GAUSS-BONNET FOR MATRIX CONFORMALLY RESCALED DIRAC

MASOUD KHALKHALI AND ANDRZEJ SITARZ

ABSTRACT. We derive an explicit formula for the scalar curvature over a two-torus with a Dirac operator conformally rescaled by a globally diagonalizable matrix. We show that the Gauss-Bonnet theorem holds and extend the result to all Riemann surfaces with Dirac operators modified in the same way.

1. INTRODUCTION

In this paper we consider a new class of noncommutative algebras that lend themselves to spectral and geometric analysis. In noncommutative geometry the metric structure on a noncommutative space \( \mathcal{A} \) is encoded in a spectral triple on \( \mathcal{A} \) and under suitable conditions, using the spectral data, one can formulate a notion of scalar and Ricci curvature for \( \mathcal{A} \) [5,15]. In recent years much progress has been made in understanding and computing the curvature invariants of curved noncommutative tori using spectral properties of the Dirac operator [4,5,6,9,10,12,13,15,11]. In this paper we consider a different class of noncommutative algebras, namely algebras of matrix valued functions on a classical 2 dimensional Riemannian manifold and prove a Gauss-Bonnet theorem for them.

Let \( M \) be a two dimensional Riemannian manifold which we assume to be closed, connected and oriented with a fixed spin structure. Let \( D : L^2(S) \to L^2(S) \) denote the Dirac operator of \( M \) acting on the space of spinors. Consider the algebra \( \mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C}) \) of smooth matrix valued functions on \( M \) and its diagonal representation on \( \mathcal{H} = L^2(S) \otimes \mathbb{C}^n \). Taking the diagonal action of the Dirac operator we obtain a spectral triple on \( \mathcal{A} \). Let \( h \in \mathcal{A} \) be a positive element. We use \( h \) to perturb the spectral triple of \( \mathcal{A} \) in the following way. Consider the operator \( D_h = hDh \) as a conformally rescaled Dirac operator. Although \( D_h \) does not have bounded commutators with the elements of \( \mathcal{A} \) we might still consider the resulting geometry as a twisted spectral triple [14] or pass to \( h \) in the commutant of \( \mathcal{A} \) [2]. In either case the perturbation is of the same type. We would like to address the question: does the Gauss-Bonnet theorem hold for \( D_h \)? We provide a positive answer to this question.

Recall that the Gauss-Bonnet theorem for two dimensional manifolds can be stated in terms of spectral zeta functions. For \( n = 1 \), that is \( \mathcal{A} = C^\infty(M) \), where one deals with a commutative algebra, the spectral zeta function \( \zeta_D(s) \) associated to the Dirac Laplacian \( D^*D \) is defined by

\[
\zeta_D(s) = \sum \lambda_j^{-s}, \quad \text{Re}(s) > 1,
\]

where the summation is over non-zero eigenvalues of \( D^*D \). The zeta function is absolutely convergent and holomorphic for \( \text{Re}(s) > 1 \) and has a meromorphic continuation to \( \mathbb{C} \) with a unique (simple) pole at \( s = 1 \). In particular \( \zeta(0) \) is defined and it is well know that it is a topological invariant. For example, in the special case where \( D = \partial \) is the Cauchy-Riemann...
operator of the complex structure defined by the conformal class of the metric, we have

\[ \zeta_D(0) + 1 = \frac{1}{12\pi} \int_M R = \frac{1}{6} \chi(M), \]

where \( R \) is the scalar curvature and \( \chi(M) \) is the Euler-Poincaré characteristic. Thus \( \zeta_D(0) \) is a topological invariant, and, in particular, it remains invariant under the conformal perturbation \( g \to e^{\xi} g \) of the metric [4].

For technical reasons, we assume there is a unitary element \( U \in \mathcal{A} \) such that \( h = UHU^* \), where \( H \) is a diagonal matrix. Then:

\[ hDh = UHU^* D UHU^* = U (H (D + U^*[D, U]) H) U^*. \]

Therefore the spectrum of \( Dh \) is the same as the spectrum of \( D_{A,H} = H (D + A) H \), where \( A \) is a matrix valued one-form.

In this paper we show that the Gauss-Bonnet theorem holds for the family of conformally rescaled Dirac operators with possible fluctuations \( D_{A,H} = H (D + A) H \) where the rescaling is a diagonalizable matrix and we compute the local expressions for the scalar curvature. The main point of using the diagonal matrix \( H \) as the conformal rescaling is that \( H \) commutes then with all its derivatives thus making the computations feasible.

In the computations we use the matrix valued pseudodifferential operators over the manifold as opposed to the methods used in [18]. The results demonstrate that unlike in the case of higher residues there, the expressions for the value of the \( \zeta \) function at 0 are complicated also in the matrix case.

Among other examples of curvature computations for noncommutative manifolds we should mention the Moyal sphere [8] where the algebraic approach of [26] was applied, and the differential geometry based construction for the 4-sphere [1], as well as the toric noncommutative manifolds [22].

2. Computations for the torus

Consider the canonical spectral triple for a flat two torus \( M = \mathbb{R}^2 / \mathbb{Z}^2 \). Its spin structure is defined by the Pauli spin matrices \( \sigma^1, \sigma^2 \) and its Dirac operator is

\[ D = \sigma^1 \delta_1 + \sigma^2 \delta_2. \]

Here \( \delta_1, \delta_2 \) are the partial derivatives \( \frac{1}{i} \frac{\partial}{\partial x} \) and \( \frac{1}{i} \frac{\partial}{\partial y} \).

To compute the resolvent kernel we work in the algebra of matrix valued pseudodifferential operators obtained by tensoring the algebra \( \Psi \) of pseudodifferential operators on a smooth manifold \( M \) by the algebra of \( n \) by \( n \) matrices. Let \( h = UHU^* \) be a conformal factor where \( H \) is a diagonal matrix-valued function and \( U \) a unitary matrix-valued function on a torus.

2.1. Computing the resolvent. We have the following form of the symbol of the operator \( D_{A,H}^2 = H (D + A) H^2 (D + A) H \),

\[ \sigma_{D_{A,H}^2} = a_2 + a_1 + a_0, \]
where:
\[ a_2 = H^4 \xi^2, \]
\[ a_1 = i \varepsilon_{ij} \sigma^3 2H^3 \delta_i(H) \xi^j + 4H^3 \delta_i(H) \xi^i - i \varepsilon_{ij} \sigma^3 H A_i H \xi^j \]
\[ + H^3 A_i H \xi^i + i \varepsilon_{ij} \sigma^3 H A_i H \xi^j \]  
\[ + H^3 A_i H \xi^i, \]
\[ a_0 = H^4(\Delta H) + H^3 A_j \delta_i(H) - H^3 i \sigma^3 \varepsilon_{ij} A_i \delta_j(H) + H^3 \delta_i(A_i) H + i \sigma^3 H^3 \varepsilon_{ij} \delta_j(A_i) H \]
\[ + 2H^2 \delta_i(H) \delta_j(H) + 2H^2 \delta_i(H) A_i H + 2H A_i H^2 \delta_i(H) + 2i \sigma^3 H^2 \varepsilon_{ij} \delta_j(H) A_j H \]
\[ + i \sigma^3 \varepsilon_{ij} HA_i H^2 \delta_i(H) + i \sigma^3 \varepsilon_{ij} HA_i H^2 A_j + HA_i H^2 A_j H. \]

Then the first symbols of \((D_{H,a})^{-2} = b_0 + b_1 + b_2 + \cdots\) are:

\[ b_0 = (a_2 + 1)^{-1}, \]
\[ b_1 = - (b_0 a_1 + \partial_k(b_0) \delta_k(a_2)) b_0, \]
\[ b_2 = - (b_1 a_1 + b_0 a_0 + \partial_k(b_0) \delta_k(a_1) + \partial_k(b_1) \delta_k(a_2) + \frac{1}{2} \partial_k \partial_j(b_0) \delta_k \delta_j(a_2)) b_0. \]

Here:
\[ b_0 = (1 + H^4 \xi^2)^{-1}, \]

and in the above formulas we have used that \(H\) commutes with \(\delta_i(H)\).

The computations of \(b_2\) yield three parts, which we treat separately: independent of \(A\), linear in \(A\), quadratic in \(A\) and terms depending on the derivate of \(A\).

### 3. Curvature

Apart from the computations of the Gauss-Bonnet term, it is interesting to obtain an explicit formula for the local curvature term (or more accurately, for the curvature modified by the local measure). We use the noncommutative geometry setup to compute the scalar curvature functional in the sense defined below.

A matrix-valued function \(R : T^2 \to M_n(\mathbb{C})\) we call a local curvature functional if for any \(f\) from the algebra of matrix valued functions on the torus:

\[ \zeta_{f,D}(0) = \int_{T^2} \text{Tr} f R, \]

where
\[ \zeta_{f,D} = \text{Tr} f |D|^{-s}. \]

Note that since we are working with the operator that is defined on the flat torus, the integral is with respect to the standard flat metric, hence the matrix valued function \(R\) contains both the scalar curvature as well as the volume form arising from the metric defined by the Dirac operator \(D\).

Using our assumptions and notation, for the Dirac operator, which has been conformally modified by a globally diagonalizable function, we obtain four contributions to the curvature functional term. In all computations below we use the fact that \(H, \delta_i(H)\) and \(b_0\) commute with each other.
3.1. Terms not depending on $A$. First term, which is independent of $A$ and contains only functions of $H$ and its derivatives $\delta_i(H)$, is therefore identical with the classical contribution,

$$b_2(H, \xi) = 96 b_0^2 \delta_i(H) \delta_i(H) H^{14} (\xi^2)^3 - 136 b_0^4 \delta_i(H) \delta_i(H) H^{10} (\xi^2)^2 + 46 b_0^3 \delta_i(H) \delta_i(H) H^6 (\xi^2) - 2 b_0^2 \delta_i(H) \delta_i(H) H^2 - 8 b_0 \Delta(H) H^{11} (\xi^2)^2 + 8 b_0^3 \Delta(H) H^7 (\xi^2) - b_0^2 \Delta(H) H^3,$$

(3.1)

Integrating out over the $\xi$ space and using

$$\int_0^{\infty} r^{2k+1} dr = \frac{1}{2(k+1)a^{2(k+1)}}$$

we obtain:

$$R(H) = -\pi \left( \frac{1}{3} H^{-2} \delta_i(H) \delta_i(H) + \frac{1}{3} H^{-1} \Delta(H) \right).$$

(3.2)

In the subsequent computation we use Lesch rearrangement lemma [21] and denote the conjugation by $\Delta$ conjugation by $H^4$:

$$\Delta(x) = H^{-4} x H^4.$$

3.2. Terms linear in $A$. We have,

$$b_2^{(1)}(H, A) = -b_0 h A_i b_0 \delta_i(H) H^2 + 5 b_0 h A_i b_0^2 \delta_i(H) H^6 \xi^2 - 4 b_0 h A_i b_0^3 \delta_i(H) H^{10} (\xi^2)^2 - b_0 H^3 A_i b_0 \delta_i(H) + 7 b_0 H^3 A_i b_0^2 \delta_i(H) H^4 \xi^2 - 4 b_0 H^3 A_i b_0^3 \delta_i(H) H^8 (\xi^2)^2 + 3 b_0^2 H^5 A_i b_0 \delta_i(H) H^2 \xi^2 - 4 b_0^2 H^5 A_i b_0^2 \delta_i(H) H^6 (\xi^2)^2 + b_0^2 H^7 A_i b_0 \delta_i(H) \xi^2 - 4 b_0^2 H^7 A_i b_0^2 \delta_i(H) H^4 (\xi^2)^2$$

and

$$b_2^{(1)}(H, A) = -2 b_0 \delta_i(H) H^2 A_i b_0 H + 2 b_0^2 \delta_i(H) H^4 A_i b_0 H^3 \xi^2 + 6 b_0^2 \delta_i(H) H^6 A_i b_0 H \xi^2 - 4 b_0^3 \delta_i(H) H^8 A_i b_0 H^3 (\xi^2)^2 - 4 b_0^3 \delta_i(H) H^{10} A_i b_0 H (\xi^2)^2.$$

Explicit computations give first:

$$R^{(1)}(H, A) = \sum_{i=1,2} 2\pi H G(\Delta)(A_i) \delta_i(H),$$

where $G$ is the following function:

$$G(s) = \frac{(1 + \sqrt{s}) \sqrt{s}}{(s - 1)^3} \left( (s + 1) \ln(s) - 2(s - 1) \right),$$

and a second term,

$$R^{(2)}(H, A) = \sum_{i=1,2} -2\pi H^{-2} \delta_i(H) G(\Delta)(A_i) H,$$

surprisingly with the same function $G(s)$.

Note that after taking the trace both terms shall cancel each other independently of its value at $s = 1$, which is $G(1) = \frac{2}{3}$. 
Therefore

\[
\text{Tr } (R^{(1)}(H, A) + R^{(2)}(H, A)) = 0.
\]

3.3. **Terms linear in** \(\delta_i(A_j)\). In this case we have:

\[
b_2(H, \delta_i(A_j)) = -b_0 H^3 \delta_i(A_i) b_0 H + b_0^2 H^5 \delta_i(A_i) b_0 H^3 \xi^2
+ b_0^2 H^7 \delta_i(A_i) b_0 H \xi^2,
\]

and again explicit integration over \(\xi\) gives:

\[
\pi H^{-1} F(\Delta)(\delta_i(A_i)) H,
\]

where

\[
F = -\frac{(1 + \sqrt{s})\sqrt{s}}{(s - 1)^2} \ln(s) + \frac{\sqrt{s} + 1}{s - 1}.
\]

Again, it is not difficult to check that \(F(1) = 0\) and the expression vanishes after we take the trace of it, so:

\[
\text{Tr } (R(H, \delta_i(A_j))) = 0.
\]

3.4. **Quadratic terms in** \(A_i\). We have:

\[
b_2(H, A^2) = -b_0 H A_i H^2 A_i b_0 H + b_0 H A_i b_0 H^6 A_i b_0 H \xi^2
+ b_0 H^3 A_i b_0 H^2 A_i b_0 H^3 \xi^2.
\]

Integrating over \(\xi\) we obtain:

\[
R(H, A^2) = -\pi H^{-1} Q(\Delta^{(1)}, \Delta^{(2)})(A_i \cdot A_i) H
\]

where

\[
Q(s, t) = \frac{\sqrt{s}(\sqrt{t} + s)}{(s - 1)(s - t)} \ln s - \frac{\sqrt{s} \sqrt{s}}{(s - t) \sqrt{t}} \ln t.
\]

To compute the trace we first take \(t = 1\):

\[
F(s) = Q(s, 1) = \frac{(s + 1) \ln s + 2(1 - s)}{(s - 1)^2},
\]

and observe that due to the trace property:

\[
\text{Tr } (H^{-1} F(\Delta)(A_i) A_i H) = \text{Tr } (A_i F(\Delta)(A_i))
= \text{Tr } (F(\Delta^{-1})(A_i) A_i).
\]

But:

\[
F(\frac{1}{s}) = -\frac{(\frac{1}{s} + 1) \ln s + 2(1 - \frac{1}{s})}{(\frac{1}{s} - 1)^2}
= \frac{-(s + 1) \ln s - 2(1 - s)}{(s - 1)^2}
= -F(s),
\]

and therefore

\[
\text{Tr } R(H, A^2) = 0,
\]

so the quadratic term vanishes as well.
4. THE GAUSS-BONNET THEOREM

The term which does not depend on $A$ is a total derivative term:

$$\frac{1}{3} \delta_1 \left( H^{-1} \delta_1 (H) \right)$$

Since the trace is closed on the torus this contribution to the Gauss-Bonnet term vanishes. For the linear terms as well as for the quadratic we have already demonstrated that taking the trace gives 0, similarly as for the term depending on the derivative of $A$. Hence the conclusion:

**Proposition 4.1.** For the matrix conformally rescaled Dirac operator on the two-dimensional torus, $D_h = hDh$, where $h$ is a globally diagonalizable positive matrix, the Gauss-Bonnet theorem holds:

$$\zeta_{D_h}(0) = \zeta_D(0).$$

5. MATRIX GAUSS-BONNET FOR AN ARBITRARY TWO-DIMENSIONAL MANIFOLD

Let $M$ be a closed, connected, two-dimensional Riemannian manifold and $D$ a Dirac operator for a fixed metric $g$ on $M$. Consider the operator

$$D_{H,A} = H(D + A)H,$$

for $H$ a diagonal matrix valued function on $M$ and $A$ a matrix-valued one-form (identified here with their Clifford image).

We again aim to compute the value of $\zeta_{D_{H,A}}(0)$ using again the methods of pseudodifferential calculus. Of course, using the pseudodifferential calculus on a compact Riemannian manifold requires some care, as the formulation depends on local coordinates and requires patching together local data (see [24] for a recent review and literature) using partition of unity.

In particular, for a curved manifold the expressions for the product of the symbols that were used in (2.2) become complicated (even using local charts and local coordinates) as the metric tensor depends on them. An example of the complexity for the products in the pseudodifferential calculus that uses normal symbols (based on the normal coordinates) is given in [25].

In our case, however, we are interested only in the contributions that contain the term $A$ and therefore we can easily use the local arguments. Moreover, due to seminal results of Guillemin and Wodzicki [17, 27], the trace of the integral over the cotangent space of a classical symbol of order $-2$ provides a local density on a 2-dimensional manifold and thus our computations are indeed coordinate independent.

Let us denote the (local) symbols of $D^2_{H}$ as:

$$D^2_H = (HDH)^2 = a^H_2 + a^H_1 + a^H_0,$$

and the symbols of $D^2$ alone as:

$$D^2 = a^o_2 + a^o_1 + a^o_0.$$

Similarly, like in the case of the torus we split the computation into the case of terms not depending on $A$, linear in $A$ and quadratic in $A$. 
5.1. **Terms independent of** $A$. As the matrix $H$ is diagonal, we can treat the case as $H$ were a scalar function. Thus the problem is reduced to the problem of a usual conformal rescaling of the classical Dirac operator.

It is well known that for any conformal rescaling of the metric the Gauss-Bonnet theorem holds, therefore the contribution of the part of $b_2$ that does not contain $A$ guarantees that the value of the zeta function will remain unchanged provided that all contributions depending on $A$ shall vanish. We examine the linear and quadratic contributions at each point $x \in M$ of the manifolds using normal coordinates.

5.2. **Terms linear in** $A$. Linear terms do arise in $b_2$ from the following terms (using local coordinates in a given chart):

$$b_0a_1^Hb_0a_1(A)b_0 + \partial_k^x(b_0)\partial_k^x(a_2^H)b_0a_1(A)b_0 + b_0a_1(A)b_0a_1^Hb_0$$

$$- b_0a_0(A)b_0 - \partial_k^x(b_0)\partial_k^x(a_1(A))b_0 - \partial_k^x(b_0a_1(A)b_0)\partial_k^x(a_2^H)b_0,$$

where by $a_1(A)$, $a_0(A)$ we denote the terms containing linear $A$ in $a(D_{H,A}^2)$, respectively. First of all, observe that the expression in much simpler as it involves only (at most) first-order derivatives. As to compute the density we can use any coordinate system let us choose the normal coordinates around $x \in M$ with respect to the metric $g$. First, the terms without derivatives reduce easily to the torus case (at point $x$). The only difficulty arises from terms with derivatives with respect to normal coordinates, that is, $\partial_k^x(a_1^H)$ and $\partial_k^x(a_1(A))$.

However, since $a_2^H = H^3g_{ij}\xi^i\xi^j$, then we use the fact that normal coordinates the first derivatives of the metric vanish at the point $x$ and therefore the only remaining term would be the derivative of $H^3$. Therefore, the term that contains the derivative of $a_2^H$ would be reduced to the term linear in $A$ from the torus case (a point $x$ and with the derivatives taken with respect to the normal coordinates).

Similar argument works also for the other term, $a_1(A)$, which reads:

$$(H^3A_iH + H A_kH^3) \sigma^k \sigma^i \xi^i,$$

and because in the term $\partial_k^x(b_0)\partial_k^x(a_1(A))b_0$ there are no further $\sigma$ matrices we can compute first the trace over the Clifford algebra and rephrase is as

$$\frac{1}{2} (H^3A_iH + H A_kH^3) g^{ki} \xi^i,$$

obtaining again the metric. Hence, in normal coordinates at point $x$ the expression is again identical (in the sense of the dependence on $A$ and $H$) to the one for the flat torus.

5.3. **Quadratic terms.** Now, let us concentrate on quadratic terms in $A$ in the formula for $b_2$. They can arise only from two terms and are:

$$b_0a_1(A)b_0a_1(A)b_0 - b_0a_0(A^2)b_0,$$

where $a_1(A)$ denotes the part of this symbol $a_1$ which contains a term linear in $A$ and $a_0(A^2)$ similarly denotes part of $a_0$ symbol containing the quadratic term. Note that this case is even simpler as there are no derivatives and it is easy to see that when written in normal coordinates the expression becomes:

$$a_1(A) = (H^3\sigma^j \xi^j(\sigma^i A_i)H + \sigma^i H A_iH^3 \sigma^j \xi^j),$$
and
\[ a_0(A) = (\sigma^i H A_i H)(\sigma^k H A_k H), \]
which again is naturally the same as in the case of torus.

5.4. The local density and the Gauss-Bonnet. As we have pointed out earlier, the symbol that we compute is of order \(-2\) and integrated over the variables \(\xi\) contributes to a local density, so we can compute the value at each point in any coordinate system. Using the normal coordinate system we have shown that at each point \(x\) using the local normal coordinate systems the dependence on the \(A\) term is given through identical expressions as in the case of the flat torus. Therefore, using the arguments from previous section, we see that at each point \(x \in M\) the linear and quadratic terms in \(A\) give no contributions to the local density and hence to the value of the zeta function of \(D^2 H\). We note in passing that the use of normal coordinates in similar problems related to the computations of the density of Wodzicki residue was used, for example, in [19].

On the other hand, since the matrix-valued function \(H\) is assumed to be diagonal, the \(H\) dependent terms can be treated in the same way as the scalar modification of the Dirac operator, that is, a case where \(H\) is just a function on \(M\). The latter case, however, obviously does not change the Gauss-Bonnet term, hence as a consequence, combining these two results we conclude that the Gauss-Bonnet theorem holds for the arbitrary conformal rescaling of the Dirac operator over a two-dimensional Riemann surface if the rescaling matrix is diagonalizable. We shall briefly discuss in the next section when such situation is possible.

6. Diagonalizability of matrix functions

Having demonstrated that for the special case of globally diagonalizable matrix the Gauss-Bonnet theorem holds, a natural question arises as to what extent the diagonalizability condition \(h = U H U^*\) is general. Is it always possible to find the unitary \(U\)? In what follows we analyze the question in more detail.

Let \(X\) be a compact Hausdorff space and let \(H : X \to M_n(C)\) be a continuous map with values in positive definite matrices. We also assume that for all \(x \in X\), \(H(x)\) has simple spectrum. A natural question is if \(H\) can be continuously diagonalized. That is, if there is a decomposition \(h = U H U^*\) with \(h\) diagonal, \(U\) unitary, and both \(h\) and \(U\) continuous. A similar question has been studied for normal and selfadjoint maps in [16] where obstructions to continuous diagonalizability are identified. In our case of positive definite matrices, these obstructions are much easier to identify and the necessity of their vanishing are directly proved in this section.

We give now the complete obstruction for diagonalizability of \(H\) in terms of first Chern classes. Given a map \(H : X \to M_n(C)\) as above, let \(\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_n(x), x \in X\) denote the eigenvalues of \(H(x)\). Since \(H\) is continuous and has a simple real spectrum, \(\lambda_i\)'s are continuous functions on \(X\). Let \(E_i(x)\) denote the corresponding eigenspaces. We obtain complex line bundles \(L_i\) on \(X\),
\[
L_i \subset X \times \mathbb{C}^n = \{(x,v); v \in E_i(x)\}.
\]

It is clear that \(H\) is continuously diagonalizable if and only if the line bundles \(L_i\) are trivial for all \(i\). The obstruction for triviality of complex line bundle \(L\) is therefore given by its first
Chern class: $L$ is trivial if and only if $c_1(L) = 0$. Thus the obstruction to diagonalizability of $H$ lies in $\bigoplus_{i=1}^n H^2(X, \mathbb{Z})$ so that $H$ is diagonalizable if and only if

$$c_1(L_i) = 0, \quad 1 \leq i \leq n.$$ 

In particular we get the following:

**Proposition 6.1.** Let $X$ be a compact connected Hausdorff space such that $H^2(X, \mathbb{Z}) = 0$. Then any continuous map $H : X \to M_2(\mathbb{C})$ with values in positive definite matrices with simple spectrum is continuously diagonalizable.

It should be noted that if $X$ is a smooth manifold and $H$ is smooth as well, then the roots $\lambda_i(x)$ will be smooth functions on $X$. It follows that the line bundles $L_i$ will be smooth and in that case $H$ will be smoothly diagonalizable, provided of course $H^2(X, \mathbb{Z}) = 0$.

On the other hand it is not difficult to give examples to show that continuous diagonalizability is not always possible. The simplest example is a sphere with the function

$$F : S^2 \to M_2(\mathbb{C}), \quad F(x, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices. They are selfadjoint and satisfy the relations $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$. Then $F^2(x) = 1$, the identity matrix, for all $x \in S^2$ and therefore $p = \frac{1+F}{2}$ is a projection in $M_2(C(S^2))$. We have

$$p(x_1, x_2, x_3) = \frac{1}{2}\left( \begin{array}{cc} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{array} \right).$$

Thus $p$ has rank 1 and hence defines a complex line bundle over $S^2$. This line bundle is non-trivial in the sense that it admits no nowhere zero section, hence $p$ cannot be continuously diagonalized. In fact, it can be shown that this line bundle is the line bundle associated to the Hopf fibration

$$S^1 \to S^3 \to S^2,$$

which has no nowhere zero section hence cannot be a trivial line bundle. Alternatively, the first Chern class of this bundle can be explicitly computed as

$$c_1(p) = \frac{1}{2\pi i} \text{Tr}(p dp dp) = \frac{1}{4\pi}(x_1dx_2dx_3 - x_2dx_1dx_3 + x_3dx_1dx_2),$$

which is a multiple of the volume form of the round sphere. In particular $\int_{S^2} c_1(p) = -1$. Now to get a non-diagonalizable positive definite map, let $H = 1 + p$. It has a simple spectrum for all $x \in S^2$ and is positive definite. Since $p$ is not continuously diagonalizable, it follows that $H$ is not continuously diagonalizable either.

In fact, one can extend the result to all compact Riemann surfaces.

**Proposition 6.2.** If $X$ is a compact Riemann surface then there exists a non-diagonalizable smooth positive $2 \times 2$ matrix.

**Proof.** To show that we shall find a hermitian projection $p$ that has a nontrivial Chern class and therefore cannot be diagonalized globally. Then, similarly as above, $1 + p$ is the desired positive non-diagonalizable matrix. The construction below extends the method of finding an explicit nontrivial projection on the torus as shown in [23].

Let us take a product $S^1 \times (0, 1)$ and define the following matrix valued function:

$$p = \left( \begin{array}{cc} f(t) & h(t) + g(t)e^{2\pi is} \\ h(t) + g(t)e^{-2\pi is} & 1 - f(t) \end{array} \right),$$

where
where $0 < t < 1$ parametrizes the interval and $0 \leq s < 1$ parametrizes the circle. The matrix $p(t, s)$ is a projection iff:

$$g(t)h(t) = 0, \quad f(t)^2 + g(t)^2 + h(t)^2 = f(t).$$

The first Chern class of $p$ could be explicitly computed as:

$$c_1(p) = \frac{1}{2\pi i} \text{Tr}(p dp dp) = (4gg'f - 4g^2f' - 2gg')\ dt \wedge ds = (2(g^2)'f - 4g^2f' - (g^2)'')\ dt \wedge ds.$$

Now, choose the functions $g, h$ in such a way, so that support of $g$ is in $(\epsilon, \frac{1}{2})$ and support of $h$ in $(\frac{1}{2}, 1 - \epsilon)$ and they vary between 0 and 1. Further, take $f$ to be increasing on support of $g$ from 0 to 1 and decreasing on support of $h$, we can check that the integral of the above form:

$$\int_{\text{supp}(g)} (2(g^2)'f - 4g^2f' - (g^2)')\ dt \wedge ds = -1,$$

so that the projection $p$ is nontrivial.

Take now an arbitrary Riemann surface and find a circle and its tubular neighborhood. Then, as this tubular neighborhood is diffeomorphic with $(0, 1) \times S^1$ we can use that diffeomorphism to define a matrix value map on it, which arises from the projection $p$. However, note that the projection $p(t, s)$ is constant:

$$p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

for $t < \epsilon$ and $t > 1 - \epsilon$, and so is its image. Therefore we can extend it in a smooth way to a matrix valued map on the entire Riemann surface. As the Chern number of the projection does not change it defines a nontrivial line bundle. Taking $1 + p$ we obtain a positive matrix (with constant eigenvalues 1 and 2), so that both eigenspaces are nontrivial line bundles. □

The above result does not generalize in a natural way to higher dimensional manifolds, and we can formulate a question as follows. For a compact manifold $X$ of dimension bigger than 2, if $H^2(X, \mathbb{Z})$ is non-trivial, can one always find a positive matrix-valued $H$, which is not diagonalizable? If so, what is the minimal size of such a matrix? Another interesting problem is of different type: having a smooth matrix valued function $H$ on a manifold $X$, is there a Chern-Weil type description of obstructions to diagonalizability (which are obviously related to classes $c_1(L_i)$) in terms of differential forms obtained from $H$?

Finally, let us mention that although for the 2-dimensional manifolds the action remains purely topological, the study of the matrix-conformal rescaling in the higher-dimensional case leads to a question of the minima of the corresponding Einstein-Hilbert functional and the corresponding equations of motion.

**Acknowledgements:** The authors thank the referee for careful reading and remarks that improved the presentation. The research was partially supported through H2020-MSCA-RISE-2015-691246-QUANTUM DYNAMICS and through Polish support grant for the international cooperation project 3542/H2020/2016/2 and 328941/PnH/2016.
REFERENCES

[1] J. Arnlind, M. Wilson. On the Chern-Gauss-Bonnet theorem for the noncommutative 4-sphere. J. Geom. Phys. 111 (2017), 126-141.

[2] T. Brzezinski, N. Ciccoli, L. Dąbrowski, A. Sitarz, Twisted reality condition for Dirac operators. Math. Phys. Anal. Geom. 19 (2016), no. 3, Art. 16.

[3] A. Connes, F. Fathizadeh, The term \( a_4 \) in the heat kernel expansion of noncommutative tori, arXiv:1611.09815.

[4] A. Connes, H. Moscovici, Modular curvature for noncommutative two-tori, J. Amer. Math. Soc. 27 (2014) 639.

[5] A. Connes, P. Tretkoff, The Gauss–Bonnet theorem for the noncommutative two torus, in: Noncommutative geometry, arithmetic, and related topics, 141–158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[6] L. Dąbrowski and A. Sitarz, Curved noncommutative torus and Gauss–Bonnet, Journal of Mathematical Physics 54 (2013) 013518.

[7] L. Dąbrowski, A. Sitarz, Asymmetric noncommutative torus, SIGMA Symmetry Integrability Geom. Methods Appl. 11, Paper 075, (2015).

[8] M. Eckstein, A. Sitarz, R. Wulkenhaar, The Moyal Sphere, Journal of Mathematical Physics, 57, 112301 (2016) (2016).

[9] A. Fathi, M. Khalkhali, On Certain Spectral Invariants of Dirac Operators on Noncommutative Tori, arXiv:1504.01174v1.

[10] A. Fathi, A. Ghorbanpour, M. Khalkhali, The Curvature of the Determinant Line Bundle on the Noncommutative Two Torus, Math. Phys. Anal. Geom. 20 (2017), no. 2.

[11] F. Fathizadeh, On the Scalar Curvature for the Noncommutative Four Torus, Journal of Mathematical Physics, 56(6):062303, 2015.

[12] F. Fathizadeh, M. Khalkhali, Scalar curvature for the noncommutative two torus, J. Noncommut. Geom. 7 (2013), 1145–1183.

[13] F. Fathizadeh, M. Khalkhali, The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure, J. Noncommut. Geom., 6, no. 3, 457-480, (2012).

[14] F. Fathizadeh, G. Olivier, On the Chern-Gauss-Bonnet theorem and conformally twisted spectral triples for \( C^* \)-dynamical systems. SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016), Paper No. 016.

[15] R. Floricel, A. Ghorbanpour, M. Khalkhali, The Ricci Curvature in Noncommutative Geometry, arXiv:1612.06688.

[16] G. Friedman and E. Park, Unitary equivalence of normal matrices over topological spaces, Journal of Topology and Analysis Vol. 8, No. 2 (2016) 313–348.

[17] V. W. Guillemin, A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues, Adv. Math. 55(2), (1985) 131–160.

[18] B. Iochum, T. Masson, Heat trace for Laplace type operators with non-scalar symbols. J. Geom. Phys. 116 (2017), 90–118.

[19] W. Kalau and M. Walze, Gravity, noncommutative geometry and the Wodzicki residue, J. Geom. Phys. 16, 327-344 (1995).

[20] M. Khalkhali, A. Moatadelro, S. Sadeghi, A Scalar Curvature Formula For the Noncommutative 3-Torus, arXiv:1610.04740.

[21] M. Lesch. Divided differences in noncommutative geometry: Rearrangement lemma, functional calculus and expansional formula, J. Noncommut. Geom., (to appear), arXiv:1405.0863v2.

[22] Y. Liu. Modular curvature for toric noncommutative manifolds, arXiv:1510.04668.

[23] T. Loring, The torus and noncommutative topology. Ph.D. Dissertation, Univ. of Calif., Berkeley, 1986.

[24] V. Nazaikinskii, A. Savin, B-W. Schulze, B. Sternin Pseudodifferential Operators in: Differential Operators on Manifolds with Singularities. Analysis and Topology, Francic and Taylor to appear.

[25] M. Pflaum, The Normal Symbol on Riemannian Manifolds, New York J. Math 4, (1998), 97–125.

[26] J. Rosenberg, Levi-Civita’s Theorem for Noncommutative Tori, SIGMA 9 (2013) 071.

[27] M. Wodzicki, Local invariants of spectral asymmetry, Invent. Math. 75(1), (1989) 143–178.
Department of Mathematics, University of Western Ontario, London Ontario, N6A 5B7, Canada.
E-mail address: masoud@uwo.ca

Institute of Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland.
Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-950 Warszawa, Poland.
E-mail address: andrzej.sitarz@uj.edu.pl