On Low-Space Differentially Private Low-rank Factorization in the Spectral Norm

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Abstract

Low-rank factorization is used in many areas of computer science where one performs spectral analysis on large sensitive data stored in the form of matrices. In this paper, we study differentially private low-rank factorization of a matrix with respect to the spectral norm in the turnstile update model. In this problem, given an input matrix \( A \in \mathbb{R}^{m \times n} \) updated in the turnstile manner and a target rank \( k \), the goal is to find two rank-\( k \) orthogonal matrices \( U_k \in \mathbb{R}^{m \times k} \) and \( V_k \in \mathbb{R}^{n \times k} \), and one positive semidefinite diagonal matrix \( \Sigma_k \in \mathbb{R}^{k \times k} \) such that

\[
A \approx U_k \Sigma_k V_k^T
\]

with respect to the spectral norm.

Our main contributions are two computationally efficient and sub-linear space algorithms for computing a differentially private low-rank factorization. We consider two levels of privacy. In the first level of privacy, we consider two matrices neighboring if their difference has a Frobenius norm at most 1. In the second level of privacy, we consider two matrices as neighboring if their difference can be represented as an outer product of two unit vectors. Both these privacy levels are stronger than those studied in the earlier papers such as Dwork et al. (STOC 2014), Hardt and Roth (STOC 2013), and Hardt and Price (NIPS 2014).

As a corollary to our results, we get non-private algorithms that compute low-rank factorization in the turnstile update model with respect to the spectral norm. We note that, prior to this work, no algorithm that outputs low-rank factorization with respect to the spectral norm in the turnstile update model was known; i.e., our algorithm gives the first non-private low-rank factorization with respect to the spectral norm in the turnstile update mode.

Our algorithms generate private linear sketches of the input matrix. Therefore, using the binary tree mechanism of Chan et al. (TISSEC: 14(3)) and Dwork et al. (STOC 2010), we get algorithms for continual release of low-rank factorization under both these privacy levels. This gives the first instance of differentially private algorithms with continual release that guarantees a stronger level of privacy than event-level privacy.

Keywords. Differential privacy, low-rank factorization, continual release, turnstile update model.

1 Introduction

Spectral analysis of matrices is used in many areas where large sensitive data are stored in the form of matrices. A partial list includes data mining [4], recommendation systems [17], information retrieval [43, 48], web search [11, 37], clustering [12, 15, 39], and learning distributions [2, 34]. A general technique used in the literature involving such analysis first computes an approximate
low-rank factorization (LRF) of the matrix to avoid the curse of dimensionality with the hope that the output from using the LRF does not “differ by much” from the output from using the original data matrix. These algorithms often do not consider privacy (or use ad-hoc anonymization methods) and do not account for dynamic data matrices. On the other hand, the limitations of ad-hoc anonymization of these data has been exemplified by the denonymization of the Netflix datasets [41] and the real-world data changes dynamically. This raises two natural questions:

(i) Can we efficiently perform LRF under the spectral norm while preserving the privacy of the dynamic data matrix?

(ii) Can we release LRF under the continual release model [21]?

We answer these questions in the affirmative. Specifically, we give two differentially private algorithms that receive an $m \times n$ matrix in the turnstile update model and output a $k$-rank factorization with respect to the spectral norm. Our algorithms efficiently compute a small space linear sketches of the private matrix. We use the private matrix only when generating the linear sketches; therefore, our algorithms can be easily modified to get differentially private algorithms under continual release (i.e., the output is released at every time epoch) by using the binary tree mechanism [8, 21].

In the recent past, there has been a lot of attention to a related problem of differentially private low-rank approximation (LRA) with respect to the spectral norm [24, 27, 29, 35] leading to an optimal approximation error. However, these algorithms run in $O(mnk)$ time, use $O(mn)$ space, and assume that the matrix is static. It is easy to see that, if there are no time or space constraints, then LRA can be used to give LRF in polynomial time. However, it is not known how to efficiently factorize a matrix in sub-linear space. Even in the non-private setting, there is no known algorithm that computes spectral approximation in the turnstile update model — the only algorithm that computes an LRA is under Frobenius norm by Boutsidis et al. [7] and Clarkson and Woodruff [11]. Unfortunately, the algorithm of Boutsidis et al. [7] is not robust against noise. In a closely related work, Upadhyay [54] gave a differentially private algorithm for LRF in the turnstile update model when the approximation metric is the Frobenius norm. This algorithm can be seen as a robust version of the algorithm of Boutsidis et al. [7].

One natural point to start would be to see if one could use the existing algorithm that gives approximation under Frobenius norm. Unfortunately, Karnin and Liberty [36] observed that even though the exact solution of LRF is the same with respect to both the Frobenius and the spectral norm, the same cannot be said about their approximate solutions. Moreover, even though the spectral norm and Frobenius norm are trivially within the factor of each other, the same cannot be said about their approximate solutions. Therefore, it is not clear if we can use the algorithms of Upadhyay [54] and Boutsidis et al. [7] to give algorithms for LRF with respect to the spectral norm. On the other hand, there is an instance of algorithm by Halko et al. [26] that gives approximation under both the spectral as well as the Frobenius norm. However, their algorithm is not in a turnstile update model.

### 1.1 Formal Problem Description

In this paper, we study differentially private LRF in the turnstile update model when the approximation metric is the spectral norm. In the turnstile update model, an update is in the form of triples $\{i, j, s_\tau\}$, where $1 \leq i \leq m, 1 \leq j \leq n, s_\tau \in \mathbb{R}$ for all $\tau \geq 1$. This leads to a change in
the \((i, j)\)-th entry of the private matrix \(A\) as follows: 
\[A_{i,j} \leftarrow A_{i,j} + s_\tau.\]
We first give the formal definition of differential privacy.

**Definition 1.** A randomized algorithm \(M\) gives \((\varepsilon, \delta)\)-differential privacy, if for all neighbouring databases (presented in the form of matrices) \(A\) and \(A'\), and all subsets \(S\) in the range of \(M\), 
\[
\Pr[M(A) \in S] \leq e^\varepsilon \Pr[M(A') \in S] + \delta
\]
over the coin tosses of \(M\).

We can now formally define the main problem we address in this paper.

**Problem 1.** \(((\varepsilon, \delta, \beta, \gamma, \zeta, k)\)-LRF). Given parameters \(\varepsilon, \delta, \gamma, \beta, \zeta\), the target rank \(k\), and a private \(m \times n\) matrix \(A\) updated in the general turnstile update model, output (under continual release) an \((\varepsilon, \delta)\)-differentially private rank-\(k\) factorization \(\tilde{U}_k, \tilde{\Sigma}_k, \tilde{V}_k\) such that 
\[
\Pr\left[\|A - \tilde{U}_k\tilde{\Sigma}_k\tilde{V}_k^T\|_2 \leq \gamma\|A - [A]_k\|_2 + \zeta\right] \geq 1 - \beta,
\]
where \(\|\cdot\|_2\) denotes the spectral norm and \([A]_k\) is the best rank-\(k\) approximation of \(A\). We refer the term \(\gamma\) as the multiplicative error and the term \(\zeta\) as the additive error. We call the tuples \((\varepsilon, \delta)\) the privacy parameters.

Our problem statement is more general than principal component analysis in the sense that we require approximation to both the left and the singular vectors. We discuss this in more details in Appendix B.

**Granularity of Privacy:** In the past, differentially private LRA with respect to the spectral norm have been proposed with varying levels of privacy. Kapralov and Talwar [35] considered two matrices neighboring if the difference of their spectral norm is at most 1. Dwork et al. [24] considered two row-normalized matrices neighboring if they differ in one row. Hardt and Price [27] and Hardt and Roth [29] considered two matrices as neighboring if they differ exactly in one entry. In this paper, we consider even stronger levels of privacy. This matrices deals with streaming matrices; therefore, for the sake of discussion and ease of comparison of privacy model, below we define the granularity of privacy with respect to neighboring matrices. Two streams are said to be neighboring if they are formed using neighboring matrices.

In the first privacy level, \(Priv_1\), we call two matrices \(A\) and \(A'\) as neighboring if \(\|A - A'\|_F \leq 1\), where \(\|\cdot\|_F\) denotes the Frobenius norm. In the second privacy level, \(Priv_2\), we consider two matrices are neighboring if \(\|A - A'\|_F \leq 1\) and \(A - A'\) is a rank-1 matrix. In other words, the matrices differ in only one spectrum by at most 1. Both these privacy notions are motivated by natural scenarios that are not captured by the privacy level studied in previous works [24, 27, 29]. For example, \(Priv_2\) is a natural choice where the spectrum of the input matrix is the key feature, which is the case in scenarios where spectral analyses are performed. The choice of \(Priv_1\) is motivated by the examples listed in Upadhyay [54], where the presence or absence of an individual in a social graph can lead to a change of at most 1 in the Frobenius norm of the corresponding adjacency matrix. We refer the readers to Upadhyay [54] for more details. Another notable example where \(Priv_2\) make sense is the word2vec model.

**1.2 Contributions of This Paper**

All the earlier works [27, 29, 35] use an approach known as the *subspace iteration*. A subspace iteration algorithm runs for \(k\) rounds. In every iteration, the algorithm computes (and stores) the
Our key insight is that we can compute the top-$k$ singular vector in one step (even in the turnstile update model) by using linear sketches. This also allows us to reduce the additive error as we do not have to add noise $k$ times to preserve the privacy in every iteration, giving us $\sqrt{k}$ improvement over private algorithms that use subspace iteration.

Throughout this section, we assume $\delta = o(1/n^2)$ and let $\tilde{O} (\cdot)$ hides a log $n$ factor and denote by $\Delta_k (A) := \|A - [A]_k\|_2$. We state our results assuming $m \geq n$. All our results hold true for $m < n$ with the roles of $m$ and $n$ reversed. Our algorithms are also efficient because all the costly computations are done on low-dimensional sketches and the sketches themselves can be generated and updated efficiently using known techniques [11, 33].

**LRF With Respect to Spectral Norm Under Priv1:** Our first algorithm, SPECTRAL-LRF, is based on the following intuition. Let $[U]_k [\Sigma]_k [V]_k^T$ be a singular-value decomposition of $[A]_k$. Therefore, if an orthonormal column matrix $U$ is a “faithful” representation of $[U]_k$, then $X := [\Sigma]_k [V]_k^T \approx \arg\min_{r(X) \leq k} \|UX - A\|$ and $\|UX - A\|_2 \approx \|([A]_k - A)\|_2$. We show that the matrix $U$ can be computed from a linear sketch. If we pick $S$ to be a random projection matrix such that, simultaneously for all $X$, $\|S(UX - A)\|_2 \approx \|(UX - A)\|_2$ with high probability, then $\|S(UX - A)\|_2 \approx \|([A]_k - A)\|_2$ with high probability. In other words, we need to store $SA$ and the linear sketch used to compute $U$, and output the product of $U$ and $\arg\min_{r(X) \leq k} \|S(UX - A)\|_2$. We show the following.

**Theorem 2.** (Informal statement of Theorem [11]). Let $m, n \in \mathbb{N}$ (where $m \geq n$), $(\varepsilon, \delta)$ be the privacy parameters, and $k$ be the desired rank of the factorization. Let $\alpha \in (0, 1)$ be an arbitrary constant and $\eta = \max \{k^2, \alpha^{-1}\}$. Given an $m \times n$ matrix $A$ in the turnstile update model, there exists an efficient $(\varepsilon, \delta)$-differentially private algorithm that uses $O_\delta (mn\alpha^{-1} + n\eta\alpha^{-3})$ space and outputs a $k$-rank factorization $U_k, \Sigma_k, V_k$ such that, with probability at least $9/10$ over the coin tosses of the algorithm,

$$\|A - U_k \Sigma_k V_k^T\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \Delta_k (A) + \tilde{O} \left( \left( \frac{\sqrt{m} + \sqrt{n} + \alpha^{-2}}{(1 - \alpha)^{3/2} \varepsilon} \right) \right).$$

In practice, $k \ll \max \{m, n\}$ and $\alpha$ is a small constant. In that case, we have $(1 - \alpha)^{-1} \approx 1 + \alpha$. If we scale the value of $\alpha$ appropriately, then we have the following corollary to Theorem 2.

**Corollary 3.** (Informal statement of Corollary [17]). Let $m, n, \varepsilon, \delta$ be as in Theorem 2 with $m \geq n$. Then there exists an efficient $(\varepsilon, \delta)$-differentially private algorithm under Priv1 that solves $(\varepsilon, \delta, \gamma, 1/10, \zeta, k)$-LRF in $O_\delta (mn\gamma^{-1} + n\zeta\gamma^{-3})$ space, where $\gamma := (1 + \alpha)$ and $\zeta := \tilde{O} \left( \left( \sqrt{m} + \sqrt{n} \varepsilon^{-1} \right) \right)$.

**Remark 1.** Hardt and Roth [22, Thm 1.2] showed that any differentially private algorithm incurs an additive error $\zeta = \Omega (\varepsilon^{-1} \sqrt{n})$ for an $n \times n$ matrix. Their lower bound holds even when the algorithm can access the input matrix any number of times. Corollary 3 shows that we can achieve the same bound, up to a logarithmic factor, even when the matrix is updated in the turnstile manner at the cost of small multiplicative error that depends only on the $(k + 1)$-th singular value.

**Improving the Space Bound Under Priv2:** The algorithm SPECTRAL-LRF had a space requirement that depends on the dimension of both $S$ and $U$. However, if the matrix is almost square, then the storage required to store $SA$ is much higher than that to store the sketch that is used to compute
A direct observation then is that, if we can simultaneously generate an orthonormal matrix \( V \) which is a “faithful” representation of \( |V|_k \), then \( \tilde{X} := \Sigma |_{X|_k} \approx \text{argmin}_{|X|_{k}} \| UXV^T - A \|_2 \). Further, if we pick \( Q \) and \( R \) that satisfy similar properties as \( S \) in Section 1.2, then \( \| Q(UXV^T - A)R \|_2 \approx \| |A|_{k} - A \| \). Therefore, we can output the product of matrices \( U, \text{argmin}_{|X|_{k}} \| Q(UXV^T - A)R \|_2 \), and \( V^T \). This forms the basis of LOW-SPACE-LRF.

**Theorem 4.** (Informal statement of Theorem 2). Let \( m, n \in \mathbb{N} \) (where \( m \leq n \)) and \( \varepsilon, \delta \) be the input parameters. Let \( k \) be the desired rank of the factorization. Let \( 0 < \alpha < 1 \) be an arbitrary constant and \( \eta = \max \{ k^2, \alpha^{-1} \} \). Given an \( m \times n \) matrix \( A \) in the turnstile update model, there exists an efficient \( (\varepsilon, \delta) \) differentially private algorithm that outputs a \( k \)-rank factorization \( U_k, \Sigma_k, V_k \) using \( O((m+n)\eta \alpha^{-1} \log(1/\delta)) \) space, such that with probability at least \( 9/10 \) over the coin tosses of the algorithm,

\[
\| A - U_k \Sigma_k V_k^T \|_2 \leq \frac{(1 + \alpha)^2}{(1 - \alpha)^2} \Delta_k(A) + \tilde{O} \left( \frac{k \alpha^{-1} \ln(1/\delta) + \sqrt{m} + \sqrt{n}}{\varepsilon(1 - \alpha)^3} \right).
\]

As before we get analogous results to Corollary 3.

**Corollary 5.** (Corollary 2 informal). Let \( m, n, \varepsilon, \delta \) be as in Theorem 2 and \( \alpha \in (0, 1) \) be a small constant. Then there exists an efficient \( (\varepsilon, \delta) \) differentially private algorithm that solves \( (\varepsilon, \delta, \gamma, 1/10, \zeta, k) \)-LRF under \( \text{DPP} \) of rank \( k + 1 \). In this case, approximation with respect to the Frobenius norm and spectral norm are equivalent. However, due to the result of Upadhyay [54], any algorithm that achieves \( (1 + \alpha) \) multiplicative error and \( O(n) \) additive error has to use \( \Omega(nk/\alpha) \) space. Therefore, one cannot hope to get an LRA under spectral norm with no multiplicative error and low space under the turnstile update model.

### 1.3 Applications of Our Results

**Application in Continual Release:** One of the characteristics of our algorithms is that \( U \) is computed from a linear input matrix and \( SA \) is a linear sketch. This implies that we can use the binary tree mechanism [3] to get a low-rank factorization under continual release by paying an \( O(\log T) \) factor in the additive error, where \( T \) is the number of updates. This is stated in the form of the following results (for a small constant \( \alpha \in (0, 1) \)).

**Theorem 6.** (Informal statement of Theorem 3). Let \( m, n, \varepsilon, \delta \) be as in Theorem 2. Then there exists an efficient \( (\varepsilon, \delta) \)-differentially private algorithm under \( \text{Priv}_1 \) that solves \( (\varepsilon, \delta, \gamma, 1/10, \zeta, k) \)-LRF under continual release for \( T \) time epochs, where \( \gamma := (1 + \alpha) \) and \( \zeta := \tilde{O} \left( \frac{\sqrt{m} + \sqrt{n}}{\varepsilon^{-1} \log T} \right) \).

**Theorem 7.** (Informal statement of Theorem 4). Let \( m, n, \varepsilon, \delta \) be as in Theorem 4. Then there exists an efficient \( (\varepsilon, \delta) \)-differentially private algorithm under \( \text{Priv}_2 \) that solves \( (\varepsilon, \delta, \gamma, 1/10, \zeta, k) \)-LRF under continual release for \( T \) time epochs, where \( \gamma := (1 + \alpha) \) and \( \zeta := \tilde{O} \left( \frac{\sqrt{m} + \sqrt{n}}{\varepsilon^{-1} \log T} \right) \).

**Application in Non-private Setting:** As an immediate result of Theorem 2, when \( \varepsilon \to \infty \), we get a non-private algorithm that achieves low-rank factorization in the turnstile update model with respect to the spectral norm. Prior to this work, no algorithm that outputs LRA with respect to the spectral norm in the turnstile update model was known.
Theorem 8. Let \( m, n \) be as in Theorem 2 and \( \alpha \in (0, 1) \). Given a matrix \( A \in \mathbb{R}^{m \times n} \) in the turnstile update model, there exists an efficient algorithm that uses \( O(mk\alpha^{-1} + n\alpha^{-4}) \) space and outputs a \( k \)-rank factorization \( U_k, \Sigma_k, V_k \) such that \( \|A - U_k\Sigma_kV_k^T\|_2 \leq (1 + \alpha)\Delta_k(A) \).

Similarly, we can improve the space bound of the algorithm by setting \( \varepsilon \to \infty \) in Theorem 2.

Theorem 9. Let \( m, n \) be as in Theorem 2 and \( \alpha \in (0, 1) \). Given a matrix \( A \in \mathbb{R}^{m \times n} \) in the turnstile update model, there exists an efficient algorithm that uses \( O((m + n)k\alpha^{-1}) \) space and outputs a \( k \)-rank factorization \( U_k, \Sigma_k, V_k \), such that \( \|A - U_k\Sigma_kV_k^T\|_2 \leq (1 + \alpha)\Delta_k(A) \).

### 1.4 Technical Overview

In this section, we only present the proof idea for the correctness of SPECTRAL-LRF. The proof idea for the correctness of LOW-SPACE-LRF and the algorithms for continual releases follows from the intuition given in Section 1.2. The basic idea behind SPECTRAL-LRF is to compute matrix \( U \) and find \( S \) with certain properties. Proving that our candidate choices for \( S \) and \( U \) satisfy the desired properties turns out to be the main technical challenge of this paper. The main complication here is that the spectral norm does not satisfy the Pythagorean theorem, which forms a key ingredient in the proof when the approximation guarantees are in the Frobenius norm [54]. In what follows, we use the private matrix \( A \) to give the intuition. In the actual proof, there is an extra Gaussian mechanism due to use of the Gaussian mechanism to preserve privacy. We show that if we use the Gaussian mechanism appropriately, the additive error scales optimally with the dimensions of the private matrix \( A \).

Let \( [U]_k \Sigma_k [V]_k^T \) be a singular-value decomposition of \( [A]_k \) and \( r(\cdot) \) denote the rank of the matrix. Let \( \Phi \) be a matrix that satisfies Lemma 10 for parameters \((\sqrt{\alpha}/k, 1/10)\). Then our choice of \( U \) is an orthonormal basis of the column space of \( A\Phi \). This implies that \( \min_{r(X) \leq k} \|UX - A\|_2 \leq \min_{r(X) \leq k} \|A\Phi X - A\|_2 \). We use two optimization problems to prove that \( U \) is a good approximation to \([U]_k: \min_{X} \|\Phi^T([V]_kX - A^T)\|_2 \) (PROBLEM 1) and \( \min_{X} \|([V]_kX - A^T)\|_2 \) (PROBLEM 2). The result follows by combining equation (1), equation (2), and equation (3) below listed.

Since PROBLEM 1 is a minimization problem over all allowable matrices \( X \), using Theorem 44 and the singular value decomposition of \( \Phi^T[V]_k \), we get \( \tilde{X} = (\Phi^T[V]_k)^T\Phi^T A^T \). Since the rank of \( ([V]_k(\Phi^T[V]_k)^T) \) is at most \( k \),

\[
\min_{r(X) \leq k} \|UX - A\|_2 \leq \min_{r(X) \leq k} \|A \Phi X - A\|_2 \\
\leq \|A \Phi ([V]_k(\Phi^T[V]_k)^T)^T - A\|_2 \\
= \|[V]_k \tilde{X} - A^T\|_2. \tag{1}
\]

We then show that, if \( \tilde{X} \) is a solution to PROBLEM 1 and \( \tilde{X} \) is a solution to PROBLEM 2, then

\[
\|[V]_k \tilde{X} - A^T\|_2 \leq (1 - \alpha)^{-1} \|([V]_k \tilde{X} - A^T)\|_2. \tag{2}
\]

Now if we pick \( X = \Sigma_k [U]_k^T \) in PROBLEM 2, then

\[
\|[V]_k \tilde{X} - A^T\|_2 = \min_{X} \|([V]_kX - A^T)\|_2 \\
\leq \|A^T - [V]_k \Sigma_k \Sigma_k [U]_k^T\|_2 = \Delta_k(A). \tag{3}
\]
We sample $S$ from a distribution of random projection matrices that satisfies the Johnson-Lindenstrauss lemma. To prove that our choice of $S$ satisfies the required property, we introduce the optimization problem: \[ \min_X \|UX - A\|_2. \] Since $X$ can have rank at most the rank of $A$, using Theorem 44, we have that \[ \hat{X} = U^T[UU^T A] = \arg\min_X \|UX - A\|_2. \] This implies that \[ U^T(U\hat{X} - A) = U^T(UU^T - I)A = 0. \] We use this fact, the sub-additivity and sub-multiplicativity of the spectral norm, and the fact that $\|S\|_2 \approx 1$ if $S$ satisfies the Johnson-Lindenstrauss lemma to prove that $S$ satisfies the required property.

To preserve the privacy of SPECTRAL-LRF, we note that the private matrix is used twice: when we compute the matrix $U$ and when we compute $SA$. Therefore, if we add appropriately scaled Gaussian matrices at these stages, then the privacy of the algorithm follows from the fact that differential privacy is preserved under post-processing. For the privacy proof of LOW-SPACE-LRF, we use the idea used by Upadhyay [54] to use both the input and the output perturbation.

### 1.5 Comparison with Previous Works

All the previous private algorithms compute low-rank approximation (either of the matrix or its covariance). Though it is possible to compute the factorization of their outputs, this would incur an extra $O(mn^2)$ running time for the factorization of an $m \times n$ rank-$k$ matrix. Moreover, they require $O(mn)$ space just to store the output (Dwork et al. [24] requires $O(n^2)$, but they work with covariance matrices). Unlike this work, previous works also assume structured private matrix, like matrices with singular value separation [29, 27, 35]. Previous works have studied three variants of LRA. Dwork et al. [24] only approximate the right-singular vector while Hardt and Roth [29] (and subsequently, Hardt and Price [27]) who improved the result of Hardt and Roth [29] and Upadhyay [52] approximates both right and left singular vectors. Kapralov and Talwar [35] and Jiang et al. [31] considered $(\varepsilon, 0)$-differential privacy. On the other hand, our algorithm outputs a factorization with $(\varepsilon, \delta)$-differential privacy. In this sense, our problem is closely related to Hardt and Price [27]. We give the technical descriptions and the differences between Problem 1 and all the previously studied problems in Appendix B. Here, we only compare this work with the results of Hardt and Price [27] and Upadhyay [52]. We give a comparison of the results in Table 1. Below, we enumerate the key differences between our result and that of Hardt and Price [27].

1. **Space efficiency.** The algorithms of Hardt and Price [27] and Upadhyay [52] use $O(mn)$ space. On the other hand, both of our algorithms use sub-linear space.

2. **Approximation error.** Both of our bounds improve the additive error of Upadhyay [52] by a factor of $k^{3/2} \alpha^2 \sqrt{\log(1/\delta)}$. To make a reasonable comparison with Hardt and Price [27], we consider their result without coherence assumption. Hardt and Price [27] incurs an additive error $\tilde{O}(\sigma_1 \sqrt{k(m + n)\varepsilon^{-1}})$, where $\sigma_1$ is the maximum singular value of the input matrix. We can rewrite our results to say that we incur an additive error that depends on $\alpha \sigma_{k+1} + \tilde{O}(\sqrt{m + n}/\varepsilon)$. In most real world scenarios, $\sigma_{k+1} \ll \sigma_1$. In other words, we improve the result of Hardt and Price [27] if $\sigma_1$ is large. Recall that the dependency on $\alpha$ is unavoidable for low-space algorithm (see, Remark 2).

3. **Streaming constraints.** Hardt and Price [27] is a private version of the iterative algorithm of Halko et al. [26]. The algorithm of Upadhyay [52] is one-pass, but assumes that the matrix is streamed row-wise. A row-wise update is an easier problem (with respect to the
space required) compared to the turnstile update model even in the non-private setting as illustrated by Clarkson and Woodruff [10].

### 1.6 Related Works

Low-rank approximation of large data-matrices has received a lot of attention in the recent past in the private as well as the non-private setting. In what follows, we give a brief exposition of those that are most relevant to this work.

In the private setting, previous works have either used random projection [10] [33] [47] or random sampling (at a cost of a small additive error) to give low-rank approximation [3] [16] [18] [43] [25] [44] [45]. Many of the latter algorithms were improved independently by Deshpande and Vempala [14] and Sarlos [47]. Subsequent works [11] [38] [40] [42] achieved a run-time that depends linearly on the input sparsity of the matrix. In a series of works, Clarkson and Woodruff [10] [11] showed space lower bounds and almost matching space algorithms. Recently, Boutsidis et al. [7] gave the first space-optimal algorithm for low-rank approximation, but they do not optimize for run-time.

In the private setting, LRA has been studied under a privacy guarantee called differential privacy. Differential privacy was introduced by Dwork et al. [20]. The Gaussian variant of this basic sanitizer was proven to preserve differential privacy by [19] in a follow-up work. Since then, many algorithms for preserving differential privacy have been proposed in the literature [22]. All these mechanisms have a common theme: they perturb the output before responding to queries. Recently, Blocki et
al. [5] and Upadhyay [51] took a complementary approach. They perturb the input reversibly and then perform a random projection of the perturbed matrix.

Blum et al. [6] first studied the problem of differentially private LRA in the Frobenius norm. This was improved by Hardt and Roth [28] under the low coherence assumption. Upadhyay [52] later made it a single-pass. Differentially-private LRA has been studied in the spectral norm as well by many works [9, 35, 29, 27]. Kapralov and Talwar [35] and Chaudhary et al. [5] and Upadhyay [51] took a complementary approach. They perturb the input reversibly and LRA later made it a single-pass. Differentially-private learning [24, 30, 50].

2 Notations and Previous Known Results Used in This Paper

We let $\mathbb{N}$ denote the set of natural numbers. We use bold-face capital letters to denote matrices and bold-face small letters to denote vectors. We denote by $0^{m\times n}$ the all-zero $m \times n$ matrix and by $I_n$ the $n \times n$ identity matrix. For a matrix $A$, we denote its best $k$-rank approximation by $[A]_k$, its Frobenius norm by $\|A\|_F$, and its spectral norm by $\|A\|_2$. The singular-value decomposition (SVD) of an $m \times n$ rank-$r$ matrix $A$ is a decomposition of $A$ as a product of three matrices, $A = U\Sigma V^T$ such that $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$ have orthonormal columns and $\Sigma \in \mathbb{R}^{r \times r}$ is a diagonal matrix with singular values of $A$ on its diagonal. For a matrix $A$, we use the symbol $r(A)$ to denote its rank and $\det(A)$ to denote its determinant. The Moore-Penrose pseudo-inverse of a matrix $A = U\Sigma V^T$ is denoted by $A^\dagger$ and has a SVD $A^\dagger = V\Sigma^\dagger U^T$, where $\Sigma^\dagger$ consists of inverses of only non-zero singular values of $A$. Given a random variable $x$, we denote by $\mathcal{N}(\mu, \rho^2)$ the fact that $x$ has a normal Gaussian distribution with mean $\mu$ and variance $\rho^2$.

Let $\alpha, \beta > 0$. A distribution $\mathcal{D}$ over $p \times n$ random matrices satisfies $(\alpha, \beta)$-Johnson-Lindenstrauss property (JLP) if, for any unit vector $x \in \mathbb{R}^n$, we have $(1 - \alpha) \leq \|\Phi x\|_2^2 \leq (1 + \alpha)$ with probability $1 - \beta$ over $\Phi \sim \mathcal{D}$. A distribution $\mathcal{D}$ of $p \times m$ matrices satisfies $(\alpha, \beta)$-subspace embedding for a matrix $A \in \mathbb{R}^{m \times n}$ if, for all $x \in \mathbb{R}^n$, with probability $1 - \beta$ over $\Phi \sim \mathcal{D}$, $(1 - \alpha)\|Ax\|_2 \leq \|\Phi Ax\|_2 \leq (1 + \alpha)\|Ax\|_2$.

Lemma 10. (Clarkson and Woodruff [11]) Let $B_1$ and $B_2$ be arbitrary matrices with $m$ rows such that $B_1$ has rank-$r$. Let $\mathcal{D}$ be a distribution over $p \times m$ random matrices that satisfies the $(\alpha, \beta)$-subspace embedding for $B_1$. Then there exists a $p = \Theta(\alpha^{-2})$ such that, with probability at least $1 - \beta$ over $\Phi \sim \mathcal{D}$,

$$\|B_1^T \Phi^T \Phi B_2 - B_1 \|_F^2 \leq \alpha^2 \|B_1\|_F^2 \|B_2\|_F^2.$$ 

The tuple $(\alpha, \beta)$ is called the error parameters.
Initialization. Let \( \eta := \max \{k^2, 1/\alpha\} \), \( \rho := \sqrt{(1 + \alpha) \ln(1/\delta)}/\varepsilon \), \( t = O(\eta \alpha^{-1} \log(k/\alpha) \log(1/\delta)) \), and \( v = O(\eta \alpha^{-3} \log(t/\alpha) \log(1/\delta)) \). Sample \( N_1 \sim \mathcal{N}(0, \rho^2)^{m \times t} \) and \( N_2 \sim \mathcal{N}(0, \rho^2)^{v \times n} \) Let \( \Phi \in \mathbb{R}^{n \times t} \) be such that \( \Phi \sim \mathcal{D}_R \) satisfies the statement of Lemma 10 for the error parameters \((\alpha/k, \delta)\). Let \( S \in \mathbb{R}^{v \times n} \) such that \( S \sim \mathcal{D}_A \) satisfies \((\alpha^2, \delta)\)-JLP. Initialize all \( m \times t \) zero matrix \( \hat{Y} \) and an all zero \( v \times n \) matrix \( \hat{Z} \). Publish \( S \) and \( \Phi \). \( N_1 \) and \( N_2 \) are private matrices.

Update rule and computing the factorization. Suppose at time \( \tau \), the stream is \((i_\tau, j_\tau, s_\tau)\), where \((i_\tau, j_\tau) \in [m] \times [n]\). Let \( A_\tau \) be a matrix with the only non-zero entry \( s_\tau \) in the position \((i_\tau, j_\tau)\). Update the matrices by the following rule: \( \hat{Y} \leftarrow \hat{Y} + A_\tau \Phi \) and \( \hat{Z} \leftarrow \hat{Z} + SA_\tau \).

Once the matrix is streamed, we follow the following steps.

1. Compute the singular value decomposition \( \tilde{U} \Sigma \tilde{V}^T \) of \( SU \in \mathbb{R}^{v \times t} \), where \( U \in \mathbb{R}^{m \times t} \) is a matrix whose columns are an orthonormal basis for the column space of \((\hat{Y} + N_1)\).
2. Compute the singular value decomposition of \( \tilde{V} \Sigma \tilde{U}^T[\tilde{U} \tilde{U}^T(\hat{Z} + N_2)]_k \in \mathbb{R}^{t \times n} \). Let it be \( U' \Sigma' \nu \).
3. Output \( U_k := UU', \Sigma_k := \Sigma' \) and \( V_k := V' \). Let \( M_k = U_k \Sigma_k V_k^T \).

![Figure 1: Differentially private Low-rank Factorization (Spectral-LRF)](image)

### 3 Differentially Private Algorithm for Spectral LRF in the Turnstile Model

In this section, we present our space-efficient private algorithm for LRF under \( \text{Priv}_1 \) with respect to the spectral norm in the turnstile update model. Our privacy level, \( \text{Priv}_1 \), is stronger than that considered by previous \((\varepsilon, \delta)\)-differentially private algorithms \[24, 29, 27\]. Our algorithm formalizes the ideas mentioned in Section 1.4. In particular, we show that we can compute a set of \( k \)-orthonormal basis vectors that approximates the top-\( k \) left singular space of the private matrix in the turnstile update model. The details of our algorithm \text{Spectral-LRF} is presented in Figure 1. We prove the following bound about \text{Spectral-LRF}.

**Theorem 11.** Let \( m, n \in \mathbb{N} \) and \( \varepsilon, \delta \) be the input parameters with \( m \geq n \). Let \( k \) be the desired rank of the factorization. Let \( \alpha \in (0, 1) \) be an arbitrary constant. Given an \( m \times n \) matrix \( A \) in the turnstile update model, the algorithm \text{Spectral-LRF}, presented in Figure 1, satisfies the following properties:

1. Let \( \eta = \max \{k^2, 1/\alpha\} \). Then \text{Spectral-LRF} uses \( O((m \eta \alpha^{-1} + n \eta \alpha^{-3}) \log(k/\alpha^2) \log(1/\delta)) \)
space.

2. \text{Spectral-LRF} is \((\varepsilon, \delta)\)-differentially private under \( \text{Priv}_1 \).

3. With probability at least 9/10 over the coin tosses of \text{Spectral-LRF},

\[
\|A - M_k\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \Delta_k(A) + O \left( \frac{\sqrt{m} + \sqrt{n} + \alpha^{-2}}{(1 - \alpha)^2 \varepsilon} \sqrt{\log(1/\delta)} \right),
\]

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where $M_k = U_k \Sigma_k V_k^T$.

When $m < n$, we can get similar bounds with the roles of $m$ and $n$ reversed and by using $A^T$ in Figure 1. The key technical point in the proof is to prove that $U$ is indeed a good approximation to the top-$k$ left singular vectors of $A$ and $S$ has the property that, simultaneously for all matrices $X$ of appropriate dimensions, $\|S(UX - A)\|_2 \leq \frac{(1+\alpha)}{(1-\alpha)^2} \|UX - A\|_2 + \|X\|_2 \left(\frac{1+\alpha}{1-\alpha}\|Y^T\|_{\frac{1}{1+k}}\|V^T\|_{\frac{1}{1+k}}\right)$.

**Proof of Theorem 11.** The space complexity of the algorithm is straightforward from the choice of $v$ and $t$. The proof of correctness consists of two main lemmas (Lemma 12 and Lemma 15). In Lemma 12, we bound $\|M_k - A\|_2$ by $\|A - [A]_k\|_2$ and two additive terms. In Lemma 15, we bound the two additive terms.

For all our correctness proofs, we make a standard assumption that $\delta \ll 1/100$. Let $Y = \hat{Y} + N_1$, $Z = \hat{Z} + N_2$, and $B = A + S^T N_2$ for $\hat{Y}, \hat{Z}, S, N_1, N_2$ as in Figure 1.

**Lemma 12.** Let $U_k, \Sigma_k, V_k$ be the output of the algorithm SPECTRAL-LRF presented in Figure 1 such that $M_k := U_k \Sigma_k V_k^T$. Then with probability $1 - O(\delta)$ over $\Phi \sim \mathcal{D}_R$ and $S \sim \mathcal{D}_A$,

$$\|M_k - A\|_2 \leq \frac{(1+\alpha)}{(1-\alpha)^2} \|A - [A]_k\|_2 + \frac{2\|N_2 S^T\|_2}{1-\alpha} + \frac{1+\alpha}{1-\alpha} \|N_1 ([V]_k^T \Phi)^T [V]^T\|_2.$$ 

**Proof.** We prove Lemma 12 by proving two separate claims. Claim 13 shows that $U$ is a “faithful” representation of the top-$k$ left singular vectors of $[A]_k$ and Claim 14 shows that $S$ satisfies the required properties with the choice of error parameters.

**Claim 13.** Let $A$ be the input matrix. Let $\Phi \sim \mathcal{D}_R$ be a random matrix that satisfies Lemma 11 with error parameters $(\sqrt{\alpha}/k, \delta)$ with respect to a rank-$k$ matrix $[V]_k$. Then with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,

$$\min_{x, r(x) \leq k} \|UX - B\|_2 \leq \frac{(1+\alpha)}{(1-\alpha^2)} \Delta_k(A) + \|N_1 ([V]_k^T \Phi)^T [V]^T\|_2 + \|S^T N_2\|_2.$$ 

Before, we prove Claim 13 we prove some auxiliary results. Let us denote by $[U]_k [\Sigma]_k [V]^T$ the SVD of $[A]_k$. We first note that $U$ is an orthonormal basis for the column space of $Y$, i.e.,

$$\min_{x, r(x) \leq k} \|UX - B\|_2 \leq \min_{x, r(x) \leq k} \|YX - B\|_2.$$ 

(4)

We also note that the choice of $t$ allows $\Phi$ to satisfy Lemma 11 with error parameters $(\sqrt{\alpha}/k, \delta)$ [11]. Therefore, in order to show that $U$ approximates the top-$k$ left singular vectors of the matrix $A$, we need to find an appropriate matrix $X$ such that $\|YX - A\|_2$ is bounded by a multiplicative factor of $\|A - [A]_k\|_2$. Towards this goal, consider the following two optimization problems:

$$\min_X \| \Phi^T ([V]_k X - A^T) \|_2 \quad \text{and} \quad \min_X \| [V]_k X - A^T \|_2.$$ 

(5)

with

$$\tilde{X} := \arg \min_X \| \Phi^T ([V]_k X - A^T) \|_2$$

$$\hat{X} := \arg \min_X \| [V]_k X - A^T \|_2$$
as one of the solutions. We first give the sketch of the proof. Our goal is to show that

\[
\|[V]_k \hat{X} - A^T\|_2 \leq (1 - \alpha)^{-1}\|[V]_k \hat{X} - A^T\|_2.
\]

(6)

This would give us Claim 13 after substituting the value of \(B\) and \(Y\) because of the following argument.

Since \(\hat{X}\) minimizes Problem 2, \(X = [\Sigma]_k[U]_k^T\) would only increase the value of \(\|[V]_k X - A^T\|_2\).

That is,

\[
\|[V]_k \hat{X} - A^T\|_2 = \min_{X} \|[V]_k X - A^T\|_2 \\
\leq \|[A^T - [\Sigma]_k[U]_k^T]\|_2 \\
= \Delta_k(\Lambda). \tag{7}
\]

Now since Problem 1 is a minimization problem over all possible matrices \(X\), using Theorem 44 and the singular value decomposition of \(\Phi^T[V]_k\), we get \(\hat{X} = (\Phi^T[V]_k)^\dagger \Phi^T A^T\). Furthermore, \(r((V)_k(\Phi^T[V]_k)^\dagger) \leq k\). This implies that

\[
\min_{r(X) \leq k} \|[U]X - A\|_2 \leq \min_{r(X) \leq k} \|[A\Phi X - A]\|_2 \\
\leq \|[A\Phi ([V]_k(\Phi^T[V]_k)^\dagger)^T - A]\|_2 \\
= \|[V]_k \bar{X} - A^T\|_2 \tag{8}
\]

We get the desired result by combining equation (6), equation (7), equation (8), and substituting the value of \(B = A + S^T N_2\) and \(Y = \hat{A} + N_1\). Therefore, the key to our proof is to prove equation (6).

Proving equation (6). We now prove equation (6). Using the sub-additivity of the spectral norm, we have

\[
\|[A^T - [V]_k \bar{X}]\|_2 \leq \|[A^T - [V]_k \bar{X}]\|_2 + \|[V]_k (\bar{X} - \hat{X})\|_2.
\]

The first term in the above expression has the form that we desire. Therefore, towards the goal we set forth, we need to bound \(\|[V]_k (\bar{X} - \hat{X})\|_2\). Let \(C = [V]_k (\bar{X} - \hat{X})\) and \(D = [V]_k (\bar{X} - \hat{X})\). It is easy to see that \(C^T C = D^T D\); therefore \(\|[D]\|_2 = \|[C]\|_2\). This implies that instead of \(D\) if we bound \(C\), then we are done. We first make an observation that \([V]_k \Phi \Phi^T \Phi^T - [V]_k \hat{X}\) is a projection on to the space orthogonal to \([V]_k \Phi\) and \([V]_k \Phi \Phi^T \Phi^T - [V]_k \hat{X}\) equals

\[
[V]_k \Phi \Phi^T \Phi^T - [V]_k \Phi \Phi^T \Phi^T \Phi^T \Phi^T \Phi^T A^T = 0. \tag{9}
\]

Now, we return to the proof of equation (6). The sub-additivity and sub-multiplicativity of the spectral norm gives us the following:

\[
\|[C]\|_2 = \|[V]_k \Phi \Phi^T \Phi^T \Phi^T [V]_k C - [V]_k \Phi \Phi^T \Phi^T [V]_k C + C\|_2 \\
\leq \|[V]_k \Phi \Phi^T \Phi^T [V]_k C\|_2 + \|[V]_k \Phi \Phi^T \Phi^T \Phi^T \Phi^T [V]_k - \Phi \Phi^T \Phi^T \Phi^T \Phi^T [V]_k\|_2 \\
\leq \|[V]_k \Phi \Phi^T \Phi^T [V]_k C\|_2 + \|[\Phi \Phi^T - \Phi \Phi^T \Phi^T \Phi^T \Phi^T] [V]_k\|_2. 
\]
The choice of the dimension of $\Phi$ allows us to use the result of Clarkson and Woodruff [10] (Lemma [13]).

\[
\|C\|_2 \leq (1 - \alpha^2)^{-1}\|\Phi^T[V][k]C\|_2 \\
\leq (1 - \alpha^2)^{-1}\|\Phi^T[V][k]C\|_F \\
= (1 - \alpha^2)^{-1}\|\Phi^T[V][k](\bar{X} - \hat{X}) + [V][k]\Phi^T(A^T - [V][k]\bar{X})\|_F \\
= (1 - \alpha^2)^{-1}\|\Phi^T(A^T - [V][k]\bar{X})\|_F \\
\leq (1 - \alpha^2)^{-1}\alpha\|\Phi^T(A^T - [V][k]\bar{X})\|_F/\sqrt{k}. \\
\leq (1 - \alpha^2)^{-1}\alpha\|\Phi^T(A - [V][k]\bar{X})\|_2,
\]

the equality follows from equation (9), the second and last inequality follows from Fact [40].

In the above inequalities, we were able to apply Lemma [10] because $[V][k]$ is a rank-$k$ matrix and $\Phi$ satisfies $(\sqrt{\alpha/k}, \delta)$-subspace embedding with respect to $[V][k]$. Now, from the sub-additivity of the spectral norm,

\[
\|A^T - [V][k]\bar{X}\|_2 \leq \|A^T - [V][k]\hat{X}\|_2 + \|[V][k](\bar{X} - \hat{X})\|_2 \\
\leq (1 + \alpha - \alpha^2)(1 - \alpha)^{-1}\|A^T - [V][k]\hat{X}\|_2 \\
\leq (1 + \alpha)(1 - \alpha)^{-1}\|A^T - [V][k]\hat{X}\|_2 \\
= (1 - \alpha)^{-1}\|A^T - [V][k]\hat{X}\|_2. \tag{10}
\]

This gives us the desired expression stated in equation (6).

Proof of Claim [13] We now complete the proof of Claim [13]. We follow the sketch of the proof given earlier. By the definition of $\bar{X}$, we have

\[
\|A^T - [V][k]\bar{X}\|_2 = \min_{\hat{X}}\|[V][k]\hat{X} - A^T\|_2 \leq \|[V][k][\Sigma][k][U][k]^T - A^T\|_2 = \Delta_k(A)
\]

by setting $X = [\Sigma][k][U][k]^T$. Using equation (6), substituting the value of $\bar{X}$, taking the transpose, and the fact that the spectral norm is preserved under transpose, we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,

\[
\|A\Phi([V]^T\Phi)^\dagger[V][k]^T - A\|_2 \leq (1 - \alpha)^{-1}\Delta_k(A). \tag{11}
\]

Moreover, since $([V][k]\Phi)^\dagger[V][k]$ has rank at most $k$ and $Y = A\Phi + N_1$, with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,

\[
\min_{X, r(X) \leq k} \|(YX - B)\|_2 \leq \|Y([V][k]^T\Phi)^\dagger[V][k]^T - B\|_2 \\
= \|A\Phi([V][k]^T\Phi)^\dagger[V][k]^T + N_1([V][k]^T\Phi)^\dagger[V][k]^T - B\|_2 \\
= \|A\Phi([V][k]^T\Phi)^\dagger[V][k]^T + N_1([V][k]^T\Phi)^\dagger[V][k]^T - A - S^\dagger N_2\|_2 \\
\leq \|A\Phi([V][k]^T\Phi)^\dagger[V][k]^T - A\|_2 \\
+ \|N_1([V][k]^T\Phi)^\dagger[V][k]^T\|_2 + \|S^\dagger N_2\|_2 \tag{12}
\]

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Combining equation (11) and equation (12), we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,
\[
\min_{X, r(X) \leq k} \| YX - B \|_2 \leq (1 - \alpha)^{-1} \Delta_k(A) + \| N_1([V]_k^T \Phi)^\dagger[V]_k^T \|_2 + \| S^\dagger N_2 \|_2. \tag{13}
\]

The above expression relates $\min_{X, r(X) \leq k} \| YX - B \|$ with $(1 - \alpha)^{-1} \Delta_k(A)$, $\| N_1([V]_k^T \Phi)^\dagger[V]_k^T \|_2$, and $\| S^\dagger N_2 \|_2$. Combining equation (13) and equation (1), we have with probability $1 - \delta$ over $\Phi \sim \mathcal{D}_R$,
\[
\min_{X, r(X) \leq k} \| UX - B \|_2 \leq (1 - \alpha)^{-1} \Delta_k(A) + \| N_1([V]_k^T \Phi)^\dagger[V]_k^T \|_2 + \| S^\dagger N_2 \|_2. \tag{14}
\]

This completes the proof of the Claim 13.

Note that the terms $\| N_1([V]_k^T \Phi)^\dagger[V]_k^T \|_2$ and $\| S^\dagger N_2 \|_2$ are due to the addition of Gaussian matrices to preserve privacy. We later bound them in Lemma 15. We next bound that multiplying $S$ from the right does not change our bound by a lot. More concretely, we prove the following claim.

**Claim 14.** Let $U, B, A, S, N_1,$ and $N_2$ be as above, and let $\bar{X}_k = \arg\min_{X, r(X) = k} \| S(UX - B) \|_2$. Let $\mathcal{D}_A$ be a distribution that satisfies $(\alpha^2, \delta)$-JLP. Then with probability $1 - 3\delta$ over $S \sim \mathcal{D}_A$ and $\Phi \sim \mathcal{D}_R$, we have
\[
\frac{1 - \alpha}{1 + \alpha} \| (U \bar{X}_k - B) \|_2 \leq \frac{1}{(1 - \alpha)} \Delta_k(A) + \| N_2 S^\dagger \|_2 + \| N_1([V]_k^T \Phi)^\dagger[V]_k^T \|_2. \tag{15}
\]

**Proof.** First note that the choice of $v$ allows $S$ to satisfy $(\alpha^2, \delta)$-JLP [11]. Let $\bar{X}_k = \arg\min_{X, r(X) \leq k} \| UX - B \|_2$. We want to bound $\| U\bar{X}_k - B \|_2$ in terms of $\| U\bar{X}_k - B \|_2$. We give an even stronger result that states that, simultaneously for all $X$, $\| S(UX - B) \|_2$ is bounded by a multiplicative factor of $\| UX - B \|_2$.

For this consider the following optimization problem: $\min_X \| UX - B \|_2$. Since $X$ can have rank at most the rank of $B$, using Theorem 44 we have that
\[
\hat{X} = U^T [U^T B] = U^T B
\]
is a solution to $\min_X \| UX - B \|_2$. This implies that
\[
U^T (U\hat{X} - B) = U^T (U^T B - B) = 0. \tag{16}
\]

Therefore, using sub-additivity and sub-multiplicativity of the spectral norm and $(\alpha^2, \delta)$-JLP of $\mathcal{D}_A$, we have with probability $1 - \delta$,
\[
\| S(UX - B) \|_2^2 = \| SU(X - \hat{X}) + S(U\hat{X} - B) \|_2^2
\leq 2(\| SU(X - \hat{X}) \|_2^2 + \| S(U\hat{X} - B) \|_2^2)
\leq 2(1 + \alpha^2)^2 \left( \| U(X - \hat{X}) \|_2^2 + \| (U\hat{X} - B) \|_2^2 \right)
\leq 2(1 + \alpha^2)^2 \left( \| U(X - \hat{X}) \|_2^2 + \| (UX - B) \|_2^2 \right)
= 2(1 + \alpha^2)^2 \left(\| U^T U(X - \hat{X}) \right)
+ 2(1 + \alpha^2)^2 \left( U^T (U\hat{X} - B) \right) + \| (UX - B) \|_2^2
\leq (4 + 12\alpha^2) \| (UX - B) \|_2^2,
\]

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where the second equality follows from equation (16), the second inequality follows from Lemma 12, the third inequality follows from the definition of $\tilde{X}$, and the last inequality follows because $\alpha < 1$. This in particular implies that

$$\|S(UX - B)\|_2^2 - 4\|(UX - B)\|_2^2 \leq 12\alpha^2\|(UX - B)\|_2^2.$$  

Taking square root and the fact that

$$|a - b| \leq \sqrt{(a + b)(a - b)} = \sqrt{a^2 - b^2}$$

for positive integers $a$ and $b$, this gives the inequality

$$\|S(UX - B)\|_2 - 2\|(UX - B)\|_2 \leq 2\sqrt{3}\alpha\|(UX - B)\|_2$$

By rescaling the value of $\alpha$, we have the following:

$$2(1 - \alpha)\|(UX - B)\|_2 \leq \|S(UX - B)\|_2 \leq 2(1 + \alpha)\|(UX - B)\|_2 \quad (17)$$

We now return to the proof of Claim 14. Using the definitions of $\tilde{X}_k$ and $\hat{X}_k$ along with equation (17), we have the following set of inequalities:

$$\min_{X, r(X) = k} \|UX - B\|_2 = \|UX_k - B\|_2 \geq (2 + 2\alpha)^{-1} \|S(UX_k - B)\|_2 \geq (2 + 2\alpha)^{-1} \min_{X, r(X) \leq k} \|S(UX - B)\|_2 = (2 + 2\alpha)^{-1} \|S(UX_k - B)\|_2 \geq \frac{1 - \alpha}{1 + \alpha} \|(UX_k - B)\|_2. \quad (18)$$

Combining equation (18) with equation (14), we have with probability $1 - 3\delta$ over $\Phi \sim D_R$ and $S \sim D_A$,

$$\frac{1 - \alpha}{1 + \alpha} \|(UX_k - B)\|_2 \leq (1 - \alpha)^{-1} \Delta_k(A) + \|S^T N_2\|_2 + \|N_1([V]_k^T \Phi)^T |V|_k^T\|_2. \quad (19)$$

This completes the proof of Claim 14.

We can now complete the proof of Lemma 12. Since $B = A + S^T N_2$, using sub-additivity of norm,

$$\|UX_k - A\|_2 - \|S^T N_2\|_2 \leq \|UX_k - B\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \|A - [A]_k\|_2 + \frac{1 + \alpha}{1 - \alpha} \|S^T N_2\|_2 + \|N_1([V]_k^T \Phi)^T |V|_k^T\|_2.$$  

This implies that

$$\|UX_k - A\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \|A - [A]_k\|_2 + \frac{2\|S^T N_2\|_2}{1 - \alpha} + \frac{1 + \alpha}{1 - \alpha} \|N_1([V]_k^T \Phi)^T |V|_k^T\|_2.$$  

Setting $L := SU$, $R = I$, and $O := Z$ Theorem 44 gives that $\bar{X} = V\Sigma^U \hat{U}^T \hat{U}^{T}\bar{Z}$. Using this value of $\bar{X}$ completes the proof of Lemma 12.
Now all that remains to complete the proof of correctness of Theorem 11 are the two expressions in the additive term: \( \|N_1([V]_k^T \Phi)^\dagger [V]_k^T \|_2 \) and \( \|S^\dagger N_2\|_2 \). We next bound them in Lemma 15.

**Lemma 15.** Let \( \rho = \sqrt{(1 + \alpha) \log(1/\delta) / \varepsilon} \) and \( N_1 \sim \mathcal{N}(0, \rho^2)^{m \times t} \). Then with probability \( 1 - 2\delta \) over \( \Phi \sim \mathcal{D}_R \), \( \|N_1([V]_k^T \Phi)^\dagger [V]_k^T \|_2 \) is \( O(\rho(1 - \alpha)^{-1/2}(\sqrt{k} + \sqrt{m})) \) and \( \|N_2 S\|_2 \) is \( O(\rho(\sqrt{v} + \sqrt{m})) \).

**Proof.** We first bound \( \|N_1([V]_k^T \Phi)^\dagger [V]_k^T \|_2 \). Let \( C = N_1([V]_k^T \Phi)^\dagger [V]_k^T \). Then \( C \Phi = N_1([V]_k^T \Phi)^\dagger [V]_k^T \Phi \). Now \( ([V]_k \Phi)^\dagger [V]_k \Phi \) is a projection unto a random subspace of dimension \( k \). Since every entries of \( N_1 \) is picked i.i.d. from \( \mathcal{N}(0, \rho^2) \), \( C \Phi = N_1([V]_k^T \Phi)^\dagger [V]_k^T \Phi = \begin{pmatrix} \tilde{N}_1 & 0 \end{pmatrix} \), where \( \tilde{N}_1 \) is an \( m \times k \) matrix with every entries picked i.i.d. from \( \mathcal{N}(0, \rho^2) \). Using Rudelson and Vershynin [36] Proposition 2.4, we have \( \|C \Phi\|_2 = O(\rho(\sqrt{m} + \sqrt{k})) \) with probability 99/100. For our choices of \( t \), Sarlos [37] showed that, for any matrix \( D \), all the singular values of \( D \Phi \) lies in between \( (1 \pm \alpha) \) of the singular values of \( D \). This in particular implies that \( \|C \Phi\|_2 \geq (1 - \alpha)^{1/2}\|C\|_2 \), and, therefore, \( \|C\|_2 \leq (1 - \alpha)^{-1/2}O(\rho(\sqrt{k} + \sqrt{m})) \). This completes the proof of Lemma 15. For the second part, if we instantiate \( S \) with a subsampled Hadamard matrices, it is known that it satisfies \( (\alpha^2, \delta) \)-JLP for the values of \( v \). Applying Lemma 34, we have \( \|S^\dagger N_2\|_2 = \|N_2\|_2 = O(\rho(\sqrt{v} + \sqrt{m})) \) using Rudelson and Vershynin [16] with probability 99/100. This completes the proof of Lemma 15. \( \square \)

Combining Lemma 12 and Lemma 15 completes the correctness proof of Theorem 11. We next prove the privacy part of Theorem 11.

**Lemma 16.** If \( \rho = \sqrt{(1 + \alpha) \log(1/\delta) / \varepsilon} \), then publishing \( Y_r \) and \( Z \) preserves \((2\varepsilon, 4\delta)\)-differential privacy.

**Proof.** First note that Clarkson and Woodruff [11] showed that for the choice of \( t \) and \( v \), with probability \( 1 - \delta \), we have \( \|SD\|_F^2 \leq (1 + \alpha)\|D\|_F^2 \) and \( \|D \Phi \|_F^2 \leq (1 + \alpha)\|D\|_F^2 \) for all \( D \). Let \( A \) and \( A' \) be two neighboring matrices such that \( E = A - A' \). Then \( \|SE\|_F^2 \leq (1 + \alpha)\|E\|_F^2 \leq (1 + \alpha)\|E\|_F^2 \). Publishing \( Z \) preserves \((\varepsilon, \delta)\)-differential privacy follows from considering the vector form of the matrix \( SA \) and \( N_2 \) and applying Theorem 37. Similarly, we use Theorem 37 and the fact that, for any matrix \( C \) of appropriate dimension, \( \|C \Phi\|_2^2 \leq (1 + \alpha)\|C\|_2^2 \), to prove that publishing \( A \Phi + N_1 \) preserves differential privacy. \( \square \)

Combining Lemma 12, Lemma 15, and Lemma 16 gives Theorem 11. \( \square \)

In the most natural scenarios, \( k \ll \max\{m, n\} \) and \( \alpha \) is a small constant. Then we have \( (1 - \alpha)^{-1} \approx (1 + \alpha) \). If we scale the value of \( \alpha \) appropriately, then we have the following corollary.

**Corollary 17.** Under the assumptions of Theorem 17 and \( \alpha \in (0, 1) \) a small constant, given an \( m \times n \) matrix \( A \) in the turnstile update model, the algorithm **Spectral-LRF**, presented in Figure 1, is \((\varepsilon, \delta)\)-differentially private under \( \mathbb{P}_{\text{Priv}} \) and outputs a \( k \)-rank factorization \( U_k, \Sigma_k, \) and \( V_k^T \) (with \( M_k = U_k \Sigma_k V_k^T \)), such that, with probability \( 9/10 \) over the coin tosses of **Spectral-LRF**,

\[
\|A - M_k\|_2 \leq (1 + \alpha)\Delta_k(A) + O \left( \left( \sqrt{m} + \sqrt{n} \right) \varepsilon^{-1} \sqrt{\log(1/\delta)} \right),
\]

As another corollary of Theorem 11 when \( \varepsilon \to \infty \), we get a non-private algorithm that computes low-rank factorization under turnstile update model with respect to the spectral norm.
Improving the Space Bound in the Turnstile Model

In this section, we improve the space bound of Spectral-LRF under Priv$_2$ in the turnstile update model. This privacy level is also stronger than that considered by previous ($\varepsilon, \delta$)-differentially private algorithms, but weaker than that in Section 3. Our algorithm formalizes the ideas mentioned in Section 1.2. In particular, we show that we can simultaneously compute two sets of $k$-orthonormal basis vectors that approximates the top-$k$ left and top-$k$ right singular vectors of the private matrix in the turnstile update model.

We have to be careful with how we introduce noise to preserve privacy. If we use output perturbation to compute the sketches $Y_c = A\Phi + N$ and $Y_r = \Psi A + N'$, then one of the error terms is $\|N([A]_k^{\top}\Phi)\|_2$. This term can cause the additive error to be $\Omega(n)$ if the top singular values of $A$ is $1/n$. If we only use input perturbation of the sketches followed by a multiplication by Gaussian matrices as in[5, 52, 53], the multivariate Gaussian distribution corresponding to one of the sketches is not defined as one of the sketches is not a full rank matrix. Moreover, we cannot guarantee that the kernel space are the same for neighboring matrices $A$ and $A'$. Therefore, the privacy proof would not follow. Our algorithm uses both the input perturbation with a careful choice of parameters and output perturbation to the other two sketches. This preserves privacy as well as keep the additive error bounded. We show the following theorem.

**Theorem 19.** Let $m, n, k \in \mathbb{N}$ and $\varepsilon, \delta$ be the input parameters. Let $\alpha \in (0, 1)$ be an arbitrary constant, $\kappa = (1 + \alpha)/(1 - \alpha)$, $\sigma_{\text{min}} = 16\ln(1/\delta)\sqrt{\kappa \ln(4/\delta)/\varepsilon}$, $\rho_1 = \sqrt{(1 + \alpha)\ln(1/\delta)/\varepsilon}$, and $\rho_2 = \rho_1\sqrt{1 + \alpha}$. Given an $m \times n$ matrix $A$ in the turnstile update model, the algorithm LOW-Space-LRF, presented in Figure 2, satisfies the following properties:

1. Let $\eta = \max\{k^2, 1/\alpha\}$. Then Spectral-LRF uses $O((m+n)\eta\alpha^{-1}\log(k/\alpha)\log(1/\delta))$ space.

2. Spectral-LRF is $(\varepsilon, \delta)$-differentially private under Priv$_2$.

3. With probability at least 9/10 over the coin tosses of Spectral-LRF,

$$\|M_k - (A \ 0)\|_2 \leq \frac{(1 + \alpha)^2}{(1 - \alpha)^4} \Delta_k(A) + O \left( \frac{\kappa \alpha^{-1} \ln(1/\delta) + \sqrt{m + \sqrt{n}}}{\varepsilon(1 - \alpha)^3} \right) \sqrt{\ln(1/\delta)}.$$

For the sake of simplicity, we present the algorithm when $m \leq n$. When $m > n$, we change the algorithm in Figure 2 as follows: (i) use $A^T$ instead of $A$, (ii) use $(0 \ \sigma_{\text{nn}})^T$ instead of $(0 \ \sigma_{\text{mm}})$ to initialize $\hat{Y}_c, \hat{Y}_r$ and $\hat{Z}$, (iii) the matrices $S, T, \Phi$, and $\Psi$ are picked using the same distribution but with appropriate dimensions to allow matrix multiplication.

The space complexity of the algorithm is straightforward from the choice of $t$ and $v$. We first prove part 3 of Theorem 19. Our proof consists of few main lemmas (Lemma 20, Lemma 21, Lemma 25, and Lemma 24). Lemma 20 bounds $\|M_k - (A \ 0)\|_2$ by $\|M_k - \hat{A}\|_2$ and a fixed additive term. Lemma 21 bounds $\|M_k - \hat{A}\|_2$ by $\|\hat{A} - \hat{\hat{A}}\|_2$ and two additive terms. Lemma 25
Let $\Lambda$ := max $\{k^2/\alpha\}$, $\rho_1 := (1 + \alpha)\ln(1/\delta)/\varepsilon$, $t = O(\eta \alpha^{-1} \log(k/\alpha) \log(1/\delta))$, $v = O(\eta \alpha^{-3} \log(t/\alpha) \log(1/\delta))$, $\kappa = (1 + \alpha)/(1 - \alpha)$, and $\sigma_{\min} = 16 \ln(1/\delta) / (\varepsilon k)$. Let $\Phi_x = (0 \sigma_{\min} I_m)$. Sample $N_1 \sim N(0, \rho^2_2 I_{t(x(m+n)})$ and $N_2 \sim N(0, \rho^2_2 I_{v^x v^x})$. Let $\Phi$, $\Psi \in \mathbb{R}^{t \times t}$ be such that $\Phi, \Psi \sim D_R$ satisfies the statement of Lemma [14] for the error parameter $\sqrt{k}/\delta$. Let $S \in \mathbb{R}^{v \times m}$, $T \in \mathbb{R}^{v \times t}$ such that $S, T$ satisfies $(\alpha^2, \delta)$-JLP. Sample $\Omega \sim N(0, 1)^{m \times t}$ and $\Phi = \frac{1}{\sqrt{v}} \Phi \Omega$. Initialize $\hat{Y}_c = A \hat{A} \in \mathbb{R}^{m \times t}$, $\hat{Y}_r = \Psi \hat{A} \in \mathbb{R}^{t \times (m+n)}$, and $\hat{Z} = SAT \in \mathbb{R}^{v \times v}$.

Update rule and computing the factorization. Suppose at time $\tau$, the stream is $(i, j, s, r)$, where $(i, j, s, r) \in [m] \times [n]$. Let $A_r$ be an $m \times (m+n)$ matrix with the only non-zero entry $s_r$ in the position $(i, j)$. Update the matrices by the following rule: $\hat{Y}_c \leftarrow \hat{Y}_c + A_r \hat{\Phi}, \hat{Y}_r \leftarrow \hat{Y}_r + \Psi A_r$, and $\hat{Z} \leftarrow \hat{Z} + SA_r T$.

Once the matrix is streamed, we follow the following steps.

1. Compute a matrix $U \in \mathbb{R}^{m \times t}$ whose columns are an orthonormal basis for the column space of $\hat{Y}_c$.
2. Compute a matrix $V \in \mathbb{R}^{(m+n) \times t}$ whose rows are an orthonormal basis for the row space of $\hat{Y}_r + N_1$.
3. Let $\hat{U}_c \hat{\Sigma}_c \hat{V}_c^\top$ be the SVD of $SU \in \mathbb{R}^{v \times t}$. Let $\hat{U}_r \hat{\Sigma}_r \hat{V}_r^\top$ be the SVD of $T^\top V \in \mathbb{R}^{v \times t}$.
4. Let $U' \hat{\Sigma}' V'^\top$ be the SVD of $\hat{V}_c \hat{\Sigma}_c \hat{V}_c^\top \hat{U}_c \hat{\Sigma}_c \hat{V}_c^\top (\hat{Z} + N_2) \hat{V}_r \hat{\Sigma}_r \hat{V}_r^\top \hat{U}_r \hat{\Sigma}_r \hat{V}_r^\top$.
5. Output $M_k = U_k \Sigma_k V_k^\top$.

Figure 2: Differentially private Low-rank Factorization With Improved Space (LOW-SPACE-LRF)

bounds $\|\hat{A} - [\hat{A}]_k\|_2$ in the terms of $\|A - [A]_k\|_2$ and a fixed additive term. Lemma [24] bounds the two additive terms in Lemma [21] completing the proof.

For all our correctness arguments, we make a standard assumption that $\delta \ll 1/100$. Let $Z := \hat{Z} + N_2$, $Y_c := \hat{Y}_c$, $Y_r := \hat{Y}_r + N_1$, and $B = \hat{A} + S^\top N_2 T^\top$, where all the variables on the right hand side of the expressions are as in Figure 2.

We start by proving a bound on $\|M_k - A\|_2$ by $\|M_k - \hat{A}\|_2$ and a small additive term. The following lemma provides such a bound.

**Lemma 20.** Let $A$ be an $m \times n$ input matrix, and let $\hat{A} = (A \sigma_{\min} I_m)$ for $\sigma_{\min}$ defined in Figure 2. Let $M_k = U_k \Sigma_k V_k^\top$, where $U_k, \Sigma_k, V_k$ is the factorization outputted by LOW-SPACE-LRF. Then

$\|M_k - (A 0)\|_2 \leq \|M_k - \hat{A}\|_2 + \sigma_{\min}$.

**Proof.** Since $\hat{A} = (A \sigma_{\min} I_m)$, the lemma is immediate from the following.

$\|M_k - (A 0)\|_2 - \sigma_{\min} \|\sigma_{\min} I_m\|_2 \leq \|M_k - (A 0) - (0 \sigma_{\min} I_m)\|_2 = \|M_k - \hat{A}\|_2$,

where the first inequality follows from the sub-additivity of the spectral norm.

We next prove a bound on $\|M_k - \hat{A}\|_2$ by $\|\hat{A} - [\hat{A}]_k\|_2$ and some additive terms.
Lemma 21. Let $M_k := U_k \Sigma_k V_k^T$, where $U_k, \Sigma_k, V_k$ is the factorization outputted by LOW-SPACE-LRF presented in Figure 2. Then with probability $1 - O(\delta)$ over $\Phi, \Psi \sim \mathcal{D}_R$ and $S, T \sim \mathcal{D}_A$,

$$\|M_k - \hat{A}\|_2 \leq \frac{(1 + \alpha)^2}{(1 - \alpha)^2} \|\hat{A} - [\hat{A}]_k\|_2 + \frac{2(1 + \alpha^2)\|N_2 S^\dagger\|_2}{(1 - \alpha)^2} + \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 \|N_1 ([V]_k^T \Phi)^\dagger [V]_k^T\|_2.$$

Proof. Our proof consists of two main claims (Claim 22 and Claim 23). Claim 22 shows that $U$ and $V$ are “faithful” representations of the top-$k$ left and right singular vectors of $[A]_k$ and Claim 23 shows that $S$ and $T$ satisfies the required properties with the choice of error parameters.

Claim 22. Let $A$ be the input matrix. Let $\Phi \sim \mathcal{D}_R$ be a random matrix that satisfies Lemma 10 with error parameters $(\sqrt{\alpha}/k, \delta)$ with respect to $[V]_k$ and $\Psi \sim \mathcal{D}_R$ satisfies Lemma 14 with error parameters $(\sqrt{\alpha}/k, \delta)$ with respect to $\tilde{A} \Phi([V]_k^T \Phi)^\dagger$. Then with probability $1 - 2\delta$ over the choices of $\Phi$ and $\Psi$,

$$\min_{X, r(X) \leq k} \|UXV^T - B\|_2 \leq (1 - \alpha)^{-2}\|\hat{A} - [\hat{A}]_k\|_2 + \|\tilde{A} \Phi D N_1\|_2 + \|S^\dagger N_2 T^\dagger\|_2. \quad (20)$$

Proof. Let us denote by $[U]_k [\Sigma]_k [V]_k^T$ the SVD of $[\hat{A}]_k$. By construction, $U$ is an orthonormal basis for the column space of $Y_c$ and $V$ is an orthonormal basis for the row space of $Y_r$, i.e.,

$$\min_{X, r(X) \leq k} \|UXV^T - B\|_2 \leq \min_{X, r(X) \leq k} \|Y_c X Y_r - B\|_2. \quad (20)$$

We also note that the choice of $t$ allows $\Phi$ and $\Psi$ to satisfy Lemma 10 with error parameters $(\sqrt{\alpha}/k, \delta)$ [11]. Therefore, in order to show that the matrix $U$ approximates the top-$k$ left singular vectors of the matrix $\hat{A}$ and $V$ approximates the top-$k$ right singular vectors of the matrix $\hat{A}$, we need to find a matrix $X$ such that $\|Y_c X Y_r - \tilde{A}\|_2$ is bounded by $(1 - \alpha)^{-2}\|\hat{A} - [\hat{A}]_k\|_2$ up to some additive terms. Towards this goal, first consider the following two optimization problems:

$$\min_X \|\Phi^T ([V]_k X - \hat{A}^T)\|_2 \quad \text{and} \quad \min_X \|([V]_k X - \hat{A}^T)\|_2. \quad (21)$$

with

$$\tilde{X} := \arg\min_X \|\Phi^T ([V]_k X - \hat{A}^T)\|_2$$

$$\bar{X} := \arg\min_X \|([V]_k X - \hat{A}^T)\|_2$$

being one of the solutions of equation (21).

Recall that $\Phi$ satisfies Lemma 10 with respect to $[V]_k$. Therefore, if we substitute $A$ by $\hat{A}$ in equation (3), then $\|[V]_k X - \hat{A}^T\|_2 \leq (1 - \alpha)^{-1}\|([V]_k \bar{X} - \hat{A}^T)\|_2$, where $\bar{X} = (\Phi^T [V]_k)^\dagger \Phi^T \hat{A}^T$. This is the same as

$$\|\tilde{A} \Phi ([V]_k^T \Phi)^\dagger [V]_k^T - \hat{A}\|_2 = \|\bar{X}^T [V]_k^T - \hat{A}\|_2 \leq (1 - \alpha)^{-1}\|([V]_k \bar{X} - \hat{A}^T)\|_2.$$

Define a matrix $C := \tilde{A} \Phi ([V]_k^T \Phi)^\dagger$. The matrix $C$ has a rank at most $k$ and the above equation can be rewritten as

$$\|C [V]_k^T - \hat{A}\|_2 \leq (1 - \alpha)^{-1}\|([V]_k \bar{X} - \hat{A}^T)\|_2. \quad (22)$$

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Now consider the following optimization problem:

\[
\min \| \Psi(CX - \hat{A}) \|_2 \quad \text{and} \quad \min \| (CX - \hat{A}) \|_2.
\]

(23)

Setting \( L := \Phi C \) and \( A = \hat{A} \) Theorem 14 and a singular value decomposition of \( \Psi C \), we have

\[
\hat{X}' := (\Psi C)^t \Psi \hat{A} = \arg\min \| \Psi(CX - \hat{A}^T) \|_2.
\]

Recall that \( \Psi \) satisfies Lemma 10 with respect to \( C \). Therefore, using equation (6) on problems (23), we have

\[
\| CX' - \hat{A} \|_2 \leq (1 - \alpha)^{-1} \min \| CX - \hat{A} \|_2
\]

\[
\leq (1 - \alpha)^{-1} \| C[V]_k^T - \hat{A} \|_2
\]

\[
\leq (1 - \alpha)^{-2} \| [(V][_k \hat{X} - \hat{A}^T) \|_2
\]

\[
= (1 - \alpha)^{-2} \min \| [(V][_k X - \hat{A}^T) \|_2
\]

\[
\leq (1 - \alpha)^{-2} \| [U][_k \Sigma [V]_k^T - \hat{A} \|_2
\]

\[
= (1 - \alpha)^{-2} \| [\hat{A}]_k - \hat{A} \|_2.
\]

(24)

In the above, the third inequality follows from using equation (22) and fourth follows from the definition of \( \hat{X} \).

Let \( D = ([V]_k \Phi)^t (\Psi \Phi ([V]_k \Phi)^t)^t \), i.e., \( CX' = \hat{A} \Phi D \Psi \hat{A} \). Since \( ([V]_k \Phi)^t [V]_k \) has rank at most \( k \) and \( Y = \Psi \hat{A} + N_1 \), with probability \( 1 - \delta \) over \( \Phi \sim \mathcal{D}_R \),

\[
\min_{X, r(X) \leq k} \| Y_c X Y_r - B \|_2 \leq \| \hat{A} \Phi D \Psi \hat{A} + \hat{A} \Phi D N_1 - B \|_2
\]

\[
= \| \hat{A} \Phi D \Psi \hat{A} + \hat{A} \Phi D N_1 - \hat{A} - S^t N_2 T^t \|_2
\]

\[
\leq \| \hat{A} \Phi D \Psi \hat{A} - \hat{A} \|_2 + \| \hat{A} \Phi D N_1 \|_2 + \| S^t N_2 T^t \|_2
\]

\[
\leq \| CX' - \hat{A} \|_2 + \| \hat{A} \Phi D N_1 \|_2 + \| S^t N_2 T^t \|_2
\]

(25)

Combining equation (24) and equation (25), we have with probability \( 1 - 2\delta \) over the choice of \( \Phi \) and \( \Psi \),

\[
\min_{X, r(X) \leq k} \| Y_c X Y_r - B \|_2 \leq (1 - \alpha)^{-2} \| \hat{A} - \hat{A} \|_2 + \| \hat{A} \Phi D N_1 \|_2 + \| S^t N_2 T^t \|_2.
\]

(26)

Combining equation (26) and equation (20), we have with probability \( 1 - 3\delta \) over \( \Phi \) and \( \Psi \),

\[
\min_{X, r(X) \leq k} \| UXV^T - B \|_2 \leq (1 - \alpha)^{-2} \| \hat{A} - \hat{A} \|_2 + \| \hat{A} \Phi D N_1 \|_2 + \| S^t N_2 T^t \|_2.
\]

(27)

This completes the proof of Claim 22.

\[\square\]

We would have been done if we had not generated a noisy version of the sketch \( S \hat{A} T \). The next claim bounds the effect of generating this sketch.
Claim 23. Let $U, B, \hat{A}, S, N_1$, and $N_2$ be as above, and let $\tilde{X}_k = \text{argmin}_{X, r(X) = k} \|S(UXV^T - B)\|_2$. Let $D_A$ be a distribution that satisfies $(\alpha^2, \delta)$-JLP. Then with probability $1 - 6\delta$ over $S, T \sim D_A$ and $\Phi, \Psi \sim D_R$, we have

\[
\left(\frac{1 - \alpha}{1 + \alpha}\right)^2 \|U\tilde{X}_k V^T - B\|_2 \leq (1 - \alpha)^{-2} \|\hat{A} - [\hat{A}]_k\|_2 + \|S^\dagger N_2^T\|_2 + \|\hat{A} \Phi D N_1\|_2.
\]

Proof. Note that the choice of $v$ allows $S$ and $T$ to satisfy $(\alpha^2, \delta)$-JLP.

Let $\tilde{X}_k = \text{argmin}_{X, r(X) \leq k} \|U(XV^T - B)\|_2$. We want to bound $\|U\tilde{X}_k V^T - B\|_2$ in terms of $\|U\tilde{X}_k V^T - B\|_2$. Using the definitions of $\tilde{X}_k$ and $\hat{X}_k$ along with equation (17), we have

\[
\min_{X, r(X) = k} \|U XV^T - B\|_2 = \|U \tilde{X}_k V^T - B\|_2
\]

\[
\geq (2 + 2\alpha)^{-1} \|S(U \tilde{X}_k V^T - B)\|_2
\]

\[
\geq (2 + 2\alpha)^{-2} \|S(U \tilde{X}_k V^T - B) T\|_2
\]

\[
\geq (2 + 2\alpha)^{-2} \min_{X, r(X) \leq k} \|S(U XV^T - B) T\|_2
\]

\[
= (2 + 2\alpha)^{-1} \|S(U \tilde{X}_k V^T - B) T\|_2
\]

\[
\geq \left(\frac{1 - \alpha}{1 + \alpha}\right)^2 \|U \tilde{X}_k V^T - B\|_2.
\]

Combining equation (28) with equation (27), we have with probability $1 - 3\delta$ over $\Phi \sim D_R$ and $S \sim D_A$,

\[
\left(\frac{1 - \alpha}{1 + \alpha}\right)^2 \|U \tilde{X}_k V^T - B\|_2 \leq (1 - \alpha)^{-2} \|\hat{A} - [\hat{A}]_k\|_2 + \|S^\dagger N_2^T\|_2 + \|\hat{A} \Phi D N_1\|_2.
\]

This completes the proof of Claim 23. 

We can now complete the proof of Lemma 21. Since $B = \hat{A} + S^\dagger N_2^T$

\[
\|U \tilde{X}_k V^T - \hat{A}\|_2 - \|S^\dagger N_2^T\|_2 \leq \|U \tilde{X}_k V^T - B\|_2
\]

\[
\leq \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 \|\hat{A} - [\hat{A}]_k\|_2 + \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 \|S^\dagger N_2^T\|_2 + \|\hat{A} \Phi D N_1\|_2.
\]

This implies that

\[
\|U \tilde{X}_k V^T - \hat{A}\|_2 \leq \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 \|\hat{A} - [\hat{A}]_k\|_2 + \left(\frac{1 + \alpha^2}{(1 - \alpha)^2}\right) \|S^\dagger N_2^T\|_2 + \left(\frac{1 + \alpha}{1 - \alpha}\right)^2 \|\hat{A} \Phi D N_1\|_2.
\]

Setting $O := Z_s$, $L := SU$ and $R := V^T T$ in the statement of Theorem 43, we have

\[
\tilde{X}_k = \hat{V}_s \hat{\Sigma}_l \hat{U}_s^T \hat{U}_s^T [U_s \hat{U}_s^T Z V_t \hat{V}_t^T] [V_t \hat{V}_t^T] \tilde{X}_k \hat{\Sigma}_l \hat{U}_t^T
\]

is one of the candidate solutions to

\[
\text{argmin}_{X, r(X) = k} \|S(U XV^T - B)\|_2.
\]

Using this value of $\tilde{X}_k$ completes the proof of Lemma 21. 

\[
\square
\]
Lemma 21 contains two additive terms \( \|S^TV_2^T\|_2 \) and \( \|\hat{A}\Phi DN_1\|_2 \). We next bound both these terms.

**Lemma 24.** Let \( \rho = \sqrt{(1 + \alpha)\log(1/\delta)/\varepsilon} \) and \( N_1 \) \( \sim N(0, \rho^2)^{(m+n)\times t} \). Then with probability \( 1 - 2\delta \) over \( \Phi \sim D_R \), \( \|\hat{A}\Phi DN_1\|_2 = O(\rho(1 - \alpha)^{-1/2}(\sqrt{k} + \sqrt{m + n})) \) and \( \|S^TV_2^T\|_2 = O(\rho^2\sqrt{\varepsilon}) \).

**Proof.** Let \( C = \hat{A}\Phi DN_1 \). Then

\[
\Psi C = \Psi \hat{A}\Phi (|V|^T|\Phi|)(\Psi \hat{A}\Phi (|V|^T|\Phi|))^T N_1.
\]

Now \( \Psi \hat{A}\Phi D \) is a projection unto a random subspace of dimension at most \( k \). Since every entries of \( N_1 \) is picked i.i.d. from \( N(0, \rho^2) \), \( \Psi C \sim \tilde{N}_1 \), where \( \tilde{N}_1 \) is an \( m \times k \) matrix with every entries picked i.i.d. from \( N(0, \rho^2) \). Using Rudelson and Vershynin [46, Proposition 2.4], we have \( \|\Psi C\|_2 = O(\rho(\sqrt{m + n} + \sqrt{k})) \) with probability 99/100. For our choices of \( t \), Sarlos [47] showed that, for any matrix \( E \), all the singular values of \( \Psi E \) lies in between \( (1 \pm \alpha) \) of the singular values of \( E \). This in particular implies that \( \|\Psi C\|_2 \geq (1 - \alpha)^{1/2}\|C\|_2 \). That is, \( \|C\|_2 = (1 - \alpha)^{-1/2}O(\rho_1(\sqrt{k} + \sqrt{m + n})) \).

For the second part, if we instantiate \( S \) with a subsampled Hadamard matrices, it is known that it satisfies \( (\alpha^2, \delta) \)-JLP for the values of \( \rho_2 \) and \( \delta \). Applying Lemma 24 twice, we have \( \|S^TV_2^T\|_2 = \|N_2\|_2 = O(\rho_2\sqrt{\varepsilon}) \) using Rudelson and Vershynin [48] with probability 99/100. This completes the proof of Lemma 24.

We finally compute an upper bound on \( \|\hat{A} - \hat{[A]}_k\|_2 \) in terms of \( \|A - [A]_k\|_2 \).

**Lemma 25.** Let \( d \) be the maximum of the rank of \( A \) and \( \hat{A} \). Let \( \sigma_1, \cdots, \sigma_d \) be the singular values of \( A \) and \( \hat{\sigma}_1, \cdots, \hat{\sigma}_d \) be the singular values of \( \hat{A} \). Then \( \|\hat{A} - \hat{[A]}_k\|_2 \leq \|A - [A]_k\|_2 + \sigma_{\min} \).

**Proof.** We first prove that \( |\sigma_i - \hat{\sigma}_i| \leq \sigma_{\min} \) for all \( 1 \leq i \leq d \). We can write \( \hat{A} = \begin{pmatrix} A & 0 \\ 0 & \sigma_{\min}I_m \end{pmatrix} \). By construction, all the singular values of \( \begin{pmatrix} 0 & \sigma_{\min}I_m \end{pmatrix} \) are \( \sigma_{\min} \); therefore, Theorem 39 implies that \( |\sigma_i - \hat{\sigma}_i| \leq \sigma_{\min} \) for all \( 1 \leq i \leq d \). In particular, since \( \|A - [A]_k\|_2 = \hat{\sigma}_{k+1} \) and \( \|A - [A]_k\|_2 = \sigma_{k+1} \), this implies that \( \|\sigma_{k+1} - \hat{\sigma}_{k+1}\| \leq \sigma_{\min} \). The lemma follows.

Combining Lemma 24, Lemma 21, Lemma 25, and Lemma 24 completes the proof of part 3 of Theorem 19 after noting that \( \alpha \in (0, 1) \). We next prove part 2 of Theorem 19. This is essentially the same as in Upadhay [54]. We include it for the sake of completion.

**Lemma 26.** If \( \sigma_{\min}, \rho_1 \) and \( \rho_2 \) be as in Theorem 19, then the algorithm presented in Figure 2 is \( (3\varepsilon, 6\delta) \)-differentially private.

We prove the lemma when \( m \leq n \). The case for \( m \geq n \) is analogous after inverting the roles of \( \hat{\Phi} \) and \( \hat{\Psi} \). Let \( A \) and \( A' \) be two neighboring matrices, i.e., \( E = A - A' = uv^T \). Then \( A \) and \( A' \), constructed by Low-Space-LRF, has the following property: \( A' = A + (E - 0) \).

**Claim 27.** If \( \rho_1 = \sqrt{(1 + \alpha)\log(1/\delta)} / \varepsilon \), then publishing \( Y_r \) preserves \( (\varepsilon, 2\delta) \)-differential privacy. If \( \rho_2 = (1 + \alpha)\sqrt{\log(1/\delta)} / \varepsilon \), then publishing \( Z \) preserves \( (\varepsilon, 2\delta) \)-differential privacy.

**Proof.** First note that by the choice of \( t \) and \( v \), using Clarkson and Woodruff [11] gives that \( \|SD\|_F^2 \leq (1 + \alpha)^2\|D\|_F^2 \) and \( \|\hat{\Psi} D\|_F^2 \leq (1 + \alpha)^2\|D\|_F^2 \) for all \( D \) with probability \( 1 - \delta \). Let \( A \) and \( A' \) be two neighboring matrices such that \( E = A - A' = uv^T \). Then \( \|S(0) T^T\|_F^2 \leq (1 + \alpha)^2 \). Publishing \( Z \) preserves \( (\varepsilon, \delta) \)-differential privacy follows from considering the vector form of the matrix \( S^T A T^T \) and \( N_2 \) and applying Theorem 37. Similarly, we use Theorem 37 and the fact that, for any matrix \( C \) of appropriate dimension, \( \|\hat{\Psi} C\|_F^2 \leq (1 + \alpha)^2\|C\|_F^2 \), to prove that publishing \( \hat{\Psi} A + N_1 \) preserves differential privacy.  

We next give a proof sketch that $Y_c$ is $(\varepsilon, \delta)$-differentially private. This would complete the proof of Lemma 26 as the lemma would follow by combining Lemma 35 and Theorem 36.

**Claim 28.** Let $\sigma_{\min}$ be as in Theorem 74. Then publishing $Y_c$ preserves $(\varepsilon, 2\delta)$-differential privacy.

**Proof.** Our proof uses an analysis of multivariate Gaussian distribution. The multivariate Gaussian distribution is a generalization of univariate Gaussian distribution. Let $\mu$ be an N-dimensional vector. An $N$-dimensional multivariate random variable, $x \sim \mathcal{N}(\mu, \Lambda)$, where $\Lambda = \mathbb{E}[(x - \mu)(x - \mu)^T]$ is the $N \times N$ covariance matrix, has the probability density function given by $PDF_x(x) := \frac{e^{-\frac{1}{2}x^T\Lambda^{-1}x}}{\sqrt{(2\pi)^N\det(\Lambda)}}$, where $\det(\cdot)$ denotes the determinant of the matrix. If $\Lambda$ has a non-trivial kernel space, then the multivariate distribution is undefined. However, in this proof, all our covariance matrices have only trivial kernel. Multivariate Gaussian distributions is invariant under affine transformation, i.e., if $y = Ax + b$, where $A \in \mathbb{R}^{M \times N}$ is a rank-$M$ matrix and $b \in \mathbb{R}^M$, then $y \sim \mathcal{N}(A\mu + b, \Lambda A \Lambda^T)$.

Let $A - A' = E = uv^T$ and let $\hat{v} = (v \ 0^n)$. Then $\hat{A} - \hat{A}' = uv^T$. Since $\Phi^T$ is sampled from $D_R$, using [11], we have $\|\Phi^T W\|_F = (1 + \alpha)\|W\|_F$ for any matrix $W$ with probability $1 - \delta$. Therefore, $u \hat{v}^T = (1 + \alpha)\|u\|_2^2u\hat{v}^T = u\hat{v}^T$ for some unit vectors $u, \hat{v}$ and $\hat{u} = (1 + \alpha)^{1/2}u$. We now show that $\hat{A}\Phi\Omega_1$ preserves privacy. We prove that each row of the published matrix preserves $(\varepsilon_0, \delta_0)$-differential privacy for some appropriate $\varepsilon_0, \delta_0$, and then invoke Theorem 35 to prove that the published matrix preserves $(\varepsilon, \delta)$-differential privacy.

It may seem that the privacy of $Y_c$ follows from the result of Blocki et al. [5], but this is not the case because of the following reasons.

1. The definition of neighboring matrices considered in this paper is different from that of Blocki et al. [5]. To recall, Blocki et al. [5] considered two matrices neighboring if they differ in at most one row by a unit norm. In our case, we consider two matrices are neighboring if they have the form $uv^T$ for unit vectors $u, v$.

2. We multiply the Gaussian matrix to a random projection of $\hat{A}$ and not to $A$ as in the case of Blocki et al. [5], i.e., to $A\Phi$ and not to $A$.

We first give a brief overview of how to deal with these issues. The first issue is resolved by analyzing $(\hat{A} - \hat{A}')\Phi$. We observe that this expression can be represented in the form of $\hat{u}\hat{v}^T$, where $\hat{u} = (1 + \alpha)^{1/2}u$ for some $\|u\|_2 = 1, \|\hat{v}\|_2 = 1$. The second issue can be resolved by observing that $\Phi$ satisfies $(\alpha, \delta)$-LJP because of the choice of $t$. Since the rank of $\hat{A}$ and $\hat{A}\Phi$ are the same, the singular values of $\hat{A}\Phi$ are within a multiplicative factor of $(1 + \alpha)^{1/2}$ of the singular values of $\Phi$ with probability $1 - \delta$ due to Sarlos [47]. Therefore, our proof goes through if we scale the singular values of $\hat{A}$ appropriately.

In practical scenarios, $k \ll \max\{m, n\}$ and $\alpha$ is a small constant. This implies that $(1 - \alpha)^{-1} \approx (1 + \alpha)$. If we scale the value of $\alpha$ appropriately, then we have the following corollary:

**Corollary 29.** Under the assumptions of Theorem 74 and $0 < \alpha < 1$ a small constant, given an $m \times n$ matrix $A$ in the turnstile update model, the algorithm SPECTRAL-LRF, presented in Figure 7, is $(\varepsilon, \delta)$ differentially private under Priv2 and outputs a $k$-rank factorization $U_k, \Sigma_k, V_k^T$ (with $M_k = U_k\Sigma_kV_k^T_k$, such that, with probability $9/10$ over the coin tosses of LOW-SPACE-LRF,

$$\|(A \ 0) - M_k\|_2 \leq (1 + \alpha)\Delta_k(A) + O\left(\sqrt{m} + \sqrt{n}\varepsilon^{-1}\sqrt{\ln(1/\delta)}\right).$$
Input: A time upper bound $T$, privacy parameters $\varepsilon, \delta$, and a stream $s \in \mathbb{R}^T$.
Output: At each time step $\tau$, output a factorization $\hat{U}_k(\tau), \hat{\Sigma}_k(\tau),$ and $\hat{V}_k(\tau)$.

Initialization: Set $t, \nu, \Phi, S$ as in Figure 3. Every $\hat{Y}_i$ and $\hat{Z}_i$ are initialized to an all zero matrices for $i \in [\log T]$. Set $\varepsilon' = \varepsilon/\sqrt{\log T}, \delta' = \delta/2\log T$ and $\rho = \sqrt{(1 + \alpha) \ln(1/\delta)}/\varepsilon'$.

Estimating the LRF at time $t$. On receiving an input $(r, c, s_r)$ where $s_r \in \mathbb{R}$ at $1 \leq \tau \leq T$, form a matrix $A_\tau \in \mathbb{R}^{m \times n}$ which is an all zero matrix except with only non-zero entry $s_r$ at location $(r, c) \in [m] \times [n]$. Compute $i := \min \{ j : \tau_j \neq 0 \}$, where $\tau = \sum_j \tau_j \cdot 2^j$ is the binary expansion of $\tau$. Then

1. Compute $\hat{Y}_i := A_\tau \Phi + \sum_{j<i} \hat{Y}_j$ and $\hat{Z}_i := SA_\tau + \sum_{j<i} \hat{Z}_j$.

2. For $j := 0, \ldots, i - 1$, set $Y_j = \hat{Y}_j = 0$ and $Z_j = \hat{Z}_j = 0$. Compute $Y_i = \hat{Y}_i + \mathcal{N}(0, \rho^2)^{\times n}$ and $Z_i = \hat{Z}_i + \mathcal{N}(0, \rho^2)^{\times n}$. Compute $Y(\tau) = \sum_{j: \tau_j = 1} Y_j$ and $Z(\tau) = \sum_{j: \tau_j = 1} Z_j$.

3. Compute a matrix $U \in \mathbb{R}^{n \times t}$ whose columns are an orthonormal basis for the column space of $Y(\tau)$.

4. Compute the singular value decomposition of $SU \in \mathbb{R}^{n \times t}$. Let it be $\hat{U}\hat{\Sigma}\hat{V}^T$.

5. Compute the singular value decomposition of $\hat{V}\hat{\Sigma}^\dagger\hat{U}^T[\hat{U}\hat{U}^TZ(\tau)]_k$. Let it be $U'\Sigma'V'^T$.

6. Output $\hat{U}_k(\tau) := UU', \hat{\Sigma}_k(\tau) := \Sigma'$ and $\hat{V}_k(\tau) := V'$. Let $M_k(\tau) := U_k(\tau)\Sigma_k(\tau)V_k(\tau)^T$.

Figure 3: Differentially private Low-rank Factorization Under Continual Release (CONTINUAL-LRF)

5 A Generic Transformation for Continual Release

Our algorithms for the continual release use the binary tree mechanism [8, 21] to store the sketches of matrix generated at various time epochs. We present the algorithm in Figure 3 for the sake of completion. We give an intuition of the algorithm by visualizing the data-structure that stores all the sketches in the form of a binary tree. Every leaf node $\tau$ stores the sketches of $A_\tau$, the root note stores the sketch of the entire matrix streamed in $[0, T]$, and every other node $n$ stores the sketch corresponding to the updates in a time range represented by the leaves of the subtree rooted at $n$, i.e., $\hat{Y}_i$ and $\hat{Z}_i$ stores sketches involving $2^i$ updates to $A$. If a query is to compute the low-rank factorization of the matrix from a particular time range $[1, \tau]$, we find the nodes that uniquely cover the time range $[1, \tau]$. We then use the value of $Y(\tau)$ and $Z(\tau)$ formed using those nodes to compute the low-rank factorization. From the binary tree construction, every time epoch appears in exactly $O(\log T)$ nodes (from the leaf to the root node). Moreover, every range $[1, \tau]$ appears in at most $O(\log T)$ nodes of the tree (including leaves and root node). A straightforward application of the analysis of Chan et al. [8] to the proof of Corollary 17 gives us the following result.

Theorem 30. Let $m, n \in \mathbb{N}$ and $\varepsilon, \delta$ be the input parameters. Let $0 < \alpha < 1$ be an arbitrary constant. Let $A(t) \in \mathbb{R}^{m \times n}$ be the matrix formed until time $t$. Then the algorithm CONTINUAL-
LRF, presented in Figure 3, outputs an \((\epsilon, \delta)\)-differentially private \(k\)-rank approximation \(M_k(t)\) under continual release and under \(\text{Priv}_1\), such that, with probability at least \(9/10\) over the coin tosses of \(\text{Continual-LRF}\),

\[
\|A(t) - M_k(t)\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \Delta_k(A(t)) + O\left(\left(\sqrt{m} + \sqrt{n}\right)\epsilon^{-1} \log T \sqrt{\log(1/\delta)}\right).
\]

We can also convert the algorithm \(\text{LOW-SPACE-LRF}\) to one that outputs a low-rank factorization under continual release by using less space than \(\text{Continual-LRF}\) and secure under \(\text{Priv}_2\). We make the following changes to \(\text{Continual-LRF}\): (i) Initialize \(\hat{Y}_c, \hat{Y}_r\), and \(\hat{Z}\) as we initialize \(Y_c, \hat{Y}_c, \hat{Y}_r\), and \(\hat{Z}\) in Figure 2 for all \(i \in \left[\log T\right]\), (ii) we maintain \(Y_c, \hat{Y}_c, Y_r, \hat{Y}_r, Z_j, \hat{Z}_j\).

**Theorem 31.** Let \(m, n \in \mathbb{N}\) and \(\epsilon, \delta\) be the input parameters. Let \(k\) be the desired rank of the factorization. Let \(0 < \alpha < 1\) be an arbitrary constant. Let \(A(t) \in \mathbb{R}^{m \times n}\) be the matrix formed until time \(t\). Then there is an efficient algorithm that outputs an \((\epsilon, \delta)\)-differentially private \(k\)-rank approximation \(M_k(t)\) under continual release and under \(\text{Priv}_2\), such that, with probability at least \(9/10\),

\[
\|A(t) - M_k(t)\|_2 \leq \frac{(1 + \alpha)}{(1 - \alpha)^2} \Delta_k(A(t)) + O\left(\left(\sqrt{m} + \sqrt{n}\right)\epsilon^{-1} \log T \sqrt{\log(1/\delta)}\right).
\]

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A Proof of Claim 28

We now return to the proof. Denote by \( \hat{A} = (A \ \sigma_{\text{min}} I_m) \) and by \( \hat{A}' = (A' \ \sigma_{\text{min}} I_m) \), where \( A - A' = uv^\top \). Then \( \hat{A}' - \hat{A} = (uv^\top \ 0) \). Let \( U_C \Sigma_C V_C^\top \) be the SVD of \( C = \hat{A} \Phi \) and \( \hat{U}_C \hat{\Sigma}_C \hat{V}_C^\top \) be the SVD of \( \hat{C} = \hat{A}' \Phi \). From above discussion, we know that if \( A - A' = uv^\top \), then \( C - \hat{C} = (1 + \alpha)^{1/2} \hat{u} \hat{v}^\top \) for some unit vectors \( \hat{u} \) and \( \hat{v} \). For notational brevity, in what follows we write \( u \) for \( \hat{u} \) and \( v \) for \( \hat{v} \).

Note that both \( C \) and \( \hat{C} \) is a full rank matrix because of the construction; therefore \( CC^\top \) is a full dimensional \( m \times m \) matrix. This implies that the affine transformation of the multi-variate Gaussian is well-defined (both the covariance \( CC^\top \)^{-1} has full rank and \( \text{det}(CC^\top) \) is non-zero). That is, the PDF of the distributions of the rows, corresponding to \( C \) and \( \hat{C} \), is just a linear transformation of \( \mathcal{N}(0, I_{m \times m}) \). Let \( y \sim \mathcal{N}(0, 1)^t \).

\[
\text{PDF}_{C_Y}(x) = \frac{1}{\sqrt{(2\pi)^d \text{det}(CC^\top)}} e^{\left(-\frac{1}{2}x(\Sigma_C^{-1}x)^\top\right)}
\]

\[
\text{PDF}_{\hat{C}_Y}(x) = \frac{1}{\sqrt{(2\pi)^d \text{det}(\hat{C}C^\top)}} e^{\left(-\frac{1}{2}x(\hat{C}\hat{C}^\top)^{-1}x\right)}
\]

Let \( \varepsilon_0 = \sqrt{\text{ln}(1/\delta) \log(1/\delta)} \) and \( \varepsilon_0 = \delta/2t \), We prove that every row of the published matrix is \((\varepsilon_0, \delta_0)\) differentially private; the theorem follows from Theorem 66. Let \( x \) be sampled either from \( \mathcal{N}(0, CC^\top) \) or \( \mathcal{N}(0, \hat{C}C^\top) \). It is straightforward to see that the combination of Claim 32 and Claim 33 proves differential privacy for a row of published matrix. The lemma then follows by an application of Theorem 36 and our choice of \( \varepsilon_0 \) and \( \delta_0 \).

Claim 32. Let \( C \) and \( \varepsilon_0 \) be as defined above. Then

\[
e^{-\varepsilon_0} \leq \sqrt{\text{det}(CC^\top) / \text{det}(\hat{C}C^\top)} \leq e^{\varepsilon_0}.
\]

Proof. The claim follows simply as in [5] after a slight modification. More concretely, we have \( \text{det}(CC^\top) = \prod_i \sigma_i^2 \), where \( \sigma_1 \geq \cdots \geq \sigma_m \geq \sigma_{\text{min}}(C) \) are the singular values of \( C \). Let \( \bar{\sigma}_1 \geq \cdots \geq \bar{\sigma}_m \geq \sigma_{\text{min}}(\hat{C}) \) be its singular value for \( \hat{C} \). The matrix \( E \) has only one singular value \( \sqrt{1 + \alpha} \). This is because \( EE^\top = (1 + \alpha)vv^\top \). To finish the proof of this claim, we use Theorem 41.

Since the singular values of \( C - \hat{C} \) and \( \hat{C} - C \) are the same, Lidskii’s theorem (Theorem 41) gives \( \sum_i (\sigma_i - \bar{\sigma}_i) \leq \sqrt{1 + \alpha} \). Therefore, with probability \( 1 - \delta \),

\[
\prod_{i: \bar{\sigma}_i \geq \sigma_i} \frac{\sigma_i^2}{\bar{\sigma}_i^2} = \prod_{i: \bar{\sigma}_i \geq \sigma_i} \left( 1 + \frac{\bar{\sigma}_i - \sigma_i}{\sigma_i} \right) \leq \exp \left( \frac{\varepsilon \sum_i (\bar{\sigma}_i - \sigma_i)}{32 \sqrt{(1 + \alpha) t \log(2/\delta) \log(t/\delta)}} \right) \leq e^{\varepsilon_0/2}.
\]

The first inequality holds because \( \Phi \sim D_R \) satisfies \((\alpha, \delta)\)-JLP due to the choice of \( t \). Since \( C \) and \( A \) have same rank, this implies that all the singular values of \( C \) are within \((1 \pm \alpha)^{1/2}\)}
multiplicative factor of $\hat{A}$ due to a result by Sarlos [47]. In other words, $\sigma_i \geq \sigma_{\text{min}}(C) \geq (1 - \alpha)^{1/2}\sigma_{\text{min}}$. The case for all $i \in [m]$ when $\tau_i \leq \sigma_i$ follows similarly as the singular values of $E$ and $-E$ are the same. This completes the proof of Claim 32. \hfill $\square$

**Claim 33.** Let $C, \varepsilon_0$, and $\delta_0$ be as defined earlier. Let $y \sim \mathcal{N}(0,1)^m$. If $x$ is sampled either from $Cy$ or $\bar{C}y$, then we have

$$\Pr \left[ \left\| x^T(\bar{C} \bar{C}^T)^{-1}x - x^T(C C^T)^{-1}x \right\| \leq \varepsilon_0 \right] \geq 1 - \delta_0.$$

**Proof.** Without any loss of generality, we can assume $x = Cy$. The case for $x = \bar{C}y$ is analogous. Let $C - \bar{C} = \nu u^T$. Note that $E[(\Omega)_{i,j}] = 0^n$ for all $1 \leq i, j \leq m$ and $\text{COV}((\Omega)_{i,j}) = 1$ if and only if $i = j$. Then

$$x^T(C C^T)^{-1}x - x^T(\bar{C} \bar{C}^T)^{-1}x = x^T((C C^T)^{-1}(\bar{C} \bar{C}^T)^{-1} - (\bar{C} \bar{C}^T)^{-1})(\bar{C} \bar{C}^T)^{-1}1 x.$$ 

Using the singular value decomposition of $C = UC\Sigma$, and $\bar{C} = U\tilde{C}\tilde{\Sigma}$, we have

$$x^T((C C^T)^{-1}(\bar{C} \bar{C}^T)^{-1} - (\bar{C} \bar{C}^T)^{-1})(\bar{C} \bar{C}^T)^{-1}1 x = x^T(\Sigma^{-1}1 \Sigma^{-1}1 x) + \left( x^T(\Sigma^{-2}1 \Sigma^{-2}1 x) \right) (\Sigma^{-1}1 \Sigma^{-1}1 x).$$

Since $x \sim Cy$, where $y \sim \mathcal{N}(0,1)^m$, we can write the above expression as $\tau_1 + \tau_2 + \tau_3 + \tau_4$, where

$$\tau_1 = \left( y^T C (U \Sigma C^{-1}1 \Sigma^{-1}1 x) \right), \quad \tau_2 = \left( v^T (\tilde{U} \Sigma^{-2}1 \tilde{U}^T) Cy \right),$$

$$\tau_3 = \left( y^T C (U \Sigma C^{-2}1 \Sigma^{-2}1 x) \right), \quad \tau_4 = \left( u^T (\tilde{V} \Sigma^{-1}1 \tilde{V}^T) Cy \right).$$

Now since $\|\Sigma\|_2, \|\Sigma C\|_2 \geq \sigma_{\text{min}}(C)$, plugging in the SVD of $C$ and $C - \bar{C} = \nu u^T$, and that every term $\tau_i$ in the above expression is a linear combination of a Gaussian, i.e., each term is distributed as per $\mathcal{N}(0, \|\tau_i\|^2)$, we calculate $\|\tau_i\|$ as below.

$$\|\tau_1\|_2 = \| (V \Sigma C U^T) (U \Sigma C^{-1}1 \Sigma^{-1}1 x) \|_2 \leq \| u \|_2 \leq \sqrt{1 + \alpha},$$

$$\|\tau_2\|_2 = \| v^T (\tilde{U} \Sigma^{-2}1 \tilde{U}^T) (\tilde{U} \Sigma^{-1}1 \tilde{U}^T C - \nu u^T) \|_2 \leq \| v^T (\tilde{U} \Sigma^{-2}1 \tilde{U}^T C - \nu u^T) v^T \|_2 \leq \frac{1}{\sigma_{\text{min}}(C)} + \frac{\sqrt{1 + \alpha}}{\sigma_{\text{min}}(C)^2}.$$

$$\|\tau_3\|_2 = \| (V \Sigma C U^T) (U \Sigma C^{-2}1 \Sigma^{-2}1 x) \|_2 \leq \| \Sigma C^{-1}1 \Sigma^{-1}1 x \|_2 \leq \frac{1}{\sigma_{\text{min}}(C)},$$

$$\|\tau_4\|_2 = \| u^T (\tilde{V} \Sigma^{-1}1 \tilde{V}^T C - \nu u^T) \|_2 \leq \| u^T (\tilde{V} \Sigma^{-1}1 \tilde{V}^T C - \nu u^T) v^T \|_2 \leq \sqrt{1 + \alpha} + \frac{\sqrt{1 + \alpha}}{\sigma_{\text{min}}(C)}.$$

Using the concentration bound on the Gaussian distribution, each term, $\tau_1, \tau_2, \tau_3, \text{and } \tau_4$, is less than $\|\tau_i\| \ln(4/\delta_0)$ with probability $1 - \delta_0/2$. The second claim follows because with probability $1 - \delta_0$,

$$\left| x^T(C C^T)^{-1}x - x^T(\bar{C} \bar{C}^T)^{-1}x \right| \leq 2 \left( \frac{\sqrt{1 + \alpha}}{\sigma_{\text{min}}(C)} + \frac{1 + \alpha}{\sigma_{\text{min}}(C)^2} \right) \ln(4/\delta_0) \leq \varepsilon_0.$$
where the second inequality follows from the choice of $\sigma_{\min}$ and the fact that $\sigma_{\min}(C) \geq (1 - \alpha)^{1/2}\sigma_{\min}$. 

Lemma 26 follows by combining Claims 32 and 33.

B Related Problems Studied in Previous Works

Differential privacy was introduced by Dwork et al. [20]. Since then, many algorithms for preserving differential privacy have been proposed in the literature (see, Dwork and Roth [22]). Dwork et al. [21] first considered streaming algorithms with privacy under the model of \textit{pan-privacy}, where the internal state is known to the adversary. Subsequently, there have been some works on online private learning [24] [20] [50] for various tasks. There are some recent works on differentially private LRA as well. Blum et al. [6] first studied this problem in the Frobenius norm. This was improved by Hardt and Roth [28] under the low coherence assumption. Upadhyay [52] later showed that one can make the two-pass algorithm of Hardt and Roth [28] single-pass. differentially private LRA has been studied in the spectral norm as well by many works [9] [35] [29] [27]. Recently, Dwork et al. [24] gave a tighter analysis of the algorithm of Blum et al. [6] and used it to give a private online algorithm for covariance matrices. We now define the problem studied in each of the work listed above and draw the contrast with Problem 2.

\begin{itemize}
    \item \textbf{Spectral low-rank approximation} Kapralov and Talwar [35], Hardt and Roth [29], Hardt and Price [27], and Upadhyay [52] studied low-rank approximation.
    \item \textbf{Problem 2.} (Approximation with respect to the spectral norm). Given parameters $\alpha, \beta, \tau$, a private $m \times n$ matrix $A$ (where $m \ll n$) and the target rank $k$, find a rank-$k$ matrix $B$ such that
    \[ \Pr[\|A - B\|_2 \leq \gamma \Delta_k(A) + \zeta] \geq 1 - \beta. \]
    \textit{Hardt and Price [27] and Jiang et al. [31] consider two matrices as neighboring if they differ in exactly one entry by at most 1. Upadhyay [52] considered two matrices are neighboring if their differ in at most one row or column of norm 1. Kapralov and Talwar [35] considered two matrices as neighboring if the difference of their spectral norm is at most 1. The result of Kapralov and Talwar [35] and Jiang et al. [31] is for \(\varepsilon\)-differential privacy (i.e., \(\delta = 0\)).}
    \item \textbf{Difference from this paper.} We consider low-rank factorization with respect to the spectral norm while Problem 2 studied only low-rank approximation. Moreover, granularity of privacy we consider is more general than theirs.
    \item \textbf{Approximating the right singular vectors} Dwork et al. [24] studied the following problem.
    \item \textbf{Problem 3.} Given parameters $\alpha, \beta, \tau$ and an $m \times n$ private matrix $A$ (where $m \gg n$), compute a rank-$k$ matrix $B$ such that with probability $1 - \beta$
    \[ \|A^T A - B\|_2 \leq \min_{\text{rank}(B_k) \leq k} \|A^T A - B_k\| + \zeta. \]
\end{itemize}
Dwork et al. [24] consider two matrices neighbouring if they differ by at most one row. They further assume that the rows are normalized.

**Difference from this paper.** We consider low-rank factorization of both the right and the left singular vectors while Problem 3 studied low-rank “approximation” of the right singular vectors. Moreover, granularity of privacy we consider is more general than theirs.

## C Useful Facts

We use the notation \(\text{Rad}(p)\) to denote a distribution with support \(\pm 1\) such that +1 is sampled with probability \(p\) and −1 is sampled with probability \(1−p\). An \(n \times n\) Walsh-Hadamard matrix \(H_n\) is constructed recursively as follows:

\[
H_n = \begin{pmatrix}
H_{n/2} & H_{n/2} \\
H_{n/2} & -H_{n/2}
\end{pmatrix}
\quad \text{and} \quad H_1 := 1.
\]

A randomized Walsh-Hadamard matrix \(W_n\) is formed by multiplying \(H_n\) with a diagonal matrix whose diagonal entries are picked i.i.d. from \(\text{Rad}(1/2)\). We drop the subscript \(n\) where it is clear from the context. A \(r \times n\) subsampled randomized Hadamard matrix is constructed by multiplying \(\Pi_{1..r}\) from the left to a randomized Hadamard matrix, where \(\Pi_{1..r}\) is the matrix formed by the first \(r\) rows of a random permutation matrix.

**Lemma 34.** Let \(S\) be a \(v \times m\) subsampled randomized Hadamard matrix, where \(v \leq m\) and \(N_2 \in \mathbb{R}^{v \times n}\). Then we have,

\[
\|S^\dagger N_2\|_2 = \|N_2\|_2.
\]

**Proof.** One way to look at the action of \(S\) when it is a subsampled Hadamard transform is that it is a product of matrices \(W\) and \(\Pi_{1..v}\), where \(\Pi_{1..v}\) is the matrix formed by the first \(r\) rows of a random permutation matrix and \(W\) is a randomized Walsh-Hadamard matrix formed by multiplying a Walsh-Hadamard matrix with a diagonal matrix whose non-zero entries are picked i.i.d. from \(\text{Rad}(1/2)\).

Since \(WD\) has orthonormal rows, \(S^\dagger = (\Pi_{1..v}WD)^\dagger = (WD)^T(\Pi_{1..v})^\dagger\).

\[
\|S^\dagger N\|_2 = \|((\Pi_{1..v}WD)^\dagger)N\|_2 = \|(WD)^T\Pi_{1..v}^\dagger N\|_2
\]

Using the fact that \(\Pi_{1..v}\) is a full row rank matrix and \(\Pi_{1..v}\Pi_{1..v}^T\) is an identity matrix, we have \(\Pi_{1..v}^\dagger = \Pi_{1..v}^T(\Pi_{1..v}\Pi_{1..v}^T)^{-1} = \Pi_{1..v}^T\). The result follows. \(\square\)

**Differential privacy.** We use the following results about differential privacy in this paper.

**Lemma 35.** (Post-processing [19].) Let \(M(D)\) be an \((\varepsilon, \delta)\)-differential private mechanism for a database \(D\), and let \(h\) be any function, then any mechanism \(M' := h(M(D))\) is also \((\varepsilon, \delta)\)-differentially private.
Theorem 36. (Composition [23]) Let $\varepsilon_0, \delta_0 \in (0, 1)$, and $\delta' > 0$. If $\mathcal{M}_1, \cdots, \mathcal{M}_\ell$ are each $(\varepsilon, \delta)$-differential private mechanism, then the mechanism
\[
\mathcal{M}(D) := (\mathcal{M}_1(D), \cdots, \mathcal{M}_\ell(D))
\]
releasing the concatenation of each algorithm is $(\varepsilon', \ell\delta + 0 + \delta')$-differentially private for $\varepsilon' < \sqrt{2\ell\ln(1/\delta')\varepsilon_0 + 2\ell\varepsilon_0}$.

Theorem 37. (Gaussian mechanism [12].) Let $x, y \in \mathbb{R}^n$ be any two vectors such that $\|x - y\|_2 \leq c$. Let $\rho = c\varepsilon^{-1}\sqrt{\log(1/\delta)}$ and $g \sim \mathcal{N}(0, \rho^2 I)$. Then for any $s \subseteq \mathbb{R}^n$, \(\Pr[x + g \in s] \leq e^\varepsilon \Pr[y + g \in s] + \delta\).

Properties of Gaussian distribution. We need the following property of a random Gaussian matrices.

Fact 38. ([22, 47]) Let $P \in \mathbb{R}^{m \times n}$ be a matrix of rank $r$ and $Q \in \mathbb{R}^{m \times n'}$ be an $m \times n'$ matrix. Let $D$ be a distribution of matrices over $\mathbb{R}^{m \times n'}$ with entries sampled i.i.d. from $\mathcal{N}(0, 1/t)$. Then there exists a $t = O(r/\alpha \log(r/\beta))$ such that $D$ is an $(\alpha, \beta)$-subspace embedding for generalized regression.

Matrix Theory. We use some basic results from matrix theory.

Theorem 39. (Weyl’s perturbation theorem) For any $m \times n$ matrices $P, Q$, we have $|\sigma_i(P + Q) - \sigma_i(P)| \leq \|Q\|_2$, where $\sigma_i(\cdot)$ denotes the $i$-th singular value and $\|Q\|_2$ is the spectral norm of the matrix $Q$.

Fact 40. Let $A$ be a rank-$r$ matrix, then $\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\|A\|_2$.

We also need the following results for the privacy proof.

Theorem 41. (Lidskii Theorem). Let $A, B$ be $n \times n$ Hermittian matrices. Then for any choice of indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$,
\[
\sum_{j=1}^k \lambda_{i_j}(A + B) \leq \sum_{j=1}^k \lambda_{i_j}(A) + \sum_{j=1}^k \lambda_{i_j}(B),
\]
where $\{\lambda_i(A)\}_{i=1}^n$ are the eigen-values of $A$ in decreasing order.

We need the following theorem in our proofs.

Lemma 42. (Sarlos [47]) Let $\Phi$ be a matrix that satisfies $(\alpha, \beta)$-subspace embedding for $I$, then $(1 - \alpha) \leq \|\Phi\|_2 \leq (1 + \alpha)$ with probability at least $1 - \beta$.

Lemma 43. Given integer $k$ and $\alpha, \delta > 0$, there is $p = O(k\log(1/\delta)/\alpha)$ such that if $S$ is an $p \times n$ $(\alpha, \beta)$-subspace embedding matrix, then for $n \times k$ matrix $U$ with orthonormal columns, with probability at least $1 - \delta$, the spectral norm $\|U^T \Phi \Phi^T U - I\|_2 \leq \alpha$.

Theorem 44. (Sou and Rantzer [72]). Let matrices $O \in \mathbb{R}^{m \times n}, L \in \mathbb{R}^{m \times p}$ and $R \in \mathbb{R}^{q \times n}$ be given. Then
\[
L^T [U_L U_L^T O V_R V_R^T] R^T = \arg\min_{X, r(x) \leq k} \|O - L X R\|_2,
\]
where $L := U_L \Sigma_L V_L^T$ and $R := U_R \Sigma_R V_R^T$.

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