Interplay Between Structural Randomness, Composite Disorder, and Electrical Response: Resonances and Transient Delays in Complex Impedance Networks

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We study the interplay between structural and conductivity (composite) disorder and the collective electrical response in random networks models. Translating the problem of time-dependent electrical response (resonance and transient relaxation) in binary random composite networks to the framework of generalized eigenvalues, we study and analyze the scaling behavior of the density of resonances in these structures. We found that by controlling the density of shortcuts (topological randomness) and/or the composite ratio of the binary links (conductivity disorder), one can effectively shape resonance landscapes, or suppress long transient delays in the corresponding random impedance networks.

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Resistor networks have been widely studied since the 70’s as models for conductivity problems and classical transport in disordered media. With the recent surge of research on complex networks, resistor-networks and related flow models have been employed to study and explore community structures in social networks and to construct recommendation models for community networks. Also, resistor networks, as abstract models for network flows with a fundamental conservation law, were utilized to study transport in scale-free (SF) and small-world (SW) networks, and in tree structures and hierarchical lattices.

Complex impedance networks have been investigated to study electrical and optical properties of two-dimensional thin films, dielectric resonances of two-dimensional regular lattice structures, lattice animals, and other fractal clusters. In this Letter, we investigate electrical response (resonances and delays) when both the structure and the composition of the local conductances can be random, and we focus on the interplay between structural and composite disorder, and response. Random structures, in particular, random nanowire networks, can play a key role in the design and fabrication of future electronics devices, such as transistors or interconnects. Assessing performance and reliability of these systems requires to understand the time-dependent intrinsic electrical response (resonances and transient delays) of these devices which have to switch electric currents on and off, and driven by high clock speeds. For example, in a random nanowire network made of single-wall carbon nanotubes, the individual wires can be either conductors or semiconductors (based on their individual chiralities), resulting in links with (binary) composite disorder; their natural composition comes with a dominance of the semiconducting tubes. Likewise, inherent delays in electrical signal propagation can have crucial effects on processes in neuronal networks. The compartmental-model representation of passive dendritic trees is an R1-C-R2 network (each compartment consists of an R1 membrane leakage resistor in parallel with a capacitance C, and compartments are connected with an R2 junctional resistor). The framework employed here can be employed to study the effects of local defects (damaged or destroyed links) on global signal delays, ultimately governed by the structure and link disorder in the network.

Here, we focus on the interplay between topological randomness, conductivity disorder, and system response. While the resonance and relaxation properties are well understood in low-dimensional structures with conductivity (bond) disorder, and recently on the complete graph, to our knowledge, a similar investigation on complex random network structures with link conductivity disorder have not been initiated or explored. We employ the framework applicable to binary link disorder. The powerful feature of the framework is that it can be employed to study the singularities of the electrical response associated with any kind of binary link disorder on any graph (random L-C, RL-C, R-C, or more complicated composite circuits, involving two, but individually arbitrarily complicated building blocks).

The equations governing current flows in any network can be written as $\sum_j \sigma_{ij}(V_i - V_j) = I_{it}$, where $\sigma_{ij}$ are now the possibly complex link conductances (or admittances). Nodes $s$ and $t$ are the nodes where a current $I$ enters and leaves the network, respectively. The above equation can be rewritten as $\sum_i L_{ij} V_i = I_{is} - I_{it}$, where $L_{ij} = \delta_{ij} \sum_{\ell \neq i} \sigma_{it} - \sigma_{ij}$ is the Laplacian of the underlying graph with complex couplings, also referred to as the admittance matrix in the present context.

For example, in L-C composite networks (or in RL-C composite networks with weak dissipation), in order to find resonance frequencies one can identify the non-trivial singularities, the “zeros” of the admittance matrix (corresponding to the “poles” of the complex impedance matrix), i.e., requiring that zero input current gives rise to finite potential.
In what follows, for brevity, we use the “resonance” terminology in composite networks, and will also refer to \( \lambda \) as “frequency”. The eigenvalues of the above system always fall in the \([1, \infty)\) interval, \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\), with true resonances corresponding to \(-1 < \lambda_j < 1\). We focused on two important observables, the density of resonances \(\rho(\lambda)\) and number of resonances per node \(\varpi\):}

\[
\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{n_R} \delta(\lambda - \lambda_\alpha), \quad \varpi = \int \rho(\lambda) d\lambda = \frac{n_R}{N},
\]

where \(n_R\) is the total number of true resonances (not associated with \(\lambda_j = \pm 1\)). In this work, we determined the spectrum of the generalized eigenvalue problem Eq. (1) numerically, and constructed the above observables by averaging over 10,000 realizations (1,000 realizations for the largest system size) of both structural and composite disorder.

Before studying random structures with binary composite disorder, we recall two known extreme cases: the one-dimensional ring [Fig. (a)] and the complete graph [Fig. (c)], both with the same composite (binary link) disorder. For a one-dimensional ring, there is a single resonance frequency \(\lambda_{1 \to 2}\) (which can also be obtained via elementary considerations). More specifically, \(n_R = 1\) and the frequency is distributed binomially about \(\langle \lambda \rangle = 2q - 1\). Thus, in the large-N limit, the density of resonances Eq. (3) approaches a Gaussian distribution with the above mean and vanishing width, i.e., a delta function. For example, for \(q=1/2\), \(\rho(\lambda) \simeq \frac{1}{N} \delta(\lambda)\) and \(\varpi = \frac{1}{N}\).
FIG. 2: (a) Average number of resonances per node vs density of shortcuts \( p \) in a 1d SW network with composite ratio \( q=1/2 \). The inset shows the same data on log-log scales, with the straight dashed line indicating the asymptotic large-\( N \) behavior, \( \overline{\rho} \sim p \). (b) Density of resonances for fixed number of nodes \( N=1000 \) with \( q=1/2 \), and for different values of \( p \). (c) Density of resonances in the large-\( p \) regime for \( q=1/2 \), for various system sizes. The inset shows the scaled plot of the same data, \( \rho(\lambda)/p^{1/2} \) vs \( p^{1/2}/\lambda \). These and the following plots all show ensemble averages over 10,000 network- and composite-disorder realizations (1,000 for the largest system size).

In the other extreme case where all nodes are connected to all others (i.e., the complete graph), using a path-integral approach \( \rho_0(p) \sim O(1) \) in the large-\( N \) limit, Fyodorov obtained (without loss of generality, for \( q=1/2 \)) that
\[
\rho(\lambda) \xrightarrow{N \to \infty} \delta(\lambda) \quad \text{and} \quad \overline{\rho} \xrightarrow{N \to \infty} 1 \quad \text{i.e., the total number of resonances approaches the number of nodes, but they are all narrowly centered about the same frequency (becoming fully degenerate as \( N \to \infty \)).}
\]

Now, we consider small-world (SW) networks \( \overline{\rho}(\lambda) \) as random structures, where random shortcuts were added to a one-dimensional ring (1d SW) \( \overline{\rho}(\lambda) \) [Fig. 1(b)], resulting in an average number of random shortcuts per node \( p \). For comparison with the previous two extreme cases, we show results for the same composite ratio \( q=1/2 \). The results show that for any nonzero value of \( p \), the number of resonances per node will approach a nonzero \( \overline{\rho} \) \( > 0 \) value in the large-\( N \) limit, as opposed to the pure 1d ring where it vanishes as \( 1/N \) [Fig. 2(a)]. Further, as the number of random links per node \( p \) increases, \( \overline{\rho} \) increases monotonically, and the density of resonances initially \( [0 < p \leq O(1)] \) widens; at around \( p \sim O(1) \), the spectrum becomes extended [Fig. 2(b)]. As we further increase \( p \), the number of resonances per node continues to increase monotonically as a function \( p \), quickly “saturating” to its maximum value \( \overline{\rho} = 1 \) [Fig. 2(a)], while the density of resonances becomes progressively centered about \( \lambda = 0 \) [Fig. 2(b,c)], eventually converging to a delta-function (if both \( p \to \infty \), \( N \to \infty \)). Indeed, one can recall for the complete graph, that the average number of resonance per node approaches \( \overline{\rho} = 1 \), but all frequencies are centered about the same value \( \rho_0(p) \). Note that in both the low shortcut density \( [0 < p \ll O(1), \overline{\rho}(\lambda)] \) and the high shortcut density \( [p \gg O(1)] \) (not shown) regimes, for fixed \( p \), the density of resonances becomes independent of the size of the network for large \( N \). Finite-size effects are very strong, however, for \( p \sim O(1) \), in particular in the low- and high-frequency regime Fig. 2(c). Further analysis in the high-connectivity \( [p \gg O(1)] \) regime also reveals that the limit density of resonances has the scaling form \( \rho(\lambda) = p^{1/2} \delta(p^{1/2}/\lambda) \) [Fig. 2(c), and inset]. This scaling form, valid for all \( p \gg O(1) \), is identical to the one found for regular long-range connectivity graphs \( \overline{\rho}(\lambda) = 1/N \), including the limit of complete graph \( (p=CN) \).

Next, we provide more details for the low shortcut-density regime (also referred to as the SW regime), \( 0 < p \ll O(1) \). The scaling of the number of resonances per node \( \overline{\rho}(\lambda) \) in this regime [Fig. 2(a)], can be extracted from the finite-size behavior, typical in 1d SW networks \( \overline{\rho}(\lambda) \). Since the number of random links per node (density of shortcuts) is \( p \), the typical (Euclidean) distance between nodes with shortcuts emanating from them scales as \( x \sim p^{-1} \). Thus, for \( N \ll \xi \) (\( Np \ll 1 \), there are no random links in almost any realization of the network, and the resonance structure will essentially be identical to that of the pure ring. A crossover, governed by the emerging SW structure, can be expected when \( N \gg \xi \) (\( Np \gg 1 \)). Thus, in the SW regime, \( 0 < p \ll O(1) \), for arbitrary \( N \), the above crossover behavior of \( \overline{\rho}(N,p) \) can be expressed in terms of \( N \) and the scaled variable \( x = Np \) with the help of a scaling function \( f(x) \), such that
\[
\overline{\rho}(N,p) = \frac{1}{N} f(Np)
\]
obeys the scaling form \( \rho(\lambda) = \bar{\rho}(p)p^{-1/2}\Psi(\lambda/p^{1/2}) \sim p^{1/2}\Psi(\lambda/p^{1/2}) \). Further, the scaling function \( \Psi(s) \) is reasonably well approximated by \( \sim e^{-cs^2} \) in the vicinity of the center as \( N \rightarrow \infty \), i.e., the density of resonances approaches a Gaussian shape [Fig. 3(a,b)].

To model more complicated spatially-embedded random structures [Fig. 4(a)], we considered when both the value of the complex link conductivity and the probability to have a link between two nodes can depend on Euclidean distance between the two nodes it connects. From elementary length scale considerations, for the distance-dependent link conductivity, one has \( \sigma_{ij} \sim 1/d_{ij} \), where \( d_{ij} \) is the Euclidean distance between nodes \( i \) and \( j \) (assuming uniform “wire” cross-sections) [Eq. (1)] easily generalizes to this case. Whereas the probability of having a link (shortcut) between node \( i \) and \( j \), can also be suppressed e.g., \( p_{ij} \sim 1/d_{ij}^\alpha \) (power-law-suppressed SW networks due to “wiring”-cost considerations or topological constraints [30, 51]). In Fig. 4(b), we show the resonance spectrum of a two-dimensional power-law-suppressed SW network (2d SW) with open boundaries with \( \alpha=1 \) and \( p=1.00 \) (random shortcuts with distance-dependent conductivities were added on top of a two-dimensional regular “substrate” [Fig. 4(a)]), with composite ratio \( q=1/2 \), together with the known results [16, 17] of the regular two-dimensional topological structures with the same composite disorder. For regular two-dimensional structures, in the large-system size limit, \( \Psi(0) \sim \mathcal{O}(1) \) and the spectrum is known to be extended [16, 17]. The addition of distance-dependent shortcuts, however, strongly modifies the density of resonances in the vicinity of \( \Lambda=\pm 1 \) (strong peaks for low and high frequencies). Further, the structure of the peaks do not approach a limit density in that region, but diverge with system size (with \( p \) fixed).

An analogous plot for an asymmetric link disorder with \( q=2/3 \) [Fig. 4(c)] shows strong (diverging) peaks only in the small-frequency regime [also translating to large transient relaxation times or delays in \( RC \) networks Eq. (2)]. Our analyses also indicate that the main qualitative features (articulated peaks for low and/or high frequencies) of structures with distance-dependent shortcuts prevail for a range of \( \alpha, 0 \leq \alpha \leq \alpha_c \approx 2 \pm 0.5 \).

In summary, we have shown that in random composite networks, by controlling the density of shortcuts \( p \) (topological randomness) and/or the composite ratio \( q \) of the binary links (conductivity disorder), one can effectively shape the resonance landscape, or suppress long transient delays in electrical signal propagation. Here, we have highlighted the interplay between structural and composite (conductivity) disorder and the collective electrical response in spatially-embedded random networks models. The electrical response of more realistic off-lattice random structures, embedded in two- and three-dimensions, reflecting relevant wiring cost and topological constraints [19, 20, 30] will be considered in future works. A detailed analyses on such structures will help one understand electrical response (resonances and signal delays) in complex materials and biological networks.

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FIG. 4: (a) Schematic plot of a 2d SW graph. (b) Density of resonances in regular 2d \((p=0, \text{open symbols})\) and power-law-suppressed 2d SW networks \((p=1.00, \alpha=1.00, \text{solid symbols})\) for different system sizes \((N=L\times L)\), and for composite ratio \(q=1/2\). The insets show the same data enlarged for the 2d SW network in the vicinity of one of the peaks. (c) The same plot as (b) for composite ratio \(q=2/3\).

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