Some formulas for Legendre functions related to the Poisson transform and Lorentz group representation

I A Shilin\textsuperscript{1,2} and A I Nizhnikov\textsuperscript{3}

\textsuperscript{1} Department of Higher Mathematics, Sholokhov Moscow State University for the Humanities, V. Radishevskaya 16 – 18, Moscow 109240, Russia
\textsuperscript{2} Department of Mathematical Simulation, Moscow Aviation Institute, Volokolamskoe shosse, 4, Moscow 125993, Russia
E-mail: ilyashilin@li.ru
\textsuperscript{3} Department of Mathematical Physics, Moscow Pedagogical State University, M. Pirogovskaya 1, Moscow 119991, Russia
E-mail: tensit@mail.ru

Abstract. We consider the representation of Lorentz group in the spaces of homogeneous functions on cone and hyperboloid. Using the Poisson transform, which maps a function from the first space into the corresponding function from the second space, we derive some formulas containing Legendre functions. Using the restriction of the representation in the first space on a one-parameter subgroup of hyperbolic rotations, we obtain another formula for Legendre functions. All obtained formulas are new.

1. Introduction

Let the linear space $\mathbb{R}^{n+1}$ is endowed with the quadratic form $q(x) := x_0^2 - x_1^2 - \ldots - x_n^2$ and $\hat{q}$ be a polar bilinear form for $q$. We will deal with two kinds of orbits of the Lorentz group $SO(n,1)$. One of them is cone $O_0 := \{x : q(x) = 0\}$. The second kind of orbits consists of two-sheet hyperboloids $O_r := \{q(x) = r\}$ for any $r > 0$.

Let $G$ be a connected component of $SO(n,1)$, which contains the identical element. For any $\sigma \in \mathbb{C}$, let $\mathcal{C}_\sigma$ be a linear subspace in $C^\infty(O_0)$ consisting of $\sigma$-homogeneous functions on the $O_1$. It is useful to suppose throughout this paper that $1 - n < re \sigma < 0$. We define the $G$-representations $T_\sigma$ in $\mathcal{C}_\sigma$ and $T_\sigma$ in $\mathcal{D}_\sigma$ by left shifts: $T_\sigma(g)[f(x)] := f(g^{-1}x)$ and $S_\sigma(g)[f(x)] := f(g^{-1}x)$.

Suppose that $\gamma$ is a contour on $O_0$ intersecting all generatrices. Every point $x \in O_0$ can be represented as $x_i = iF_i(\xi_1, \ldots, \xi_{n-1})$, $i = 0, 1, \ldots, n$, where $\xi_1, \ldots, \xi_{n-1}$ are parameters of the contour $\gamma$. For the subgroup $H_\gamma$ of $G$ which acts transitively on $\gamma$, we have $dx = t^{n-2} dt \ d\gamma$, where $dx$ is a $G$-invariant measure on $O_0$ and $\frac{d\gamma}{\gamma}$ is a $H_\gamma$-invariant measure on $\gamma$.

For any pair $(\mathcal{C}_\sigma, \mathcal{C}_{\sigma^*})$, we define the bilinear functionals

$$F_\gamma : (\mathcal{C}_\sigma, \mathcal{C}_{\sigma^*}) \rightarrow \mathbb{C}, \ (f_1, f_2) \mapsto \int_\gamma f_1(x) f_2(x) \ d\gamma.$$ 

The functional $F_\gamma$ does not depend on $\gamma$ if $\sigma^* = 1 - n - \sigma$ because of the homogeneity of $f_1$ and $f_2$. It follows from the relation between $dx$ and $d\gamma$.

Published under licence by IOP Publishing Ltd
Let $f \in \mathcal{C}_\sigma$ and $y \in O_1$. The integral map $\Pi(f)(y) := F_\gamma(f, (q(x, y)))^{1-n-\sigma}$ is said to be the Poisson transform (see, for instance, page 194 in [5]). This map intertwines the representations $T_\sigma$ and $S_\sigma$, moreover, $\text{Im} \Pi \subset \text{Ker} \Box_n$, where $\Box_n = \frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace – Beltrami operator (see, e.g. [3]). The asymptotical properties of the Poisson transform was researched in [1]. The aim of our paper is to use the above transform for the obtaining of formulae for Legendre functions.

Let us consider the spherical, parabolic and hyperbolic contours on $O_0$. Each of them intersect any or almost any generatrix. Let $\gamma_1$ be the intersection of cone $O_0$ and plane $x_0 = 1$. Then $H_{\gamma_1} \cong SO(n)$ and $\text{d} \gamma_1 = \frac{\partial x_1 \ldots \partial x_{n-1}}{\partial x_{n}}$, where $\gamma \in S_n$ (symmetric group). We denote the intersection of $O_0$ and plane $x_0 + x_n = 1$ by symbol $\gamma_2$. The subgroup $H_{\gamma_2}$ consists of matrices

$$n(b) = \begin{pmatrix} \text{diag}(1, \ldots, 1) & b^T & b^T \\ -b & 1 - b^* & -b^* \\ b & b^* & b^* \end{pmatrix},$$

where $b = (b_1, \ldots, b_{n-1})$ and $b^* = \frac{1}{2}(b^2_1 + \ldots + b^2_{n-1})$. Here $\text{d} \gamma_2 = \partial x_1 \ldots \partial x_{n-1}$. Let $\gamma_3, \ldots$ be the intersection of cone $O_0$ and plane $x_0 + x_n = 1$. If $\gamma_3 = \gamma_{3+} \cup \gamma_{3-}$, then $H_{\gamma_3} = SO(n-1, 1)$ and

$$\text{d} \gamma_3 = \frac{\partial x_0 \partial x_1 \ldots \partial x_{n-2}}{|x_{n-1}|},$$

where $\gamma \in S_{n-1}$.

For any contour, we define a basis in the space of restrictions of functions $f \in \mathcal{C}_\sigma$ to this contour. This basis can be continued on $O_0$ via a homogeneity. Finally, we have the following bases (named as spherical, parabolic, and hyperbolic basis, respectively): \{\text{f}_{k, K} : k \geq k_1\}, \{\text{f}_{\lambda, L} : \lambda \geq 0\}, \{\text{f}_{\mu, M, \pm} : \mu \in \mathbb{R}\}$, where the short notation $T$ denotes $T = (t_1, \ldots, t_{n-2})$, integers $t_i$ satisfy the conditions $t_i \geq t_{i+1} \geq |t_{n-2}|$, and

$$f_{k, K}(x) = N_{(k, K)} x_0^{\sigma-k} \Xi_{(k, K)}(x),$$

$$f_{\lambda, L}(x) = N_{(\lambda, L)} x_0^{\sigma-n+3} \left(\frac{\lambda}{2}\right)^{\frac{n}{2}-1} x_1^{\frac{n}{2}-1} \left(\frac{\lambda x_n-1}{2}\right)^{\frac{3-n}{2}-1} \frac{x_{n-1}}{x_n} \Xi_L(x),$$

$$f_{\mu, M, \pm}(x) = N_{(\mu, M, \pm)} x_0^{\sigma+3} \frac{\lambda}{2} \frac{\lambda x_n-1}{2} x_1^{\frac{3-n}{2}-m_1} P_{\frac{3-n}{2}-m_1}^\pm \left(\frac{1}{2} \frac{\lambda x_n-1}{2} \frac{x_n}{x_n}\right) \Xi_M(x),$$

$$\Xi_{(t_1, \ldots, t_s)}(x) = \sum_{i=1}^{s-3} r_{s-i}^{t_i-t_{i+1}} C_{t_i-t_{i+1}} x_{s-i} r_{s-i} (x_2 + i x_1)^{\lambda x_n},$$

where $[4]$

$$t_1 \geq \ldots \geq t_{s-3} \geq |t_{s-2}| \geq 0, \quad r_j^2 = x_1^2 + \ldots + x_j^2,$$

and

$$(x_n)^{\mu}_\pm = \left\{ \begin{array}{ll} |x_n|^\mu, & \text{for } x \in \gamma_{3\pm}, \\
0, & \text{for } x \in \gamma_{3\mp}, \end{array} \right.$$
2. Formulae induced by the Poisson transform

Let us consider the case \( SO(3,1) \) of Lorentz group. Moving to this group, let us note that length of the multi-indexes \( K, L, M \) is equal to 1, therefore, \( K \equiv k_1 \) and so on. We have the distribution

\[
 f_{k_1}(x) = \sum_{k=0}^{\infty} \sum_{k_1=-|k|} c_{k_1,k_1} f_{k_1}(x).
\]

If \( x \in \gamma_1 \), then \( x = (1, \sin \alpha \sin \beta, \sin \alpha \cos \beta, \cos \alpha) \), where \( \alpha \in [0; \pi) \) and \( \beta \in [0; 2\pi) \). For any \( x \in \gamma_2 \), we can write \( x = \left( \frac{1+u^2}{2}, u \sin \iota, u \cos \iota, \frac{1-u^2}{2} \right) \). Here \( u \geq 0, \iota \in [0; 2\pi) \). We will deal with the restrictions

\[
 f_{k,k_1}(\alpha, \beta) = C_{k-k_1}^{-1}(\cos \alpha) \sin^{k_1} \alpha e^{ik_1\beta},
\]

\[
 f_{\lambda,l_1}(u, \iota) = 2^{-l_1} e^{\iota l_1} J_{l_1}(\lambda u).
\]

From the formula

\[
 \int_{-1}^{1} (1-s^2)^{\nu-\frac{1}{2}} C_{a}^\nu(s) C_{b}^\nu(s) \, ds = 0,
\]

where \( a \neq b, \text{re} \nu > -\frac{1}{2} \) (see, e.g. 7.313(1) in [2]), we derive

\[
 f_{\lambda,l_1}(x) = \sum_{k=0}^{\infty} c_{\lambda,l_1,k,-l_1} f_{k,-l_1}(x).
\]

Let \( l_1 = 0 \). Then

\[
 \Pi(f_{\lambda,0}) = \sum_{k=0}^{\infty} c_{\lambda,0,k,0} \Pi(f_{k,0}).
\]

We choose \( \gamma_2 \) in the left side of this equality and \( \gamma_1 \) in the right side. Let \( y(v) = (\frac{v-1+v}{2}, 0, \ldots, 0, \frac{v-1-v}{2}) \) in the left side and \( y(p) = (\cosh p, 0, \ldots, 0, \sinh p) \) in the opposite side, so \( v = e^{-p} \). In addition, \( c_{\lambda,l_1,k,k_1} = F_{\gamma}(f_{\lambda,l_1}, f_{k,k_1}) \), where we choose \( \gamma = \gamma_2 \). Finally, we have

\[
 G_{13}^{21} \left( \left( \frac{\lambda}{2e^{p}} \right)^2 \right| \begin{array}{c} 0 \ 0 \ \sigma \end{array} \right) = 2^{-\sigma-\frac{1}{2}} \pi^\frac{3}{2} e^{\sigma p} \sinh^\frac{1}{2} p \sin^{-1}[\pi(\sigma + 1)]
\]

\[
 \cdot \sum_{k=0}^{\infty} (-1)^k (k!)^{-1} c_{\lambda,0,k,0} \Gamma(k+1) \Gamma^{-1}(\sigma - k + 1) P_{\sigma+\frac{1}{2}}^{-k+\frac{1}{2}}(\cosh p).
\]

Here \( G_{pq}^{mn} \) means the Meijer \( G \)-function.

By analogy, from the distribution

\[
 f_{k,k_1}(x) = \int_{0}^{+\infty} c_{k,k_1,k,-k_1} f_{\lambda,-k_1}(x) \, d\lambda,
\]

we obtain

\[
 P_{-\sigma-\frac{1}{2}}^{-\frac{3}{2}}(\cosh p) = 2\pi^{-\frac{3}{2}} \sinh^\frac{1}{2} p \Gamma^{-1}(-\sigma)
\]

\[
 \cdot \int_{0}^{+\infty} G_{13}^{21} \left( \left( \frac{\lambda^2}{4} \right| \begin{array}{c} 0 \ 0 \ \sigma - 1 \end{array} \right) \right) G_{13}^{21} \left( \left( \frac{\lambda}{2e^{p}} \right)^2 \right| \begin{array}{c} 0 \ 0 \ \sigma - 1 \end{array} \right) d\lambda.
\]
In the same way,

\[ f_{\mu, m_1, \pm}(x) = \int_{0}^{+\infty} c_{\mu, m_1, \pm, \lambda, -m_1} f_{\lambda, -m_1}(x) \, d\lambda \]

and, consequently, for instance,

\[ \Pi(f_{\mu, m_1, +}) = \int_{0}^{+\infty} c_{\mu, m_1, +, \lambda, -m_1} \Pi(f_{\lambda, -m_1}) \, d\lambda. \]

We choose \( \gamma_4 \) (in fact, \( \gamma_3^+ \)) in the left side of this equality and \( \gamma_2 \) in the right side. Let us choose the following parametrization in \( \gamma_3^+ \): \( x = (\cosh \xi, \sinh \xi \sin \eta, \sinh \xi \cos \eta, \pm 1) \). Then

\[ f_{\mu, m_1, \pm}(\xi, \eta) = \begin{cases} \frac{P^{m_1}}{\xi^{3/2}} (\cosh \xi) \sinh m_1 \xi e^{i m_1 \eta}, & \text{for } x(\xi, \eta) \in \gamma_3^\pm, \\ 0, & \text{for } x(\xi, \eta) \in \gamma_3^\mp. \end{cases} \]

It follows to

\[ P^{\sigma + 1/2 + i \mu} (\tanh p) = \sqrt{2} \pi^{3/2} \cosh p \sin^{-1}[\pi(\sigma + 1)] \]

\[ \cdot \Gamma^{-1} \left( i \mu - \sigma - \frac{1}{2} \right) \Gamma^{-1} \left( -\frac{3}{2} - \sigma - i \mu \right) \sum_{k=0}^{\infty} (-1)^k (k!)^{-1} \]

\[ \cdot N(k, k_1) \sinh^{1/2} p \Gamma(k + 1) \Gamma^{-1}(\sigma - k + 1) P_{\sigma + 1/2}^{-1}(\cosh p). \]

In the simplest case \( SO(2, 1) \) of Lorentz group, we have \( x(\alpha) = (1, \cos \alpha, \sin \alpha) \in \gamma_1 \) and \( x(z) = \left( \frac{1 + z^2}{2}, \frac{1 - z^2}{2}, z \right) \in \gamma_2 \), and, for the decomposition

\[ \Pi(f_\lambda) = \sum_{k \in \mathbb{Z}} c_{\lambda k} \Pi(f_k), \]

we obtain, if \( \lambda \neq 0 \),

\[ c_{\lambda k} = F_{\gamma_2}(f_k, f_k) = 2^{2\sigma + 1} e^{-|\lambda|} \sin \gamma \Gamma^{-1}(\sigma - k \sin \gamma) \gamma_1 \Gamma^{-1}(\sigma - k \sin \gamma) \Gamma^{-1}(\sigma + k \sin \gamma + 1) \Gamma^{-1}(\sigma - k \sin \gamma) G_{12}^{21} \left( 2|\lambda| \right) \left( \frac{-\sigma - k \sin \gamma}{0, -2\sigma - 1} \right). \]

Further,

\[ \Pi(f_\lambda) = F_{\gamma_2}(f_\lambda, \hat{q}^{-\sigma - 1}) = 2^{-\sigma - 1/2} \pi^{-1} \cosh \gamma \int_{-\infty}^{+\infty} \left( z^2 - 2z \tanh p + 1 \right)^{-\sigma - 1} e^{i k z} \, dz, \]

\[ \Pi(f_k) = F_{\gamma_1}(f_k, \hat{q}^{-\sigma - 1}) = \int_{0}^{2\pi} (\cosh p - \sin \alpha \sinh p)^{-\sigma - 1} e^{i k \alpha \sin \gamma} \, d\alpha. \]

It follows to

\[ K_{\sigma + 1/2} \left( \frac{|\lambda|}{\tanh p} \right) = \sqrt{\frac{\pi \cosh p}{2}} e^{-i \lambda \cosh p} \sum_{k \in \mathbb{Z}} (-i \sin \lambda)^k P_{\sigma}^k (\cosh p) W_{-i \sin \lambda, \sigma + 1/2, \text{second kind}} (2|\lambda|). \]

In this formula, \( K_{\sigma + 1/2} \left( \frac{|\lambda|}{\tanh p} \right) \) denotes the modified Bessel function of the second kind.
3. Formulae induced by the subrepresentation

For example, let us consider the subgroup of hyperbolic rotations in plane $x_0x_{n-1}$. This one-parameter subgroup consists of matrices

$$h(\varrho) =\begin{pmatrix} \cosh \varrho & 0 & \sinh \varrho & 0 \\ 0 & \text{diag}(1,\ldots,1) & 0 & 0 \\ \sinh \varrho & 0 & \cosh \varrho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Let $t_{\ldots \ldots \ldots}(\varrho)$ be the matrix element of the restriction of $T_\sigma$ to above subgroup with respect to basis $\{f_{\ldots \ldots \ldots} : \lambda \geq 0\}$. This matrix element can be obtained via the functionals $F_\gamma$, i.e.

$$t_{\ldots \ldots \ldots}(\varrho) = F_\gamma(T_\sigma(h(\varrho))[f_{\ldots \ldots \ldots}, f_{\ldots \ldots \ldots}].$$

Let us choose $x \in O_0$ such that $x_i = 0$ for $1 \leq i \leq n-2$. Then $f_{\ldots \ldots \ldots}(x) = 0$ under the condition $l_i \neq 0$ for $i \leq 2$. We choose $\gamma = \gamma_2$ and use the addition theorem for Bessel functions (see, e.g. 9.3.6(6) in [4]).

It is possible to prove that $F_\gamma$ is invariant with respect to any pair $(T_\sigma(h(\varrho)), T_{1-\sigma-n}(h(\varrho)))$, i.e.

$$t_{\ldots \ldots \ldots}(\varrho) = F_\gamma(f_{\ldots \ldots \ldots}, T_{1-\sigma-n}(h(\varrho))[f_{\ldots \ldots \ldots}]).$$

It follows from the decomposition $G = KAK$, where $K$ is a maximal compact subgroup (indeed, $K = H_{n_1}$) and $A$ is an maximal Abelian subgroup (see, for instance, [6]). Let us express the functions $f_{\ldots \ldots \ldots}$ and $f_{\ldots \ldots \ldots}$ via functions of the basis $\{f_{\ldots \ldots \ldots} : \mu \in \mathbb{R}\}$. We need the notation $E := c_{\ldots \ldots \ldots} + c_{\ldots \ldots \ldots} + c_{\ldots \ldots \ldots} + c_{\ldots \ldots \ldots} + c_{\ldots \ldots \ldots}$. Suppose $n > 3$, $\varrho > 0$ and $0 < \lambda \cosh^2 \varrho < \lambda^2$. From the comparison of two approaches for $t_{\ldots \ldots \ldots}(\varrho)$, we have

$$\int_{-\infty}^{+\infty} E \Gamma^2 \left( \frac{4-n}{2} - \mu i \right) P_{\frac{3-n}{2} + \mu i}^\frac{3-n}{2} (\cosh \varrho) d\mu$$

$$= 2 \frac{3-n}{2} n^{-3} \left( \pi(n-3) \right)^{-1} \lambda^{\frac{1-n}{2}} \sinh \frac{n-3}{2} \varrho \cosh \varrho \left( \frac{n-3}{2} \right) \sum_{s=0}^\infty (-1)^s (s!)^{-1}$$

$$\cdot \left( s + \frac{n-3}{2} \right) \Gamma(n+s-3) \Gamma \left( \frac{s+1}{2} \right) \Gamma^{-1} \left( \frac{n-s}{2} - 1 \right)$$

$$\cdot J_{\frac{n-3}{2}+s} \left( \frac{\lambda \sinh \varrho}{\cosh^2 \varrho} \right) 2F_1 \left( \frac{s+1}{2}, 2 - \frac{n-3}{2}; 2 - \frac{n-1}{2}; \left( \lambda \cosh^2 \varrho \right)^2 \right).$$

Here $2F_1$ denotes the Gauss hypergeometric function.

Acknowledgments

The authors wish to acknowledge the Ministry of Education and Science of the Russian Federation: the research described in this paper is supported by grant NK 586-P30.
Appendix

The logical continuation of this paper follows to the consideration of bilinear functionals of more general type. For instance, recently, for the rudimentary Lorentz group $SO(2,1)$, the authors study the functionals

$$D_{\gamma_i} : \mathbb{C}_2^\sigma \rightarrow \mathbb{C},\ (u,v) \mapsto \int_{\gamma_i} \int_{\gamma_i} k(x, \hat{x}) u(x) v(\hat{x}) \, dx \, d\hat{x},$$

where the kernel $k$ provides that $D_{\gamma_i}$ is invariant with respect to $T_\sigma$, i.e., for any $g \in SO(2,1)$,

$$D_{\gamma_i}(T_\sigma(g)[u], T_\sigma(g)[v]) = D_{\gamma_i}(u,v).$$

The authors obtained recently the restrictions of the kernel $k$ to the direct products $\gamma_i \times \gamma_i$ of contours. Using these functionals, the authors derived some new formulas for Whittaker functions $M_{\mu,\nu}$ and $W_{\mu,\nu}$. At present time, the corresponding paper is preparing for publication.

References

[1] Artemov A A 2004 The Poisson transform for a hyperboloid of one sheet Sb. Math. 195 643
[2] Gradshteyn I S and Ryzhik I M 2007 Table of Integrals, Series, and Products (New York: Academic Press)
[3] Shtepina T V 2004 Generalization of the Funk-Hecke theorem to the case of a hyperbolic space Izv. Math. 68 1051
[4] Vilenkin N Ja 1968 Special Functions and Theory of Group Representations (Providence, R. I.: AMS)
[5] Vilenkin N Ja and Klimyk A U 1993 Representation of Lie Groups and Special Functions vol 2 (Dordrecht: Kluwer Academic Publishers)
[6] Wehrhahn R F, Smirnov Yu F and Shirokov A M 1992 Symmetry scattering on the hyperboloid $SO(2,1)/SO(2)$ in different coordinate systems J. Math. Phys. 33 2384