Abstract

We obtain explicit expressions for the $\eta$ and $\eta'$ masses and decay constants using $U(3)_L \otimes U(3)_R$ chiral perturbation theory at next-to-leading order in a combined expansion in $p^2$ and $1/N_c$. A numerical fit of the parameters appearing to this order of the expansion is also discussed.
The masses of the I=0 pseudoscalar $\eta$, $\eta'$ particles as well as the value of their mixing angle have long been the subject of discussion and theoretical interest, also in recent years. Phenomenologically, the situation of the $\eta - \eta'$ mixing remains not completely settled, the reason being the high sensitivity of the mixing angle $\theta$ on the deviation $(\Delta)$ of the $\eta_8$ mass from the Gell Mann-Okubo $(\Delta = 0)$ relation

$$m_{\eta_8}^2 = \frac{1}{3}(4m_K^2 - m_{\pi}^2)(1 + \Delta).$$

Indeed, if $\Delta = 0$ the singlet-octet mixing yields $\theta \sim -10^\circ$, whereas data on $J/\Psi \rightarrow \eta(\eta')\gamma$ and $\eta(\eta', \pi^0) \rightarrow \gamma\gamma$ favour a higher mixing $\theta \sim -20^\circ$ [3], a factor of two bigger, that can be accomodated from (1) with just a small $\Delta \sim 0.16$, (see also ref. [3], which advocates for a smaller mixing).

The value of the mixing angle is linked to the value of the $\eta'$ mass, which is heavier than the octet of light pseudoscalars, $M_{\eta'} = 958$ MeV $\sim 2M_K$, but still lighter than the next I=0 candidate $\eta(1279)$, which is so distant from the $\eta$ mass that it might not significantly mix with it. Therefore, we only consider $\eta - \eta'$ mixing.

Within the $SU(3)_L \otimes SU(3)_R$ Chiral Lagrangian [4] analysis, the shift downwards of the $\eta$ mass due to this mixing is an $O(p^4)$ effect of $SU(3)_{R+L}$ breaking, proportional to $(m_s - \frac{m_u + m_d}{2})^2$ and multiplied by the constant $\tilde{L}_\eta^{SU}$, which contains information of the $\eta'$ state that has been integrated out.

In this article we re-analyse these issues on the basis of large-$N_c$ Chiral Perturbation Theory ($\chi$PT). The $\eta'$ is regarded as the ninth goldstone boson of a nonet, together with the octet of light pseudoscalars, and the corrections are treated in a double expansion, both in powers of quark masses $m_{\text{quark}}$ and $1/N_c$. Such a formulation for this problem is plausible because $M_{\eta'}^2 - M_{\text{octet}}^2$ is an effect induced by the $U_A(1)$ anomaly, which vanishes in the large-$N_c$ limit as $1/N_c$. Furthermore, the fact that phenomenologically it is found that $M_{\eta'} - M_{\text{octet}} \sim M_{\text{octet}}$, together with the usual bookkeeping of quark masses as $m_{\text{quark}} \sim O(M_{\text{octet}}^2)$, suggests that the relative magnitude of the double expansion may be regarded as

$$m_q \sim 1/N_c \sim M_{\text{octet}}^2 \sim p^2 \sim O(\delta);$$

this is the approach that we adopt and henceforth we shall refer to it as the combined expansion. It has already been used in the literature [3], in particular Leutwyler has forcefully pursued the analysis of the masses of $\eta$ and $\eta'$ in order to discard the possibility that $m_u = 0$. In this note we reproduce Leutwyler’s results and proceed further to obtain a fit for the masses and decay constants of the goldstone boson nonet [3].

Let us start from the $U_L(3) \otimes U_R(3)$ chiral lagrangian which has been recently proposed to $O(p^4)$ [4], from which we share the notation and conventions, and which we refer to for references therein. The nonet of fields are gathered in a unitary $3 \times 3$ matrix $U \in U(3)$, whose determinant $\det U = \exp(i\sqrt{6}\eta^0/f)$ differs from unity by the presence of the $\eta^0$ field. The chiral lagrangian $\mathcal{L}(\rho^0) + \mathcal{L}(\rho^2) + \ldots$ reads

$$\mathcal{L}(\rho^0) = -W_0(X),$$

$$\mathcal{L}(\rho^2) = W_1(X)\langle D_\mu U^\dagger D_\mu U \rangle + W_2(X)\langle U^\dagger \chi + \chi^\dagger U \rangle + iW_3(X)\langle U^\dagger \chi - \chi^\dagger U \rangle + W_5(X)\langle U^\dagger (D_\mu U) (ia_\mu) \rangle.$$
We have only kept the axial source because we take the divergence of the axial current as the interpolating fields for the nonet. The axial source \( a_\mu \) appears in the last term as well as through the covariant derivatives of the fields:

\[
D_\mu U = \partial_\mu U - \frac{i}{2} (a_\mu U + U a_\mu).
\]

The quark masses appear in

\[
\chi = 2B \text{ diag}(m_u, m_d, m_s).
\]

As a consequence of the \( U_A(1) \) anomaly, the lagrangian has coefficient functions that depend on \( X = \log(\det U) = i\sqrt{6}\eta_0/f \), which are not fixed by symmetry arguments. From the large-\( N_c \) perspective however, only the lowest powers of \( X \) survive, for each new power of \( X \) is suppressed by a factor of \( 1/N_c \). Of interest to us, the only terms that prevail are:

\[
W_0(X) = \text{Constant} + \frac{f^2}{4} v_{02} X^2 + \ldots, \quad W_1(X) = \frac{f^2}{4} + \ldots, \\
W_2(X) = \frac{f^2}{4} + \ldots, \quad W_3(X) = -i\frac{f^2}{4} v_{31} X + \ldots, \\
W_5(X) = \frac{f^2}{4} v_{50} + \ldots,
\]

with \( v_{02}, v_{31}, v_{50} \sim O(1/N_c) \). We shall expand the chiral lagrangian and keep terms up to \( O(\delta^2) \), according to the combined power counting (2). Recall that \( f^2 \sim O(N_c) \), \( B \sim O(1) \). Within this approximation the chiral logarithms are suppressed (they appear at \( O(\delta^3) \)), although some terms of the same order \( O(p^4) \) have to be included at tree level. More precisely, the contributions

\[
L_{(p^4)} = L_5(D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)) + L_8(\chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi)
\]

should be included, since \( L_5 \) and \( L_8 \) are \( O(N_c) \). Notice that a term \( L_7(U^\dagger \chi - \chi^\dagger U)^2 \) appears at \( O(\delta^4) - L_7 \) is \( O(1) \) in \( 1/N_c \) (see [8]). The rest of terms all remain subleading.

The masses and the decay constants can be obtained from the two-point function of axial currents, which is most easily done by taking functional derivatives with respect to \( a_\mu \). We shall thus keep only terms quadratic in \( a_\mu \). The part of the effective action quadratic in the nonet fields finally reads

\[
S = \frac{1}{2} \int d^4x \left( (\partial_\mu \phi^a - b_\mu^a) A_{ab}(\partial_\mu \phi^b - b_\mu^b) - \phi^a B_{ab} \phi^b + \phi^a C_{ab} \partial_\mu b_\mu^b \right),
\]

where \( b_\mu = 2fa_\mu \), \( \phi^a (a = 0, 1, \ldots, 8) \) are the fields with \( SU(3)_{R+L} \) quantum numbers and

\[
A = I + \Delta A , \quad B = 2mB(D + \Delta D) , \quad C = \Delta C
\]

\footnote{In the \( U(3)_L \otimes U(3)_R \) chiral lagrangian, instead of constants the \( L_i \)'s are functions of the \( \eta_0 \) field. When we write \( L_5 \) or \( L_8 \) throughout the article we mean the value of those functions with the argument set to zero, \( L_5(0), L_8(0) \). They are not to be confused with the constants in \( [SU] \), for which we reserve a superscript of \([SU] \).}
The mass matrix $D$ is non-diagonal already at leading order:

$$D_{11} = D_{22} = D_{33} = 1, \quad D_{44} = D_{55} = D_{66} = D_{77} = 1 + \frac{x}{2},$$

$$D_{88} = 1 + \frac{2}{3}x, \quad D_{08} = -\frac{\sqrt{2}}{3}x, \quad D_{00} = 1 + \frac{1}{3}x - \frac{3}{2}v_{02}mB,$$

(9)

with $m = (m_u + m_d)/2$, $x = (m_s - m)/m$. On the other hand, the matrix $\Delta A$ brings the next-to-leading order corrections to the kinetic term,

$$\Delta A_{11} = \Delta A_{22} = \Delta A_{33} = 16\frac{mB}{f^2}L_5,$$

$$\Delta A_{44} = \Delta A_{55} = \Delta A_{66} = \Delta A_{77} = 16\frac{mB}{f^2}L_5\left(1 + \frac{x}{2}\right),$$

$$\Delta A_{88} = 16\frac{mB}{f^2}L_5\left(1 + \frac{2}{3}x\right),$$

$$\Delta A_{08} = -\frac{16\sqrt{2}mB}{3f^2}L_5x,$$

$$\Delta A_{00} = 16\frac{mB}{f^2}L_5\left(1 + \frac{1}{3}x\right),$$

(10)

and $\Delta D$ corrects the mass matrix,

$$\Delta D_{11} = \Delta D_{22} = \Delta D_{33} = 32\frac{mB}{f^2}L_8,$$

$$\Delta D_{44} = \Delta D_{55} = \Delta D_{66} = \Delta D_{77} = 32\frac{mB}{f^2}L_8\left(1 + x + \frac{1}{4}x^2\right),$$

$$\Delta D_{88} = 32\frac{mB}{f^2}L_8\left(1 + \frac{4}{3}x + \frac{2}{3}x^2\right),$$

$$\Delta D_{08} = -\frac{32\sqrt{2}mB}{3f^2}L_8x(2 + x) + \sqrt{2}v_{31}x,$$

$$\Delta D_{00} = 32\frac{mB}{f^2}L_8\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) - 2v_{31}(3 + x).$$

(11)

$\Delta C$ comes entirely from the $U_A(1)$ anomaly:

$$\Delta C_{00} = \frac{3}{2}v_{50}.$$  

(12)

From the correlator of two axial currents we can read off the physical masses and the decay constants, the former from the location of the poles, the latter from their residues. Following the steps of ref. [4], the correlator is the second derivative of (7) with respect to $a_\mu$ when evaluated for $a_\mu$ that minimises the effective action; it boils down to a simultaneous diagonalisation of $A$ and $B$ for the masses,

$$A = F^\dagger I F, \quad B = F^\dagger M^2 F,$$

(13)

whereas the decay constants read ($P$ is a diagonal index, $\alpha$ is non-diagonal),

$$f_{P\alpha} = f \left((F^\dagger)^{-1}\left(A + \frac{C}{2}\right)\right)_{P\alpha}.$$  

(14)
Consistently, we perform a perturbative diagonalization, order by order in the expansion. First, we diagonalize to leading order and only $D$ has a non-diagonal part; it is already diagonal in $\pi$ and $K$ (we shall neglect the isospin breaking),

$$2mB = M_{\pi}^2$$
$$2mB \left(1 + \frac{x}{2}\right) = M_{K}^2,$$

only the 0 and 8 components do mix,

$$2mB \ F_0 D F_0^+ = M^2,$$

with

$$F_0 = \begin{pmatrix} \cos \theta_0 & - \sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix}.$$  

(17)

From (15) we can fix $m_B$ and $x$ unambiguously. Upon diagonalization of (16) we fix $v_0^2$ so as to describe correctly $M_{\eta'}^2$. This implies

$$\theta_0 \simeq -20^\circ, \quad v_0^2 \simeq -0.22 \text{ GeV}^2.$$  

(18)

In this way the prediction for $M_{\eta'}^2 \simeq (549 \text{ MeV})^2$ which is only a 10% off. This is a known feature of the leading $1/N_c$ approximation: it was proven in ref. [9] that the ratio $M_{\eta}/M_{\eta'}$ always comes out too small; the next-to-leading corrections are thus needed to reconcile the $1/N_c$ expansion with the observed masses [10].

Let us emphasize that we had two masses, $M_{\eta}^2$ and $M_{\eta'}^2$, to fit and just one parameter, $v_0^2$, to tune. In principle, we had the choice to fit either one to its experimental value. We could have, instead, fixed the value of $v_0^2$ in order to describe correctly $M_{\eta}^2$, obtaining a prediction for $M_{\eta'}^2$. This choice would give, though, a poor prediction for $M_{\eta'}^2$ since it is the $\eta'$ that gets the bulk of the contribution from $v_0^2$ being, therefore, more sensitive to it.

The next-order expressions for the masses can also be found in [5]. The $\Delta A$ corrections are absorbed into a wave-function renormalization and the $\pi$ and $K$ are still diagonal

$$M_{\pi}^2 = 2mB \left(1 + 16 \frac{mB}{f^2} (2L_8 - L_5)\right),$$
$$M_{K}^2 = 2mB \left(1 + \frac{x}{2}\right) \left(1 + 16 \frac{mB}{f^2} (2L_8 - L_5) \left(1 + \frac{x}{2}\right)\right),$$

(19)

whereas the 0 and 8 components remain to be diagonalized and can be written as

$$m_{88}^2 = \frac{4M_{K}^2 - M_{\pi}^2}{3} + \frac{4}{3}(M_{K}^2 - M_{\pi}^2) \Delta_M,$$
$$m_{08}^2 = -\frac{2\sqrt{2}}{3}(M_{K}^2 - M_{\pi}^2)(1 + \Delta_M - \Delta_N),$$
$$m_{00}^2 = \frac{1}{3}(2M_{K}^2 + M_{\pi}^2)(1 - 2\Delta_N) + \frac{2}{3}(M_{K}^2 - M_{\pi}^2) \Delta_M - 3v_0^2,$$

(20)

where

$$\Delta_M = \frac{8}{f^2}(M_{K}^2 - M_{\pi}^2)(2L_8 - L_5), \quad \Delta_N = 3v_{31} - \frac{12}{f^2}v_0^2 L_5.$$  

(21)
Note the explicit violation of the Kaplan-Manohar symmetry \[ [11] \] in \( SU(3)_L \otimes SU(3)_R \) - usually stated as the fact that in the expressions for the masses \( L_8 \) and \( L_5 \) always come through the combination \( (2L_8 - L_5) \) - in \( U(3) \) chiral perturbation theory which is apparent in \( \Delta_N \): this correction brings about a different dependence, on \( L_5 \) only, due to the presence of \( v_{02} \), i.e., the \( U_A(1) \) anomaly.

Let us now proceed to diagonalise to next-to-leading order. In order to preserve the form of the above partial result, we write the transformation matrix as

\[
F = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
1 + \frac{\Delta_{\text{ass}}}{2} & \frac{\Delta_{\text{ass}}}{2} \\
\frac{\Delta_{\text{ass}}}{2} & 1 + \frac{\Delta_{\text{ass}}}{2}
\end{pmatrix} ,
\]

where we have explicitly separated the rotation that diagonalizes the kinetic term. Note that \( \theta \) also gets corrections with respect to \( \theta_0 \). For the physical masses we find,

\[
M_\eta^2 = (1 - \frac{y}{3})M_K^2 + \frac{y}{3}M_\pi^2 - \frac{\sqrt{9 + 2y + y^2}}{3}(M_K^2 - M_\pi^2) + \left(1 - \frac{9 + y}{3\sqrt{9 + 2y + y^2}}\right)(M_K^2 - M_\pi^2)\Delta_M \\
+ \left(-\frac{1}{3}(2M_K^2 + M_\pi^2) + \frac{1}{3\sqrt{9 + 2y + y^2}}(3(2M_K^2 - 3M_\pi^2) + y(-2M_K^2 - M_\pi^2))\right)\Delta_N ,
\]

where

\[
y \equiv \frac{9}{2} \frac{v_{02}}{M_K^2 - M_\pi^2}
\]

and

\[
M_\eta^2 + M_\eta'^2 = \frac{-2}{3}(-3 + y)M_K^2 + \frac{2}{3}yM_\pi^2 + 2(M_K^2 - M_\pi^2)\Delta_M - \frac{2}{3}(2M_K^2 + M_\pi^2)\Delta_N ;
\]

the angle \( \theta \) being,

\[
\tan 2\theta = \frac{2\sqrt{2}}{1 + y} \left(1 + \frac{y}{1 + y} \Delta_M - \Delta_N - \frac{1}{1 + y} \frac{2M_K^2 + M_\pi^2}{M_K^2 - M_\pi^2} \Delta_N \right).
\]

Let us now turn to the numerical exploitation of these results. The global counting of parameters versus data goes as follows: we are left with \( y \) (which is proportional to \( v_{02} \)), \( \Delta_M \) and \( \Delta_N \) to fix \( M_\eta^2 \) and \( M_\eta'^2 \). Thus, no prediction can be made. Yet, if we fix a rotation angle, then there is a fit for all three parameters. The results for angles which are close to the zeroth order \( \theta_0 \) are displayed in the following table:

| \( \theta \) | \( \Delta_M \) | \( \Delta_N \) | \( \frac{\Delta_{\text{ass}}}{2} \) |
|-----------|----------|----------|----------------|
| \(-18^\circ\) | 0.113 | 0.270 | 30.8 |
| \(-20^\circ\) | 0.156 | 0.220 | 29.3 |
| \(-22^\circ\) | 0.203 | 0.178 | 27.9 |
| \(-24^\circ\) | 0.254 | 0.143 | 26.5 |

The best fit, if \( U(3) \) chiral perturbation theory is to make sense, is expected to be near the zeroth order result. One could play with different convergence criteria to argue in favor of a given angle. A nice feature of our table is that \( \Delta_M \) and \( \Delta_N \) move in different directions as we change the mixing angle. This leads to a minimization of corrections around \( \theta \sim -20^\circ \). Moreover, one may
select the optimal $\theta$ where the masses of $\eta$ and $\eta'$ receive the smallest correction in the sense that corrections proportional to $\Delta_M$ and $\Delta_N$ tend to cancel. Interestingly enough, we find it to be very close to $-20^\circ$ as well.

Of course none of these arguments is a substitute for a global fit to data. However, our results agree in hinting at a mixing angle close to $-20^\circ \geq \theta \geq -21^\circ$, which correspond to values of $\Delta_M$ and $\Delta_N$ compatible with those given in [6] ($\Delta_M \simeq 0.18$, $\Delta_N \simeq 0.24$). We stress that since from this point of view the angle $\theta$ is the only variable, the values of $\Delta_M$ and $\Delta_N$ are correlated.

At this point let us comment on the Gell-Mann-Okubo formula, with corrections from chiral perturbation theory. The violation of the original formula is related to $\Delta_M$,

$$4M_K^2 - M_\pi^2 - 3M_\eta^2 - 3\sin^2 \theta (M_{\eta'}^2 - M_\eta^2) = -4(M_K^2 - M_\pi^2)\Delta_M.$$ \hspace{1cm} (27)

The leading order equation, thus, corresponds to setting the r.h.s. to zero. At this order, $\theta \sim -10^\circ$ in order to fulfill the equation. This is not the procedure we chose in Eq.18, as we took $M_{\eta'}^2$ to fix $\theta \sim -20^\circ$ at leading order. Eq. (27) thus provides a different way of fitting the mixing, which is strongly corrected by $\Delta_M$. Of course, at next-order, the best fit at e.g. $\Delta_M \sim 0.156$ is $\theta \sim -20^\circ$ as well.

We may also use our analytical results to get the form of the decay constants for the whole nonet. The diagonal elements of relevance are given by the combination

$$f_{physical} = f \left(\frac{\cos \theta}{\sin \theta} - \frac{-\sin \theta}{\cos \theta}\right) \left(1 + \frac{\Delta_{\eta_8}}{\Delta_{\eta_0}} \frac{\Delta_{\eta_0}}{2} + \frac{\Delta_{\eta_0}}{2} + \frac{\Delta_{E_0}}{2}\right) \left(\cos \theta \quad \sin \theta \quad -\sin \theta \quad \cos \theta\right).$$ \hspace{1cm} (28)

This expression yields

$$f_\pi = f \left(1 + 4\frac{L_5}{f_\pi^2} M_\pi^2\right),$$

$$f_K = f \left(1 + 4\frac{L_5}{f_K^2} M_K^2\right),$$

$$f_\eta = f \left(1 + 4\frac{L_5}{f_\eta^2} \left(M_K^2 - M_\pi^2 - 3\sin^2 \theta \left(M_{\eta'}^2 - M_\eta^2\right)\right) + \frac{3}{8} v_{50} \left(1 + \frac{1 + y}{\sqrt{9 + 2 y + y^2}}\right)\right),$$

$$f_{\eta'} = f \left(1 + 4\frac{L_5}{f_{\eta'}}^2 \left(M_K^2 + \frac{M_\pi^2 - M_K^2}{3} \frac{9 + y}{\sqrt{9 + 2 y + y^2}}\right) + \frac{3}{8} v_{50} \left(1 - \frac{1 + y}{\sqrt{9 + 2 y + y^2}}\right)\right).$$ \hspace{1cm} (29)

Note that in the exact $SU(3)_{R+L}$ limit of quarks degenerate in mass there is no mixing and one finds

$$f_\eta \rightarrow f \left(1 + 4\frac{L_5}{f_\eta^2} 4M_K^2 - M_\pi^2\right),$$

$$f_{\eta'} \rightarrow f \left(1 + 4\frac{L_5}{f_{\eta'}^2} 2M_K^2 + M_\pi^2\right) + \frac{3}{4} v_{50},$$ \hspace{1cm} (30)

which is the $y \rightarrow -\infty$ limit of (29), as expected.

Let us compare these results with data. We can take $f_\pi$ and $f_K$ to fix $f$ and $L_5$, ($f_\pi = 92.4$ MeV, $f_K = 1.223 f_\pi$)

$$f = 90.8 \text{MeV}, \quad L_5 = 2.0 \times 10^{-3}.$$ \hspace{1cm} (31)
This is the same value as obtained for $L_5^{[SU]} r$ in [4] for it is extracted from the same source. In our approximation the issue of the running of the $L_5$, $L_8$ cannot be addressed because the loop corrections are dropped altogether. As a matter of principle one expects $L_5 \sim L_5^{[SU]}$ because integrating out the $\eta'$ does not give any contribution to $L_5^{[SU]}$, $L_8^{[SU]}$ at tree level, except for $L_7^{[SU]}$.

We are left with $f_\eta$ and $f_{\eta'}$ to be described by $y$ (again, proportional to $v_0^2$) and $v_5^{0}$. Assuming that convergence criteria fix a rotation angle around $\theta \sim -20^\circ$, this leaves room for a fit of $v_5^{0}$ and a prediction. The values of $\eta$ and $\eta'$ decay constants are, though, poorly determined. Indeed, the experimental values of $f_\eta$ and $f_{\eta'}$ [6] have large errors:

$$0.943 \leq \frac{f_\eta}{f_\pi} \leq 1.091,$$

$$0.912 \leq \frac{f_{\eta'}}{f_\pi} \leq 1.015.$$  

Let us present the output of our formulae as a function of the mixing angle $\theta$. At each given $\theta$, the experimental values of $\frac{f_\eta}{f_\pi}$ allow for a range of values of the parameter $v_5^{0}$ and of $\frac{f_{\eta'}}{f_\pi}$. We again display the results for angles which are close to the zeroth order $\theta_0$.

| $\theta$ | $-v_5^{0}$ min/max | $\frac{f_{\eta'}}{f_\pi}$ min/max |
|----------|---------------------|----------------------------------|
| $-20^\circ$ | 2.293/0.601 | -0.218/0.902 |
| $-22^\circ$ | 1.775/0.365 | 0.171/1.080 |
| $-24^\circ$ | 1.394/0.197 | 0.457/1.205 |

A word of caution should be added as regards the use of the $f_{\eta'}$ value, as made in this paper. As pointed out by Shore and Veneziano [12], from the decay rate of $\eta' \to \gamma \gamma$ one cannot obtain the value of $f_{\eta'}$ because the singlet axial current $A^{(5)}_\mu$ is not conserved in the chiral limit, due to the $U_A(1)$ anomaly, which makes $f_{\eta'}$ from $\langle 0|A^{(5)}_\mu|\eta'(k)\rangle = ik_\mu f_{\eta'}$ depend on a subtraction point and thus not be observable. Of course this dependence is $1/N_c$ suppressed because in the large-$N_c$ limit the anomaly is a subleading effect. The relation between our $f_{\eta'}$ and what is measured in $\eta' \to \gamma \gamma$ and given in [3], [13] can be worked out and amounts to include one further term in the effective action coupled to an external electromagnetic source [7]. This would still bring in a new unknown constant and we would face the problem of having one more constant to fit than measured constants available. Nevertheless, we overcome this shortcoming by using the criterium of minimum sensitivity that minimizes the size of the corrections to a given order and conclude that the combined expansion [4], within the framework of $U_L(3) \otimes U_R(3)$ chiral perturbation theory, is able to accommodate very naturally the observed values of masses and decay constants $f_\pi$, $f_K$.

The big uncertainty in $f_\eta$ makes the method inconclusive for $f_{\eta'}$ (which, strictly speaking, is not experimentally known), as can be seen from the last table. If data from $\eta' \to \gamma \gamma$ were more precise, we could with our method pursue to determine the new constant that is involved in the process, and how well it would accommodate in the framework of the combined expansion.

We finish by quoting the values of all the parameters in one batch, as functions of the mixing angle in the range $20^\circ < \theta < 24^\circ$:

$$0.15 \leq \Delta_M \leq 0.26,$$  

(33)
\[0.980 \leq \frac{2m_B}{M^2} \leq 0.988,\]
\[18.3 \leq x \leq 20.9,\]
\[-4.7 \leq y \leq -4.2,\]
\[26 \leq -v_{02}/f^2 \leq 29,\]
\[0.14 \leq \Delta_N \leq 0.22,\]
\[1.35 \times 10^{-3} \leq L_S \leq 1.57 \times 10^{-3},\]
\[-0.164 \leq v_{31} \leq -0.161.\]

We should like to add that after this work was finished there appeared the document of ref. \[14\] in which the author reports some results, as yet unpublished, on the same issue that concerns our study.

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