New Characterizations of the Whitney Spheres and the Contact Whitney Spheres

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Abstract. In this paper, based on the classical Yano’s formula, we first establish an optimal integral inequality for compact Lagrangian submanifolds in the complex space forms, which involves the Ricci curvature in the direction $J \vec{H}$ and the norm of the covariant differentiation of the second fundamental form $h$, where $J$ is the almost complex structure and $\vec{H}$ is the mean curvature vector field. Second and analogously, for compact Legendrian submanifolds in the Sasakian space forms with Sasakian structure $(\varphi, \xi, \eta, g)$, we also establish an optimal integral inequality involving the Ricci curvature in the direction $\varphi \vec{H}$ and the norm of the modified covariant differentiation of the second fundamental form. The integral inequality is optimal in the sense that all submanifolds attaining the equality are completely classified. As direct consequences, we obtain new and global characterizations for the Whitney spheres in complex space forms as well as the contact Whitney spheres in Sasakian space forms. Finally, we show that, just as the Whitney spheres in complex space forms, the contact Whitney spheres in Sasakian space forms are locally conformally flat manifolds with sectional curvatures non-constant.

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1. Introduction

In this paper, we consider compact Lagrangian submanifolds in the $n$-dimensional complex space form $N^n(4c)$ of constant holomorphic sectional curvature $4c$, $c \in \{0, 1, -1\}$; and analogously, we also consider compact Legendrian submanifolds in the $(2n + 1)$-dimensional Sasakian space form $\tilde{N}^{2n+1}(\tilde{c})$ with constant $\varphi$-sectional curvature $\tilde{c}$. As our main achievements, 

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we shall establish an integral inequality for either class of such submanifolds. Then, as direct consequences, we can get new and integral characterizations for the Whitney spheres in the complex space forms and also the contact Whitney spheres in the Sasakian space forms.

Recall that the complex space form $N^n(4c)$ with almost complex structure $J$ and Riemannian metric $g$ is the complex Euclidean space $\mathbb{C}^n$ for $c = 0$, the complex projective space $\mathbb{C}P^n(4)$ for $c = 1$, and the complex hyperbolic space $\mathbb{C}H^n(-4)$ for $c = -1$. Let $M^n \hookrightarrow N^n(4c)$ be a Lagrangian immersion of an $n$-dimensional differentiable manifold $M^n$ ($n \geq 2$), i.e., $J$ carries each tangent space of $M^n$ into its corresponding normal space. To state our first

main result, we shall recall the notion of Whitney spheres in each complex space form.

**Example 1.1.** Whitney spheres in $\mathbb{C}^n$ (cf. [2, 5, 7, 15, 17, 18]).

As the most classical notion of Whitney spheres, these are usually defined as a family of Lagrangian immersions from the unit sphere $\mathbb{S}^n$, centered at the origin $O$ of $\mathbb{R}^{n+1}$, into the complex Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$, given by $\Psi_r, B : \mathbb{S}^n \rightarrow \mathbb{C}^n$ with

$$
\Psi_r, B(u_1, \ldots, u_{n+1}) = \frac{r}{1+u_{n+1}^2}(u_1, u_1u_{n+1}, \ldots, u_n, u_nu_{n+1}) + B, \quad (1.1)
$$

where $r$ is a positive number and $B$ is a vector of $\mathbb{C}^n$. The number $r$ and the vector $B$ are called the radius and the center of the Whitney spheres, respectively. Up to translation and scaling of $\mathbb{C}^n$, all the Whitney spheres are congruent with the standard one corresponding to $r = 1$ and $B = O$. According to Gromov [12], the sphere cannot be embedded into $\mathbb{C}^n$ as a Lagrangian submanifold. This fact implies that the Whitney spheres in (1.1) have the best possible behavior, because it is embedded except at the poles of $\mathbb{S}^n$ where it has a double points. Indeed, in a certain sense, the Whitney spheres in $\mathbb{C}^n$ play the role of umbilical hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ inside the family of Lagrangian submanifolds and have been characterized in several ways as done for the Euclidean spheres (cf. [17]).

**Example 1.2.** Whitney spheres in $\mathbb{C}P^n(4)$ (cf. [6, 8, 10, 15]).

In this case, the Whitney spheres are a one-parameter family of Lagrangian sphere immersions into $\mathbb{C}P^n(4)$, given by $\Psi_\theta : \mathbb{S}^n \rightarrow \mathbb{C}P^n(4)$ for $\theta > 0$ with

$$
\Psi_\theta(u_1, \ldots, u_{n+1}) = \Pi \left( \frac{(u_1, \ldots, u_n)}{\cosh \theta + i \sinh \theta u_{n+1}} ; \frac{\sinh \theta \cosh \theta (1+u_{n+1}^2)+iu_{n+1}}{\cosh^2 \theta + \sinh^2 \theta u_{n+1}^2} \right), \quad (1.2)
$$

where $\Pi : \mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n(4)$ is the Hopf projection. We notice that $\Psi_\theta$ are embeddings except at the poles of $\mathbb{S}^n$ where it has a double points, and that $\Psi_0$ is the totally geodesic Lagrangian immersion of $\mathbb{S}^n$ into $\mathbb{C}P^n(4)$.

**Example 1.3.** Whitney spheres in $\mathbb{C}H^n(-4)$ (cf. [6, 8, 10, 15]).

Let $(\cdot, \cdot)$ denote the hermitian form of $\mathbb{C}^{n+1}$, i.e., $(z, w) = \sum_{i=1}^{n} z_i \overline{w}_i - z_{n+1} \overline{w}_{n+1}$ for $z, w \in \mathbb{C}^{n+1}$, and $\mathbb{H}_1^{2n+1}(-1) = \{ z \in \mathbb{C}^{n+1} : (z, z) = -1 \}$ be
the Anti-de Sitter space of constant sectional curvature $-1$. Then, the Whitney spheres in $\mathbb{CH}^n(-4)$ are a one-parameter family of Lagrangian sphere immersions into $\mathbb{CH}^n(-4)$, given by $\Phi_\theta : \mathbb{S}^n \to \mathbb{CH}^n(-4)$ for $\theta > 0$ with

$$\Phi_\theta(u_1, \ldots, u_{n+1}) = \Pi \left( \frac{(u_1, \ldots, u_n)}{\sinh \theta + i \cosh \theta u_{n+1}^2}; \frac{\sinh \theta \cosh \theta (1 + u_{n+1}^2) - i u_{n+1}}{\sinh^2 \theta + \cosh^2 \theta u_{n+1}^2} \right), \quad (1.3)$$

where $\Pi : \mathbb{H}^{2n+1}_1(-1) \to \mathbb{CH}^n(-4)$ is the Hopf projection. We also notice that $\Phi_\theta$ are embeddings except in double points.

The remarkable properties of the Whitney spheres are summarized as follows:

**Theorem 1.1.** (cf. [2,3,6–8,10,15,17]) Let $x : M^n \to N^n(4c)$ be an $n$-dimensional compact Lagrangian submanifold that is neither totally geodesic nor of parallel mean curvature vector field. Then, $x(M^n)$ is the Whitney sphere in $N^n(4c)$ if and only if one of the following pointwise relations holds:

1. The squared mean curvature $|\vec{H}|^2$ and the scalar curvature $R$ of $M^n$ satisfy the relation $|\vec{H}|^2 = \frac{n+2}{n(n-1)}R - \frac{n+2}{n}c$;
2. The second fundamental form $h$ and the mean curvature vector field $\vec{H}$ of $M^n$ satisfy $h(X,Y) = \frac{n}{n+2} \left[ g(X,Y)\vec{H} + g(JX,\vec{H})JY + g(JY,\vec{H})JX \right]$ for $X,Y \in TM^n$;
3. The second fundamental form $h$ and the mean curvature vector field $\vec{H}$ of $M^n$ satisfy $\|\nabla h\|^2 = \frac{3n^2}{n+2} \|\nabla \vec{H}\|^2$. Here, $\nabla h$ denotes the covariant differentiation of $h$ with respect to the van der Waerden-Bortolotti connection of $x : M^n \to N^n(4c)$.

Moreover, Castro–Montealegre–Urbano [6] and Ros-Urbano [17] further proved that the Whitney spheres in $N^n(4c)$ can be characterized by some other relations about the global geometric and topological invariants.

As the first main result of this paper, we have obtained an optimal integral inequality that involves the Ricci curvature $\text{Ric} (J\vec{H}, J\vec{H})$ in the direction $J\vec{H}$ and the norm of the covariant differentiation $\nabla h$ of the second fundamental form:

**Theorem 1.2.** Let $x : M^n \to N^n(4c)$ ($n \geq 2$) be an $n$-dimensional compact Lagrangian submanifold. Then, it holds that

$$\int_{M^n} \text{Ric} (J\vec{H}, J\vec{H}) \, dV_{M^n} \leq \frac{(n-1)(n+2)}{3n^2} \int_{M^n} \|\nabla h\|^2 \, dV_{M^n}, \quad (1.4)$$

where $\| \cdot \|$ and $dV_{M^n}$ denote the tensorial norm and the volume element of $M^n$ with respect to the induced metric, respectively.

Moreover, the equality in (1.4) holds if and only if either $x(M^n)$ is of parallel second fundamental form, or it is one of the Whitney spheres in $N^n(4c)$.

**Remark 1.1.** The classification of Lagrangian submanifolds with parallel second fundamental form in $N^n(4c)$ has been fulfilled for each $c$, see [11,14] for details.

From Theorem 1.2, we get a new and global geometric characterization of the Whitney spheres in $N^n(4c)$:
Corollary 1.1. Let \( x : M^n \to N^n(c) \) \((n \geq 2)\) be an \( n \)-dimensional compact Lagrangian submanifold with non-parallel mean curvature vector field. Then,

\[
\int_{M^n} \text{Ric}(J\vec{H}, J\vec{H}) \, dV_{M^n} = \frac{(n-1)(n+2)}{3n^2} \int_{M^n} \|\vec{\nabla} h\|^2 \, dV_{M^n} \tag{1.5}
\]

holds if and only if \( x(M^n) \) is a Whitney sphere in \( N^n(c) \).

Next, before stating our second main result, we shall first review the standard models of the Sasakian space form \( \tilde{N}^{2n+1}(\bar{c}) \) with Sasakian structure \((\varphi, \xi, \eta, g)\) possessing constant \( \varphi \)-sectional curvature \( \bar{c} \), then for each value \( \bar{c} \), we introduce the canonical Legendrian (i.e., the \( n \)-dimensional \( C \)-totally real, or equivalently, integral) submanifolds: The contact Whitney spheres in \( \tilde{N}^{2n+1}(\bar{c}) \).

Example 1.4. Contact Whitney spheres in \( \tilde{N}^{2n+1}(-3) = (\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g) \).

Here, for the Cartesian coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \) of \( \mathbb{R}^{2n+1} \),

\[
\begin{cases}
\xi = 2 \frac{\partial}{\partial z}, & \eta = \frac{1}{2} \left( dz - \sum_{i=1}^n y_i \, dx_i \right), & g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i),

\varphi \left( \sum_{i=1}^n (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z} \right) = \sum_{i=1}^n (Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{i=1}^n Y_i y_i \frac{\partial}{\partial z},
\end{cases}
\]

define the standard Sasakian structure \((\varphi, \xi, \eta, g)\) on \( \mathbb{R}^{2n+1} \).

As were introduced by Blair and Carriazo in [1], the contact Whitney spheres in \( \tilde{N}^{2n+1}(-3) \) were the Legendrian imbeddings \( \tilde{\Psi}_{B,a,r} : S^n \to \mathbb{R}^{2n+1} \) defined by

\[
\tilde{\Psi}_{B,a,r}(u_0, u_1, \ldots, u_n) = \frac{r}{1+u_0^2} (u_0u_1, \ldots, u_0u_n, u_1, \ldots, \frac{r^{n-1}}{1+u_0^2} + a(1+u_0^2)) + B, \tag{1.6}
\]

where \( r \) is a positive number, \( a \) is a real constant and \( B \) is a vector of \( \mathbb{R}^{2n+1} \).

Example 1.5. Contact Whitney spheres in \( \tilde{N}^{2n+1}(\bar{c}) = (S^{2n+1}, \varphi, \xi, \eta, g) \) with \( \bar{c} > -3 \). Note that the unit sphere \( S^{2n+1} \), as a real hypersurface of the complex Euclidean space \( \mathbb{C}^{n+1} \), has a natural Sasakian structure \((\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})\); \( \bar{g} \) is the induced metric; \( \bar{\xi} = JN \), where \( J \) is the natural complex structure of \( \mathbb{C}^{n+1} \) and \( N \) is the unit normal vector field of the inclusion \( S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}; \)

\( \bar{\eta}(X) = \bar{g}(X, \bar{\xi}) \) and \( \varphi(X) = JX - \langle JX, N \rangle N \) for any tangent vector field \( X \) of \( S^{2n+1} \), where \( \langle \cdot, \cdot \rangle \) denotes the standard Hermitian metric on \( \mathbb{C}^{n+1} \). Then, the standard Sasakian structure \((\varphi, \xi, \eta, g)\) on \( S^{2n+1} \) is given by applying a \( D_a \)-homothetic deformation as follows:

\[
\eta = a\bar{\eta}, \quad \xi = \frac{1}{a} \bar{\xi}, \quad \varphi = \bar{\varphi}, \quad g = a\bar{g} + a(a-1)\bar{\eta} \otimes \bar{\eta},
\]

where \( a \) is a positive real number and \( \bar{c} = \frac{4}{a} - 3 \).
Then, as were introduced in [14], the contact Whitney spheres in $\tilde{N}^{2n+1}(\tilde{c})$ for $\tilde{c} > -3$ are a family of Legendrian immersions $\tilde{\Psi}_\theta : S^n \to S^{2n+1}$ for $\theta > 0$, that are explicitly given by

$$\tilde{\Psi}_\theta(u_1, u_2, \ldots, u_{n+1}) = \left(\frac{(u_1, \ldots, u_n)}{\cosh \theta + i \sinh \theta u_{n+1}}, \frac{\sinh \theta \cosh (1 + u_{n+1}^2) + i u_{n+1}}{\cosh^2 \theta + \sinh^2 \theta u_{n+1}^2}\right).\quad (1.7)$$

**Example 1.6. Contact Whitney spheres** in $\tilde{N}^{2n+1}(\tilde{c}) = (\mathbb{B}^n \times \mathbb{R}, \varphi, \xi, \eta, g)$ with $\tilde{c} < -3$. Here, $\mathbb{B}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \|z\|^2 = \sum_{i=1}^n |z_i|^2 < 1\}$ equipped with the usual complex structure and the canonical Bergman metric

$$\tilde{g} = 4\left\{1 - \frac{1}{\|z\|^2} \sum_{i=1}^n dz_i d\bar{z}_i + \frac{1}{(1 - \|z\|^2)^2} \sum_{i,j=1}^n z_i \bar{z}_j d\bar{z}_j \right\}$$

is a Kähler manifold with constant holomorphic sectional curvature $-1$. Let $t$ be the coordinate of $\mathbb{R}$ and $\omega = \frac{2\sqrt{-1}}{1 - \|z\|^2} \sum_{j=1}^n (\bar{z}_j dz_j - z_j d\bar{z}_j)$. Then, $\mathbb{B}^n \times \mathbb{R}$ has a Sasakian structure $\{\varphi, \xi, \bar{\eta}, \tilde{g}\}$ with constant $\varphi$-sectional curvature $-4$, defined as follows:

$$\begin{align*}
\bar{\eta} &= \omega + dt, \quad \bar{\xi} = \frac{\partial}{\partial t}, \quad \bar{g} = \tilde{g} + \bar{\eta} \otimes \bar{\eta}, \\
\varphi \left( \sum_{i=1}^n (a_i \frac{\partial}{\partial z_i} + b_i \frac{\partial}{\partial \bar{z}_i}) + c \frac{\partial}{\partial t} \right) &= \sqrt{-1} \sum_{i=1}^n (b_i \frac{\partial}{\partial z_i} - a_i \frac{\partial}{\partial \bar{z}_i}) + \frac{2}{1 - \|z\|^2} \sum_{i=1}^n (b_i \bar{z}_i + a_i z_i) \frac{\partial}{\partial t}.
\end{align*}$$

Then, $\tilde{N}^{2n+1}(\tilde{c}) = (\mathbb{B}^n \times \mathbb{R}, \varphi, \xi, \eta, g)$ is given by the $D_a$-homothetic deformation

$$\eta = a\bar{\eta}, \quad \xi = \frac{1}{a} \bar{\xi}, \quad g = a\bar{g} + a(a - 1)\bar{\eta} \otimes \bar{\eta},$$

and $\tilde{c} = -\frac{1}{a} - 3$, where $a$ is a positive number.

As were introduced in [14], the contact Whitney spheres in $\tilde{N}^{2n+1}(\tilde{c})$ for $\tilde{c} < -3$ are a one-parameter family of Legendrian immersions $\tilde{\Phi}_\theta : S^n \to \mathbb{B}^n \times \mathbb{R}$ for $\theta > 0$, that are explicitly given by

$$\pi(\tilde{\Phi}_\theta(u_1, u_2, \ldots, u_{n+1})) = \Pi \left(\frac{(u_1, \ldots, u_n)}{\cosh \theta + i \sinh \theta u_{n+1}}, \frac{\sinh \theta \cosh (1 + u_{n+1}^2) - i u_{n+1}}{\cosh^2 \theta + \sinh^2 \theta u_{n+1}^2}\right),\quad (1.8)$$

where $\pi : \tilde{N}^{2n+1}(\tilde{c}) \to N^n(c)$ with $c = \tilde{c} + 3$ is the canonical projection and $\Pi : \mathbb{H}^{2n+1}(-1) \to \mathbb{C}H^n(-4)$ is the Hopf projection.

According to Proposition 2 of Blair-Carriazo [1] and Theorem 4.2 of Hu-Yin [14], for each contact Whitney sphere $M^n$ in any Sasakian space form $\tilde{N}^{2n+1}(\tilde{c})$, the second fundamental form $h$ and the mean curvature vector field $\tilde{H}$ satisfy the relation

$$h(X, Y) = \frac{n}{n+2} \left[g(X, Y)\tilde{H} + g(\varphi X, \tilde{H})\varphi Y + g(\varphi Y, \tilde{H})\varphi X\right].\quad (1.9)$$
for any tangent vector fields \(X, Y \in TM^n\). Without introducing the notion of contact Whitney spheres as canonical examples, Pitiş [16] proved that results for Lagrangian submanifolds of the complex space forms in [6,17] hold analogously for those Legendrian submanifolds of the Sasakian space forms which satisfy (1.9). Moreover, an analogue of the result for Whitney spheres in complex space forms by Li-Vrancken [15] was established for contact Whitney spheres in Sasakian space forms by Hu-Yin [14]. It follows that a corresponding version of Theorem 1.1 for contact Whitney spheres in Sasakian space forms is already known.

Next, along a similar spirit as above, we get the second main result of this paper. Actually, we can show that an optimal integral inequality, as in Theorem 1.2, that involves the Ricci curvature \(\text{Ric}(\varphi \bar{H}, \varphi \bar{H})\) in the direction \(\varphi \bar{H}\) and the norm of the modified covariant differentiation \(\bar{\nabla}^{\xi}h\) of the second fundamental form holds also for compact Legendrian submanifolds in the Sasakian space forms:

**Theorem 1.3.** Let \(x : M^n \to \tilde{N}^{2n+1}(\tilde{c}) (n \geq 2)\) be an \(n\)-dimensional compact Legendrian submanifold. Then, it holds that

\[
\int_{M^n} \text{Ric}(\varphi \bar{H}, \varphi \bar{H}) \, dV_{M^n} \leq \frac{(n-1)(n+2)}{3n^2} \int_{M^n} \|\bar{\nabla}^{\xi}h\|^2 \, dV_{M^n},
\]

(1.10)

where, \(\bar{\nabla}^{\xi}h\) denotes the projection of \(\bar{\nabla}h\) onto \(TM^n \oplus \varphi(TM^n)\), whereas \(\bar{\nabla}h\) denotes the covariant differentiation of \(h\) with respect to the van der Waerden-Bortolotti connection of \(M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})\), \(\|\cdot\|\) and \(dV_{M^n}\) denote the tensorial norm and the volume element of \(M^n\) with respect to the induced metric, respectively.

Moreover, the equality in (1.10) holds if and only if either \(\bar{\nabla}^{\xi}h = 0\) (i.e. \(x(M^n)\) is of \(C\)-parallel second fundamental form), or \(x(M^n)\) is one of the contact Whitney spheres in \(\tilde{N}^{2n+1}(\tilde{c})\).

**Remark 1.2.** The classification of Legendrian submanifolds with \(C\)-parallel second fundamental form in the Sasakian space forms has been fulfilled. For the details, see Theorem 4.1 in [14].

From Theorem 1.3, we get a new and global geometric characterization of the contact Whitney spheres in \(\tilde{N}^{2n+1}(\tilde{c})\):

**Corollary 1.2.** Let \(x : M^n \to \tilde{N}^{2n+1}(\tilde{c}) (n \geq 2)\) be an \(n\)-dimensional compact Legendrian submanifold with non-\(C\)-parallel mean curvature vector field. Then,  

\[
\int_{M^n} \text{Ric}(\varphi \bar{H}, \varphi \bar{H}) \, dV_{M^n} = \frac{(n-1)(n+2)}{3n^2} \int_{M^n} \|\bar{\nabla}^{\xi}h\|^2 \, dV_{M^n}
\]

(1.11)

holds if and only if \(x(M^n)\) is a contact Whitney sphere in \(\tilde{N}^{2n+1}(\tilde{c})\).

2. Preliminaries

In this section, we first briefly review some of the basic notions about Lagrangian submanifolds in the complex space form \(N^n(4c)\) and Legendrian
submanifolds in the Sasakian space form $\tilde{N}^{2n+1}(\tilde{c})$, respectively. Then, we state a classical formula due to K. Yano that we need in the proof of our theorems.

Let $M^n \hookrightarrow N^n(4c)$ (resp. $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$) be an isometric immersion from an $n$-dimensional Riemannian manifold $M^n$ into the $n$-dimensional complex space form $N^n(4c)$ of constant holomorphic sectional curvature $4c$ (resp. the $(2n+1)$-dimensional Sasakian space form $\tilde{N}^{2n+1}(\tilde{c})$ of constant $\varphi$-section curvature $\tilde{c}$). For simplicity, we denote by the same notation $g$ the Riemannian metric on $M^n$, $N^n(4c)$ and $\tilde{N}^{2n+1}(\tilde{c})$. Let $\nabla$ (resp. $\tilde{\nabla}$) be the Levi-Civita connection of $M^n$ (resp. $N^n(4c)$ and $\tilde{N}^{2n+1}(\tilde{c})$). Then, for both $M^n \hookrightarrow N^n(4c)$ and $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$, we have the Gauss and Weingarten formulas:

$$\tilde{\nabla}_XY = \nabla_XY + h(X,Y), \quad \tilde{\nabla}_XV = -A_VX + \nabla^\perp_XV$$  \hspace{1cm} (2.1)

for any tangent vector fields $X,Y \in TM^n$ and normal vector field $V \in T^\perp M^n$. Here, $\nabla^\perp$ denotes the normal connection in the normal bundle $T^\perp M^n$. $h$ (resp. $A_V$) denotes the second fundamental form (resp. the shape operator with respect to $V$) of $M^n \hookrightarrow N^n(4c)$ (resp. $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$). From (2.1), we have the relation

$$g(h(X,Y),V) = g(A_VX,Y).$$  \hspace{1cm} (2.2)

### 2.1. Lagrangian Submanifolds of the Complex Space Form $N^n(4c)$

The curvature tensor $\tilde{R}(X,Y)Z := \tilde{\nabla}_X\tilde{\nabla}_YZ - \tilde{\nabla}_Y\tilde{\nabla}_XZ - \tilde{\nabla}_{[X,Y]}Z$ of $N^n(4c)$ has the following expression:

$$\tilde{R}(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ].$$  \hspace{1cm} (2.3)

Let $M^n \hookrightarrow N^n(4c)$ be a Lagrangian immersion. Then, we have (cf. e.g. [15])

$$\nabla^\perp_XJY = J\nabla^\perp_XY, \quad A_{JX}Y = -Jh(X,Y) = A_JYX,$$  \hspace{1cm} (2.4)

and thus $g(h(X,Y),JZ)$ is totally symmetric in $X$, $Y$ and $Z$:

$$g(h(X,Y),JZ) = g(h(Y,Z),JX) = g(h(Z,X),JY).$$  \hspace{1cm} (2.5)

We choose a local adapted Lagrangian frame field $\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$ such that $e_1, \ldots, e_n$ are orthonormal tangent vector fields, and $e_1^* = Je_1, \ldots, e_n^* = Je_n$ are orthonormal normal vector fields of $M^n \hookrightarrow N^n(4c)$, respectively. In follows we shall make use of the indices convention: $i^* = n + i, \quad 1 \leq i, j, k, \ldots \leq n$.

Denote by $\{\omega^1, \ldots, \omega^n\}$ the dual frame of $\{e_1, \ldots, e_n\}$. Let $\omega^i_j$ and $\omega^{i*}_j$ denote the connection 1-forms of $TM^n$ and $T^\perp M^n$, respectively:

$$\nabla e_i = \sum_{j=1}^n \omega^j_i e_j, \quad \nabla^\perp e_i^* = \sum_{j=1}^n \omega^{j*}_i e_j^*, \quad 1 \leq i \leq n,$$
where $\omega^j_i + \omega^i_j = 0$ and by (2.4) it holds that $\omega_i^j = \omega_i^j\ast$. Put $h_i^j\ast = g(h(e_i, e_j), J e_k)$. From (2.5), we see that

$$h_i^j\ast = h_i^k\ast = h_{jk}^i, \quad 1 \leq i, j, k \leq n.$$ \hfill (2.6)

Let $R_{ijkl} := g(R(e_i, e_j)e_l, e_k)$ and $R_{i,kl} \ast := g(R^\perp(e_i, e_j)e_l\ast, e_k\ast)$ be the components of the curvature tensors of $\nabla$ and $\nabla^\perp$, respectively. Then the equations of Gauss, Ricci and Codazzi of $M^n \hookrightarrow N^n(4c)$ are given by

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{m=1}^n (h_{ik\ast}^m h_{jl\ast}^m - h_{il\ast}^m h_{jk\ast}^m),$$ \hfill (2.7)

$$R_{ijk\ast l\ast} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{m=1}^n (h_{ik\ast}^m h_{jl\ast}^m - h_{il\ast}^m h_{jk\ast}^m) = R_{ijkl},$$ \hfill (2.8)

$$h_i^j\ast, k = h_i^j\ast, k,$$ \hfill (2.9)

where $h_i^j\ast, k$ denotes the components of the covariant differentiation of $h$, namely $\nabla h$, defined by

$$\sum_{l=1}^n h_{ij\ast, k}^l e_l^\ast := \nabla_{e_l^\ast} h(e_i, e_j) - h(\nabla_{e_l^\ast} e_i, e_j) - h(e_i, \nabla_{e_l^\ast} e_j).$$ \hfill (2.10)

The mean curvature vector field $\vec{H}$ of $M^n \hookrightarrow N^n(4c)$ is defined by

$$\vec{H} := \frac{1}{n} \sum_{i=1}^n h(e_i, e_i) =: \sum_{j=1}^n H_j^\ast e_j\ast, \quad H_j^\ast = \frac{1}{n} \sum_{i=1}^n h_i^j\ast, \quad 1 \leq j \leq n.$$ \hfill (2.11)

Put $\nabla_{e_i}^\perp \vec{H} = \sum_{j=1}^n H_i^j\ast e_j\ast, \quad 1 \leq i \leq n$. From (2.6) and (2.9), we obtain

$$H_i^j\ast = H_i^j\ast, \quad 1 \leq i, j \leq n.$$ \hfill (2.12)

2.2. Legendrian Submanifolds of the Sasakian Space Form $\tilde{N}^{2n+1}(\tilde{c})$

The following facts of this subsection are referred to e.g. [14]. The curvature tensor of the Sasakian space form $\tilde{N}^{2n+1}(\tilde{c})$ is given by

$$\vec{R}(X, Y)Z = \frac{\tilde{c} + 3}{4} [g(Y, Z)X - g(X, Z)Y] + \frac{\tilde{c} - 1}{4} \left[ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \right]$$

$$+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z].$$ \hfill (2.13)

Moreover, for tangent vector fields $X, Y$ of $\tilde{N}^{2n+1}(\tilde{c})$, the Sasakian structure $(\varphi, \xi, \eta, g)$ of $\tilde{N}^{2n+1}(\tilde{c})$ satisfy:

$$\begin{cases}
\eta(X) = g(X, \xi), & \varphi \xi = 0, \quad \eta(\varphi X) = 0, \\
\varphi X = -X + \eta(X)\xi, & d\eta(X, Y) = g(X, \varphi Y), \\
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), & \text{rank } (\varphi) = 2n, \\
\nabla_X \xi = -\varphi X, & (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.
\end{cases}$$ \hfill (2.14)
Let $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$ be a Legendrian immersion. Then, we have

$$A_\varphi Y = -\varphi h(X, Y), \quad \nabla_X^\perp \varphi Y = \varphi \nabla_X Y + g(X, Y)\xi.$$  \hfill (2.15)

In follows we shall make the following convention on range of indices:

$$\alpha^* = \alpha + n; \quad 1 \leq i, j, k, l, m \leq n; \quad 1 \leq \alpha, \beta \leq n + 1.$$

We choose a local Legendre frame field \{\emph{e}_1, \ldots, \emph{e}_n, \emph{e}_{n+1}, \ldots, \emph{e}_{2n+1} = \xi\} along $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$ such that \{\emph{e}_i\}_{i=1}^n is an orthonormal frame field of $M^n$, and \{\emph{e}_{n+1} = \varphi \emph{e}_1, \ldots, \emph{e}_{2n+1} = \xi\} are the orthonormal normal vector fields of $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$. Let $\omega_i^j$ and $\omega_{\alpha^*}^\beta$ denote the connection \-1-forms of $TM^n$ and $T^\perp M^n$, respectively:

$$\nabla_{\emph{e}_j} = \sum_{i=1}^n \omega_i^j \emph{e}_j, \quad \nabla^\perp_{\emph{e}_j} e_\alpha^* = \sum_{\beta=1}^{n+1} \omega_{\alpha^*}^\beta e_\beta^*, \quad 1 \leq i \leq n, \quad 1 \leq \alpha \leq n + 1,$$

where $\omega_i^j + \omega_j^i = 0$ and $\omega_{\alpha^*}^\beta + \omega_{\beta^*}^\alpha = 0$. Moreover, by (2.15), we have $\omega_i^j = \omega_i^j$, and $\omega_{i^*}^{2n+1} = \omega_i$. Put $h_{i^*}^{k^*} = g(h(\emph{e}_i, \emph{e}_j), \varphi \emph{e}_k)$ and $h_{i^*}^{j^*} = g(h(\emph{e}_i, \emph{e}_j), \emph{e}_{2n+1})$. From (2.14) and (2.15), we have

$$h_{i^*}^{k^*} = h_{ik}^{j^*} = h_{jk}^{i^*}, \quad h_{i^*}^{j^*} = 0, \quad 1 \leq i, j, k \leq n.$$  \hfill (2.16)

Now, with the same notations as in the preceding subsection, the equations of Gauss, Ricci and Codazzi of $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$ are as follows:

$$R_{ijkl} = \frac{\tilde{c}+3}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{m=1}^n (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}),$$  \hfill (2.17)

$$R_{ijkl^*l^*} = \frac{\tilde{c}-1}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{m=1}^n (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \quad R_{ijkl^*_{(2n+1)}} = 0,$$  \hfill (2.18)

$$h_{ij,k}^{\alpha^*} = h_{ik,j}^{\alpha^*}. \quad \hfill (2.19)$$

where as usual $h_{ij,k}^{\alpha^*}$ is defined by

$$\sum_{\alpha=1}^{n+1} h_{ij,k}^{\alpha^*} e_\alpha^* := \nabla_{e_k}^\perp (h(e_i, e_j)) - h(\nabla_{e_k} e_i, e_j) - h(e_i, \nabla_{e_k} e_j), \quad 1 \leq i, j, k \leq n.$$  \hfill (2.20)

Moreover, associated to $\nabla$, $\nabla^\perp$ and $\tilde{\nabla}$, we can naturally define a modified covariant differentiation $\tilde{\nabla}\xi h$ of the second fundamental form by

$$(\tilde{\nabla}_X^\xi h)(Y, Z) := \nabla_{\xi}^X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) - g(h(Y, Z), \varphi X)\xi.$$  \hfill (2.21)

Recall that the second fundamental form $h$ of $M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{c})$ is said to be $C$-parallel if and only if $\nabla^\xi h = 0$ (cf. [14]). Actually, we have $g((\nabla^\xi_X h)(Y, Z), \xi) = 0$ for any $X, Y, Z \in TM^n$. Thus, we can denote

$$((\nabla^\xi_{ek} h)(e_i, e_j) := \sum_{l=1}^n h_{ij,k}^{l^*} e_l^*, \quad 1 \leq i, j, k \leq n.$$  \hfill (2.22)
Then, by (2.20), (2.21) and the above discussions, we have
\[ h^{(n+1)*}_{ij,k} = h^{k*}_{ij}, \quad h^{l*}_{ij,k} = \bar{h}^{l*}_{ij,k}, \quad \forall i, j, k, l. \] (2.23)

From (2.16), the mean curvature vector \( \vec{H} \) of \( M^n \hookrightarrow \tilde{N}^{2n+1}(\tilde{\epsilon}) \) becomes:
\[ \vec{H} = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \sum_{k=1}^{n} H^{k*} e_k, \quad H^{k*} := \frac{1}{n} \sum_{i=1}^{n} h^{k*}_{ii}, \quad 1 \leq k \leq n. \] (2.24)

Put
\[ \nabla_{e_i}^\perp \vec{H} = \sum_{\alpha=1}^{n+1} H^{\alpha*}_{i} e_{\alpha*}, \quad \nabla_{e_i} \vec{H} := \nabla_{e_i}^\perp \vec{H} - g(\vec{H}, e_{i*}) =: \sum_{k=1}^{n} \bar{H}^{k*}_{i} e_{k*}, \quad 1 \leq i \leq n. \]

From (2.16), (2.19) and (2.23), we get
\[ H^*_{j,i} = H^*_{i,j}, \quad \bar{H}^{*}_{j,i} = \bar{H}^{*}_{i,j}, \quad 1 \leq i, j \leq n. \] (2.25)

2.3. Yano’s formula

To prove Theorem 1.2 and Theorem 1.3, we still need the following useful formula due to Yano [19]. A simply proof is referred also to [13].

Lemma 2.1. (cf. Lemma 5.1 of [13]) Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \( \nabla \). Then, for any tangent vector field \( X \) on \( M \), it holds that
\[ \text{div}(\nabla X X - (\text{div} X) X) = \text{Ric}(X, X) + \frac{1}{2} \| L_X g \|^2 - \| \nabla X \|^2 - (\text{div} X)^2, \] (2.26)
where \( L_X g \) is the Lie derivative of \( g \) with respect to \( X \) and \( \| \cdot \| \) denotes the length with respect to \( g \).

3. Proof of Theorem 1.2

First of all, we state the following simple fact without proof.

Lemma 3.1. Let \( x : M^n \rightarrow N^n(4c) \) be an \( n \)-dimensional Lagrangian submanifold with mean curvature vector field \( \vec{H} \). Then, it holds that
\[ \| \nabla J \vec{H} \|^2 \geq \frac{1}{n} (\text{div} J \vec{H})^2. \] (3.1)
Moreover, the equality in (3.1) holds if and only if \( \nabla J \vec{H} = \frac{1}{n} (\text{div} J \vec{H}) \text{id} \), i.e., \( J \vec{H} \) is a conformal vector field on \( M^n \), or equivalently, \( M^n \) is a Lagrangian submanifold with conformal Maslov form.

We also need the following result due to Li-Vrancken [15]:

Lemma 3.2. (cf. Lemma 3.2 in [15]) Let \( x : M^n \rightarrow N^n(4c) \) be an \( n \)-dimensional Lagrangian submanifold with mean curvature tensor \( \vec{H} \). Then, it holds that
\[ \| \nabla h \|^2 \geq \frac{3n^2}{n+2} \| \nabla^\perp \vec{H} \|^2. \] (3.2)
Moreover, the equality in (3.2) holds if and only if
\[
(\nabla_Z h)(X,Y) = \frac{n}{n+2} \left[ g(Y,Z)\nabla_X^\perp \vec{H} + g(X,Z)\nabla_Y^\perp \vec{H} + g(X,Y)\nabla_Z^\perp \vec{H} \right].
\] (3.3)

Now, we are ready to complete the proof of Theorem 1.2.

\textbf{Proof of Theorem 1.2.} Let \(M^n \hookrightarrow N^n(4c)\) be a compact Lagrangian submanifold and \(\{e_1,\ldots,e_n, e_1^*,\ldots, e_n^*\}\) be a local adapted Lagrangian frame field along \(M^n\). From (2.4) and that \(\nabla^\perp e_i^* \vec{H} = n \sum_{j=1}^n H^*_i,j e_j^*\), we have
\[
\|\nabla J \vec{H}\|^2 = \|\nabla^\perp \vec{H}\|^2 = \sum_{i,j=1}^n (H^*_i,j)^2.
\] (3.4)

Then, by (2.12), calculating the squared length of the Lie derivative \(\mathcal{L}_{J \vec{H}} g\) of \(g\) with respect to \(J \vec{H}\), we obtain
\[
\|\mathcal{L}_{J \vec{H}} g\|^2 = \sum_{i,j=1}^n \left[ (\mathcal{L}_{J \vec{H}} g)(e_i,e_j) \right]^2 = \sum_{i,j=1}^n (H^*_i,j + H^*_j,i)^2 = 4\|\nabla^\perp \vec{H}\|^2.
\] (3.5)

Thus, we can apply Lemma 2.1 and (3.1) to obtain that
\[
\text{div}(\nabla J \vec{H} - (\text{div} J \vec{H}) J \vec{H}) = \text{Ric}(J \vec{H}, J \vec{H}) + \|\nabla J \vec{H}\|^2 - (\text{div} J \vec{H})^2 \geq \text{Ric}(J \vec{H}, J \vec{H}) - (n-1)\|\nabla^\perp \vec{H}\|^2,
\] (3.6)

where the equality in (3.6) holds if and only if \(\nabla J \vec{H} = \frac{1}{n}(\text{div} J \vec{H}) \text{id}\), or equivalently, \(M^n\) is a Lagrangian submanifold with conformal Maslov form.

From (3.6), by further applying Lemma 3.2, we get
\[
\text{div}(\nabla J \vec{H} - (\text{div} J \vec{H}) J \vec{H}) \geq \text{Ric}(J \vec{H}, J \vec{H}) - \frac{1}{n}(\text{div} J \vec{H})^2 - \frac{n-1}{3n^2}\|\nabla h\|^2,
\] (3.7)

where the equality holds if and only if both \(\nabla J \vec{H} = \frac{1}{n}(\text{div} J \vec{H}) \text{id}\) and (3.3) hold.

By the compactness of \(M^n\), we can integrate the inequality (3.7) over \(M^n\). Then, applying for the divergence theorem, we obtain the integral inequality (1.4).

It is easily seen that the equality holds in (1.4) if and only if the equality in (3.2) holds identically. Thus, according to Main Theorem in [15], equality in (1.4) holds if and only if \(x(M^n)\) is of parallel second fundamental form, or \(x(M^n)\) is one of the Whitney spheres in \(N^n(4c)\).

This completes the proof of Theorem 1.2. \(\square\)

4. \textbf{Proof of Theorem 1.3}

Let \(x : M^n \rightarrow \tilde{N}^{2n+1}(c)\) be an \(n\)-dimensional Legendrian submanifold in the Sasakian space form \(\tilde{N}^{2n+1}(c)\) with Sasakian structure \((\varphi, \xi, \eta, g)\). First of all, similar to Lemma 3.1, we have the following simple result.
Lemma 4.1. Let $x : M^n \to \tilde{N}^{2n+1}(\tilde{c})$ be an $n$-dimensional Legendrian submanifold with mean curvature vector field $\bar{H}$. Then, it holds that
$$\|\nabla(\varphi \bar{H})\|^2 \geq \frac{1}{n}(\text{div} \, \varphi \bar{H})^2. \quad (4.1)$$
Moreover, the equality in (4.1) holds if and only if $\nabla(\varphi \bar{H}) = \frac{1}{n}(\text{div} \, \varphi \bar{H}) \text{id}$, i.e., $\varphi \bar{H}$ is a conformal vector field on $M^n$.

We also need the following result:

Lemma 4.2. (cf. Lemma 3.3 in [14]) Let $x : M^n \to \tilde{N}^{2n+1}(\tilde{c})$ be an $n$-dimensional Legendrian submanifold with second fundamental form $h$ and mean curvature vector field $\bar{H}$. Then, it holds that
$$\|\nabla^\xi h\|^2 \geq \frac{3n^2}{n+2} \|\nabla^\xi \bar{H}\|^2, \quad (4.2)$$
where, with respect to a local Legendre frame field $\{e_A\}_{A=1}^{2n+1}$,
$$\|\nabla^\xi h\|^2 = \sum_{i,j,k,l=1}^n (h^e_{i,j,k})^2, \quad \|\nabla^\xi \bar{H}\|^2 = \sum_{i,j=1}^n (H^e_{i,j})^2.$$

Moreover, the equality in (4.2) holds if and only if
$$h^e_{i,j,k} = \frac{n}{n+2} (H^e_{i,j} \delta_{jk} + H^e_{j,k} \delta_{ik} + H^e_{k,i} \delta_{ij}), \quad 1 \leq i, j, k, l \leq n. \quad (4.3)$$

Now, we are ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $x : M^n \to \tilde{N}^{2n+1}(\tilde{c})$ be a compact $n$-dimensional Legendrian submanifold and $\{e_1, \ldots, e_n, e_1^*, \ldots, e_{n^*}, e_{2n+1} = \xi\}$ be a local adapted Legendre frame field along $M^n$. By definition, we have
$$\|\nabla(\varphi \bar{H})\|^2 = \sum_{i,j=1}^n (g(\nabla_{e_i}(\varphi \bar{H}), e_j))^2 = \sum_{i,j=1}^n (H^e_{i,j})^2 = \|\nabla^\xi \bar{H}\|^2. \quad (4.4)$$

Then, by (2.25), calculating the squared length of the Lie derivative $\mathcal{L}_{\varphi \bar{H}}g$ of $g$ with respect to $\varphi \bar{H}$, we obtain
$$\|\mathcal{L}_{\varphi \bar{H}}g\|^2 = \sum_{i,j=1}^n [(\mathcal{L}_{\varphi \bar{H}}g)(e_i, e_j)]^2 = \sum_{i,j=1}^n (H^e_{i,j} + H^e_{i,j})^2 = 4\|\nabla(\varphi \bar{H})\|^2. \quad (4.5)$$

Thus, we can apply Lemma 2.1 and (4.1) to obtain that
$$\text{div}(\nabla_{\varphi \bar{H}}(\varphi \bar{H}) - (\text{div} \, \varphi \bar{H}) \varphi \bar{H}) = \text{Ric} \, (\varphi \bar{H}, \varphi \bar{H}) + \|\nabla(\varphi \bar{H})\|^2 - (\text{div} \, \varphi \bar{H})^2 \geq \text{Ric} \, (\varphi \bar{H}, \varphi \bar{H}) - (n-1)\|\nabla(\varphi \bar{H})\|^2, \quad (4.6)$$
where the equality in (4.6) holds if and only if $\nabla(\varphi \bar{H}) = \frac{1}{n}(\text{div} \, \varphi \bar{H}) \text{id}$.

From (4.6) and that $\|\nabla(\varphi \bar{H})\|^2 = \|\nabla^\xi \bar{H}\|^2$, by further applying Lemma 4.2, we get
$$\text{div}(\nabla_{\varphi \bar{H}}(\varphi \bar{H}) - (\text{div} \, \varphi \bar{H}) \varphi \bar{H}) \geq \text{Ric} \, (\varphi \bar{H}, \varphi \bar{H}) - \frac{(n-1)(n+2)}{3n^2} \|\nabla^\xi h\|^2, \quad (4.7)$$
where the equality holds if and only if both $\nabla(\varphi \vec{H}) = \frac{1}{n}(\text{div} \varphi \vec{H}) \text{id}$ and (4.3) hold.

By the compactness of $M^n$, we can integrate the inequality (4.7) over $M^n$. Then, applying for the divergence theorem, we obtain the integral inequality (1.10).

It is easily seen from the above arguments that the equality in (1.10) holds if and only if the equality in (4.2) holds identically. Thus, according to Theorem 1.1 in [14], equality in (1.10) holds if and only if either $x(M^n)$ is of $C$-parallel second fundamental form, or $x(M^n)$ is one of the contact Whitney spheres in $\tilde{N}^{2n+1}(c)$.

This completes the proof of Theorem 1.3. □

As final remarks, we would mention that all the Whitney spheres in the complex space forms are conformally equivalent to the round sphere (cf. [17] and [6]). Now, for any one of the contact Whitney spheres, $x : S^n \to \tilde{N}^{2n+1}(c)$, its second fundamental form $h$ has the expression (1.9). Thus, by using the Gauss equation and direct calculations, we can immediately obtain the following

**Theorem 4.1.** The sectional curvatures of the contact Whitney spheres are not constant. Nevertheless, all the contact Whitney spheres in each of the Sasakian space forms are conformally equivalent to the round sphere.

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