STABILIZERS IN HIGMAN–THOMPSON GROUPS

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Abstract. We investigate stabilizers of finite sets of rational points in Cantor space for the Higman–Thompson groups $V_{n,r}$. We prove that the pointwise stabilizer is an iterated ascending HNN extension of $V_{n,q}$ for any $q \geq 1$. We also prove that the commutator subgroup of the pointwise stabilizer is simple, and we compute the abelianization. Finally, for each $n$ we classify such pointwise stabilizers up to isomorphism.

In recent work, Golan and Sapir use iterated ascending HNN extensions to describe the stabilizer of any finite set of rational points in $(0, 1)$, and they classify such stabilizers up to isomorphism [4]. Here we record analogous results for stabilizers in Thompson’s group $V$ and more generally for the Higman–Thompson group $V_{n,r}$.

For $n \geq 2$ and $r \geq 1$, let $C_{n,r}$ denote the Cantor space $\{1, \ldots, r\} \times \{1, \ldots, n\}^\omega$, and let $V_{n,r}$ be the associated Higman–Thompson group (denoted $G_{n,r}$ in [2, 5]). A point in $C_{n,r}$ is said to be rational if the associated sequence is eventually repeating. If $S$ is a finite set of rational points in $C_{n,r}$, the stabilizer $\text{Stab}(S)$ is the group of elements of $V_{n,r}$ that fix $S$ setwise, the pointwise stabilizer $\text{Fix}(S)$ is the subgroup of elements that fix $S$ pointwise. Let $\text{Fix}_0(S)$ be the subgroup consisting of elements that are the identity in some neighborhood of $S$.

Our main theorem is the following:

**Theorem 1.** Let $n \geq 2$ and $q, r \geq 1$, and let $S \subset C_{n,r}$ be a finite, nonempty set of rational points. Then:

1. $\text{Fix}(S)$ is an iterated ascending HNN extension of $V_{n,q}$.
2. $\text{Fix}(S)$ and $\text{Stab}(S)$ have type $F_\infty$.
3. The commutator subgroup of $\text{Fix}(S)$ is simple, and is the intersection of $\text{Fix}_0(S)$ with the commutator subgroup of $V_{n,r}$.
4. The abelianization of $\text{Fix}(S)$ is the direct sum of $\mathbb{Z}^{|S|}$ with the abelianization of $V_{n,r}$.

For (1), recall that a group $G$ is an ascending HNN extension of a subgroup $H$ if $G$ has presentation

$$G = \langle H, t \mid t^h t^{-1} = \varphi(h) \text{ for all } h \in H \rangle$$

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where $\varphi: H \to H$ is a monomorphism. More generally, $G$ is an \textbf{iterated ascending HNN extension} of $H$ if there exists a chain

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

of subgroups of $H$ such that each $H_i$ is an ascending HNN extension of $H_{i-1}$. We prove (1) in Section 2 below.

For (2), it is well known that an ascending HNN extension of a group of type $F_\infty$ has type $F_\infty$. Since the groups $V_{n,r}$ all have type $F_\infty$ by a result of Brown [2, Theorem 7.3.1], it follows from statement (1) above that Fix($S$) always has type $F_\infty$ when $S$ is a finite set of rational points. Since Fix($S$) has finite index in Stab($S$), the stabilizer has type $F_\infty$ as well, and thus (2) follows from (1). The authors will use these results in [1] to prove finiteness properties for certain interesting groups, including a class of Röver–Nekrashevych groups.

For parts (3) and (4), recall that Higman proved that the commutator subgroup of $V_{n,r}$ is always simple, and that the abelianization of $V_{n,r}$ is trivial if $n$ is even and cyclic of order two if $n$ is odd [5, Chapter 5]. Thus the abelianization of Fix($S$) is $\mathbb{Z}^{|S|}$ if $n$ is even and $\mathbb{Z}^{|S|} \oplus \mathbb{Z}_2$ if $n$ is odd. We prove (3) and (4) in Theorems 10 and 11 below.

In addition to the above results, the following theorem classifies the isomorphism types of the groups Fix($S$).

\textbf{Theorem 2.} Let $n \geq 2$ and $q, r \geq 1$, and let $S \subseteq C_{n,q}$ and $S' \subseteq C_{n,r}$ be finite, nonempty sets of rational points. Then Fix($S$) and Fix($S'$) are isomorphic if and only if $|S| = |S'|$.

The proof of this theorem can be found at the end of Section 3.

1. Background

1.1. \textbf{Thompson-like homeomorphisms.} We view $C_{n,r}$ as a space of infinite sequences of digits, where the first digit of each sequence lies in $\{1, \ldots, r\}$ and the remaining digits lie in $\{1, \ldots, n\}$. If $\alpha \in \{1, \ldots, r\} \times \{1, \ldots, n\}^*$ is any finite prefix, the associated cone $C_\alpha$ consists of all sequences in $C_{n,r}$ that have $\alpha$ as a prefix. Note that the clopen sets in $C_{n,r}$ are precisely the finite unions of cones.

Given any two cones $C_\alpha \subseteq C_{n,q}$ and $C_\beta \subseteq C_{n,r}$, the corresponding \textbf{prefix-replacement homeomorphism} from $C_\alpha$ to $C_\beta$ is the function that maps any word $\alpha \psi \in C_\alpha$ to the corresponding word $\beta \psi \in C_\beta$. If $U \subseteq C_{n,q}$ and $U' \subseteq C_{n,r}$ are open, we say that a homeomorphism $f: U \to U'$ is \textbf{Thompson-like} if each point of $U$ is contained in a cone on which $f$ restricts to a prefix replacement. The \textbf{Higman–Thompson group} $V_{n,r}$ is the group of all Thompson-like homeomorphisms of $C_{n,r}$.

1.2. \textbf{Types of clopen sets.} Every clopen set $E \subseteq C_{n,r}$ can be expressed as a finite, disjoint union $C_{\alpha_1} \uplus \cdots \uplus C_{\alpha_t}$ of cones. Such a decomposition is not unique, since we can replace any cone $C_{\alpha_j}$ by the disjoint union of the $n$ subcones $\{C_{\alpha_{j,i}}\}_{i=1}^n$. However, any two decompositions of the same clopen set into cones must have the same number of cones modulo $n-1$. We refer to this element of $\mathbb{Z}_{n-1}$ as the \textbf{type} of $E$, denoted type($E$). Note that any cone has type 1, the whole Cantor space $C_{n,r}$ has type $r$ (modulo $n-1$), and

$$\text{type}(E_1 \cup E_2) = \text{type}(E_1) + \text{type}(E_2) - \text{type}(E_1 \cap E_2)$$
for any two clopen sets \(E_1, E_2 \subseteq \mathcal{C}_{n,r}\). As an immediate consequence, we see that
\[\text{type}(E \setminus F) = \text{type}(E) - \text{type}(E \cap F),\]
for any clopen sets \(E, F \subseteq \mathcal{C}_{n,r}\).

**Proposition 3.** Let \(n \geq 2\), let \(q, r \geq 1\), and let \(E \subseteq \mathcal{C}_{n,q}\) and \(E' \subseteq \mathcal{C}_{n,r}\) be clopen sets. Then there exists a Thompson-like homeomorphism \(f : E \rightarrow E'\) if and only if \(\text{type}(E) = \text{type}(E')\).

**Proof.** If a Thompson-like homeomorphism \(f : E \rightarrow E'\) exists, then there is a suitable subdivision of \(E\) into cones so that their images are cones too, and therefore \(\text{type}(E) = \text{type}(E')\). Conversely, if \(\text{type}(E) = \text{type}(E')\), there exists a positive integer \(m\) with
\[m \equiv \text{type}(E) \pmod{n - 1}\]
and suitable cones \(C_1, \ldots, C_m \subseteq \mathcal{C}_{n,q}\) and \(C'_1, \ldots, C'_m \subseteq \mathcal{C}_{n,r}\) so that \(E = \bigcup_{i=1}^m C_i\) and \(E' = \bigcup_{i=1}^m C'_i\). In this case, we can define a Thompson-like homeomorphism \(f : E \rightarrow E'\) by mapping each \(C_i\) to \(C'_i\) by a prefix replacement. \(\square\)

**Corollary 4.** Let \(E \subseteq \mathcal{C}_{n,r}\) be a proper, nonempty clopen set, and let \(q = \text{type}(E)\). Then \(\text{Fix}(\mathcal{C}_{n,r} \setminus E) \cong \mathcal{V}_{n,q}\).

**Proof.** By Proposition 3 there is a Thompson-like homeomorphism \(h : E \rightarrow \mathcal{C}_{n,q}\). Then \(f \mapsto h(f|_E)h^{-1}\) defines an isomorphism \(\text{Fix}(\mathcal{C}_{n,r} \setminus E) \rightarrow \mathcal{V}_{n,q}\). \(\square\)

1.3. **Germs at fixed points.** If \(p \in \mathcal{C}_{n,r}\) is a rational point, the group of germs at \(p\) is the quotient
\[(V_{n,r})_p = \text{Fix}(p)/\text{Fix}_0(p).\]
If \(f \in \text{Fix}(p)\), we let \((f)_p\) denote its image in \((V_{n,r})_p\). Note that if \(f, g \in \text{Fix}(p)\), then \((f)_p = (g)_p\) if and only if \(f\) and \(g\) agree on a neighborhood of \(p\).

**Proposition 5.** Let \(p \in \mathcal{C}_{n,r}\) be a rational point, and write \(p = \alpha \beta\) for some finite sequences \(\alpha\) and \(\beta\), where \(\beta\) is not a power of a shorter sequence. Then there exists an \(f \in V_{n,r}\) that agrees with the prefix replacement \(\alpha \psi \mapsto \alpha \beta \psi\) in a neighborhood of \(p\), and \((V_{n,r})_p\) is the infinite cyclic group generated by \((f)_p\).

**Proof.** Note that \(C_{\alpha \beta}\) has nonempty complement. Since \(C_{\alpha \beta}\) and \(C_{\alpha \beta^2}\) have type 1, their complements have type \(r - 1\), so by Proposition 3 there exists a Thompson-like homeomorphism \(h : \mathcal{C}_{n,r} \setminus C_{\alpha \beta} \rightarrow \mathcal{C}_{n,r} \setminus C_{\alpha \beta^2}\). Let \(f\) be the element of \(V_{n,r}\) which agrees with \(h\) on \(\mathcal{C}_{n,r} \setminus C_{\alpha \beta}\) and maps \(C_{\alpha \beta}\) to \(C_{\alpha \beta^2}\) by a prefix replacement. Then \(f\) agrees with the prefix replacement \(\alpha \psi \mapsto \alpha \beta \psi\) on \(C_{\alpha \beta}\).

Now observe that if \(g \in \text{Fix}(p)\), then \(g\) must act as a prefix replacement of the form \(\alpha \psi \mapsto \alpha \beta^k \psi\) or \(\alpha \beta^k \psi \mapsto \alpha \psi\) in a neighborhood of \(p\) for some \(k \geq 0\). In the first case, \(g\) agrees with \(f^k\) in a neighborhood of \(p\), and hence \((g)_p = (f)_p^k\). In the second case, \(g\) agrees with \(f^{-k}\) in a neighborhood of \(p\), and hence \((g)_p = (f)_p^{-k}\). Thus \((V_{n,r})_p\) is precisely the infinite cyclic group generated by \((f)_p\). \(\square\)

We refer to the germ \((f)_p\) described in Proposition 5 as the attracting generator for \((V_{n,r})_p\).

We will need the following generalization of Proposition 5.

**Proposition 6.** Let \(S \subseteq \mathcal{C}_{n,r}\) be a finite set of rational points, and let \(s \in S\). Then there exists an \(f \in \text{Fix}_0(S \setminus \{s\})\) that fixes \(s\) such that \((f)_s\) is the attracting generator for \((V_{n,r})_s\).
Proof. Let $E$ be a clopen neighborhood of $S \setminus \{s\}$ that does not contain $s$. Write $s = \alpha \beta$, where $\alpha$ and $\beta$ are finite words and $\beta$ is not a power of a shorter word. Replacing $\alpha$ with $\alpha \beta^k$ for some $k$, we may assume that $C_\alpha$ is properly contained in $\mathcal{C}_{n,r} \setminus E$. Since $\text{type}(C_\alpha) = \text{type}(C_{\alpha \beta}) = 1$, by the argument at the end of Subsection 1.2, we see that

$$\text{type}(\mathcal{C}_{n,r} \setminus (E \sqcup C_\alpha)) = \text{type}(\mathcal{C}_{n,r} \setminus (E \sqcup C_{\alpha \beta}))$$

so by Proposition 3 there is a Thompson-like homeomorphism

$$h: \mathcal{C}_{n,r} \setminus (E \sqcup C_\alpha) \to \mathcal{C}_{n,r} \setminus (E \sqcup C_{\alpha \beta}).$$

Then the element $f \in V_{n,r}$ that is the identity on $E$, maps $C_\alpha$ to $C_{\alpha \beta}$ by a prefix replacement, and agrees with $h$ on $\mathcal{C}_{n,r} \setminus (E \sqcup C_\alpha)$ has the desired properties. □

**Corollary 7.** If $S \subset \mathcal{C}_{n,r}$ is a finite set of rational points, then there exists an $f \in \text{Fix}(S)$ so that $(f)_s$ is the attracting generator for $(V_{n,r})_s$ for each $s \in S$.

**Proof.** Let $S = \{s_1, \ldots, s_m\}$. By Proposition 4, there exists for each $i$ an element $f_i \in \text{Fix}_0(S \setminus \{s_i\})$ that fixes $s_i$ such that $(f_i)_s$ is the attracting generator for $(V_{n,r})_s$. Then the product $f = f_1 \cdots f_m$ has the desired property. □

2. ASCENDING HNN EXTENSIONS

In this section we prove that pointwise stabilizers are ascending HNN extensions. It follows from Bass-Serre theory (see [6]) that a group $G$ is an ascending HNN extension of a group $H$ if and only if $G$ acts by automorphisms on some directed tree $\Gamma$ with the following properties:

1. $\Gamma$ has exactly one orbit of vertices under $G$.
2. The stabilizer of each vertex of $\Gamma$ is isomorphic to $H$.
3. Each vertex of $\Gamma$ has exactly one outgoing edge.

The following proposition translates this geometry into algebra.

**Proposition 8.** Let $G$ be a group. Let $t \in G$ have infinite order, let $H \leq G$, and suppose that

1. $\langle t \rangle \cap H = 1$,
2. $t^{-1} H t \subseteq H$,
3. $\bigcup_{i,j \in \mathbb{N}} t^i H t^{-j} = G$.

Then $G$ is an ascending HNN extension of $H$ by $t$.

**Proof.** Let $\Gamma$ be the directed graph whose vertices are the left cosets of $H$ in $G$, with a directed edge from $gH$ to $gtH$ for every vertex $gH$. Note that if $gH = g'H$ for some $g, g' \in G$, then since $t^{-1} H t \subseteq H$ we have $gtH = g'tH$, and thus $G$ has only one directed edge emanating from each vertex. Clearly $G$ acts on $\Gamma$ by automorphisms with one orbit of vertices and one orbit of edges, so it suffices to prove that $\Gamma$ is a directed tree.

Note first that $\Gamma$ is connected, for by condition (3) if $gH$ is any vertex of $\Gamma$ then $gt^i H = t^i H$ for some $i, j \in \mathbb{N}$, and hence $gH$ lies in the same component of $\Gamma$ as the vertex $H$. To prove that $\Gamma$ has no directed cycles, let $K$ be the ascending union of the subgroups $\{t^n H t^{-n}\}_{n \in \mathbb{N}}$. It follows from condition (1) that $K \cap \langle t \rangle = 1$, so by condition (3) the group $G$ is the disjoint union of the cosets $\{t^j K\}_{j \in \mathbb{Z}}$, each of which is a disjoint union of cosets of $H$. Since $gtH \subseteq t^{j+1} K$ whenever $gH \subseteq t^j K$, we conclude that $\Gamma$ contains no directed cycle.
we conclude that the graph $\Gamma$ has no directed cycles. Since every vertex of $\Gamma$ has only one outgoing edge, it follows that $\Gamma$ is a directed tree.

We can now give a proof of the first part of Theorem 1.

**Theorem 9.** Let $n \geq 2$ and $q, r \geq 1$, and let $S$ be a finite, nonempty set of rational points in $\mathcal{C}_{n,r}$. Then $\text{Fix}(S)$ is an iterated ascending HNN extension of $V_{n,q}$.

**Proof.** We prove the result by induction on $|S|$. Fix a point $s \in S$. By Proposition 4 there exists an element $f \in \text{Fix}(S)$ such that $(f)_s$ is the attracting generator for $(V_{n,r})_s$. Let $C_\alpha$ be a cone containing $s$ on which $f$ acts as a prefix replacement $\alpha \psi \mapsto \alpha \beta \psi$, and note that $C_\alpha$ is disjoint from $S \setminus \{s\}$. Let $T$ be a clopen subset of $C_\alpha$ that contains $C_{\alpha \beta}$ and satisfies

$$\text{type}(T) \equiv r - q \pmod{n - 1},$$

and let $H = \text{Fix}(S \cup T)$. Since

$$\text{type}(\mathcal{C}_{n,r} \setminus T) \equiv q \pmod{n - 1}$$

it follows from Proposition 8 that there exists a Thompson-like homeomorphism $k: \mathcal{C}_{n,r} \setminus T \to \mathcal{C}_{n,q}$, which determines an isomorphism $\text{Fix}(T) \to V_{n,q}$ as in Corollary 4. If $|S| = 1$, then $H = \text{Fix}(T)$ and hence $H \cong V_{n,q}$. If $|S| \geq 2$ then this isomorphism instead maps $H$ to the subgroup $\text{Fix}(k(S \setminus \{s\}))$ of $V_{n,q}$, which by our induction hypothesis is an iterated ascending HNN extension of $V_{n,q}$. Thus it suffices to prove that $\text{Fix}(S)$ is an ascending HNN extension of $H$.

We verify the three conditions in Proposition 8. Clearly $f$ has infinite order. Next, since $f^{-1}(T)$ contains $T$ we have that $f^{-1}Hf = \text{Fix}(S \cup f^{-1}(T)) \subseteq H$. Finally, if $g \in \text{Fix}(S)$ then $(g)_s = (f)_s^i$ for some integer $i \in \mathbb{Z}$. Let $U$ be a neighborhood of $s$ on which $g$ agrees with $f^i$, and let $j \geq |i|$ so that $C_{\alpha \beta} \subseteq U$ and hence $f^j(T) \subseteq U$. Since $f^{-i}g$ is the identity on $U$, it follows that $f^{-i-j}gf^j$ is the identity on $T$. Then $f^{-i-j}gf^j$ lies in $H$, and therefore $g \in f^{i+j}Hf^{-j}$. □

3. The Commutator Subgroup

In this section we analyze the structure of the commutator subgroup $[\text{Fix}(S), \text{Fix}(S)]$ as well as the abelianization of $\text{Fix}(S)$.

**Theorem 10.** Let $n \geq 2$ and $r \geq 1$, and let $S \subset \mathcal{C}_{n,r}$ be a finite set of rational points. Then

$$[\text{Fix}(S), \text{Fix}(S)] = \text{Fix}_0(S) \cap [V_{n,r}, V_{n,r}],$$

and this group is simple.

**Proof.** Let $S = \{s_1, \ldots, s_m\}$, and let $\pi: \text{Fix}(S) \to \prod_{i=1}^m (V_{n,r})_{s_i}$ be the homomorphism defined by $\pi(f) = ((f)_{s_1}, \ldots, (f)_{s_m})$. Then the kernel of $\pi$ is $\text{Fix}_0(S)$ and the codomain is a free abelian group, so it follows that $[\text{Fix}(S), \text{Fix}(S)] \subseteq \text{Fix}_0(S)$, and hence $[\text{Fix}(S), \text{Fix}(S)] \subseteq \text{Fix}_0(S) \cap [V_{n,r}, V_{n,r}]$.

For the opposite inclusion, let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of nonempty clopen sets whose union is $\mathcal{C}_{n,r} \setminus S$, and let $G_i$ denote the subgroup of $V_{n,r}$ consisting of elements that are supported on $E_i$. Note that $\text{Fix}_0(S)$ is the ascending union of the subgroups $G_i$, and that each $G_i$ is isomorphic to $V_{n,1}$, where $t_i = \text{type}(E_i)$. We claim that $[G_i, G_i] = [V_{n,r}, V_{n,r}] \cap G_i$ for each $i$.

The inclusion $[G_i, G_i] \subseteq [V_{n,r}, V_{n,r}] \cap G_i$ is clear. For the opposite inclusion, if $n$ is even then $[G_i, G_i] = G_i$ and $[V_{n,r}, V_{n,r}] = V_{n,r}$ and we are done. If $n$ is odd,
then \([V_{n,r}, V_{n,r}]\) has index two in \(V_{n,r}\), and its complement contains all elements of order two (see [5, Chapter 5]). Then any element of order two in \(G_i\) is not contained in \([V_{n,r}, V_{n,r}]\), which means that \([V_{n,r}, V_{n,r}] \cap G_i\) is a proper subgroup of \(G_i\). Since \([G_i, G_j]\) has index two in \(G_i\) and \([G_i, G_i] \subseteq [V_{n,r}, V_{n,r}] \cap G_i\), it follows that \([G_i, G_j] = [V_{n,r}, V_{n,r}] \cap G_i\).

We conclude that

\[
\text{Fix}(S) \cap [V_{n,r}, V_{n,r}] = \bigcup_{i \in \mathbb{N}} G_i \cap [V_{n,r}, V_{n,r}] = \bigcup_{i \in \mathbb{N}} [G_i, G_i] \subseteq [\text{Fix}(S), \text{Fix}(S)]
\]

and therefore \([\text{Fix}(S), \text{Fix}(S)] = \text{Fix}(S) \cap [V_{n,r}, V_{n,r}]\). Moreover, since each \(G_i\) is isomorphic to \(V_{n,q}\) for some \(q \geq 1\), each of the commutator subgroups \([G_i, G_i]\) is simple [4], so the ascending union \([\text{Fix}(S), \text{Fix}(S)]\) must be simple as well. \(\square\)

**Theorem 11.** Let \(S \subseteq \mathbb{C}_{n,r}\) be a finite set of rational points. Then

\[
\text{Fix}(S)/[\text{Fix}(S), \text{Fix}(S)] \cong \begin{cases} 
\mathbb{Z}[S] & \text{if } n \text{ is even}, \\
\mathbb{Z}[S] \oplus \mathbb{Z}_2 & \text{if } n \text{ is odd}.
\end{cases}
\]

*Proof.* Let \(S = \{s_1, \ldots, s_m\}\) and let \(\pi: \text{Fix}(S) \to \prod_{i=1}^m (V_{n,r})_{s_i}\) be the homomorphism defined by

\[
\pi(f) = ((f)_{s_1}, \ldots, (f)_{s_m}).
\]

Let \(A = V_{n,r}/[V_{n,r}, V_{n,r}]\), and let \(\rho: V_{n,r} \to A\) be the quotient homomorphism. Note that \(\prod_{i=1}^m (V_{n,r})_{s_i} \cong \mathbb{Z}^m\), and that \(A\) is trivial if \(n\) is even and \(A \cong \mathbb{Z}_2\) if \(n\) is odd (see [5, Chapter 5]). Let \(\sigma: \text{Fix}(S) \to \prod_{i=1}^m (V_{n,r})_{s_i} \times A\) be the homomorphism \(\pi \times \rho\). Then

\[
\ker(\sigma) = \ker(\pi) \cap \ker(\rho) = \text{Fix}_0(S) \cap [V_{n,r}, V_{n,r}] = [\text{Fix}(S), \text{Fix}(S)]
\]

by Theorem 10, so it suffices to prove that \(\sigma\) is surjective.

By Proposition 6, there exists for each \(i\) an element \(f_i \in \text{Fix}_0(S) \setminus \{s_i\}\) so that \((f_i)_{s_i}\) is a generator for \((V_{n,r})_{s_i}\). Then \(\pi\) maps \(f_1, \ldots, f_m\) to a generating set for \(\prod_{i=1}^m (V_{n,r})_{s_i}\), and therefore \(\pi\) is surjective. Thus, all the remains is to prove that \(\text{Fix}_0(S) = \ker(\pi)\) maps onto \(A\) under \(\rho\).

If \(n\) is even then \(A\) is trivial and we are done. If \(n\) is odd, then any element of order two in \(V_{n,r}\) maps to the nontrivial element of \(A\) under \(\rho\). In particular, if we choose disjoint clopen sets \(E, E'\) of the same type that are disjoint from \(S\) and let \(h: E \to E'\) be a Thompson-like homeomorphism, then the element \(f \in V_{n,r}\) that agrees with \(h\) on \(E\), agrees with \(h^{-1}\) on \(E'\), and is the identity elsewhere lies in \(\text{Fix}_0(S)\) and maps to the nontrivial element of \(A\) under \(\rho\). \(\square\)

Finally, we are ready to prove the following

**Theorem 2.** Let \(n \geq 2\), let \(q, r \geq 1\), and let \(S \subseteq \mathbb{C}_{n,q}\) and \(S' \subseteq \mathbb{C}_{n,r}\) be finite sets of rational points. Then \(\text{Fix}(S)\) is isomorphic to \(\text{Fix}(S')\) if and only if \(|S| = |S'|\).

*Proof.* If \(\text{Fix}(S) \cong \text{Fix}(S')\) then \(|S| = |S'|\) by Theorem 11. For the converse, suppose \(|S| = |S'|\), and choose a bijection \(\varphi: S \to S'\).

By Corollary 4 there exists an \(f \in \text{Fix}(S)\) so that for each \(s \in S\), the germ \((f)_s\) is the attracting generator for \((V_{n,q})_s\). For each \(s \in S\), choose a cone containing \(s\) on which \(f\) acts as a prefix replacement, and let \(E\) be the complement of the union of these cones. Shrinking the cones if necessary, we may assume that \(E\) is nonempty. Note then that \(E\) is clopen, \(f(E) \supset E\), and \(\bigcup_{n \in \mathbb{N}} f^n(E) = \mathbb{C}_{n,q} \setminus S\).
Repeating the last paragraph for \( S' \), we obtain an element \( f' \in \text{Fix}(S') \) and a clopen set \( E' \subset C_{n,r} \) so that \((f')_s \) is the attracting generator for \((V_{n,r})_s \) for each \( s \in S' \), and moreover \( f'(E') \supset E' \) and \( \bigcup_{n \in \mathbb{N}} (f')^n(E') = C_{n,r} \backslash S' \). Replacing \( E' \) by a larger clopen subset of \( f(E') \) if necessary, we may assume that \( \text{type}(E') = \text{type}(E) \).

Now, observe that \( C_{n,q} \backslash S \) is the disjoint union of \( E \) and the sets

\[
E_k = f^k(E) \setminus f^{k-1}(E)
\]

for \( k \geq 1 \). Similarly, \( C_{n,r} \backslash S' \) is the disjoint union of \( E' \) and analogous sets \( E'_k \) \( (k \geq 1) \). Note that \( \text{type}(E_k) = \text{type}(E'_k) = 0 \) for each \( k \geq 1 \). Choose Thompson-like homeomorphisms \( h_0: E \to E' \) and \( h_1: E_1 \to E'_1 \), and let \( h: C_{n,q} \to C_{n,r} \) be the homeomorphism that agrees with \( \varphi \) on \( S \), agrees with \( h_0 \) on \( E \), and agrees with \( (f')^{n-1}h_1f^{1-n} \) on \( E_n \) for each \( n \geq 1 \). We claim that \( h \text{Fix}(S)h^{-1} = \text{Fix}(S') \), and hence \( \text{Fix}(S) \cong \text{Fix}(S') \).

By symmetry, it suffices to prove that \( h \text{Fix}(S)h^{-1} \subset \text{Fix}(S') \). If \( g \in \text{Fix}(S) \), then clearly \( hgh^{-1} \) fixes \( S' \) pointwise, so it suffices to prove that \( hgh^{-1} \) is an element of \( V_{n,r} \). Since \( h \) restricts to a Thompson-like homeomorphism \( C_{n,q} \backslash S \to C_{n,r} \backslash S' \), it suffices to prove that \( hgh^{-1} \) is Thompson-like in a neighborhood of \( S' \).

To see this, observe first that \( hgh^{-1} \) agrees with \( f' \) on \( C_{n,r} \backslash E' \). More generally, for every \( k \in \mathbb{Z} \) the functions \( hgh^{-1} \) and \( (f')^k \) agree in a neighborhood of \( S' \). For each \( s \in S \), we know that \( (f)_s \) is a generator for the infinite cyclic group \((V_{n,q})_s \), so \( g \) must agree with some power of \( f \) in a neighborhood of \( s \). It follows that \( hgh^{-1} \) agrees with some power of \( f' \) in a neighborhood of \( h(s) \), and therefore \( hgh^{-1} \) is Thompson-like in a neighborhood of \( h(s) \).

**Remark 12.** Note that \( \text{Fix}_0(S) \cong \text{Fix}_0(S') \) for any finite sets \( S \) and \( S' \) (regardless of size), since any Thompson-like homeomorphism \( C_{n,q} \backslash S \to C_{n,r} \backslash S' \) conjugates \( \text{Fix}_0(S) \) to \( \text{Fix}_0(S') \). (More generally, if \( U \subset C_{n,q} \) and \( U' \subset C_{n,r} \) are nonempty, non-compact open sets, then the subgroup of \( V_{n,q} \) consisting of elements supported on a compact subset of \( U \) is isomorphic to the subgroup of \( V_{n,r} \) consisting of elements supported on a compact subset of \( U' \).) Since \([\text{Fix}(S), \text{Fix}(S)]\) is the commutator subgroup of \( \text{Fix}_0(S) \), it follows that \([\text{Fix}(S), \text{Fix}(S)] \cong [\text{Fix}(S'), \text{Fix}(S')]\) for any nonempty finite sets \( S \subset C_{n,q} \) and \( S' \subset C_{n,r} \).

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