NEWTON-KANTOROVITCH METHOD FOR DECOUPLED
FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL
EQUATIONS

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ABSTRACT. We present and prove a Newton-Kantorovitch method for solving decoupled forward-backward stochastic differential equations (FBSDEs) involving smooth coefficients with uniformly bounded derivatives. As Newton’s method is required a suitable initial condition to converge, we show that such initial conditions are solutions of a linear backward stochastic differential equation. In addition, we show that converges linearly to the solution.

1. INTRODUCTION

In this paper, we study a method of approximating non-Markovian and decoupled forward-backward stochastic differential equations (FBSDEs) of the following form for arbitrary \( T > 0 \):

\[
\begin{align*}
X(t) &= X(0) + \int_0^t b(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s), \\
Y(t) &= \varphi(X(T)) + \int_t^T f(s, X(s), Y(s), Z(s)) \, ds - \int_t^T Z(s) \, dW(s),
\end{align*}
\]

where the solution triples \((X, Y, Z) \equiv (X(t), Y(t), Z(t))_{t \in [0,T]}\) take values in \( \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k}\), \(W\) is a \(k\)-dimensional Wiener process, and \(b, f, \sigma, \varphi\) are measurable functions that could in general be random and defined on a probability space.

We propose a scheme for approximating such FBSDEs with random coefficients based on applying the Newton-Kantorovitch theory. Unlike a four-step scheme that relies on the Markov structure (see e.g., Ma, Protter and Yong [13], and Delarue [6]), this method allows us to find a non-Markovian approximation. In addition, as our approach is a type of contraction mapping [3, 20], the approximation works for arbitrary large durations with smooth random coefficients, and with monotonicity conditions; for details, see Hu and Peng [9], Peng and Wu [21], and Yong [25]. We also refer readers to [27, 15] for more details about the uniqueness and the existence of FBSDE solutions.

Most numerical algorithms for FBSDEs are based on time-space discretization schemes [7] for quasi-linear parabolic partial differential equations for coupled FB-SDEs with monotonicity condition [5]. In contrast, to the best of our knowledge,
very little has been published on state space discretization based approaches. Vi-dossich proved that the Chaplygin and Newton methods are equivalent for ordinary differential equations [22], and Kawabata and Yamada studied a state space discretization based on Newton-Kantorovitch approach for stochastic differential equations (SDEs) [12]. Ouknine [17] showed that the coefficients can be relaxed by a linear growth condition. Amano obtained a probabilistic second-order error estimate [2]. Wrzosek [23] extended to stochastic functional differential equations.

The aim of this paper is to formulate Newton-Kantorovitch approximation of the decoupled multi-dimensional FBSDEs with random coefficients and to show that it converges globally (2) as follows:

**Theorem 1.1.** Suppose that \( b, \sigma, f, \) and \( \varphi \) are all \( C^1 \), their derivatives are uniformly bounded \( (s, \omega) \)-a.e., and \( b, \sigma, \) and \( f \) are square-integrable w.r.t. their time variables. Then, there exists a positive constant \( C > 0 \) such that

\[
\|(X - X_{n+1}, Y - Y_{n+1}, Z - Z_{n+1})\| \leq C2^{-n} \quad n \in \mathbb{N} \cup \{0\}.
\]

As pointed out in [8, 4], solutions \((X, Y, Z)\) of decoupled FBSDEs can be viewed as fixed points of a mapping. Here, we consider an alternative mapping \( F_\varphi : \mathbb{S}_d^2 \times \mathbb{S}_m^2 \times \mathbb{H}^2 \rightarrow \mathbb{S}_d^2 \times L_T^2 \) inspired by a one-dimensional analog of [12, 26], as follows. For given \( \varphi \) and \( u = (x, y, z) \in \mathbb{S}_d^2 \times \mathbb{S}_m^2 \times \mathbb{H}^2 \), we define

\[
F_\varphi(u)(t) = \begin{pmatrix}
(x(t) - x(0) - \int_0^t b(s, x(s)) \, ds - \int_0^t \sigma(s, x(s)) \, dW(s)) \\
y(t) - \varphi(x(T)) - \int_t^T f(s, u(s)) \, ds + \int_t^T z(s) \, dW(s)
\end{pmatrix}^T.
\]

Then, for any \((X_0, Y_0, Z_0) \in \mathbb{S}_d^2 \times \mathbb{S}_m^2 \times \mathbb{H}^2\) with \(X_0(0) = X(0)\) and assuming that the driver \( f \) is smooth, the Newton-Kantorovitch approximation process is given by

\[
F_\varphi(X_n, Y_n, Z_n) + F_\varphi'(X_n, Y_n, Z_n)(X_{n+1} - X_n, Y_{n+1} - Y_n, Z_{n+1} - Z_n) = 0,
\]

\[
X_{n+1}(0) = X(0),
\]

for \( n \in \mathbb{N} \cup \{0\} \), where \( F_\varphi' \) stands for Gâteaux derivative. This sequence is well-posed iff a unique system of linear backward stochastic differential equations (BSDEs) formally by Theorem 3.6. Note that, if the given duration \( T \) is arbitrarily, then, even for linear FBSDEs with constant coefficients, the necessary and sufficient conditions become more complicated when the diffusion coefficients also depend on \( z \), as was pointed out by Ma, Wu, Zhang and Zhang [14]. In this paper, we focus on decoupled FBSDEs and obtain the convergence result, Theorem 4.6.

The rest of this paper is structured as follows. Section 2 introduces the notations and assumptions used. Section 3 is devoted to formulating the Newton-Kantorovitch approximation process. Finally, Section 4 proves the main theorem for the decoupled FBSDEs.

2. Preliminaries

For \( x, y \in \mathbb{R}^d \), \( |x| \) denotes the Euclidian norm and \( \langle x, y \rangle \) denotes the inner product. Matrices size of \( m \times k \) will be represented as an element \( y \in \mathbb{R}^{m \times k} \), whose Euclidean norms are also given by \( |y| = \sqrt{yy^T} \) and for which \( \langle y, z \rangle = \operatorname{trace}(yz^T) \) where \( A^T \) is the transpose matrix of \( A \). In this paper, we only consider the derivatives with respect to the space variable.
where $\Delta x^\alpha$ For all $s, x, y, z \in \mathbb{R}$, we introduce the weighted norm
\[
\|Y\|^2 = \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y(s)|^2 \right] + \mathbb{E} \left[ \int_0^T |Z(s)|^2 \, ds \right].
\]
For the sake of simplicity, we also write the operator norm of the operator $A$ as $\|A\|$. The Banach spaces $S^2_m \times \mathbb{H}^2$ and $S^2_2$ are defined by
\[
\|Y, Z\|^2 = \|Y\|^2_{S^2_m} + \|Z\|^2_{\mathbb{H}^2}, \quad \|X\|^2 = \|X\|^2_{S^2_2}.
\]
For $\alpha \in \mathbb{R}$, we introduce the weighted norm
\[
\|Y, Z\|^2_\alpha = \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\alpha s} |Y(s)|^2 \right] + \mathbb{E} \left[ \int_0^T e^{\alpha s} |Z(s)|^2 \, ds \right].
\]
For $p, q \in \mathbb{N}$, $C^k((\mathbb{R}^p, \mathbb{R}^q))$ is the set of functions of class $C^k$ from $\mathbb{R}^p$ to $\mathbb{R}^q$, and $C^k((\mathbb{R}^p, \mathbb{R}^q))$ is the subset of these functions whose partial derivatives of order at most $k$ are bounded. When the domain and range dimensions are clear based on context and when there is no risk of confusion, we often eliminate the spaces to simplify the notation.

For a smooth $g$ such that $g^i(\cdot, \cdot, \cdot) \in C^1(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r, \mathbb{R}^q)$, it is convenient to obtain a concrete representation of $g' : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \rightarrow \mathbb{R}^q$. The Fréchet derivative at $u \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$ is the matrix representation of $g'(u)$ [16, page 60] and it can be obtained using the Jacobian matrix:
\[
g(u + \Delta u) - g(u) = g'(u) \Delta u + R_g(u) \Delta u, \quad \Delta u \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r
\]
where the Lagrange remainder is given by
\[
R_g(u) \Delta u = \left( \int_0^1 \{g'(u + \theta \Delta u) - g'(u)\} \, d\theta \right) \Delta u.
\]
Notice that, for all $u = (x, y, z)^T$, $\Delta u = (\Delta x, \Delta y, \Delta z)^T \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$,
\[
g'(u) \Delta u = g_x'(u) \Delta x + g_y'(u) \Delta y + g_z'(u) \Delta z,
\]
where we obtain set $g_x'(u) : \mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^p$ such that $g_x'(u) \Delta x = \left( \frac{\partial}{\partial x_i} g^{(i)}(x, y, z) \Delta x^{(i)} \right)_{i \leq p, j \leq p}$ where $\Delta x^{(i)}$ stands for the $i$th component for $i \leq p$. We define $g_y'$ and $g_z'$ similarly. Finally, for all $s \in [0, T]$, we define
\[
\|g'\|_{\infty} = \sup_{(s, x, y, z) \in [0, T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r} |g'(s, x, y, z)|,
\]
for $g(s, \cdot, \cdot, \cdot) \in C^1_b(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r, \mathbb{R})$.

3. Newton-Kantorovitch scheme for FBSDEs

In this section, we formulate the Newton-Kantorovitch scheme of the following system of FBSDEs:

$$
X(t) = X(0) + \int_0^t b(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s),
$$

where the solutions $(X(t), Y(t), Z(t)) = (X(t), Y(t), Z(t))_{t \in [0,T]}$ take values in $\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k}$, $W$ is the $k$-dimensional Wiener process, and the (progressively) measurable functions $b, f, \sigma$ and $\varphi$ are defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

In this paper, we also assume the following.

- $X(0) : \Omega \to \mathbb{R}^d$ is an $\mathcal{F}_0$-measurable and square-integrable random vector.
- $b : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is $\mathcal{P} \otimes B(\mathbb{R}^d)$-measurable.
- $\sigma : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$ is $\mathcal{P} \otimes B(\mathbb{R}^d)$-measurable.
- There exists a constant $C > 0$ such that for any $x \in \mathbb{R}^d$,
  $$
  |b(s, x)| \leq |b(s, 0)| + C|x|, \quad (s, \omega)\text{-a.e.},
  $$
  $$
  |\sigma(s, x)| \leq |\sigma(s, 0)| + C|x|, \quad (s, \omega)\text{-a.e.}
  $$
- $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^m$ is $\mathcal{F}_T \otimes B(\mathbb{R}^d)$-measurable.
- $f : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k} \to \mathbb{R}^m$ is $\mathcal{P} \otimes B(\mathbb{R}^m) \otimes B(\mathbb{R}^{m \times k})$-measurable.
- There exists a constant $C > 0$ such that for any $(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k}$,
  $$
  |f(s, x, y, z)| \leq |f(s, 0, 0, 0)| + C(|x| + |y| + |z|), \quad (s, \omega)\text{-a.e.}
  $$
  $$
  |\varphi(x)| \leq |\varphi(0)| + C|x|, \quad \omega\text{-a.e.}
  $$

In addition, in the BSDE,

$$
Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) \, ds - \int_t^T Z(s) \, dW(s),
$$

we replace the above assumption on $\varphi$ with the following condition:

- $\xi : \Omega \to \mathbb{R}^m$ is an $\mathcal{F}_T$-measurable and square integrable random vector; $\xi \in L^2$.

We also introduce the following assumptions.

**Assumption 3.1.** $b(\cdot, 0), \sigma(\cdot, 0), f(\cdot, 0, 0, 0) \in \mathbb{H}^2$, i.e.,

$$
\mathbb{E} \left[ \int_0^T \{ |b(s, 0)|^2 + |\sigma(s, 0)| + |f(s, 0, 0, 0)|^2 \} \, ds \right] < \infty
$$

and $\varphi(0) \in L^2$.

**Assumption 3.2.** $b(s, \cdot) \in C^1_b(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma(s, \cdot) \in C^1_b(\mathbb{R}^d, \mathbb{R}^{d \times k})$, $f(s, \cdot, \cdot, \cdot) \in C^1_b(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k}, \mathbb{R}^m)$, $(s, \omega)$-a.e. and $\varphi \in C^1_b(\mathbb{R}^d, \mathbb{R}^m)$, $\omega$-a.e.
For given \( \varphi \), we consider the operator \( F_\varphi \) (defined by (3)), namely,

\[
F_\varphi(u)(t) = \left( x(t) - x(0) - \int_0^t b(s, x(s)) \, ds - \int_0^t \sigma(s, x(s)) \, dW(s) \right)^T
\]

\[
y(t) - \varphi(x(T)) - \int_t^T f(s, u(s)) \, ds + \int_t^T z(s) \, dW(s)
\]

for \( u = (x, y, z) \in S^2_d \times S^2_m \times H^2 \). As an immediate consequence of the result given Lemma 3.1 of [12], we obtain the following corresponding result for FBSDEs.

**Lemma 3.3.** If Assumption 3.1 holds, then the operator \( F_\varphi \) defined by (3) maps the space \( S^2_d \times S^2_m \times H^2 \) into \( S^2_d \times L^2_T \) and \( t \mapsto F_\varphi(u)(t) \) is a continuous modification for \( u \in S^2_d \times S^2_m \times H^2 \).

**Proof.** For any \( u = (x, y, z) \in S^2_d \times S^2_m \times H^2 \), it follows, from \( \|x\|_{S^2_d} < \infty \) and \( \|z\|_{H^2} < \infty \), that the Itô integrals \( t \mapsto \int_0^t \sigma(s, x(s)) \, dW(s) \) and \( t \mapsto \int_t^T z(s) \, dW(s) \), respectively, are continuous modifications. By the Jensen inequality, we obtain

\[
E \left[ \int_0^t |b(s, x(s))|^2 \, ds \right] \leq 2E \left[ \int_0^t |b(s, 0)|^2 \, ds \right] + 2C^2 T \|x\|_{S^2_d}^2 < \infty
\]

and

\[
E \left[ \int_t^T |f(s, x(s), y(s), z(s))|^2 \, ds \right] \leq 4E \left[ \int_t^T |f(s, 0, 0, 0)|^2 \, ds \right] + 4C^2 \left( T \|x\|_{S^2_d}^2 + \|y\|_{S^2_m}^2 + \|z\|_{H^2}^2 \right) < \infty.
\]

By Doob’s inequality and Itô’s isometry property, there exists a \( c > 0 \) such that

\[
E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, x(s)) \, dW(s) \right|^2 \right] \leq cE \left[ \int_0^T \sigma(s, x(s)) \, dW(s) \right]^2 \leq 2cE \left[ \int_0^T |\sigma(s, 0)|^2 \, ds \right] + 2cC^2 T \|x\|_{S^2_d}^2 < \infty
\]

and

\[
E \left[ \sup_{0 \leq t \leq T} \left| \int_t^T z(s) \, dW(s) \right|^2 \right] \leq cE \left[ \int_t^T z(s) \, dW(s) \right]^2 = cE \left[ \int_0^T |z(s)|^2 \, ds \right] < \infty.
\]
Using the Jensen inequality, we further obtain that
\[
\frac{1}{9} E \left[ \sup_{0 \leq t \leq T} |F_\varphi(u)(t)|^2 \right] \\
\leq \|x\|_2^2 + E[|x(0)|^2] + E \left[ \int_0^T |b(s, x(s))|^2 \, ds \right] + E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, x(s)) \, dW(s) \right|^2 \right] \\
+ \|y\|_2^2 + E \left[ |\varphi(0)|^2 + C|x(T)|^2 \right] \\
+ E \left[ \int_t^T f(s, x(s), y(s), z(s)) \, ds \right] + E \left[ \sup_{0 \leq t \leq T} \left| \int_t^T z(s) \, dW(s) \right|^2 \right],
\]
which is bounded by the above estimates. This completes the proof. \(\square\)

The following lemma shows that the operator \(F_\varphi : S^2_d \times S^2_m \times \mathbb{H}^2 \to S^2_d \times S^2_m \times \mathbb{H}^2\) has a Gâteaux derivative.

**Lemma 3.4.** If Assumptions 3.1 and 3.2 hold, then, for all \(u = (x, y, z) \in S^2_d \times S^2_m \times \mathbb{H}^2\), the Gâteaux derivative \(F'_\varphi(u) : S^2_d \times S^2_m \times \mathbb{H}^2 \to S^2_d \times S^2_m \times \mathbb{H}^2\) of \(F_\varphi\) at \(u \in S^2_d \times S^2_m \times \mathbb{H}^2\) in the direction \(\overline{u} = (\overline{x}, \overline{y}, \overline{z}) \in S^2_d \times \mathbb{H}^2\) exists and is given for any \(t \in [0, T]\), by

\[
F'_\varphi(u)\overline{u}(t) = \begin{pmatrix}
\overline{x}(t) - x(0) - \int_0^t b'_x(s, x(s))\overline{x}(s) \, ds - \int_0^t \sigma'_x(s, x(s))\overline{x}(s) \, dW(s) \\
\overline{y}(t) - \varphi'(x(T))\overline{y}(T) - \int_t^T f'(s, u(s))h(s) \, ds + \int_t^T \overline{z}(s) \, dW(s)
\end{pmatrix}^T.
\]

**Proof.** We denote the right-hand side of (8) by \(A(u)\overline{u}(t)\) for \(\overline{u} = (\overline{x}, \overline{y}, \overline{z}) \in S^2_d \times S^2_m \times \mathbb{H}^2\). We note that because
\[
E \left[ \int_t^T \left\{ |b'_x(s, x(s))\overline{x}(s)|^2 + |\sigma'_x(s, x(s))\overline{x}(s)|^2 + |f'(s, u(s))h(s)|^2 \right\} \, ds \right] \\
\leq (1 \vee T) \{ \|b'\|_\infty^2 + \|\sigma'\|_\infty^2 + \|f'\|_\infty^2 \} \|h\|^2,
\]
we obtain that \(A(u)\overline{u} \in \mathbb{L}_T^2\) by the same argument of Lemma 3.3. It follows from (5) that, for all \(s \in [0, T]\),
\[
b(s, x(s) + \delta \overline{x}(s)) - b(s, x(s)) = \delta b'_x(s, x(s))\overline{x}(s) + R_b (x(s)) (\delta \overline{x})(s),
\]
\[
\sigma(s, x(s) + \delta \overline{x}(s)) - \sigma(s, x(s)) = \delta \sigma'_x(s, x(s))\overline{x}(s) + R_\sigma (x(s)) (\delta \overline{x})(s),
\]
\[
f(s, u(s) + \delta \overline{u}(s)) - f(s, u(s)) = \delta f'(s, u(s))\overline{u}(s) + R_f (u(s)) (\delta \overline{u})(s),
\]
and
\[
\varphi(x(T) + \delta \overline{x}(T)) - \varphi(x(T)) = \delta \varphi'(x(T))\overline{x}(T) + R_\varphi (x(T)) (\delta \overline{x})(T).
\]
Hence, for all $\delta > 0$ and $t \in [0, T]$, 
\[
\frac{F_\varphi(u + \delta \Pi)(t) - F_\varphi(u)(t)}{\delta} = A(u)\Pi(t) + \frac{1}{\delta} \left( \int_0^t R_b (x(s)) (\delta \mathcal{T})(s) \, ds - \int_t^T R_b (x(s)) (\delta \mathcal{T})(s) \, dW(s) \right) + \frac{1}{\delta} \left( \int_0^t R_\varphi(x(T)) (\delta \mathcal{T})(T) - \int_t^T R_\varphi(u(s)) (\delta \mathcal{T})(s) \, ds. \right)
\]

Since $b'_\varphi(s, \cdot)$, $\sigma'_\varphi(s, \cdot)$, $f'(s, \cdot)$ and $\varphi'$ are bounded and continuous $(s, \omega)$-a.e., by using the dominated convergence theorem, we obtain
\[
\lim_{\delta \to 0} \left\| \frac{F_\varphi(u + \delta \Pi) - F_\varphi(u)}{\delta} - A(u)\Pi \right\|_{\mathcal{S}_d^2 \times \mathcal{L}_m^2} = 0,
\]
thus completing the proof. \(\Box\)

The following lemma is a key in order to define the Newton-Kantorovitch approximation process.

**Lemma 3.5.** Suppose that Assumptions 3.1 and 3.2 hold and let $u = (x, y, z) \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$. Then, there exists a unique $\tilde{u} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$ such that $F_\varphi(u) + F'_\varphi(u)(\tilde{u} - u) = 0$ with initial condition $\tilde{x}(0) = x(0)$.

**Proof.** Let $u = (x, y, z)$ be in $\mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$. We show that there exists a unique $\tilde{u} = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$ such that $F_\varphi(u) + F'_\varphi(u)(\tilde{u} - u) = 0$ with $\tilde{x}(0) = x(0)$, i.e.,
\[
\begin{align*}
\tilde{x}(t) &= x(0) + \int_0^t \{b(s, x(s)) + b'_\varphi(s, x(s))(\tilde{x}(s) - x(s))\} \, ds \\
&\quad + \int_0^t \{\sigma(s, x(s)) + \sigma'_\varphi(s, x(s))(\tilde{x}(s) - x(s))\} \, dW(s), \\
\tilde{y}(t) &= \varphi(x(T)) + \varphi'(x(T))(\tilde{x}(T) - x(T)) \\
&\quad + \int_t^T \{f(s, u(s)) + f'(s, u(s))(\tilde{u}(s) - u(s))\} \, ds - \int_t^T \tilde{z}(s) \, dW(s),
\end{align*}
\]
Because the above equation is a linear decoupled FBSDE with uniformly bounded coefficients, there exists a unique $\tilde{u} \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$ as required. \(\Box\)

From Lemma 3.5, we can conclude that, for any initial condition $(X_0, Y_0, Z_0) \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$, we can define the Newton-Kantorovitch approximation process $(X_{n+1}, Y_{n+1}, Z_{n+1}) \in \mathcal{S}_d^2 \times \mathcal{S}_m^2 \times \mathbb{H}^2$ as solving the equation
\[
F_\varphi(X_n, Y_n, Z_n) + F'_\varphi(X_n, Y_n, Z_n)(X_{n+1} - X_n, Y_{n+1} - Y_n, Z_{n+1} - Z_n) = 0,
\]
for $n \in \mathbb{N} \cup \{0\}$, which is equivalent to
\[
(X_{n+1}, Y_{n+1}, Z_{n+1}) = (X_n, Y_n, Z_n) - F'_\varphi(X_n, Y_n, Z_n)^{-1}F_\varphi(X_n, Y_n, Z_n).
\]

The following theorem shows that (10) has a unique solution that satisfies a linear decoupled FBSDE with uniformly bounded coefficients.
Theorem 3.6. If Assumptions 3.1 and 3.2 hold. Then \((X_{n+1}, Y_{n+1}, Z_{n+1})\) satisfies the following linear decoupled FBSDE for 0 \(\leq t \leq T\):

\[
X_{n+1}(t) = X_{n+1}(0) + \int_0^t b_n(s, X_{n+1}(s)) \, ds + \int_0^t \sigma_n(s, X_{n+1}(s)) \, dW(s),
\]

\[
Y_{n+1}(t) = \varphi_n(X_{n+1}(T)) + \int_t^T f_n(s, X_{n+1}(s), Y_{n+1}(s), Z_{n+1}(s)) \, ds - \int_t^T Z_{n+1}(s) \, dW(s),
\]

where we define for 0 \(\leq s \leq T\) and \((x, y, z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times k}\),

\[
b_n(s, x) = b(s, X_n(s)) + b'_z(s, X_n(s))(x - X_n(s)),
\]

\[
\sigma_n(s, x) = \sigma(s, X_n(s)) + \sigma'_z(s, X_n(s))(x - X_n(s)),
\]

\[
f_n(s, x, y, z) = f(s, X_n(s), Y_n(s), Z_n(s)),
\]

\[
+ f'(s, X_n(s), Y_n(s), Z_n(s))(x - X_n(s), y - Y_n(s), z - Z_n(s)),
\]

\[
\varphi_n(x) = \varphi(X_n(T)) + \varphi'_z(X_n(T))(x - X_n(T)).
\]

In particular, if \(X_0(0) = X(0), then X_{n+1}(0) = X(0)\) for all \(n \in \mathbb{N} \cup \{0\}\).

Proof. The existence and uniqueness of \((X_{n+1}(t), Y_{n+1}(t), Z_{n+1}(t))_{t \in [0, T]}\) follows from Lemma 3.5, where, for all \(n \in \mathbb{N} \cup \{0\}\). The equation (9) can be re-expressed as the desired FBSDE for 0 \(\leq t \leq T\). Finally, by the initial condition \(X_{n+1}(0) = X_n(0)\), if \(X_0(0) = X(0)\), then \(X_{n+1}(0) = X(0)\), for all \(n \in \mathbb{N} \cup \{0\}\). \(\square\)

4. CONVERGENCE OF THE NEWTON-KANTOROVITCH APPROXIMATION PROCESS FOR FBSDES

In this section, we investigate the convergence of the Newton-Kantorovich approximation processes defined by (10).

4.1. Convergence of the Newton-Kantorovich approximation process for SDEs. In this subsection, we consider the Newton-Kantorovich scheme for the forward process \(X\) and extend earlier studies of Newton-Kantorovich methods for SDEs,

\[
X(t) = X(0) + \int_0^t b(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s).
\]

Theorem 4.1. Let \(X\) be a solution of (12). If Assumptions 3.1 and 3.2 hold, then, for any \(X_0 \in \mathcal{S}_d^2\) where \(X_0(0) = X(0)\), we obtain that, for all \(n \in \mathbb{N} \cup \{0\}\) and \(\epsilon \in (0, 1)\),

\[
\|X - X_{n+1}\| \leq \frac{C_0^{n+1}}{(n + 1)!} \|X - X_0\| \leq \epsilon^{n+1} e^{C_0^T/\epsilon} \|X - X_0\|,
\]

where the constant \(C_0\) is given by

\[
C_0 = 8c_{b, \sigma}T \exp(4c_{b, \sigma}T), \quad c_{b, \sigma} = \|b'\|_{\infty} + 18\|\sigma'\|_{\infty} + \|\sigma''\|_{\infty}.
\]

Remark 4.2. An estimate was initially given by [12] for one-dimensional SDEs with uniformly bounded coefficients and an alternative estimation was proposed by [1].
Proof of Theorem 4.1. The proof is based on [1]. By a fundamental result in [10], $X$ exists and is unique. For $0 \leq s \leq T$ and $n \in \mathbb{N}$, define

$$\overline{X}_n(s) = X(s) - X_n(s),$$

we obtain for all $s \in [0, T]$ and $n \in \mathbb{N}$, $\mathbb{P}$-a.s. By the mean value theorem, for $s \in [0, T]$ and $n \in \mathbb{N}$, we obtain

$$b(s, X(s)) - b_n(s, X_{n+1}(s))$$
$$= \{b(s, X(s)) - b(s, X_n(s))\} + b'_n(s, X_n(s))\overline{X}_n(s)$$
$$= R_b (X_n(s)) \overline{X}_n(s) + b'_n(s, X_n(s))\overline{X}_{n+1}(s)$$

and

$$\sigma(s, X(s)) - \sigma_n(s, X_{n+1}(s))$$
$$= \{\sigma(s, X(s)) - \sigma(s, X_n(s))\} + \sigma'_n(s, X_n(s))\overline{X}_n(s)$$
$$= R_\sigma (X_n(s)) \overline{X}_n(s) + \sigma'_n(s, X_n(s))\overline{X}_{n+1}(s).$$

Recall (5) and we have, for $s \in [0, T]$, $n \in \mathbb{N}$ and $h \in \mathbb{R}_2$,

$$R_b (X_n(s)) h(s) = \left\{ \int_0^1 b'_x (s, X_n(s)) + \theta h(s) \, d\theta - b'_x (s, X_n(s)) \right\} h(s),$$
$$R_\sigma (X_n(s)) h(s) = \left\{ \int_0^1 \sigma'_x (s, X_n(s)) + \theta h(s) \, d\theta - \sigma'_x (s, X_n(s)) \right\} h(s).$$

This allows us to obtain

$$\overline{X}_{n+1}(t) = \int_0^t \left\{ b'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_b (X_n(s)) \overline{X}_n(s) \right\} \, ds + \int_0^t \left\{ \sigma'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_\sigma (X_n(s)) \overline{X}_n(s) \right\} \, dW(s),$$

which, by applying Itô’s formula yields

$$|\overline{X}_{n+1}(t)|^2 = 2 \int_0^t \langle \overline{X}_{n+1}(s), b'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_b (X_n(s)) \overline{X}_n(s) \rangle \, ds + 2 \int_0^t \langle \overline{X}_{n+1}(s), \sigma'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_\sigma (X_n(s)) \overline{X}_n(s) \rangle \, dW(s)$$
$$+ \int_0^t \sigma'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_\sigma (X_n(s)) \overline{X}_n(s) \rangle \, dW(s) + 2 \int_0^t |\sigma'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_\sigma (X_n(s)) \overline{X}_n(s) |^2 \, ds.$$ 

Note that we obtain

$$2 \langle \overline{X}_{n+1}(s), b'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_b (X_n(s)) \overline{X}_n(s) \rangle \leq 2 \|b'\| |\overline{X}_{n+1}(t)|^2 + 4 \|b'\|^2 |\overline{X}_n(t)|^2$$

and

$$|\sigma'_x (s, X_n(s))\overline{X}_{n+1}(s) + R_\sigma (X_n(s)) \overline{X}_n(s) |^2 \leq 2 \|\sigma'\|^2 |\overline{X}_{n+1}(t)|^2 + 4 \|\sigma'\|^2 |\overline{X}_n(t)|^2.$$
The Burkholder-Davis-Gundy’s inequality implies that there exists a $c_0$ such that
\[
\mathbb{E} \left[ 2 \int_0^t (\overline{X}_{n+1}(s), \sigma'(s, X_n(s)) X_{n+1}(s) \, dW(s)) \right] \\
\leq \mathbb{E} \left[ 2 \left( \frac{1}{4} \sup_{0 \leq s \leq t} |\overline{X}_{n+1}(s)|^2 \right)^{1/2} \left( 4c_0^2 \|\sigma'\| \int_0^t |X_{n+1}(s)|^2 \, ds \right)^{1/2} \right] \\
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_{n+1}(s)|^2 \right] + 4c_0^2 \|\sigma'\| \mathbb{E} \left[ \int_0^t |X_{n+1}(s)|^2 \, ds \right],
\]
where we obtain the last inequality by applying the inequality $ab \leq (a^2/2) + (b^2/2)$ for all $a, b \in \mathbb{R}$. Similarly, we have
\[
\mathbb{E} \left[ 2 \int_0^t (\overline{X}_{n+1}(t), R_{\sigma}(X_n(s)) \overline{X}_n(s) \, dW(s)) \right] \\
\leq \mathbb{E} \left[ 2 \left( \frac{1}{4} \sup_{0 \leq s \leq t} |\overline{X}_{n+1}(s)|^2 \right)^{1/2} \left( 8c_0^2 \|\sigma'\| \int_0^t |X_{n+1}(s)|^2 \, ds \right)^{1/2} \right] \\
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_{n+1}(s)|^2 \right] + 8c_0^2 \|\sigma'\| \mathbb{E} \left[ \int_0^t |X_{n+1}(s)|^2 \, ds \right],
\]
where we note that an explicit upper bounded of 3 can be obtained for the constant $c_0$; see Theorem 3.28 in [11]. By setting $c_{b, \sigma} = \|b\|_\infty + 2c_0^2 \|\sigma'\|_\infty + \|\sigma''\|_\infty$, we obtain, for any $t' \in [0, T]$,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t'} |\overline{X}_{n+1}(t)|^2 \right] \leq 4c_{b, \sigma} \int_0^{t'} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_{n+1}(t)|^2 \right] \, ds \\
+ 8c_{b, \sigma} \int_0^{t'} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_n(t)|^2 \right] \, ds.
\]
Gronwall’s inequality further implies that
\[
(14) \quad \mathbb{E} \left[ \sup_{0 \leq s \leq t'} |\overline{X}_{n+1}(t)|^2 \right] \leq C_0 \int_0^{t'} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_n(t)|^2 \right] \, ds,
\]
where $C_0 = 8c_{b, \sigma} \exp(4c_{b, \sigma} T)$. Iterating (14) yields
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\overline{X}_{n+1}(t)|^2 \right] \leq C_0^n \int_0^{T} ds_1 \int_0^{s_1} ds_2 \mathbb{E} \left[ \sup_{0 \leq t \leq s_1} |\overline{X}_{n-1}(t)|^2 \right] \\
\leq C_0^{n+1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\overline{X}_n(t)|^2 \right] \int_0^{T} ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_{n+1} \\
= \frac{(T C_0)^{n+1}}{(n+1)!} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\overline{X}_n(t)|^2 \right].
\]
In addition, we obtain that
\[
\varepsilon^{-(n+1)} \frac{(T C_0)^{n+1}}{(n+1)!} \leq \sum_{k=0}^{\infty} \frac{(T C_0/\varepsilon)^k}{k!} = e^{C_0 T/\varepsilon},
\]
thus completing the proof. \qed

**Remark 4.3.** We can also use the same argument to show that the approximation process is a Cauchy sequence in $S^2_{\varepsilon}$.
4.2. Toward decoupled forward-backward stochastic differential equations.

The inequality in Theorem 4.4 below indicates that approximating the terminal condition is the key to estimating the error between the solution and the Newton-Kantorovitch approximation process. In this section, we consider the Newton-Kantorovitch approximation with respect to the terminal condition. First, we show that it converges with respect to the weighted norm $\| \cdot \|_\alpha$.

**Theorem 4.4.** Let $(X, Y, Z)$ be a solution of the FBSDE (1). If Assumptions 3.1 and 3.2 hold and $(X_0, Y_0, Z_0) \in \mathbb{S}_d^2 \times \mathbb{S}_m^2 \times \mathbb{H}^2$ such that $X_0(0) = X(0)$, then, for all $n \in \mathbb{N} \cup \{0\}$, $\epsilon \in (0, 1)$ and $T > 0$, we obtain

$$
\|(Y - Y_{n+1}, Z - Z_{n+1})\|_\alpha^2 \leq e^{n+1} \left\{ C_1 \|X - X_0\|_{\mathbb{S}_d^2}^2 + \|(Y - Y_0, Z - Z_0)\|_{\alpha}^2 \right\}
$$

where $C_1 = e^{-1} (1 + (1 + 4(2 + TC_0)\|\varphi\|_{\infty}^2) (1 + 4c_0^2)) \epsilon^{1+TC_0/\epsilon}$, $c_0 = 3$ and $C_0$ is given by Theorem 4.1 and $\alpha = 2\|f\|_{\infty} + 4\|f\|_{\infty}^2 + 12\|f\|_{\infty} (1 + 4c_0^2) (1 \vee T) \epsilon^{-1}$.

**Proof.** By the fundamental result in [18], the solution $(X, Y, Z)$ exists and is unique. For $s \in [0, T]$ and $n \in \mathbb{N}$, we define

$$
h_n(s) = (\dot{X}_n(s), \dot{Y}_n(s), \dot{Z}_n(s)) = (X(s) - X_n(s), Y(s) - Y_n(s), Z(s) - Z_n(s)),
$$

and apply the mean value theorem, yielding

$$
f(s, X_n(s), Y_n(s), Z_n(s)) - f_n(s, X_n+1(s), Y_{n+1}(s), Z_{n+1}(s))
\begin{align*}
&= f'(s, X_n(s), Y_n(s), Z_n(s))h_{n+1}(s) \\
&+ \{ f'(s, X_n(s), Y_n(s), Z_n(s)) - f'(s, X_n(s), Y_n(s), Z_n(s))h_n(s) \} \\
&= f'(s, X_n(s), Y_n(s), Z_n(s))h_{n+1}(s) + R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s). \\
\end{align*}
$$

Recall (5) and we have, for $s \in [0, T]$, $n \in \mathbb{N}$ and $h \in \mathbb{S}_d^2 \times \mathbb{S}_m^2 \times \mathbb{H}^2$,

$$
R_f(X_n(s), Y_n(s), Z_n(s)) h(s)
\begin{align*}
&= \left\{ \int_0^1 f'(s, X_n(s), Y_n(s), Z_n(s)) + \theta h(s) \right\} d\theta - f'(s, X_n(s), Y_n(s), Z_n(s)) \\
&= \left\{ \int_0^1 f'(s, X_n(s), Y_n(s), Z_n(s)) + \theta h(s) \right\} d\theta - f'(s, X_n(s), Y_n(s), Z_n(s)). \\
\end{align*}
$$

This allows us to show that $(\dot{Y}_{n+1}, \dot{Z}_{n+1})$ satisfies the following linear BSDE

$$
\dot{Y}_{n+1}(t) - \dot{Y}_{n+1}(T) + \int_t^T \dot{Z}_{n+1}(s) dW(s)
= \int_t^T \{ f'(s, X_n(s), Y_n(s), Z_n(s))h_{n+1}(s) + R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s) \} \, ds.
$$

Applying Itô’s formula to $e^{\alpha t} |\dot{Y}_{n+1}(t)|^2$ for all $\alpha \in \mathbb{R}$, we obtain

$$
\begin{align*}
e^{\alpha t} |\dot{Y}_{n+1}(t)|^2 - e^{\alpha T} |\dot{Y}_{n+1}(T)|^2 + \int_t^T e^{\alpha s} |\dot{Z}_{n+1}(s)|^2 \, ds \\
= \int_t^T e^{\alpha s} (-\alpha) |\dot{Y}_{n+1}(s)|^2 \, ds - 2 \int_t^T e^{\alpha s} (\dot{Y}_{n+1}(s), \dot{Z}_{n+1}(s)) dW(s) \\
+ 2 \int_t^T e^{\alpha s} (\dot{Y}_{n+1}(s), f'(s, X_n(s), Y_n(s), Z_n(s))h_{n+1}(s) + R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s)) \, ds.
\end{align*}
$$
From the Cauchy-Schwarz inequality and the inequality $2ab \leq \delta^{-1}|a|^2 + \delta|b|^2$ for all $\delta > 0$, we have

$$2 |\langle \nabla_n+1(s), f_n(s), X_n(s), Y_n(s), Z_n(s) \rangle \nabla_n+1(s) | \leq 2 \| f' \|_\infty^2 |\nabla_n+1(s)|^2 + \frac{1}{2} |\nabla_n+1(s)|^2,$$

$$2 |\langle \nabla_n+1(s), f_n(s), X_n(s), Y_n(s), Z_n(s) \rangle \nabla_n+1(s) | \leq 2 \| f' \|_\infty^2 |\nabla_n+1(s)|^2,$$

$$2 |\langle \nabla_n+1(s), f_n(s), X_n(s), Y_n(s), Z_n(s) \rangle \nabla_n+1(s) | \leq 2 \| f' \|_\infty^2 |\nabla_n+1(s)|^2 + \frac{1}{2} |\nabla_n+1(s)|^2,$$

$$2 |\langle \nabla_n+1(s), Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s) | \leq \delta^{-1} |\nabla_n+1(s)|^2 + \delta \| Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2,$$

and

$$\text{(16b)} \quad \| Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s) \| \leq 2 \| f' \|_\infty \| h_n(s) \|.$$

The right-hand side of (16a) is therefore less than or equal to

$$\int_t^T e^{\alpha s} \left\{ (-\alpha) + 2 \| f' \|_\infty + 4 \| f' \|_\infty^2 + \delta^{-1} \right\} |\nabla_n+1(s)|^2 \, ds$$

$$+ \int_t^T e^{\alpha s} \left\{ \frac{1}{2} |\nabla_n+1(s)|^2 + \frac{1}{2} |\nabla_n+1(s)|^2 + \delta \| Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \right\} \, ds$$

$$- 2 \int_t^T e^{\alpha s} (\nabla_n+1(s), Z_n+1(s) dW(s)).$$

Setting $\alpha \equiv \alpha(\delta) = 2 \| f' \|_\infty + 4 \| f' \|_\infty^2 + \delta^{-1}$, we obtain

$$\text{(16c)} \quad e^{\alpha t} \| \nabla_n+1(t) \|^2 - e^{\alpha T} \| \nabla_n+1(T) \|^2 + \frac{1}{2} \int_t^T e^{\alpha s} \left\{ |\nabla_n+1(s)|^2 - |\nabla_n+1(s)|^2 \right\} \, ds$$

$$\leq \delta \int_t^T e^{\alpha s} |Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \, ds - 2 \int_t^T e^{\alpha s} (\nabla_n+1(s), Z_n+1(s) dW(s)).$$

As $(Y, Z), (Y_n, Z_n) \in \mathbb{S}^2_n \times \mathbb{H]^2}$, the local martingale

$$\left\{ \int_0^t e^{\alpha s} (\nabla_n+1(s), Z_n+1(s) dW(s)) \right\}_{t \in [0, T]}$$

vanishes at 0. Thus, in particular, by setting $t = 0$, we obtain

$$\text{(16d)} \quad \mathbb{E} \left[ \int_0^T e^{\alpha s} \left\{ |\nabla_n+1(s)|^2 - |\nabla_n+1(s)|^2 \right\} \, ds \right]$$

$$\leq 2 \delta \mathbb{E} \left[ \int_0^T e^{\alpha s} |Rf (X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \, ds \right] + 2 \mathbb{E} \left[ e^{\alpha T} |\nabla_n+1(T)|^2 \right].$$

Note that

$$\int_t^T e^{\alpha s} (\nabla_n+1(s), Z_n+1(s) dW(s)) = \int_0^T 1_{[t, T]}(s)e^{\alpha s} (\nabla_n+1(s), Z_n+1(s) dW(s)).$$
Applying the Burkholder-Davis-Gundy inequality indicates that there is a universal \( c_0 \) such that

\[
2E \left[ \sup_{0 \leq t \leq T} \left| \int_t^T e^{\alpha s} (\mathbf{Y}_{n+1}(s), \mathbf{Z}_{n+1}(s)) \, dW(s) \right| \right] \\
\leq 2c_0 \mathbb{E} \left( \int_0^T 1_{[s,T]}(s) e^{2\alpha s} |\mathbf{Y}_{n+1}(s)|^2 |\mathbf{Z}_{n+1}(s)|^2 \, ds \right)^{1/2} \\
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_{n+1}(t)| \left( 4c_0^2 \int_0^T e^{\alpha s} |\mathbf{Z}_{n+1}(s)|^2 \, ds \right)^{1/2} \right],
\]

where we note that an explicit upper bounded of 3 can be obtained on the constant; refer to Theorem 3.28 in [11]. Hence, by considering the supremum of (16c) and using the inequality \( ab \leq (a^2/2) + (b^2/2) \) for all \( a, b \in \mathbb{R} \), we obtain

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_{n+1}(t)|^2 - e^{\alpha T} |\mathbf{Y}_{n+1}(T)|^2 + \frac{1}{2} \int_0^T e^{\alpha s} \left\{ |\mathbf{Z}_{n+1}(s)|^2 - |\mathbf{X}_{n+1}(s)|^2 \right\} \, ds \right] \\
\leq \delta \int_0^T e^{\alpha s} \mathbb{E} \left[ |R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \right] \, ds \\
+ \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_{n+1}(t)|^2 \right] + 2c_0^2 \mathbb{E} \left[ \int_0^T e^{\alpha s} |\mathbf{Z}_{n+1}(s)|^2 \, ds \right],
\]

which implies that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_{n+1}(t)|^2 - 2e^{\alpha T} |\mathbf{Y}_{n+1}(T)|^2 + \int_0^T e^{\alpha s} \left\{ |\mathbf{Z}_{n+1}(s)|^2 - |\mathbf{X}_{n+1}(s)|^2 \right\} \, ds \right] \\
\leq 2\delta \int_0^T e^{\alpha s} \mathbb{E} \left[ |R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \right] \, ds + 4c_0^2 \mathbb{E} \left[ \int_0^T e^{\alpha s} |\mathbf{Z}_{n+1}(s)|^2 \, ds \right].
\]

By then taking the inequality (16d) into consideration, we observe

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_{n+1}(t)|^2 + \int_0^T e^{\alpha s} \left\{ |\mathbf{Z}_{n+1}(s)|^2 - |\mathbf{X}_{n+1}(s)|^2 \right\} \, ds \right] \\
\leq 2\delta \left( 1 + 4c_0^2 \right) \int_0^T e^{\alpha s} \mathbb{E} \left[ |R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \right] \, ds \\
+ 2 \left( 1 + 4c_0^2 \right) e^{\alpha T} \mathbb{E} \left[ |\mathbf{Y}_{n+1}(T)|^2 \right] + 4c_0^2 \mathbb{E} \left[ \int_0^T e^{\alpha s} |\mathbf{X}_{n+1}(s)|^2 \, ds \right].
\]

If we also consider the inequality (16b), we find

\[
\int_0^T e^{\alpha s} \mathbb{E} \left[ |R_f(X_n(s), Y_n(s), Z_n(s)) h_n(s)|^2 \right] \, ds \\
\leq 6\|f'\|_{\infty} (1 \vee T) \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{X}_n(t)|^2 + \sup_{0 \leq t \leq T} e^{\alpha t} |\mathbf{Y}_n(t)|^2 + \int_0^T e^{\alpha s} |\mathbf{Z}_n(s)|^2 \, ds \right] \right\}.
\]

Selecting \( \delta \) such that \( 12\|f'\|_{\infty} (1 + 4c_0^2) (1 \vee T) \delta = \epsilon \), we obtain

\[
\alpha \equiv \alpha(\delta) = 2\|f'\|_{\infty} + 4\|f'\|_{\infty}^2 + 12\|f'\|_{\infty} (1 + 4c_0^2) (1 \vee T) \epsilon^{-1}.
\]
This leads to
\[ \|(Y_{n+1}, Z_{n+1})\|_\alpha^2 \]
\[ \leq \varepsilon \|(X_{n}, Y_{n}, Z_{n})\|_\alpha^2 + 2 (1 + 4c_0^2) e^{\alpha T} \|(Y_{n+1}(T))\|_{L^2}^2 + (1 + 4c_0^3) \|X_{n+1}\|_\alpha^2. \]
Hence, using (14),
\[ \mathbb{E} \left( Y_{n+1}^2(T) \right) \leq \|\varphi'\|_{L^2}^2 \left\{ 4 \mathbb{E} \left( |X_n|^2 \right) + 2 \mathbb{E} \left( |X_{n+1}(T)|^2 \right) \right\} \]
\[ \leq 2 (2 + TC_0) \|\varphi'\|_{L^2}^2 \mathbb{E} \left( |X_n|^2 \right). \]
Thus, we obtain
\[ 2 (1 + 4c_0^2) e^{\alpha T} \|(Y_{n+1}(T))\|_{L^2}^2 \leq (1 + 4TC_0) \|\varphi'\|_{L^2}^2 \left( 1 + 4c_0^3 \right) e^{\alpha T}, \]
Applying inequality (13) then yields for \( c_1 = \varepsilon^{\alpha + 1} \left( 1 + 4(2 + TC_0) \|\varphi'\|_{L^2}^2 \right) \left( 1 + 4c_0^3 \right) e^{\alpha T}, \)
\[ \|(Y_{n+1}, Z_{n+1})\|_\alpha^2 \leq \varepsilon \|(Y_n, Z_n)\|_\alpha^2 + c_1 \|X - X_0\|_\alpha^2 \frac{C_0}{n!}. \]
For any positive sequence \( \{a_n\}, \{b_n\} \) and \( \varepsilon > 0 \) such that \( a_{n+1} \leq b_n + \varepsilon a_n \) for all \( n \in \mathbb{N} \cup \{0\} \), we obtain
\[ a_{n+1} \leq b_n + \varepsilon (a_{n-1} + b_{n-1}) = \varepsilon^2 (b_{n-1} + a_{n-1}) \]
\[ \leq \cdots \leq \varepsilon^{n+1} \left\{ \sum_{k=0}^{n} \varepsilon^{k-(n+1)} b_{n-k} + a_0 \right\}. \]
Replacing \( b_n = c_1 \|X - X_0\|_\alpha^2 \frac{C_0}{n!} \) for all \( n \in \mathbb{N} \), we obtain
\[ \sum_{k=0}^{n} \varepsilon^{k-(n+1)} b_{n-k} \leq \varepsilon^{-1} c_1 \|X - X_0\|_\alpha^2 \sum_{k=0}^{n} \frac{(C_0/\varepsilon)^k}{k!} \leq \varepsilon^{-1} c_1 \|X - X_0\|_\alpha^2 e^{C_0/\varepsilon}, \quad n \in \mathbb{N} \cup \{0\}. \]
Thus, we obtain
\[ \|(Y_{n+1}, Z_{n+1})\|_\alpha^2 \leq \varepsilon^{n+1} \left( C_1 \|X - X_0\|_\alpha^2 + \|(Y_0, Z_0)\|_\alpha^2 \right), \]
where we define \( C_1 = \varepsilon^{-1} c_1 e^{C_0/\varepsilon} = \varepsilon^{-1} \left\{ 1 + (1 + 4(2 + TC_0) \|\varphi'\|_{L^2}^2) \left( 1 + 4c_0^3 \right) \right\} e^{\alpha T + C_0/\varepsilon}. \)
This concludes the proof. \( \square \)

**Remark 4.5.** We can also prove the so-called "semilocal theorem," which states that the approximation process is a Cauchy sequence in \( \mathbb{S}_m^2 \times \mathbb{H}^2 \) by the same argument. For the definition of "semilocal," refer to [24].

We can now prove our main result based on the following theorem.

**Theorem 4.6.** Let \( (X, Y, Z) \) be a solution of the FBSDE (1). If Assumptions 3.1 and 3.2 hold and \( (X_0, Y_0, Z_0) \in \mathbb{S}_m^2 \times \mathbb{S}_n^2 \times \mathbb{H}^2 \) with \( X_0(0) = X(0) \), then, there exists a \( C_3 > 0 \) such that, for all \( n \in \mathbb{N} \cup \{0\} \), we obtain
\[ \|(X - X_{n+1}, Y - Y_{n+1}, Z - Z_{n+1})\|^2 \leq \varepsilon^{n+1} C_3 \|(X - X_0, Y - Y_0, Z - Z_0)\|^2, \]
where the constant \( C_3 \equiv C_3(\varepsilon) \) is bounded by the coefficients and independent of \( n \).
Proof. By inequality (13) in Theorem 4.1 and by inequality (15) in Theorem 4.4, we obtain
\[ \|X - X_{n+1}\|_2^2 \leq \epsilon^{n+1} e^{C_0 T/\epsilon} \|X - X_0\|_2^2, \]
and
\[ \|(Y - Y_{n+1}, Z - Z_{n+1})\|_2^2 \leq \|(Y - Y_0, Z - Z_0)\|_2^2 \]
respectively, where
\[ C_0 = 8c_{b,\sigma} \exp(4c_{b,\sigma} T), \quad c_{b,\sigma} = \|b'\|_\infty + 18\|\sigma'\|_\infty + \|\sigma''\|_\infty, \]
\[ C_1(\epsilon) = \epsilon^{-1} \left(1 + \left(1 + 4(2 + TC_0)\|\sigma'\|_\infty^2 \right) \left(1 + 4c_0^2 \right) \right) e^{\alpha T + C_0 / \epsilon}, \quad c_0 = 3, \]
\[ \alpha = 2\|f''\|_\infty + 4\|f''\|_\infty^2 + 12\|f''\|_\infty \left(1 + 4c_0^2 \right) (1 \vee T) \epsilon^{-1}. \]
Defining
\[ C_3 = \left\{ (C_1 + e^{C_0 T/\epsilon}) \vee e^{\alpha T} \right\}, \]
concludes the proof. \(\square\)

This finally allows us to prove our main theorem, Theorem 1.1.

Proof of Theorem 1.1. By setting \(\epsilon = 1/2\), the desired result can be obtained from Theorem 4.6. \(\square\)

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