Off-Diagonal Decay of Toric Bergman Kernels

Dedicated to the memory of Louis Boutet de Monvel

STEVE ZELDITCH
Department of Mathematics, Northwestern University, Evanston, IL 60208 USA.
e-mail: zelditch@math.northwestern.edu

Received: 11 November 2015 / Revised: 7 October 2016 / Accepted: 7 October 2016
Published online: 18 October 2016 – © Springer Science+Business Media Dordrecht 2016

Abstract. We study the off-diagonal decay of Bergman kernels $\Pi_{hk}(z, w)$ and Berezin kernels $P_{hk}(z, w)$ for ample invariant line bundles over compact toric projective Kähler manifolds of dimension $m$. When the metric is real analytic, $P_{hk}(z, w) \simeq k^m \exp(-kD(z, w))$ where $D(z, w)$ is the diastasis. When the metric is only $C^\infty$ this asymptotic cannot hold for all $(z, w)$ since the diastasis is not even defined for all $(z, w)$ close to the diagonal. Our main result is that for general toric $C^\infty$ metrics, $P_{hk}(z, w) \simeq k^m \exp(-kD(z, w))$ as long as $w$ lies on the $R_m^+$-orbit of $z$, and for general $(z, w)$, $\limsup_{k \to \infty} \frac{1}{k} \log P_{hk}(z, w) \leq -D(z^*, w^*)$ where $D(z, w^*)$ is the diastasis between $z$ and the translate of $w$ by $(S^1)_m$ to the $R_m^+$ orbit of $z$. These results are complementary to Mike Christ’s negative results showing that $P_{hk}(z, w)$ does not have off-diagonal exponential decay at “speed” $k$ if $(z, w)$ lies on the same $(S^1)_m$-orbit.

Mathematics Subject Classification 32A25, 14M25.

Keywords. toric Kähler manifold, line bundle, Bergman kernel.

1. Introduction

The problem we are concerned with in this note is to find conditions on a positive Hermitian line bundle $(L, h) \to (M, \omega)$ over a Kähler manifold so that the Szegö kernel $\Pi_{hk}(z, w)$ for $H^0(M, L^k)$ has exponential decay at speed $k$. We denote the Berezin kernel or normalized Szegö kernel by

$$P_{hk}(z, w) := \frac{||\Pi_{hk}(z, w)||}{\Pi_{hk}(z, z)^{1/2} \Pi_{hk}(w, w)^{1/2}}.$$  \hfill (1)

Problem Let $D_h^*(z, w)$ be the upper semi-continuous regularization of

$$\limsup_{k \to \infty} \frac{1}{k} (-\log P_{hk}(z, w)).$$ \hfill (2)

Determine $D_h^*(z, w)$ and in particular determine when it is non-zero.

Research partially supported by NSF Grant DMS-1541126.
The minus sign is due to the fact that (1) is pluri-superharmonic in $z$ and we prefer to deal with pluri-subharmonic functions. It is known that for real analytic metrics, $P_{h^k}(z, w) \leq C k^m e^{-kD(z, w)}$ for points $(z, w)$ sufficiently close to the diagonal, where $D(z, w)$ is the so-called Calabi diastasis (Section 2.1). Near the diagonal, $D(z, w) \sim |z - w|^2$. For general smooth metrics, the sharpest general result is that $P_{h^k}(z, w) \leq C e^{-A \sqrt{k} \log k}$ for all $A < \infty$ [2,3]. This raises the question of whether, for $C^\infty$ but not real analytic metrics, $D_h^*(z, w)$ can be strictly negative off the diagonal.

A stronger condition which arises in several problems (see [6]) is whether there exists a pointwise limit

$$
\frac{1}{k} \log P_{h^k}(z, w) \rightarrow -\tilde{D}(z, w) \tag{3}
$$

for some function $\tilde{D}(z, w)$ defined near the diagonal in $M \times \bar{M}$. If the metric is real analytic, then such a limit does exist and $\tilde{D}(z, w) = D(z, w)$ is the Calabi diastasis of the metric (see Section 2.1). The diastasis is the real part of the off-diagonal analytic continuation of a local Kähler potential of $\omega$ [1]. Existence of regular pointwise limit near the diagonal would be surprising if the metric is $C^\infty$ but not real analytic, since it would define a Calabi diastasis even though the Kähler potential admits no analytic continuation. One might, therefore, expect the neighborhood of $z$ in which the limit (3) exists to be the largest neighborhood of $z$ in which the Kähler potential $\phi$ admits an analytic continuation.

In this note, we study these questions in the case of a positive Hermitian holomorphic toric line bundle $(L, h) \rightarrow (M, \omega_h)$ with $C^\infty$ metric $h$. As recalled in Section 3, a toric Kähler manifold is a Kähler manifold on which the complex torus $(\mathbb{C}^*)^m$ acts holomorphically with an open-orbit $M^\alpha$. We denote by $T^m$ the underlying real torus and by $\mathbb{R}^m_+$ the real subgroup of $(\mathbb{C}^*)^m$. We denote a point by $z = e^{\rho/2 + i\theta} m_0$ where $e^{\rho/2}$ denotes the $\mathbb{R}^m_+$ action and $e^{i\theta}$ denotes the $T^m$ action. Let $h = e^{-\phi}$ in a toric holomorphic frame over $M^\alpha$. As recalled in Section 3.3, $\phi(e^{\rho/2}) = \tilde{\phi}(\rho)$ on the open orbit, where $\tilde{\phi}$ is convex.

Given two points $z = e^{\rho_1/2 + i\theta_1}, w = e^{\rho_2/2 + i\theta_2}$ we denote by $z^* = e^{\rho_1/2}$, resp. $w^* = e^{\rho_2/2}$ the unique point on the $\mathbb{R}^m_+$ orbit of $m_0$ which lie on the same $T^m$ orbit as $z$, resp. $w$. Our main result is that $D_h^*(z, w) \leq -D(z^*, w^*)$ where $D(z, w)$ is the Calabi diastasis (see Sections 2.1, 3.3).

**THEOREM 1.** Let $(L, h) \rightarrow (M, \omega)$ be a positive Hermitian toric line bundle over a toric Kähler manifold. Then if $z, w \in M^\alpha$,

$$
\limsup_{k \rightarrow \infty} \frac{1}{k} \log P_{h^k}(z, w) \leq -D(z^*, w^*) \leq 0,
$$

with $D(z^*, w^*) = 0$ if and only if $z^* = w^*$. Furthermore, if $z = e^{\rho_1/2 + i\theta}, w = e^{\rho_2/2 + i\theta}$ lie on the same $\mathbb{R}^m_+$ orbit, then one has the pointwise limit (3).