ODDS-ON TREES

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Abstract. Let \( \mathcal{P} : \mathbb{R}^d \rightarrow A \) be a query problem over \( \mathbb{R}^d \) for which there exists a data structure \( S \) that can compute \( \mathcal{P}(q) \) in \( O(\log n) \) time for any query point \( q \in \mathbb{R}^d \). Let \( D \) be a probability measure over \( \mathbb{R}^d \) representing a distribution of queries. We describe a data structure \( T = T_{\mathcal{P}, D} \), called the odds-on tree, of size \( O(n^\epsilon) \) that can be used as a filter that quickly computes \( \mathcal{P}(q) \) for some query values in \( \mathbb{R}^d \) and relies on \( S \) for the remaining queries. With an odds-on tree, the expected query time for a point drawn according to \( D \) is \( O(H^* + 1) \), where \( H^* \) is a lower-bound on the expected cost of any linear decision tree that solves \( \mathcal{P} \).

Odds-on trees have a number of applications, including distribution-sensitive data structures for point location in 2-d, point-in-polytope testing in \( d \) dimensions, ray shooting in simple polygons, ray shooting in polytopes, nearest-neighbour queries in \( \mathbb{R}^d \), point-location in arrangements of hyperplanes in \( \mathbb{R}^d \), and many other geometric searching problems that can be solved in the linear-decision tree model. A standard lifting technique extends these results to algebraic decision trees of constant degree. A slightly different version of odds-on trees yields similar results for orthogonal searching problems that can be solved in the comparison tree model.

1 Introduction

Geometric search problems have a long and rich history \cite{2, 26, 36}. In these problems, we are typically given a collection \( S \) of \( n \) geometric objects in \( \mathbb{R}^d \) and asked to store them in a data structure so that we can efficiently answer queries about these objects. Until quite recently, the performance of these data structures was typically measured, at least theoretically, in terms of the worst-case (over all possible choices of \( S \) and \( q \)) query time as a function of \( n \). A worst-case query time of \( O(\log n) \) is typically viewed as the gold standard for such search problems, at least in models of computation that only allow binary decisions.

Somewhat more recently, researchers have begun studying geometric search problems under the lens of distribution-sensitivity. In this setting, one assumes that there is a probability measure \( D \) over the set of possible queries, and one attempts to optimize the expected query time when queries are distributed according to \( D \). For example, given a planar triangulation \( G \) and a distribution \( D \) over \( \mathbb{R}^2 \), one can construct an \( O(n) \) sized data structure that can determine the face of \( G \) that contains any query point \( q \in \mathbb{R}^2 \). The
expected query time of this structure is $O(H + 1)$, where

$$H = \sum_i p_i \log(1/p_i)$$

(1)

and $p_i$ denotes the probability that $q$ is contained in the $i$th face of $G$.\cite{1, 5, 6, 7, 8, 28, 29} Information theory tells us that this result is optimal in any dichotomous model of computation where any execution branching has at most 2 possible outcomes.

More recently, Collette et al.\cite{17, 18} have shown that a similar result holds for point location in any simple connected planar subdivision $G$. Note that this involves more than simply triangulating $G$ and applying the results for triangulations. Triangulating $G$ increases the number of faces and therefore also increases the value of $H$ in (1). On the other hand, we can not expect to always achieve a query time of $O(H + 1)$ since this would imply, for example, an $O(1)$ query-time data structure for testing if a point is contained in a simple polygon. The result of Collette et al. is a data structure whose expected query time is $O(H^* + 1)$ where $H^*$ is a lower-bound on the expected cost of any linear decision tree for point location in $G$ with queries distributed according to $D$.

Dujmović et al.\cite{23} study distribution-sensitive 2-sided 2-dimensional orthogonal range counting. They describe a data structure, called a biased range tree, that preprocesses an $n$ point set $S \subset \mathbb{R}^2$ and a query distribution $D$ over $\mathbb{R}^2$. A biased range tree uses $O(n \log n)$ preprocessing time and space and can answer 2-sided 2-dimensional orthogonal range counting queries over $S$ in expected time $O(H^* + 1)$, where $H^*$ is a lower-bound on the expected cost of any comparison tree for 2-sided range queries over $S$ with queries distributed according to $D$.

The results of Collette et al.\cite{17, 18} and Dujmović et al.\cite{23} use similar techniques to prove their optimality, but each requires their own ad hoc arguments. For example, the point location results are achieved by finding near-minimum-entropy Steiner triangulations and then applying existing distribution-sensitive results for point location in triangulations. Biased range trees on the other hand, mix $k$-d trees, fractional cascading, and biased binary search trees to achieve their running time.

In the current paper, we describe a general and low-overhead method of taking any $O(\log n)$ query-time data structure and making it distribution-sensitive. Let $\mathcal{P} : \mathbb{R}^d \rightarrow A$ be a query problem over $\mathbb{R}^d$ for which there exists a data structure $S$ that can compute $\mathcal{P}(q)$ in $O(\log n)$ time for any query point $q \in \mathbb{R}^d$. Let $D$ be a probability measure over $\mathbb{R}^d$ representing a distribution of queries. We describe a data structure $T = T_{\mathcal{P}, D}$, called the odds-on tree\footnote{There is no natural definition of $n$ for the abstract problem $\mathcal{P}$. Nevertheless, all the problems we will eventually consider arise from a set of $n$ objects in $\mathbb{R}^d$ and, for all these problems $O(\log n)$ is the optimal worst-case query time in the models of computation we will consider.} of size $O(n^*)$ that can be used as a filter that quickly computes $\mathcal{P}(q)$ for some query values in $\mathbb{R}^d$ and relies on $S$ for the remaining queries. With an odds-on tree, the expected query time for a point drawn according to $D$ is $O(H^* + 1)$, where $H^*$ is a lower-bound on the expected cost of any linear decision tree that solves $\mathcal{P}$.

For any constant integer $p \geq 1$, a standard lifting technique allows us to lift queries from $\mathbb{R}^d$ into $\mathbb{R}^{d'}$, with $d' = (p+d)/p - 1$ so that there is a correspondence between $d'$-variate

\footnote{odds-on · adj. having a better than even chance of success “the odds-on favourite”}
linear inequalities (halfspaces in $\mathbb{R}^d$) and $d$-variate polynomial inequalities of maximum degree $p$ \[44\]. This technique yields the same result, except that $H^*$ becomes a lower-bound on the expected cost of any degree $p$ algebraic decision tree that solves $P$.

Odds-on trees have a plethora of applications, including distribution-sensitive data structures for point location in 2-d, point-in-polytope testing in $d$ dimensions, ray shooting in simple polygons, ray shooting in polytopes, nearest-neighbour queries in $\mathbb{R}^d$, point-location in arrangements of hyperplanes in $\mathbb{R}^d$, and many other geometric searching problems that can be solved in the linear-decision tree model. Furthermore, a variant of the odds-on tree that works in the comparison tree model provides distribution-sensitive data structures for orthogonal searching problems in the comparison tree model.

The remainder of this paper is organized as follows: Section 2 presents some preliminary definitions and background material. Section 3 presents the odds-on tree and algorithms for constructing it. Section 4 proves that the odds-on tree matches the query time of any linear decision tree. Section 5 presents some of the geometric applications of this data structure. Finally, Section 6 summarizes and concludes with directions for future work.

2 Preliminaries

Throughout this paper, the underlying dimension, $d$, is a constant, and other constants defined in the paper may (implicitly) depend on $d$. A simplex in $\mathbb{R}^d$ is the common intersection of a set of at most $d + 1$ closed halfspaces in $\mathbb{R}^d$. Note that, under this definition, simplices need not be bounded and $\mathbb{R}^d$, as well as $\emptyset$, are both simplices.

Throughout this paper, we assume an underlying probability measure $D$ over $\mathbb{R}^d$. All expectations and probabilities are (implicitly) with respect to $D$. For any subset $X \subseteq \mathbb{R}^d$, $\text{Pr}(X)$ refers to $D(X)$. We use the notation $D|_{X}$ to denote the distribution $D$ conditioned on $X$, i.e., $D|_{X}(Y) = \text{Pr}(Y \mid X) = \text{Pr}(X \cap Y) / \text{Pr}(X)$ for all $Y \subseteq \mathbb{R}^d$. If $F = \{X_1, \ldots, X_k\}$ are non-overlapping subsets of $\mathbb{R}^d$ then the entropy of $F$, denoted $H(F)$ is

$$H(F) = \sum_{i=1}^{k} \text{Pr}(X_i \mid \cup F) \log \left( \frac{1}{\text{Pr}(X_i \mid \cup F)} \right),$$

where for any set $S$, $\cup S$ denotes $\bigcup_{s \in S} s$. The probability measure $D$ is used as an input to our algorithms. We assume that the algorithm has access to $D$ through a Sampling Oracle that allows us to draw a random sample $q \in \mathbb{R}^d$ from the distribution $D$.

A query problem over $\mathbb{R}^d$ is a function $P : \mathbb{R}^d \rightarrow A$ where $A$ is some set of answers. We assume the existence of two oracles that allow access to $P$. The Backup Oracle allows us to compute $P(q)$ for any query point $q$, but requires $O(\log n)$ time to do so. The Interference Oracle allows us to test, for any simplex $\Delta$, if there exists $p, q \in \Delta$ with $P(p) \neq P(q)$. The running time of the Interference Oracle will be unspecified.

The oracles are used in the following ways: The Backup Oracle is used to answer any queries that can not be answered directly by the odds-on tree. The Sampling Oracle and the Interference Oracle are used only during the construction of the odds-on tree. The
running times in theorems in Sections 3 and 4 all specify the number of invocations of the Sampling and Interference Oracle used. When discussing applications in Section 5 efficient implementations of the Interference Oracle for specific problems will be described.

A decision tree for \( P \) is a rooted ordered binary tree in which each internal node \( v \) is labelled with a function \( v_f : \mathbb{R}^d \to \{0, 1\} \) and each leaf \( w \) is labelled with an element \( w_A \in A \). A query point \( q \in \mathbb{R}^d \) follows a root-to-leaf path, proceeding to the right child of \( v \) if \( v_f(q) = 1 \) and the left child of \( v \) if \( v_f(q) = 0 \). For a decision tree \( T \) and a point \( q \in \mathbb{R}^d \), we denote by \( T(q) \) the label of the leaf on the root-to-leaf path for \( p \) in \( T \). A decision tree solves \( P \) if \( T(q) = P(q) \) for all \( q \in \mathbb{R}^d \). The depth of a node \( v \) in a rooted tree \( T \), denoted \( \text{depth}(v) \), is the number of edges on the path from \( v \) to the root of \( T \). The (expected) cost of a decision tree, denoted \( \mu_D(T) \), is the expected depth of the leaf reached when \( p \) is drawn according to the probability measure \( D \).

Decision trees are classified based on the types of functions, \( v_f \), used at their nodes. In a linear decision tree, each \( v_f \) is a linear inequality. In a degree \( p \) algebraic decision tree, each \( v_f \) is a \( d \)-variate polynomial inequality of degree \( p \). In a comparison tree, \( v_f \) is a simple comparison that compares one coordinate of a query point \( q \) to some value.

Entropy, query problems, and decision trees are all related by (half of) Shannon’s Source Coding Theorem [42]:

**Theorem 1** (Shannon 1948). Let \( P : \mathbb{R}^d \to A \) be a query problem and define \( P^{-1}(a) = \{ q \in \mathbb{R}^d : P(q) = a \} \), for any \( a \in A \). Then, for any decision tree \( T \) that solves \( P \),

\[
\mu_D(T) \geq H(\{ P^{-1}(a) : a \in A \}) .
\]

### 3 The Data Structure

In this section we describe a data structure \( T_{P,D} \) called the odds-on tree that, in conjunction with a Backup Oracle, yields a data structure that solves \( P \) and has expected query time that is within a constant factor of the expected cost of any linear decision tree \( T^* \) for \( P \).

For a reader familiar with Matoušek’s efficient partition trees [33], the executive summary of this section is as follows: An odds-on tree is essentially a partition tree on a sample of \( O(n^4) \) points drawn according to the distribution \( D \). A complete description follows.

**Theorem 2** (Matoušek 1992). There exists a constant \( c \) such that, for any set \( S \) of \( m \) points in \( \mathbb{R}^d \) and any constant \( r \leq m \), there exists a sequence \( \{ \Delta_1, \ldots, \Delta_r \} \) of closed simplices such that \( \bigcup_{i=1}^r \Delta_i = \mathbb{R}^d \),

1. \( |\Delta_i^s \cap S| \leq 2m/r \), where \( \Delta_i^s = \Delta_i \setminus \left( \bigcup_{j=1}^{i-1} \Delta_j \right) \), and

2. For any hyperplane \( \ell \), there are at most \( cr^{1-1/d} \) elements of \( \{ \Delta_1, \ldots, \Delta_r \} \) whose interiors intersect \( \ell \).

The sequence of simplices \( \Delta_1, \ldots, \Delta_r \) can be computed in \( O(m) \) time.
Note that Condition 1 of Theorem 3 is not in the original statement of the theorem, but follows from Matoušek’s incremental construction of $\Delta_1, \ldots, \Delta_r$ [33].

Let $S$ be a set of $m$ points in $\mathbb{R}^d$. Then the partition tree $T_S$ for $S$ is a rooted ordered tree obtained by recursively applying Theorem 2. The root of $T_S$ has $r$ children corresponding to the simplices $\Delta_1, \ldots, \Delta_r$ obtained by applying Theorem 2 to $S$. The $i$th child of the root is itself the root of the partition tree $T_{S \cap \Delta_i}$ for $S \cap \Delta_i$. This recursive process stops when the set $S$ contains at most 1 point or when the depth exceeds some pre-specified maximum depth $k$.

Next we define some regions, $\Delta(v)$, $\mathcal{O}(v)$, and $\Xi(v)$, that are associated with each node $v$ of $T_S$. Every node $v$ in $T_S$, except the root of $T_S$, is naturally associated with a simplex $\Delta(v)$ that was obtained from Theorem 2 and that generated $v$. For the root of $T_S$, we define $\Delta(v) = \mathbb{R}^d$.

For a node $v$ of $T$ whose ancestors are $v_1, \ldots, v_i$ we define $\mathcal{O}(v) = \Delta(v) \cap \bigcap_{i=1}^i \Delta(v_i)$. Note that $\mathcal{O}(v) \subseteq \Delta(v)$, and that $\mathcal{O}(v)$ is a convex polytope that has $O((d+1)(i+1)^{d/2})$ vertices since it is the intersection of at most $(d+1)(i+1)$ halfspaces [34].

For a point $q \in \mathbb{R}^d$, the search path for $q$ in $T_S$ starts at the root and proceeds to the first child $i$ such that $q \in \Delta_i$ (note that this implies $q \in \Delta_i^*$) and this process is applied recursively until reaching a leaf of $T_S$. In this way, for every node $v$ of the partition tree there is a maximal subset $\Xi(v) \subseteq \mathbb{R}^d$ such that the search path for every point $q \in \Xi(v)$ contains $v$. Note that $\Xi(v) \subseteq \mathcal{O}(v) \subseteq \Delta(v)$, but that $\Xi(v)$ is not necessarily convex or even connected.

We extend the definitions of $\Xi$, $\mathcal{O}$, and $\Delta$ to sets of nodes in $T$ in the natural way: if $V$ is a set of nodes in $T$, then $\Xi(V) = \{\Xi(v) : v \in V\}$, $\mathcal{O}(V) = \{\mathcal{O}(v) : v \in V\}$, and $\Delta(V) = \{\Delta(v) : v \in V\}$.

The following theorem summarizes the properties of the partition tree $T_S$ [33] (each property is inherited from the corresponding property of the simplicial partition in Theorem 2):

**Theorem 3.** Let $S$ be a set of $m$ points in $\mathbb{R}^d$, let $T_S$ denote the partition tree described above, and let $V_i$ denote the set of at most $r^i$ nodes of $T_S$ at depth $i$. There exists a constant $c$, independent of $r$ and $m$, such that the partition tree $T_S$ has the following properties, for every $i \in \mathbb{Z}$:

1. For every node $v \in V_i$, $|S \cap \Xi(v)| \leq m(2/r)^i$, and

2. For any hyperplane $\ell$, the number of elements in $\mathcal{O}(V_i)$ whose interiors intersect $\ell$ is at most $(cr^{i-1/d})^i$.

The partition tree $T_S$ can be constructed from $S$ in $O(m \log m)$ time.

The following is a sampling version of Theorem 3 that we will use in the construction of an odds-on tree:

**Theorem 4.** Let $S$ be a sample of $m$ points in $\mathbb{R}^d$ i.i.d. according to $D$, let $T_D = T_S$ denote the partition tree given by Theorem 3, and let $V_i$ denote the set of at most $r^i$ nodes of $T_D$ at
There exists a constant $c$, independent of $r$, $m$ and $D$, such that with probability at least $1 - O(e^{-m^{1/2}})$, $T_D$ has the following properties, for every $i \in \{0, \ldots, \lfloor (1/4) \log m \rfloor \}$:

1. For every node $v \in V_i$, $\Pr(\Xi(v)) \leq (3/r)^i$, and

2. For any hyperplane $\ell$, the number of elements in $Q(V_i)$ whose interiors intersect $\ell$ is at most $(cr^{-1-1/d})^i$.

The sample partition tree $T_D$ can be constructed in $O(m \log m)$ time plus the cost of $O(m)$ calls to the Sampling Oracle.

**Proof.** Condition 2 follows with certainty from Theorem 3. In this proof we bound the probability of failure for Condition 1. Let $k = \lfloor (1/4) \log m \rfloor$.

We will use $D_m(A)$ to denote the empirical measure of a set $A \subseteq \mathbb{R}^d$:

$$D_m(A) \overset{\text{def}}{=} \frac{|S \cap A|}{m} .$$

From Theorem 3 we have

$$\sup_{v \in V_i} D_m(\Xi(v)) \leq (2/r)^i .$$

Now,

$$\Pr \left( \sup_{v \in V_i} D(\Xi(v)) > \left( \frac{3}{r} \right)^i \right) = \Pr \left( \bigcup_{v \in V_i} \left[ D(\Xi(v)) - D_m(\Xi(v)) > \left( \frac{3}{r} \right)^i - D_m(\Xi(v)) \right] \right) \leq \Pr \left( \bigcup_{v \in V_i} \left[ D(\Xi(v)) - D_m(\Xi(v)) > \left( \frac{3}{r} \right)^i - \left( \frac{2}{r} \right)^i \right] \right) \leq \Pr \left( \sup_{v \in V_i} (D(\Xi(v)) - D_m(\Xi(v))) > r^{-i} \right) \leq \Pr \left( \sup_{A \in A} (D(A) - D_m(A)) > r^{-i} \right)$$

where $A$ are sets formed by taking the intersection of $k$ closed simplices and subtracting $k(r-1)$ (possibly empty) simplices from this intersection. This is because $\Xi(V) \subseteq A$. We can actually handle all levels $i \in \{0, \ldots, k\}$ of the tree at once with the inequality

$$\Pr \left( \bigcup_{i \in \{1, \ldots, k\}} \left[ \sup_{v \in V_i} D(\Xi(v)) > \left( \frac{3}{r} \right)^i \right] \right) \leq \Pr \left( \sup_{A \in A} (D(A) - D_m(A)) > r^{-k} \right) .$$

The class $A$ for $k = 1$ and $r = 1$ is the class of all simplices in $\mathbb{R}^d$. Since a simplex in $\mathbb{R}^d$ is the common intersection of $d+1$ halfspaces in $\mathbb{R}^d$, it helps to first consider halfspaces. The set of halfspaces in $\mathbb{R}^d$ has Vapnik-Chervonenkis dimension $d+1$ (this follows from Radon’s Theorem [24]). By Sauer’s lemma [10, 20, pages 28–29], the number of subsets of an $m$-point set that can be obtained by intersections with halfspaces does not exceed $(m+1)^{d+1}$. A simple combinatorial argument then implies that the number of subsets of an $m$-point set that can be obtained by intersections with simplices does not exceed $(m+1)^{(d+1)^2}$. 


Assume now general $r$ and $k$. Then the number of subsets of an $m$-point set that can be obtained by intersections with sets from $\mathcal{A}$ does not exceed $(m+1)^{rk(d+1)^2}$, by the same combinatorial argument. By a version of the Vapnik-Chervonenkis inequality [43] shown by Devroye [10],

$$\Pr\left(\sup_{A \in \mathcal{A}} |D(A) - D_m(A)| \geq t \right) \leq 4 \left(m^2 + 1\right)^{rk(d+1)^2} \exp\left(4t + 4t^2 - 2mt^2\right),$$

for all $t > 0$. Thus,

$$\Pr\left(\bigcup_{i \in \{1, \ldots, k\}} \left[\sup_{v \in V_i} D(\Xi(v)) > \left(\frac{3}{r}\right)^i\right]\right) \leq 4 \left(m^2 + 1\right)^{rk(d+1)^2} \exp\left(4r^{-k} + 4r^{-2k} - 2mr^{-2k}\right)$$

$$\leq \exp\left(2 + 4r^{-k} + 4r^{-2k} + rk(d + 1)^2 \ln(m^2 + 1) - 2mr^{-2k}\right)$$

$$\leq \exp\left(10 + r(d + 1)^2 \log_r(m) \ln(m^2 + 1) - 2m^{1/2}\right),$$

$$= O\left(e^{-m^{1/2}}\right)$$

since $k = (1/4) \log_r m$. \hfill \qed

Note that, so far, we have not considered the query problem $\mathcal{P}$ at all; the sample partition tree $T_D$ of Theorem 4 is defined completely in terms of the probability measure $D$. The odds-on tree $T_{\mathcal{P}, D}$ for $(\mathcal{P}, D)$ is obtained in the following way: We start by constructing a sample partition tree $T_D$ as described in Theorem 4 using the value $m = n^\tau$ for some parameter $\tau > 0$ and setting the maximum depth to $k = \lfloor(1/4) \log_r m\rfloor$.

Next, we trim some nodes of $T_{\mathcal{P}, D}$. We use the Interference Oracle to test, for each node $v$ of $T_D$, if $\mathcal{P}(p) = \mathcal{P}(q)$ for all pairs of points $p, q \in \mathcal{O}(v)$. If so, we remove all the subtrees rooted at the children of $v$, we call $v$ a terminal leaf, and we label $v$ with the label $\ell(v) = \mathcal{P}(q)$. Note that the Interference Oracle works for simplices, but $\mathcal{O}(v)$ is a polytope. Thus, this test requires decomposing $\mathcal{O}(v)$ into simplices and using the Interference Oracle on each simplex. Using the bottom vertex triangulation [15] for this decomposition allows us to decompose $\mathcal{O}(v)$ into $O((\text{depth}_T(v))^{d/2}) = O(\log^{O(1)} n)$ simplices in $O(\log^{O(1)} n)$ time. This yields the following lemma:

**Lemma 1.** For any $n > 0$ and any $\tau > 0$, an odds-on tree of size $m = n^\tau$, with maximum depth $k = \lfloor(1/4) \log_r m\rfloor$, and having the properties of Theorem 4 can be constructed in $O(n^\tau \log^{O(1)} n)$ time plus the cost of $O(n^\tau)$ calls to the Sampling Oracle and $O(n^\tau \log^{O(1)} n)$ calls to the Interference Oracle.

Using an odds-on tree to answer a query $q \in \mathbb{R}^d$, is easy: We follow the search path for $q$ in $T_{\mathcal{P}, D}$ until we either reach a terminal leaf $v$, in which case we output $\ell(v)$, or we reach a non-terminal leaf $w$ after $O(\log n)$ steps, in which case we rely on the Backup Oracle to report $\mathcal{P}(v)$ in $O(\log n)$ time. The correctness of this procedure follows immediately from the definition of terminal and non-terminal nodes. In the next section, we analyze the performance of odds-on trees.
4 Analysis

In this section, our goal is to lower-bound the expected cost of any linear decision tree that solves $\mathcal{P}$ in terms of the odds-on tree $T_{\mathcal{P},D}$. We accomplish this by decomposing the nodes of $T_{\mathcal{P},D}$ into subsets with some helpful combinatorial properties.

An $i$-set of a rooted tree $T$ is a set of vertices in $T$ all of which are at depth at most $i$ and in which no vertex in the set is the ancestor of any other vertex in the set. We say that a set of regions $F = \{X_1, \ldots, X_t\}$, $X_i \subseteq \mathbb{R}^d$, is in $k$-general position if there is no hyperplane that intersects $k$ or more elements of $F$.

**Lemma 2.** Let $T_{\mathcal{P},D}$ be the odds-on tree defined in Section 3, let $V$ be an $i$-set of $T_{\mathcal{P},D}$, and let $k > 1$ be a constant. Then $V$ contains a subset $V' \subseteq V$ such that the elements of $\mathcal{O}(V')$ are pairwise disjoint and in $k$-general position and $|V'| = \Omega(|V|/r^{i(d/k+1-1/d+\delta)})$, where $\delta > 0$ is a decreasing function of $r$.

**Proof.** We will first use the probabilistic method [3] to establish the existence of a (not necessarily disjoint) set $V''$ satisfying the size and $k$-general position requirements and then show that $V''$ contains a large subset $V'$ whose elements are also pairwise disjoint.

Let $V''$ be a Bernoulli sample of $V$ where each element is selected independently with probability $p = r^{-i(d/k+1-1/d+\delta)}$. We will prove that

$$\Pr\{\mathcal{O}(V'') \text{ is in } k\text{-general position and } |V''| = \Omega(p|V|)\} > 0.$$  

Consider any hyperplane $\ell$. Condition 2 of Theorem 4 implies that $\ell$ intersects the interior of at most $(cr^{1-1/d})^i$ elements of $V$ for some constant $c$. The probability that $\ell$ intersects the interior of $k$ or more elements of $V''$ is therefore no more than

$$\left(\binom{(cr)^{1-1/d}}{k}\right) \cdot p^k \leq (cr)^{i(k/d)}p^k.$$  

For each node $v \in V$, $\mathcal{O}(v)$ has $O(i^{d/2})$ vertices, and the number of nodes in $V$ is at most $r^i$. Therefore, the elements of $\mathcal{O}(V)$ define a test set $L$ of $O((i^{d/2})r^i) = O(i^{d/2}r^{di})$ hyperplanes such that $\mathcal{O}(V'')$ is in $k$-general position if and only if no hyperplane in $L$ intersects $k$ or more elements of $\mathcal{O}(V'')$. The probability that any hyperplane in $L$ intersects $k$ or more elements of $\mathcal{O}(V'')$ is therefore at most

$$O(i^{d/2}r^{di}) \cdot (cr)^{i(k/d)}p^k = O(r^{i(d+k/d+\delta)})p^k = O(r^d-k\delta) = o(1)$$

for any $\delta > (k - k/d)\log r$. The above argument shows that the nodes in $V''$ are quite likely to be in $k$-general position. To see that $V''$ is sufficiently large, we simply observe that $|V''|$ is a binomial($|V|$, $p$) random variable and therefore has median value at least $|p|V| = \Omega(|V|/r^{i(d/k+1-1/d+\delta)})$. Therefore,

$$\Pr\{\mathcal{O}(V'') \text{ is in } k\text{-general position and } |V''| = \Omega(|V|/r^{i(d/k+1-1/d+\delta)})\} \geq 1 - o(1)+1/2 > 0.$$  

This establishes the existence of a sufficiently large set $V''$ such that $\mathcal{O}(V'')$ is in $k$-general position.
Finally, we select $V' \subseteq V''$ so that the elements of $\mathcal{O}(V')$ are pairwise disjoint. To do this, imagine sweeping a hyperplane $\ell(t) = \{(x_0, \ldots, x_d) \in \mathbb{R}^d : x_0 = t\}$ from $t = -\infty$ to $+\infty$. Associate with each element $v \in V'$, the maximal interval $[a_v, b_v]$ such that $\mathcal{O}(v)$ intersects $\ell(t)$ for all $t \in [a_v, b_v]$. This yields a set $S_{V''}$ of real intervals such that no point is contained in $k$ or more elements of $S_{V''}$. Dilworth’s Theorem \cite{Dilworth50} implies that $S_{V''}$ contains a subset of size at least $|V''|/k$ of non-overlapping intervals. This subset corresponds to a subset $V' \subseteq V''$ with $|V'| \geq |V''|/k$ and such that the elements of $\mathcal{O}(V')$ are pairwise disjoint. The set $V'$ satisfies all the conditions of the lemma. \hfill \Box

We are now ready to show that the expected search time in the odds-on tree is a lower bound on the expected cost of any linear decision tree that solves $P$.

**Lemma 3.** Let $T_{P,D}$ be the odds-on tree defined in Section 3 and assume $T_{P,D}$ satisfies Conditions 1 and 2 of Theorem 4. Let $L$ denote the set of leaves of $T_{P,D}$, and let $T^*$ be any linear decision tree that solves $P$. Then

$$\mu_D(T^*) = \Omega(H(\Xi(L)) - 1) .$$

**Proof.** This proof mixes the ideas from the proofs of Lemma 3 by Dujmović et al. \cite{Dujmovic04} and Lemma 4 by Collette et al. \cite{Collette17}. For an $i$-set $V$ of nodes in $T_{P,D}$, we define the shorthands $Pr(V) = Pr(\Xi(V))$ and $H(V) = H(\Xi(V))$. Let $T'$ be the tree obtained from $T_{P,D}$ by removing all terminal leaves, and let $L'$ denote the set of leaves of $T'$. Note that

$$H(L') = H(L) - O(\log r) = H(L) - O(1)$$

since each leaf in $L'$ has at most $r = O(1)$ children in $L$.

We first partition $L'$ into groups $G_1, G_2, \ldots$, where $G_i$ contains all leaves $v$ such that $1/2^{i-1} \geq Pr(\Xi(v)) \geq 1/2^i$. Note that Condition 1 of Theorem 4 implies that the depth of a node in $G_i$ is at most $i/\log(r/3)$.

We then further partition each group $G_i$ into subgroups $G_{i,1}, \ldots, G_{i,t_i}$. For each $j \in \{1, \ldots, G_{i,t_i-1}\}$, $|G_{i,j}| = \Omega(2^{\alpha i})$, for some constant $\alpha > 0$ and the elements of $\mathcal{O}(G_{i,j})$ are pairwise disjoint and in $k$-general position. Furthermore, the final subgroup, $G_{i,t_i}$, has size at most $O(2^{\beta i})$, for some constant $\beta < 1$.

This partitioning is accomplished by repeatedly applying Lemma 2 to remove a subset $G_{i,j} \subseteq G_i$ that is in $k$-general position and has size $\Omega(2^{\alpha i})$, stopping the process once the size of $G_i$ drops below $O(2^{\beta i})$. This works provided that we choose $\beta$, $k$, and $r$ so that $\beta > ((\log r)/(\log(r/3)))((d/k+1-1/d+\delta))$ and set $\alpha \leq \beta - ((\log r)/(\log(r/3)))((d/k+1-1/d+\delta))$. For example, by choosing $r$ and $k$ to be sufficiently large, $\beta = 1 - 1/(3d)$ and $\alpha = 1/(3d)$.

Consider the linear decision tree $T^*$ that solves $P$. The leaves of $T^*$ partition $\mathbb{R}^d$ into cells whose closures are convex polytopes. For a leaf $w$ of $T^*$ we denote its polytope by $\mathcal{O}(w)$. If the depth of $w$ is $i$, then $\mathcal{O}(w)$ is the intersection of at most $i$ halfspaces. This implies that $\mathcal{O}(w)$ intersects at most $ik$ elements of $\mathcal{O}(G_{i,j})$ since, otherwise, $\mathcal{O}(w)$ contains $\mathcal{O}(v)$ for some non-terminal node $v \in G_{i,j}$. (This would contradict the assumption that $T^*$ solves $P$.)
Let \( w \) be some leaf of \( T^* \) such that \( \mathcal{O}(w) \) intersects \( t \) elements of \( \mathcal{O}(G_{i,j}) \). Then, by the discussion in the previous paragraph, depth_{\mathcal{O}}(w) \geq \lceil t/k \rceil. We can easily create a subtree \( T^*_w \) of \( w \) whose height is at most \( t \) and with the property that \( \mathcal{O}(w') \) intersects at most one element of \( \mathcal{O}(G_{i,j}) \) for each leaf \( w' \) of \( T^*_w \). If we do this for every leaf \( w \) of \( T^* \) we obtain a tree \( T^*_{i,j} \) such that every leaf of \( T^*_{i,j} \) intersects at most one element of \( \mathcal{O}(G_{i,j}) \).

Let \( D_{i,j} = D_{|\cup \mathcal{E}(G_{i,j})} \) denote the distribution \( D \) conditioned on \( \cup \mathcal{E}(G_{i,j}) \). Note that the leaves of \( T^*_w \) could be relabeled so that they indicate which element of \( \mathcal{O}(G_{i,j}) \) (if any) that they intersect. By Shannon’s Theorem (Theorem 13), this implies that

\[
(k + 1) \cdot \mu_{D_{i,j}}(T^*) \geq \mu_{D_{i,j}}(T^*_{i,j}) \geq H(G_{i,j}) \, .
\]

It follows [13, Lemma 3] that

\[
(k + 1) \cdot \mu_D(T^*) \geq H(L') - H(\{\cup \mathcal{E}(G_{i,j}) : i \in \mathbb{N}, j \in \{1, \ldots, t_i\}\}) - O(1) \, .
\]

Thus, all that remains is to upper-bound the contribution of \( \bar{H} = H(\{\cup \mathcal{E}(G_{i,j}) : i \in \mathbb{N}, j \in \{1, \ldots, t_i\}\}) \), as follows:

\[
\begin{align*}
\bar{H} & = H(\{\cup \mathcal{E}(G_{i,j}) : i \in \mathbb{N}, j \in \{1, \ldots, t_i\}\}) \\
& = \sum_{i=1}^{\infty} \sum_{j=1}^{t_i} \Pr(G_{i,j}) \log(1/\Pr(G_{i,j})) \\
& = \sum_{i=1}^{\infty} \sum_{j=1}^{t_i-1} \Pr(G_{i,j}) \log(1/\Pr(G_{i,j}) + \Pr(G_{i,t_i}) \log(1/\Pr(G_{i,t_i}))) \\
& \leq \sum_{i=1}^{\infty} \sum_{j=1}^{t_i-1} \Pr(G_{i,j}) \log(2^i - 2^j + 1) + 2^{2i-2} + O(1) \\
& \leq (1 - \alpha)H(L') + O(1) \, .
\end{align*}
\]

Thus, we have

\[
(k + 1) \mu_D(T^*) \geq H(L') - \bar{H} - O(1) \\
\geq \alpha H(L') - O(1) \\
\geq \alpha H(L) - O(1) \\
= \Omega(H(L) - 1) \, ,
\]

so \( \mu_D(T^*) = \Omega(H(L) - 1) \), as required. \( \square \)

**Remark:** By more carefully handling the constants in the proof of Lemma 3 and allowing \( k \) and \( r \) to be arbitrarily large, one can obtain the tighter lower-bound

\[
\mu_D(T^*) + \log(\mu_D(T^*)) \geq \alpha H - O(\log(kr))
\]

where \( \alpha \) can be made arbitrarily close to \( 1/d \) by increasing \( k \) and \( r \).

---

*For example, making the root of \( T_w^* \) correspond to the bisector of two polyhedra of \( \mathcal{O}(G_{i,j}) \) that intersect \( \mathcal{O}(w) \) means that each of the children of the root intersect at most \( t - 1 \) elements of \( \mathcal{O}(G_{i,j}) \). Applying this recursively yields a subtree of height at most \( t - 1 \). A slightly more involved argument can produce a tree \( T_w^* \) of depth \( O(\log t) \).*
Theorem 5. Let $\mathcal{P} : \mathbb{R}^d \rightarrow A$ be a decision problem for which we have a $(O(\log n)$ time) Backup Oracle and an Interference Oracle, and let $D$ be any probability measure over $\mathbb{R}^d$ for which we have a Sampling Oracle. Then, for any constant $\epsilon > 0$, an odds-on tree of size $O(n^\epsilon)$ can be constructed in $O(n^\epsilon)$ time plus the cost of $O(n^\epsilon)$ calls to the Sampling Oracle and $O(n^\epsilon)$ calls to the Interference Oracle.

This odds-on tree can, in conjunction with the Backup Oracle, compute $\mathcal{P}(q)$ for any $q \in \mathbb{R}^d$ drawn according to $D$ in $O(H^* + 1)$ expected time, where $H^* \leq \mu_D(T^*)$ for any linear decision tree $T^*$ that solves $\mathcal{P}$.

Proof. Applying Lemma 1 with $\tau < \epsilon$ yields the stated bounds on the construction time and the use of the Sampling and Interference Oracles.

If $T_{\mathcal{P}, D}$ satisfies Conditions 1 and 2 of Theorem 4 then, by Lemma 3, the expected time to answer queries using $T_{\mathcal{P}, D}$ is

$$\sum_{t \in L} \Pr(t) O(\text{depth}_T(t)) = \sum_{t \in L} \Pr(t) O(\log(1/\Pr(t))) = O(H(L)) .$$

Otherwise, the expected time to answer queries using $T_{\mathcal{P}, D}$ is $O(\log n)$. Therefore, the expected time (where the expectation is taken over $D$ and the random sampling used to generate $S$ in Theorem 4) to answer a query using an odds-on tree is

$$(1 - O(e^{-n^{\epsilon/2}})) \cdot O(H(L)) + O(e^{-n^{\epsilon/2}}) \cdot O(\log n) = O(H(L) + 1) .$$

On the other hand, by Lemma 3 the expected cost of any linear decision tree $T^*$ that solves $\mathcal{P}$ is

$$\mu_D(T^*) = \Omega(H(L) - 1) ,$$

which completes the proof. \hfill \square

Using a standard lifting of query points [44], any degree $p$ algebraic decision tree that solves a problem $\mathcal{P} : \mathbb{R}^d \rightarrow A$ can be implemented as a linear decision tree in $\mathbb{R}^{d'}$, with $d' = \binom{p+d}{p} - 1$. Applying this yields the following corollary:

Corollary 1. Let $\mathcal{P} : \mathbb{R}^d \rightarrow A$ be a decision problem for which we have an $(O(\log n)$ time) Backup Oracle and an Interference Oracle, and let $D$ be any probability measure over $\mathbb{R}^d$ for which we have a Sampling Oracle. Then, for any constants $\epsilon > 0$ and $p \geq 1$, using $O(n^\epsilon)$ calls to the Sampling Oracle and $O(n^\epsilon)$ calls to the Interference Oracle, an odds-on tree can be constructed in $O(n^\epsilon)$ time and $O(n^\epsilon)$ space. This odds-on tree can, in conjunction with the backup oracle, answer $\mathcal{P}$-queries drawn according to $D$ in $O(H^* + 1)$ expected time, where $H^* \leq \mu_D(T^*)$ for any degree $p$ algebraic decision tree $T^*$ that solves $\mathcal{P}$.

5 Applications

In this section, we discuss a few of the many potential applications of odds-on trees. In all of our applications, the problem $\mathcal{P}$ is a query problem over some set of $n$ geometric objects
in \( \mathbb{R}^d \). In order to shorten the statements of the theorems in this section, we say that a data structure for a problem \( P \) is distribution-sensitive (in the linear decision tree model) if the expected query time of the data structure is \( O(\mu_D(T^*) + 1) \) for any linear decision tree \( T^* \) that solves \( P \). Before delving into the details of the applications, we first outline some general strategies for implementing the Interference Oracle needed to build an odds-on tree.

An odds-on tree can be made to have size \( O(n^{\epsilon}) \) for any constant \( \epsilon > 0 \). Furthermore, the number of calls to the Interference Oracle made during the construction of an odds-on tree is \( O(n^{\epsilon}) \). In almost all of our applications, the Interference Oracle can be trivially implemented to run in \( O(n) \) time by testing the query simplex \( \Delta \) against each input element. This means that, even with no preprocessing, the contribution of calls to the Interference Oracle to the construction time of an odds-on tree is no more than \( O(n^{1+\epsilon}) \). Some of the subsequent theorems in this paper will use this fact implicitly.

In other cases, the Interference Oracle corresponds to a natural query for which there exists (or we can develop) an \( O(n^{1-\epsilon}) \) query-time \( O(n \log n) \) preprocessing-time data structure. In these cases, the time to construct the odds-on tree becomes \( O(n \log n) \). One particularly common instance of this occurs when Interference Oracle queries can be reduced to \( O(1) \) simplex range counting queries in \( \mathbb{R}^d \) for some constant dimension \( d' \). In this case, Matoušek’s partition trees \( [33] \) yield an Interference Oracle that can be constructed in \( O(n \log n) \) time and that can answer queries in \( O(n^{1-1/d'+\epsilon}) \) time.

### 5.1 Point Location Problems

The **planar point location problem** is to determine which face of a planar straight line graph \( G \) contains a query point \( q \in \mathbb{R}^2 \). A number of authors have considered distribution-sensitive algorithms for this problem \( [4, 5, 6, 7, 8, 17, 28, 29] \) and have mostly solved it. The most general such result is due to Collette et al. \( [17] \) and gives a distribution-sensitive data structure for point location in connected planar subdivisions. This leaves open the case where the graph, \( G \), of the subdivision is not connected. Since there are several \( O(\log n) \) query-time, \( O(n \log n) \) preprocessing-time data structures for planar point location in (possibly disconnected) planar subdivision \( [1, 25, 30, 37, 39] \) we can apply odds-on trees. To obtain a fast preprocessing time, we can use an implementation of the Interference Oracle based on partition trees. For this problem, the Interference Oracle must answer queries of the form: “Does the interior of query triangle \( \Delta \) intersect more than one face of \( G \)?” We can build a data structure for this problem by first removing from \( G \) any edges and vertices not on the boundary of more than 1 face to obtain a graph \( G' \). The interior of \( \Delta \) intersects more than one face of \( G \) if and only if the interior of \( \Delta \) intersects an edge of \( G' \).

If \( \Delta \) intersects an edge of \( G' \) then \( \Delta \) contains a vertex of \( G' \) or some edge of \( \Delta \) intersects some edge of \( G' \). Determining if \( \Delta \) contains a vertex of \( G' \) is the classic 2-dimensional simplex-range counting problem that can be solved in \( O(n^{1/2+\epsilon}) \) time after \( O(n \log n) \) preprocessing using partition trees. To determine if some edge \( uw \) of \( \Delta \) intersects some edge \( xy \) of \( G' \), we observe that \( uw \) and \( xy \) intersect if and only if

\[
L(u, w, y) \land L(x, y, u) \land L(w, u, x) \land L(y, x, w)
\]
or

\[ R(u, w, y) \land R(x, u, y) \land R(w, x, y) \land R(y, u, x) \]

where \( L(a, b, c) \) and \( R(a, b, c) \), are predicates that are true if and only if the sequence of points \( abc \) form a left turn, respectively, a right turn. If we fix \( x \) and \( y \), then each of the above predicates is the sign of a linear function over the variables \( u_1, u_2, w_1, \) and \( w_2 \). Therefore, we can treat each edge of \( G' \) as a point in \( \mathbb{R}^4 \), yielding a point set \( S_{G'} \subset \mathbb{R}^4 \) such that testing if edge \( uw \) intersects some edge of \( G' \) can be reduced to two simplex range counting queries over \( S_{G'} \). Again, partition trees allow us to do this in \( O(n^{3/4 + \epsilon}) \) time after \( O(n \log n) \) preprocessing. Therefore, the Interference Oracle for point location can be implemented to run in \( O(n^{3/4 + \epsilon}) \) time after \( O(n \log n) \) preprocessing. This yields our first theorem:

**Theorem 6.** There exists a distribution-sensitive data structure for the planar point location problem that uses \( O(n \log n) \) preprocessing time and \( O(n) \) space.

The 3-d point in polytope problem is the problem of determining if a query point \( q \in \mathbb{R}^3 \) is contained in a 3-dimensional polytope \( P \). The Dobkin-Kirkpatrick hierarchy [22] gives an \( O(\log n) \) query-time, \( O(n) \) preprocessing-time data structure for this problem. Furthermore, Interference Oracle queries (which involve testing if a tetrahedron intersects the boundary of \( P \)) can also be answered in \( O(\log n) \) time by the Dobkin-Kirkpatrick hierarchy.

**Theorem 7.** There exists a distribution-sensitive data structure for the 3-dimensional point in polytope problem that uses \( O(n) \) preprocessing time and space.

These results can be extended to higher dimensions, though with more space [15]:

**Theorem 8.** There exists a distribution-sensitive data structure for the \( d \)-dimensional point in polytope problem that uses \( O(n^{\lfloor d/2 \rfloor + \epsilon}) \) preprocessing time and space.

The problem of point-location in an arrangement of hyperplanes is the problem of determining which cell in an arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \) contains a query point \( q \in \mathbb{R}^d \). Liu gives an \( O(n^d) \) space and preprocessing-time, \( O(\log n) \) query-time data structure for this problem [31].

**Theorem 9.** There exists a distribution-sensitive data structure for point location in an arrangement of hyperplanes that uses \( O(n^d) \) preprocessing time and space.

### 5.2 Post-Office Queries

For an \( n \) point set \( S \subset \mathbb{R}^d \), the \( d \)-dimensional post-office problem asks for a point of \( p \in S \) that minimizes the Euclidean distance \( \|pq\| \) for a query point \( q \in \mathbb{R}^d \).

The post-office problem can be solved efficiently through the use of point location in Voronoi diagrams. In 2-dimensions, Voronoi diagrams are planar graphs of size \( O(n) \) and can be computed in \( O(n \log n) \) time [38]. Combining this with Theorem 6 gives a distribution-sensitive data structure for the 2-d post-office problem:
Theorem 10. There exists a distribution-sensitive data structure for the 2-dimensional post office problem that uses $O(n \log n)$ preprocessing time and $O(n)$ space.

In $d > 2$ dimensions, the post-office problem can still be solved using Voronoi diagrams, but the space and preprocessing costs are higher [16]:

Theorem 11. There exists a distribution-sensitive data structure for the 2-dimensional post office problem that uses $O(n^{\lceil d/2 \rceil + \epsilon})$ preprocessing time and space.

5.3 Ray Shooting

The ray shooting in a polygon problem asks for the first point on the boundary of a polygon $P$ intersected by a query ray. A query ray can be represented as a pair of points in $\mathbb{R}^2$ (the source and any other point on the ray), so this is a query problem over $\mathbb{R}^3$. Several $O(n)$ preprocessing time $O(\log n)$ query time solutions to this problem exist [12, 27].

Theorem 12. There exists a distribution-sensitive data structure for the ray shooting in a polygon problem that uses $O(n^{1+\epsilon})$ preprocessing time and $O(n)$ space.

The ray shooting in a $d$-dimensional polytope problem asks for the first point on the boundary of a convex polytope $P \subseteq \mathbb{R}^d$ intersected by a query ray. For polytopes in $\mathbb{R}^3$, the Dobkin-Kirkpatrick hierarchy [22] can perform ray shooting in $O(\log n)$ time per query.

Theorem 13. There exists a distribution-sensitive data structure for the ray shooting in a 3-dimensional polytope problem that uses $O(n^{1+\epsilon})$ preprocessing time and $O(n)$ space.

Schwarzkopf [41] gives an $O(n^{\lfloor d/2 \rfloor + \epsilon})$ space, $O(\log n)$ query time solution for the problem of ray shooting in a $d$-dimensional polytope with $d \geq 4$.

Theorem 14. There exists a distribution-sensitive data structure for the ray shooting in a $d$-dimensional polytope problem that uses $O(n^{\lfloor d/2 \rfloor + \epsilon})$ preprocessing time and space.

5.4 Orthogonal Range Counting

The 2-d orthogonal range counting problem is the problem of counting the number of points of a data set $S \subseteq \mathbb{R}^2$ that are contained in a query rectangle $[q_1, q_2] \times [q_3, q_4]$. (Note that this produces a query point $q \in \mathbb{R}^4$.) Bentley’s range trees [10], with fractional cascading [13, 32], yield an $O(n \log n)$ preprocessing time and space, $O(\log n)$ query time data structure for this problem. Interference Oracle queries for 2-d orthogonal range counting require determining if any rectangle represented by a point in a 4-dimensional simplex has some point of $S$ on its boundary. This problem can be decomposed into $O(1)$ simplex range counting queries in $\mathbb{R}^4$ in a manner similar to that used for the point location problem. Thus, an Interference Oracle for this problem can be implemented in $O(n^{3/4 + \epsilon})$ time after $O(n \log n)$ preprocessing.

Theorem 15. There exists a distribution-sensitive data structure for the 2-d orthogonal range counting problem that uses $O(n \log n)$ preprocessing time and $O(n \log n)$ space.
Range trees satisfy the constraints of the comparison tree model of computation, which is considerably weaker than the linear decision tree model. Dujmović et al. [23] give a distribution-sensitive 2-d orthogonal range counting data structure that works in the comparison tree model. However, their technique can only answer 2-sided queries, i.e., queries of the form \([q_1, \infty) \times (q_2, \infty)\).

Filter trees can be made to work in the comparison tree model and handle full (4-sided) 2-d orthogonal range counting queries. The key modification required is the use of \(k\)-d trees in Theorem 7 rather than Matoušek’s partition trees. (This method, in 2 dimensions, is essentially how Dujmović et al. obtain their results.)

In the comparison tree model, an Interference Oracle query is a \(d\)-dimensional box, rather than a simplex. In the special case of 2-d orthogonal range counting, Interference Oracle queries can be reduced to a constant number of rectangular range counting queries among the input set \(S\). Therefore, Interference Oracle queries can, in this case, be answered in \(O(\log n)\) time after \(O(n \log n)\) preprocessing using range trees.

**Theorem 16.** There exists a distribution-sensitive data structure in the comparison tree model for the 2-d orthogonal range counting problem that uses \(O(n \log n)\) preprocessing time and space.

### 6 Summary and Conclusions

We have presented a data structure — the odds-on tree — that can be added on to any data structure that answers queries drawn from some distribution \(D\) over \(\mathbb{R}^d\). If the underlying data structure has query time \(O(\log n)\), then the combined data structure will have optimal expected query time in the linear decision tree model. A variant of the odds-on tree based on \(k\)-d trees provides a similar result in the comparison tree model. Section 5 contains a smattering of applications of the odds-on tree. Many more are possible.

Note that Corollary 1 applies to all of the problems in Theorems 6–16 to yield, for any constant \(p\), data structures that are distribution-sensitive in the degree \(p\) algebraic decision tree model. However, when applying Corollary 1 the Interference Oracle becomes considerably more complicated, so that the fast preprocessing times in Theorems 6, 7, 10, and 15 increase to \(O(n^{1+\epsilon})\). However, the real power of Corollary 1 is its applications to problems that cannot be solved by finite linear decision trees. Examples of such problems include fixed-radius circular ray shooting in polygons [14], point-location in 2-dimensional power diagrams [9], point-location in planar subdivisions whose edges are defined by algebraic curves (such as arrangements of circles), and many others.

We believe that the odds-on tree may actually be practical in some settings. The comparison-tree version of the odds-on tree is based on \(k\)-d trees, which are simple and widely used in practice. At worst, a comparison-based odds-on tree will perform \(O(\epsilon \log n)\) extra comparisons before falling back to the backup data structure.

In low dimensions, simpler alternatives to Matoušek’s Partition Theorem (Theorem 2) are available and may be more practical. For example, in 2-d one can, \(O(n)\) time, find two lines that partition any \(m\) point set into four point sets each of size at most \(\lceil m/4 \rceil\)
and such that no line intersects more than the 3 of the resulting sets. This result is
strong enough that it can be used in place of Theorem to prove all our results (for 2-d
problems) and yields an odds-on tree that only requires 2 point-line comparisons at each
node.

Because the odds-on tree is so small (of size $O(n^c)$) it may be useful to speed up
searching in environments where memory is constrained. For example, it can be applied
to the succinct point location data structures of Bose et al. to obtain a distribution-
sensitive and succinct data structure for point location. In external-memory settings, an
odds-on tree may also be useful as a filter that is sufficiently small to fit into internal memory
while the backup structure lives in (larger but slower) external memory.

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