Symmetry and phase transitions: Quantum phase transitions in algebraic models

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Abstract. A brief description of the role of symmetry in phase transitions is given. Quantum phase transitions (QPT) in algebraic models are discussed, both at the semi-classical and quantal level. The newly introduced concept of excited states quantum phase transitions (ESQPT) is presented.

1. Introduction

Quantum phase transitions (QPT) are phase transitions that occur as a function of coupling constants, $\xi_1, \xi_2, \ldots$, the so-called control parameters, that appear in the quantum Hamiltonian, $H$, that describes the system

$$H = \varepsilon (H_1 + \xi_1 H_2 + \xi_2 H_3 + \ldots).$$

Associated with phase transitions there are order parameters, the expectation values of suitable chosen operators that describe the state of the system $\langle P \rangle$. A simple example is the Ising model in a transverse magnetic field

$$H = -V \left( \xi \sum_i \hat{\sigma}_i^z - \sum_{\langle ij \rangle} \hat{\sigma}_i^+ \hat{\sigma}_j^+ \right)$$

which has a second order QPT at $\xi = \xi_c = 1/2$. The order parameter is the magnetization, $M \propto \langle \hat{\sigma}_z \rangle$. The two phases are: (i) ferromagnet for $\xi < \xi_c$ and (ii) paramagnet for $\xi > \xi_c$. QPTs are also called ground state phase transitions [1] and/or zero temperature phase transitions.

An algebraic model of a physical many-body system is a mapping of all operators of the system onto an algebraic space, $\mathfrak{g}$, usually taken as the unitary (compact) algebra in $n$ dimensions, $\mathfrak{u}(n)$. The representations of $\mathfrak{u}(n)$ are labelled by the total number of particles $N$. They are totally symmetric for bosons and totally antisymmetric for fermions with a one-row (boson) or one-column (fermions) Young tableau. Since all physical operators are expanded onto elements of $\mathfrak{g}$, the algebra $\mathfrak{g}$ is called the spectrum generating algebra (SGA). Many-body systems composed of a mixture of bosons and fermions can also be treated by algebraic models by mapping onto an algebraic superspace, $\mathfrak{g}^*$, usually taken as the unitary (compact) superalgebra in $n$ bosonic and $m$ fermionic dimensions, $\mathfrak{u}(n/m)$. States are characterized by the totally supersymmetric representations of $\mathfrak{u}(n/m)$, labeled by the total number of bosons plus fermions, $N = N_B + N_F$. The algebra $\mathfrak{g}^*$ is called the spectrum generating superalgebra (SGSA).
In this contribution a study of quantum phase transitions in algebraic models will be presented.

2. QPT in algebraic models
The generic Hamiltonian $H$ is written in terms of elements $G_{\alpha \alpha'}$ of $g$,

$$H = \sum_{\alpha} \epsilon_{\alpha} G_{\alpha \alpha'} + \sum_{\alpha \alpha' \beta \beta'} u_{\alpha \alpha' \beta \beta'} G_{\alpha \alpha'} G_{\beta \beta'} + \ldots ; \quad G_{\alpha \alpha'} \in g .$$  \hspace{1cm} (3)

In most applications, only terms up to quadratic in the elements of $g$ are kept. In this presentation, bosonic systems will be considered for which the elements of $g$ are bilinear products of boson creation and annihilation operators,

$$G_{\alpha \alpha'} = b^\dagger_{\alpha} b_{\alpha'} \quad (\alpha, \alpha' = 1, \ldots, n).$$  \hspace{1cm} (4)

The algebra $g$ is $U(n)$. The basis states are totally symmetric $[N] \equiv \Box \Box \ldots \Box$ and can be constructed by acting with boson operators on the vacuum

$$|N\rangle = \frac{1}{\sqrt{N!}} b_{\alpha}^\dagger b_{\alpha'}^\dagger \ldots |0\rangle .$$  \hspace{1cm} (5)

2.1. Phases
Phases in algebraic models are all possible breakings of the algebra $g$ into its subalgebras (subject to some conditions, for example, rotational invariance, i.e. the chain must contain $so(3)$). The subalgebras $g_1, g_2, \ldots, g_\phi$ are often called dynamic symmetries

$$g \supset g_1 \supset g_1' \supset \ldots$$
$$g \supset g_2 \supset g_2' \supset \ldots$$
$$\ldots$$
$$g \supset g_\phi \supset g_\phi' \supset \ldots$$  \hspace{1cm} (7)

A classification of phases of interacting boson models in arbitrary number of dimensions $n$ is given in [2]. Whether or not there is a phase transition between two phases will be determined by further analysis as shown in the following sections. Throughout this article, two examples will be discussed, the Interacting Boson Model of nuclei and the Vibron Model of molecules.

2.1.1. The Interacting Boson Model (IBM) The Interacting Boson Model [3] is a model of atomic nuclei in terms of correlated pairs of nucleons with angular momentum $J = 0$ and $J = 2$, treated as bosons and called $s$ and $d$ bosons [4]. The model can also be seen as a treatment of $S$ and $D$-wave pairing in 3-dimensions. There are six boson operators $b_{\alpha}(\alpha = 1, \ldots, 6) = s, d_{\mu}(\mu = 0, \pm 1, \pm 2)$, spanning a six-dimensional space. The algebraic structure is $u(6)$, with $6 \times 6 = 36$ elements. The phases are:

$$u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2) \quad (I)$$
$$u(6) \supset su(3) \supset so(3) \supset so(2) \quad (II)$$
$$u(6) \supset so(6) \supset so(5) \supset so(3) \supset so(2) \quad (III).$$  \hspace{1cm} (8)
2.1.2. The Vibron Model (VM)  The Vibron Model [5] is a model of vibrations of molecules in terms of quanta of vibrations, called vibrons [6]. It can be formulated in 1, 2 or 3 dimensions. In this contribution, the vibron model in 2-dimensions will be considered [7]. The vibron operators are a singlet $\sigma$ and a doublet $\tau^+, \tau^-$. They span a three-dimensional space. The algebraic structure is $u(3)$ with $3 \times 3 = 9$ elements. The phases are:

\[
\begin{align*}
  u(3) & \supset u(2) \supset so(2) \quad (I) \\
  u(3) & \supset so(3) \supset so(2) \quad (II)
\end{align*}
\]

The vibron model in 2-dimensions is isomorphic to, i.e. has the same algebraic structure of, a model of high-$T_c$ superconductors in terms of correlated electron pairs in two dimensions, called $s$, $d_+$, $d_-$ bosons which can also be viewed as a treatment of S and D-wave pairing in two dimensions [8]. [A discrete version of this model on a square lattice, with symmetry $D_{4h}$ is given in [9]. The boson operators that transform as representations of $D_{4h}$ are $\Gamma_1^\dagger \equiv s^\dagger$, $\Gamma_3^\dagger \equiv (d_+^\dagger - d_-^\dagger)$, $\Gamma_4^\dagger \equiv (d_+^\dagger - d_-^\dagger)$.]

2.2. Phase diagram

Quantum phase transitions in algebraic models are best studied by writing the Hamiltonian in terms of (invariant) Casimir operators of the algebras $g_i (i = 1, 2, \ldots, \phi)$ that define the phases

\[
H = \alpha_1 C(g_1) + \alpha_2 C(g_2) + \ldots + \alpha_\phi C(g_\phi).
\]

(10)

Taking out a scale factor this can be rewritten as

\[
H = \alpha_1 \left[ C(g_1) + \frac{\alpha_2}{\alpha_1} C(g_2) + \ldots + \frac{\alpha_\phi}{\alpha_1} C(g_\phi) \right].
\]

(11)

The control parameters are the ratios

\[
\xi_1 = \frac{\alpha_2}{\alpha_1}, \quad \xi_2 = \frac{\alpha_3}{\alpha_1}, \ldots, \quad \xi_{\phi-1} = \frac{\alpha_\phi}{\alpha_1}.
\]

(12)

The number of control parameters is equal to the number of phases $\phi$ minus one. It is sufficient to take one Casimir operator for each $g_i$. The corresponding Hamiltonian is called the ”essential” Hamiltonian.

In the two cases introduced in the previous subsection, the essential Hamiltonian is, for the interacting boson model, $g \equiv u(6)$

\[
H = \varepsilon \left[ C_1(u(5)) + \xi_1 C_2(su(3)) + \xi_2 C_2(so(6)) \right]
\]

(13)

and for the two-dimensional vibron model, $g \equiv u(3)$,

\[
H = \varepsilon \left[ C_1(u(2)) + \xi C_2(so(3)) \right].
\]

(14)

Here $\varepsilon$ is a scale factor, and $C_p(g)$ denotes the Casimir operator of order $p$ of the algebra $g$.

The essential Hamiltonian is often parametrized in such a way that the control parameters vary in a finite interval, usually $0 - 1$ instead of $0 - \infty$, and the Casimir operators are scaled by appropriate powers of the number of particles $N$ to guarantee that the location of the critical points is independent of $N$ in the large $N$-limit. Since the eigenvalues of the a Casimir operator of order $p$ are in the large $N$-limit $\propto N^p$, this scaling is often adopted. For example, for the two-dimensional vibron model, the scaled essential Hamiltonian is

\[
H = \varepsilon \left[ \frac{1 - \xi}{N} C_1(u(2)) + \frac{\xi}{N^2} C_2(so(3)) \right]; \quad 0 \leq \xi \leq 1.
\]

(15)

In practice, combinations of Casimir operators are often used.
2.2.1. *Essential Hamiltonian of the interacting boson model* In most studies, the essential Hamiltonian of this model is taken as

\[
H = \varepsilon \left[ (1 - \xi) \hat{n}_d - \frac{\xi}{4N} \hat{\mathcal{Q}}^x \cdot \hat{\mathcal{Q}}^x \right]
\]

\[
\hat{n}_d = \left( d^\dagger \cdot \hat{d} \right)
\]

\[
\hat{\mathcal{Q}}^x = \left( d^\dagger \times \hat{s} + s^\dagger \times \hat{d} \right)^{(2)} + \chi \left( d^\dagger \times d \right)^{(2)}.
\]

(16)

There are three phases, characterized by the first algebra in the subalgebra chains of Eq.(8), \( g_1 \equiv u(5) \), \( g_2 \equiv su(3) \), \( g_3 \equiv so(6) \), and two control parameters, \( \xi_1, \xi_2 \), in the parametrization of Eq.(13) or \( \xi, \chi \) in the parametrization of Eq.(16). The phase diagram of this model is thus two-dimensional and usually represented as a triangle.

2.2.2. *Essential Hamiltonian of the two-dimensional vibron model* In most studies, the Hamiltonian of the two-dimensional vibron model is parametrized as

\[
H = \varepsilon \left[ (1 - \xi) \hat{n} + \frac{\xi}{N - 1} \hat{\mathcal{P}} \right]
\]

\[
\hat{n} = \tau^+_+ \tau_+ + \tau^-_-; \quad \hat{\mathcal{P}} = N(N + 1) - \frac{1}{2} \left( \hat{\mathcal{D}}_+ \hat{\mathcal{D}}_- + \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ \right) + \hat{\ell}^2
\]

\[
\hat{\mathcal{D}}_+ = \sqrt{2} \left( \tau^+_+ \sigma - \sigma^+ \tau^-_- \right)
\]

\[
\hat{\mathcal{D}}_- = \sqrt{2} \left( -\tau^+_- \sigma + \sigma^+ \tau^+_+ \right).
\]

(17)

Similarly, in the s-d model of high-T\(_c\) superconductivity, the essential Hamiltonian is parametrized as

\[
H = \varepsilon \left[ (1 - \xi) \hat{n}_d + \frac{\xi}{2N} \left( \hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+ \right) \right]
\]

\[
\hat{n}_d = d^\dagger \times \hat{s} + s^\dagger \times \hat{d}_-
\]

\[
\hat{T}_+ = \sqrt{2} \left( d^\dagger s - s^\dagger d_- \right)
\]

\[
\hat{T}_- = \sqrt{2} \left( -d^\dagger s + s^\dagger d_- \right).
\]

(18)

In either parametrization there are two phases, characterized by the algebras, \( g_1 \equiv u(2) \) and \( g_2 \equiv so(3) \), and one control parameter, \( \xi \). The phase diagram is one dimensional (line).

2.3. *Construction of the phase diagram. "Semiclassical" mean field method* The construction of the phase diagram of algebraic models can be done by a semiclassical method using an algorithm developed by Gilmore [1].

2.3.1. *Coset spaces* The first step is the introduction of a geometry of algebraic models through coset spaces. A review of coset spaces is given in [10]. The algebra \( g \) is split into a stability algebra, \( h \), and a remainder, \( p \),

\[
g = h \oplus p, \quad g \supset h
\]

(19)
with factor algebra $g/h$. For bosonic systems described by $u(n)$, a convenient coset is $u(n)/u(n-1) \oplus u(1)$, with

$$h = b_1^\dagger b_1, b_\alpha^\dagger b_\beta \quad (\alpha, \beta = 2, ..., n)$$

$$p = b_\alpha^\dagger b_\alpha, b_\beta^\dagger b_1 \quad (\alpha = 2, ..., n).$$

(20)

The coset space $u(n)/u(n-1) \oplus u(1)$ has dimension $2(n-1)$. There are $(n-1)$ complex variables in this space.

2.3.2. Coherent states

The second step is the introduction of coherent or intrinsic states

$$|\eta_i\rangle = \exp \left[ \sum_i p_i \eta_i \right] |\Lambda_{\text{ext}}\rangle.$$  (21)

For bosonic systems a convenient extremal state is

$$|\Lambda_{\text{ext}}\rangle = \frac{1}{\sqrt{N!}} \left( b_1^\dagger \right)^N |0\rangle.$$  (22)

and the coherent states are written as

$$|N, \eta_\alpha\rangle = \exp \left[ \sum_\alpha \eta_\alpha b_\alpha^\dagger b_1 + \sum_\alpha \eta_\alpha^* b_\alpha^\dagger b_1 \right] \frac{1}{\sqrt{N!}} \left( b_1^\dagger \right)^N |0\rangle.$$  (23)

It is convenient to use projective coherent states with a fixed number of particles, $N$. For the interacting boson model, the coset space is five dimensional, and the projective coherent states are

$$|N; \alpha_\mu\rangle = \left( s^\dagger + \sum_\mu \alpha_\mu d_\mu^\dagger \right)^N |0\rangle \quad (\mu = 0, \pm 1, \pm 2).$$  (24)

The variables $\alpha_\mu$ are in general complex, the real part representing the coordinates themselves, $\alpha_\mu$, and the imaginary part the associated momenta, $\pi_\mu$. However, for the construction of the phase diagram discussed here it is sufficient to take them real. The variables $\alpha_\mu$ are then transformed to the so-called Bohr variables $\beta, \gamma, \theta_1, \theta_2, \theta_3$, where $\beta, \gamma$ are called intrinsic variables and $\theta_1, \theta_2, \theta_3$ are Euler angles characterizing the orientation of the intrinsic frame in the laboratory frame.

For the two-dimensional vibron model and the model of high-T$_c$ superconductors the space is two dimensional, and

$$|N; \eta_\mu\rangle = \left( s^\dagger + \sum_\mu \eta_\mu d_\mu^\dagger \right)^N |0\rangle \quad (\mu = \pm).$$  (25)

The real variables $\eta_\mu$ can be transformed into polar coordinates $r, \theta$ with $r$ the intrinsic variable and $\theta$ the polar angle.

2.3.3. Ground state energy functional

The ground state energy functional, sometimes called the Landau potential, is the expectation value of the Hamiltonian in the coherent state of Eq.(23)

$$E(\eta_i) = \frac{\langle \eta_i | H | \eta_i \rangle}{\langle \eta_i | \eta_i \rangle}.$$  (26)
By a rotation to the intrinsic frame the angles $\theta$s can be eliminated and the energy functional can be written in terms only of the intrinsic variables. For the interacting boson model,

$$E(N; \beta, \gamma) = \frac{\langle N; \beta, \gamma | H | N; \beta, \gamma \rangle}{\langle N; \beta, \gamma | N; \beta, \gamma \rangle}. \quad (27)$$

Explicit evaluation of the expectation value of the Hamiltonian Eq.(16) gives the Landau potential, or scaled energy per particle, $V(\beta, \gamma) \equiv E(\beta, \gamma)/\varepsilon N$

$$V(\beta, \gamma) = \frac{\beta^2}{1 + \beta^2} \left[ (1 - \xi) - (\chi^2 + 1) \frac{\xi}{4N} \right]$$

$$- \frac{5\xi}{4N(1 + \beta^2)} - \frac{\xi(N - 1)}{4N(1 + \beta^2)^2}$$

$$\times \left[ 4\beta^2 - 4\sqrt{\frac{2}{7}} \chi\beta^3 \cos 3\gamma + \frac{2}{7} \chi^2 \beta^4 \right]. \quad (28)$$

For the two-dimensional vibron model

$$E(N; r) = \frac{\langle N; r | H | N; r \rangle}{\langle N; r | N; r \rangle}. \quad (29)$$

and the Landau potential is

$$V(r) = \left[ (1 - \xi) \frac{r^2}{1 + r^2} + \xi \left( \frac{1 - r^2}{1 + r^2} \right)^2 \right]. \quad (30)$$

2.3.4. Classical order parameters A minimization of $E$ with respect to the $\eta$s gives the equilibrium values $\eta_e$. These are the classical order parameters. The value of $E$ at equilibrium $E_{\text{min}}$ is the ground state energy

$$E_{\text{min}} = E(N; \eta_e). \quad (31)$$

For the interacting boson model,

$$E_{\text{min}} = E(N; \beta_e, \gamma_e). \quad (32)$$

The equilibrium value is $\gamma_e = 0^\circ$ for $\chi < 0$ and $\gamma_e = 60^\circ$ for $\chi > 0$. For $\chi = 0$, $E$ does not depend on $\gamma$.

For the two-dimensional vibron model

$$E_{\text{min}} = E(N; r_e). \quad (33)$$

2.3.5. Order of the phase transition The ground state energy $E_{\text{min}}$ depends on the number of particles $N$ and the control parameter(s) $\xi$, $E_{\text{min}}(N; \xi)$. When $N \to \infty$, discontinuities (phase transitions) may appear in $E_{\text{min}}$ and/or its derivatives with respect to $\xi$ for some value of the control parameter(s) $\xi$, called critical values, $\xi_c$. It has become customary to classify phase transitions according to Ehrenfest classification

$$\frac{E_{\text{min}}}{\partial E_{\text{min}}} \quad 0\text{-th order}$$

$$\frac{\partial E_{\text{min}}}{\partial \xi} \quad 1\text{-st order}$$

$$\cdots$$

$$\frac{\partial^n E_{\text{min}}}{\partial \xi^n} \quad n\text{-th order}$$

No discontinuity crossover \quad (34)
2.3.6. Phase diagrams Phase diagrams of bosonic models of the type described in the subsections above (called s-b boson models) in an arbitrary number of dimensions have been constructed [2]. For the interacting boson model the phase diagram is a triangle [11], shown in figure 1. In this triangle, the three dynamic symmetries are placed at the vertices and the parameters $\xi$ and $\chi$ are as shown. There is a line of first order transitions ending in a point of second order transitions. For the two-dimensional vibron model the phase diagram is a line, as shown in the insert to figure 2. In this case there is only a point of second order transitions.

2.3.7. Behavior of the order parameters The behavior of the order parameters as a function of the control parameters is of great interest in the theory of phase transitions. For second order transitions this behavior has been extensively investigated. Around the critical point, the order parameter, generically denoted by $r_e$ here, behaves as

$$r_e \propto (\xi - \xi_c)^\mu \quad \xi \geq \xi_c$$

$$r_e = 0 \quad \xi < \xi_c$$

(35)

![Figure 1. Phase diagram of IBM. A curve of first-order transitions and point of second-order phase transitions separate spherical and deformed phases.](image1)

![Figure 2. Classical and quantum order parameters for the $u(n-1)$-so$(n)$ phase transition. (Left) Evolution of classical equilibrium coordinate $r_e$. (Right) Quantum calculations of order parameter $\omega$ at finite $N$ and the classical limit ($N \to \infty$). Left part adapted from Ref. [16].](image2)
The exponent $\mu$ is called the critical exponent. This behavior is shown in the left panel of figure 2 for the two-dimensional vibron model. The critical exponent is $\mu = 1/2$. All second order phase transitions of the type $u(n-1)$-so($n$) in s-b boson models have the same behavior and the same critical exponent. For first order transitions, the situation is much more complex. The energy $E$ as a function of the intrinsic variables may have more than one minimum. The energy functional of the interacting boson model, Eq.(28), has, as a function of $\beta$, one minimum (at $\beta_e = 0$) for $\xi \leq \xi^*$ ($\xi^*$ is called spinodal point), two minima for $\xi^* < \xi < \xi^{**}$ ($\xi^{**}$ is called antispinodal point) and one minimum for $\xi \geq \xi^{**}$. Within the coexistence region, $\xi^* < \xi < \xi^{**}$, the potential at the minima has equal value for $\xi = \xi^c$ (called critical point). It has been suggested to define a spinodal exponent that characterizes the behavior of the order parameter around the spinodal point as

$$
(\beta_e - \beta_e^*) \propto (\xi - \xi_e)^{\mu^*} \quad \xi \geq \xi^*,
$$

$$
\beta_e = 0 \quad \xi < \xi^c.
$$

The interacting boson model with up to quadratic interaction does not have phase transitions in the order parameters $\gamma_e$ and therefore critical exponents for this order parameter will not be discussed here.

Critical exponents have been evaluated for $u(n)$ models with at most quadratic Hamiltonians. For the two-dimensional vibron model, as discussed above, the critical exponent is $\mu = 1/2$. For the interacting boson model, $\mu = 1/2$ for the $u(5)$-so(6) transition and $\mu^* = 1/2$ for the $u(5)$-su(3) transition. To understand this behavior, it is instructive to consider a simple, analytically solvable case [12], that of the Landau potential

$$
V(\beta) = \beta^2 + \xi \left[ (2 - \beta^2)^2 - \eta \beta^3 \right], \quad \beta \geq 0,
$$

with two control parameters $\xi$ and $\eta$. Up to a change of variables, this is the generic form of the potential for s-b boson models with at most two-body interactions. Using the method discussed above, one finds that for $\eta \neq 0$, a phase transition occurs as a function of $\xi$. The spinodal point, $\xi^*$, the critical point, $\xi^c$, and the antispinodal point, $\xi^{**}$, can be calculated to be at

$$
\xi^* = \frac{32}{128 + 9\eta^2}; \quad \xi^c \cong \frac{32}{128 + 8\eta^2}; \quad \xi^{**} = \frac{1}{4}.
$$

The upper branch of the order parameter is given by

$$
\beta_e = \frac{3}{8} \eta + \frac{1}{2} \sqrt{\frac{9}{16} \eta^2 + 8 \left( 1 - \frac{1}{4\xi} \right)}, \quad \xi \geq \xi^*,
$$

with spinodal exponent $\mu^* = 1/2$. For $\eta = 0$, the transition becomes second order. The spinodal, critical and antispinodal points coincide and the order parameter is given by

$$
\beta_e = \sqrt{2 \left( 1 - \frac{1}{4\xi} \right)}, \quad \xi \geq \xi^c,
$$

with critical exponent $\mu = 1/2$. The ground state energy is

$$
E_0 = \begin{cases} 
4\xi, & \xi \leq \xi^c = \frac{1}{4} \\
2 - \frac{1}{4\xi}, & \xi \geq \xi^c = \frac{1}{4}.
\end{cases}
$$
The ground state energy $E(0^+_1)$ and its derivatives across first-order (left) and second-order QPTs for the IBM at finite $N$. The control parameter used for these plots is $\zeta \equiv 4\xi / (1 + 3\xi)$. Figure from Ref. [13].

2.4. Construction of the phase diagram. Quantal method

2.4.1. Diagonalization of $H$ for finite $N$

In this method, the essential Hamiltonian is diagonalized numerically as a function of the control parameters. The ground state energy and its derivatives are computed [13]. If the diagonalization is done for sufficiently large $N$, one obtains indications of phase transitional behavior and its order. An example is shown in figure 3 where the ground state energy of the interacting boson model and its derivatives are studied as a function of the control parameter $\xi$ for the transition $u(5)$-$su(3)$, $\chi = -\sqrt{7}/2$, and $u(5)$-$so(6)$, $\chi = 0$. From these plots, one can see that there are indications of a first order transition for $u(5)$-$su(3)$ and of a second order transition for $u(5)$-$so(6)$.

2.4.2. Order parameters. Computation for finite $N$

The quantal order parameters are the expectation values of some suitable chosen operators constructed from the elements $G_{\alpha\alpha'}$ of the algebra $g$. In general, for $u(n)$ models, the number of independent order parameters is $n - 1$. If constraints are imposed, the actual number is less than $n - 1$. In $s$-$b$ boson models, one of the quantal order parameters is always taken as the expectation value in the ground state of the number operator for $b$-bosons, $\hat{n}_b$,

$$
\omega = \frac{\langle \hat{n}_b \rangle}{N}.
$$

Several choices are discussed in the literature for the other order parameters, if needed.

For the interacting boson model, there are in general five order parameters. However, the constraint of rotational invariance reduces this number to two. The classical order parameters
are $\beta_e, \gamma_e$. The quantal order parameters can be chosen as

\[
\omega_1 = \frac{\langle \hat{n}_d \rangle}{N} \rightarrow \frac{\beta_e^2}{1 + \beta_e^2}
\]

\[
\omega_2 = \frac{1}{N^2} \left\langle \left( d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d} \right)^{(2)} \cdot \left( d^\dagger \times \tilde{d} \right)^{(2)} + h.c. \right\rangle \rightarrow \frac{\beta_e^3 \cos 3\gamma_e}{(1 + \beta_e^2)^2}.
\]

(43)

(The arrow indicates the classical order parameters that correspond to the quantal order parameters.) For the two-dimensional vibron model, there are in general two order parameters. However, two-dimensional rotational invariance reduces this number to one

\[
\omega = \frac{\langle \hat{n} \rangle}{N} \rightarrow \frac{r_e^2}{1 + r_e^2}.
\]

(44)

The behavior of the order parameter with $\xi$ for this model obtained numerically for finite $N$ is shown in the right part of figure 2 where it is compared with the classical behavior. This behavior is common to all second order phase transitions of the type, $u(n-1)so(n)$. A similar study has been done for the first order transition $u(5)so(3)$ in the interacting boson model. Qualitatively, in finite systems, the discontinuities are smoothed out, a zeroth order transition looks like a first order, a first like a second and a second like a crossover.

2.4.3. Finite size scaling

The scaling behavior of various quantities of physical interest as a function of the number of particles, $N$, has been investigated both numerically and analytically. For second order transitions, all physical quantities $\Phi$ are expected to scale as a power law [14]

\[
\Phi(\xi^c) \sim N^{-\left( A\Phi + 2B\Phi/3 \right)}.
\]

(45)

Particularly important is the behavior of the ground state energy, $E_0$, and the order parameter $\omega$. The scaling exponents $S_{\Phi} = -(A\Phi + 2B\Phi/3)$ for the $u(n-1)so(n)$ phase transition have been determined to be [15], for $E_0$, $S_{\Phi} = -\frac{4}{3}$ and, for $\omega$, $S_{\Phi} = -\frac{2}{3}$. The numerical determination [16] of the scaling exponent for $\omega$ for the two-dimensional vibron model gives $-0.623(5)$.

3. Excited state quantum phase transitions (ESQPT)

In recent years, interest has grown in the study of excited states QPT. Discontinuities in certain quantities related to excited states are called excited state quantum phase transitions (ESQPT) [17]. ESQPT are best studied by constructing the so-called correlation diagram, that is how energies of excited states evolve as a function of the control parameter. In figure 4, the correlation diagram for a second order phase transition in a two-level pairing model is shown. One can see from this figure that levels accumulate along a curve separating the two phases. The level density diverges along this curve. Phase transitions thus appear to occur as one traverses the diagram both in the horizontal direction, as a function of the control parameter $\xi$, and in the vertical direction as a function of the excitation energy, $E$, for fixed value of $\xi$. One can also see that, at the critical value $\xi = \xi_c$, all levels accumulate at zero energy. For finite $N$, the divergence in the level density is smoothed out and a gap $\Gamma$ appears. Finite size scaling for the gap in second order transitions has been investigated both analytically and numerically. The scaling exponent for the gap $\Gamma$ in $u(n-1)so(n)$ transitions is $S_{\Phi} = -\frac{1}{3}$. A numerical investigation [16] of the scaling exponent for the gap in the two-dimensional vibron model gives $-0.335(1)$. 

10
Figure 4. Excitation energies for a two-level fermionic pairing model, illustrating the compression of energy levels associated with a ESQPT. Figure from Ref. [17].

Figure 5. Experimental data for Sm isotopes (circles) and Gd isotopes (triangles) (a) two-neutron separation energy $S_{2n}$, (b) electromagnetic transition strength $B(E2; 2_1^+ \rightarrow 0_1^+)$, and (c) energy of the lowest $J = 0$ excitation $E(0^+_2)$.

4. Experimental evidence for QPT in nuclei
Quantum phase transitions have been observed in nuclei. Medium-mass and heavy nuclei can be well described by the interacting boson model with Hamiltonian

$$H = \varepsilon \hat{n}_d - \kappa \hat{Q}^x \cdot \hat{Q}^x.$$  \hspace{1cm} (46)

The interactions in (46) are effective interactions that depend on proton and neutron number. The "control parameters" thus vary from nucleus to nucleus and can thus be swept by varying neutron and proton number. The Hamiltonian can be converted to the form (16) with a control parameter $\xi$ that depends linearly (to first approximation) on neutron and proton number. Figure 5 shows the behavior of several quantities as a function of neutron number in the Sm isotopes. Panel (a) shows the two-neutron separation energies, $S_{2n}(N) = E_0(N + 1) - E_0(N) \propto$
Figure 6. Phase diagram of the proton-neutron interacting boson model (IBM-2). Figure from Ref. [18].

\[ \frac{\partial E_0}{\partial \xi} \] Panel (b) shows \( B(E_2; 2_1 \to 0_1) \propto \langle \tilde{n}_d \rangle^2 \) and panel (c) \( E(0_2) - E(0_1) \propto \Gamma \). The discontinuity in \( S_{2n} \) is clearly visible as well as the sudden increase in the order parameter \( \langle \tilde{n}_d \rangle \).

5. Symmetry and phase transitions

Dynamic symmetries

\[
\begin{align*}
g_1 & \supset g_1 \supset \ldots \supset g_\varphi \supset g'_\varphi \supset \ldots \\
g & \supset \ldots \\
g_\varphi & \supset g'_\varphi \supset \ldots
\end{align*}
\]

(47)
determine the phases of the system and the structure of the corresponding phase diagram. In this presentation, two examples have been discussed, the interacting boson model with algebraic structure

\[
\begin{align*}
u(5) & \supset so(5) \supset so(3) \supset so(2) \\
u(6) & \supset su(3) \supset so(3) \supset so(2) \\
su(6) & \supset su(5) \supset so(3) \supset so(2)
\end{align*}
\]

(48)

and the two-dimensional vibron model with algebraic structure

\[
\begin{align*}
u(3) & \supset u(2) \supset so(2) \\
su(3) & \supset so(3) \supset so(2)
\end{align*}
\]

(49)

In the last few years, several other cases have been investigated. Results are available for:

(A) Quantum phase transitions in two-fluid bosonic systems [18, 19].

(B) Quantum phase transitions in two-level pairing models (bosonic and fermionic) [20].

The algebra \( g \) is, in these cases,

\[
g \equiv u(n_1) \oplus u(n_2).
\]

(50)
The phase diagrams for these systems are more complicated, as there are more phases and more control parameters. The phase structure of the proton-neutron interacting boson model (IBM-2) is shown in figure 6. The phase diagram is here three dimensional, represented as a tetrahedron. There is a plane of first order transitions and a sheet of second order transitions.

Investigations have started for:

(C) Quantum phase transitions in mixed Bose-Fermi systems [21, 22].

The algebra \( g \) is here

\[
g \equiv u_B(n_1) \oplus u_F(n_2).
\]

It has been found that supersymmetry (Bose-Fermi symmetry) plays a role in this case, if \( g \) is embedded onto \( g^* \equiv u(n_1/n_2) \).

6. Conclusions

Major progress has been made in recent years in the study of QPT in algebraic models, both in the ground state (QPT) and in excited states (ESQPT). In this presentation the general framework for QPT in algebraic models has been given, including both semiclassical and quantal analysis. The phase diagrams of two bosonic models, the interacting boson model with algebraic structure U(6) and the two-dimensional vibron model U(3) have been constructed and their properties (control parameters, order parameters, critical exponents and scaling exponents) discussed. The analysis has been completed for second order transitions. For first order transitions it is to a large extent completed although some problems still remain (finite size scaling behavior). QPT have been verified experimentally to occur in nuclei and in molecules. Evidence for QPT in nuclei (interacting boson model) has been presented. ESQPT have been introduced and briefly presented. In particular the correlation diagrams and gaps of the interacting boson model and of the two-dimensional vibron model have been discussed. Experimental evidence for ESQPT is not yet available, although some indications show that they occur in molecules.

Dynamic symmetry plays a crucial role in the study of QPT in algebraic models, since it determines the phases of the system and the corresponding phase transitions between them.

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