Abstract

In this paper, we investigate a class of domains $\Omega_{\gamma}^{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : |z|^{\gamma} < |w| < 1\}$ for $\gamma > 0$ that generalizes the Hartogs triangle. We obtain a sharp range of $p$ for the boundedness of the Bergman projection on the domain considered here. It generalizes the results by Edholm and McNeal [1] for $n = 1$ to any dimension $n$.

Key words: Hartogs triangle, $L^p$ regularity, Bergman kernel, Bergman projection.

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1 Introduction

Let $\Omega$ be a domain in the $n$-dimensional complex space $\mathbb{C}^n$. For $p > 0$, denote

$$L^p(\Omega) = \left\{ f : \left( \int_\Omega |f|^p dV \right)^{\frac{1}{p}} := \|f\|_p < \infty \right\},$$

where $dV(z)$ is the ordinary Lebesgue volume measure on $\Omega$. As we know, $L^p(\Omega)$ is a Banach space when $p > 1$. For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product:

$$\langle f, g \rangle = \int_\Omega f(z) \overline{g(z)} dV(z). \quad (1.1)$$

Let $\mathcal{O}(\Omega)$ denote the holomorphic functions on $\Omega$, and let $A^p(\Omega) = \mathcal{O}(\Omega) \cap L^p(\Omega)$. The Bergman projection associated with $\Omega$ will be written as $P_\Omega$, or $P$ if $\Omega$ is clear and is the...
orthogonal projection operator $P : L^2(\Omega) \rightarrow A^2(\Omega)$. It is elementary that $P$ is self-adjoint with respect to the inner product (1.1). The Bergman kernel, denoted as $B_\Omega(z, w)$, satisfies

$$P_\Omega f(z) = \int_\Omega B_\Omega(z, w) f(w) dV(w), \quad f \in L^2(\Omega).$$

Given an orthonormal Hilbert space basis $\{\phi_\alpha\}_{\alpha \in A}$ for $A^2(\Omega)$, the Bergman kernel is given by the following formula:

$$B_\Omega(z, w) = \sum_{\alpha \in A} \overline{\phi_\alpha(z)} \phi_\alpha(w).$$

The Hartogs triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$ is a pseudoconvex domain that is the source of many counterexamples in several complex variables; see [17]. The generalized Hartogs triangles recently studied by Edholm and McNeal [1, 3] are a class of pseudoconvex domains in $\mathbb{C}^2$ defined for $\gamma > 0$ by

$$\mathbb{H}_\gamma = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{\gamma} < |z_2| < 1\},$$

and further Edholm, McNeal [4] and Chen [13] continue their study of the Bergman projection on $\mathbb{H}_\gamma$ and obtain Sobolev estimates.

The generalized Hartogs triangles exhibit the same pathological behavior as the classical Hartogs triangle due to the singularity of the boundary, which is non-Lipschitz at the origin, with the additional surprising dependence on the rationality or irrationality of the power $\gamma$.

In general, the regularity of $P_\Omega$ depends closely on the geometry of $\Omega$. For various geometric conditions on $\Omega$, understanding the range of $p$ for which $P_\Omega$ is $L^p$ bounded is an active area of research. In [10], the author constructs pseudoconvex domains in $\mathbb{C}^2$, where $P_\Omega$ is bounded if and only if $p = 2$. Beberok [8] also considered the $L^p$ boundedness of the Bergman projection on the following generalization of the Hartogs triangle:

$$\mathcal{H}_{k+1}^n := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : ||z|| < |w|^k < 1\},$$

where $k \in \mathbb{Z}^+$ and $|| \cdot ||$ is the Euclidean norm in $\mathbb{C}^n$. On some other domains, the projection has only a finite range of mapping regularity (see, for example, [2, 6, 12]).

In this article, we mainly study the following bounded regions. For $\gamma > 0$, we define the domain $\Omega_{\gamma+1}^{n+1} \subset \mathbb{C}^n \times \mathbb{C}$ by

$$\Omega_{\gamma+1}^{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : |z|^{\gamma} < |w| < 1\}$$

and call $\Omega_{\gamma+1}^{n+1}$ the generalized Hartogs triangle of exponent $\gamma, n$. On the generalized Hartogs triangle, we denote the Bergman projection by $P_\gamma$ and the Bergman kernel as $B_{\gamma,n}((z, w), (s, t))$. 

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The primary purpose of this paper is to show that the Bergman projection of $\Omega^{n+1}_\gamma$, $P_\gamma$, is $L^p$ bounded for only a restricted range of $p \in (1, \infty)$. Modeled on Edholm’s results [1], we will discuss $\gamma$ dividing into rational and irrational numbers. In the former case, we decompose $A^2(\Omega^{n+1}_\gamma)$ to obtain an estimate of the Bergman kernel. For more discussion of the Bergman kernel, see [11, 15, 16]. We then use Schur’s lemma to discuss the boundedness of the projection operator. For the proof of unboundedness, we exhibit a single function $f \in L^\infty(\Omega^{n+1}_\gamma)$ such that $P_\gamma f \notin L^p(\Omega^{n+1}_\gamma)$.

We use the following notation to simplify writing various inequalities. If $A$ and $B$ are functions depending on several variables, we write $A \lesssim B$ to signify that there exists a constant $K > 0$, independent of relevant variables, such that $A \leq KB$. The independence of which variables will be clear in context. We also write $A \approx B$ to mean that $A \lesssim B \lesssim A$.

If $x \in \mathbb{R}$, $\lfloor x \rfloor$ will denote the greatest integer not exceeding $x$.

## 2 Main results

The precise statement of our main result is as follows.

**Theorem 2.1.** Let $m, l \in \mathbb{Z}^+$ with $\gcd(m, l) = 1$. The Bergman projection $P_{m/l}$ is a bounded operator from $L^p(\Omega^{n+1}_{m/l})$ to $L^p(\Omega^{n+1}_{m/l})$ if and only if $p \in \left(\frac{2m+2nl}{m+nl+1}, \frac{2m+2nl}{m+nl-1}\right)$.

However, when $\gamma \notin \mathbb{Q}$, the $L^p$ mapping of $P_\gamma$ completely degenerates:

**Theorem 2.2.** Let $\gamma > 0$ be irrational. The Bergman projection $P_\gamma$ is a bounded operator from $L^p(\Omega^{n+1}_\gamma)$ to $L^p(\Omega^{n+1}_\gamma)$ if and only if $p = 2$.

## 3 The Rational Case: $L^p$ Boundedness

### 3.1 Decomposing the Bergman Space

**Lemma 3.1.** [6] For any $v_1, \cdots, v_n \geq 0$,

$$
\int_{S^{2n-1}} |\zeta_1|^{2v_1} \cdots |\zeta_n|^{2v_n} d\sigma(\zeta) = \frac{2^n \pi^n}{\Gamma(n + |v|)},
$$

where $|v| = v_1 + \cdots + v_n$, $v! = \Gamma(v_1 + 1) \cdots \Gamma(v_n + 1)$, and $S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$ with respect to the surface measure $d\sigma$. 

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Lemma 3.2. If the monomial $z^\alpha w^\beta \in A^2(\Omega^{n+1}_\gamma)$, then multi-indices $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{Z}$ should meet $|\alpha| + \gamma(\beta + 1) > -n$, and

$$\|z^\alpha w^\beta\|^2_{L^2(\Omega^{n+1}_\gamma)} = \frac{\pi^{n+1} \alpha!}{(\beta + \frac{|\alpha| + n}{\gamma} + 1) \Gamma(|\alpha| + n + 1)}.$$ 

**Proof.** Using polar coordinates and Lemma 3.1,

$$\|z^\alpha w^\beta\|^2_{L^2(\Omega^{n+1}_\gamma)} = \int_{\Omega^{n+1}_\gamma} |z^\alpha|^2 |w|^2 \beta \ dV(z, w)$$

$$= \int_{0<|w|<1} |w|^{2\beta} \int_{|\zeta|=1} |\zeta|^2 |\alpha_1| \cdots |\alpha_n| d\sigma(\zeta) r^{2|\alpha|+2n-1} dr dV(w)$$

$$= \frac{\pi^n \alpha!}{(|\alpha| + n) \Gamma(|\alpha| + n)} \int_{0<|w|<1} |w|^{2\beta+2|\alpha|+2n} \ dV(w)$$

$$= \frac{2\pi^{n+1} \alpha!}{\Gamma(|\alpha| + n + 1)} \int_{0}^{1} r^{2\beta+2|\alpha|+2n+1} dr. \quad (3.2)$$

This integral converges if and only if $2\beta + \frac{2|\alpha|+2n}{\gamma} + 1 > -1$, i.e., $|\alpha| + \gamma(\beta + 1) > -n$. Furthermore, when the integral (3.2) converges, it equals

$$\frac{2\pi^{n+1} \alpha!}{\Gamma(|\alpha| + n + 1)} \cdot \frac{1}{2\beta + \frac{2|\alpha|+2n}{\gamma} + 2} = \frac{\pi^{n+1} \alpha!}{(\beta + \frac{|\alpha| + n}{\gamma} + 1) \Gamma(|\alpha| + n + 1)}.$$

Now, we consider the Bergman kernel $B_{\gamma,n}((z, w), (s, t))$, $z, s \in \mathbb{C}^n, w, t \in \mathbb{C}$. Choose one unitary matrix $U$ such that $z = |z| \mathbf{1} U^{-1}$, where $\mathbf{1} = (1, 0, \cdots, 0)$. Then, from lemma 3.2 we obtain

$$B_{\gamma,n}((z, w), (s, t)) = \sum_{|\alpha|+\gamma(\beta+1)>-n} \frac{z^\alpha w^\beta (s^\alpha \overline{t}^\beta)}{\|z^\alpha w^\beta\|^2_{L^2(\Omega^{n+1}_\gamma)}} = \sum_{|\alpha|+\gamma(\beta+1)>-n} \frac{(|z| \mathbf{1})^\alpha w^\beta ((sU)^\alpha \overline{t}^\beta)}{\|z^\alpha w^\beta\|^2_{L^2(\Omega^{n+1}_\gamma)}}$$

$$= \sum_{(\alpha, \beta) \in \Lambda_{\gamma,n}} \frac{(|z| \mathbf{1}^T \overline{S}^T)^\alpha (w \cdot \overline{T})^\beta}{N_{\gamma,n}(\alpha, \beta)} = \sum_{(\alpha, \beta) \in \Lambda_{\gamma,n}} \frac{(z \cdot \overline{s})^\alpha (w \cdot \overline{t})^\beta}{N_{\gamma,n}(\alpha, \beta)}, \quad (3.3)$$

where

$$\Lambda_{\gamma,n} = \{(\alpha, \beta) \in \mathbb{N} \times \mathbb{Z} : \alpha_1 + \gamma(\beta + 1) > -n\}, \quad (3.4)$$

$$N_{\gamma,n}(\alpha, \beta) = \int_{\Omega^{n+1}_\gamma} |z_1|^{2\alpha_1} |w|^2 \beta \ dV(z, w) = \frac{\pi^{n+1} \Gamma(\alpha_1 + 1)}{(\beta + \frac{\alpha_1 + n}{\gamma} + 1) \Gamma(\alpha_1 + n + 1)}. \quad (3.5)$$
Therefore, below, we only need to consider a two-dimensional array that satisfies the inequality (3.4).

When \( \gamma = \frac{m}{l} \in \mathbb{Q}^+ \), \( \gcd(m, l) = 1 \), the strict inequality defining \((\alpha_1, \beta) \in \Lambda_{\gamma,n}\) can be re-expressed as a non-strict inequality:

\[
\Lambda_{\Phi, n} = \{(\alpha_1, \beta) \in \mathbb{N} \times \mathbb{Z} : \alpha_1 + \frac{m}{l}(\beta + 1) > -n \} = \{(\alpha_1, \beta) \in \mathbb{N} \times \mathbb{Z} : l\alpha_1 + m\beta \geq -m - nl + 1 \}. \tag{3.6}
\]

We split the Bergman space into \( m \) orthogonal subspaces

\[
A^2 \left( \Omega_{m/l}^{n+1} \right) = S_0 \oplus S_1 \oplus \cdots \oplus S_{m-1}, \tag{3.7}
\]

where \( S_j \) is the subspace spanned by monomials of the form \( z^{\alpha}w^\beta \in A^2 \left( \Omega_{m/l}^{n+1} \right) \), where \( \alpha_1 = j \mod m \). Let

\[
G_j = \{(\alpha_1, \beta) \in \Lambda_{\Phi, n} : \alpha_1 = j \mod m \}, \tag{3.8}
\]

\( G_j \cap G_k = \emptyset \) if \( j \neq k \). Each \( S_j \) is a closed subspace of \( A^2 \left( \Omega_{m/l}^{n+1} \right) \), and thus a Hilbert space. Therefore the orthogonal projection, \( L^2 \left( \Omega_{m/l}^{n+1} \right) \rightarrow S_j \), is well defined and represented by integration against a kernel, \( K_j \). It follows that

\[
B_{\frac{m}{l}, n}((z, w), (s, t)) = \sum_{j=0}^{m-1} K_j((z, w), (s, t)). \tag{3.9}
\]

Call each \( K_j \) a sub-Bergman kernel. In the next subsection, we shall focus on the subspaces \( S_j \) and estimate each \( K_j \).

### 3.2 Estimation of the Bergman Kernel

Let \( \gamma = \frac{m}{l} \in \mathbb{Q}^+ \), \( \gcd(m, l) = 1 \). For each \( j = 0, \ldots, m - 1 \), let \( K_j \) be the sub-Bergman kernel of \( B_{\Phi, n} \). By definition, \( \{ z^{\alpha}w^\beta : (\alpha_1, \beta) \in G_j \} \) is an orthonormal basis for \( S_j \), where \( G_j \) is given by (3.8). Then, (3.3) follows that \( K_j \) can be written as the following sum:

\[
K_j((z, w), (s, t)) = \sum_{(\alpha_1, \beta) \in G_j} \frac{(z \cdot \overline{s})^{\alpha_1}(w \cdot \overline{t})^\beta}{N_{\frac{m}{l}, n}(\alpha_1, \beta)}, \tag{3.10}
\]

where \( N_{\frac{m}{l}, n}(\alpha_1, \beta) \) by (3.5).
Theorem 3.3. Let $m, l \in \mathbb{Z}^+$ be relatively prime. The sub-Bergman kernel $K_j$ of domain $\Omega_{m/l}$ satisfies the estimate

$$|K_j((z, w), (s, t))| \lesssim \frac{|b|^{\frac{j}{m} + (n+1)l - E_j - 1}}{1 - b|t|^2|b' - a^m|n+1},$$

where $a = z \cdot \overline{s}$, $b = w \cdot \overline{t}$, $E_j = \left\lfloor \frac{(j+n)l - 1}{m} \right\rfloor$.

Proof. First we find $K_j((z, w), (z, w))$ and then use polarization to move off the diagonal. Let $a = |z|^2, b = |w|^2$. Starting from (3.10) and using (3.6),

$$K_j((z, w), (z, w)) = \sum_{(\alpha_1, \beta) \in G_j} \frac{a^{\alpha_1}b^{\beta}}{N_{\pi, n}(\alpha_1, \beta)} = \frac{1}{\pi^{n+1}} \sum_{\alpha_1 \in \mathcal{R}_j} \sum_{\beta \geq -\frac{\alpha_1}{m} + \frac{1-nl-m}{m}} \frac{\Gamma(\alpha_1 + n + 1)(\alpha_1 + n)l}{\Gamma(\alpha_1 + 1)m} \frac{\Gamma(\alpha_1 + n + 1)(\beta + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}b^{\beta}, \quad (3.11)$$

where $\mathcal{R}_j := \{\alpha_1 \geq 0 : \alpha_1 = j \text{ mod } m\}$ and the inner sum is taken over integers $\beta$ with $\beta \geq -\frac{\alpha_1}{m} + \frac{1-nl-m}{m}$. We want to compute the smallest such integer, called $\ell(j)$. Notice that

$$-\frac{l\alpha_1}{m} + \frac{1-nl-m}{m} = -1 - \frac{l(\alpha_1 - j)}{m} = \frac{(j+n)l - 1}{m}$$

and since $\alpha_1 = j \text{ mod } m$, it follows that

$$\ell(j) = -1 - \frac{l(\alpha_1 - j)}{m} - E_j, \quad (3.12)$$

where $E_j = \left\lfloor \frac{(j+n)l - 1}{m} \right\rfloor$. Therefore,

$$= \frac{1}{\pi^{n+1}} \sum_{\alpha_1 \in \mathcal{R}_j} \sum_{\beta = \ell(j)}^{\infty} \frac{\Gamma(\alpha_1 + n + 1)(\alpha_1 + n)l}{\Gamma(\alpha_1 + 1)m} \frac{\Gamma(\alpha_1 + n + 1)(\beta + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}b^{\beta}$$

$$= \frac{l}{\pi^{n+1}m} \sum_{\alpha_1 \in \mathcal{R}_j} \sum_{\beta = \ell(j)}^{\infty} \frac{\Gamma(\alpha_1 + n + 1)(\alpha_1 + n)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}b^{\beta}$$

$$+ \frac{1}{\pi^{n+1}} \sum_{\alpha_1 \in \mathcal{R}_j} \sum_{\beta = \ell(j)}^{\infty} \frac{\Gamma(\alpha_1 + n + 1)(\beta + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}b^{\beta}$$

$$:= \frac{l}{m\pi^{n+1}} I(j) + \frac{1}{\pi^{n+1}} J(j).$$

It remains to compute the sums $I(j)$ and $J(j)$. Let $u := ab^{-l/m}$, and note that both $a^m < b^l < 1$ and $|u| < 1$. The summation of $I(j)$ is straightforward:

$$I(j) = \sum_{\alpha_1 \in \mathcal{R}_j} \frac{\Gamma(\alpha_1 + n + 1)(\alpha_1 + n)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} \sum_{\beta = \ell(j)}^{\infty} b^{\beta}$$
The summation of $J(j)$ is slightly more involved. First, the sum is split into two pieces:

$$J(j) = \frac{1}{\pi^{n+1}} \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} \sum_{\beta = \ell(j)}^{\infty} (\beta + 1) b^\beta$$

$$= \frac{1}{\pi^{n+1}} \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} \sum_{\beta = \ell(j)}^{\infty} \frac{d}{db} (b^{\beta+1})$$

$$= \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} \left[ \frac{b^{\ell(j)+1}}{(1-b)^2} + (\ell(j) + 1) b^{\ell(j)} \right]$$

$$= \frac{b}{(1-b)^2} \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} b^{\ell(j)} + \frac{1}{1-b} \sum_{\alpha_1 \in R_j} (\ell(j) + 1) \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} b^{\ell(j)}$$

$$:= J_1(j) + J_2(j).$$

For the first piece, it follows

$$J_1(j) = \frac{b^{\ell(j)-E_j}}{(1-b)^2} \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}$$

$$= \frac{b^{\ell(j)-E_j}}{(1-b)^2} \frac{d^n}{du^n} \left( \frac{u^{j+n}}{1-u^m} \right). \quad (3.14)$$

For the second piece,

$$J_2(j) = \frac{b^{\ell(j)-E_j}}{1-b} \sum_{\alpha_1 \in R_j} (\ell(j) + 1) \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1}$$

$$= \frac{b^{\ell(j)-E_j}}{1-b} \left[ (\frac{j}{m} - E_j) \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} - \frac{j}{m} \sum_{\alpha_1 \in R_j} \frac{\Gamma(\alpha_1 + n + 1)}{\Gamma(\alpha_1 + 1)} a^{\alpha_1} \right]$$

$$= \frac{b^{\ell(j)-E_j}}{1-b} \left[ (\frac{j}{m} - E_j) \frac{d^n}{du^n} \left( \frac{u^{j+n}}{1-u^m} \right) - \frac{j}{m} \cdot u \frac{d^{n+1}}{du^{n+1}} \left( \frac{u^{j+n}}{1-u^m} \right) \right]. \quad (3.15)$$

(3.13) and (3.15) can be combined more simply as

$$I(j) + \frac{m}{l} J_2(j) = \frac{b^{\ell(j)-E_j}}{1-b} \left( j + n - \frac{m}{l} E_j \right) \frac{d^n}{du^n} \left( \frac{u^{j+n}}{1-u^m} \right)$$
Combining this with (3.14), we now have

\[ K_j((z, w), (z, w)) = \frac{l}{m \pi^{n+1}} \left[ I(j) + \frac{m}{l} J_2(j) + \frac{m}{l} J_1(j) \right] \]

\[ = \frac{l}{m \pi^{n+1}} \cdot g_j(b) \cdot \frac{b^{j-1-E_j}}{(1 - b)^2} \cdot \frac{d^n}{dw^n} \left( \frac{u^{j+n}}{1 - u^m} \right), \tag{3.16} \]

where \( g_j(b) := j + n - \frac{m}{l} E_j + \left( \frac{m}{l} + \frac{m}{l} E_j - j - n \right) b \). Using Leibniz's rule,

\[ \frac{d^n}{du^n} \left( \frac{u^{j+n}}{1 - u^m} \right) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{1 - u^m} \right)^{(k)} (u^{j+n})^{(n-k)} = \frac{u^j \cdot Q(u^m)}{(1 - u^m)^{n+1}}, \]

where \( Q(u^m) \) is a polynomial of no more than \( n \) degrees with respect to \( u^m \). Note that \( u = ab^{-l/m} \),

\[ \frac{d^n}{du^n} \left( \frac{u^{j+n}}{1 - u^m} \right) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{1 - u^m} \right)^{(k)} (u^{j+n})^{(n-k)} = \frac{u^j \cdot Q(a^m b^{-l})}{(1 - a^m)^{n+1}}. \tag{3.17} \]

Polarization now gives the formula for \( K_j((z, w), (s, t)) \), substituting \( a = z \cdot \bar{s} \) and \( b = w \cdot \bar{t} \) into equation (3.17). Finally, note that \( \Omega_{m/l}^{n+1} \) is a bounded domain where \( |z|^m < |w|^l < 1 \) and the estimates

\[ |g_j(b)| \lesssim 1, \quad |Q(a^m b^{-l})| \lesssim 1, \]

then the sub-Bergman kernel \( K_j \) satisfies the estimate

\[ |K_j((z, w), (s, t))| \lesssim \frac{|b|^{j+n(l-1-E_j-1)}}{|1 - b|^2 |b^l - a^m|^{n+1}}. \]

Recall that \( E_j = \left\lfloor \frac{(j+n)l-1}{m} \right\rfloor \), so

\[ \frac{(j+n)l - 1}{m} - 1 < E_j \leq \frac{(j+n)l - 1}{m} \quad \forall j \in \{0, \ldots, m-1\}. \]

Then, Theorem 3.3 and (3.9) yield the following estimate on the full Bergman kernel:

\[ |B_{m,n}((z, w), (s, t))| \lesssim \frac{|b|^{(n+1)(l-1)-\frac{m-1}{m}}}{|1 - b|^2 |b^l - a^m|^{n+1}}. \tag{3.18} \]
3.3 Boundedness of operators on $\Omega^{n+1}_{m/l}$

If $\Omega \subset \mathbb{C}^{n+1}$ is a domain and $K$ is an a.e. positive, measurable function on $\Omega \times \Omega$, let $K$ denote the integral operator with kernel $K$:

$$K(f)(z, w) = \int_{\Omega} K((z, w), (s, t)) f(s, t) dV(s, t).$$

The basic $L^p$ mapping result is the following:

**Proposition 3.4.** If the kernel of the domain $\Omega^{n+1}_{m/l}$ satisfies the estimate

$$|K((z, w), (s, t))| \lesssim |b|^4 |1 - b|^2 |b' - a|^{n+1},$$

(3.19)

where $a = z \cdot \overline{\tau}, b = w \cdot \overline{\tau}$, then, $K : L^p(\Omega^{n+1}_{m/l}) \rightarrow L^p(\Omega^{n+1}_{m/l})$ boundedly if

$$\frac{2nl + 2m}{Am + 2nl + 2m - (n+1)lm} < p < \frac{2nl + 2m}{(n+1)lm - Am},$$

(3.20)

when both denominators in (3.20) are positive and $Am + 2nl + 2m - (n+1)lm > (n+1)lm - Am$.

Some lemmas are needed before proving Proposition 3.4.

**Lemma 3.5.** (Schur’s Lemma [15]) Let $\Omega \subset \mathbb{C}^n$ be a domain, $K$ be an a.e. positive, measurable function on $\Omega \times \Omega$, and $K$ be the integral operator with kernel $K$. Suppose there exists a positive auxiliary function $h$ on $\Omega$, and numbers $0 < a < b$ such that for all $\epsilon \in (a, b)$, the following estimates hold:

$$K(h^{-\epsilon})(z) := \int_{\Omega} K(z, w) h(w)^{-\epsilon} dV(w) \lesssim h(z)^{-\epsilon}$$

$$K(h^{-\epsilon})(w) := \int_{\Omega} K(z, w) h(z)^{-\epsilon} dV(z) \lesssim h(w)^{-\epsilon}.$$  

Then, $K$ is a bounded operator on $L^p(\Omega)$ for all $p \in (\frac{a+b}{b}, \frac{a+b}{a})$.

**Lemma 3.6.** [2] Let $D \subset \mathbb{C}$ be the unit disk, $\epsilon \in (0, 1)$ and $\beta \in (-\infty, 2)$. Then, for $z \in D$,

$$I_{\epsilon, \beta}(z) := \int_D (1 - |w|^2)^{-\epsilon} |w|^{-\beta} dV(w) \lesssim (1 - |z|^2)^{-\epsilon}$$

with a constant independent of $z$.

**Lemma 3.7.** [14] Let $D_n \subset \mathbb{C}^n$ be the unit ball, $k \in \mathbb{Z}^+$, $\epsilon \in (0, 1)$ and $\Delta \in \mathbb{C}^n, |\Delta| < 1$. Then

$$\int_{D_n} \frac{(1 - |\eta|^{2k})^{-\epsilon}}{1 - (\eta \cdot \Delta)^{n+1}} dV(\eta) \approx (1 - |\Delta|^{2k})^{-\epsilon}. $$

(3.21)
3.4 Proof of Proposition 3.4

Proof. Let $h(z, w) := (|w|^{2l} - |z|^{2m})(1 - |w|^2)$. This function (essentially) measures the distance of $(z, w) \in \Omega^{n+1}_{m/l}$ to $b(\Omega^{n+1}_{m/l})$. We will prove that for all $\epsilon \in \left(\frac{(n+1)}{2} - \frac{A}{2l}, \frac{n}{m} + \frac{1}{l} + \frac{A}{2l} - \frac{n+1}{2}\right)$, and any $(z, w) \in \Omega^{n+1}_{m/l}$,

$$|K| (h^{-\epsilon})(z, w) := \int_{\Omega^{n+1}_{m/l}} |K((z, w), (s, t))| h(s, t)^{-\epsilon} dV(s, t) \lesssim h(z, w)^{-\epsilon}. \quad (3.22)$$

From estimate (3.19), we see that

$$|K| (h^{-\epsilon})(z, w) \lesssim \int_{0<|t|<1} \int_{|s|^2 + |t|} \int_{0}^{\pi} \frac{|w| \rho^A (|t|^{2l} - |s|^{2m})^{-\epsilon} (1 - |t|^2)^{-\epsilon}}{|1 - w \cdot \rho|^{2l}|w|^{l} \cdot (z \cdot \rho)^m |n+1|} dV(s) dV(t).$$

Let

$$t = \rho e^{i \varphi}, \quad w = |w| e^{i \varphi_0}, \quad s = (r_1 e^{i \theta_1}, \ldots, r_n e^{i \theta_n}), \quad z = (|z_1| e^{i \vartheta_1}, \ldots, |z_n| e^{i \vartheta_n}),$$

then

$$|K| (h^{-\epsilon})(z, w) \leq \int_{0}^{1} \int_{|\rho| < \rho_{\pi}} \left(\prod_{k=1}^{n+1} \int_{-\pi}^{\pi} \frac{|w| \rho^A (|\rho|^{2l} - |\rho|^{2m})^{-\epsilon} (1 - \rho^2)^{-\epsilon} r_1 \cdots r_n d\varphi \theta_1 \cdots d\theta_n \rho dr \rho d\varphi}{|1 - |w| \rho e^{i \varphi}|^{2l}|w|^l \rho^l - (|z_1| r_1 e^{i \theta_1} + \cdots + |z_n| r_n e^{i \theta_n})^m |n+1|}\right).$$

The last equation we used the periodicity of the $\theta_1, \ldots, \theta_n$ and $\varphi$ integrals. Next, we first consider the following integral

$$\int_{|\rho| < \rho_{\pi}} \left(\prod_{k=1}^{n+1} \int_{-\pi}^{\pi} \frac{(\rho^{2l} - |\rho|^{2m})^{-\epsilon} r_1 \cdots r_n d\theta_1 \cdots d\theta_n dr}{|w|^l \rho^l - (|z_1| r_1 e^{i \theta_1} + \cdots + |z_n| r_n e^{i \theta_n})^m |n+1|}\right).$$

Make the substitution $\tilde{r}_k = r_k \rho^{-l/m}$ ($k = 1, \ldots, n$), and let

$$\eta = (\tilde{r}_1 e^{i \theta_1}, \ldots, \tilde{r}_n e^{i \theta_n}), \quad \Delta = (|z_1||w|^{-l/m}, \ldots, |z_n||w|^{-l/m}),$$

then

$$\int_{|\rho| < \rho_{\pi}} \left(\prod_{k=1}^{n} \int_{-\pi}^{\pi} \frac{\rho^{2n\frac{l}{m} - 2\epsilon - (n+1)l} (1 - |\tilde{r}|^{2m})^{-\epsilon} \tilde{r}_1 \cdots \tilde{r}_n d\theta_1 \cdots d\theta_n d\tilde{r}}{|w|^{(n+1)l} |1 - (|z_1||w|^{-l/m} \tilde{r}_1 e^{i \theta_1} + \cdots + |z_n||w|^{-l/m} \tilde{r}_n e^{i \theta_n})^m |n+1|}. \quad (3.23)$$
From Lemma 3.7 we have

\[
(3.23) \approx \frac{2nl}{m} \frac{2\epsilon - (n+1)l}{|w|^{n+1}l} (1 - |\Delta|^{2m})^{-\epsilon} = \rho^{2nL - 2\epsilon - (n+1)l} |w|^{2\epsilon - (n+1)l} (|w|^{2l} - |z|^{2m})^{-\epsilon}.
\]

This means that

\[
|K| (h^{-\epsilon}) (z, w) \lesssim |w|^{2\epsilon + A - (n+1)l} (|w|^{2l} - |z|^{2m})^{-\epsilon} \int_0^1 \int_0^\pi \frac{\rho^{2nL - 2\epsilon - (n+1)l} + A + 1 (1 - \rho^2)^{-\epsilon} d\varphi dp}{1 - |w| \rho \rho |z|}.
\]

From Lemma 3.6

\[
|K| (h^{-\epsilon}) (z, w) \lesssim |w|^{2\epsilon + A - (n+1)l} (|w|^{2l} - |z|^{2m})^{-\epsilon} (1 - |w|^2)^{-\epsilon},
\]

when \( \frac{2nl}{m} - 2\epsilon - (n+1)l + A > -2, \) i.e., \( \epsilon < \frac{n}{m} + \frac{1}{l} + \frac{A}{2l} - \frac{n+1}{2} ). \) Then,

\[
|K| (h^{-\epsilon}) (z, w) \lesssim (|w|^{2l} - |z|^{2m})^{-\epsilon} (1 - |w|^2)^{-\epsilon} = h(z, w)^{-\epsilon},
\]

when \( 2\epsilon + A - (n+1)l \geq 0, \) i.e., \( \frac{n+1}{2} - \frac{A}{2l} \leq \epsilon. \) This completes the proof of (3.22). Finally, combining (3.22) and Schur’s lemma (Lemma 3.5) yields that the operator \( |K| \) is bounded from \( L^p(\Omega^{n+1}_{m/l}) \) to \( L^p(\Omega^{n+1}_{m/l}) \) when

\[
\frac{2nl + 2m}{Am + 2nl + 2m - (n+1)lm} < p < \frac{2nl + 2m}{(n+1)lm - Am}.
\]

Note that because of the conjugate symmetry of \( K, \) it is sufficient to establish just one of the estimates to apply Lemma 3.5. A fortiori, \( K \) is bounded from \( L^p(\Omega^{n+1}_{m/l}) \) to \( L^p(\Omega^{n+1}_{m/l}) \) for \( p \) in the same range. This completes the proof.

\[\square\]

**Corollary 3.8.** The Bergman projection \( P_{m/n} \) is a bounded operator on \( L^p(\Omega^{n+1}_{m/l}) \) for all \( p \in \left( \frac{2m+2nl}{m+nl+1}, \frac{2m+2nl}{m+nl+1} \right). \)

**Proof.** This comes immediately from Proposition 3.3 by taking \( A = (n+1)l - 1 - \frac{n-1}{m}. \) \[\square\]
4 The Rational Case: $L^p$ Non-Boundedness

We shall show that $P_{m/l}$ fails to be $L^p$ bounded by exhibiting a single function $f \in L^\infty(\Omega_{m/l}^{n+1})$ such that $P_{m/l}f \notin L^p(\Omega_{m/l}^{n+1})$.

**Proposition 4.1.** If both $(\eta_1, \eta_2)$ and $(\eta_1, -\eta_2)$ belong to $\Lambda_{\gamma,n}$, and $f(z, w) := z_1^{\eta_1} \overline{w_2}$. Then there exists a constant $C$ such that

$$P_\gamma(f)(z, w) = C z_1^{\eta_1} w^{-\eta_2}.$$

**Proof.** From (3.3), we can write the Bergman kernel as

$$B_{\gamma,n}((z, w), (s, t)) = \sum_{(\alpha_1, \beta) \in \Lambda_{\gamma,n}} \frac{(z \cdot \overline{s})^\alpha (w \cdot \overline{t})^\beta}{N_{\gamma,n}(\alpha_1, \beta)}.$$

where $\Lambda_{\gamma,n}$ is given by (3.4) and $N_{\gamma,n}(\alpha_1, \beta)$ by (3.5). It follows that

$$P_\gamma(f)(z, w) = \int_{\Omega_{m/l}^{n+1}} B_{\gamma,n}((z, w), (s, t)) f(s, t) dV(s, t) = \int_{\Omega_{m/l}^{n+1}} \sum_{(\alpha_1, \beta) \in \Lambda_{\gamma}} \frac{(z \cdot \overline{s})^\alpha (w \cdot \overline{t})^\beta}{N_{\gamma,n}(\alpha_1, \beta)} s_1^{\eta_1} t_2^{\eta_2} dV(s, t)$$

$$= \sum_{(\alpha_1, \beta) \in \Lambda_{\gamma,n}} \frac{1}{N_{\gamma,n}(\alpha_1, \beta)} \int_{|s| < 1} \int_0^1 \int_0^{2\pi} (z \cdot \overline{s})^\alpha s_1^{\eta_1} t_2^{\eta_2} (r e^{-i\theta})^\beta + \eta_2 r \theta d\theta dr dV(s)$$

$$= \sum_{\alpha_1 \in \mathbb{N}} \frac{\pi}{N_{\gamma,n}(\alpha_1, -\eta_2) w_{\eta_2}} \int_{|s| < 1} (z \cdot \overline{s})^\alpha s_1^{\eta_1} (1 - |s|^{2\gamma}) dV(s)$$

$$= \sum_{\alpha_1 \in \mathbb{N}} \frac{\pi}{N_{\gamma,n}(\alpha_1, -\eta_2) w_{\eta_2}} \int_0^{1} r^{\alpha_1 m + 2n - 1} (1 - r^{2\gamma}) dr \int_{|\xi| = 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (z \cdot \overline{s})^\alpha s_1^{\eta_1} e^{(n-\alpha_1)i\gamma} d\theta d\sigma(\xi)$$

Notice that

$$\int_{|s| < 1} (z \cdot \overline{s})^\alpha s_1^{\eta_1} (1 - |s|^{2\gamma}) dV(s) = \int_{|s| < 1} (z_1 s_1 + z' \cdot \overline{s})^\alpha s_1^{\eta_1} (1 - |s|^{2\gamma}) dV(s)$$

$$= \int_{|s| < 1} \int_0^{(1-|s'|^2)^{1/2}} \int_0^{2\pi} \sum_{k=0}^{\eta_1} \binom{\eta_1}{k} (z_1 r e^{-i\theta})^k (z' \cdot \overline{s})^{n-k} (r e^{i\theta})^m d\theta (1 - (r^2 + |s'|^{2\gamma}) dr dV(s')$$

$$= \int_{|s| < 1} z_1^{\eta_1} |s_1|^{2n} (1 - |s|^{2\gamma}) dV(s) = \frac{\eta_1 \pi^m B(\frac{n+2}{\gamma}, 2) z_1^{\eta_1}}{\gamma \Gamma(n + \eta_1)}.$$ 

Therefore, $P_\gamma(f)(z, w) = C z_1^{\eta_1} w^{-\eta_2}$, where $C = \frac{\pi^{n+1} \Gamma(n+\eta_1) B(\frac{n+2}{\gamma}, 2)}{\gamma \Gamma(n+\eta_1) \Gamma(n+\eta_1)}$ is a constant. This completes the proof. \[\square\]
When $\gamma \in \mathbb{Q}^+$, a similar result on subspaces $\mathcal{S}_j$ holds by the same proof:

**Proposition 4.2.** If both $(\eta_1, \eta_2)$ and $(\eta_1, -\eta_2)$ belong to $\mathcal{G}_j$ for some $j \in \{0, 1, \ldots, m-1\}$, and $f(z, w) := z_1^{\eta_1} \bar{w}^{\eta_2}$, then there exists a constant $C$ such that

$$\mathcal{K}_l(f)(z, w) = \begin{cases} C z_1^{\eta_1} w^{-\eta_2}, & l = j \\ 0, & l \neq j \end{cases}$$

for all $l \in \{0, 1, \ldots, m-1\}$.

**Proposition 4.3.** For each $j \in \{0, 1, \ldots, m-1\}$, the sub-Bergman projection $\mathcal{K}_j$ does not map $L^\infty(\Omega_{m/l}^{n+1})$ to $L^p(\Omega_{m/l}^{n+1})$ for any $p > \frac{2m+2nl}{m+mE_j-lj}$.

**Proof.** Fix $j$, and take $\eta_1 = j + km$ for some $k \in \mathbb{Z}^+ \cup \{0\}$. Let $\eta_2 = \ell(\eta_1)$, and note that (3.12) says that

$$\eta_2 = -1 - lk - E_j < 0.$$  

Thus, $(\eta_1, \eta_2), (\eta_1, -\eta_2) \in \mathcal{G}_j$. Let $f(z, w) := z_1^{\eta_1} / \bar{w}^{\eta_2}$; clearly $f \in L^\infty(\Omega_{m/l}^{n+1})$. Proposition 4.1 states that $\mathcal{K}_j(f)(z, w) = C z_1^{\eta_1} w^{\eta_2}$. Computing in polar coordinates,

$$\int_{\Omega_{m/l}^{n+1}} |z_1|^{\eta_1 p} |w|^{\eta_2 p} dV(z, w) = \int_{0<|w|<1} |w|^{\eta_1 p} \int_0^{r=1} |\xi|^{|\eta_1| p + 2n - 2} |\xi|^{|\eta_1| p + 2n - 2} |w|^{\eta_2 p} d\sigma(\xi) dr dV(w)$$

$$= \frac{1}{2n + \eta_1 p} \int_{|\xi|=1} |\xi|^{|\eta_1| p} d\sigma(\xi) \int_{0<|w|<1} |w|^{\eta_2 p} \frac{1}{m} (2n + \eta_1 p) dV(w)$$

$$= \frac{2\pi}{2n + \eta_1 p} \int_{|\xi|=1} |\xi|^{|\eta_1| p} d\sigma(\xi) \int_0^1 r^{\eta_2 p} \frac{1}{m} (2n + \eta_1 p + 1) dr.$$

This integral diverges when

$$\eta_2 + \frac{l}{m} (2n + \eta_1 p) + 1 \leq -1.$$  

(4.24)

Substituting $\eta_1 = j + km$ and $\eta_2 = -1 - lk - E_j$, (4.24) becomes

$$-p (m + mE_j - lj) \leq -2nl - 2m.$$  

(4.25)

However, since $E_j = \left\lceil \frac{(j + n)l - 1}{m} \right\rceil$,

$$m + mE_j - lj > m + m \left\{ \frac{(j + n)l - 1}{m} - 1 \right\} - lj$$

$$= nl - 1 \geq 0.$$  

Therefore, (4.25) is equivalent to $p \geq \frac{2m+2nl}{m+mE_j-lj}$, which completes the proof. \qed
Proposition 4.4. For $p \geq \frac{2m + 2nl}{m + nl - 1}$, $P_{m/l}$ fails to map $L^\infty(\Omega_{m/l}^{n+1})$ to $L^p(\Omega_{m/l}^{n+1})$.

Proof. As $m$ and $l$ are relatively prime, according to elementary number theory, there is a unique $x \in \{n, \ldots, n + m - 1\}$ such that

$$lx = 1 \mod m.$$  \hfill (4.26)

Setting $j_0 = x - n$, it follows that

$$E_{j_0} = \frac{(j_0 + n)l - 1}{m}$$  \hfill (4.26)

and

$$\ell(j_0) = -1 - \frac{l(j_0 - j_0)}{m} - E_{j_0} = -\frac{l}{m}j_0 - 1 + \frac{1 - nl}{m}.  \hfill (4.27)$$

Thus, $(j_0, \ell(j_0)), (j_0, -\ell(j_0)) \in G_{j_0}$. Proposition 4.3 says that $K_{j_0}$ does not map the bounded function $g(z, w) = z^{j_0}\bar{w}^{\ell(j_0)}$ to $L^p(\Omega_{m/l}^{n+1})$ for $p \geq \frac{2m + 2nl}{m + nl - 1}$. On the other hand, Proposition 4.2 states that $K_j(g) = 0$ for all $j \neq j_0$. Thus, (3.9) gives the claimed result. \hfill \square

To obtain $L^p$ nonboundedness for $p < 2$, recall an elementary consequence of the self-adjointness of the Bergman projection:

Lemma 4.5. [2] Let $\Omega$ be a bounded domain and $p > 1$. If $P$ maps $L^p(\Omega)$ to $L^p(\Omega)$ boundedly, then it also maps $L^q(\Omega)$ to $L^q(\Omega)$ boundedly, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 4.4 and Lemma 4.5 give the other half of Theorem 1.1:

Proposition 4.6. $P_{m/l}$ is not a bounded operator on $L^p(\Omega_{m/l}^{n+1})$ for $p \notin \left(\frac{2m + 2nl}{m + nl + 1}, \frac{2m + 2nl}{m + nl - 1}\right)$.

5 The Irrational Case: Degenerate $L^p$ Mapping

Lemma 5.1. [18] (Dirichlet) If $\gamma$ is irrational, $n \in \mathbb{Z}^+$, there exists a sequence of rational numbers $\left\{\frac{m_k}{n_k}\right\}$, with $\frac{m_k}{n_k} \to \gamma$, such that

$$\left|\frac{l_k}{m_k} - \frac{1}{\gamma}\right| < \frac{1}{2nm_k^2}.$$  \hfill \hfill (5.5)

Next, we use the above lemma to prove Theorem 2.2.

Proof. Fix $p > 2$. We will exhibit an $f \in L^\infty(\Omega_{\gamma}^{n+1})$ such that $P_\gamma(f) \notin L^p(\Omega_{\gamma}^{n+1})$. \hfill 14
Let \( \left\{ \frac{m_k}{l_k} \right\} \) be a sequence of rational numbers given by Lemma 5.1. Temporarily fix the index \( k \). From (4.27), there exists a unique \( \eta = (\eta_1, \eta_2) \in \Lambda_{m_k/l_k,n} \) with \( 0 \leq \eta_1 \leq m_k - 1 \) such that
\[
\eta_2 = \frac{1 - l_k \eta_1 - nl_k - m_k}{m_k} \in \mathbb{Z}.
\] (5.28)

Assume for the moment that this multi-index \( \eta \in \Lambda_{\gamma,n} \). We will briefly show that this is always the case.

Let \( f_k(z,w) := \frac{z_1^{m}}{\bar{w}^{m}} \); as \( \eta_2 < 0 \), \( f_k \in L^\infty (\Omega_\gamma^{n+1}) \). Since we are assuming \( \eta \in \Lambda_{\gamma,n} \), Proposition 4.1 implies \( P_{\gamma} (f_k)(z,w) \approx z_1^{m} w^{m} \).

It follows that
\[
\| P_{\gamma} (f_k) \|_{L^p (\Omega_\gamma^{n+1})} \approx \int_{\Omega_\gamma^{n+1}} |z_1|^{[n]p} |w|^{[n]p} dV(z,w)
\]
\[
= \int_{0<|w|<1} |w|^{[n]p} \int_{0}^{|w|^\frac{n}{p}} \int_{|\xi|=1} r^{[n]p + 2n - 1} |\xi|^{[n]p} d\sigma(\xi) dr dV(w)
\]
\[
\approx \int_{0<|w|<1} |w|^{[n]p + \frac{1}{\gamma}(2n + [n]p)} dV(w)
\]
\[
\approx \int_{0}^{1} r^{[n]p + \frac{1}{\gamma}(2n + [n]p) + 1} dr.
\]

This diverges if the exponent is \( \eta_2 p + \frac{1}{\gamma}(2n + [n]p) + 1 \leq -1 \). Substituting the expression for \( \eta_2 \) in (5.28) and rearranging terms, this happens exactly when
\[
p \left( 1 + \frac{nl_k - 1}{m_k} + \eta_1 \left( \frac{l_k}{m_k} - \frac{1}{\gamma} \right) \right) \geq 2 + \frac{2n}{\gamma}.
\] (5.29)

Consider the left hand side of (5.29). Since \( 0 \leq \eta_1 \leq m_k - 1 \),
\[
\eta_1 \left| \frac{l_k}{m_k} - \frac{1}{\gamma} \right| < \frac{1}{m_k},
\]
by Lemma 5.1. Thus
\[
p \left( 1 + \frac{nl_k - 1}{m_k} + \eta_1 \left( \frac{l_k}{m_k} - \frac{1}{\gamma} \right) \right) \geq p \left( 1 + \frac{nl_k - 1}{m_k} - \eta_1 \left| \frac{l_k}{m_k} - \frac{1}{\gamma} \right| \right)
\]
\[
> p \left( 1 + \frac{nl_k - 2}{m_k} \right).
\]

However since \( p > 2 \), we can always choose \( k \) large enough so that
\[
p \left( 1 + \frac{nl_k - 2}{m_k} \right) > 2 + \frac{2n}{\gamma}.
\]

Thus, (5.29) is satisfied for such \( k \), which shows \( P_{\gamma} (f_k) \notin L^p (\Omega_\gamma^{n+1}) \). We now show that the unique multi-index \( \eta = (\eta_1, \eta_2) \in \Lambda_{m_k/l_k,n} \) with \( 0 \leq \eta_1 \leq m_k - 1 \) and \( \eta_2 \) given by (5.28) is necessarily in \( \Lambda_{\gamma,n} \). We leave off the subscript \( k \) in what follows.
Again, the rational approximation \( \left| \frac{l}{m} - \frac{1}{\gamma} \right| < \frac{1}{2nm} \) is essential. If \( \frac{m}{l} > \gamma \), then \( A^2 \left( \Omega^{n+1}_{m/l} \right) \subset A^2 \left( \Omega^{n+1}_{m/l} \right) \), so automatically, \( \eta \in \Lambda_{\gamma,n} \). Suppose instead that \( \frac{m}{l} < \gamma \). \( \text{(3.4)} \) implies that \( \eta \in \Lambda_{\gamma,n} \) if and only if \( \eta_1 \geq 0 \) and the lattice point corresponding to \( \eta \) lies strictly above the line

\[ g(\eta) := -\frac{\eta_1}{\gamma} - \frac{n}{\gamma} - 1. \]

However, since \( \frac{m}{l} \in \mathbb{Q}^+ \), a multi-index \( \eta \in \Lambda_{m/l,n} \) if and only if both \( \eta_1 \geq 0 \) and the lattice point corresponding to \( \eta \) lies on or above the line

\[ h(\eta_1) := -\frac{l}{m} \eta_1 + \frac{1-nl}{m} - 1. \]

Now for \( 0 \leq \eta_1 \leq m - 1 \),

\[
\begin{align*}
h(\eta_1) - g(\eta_1) & = \frac{1}{m} - (\eta_1 + n) \left( \frac{l}{m} - \frac{1}{\gamma} \right) \\
& \geq \frac{1}{m} - (m + n - 1) \frac{1}{2nm^2} \\
& > 0.
\end{align*}
\]

From this, it follows that \( \eta = (\eta_1, \eta_2) \in \Lambda_{\gamma,n} \). Since \( p > 2 \) was arbitrary, the above shows that \( P_\gamma \) is not \( L^p \) bounded for any \( p > 2 \). Lemma \( \text{[4.5]} \) now shows that \( P_\gamma \) is not \( L^p \) bounded for any \( 1 < p < 2 \), which completes the proof.

\[ \square \]

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