (59)

(each continuous group of boxes stands for an $SU(N)$ trace); explicit formulae for highest $SU(4)$ weight operators of this form are given in Appendix A.6. In the large $N$ limit the linear combinations

$$\tilde{Y}_{[p,q,p]} = K_{[p,q,p]} + O(N^{-2})$$ (60)

$$Y_{[p,q,p]} = O_{[p,q,p]} - \frac{p(p + q)}{N} K_{[p,q,p]} + O(N^{-2})$$ (61)

are eigenstates of the dilatations operator. $Y_{[p,q,p]}$ have protected normalization and scaling dimension ($\Delta_Y = 2p + q$) at order $g^2$, and were argued to be $\frac{1}{4}$-BPS.

We did not specify the $SU(4)$ weights of operators $O_{[p,q,p]}$ and $K_{[p,q,p]}$ in (59). The choice of weights will depend on the representations in the triple product $[p,q,p] \otimes [r,s,r] \otimes [0,k,0]$ in the following way. Assume $p \leq r$; then it is convenient to choose

$$Y_{[p,q,p]}(x) \sim [z_1^l z_2^n z_3^p]$$ (62)

$$Y_{[r,s,r]}(y) \sim [z_1^m z_2^n z_3^s]$$ (63)

$$O_{[0,k,0]}(w) \sim [z_1^k]$$ (64)

with $l \equiv \frac{1}{2}(2p + q) + k - (2r + s)$, $m \equiv k - l$ and $n \equiv p + q - l$. We will also assume that none of the simplifications (39-41) apply, since those cases were already discussed in Section 7.1.3.

8.2 $\langle KKO_\frac{1}{2} \rangle_{\text{free}}, \langle KOO_\frac{1}{2} \rangle_{\text{free}}, \langle OKO_\frac{1}{2} \rangle_{\text{free}}$, and $\langle OOO_\frac{1}{2} \rangle_{\text{free}}$

We can estimate the leading large $N$ behavior of the combinatorial factors $\alpha_{\text{free}}$ and $\tilde{\beta}_{xy}$ using the “trace merging formula”

$$2 (\text{tr} A t^c) (\text{tr} B t^c) = \text{tr} AB - \frac{1}{N} (\text{tr} A) (\text{tr} B)$$ (65)

where $A$ and $B$ are arbitrary $N \times N$ matrices and $t^c$ are $SU(N)$ generators in the fundamental (sums on repeated indices are implied). With (65) and the expression for the quadratic Casimir

$$T_r^c T_r^c = C_2(r) \mathbf{1}$$ (66)

21 In particular, the adjoint and fundamental representations will be of interested, and for these $C_2(\text{adj}) = N$, $C_2(\text{fund}) = \left(\frac{N^2 - 1}{2N}\right)$, $C(\text{adj}) = N$, $C(\text{fund}) = \frac{1}{2}$. 20
of SU(N), we find for example that for $k \geq 2$

\[
(tr t^{a_1} t^{a_2} \cdots t^{a_k})(tr t^{b_1} \cdots t^{b_n}) = \left( \frac{N^2 - 1}{2N} \right)^k - \left( \frac{N^2 - 1}{2N} \right) \left( \frac{1}{2N} \right)^{k-1}
\]

\[
= \left( \frac{N}{2} \right)^k [1 + O(1/N^2)]
\]

(67)

To have this large $N$ behavior, generators in the two traces should appear in opposite order. When the generators are taken in any other order (except cyclic permutations inside the traces), such products are suppressed by at least a power of $N^2$.

Calculations proceed along the same lines as in [19]. We begin by considering correlators of the form $\langle OOO \rangle$ with the two $O$-s in the same representation $[p, q, p]$. In this case the leading contribution to $\alpha_{\text{free}}$ comes from terms like

\[
(tr t^{a_1} \cdots t^{a_l} t^{b_1} \cdots t^{b_n})(tr t^{c_1} \cdots t^{c_p}) (tr t^{d_1} \cdots t^{d_m} t^{b_n} \cdots t^{b_1})(tr t^{e_1} \cdots t^{e_1})
\]

\[
\sim \left( \frac{1}{2} \right)^2 N \left( \frac{N}{2} \right)^{2l-2+n} \times \left( \frac{N}{2} \right)^p = \left( \frac{1}{2} \right) \left( \frac{N}{2} \right)^{2l+n+p-1}
\]

(68)

The factor of $\left( \frac{1}{2} \right)^2$ comes about because we merge traces twice; the exponent $2l + n - 2$ counts how many generators collapse using $t^c t^c \sim \frac{1}{2} N \mathbf{1}$; the extra factor of $N$ is due to $tr \mathbf{1} = N$; and finally $(N/2)^p$ is from contracting the traces of equal length containing the $t^{c_i}$-s. All remaining calculations of this and next Sections are analogous, and we won’t spell things out as much.

If the representations of $[p, q, p]$ and $[r, s, r]$ are different, a similar situation occurs when for example $p = r + s$, i.e. $O_{[p, q, p]}$ and $O_{[r, s, r]}$ contain traces of equal length. Then we merge traces twice, and one set of traces collapses completely as in (67). Otherwise, we have to merge traces three times, so the leading contributions to $\langle OOO \rangle_{\text{free}}$ are of the form

\[
(tr t^{a_1} \cdots t^{a_l} t^{b_1} \cdots t^{b_n})(tr t^{c_1} \cdots t^{c_p}) (tr t^{d_1} \cdots t^{d_m} t^{b_n} \cdots t^{b_1})(tr t^{e_1} \cdots t^{e_1})
\]

\[
\sim \left( \frac{1}{2} \right)^3 \left( \frac{N}{2} \right)^{p+l+m+n-3} \text{ otherwise}
\]

(69)

For the other three types of correlators, no pair of traces ever collapses completely, so the answers are more uniform. We find that the large $N$ behavior

\[\text{One can derive (67) from a one-term recursion relation defined by (65), (66), and tr } t^n \text{ = 0.}\]
of $\langle KK\mathcal{O}_g \rangle_{\text{free}}$ is defined by the terms like

$$(\text{tr} \, t^{l_1} \ldots t^{l_m} \ldots t^{l_n} t^{c_1} \ldots t^{c_p}) (\text{tr} \, t^{d_1} \ldots t^{d_m} \ldots t^{d_n} t^{c_1} \ldots t^{c_p}) (\text{tr} \, t^{a_1} \ldots t^{a_m} \ldots t^{a_n} t^{c_1} \ldots t^{c_p})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p-l+m+n-2} N = \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n-1}$$

(70)

as we merge traces twice. Similarly, $\langle O\mathcal{K}\mathcal{O}_g \rangle_{\text{free}}$ scales as the terms

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n-2}$$

(71)

since traces have to be merged three times now. The three-point functions $\langle K\mathcal{K}\mathcal{O}_g \rangle_{\text{free}}$ also have the leading large $N$ dependence [7].

8.3 $\langle KK\mathcal{O}_g \rangle_{g^2}$, $\langle K\mathcal{O}\mathcal{O}_g \rangle_{g^2}$, $\langle O\mathcal{K}\mathcal{O}_g \rangle_{g^2}$, and $\langle O\mathcal{O}\mathcal{O}_g \rangle_{g^2}$

Here there are no special cases to consider. We have to merge traces twice for $\langle KK\mathcal{O}_g \rangle_{g^2}$, three times for $\langle K\mathcal{O}\mathcal{O}_g \rangle_{g^2}$ or $\langle O\mathcal{K}\mathcal{O}_g \rangle_{g^2}$, and four times for $\langle O\mathcal{O}\mathcal{O}_g \rangle_{g^2}$. The leading behavior of the $\beta_{xy}$ combinatorial coefficient for the three-point functions $\langle KK\mathcal{O}_g \rangle_{g^2}$ is the same for terms of the form

$$(\text{tr} \, t^{a_1} \ldots t^{a_m} \ldots t^{b_n} \ldots t^{c_p} [t^{c_1}, t^{c_2} \ldots t^{c_p}) (\text{tr} \, t^{d_1} \ldots t^{d_m} \ldots t^{d_n} \ldots t^{c_1} \ldots t^{c_p}) (\text{tr} \, t^{e_1} \ldots t^{e_m} \ldots t^{e_n} \ldots t^{c_1} \ldots t^{c_p})$$

$$\times (\text{tr} \, t^{f_1} \ldots t^{f_m} \ldots t^{f_n} \ldots t^{c_1} \ldots t^{c_p})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{l+m-2} (\text{tr} \, t^{b_1} \ldots t^{b_n} \ldots t^{c_1} \ldots t^{c_2} \ldots t^{c_1})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n-4} (\text{tr} \, t^{c_1} \ldots t^{c_1} \ldots t^{c_1} \ldots t^{c_2} \ldots t^{c_1})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n}$$

(72)

which give the leading large $N$ contributions to it. In the same fashion, the most significant terms in the correlators $\langle O\mathcal{K}\mathcal{O}_g \rangle_{g^2}$ are

$$(\text{tr} \, t^{a_1} \ldots t^{a_m} \ldots t^{b_n} \ldots t^{c_p} \ldots t^{c_1} \ldots t^{c_1}) (\text{tr} \, t^{c_1} \ldots t^{c_2} \ldots t^{c_1} \ldots t^{c_1}) (\text{tr} \, t^{d_1} \ldots t^{d_m} \ldots t^{d_n} t^{c_1})$$

$$\times (\text{tr} \, t^{e_1} \ldots t^{e_m} \ldots t^{e_n} \ldots t^{c_1})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n-1} \sim \langle K\mathcal{O}\mathcal{O}_g \rangle_{g^2}$$

(73)

while $\langle O\mathcal{O}\mathcal{O}_g \rangle_{g^2}$ gets it leading $N$ behavior from terms like

$$(\text{tr} \, t^{a_1} \ldots t^{a_m} \ldots t^{b_n} \ldots t^{c_p} \ldots t^{c_1} \ldots t^{c_1}) (\text{tr} \, t^{c_1} \ldots t^{c_2} \ldots t^{c_1} \ldots t^{c_1}) (\text{tr} \, t^{d_1} \ldots t^{d_m} t^{c_1} \ldots t^{c_1}) (\text{tr} \, t^{e_1} \ldots t^{e_m} \ldots t^{e_n} \ldots t^{c_1})$$

$$\sim \left( \frac{1}{2} \right)^2 \left( \frac{N}{2} \right)^{p+l+m+n-2} \sim \langle O\mathcal{O}\mathcal{O}_g \rangle_{g^2}$$

(74)
×(tr \, t^n \, t^{n+1} \, t^{d_m} \, t^{d_1})
\sim \left(\frac{1}{2}\right)^3 \frac{N}{2}^{p+l+m+n-2} \tag{74}

\section{8.4 \{O_\frac{1}{2} O_\frac{1}{2} O_\frac{1}{2}\} correlators are protected}

With just a little more work, we can find the ratios of the order $g^2$ corrections to the three-point functions $\langle O O O \rangle_{g^2}$, $\langle O KO O \rangle_{g^2}$, $\langle K OO O \rangle_{g^2}$, and $\langle KKO O \rangle_{g^2}$. The argument proceeds along the same lines as in [19]. Given a term with generators in a particular order, contributing to $\langle KKO O \rangle_{g^2}$, such as the one shown in (72), we know that a term with the same order of generators also gives a leading contribution to $\langle OKO O \rangle_{g^2}$ as in (73). However, cyclic permutations within the two traces (of length $p$ and $p+q$) of $O$, contribute to $\langle OKO O \rangle_{g^2}$ in the same amount as the term (73). Therefore,

$$\langle OKO O \rangle_{g^2}/\langle KKO O \rangle_{g^2} = \frac{p(p+q)}{N} + O(N^{-3}) \equiv \beta \tag{75}$$

In the same fashion

$$\langle KOO O \rangle_{g^2}/\langle KKO O \rangle_{g^2} = \frac{r(r+s)}{N} + O(N^{-3}) \equiv \beta' \tag{76}$$

and

$$\langle OOO O \rangle_{g^2}/\langle KKO O \rangle_{g^2} = \frac{p(p+q)}{N} + O(N^{-3}). \tag{77}$$

Next consider the Born level correlators of Section 8.2. When a pair of traces collapses completely (see equations 69-71), we get

$$\langle Y_{[p,q,p]} Y_{[r,s,r]} O_{\frac{1}{2}} \rangle_{\text{free}} \sim \langle OOO O \rangle_{\text{free}} \sim N^{p+l+m+n-1} \tag{78}$$

Otherwise, the contributions add up to

$$\langle OOO O \rangle_{\text{free}} - \beta \langle KKO O \rangle_{\text{free}} - \beta' \langle OKO O \rangle_{\text{free}} + \beta \beta' \langle KKO O \rangle_{\text{free}} \sim N^{p+l+m+n-3} \tag{79}$$

The terms in (79) are all of the same order and do not cancel. The factors of $\beta$ and $\beta'$ discussed above are still present, but there are other complications. First, the string of $t^c$-s can be inserted anywhere in the third trace in (79), and cyclic permutations of the $t^l$-s in the same trace give terms of the same order in $N$. This results in an extra factor of $(r-p)^2$. Second, different terms in the sum over antisymmetrizations (as in equation (123), for example) contribute differently. The combinatorics is more involved, and we do not discuss this case in detail.
Bringing everything together, we see that the order $g^2$ corrections to the three-point function of the BPS operators in the large $N$ limit add up to

$$\langle \mathcal{Y}_{[p,q,p]} \mathcal{Y}_{[r,s,r]} \mathcal{O}_{1/2} \rangle g^2 \propto \left( \frac{1}{2} \right) \left( \frac{N}{2} \right)^{p+l+m+n-1} \tilde{B}(x,y) N \times \left( \frac{-\beta}{1} \right) \left( \frac{1 + \mathcal{O}(N^{-2})}{\beta} \right) \left( \frac{\beta'}{\beta'} \right) \left( \frac{-\beta'}{1} \right)$$

$$= \left( \frac{1}{2} \right) \left( \frac{N}{2} \right)^{p+l+m+n-1} \tilde{B}(x,y) N \times \mathcal{O}(N^{-4})$$

A comparison of (80) with (78) or (79) shows that order $g^2$ corrections to three-point functions of one $1/2$-BPS operator with two $1/4$-BPS operators vanish in the large $N$ limit, within working precision.

9 Supergravity considerations

In the AdS/CFT correspondence, there is a duality mapping single trace $1/2$-BPS primary operators $\text{tr} X^k$ of the SYM theory onto canonical supergravity fields, $[2], [3]$. Given a set of such $1/2$-BPS primary operators, one can compute their two- and three-point functions in SYM. The two-point functions define the normalization of operators, and the three-point functions probe the interactions between them. Independently, both the normalization of the operators and the couplings, can be read off from the supergravity action (or supergravity equations of motion), $[4]$. So as a check of the AdS/CFT correspondence, one can compare the unambiguously defined three- and higher $n$-point functions of normalized $1/2$-BPS operators in SYM, with the correlators of the corresponding elementary excitations in supergravity, $[4], [13], [26], [27]$.

We would like to proceed, in the same spirit, with the $1/4$-BPS chiral primaries of the $\mathcal{N}=4$ Super Yang Mills calculated in this paper and in $[19]$. We argued that these two- and three-point functions are independent of the SYM coupling constant (at least to order $g^2$), so it is reasonable to expect these correlators to agree with their dual AdS description. However, multiple trace operators do not correspond to any of the fields appearing in the supergravity action, so the discussion will be different than in the case of the previously studied operators $\text{tr} X^k$.

9.1 OPE definition of $1/4$-BPS chiral primaries

One of the ways to see $1/4$-BPS chiral primaries is to consider higher $n$-point correlators of $1/4$-BPS operators. For example, four-point functions of $[0,2,0]$ operators reveal a pole corresponding to the exchange of a $[2,0,2]$ operator with a protected dimension $\Delta = 4$, $[7]$. In general, the $1/4$-BPS primaries $\mathcal{Y}_{[p,q,p]}$ show
up in the four-point functions
\[ \langle \text{tr} X^{(p+q)}(x) \text{tr} X^p(x+\epsilon) \text{tr} X^{(p+q)}(y) \text{tr} X^p(w) \rangle \] (81)
in the limit \( \epsilon \to 0 \), as the \([p,q,p]\) operators with the threshold value of scaling dimension \( \Delta = 2p + q = \dim[\text{tr} X^{(p+q)}] + \dim[\text{tr} X^p] \). In other words, \( \frac{1}{4}\)-BPS chiral primaries can be defined by the OPE-s of \( \frac{1}{2}\)-BPS operators as
\[ \mathcal{P}_{[p,q,p]}^{\Delta=2p+q} \left[ \lim_{\epsilon \to 0} \text{tr} X^{(p+q)}(x) \text{tr} X^p(x+\epsilon) \right] = \sum_i c_i \mathcal{Y}^{[p,q,p]}_i(x) \] (82)
Here, \( \mathcal{P}_{[p,q,p]}^{\Delta=2p+q} \) projects onto the \([p,q,p]\) representation of \( SU(4) \), and eliminates operators with scaling dimension other than \( \Delta \) (e.g. the non-chiral descendants with the same \( SU(4) \) quantum numbers). Singular terms normally subtracted from an OPE such as (82), are automatically removed by applying \( \mathcal{P}_{[p,q,p]}^{\Delta=2p+q} \).

On the other hand, one can see by calculating three-point correlators that all \( \frac{1}{4}\)-BPS primary operators \( \mathcal{Y}^{[p,q,p]}_i \) are present in the OPE (82). It appears that for general \( N \), there is no canonical definition of the special \( \mathcal{Y}^{[p,q,p]}_i \) that is a linear combination of the single and double-trace scalar composite \([p,q,p]\) operators only. However, this \( \mathcal{Y}^{[p,q,p]}_i \) dominates in the \( N \to \infty \) limit. For large \( N \), all other terms in the right hand side of (82) are suppressed by at least a factor of \( 1/N \), and the predominantly double-trace \( \frac{1}{4}\)-BPS chiral primary operator
\[ \mathcal{Y}^{[p,q,p]} = \left( \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{array} \right) + \mathcal{O}(1/N) \] (83)
is uniquely defined by the OPE of \( \frac{1}{4}\)-BPS primaries.

When translated into the SUGRA language, the definition (82) implies that \( \frac{1}{4}\)-BPS primary operators of SYM should be thought of as threshold bound states of elementary SUGRA excitations. A threshold bound state is a state whose mass is precisely equal to the sum of the masses of all its constituents, and thus occurs at the lower end of the spectrum. Any bound state of BPS states which is itself BPS is automatically a threshold bound state. A familiar example is provided by an assembly of like sign charged Prasad-Sommerfield magnetic monopoles, whose classical static solution forms a threshold bound state of monopole constituents.

9.2 \( \langle \mathcal{O}_+ \mathcal{O}_+ \rangle \) and \( \langle \mathcal{O}_+ \mathcal{O}_+ \mathcal{O}_+ \rangle \) correlators

We are now going to illustrate the consistency of this dictionary. Specifically, we will look at two- and three-point functions involving \( \frac{1}{2}\)- and \( \frac{1}{4}\)-BPS operators in the large \( N \) limit, that we calculated earlier in this paper and in [19], in \( \mathcal{N}=4 \) SYM. Then we will compare these correlators with their dual supergravity description.
The normalization of $\frac{1}{2}$- and $\frac{1}{4}$-BPS operators comes from their two-point functions, whose leading large $N$ behavior is \cite{4, 19}.

$$\langle \text{tr} \, X^p(x) \; \text{tr} \, X^q(y) \rangle \sim \frac{N^q}{(x-y)^{2q}}$$  \hspace{1cm} (84)

$$\langle \mathcal{Y}^{[p,q,p]}(x) \; \mathcal{Y}^{[p,q,p]}(y) \rangle \sim \frac{N^{(2p+q)}}{(x-y)^{2(2p+q)}}$$  \hspace{1cm} (85)

times some $N$-independent factors which we omit.

The simplest three point functions involving $\frac{1}{2}$- and $\frac{1}{4}$-BPS operators are of the form $\langle O_{\frac{1}{2}}(x) O_{\frac{1}{4}}'(y) O_{\frac{1}{4}}(w) \rangle$. If the $SU(N)$ traces collapse completely (in which case $\langle O_{\frac{1}{2}} O_{\frac{1}{4}}' O_{\frac{1}{4}} \rangle$ are extremal), the normalized three point-functions are then

$$\frac{1}{\sqrt{N^{(2p+q)+(p+q)+p}}} \langle \text{tr} \, X^{(p+q)}(x) \; \text{tr} \, X^{p}(y) \; \mathcal{Y}^{[p,q,p]}(w) \rangle \sim 1$$  \hspace{1cm} (86)

from a field theory calculation; the space-time coordinate dependence is fixed by conformal invariance, so we will not exhibit it anymore. If the traces do not collapse completely, the correlator is suppressed by $1/N^2$ (see the discussion around equations (68-69) of Section \ref{section}), and

$$\frac{1}{\sqrt{N^{(2p+q)+(k+l)+k}}} \langle \text{tr} \, X^{(k+l)}(x) \; \text{tr} \, X^{k}(y) \; \mathcal{Y}^{[p,q,p]}(w) \rangle \sim \frac{1}{N^2}$$  \hspace{1cm} (87)

whenever $k \neq p$ of $l \neq q$. All this matches nicely with the corresponding supergravity diagrams:

\begin{enumerate}
  \item (a)
  \item (b)
  \item (c)
\end{enumerate}

Leading AdS diagrams for (a) equation (85); (b) equation (84); (c) equation (87). Each cubic bulk interaction vertex goes like $1/N$.

We denoted the $\frac{1}{2}$-BPS operators by “•”; and the predominantly double trace $\frac{1}{4}$-BPS primaries which arise from bringing two $\frac{1}{2}$-BPS operators together by “○”.

There are also AdS diagrams with quartic interactions in the bulk, which have the same large $N$ dependence as (88c); we will not show these.
9.3 $\langle O^4 O'_4 O'_{4} \rangle$ correlators

Other three point functions involving the $1/4$-BPS as well as the $1/2$-BPS operators can be analyzed similarly. Whenever traces of the SYM operators do not collapse completely, the supergravity counterparts of such correlators have extra bulk interaction vertices. The leading dependence of such correlators is then suppressed by the corresponding power of $1/N$. For example, correlators of the form $\langle O^4(x)O'_4(y)O_{4}(w)\rangle$, discussed in Section 8, behave like

$$\frac{1}{\sqrt{N(2p+q)+(2r+s)+k}} \langle Y^{[p,q,p]}_r Y^{[s,r]} Y^{[l,k,l]} \rangle \sim \begin{cases} 1/N & \text{(a) if one pair of traces collapses completely} \\ 1/N^3 & \text{(b) otherwise} \end{cases}$$

From the AdS point of view, this difference is captured by the following diagrams

(a) \hspace{2cm} (b)

AdS description of equation (89).

9.4 $\langle O^4 O'_4 O''_{4} \rangle$ correlators

Similar arguments show that when all operators are $1/4$-BPS, the normalized three-point functions are

$$\frac{1}{\sqrt{N(2p+q)+(2r+s)+(2l+k)}} \langle Y^{[p,q,p]}_r Y^{[s,r]} Y^{[l,k,l]} \rangle \sim \begin{cases} 1 & \text{(a) if all traces collapse pairwise} \\ 1/N^2 & \text{(b) if only one pair of traces collapses} \\ 1/N^4 & \text{(c) otherwise} \end{cases}$$
and the corresponding AdS diagrams

\[ (a) \quad (b) \quad (c) \]

AdS description of equation (91).

show the correct leading large $N$ behavior.

9.5 Detailed agreement between SYM and AdS

Unlike the $\frac{1}{2}$-BPS calculations (e.g. [4]), this study does not provide a new independent check or application of the AdS/CFT correspondence. On the one hand, the definition of the predominantly double-trace $\frac{1}{4}$-BPS operators in the SYM theory (in the large $N$ limit) is based on the OPE of $\frac{1}{2}$-BPS primaries. On the other hand, AdS correlators of the duals of the $\frac{1}{4}$-BPS operators (bound states of elementary SUGRA excitations) are defined by the corresponding correlators of primary supergravity fields. Therefore, SYM correlators involving $\frac{1}{4}$-BPS operators agree by construction with their SUGRA counterparts.

This is especially clear in the cases show in diagrams (88a,b), (90a), and (92a). To leading order in $N$, these two- and three-point functions of $\frac{1}{4}$-BPS scalar composite operators are expressed in terms of the previously studied two- and three-point functions of $\frac{1}{2}$-BPS chiral primary operators.

10 Conjectures

Let us summarize what has been done so far. First, $\frac{1}{4}$-BPS primary operators were identified in [13]: for general representations $[p,q,p]$ in the large $N$ limit; and for general $N$ in the case when $2p + q \leq 7$. Second, three-point functions involving $\frac{1}{2}$-BPS operators as well as several infinite families of $\frac{1}{4}$-BPS operators were considered in this paper, also for arbitrary $N$. It was found that there are no $O(g^2)$ corrections to such correlators. Next, all three-point functions involving the $\frac{1}{4}$-BPS primaries with $2p + q \leq 7$ were computed for general $N$, and were shown to be protected at order $g^2$. In the large $N$ approximation, three point functions involving two $\frac{1}{4}$-BPS primaries and one $\frac{1}{2}$-BPS primary were shown to receive no $O(g^2)$ corrections, for general representations of the operators involved. Finally, we presented AdS considerations which reproduced many features of the CFT two- and three-point functions in the large $N$ limit.
Collecting all the non-renormalization effects established above generates strong evidence for a number of natural conjectures, which we now state:

(1) We conjecture that on the CFT side, for every \([p,q,p]\) representation of SU(4) and arbitrary \(N\), there are \(1\over 4\)-BPS chiral primaries. Within each \([p,q,p]\), one of these operators is a linear combination of double and single trace scalar composites only; the other \(1\over 4\)-BPS chiral primaries in \([p,q,p]\) also involve operators with higher numbers of traces.

(2) We speculate that two-point functions of \(1\over 4\)-BPS operators, as well as three-point functions involving \(1\over 2\)-BPS and \(1\over 4\)-BPS operators, do not depend on the coupling \(g^2\) of \(\mathcal{N}=4\) SYM. This non-renormalization also persist for all \(N\), and is not just a large \(N\) approximation.

(3) One of the group theory arguments of Section 6.2, and the analysis of Section 8, generalize straightforwardly to extremal correlators, i.e. \((n+1)\)-point functions of the form \(\langle \mathcal{O}_0(x_0)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n) \rangle\) with \(\Delta_0 = \Delta_1 + \ldots + \Delta_n\). So do the AdS considerations of Section 9. Therefore, we conjecture that arbitrary extremal correlators of \(1\over 2\)- and \(1\over 4\)-BPS chiral primaries are also protected.

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Appendix

A \(\frac{1}{4}\)-BPS operators explicitly

Scalar composite gauge invariant operators \(O \sim [(\phi^1)^{p+q}(\phi^2)^p]\) corresponding to highest weights of representations \([p, q, p]\) of the \(R\)-symmetry group \(SU(4) \sim SO(6)\) were studied in [19]. It was found that there are many such gauge invariant operators \(O_i\) for a given set of fields. They depend on how we partition the products into traces.

In general, none of the \(O_i\)-s are eigenstates of the dilatations operator. To construct operators with a well defined scaling dimension, one has to take particular linear combinations, \(\gamma_j = \sum_i C_{ij}O_i\). Some of the \(\gamma_j\)-s are \(\frac{1}{4}\)-BPS: they are annihilated by a quarter of the Poincaré supercharges, they are not descendants of \(\frac{1}{8}\)-BPS primaries, and at order \(g^2\) they have a protected scaling dimension \(\Delta = 2p + q\); their normalization also receives no \(O(g^2)\) corrections.

In this Appendix we list some of the results of [19]. In particular, we define the operators \(O_i\) for each representation \([p, q, p]\) with \(2p + q \leq 7\), and list their linear combinations that are \(\frac{1}{4}\)-BPS. (We do not show the other pure operators here.) This is done for general \(N\). We also write out the two special scalar composite operators which contribute to the \(\frac{1}{4}\)-BPS primaries in the large \(N\) limit, for all \([p, q, p]\) representations.

A.1 \([2,0,2]\)

There are two linearly independent gauge invariant scalar composite operators with the highest \([2,0,2]\) weight,

\[
O^{[2,0,2]}_1 = \mathrm{tr} \; z_1 z_2 z_2 z_2 - \mathrm{tr} \; z_1 z_2 z_1 z_2 = -\frac{1}{2} \mathrm{tr} \; [z_1, z_2] [z_1, z_2]
\]

\[
O^{[2,0,2]}_2 = 2 (\mathrm{tr} \; z_1 z_1 \; \mathrm{tr} \; z_2 z_2 - \mathrm{tr} \; z_1 z_2 \; \mathrm{tr} \; z_1 z_2)
\]

The operator whose two-point functions with itself and other scalar composites receives no \(O(g^2)\) corrections is

\[
\gamma_{[2,0,2]} = O^{[2,0,2]}_2 - \frac{4}{N} O^{[2,0,2]}_1
\]

Other \(\frac{1}{4}\)-BPS operators in the same representation (the ones corresponding to different \(SU(3) \times U(1)\) weights) have the same coefficients in front of double and single trace operators (provided they are normalized in the same way). For example, such are (50,52).
\section*{A.2 \: \[2,1,2\]}

Here there are also two partitions of integer $2p + q$ which result in linearly independent scalar composite operators

\begin{align*}
O^{[2,1,2]}_1 &= \text{tr} z_1 z_1 z_2 z_2 - \text{tr} z_1 z_2 z_2 z_1 - \frac{1}{2} \text{tr} [z_1, z_2][z_1^2, z_2] \quad (96) \\
O^{[2,1,2]}_2 &= \text{tr} z_1 z_2 z_1 z_2 - 2 \text{tr} z_1 z_2 z_1 z_2 + \text{tr} z_1 z_2 z_3 + \text{tr} z_1 z_2 z_2 \text{tr} z_1 z_1 \quad (97)
\end{align*}

The single $\frac{1}{4}$-BPS operators is

\[ \mathcal{Y}^{[2,1,2]} = \frac{6}{N} O^{[2,1,2]}_1 \]  

\section*{A.3 \: \[2,2,2\]}

$2p + q = 6$ is the lowest dimension for which there are more than two linearly independent gauge invariant operators. Here we have five:

\begin{align*}
O^{[2,2,2]}_1 &= \text{tr} z_1 z_2 z_1 z_2 z_2 - \frac{2}{3} \text{tr} z_1 z_2 z_1 z_2 z_2 - \frac{1}{3} \text{tr} z_1 z_2 z_1 z_2 z_2 \quad (99) \\
O^{[2,2,2]}_2 &= \text{tr} z_1 z_2 z_1 z_2 + 2 \text{tr} z_1 z_2 z_1 z_2 \text{tr} z_1 z_2 + \frac{1}{2} (2 \text{tr} z_1 z_2 z_2 z_2 + \text{tr} z_1 z_2 z_1 z_2) \text{tr} z_1 z_1 \quad (100) \\
O^{[2,2,2]}_3 &= \frac{1}{2} \text{tr} z_1 z_1 z_1 z_2 z_2 - \text{tr} z_1 z_1 z_2 z_2 \text{tr} z_1 z_2 - 3 \text{tr} z_1 z_2 z_2 \text{tr} z_1 z_2 - 3 \text{tr} z_1 z_2 z_2 z_2 \text{tr} z_1 z_2 \\
O^{[2,2,2]}_4 &= \frac{1}{2} \text{tr} z_1 z_1 z_2 z_2 \text{tr} z_1 z_2 \text{tr} z_1 z_2 + \frac{1}{2} (4 \text{tr} z_1 z_2 z_2 z_2 - 3 \text{tr} z_1 z_2 z_2 z_2) \text{tr} z_1 z_2 \quad (100) \\
O^{[2,2,2]}_5 &= \text{tr} z_1 z_1 (\text{tr} z_1 z_2 z_2) - \text{tr} z_1 z_2 z_2 \text{tr} z_1 z_2 \quad (103)
\end{align*}

The two linear combinations of operators which satisfy $\langle \mathcal{Y} O_i \rangle = 0$ for all $i$, are

\[ \mathcal{Y}^{[2,2,2]}_1 = \frac{8N}{(N^2 - 4)} O^{[2,2,2]}_1 + \frac{8}{3(N^2 - 4)} \left( 2O^{[2,2,2]}_3 + O^{[2,2,2]}_4 \right) \]  

and the one orthogonal to it (in the sense that $\langle \mathcal{Y}^{[2,2,2]}_1(x) \mathcal{Y}^{[2,2,2]}_2(y) \rangle = 0$)

\[ \mathcal{Y}^{[2,2,2]}_2 = \frac{144 \left( N^2 - 4 \right) \left( N^2 - 2 \right) \left( 3N^2 - 7 \right) \left( 3N^2 + 8 \right)}{3N^6 - 47N^4 + 248N^2 - 192} O^{[2,2,2]}_1 - \frac{3N \left( 3N^2 - 7 \right) \left( 3N^2 + 8 \right)}{3N^6 - 47N^4 + 248N^2 - 192} O^{[2,2,2]}_2 \\
- \frac{2 \left( 3N^4 - 23N^2 + 104 \right)}{3N^6 - 47N^4 + 248N^2 - 192} \left( 2O^{[2,2,2]}_3 + O^{[2,2,2]}_4 \right) + O_5 \]  

\section*{A.4 \: \[3,1,3\]}

Two $[p, q, p]$ representations have $2p + q = 7$. These are $[3,1,3] = 960$ and $[2,3,2] = 1470$. In the first case, the scalar composite operators are

\begin{align*}
O^{[3,1,3]}_1 &= \frac{1}{2} \text{tr} z_1 z_1 z_1 z_2 z_2 z_2 - \frac{1}{2} \text{tr} z_1 z_1 z_2 z_2 z_2 z_2 - \frac{1}{2} \text{tr} z_1 z_1 z_2 z_2 z_1 z_2 
\end{align*}
The $O(2,3,2)$ protected operators work out to be

\begin{align}
\mathcal{O}_1^{[2,3,2]} &= + \frac{1}{3} \mathrm{tr} z_1 z_1 z_2 z_2 z_2 z_2 + \frac{1}{3} \mathrm{tr} z_1 z_1 z_2 z_2 z_2 z_1 + \frac{1}{3} \mathrm{tr} z_1 z_2 z_2 z_1 z_2 z_2 \\
\mathcal{O}_2^{[2,3,2]} &= \mathrm{tr} z_1 z_1 z_2 z_2 z_2 z_2 - 3 \mathrm{tr} z_1 z_1 z_2 z_2 - 3 \mathrm{tr} z_1 z_2 z_2 z_1 z_2 \\
&\quad + (2 \mathrm{tr} z_1 z_2 z_2 z_2 + \mathrm{tr} z_1 z_2 z_2 z_2) \mathrm{tr} z_1 z_2 z_2 - \mathrm{tr} z_1 z_2 z_2 z_2 \mathrm{tr} z_1 z_1 z_2 \\
\mathcal{O}_3^{[2,3,2]} &= - \left( \mathrm{tr} z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_2 z_2 z_2 \right) z_1 z_2 \\
&\quad + (\mathrm{tr} z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_2 z_2 z_2) \mathrm{tr} z_1 z_2 + \mathrm{tr} z_1 z_2 z_2 z_2 + \mathrm{tr} z_1 z_2 \mathrm{tr} z_1 z_2 z_2 \\
\mathcal{O}_4^{[2,3,2]} &= \mathrm{tr} z_1 z_2 (2 \mathrm{tr} z_1 z_2 z_2 - \mathrm{tr} z_2 z_2) \mathrm{tr} z_1 z_1 z_1 - 3 \mathrm{tr} z_1 z_1 \mathrm{tr} z_1 z_2 z_2 \\
&\quad + \mathrm{tr} z_1 z_1 (\mathrm{tr} z_2 z_2 \mathrm{tr} z_1 z_2 z_2 + \mathrm{tr} z_1 z_1 \mathrm{tr} z_2 z_2 z_2) \\
\end{align}

The $O(g^2)$ protected operators work out to be

\begin{align}
\mathcal{Y}_1^{[2,3,2]} &= - \frac{12N}{N^2 - 2} \mathcal{O}_1^{[2,3,2]} + \frac{5}{N^2 - 2} \mathcal{O}_3^{[2,3,2]} \\
\mathcal{Y}_2^{[2,3,2]} &= \frac{96}{N^2 - 4} \mathcal{O}_1^{[2,3,2]} - \frac{4N}{N^2 - 4} \mathcal{O}_2^{[2,3,2]} + \frac{10N}{N^2 - 4} \mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]} \\
\end{align}

**A.5  [2,3,2]**

The other $[p,q,p]$ of $SU(4)$ with $2p + q = 7$ is the [2,3,2]. Here we have seven linearly independent operators corresponding to the highest weight state:

\begin{align}
\mathcal{O}_1^{[2,3,2]} &= 2 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_1 z_2 z_2 z_1 z_2 \\
\mathcal{O}_2^{[2,3,2]} &= 2 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 - 4 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 - \mathrm{tr} z_1 z_1 z_1 z_2 z_2 \\
&\quad + (\mathrm{tr} z_1 z_1 z_1 z_2 z_2 + \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2) \mathrm{tr} z_1 z_1 \\
\mathcal{O}_3^{[2,3,2]} &= \mathrm{tr} z_1 z_1 z_1 z_1 z_1 z_2 z_2 - 2 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 - \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 \\
&\quad + (8 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 - 7 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2) \mathrm{tr} z_1 z_1 \\
\mathcal{O}_4^{[2,3,2]} &= 3 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 z_2 - 6 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 \\
&\quad + (2 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_2 + \mathrm{tr} z_1 z_1 z_1 z_1 z_1 z_2 z_2) \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 \\
\mathcal{O}_5^{[2,3,2]} &= 3 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 z_2 - 6 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_2 - \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 \\
&\quad + (7 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 - 4 \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2) \mathrm{tr} z_1 z_1 z_1 z_1 z_2 z_2 \\
\mathcal{O}_6^{[2,3,2]} &= -8 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_1 z_1 z_1 - 6 \mathrm{tr} z_1 z_1 z_1 z_2 z_2 z_1 z_1 z_1 \\
&\quad + \mathrm{tr} z_1 z_1 (11 \mathrm{tr} z_1 z_1 z_1 z_1 z_1 + 3 \mathrm{tr} z_1 z_1 z_1 z_1 z_1 z_1) \mathrm{tr} z_1 z_2 z_1 z_2 z_2 \\
\mathcal{O}_7^{[2,3,2]} &= 7 \mathrm{tr} z_1 z_1 z_1 z_2 z_1 z_2 z_1 z_1 z_1 - 6 \mathrm{tr} z_1 z_1 z_1 z_2 z_1 z_2 z_1 z_1 z_1 \\
&\quad + \mathrm{tr} z_1 z_1 (-4 \mathrm{tr} z_2 z_2 z_2 z_1 z_1 z_1 + 3 \mathrm{tr} z_1 z_1 z_1 z_1 z_1 z_1) \mathrm{tr} z_1 z_2 z_1 z_2 z_2 \\
\end{align}

The $-\frac{1}{4}$-BPS operators are

\begin{align}
\mathcal{Y}_1^{[2,3,2]} &= - \frac{10N}{N^2 - 7} \mathcal{O}_1^{[2,3,2]} + \mathcal{O}_2^{[2,3,2]} + \frac{2}{N^2 - 7} \left( \mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]} + \mathcal{O}_5^{[2,3,2]} \right) \\
\end{align}
\[ Y_{2,3,2}^{2,3,2} = -20 O_{1,3,2}^{2,3,2} + \frac{2}{N} (N^2 + 2) O_{2,3,2}^{2,3,2} - \frac{2}{N} \left( O_{3,3,2}^{2,3,2} + O_{4,3,2}^{2,3,2} \right) + O_{6,3,2}^{2,3,2} \]  

(120)

\[ Y_{3,2,3}^{2,3,2} = 10 O_{1,3,2}^{2,3,2} - \frac{(N^2 - 4)}{N} O_{2,3,2}^{2,3,2} - \frac{2}{N} \left( O_{3,3,2}^{2,3,2} + O_{4,3,2}^{2,3,2} \right) + O_{7,3,2}^{2,3,2} \]  

(121)

(The \( Y_{1,2,3}^{2,3,2} \) are not orthogonal. Although orthonormal linear combinations are easy to find, they look rather messy and we don’t list them here.)

### A.6 \( O_{[p,q,p]} \) and \( K_{[p,q,p]} \) in the large \( N \) limit

Recall that the \( SO(6) \) Young tableau for the \([p,q,p]\) of \( SU(4) \) consists of two rows (one of length \( p + q \), and the other of length \( p \)). Among the possible partitions of the highest weight tableau, there are two special ones

\[ \begin{align*}  
O_{[p,q,p]} & \sim \left( \begin{array}{cccc}
1 & \ldots & 1 & 1 \\
p & \ldots & 2 & q 
\end{array} \right), \\
K_{[p,q,p]} & \sim \left( \begin{array}{cccc}
1 & \ldots & 1 & 1 \\
p & \ldots & 2 & q 
\end{array} \right) 
\end{align*} \]  

(122)

(each continuous group of boxes stands for a single trace). Explicitly, the corresponding operators are

\[ O_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k! (p-k)!} \text{tr} \left( z_1^{p+q-k} z_2^k \right) \text{tr} \left( z_1^k z_2^{p-k} \right) \]  

(123)

\[ K_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k! (p-k)!} \text{tr} \left( z_1^{p+q-k} z_2^k \right) \left( z_1^k z_2^{p-k} \right) \]  

(124)

after projecting \( SU(4) \to SU(3) \times U(1) \) and keeping only the highest \( U(1) \)-charge pieces. Made of only \( z_1 \) and \( z_2 \), both types of operators are annihilated by four out of the sixteen Poincaré supersymmetry generators.

\( K_{[p,q,p]} \) is special because it is the only single trace \([p,q,p]\) operator which can be constructed out of these fields. On the other hand, \( O_{[p,q,p]} \) is “the most natural” double trace composite operator in this representation. We also recognize it as the free theory chiral primary from the classification of [18].

With a slight abuse of notation, we will use the same name for operators with different \( SU(4) \) weights; e.g. all \([p,q,p]\) single trace scalar composites will be referred to as \( K_{[p,q,p]} \), etc.

### B \([p,q,p] \otimes [r,s,r] \otimes [0,k,0]\) and BZ triangles

Tensoring irreducible representations using Young tableaux can get quite tedious. Berenstein-Zelevinsky (BZ) triangles [24] provide a powerful way to cal-
calkulate the multiplicity of the scalar representation in $\lambda \otimes \mu \otimes \nu$. We will discuss the construction for $SU(3)$ and $SU(4)$, and the generalization to higher $SU(N)$ (but not to other Lie algebras, which is not currently known) being straightforward.

For $SU(3)$, the triangles are constructed according to the following rules:

$$
\begin{array}{cccc}
  & m_{13} & n_{12} & l_{23} \\
 m_{23} & n_{13} & l_{12} & n_{23} \\
 n_{13} & m_{23} & l_{23} & m_{13}
\end{array}
$$

where the nine non-negative integers $l_{ij}$, $m_{ij}$, $n_{ij}$ are related to the Dynkin labels $(\lambda_1, \lambda_2)$, $(\mu_1, \mu_2)$, $(\nu_1, \nu_2)$ of the highest weights of the three representations by

$$
\begin{align}
  m_{13} + n_{12} &= \lambda_1 \\
  m_{23} + n_{13} &= \lambda_2 \\
  m_{23} + n_{12} &= \mu_1 \\
  n_{23} + l_{13} &= \mu_2 \\
  n_{13} + l_{12} &= \nu_1 \\
  l_{23} + m_{13} &= \nu_2
\end{align}
$$

They must further satisfy the so-called hexagon conditions

$$
\begin{align}
  n_{12} + m_{23} &= m_{12} + n_{23} \\
  l_{12} + m_{23} &= m_{12} + l_{23} \\
  l_{12} + n_{23} &= n_{12} + l_{23}
\end{align}
$$

This means that the length of opposite sides in the hexagon formed by $n_{12}$, $l_{23}$, $m_{12}$, $n_{23}$, $l_{12}$, and $m_{23}$ in (125) are equal, the length of a segment being the sum of its two vertices.

The number of such triangles gives the multiplicity $N_{\lambda \mu \nu}$; if it is not possible to construct such a triangle, $\nu^*$ does not occur in the tensor product $\lambda \otimes \mu$.

The integers in the BZ triangles have the following origin. Each pair of indices $ij$, $i < j$, on the labels of the triangle is related to a positive root of $SU(3)$. For $SU(N)$, positive roots can be written as $\epsilon_i - \epsilon_j$, $1 \leq i \leq j \leq N$, in terms of orthonormal vectors $\epsilon_i$ in $\mathbb{R}^N$.

The triangle encodes three sums of positive roots:

$$
\begin{align}
  \mu + \nu - \lambda^* &= \sum_{i<j} l_{ij}(\epsilon_i - \epsilon_j) \\
  \nu + \lambda - \mu^* &= \sum_{i<j} m_{ij}(\epsilon_i - \epsilon_j) \\
  \lambda + \mu - \nu^* &= \sum_{i<j} n_{ij}(\epsilon_i - \epsilon_j)
\end{align}
$$

The hexagon relations (127) can be seen as consistency conditions for these three expansions.

For $SU(4)$, the BZ triangles are defined in a similar way, in terms of

$$
\begin{array}{cccc}
  & m_{14} & n_{12} & l_{34} \\
 m_{24} & l_{13} & m_{13} & n_{23} \\
 m_{34} & l_{24} & m_{24} & l_{34}
\end{array}
$$

Note: It is conventional to choose $\nu^*$ instead of $\nu$ for the third weight.
eighteen non-negative integers, related to the Dynkin labels by
\[
\begin{align*}
    m_{14} + n_{12} &= \lambda_1 \\
    n_{14} + l_{12} &= \mu_1 \\
    l_{14} + m_{12} &= \nu_1 \\
    m_{24} + n_{13} &= \lambda_2 \\
    n_{24} + l_{13} &= \mu_2 \\
    l_{24} + m_{13} &= \nu_2 \\
    m_{34} + n_{14} &= \lambda_3 \\
    n_{34} + l_{14} &= \mu_3 \\
    l_{34} + m_{14} &= \nu_3
\end{align*}
\] (130)

Furthermore, an \( SU(4) \) BZ triangle has three hexagons:24
\[
\begin{align*}
    n_{12} + m_{24} &= m_{13} + n_{23} \\
    n_{13} + l_{23} &= l_{12} + n_{24} \\
    l_{24} + n_{23} &= l_{13} + n_{34} \\
    n_{12} + l_{34} &= l_{23} + n_{23} \\
    n_{13} + m_{34} &= n_{24} + m_{23} \\
    n_{23} + m_{24} &= m_{12} + n_{14} \\
    m_{24} + l_{23} &= l_{34} + m_{13} \\
    m_{34} + l_{12} &= l_{23} + m_{23} \\
    l_{13} + m_{23} &= l_{24} + m_{12}
\end{align*}
\] (131)

As an application, consider \( \nu = [0, k, 0] \subset [p, q, p] \otimes [r, s, r] = \lambda \otimes \mu \) of \( SU(4) \); here all representations are self-conjugate. The restrictions on the \( l_{ij}, m_{ij} \), \( n_{ij} \) (these integers must all be non-negative) imply that the entries of the BZ triangle are actually
\[
\begin{align*}
    m_{14} &= l_{14} = m_{12} = l_{34} = 0, \\
    n_{12} &= p, \\
    n_{23} &= n_{14}, \\
    n_{34} &= r, \\
    l_{23} &= m_{34} = p - n_{14}, \\
    l_{12} &= m_{23} = r - n_{14}, \\
    l_{13} &= \frac{1}{2}(s + k - (2p + q) + 2n_{14}), \\
    m_{24} &= \frac{1}{2}(q + k - (2r + s) + 2n_{14}), \\
    n_{13} &= \frac{1}{2}(q - k + (2r + s) - 2n_{14}), \\
    n_{24} &= \frac{1}{2}(s - k + (2p + q) - 2n_{14}), \\
    m_{13} &= \frac{1}{2}((2p + q) + k - (2r + s)), \\
    l_{24} &= \frac{1}{2}((2r + s) + k - (2p + q)).
\end{align*}
\] (132)

All entries thus depend on a single parameter \( n_{14} \) which is subject to restrictions \( 0 \leq n_{14} \leq p, r, \frac{1}{2}(p + r - k) \); plus we get further constraints \( k \geq |(2p + q) - (2r + s)|, \)
\( p + q \geq r, \) and \( r + s \geq p, \) etc.

Now, recall that \( SO(6) \sim SU(4) \), and all our operators are in fact made of the scalars which are in the fundamental 6 of \( SO(6) \). In terms of the Young diagrams for \( SO(6) \), the representations involved are partitioned as
\[
\begin{array}{cccccc}
    n_{14} & m_{34} & m_{24} & n_{13} & n_{14} & m_{23} & l_{13} & n_{24} \\
    n_{14} & m_{34} & q & n_{14} & m_{23} & & s \\
    p & r & & & & k & l_{13}
\end{array}
\] (133)

24 The \( SU(N) \) generalization is straightforward; the BZ triangles are built out of three corner vertices and \( \frac{1}{2}(N - 1)(N - 2) \) hexagons.
An especially convenient decomposition is when the \([0,k,0]\) state is made up of say only 1-s and 2-s. In which case, by symmetry in the vertices, there will be no contributions proportional to \(C\) provided only two flavors are involved in the diagram. Unfortunately, this can be achieved only when \(s+q \geq k\).

Alternatively, we can take a \([p,q,p]\) state made up of \(n_{14} + m_{34} + m_{24} 1\)-s and \(n_{14} + m_{34} + n_{13}\) 2-s; \([r,s,r]\) state made up of \(n_{14} + m_{23} + l_{13} 1\)-s and \(n_{14} + m_{23} + n_{24} 2\)-s; \([0,k,0]\) state made up of \(m_{34} + m_{24} 1\)-s and \(m_{23} + l_{13} 1\)-s, minus contractions. Unlike the previous decomposition, this one works for any \([0,k,0] \otimes [p,q,p] \otimes [r,s,r]\) containing the singlet.

C Partitioning a tableau into 2 flavors

It is often convenient to choose the operators to have only two distinct flavors. Here we shall see that it can always be done.

Consider an operator in the \([p,q,p]\) of \(GL(6)\) made of \(n_1\) 1-s and \(n_2\) 2-s, to be concrete. We have the following constraints: \(p \leq n_2, n_1 \leq p + q\). This state can be assigned an \(SU(4)\) weight \(w = (n_2, n_1, 0) + (n_2, 1 - n_1, 1) = (n_2, n_1 - n_2, n_2)\).

Next, we project this onto the \(SU(3) \times U(1)\); for example, we can choose \(1 \rightarrow 1 + 1, 2 \rightarrow 2 + 2\). Then \(w\) contains terms with \(b\) 1-s, \(n_1 - b\) 1-s, \(c\) 2-s, \(n_2 - c\) 2-s; these have weights \(w'_{b,c} = (n_2 - n_1 + 2(b - c), 2c - n_2)^2\).

To make an irrep of \(GL(6)\) into one of \(so(6)\), we must subtract traces. Since traces have weight zero, contributions with \(n\) contractions instead of 1 and \(m\) instead of 2, are equivalent to \(n' = n_1 - 2n, n'' = n_2 - 2m\). They are projected onto \(w'^{n,m}_{b,c} = (n_2 - n_1 + 2(b + n) - 2(c + m), 2(c + m) - n_2)^2\).

We see that for fixed \(n_1\) and \(n_2\), \(w^{n,m}_{b,c} = w^{n',m'}_{b',c'}\) iff \(b + n = b' + n', c + m = c' + m'\).

We are interested in having \(b = n_1, c = n_2\), for example; then \(w^{n_1,n_2}_{b,c} = (n_1 - n_2, n_2)^{n_1 + n_2}\). In order for \(w^{n,m}_{b,c}\) to have the same weight we must have \(n_1 - n \geq 2, n = n + m + c = m = n_2\), or \(m = 0\); likewise, traces also do not contribute to the projection onto \(b = n_1, c = 0\).

This means that \([p,q,p]\) states of \(so(6) \sim SU(4)\) which consist of \(n_1\) of any \(\phi_a\) and \(n_2\) of any other \(\phi_b\) minus various contractions, project onto pure states of \(SU(3) \times U(1)\), ones containing \(n_1\) \(z_\alpha\)-s and \(n_2\) \(z_b\)-s or \(n_1\) \(\bar{z}_\alpha\)-s and \(n_2\) \(\bar{z}_b\)-s, etc. without having to subtract any traces.

D Details of \(\langle \mathcal{O}_+ \mathcal{O}_+ \mathcal{O}_+ \rangle\) calculations

In Section 7.3 we needed to explicitly calculate several three-point functions \(\langle \mathcal{O}_{l,k,l}(x) \mathcal{O}_{p,q,p}(y) \mathcal{O}_{r,s,r}(w) \rangle\) of \(\frac{1}{2}\)-BPS operators. The flavor breakdown is discussed in Section 7.3, see equations (65-68). For the five cases of (54), we list the values of combinatorial coefficients multiplying the Born diagram, as well as the ones in front of \(\tilde{B}(x, w)\) and \(\tilde{B}(w, y)\).
When operators $O_w$ are properly symmetrized, we can mark the $z_1$-s exchanged between $O_w$ and $O_z$ separately from the $z_1$-s exchanged between $O_w$ and $O_y$. As far as the combinatorial factors $\alpha_{\text{free}}$, $\tilde{\beta}_{xw}$ and $\tilde{\beta}_{yw}$ are concerned, the difference is just a multiplicative factor. This would be equivalent to calculating the three-point functions with

\[ O_{[l,k,l]}(x) \sim [z_1^ax_2^b z_3^c] \] (134)

\[ O_{[p,q,p]}(y) \sim [z_1^c z_2^b z_3^a] \] (135)

\[ O_{[r,s,r]}(w) \sim [z_1^c z_2^b z_3^a] \] (136)

rather than \([\underline{54}, \underline{28}]\) instead,\(^{25}\) with the same $e \equiv \frac{1}{2}((2l+k) + (2p+q) - (2r+s)) \leq l + k, p + q$, and integers $a, b, c, d$ partitioning $r + s = a + c$, $r = b + d$. This simplifies the calculations dramatically.

For operators in the [2,0,2] or [2,1,2] representations, we chose the $\frac{1}{2}$-BPS operator from the beginning. In the other cases, several $\frac{1}{4}$-BPS chiral primaries exist in each representation, so instead we choose the operators as $O_w = \sum C^j_i O^j_i$ for example (see Appendix A for the definitions). In all cases \([137, 141]\) listed above, Born level correlators are nonzero for general $\alpha$ and $\tilde{\beta}_{xw}$, and integers $a, b, c, d$ partitioning $r + s = a + c$, $r = b + d$. This simplifies the calculations dramatically.

With $O_x, O_y, O_w$ as in \([\underline{34}, \underline{33}]\), we list the representations, choices of flavors, Born level and order $g^2$ results below.

\[ [2, 0, 2] \otimes [2, 0, 2] \otimes [2, 2, 2] : \langle [z_1^2 z_2^2 z_3^2]_x [z_1^2 z_2^2 z_3^2]_y [z_1^2 z_2^2 z_1^2 z_2^2]_w \rangle \] (137)

\[
\alpha_{\text{free}} = \frac{3(N^2 - 1)}{2048N^3} \left( 12288C_w^2 + 2048C_w^3 + 6144C_w^4 - 5120C_w^1 N - 4096C_w^5 N - 12800C_w^2 N^2 - 1280C_w^3 N^2 - 768C_w^4 N^2 + 2688C_w^5 N^3 - 4560C_w^5 N^3 + 3120C_w^4 N^4 + 1096C_w^3 N^4 - 168C_w^4 N^4 - 176C_w^1 N^5 + 1616C_w^5 N^5 + 8C_w^2 N^6 - 46C_w^3 N^6 - 6C_w^4 N^6 + C_w^5 N^7 \right)
\]

\[
\tilde{\beta}_{xw} = \frac{9(N^2 - 1)(N^2 - 16)}{4096N^2} \left( 768C_w^2 + 128C_w^3 + 384C_w^4 + 64C_w^5 N - 256C_w^5 N - 464C_w^4 N^2 - 216C_w^3 N^2 - 72C_w^4 N^2 - 80C_w^1 N^3 - 176C_w^5 N^3 + \right)
\]

\(^{25}\) As written in \([\underline{33}]\), $O_w$ is not even a $[r, s, r]$ operator; we need to subtract $SO(6)$ traces. But when calculating whether the three coefficients $\alpha_{\text{free}}$, $\tilde{\beta}_{xw}$ and $\tilde{\beta}_{yw}$ are zero or not, the answers are the same as if we had done it properly.

\(^{26}\) The representations involved are quite large, so calculating directly Clebsch-Gordan coefficients for the states involved in a particular tensor product is difficult. Instead, we compute the Born level correlator and if it doesn’t vanish, we know the CG is not zero. This is not necessarily the case: for example if we chose the flavors as $\langle [z_1^2 z_2^2 z_3^2]_x [z_2^2 z_3^2 z_1^2]_y [z_2^2 z_1^2 z_3^2]_w \rangle$ in \([\underline{38}]\), the correlators would vanish both at Born level and to order $g^2$. 

37
\[ \alpha_{\text{free}} = \frac{3(N^2 - 1)(N^2 - 4)}{10240N^4} (34560C_w^2 + 17280C_w^3 + \\
51840C_w^4 + 51840C_w^5 - 17280C_w^6 N - 86400C_w^6 N + \\
17280C_w^7 N - 24720C_w^8 N^2 + 13560C_w^9 N^2 - \\
11160C_w^{10} N^2 + 6120C_w^{11} N^2 + 6528C_w^{12} N^3 + \\
1320C_w^{13} N^3 - 3720C_w^{14} N^3 + 3840C_w^{15} N^4 + \\
6240C_w^{16} N^4 + 3440C_w^{17} N^4 + 1860C_w^{18} N^4 - 254C_w^{19} N^5 + \\
4140C_w^{20} N^5 + 240C_w^{21} N^5 + 10C_w^{22} N^6 - 160C_w^{23} N^6 - \\
10C_w^{24} N^6 - 105C_w^{25} N^6 + C_w^{26} N^7) \]

\[ \beta_{yw} = \beta_{xw} = 8C_w^2 N^4 - 14C_w^3 N^4 - 6C_w^4 N^4 + C_w^5 N^5 \]

\[ [2, 0, 2] \otimes [2, 1, 2] \otimes [2, 3, 2] : \langle[\bar{z}_1 z_2 z_3]_x [\bar{z}_1 z_2^3 z_3]_y [\bar{z}_1^3 z_2^2 z_3^2]_w \rangle \]  \hspace{1cm} (138)

\[ \beta_{xw} = \frac{9(N^2 - 1)(N^2 - 4)(N^2 - 16)}{20480N^3} (2160C_w^2 + 1080C_w^3 + \\
3240C_w^4 + 3240C_w^5 - 5400C_w^6 N + 1080C_w^7 N - \\
960C_w^8 N^2 - 1920C_w^9 N^2 - 540C_w^{10} N^2 - 1980C_w^{11} N^2 - \\
126C_w^{12} N^3 - 660C_w^{13} N^3 + 240C_w^{14} N^3 + 10C_w^{15} N^4 - \\
10C_w^{16} N^4 - 105C_w^{17} N^4 + C_w^{18} N^5) \]

\[ \beta_{yw} = \frac{3(N^2 - 1)(N^2 - 4)(N^2 - 36)}{5120N^3} (1920C_w^2 + 960C_w^3 + 2880C_w^4 + \\
2880C_w^5 - 4800C_w^6 N + 960C_w^7 N - 840C_w^8 N^2 - \\
1380C_w^9 N^2 - 780C_w^{10} N^2 - 1740C_w^{11} N^2 - 98C_w^{12} N^3 - \\
540C_w^{13} N^3 - 120C_w^{14} N^3 + 10C_w^{15} N^4 - 100C_w^{16} N^4 - \\
10C_w^{17} N^4 + 75C_w^{18} N^4 + C_w^{19} N^5) \]

\[ [2, 0, 2] \otimes [2, 1, 2] \otimes [3, 1, 3] : \langle[\bar{z}_1 z_2 z_3]_x [\bar{z}_1^2 z_2^2 z_3]_y [\bar{z}_1^2 z_2^2 z_3^2]_w \rangle \]  \hspace{1cm} (139)

\[ \alpha_{\text{free}} = \frac{(N^2 - 1)(N^2 - 4)}{10240N^2} (-103680C_w^2 + 11520C_w^5 N + 18000C_w^7 N^2 - 9504C_w^8 N^2 - 545C_w^9 N^3 + \\
8520C_w^{10} N^3 + 60C_w^{11} N^4 + 276C_w^{12} N^4 + 5C_w^{13} N^5) \]

\[ \beta_{xw} = \frac{3(N^2 - 1)(N^2 - 4)(N^2 - 16)}{20480N} (-2160C_w^2 + 864C_w^3 - 225C_w^4 N - \\
1080C_w^4 N + 60C_w^5 N^2 + 24C_w^6 N^2 + 5C_w^7 N^3) \]

\[ \beta_{yw} = \frac{(N^2 - 1)(N^2 - 4)(N^2 - 36)}{10240N} (-4320C_w^2 + 1728C_w^3 - 465C_w^4 N - \\
2520C_w^4 N + 60C_w^5 N^2 + 228C_w^6 N^2 + 5C_w^7 N^3) \]
\[ \alpha_{\text{free}} = \frac{(N^2 - 1)(N^2 - 4)}{57600N^2} \left( -129600C_{w}^{2}C_{y}^{2} - 64800C_{w}^{3}C_{y}^{3} - 194400C_{w}^{3}C_{y}^{4} - 
\]

\[
194400C_{y}^{2}C_{w}^{5} + 14400C_{y}^{4}C_{w}^{2}N + 45900C_{y}^{2}C_{y}^{2}N + 7200C_{y}^{3}C_{w}^{3}N - 8640C_{y}^{3}C_{y}^{3}N + 21600C_{y}^{3}C_{y}^{4}N - 
\]

\[
43200C_{w}^{-2}C_{y}^{4}N - 21600C_{w}^{-3}C_{y}^{4}N - 64800C_{w}^{4}C_{y}^{4}N + 
\]

\[
21600C_{w}^{4}C_{y}^{5}N - 64800C_{w}^{5}C_{y}^{5}N + 324000C_{y}^{2}C_{w}^{6}N - 
\]

\[
64800C_{y}^{2}C_{w}^{6}N + 88200C_{y}^{2}C_{y}^{2}N^2 - 3150C_{y}^{2}C_{w}^{3}N^2 - 
\]

\[
14400C_{w}^{2}C_{y}^{3}N^2 + 14400C_{w}^{3}C_{y}^{3}N^2 + 67500C_{y}^{2}C_{w}^{4}N^2 - 
\]

\[
12960C_{y}^{2}C_{y}^{5}N^2 + 18900C_{y}^{3}C_{w}^{4}N^2 + 43200C_{y}^{3}C_{y}^{5}N^2 + 
\]

\[
43200C_{y}^{5}C_{w}^{5}N^2 - 36000C_{y}^{5}C_{y}^{6}N^2 + 108000C_{y}^{6}C_{w}^{6}N^2 + 
\]

\[
7200C_{y}^{4}C_{w}^{2}N^2 - 21600C_{y}^{4}C_{y}^{6}N^2 - 6600C_{y}^{3}C_{w}^{3}N^3 - 
\]

\[
13965C_{w}^{2}C_{y}^{2}N^3 - 9675C_{y}^{5}C_{w}^{3}N^3 + 3696C_{y}^{3}C_{y}^{3}N^3 - 
\]

\[
7125C_{y}^{4}C_{y}^{4}N^3 + 34200C_{y}^{2}C_{y}^{6}N^3 - 7650C_{y}^{3}C_{y}^{4}N^3 + 
\]

\[
20700C_{y}^{3}C_{y}^{4}N^3 - 13500C_{y}^{5}C_{y}^{5}N^3 + 5400C_{y}^{4}C_{y}^{5}N^3 - 
\]

\[
4500C_{y}^{2}C_{y}^{6}N^3 + 31680C_{y}^{3}C_{y}^{6}N^3 + 11700C_{y}^{2}C_{y}^{7}N^3 - 
\]

\[
6120C_{y}^{3}C_{y}^{7}N^3 - 820C_{y}^{3}C_{y}^{7}N^4 - 12750C_{y}^{2}C_{y}^{7}N^4 - 
\]

\[
16050C_{y}^{2}C_{y}^{7}N^4 + 6420C_{y}^{3}C_{y}^{8}N^4 + 9930C_{y}^{2}C_{y}^{9}N^4 - 
\]

\[
10500C_{y}^{3}C_{y}^{8}N^4 + 5490C_{y}^{3}C_{y}^{8}N^4 - 6750C_{y}^{2}C_{y}^{9}N^4 - 
\]

\[
13500C_{y}^{2}C_{y}^{9}N^4 + 8595C_{y}^{2}C_{y}^{10}N^4 - 5100C_{y}^{2}C_{y}^{10}N^4 - 
\]

\[
9900C_{y}^{2}C_{y}^{10}N^4 - 300C_{y}^{2}C_{y}^{11}N^4 + 5400C_{y}^{2}C_{y}^{11}N^4 + 
\]

\[
50C_{y}^{2}C_{y}^{11}N^5 + 20C_{y}^{1}C_{y}^{12}N^5 - 925C_{y}^{1}C_{y}^{13}N^5 + 
\]

\[
471C_{y}^{2}C_{y}^{13}N^5 + 75C_{y}^{2}C_{y}^{14}N^5 - 6300C_{y}^{2}C_{y}^{14}N^5 - 
\]

\[
10350C_{y}^{2}C_{y}^{14}N^5 - 4200C_{y}^{2}C_{y}^{15}N^5 + 225C_{y}^{2}C_{y}^{15}N^5 - 
\]

\[
5850C_{y}^{2}C_{y}^{15}N^5 - 12600C_{y}^{2}C_{y}^{16}N^5 + 5580C_{y}^{3}C_{y}^{6}N^5 + 
\]

\[
180C_{y}^{3}C_{y}^{7}N^5 + 5C_{y}^{1}C_{y}^{11}N^5 - 30C_{y}^{2}C_{y}^{11}N^6 + 
\]

\[
630C_{y}^{2}C_{y}^{11}N^6 + 150C_{y}^{2}C_{y}^{12}N^6 + 225C_{y}^{2}C_{y}^{13}N^6 - 
\]

\[
6300C_{y}^{4}C_{y}^{6}N^6) \]

\[
\tilde{\beta}_{gw} = \frac{(N^2 - 4)}{115200N^2} \left( 1036800C_{w}^{2}C_{y}^{2} + 518400C_{y}^{2}C_{w}^{3} - 
\]

\[
414720C_{w}^{3}C_{y}^{3} - 207360C_{w}^{3}C_{y}^{3} + 1555200C_{y}^{2}C_{y}^{4} - 
\]

\[
622080C_{y}^{3}C_{y}^{4} + 1555200C_{y}^{2}C_{y}^{5} - 622080C_{y}^{3}C_{y}^{5} + 
\]

\[
115200C_{y}^{4}C_{y}^{5}N + 57600C_{y}^{3}C_{y}^{6}N + 172800C_{y}^{4}C_{y}^{6}N + 
\]

\[
\]
\[ \alpha_{\text{free}} = \frac{(N^2 - 1)(N^2 - 4)}{1036800N^2} (2332800C_{w}^{2}C_{y}^{2} - 2332800C_{w}^{2}C_{y}^{2}N - 2332800C_{w}^{2}C_{y}^{2}N + 10800C_{w}^{1}C_{y}^{1}N^2 + 324000C_{w}^{2}C_{y}^{2}N^2 - 492480C_{w}^{2}C_{y}^{2}N^2 + 114048C_{w}^{3}C_{y}^{3}N^2 - 259200C_{w}^{1}C_{y}^{1}N^2 + 259200C_{w}^{1}C_{y}^{1}N^2 - 777600C_{w}^{2}C_{y}^{2}N^2 - 44100C_{w}^{2}C_{y}^{2}N^2 - 44100C_{w}^{2}C_{y}^{2}N^2 - 8280C_{w}^{1}C_{y}^{1}N^3 - 8280C_{w}^{1}C_{y}^{1}N^3 + 950400C_{w}^{2}C_{y}^{2}N^3 - 250560C_{w}^{2}C_{y}^{2}N^3 - 5875C_{w}^{1}C_{y}^{1}N^4 - 13320C_{w}^{1}C_{y}^{1}N^4 + 90720C_{w}^{2}C_{y}^{2}N^4 - 90720C_{w}^{2}C_{y}^{2}N^4 - 22752C_{w}^{3}C_{y}^{3}N^4 - 52800C_{w}^{2}C_{y}^{2}N^4 - 52800C_{w}^{2}C_{y}^{2}N^4 + 396000C_{w}^{2}C_{y}^{2}N^4 + 900C_{w}^{1}C_{y}^{1}N^5 + 900C_{w}^{1}C_{y}^{1}N^5 - 42840C_{w}^{2}C_{y}^{2}N^2 + 248400C_{w}^{2}C_{y}^{2}N^2 - 51600C_{w}^{3}C_{y}^{3}N^2 + 102960C_{w}^{3}C_{y}^{3}N^2 + 18000C_{w}^{1}C_{y}^{1}N^2 - 982800C_{w}^{2}C_{y}^{2}N^2 + 289440C_{w}^{3}C_{y}^{3}N^2 - 129600C_{w}^{3}C_{y}^{3}N^2 + 53100C_{w}^{1}C_{y}^{1}N^2 + 237600C_{w}^{2}C_{y}^{2}N^2 - 52850C_{w}^{1}C_{y}^{1}N^3 - 60720C_{w}^{2}C_{y}^{2}N^3 - 76800C_{w}^{3}C_{y}^{3}N^3 + 22488C_{w}^{4}C_{y}^{4}N^3 - 56200C_{w}^{1}C_{y}^{1}N^3 - 303600C_{w}^{2}C_{y}^{2}N^3 - 428400C_{w}^{3}C_{y}^{3}N^3 - 315600C_{w}^{4}C_{y}^{4}N^3 - 105825C_{w}^{5}C_{y}^{5}N^3 - 574200C_{w}^{4}C_{y}^{4}N^3 - 392400C_{w}^{4}C_{y}^{4}N^3 + 286560C_{w}^{5}C_{y}^{5}N^3 - 18000C_{w}^{2}C_{y}^{2}N^3 - 16560C_{w}^{3}C_{y}^{3}N^3 - 6755C_{w}^{1}C_{y}^{1}N^4 + 3000C_{w}^{2}C_{y}^{2}N^4 - 70200C_{w}^{2}C_{y}^{2}N^4 + 29160C_{w}^{3}C_{y}^{3}N^4 + 70920C_{w}^{3}C_{y}^{3}N^4 + 4800C_{w}^{4}C_{y}^{4}N^4 + 29640C_{w}^{5}C_{y}^{5}N^4 - 40680C_{w}^{1}C_{y}^{1}N^4 + 13500C_{w}^{2}C_{y}^{2}N^4 + 52020C_{w}^{3}C_{y}^{3}N^4 - 45600C_{w}^{5}C_{y}^{5}N^4 - 309600C_{w}^{4}C_{y}^{4}N^4 - 1500C_{w}^{1}C_{y}^{1}N^4 + 250C_{w}^{2}C_{y}^{2}N^5 + 300C_{w}^{2}C_{y}^{2}N^5 - 8000C_{w}^{3}C_{y}^{3}N^5 + 3948C_{w}^{5}C_{y}^{5}N^5 + 400C_{w}^{4}C_{y}^{4}N^5 - 51600C_{w}^{5}C_{y}^{5}N^5 + 1125C_{w}^{1}C_{y}^{1}N^6 + 3960C_{w}^{2}C_{y}^{2}N^6 + 25C_{w}^{3}C_{y}^{3}N^6 + 5160C_{w}^{5}C_{y}^{5}N^6)\]

\[
[2, 0, 2] \otimes [3, 1, 3] \otimes [3, 1, 3] : \langle [z_1^2 z_3^2] [z_1 z_2 z_3] [z_1^2 z_2 z_3^2] \rangle (141)\]
\[
\begin{align*}
\beta_{yw} &= \frac{(N^2-1)(N^2-4)}{51840} (172800C_1^1C_2^2 + 172800C_1^1C_2^2 - 69120C_1^2C_4^3 - 69120C_1^1C_3^4 + 4147200C_y^2C_w^4 - 1658880C_y^2C_w^4 + 197856C_y^2C_w^4 - 1658880C_y^2C_w^4 + 33450C_y^1C_3^1N - 837000C_y^2C_4^2N - 79920C_y^2C_4^2N - 79920C_y^2C_4^2N - 457200C_y^1C_4^1N + 2678400C_y^1C_4^1N - 107700C_y^1C_4^1N^2 - 107700C_y^1C_4^1N^2 - 2640C_y^2C_4^3N^2 - 2640C_y^2C_4^3N^2 - 910800C_y^2C_4^3N^2 - 910800C_y^2C_4^3N^2 - 910800C_y^2C_4^3N^2 + 96480C_y^3C_4^1N^2 - 910800C_y^2C_4^3N^2 + 96480C_y^3C_4^1N^2 - 910800C_y^2C_4^3N^2 + 13175C_y^1C_4^1N^3 + 18000C_y^2C_4^3N^3 + 83880C_y^2C_4^3N^3 + 83880C_y^2C_4^3N^3 - 43200C_y^3C_4^1N^3 - 100800C_y^1C_4^1N^3 - 100800C_y^1C_4^1N^3 - 597600C_y^1C_4^1N^3 + 1500C_y^1C_4^1N^4 + 1500C_y^1C_4^1N^4 + 9480C_y^1C_4^1N^4 + 9480C_y^1C_4^1N^4 + 59760C_y^3C_4^1N^4 + 59760C_y^3C_4^1N^4 + 125C_y^1C_4^1N^5 - 5976C_y^1C_4^1N^5)
\end{align*}
\]

The last two sets of calculations are very formidable. Getting the coefficients \(\beta_{yw}\) took 2430 hrs for \(\text{[2,0,2]} \otimes [3,1,3] \otimes [3,1,3] \otimes [2,3,2]\) and 2830 hrs for \(\text{[2,0,2]} \otimes [3,1,3] \otimes [2,3,2]\) of a Mathematica computation on a Pentium-III with 1.4MHz speed.

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