Gauge fluctuations and transition temperature for superconducting wires

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We consider the Ginzburg-Landau model, confined in an infinitely long rectangular wire of cross-section $L_1 \times L_2$. Our approach is based on the Gaussian effective potential in the transverse unitarity gauge, which allows to treat gauge contributions in a compact form. The contributions from the scalar self-interaction and from the gauge fluctuations are clearly identified. Using techniques from dimensional and zeta-function regularizations, modified by the confinement conditions, we investigate the critical temperature for a wire of transverse dimensions $L_1, L_2$. Taking the mass term in the form $m_0^2 = a(T/T_0 - 1)$, where $T_0$ is the bulk transition temperature, we obtain equations for the critical temperature as a function of the $L_i$'s and of $T_0$, and determine the limiting sizes sustaining the transition. A qualitative comparison with some experimental observations is done.

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I. INTRODUCTION

In the present work we discuss the critical behavior of the Ginzburg-Landau model compactified in spatial dimensions and we implement the gauge contributions using the Gaussian effective potential formalism. Spontaneous symmetry breaking is obtained by taking the bare mass coefficient in the Hamiltonian parametrized as $m_0^2 = a(T/T_0 - 1)$, with $a > 0$ and the parameter $T$ varying in an interval containing the bulk transition temperature $T_0$. With this choice, considering the system confined in an infinitely long rectangular cylinder with cross-section area $A = L_1 \times L_2$, we obtain the Ginzburg-Landau model describing phase transitions in samples of a material in the form a wire. This generalizes to wires recent results obtained for materials in the form of films [1]. We investigate the behavior of the system taking into account gauge fluctuations, which means that charged transitions are included in our work. We are particularly interested in the problem of how the critical behavior depends on the dimensions $L_1, L_2$. This study is done by means of the Gaussian Effective Potential (GEP) as developed in Refs. [2–5], together with a spatial compactification mechanism introduced in recent publications [8,9].

A central ingredient in our approach relies on the topological nature of the Matsubara imaginary-time formalism. In order to perform the calculation of the partition function in a quantum field theory, the Matsubara prescription results to be equivalent to a path-integral calculated on $R^{D-1} \times S^1$, where $S^1$ is a circle of circumference $\beta = 1/T$. This result was demonstrated at the one-loop level in Ref. [10] and has been assumed to be valid for higher orders [11]. Relying on the fact that for Euclidean field theories inverse imaginary time and spatial coordinates are on the same footing, such a topological result has been generalized to treat different physical situations in which fields are confined in higher purely spatial dimensions, considering the Matsubara mechanism on a $R^{D-2} \times S^1 \times S^{12} \times S^{14}$ topology, describing space confinement in a $d$-dimensional subspace [8,9,12]. As a consequence the Matsubara formalism can be thought, in a generalized way, as a mechanism to deal with spatial constraints in a field theory model. In this situation, for consistency, the fields fulfill periodic (antiperiodic) boundary conditions for bosons (fermions). In any case we infer from the above discussion that we are justified to consider in this paper the Matsubara mechanism as a path-integral formalism on $R^{D-2} \times S^1 \times S^1$ to deal simultaneously with spatial constraints in a subspace of dimension $d = 2$ (the transverse dimensions of a wire having a rectangular cross-section). These ideas have been applied in different physical situations [8,12,9,13].

We begin by briefly presenting the study of the U(1) Scalar Electrodynamics in the transverse unitarity gauge, along the lines developed in refs. [1,6], and afterwards presenting a general procedure to treat a massive field theory in a $D$-dimensional Euclidean space, compactified in a $d$-dimensional subspace, with $d \leq D$. This permits to extend to an arbitrary subspace some results in the literature for the behavior of field theories in presence of spatial boundaries [9,8,7]. After describing the general formalism, we focus on the particularly interesting case of $d = 2$. We investigate the behavior of the system taking into account gauge fluctuations, which means that charged transitions are included in our work. We are particularly interested in the problem of how the critical behavior depends on the dimensions...
This study is done by means of the Gaussian Effective Potential (GEP) as developed in Refs. [2–5], together with the spatial compactification mechanism mentioned above, introduced in recent publications [8,9].

II. THE GAUSSIAN EFFECTIVE POTENTIAL FOR THE GINZBURG-LANDAU MODEL

We start from the Hamiltonian density of the GL model in Euclidean $d$-dimensional space (unless explicitly stated we work in Natural Units (NU), $\hbar = c = k = 1$) written in the form [14],

$$H' = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} [(\partial_\mu - ieA_\mu) \Psi]^2 + \frac{1}{2} m_0^2 |\Psi|^2 + \lambda (|\Psi|^2)^2, \quad (2.1)$$

where $\Psi$ is a complex field, and $m_0$ is the bare mass. The components of the transverse magnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ($\mu, \nu = 1, \ldots, d$) are related to the $d$-dimensional potential vector by the well known equation,

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = |\nabla \times A|^2. \quad (2.2)$$

In order to obtain only physical degrees of freedom, we can introduce two real fields instead of the complex field $\Psi$, assuming a transverse unitarity gauge. We can define the field in terms of two real fields, as $\Psi = \phi e^{i\gamma}$, together with the gauge transformation $A \to A - 1/e \nabla \gamma$. The unitarity gauge makes the original transverse field to acquire a longitudinal component $A_L$ proportional to $\nabla \gamma$. Then the original functional integration over $\Psi$ and $\Psi^*$ in the generating functional of correlation functions, becomes an integration over $\phi$, $A_T$ and $A_L$. The longitudinal component of the vector potential can be integrated out, leading to the generating functional (up to constant terms),

$$Z[j] = \int D\phi \ DA_T \exp \left[ - \int d^d x H + \int d^d x \ j \phi \right], \quad (2.3)$$

where the Hamiltonian is

$$H = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \lambda \phi^4 + \frac{1}{2} e^2 \phi^2 A^2 + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2 \epsilon} (\nabla \cdot A)^2. \quad (2.4)$$

We have introduced above a gauge fixing term, the limit $\epsilon \to 0$ being taken later on after the calculations have been done. In Eq.(2.4) and in what follows, unless explicitly stated, $A$ stands for the transverse gauge field.

The Gaussian effective potential can be obtained from Eq.(2.4), performing a shift in the scalar field in the form $\phi = \tilde{\phi} + \phi$, which allows to write the Hamiltonian in the form

$$H = H_0 + H_{int}, \quad (2.5)$$

with $H_0$ being the free part and $H_{int}$ the interaction part, given respectively by

$$H_0 = \left[ \frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{1}{2} \Omega^2 \tilde{\phi}^2 \right] + \left[ \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} \Delta^2 A_\mu A^\mu + \frac{1}{2 \epsilon} (\nabla \cdot A)^2 \right], \quad (2.6)$$

and

$$H_{int} = \sum_{n=0}^{4} v_n \tilde{\phi}^n + \frac{1}{2} (e^2 \phi^2 - \Delta^2) A_\mu A^\mu + \frac{1}{2} e^2 \phi A_\mu A^\mu \phi + \frac{1}{2} e^2 A_\mu A^\mu \phi^2, \quad (2.7)$$

where

$$v_0 = \frac{1}{2} m_0^2 \phi^2 + \lambda \phi^4, \quad (2.8)$$
$$v_1 = m_0^2 \phi + 4 \lambda \phi^3, \quad (2.9)$$
$$v_2 = \frac{1}{2} m_0^2 \phi^2 + 6 \lambda \phi^2 - \frac{1}{2} \Omega^2, \quad (2.10)$$
$$v_3 = 4 \lambda \phi, \quad (2.11)$$
$$v_4 = \lambda. \quad (2.12)$$
It is clear from Eqs. (2.5, (2.6) and (2.7), that \( H \) describes two interacting fields, a real scalar field \( \phi \) of mass \( \Omega \) and a real vector gauge field \( A \) of mass \( \Delta \).

The effective potential, which is defined by

\[
V_{\text{eff}}[\phi] = \frac{1}{V} \left[ -\ln Z + \int d^d x j(x) \phi + \int d^d x J \cdot A \right],
\]  

(2.13)

where \( V \) is the total volume, can be obtained at first order from standard methods from perturbation theory. One can find, from Eqs. (3.1), (2.6) and (2.7),

\[
V_{\text{eff}}[\phi] = I_1^d(\Omega) + 2I_1^d(\Delta) + \frac{1}{2} m_0^2 \phi^2 + \phi^4 + \frac{1}{2} \left[m_0^2 - \Omega^2 + 12 \lambda \phi^2 + 6 \lambda I_0^d(\Omega)\right] I_0^d(\Omega) \\
+ \left[e^2 (\phi^2 + I_0^d(\Omega)) - \Delta^2\right] I_0^d(\Delta),
\]  

(2.14)

where,

\[
I_0^d(M) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2},
\]  

(2.15)

and

\[
I_1^d(M) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln(k^2 + M^2),
\]  

(2.16)

with \( k = (k_1, \ldots, k_d) \) being the \( d \)-dimensional momentum.

The Gaussian effective potential is derived by the requirement that \( V_{\text{eff}}[\phi] \) must be stationary under variations of the masses \( \Delta \) and \( \Omega \). This means that values \( \Omega \) and \( \Delta \) for the masses \( \Omega \) and \( \Delta \) should be found such that the conditions,

\[
\left( \frac{\partial V_{\text{eff}}}{\partial \Omega^2} \right)_{\Omega^2 = \Omega^2} = 0,
\]  

(2.17)

\[
\left( \frac{\partial V_{\text{eff}}}{\partial \Delta^2} \right)_{\Delta^2 = \Delta^2} = 0,
\]  

(2.18)

be simultaneously satisfied. These conditions generate the gap equations,

\[
\Pi = m_0^2 + 12 \lambda \phi^2 + 12 \lambda I_0^d(\Pi) + 2e^2 I_0^d(\Pi),
\]  

(2.19)

\[
\Delta = e^2 \phi^2 + e^2 I_0^d(\Pi).
\]  

(2.20)

Replacing \( \Omega \) and \( \Delta \) in Eq.(2.14) by the solutions \( \Pi \) and \( \Delta \), of Eqs.(2.19) and (2.20) we obtain for the GEP the formal expression,

\[
V_{\text{eff}}[\phi] = I_1^d(\Pi) + 2I_1^d(\Delta) + \frac{1}{2} m_0^2 \phi^2 + \phi^4 - \lambda I_0^d(\Pi)^2 - e^2 I_0^d(\Pi) I_0^d(\Delta).
\]  

(2.21)

Notice that Eqs.(2.19) and (2.20) are a pair of very involved coupled equations, and no analytical solution for them has been found, they can be solved only by numerical methods. Later on we will see that this difficulty, in the limit of criticality can be surmounted.

Next we intend to write an expression for the Gaussian mass, \( m \), obtained in our case from the standard prescription, as the second derivative of the Gaussian effective potential for \( \phi = 0 \). To calculate the second derivative of \( V_{\text{eff}} \) with respect to \( \phi \), we remark from Eqs.(2.19) and (2.20) that \( \Pi^2 \) and \( \Delta^2 \) also depend on \( \phi \) according to the relations

\[
\frac{d\Pi^2}{d\phi} = 24 \lambda \phi - 2e^2 I_{-1}^d(\Delta) \frac{d\Delta}{d\phi},
\]  

(2.22)

\[
\frac{d\Delta^2}{d\phi} = 2e^2 \phi - \frac{1}{2} e^2 I_{-1}^d(\Delta) \frac{d\Pi^2}{d\phi},
\]  

(2.23)
where
\[ I^d_{1}(M) = 2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + M^2)^2}. \] (2.24)

Replacing Eq.(2.23) in (2.22) we get,
\[ \frac{d\Omega^2}{d\varphi} = \left[ 24\lambda - 2e^4 I^d_{1}(\Omega) \right] \varphi \left[ 24\lambda - 2e^4 I^d_{1}(\Omega) \right] \Omega. \] (2.25)

and the second derivative of the GEP with respect to \( \varphi \) is given by,
\[ \frac{d^2V_{eff}}{d\varphi^2} = m^2 + 12\lambda \varphi^2 + 2e^2 I^d_{0}(\Omega_0) + 2e^4 \varphi^2 I^d_{-1}(\Omega). \] (2.26)

Thus we have the formula for the Gaussian mass,
\[ m^2 \equiv \frac{d^2V_{eff}}{d\varphi^2} \bigg|_{\varphi=0} = m^2_0 + 12\lambda I^d_{0}(\Omega_0) + 2e^2 I^d_{-1}(\Omega_0). \] (2.27)

where \( \Omega_0 \) and \( \Omega_0 \) are respectively solutions for \( \Omega \) and \( \Omega \) at \( \varphi = 0 \), explicitly,
\[ \Omega^2_0 = m^2_0 + 12\lambda I^d_{0}(\Omega_0) + 2e^2 I^d_{-1}(\Omega_0). \] (2.28)
\[ \Omega_0 = e^2 I^d_{-1}(\Omega_0). \] (2.29)

Therefore, from Eqs.(2.27) and (2.28) we get simply,
\[ m^2 = \Omega^2_0. \] (2.30)

Hence, we see from the gap equation (2.27) that the Gaussian mass obeys a generalized "Gaussian" Dyson-Schwinger equation,
\[ m^2 = m^2_0 + 12\lambda I^d_{0}(m) + 2e^2 I^d_{-1}(m). \] (2.31)

This expression will be used later to describe the system in the neighbourhood of criticality.

**III. CRITICAL BEHAVIOR OF THE GINZBURG-LANDAU MODEL COMPACTIFIED ON TWO SPATIAL DIMENSIONS**

We will work in the approximation of neglecting boundary corrections to the coupling constant. A precise definition of the boundary-modified mass parameter will be given later for the situation of \( D = 3 \) with \( d = 2 \), corresponding to a wire of rectangular section \( L_1 \times L_2 \). We also consider the limiting situation of one of the transverse dimensions, \( L_2 \to \infty \), which corresponds to a film of thickness \( L_1 \). We use Cartesian coordinates \( \mathbf{r} = (x_1, ..., x_d, z) \), where \( z \) is a \( (D - d) \)-dimensional vector, with corresponding momentum \( \mathbf{k} = (k_1, ..., k_d, q) \), \( q \) being a \( (D - d) \)-dimensional vector in momentum space. Then the generating functional of correlation functions has the form,
\[ Z = \int \mathcal{D}\varphi^d \mathcal{D}\varphi \exp \left( -\int_0^L d^d r \int d^{D-d} z H(\varphi, \nabla \varphi) \right), \] (3.1)

where \( L = (L_1, ..., L_d) \), and we are allowed to introduce a generalized Matsubara prescription, performing the following multiple replacements (compactification of a \( d \)-dimensional subspace).
\[ \int \frac{dk_i}{2\pi} \rightarrow \frac{1}{L_i} \sum_{n_i=-\infty}^{+\infty} ; \quad k_i \rightarrow \frac{2n_i \pi}{L_i}, \quad i = 1, 2, \ldots, d. \]  

(3.2)

A simpler situation is the system confined simultaneously between two parallel planes a distance \( L_1 \) apart from one another normal to the \( x_1 \)-axis and two other parallel planes, normal to the \( x_2 \)-axis separated by a distance \( L_2 \) (a “wire” of rectangular section). We emphasize however, that here we are considering an Euclidean field theory in \( D \) purely spatial dimensions, we are not working in the framework of finite temperature field theory. Temperature is introduced in the mass term of the Hamiltonian by means of the usual Ginzburg-Landau prescription.

For our proposes we only need the calculation of the integral given in equation (2.15) in the situation of confinement of the present section. With the prescription (3.2), the expression corresponding to equation (2.15) for the confined system can be written in the form,

\[ I_0^D (M) = \frac{1}{4 \pi^2 L_1 L_2 \ldots L_d} \sum_{n_1, \ldots, n_d = -\infty}^{+\infty} \int \frac{d^{D-d} q}{(a_1^2 n_1^2 + a_2^2 n_2^2 + \ldots + a_d^2 n_d^2 + c^2 + q^2)^s}, \]  

(3.3)

In the following, to deal with dimensionless quantities in the regularization procedures, we introduce the parameters

\[ c = \frac{m}{2\pi}, \quad a_i = \frac{1}{L_i}, \quad q_i = \frac{k_i}{2\pi} \]  

(3.4)

and using a well-known dimensional regularization formula [15] to perform the integration over the \( (D - d) \) non-compactified momentum variables, we have,

\[ \int \frac{d^{D-d} q}{(a_1^2 n_1^2 + a_2^2 n_2^2 + \ldots + a_d^2 n_d^2 + c^2 + q^2)^s} = \frac{\pi^{D-d}}{2} \Gamma(s - \frac{D-d}{2}) \frac{1}{\Gamma(s)} \frac{1}{(a_1^2 n_1^2 + a_2^2 n_2^2 + \ldots + a_d^2 n_d^2 + c^2)^s - \frac{D-d}{2}}. \]  

(3.5)

with which Eq.(3.3) is written,

\[ I_0^D (M) = \frac{\pi^{D-d}}{4 \pi^2 L_1 L_2 \ldots L_d} \sum_{n_1, \ldots, n_d = -\infty}^{+\infty} \frac{1}{(a_1^2 n_1^2 + a_2^2 n_2^2 + \ldots + a_d^2 n_d^2 + c^2)^s - \frac{D-d}{2}}. \]  

(3.6)

The sum in Eq.(3.6) is one of the Epstein-Hurwitz zeta-functions, defined by,

\[ A_d^\nu (\nu; a_1, \ldots, a_d) = \sum_{n_1, \ldots, n_d = -\infty}^{+\infty} (a_1^2 n_1^2 + \cdots + a_d^2 n_d^2 + c^2)^{-\nu} \]  

(3.7)

where \( \nu = s - \frac{D-d}{2} \). Next we can proceed generalizing to several dimensions the mode-sum regularization prescription described in Ref. [16]. This generalization has been done in [8] and we briefly describe here its main steps. From the identity,

\[ \frac{1}{\Delta^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt \ t^{\nu-1} e^{-\Delta t}, \]  

(3.8)

and using the representation for Bessel functions of the third kind, \( K_{\nu} \),

\[ 2(a/b)^2 K_{\nu} (2\sqrt{ab}) = \int_0^\infty dx \ x^{\nu-1} e^{-(a/x) - bx}, \]  

(3.9)

we obtain after some rather long but straightforward manipulations [8],

\[ A_d^\nu (\nu; a_1, \ldots, a_d) = \frac{2^{\nu-d} + 1}{a_1 \cdots a_d \Gamma(\nu)} \left[ 2^{\nu-d-1} \Gamma \left( \nu - \frac{d}{2} \right) \left( \frac{2\pi c}{d-2\nu} \right)^{d-2\nu} + 2 \sum_{i=1}^{d} \sum_{n_i = 1}^{+\infty} \left( \frac{n_i}{2\pi c a_i} \right)^{\nu-d} K_{\nu-d} \left( \frac{2\pi c n_i}{a_i} \right) \right] \]  

(3.10)
In our case \( s = 1 \) and for wires \( d = 2 \), and Eq.(3.10) becomes,

\[
A_2^2 (2 - \frac{D}{2}; L_1, L_2) = \frac{L_1 L_2 2^{2-D/2} \pi^{3-D}}{\Gamma(2 - \frac{D}{2})} \left[ 2^{-\frac{D}{2}} \Gamma(1 - \frac{D}{2}) M^{D-2} + \sum_{n_1=1}^{\infty} \left( \frac{L_1^2 n_1^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (ML_1 n_1) + 2 \sum_{n_2=1}^{\infty} \left( \frac{L_2^2 n_2^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (ML_2 n_2) + 2^2 \sum_{n_1, n_2=1}^{\infty} \left( \frac{L_1^2 n_1^2 + L_2^2 n_2^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (M \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}) \right].
\]

(3.11)

Thus, inserting the equation (3.11) into equation (3.6) and making some algebraic manipulations, we obtain

\[
I_0^D (M) = 2^{-D} \pi^{D-\frac{D}{2}} \Gamma(1 - \frac{D}{2}) M^{D-2} + 2^{-\frac{D}{2}} \pi^{D-\frac{D}{2}} \left[ \sum_{n_1=1}^{\infty} \left( \frac{L_1^2 n_1^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (ML_1 n_1) + 2 \sum_{n_2=1}^{\infty} \left( \frac{L_2^2 n_2^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (ML_2 n_2) + 2^2 \sum_{n_1, n_2=1}^{\infty} \left( \frac{L_1^2 n_1^2 + L_2^2 n_2^2}{M} \right)^{1-\frac{D}{2}} K_{1-\frac{D}{2}} (M \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}) \right].
\]

(3.12)

where \( K_\nu \) are the Bessel function of third kind.

**IV. CRITICAL BEHAVIOR FOR WIRES**

In our case, we are analyzing a superconducting type-I wire, which obeys the following conditions [14],

\[
\xi(T) = (\frac{\mu}{m})^{-1} \gg \lambda(T) \gg L,
\]

(4.1)

where \( L \) is the smallest one of the linear dimensions in the rectangular cross-section area of the wire \( A = L_1 \times L_2 \), and \( \xi(T) \) and \( \lambda(T) \) are respectively the critical correlation length and the critical penetration depth defined by,

\[
\xi(T) = \frac{\xi_0}{|t|^{1/2}} ; \quad \lambda(T) = \frac{\lambda_0}{|t|^{1/2}} ; \quad t = \frac{T - T_c}{T_c},
\]

(4.2)

where \( T_c \) is the transition temperature, \( \xi_0 \) and \( \lambda_0 \) the intrinsic coherence length and London penetration depth, respectively. In the case where one of the linear dimensions is much larger than the other, \( L_1 \geq L_2 \) or \( L_2 \geq L_1 \), we should retrieve a film-like behavior, and \( L \) in Eq.(4.1) would be the thickness of the film. After that, we can take \( M = \mu \) in Eq.(3.12) and restrict ourselves to the neighborhood of criticality, that is to the region defined by \( \mu \approx 0 \). Then we can use the asymptotic formula,

\[
K_{|\nu|}(z) = \frac{1}{2} \Gamma(|\nu|) \left( \frac{2}{z} \right)^{|\nu|} ; \quad z \sim 0
\]

(4.3)

which allows, near criticality, to write equation (3.12) in the form

\[
I_0^D (\mu \approx 0) = 2^{-D} \pi^{D-\frac{D}{2}} \Gamma(1 - \frac{D}{2}) \left[ \sum_{n_1=1}^{\infty} \frac{2^{\frac{D}{2}-1}}{(L_1 n_1)^{\frac{D}{2}-1}} + \sum_{n_2=1}^{\infty} \frac{2^{\frac{D}{2}-1}}{(L_2 n_2)^{\frac{D}{2}-1}} + 2 \sum_{n_1, n_2=1}^{\infty} \frac{2^{\frac{D}{2}-1}}{(L_1^2 n_1^2 + L_2^2 n_2^2)^{\frac{D}{2}-1}} \right]
\]

(4.4)

or,

\[
I_0^D (\mu \approx 0) = \frac{\Gamma \left( \frac{D}{2} - 1 \right)}{2\pi^{D/2}} \left[ \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2) + 2E_2 \left( \frac{D-2}{2}; L_1, L_2 \right) \right]
\]

(4.5)

where \( E_2 \left( \frac{D-2}{2}; L_1, L_2 \right) \) is the generalized 2-dimensional Epstein zeta-function defined by [27]

\[
E_2 \left( \frac{D-2}{2}; L_1, L_2 \right) = \sum_{n_1, n_2=1}^{\infty} \left[ L_1^2 n_1^2 + L_2^2 n_2^2 \right]^{-\frac{D-2}{2}}.
\]

(4.6)
In an analogous way as it is done for the Riemann zeta-function, one can also construct analytical continuations (and recurrence relations) for the multidimensional Epstein-Hurwitz zeta-functions which permit to write them in terms of Kelvin and Riemann zeta functions. To start one considers the analytical continuation of the Epstein-Hurwitz zeta-function given by [16]

$$\sum_{n=1}^{\infty} (n^2 + p^2)^{-\nu} = -\frac{1}{2\pi} \frac{2\nu}{2\pi - 1\Gamma(\nu)} \left[ \Gamma \left( \nu - \frac{1}{2} \right) + 4 \sum_{n=1}^{\infty} (\pi pn)^{-\nu} \frac{1}{2\pi pm} K_{\nu-\frac{1}{2}}(2\pi pm) \right].$$

Using this relation to perform one of the sums in (4.6) leads immediately to the question of which sum is firstly evaluated. Whatever the sum one chooses to perform firstly, the manifest order to preserve this symmetry, we adopt here a symmetrized summation. Generalizing the prescription introduced in [8], we consider the multidimensional Epstein function defined as the symmetrized summation

$$E_d(\nu; L_1, \ldots, L_d) = \frac{1}{d!} \sum_{\sigma} \prod_{i=1}^{d} \sum_{n_i=1}^{\infty} \sum_{n_{d_i}=1}^{\infty} \left[ \sigma_i n_i^2 + \cdots + \sigma_d n_d^2 \right]^{-\nu},$$

where \(\sigma_i = \sigma(L_i)\), with \(\sigma\) running in the set of all permutations of the parameters \(L_1, \ldots, L_d\), and the summations over \(n_1, \ldots, n_d\) being taken in the given order. Applying (4.7) to perform the sum over \(n_d\), one gets

$$E_d(\nu; L_1, \ldots, L_d) = -\frac{1}{2d} \sum_{i=1}^{d} E_{d-1}(\nu; \ldots, \hat{L}_i, \ldots)$$

$$+ \frac{\sqrt{\pi}}{2d \Gamma(\nu)} \left( \nu - 1 \right) \sum_{i=1}^{d} \frac{1}{L_i} E_{d-1} \left( \nu - \frac{1}{2}; \ldots, \hat{L}_i, \ldots \right) + \frac{2\sqrt{\pi}}{d \Gamma(\nu)} \left( \nu - \frac{1}{2} \right) L_i^{\nu} + \cdots + L_i^{\nu} \right) W_d \left( \nu - \frac{1}{2}; L_1, \ldots, L_d \right),$$

where the hat over the parameter \(L_i\) in the functions \(E_{d-1}\) means that it is excluded from the set \(\{L_1, \ldots, L_d\}\) (the others being the \(d-1\) parameters of \(E_{d-1}\)), and

$$W_d(\eta; L_1, \ldots, L_d) = \sum_{i=1}^{d} \frac{1}{L_i} \sum_{n_{d_i}=1}^{\infty} \left( \frac{\pi n_i}{L_i} \right)^{\frac{\eta}{\nu}} \left( \frac{2\pi n_i}{L_i} \sqrt{\cdot \cdot \cdot + L_i^{\nu} n_i^2 + \cdots} \right),$$

with \(\cdots + \hat{L}_i^{\nu} n_i^2 + \cdots\) representing the sum \(\sum_{j=1}^{d} L_j^{\nu} n_i^2 - L_i^{\nu} n_i^2\). In particular, noticing that \(E_1(\nu; L_j) = L_j^{-2\nu} \zeta(2\nu)\), one finds

$$E_2 \left( \frac{D-3}{2}; L_1, L_2 \right) = -\frac{1}{4} \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2)$$

$$+ \frac{\sqrt{\pi} \Gamma(D-3)}{4 \Gamma(D-2)} \left( \frac{1}{L_1 L_2^{D-2}} + \frac{1}{L_1^{D-3} L_2} \right) \zeta(D-3) + \frac{\sqrt{\pi} \Gamma(D-3)}{4 \Gamma(D-2)} W_2 \left( \frac{D-3}{2}; L_1, L_2 \right).$$

Inserting (4.11) into (4.5) we get,

$$I_0^D(m \approx 0) = \frac{\Gamma(D-2)}{4\pi^{D/2}} \left( \frac{1}{L_1^{D-2}} + \frac{1}{L_2^{D-2}} \right) \zeta(D-2) + \frac{\Gamma(D-3)}{4\pi^{D-1/2}} \left( \frac{1}{L_1 L_2^{D-3}} + \frac{1}{L_2 L_2^{D-3}} \right) \zeta(D-3)$$

$$+ \frac{W_2(D-3/2; L_1, L_2)}{\pi^{D-1/2}},$$

where,

$$W_2(0; L_1, L_2) = \sum_{n_1, n_2=1}^{\infty} \left\{ \frac{1}{L_1} K_0 \left( 2\pi L_2 n_1 n_2 \right) + \frac{1}{L_2} K_0 \left( 2\pi L_1 n_1 n_2 \right) \right\}.$$

For \(D = 3\), the first and second terms between the brackets in Eq. (4.12) are divergent due to the poles of the \(\zeta\)-function and \(\Gamma\)-function, respectively. However, the whole expression in (4.12) is not divergent, in fact these divergences cancel between themselves leaving a finite result for \(I_0^D(m \approx 0)\). This can be easily seen if we consider the following properties of the \(\zeta\)-function,
\[
\zeta(z) = \frac{1}{\Gamma(z/2)} \Gamma\left(\frac{1-z}{2}\right) \pi^{-\frac{1}{2}} \zeta(1-z); \quad \lim_{z \to 1} \left[ \zeta(z) - \frac{1}{z-1} \right] = \gamma, \tag{4.14}
\]

where \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant.

The quantity \( W_2(0; L_1, L_2) \), appearing in Eq. (4.12), involves complicated double sums; in particular it is very difficult to be analytically handled for \( L_1 \neq L_2 \), what means that it is not possible to take analytically limits such as \( L_2 \to \infty \) for finite \( L_1 \). Such a limit (which corresponds to a film of thickness \( L_1 \)) will be considered numerically. Using Eq. (4.14) in Eq. (4.12) we obtain for \( D = 3 \),

\[
I_0^3(\overline{m} \approx 0) \approx \frac{\gamma}{2\pi} \left( \frac{1}{L_1} + \frac{1}{L_2} \right) + \frac{1}{\pi} W_2(0; L_1 L_2), \tag{4.15}
\]

and Eq. (3.12) becomes,

\[
I_0^3(\Delta_0) = \frac{\Gamma\left(-\frac{1}{2}\right) \Delta_0}{2^3 \pi^{3/2}} + \frac{1}{2^{1/2} \pi^{3/2}} \left[ \sum_{n_1=1}^{\infty} \left( \frac{\Delta_0}{L_1 n_1} \right)^{1/2} K_{1/2}(\Delta_0 L_1 n_1) + \sum_{n_2=1}^{\infty} \left( \frac{\Delta_0}{L_2 n_2} \right)^{1/2} K_{1/2}(\Delta_0 L_2 n_2) + \sum_{n_1, n_2=1}^{\infty} \left( \frac{\Delta_0}{\sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}} \right)^{1/2} K_{1/2}(\Delta_0 \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}) \right]. \tag{4.16}
\]

Using the formula,

\[
K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \tag{4.17}
\]

and remembering the notation, \( \Delta_0 = \sqrt{e I_0^3(\overline{m})} \) we obtain, after some rearrangements,

\[
I_0^3(\sqrt{e I_0^3(\overline{m})}) = -\frac{e \sqrt{I_0^3(\overline{m} \approx 0)}}{4\pi} + \frac{\ln(1 - e^{-e L_1 \sqrt{I_0^3(\overline{m} \approx 0)}})}{2 \pi L_1} + \frac{\ln(1 - e^{-e L_2 \sqrt{I_0^3(\overline{m} \approx 0)}})}{2 \pi L_2} + \frac{1}{\pi} \sum_{n_1, n_2=1}^{\infty} e^{-e \sqrt{I_0^3(\overline{m} \approx 0)}} \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}. \tag{4.18}
\]

Thus we can write the Gaussian gap equation (2.31) in the neighbourhood of criticality in the form,

\[
\overline{m}^2 \approx m_0^2 + 12 \lambda I_0^3(\overline{m} \approx 0) - \frac{e^2}{2 \pi} \sqrt{I_0^3(\overline{m} \approx 0)} - \frac{e^2}{\pi L_1} \ln(1 - e^{-e L_1 \sqrt{I_0^3(\overline{m} \approx 0)}}) + \frac{e^2}{\pi L_2} \ln(1 - e^{-e L_2 \sqrt{I_0^3(\overline{m} \approx 0)}}) + \sum_{n_1, n_2=1}^{\infty} e^{-e \sqrt{I_0^3(\overline{m} \approx 0)}} \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2}. \tag{4.19}
\]

Taking \( m_0^2 = a(T/T_0 - 1) \), with \( a > 0 \) we obtain from the above equation for \( \overline{m}^2 = 0 \), the critical temperature as a function of the wire transverse dimensions, \( L_1, L_2 \) and the bulk transition temperature, \( T_0 \),

\[
T_c(L_1, L_2) = T_0 \left[ 1 - \frac{12 \lambda}{a} I_0^3(\overline{m} \approx 0) + \frac{e^3}{2 \alpha \pi} \sqrt{I_0^3(\overline{m} \approx 0)} + \frac{e^2}{\alpha \pi L_1} \ln(1 - e^{-e L_1 \sqrt{I_0^3(\overline{m} \approx 0)}}) + \frac{e^2}{\alpha \pi L_2} \ln(1 - e^{-e L_2 \sqrt{I_0^3(\overline{m} \approx 0)}}) - 2 \frac{e^2}{\alpha \pi} \sum_{n_1, n_2=1}^{\infty} e^{-e \sqrt{I_0^3(\overline{m} \approx 0)}} \sqrt{L_1^2 n_1^2 + L_2^2 n_2^2} \right]. \tag{4.20}
\]

For reasons that will become apparent later, let us take \( L_2 = p L_1 \), \( L_1 \equiv L \), with \( p \) a positive constant; then we get from Eqs. (4.13) and (4.15),

\[
I_0^3(\overline{m}) = C_L, \tag{4.21}
\]

where
\[ C_p = \frac{\gamma}{2\pi} \left( \frac{p+1}{p} \right) + \frac{1}{\pi} \left[ \sum_{n_1, n_2=1}^{\infty} \left( K_0(2\pi pn_1n_2) + \frac{1}{p} K_0(2\pi n_1n_2/p) \right) \right]. \quad (4.22) \]

and the critical temperature (4.20) is rewritten in the form,

\[ T_c(L, p) = T_0 \left[ 1 - \frac{12\lambda C_p}{aL} + \frac{\epsilon^3 \sqrt{C_p}}{2\pi a \sqrt{L}} + \frac{\epsilon^2}{a\pi L} \ln(1 - e^{-\epsilon \sqrt{L C_p}}) + \right. \]
\[ \left. + \frac{\epsilon^2}{a\pi pL} \ln(1 - e^{-\epsilon \sqrt{L C_p}}) - \frac{2\epsilon^2}{a\pi L} \sum_{n_1, n_2=1}^{\infty} \frac{e^{-\epsilon \sqrt{L C_p} \sqrt{n_1^2 + p^2 n_2^2}}}{\sqrt{n_1^2 + p^2 n_2^2}} \right]. \quad (4.23) \]

This equation describes the behavior of the critical temperature of a type-I superconducting wire having a rectangular cross-section, in terms of one of its transverse dimensions \((L_1 = L)\) and of the parameter \(p = L_2/L_1\). We recall that in this expression all gauge fluctuations have been taken into account by means of the Gaussian effective potential. These contributions are given by the last four terms in Eq.\((4.23)\).

**Comparison to some experimental results:**

Up to now we have used Natural Units, \(c = \hbar = k_B = 1\). To proceed, we restore SI units, remembering that a factor \(1/k_B T_0\) is implicit in the exponent of equation (3.1). Next we rescale the fields and coordinates by \(\phi \rightarrow \phi_{\text{new}} = \sqrt{\xi_0/k_B T_0} \phi, A \rightarrow A_{\text{new}} = \sqrt{\xi_0/k_B T_0} \phi, x \rightarrow x_{\text{new}} = x/\xi_0\), with \(\xi_0 = 0.18 \hbar v_F/k_B T_0\), being the intrinsic coherent length and \(v_F\) the Fermi velocity of a particular material. This implies that, in the Ginzburg-Landau model, the parameters \(a, \lambda, \epsilon\) become dimensionless and are given by [14],

\[ a = 1, \quad \lambda = 111.08 \left( \frac{T_0}{T_F} \right)^2, \quad \epsilon \approx 2.59 \sqrt{\frac{\alpha v_F}{c}}, \quad (4.24) \]

where \(T_F\) and \(\alpha\) are respectively the Fermi temperature and the fine structure constant. Thus, equation (4.23) is rewritten as,

\[ T_c(L, pL) = T_0 \left[ 1 - 1333.0 \left( \frac{\xi_0}{L} \right) \left( \frac{T_0}{T_F} \right)^2 C_p + 17.253 \times 10^{-4} \sqrt{\frac{\xi_0}{L}} \sqrt{C_p v_F^3} + \right. \]
\[ + 15.594 \times 10^{-3} \left( \frac{\xi_0}{L} \right) v_F \ln(1 - e^{-0.22128 \sqrt{\frac{\xi_0}{L}} C_p v_F}) + \right. \]
\[ + 15.594 \times 10^{-3} \left( \frac{\xi_0}{L} \right) v_F \ln(1 - e^{-0.22128 \sqrt{\frac{\xi_0}{L}} C_p v_F}) + \]
\[ - 0.031187 \left( \frac{\xi_0}{L} \right) v_F \sum_{n_1, n_2=1}^{\infty} \frac{e^{-0.22128 \sqrt{\frac{\xi_0}{L}} C_p v_F \sqrt{n_1^2 + p^2 n_2^2}}}{\sqrt{n_1^2 + p^2 n_2^2}} \right]. \quad (4.25) \]

As an application of this equation we plot the behavior of the critical temperature \(T_c(L, p)\) as a function of \(1/L\) to a superconducting wire made from niobium, characterized by \(v_F = 1.37 \times 10^6 \text{ m/s}, T_0 = 9.3 \text{ K},\) and \(T_F = 6.18 \times 10^4\) K. We have considered three cases, with \(p = 1, p = 4\) and \(p = 10\) corresponding to wires of different rectangular cross-sections.
FIG. 1. Critical temperature $T_c$ as function of one of the transverse dimensions $1/L$, from Eq.(4.25) for a superconducting wire made from Niobium of rectangular section cross $A = L \times pL$. The figure shows the behavior of the critical temperature for the cases, $p = 1$, $p = 4$ and $p = 10$, respectively.

In Fig. 1, we show the quasi linear behavior of the critical temperature of a niobium wire as a function of $L$, for $p = 1$, $p = 4$ and $p = 10$. These curves suggest the existence of minimal values of $L$ (or of the rectangular wire cross-sections), for which the superconducting phase transition is suppressed. The approximate values for each case are: $L_{p=1}^{min} \approx 2.5$ nm, $L_{p=4}^{min} \approx 1.0$ nm and $L_{p=10}^{min} \approx 0.5$ nm respectively.

Also, it can be shown numerically that as we take larger and larger values of $p$ ($p \to \infty$) in Eq.(4.25) all the curves tend to coincide, corresponding to a sample in the form of a film of thickness $L$. A numerical analysis of Eq.(4.22) shows that as $p \to \infty$, the quantity $C_p$ tends to a constant value, $C_p = \gamma$. Replacing this value in Eq.(4.25) we get the critical temperature for a film. In Fig.2 we plot Eq.(4.25) in this limiting situation, which corresponds to a superconducting film of thickness $L$. In this case we find a minimal film thickness $L_{film}^{min} \approx 0.13$ nm.

V. CONCLUSIONS

In this paper we have considered the confined Ginzburg-Landau model, in the transverse unitarity gauge, as a model to describe samples of superconducting materials not in bulk form. To generate the contributions from gauge fluctuations, we have used the Gaussian effective potential [2–5], which allows to obtain a gap equation that can be treated with the method of recent developments [8,9]. We have derived a critical equation that describes the changes in the critical temperature due to confinement. Contributions from the self interaction of the scalar field and from the gauge field fluctuations can be clearly identified. Our approach suggests a minimal size sustaining the existence of charged and non-charged transitions for both superconducting wires and films. Such a kind of result could be of practical interest to define limits of miniaturization of superconducting wires in manufacturing electronic circuits. It is interesting to compare the minimal allowed value of $L \approx 2.5$ nm for a wire having a square cross-section that we have obtained, to the experimentally observed superconducting state in nanowires having a cross-section diameter of $\approx 10$ nm [29].

Also, the behavior described in Eq.(4.25) for $p \to \infty$ (a film) could be contrasted with the linear decreasing of the critical temperature with the inverse of the film thickness, that has been found experimentally in materials containing transition metals, for example, in PB [20], in W-Re alloys [21], in Nb [22,24,26], Mo-Ge [23]; for some of these cases, this behavior has been explained in terms of proximity, localization and Coulomb-interaction effects. We can clearly see from Eq.(4.25), as noticed above, that a linear decreasing of the critical temperature with the inverse of film thickness is recovered when we take $e = 0$. Even though, with $e \neq 0$ a quasi-linear decreasing of $T_c$ with $1/L$ can be directly obtained from Eq.(4.25), since the terms coming from effects of gauge fluctuations are very small as compared to the term generated from the self coupling. Also a comparison may be done with recent theoretical results for type II superconductors [28], where a similar behavior of the critical temperature with the film thickness has been found for non-charged transitions.

However, it should be noticed that we have used 3-dimensional values for the phenomenological parameters. This means that we should restrict ourselves only to relatively thin wires and films, which could be considered as essentially 3-dimensional objects; very thin wires and films cannot be physically accommodated in the context of our model. For
instance, the value we have obtained for $L_{\text{min}}^{\text{film}}(Nb)(\approx 0.13 \, \text{nm})$ is of the order of magnitude of a few Bohr radii and should be considered to correspond to a 2-dimensional system. So, on physical grounds, it is beyond the domain of validity of the model. Nevertheless our results obtained from the "pure" GL model are in qualitative agreement with the experimentally observed behaviors mentioned above. It is worth to emphasize that the quasi-linear character of the decreasing of the transition temperature obtained in this paper, emerge solely as a topological effect of the spatial compactification of the Ginzburg-Landau model.

Also, it should be remarked that our formalism can not account for microscopic details present in real samples of materials nor for the influence of manufacturing aspects, like the kind of the substrate on which a film is deposited, or the presence of impurities. This means that our expression for the transition temperature should be further improved (in particular the GL parameters), and more work is needed to better understand the behavior of real samples. This will be the subject of future investigation.

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