Abstract—This paper considers variational inequalities (VI) defined by the conditional value-at-risk (CVaR) of uncertain functions and provides three stochastic approximation schemes to solve them. All methods use an empirical estimate of the CVaR at each iteration. The first algorithm constrains the iterates to the feasible set using projection. To overcome the computational burden of projections, the second one handles inequality and equality constraints defining the feasible set differently. Particularly, projection onto the affine subspace defined by the equality constraints is achieved by matrix multiplication and inequalities are handled by using penalty functions. Finally, the third algorithm discards projections altogether by introducing multiplier updates. We establish asymptotic convergence of all our schemes to any arbitrary neighborhood of the solution of the VI. A simulation example concerning a network routing game illustrates our theoretical findings.

I. INTRODUCTION

A variety of equilibrium-seeking problems in game theory can be cast as a variational inequality (VI) problem [1]. For example, Nash equilibria of a game and Wardrop equilibria of a network routing game both correspond to solutions of a VI under mild conditions. Inspired by real-life, it is natural to perceive the costs or utilities that players wish to optimize in such games as uncertain. Faced with randomness, risk-preferences of players often determines their decisions. Players optimize a risk-measures of random costs in such scenarios and consequently, equilibria can be found by solving a VI involving risk-measures of uncertain costs. Motivated by this setup, we consider VIs defined by the conditional value-at-risk (CVaR) of random costs and develop stochastic approximation (SA) schemes to solve them.

Literature review: The most popular way of incorporating uncertainty in VIs is the stochastic variational inequality (SVI) problem, see e.g. [2] and references therein. Here, the map associated to the VI is the expectation of a random function. SA methods for solving SVI are well studied [2], [3]. A key feature of such schemes is the use of an unbiased estimators of the map using any number of sample of the uncertainty. This leads to strong convergence guarantees under a mild set of assumptions. However, the empirical estimator of CVaR, while being consistent, is biased [4]. Therefore, depending on the required level of precision, more samples are required to estimate the CVaR. This biasedness poses challenges in the convergence analysis of SA schemes. For a general discussion on risk-based VIs, including CVaR, and their potential applications, see [5]. In [6], a sample average approximation method for estimating the solution of CVaR-based VI was discussed. Our work also broadly relates to [7] and [8] where sample-based methods are used for optimizing the CVaR and other risk measures, respectively. The convergence analysis of our iterative methods consists of approximating the asymptotic behavior of iterates with a trajectory of a continuous-time dynamical system and studying their stability. See [9] and [10] for a comprehensive account of such analysis.

Statement of Contributions: Our starting point is the definition of the CVaR-based variational inequality (VI), where the map defining the VI consists of components that are the CVaR of random functions. The feasibility set, assumed to be a convex compact set, is defined by a set of inequality and linear equality constraints. We motivate the VI problem using two examples from noncooperative games. Our first contribution is the first SA scheme termed as the projected method. This iterative method consists of moving along the empirical estimate of the map defining the VI and projecting each iterate onto the feasibility set. We show that under strict monotonicity, the projected algorithm asymptotically converges to any arbitrary neighborhood of the solution of the VI, where the size of the neighborhood influences the number of samples required to form the empirical estimate in each iteration. Our second contribution is the subspace-constrained method that overcomes the computational burden of calculating projections onto the feasibility set by dealing with equality and inequalities differently. In particular, the proximity to satisfying inequality constraints is ensured using penalty functions and iterates are constrained on the subspace generated by linear equality constraints by pre-multiplying the iteration step by an appropriate matrix. We establish that under strict monotonicity, the algorithm converges asymptotically to any neighborhood of the solution of the VI. Our third contribution is the multiplier-driven method where projections are discarded altogether by introducing a multiplier for the inequality constraints. The iterates satisfy equality constraints throughout the execution as is the case with the subspace-constrained method by using matrix pre-multiplication. The iterates are shown to converge asymptotically under strict monotonicity to any neighborhood of the solution of the VI. Finally, we demonstrate the behavior of the algorithms using a network routing example. Our work here is an extension of [11]. Most notably, we have extended the result for the first projection-based algorithm to the case with only a finite
number of samples of the uncertainty available in each iteration. In addition our second and third algorithm have been modified by introducing a matrix pre-multiplication which constrains iterates to lie in the affine subspace defined by the equality constraints.

II. PRELIMINARIES

1) Notation: Let $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers, respectively. For $N \in \mathbb{N}$, we write $[N] := \{1, 2, \ldots, N\}$. For a given $x \in \mathbb{R}$, we use the notation $|x| := \max\{x, 0\}$. For $x, y \in \mathbb{R}$, the operator $|x|^+ := \max\{x, y\}$.

2) Variational inequalities and KKT points: For a given map $F : \mathbb{R}^n \to \mathbb{R}^n$ and a closed set $\mathcal{H} \subseteq \mathbb{R}^n$, the variational inequality (VI) problem, $VI(\mathcal{H}, F)$, is to find $h^* \in \mathcal{H}$ such that $(h - h^*)^T F(h^*) \geq 0$ holds for all $h \in \mathcal{H}$. Let $\{\mathcal{H}_n\}$ be a sequence of closed sets in $\mathbb{R}^n$. A sequence $\{h_n\}$ in $\mathbb{R}^n$ is said to converge strongly to $h \in \mathbb{R}^n$ if $\lim_{n \to \infty} \|h_n - h\| = 0$.

3) Projected dynamical systems: For a given map $F : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ and a closed set $\mathcal{H} \subseteq \mathbb{R}^n$ the associated projected dynamical system is given by

$$\dot{h}(t) = \Pi_{\mathcal{H}_n}(h(t))(F(h(t), t)).$$

Here $\Pi_{\mathcal{H}}(h)$ is the tangent cone of $\mathcal{H}$ at $h$ (see Section II.1).

III. PROBLEM STATEMENT AND MOTIVATING EXAMPLES

Consider a set of functions $C_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \mathbb{N}$, $\mathcal{H} \subseteq \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, and the functions $q_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathbb{N}$, $s \in \mathbb{N}$ are convex and continuously differentiable. Define the map $F_i : \mathbb{R}^n \to \mathbb{R}$ as the CVaR of $C_i$ at level $\alpha \in (0, 1]$:

$$F_i(h) := \text{CVaR}_\alpha \{C_i(h, \xi)\}, \quad \text{for all } i \in \mathbb{N}. \quad (3)$$

For notational convenience, let $C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the element-wise concatenation of the maps $\{C_i\}_{i \in \mathbb{N}}$ and $\{F_i\}_{i \in \mathbb{N}}$, respectively. Let $\mathcal{H} \subseteq \mathbb{R}^n$ be a nonempty closed set of the form

$$\mathcal{H} := \{h \in \mathbb{R}^n | Ah = b, \ q_i(h) \leq 0, \forall i \in \mathbb{N}\}, \quad (4)$$

where $A \in \mathbb{R}^{l \times n}$, $b \in \mathbb{R}^l$ and $l \in \mathbb{N}$, and the functions $q_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \mathbb{N}$, $s \in \mathbb{N}$ are convex and continuously differentiable. The objective of this paper is to provide stochastic approximation (SA) algorithms to solve the variational inequality problem $VI(\mathcal{H}, F)$. Our strategy will be to use an empirical estimator, derived from samples of $C(h, \xi)$, of the map $F$ at each iteration of the algorithm. Below we discuss two motivating examples for our setup.
A. CVaR-based routing games

Consider a directed graph \( \mathcal{G} = (V, E) \), where \( V = [n] \) is the set of vertices, and \( E \subseteq V \times V \) is the set edges. To such a graph we associate a set \( \mathcal{W} \subseteq \mathcal{V} \times \mathcal{V} \) of origin-destination (OD) pairs. An OD-pair \( w \) is given by an ordered pair \((v^w_u, v^w_d)\), where \( v^w_u \) and \( v^w_d \) are the origin and the destination of \( w \), respectively. The set of all paths in \( \mathcal{G} \) from the origin to the destination of \( w \) is denoted \( \mathcal{P}_w \). The set of all paths is given by \( \mathcal{P} = \bigcup_{w \in \mathcal{W}} \mathcal{P}_w \). Each of the participants, or agents, of the routing game is associated to an OD-pair, and can choose which path to take to travel from its origin to its destination. The choices of all agents give rise to a flow vector \( h \in \mathbb{R}^{\mathcal{P}} \). We consider a non-atomic routing game and so \( h \) is a continuous variable.

For each (OD)-pair \( w \), a value \( D_w \geq 0 \) defines the demand associated to it. The feasible set \( \mathcal{H} \subseteq \mathbb{R}^{\mathcal{P}} \) is then given by \( \mathcal{H} = \{ h \mid \sum_{p \in \mathcal{P}_w} h_p = D_w, \forall w \in \mathcal{W}, \ h_p \geq 0 \ \forall p \in \mathcal{P} \} \). To each of the paths \( p \in \mathcal{P} \), we associate a cost function \( C_p : \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^m \to \mathbb{R}, (h, \xi) \mapsto C_p(h, \xi) \), which depends on the flow \( h \), as well as on the uncertainty \( \xi \in \mathbb{R}^m \). Each agent chooses \( p \in \mathcal{P}_w \) that minimizes CVaR \( \alpha \) \( \left[ C_p(h^*, \xi) \right] \). These elements define the CVaR-based routing game [6] to which we assign the following notion of equilibrium: the flow \( h^* \in \mathcal{H} \) is said to be a CVaR-based Wardrop equilibrium (CWE) of the CVaR-based routing game if, for all \( w \in \mathcal{W} \) and all \( p, p' \in \mathcal{P}_w \) such that \( h^*_p > 0 \), we have

\[
\text{CVaR}_\alpha \left[ C_p(h^*, \xi) \right] \leq \text{CVaR}_\alpha \left[ C_{p'}(h^*, \xi) \right].
\]

The intuition is that at equilibrium, for each agent, there is no other path than the selected one that has a smaller value of conditional value-at-risk. Under continuity of \( C_p \), the set of CWE is equal to the set of solutions of VI(\( \mathcal{H}, F \)), where \( F : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}} \) takes the form (3).

B. CVaR-based Nash equilibrium

A more general example of our setup would be in finding the Nash equilibrium of a non-cooperative game [1, section 1.4.2]. Let there be \( N \) players with individual cost functions \( \theta_i : \mathbb{R}^n \to \mathbb{R}, x \mapsto \theta_i(x) \) and possible strategy sets \( \mathcal{X}_i \subseteq \mathbb{R}^n \). Here \( x \in \mathbb{R}^n \) denotes the vector containing the strategies of all players, where \( x_i \in \mathbb{R}^n \) is the strategy of player \( i \). We assume without loss of generality that the action/strategy sets of each player are of the same dimension \( n \). An alternative notation is \( \theta_i(x) = \theta_i(x_i, x_{-i}) \), where \( x_{-i} \) is the vector containing the strategies of all players except \( i \). Each player \( i \) aims to minimize its cost \( \theta_i \) by choosing its own strategy optimally. That is, for any fixed \( \bar{x}_{-i} \) they solve

\[
\begin{align*}
\min_{x_i \in \mathcal{X}_i} & \quad \theta_i(x_i, \bar{x}_{-i}) \\
\text{s.t.} & \quad x_i \in \mathcal{X}_i.
\end{align*}
\]

A Nash equilibrium of such a game is a solution vector \( x^* \) such that none of the players can reduce their cost by changing their strategy. Under the assumption that the sets \( \mathcal{X}_i \) are convex and closed, and the functions \( x_i \mapsto \theta_i(x_i, \bar{x}_{-i}) \) are convex and continuously differentiable for any \( \bar{x}_{-i} \), a joint strategy vector \( x^* \) is a Nash equilibrium if and only if it is a solution to VI(\( \mathcal{X}, F \)), where \( F(x) := (\nabla_x \theta_i(x))_{i=1}^N \) is the concatenation of the gradients of \( \theta_i \) functions, and \( \mathcal{X} = \prod_{i=1}^N \mathcal{X}_i \). Consider the functions \( \theta_i \) of the form

\[
\theta_i(x) := \text{CVaR}_\alpha \left[ f_i(x_i, x_{-i})g(\xi) + \tilde{f}_i(x_i, x_{-i}) \right],
\]

where functions \( f_i, g \) and \( \tilde{f}_i \) are real-valued, \( \xi \) models the uncertainty, and \( f_i(x_i, x_{-i}) \geq 0 \) for all \( x \). Then, VI(\( \mathcal{X}, F \)) is a CVaR-based variational inequality as discussed in our paper. Specifically, in this case, since CVaR is positive-homogeneous and shift-invariant [14, Chapter 6], we have

\[
\theta_i(x) = \text{CVaR}_\alpha \left[ g(\xi) \right] f_i(x_i, x_{-i}) + \tilde{f}_i(x_i, x_{-i}).
\]

As a consequence, we get

\[
\nabla \theta_i(x) = \text{CVaR}_\alpha \left[ g(\xi) \right] \nabla_x f_i(x_i, x_{-i}) + \nabla_x \tilde{f}_i(x_i, x_{-i}).
\]

Under the assumption that \( \nabla_x f_i \) is nonnegative for all \( x \in \mathcal{X} \), we get

\[
\nabla \theta_i(x) = \text{CVaR}_\alpha \left[ g(\xi) \right] \nabla_x f_i(x_i, x_{-i}) + \nabla_x \tilde{f}_i(x_i, x_{-i}).
\]

where CVaR is understood component-wise. Thus, \( F \) can be written as concatenation of CVaR of various functions and finding the Nash equilibrium of this game is equivalent to solving VI(\( \mathcal{X}, F \)), which fits into our presented framework.

IV. Stochastic approximation algorithms for solving VI(\( \mathcal{H}, F \))

In this section, we introduce the SA algorithms along with their convergence analysis. All introduced schemes approximate \( F \) with the estimator given in (4). Given \( N \) independently and identically distributed samples \( \{(C_i(h, \xi))_j\}_{j=1}^N \) of the random variable \( C_i(h, \xi) \), let

\[
\hat{F}^N_i(h) := \inf_{\xi \in \mathbb{R}} \left\{ t + (N\alpha)^{-1} \sum_{j=1}^N \left[ (C_i(h, \xi))_j - t \right]^+ \right\}
\]

stand for the estimator of \( F_i(h) \). Analogously, the estimator of \( F(h) \) formed using the element-wise concatenation of \( \hat{F}^N_i(h) \), \( i \in [n] \), is denoted by \( \hat{F}^N(h) \). We assume that the \( N \) samples of each cost function are a result of the same set of \( N \) events, that is, the distribution of \( \hat{F}^N(h) \) depends on \( \mathcal{P}^N \). We next present our first algorithm.

A. Projected algorithm

For a given sequence of step-sizes \( \{\gamma^k\}_{k=0}^\infty \), with \( \gamma^k > 0 \) for all \( k \), a sequence \( \{N_k\}_{k=0}^\infty \subseteq \mathbb{N} \), and an initial vector \( h^0 \in \mathcal{H} \), the first algorithm under consideration, which we will refer to as the projected algorithm, is given by

\[
h^{k+1} = \Pi_{\mathcal{H}} (h^k - \gamma^k \hat{F}^{N_k}(h^k)),
\]

where \( \Pi_{\mathcal{H}} \) is the projection operator (see Section 1.1).
convergence of the algorithms. We can write the projected algorithm (5) equivalently as

$$h^{k+1} = \Pi_H\left(h^k - \gamma_k (F(h^k) + \beta N_k)\right),$$

(6)

where $\beta N_k := \tilde{F}(h^k) - F(h^k)$ is the error introduced by the estimation. For this and the upcoming algorithms, common assumptions on the sequence $\{\gamma_k\}$ are

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} (\gamma_k)^2 < \infty.$$  

(7)

Our first result gives sufficient conditions for convergence of (6) to any neighborhood of the solution $h^*$ of VI$(H, F)$.

**Proposition IV.1.** (Convergence of the projected algorithm (5)): Let $F$ as defined in (3) be a strictly monotone, continuous function, and let $H$ be a compact convex set of the form (4). For the algorithm (6), assume that step-sizes $\{\gamma_k\}$ satisfies (7) and the sequence $\{N_k\}$ is such that $\{\beta N_k\}$ is bounded with probability one. Then, for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $N_k \geq N_\epsilon$ for all $k$ implies, with probability one,

$$\lim_{k \to \infty} \|h^k - h^*\| \leq \epsilon.$$

**Proof.** To ease the exposition of this proof, we split the error as $\beta N_k = e_{N_k} + \tilde{e}_{N_k}$, where $e_{N_k} = \mathbb{E}[\beta N_k]$. Note that we then have $\mathbb{E}[\tilde{e}_{N_k}] = 0$, and by the boundedness assumption, there exists a constant $B_\epsilon > 0$ such that $\|e_{N_k}\| \leq B_\epsilon$ for all $k$. The first step of the proof is to show that the sequence $\{h^k\}$ converges to a trajectory of the following continuous-time projected dynamical system:

$$\dot{h}(t) = \Pi_{T_{h}(\bar{h}(t))} (-F(h(t)) - e(t)), \quad \bar{h}(0) \in H.$$

(8)

Here $e(t)$ is a uniformly bounded measurable map satisfying $\|e(t)\| \leq B_\epsilon$ for all $t$ (see Section 11-C for further details on how solutions to projected dynamical systems are defined). For the sake of rigor, we note that the existence of a trajectory of (8) starting from any point in $H$ is guaranteed by [15, Lemma A.1]. To make precise the convergence of the sequence generated by (5) to a trajectory of (8), we say that $\{h^k\}$ converges to a trajectory $\bar{h}(\cdot)$ of (8) if

$$\lim_{i \to \infty} \sup_{j \geq 1} \left\|h^{j-1} - \left(\sum_{k=i}^{j-1} \gamma_k\right)\right\| = 0.$$  

(9)

That is, the discrete-time trajectory formed by the linear interpolation of the iterates $\{h^k\}$ approaches the continuous time trajectory $t \to \bar{h}(t)$. The proof of the existence of a map $\bar{h}(\cdot)$ satisfying (9) is similar to that of [9, Theorem 5.3.1], with the only change being the existence of an error term $e(t)$ in dynamics (8) which is absent in the cited reference. The inclusion of the error term is facilitated by reasoning presented in the proof of [9, Theorem 5.2.2]. We avoid repeating these arguments here in the interest of space.

Convergence of the sequence $\{h^k\}$ can now be analyzed by studying the asymptotic stability of (8). To this end, we consider the candidate Lyapunov function

$$V(\bar{h}) = \frac{1}{2} \|\bar{h} - h^*\|^2,$$

where $h^*$ is the unique solution of VI$(H, F)$. We first look at the case $e(\cdot) \equiv 0$. For notational convenience, define the right-hand side of (8) in such a case by the map $X_{e \equiv 0} : \mathbb{R}^n \to \mathbb{R}^n$. The Lie derivative of $V$ along $X_{e \equiv 0}$ is then given by

$$\nabla V(\bar{h})^T X_{e \equiv 0}(\bar{h}) = (\bar{h} - h^*)^T \Pi_{T_{h}(\bar{h})}(-F(\bar{h})).$$

(10)

We want to show that the right-hand side of the above equation is negative for all $\bar{h} \neq h^*$. We first note that by Moreau’s decomposition theorem [16, Theorem 3.2.5], for any $v \in \mathbb{R}^n$ and $\bar{h} \in H$, we have $\Pi_{T_{h}(\bar{h})}(v) = v - \Pi_{N_{\bar{h}}(\bar{h})}(v)$, where $N_H(\bar{h})$ is the normal cone to $H$ at $\bar{h}$. Using the above relation in (10) gives

$$\nabla V(\bar{h})^T X_{e \equiv 0}(\bar{h}) = - (\bar{h} - h^*)^T F(\bar{h})$$

$$+ (h^* - h^*)^T \Pi_{N_{h^*}(\bar{h})}(-F(\bar{h}))$$

$$\leq - (\bar{h} - h^*)^T F(\bar{h}),$$

(11)

where the inequality is due to the definition of the normal cone (see Section 11-C) and the fact that $h^* \in H$. Due to strict monotonicity of $F$, we have $(\bar{h} - h^*^T) F(\bar{h}) > (\bar{h} - h^*)^T F(h^*)$ whenever $\bar{h} \neq h^*$. Furthermore, since $h^* \in \text{SOL}(H, F)$ we know that $(\bar{h} - h^*)^T F(h^*) \geq 0$ for all $\bar{h} \in H$. Combining these two facts implies that the function $W(\bar{h}) := (\bar{h} - h^*)^T F(\bar{h})$ satisfies $W(\bar{h}) > 0$ whenever $\bar{h} \neq h^*$. Using this in the inequality (11) yields

$$\nabla V(\bar{h})^T X_{e \equiv 0}(\bar{h}) \leq -W(\bar{h}) < 0$$

(12)

whenever $\bar{h} \neq h^*$. Now let $\bar{H}_\epsilon := \{h \in H : \|h - h^*\| \geq \epsilon\}$. Since $H$ is compact, $\bar{H}_\epsilon$ is compact. Since $W$ is continuous, there exists a $\delta > 0$ such that $W(\bar{h}) \geq \delta$ for all $\bar{h} \in \bar{H}_\epsilon$. Therefore we get, from (12),

$$\nabla V(\bar{h})^T X_{e \equiv 0}(\bar{h}) \leq -\delta, \quad \text{for all } \bar{h} \in \bar{H}_\epsilon.$$  

(13)

Next, we drop the assumption $e(\cdot) \equiv 0$ and use the map $X : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ to denote the right-hand side of (8). Consider any trajectory $t \to h(t)$ of (8). Since the map is absolutely continuous and $V$ is differentiable, we have for almost all $t \geq 0$ and for $h(t) \in \bar{H}_\epsilon$

$$\frac{dV}{dt}(t) = \nabla V(\bar{h}(t))^T X(h(t), t) \leq -\delta - (\bar{h}(t) - h^*)^T e(t),$$

where for obtaining the above inequality we have first used Moreau’s decomposition as before to get rid of the projection operator in $X$ and then employed (13). Next we bound the error term in the above inequality. Since $H$ is compact and $h^* \in H$, there exists $B_\epsilon > 0$ such that $\|h - h^*\| \leq B_\epsilon$ for all $\bar{h} \in H$. In addition $\|e(t)\| \leq B_\epsilon$ for all $t$, where $B_\epsilon$ is the bound satisfying $\|e_{N_k}\| \leq B_\epsilon$. Since the empirical estimate of the CVAr is consistent, we know that $B_\epsilon$ can be made arbitrarily small by selecting $N_k$ to be appropriately large for all $k$. That is, there exists $N_\epsilon \in \mathbb{N}$ such that when $N_k > N_\epsilon$ we have $\|e_{N_k}\| < \frac{1}{B_\epsilon}$. Consequently, if $N_k > N_\epsilon$ for all $k$, then $\|e(t)\| < \frac{\delta}{B_\epsilon}$ for all $t$. By selecting such a sample size at each iteration and thus bounding the error term, we obtain

$$\frac{dV}{dt}(t) \leq -\delta - (\bar{h}(t) - h^*)^T e(t)$$

$$\leq -\delta + \|\bar{h}(t) - h^*\| \|e(t)\| < -\delta + \frac{\delta B_\epsilon}{B_\epsilon} \leq 0,$$
which holds for almost all \( t \) whenever \( \hat{h}(t) \in \mathcal{H}_r \). That is, the trajectory converges to the set \( \{ h \in \mathcal{H} \mid \| h - h^* \| \leq \epsilon \} \) as \( t \to \infty \). This concludes the proof. \( \square \)

In the above result, the restriction \( N_k \geq N_{\epsilon} \) does not need to hold for all \( k \). The result also holds if there exists a \( K \in \mathbb{N} \) such that \( N_k \geq N_{\epsilon} \) for all \( k \geq K \). Regarding boundedness of \( \{ \beta^N_k \} \), it is ensured if for example each \( C_i \) is bounded over the set \( \mathcal{H} \times \Xi \), where \( \Xi \) is the support of \( \xi \).

Despite the convergence property established in Proposition [IV.1] the algorithm in (6) suffers from some disadvantages. Most notably, the algorithm requires computing projections onto the set \( \mathcal{H} \) at each iteration, which can be computationally expensive. To address these issues we propose two algorithms that achieve similar convergence to any neighborhood of the solution of the VI(\( \mathcal{H}, F \)). The first requires projection onto inequality constraints only and the second does not involve any projection on the primal iterates and instead ensures feasibility using dual variables. As in Proposition [IV.1] we will impose continuity and monotonicity assumptions on \( F \) in the upcoming results. We provide the following general result on the continuity and monotonicity properties of \( F \).

**Lemma IV.2. (Sufficient conditions for monotonicity and continuity of \( F \)):** The following hold:

- If for any \( \epsilon > 0 \) there exist a \( \delta > 0 \) such that \( \| h - h^\prime \| \leq \delta \) implies \( \| C_i(h, \xi) - C_i(h^\prime, \xi) \| \leq \epsilon \) for all \( i \in [n] \) and all \( \xi \), then \( F \) is continuous.
- Assume that there exist functions \( f_i : \mathbb{R}^n \to \mathbb{R} \) and \( g_i : \mathbb{R}^m \to \mathbb{R} \) such that \( C_i(h, \xi) = f_i(h) + g_i(\xi) \), for all \( i \in [n] \). Let \( f(h) := (f_1(h), \ldots, f_n(h)) \). Then, \( F \) is monotone (resp. strictly) if \( f \) is monotone (resp. strictly monotone).

**Proof.** Continuity follows by arguments similar to the proof of [6, Lemma IV.8]. For the second part, note that CVaR satisfies \( \text{CVaR}_\alpha [C_i(h, \xi)] = f_i(h) + \text{CVaR}_\alpha [g_i(\xi)] \), for all \( h \) and \( i \in [n] \) [14, Page 261]. The proof then follows from the fact that \( F(h) - F(h^\star) = f(h) - f(h^\star) \). \( \square \)

In the above result, the continuity condition, that may be difficult to check in practice, holds if \( \xi \) has a compact support and for any fixed \( \xi \), the functions \( C_i \) are continuous with respect to \( h \). We now introduce our next algorithm.

**B. Subspace-constrained algorithm**

In this section, we take a closer look at the form of \( \mathcal{H} \) given in (4) and design an algorithm that handles inequality and equality constraints independently. To this end, we write \( \mathcal{H}_{\text{aff}} := \{ h \in \mathbb{R}^n \mid Ah = b \} \), and \( \mathcal{H}_{\text{ineq}} := \{ h \in \mathbb{R}^n \mid g^i(h) \leq 0, \forall i \in [s] \} \) for the sets of points satisfying the equality and inequality constraints, respectively. We then have \( \mathcal{H} = \mathcal{H}_{\text{aff}} \cap \mathcal{H}_{\text{ineq}} \). It turns out that, using matrix operation, we can ensure that the iterates of our algorithm always remain in \( \mathcal{H}_{\text{aff}} \). The method works as follows. Let \( \{a_1, \ldots, a_i\} \) be the row vectors of \( A \), and let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis for \( \mathbb{R}^n \) such that the first \( M \in \mathbb{N} \) vectors \( \{u_1, \ldots, u_M\} \) form a basis for the span of vectors \( \{a_1, \ldots, a_i\} \). Then, for the subspace \( S = \{ g \in \mathbb{R}^n \mid Ag = 0 \} \), we have

\[
\Pi_S(v) = \left( I - \sum_{i=1}^M u_i u_i^\top \right) v, \quad \text{for any } v \in \mathbb{R}^n.
\]

This well known fact follows from [17, Theorem 7.10] and noting that \( \Pi_S(v) = v - \Pi_{S^\perp}(v) \), where \( S^\perp \) is the set of vectors orthogonal to the subspace \( S \). Thus, the projection onto \( S \) is achieved by pre-multiplying with the matrix

\[
L := I - \sum_{i=1}^M u_i u_i^\top.
\]

Consequently, for any vector \( z \) of the form \( z = Lv, v \in \mathbb{R}^n \) we have \( Az = 0 \). To construct \( L \) one can find the orthonormal basis vectors \( \{u_i\}_{i \in [r]} \) for the span of \( \{a_j\}_{j \in [s]} \) and \( \mathbb{R}^n \) by using Gram-Schmidt orthogonalization process [17, Section 6.4]. Alternatively, if \( A \) has full row rank one can use \( L := I - A^\top (AA^\top)^{-1}A \), see e.g., [18]. We use this projection operator to define our next method called the subspace-constrained algorithm:

\[
h_k^\prime = h_k - \gamma_k L \left( F(h_k) + c(h_k - \Pi_{\text{ineq}}(h_k)) + \beta^N_k \right),
\]

where the initial iterate \( h_0 \in \mathcal{H}_{\text{aff}} \). In the above, \( c > 0 \) is a parameter to be specified later in the convergence result, the error sequence \( \{\beta^N_k\} \) is as defined in (6), and \( L \in \mathbb{R}^{n \times n} \) is as defined in (4).

Due to the presence of \( L \) in the above algorithm, the direction in which the iterate moves in each iteration is projected onto the subspace \( S \). Hence, \( h_k^\prime \in \mathcal{H}_{\text{aff}} \) for all \( k \). We formally establish this in the below result. Furthermore, convergence to a neighbourhood of the set \( \mathcal{H}_{\text{ineq}} \) is achieved through the term \( h_k - \Pi_{\text{ineq}}(h_k) \). That is, the higher the value of the design parameter \( c \), the closer the limit of \( \{h_k\} \) is to \( \mathcal{H}_{\text{ineq}} \). Together, these mechanisms ensure that we keep iterates close to \( \mathcal{H} \) and ultimately drive them to a neighbourhood of \( h^\star \).

**Proposition IV.3. (Convergence of the subspace-constrained algorithm (15)):** Let \( F \) as defined in (3) be a strictly monotone, continuous function, and let \( \mathcal{H} \) be a compact convex set of the form (4). For the algorithm (15) assume that step-sizes \( \{\gamma_k\} \) satisfies (7) and that the sequence \( \{N^k\} \) is such that there exists \( B_{\text{traj}} \in \mathbb{R} \) satisfying \( \|h^k\| \leq B_{\text{traj}} \) and also \( \{\beta^N_k\} \) is bounded with probability one. Then, for any \( \epsilon > 0 \), there exist \( c_B(B_{\text{traj}}) > 0 \) and \( N_{\epsilon}(B_{\text{traj}}) \in \mathbb{N} \) such that \( c \geq c_B(B_{\text{traj}}) \) and \( N_k \geq N_{\epsilon}(B_{\text{traj}}) \) for all \( k \) imply that the iterates of (15) satisfy, with probability one,

\[
\lim_{k \to \infty} \| h_k - h^\star \| \leq \epsilon.
\]

**Proof.** First we show that \( h_k \in \mathcal{H}_{\text{aff}} \) for all \( k \). To see this, recall that \( ALv = 0 \) for any \( v \in \mathbb{R}^n \). Using this in (15) implies \( Ah_k^\prime = Ah_k \) for all \( k \). Consequently, for all \( k \), we have \( Ah_k^\prime = Ah_0 = b \) and therefore \( h_k^\prime \in \mathcal{H}_{\text{aff}} \).

Analogous to the proof of Proposition [IV.1] it can be established that \( \{h_k\} \) converges with probability one, in the sense of (9), to a trajectory of the following dynamics

\[
\dot{h}(t) = -L \left( F(\hat{h}(t)) + c \left( \hat{h}(t) - \Pi_{\text{ineq}}(\hat{h}(t)) \right) - e(t) \right),
\]
with the initial state $\bar{h}(0) \in \mathcal{H}_{aff}$. Here, $e(\cdot)$ is a uniformly bounded measurable map satisfying $\|e(t)\| \leq B$ for all $t$. We will use the above fact to establish convergence of the sequence $\{h^k\}$ by analyzing the asymptotic stability of (16).

Note that $Ah(t) = 0$ for all $t$, and therefore a trajectory $h(\cdot)$ of (16) satisfies $h(t) \in \mathcal{H}_{aff}$ for all $t \geq 0$ as $\bar{h}(0) \in \mathcal{H}_{aff}$. Now consider the Lyapunov candidate

$$V(h) = \frac{1}{2}\|h - h^*\|^2,$$

where $h^*$ is the unique solution of VI$(\mathcal{H}, F)$, that follows from strict monotonicity. As was the case for the previous result, we will first analyze the evolution of $V$ along (16) when $e \equiv 0$. Therefore, we define the notation $X_{e \equiv 0} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to represent the right-hand side of (16) with $e \equiv 0$. The Lie derivative of $V$ along $X_{e \equiv 0}$ is

$$\nabla V(\bar{h})^\top X_{e \equiv 0}(\bar{h}) = -(\bar{h} - h^*)^\top L\left(F(\bar{h}) + e(\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}))\right).$$

Since $\bar{h}, h^* \in \mathcal{H}_{aff}$, we have $A(\bar{h} - h^*) = 0$ and so $(\bar{h} - h^*) \in S$. Consequently, for any vector $v \in \mathbb{R}^n$, we have

$$(\bar{h} - h^*)^\top v = (\bar{h} - h^*)^\top (\Pi_{\mathcal{S}}(v) + \Pi_{\mathcal{S}^\perp}(v)) = (\bar{h} - h^*)^\top \Pi_{\mathcal{S}}(v) = (\bar{h} - h^*)^\top L_v.$$

Using the above equality in (17) gives

$$\nabla V(\bar{h})^\top X_{e \equiv 0}(\bar{h}) = -(\bar{h} - h^*)^\top \left(F(\bar{h}) + e(\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}))\right).$$

We first upper bound the second term on the right-hand side of the above equality. We have

$$-e(\bar{h} - h^*)^\top (\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h})) = -e(\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}) + \Pi_{\mathcal{H}_{ineq}}(\bar{h}) - h^*)^\top (\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}))$$

$$= -e\|\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h})\|^2 + e(\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}))^\top (\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h})) \leq 0,$$

where for the inequality we have used the fact that $(\bar{h} - \Pi_{\mathcal{H}_{ineq}}(\bar{h}))^\top (\bar{h}^* - \Pi_{\mathcal{H}_{ineq}}(\bar{h})) \leq 0$ for any $h \in \mathbb{R}^n$ (see [19, Thm. 3.1.1]). Note that the inequality (19) is strict whenever $\bar{h} \neq \Pi_{\mathcal{H}_{ineq}}(\bar{h})$. We now turn our attention towards the first term in (18). Due to strict monotonicity of $F$ and the fact that $h^* \in SOL(\mathcal{H}, F)$, we obtain

$$-(\bar{h} - h^*)^\top F(\bar{h}) \leq -(\bar{h} - h^*)^\top F(h^*) \leq 0$$

whenever $\bar{h} \in \mathcal{H}$ and $\bar{h} \neq h^*$. The above inequality along with (19) shows $\nabla V(\bar{h})^\top X_{e \equiv 0}(\bar{h}) \leq 0$ for any $h \in \mathcal{H}$. However, recalling the approach in the proof of Proposition IV.1 what we require in order to establish convergence is the existence of $\delta > 0$ such that

$$\nabla V(\bar{h})^\top X_{e \equiv 0}(\bar{h}) \leq -\delta$$

for all $\bar{h} \in \mathcal{H}_e$, where

$$\mathcal{H}_e := \{h \in \mathcal{H}_{aff} \mid \|h - h^*\| \geq \epsilon\}.$$
C. Multiplier-driven algorithm

Algorithms (6) and (15) involve projection onto $\mathcal{H}$ or $\mathcal{H}_{\text{ineq}}$ at each iteration, which can be computationally burdensome. Our next algorithm overcomes this limitation. We assume $\mathcal{H}$ to be of the form (4) and introduce a multiplier variable $\lambda \in \mathbb{R}_+^n$ that enforces satisfaction of the inequality constraint as the algorithm progresses. In order to simplify the coming equations we introduce the notation $H(h, \lambda) := F(h) + Dq(h)\top \lambda$, where $Dq(h)$ is the Jacobian of $q$ at $h$. The **multiplier-driven algorithm** is now given as

$$
\begin{align*}
L & h^{k+1} = h^k - \gamma^k L(H(h^k, \lambda^k) + \beta^k N_k), \\
\lambda^{k+1} & = [\lambda^k + \gamma^k q(h^k)]^+.
\end{align*}
$$

(27)

Here $L$ is as defined in (14). Also recall that $\beta^k N_k$ is the error due to empirical estimation of $F$. The next result establishes the convergence properties of (27) to a KKT point of the VI (see Section III for definitions) and so to a solution of the VI.

**Proposition IV.5.** (Convergence of the multiplier-driven algorithm (27)): Let $F$, as defined in (3), be a strictly monotone, continuous function, and let $\mathcal{H}$ be a compact convex set of the form (4), where functions $q^i$, $i \in [s]$, are affine. Assume that the LICQ holds for $h^* \in \text{SOL}(\mathcal{H}, F)$, and let $(h^*, \lambda^*, \mu^*)$ be an associated KKT point. For algorithm (27), assume that step-sizes $\{\gamma^k\}$ satisfies (1) and let $\{N_k\}$ be such that $\{\beta^k N_k\}$, $\{h^k\}$, and $\{\lambda^k\}$ are bounded with probability one. Then, for any $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that if $N_k \geq N_\epsilon$ for all $k$, then, with probability one,

$$
\lim_{k \to \infty} \|h^k - h^*\| \leq \epsilon.
$$

Proof. Analogous to the proof of Proposition IV.1, the first step establishes convergence with probability one of the sequence $\{(h^k, \lambda^k)\}$, in the sense of (9), to a trajectory $(\hat{h}(\cdot), \hat{\lambda}(\cdot))$ of the following dynamics

$$
\begin{align*}
\dot{h}(t) & = -L \left( H(h(t), \hat{\lambda}(t)) + e(t) \right), \\
\dot{\lambda}(t) & = \left[ q(h(t)) \right]^+ \lambda(t),
\end{align*}
$$

(28a, 28b)

with initial condition $h(0) \in \mathbb{R}^n$ and $\hat{\lambda}(0) \in \mathbb{R}_+^n$. Note that due to the presence of this operator in (28b), $\hat{\lambda}$ is contained in the nonnegative orthant along any trajectory of the system. The map $\bar{e}(\cdot)$ is uniformly bounded and so, as before, we have $\|e(t)\| \leq B_\epsilon$, for all $t$. The proof of convergence of the iterates to a continuous trajectory is similar to that of [9, Theorem 5.2.2] and is not repeated here for brevity. Note that, as was the case for Proposition IV.3, multiplication with the matrix $L$ ensure that $h^k, \hat{h}(t) \in \mathcal{H}_{\text{aff}}$ for all $k$ and $t \geq 0$. Next, we analyze the convergence of (28). We will occasionally use $\bar{x}$ as shorthand for $(\bar{h}, \bar{\lambda})$. Define the candidate Lyapunov function

$$
V(\hat{h}, \hat{\lambda}) := \frac{1}{2} \left[ \|\hat{h} - h^*\|^2 + \|\hat{\lambda} - \lambda^*\|^2 \right],
$$

(29)

where $h^*$ is the unique solution of VI($\mathcal{H}, F$) and there exist $\mu^* \in \mathbb{R}^l$ such that $(h^*, \lambda^*, \mu^*)$ is an associated KKT point. We analyze the evolution of (29) for the case $\epsilon \equiv 0$. Denoting the right-hand side of (28) for this case by $X_{e=0}$, the Lie derivative of $V$ along (28) is

$$
\nabla V(\bar{x})^\top X_{e=0}(\bar{x}) = -(\hat{h} - h^*)^\top H(\bar{h}, \bar{\lambda}) + (\lambda - \lambda^*)^\top (q(\bar{h}) + (q(\bar{h}))_+^\top - q(\bar{h})).
$$

(30)

Here we have dropped the matrix $L$ from the term $(\bar{h} - h^*)^\top L \bar{H}(\bar{h}, \bar{\lambda})$, which is justified by the same argument used for deriving (18). Note that for any $i$, $\|q(\bar{h})_+^\top h_i - q(\bar{h})_i\|_2 < \lambda_i = 0$.

Since $q$ is affine, we have $Dq(h) = Dq(h^*)$ for all $h \in \mathbb{R}^n$. Combined with strict monotonicity this gives, for $h \neq h^*$,

$$
0 < (\hat{h} - h^*)^\top (H(h, \bar{\lambda}) - H(h^*, \bar{\lambda})) = (\hat{h} - h^*)^\top (H(h, \bar{\lambda}) - H(h^*, \lambda^*)) + Dq(h^*)^\top (\lambda^* - \lambda^*) \leq 0.
$$

(31)

From (31) we have $-H(h^*, \lambda^*) = A^\top \mu^*$. Since we have $\hat{h}, h^* \in \mathcal{H}_{\text{aff}}$ it follows that $-(\hat{h} - h^*)^\top H(h^*, \lambda^*) = 0$. Then, using the assumption that $q$ is affine, we have $-H(h^*, \lambda^*)^\top (\lambda^* - \lambda^*) < 0$. Combining these derivations and writing $W(h)$ for the right-hand side of (30) we get that for $h \neq h^*$, $W(h) < (\lambda - \lambda^*)^\top q(h^*)$. From (31) we have $\lambda^\top q(h^*) = 0$ and $\lambda^\top q(h^*) \leq 0$, then we implies $\nabla V(h, \bar{\lambda})X_{e=0}(\bar{h}, \bar{\lambda}) < 0$ for almost all $t$ with $\hat{h}(t) \neq h^*$. The rest of the proof is analogous to the corresponding section of the proof of Proposition IV.1. Therefore, $\sqrt{\frac{1}{N}} \sum_{i=1}^N \langle \lambda^\top q(\bar{h}) \rangle \to 0$ as $N \to \infty$.

**Remark IV.6.** (Implementation aspect of Proposition IV.5): In Proposition IV.5 we require boundedness of $\{h^k\}, \{\lambda^k\}$. When upper bounds on $\|\lambda^k\|$ are known beforehand, projection onto hyper-rectangles can ensure boundedness of $\{\lambda^k\}$, while the result remains valid, (cf. [9, Page 40, Theorem 5.2.2]). For boundedness of $\{h^k\}$, see Remark IV.4.

V. Simulations

Here we demonstrate an application of the presented stochastic approximation algorithms for finding the solutions of a CVaR-based variational inequality. The example is an
instance of a CVaR-based routing game (see Section III-A) based on the Sioux Falls network [20]. The network consists of 24 nodes and 76 edges. To each of the edges, we associate an affine cost function given by $C_e(f_e, u_e) = t_e(1 + u_e f_e)$, where $f_e$ is the flow over edge $e$, and $t_e$ and $c_e$ are the free-flow travel time and capacity of edge $e$, respectively, as obtained from [20]. The uncertainty $u_e$ has the uniform distribution over the interval $[0,0.5]$ for all edges connected to the vertices 10, 16, or 17. For the rest of the edges, $u_e$ is set to zero. This defines the cost functions for all edges, and consequently defines the costs of all paths through the network as well. We consider three origin destination(OD) pairs $W = \{(1,19), (13,8), (12,18)\}$, and for each of these paths we select the ten paths that have the smallest free-flow travel time associated to them. The set of these 30 paths we denote as $P$. The demands for each OD-pair are given by $d_{1,19} = 300, d_{13,8} = 600, d_{12,18} = 200$. We aim to find a CVaR-based Wardrop equilibrium which is equivalent to finding a solution of the VI problem defined by a map $F(h) := Ah + b + CVaR_{\alpha}(c)$, and a feasible set $\mathcal{H} = \{h \in \mathbb{R}^{30} | h \geq 0, \sum_{i=1}^{11} h_i = 300, \sum_{i=11}^{20} h_i = 600, \sum_{i=21}^{30} h_i = 200\}$. Here, $h, b \in \mathbb{R}^{30}$, $A \in \mathbb{R}^{30 \times 30}$, and $\alpha = 0.05$. The exact values of $A$ and $b$ and the distribution $\xi$ are constructed using the cost functions and the network structure, see [21, Section 6] for details.

In Figure 1 we see the evolution of the error for each of the different algorithms. The stepsize sequence for the projected, subspace-constrained, and multiplier-driven algorithms are $\gamma_h = \frac{1}{10^4}$, $\gamma_k = \frac{1}{10^3}$, and $\gamma_k = \min\left(\frac{1}{10^4}, \frac{1}{200}\right)$, respectively. In addition, for the subspace-constrained algorithm we initially let $\gamma$ depend on $k$, to prevent unstable behaviour. We used $\gamma = \min\left(\frac{1}{200}, \frac{1}{200}\right)$. For the multiplier driven algorithm, for similar reasons, we used a modified step-size sequence for updating the multipliers $\lambda$ given by $\gamma^k_{\lambda} = 2 \gamma^k$ for $k < 1000$ and $\gamma^k_{\lambda} = 0.5 \gamma^k$, otherwise. The figure shows that all algorithms converge to a neighbourhood of the solution of the variational inequality, albeit requiring a different number of iterations. Specifically, the number of iterations taken by the projected algorithm to converge is two orders of magnitude less than that of the subspace-constrained and multiplier-driven algorithms. The quality of convergence is summarized in Table I where we can see both the accuracy of the achieved convergence as well as the effect of increasing the sample sizes. It is important to note that the errors shown in Fig. I and Table I are in terms of the deviation in the value of the map $\| F(h^*) - F(h^*) \|$, rather than deviation in the solution $\| h^* - h^* \|$. This is because the solution $h^*$ is not unique for the formulated VI. However, for any two solutions $h^*, h^* \in SOL(F, \mathcal{H})$ we do have $F(h^*) = F(h^*)$.

### VI. Conclusions

We have considered variational inequalities defined by the CVaR of cost functions and provided stochastic approximation algorithms for solving them. We have analyzed the asymptotic convergence of these algorithms when, at each iteration, only a finite number of samples are used to estimate the CVaR. Future work will focus on analyzing the finite-time properties of the introduced algorithms and the sample complexity for a desired error tolerance of the last iterate. Finally, we wish to explore accelerated methods to solve the problem.

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### TABLE I

| Samples per iteration | 25 | 50 | 100 |
|-----------------------|----|----|-----|
| Projected             | 0.3875 | 0.2062 | 0.1157 |
| Subspace constrained  | 0.3780 | 0.2015 | 0.1332 |
| Multiplier-driven     | 0.3889 | 0.1987 | 0.1064 |