On the eigenvalue problem for a particular class of finite Jacobi matrices

F. Štampach¹, P. Štovíček²

Department of Mathematics, Faculty of Nuclear Science, Czech Technical University in Prague, Trojanova13, 12000 Praha, Czech Republic

¹stampfri@fjfi.cvut.cz
²stovicek@kmlinux.fjfi.cvut.cz

Abstract

A function $\mathcal{F}$ with simple and nice algebraic properties is defined on a subset of the space of complex sequences. Some special functions are expressible in terms of $\mathcal{F}$, first of all the Bessel functions of first kind. A compact formula in terms of the function $\mathcal{F}$ is given for the determinant of a Jacobi matrix. Further we focus on the particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. A formula is derived for the characteristic function. Yet another formula is presented in which the characteristic function is expressed in terms of the function $\mathcal{F}$ in a simple and compact manner. A special basis is constructed in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. A vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.

Keywords: tridiagonal matrix, finite Jacobi matrix, eigenvalue problem, characteristic function

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1 Introduction

The results of the current paper are related to the eigenvalue problem for finite-dimensional symmetric tridiagonal (Jacobi) matrices. Notably, the eigenvalue problem for finite Jacobi matrices is solvable explicitly in terms of generalized hypergeometric series [7]. Here we focus on a very particular class of Jacobi matrices which makes
it possible to derive some expressions in a comparatively simple and compact form. We do not aim at all, however, at a complete solution of the eigenvalue problem. We restrict ourselves to derivation of several explicit formulas, first of all that for the characteristic function, as explained in more detail below. We also develop some auxiliary notions which may be, to our opinion, of independent interest.

First, we introduce a function, called $\mathcal{F}$, defined on a subset of the space of complex sequences. In the remainder of the paper it is intensively used in various formulas. The function $\mathcal{F}$ has remarkably simple and nice algebraic properties. Among others, with the aid of $\mathcal{F}$ one can relate an infinite continued fraction to any sequence from the definition domain on which $\mathcal{F}$ takes a nonzero value. This may be compared to the fact that there exists a correspondence between infinite Jacobi matrices and infinite continued fractions, as explained in [2, Chp. 1]. Let us also note that some special functions are expressible in terms of $\mathcal{F}$. First of all this concerns the Bessel functions of first kind. We examine the relationship between $\mathcal{F}$ and the Bessel functions and provide some supplementary details on it.

Further we introduce an infinite antisymmetric matrix, with entries indexed by integers, such that its every row or column obeys a second-order difference equation which is very well known from the theory of Bessel functions. With the aid of function $\mathcal{F}$ one derives a general formula for entries of this matrix. The matrix also plays an essential role in the remainder of the paper.

As an application we present a comparatively simple formula for the determinant of a Jacobi matrix of odd dimension under the assumption that the neighboring parallels to the diagonal are constant. As far as the determinant is concerned this condition is not very restrictive since a Jacobi matrix can be written as a product of another Jacobi matrix with all units on the neighboring parallels which is sandwiched with two diagonal matrices. The formula further simplifies in the particular case when the diagonal is antisymmetric (with respect to its center). In that case zero is always an eigenvalue and we give an explicit formula for the corresponding eigenvector.

Finally we focus on the rather particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. Within this class it suffices to consider matrices whose diagonal is, in addition, antisymmetric. In this case we derive a formula for the characteristic function. Yet another formula is presented in which the characteristic function is expressed in terms of the function $\mathcal{F}$ in a very simple and compact manner. Moreover, we construct a basis in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. This form is rather suitable for various computations. Particularly, one can readily derive a formula for the resolvent. In addition, a vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.

2 The function $\mathcal{F}$

We introduce a function $\mathcal{F}$ defined on a subset of the linear space formed by all complex sequences $x = \{x_k\}_{k=1}^{\infty}$. 

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Definition 1. Define $\mathfrak{F}: D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1} \cdots x_{k_m}x_{k_m+1}$$  \hspace{1cm} (1)

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty}; \sum_{k=1}^{\infty} |x_kx_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Remark 2. Note that the domain $D$ is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

To see that the series on the RHS of (1) converges absolutely whenever $x \in D$ observe that the absolute value of the $m$th summand is majorized by the expression

$$\sum_{k \in \mathbb{N}_m \atop k_1 < k_2 < \ldots < k_m} |x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1} \cdots x_{k_m}x_{k_m+1}| \leq \frac{1}{m!} \left( \sum_{j=1}^{\infty} |x_jx_{j+1}| \right)^m.$$

Obviously, if all but finitely many elements of a sequence $x$ are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. Thus

$$\mathfrak{F}(x_1) = 1, \quad \mathfrak{F}(x_1, x_2) = 1 - x_1x_2, \quad \mathfrak{F}(x_1, x_2, x_3) = 1 - x_1x_2 - x_2x_3,$$

$$\mathfrak{F}(x_1, x_2, x_3, x_4) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4, \text{ etc.}$$

Let $T$ denote the truncation operator from the left defined on the space of all sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$  

$T^n, n = 0, 1, 2, \ldots,$ stands for a power of $T$. Hence $T^n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$.

The proof of the following proposition is immediate.

Proposition 3. For all $x \in D$ one has

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1x_2 \mathfrak{F}(T^2x).$$  \hspace{1cm} (2)

Particularly, if $n \geq 2$ then

$$\mathfrak{F}(x_1, x_2, x_3, \ldots, x_n) = \mathfrak{F}(x_2, x_3, \ldots, x_n) - x_1x_2 \mathfrak{F}(x_3, \ldots, x_n).$$  \hspace{1cm} (3)

Remark 4. Clearly, given that $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$, relation (3) determines recursively and unambiguously $\mathfrak{F}(x_1, \ldots, x_n)$ for any finite number of variables $n \in \mathbb{Z}_+$ (including $n = 0$).
Remark 5. One readily verifies that
\[ \mathcal{F}(x_1, x_2, \ldots, x_n) = \mathcal{F}(x_n, \ldots, x_2, x_1). \]  
Hence equality (3) implies, again for \( n \geq 2 \),
\[ \mathcal{F}(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) = \mathcal{F}(x_1, \ldots, x_{n-2}, x_{n-1}) - x_{n-1}x_n \mathcal{F}(x_1, \ldots, x_{n-2}). \]  

Remark 6. For a given \( x \in D \) such that \( \mathcal{F}(x) \neq 0 \) let us introduce sequences \( \{P_k\}_{k=0}^{\infty} \) and \( \{Q_k\}_{k=0}^{\infty} \) by \( P_0 = 0 \) and \( P_k = \mathcal{F}(x_2, \ldots, x_k) \) for \( k \geq 1 \), \( Q_k = \mathcal{F}(x_1, \ldots, x_k) \) for \( k \geq 0 \). According to (5), the both sequences obey the difference equation
\[ Y_{k+1} = Y_k - x_k x_{k+1} Y_{k-1}, \quad k = 1, 2, 3, \ldots, \]
with the initial conditions \( P_0 = 0 \), \( P_1 = 1 \), \( Q_0 = Q_1 = 1 \), and define the infinite continued fraction
\[ \frac{\mathcal{F}(Tx)}{\mathcal{F}(x)} = \lim_{k \to \infty} \frac{P_k}{Q_k} = \frac{1}{1 - \frac{x_1 x_2}{1 - \frac{x_2 x_3}{1 - \frac{x_3 x_4}{1 - \ldots}}}}. \]

Proposition 3 admits a generalization.

Proposition 7. For every \( x \in D \) and \( k \in \mathbb{N} \) one has
\[ \mathcal{F}(x) = \mathcal{F}(x_1, \ldots, x_k) \mathcal{F}(T^k x) - \mathcal{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathcal{F}(T^{k+1} x). \]  

Proof. Let us proceed by induction in \( k \). For \( k = 1 \), equality (6) coincides with (2). Suppose (6) is true for \( k \in \mathbb{N} \). Applying Proposition 3 to the sequence \( T^k x \) and using (5) one finds that the RHS of (6) equals
\[ \mathcal{F}(x_1, \ldots, x_k) \mathcal{F}(T^{k+1} x) - \mathcal{F}(x_1, \ldots, x_k) x_{k+1} x_{k+2} \mathcal{F}(T^{k+2} x) - \mathcal{F}(x_1, \ldots, x_k) x_k x_{k+1} \mathcal{F}(T^{k+1} x) = \mathcal{F}(x_1, \ldots, x_k) \mathcal{F}(T^{k+1} x) - \mathcal{F}(x_1, \ldots, x_k) x_{k+1} x_{k+2} \mathcal{F}(T^{k+2} x). \]
This concludes the verification. \( \square \)

Remark 8. With the aid of Proposition 3 one can rewrite equality (6) as follows
\[ \mathcal{F}(x) = \mathcal{F}(x_1, \ldots, x_k) \mathcal{F}\left( \frac{\mathcal{F}(x_1, \ldots, x_{k-1})}{\mathcal{F}(x_1, \ldots, x_k)} x_k, x_{k+1}, x_{k+2}, x_{k+3}, \ldots \right). \]  

Later on, we shall also need the following identity.

Lemma 9. For any \( n \in \mathbb{N} \) one has
\[ u_1 \mathcal{F}(u_2, u_3, \ldots, u_n) \mathcal{F}(v_1, v_2, v_3, \ldots, v_n) - v_1 \mathcal{F}(u_1, u_2, u_3, \ldots, u_n) \mathcal{F}(v_2, v_3, \ldots, v_n) = \sum_{j=1}^{n} \prod_{k=1}^{j-1} u_k v_k (u_j - v_j) \mathcal{F}(u_{j+1}, u_{j+2}, \ldots, u_n) \mathcal{F}(v_{j+1}, v_{j+2}, \ldots, v_n). \]
Proof. The equality can be readily proved by induction in $n$ with the aid of (3).

Example 10. For $t, w \in \mathbb{C}$, $|t| < 1$, a simple computation leads to the equality

$$\mathcal{F}\left(\{t^{k-1}w\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m t^{m(2m-1)} w^{2m}}{(1 - t^2)(1 - t^4) \cdots (1 - t^{2m})}. \quad (9)$$

This function can be identified with a basic hypergeometric series (also called $q$-hypergeometric series) defined by

$$r \phi_s(a; b; q, z) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \left((-1)^k q^k (k-1)^{1+s-r} z^k \right) \frac{(q; q)_k}{(q; q)_k},$$

where $r, s \in \mathbb{Z}_+$ (nonnegative integers) and

$$(\alpha; q)_k = \prod_{j=0}^{k-1} (1 - \alpha q^j), \quad k = 0, 1, 2, \ldots,$$

see [5]. In fact, the RHS in (9) equals $0 \phi_1(0; t^2, tw^2)$ where

$$0 \phi_1(0; q, z) = \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} z^k = \sum_{k=0}^{\infty} \frac{q^k}{(1 - q)(1 - q^2) \cdots (1 - q^k)} z^k,$$

with $q, z \in \mathbb{C}$, $|q| < 1$, and the recursive rule (2) takes the form

$$0 \phi_1(0; 0, q, z) = 0 \phi_1(0; 0, qz) + z 0 \phi_1(0; 0, q^2z). \quad (10)$$

Put $e(q; z) = 0 \phi_1(0; q, 1 - q)z$. Then $\lim_{q \uparrow 1} e(q; z) = \exp(z)$. Hence $e(q; z)$ can be regarded as a $q$-deformed exponential function though this is not the standard choice (compare with [5] or [6] and references therein). Equality (10) can be interpreted as the discrete derivative

$$\frac{e(q; z) - e(q; qz)}{(1 - q)z} = e(q; q^2z).$$

Moreover, in view of Remark 3 one has

$$\frac{1}{1 + \frac{z}{q^2z}} = \frac{1}{1 + \frac{z}{qz}} = \frac{1}{1 + \frac{z}{q^2z}}.$$

This equality is related to the Rogers-Ramanujan identities, see the discussion in [3, Chp. 7].
Example 11. The Bessel functions of the first kind can be expressed in terms of function $\mathfrak{F}$. More precisely, for $\nu \notin \mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu + 1)} \mathfrak{F}\left(\left\{ \frac{w}{\nu + k} \right\}_{k=1}^{\infty}\right). \quad (11)$$

The recurrence relation (2) transforms to the well known identity

$$zJ_\nu(z) - 2(\nu + 1)J_{\nu+1}(z) + zJ_{\nu+2}(z) = 0.$$  

To prove (11) one can proceed by induction in $j = 0, 1, \ldots, m - 1$, to show that

$$\sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} \frac{1}{(\nu + k_1)(\nu + k_1 + 1)(\nu + k_2 + 1) \cdots (\nu + k_m)(\nu + k_m + 1)}$$

$$= \frac{1}{j!} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_{m-j}=k_{m-j-1}+2}^{\infty} \frac{1}{(\nu + k_1)(\nu + k_1 + 1)(\nu + k_2 + 1) \cdots (\nu + k_{m-j})(\nu + k_{m-j} + 1)$$

$$\times \left(\nu + k_{m-j} + 2\right)(\nu + k_{m-j} + 3) \cdots (\nu + k_{m-j} + j + 1).$$

In particular, for $j = m - 1$, the RHS equals

$$\frac{1}{(m-1)!} \sum_{k_1=1}^{\infty} \frac{1}{(\nu + k_1)(\nu + k_1 + 1)(\nu + k_1 + 2) \cdots (\nu + k_1 + m)}$$

$$= \frac{1}{m! (\nu + 1)(\nu + 2) \cdots (\nu + m)} = \frac{\Gamma(\nu + 1)}{m! \Gamma(\nu + m + 1)}$$

and so

$$\frac{w^\nu}{\Gamma(\nu + 1)} \mathfrak{F}\left(\left\{ \frac{w}{\nu + k} \right\}_{k=1}^{\infty}\right) = \sum_{m=0}^{\infty} (-1)^m \frac{w^{2m+\nu}}{m! \Gamma(\nu + m + 1)},$$

as claimed. Furthermore, Remark 6 provides us with the infinite fraction

$$\frac{\nu + 1}{w} J_{\nu+1}(2w) = \frac{1}{w^2} \frac{w\nu}{(\nu + 1)(\nu + 2)} \left(1 - \frac{w^2}{(\nu + 2)(\nu + 3)} \left(1 - \frac{w^2}{(\nu + 3)(\nu + 4)} \left(1 - \ldots \right)\right)\right).$$  

This can be rewritten as
\[
\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{z}{2(\nu + 1)} - \frac{z^2}{2(\nu + 2)} - \frac{z^2}{2(\nu + 3)} - \frac{z^2}{2(\nu + 4)} - \ldots
\]

Comparing to Example 11, one can also find the value of \( \tilde{g} \) on the truncated sequence \( \{w/(\nu + k)\}_{k=1}^n \).

**Proposition 12.** For \( n \in \mathbb{Z}_+ \) and \( \nu \in \mathbb{C} \setminus \{-n, -n + 1, \ldots, -1\} \) one has
\[
\tilde{g}\left(\frac{w}{\nu + 1}, \frac{w}{\nu + 2}, \ldots, \frac{w}{\nu + n}\right) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + n + 1)} \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \frac{(n-s)!}{s!(n-2s)!} w^{2s} \prod_{j=s}^{n-1-s} (\nu+n-j). \tag{12}
\]

In particular, for \( m, n \in \mathbb{Z}_+ \), \( m \leq n \), one has
\[
\tilde{g}\left(\frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n}\right) = \frac{m!}{n!} \sum_{s=0}^{\lfloor (n-m)/2 \rfloor} (-1)^s \frac{(n-s)!}{s!(m+s)!} (n-m-s)! \frac{w^{2s}}{(\nu+n-j)}. \tag{13}
\]

**Proof.** Firstly, the equality
\[
\sum_{k=1}^{n} \frac{(n+1-k)(n+2-k)\ldots(n+s-1-k)}{(\nu+k)(\nu+k+1)\ldots(\nu+k+s)} = \frac{n(n+1)\ldots(n+s-1)}{s(\nu+n+s)(\nu+1)(\nu+2)\ldots(\nu+s)} \tag{14}
\]
holds for all \( n \in \mathbb{Z}_+ \), \( \nu \in \mathbb{C} \), \( \nu \notin -\mathbb{N} \), and \( s \in \mathbb{N} \). To show (14) one can proceed by induction in \( s \). The case \( s = 1 \) is easy to verify. For the induction step from \( s-1 \) to \( s \), with \( s > 1 \), let us denote the LHS of (14) by \( Y_s(\nu, n) \). One observes that
\[
Y_s(\nu, n) = \frac{\nu + n + s - 1}{s} Y_{s-1}(\nu, n) - \frac{\nu + n + 2s - 1}{s} Y_{s-1}(\nu + 1, n).
\]

Applying the induction hypothesis the equality readily follows.

Next one shows that
\[
\sum_{k_1=1}^{n-2s+2} \sum_{k_2=k_1+2}^{n-2s+4} \ldots \sum_{k_s=k_{s-1}+2}^{n} \times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\ldots(\nu+k_s)(\nu+k_s+1)} \times \frac{1}{(n-2s+2)(n-2s+3)\ldots(n-s+1)} \tag{15}
\]

Next one shows that
\[
\sum_{k_1=1}^{n-2s+2} \sum_{k_2=k_1+2}^{n-2s+4} \ldots \sum_{k_s=k_{s-1}+2}^{n} \times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\ldots(\nu+k_s)(\nu+k_s+1)} \times \frac{1}{(n-2s+2)(n-2s+3)\ldots(n-s+1)} \tag{15}
\]

Next one shows that
\[
\sum_{k_1=1}^{n-2s+2} \sum_{k_2=k_1+2}^{n-2s+4} \ldots \sum_{k_s=k_{s-1}+2}^{n} \times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\ldots(\nu+k_s)(\nu+k_s+1)} \times \frac{1}{(n-2s+2)(n-2s+3)\ldots(n-s+1)} \tag{15}
\]
holds for all \( n \in \mathbb{Z}_+, s \in \mathbb{N}, 2s \leq n + 2 \). To this end, we again proceed by induction in \( s \). The case \( s = 1 \) is easy to verify. In the induction step from \( s - 1 \) to \( s \), with \( s > 1 \), one applies the induction hypothesis to the LHS of (15) and arrives at the expression

\[
\sum_{k=1}^{n-2s+2} \frac{1}{(\nu + k)(\nu + k + 1)(s-1)!} \\
\times \frac{(n - k - 2s + 3)(n - k - 2s + 4) \ldots (n - k - s + 1)}{\nu + 2s + 3)(\nu + 2s + 4) \ldots (\nu + n + 1)}.
\]

Using (14) one obtains the RHS of (15), as claimed.

Finally, to conclude the proof, it suffices to notice that

\[
F(w^{\nu + 1}, w^{\nu + 2}, \ldots, w^{\nu + n}) = 1 + \sum_{s=1}^{[n/2]} (-1)^s \sum_{k_1=1}^{n-2s+1} \sum_{k_2=k_1+2}^{n-2s+3} \sum_{k_s=k_{s-1}+2}^{n-1} \frac{w^{2s}}{(\nu + k_1)(\nu + k_1 + 1)(\nu + k_2)(\nu + k_2 + 1) \ldots (\nu + k_s)(\nu + k_s + 1)}
\]

and to use equality (15).

One can complete Proposition 12 with another relation to Bessel functions.

**Proposition 13.** For \( m, n \in \mathbb{Z}_+, m \leq n \), one has

\[
\pi J_m(2w)Y_{n+1}(2w) = -\frac{n!}{m!} w^{m-n-1} \delta \left( \frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n} \right) + \sum_{s=0}^{m-1} \frac{(m-s-1)! (n-m+2s+1)!}{s! (n+s+1)! (n-m+s+1)!} w^{n-m+2s+1} + O(w^{n+1+1} \log(w)).
\]

**Proof.** Recall the following two facts from the theory of Bessel functions (see, for instance, [4, Chapter VII]). Firstly, for \( \mu, \nu \notin \mathbb{N} \), one has

\[
J_\mu(z)J_\nu(z) = \sum_{s=0}^{\infty} (-1)^s \frac{(s+\mu+\nu+1)_s}{s! \Gamma(\mu+s+1)\Gamma(\nu+s+1)} \left( \frac{z^2}{2} \right)^{\nu+\mu+2s}
\]

where \((a)_s = a(a+1) \ldots (a+s-1)\) is the Pochhammer symbol. Secondly, for \( n \in \mathbb{Z}_+ \),

\[
\pi Y_n(z) = \frac{\partial}{\partial \nu} (J_\nu(z) - (-1)^n J_{-\nu}(z)) \bigg|_{\nu=n}.
\]
For \( m, n \in \mathbb{Z}_+ \), \( m \leq n \), a straightforward computation based on these facts yields

\[
\pi J_m(z)Y_n(z) = - \sum_{s=0}^{\lfloor (m-n-1)/2 \rfloor} (-1)^s \frac{(n-s-1)!(n-m-s-1)!}{s!(m+s)!(n-m-2s-1)!} \left( \frac{z}{2} \right)^{m-n+2s}
\]

\[
- \sum_{s=0}^{m-1} \frac{(m-s-1)!(n-m+2s)!}{s!(n+s)!(n-m+s)!} \left( \frac{z}{2} \right)^{n-m+2s} + 2J_n(z)J_m(z) \log \left( \frac{z}{2} \right)
\]

\[
+ \sum_{s=0}^{\infty} (-1)^s \frac{(m+n+2s)!}{s!(m+s)!(n+s)!(m+n+s)!} \left( \frac{z}{2} \right)^{m+n+2s} \left( 2\psi(m+n+2s+1) - \psi(m+s+1) - \psi(n+s+1) - \psi(m+n+s+1) - \psi(s+1) \right)
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function. The proposition follows from \((17)\) and \((13)\).

Remark 14. Note that the first term on the RHS of \((16)\) contains only negative powers of \( w \). One can extend \((16)\) to the case \( n = m - 1 \). Then

\[
\pi J_m(2w)Y_m(2w) = - \sum_{s=0}^{m-1} \frac{(m-s-1)!(2s)!}{(s!)^2(m+s)!} w^{2s} + O(w^{2m} \log(w)).
\]

3 The matrix \( \mathbf{J} \)

In this section we introduce an infinite matrix \( \mathbf{J} \) that is basically determined by two simple properties – it is antisymmetric and its every row satisfies a second-order difference equation known from the theory of Bessel functions. Of course, in that case every column of the matrix satisfies the difference equation as well.

Lemma 15. Suppose \( w \in \mathbb{C} \setminus \{0\} \). The dimension of the vector space formed by infinite-dimensional matrices \( A = \{A(m,n)\}_{m,n \in \mathbb{Z}} \) satisfying, for all \( m,n \in \mathbb{Z} \),

\[
wA(m,n-1) - nA(m,n) + wA(m,n+1) = 0
\]

and

\[
A(n,m) = -A(m,n)
\]

equals 1. Every such a matrix is unambiguously determined by the value \( A(0,1) \), and one has

\[
\forall n \in \mathbb{Z}, \ A(n,n+1) = A(0,1).
\]

Proof. Suppose \( A \) solves \((18)\) and \((19)\). Then \( A(m,m) = 0 \). Equating \( m = n \) in \((18)\) and using \((19)\) one finds that \( A(n,n+1) = -A(n,n-1) = A(n-1,n) \). Hence \((20)\) is fulfilled. Clearly, the matrix \( A \) is unambiguously determined by the second-order difference equation \((18)\) in \( n \) and by the initial conditions \( A(m,m) = 0 \), \( A(m,m+1) = A(0,1) \), when \( m \) runs through \( \mathbb{Z} \).
Conversely, choose $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Let $A$ be the unique matrix determined by (18) and the initial conditions $A(m, m) = 0$, $A(m, m + 1) = \lambda$. It suffices to show that $A$ satisfies (19) as well. Note that $A(m, m - 1) = -\lambda$. Furthermore,

$$
wA(m - 1, m + 1) - mA(m, m + 1) + wA(m + 1, m + 1) = 0.
$$

From (18) and the initial conditions it follows that $A(m, m + 2) = (m + 1)\lambda / w$, and so $mA(m, m + 2) = (m + 1)A(m - 1, m + 1)$. Consequently,

$$
wA(m - 1, m + 2) - mA(m, m + 2) + wA(m + 1, m + 2) = 0.
$$

One observes that, for a given $m \in \mathbb{Z}$, the sequence

$$
x_n = -A(m - 1, n) + \frac{m}{w} A(m, n), \quad n \in \mathbb{Z},
$$

solves the difference equation

$$
wx_{n-1} - nx_n + wx_{n+1} = 0
$$

with the initial conditions $x_{m+1} = A(m + 1, m + 1)$, $x_{m+2} = A(m + 1, m + 2)$. By the uniqueness, $x_n = A(m + 1, n)$. This means that, for all $m, n \in \mathbb{Z}$,

$$
wA(m - 1, n) - mA(m, n) + wA(m + 1, n) = 0.
$$

Put $B(m, n) = -A(n, m)$. Then $B$ fulfills (18) and $B(m, m) = 0$, $B(m, m + 1) = \lambda$. Whence $B = A$.

**Lemma 16.** Suppose $w \in \mathbb{C} \setminus \{0\}$. If a matrix $A = \{A(m, n)\}_{m,n \in \mathbb{Z}}$ satisfies (18) and (19) then

$$
\forall m, n \in \mathbb{Z}, \quad A(m, -n) = (-1)^n A(m, n), \quad A(-m, n) = (-1)^m A(m, n). \quad (22)
$$

**Proof.** For any sequence $\{x_n\}_{n \in \mathbb{Z}}$ satisfying the difference equation (21) one can verify, by mathematical induction, that $x_{-n} = (-1)^n x_n$, $n = 0, 1, 2, \ldots$. 

**Definition 17.** For a given parameter $w \in \mathbb{C} \setminus \{0\}$ let $\mathfrak{F} = \{\mathfrak{F}(m, n)\}_{m,n \in \mathbb{Z}}$ denote the unique matrix satisfying (18), (19) and $\mathfrak{F}(m, m + 1) = 1$, $\forall m \in \mathbb{Z}$.

**Remark 18.** Here are several particular entries of the matrix $\mathfrak{F}$,

$$
\mathfrak{F}(m, m) = 0, \quad \mathfrak{F}(m, m+1) = 1, \quad \mathfrak{F}(m, m+2) = \frac{m+1}{w}, \quad \mathfrak{F}(m, m+3) = \frac{(m+1)(m+2)}{w^2} - 1,
$$

with $m \in \mathbb{Z}$. Some other particular values follow from (19) and (22). Below, in Proposition 22 we derive a general formula for $\mathfrak{F}(m, n)$. 

10
Lemma 19. For $0 \leq m < n$ one has (with the convention $\mathfrak{F}(\emptyset) = 1$)
\[
J(m,n) = \frac{(n-1)!}{m!} w^{m-n+1} \mathfrak{F}\left(\frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n-1}\right).
\]

Proof. The RHS of (23) equals 1 for $n = m + 1$, and $(m+1)/w$ for $n = m + 2$. Moreover, in view of (5), the RHS satisfies the difference equation (21) in the index $n$.

Remark 20. From (23) and (11) it follows that
\[
\forall m \in \mathbb{Z}, \lim_{n \to \infty} \frac{w^{n-1}}{(n-1)!} J(m,n) = J_{m}(2w).
\]
This is in agreement with the well known fact that, for any $w \in \mathbb{C}$, the sequence $\{J_{n}(2w)\}_{n \in \mathbb{Z}}$ fulfills the second-order difference equation (21).

Remark 21. Rephrasing Proposition 13 and Remark 14 one has, for $m, n \in \mathbb{Z}_+, m \leq n$,
\[
\pi J_{m}(2w) Y_{n}(2w) = -w^{-1} J(m,n) - \sum_{s=0}^{m-1} \frac{(n-s-1)! (n-m+2s)!}{s! (n+s)! (n-m+s)!} w^{n-m+2s} + O(w^{m+n} \log(w)).
\]

Proposition 22. For $m, n \in \mathbb{Z}$, $m \leq n$, one has
\[
J(m,n) = \sum_{s=0}^{\lfloor (n-m-1)/2 \rfloor} (-1)^s \frac{(n-s-1)!}{(n-m-2s-1)!} \frac{(n-m-s-1)!}{s!} w^{m-n+2s+1}.
\]

Proof. We distinguish several cases. First, consider the case $0 \leq m < n$. Then (24) follows from (23) and (13). Observe also that for $m = n, m, n \in \mathbb{Z}$, the RHS of (24) is an empty sum and so the both sides in (24) are equal to 0.

Second, consider the case $m \leq 0 \leq n$. Put $m = -k, k \in \mathbb{Z}_+$. The RHS of (24) becomes
\[
\sum_{s=0}^{\lfloor (n+k-1)/2 \rfloor} (-1)^s \frac{(n-s-1)!}{(n+k-2s-1)!} \frac{(n+k-s-1)!}{s!} w^{k-n+2s+1}.
\]
Suppose $k \leq n$. Then the summands in (25) vanish for $s = 0, 1, \ldots, k-1$, and so the sum equals
\[
\sum_{s=0}^{\lfloor (n-k-1)/2 \rfloor} (-1)^{s+k} \frac{(n-k-s-1)!}{(n-k-2s-1)!} \frac{(n-s-1)!}{s!} \frac{(n-s-1)!}{(s+k)!} w^{k-n+2s+1}.
\]
By the first step, this expression is equal to \((-1)^k \mathfrak{J}(k, n) = \mathfrak{J}(-k, n)\) (see Lemma 16). Further, suppose \(k \geq n\). Then the summands in (25) vanish for \(s = 0, 1, \ldots, n-1\), and so the sum equals

\[
\sum_{s=0}^{\lfloor (k-n-1)/2 \rfloor} (-1)^{n+s} {k-s-1 \choose k-n-2s-1} \frac{(k-s-1)!}{(n+s)!} w^{n-k+2s+1}.
\]

Using once more the first step, this expression is readily seen to be equal to \((-1)^{k+1} \mathfrak{J}(n, k) = \mathfrak{J}(-k, n)\).

Finally, consider the case \(m \leq n \leq 0\). Put \(m = -k, n = -\ell, k, \ell \in \mathbb{Z}_+\). Hence \(0 \leq \ell \leq k\). The RHS of (24) becomes

\[
\sum_{s=0}^{\lfloor (k-\ell-1)/2 \rfloor} (-1)^s {k-\ell-s-1 \choose k-\ell-2s-1} \frac{(k-\ell-s-1)!}{s!} w^{\ell-k+2s+1}.
\]

Using again the first step, this expression is readily seen to be equal to \((-1)^{k+\ell+1} \mathfrak{J}(\ell, k) = \mathfrak{J}(-k, -\ell)\).

4 The characteristic function for the antisymmetric diagonal

For a given \(d \in \mathbb{Z}_+\) let \(E_\pm\) denote the \((2d+1) \times (2d+1)\) matrix with units on the upper (lower) parallel to the diagonal and with all other entries equal to zero. Hence

\[
(E_+)_{j,k} = \delta_{j+1,k}, \quad (E_-)_{j,k} = \delta_{j,k+1}, \quad j, k = -d, -d+1, -d+2, \ldots, d.
\]

For \(y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}\) let \(\text{diag}(y)\) denote the diagonal \((2d+1) \times (2d+1)\) matrix with the sequence \(y\) on the diagonal. Everywhere in what follows, \(I\) stands for a unit matrix.

First a formula is presented for the determinant of a Jacobi matrix with a general diagonal but with constant neighboring parallels to the diagonal. As explained in the subsequent remark, however, this formula can be extended to the general case with the aid of a simple decomposition of the Jacobi matrix in question.

**Proposition 23.** For \(d \in \mathbb{N}\), \(w \in \mathbb{C}\) and \(y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}\), \(\prod_{k=1}^d y_k y_{-k} \neq 0\), one has

\[
\det(\text{diag}(y) + wE_+ + wE_-) = \left( \prod_{k=1}^d y_k y_{-k} \right) \left( y_0 \mathfrak{H} \left( \frac{w}{y_1}, \ldots, \frac{w}{y_d} \right) \mathfrak{H} \left( \frac{w}{y_{-1}}, \ldots, \frac{w}{y_{-d}} \right) \right.
\]

\[
- w^2 \mathfrak{H} \left( \frac{w}{y_2}, \ldots, \frac{w}{y_d} \right) \mathfrak{H} \left( \frac{w}{y_{-2}}, \ldots, \frac{w}{y_{-d}} \right)\left. - w^2 \mathfrak{H} \left( \frac{w}{y_2}, \ldots, \frac{w}{y_d} \right) \mathfrak{H} \left( \frac{w}{y_{-2}}, \ldots, \frac{w}{y_{-d}} \right) \right). \tag{26}
\]
Proof. Let us proceed by induction in $d$. The case $d = 1$ is easy to verify. Put $N_d(w; y) = \det \left( \text{diag}(y) + wE_+ + wE_- \right)$. Suppose (26) is true for some $d \geq 1$. For given $w \in \mathbb{C}$ and $y \in \mathbb{C}^{2d+3}$ consider the quantity $N_{d+1}(w; y)$. Let us split the corresponding $(2d+3) \times (2d+3)$ Jacobi matrix into four blocks by splitting the set of indices into two disjoint sets $\{-d-1, d+1\}$ and $\{-d, -d+1, -d+2, \ldots, d\}$. Applying the rule
\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)
\]
one derives the recurrence relation
\[
N_{d+1}(w; y_{-d}; y_{-d+1}, \ldots, y_{d+1}) = y_{d+1}y_{-d-1}N_d(w; y_{-d}; y_{-d+1}, \ldots, y_{d-1}, y_{d})
\]
where
\[
y_d' = \left(1 - \frac{w^2}{y_{d}y_{d+1}}\right)y_{d}, \quad y_{-d}' = \left(1 - \frac{w^2}{y_{-d}y_{-d-1}}\right)y_{-d}.
\]
Now it is sufficient to use the induction hypothesis jointly with the equality
\[
(1 - x_{n-1}x_{n}) \mathfrak{F}(x_1, x_2, \ldots, x_{n-2}, \frac{x_{n-1}}{1 - x_{n-1}x_{n}}) = \mathfrak{F}(x_1, x_2, \ldots, x_{n-1}, x_{n})
\]
which is valid for $n \geq 2$ and which follows from relations (4) and (7), with $k = 2$. \qed

Remark 24. Let us consider a general finite symmetric Jacobi matrix $J$ of the form
\[
J = \begin{pmatrix}
\lambda_1 & w_1 \\
1 & \lambda_2 & w_2 \\
& \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
w_{n-2} & \lambda_{n-1} & w_{n-1} \\
w_{n-1} & \lambda_n
\end{pmatrix}
\]
such that $\prod_{k=1}^{n-1} w_k \neq 0$. The Jacobi matrix can be decomposed into the product
\[
J = G \tilde{J} G
\]
where $G = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a diagonal matrix and $\tilde{J}$ is a Jacobi matrix with all units on the neighboring parallels to the diagonal,
\[
\tilde{J} = \begin{pmatrix}
\tilde{\lambda}_1 & 1 & & & & \\
1 & \tilde{\lambda}_2 & 1 & & & \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
1 & \tilde{\lambda}_{n-1} & 1 & & & & \\
1 & \tilde{\lambda}_n
\end{pmatrix}.
\]
Proposition 25. 

Hence det$(J) = \left( \prod_{k=1}^{d} \gamma_{k}^{2} \right) \det(J)$, and one can employ formula (26) to evaluate det$(J)$ (in the case of odd dimension). In more detail, one can put

\[
\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \quad \gamma_{2k} = w_{1} \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \quad k = 1, 2, 3, \ldots.
\]

Alternatively, the sequence \( \{\gamma_{k}\}_{k=1}^{n} \) is defined recursively by \( \gamma_{1} = 1, \gamma_{k+1} = \frac{w_{k}}{\gamma_{k}} \). Furthermore, \( \lambda_{k} = \frac{\lambda_{k}}{\gamma_{k}^{2}} \). With this choice, (27) is clearly true.

Next we aim to derive a formula for the characteristic function of a Jacobi matrix with an antisymmetric diagonal. Suppose \( \lambda = (\lambda_{-d}, \lambda_{-d+1}, \lambda_{-d+2}, \ldots, \lambda_{d}) \in \mathbb{C}^{2d+1} \) and \( \lambda_{-k} = -\lambda_{k} \) for \( -d \leq k \leq d \); in particular, \( \lambda_{0} = 0 \). We consider the Jacobi matrix \( K = \text{diag}(\lambda) + wE_{+} + wE_{-} \). Let us denote, temporarily, by \( S \) the diagonal matrix with alternating signs on the diagonal, \( S = \text{diag}(1, -1, 1, \ldots, 1) \), and by \( Q \) the permutation matrix with the entries \( Q_{j,k} = \delta_{j+k,0} \) for \( -d \leq j, k \leq d \). The commutation relations

\[ SQKQS = -K, \quad S^{2} = Q^{2} = I, \]

imply

\[
det(K - zI) = det(SQ(K - zI)QS) = -det(K + zI).
\]

Hence the characteristic function of \( K \) is an odd polynomial in the variable \( z \). This can be also seen from the explicit formula (28) derived below.

**Proposition 25.** Suppose \( d \in \mathbb{N}, \ w \in \mathbb{C}, \ \lambda \in \mathbb{C}^{2d+1} \) and \( \lambda_{-k} = -\lambda_{k} \) for \( k = -d, -d+1, -d+2, \ldots, d \). Then

\[
\frac{(-1)^{d+1}}{z} \det \left( \text{diag}(\lambda) + wE_{+} + wE_{-} - zI \right)
= \left( \prod_{k=1}^{d} (\lambda_{k}^{2} - z^{2}) \right) \mathfrak{F} \left( \frac{w}{\lambda_{1} - z}, \ldots, \frac{w}{\lambda_{d} - z} \right) \mathfrak{F} \left( \frac{w}{\lambda_{1} + z}, \ldots, \frac{w}{\lambda_{d} + z} \right)
+ 2 \sum_{j=1}^{d} w^{2j} \left( \prod_{k=j+1}^{d} (\lambda_{k}^{2} - z^{2}) \right) \mathfrak{F} \left( \frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_{d} - z} \right) \mathfrak{F} \left( \frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_{d} + z} \right).
\]

**Proof.** This is a particular case of (26) where one has to set \( y_{k} = \lambda_{k} - z \) for \( k > 0 \), \( y_{0} = -z \), \( y_{k} = -(\lambda_{k} + z) \) for \( k < 0 \). To complete the proof it suffices to verify that

\[
\frac{w^{2}}{z (\lambda_{1} - z)} \mathfrak{F} \left( \frac{w}{\lambda_{2} - z}, \ldots, \frac{w}{\lambda_{d} - z} \right) \mathfrak{F} \left( \frac{w}{\lambda_{1} + z}, \frac{w}{\lambda_{2} + z}, \ldots, \frac{w}{\lambda_{d} + z} \right)
- \frac{w^{2}}{z (\lambda_{1} + z)} \mathfrak{F} \left( \frac{w}{\lambda_{2} - z}, \ldots, \frac{w}{\lambda_{d} - z} \right) \mathfrak{F} \left( \frac{w}{\lambda_{2} + z}, \ldots, \frac{w}{\lambda_{d} + z} \right)
= 2 \sum_{j=1}^{d} w^{2j} \left( \prod_{k=1}^{j} \frac{1}{\lambda_{k}^{2} - z^{2}} \right) \mathfrak{F} \left( \frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_{d} - z} \right) \mathfrak{F} \left( \frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_{d} + z} \right).
\]

To this end, one can apply (5), with \( n = d, \ u_{k} = w/(\lambda_{k} - z), \ v_{k} = w/(\lambda_{k} + z) \). Note that \( u_{j} - v_{j} = 2zw_{j}w_{j}/w \). \( \square \)
Zero always belongs to spectrum of the Jacobi matrix $K$ for the characteristic function is odd. Moreover, as is well known and as it simply follows from the analysis of the eigenvalue equation, if $w \neq 0$ then to every eigenvalue of $K$ there belongs exactly one linearly independent eigenvector.

**Proposition 26.** Suppose $w \in \mathbb{C}$, $\lambda \in \mathbb{C}^{2d+1}$, $\lambda_{-k} = -\lambda_k$ for $-d \leq k \leq d$, and $\prod_{k=1}^{d} \lambda_k \neq 0$. Then the vector $v \in \mathbb{C}^{2d+1}$, $v^T = (\theta_{-d}, \theta_{-d+1}, \theta_{-d+2}, \ldots, \theta_d)$, with the entries

$$\theta_k = (-1)^k w^k \left( \prod_{j=k+1}^{d} \lambda_j \right) \frac{w}{\lambda_{k+1}} \frac{w}{\lambda_{k+2}} \cdots \frac{w}{\lambda_d} \quad \text{for } k = 0, 1, 2, \ldots, d,$$

$$\theta_{-k} = (-1)^k \theta_k \quad \text{for } -d \leq k \leq d,$$

belongs to the kernel of the Jacobi matrix $\text{diag}(\lambda) + wE_+ + wE_-$. In particular, $\theta_0 = \lambda_1 \lambda_2 \cdots \lambda_d \frac{w}{\lambda_1}, \frac{w}{\lambda_2}, \ldots, \frac{w}{\lambda_d}$, $\theta_d = (-1)^d w^d$, and so $v \neq 0$.

**Remark.** Clearly, formulas (29) can be extended to the case $\prod_{k=1}^{d} \lambda_k = 0$ as well provided one makes the obvious cancellations.

**Proof.** One has to show that

$$w \theta_{k-1} + \lambda_k \theta_k + w \theta_{k+1} = 0, \quad k = -d + 1, -d + 2, \ldots, d - 1,$$

and $\lambda_{-d} \theta_{-d} + w \theta_{-d+1} = 0, \quad w \theta_{-d+1} + \lambda_d \theta_d = 0$. Owing to the symmetries $\lambda_{-k} = -\lambda_k$, $\theta_{-k} = (-1)^k \theta_k$, it suffices to verify the equalities only for indices $0 \leq k \leq d$. This can be readily carried out using the explicit formulas (29) and the rule (3).

\[\blacksquare\]

### 5 Jacobi matrices with a linear diagonal

Finally we focus on finite-dimensional Jacobi matrices of odd dimension whose diagonal depends linearly on the index and whose parallels to the diagonal are constant. Without loss of generality one can assume that the diagonal equals $(-d, -d+1, -d+2, \ldots, d), \ d \in \mathbb{Z}_+$. For $w \in \mathbb{C}$ put

$$K_0 = \text{diag}(-d, -d+1, -d+2, \ldots, d), \ K(w) = K_0 + wE_+ + wE_-.$$

Concerning the characteristic function $\chi(z) = \det(K(w) - z)$, we know that this is an odd function. Put

$$\chi_{\text{red}}(z) = \frac{(-1)^{d+1}}{z} \det(K(w) - z).$$

Hence $\chi_{\text{red}}(z)$ is an even polynomial of degree $2d$. Further, denote by $\{e_{-d}, e_{-d+1}, e_{-d+2}, \ldots, e_d\}$ the standard basis in $\mathbb{C}^{2d+1}$.

Suppose $w \neq 0$. Let us consider a family of column vectors $x_{s,n} \in \mathbb{C}^{2d+1}$ depending on the parameters $s, n \in \mathbb{Z}$ and defined by

$$x_{s,n}^T = \left( \mathfrak{J}(s+d, n), \mathfrak{J}(s+d-1, n), \mathfrak{J}(s+d-2, n), \ldots, \mathfrak{J}(s-d, n) \right).$$
From the fact that the matrix $J$ obeys (18), (19) one derives that
\[ \forall s, n \in \mathbb{Z}, \quad K(w)x_{s,n} = s x_{s,n} - w J(s + d + 1, n)e_{-d} - w J(s - d - 1, n)e_d. \]

Put
\[ v_s = x_{s,s+d+1}, \quad s \in \mathbb{Z}. \]
Recalling that $J(m, m) = J(-m, m) = 0$ one has
\[ K(w)v_s = s v_s - w J(s - d - 1, s + d + 1)e_d. \quad (30) \]

**Remark 27.** Putting $s = 0$ one gets $K(w)v_0 = 0$, and so $v_0$ spans the kernel of $K(w)$.

**Lemma 28.** For every $\ell = -d, -d+1, -d+2, \ldots, d$, one has
\[ w^{d+\ell} \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)! (\ell-s)!} v_s \in e_{\ell} + \text{span}\{e_{\ell+1}, e_{\ell+2}, \ldots, e_d\}. \]

In particular,
\[ e_d = w^{2d} \sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)! (d-s)!} v_s. \quad (31) \]

Consequently, $V = \{v_{-d}, v_{-d+1}, v_{-d+2}, \ldots, v_d\}$ is a basis in $\mathbb{C}^{2d+1}$.

**Proof.** One has to show that
\[ w^{d+\ell} \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)! (\ell-s)!} J(s - k, s + d + 1) = \delta_{\ell,k} \quad \text{for} \quad -d \leq k \leq \ell. \]

Note that for any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_+$,
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+k}{r} = 0, \quad r = 0, 1, 2, \ldots, n - 1, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+k}{n} = (-1)^n. \]
Using these equalities and (24) one can readily show, more generally, that
\[ \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)! (\ell-s)!} J(m + s, n + s) = 0 \quad \text{for} \quad m, n \in \mathbb{Z}, m \leq n \leq m + d + \ell, \]
and
\[ \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)! (\ell-s)!} J(m + s, m + d + \ell + s + 1) = w^{-d-\ell}. \]

This proves the lemma. \(\square\)
Denote by \( \tilde{K}(w) \) the matrix of \( K(w) \) in the basis \( V \) introduced in Lemma 28.

Let \( a, b \in \mathbb{C}^{d+1} \) be the column vectors defined by \( a^T = (\alpha_{-d}, \alpha_{-d+1}, \alpha_{-d+2}, \ldots \alpha_d) \), \( b^T = (\beta_{-d}, \beta_{-d+1}, \beta_{-d+2}, \ldots \beta_d) \),

\[
\alpha_s = J(s-d-1, s+d+1), \quad \beta_s = (-1)^{d+s} \frac{w^{2d+1}}{(d+s)!(d-s)!}, \quad s = -d, -d+1, -d+2, \ldots, d.
\] (32)

Note that \( \alpha_{-s} = -\alpha_s \), \( \beta_{-s} = \beta_s \). (33)

The former equality follows from (22) and (19). From (30) and (31) one deduces that

\( \tilde{K}(w) = K_0 - ba^T \). (34)

Note, however, that the components of the vectors \( a \) and \( b \) depend on \( w \), too, though not indicated in the notation.

According to (34), \( \tilde{K}(w) \) differs from the diagonal matrix \( K_0 \) by a rank-one correction. This form is suitable for various computations. Particularly, one can express the resolvent of \( \tilde{K}(w) \) explicitly,

\[
(\tilde{K}(w) - z)^{-1} = (K_0 - z)^{-1} + \frac{1}{1 - a^T(K_0 - z)^{-1}b} (K_0 - z)^{-1}ba^T(K_0 - z)^{-1}.
\]

The equality holds for any \( z \in \mathbb{C} \) such that \( z \notin \text{spec}\{K_0\} = \{-d, -d+1, -d+2, \ldots, d\} \) and \( 1 - a^T(K_0 - z)^{-1}b \neq 0 \). Clearly, this set of excluded values of \( z \) is finite.

Let us proceed to derivation of a formula for the characteristic function of \( K(w) \). Proposition 25 is applicable to \( K(w) \) and so

\[
\chi_{\text{red}}(z) = \left( \prod_{k=1}^{d}(k^2 - z^2) \right) \left( \frac{w}{1-z}, \ldots, \frac{w}{d-z} \right) F \left( \frac{w}{1+z}, \ldots, \frac{w}{d+z} \right) + 2 \sum_{j=1}^{d} w^{2j} \left( \prod_{k=j+1}^{d}(k^2 - z^2) \right) F \left( \frac{w}{j+1-z}, \ldots, \frac{w}{d-z} \right) F \left( \frac{w}{j+1+z}, \ldots, \frac{w}{d+z} \right). \] (35)

Below we derive a more convenient formula for \( \chi_{\text{red}}(z) \).

**Lemma 29.** One has

\[
\chi_{\text{red}}(0) = \sum_{s=0}^{d} \frac{((d-s)!)^2 (2d-s+1)!}{s!(2d-2s+1)!} w^{2s} \] (36)

and

\[
\chi_{\text{red}}(n) = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) \binom{n+k}{2k+1} \binom{d+k+1}{2k+1} w^{2d-2k} \] (37)

for \( n = 1, 2, \ldots, d \).
Proof. Let us first verify the formula for $\chi_{\text{red}}(0)$. From (35) it follows that

$$
\chi_{\text{red}}(0) = (d!)^2 \mathfrak{F}\left(w, \frac{w}{d}, \ldots, \frac{w}{d}\right)^2 + 2 \sum_{j=1}^{d} w^{2j} \left(\frac{d!}{j!}\right)^2 \mathfrak{F}\left(\frac{w}{j+1}, \frac{w}{j+2}, \ldots, \frac{w}{d}\right)^2.
$$

By Proposition 13,

$$
\chi_{\text{red}}(0) = \pi^2 w^{2d+2} Y_{d+1}(2w)^2 \left(J_0(2w)^2 + 2 \sum_{j=1}^{d} J_j(2w)^2\right) + O\left(w^{2d+2} \log(w)\right).
$$

Further we need some basic facts concerning Bessel functions; see, for instance, [1, Chp. 9]. Recall that

$$
J_0(z)^2 + 2 \sum_{j=1}^{\infty} J_j(z)^2 = 1.
$$

Hence

$$
\chi_{\text{red}}(0) = \pi^2 w^{2d+2} Y_{d+1}(2w)^2 \left(\left(\sum_{k=0}^{d} \frac{(d-k)!}{k!} w^{2k}\right)^2 + O\left(w^{2d+2} \log(w)\right)\right).
$$

Note that $\chi_{\text{red}}(0)$ is a polynomial in the variable $w$ of degree $2d$, and so

$$
\chi_{\text{red}}(0) = \sum_{s=0}^{d} \sum_{k=0}^{s} \frac{(d-k)! (d-s+k)!}{k! (s-k)!} w^{2s}.
$$

Using the identity

$$
\sum_{k=0}^{s} \frac{(d-k)! (d-s+k)!}{k! (s-k)!} = (d-s)!^2 \sum_{k=0}^{s} \frac{(d-k)!}{d-s} \frac{(d-s+k)!}{d-s} = (d-s)!^2 \frac{(2d-s+1)!}{s! (2d-2s+1)!},
$$

one arrives at (36).

To show (37) one can make use of (34). One has

$$
\chi_{\text{red}}(z) = \frac{(-1)^{d+1}}{z} \det(\tilde{K}(w) - z) = \frac{(-1)^{d+1}}{z} \det(K_0 - z) \det\left(I - (K_0 - z)^{-1}ba^T\right).
$$

Note that $\det(I + ba^T) = 1 + a^T b$. Hence, in view of (33),

$$
\chi_{\text{red}}(z) = \prod_{k=1}^{d} \left(k^2 - z^2\right) \left(1 - \sum_{s=-d}^{d} \frac{s \beta_s \alpha_s}{s - z}\right) = \prod_{k=1}^{d} \left(k^2 - z^2\right) \left(1 - 2 \sum_{s=1}^{d} \frac{s \beta_s \alpha_s}{s^2 - z^2}\right).
$$
Hence
\[ \xi w x 31 \]

**Remark** For every Proposition 30.

Using (32) one gets
\[ 33 \]

**Remark**

\[ n \]

simple, and formula (38) implies that the interval \(-\infty, 0\). With the aid of (3) one derives the equality
\[ \chi \prod \]

If \( w \in \mathbb{R} \), \( w \neq 0 \), then the spectrum of the Jacobi matrix \( K(w) \) is real and simple, and formula (38) implies that the interval \([-1, 1]\) contains no other eigenvalue except of 0.

Eigenvectors of \( K(w) \) can be expressed in terms of the function \( \mathfrak{F} \), too. Suppose \( w \neq 0 \). Let us introduce the vector-valued function \( x(z) \in \mathbb{C}^{2d+1} \) depending on \( z \in \mathbb{C} \), \( x(z)^T = (\xi_{-d}(z), \xi_{-d+1}(z), \xi_{-d+2}(z), \ldots, \xi_d(z)) \),
\[ \xi_k(z) = w^{-d-k} \frac{\Gamma(z+d+1)}{\Gamma(z-k+1)} \mathfrak{F} \left( \frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d} \right), \quad -d \leq k \leq d. \]

With the aid of (34) one derives the equality
\[ (K(w) - z)x(z) = -w^{-2d} \frac{\Gamma(z+d+1)}{\Gamma(z-d)} \mathfrak{F} \left( \frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d} \right) e_d. \]

**Remark 33.** According to (12),
\[ \xi_k(z) = w^{-d-k} \sum_{s=0}^{(d+k)/2} (-1)^s \frac{(d+k-s)!}{s!(d+k-2s)!} w^{2s} \prod_{j=s}^{d+k-s-1} (z+d-j). \]

Hence \( \xi_k(z) \) is a polynomial in \( z \) of degree \( d + k \). In particular, \( \xi_{-d}(z) = 1 \), and so \( x(z) \neq 0 \).

**Proposition 34.** One has
\[ \chi(z) = -z \left( \prod_{k=1}^{d} (z^2 - k^2) \right) \mathfrak{F} \left( \frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d} \right). \]

If \( w \in \mathbb{C} \), \( w \neq 0 \), then for every eigenvalue \( \lambda \in \text{spec}(K(w)) \), \( x(\lambda) \) is an eigenvector corresponding to \( \lambda \).
Proof. Denote by $P(z)$ the RHS of (40). By (39), if $P(\lambda) = 0$ then $x(\lambda)$ is an eigenvector of $K(w)$. Thus it suffices to verify (40). The both sides depend on $w$ polynomially and so it is enough to prove the equality for $w \in \mathbb{R} \setminus \{0\}$. Note that $P(z)$ is a polynomial in $z$ of degree $2d + 1$, and the coefficient standing at $z^{2d+1}$ equals $-1$. The set of roots of $P(z)$ is contained in $\text{spec}(K(w))$. One can show that $P(z)$ has no multiple roots. In fact, suppose $P(\lambda) = P'(\lambda) = 0$ for some $\lambda \in \mathbb{R}$. From (39) one deduces that $(K(w) - \lambda)x(\lambda) = 0$, $(K(w) - \lambda)x'(\lambda) = x(\lambda)$ (here $x'(z)$ is the derivative of $x(z)$). Hence

$$(K(w) - \lambda)^2 x(\lambda) = (K(w) - \lambda)^2 x'(\lambda) = 0.$$  

Note that $x'(\lambda) \neq 0$, and $x'(\lambda)$ differs from a multiple of $x(\lambda)$ for $\xi_{-d}(z) = 1$. This contradicts the fact, however, that the spectrum of $K(w)$ is simple. One concludes that the set of roots of $P(z)$ coincides with $\text{spec}(K(w))$. Necessarily, $P(z)$ is equal to the characteristic function of $K(w)$.

Remark 35. With the aid of (40) one can rederive equality (37). For $1 \leq n \leq d$, a straightforward computation gives

$$\chi(n) = (-1)^{d+n} w^{2d+1} \mathfrak{J}(d - n + 1, d + n + 1).$$

Equality (37) then follows from (24).

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