Concentration phenomena in high dimensional geometry.

Olivier Guédon

Université Paris-Est Marne-la-Vallée

Workshop Algorithmic Geometry.
October 2015
Conjecture and thematics.

Let $X$ be a random vector uniformly distributed on an isotropic (choice of the Euclidean structure) convex body in $\mathbb{R}^n$.

Conjecture
All the volume is concentrated in a thin Euclidean shell.

$$\mathbb{P} \left( \left| ||X||_2 - \sqrt{n} \right| \geq t\sqrt{n} \right) \leq C \exp(-c t \sqrt{n})$$

Hölder or reverse Hölder inequalities.
Convex body in "isotropic position".
Intersection with a Euclidean ball of radius $\sqrt{n}$. 

Pictures - Intuition in high dimension.
volume in a shell of radius $\sqrt{n}$ and width 1
Brunn-Minkowski inequality.

Let $A$ and $B$ be two compacts in $\mathbb{R}^n$ such that $|A| \cdot |B| > 0$ then

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

Geometry of convex bodies:
Let $K$ be a convex body with non empty interior, $\lambda \in [0, 1]$

$$|(1 - \lambda)(K \cap A) + \lambda(K \cap B)|^{1/n} \geq (1 - \lambda)|K \cap A|^{1/n} + \lambda|K \cap B|^{1/n}$$

whenever $|K \cap A| \cdot |K \cap B| > 0$
Brunn-Minkowski inequality.

Let $A$ and $B$ be two compacts in $\mathbb{R}^n$ such that $|A| \cdot |B| > 0$ then

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

Geometry of convex bodies:

Let $K$ be a convex body with non empty interior, $\lambda \in [0, 1]$

$$|(1 - \lambda)(K \cap A) + \lambda(K \cap B)|^{1/n} \geq (1 - \lambda)|K \cap A|^{1/n} + \lambda|K \cap B|^{1/n}$$

whenever $|K \cap A| \cdot |K \cap B| > 0$

Consequence. Let $\mu$ be the uniform measure on $K$ then

$$\mu((1 - \lambda)A + \lambda B)^{1/n} \geq (1 - \lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}$$

when $\mu(A)\mu(B) > 0$. 
Brunn-Minkowski inequality.

Let $A$ and $B$ be two compacts in $\mathbb{R}^n$ such that $|A| \cdot |B| > 0$ then

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

Geometry of convex bodies:
Let $K$ be a convex body with non empty interior, $\lambda \in [0, 1]$

$$|(1 - \lambda)(K \cap A) + \lambda(K \cap B)|^{1/n} \geq (1 - \lambda)|K \cap A|^{1/n} + \lambda|K \cap B|^{1/n}$$

whenever $|K \cap A| \cdot |K \cap B| > 0$

$\mu$ uniform measure on $K$, for every compact $A, B$

$$\mu \left( (1 - \lambda)A + \lambda B \right) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}$$
Brunn-Minkowski inequality.

Let \( A \) and \( B \) be two compacts in \( \mathbb{R}^n \) such that \( |A| \cdot |B| > 0 \) then

\[
|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}
\]

Geometry of convex bodies:
Let \( K \) be a convex body with non empty interior, \( \lambda \in [0, 1] \)

\[
|(1 - \lambda)(K \cap A) + \lambda(K \cap B)|^{1/n} \geq (1 - \lambda)|K \cap A|^{1/n} + \lambda|K \cap B|^{1/n}
\]

whenever \( |K \cap A| \cdot |K \cap B| > 0 \)

\( \mu \) uniform measure on \( K \), for every compact \( A, B \)

\[
\mu ( (1 - \lambda)A + \lambda B ) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}
\]

We say that \( \mu \) is log-concave.
Log-concave measures.

Let $f : \mathbb{R}^n \to \mathbb{R}^+$ such that $\forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1],

\[ f((1 - \theta)x + \theta y) \geq f(x)^{1-\theta} f(y)^{\theta} \]

A measure with density $f \in L^1_{\text{loc}}$ is said to be log-concave and satisfies $\forall A, B \subset \mathbb{R}^n, \forall \theta \in [0, 1],

\[ \mu((1 - \theta)A + \theta B) \geq \mu(A)^{1-\theta} \mu(B)^{\theta} \]

60’s and 70’s : Henstock-McBeath, Borell, Prékopa-Leindler...
Log-concave measures.

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) such that \( \forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1], \)

\[
f((1 - \theta)x + \theta y) \geq f(x)^{1-\theta} f(y)\theta
\]

A measure with density \( f \in L_1^{\text{loc}} \) is said to be log-concave and satisfies \( \forall A, B \subset \mathbb{R}^n, \forall \theta \in [0, 1], \)

\[
\mu((1 - \theta)A + \theta B) \geq \mu(A)^{1-\theta} \mu(B)^\theta
\]

60’s and 70’s : Henstock-Mc Beath, Borell, Prékopa-Leindler...

Classical examples :
1) Probabilistic : \( f(x) = \exp(-|x|^2), f(x) = \exp(-|x|_1) \)
2) Geometric : \( f(x) = 1_K(x) \) where \( K \) is a convex body.
Properties of log-concave measures.

**Marginals**

Let \( w : \mathbb{R}^2 \to \mathbb{R}_+ \) be a log-concave function. Then

\[
x \mapsto \int w(x, y) \, dy
\]

is log-concave on \( \mathbb{R} \).

In other words, when \( \mu \) is log-concave, for every subspace \( F \), the marginal \( \pi_F \mu \) is log-concave.
Properties of log-concave measures.

Marginals
Let \( w : \mathbb{R}^2 \to \mathbb{R}_+ \) be a log-concave function. Then

\[
x \mapsto \int w(x, y) dy
\]

is log-concave on \( \mathbb{R} \).

In other words, when \( \mu \) is log-concave, for every subspace \( F \), the marginal \( \pi_F \mu \) is log-concave.

Convolution
If \( f \) and \( g \) are two log-concave functions on \( \mathbb{R} \) then

\[
x \mapsto \int f(x - y) g(y) dy
\]

is log-concave on \( \mathbb{R} \).

In other words, if \( X \) et \( Y \) are random vectors with log-concave law then \( X + Y \) is log-concave.
K. Ball
*Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$.* Studia Math. 88 (1988), no. 1, 69–84
and more recent ones of Klartag, Paouris . . .
L. Lovász, M. Simonovits
*Random walks in a convex body and an improved volume algorithm.* Random Structures Algorithms 4 (1993), no. 4, 359–412.
R. Kannan, L. Lovász, M. Simonovits
*Isoperimetric problems for convex bodies and a localization lemma.* Discrete Comput. Geom. 13 (1995), no. 3-4, 541–559.
*Random walks and an $O^*(n^5)$ volume algorithm for convex bodies.* Random Structures Algorithms 11 (1997), no. 1, 1–50.
The hyperplane conjecture:

does there exist a constant $C > 0$ such that:

for every $n$ and every convex body $K \subset \mathbb{R}^n$ of volume 1 and barycenter at the origin, there is a direction $\xi$ such that $\text{Vol}(K \cap \xi^\perp) \geq C$?

let $K_1$ and $K_2$ be two convex bodies with barycenter at the origin such that for every $\xi \in S^{n-1}$

$$\text{Vol}(K_1 \cap \xi^\perp) \leq \text{Vol}(K_2 \cap \xi^\perp)$$

then $\text{Vol}(K_1) \leq C \text{Vol}(K_2)$?
Convex geometry - Log-concave measures.

The hyperplane conjecture : equivalent formulation

\[ n L_K^2 = \min_{\mathcal{E}, \text{Vol} \mathcal{E} = \text{Vol} B^n_2} \frac{1}{(\text{Vol} K)^{1 + \frac{2}{n}}} \int_K \|x\|^2_\mathcal{E} \, dx, \quad \sup_{n, K} L_K \leq C? \]
Convex geometry - Log-concave measures.

The hyperplane conjecture: equivalent formulation

\[ nL_K^2 = \min_{\mathcal{E}, \text{Vol } \mathcal{E} = \text{Vol } B^n_2} \frac{1}{(\text{Vol } K)^{1 + \frac{2}{n}}} \int_K \|x\|^2 \mathcal{E} \, dx, \quad \sup_{n,K} L_K \leq C? \]

Attained when \( K \) is in isotropic position:
\( K \) has barycenter at the origin and the inertia matrix is the identity

\[ \frac{1}{\text{Vol } K} \int_K x_i x_j \, dx = \delta_{i,j}. \quad L_K = \frac{1}{(\text{Vol } K)^{\frac{1}{n}}} \]
Convex geometry - Log-concave measures.

The hyperplane conjecture: equivalent formulation

\[ nL^2_K = \min_{\mathcal{E}, \Vol \mathcal{E} = \Vol B^n_2} \frac{1}{(\Vol K)^{1 + \frac{2}{n}}} \int_K \|x\|^2 \, dx, \quad \sup_{n, K} L_K \leq C? \]

Attained when \( K \) is in isotropic position:

\( K \) has barycenter at the origin and the inertia matrix is the identity

\[ \frac{1}{\Vol K} \int_K x_i x_j \, dx = \delta_{i,j}, \quad L_K = \frac{1}{(\Vol K)^{\frac{1}{n}}} \]

Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \) be a log-concave isotropic function,

\[ \int f(x) \, dx = 1, \quad \int x f(x) \, dx = 0, \quad \int x_i x_j f(x) \, dx = \delta_{i,j}. \]

\[ \sup_{f \text{ isotropic}} f(0)^{1/n} \leq C? \]
Convex geometry - Log-concave measures.

The hyperplane conjecture: equivalent formulation

\[ nL_K^2 = \min_{E, \text{Vol } E = \text{Vol } B^n_2} \frac{1}{(\text{Vol } K)^{1 + \frac{2}{n}}} \int_K \|x\|_E^2 \, dx, \quad \text{sup}_{n, K} L_K \leq C? \]

Attained when \( K \) is in isotropic position:
\( K \) has barycenter at the origin and the inertia matrix is the identity

\[ \frac{1}{\text{Vol } K} \int_K x_i x_j \, dx = \delta_{i,j}, \quad L_K = \frac{1}{(\text{Vol } K)^{\frac{1}{n}}} \]

Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \) be a log-concave isotropic function,

\[ \int f(x) \, dx = 1, \quad \int x f(x) \, dx = 0, \quad \int x_i x_j f(x) \, dx = \delta_{i,j}. \]

\[ \sup_{f \text{ isotropic}} f(0)^{1/n} \leq C? \]

Theorem (Ball). These two questions are equivalent.
**Theorem** (Ball, ’85). Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function. Then for every $p > 0$, the function $F : \mathbb{R}^n \to \mathbb{R}_+$

$$x \mapsto \left( \int_0^{+\infty} f(rx) r^{p-1} dr \right)^{-1/p}$$

is convex.
Theorem (Ball, ’85). Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function. Then for every $p > 0$, the function $F : \mathbb{R}^n \to \mathbb{R}_+$

$$x \mapsto \left( \int_0^{+\infty} f(rx) r^{p-1} dr \right)^{-1/p}$$

is convex. And homogeneous.
Theorem (Ball, ’85). Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a log-concave function. Then for every $p > 0$, the function $F : \mathbb{R}^n \to \mathbb{R}_+$

$$x \mapsto \left( \int_0^{+\infty} f(rx) r^{p-1} dr \right)^{-1/p}$$

is convex. And homogeneous.

When $f(0) > 0$, we define a family of convex sets

$$K_p(f) = \left\{ x \in \mathbb{R}^n, \ p \int_0^{+\infty} f(rx) r^{p-1} dr \geq f(0) \right\}$$
Computing the volume of a convex body

$K \subset \mathbb{R}^n$ is given by a separation oracle
$K \subset \mathbb{R}^n$ is given by a separation oracle

Elekes (’86), Bárány-Füredi (’86) : it is not possible to compute with a deterministic algorithm in polynomial time the volume of a convex body (even approximately)
Computing the volume of a convex body

$K \subset \mathbb{R}^n$ is given by a separation oracle

Elekes (’86), Bárány-Füredi (’86): it is not possible to compute with a deterministic algorithm in polynomial time the volume of a convex body (even approximately)

**Randomization** - Given $\varepsilon$ and $\eta$, Dyer-Frieze-Kannan (’89) established randomized algorithms returning a non-negative number $\zeta$ such that

$$(1 - \varepsilon)\zeta < \text{Vol } K < (1 + \varepsilon)\zeta$$

with probability at least $1 - \eta$. The running time of the algorithm is polynomial in $n$, $1/\varepsilon$ and $\log(1/\eta)$. 
Computing the volume of a convex body

$K \subset \mathbb{R}^n$ is given by a separation oracle

Elekes (’86), Bárány-Füredi (’86) : it is not possible to compute with a deterministic algorithm in polynomial time the volume of a convex body (even approximately)

**Randomization** - Given $\varepsilon$ and $\eta$, Dyer-Frieze-Kannan (’89) established randomized algorithms returning a non-negative number $\zeta$ such that

$$(1 - \varepsilon)\zeta < \text{Vol } K < (1 + \varepsilon)\zeta$$

with probability at least $1 - \eta$. The running time of the algorithm is polynomial in $n$, $1/\varepsilon$ and $\log(1/\eta)$.

The number of oracle calls is a random variable and the bound is for example on its expected value.
Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.
Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

Rounding - Put the convex body in a position where

\[ B_2^n \subset K \subset d B_2^n \]

where \( d \leq n^{\text{const}} \).
Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

**Rounding** - Put the convex body in a position where

\[ B_2^n \subset K \subset d B_2^n \]

where \( d \leq n^{\text{const}} \).

- John ('48): \( d \leq n \) (or \( d \leq \sqrt{n} \) in the symmetric case).

How to find an algorithm to do so?
Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

**Rounding** - Put the convex body in a position where

\[ B_n^2 \subset K \subset d \, B_n^2 \]

where \( d \leq n^{\text{const}} \).

- Idea: find an algorithm which produces in polynomial time a matrix \( A \) such that \( AK \) is in an approximate isotropic position.  

Conjecture 2 of KLS (’97) : solved in 2010 by Adamczak, Litvak, Pajor, Tomczak-Jaegermann
Computing the volume of a convex body

The randomized algorithm proposed by Kannan, Lovász and Simonovits improves significantly the polynomial dependence.

**Rounding** - Put the convex body in a position where

\[ B_2^n \subset K \subset d B_2^n \]

where \( d \leq n^{\text{const}} \).

- Idea : find an algorithm which produces in polynomial time a matrix \( A \) such that \( AK \) is in an approximate isotropic position.

Conjecture 2 of KLS ('97) : solved in 2010 by Adamczak, Litvak, Pajor, Tomczak-Jaegermann

Computing the volume - Monte Carlo algorithm, estimates of local conductance.

Conjecture 1 of KLS ('95) : isoperimetric inequality - open!
Isoperimetric problem.
Isoperimetric problem.

Define

$$\mu^+ (S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B_2^n) - \mu(S)}{\varepsilon}$$
Isoperimetric problem.

Define

\[ \mu^+(S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B^n_2) - \mu(S)}{\varepsilon} \]

**Question.** Find the largest \( h \) such that

\[ \forall S \subset K, \quad \mu^+(S) \geq h \mu(S)(1 - \mu(S)) \]

\( \mu \) is log-concave with log concave density \( f \).
Isoperimetric problem.

Define
\[ \mu^+(S) = \lim_{\varepsilon \to 0} \inf \frac{\mu(S + \varepsilon B^n_2) - \mu(S)}{\varepsilon} \]

Question. Find the largest \( h \) such that
\[ \forall S \subset K, \mu^+(S) \geq h \mu(S)(1 - \mu(S)) \]

\( \mu \) is log-concave with log concave density \( f \).

The probability \( d\mu(x) = f(x)dx \) is log-concave isotropic. Poincaré type inequality. For every regular function \( F \),
\[ h^2 \text{ Var}_\mu F \leq \int |\nabla F(x)|^2 f(x)dx. \]

The conjecture is that \( h \) is a universal constant.
Kannan, Lovász, Simonovits [’95],

$$h \geq \frac{c}{\int_K |x - g_K|_2^2 dx}$$

Bobkov [’07]:

$$h \geq \frac{c}{(\text{Var } |X|_2^2)^{1/4}}.$$
Kannan, Lovász, Simonovits ['95], Bobkov ['07]:

\[ h \geq \frac{c}{\int_K |x - g_K|^2 dx} \quad h \geq \frac{c}{(\text{Var} |X|^2)^{1/4}}. \]

Poincaré type inequality. For every regular function \( F \),

\[ h^2 \text{Var}_\mu F \leq \int |\nabla F(x)|^2 f(x) dx. \]

KLS conjecture is that \( h \) is a universal constant.

Take \( F(x) = |x|_2 \) or \( F(x) = |x|_2^p \)

Strong concentration of the Euclidean norm

\[ P \left( \left| |X|_2 - \sqrt{n} \right| \geq t\sqrt{n} \right) \leq C \exp(-c t \sqrt{n}) \]
Kannan, Lovász, Simonovits ['95], Bobkov ['07]:

\[ h \geq \frac{c}{\int_K |x - g_K|_2^2 dx} \]

\[ h \geq \frac{c}{(\text{Var} |X|^2)^{1/4}}. \]

**Poincaré type inequality.** For every regular function \( F \),

\[ h^2 \text{Var}_\mu F \leq \int |\nabla F(x)|_2^2 f(x) dx. \]

**KLS conjecture** is that \( h \) is a universal constant.

Take \( F(x) = |x|_2 \) or \( F(x) = |x|^p \)

**Strong concentration of the Euclidean norm**

\[ \mathbb{P} \left( \|X\|_2 - \sqrt{n} \geq t\sqrt{n} \right) \leq C \exp(-c t \sqrt{n}) \]

Large and medium scales!
Proposition. (Borell ’73) Let \( \mu \) be a log-concave probability, \( C \) a symmetric convex set in \( \mathbb{R}^n \) such that \( \mu(C) \geq 2/3 \). Then for every \( t \geq 1 \),

\[
\mu \left( \mathbb{R}^n \setminus (tC) \right) \leq \left( \frac{1}{2} \right)^{\frac{t+1}{2}}
\]
Proposition. (Borell ’73) Let $\mu$ be a log-concave probability, $C$ a symmetric convex set in $\mathbb{R}^n$ such that $\mu(C) \geq 2/3$. Then for every $t \geq 1$,

$$\mu(\mathbb{R}^n \setminus (tC)) \leq \left(\frac{1}{2}\right)^{t+1}$$

Indeed for $\alpha = \frac{t-1}{t+1}$ we have: $1 - \alpha = \frac{2}{t+1}$ and

$$(1 - \alpha)(\mathbb{R}^n \setminus (tC)) + \alpha C \subset (\mathbb{R}^n \setminus C)$$
Concentration - Khintchine

**Proposition.** (Borell ’73) Let $\mu$ be a log-concave probability, $C$ a symmetric convex set in $\mathbb{R}^n$ such that $\mu(C) \geq 2/3$. Then for every $t \geq 1$,

$$\mu(\mathbb{R}^n \setminus (tC)) \leq \left(\frac{1}{2}\right)^{t+1} t^{1/2}$$

**Consequences:** reverse Hölder inequality. If $X$ is a log-concave random vector then for every $\theta \in \mathbb{R}^n$, for every $p \geq 2$,

$$\left(\mathbb{E}|\langle X, \theta \rangle|^p\right)^{1/p} \leq C p \left(\mathbb{E}|\langle X, \theta \rangle|^2\right)^{1/2}.$$
**Proposition.** (Borell ’73) Let $\mu$ be a log-concave probability, $C$ a symmetric convex set in $\mathbb{R}^n$ such that $\mu(C) \geq 2/3$. Then for every $t \geq 1$,

$$\mu(\mathbb{R}^n \setminus (tC)) \leq \left(\frac{1}{2}\right)^{t+1}$$

**Consequences:** reverse H"{o}lder inequality. If $X$ is a log-concave random vector then for every $\theta \in \mathbb{R}^n$, for every $p \geq 2$,

$$\left(\mathbb{E}|\langle X, \theta \rangle|^p\right)^{1/p} \leq C p \left(\mathbb{E}|\langle X, \theta \rangle|^2\right)^{1/2}.$$  

$$C = \left\{ x \in \mathbb{R}^n, |\langle x, \theta \rangle| \leq 3 \left(\mathbb{E}|\langle X, \theta \rangle|^2\right)^{1/2} \right\}$$
Proposition. (Borell ’73) Let $\mu$ be a log-concave probability, $C$ a symmetric convex set in $\mathbb{R}^n$ such that $\mu(C) \geq 2/3$. Then for every $t \geq 1$,

$$
\mu(\mathbb{R}^n \setminus (tC)) \leq \left(\frac{1}{2}\right)^{\frac{t+1}{2}}
$$

Consequences: reverse Hölder inequality. If $X$ is a log-concave random vector then for every $\theta \in \mathbb{R}^n$, for every $p \geq 2$,

$$
(\mathbb{E}|\langle X, \theta \rangle|^p)^{1/p} \leq C p \left(\mathbb{E}|\langle X, \theta \rangle|^2\right)^{1/2}.
$$

$$
C = \left\{ x \in \mathbb{R}^n, |\langle x, \theta \rangle| \leq 3 \left(\mathbb{E}|\langle X, \theta \rangle|^2\right)^{1/2} \right\}
$$

norm, Khintchine-Kahane
Evidence : in isotropic position, $\mathbb{E}|X|^2_2 = n$. Take the proposition with

$$C = \left\{ x \in \mathbb{R}^n, \ |x|_2 \leq \sqrt{3n} \right\}$$

then $\mu(C) \geq 2/3$ and for every $t \geq 1$,

$$\mu(\mathbb{R}^n \setminus (tC)) \leq \left( \frac{1}{2} \right)^{t+1}$$
Results.

**Evidence**: $X$ log-concave then for every $t \geq 1$, \[
P\left\{ |X|_2 \geq t\sqrt{3n} \right\} \leq e^{-t}
\]
Results.

**Evidence**: $X$ log-concave then for every $t \geq 1$,

$$\mathbb{P}\left\{ |X|_2 \geq t\sqrt{3n} \right\} \leq e^{-t}$$

**Theorem** (Paouris 2006). For every $t \geq 10$

$$\mathbb{P}\left\{ |X|_2 \geq t\sqrt{n} \right\} \leq Ce^{-ct\sqrt{n}}$$
Evidence: $X$ log-concave then for every $t \geq 1$,

$$\mathbb{P} \left\{ |X|_2 \geq t\sqrt{3n} \right\} \leq e^{-t}$$

Theorem (Paouris 2006). For every $t \geq 10$

$$\mathbb{P} \left\{ |X|_2 \geq t\sqrt{n} \right\} \leq Ce^{-c t \sqrt{n}}$$

After works of Klartag, Fleury-G-Paouris, Fleury

Theorem (G-Milman 2011). For every $t \in (0, 1)$

$$\mathbb{P} \left\{ ||X||_2 - \sqrt{n} \geq t\sqrt{n} \right\} \leq Ce^{-c t^3 \sqrt{n}}$$
Pictures - Intuition in high dimension.

convex body in "isotropic position".
Pictures - Intuition in high dimension.

intersection with a ball of radius $\sqrt{n}$.
volume inside a ball of radius $100\sqrt{n}$
volume inside a shell of width $\sqrt{n}/n^{1/6}$
Thin shell and central limit theorem

CLT: classical case. $x_1, \ldots, x_n$, $n$ i.i.d random variables,

$E x_i^2 = 1, E x_i = 0, E x_i^3 = \tau$

then $\forall \theta \in S^{n-1}$

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \tau \left| \theta \right|_4^2 = \frac{\tau}{\sqrt{n}}.$$
Question. [Ball ’97], [Brehm-Voigt ’98] Let \( K \) be an isotropic convex body, find a direction \( \theta \in S^{n-1} \) such that

\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \alpha_n
\]

with \( \lim_{+\infty} \alpha_n = 0 \)?
Question. [Ball ’97], [Brehm-Voigt ’98] Let $K$ be an isotropic convex body, find a direction $\theta \in S^{n-1}$ such that

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \alpha_n$$

with $\lim_{+\infty} \alpha_n = 0$?

Conjecture. [Anttila-Ball-Perissinaki ’03]

Thin shell conjecture : $\forall n, \exists \varepsilon_n$ such that for every random vector uniformly distributed in an isotropic convex body

$$\mathbb{P} \left( \left| \frac{|X|_2^2}{\sqrt{n}} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n$$

with $\lim_{+\infty} \varepsilon_n = 0$. Or more vaguely, does $\text{Var} \frac{|X|_2}{n}$ go to zero as $n \to \infty$?
Question. [Ball ’97], [Brehm-Voigt ’98] Let $K$ be an isotropic convex body, find a direction $\theta \in S^{n-1}$ such that

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \alpha_n$$

with $\lim_{+\infty} \alpha_n = 0$?

Conjecture. [Anttila-Ball-Perissinaki ’03]

Thin shell conjecture : $\forall n, \exists \varepsilon_n$ such that for every random vector uniformly distributed in an isotropic convex body

$$\mathbb{P} \left( \left| \frac{|X|^2}{\sqrt{n}} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n$$

with $\lim_{+\infty} \varepsilon_n = 0$. Or more vaguely, does $\text{Var} \left| X \right|^2/n$ go to zero as $n \to \infty$?

Theorem[ABP]. Thin shell $\Rightarrow$ CLT
Concentration of the mass in a Euclidean ball or shell

$\iff$

Behavior of $(\mathbb{E}|X|^p_2)^{1/p}$ for some values of $p$. 

Concentration of the mass in a Euclidean ball or shell \iff Behavior of \( (\mathbb{E}|X|^p)^{1/p} \) for some values of \( p \).

- \( X \) log-concave random vector. Paouris Theorem (large deviation) may be written as (ALLOPT '12)

\[
\forall p \geq 1, \quad (\mathbb{E}|X|^p)^{1/p} \leq C \mathbb{E}|X|^2 + c \sigma_p(X)
\]

where \( \sigma_p(X) = \sup_{|z|^2 \leq 1} (\mathbb{E}\langle z, X \rangle^p)^{1/p} \).
Concentration of the mass in a Euclidean ball or shell
⇔
Behavior of \((\mathbb{E}|X|_2^p)^{1/p}\) for some values of \(p\).

- \(X\) log-concave random vector. Paouris Theorem (large deviation) may be written as (ALLOPT ’12)

\[ \forall p \geq 1, \quad (\mathbb{E}|X|_2^p)^{1/p} \leq C \mathbb{E}|X|_2 + c \sigma_p(X) \]

where \(\sigma_p(X) = \sup_{|z|_2 \leq 1} \left( \mathbb{E}<z, X>^p \right)^{1/p} \).

→ In isotropic position, \(\mathbb{E}|X|_2 \leq (\mathbb{E}|X|_2^2)^{1/2} = \sqrt{n}\).

By Borell’s inequality (Khintchine type inequality)

\[ \forall p \geq 1, \quad (\mathbb{E}<z, X>^p)^{1/p} \leq C p \left( \mathbb{E}<z, X>^2 \right)^{1/2} = C p |z|_2 \]

Hence

\[ \forall p \geq 1, \quad (\mathbb{E}|X|_2^p)^{1/p} \leq C \sqrt{n} + cp \]
Concentration of the mass in a Euclidean ball or shell

\[ \equiv \]

Behavior of \((\mathbb{E}|X|^p_2)^{1/p}\) for some values of \(p\).

- \(X\) log-concave random vector. Paouris Theorem (large deviation) may be written as (ALLOPT '12)

\[ \forall p \geq 1, \quad (\mathbb{E}|X|^p_2)^{1/p} \leq C \mathbb{E}|X|_2 + c \sigma_p(X) \]

where \(\sigma_p(X) = \sup_{|z|_2 \leq 1} (\mathbb{E}\langle z, X \rangle^p)^{1/p}\).

→ In isotropic position, \(\mathbb{E}|X|_2 \leq (\mathbb{E}|X|_2^2)^{1/2} = \sqrt{n}\).

By Borell’s inequality (Khintchine type inequality)

\[ \forall p \geq 1, \quad (\mathbb{E}\langle z, X \rangle^p)^{1/p} \leq C p (\mathbb{E}\langle z, X \rangle^2)^{1/2} = C p |z|_2 \]

Hence

\[ \forall p \geq 1, \quad (\mathbb{E}|X|^p_2)^{1/p} \leq C \sqrt{n} + c p \]

Take \(p = t \sqrt{n}\), Markov gives

\[ \forall t \geq 10, \quad \mathbb{P}(|X|_2 \geq t \sqrt{n}) \leq e^{-c t \sqrt{n}}. \]
KLS conjecture and consequences.

- **Strong concentration of the Euclidean norm**

\[
P \left( \left| \|X\|_2 - \sqrt{n} \right| \geq t\sqrt{n} \right) \leq C \exp(-c t \sqrt{n})
\]

\[
\forall p \in [-c\sqrt{n}, c\sqrt{n}], \quad (\mathbb{E}|X|^p)^{1/p} \leq (\mathbb{E}|X|^2)^{1/2} \left(1 + \frac{c|p|}{n}\right).
\]
KLS conjecture and consequences.

- **Strong concentration of the Euclidean norm**

\[ \mathbb{P} \left( \left| |X|_2 - \sqrt{n} \right| \geq t\sqrt{n} \right) \leq C \exp\left( -c t \sqrt{n} \right) \]

\[ \forall p \in [-c\sqrt{n}, c\sqrt{n}], \quad (\mathbb{E}|X|_2^p)^{1/p} \leq (\mathbb{E}|X|_2^2)^{1/2} \left( 1 + \frac{c|p|}{n} \right). \]

- **KLS conjecture**: \( \exists h > 0 \) a universal constant such that for every regular function \( F \),

\[ h^2 \text{Var}_\mu F \leq \int |\nabla F(x)|_2^2 f(x) dx. \]
KLS conjecture and consequences.

- **Strong concentration of the Euclidean norm**
  \[
  
  \mathbb{P} \left( \left| \left| X \right|_2 - \sqrt{n} \right| \geq t \sqrt{n} \right) \leq C \exp(-c t \sqrt{n}) 
  \]

- **KLS conjecture**: \( \exists h > 0 \) a universal constant such that for every regular function \( F \),
  \[
  h^2 \Var_\mu F \leq \int |\nabla F(x)|_2^2 f(x) dx.
  \]

- **Take** \( F(x) = |x|_2 \) or \( F(x) = |x|_2^2 \)

**Variance conjecture**: \( \Var |X|_2 \leq C \) or \( \Var |X|_2^2 \leq Cn. \)
KLS conjecture and consequences.

- **Strong concentration of the Euclidean norm**

\[ \mathbb{P} \left( \|X\|_2 - \sqrt{n} \geq t\sqrt{n} \right) \leq C \exp(-c t \sqrt{n}) \]

\[ \forall p \in [-c\sqrt{n}, c\sqrt{n}], \quad (\mathbb{E}|X|^p)^{1/p} \leq (\mathbb{E}|X|_2^2)^{1/2} \left(1 + \frac{c|p|}{n}\right). \]

- **KLS conjecture**: \( \exists h > 0 \) a universal constant such that for every regular function \( F \),

\[ h^2 \text{Var}_\mu F \leq \int |\nabla F(x)|_2^2 f(x) dx. \]

- **Take** \( F(x) = |x|_2 \) or \( F(x) = |x|^2_2 \)

**Variance conjecture**: \( \text{Var}|X|_2 \leq C \) or \( \text{Var}|X|^2_2 \leq Cn. \)

- **Eldan-Klartag ['11], Eldan ['12].**
Idea to attack the problem

We replace a simple quantity : $|X|_2$ by a more complicated

$$\mathbb{E}|X|^p_2 = c_{n,k,p} \mathbb{E}\mathbb{E}_F |P_F X|^p_2$$
Idea to attack the problem

We replace a simple quantity : $|X|_2$ by a more complicated

$$
\mathbb{E}|X|^p = c_{n,k,p} \quad \mathbb{E}\mathbb{E}_F |P_F X|^p
$$

Next, we compare to something we know how to compute : $G_n$ Gaussian vector

$$
\mathbb{E}|G_n|^p = c_{n,k,p} \quad \mathbb{E}\mathbb{E}_F |P_F G_n|^p
$$
Idea to attack the problem

We replace a simple quantity : $|X|_2$ by a more complicated
\[ \mathbb{E}|X|^p_2 = c_{n,k,p} \mathbb{E}\mathbb{E}_F|PFX|^p_2 \]

Next, we compare to something we know how to compute : $G_n$ Gaussian vector
\[ \mathbb{E}|G_n|^p_2 = c_{n,k,p} \mathbb{E}\mathbb{E}_F|PFGn|^p_2 \]

But for a $k$-dimensional $F$, $PFG_n \sim G_k$ hence
\[ \mathbb{E}|X|^p_2 = \frac{\mathbb{E}|G_n|^p_2}{\mathbb{E}|G_k|^p_2} \mathbb{E}\mathbb{E}_F|PFX|^p_2. \]

An integration in polar coordinates proves that
\[ \mathbb{E}_F\mathbb{E}|PFX|^p_2 = \mathbb{E}_U h_{k,p}(U) \]

where for every $u \in SO(n)$,
\[ h_{k,p}(u) = |S^{k-1}| \int_0^{+\infty} t^{k+p-1} \pi_{u(F_0)} w(tu(\theta_0)) dt \]
Log-Sobolev inequality
For every function $h \geq 0$, define $\text{Ent} h = \int h \log h - \int h \log \int h$.
We have a log-Sobolev inequality when there exists $C$ such that for all smooth functions $h$,

$$\text{Ent} h \leq C \int h |\nabla \log h|^2$$
Other reverse Hölder inequalities?

Consequences: reverse Hölder inequality. Let

\[ M : p \mapsto \left( \int h^p \right)^{1/p} = \exp \left( \frac{1}{p} \log \int h^p \right) \]

Then

\[ M'(p) = M(p) \left( \frac{-1}{p^2} \log \int h^p + \frac{1}{p} \frac{\int h^p \log h}{\int h^p} \right) \]

\[ = \frac{1}{p^2} \left( \int h^p \right)^{\frac{1}{p} - 1} \left( - \int h^p \log \int h^p + \int h^p \log h^p \right) \]

\[ = \frac{1}{p^2} \left( \int h^p \right)^{\frac{1}{p} - 1} \text{Ent } h^p \]

\[ \leq C_{LS} \left( \int h^p \right)^{\frac{1}{p} - 1} \int h^p \| \nabla \log h \|_2^2 \]
Other reverse Hölder inequalities?

Consequences: reverse Hölder inequality. Let

\[ M : p \mapsto \left( \int h^p \right)^{1/p} = \exp \left( \frac{1}{p} \log \int h^p \right) \]

If the function \( h \) has a log-Lipschitz constant bounded by \( L \), then we have

\[ M'(p) \leq C_{LS} L^2 M(p) \]

hence for every \( p > r \)

\[ \frac{M(p)}{M(r)} \leq \exp \left( C_{LS} L^2 (p - r) \right) \]
Consequences: reverse Hölder inequality. Let

\[ M : p \mapsto \left( \int h^p \right)^{1/p} = \exp \left( \frac{1}{p} \log \int h^p \right) \]

If the function \( h \) has a log-Lipschitz constant bounded by \( L \), then we have

\[ M'(p) \leq C_L S L^2 M(p) \]

hence for every \( p > r \)

\[ \frac{M(p)}{M(r)} \leq \exp \left( C_L S L^2 (p - r) \right) \]

And \( SO(n) \) satisfies the criteria curvature-dimension of Bakry-Émery and in this case, \( C_{LS} \leq \frac{c}{n} \)
Where disappears the geometry of convex bodies?

Everything is hidden in the study of the log-Lipschitz constant of the function $h_{k,p}$ defined on $SO(n)$ by

$$u \mapsto |S^{k-1}| \int_0^{+\infty} t^{k+p-1} \pi_{u(F_0)} w(tu(\theta_0)) dt$$
Where disappears the geometry of convex bodies?

Everything is hidden in the study of the log-Lipshitz constant of the function $h_{k,p}$ defined on $SO(n)$ by

$$u \mapsto |S^{k-1}| \int_0^{+\infty} t^{k+p-1} \pi_{u(F_0)} w(tu(\theta_0)) dt$$

- Marginals of log-concave measure are log-concave
- The Ball’s bodies
- Some reverse Hölder inequality of Borell in a log-concave setting
THANK YOU