UN SUS PENDED CONNECTIVE E-THEORY

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ABSTRACT. We prove connective versions of results by Shulman [Shu10] and Dădărlat-Loring [DL94]. As a corollary, we see that two separable C*-algebras of the form $C_0(X) \otimes A$, where $X$ is a based, connected, finite CW-complex and $A$ is a unital properly infinite algebra, are $bu$-equivalent if and only if they are asymptotic matrix homotopy equivalent.

0. INTRODUCTION

Let $S$ denote the Connes-Higson asymptotic homotopy category of separable C*-algebras (c.f. [CH90, GHT00]). Let $\Sigma$ denote the suspension functor $\Sigma B := C_0(\mathbb{R}) \otimes B$ and let $K$ denote the algebra of compact operators on a separable Hilbert space.

$E$-theory is the bivariant $K$-theory defined by

$$E(A, B) := S(\Sigma A, \Sigma B \otimes K).$$

In this paper, we prove connective extensions of the following two closely related results.

Theorem 0.1 (Shulman [Shu10]). Let $A$ be a separable C*-algebra. Then $qA \otimes K$ is $S$-equivalent to $\Sigma^2A \otimes K$.

Theorem 0.2 (Dădărlat-Loring [DL94, Theorem 4.3]). Let $A$ and $B$ be separable C*-algebras. If the abelian monoid $S(A, A \otimes K)$ is a group, then the suspension functor induces an isomorphism

$$S(A, B \otimes K) \cong E(A, B \otimes K).$$

See Theorem 3.8 and Theorem 3.11 for the precise statements. Considering stable algebras, we obtain Theorem 0.1 and Theorem 0.2 respectively. We note that this gives new and more conceptual, if not simpler, proofs of the theorems.

We refer to [Tho03] and references therein for details of connective $E$-theory and its applications.

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\footnote{Our proof of Theorem 0.2 is closely related to the remark at end of Section 4 in [DL94, Theorem 4.3].}
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1. Asymptotic Matrix Homotopy Category

We start by fixing some notation.

**Notation 1.1.**

(i) Let $A$ and $B$ be $C^*$-algebras. We write $A \ast B$, $A \times B$ and $A \otimes B$ for the free product, direct product/sum and maximal tensor product of $A$ and $B$, respectively.

(ii) For $n \geq 1$, let $M_n$ denote the $C^*$-algebra of $n \times n$ complex matrices. For $n, m \geq 1$, we write $\oplus$ for the operation $\oplus : M_n \times M_m \rightarrow M_{n+m}$, $(a, b) \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$ and, $i_{m,n}$, for $m \geq n$, for the inclusion $i_{m,n} : M_n \hookrightarrow M_m$, $a \mapsto a \oplus 0$.

(iii) For $k \geq 0$, let $\Sigma^k$ denote the $C^*$-algebra $C_0(\mathbb{R}^k)$ of continuous functions on $\mathbb{R}^k$ vanishing at infinity. We identify $\Sigma^0$ with $\mathbb{C}$ and $\Sigma^k \otimes \Sigma^l$ with $\Sigma^{k+l}$ for $k, l \geq 0$.

**Definition 1.2.** Let $A$ and $B$ be separable $C^*$-algebras. We define $m(A, B)$ as the colimit $m(A, B) := \colim_{n \to \infty} S(A, B \otimes M_n)$ along $(\text{id}_B \otimes i_{m,n})_\ast$.

We summarize some properties of $m$ that are well-known and/or easy to check. Statements (i)-(iii) say, essentially, that $m$ is a homotopy invariant, matrix stable category enriched over the abelian monoids.

**Proposition 1.3.** Let $A$, $B$, $C$ and $D$ stand for separable $C^*$-algebras and let $m, n \geq 1$.

(i) Homotopic $\ast$-homo morphisms $A \rightarrow B$ define the same class in $m(A, B)$.

(ii) The composition $S(B, C \otimes M_m) \times S(A, B \otimes M_n) \rightarrow S(A, C \otimes M_{mn})$ gives $m$ a category structure, with the identity morphism on $A$ represented by $\text{id}_A \otimes i_{n,1} : A \rightarrow A \otimes M_n$.

(iii) The addition $S(A, B \otimes M_n) \times S(A, B \otimes M_m) \rightarrow S(A, B \otimes M_{n+m})$.
gives \( m(A, B) \) the structure of an abelian monoid, bilinear with respect to composition.

(iv) The tensor product

\[
\begin{array}{ll}
S(A, B \otimes M_n) \times S(C, D \otimes M_m) & \to S(A \otimes C, B \otimes D \otimes M_{nm}) \\
(f, g) & \mapsto f \otimes g
\end{array}
\]  

defines a natural bilinear functor

\[
\otimes : m(A, B) \times m(C, D) \to m(A \otimes C, B \otimes D).
\]

(v) For any \( A \) and \( C \), the functor \( F(B) := m(A, B \otimes C) \) is split exact.

Proof. We prove only the last statement. This follows from [DL94, Proposition 3.2] and [Wei94, Theorem 2.6.15]. \( \square \)

Definition 1.4 (c.f. [Tho03, Definition 4.4.14]). We call \( m \) the asymptotic matrix homotopy category of separable C*-algebras.

Lemma 1.5 (Cuntz [Cun87, Proposition 3.1(a)]). For any separable C*-algebras \( B \) and \( C \), the natural map

\[
B \ast C \to B \times C
\]

is an \( m \)-equivalence. \( \square \)

Corollary 1.6. For any separable C*-algebras \( B, C \) and \( D \), the natural map

\[
(B \otimes D) \ast (C \otimes D) \to (B \ast C) \otimes D
\]

is an \( m \)-equivalence.

Proof. The following diagram is commutative

\[
\begin{array}{ccc}
(B \otimes D) \ast (C \otimes D) & \longrightarrow & (B \ast C) \otimes D \\
\downarrow & & \downarrow \\
(B \otimes D) \times (C \otimes D) & \longrightarrow & (B \times C) \otimes D
\end{array}
\]

The vertical maps are \( m \)-equivalences by Lemma 1.5 and the bottom horizontal map is an isomorphism. \( \square \)

Notation 1.7. Let \( B \) be a separable C*-algebra. Following Cuntz, we write \( qB \) for the kernel of the folding map \( B \ast B \xrightarrow{\text{id} \times \text{id}} B \).

We note that the short exact sequence

\[
0 \longrightarrow qB \longrightarrow B \ast B \longrightarrow B \longrightarrow 0
\]

is split-exact.

Proposition 1.8. For any separable C*-algebras \( B \) and \( D \), the natural map

\[
\sigma_{B, D} : q(B \otimes D) \to qB \otimes D
\]

is an \( m \)-equivalence.
Proof. Fix $A$ and let $F$ denote the functor $F(B) := m(A, B)$.

We apply $F$ to the following commutative diagram of split-exact sequences:

$$0 \rightarrow q(B \otimes D) \rightarrow B \otimes D \ast B \otimes D \rightarrow B \otimes D \rightarrow 0.$$  \hspace{1cm} (18)

By Corollary 1.6, $F$ induces isomorphism on the middle map. Since $F$ is split exact, it follows that $F(\sigma_{B,D})$ is an isomorphism. Now the proof follows from Yoneda’s Lemma. $\Box$

Remark 1.9. Let $\text{Ho}$ denote the homotopy category of $C^*$-algebras and let $n$ denote the matrix homotopy category with morphisms

$$n(A, B) := \text{colim}_n \text{Ho}(A, B \otimes M_n).$$  \hspace{1cm} (19)

Then, in Lemma 1.5 and Corollary 1.6, we actually have $n$-equivalences. However, the map $\sigma_{B,D}$ from Proposition 1.8 is not an $n$-equivalence in general. For instance, let $T_0$ denote the reduced Toeplitz algebra. Then $T_0$ is $KK$-contractible, hence $q(T_0) \otimes K$ is contractible i.e. homotopy equivalent to the zero algebra 0 (c.f. [Cun84]). However, $q\mathbb{C} \otimes T_0 \otimes K$ has a non-trivial projection, hence not contractible. It follows that $\sigma_{\mathbb{C},T_0} : q(T_0) \rightarrow q\mathbb{C} \otimes T_0$ is not an $n$-equivalence.

Indeed, for any $A$ and $B$, we have a natural isomorphism

$$\text{Ho}(A, B \otimes K) \cong n(A, B \otimes K).$$  \hspace{1cm} (20)

Hence if $f : A \rightarrow B$ is an $n$-equivalence, then $f \otimes \text{id}_K : A \otimes K \rightarrow B \otimes K$ is a homotopy equivalence.

Remark 1.10. Let $A$ and $B$ be separable $C^*$-algebras.

(i) We have a natural isomorphism

$$S(A, B \otimes K) \cong m(A, B \otimes K).$$  \hspace{1cm} (21)

It follows that if $f \in m(A, B)$ is an $m$-equivalence, then $f \otimes \text{id}_K$ is an $S$-equivalence from $A \otimes K$ to $B \otimes K$.

(ii) Tensoring with $K$ gives an isomorphism

$$S(A, B \otimes K) \cong S(A \otimes K, B \otimes K).$$  \hspace{1cm} (22)

2. Matrix Homotopy Symmetry

The following definition is inspired by [DL94].

Definition 2.1. A separable $C^*$-algebra $A$ is matrix homotopy symmetric if $\text{id}_A \in m(A, A)$ has an additive inverse: there is $n \geq 1$ and $\eta : A \rightarrow A \otimes M_m$ such that $i_{n,1} \oplus \eta : A \rightarrow A \otimes M_{n+m}$ is null-homotopic.
Remark 2.2. (i) If the monoid \( m(A, A) \) is a group, then \( A \) is matrix homotopy symmetric. Conversely, if \( A \) is matrix homotopy symmetric, then \( m(A, B) \) and \( m(B, A) \) are abelian groups for any \( B \).

(ii) If \( A \) is matrix homotopy symmetric, then so is \( A \otimes D \) for any \( D \).

(iii) If \( A \) is \( m \)-equivalent to \( B \) and \( A \) is matrix homotopy symmetric, then so is \( B \).

Example 2.3. (i) The algebra \( \Sigma^1 \) is matrix homotopy symmetric. In fact, the algebra \( C_0(X) \), of continuous functions vanishing at the base point, is matrix homotopy symmetric for any based, connected, finite CW-complex \( X \) (c.f. [DN90, Proposition 3.1.3] and the discussion preceding it).

(ii) The algebra \( qB \) is matrix homotopy symmetric for any \( B \), by taking \( n = m = 1 \) and \( \eta = \tau \) the flip-map on \( qA \) (c.f. [Cum87, Proposition 1.4]).

Notation 2.4. Let \( B \) be a separable \( C^* \)-algebra. Let \( \pi_B: qB \to B \) denote the composition

\[
\pi_B: qB \xrightarrow{\cong} B \ast B \xrightarrow{id \ast 0} B.
\]

We remark that \( q \) is functorial (for \( * \)-homomorphisms) and for any \( * \)-homomorphism \( f: A \to B \), we have a commutative diagram

\[
\begin{array}{ccc}
qA & \xrightarrow{q(f)} & qB \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
A & \xrightarrow{f} & B \\
\end{array}
\]

From our point of view, the following is the key ingredient that underlies both Theorem 0.1 and Theorem 0.2.

Proposition 2.5. Let \( A \) be a separable \( C^* \)-algebra. Then the following statements are equivalent:

(a) The algebra \( A \) is matrix homotopy symmetric.

(b) For any \( B \) and \( D \), we have

\[
(\pi_B \otimes \text{id}_D)_*: m(A, qB \otimes D) \cong m(A, B \otimes D).
\]

(c) The map \( \pi_A: qA \to A \) is an \( m \)-equivalence.

(d) The map \( \pi_C \otimes \text{id}_A: qC \otimes A \to A \) is an \( m \)-equivalence.

Proof. The statements (c) and (d) are equivalent by Proposition 1.8.

Since \( qA \) is matrix homotopy symmetric (c.f. Example ii), it follows from Remark 2.2 that (c) implies (a).

Suppose that \( A \) is matrix homotopy symmetric. Then the functor \( F(B) := m(A, B \otimes D) \) is a homotopy invariant, split exact, matrix stable functor with values in abelian groups. Hence \( (\pi_B)_*: F(qB) \to F(B) \) is an isomorphism for all \( B \), by [Cum87, Proposition 3.1], i.e. (a) implies (b).

The remaining implication, (b) \( \Rightarrow \) (c), follows from Yoneda’s Lemma. \( \square \)
As a corollary, we now prove Theorem 0.1. In view of Proposition 1.8, it is enough to prove the following. See also Theorem 3.8.

**Theorem 2.6** (Bott Periodicity). Let \( u: qC \to \Sigma^2 \otimes M_2 \in m(qC, \Sigma^2) \) denote the Bott element. Then

\[
u \otimes \text{id}_K: qC \otimes K \to \Sigma^2 \otimes M_2 \otimes K \cong \Sigma^2 \otimes K
\]

is an \( m \)-equivalence (equivalently, an \( S \)-equivalence).

Proof. We have a commutative diagram

\[
\begin{array}{ccc}
q(qC) & \xrightarrow{q(u)} & q(\Sigma^2 \otimes M_2) \\
\downarrow \pi_{qC} & & \downarrow \pi_{\Sigma^2 \otimes M_2} \\
qC & \xrightarrow{u} & \Sigma^2 \otimes M_2 \\
\end{array}
\]

The vertical maps are \( m \)-equivalences by Proposition 2.5 and the map \( q(u) \otimes \text{id}_K \) is a homotopy equivalence (in particular, an \( m \)-equivalence) by \( KK \)-theoretic Bott Periodicity. It follows that \( u \otimes \text{id}_K \) is an \( m \)-equivalence. \( \square \)

3. Bott Invertibility

**Definition 3.1.** Let \( u: qC \to \Sigma^2 \otimes M_2 \in m(qC, \Sigma^2) \) denote the Bott element. We say that a separable \( C^* \)-algebra \( D \) is Bott inverting if the element

\[
u \otimes \text{id}_D \in m(qC \otimes D, \Sigma^2 \otimes D)
\]

is an \( m \)-equivalence.

**Remark 3.2.** (i) If \( D \) is Bott inverting, then so is \( D \otimes B \) for any \( B \).

(ii) If \( D \) is \( m \)-equivalent to \( B \) and \( D \) is Bott inverting, then so is \( B \).

First we show that there are plenty of algebras that are Bott inverting. See Example 3.10 for an example that is not Bott inverting.

**Lemma 3.3.** Let \( D \) be a separable \( C^* \)-algebra. Suppose that for some \( n \geq 1 \), the inclusion

\[
\text{id}_D \otimes i_{n,1}: D \hookrightarrow D \otimes M_n
\]

factors in \( S \) through a Bott inverting algebra. Then \( D \) is Bott inverting.

Proof. Let

\[
\begin{array}{ccc}
D & \xrightarrow{f} & B \\
\downarrow g & & \downarrow \ \\
D \otimes M_n
\end{array}
\]

\( \text{The map } \pi_{qC} \text{ is in fact an } n \text{-equivalence by [Cum87] Theorem 1.6}. \)
be a factorization of the inclusion $\text{id}_D \otimes i_{n,1} : D \rightarrow D \otimes M_n$, with $B$ Bott inverting. Then the following diagram is commutative in $\mathbf{S}$:

\[
\begin{array}{ccc}
q \mathcal{C} \otimes D & \xrightarrow{id_{q \mathcal{C}} \otimes f} & q \mathcal{C} \otimes B \\
\downarrow u \otimes id_D & & \downarrow u \otimes id_B \\
\Sigma^2 \otimes M_2 \otimes D & \xrightarrow{id_{\Sigma^2 \otimes M_2} \otimes f} & \Sigma^2 \otimes M_2 \otimes B \\
\downarrow u \otimes id_{\Sigma^2 \otimes M_2} & & \downarrow u \otimes id_{\Sigma^2 \otimes M_2} \\
\Sigma^2 \otimes M_2 \otimes D \otimes M_n & \xrightarrow{id_{\Sigma^2 \otimes M_2} \otimes g} & \Sigma^2 \otimes M_2 \otimes D \otimes M_n
\end{array}
\]  

(30)

Since $i_{n,1}$ is invertible in $\mathbf{m}$, and $u \otimes id_B$ is invertible by assumption, it follows that $u \otimes id_D$ is invertible. □

**Definition 3.4.** We say that a $C^*$-algebra $D$ is **stable** if $D \cong D \otimes K$.

By Bott Periodicity (Theorem 2.6) and Remark 3.2, stable algebras are Bott inverting.

**Lemma 3.5** (Kirchberg). Let $D$ be a separable $C^*$-algebra. If $D$ contains a stable full $C^*$-subalgebra, then map $\text{id}_D \otimes i_{4,1} : D \hookrightarrow D \otimes M_4$ factors through a stable algebra.

**Proof.** See the proof of [Tho03, Lemma 4.4.7]. □

Combining Lemma 3.3 and Lemma 3.5, we get the following.

**Corollary 3.6.** All separable $C^*$-algebras that contain a stable full $C^*$-subalgebra are Bott inverting. In particular, all separable unital properly infinite $C^*$-algebras are Bott inverting. □

**Remark 3.7.** Same methods show that comparison map from algebraic to topological $K$-theory

\[
K_*^{\text{alg}}(D) \rightarrow K_*^{\text{top}}(D)
\]

is an isomorphism if $D$ has a stable full $C^*$-subalgebra (c.f. [SW90]).

Now we are ready to state and prove the connective versions of Theorem 0.1 and Theorem 0.2, which we recover by considering stable algebras.

**Theorem 3.8.** Let $A$ be a separable $C^*$-algebra. If $D$ is Bott inverting, then we have equivalences

\[
qA \otimes D \cong_m q \mathcal{C} \otimes A \otimes D \cong_m \Sigma^2 \otimes A \otimes D.
\]

(33)

**Proof.** Follows from Proposition 1.8 and Bott invertibility. □

**Definition 3.9** (A. Thom [Tho03, Theorem 4.2.1]). Let $A$ and $B$ be separable $C^*$-algebras. For $n \in \mathbb{Z}$, we define $\text{bu}_n(A, B)$ as the colimit

\[
\text{bu}_n(A, B) := \lim_{k \rightarrow \infty} \mathbf{m}(\Sigma^k \otimes A, \Sigma^{k+n} \otimes B)
\]

(34)

along the suspension maps. The **connective $E$-category** $\text{bu}$ is the category with morphisms $\text{bu}_0(A, B)$. 
Let $X$ and $Y$ be based, connected, finite CW-complexes. Then from the proof of Theorem [Tho03, Theorem 4.2.1], we see that
\[
\text{bu}_n(C_0(X), C_0(Y)) \cong \text{kk}_n(Y, X) \tag{35}
\]
in the notation of [DN90, DM00].

**Example 3.10.** Let $X$ be a based, connected, finite CW-complex and let $D = C_0(X)$. Then, for any $k \leq 0$, we have $\text{bu}_k(D, \mathbb{C}) \cong 0$ by [DN90, Corollary 3.4.3].

We claim that $D$ is Bott inverting if and only if $D$ is $m$-contractible. Indeed, first note that, by Proposition 2.5, the map
\[
\id_{\Sigma^1} \otimes \pi_C: \Sigma^1 \otimes q\mathbb{C} \to \Sigma^1 \tag{36}
\]
is an $m$-equivalence, thus $\pi_C: q\mathbb{C} \to \mathbb{C}$ is a $bu$-equivalence. Now suppose that $D$ is Bott inverting. Then
\[
\text{bu}_k(D, \mathbb{C}) \cong \text{bu}_k(q\mathbb{C} \otimes D, \mathbb{C}) \cong \text{bu}_{k-2}(D, \mathbb{C}). \tag{37}
\]
for any $k \in \mathbb{Z}$. Hence the map $0: D \to 0$ induces an $m$-equivalence by [DM00] Theorem 2.4. The converse is clear.

In particular, for any $k \geq 0$, the algebra $\Sigma^k$ is not Bott inverting.

**Theorem 3.11.** Let $A$ and $B$ be a Bott inverting separable $\mathbb{C}^*$-algebras. If $A$ is matrix homotopy symmetric, then we have a natural isomorphism
\[
m(A, B) \cong \text{bu}(A, B). \tag{38}
\]

**Proof.** Suppose that $A$ is matrix homotopy symmetric. By Proposition 2.5, we have isomorphisms
\[
\begin{array}{ccc}
m(A, q\mathbb{C} \otimes B) & \cong & m(q\mathbb{C} \otimes A, q\mathbb{C} \otimes B) \\ \cong & m(A, B) & \cong m(q\mathbb{C} \otimes A, B)
\end{array} \tag{39}
\]
and by Bott invertibility, we have
\[
m(q\mathbb{C} \otimes A, q\mathbb{C} \otimes B) \cong m(\Sigma^2 \otimes A, \Sigma^2 \otimes B). \tag{40}
\]
Now it is easy to check that the composition
\[
m(A, B) \to m(q\mathbb{C} \otimes A, q\mathbb{C} \otimes B) \to m(\Sigma^2 \otimes A, \Sigma^2 \otimes B) \tag{41}
\]
is the double suspension $\Sigma^2$. □

**Corollary 3.12.** Let $A$ and $B$ be a matrix homotopy symmetric, Bott inverting separable $\mathbb{C}^*$-algebras. Then $A$ and $B$ are $bu$-equivalent if and only if they are $m$-equivalent. □
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