SPANNING TREES OF 3-UNIFORM HYPERGRAPHS

ANDREW GOODALL\textsuperscript{1} AND ANNA DE MIER\textsuperscript{2}

Abstract. Masbaum and Vaintrob’s “Pfaffian matrix tree theorem” implies that counting spanning trees of a 3-uniform hypergraph (abbreviated to 3-graph) can be done in polynomial time for a class of “3-Pfaffian” 3-graphs, comparable to and related to the class of Pfaffian graphs. We prove a complexity result for recognizing a 3-Pfaffian 3-graph and describe two large classes of 3-Pfaffian 3-graphs — one of these is given by a forbidden subgraph characterization analogous to Little’s for bipartite Pfaffian graphs, and the other consists of a class of partial Steiner triple systems for which the property of being 3-Pfaffian can be reduced to the property of an associated graph being Pfaffian. We exhibit an infinite set of partial Steiner triple systems that are not 3-Pfaffian, none of which can be reduced to any other by deletion or contraction of triples.

We also find some necessary or sufficient conditions for the existence of a spanning tree of a 3-graph (much more succinct than can be obtained by the currently fastest polynomial-time algorithm of Gabow and Stallmann for finding a spanning tree) and a superexponential lower bound on the number of spanning trees of a Steiner triple system.

1. Introduction

1.1. Spanning trees of 3-uniform hypergraphs. In this paper we investigate the problem of the existence, finding and counting of spanning trees of 3-uniform hypergraphs (henceforth called \emph{3-graphs} for short). The initial motivation for our work was Masbaum and Vaintrob’s Pfaffian matrix tree theorem \cite{18}. They introduce the notion of an orientation (or equivalently a sign) of a spanning tree of a 3-graph. The Pfaffian matrix tree theorem gives a generating function for signed spanning trees of a 3-graph. We shall be particularly interested in how this spanning tree orientation can be used to identify a large class of 3-graphs for which the problem of counting the number of spanning trees can be done in polynomial time. This class is comparable to that of Pfaffian graphs, for which there is a polynomial-time algorithm for counting the number of perfect matchings. A classical theorem of Kasteleyn \cite{12} is that planar graphs are Pfaffian: can we find a similar class of 3-graphs for which counting the number of spanning trees can be done in polynomial time?

We should be clear at the outset about how we are defining a spanning tree of a 3-graph, for there are various natural alternatives. (More detailed definitions of these and other terms from the theory of hypergraphs are given in Section 2 below.) A spanning tree of a 3-graph \(H\) is an inclusion-maximal subset \(T\) of the hyperedges of \(H\) that covers all the vertices subject to the condition that \(T\) does not contain a cycle of hyperedges. If \(B_H\) is the usual bipartite vertex-hyperedge incidence graph associated with \(H\), then a spanning tree of \(H\) in this sense corresponds precisely to a spanning tree of \(B_H\) with the property that either all three edges of \(B_H\) incident with a given hyperedge belong to the tree or none of them do. Alternatively, if each hyperedge \(\{a, b, c\}\) of \(H\) is represented as a triangle of edges \(ab, bc, ca\) in a graph \(G_H\) on the same vertex set as \(H\), then a spanning tree of \(H\) corresponds to a cactus subgraph of \(G_H\) covering all vertices. See \cite{1} for a generalization of the Masbaum–Vaintrob theorem to arbitrary 3-graphs.

\textsuperscript{1}Research supported in part by the hosting department while visiting the other author and Marc Noy at various times in 2008-10.

\textsuperscript{2}Research supported in part by projects MTM2008-03020 and DGR2009-SGR1040.
hypergraphs in which spanning trees are now cacti with cycles of any odd length and not just triangles.

Spanning trees of 3-graphs differ in fundamental ways from spanning trees of ordinary graphs: a closer correspondence is to be found with perfect matchings. Whereas for spanning trees of graphs the problems of the existence, finding and counting of spanning trees each have a straightforward polynomial-time algorithm, the same is not true for spanning trees of 3-graphs.

The augmenting path algorithm finds a maximum matching of a bipartite graph in polynomial time. Consequently, both the problem of whether there is a perfect matching of a bipartite graph and the problem of finding one can be solved in polynomial time. Edmonds’ maximum matching algorithm [6] solves in polynomial time the existence and search problems for whether an arbitrary graph has a perfect matching.

Lovász’s matroid matching algorithm [14, 15] provides a polynomial-time algorithm solving the problem of the existence and finding of a spanning tree of a 3-graph. However, since it solves such a general and complicated problem, the algorithm is involved, has running time a polynomial of high degree and is not optimal when restricting attention from linear matroids to the graphic matroids underlying the case of 3-graphs. The augmenting path algorithm for linear matroids of Gabow and Stallmann [8] has running time $O(mn^2)$ with $O(mn)$ space for graphic matroids of rank $n$ and size $m$, improved to using $O(m)$ space (alternatively $O(mn \log n)$ time using $O(m \log^4 n)$ space) by the same authors in [7]. In this paper we give some straightforward necessary or sufficient conditions that give simple criteria for the existence of a spanning tree of a 3-graph and in the case of Steiner triple systems a superexponential lower bound on the number of spanning trees.

Our focus then turns to the problem of counting spanning trees of 3-graphs. This problem is #P-complete even for a very restricted class of 3-graphs, which is a consequence of the fact that counting perfect matchings is #P-complete for general graphs [23]. Masbaum and Vaintrob define an orientation or sign of a spanning tree of a 3-graph using orientations of hyperedges in a way that closely follows the definition of the sign of a perfect matching, as elucidated by Hirschman and Reiner [10]. Just as the existence of a Pfaffian orientation of the edges of a graph enables the number of perfect matchings of a graph to be computed in polynomial time, so the existence of what we shall call a “3-Pfaffian orientation” of a 3-graph allows the number of spanning trees to be calculated in polynomial time. This observation was made by Caracciolo et al. in the conclusion of their paper [5].

Having identified a property of 3-graphs that enables counting of spanning trees to be done in polynomial time, how quickly can we verify that a graph has this property? Compare the case of Pfaffian graphs: it is not known whether there is a polynomial-time checkable certificate for a graph to have a Pfaffian orientation. Vazirani and Yannakakis [24] show that the problem of determining whether a graph $G$ has a Pfaffian orientation and that of determining whether a given orientation of $G$ is Pfaffian are polynomial-time equivalent. They appeal to Lovász’ polynomial-time algorithm [17] for computing the binary rank and finding a basis of the vector space of matchings of a graph. They also show that the problem of deciding whether a graph has a Pfaffian orientation is in co-NP. We show that the problem of deciding the existence of a 3-Pfaffian orientation is also in co-NP, but we do not know if it is equivalent to deciding if a given orientation of hyperedges is 3-Pfaffian.

Although checking whether a graph is Pfaffian is not known to be polynomial time, Little [13] gave a structural characterization of Pfaffian bipartite graphs as those that do not contain an even subdivision of $K_{3,3}$ with a perfect matching in the complement. A natural question is whether there is any similar characterization of 3-Pfaffian 3-graphs: we prove such a characterization for a special subclass of tripartite 3-graphs. Whether tripartite 3-Pfaffian 3-graphs in general have a similar description in terms of forbidden subgraphs remains open.
1.2. Outline of the paper. In Section 2 we introduce some of the basic notions and notation required in the paper. We refer to [22] for a recent survey of the topic of Pfaffian orientations, and [16] for matching theory.

In Section 3 we present some elementary results about the problem of deciding if there is a spanning tree of a 3-graph and about the problem of counting them. We begin in Subsection 3.1 with a summary of what is known about the complexity of these problems in general. In Subsection 3.2 we consider the case of the complete 3-graph, for which we can enumerate the number of spanning trees, and, more importantly, thereby establish in Lemma 3.2 a correspondence between spanning trees of a 3-graph and perfect matchings of a graph that is basic to the rest of the paper. In Subsection 3.3 we describe some straightforward necessary or sufficient conditions for the existence of a spanning tree of a 3-graph. Theorem 3.4 gives an explicit formula for the number of positively and negatively oriented spanning trees of a Steiner triple system.

In Section 4 we initiate our study of orientations of spanning trees of 3-graphs and the property of a 3-graph having an "3-Pfaffian orientation," which by Masbaum and Vaintrob’s Pfaffian matrix-tree theorem [18] implies a polynomial-time algorithm for counting spanning trees. We begin in Subsection 4.1 by defining orientations of spanning trees, which are defined relative to an orientation of triples. Theorem 4.3 gives an explicit formula for the number of positively and negatively oriented spanning trees of the complete 3-graph under a canonical orientation of its triples. In Subsection 4.2 we introduce the notion of a "3-Pfaffian orientation", analogous to a Pfaffian orientation of a graph. In fact in Theorem 4.7 we see that if we make a 3-graph by adding an extra vertex to every edge of a graph then a 3-Pfaffian orientation of H corresponds exactly to a Pfaffian orientation of G. In Subsection 4.3 we prove that deciding if a 3-graph has a 3-Pfaffian orientation is in co-NP.

In Section 5 we consider a family of 3-graphs for which we can characterize the property of having a 3-Pfaffian orientation in terms of forbidden subgraphs, similar to Little’s characterization of Pfaffian bipartite graphs (Theorem 5.11 and Corollary 5.12).

In Section 6 we find a large class of partial Steiner triple systems that have 3-Pfaffian orientations (Theorem 6.2) and also describe an infinite family of partial Steiner triple systems that do not have a 3-Pfaffian orientation (Theorem 6.3). Furthermore, we prove that this second family cannot be reduced by deletion and contraction of triples to a finite set of non-3-Pfaffian 3-graphs.

Finally, in Section 7 we highlight some open problems.

2. Notation and terminology

A 3-graph is a 3-uniform hypergraph $H = (V, \Delta)$, where $\Delta \subseteq \binom{V}{3}$. There are no repeated hyperedges and no hyperedges of size 2 or 1. We shall use the name triple for a hyperedge of H. The underlying graph of a 3-graph $H = (V, \Delta)$ is the multigraph $G = (V, E)$ with edge set $E = \{\{a, b\} : \exists c \in V \{a, b, c\} \in \Delta\}$, an edge $\{a, b\}$ occurring with multiplicity $|\{c \in V : \{a, b, c\} \in \Delta\}|$. We identify a triangle of $H$ with its corresponding triangle in the underlying graph $G$. We write $abc$ for the triple $\{a, b, c\}$ of $H$ or triangle of $G$ and $ab$ for the edge $\{a, b\}$ of $G$.

Deleting a triple $abc \in \Delta$ gives the 3-graph $H \setminus abc = (V, \Delta \setminus abc)$. A sub-3-graph of $H$ is a 3-graph obtained from $H$ by deleting some subset of triples. Contracting a triple $abc$ gives the 3-graph $H/abc = (V \setminus \{b, c\}, \Delta')$ where $\Delta'$ is defined as follows. A triple $ijk$ belongs to $\Delta'$ if (i) $ijk$ and $abc$ are disjoint, or (ii) $ijk$ is obtained from a triple that meets $abc$ in one vertex by relabelling this common vertex by $a$ if it is equal to $b$ or $c$. In other words, to form $H/abc$ from $H$ we set $a = b = c$ and remove all triples that have decreased in size to a pair or singleton and also any repeated triples.

In terms of the underlying graph $G$ of $H$, deleting a triple $abc$ of $H$ corresponds to deleting the edges $ab, bc, ca$ of $G$. Contracting $abc$ corresponds to contracting $ab, bc, ca$ and removing any edges that are no longer an edge of a triangle.
The degree of a vertex \( a \in V \) in \( H \) is defined by \( d(a) = \# \{ t \in \Delta : a \in t \} \), equal to half the degree of \( a \) in the underlying graph \( G \). The multiplicity of a pair \( ab \in \binom{V}{2} \) in \( H \) is defined by \( m(ab) = \# \{ t \in \Delta : \{a, b\} \subseteq t \} \), equal to the multiplicity of the edge \( ab \) in the underlying graph \( G \).

A path in a 3-graph \( H = (V, \Delta) \) is an alternating sequence of \( \ell + 1 \) distinct vertices and \( \ell \) distinct triples, \( a_0, t_1, a_1, \ldots, a_{\ell-1}, t_\ell, a_\ell \), with the property that \( a_{i-1} \in t_i \ni a_i \) for \( i \in [\ell] \). A path is usually identified with its set of triples \( \{t_1, \ldots, t_\ell\} \). Observe that it is not required that a path with \( \ell \) triples spans \( 2\ell + 1 \) vertices, although most of the paths that appear in the paper have this property.

The 3-graph \( H \) is connected if for each pair of vertices \( u, v \in V \) there is a path \( u, t_1, \ldots, t_\ell, v \) in \( H \) that joins them. \( H \) is connected if and only if its underlying graph is connected.

A cycle in \( H \) is a closed path, i.e., an alternating sequence of \( \ell \) distinct vertices and \( \ell \) distinct triples \( a_0, t_1, \ldots, a_{\ell-1}, t_\ell \) terminated by the starting vertex \( a_\ell = a_0 \), with the property that \( a_{i-1} \in t_i \ni a_i \). A cycle is usually identified with its set of triples \( \{t_1, \ldots, t_\ell\} \). Two triples sharing two vertices form a cycle.

A forest of \( H \) is a set of triples \( T \subseteq \Delta \) with the property that there is no cycle \( C \subseteq T \). Between any two vertices in a forest there is at most one path. A spanning tree of \( H \) is a sub-3-graph \( T \) containing no cycles such that \( \bigcup T = V \), i.e., a connected forest spanning \( V \). If \( H \) has a spanning tree then \( |V| \) is necessarily odd and \( T \) contains \( \frac{|V|-1}{2} \) triples. The connected 3-graph on \( \{u, v, a, b, c\} \) with triples \( uwa, uwb, uwc \) has no spanning tree. A leaf of a tree \( T \) is a triple with two vertices of degree 1 (belonging to no other triple of \( T \)). A spanning tree of \( H \) has at least one leaf \( abc \), and at least two leaves if \(|V| \geq 5 \). The 3-graph \( T - \{b, c\} \) obtained by deleting vertices \( b, c \) is a spanning tree of \( H - \{b, c\} \) if and only if \( abc \) is a leaf of \( T \) for some \( a \) and where \( b, c \) have degree 1.

### 3. Elementary results on the existence and counting of spanning trees of 3-graphs

#### 3.1. Complexity of existence, finding and counting of spanning trees of 3-graphs.

Given a 3-graph \( H = (V, \Delta) \) and triples \( abc \) put in arbitrary linear order \( a < b < c \), define the subgraph \( G' \) of its underlying graph \( G = (V, E) \) on edge set \( E' = \{ab, ac : abc \in \Delta, a < b < c\} \) of size \( 2|\Delta| \). Partition \( E' \) into pairs \( ab, ac \) with \( abc \in \Delta, a < b < c \). A matching of the graphic matroid defined by \( G' \) is a forest of \( G' \) such that for each \( abc \in \Delta \) with \( a < b < c \) if \( ab \) belongs to the forest then so does \( ac \). A maximum matching has the greatest number of pairs possible. The 3-graph \( H \) has a spanning tree if and only if the maximum matching has size \( \frac{|\Delta|-1}{2} \). Thus the problem of determining whether a 3-graph has a spanning tree is a special case of the matroid matching problem. As mentioned in the introduction, this gives a polynomial-time algorithm for finding a spanning tree of a 3-graph.

For \( k \geq 4 \) the problem of deciding if a \( k \)-uniform hypergraph has a spanning tree is \#P-complete \( \Box \).

Counting spanning trees of a 3-graph is \#P-complete. This follows since counting perfect matchings of a graph is a \#P-complete problem in general [23,23' and this reduces to the problem of counting spanning trees for the class of 3-graphs with the property that there is a vertex that is contained in all triples.

On the other hand, counting perfect matchings is polynomial time for the class of graphs that have a Pfaffian orientation. One of the aims of this paper is to develop the analogous notion of a Pfaffian orientation for 3-graphs and thereby characterize a class of 3-graphs with the property that counting spanning trees has a polynomial-time algorithm.

#### 3.2. Spanning trees of complete 3-graphs.

For a 3-graph \( H = (V, \Delta) \) let \( \mathcal{T}(H) = \{T \subseteq \Delta : T \text{ is a spanning tree of } H\} \). Note that \( \mathcal{T}(H \setminus abc) = \{T \in \mathcal{T}(H) : abc \notin T\} \) and there is a bijection between \( \mathcal{T}(H/abc) \) and \( \{T \in \mathcal{T}(H) : abc \in T\} \). If \( abc \) is in no spanning tree of \( H \) then
Theorem 3.1. The number of spanning trees of $K_{2n+1}^{(3)}$ is given by

$$|\mathcal{T}| = (2n-1)!!(2n+1)^{n-1}.$$ 

Proof. The proof uses a similar construction to the Prüfer code for spanning trees of ordinary graphs.

A tree spanning at least five vertices always has at least two leaves; a rooted tree spanning five or more vertices has at least one leaf not containing the root as a vertex of degree 1.

Suppose we are given a spanning tree $T$ on $[2n+1]$. We remove triples from $T$ leaf by leaf in a canonical way until we are left with a tree consisting of just one triple. At the end of the algorithm described below we obtain a sequence $\gamma = \gamma_n \in [2n+1]^{n-1}$ and a perfect matching $M = M_n$ of $[2n]$. If $n = 1$, we take $\gamma$ to be the empty sequence and $M = \{12\}$. For $n \geq 2$, the algorithm proceeds as follows.

1. Initialize $\gamma_1$ as the empty sequence, $M_0$ as the empty matching and $T_1 = T$ as the spanning tree of $K_{2n+1}^{(3)}$ that is to be encoded. Root $T$ at vertex $2n+1$.
   Start with $i = 1$.

2. At step $i$ consider the rooted tree $T_i$. Remove the leaf containing the smallest vertex label in $T_i$ while not containing the root $2n+1$ as a vertex of degree 1, thereby obtaining the next rooted tree $T_{i+1}$. (If a leaf contains $2n+1$ as a vertex of degree 1 it is ignored and the leaf with the next smallest vertex is taken.) Record as $c_i$ the vertex of degree greater than 1 in this leaf and set $\gamma_{i+1} = \gamma_i c_i$. The other two vertices of degree 1 in the leaf $a_i b_i c_i$ are paired in the matching $M_i = M_{i-1} \cup \{a_i b_i\}$.

3. If the remaining tree $T_{i+1}$ has only one triple (i.e., $i = n-1$) then this triple takes the form $a_n b_n (2n+1)$; in this case set $M = M_n = M_{n-1} \cup \{a_n b_n\}$, $\gamma = \gamma_n$, and stop. Otherwise increment $i$ to $i + 1$ and go to (2).

Conversely, given a sequence $\gamma = c_1 c_2 \ldots c_{n-1} \in [2n+1]^{n-1}$ and a perfect matching $M = \{a_1 b_1, \ldots, a_n b_n\}$ of $[2n]$ a unique spanning tree of $K_{2n+1}^{(3)}$ is constructed as follows.

1. Initialize $i = 1$, $\gamma_1 = \gamma$, $M_1 = M$, $T_1$ the empty tree (no triples or vertices).

2. Find the vertex $a_i$ with smallest label that does not occur as an element of the sequence $\gamma_i$ and that occurs in the matching $M_i$, but is not paired with $c_i$. Let $b_i$ be the vertex such that $a_i b_i \in M_i$. Set $T_{i+1} = T_i \cup \{a_i b_i c_i\}$, $M_{i+1} = M_i \setminus \{a_i b_i\}$ and $\gamma_{i+1} = \gamma_i c_i b_i e_i \ldots c_{n-1}$.

3. After step $i = n-1$ the sequence $\gamma_n$ is empty and $M_n = \{a_n b_n\}$. Set $T = T_n \cup \{a_n b_n (2n+1)\}$ and stop. Otherwise, increment $i$ to $i + 1$ and go to (2).

Spanning trees of $K_{2n+1}^{(3)}$ are thus in bijection with pairs $(\gamma, M)$, where $\gamma \in [2n+1]^{n-1}$ and $M$ is a perfect matching of $[2n]$. Since there are $(2n-1)!!$ such perfect matchings, the result follows. \hfill \Box

The first part of the proof of Theorem 5.3 can be applied to any 3-graph $H$, yielding a correspondence between spanning trees of $H$ and pairs $(M, f)$, where $M$ is a perfect matching of $H - v$ and $f : M \to V$ is a function satisfying a certain condition.

Lemma 3.2. Let $H = (V, \Delta)$ be a 3-graph with underlying graph $G$, and let $v \in V$. Given a spanning tree $T$ of $H$, there is a unique perfect matching $M$ of $G - v$ and a function $f : M \to V$ such that the set of triples of $T$ is equal to $\{ij : f(ij) : ij \in M\}$. Conversely, a perfect matching $M$
of \( G-v \) and a function \( f: M \rightarrow V \) determine a spanning tree of \( H \) if \( \{ijf(ij) : ij \in M\} \subseteq \Delta \) and there is no set of edges \( \{i_0j_0, \ldots, i_{\ell-1}j_{\ell-1}, i_{\ell}j_{\ell} = i_0j_0\} \subseteq M \) such that \( f(i_{m-1}j_m) \in \{i_m, j_m\} \) for \( m \in [\ell] \).

**Proof.** Rooting a spanning tree \( T \) of \( H \) at the vertex \( v \), we construct a unique perfect matching \( M \) of \( G-v \) and associated function \( f: M \rightarrow V \) as follows.

If \( |V| = 3 \) then \( T = \{ij\} \) and set \( M = \{ij\} \) and \( f(ij) = v \). Assume now that \( |V| > 3 \). Then every leaf of \( T \) has one vertex of degree greater than 1, by which it is attached to the rest of the tree, and the remaining two vertices are of degree 1. Let \( ijk \) be a leaf of \( T \) with vertices \( i, j \) of degree 1. Remove this leaf from \( T \). Inductively the remaining tree \( T \setminus ijk \) determines a unique perfect matching \( M' \) of \( G-\{v, i, j\} \) and function \( f: M' \rightarrow V \setminus \{i, j\} \). Extend \( M' \) to a perfect matching \( M \) of \( G-v \) by adding the edge \( ij \) and the function \( f \) by setting \( f(ij) = k \).

Conversely, given a perfect matching \( M \) of \( G-v \) and a function \( f: M \rightarrow V \), the 3-graph on \( V \) having as set of triples \( T = \{ijf(ij) : ij \in M\} \) is a spanning tree of \( H \) if \( T \subseteq \Delta \) and there is no cycle of triples. It is easy to see that this amounts to the condition on \( f \) in the statement of the theorem. For such an \( f \), the 3-graph \( T \) is a tree with \((|V|-1)/2\) triples, and therefore it spans the \( |V| \) vertices of \( H \). \( \square \)

### 3.3. Necessary or sufficient conditions for the existence of spanning trees

The most straightforward necessary conditions for the existence of a spanning tree of a 3-graph \( H = (V, \Delta) \) is that \( H \) is connected and that \( |V| \) is odd. The 3-graph in Figure 1 shows that these conditions are not sufficient.

**Figure 1.** Smallest connected 3-graph on an odd number of vertices without a spanning tree. (Shaded triangles are triples.)

Our first non-trivial condition for the existence of spanning trees is a sufficient one and is as follows.

**Theorem 3.3.** Suppose \( H = (V, \Delta) \) is a 3-graph such that \( |V| \) is odd and each pair of vertices has multiplicity at least 1 in \( H \). Then \( H \) has a spanning tree.

**Proof.** Assume \( T \subseteq \Delta \) is a tree of maximum size and suppose that \( |T| < \frac{|V|-1}{2} \). Let \( U \subset V \) be the set of vertices not spanned by \( T \). Then \( |U| \) is even, containing at least two vertices \( u, v \). Since there is some triple containing \( \{u, v\} \), there is \( w \in V \) such that \( u,v,w \in \Delta \) and in fact \( w \in U \) for otherwise we could add the triple \( uvw \) as a leaf to \( T \) and obtain a larger tree of \( H \).

Set \( S = \{uvw\} \), vertex-disjoint from \( T \). For any leaf \( abc \) of \( T \) with vertices \( a, b \) of degree 1 in \( T \) there is a triple \( uai \) containing the pair \( \{u, a\} \). By the remark in the previous paragraph \( i \in V \setminus U \). If \( uai \) is a triple for some \( i \neq b \) then deleting \( abc \) from \( T \) and adding the triples \( uai \) and \( uvw \) gives a larger tree, contradicting the fact that \( T \) has maximum size. So we may assume that the only triple that contains \( u \) and at least one of \( a, b \) is \( uab \), and that this is true for every leaf \( abc \) of \( T \). We then remove all the leaves \( abc \) of \( T \) and put the triples \( uab \) in \( S \).

We repeat this argument, at each stage looking at triples containing \( u \) and vertices of degree 1 in the leaves of what is left of the initial tree \( T \). There are just two possible outcomes: either (i) at some stage we can join the remaining subtree of \( T \) and the tree \( S \) containing \( uvw \) by a triple to make a larger tree than the original tree \( T \), or (ii) we remove all the leaves of \( T \) and end up with a larger tree \( S \) that spans all but one of the vertices that are spanned by \( T \) and also the vertices \( u, v, w \). Both possibilities contradict the hypothesis that \( T \) has maximum size.
Hence the maximum tree $T$ spans all the vertices of $H$, i.e., $T$ is a spanning tree of $H$. □

An extremal case of Theorem 3.3 is when each pair of vertices is contained in exactly one triple, i.e., $H$ is a Steiner triple system. The condition on the multiplicity of pairs of vertices implies that a Steiner triple system on $n$ points also has the property that every vertex is of degree $\frac{n-2}{2}$, and that $n$ is congruent with 1 or 3 modulo 6. R.M. Wilson [25] showed that the number of non-isomorphic Steiner triple systems on $n \equiv 1$ or $3 \pmod{6}$ points lies between $(e^{-5}n)^{n^2/12}$ and $(e^{-\frac{2}{3}}n)^{n^2/6}$. (Given the truth of the then conjecture of Van der Waerden on the size of permanents, Wilson improved the lower bound, and further conjectured that the actual number is in fact asymptotically $(e^{-\frac{2}{3}}n)^{n^2/6}$.) There is just one isomorphism class for $n \in \{3, 7, 9\}$, two for $n = 13$, eighty for $n = 15$

For Steiner triple systems we can not only assert the existence of a spanning tree but also give a superexponential lower bound on the number of spanning trees.

**Theorem 3.4.** If $H = (V, \Delta)$ is a Steiner triple system on $|V| = n$ vertices then $H$ has $\Omega((n/6)^{n/12})$ spanning trees.

**Proof.** Brouwer [4] proved that any Steiner triple system on $n$ vertices has a transversal (set of pairwise disjoint triples) covering all but $5n^2/3$ vertices, and Alon, Kim and Spencer [2] improved this to all but $O(n^{1/2} \ln^{3/2}n)$ vertices. Let $P \subseteq \Delta$ be such a set of pairwise disjoint triples that together cover $U \subseteq V$, with $|U| = n - k$ and $k = o(n)$. Let $r = (n - 1)/2$.

We give a procedure that generates $\prod_{i=0}^{s} (r - k - 1 - 6i)$ spanning trees, where $s$ is the largest integer such that $r - k - 1 - 6s > 0$ ($s$ is $n/12 - o(n)$). Unfortunately, this procedure may give repeated trees; we then show that each tree cannot appear more than $n/6$ times. Recall that in a Steiner triple system every vertex belongs to $r$ triples. In $H_U$ every vertex belongs to at least $r - k$ triples. Let $u_0$ be a vertex of $U$. The construction of a spanning tree consists in first using $P$ to construct a “comb-like” tree of $H_U$ and then extending this tree to a spanning tree of $H$. So let us begin by considering the restriction $H_U$. Let $t_0$ be any triple containing $u_0$, subject only to the condition that $t_0 \notin P$. Say $t_0 = \{u_0, u'_0, u''_0\}$. Let $p_0, p_1, p_2$ be the triples in $P$ that contain $u_0, u'_0, u''_0$, respectively. Clearly the triples $t_0, p_0, p_1, p_2$ form a tree $T_0$. Let $w_1$ be any of the four vertices in $(p_1 \cup p_2) \setminus t_0$. There are at least $r - k - 7$ triples that contain $w_1$ but no other vertex of $T_0$. Let $t_1 = \{u_1, u'_1, u''_1\}$ be any one of them. Let $p_3$ and $p_4$ be the triples in $P$ that contain $u'_1$ and $u''_1$, respectively. Let $T_1 = T_0 \cup \{t_1, p_3, p_4\}$. We proceed recursively in this way as long as $r - k - 1 - 6i$ is positive: we choose $u_i$ to be any of the four vertices in $p_{2i-1} \cup p_{2i}$ that are not in $t_{i-1}$ and we choose $t_i = \{u_i, u'_i, u''_i\}$ a triple containing $u_i$ and no other vertex in $T_{i-1}$. Then we take the two triples $p_{2i+1}, p_{2i+2}$ in $P$ that contain $u'_i, u''_i$ and set $T_i = T_{i-1} \cup \{t_i, p_{2i+1}, p_{2i+2}\}$.

Once we have a tree $T_s$, covering $6s + 9$ vertices, we need to complete it to a spanning tree of $H$. We repeatedly use the following claim.

**Claim.** Let $T$ be a tree of $H$ and let $W$ be the set of vertices not spanned by $T$. Then there are vertices $a, b$ of $W$ such that the triple that contains them has its third vertex in $T$.

**Proof of the claim.** Suppose it were not the case. Then the triples on $W$ would form a Steiner triple system. But since $W$ has even cardinality this is impossible. □

Therefore, by adding a leaf at a time, we can complete $T_s$ to a spanning tree of $H$. There may be many ways of completing $T_s$, but we just take one of them arbitrarily.

If we fix the starting vertex $u_0$, by applying the procedure just described we obtain $4^s \prod_{i=0}^{s} (r - k - 1 - 6i)$ spanning trees of $H$. Indeed, at step $i$ we need to choose one of four vertices and then we know that this vertex belongs to at least $r - k - 1 - 6i$ triples that are contained in $H_U$ but do not contain any vertex already in the tree. It could be, however, that the same tree is produced several times. For instance, the tree in Figure 2 could appear in two different ways.

Next we bound the number of possible repetitions of a given spanning tree $T$. Let us first colour the triples of $T$ in the following way. The triples from $P$ are coloured blue; the triples
entirely contained in $U$ and that intersect three triples of $P$ are coloured red, and the remaining triples are coloured green. Observe that green triples are only included in the final stage of the construction of a tree (when the claim is used), whereas blue and red triples can appear both during the first steps of the construction and also at the end.

The skeleton of $T$ is the graph whose vertices are the red triples and where two vertices are adjacent if there is a blue edge in the tree intersecting the corresponding red triples in different vertices. The skeleton is a forest (it will be a tree if there is only one red triple containing $u_0$); root each component of the forest at the vertex corresponding to the triple that contains $u_0$. Observe that the skeleton contains at least one rooted path of length $s$. If by the above procedure the same tree is produced more than once, the corresponding skeleton has at least two different rooted paths of length $s$. The skeleton of a spanning tree contains at most $n/6$ vertices, since in the tree every red triple has two blue triples attached. There are at most $n/6 - s$ vertices that can be the end of a rooted path of length $s$. Since we are only interested in a lower bound for the number of trees, certainly there are no more than $n/6$ rooted paths of length $s$ in the skeleton, so each tree is produced at most $n/6$ times.

Therefore the number of spanning trees of a Steiner triple system is a least

$$4^s \prod_{i=0}^{s} (n-k-1-6i)$$

This is $\Omega((n-k-1)!^{1/6})$ and since $k = o(n)$ we thus have $\Omega((n/2)^{1/6})$ spanning trees, which by Stirling’s approximation gives the statement of the theorem.

We now return to the question of the existence of spanning trees and will this time present a necessary condition. Consider again a 3-graph $H = (V, \Delta)$ with underlying graph $G = (V, E)$. The hypergraph obtained from $H$ by deleting vertices in $S \subseteq V$ is denoted by $H - S$. This may contain hyperedges of size 1, 2 or 3. The underlying graph $G - S$ consists of triangles for each triple, edges for each pair, and isolated vertices for each singleton of $H - S$. A connected component of $H - S$ corresponds exactly to a connected component of the graph $G - S$. Let $q(H - S)$ denote the number of connected components of $H - S$ spanning an odd number of vertices, which is also equal to $q(G - S)$, the number of odd connected components of $G - S$. We shall use $q(H - S)$ and $q(G - S)$ interchangeably.

**Theorem 3.5.** If $H = (V, \Delta)$ has a spanning tree then $q(H - S) \leq |S| - 1$ for each non-empty $S \subseteq V$.

**Proof.** Given that $H$ has a spanning tree $T$, $|V|$ is odd. Since $q(H - S) \leq q(T - S)$ it suffices to prove that $q(T - S) \leq |S| - 1$ for each non-empty $S \subseteq V$. Beginning with $|S| = 1$, take $S = \{v\}$.

---

**Figure 2.** A spanning tree and its skeleton (here $s = 2$). Red triples are shown as thick long lines, blue triples are thin dashed lines and green triples are depicted as bags.
and root $T$ at $v$. To each triple $abv$ of $T$ rooted at $v$ there corresponds a branch of $T$ comprising all triples that lie on a path from $v$ that starts with the triple $abv$. Denote this branch by $T_{ab}$. The 3-graph $T_{ab}$ is a tree. Removing $v$ from $T$ creates a connected component $T_{ab} - v$ for each $abv \in T$. Each hypergraph $T_{ab} - v$ spans an even number of vertices. Hence the statement of the theorem is true for any 3-graph $H$ when $|S| = 1$. Assume as induction hypothesis that the statement is true for all 3-graphs $H$ and sets $S$ with $|S| \leq k$, where $1 \leq k \leq |V| - 1$. Suppose $|S| = k$ and take $v \in V \setminus S$. By hypothesis $q(H - S) \leq q(T - S) \leq k - 1$ and we wish to prove that $q(T - S - v) \leq k$.

Root $T$ at $v$ as before. For each $abv \in T$ define $S_{ab} = V(T_{ab}) \cap S$. Possibly $S_{ab} = \emptyset$, in which case $q(T_{ab} - S_{ab}) = 1$ and upon removing $v$ we obtain one even component $T_{ab} - v$. The non-empty sets $S_{ab}$ partition $S$. By induction hypothesis, if $S_{ab} \neq \emptyset$ then $q(T_{ab} - S_{ab}) \leq |S_{ab}| - 1$ for each $abv \in T$. The vertex $v$ belongs to a unique component of $T_{ab} - S_{ab}$ for each $abv \in T$ and furthermore has degree 1 in $T_{ab}$. Removing $v$ from $T_{ab}$ therefore creates no new components in $T_{ab} - S_{ab}$ and switches the parity of the size of the component of $T_{ab} - S_{ab}$ that contains $v$. Hence

$$q(T - S - v) \leq \sum_{ab : abv \in T} (|S_{ab}| - 1) + \# \{ab : abv \in T, S_{ab} \neq \emptyset\} = \sum_{ab : abv \in T, S_{ab} \neq \emptyset} |S_{ab}| = |S| = k.$$ 

This completes the inductive step. \hspace{1cm} \Box

The condition of Theorem 3.5, although necessary for the existence of a spanning tree of a 3-graph is not sufficient, unlike its counterpart for perfect matchings of graphs (Tutte’s 1-factor theorem). The following lemma implies that if we can find a 3-graph $H$ whose underlying graph $G$ is Hamiltonian then $H$ satisfies the conclusion of Theorem 3.5.

**Lemma 3.6.** Let $G = (V, E)$ be a graph with an odd number of vertices. If $G$ is Hamiltonian then $q(G - S) \leq |S| - 1$ for each non-empty $S \subseteq V$.

**Proof.** Removing $S$ from $G$ creates at most $|S|$ connected components since this is true of the Hamiltonian cycle of $G$. Therefore if the condition $q(G - S) \geq |S|$ holds for some $S$ then there must be equality. Since $q(G - S)$ has the same parity as $|V| - |S|$ and $|V|$ is odd, $q(G - S)$ has parity opposite to $|S|$, and hence equality is impossible. \hspace{1cm} \Box

Figure 3 gives examples of Hamiltonian graphs that underlie 3-graphs without a spanning tree, thereby showing that the condition of Theorem 3.5 is not sufficient. We will see in a moment why these 3-graphs have no spanning trees.

Recall that for $H$ to have a spanning tree its underlying graph $G$ must be connected. A block of a connected graph $G$ is a maximal 2-connected subgraph.

**Proposition 3.7.** Suppose the underlying graph $G$ of a 3-graph $H$ has a block that spans an even number of vertices. Then $H$ has no spanning tree.

**Proof.** Given a spanning tree $T$ of $H$ and block $B$ of $G$, the restriction of $T$ to the block $B$ is a tree spanning the vertices of $B$. Therefore $B$ has an odd number of vertices. \hspace{1cm} \Box

The parity observation behind Proposition 3.7 can be extended to give a more general necessary condition for the existence of a spanning tree.

Let $H = (V, \Delta)$ be a 3-graph. Given subsets $V_1, \ldots, V_k$ of $V$, consider the induced sub-3-graphs $H_i = (V_i, \Delta_i)$, where $\Delta_i = \{abc \in \Delta : a, b, c \in V_i\}$. Suppose moreover that the $\Delta_i$ form a partition of $\Delta$. If $H$ has a spanning tree $T$, this spanning tree restricted to $H_i$ yields a spanning
A particularly simple case is when all the sets \( U_i \) have size two. Since the only star-partitions of a set of two vertices are either two isolated vertices or an edge, depending on the parity, the converse can be used to determine whether a 3-graph has no spanning tree. Given \( U_i \) as above, if there is no tree on \( U \) that is a union of star-partitions then \( H \) has no spanning tree. Even if there is such a tree, sometimes the non-existence of a spanning tree can be inferred by showing that the structure of the required star-partitions cannot be obtained from the 3-graph.

For instance, let us use this method to show that the 3-graph on the right of Figure 3 has no spanning tree. Let \( H_1 \) and \( H_2 \) be the two sub-3-graphs isomorphic to the 3-graph illustrated on Figure 3. Then the sets \( U_1 \) and \( U_2 \) are equal, and consist of the three vertices that are common to both sub-3-graphs. Since both \( H_1 \) and \( H_2 \) have an odd number of vertices, the only possible star-partitions for \( U_1 \) and \( U_2 \) are a star \( K_{1,2} \) or three isolated vertices. For the union of two such star-partitions to be a tree, the only possibility is to take one of each. Hence there must be a spanning forest of the 3-graph in Figure 3 in which the three white vertices belong to the same component. But this forces the spanning forest to contain three triples that form a cycle.

A particularly simple case is when all the sets \( U_i \) have size two. Since the only star-partitions of a set of two vertices are either two isolated vertices or an edge, depending on the parity,
it is straightforward to check whether there is a tree that is a union of star-partitions. (This argument applies to the 3-graph on the left of Figure 3.)

4. Oriented spanning trees and 3-Pfaffian orientations

4.1. Orientations of spanning trees. An orientation of a finite subset of $\mathbb{N}$ is an order up to even permutation. The canonical orientation takes elements in the order consistent with the order $1 < 2 < 3 < \cdots$ on $\mathbb{N}$.

A $(2n+1)$-cycle $(s(1) \, s(2) \, \cdots \, s(2n+1))$ determines an orientation $s(1), s(2), \ldots, s(2n+1)$ of $[2n+1]$ given by the permutation $s$. The permutation $s$ is an even or odd permutation according as it determines the same or opposite orientation of $[2n+1]$ to the canonical orientation $1 < 2 < \cdots < 2n+1$.

Suppose $H = ([2n+1], \Delta)$ is a 3-graph. A triple $ijk \in \Delta$ can be assigned one of two orientations (order up to even permutation), or, what is the same thing here, a cyclic order, either $(i \, j \, k)$ or $(j \, i \, k)$. If $i < j < k$ then the canonical orientation is defined by taking the cyclic order $(i \, j \, k)$. In other words, given $i < j$, the triple orientation $(i \, j \, k)$ is the canonical one if $k \notin \{i+1, \ldots, j-1\}$, while if $k$ lies between $i$ and $j$ then the orientation $(i \, j \, k)$ is opposite to the canonical orientation of $ijk$.

Definition 4.1. Suppose that we are given an orientation $\omega$ of the triples of a 3-graph $H = ([2n+1], \Delta)$. To each $t = ijk \in \Delta$ is associated a cyclic permutation $\sigma(t, \omega)$ given by $(s(1) \, s(2) \, \cdots \, s(2n+1))$. As shown by Masbaum and Vaintrob [18], if $T$ is a spanning tree of $H$, the product

$$\prod_{t \in T} \sigma(t, \omega)$$

is a $(2n+1)$-cycle $(s(1) \, s(2) \, \cdots \, s(2n+1))$. The orientation of $T \in T(H)$ associated with the triple orientation $\omega$ is the order up to even permutation of vertices taken in the order $s(1), s(2), \ldots, s(2n+1)$ given by the cycle. The sign of the spanning tree $T$, $\text{sgn}(T, \omega)$ is the sign of the permutation $\sigma$.

It is also shown by Masbaum and Vaintrob [18] that the permutation $s$ in Definition 4.1 is determined up to conjugation by even permutations: not only does it not matter which of the $2n+1$ ways the cycle $(s(1) \, s(2) \, \cdots \, s(2n+1))$ is written, but its sign is also independent of the order in which the factors are taken in the product over triples of $T$.

We fix the notation $\omega_0$ for the canonical orientation on each triple $ijk$ given by the cycle $(i \, j \, k)$ consistent with the natural order $i < j < k$.

For two triple orientations $\omega_1$ and $\omega_2$ of $H$ and $T \in T(H)$ we have

$$\text{sgn}(T, \omega_2) = (-1)^{\# \{t \in T : \sigma(t, \omega_1) \neq \sigma(t, \omega_2) \}} \text{sgn}(T, \omega_1).$$

A convenient way to calculate the sign of a spanning tree is as follows and illustrated in Figure 6. Given a spanning tree $T$ of $H$ and a triple orientation $\omega$, embed the underlying graph...
of $T$ in the plane so that boundaries of the interior faces are the triples of $T$ and so that the vertices of a triple $ijk$ appear in anticlockwise order consistent with the triple orientation $\omega$. Starting at an arbitrary vertex, tour the tree in an anticlockwise sense, reading off a cyclic string of 3 vertex labels. Remove repeated vertex labels until a cyclic string of length $2n+1$ remains, equal to $(s(1) s(2) \cdots s(2n+1))$ for some permutation $s$ of $[2n+1]$. Then the sign of $s$ as a permutation is equal to $\text{sgn}(T, \omega)$.

**Figure 6.** Embeddings of two spanning trees of $K_7^{(3)}$ on vertex set $[7] = \{1, 2, 3, 4, 5, 6, 7\}$. The left-hand tree has oriented triples $(1 2 4)$, $(2 7 6)$ and $(3 6 5)$. The right-hand tree has oriented triples $(1 2 4)$, $(3 7 4)$ and $(4 6 5)$. The linear order on $[7]$ given below each tree is obtained by taking the vertex labels the first time we encounter them, but other orders are possible by taking vertices later than at their first appearance (there is an even number of intermediate vertices between any two appearances of a given vertex). This order of appearance is then written as a permutation of $[7]$, whose sign gives the sign of the tree under the given triple orientation.

For a given orientation $\omega$ of triples of $H = ([2n+1], \Delta)$, let $T^+(H) = \{T \in \mathcal{T}(H) : \text{sgn}(T, \omega) = +1\}$ and $T^-(H) = \{T \in \mathcal{T}(H) : \text{sgn}(T, \omega) = -1\}$. These sets will be denoted by $T^+$ and $T^-$ respectively when $H = K_{2n+1}^{(3)}$ is complete.

For $S \subseteq \Delta$ define $(i j)S$ to be the set obtained from $S$ by switching $i$ and $j$ in triples containing either of these two vertices. If $i$ and $j$ have the property that $\{t-\{i\} : t \in \Delta, i \in t\} = \{t-\{j\} : t \in \Delta, j \in t\}$ then this action set-stabilizes $\Delta$. Furthermore, if $T$ is a spanning tree of $H$ then in this situation $(i j)T$ is also a spanning tree of $H$. Under the canonical orientation, the sign of $(i j)T$ is related to the sign of $T$ in a particularly straightforward way when $j = i+1$:

**Lemma 4.2.** Let $H = ([2n+1], \Delta)$ be a 3-graph with canonical orientation of its triples. Suppose that $i \in [2n]$ has the property that $\{t-\{i\} : t \in \Delta, i \in t\} = \{t-\{i+1\} : t \in \Delta, i+1 \in t\}$. Then $(i i+1)T$ is a spanning tree of $H$ with opposite sign to that of $T$ if $\{i, i+1\}$ is not contained in any triple of $T$, while $(i i+1)T$ has the same sign as $T$ if some triple of $T$ contains $\{i, i+1\}$.

**Proof.** Let us start with a fixed embedding of $T$ in the plane, as described above. Interchanging the labels $i$ and $i+1$ in the embedding gives an embedding of $(i i+1)T$ in which all triples appear in anticlockwise order if and only if there is no triple containing both $i$ and $i+1$. If
this is the case, when touring the embedding of \((i, i + 1)T\) in anticlockwise order, we obtain the same permutation of the vertices as when touring \(T\), except that elements \(i\) and \(i + 1\) are transposed. Hence clearly the sign of \((i, i + 1)T\) is opposite to that of \(T\). If \(T\) contains a (necessarily unique) triple \(\{i, i + 1, j\}\), consider the orientation \(\omega\) that agrees with \(\omega_0\) everywhere except in the triple \(\{i, i + 1, j\}\). Then \(\text{sgn}((i, i + 1)T, \omega_0) = -\text{sgn}((i, i + 1)T, \omega)\) by equation (1) and \(\text{sgn}((i, i + 1)T, \omega) = -\text{sgn}(T, \omega_0)\) by the same argument about touring the embedding as before.

The involution \(T \mapsto (i, i + 1)T\) of Lemma 4.2 specializes to the sign-reversing involution on perfect matchings on \([2n]\) of [21] Lemma 2.1 when applied to 3-graphs in which every triple contains the vertex \(2n + 1\) (where spanning trees of the 3-graph correspond precisely to perfect matchings on \([2n]\)).

**Theorem 4.3.** The distribution of positive and negative spanning trees of \(K^{(3)}_{2n+1}\) under the canonical orientation is given by

\[|T^+| - |T^-| = (2n + 1)^{n-1}.\]

**Proof.** Let \(T_i\) denote the set of trees that have a triple containing \(\{2i-1, 2i\}\). By Lemma 4.2 the involution \(\tau_i : T \mapsto (2i-1 2i)T\) reverses the sign of trees in \(T \setminus T_i\). If \(T \in T_j\) then \((2i-1 2i)T \in T_j\), since the pairs \(\{2i-1, 2i : i \in [n]\}\) are pairwise disjoint. So for each \(j \in [n]\) the restriction of \(\tau_i\) to \(T_j\) is a map \(T_j \to T_j\) reversing the sign of trees in \(T_j \setminus T_i\). (On \(T_i\) itself \(\tau_i\) fixes the sign of every tree.) Hence

\[|T^+| - |T^-| = \sum_{T \in T} \text{sgn}(T, \omega_0) = \sum_{T \in T \cap T_0 \cap \cdots \cap T_n} \text{sgn}(T, \omega_0).\]

A tree belonging to \(\bigcap_{i \in [n]} T_i\) has set of triples equal to \(\{\{2i-1, 2i, f(i) : i \in [n]\}\) for some function \(f : [n] \to [2n+1]\), (with \(f(i) = 2n + 1\) for at least one value of \(i\)). The canonical orientation of a triple \(\{2i-1, 2i, f(i)\}\) is \((2i-1 2i f(i))\), no matter whether \(f(i) > 2i\) or \(f(i) < 2i-1\). To show that a tree \(T \in \bigcap_{i \in [n]} T_i\) is positively oriented under the canonical orientation \(\omega_0\), we embed it \(T\) in the plane so that the vertices of a triple appear in anticlockwise order consistent with the orientation \(\omega_0\). Traversing the tree anticlockwise starting at vertex \(2n + 1\), the vertices appear, for some permutation \(\pi\) of \([n]\), in the order \(2n + 1, 2\pi(1)-1, 2\pi(1), 2\pi(2)-1, 2\pi(2), \ldots, 2\pi(n)-1, 2\pi(n)\).

This is an even permutation of \(1, 2, 3, 4, \ldots, 2n - 1, 2n, 2n + 1\).

To evaluate \(\bigcap_{i \in [n]} T_i\), use the “Prüfer code” described in the proof of Theorem 3.1, in which the perfect matching \(M\) is fixed equal to \(\{\{2i-1, 2i\} : i \in [n]\}\). Trees in \(\bigcap_{i \in [n]} T_i\) are in bijective correspondence with sequences \(\gamma \in [2n+1]^{n-1}\).

**4.2. Tree generating polynomials.** Let \(y = (y_t : t \in \Delta)\) be a set of commuting indeterminates indexed by triples of the sub-3-graph \(H = ([2n+1], \Delta)\) of \(K^{(3)}_{2n+1}\). (Here we depart from Masbaum and Vaintrob [18], but follow for example Caracciolo et al. [5], by indexing the indeterminates by triples rather than oriented triples. In other words, \(y_{ijk} = y_{i,j,k} = y_{jik}\), and so on.) The tree generating polynomial of \(H\) is defined by

\[\mathcal{P}(H, y) = \sum_{T \in T(H)} \prod_{t \in T} y_t.\]

The signed tree generating polynomial associated with an orientation \(\omega\) of the edges is defined by

\[\mathcal{P}^{\omega}(H, y) = \sum_{T \in T(H)} \text{sgn}(T, \omega) \prod_{t \in T} y_t.\]

By equation (1) in the previous subsection, the polynomial \(\mathcal{P}^{\omega}(H, y)\) is related to \(\mathcal{P}^{\omega_0}(H, y)\) by substituting \(-y_t\) for \(y_t\) for triples \(t\) on which \(\omega\) is opposite to \(\omega_0\).
Define the antisymmetric \((2n+1) \times (2n+1)\) matrix \(\Lambda\) with \((i,j)\) entry given by

\[
\Lambda_{i,j} = \sum_{k \neq i,j} \epsilon_{i,j,k} y_{i,j,k},
\]

where \(\epsilon_{i,j,k} = +1\) if \((i \ j \ k)\) is a cyclic permutation of \(i < j < k\), \(\epsilon_{i,j,k} = -1\) if \((i \ j \ k)\) is opposite to this canonical orientation, and \(\epsilon_{i,j,k} = 0\) if two of the indices are equal. Let \(\Lambda^{(k)}\) denote the matrix obtained from \(\Lambda\) by deleting row \(k\) and column \(k\). The following is the Pfaffian matrix-tree theorem of Masbaum and Vaintrob.

**Theorem 4.4.** [18] For any \(k \in [2n+1]\) the signed tree polynomial associated with the canonical orientation \(\omega_0\) is given by

\[
\mathcal{P}^{\omega_0}(K^{(3)}_{2n+1}, y) = (-1)^{k-1} \operatorname{Pf}(\Lambda^{(k)}).
\]

An orientation \(\omega\) of the triples of \(K^{(3)}_{2n+1}\) restricted to \(\Delta \subseteq \binom{[2n+1]}{3}\) gives an orientation of the sub-3-graph \(H = ([2n+1], \Delta)\); the signed tree polynomial \(\mathcal{P}^{\omega}(H, y)\) is obtained from \(\mathcal{P}^{\omega_0}(K^{(3)}_{2n+1}, y)\) upon setting \(y_t = 0\) if \(t \not\in \Delta\).

**Definition 4.5.** An orientation \(\omega\) of the triples of a 3-graph \(H\) is 3-Pfaffian if \(\operatorname{sgn}(T, \omega)\) is constant for \(T \in \mathcal{T}(H)\). A 3-graph is said to be 3-Pfaffian if there exists some 3-Pfaffian orientation of its triples.

See Subsection 6.2 for some examples of 3-Pfaffian and non-3-Pfaffian 3-graphs.

For a 3-Pfaffian orientation \(\omega\) of \(H\), \(\mathcal{P}^{\omega}(H, y) = \pm \mathcal{P}(H, y)\); in particular, in this case by Theorem 4.4 the number of spanning trees of \(H\) will be computable in polynomial time by the evaluation of \(\mathcal{P}^{\omega}(H; 1)\) (setting \(y_t = 1\) for each \(t \in \Delta\)). To evaluate \(\mathcal{P}^{\omega}(H, 1)\) from \(\mathcal{P}^{\omega_0}(H, y)\) set \(y_t = +1\) for triples \(t\) on which \(\omega\) is the same as \(\omega_0\) and \(y_t = -1\) when \(\omega\) is opposite to \(\omega_0\) on \(t\).

The correspondence from Lemma 6.2 between spanning trees and perfect matchings \(M\) of \(G - \{2n+1\}\) together with an “apex-choosing” function \(f : M \to [2n]\) is used by Hirschman and Reiner [10] to prove a useful alternative formulation of the Masbaum–Vaintrob theorem.

**Theorem 4.6.** [10] For a 3-graph \(H = ([2n+1], \Delta)\),

\[
\mathcal{P}^{\omega_0}(H, y) = \prod_{M \in \mathcal{M}} \operatorname{sgn}(M) \prod_{i<j, f(ij) \in M} \epsilon_{i,j,f(ij)} y_{i,j,f(ij)},
\]

where \(\operatorname{sgn}(M)\) is the sign of the perfect matching \(M\), given by

\[
\operatorname{sgn}(M) = (-1)^{\text{cross}(M)},
\]

\[
\text{cross}(M) = \#\{i < j < k < l : \{i, k\}, \{j, l\} \in M\}.
\]

An orientation of the edges of a graph \(G = ([2n], E)\) is Pfaffian if for all perfect matchings \(M\) of \(G\) the quantity

\[
\operatorname{sgn}(M) \cdot (-1)^{\#\{i < j : j \to i\}}
\]

is constant, where \(\operatorname{sgn}(M)\) is defined as in the previous theorem and \(j \to i\) denotes an oriented edge with \(j\) directed towards \(i\).

As a straightforward application, Theorem 4.6 yields a simple criterion for 3-graph \(H\) to be 3-Pfaffian when \(H\) has the property that all its triples contain a common vertex — in the terminology of Section 5 below, that is to say when the 3-graph \(H\) is the 1-suspension of an ordinary graph.

**Theorem 4.7.** Let \(G\) be a graph on vertex set \([2n]\) and edge set \(E \subseteq \binom{[2n]}{2}\). Let \(H\) be the 3-graph on vertex set \([2n+1]\) and triple set \(\Delta = \{ij(2n+1) : ij \in E\}\).

Then \(H\) has a 3-Pfaffian orientation if and only if \(G\) has a Pfaffian orientation: if edge \(ij\) has orientation \(i \to j\) in the Pfaffian orientation of \(G\) then the triple orientation given by \((i \ j \ 2n+1)\) defines a 3-Pfaffian orientation of \(H\), and conversely.
Proof. Spanning trees of $H$ are in one-one correspondence with perfect matchings of $G$. As described in the statement of the theorem, orientations of the edges of $G$ are also in one-one correspondence with orientations of the triples of $H$. Let us denote by $\overset{\omega}{\rightarrow}$ the orientation of $G$ corresponding to orientation $\omega$ of $H$, i.e., $i \overset{\omega}{\rightarrow} j$ if and only if the triple $ij(2n + 1)$ is oriented $(i \ j \ 2n + 1)$ in $\omega$. Recall that $P^\omega(H, y) = P^\omega(H, y_t)$, where $y_t$ equals or is opposite to $y_t$ depending on whether $\omega$ and $\omega_0$ agree or not in $t$.

In the expansion of Theorem 4.6, the term $\epsilon_{i,j,2n+1}$ is constant equal to 1 if $i < j$, therefore,

$$P^\omega(H, y) = \sum_{\text{perfect matchings } M \text{ of } [2n]} \text{sgn}(M) \prod_{ij \in M \text{~or~ reverses all their signs.}} y_{ij(2n+1)}$$

$$= \sum_{\text{perfect matchings } M \text{ of } [2n]} \text{sgn}(M)(-1)^\# \{i < j \rightarrow i\}.$$

It is clear then that $\omega$ is a 3-Pfaffian orientation of $H$ if and only if $\overset{\omega}{\rightarrow}$ is a Pfaffian orientation of $G$. □

Recall that $\mathcal{T}(H)$ denotes the set of spanning trees of a 3-graph $H$ and that $\mathcal{T}(H \setminus abc) = \{T \in \mathcal{T}(H) : abc \notin T\}$ and there is a bijection between $\mathcal{T}(H/abc)$ and $\{T \in \mathcal{T}(H) : abc \in T\}$. If $abc$ is in no spanning tree of $H$ then $\mathcal{T}(H) = \mathcal{T}(H/abc)$. If $abc$ is in every spanning tree of $H$ then contracting the triple $abc$ defines a bijection from $\mathcal{T}(H)$ to $\mathcal{T}(H/abc)$.

Lemma 4.8. If a triple $abc$ occurs in no spanning tree of $H$ then $H$ is 3-Pfaffian if and only if $H/abc$ is 3-Pfaffian. Similarly, if $abc$ occurs in every spanning tree of $H$ then $H$ is 3-Pfaffian if and only if $H/abc$ is 3-Pfaffian.

Proof. The only thing to prove is that contracting a triple $abc$ either preserves the sign of all spanning trees $\{T \in \mathcal{T}(H) : abc \in T\}$ or reverses all their signs.

Let $V = [2n + 1]$ and $abc$ the triple to be contracted. By labelling the vertices suitably we may assume $\{b, c\} = \{2n, 2n + 1\}$: the property that all spanning trees have the same sign is unaffected by a permutation of vertex labels.

Embed a given tree $T \in \mathcal{T}(H)$ in the plane so that the orientation of triples corresponds to the anticlockwise order of its vertices. The anticlockwise appearance of vertices around $T$ up to cyclic permutation takes the form $AaBbCc$, where $A, B, C$ are each an even length sequence of vertices, and $A \cup B \cup C \cup \{a, b, c\}$ is a partition of $[2n + 1]$. Upon contracting $abc$ to a vertex with label $a$, a spanning tree of $H/abc$ on vertex set $[2n - 1]$ is obtained with vertices around the tree appearing in the order $AaBC$ up to cyclic permutation. The parity of $AaBbCc$ as a permutation of $[2n + 1]$ is the same as the parity of $AaBbCc$ since $B$ has even length. Since $b, c$ are greater than all the other vertex labels the parity of $AaBbCc$ is equal to that of $AaBC$ plus that of $bc$. Hence all spanning trees of $H$ have their sign multiplied by the sign of $bc$ as a permutation of $\{2n, 2n + 1\}$ when contracting the triple $abc$. □

Definition 4.9. A 3-graph $H = (V, \Delta)$ is minimally non-3-Pfaffian with respect to triple deletion and contraction if $H$ is non-3-Pfaffian and there is no $t \in \Delta$ such that $H \setminus t$ or $H/t$ is non-3-Pfaffian. (A 3-graph that has no spanning trees is vacuously 3-Pfaffian.)

Lemma 4.8 implies that in a minimal non-3-Pfaffian 3-graph (with respect to triple deletion and contraction) each triple occurs in at least one spanning tree and no triple occurs in all spanning trees.

Since the property of being 3-Pfaffian is preserved by deletion and contraction, if a 3-graph $H$ after deletion and contraction of triples gives a non-3-Pfaffian graph then $H$ must be non-3-Pfaffian. This is the same as restricting attention to spanning trees of $H$ that contain a given subset of triples (those that are contracted) and disjoint from another subset of triples (those deleted). More generally, if some subset of the class $\mathcal{T}(H)$ of all spanning trees of $H$ can be
shown to be impossible to make all the same sign then the same is true of the whole class \( T(H) \), i.e., \( H \) is non-3-Pfaffian.

4.3. Complexity results for orientations. As observed in the previous subsection, \( \mathcal{P}(H; 1) = |T(H)|, \) and \( \mathcal{P}^\omega(H; 1) = |T^+(H)| - |T^-(H)|. \) Consider the distribution of \( \mathcal{P}^\omega(H; 1) \) when \( \omega \) is a triple orientation chosen uniformly at random (u.a.r.). By equation (1), if we let \( y_t \) take values in \( \{-1, +1\} \) u.a.r. for \( t \in \Delta \) then the random variable \( \mathcal{P}^\omega(H; y) \) is equal to the random variable \( \mathcal{P}^\omega(H; 1) \) under an orientation \( \omega \) of triples taken u.a.r. The following lemma is analogous to the well-known result [10] that the expected value of the determinant of the skew adjacency matrix of a graph \( G \) (under all possible orientations of its edges) is equal to the number of perfect matchings of \( G \).

**Lemma 4.10.** Suppose \( H = ([2n+1], \Delta) \) is a 3-graph with a fixed orientation \( \omega \) of its triples. For each \( t \in \Delta \) let \( y_t \) take values in \( \{-1, +1\} \) independently uniformly at random, while \( y_t = 0 \) when \( t \notin \Delta \). Then

\[
\mathbb{E}[\mathcal{P}^\omega(H; y)] = 0,
\]

\[
\mathbb{E}[\mathcal{P}^\omega(H; y)^2] = |T(H)|.
\]

**Proof.** The random variables \( y_t \) for \( t \in \Delta \) are independent, each with expected value \( \mathbb{E}(y_t) = 0 \). For \( S \subseteq \Delta \) let \( y_S = \prod_{t \in S} y_t \). Then \( \mathbb{E}(y_T) = 0 \) for each spanning tree \( T \) and

\[
\mathbb{E}\left[ \sum_{T \in T(H)} \text{sgn}(T, \omega)y_T \right] = \sum_{T \in T(H)} \text{sgn}(T, \omega)\mathbb{E}(y_T) = 0.
\]

Also,

\[
\mathbb{E}\left[ \left( \sum_{T \in T(H)} \text{sgn}(T, \omega)y_T \right)^2 \right] = \sum_{S,T \in T(H)} \text{sgn}(S, \omega)\text{sgn}(T, \omega)\mathbb{E}(y_{S\Delta T}),
\]

where \( \mathbb{E}(y_{S\Delta T}) = \mathbb{E}(y_{S\Delta T}) \), for if \( t \in S \cap T \) then \( y_t^2 = 1 \). Since \( \mathbb{E}(y_{S\Delta T}) = 0 \) unless \( S \Delta T = \emptyset \) in which case \( \mathbb{E}(y_\emptyset) = 1 \) this yields

\[
\mathbb{E}\left[ \left( \sum_{T \in T(H)} \text{sgn}(T, \omega)y_T \right)^2 \right] = \sum_{T \in T(H)} \text{sgn}(T, \omega)^2 = |T(H)|.
\]

\( \square \)

Whereas counting (unsigned) spanning trees of 3-graphs (evaluating \( |T(H)| \)) is \#P-complete in general, the problem of evaluating \( |T^+(H)| - |T^-(H)| \) under any given triple orientation turns out to be polynomial time by Theorem 4.2 above, as it is the evaluation of the Pfaffian of a polynomial-size matrix with integer entries (each bounded in absolute value by \( 2n - 1 \)).

**Corollary 4.11.** A 3-graph \( H \) has a spanning tree, i.e., \( P(H; 1) = |T(H)| \neq 0 \), if and only if there is some triple orientation \( \omega \) of \( H \) such that \( \mathcal{P}^\omega(H; 1) = |T^+(H)| - |T^-(H)| \neq 0 \).

**Proof.** Clearly \( |T^+(H)| - |T^-(H)| \neq 0 \) implies the existence of a spanning tree. By Lemma 4.10 the variance of \( |T^+(H)| - |T^-(H)| \) is positive if and only if \( |T(H)| \neq 0 \). \( \square \)

If there is a point \( y \) such that \( \mathcal{P}^\omega(H; y) \neq 0 \) then \( H \) has a spanning tree. Caracciolo et al. [5] give an algorithm that runs in expected polynomial time for deciding the existence of a spanning tree. Since the polynomial \( \mathcal{P}^\omega(H; y) \) has \( \Delta \leq \binom{2n+1}{3} \) variables and total degree \( n \) the problem of deciding if it is non-zero can be solved in expected polynomial time by evaluating it at random points in a field \( \mathbb{F}_q \) of sufficiently large order \( q \geq 2n \).

We turn from the problem of deciding if there is a triple orientation for which the difference between positively and negatively oriented spanning trees is non-zero to the problem of whether there is a 3-Pfaffian orientation (for which all spanning trees have the same sign). The former
problem is polynomial time by Corollary 4.11 and the fact that deciding if there is a spanning tree is polynomial time. We do not know whether the problem of whether a 3-graph is 3-Pfaffian can be solved in polynomial time. (It is also unknown whether the problem of deciding if a graph is Pfaffian can be solved in polynomial time.)

However, a similar method of proof to that of Vazaran and Yannakakis [24] for Pfaffian orientations of graphs shows that the problem of deciding the existence of a 3-Pfaffian orientation is in co-NP. The main idea is to write a system of linear equations whose solutions are the 3-Pfaffian orientations of a 3-graph. We start by explaining this construction, which will be also be used later in the paper.

Let \( H = (V, \Delta) \) be a 3-graph and let \( \mathcal{T}(H) \) be its collection of spanning trees. Consider the triple–spanning tree incidence matrix \( M \in \mathbb{F}_2^{\mathcal{T}(H) \times \Delta} \) with \( (T, t) \) entry equal to 1 if \( t \in T \) and 0 otherwise. The rows of \( M \) are the indicator vectors in \( \mathbb{F}_2^\Delta \) of the triple sets of spanning trees \( T \in \mathcal{T}(H) \). The columns of \( M \) are the indicator vectors in \( \mathbb{F}_2^{\mathcal{T}(H)} \) of those trees that change orientation when the orientation of triple \( t \) is reversed (i.e., those trees containing \( t \)). Let \( c \in \mathbb{F}_2^{\mathcal{T}(H)} \) denote the indicator vector of tree orientations under the canonical orientation of edges, that is, for \( T \in \mathcal{T}(H) \) the \( T \)-component of \( c \) is 0 if \( \text{sgn}(T, \omega_0) = 1 \) and is 1 if \( \text{sgn}(T, \omega_0) = -1 \). There is some orientation of edges that leads to all trees \( T \in \mathcal{T}(H) \) having the same sign if and only if either of the equations

\[
Mx = c, \quad Mx = c + 1
\]

has a solution (\( x \) is the indicator vector of a subset of triples which when flipped in orientation change the tree orientations to have all positive signs or all negative signs, respectively).

**Theorem 4.12.** The problem of deciding whether a 3-graph has a 3-Pfaffian orientation is in co-NP.

**Proof.** According to the previous discussion, deciding whether a 3-graph \( H \) has a 3-Pfaffian orientation is equivalent to finding a solution of either of the equations \( Mx = c, \quad Mx = c + 1 \). The length of the vector \( c \) and the number of rows of \( M \) is \( |\mathcal{T}(H)| \), typically exponential in the number of vertices, say \( 2n + 1 \). However, the rank of \( M \) is polynomial on \( n \), since \( M \) has \( O(n^3) \) columns (one for each triple). If the system is inconsistent, basic linear algebra implies that there is a subset of rows of \( M \), say \( M' \), such that \( \text{rank}(M') < \text{rank}(M'|c') \), where \( c' \) is the restriction of \( c \) to the rows of \( M' \). Since \( \text{rank}(M'|c') \) cannot be more than the number of columns of \( M \) plus one, there is a polynomial time certificate that the equation \( Mx = c \) is inconsistent. Doing the same for the equation \( Mx = c + 1 \), one can verify in polynomial time that \( H \) has no 3-Pfaffian orientation.

In the next two sections we consider two special families of 3-graphs for which we can say more about the existence of 3-Pfaffian orientations.

### 5. Suspensions of Graphs and 3-Pfaffian Orientations

**Definition 5.1.** Let \( G = (V, E) \) be a graph and \( U \) a finite set disjoint from \( V \). Then the suspension of \( G \) from \( U \) is the 3-graph \( G^U = (U \cup V, \Delta) \) with set of triples \( \Delta = \{iju : ij \in E, u \in U\} \). If \( U \) has \( k \) elements then \( G^U \) is called a \( k \)-suspension of \( G \). (All \( k \)-suspensions of \( G \) are isomorphic.)

For the 3-graph \( G^U \) to have a spanning tree it is necessary that \( G \) has no isolated vertices and for \( |U| \) to have opposite parity to \( |V| \).

In this section we characterize those graphs \( G \) whose \( k \)-suspension has a 3-Pfaffian orientation.

The case of 1-suspensions has already been dealt with at the end of Subsection 4.2. A spanning tree \( T \) of a 1-suspension \( G^{(u)} \) consists of triples \( \{iju : ij \in M\} \), where \( M \) is a perfect matching of \( G \). In particular, if \( G \) has no perfect matching then \( G^{(u)} \) has no spanning trees. There is a bijective correspondence between orientations of triples of the 1-suspension \( G^{(u)} \) and orientations
of edges of $G$. If an edge $ij$ of $G$ is oriented $i \to j$, then the triple $iju$ has orientation given by the cyclic order $(i \ j \ u)$. By Theorem 4.7, the 1-suspension $G^{(u)}$ has a 3-Pfaffian orientation if and only if $G$ has a Pfaffian orientation.

For 3-suspensions and upwards, there is no orientation that makes all spanning trees have the same sign, unless of course there is no spanning tree.

**Theorem 5.2.** Let $G$ be a graph and $u,v,w \not\in V(G)$. If the 3-suspension $G^{(u,v,w)}$ has a spanning tree then it has no 3-Pfaffian orientation. For $k \geq 4$, the analogous result holds for the $k$-suspension of $G$.

**Proof.** Up to symmetry in $u,v,w$, a spanning tree of $G^{(u,v,w)}$ takes one of the following two forms:

(i) $\{uxa, vxb, wxc\} \cup \{iju : ij \in M_1\} \cup \{ijv : ij \in M_2\} \cup \{ijw : ij \in M_3\}$, where $M_1, M_2, M_3$ are matchings together spanning $G - \{a,b,c,x\}$, or

(ii) $\{uxa, vxb, vyc, wyd\} \cup \{iju : ij \in M_1\} \cup \{ijv : ij \in M_2\} \cup \{ijw : ij \in M_3\}$, where $M_1, M_2, M_3$ are matchings together spanning $G - \{a,b,c,d,x,y\}$.

Recall that a 3-Pfaffian 3-graph remains 3-Pfaffian after the deletion and contraction of triples; therefore, it is enough to show that after suitable contractions and deletions of $G^{(u,v,w)}$ we obtain a 3-graph that is not 3-Pfaffian.

Suppose first that $G^{(u,v,w)}$ has a spanning tree as in case (i). Fix the matchings $M_1, M_2, M_3$. Let $G_1$ be the graph on $\{a,b,c,x\}$ with edges $\{ax, bx, cx\}$ and let $H_1 = G_1^{(u,v,w)}$. The 3-graph $H_1$ is obtained from $G^{(u,v,w)}$ by contracting the triples $\{iju : ij \in M_1\} \cup \{ijv : ij \in M_2\} \cup \{ijw : ij \in M_3\}$ and deleting the triples that remain and do not belong to $G_1^{(u,v,w)}$.

The spanning trees of $G_1^{(u,v,w)}$ are $S_\pi = \{a\pi(u)x, b\pi(v)x, c\pi(w)x\}$, where $\pi$ ranges over the six permutations of $\{u,v,w\}$. Consider the order $a < b < c < u < v < w < x$ on the vertices of $G_1^{(u,v,w)}$ and let $\omega_0$ be the canonical orientation associated with this order. It is easy to see that the sign of $S_\pi$ under this orientation is the sign of the permutation

$$a \pi(u) b \pi(v) c \pi(w) x.$$

Therefore, three of the trees $S_\pi$ are positive and three are negative. Since each of the triples appears in exactly two of the trees, changing the orientation of any of the triples keeps the parity of the number of positive and negative trees. Thus, there is no orientation that makes all six trees the same sign, as needed.

If $G^{(u,v,w)}$ has a spanning tree as in case (ii), one argues analogously by considering the 3-graph $G_2^{(u,v,w)}$, where $G_2$ has edges $\{ax, bx, cy, dy\}$.

Finally, that a $k$-suspension $G^U$ of a graph is non-3-Pfaffian for $k \geq 3$ follows by a similar argument by permuting 3 of the vertices in $U$ while fixing the rest of the spanning tree.

The case of 2-suspensions is the richer one and occupies the rest of this section. The main result (Theorem 5.11) is a characterization of those graphs for which the 2-suspension is 3-Pfaffian in terms of forbidden subgraphs. This is similar in spirit to the result by Little [13] characterizing Pfaffian bipartite graphs as those without an even subdivision of $K_{3,3}$ with a perfect matching in the complement. Before stating and proving our characterization, we need another result akin to the theory of Pfaffian orientations. Recall that all perfect matchings of a graph have the same sign in a given orientation if and only if any cycle of even length whose complement has a perfect matching has an odd number of edges in each direction. Our goal is to establish a similar characterization of 3-Pfaffian orientations of 2-suspensions (Theorem 5.10). For this we need first to describe spanning trees of $H = G^{(u,v)}$ and their unions in terms of the graph $G$, so that conditions arise for all trees to have the same sign under a given orientation. From now on the 2-suspension is denoted by $G^{u,v}$. 

\[\square\]
Spanning trees of $G^{u,v}$ correspond to matchings $M_u$ and $M_v$ of $G = (V, E)$ with the property that $V(M_u) \cap V(M_v) = \{i\}$ for some single vertex $i$ and $V(M_u) \cup V(M_v) = V$. We call the subgraph $M_u \cup M_v$ a quasi-perfect matching of $G$. A quasi-perfect matching consists of a collection of independent edges and a single path on two edges, which together partition the vertices of $G$. A spanning tree $T$ of $G^{u,v}$ has triple set
$$\{iju : ij \in M_u\} \cup \{ijv : ij \in M_v\},$$
for some quasi-perfect matching $M_u \cup M_v$ of $G$.

Having described the spanning trees of $G^{u,v}$, we calculate their sign under a given orientation of $G^{u,v}$. Triple orientations of $G^{u,v}$ can be obtained from orientations of $G$, and vice versa. To do this we assume that the vertex set of $G$ is $[2n + 1]$ and that the vertex set of $G^{u,v}$ is ordered $1 < 2 < \cdots < 2n + 1 < u < v$. Recall that the canonical orientation of a triple $ijk$ takes $i, j, k$ in linear order up to even permutation.

Suppose we are given an orientation of triples of $G^{u,v}$. This orientation of triples is determined by its sign relative to the canonical orientation, positive or negative according as it has the same or opposite sense respectively. For each edge $ij$ of $G$ there are two triples of $G^{u,v}$, namely $iju$ and $ijv$. We call the edge $ij$ agreeing if the orientations of $iju$ and $ijv$ are both equal or both contrary to the canonical orientation; we call it opposite otherwise. The $u$-orientation of $G$ is the orientation of $G$ that orients the edge $ij$ with $i < j$ as $i \rightarrow j$ if the triple $iju$ has orientation $(ijk)$ and $j \rightarrow i$ otherwise. The notation $i \overset{u}{\rightarrow} j$ means that the $u$-orientation of the edge $ij$ is $i \rightarrow j$. Analogous notions are defined with respect to $v$.

**Lemma 5.3.** Let $T$ be a tree of $G^{u,v}$ with associated quasi-perfect matching $M_u \cup M_v$. Let $x y$ and $y z$ be the edges of the path of length 2 in $M_u \cup M_v$, with $x y \in M_u$, and let $i_1, j_1, \ldots, i_{n-1}, j_{n-1}$ be the other edges of $M_u \cup M_v$, written such that $i_\ell \overset{u}{\rightarrow} j_\ell$ or $i_\ell \overset{v}{\rightarrow} j_\ell$ depending on whether $i_\ell j_\ell$ belongs to $M_u$ or to $M_v$.

Then the sign of $T$ is the product of the sign of the permutation
$$\left(\begin{array}{cccccccc}
1 & 2 & 3 & \ldots & 2(n-1) & 2n-1 & 2n & 2n+1 & u & v \\
i_1 & j_1 & i_2 & j_2 & \ldots & j_{n-1} & u & x & y & z & v
\end{array}\right)$$
and $(-1)^{\alpha_u(xy) + \alpha_v(yz)}$ where $\alpha_u(xy) = 0$ if $x \overset{u}{\rightarrow} y$ and $\alpha_u(xy) = 1$ otherwise, and similarly for $\alpha_v(yz)$.

**Proof.** The formula follows from the definition of the sign of a tree in terms of the traversal of a planar embedding together with the fact that switching the orientation of one edge switches the sign of the tree.

More concretely, if we draw the planar embedding of the tree assuming that the triples $x y u$ and $y z v$ are oriented $(x y u)$ and $(y z v)$, respectively, and then we traverse the tree in anticlockwise sense, the permutation whose sign we need is
$$\left(\begin{array}{cccccccc}
1 & 2 & \ldots & 2\ell & 2\ell + 1 & \ldots & 2n+1 & u & v \\
i_1 & j_1 & \ldots & j_\ell & u & x & y & z & v & i_{\ell+1} & \ldots & j_{n-2} & i_{n-1} & j_{n-1}
\end{array}\right),$$
where we assume that the edges $i_1 j_1, \ldots, i_\ell j_\ell$ are the ones in $M_u$. This permutation and the one in the statement differ in an even number of transpositions hence they have the same sign. The term $(-1)^{\alpha_u(xy) + \alpha_v(yz)}$ collects the change of sign if the triples $x y u$ and $y z v$ are oriented differently.

The following lemma is an easy consequence, but it will be used often in the sequel. Given a subgraph $G'$ of $G$, its complement is the graph induced by the vertices not in $G'$, i.e., $G - V(G')$.

**Lemma 5.4.** If an edge $ij$ of $G$ is such that its complement has a quasi-perfect matching and $G^{u,v}$ has a $3$-Pfaffian orientation, then $ij$ is agreeing (in that orientation).
Lemma 5.4. Let \( Q \) be the quasi-perfect matching in the complement of \( ij \). There are many spanning trees of \( G^{u,v} \) that correspond to the quasi-perfect matching \( Q \cup \{ij\} \) of \( G \). Of all these trees, let \( T_1 \) and \( T_2 \) be two of them such that they only differ in that \( T_1 \) contains the triple \( iju \) and \( T_2 \) contains the triple \( ijv \). By Lemma 5.3 any orientation that gives the same sign to \( T_1 \) and \( T_2 \) must agree on \( ij \). □

In order to compare the sign of two spanning trees, we look at their union, which we next describe in terms of the associated quasi-perfect matchings. For the rest of this section, it will be convenient to consider that an edge is a cycle of length two. See Figure 7 for an illustration of the statement of the following lemma.

Lemma 5.5. Let \( Q_1 \) and \( Q_2 \) be two quasi-perfect matchings. Then the connected components of \( Q_1 \cup Q_2 \) are of the following types.

(C) A cycle of even length.
(H) Two edge-disjoint cycles with a path (possibly empty) with ends in the cycles.
(T) Three internally vertex-disjoint paths having common endpoints (including a cycle of odd length as a degenerate case).

Moreover, all connected components except one are of type (C), and the component of type (H) or (T) has an odd number of vertices.

Proof. Let \( p_1 \) and \( p_2 \) be the 2-paths in \( Q_1 \) and \( Q_2 \). If a connected component of \( Q_1 \cup Q_2 \) contains no edges from \( p_1 \) or \( p_2 \) then we are in case (C), since the component will result from the union of two matchings. So we focus on the component containing \( p_1 \). Colour the edges \( Q_1 \) blue and the edges of \( Q_2 \) red. An edge in \( Q_1 \cap Q_2 \) edge is both red and blue. Let \( xy, yz \) be the blue edges of \( p_1 \). If \( xy \) is also red, \( xy \) is a cycle of length two. Otherwise \( x \) must be incident to some red edge \( xx_1 \), since \( Q_2 \) is a quasi-perfect matching of \( G \). Similarly \( x_1 \) is incident to a blue edge \( x_1x_2 \), and so on, until some vertex \( x_k \) is repeated. (There may be a choice between two red edges along the way if the path \( p_2 \) is encountered when forming this cycle. If this is the case then an arbitrary choice of red edge is made.) The edge \( x_{k-1}x_k \) must be red, since every vertex is incident to at least one blue edge, and the only vertex incident to two blue edges is \( y \). If \( x_k = y \), we continue to explore the connected component from a red edge incident with \( x \) and eventually another cycle is closed. Otherwise we continue the component from \( y \). In both cases a second cycle is closed; as before the last edge added must be red, hence the vertex at which the second cycle is closed is the middle vertex of the path \( p_2 \). At this point all vertices in the component are incident with one edge of each colour, except for one or two vertices which are adjacent to two edges of the same colour and one or two of the other. So these are all the edges of \( Q_1 \cup Q_2 \) in this component.

Note that in particular both paths \( p_1 \) and \( p_2 \) are always in the same component. We are in case (T) or (H) according to whether the two cycles meet in an edge or not. Note that a particular example of (T) consists of a cycle of odd length, considering that one of the edges is a cycle of length 2.

The claim on the number of vertices follows from the fact that the total number of vertices is odd and components of type (C) have an even number of vertices. □

Note that if the graph \( G \) is bipartite the paths and cycles in the statement of Lemma 5.5 are all of even length.

By inspecting the \( u \)- and \( v \)-orientations of the edges in a component of the union of two quasi-perfect matchings of \( G \), we are able to characterize 3-Pfaffian orientations of \( G^{u,v} \) in terms of their behaviour on even cycles and some other small subgraphs of \( G \). To reach this characterization we require some further lemmas.

Given a graph with an orientation of its edges, a cycle of even length is said to be oddly oriented if when traversing it cyclically we encounter an odd number of edges oriented forward (and hence an odd number oriented backwards). By allowing cycles of length two, the next lemma is a generalization of Lemma 5.4 (A cycle of length two is always oddly oriented.)
**Figure 7.** Some examples of connected components of type (H) and (T) given in Lemma 5.3. For clarity, one quasi-perfect matching is depicted by a solid line, the other by a dashed line.

**Lemma 5.6.** Let $C$ be a cycle of even length in $G$ such that its complement contains a quasi-perfect matching $Q$. If a given orientation of $G^{u,v}$ is 3-Pfaffian, then all the edges of $C$ are agreeing and the cycle is oddly oriented (with respect to the given orientation).

**Proof.** That all the edges of $C$ are agreeing follows from Lemma 5.4 so we focus on the second claim. Let $a_1, b_1, \ldots, a_k, b_k$ be the vertices of $C$ in cyclic order. Construct a (partial) tree $T_Q$ of $G^{u,v}$ from $Q$ in the following way: if $Q = M \cup N$ for some matchings $M$ and $N$ of $G - C$, let $T_Q = \{ij : ij \in M\} \cup \{ij : v \in N\}$.

Now let $T_1$ be the tree having as triples $T_Q$ plus the triples $\{a_i b_i u : 1 \leq i \leq k\}$ and let $T_2$ be the tree whose triples are $T_Q$ together with $\{a_i b_{i-1} u : 2 \leq i \leq k\} \cup \{a_1 b_k u\}$. Assume the edges in $C$ are oriented cyclically, that is, $a_i \rightarrow b_i$ and $b_i \rightarrow a_{i+1}$. Then by Lemma 5.3 the trees $T_1$ and $T_2$ have opposite signs. Hence if an orientation gives both of them the same sign, an odd number of the edges in $C$ need to be reversed. \qed

**Lemma 5.7.** Let $xy$ and $yz$ be two edges of $G$ such that the complement of their union contains a perfect matching. In any 3-Pfaffian orientation of $G^{u,v}$, one of the two edges is agreeing and the other is opposite.

**Proof.** Let $M$ denote the perfect matching, and let $T_M$ be the collection of triples obtained by adding $u$ to the edges of $M$. Let $T_1 = T_M \cup \{xyu, yzu\}$ and $T_2 = T_M \cup \{xyv, yzu\}$. The conclusion follows again by comparing the expressions for the signs of $T_1$ and $T_2$ given in Lemma 5.3. \qed

**Corollary 5.8.** If $G^{u,v}$ is 3-Pfaffian, then $G$ does not contain a path of length 6 whose complement has a perfect matching.

**Proof.** Suppose for a contradiction that $a_1 a_2 \ldots a_7$ is path of length 6 in $G$. Take a 3-Pfaffian orientation of $G$. The complement of the edge $a_3 a_4$ contains a quasi-perfect matching, hence this edge is agreeing. Similarly, $a_4 a_5$ is also agreeing. But Lemma 5.7 implies that only one of $a_3 a_4$ and $a_4 a_5$ can be agreeing. \qed

The following lemma describes how a 3-Pfaffian orientation behaves in a path of length 4.

**Lemma 5.9.** Let $x_1x_2x_3x_4x_5$ be a path of length 4 in $G$ whose complement has a perfect matching. In any 3-Pfaffian orientation, the edges $x_1x_2$ and $x_4x_5$ are agreeing and the other two are opposite. Moreover, $x_2 \rightarrow x_3$ if and only if $x_4 \rightarrow x_3$, and analogously for the $u$-orientation.

**Proof.** Which edges are agreeing and which ones are opposite follows from Lemmas 5.4 and 5.7. Now we proceed as in the proof of Lemma 5.7. Let $M$ denote the perfect matching in the complement of the path, and let $T_M$ be the collection of triples obtained by adding $u$ to the edges of $M$. Let $T_1 = T_M \cup \{x_1x_2u, x_2x_3v, x_4x_5u\}$ and $T_2 = T_M \cup \{x_1x_2u, x_3x_4v, x_4x_5u\}$.
The conclusion follows again by comparing the expressions for the sign of $T_1$ and $T_2$ given in Lemma 5.3.

The necessary conditions for an orientation to be 3-Pfaffian given in the previous lemmas turn out to be sufficient. Recall that an edge is considered to be a cycle of length two.

**Theorem 5.10.** The following are equivalent for an orientation of $G_{u,v}$.

(i) The orientation is 3-Pfaffian.

(ii) With respect to this orientation,

(a) if $C$ is an even cycle of $G$ whose complement has a quasi-perfect matching, all its edges are agreeing and $C$ is oddly oriented;

(b) if $xyz$ is a path of length 2 in $G$ whose complement has a perfect matching, one of the edges is agreeing and the other is opposite;

(c) if $x_1x_2x_3x_4x_5$ is a path of length 4 in $G$ whose complement has a perfect matching, then $x_2 \xrightarrow{u} x_3$ if and only if $x_4 \xrightarrow{u} x_3$, and analogously for the $v$-orientation.

**Proof.** The implication (i)⇒(ii) follows from Lemmas 5.6, 5.7 and 5.9. For the converse, let $T_1$ and $T_2$ be two spanning trees of $G_{u,v}$. We need to prove that they get the same sign if the orientation satisfies the conditions in (ii).

We first show that certain subgraphs cannot appear in $G$ if there is an orientation satisfying (ii). A $P_6$ is a path with 6 edges and $K_{2,3}$ denotes the graph $K_{2,3}$ with one edge removed.

**Claim 1.** If $G$ has an orientation satisfying (ii), then $G$ has no subgraph isomorphic to an odd cycle, a $P_6$ or a $K_{2,3}$ whose complement contains a perfect matching.

**Proof of Claim 1.** Let $C$ be an odd cycle in $G$. Since $C$ is not 2-edge-colourable, there are two consecutive edges of $G$ that are either both opposite or both agreeing. If $C$ has a perfect matching in the complement, condition (ii).(b) applied to this pair of edges yields a contradiction.

That $G$ contains no path of length 6 with a perfect matching in the complement follows from the same argument as in Corollary 5.8 using (ii).(a) and (ii).(b).

Finally, suppose $x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_2$ are the edges of a $K_{2,3}$ with a perfect matching in the complement. By (ii).(a), the edges $x_3x_4$ and $x_4x_5$ are agreeing, since each contains a quasi-perfect matching in the complement. But by (ii).(b) one of them must be opposite.

We next see which are the connected components of $Q_1 \cup Q_2$, where $Q_1$ and $Q_2$ are the quasi-perfect matchings associated to $T_1$ and $T_2$. 

**Claim 2.** The connected components of $Q_1 \cup Q_2$ are cycles of even length and a path of length 2 or 4.

**Proof of Claim 2.**

It will be used throughout the proof that a graph whose connected components are even cycles and paths of odd length contains a perfect matching.

Lemma 5.6 gives the three types of components that can arise. They are all even cycles (including edges), except for one of the components that is of type (H) or (T). Let us take a connected component $D$ of type (H). It consists of two cycles joined by a possibly empty path. Due to the restriction on the order of $D$, only the following two combinations can arise: the two cycles have the same parity and the path has even length, or the two cycles have different parity and the path has odd length. In this last case, it is easy to see that $D$ contains a spanning subgraph consisting of an odd cycle and a perfect matching, which together with a perfect matching in the type (C) components contradicts Claim 1. Hence $D$ consists of two even cycles joined by a path of even length. If one of the cycles has length six or more, Claim 1 is again contradicted by finding a $P_6$ with a perfect matching in the complement. Finally, if one of the cycles has length 4, it is easy to find a $K_{2,3}$ with a perfect matching in the complement. We have thus reached the conclusion that a component of type $H$ consists of two cycles of length 2 for $Q_1 \cup Q_2$. 

□
joined by a path of even length, that is, the component is a path of even length, and this path can only have length 2 or 4 by Claim 1.

Next we look at possible components of type (T), that is, three paths with common endpoints. Since the total number of vertices is odd, there are two paths of the same parity, which together form a cycle of even length, and the other path has necessarily even length. Reasoning as in the preceding paragraphs, we conclude that the cycle has length 2. Thus, in fact the (T) component is an odd cycle, which is impossible, so there are is no component of type (T).

The only thing left is to conclude that both trees have the same sign. This follows from Lemma 5.3.

More concretely, suppose that the component of type (H) in the union of $Q_1 \cup Q_2$ is a path of length 2, say $a b c$. It could be that both trees contain the triple $abv$, or that both contain the triple $abv$, or that one of them, say $T_1$, contains $abu$ and the other one $abv$. Let us focus first on the latter case. To compute the sign of $T_1$, we compute first the sign of the permutation $\pi u a b c v$, where $\pi$ are the entries that correspond to vertices that do not belong to the path of length 2. To get the sign of $T_1$ we may need to modify the sign according to the orientation of the path of length 2. The corresponding permutation for $T_2$ can be split similarly as $\pi' u c b a v$. The two permutations $\pi$ and $\pi'$ differ in an even number of transpositions, since all even cycles in $Q_1 \cup Q_2$ are oddly oriented. The permutations $a b c$ and $c b a$ have clearly opposite signs, so $T_1$ and $T_2$ have the same sign if and only if $\alpha_u(ab) + \alpha_v(bc) + \alpha_w(cb) + \alpha_c(ba)$ is odd, and this is implied by (ii).(b). The case that both $T_1$ and $T_2$ contain $abu$ (or $abv$) is simpler and dealt with in the same way.

We now suppose that the component of type (H) in the union of $Q_1 \cup Q_2$ is a path of length 4, say $a b c d e$, with $a b c$ being the path of length 2 in $Q_1$ and $c d e$ that in $Q_2$. To compute the sign of $T_1$, we compute first the sign of the permutation $\pi d e u a b c v$, where $\pi$ are the entries that correspond to vertices that do not belong to the path of length 4. We assume that $T_1$ contains triples $abu$ and $bcv$; this is no restriction since the tree $(ac)T_1$ has the same sign as $T_1$ by the conclusion of the previous paragraph. To get the sign of $T_1$ we may need to modify the sign according to the orientation of the path of length 2 and to that of edge $de$. The corresponding permutation for $T_2$ can be split similarly as $\pi' a b u c d e v$. The two permutations $\pi$ and $\pi'$ differ in an even number of transpositions, since all even cycles in $Q_1 \cup Q_2$ are oddly oriented. Note also that $d e u a b c v$ and $a b u c d e v$ have the same sign. Hence both trees have the same sign if and only if $\alpha_u(ab) + \alpha_v(bc) + \alpha_u(cd) + \alpha_v(de) + \alpha_u(de) + \alpha_v(ab)$ is even. The edges $ab$ and $de$ are agreeing by (ii).(a), therefore we only need to worry about $\alpha_v(bc) + \alpha_u(cd)$. That this is even follows by combining the fact that both $bc$ and $cd$ are opposite and the condition in (ii).(c).

The conditions of Theorem 5.10 for an orientation of a 2-suspension to be 3-Pfaffian are quite restrictive and suggest that there are few of them. This is confirmed by the following characterization by forbidden subgraphs. As usual $C_\ell$ denotes the cycle with $\ell$ edges.

**Theorem 5.11.** Let $G$ be a graph and $u, v \notin V(G)$. Then the 2-suspension $G^{u,v}$ has a 3-Pfaffian orientation if and only if

(i) the graph $G - \{i\}$ is Pfaffian for each vertex $i$,

(ii) $G$ has no subgraph isomorphic to $C_3, C_5, P_6$ or $K_{2,3}$ whose complement has a perfect matching.

**Proof.** If $G^{u,v}$ has a 3-Pfaffian orientation, Claim 1 in the proof of Theorem 5.10 shows that $G$ contains no subgraph isomorphic to an odd cycle, a $P_6$ or a $K_{2,3}$ whose complement contains a perfect matching, hence (ii) holds. (Observe that excluding $P_6$ automatically excludes all odd cycles of length at least 7.) To show (i) holds, consider the $u$-orientation of $G$ corresponding to the 3-Pfaffian orientation of $G^{u,v}$. Let $C$ be a cycle of even length $\ell \geq 4$ in $G - \{i\}$ whose complement has a perfect matching $M$. We need to show that $C$ is oddly oriented with respect
to the $u$-orientation. If vertex $i$ is adjacent to some vertex in $C$, then $G$ would contain a copy of $K_{2,3}$ or of $P_6$ with a perfect matching in the complement, so we conclude that $i$ is only adjacent to vertices covered by the perfect matching $M$. Hence, $C$ is an even cycle whose complement in $G$ contains a quasi-perfect matching. Since the orientation of $G^{u,v}$ is 3-Pfaffian, Lemma 5.6 implies that $C$ is oddly oriented, hence $G - \{i\}$ is Pfaffian.

For the converse, let $B$ be a minimal graph with respect to edge deletion such that the 2-suspension $B^{u,v}$ is non-3-Pfaffian. In particular, any triple belongs to some spanning tree of $B^{u,v}$, otherwise the corresponding edge in $B$ could have been deleted.

Choose $ab \in E(B)$ such that there is some spanning tree of $B^{u,v}$ containing neither $abu$ nor $abv$. (If every edge $ab$ of $B$ has the property that each spanning tree of $B^{u,v}$ contains $abu$ or $abe$ then $ab$ is in every quasi-perfect matching of $B$. It is not difficult to see that this can only happen if $B$ is a set of vertex disjoint edges and one path of length 2. However in this case $B^{u,v}$ is 3-Pfaffian.)

Let $G = B \setminus ab$. By minimality of $B$, the 3-graph $G^{u,v}$ is 3-Pfaffian. Then there is a $u$-orientation and a $v$-orientation of the edges of $G$ with the property that all the spanning trees of $G^{u,v}$ have the same sign when triples $iju$ are oriented according to the $u$-orientation of $ij$ and triples $ijv$ according to the $v$-orientation of $ij$.

Extend both the $u$- and $v$-orientation of $G$ to orientations of $B$ by orienting the edge $ab$ in any way. Since the resulting orientation of $B^{u,v}$ is not 3-Pfaffian, there exist two quasi-perfect matchings $Q^+$ and $Q^-$ such that they both contain $ab$ and the associated spanning trees $T^+$ and $T^-$ have opposite signs.

Let $Q = M_a \cup M_b$ be an arbitrary quasi-perfect matching of $G$ and consider the graphs $H^+ = Q \cup Q^+$ and $H^- = Q \cup Q^-$. Lemma 5.5 gives the possible subgraphs that can arise as connected components of $Q \cup Q^+$ and $Q \cup Q^-$. If one of the connected components of type (H) or (T) is not a path of length 2 or 4, then we can find one of the excluded subgraphs in condition (ii), just as in Claim 2 in the proof of Theorem 5.10. If this is the case we are done, so suppose that all the connected components are even cycles or paths of length 2 or 4.

Our next goal is to show that the edge $ab$ belongs to one of these even cycles and not to the paths. We look at $H^+$ since the argument is symmetric. If $H^+$ contains a path of length 2, then the paths of length 2 in the quasi-perfect matchings $Q$ and $Q^+$ coincide and, since $Q$ does not contain $ab$, it follows that in this case $ab$ must belong to one of the cycles of $H^+$. If $H^+$ contains a path of length four $x_1x_2x_3x_4x_5$, it means that one of $Q$ or $Q^+$ contains the edges $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\}$ and the other contains the edges $\{x_1x_2, x_2x_3, x_4x_5\}$. Thus if the edge $ab$ belongs to this path, it is either $x_2x_3$ or $x_3x_4$. We assume it is $x_2x_3$, and hence that $Q$ contains $x_1x_2$, $x_3x_4$ and $x_4x_5$. Therefore the component of type (H) in $H^-$ is also the path $x_1x_2x_3x_4x_5$. The other connected components in $H^+$ and $H^-$ are cycles of even length that do not contain $ab$ and whose complement contains a quasi-perfect matching in $G$ (having $x_3x_4x_5$ as its path of length 2). Since the orientation is 3-Pfaffian in $G^{u,v}$, all even cycles in $H^+$ and $H^-$ are oddly oriented. Then by Lemma 5.3 it is easy to see that $T^+$ and $T^-$ either have both the same sign or both the opposite as the tree associated to $Q$, which is not possible by the choice of $T^+$ and $T^-$. So we can conclude that $H^+$ is a collection of cycles of even length and a path of length 2 or 4, and that the edge $ab$ belongs to one of the cycles.

Since $H^+$ is a spanning subgraph of $B$, it is only left to decide which other edges we can have in addition to those of $H^+$. We show that if $B$ does not contain any of the subgraphs in (ii) then there is a vertex $i$ for which $B - \{i\}$ is not a Pfaffian graph. Let us start by analysing what happens if the path of $H^+$ has length 4. Let $x_1y_1y_2x_2$ be this path. Observe that vertex $x_1$ (and similarly $x_2$) has degree one in $B$. Indeed, if $x_1$ was joined to a vertex other than $y_1$ it would create a $P_6$, a $K_{2,3}$, a $C_5$ or a $C_3$, all of them with a perfect matching in the complement. The vertex $x$ cannot be adjacent to any of the even cycles of length at least 4, since this would create either a $P_6$ or a $K_{2,3}$ with a perfect matching in the complement. There can be edges joining
z and some of the isolated edges (cycles of length 2) of $H^+$. Let $y_1, \ldots, y_i$ ($i \geq 2$) be all the neighbours of $z$. There are edges $x_iy_i$, and all the $x_i$ have degree one. Therefore the edges $x_iy_i$ belong to every quasi-perfect matching of $B$ and, in order to cover $z$, each quasi-perfect matching contains exactly one of the edges $zy_i$. Therefore, if a vertex $y_i$ was in other edges than $zy_i$, then these other edges would belong to no quasi-perfect matching. By minimality of $B$ we conclude that $B$ has a connected component that is isomorphic to a star with every edge subdivided; let us call this component $S$. The case where the (H) component of $H^+$ is a path of length 2 is argued similarly and the same conclusion reached (i.e., that there is a component isomorphic to a star with every edge subdivided).

It is easy to see that $S^{|u, v}$ is a 3-Pfaffian graph. Indeed, take $x_i \xrightarrow{u} y_i, y_i \xrightarrow{u} z$ and $x_i \xrightarrow{v} y_i, z \xrightarrow{v} x_i$. This orientation satisfies the conditions described in Theorem 5.10. If the rest of $B$, that is, $B - S$, had a Pfaffian orientation, we could use it to extend the orientation of $S$ just described to an orientation satisfying the conditions of Theorem 5.10 and therefore $B^{u, v}$ would be 3-Pfaffian. Hence, $B - S$, or $B - \{z\}$ in particular, is not a Pfaffian graph. □

By combining Theorem 5.11 and Little’s characterization of Pfaffian bipartite graphs we obtain a characterization of 3-Pfaffian 2-suspensions of bipartite graphs.

**Corollary 5.12.** Let $G$ be a bipartite graph and $u, v \notin V(G)$. Then the 2-suspension $G^{u, v}$ has a 3-Pfaffian orientation if and only if $G$ has none of the following as subgraphs:

(i) an even subdivision of $K_{3, 3}$ whose complement in $G$ has a quasi-perfect matching;
(ii) a $P_6$ or $K_{2, 3}$ whose complement in $G$ has a perfect matching.

6. Partial Steiner triple systems and 3-Pfaffian orientations

6.1. Partial Steiner triple systems. In this section we consider 3-graphs $H$ with the property that the multiplicity of every pair of vertices is at most 1. Such a 3-graph will be called a partial Steiner triple system.

Let $G$ be the underlying graph of a partial Steiner triple system $H = (V, \Delta)$. For an edge $ij \in E(G)$, the only $k \in V$ such that $ijk \in \Delta$ is denoted $n(ij)$. Recall that Lemma 5.2 assigns to every spanning tree of $H$ a pair $(M, f)$, where $M$ is a perfect matching of $G - v$ and $f : M \rightarrow V$ is such that the triples of $T$ are $\{ijf(ij)\}$. If $H$ is a partial Steiner triple system, the function $f$ is necessarily $n|_M$. In order to describe the perfect matchings that arise we need some further definitions.

Let $t_1, t_2, \ldots, t_\ell$ be the triples of a cycle spanning $2\ell$ vertices, that is, there are $2\ell$ different vertices $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ such that $t_i = \{a_i, b_i, a_{i+1}\}$ ($a_{\ell+1} = a_1$). The $2\ell$-cycle of the underlying graph $G$ with edges $a_1b_1, b_1a_2, \ldots, a_\ell b_\ell, b_\ell a_1$ will be called a switching cycle. We say that a perfect matching $M$ of $G$ alternates around a switching cycle if there is another perfect matching $N$ such that the symmetric difference $M \Delta N$ is a switching cycle.

**Corollary 6.1.** Suppose $H = (V, \Delta)$ is a partial Steiner triple system. For any fixed $v \in V$, spanning trees of $H$ are in bijective correspondence with perfect matchings of $G - v$ that do not alternate around a switching cycle.

**Proof.** It follows from Lemma 5.2 that for any 3-graph $H$ spanning trees are in one-one correspondence with pairs $(M, f)$ where $M$ is a perfect matching of $G - v$ and $f : M \rightarrow V$ is a function with the property that there are no cycles in $\{ijf(ij) : ij \in M\}$. As noted above, the function $f$ is uniquely determined from the matching, since each pair is in at most one triple. The condition that there are no cycles in $\{ijf(ij) : ij \in M\}$ translates directly to the fact that $M$ does not alternate around a switching cycle. □

**Theorem 6.2.** If $H = (V, \Delta)$ is a partial Steiner triple system with the property that $H - v$ has no cycles for some $v \in V$ then the number of spanning trees of $H$ is equal to the number of perfect matchings of $G - v$. Furthermore, $H$ is 3-Pfaffian if and only if $G - v$ is Pfaffian.
Proof. Let \( V = [2n+1] \). If \( G - v \) has no switching cycles, i.e., if \( H - v \) has no cycles, then by Corollary 6.1 perfect matchings of \( G - v \) are in bijective correspondence with spanning trees of \( H \).

To prove the second part, we relate orientations of triples in \( H - v \) to orientations of edges in \( G - v \) so that we can express the Masbaum-Vaintrob theorem in terms of edge orientations.

A triangle \( abc \) in \( G - v \) is called black if \( abc \) is a triple of \( H - v \). An edge of \( G - v \) is black if it belongs to a black triangle; it is white otherwise (if \( ab \) is white, then \( abc \) is a triple of \( H \)).

Given an orientation \( \omega \) of \( H \), we define an orientation of \( G - v \) in the following way. If \( abc \) is a black triangle with \( a < b < c \) and the corresponding triple \( abc \) is oriented \((a,b,c)\), orient \( a \to b, b \to c, c \to a \). Otherwise, if \( abc \) is oriented \((a,c,b)\), orient \( a \to c, c \to b, b \to a \).

White edges are arbitrarily oriented.

The Hirschman-Reiner formulation of the Masbaum-Vaintrob theorem (Theorem 4.1) gives

\[
\mathcal{P}^\omega (H, y) = \sum_{\text{perfect matchings } M} \text{sgn}(M) \prod_{ij \in M} \epsilon_{i,j,n(ij)} y_{ijn(ij)},
\]

where \( n(ij) \) denotes the only vertex such that \( ijn(ij) \in \Delta \) and \( y_{ijn(ij)} \) equals \( y_{ijn(ij)} \) or \(-y_{ijn(ij)}\) according to whether the orientation of \( ijn(ij) \) equals or is opposite to the canonical orientation.

It is straightforward to check that, for \( i < j \),

\[
\epsilon_{i,j,n(ij)} y_{ijn(ij)} = \begin{cases} y_{ijn(ij)} & \text{if } i \to j; \\ -y_{ijn(ij)} & \text{if } j \to i. \end{cases}
\]

Therefore,

\[
\mathcal{P}^\omega (H, y) = \sum_{\text{perfect matchings } M} \text{sgn}(M) (-1)^{\#(i < j \to i)} \prod_{ij \in M} y_{ijn(ij)}.
\]

Thus if the orientation of \( G - v \) is Pfaffian, the orientation \( \omega \) is a 3-Pfaffian orientation of \( H \), and conversely. Therefore if \( G - v \) has a Pfaffian orientation with the property that each black triangle \( ijk \) of \( G - v \) is cyclically oriented then \( H \) is 3-Pfaffian.

We show that any Pfaffian orientation of \( G - v \) can be converted into a Pfaffian orientation cyclic on black triangles of \( G - v \).

Let \( abc \) be a black triangle of \( G - v \) with some orientation of its edges. Suppose this orientation of \( abc \) is not already cyclic. Two of the edges of \( abc \) must be in the same direction when traversing the triangle, say \( ab \) and \( bc \). Then \( abc \) can be cyclically oriented by reversing the directions of all edges incident with \( b \) or by reversing the direction of all edges incident with \( a \) and then of those incident with \( c \). Reversing the direction of all the edges incident with a given vertex of \( G - v \) preserves the property of being a Pfaffian orientation, since any even cycle has its parity of forward edges preserved.

We next show how to combine these movements to make all black triangles cyclic. Since \( H - v \) is a forest, there is some ordering \( \tau_1, \ldots, \tau_\ell \) of the black triangles such that \( \text{card}(\cup_{j \leq \ell} \tau_j) \cap \tau_{\ell+1} \leq 1 \). Inductively, suppose that the first \( i \) black triangles are cyclically oriented. Let \( a, b \) be two vertices of \( \tau_{\ell+1} \) that do not belong to \( \cup_{j \leq \ell} \tau_j \). Then if \( \tau_{\ell+1} \) is not cyclically oriented it can be made so by reversing the orientation of all edges incident with \( a \), or with \( b \), or with both. This clearly leaves all black triangles already processed unaltered, so eventually all black triangles are cyclically oriented, as needed.

\( \square \)

In particular, in Theorem 6.2 if \( H \) is such that \( H - v \) has no cycles and \( G - v \) is Pfaffian. In this case the Pfaffian tree polynomial \( P^\omega (H, y) \) is up to sign equal to the tree generating polynomial \( P(H, y) \) when \( \omega \) is a 3-Pfaffian orientation of \( H \). Galluccio and Loebl [9] prove a statement first made by Kasteleyn that the generating function for perfect matchings of a graph embeddable in an orientable surface of genus \( g \) may be written as a linear
combination of $4^g$ Pfaffians (with coefficients independent of the graph). Suppose we have a
3-graph $H = (V, \Delta)$ with the property that there is $v \in V$ such that the graph $G - v$ underlying
$H - v$ is without triple cycles and is of genus $g$. Then we can use the one-one sign-preserving
 correspondence between spanning trees of $H$ and perfect matchings of $G - v$ to deduce a similar
result: there are $4^g$ triple orientations of $H$ such that the tree generating polynomial $P(H, y)$
can be expressed as a linear combination of $4^g$ signed tree generating polynomials $P^{\omega}(H, y)$,
where $\omega$ ranges over $4^g$ triple orientations.

6.2. Minimal non-3-Pfaffian 3-graphs. By Theorem 4.12 there is a polynomial-size certifi-
 cate witnessing a non-3-Pfaffian 3-graph. Even if the number of spanning trees is exponential in
$n$, there is a polynomial-size subset of spanning trees of $H$ whose elements cannot be made all
the same sign. In view of the fact that a non-minimal non-3-Pfaffian 3-graph can be reduced to
a minimal non-3-Pfaffian sub-3-graph by deletion and contraction of triples, it is natural to ask
whether there is a finite set of obstructions to being 3-Pfaffian, such as given by Corollary 5.12
for 2-suspensions of graphs. In this subsection we show that this is not the case by giving an
infinite collection of minimal non-3-Pfaffian graphs (see Theorem 6.4).

Figure 8. Some non-3-Pfaffian 3-graphs $H$ minimal with respect to deletion
and contraction of triples, given by their underlying graph with one vertex
deleted. Edges not in shaded triangles are pairs of vertices in a triple containing
the removed vertex.

The 3-graphs $H$ in Table 1 are minimally non-3-Pfaffian and $H - \{0\}$ has underyling graph
$G - \{0\}$ of the form illustrated in Figure 8. The orientation of a spanning tree is given as the
cyclic permutation of the vertex set obtained as product of 3-cycles; to form this product the
oriented triples of the spanning tree are taken in the order given in the previous column of the
table. The sign of the orientation is relative to the order of vertices given in the first column.
In each case there are an odd number of negative spanning trees. It is readily checked that a
given triple belongs to an even number of spanning trees, and therefore that it is not possible to
change triple orientations to obtain spanning trees all of the same sign.

Proposition 6.3. Let $H$ be a 3-graph on vertices $0, 1, 2, \ldots, 2k$ with triples
$\{2k, 1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}, \ldots, \{2k - 2, 2k - 1, 2k\}$
and containing two triples of the form
$\{0, 2x - 1, 2y - 1\}, \{0, 2z - 1, 2t - 1\}$
for some distinct $x, y, z, t$. Then $H$ is non-3-Pfaffian. Similarly, a 3-graph with triples
$\{2k, 1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}, \ldots, \{2k - 2, 2k - 1, 2k\}$
Oriented triples

1

( 0 2 1

( 0 2

( 0 1 2

Orientation

+ 

Spanning tree

012, 023, 031, 1ab, 2bc, 3ca

{ 012, 1ab, 3ca }

{ 012, 2bc, 3ca }

{ 023, 2bc, 1ab }

{ 023, 3ca, 1ab }

{ 031, 3ca, 2bc }

{ 031, 1ab, 2bc }

( 0 1 2

( 0 1

( 0 3

( 0 4

( 0 1 2

{ 0 2 3 1 }

{ 0 3 a c 2 b 1 }

( 0 1 2 

{ 0 2 c b 1 a 3 }

( 0 2 b c a 3 1 )

( 0 3 a c 2 b 3 )

( 0 2 b c 3 2 4 )

( 0 3 a c 2 b 1 )

{ 0 4 d a c 2 b 1 3 } 

{ 0 1 2 b c a 4 d 3 }

{ 0 1 a b d 3 c 2 4 }

{ 0 2 3 c b 1 a 4 }

{ 0 1 3 b c 2 d a 4 }

{ 0 a b 1 c d 3 2 4 }

1

Sign

+ 

− 

− 

− 

− 

− 

− 

+ 

− 

− 

− 

− 

− 

− 

+ 

− 

− 

− 

+ 

Table 1. Three non-3-Pfaffian 3-graphs minimal with respect to deletion and contraction of triples.

and three triples of the form

\{0, 2x−1, 2y−1\}, \{0, 2y−1, 2z−1\}, \{0, 2z−1, 2x−1\}

for some distinct x, y, z, is non-3-Pfaffian.

Proof. Since the property of being 3-Pfaffian is preserved by deletion and contraction of triples we may assume in the first case that k = 4 and \{x, y, z, t\} = \{1, 2, 3, 4\} and in the second case that k = 3 and \{x, y, z\} = \{1, 2, 3\}. These cases are the non-3-Pfaffian 3-graphs given in Table 1.

The 3-graphs in Table 2 are illustrated in Figure 9.

Table 2. A 3-Pfaffian and a non-3-Pfaffian 3-graph.
In this case we cannot determine directly from the table whether the second graph is 3-Pfaffian or not. To do so, we transform the problem into an algebraic one. The incidence matrix for triples (rows) and spanning trees (columns) is as follows (with spanning trees in the same order as in the table and each column labelled by the sign of the corresponding tree):

\[
\begin{array}{cccccc}
01c & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
02d & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
03a & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
04b & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1ab & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
2bc & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
3cd & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
4da & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]

The non-zero positions in the row indexed by triple \( t \) correspond to those trees that will change sign if triple \( t \) changes orientation. Therefore, finding a 3-Pfaffian orientation is equivalent to finding a subset of rows whose sum (modulo 2) is either \( (1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1) \) or \( (0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0) \) (in the first case all trees would be negative and in the second case they would be positive). In other words, we need to check whether either of the two vectors belongs to the row span of the matrix over \( \mathbb{F}_2 \).

Since row \( i \) and row \( i+4 \) for each \( i = 1, 2, 3, 4 \) sum to the all-one vector \( (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1) \), the row span is the rank 5 subspace of \( \mathbb{F}_2^7 \), generated by the rows of the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

A 3-Pfaffian orientation exists if and only if the vector \( (0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0) \) is spanned by the rows of this matrix. This is easily seen not to be the case. Hence no orientation of triples can make all spanning trees have the same sign, i.e., the second graph in Table \( \text{2} \) is a non-3-Pfaffian 3-graph.

The two graphs in Table \( \text{2} \) are the first members of an infinite family. The next member is given in Table \( \text{3} \); it is the 3-graph \( H \) for which the underlying graph \( G - \{0\} \) of \( H - \{0\} \) consists of a 5-cycle of triangles \( 1ab, 2bc, 3cd, 4de, 5ea \) with edges \( 1c, 2d, 3e, 4a, 5b \).
The triple–spanning tree incidence matrix is here—taking spanning trees in the order given in Table 3—given by

\[
\begin{array}{cccccccccc}
+ & + & + & + & - & - & - & - & + \\
01c & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
02d & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
03e & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
04a & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
05b & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1ab & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
2bc & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
3cd & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
4de & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
5ea & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

Simple inspection shows that the sum of rows 2 to 6 is the vector 
\((0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0)\); therefore by changing the orientation of the triples 02d, 03e, 04a, 05b, 1ab all the trees become positive.

These examples concern the case of a 3-graph \(H\) for which the underlying graph \(G - \{0\}\) of \(H - \{0\}\) is a cycle of triangles together with edges each joining an “inner” vertex (degree 4) to an “outer” vertex (degree 2). Theorem 6.3 below says that these graphs are non-3-Pfaffian if and only if the cycle of triangles is even. Moreover, they are all minimal non-3-Pfaffian graphs. Recall from Proposition 6.3 that if the graph \(G - \{0\}\) underlying \(H - \{0\}\) has two independent edges joining pairs of “outer” vertices of a cycle of triangles, or \(G - \{0\}\) has a 3-cycle of edges joining three “outer” vertices of such a cycle of triangles, then the 3-graph \(H\) is non-3-Pfaffian, but in this case it is non-minimal (the minimal examples being those in Figure 8).

The Lucas numbers \(L_k\) are defined for \(k \geq 3\) by \(L_k = L_{k-2} + L_{k-1}\) and \(L_1 = 1, L_2 = 3\). This sequence is given explicitly by \(L_k = \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k\).

**Theorem 6.4.** Let \(H\) be the 3-graph on vertices \(0, 1, 1', 2, 2', \ldots, k, k'\) with triples
\[
\{k - 1, k, 1'\}, \{k, 1, 2'\}, \{1, 2, 3'\}, \ldots, \{k - 2, k - 1, k'\}
\]
and
\[
\{0, 1, 1'\}, \{0, 2, 2'\}, \ldots, \{0, k, k'\}.
\]
Then $H$ has $L_k$ spanning trees. For odd values of $k$ the 3-graph $H$ is 3-Pfaffian but for even values of $k \geq 4$ it is non-3-Pfaffian. Furthermore, when $k \geq 4$ is even $H$ is a minimal non-3-Pfaffian graph.

**Proof.** The 3-graph $H$ in the case $k = 3$ is shown by direct calculation to have a 3-Pfaffian orientation (see the first entry of Table 2) and the cases $k = 1$ and 2 trivially also give 3-Pfaffian 3-graphs. So we assume $k \geq 4$.

Let $s_i = \{0, i, i'\}$ and $t_i = \{i - 2, i - 1, i'\}$ for $i = 1, \ldots, k$ (in which $t_1 = \{k - 1, k, 1'\}$, $t_2 = \{k, 1, 2'\}$). If successive triples $s_i, s_{i+1}$, reading subscripts modulo $k$, belong to a spanning tree $T$ of $H$ then $s_{i-1}$ must also belong to $T$. This is because the only triples containing vertex $(i-1)'$ are $s_{i-1}$ and $t_{i-1}$, and the latter makes a cycle with $s_i$ and $s_{i+1}$. Therefore if there are any successive triples $s_i$ and $s_{i+1}$ in $T$ then $T$ consists of all the triples $s_1, s_2, \ldots, s_k$. For any other spanning tree there are no two consecutive triples $s_i, s_{i+1}$. On the other hand, given a non-empty subset $I$ of $\{1, 2, \ldots, k\}$ with the property that no two elements are consecutive (modulo $k$) the triples $\{s_i : i \in I\} \cup \{t_j : j \notin I\}$ form a spanning tree of $H$, which we shall denote by $T_I$. The singleton subsets $I = \{i\}$ vacuously satisfy the consecutiveness condition. Since $k \geq 4$ there is at least one such set $I$ with 2 or more elements. There are $L_k - 1$ such non-empty subsets $I$ uniquely determining spanning trees $T_I$ in this way. Together with the spanning tree $S$ consisting of triples $\{s_1, s_2, \ldots, s_k\}$, they account for all $L_k$ spanning trees of $H$.

Take the vertices of $H$ in the order $0, 1', 2', \ldots, k', k$. We shall choose triple orientations $(0 1')$ for the $s_i$ and $(i - 2 1 - i')$ for the $t_i$ and calculate directly the sign of the spanning tree $T_I$ with set of triples $\{s_i : i \in I\} \cup \{t_j : j \notin I\}$ for $I \subseteq \{1, 2, \ldots, k\}$ having no two consecutive elements modulo $k$. Then we shall argue that when $k$ is even no switches of triple orientations can make all the spanning trees the same sign, whereas the reverse is true when $k$ is odd. First however we observe that the spanning tree $S$ with set of triples $\{s_1, s_2, \ldots, s_k\}$ gives the following cyclic permutation of the vertex set:

$$(0 1' 1) (0 2' 2) \cdots (0 k' k) = (0 1' 1 2' 2 \cdots k' k).$$

Hence the spanning tree $S$ has positive orientation.

**Claim.** A spanning tree $T$ of $H$ has sign given by

$$\begin{cases} (-1)^{|I|-1} & T = T_I = \{s_i : i \in I\} \cup \{t_j : j \notin I\}, \\ +1 & T = S = \{s_1, \ldots, s_k\}. \end{cases}$$

We delay the proof of this claim and proceed to determine whether $H$ is 3-Pfaffian or not. If $k$ is odd, switching the orientation of all triples $s_i$ clearly makes all trees negative, so in this case $H$ is 3-Pfaffian. To treat the case $k$ even it is necessary to look more carefully at the effect that switching orientations has on the sign of the trees. Since each tree contains exactly one of $s_i$ and $t_i$, switching both of them has the effect of switching the signs of all trees, hence at most one of $s_i$ and $t_i$ has to be switched. Observe also that if in a given orientation we switch the triple $t_i$ the signs of the trees are opposite to those obtained by switching $s_i$. Therefore, we can assume that if $H$ has a 3-Pfaffian orientation, then this orientation can be obtained from the initial one by switching a subset of the triples $s_i$. Now, since we want all the trees $T_{I_{\{i\}}}$ to have the same sign, the only options are to either switch no $s_i$ or to switch all of them. The first option is clearly not 3-Pfaffian and the second one makes all trees but $S$ negative, hence it is also non-3-Pfaffian.

We next show that for $k$ even the graph $H$ is minimally non-3-Pfaffian. By symmetry, it is enough to consider the deletions $H \setminus s_1$ and $H \setminus t_1$ and the contractions $H/s_1$ and $H/t_1$. The 3-graph $H \setminus s_1$ is easily seen to be 3-Pfaffian by switching the orientation of $s_2, s_3, \ldots, s_k$ (the spanning tree $S$ of $H$ is no longer a spanning tree of $H \setminus s_1$). For $H_1 = H \setminus t_1$, we observe that $H_1 - 0$ has no cycles, and hence satisfies the hypothesis of Theorem 6.2. Since the underlying graph of $H_1 - 0$ is planar, it is a Pfaffian graph, hence the theorem implies that $H_1/t_1$ is 3-Pfaffian. The contraction $H/s_1$ is shown to be 3-Pfaffian by a similar argument. Finally, the contraction
Problem 7.1. Find a characterization of 3-graphs that have a spanning tree.
In Section 3 we found a superexponential lower bound of the type asked for in the following problem for the cases \( d = \frac{n-1}{2} \), \( m = 1 \) (Steiner triple systems, Theorem 3.4) and \( d = \binom{n}{2} \), \( m = n-2 \) (the complete 3-graph, Theorem 3.5).

**Problem 7.2.** Suppose that \( H = ([2n+1], \Delta) \) is a 3-graph such that each vertex is of degree at least \( d \) and each pair of vertices has multiplicity at least \( m \). Find a lower bound on the number of spanning trees of \( H \) (as a function of \( n, d \) and \( m \)).

In Section 5 we considered suspensions of graphs. A suspension of a bipartite graph \( G = (A \cup B, E) \) is a special form of tripartite 3-graph \( H = (A \cup B \cup C, \Delta) \), where all triples are of the form \( abc \) for \( a \in A, b \in B, c \in C \). For \( k \)-regular bipartite graphs Schrijver \([19]\) established exponential lower bounds on the number of perfect matchings. It may be easier to solve Problem 7.2 when restricted to the case when \( H \) is a tripartite 3-graph.

Little \([13]\) gave a forbidden subgraph characterization of bipartite Pfaffian graphs: if there is an even subdivision of \( K_{3,3} \) whose complement has a perfect matching then the graph is non-Pfaffian.

**Problem 7.3.** Is there a forbidden subgraph characterization for 3-Pfaffian tripartite 3-graphs?

Corollary 5.12 provides an affirmative answer to the question raised in Problem 7.3 for 2-suspensions and likewise Theorem 4.7 for 1-suspensions.

Theorem 6.4 provides an example of an infinite set of non-3-Pfaffian partial Steiner triple systems, no two of which can be obtained from the other by deletion or contraction of triples. Each partial Steiner triple system \( H \) belonging to this set has the property that if \( v \) is a vertex such that \( H - v \) has no cycles, then the underlying graph \( G - v \) is non-planar.

**Problem 7.4.** Let \( \mathcal{H} \) be the set of non-3-Pfaffian partial Steiner triple systems \( H = (V, \Delta) \) with the property that there is \( v \in V \) such that \( H - v \) has no cycles and the underlying graph \( G - v \) of \( H - v \) is planar. Is there an infinite set of 3-graphs in \( \mathcal{H} \) that are minimal with respect to deletion and contraction of triples?

Acknowledgements

We would like to thank Marc Noy for his generosity and the impetus he gave to our work on the subject of this paper – many of the results would not have been obtained without him. Also, we thank Martin Loebl for the insights he shared on the topic of Pfaffian orientations and the many ideas he produced that inspired our work.

References

[1] A. Abdesselam, Grassmann–Berezin calculus and theorems of the matrix-tree type, Adv. in Appl. Math. 33 (2004), 51–70.
[2] N. Alon, J.-H. Kim, J. Spencer, Nearly perfect matchings in regular simple hypergraphs, Israel J. Math. 100 (1) (1997) 171–187.
[3] L. Andersen, H. Fleischner, The NP-completeness of finding A-trails in Eulerian graphs and of finding spanning trees in hypergraphs, Discrete Appl. Math. 59 (1995) 203–214.
[4] A. E. Brouwer, On the size of a maximum transversal in a Steiner triple system, Canadian J. Math. 33 (1981) 1202–1204.
[5] S. Caracciolo, G. Masbaum, A. Sokal, A. Sportiello, A randomized polynomial-time algorithm for the spanning hypertree problem on 3-uniform hypergraphs, arXiv:math/0812.3593v1 [cs.CC] (2008).
[6] J. Edmonds, Paths, trees, and flowers, Canad. J. Math. 17 (1965), 449–467.
[7] H. Gabow, M. Stallmann, Efficient algorithms for graphic matroid intersection and parity, Automata, Languages and Programming; Lecture Notes in Computer Science 194, (ed. W. Brauer), Springer-Verlag, New York, 1985, 210–220.
[8] H. Gabow, M. Stallmann, Augmenting path algorithm for linear matroid parity, Combinatorica 6 (2) (1986) 123–150.
[9] A. Galluccio, M. Loebl, On the theory of Pfaffian orientations. I. Perfect matchings and permanents, Electron. J. Combin. 6 (1), 1999, R6.
[10] S. Hirschman, V. Reiner, Note on the Pfaffian matrix-tree theorem, Graphs Combin. 20 (1) (2004) 59–63.
[11] P. Jensen, B. Korte, Complexity of matroid property algorithms, SIAM J. Comput. 11 (1982) 184–190.
[12] P.W. Kasteleyn, Graph theory and crystal physics. In: Graph Theory and Theoretical Physics (ed. F. Harary), Academic Press, New York, 1967, 43–110.
[13] C. Little, A characterization of convertible (0, 1)-matrices, J. Combin. Theory B 18 (1975) 187–208.
[14] L. Lovász, The matroid matching problem, in: Algebraic Methods in Graph Theory, Vol. I, II, Colloquia Mathematica Societatis János Bolyai, Szeged, Hungary, 1978, pp. 495–517.
[15] L. Lovász, Matroid matching and some applications, J. Combin. Theory Ser. B 28 (2) (1980) 208–236.
[16] L. Lovász, M. Plummer, Matching Theory, Vol. 29 of Annals of Discrete Mathematics, North-Holland, 1986, also at: Akadémia Kiadó, Budapest, 1986.
[17] L. Lovász, Matching structure and the matching lattice, J. Combin. Theory Ser. B 43 (2) (1987) 187–222.
[18] G. Masbaum, A. Vaintrob, A new matrix-tree theorem, Internat. Math. Res. Notices 27 (2002) 1397–1426.
[19] A. Schrijver, Counting 1-factors in regular bipartite graphs, J. Combin. Theory B 72 (1) (1998) 122–135.
[20] S. Sivasubramanian, Spanning trees in complete uniform hypergraphs and a connection to extended r-Shi hyperplane arrangements, [arXiv:math/0605083v2 [math.CO]] (2006).
[21] J. Stembridge, Nonintersecting paths, Pfaffians and plane partitions, Adv. Math. 83 (1990) 96–131.
[22] R. Thomas, A survey of Pfaffian orientations of graphs, in: Proceedings of the International Congress of Mathematicians, Madrid, Spain, European Mathematical Society, Zürich, 2006.
[23] L. Valiant, The complexity of enumeration and reliability problems, SIAM J. Comput. 8 (3) (1979) 410–421.
[24] V. Vazirani, M. Yannakakis, Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, Discrete Appl. Math. 25 (1–2) (1989) 179–190.
[25] R. Wilson, Nonisomorphic Steiner triple systems, Math. Z. 135 (4) (1974) 303–313.

Unaffiliated.
E-mail address: goodall.aj@googlemail.com

Départament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Jordi Girona 1-3, 08034 Barcelona, Spain.
E-mail address: anna.de.mier@upc.edu