Self-force of a scalar charge in the space-time of extreme charged anti-dilatonic wormhole

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The self-interaction for a static scalar charge in the space-time of extreme charged anti-dilatonic wormhole is calculated. We assume that the scalar charge is the source of massless scalar field with minimal coupling of the scalar field to the curvature of spacetime.

I. INTRODUCTION

Wormholes are topological handles in spacetime linking different universes or different parts of the same universe. Interest in these configurations dates back to at least 1916 [1] with revivals of activity following the works of Einstein and Rosen in 1935 [2] and the later series of works initiated by Wheeler in 1955 [3]. A fresh interest in this topic has been rekindled by the works of Morris and Thorne [4] and of Morris, Thorne and Yurtsever [5]. In classical general relativity, it is well known that traversable wormholes as solutions to the Einstein equations can only exist with exotic matter which violates the null energy condition $T^{\mu\nu}u_\mu u_\nu \geq 0$ for any null vector field $u^\mu$. Various models providing the wormhole existence include scalar fields [6, 7]; wormhole solutions in Einstein-Gauss-Bonnet theory [8, 9]; wormholes geometries induced by quantum effects [10, 11]; wormhole solutions in semi-classical theory of gravity [12–14]; solutions in modified theories of gravity [15–19]; modified teleparallel theories [20] and their extensions [21], etc. The geometry of wormhole as well as a good introduction in the subject may be found in the Visser book [22] and in the review by Lobo [23]. Static spherically symmetric wormholes would look observationally almost like black holes. One of the effects which can differ these two spacetimes is the self-interaction for the charge. Self-interaction effects in curved spacetime have been vigorously explored; for an extensive review, see [24–27]. The origin of this induced self-interaction resides on the nonlocal structure of the field caused by the space-time curvature or nontrivial topology. In flat space-time, the effect is determined by the derivative of acceleration of the charge. For electrically charged particles in flat spacetime, the self-force is given by the Abraham-Lorentz-Dirac formula [28, 29]. In static curved space-times and space-times with nontrivial topology the self-force can be nonzero even for the charge at rest (here and below the words “at rest” mean that the velocity of charge is collinear to the timelike Killing vector which always exists in a static space-time). The geometry of wormhole as well as a good introduction in the subject may be found in the Visser book [22] and in the review by Lobo [23].

The self-force on a charge interacting with a massless minimally coupled scalar field was considered by Quinn [41]. More recently, a fresh interest in the topic has been focused on the gravitational self-force, in an effort to model the insipral and gravitational-wave emissions of a binary system with a small mass ratio [42–44]. This interest has been prompted by the preparation of gravitational wave detectors which are capable of detecting gravitational waves emitted when a compact object falls into a supermassive black hole. For the Schwarzschild black hole the self-force on a static charge $q$ is repulsive, and it has the dependence $f \sim q^2/r^3$, where $r$ is the Schwarzschild radial coordinate of the charge. Significant efforts were made to calculate the self-force on the background of different types of the black holes [47, 48]. The self-force for a charge at rest in static spherically symmetric wormhole spacetimes is determined by the profile of wormhole throat and coupling of the field of charge with curvature of spacetime [70, 71]. It was shown [73, 77] that there is an infinite set of values for the coupling constant of the scalar field to the curvature of the spacetime for which the self-force on a static scalar charge diverges. This set of values depends on the profile of the throat. The nature of this divergence is not entirely clear.

The aim of this paper is to analyze the self-force problem for a static pointlike scalar charge in an extreme charged anti-dilatonic wormhole [79, 80]. This problem has mathematical difficulties. The field of charge in the considered problem is determined by the Green’s function of some ordinary inhomogeneous differential equation of second order.
where describes the procedure of renormalization of the self potential and the final result. the unrenormalized expression for self potential of the static scalar charge on the considered background. Section IV where we take into account 

\[ \frac{d\tau}{dt} \]

which follows relations are valid

\[ Q_\phi = M = \frac{\pi^2}{2} |Q|, \]

where \( Q \) - electric charge of the wormhole, \( M \) - mass of the wormhole and \( Q_\phi \) - dilatonic charge of the wormhole.

III. SELF POTENTIAL OF THE STATIC SCALAR CHARGE

Let us consider a scalar field \( \phi \) with scalar source \( j \). The corresponding field equation has the form

\[ \phi^{(\mu)} = -4\pi j = -4\pi q \int \delta^{(4)}(x^\mu, \hat{x}^\mu(\tau)) \frac{d\tau}{\sqrt{-g^{(4)}},} \]

where \( g^{(4)} \) is the determinant of the metric \( g_{\mu\nu} \), \( q \) is the scalar charge and \( \tau \) is its proper time. The world line of the charge is given by \( \hat{x}^\mu(\tau) \). For a particle at rest in static space-time field equation (4) can be rewritten as follows

\[ \left\{ e^{-2\alpha} \left[ \frac{\partial^2}{\partial r^2} + \frac{2r}{(r^2 + Q^2)} \frac{\partial}{\partial r} + \frac{1}{(r^2 + Q^2)} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right) \right] \phi(r, \theta, \phi; \tilde{r}, \tilde{\theta}, \tilde{\phi}) = -4\pi q \delta(r, \tilde{r}) \delta(\theta, \tilde{\theta}) \delta(\phi, \tilde{\phi}), \right\} \]

where we take into account \( d\tau/dt = \sqrt{-g_{\mu\nu}} = e^{-\alpha(r)} \) for the static charge. Due to spherical symmetry of the problem under consideration we represent the potential in the following form

\[ \phi = 4\pi q \sum_{l, m} Y_{lm}(\Omega) Y^*_{lm}(\hat{\Omega}) g_l(r, \tilde{r}) = q \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \gamma) g_l(r, \tilde{r}), \]

where \( Y_{lm}(\Omega) \) is the spherical functions of argument \( \Omega = (\theta, \phi), \) \( \cos \gamma = \cos \theta \cos \tilde{\theta} + \sin \theta \sin \tilde{\theta} \cos(\phi - \tilde{\phi}). \) The radial part, \( g_l, \) satisfies the equation

\[ \frac{\partial^2 g_l(r, \tilde{r})}{\partial r^2} + \frac{2r}{(r^2 + Q^2)} \frac{\partial g_l(r, \tilde{r})}{\partial r} - \frac{l(l + 1)}{(r^2 + Q^2)} g_l(r, \tilde{r}) = - \frac{\delta(r, \tilde{r})}{e^{\alpha(r)}(r^2 + Q^2)}. \]

Introducing a new function

\[ G_l(r, \tilde{r}) = e^{\alpha(r)} g_l(r, \tilde{r}) \]
and taking into account

\[ \frac{\delta(r, \bar{r})}{e^{\alpha(r)}} = \frac{\delta(r, \bar{r})}{e^{\alpha(\bar{r})}} \]  

(9)

one can obtain the following equation for \( G_{1}(r, \bar{r}) \)

\[ \frac{\partial^{2} G_{1}(r, \bar{r})}{\partial r^{2}} + \frac{2r}{(r^2 + Q^2)} \frac{\partial G_{1}(r, \bar{r})}{\partial r} - \frac{l(l+1)}{(r^2 + Q^2)} G_{1}(r, \bar{r}) = - \frac{\delta(r, \bar{r})}{(r^2 + Q^2)}. \]  

(10)

Two independent solutions of corresponding homogeneous equation will be denoted by \( \Psi_{1}(r) \) and \( \Psi_{2}(r) \)

\[ \frac{d^{2}\Psi}{dr^{2}} + \frac{2r}{(r^2 + Q^2)} \frac{d\Psi}{dr} - \frac{l(l+1)}{(r^2 + Q^2)} \Psi = 0. \]  

(11)

\( \Psi_{1}(r) \) is chosen to be the solution which is equal to zero at \( r \to +\infty \) and divergent at \( r \to -\infty \). \( \Psi_{2}(r) \) is chosen to be the solution which is equal to zero at \( r \to -\infty \) and divergent at \( r \to +\infty \). That is,

\[ \lim_{r \to +\infty} \Psi_{1} = 0, \quad \lim_{r \to +\infty} \Psi_{2} = \infty, \]

\[ \lim_{r \to -\infty} \Psi_{1} = \infty, \quad \lim_{r \to -\infty} \Psi_{2} = 0. \]  

(12)

Then one can represent the solution of equation (11) in the following form

\[ G_{1} = \theta(r - \bar{r})\Psi_{1}(r)\Psi_{2}(\bar{r}) + \theta(\bar{r} - r)\Psi_{1}(\bar{r})\Psi_{2}(r). \]  

(13)

Normalization of \( \Psi \) is achieved by integrating (10) with respect to \( r \) from \((\bar{r} - \epsilon)\) to \((\bar{r} + \epsilon)\) and letting \( \epsilon \to 0 \). This results in the Wronskian condition

\[ W(\Psi_{1}, \Psi_{2}) = \Psi_{1} \frac{d\Psi_{2}}{dr} - \Psi_{2} \frac{d\Psi_{1}}{dr} = \frac{1}{r^2 + Q^2}. \]  

(14)

We consider the radial equation (11) in domains \( r > 0 \) and \( r < 0 \). We may easily construct independent solutions of this equation for these two domains separately

\[ \phi^{1}_{+}(r) = P_{l} \left( \frac{ir}{|Q|} \right), \quad \phi^{2}_{+}(r) = Q_{l} \left( \frac{ir}{|Q|} \right), \quad r > 0, \]

\[ \phi^{1}_{-}(r) = P_{l} \left( -\frac{ir}{|Q|} \right), \quad \phi^{2}_{-}(r) = Q_{l} \left( -\frac{ir}{|Q|} \right), \quad r < 0, \]  

(15)

where \( P_{l} \) and \( Q_{l} \) are the Legendre polynomials of the first and second kind. Asymptotically

\[ \phi^{1}_{\pm}|_{r \to \pm \infty} \sim r^{l}, \quad \phi^{2}_{\pm}|_{r \to \pm \infty} \sim r^{-l-1}. \]  

(16)

The Wronskian of these solutions has the following form (see Ref. [81])

\[ W(\phi^{1}_{\pm}, \phi^{2}_{\pm}) = \frac{\pm i|Q|}{r^2 + Q^2}. \]  

(17)

The solutions over all space can be written as follows

\[ \Psi_{1} = \begin{cases} \alpha^{1}_{+}\phi^{1}_{+} + \beta^{1}_{+}\phi^{2}_{+}, & r > 0 \\ \alpha^{1}_{-}\phi^{1}_{-} + \beta^{1}_{-}\phi^{2}_{-}, & r < 0 \end{cases}, \]

\[ \Psi_{2} = \begin{cases} \alpha^{2}_{+}\phi^{1}_{+} + \beta^{2}_{+}\phi^{2}_{+}, & r > 0 \\ \alpha^{2}_{-}\phi^{1}_{-} + \beta^{2}_{-}\phi^{2}_{-}, & r < 0 \end{cases}, \]  

(18)

where \( \alpha^{1,2}_{\pm}, \beta^{1,2}_{\pm} \) are constants. Applying the boundary conditions (12) in (13) and using (14) we get

\[ \alpha^{1}_{\pm} = 0, \quad \alpha^{2}_{\pm} = 0. \]  

(19)
Then the solutions \[18\] reduce to the following form

\[
\Psi_1 = \begin{cases} 
\beta_1^+ \phi_1^+, & r > 0 \\
\alpha_1^+ \phi_1^- + \beta_1^- \phi_2^+, & r < 0
\end{cases},
\]

\[
\Psi_2 = \begin{cases} 
\alpha_2^+ \phi_1^+ + \beta_2^+ \phi_2^+, & r > 0 \\
\beta_2^- \phi_2^-, & r < 0
\end{cases}.
\tag{20}
\]

Substituting these expressions into the Wronskian and taking into account \[14\] we get

\[
W(\Psi_1, \Psi_2) = \begin{cases} 
-\alpha_2^+ \beta_1^- \frac{i|Q|}{(r^2 + Q^2)}, & r > 0 \\
-\alpha_1^+ \beta_2^+ \frac{i|Q|}{(r^2 + Q^2)}, & r < 0
\end{cases}
\tag{21}
\]

The Wronskian condition \[14\] implies the constraints on the coefficients:

\[
\alpha_2^+ \beta_1^- = \alpha_1^+ \beta_2^+ = -\frac{1}{i|Q|} = \frac{i}{|Q|}.
\tag{22}
\]

The matching conditions of solutions \(\Psi_1(r), \Psi_2(r)\) at \(r = 0\) are the following

\[
\lim_{r \to +0} \Psi_1(r) = \lim_{r \to +0} \Psi_1(r), \quad \lim_{r \to -0} \frac{d\Psi_1(r)}{dr} = \lim_{r \to -0} \frac{d\Psi_1(r)}{dr},
\]

\[
\lim_{r \to +0} \Psi_2(r) = \lim_{r \to +0} \Psi_2(r), \quad \lim_{r \to -0} \frac{d\Psi_2(r)}{dr} = \lim_{r \to -0} \frac{d\Psi_2(r)}{dr}.
\tag{23}
\]

Using these conditions one can obtain the following relations \[19\]

\[
\beta_1^+ \phi_1^+(0) = \alpha_1^+ \phi_1^+(0) + \beta_1^- \phi_2^+(0), \quad \alpha_2^+ \phi_1^+(0) + \beta_2^+ \phi_2^+(0) = \beta_2^- \phi_2^+(0),
\tag{24}
\]

\[
\beta_1^+ \frac{d\phi_1^+}{dr} \bigg|_0 = \alpha_1^+ \frac{d\phi_1^+}{dr} \bigg|_0 + \beta_1^- \frac{d\phi_2^+}{dr} \bigg|_0, \quad \alpha_2^+ \frac{d\phi_1^+}{dr} \bigg|_0 + \beta_2^+ \frac{d\phi_2^+}{dr} \bigg|_0 = \beta_2^- \frac{d\phi_2^+}{dr} \bigg|_0.
\tag{25}
\]

These relations can be rewritten as follows

\[
\alpha_1^+ = \beta_1^+ \frac{W(\phi_2^+, \phi_2^+)}{W(\phi_1^+, \phi_2^+)} \bigg|_0, \quad \beta_1^+ = \beta_1^+ \frac{W(\phi_1^+ + \phi_2^+)}{W(\phi_1^+, \phi_2^+)} \bigg|_0,
\]

\[
\alpha_2^+ = -\beta_2^+ \frac{W(\phi_2^+, \phi_2^+)}{W(\phi_1^+, \phi_2^+)} \bigg|_0, \quad \beta_2^+ = +\beta_2^+ \frac{W(\phi_1^+ + \phi_2^+)}{W(\phi_1^+, \phi_2^+)} \bigg|_0.
\tag{26}
\]

Taking into account \[15\] and using the relations (see Ref. \[81\])

\[
P_1(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{i}{2}\right) \Gamma\left(1 + \frac{i}{2}\right)}, \quad P_1'(0) = -\frac{2\sqrt{\pi}}{\Gamma\left(\frac{1}{2} + \frac{i}{2}\right) \Gamma\left(-\frac{i}{2}\right)},
\tag{27}
\]

\[
Q_1(0) = \frac{\sqrt{\pi}}{2} e^{-\frac{i2\pi}{4}(l+1)} \frac{\Gamma\left(\frac{1}{2} + \frac{i}{2}\right)}{\Gamma\left(1 + \frac{i}{2}\right)}, \quad Q_1'(0) = \sqrt{\pi} e^{-\frac{i2\pi}{4}} \frac{\Gamma\left(1 + \frac{i}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{i}{2}\right)},
\tag{28}
\]

the Wronskians in \[20\] can be calculated as:

\[
W(\phi_1^+, \phi_2^+)|_0 = -\frac{\pi(1)^{l}}{|Q|}, \quad W(\phi_1^+ + \phi_2^+)|_0 = \frac{i\pi}{|Q|} (1)^{l}, \quad W(\phi_1^+, \phi_2^+)|_0 = -\frac{i}{|Q|},
\]

\[
W(\phi_1^+, \phi_2^+)|_0 = \frac{i}{|Q|}, \quad W(\phi_1^+ + \phi_2^+)|_0 = -\frac{i(1)^{l}}{|Q|}.
\tag{29}
\]

Then, we can rewrite the equations \[20\] as

\[
\alpha_1^+ = -i\pi(1)^{l} \beta_1^+, \quad \beta_1^+ = -i\pi(1)^{l} \beta_1^+, \\
\alpha_2^+ = i\pi(1)^{l} \beta_2^+, \quad \beta_2^+ = -(1)^{l} \beta_2^+.
\tag{30}
\]
Using these relations and \([13, 15, 20, 22]\) we obtain for \(r > \hat{r} > 0\)

\[
G_t(r, \hat{r}) = \frac{i}{|Q|} \phi_+^2(r)\phi_+^1(\hat{r}) + \frac{i}{|Q|} W(\phi_+^2, \phi_+^1) |_{\phi_+^2(r)} \phi_+^2(\hat{r}),
\]

\[
= \frac{i}{|Q|} \phi_+^2(r)\phi_+^1(\hat{r}) + \frac{1}{\pi|Q|} \phi_+^2(r)\phi_+^2(\hat{r})
\]

\[
= \frac{i}{|Q|} Q_l(\hat{r}) + \frac{1}{\pi|Q|} Q_l(\hat{r}) Q_l(\hat{r})
\]  

(31)

where \(z = \frac{ir}{|Q|}\) and \(\tilde{z} = \frac{i\hat{r}}{|Q|}\).

After substituting this expression into (8) and then into (6), one can obtain, for the case \(r > \hat{r} > 0\) and \(\theta = \tilde{\theta}, \varphi = \tilde{\varphi}\)

\[
\phi(r, \hat{r}) = q e^{-\alpha(\tilde{r})} \sum_{l=0}^{\infty} (2l + 1) \left\{ iP_l(\hat{r})Q_l(z) + \frac{1}{\pi} Q_l(\hat{r}) Q_l(\hat{r}) \right\}.
\]  

(32)

To find \(\sum_{l=0}^{\infty} (2l + 1) P_l(z) Q_l(\hat{r})\) we use the Heine formula \([81]\)

\[
\sum_{l=0}^{\infty} (2l + 1) P_l(z) Q_l(\hat{r}) = \frac{1}{\hat{z} - z}.
\]

To find \(\sum_{l=0}^{\infty} (2l + 1) Q_l(i\hat{r}) Q_l(i\hat{r})\) we use the integral representation for the Legendre function of the second kind \([81]\)

\[
Q_l(z) = \frac{1}{\pi} \int_{-1}^{1} \frac{P_l(t)}{z - t} dt,
\]

(33)

and get

\[
\sum_{l=0}^{\infty} (2l + 1) Q_l(i\hat{r}) Q_l(i\hat{r}) = \arctan x - \arctan \frac{x}{x - \hat{x}}.
\]  

(34)

Therefore for \(r > \hat{r} > 0\) and \(\theta = \tilde{\theta}, \varphi = \tilde{\varphi}\)

\[
\phi(r, \hat{r}) = q e^{-\alpha(\tilde{r})} \left[ \frac{1}{r - \hat{r}} - \arctan \frac{\hat{r}}{|Q|} - \arctan \frac{\hat{r}}{|Q|} \right].
\]  

(35)

IV. RENORMALIZATION AND RESULT

The procedure of the self-force evaluation requires the renormalization of a scalar potential \(\phi(x; \hat{x})\) which is diverged in the limit \(x \to \hat{x}\) (see, for example, papers \([82, 83]\)). This renormalization is achieved by subtracting the DeWittSchwinger counterterm \(\phi_{\text{os}}(x; \hat{x})\) from \(\phi(x; \hat{x})\) and then letting \(x \to \hat{x}\)

\[
\phi_{\text{ren}}(x) = \lim_{\hat{x} \to x} \left( \phi(x; \hat{x}) - \phi_{\text{os}}(x; \hat{x}) \right).
\]  

(36)

For a scalar charge at rest in static curved space-time the DeWittSchwinger counterterm \(\phi_{\text{os}}(x; \hat{x})\), which must be subtracted, has the following form \([84]\)

\[
\phi_{\text{os}}(x^i; \hat{x}^i) = q \left( \frac{1}{\sqrt{2\sigma}} + \frac{\partial g_{tt}(\hat{x})}{\partial \hat{x}^i} \frac{\sigma^i}{4g_{tt}(\hat{x})\sqrt{2\sigma}} \right),
\]

(37)
where \[85, 86\]

\[
\sigma^i = - \left( x^i - \tilde{x}^i \right) - \frac{1}{2} \Gamma^i_{jk} \left( x^j - \tilde{x}^j \right) \left( x^k - \tilde{x}^k \right) \\
\left( - \frac{1}{6} \left( \Gamma^i_{jm} \Gamma^m_{kl} + \frac{\partial \Gamma^i_{jk}}{\partial x^l} \right) \left( x^j - \tilde{x}^j \right) \left( x^k - \tilde{x}^k \right) \left( x^l - \tilde{x}^l \right) + O \left( (x - \tilde{x})^4 \right) \right),
\]

\[
\sigma = \frac{g_{ij}(\tilde{x})}{2} \sigma^i \sigma^j,
\]

(38)

\(\Gamma^i_{jk}\) are the Christoffel symbols calculated at point \(\tilde{x}\). The DeWitt-Schwinger counterterm \(\phi_{DS}(x; \tilde{x})\) in the limit \(\theta = \tilde{\theta}, \varphi = \tilde{\varphi}\) can be easily calculated using the metric (1)


\[
\phi_{DS}(x^i; \tilde{x}^i) = \frac{q e^{-\alpha(\tilde{r})}}{|r - \tilde{r}|}
\]

(39)

Using the expression (35) for \(\phi(\tilde{r}, r)\) we obtain the renormalized expression for \(\phi\) in domain \(r > 0\)

\[
\phi_{ren}(r) = \lim_{\tilde{r} \to r} \left[ \phi(r, \tilde{r}) - \phi_{DS}(r, \tilde{r}) \right] = -\frac{q|Q| e^{-\alpha(r)}}{\pi(r^2 + Q^2)}.
\]

(40)

\(\phi_{ren}\) in domain \(r < 0\) coincides with this expression because of the symmetry \(r \leftrightarrow -r\) of the problem. The limiting case \(r \to \infty\) gives us the following result

\[
\phi_{ren}(r) \approx -\frac{q|Q|}{\pi r^2}.
\]

(41)

The only nonzero component of the self-force is

\[
F^r(r) = -\frac{q}{2} g_{rr} \partial_{\phi_{ren}}(r) \quad = \quad -\frac{q^2 e^{-3\alpha(r)} \left( \frac{r}{|Q|} - \frac{1}{2} \arctan \left( \frac{r}{|Q|} \right) \right)}{\pi Q^2 (1 + r^2/Q^2)^2}.
\]

(42)

Thus, we have obtained an analytic expression [42] for the self-force on a static scalar charge in an extreme charged anti-dilatonic wormhole spacetime [11].

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