From spin to anyon notation: the XXZ Heisenberg model as a $D_3$ (or $su(2)_4$) anyon chain

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Abstract

We discuss a relationship between certain one-dimensional quantum spin chains and anyon chains. In particular, we show how the XXZ Heisenberg chain is realized as a $D_3$ (alternatively $su(2)_4$) anyon model. We find that the difference between the models lies primarily in the choice of boundary conditions.

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1. Introduction

Formulations of many-body systems considered in quantum information theory differ from what is traditionally used in condensed matter physics. It is important that the different approaches are compared and that communication between the fields occurs. Here, we aim to shed light on and discuss the relationship between the recently proposed anyon chains from topological quantum information [12] and their quantum spin chain cousins from condensed matter physics. Specifically, we will provide conditions which when met imply a complete equivalence between the two different formalisms.

The one-dimensional anyon chains were constructed analogously to spin chains to provide a stepping stone to the understanding of higher dimensional anyonic models. These models were successful at demonstrating the presence of topological symmetries [5, 12]. The anyons used to construct these models typically exhibit non-standard braiding statistics and are not required to have integer quantum dimension\(^1\). The global Hilbert spaces used for the anyon chains rely on fusion paths [24, 28] and often have no tensor product structure. One fruitful method of constructing anyonic theories utilizes quasi-triangular Hopf algebras [10]. These anyon chains can also be constructed as the Hamiltonian limits of interaction-round-a-face or restricted solid-on-solid models, albeit not restricted to the ADE classification.

On the other hand, there are quantum spin chains which have a well-established place in modern condensed matter physics, providing insight into critical behaviour of correlated physical systems and describing quasi-one-dimensional materials [11]. As we are discussing

\(^1\) Roughly speaking, the quantum dimension is the dimension of the internal Hilbert space of the particle and determines the probability that fusion leads to annihilation or the creation of other anyons [28].
their connection to anyon chains, we will consider spin chains that have the underlying symmetry of a quasi-triangular Hopf algebra (e.g. a quantum group) [7, 23]. Consequently, it is natural to discuss models constructible from the quantum inverse scattering method (QISM) [19, 31] and its variants, although all results will have no dependence on integrability.

An equivalence between spin and anyon chains occurs when the underlying symmetry of each is that of the same quasi-triangular Hopf algebra. This equivalence has previously appeared as a face-vertex correspondence for integrable two-dimensional lattice models [26, 29]. The correspondence for one-dimensional quantum models will be illustrated by presenting the nearest-neighbour XXZ Heisenberg chain viewed as a $D_3$ anyon model. We use this model as it is a well-known and simple model for which we can calculate operators and energy spectra explicitly. The generalization to other models with the Hopf algebra symmetries is straightforward.

While the local XXZ Hamiltonian has the complete $D_3$ symmetry, the symmetry of the global Hamiltonian depends upon the boundary conditions imposed. Thus, the correspondence depends upon the boundary conditions, an aspect of these models not previously discussed2. We consider open boundaries with free ends [4, 30], periodic boundaries of both spin type [8, 25, 36] and anyon type [12, 33] and braided boundaries [14, 17, 22]. Of these, only the open and braided boundary conditions always have an equivalent description in the spin and anyon pictures.

It is also possible to present the XXZ model using other underlying symmetries, e.g. $su(2)_4$, $D_5$ or $U_q(su(2))$; however, using $D_3$ has certain advantages. There are no superfluous anyons, like the anyons in a half-integer subsector of $su(2)_4$ or an additional anyon of quantum dimension 2 in $D_5$. The anisotropy parameter is not dependent upon the algebra like $U_q(su(2))$, where $\frac{J}{J_z} = \cosh(\ln(q))$ ($J$ and $J_z$ are the coupling constants of the model). We also note that the XXZ Heisenberg chain has appeared in other papers in an anyonic form, specifically as the spin-1 $su(2)_4$ chain [13, 34], although not discussed as such.

2. Background

Here, we present the background information for the XXZ Heisenberg model, the $D_3$ algebra and the spin and anyon bases for the models. We also discuss when the operators in each of the bases are said to have the symmetry of $D_3$.

2.1. The algebra

$D_3$ is the group of symmetries on a triangle consisting of a rotation $\sigma$ and flip $\tau$. The group has the presentation

$$D_3 = \{ \sigma, \tau | \sigma^3 = \tau^2 = \sigma \tau \sigma \tau = 1 \}.$$ 

Its group algebra is the linear combination of its elements over the complex numbers. It is also possible to embed this algebra into a $k$-fold space by the use of the general coproduct,

$$\Delta^{(k)}(g) = \underbrace{g \otimes \ldots \otimes g}_{k\text{-times}}, \quad g \in D_3,$$

extended linearly to the algebra. This algebra is known to form a quasi-triangular Hopf algebra [7, 23]. As it is cocomutative, the universal $R$-matrix is just the identity operator.

2 While this correspondence has not been discussed in terms of one-dimensional quantum chains, similar observations relating to the partition function of certain two-dimensional lattice models have been made [9].
The representation theory of this algebra is also known; it has three irreducible representations (irreps): two are one-dimensional,

\[ \pi_\pm(\sigma) = 1, \quad \pi_\pm(\tau) = \pm 1, \]

and one is two-dimensional,

\[ \pi_2(\sigma) = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}, \quad \pi_2(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

For each irrep \( \pi_a \), we will associate a space (module) \( V_a \). The fusion rules are as follows:

\[ V_- \otimes V_- \cong V_+, \quad V_- \otimes V_2 \cong V_2 \quad \text{and} \quad V_2 \otimes V_2 \cong V_2 \oplus V_+ \oplus V_. \]

The space \( V_+ \) is the vacuum or trivial space and fusion with it is trivial, i.e., \( V_+ \otimes V_a \cong V_a \).

As fusion is associative, i.e., \( (V_a \otimes V_b) \otimes V_c \cong V_a \otimes (V_b \otimes V_c) \), there exists \( F \)-moves (generalized 6-j symbols) which relate the two different ways to decompose the threefold tensor product space [24, 33]. The interpretation of these operators can be understood via the diagrammatic interpretation presented later. The \( F \)-moves can be explicitly constructed from the representations above and are found in appendix B.

To construct an anyonic model, we associate with each irreducible \( D_3 \) module an anyon [10]. The fusion of anyons is governed by the fusion of the respective modules. Similarly, the other properties of the anyons are inherited from the representation theory of the algebra 3.

2.2. The bases

The Hilbert space for the spin formalism consists of the tensor product of \( L \) sites containing spin-\( \frac{1}{2} \) particles, alternatively qubits or other two-state systems, coupled to an auxiliary \( C^4 \) space, specifically

\[ [V_2 \oplus V_+ \oplus V_-] \otimes V_2^{\otimes L}. \]

This has a natural basis with \( 2^{L+2} \) vectors, which we shall refer to as the spin basis. We also note that if we project onto the \( V_+ \) component of the auxiliary space, then we are just left with \( L \) spin-\( \frac{1}{2} \) sites 4.

To form the Hilbert space for the anyon formalism, we need to consider fusion paths [24, 28] and as such we will refer to its basis as the fusion path basis. As this Hilbert space will be equivalent to the spin formalism, we again have an auxiliary space of \( V_2 \oplus V_+ \oplus V_- \) followed by \( L \) copies of \( V_2 \). To form a fusion path, working from left to right, we first fuse an irreducible subspace in the auxiliary space with \( V_2 \), choosing which irreducible space we project onto. This is then fused to another \( V_2 \), so on and so forth. We record the irreducible subspace of the auxiliary space and the subsequent irreducible subspaces which appear after fusion,

\[
\begin{array}{cccccc}
& 2 & 2 & 2 & 2 & 2 \\
a_0 & a_1 & a_2 & a_3 & a_{L-1} & a_L \\
\end{array}
\]

\[ \equiv ((V_{a_0} \otimes V_2)_{a_1} \otimes V_2)_{a_2} \cdots (V_{a_{L-1}} \otimes V_2)_{a_L} \]

\[ \equiv [a_0 a_1 a_2 \cdots a_{L-1} a_L]. \]

3 As the quantum dimension of the anyon equals the dimension of the associated irrep, all the anyons will have an integer quantum dimension. This will also be true for other anyonic theories defined via matrix representations of Hopf algebras.

4 For more general models, the global Hilbert space will be \( L \) copies of a certain irrep, corresponding to an anyon, coupled to an auxiliary space which is the direct sum of all irreps, i.e. a reservoir of all anyons.
Here, the use of the subscript of the bracket, \((\cdots)\), denotes the subspace of the space inside the bracket isomorphic to \(V_a\). These sequences of labels form the basis vectors of the anyon Hilbert space. Using this formulation, we observe that these fusion path basis vectors correspond to a subspace in the spin basis, whose dimensionality is equal to that of dimension of the outgoing anyon\(^5\). We remark that each label \(a_i\), as produced by fusion, is limited by the preceding label. We find that neighbouring pairs must belong to the following set:

\[a_i a_{i+1} \in \{+2, -2, 2+, 2-, 22\}\]

We now have an appropriate fusion path basis.

Diagrammatically, we have fused left to right; however, we can rearrange fusion adopting the additional convention of also fusing top to bottom. The reordering of fusion is done by the aforementioned \(F\)-moves. In terms of fusion diagrams, we have

\[\begin{array}{c}
\text{a} \\
\text{d} \\
\text{e}
\end{array} \quad \begin{array}{c}
\text{b} \\
\text{c}
\end{array} = \sum_{d'} (F_{abc})_{d'} a \begin{array}{c}
\text{d'} \\
\text{e}
\end{array}.
\]

On the left, the anyons \(a\) and \(b\) are fused with the result fusing to the anyon \(c\), while on the right, the anyons \(b\) and \(c\) are fused with the result fusing to \(a\). The \(F\)-moves must satisfy a pentagon equation, although in the \(D_3\) case this is automatically satisfied as \(D_3\) forms a Hopf algebra.

We also want to determine the dimensionality of the anyon Hilbert space. We define the number \(N^{(ab)}_{L}\) to be the number of basis vectors with \(a_0 = a\) and \(a_L = b\). The numbers are determined by the relations

\[N^{(a+)}_{L} = N^{(a-)}_{L-1} = N^{(a2)}_{L} = N^{(ba)}_{L} \quad \text{and} \quad N^{(a2)}_{L-1} = 2N^{(a+)}_{L-1}.
\]

We can recognize that these numbers are those appearing in the Jacobsthal sequence (A001045 [1]) which have the form

\[N^{(22)}_{L} = \frac{1}{2} (2^L + 1 + (-1)^L).
\]

Using this, we can determine the dimension of the anyon Hilbert space

\[\text{Anyon dimension} = \sum_{a,b} N^{(ab)}_{L} = N^{(22)}_{L+2} = \frac{1}{3} (2^{L+3} + (-1)^L).
\]

We can also determine that we indeed have the correct dimension of the spin Hilbert space

\[\text{Spin dimension} = \sum_{a,b} N^{(ab)}_{L} \times \dim(V_b) = 2^{L+2}.
\]

2.3. Equivalent operators

The global Hilbert spaces were both constructed using \(D_3\) models and therefore they should be \(D_3\) modules themselves. It follows that for an operator to be expressible in both the spin and anyon bases, it must have the underlying symmetry of the \(D_3\) algebra. In the spin formalism, this means that the operator must commute with the action of the algebra, while in the anyon formalism, we can invoke Schur’s lemma which will constrain the action on the outgoing anyon\(^6\).

\(^5\) While an individual fusion path basis vector will correspond to a subspace in the spin basis, generic vectors in the anyon Hilbert space will have no such correspondence.

\(^6\) This can be generalized to any quasi-triangular Hopf \(H\) with a corresponding anyonic theory. In the spin formalism, the operator must commute with the action of \(H\), which by Schur’s lemma will constrain the operator’s action on the outgoing anyon in the anyon formalism.
Suppose we have an operator $\mathcal{O}$ in the spin basis. It is said to have the $D_3$ symmetry if it commutes with the action of the algebra,

$$[\Pi(g), \mathcal{O}] = 0,$$

where

$$\Pi = \{(\tau_2 \oplus \tau_+ \oplus \tau_-) \oplus \tau_2^{\otimes \ell} \} \circ \Delta^{(\ell+1)},$$

for all $g \in D_3$. This operator will have a counterpart in the anyon fusion path which we will denote $\tilde{\mathcal{O}}$. The $D_3$ symmetry is both sufficient and necessary. As the operator commutes with the action of the algebra, Schur’s lemma requires

$$\text{If } a_{\mathcal{L}} \neq a'_{\mathcal{L}} \text{ then } \langle a'_0 a'_1 \ldots a'_{\mathcal{L}} | \tilde{\mathcal{O}} | a_0 a_1 \ldots a_{\mathcal{L}} \rangle = 0.$$  

(2)

Thus for an operator $\tilde{\mathcal{O}}$ in the fusion path basis to have a spin counterpart, the last label $a_{\mathcal{L}}$ must be invariant under the action of $\tilde{\mathcal{O}}$. Similarly, from construction, the auxiliary space must be also invariant under the action of the operator and thus the first label $a_0$ is invariant under $\tilde{\mathcal{O}}$. The fixing of $a_0$ and $a_{\mathcal{L}}$ is necessary and sufficient for an operator in the fusion path basis to have a counterpart in the spin basis.

As stated previously, this has a natural generalization to other models with a Hopf algebra symmetry. Condition (1) is unchanged while condition (2) requires that the outgoing anyon only remains of the same type. This modification is necessary when multiple copies of the same anyon can appear after the fusion of two anyons.

2.4. Projection operators and the local Hamiltonian

As we are dealing with a model with the $D_3$ symmetry, we expect that the global Hamiltonian will just be composed of projection operators. Additionally, we only consider models with nearest-neighbour interactions, so we further restrict ourselves to projection operators on two sites. In the spin basis, we have the two-site projection operator that is given by

$$P^{(b)} = \frac{\dim(V_b)}{6} \sum_{g \in D_3} \text{Trace}(\pi_6(g^{-1})) \pi_2(g) \otimes \pi_2(g).$$

By construction, this local operator commutes with the action of the algebra and has a corresponding operator in the fusion path basis. We can diagrammatically determine the projection operators in the following way [12]:

$$\tilde{P}^{(b)}_{i-1,i+1} \left\{ \begin{array}{ccc} 2 & 2 \\ a_{i-1} & a_i & a_{i+1} \end{array} \right\} = \sum_{b'} \left\{ \begin{array}{ccc} 2 & 2 \\ a_{i-1} & a_i & a_{i+1} \end{array} \right\} \left\{ \begin{array}{ccc} 2 & 2 \\ a'_{i-1} & a'_i & a'_{i+1} \end{array} \right\} \left\{ \begin{array}{ccc} 2 & 2 \\ a'_{i-1} & a'_i & a'_{i+1} \end{array} \right\}$$

$$= \sum_{a'_i} \left\{ \left( F_{a_{i+1}}^{a_{i-1},22} \right)_{a'_i} \right\} \left( F_{a_{i+1}}^{a_{i-1},22} \right)_{a'_i} \left\{ \begin{array}{ccc} 2 & 2 \\ a'_{i-1} & a'_i & a'_{i+1} \end{array} \right\},$$

provided the $F$-moves are unitary. Alternatively, we can write this as [13, 34]

$$\tilde{P}^{(b)}_{i-1,i+1} = \sum_{a_{i-1},a_i,a_{i+1}} \left( F_{a_{i+1}}^{a_{i-1},22} \right)_{a'_i}^{a_i} \left( F_{a_{i+1}}^{a_{i-1},22} \right)_{a'_i}^{a_i} \left(\ldots a_{i-1}a'_i a'_i a_{i+1}\ldots\right).$$

As expected, this two-site operator acts upon three labels in the fusion path basis and leaves the first and last anyons invariant under its action.

The original isotropic or XXX Heisenberg local Hamiltonian was defined as the exchange interaction on neighbouring sites. This was generalized to allow the strength of the interaction

7 As stated previously, this has a natural generalization to other models with a Hopf algebra symmetry. Condition (1) is unchanged while condition (2) requires that the outgoing anyon only remains of the same type. This modification is necessary when multiple copies of the same anyon can appear after the fusion of two anyons.
for spins in the $z$-direction to differ from those in the $x$- and $y$-directions resulting in the XXZ Hamiltonian below:

\[ h = \frac{J_z}{2} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + \frac{J_x}{2} (\sigma^z \otimes \sigma^z) + \left( \frac{J_z}{2} - J \right) I \otimes I, \]

\[ = \begin{pmatrix} J_z - J & 0 & 0 & 0 \\ 0 & -J & J & 0 \\ 0 & J & -J & 0 \\ 0 & 0 & 0 & J_z - J \end{pmatrix} = -2JP^(-) + (J_z - J) P^{(2)}, \tag{3} \]

where $\sigma^j$ are the usual Pauli matrices. This local Hamiltonian commutes with the action of $D_3$ as it is expressible in terms of projection operators. Furthermore, we can use the natural anyon analogues of the projection operators to determine its equivalent operator in the fusion path basis:

\[ \tilde{h} = -2JP^(-) + (J_z - J) P^{(2)}. \]

This is equivalent to the known local Hamiltonian for the ‘spin-1’ $su(2)_4$ model, up to a gauge transformation, mapping the anyons $(+, -)$ to $(0, 1, 2)$ [13, 34]. The anyons 0, 1 and 2 of $su(2)_4$ are the analogues of the spin-0, spin-1 and spin-2 particles of $su(2)$.

Other two-site $D_3$ invariant operators can be mapped between the two formalisms by

\[ o = c_+ P^{(+)} + c_- P^{(-)} + c_2 P^{(2)} \Leftrightarrow \tilde{o} = c_+ \tilde{P}^{(+)} + c_- \tilde{P}^{(-)} + c_2 \tilde{P}^{(2)}, \tag{4} \]

where $c_k \in \mathbb{C}$.

3. Quantum chains

To illustrate how the boundary conditions of a quantum chain affect the global symmetry, we provide an account of a variety of models. For the spin chains, we use models constructible via the QISM and its variants as these are commonly associated with quasi-triangular Hopf algebras. Open spin chains with free ends are seen to be in correspondence with open anyon chains, while closed models of either type are more complicated as the global symmetry can be broken. We find that among the closed models, braided models have a clear correspondence between the spin and anyonic formulations. It is then shown that while the periodic XXZ spin chain has an anyon counterpart, generic periodic spin chains do not. Likewise, we show that generic periodic anyon chains have no spin chain counterparts; this includes $D_3$ anyons.

3.1. Open chains

The simplest (and somewhat trivial) example of a direct equivalence between chains is the open chain with free end (non-interacting boundary fields) case. Whatever symmetry is contained by the local Hamiltonian is inherited by the global Hamiltonian (using the condition of coassociativity). The spin and anyon versions are of a very similar form:

\[ \mathcal{H} = \sum_{i=1}^{L-1} h_{(i+1)} \Leftrightarrow \tilde{\mathcal{H}} = \sum_{i=1}^{L-1} \tilde{h}_{(i+1)(i+1)}. \tag{5} \]

These provide models with identical energy spectra. This chain does not have a quantum group symmetry [20, 27]; however, its invariance under the action of $D_3$ will guarantee special degeneracies in the spin basis. To match up the degeneracies of each of the energies, the spin
dimension of each vector in the fusion path basis must be considered\(^8\). Here, we can see that the open XXZ chain is equivalent to the open \(D_3\) chain or the ‘spin-1’ \(su(2)\) chain restricted to the integer sector.

The introduction of non-trivial boundary fields will break the \(D_3\) in either basis removing the correspondence between the two formalisms.

3.2. Braided chains

Closed boundary conditions are more complicated due to the interaction between the first and last sites. One type of closed model which can be realized equivalently in both the spin and anyon bases is braided models \([14, 17, 22]\). These are guaranteed to have the full symmetry of the underlying algebra. In the case of the \(D_3\) chain, a braided model in the spin formalism requires the existence of an invertible operator \(b \in V_2 \otimes V_2\) satisfying the following\(^9\).

1. It is invertible and expressible in terms of projection operators of \(\pi_2 \otimes \pi_2\), i.e. commutes with the action of the algebra on the twofold tensor product space.
2. It satisfies the braid equation \(b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23}\).
3. It braids the local Hamiltonian \(h_{12}b_{23}h_{12} = b_{23}h_{12}b_{23}\) and \(b_{12}b_{23}h_{12} = h_{23}b_{12}b_{23}\).

Once such an operator is found, we can define a global braiding operator and global Hamiltonian:

\[
B = b_{12}b_{23} \ldots b_{(L-1)L} \quad \text{and} \quad \mathcal{H} = B^{-1}h_{12}B + \sum_{i=1}^{L-1} h_{(i+1)}\]

It follows that the global braiding operator and global Hamiltonian must commute with the action of the algebra\(^10\). The global braiding operator plays the role of a generalized translation operator, satisfying,

\[
Bh_{(i+1)}B^{-1} = h_{(i+1)(i+2)} \quad \text{and} \quad [B, \mathcal{H}] = 0,
\]

for \(1 \leq i \leq L - 2\). The additional term in this model, although it acts globally, is viewed as a local interaction as it commutes with all local Hamiltonians not acting on either site 1 or \(L\). Thus compared to the open chain, the additional term only gives a finite correction to the energy. As the global Hamiltonian commutes with the action of the algebra, this model has a natural anyonic counterpart. The anyonic counterpart is obtained by interchanging the local spin and anyon operators, i.e. using relation (4) to obtain \(h \leftrightarrow ˜h\) and \(b \leftrightarrow ˜b\) yielding

\[
B = ˜b_{012}b_{123} \ldots ˜b_{(L-2)(L-1)L} \quad \text{and} \quad ˜\mathcal{H} = B^{-1}h_{012}B + \sum_{i=1}^{L-1} ˜h_{(i-1)(i+1)}.
\]

For the XXZ chain, we find many different operators satisfying conditions 2 and 3; however, only one also satisfies condition 1. This operator corresponds to the representation of the universal \(R\)-matrix of \(D_3\) and gives rise to the periodic spin chain, which we discuss in the next section. The other operators, satisfying conditions 2 and 3 but not condition 1, may correspond to different anyonic theories.

\(^8\) To assist the reader, we have included the spectrum of the \(L = 4\) open chain in the appendix.

\(^9\) These conditions have been adapted from \([14, 17, 22]\) to construct a model with the \(D_3\) symmetry, which is also invariant under the action of the global braiding operator \(B\) but not necessarily integrable. It should be noted that we also require \(b\) and \(b_0\) to commute but for the \(D_3\) case, this is ensured by condition 1.

\(^10\) As the case with the open chain, the proof of this relies on each local operator, \(b_{(i+1)}\) and \(b_{(i+1)}\), commuting with the Hopf algebra due to its coassociativity. This is discussed in \([14, 17, 22]\).
3.3. The periodic XXZ spin chain

The periodic XXZ chain can be realized in the fusion path basis as it is also a braided model. This occurs because the permutation operator is also expressible in terms of projection operators and is consequently a suitable braiding operator: explicitly, this is

\[ \Pi = P^{(+)} - P^{(-)} + P^{(2)}, \quad \text{where} \quad \Pi(v \otimes w) = w \otimes v. \]

This allows the use of the braided model formalism to consider a periodic XXZ spin chain in the fusion path basis.

We remark that it is in general not possible to represent periodic spin chains in the fusion path basis as periodicity can break the underlying symmetry. The breaking of this underlying symmetry is related to the (lack of) cocomutativity of the quasi-triangular Hopf algebra in question. However, irrespective of whether the symmetry is broken, certain bulk properties including energy per site and the central charge are consistent with the open chain [2, 6].

3.4. The periodic D3 anyon chain

Now we consider the periodic D3 anyon chain starting from the viewpoint of an open chain. Using the L sites with the additional auxiliary space, we have the global Hamiltonian given by equation (5). We then impose periodicity in the basis by requiring that the incoming anyon is equal to the outgoing anyon, i.e. \( a_0 = a_L \); however, the model itself is not yet translationally invariant. We are only considering an invariant subspace of the full Hilbert space and now have that the auxiliary space is coupled to the rest. We can calculate both the anyon and spin dimensions

\[ \text{Anyon dimension} = \sum_a N^{(aa)}_L = 2^L + (-1)^L, \]

\[ \text{Spin dimension} = \sum_a N^{(aa)}_L \dim(V_a) = \frac{1}{3}[5 \cdot 2^L + 4 \cdot (-1)^L]. \]

At this stage, we have that \( a_L \) is still invariant under the action of the Hamiltonian and subsequently, there still exists a corresponding model in the spin basis.

To obtain the periodic anyon models as presented in [12], which are translationally invariant, we need to include the term \( \tilde{h}_{L-1,L1} \) yielding the global Hamiltonian

\[ \tilde{H} = \tilde{h}_{L-1,L1} + \tilde{h}_{L12} + \sum_{i=2}^{L-1} \tilde{h}_{i-1,i+1}. \]

Once this term is included, we no longer have that the outgoing (now also incoming) anyon is unchanged by the Hamiltonian implying that the D3 symmetry is lost and this model has no spin model counterpart\(^{11}\).

The periodic anyon boundary condition for this model has yet to be studied in substantial detail. It follows that the ground-state energy density must be the same as the periodic spin case. Also, the central charge will match the periodic and open spin chains. It is of interest then to compare the low-lying excitations of the XXZ chain [3, 16] with their anyon counterparts. While Verbus et al [34] have already constructed the same model, there is no discussion of its correspondence to the spin-1/2 XXZ model.

\(^{11}\) Alternatively, we could have considered a \( \mathcal{L} + 1 \) site model and required \( a_0 = a_L \) and \( a_1 = a_{L+1} \). This would not have demonstrated as clearly how the D3 invariance is lost.

\(^{12}\) We remark that while from the perspective of this paper the D3 symmetry has been lost, there are other notions of D3 symmetry which can be applied, for example, when the periodic anyon chain is viewed as living on a torus, eigenstates are classified by an associated flux, labelled by a D3 anyon, through the torus, rather than by an outgoing anyon [5, 12, 13].
4. Discussion

A correspondence between quantum spin and anyon chains exists when there is the underlying symmetry of a quasi-triangular Hopf algebra present. The symmetry inherited by the global Hamiltonian from the local Hamiltonian will depend upon the choice of boundary conditions. Open and braided models have a natural correspondence between the spin and anyon formalisms. On the other hand, periodic models generally do not have a natural correspondence in either formalism. In the spin language, the symmetry is present if the global Hamiltonian commutes with the action of the algebra, while in the anyon language, we require the incoming and outgoing anyons to be invariant under the action of the global Hamiltonian.

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Appendix A. Basic concepts of an anyonic system

Anyons are particles that are generalizations of bosons and fermions. The first generalization proposed the existence of anyons in a continuous two-dimensional space and was characterized by new braiding relations. Specifically, the interchange of two indistinguishable particles was allowed to result in an arbitrary phase shift $e^{i\theta}$ of the wavefunction of the particles in contrast to the restricted phases $\theta = 0, \pi$ associated with bosons and fermions [21, 35]. These braiding relations were extended to also include unitary transformations on a degenerate subspace of the many-particle wavefunction generated by the permutation of particles. Different definitions were also put forth which described anyons in any number of spatial dimensions, such as alternative exclusion principles [15]. In this paper, we are concerned with algebraic descriptions of anyonic systems given by finite monoidal categories equipped with braiding rules. These are useful for describing low-dimensional anyonic lattice models [18, 33].

An anyonic system will consist of a set of anyon types $\{x_i\}_{k=1}^K$ which are closed under fusion of particles,

$$x_i \otimes x_j = \bigoplus_{k=1}^K n_{i,j}^k x_k,$$

where $n_{i,j}^k$ are non-negative integers and satisfy an associativity condition. In the case of the $D_3$ anyonic theory, the anyon types are $\{2, +, -\}$ and the fusion rules are the same as fusion rules for the tensor product decomposition. Each anyon $x_i$ will have an associated quantum dimension, $d_i \in \mathbb{R}$, which satisfy

$$d_i d_j = \sum_{k=1}^K n_{i,j}^k d_k.$$

The anyonic systems considered here also require braiding rules which define the interchange of anyons and are given by a mapping:

$$R : x_i \otimes x_j \rightarrow x_j \otimes x_i.$$
This mapping defines the braiding statistics of the anyons. For further details, an introduction to the subject can be found in [33] and references therein.

Appendix B. F-moves and projection operators

We have calculated the $F$-moves though explicitly decomposing the space $V_a \otimes V_b \otimes V_c$ in the two different manners mentioned previously and by then looking at the transformations between them. The $F$-moves which deal with only one-dimensional irreps are

$$F_a^{bc} = \delta^{a_{xy}} \delta^{b}_{xy},$$

where $a, b, c \in \{+, -\}$. The $F$-moves with precisely one 2-particle present are

$$F_a^{b2} = \delta^{a_{xy}} \delta^{2}_{xy}, \quad (F_a^{2b})^y = \delta^{2}_{xy}, \quad (F_a^{2c})^y = \delta^{2}_{xy},$$

where $a, b, c \in \{+, -\}$. The $F$-moves with precisely two 2-particles present and one $+$-particle are

$$(F_+^{+2})^y = \delta^{2}_{xy}, \quad (F_+^{+2})^y = \delta^{2}_{xy}, \quad (F_+^{+2})^y = \delta^{2}_{xy},$$

Here are the $F$-moves with precisely two 2-particles present and one $-$-particle:

$$(F_-^{+2})^y = \delta^{2}_{xy}, \quad (F_-^{+2})^y = \delta^{2}_{xy}, \quad (F_-^{+2})^y = \delta^{2}_{xy},$$

Here are the other $F$-moves with all 2-particles:

$$(F_+^{22})^y = \delta^{2}_{xy}, \quad (F_-^{22})^y = \delta^{2}_{xy},$$

and

$$(F_0^{22})^y = \delta^{2}_{xy}.$$
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Table C1. The spectrum of the $L = 4$ open chain, restricted to the $V_+$ component of the auxiliary space, with $J = 1$ and $J_z = \cosh(\frac{2i\pi}{3})$. Spin mult. and Anyon mult. refer, respectively, to the multiplicity of the energy occurring in the spin and anyon formalisms. The symmetry sector is defined as the outgoing anyon or the subspace appearing in the decomposition of $\pi_+ \otimes \pi_2^{\otimes 4}$ which the eigenstate belongs to.

| Symmetry sector | Energy   | Spin mult. | Anyon mult. |
|-----------------|----------|------------|-------------|
| $\pi_+$         | $-3.9050$ | 1          | 1           |
|                 | $1.8924$  | 1          | 1           |
|                 | $5.3781$  | 1          | 1           |
| $\pi_-$         | $-3.1196$ | 1          | 1           |
|                 | $1.1218$  | 1          | 1           |
|                 | $5.3632$  | 1          | 1           |
| $\pi_2$         | $-0.0665$ | 2          | 1           |
|                 | $1.8290$  | 2          | 1           |
|                 | $5.4320$  | 2          | 1           |
|                 | $5.5365$  | 2          | 1           |
|                 | $9.3655$  | 2          | 1           |

Appendix C. Energies for the open chain with free ends

There is an equivalence between the spin and anyon formalisms for the open chains with free ends. Here, we have provided a concrete example to demonstrate this. For the open XXZ chain with four sites, we set the coupling parameters to $J = 1$ and $J_z = \cosh(\frac{2i\pi}{3})$. Furthermore, we restrict our auxiliary space to $V_+$, implying a spin dimension of $2^4 = 16$ and an anyon dimension of $\sum_a N_3^{(a)} = 11$. Numerically, we find the energies and multiplicities presented in table C1.

We see that the eigenspectra of the two formalisms are the same and that the multiplicity in the spin formulation is simply the product of the dimension of irrep of the symmetry sector and the anyon multiplicity. Projection onto the $\pi_+$ symmetry sector in a spin picture is achieved by applying the global projection operator $\frac{\dim(V_+)}{6} \sum_{g \in D_3} \text{Trace}(\pi_+(g^{-1})) \Pi(g)$.

References

[1] 2011 The On-Line Encyclopedia of Integer Sequences http://oeis.org
[2] Affleck I 1986 Universal term in the free energy at a critical point and the conformal anomaly Phys. Rev. Lett. 56 746–8
[3] Alcaraz F C, Barber M N and Batchelor M T 1988 Conformal invariance, the XXZ chain and the operator content of two-dimensional critical systems Ann. Phys. 182 280–343
[4] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 Surface exponents of the quantum XXZ, Ashkin–Teller and Potts models J. Phys. A: Math. Gen. 20 6397–409
[5] Ardonne E, Gukelberger J, Ludwig A W W, Trebst S and Troyer M 2011 Microscopic models of interacting Yang–Lee anyons New J. Phys. 13 045006
[6] Blote H W J, Cardy J L and Nightingale M P 1986 Conformal invariance, the central charge, and universal finite-size amplitudes at criticality Phys. Rev. Lett. 56 742–5
[7] Chari V and Pressley A 1994 A Guide to Quantum Groups (Cambridge: Cambridge University Press)
[8] Cloizeaux J D and Gaudin M 1986 Anisotropic linear magnetic chain J. Math. Phys. 7 1384
[9] de Vega H J and Giaconia H J 1989 Intertwining vectors and the connection between critical vertex and SOS models J. Phys. A: Math. Gen. 22 2759–79
[10] de Wild Propitius M and Bais F A 1998 Discrete Gauge Theories in Particles and Fields (CRM Series in Mathematical Physics) ed G Semenoff and L Vinet (New York: Springer) p 353
[11] Essler F, Frahm H, Göhmann F, Klümper A and Korepin V E 2005 The One-Dimensional Hubbard Model (Cambridge: Cambridge University Press)
[12] Feiguin A, Trebst S, Ludwig A W W, Troyer M, Kitaev A Y, Wang Z and Freedman M H 2007 Interacting anyons in topological quantum liquids: the golden chain Phys. Rev. Lett. 98 160409
[13] Gils C, Ardonne E, Trebst S, Huse D A, Ludwig A W W, Troyer M and Wang Z 2013 Anyonic quantum spin chains: spin-1 generalizations and topological stability in preparation
[14] Grosse H, Pallua S, Prester P and Raschhofer E 1994 On a quantum group invariant spin chain with non-local boundary conditions J. Phys. A: Math. Gen. 27 4761–71
[15] Haldane F D M 1991 ‘Fractional statistics’ in arbitrary dimensions: a generalization of the Pauli principle Phys. Rev. Lett. 67 937–40
[16] Hamer C J 1986 Finite-size corrections for ground states of the XXZ Heisenberg chain J. Phys. A: Math. Gen. 19 3335
[17] Karowski M and Zapletal A 1994 Quantum group invariant integrable n-state vertex models with periodic boundary conditions Nucl. Phys. B 419 567–88
[18] Kitaev A 2006 Anyons in an exactly solved model and beyond Ann. Phys. 321 2–111
[19] Korepin V E, Bogoliubov N M and Izergin A G 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)
[20] Kulish P P and Sklyanin E K 1991 The general $U_q(\mathfrak{sl}(2))$ invariant XXZ integrable quantum spin chain J. Phys. A: Math. Gen. 24 L435–9
[21] Leinaas J M and Myrheim J 1977 On the theory of identical particles Il Nuovo Cimento B 37 1–23
[22] Links J and Foerster A 1997 On the construction of integrable closed chains with quantum supersymmetry J. Phys. A: Math. Gen. 30 2483–7
[23] Majid S 1995 Foundations of Quantum Group Theory (Cambridge: Cambridge University Press)
[24] Moore G and Seiberg N 1989 Classical and quantum conformal field theory Commun. Math. Phys. 123 171–254
[25] Orbach R 1958 Linear antiferromagnetic chain with anisotropic coupling Phys. Rev. 112 309–16
[26] Pasquier V 1988 Etiology of IRF models Commun. Math. Phys. 118 355–64
[27] Pasquier V and Saleur H 1990 Common structures between finite systems and conformal field theories through quantum groups Nucl. Phys. B 330 523–56
[28] Preskill J 2004 Lecture Notes for Physics 219: Quantum Computation (Pasadena, CA: California Institute of Technology) www.theory.caltech.edu/~preskill/ph219/topological.pdf
[29] Roche P 1990 Ocneanu cell calculus and integrable lattice models Commun. Math. Phys. 127 395–424
[30] Sklyanin E K 1988 Boundary conditions for integrable quantum systems J. Phys. A: Math. Gen. 21 2375–89
[31] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 Quantum inverse problem method. 1 Theor. Math. Phys. 40 688–706
[32] Slingerland J K and Bais F A 2001 Quantum groups and non-Abelian braiding in quantum Hall systems Nucl. Phys. B 612 229–90
[33] Trebst S, Troyer M, Wang Z and Ludwig A W W 2008 A short introduction to Fibonacci anyon models Prog. Theor. Phys. Suppl. 176 384
[34] Verbus V, Martina L and Protogenov A P 2011 Chain of interacting $SU(2)_4$ anyons and quantum $SU(2)_4 \times SU(2)_4$ doubles Theor. Math. Phys. 167 843–55
[35] Wilczek F 1982 Quantum mechanics of fractional-spin particles Phys. Rev. Lett. 49 957–9
[36] Yang C N and Yang C P 1966 One-dimensional chain of anisotropic spin–spin interactions: I. Proof of Bethe’s hypothesis for ground state in a finite system Phys. Rev. 150 321–7