The $\tau_2$ model and parafermions

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Abstract

Paul Fendley has recently found a ‘parafermionic’ way to diagonalize a simple solvable Hamiltonian associated with the chiral Potts model. Here we indicate how this method generalizes to the $\tau_2$ model with open boundaries and make some comments.

Keywords: statistical mechanics, lattice models, chiral Potts model, solvable Hamiltonians, parafermions

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1. Introduction

For a given integer $N$, let

$$\omega = e^{2\pi i/N}. \quad (1.1)$$

Let $I, Z, X$ be $N$-by-$N$ matrices, $I$ the identity and $Z, X$ having elements

$$Z_{jk} = \omega^{j-1}\delta_{jk}, \quad X_{jk} = \delta_{j,k+1}, \quad (1.2)$$

and $Z_m$ the $N^L$-dimensional matrix

$$Z_m = I \otimes \cdots I \otimes Z \otimes I \cdots \otimes I \quad (1.3)$$

where there are $L$ terms in the direct product and $Z$ is in position $m$. Similarly, let

$$X_m = I \otimes \cdots I \otimes X \otimes I \cdots \otimes I. \quad (1.4)$$

In 1989 [1, 2] the author posed a puzzle by showing\(^1\), via a quite roundabout route that the Hamiltonian

$$\mathcal{H} = -\sum_{j=1}^{L} \alpha_j X_j - \sum_{j=1}^{L-1} \gamma_j Z_j Z_{j+1}^{-1} \quad (1.5)$$

\(^1\) I have slightly simplified the puzzle by taking $r = L + 1$ and $\gamma_1, \gamma_r = 0$ in [1], then re-labelling $\gamma_2, \ldots, \gamma_{r-1}$ as $\gamma_1, \ldots, \gamma_{r-1}$.
has the same eigenvalues as the diagonal matrix

$$\mathcal{H} = - \sum_{j=1}^{L} \zeta_j Z_j$$

where $\zeta_1, \ldots, \zeta_L$ are the solutions of the equation

$$\det \begin{pmatrix} -\zeta^{N/2} & 0 & 0 & \ldots & 0 & 0 \\ \frac{N}{2} & -\zeta^{N/2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{N}{2} & -\zeta^{N/2} & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -\zeta^{N/2} & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \frac{N}{2} \\ 0 & 0 & 0 & \ddots & \ddots & \frac{N}{2} \\ -\zeta^{N/2} & \frac{N}{2} & \cdots & \cdots & \frac{N}{2} & \frac{N}{2} \end{pmatrix} = 0.$$ (1.7)

Despite appearances, this determinant is a multinomial in integer powers of $\zeta_N, g_{N1}, \ldots, g_{NL}$. I then said that this suggested there might be a simpler way of obtaining this result, similar to the spinor operator or Clifford algebra method used by Kaufman [3] for the Ising model. I repeated this suggestion in 2004 [4].

This puzzle has now been solved by Paul Fendley, using parafermion operators [5]. Here I wish to suggest an extension of Fendley's method to the more general problem of the $\tau_2$ model with open boundaries [4].

The equations of sections 4–6 have not been rigorously proved, but are conjectures based on algebraic computer calculations for small values (between 2 and 6) of $N$ and $L$. Since my original presentation of this work, [6] equations (4.3) and (4.8) have been proved by Helen Au-Yang and Jacques Perk [7]. Their paper follows this.

2. $\tau_2$ model

The $\tau_2$ model was first introduced by Bazhanov and Stroganov [8] in their paper that showed how the chiral Potts model was related, via the $\tau_2$ model, to the six-vertex model of Lieb [9]. Here we use the definition of [4, 10], generalized to allow the parameters $\alpha_p, \beta_p, \gamma_p, \delta_p$ of equation (2.2) below to all vary without restriction from column to column.

Consider the square lattice with $M$ rows of $L + 1$ sites and periodic boundary conditions. (In section 3 we effectively remove the end column, which is the reason for this choice of notation.) At each site $i$ (numbered $i = 0, 1, 2, \ldots, L$) there is a ‘spin’ $\sigma_i$, which takes the values $0, \ldots, N - 1$. Constrain them so that if $j$ is the site immediately above $i$, then $\sigma_j = \sigma_i$ or $\sigma_j = \sigma_i - 1$. (2.1)

Let $\omega = e^{2\pi i/N}$ be the primitive $N$th root of unity. Similarly to equation (15) of [4], define a function $F_{pq}(a, m)$, for $a, m = 0, 1, b, c, d$ are the four spins round a face, arranged as in figure 1, then to that face assign a Boltzmann weight

$$W_{pq}(a, b, c, d) = \sum_{m=0}^{\frac{N}{2}} \omega^{m(d-b)}(-\omega_{pq})^{\frac{a(N-a)}{2} - d} F_{pq}(a - d, m) F_{pq}(b - c, m)$$
as in equation (14) of [4]. Hence
\[ W_{pp'q}(a, b, b, a) = b_p b_{p'} - \omega^{a_{b+1}} t_q c_p c_{p'} \]
\[ W_{pp'q}(a, b, b, a - 1) = -\omega t_q d_p b_{p'} + \omega^{a_{b+1}} t_q a_p c_{p'} \]
\[ W_{pp'q}(a, b, b - 1, a) = b_p d_{p'} - \omega^{a_{b+1}} c_p a_{p'} \]
\[ W_{pp'q}(a, b, b - 1, a - 1) = -\omega t_q d_p d_{p'} + \omega^{a_{b+1}} a_p a_{p'} . \]
(2.3)

Note that here \(a, b, c, d\) are complex numbers.

Take the parameters \(p, p'\) of the face between columns \(j\) and \(j + 1\) to be \(p = p_{2j-1}, p' = p_{2j}\). If \(p = p_m\), write
\[ a_p = a_m, \quad b_p = b_m, \quad c_p = c_m, \quad d_p = d_m \]
(2.4)
and similarly for \(p'\). Then the weight of the face between columns \(j\) and \(j + 1\) is
\[ W_j(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) | t_q \) = \[ W_{p_{2j-1}, p_{2j}, q}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j). \]
(2.5)

Thus there are \(2L + 2\) sets of parameters \((a_p, b_p, c_p, d_p)\), for \(p = p_{-1}, p_0, \ldots, p_{2L}\). We do not impose any constraints (such as equation (2.4) of [10]) on \(a_p, b_p, c_p, d_p\). As in [4], all these \(2L + 2\) parameters can be chosen arbitrarily.

With these definitions, the transfer matrix of the \(r_2\) model is the \(N^{L+1}\) by \(N^{L+1}\) matrix \(\tau_2(q)\) with elements
\[ [\tau_2(q)]_{\sigma, \sigma'} = \prod_{j=0}^{L} W_j(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) | t_q \). \]
(2.6)

using the cyclic boundary conditions \(\sigma_{L+1} = \sigma_0, \sigma'_{L+1} = \sigma'_0\). The RHS is just the product of the face weights of a typical row of the lattice, as shown in figure 1.

We shall also use the \(N^{L+1}\)-dimensional matrices \(X_j, Z_j\), with entries
\[ [Z_j]_{\sigma, \sigma'} = \omega^{t_q} \prod_{k=0}^{L} \delta(\sigma_k, \sigma'_k), \quad [X_j]_{\sigma, \sigma'} = \delta(\sigma_j, \sigma'_{j+1}) \prod_{k \neq j}^{L} \delta(\sigma_k, \sigma'_k) \]
(2.7)
the second product being over all values of \(k\) from 0 to \(L\), except \(j\). We also take \(I\) to be the identity matrix, and set \(X = X_0 X_1 X_2 \cdots X_L\).
**Functional relations for \( \tau_2(t_q) \)**

Write \( t_q \) simply as \( t \): \( \quad t_q = t. \) \( \text{(2.8)} \)

From the relations of \([10]\), two matrices \( \tau_2(t), \tau_2(t') \), with the same parameters \( p_{-1}, p_0, \ldots, p_{2L} \) but different \( t \), commute:

\[ \tau_2(t) \tau_2(t') = \tau_2(t') \tau_2(t). \] \( \text{(2.9)} \)

Define two-by-two matrices \( A_0, A_4, \ldots, A_{2L} \) and \( B_{-1}, B_1, \ldots, B_{2L-1} \) by

\[ A_j = \begin{pmatrix} c_j^N t^N & a_j^N \\ b_j^N & d_j^N \end{pmatrix}, \quad B_j = \begin{pmatrix} -c_j^N & b_j^N \\ a_j^N & -d_j^N \end{pmatrix} \] \( \text{(2.10)} \)

and set

\[ U = B_{-1} A_0 B_1 A_2 \cdots A_{2L}. \] \( \text{(2.11)} \)

Also, from equation (47) of \([4]\), one can define related matrices \( \tau_3(t_q), \ldots, \tau_{N+1}(t_q) \) so that, for \( j = 2, \ldots, N \),

\[ \begin{align*}
\tau_2(\omega^{j-1}) \tau_j(t) &= z(\omega^{j-1}) X \tau_{j-1}(t) + \tau_{j+1}(t), \\
\tau_j(\omega t) \tau_2(t) &= z(\omega t) X \tau_{j-1}(\omega^2 t) + \tau_{j+1}(t), \\
\tau_{N+1}(t) &= z(t) X \tau_{N-1}(\omega t) + (\alpha + \overline{\alpha}) I,
\end{align*} \] \( \text{(2.12)} \)

where

\[ t_1(t) = I, \quad z(t) = \omega^L \prod_{j=1}^{2L} (c_j d_j - a_j b_j t), \] \( \text{(2.13)} \)

and \( \alpha, \overline{\alpha} \) are the two eigenvalues of \( U \), so

\[ \alpha + \overline{\alpha} = \text{Trace } U. \] \( \text{(2.14)} \)

Clearly \( z(t) \) is a polynomial of degree \( 2L + 2 \). Each element of the matrix \( \tau_2(t) \) is a polynomial in \( t = t_q \) of degree at most \( L + 1 \). It follows that \( \tau_j(t) \) is at most of degree \( (j - 1)(L + 1) \), and this is consistent with the fact that \( U, \alpha + \overline{\alpha} \) are of degree \( N(L + 1) \).

Since \( \tau_2(t) \) commutes with \( X \), one can choose a representation where the \( \tau_j(t) \) (for all \( j \)) are diagonal, with elements (eigenvalues) that are polynomials in \( t \) of degree \( (j - 1)(L + 1) \). Then equation (2.12) defines all the eigenvalues of \( \tau_2(t), \ldots, \tau_{N+1}(t) \). However, the situation is similar to Bethe ansatz calculations: in general the best one can do is a brute force numerical calculation for each of the \( N^{L+1} \) eigenvalues.

**3. \( \tau_2 \) model with open boundaries**

The problem dramatically simplifies if one imposed fixed boundary conditions, instead of cyclic ones. As is remarked in \([4]\), we can do this easily by taking

\[ a_{-1} = d_{-1} = 0. \] \( \text{(3.1)} \)

Then for the left-hand weight function \( W_0 \), the \( a_p, d_p \) in equation (2.3) are zero. Hence \( W_0(a, b, c, d) \) vanishes unless \( d = a \), i.e.

\[ \sigma_0' = \sigma_0. \] \( \text{(3.2)} \)

Also, from equations (3.1) and (2.13),

\[ z(t) = 0 \] \( \text{(3.3)} \)
so the terms in equation (2.12) involving $X$ do not occur and all the $\tau_j(t)$ matrices are block-diagonal, having non-zero entries only when $\sigma'_0 = \sigma_0$. If we also choose
\begin{equation}
c_{-1} = c_{2L} = 0, \quad (3.4)
\end{equation}
then $\sigma_0$, $\sigma'_0$ no longer enter the RHS of equation (2.6), so all the diagonal blocks are the same and without further loss of generality we can focus on the case
\begin{equation}
\sigma'_0 = \sigma_0 = 0. \quad (3.5)
\end{equation}

This reduces the dimensionality of the transfer matrices: the $\tau_j(t)$ are now of dimension $N^L$. Hereinafter we take $I$ to be the $N^L$-dimensional identity matrix, and re-define $Z_j$, $X_j$ to be $N^L$-dimensional matrices given by equation (2.7), the products being from $k = 1$ to $k = L$.

We choose the normalization of the weights so that
\begin{equation}
b_{-1} = b_0 = b_1 = \cdots = b_{2L} = 1 \quad (3.6)
\end{equation}
which ensures that the allowed values of the end weights $W_0$, $W_L$ are
\begin{align*}
W_0(a, b, b, a|t) &= 1, \quad W_0(a, b, b - 1, a|t) = d_0, \\
W_L(a, b, b, a|t) &= 1, \quad W_L(a, b, b - 1, a|t) = -\omega \alpha d_{2L-1}.
\end{align*}
\begin{equation}
(3.7)
\end{equation}
Effectively each row of the lattice loses the first column of spins and the bordering half-faces, leaving the section of figure 1 between the zig–zag vertical lines.

The relations (2.12) now greatly simplify, so equation (2.12) reduces to
\begin{equation}
\tau_j(t) = \tau_2(t) \tau_2(\omega t) \cdots \tau_2(\omega^{j-1} t) 
\end{equation}
\begin{equation}
\tau_2(t) \tau_2(\omega t) \cdots \tau_2(\omega^{N-1} t) = (\alpha + \overline{\alpha}) I. \quad (3.8)
\end{equation}
From equation (2.10),
\begin{equation}
B_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{2L} = \begin{pmatrix} 0 & a_{NL}^N \\ 1 & d_{2L}^N \end{pmatrix}, \quad (3.9)
\end{equation}
so if
\begin{equation}
V = A_0 B_1 \cdots B_{2L-1}, \quad (3.10)
\end{equation}
then from equation (2.11), $U$ is the two-by-two matrix
\begin{equation}
U = \begin{pmatrix} V_{22} & \cdots \\ V_0 & 0 \end{pmatrix} \quad (3.11)
\end{equation}
so its two eigenvalues $\alpha$, $\overline{\alpha}$ can be taken to be
\begin{equation}
\alpha = V_{22}, \quad \overline{\alpha} = 0 \quad (3.12)
\end{equation}
and equation (3.8) becomes
\begin{equation}
\tau_2(t) \tau_2(\omega t) \cdots \tau_2(\omega^{N-1} t) = V_{22} I. \quad (3.13)
\end{equation}
The element $V_{22}$ is a polynomial in $t^N$ of degree $L$. Write it as
\begin{equation}
V_{22} = f(t^N) = s_0 + s_1 t^N + s_2 t^{2N} + \cdots + s_L t^{NL}. \quad (3.14)
\end{equation}
From equations (2.10) and (3.6), it readily follows that
\begin{equation}
f(0) = s_0 = 1. \quad (3.15)
\end{equation}
Let $1/r_1$, \ldots, $1/r_L$ be the zeros of $f(t^N)$, so
\begin{equation}
s_{r_1}^{NL} + s_{r_1}^{N(L-1)} + s_{r_2}^{N(L-2)} + \cdots + s_L = 0, \quad (3.16)
\end{equation}
then
\[ f(t^N) = \prod_{j=1}^{L} (1 - t_j^N r_j). \] (3.17)

The matrix \( \tau_2(t) \) is a polynomial of degree \( L \), equal to the identity matrix when \( t = 0 \):
\[ \tau_2(0) = I. \] (3.18)

Going to the diagonal representation, it follows from equation (3.13) that each eigenvalue of \( \tau_2(t) \) must be of the form
\[ [\tau_2(t)]_p = \prod_{k=1}^{L} (1 - \omega^{p_k} r_k t), \] (3.19)
where \( p_1, \ldots, p_L \) are integers with values \( 0, \ldots, N-1 \) and \( r_1, \ldots, r_L \) are independent of \( t \). Thus there are \( N^L \) distinct possible forms for the eigenvalues. Numerical evidence suggests that there is a one-to-one correspondence between the \( N^L \) eigenvalues of \( \tau_2(t) \) and the expressions (3.19).

Making this assumption, it follows that there is a similarity transformation that takes \( \tau_2(t) \) to
\[ P^{-1} \tau_2(t) P = \prod_{k=1}^{L} (I - \omega r_k Z_j), \] (3.20)
the eigenvector matrix \( P \) being independent of \( t \).

### 3.1. The associated Hamiltonian \( \mathcal{H} \)

We define the associated Hamiltonian \( \mathcal{H} \) to be the coefficient of \( \omega t \) in the Taylor expansion of \( \tau_2(t) \), so
\[ \tau_2(t) = I + \omega t \mathcal{H} + O(t^2). \] (3.21)
then from equations (2.3) and (2.6),
\[ \mathcal{H} = - \sum_{j=1}^{L} \sum_{k=j}^{L} \omega^{k-j} d_{j-2} a_{j-1} \cdots d_{k-2} a_{k-1} Z_j Z_k^{-1} X_j \cdots X_k \]
\[ + \sum_{j=1}^{L-1} \sum_{k=j+1}^{L} \omega^{k-j} c_{j-2} a_{j-1} \cdots a_{k-1} d_{k-1} Z_j Z_k^{-1} X_j+1 \cdots X_k \]
\[ - \sum_{j=1}^{L} \sum_{k=j}^{L-1} \omega^{k-j} c_{j-2} a_{j-1} \cdots c_{k-1} Z_j Z_k^{-1} X_j \cdots X_k \]
\[ + \sum_{j=1}^{L-1} \sum_{k=j}^{L} \omega^{k-j} d_{j-2} a_{j-1} \cdots a_{k-1} c_{k} Z_j Z_{k+1}^{-1} X_j \cdots X_k \] (3.22)
From equation (3.20) its \( N^L \) eigenvalues must be
\[ - \sum_{j=1}^{L} r_j \omega^j, \] (3.23)
so we can write the diagonal form of \( \mathcal{H} \) as
\[ \mathcal{H}_d = - \sum_{j=1}^{L} r_j Z_j. \] (3.24)
A particularly simple case. This Hamiltonian $\mathcal{H}$ simplifies if we further specialize to the case when

$$a_1 = a_2 = \cdots = a_{2L-2} = 0.$$  \hfill (3.25)

Define $\alpha_1, \ldots, \alpha_L, \gamma_1, \ldots, \gamma_{L-1}$ by

$$\alpha_i = d_{2i-3}d_{2i-1}, \quad \gamma_i = c_{2i-1}c_2,$$

then equation (3.22) reduces to

$$\mathcal{H} = -\sum_{j=1}^{L-1} \alpha_j X_j - \sum_{j=1}^{L-1} \gamma_j Z_j Z_{j+1}^{-1},$$  \hfill (3.27)

which is equation (1.5).

Set

$$g_{2j-1} = \alpha_j, \quad g_{2j} = \gamma_j,$$

and take $D(t^N)$ to be the $2L$ by $2L$ tridiagonal matrix:

$$D(t^N) = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\frac{g_{2L} t^N}{g_{2L-1} t^N} & \frac{g_{2L-2} t^N}{g_{2L-1} t^N} & 1 & 1 \\
0 & 0 & 0 & 0 & \frac{g_{2L-2} t^N}{g_{2L-1} t^N} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{g_{2L-2} t^N}{g_{2L-1} t^N} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{g_{2L-2} t^N}{g_{2L-1} t^N}
\end{pmatrix}. \hfill (3.29)$$

Then we can write $V_{22}$ as

$$V_{22} = f(t^N) = \det D(t^N)$$  \hfill (3.30)

and $1/r_1^N, \ldots, 1/r_N^N$ are the zeros of $D(t^N)$. This is the result mentioned in the Introduction and obtained in 1989 by a less direct method [1, section 8], [2].

4. Method using parafermions

For $N = 2$ the model is equivalent to the Ising model, whose associated Hamiltonian also has the sum structure (3.23) for its eigenvalues. I remarked that it would be interesting to find a method to obtain this structure that somehow parallels the simple free-fermion or Clifford algebra method used when $N = 2$ by Kaufman [3]. This has recently been achieved for general $N$ by Paul Fendley [5] for the case when (3.25) is satisfied. Here we no longer use this restriction: we allow $a_1, a_2, \ldots, a_{2L-2}$ to be arbitrary.

In Kaufman’s method, as applied to the Hamiltonian $\mathcal{H}$ of equation (3.22) one constructs a set of $2L$ operators $\Gamma_0, \ldots, \Gamma_{2L-1}$ such that the commutator of $\mathcal{H}$ with each $\Gamma_j$ is a linear combination of $\Gamma_0, \ldots, \Gamma_{2L-1}$. We can take

$$\Gamma_0 = Z_1^{-1}. \hfill (4.1)$$

Fendley found, for general $N$, that if we generate a set of matrices $\Gamma_j$ by successive commutation with $\mathcal{H}$, then this set closes after $NL$ members. He then went on to obtain relations that give the eigenvalues (3.23).

Fendley considered the Hamiltonian (3.22). Here I shall indicate how his method can be generalized to the more general $t_2$ model with open boundaries discussed above, and is associated Hamiltonian $\mathcal{H}$ as defined by equation (3.21). I continue to impose the restrictions (3.1), (3.4), (3.6), but not (3.25). The results are based on numerical calculations for small $N$ and $L$ (no bigger than 5), so are just plausible conjectures.

Both $t_2(t)$ and $\mathcal{H}$ depend on the $6L - 4$ independent parameters $d_0, a_1, c_1, d_1, \ldots, a_{2L-2}, c_{2L-2}, d_{2L-2}, d_{2L-1}$. Remarkably, all the following equations will involve these parameters only via the coefficients $s_0, s_1, \ldots, s_L$ in equation (3.14).
4.1. Relations involving \( \Gamma \)

Defining \( \Gamma_1, \ldots, \Gamma_{NL} \) by

\[
\Gamma_{j+1} = \frac{\omega}{1 - \omega} (H \Gamma_j - \Gamma_j H), \quad j = 0, 1, \ldots,
\]

we find that

\[
s_0 \Gamma_{NL} + s_1 \Gamma_{N(L-1)} + s_2 \Gamma_{N(L-2)} + \cdots + s_L \Gamma_0 = 0.
\]

4.2. Relations involving \( \tau_2(t) \)

Set

\[
q = NL - 1.
\]

We find for all \( j \) that \( \tau_2(t) \Gamma_j \tau_2(t)^{-1} \) is a linear combination of \( \Gamma_0, \ldots, \Gamma_q \). Set

\[
X = X_1 X_2 \cdots X_L.
\]

From equations (2.3) and (2.6), the matrix \( \tau_2(t) \) is a polynomial in \( t \) of degree \( L \), its coefficient of \( t^0 \) being \( I \) and the coefficient of \( t^L \) being \( \omega d_0 \cdots d_{2L-1} X^{-1} \). The matrix \( \tau_2(t) \) commutes with \( X \). Each \( \Gamma_j \) is independent of \( t \) and satisfies the \( \omega \)-commutation relation

\[
X \Gamma_j = \omega \Gamma_j X
\]

so, for \( 0 \leq j \leq NL \),

\[
\mu_j = \Gamma_j \tau_2(t) - \tau_2(t) \Gamma_j
\]

is a polynomial in \( t \) whose coefficient of \( t^0 \) is zero and

\[
\nu_j = \omega \Gamma_j \tau_2(t) - \tau_2(t) \Gamma_j
\]

is a polynomial in \( t \) whose coefficient of \( t^L \) is zero. We observe numerically that

\[
\tau_2(t) \mu_j = \mu_{j-1}
\]

for \( 0 < j \leq NL \), and find an equation very similar to equation (4.3), namely

\[
s_0 \nu_{NL} + s_1 \nu_{N(L-1)} + s_2 \nu_{N(L-2)} + \cdots + s_L \nu_0 = 0.
\]

Let \( H \) be the \( NL \) by \( NL \) matrix, with elements \( h_{jk} \) and rows and columns labelled \( 0, \ldots, q \):

\[
H = \\
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
-s_L & 0 & 0 & \cdots & -s_{L-1} & -s_j & 0 \\
\end{pmatrix}
\] (The last row is shown for the case \( N = 3 \); in general there are \( N - 1 \) zeros between \( -s_j \) and \( -s_{j-1} \), and after \( -s_1 \).)

Then equations (4.2) and (4.3) are equivalent to

\[
\Gamma_j H - H \Gamma_j = (1 - \omega^{-1}) \sum_{k=0}^{NL-1} h_{jk} \Gamma_k
\]

and equations (4.7) and (4.9) to

\[
\mu_j = t \sum_{k=0}^q h_{jk} v_k,
\]
\[ \Gamma_j t_2(t) - t_2(t) \Gamma_j = \sum_{k=0}^q \delta_{jk} \omega \Gamma_k t_2(t) - t_2(t) \Gamma_k \]  
(4.13)

for \( 0 \leq j < NL \). If we define the NL by NL matrix
\[ M = (I - tH)^{-1} (I - \omega tH), \]  
(4.14)

then equation (4.13) can be written
\[ t_2(t) \Gamma_j t_2(t)^{-1} = \sum_{k=0}^q M_{jk} \Gamma_k. \]  
(4.15)

Thus \( M \) is the representative of \( t_2(t) \), in the sense of the Kaufman–Onsager representative matrices for the Ising model [11].

4.3. The eigenvalues of \( \hat{t}, t_2(t) \)

Let \( P \) be the NL by NL matrix that diagonalizes \( H \):
\[ HP = PH_d, \]  
(4.16)

where \( H_d \) is the diagonal matrix with elements \( \lambda_1, \ldots, \lambda_{NL} \). Set
\[ \hat{\Gamma}_j = \sum_{j=0}^q (P^{-1})_{ij} \Gamma_j. \]  
(4.17)

Multiplying equation (4.11) by \( (P^{-1})_{ij} \) and summing over \( j \), it becomes
\[ \hat{\Gamma}_j \hat{t} - \hat{t} \hat{\Gamma}_j = (1 - \omega^{-1}) \lambda_i \hat{\Gamma}_i. \]  
(4.18)

Similarly, equation (4.13) becomes
\[ \hat{\Gamma}_j t_2(t) - t_2(t) \hat{\Gamma}_i = t \lambda_i [\omega \hat{\Gamma}_j t_2(t) - t_2(t) \hat{\Gamma}_j]. \]  
(4.19)

Here \( i = 1, \ldots, NL \).

The characteristic polynomial of \( H \) is
\[ |H - \lambda I| = s_0 \lambda^{NL} + s_1 \lambda^{N(L-1)} + s_2 \lambda^{N(NL-2)} + \cdots + s_L \lambda^N + s_L, \]
which is the RHS of equation (3.16). The eigenvalues of \( H \) are therefore \( \lambda_{p,k} = \omega^p r_k \), where \( p = 0, 1, \ldots, N-1 \) and \( k = 1, \ldots, L \) as in equation (3.19). We can therefore naturally replace \( i \) in equations (4.18) and (4.19) by the pair of numbers \( p, k \).

The matrices \( \hat{\mathcal{H}}, t_2(t) \) are of dimension \( NL \). They commute (for all \( t \)), so there is a similarity transformation, independent of \( t \), that diagonalizes both. Write their elements in this representation as \( \hat{\mathcal{H}}_mn, \) \( \hat{t}_mn, \) \( \hat{\Gamma}_mn \).

Let us go this representation. The \( \hat{\Gamma}_j \) are also of dimension \( NL \), but do not commute with either \( \hat{\mathcal{H}} \) or \( t_2(t) \), so are not diagonal in this representation. Write their elements as \( (\hat{\Gamma}_{p,k})mn \). Then equations (4.18) and (4.19) become
\[ [\hat{\mathcal{H}}_mn - \hat{\mathcal{H}}_mn - (1 - \omega^{-1}) \omega^p r_k] (\hat{\Gamma}_{p,k})mn = 0, \]  
(4.20)

\[ [(1 - \omega^p t \Gamma_j) \hat{\mathcal{H}}_mn - (1 - \omega^p t \Gamma_k) \hat{\mathcal{H}}_mn] (\hat{\Gamma}_{p,k})mn = 0. \]  
(4.21)

From equation (3.21), \( \hat{\mathcal{H}}_m = 1 + \omega t \hat{\mathcal{H}}_m + O(t^2) \). Expanding equation (4.21) to first order in \( t \), we obtain equation (4.20).

Let us use the known values equations (3.19) and (3.23) of the eigenvalues of \( t_2(t) \) and \( \hat{t} \). Write
\[ m = \{m_1, \ldots, m_L\}, \quad n = \{n_1, \ldots, n_L\}, \]  
(4.22)
where the \( m_i, n_i \) take the values 0, 1, \ldots, \( N - 1 \), ordering the elements so that equation (3.24) is equivalent to

\[
\mathcal{H}_m = - \sum_{j=1}^{L} \omega^{m_j} r_j
\]  

(4.23)

and equation (3.19) to

\[
\tau_m = \prod_{k=1}^{L} (1 - \omega^{m_k+1} r_k t).
\]  

(4.24)

Then equations (4.20) and (4.21) are satisfied if, when \( (\hat{\Gamma}_{p,k})_{mn} \) is non-zero,

\[
m_k = n_k + 1 = p \pmod{N}, \quad m_j = n_j \quad \text{for} \quad j \neq k.
\]

This means that \( X_k^{-1} \hat{\Gamma}_{p,k} \) is a diagonal matrix, of rank \( N^{L-1} \).

We observe that the elements of \( (\hat{\Gamma}_{p,k})_{mn} \) are indeed non-zero when equation (4.25) is satisfied. If we assume that this is so, then equation (4.20) defines the eigenvalues of \( \mathcal{H} \) to within the addition of a single constant to all. From equation (3.22), \( \mathcal{H} = 0 \), which fixes this constant. It follows that the eigenvalues of \( \mathcal{H} \) are indeed given by equation (4.23), so Fendley’s method is indeed a new way of obtaining them.

Similarly equation (4.21) gives the eigenvalues of \( \tau_2(t) \) to within an overall multiplicative factor. One way of determining this would be to use equation (3.18) and the fact that \( \tau_2(t) \) is a polynomial in \( t \) of degree \( L \). This would necessarily give equation (4.24).

5. Parafermionic properties of the \( \Gamma_j \)

Fendley noted various relations between the \( \hat{\Gamma}_j \). A significant one is that if \( G \) is any linear combination of the \( \hat{\Gamma}_{p,k} \), i.e.

\[
G = \sum_{p=0}^{N-1} \sum_{k=1}^{L} \rho_{p,k} \hat{\Gamma}_{p,k}
\]  

(5.1)

where the coefficients \( \rho_{p,k} \) are arbitrary, then \( G^N \) is proportional to the identity matrix \( I \). More precisely,

\[
G^N = \left( \sum_{k=1}^{L} \beta_k \rho_{0,k} \cdots \rho_{N-1,k} \right) I
\]  

(5.2)

where \( \beta_1, \ldots, \beta_L \) are scalar factors, independent of the \( \rho_{p,k} \). From equation (4.17) each \( \hat{\Gamma}_{p,k} \) is a linear combination of the \( \hat{\Gamma}_j \), so it is also true that the \( N \)th power of any linear combination of the \( \Gamma_j \) matrices is proportional to the identity matrix. This is a natural extension of the anti-commutation property of the free-fermion or Clifford algebra used by Kaufman [3] for the Ising model.

There are also quadratic relations between the \( \hat{\Gamma}_j \). As above, we write \( \hat{\Gamma}_j \) as \( \hat{\Gamma}_{p,k} \). Then

\[
\hat{\Gamma}_{p,k} \hat{\Gamma}_{p',k} = 0 \quad \text{if} \quad p' \neq p - 1 \pmod{N},
\]  

(5.3)

and, for all \( p, k, p', k' \),

\[
(r_k \omega^p - r_k \omega^{p'+1}) \hat{\Gamma}_{p,k} \hat{\Gamma}_{p',k'} + (r_k \omega^{p'} - r_k \omega^{p'+1}) \hat{\Gamma}_{p,k} \hat{\Gamma}_{p',k'} = 0.
\]  

(5.4)

The relation (5.3) implies that both sides of equation (5.4) are zero if \( k = k' \), for all \( p, p' \). For \( N = 2, \ldots, 6 \), we have verified that equation (5.4) implies equation (5.2).
6. Group property of the $\tau_2(t)$ matrices

Going back to equation (4.15), consider the set of all $N^L$-dimensional invertible matrices $V$ such that $V \Gamma_j V^{-1}$ is a linear combination of $\Gamma_0, \ldots, \Gamma_N$, i.e. $\exists$ an $NL$ by $NL$ matrix $v$ with elements $v_{jk}$ such that

$$V \Gamma_j V^{-1} = \sum_{k=0}^{q} v_{jk} \Gamma_k.$$  \hfill (6.1)

Such a set is necessarily a group $G$. If $V$ can be arbitrarily close to the identity $I$, then we can write

$$V = I + \epsilon \tilde{H} + O(\epsilon^2),$$  \hfill (6.2)

expanding equation (6.1) to first order in $\epsilon$, we obtain a generalization of equation (4.11):

$$\tilde{H} \Gamma_j - \Gamma_j \tilde{H} = \sum_{k=0}^{q} h_{jk} \Gamma_k.$$ \hfill (6.3)

Given the $\Gamma_j$, this is a linear equation for $\tilde{H}, h_{jk}$. For $N > 2$, if we work in the representation where $\tau_2(t)$, $\mathcal{H}$ are diagonal, then we find numerically that the only solutions for $\tilde{H}, h$ of equation (6.3) are also diagonal. Thus in any representation the only solutions commute with $\tau_2(t)$ and this in turn suggests that the only solutions of equation (6.1) may be matrices $V$ that commute with $\tau_2(t)$. This is not true for $N = 2$, where $\log V, \mathcal{H}$ can be arbitrary quadratic forms in the $2L \Gamma$'s: such forms do not in general all commute. In this sense the group $G$ is more restricted when $N > 2$ than when $N = 2$.

7. Summary

We have indicated how Fendley’s method can be generalized from the simple Hamiltonian (3.27) to the more general Hamiltonian (3.22), and to the corresponding $\tau_2$ model with open boundaries. We have defined a set of operators $\Gamma_0, \ldots, \Gamma_{NL-1}$ that satisfy the commutation relations (4.11) with $\mathcal{H}$, as well as the relations (4.13) and (4.15) with $\tau_2(t)$. They have the parafermionic properties (5.2)–(5.4) observed by Fendley.

We emphasize again that most of the equations in sections 4–6 were conjectured by the author, based on the results of computer algebra calculations for small $N$ and $L$. Proofs of equations (4.3) and (4.8) are given in the following paper [7].

A significant difference from the usual Clifford algebra is that the $\Gamma$ matrices depend on the Hamiltonian $\mathcal{H}$, or on the transfer matrix $\tau_2(t)$. Also, as discussed in section 6, for $N > 2$ the group $G$ appears to be restricted to matrices that commute with $\tau_2(t)$. Even so, as Fendley says, this method opens up some very interesting possibilities. In particular, can it be extended to an Hamiltonian that has some kind of cyclic property?²

The solvable chiral Potts model has a superintegrable case that is the $\tau_N$ model turned through 90°. Does this parafermion algebra play a useful role in that more general model? The corner transfer matrix method of calculating order parameters works well for solvable two-dimensional models with the rapidity-difference property, but seems to utterly fail for the chiral Potts model [12]. Does this parafermion algebra provide a way of working with $NL$-dimensional representatives instead of the full $N^L$-dimensional corner transfer matrices, as can be done for the Ising case $N = 2$ [11]?

² If one simply extends the second summation in (1.5) to include $j = L$, taking $Z_{2,N}$ to be $Z_2$, one does not appear to obtain an eigenvalue spectrum with the simple form (1.6), but perhaps there is another way of making the extension that does preserve that simple ‘direct sum’ form of the eigenvalue spectrum.
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