Calculation of oscillation probabilities of atmospheric neutrinos using nuCraft

Marius Wallraff\textsuperscript{a,*}, Christopher Wiebusch\textsuperscript{a}

\textsuperscript{a}III. Physikalisches Institut, RWTH Aachen University, Germany

Abstract

NuCraft (\texttt{nucraft.hepforge.org}) is an open-source Python project that calculates neutrino oscillation probabilities for neutrinos from cosmic-ray interactions in the atmosphere for their propagation through Earth. The solution is obtained by numerically solving the Schrödinger equation. The code supports arbitrary numbers of neutrino flavors including additional sterile neutrinos, CP violation, arbitrary mass hierarchies, matter effects with a configurable Earth model, and takes into account the production height distribution of neutrinos in the Earth’s atmosphere.

Keywords: neutrino oscillation; atmospheric neutrinos; sterile neutrinos; nuCraft

PROGRAM SUMMARY

\textit{Manuscript Title:} Calculation of oscillation probabilities of atmospheric neutrinos using nuCraft
\textit{Authors:} Marius Wallraff, Christopher Wiebusch
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\textit{Licensing provisions:} Revised BSD License
\textit{Programming language:} Python
\textit{Computer:} IA32/x86-64 compatible
\textit{Operating system:} all that are supported by SciPy, e.g., Linux, Windows, OS X
\textit{RAM:} 134217728 bytes
\textit{Keywords:} neutrino oscillation, sterile neutrino, atmospheric neutrino
\textit{Classification:} 1.1 Cosmic Rays, 11.1 General, High-Energy Physics and Computing, 11.6 Phenomenological and Empirical Models and Theories
\textit{External routines/libraries:} NumPy, SciPy
\textit{Nature of problem:} Calculation of oscillation probabilities of neutrinos that originate in cosmic-ray interactions in the Earth’s atmosphere and propagate through the Earth, for realistic Earth and atmospheric models and multiple flavors (optionally including sterile neutrinos and CP violation).
\textit{Solution method:} Direct solution of the Schrödinger equation for \( n \) flavors including matter effects, with sampling of the atmosphere.
\textit{Restrictions:} Energy loss and absorption of neutrinos inside the Earth is not modeled; they have to be handled independently.

\textit{Unusual features:} Completely configurable oscillation parameters (including optional sterile flavors), configurable and realistic Earth model.
\textit{Running time:} Roughly 100 neutrinos per second and CPU core (depends on energy and oscillation parameters).

1. Introduction

Neutrino oscillations have been a major research topic for many particle and astroparticle physicists over the last decades. While neutrinos do not possess a mass in the minimum Standard Model of Particle Physics, many oscillation experiments have demonstrated that there are non-zero neutrino masses, and that their mass eigenstates differ from their flavor eigenstates.

Despite the large progress that has been made in this field, there are still many open questions regarding neutrinos, including their absolute mass scale, their mass hierarchy, CP-violation, whether there are more than the three known flavors, and whether neutrinos are Majorana particles. Additionally, some neutrino properties are not yet very well measured, and neutrino oscillations are a good phenomenon to improve our knowledge of those.

NuCraft is a Python project designed to compute oscillation probabilities of neutrinos that originate from cosmic-ray interactions in the Earth’s atmosphere, so-called atmospheric neutrinos. Many experiments are able to detect and measure atmospheric neutrinos, and
nuCraft can help to utilize those large numbers for oscillation analyses.

2. Theory

2.1. Neutrino oscillation in vacuum

Neutrinos change their flavor during propagation in spacetime as a consequence of their weak-interaction flavor eigenstates $\nu_j$ not being identical to their mass eigenstates $\nu_{\alpha}$, instead, they are a linear combination of each other, described by the unitary Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $U$:

$$
\nu_{\alpha} = \sum_j U_{\alpha j} \nu_j
$$

The PMNS matrix for $n \in \mathbb{N}$ neutrino flavors can be parameterized as a product of rotation matrices $R_{\theta, \delta} = R_{\theta}(\theta_{\theta, \delta} \delta_{\theta, \delta}) \in \mathbb{C}^{n \times n}$,

$$
U = \prod_{j=1}^{n-1} \prod_{k=j+1}^n R_{\theta_{jk}} \delta_{\theta_{jk}} = \delta_{\theta_{mn}} \left( \left( \delta_{jm} + \delta_{km} \right) \cos(\theta_{jk}) + 1 - \left( \delta_{jm} + \delta_{km} \right) \right) + \left( \delta_{jm} \delta_{nk} - \delta_{j} \delta_{nk} \right) \sin(\theta_{jk}) \exp \left( -\frac{i}{\Delta \nu} \delta_{jk} \right),
$$

with mixing angles $\theta_{jk} \in \mathbb{R}$, CP-violating Dirac phases $\delta_{jk} \in \mathbb{Z} \cup \{ \infty \}$, and Kronecker’s delta $\delta_{mn}$. This parameterization is given explicitly, because there is no clear canonical version for cases with $n > 3$, and rotation matrices do not commute in general. If neutrinos are Majorana particles, one also has to add CP-violating Majorana phases to the diagonal of the PMNS matrix, but those do not influence oscillation phenomena and are therefore omitted for this work. For antineutrinos, $U$ has to be replaced by its complex conjugate.

Neutrinos originate from weak interactions, so they are generated in definite flavor eigenstates. Their migration from one eigenstate into another and can be omitted; using the PMNS matrix to translate the Schrödinger equation into the flavor bases yields

$$
\frac{d}{dx} \left| \nu_{\alpha} \right\rangle = -\frac{-i}{2E_{\nu}} H_0 \left| \nu_{\alpha} \right\rangle
$$

$$
H_0 = U \text{ diag } \left( 0, \Delta m^2_{\alpha 1}, \ldots, \Delta m^2_{\alpha 3} \right) U^\dagger.
$$

Transition probabilities can then be obtained by

$$
P_{\alpha \rightarrow \beta}(x) = \left\| \left( \nu_{\beta}(0) \left| \nu_{\alpha}(x) \right\|^2. (1)
$$

2.2. Matter effects

When propagating through matter, neutrinos are subject to coherent forward scattering, which can strongly influence the oscillation behavior. The three known flavors of neutrinos can scatter on all particles via Neutral Current (NC) interactions, and $\nu_e$ and $\nu_\mu$ can additionally scatter via Charged Current (CC) interactions on electrons without being absorbed. In contrast, sterile neutrinos do not interact via weak interactions per definition.

These processes induce an effective squared mass:

$$
\frac{d}{dx} \left| \nu_{\alpha} \right\rangle = -\frac{-i}{2E_{\nu}} \left( H_0 + A \right) \left| \nu_{\alpha} \right\rangle
$$

$$
A = \text{ diag } (A_{CC} + A_{NC}, A_{NC}, A_{NC}, 0, \ldots)
$$

$$
A_{NC} = \pm 2 \sqrt{2} G_F E_e Y_e \rho / m_N
$$

$$
A_{CC} = \pm \sqrt{2} G_F E_e Y_e \rho / m_N,
$$

where $G_F$ is Fermi’s constant, $Y_e$ is the electron fraction ($Y_e = Y_\rho = 1 - Y_\bar{\nu}$), $\rho$ is the mass density, and $m_N$ is the mean of the proton mass and the neutron mass. The upper signs hold for particles, the lower for antiparticles. The reason for $A_{NC}$ only to depend on the neutron density is that in electrically neutral and unpolarized matter, proton and electron potentials cancel out.

In some publications, matter effects are classified as either caused by the Mikheyev-Smirnov-Wolfenstein (MSW) effect or by parametric enhancement to gain phenomenological insights. The work presented here correctly handles both, but the effects cannot be separated because they both originate from $A$ when numerically solving the Schrödinger equation.
2.3. The interaction picture

Numerical algorithms for solving ordinary differential equations generally work the better the smoother the solutions are, such that the internal time steps can be chosen large without losing precision. The solution for the Schrödinger equation (2) however is in most cases similar to the plane-wave solution for vacuum oscillations. To significantly reduce the number of time steps the solvers need in these cases, the Schrödinger equation can be transformed into the interaction basis, in which the vacuum solution is a constant function [3]:

\[
\frac{d}{dx} |ν_{\text{inter}}\rangle = -\frac{i}{2E_ν} \tilde{A} |ν_{\text{inter}}\rangle
\]

The solution for the Schrödinger equation (2) is given by:

\[
|ν_{\text{inter}}\rangle = \exp \left( -iH_0 x \right) |ν_α\rangle
\]

\[
\tilde{A} = \exp \left( -iH_0 x \right) A \exp \left( iH_0 x \right)
\]

The additional computations that are needed per time step are relatively expensive, but in most cases they are more than compensated for by the reduced number of steps required.

3. Program

NuCraft is fully written in Python and is compatible with both Python 2 (tested with 2.6 and above) and Python 3 (tested with 3.3). It relies on the libraries NumPy [4] and SciPy [5] and especially uses a SciPy wrapper around the ODE solver ZVODE [6]. Using ZVODE, it directly solves the Schrödinger equation (3) in the interaction picture. The nuCraft source code is available at [7] under the revised BSD license (3-clause version).

The project nuCraft consists of two classes, NuCraft and EarthModel. NuCraft is a class with three helper methods ConstructMassMatrix, ConstructMixingMatrix and InteractionAlt as well as the main methods CalcWeights and CalcWeightsLegacy, with the latter solving the Schrödinger equation not in the interaction picture. The legacy method does not perform the transformations described in subsection 2.3 and is therefore easier to read and faster in cases where vacuum oscillations are weak in comparison to matter effects (e.g., for sterile neutrinos at high neutrino energies), but generally, it is substantially slower and does not offer all features, so its use is discouraged.

EarthModel is an auxiliary class that allows for convenient and flexible specification of the parameters of the Earth that are relevant for oscillation effects of atmospheric neutrinos; see section 4.

4. Earth Model

As detailed in subsection 2.2 the mass density \( \rho \) and the electron fraction \( Y_e \) are important for the proper calculation of oscillation probabilities. NuCraft assumes the Earth to be spherical and by default uses the mass density values given by the Preliminary Reference Earth Model (PREM; see figure 1) [8]. The electron fraction is assumed to be 0.4957 in the mantle (including the crust) and 0.4656 in the inner and outer core. The electron fractions of these three regions can be adjusted independently.

The customization of the Earth model can be done by the aforementioned class EarthModel. By default, NuCraft only offers the default PREM density profile. Electron fractions can be modified with a keyword argument, new density profiles can either be added to the dictionary models inside EarthModel, or can be loaded from a text file; an example file is provided with the code. Together with a trivial change in nuCraft’s main class, this class can also be used to employ non-symmetrical Earth models for use with reactor neutrino experiments.

5. Atmosphere

The atmosphere is relevant because it is the region where the atmospheric neutrinos are produced, thereby influencing the neutrino path length. NuCraft uses the atmospheric model described in [9]. The original model
Figure 2: Expectation values and widths (defined as the 68% probability range around the expectation value) of the neutrino production height as function of the cosine of the zenith angle at two energies. Black corresponds to 200 GeV, gray to 2 GeV; gray has been shifted to the right by 0.005.

Figure 3: Exemplary unnormalized probability density function for the neutrino production height at 2 GeV and \( \cos(\theta) = 0.75 \) (black; corresponding to a path length of 9556 km without atmosphere), and log-normal distribution fitted to it (gray).

Figure 4: Time in seconds needed per oscillation probability computation in dependence of neutrino energy for the standard three-flavor model, for muon neutrinos (gray) and muon antineutrinos (black), distributed uniformly in \( \cos(\theta) \). The feature at about 10 GeV is caused by matter effects. Times were measured using a single core of an AMD Phenom II X6 1055T CPU.

6. Performance

By default, nuCraft computes eight equally likely production heights by evaluating the quantile function of the log-normal parametrization at the central values of the partition of \([0, 1]\) into eight equally-sized intervals. It then calculates the oscillation probabilities at the detector for the neutrino with the lowest production height. For the other seven heights, vacuum oscillation probabilities are computed analytically for the path length differences, and the average of all eight heights is returned. Alternatively, nuCraft can assume production at a fixed height, randomly draw a single height from the described model, or fully propagate neutrinos at eight equally likely production heights to the detector, which is slow and meant for comparison only.

Eight is a good compromise between precision and speed.
functions. Figure 4 can be used to estimate the calculation speed, which does also depend on the neutrino parameters, zenith angle distribution, and to a smaller extent on the Earth model.

Figure 5 shows an example oscillogram computed with nuCraft by calculating probabilities for one neutrino per grid point.

7. Acknowledgments

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References

[1] T. K. Kuo, J. Pantaleone, Neutrino oscillations in matter, Rev. Mod. Phys. 61 (1989) 937–979. doi:10.1103/RevModPhys.61.937. URL http://www.nikhef.nl/~h84/matterosc.pdf
[2] E. K. Akhmedov, Parametric resonance in neutrino oscillations in matter, Pramana 54 (2000) 47. URL http://arxiv.org/abs/hep-ph/9907435
[3] C. A. Argüelles, J. Kopp, Sterile neutrinos and indirect dark matter searches in IceCube, JCAP 1207 (2012) 016. doi:10.1088/1475-7516/2012/07/016. URL http://arxiv.org/abs/1202.3431
[4] D. Ascher, P. F. Dubois, K. Hinsen, J. Hugunin, T. Oliphant, et al., NumPy: Scientific computing with Python (1995–). URL http://www.numpy.org/
[5] E. Jones, T. Oliphant, P. Peterson, et al., SciPy: Open source scientific tools for Python (2001–). URL http://www.scipy.org/
[6] P. N. Brown, G. D. Byrne, A. C. Hindmarsh, Vode: a variable-coefficient ode solver, SIAM J. Sci. Stat. Comput. 10 (5) (1989) 1038–1051. doi:10.1137/0910062. URL http://dx.doi.org/10.1137/0910062
[7] M. Wallraff, nuCraft repository (2013–). URL http://nucraft.hepforge.org/
[8] A. M. Dziewonski, D. L. Anderson, Preliminary reference earth model, Physics of the Earth and Planetary Interiors 25 (4) (1981) 297–356. doi:10.1016/0031-9201(81)90046-7. URL http://inspirehep.net/record/18621
[9] T. K. Gaisser, T. Stanev, Path length distributions of atmospheric neutrinos, Phys. Rev. D 57 (1998) 1977–1982. doi:10.1103/PhysRevD.57.1977. URL http://arxiv.org/abs/astro-ph/9708146