Flows, coalescence and noise

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Summary. We are interested in stationary “fluid” random evolutions with independent increments. Under some mild assumptions, we show they are solutions of a stochastic differential equation (SDE). There are situations where these evolutions are not described by flows of diffeomorphisms, but by coalescing flows or by flows of probability kernels.

In an intermediate phase, for which there exists a coalescing flow and a flow of kernels solution of the SDE, a classification is given: All solutions of the SDE can be obtained by filtering a coalescing motion with respect to a sub-noise containing the Gaussian part of its noise. Thus, the coalescing motion cannot be described by a white noise.

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Introduction.

A stationary motion on the real line with independent increments is described by a Levy process, or equivalently by a convolution semigroup of probability measures. This naturally extends to “rigid” motions represented by Levy processes on Lie groups. If one assumes the continuity of the paths, a convolution semigroup on a Lie group $G$ is determined by an element of the Lie algebra $\mathfrak{g}$ (the drift) and a scalar product on $\mathfrak{g}$ (the diffusion matrix) (see for example [31]). We call them the local characteristics of the convolution semigroup.

We will be interested in stationary “fluid” random evolutions which have independent increments. Strong solutions of stochastic differential equations (SDEs) driven by smooth vector fields define such evolutions. Those are of a regular type, namely

(a) The probability that two points thrown in the fluid at the same time and at distance $\varepsilon$ separate at distance one in one unit of time tends to 0 as $\varepsilon$ tends to 0.

(b) Such points will never hit each other.

Their laws can be viewed as convolution semigroups of probability measures on the group of diffeomorphisms.
On a compact manifold, let \( V_0, V_1, \ldots, V_n \) be vector fields and \( B^1, \ldots, B^n \) be independent Brownian motions. Consider the SDE
\[
dX_t = \sum_{k=1}^n V_k(X_t) \circ dB^k_t + V_0(X_t) \, dt,
\]
which equivalently can be written
\[
df(X_t) = \sum_{k=1}^n V_k f(X_t) dB^k_t + \frac{1}{2} Af(X_t) dt
\]
for all smooth function \( f \) and \( Af = \sum_{k=1}^n V_k(V_k f) + V_0 f \). Note that \( Af^2 - 2fAf = \sum_{k=1}^n (V_k f)^2 \). Then, strong solutions (when they exist), as defined for example in [39], of this SDE produce a flow of maps \( \varphi_t \), such that for all \( x \), \( \varphi_t(x) \) is a strong solution of the SDE with \( \varphi_0(x) = x \), which means that \( \varphi_t \) is a function of the Brownian paths \( B^1, \ldots, B^n \) up to time \( t \). When the vector fields are smooth, strong solutions are known to exist, and to be unique. The framework can be extended to include flows of maps driven by vector fields valued Brownian motions, which means essentially that \( n = \infty \) (see for example [3, 17, 20, 21, 27]).

In a previous work [23], this was extended again to include flows of Markovian operators \( S_t \) solutions of the SPDE
\[
dS_t f = \sum_{k=1}^\infty S_t(V_k f) dB^k_t + \frac{1}{2} S_t(Af) dt,
\]
assuming the covariance function \( C = \sum_{k=1}^\infty V_k \otimes V_k \) of the Brownian vector field \( \sum_{k=1}^\infty V_k B^k \) is compatible with \( A \), namely that
\[
Af^2 - 2fAf \leq \sum_{k=1}^\infty (V_k f)^2.
\]
Existence and uniqueness of a flow of Markovian operators \( S_t \), which is a strong solution of the previous SPDE in the sense that \( S_t \) is a function of the Brownian paths \( (B^i)_{i \geq 1} \) up to time \( t \) holds under rather weak assumptions. However it is assumed in [23] that \( A \) is self-adjoint with respect to a measure \( m \) and the Markovian operators act on \( L^2(m) \) only.

The local characteristics of these flows are given by \( A \) and the covariance function \( C \), and they determine the SDE or the SPDE. But it was shown in [23] that covariance functions which are not smooth on the diagonal (e.g. covariance associated with Sobolev norms of order between \( d/2 \) and \( (d + 2)/2 \), \( d \) being the dimension of the space) can produce strong solutions, which define random evolutions of different type:

- turbulent evolutions where (a) is not satisfied, which means that two points thrown initially at the same place separate, though there is no pure diffusion, i.e. that \( Af^2 - 2fAf = \sum_{k=1}^\infty (V_k f)^2 \).
- coalescing evolutions where (b) does not hold.
In this paper, we adopt a different approach based on consistent systems of $n$ point Markovian Feller semigroups which can be viewed as determining the law of the motion of $n$ unsecable points thrown into the fluid. Regular and coalescing evolutions are represented by flows of maps. Turbulent evolutions by flows of probability kernels $K_{s,t}(x,dy)$ describing how a point mass (made of a continuum of unsecable points) in $x$ at time $s$ is spread at time $t$. (Note that in that case, the motion of an unsecable point is not fully determined by the flow).

Among turbulent evolutions, we can distinguish the intermediate ones where two points thrown in the fluid at the same place separate but can meet after, i.e. where (a) and (b) are both not satisfied.

In the intermediate phase, it has been shown in [9] (for gradient fields) and (at a physical level) in [10, 11, 14] that a coalescing solution of the SDE can be defined in law, i.e. in the sense of the martingale problems for the $n$-point motions. We present a construction of a coalescing flow in the intermediate phase. This flow obviously differs from the strong solution $(S_{s,t}, s \leq t)$ and corresponds to an absorbing boundary condition on the diagonal for the two-point motion.

This flow generates a vector field valued white noise $W$ and we can identify the strong solution to the coalescing flow $(\varphi_{s,t}, s \leq t)$ filtered by the velocity field $\sigma(W)$. The noise, in Tsirelson sense (see [41]), associated to the coalescing flow, is not linearizable, i.e. cannot be generated by a white noise though it contains $W$.

A classification of the solutions of the SDE (or of the SPDE) can be given: They are obtained by filtering a coalescing motion defined on an extended probability space with respect to a sub-noise containing the Gaussian part of its noise.

Let us explain in more details the contents of the paper. We give in section 1 and 2 construction results, which generalize a theorem by De Finetti on exchangeable variables (see for example [18]). A stochastic flow of kernels $K$ is associated with a general compatible family $(P_t^{(n)}, n \geq 1)$ of Feller semigroups. The flow $K$ is induced by a flow of measurable mappings when

$$P_t^{(2)} f \otimes^2(x, x) = P_t f^2(x),$$

for all $f \in C(M)$, $x \in M$ and $t \geq 0$. The Markov process associated with $P_t^{(n)}$ represents the motion of $n$ unsecable points thrown in the fluid. The two notions are shown to be equivalent: the law of a stochastic flow of kernels is uniquely determined by the compatible system of $n$-point motions. This construction is related to a recent result of Ma and Xiang [28] where an associated measure valued process was constructed in a special case (the flow can actually be viewed as giving the genealogy of this process, i.e. as its “historical process”) and to a result of Darling [9]. Note however that Darling did not get flows of measurable maps except in very special cases. See also Tsirelson [44] for an alternative approach to this construction.

In section 3 we define the noise associated with $K$ and introduce the notion of “filtering with respect to a sub-noise”.

In section 4 coalescing flows are constructed and briefly studied. They can be obtained from any flow whose two-point motion hits the diagonal. Then the original flow is shown to be recovered by filtering the coalescing flow with respect to a sub-noise.
In section 5 we restrict our attention to diffusion generators. We define the vector field valued white noise \( W \) associated with the stochastic flow of kernels \( K \) and prove that the flow solves the SDE driven by the white noise \( W \).

In section 6 under some off diagonal uniqueness assumption for the law of the \( n \)-point motion, we show there is only one strong solution of the SDE. In the intermediate phase described above, the classification of other solutions by filtering of the coalescing solution is established. Then we identify the linear part of the noise generated by these solutions to the noise generated by \( W \).

The examples related to our previous work (see [23]) are presented in section 7, with an emphasis on the verification of the Feller property for the semigroups \( P_t^{(n)} \), the classification of the solutions and the appearance of non-classical noise, i.e. predictable noises which cannot be generated by white noises.

1 Stochastic flow of measurable mappings.

1.1 Compatible family of Feller semigroups.

Let \( M \) be a separable compact metric space and \( d \) a distance on \( M \).

Definition 1.1.1. Let \( (P_t^{(n)}, n \geq 1) \) be a family of Feller semigroups, respectively defined on \( M^n \) and acting on \( C(M^n) \). We say that this family is compatible as soon as for all \( k \leq n \),

\[
P_t^{(k)} f(x_1, \ldots, x_k) = P_t^{(n)} g(y_1, \ldots, y_n)
\]

where \( f \) and \( g \) are any continuous functions such that

\[
g(y_1, \ldots, y_n) = f(y_{i_1}, \ldots, y_{i_k})
\]

with \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) and \( (x_1, \ldots, x_k) = (y_{i_1}, \ldots, y_{i_k}) \).

We will denote by \( P_t^{(n)} \) the law of the Markov process associated with \( P_t^{(n)} \) starting from \( (x_1, \ldots, x_n) \). This Markov process will be called the \( n \)-point motion of this family of semigroups. It is defined on the set of càdlàg paths on \( M^n \).

Remark 1.1.2. \( P_t^{(n)} \) is a Feller semigroup on \( M^n \) if and only if \( P_t^{(n)} \) is positive (i.e. \( P_t^{(n)} f \geq 0 \) for all \( f \geq 0 \)), \( P_t^{(n)} 1 = 1 \) and for all continuous function \( f \), \( \lim_{t \to 0} P_t^{(n)} f(x) = f(x) \) which implies the uniform convergence of \( P_t^{(n)} f \) towards \( f \) (see theorem 9.4 in chapter I of [7]).

1.2 Convolution semigroups on the space of measurable mappings.

We equip \( M \) with its Borel \( \sigma \)-field \( \mathcal{B}(M) \). Let \( (F, \mathcal{F}) \) be the space of measurable mappings on \( M \) equipped with the \( \sigma \)-field generated by the mappings \( \varphi \mapsto \varphi(x) \) for all \( x \in M \).

Definition 1.2.1. A probability measure \( Q \) on \( (F, \mathcal{F}) \) is called regular if there exists a measurable mapping \( J : (F, \mathcal{F}) \to (F, \mathcal{F}) \) such that

\[
(M \times F, \mathcal{B}(M) \otimes \mathcal{F}) \rightarrow (M, \mathcal{B}(M)) \quad (x, \varphi) \rightarrow J(\varphi)(x)
\]
is measurable and for all \( x \in M \),
\[
Q(d\varphi) - a.s., \quad J(\varphi)(x) = \varphi(x), \tag{1.3}
\]
i.e. \( J \) is a measurable modification of the identity mapping on \((F, \mathcal{F}, Q)\). We call it a measurable presentation of \( Q \).

**Remark 1.2.2.** If \( J \) and \( J' \) are two measurable presentations of a regular probability measure \( Q \) and if \( \mu \in \mathcal{P}(M) \), then (using Fubini’s theorem),
\[
\mu(dx) \otimes Q(d\varphi) - a.s., \quad J(\varphi)(x) = J'(\varphi)(x).
\]
This remark will be used to prove (1.4) below.

**Proposition 1.2.3.** Let \( Q_1 \) and \( Q_2 \) be two probability measures on \((F, \mathcal{F})\). Assume \( Q_1 \) is regular. Let \( J \) be a measurable presentation of \( Q_1 \). Then the mapping
\[
(F^2, \mathcal{F}^\otimes 2) \rightarrow (F, \mathcal{F})
\]
\[ (\varphi_1, \varphi_2) \mapsto J(\varphi_1) \circ \varphi_2 \]
is measurable. Moreover, if \( J' \) is another measurable presentation of \( Q_1 \), then for all \( x \in M \)
\[
Q_1(d\varphi_1) \otimes Q_2(d\varphi_2) - a.s., \quad J(\varphi_1) \circ \varphi_2(x) = J'(\varphi_1) \circ \varphi_2(x). \tag{1.4}
\]

**Remark 1.2.4.** (i) \( (\varphi_1, \varphi_2) \mapsto J(\varphi_1) \circ \varphi_2 \) is measurable but \( (\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2 \) is not measurable.

(ii) The law of \( J(\varphi_1) \circ \varphi_2 \) does not depend of the chosen presentation \( J \).

**Proof of proposition 1.2.3.** Let \( J \) be a measurable presentation of \( Q_1 \). For all \( x \in M \), the mapping \( (\varphi_1, \varphi_2) \mapsto J(\varphi_1) \circ \varphi_2(x) \) is measurable since it is the composition of the measurable mappings \( (\varphi_1, \varphi_2) \mapsto (\varphi_1, \varphi_2(x)) \) and \( (\varphi_1, y) \mapsto J(\varphi_1)(y) \). By definition of \( \mathcal{F} \), the mapping \( (\varphi_1, \varphi_2) \mapsto J(\varphi_1) \circ \varphi_2 \) is measurable.

For \( x \in M \), we have
\[
Q_1(d\varphi_1) - a.s., \quad J(\varphi_1)(x) = \varphi_1(x).
\]
Thus, for all \( x \in M \) and all \( \varphi_2 \in F \),
\[
Q_1(d\varphi_1) - a.s., \quad J(\varphi_1) \circ \varphi_2(x) = \varphi_1 \circ \varphi_2(x) = J'(\varphi_1) \circ \varphi_2(x).
\]
Therefore, using Fubini’s theorem,
\[
Q_1(d\varphi_1) \otimes Q_2(d\varphi_2) - a.s., \quad J(\varphi_1) \circ \varphi_2(x) = J'(\varphi_1) \circ \varphi_2(x). \quad \square
\]

**Definition 1.2.5.** We denote \( Q_1 \ast Q_2 \), and we call the convolution product of \( Q_1 \) and \( Q_2 \), the law of the random variable \( (\varphi_1, \varphi_2) \mapsto J(\varphi_1) \circ \varphi_2 \) defined on the probability space \((F^2, \mathcal{F}^\otimes 2, Q_1 \otimes Q_2)\).

**Definition 1.2.6.** A convolution semigroup on \((F, \mathcal{F})\) is a family \((Q_t)_{t \geq 0}\) of regular probability measures on \((F, \mathcal{F})\) such that for all nonnegative \( s \) and \( t \), \( Q_{s+t} = Q_s \ast Q_t \).
Definition 1.2.7. A convolution semigroup \((Q_t)_{t \geq 0}\) on \((F, \mathcal{F})\) is called Feller if

(i) \(\forall f \in C(M), \lim_{t \to 0} \sup_{x \in M} \int (f \circ \varphi(x) - f(x))^2 Q_t(d\varphi) = 0.\)

(ii) \(\forall f \in C(M), \forall t \geq 0, \lim_{d(x,y) \to 0} \int (f \circ \varphi(x) - f \circ \varphi(y))^2 Q_t(d\varphi) = 0.\)

Proposition 1.2.8. Let \((Q_t)_{t \geq 0}\) be a Feller convolution semigroup on \((F, \mathcal{F})\). For all \(n \geq 1, f \in C(M^n)\) and \(x \in M^n\), set

\[
P_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x) \, Q_t(d\varphi). \tag{1.5}
\]

Then \((P_t^{(n)}, n \geq 1)\) is a compatible family of Feller semigroups on \(M\) satisfying

\[
P_t^{(2)} f^{\otimes 2}(x,x) = P_t f^2(x), \tag{1.6}
\]

for all \(f \in C(M), x \in M\) and \(t \geq 0.\)

Proof. It is easy to see that this family is compatible and that for all \(n \geq 1\) and all \(t \geq 0, P_t^{(n)}\) is Markovian. Let \(s\) and \(t\) in \(\mathbb{R}^+\), \(f \in C(M^n)\) and \(x \in M^n\), then

\[
P_{s+t}^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x) \, Q_{s+t}(d\varphi) \\
= \int f \circ \mathcal{J}(\varphi_1) \otimes n \circ \varphi^{\otimes n}_2(x) \, Q_t(d\varphi_1) \otimes Q_s(d\varphi_2) \\
= \int P_t^{(n)} f \circ \varphi^{\otimes n}_2(x) \, Q_s(d\varphi_2) \\
= P_s^{(n)} P_t^{(n)} f(x)
\]

where \(\mathcal{J}\) is a measurable presentation of \(Q_t\). This proves that \(P_t^{(n)}\) is a semigroup.

Let us now prove the Feller property. Let \(h \in C(M^n)\) be in the form \(f_1 \otimes \cdots \otimes f_n, x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\). We have for \(M\) large enough

\[
|P_t^{(n)} h(y) - P_t^{(n)} h(x)| \leq M \sum_{k=1}^n \left( \int (f_k \circ \varphi(y_k) - f_k \circ \varphi(x_k))^2 Q_t(d\varphi) \right)^{\frac{1}{2}} \tag{1.7}
\]

which converges towards 0 as \(d(x,y)\) goes to 0 since (ii) in definition 1.2.7 is satisfied. We also have

\[
|P_t^{(n)} h(x) - h(x)| \leq M \sum_{k=1}^n \left( \int (f_k \circ \varphi(x_k) - f_k(x_k))^2 Q_t(d\varphi) \right)^{\frac{1}{2}} \tag{1.8}
\]

which converges towards 0 as \(t\) goes to 0 since (i) in definition 1.2.7 is satisfied. These properties extend to all function \(h\) in \(C(M^n)\) by an approximation argument. This proves the Feller property of the Markovian semigroups \(P_t^{(n)}\).

It remains to prove (1.6). This follows from

\[
P_t^{(2)} f^{\otimes 2}(x,x) = \int f^{\otimes 2} \circ \varphi^{\otimes 2}(x,x) \, Q_t(d\varphi) \\
= \int f^2 \circ \varphi(x) \, Q_t(d\varphi) = P_t^{(1)} f^2(x). \tag*{\Box}
\]
Remark 1.2.9. Let \((Q_t)_{t \geq 0}\) be a Feller convolution semigroup on \((F, \mathcal{F})\).

- The semigroup \((Q_t)_{t \geq 0}\) is uniquely determined by \((P_t^{(n)}, n \geq 1)\).

- Let \(X\) and \(\varphi\) be independent random variables respectively in \(M^n\) and in \(F\). Denote by 
  \(\mu\) the law of \(X\) and suppose that the law of \(\varphi\) is \(Q_t\). Then Fubini’s theorem implies that 
  for all measurable presentation \(\mathcal{J}\) of \(Q_t\), the random variable \(\mathcal{J}(\varphi)^{\otimes n}(X)\) is distributed 
  as \(\mu P_t^{(n)}\), where \(P_t^{(n)}\) is defined by (1.5).

- For all \(x \in M\), \(Q_0(d\varphi)\)-almost surely, \(\varphi(x) = x\).

1.3 Stochastic flows of mappings.

Definition 1.3.1. Let \((\Omega, \mathcal{A}, P)\) be a probability space and let \(\varphi = (\varphi_{s,t}, s \leq t)\) be a family 
of \((F, \mathcal{F})\)-valued random variables such that for all \(x \in M\) and all \(t \in \mathbb{R}\), \(P\)-a.s. \(\varphi_{t,t}(x) = x\). 
For \(t \geq 0\), denote by \(Q_t\) the law of \(\varphi_{0,t}\). The family \(\varphi\) is called a stochastic flow of mappings 
if for all \(t \geq 0\), \(Q_t\) is regular and if the following properties are satisfied by \(\varphi\):

(a) For all \(s \leq u \leq t\), all \(x \in M\) and all measurable presentation \(\mathcal{J}_{t-u}\) of \(Q_{t-u}\), 
  \(P\)-almost surely, \(\varphi_{s,t}(x) = \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi_{s,u}(x)\). (cocycle property)

(b) For all \(s \leq t\), the law of \(\varphi_{s,t}\) is \(Q_{t-s}\). (Stationarity)

(c) The flow has independent increments, i.e. for all \(t_1 \leq t_2 \leq \cdots \leq t_n\), the family 
  \(\{\varphi_{t_i, t_{i+1}}, 1 \leq i \leq n−1\}\) is independent.

(d) For all \(f \in C(M)\) and all \(s \leq t\), \(\lim_{(u,v)\to (s,t)} \sup_{x \in M} \mathbb{E}[(f \circ \varphi_{s,t}(x) − f \circ \varphi_{u,s}(x))^2] = 0\).

(e) For all \(f \in C(M)\) and all \(s \leq t\), \(\lim_{d(x,y)\to 0} \mathbb{E}[(f \circ \varphi_{s,t}(x) − f \circ \varphi_{s,t}(y))^2] = 0\).

Remark 1.3.2. Item (a) holds for all set of measurable presentations as soon as (a) holds for one of them.

- If \(\psi\) is equal in law to a stochastic flow of mappings \(\varphi\), then \(\psi\) is also a stochastic flow 
of mappings. Indeed, it is straightforward to check that \(\psi\) satisfies (b), (c), (d) and (e), 
  and after having remarked that for all \(x \in M\), \((\varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_3(x), \mathcal{J}_{t-u}(\varphi_2) \circ \varphi_1(x))\) 
is measurable, we prove that \(\psi\) satisfies (a).

Proposition 1.3.3. Let \(\varphi\) be a stochastic flow of mappings, and for \(t \geq 0\), let \(\mathcal{J}_t\) be a measurable 
presentation of the law of \(\varphi_{0,t}\). Then \(\varphi' = (\mathcal{J}_{t-s}(\varphi_{s,t}), s \leq t)\) is a stochastic flow 
of mappings satisfying

(i) For all \(s \leq t\) and all \(x \in M\), a.s. \(\varphi'_{s,t}(x) = \varphi_{s,t}(x)\).

(ii) For all \(s \leq u \leq t\) and all \(x \in M\), \(P\)-almost surely, \(\varphi'_{s,t}(x) = \varphi'_{u,t} \circ \varphi'_{s,u}(x)\).
Proof. Item (i) is a consequence of the fact that \( \mathcal{J}_{t-s} \) is a measurable presentation of the law of \( \varphi_{s,t} \). Then \( \varphi \) and \( \varphi' \) share the same law and \( \varphi' \) is a stochastic flow of mappings. Let us now prove (ii). For \( s \leq u \leq t \) and \( x \in M \), it holds that \( P \)-almost surely,

\[
\varphi'_{s,t}(x) = \mathcal{J}_{t-u}(\varphi'_{u,t}) \circ \varphi'_{s,u}(x) = \mathcal{J}_{t-u} \circ \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x).
\]

Since \( \mathcal{J}_{t-u} \circ \mathcal{J}_{t-u} \) is also a measurable presentation of \( Q_{t-u} \), using remark \ref{remark:1.2.2} \( P \)-almost surely,

\[
\mathcal{J}_{t-u} \circ \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x) = \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x) = \varphi'_{u,t} \circ \varphi'_{s,u}(x).
\]

This proves (ii). □

Definition 1.3.4. • The stochastic flow of mappings \( \varphi' \) defined in proposition \ref{proposition:1.3.3} will be called a measurable modification of \( \varphi \).

• A stochastic flow of mappings which is a measurable modification of a stochastic flow of mappings is called a measurable stochastic flow of mappings.

Remark 1.3.5. In the proof of theorem \ref{theorem:1.4.2} below, the measurable presentations we construct satisfy \( \mathcal{J}_{t} \circ \mathcal{J}_{t} = \mathcal{J}_{t} \) for all \( t \geq 0 \), and the measurable stochastic flow \( \varphi \) we construct satisfies \( \mathcal{J}_{t-s}(\varphi_{s,t}) = \varphi_{s,t} \) for all \( s \leq t \). But, we will see that this property doesn’t hold for stochastic flows of kernels studied in section 2.

Proposition 1.3.6. Let \( \varphi = (\varphi_{s,t}, s \leq t) \) be a stochastic flow of mappings. For all \( n \geq 1 \), \( f \in C(M^n) \) and \( x \in M^n \), set

\[
P^{(n)}_t f(x) = \mathbb{E}[f \circ \varphi^{(n)}_{0,t}(x)]. \tag{1.9}
\]

Then \( (P^{(n)}_t, n \geq 1) \) is a compatible family of Feller semigroups on \( M \) satisfying \ref{condition:1.6}.

Proof. For \( t \geq 0 \), denote by \( Q_t \) the law of \( \varphi_{0,t} \). Then \( Q_t \) is regular and there is \( \mathcal{J}_t \) a measurable presentation of \( Q_t \). With (a), (b) and (c), we show that for all \( (s,t) \in \mathbb{R}^2_+ \), \( Q_s * Q_t = Q_{s+t} \), i.e. \( (Q_t)_{t \geq 0} \) is a convolution semigroup. Finally (d) and (e) imply that it is Feller. To conclude, we apply proposition \ref{proposition:1.2.8} □

1.4 Construction and characterization.

In this section, we present a theorem stating that to any compatible family \( (P^{(n)}_t, n \geq 1) \) of Feller semigroups, one can associate a Feller convolution semigroup on \( (F, \mathcal{F}) \) and a stochastic flow of mappings.

Let \( (\Omega^0, \mathcal{A}^0) \) denote the measurable space \( (\prod_{s \leq t} F, \otimes_{s \leq t} \mathcal{F}) \). For \( s \leq t \), let \( \varphi^0_{s,t} \) denote the random variable \( \omega \mapsto \omega(s,t) \). Let \( \varphi^0 \) be the random variable \( (\varphi^0_{s,t}, s \leq t) \). Then \( \varphi^0(\omega) = \omega \). Let \( (T_h)_{h \in \mathbb{R}} \) be the one-parametric group of transformations of \( \Omega^0 \) defined by \( T_h(\omega)(s,t) = \omega(s + h, t + h) \), for all \( s \leq t, h \in \mathbb{R} \) and \( \omega \in \Omega^0 \).

Definition 1.4.1. A probability space \( (\Omega, \mathcal{A}, P) \) is said separable if the Hilbert space \( L^2(\Omega, \mathcal{A}, P) \) is separable. (Note that this implies that for all \( 1 \leq p < \infty \), \( L^p(\Omega, \mathcal{A}, P) \) is separable.)
Theorem 1.4.2. (i) Let \((P_t^{(n)}, n \geq 1)\) be a compatible family of Feller semigroups on \(M\) satisfying
\[
P_t^{(2)} f^\oplus_2(x, x) = P_t f^2(x),
\]
for all \(f \in C(M), x \in M\) and \(t \geq 0\). Then there exists a unique Feller convolution semigroup \((Q_n)_{t \geq 0}\) on \((F, \mathcal{F})\) such that for all \(n \geq 1, t \geq 0, f \in C(M^n)\) and \(x \in M^n\),
\[
P_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x) Q_t(d\varphi).
\]

(ii) For all Feller convolution semigroup \(Q = (Q_t)_{t \geq 0}\) on \((F, \mathcal{F})\), there exists a unique \((T_h)_{h \in \mathbb{R}}\)-invariant probability measure \(P_Q\) on \((\Omega^0, \mathcal{A}^0)\) such that \((\Omega^0, \mathcal{A}^0, P_Q)\) is separable, the family of random variables \(\varphi^0 = (\varphi_{s,t}^0, s \leq t)\) is a stochastic flow of mappings and for all \(s \leq t\), the law of \(\varphi_{s,t}^0\) is \(Q_{t-s}\). Every measurable modification \(\varphi'\) of \(\varphi^0\) satisfies \(\varphi'_{s+h,t+h} = \varphi'_{s,t} \circ T_h\) for all \(s \leq t\) and all \(h \in \mathbb{R}\).

The flow \(\varphi^0\) is called the canonical stochastic flow of mappings associated with \(Q\) (or equivalently with \((P_t^{(n)}, n \geq 1)\)).

Remark 1.4.3. Theorem 1.4.2 is also satisfied when \(M\) is a locally compact separable metric space. In this case, \((P_t^{(n)}, n \geq 1)\) is a compatible family of Markovian semigroups acting continuously on \(C_0(M^n)\), the set of continuous functions on \(M^n\) converging towards 0 at \(\infty\) (we call them Feller semigroups). In the previous definitions \((1.2.7)\) and \((1.3.1)\) and in the statement of the theorem the function \(f\) has to be taken in \(C_0(M)\) or in \(C_0(M^n)\). Moreover (ii) of definition \((1.2.7)\) must be modified by: for all \(x \in M, f \in C_0(M)\) and \(t \geq 0\),
\[
\begin{align*}
\lim_{y \to x} \int (f \circ \varphi(y) - f \circ \varphi(x))^2 Q_t(d\varphi) &= 0, \\
\lim_{y \to \infty} \int (f \circ \varphi(y))^2 Q_t(d\varphi) &= 0.
\end{align*}
\]
In definition \((1.3.1)\) (e) must be modified by: for all \(x \in M\) and \(s \leq t\),
\[
\begin{align*}
\lim_{y \to x} E[(f \circ \varphi_{s,t}(y) - f \circ \varphi_{s,t}(x))^2] &= 0, \\
\lim_{y \to \infty} E[(f \circ \varphi_{s,t}(y))^2] &= 0.
\end{align*}
\]

Proof. In order to prove this remark, note that the one-point compactification of \(M\), \(\hat{M} = M \cup \{\infty\}\), is a separable compact metric space. On \(\hat{M}\), we define the compatible family of Feller semigroups, \((\hat{P}_t^{(n)}, n \geq 1)\), by the following relations:

for all \(n \geq 2\) and all family of continuous functions on \(\hat{M}\), \(\{f_i, i \geq 1\}\),
\[
\hat{P}_t^{(n)} f_1 \otimes \cdots \otimes f_n = P_t^{(n)} g_1 \otimes \cdots \otimes g_n
\]
\[
+ \sum_{i=1}^n f_i(\infty) \hat{P}_t^{(n-1)} f_1 \otimes \cdots \otimes f_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n
\]
and
\[
\hat{P}_t^{(1)} f_1 = f_1(\infty) + P_t^{(1)} g_1,
\]
where \(g_i = f_i - f_i(\infty) \in C_0(M)\) and with the convention \(P_t^{(n)} g_1 \otimes \cdots \otimes g_n(x_1, \ldots, x_n) = 0\) if there exists \(i\) such that \(x_i = \infty\). We apply theorem \((1.4.2)\) to \(\hat{M}\) and to the family \((\hat{P}_t^{(n)}, n \geq 1)\) to construct a Feller convolution semigroup \(Q\) and a stochastic flow of mappings \((\hat{\varphi}_{s,t}, s \leq t)\) on \(\hat{M}\). This stochastic flow of mappings satisfies
(i) \( \hat{\varphi}_{s,t}(\infty) = \infty \) for all \( s \leq t \) and

(ii) \( \hat{\varphi}_{s,t}(x) \neq \infty \) for all \( x \in M \) and \( s \leq t \).

**Proof of (i).** For all \( f \in C(\hat{M}) \),

\[
E[(f \circ \hat{\varphi}_{s,t}(\infty) - f(\infty))^2] = \hat{P}^{(2)}_{t-s}f^{\otimes 2}(\infty, \infty) - 2f(\infty)\hat{P}^{(1)}_{t-s}f(\infty) + f(\infty)^2
\]

since \( \hat{P}^{(2)}_{t-s}f^{\otimes 2}(\infty, \infty) = f(\infty)^2 \) and \( \hat{P}^{(1)}_{t-s}f(\infty) = f(\infty) \). This implies (i). \( \square \)

**Proof of (ii).** Let \( g_n \) be a sequence in \( C_0(M) \) such that \( g_n \in [0,1] \) and simply converging towards 1. Then \( f_n = 1 - g_n \in C(\hat{M}) \) is such that \( f_n(\infty) = 1 \) and for all \( x \in M \)

\[
E[(f_n \circ \hat{\varphi}_{s,t}(x))^2] = \hat{P}^{(2)}_{t-s}g_n^{\otimes 2}(x,x) + 1 - 2\hat{P}^{(1)}_{t-s}g_n(x).
\]

This implies that \( \lim_{n \to \infty} E[(f_n \circ \hat{\varphi}_{s,t}(x))^2] = 0 \). Assertion (ii) follows since \( 1_{\{\hat{\varphi}_{s,t}(x) = \infty\}} = \lim_{n \to \infty} f_n \circ \hat{\varphi}_{s,t}(x). \) \( \square \)

For all \( x \in M \), let us denote \( \hat{\varphi}_{s,t}(x) \) by \( \varphi_{s,t}(x) \). Assertions (i) and (ii) implies that \( \varphi_{s,t} \in F \) and that \( (\varphi_{s,t}, s \leq t) \) is a stochastic flow of mappings on \( M \). In a similar way, one can show that \( \hat{Q} \) induces a Feller convolution semigroup on \( (F, \mathcal{F}) \). \( \square \)

Let us explain briefly the method we use to prove theorem \[1.4.2\]. We first suppose we are given a compatible family of Feller semigroups satisfying \[1.6\]. Then we define a convolution semigroup \((Q_t, t \geq 0)\) on measurable mappings on \( M \). For all \( t \), to define \( Q_t \), we define \( P^{(n)}_t \), the law of \( (\varphi(z_l), l \in \mathbb{N}) \), where the law of \( \varphi \) is \( Q_t \), for some dense family \((z_l, l \in \mathbb{N}) \) in \( M \) and get \( Q_t \) by an approximation. Hence \( Q_t \) is defined as the law of a random variable, which takes its values in the “bad” space \( F \), but is defined on a “nice” space \( M^\mathbb{N} \).

The approximation used to construct this convolution semigroup allows us to define a stochastic flow of mappings on \( M \) in such a way that these mappings are measurable, defining it first on the dyadic numbers. We get a measurable flow defined on a “nice” space. Note that a difficulty to get this measurability comes from the fact that the composition of mappings from \( M \) onto \( M \) is not measurable with respect to the natural \( \sigma \)-field.

### 1.5 Proof of the first part of theorem \[1.4.2\]

In the following we assume we are given \((P^{(n)}_t, n \geq 1)\), a compatible family of Feller semigroups satisfying \[1.6\]. And we intend to construct a Feller convolution semigroup \((Q_t)_{t \geq 0}\) on \((F, \mathcal{F})\) satisfying \[1.11\]. The uniqueness of such a convolution semigroup is immediate since \[1.11\] characterizes \( Q_t \).

#### 1.5.1 A measurable choice of limit points in \( M \).

It is known that, as a separable compact metric space, \( M \) is homeomorphic to a closed subset of \([0, 1]^\mathbb{N}\) (see corollaire 1 6.1 of chapter 9 in \[3\]). A point \( y \) can be represented by a sequence \((y^n)_{n \in \mathbb{N}} \in [0, 1]^\mathbb{N} \). Let \( y = (y_i)_{i \in \mathbb{N}} \) be a sequence of elements of \( M \).
Let $y^l = \limsup_{i \to \infty} y^l_i$. Let $i^l_k = \inf \{i, \ |y^l_i - y^l_k| < 1/k \}$. By induction, for all integer $j$, we construct $y^j$ and $\{i^l_k, \ k \in \mathbb{N} \}$ by the relations

$$y^j = \limsup_{k \to \infty} y^j_{k-1} \quad \text{and} \quad i^l_k = \inf \{i \in \{i^l_{k-1}, \ k \in \mathbb{N} \}, \ |y^j_i - y^j_k| < 1/k \}.$$  

We denote $(y^n)_{n \in \mathbb{N}}$ by $l(y)$. Note that $l(y)^j = \lim_{n \to \infty} y^j_{n_i}$. Hence $l(y)$ belongs to $M$. It is easy to see that $l$ satisfies the following lemma.

**Lemma 1.5.1.** $l : M^N \to M$ is a measurable mapping, $M$ being equipped with the Borel $\sigma$-field $\mathcal{B}(M)$ and $M^N$ with the product $\sigma$-field $\mathcal{B}(M)^{\otimes N}$. Moreover $l((y_i)_{i \in \mathbb{N}}) = y_\infty$ when $y_i$ converges towards $y_\infty$.

**1.5.2 Notation and definitions.**

Let $\{z_l, \ l \in \mathbb{N} \}$ be a dense family in $M$, which will be fixed in the following. We wish to define a measurable mapping $i : M^N \to F$ such that $i((y_j)_{j \in \mathbb{N}})(z_l) = y_l$ for all integer $l$.

Let $(\varepsilon_k)_{k \in \mathbb{N}}$ be a positive sequence decreasing towards 0 (this sequence will be fixed later). Let $i : M^N \to F$ be the injective mapping defined by

$$i(y)(x) = l((y^n_{n_k})_{k \in \mathbb{N}}) \quad \text{(1.16)}$$

where

$$n^x_k = \inf \{n, \ d(z_n, x) \leq \varepsilon_k \}, \quad \text{(1.17)}$$

for $(y, x) \in M^N \times M$. Note that $i(y)$ defined this way is a measurable mapping since $l$ is measurable and $x \mapsto (y^x_{n_k})_{k \in \mathbb{N}}$ is measurable. Note also that the relation $i(y)(z_l) = y_l$ is satisfied for all integer $l$.

**Lemma 1.5.2.** For $n \geq 1$, the mappings $\Phi_n : (M^N)^n \to F$ and $\Psi_n : M \times (M^N)^n \to M$, defined by

$$\Phi_n(y^1, \ldots, y^n) = i(y^n) \circ i(y^{n-1}) \circ \cdots \circ i(y^1)$$

$$\Psi_n(x, y^1, \ldots, y^n) = \Phi_n(y^1, \ldots, y^n)(x)$$

are measurable. ($(M^N)^n$ and $M \times (M^N)^n$ are equipped with the product $\sigma$-field.) In particular, $i$ is measurable.

**Proof.** Note that $\Psi_1$ is the composition of the mappings $l$ and $(x, y) \mapsto (y^x_{n_k})_{k \in \mathbb{N}}$. Since these mappings are measurable, $\Psi_1$ is measurable. By induction, we prove that $\Psi_n$ is measurable since, for $n \geq 2$,

$$\Psi_n(x, y^1, \ldots, y^n) = \Psi_1(\Psi_{n-1}(x, y^1, \ldots, y^{n-1}), y^n).$$

For all $A \in \mathcal{B}(M)$ and $x \in M$,

$$\Phi_n^{-1}(\{\varphi \in F, \ \varphi(x) \in A \}) = \{y \in (M^N)^n, \ (x, y) \in \Psi_n^{-1}(A) \}.$$ 

This event belongs to $(\mathcal{B}(M)^{\otimes N})^n$ since $\Psi_n$ is measurable. This shows the measurability of $\Phi_n$. □

We need to introduce $\Phi_n$ because the composition application $F^n \to F$, $(\varphi_1, \ldots, \varphi_n) \mapsto \varphi_n \circ \cdots \circ \varphi_1$ is not $\mathcal{F}^{\otimes n}$-measurable in general.

Let $j : F \to M^N$ be the mapping defined by

$$j(\varphi) = (\varphi(z_l))_{l \in \mathbb{N}}. \quad \text{(1.18)}$$
Lemma 1.5.3. The mapping \( j \) is measurable and satisfies \( j \circ i(y) = y \) for all \( y \in M^N \).

Proof. We have for all \( A \in \mathcal{B}(M)^{\otimes n} \),
\[
j^{-1}(\{y \in M^N, (y_1, \ldots, y_n) \in A\}) = \{\varphi \in F, (\varphi(z_1), \ldots, \varphi(z_n)) \in A\}.
\]
This set belongs to \( \mathcal{F} \). \( \square \)

Note that for all \( l \in \mathbb{N} \) and \( \varphi \in F \), \( i \circ j(\varphi)(z_l) = \varphi(z_l) \).

Remark 1.5.4. Set \( \mathcal{J} = i \circ j \). Lemmas [1.5.2 and 1.5.3] imply that the mappings \( (\varphi_1, \ldots, \varphi_n) \mapsto \mathcal{J}(\varphi_n) \circ \cdots \circ \mathcal{J}(\varphi_1) \) and \( (x, \varphi_1, \ldots, \varphi_n) \mapsto \mathcal{J}(\varphi_n) \circ \cdots \circ \mathcal{J}(\varphi_1)(x) \) are measurable.

1.5.3 Constructions of probabilities on \( M^N \) and on \( F \).

By Kolmogorov’s theorem, we construct on \( M^N \) a probability measure \( P^{(\infty)}_t \) such that
\[
P^{(\infty)}_t(A \times M^N) = P^{(n)}_1(A(z_1, \ldots, z_n)) \text{ for any } A \in \mathcal{B}(M)^{\otimes n}.
\]
We now prove useful lemmas satisfied by \( P^{(\infty)}_t \):

Lemma 1.5.5. For all positive \( T \), there exists a positive function \( \varepsilon_T(r) \) converging towards 0 as \( r \) goes to 0 such that
\[
\sup_{t \in [0, T]} E^{(2)}_{(x,y)}[(d(X_t, Y_t))^2] \leq \varepsilon_T(d(x, y)). \tag{1.19}
\]

Proof. For all continuous function \( f \), we have
\[
E^{(2)}_{(x,y)}[(f(X_t) - f(Y_t))^2] = P^2_t f^2(x) + P^2_t f^2(y) - 2P^2_t f \otimes^2 (x, y).
\]

since \([1.6]\) is satisfied. Let \((f_n)_{n \geq 1}\) be a dense sequence in \( \{f \in C(M), \|f\|_\infty \leq 1\} \). Then \( d''(x, y) = (\sum_{n \geq 1} 2^{-n}(f_n(x) - f_n(y))^2)^{\frac{1}{2}} \) is a distance equivalent to \( d \) and we have
\[
E^{(2)}_{(x,y)}[(d''(X_t, Y_t))^2] = P^2_t h(x, x) + P^2_t h(y, y) - 2P^2_t h(x, y),
\]

where \( h \) is the continuous function \( \sum_{n \geq 1} 2^{-n} f_n \otimes f_n \). We conclude the lemma after remarking that this function is uniformly continuous in \((t, x, y)\) on \([0, T] \times M^2 \). \( \square \)

From now on we fix \( T \) and define the sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) (which defines the sequence \((n_k^x)_{k \in \mathbb{N}}\) for all \( x \in M \) by equation \([1.17]\)) such that \( 0 \leq r \leq 2\varepsilon_k \) implies \( \varepsilon_T(r) \leq 2^{-3k} \). The sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) is well defined since \( \lim_{r \to 0} \varepsilon_T(r) = 0 \). Since \( i \) depends on \( T \), we now denote \( i \) by \( i_T \), \( \Phi_n \) by \( \Phi^T_n \) and \( \Psi_n \) by \( \Psi^T_n \).

Remark 1.5.6. One can construct \( i_t \) such that \( i_t = i_1 \) for all \( t \leq 1 \).

Lemma 1.5.7. For all \( t \in [0, T] \) and for any independent random variables \( X \) and \( Y \) respectively in \( M \) and \( M^N \), such that the law of \( Y \) is \( P^{(\infty)}_t \), then \( Y_{n_k^x} \) converges almost surely towards \( l((Y_{n_k^x})_{k \in \mathbb{N}}) = i_T(Y)(X) \) as \( k \) goes to \( \infty \).
Proposition 1.5.9. The following proposition.

Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of random variables in \(M\) converging in probability towards a random variable \(X\). Let \(Y\) a random variable in \(M^\infty\) independent of \((X_n)_{n \in \mathbb{N}}\). Then \(i_T(Y)(X_n) = l((Y_{n_k})_{k \in \mathbb{N}})\) converges in probability towards \(i_T(Y)(X) = l((Y_{n_k})_{k \in \mathbb{N}})\) as \(n\) tends to \(\infty\).

Proof. Let \(Z_n = l((Y_{n_k})_{k \in \mathbb{N}})\) and \(Z = l((Y_{n_k})_{k \in \mathbb{N}})\). For all integer \(k\), we have

\[
P[d(Z_n, Z) > \varepsilon] \leq P[d(Z_n, Y_{n_k}) > \varepsilon/3] + P[d(Y_{n_k}/3, Y_{n_k}) > \varepsilon/3] + P[d(Y_{n_k}/3, Z) > \varepsilon/3].
\]

Lemma 1.5.7 implies that the first and last terms of the right hand side of the preceding equation converge towards 0 as \(k\) goes to \(\infty\). The second term is lower than \(\frac{9}{\varepsilon}E[\varepsilon_T(d(z_{n_k}, z_{n_k})]]\). Since for all positive \(\alpha\), there exists a positive \(\eta\) such that \(|r| < \eta\) implies \(\frac{9}{\varepsilon}|\varepsilon_T(r)| < \alpha\), we get

\[
P[d(Y_{n_k}, Y_{n_k}) > \varepsilon/3] \leq \alpha + CP[d(z_{n_k}, z_{n_k}) > \eta] \leq \alpha + CP[d(X_n, X) > \eta - 2\varepsilon_k),
\]

where \(C = 9D^2/\varepsilon^2\), where \(D\) is the diameter of \(M\) (one can choose \(\varepsilon_T\) such that \(\varepsilon_T(r) \leq D^2\) for all \(r\)). Therefore, we get \(P[d(Z_n, Z) > \varepsilon] \leq \alpha + CP[d(X_n, X) \geq \eta]\) and for all positive \(\alpha\), \(\lim_{n \to \infty} P[d(Z_n, Z) > \varepsilon] \leq \alpha\). Thus we prove that \(Z_n\) converges in probability towards \(Z\). □

For all \(t \in [0, T]\), set \(Q_t = i_T(P_t^{\infty})\). It is a probability measure on \((F, \mathcal{F})\) and it satisfies the following proposition.

Proposition 1.5.9. \(Q_t\) is the unique probability measure on \((F, \mathcal{F})\) such that for any continuous function \(f\) on \(M^n\) and any \(x \in M^n\),

\[
\int_F f \circ \varphi^{\infty}(x) Q_t(d\varphi) = P_t^{(n)} f(x).
\]  

Moreover, \(j^*(Q_t) = P_t^{\infty}\) and \((i_T \circ j)^*(Q_t) = i_T^*(P_t^{\infty}) = Q_t\).

Proof. The unicity is obvious since \((1.21)\) characterizes \(Q_t\). Let us check that \(Q_t = i_T^*(P_t^{\infty})\) satisfies \((1.21)\). Let \(Y\) be a random variable of law \(P_t^{\infty}\) then for all \(f \in C(M^n)\) and all \(x \in M^n\),

\[
\int_F f \circ \varphi^{\infty}(x) Q_t(d\varphi) = E[f(i_T(Y)(x_1), \ldots, i_T(Y)(x_n))] = \lim_{k \to \infty} E[f(Y_{n_k}^{x_1}, \ldots, Y_{n_k}^{x_n})] = \lim_{k \to \infty} P_t^{(n)} f(z_{n_k}^{x_1}, \ldots, z_{n_k}^{x_n}) = P_t^{(n)} f(x),
\]
using first dominated convergence theorem and lemma \ref{lem:1.5.7}, then the definition of $P_t^{(\infty)}$ and the fact that $P_t^{(n)}$ is Feller. □

Remark 1.5.10. Since $T$ can be taken arbitrarily large, we can define $Q_t$ for all positive $t$ and the definition of $Q_t$ is independent of the chosen $T$, since $Q_t$ satisfies proposition \ref{lem:1.5.9}.

1.5.4 A convolution semigroup on $(F, \mathcal{F})$.

Lemma 1.5.11. For all $t \geq 0$, $Q_t$ is regular. And for all $T \geq t$, $i_T \circ j$ is a measurable presentation of $Q_t$.

Proof. Let $0 \leq t \leq T$. For all $x \in M$ and $\varphi \in F$, $i_T \circ j(\varphi)(x) = \Psi_1^T(x, j(\varphi))$. Since $\Psi_1^T$ and $j$ are measurable, the mapping $(x, \varphi) \mapsto i_T \circ j(\varphi)(x)$ is measurable.

Let $x \in M$. Since $Q_t = i_T^*(P_t^{(\infty)})$, if $Y$ is a random variable of law $P_t^{(\infty)}$,

$$Q_t[d(\varphi(z_{n_k^l}), \varphi(x)) > 2^{-k}] = P[d(Y_{n_k^l}, i_T(Y)(x)) \geq 2^{-k}] = \lim_{l \to \infty} P[d(Y_{n_k^l}, Y_{n_k^l}) \geq 2^{-k}] \leq 2^{-k}$$

since for all $l \geq k$, $d(z_{n_k^l}, z_{n_k^l}) \leq 2\varepsilon_k$ (see equation (1.20)). Using Borel-Cantelli’s lemma, we prove that $\varphi(z_{n_k^l})$ converges almost surely towards $\varphi(x)$. Therefore,

$$Q_t(d\varphi) - a.s., \quad i_T \circ j(\varphi)(x) = \varphi(x).$$

This proves the lemma. □

In the published version of this paper \cite{26}, we have made the following incorrect remark (Fubini’s theorem cannot be applied here since $(x, \varphi) \mapsto \varphi(x)$ is not measurable):

Remark 1.5.12. Let $\varphi$ and $X$ be independent random variables respectively $F$-valued and $M$-valued. Then, if the law of $\varphi$ is $Q_t$ and if $M \times \Omega \ni (x, \omega) \mapsto \varphi(x, \omega) \in M$ is measurable, Fubini’s theorem implies that for all $T \geq t$,

$$P - a.s., \quad i_T \circ j(\varphi)(X) = \varphi(X). \quad (1.22)$$

Counterexample to remark \ref{rem:1.5.12} (labelled 1.7 in [26]) This counterexample was communicated to us by G. Riabov (see \cite{33}). Let $\varphi$ be a random variable in $F$ of law $Q$ such that $M \times \Omega \ni (x, \omega) \mapsto \varphi(x, \omega) \in M$ is measurable. Suppose that $Q$ is regular and let $\mathcal{J}$ be a regular presentation of $Q$. Let $X$ be a random variable in $M$ independent of $\varphi$. Out of $\varphi$ and $X$, define $\psi \in F$ by $\psi(x) = \varphi(x)$ is $x \neq X$ and $\psi(x) = X$ is $x = X$. Then $M \times \Omega \ni (x, \omega) \mapsto \psi(x, \omega) \in M$ is measurable. Suppose also that the law of $X$ has no atoms, then (reminding the definition of $\mathcal{F}$) $\psi$ and $X$ are independent and the law of $\psi$ is $Q$. Note that $\psi(X) = X$ and (except for very special cases) we won’t have that a.s.

$\mathcal{J}(\psi)(X) = \psi(X) = X$.

Remark 1.5.13. Let $\varphi$ and $X$ be independent random variables respectively $F$-valued and $M$-valued. If $Q$, the law of $\varphi$, is regular and if $\mathcal{J}$ and $\mathcal{J}'$ are two measurable presentations of $Q$. Then $P$-a.s., $\mathcal{J}(\varphi)(X) = \mathcal{J}'(\varphi)(X)$. Lemma \ref{lem:1.5.7} also shows that if $Q = Q_t$ ($t \leq T$), $\varphi(z_{n_k^l})$ converges a.s. towards $\mathcal{J}(\varphi)(X)$.\16
Lemma 1.5.14. For all $t_1, \ldots, t_n$ in $[0, T]$,

$$\left(\Phi_n^T\right)^c(P_{t_1}^{(\infty)} \otimes \cdots \otimes P_{t_n}^{(\infty)}) = Q_{t_1 + \cdots + t_n}.$$  \hfill (1.23)

Proof. Let us prove that $(\Phi_n^T)^c(P_{t_1}^{(\infty)} \otimes \cdots \otimes P_{t_n}^{(\infty)})$ satisfies (1.21) for all $f \in C(M^k)$, all $x \in M^k$ and $t = t_1 + \cdots + t_n$. To simplify we prove this for $k = 1$. Let $f \in C(M)$ and $x \in M$, then applying Fubini’s theorem,

$$\int_F f(\varphi(x)) \left(\Phi_n^T\right)^c(P_{t_1}^{(\infty)} \otimes \cdots \otimes P_{t_n}^{(\infty)})(d\varphi)$$

$$= \int f(i_T(y^n) \circ i_T(y^{n-1}) \circ \cdots \circ i_T(y^1)(x)) P_{t_1}^{(\infty)}(dy^1) \otimes \cdots \otimes P_{t_n}^{(\infty)}(dy^n)$$

$$= \int P_{t_1}^{(1)} f(i(y^{n-1}) \circ \cdots \circ i_T(y^1)(x)) P_{t_1}^{(\infty)}(dy^1) \otimes \cdots \otimes P_{t_{n-1}}^{(\infty)}(dy^{n-1})$$

$$= \cdots = P_{t_1 + \cdots + t_n} f(x).$$

The proof is similar for $f \in C(M^k)$ and $x \in M^k$. We conclude using proposition 1.5.9.

Proposition 1.5.15. $(Q_t)_{t \geq 0}$ is a Feller convolution semigroup on $(F, \mathcal{F})$.

Proof. For all nonnegative $s$ and $t$, $\Phi_n^T \circ j^{\otimes 2}$ is measurable. Proposition 1.5.9 and lemma 1.5.14 implies that $(\Phi_n^T \circ j^{\otimes 2})(Q_s \otimes Q_t) = Q_{s+t}$. Since $(\Phi_n^T \circ j^{\otimes 2})(\varphi_1, \varphi_2) = (i_T \circ j)(\varphi_1) \circ (i_T \circ j)(\varphi_2)$, we have easily that $Q_s * Q_t = Q_{s+t}$. The Feller property for $Q$ is easy to prove.

This proves the first part of theorem 1.4.2.

1.6 Proof of the second part of theorem 1.4.2.

We now assume we are given a Feller convolution semigroup $Q = (Q_t)_{t \geq 0}$. With $Q$, we associate a compatible family of Feller semigroups $(P_t^{(n)}, n \geq 1)$ and construct $P_t^{(\infty)}$ like in section 1.5.3.

1.6.1 Construction of a probability space.

For all $n \in \mathbb{N}$, let $D_n = \{j2^{-n}, j \in \mathbb{Z}\}$ and $D = \cup_{n \in \mathbb{N}}D_n$ the set of the dyadic numbers. We take $T = 1$ and set $i = i_1$ and $\Phi_n = \Phi_1^n$.

For all integer $n \geq 1$, let $(S_n, \mathcal{S}_n, P_n)$ denote the probability space $(M^N, \mathcal{B}(M)^{\otimes N}, P_2^{(\infty)} \otimes \mathbb{Z})$. Let $\pi_{n-1,n} : S_n \rightarrow S_{n-1}$, $\omega^{n-1} \mapsto \omega^{n-1}$, where

$$\omega^{n-1}_{i/2^{n-1}} = \omega^{n}_{i/2^n} \circ j \Phi_2(\omega^{2^n}_{2i+1/2^n}, \omega^{2^n}_{2i+1/2^n}) = j(i(\omega^{n}_{2i+1/2^n}) \circ i(\omega^{n}_{2i+1/2^n})).$$  \hfill (1.24)

From lemma 1.5.14, $\pi_{n-1,n}^*(P_n) = P_{n-1}$. Let $\Omega = \{(\omega^n)_{n \in \mathbb{N}} \in \prod S_n, \pi_{n-1,n}(\omega^n) = \omega^{n-1}\}$ and $\mathcal{A}$ be the $\sigma$-field on $\Omega$ generated by the mappings $\pi_n : \Omega \rightarrow S_n$, with $\pi_n((\omega^k)_{k \in \mathbb{N}}) = \omega^n$. Let $\mathcal{P}$ be the unique probability on $(\Omega, \mathcal{A})$ such that $\pi_n^*(\mathcal{P}) = \mathcal{P}_n$ (see theorem 3.2 in [34]).

For all dyadic numbers $s < t$, let $\mathcal{F}_{s,t}$ be the $\sigma$-field generated by the mappings $(\omega^k)_{k \in \mathbb{N}} \mapsto \omega_u^n$ for all $(n, u)$ such that $(s, t) \in D_n^2$ and $u \in D_n \cap [s, t[$.

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1.6.2 A measurable stochastic flow of mappings on $M$.

For $t \geq 0$, set $\mathcal{J}_t = i_t \circ j$. Then, $\mathcal{J}_t$ is a measurable presentation of $Q_s$ for all $s \leq t$. Recall that $i_t$ can be chosen such that $i_t = i$ for $t \leq 1$, so that $\mathcal{J}_t = \mathcal{J} := i \circ j$ if $t \leq 1$. Note also that $\mathcal{J}_t \circ \mathcal{J}_s = i_t \circ j \circ i_s \circ j = \mathcal{J}_t$ since $j \circ i_s(y) = y$.

Definition 1.6.1. On $(\Omega, \mathcal{A}, P)$, we define the following random variables

1. For all $s < t \in D_n$, let $\varphi^t_{n,k}(\omega) = \Phi_{(t-s)}^{(k)}(\omega_s^n, \ldots, \omega_{t-2-n}^n)$.

2. For all $s < t \in D$, let $\varphi_{s,t} = \mathcal{J}_{t-s}(\varphi^n_{s,t})$ where $n = \inf \{k \mid (s, t) \in D^2_k\}$.

3. For all $t \in D$, let $\varphi_{t,t} \in F$ be defined by $\varphi_{t,t}(x) = x$ for all $x \in M$.

Let us remark that for all $s < t \in D_n$, the law of $\varphi^n_{s,t}$ is $Q_{t-s}$ (this is a consequence of lemma [1.5.14]), and therefore the law of $\varphi_{s,t}$ is also $Q_{t-s}$. Note also that for all $s \leq u \leq t \in D_n$, we have $\varphi^n_{s,t} = \varphi^n_{u,t} \circ \varphi^n_{s,u}$.

Remark 1.6.2.

- If $t \in D$, since $\varphi_{t,t}$ is continuous, $\mathcal{J}_0(\varphi_{t,t}) = \varphi_{t,t}$. And if $s < t \in D$, $\mathcal{J}_{t-s}(\varphi_{s,t}) = \varphi_{s,t}$.

- If $u \in D_n$, then $\varphi_{u,u+2-n} = \varphi^n_{u,u+2-n}$. Indeed, using that $j \circ i(y) = y$ for all $y \in M^n$, $\varphi_{u,u+2-n}(\omega) = i \circ j \circ i(\omega_u^n) = i(\omega_u^n) = \varphi^n_{u,u+2-n}(\omega)$.

- If $s < t \in D_n$, then by definition,
  
  $\varphi^n_{s,t} = \varphi^n_{t-2-n,t} \circ \cdots \circ \varphi^n_{s,s+2-n}$
  
  $= \mathcal{J}(\varphi^n_{t-2-n,t}) \circ \cdots \circ \mathcal{J}(\varphi^n_{s,s+2-n})$
  
  and $\varphi_{s,t}$ is a measurable function of $(\varphi^n_{u,u+2-n})_{u \in D_n}$. Hence, for all $s < t \in D$, $\varphi_{s,t}$ is a measurable function of $(\varphi^n_{u,u+2-n})_{(n, u) \in \cup k \in \mathbb{N} \times D_k}$.

- If $u \in D_n$,
  
  $\varphi_{u,u+2-n} = \varphi^n_{u,u+2-n} = i(\omega^n_u) = i \circ j \circ i(\omega^n_u + 1) \circ i(\omega^n_u + 1)$
  
  $= \mathcal{J}(\varphi^n_{u,u+2-n+1} \circ \varphi^n_{u+2-n+1,u+2-n})$
  
  $= \mathcal{J}(\varphi^n_{u,u+2-n+1} \circ \varphi_{u+2-n+1,u+2-n})$.

Proposition 1.6.3. For all $s < t \in D_n$ and all $M$-valued random variable $X$ independent of $\mathcal{F}_{s,t}$,

$\varphi^n_{s,t}(X) = \varphi_{s,t}(X)$ $P$-almost surely.

Proof. To prove this proposition we will apply the following lemma:

Lemma 1.6.4. Let $X$ be an $M$-valued random variable and let $\varphi_1$ and $\varphi_2$ be two $F$-valued random variables. Suppose that $X$, $\varphi_1$ and $\varphi_2$ are independent and that the laws of $\varphi_1$ and $\varphi_2$ are respectively $Q_s$ and $Q_t$, with $(s, t) \in \mathbb{R}^2_+$. Set $\varphi = \mathcal{J}_{s+t}(\mathcal{J}_t(\varphi_2) \circ \mathcal{J}_s(\varphi_1))$. Then the law of $\varphi$ is $Q_{s+t}$ and a.s.,

$\varphi(X) = \mathcal{J}_t(\varphi_2) \circ \mathcal{J}_s(\varphi_1)(X)$. 

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Proof. First \((x, f, g) \mapsto J_t(g) \circ J_s(f)(x)\) and \((x, f, g) \mapsto J_{s+t}(J_t(g) \circ J_s(f))(x)\) are measurable. It holds that the law of \(J_t(\varphi_2) \circ J_s(\varphi_1)\) is \(Q_s \ast Q_t = Q_{s+t}\) and that for all \(x \in M\), a.s. \(\varphi(x) = J_t(\varphi_2) \circ J_s(\varphi_1)(x)\). We conclude using Fubini’s theorem. □

Proof of Proposition 1.6.3. Fubini’s theorem implies that for all \(s < t \in D_n\), a.s. \(\varphi^n_{s,t}(X) = J_{t-s}(\varphi^n_{0,t})(X)\) (using the fact that \(\varphi^n_{0,t} = \Phi^{(2n)}(\omega^n_{s}, \ldots, \omega^n_{t-2-n})\) and that \(\varphi^n_{s,t}(x, \omega) = \Psi^{(2n)}(x, \omega^n_{s}, \ldots, \omega^n_{t-2-n})\)). Choosing \(n = \inf\{k : (s, t) \in D_k\}\), we have that \(\varphi^n_{s,t}(X) = \varphi_{s,t}^{n+1}(X)\) a.s.

Therefore, it is enough to prove that for all \(n \geq 1\) such that \((s, t) \in D^2_n\), \(\varphi_{s,t}^n(X) = \varphi_{s,t}^{n+1}(X)\) almost surely. This holds since

\[
\varphi_{s,t}^n(X) = J(\varphi_{t-s}^{n-1})(\varphi_{s,t}^n(X)) = J(\varphi_{s,s+2-n}^{n+1}) \circ \cdots \circ J(\varphi_{t-s}^{n+1})(X).
\]

Since for \(u \in D_{n+1}\), \(\varphi_{u,u+2-n}^{n+1} = J(\varphi_{u,u+2-n}^{n+1})\), using the independence of the family of random variables \(\varphi_{u,u+2-n}^{n+1} : u \in D_{n+1} \cap [s, t]\) that \(X\) is independent of this family and the lemma \((t-s)^2\) times, we prove that the last term is a.s. equal to \(J(\varphi_{t-s}^{n+1}) \circ \cdots \circ J(\varphi_{s,s+2-n}^{n+1})(X) = \varphi_{s,t}^{n+1}(X)\). □

Remark 1.6.5. The preceding proposition implies that for all \(s < u < t \in D\) and all \(M\)-valued random variable \(X\) independent of \(\mathcal{F}_{s,t}\),

\[
\varphi_{s,t}(X) = \varphi_{u,t} \circ \varphi_{s,u}(X) \quad \mathbb{P} \text{ – almost surely.} \tag{1.25}
\]

Proof. There is \(n\) such that \((s, t, u) \in D^3_n\), then a.s. \(\varphi_{s,t}(X) = \varphi_{s,t}^n(X) = \varphi_{u,t} \circ \varphi_{s,u}^n(X) = \varphi_{u,t} \circ \varphi_{s,u}(X)\). □

We now intend to define by approximation for all \(s < t \in \mathbb{R}\) a \((F, \mathcal{F})\)-valued random variable \(\varphi_{s,t}\) of law \(Q_{t-s}\). In order to do this, we prove the following lemma.

Lemma 1.6.6. For all continuous function \(f\) on \(M^2\), the mapping

\[
(s, t, u, v, x, y) \mapsto \mathbb{E}[f(\varphi_{s,t}(x), \varphi_{u,v}(y))] \tag{1.26}
\]

is continuous on \(\{(s, t) \in D^2, s \leq t\} \times M^2\). (And therefore uniformly continuous on every compact.)

Proof. For all \(s \leq u \leq t \leq v\) in \(D\), using the cocycle property, we have (In order to apply Fubini’s theorem, we recall that for all \(s \leq u \leq t\), \(\varphi_{s,t} = J_{t-s}(\varphi_{s,t})\))

\[
\mathbb{E}[f(\varphi_{s,t}(x), \varphi_{u,v}(y))] = \mathbb{E}[f(\varphi_{u,t} \circ \varphi_{s,u}(x), \varphi_{t,v} \circ \varphi_{v,t}(y))] = (\mathbb{P}_{u-s}^{(1)} \otimes I)(\mathbb{P}_{t-u}^{(2)}(I \otimes \mathbb{P}_{v-t}^{(1)}))f(x, y).
\]

For all \(s \leq u \leq v \leq t\) in \(D\), using the cocycle property, we have

\[
\mathbb{E}[f(\varphi_{s,t}(x), \varphi_{u,v}(y))] = \mathbb{E}[f(\varphi_{v,t} \circ \varphi_{u,v} \circ \varphi_{s,u}(x), \varphi_{u,v}(y))] = (\mathbb{P}_{u-s}^{(1)} \otimes I)(\mathbb{P}_{v-u}^{(2)}(\mathbb{P}_{t-v}^{(1)} \otimes I))f(x, y).
\]
For all \( s \leq t \leq u \leq v \) in \( D \),
\[
\mathbb{E}[f(\varphi_{s,t}(x), \varphi_{u,v}(y))] = (P_{t-s}^{(1)} \otimes P_{u-t}^{(1)}) f(x, y).
\]

All these functions are continuous and they join. This implies the lemma. \( \square \)

For all real \( t \) and all integer \( n \), let \( t_n = \sup\{ u \in D_n, \ u \leq t \} \). For all \( s < t \in \mathbb{R} \), we define the increasing sequences \( (s_n)_{n \in \mathbb{N}} \) and \( (t_n)_{n \in \mathbb{N}} \). Using lemma 1.6.6 for \( f(x, y) = d(x, y) \) and the Markov inequality, for all positive \( \varepsilon \), we have
\[
\lim_{n \to \infty} \sup_{k > n} \sup_{x \in M} P[d(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)) \geq \varepsilon] = 0. 
\] (1.27)

Set \( J^n = J_{t_n-s_n} \) and define the following measurable mappings \( \Phi : F^N \to F \) and \( \Psi : M \times F^N \to M \) by \( \Psi(x, (\varphi_n)_{n \in \mathbb{N}}) = l((J^n(\varphi_n)(x))_{n \in \mathbb{N}}) \) and \( \Phi((\varphi_n)_{n \in \mathbb{N}})(\cdot) = \Psi(\cdot, (\varphi_n)_{n \in \mathbb{N}}) \).

We then set for all \( s \leq t \), \( \psi_{s,t} = \Phi((\varphi_{s_n,t_n})_{n \in \mathbb{N}}) \) and \( \varphi_{s,t} = J_{t-s}(\psi_{s,t}) \). Note that

- for all \( x \in M \), a.s. \( \psi_{s,t}(x) = l((\varphi_{s_n,t_n}(x))) \) and when \( s = t \), a.s. \( \psi_{t,t}(x) = x \).
- \( \varphi_{s,t} \) is a measurable function of \( (\varphi_{u,u+2^{-n}}(n,n))_{n \in \mathbb{N}} \times D_k \).

**Lemma 1.6.7.** For all positive \( \varepsilon \) and all \( s \leq t \),
\[
\lim_{n \to \infty} \sup_{x \in M} P[d(\varphi_{s_n,t_n}(x), \psi_{s,t}(x)) \geq \varepsilon] = 0. 
\] (1.28)

**Proof.** Equation (1.27) implies that \( \varphi_{s_n,t_n}(x) \) converges in probability towards \( \psi_{s,t}(x) \). Thus, for all positive \( \varepsilon \),
\[
P[d(\varphi_{s_n,t_n}(x), \psi_{s,t}(x)) \geq \varepsilon] = \lim_{k \to \infty} P[d(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)) \geq \varepsilon].
\]

Therefore,
\[
\sup_{x \in M} P[d(\varphi_{s_n,t_n}(x), \psi_{s,t}(x)) \geq \varepsilon] \leq \sup_{k > n} \sup_{x \in M} P[d(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)) \geq \varepsilon],
\]
which implies the lemma. \( \square \)

**Proposition 1.6.8.** For all \( s < t \in \mathbb{R} \), the law of \( \varphi_{s,t} \) is \( Q_{t-s} \).

**Proof.** For all \( k \geq 1 \), \( f \in C(M^k) \) and \( x \in M^k \), lemma 1.6.7 and dominated convergence theorem implies that
\[
\mathbb{E}[f(\psi_{s,t}^{\otimes k}(x))] = \lim_{n \to \infty} \mathbb{E}[f(\varphi_{s_n,t_n}^{\otimes k}(x))] = \lim_{n \to \infty} P_{t_n-s_n}^{(k)} f(x) = P_{t-s}^{(k)} f(x)
\]

since \( P_{t}^{(k)} \) is Feller. This implies that the law of \( \psi_{s,t} \) is \( Q_{t-s} \), and therefore that the law of \( \varphi_{s,t} = J_{t-s}(\psi_{s,t}) \) is \( Q_{t-s} \). \( \square \)

Let us now prove the cocycle property.

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Proposition 1.6.9. For all $s < u < t$ and all $x \in M$, $\mathbb{P}$-almost surely,

$$\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x). \quad (1.29)$$

Proof. By construction, for all $s \leq t$, $\mathcal{J}_{t-s}(\varphi_{s,t}) = \varphi_{s,t}$ (since $\mathcal{J}_t \circ \mathcal{J}_t = \mathcal{J}_t$).

Almost surely, we have $\varphi_{s_n,t_n}(x) = \varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x)$ since $s_n < u_n < t_n$ belong to $D$. Proposition 1.7 implies that $\varphi_{s,t}(x) = \psi_{s,t}(x)$ a.s. and Lemma 1.10 implies that $\varphi_{s_n,t_n}(x)$ converges in probability towards $\varphi_{s,t}(x)$.

Let us now show that $\varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x)$ converges in probability toward $\varphi_{u,t} \circ \varphi_{s,u}(x)$. For $n \geq 1$, set $X_n = \varphi_{s,u_n}(x)$ and $X = \varphi_{s,u}(x)$. Then $X_n$ converges in probability towards $X$ and for $n \geq 1$ and $\varepsilon > 0$,

$$\mathbb{P}[d(\varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x), \varphi_{u,t} \circ \varphi_{s,u}(x)) \geq \varepsilon]$$

$$= \mathbb{P}[d(\mathcal{J}_{t-u}(\varphi_{u_n,t_n})(X_n), \mathcal{J}_{t-u}(\varphi_{u,t})(X)) \geq \varepsilon]$$

$$\leq \mathbb{P}[d(\mathcal{J}_{t-u}(\varphi_{u_n,t_n})(X_n), \mathcal{J}_{t-u}(\varphi_{u,t})(X_n)) \geq \varepsilon/2] + \mathbb{P}[d(\mathcal{J}_{t-u}(\varphi_{u,t})(X_n), \mathcal{J}_{t-u}(\varphi_{u,t})(X)) \geq \varepsilon/2].$$

Since $X_n$ is independent of $\mathcal{F}_{u_n,t_n}$, the first term is equal to

$$\int \mathbb{P}[d(\mathcal{J}_{t-u}(\varphi_{u_n,t_n})(y), \mathcal{J}_{t-u}(\varphi_{u,t})(y)) \geq \varepsilon/2] \mathbb{P}_{X_n}(dy)$$

$$\leq \sup_{y \in M} \mathbb{P}[d(\varphi_{u_n,t_n}(y), \varphi_{u,t}(y)) \geq \varepsilon/2]$$

which converges to 0 as $n \to \infty$ (using Lemma 1.10). We show that the second term converges to 0 using Lemma 1.6 (since the sequence $(X_n)$ is independent of $\varphi_{u,t}$).

Thus we have constructed a measurable stochastic flow of measurable mappings on $M$ associated with the compatible family of Feller semigroups $(\mathbb{P}_t^{(k)}, k \geq 1)$ and with the Feller convolution semigroup $(\mathbb{Q}_t, t \geq 0)$.

Let $\varphi$ be the $(\Omega^0, \mathcal{A}^0)$-valued random variable defined by $\varphi = (\varphi_{s,t}, s \leq t)$. Let $\mathbb{P}_Q = \varphi^*(\mathbb{P})$ be the law of $\varphi$. Then by a monotone class argument we show that $T_h^* (\mathbb{P}_Q) = \mathbb{P}_Q$ for all $h \in \mathbb{R}$.

On $(\Omega^0, \mathcal{A}^0, \mathbb{P}_Q)$, let $\varphi'$ be a measurable modification of the canonical stochastic flow $\varphi^0$. Then for all $t \geq 0$, there is a measurable presentation $\mathcal{J}'_t$ of $\mathbb{Q}_t$ such that for all $s \leq t$, $\varphi_{s,t}' = \mathcal{J}'_{t-s}(\varphi_{s,t})$. This modification is $(T_h)_h \in \mathbb{R}$-invariant, since for all $s \leq t$ and all $h \in \mathbb{R}$, $\varphi_{s+h,t+h}' = \mathcal{J}'_{t-s}(\varphi_{s+h,t+h}) = \mathcal{J}_{t-s}(\mathcal{T}_{s+h,t+h}^0) = \mathcal{T}_{h} \circ \mathcal{T}_{s+h,t+h}^0 = \varphi_{s,t}' \circ \mathcal{T}_h$.

The fact that $(\Omega^0, \mathcal{A}^0, \mathbb{P}_Q)$ is separable is a consequence of the construction of $\varphi$. The proof of Theorem 1.4.2 is finished.

1.7 The example of Lipschitz SDEs.

We first show a sufficient condition for a compatible family of Markovian kernels semigroups to be constituted of Feller semigroups.

Lemma 1.7.1. A compatible family $(\mathbb{P}_t^{(n)}, n \geq 1)$ of semigroups of Markovian kernels is constituted of Feller semigroups when the following condition is satisfied
Remark 1.7.2. • which converges towards 0 as $t \to 0$, $\lim_{t \to 0} P_t(1) f(x) = f(x)$ and for all $x \in M$, $\varepsilon > 0$ and $t > 0$, $\lim_{y \to x} P_t(2) f(x, y) = 0$, where $f(x, y) = 1_{d(x,y) > \varepsilon}$.

**Proof.** Let $h \in C(M^n)$ be in the form $f_1 \otimes \cdots \otimes f_n$ and $x = (x_1, \ldots, x_n)$ in $M^n$. We have for $M$ large enough

$$|P_{t}^{(n)} h(x) - h(x)| \leq M \sum_{k=1}^{n} (P_{t}^{(1)} f_{k}^{2} + f_{k}^{2} - 2 f_{k} P_{t}^{(1)} f_{k})^{\frac{1}{2}}(x_k)$$  \hspace{1cm} (1.30)

which converges towards 0 as $t$ goes to 0 since for all $f \in C(M)$ and all $x \in M$, $\lim_{t \to 0} P_{t}^{(1)} f(x) = f(x)$. We also have for $y = (y_1, \ldots, y_n)$ in $M^n$,

$$|P_{t}^{(n)} h(y) - P_{t}^{(n)} h(x)| \leq M \sum_{k=1}^{n} P_{t}^{(2)}(1 \otimes f_{k} - f_{k} \otimes 1)(y_k, x_k)$$  \hspace{1cm} (1.31)

which converges towards 0 as $y$ tends to $x$ since for all $f \in C(M)$ and all $x \in M$, $\lim_{y \to x} P_{t}^{(2)}(1 \otimes f - f \otimes 1)(y, x) = 0$. Indeed, $\forall \alpha > 0$, $\exists \varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $|f(y) - f(x)| < \alpha$. This implies

$$P_{t}^{(2)}(1 \otimes f - f \otimes 1)(y, x) \leq \alpha + 2\|f\|_{\infty} P_{t}^{(2)} f_{\varepsilon}(x, y).$$  \hspace{1cm} (1.32)

This implies $\limsup_{y \to x} P_{t}^{(2)}(1 \otimes f - f \otimes 1)(y, x) \leq \alpha$ for all $\alpha > 0$. □

**Remark 1.7.2.** • The previous result extends to the locally compact case (using the fact that $C_{0}(M)$ is constituted of uniformly continuous functions).

• When (F) is satisfied, for all positive $t$, $f \in C_{0}(M)$ and $x \in M$, $P_{t}^{(1)} f^{\otimes 2}(x, x) = P_{t}^{(1)} f^{2}(x)$. This implies that (F) is not a necessary condition. Theorem 1.4.2 shows that a stochastic flow of mappings is associated with this family of semigroups.

**Definition 1.7.3.** A two-parametric family $(W_{s,t}, s \leq t)$ of real random variables is called a real white noise if

(i) for all $s < t$, $W_{s,t}$ is a centered Gaussian variable with variance $t - s$,

(ii) for all $((s_{i}, t_{i}), 1 \leq i \leq n)$ with $s_{i} \leq t_{i} \leq s_{i+1}$, the random variables $(W_{s_{i},t_{i}}, 1 \leq i \leq n)$ are independent and

(iii) for all $s \leq t \leq u$, $W_{s,u} = W_{s,t} + W_{t,u}$.

Let $V, V_{1}, \ldots, V_{k}$ be bounded Lipschitz vector fields on a smooth locally compact manifold $M$. We also assume that $V_{1}, \ldots, V_{k}$ are $C^{1}$. Let $W^{1}, \ldots, W^{k}$ be $k$ independent real white noises. We consider the SDE on $M$

$$dX_{t} = \sum_{i=1}^{k} V_{i}(X_{t}) \circ dW_{i} + V(X_{t}) \, dt, \quad t \in \mathbb{R}. \hspace{1cm} (1.33)$$

From the usual theory of strong solutions of SDEs (see for example [20]), it is possible to construct a stochastic flow of diffeomorphisms $(\varphi_{s,t}, s \leq t)$ such that for all $x \in M$, $\varphi_{s,t}(x)$ is a strong solution of the SDE (1.33) with $\varphi_{s,s}(x) = x$.  

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Using this stochastic flow, it is possible to construct a compatible family of Markovian semigroups \( (P_t^{(n)}, n \geq 1) \) with\(P_t^{(n)}h(x_1, \ldots , x_n) = \mathbb{E}[h(\varphi_{0,t}(x_1), \ldots , \varphi_{0,t}(x_n))]\) (1.34) for \( h \in C(M^n) \) and \( x_1, \ldots , x_n \) in \( M \). Using lemma 1.7.1, it is easy to check that these semigroups are Feller (these properties were previously observed by P. Baxendale in [3]).

It can easily be shown that the canonical stochastic flow of maps associated with this family of semigroups is equal in law to \((\varphi_{s,t}, s \leq t)\).

2 Stochastic flow of kernels.

2.1 Notation and definitions.

In order to simplify, we suppose in this section that \( M \) is a compact metric space. But, as it is explained in section 3 all the results of this section extend to locally compact separable metric spaces. We denote by \( \mathcal{P}(M) \) the space of probability measures on \( M \), equipped with the weak convergence topology. Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of functions dense in \( \{ f \in C(M), \| f \|_\infty \leq 1 \} \). We will equip \( \mathcal{P}(M) \) with the distance \( \rho(\mu, \nu) = (\sum_n 2^{-n}(\int f_n \, d\mu - \int f_n \, d\nu)^2)^{1/2} \) for all \( \mu \) and \( \nu \) in \( \mathcal{P}(M) \). Thus \( \mathcal{P}(M) \) is a separable compact metric space.

Let us recall that a kernel \( K \) on \( M \) is a measurable mapping from \( M \) into \( \mathcal{P}(M) \), \( M \) and \( \mathcal{P}(M) \) being equipped with their Borel \( \sigma \)-fields. For all \( f \in C(M) \) and \( x \in M \), \( Kf(x) \) denotes \( \int f(y) \, K(x, dy) \). For all \( \mu \in \mathcal{P}(M) \), \( \mu K \) denotes the probability measure defined by \( \int f(y) \, \mu K(dy) = \int Kf(x) \, \mu(dx) \). We denote by \( E \) the space of all kernels on \( M \) and we equip \( E \) with the \( \sigma \)-field generated by the mappings \( K \mapsto \mu K \), for all \( \mu \in \mathcal{P}(M) \) (\( \mathcal{P}(M) \) is equipped with its Borel \( \sigma \)-field \( \mathcal{B}(\mathcal{P}(M)) \)). We denote this \( \sigma \)-field by \( \mathcal{E} \).

Let \( \Gamma \) denote the space of measurable mappings on \( \mathcal{P}(M) \). We equip \( \Gamma \) with the \( \sigma \)-field generated by the mappings \( \Phi \mapsto \Phi(\mu) \) for all \( \mu \in \mathcal{P}(M) \). Note that \( (\Gamma, \mathcal{G}) = (F, \mathcal{F}) \) once we have replaced \( M \) by \( \mathcal{P}(M) \).

2.2 Convolution semigroups on the space of kernels.

Let \( I \) denote the measurable mapping from \((E, \mathcal{E})\) on \((\Gamma, \mathcal{G})\) defined by \( I(K)(\mu) = \mu K \). Note that \( I(E) \) is not measurable in \( \Gamma \) but \( I \) is measurable.

Definition 2.2.1. (i) A probability measure \( \nu \) on \((E, \mathcal{E})\) is called regular if \( I^*(\nu) \) is a regular probability measure on \((\Gamma, \mathcal{G})\).

(ii) A convolution semigroup on \((E, \mathcal{E})\) is a family \((\nu_t)_{t \geq 0}\) of regular probability measures on \((E, \mathcal{E})\) such that \((I^*(\nu_t))_{t \geq 0}\) is a convolution semigroup on \((\Gamma, \mathcal{G})\).

Let \( \delta : \Gamma \to E \) be the mapping defined by \( \delta(\Phi)(x) = \Phi(\delta_x) \). Note that \( \delta \) is not measurable in general.

Proposition 2.2.2. Let \( Q \) be a regular probability measure on \((\Gamma, \mathcal{G})\) and \( J \) a measurable presentation of \( Q \). Then \( \delta \circ J \) is measurable and the probability measure \( \nu = (\delta \circ J)^*(Q) \) is a regular probability measure on \((E, \mathcal{E})\) if \( I^*(\nu) = Q \).
Proof. Let $Q$ be a regular probability measure on $(\Gamma, \mathcal{G})$ and $J$ a measurable presentation of $Q$. The mapping $\mathcal{P}(M) \times \Gamma \ni (\mu, \Phi) \mapsto J(\Phi)(\mu) \in \mathcal{P}(M)$ and $M \ni x \mapsto \delta_x \in \mathcal{P}(M)$ are measurable. Thus $M \times \Gamma \ni (x, \Phi) \mapsto \delta \circ J(\Phi)(x) = J(\Phi)(\delta_x) \in \mathcal{P}(M)$ is measurable, which implies that $\delta \circ J$ is measurable. □

Remark 2.2.3. The probability measure $\nu$ defined in proposition 2.2.2 depends only of $Q$. Indeed, if $J'$ is another measurable presentation of $Q$, for all $x \in M$, $Q(d\Phi)$-a.s., $\delta \circ J(\Phi)(x) = \delta \circ J'(\Phi)(x)$, which implies by Fubini theorem that for all $\mu \in \mathcal{P}(M)$, $Q(d\Phi)$-a.s., $\mu(\delta \circ J(\Phi)) = \mu(\delta \circ J'(\Phi))$ and then that $(\delta \circ J)^*(Q) = (\delta \circ J')^*(Q)$.

Definition 2.2.4. A measurable presentation of a regular probability measure $\nu$ is a measurable mapping $p : (E, \mathcal{E}) \to (E, \mathcal{E})$ such that $(x, K) \mapsto p(K)(x)$ is measurable and such that for all $x \in M$, $\nu(dK)$-a.s. $p(K)(x) = K(x)$.

Remark 2.2.5. • If $p$ is a measurable presentation of a regular probability measure $\nu$, then $(\mu, K) \mapsto \mu(p(K))$ is measurable and for all $\mu \in \mathcal{P}(M)$, $\nu(dK)$-a.s. $\mu(p(K)) = \mu K$.

• If $\nu$ is a regular probability measure on $(E, \mathcal{E})$ and $J$ is a measurable presentation of $Q := \mathcal{I}^*(\nu)$, then for all $x \in M$, if $\nu(dK)$-a.s. $\delta \circ J \circ \mathcal{I}(K)(x) = K(x)$. Therefore, the mapping $p = \delta \circ J \circ \mathcal{I}$ is a measurable presentation of $\nu$.

• Let $(\nu_t)_{t \geq 0}$ be a convolution semigroup on $(E, \mathcal{E})$. If $K_1$ and $K_2$ are random kernels with laws $\nu_s$ and $\nu_t$ and if $p_t$ is a measurable presentation of $\nu_t$, then $K_1(p_t(K_2))$ is a random kernel with law $\nu_{s+t}$ (note that $(K_1, K_2) \mapsto K_1(p_t(K_2))$ is measurable).

Definition 2.2.6. A convolution semigroup $(\nu_t)_{t \geq 0}$ on $(E, \mathcal{E})$ is called Feller if

(i) for all $f \in C(M)$, $\lim_{t \to 0} \sup_{x \in M} \int (Kf(x) - f(x))^2 \nu_t(dK) = 0$,

(ii) for all $f \in C(M)$ and all $t \geq 0$, $\lim_{d(x,y) \to 0} \int (Kf(x) - Kf(y))^2 \nu_t(dK) = 0$.

Proposition 2.2.7. Let $(\nu_t)_{t \geq 0}$ be a Feller convolution semigroup on $(E, \mathcal{E})$. For all $n \geq 1$, $f \in C(M)$ and $x \in M^n$, set

$$P_t^{(n)} f(x) = \int K^\otimes_n f(x) \nu_t(dK). \quad (2.1)$$

Then $(P_t^{(n)}, n \geq 1)$ is a compatible family of Feller semigroups on $M$.

Proof. This is the same proof as the one of proposition 1.2.8 □

Proposition 2.2.8. Let $(Q_t)_{t \geq 0}$ be a convolution semigroup on $(\Gamma, \mathcal{G})$. Let $J_t$ be a measurable presentation of $Q_t$ and $\nu_t = (\delta \circ J_t)^*(Q_t)$. If $Q_t = \mathcal{I}^*(\nu_t)$, $(\nu_t)_{t \geq 0}$ is a convolution semigroup on $(E, \mathcal{E})$. Then, $(Q_t)_{t \geq 0}$ is Feller if and only if $(\nu_t)_{t \geq 0}$ is Feller.

Proof. The fact that $(\nu_t)_{t \geq 0}$ is a convolution semigroup follows from the definition 2.2.1.
Remark 2.3.2. • The flow has independent increments, i.e. for all

\[ \lim_{t \to 0} \sup_{\mu \in \mathcal{P}(M)} \int (\Phi(\mu)f - \mu f)^2 Q_t(d\Phi) = 0 \]  

(2.2)

\[ \lim_{\rho(\mu,\nu) \to 0} \int (\Phi(\mu)f - \Phi(\nu)f)^2 Q_t(d\Phi) = 0. \]  

(2.3)

We first prove (2.2) and (i) in definition 2.2.6 are equivalent. Equation (2.2) implies (i) since \( \int (Kf(x) - f(x))^2 \nu_t(dK) = \int (\Phi(\delta_x)f - \Phi(\delta_x)f)^2 Q_t(d\Phi). \) And (i) implies (2.2) since

\[ \int (\Phi(\mu)f - \mu f)^2 Q_t(d\Phi) = \int (\mu Kf - \mu f)^2 \nu_t(dK) \]

\[ \leq \int \left( \int (Kf(x) - f(x))^2 \nu_t(dK) \right) \mu(dx). \]

We now prove (2.3) and (ii) in definition 2.2.6 are equivalent. Equation (2.3) implies (ii) since \( \int (Kf(x) - Kf(y))^2 \nu_t(dK) = \int (\Phi(\delta_x)f - \Phi(\delta_x)f)^2 Q_t(d\Phi) \) and \( \lim_{d(x,y) \to 0} \rho(\delta_x, \delta_y) = 0. \) Assume (ii) holds. For \( \mu \) and \( \nu \) in \( \mathcal{P}(M), \) we have

\[ \int (\Phi(\mu)f - \Phi(\nu)f)^2 Q_t(d\Phi) = \int (\mu Kf - \nu Kf)^2 \nu_t(dK) \]

\[ = (\mu - \nu)^2 \int K^{\otimes 2} f^{\otimes 2} \nu_t(dK). \]

We conclude since \( \int K^{\otimes 2} f^{\otimes 2} \nu_t(dK) \) is a continuous function (see proposition 2.2.7). \( \square \)

2.3 Stochastic flows of kernels.

**Definition 2.3.1.** Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be a probability space and let \( K = (K_{s,t}, s \leq t) \) be a family of \( (E, \mathcal{E}) \)-valued random variables such that for all \( x \in M \) and all \( t \in \mathbb{R}, \) \( \mathbb{P}-a.s. \) \( K_{t,t}(x) = \delta_x. \) For \( t \geq 0, \) denote by \( \nu_t \) the law of \( K_{0,t}. \) The family \( K \) is called a stochastic flow of kernels if for all \( t \geq 0, \) \( \nu_t \) is regular and if the following properties are satisfied by \( K \)

(a) For all \( s \leq u \leq t, \) for all \( x \in M, \) for all \( f \in C(M) \) and for all measurable presentation \( p_{t-u} \) of \( \nu_{t-u}, \) \( \mathbb{P}-a.s. \) \( K_{s,t}f(x) = K_{s,u}(p_{t-u}(K_{u,t})f)(x). \) (cocycle property)

(b) For all \( s \leq t, \) the law of \( K_{s,t} \) is \( \nu_{t-s}. \) (Stationarity)

(c) The flow has independent increments, i.e. for all \( t_1 \leq t_2 \leq \cdots \leq t_n, \) the family \( \{K_{t_i,t_{i+1}}, 1 \leq i \leq n - 1\} \) is independent.

(d) For all \( f \in C(M) \) and all \( s \leq t, \) \( \lim_{(u,v) \to (s,t)} \sup_{x \in M} \mathbb{E}[(K_{s,t}f(x) - K_{u,v}f(x))^2] = 0. \)

(e) For all \( f \in C(M) \) and all \( s \leq t, \) \( \lim_{d(x,y) \to 0} \mathbb{E}[(K_{s,t}f(x) - K_{s,t}f(y))^2] = 0. \)

**Remark 2.3.2.** • Item (a) holds for all set of measurable presentations as soon as (a) holds for one of them.
• If \( K' \) is equal in law to a stochastic flow of kernels \( K \), then \( K' \) is also a stochastic flow of kernels.

**Proposition 2.3.3.** Let \( K \) be a stochastic flow of kernels, and for \( t \geq 0 \), let \( p_{t} \) be a measurable presentation of the law of \( K_{0,t} \). Then \( K' = (p_{t-s}(K_{s,t}), s \leq t) \) is a stochastic flow of kernels satisfying

(i) For all \( s \leq t \) and \( \mu \in \mathcal{P}(M) \), a.s. \( \mu K'_{s,t}(x) = \mu K_{s,t}(x) \).

(ii) For all \( s \leq u \leq t \) and for all \( \mu \in \mathcal{P}(M) \), \( \mathbb{P} \)-almost surely, \( \mu K'_{s,t} = \mu K'_{s,u} K'_{u,t} \).

**Proof.** Follow the proof of proposition at page 10 in section 2. \( \square \)

**Definition 2.3.4.** • The stochastic flow of kernels \( K' \) defined in the proposition 2.3.3 will be called a measurable modification of \( K \).

• A stochastic flow of kernels which is a measurable modification of a stochastic flow of kernels is called a measurable stochastic flow of kernels.

**Proposition 2.3.5.** Let \( (K_{s,t}, \ s \leq t) \) be a stochastic flow of kernels. For all \( n \geq 1 \), \( f \in C(M^n) \) and \( x \in M^n \), set

\[
P^{(n)}_t f(x) = \mathbb{E}[K^n f(x)].
\]  

(2.4)

Then \( (P^{(n)}_t, \ n \geq 1) \) is a compatible family of Feller semigroups on \( M \).

**Proof.** This is the same proof as the one to prove proposition 2.2.8 \( \square \)

### 2.4 Construction and characterization.

Let \( (\Omega^0, \mathcal{A}^0) \) denote the measurable space \( (\prod_{s \leq t} E, \otimes_{s \leq t} \mathcal{E}) \). For \( s \leq t \), let \( K^0_{s,t} \) denote the random variable \( \omega \mapsto \omega(s,t) \). Let also \( K^0 \) be the random variable \( (K^0_{s,t}, \ s \leq t) \). Then \( K^0(\omega) = \omega \). Let \( (T_h)_{h \in \mathbb{R}} \) be the one-parametric group of transformations of \( \Omega^0 \) defined by \( T_h(\omega)(s,t) = \omega(s+h, t+h) \), for all \( s \leq t, h \in \mathbb{R} \) and \( \omega \in \Omega^0 \).

**Theorem 2.4.1.** 1- For all compatible family \( (P^{(n)}_t, \ n \geq 1) \) of Feller semigroups on \( M \), there exists a unique Feller convolution semigroup \( (\nu_t)_{t \geq 0} \) on \( (E, \mathcal{E}) \) such that for all \( n \geq 1 \), \( t \geq 0 \), \( f \in C(M^n) \) and \( x \in M^n \),

\[
P^{(n)}_t f(x) = \int K^n f(x) \nu_t(dK).
\]  

(2.5)

2- For all Feller convolution semigroup \( \nu = (\nu_t)_{t \geq 0} \) on \( (E, \mathcal{E}) \), there exists a unique \( (T_h)_{h \in \mathbb{R}} \)-invariant probability measure \( \mathbb{P}_\nu \) on \( (\Omega^0, \mathcal{A}^0) \) such that \( (\Omega^0, \mathcal{A}^0, \mathbb{P}_\nu) \) is separable, the family of random variables \( (K^0_{s,t}, \ s \leq t) \) is a stochastic flow of kernels and for all \( s \leq t \), the law of \( K^0_{s,t} \) is \( \nu_{t-s} \). Every measurable modification \( K' \) of \( K^0 \) satisfies \( K'_{s+h,t+h} = K'_{s,t} \circ T_h \) for all \( s \leq t \) and all \( h \in \mathbb{R} \).

The flow \( K^0 \) is called the canonical stochastic flow of kernels associated with \( \nu \) (or equivalently with \( (P^{(n)}_t, \ n \geq 1) \)).

**Remark 2.4.2.** In the case \( (1.6) \) is satisfied, \( (P^{(n)}_t, \ n \geq 1) \) is associated to a stochastic flow of mappings \( \varphi \) by Theorem 1.1. Set \( \delta_{\varphi} = (\delta_{\varphi_{s,t}}, \ s \leq t) \). Then, for all \( s \leq t \), \( \delta_{\varphi_{s,t}} \) is a random kernel of law \( \nu_{s,t} \) (since \( \mathbb{E}[\delta_{\varphi_{s,t}} f(x)] = \mathbb{E}[f \circ \varphi_{s,t}(x)] = (P^{(n)}_{t-s} f(x)) \)) and one can check that \( \delta_{\varphi} \) is a stochastic flow of kernels of law \( \mathbb{P}_\nu \).
2.5 Proof of theorem 2.4.1

Let \((P_t^{(n)}, n \geq 1)\) be a compatible family of Feller semigroups on \(M\). Starting with this family of semigroups, we intend to construct a Feller convolution semigroup \(\nu = (\nu_t)_{t \geq 0}\) on \((E, \mathcal{E})\). The idea is to construct a compatible family of Feller semigroups on \(P(M)\), then to apply theorem 1.4.2 to construct a Feller convolution semigroup \(Q = (Q_t)_{t \geq 0}\) on \((\Gamma, \mathcal{G})\) and to construct \(\nu\) using the mappings \(\delta \circ J_t\), where \(J_t\) is a measurable presentation of law \(Q_t\).

2.5.1 Construction of a compatible family of Feller semigroups on \(P(M)\).

For all integer \(k\), we define a Feller semigroup \(\Pi_t^{(k)}\) acting on the continuous functions on \(P(M)^k\) (see [28] for a similar construction when \(k = 1\)).

Let \(\mathcal{A}_k\) denote the algebra of functions \(g : P(M)^k \to \mathbb{R}\) such that \(^{1}\)

\[ g(\mu_1, \ldots, \mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle \tag{2.6} \]

for \(f \in C(M^n)\) and \(n_1, \ldots, n_k\) integers such that \(n = n_1 + \cdots + n_k\) \((\mathcal{A}_k\) is the union of an increasing family of algebras \(\mathcal{A}_{n_1, \ldots, n_k}\)). For all \(g \in \mathcal{A}_k\), given by equation (2.6), let

\[ \Pi_t^{(k)} g(\mu) = \langle P_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle. \tag{2.7} \]

with \(\mu = (\mu_1, \ldots, \mu_k) \in P(M)^k\). Since the family of semigroups \((P_t^{(n)}, n \geq 1)\) is compatible, \(\Pi_t^{(k)}\) is independent of the expression of \(g\) in (2.6).

Let us notice that \(\Pi_t^{(k)}\) acts on \(\mathcal{A}_k\) and that, by the theorem of Stone-Weierstrass, the algebra \(\mathcal{A}_k\) is dense in \(C(P(M)^k)\).

**Lemma 2.5.1.** \(\Pi_t^{(k)}\) is a Markovian operator acting on \(\mathcal{A}_k\).

**Proof.** The only thing to be proved is the positivity property (it is obvious that \(\Pi_t^{(k)} 1 = 1\)).

For all integer \(N\), let \((X_{i,j}^t, 1 \leq i \leq k, 1 \leq j \leq N)\) be a Markov process associated with the Markovian semigroup \(P_t^{(N)}\) such that the random variables \((X_{i,j}^t, 1 \leq i \leq k, 1 \leq j \leq N)\) are independent and the law of \(X_{i,j}^0\) is \(\mu_i\), where \((\mu_1, \ldots, \mu_k) \in P(M)^k\). Let us introduce the following Markov process on \(P(M)^k\), \(\mu_t^N = (\mu_t^{N,1}, \ldots, \mu_t^{N,k})\) where

\[ \mu_t^{N,i} = \frac{1}{N} \sum_{j=1}^N \delta_{X_{i,j}^t}, \quad \text{for } 1 \leq i \leq k. \tag{2.8} \]

For \(g(\mu_1, \ldots, \mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle\), we have

\[ \mathbb{E}[g(\mu_t^N)] = \mathbb{E}[\langle f, (\mu_t^{N,1})^{\otimes n_1} \otimes \cdots \otimes (\mu_t^{N,k})^{\otimes n_k} \rangle] \]

\[ = \frac{1}{N^n} \sum_{i_1=1}^k \sum_{l_1=1}^n \cdots \sum_{i_N=1}^k \sum_{l_N=1}^n \mathbb{E}[f(X_{i_1}^{l_1,1}, X_{i_2}^{l_2,1}, \ldots, X_{i_{N-1}}^{l_{N-1},1}, X_{i_N}^{l_N,1}, \ldots, X_{i_N}^{l_N,n_k})] \]

\[ = \langle P_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle + R_N. \]

\(^{1}\)Here and in the following, for all measure \(\mu\) and \(f \in L^1(\mu)\), we denote \(\int f \, d\mu \) by \(\langle f, \mu \rangle\), \(\langle \mu, f \rangle\) or \(\mu f\).
The remainder term \( R_N \) comes from terms in which \( j_i^a = j_i^b \) for some \( a \neq b \) and some \( i \) and is therefore dominated by \( 2\|f\|_\infty(1 - \prod_{i=1}^k(N(N-1)\cdots(n_i-n+1)/N^{n_i})) \). Thus

\[
\lim_{N \to \infty} E[g(\mu_i^N)] = \langle P_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle \\
= \Pi_t^{(k)} g(\mu_1, \ldots, \mu_k).
\] (2.9)

This shows that \( \Pi_t^{(k)} \) is positive. \( \Box \)

Using this lemma, it is easy to define \( \Pi_t^{(k)} g \) for all continuous function \( g \) and to show that \( \Pi_t^{(k)} \) is a Markovian semigroup acting on \( C(M^n) \).

**Lemma 2.5.2.** (\( \Pi_t^{(k)}, k \geq 1 \)) is a compatible family of Feller semigroups on \( \mathcal{P}(M) \) satisfying \( (1.6) \).

**Proof.** Since the semigroups \( P_t^{(n)} \) are Feller, the semigroups \( \Pi_t^{(k)} \) are also Feller : for all \( g \) in \( \mathcal{A}_k \), then \( \Pi_t^{(k)} g \) is continuous and \( \lim_{t \to 0} \Pi_t^{(k)} g = g \) and these properties extend to every continuous functions.

It is clear that the family of semigroups \( (\Pi_t^{(k)}, k \geq 1) \) is compatible (in the sense given in section 1.1). Thus \( (\Pi_t^{(k)}, k \geq 1) \) is a compatible family of Feller semigroups on \( \mathcal{P}(M) \). We denote \( \Pi_t^{(2)} \otimes \otimes \) the law of the Markov process associated with \( \Pi_t^{(2)} \) starting from \( (\mu, \nu) \) and we denote this process by \( (\mu_t, \nu_t) \).

For \( g \in \mathcal{A}_1 \) in the form (2.6), \( t \geq 0 \) and \( \mu \in \mathcal{P}(M) \), we have

\[
\Pi_t^{(2)} g \otimes \otimes (\mu, \mu) = \langle P_t^{(2n)} f \otimes \otimes , \mu \otimes \otimes 2n \rangle = \Pi_t^{(1)} g^2(\mu).
\]

Thus (1.6) is satisfied for \( g \in \mathcal{A}_1 \) and this extends to \( C(\mathcal{P}(M)) \). \( \Box \)

**2.5.2 Proof of the first part of theorem 2.4.1.**

Using theorem 1.4.2 we construct \( (Q_t)_{t \geq 0} \) a Feller convolution semigroup on \( (\Gamma, \mathcal{G}) \). Let \( J_t \) be a measurable presentation of \( Q_t \). Set \( \nu_t = (\delta \circ J_t)^* Q_t \).

**Lemma 2.5.3.** For all \( \mu \in \mathcal{P}(M) \) and all \( t \geq 0 \),

\[
Q_t(d\Phi) - a.s., \quad \Phi(\mu) = \mu(\delta \circ J_t(\Phi)).
\] (2.11)

And for all \( t \geq 0 \), \( \mathcal{I}^*(\nu_t) = Q_t \).

**Proof.** For all \( f \in C(M) \), set \( g(\mu) = \mu f \), then

\[
E[(\mu(\delta \circ J_t(\Phi)) f - \Phi(\mu)f)^2] = E \left[ \left( \int g(\Phi(\delta_x)) \mu(dx) - g(\Phi(\mu)) \right)^2 \right] \\
= \int \Pi_t^{(2)} g^2(\delta_x, \delta_y) \mu(dx) \mu(dy) + \Pi_t^{(2)} g^2(\mu, \mu) \nonumber - 2 \int \Pi_t^{(2)} g^2(\delta_x, \mu) \mu(dx).
\]

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Lemma 2.5.4.
For all \(K\) of presentation \(Q\), we construct \(P\) such that \(J\). Let (\(\Phi\) \(P\). 1.4.2, we construct \(P\).

Proof. For all \(Q\), we use \(J\). This proves the lemma.

Lemma 2.5.3 implies that \(\nu = (\nu_t)_{t \geq 0}\) is a Feller convolution semigroup on \((E, \mathcal{E})\) (we apply proposition 2.2.8) and (2.5) holds. This proves the first part of theorem 2.4.1.

2.5.3 Proof of the second part of theorem 2.4.1.
Suppose now we are given \(\nu = (\nu_t)_{t \geq 0}\) a Feller convolution semigroup on \((E, \mathcal{E})\). For \(t \geq 0\), set \(Q_t = T^t(\nu_t)\). Then \(Q = (Q_t)_{t \geq 0}\) is a Feller convolution semigroup on \((\Gamma, \mathcal{G})\). Using theorem 1.4.2, we construct \(P\) the law of a stochastic flow of mappings on \(P(M)\) associated with \(Q\). Let \((\Phi_{s,t}, s \leq t)\) be a stochastic flow of mappings of law \(P\). For \(t \geq 0\), there is a measurable presentation \(J\) of \(Q\) and set \(p_t = \delta \circ J_t \circ I\). Then \(p_t\) is a measurable presentation of \(\nu_t\). For \(s \leq t\), set \(K_{s,t} = \delta \circ J_t \circ (\Phi_{s,t})\).

We now show that \(K = (K_{s,t}, s \leq t)\) is a stochastic flow of kernels. Note that the law of \(K_{s,t}\) is \(\nu_{t-s}\). Thus it is easy to check that \(K\) satisfies (b), (c), (d) and (e). In order to show (a), we use

Lemma 2.5.4. For all \(\mu \in \mathcal{P}(M)\) and all \(s \leq t\),

\[P \text{- a.s., } \mu K_{s,t} = \Phi_{s,t}(\mu).\]  

(2.12)

Proof. For all \(f \in M\), set \(g(\mu) = \mu f\), then like in the proof of lemma 2.5.3,

\[E[(\mu K_{s,t} f - \Phi_{s,t}(\mu) f)^2] = E\left[\left(\int g(\Phi_{s,t}(\delta_x)) \mu(dx) - g(\Phi_{s,t}(\mu))\right)^2\right] = \int \Pi^{(2)}_t g^{\otimes 2}(\delta_x, \delta_y) \mu(dx) \mu(dy) + \Pi^{(2)}_t g^{\otimes 2}(\mu, \mu), \]

\[- 2 \int \Pi^{(2)}_t g^{\otimes 2}(\delta_x, \mu) \mu(dx) = 0.\]

This proves the lemma.

Remark 2.5.5. If \(\Lambda\) is a \(P(M)\)-valued random variable independent of \(\Phi_{s,t}\), then (using that \((\mu, K) \mapsto \mu(p_{t-s}(K_{s,t}))\) and \((\mu, \Phi) \mapsto J_{t-s}(\Phi_{s,t})(\mu)\) are measurable)

\[P \text{- a.s., } \Lambda(p_{t-s}(K_{s,t})) = J_{t-s}(\Phi_{s,t})(\Lambda).\]

Let \(s \leq u \leq t\) and \(\mu \in \mathcal{P}(M)\). Lemma 2.5.4 and the cocycle property for \(\Phi\) imply that a.s.,

\[\mu K_{s,t} = \Phi_{s,t}(\mu) = J_{t-u}(\Phi_{u,t}) \circ \Phi_{s,u}(\mu).\]

Lemma 2.5.4 implies that a.s., \(J_{t-u}(\Phi_{u,t}) \circ \Phi_{s,u}(\mu) = J_{t-u}(\Phi_{u,t})(\mu K_{s,u})\). Fubini’s theorem, lemma 2.5.4 and the fact that \(\mu K_{s,u}\) and \(\Phi_{u,t}\) are independent imply that a.s.,

\[J_{t-u}(\Phi_{u,t})(\mu K_{s,u}) = \mu K_{s,u}(p_{t-u}(K_{u,t})).\]

This proves (a), i.e., a.s. \(\mu K_{s,t} = \mu K_{s,u}(p_{t-u}(K_{u,t}))\). We let \(P\nu\) be the law of \(K\). Then \(T^*_h(P\nu) = P\nu\). The rest of the proof is similar to the end of the proof of theorem 1.4.2.
2.6 Sampling the flow.

Let \((K_{s,t}, s \leq t)\) be a measurable stochastic flow of kernels defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and \((T_h)_{h \in \mathbb{R}}\) a one-parametric group of transformations of \(\Omega\) preserving \(\mathbb{P}\) and such that \(K_{s,t} \circ T_h = K_{s+h, t+h}\). In this section, we construct on an extension of \((\Omega, \mathcal{A}, \mathbb{P})\) a random path \(X_t\) starting at \(x\) such that for all positive \(t\),

\[
K_{0,t}f(x) = \mathbb{E}[f(X_t) | \mathcal{A}].
\]

(2.13)

For \(x \in M\) and \(\omega \in \Omega\), by Kolmogorov’s theorem, we define on \(M^{\mathbb{R}^+}\), a probability \(\mathbb{P}^0_{x,\omega}\) such that

\[
\mathbb{E}^0_{x,\omega} \left[ \prod_{i=1}^{n} f_i(X^0_{t_i}) \right] = K_{0,t_1}(f_1(K_{t_1,t_2}f_2(\cdots(f_{n-1}K_{t_{n-1},t_n}f_n))))(x),
\]

(2.14)

for all \(f_1, \ldots, f_n\) in \(C(M)\), \(0 < t_1 < t_2 < \cdots < t_n\).

With \(\mathbb{P}\) and \(\mathbb{P}^0_{x,\omega}\), we construct a probability \(\mathbb{P}^0_x(\text{d}\omega, \text{d}\omega') = \mathbb{P}(\text{d}\omega) \otimes \mathbb{P}^0_{x,\omega}(\text{d}\omega')\) on \(\Omega \times M^{\mathbb{R}^+}\). Then, on the probability space \((\Omega \times M^{\mathbb{R}^+}, \mathcal{A} \otimes \mathcal{B}(M) \otimes \mathbb{R}^+, \mathbb{P}^0_x)\), the random process \((X^0_t, t \geq 0)\), defined by \(X^0_t(\omega, \omega') = \omega'(t)\), is a Markov process starting at \(x\) with semigroup \(\mathbb{P}^{(1)}_t\) since

\[
\mathbb{E}^0_x \left[ \prod_{i=1}^{n} f_i(X^0_{t_i}) \right] = \mathbb{P}^{(1)}_{t_1}(f_1(\mathbb{P}^{(1)}_{t_2-t_1}f_2(\cdots(f_{n-1}\mathbb{P}^{(1)}_{t_{n-1}-t_{n-2}}f_{n-2})(\cdots(f_{n-1}K_{t_{n-1},t_n}f_n)))))(x),
\]

(2.15)

for all \(f_1, \ldots, f_n\) in \(C(M)\), \(0 < t_1 < t_2 < \cdots < t_n\).

Therefore, there is a c càdlàg (or continuous when \(\mathbb{P}^{(1)}_t\) is the semigroup of a continuous Markov process) modification \(X = (X_t, t \geq 0)\) of \((X^0_t, t \geq 0)\). Let now \(\mathbb{P}_{x,\omega}\) be the law of \(X\) knowing \(\mathcal{A}\). It is a law on \(D(\mathbb{R}^+, M)\), the space of c càdlàg functions (or \(C(\mathbb{R}^+, M)\) when \(\mathbb{P}^{(1)}_t\) is the semigroup of a continuous Markov process). Equipped with the Skorohod topology (see [29] or [3]), \(D(\mathbb{R}^+, M)\) becomes a Polish space (respectively \(C(\mathbb{R}^+, M)\) is equipped with the topology of uniform convergence on every compact on \(\mathbb{R}^+\)).

On the probability space \((\Omega \times D(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(D(\mathbb{R}^+, M)), \mathbb{P}_x)\) (respectively on \((\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), \mathbb{P}_x)\)), where \(\mathbb{P}_x(\text{d}\omega, \text{d}\omega') = \mathbb{P}(\text{d}\omega) \otimes \mathbb{P}_{x,\omega}(\text{d}\omega')\), let \(X\) be the random process \(X(\omega, \omega') = \omega'.\) Then \(X\) is a c càdlàg (respectively continuous) process and

\[
\mathbb{E}_x \left[ \prod_{i=1}^{n} f_i(X_{t_i}) | \mathcal{A} \right] = \mathbb{E}_{x,\omega} \left[ \prod_{i=1}^{n} f_i(X_{t_i}) \right] = K_{0,t_1}(f_1(K_{t_1,t_2}f_2(\cdots(f_{n-1}K_{t_{n-1},t_n}f_n))))(x),
\]

(2.16)

where \(\mathbb{E}_x\) denotes the expectation with respect to \(\mathbb{P}_x\).

Let \((K'_{s,t}, s \leq t)\) be the stochastic flow of kernels defined on \((\Omega, \mathcal{A}, \mathbb{P})\) by

\[
K'_{s,t}f(x, \omega) = K'_{0,t-s}f(x, T_s\omega)
\]

(2.17)

where

\[
K'_{0,t}f(x) = \mathbb{E}_x[f(X_t) | \mathcal{A}] = \int f(X_t(\omega, \omega')) \mathbb{P}_{x,\omega}(\text{d}\omega')
\]

(2.18)

for \(f \in C(M), x \in M\). Then \((K'_{s,t}, s \leq t)\) is a càdlàg in \(t\) (respectively continuous in \(t\)) modification of \((K_{s,t}, s \leq t)\).
Remark 2.6.1. The concept of sampling will be used in section 3.4.

Replacing $K_{0,t}$ by $K_{0,t}^{\otimes n}$ and $P_t^{(1)}$ by $P_t^{(n)}$ in the above, we obtain a random process $X^{(n)}$ in $M^n$ which represents an $n$-sampling of the flow. The coordinates of $X^{(n)}$ are independent given the flow $K$.

Let $(x_t)_{t \geq 1}$ be a sequence in $M$. For $\omega \in \Omega$, let $P_{x_1,...,x_n,\omega} = \otimes_{i=1}^{n} P_{x_i,\omega}$, $P_{(x_t)_{t \geq 1},\omega} = \otimes_{i=1}^{n} P_{x_i,\omega}$, $P_{x_1,...,x_n}(d\omega, d\omega_1',...,d\omega_n') = P(d\omega) \otimes P_{x_1,...,x_n,\omega}(d\omega_1',...,d\omega_n')$ and $P_{(x_t)_{t \geq 1}}(d\omega, d\omega') = P(d\omega) \otimes P_{(x_t)_{t \geq 1},\omega}(d\omega')$. Then the process $X^{(n)}(\omega, \omega') = (\omega_1',...,\omega_n')$ defines an $n$-sampling of the flow (under $P_{x_1,...,x_n}$ or $P_{(x_t)_{t \geq 1}}$). Let $X_t^i(\omega, \omega') = \omega_i'$. Then, under $P_{(x_t)_{t \geq 1}}$, the sequence $(X_t^i)_{i \geq 1}$ are independent conditionaly to $A$. Moreover, if for all $i \geq 1$, $x_i = x$, this sequence is identically distributed and the law of large numbers implies that for all $f \in C_0(M)$,

$$\frac{1}{n} \sum_{i=1}^{n} f(X_t^i)$$

converges a.s. towards $E_x[f(X_t^i)|A] = K_{0,t} f(x)$.

Since, under $P_{(x_t)_{t \geq 1}}$, $X^{(n)}$ is equal in law to the $n$-point motion of $K$ starting from $(x_1,...,x_n)$, if for all $n \geq 1$, we let $X^{(n)}$ denote the $n$-point motion starting from $(x,...,x)$, we have that $\frac{1}{n} \sum_{i=1}^{n} f(X_t^i)$ converges in law towards $K_{0,t} f(x)$ for all $f \in C^0(M^n)$. This gives an intuitive way to recover $K_{0,t}(x)$ out of the $n$-point motions.

3 Noise and stochastic flows.

3.1 The noise generated by a stochastic flow of kernels.

The definition of a noise we give here is very close to the one given by Tsirelson in [41].

Definition 3.1.1. A noise consists of a separable probability space $(\Omega, \mathcal{A}, P)$, a one-parametric group $(T_h)_{h \in \mathbb{R}}$ of $P$-preserving $L^2$-continuous transformations of $\Omega$ and a family $\{\mathcal{F}_{s,t}, -\infty \leq s \leq t \leq \infty\}$ of sub-$\sigma$-fields of $\mathcal{A}$ such that

(a) $T_h$ sends $\mathcal{F}_{s,t}$ onto $\mathcal{F}_{s+t,h+t}f$ for all $h \in \mathbb{R}$ and all $s \leq t$,

(b) $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for all $s \leq t \leq u$,

(c) $\mathcal{F}_{s,t} \lor \mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ for all $s \leq t \leq u$.

Moreover, we will assume that, for all $s \leq t$, $\mathcal{F}_{s,t}$ contains all $P$-negligible sets of $\mathcal{F}_{-\infty,\infty}$, denoted $\mathcal{F}$.

In the following, $(\Omega^0, \mathcal{A}^0, P_\nu)$ denotes the canonical probability space of a stochastic flow of kernels on $M$, a locally compact separable metric space, associated with a Feller convolution semigroup $\nu$. And $K^0 = (K^0_{s,t}, s \leq t)$ denotes this canonical flow. When this stochastic flow is induced by a flow of maps, one can take for $(\Omega^0, \mathcal{A}^0, P_\nu)$, the canonical probability space associated to this stochastic flow of mappings.

For all $-\infty \leq s \leq t \leq \infty$, let $\mathcal{F}_{s,t}^\nu$ be the sub-$\sigma$-field of $\mathcal{A}^0$ generated by the random variables $K^0_{u,v}$ for all $s \leq u \leq v \leq t$ completed by all $P_\nu$-negligible sets of $\mathcal{A}^0$. Then the cocycle property of $K^0$ implies that $N_\nu := (\Omega^0, \mathcal{A}^0, (\mathcal{F}_{s,t}^\nu)_{s \leq t}, P_\nu, (T_h)_{h \in \mathbb{R}})$ is a noise ($T_h$ is $L^2$-continuous because of the Feller property). We call it the noise generated by the canonical flow $K^0$. 

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Definition 3.1.2. Let \( \nu \) be a Feller convolution semigroup, \( N = (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s \leq t}, P, (T_h)_{h \in \mathbb{R}}) \) be a noise and \( K \) be a measurable stochastic flow of kernels of law \( P_{\nu} \) defined on \( (\Omega, \mathcal{A}, P) \) such that for all \( s < t \), \( K_{s,t} \) is \( \mathcal{F}_{s,t} \)-measurable and for all \( h \in \mathbb{R} \),

\[
K_{s+h,t+h} = K_{s,t} \circ T_h, \quad \text{a.s.}
\] (3.1)

We will call \((N, K)\) an extension of the noise \( N_{\nu} \).

Let \((N_1, K_1)\) and \((N_2, K_2)\) be two extensions of the noise \( N_{\nu} \). Let \( \Omega = \Omega_1 \times \Omega_2 \), \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) and \( P \) be the probability measure on \( (\Omega, \mathcal{A}) \) defined by

\[
E[Z] = \int E_1[Z_1|K_1 = K']E_2[Z_2|K_2 = K'] \, dP_{\nu}(dK),
\] (3.2)

for any bounded random variable \( Z(\omega_1, \omega_2) = Z_1(\omega_1)Z_2(\omega_2) \). Let \((T_h)_{h \in \mathbb{R}}\) be the one-parametric group of \( P \)-preserving transformations of \( \Omega \) defined by \( T_h(\omega_1, \omega_2) = (T_h(\omega_1), T_h(\omega_2)) \). For all \( s < t \), let \( \mathcal{F}_{s,t} = \mathcal{F}_{s,t}^1 \otimes \mathcal{F}_{s,t}^2 \). Then \( N := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s \leq t}, P, (T_h)_{h \in \mathbb{R}}) \) is a noise. And if \( K \) denotes the random variable \( K(\omega_1, \omega_2) = K_1(\omega_1) = K_2(\omega_2) \) \( P \)-a.s., then \((N, K)\) is an extension of \( N_{\nu} \). We will call \((N, K)\) the product of the extensions \((N_1, K_1)\) and \((N_2, K_2)\). Note that \( N_1 \) and \( N_2 \) are isomorphic to sub-noises of \( N \).

### 3.2 Filtering kernels

Let \( K \) be a random kernel defined on a probability space \( (\Omega, \mathcal{A}, P) \) and let \( \bar{\mathcal{A}} \) be a sub-\( \sigma \)-field. Denote by \( \nu \) the law of \( K \) and set \( Q = T^*(\nu) \). Then \( Q \) is a law on \((\Gamma, \mathcal{G})\).

**Lemma 3.2.1.** Suppose that

\[
\lim_{\rho(\mu_1, \mu_2) \to 0} E[\rho(\mu_1 K, \mu_2 K)^2] = 0.
\] (3.3)

Then \( \nu \) is regular and there is \( \bar{K} \) an \( \bar{\mathcal{A}} \)-measurable random kernel, with law denoted by \( \bar{\nu} \), such that

(i) For all \( \mu \in \mathcal{P}(M) \), \( \mu \bar{K} = E[\mu K | \bar{\mathcal{A}}] \);

(ii) \( \bar{\nu} \) is regular;

(iii) Let \( p \) and \( \bar{p} \) be measurable presentations respectively of \( \nu \) and of \( \bar{\nu} \). Let \( \Lambda \) be a \( \mathcal{P}(M) \)-valued random variable and \( \bar{\mathcal{A}}' \) be another sub-\( \sigma \)-field of \( \mathcal{A} \). Suppose that \( \sigma(\Lambda) \lor \bar{\mathcal{A}}' \) and \( \sigma(K) \lor \bar{\mathcal{A}} \) are independent. Then, if \( \bar{\Lambda} = E[\Lambda | \bar{\mathcal{A}}] \),

\[
\bar{\Lambda} \bar{p}(\bar{K}) = E[\Lambda p(K) | \bar{\mathcal{A}}' \lor \bar{\mathcal{A}}].
\] (3.4)

**Proof.** In the proof of this lemma, we will use regular probability measures: Let \( Y \) be a random variable taking its values in a Borel space \( S \) equipped with its Borel \( \sigma \)-field \( \mathcal{S} \). Applying theorems 6.3 and 6.4 in [18], there is a regular probability measure \( \nu : \Omega \times S \to [0, 1] \) such that
(i) \( \nu(\omega, \cdot) \) is a probability measure on \( S \);

(ii) \( \nu(\cdot, A) \) is \( \mathcal{A} \)-measurable for all \( A \in S \);

and such that for all bounded measurable function \( g \), and \( \Lambda \) an \( \mathcal{A} \)-measurable random variable,

\[
E[g(\Lambda, Y)|\mathcal{A}] = \int_S g(\Lambda(\omega), y)\nu(\omega, dy).
\]

In other words, \( \nu \) is the conditional law of \( Y \) given \( \mathcal{G} \). (Regular conditional distributions were actually implicitly used in section 2.6).

In general, if \( Y \) is a \( \mathcal{P}(M) \)-valued random variable of law \( P_Y \), \( f \) an element of \( C_c(M) \), \( \varphi(f) = E[Yf] \) is a positive linear form on \( C_c(M) \). By the Riesz-Markov-Kakutani representation theorem there is \( \mu \in \mathcal{P}(M) \) such that \( E[Yf] = \mu f \). Then \( \mu \) can be denoted \( \int_{\mathcal{P}(M)} yP_Y(dy) \).

Using regular probability measures as above, one can define \( E[Y|\mathcal{A}] = \int_{\mathcal{P}(M)} y\nu(\cdot, dy) \), with \( \nu \) the conditional law of \( Y \) given \( \mathcal{A} \).

We now prove the lemma. Let us first show that \( \nu \) is regular, i.e. that \( Q \) is regular. Let \( (\mu_k, k \in \mathbb{N}) \) be a dense sequence in \( \mathcal{P}(M) \) and let \( j : \Gamma \rightarrow \mathcal{P}(M)^{\otimes \mathbb{N}} \) be the measurable function defined by \( j(\Phi) = (\Phi(\mu_k), k \in \mathbb{N}) \). Following section 1, (3.3) allows to construct \( i : \mathcal{P}(M)^{\otimes \mathbb{N}} \rightarrow \Gamma \) a measurable function such that \( J = i \circ j \) is a measurable presentation of \( Q \). This shows that \( Q \) and therefore \( \nu \) are regular. With this construction, as is shown in section 2, \( \delta \circ J \) is measurable, \( (\delta \circ J)^*(Q) = \nu \) and \( p = \delta \circ J \circ I \) is a measurable presentation of \( \nu \).

Set \( P^{(\infty)} = j^*(Q) \). Then \( i^*(P^{(\infty)}) = Q \) and \( (\delta \circ i)^*(P^{(\infty)}) = \nu \) (indeed \( \delta \circ i = (\delta \circ J) \circ i \) is measurable and \( (\delta \circ i)^*(P^{(\infty)}) = (\delta \circ J \circ i)^*(P^{(\infty)}) = (\delta \circ J)^*(Q) = \nu ) \).

Set \( Y = j \circ I(K) \). Then \( Y \) is a random variable of law \( P^{(\infty)} \). Since \( M^{\mathbb{N}} \) is a Polish space, one can define \( \bar{P}^{(\infty)}(\omega, dy) \) the conditional law of \( Y \) given \( \mathcal{A} \). In particular for all bounded measurable function \( g \) and \( \Lambda \) an \( \mathcal{A} \)-measurable random variable,

\[
E[g(\Lambda, Y)|\mathcal{A}] = \int_{\mathcal{P}(M)^{\mathbb{N}}} g(\Lambda(\omega), y)\bar{P}^{(\infty)}(\omega, dy).
\]

Therefore, if \( K' = p(K) = \delta \circ i(Y) \), we have for all bounded measurable function \( g \) and \( \Lambda \) an \( \mathcal{A} \)-measurable random variable,

\[
E[g(\Lambda, p(K))|\mathcal{A}] = \int_{\mathcal{P}(M)^{\mathbb{N}}} g(\Lambda(\omega), (\delta \circ i)(y))\bar{P}^{(\infty)}(\omega, dy).
\]

For all \( \omega \in \Omega \), define \( \tilde{\nu}(\omega, \cdot) = (\delta \circ i)^*(\bar{P}^{(\infty)}(\omega, \cdot)) \). Then \( \tilde{\nu}(\omega, dk) \) is a regular probability measure and we have for all bounded measurable function \( g \) and \( \Lambda \) an \( \mathcal{A} \)-measurable random variable,

\[
E[g(\Lambda, p(K'))|\mathcal{A}] = \int_{E} g(\Lambda(\omega), k)\tilde{\nu}(\omega, dk).
\]

For \( x \in M \), set \( \bar{K}(x) = \int_{E} p(k)(x)\tilde{\nu}(\omega, dk) \).
Then $K$ is a random kernel is $\mathcal{A}$-measurable and a.s. $K(x) = \int_E k(x) \nu(\omega, dk)$. Denote by $\tilde{\nu}$ the law of $K$ and set $\bar{Q} = \mathcal{I}^*(\tilde{\nu})$. Let us now show (i), (ii) and (iii) are satisfied. For $\mu \in \mathcal{P}(M)$, a.s. $\mu p(K) = \mu K$ and therefore, a.s.

$$E[\mu K|\bar{A}] = E[\mu p(K)|\bar{A}] = \int_E \mu k \tilde{\nu}(\omega, dk) = \mu K$$

where in the last equality we have used Fubini’s theorem. This proves (i).

To prove (ii), observe that for $f \in C(M)$ and $(\mu_1, \mu_2) \in \mathcal{P}(M)^2$,

$$E[(\mu_2 K f - \mu_1 K f)^2] = E\left[E[\mu_2 K f - \mu_1 K f|\bar{A}]^2\right] \leq E\left[(\mu_2 K f - \mu_1 K f)^2\right].$$

This implies that $\tilde{\nu}$ satisfies $[3.3]$ and as a consequence, by construction, that $\mathcal{J}$ and $p$ are also measurable presentations respectively of $\bar{Q}$ and of $\tilde{\nu}$.

Let us finally prove (iii). Note first that item (i) implies that (3.4) holds if $\Lambda$ is not random (indeed, for all $\mu \in \mathcal{P}(M)$, a.s. $\mu K = \mu p(K)$ and $\mu K = \mu p(K)$). Let now $\bar{Z}'$ and $\bar{Z}$ be bounded random variables, respectively $\mathcal{A}'$-measurable and $\mathcal{A}$-measurable. Then (using that $\sigma(\Lambda) \vee \mathcal{A}'$ and $\sigma(K) \vee \mathcal{A}$ are independent and using the notation $P_{\bar{Z}}$ for the law of a random variable $\bar{Z}$),

$$E[\Lambda p(K) \bar{Z}' \bar{Z}] = \int_{\mathcal{P}(M) \times \mathbb{R}} E[\mu p(K) \bar{Z}] \bar{Z}' P_{(\Lambda, \bar{Z})}(d\mu, d\bar{Z}')
= \int_{\mathcal{P}(M) \times \mathbb{R}} E[\mu p(K) \bar{Z}] \bar{Z}' P_{(\Lambda, \bar{Z})}(d\mu, d\bar{Z}')
= E[\Lambda p(K) \bar{Z}' \bar{Z}]
= \int_{E \times \mathbb{R}} E[\Lambda \tilde{p}(k) \bar{Z}'] \bar{Z} P_{(\bar{Z}, \bar{Z})}(d\bar{k}, d\bar{z})
= \int_{E \times \mathbb{R}} E[\Lambda \tilde{p}(k) \bar{Z}'] \bar{Z} P_{(\bar{Z}, \bar{Z})}(d\bar{k}, d\bar{z})
= E[\Lambda \tilde{p}(K) \bar{Z}' \bar{Z}]$$

This implies (iii). □

### 3.3 Filtering by a sub-noise.

Let $\bar{N}$ be a sub-noise of an extension $(N, K)$ of $N$, i.e. $\bar{N}$ is a noise $(\Omega, \mathcal{A}, (\bar{F}_{s,t})_{s \leq t}, P, (T_h)_{h \in \mathbb{R}})$ such that $\bar{F}_{s,t} \subset F_{s,t}$ for all $s \leq t$.

**Remark 3.3.1.** A sub-noise is characterized by $\bar{F}_{-\infty,\infty}$, denoted $\bar{F}$. This $\sigma$-field has to be stable under $T_h$, to contain all $\mathcal{P}$-negligible sets of $\mathcal{F}$, and be such that $\mathcal{F} = (\mathcal{F} \cap \mathcal{F}_{-\infty,0}) \vee (\bar{F} \cap \mathcal{F}_{0,\infty})$.

For all $n \geq 1$, let $\bar{P}^{(n)}_t$ be the operator acting on $C(M^n)$ defined by

$$\bar{P}^{(n)}_t(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = E \left[ \prod_{i=1}^n E[K_{0,t} f_i(x_i)|\bar{F}_{0,t}] \right], \quad (3.5)$$

for all $x_1, \ldots, x_n$ in $M$ and all $f_1, \ldots, f_n$ in $C(M)$.
Lemma 3.3.2. The family \((\bar{P}^{(n)}_t, n \geq 1)\) is a compatible family of Feller semigroups.

**Proof.** The semigroup property of \(\bar{P}^{(n)}_t\) follows directly from the independence of the increments of the flow. The Markovian property and in particular the positivity property holds since for all \(h \in C(M^n),\)

\[
\bar{P}^{(n)}_t h(x_1, \ldots, x_n) = E[(h, \otimes^{n}_{i=1} E[K_{0,t}(x_i)|\mathcal{F}_{0,t})].
\]  

(3.6)

From this, it is clear that \((\bar{P}^{(n)}_t, n \geq 1)\) is a compatible family of Markovian semigroups respectively acting on \(C(M^n)\).

It remains to prove the Feller property. For all continuous functions \(f_1, \ldots, f_n, h = f_1 \otimes \cdots \otimes f_n, x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) in \(M^n\), for \(M\) large enough,

\[
|\bar{P}^{(n)}_t h(x) - \bar{P}^{(n)}_t h(y)| \leq M \sum_{i=1}^{n} E[(E[|K_{0,t} f_i(x_i) - K_{0,t} f_i(y_i)|\mathcal{F}_{0,t})^2]^{\frac{1}{2}}
\]

\[
\leq M \sum_{i=1}^{n} E[(K_{0,t} f_i(x_i) - K_{0,t} f_i(y_i))^2]^{\frac{1}{2}}
\]  

(3.7)

which converges towards 0 as \(y\) tends to \(x\) since \((e)\) in definition 2.3.1 is satisfied.

We also have, for all \(h = f_1 \otimes \cdots \otimes f_n\) and \(x = (x_1, \ldots, x_n)\) in \(M^n\), for \(M\) large enough,

\[
|\bar{P}^{(n)}_t h(x) - h(x)| \leq M \sum_{i=1}^{n} E[(E[|K_{0,t} f_i(x_i) - f_i(x_i)|\mathcal{F}_{0,t})^2]^{\frac{1}{2}}
\]

\[
\leq M \sum_{i=1}^{n} E[(K_{0,t} f_i(x_i) - f_i(x_i))^2]^{\frac{1}{2}}
\]  

(3.8)

which converges towards 0 as \(t\) tends to 0 since \((d)\) in definition 2.3.1 is satisfied. Hence, for all function \(h \in C(M^n)\) such that \(h\) is a linear combination of functions of the type \(f_1 \otimes \cdots \otimes f_n\), we have \(\bar{P}^{(n)}_t h\) is continuous and \(\lim_{t \to 0} \bar{P}^{(n)}_t h(x) = h(x)\) for all \(x \in M^n\). This extends to all functions \(h \in C(M^n)\). \(\Box\)

Let us denote by \(\bar{\nu} = (\bar{\nu}_t)_{t \geq 0}\) the Feller convolution semigroup on \((E, \mathcal{E})\) associated with \((\bar{P}^{(n)}_t, n \geq 1)\). Note that the one-point motion of \(\nu\) and \(\bar{\nu}\) are the same, i.e. \(\bar{P}^{(1)}_t = P^{(1)}_t\).

Lemma 3.3.3. Let \((N, K)\) be an extension of \(N\) and \(\bar{N}\) be a sub-noise of \(N\). Then there exists \(K = (\bar{K}_{s,t}, s \leq t)\) a stochastic flow of kernels of law \(P_\nu\) such that \((\bar{N}, \bar{K})\) is an extension of \(N_\nu\) and

\[
\bar{K}_{s,t} f(x) = E[K_{s,t} f(x)|\mathcal{F}_{s,t}] = E[K_{s,t} f(x)|\bar{F}_s]  \quad P - a.s.
\]

(3.9)

for all \(s \leq t, x \in M\) and \(f \in C_0(M)\). We say \(\bar{K}\) is obtained by filtering \(K\) with respect to \(\bar{N}\).

**Proof.** For \(s \leq t, K_{s,t}\) satisfies \((3.3)\) and there is \(\bar{K}_{s,t}\), the random kernel obtained by filtering \(K_{s,t}\) with respect to \(\mathcal{F}_{s,t}\) (Remark: one can first define \(\bar{K}_{0,t}\) for \(t \geq 0\) and set \(\bar{K}_{s,t} = \bar{K}_{0,t-s} \circ T_s\) in order to ensure that \(\bar{K}_{s,t} \circ T_h = \bar{K}_{s+h,t+h}\)). For all \(s \leq t\), the law of \(\bar{K}_{s,t}\) is \(\bar{\nu}_{t-s}\) and \(K\) is
Therefore, if moreover \( \nu \) since \( \sigma(\mu K_{s,t}) \vee \tilde{F}_{s,u}(\subset F_{s,u}) \) and \( \sigma(\mu K_{u,t}) \vee \tilde{F}_{u,t}(\subset F_{u,t}) \) are independent, one can apply (iii) of lemma \( \ref{lema:3.2.1} \) and easily obtain (a). \( \square \)

**Definition 3.3.4.** Given two Feller convolution semigroups on \((E, \mathcal{E}), \nu^1 \) and \( \nu^2 \), we say that \( \nu^1 \) dominates (respectively weakly dominates) \( \nu^2 \), denoted \( \nu^1 \preceq \nu^2 \) (respectively \( \nu^1 \preceq w \nu^2 \)), if there exists a sub-noise of \( N_{\nu^1} \) (respectively of an extension \((N^1, K^1)\) of \( N_{\nu^1} \)) such that \( P_{\nu^2} \) is the law of the flow obtained by filtering the canonical flow of law \( P_{\nu^1} \) (respectively by filtering \( K^1 \)) with respect to this sub-noise.

Notice that in lemma \( \ref{lema:3.3.3} \) \( \nu \) weakly dominates \( \bar{\nu} \) and \( \nu \) dominates \( \bar{\nu} \) if \( \bar{N} \) is a sub-noise of \( N_{\nu} \). Note that the domination relation is in fact an extension of the notion of barycenter.

**Lemma 3.3.5.** Let \( \nu \) and \( \bar{\nu} \) be two Feller convolution semigroups such that \( \nu \) dominates \( \bar{\nu} \). Let \( (N, K) \) be an extension of \( N_{\nu} \). Let \( \bar{N}_{\nu} \) be the sub-noise (isomorphic to \( N_{\nu} \)) of \( N \) generated by \( K \). Then there exists a sub-noise \( \bar{N} \) of \( \bar{N}_{\nu} \) such that \( P_\nu \) is the law of the flow obtained by filtering \( K \) with respect to \( \bar{N} \).

**Proof.** Let \( N_{\nu} := (\Omega^0, \mathcal{A}^0, (\mathcal{F}^\nu_{s,t})_{s\leq t}, P_{\nu}, (T_h)_{h \in \mathbb{R}}) \) be the noise generated by the canonical flow associated with \( \nu \). Notice that \( \nu \succeq \bar{\nu} \) means the existence of \( \bar{N}^0 \) a sub-noise of \( N_{\nu} \) such that \( P_\bar{\nu} \) is the law of \( \tilde{K}^0 \), the flow obtained by filtering the canonical flow of law \( P_\nu \) with respect to \( \bar{N}^0 \).

Note that the mapping \( K : (\Omega, \mathcal{A}) \to (\Omega^0, \mathcal{A}^0) \) is measurable. Let \( \mathcal{F} \) be the completion of \( K^{-1}(\mathcal{F}^0) \) by all \( P \)-negligible sets of \( \mathcal{A} \) and, for all \( s \leq t \), set \( \mathcal{F}_{s,t} = \mathcal{F} \cap \mathcal{F}_{s,t} \). Then \( \bar{N} = (\Omega, \mathcal{A}, (\mathcal{F}^\nu_{s,t})_{s\leq t}, P, (T_h)_{h \in \mathbb{R}}) \) is a sub-noise of \( N \). Lemma \( \ref{lema:3.3.3} \) allows us to define \( \bar{K} \) the flow obtained by filtering \( K \) with respect to \( \bar{N} \). One can check that \( \bar{K} = \tilde{K}^0(K) \). This implies that the law of \( \bar{K} \) is \( P_\bar{\nu} \). Thus the proposition is proved. \( \square \)

**Proposition 3.3.6.** The domination relation and the weak domination relation are partial orders on the class of Feller convolution semigroups.

**Proof.** 1) The transitivity of the domination relation follows from lemma \( \ref{lema:3.3.5} \) by the chain rule for conditional expectations.

Let us observe that if \( \nu^1 \preceq \nu^2 \) and \( \nu^2 \preceq \nu^1 \) then \( \nu^1 = \nu^2 \). Indeed, if \( \nu^1 \succeq \nu^2 \), Jensen’s inequality shows that for all \( f_1, \ldots, f_n \) in \( C(M) \), \( x_1, \ldots, x_n \) in \( M \) and \( t \geq 0 \),

\[
E_{\nu^1} \left[ \exp \left( \sum_{i=1}^{n} K_{0,t} f_i(x_i) \right) \right] \geq E_{\nu^2} \left[ \exp \left( \sum_{i=1}^{n} K_{0,t} f_i(x_i) \right) \right]. \tag{3.10}
\]

Therefore, if moreover \( \nu^1 \preceq \nu^2 \), the preceding inequality becomes an equality. This proves \( \nu^1 = \nu^2 \).
For the weak domination relation, the proof is similar. We prove the transitivity using the product of extensions. Indeed, if \( \bar{\nu} \preceq w \nu \), given any extension \((N^1, K^1)\) of \( N_\nu \), there exist a larger extension \((N, K)\) and a subnoise \( \bar{N} \) of \( N \) such that \( \bar{K} \) has law \( P_{\bar{\nu}} \); let \( \bar{N}^2 \) be a sub-noise of an extension \((N^2, K^2)\) of \( N_{\nu} \) such that \( \bar{K}^2 \) has law \( P_{\bar{\nu}} \). Then \((N, K)\) is taken as the product of the extensions \((N^1, K^1)\) and \((N^2, K^2)\), and \( \bar{N} \) is induced by \( \bar{N}^2 \).

Remark 3.3.7. The concept of filtering will be used in sections 4.3, 5.5, 6.2 and an example is given in the following section.

3.4 An example of filtering.

Let \( M = \{0, 1\} \). Then \( F \), the set of maps from \( \{0, 1\} \) on \( \{0, 1\} \) is constituted of the maps \( \sigma, I, f_0 \) and \( f_1 \), with \( I \) the identity, \( \sigma(0) = 1 \), \( \sigma(1) = 0 \), \( f_0 = 0 \) and \( f_1 = 1 \). Let \( (N_t) \) be a Poisson process on \( \mathbb{R} \) and \( (\varphi_n)_{n \in \mathbb{Z}} \) be a sequence, independent of the Poisson process, of independent random variables taking their values in \( F \) with law 

\[
\frac{1}{4}(\delta_{f_0} + \delta_{f_1} + \delta_I + \delta_\sigma).
\]

We then define a stochastic flow of mappings on \( \{0, 1\} \) by

\[
\begin{align*}
\varphi_{s,t} &= I & & \text{if } N_t - N_s = 0 \\
\varphi_{s,t} &= \varphi_{N_t-1} \circ \cdots \circ \varphi_{N_s} & & \text{if } N_t - N_s > 0
\end{align*}
\]

for all \( s \leq t \). Note that \( \varphi \) is a coalescing flow since for all \( s \), there is a.s. a finite time \( T \) such that for all \( t \geq T \), \( \varphi_{s,t}(0) = \varphi_{s,t}(1) \). The one-point motion of this flow is given by the symmetric random walk with generator \( A^{(1)} \) given by

\[
A^{(1)} = \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2
\end{pmatrix}.
\]

Note also that, since the \( \{0, 1\} \) has only two points, the \( n \)-point motions associated with this stochastic flow of mappings are determined by the two-point motion. The generator \( A^{(2)} \) of the two-point motion is (the state space is \( \{(0,0), (1,1), (0,1), (1,0)\} \))

\[
A^{(2)} = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & 1/4 & 1/4
\end{pmatrix}.
\]

With the stochastic flow \( \varphi \) and an independent sequence of random variables \( (Y_n)_{n \in \mathbb{Z}} \) with \( P[Y_n = 1] = p = 1 - P[Y_n = 0] \), we define a stochastic flow of kernels \( K \), by

\[
\begin{align*}
K_{s,t}(i) &= \delta_i & & \text{if } N_t - N_s = 0 \\
K_{s,t} &= K_{N_t-1} \cdots K_{N_s} & & \text{if } N_t - N_s > 0
\end{align*}
\]

where \( K_n = Y_n \delta_{\varphi_n} + (1 - Y_n)\frac{1}{2}(\delta_0 + \delta_1) \).
Denote by $N^c$ the noise of $\varphi$, by $N$ the noise of $K$ and by $\hat{N}$ the noise of $(\varphi, Y)$. Then $N^c$ is the noise of $(N_t, \varphi_{N_t})$, $\hat{N}$ is the noise of $(N_t, \varphi_{N_t}, Y_{N_t})$ and $N$ is the noise of $(N_t, K_{N_t})$. The noises $N^c$ and $N$ are subnoises of $\hat{N}$. And $N$ cannot be isomorphic to a subnoise of $N^c$.

Indeed for $\varepsilon M$ in this form are dense in $L^F$. Then we can rewrite modification in $t$ with $\varepsilon F$ paths, then all martingales of $M$.

Let $(\varphi_{s,t}, s \leq t)$ be a stochastic flow of kernels. For all $s \leq t$ set $\mathcal{F}_{s,t} = \sigma(K_{u,v}, s \leq u \leq v \leq t)$. Let $\mathcal{F}$ be the filtration $(\mathcal{F}_{0,t})_{t \geq 0}$. Let $\mathcal{M}(\mathcal{F})$ be the space of locally square integrable $\mathcal{F}$-martingales.

**Proposition 3.5.1.** Suppose that $\mathcal{P}_t$ is the semigroup of a Markov process with continuous paths, then all martingales of $\mathcal{M}(\mathcal{F})$ are continuous.

**Proof.** Let $M \in \mathcal{M}(\mathcal{F})$ be a martingale in the form $E[F|\mathcal{F}_{0,t}]$ where $F = \prod_{i=1}^n K_{s_i,t_i} f_i(x_i)$, with $f_1, \ldots, f_n$ in $C(M)$, $x_1, \ldots, x_n$ in $M$ and $0 \leq s_i < t_i$ (we take here the continuous modification in $t$ of the stochastic flow of kernels). By definition of the filtration, functions in this form are dense in $L^2(\mathcal{F}_{0,\infty})$. This implies that martingales of this form are dense in $\mathcal{M}(\mathcal{F})$. Since the space of continuous martingales is closed in $\mathcal{M}(\mathcal{F})$, it is enough to prove the continuity of these martingales.

For all $t$, let $\tilde{K}_t$ be the kernel defined on $\mathbb{R}^+ \times M$ by

$$\tilde{K}_t(s, x) = \begin{cases} 
\delta_{s-t} \otimes \delta_x & \text{for } s \geq t, \\
\delta_0 \otimes K_{s,t}(x) & \text{for } s \leq t.
\end{cases}$$

(3.11)

Then we can rewrite $F$ in the form $\prod_{i=1}^n \tilde{K}_t f_i(s_i, x_i)$, where $\tilde{f}_i(s, x) = f_i(x)$.

Note that $(\tilde{K}_t(s_i, x_i), 1 \leq i \leq n)$ is a Markov process on $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{P}(M))^n$. This Markov process is continuous and Feller (the Feller property follows from the Feller property of the semigroups $(\Pi_t^{(k)}, k \geq 1)$). It is well known that the martingales relative to the filtration denoted here $(\mathcal{F}_t^{\{s_i, x_i\}_{1 \leq i \leq n}}, t \geq 0)$ generated by such a process are continuous (see [39] tome II).

This proves that $E[F|\mathcal{F}_t^{\{s_i, x_i\}_{1 \leq i \leq n}}]$ is a continuous martingale. We conclude after remarking that $M_t = E[F|\mathcal{F}_t^{\{s_i, x_i\}_{1 \leq i \leq n}}]$, which holds since the $\sigma$-field $\mathcal{F}_t^{\{s_i, x_i\}_{1 \leq i \leq n}}$ is a sub-$\sigma$-field of $\mathcal{F}_t$ and $M_t$ is easily seen to be $\mathcal{F}_t^{\{s_i, x_i\}_{1 \leq i \leq n}}$-measurable. \(\square\)

## 4 Stochastic coalescing flows.

In this section we study stochastic coalescing flows, we denote by $(\varphi_{s,t}, s \leq t)$. In section 4.2 it is shown that for all $s < t$, $\varphi_{s,t}^s(\lambda)$ is atomic (where $\lambda$ denotes any positive Radon measure on $M$). We study this point measure valued process which provides a description of the coalescing flow.
In section 4.3, starting from a compatible family of Feller semigroups, under the hypothesis that starting close to the diagonal the two-point motion hits the diagonal with a probability close to 1, we construct another compatible family of Feller semigroups to which is associated a stochastic coalescing flow. We then show that the stochastic flow of kernels associated with the first family of semigroups can be defined by filtering the stochastic coalescing flow with respect to a sub-noise of an extension of its canonical noise.

Finally, we give three examples. The first one, due to Arratia [2], describes the flow of independent Brownian motions sticking together when they meet. The second one is due to Propp and Wilson [35] in the context of perfect simulation of the invariant distribution of a finite state irreducible Markov chain, their stochastic flows being indexed by the integers. The third one is the construction of a stochastic coalescing flow solution of Tanaka’s SDE

\[ dX_t = \text{sgn}(X_t)dW_t, \quad (4.1) \]

where \( W \) is a real white noise. This coalescing flow was constructed by Watanabe in [46] and Warren in [45]. In [23], a stochastic flow of kernels solution of this SDE was constructed as the only strong solution of this SDE.

4.1 Definition.

Let \( M \) be a locally compact separable metric space.

**Definition 4.1.1.** A stochastic flow of mappings on \( M \), \( (\varphi_{s,t}, s \leq t) \), is called a stochastic coalescing flow if for some \( (x,y) \in M^2 \), \( T_{x,y} = \inf\{ t \geq 0, \varphi_{0,t}(x) = \varphi_{0,t}(y) \} \) is finite with a positive probability and for all \( t \geq T_{x,y} \), \( \varphi_{0,t}(x) = \varphi_{0,t}(y) \). In other words, a pair of points stick together after a finite time with a positive probability.

**Remark 4.1.2.** This definition depends only on the two-point motion.

Let \((P_t^{(n)}, n \geq 1)\) be a compatible family of Feller semigroups. We denote by \( P_t^{(2)}(x,y) \) the law of the Markov process associated with \( P_t^{(2)} \) starting from \((x,y)\) and we denote this process \((X_t, Y_t)\) or \(X_t^{(2)}\). Let \( T_\Delta = \inf\{ t \geq 0, X_t = Y_t \} \).

**Remark 4.1.3.** A compatible family \((P_t^{(n)}, n \geq 1)\) of Feller semigroups defines a stochastic coalescing flow if and only if for all \((x,y) \in M^2\), for all \( t \geq T_\Delta \), \( X_t = Y_t \), \( P_t^{(2)}(x,y)\)-almost surely, and for some \((x,y) \in M^2\), \( P_t^{(2)}(x,y)[T_\Delta < \infty] > 0 \).

4.2 A point measure valued process associated with a stochastic coalescing flow.

In this subsection, we suppose we are given a compatible family of Feller semigroups \((P_t^{(n)}, n \geq 1)\) such that

\[
\begin{align*}
\forall x \in M, \forall t > 0, \quad \lim_{y \to x} P_t^{(2)}(x,y)[X_t \neq Y_t] &= 0, \\
\forall (x,y) \in M^2, \quad P_t^{(2)}(x,y)[T_\Delta < \infty] &= 0.
\end{align*}
\]  

\[(4.2)\]
Lemma 4.2.3. For all positive $y \geq 0$ as $y$ tends to $x$ when $x \neq 0$.

Let $\varphi = (\varphi_{s,t}, s \leq t)$ be a measurable stochastic coalescing flow associated with $(\mathbb{P}_t^{(n)}$, $n \geq 1)$. For all $s < t \in \mathbb{R}$, let $\mu_{s,t} = \varphi^*_{s,t}(\lambda)$, where $\lambda$ is any positive Radon measure on $M$.

Proposition 4.2.2. (a) For all $s < t \in \mathbb{R}$, almost surely, $\mu_{s,t}$ is atomic.

(b) For all $s < u < t \in \mathbb{R}$, almost surely, $\mu_{s,t}$ is absolutely continuous with respect to $\mu_{u,t}$.

Proof. Fix $s < t \in \mathbb{R}$. For all positive $\varepsilon$ and all $x \in M$, let $m_\varepsilon^x = \int_{B(x,\varepsilon)} \varphi_{s,t}(x) \lambda(dy)$ ($m_\varepsilon^x$ is well defined since $(x,\omega) \mapsto \varphi_{s,t}(x,\omega)$ is measurable). For all $\alpha \in ]0,1[$ and $x \in M$, let

$$A^{\alpha,x}_n = \{ m_\varepsilon^x < (1-\alpha)\lambda(B(x,\varepsilon_n^x)) \}, \quad (4.3)$$

where $\varepsilon_n^x$ is a positive sequence such that $d(x,y) \leq \varepsilon_n^x$ implies

$$\mathbb{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}] \leq 2^{-n}.$$

Lemma 4.2.3. For all positive $\alpha$, $x \in M$ and $n \in \mathbb{N}$, $\mathbb{P}(A^{\alpha,x}_n) \leq \frac{1}{\alpha 2^n}$.

Proof. For all integer $n$, we have

$$\mathbb{E}[m_\varepsilon^x] = \int_{B(x,\varepsilon_n^x)} \mathbb{P}^{(2)}_{(x,y)}[X_{t-s} = Y_{t-s}] \lambda(dy)$$

$$= \int_{B(x,\varepsilon_n^x)} (1 - \mathbb{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}]) \lambda(dy)$$

$$\geq (1 - 2^{-n})\lambda(B(x,\varepsilon_n^x)).$$

And we conclude since

$$\mathbb{E}[m_\varepsilon^x] \leq \mathbb{P}(A^{\alpha,x}_n)(1-\alpha)\lambda(B(x,\varepsilon_n^x)) + (1-\mathbb{P}(A^{\alpha,x}_n))\lambda(B(x,\varepsilon_n^x))$$

we use the fact that $m_\varepsilon^x \leq \lambda(B(x,\varepsilon_n^x))). \Box$

Lemma 4.2.4. For all $x \in M$, almost surely, $m_\varepsilon^x \sim \lambda(B(x,\varepsilon_n^x))$ as $n \to \infty$.

Proof. Using Borel-Cantelli’s lemma, for all $\alpha \in ]0,1[$

$$1 - \alpha \leq \lim inf_{n \to \infty} \frac{m_\varepsilon^x}{\lambda(B(x,\varepsilon_n^x))} \leq \lim sup_{n \to \infty} \frac{m_\varepsilon^x}{\lambda(B(x,\varepsilon_n^x))} \leq 1$$

almost surely. This implies $\lim_{n \to \infty} \frac{m_\varepsilon^x}{\lambda(B(x,\varepsilon_n^x))} = 1$ a.s. $\Box$

Since for all $(x,\omega) \in M \times \Omega$,

$$\mu_{s,t}(\{\varphi_{s,t}(x)\}) = \lambda(\{y, \varphi_{s,t}(y) = \varphi_{s,t}(x)\})$$

$$\geq \lambda(\{y \in B(x,\varepsilon_n), \varphi_{s,t}(y) = \varphi_{s,t}(x)\}),$$
Lemma 4.2.4 implies that for all \( x \in M \),
\[
\mu_{s,t}(\{\varphi_{s,t}(x)\}) > 0
\]  
(4.4)
almost surely. Since \( (x, \omega) \mapsto \mu_{s,t}(\{\varphi_{s,t}(x)\}) \) is measurable,
\[
\lambda(dx) \otimes P(d\omega) \text{-a.e., } \quad \mu_{s,t}(\{\varphi_{s,t}(x)\}) > 0.
\]  
(4.5)
This equation implies (since \( \mu_{s,t} = \varphi_{s,t}^*(\lambda) \))
\[
\mu_{s,t}(dy) \text{-a.e., } \quad \mu_{s,t}(\{y\}) > 0
\]  
(4.6)
almost surely. This last equation is one characterization of the atomic nature of \( \mu_{s,t} \) and (a) is proved.

To prove (b), note first that \( \lambda(dx) \otimes P(d\omega) \text{-a.e., } \varphi_{u,t}^*(\delta_x) = \delta_{\varphi_{u,t}(x)} \) is absolutely continuous with respect to \( \varphi_{u,t}^*(\lambda) = \mu_{u,t} \) since (4.4) holds. Note also that \( \lambda(dx) \otimes P(d\omega) \text{-a.e., } \varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x) \). This implies
\[
\mu_{s,t} = \varphi_{u,t}^*(\mu_{s,u}) \quad \text{a.s.}
\]  
(4.7)
Since \( \mu_{s,u} \) is atomic, independent of \( \varphi_{u,t} \) and \( E[\mu_{s,u}] = \lambda \), it follows that \( \mu_{s,t} \) is absolutely continuous with respect to \( \mu_{u,t} \). This proves (b). \( \square \)

**Remark 4.2.5.** \( \bullet \) \((\mu_{s,t}, s \leq t)\) is Markovian in \( t \).

\( \bullet \) Since \( \mu_{s,t} \) is atomic for \( t > s \), there exists a point process \( \xi_{s,t} = \{\xi^i_{s,t}\} \) and weights \( \{\alpha^i_{s,t}\} \in \mathbb{R}^N \) such that \( \mu_{s,t} = \sum_i \alpha^i_{s,t} \delta_{\xi^i_{s,t}} \). The point process \( \xi_{s,t} \) and the marked point process \( (\xi_{s,t}, \alpha_{s,t}) \) are Markovian in \( t \) since for all \( s < u < t \), \( \xi_{s,t} = \varphi_{u,t}(\xi_{s,u}) \) and \( \alpha^i_{s,t} = \sum_j (\xi^j_{s,t} = \varphi_{u,t}(\xi^j_{s,u})) \alpha^j_{s,u} \).

\( \bullet \) Let \( A^i_{s,t} = \varphi_{u,t}^{-1}(\xi^i_{s,t}) \) and \( \Pi_{s,t} \) be the collection of the sets \( A^i_{s,t} \). Note that \( \cup_i A^i_{s,t} = M \) \( \lambda \)-a.e, the union being disjoint. Note also that \( \xi_{s,t} \) and \( \Pi_{s,t} \) determines \( \varphi_{s,t} \) \( \lambda \)-a.e. Note finally that \( \Pi_{s,t} \) is Markovian in \( s \) when \( s \) decreases, since for all \( s < u < t \), \( \Pi_{s,t} = \{\varphi_{s,u}^{-1}(A^i_{u,t})\} \).

This Markov process has also a coalescence property: one can have for \( i \neq j \), \( \varphi_{s,u}^{-1}(A^i_{u,t}) = \varphi_{s,u}^{-1}(A^j_{u,t}) \). When \( s \) decreases, the partition \( \Pi_{s,t} \) becomes coarser.

### 4.3 Construction of a family of coalescent semigroups.

Let \( (P_t^{(n)}, n \geq 1) \) be a compatible family of Feller semigroups on a separable locally compact metric space \( M \) and \( \nu = (\nu_t)_{t \in \mathbb{R}} \) the associated Feller convolution semigroup on \( (E, \mathcal{E}) \). Let \( \Delta_n = \{x \in M^n, \exists i \neq j, x_i = x_j\} \) and \( T_{\Delta_n} = \inf\{t \geq 0, X^{(n)}_t \in \Delta_n\} \), where \( X^{(n)}_t \) denotes the \( n \)-point motion, i.e. the Markov process on \( M^n \) associated with the semigroup \( P_t^{(n)} \). We will denote \( \Delta_2 \) by \( \Delta \).

**Theorem 4.3.1.** There exists a unique compatible family \( (P_t^{(n),c}, n \geq 1) \) of Markovian semigroups on \( M \) such that if \( X^{(n),c} \) is the associated \( n \)-point motion and \( T_{\Delta_n}^c = \inf\{t \geq 0, X^{(n),c}_t \in \Delta_n\} \), then

\( \bullet \) \((X^{(n),c}_t, t \leq T_{\Delta_n}^c)\) is equal in law to \((X^{(n)}_t, t \leq T_{\Delta_n})\).
• for \( t \geq T_{\Delta_n} \), \( X_t^{(n),c} \in \Delta_n \).

Moreover, this family is constituted of Feller semigroups if condition (C) below is satisfied,

(C) For all \( t > 0, \varepsilon > 0 \) and \( x \in M \),

\[
\lim_{y \to x} P^{(2)}_{(x,y)} \{ \{(T_{\Delta} > t) \cap \{d(X_t, Y_t) > \varepsilon\} \} = 0
\]

where \((X_t, Y_t) = X_t^{(2)}\). And for some \(x\) and \(y\) in \(M\), \(P^{(2)}_{(x,y)}[T_{\Delta} < \infty] > 0\).

In this case, \((P_t^{(n),c}, n \geq 1)\) satisfies (1.6) and is associated with a coalescing flow.

**Proof.** For all \( n \geq 1 \), let \( P_n \) be the set of all partitions of \(\{1, \ldots, n\}\). The number of elements of \( \pi \in P_n \) is denoted \( |\pi| \). For all \( \pi \in P_n \), we define the equivalent relation \( i \sim_{\pi} j \) if \( i \) and \( j \) belong to the same element of \( \pi \). We define a partial order on \( P_n \) by \( \pi' \leq \pi \) if \( i \sim_{\pi'} j \) implies \( i \sim_{\pi} j \) (\( \pi \) is finer than \( \pi' \)).

For all \( \pi \in P_n \), we let \( E_\pi \) be the set of elements \( x \in M^n \) such that \( x_i = x_j \) if \( i \sim_{\pi} j \) and \( \partial E_\pi = \bigcup_{\pi' < \pi} E_{\pi'} \), the set of elements \( x \in E_\pi \) such that there exists \( i \) and \( j \) with \( i \sim_{\pi} j \) and \( x_i = x_j \). Let \( j_\pi \) be an isometry between \( M[|\pi|] \) and \( E_\pi \).

By induction on \( k = |\pi| \), we define a Markov process \( X^\pi \) on \( E_\pi \). For \( k = 1 \), we let \( X^\pi = j_\pi(X^{(1)}) \). Assume now we have defined a Markov process on \( E_\pi \) for all \( \pi \) such that \(|\pi| \leq k \). Let \( \pi \in P_n \) with \(|\pi| = k + 1 \), we define \( X^\pi \) concatenating the process \( j_\pi(X^{(k+1)}) \) stopped at the entrance time \( T \) in \( \partial E_\pi \) with the process \( X^\pi' \) starting from the corresponding point and where \( \pi' \) is the finest partition such that \( X^\pi' \in E_{\pi'} \). This way, we construct a Markov process on \( M^n \), \( X^{(n),c} = X^\pi \) for \( \pi = \{\{1\}, \ldots, \{n\}\} \).

For all integer \( n \), let \( P_t^{(n),c} \) be the Markovian semigroup associated with the Markov process \( X^{(n),c} \). From the above construction, it is clear that the family \((P_t^{(n),c}, n \geq 1)\) of Markovian semigroups is compatible.

It remains to prove that when (C) is satisfied, this family of Markovian semigroups is constituted of Feller semigroups. This holds since (C) implies (F) in lemma 1.7.1: for all positive \( \varepsilon \), \( P^{(2),c}_{(x,y)}[d(X_t, Y_t) > \varepsilon] \leq P^{(2)}_{(x,y)}[(T_{\Delta} > t) \cap \{d(X_t, Y_t) > \varepsilon\}] \) which converges towards 0 as \( y \to x \). Note that when (C) holds, it is easy to see that the canonical flow is a coalescing flow. \( \square \)

We now suppose that \((P_t^{(n),c}, n \geq 1)\) is constituted of Feller semigroups (which is true when (C) holds). We denote by \( \nu^c \) the associated Feller convolution semigroup.

**Theorem 4.3.2.** The convolution semigroup \( \nu^c \) weakly dominates \( \nu \).

**Proof.** The idea of the proof is to define a coupling between the flows of kernels \( K \) and \( K^c \) respectively of law \( P_\nu \) and \( P_{\nu^c} \). (Since we did not assume (C) holds, it is not clear that \( K^c \) is a flow of mappings.)

In a way similar to the construction of the Markov process \( X^{(n),c} \) in the proof of theorem 4.3.1, for all integer \( n \geq 1 \), we construct a Markov process \( \tilde{X}^{(n)} \) on \((M \times M)^n\) identified with \( M^n \times M^n \) such that:

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• $(\hat{X}_1^{(n)}, \ldots, \hat{X}_n^{(n)})$ is the $n$-point motion of $\nu^c$;

• $(\hat{X}_{n+1}^{(n)}, \ldots, \hat{X}_{2n}^{(n)})$ is the $n$-point motion of $\nu$;

• between the coalescing times, $\hat{X}^{(n)}$ is described by the $(k+n)$-point motion of $\nu$ (when $(\hat{X}_1^{(n)}, \ldots, \hat{X}_n^{(n)})$ belongs to $E_\pi$, with $|\pi| = k$).

Let $\hat{P}_t^{(n)}$ be the Markovian semigroup associated with $\hat{X}^{(n)}$. One easily checks that this semigroup is Feller using the fact that $P_t^{(n)}$ and $P_t^{(n)c}$ are Feller. Then $(\hat{P}_t^{(n)}, n \geq 1)$ is a compatible family of Feller semigroups, associated with a Feller convolution semigroup $\hat{\nu}$.

Let $\hat{K}$ be the canonical stochastic flow associated with this family of semigroups. Straightforward computations show that for all $s < t$, $(f, g) \in C(M)^2$ and $(x, y) \in M^2$,

$$E[(\hat{K}_{s,t}(f \otimes g)(x,y))^2] = \hat{P}_{t-s}^{(2)} f \otimes g(x,y),$$

$$E[(\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))(x,y)] = \hat{P}_{t-s}^{(3)} f \otimes g(x,y),$$

$$E[(\hat{K}_{s,t}(f \otimes g)\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))(x,y)] = \hat{P}_{t-s}^{(3)} f^2 \otimes g(x,y).$$

This implies that

$$E[(\hat{K}_{s,t}(f \otimes g) - \hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))(x,y)] = 0. \quad (4.8)$$

Thus we have $\hat{K}_{s,t}(x,y) = K_{s,t}^c(x) \otimes K_{s,t}^c(y)$ and it is easy to check that the laws of $K^c$ and $K$ are respectively $P_{t\nu^c}$ and $P_{t\nu}$. Thus $(N_\nu, K^c)$ is an extension of $N_{\nu^c}$. Let $\tilde{N}_\nu$ be the sub-noise of $N_\nu$ generated by $\tilde{K}$.

Let us notice now that for all $g, f_1, \ldots, f_n$ in $C_0(M)$, all $y, x_1, \ldots, x_n$ in $M$ and all $s < t$, we have (setting $y_i = x_{n+1} = y$ and for $i \leq n$, $h_i = f_i \otimes 1$ and $h_{n+1} = 1 \otimes g$)

$$E \left[ K_{s,t}^c g(y) \prod_{i=1}^{n+1} K_{s,t} f_i(x_i) \right] = E \left[ \prod_{i=1}^{n+1} \hat{K}_{s,t} h_i(x_i, y_i) \right] = \hat{P}_{t-s}^{(n+1)} f_1 \otimes \cdots \otimes f_n \otimes g(x_1, \ldots, x_n, y).$$

More generally one can prove in a similar way that for all $g, f_1, \ldots, f_n$ in $C_0(M)$, all $y, x_1, \ldots, x_n$ in $M$, all $s < t$ and all $(s_i, t_i)_{1 \leq i \leq n}$ with $s_i \leq t_i$ that

$$E \left[ K_{s,t}^c g(y) \prod_{i=1}^{n} K_{s_i, t_i} f_i(x_i) \right] = E \left[ K_{s,t} g(y) \prod_{i=1}^{n} K_{s_i, t_i} f_i(x_i) \right]. \quad (4.9)$$

This implies that $K_{s,t} g(y) = E[K_{s,t}^{c} g(y) | \sigma(K)]$ and therefore that $\nu^c \succeq^{\nu} \nu$. □

**Remark 4.3.3.** Let $(X^{(n)}, n \geq 1)$ be a family of strong Markov processes respectively taking their values in $M^n$. We suppose that the associated family of Markovian semigroups $(P_t^{(n)}, n \geq 1)$ is compatible and that for all $x \in M$,

$$\lim_{y \to x} P_{x,y}^{(2)} \{ \{ T_\Delta > t \} \cap \{ d(X_t, Y_t) > \varepsilon \} \} = 0 \quad (4.10)$$
for all $\varepsilon > 0$ and $t > 0$. Then $(P_t^{(n)}, n \geq 1)$ (and $(P_t^{(n),c}, n \geq 1)$) are Feller semigroups.

One can prove this with a coupling similar to the coupling given in the proof of the previous theorem: the idea is to construct on the same probability space two Markov processes $X^{(n)}(s)$ and $Y^{(n)}(s)$ associated to $P_t^{(n)}$ and such that $X_i^{(n)}(t) = Y_i^{(n)}(t)$ if $t \geq \inf\{s, \ X_i^{(n)}(s) = Y_i^{(n)}(s)\}$.

**Remark 4.3.4.** The example given in section 3.4 gives an illustration of the two theorems of this section, first with $P_t^{(n)} = P_t^{\otimes n}$ then with $P_t^{(n)}$ the semigroup of the $n$-point motion of $K$. This example shows in particular that one can have $\nu \preceq \nu^c$ and $\nu \not\preceq \nu^c$.

### 4.4 Examples.

#### 4.4.1 Arratia’s coalescing flow of independent Brownian motions.

The first example of coalescing flows was given by Arratia [2]. On $\mathbb{R}$, let $P_t$ be the semigroup of a Brownian motion. With this semigroup we define the compatible family $(P_t^{\otimes n}, n \geq 1)$ of Feller semigroups. Note that the $n$-point motion of this family of semigroups is given by $n$ independent Brownian motions. Let us also remark that the canonical stochastic flow of kernels associated with this family of semigroups is not random and is given by $(P_t^{\otimes n}, n \geq 1)$.

Let $(P_t^{(n)}, n \geq 1)$ be the compatible family of Markovian coalescent semigroups associated with $(P_t^{\otimes n}, n \geq 1)$ (see section 4.3). Note that the $n$-point motion of this family of semigroups is given by $n$ independent Brownian motions who stick together when they meet.

**Proposition 4.4.1.** The family $(P_t^{(n)}, n \geq 1)$ is constituted of Feller semigroups and is associated with a coalescing flow.

**Proof.** It is obvious after remarking that two real independent Brownian motions meet each other almost surely (condition (C) is verified). \(\square\)

#### 4.4.2 Propp-Wilson algorithm.

Similarly to Arratia’s coalescing flow, let $P_t$ be the semigroup of an irreducible aperiodic Markov process on a finite set $M$, with invariant probability measure $m$. Let $(P_t^{(n)}, n \geq 1)$ be the compatible family of Markovian coalescent semigroups associated with $(P_t^{\otimes n}, n \geq 1)$ (see section 3.3). Note that the $n$-point motion of this family of semigroups is given by $n$ independent Brownian motions who stick together when they meet.

**Proposition 4.4.2.** The family $(P_t^{(n)}, n \geq 1)$ is constituted of Feller semigroups and is associated with a coalescing flow.

**Proof.** It is obvious since the two-point motion defined by $P_t^{\otimes 2}$ hits the diagonal almost surely. \(\square\)

Let $\varphi = (\varphi_{s,t}, s \leq t)$ denote this coalescing flow. Then almost surely, for all $x, y$ in $M$, $\tau_{x,y} = \inf\{t > 0, \ \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite. Therefore, after a finite time $\text{Card}\{\varphi_{0,t}(x), x \in M\} = 1$.

In Propp-Wilson [35], an algorithm to exactly simulate a random variable distributed according to the invariant probability measure of a Markov chain with finite state space is
given. The method consists in constructing a stochastic coalescing flow. We explain this in our context.

Let \( \tau = \inf\{t > 0, \, \varphi_{-1,0}(x) = \varphi_{-1,0}(y) \, \text{for all} \, (x, y) \in M^2 \} \).

**Proposition 4.4.3.** \( \tau \) is almost surely finite and the law of \( X_\tau \), the random variable \( \varphi_{-\tau,0}(x) \) (independent of \( x \in M \)), is m.

**Proof.** Let us remark that for \( t > \tau \) and all \( x \in M \), the cocycle property implies that \( \varphi_{-t,0}(x) = X_\tau \).

Since for all positive \( t \),

\[
P[\tau \geq t] = P[\exists x, y, \, \varphi_{-t,0}(x) \neq \varphi_{-t,0}(y)] \leq \sum_{(x,y) \in M^2} P[\tau_{x,y} \geq t] \tag{4.11}
\]

which converges towards 0 as \( t \) goes to infinity. Thus \( \tau < \infty \) a.s.

For all function \( f \) on \( M \) and all \( x \in M \), \( \lim_{t \to \infty} P_tf(x) = \sum_{y \in M} f(y)m(y) \) and

\[
P_tf(x) = \mathbb{E}[f(\varphi_{-t,0}(x))] = \mathbb{E}[f(\varphi_{-t,0}(x))1_{t \leq \tau}] + \mathbb{E}[f(X_\tau)1_{\tau < t}]. \tag{4.12}
\]

Since \( \tau \) is almost surely finite, as \( t \) goes to infinity, the first term of the right hand side of the preceding equation converges towards 0 and the second term converges towards \( \mathbb{E}[f(X_\tau)] \). Therefore we prove that \( \mathbb{E}[f(X_\tau)] = \sum_{y \in M} f(y)m(y) \). \( \square \)

### 4.4.3 Tanaka’s SDE.

In [23], starting from a real Brownian motion \( B \), we constructed a family of random operators \( (S_t, \, t \geq 0) \), strong solution of the SDE

\[
dX_t = \text{sgn}(X_t)dB_t, \quad t \geq 0. \tag{4.13}
\]

For \( f \) continuous,

\[
S_tf(x) = f(R^x_t)1_{t \leq T_x} + \frac{1}{2}(f(R^x_t) + f(-R^x_t))1_{t \geq T_x}, \tag{4.14}
\]

where \( R^x_t \) is the Brownian motion \( x + B_t \) reflected at 0 and \( T_x \) the first time it hits 0. For all continuous functions \( f_1, \ldots, f_n \), let

\[
P^{(n)}_t(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = \mathbb{E} \left[ \prod_{i=1}^n S_tf_i(x_i) \right]. \tag{4.15}
\]

Then it is easy to see that \( (P^{(n)}_t, \, n \geq 1) \) is a compatible family of Feller semigroups. Let \( (P^{(n),c}_t, \, n \geq 1) \) be the family of semigroups constructed by theorem 4.3.1.

Let us describe the \( n \)-point motion associated with \( (P^{(n),c}_t, \, n \geq 1) \). Let \( (X_t, \, t \geq 0) \) be a Brownian motion starting at 0. Let \( B_t = \int_0^t \text{sgn}(X_s) \, dX_s \), \( (B_t, \, t \geq 0) \) is also a Brownian motion starting at 0. For all \( x \in \mathbb{R} \), let \( \tau_x = \inf\{t \geq 0, \, |x| + B_t = 0\} \). Note that \( X_{\tau_x} = 0 \).

For all \( x \in \mathbb{R} \), let

\[
X^x_t = \begin{cases} 
  x + \text{sgn}(x)B_t & \text{if } t < \tau_x, \\
  X_t & \text{if } t \geq \tau_x.
\end{cases} \tag{4.16}
\]
Then $B_t = \int_0^t \text{sgn}(X^x_t) \, dX^x_t$ and $X^x$ is a solution of the SDE
\[ dX^x_t = \text{sgn}(X^x_t) \, dB_t, \quad t \geq 0, \quad X^x_0 = x. \] (4.17)
Thus, for all $x_1, \ldots, x_n$ in $M$, $((X^{x_1}_t, \ldots, X^{x_n}_t), \ t \geq 0)$ is the $n$-point motion of the family of semigroups $(P_t^{(n),c}, \ n \geq 1)$.

**Proposition 4.4.4.** The family $(P_t^{(n)}, \ n \geq 1)$ is constituted of Feller semigroups and is associated with a coalescing flow.

**Proof.** It is easy to see that $(P_t^{(n),c}, \ n \geq 1)$ is constituted of Feller semigroups since for all $t$ and $x_0$, $x \mapsto X^x_t$ is a.s. continuous at $x_0$ (it implies that (F) in lemma 1.7.1 is satisfied). This also implies that (1.6) is satisfied. Thus, the associated stochastic flow is a flow of mappings. And it is a coalescing flow since almost surely, every pair of point meets after a finite time. Note that condition (C) is verified. □

5 **Stochastic flows of kernels and SDEs.**

5.1 **Hypotheses.**

In this section, $M$ is a smooth locally compact manifold and we suppose we are given $(P_t^{(n)}, \ n \geq 1)$, a compatible family of Feller semigroups, or equivalently a Feller convolution semigroup $\nu = (\nu_t)_{t \geq 0}$ on $(E, \mathcal{E})$. For all positive integer $n$, we will denote by $X^{(n)}_t$ the $n$-point motion, i.e. the Markov process associated with the semigroup $P_t^{(n)}$. We denote by $A^{(n)}$ the infinitesimal generator of $P_t^{(n)}$ and by $D(A^{(n)})$ its domain. We assume that

(i) The space $C^2_K(M) \otimes C^2_K(M)^3$ of functions of the form $f(x)g(y)$, with $f, g$ in $C^2_K(M)$ and $x, y$ in $M$, is included in $D(A^{(2)})$.

(ii) The one-point motion $X^{(1)}_t$ has continuous paths.

In that case, we say that $\nu$ is a diffusion convolution semigroup on $(E, \mathcal{E})$ and that the $P_t^{(n)}$ are diffusion semigroups.

5.2 **Local characteristics of a diffusion convolution semigroup.**

Let us denote by $A$ the restriction of $A^{(1)}$ to $C^2_K(M)$. Note that it follows easily from (i) and (ii) that for all $f \in C^2_K(M)$,
\[ M_t^f = f(X^{(1)}_t) - f(X^{(1)}_0) - \int_0^t A f(X^{(1)}_s) \, ds \] (5.1)

$f$ is in the domain of the infinitesimal generator $A$ of a Feller semigroup $P_t$ if and only if $\frac{P_t^f - f}{t}$ converges uniformly as $t$ goes towards 0. Its limit is denoted $Af$. $^3C_K(M)$ (respectively $C^2_K(M)$) denotes the set of continuous (respectively $C^2$) functions with compact support.
is a martingale. Since $f^2$ also belongs to $C^2_K(M)$, using the martingale $Mf^2$, it is easy to see that
\[
\langle Mf \rangle_t = \int_0^t \Gamma(f)(X_s^{(1)}) \, ds \tag{5.2}
\]
where
\[
\Gamma(f) = Af^2 - 2f Af. \tag{5.3}
\]
In the following $\Gamma(f, g)$ will denote $A(fg) - fAg - gAf$, for $f$ and $g$ in $C^2_K(M)$.

**Lemma 5.2.1.** On a smooth local chart on an open set $U \subset M$, there exist continuous functions on $U$, $a^{i,j}$ and $b^i$ such that for all $f \in C^2_K(M)$,
\[
Af = \frac{1}{2} \partial^2 f \partial x^i \partial x^j + b^i \frac{\partial f}{\partial x^i}. \tag{5.4}
\]

**Proof.** For all $x \in U$, let $\phi^i(x) = x^i$ denote the coordinate functions of the local chart. We can extend $\phi^i$ into an element of $C^2_K(M)$. For $f \in C^2_K(M)$, using Itô’s formula, for $t < T_U$, the exit time of $U$,
\[
f(X_t^{(1)}) - f(X_0^{(1)}) - \int_0^t \left( \frac{1}{2} a^{i,j}(X_s^{(1)}) \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^{(1)}) + b^i(X_s^{(1)}) \frac{\partial f}{\partial x^i}(X_s^{(1)}) \right) \, ds
\]
is a martingale, where $b^i(x) = A \phi^i(x)$ and $a^{i,j}(x) = \Gamma(\phi^i, \phi^j)(x)$. And we get (5.4) since for all $x \in U$, $Af(x) = \lim_{t \to 0} \frac{P_t^{(f,x)}(f(x)) - f(x)}{t}$.

Note that the two-point motion $X_t^{(2)}$ has also continuous trajectories and these results also apply to functions in $C^2_K(M) \otimes C^2_K(M)$. For all $f, g$ in $C^2_K(M)$, let
\[
C(f, g) = A^{(2)}(f \otimes g) - f \otimes Ag - Af \otimes g. \tag{5.5}
\]
It is clear that on a local chart on $U \times V \subset M \times M$,
\[
C(f, g)(x, y) = c^{i,j}(x, y) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial y^j}(y), \tag{5.6}
\]
where $c^{i,j} \in C(U \times V)$. Then we can shortly write $A^{(2)} = A \otimes I + I \otimes A + C$. On a local chart on $U \times V$, for all $h \in C^2_K(M) \otimes C^2_K(M)$,
\[
A^{(2)} h(x, y) = \frac{1}{2} a^{i,j}(x) \frac{\partial^2 h(x, y)}{\partial x^i \partial x^j} + b^i(x) \frac{\partial h(x, y)}{\partial x^i}
+ \frac{1}{2} a^{i,j}(y) \frac{\partial^2 h(x, y)}{\partial y^i \partial y^j} + b^i(y) \frac{\partial h(x, y)}{\partial y^i} + c^{i,j}(x, y) \frac{\partial^2 h(x, y)}{\partial x^i \partial y^j}. \tag{5.7}
\]

We will call $\Gamma(f, g)(x) - C(f, g)(x, x) = \frac{1}{2} A^{(2)}(1 \otimes f - g \otimes 1)^2(x, x) - (1 \otimes f - g \otimes 1)(1 \otimes Af - Ag \otimes 1)(x, x)$ the pure diffusion form. It can easily be seen that it is nonnegative and it vanishes if the associated canonical flow is a flow of maps. Indeed
\[
\Gamma(f, f)(x) = \lim_{t \to 0} \frac{1}{t} \left( P_t^{(1)} f^2(x) - P_t^{(2)} f^{\otimes 2}(x, x) \right)
= \lim_{t \to 0} \frac{1}{2t} \left( P_t^{(2)} (1 \otimes f - f \otimes 1)^2(x, x) \right). \tag{5.8}
\]
The converse is not true (see examples in section [7]). Diffusive flows for which the pure diffusion form vanishes may be called turbulent.

The two-point motion \( X_t^{(2)} = (X_t, Y_t) \) solves the following martingale problem associated with \( A^{(2)} \):

\[
M_t^{f,g} := f(X_t)g(Y_t) - f(X_0)g(Y_0) - \int_0^t A^{(2)}(f \otimes g)(X_s, Y_s) \, ds
\]

is a martingale for all \( f \) and \( g \) in \( C^2_K(M) \).

Note that for all functions \( h_1 \) and \( h_2 \) in \( C^2_K(M) \otimes C^2_K(M) \), the martingale bracket \( \langle h_1(X^{(2)}), h_2(X^{(2)}) \rangle_t \) is equal to

\[
\int_0^t (A^{(2)}(h_1 h_2) - h_1 A^{(2)} h_2 - h_2 A^{(2)} h_1)(X_s^{(2)}) \, ds
\]

and for all functions \( f \) and \( g \) in \( C^2_K(M) \),

\[
\langle f(X), g(Y) \rangle_t = \int_0^t C(f, g)(X_s, Y_s) \, ds.
\]

**Definition 5.2.2.** (a) A covariance function on the space of vector fields is a symmetric map \( C \) from \( T^* M^2 \) in \( \mathbb{R} \) such that its restriction to \( T_x^* M \times T_y^* M \) is bilinear and for any \( n \)-uples \( (\xi_1, \ldots, \xi_n) \) of \( T^* M^2 \), \( \sum_{1 \leq i,j \leq n} C(\xi_i, \xi_j) \geq 0 \), (see [23]). For \( f \) and \( g \) in \( C^1_K(M) \), we denote \( C(df(x), dg(y)) \) by \( C(f, g)(x, y) \).

(b) We say the covariance function is continuous if \( C(f, g) \) is continuous for all \( f \) and \( g \) in \( C^1_K(M) \).

**Proposition 5.2.3.** (a) \( C \) is a continuous covariance function on the space of vector fields.

(b) For all \( f_1, \ldots, f_n \) in \( C^2_K(M) \), then \( g = f_1 \otimes \cdots \otimes f_n \in \mathcal{D}(A^{(n)}) \) and for \( x = (x_1, \ldots, x_n) \in M^n \),

\[
A^{(n)} g(x) = \sum_i \prod_{j \neq i} f_j(x_i) A f_i(x_i) + \sum_{i,j} C(f_i, f_j)(x_i, x_j) \prod_{k \neq i,j} f_k(x_k).
\]

**Proof.** For all \( f \) and \( g \) in \( C^2_K(M) \), \( C(f, g)(x, y) \) is a function of \( df(x) \) and of \( dg(y) \) we denote \( C(df(x), dg(y)) \). Hence \( C \) is a symmetric map from \( T^* M^2 \) in \( \mathbb{R} \) and its restriction to \( T_x^* M \times T_y^* M \) is bilinear. To prove (a), it remains to prove \( \sum_{i,j} C(\xi_i, \xi_j) \geq 0 \) for all \( \xi_1, \ldots, \xi_n \) in \( T^* M^2 \). This holds since, for all \( f_1, \ldots, f_n \) in \( C^2(M) \) and all \( x_1, \ldots, x_n \) in \( M \),

\[
\sum_{i,j} C(f_i, f_j)(x_i, x_j) = (A^{(n)} g^2 - 2g A^{(n)} g)(x_1, \ldots, x_n)
\]

where \( g(x_1, \ldots, x_n) = \sum_{i=1}^n f_i(x_i) \in \mathcal{D}(A^{(n)}) \). This expression is nonnegative since \( A^{(n)} g^2 - 2g A^{(n)} g = \lim_{t \to 0} \frac{1}{2}(P^{(n)}_t g^2 - (P^{(n)}_t g)^2 + (P^{(n)}_t g - g)^2) \).

The proof of (b) is an application of Itô’s formula. \( \square \)
Definition 5.2.4. The diffusion generator $A$ and the covariance function $C$ are called the local characteristics of the family $(P^{(n)}_t, n \geq 1)$ or of the diffusion convolution semigroup.

When there is no pure diffusion, to give the local characteristics $(A, C)$ in a system of local charts is equivalent to give a drift $b$ and $C$ (this corresponds to the usual definition of a local characteristics of a stochastic flow) since in this case $a^{ij}(x) = c^{ij}(x, x)$.

Remark 5.2.5. When $(P^{(n)}_t, n \geq 1)$ satisfies (C), (i) and (ii), then $(P^{(n),c}_t, n \geq 1)$ also satisfies (i) if and only if for all $x$ in $M$ and all $f, g$ in $C^2_K(M)$, $C(f, g)(x, x) = \Gamma(f, g)(x)$ (this holds since we have $C(f, g)(x, x) - \Gamma(f, g)(x) = \lim_{t \to 0} \frac{1}{t}(P^{(2),c}_t(f \otimes g)(x, x) - P^{(1)}_t(fg)(x)))$, i.e. when there is no pure diffusion. So that the results of this section also apply to $(P^{(n),c}_t, n \geq 1)$.

Then in this case $(P^{(n)}_t, n \geq 1)$ and $(P^{(n),c}_t, n \geq 1)$ have the same local characteristics.

Let $K = (K_{s,t}, s \leq t)$ be a measurable stochastic flow of kernels associated with $(P^{(n)}_t, n \geq 1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 5.2.6. Let $C$ be a covariance function on the space of vector fields. A two parametric family $W^s = (W^s_{s,t}, s \leq t)$ of random variables taking their values in the space of vector fields on $M$ is called a vector field valued white noise of covariance $C$ if

(i) for all $s \leq t \leq s_{i+1}$, the random variables $(W_{s,t}^i, 1 \leq i \leq n)$ are independent,

(ii) for all $s \leq u \leq t$, $W_{s,t} = W_{s,u} + W_{u,t} a.s.$ and

(iii) for all $s \leq t$, $\{\langle W_{s,t}, \xi \rangle, \xi \in T^*M \}$ is a centered Gaussian process of covariance given by

$$E[\langle W_{s,t}, \xi \rangle \langle W_{s,t}, \xi' \rangle] = (t-s)C(\xi, \xi'), \quad (5.13)$$

for $\xi$ and $\xi'$ in $T^*M$.

In this section, we intend to define on $(\Omega, \mathcal{A}, \mathbb{P})$ a vector field valued white noise $W$ of covariance $C$ such that $K$ solves a SDE driven by $W$.

In section [6], under an additional assumption, we will prove that the linear (or Gaussian) part of the noise generated by $K$ (in the case it is the canonical flow) is the noise generated by the vector field valued white noise $W$.

5.3 The velocity field $W$.

For all $s \leq t$, all $f \in C^2_K(M)$ and all $x \in M$, let

$$M_{s,t}f(x) = K_{s,t}f(x) - f(x) - \int_s^t K_{s,u}(Af)(x) \, du. \quad (5.14)$$

\footnote{when $\xi = (x, u)$, $\langle W_{s,t}, \xi \rangle = \langle W_{s,t}(x), u \rangle$.}
Lemma 5.3.1. For all \( s \in \mathbb{R}, \ f \in C^2_K(M) \) and \( x \in M, \ M^{f,x}_s = (M_{s,t}f(x), \ t \geq s) \) is a martingale with respect to the filtration \( \mathcal{F}^s = (\mathcal{F}_{s,t}, \ t \geq s) \) and

\[
\frac{d}{dt}\langle M^{f,x}_s, M^{g,y}_s \rangle_t = K^\otimes_2 C(f,g)(x,y),
\]

for all \( f, g \in C^2_K(M) \) and all \( x, y \in M \).

Proof. Since \( K \) is a measurable stochastic flow of kernels and that for all positive \( h \) and all \( f \) in \( C^2_K(M) \), a.s.

\[
M_{s,t+h}f(x) - M_{s,t}f(x) = K_{s,t}(M_{t,t+h}f)(x),
\]

\( M^{f,x}_s \) is a martingale. Note that equation (5.16) also implies that for all positive \( h \), all \( f, g \) in \( C^2_K(M) \) and all \( x, y \) in \( M \),

\[
E[(M_{s,t+h}f(x) - M_{s,t}f(x))(M_{s,t+h}g(y) - M_{s,t}g(y))|\mathcal{F}_{s,t}]
= K^\otimes_2(E[M_{t,t+h}f \otimes M_{t,t+h}g])(x,y).
\]

The stationarity implies that \( E[M_{t,t+h}f(x)M_{t,t+h}g(y)] = E[M_{0,h}f(x)M_{0,h}g(y)] \). Computation using the fact that \( P_t^{(1)}f - f = \int_0^t P_s^{(1)}Af \ ds \) and \( P_t^{(2)}(f \otimes g) - f \otimes g = \int_0^t P_s^{(2)}A^{(2)}(f \otimes g) \ ds \) gives

\[
E[M_{0,h}f(x)M_{0,h}g(y)] = \int_0^h P_s^{(2)}(C(f,g))(x,y) \ ds.
\]

Since \( P_t^{(2)} \) is Feller and \( C(f,g) \) is continuous with compact support,

\[
E[M_{0,h}f(x)M_{0,h}g(y)] = h \ C(f,g)(x,y) + o(h),
\]

uniformly in \((x,y) \in M^2\).

Therefore \( E[(M_{s,t+h}f(x) - M_{s,t}f(x))(M_{s,t+h}g(y) - M_{s,t}g(y))|\mathcal{F}_{s,t}] \) is equivalent as \( h \) tends to 0 to \( h \ K^\otimes_2 C(f,g)(x,y) \). This proves the lemma. \( \square \)

Remark 5.3.2. In the case of Arratia’s coalescing flow \( (\varphi_{s,t} \ s \leq t) \), \( C = 0 \) but \( \frac{d}{dt}\langle M^{f,x}_s, M^{g,y}_s \rangle_t = 1_{\{\varphi_{s,t}(x) = \varphi_{s,t}(y)\}} \). In this case, \( C^2_K(M) \otimes C^2_K(M) \) is not included in \( \mathcal{D}(A^{(2)}) \). This property also fails for the coalescing flow associated with Tanaka’s SDE.

For all \( s < t \), \( n \geq 1 \) and \( 0 \leq k \leq 2^n - 1 \), let \( t^n_k = s + k2^{-n}(t - s) \) and

\[
W^n_{s,t}f = \sum_{k=0}^{2^n-1} M^{t^n_k,t^n_{k+1}}_s f,
\]

where \( f \in C^2_K(M) \). Note that \((M^{t^n_k,t^n_{k+1}}_s)_{0 \leq k \leq 2^n - 1}\) are independent equidistributed random variables.
5.3.1 Convergence in law.

**Lemma 5.3.3.** For all \( s < t \) and all \(((x_i, f_i), 1 \leq i \leq m) \in (M \times C^2_K(M))^m\), then \( \sum_{i=1}^{m} W^n_{s,t}f_i(x_i) \) converges in law towards \( \sum_{i=1}^{m} W_{s,t}f_i(x_i) \) as \( n \) tends to \( \infty \), where \( W \) is a vector field valued white noise of covariance \( C \).

**Proof.** Using lemma [5.3.1] we have for all \( f, g \in C^2_K(M) \) and all \( x, y \in M \),

\[
E[M^n_{s,t,k+1} f(x) M^n_{s,t,k+1} g(y)] = \int_0^{2^{-n(t-s)}} P_u^{(2)} C(f, g)(x,y) \, du
= 2^{-n(t-s)} C(f, g)(x,y) + o(2^{-n}) \tag{5.20}
\]

and this development is uniform in \( x \) and \( y \) in \( M \).

We will only prove the proposition when \( m = 1 \) (the proof being the same for \( m > 1 \)). The proposition is just an application of the central limit theorem for arrays (see [6]), which we can apply since (5.20) is satisfied provided the Lyapounov condition

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} E[|M^n_{s,t,k+1} f(x)|^{2+\delta}] = 0 \tag{5.21}
\]

for some positive \( \delta \), is satisfied.

Using Burkholder-Davies-Gundy’s inequality and lemma [5.3.1]

\[
E[|M^n_{s,t,k+1} f(x)|^{2+\delta}] \leq C E \left[ \left( \int_0^{2^{-n(t-s)}} K^{\otimes 2}_{0,u}(C(f,f))(x,x) \, du \right)^{\frac{2+\delta}{2}} \right] 
\leq C 2^{-\frac{(2+\delta)\delta}{2}},
\]

where \( C \) is a constant (changing every line) depending only on \( f, (t-s) \) and \( \delta \). This implies

\[
\sum_{k=0}^{2^n-1} E[|M^n_{s,t,k+1} f(x)|^{2+\delta}] \leq C 2^n 2^{-\frac{(2+\delta)\delta}{2}} \leq C 2^{-\frac{n\delta}{2}}. \tag{5.22}
\]

**Remark 5.3.4.** For Arratia’s coalescing flow, one can show the convergence in law as \( n \) goes to \( \infty \) of \( (W^n_{s,t}(x_1), \ldots, W^n_{s,t}(x_k)) \) towards \( (B^1_{s,t}, \ldots, B^k_{s,t}) \), where \( (B^1, \ldots, B^k) \) is a \( k \)-dimensional white noise.

5.3.2 Convergence in \( L^2(P) \).

In the preceding section, we proved the convergence in law of \( W^n \) towards a vector field valued white noise of covariance \( C \). In this section, we prove that this convergence holds in \( L^2(P) \).

**Lemma 5.3.5.** For all \( s < t \) and all \( (x, f) \in M \times C^2_K(M) \), \( W^n_{s,t} f(x) \) converges in \( L^2(P) \).
Proof. For all $f \in C^2_k(M)$, all $x \in M$ and all $s < t$,
\[
E[(W_{s,t}^n f(x) - W_{s,t}^{n+k} f(x))^2] = E[(W_{s,t}^n f(x))^2] + E[(W_{s,t}^{n+k} f(x))^2] - 2E[W_{s,t}^n f(x)W_{s,t}^{n+k} f(x)]
\] (5.23)

Elementary computations using equation (5.18) implies
\[
E[(W_{s,t}^n f(x))^2] = (t-s) C(f,f)(x,x) + o(1) \quad (5.24)
\]
\[
E[(W_{s,t}^{n+k} f(x))^2] = (t-s) C(f,f)(x,x) + o(1) \quad (5.25)
\]
as $n$ goes to $\infty$ and this uniformly in $k \in \mathbb{N}$. Using the independence of the increments, the last term (5.23) rewrites
\[
E[W_{s,t}^n f(x)W_{s,t}^{n+k} f(x)] = \sum_{i=0}^{2^n-1} \sum_{j=i2^k}^{2^n-1} E[M_{i,i+1}^{n,n} f(x)M_{j,j+1}^{n+k,n+k} f(x)]. \quad (5.26)
\]

Note that for $s \leq u \leq v \leq t$, using first the martingale property, then equation (5.18) and the uniform continuity of $C(f,f)$, we have
\[
E[M_{s,t} f(x)M_{u,v} f(x)] = E[M_{s,t} f(x)M_{u,v} f(x)] = E[(K_{s,u} \otimes I)(M_{u,v} f \otimes M_{u,v} f)(x,x)] = E[(K_{s,u} \otimes I)(E[M_{u,v} f \otimes M_{u,v} f])(x,x)] = (v-u) C(f,f)(x,x) + o(v-u), \quad (5.27)
\]
uniformly in $x \in M$. This implies
\[
E[W_{s,t}^n f(x)W_{s,t}^{n+k} f(x)] = (t-s) C(f,f)(x,x) + o(1) \quad (5.28)
\]
as $n$ tends to $\infty$ and uniformly in $k \in \mathbb{N}$. We therefore have
\[
\lim_{n \to \infty} \sup_{k \in \mathbb{N}} E[(W_{s,t}^n f(x) - W_{s,t}^{n+k} f(x))^2] = 0, \quad (5.29)
\]
i.e. $(W_{s,t}^n f(x), \ n \in \mathbb{N})$ is a Cauchy sequence in $L^2(\mathbb{P})$. This proves the lemma. □

Remark 5.3.6. For Arratia’s coalescing flow, this lemma is not satisfied since $(W_{s,t}^n f(x), \ n \in \mathbb{N})$ fails to be a Cauchy sequence in $L^2(\mathbb{P})$.

Thus, for all $s < t$, we have defined the vector field valued random variable $W_{s,t}$ such that $W_{s,t} f(x)$ is the $L^2(\mathbb{P})$-limit of $W_{s,t}^n f(x)$ for all $x \in M$ and all $f \in C(M)$. Then, using lemma 5.3.3, it is easy to see that $W = (W_{s,t}, \ s \leq t)$ is a vector field valued white noise of covariance $C$. 

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5.4 The stochastic flow of kernels solves a SDE.

In [23], it is shown that a vector field valued white noise \( W \) of covariance \( C \) can be constructed with a sequence of independent real white noises \( (W^\alpha_\alpha) \) by the formula \( W = \sum_\alpha V_\alpha W^\alpha_\alpha \), where \( (V^\alpha_\alpha) \) is an orthonormal basis of \( H_C \), the self-reproducing space associated with \( C \).

For all predictable (with respect to the filtration \( (\mathcal{F}_{-\infty,t}, t \in \mathbb{R}) \)) process \( (H_t)_{t \in \mathbb{R}} \) taking its values in the dual of \( H_C \), we define the stochastic integral of \( H \) with respect to \( W \) by the formula

\[
\int_s^t H_u(W(du)) = \sum_\alpha \int_s^t \langle H_u, V_\alpha \rangle W^\alpha(du),
\]

for \( s < t \). Note that the above definition is independent of the choice of the orthonormal basis \( (V^\alpha_\alpha) \).

In particular this applies to \( H_u(V) = K_{s,u}(Vf)(x)1_{s \leq u < t} \) for \( f \in C^2_K(M) \) and \( x \in M \). Then the stochastic integral \( \sum_\alpha \int_s^t K_{s,u}(V^\alpha f)W^\alpha(du) \) is denoted

\[
\int_s^t K_{s,u}(Wf(du))(x).
\]

**Remark 5.4.1.** The stochastic integral (5.31) is equal to the limit in \( L^2(\mathbb{P}) \) of

\[
\sum_{k=0}^{2^n-1} K_{s,t^n_k}(W_{t^n_{k+1}} f)(x)
\]

as \( n \) tends to \( \infty \), where \( t^n_k = s + k2^{-n}(t - s) \). Indeed,

\[
\begin{align*}
\mathbb{E} \left[ \left( \int_s^t K_{s,u}(Wf(du))(x) - \sum_{k=0}^{2^n-1} K_{s,t^n_k}(W_{t^n_{k+1}} f)(x) \right)^2 \right] &= \\
&= \sum_{k=0}^{2^n-1} \int_{t^n_k}^{t^n_{k+1}} \mathbb{P}^{(2)}(I + \mathbb{P}^{(2)} - 2I \otimes \mathbb{P}^{(1)})(f,f)(x,x) \, du
\end{align*}
\]

which tends to 0 as \( n \) tends to \( \infty \).

**Proposition 5.4.2.** \( W \) is the unique vector field valued white noise such that for all \( s < t \), \( x \in M \) and \( f \in C^2_K(M) \), \( \mathbb{P} \)-almost surely,

\[
K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(Wf(du))(x) + \int_s^t K_{s,u}(Af)(x) \, du.
\]

Note that giving the local characteristics of the flow is equivalent to giving this SDE. This SDE will be called the \( (A,C) \)-SDE.

**Proof.** For all \( s < t \), from remark 5.4.1

\[
\int_s^t K_{s,u}(Wf(du))(x) = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} K_{s,t^n_k}(W_{t^n_{k+1}} f)(x)
\]
in $L^2(P)$, where $t^n_k = s + k2^{-n}(t-s)$.

For all integers $i, l, k$ and $n$ such that $l \geq n$ and $k2^{l-n} \leq i \leq (k+1)2^{l-n} - 1$, the development (5.27) implies

$$
E[M_{t^n_i,t^n_{i+1}} f(x)M_{t^n_k,t^n_{k+1}} f(x)] = 2^{-l}(t-s)C(f,f)(x,x) + o(2^{-l}),
$$

(5.34)

uniformly in $x \in M$. This implies that for $l \geq n$,

$$
E \left[ \left( \sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1} M_{t^n_i,t^n_{i+1}} f(x) - M_{t^n_k,t^n_{k+1}} f(x) \right)^2 \right] = o(2^{-n}),
$$

(5.35)

uniformly in $x \in M$. Taking the limit as $l$ goes to $\infty$, we get

$$
E[(W_{t^n_k,t^n_{k+1}} f(x) - M_{t^n_k,t^n_{k+1}} f(x))^2] = o(2^{-n}),
$$

(5.36)

uniformly in $x \in M$. We use this estimate to prove that

$$
\int_s^t K_{s,u}(W(du)f)(x) = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} K_{s,t^n_k}(M_{t^n_k,t^n_{k+1}} f)(x)
$$

(5.37)

in $L^2(P)$. This holds since

$$
E \left[ \left( \sum_{k=0}^{2^n-1} K_{s,t^n_k}(W_{t^n_k,t^n_{k+1}} f) - \sum_{k=0}^{2^n-1} K_{s,t^n_k}(M_{t^n_k,t^n_{k+1}} f) \right)^2(x) \right] =
$$

$$
= \sum_{k=0}^{2^n-1} E[(K_{s,t^n_k}(W_{t^n_k,t^n_{k+1}} f - M_{t^n_k,t^n_{k+1}} f))^2(x)]
$$

$$
\leq \sum_{k=0}^{2^n-1} P_{t^n_k-s} \left( E[(W_{t^n_k,t^n_{k+1}} f - M_{t^n_k,t^n_{k+1}} f)^2]\right)(x)
$$

$$
\leq 2^n o(2^{-n}) = o(1).
$$

Note now that

$$
\sum_{k=0}^{2^n-1} K_{s,t^n_k}(M_{t^n_k,t^n_{k+1}} f)(x) = \sum_{k=0}^{2^n-1} K_{s,t^n_k} \left( K_{t^n_k,t^n_{k+1}} f - f - \int_{t^n_k}^{t^n_{k+1}} K_{t^n_k,u}(Af) \, du \right)(x)
$$

$$
= K_{s,t} f(x) - f(x) - \int_s^t K_{s,u}(Af)(x) \, du.
$$

This proves that $K$ solves the $(A,C)$-SDE driven by $W$. Finally, note that if $K$ solves the $(A,C)$-SDE driven by a vector field valued white noise $W'$ then we must have $W' = W$. □

Let $X = (X_t, t \geq 0)$ be the Markov process defined in section 2.6 on $(\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), P(\omega) \otimes P_{x,\omega}(\omega'))$ by $X(\omega, \omega') = \omega'$.
Proposition 5.4.3. Assume there is no pure diffusion (i.e. for all \( f \in C^2_\Gamma(M) \) and all \( x \in M \), \( \Gamma(f)(x) = C(f, f)(x, x) \)). Then, for all \( t \geq 0 \), \( x \in M \) and \( f \in C^2_\Gamma(M) \), \( \mathbb{P}(d\omega) \otimes \mathbb{P}_{x,\omega}(d\omega') \)-almost surely,

\[
f(X_t) = f(x) + \int_0^t W(du) f(X_u) + \int_0^t Af(X_u) \, du,
\]

i.e. \( X \) is a weak solution of this SDE (in the sense given in \( \text{(5.39)} \)).

Proof. Like in the proof of \( \text{(5.37)} \) in proposition \( \text{5.4.2} \) we show that

\[
\int_0^t W(du) f(X_u) = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} M^f_{t_{k+1}} \, \times \, M^f_{t_k} \, f(X_{t_k}^n)
\]

(5.39)
in \( L^2(\mathbb{P}_x) \), with \( \mathbb{P}_x = \mathbb{P}(d\omega) \otimes \mathbb{P}_{x,\omega}(d\omega') \). Let \( M^f_t = f(X_t) - f(x) - \int_0^t Af(X_u) \, du \), then \( (M^f_t, t \geq 0) \) is a martingale relative to the filtration \( (\mathcal{F}^X_t, t \geq 0) \) generated by the Markov process \( X \). We now prove that \( \mathbb{E}_x[(M^f_t - \int_0^t W(du) f(X_u))^2] = 0 \), where \( \mathbb{E}_x \) denotes the expectation with respect to \( \mathbb{P}_x \). It is easy to see that, since there is no pure diffusion,

\[
\mathbb{E}_x[(M^f_t)^2] = \mathbb{E}_x[(\int_0^t W(du) f(X_u))^2] = \mathbb{E}_x[\int_0^t C(f, f)(X_u, X_u) \, du].
\]

(5.40)

Equation \( \text{(5.39)} \) and the martingale property of \( M^f_t \) implies that

\[
\mathbb{E}_x[M^f_t \int_0^t W(du) f(X_u)] = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mathbb{E}_x[M^f_{t_{k+1}} \times M^f_{t_k} \, f(X_{t_k}^n)].
\]

(5.41)

Since for all \( 0 \leq s < t \), \( \mathbb{E}_x[M^f_t - M^f_s | \mathcal{F}^X_s] = M^f_s f(X_s) \), we get

\[
\mathbb{E}_x[M^f_t \int_0^t W(du) f(X_u)] = \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mathbb{E}_x[(M^f_{t_{k+1}} - M^f_{t_k}) \times M^f_{t_k} \, f(X_{t_k}^n)]
\]

(5.42)

Therefore \( \mathbb{E}_x[(M^f_t - \int_0^t W(du) f(X_u))^2] = 0 \). \( \square \)

5.5 The \( (A, C) \)-SDE.

In this section and in the following, let \( A \) be a second order differential operator mapping \( C^2_\Gamma(M) \) in \( C_\Gamma(M) \) and \( C \) a continuous covariance on vector fields.

Definition 5.5.1. Let \( K \) be a stochastic flow of kernels and \( W \) a vector field valued white noise, defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\).
(i) \((K, W)\) is a solution of the \((A, C)\)-SDE if the covariance of \(W\) is \(C\) and \((K, W)\) satisfies (5.32) for all \(s < t, x \in M\) and \(f \in C^2(K(M))\).

(ii) \((K, W)\) is called a strong solution of the \((A, C)\)-SDE if moreover for all \(s \leq t, K_{s,t}\) is \(\mathcal{F}_{s,t}^{W}\)-measurable, where \(\mathcal{F}_{s,t}^{W}\) is the completion by all \(P\)-negligible sets of \(\mathcal{A}\) of the \(\sigma\)-field \(\sigma(W_{u,v}, s \leq u \leq v \leq t)\).

(iii) When a solution \((K, W)\) of the \((A, C)\)-SDE is not a strong solution, we say it is a weak solution.

Remark 5.5.2. Let \((K, W)\) be a solution of the \((A, C)\)-SDE and \(\nu\) the Feller convolution semigroup associated with \(K\). Then \(\nu\) is a diffusion convolution semigroup with local characteristics \((A, C)\).

The proof of this remark is left to the reader.

Remark 5.5.3. The fact that \((K, W)\) is a strong (respectively a weak) solution of the \((A, C)\)-SDE only depends on the law of \(K\). So that we can say shortly that \(K\) is a strong (respectively a weak) solution of the \((A, C)\)-SDE.

Definition 5.5.4. We will say that \((P_n^{(n)}, n \geq 1)\), a compatible family of Feller semigroups, or \(\nu = (\nu_t)\), a Feller convolution semigroups, defines a strong (respectively a weak) solution of the \((A, C)\)-SDE if \(P_{\nu}\) is the law of a stochastic flow of kernels, which is a strong (respectively a weak) solution of the \((A, C)\)-SDE.

Definition 5.5.5. We say that (strong) uniqueness holds for the \((A, C)\)-SDE when there is only one diffusion convolution semigroup with local characteristics \((A, C)\) defining a (strong) solution.

5.6 Strong solution and filtering.

Let us now consider the canonical flow associated with \(\nu\), a diffusion convolution semigroup, with local characteristics \((A, C)\). Let \(N_{\nu}^W := (\Omega^0, A^0, (\mathcal{F}_{s,t}^W)_{s \leq t}, P_{\nu}, (T_h)_{h \in \mathbb{R}})\) be the noise generated by the vector field valued white noise \(W\). Note that \(N_{\nu}^W\) is a linear or Gaussian sub-noise of \(N_{\nu}\), the noise generated by the canonical flow.

Let \(\bar{K} = (\bar{K}_{s,t}, s \leq t)\) be the stochastic flow of kernels obtained by filtering the canonical flow with respect to the sub-noise \(N_{\nu}^W\) (see section 3.3). It is easy to see that \(\bar{K}\) also solves the \((A, C)\)-SDE (see the proof of lemma 3.9 in [23]) and has the same local characteristics as the canonical flow. Since, for all \(s \leq t, \bar{K}_{s,t}\) is \(\mathcal{F}_{s,t}^{W}\)-measurable, \((\bar{K}, W)\) is a strong solution of the \((A, C)\)-SDE. Let \(\nu^*\) denote the associated diffusion convolution semigroup.

5The noise \((G_{s,t})_{s \leq t}\) is Gaussian if and only if there exists a countable family of independent real white noises \(\{W^\alpha\}\) such that, up to negligible sets, \(G_{s,t}\) is generated by the random variables \(W^\alpha_{u,v}\) for all \(s \leq u \leq v \leq t\) and all \(\alpha\).
For any $f \in C_0(M)$ and $x \in M$, $\bar{K}_{s,t}f(x)$ can be expanded into a sum of Wiener chaos elements, i.e. iterated Wiener integrals of the form

$$\sum_{\alpha_1, \ldots, \alpha_n} \int C^{\alpha_1, \ldots, \alpha_n}(s_1, \ldots, s_n) \, dW_{s_n}^{\alpha_n} \cdots dW_{s_1}^{\alpha_1}. \quad (5.43)$$

Since $W$ was constructed from the flow, it is clear that the functions $C^{\alpha_1, \ldots, \alpha_n}$ are determined by the law of the flow (we will give, under some additional assumptions, an explicit form of them in the following section).

### 5.7 The Krylov-Veretennikov expansion.

We still assume we are given $\nu = (\nu_t)_{t \geq 0}$ a diffusion convolution semigroup, in the sense of section 5.1 associated with a set of local characteristics $(A, C)$.

We suppose in this section the existence of a Radon measure $m$ on $M$ such that $A$ is symmetric with respect to $m$.

Moreover, we assume that $\text{Im}(I - A)$ is dense in $C_0(M)$ (it implies that $P_t^{(1)}$ is symmetric with respect to $m$ and is the unique Feller semigroup whose generator extends $A$).

Following [23], starting from the vector field valued white noise $W$, one can define $(S_{s,t}, \ s \leq t)$ a stochastic flow of Markovian operators (acting on $L^2(m)$) such that for all $s \leq t$, $S_{s,t}$ is $\sigma(W)$-measurable and for $f \in L^2(m)$ and $s \leq u \leq t$,

$$S_{s,t}f = S_{s,u}S_{u,t}f,$$

$$S_{s,t}f = P_{t-s}^{(1)}f + \int_s^t S_{s,u}W(du)P_{t-u}^{(1)}f,$$

where both equalities hold in $L^2(m \otimes P)$. These operators are given by the Wiener chaos expansion (called Krylov-Veretennikov expansion)

$$S_{s,t}f = P_{t}^{(1)}f + \sum_{n \geq 1} J_{s,t}^n f, \quad (5.44)$$

with

$$J_{s,t}^n f = \int_{s \leq s_1 \leq \cdots \leq s_n \leq t} P_{s_1-s}^{(1)}W(ds_1)P_{s_2-s_1}^{(1)} \cdots P_{s_n-s_{n-1}}^{(1)} W(ds_n)P_{t-s_n}^{(1)}f. \quad (5.45)$$

They can be characterized (theorem 3-2 in [23]) as the unique flow of random operators on $L^2(m)$, $\sigma(W)$-measurable, such that $\mathbb{E}[(S_{s,t}f)^2] \leq P_{t-s}^{(1)}f^2$ and

$$S_{s,t}f - f = \int_s^t S_{s,u}W(du)f + \frac{1}{2} \int_s^t S_{s,u}\bar{A}f \, du \quad \text{in } L^2(m \otimes P) \quad (5.46)$$

for all $f$ in the domain of the $L^2$-generator $\bar{A}$, denoted $\mathcal{D}(\bar{A})$. It implies the following

Proposition 5.7.1. (a) If $\nu$ defines a strong solution $(K, W)$ of the $(A, C)$-SDE, then for all $s \leq t$, $m \otimes P$-a.e., for all $f \in C_K(M)$,

$$K_{s,t}f = S_{s,t}f \quad (5.47)$$
(b) Strong uniqueness holds.

Proof. (a) It is clear that $K$ induces a flow of Markovian operators on $L^2(m)$ which verifies \eqref{5.46} for $f \in C^2_K(m)$. Then \eqref{5.46} extends to functions in the domain of the Feller generator and finally to $\mathcal{D}(A)$.

(b) From (a), it is clear that $m^{\otimes n}$-a.e., $P_t^{(n)} = E[S_{0,t}^{\otimes n}]$. Since it is a Feller semigroup, it is uniquely determined. ☐

6 Noise and classification.

6.1 Assumptions.

In this section, as before $M$ denotes a smooth locally compact manifold. We fix a pair of local characteristics $(A, C)$ on $M$. $A$ is a second order differential operator mapping $C^2_K(M)$ in $C_K(M)$ and $C$ a continuous covariance on vector fields. The associated differential operators $A^{(n)}$ on $C^2_K(M)^{\otimes n}$ are defined by equation \eqref{5.11}.

Let $\mathcal{M}(n, x)$ be the following martingale problem associated with $A^{(n)}$ and $x \in M^n$:

There exists a probability space on which is constructed a $M^n$-valued stochastic process $X^{(n)} = (X_t^{(n)}, t \geq 0)$ such that

$$f(X_t^{(n)}) - f(x) - \int_0^t A^{(n)} f(X_s^{(n)}) \, ds$$

(6.1)

is a martingale for all test function $f$ in $C^2_K(M) \otimes \cdots \otimes C^2_K(M)$.

We suppose that the local characteristics $(A, C)$ verify the following assumption

(U) For all $n \geq 1$, the martingale problem $\mathcal{M}(n, x)$ has a unique solution in law on the set of continuous trajectories stopped at $\Delta_n$.

Remark 6.1.1. Condition (U) is satisfied when the coefficients of the local characteristics are $C^2$ outside of $\Delta_n$ (see theorem 12.12 and section V.19 in \cite{39}) or when $A^{(n)}$ is elliptic outside of $\Delta_n$ (see section V.24 in \cite{39}).

Our purpose is to classify Feller convolution semigroups associated with these local characteristics. We will treat two cases

(A) The non coalescing case where the solution of the martingale problem $\mathcal{M}(2, x)$ does not hit the diagonal when $x = (x_1, x_2)$ with $x_1 \neq x_2$.

(B) The coalescing case where

there is no pure diffusion (i.e. $(\frac{1}{2}Af^2 - fAf)(x) = C(f, f)(x, x)$ for all $f \in C^2_K(M)$ and $x \in M$)

and where assumption

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(C) For all \( t > 0, \varepsilon > 0 \) and \( x \in M \), \( \lim_{y \to x} P_{(x,y)}^{(2)}([T_\Delta > t] \cap \{d(X_t, Y_t) > \varepsilon\}] = 0 \) and for some \((x, y) \in M^2, P_{(x,y)}^{(2)}[T_\Delta < \infty] > 0\)

holds for \( X_t^{(2)} = (X_t, Y_t) \) a solution of \( \mathcal{M}(2, x) \).

When the local characteristics are non coalescing (case (A)), these local characteriscs are associated with at most a unique convolution semigroup and a unique canonical flow (which is not always a flow of maps). From section 5.5 we know the latter has to be a strong solution of the SDE (otherwise uniqueness would be violated). Assumption (F) (see section 1.7) is a sufficient (but not necessary) condition for existence. The family of semigroups given in the example of Lipschitz SDE’s (see section 1.7) satisfies these assumptions.

In sections 6.2, 6.3 and 6.4 we assume (B) is satisfied.

6.2 The coalescing case : classification.

Following Harris [15], \( \mathcal{M}(n, x) \) has a unique solution in law on the set of coalescing trajectories, i.e. \( X^{(n)}(\omega) \in C^{(n)} \) where \( C^{(n)} \) is the set of continuous functions \( f : \mathbb{R}^+ \to M^n \) such that if \( f_i(s) = f_j(s) \) for \( 1 \leq i, j \leq n \) and \( s \geq 0 \) then for all \( t \geq s \), \( f_i(t) = f_j(t) \) (In [15], this martingale problem is solved when \( M = \mathbb{R} \), but the proof can obviously be adapted to our framework). Since (C) holds, remark 4.3.3 implies that the associated semigroups are Feller.

Hence all coalescing flows with these local characteristics have the same law \( P_{\nu^c} \). They induce the same family of semigroups \( (P_t^{(n), c}, n \geq 1) \) and the same diffusion convolution semigroup \( \nu^c \).

Let \( N_{\nu^c} \) be the noise generated by the canonical coalescing flow associated with the local characteristics \((A, C)\).

Let \( W \) be the vector field valued white noise defined on \((\Omega^0, \mathcal{A}^0, P_{\nu^c})\) in section 5 and \( N_{\nu^c}^W \) the sub-noise of \( N_{\nu^c} \) generated by \( W \). Then \( N_{\nu^c}^W \) is a Gaussian sub-noise of \( N_{\nu^c} \) and it is possible to represent it by a countable family of independent real white noises \( \{W^\alpha\} \) such that \( W = \sum \alpha V^\alpha W^\alpha \), where \( \{V^\alpha\} \) is a countable family of vector fields on \( M \).

We denote by \( \nu^s \) the diffusion convolution semigroup associated with the flow obtained by filtering the canonical coalescing flow of law \( P_{\nu^c} \) with respect to \( N_{\nu^c}^W \).

The following theorem gives a representation of all flows with the same local characteristics. They lie “between” the strong solution and the coalescing solution of the SDE which are distinct when the coalescing solution is not a strong solution of the SDE.

**Theorem 6.2.1.** Suppose we are given a set of local characteristics \((A, C)\) and that assumption (B) is verified.

(a) \( \nu^c \) is the unique diffusion convolution semigroup associated with \((A, C)\) and defining a flow of maps (which is coalescing).

(b) \( \nu^s \) is the unique diffusion convolution semigroup associated with \((A, C)\) and defining a strong solution of the \((A, C)\)-SDE.
(c) The diffusion convolution semigroups associated with \((A,C)\) are all the Feller convolution semigroups weakly dominated by \(\nu^c\) and dominating \(\nu^s\).

Note that \(\nu^c\) and \(\nu^s\) are not necessarily distinct.

**Proof.** We have already proved (a) at the beginning of this section. Theorem 4.3.2 implies that every diffusion convolution semigroup \(\bar{\nu}\) with local characteristics \((A,C)\) is weakly dominated by \(\nu^c\) so that a stochastic flow \(\bar{K}\) of law \(P_{\bar{\nu}}\) can be obtained by filtering on an extension \((N,\varphi)\) of \(N_{\nu^c}\) the coalescing flow \(\varphi\) with respect to a sub-noise \(\bar{N}\) of \(N\).

Let \(\bar{W}\) be the velocity field associated with \(\bar{K}\). Proposition 5.4.2 shows that \((\bar{K},\bar{W})\) solves the \((A,C)\)-SDE. Notice that \(\bar{W}\) can be obtained by filtering \(W\) with respect to \(\bar{N}\). Indeed, section 5.3 shows that \(W^n_{s,t}\) (defined from \(K\)) converges (in \(L^2\)) towards \(\bar{W}_{s,t}\) and we have that for all \(s \leq t\), \(f \in C^2_F(M)\) and \(x \in M\), \(\bar{W}^n_{s,t}f(x) = E[W^n_{s,t}f(x)|\mathcal{F}_{s,t}]\) a.s. and therefore that \(\bar{W}_{s,t}f(x) = E[W_{s,t}f(x)|\mathcal{F}_{s,t}]\) a.s. Since \(\bar{W}\) and \(W\) have the same law, we must have \(W_{s,t} = \bar{W}_{s,t}\) a.s. This proves that \(\bar{\nu}\) dominates \(\nu^s\).

Let us now suppose that \((\bar{K},\bar{W})\) is a strong solution of the \((A,C)\)-SDE. Then, since \(\bar{W} = W\), we must have \(N_{\nu^c} = \bar{N}\) (since \(\bar{K}_{s,t}\) is \(\mathcal{F}^W_{s,t}\)-measurable) and thus \(\nu^s = \bar{\nu}\). This proves the strong uniqueness for the \((A,C)\)-SDE.

Finally let \(\bar{\nu}\) be a Feller convolution semigroup weakly dominated by \(\nu^c\) and dominating \(\nu^s\). The fact that \(\bar{\nu} \preceq \nu^c\) implies that a stochastic flow \(\bar{K}\) of law \(P_{\bar{\nu}}\) can be obtained by filtering on an extension \((N,\varphi)\) of \(N_{\nu^c}\) the coalescing flow \(\varphi\) with respect to a sub-noise \(\bar{N}\) of \(N\). Then section 5.3 shows that \(W^n_{s,t}\) (defined from \(K\)) converges (in \(L^2\)) towards \(W_{s,t} = E[W_{s,t}|\mathcal{F}_{s,t}]\). Now, since \(\bar{\nu} \succeq \nu^s\), there exists (see lemma 3.3.5) a sub-noise \(\bar{N}\) of \(N\) such that the flow obtained by filtering \(\bar{K}\) or equivalently, the coalescing flow, with respect to \(\bar{N}\) has law \(P_{\bar{\nu}}\). The associated white noise \(\bar{W}\) verifies for all \(s \leq t\), \(x \in M\) and \(f \in C^2_F(M)\)

\[
\bar{W}_{s,t}f(x) = E[W_{s,t}f(x)|\bar{\mathcal{F}}_{s,t}] = E[W_{s,t}|\bar{\mathcal{F}}_{s,t}].
\]

Since \(\bar{W}\) has covariance \(C\), it has to coincide with \(W\) and \(\bar{W} = W\).

Thus, \((\bar{K},\bar{W})\) solves the \((A,C)\)-SDE so that \(\bar{\nu}\) is a diffusion convolution semigroup whose local characteristics are \((A,C)\). □

### 6.3 The coalescing case: martingale representation.

On the probability space \((\Omega^0,\mathcal{A}^0,\mathbb{P}_{\nu^c})\), let \(\mathcal{F}^{\nu^c}\) be the filtration \((\mathcal{F}^{\nu^c}_t)_{t \geq 0}\) and \(\mathcal{M}(\mathcal{F}^{\nu^c})\) be the space of locally square integrable \(\mathcal{F}^{\nu^c}\)-martingales.

**Proposition 6.3.1.** For all \(\mathcal{F}^{\nu^c}\)-martingale \(M = (M_t)_{t \in \mathbb{R}^+}\), there exist predictable processes \(\Phi^\alpha = (\Phi^\alpha_s)_{s \geq 0}\) such that

\[
M_t = \sum \int_0^t \Phi^\alpha_s W^\alpha(ds).
\]

**Remark 6.3.2.** Of course, this does not imply that \(\mathcal{F}^{\nu^c}\) is generated by \(W\).

**Proof.** We follow an argument by Dellacherie (see Rogers-Williams (V-25)). Suppose there exists \(F \in L^2(\mathcal{F}^{\nu^c}_{0,\infty})\) orthogonal in \(L^2(\mathcal{F}^{\nu^c}_{0,\infty})\) to all stochastic integrals of \(W^\alpha\) of the form (6.3), then \(M_t = E[F|\mathcal{F}^{\nu^c}_0]\) is orthogonal to \(W^\alpha\) for all \(\alpha\), i.e. \((M,W^\alpha) = 0\).

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Let $\tau = \inf\{t, \ |M_t| = 1/2\}$ and $\hat{P}_{\nu^c} = (1 + M_\tau) \cdot P_{\nu^c}$. Since $M$ is a uniformly integrable martingale and $\tau$ a stopping time (with $1 + M_\tau \geq 1/2$), $\hat{P}_{\nu^c}$ is a probability measure on $(\Omega^0, \mathcal{A}^0)$. Since $\langle M, W^\alpha_0 \rangle_t = 0$, we get that under $\hat{P}_{\nu^c}$, $(W^\alpha_0)_\alpha$ is a family of independent Brownian motions.

We are now going to prove that since $(\mathcal{U})$ is satisfied, we must have $P_{\nu^c} = \hat{P}_{\nu^c}$, which implies $M_t = 0$ and a contradiction.

Let $F = \prod_{i=1}^n f_i(\varphi_{0,t_i}(x_i))$, for $f_1, \ldots, f_n$ in $C^2_0(M)$, $t_1, \ldots, t_n$ in $\mathbb{R}^+$ and $x_1, \ldots, x_n$ in $M$. We know that under $P_{\nu^c}$, for all $1 \leq i \leq n$, $(\varphi_{0,t_i}(x_i), \ t \geq 0)$ is a solution of the SDE

$$dg_i(\varphi_{0,t_i}(x_i)) = \sum_\alpha V_\alpha g_i(\varphi_{0,t_i}(x_i)) W^\alpha(dt) + Af(\varphi_{0,t_i}(x_i))dt,$$

(6.4)

for all $g_1, \ldots, g_n$ in $C^2_0(M)$. Note that under $\hat{P}_{\nu^c}$, these SDEs are also satisfied. Since under $\hat{P}_{\nu^c}$, $(W^\alpha)_\alpha$ is a family of independent Brownian motions, $((\varphi_{0,t_i}(x_i), \ t \geq 0), \ 1 \leq i \leq n)$ is a coalescing solution of the martingale problem associated with $A^{(n)}$ and $(\mathcal{U})$ implies that the law of $((\varphi_{0,t_i}(x_i), \ t \geq 0), \ 1 \leq i \leq n)$ is the same under $P_{\nu^c}$ and under $\hat{P}_{\nu^c}$. Therefore $\hat{E}[F] = E[F]$, where $\hat{E}$ denotes the expectation with respect to $\hat{P}_{\nu^c}$.

To conclude that $\hat{P}_{\nu^c} = P_{\nu^c}$, we need to prove $\hat{E}[F] = E[F]$ with $F = \prod_{i=1}^n f_i(\varphi_{s,t_i}(x_i))$ for all $f_1, \ldots, f_n$ in $C^2_0(M)$, $0 \leq s_i < t_i$ in $\mathbb{R}^+$ and $x_1, \ldots, x_n$ in $M$. This can be proved the same way but using the kernel $\tilde{K}_t$ introduced in section 2.6. In this case $\tilde{K}_t = \delta_{\tilde{\varphi}_t}$, where $\tilde{\varphi}_t : \mathbb{R}^+ \times M \to \mathbb{R}^+ \times M$ is measurable. Then $F = \prod_{i=1}^n f_i(\tilde{\varphi}_{t_i}(s_i, x_i))$ and $(\tilde{\varphi}_t(s_i, x_i), \ t \geq 0)$ is a solution of an SDE on $\mathbb{R}^+ \times M$. \[\Box\]

6.4 The coalescing case : the linear noise.

Let us remark that if $\nu$ is a diffusion convolution semigroup, then $N_{\nu}$ is a predictable noise (see proposition 3.5.1), i.e. $\mathcal{M}(\mathcal{F}^\nu)$ is formed of continuous martingales (in particular, a Gaussian noise is predictable). Following Tsirelson [41], a linear representation of a predictable noise $N = (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s \leq t}, P, (T_h)_{h \in \mathbb{R}})$ is a family of real random variables $X = (X_{s,t}, \ s \leq t)$ such that

(a) $X_{s,t} \circ T_h = X_{s+h,t+h}$ for all $s \leq t$ and all $h \in \mathbb{R},$

(b) $X_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for all $s \leq t$,

(c) $X_{r,s} + X_{s,t} = X_{r,t}$ a.s., for all $r \leq s \leq t$.

The space of linear representations is a vector space. Equipped with the norm $\|X\| = (\mathbb{E}[|X_{0,1}|^2])^{1/2}$, it is a Hilbert space we denote by $H_{\text{lin}}$. Let $H^0_{\text{lin}}$ be the orthogonal in $H_{\text{lin}}$ of the one-dimensional vector space constituted of the representation $X_{s,t} = v(t-s)$ for $v \in \mathbb{R}$, then $H^0_{\text{lin}}$ is constituted with the centered linear representations. Note that if $X \in H^0_{\text{lin}}$ with $\|X\| = 1$, then $(X_{0,t})_{t \geq 0}$ is a standard Brownian motion. The Hilbert space $H^0_{\text{lin}}$ is a Gaussian system and every $X \in H^0_{\text{lin}}$ is a real white noise.

Note that if $X$ and $Y$ are orthogonal linear representations then $X$ and $Y$ are independent.
For all \(-\infty \leq s \leq t \leq \infty\), let \(\mathcal{F}_{s,t}^{\text{lin}}\) be the \(\sigma\)-field generated by the random variables \(X_{u,v}\) for all \(X \in H_0^{\text{lin}}\) and \(s \leq u \leq v \leq t\), and completed by all \(\mathbb{P}\)-negligible sets of \(\mathcal{F}_{-\infty,\infty}\). Then \(N_{\text{lin}} := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t}^{\text{lin}})_{s \leq t}, \mathbb{P}, (T_h)_{h \in \mathbb{R}})\) is a noise. It is called the linearizizable part of the noise \(N\). The noise \(N_{\text{lin}}\) is a maximal Gaussian sub-noise of \(N\), hence \(N\) is Gaussian if and only if \(N_{\text{lin}} = N\). When \(N_{\text{lin}}\) is trivial (i.e. constituted of trivial \(\sigma\)-fields), one says that \(N\) is a black noise (when \(N\) is not trivial).

**Theorem 6.4.1.** \(N_{\nu}^{\text{lin}} = N_{\nu}^{\text{lin}}\).

**Proof.** Let \(H^W\) be the space of centered linear representations of the noise \(N_{\nu}^{\text{lin}}\). Then \(H^W\) is an Hilbert space (an orthonormal basis of \(H^W\) is given by \(\{(W_0^\alpha)_{s \leq t}\}\) and we have \(H^W \subset H_0^{\text{lin}}\). This implies that \(N_{\nu}^{\text{lin}}\) is a Gaussian sub-noise of \(N_{\nu}^{\text{lin}}\).

If \(N_{\nu}^{\text{lin}} \neq N_{\nu}^{\text{lin}}\) then there exists a linear representation \(X \neq 0 \in H_0^{\text{lin}}\) orthogonal to \(H^W\) and therefore independent of \(\{W^\alpha\}\). Since \((X_{0,t})_{t \geq 0} \in \mathcal{M}(\mathcal{F})\), proposition 6.3.1 implies that the martingale bracket of \(X_{0,t}\) equals 0. This is a contradiction. \(\square\)

In section 4.4.3 we give an example of a stochastic coalescing flow whose noise is predictable but not Gaussian. It is an example of non-uniqueness of the diffusion convolution semigroup associated with a set of local characteristics.

**Remark 6.4.2.** In example 4.4.3 although the covariance function \(C\) is not continuous, it is still possible to construct a white noise \(W\) from the coalescing flow \((\varphi_{s,t}, s \leq t)\). For all \(s < t\), we set \(W_{s,t} = \int_s^t \text{sgn}(\varphi_{s,u}(0)) \, d\varphi_{s,u}(0)\). Then we have \(W_{s,t} = \int_s^t \text{sgn}(\varphi_{s,u}(x)) \, d\varphi_{s,u}(x)\) for all \(x \in \mathbb{R}\). Therefore one can check that \(W = (W_{s,t}, s \leq t)\) is a real white noise.

The coalescing flow \((\varphi_{s,t}, s \leq t)\) solves the SDE

\[
\varphi_{s,t}(x) = \int_s^t \text{sgn}(\varphi_{s,u}(x)) \, dW_u, \quad \text{for } s < t \text{ and } x \in \mathbb{R}.
\]  

(6.5)

The results of this subsection apply since proposition 6.3.1 is also satisfied if we only assume the uniqueness in law of the coalescing solutions \(\varphi\) of the SDE satisfied by the \(n\)-point motion (i.e. the SDE (6.4)), which here is almost obvious. Therefore, the linear part of the noise generated by the coalescing flow is given by the noise generated by \(W\). But since the strong solution of the SDE (6.5) is not a flow of mappings, the coalescing flow is not a strong solution. Therefore, we recover the result of Warren [45] and Watanabe [46] that the noise of this stochastic coalescing flow is predictable but not Gaussian.

The strong solution given in section 4.4.3 can be recovered by filtering the coalescing solution with respect to the noise generated by \(W\).

### 7 Isotropic Brownian flows.

In this section, we give examples of compatible families of Feller semigroups. They are constructed on \(M\), a two-point symmetric space, with \(C\) an isotropic covariance function on the space of vector fields and the semigroup of a Brownian motion on \(M\).

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Footnote: i.e. such that if \((X^1, \ldots, X^n)\) solves the SDE then if for \(i \neq j\) and \(s \geq 0\), \(X^i_s = X^j_s\) then \(X^i_t = X^j_t\) for all \(t \geq s\).
7.1 Isotropic covariance functions.

Let $M = G/K$ be a two-point symmetric space. This class of spaces includes euclidean spaces, hyperbolic spaces and spheres, see [16], chapter III. $G$ is the group of isometries on $M$. A covariance function $C$ is said isotropic if

$$C(g \cdot \xi, g \cdot \xi') = C(\xi, \xi') \quad (7.1)$$

for all $g \in G$ and $(\xi, \xi') \in (T^*M)^2$ and where $g \cdot \xi = Tg(\xi)$ (or $g \cdot (x, u) = (gx, Tgxu)$ for $(x, u) \in T^*M$).

Examples of isotropic covariances are given by Monin and Yaglom in [32] on $\mathbb{R}^d$ and by Raimond [36, 37] on the sphere and on the hyperbolic plane. In these examples, the group $G$ of isometries on $\mathbb{R}^d$ (making $\mathbb{R}^d$ homogeneous) is generated by $O(d)$ and by the translations. For the sphere $\mathbb{S}^d$, this group is $O(d + 1)$ and for the hyperbolic space, it is $O(d, 1)$.

7.2 A compatible family of Markovian semigroups.

Let $C$ be an isotropic covariance on $\mathcal{X}(M)$, the space of vector fields on the two-point symmetric space $M = G/K$. To this isotropic covariance function is associated a Brownian vector field on $M$ (i.e. a $\mathcal{X}(M)$-valued Brownian motion $W$ such that $E[\langle W_t, \xi \rangle \langle W_s, \xi' \rangle] = t \wedge s C(\xi, \xi')$). Let $\mathcal{P}$ be the associated Wiener measure, constructed on the canonical space $\Omega = \{\omega : \mathbb{R}^+ \to \mathcal{X}(M)\}$, equipped with the $\sigma$-field $\mathcal{A}$ generated by the coordinate functions.

We denote by $W$ the random variable $W(\omega) = \omega$. $W$ is a Brownian vector field of covariance $C$ which is isotropic in the sense that for all $g \in G$, $(Tg^{-1}W_t(gx), t \in \mathbb{R}^+, x \in M)$ is a Brownian vector field of covariance $C$.

Let $P_t$ be the heat semigroup on $M$, $m$ the volume element and $\Delta$ the Laplacian.

Let $(S_t, t \geq 0)$ be the family of random operators defined in [23], associated with $W$ and to the heat semigroup $P_t$. Following [23], we define the associated semigroups of the $n$-point motion, $P_t^{(n)} = E[S_t^{\otimes n}]$ (with $P_t^{(1)} = P_t$). Then, it is obvious that $(P_t^{(n)}, n \geq 1)$ is a compatible family of Markovian semigroups of operators acting on $L^2(m^{\otimes n})$. We now prove that these semigroups are induced by Feller semigroups (the question was raised in [28]).

One can extend $(W_t)_{t \geq 0}$ into a vector field valued white noise $(W_{s,t}, s \leq t)$ of covariance $C$ such that $W_t = W_{0,t}$ for $t \geq 0$ and associate to it a stationary cocycle of random operators $(S_{s,t}, s \leq t)$ such that $S_{0,t} = S_t$ for $t \geq 0$.

7.3 Verification of the Feller property.

For all $g \in G$, let $L_g : \Omega \to \Omega$ defined by $L_g \omega(t) = Tg^{-1}(\omega_t(g\cdot))$, for all $t \in \mathbb{R}$ and $x \in M$. Then $L_g$ is linear and for all $g_1$ and $g_2$ in $G$, $L_{g_1g_2} = L_{g_1}L_{g_2}$ (i.e. $g \to L_g$ is a representation of $G$). It is easy to check that for all $g \in G$, $(L_g)^* \mathcal{P} = \mathcal{P}$. Note that this last condition is also a characterization that $C$ is isotropic.

For all $g \in G$, $L_g$ induces a linear transformation on $L^2(\Omega, \mathcal{A}, \mathcal{P})$ we will also denote by $L_g$. Then for all $f \in L^2(\Omega, \mathcal{A}, \mathcal{P})$, we have $L_g f(\omega) = f(L_g \omega)$. This transformation is unitary since

$$\|L_g f\|^2 = \int f^2(L_g \omega) \mathcal{P}(d\omega) = \int f^2(\omega) ((L_g)^* \mathcal{P})(d\omega) = \|f\|^2,$$
(where \( \| \cdot \| \) denotes the \( L^2(\mathcal{P}) \)-norm).

**Proposition 7.3.1.** For all \( v \in L^2(\Omega, \mathcal{A}, \mathcal{P}) \), the mapping \( g \mapsto L_g v \) is continuous.

**Proof.** Note that, since \( L \) is a representation, it is enough to prove the continuity at \( e \), the identity element in \( G \).

**Remark 7.3.2.** Let \( (v_n, n \in \mathbb{N}) \) be a sequence in \( L^2(\Omega, \mathcal{A}, \mathcal{P}) \) converging towards \( v \in L^2(\Omega, \mathcal{A}, \mathcal{P}) \) as \( n \to \infty \) such that \( \lim_{g \to e} L_g v_n = v_n \) for all integer \( n \), then \( \lim_{g \to e} L_g v = v \).

Indeed, since for all \( g \in G \), \( L_g \) is unitary, \( \| L_g v - v \| \leq 2\| v_n - v \| + \| L_g v_n - v_n \| \). Hence \( \lim \sup_{g \to e} \| L_g v - v \| \leq 2\| v_n - v \| \) for all integer \( n \).

We first prove that \( \lim_{g \to e} L_g v = v \) for every \( v \) of the form \( \sum_i W_t(\xi_i) \) (with \( W_t(x, u) = \langle W_t(x), u \rangle \), where \( \langle \cdot , \cdot \rangle \) denotes the Riemannian metric):

\[
\| L_g \left( \sum_i W_t(\xi_i) \right) - \sum_i W_t(\xi_i) \|^2 = 2 \sum_{i,j} t_i \wedge t_j (C(\xi_i, \xi_j) - C(g \cdot \xi_i, \xi_j))
\]

which converges towards 0 as \( g \) tends to \( e \).

Let \( H \) denote the closure (in \( L^2(\Omega, \mathcal{A}, \mathcal{P}) \)) of the class of all \( v \) of the form \( \sum_i W_t(\xi_i) \). Remark 7.3.2 implies that \( \lim_{g \to e} L_g v = v \) holds for all \( v \in H \).

It is well known that \( L^2(\Omega, \mathcal{A}, \mathcal{P}) \) is the orthogonal sum of the Wick powers \( H^n \) of \( H \) (See [40]), also called the \( n \)-th Wiener chaos (see [33]), \( H^0 \) is constituted by the constants. The space \( H^n \) is isometric to the symmetric tensor product Hilbert space \( H^\otimes n \). We now prove that \( \lim_{g \to e} L_g v = v \) holds for all \( v \in H^n \). For all \( v = v_1 \otimes^s \cdots \otimes^s v_n \in H^n \) (or : \( v_1 v_2 \cdots v_n \) : in wick notation), we have

\[
\| L_g v - v \| \leq \sum_j \| L_g v_1 \otimes^s \cdots \otimes^s L_g v_{j-1} \otimes^s (L_g v_j - v_j) \otimes^s v_{j+1} \otimes^s \cdots \otimes^s v_n \|
\]

\[
\leq \sqrt{n!} \sum_j \| L_g v_j - v_j \| \times \prod_{i \neq j} \| v_i \|
\]

which converges towards 0 as \( g \) tends to \( e \). Since the class of linear combinaisons of elements of the form \( v_1 \otimes^s \cdots \otimes^s v_n \) is dense in \( H^n \), we have \( \lim_{g \to e} L_g v = v \) for all \( v \in H^n \). And we conclude since \( L^2(\Omega, \mathcal{A}, \mathcal{P}) = \oplus_{n \geq 0} H^n \).

For all \( x \in M, s \leq t \) and \( f \in C_0(M) \), since \( P^{(1)}_{\varepsilon} \) is absolutely continuous with respect to \( m \), we have

\[
P^{(1)}_{\varepsilon} S_{s+\varepsilon',t} f(x) = E[P^{(1)}_{\varepsilon'} S_{s+\varepsilon',t} f(x) | \mathcal{F}_{s+\varepsilon',t}],
\]

for \( 0 < \varepsilon' \leq \varepsilon \). Thus, for all \( s < t \), \( P^{(1)}_{\varepsilon} S_{s+\varepsilon',t} f(x) \) is a martingale as \( \varepsilon \) decreases. This martingale converges and we denote its limit by \( K_{s,t} f(x) \). Then \( S_{s,t} f = K_{s,t} f \) in \( L^2(m \otimes \mathcal{P}) \) and \( P^{(n)}_t = \tilde{P}^{(n)}_t m^{\otimes n}\)-a.e., where \( \tilde{P}^{(n)}_t \) denotes \( E[K_{s,t}^{\otimes n}] \).

**Lemma 7.3.3.** The mapping \( x \mapsto K_{s,t} f(x) \) is continuous for all Lipschitz function \( f \) and all \( s \leq t \).
Proof. Note that for all \( g \in G \) and all \( x \in M \),
\[
L_g K_{s,t} f(x) = K_{s,t} f^g(x)
\]
where \( f^g(x) = f(g^{-1}x) \). We then have
\[
\|K_{s,t} f(gx) - K_{s,t} f(x)\| \leq \|K_{s,t} f(gx) - K_{s,t} f^g(x)\| + \|L_g K_{s,t} f(x) - K_{s,t} f(x)\|.
\]

Hence \( \lim_{g \to e} K_{s,t} f(gx) = K_{s,t} f(x) \) since \( \lim_{g \to e} L_g K_{s,t} f(x) = K_{s,t} f(x) \) and \( \|K_{s,t} f(gx) - K_{s,t} f^g(x)\| \leq \|f - f^g\|_\infty \) which converges towards 0 (since \( |f(x) - f^g(x)| \leq Cd(x, g^{-1}x) \), which converges towards 0 as \( g \to e \)). This implies the lemma. \( \square \)

Proposition 7.3.4. (a) \( (\tilde{P}^{(n)}_t, \ n \geq 1) \) is a compatible family of Feller semigroups.

(b) The associated convolution semigroup \( \nu^* = (\nu^*_t)_{t \geq 0} \) is a diffusion convolution semigroup with local characteristics \( (\frac{1}{2} \Delta, C) \).

Proof. For all bounded Lipschitz functions \( f_1, \ldots, f_n \), lemma 7.3.3 implies that \( (x_1, \ldots, x_n) \mapsto \tilde{P}^{(n)}_t f_1 \otimes \cdots \otimes f_n(x_1, \ldots, x_n) = \mathbb{E}[\prod_{i=1}^n K_{s,t} f_i(x_i)] \) is continuous. This suffices to prove (a) (the proof that \( \lim_{t \to 0} P^{(n)}_t h(x) = h(x) \) for all \( h \in C(M^n) \) is the same as in lemma 1.7.1).

To prove (b), notice that Itô’s formula for \( (S_{s,t}, \ s \leq t) \) (see theorem 3.2 in [23]) implies that for all \( f \in C^2_K(M) \) and \( s \leq t \),
\[
K_{s,t} f(x) = f(x) + \int_s^t K_{s,u}(W f(du))(x) + \frac{1}{2} \int_s^t K_{s,u}(\Delta f)(x) \, du,
\]
i.e. \((K, W)\) solves the \((\frac{1}{2} \Delta, C)\)-SDE. \( \square \)

7.4 Classification.

Let \( \nu^* \) be the diffusion convolution semigroup constructed above. It defines a strong solution of the \((\frac{1}{2} \Delta, C)\)-SDE. Note that there is no pure diffusion.

Let \( (d_t)_{t \geq 0} \) denote the distance process induced by the 2-point motion \( X^{(2)}_t = (X_t, Y_t) \) (then \( d_t = d(X_t, Y_t) \)). The isotropy condition and the fact that in two point homogeneous spaces, pairs of equidistant points, can be exchanged by an isometry imply that \( d_t \) is a real diffusion. We denote in the following the law of this diffusion starting from \( r \geq 0 \) by \( P_r \). Let \( H_r = \inf\{t > 0 \mid d_t = r\} \).

Proposition 7.4.1. (1) \( \nu^* \) defines a non-coalescing flow of maps (i.e. such that the 2-point motion starting outside of the diagonal never hits the diagonal) if and only if \( 0 \) is a natural boundary point, i.e. if
\[
\forall r > 0, \ P_r[H_0 < \infty] = 0 \text{ and } P_0[H_r < \infty] = 0.
\]

(2) \( \nu^* \) defines a coalescing flow of maps if and only if \( 0 \) is a closed exit boundary point, i.e. if
\[
\exists r > 0, \ P_r[H_0 < \infty] > 0 \text{ and } \forall r > 0, \ P_0[H_r < \infty] = 0.
\]
\( \nu^s \) defines a turbulent flow without hitting (i.e. such that the 2-point motion starting outside of the diagonal never hits the diagonal) if and only if 0 is an open entrance boundary point, i.e. if
\[
\forall r > 0, \ P_r[H_0 < \infty] = 0 \text{ and } \exists r > 0, \ P_0[H_r < \infty] > 0. \tag{7.7}
\]

\( \nu^s \) defines a turbulent flow with hitting (i.e. such that the 2-point motion starting outside of the diagonal hits the diagonal with a positive probability) if and only if 0 is a reflecting regular boundary point, i.e. if
\[
\exists r > 0, \ P_r[H_0 < \infty] > 0 \text{ and } \exists r > 0, \ P_0[H_r < \infty] > 0. \tag{7.8}
\]

In all cases except (4), \( \nu^s \) is the unique diffusion convolution semigroup with local characteristics \((\frac{1}{2}\Delta, C)\).

In case (4), called the intermediate phase, \( \nu^c \neq \nu^s \) and theorems 6.2.1 and 6.4.1 apply. Thus \( N_{\nu^c} \) is a predictable non-Gaussian noise.

**Proof.** The proof of (1), (2), (3) and (4) is straightforward. Notice that the local characteristics satisfy (U). In all cases, \( \nu^s \) defines a strong solution of the \((\frac{1}{2}\Delta, C)\)-SDE. This with theorem 6.2.1 implies that in the coalescing case (2), since \( \nu^s = \nu^c \), \( \nu^s \) is the unique diffusion convolution semigroup whose local characteristics are \((\frac{1}{2}\Delta, C)\).

In the non-coalescing case (1) and in the turbulent case without hitting (3), the fact that \( \nu^s \) is the unique diffusion convolution semigroup whose local characteristics are \((\frac{1}{2}\Delta, C)\) follows directly from (U).

In the intermediate phase (4), we must have \( \nu^c \neq \nu^s \) since \( \nu^s \) defines a turbulent flow and \( \nu^c \) a flow of maps. Moreover, condition (B) holds so that we can conclude using theorems 6.2.1 and 6.4.1.

**Remark 7.4.2.** The \((\frac{1}{2}\Delta, C)\)-SDE has a solution, unique in law except in the intermediate phase, in which case all solutions are obtained by filtering, on an extension \((N, \varphi)\) of the noise of the coalescing solution, this coalescing solution \( \varphi \) with respect to a sub-noise of \( N \) containing \( W \).

**Remark 7.4.3.** The conditions involving the distance process can be verified using the speed and scale measures of this process which are explicitly determined by the spectral measures of the isotropic fields (cf [23] for \( \mathbb{R}^d \) and for \( S^d \)).

### 7.5 Sobolev flows.

In [23], Sobolev flows \((S_{s,t}, s \leq t)\) on \( \mathbb{R}^d \) and on \( S^d \) are studied. The Sobolev covariances are described with two parameters \( \alpha > 0 \) and \( \eta \in [0, 1] \). The associated self-reproducing spaces are Sobolev spaces of vector fields of order \((d + \alpha)/2\). The incompressible and gradient subspaces are orthogonal and respectively weighted by factors \( \eta \) and \( 1 - \eta \).

Let us apply the results obtained in [23]. We will call the stochastic flow associated with \((S_{s,t}, s \leq t)\) (see section 5.7 and 7.3) Sobolev flow as well. When \( \alpha > 2 \), we are in case (1) and Sobolev flows are flows of diffeomorphisms. More interestingly, when \( 0 < \alpha < 2 \) then

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\( ^7 \) We recall that a turbulent flow was defined as a stochastic flow of kernels which is not a flow of maps and without pure diffusion.
If \( d \in \{2, 3\} \) and \( \eta < 1 - \frac{d}{\alpha^2} \), we are in case (2) and the Sobolev flow is a coalescing flow.

If \( d \geq 4 \) or if \( d \in \{2, 3\} \) and \( \eta > \frac{1}{2} - \frac{(d-2)}{2\alpha} \), we are in case (3) and the Sobolev flow is turbulent without hitting.

if \( d \in \{2, 3\} \) and \( 1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha} \), we are in case (4) (i.e. the intermediate phase) and the Sobolev flow is turbulent with hitting.

In dimension 1, the parameter \( \eta \) vanishes. The critical case was studied in [1, 12, 30]. There is a strong coalescing solution for \( \alpha \in [1, 2[ \) and an intermediate phase for \( \alpha \in ]0, 1[ \).

By construction, in all these cases, the noise generated by the Sobolev flows are Gaussian noises. For the intermediate phase, in which there exist two different solutions to the \((\frac{1}{2}\Delta, C)\)-SDE (namely the coalescing one and the turbulent one), the noise of the associated coalescing flow is predictable but not Gaussian.

These different cases are represented by the phase diagram below, for the homogeneous space \( S^3 \). Recall that a flow of diffeomorphisms is called stable (respectively unstable) when the first Lyapounov exponent is negative (respectively positive). These exponents actually converge actually towards \(-\infty\) or to \(+\infty\) as \( \alpha \) approaches the critical value 2.
8 Conclusion.

Looking at the phase diagram above, it looks as this case has been fully analysed.

The three different types of motion which can be defined by a consistent system of Feller semigroups appear in this picture: Flows of non coalescing maps occur when, for the two point motion, the diagonal and the complement of the diagonal are absorbing.

When the first condition fails, i.e. when the diagonal is not absorbing, we get a diffusive flow, i.e. a flow of non trivial Markov kernels. We see in this example that this can happen without pure diffusion, i.e. when the evolution equation has no dissipative term. In that case we say that the flow is turbulent. It can be viewed as an effect of extreme unstability due to the importance of very high frequency divergence free components in the velocity field near the diagonal.

When the second condition fails, i.e. when the complement of the diagonal is not absorbing, we get flows of coalescing maps. We see, in the intermediate phase, that a turbulent and a coalescing flow can have the same local characteristics. This happens when both conditions fail for the two point motion associated with the turbulent flow.

Moreover, it is likely that at least in the other isotropic situations, a very similar picture will occur, the parameters being the singularity of the covariance on the diagonal and the balance between gradient and incompressible velocity fields.

Yet there is still some important work to do about the intermediate phase. We know there exists two remarkable distinct solutions in that case for the SDE: the coalescing flow, the noise of which is not linear but for which the linear part has been identified as the velocity white noise $W$, and the unique strong solution which is a flow of non trivial kernels obtained by averaging the coalescing flow with respect to $W$. Other solutions do exist and we have shown that their associated convolution semigroups are weakly dominated by the “coalescing” convolution semigroup and dominate the “strong” or “linear” one. But this classification should be made analytically precise and one can conjecture it involves a “gluing” parameter on the diagonal (see section 3.4 arXiv math.PR/0203221 and math.PR/0212269 for first steps in this direction.) Moreover, the non linear part of the relevant noises remains to be fully analysed. Finally, one can expect that more complex phenomena occur for SDEs in which a multiplicity of weak solutions with different one-point motions do exist. Hence this paper can only be a step in the understanding of the multiplicity of flows with given velocity field, or given local characteristics.

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