On the projections of mutual multifractal spectra

DOUZI Zied and SELMI Bilel

Abstract

The aim of this article is to study the behaviour of the relative multifractal spectrum under projections. First of all, we depict a relationship between the mutual multifractal spectra of a couple of measures\((\mu, \nu)\) and its orthogonal projections in Euclidean space. As an application, we improve Svetova’s result (Tr. Petrozavodsk. Gos. Univ. Ser. Mat., 11 (2004), 41-46) and study the mutual multifractal analysis of the projections of measures.

Keywords: Mutual Hausdorff dimension; Mutual packing dimension; Projection; Multifractal analysis.

2010 MSC: 28A20, 28A80

1. Introduction

In the previous years, there has been great interest in understanding the fractal dimensions of projections of sets and measures. The first significant work in this area was the result of Marstrand [28] who showed a well-known theorem according to which the Hausdorff dimension of a planar set is preserved under orthogonal projections. In [27], Kaufman had employed potential theoretic methods in order to prove Marstrand result, which has been generalized later by Mattila in [29]. Let us mention that Falconer et al [23, 24] have proved that the packing dimension of the projected set or measure will be the same for almost all projections. Other works were carried out in this sense for classes of similar measures in euclidean and symbolic spaces [8, 25, 26, 42, 43]. However, despite these substantial advances for fractal sets, only very little is known about the multifractal structure of projections of measures [7, 21, 37, 39, 40, 41].

Based on some ideas of multifractal formalism given by Olsen and Peyrière [32, 38], Svetova introduced in [45, 46, 47, 48] a new formalism for a multifractal analysis of one measure with respect to an other. This formalism is called by the mutual multifractal formalism and for which Svetova studied some basic properties. More specifically, given two compactly supported Borel probability measures \(\mu\) and \(\nu\) on \(\mathbb{R}^n\) and \(\alpha, \beta \in \mathbb{R}\), Svetova estimated the size of the iso-Hölder set

\[
E_{\mu,\nu}(\alpha, \beta) = \left\{ x \in \text{supp} \mu \cap \text{supp} \nu; \quad \alpha_\mu(x) = \alpha \quad \text{and} \quad \alpha_\nu(x) = \beta \right\},
\]

1Faculty of sciences of Monastir, Department of mathematics, 5000-Monastir, Tunisia. 
E-mail addressse: zied.douzi@fsm.rnu.tn, bilel.selmi@fsm.rnu.tn
where \( \alpha_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \) and \( B(x, r) \) is the closed ball of center \( x \) and radius \( r \). The mutual multifractal analysis of a measure allows to relate the Hausdorff and packing dimensions of these levels sets to the Legendre transform of some multifractal functions. There has recently been a great interest for this subject and positive results have been written in various situations in the dynamic contexts \([2, 3, 4, 5, 6, 34]\). Recently, many authors were interested in mutual (mixed) multifractal spectra, see for example \([15, 16, 17, 18, 19, 20, 31, 33, 35, 44]\). We write for \( \gamma \geq 0 \),

\[
B_{\mu, \nu}(\gamma) = \left\{ x \in \text{supp} \mu \cap \text{supp} \nu; \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} = \gamma \right\}.
\]

It is clear that

\[
\bigcup_{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^* \atop \text{for } \gamma} E_{\mu, \nu}(\alpha, \beta) \subset B_{\mu, \nu}(\gamma).
\]

The latter union is composed by an uncountable number of pairwise disjoint nonempty sets. Then, the Hausdorff and packing dimension of \( B_{\mu, \nu}(\gamma) \) is fully carried by some subset \( E_{\mu, \nu}(\alpha, \beta) \). Also, Selmi et al. investigated the projection properties of the \( \nu \)-Hausdorff, and the \( \nu \)-packing dimensions of \( B_{\mu, \nu}(\gamma) \) in \([21]\). In this article, they derived global bounds on the relative multifractal dimensions of a projection of a measures in terms of its original relative multifractal dimensions. It is more difficult to obtain a lower and upper bound for the dimension of the set \( B_{\mu\nu, \nu\nu}(\gamma) \), where \( V \) is a linear subspace of \( \mathbb{R}^n \).

The purpose of this paper is to improve Svetova’s result and to propose a sufficient condition that gives the lower bound for the Hausdorff and the packing dimensions of \( B_{\mu\nu, \nu\nu}(\gamma) \). Our first aim is to study the behavior of the mutual Hausdorff, packing and pre-packing dimensions under projections. The second aim is to investigate a relationship between the mutual multifractal spectra and its projections onto a lower dimensional linear subspace.

2. Preliminaries and Results

Let us recall the multifractal formalism introduced by Svetova in \([46]\). Let \( \mu \) and \( \nu \) be two compactly supported Borel probability measures on \( \mathbb{R}^n \). We denote by \( \text{supp} \mu \) the topological support of \( \mu \).

**Definition 2.1.** For \( q, t, s \in \mathbb{R} \), \( E \subseteq \mathbb{R}^n \) and \( \delta > 0 \), we define

\[
\mathcal{P}_{\mu, \nu, \delta}^{q, t, s}(E) = \sup \sum_{i} \mu(B(x_i, r_i)^q \nu(B(x_i, r_i)^t)(2r_i)^s),
\]

where the supremum is taken over all centered \( \delta \)-packing of \( E \),

\[
\mathcal{P}_{\mu, \nu}^{q, t, s}(E) = \inf_{\delta > 0} \mathcal{P}_{\mu, \nu, \delta}^{q, t, s}(E),
\]

and we introduce the generalized packing measure relatively to \( \mu \) and \( \nu \)

\[
\mathcal{P}_{\mu, \nu}^{q, t, s}(E) = \inf_{E \subseteq \bigcup_{i} E_i} \sum_{i} \mathcal{P}_{\mu, \nu}^{q, t, s}(E_i).
\]
In a similar way we define the generalized Hausdorff measure relatively to \( \mu \) and \( \nu \) by

\[
H_{\mu,\nu}^{q,t,s}(E) = \inf \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t (2r_i)^s,
\]

where the infimum is taken over all centered \( \delta \)-covering of \( E \),

\[
H_{\mu,\nu}^{q,t,s}(E) = \sup_{\delta > 0} \overline{H}_{\mu,\nu}^{q,t,s}(E),
\]

and we introduce the generalized Hausdorff measure relatively to \( \mu \) and \( \nu \)

\[
H_{\mu,\nu}^{q,t,s}(E) = \sup_{F \subseteq E} \overline{H}_{\mu,\nu}^{q,t,s}(F),
\]

with the conventions \( 0^q = \infty \) for \( q \leq 0 \) and \( 0^q = 0 \) for \( q > 0 \).

Remarks 2.1.

1. The functions \( H_{\mu,\nu}^{q,t,s} \) and \( P_{\mu,\nu}^{q,t,s} \) are metric outer measures and thus measures on the Borel family of subsets of \( \mathbb{R}^n \). An important feature of the Hausdorff and packing measures is that \( \mathcal{P}_{\mu,\nu}^{q,t,s} \leq \overline{\mathcal{P}}_{\mu,\nu}^{q,t,s} \) and that there exists an integer \( \xi \in \mathbb{N} \), such that \( H_{\mu,\nu}^{q,t,s} \leq \xi \mathcal{P}_{\mu,\nu}^{q,t,s} \) (see [49]).

2. In the special case where \( q = 0 \) or \( t = 0 \), the mutual multifractal spectra is strictly related to Olsen’s multifractal formalism [32].

3. The mutual multifractal spectra represents the relative multifractal analysis introduced by Cole [11] in the case where \( s = 0 \). Other works were carried out in this sense in probability and symbolic spaces [1, 12, 13, 14].

Proposition 2.1. ([46, 49])

1. There exists a unique number \( b_{\mu,\nu}^{q,t}(E) \in [-\infty, +\infty] \) such that

\[
H_{\mu,\nu}^{q,t,s}(E) = \begin{cases} 
\infty & \text{if } s < b_{\mu,\nu}^{q,t}(E), \\
0 & \text{if } b_{\mu,\nu}^{q,t}(E) < s.
\end{cases}
\]

2. There exists a unique number \( B_{\mu,\nu}^{q,t}(E) \in [-\infty, +\infty] \) such that

\[
\mathcal{P}_{\mu,\nu}^{q,t,s}(E) = \begin{cases} 
\infty & \text{if } s < B_{\mu,\nu}^{q,t}(E), \\
0 & \text{if } B_{\mu,\nu}^{q,t}(E) < s.
\end{cases}
\]

3. There exists a unique number \( \Lambda_{\mu,\nu}^{q,t}(E) \in [-\infty, +\infty] \) such that

\[
\overline{\mathcal{P}}_{\mu,\nu}^{q,t,s}(E) = \begin{cases} 
\infty & \text{if } s < \Lambda_{\mu,\nu}^{q,t}(E), \\
0 & \text{if } \Lambda_{\mu,\nu}^{q,t}(E) < s.
\end{cases}
\]
3 PROJECTION RESULTS

Let $E \subseteq \mathbb{R}^n$ and $q, t \in \mathbb{R}$. We can remark that

$$b_{\mu,\nu}^q(E) \leq B_{\mu,\nu}^q(E) \leq \Lambda_{\mu,\nu}^q(E).$$

Then we are able to define the multifractal dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$: $\mathbb{R}^2 \to [-\infty, +\infty]$ by

$$b_{\mu,\nu}(q, t) = b_{\mu,\nu}^q(\text{supp} \mu \cap \text{supp} \nu), \quad B_{\mu,\nu}(q, t) = B_{\mu,\nu}^q(\text{supp} \mu \cap \text{supp} \nu)$$

and $\Lambda_{\mu,\nu}(q, t) = \Lambda_{\mu,\nu}^q(\text{supp} \mu \cap \text{supp} \nu)$.

It is well known that the functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$ are decreasing and $B_{\mu,\nu}$, $\Lambda_{\mu,\nu}$ are convex (see [49]).

3. Projection results

Let $m$ be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmannian manifold of all $m$-dimensional linear subspaces of $\mathbb{R}^n$. We denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map, $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto $V$. Now, for a Borel probability measure $\mu$ on $\mathbb{R}^n$, supported on the compact set $\text{supp} \mu$ and for $V \in G_{n,m}$, we define $\mu_V$, the projection of $\mu$ onto $V$, by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)), \quad \forall A \subseteq V.$$ 

Since $\mu$ has a compact support, $\text{supp} \mu_V = \pi_V(\text{supp} \mu)$ for all $V \in G_{n,m}$.

In the following, we are interested about the behavior of mutual Hausdorff, packing and pre-packing dimensions under projections. Throughout this paper, we suppose that $\text{supp} \mu = \text{supp} \nu$. We are based on ideas of Selmi et al in [21], to show the following results.

**Theorem 3.1.** Let $\mu$ and $\nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$ and $E \subseteq \text{supp} \mu$. Then, for $(q, t) \in [(1 - \infty, 0]^2 \cup ((1 - \infty, 0) \times [0, 1]) \cup ([0, 1] \times 1 - \infty, 0]$ and for all $V \in G_{n,m}$, we have

$$\Lambda_{\mu,\nu}^{q,t}(E) \leq \Lambda_{\mu,\nu}^{q,t}(E).$$

**Proof.** Let $s \in \mathbb{R}$ such that $\Lambda_{\mu,\nu}^{q,t}(E) < s$. Consider $V \in G_{n,m}$ and fix $\delta > 0$.

Let $\left( B_i = B(x_i, r_i) \right)_i$ be a $\delta$-centered packing of $\pi_V(E)$. There exists an integer $K_m$ depending on $m$ only such that we can divide up the balls $B(x_i, 2r_i)$ into $K \leq K_m$ families of disjoint balls $B_1, \ldots, B_K$. Let $1 \leq l \leq K$. For each $B(x_i, r_i) \in B_l$, denote $E_l = E \cap \pi_V^{-1}(B(x_i, r_i))$.

We have $E_l \subseteq \bigcup_{y \in E_l} B(y, r_i)$, so Besicovitch’s covering theorem [30] provides a positive integer $K_n$ as well as $K_i \leq K_n$ families of pairwise disjoint balls $B_{r,k} = \left\{ B_j^{(i,k)} = B(y_j^{(i,k)}, r_{ijk}); \ r_{ijk} = \frac{r_i}{2} \right\}$, $1 \leq k \leq K_i$, extracted from $\left\{ B(y, r_i) \right\}_{y \in E_l}$ such that

$$E_l \subseteq \bigcup_{k=1}^{K_i} \bigcup_{j} B_j^{(i,k)}. $$
3 PROJECTION RESULTS

- **Case 1**: For $q \leq 0$ and $t \leq 0$, we have
  \[
  \sum \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leq 2^s \sum \mu(B_j^{(i,k)})^q \nu(B_j^{(i,k)})^t (2r_{ijk})^s \\
  \leq 2^s \sum_{i,j,k} \mu(B_j^{(i,k)})^q \nu(B_j^{(i,k)})^t (2r_{ijk})^s.
  \]

- **Case 2**: For $q \leq 0$ and $0 \leq t \leq 1$, we have
  \[
  \sum \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leq 2^s \sum \mu(B_j^{(i,k)})^q \left( \bigcup_{k=1}^{K_i} \bigcup_{j} B_j^{(i,k)} \right)^t (2r_{ijk})^s \\
  \leq 2^s \sum_{i,j,k} \mu(B_j^{(i,k)})^q \nu(B_j^{(i,k)})^t (2r_{ijk})^s.
  \]

- **Case 3**: For $0 \leq q \leq 1$ and $t \leq 0$, we have
  \[
  \sum \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leq 2^s \sum \mu \left( \bigcup_{k=1}^{K_i} B_j^{(i,k)} \right)^q (B_j^{(i,k)})^t (2r_{ijk})^s \\
  \leq 2^s \sum_{i,j,k} \mu(B_j^{(i,k)})^q \nu(B_j^{(i,k)})^t (2r_{ijk})^s.
  \]

In all cases and by construction, since the balls $B(x_i, 2r_i) \in B_i$ are pairwise disjoint, if $B(y, r) \in B_{i,k}$ and $B(y', r') \in B_{i',k'}$ with $i \neq i'$, then $B(y, r) \cap B(y', r') = \emptyset$. Consequently, we can collect the balls $B(y, r)$ invoked in the above sum into at most $K_n$ centered packing of $E$. This holds for all $1 \leq l \leq K$, so

\[
\sum \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leq 2^s K_n K_n \sup \left\{ \sum_j \mu(B(y_j, r_j))^q \nu(B(y_j, r_j))^t (2r_j)^s \right\},
\]

where the supremum is taken over all centered packing of $E$ by closed balls of radius $r$. Thus

\[
\mathcal{P}_{\mu_V, \nu_V, \delta}^q \left( \pi_V(E) \right) \leq 2^s K_n K_n \mathcal{P}_{\mu, \nu, \delta}^q (E).
\]

Letting $\delta \to 0$, we obtain

\[
\mathcal{P}_{\mu_V, \nu_V}^{q, t, s} \left( \pi_V(E) \right) \leq 2^s K_n K_n \mathcal{P}_{\mu, \nu}^{q, t, s} (E), \tag{3.1}
\]

and the result yields.

**Corollary 3.1.** Let $\mu$ and $\nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q, t) \in ([-\infty, 0]^2) \cup ([-\infty, 0] \times [0, 1]) \cup ([0, 1] \times [-\infty, 0])$ and for all $V \in G_{n,m}$, we have

\[
\Lambda_{\mu_V, \nu_V}^{q, t} \leq \Lambda_{\mu, \nu}^{q, t}(q, t).
\]
3 PROJECTION RESULTS

Proof. The proof is a consequence of Theorem 3.1. □

Theorem 3.2. Let $\mu$ and $\nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q, t) \in ([-\infty, 0] \cup ([1, 0] \times [0, 1]) \cup ([0, 1] \times -\infty, 0])$ and for all $V \in G_{n,m}$, we have

$$B_{\mu, \nu}(q, t) \leq B_{\mu, \nu}(q, t).$$

Proof. Let $s \in \mathbb{R}$ such that $B_{\mu, \nu}(q, t) < s$.

Consider $F \subseteq \mathbb{R}^n$ and $V \in G_{n,m}$. Due to inequality (3.1), we have

$$P_{\mu, \nu}(F) \leq b_{\mu, \nu}(F).$$

Since $P_{\mu, \nu}(\text{supp } \mu) = 0$, there exists $(E_i)_i$ a covering of $\text{supp } \mu$ such that

$$\sum_i P_{\mu, \nu}(E_i) < 1.$$

So, $\pi_V(\text{supp } \mu) \subseteq \bigcup_i \pi_V(E_i)$ and we have

$$P_{\mu, \nu}(\text{supp } \mu) \leq \sum_i P_{\mu, \nu}(\pi_V(E_i)) \leq 2^s K_n K_m P_{\mu, \nu}(F) < \infty.$$

Thus $B_{\mu, \nu}(q, t) \leq s$. □

Theorem 3.3. Let $\mu, \nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q, t) \in ([-\infty, 0^2] \cup ([1, \infty, 0[ \times [0, 1]) \cup ([0, 1] \times -\infty, 0])$ and for all $V \in G_{n,m}$, we have

$$b_{\mu, \nu}(q, t) = b_{\mu, \nu}(q, t).$$

Proof. Let’s prove that $b_{\mu, \nu}(q, t) \leq b_{\mu, \nu}(q, t)$.

Let $s \in \mathbb{R}$ such that $s < b_{\mu, \nu}(q, t)$. Choose $F \subseteq \text{supp } \mu$ and $V \in G_{n,m}$. Fix $\delta > 0$ and let $\left( B_i = B(x_i, r_i) \right)$ be a $\delta$-centered covering of $F$. Let $E_i$ such that $\pi_V^{-1}(E_i) = F \cap B(x_i, r_i)$. We have $E_i \subseteq \bigcup_{y \in E_i} B(y, r_i)$, so Besicovitch’s covering theorem provides a positive integer $K_n$ as well as $K_i \leq K_n$ families of pairwise disjoint balls $B_{i,k} = \left\{ B^{(i,k)}_j = B(y^{(i,k)}_j, r_{ijk}); r_{ijk} = \frac{r_i}{2} \right\}$, $1 \leq k \leq K_i$, extracted from $\left\{ B(y, r_i) \right\}_{y \in E_i}$ and such that

$$E_i \subseteq \bigcup_{k=1}^{K_i} \bigcup_{j} B^{(i,k)}_j.$$
3 PROJECTION RESULTS

• **Case 1** : For \( q < 0 \) and \( t < 0 \), we have

\[
\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu_V(B_{j(i,k)})^q \nu_V(B_{j(i,k)})^t (2r_{ijk})^s \\
\leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu_V(B_{j(i,k)})^q \nu_V(B_{j(i,k)})^t (2r_{ijk})^s.
\]

• **Case 2** : For \( q < 0 \) and \( 0 < t \leq 1 \), we have

\[
\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu_V \left( \bigcup_{k=1}^{k_i} \bigcup_j B_{j(i,k)} \right)^t (2r_{ijk})^s \\
\leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu_V(B_{j(i,k)})^q \nu_V(B_{j(i,k)})^t (2r_{ijk})^s.
\]

• **Case 3** : For \( 0 < q \leq 1 \) and \( t < 0 \), we have

\[
\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu_V \left( \bigcup_{k=1}^{k_i} \bigcup_j B_{j(i,k)} \right)^q (2r_{ijk})^s \\
\leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu_V(B_{j(i,k)})^q \nu_V(B_{j(i,k)})^t (2r_{ijk})^s.
\]

Thus

\[
\overline{H}_{q,t,s}^{l,t,s}(F) \leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\pi_V(F)).
\]

Letting \( \delta \downarrow 0 \), we obtain

\[
\overline{H}_{q,t,s}^{l,t,s}(F) \leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\pi_V(F)).
\]

Thus

\[
\overline{H}_{q,t,s}^{l,t,s}(F) \leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\pi_V(F)) \\
\leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\pi_V(\text{supp } \mu)) \\
\leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\text{supp } \mu).\]

The arbitrary on \( F \) implies that

\[
\overline{H}_{q,t,s}^{l,t,s}(\text{supp } \mu) \leq 2^s \overline{H}_{q,t,s}^{l,t,s}(\text{supp } \mu_V) \tag{3.2}
\]

and the result holds.

In order to prove the other inequality, let \( E \subseteq \mathbb{R}^n \) and \( s \in \mathbb{R} \) such that \( b_{q,t,s}^{l,t,s}(E) < s \). Fix \( V \in G_{n,m}, \delta > 0 \) and suppose that \( \left( B(x_i, r_i) \right)_i \) is a \( \delta \)-cover of \( \pi_V(E) \). Denote \( E_i = \)
3 PROJECTION RESULTS

$E \cap \pi_V^{-1}(B(x_i, r_i))$. We have $E_i = \bigcup_{y \in E_i} \cap \pi_V^{-1}(\{x_i\}) B(y, \frac{r}{n})$. By applying Besicovitch covering theorem, we find an integer $K_n$, depending only on $n$ as well as $K_i \leq K_n$ families of pairwise disjoint balls $B_{i,k} = \{ B_{i,k}^{(i,k)} = B(y, \frac{r_{ijk}}{n}); r_{ijk} = \frac{r}{2} \}, 1 \leq k \leq K_i$ such that

$$E \cap \pi_V^{-1}(B(x_i, r_i)) \subseteq \bigcup_{k=1}^{K_i} \bigcup_{j} B_{i,k}^{(i,k)}.$$

**Case 1:** For $q < 0$ and $t < 0$, we have

$$\sum_i \mu V(B_i)^q \nu V(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu(B_{i,k}^{(i,k)})^q (B_{i,k}^{(i,k)})^t (2r_{ijk})^s \leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu(B_{i,k}^{(i,k)})^q \nu(B_{i,k}^{(i,k)})^t (2r_{ijk})^s.$$

**Case 2:** For $q < 0$ and $0 < t \leq 1$, we have

$$\sum_i \mu V(B_i)^q \nu V(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu(B_{i,k}^{(i,k)})^q \nu \left( \bigcup_{k=1}^{k_i} \bigcup_{j} B_{i,k}^{(i,k)} \right)^t (2r_{ijk})^s \leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu(B_{i,k}^{(i,k)})^q \nu(B_{i,k}^{(i,k)})^t (2r_{ijk})^s.$$

**Case 3:** For $0 < q \leq 1$ and $t < 0$, we have

$$\sum_i \mu V(B_i)^q \nu V(B_i)^t (2r_i)^s \leq 2^s \sum_i \mu \left( \bigcup_{k=1}^{k_i} \bigcup_{j} B_{i,k}^{(i,k)} \right)^q (B_{i,k}^{(i,k)})^t (2r_{ijk})^s \leq 2^s \sum_{i,j} \sum_{k=1}^{k_i} \mu(B_{i,k}^{(i,k)})^q \nu(B_{i,k}^{(i,k)})^t (2r_{ijk})^s.$$

Then

$$\mathcal{H}_{\mu V, \nu V, \delta}^{q,t,s}(\pi_V(E)) \leq 2^s \mathcal{H}_{\mu, \nu, \delta}^{q,t,s}(E).$$

Letting $\delta \downarrow 0$, we obtain

$$\mathcal{H}_{\mu V, \nu V}^{q,t,s}(\pi_V(E)) \leq 2^s \mathcal{H}_{\mu, \nu}^{q,t,s}(E).$$

Thus, given a subset $E$ of $\text{supp } \mu$, $\pi_V(E) \subseteq \text{supp } \mu V$ and

$$\mathcal{H}_{\mu V, \nu V}^{q,t,s}(\pi_V(E)) \leq 2^s \mathcal{H}_{\mu, \nu}^{q,t,s}(E) \leq 2^s \mathcal{H}_{\mu, \nu}^{q,t,s}(\text{supp } \mu).$$

The arbitrary on $E$ implies that

$$\mathcal{H}_{\mu V, \nu V}^{q,t,s}(\text{supp } \mu V) \leq 2^s \mathcal{H}_{\mu, \nu}^{q,t,s}(\text{supp } \mu).$$
and the result holds. This achieves the proof of Theorem 3.3. □

**Theorem 3.4.** Let \( \mu \) and \( \nu \) be two compactly supported Borel probability measures on \( \mathbb{R}^n \). Then for \( q,t \geq 1 \) and all \( V \in G_{n,m} \), we have

\[
\text{b}_{\mu,\nu,q}(q,t) \geq \text{b}_{\mu,\nu}(q,t).
\]

**Proof.** Fix \( V \in G_{n,m} \) and \( \delta > 0 \) and suppose that \( (B_i = B(x_i, r_i))_i \) is a \( \delta \)-cover of \( \pi_V(E) \). For each \( i \), we may use the Besicovitch covering theorem to find a constant \( \xi \), depending only on \( n \), and a family of balls \( (B_{ij} = B(x_{ij}, r_{ij}))_{j \in \mathbb{N}} \) with \( r_{ij} = \frac{\delta}{2} \) which is a \( \delta \)-cover of \( \pi_V^{-1}(B_i) \cap E \) such that

\[
\bigcup_j B(x_{ij}, r_{ij}) \subseteq \pi_V^{-1}\left(B(x_i, 2r_i) \cap V\right).
\]

Note that \( \tilde{B}_i = B(x_i, 2r_i) \). Then

\[
\sum_i \mu_V((\tilde{B}_i)^q \nu_V((\tilde{B}_i)^t (4r_i)^s) \geq \xi^{-(q+t)} \sum_i (4r_i)^s \left( \sum_j \mu(B_{ij}) \right)^q \left( \sum_j \nu(B_{ij}) \right)^t \geq 4^s \xi^{-(q+t)} \sum_{i,j} \mu(B_{ij})^q \nu(B_{ij})^t (2r_{ij})^s.
\]

Consequently, as \((B_i)_i\) any \( \delta \)-cover of \( \pi_V(E) \), we conclude that

\[
\mathcal{H}_{\mu,\nu,\delta}^{q,t,s}(E) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu,\delta}^{q,t,s}(\pi_V(E)).
\]

Letting \( \delta \downarrow 0 \), gives that

\[
\mathcal{H}_{\mu,\nu}^{q,t,s}(E) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu}^{q,t,s}(\pi_V(E)).
\]

Thus

\[
\mathcal{H}_{\mu,\nu}^{q,t,s}(E) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu}^{q,t,s}(\pi_V(E)) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu}^{q,t,s}(\pi_V(\text{supp } \mu)) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu}^{q,t,s}(\text{supp } \mu_V).
\]

The arbitrary on \( E \) implies that

\[
\mathcal{H}_{\mu,\nu}^{q,t,s}(\text{supp } \mu) \leq 4^{-s} \xi^{(q+t)} \mathcal{H}_{\mu,\nu}^{q,t,s}(\text{supp } \mu_V)
\]

and the result yields.

**Remark 3.1.** Notice that in the case where \( t = 0 \) or \( q = 0 \), the preceding results were treated by O’Neil in [37]. Also, when \( s = 0 \), Douzi and Selmi investigated the projection properties of the mutual Hausdorff, packing and pre-packing measures in [21]. They derived global bounds on the relative multifractal dimensions of a projection of a measures in terms of its original relative multifractal dimensions.
4. Application

This section is devoted to study the behavior of projections of measures obeying to the mutual multifractal formalism. More precisely, we prove that for \((q, t) \in \left\{ [1 - \infty, 0]^2 \right\} \cup \left( [1 - \infty, 0[ \times ]0, 1] \right) \cup \left( ]0, 1[ \times ]-\infty, 0[ \right) \), if the mutual multifractal formalism holds for the couple \((\mu, \nu)\) at \(\alpha = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial q}\) and \(\beta = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial t}\), it holds for \((\mu_V, \nu_V)\) for all \(V \in G_{n,m}\). Before detailing our results let us recall the mutual multifractal formalism introduced by Svetova [46, 48]. For \(\alpha, \beta \geq 0\), let

\[ E_{\mu, \nu}(\alpha, \beta) = \left\{ x \in \text{supp} \mu \cap \text{supp} \nu : \lim_{r \to 0} \frac{\log (\mu(B(x,r)))}{\log r} = \alpha \quad \text{and} \quad \lim_{r \to 0} \frac{\log (\nu(B(x,r)))}{\log r} = \beta \right\}. \]

We are interested to the estimation of the Hausdorff and packing dimension of \(E_{\mu, \nu}(\alpha, \beta)\). Let us mention that in the last decay there has been a great interest for the multifractal analysis and positive results have been written in various situations (see for example [9, 10, 11, 21, 32]). Our purpose in the following theorem is to prove the result of Theorem 3 in [46] under less restrictive hypotheses.

**Theorem 4.1.** Let \(\mu, \nu\) be two compactly supported Borel probability measures on \(\mathbb{R}^n\). If \(B_{\mu, \nu}\) is differentiable at \((q, t)\) and we set \(\alpha = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial q}\) and \(\beta = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial t}\). Assume that \(H_{\mu, \nu}^{q,t}(\text{supp} \mu \cap \text{supp} \nu) > 0\). Then, we have

\[ \dim_H E_{\mu, \nu}(\alpha, \beta) = \dim_P E_{\mu, \nu}(\alpha, \beta) = B_{\mu, \nu}^*(\alpha, \beta) = b_{\mu, \nu}^*(\alpha, \beta), \]

where \(f^*(\alpha, \beta) = \inf_{q,t} \left( \alpha q + \beta t + f(\alpha, \beta) \right)\) denotes the Legendre transform of the function \(f\). Here \(\dim_H\) and \(\dim_P\) denote the Hausdorff and packing dimensions (see [22] for the definitions) and we say that the mutual multifractal formalism is valid.

**Proof.** It is known (for instance, see [46]) that, for all reals \(\alpha\) and \(\beta\), one has

\[ \dim_P E_{\mu, \nu}(\alpha, \beta) \leq \alpha q + \beta t + B_{\mu, \nu}(q,t). \]

**Theorem 4.1** is a consequence from the following lemmas.

**Lemma 4.1.** Let \(\eta_1, \eta_2 > 0\). We set \(\alpha = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial q}\) and \(\beta = -\frac{\partial B_{\mu, \nu}(q,t)}{\partial t}\). Then

\[ H^{\alpha q + \beta t + B_{\mu, \nu}(q,t) - \eta_2}(E_{\mu, \nu}(\alpha, \beta)) \geq 2^{\alpha q + \beta t - m - \eta_2} H^{q,t}(E_{\mu, \nu}(\alpha, \beta)). \]

**Proof.** We treat the case \(q \leq 0\) and \(t \leq 0\). The other cases are proved similarly. The result is true for \(q = t = 0\), so we may assume that \(q < 0\) and \(t < 0\).

For \(m \in \mathbb{N}^\ast\), write

\[ E_m := \left\{ x \in E_{\mu, \nu}(\alpha, \beta) : \frac{\log (\mu(B(x,r)))}{\log r} \leq \alpha - \frac{\eta_1}{q} \quad \text{and} \quad \frac{\log (\nu(B(x,r)))}{\log r} \leq \beta - \frac{\eta_2}{t} \quad \text{for} \quad 0 < r < \frac{1}{m} \right\}. \]

Given \(F \subseteq E_m\), \(0 < \delta < \frac{1}{m}\) and \(\left( B(x_i, r_i) \right)_i\) a centered \(\delta\)-covering of \(F\), we have
Thus

\[ \log \mu(B(x_i, r_i)) \leq \alpha - \frac{\eta_1}{q} \text{ and } \log \nu(B(x_i, r_i)) \leq \beta - \frac{\eta_2}{q}. \]

Hence

\[ \mu(B(x_i, r_i))^q \leq r_i^{\alpha q - \eta_1} \text{ and } \nu(B(x_i, r_i))^t \leq r_i^{\beta t - \eta_2}. \]

We can deduce that

\[ \mathcal{H}_{\mu, \nu}^q(B(x_i, r_i))^t(2r_i)^B_{\mu, \nu}(q, t) \leq 2^B_{\mu, \nu}(q) r_i^{\alpha q + \beta t + B_{\mu, \nu}(q, t) - \eta_1 - \eta_2}. \]

Letting \( \delta \searrow 0 \) gives that

\[ \mathcal{H}_{\mu, \nu}^q(B_{\mu, \nu}(q, t))(E_m) \leq 2^{-\alpha q - \beta t + \eta_1 + \eta_2} \mathcal{H}_{\delta}^{\alpha q + \beta t + B_{\mu, \nu}(q, t) - \eta_1 - \eta_2}(F). \]

for all \( F \subseteq E_m \). Hence

\[ \mathcal{H}_{\mu, \nu}^{q, t, B_{\mu, \nu}(q, t)}(E_m) \leq 2^{-\alpha q - \beta t + \eta_1 + \eta_2} \mathcal{H}^{\alpha q + \beta t + B_{\mu, \nu}(q, t) - \eta_1 - \eta_2}(E_m) \]

and the result follows since \( E_{\mu, \nu}(\alpha, \beta) = \bigcup_m E_m \).

**Lemma 4.2.** We have \( \mathcal{H}_{\mu, \nu}^{q, t, B_{\mu, \nu}(q, t)} \left( \left( \text{supp } \mu \cap \text{supp } \nu \right) \setminus E_{\mu, \nu}(\alpha, \beta) \right) = 0. \)

**Proof.** Let us introduce, for \( \alpha, \beta \in \mathbb{R} \)

\[ F_{\alpha, \beta} = \left\{ x; \limsup_{r \to 0} \frac{\log (\mu(B(x, r)))}{\log r} > \alpha, \text{ or } \limsup_{r \to 0} \frac{\log (\nu(B(x, r)))}{\log r} > \beta \right\}, \]

\[ F_{\alpha, \beta}^t = \left\{ x; \liminf_{r \to 0} \frac{\log (\mu(B(x, r)))}{\log r} < \alpha, \text{ or } \liminf_{r \to 0} \frac{\log (\nu(B(x, r)))}{\log r} < \beta \right\}, \]
\[ F^2_{\alpha, \beta} = \left\{ x; \limsup_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r} > \alpha, \text{ or } \liminf_{r \to 0} \frac{\log(\nu(B(x, r)))}{\log r} < \beta \right\}, \]

\[ F^3_{\alpha, \beta} = \left\{ x; \liminf_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r} < \alpha, \text{ or } \limsup_{r \to 0} \frac{\log(\nu(B(x, r)))}{\log r} > \beta \right\}. \]

We have to prove that
\[
\mathcal{H}^q_{\mu, \nu}(F_{\alpha, \beta}) = 0 \text{ for every } \alpha > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial t} \quad (4.1)
\]

\[
\mathcal{H}^{q, t}_{\mu, \nu}(F^1_{\alpha, \beta}) = 0 \text{ for every } \alpha < -\frac{\partial B_{\mu, \nu}(q, t)}{\partial q} \text{ and } \beta < -\frac{\partial B_{\mu, \nu}(q, t)}{\partial t} \quad (4.2)
\]

\[
\mathcal{H}^{q, t}_{\mu, \nu}(F^2_{\alpha, \beta}) = 0 \text{ for every } \alpha > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial q} \text{ and } \beta < -\frac{\partial B_{\mu, \nu}(q, t)}{\partial t} \quad (4.3)
\]

and
\[
\mathcal{H}^{q, t}_{\mu, \nu}(F^3_{\alpha, \beta}) = 0 \text{ for every } \alpha < -\frac{\partial B_{\mu, \nu}(q, t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial t} \quad (4.4)
\]

Let us sketch the proof of assertion (4.1). Given
\[ \alpha > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu, \nu}(q, t)}{\partial t}, \]

we can choose \( h > 0 \) such that
\[ B_{\mu, \nu}(q - h, t) < B_{\mu, \nu}(q, t) + \alpha h \text{ and } B_{\mu, \nu}(q, t - h) < B_{\mu, \nu}(q, t) + \beta h. \]

Which implies,
\[ \mathcal{P}^{q-h, t}_{\mu, \nu}(B_{\mu, \nu}(q, t) + \alpha h) \left( \text{supp } \mu \cap \text{supp } \nu \right) = 0 \]

and
\[ \mathcal{P}^{q, t-h}_{\mu, \nu}(B_{\mu, \nu}(q, t) + \beta h) \left( \text{supp } \mu \cap \text{supp } \nu \right) = 0. \]

Let \( \delta > 0 \). For each \( x \in F_{\alpha, \beta} \), there exists \( 0 < r_x < \delta \) such that
\[ \mu(B(x, r_x)) \leq r_x^{\alpha} \quad \text{or} \quad \nu(B(x, r_x)) \leq r_x^{\beta}. \]

The family \( \left( B(x, r_x) \right)_{x \in F_{\alpha, \beta}} \) is then a centered \( \delta \)-covering of \( F_{\alpha, \beta} \). Using Besicovitch’s Covering Theorem, we can construct \( \xi \) finite or countable sub-families \( \left( B(x_{i1}, r_{ij}) \right)_{j}, \ldots, \left( B(x_{\xi}, r_{\xi}) \right)_{j} \)

such that each \( F_{\alpha, \beta} \subseteq \bigcup_{i=1}^{\xi} \bigcup_{j} B(x_{ij}, r_{ij}) \) and \( \left( B(x_{ij}, r_{ij}) \right)_{j} \) is a \( \delta \)-packing of \( F_{\alpha, \beta} \). Observing that
\[ \mu(B(x_{ij}, r_{ij}))^{q} \nu(B(x_{ij}, r_{ij}))^{t} (2r_{ij})^{B_{\mu, \nu}(q, t)} \leq \mu(B(x_{ij}, r_{ij}))^{q-h} \nu(B(x_{ij}, r_{ij}))^{t} (2r_{ij})^{B_{\mu, \nu}(q, t) + ah} \]
we obtain
\[ H_{\mu, \nu}(q, t, B_{\alpha, \beta}(q, t)) \leq \xi P_{q, t} - h, B_{\mu, \nu}(q, t) + \alpha h \]
or
\[ H_{\mu, \nu}(q, t, B_{\alpha, \beta}(q, t)) \leq \xi P_{q, t} - h, B_{\mu, \nu}(q, t) + \beta h. \]
Remark that, in the last inequality, we can replace \( F_{\alpha, \beta} \) by any arbitrary subset of \( F_{\alpha, \beta} \).

Then, we can finally conclude that
\[ H_{\mu, \nu}(q, t, B_{\alpha, \beta}(q, t)) \leq \xi P_{q, t} - h, B_{\nu}(q, t) + \alpha h \]
\[ \leq \xi P_{q, t} - h, B_{\nu}(q, t) + \beta h \]
\[ (\text{supp } \mu \cap \text{supp } \nu) = 0. \]

The proof of (4.2), (4.3) and (4.4) is similar to (4.1). \( \square \)

Let us return to the proof of Theorem 4.1. By Lemma 4.1 and Lemma 4.2, we have for all \( \eta_1, \eta_2 > 0 \),
\[ H^{\alpha q + \beta t + B_{\nu}(q, t) - \eta_1 - \eta_2}(E_{\mu, \nu}(\alpha, \beta)) \geq 2^{\alpha q + \beta t - \eta_1 - \eta_2} H_{\mu, \nu}(q, t, B_{\alpha, \beta}(q, t))(E_{\mu, \nu}(\alpha, \beta)) > 0. \]
Whence,
\[ \dim_H E_{\mu, \nu}(\alpha, \beta) \geq \alpha q + \beta t + B_{\nu}(q, t) - \eta_1 - \eta_2. \]

Letting \( \eta_1 \to 0 \) and \( \eta_2 \to 0 \) yields
\[ \dim_H E_{\mu, \nu}(\alpha, \beta) \geq \alpha q + \beta t + B_{\nu}(q, t). \]
Which achieves the proof of Theorem 4.1. \( \square \)

In the following we study the validity of the multifractal formalism under projection.

**Theorem 4.2.** Let \( \mu, \nu \) be two compactly supported Borel probability measures on \( \mathbb{R}^n \) such that \( \text{supp } \mu = \text{supp } \nu \). For \( (q, t) \in (] - \infty, 0[)^2 \cup (] - \infty, 0[ \times] 0, 1[) \cup (] 0, 1[ \times] - \infty, 0[) \), suppose that
\[ (H_1), \ H^{q, t, B_{\mu, \nu}(q, t)}(\text{supp } \mu \cap \text{supp } \nu) > 0, \]
\[ (H_2), \ B_{\mu, \nu} \text{ is differentiable at } (q, t), \]
Then, for all \( V \in G_{n,m} \), we have
\[
\dim_H E_{\mu,V,\nu} (\alpha, \beta) = \dim_P E_{\mu,V} (\alpha, \beta) = \dim_H E_{\mu,V} (\alpha, \beta)
\]
\[
= \dim_P E_{\mu,V} (\alpha, \beta) = B^*_{\mu,V} (\alpha, \beta) = b^*_{\mu,V} (\alpha, \beta),
\]
where \( \alpha = -\frac{\partial B_{\mu,V}(q,t)}{\partial q} \) and \( \beta = -\frac{\partial B_{\mu,V}(q,t)}{\partial t} \).

**Proof.** By using Theorem 3.2, Theorem 3.3 and \((H_1)\), we have
\[
b_{\mu,V}(q,t) = B_{\mu,V}(q,t) = b_{\mu,V,v}(q,t) = B_{\mu,V,v}(q,t), \quad \forall V \in G_{n,m}.
\] (4.5)

\((H_1), (3.2)\) and \((4.5)\) ensure that
\[
\mathcal{H}^{q,t,B_{\mu,V,v}(q,t)} (\text{supp } \mu_V) \geq \mathcal{H}^{q,t,B_{\mu,V}(q,t)} (\text{supp } \mu) > 0, \quad \forall V \in G_{n,m}.
\]

So, Theorem 4.1 and the equalities \((4.5)\) imply that
\[
\dim_H E_{\mu,V,\nu} (\alpha, \beta) \geq \alpha q + \beta t + B_{\mu,V}(q,t).
\]

The other estimation is satisfied since
\[
\dim_P E_{\mu,V,\nu} (\alpha, \beta) \leq \alpha q + \beta t + B_{\mu,V,v}(q,t)
\]
\[
= \alpha q + \beta t + B_{\mu,V}(q,t).
\]
Which achieves the proof of Theorem 4.2. \(\square\)

**Remark 4.1.** Let \( \mu \) and \( \nu \) be two compactly supported Borel probability measures on \( \mathbb{R}^n \). We write for \( \gamma \geq 0 \),
\[
\mathcal{B}_{\mu,V}(\gamma) = \left\{ x \in \text{supp } \mu \cap \text{supp } \nu : \lim_{r \to 0} \frac{\log (\mu(B(x,r)))}{\log (\nu(B(x,r)))} = \gamma \right\}.
\]

It is clear that
\[
\bigcup_{(\alpha,\beta) \in \mathbb{R}_+ \times \mathbb{R}_+^*} E_{\mu,V}(\alpha,\beta) \subset \mathcal{B}_{\mu,V}(\gamma).
\]

The latter union is composed by an uncountable number of pairwise disjoint nonempty sets. **Theorem 4.1** shows that surprisingly the Hausdorff and packing dimension of \( \mathcal{B}_{\mu,V}(\gamma) \) is fully carried by some subset \( E_{\mu,V}(\alpha,\beta) \). Together with **Theorem 4.2**, this relationship provides a lower bound to the relative multifractal spectra of the projections of a measures introduced in **Theorem 4.2** in [21].
Bibliography

[1] N. Attia, B. Selmi and Ch. Souissi. Some density results of relative multifractal analysis. Chaos, Solitons and Fractals. (2017), vol. 103, pp. 1-11.

[2] L. Barreira, B. Saussol and J. Schmeling. Higher-dimensional multifractal analysis. J. Math. Pures Appl., (2002), Vol. 81, pp. 67-91.

[3] L. Barreira and P. Doutor. Birkhoff Averages for Hyperbolic Flows: Variational Principles and Applications. Journal of Statistical Physics. (2004), Vol. 115, pp. 1567-1603.

[4] L. Barreira and P. Doutor. Almost additive multifractal analysis. J. Math. Pures Appl., (2009), Vol. 92, pp. 1-17.

[5] L. Barreira and P. Doutor. Dimension spectra of almost additive sequences. Nonlinearity. (2009), Vol. 22, pp. 2761-2773.

[6] L. Barreira, Y. Cao and J. Wang. Multifractal Analysis of Asymptotically Additive Sequences. J. Stat. Phys., (2013), Vol. 153, pp. 888-910.

[7] J. Barral and I. Bhouri. Multifractal analysis for projections of Gibbs and related measures. Ergodic Theory and Dynamic systems. (2011), vol. 31, pp. 673-701.

[8] J. Barral and D. J. Feng. Projections of planar Mandelbrot measures. arXiv:1605.09083v1, (2016).

[9] F. Ben Nasr. Analyse multifractale de mesures. CR Acad Sci Paris Ser I. (1994), vol. 319, pp. 807-10.

[10] F. Ben Nasr, I. Bhouri and Y. Heurteaux. The validity of the multifractal formalism: results and examples. Adv. in Math. (2002), vol. 165, pp. 264-284.

[11] J. Cole. Relative multifractal analysis. Chaos, Solitons and Fractals. (2000), vol. 11, pp. 2233-2250.

[12] C. Dai, Y. Li. A multifractal formalism in a probability space. Chaos Solitons Fractals. 2006, vol. 27, pp. 57-73.

[13] C. Dai, Y. Li. Multifractal dimension inequalities in a probability space. Chaos Solitons Fractals. (2007), vol. 34, pp. 213-223.

[14] M. Dai, X. Peng and W. Li. Relative Multifractal Analysis in a Probability Space. Int. J. Nonlinear Sci., (2010), vol. 10, pp. 313-319.

[15] M. Dai. Mixed self-conformal multifractal measures. Analysis in Theory and Applications. 25 (2009), 154-165.

[16] M. Dai and Y. shi. Typical behavior of mixed $L^q$-dimensions. Nonlinear Analysis: Theory, Methods & Applications. 72 (2010), 2318-2325.
[17] M. Dai and W. Li. The mixed $L^q$-spectra of self-conformal measures satisfying the weak separation condition. J. Math. Anal. Appl., 382 (2011), 140-147.

[18] M. Dai, C. Wang and H. Sun. Mixed generalized dimensions of random self-similar measures. Int. J. Nonlinear. Sci., 13 (2012), 123-128.

[19] M. Dai, J. Houa, J. Gaob, W. Suc, L. Xid and D. Ye. Mixed multifractal analysis of China and US stock index series. Chaos, Solitons & Fractals. 87 (2016), 286-275.

[20] M. Dai, S. Shao, J. Gao, Y. Sun and W. Su. Mixed multifractal analysis of crude oil, gold and exchange rate series. Fractals. 24 (2016), 1-7.

[21] Z. Douzi and B. Selmi. Multifractal variation for projections of measures. Chaos, Solitons and Fractals. (2016), vol. 91, pp. 414-420.

[22] K. J. Falconer. The Geometry of Fractal sets. Cambridge univ. Press New. York-London. (1985), vol. 85.

[23] K. J. Falconer, J. D. Howroyd. Packing Dimensions of Projections and Dimensions Profiles. Math. Proc. Cambridge Philos. Soc. (1997), vol. 121, pp. 269-286.

[24] K. J. Falconer, P. Mattila. The Packing Dimensions of Projections and Sections of Measures. Math. Proc. Cambridge Philos. Soc. (1996), vol. 119, pp. 695-713.

[25] K. J. Falconer, X. Jim. Exact dimensionality and projections of random self-similar measures and sets. J.Lond.Math. Soc. (2014), vol. 90, pp. 388-412.

[26] M. Hochman, P. Shmerkin. Local entropy averages and projections of fractal measures. arXiv : 0910.1956v1, (2009).

[27] R. Kaufman. On Hausdorff dimension of projections. Mathematika. (1968), vol. 15, pp. 153-155.

[28] J .M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. Proceedings of the London Mathematical Society. (1954), vol. 4, pp. 257-302.

[29] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. Annales Academiae Scientiarum Fennicae. Series A I. Mathematica. (1975), vol. 1, pp. 227-244.

[30] P. Mattila. The Geometry of Sets and Measures in Euclidean Spaces. Cambridge University Press, Cambridge. (1995).

[31] J. Li, L. Olsen and M. Wu. Bounds for the $L^q$-spectra of self-similar measures without any separation conditions. J. Math. Anal. Appl., 387 (2012), 77-89.

[32] L. Olsen. A multifractal formalism. Advances in Mathematics. (1995), vol. 116, pp. 82-196.
[33] L. Olsen. *Mixed generalized dimensions of self-similar measures*. J. Math. Anal. Appl., 306 (2005), 516-539.

[34] L. Olsen. *Bounds for the $L^q$-spectra of a self-similar multifractal not satisfying the open set condition*. J. Math. Anal. Appl., 355 (2005), 12-21.

[35] L. Olsen. *On the inverse multifractal formalism*. Glasgow Mathematical Journal. 52 (2010), 179-194.

[36] T. C. O’Neil. *The multifractal spectrum of quasi self-similar measures*. Journal of Mathematical Analysis and its Applications. (1997), vol. 211, pp. 233-257.

[37] T. C. O’Neil. *The multifractal spectra of projected measures in Euclidean spaces*. Chaos, Solitons and Fractals. (2000), vol. 11, pp. 901-921.

[38] J. Peyriére. *Multifractal measures*. In: Proceedings of the NATO Advanced Study Institute on Probabilistic and Stochastic Methods in Analysis with Applications. Il Ciocco, NATO ASI Series, Series C: Mathematical and physical sciences, vol. 372, Kluwer Academic Press, Dordrecht, (1992), pp. 175-186.

[39] B. Selmi. *Multifractal dimensions for projections of measures*. Preprint. 2017.

[40] B. Selmi. *A note on the effect of projections on both measures and the generalization of $q$-dimension capacity*. Probl. Anal. Issues Anal., (2016), vol. 5, pp. 38 - 51.

[41] B. Selmi and N.Yu. Svetova. *On the projections of mutual $L^{q,t}$-spectrum*. Probl. Anal. Issues Anal., (2017), vol. 6, pp. 94 - 108.

[42] P. Shmerkin. *Projections of self-similar and related fractals: A survey of recent developments*. arXiv : 1501.00875v1, (2015).

[43] P. Shmerkin, B. Solomyak. *Absolute continuity of self-similar measures, their projections and convolution*, arXiv : 1406.0204v1, (2014).

[44] M. Slimane. *Baire typical results for mixed Hölder spectra on product of continuous besov or oscillation spaces*. Mediterr. J. Math., 13 (2016), 1513-1533.

[45] N.Yu. Svetova. *Conditional and mutual multifractal spectra. Definition and basic properties*. Tr. Petrozavodsk. Gos. Univ. Ser. Mat., (2003), vol. 10, pp. 41-58.

[46] N.Yu. Svetova. *Mutual multifractal spectra I: Exact spectra*. Tr. Petrozavodsk. Gos. Univ. Ser. Mat., (2004), vol. 11, pp. 41-46.

[47] N.Yu. Svetova. *Mutual multifractal spectra II: Legendre and Hentschel-Procaccia spectra, and spectra defined for partitions*. Tr. Petrozavodsk. Gos. Univ. Ser. Mat., (2004), vol. 11, pp. 47-56.
[48] N.Yu. Svetova. *An estimate for exact mutual multifractal spectra*. Tr. Petrozavodsk. Gos. Univ. Ser. Mat., (2008), vol. 14, pp. 59-66.

[49] N.Yu. Svetova. *The property of convexity of mutual multifractal dimension*. Tr. Petrozavodsk. Gos. Univ. Ser. Mat., (2010), vol. 17, pp. 15-24.