GAUGE BOSON EXCHANGE IN AdS_{d+1}*

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ABSTRACT

We study the amplitude for exchange of massless gauge bosons between pairs of massive scalar fields in Anti-de Sitter space. In the AdS/CFT correspondence this amplitude describes the contribution of conserved flavor symmetry currents to 4-point functions of scalar operators in the boundary conformal theory. A concise, covariant, Y2K compatible derivation of the gauge boson propagator in AdS_{d+1} is given. Techniques are developed to calculate the two bulk integrals over AdS space leading to explicit expressions or convenient, simple integral representations for the amplitude. The amplitude contains leading power and sub-leading logarithmic singularities in the gauge boson channel and leading logarithms in the crossed channel. The new methods of this paper are expected to have other applications in the study of the Maldacena conjecture.

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I. Introduction

Many 2- and 3-point correlation functions have been calculated in studies of the Maldacena conjecture [1–3], and attention has recently been turned to 4-point correlators [4–8]. One of the goals is to obtain non-perturbative information about the large $N$, fixed large $g^2 N$ limit of the $\mathcal{N} = 4$ super–Yang-Mills theory with gauge group $SU(N)$. One important question is whether the theory has a simple $t$-channel OPE structure, so that 4-point functions have convergent expansions of the schematic form [9]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle = \sum_p \frac{\gamma_{13p}}{(x_1 - x_3)^{\Delta_1 + \Delta_3 - \Delta_p}} \frac{1}{(x_1 - x_2)^{2\Delta_p}} \frac{\gamma_{24p}}{(x_2 - x_4)^{\Delta_2 + \Delta_4 - \Delta_p}}$$  \hspace{1cm} (1.1)

containing the contribution of a finite number of primary operators (and descendents).

A related question is whether 4-point correlators in the large $N$ supergravity approximation are given by their free-field values as is the case for 3-point functions [10,11]. The discovery [7] of logarithmic singularities in some diagrams contributing to the correlator $\langle \mathcal{O}_\phi \mathcal{O}_c \mathcal{O}_\phi \mathcal{O}_c \rangle$ of operators corresponding to the bulk dilaton $\phi$ and axion $c$ fields suggests that the large $N$ limit is more complicated than the simple picture suggested by (1.1). However, a definite answer is not yet known because neither $\langle \mathcal{O}_\phi \mathcal{O}_c \mathcal{O}_\phi \mathcal{O}_c \rangle$ nor any other 4-point correlator has been completely calculated, and logarithms may cancel when all diagrams are included in the full amplitude.

Four-point correlators depend continuously on two conformal invariant variables. Thus they are inherently more complex than 2- and 3-point functions whose form is determined up to a small number (typically 1) of constants by conformal symmetry. The study of 4-point correlators from the AdS/CFT correspondence has been hampered by several difficulties.

1. Simple covariant expressions for the bulk-to-bulk gluon and graviton propagators in AdS$_{d+1}$ are not known.

2. The integrals required to compute diagrams containing a bulk-to-bulk propagator (even for a scalar) are difficult, and general techniques to evaluate them have not yet been developed.

3. The computation of realistic 4-point correlators in the $\mathcal{N} = 4$ super–Yang-Mills theory requires the specific values of cubic and quartic coupling in the Type IIB, $d = 10$ supergravity theory on AdS$_5 \times S^5$. Although the complete mass spectrum of this theory is known [12], there seems to be rather little information on the couplings. The cubic couplings of scalars corresponding to conformal primary operators are one happy exception [10].
In this paper we address the first two issues discussed above. In §2, we give a simple, covariant derivation of the gauge boson propagator. In §3, we consider the AdS integral for the gauge boson exchange contribution (see Fig. 1) to the 4-point function correlator $\langle O_\Delta(x_1)O_{\Delta'}(x_2)O_\Delta^*(x_3)O_{\Delta'}^*(x_4) \rangle$ of charged scalar operators of scale dimensions $\Delta$ and $\Delta'$. The required cubic coupling is determined by gauge symmetry. We develop expansion and resummation techniques to evaluate the two integrals over $\text{AdS}_{d+1}$ required to calculate the diagram in Fig. 1. This leads to a rather simple 2-parameter integral representation of the amplitude which we pursue toward explicit evaluation in §4. We establish that the correlator contains a leading logarithmic $s$-channel singularity in the variable $x_{12}$ (we use the notation $x_{ij} = x_i - x_j$) or $x_{34}$, and the same in the $u$-channel variables $x_{14}$ or $x_{23}$. In the $t$-channel we find the leading singularity $1/(x_{24})^{2\Delta' - d + 1}$ expected for a conserved current in the boundary theory, but we also find a non-leading logarithmic term

$$x_{24} \ln x_{24}. \quad (1.2)$$

The latter provides further evidence (see also [7]) that the claimed exact correspondence [6] between an AdS exchange diagram and the contribution of a single primary and its descendents in the OPE (1.1) is not completely correct.

Because of the problem of unknown couplings our new results for gauge boson exchange amplitude are still not enough to give a complete calculation of any 4-point correlators of the $\mathcal{N} = 4$ theory. It is possible that the logarithmic singularities of the gauge boson diagram cancel with those in other diagrams. Indeed we expect that the techniques developed here can be applied to the calculation of the integrals in other diagrams,* and we also hope that the approach used to obtain the gauge boson propagator can be extended to the more complicated case of the bulk-to-bulk propagator of the graviton.

![Figure 1](image)

* In a recent MIT seminar, H. Liu outlined a treatment of diagrams with scalar exchange based on the Mellin-Barnes representation of the hypergeometric function. This method appears complementary to ours.
2. The Gauge Boson Propagator

We work on the euclidean continuation of AdS$_{d+1}$ which is defined as the $Y_{-1} > 0$ sheet of the hyperboloid

$$-Y_{-1}^2 + \sum_{i=0}^{d} Y_i^2 = -r_0^2$$

(embedded in a $d+1$ dimensional space with metric of signature $(-+++\cdots +)$). The AdS scale $r_0$ is set to $r_0 = 1$ in the following. Defining $1/z_0 \equiv Y_{-1} + Y_0$ and $z_i \equiv z_0 Y_i \quad i = 1, \ldots, d$, gives a complete coordinate chart $z_\mu(Y)$ such that the induced metric on the hyperboloid takes the form (with $z_0 > 0$)

$$ds^2 = \sum_{\mu, \nu=0}^{d} g_{\mu\nu} dz_\mu dz_\nu = \frac{1}{z_0^2} (d\gamma_0^2 + \sum_{i=1}^{d} dz_i^2).$$

This is a constant curvature metric with Ricci tensor $R_{\mu\nu} = -d g_{\mu\nu}$. It is well known that scalar propagators are most simply expressed [13] as (hypergeometric) functions of the chordal distance $u$ between observation and source points on the hyperboloid. This can be written in terms of the coordinates $z_\mu(Y)$ and $w_\mu(Y')$ as

$$u \equiv \frac{1}{2}(Y - Y')^2 = \frac{(z - w)^2}{2z_0 w_0}$$

where $(z - w)^2 = \delta_{\mu\nu}(z - w)_\mu(z - w)_\nu$.

The previous literature on the gauge field propagator in constant curvature spacetimes concentrates on the positive curvature de Sitter space, although there is work, both old [14] and new [6], on AdS. In the first of these, the covariant propagator is obtained in Feynman gauge and involves transcendental functions.* These do not appear for the physical components of the field in our approach, which suggests that the method of [14] is too entwined with gauge artifacts. In [6] the Coulomb gauge is used to find a quite simple expression for the propagator. However this approach leads to an expression for the photon exchange amplitude which is not manifestly covariant under the isometries of AdS$_{d+1}$ (and appears to be non-local as well). Having criticized these earlier treatments, it must also be said that we build upon them. The emphasis of [14] on a simple basis of independent bitensors is important for us as is the observation of [6] that the transverse, spatial modes of the field have a simple scalar propagator.

* E.g., log $u$ for odd $d$ and arcsin $\sqrt{u}$ for even $d$. 
Let us begin with the action of an abelian gauge field coupled to a conserved current source in the AdS$_{d+1}$ background:

$$S_A = \int d^{d+1}z \sqrt{g} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\xi}{2} (D\mu A^\mu)^2 + A_\mu J^\mu \right].$$  \hfill (2.4)

In the gauge fixing term of $S_A$, $D\mu$ is the AdS-covariant derivative. The propagator is a bitensor $G_{\mu\nu}(z, w)$ which satisfies the AdS-covariant equation ($\partial_\mu = \frac{\partial}{\partial z^\mu}$)

$$D^\mu D_{[\mu} G_{\nu]} + \xi \partial_\nu (D^\mu G_{\mu\nu}) = \frac{g_{\nu\nu'}}{\sqrt{g}} \delta^{(d+1)}(z, w).$$  \hfill (2.5)

Any bitensor can be expressed [14] as a sum of two linearly independent forms with scalar coefficients. We shall choose as independent bitensors, two forms which are closely related to the biscalar variable $u$, namely (with $\partial_{\nu'} = \frac{\partial}{\partial w^{\nu'}}$)

$$\partial_\mu \partial_{\nu'} u = -\frac{1}{z_0 w_0} [\delta_{\mu\nu'} + \frac{1}{w_0} (z - w)_\mu \delta_{\nu'0} + \frac{1}{z_0} (w - z)_{\nu'} \delta_{\mu0} - u \delta_{\mu0} \delta_{\nu'0}]$$  \hfill (2.6)

and $\partial_\mu u \partial_{\nu'} u$ with

$$\partial_\mu u = \frac{1}{z_0} [(z - w)_{\mu} / w_0 - u \delta_{\mu0}]$$  \hfill (2.7)

$$\partial_{\nu'} u = \frac{1}{w_0} [(w - z)_{\nu'} / z_0 - u \delta_{\nu'0}].$$

One innovation of our approach is the Ansatz for the propagator

$$G_{\mu\nu'}(z, w) = - (\partial_\mu \partial_{\nu'} u) F(u) + \partial_\mu \partial_{\nu'} S(u)$$  \hfill (2.8)

where $F(u)$ and $S(u)$ are unknown scalar functions. This ansatz contains the two independent bitensors above. However $S(u)$ is clearly a gauge artifact which gives vanishing contribution when integrated against conserved currents in a Witten diagram. (We must also check that the surface term in the partial integration vanishes.) Thus we concentrate on the determination of $F(u)$, which turns out to be quite simple, and have only secondary concern for $S(u)$, just enough to be sure that the method is consistent.

To find $F(u)$, we write the equation of motion of (2.4) in full detail as

$$\partial_\mu [z_0^{-d+3} (\partial_\mu A_{\nu'} - \partial_{\nu'} A_\mu)] + \xi \partial_{\nu'} [z_0^{-d+3} \partial_{\nu'} A_{\mu} - (d - 1) z_0^{-d+2} A_0]$$

$$+ (d - 1) \xi \delta_{\nu'0} [z_0^{-d+2} \partial_{\mu} A_{\mu} - (d - 1) z_0^{-d+1} A_0] = z_0^{-d+1} J_\nu.$$  \hfill (2.9)

Consider the transverse spatial components

$$A_i^\perp \equiv (\delta_{ij} - \partial_i \frac{1}{\sqrt{2}} \partial_j) A_j$$  \hfill (2.10)
for which we use the temporary non-covariant notation $i \equiv 1, \ldots, d$ with flat $\nabla^2$. They satisfy the gauge-independent equation

$$
(\partial_0^2 + \nabla^2) A_i^\perp - (d - 3) \frac{1}{z_0} \partial_0 A_i^\perp = \frac{1}{z_0^2} J_i^\perp.
$$

(2.11)

As observed in the Coulomb gauge treatment of [6], this means that $z_0 A_i^\perp(z)$ satisfies the same equation as a scalar field of mass $m^2 = -(d - 1)$. The solution of (2.11) can then be written as

$$
z_0 A_i^\perp(z) = \int \frac{d^{d+1}w}{w_0^{d+1}} G(u(z, w)) w_0 J_i^\perp(w)
$$

(2.12)

where $G(u)$ is the scalar Green’s function, which satisfies

$$
D^\mu \partial_\mu G + (d - 1) G = \frac{1}{\sqrt{g}} \delta^{(d+1)}(z, w).
$$

(2.13)

Let us compare (2.12) with the covariant solution of (2.9), namely

$$
A_\mu(z) = \int \frac{d^{d+1}w}{w_0^{d+1}} G_{\mu\nu'}(z, w) J^{\nu'}(w)
$$

(2.14)

We insert the representation (2.8), and observe that the $\partial_\mu \partial_\nu S$ term vanishes since the current is conserved. We apply the transverse spatial projector (2.10) to both sides of (2.14), note that

$$
\frac{\partial}{\partial z_i} F(u) = - \frac{\partial}{\partial w_i} F(u)
$$

as a consequence of the translation invariance of (2.9), and use

$$
\frac{(z - w)_i}{w_0} F(u) = \frac{\partial}{\partial z_i} \int^u dv F(v).
$$

(2.15)

The result is an equation for $z_0 A_i^\perp(z)$ which is compatible with (2.12) provided we identify $F(u) \equiv G(u)$. Thus the physical part of the covariant propagator is, for any value of $\xi$, the scalar propagator of (2.13).

To find the explicit solution of (2.13), one can refer to the older AdS literature [13] (see also [4]), but we shall proceed here in a self-contained way because we will need similar techniques for other purposes below. Typically one needs to use hypergeometric transformation formulae to convert the results in the literature to the simple explicit form we will find.
We wish to express (2.13) as a differential equation in the variable \( u \). For this we need certain properties of derivatives of \( u \), which we now simply list:

\[
\begin{align*}
\Box u &= D^\mu \partial_\mu u = (d + 1)(1 + u) \\
g^{\mu\nu} \partial_\mu u \partial_\nu u &= u(2 + u) \\
D_\mu \partial_\nu u &= g_{\mu\nu}(1 + u) \\
(D^\mu u) (D_\mu \partial_\nu \partial_\nu u) &= \partial_\nu u \partial_\nu u.
\end{align*}
\]

These properties, some of which are not required immediately, can be derived with sufficient faith, perseverance, and Christoffel symbols. Using (2.16) one finds that

\[
\begin{align*}
D^\mu \partial_\mu F(u) &= g^{\mu\nu} \partial_\mu u \partial_\nu u F''(u) + \Box u F'(u) \\
&= u(2 + u) F''(u) + (d + 1)(1 + u) F'(u).
\end{align*}
\]

One can now check that the solution of (2.13) with the fastest decay on the AdS boundary \((u \to \infty)\) is

\[
F(u) = \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}} \frac{1}{[u(u + 2)]^{(d-1)/2}}
\]

where the normalization is determined [13] by matching to the short distance behavior in flat \((d + 1)\)-dimensional space.

The overall consistency of this approach remains to be checked, and that is done by substituting the ansatz (2.8) into (2.5). It is best to do this in two stages, so we first try to find a choice of gauge parameter \( \xi \) consistent with \( S(u) \equiv 0 \). Substitution of the \( F(u) \) term of (2.8) into (2.5) leads to the equation (for separated points \( z_\mu \neq w_\mu \))

\[
\begin{align*}
-(\partial_\nu \partial_\nu u) D^\mu \partial_\mu F + (\partial_\mu \partial_\nu u) D^\mu \partial_\mu F - (D^\mu \partial_\nu \partial_\nu u) D^\mu F \\
+(D^\mu \partial_\nu \partial_\nu u) F - \xi[(\partial_\nu D^\mu \partial_\mu \partial_\nu u) F + (D^\mu \partial_\mu \partial_\nu u) \partial_\nu F] \\
+(D_\nu \partial_\mu \partial_\nu \partial_\nu u) D^\mu F + (D^\mu \partial_\nu u) D_\mu \partial_\nu F] = 0.
\end{align*}
\]

This can be simplified using (2.16) and (2.17) to give

\[
\begin{align*}
-(\partial_\nu \partial_\nu u) [D^\mu \partial_\mu F + \xi (d + 1) F + (\xi - 1)(1 + u) F'] \\
- (\partial_\nu u \partial_\nu u) [(\xi (d + 2) - d) F' + (\xi - 1)(1 + u) F''] = 0.
\end{align*}
\]

The coefficients of the two independent bitensors must each vanish. For the \( \partial_\nu \partial_\nu u \) term, we can see that there is no choice of \( \xi \) for which \( F(u) \) obeys (2.13) as already established. Thus there is no value of the gauge parameter for which \( S(u) \) vanishes. So we proceed to
the second stage in which we substitute the \( S(u) \) term of (2.8) in (2.5). This leads, after similar algebra, to the superposition of bitensors

\[
\xi (\partial _\nu \partial _\nu' u)[D^\mu \partial _\mu S' + 2(1 + u)S'' + (d + 1)S'''] + \xi \partial _\nu u \partial _\nu' u[u(2 + u)S''' + (d + 5)(1 + u)S''' + 2(d + 2)S''']
\]  

(2.21)

which must be added to (2.20).

We now choose Feynman gauge \( \xi = 1 \) to simplify the equations. Looking at the combined \( \partial _\nu \partial _\nu' u \) term, we impose an inhomogeneous condition on \( S \) which will give the desired equation (2.13) for \( F \), namely

\[
\Box S' + 2(1 + u)S'' + (d + 1)S' = 2F
\]

(2.22)

or

\[
u(2 + u)S''' + (d + 3)(1 + u)S'' + (d + 1)S' = 2F
\]

(2.23)

after use of (2.16). The combined \( \partial _\nu u \partial _\nu' u \) bitensor term of (2.20) and (2.21) imposes another relation between \( S \) and \( F \) which must be compatible with (2.23), and one can check by direct differentiation of (2.23) that this is the case.

Our program is now logically complete. Eq. (2.22) is an inhomogeneous hypergeometric equation which is straightforward to solve. We shall be content to check two details of the solution. First we note that as \( u \to 0 \), the inhomogeneous solution of (2.22) behaves like \( S'(u) \sim 1/u^{(d-3)/2} \). This is less singular than \( F(u) \), so the gauge term of (2.8) does not affect the short distance behavior of the propagator. As \( u \to \infty \), one has, analogously, \( S'(u) \sim 1/u^{d-1} \). We will need this information in the next section to check that possible surface terms from \( S(u) \) actually vanish.

3. Gauge Boson Exchange Integrals

We assume a schematic form of the AdS/CFT correspondence in which a charged bulk scalar field \( \varphi_\Delta(z) \) of mass \( m \) is the source for the scalar operator \( O_\Delta(x) \) in the boundary theory, and \( \Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2}) \). Since we are interested only in the gauge boson exchange process we can restrict our attention to the following terms of the bulk action, namely

\[
S_\varphi = \int d^{d+1}z \sqrt{g} \left[ g^{\mu\nu}D_\mu \varphi_\Delta^* D_\nu \varphi_\Delta + m^2 \varphi_\Delta^* \varphi_\Delta \right] + (\Delta \to \Delta')
\]

(3.1)

with \( D_\mu = \partial_\mu + iA_\mu(z) \), plus \( S_A \) of (2.4). The gauge group of the bulk theory is \( U(1) \), but results for non-abelian groups can easily be attained from ours by supplying suitable group theory factors.
a. Ingredients of the Witten diagram of Fig. 1

(1) The bulk-to-boundary propagators [3]

\[ K_{\Delta}(z_0, \bar{z}, \bar{x}) = C_{\Delta} \left( \frac{z_0}{z_0^2 + (\bar{z} - \bar{x})^2} \right)^\Delta \]  

with [15]

\[ C_{\Delta} = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \text{ for } \Delta > d/2, \quad C_{d/2} = \frac{\Gamma(d/2)}{2\pi^{d/2}} \]  

(3.2)

(2) The bulk-to-bulk photon propagator defined in (2.8) and (2.18).

(3) An anti-symmetric derivative at each \( A_\mu \bar{\varphi} \leftrightarrow \partial_\mu \varphi \) vertex.

We factor out the normalization constants and define the amplitude to be studied by

\[ \langle O_{\Delta}(x_1)O_{\Delta'}(x_2)O_{\Delta}(x_3)O_{\Delta'}(x_4) \rangle_{\text{gauge}} = -\frac{\Gamma(d+1)}{4\pi(d+1)/2} C_{\Delta}^2 C_{\Delta'}^2 A(x_1, x_2, x_3, x_4) \]  

with

\[ A(x_i) = \int d^{d+1}z \frac{z_0}{z_0^{d-1} w_0^{d-1}} \left( \frac{z_0}{(z-x_1)^2} \right)^\Delta \partial_\mu \left( \frac{z_0}{(z-x_3)^2} \right)^\Delta' \partial_{\nu'} \left( \frac{w_0}{(w-x_4)^2} \right)^\Delta' \]  

\[ \text{where } \partial_\mu = \frac{\partial}{\partial x_\mu} \text{ and } \partial_{\nu'} = \frac{\partial}{\partial w_{\nu'}}. \]  

(3.5)

In (3.5), we have dropped the \( \partial_\mu \partial_{\nu'} S(u) \) gauge term in the gauge boson propagator (2.8) because the bulk contribution vanishes upon partial integration due to current conservation. However there is also a surface term of the form

\[ \lim_{z_0 \to 0} \int d^dz z_0^{d+1} \frac{z_0}{\bar{z}_0^2 + (\bar{z} - \bar{x}_1)^2} \bar{\Delta} \partial_{\nu'} \left( \frac{z_0}{z_0^2 + (\bar{z} - \bar{x}_3)^2} \right)^\Delta \partial_{\nu'} S(u) \]  

(3.6)

which must be shown to vanish. To do this we use the long distance behavior of \( S'(u) \) established at the end of §2 and also (2.7). One can then see that (3.6) is a difference of two integrals of the form

\[ \lim_{z_0 \to 0} \frac{1}{(z_0^2 + (\bar{z} - \bar{x}_1)^2)^{\Delta+1}} \frac{1}{(z_0^2 + (z - x_3)^2)^\Delta} \frac{N_{\nu'}}{((z_0 - w_0)^2 + (\bar{z} - \bar{w})^2)^{d-1}} \]  

(3.7)

where \( N_{\nu'} \) has a non-vanishing smooth limit as \( z_0 \to 0 \). By standard analysis of the possible singularities of the integral in the limit, one can show that the limit vanishes if \( d > 2 \) and \( \Delta \geq d/2 \), as is our case.
To prepare the way to calculate the integrals in $A(x_1, x_2, x_3, x_4)$, we observe that the derivatives of the scalar propagators produce terms of the form

\[
(z - x_i)_\mu (\partial_\mu \partial_\nu u)(w - x_j)_{\nu'} = -\frac{1}{2z_0 w_0} \left[ (z - x_i)^2 + (w - x_j)^2 - (x_i - x_j)^2 \right]
\]  

(3.8)

where we have used (2.6).

The first step is to simplify the integral by setting $x_1 = 0$, and by changing integration variables using the inversion isometry of AdS, namely $z_\mu = z'_\mu / (z')^2$ and $w_\mu = w'_\mu / (w')^2$, with boundary points $x_2, x_3, x_4$ referred to their inverses by $x_i = x'_i / (x'_i)^2$. This method was clearly described in [15]. The current at the $z$ vertex becomes

\[
J_\mu(z, x_1 = 0, x_3) = \left( \frac{z_0}{z^2} \right)^{\Delta + \t} \partial_\mu \left( \frac{z_0}{(z - x_3)^2} \right)^{\Delta} = -2\Delta (z')^2 J_{\mu \nu} (z') (x'_3)^{2\Delta} (z_0)^{2\Delta} \frac{(z' - x'_3)_{\nu}}{(z' - x'_3)^{2(\Delta + 1)}}
\]

(3.9)

where $z'^2 J_{\mu \nu}(z') = z'^2 \delta_{\mu \nu} - 2z'_\mu z'_\nu$ is the conformal jacobian [15]. The variable $u$ is inversion invariant, see (2.3), and the contraction $J_{\mu \nu} g^{\mu \nu} \partial_\nu \partial_\nu u$ in (3.5) is invariant, so the jacobian factor $J_{\mu \nu}$ cancels. Similar remarks apply to jacobian factors in $w'$. The net result of the change of variables is

\[
A(x_1, x_2, x_3, x_4) = 2^d \Delta \Delta' |x'_3|^{2\Delta} |x'_2|^{2\Delta} |x'_4|^{2\Delta'} B(x_1, x_2, x_3, x_4)
\]

(3.10)

with

\[
B(x_i) = \int \frac{d^{d+1} w}{(w - x'_2)^{2\Delta'} (w - x'_3)^{2\Delta}} \cdot \int \frac{d^{d+1} z}{(z - x'_3)^{2\Delta + 1}} \cdot \frac{1}{[(z - w)^2(z - w^*)^2]^{(d-1)/2}} \cdot \left\{ \frac{(z - x'_3)^2 - (x'_3 - x'_2)^2}{(w - x'_2)^2} - \frac{(z - x'_3)^2 - (x'_3 - x'_4)^2}{(w - x'_4)^2} \right\}
\]

(3.11)

where we have dropped the primes on the integration variables $w, z$ and have used $2 + u = (z - w^*)^2 / 2z_0 w_0$ with $w^*_\mu = (-w_0, \bar{w})$.

b. Integrals over the interaction point $z$

We now study the $z$-integrals which take the form

\[
R_{\Delta, p} = \int \frac{d^{d+1} z}{[(z - w)^2(z - w^*)^2]^{(d+1)/2}} \frac{1}{(z - x'_3)^{2p}}
\]

(3.12)

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for the cases $p = \Delta, \Delta + 1$. Our strategy is to use a power series expansion in the product $z_0 w_0$ from the gauge boson propagator denominators, to integrate, and then to resum the series. We start with

$$
\frac{1}{[(z - w)^2(z - w^*)^2]^{(d-1)/2}} = \frac{1}{[z_0^2 + w_0^2 + (\bar{z} - \bar{w})^2]^{(d-1)}} \frac{1}{(1 - Y)^{(d-1)/2}}
$$

$$
= \frac{1}{[z_0^2 + w_0^2 + (\bar{z} - \bar{w})^2]^{(d-1)}} \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{d-1}{2})}{\Gamma\left(\frac{d-1}{2}\right) k!} Y^{2k}
$$

(3.13)

with $Y \equiv 2z_0 w_0 / (z_0^2 + w_0^2 + (\bar{z} - \bar{w})^2)$. Note that $Y \leq 1$. We then have the convergent series expansion

$$
R_{\Delta,p} = \sum_{k=0}^{\infty} \frac{\Gamma\left(k + \frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right) k!} (4w_0^2)^k R_k
$$

(3.14)

with the integrals

$$
R_k = \int_0^\infty dz_0 z_0^{2(\Delta + k) - 1} \int d^dz \frac{1}{[w_0^2 + z_0^2 + (\bar{z} - \bar{w})^2]^{d-1+2k}} (z_0^2 + (\bar{z} - \bar{w})^2)^p
$$

(3.15)

The spatial integral in (3.15) can be done by introducing one Feynman parameter $\alpha$, with the result

$$
\frac{\pi^{d/2} \Gamma(\sigma)}{\Gamma(d - 1 + 2k) \Gamma(p)} \int_0^1 d\alpha \frac{\alpha^{d/2+2k}(1 - \alpha)^{p-1}}{(w_0^2 + \alpha(1 - \alpha)(\bar{w} - \bar{w}_3)^2 + \alpha w_0^2)^\sigma}
$$

(3.16)

where we use the abbreviation $\sigma = 2k + p - 1 + d/2$. The $z_0$ integral is then straightforward,

$$
\int_0^\infty \frac{dz_0 z_0^{2(\Delta + k) - 1}}{[z_0^2 + \alpha(1 - \alpha)(\bar{w} - \bar{x}_3)^2 + \alpha w_0^2]^\sigma} = \frac{\Gamma(\Delta + k) \Gamma(\sigma - k) \alpha^{k+\Delta - \sigma}}{2\Gamma(\sigma) [w_0^2 + (1 - \alpha)(\bar{w} - \bar{x}_3)^2]^{\sigma - \Delta - k}}
$$

(3.17)

Putting things together and applying the doubling formula to $\Gamma(d - 1 + 2k)$, we finally obtain

$$
R_{\Delta,p} = \frac{\pi^{(d+1)/2}}{2^{d-1} \Gamma(p) \Gamma\left(\frac{d-1}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma(\Delta + k) \Gamma\left(\frac{d}{2} + k + p - \Delta - 1\right) w_0^{2k}}{\Gamma\left(k + \frac{d}{2}\right) k!}
$$

$$
\cdot \int_0^1 d\alpha \frac{\alpha^{d/2+k+\Delta-p-1}(1 - \alpha)^{p-1}}{[w_0^2 + (1 - \alpha)(\bar{w} - \bar{x}_3)^2]^{\frac{d}{2}+k-1+p-\Delta}}
$$

(3.18)

The resummation of the series is simplest in the case $p = \Delta + 1$. There is a cancellation of $\Gamma$ functions, and we can recognize the binomial series

$$
\sum_{k=0}^{\infty} \frac{\Gamma(\Delta + k)}{\Gamma(\Delta) k!} x^k = \frac{1}{(1 - x)^\Delta}
$$

(3.19)
in the variable \( x = \alpha w_0^2/[w_0^2 + (1 - \alpha)(\bar{w} - \bar{x}_3')^2] \). We thus obtain the exact formula

\[
R_{\Delta,\Delta+1} = \frac{\pi^{(d+1)/2}}{2^{d-1} \Gamma \left( \frac{d-1}{2} \right)} \frac{1}{[w_0^2 + (\bar{w} - \bar{x}_3')^2]^\Delta} \cdot \int_0^1 d\alpha \alpha^{\frac{d}{2} - 2} [w_0^2 + (1 - \alpha)(\bar{w} - \bar{x}_3')^2]^{\Delta - d/2}
\]  

(3.20)

Direct resummation of the series for \( R_{\Delta,\Delta} \) would be more difficult, but it can be avoided if we first integrate by parts term-by-term using \( (1 - \alpha)^{\Delta - 1} = -(1/\Delta) \frac{d}{d\alpha} (1 - \alpha)^{\Delta} \). We then quickly derive the general relation

\[
R_{\Delta,\Delta} = [w_0^2 + (\bar{w} - \bar{x}_3')^2] R_{\Delta,\Delta+1}.
\]  

(3.21)

Note that when \( d \) is even and \( \Delta \) an integer satisfying the unitarity bound \( \Delta \geq d/2 \), the integrand of (3.20) is just a simple polynomial in \( \alpha \) and the integral can be evaluated as a finite sum of elementary terms.

\[
R_{\Delta,\Delta+1} = \frac{\pi^{d+1/2}}{2^{d-1} \Gamma \left( \frac{d-1}{2} \right)} \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\Gamma(\Delta + 1)} \frac{1}{(w_0^2 + (\bar{w} - \bar{x}_3')^2)^{d/2}} \cdot \sum_{\ell=0}^{\Delta - \frac{d}{2}} \frac{\Gamma(\ell + \frac{d}{2} - 1)}{\ell!} \left( \frac{w_\ell^2}{w_0^2 + (\bar{w} - \bar{x}_3')^2} \right)^\ell
\]  

(3.22)

The restriction to even \( d \) and integer \( \Delta \) includes the \( d = 4, \mathcal{N} = 4 \) super–Yang-Mills theory which has chiral primary operators of integer dimension. Our current progress may be summarized by inserting (3.21) and (3.22) into (3.11) which now reads

\[
B(x_i) = \int \frac{d^{d+1}w}{(w - x_2')^{2\Delta'}(w - x_4')^{2\Delta'}} \left[ \frac{(w - x_3')^2 - (x_3' - x_2')^2}{(w - x_2')^2} - \frac{(w - x_3')^2 - (x_3' - x_4')^2}{(w - x_4')^2} \right]
\]  

(3.23)

It remains to carry out the integrals over the interaction point \( w \).

c. Integrals over the interaction point \( w \)

We see that the remaining integrals over \( w \) are of the form

\[
S_{k}(^\ell) = \int \frac{d^{d+1}w}{(w - x')^{2\Delta'}(w - y)^{2(\Delta' + 1)}} \frac{1}{(w^2)^{d/2 - 2 + k}} \left( \frac{w_\ell^2}{w_0^2} \right)^\ell
\]  

(3.24)

for the two cases \( k = 1 \) and \( k = 2 \), and with \( x \equiv x_4' - x_3', \ y \equiv x_2' - x_3' \) minus the reverse assignment of \( x, y \) in (3.23). Specifically, we have

\[
B(x_i) = \frac{\pi^{d+1/2}}{2^{d-1} \Gamma \left( \frac{d-1}{2} \right)} \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\Gamma(\Delta + 1)} \sum_{\ell=0}^{\Delta - \frac{d}{2}} \frac{\Gamma(\ell + \frac{d}{2} - 1)}{\ell!} S_{k}(^\ell)
\]  

(3.25)

\[
S(\ell) = S_{1}(^\ell) - y^2 S_{2}(^\ell) - (x \leftrightarrow y)
\]  

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We now perform three consecutive changes of variables: (a) combine $w^2$ and $(w-x)^2$ denominators with Feynman parameter $\alpha$, (b) combine the composite denominator from the previous step with the $(w-y)^2$ denominator using Feynman parameter $\beta$, (c) carry out the $d^d w$ integral, (d) do $dw_0$ integral as in (3.16). We suppress details and directly give the result

$$S_k^{(\ell)} = \frac{\pi^{d/2} \Gamma(\Delta' + \ell) \Gamma(\Delta' + k - 1)}{2 \Gamma(\ell + k + d/2 - 2) \Gamma(\Delta') \Gamma(\Delta' + 1)} \cdot \int_0^1 d\alpha \int_0^1 d\beta \frac{\alpha^{\Delta' - 1}(1 - \alpha)^{\ell + \frac{d}{2} + k - 2} \beta^{\Delta'} (1 - \beta)^{\ell + \frac{d}{2} - 2}}{[\beta(y - \alpha x)^2 + \alpha(1 - \alpha)x^2]^{\Delta' + k - 1}}. \quad (3.26)$$

The above representation is not well-defined when $\ell + d/2 - 1 = 0$. However, we shall be most interested in the cases with $d \geq 3$, where this special configuration does not occur, and we shall henceforth assume that $d \geq 3$, so that $\ell + d/2 - 1 > 0$.

4. **Combining all integrals**

An improved integral representation may be obtained in which the four $S_k^{(\ell)}$ contributions to $S^{(\ell)}$ are combined in a single expression which directly exhibits the natural anti-symmetry of the amplitude under the interchange of $x$ and $y$. This new form will prove very useful in obtaining logarithmic and power-singular contributions to the amplitude in various channels, as well as to obtaining a complete OPE expansion of the amplitude.

First, we carry out a change of variables: $\beta = 1/(1 + \xi)$, so that

$$\int_0^1 d\beta \frac{\beta^{\Delta'} (1 - \beta)^{\ell + \frac{d}{2} - 2}}{[\beta(y - \alpha x)^2 + \alpha(1 - \alpha)x^2]^{\Delta' + k - 1}} = \int_0^\infty d\xi \frac{\xi^{\ell + \frac{d}{2} - 2}(1 + \xi)^{-\ell - \frac{d}{2} + k - 1}}{[y^2 - 2\alpha x \cdot y + \alpha x^2 + \xi \alpha(1 - \alpha)x^2]^{\Delta' + k - 1}}. \quad (3.27)$$

The exponents in the denominators for $k = 1, 2$ differ by one, but we may render these identical by integrating by parts in the variable $\xi$ for the $k = 1$ integral and using

$$\frac{d}{d\xi} \frac{\xi^{\ell + \frac{d}{2} - 1}}{(1 + \xi)^{\ell + \frac{d}{2} - 1}} = (\ell + \frac{d}{2} - 1) \frac{\xi^{\ell + \frac{d}{2} - 2}}{(1 + \xi)^{\ell + \frac{d}{2}}}. \quad (3.28)$$

It is now easy to combine the $k = 1, 2$ contributions, and we obtain

$$S_1^{(\ell)} - y^2 S_2^{(\ell)} = \frac{\pi^{d/2} \Gamma(\Delta' + \ell)}{2 \Gamma(\ell + d/2) \Gamma(\Delta')} \cdot \int_0^1 d\alpha \int_0^\infty d\xi \frac{\alpha^{\Delta' - 1}(1 - \alpha)^{\ell + \frac{d}{2} - 1} \xi^{\ell + \frac{d}{2} - 2}(1 + \xi)^{-\ell - \frac{d}{2} + 1}(\alpha \xi x^2 - y^2)}{[y^2 - 2\alpha x \cdot y + \alpha x^2 + \xi \alpha(1 - \alpha)x^2]^{\Delta' + 1}}. \quad (3.29)$$

We now perform three consecutive changes of variables:
(a) let \( \eta \equiv \xi \alpha (1 - \alpha) \);
(b) let \( \alpha \equiv \frac{1}{1 + u} \),
(c) let \( v \equiv \eta (1 + u) \).

We find that the combination \( S_1^{(\ell)} - y^2 S_2^{(\ell)} \) now automatically changes sign under the interchange of \( x \) and \( y \), so that we have \( S^{(\ell)} = 2(S_1^{(\ell)} - y^2 S_2^{(\ell)}) \). The final result is the following simple integral representation

\[
S^{(\ell)} = \frac{\pi^{d/2} \Gamma(\Delta' + \ell)}{\Gamma(\ell + \frac{d}{2}) \Gamma(\Delta')} \int_0^{\infty} du \int_0^{\infty} dv \frac{(uv)^{\ell + \frac{d}{2} - 2}}{(u + v + uv)^{\ell + \frac{d}{2} - 1}} \cdot \frac{vx^2 - uy^2}{[uy^2 + (x - y)^2 + vx^2]^\Delta'}.
\]

This double integral representation is a convenient starting point for the study of the amplitude in various limits, as will be carried out in the next section.

4. Explicit formulas and Singularity Structure

Combining all results of the preceding section for the amplitude \( A(x_i) \), we may express the reduced form \( B(x_i) \), as defined in (3.10), in terms of the following conformal invariants

\[
s \equiv \frac{1}{2} \frac{(x - y)^2}{x^2 + y^2} \quad \text{and} \quad t \equiv \frac{x^2 - y^2}{x^2 + y^2}.
\]

In terms of the original position coordinates \( x_i \) of the amplitude, these variables take the form

\[
s = \frac{1}{2} \frac{x_{12}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad \text{and} \quad t = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}.
\]

As \( x \) and \( y \) vary, we have \( 0 \leq s \leq 1 \) and \( -1 \leq t \leq 1 \). It is convenient to express \( B \) as follows

\[
B(x_i) = \frac{\pi^{d+\frac{1}{2}} \Gamma(\Delta - \frac{d}{2} + 1)}{2^{d-1} \Gamma(\Delta') \Gamma(\frac{d}{2} - \frac{1}{2}) \Gamma(\Delta + 1) (x^2 + y^2)^\Delta'} \sum_{\ell=0}^{\Delta - \frac{d}{2}} \frac{\Gamma(\Delta' + \ell) \Gamma(\ell + \frac{d}{2} - 1)}{\Gamma(\ell + \frac{d}{2}) \ell!} I^{(\ell)}
\]

where \( I^{(\ell)} \) is given in terms of \( s \) and \( t \) only:

\[
I^{(\ell)} = \int_0^{\infty} du \int_0^{\infty} dv \frac{(uv)^{\ell + \frac{d}{2} - 2}}{(u + v + uv)^{\ell + \frac{d}{2} - 1}} \cdot \frac{\frac{1}{2} (u + v) t + \frac{1}{2} (v - u)}{[\frac{1}{2} (u + v) + \frac{1}{2} (v - u) t + 2s]^\Delta' + 1}.
\]
easily, it is convenient to perform one penultimate change of variables: \( u = 2\rho(1 - \lambda) \) and 
\( v = 2\rho(1 + \lambda) \), so that

\[
I(\ell) = 2^{1-\Delta'} \int_0^\infty d\rho \int_{-1}^{+1} d\lambda \, \frac{\rho^{\ell + \frac{d}{2} - 1}(1 - \lambda^2)^{\ell + \frac{d}{2} - 2}}{[1 + \rho(1 - \lambda^2)]^{\ell + \frac{d}{2} - 1}} \cdot \frac{\lambda + t}{[\rho + \rho \lambda t + s]^\Delta'+1}. \tag{4.4}
\]

We now make use of the elementary formula

\[
\frac{1}{(s+a)^k} = \frac{(-)^k}{\Gamma(k)} \left( \frac{\partial}{\partial s} \right)^{k-2} \frac{1}{(s+a)^2} = \frac{(-)^{k+1}}{\Gamma(k)} \left( \frac{\partial}{\partial s} \right)^{k-1} \frac{1}{s+a} \tag{4.5}
\]

and obtain for \( k = \Delta' + 1 \)

\[
I(\ell) = \frac{(-2)^{1-\Delta'}}{\Gamma(\Delta'+1)} \left( \frac{\partial}{\partial s} \right)^{\Delta'-1} \int_0^\infty d\rho \int_{-1}^{+1} d\lambda \, \frac{\rho^{\ell + \frac{d}{2} - 1}(1 - \lambda^2)^{\ell + \frac{d}{2} - 2}}{[1 + \rho(1 - \lambda^2)]^{\ell + \frac{d}{2} - 1}} \cdot \frac{\lambda + t}{[\rho + \rho \lambda t + s]^2}. \tag{4.6}
\]

Letting now \( \mu \equiv s/\rho \), and noticing that by using (4.5) again, we have

\[
\frac{(1 - \lambda^2)^{\ell + \frac{d}{2} - 2}}{[\mu + s(1 - \lambda^2)]^{\ell + \frac{d}{2} - 1}} = \frac{(-)^{\ell + \frac{d}{2}}}{\Gamma(\ell + \frac{d}{2} - 1)} \left( \frac{\partial}{\partial s} \right)^{\ell + \frac{d}{2} - 2} \frac{1}{\mu + s(1 - \lambda^2)} \tag{4.7}
\]

we obtain our final formula for the amplitudes, in which all \( I^{(\ell)} \) are expressed as derivatives of a single universal function \( I \), which is independent of \( d, \Delta' \) and \( \ell \).

\[
I^{(\ell)} = \frac{2^{1-\Delta'}(-)^{\ell + \frac{d}{2} + \Delta' - 1}}{\Gamma(\Delta'+1)\Gamma(\ell + \frac{d}{2} - 1)} \left( \frac{\partial}{\partial s} \right)^{\Delta'-1} \left\{ s^{\ell + \frac{d}{2} - 2} \left( \frac{\partial}{\partial s} \right)^{\ell + \frac{d}{2} - 2} I(s,t) \right\}. \tag{4.8}
\]

\[
I(s,t) = \int_0^\infty d\mu \int_{-1}^{+1} d\lambda \, \frac{1}{\mu + s(1 - \lambda^2)} \cdot \frac{\lambda + t}{(\mu + \lambda t + 1)^2}. \tag{4.8}
\]

The \( \mu \)-integral is elementary and may be carried out right away. One obtains

\[
I(s,t) = \int_{-1}^{+1} d\lambda \frac{\lambda + t}{[1 + \lambda t - s(1 - \lambda^2)]^2} \ln \left( \frac{1 + \lambda t}{s(1 - \lambda^2)} \right) - \int_{-1}^{+1} d\lambda \frac{\lambda + t}{1 + \lambda t - s(1 - \lambda^2)} \cdot \frac{1}{1 + \lambda t}. \tag{4.9}
\]

The \( \lambda \) integration of the second term and the term proportional to \( \ln s \) in the first term are still elementary, but the other contributions in the first term involve dilog or Spence functions and are not elementary.
\textit{a. The expansion in the direct channel: } x_2 - x_4 \to 0 \\

The singularities and OPE expansion in the gauge boson channel are obtained from an expansion for small \( x'_2 - x'_4 \), which corresponds to \( x - y \) small, and thus to \( s \) and \( t \), defined in (4.1), tending to 0. This systematic expansion may be directly read off from the integral representation in (4.9), upon series expansion in powers of \( s \) and \( t \). Now, for \( d \geq 4 \) – which we shall henceforth restrict to – the power factor of \( s \) between the two derivative factors in (4.8) cannot cause a singularity. Thus, all non-analyticity of \( B \) must arise from the non-analytic part of \( I(s, t) \) as \( s \to 0 \), which we denote here by \( I_{\text{sing}}(s, t) \). It is instructive to set

\[ I(s, t) = I_{\text{sing}}(s, t) + I_{\text{reg}}(s, t) \quad (4.10) \]

where \( I_{\text{reg}}(s, t) \) now admits a Taylor series expansion in powers of \( s \).

We first examine the non-analytic part \( I_{\text{sing}}(s, t) \) which is proportional to the explicit \( \ln s \) dependence in (4.9). We give this part below as well as the full answer of the integral involved

\[ I_{\text{sing}}(s, t) = -\ln s \cdot \int_{-1}^{+1} d\lambda \frac{\lambda + t}{[1 + \lambda t - s(1 - \lambda^2)]^2} \]

\[ = -\ln s \cdot s^2(\lambda_+ - \lambda_-)^2 \ln \frac{(1 - \lambda_+)(1 + \lambda_+)(1 - \lambda_-)(1 + \lambda_-)}{2} \lambda_+ - 2 \frac{\lambda_+ + t}{1 - \lambda_+^2} - 2 \frac{\lambda_- + t}{1 - \lambda_-^2} \]

where \( \lambda_{\pm} \) are defined by

\[ \lambda_{\pm} = -\frac{t}{2s} \pm \sqrt{\frac{t^2}{4s^2} + 1 - \frac{1}{s}}. \quad (4.12) \]

A systematic expansion in powers of \( s \) is given by the following series

\[ I_{\text{sing}}(s, t) = \ln s \sum_{k=0}^{\infty} (k + 1) s^k a_k(t) \]

\[ a_k(t) = \int_{-1}^{+1} d\lambda \frac{(\lambda + t)(1 - \lambda^2)^k}{(1 + \lambda t)^{k+2}}. \quad (4.13) \]

The function \( a_k(t) \) is hypergeometric and admits the following series expansion in powers of \( t \)

\[ a_k(t) = 2^k \frac{\Gamma \left( \frac{k}{2} + 1 \right) \Gamma \left( \frac{k}{2} + \frac{3}{2} \right)}{\Gamma \left( k + \frac{3}{2} \right)} t F \left( \frac{k}{2} + 1, \frac{k}{2} + \frac{3}{2}; k + \frac{5}{2}; t^2 \right) \]

\[ = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{\Gamma (2m + k + 2)}{\Gamma (m + k + \frac{3}{2}) m!} \left( \frac{t}{2} \right)^{2m+1}. \quad (4.14) \]
This result may be used to derive the singular and logarithmic parts of $I^{(\ell)}$ as well as the associated analytic part. It is helpful to make use of the following formulas

$$
\left( \frac{\partial}{\partial s} \right)^\ell s^k = \frac{\Gamma(k + 1)}{\Gamma(k - \ell + 1)} s^{k-\ell}
$$

and

$$
\left( \frac{\partial}{\partial s} \right)^\ell \{s^k \ln s\} = \frac{\Gamma(k + 1)}{\Gamma(k - \ell + 1)} s^{k-\ell} \{\ln s + \psi(k + 1) - \psi(k - \ell + 1)\}
$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. We find

$$
I^{(\ell)}_{\text{sing}} = \frac{2^{1-\Delta'}(-)^{\ell+\frac{d}{2}+\Delta'-1}}{\Gamma(\Delta'+1)\Gamma(\ell+\frac{d}{2}-1)} \sum_{k=k_-}^{k_+} \frac{\Gamma(k + 1)\Gamma(k + 2)}{\Gamma(k - \ell - \frac{d}{2} + 3)\Gamma(k - \Delta' + 2)} \cdot s^{k-\Delta'+1}a_k(t) \{\ln s + 2\psi(k + 1) - \psi(k - \ell - \frac{d}{2} + 3) - \psi(k - \Delta' + 2)\}.
$$

This expression is still somewhat formal, since the $\Gamma$-functions in the denominator inside the sum may be infinite for sufficiently small $k$, while the $\psi$-functions may diverge there. To properly separate out this behavior, we introduce

$$
k_+ = \max\{\Delta' - 1, \ell + \frac{d}{2} - 2\}
$$

and

$$
k_- = \min\{\Delta' - 1, \ell + \frac{d}{2} - 2\}.
$$

When $k \geq k_+$, no divergences occur and the series has well-defined terms as it stands. For $k < k_-$, the divergences of both $\Gamma$ functions produces a double zero, while $\psi$-functions only produce a single pole, so that these contributions cancel. Thus, we have

$$
I^{(\ell)}_{\text{sing}} = -\frac{2^{1-\Delta'}(-)^{\ell+\frac{d}{2}+\Delta'-1}}{\Gamma(\Delta'+1)\Gamma(\ell+\frac{d}{2}-1)} \sum_{k=k_-}^{k_+} \frac{\Gamma(k + 1)\Gamma(k + 2)\psi(k + 1 - k_+)}{\Gamma(k - \ell - \frac{d}{2} + 3)\Gamma(k - \Delta' + 2)} \cdot s^{k-\Delta'+1}a_k(t)
$$

and

$$
+ \frac{2^{1-\Delta'}(-)^{\ell+\frac{d}{2}+\Delta'-1}}{\Gamma(\Delta'+1)\Gamma(\ell+\frac{d}{2}-1)} \sum_{k=k_-}^{k_+} \frac{\Gamma(k + 1)\Gamma(k + 2)}{\Gamma(k - \ell - \frac{d}{2} + 3)\Gamma(k - \Delta' + 2)} \cdot s^{k-\Delta'+1}a_k(t) \{\ln s + 2\psi(k + 1) - \psi(k - \ell - \frac{d}{2} + 3) - \psi(k - \Delta' + 2)\}.
$$

When $k_- \leq k \leq k_+ - 1$, the ratio $\psi(k + 1 - k_+)/\Gamma(k + 1 - k_+)$ is a finite number, and thus the first line of (4.18) is the most singular contribution to $I^{(\ell)}_{\text{sing}}$. The leading logarithmic contribution enters when $k = k_+$, and takes the form

$$
I^{(\ell)}_{\text{leading log}} = \frac{2^{1-\Delta'}(-)^{\ell+\frac{d}{2}+\Delta'-1}}{\Gamma(\Delta'+1)\Gamma(\ell+\frac{d}{2}-1)} \frac{\Gamma(k_+ + 1)\Gamma(k_+ + 2)}{\Gamma(k_+ - k_- + 1)} \cdot s^{k_+ - \Delta'+1} \ln s a_{k_+}(t).
$$
The regular part $I_{\text{reg}}(s, t)$ admits a similar expansion in powers of $s$, given as follows:

$$I_{\text{reg}}(s, t) = \sum_{k=0}^{\infty} s^k b_k(t)$$

The contribution of the analytic part may be inserted to obtain the regular contribution to $I^{(\ell)}$, just as we did for $I_{\text{sing}}$ above.

It is now straightforward, but somewhat tedious, to assemble all results and to extract the physical $t$-channel limit of the gauge boson exchange amplitude, namely the limit in which $x_{13}$ and $x_{24}$ are small compared to other coordinate differences such as $x_{12}$. In this limit the variable $t$ of (4.1b) can be rewritten as $t \sim -x_{13} \cdot J(x_{12}) \cdot x_{24}/x_{12}^2$, where $J_{\mu\nu}(y) = \delta_{\mu\nu} - 2y_\mu y_\nu/y^2$ is the well-known Jacobian of the conformal inversion. To incorporate the $1/(x^2 + y^2)^{\Delta'}$ factor in (4.2) we also need

$$x^2 = x_{34}^2/x_{14}^2 x_{13}^2 \sim 1/x_{13}^2$$
$$y^2 = x_{23}^2/x_{12}^2 x_{13}^2 \sim 1/x_{13}^2.$$  \hspace{1cm} (4.21)

Finally we need the conformal inversion prefactor in (3.10), which is

$$|x_3'|^{2\Delta} |x_2'|^{2\Delta'} |x_4'|^{2\Delta'} = 1/|x_{13}|^{2\Delta} |x_{12}|^{2\Delta'} |x_{14}|^{2\Delta'} \sim 1/|x_{13}|^{2\Delta} |x_{12}|^{4\Delta'}.  \hspace{1cm} (4.22)$$

The procedure is then to substitute $I_{\text{sing}}^{(\ell)}$ from (4.18) into (4.2) and then into (3.10) and finally (3.4). The extraction of a complete asymptotic series in this limit requires non-leading corrections to the kinematic equations above, so we will restrict our discussion to the power dependence of leading terms. The procedure above gives the leading power term

$$\langle O_\Delta(x_1) O_{\Delta'}(x_2) O_\Delta^*(x_3) O_{\Delta'}^*(x_4) \rangle_{\text{gauge}} \sim x_{13} \cdot J(x_{12}) \cdot x_{24}/|x_{13}|^{2\Delta-d+2} |x_{12}|^{2(d-1)} |x_{24}|^{2\Delta'-d+2}.  \hspace{1cm} (4.23)$$

The angular dependent numerator and the various exponents are exactly as expected for the contribution of a conserved current of dimension $d-1$ in the double OPE of the scalar operators in the correlator. Similarly, the leading logarithmic contribution is given by the product of (last eq) with the factor $s^{(\Delta' - d/2+1)} \ln s$. Note that this discussion has assumed that $\Delta' - 1 > d/2 + 1$ in (4.17) and that the leading terms come from $\ell = 0$. 

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b. Expansion in the crossed channel \( x_3 - x_4 \to 0 \)

A systematic expansion for \( x_3 - x_4 \to 0 \) may be derived in a way analogous to the expansion for \( x_2 - x_4 \to 0 \), carried out in §a. above. We begin by obtaining the leading term as \( x_3 - x_4 \to 0 \), and thus as \( x \to 0 \). Remarkably, the leading term is most conveniently obtained directly from the original Feynman parameter integrals in (3.25) and (3.26). As it was shown in (3.28) and (3.29) that 

\[
S(\ell) = 2(S_1(\ell) - y^2 S_2(\ell)),
\]

it suffices to analyze the asymptotics of \( S_1(\ell) \) and \( S_2(\ell) \) directly from (3.26). Concentrating on the \( \beta \)-integral as \( x \to 0 \) first, we see that \( S_2(\ell) \) is dominated by a \( \ln x^2 \) term, while \( S_1(\ell) \) is dominated by a term of order 1. Thus, the contribution of \( S_1(\ell) \) is negligible. Furthermore, using the asymptotics of the following integral

\[
\int_0^{1} d\beta \frac{\beta^\Delta}{(\beta + b)^{\Delta + 1}} = -\ln b + \mathcal{O}(1) \quad (4.24)
\]
as \( b \to 0 \), the \( \beta \)-integral in (3.26) for \( k = 2 \) becomes trivial, and the \( \alpha \)-integral reduces to a ratio of \( \Gamma \)-functions. One finds

\[
S(\ell) = \frac{\pi \frac{\Delta}{2} \Gamma(\Delta' + \ell)}{\Gamma(\ell + \frac{\Delta}{2} + \Delta')} \frac{1}{y^{2\Delta}} \ln \frac{x^2}{y^2}. \quad (4.25)
\]

This result means that for any \( \Delta, \Delta' \) and \( d \), the leading behavior of the amplitude in the crossed channel is given by a pure logarithm of \( x^2 \), with no power singularities.

A systematic expansion for \( x_{34} \to 0 \) to all orders may be obtained from the universal function \( I(s, t) \) in (4.8), just as for the case \( x_{24} \to 0 \). Clearly, from (4.1a) the limit \( x_{34} \to 0 \) corresponds to \( x \to 0 \) and thus implies that \( s \to \frac{1}{2} \) and \( t \to -1 \). To carry out the expansion, we consider the double integral representation of \( I(s, t) \) in (4.8) and perform two consecutive changes of variables

(a) \( \mu = \frac{1}{2}(1 - \lambda^2)(\eta - 1) \) with \( 1 \leq \eta \),

(b) \( \lambda = (\xi - 1)/(\xi + 1) \) with \( 0 \leq \xi \leq \infty \).

It is convenient to set \( t = -1 + 2\tau \) at intermediate stages of the calculation. The result is the following double integral representation

\[
I(s, t) = \int_0^{\infty} \frac{d\eta}{\eta - (1 - 2s)} \int_0^{\infty} d\xi \frac{\tau(\xi + 1)^2 - \xi - 1}{(\tau \xi^2 + \eta \xi + 1 - \tau)^2}. \quad (4.27)
\]

The \( \xi \)-integral is easily carried out, and equals

\[
\frac{2 - 4\tau}{\eta^2 - 4\tau(1 - \tau)} + \frac{\eta(1 - 2\tau)}{[\eta^2 - 4\tau(1 - \tau)]^{3/2}} \left\{ \ln \tau(1 - \tau) - 2\ln \left[ \frac{\eta}{2} + \frac{1}{2} \sqrt{\eta^2 - 4\tau(1 - \tau)} \right] \right\}. \quad (4.28)
\]
The expression for $I(s,t)$ in (4.27) manifestly admits a convergent Taylor series expansion in powers of $(2s - 1)$,

$$I(s,t) = \sum_{n=0}^{\infty} (1 - 2s)^n \{a_{\text{sing}}^{(n)}(t) + a_{\text{reg}}^{(n)}(t)\}. \quad (4.29)$$

In view of (4.28), the coefficients are given by

$$a_{\text{sing}}^{(n)}(t) = -t \ln \frac{1 - t^2}{4} \int_{1}^{\infty} d\eta \frac{\eta^{-n}}{[\eta^2 - (1 - t^2)]^{3/2}}. \quad (4.30)$$

$$a_{\text{reg}}^{(n)}(t) = 2t \int_{1}^{\infty} d\eta \frac{\eta^{-n} \ln \left[ \frac{\eta}{2} + \frac{1}{2} \sqrt{\eta^2 - (1 - t^2)} \right]}{[\eta^2 - (1 - t^2)]^{3/2}} - 2t \int_{1}^{\infty} d\eta \frac{\eta^{-n-1}}{\eta^2 - (1 - t^2)}. \quad (4.30)$$

The functions $a_{\text{sing}}^{(n)}(t)$ are proportional to a hypergeometric function, as well as to a logarithm of $1 - t^2$,

$$a_{\text{sing}}^{(n)}(t) = -t \ln \frac{1 - t^2}{4} \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{3}{2})}{\Gamma(\frac{3}{2}k)k!} \frac{(1 - t^2)^k}{2k + n + 2} \quad (4.31)$$

$$= -\frac{t}{2 + n} \ln \frac{1 - t^2}{4} F\left(\frac{3}{2}, \frac{n}{2} + 1; \frac{n}{2} + 2; 1 - t^2\right).$$

The functions $a_{\text{reg}}^{(n)}(t)$ admit a Taylor series expansion in powers of $1 - t^2$, given as follows

$$a_{\text{reg}}^{(n)}(t) = 2t \sum_{k=0}^{\infty} \frac{a_k}{2k + n + 2} (1 - t^2)^k \quad (4.32)$$

$$a_k = -1 + \frac{\Gamma(k + \frac{3}{2})}{\Gamma(\frac{3}{2}k)k!} \left\{ \frac{1}{2k + n + 2} - \frac{1}{\pi} \sum_{m=1}^{k} \frac{\Gamma(m + \frac{1}{2})\Gamma(k - m + \frac{3}{2})}{\Gamma(k - m + 1)\cdot m\cdot m!} \right\}.$$

It is understood that in the expression for $a_0$, the last sum term does not contribute. Using this expansion, one may now evaluate $I^{(\ell)}$, given by (4.8), and from there the reduced amplitude $B(x_i)$ using (4.2), and the full amplitude $A(x_i)$ using (3.10). These calculations are elementary and we shall not carry them out here. Suffice it to note that the leading logarithmic behavior identified in (4.25) and (4.26) is precisely recovered from the term proportional to $\ln(1 + t)$ of $I(s,t)$ in (4.29) and (4.31).

5. Conclusions and Outlook

In this paper we have studied the contribution of the gauge boson exchange diagram in $AdS_{d+1}$ to the boundary correlator $\langle O_{\Delta}(x_1)O_{\Delta'}(x_2)O^*_\Delta(x_3)O^*_{\Delta'}(x_4)\rangle_{\text{gauge}}$ of charged scalar
composite operators of arbitrary integer dimension. The calculation was feasible because we used a simple new form of the gauge boson propagator derived in §2 by a covariant method in which physical effects were separated from gauge artifacts. Although the results hold for general even boundary dimension $d$, we shall discuss them for the case of primary interest, namely $d = 4$, where the boundary conformal theory is $\mathcal{N} = 4$ super–Yang-Mills theory.

The gauge boson exchange process, for the gauge group $SU(4)$ of the bulk AdS$_5 \times S_5$ supergravity theory, is a fundamental sub-process in many 4-point correlators of interest, although it does not give the complete amplitude for any correlator. So a full analysis of 4-point functions must wait until we learn how to compute exchange diagrams with massive scalars and vectors and as well as the graviton and massive tensors. Explicit bulk-to-bulk propagators must be found for most of these cases. Once they are found we believe that the calculational methods developed here can be applied to calculate the bulk integrals. Explicit cubic and quartic couplings of the supergravity theory will also be required.

We have shown that the gauge boson exchange diagram contains logarithmic singularities of the type [7] previously found for 4-point contact interactions. The logarithmic singularities are the leading contribution in the $s$-channel limit, $|x_{12}| \to 0$, but they occur at non-leading level in the $t$-channel, $|x_{13}| \to 0$. The leading singularities in the $t$-channel are powers which correspond to a simple double OPE type 4-point function [9] in which the conserved $SU(4)$ flavor current of the boundary theory and its descendents appears in the intermediate state. However if logarithms do not cancel with those in other diagrams, complete 4-point correlators will be more complicated than the double OPE form in (1.1) (rewritten for the $t$-channel).

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