An Index for Superconformal Quantum Mechanics

Nick Dorey and Andrew Singleton
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge,
Cambridge, CB3 OWA, UK
n.dorey@damtp.cam.ac.uk, asingleton921@gmail.com

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Abstract

We study quantum mechanical systems with $\mathfrak{osp}(4^*|4)$ superconformal symmetry. We classify unitary lowest-weight representations of this superconformal algebra and define an index which receives contributions from short and semi-short multiplets only. We consider the example of a quantum mechanical $\sigma$-model with hyper-Kähler target $\mathcal{M}$ equipped with a triholomorphic homothety. The superconformal index coincides with the Witten index of a novel form of supersymmetric quantum mechanics for a particle moving on $\mathcal{M}$ in a background magnetic field in which an unbroken $\mathfrak{su}(1|2)$ subalgebra of the superconformal algebra is linearly realised as a global symmetry.

1 Introduction

Superconformal field theories (SCFTs) are of great interest, in part because they provide the best understood examples of the AdS/CFT correspondence [4]. In these theories, the spectrum of local operators naturally decomposes into irreducible representations of the superconformal algebra. Amongst these the BPS representations play a special role; they contain primary fields whose dimensions saturate a lower bound and do not receive quantum corrections. The counting of such states is complicated by the fact that BPS representations can combine to form generic representations whose dimensions can then be corrected. Although the spectrum of BPS states can change one can define a superconformal index [13], [5] which is invariant under variations of marginal couplings.

Superconformal quantum mechanics provides a simpler setting where many of the same phenomena occur. Such models are also of independent interest as they arise in the DLCQ description of higher dimensional SCFTs and should also play a role in the, so-far elusive, boundary description of $AdS_2$. In this paper we will
describe the BPS sector of superconformal quantum mechanics and formulate an index analogous to the superconformal index of [13]. A more detailed discussion of the results presented here can be found in the PhD thesis of the second-named author [1].

Although aspects of our construction should apply more generally, we will focus on a particular family of quantum-mechanical \( \sigma \)-models. For any Riemannian target \( \mathcal{M} \) admitting a closed homothety, a bosonic \( \sigma \)-model has an \( \mathfrak{so}(2,1) \) conformal symmetry [16]. If \( \mathcal{M} \) is hyper-Kähler and the homothety is triholomorphic, then the corresponding supersymmetric \( \sigma \)-model has a superconformal symmetry isomorphic to the simple Lie superalgebra \( \mathfrak{osp}(4^*|4) \) [2]. This condition is satisfied by a large class of singular spaces, known as hyper-Kähler cones [15], which arise as Higgs branches of supersymmetric gauge theories with eight supercharges.

In order to properly formulate supersymmetric quantum mechanics on a hyper-Kähler cone, a resolution of its singularities is required. In [3], one of the authors with Barns-Graham propose a regulated definition of the index, where the singular cone is replaced by its equivariant symplectic resolution. Although, the resolution breaks superconformal invariance, a smaller algebra corresponding to the stabilizer of the BPS bound is preserved, allowing the definition of a regulated superconformal index on the resolved space. In [3], evidence is presented that the index is independent of the choice of resolution and includes information about the spectrum of superconformal multiplets associated with the unresolved space. Via localisation, the resulting definition also yields a closed formula for the index in many cases. In the final section of the paper we will use the formula of [3] to discuss some examples.

This class of target spaces obeying the conditions for superconformal invariance outlined above includes the moduli space of Yang-Mills instantons on \( \mathbb{R}^4 \). This example provides further support for the existence of \( \mathfrak{osp}(4^*|4) \)-invariant superconformal quantum mechanics. In particular, such a model should provide a discrete lightcone description of the \( (2,0) \) theory in six dimensions [9], [10]. In this context, \( \mathfrak{osp}(4^*|4) \) is part of the subalgebra of the \( (2,0) \) superconformal algebra which is preserved by compactification of a null direction. Thus we expect the model to have a discrete spectrum of lowest-weight unitary representations of \( \mathfrak{osp}(4^*|4) \) arising from the branching of six-dimensional \( (2,0) \) multiplets onto the lightcone subalgebra. In this paper, we will consider the general properties of any such \( \mathfrak{osp}(4^*|4) \) invariant model. We classify the unitary irreducible representations of \( \mathfrak{osp}(4^*|4) \), identifying various types of short and semi-short representations. We define a superconformal index which by construction is invariant under the allowed recombinations of semi-short multiplets into generic ones.
The most general index for \( \mathfrak{osp}(4^*|4) \) consists of any way of counting (semi)short representations which is invariant under continuous supersymmetry-preserving deformations of the model of interest. In particular, representations which can pair into long representations must not contribute to this counting. These arguments lead to a basis of ‘elementary indices’ of which the superconformal index must be a linear combination. The superconformal index itself is defined analogously to the Witten index, as a supertrace,

\[
I(t, y) = \text{tr} \left[ (-1)^F e^{-\beta \mathcal{H}} t^T y^M \right].
\]

where the ‘Hamiltonian’ \( \mathcal{H} \) (with eigenvalue \( E \)) vanishes on states saturating a certain BPS bound. This corresponds to a choice of supercharge \( q \) and conjugate \( s = q^\dagger \) with \( \mathcal{H} = \{ q, s \} \). Here \( t \) and \( y \) are fugacities for the Cartan generators \( T \) and \( N \) of the subalgebra \( \mathfrak{su}(2|1) \subset \mathfrak{osp}(4^*|4) \) which commutes with \( \{ q, s, \mathcal{H} \} \). The index can be further refined by including fugacities for charges corresponding to the Cartan subagebra of the global symmetry group. As with the usual Witten index, \( I(t, y) \) receives contributions only from states with \( E = 0 \), and evaluating these leads to a general expression for the superconformal index as a sum of characters of \( \mathfrak{su}(2|1) \) multiplying the elementary indices.

The problem of identifying and counting states with \( E = 0 \) can also be recast as an ordinary supersymmetric quantum mechanics problem, albeit with somewhat exotic supersymmetry. In particular, starting from the original \( \sigma \)-model, one may construct a Lagrangian whose Legendre transform coincides with the classical limit of the Hamiltonian, \( \mathcal{H} \). The “little-group” algebra \( \mathfrak{su}(2|1) \) is linearly realised as a global symmetry of the Lagrangian. The resulting model can be interpreted in terms of a particle in a magnetic field proportional to a target space Kähler form. The states which contribute to the superconformal index lie in the lowest Landau level and can be isolated in the usual way by taking a strong field limit. In the context of superconformal mechanics the limit is purely kinematic and corresponds to subjecting the model to a large conformal boost.

In the final section of the paper, we consider some examples. In these cases we compute the superconformal index exactly and extract information about the corresponding spectrum of superconformal multiplets. In particular we will see that the spectrum of certain types of multiplets can be determined exactly.

## 2 Superconformal quantum mechanics

We start from the standard supersymmetric quantum mechanical \( \sigma \)-model whose target space is a Riemannian manifold \((\mathcal{M}, g)\) of dimension \( n \). The bosonic degrees of freedom corresponding to coordinates \( X^\mu \) on \( \mathcal{M} \) are accompanied by complex conjugate fermionic degrees of freedom \( \psi^\mu \) and \( \psi^\dagger_\mu = (\psi^\mu)^\dagger \) as Grassmann-odd sections of the cotangent bundle,

\[
S = \int dt \left[ \frac{1}{2} g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu + ig_{\mu\nu}(X) \psi^\dagger_\mu \frac{D}{Dt} \psi^\nu - \frac{1}{4} R_{\mu\nu\rho\sigma}(X) \psi^\dagger_\mu \psi^\dagger_\nu \psi^\rho \psi^\sigma \right].
\]
where we define the covariant derivative
\[ \frac{D}{Dt} \psi^\mu = \nabla_X \psi^\mu = \dot{\psi}^\mu + X^\nu \Gamma^\mu_{\nu\rho} \psi^\rho, \] (2.2)
and \( R_{\mu\nu\rho\sigma} \) is the Riemann tensor on \( M \). The action is invariant under \( \mathcal{N} = (1, 1) \) supersymmetry transformations,
\[ \delta_{\epsilon} X^\mu = -\epsilon^\dagger \psi^\mu + \epsilon \psi^\mu, \]
\[ \delta_{\epsilon} \psi^\mu = i\dot{X}^\mu \epsilon - \Gamma^\mu_{\nu\rho} (\delta_{\epsilon} X^\nu) \psi^\rho. \] (2.3)

The phase space is parametrised by the coordinates \( X^\mu \) and their canonically conjugate momenta \( P_\mu = \partial L / \partial \dot{X}^\mu \) together with the fermionic variables \( \psi^\mu \), \( \psi^{\dagger \mu} \). It is often convenient to work with the covariant momentum, \( \Pi_\mu = g_{\mu\nu} \dot{X}^\nu \) which transforms as a tangent vector on the manifold. Quantization proceeds by imposing the (anti-)commutation relations,
\[ [X^\mu, \Pi_\nu] = i\delta^\mu_\nu, \quad \{ \psi^\mu, \psi^{\dagger \nu} \} = g^\mu\nu. \]

As the covariant momentum is not canonical we have further non-vanishing commutators,
\[ [\Pi_\mu, \psi^{\dagger \nu}] = i\Gamma^\nu_{\mu\rho} \psi^\rho, \quad [\Pi_\mu, \psi^{\dagger \nu}] = i\Gamma^\nu_{\mu\rho} \psi^{\dagger \rho}, \quad [\Pi_\mu, \Pi_\nu] = -R_{\rho\sigma\mu\nu} \psi^{\dagger \rho} \psi^\sigma. \] (2.4)

We choose a fermionic vacuum \( |0\rangle \) annihilated by all the fermions \( \psi^\mu, \mu = 1, \ldots, n \). The resulting Hilbert space is naturally organised by the number of fermions excited. A state with \( p \) fermions,
\[ |\alpha\rangle = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} (X) \psi^{\dagger \mu_1} \cdots \psi^{\dagger \mu_p} |0\rangle, \] (2.5)
is naturally identified with the following differential form of degree \( p \) on \( M \),
\[ \alpha = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} (X) dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_p}. \]

The inner product between states correspond to the standard \( L^2 \) inner product of forms,
\[ \langle \alpha, \beta \rangle = \int_M \alpha \wedge \ast \beta = \frac{1}{p!} \int d^n X \sqrt{g} \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p}, \] (2.6)

With this identification the Hamiltonian coincides with the Laplacian acting on forms. Its explicit form, up to additional terms arising from operator reordering, is given as,
\[ \mathbb{H} = g^{\mu\nu} \Pi_\mu \Pi_\nu + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^{\dagger \mu} \psi^{\dagger \nu} \psi^\rho \psi^\sigma. \]

The supersymmetry transformations given above are generated by supercharges,
\[ Q = i\psi^{\dagger \mu} \Pi_\mu, \quad Q^\dagger = -i\Pi_\mu^\dagger \psi^\mu. \]
obeying the algebra,
\[ \{ Q, Q^\dagger \} = H \] (2.7)

The supersymmetry algebra admits a \( U(1) \) R-symmetry generated by,
\[ J = \frac{1}{2} \left( g_{\mu \nu} \psi^\dagger \gamma_\mu \psi^\nu - n/2 \right) = \frac{1}{2} \left( p - n/2 \right) \] (2.8)

So far our discussion is generic to any Riemannian target manifold. In the special case that the \( M \) is a complex Kähler manifold with complex structure \( I^a_\mu \),
\[ Q^{(I)} = i \psi^\dagger \mu I^a_\mu \Pi^\nu, \quad Q^{(I)}\dagger = -i \Pi^\dagger \mu I^a_\mu \psi^\nu \]

Further, if the manifold is hyper-Kähler, with three linearly independent complex structures \( (I^a)_\mu \), \( a = 1, 2, 3 \) obeying the \( su(2) \) algebra,
\[ I^a I^b = -\delta^{ab} \mathbb{1} + \epsilon^{abc} I^c \] (2.9)

In this case we also have additional supercharges \( Q^{(a)} = i \psi^\dagger \mu (I^a)_\mu \Pi^\nu \) and \( Q^{(a)}\dagger \) generating the \( N = (4, 4) \) supersymmetry algebra where (2.7) is supplemented by,
\[ \{ Q^a, Q^{b\dagger} \} = \delta^{ab} H \]

The \( N = (4, 4) \) supersymmetry algebra admits an \( so(5) \simeq sp(4) \) R-symmetry [17] under which the complex supercharges transform in the four dimensional spinor representation denoted \( 4 \).

In this paper we will be interested in supersymmetric \( \sigma \)-models which also admit a conformal symmetry and thus, in particular, we will consider models with scale invariance. This requires [16] the target manifold \( M \) to admit a homothetic Killing vector field \( D \) obeying,
\[ \mathcal{L}_D g = 2g \] (2.10)

where \( \mathcal{L}_D \) denotes the corresponding Lie derivative. Further, we say that \( D \) is a \textit{closed homothety} if there exists a real scalar function \( K \) on \( M \) such that,
\[ \mathcal{L}_D K = 2K, \quad D_\mu = \partial_\mu K \] (2.11)

In this case we can define a \textit{dilation operator} for the quantum mechanics defined by the action (2.1) as \( \mathbb{D} = D^\mu \Pi_\mu - in/2 \) and a \textit{Special conformal generator} \( \mathbb{K} = K \). By virtue of (2.10) and (2.11) the three generators \( \{ \mathbb{D}, \mathbb{H}, \mathbb{K} \} \) yield an \( so(2, 1) \simeq su(1, 1) \) conformal symmetry,
\[ [\mathbb{D}, \mathbb{H}] = 2i\mathbb{H}, \quad [\mathbb{D}, \mathbb{K}] = -2i\mathbb{K}, \quad [\mathbb{H}, \mathbb{K}] = -i\mathbb{D} \]

For the generic Riemannian target \( M \), the above conformal algebra together with the \( N = (1, 1) \) supersymmetry algebra close onto an \( su(1, 1|1) \) superconformal symmetry algebra [16] with maximal bosonic subalgebra \( su(1, 1) \oplus u(1) \).
The $u(1)$ factor is generated by the R-charge $\mathcal{R}$ defined above. In addition to the Poincaré supercharges $Q$ and $Q^\dagger$, the fermionic generators include the superconformal charges,

$$S = -i [K, Q], \quad S^\dagger = -i [K, Q^\dagger].$$

Thus $Q$ and $S$ are two components of a doublet of $su(1, 1)$.

Further enhancements of the superconformal symmetry can occur in the cases of complex target manifolds discussed above if the homothety $D$ is holomorphic with respect to one or more complex structures: $L_D I = 0$. We will be interested in the maximal superconformal symmetry which occurs when $\mathcal{M}$ is a hyper-Kähler manifold admitting a tri-holomorphic closed homothety [2]. In other words, in addition to the conditions discussed above, the homothety $D$ is holomorphic with respect to each of the linearly independent complex structures on $\mathcal{M}$: $L_D I^a = 0$ for $a = 1, 2, 3$. In this case we find a superconformal algebra isomorphic to the simple Lie superalgebra $osp(4^*|4)$ [11], [12]. The bosonic subalgebra,

$$g_B = so(2, 1) \oplus su(2) \oplus so(5)$$

contains the conformal algebra together with the $so(5)$ R symmetry of the $\mathcal{N} = (4, 4)$ supersymmetry algebra which acts only on the $\sigma$-model fermions. The additional $su(2)$ acts only on the bosonic coordinates. The four Poincaré supercharges $\{Q, Q^a\}$, for $a = 1, 2, 3$ together with the corresponding superconformal charges transform in the $(2, 2, 4)$ of $g_B$.

To study the spectrum of the dilatation operator it is convenient to perform a change of basis in $su(1, 1)$ as,

$$X \mapsto e^{-\mu K} e^{\frac{1}{2} \mu^{-1} \mathbb{H}} X e^{-\frac{1}{2} \mu^{-1} \mathbb{H}} e^{\mu K} := M^{-1} X M \quad \forall X \in so(2, 1) \quad (2.12)$$

with $\mu \in (0, \infty)$, under which

$$i \mathbb{D} \mapsto L_0 = \mu^{-1} (\mathbb{H} + \mu^2 K)$$
$$\mathbb{H} \mapsto 2 \mu L_- = \mu (\mu^{-1} \mathbb{H} - \mu K - i \mathbb{D})$$
$$K \mapsto -\frac{1}{2\mu} L_+ = -\frac{1}{4\mu} (\mu^{-1} \mathbb{H} - \mu K + i \mathbb{D}). \quad (2.13)$$

These satisfy

$$L_0^\dagger = L_0, \quad L_+^\dagger = L_-, \quad [L_0, L_\pm] = 2 L_\pm, \quad [L_+, L_-] = -L_0,$$

As the supercharges $S$ and $Q$ form a doublet of $su(1, 1)$, we must also perform the same rotation on the supercharges forming linear combinations which are eigenstates of $L_0$. 


For any value of $\mu$, $L_0$ is isospectral to $D$. In the flat space example it is easy to see that $L_0$ has a discrete spectrum due to the presence of the harmonic potential provided by $K$. We expect to find the same qualitative behaviour in the general case, and expect a discrete spectrum for $L_0$ and therefore also for $D$. Of course this is quite different from ordinary quantum mechanics on a non-compact space which has a continuous spectrum of scattering states.

So far we have seen that quantum mechanics on a hyper-Kähler cone exhibits an $\mathfrak{osp}(4^*|4)$ superconformal symmetry. The main hypothesis of [3] is that associated to each such cone, there is a discrete spectrum of unitary representations of this algebra. In the next section we will examine the general features of such a theory.

### 3 The superconformal index

In each representation of $\mathfrak{osp}(4^*|4)$ basis states are labelled by the eigenvalues of the Cartan generators of the bosonic subalgebra,

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(4)$$

We choose Cartan generators $J_3$ for $\mathfrak{su}(2)$ and $M, N$ for $\mathfrak{usp}(4) \simeq \mathfrak{so}(5)$. We will work with lowest-weight representations and assume the existence of a primary state $|\Delta, j, m, n\rangle$ with,

$$\mathbb{L}_0 |\Delta, j, m, n\rangle = \Delta |\Delta, j, m, n\rangle \quad J_3 |\Delta, j, m, n\rangle = -j |\Delta, j, m, n\rangle$$
$$\mathbb{M} |\Delta, j, m, n\rangle = -m |\Delta, j, m, n\rangle \quad N |\Delta, j, m, n\rangle = -n |\Delta, j, m, n\rangle .$$

which is annihilated by all the lowering operators of the superconformal algebra,

$$\mathfrak{g}^- |\Delta, j, m, n\rangle = 0 \quad (3.1)$$

Here $2j \in \mathbb{N}$ and $m, n \in \mathbb{N}$ with $m \geq n$ label the $\mathfrak{su}(2)$ and $\mathfrak{usp}(4)$ R-symmetry representations of the superconformal primary state, while $\Delta \geq 0$ is its dimension. The quantum numbers of this state are chosen such that the module generated by the action of $\mathfrak{g}_B$ is unitary. With such a choice, one can show that the analysis of reducibility and unitarity for the resulting representation of $\mathfrak{osp}(4^*|4)$ reduces to calculating the norms of states of the schematic form

$$|n, \Delta, j, m, n\rangle := Q_1^{n_1} \ldots Q_8^{n_8} |\Delta, j, m, n\rangle , \quad n_i \in \{0,1\} . \quad (3.2)$$

Where $Q_\alpha$, $\alpha = 1, 2, \ldots, 8$ denote the eight real components of the complex supercharges $Q, Q^\alpha$ defined above. If there are no states of negative norm then the representation can be made unitary and irreducible by quotienting out any states of zero norm. This construction makes the relationship between shortening of representations and the presence of BPS states, which are annihilated by one or more supercharge, manifest.
By calculating these norms for the case of one or two supercharges acting, we can easily obtain some necessary conditions for unitarity. Proving sufficiency is somewhat harder work, requiring a link between the presence of zero norm states and the atypicality conditions of [11, 23]. The details of this construction are described in [1]. The upshot is the following classification:

**Theorem 1.** Unitary, irreducible, lowest weight representations of $\mathfrak{osp}(4^*|4)$ are obtained from the Verma module generated by the action of $\mathfrak{osp}(4^*|4)$ on $|\Delta, j, m, n\rangle$, by quotienting out null states. They come in the following types:

- *Generic ‘long’ representations* $L(\Delta, j, m, n)$ with $\Delta > 2(j + m + 1)$.
- ‘Semishort’ representations $SS(j, m, n)$ with $\Delta = \Delta_{SS} = 2(j + m + 1)$, $m \geq n$.
- ‘Short’ representations $S(m, n)$ with $\Delta = 2m$ and $j = 0$, $m \geq n$. These split into $1/2$-BPS representations with $m = n$ and $1/4$-BPS otherwise.

Long representations contain no null states of the form (3.2), and consequently have no BPS states. The lowest weight state of a short representation is $1/2$-BPS for $m = n$ and $1/4$-BPS otherwise, while semishort representations contain BPS states at higher levels.

A key feature of (semi-)short representations and their BPS states is that, since their dimension is tied to their R-charges, it cannot vary continuously with parameters of a theory, and in particular is protected from quantum corrections. In terms of the representations above, one simply observes that (semi)short representations contain fewer states than long ones, so cannot continuously change their type. This argument is not quite correct, since a long representation of dimension $\Delta = \Delta_{SS} + \epsilon : 0 < \epsilon << 1$ can continuously lower $\epsilon \to 0$. When this occurs, null states appear and the representation splits into a semishort representation with manifestly positive norm, and further (semi)short representations with zero norm. Dually, certain (semi)short representations can pair into a long one and move away from the unitarity bound. There are two ways this can occur:

\[
L(\Delta_{SS} + \epsilon, j, m, n) \to SS(j, m, n) \oplus SS(j - \frac{1}{2}, m + 1, n) \quad (j > 0)
\]
\[
L(\Delta_{SS} + \epsilon, 0, m, n) \to SS(0, m, n) \oplus S(m + 2, n) \quad (j = 0).
\]

Notice in particular that short representations with $m - n \leq 1$, in particular $1/2$-BPS representations, cannot pair up so are absolutely protected.

By an *index* for $\mathfrak{osp}(4^*|4)$ we mean any counting of (semi)short representations which is invariant under continuous changes of parameters. In particular, it must

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1. Strictly speaking these decompositions are conjectural, as we do not at present have a proof that the null representations are irreducible. This would require either a computation of the characters, or a more careful analysis of representation structure as in [22].
evaluate to zero on any combination of (semi)short representations which can pair
into a long representation. That is, if $\mathcal{R}$ labels the set of possible (semi)short
representations, we can define an index as

$$I_\alpha = \sum_{R \in \mathcal{R}} \alpha(R) N(R),$$

where $N(R)$ is the number of representations of a given type $R$ present and $\alpha(R)$
are coefficients chosen such that $I_\alpha$ is an index. The decompositions $\mathbf{3.3}$ imply
that these coefficients satisfy

$$0 = \alpha(SS(j, m, n) + \alpha(SS(j - \frac{1}{2}, m + 1, n)) \ (m \geq n \geq 0, \ j > 0)$$

$$0 = \alpha(SS(0, m, n) + \alpha(S(m + 2, n)) \ (m \geq n \geq 0).$$

Solving these constraints gives a basis of elementary indices

$$I^{r,s} = \left\{ \begin{array}{l}
N(S(r, s)) \\
N(S(r, s)) + \sum_{t=0}^{r-s-2}(-1)^{t+1}N(SS(t, r - 2 - t, s)) \\
\end{array} \right. \ (r-s \leq 1)$$

$$= \left\{ \begin{array}{l}
N(S(r, s)) \\
N(S(r, s)) + \sum_{t=0}^{r-s-2}(-1)^{t+1}N(SS(t, r - 2 - t, s)) \\
\end{array} \right. \ (r-s > 1).$$

Any index for $\mathfrak{osp}(4^*|4)$ must be a linear combination of these quantities.

The superconformal index, originally defined for four-dimensional field theory
in [13] and extended to three, five and six dimensions in [14] is formulated by
selecting a supercharge $q$ (with Hermitian conjugate $s=q^\dagger$) which only vanishes
on particular states in (semi-)short multiplets. These states correspond to the
 supersymmetric ground states of the corresponding “Hamiltonian” $\mathcal{H} = \{q, s\}$
with zero energy. The superconformal index is then defined as the corresponding
Witten index of the form $I = \text{Tr}[(-1)^F \exp(-\beta \mathcal{H}) \ldots]$ where the dots denote the
possibility of inserting other operators which commute with the supercharges and
$\mathcal{H}$. By standard arguments this quantity is independent of $\beta$ (and of all other
smooth deformations which preserve supersymmetry) provided the spectrum of
$\mathcal{H}$ is discrete. The result is a quantity which receives contributions only from
certain representatives of the (semi-)short multiplets.

Here we will pick conjugate supercharges $q$ and $s=q^\dagger$ such that,

$$\{q, s\} = \mathcal{H} = \mathbb{L}_0 + 2\mathbb{J}_3 + 2\mathbb{M}$$

In the context of the $\sigma$-model defined above, this implies picking a preferred
complex structure $\mathcal{I}_\mu^\nu$ on the target space, together with the corresponding holomor-
where \( d \) is the quaternionic dimension of the target manifold \( \mathcal{M} \). With this choice of complex structure, differential forms on \( \mathcal{M} \) are graded according to their holomorphic and anti-holomorphic degrees \( p \) and \( q \), which are the eigenvalues of the operators \( \mathbb{N} + d \) and \( \mathbb{M} + d \) respectively. This identification implies bounds on the values of eigenvalues \( d - n \) and \( d - m \) of these operators. In particular, as the (anti-)holomorphic degrees of forms are bounded by the complex dimension of the target space, the integers \( m \) and \( n \) must lie in the interval \([-d, +d]\).

The choice of supercharges described above breaks the full superconformal algebra down to the subalgebra \( su(1|2) \subset \mathfrak{osp}(4^*|4) \) spanned by generators (anti-)commuting with both \( q \) and \( s \). The bosonic subalgebra,

\[
\mathfrak{u}(1) \oplus \mathfrak{su}(2) \subset \mathfrak{su}(1|2)
\]

of this “little group”, has Cartan subalgebra generated by \( T = -(\mathbb{M} + 2\mathbb{J}_3) \) and \( \mathbb{N} \). Now suppose that, in addition to these generators we have some additional global symmetry algebra of rank \( r \) with commuting generators \( J_i, i = 1, 2, \ldots, r \).

The most general superconformal index then takes the form,

\[
\mathcal{I} (t, y, \{ \mu_i \}) = \text{Tr} \left[ (-1)^F e^{-\beta \mathcal{H}} t^T y^N e^{\sum_i \mu_i J_i} \right]
\]

As the spectrum of \( L_0 \) is discrete, the resulting index is independent of \( \beta \) by construction. Because the little group generators commute with \( q \) and \( s \), they have a well defined action on the space of states annihilated by \( \mathcal{H} \) and the index can therefore be decomposed in terms of the corresponding characters. Thus,

\[
\mathcal{I} (t, y, \{ \mu_i \}) = \sum_R C_R (\{ \mu_i \}) \, \chi_R (t, y)
\]

The sum is over irreducible representations \( R \) of \( su(2|1) \) with character \( \chi_R \). The index can also be decomposed in characters of the global symmetry. Each short or semi-short representation of the full superconformal algebra contains states with \( \mathcal{H} = 0 \) which contribute to the index. In each case the resulting states form a module for a representation of the little group \( su(2|1) \) and the contribution to the index is precisely the character of this representation. In order to decode the information contained in the index we must determine the \( su(2|1) \) character corresponding to each short or semi-short representation of \( \mathfrak{osp}(4^*|4) \). The results are as follows: for 1/2-BPS short representation \( S(m, m) \) with \( m \geq 0 \) and the 1/4-BPS short representation \( S(m, n) \) with \( m > n \geq 0 \) we find characters,

\[
\mathcal{I}_{m,m} (t, y) = t^m \left[ \chi_m(y) - t \chi_{m-1}(y) \right]
\]

\[
\mathcal{I}_{m,n} (t, y) = t^m \left[ (1 + t^2) \chi_n(y) - t (\chi_{n+1}(y) + \chi_{n-1}(y)) \right]
\]

where \( \chi_n(y) \) is the character of the spin \( n/2 \) representation of \( su(2) \);

\[
\chi_n(y) = y^n + y^{n-2} + \ldots + y^{-n}
\]

Thus \( \chi_0(y) = 1 \) and we adopt the convention that \( \chi_{-1}(y) = 0 \). Similarly the semi-short representation \( SS(j, m, n) \) yields the character,

\[
\mathcal{I}_{j,m,n} (t, y) = t^{2j+2} \mathcal{I}_{m,n} (t, y) = t^{m+2j+2} \left[ (1 + t^2) \chi_n(y) - t (\chi_{n+1}(y) + \chi_{n-1}(y)) \right]
\]
Both types, of short multiplet have a lowest weight state with \( E = 0 \) which becomes the lowest weight of the corresponding \( \mathfrak{su}(2|1) \) representation. For semi-short multiplets, states with \( E = 0 \) appear instead at the first excited level of the \( \mathfrak{osp}(4^*|4) \) representation. In particular the corresponding lowest weight of \( \mathfrak{su}(2|1) \) is a level one state of the \( \mathfrak{osp}(4^*|4) \) representation with respect to the standard triangular decomposition of the Lie superalgebra.

We can now express the index in terms of the the numbers \( N^{(m,n)} = N(S(m,n)) \), and \( N^{(j,m,n)} = N(SS(j,m,n)) \) of each type of representation present in the spectrum. In the case of a \( \sigma \)-model, the multiplets appearing are subject to the usual geometrical contraint on the (anti-)holomorphic degrees of forms on the target space. Evaluating the unrefined index with \( \mu_i = 0 \) on a generic spectrum yields,

\[
\mathcal{I}(t,y) = \sum_{m=0}^{d} N^{(m,m)} \mathcal{I}_{m,m}(t,y) - \sum_{m=1}^{d} N^{(m,m-1)} \mathcal{I}_{m,m-1}(t,y) + \sum_{m=n+2}^{\infty} \sum_{n=0}^{d-1} (-1)^{m-n} \tilde{N}^{(m,n)} \mathcal{I}_{m,n}(t,y)
\]

where for \( m - n > 1 \), we have,

\[
\tilde{N}^{(m,n)} = N^{(m,n)} + \sum_{k=\max\{0,m-1-d\}}^{m-n-2} (-1)^{k+1} N^{(k/2,m-2-k,n)}
\]

Restoring dependence on the fugacities \( \{\mu_i\} \) we can also expand the index in characters of the global symmetry group and perform a similar decomposition for the coefficient of each character. The alternating signs in this last expression correspond to potential cancellations between different (semi-)short multiplets contributing to the index. Given the value of the index \( \mathcal{I} \) as a function of \( t \) and \( y \) it is possible to read off the numbers \( N^{(m,m)} \) and \( \tilde{N}^{(m,n)} \). In general this partial information is only enough to provide certain lower bounds on the degeneracies. However in some special cases there is only one non-vanishing term on the RHS of (3.7) and we can therefore uniquely determine the degeneracy of the corresponding multiplets. In particular, this is the case for half-BPS short multiplets \( S(m,m) \) \( m = 0, \ldots d \) but also for semi-shorts \( SS(j,d-1,d-1) \). As explained in \[3\], the former multiplets correspond to the Borel-Moore homology of the target while the latter are in one to one correspondence with holomorphic functions on the corresponding complex variety. We note that the numbers of 1/4 BPS multiplets \( S(m,m-1) \) \( m = 1, \ldots d \) are also uniquely determined. We will refer to these representations of \( \mathfrak{osp}(4^*|4) \) as protected representations.

Our discussion of the superconformal index is so far very similar to the corresponding discussion in higher-dimensional field theories. However, in the particular setting of a quantum mechanical sigma model there are also some new features. The Witten index of a particle moving on a Riemannian manifold is invariant under smooth changes of the metric and is essentially a topological
invariant of $\mathcal{M}$. It is natural to look for a similar interpretation of the superconformal index. As in the classic analysis of $[7,8]$ the starting point is to reinterpret the supersymmetric vacua as cohomology classes of the supercharge. Using the mapping of states in the Hilbert space to forms and working with respect to the preferred complex structure, the supercharge $s$ acts as,

$$s = \frac{1}{\sqrt{\mu}} (\bar{\partial} + \mu \bar{\partial} \mathcal{K} \wedge)$$

where $\mathcal{K}$ is the hyper-Kähler potential and $\bar{\partial}$ is the Dolbeault operator on $\mathcal{M}$. As the spectrum of $\mathcal{H}$ is discrete, the usual Hodge theoretic argument implies that we should identify the space of states contributing to the index with the cohomology of $s$. The kernel of $s$ consists of forms $\beta$ which can be written as,

$$\beta = \exp(-\mu \mathcal{K}) \alpha$$

where $\alpha$ is any $\bar{\partial}$-closed form on $\mathcal{M}$: $\bar{\partial} \alpha = 0$. This is well-defined, as we have assumed that $\mathcal{K}$ is a function on $\mathcal{M}$. More generally one should consider $s$ acting on sections of an appropriate line bundle over $\mathcal{M}$. Similarly if $\beta$ is $\bar{\partial}$-exact with $\beta = \bar{\partial} \gamma$ for some form $\gamma$ then it is easy to check that $\alpha = s.(\exp(-\mu \mathcal{K}) \gamma)$. Hence we may formally identify the cohomology of $s$ with that of the Dolbeault operator $\bar{\partial}$. Importantly the presence of the convergence factor $\exp(-\mu \mathcal{K})$ means that the $L^2$-cohomology of $s$ with respect to the inner product $\langle, \rangle$ corresponds to the cohomology of $\bar{\partial}$ acting on forms which are $L^2$ with respect to the modified inner product,

$$\langle \alpha, \beta \rangle = \int_{\mathcal{M}} \alpha \wedge * \beta \exp(-2\mu \mathcal{K})$$

This is quite different from the standard $L^2$-Dolbeault cohomology of $\mathcal{M}$. For example, in the case of flat space $\mathcal{M} = \mathbb{C}^{2n}$, the relevant Hilbert space includes all polynomials in the complex coordinates (as well as polynomial-valued forms).

The cohomology $H(\mathcal{M}, s)$ described above is graded according to holomorphic degree,

$$H[\mathcal{M}, s] = \bigoplus_{p,q=0}^{2d} H_{p,q}[\mathcal{M}, s]$$

In the superconformal index, each $(p, q)$-form is weighted with a factor $y^{p-d}$. Setting the global fugacities to unity the superconformal index becomes,

$$\mathcal{I}(y) = \frac{1}{y^d} \sum_{p,q} (-1)^{p+q} y^p \dim H_{p,q}[\mathcal{M}, s]$$

which is the analog of Hirzebruch $\chi_y$ genus in Dolbeault cohomology.

The discussion given above is really only applicable for smooth manifolds and needs to be modified to apply to singular hyper-Kähler cones. The approach described in [3], is to replace the singular cone with its equivariant symplectic
resolution which yields a smooth space. Although this breaks superconformal invariance, it preserves the two Cartan generators of the $SU(2|1)$ little group corresponding to the index. The cohomology described above can be reformulated as in terms of sheaf cohomology and the index can then be understood as an equivariant Euler character of the resolved space. This approach also yields a concrete formula for the index as a sum over fixed points of an abelian group action on this space. We will use this formula below to discuss some simple examples in Section 4 below.

At least formally, the states with $\mathcal{H} = 0$ which contribute to the index are identified with $\bar{\partial}$-cohomology classes of $(p,q)$-forms on $\mathcal{M}$. The special case $q = 0$, which occurs for eigenstates of the $\mathfrak{osp}(4)$ Cartan generator $\mathbb{M}$ with eigenvalue equal to $-d$ corresponds to cohomology classes of holomorphic forms. Interestingly one may check that, for a subset of the (semi-)short $\mathfrak{osp}(4^*|4)$ representations listed above, all the $\mathcal{H} = 0$ states correspond to holomorphic forms. We will call these holomorphic representations. They are,

$$S(d,n) \quad n \leq d$$
$$SS(j,d - 1,n) \quad n \leq d - 1 \quad (3.10)$$

So far we have described the superconformal index in the Hamiltonian formalism where it corresponds to a particular trace over the spectrum of superconformal QM. The standard approach to computing indices in supersymmetric quantum mechanics proceeds by representing the index in question as a Euclidean functional integral. This naturally requires us to pass to the Lagrangian formulation. As discussed above the superconformal index is essentially the Witten index of a quantum mechanical system with Hamiltonian,

$$\mathcal{H} = \mathbb{L}_0 + 2\mathbb{J}_3 + 2\mathbb{M}$$

In terms of the coordinates and their canonical momenta this corresponds to the classical quantity,

$$\mathcal{H}_{cl} = \frac{1}{2\mu} g^L_{\mu\nu} \Pi_\mu \Pi_\nu + \frac{1}{4\mu} R_{\mu\nu\rho\sigma} \psi^{\dagger \mu} \psi^{\dagger \nu} \psi^{\rho} \psi^{\sigma} + \frac{\mu}{2} D^\mu D_\mu$$
$$- \mathbb{I}^\mu_\nu D^\nu \Pi_\mu + i \psi^{\dagger \mu} \psi^{\nu} \nabla_\mu D^\nu_\nu + g_{\mu\nu} \psi^{\dagger \mu} \psi^{\nu}. \quad (3.11)$$

where $D^\mathbb{J}_\mu = D^\mu \mathbb{J}_\nu$. We can then obtain a corresponding Lagrangian by performing a Legendre transform,

$$\mathcal{L}' = P_\mu \dot{X}_\mu - \mathcal{H}_{cl}, \quad \dot{X}_\mu = \frac{\partial \mathcal{H}_{cl}}{\partial P_\mu};$$

to find,

$$\mathcal{L}'' = \mu \left( \frac{1}{2} g_{\mu\nu} \dot{X}_\mu \dot{X}_\nu + \omega^\mathbb{J}_{\mu\nu} \dot{D}_\nu \dot{X}_\mu \right) - \frac{1}{4\mu} R_{\mu\nu\rho\sigma} \psi^{\dagger \mu} \psi^{\dagger \nu} \psi^{\rho} \psi^{\sigma}$$
$$+ ig_{\mu\nu} \psi^{\dagger \mu} \left( \frac{D \psi^{\nu}}{Dt} + i \psi^{\nu} + \mathbb{I}^\nu_\rho \psi^{\rho} \right). \quad (3.12)$$
where $\omega_{\mu\nu}^I$ is the Kähler form on $\mathcal{M}$ in the complex structure $I$ picked out by the BPS condition. Physically, we have a supersymmetrised coupling of the original $\sigma$-model to the magnetic field with vector potential,

$$A_\mu = \omega_{\mu\nu}^I D^\nu.$$

This model has $\mathcal{N} = (1,1)$ supersymmetry generated by $\{q, s, \mathcal{H}_{cl}\}$. A novel feature of this Lagrangian is that the little group $su(2|1)$ is also linearly realised as a “global” supersymmetry with central extension. See [1] for further details.

A striking feature of the construction given in this section is that the full spectrum of the theory (not just the index) is independent of the parameter $\mu$. Changing the value of $\mu$ simply corresponds to a change of basis for the superconformal algebra. It is natural to try to exploit this feature to simplify the dynamics. In the flat space case $\mathcal{M} = \mathbb{C}^{2n}$, the eigenstates of the bosonic part of the operator $\mathcal{H}$ are the Landau levels of a particle in a constant background magnetic field proportional to $\mu$. The fermionic degrees of freedom correspond to spin degrees of freedom which couple to the magnetic field in the standard way. Taking the limit $\mu \to \infty$ isolates the states in the lowest Landau level. Taking account of the fermions, these are precisely the states with $\mathcal{H} = 0$ which contribute to the superconformal index. It is possible to take a similar limit of the Lagrangian (3.12) in the general case. After rescaling the fields, this leads to the first-order Lagrangian,

$$\mathcal{L}_{\mu \to \infty}' = \omega_{\mu\nu}^I D^\nu \dot{X}^\mu + ig_{\mu\nu} \psi'^\dagger \mu \frac{D\psi'^\nu}{Dt}. \quad (3.13)$$

First order Lagrangians of this type arise in geometric quantization of symplectic manifolds and it would be interesting to investigate this approach in the present context.

### 4 Examples and Discussion

The simplest possible example of a hyper-Kähler manifold with a triholomorphic homothety is flat $\mathbb{R}^4 = \mathbb{C}^2$. The isometry group is $H = SO(4) = SU(2)_L \times SU(2)_R$ of which the factor $SU(2)_R$ becomes an R-symmetry subgroup of $OSp(4^*|4)$. The remaining factor, $SU(2)_L$ provides a global symmetry, with Cartan generator $L$, which we can use to refine the superconformal index. Hence we compute,

$$\mathcal{I}[\mathbb{C}^2] = \text{Tr} \left[ (-1)^F e^{-\beta \mathcal{H}} t^R y^N x^L \right]$$

For flat space, superconformal quantum mechanics essentially reduces to decoupled bosonic and fermionic harmonic oscillators and the index may be computed straightforwardly to give,

$$\mathcal{I}[\mathbb{C}^2] = \frac{t}{y} \frac{(1-yx)(1-y/x)}{(1-tx)(1-t/x)} \quad (4.1)$$

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2For a more precise treatment of this limit it is necessary to resolve the singularity using the approach of [3].
States contributing to the index lie in irreducible representations of the global symmetry. Thus the index can be expanded as,

$$\mathcal{I}[\mathbb{C}^2] = \sum_{l=0}^{\infty} \mathcal{I}_l(t, y) \chi_l(x)$$

(4.2)

where $\chi_l(x) = \sum_{j=0}^{l} x^{l-2j}$ is the character of the spin $l/2$ representation of $SU(2)_L$. Comparing (4.2) with (4.1) we obtain,

$$\mathcal{I}_0 = t (\chi_1(y) - t)$$

$$\mathcal{I}_l = t^l \left( 1 + t^2 - t \chi_1(y) \right) \quad l \geq 1$$

Although we cannot determine the spectrum of (semi-)short multiplets from the index alone, in this case the index is saturated by the protected multiplets described above whose degeneracies are uniquely fixed. The minimal spectrum consists of the following direct sum of short and semi-short representations of $\mathfrak{osp}(4^*|4)$ each transforming in a particular representation of the global symmetry $SU(2)_L$;

$$[S(1, 1), 0] \oplus [S(1, 0), 1] \oplus \sum_{l=2}^{\infty} \left[ SS \left( \frac{l}{2} - 1, 0, 0 \right), l \right]$$

(4.3)

In this case it is easy to check directly that this coincides with the true spectrum of short multiplets. In particular, each of the $OSp(4^*|4)$ representations appearing, in addition to being protected, is one of the holomorphic representations described above. The resulting $E = 0$ states are in 1 : 1 correspondence with the holomorphic forms on $\mathbb{C}^2$ and the superconformal index coincides with the index of the Dolbeault operator acting on polynomial-valued forms.

The moduli space of Yang-Mills instantons on $\mathbb{R}^4$ provides an interesting and highly non-trivial example of superconformal quantum mechanics. Let $\mathcal{M}_{K,N}$ denote the moduli space of $K$ Yang-Mills instantons of gauge group $U(N)$. The ADHM construction provides a description of $\mathcal{M}_{K,N}$ as a quotient of a flat space $\mathbb{R}^{4K^2+4KN}$. The quotient space inherits three independent complex structures from those of flat space resulting in a hyper-Kähler manifold of quaternionic dimension $d = KN$. The action of the flat-space dilatation operator also descends to provide a triholomorphic homothety on the quotient space. As a result the quantum mechanical $\sigma$-model with target space $\mathcal{M}_{K,N}$ has $\mathfrak{osp}(4^*|4)$ superconformal symmetry.

The instanton moduli space also has an $SU(2)_L \times SU(N)$ global symmetry corresponding to self-dual rotations and global gauge transformations in Euclidean spacetime. Introducing a fugacity $x$ for the Cartan generator $L$ of $SU(2)_L$ and

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3 In the following, the integer $l$ appearing as the second entry in each square bracket corresponds to the $SU(2)_L$ representation with spin $l/2$. 

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15
fugacities $z_i$, with $i = 1, \ldots, N$, for the Cartan generators $J_i$ of $SU(N) \subset U(N)$ we define a refined superconformal index for $\mathcal{M}_{K,N}$ as,

$$I_{K,N} = \text{Tr} \left[ (-1)^F e^{-\beta H_t} t^T y^N x^L \prod_{i=1}^N z_i^{J_i} \right]$$

As the ADHM moduli space has singularities corresponding to small instantons, a regularisation is required to make the index well defined. This example fits into a large class of examples discussed in [3] which admit an equivariant symplectic resolution. Working on the resolved space then yields a closed formula for the index as a sum over fixed points of the maximal torus of $SU(2)_R \times SU(2)_L \times SU(N)$. The fixed points [19–21] are labelled by $N$-coloured partitions of the instanton number $K$. Thus for each fixed point we introduce a vector $\vec{Y}$ of Young tableaux with components $Y_i$, $i = 1, 2, \ldots, N$, with a total of $K$ boxes: $||\vec{Y}|| := \sum_{i=1}^N |Y_i| = K$. We define the functions of a box, $s = (a, b)$, at row $a$ and column $b$ in the Young tableau $Y_i$ as,

$$f_{ij}(s) := -a_i(s) - l_j(s) - 1, \quad g_{ij}(s) := -a_i(s) + l_j(s), \quad (4.4)$$

where $a_i(s) := Y_{ia} - b$ the arm length and $l_j(s) := (Y_j^\vee)_b - a$ the leg length relative to $Y_j$. Using the fixed point formula eqn (1.2) of [3], we can write the superconformal index of $\mathcal{M}_{K,N}$ as,

$$I_{k,N} = \sum_{\vec{Y}: ||\vec{Y}||=K} \prod_{i,j=1}^N \prod_{s \in Y_i} \text{PE} \left[ t^{g_{ij}(s)-1} x^{f_{ij}(s)} \frac{z_i z_j}{z_i z_j} (1-t/y)(1-ty) \right], \quad (4.5)$$

where the summation is over vectors of Young tableaux corresponding to all $N$-coloured partitions of $K$ and PE denotes the Plethystic exponential.

The resulting index coincides with the $K$-instanton contribution to the Nekrasov partition function for a particular supersymmetric gauge theory. Specifically, we should identify $I_{K,N}$ with the $K$-instanton contribution,

$$Z_K \{[a_i], M, \epsilon_1, \epsilon_2 \}$$

to the partition function of a $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theory in five dimensions compactified on $\mathbb{R}^4 \times S^1$ and subjected to an $\Omega$-background in the non-compact directions with deformation parameters $\epsilon_1$ and $\epsilon_2$. In addition to an $SU(N)$ vector multiplet of $\mathcal{N} = 1$ supersymmetry in five dimensions, the model includes an adjoint hypermultiplet of mass $M$. As usual the partition function depends on complex numbers $a_i$, $i = 1, \ldots, N$, corresponding to VEVs for the scalar fields in the vector multiplet which parameterise the Coulomb branch. The dictionary between the parameters of the index and those of the 5d gauge theory is,

$$z_i = \exp(a_i), \quad y = \exp(M), \quad t = \exp \left( \frac{\epsilon_1 + \epsilon_2}{2} \right), \quad x = \exp \left( \frac{\epsilon_1 - \epsilon_2}{2} \right),$$
For this model, our superconformal index is the same object studied by Kim et al in [18] where the identification with the Nekrasov partition function described above was anticipated.

The simplest non-trivial case is that of a single instanton of gauge group $U(2)$ and the moduli space is $\mathcal{M}_{1,2} = \mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_2)$. Removing the overall factors corresponding to the center of mass, the index of the centered moduli space is a sum over the contributions of two fixed points,

$$I(\mathbb{C}^2/\mathbb{Z}_2) = \sum_{l=0}^{\infty} I_{2l}(t, y) \chi_{2l}(\rho)$$

where, comparing with (4.6), we obtain,

$$I_0 = 1 + t (\chi_1(y) - t)$$

$$I_l = t^{2l} (1 + t^2 - t \chi_1(y)) \quad l \geq 1$$

As before we can determine the minimal spectrum of short and semi-short representations of $OSp(4^*|4)$ consistent with this result for the index.

$$[S(0, 0), 0] \oplus [S(1, 1), 0] \oplus \sum_{l=2}^{\infty} [SS(l-1, 0, 0), 2l]$$

All the representations appearing are of protected type. With the exception of the single state $S(0, 0)$, the representations appearing are also holomorphic and their $E = 0$ states correspond to holomorphic forms on $\mathbb{C}^2/\mathbb{Z}_2$. Indeed the resulting spectrum can be understood by starting from the minimal spectrum on $\mathbb{C}^2$ given above and keeping only those forms even under the action of $\mathbb{Z}_2$ then adding a single “extra” state $S(0, 0)$ which corresponds instead to a $(1, 1)$-form. An extra state of this type also arises in other related contexts. In particular, it is also present in the ordinary $L^2$ cohomology of the instanton moduli space.

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4 As before the integer appearing as the second entry in each square bracket corresponds to the global $SU(2)$ representation with spin $l/2$. 

17
In the case of arbitrary rank and instanton number, we can use the results of \[3\] to determine the exact spectrum of the special protected multiplets described above. In particular 1/2 BPS multiplets \(S(n,n)\), for \(n = 0, 1, \ldots, d = NK\), are all singlets under the global symmetry, and the corresponding degeneracies \(N^{(n,n)}\) are given as,

\[ N^{(n,n)} = b_{2(d-n)} \]

where the non-negative integer \(b_{2r}\) is the dimension of the Borel-Moore Homology group of degree \(2r\). These dimensions are the coefficients of the corresponding Poincare polynomial,

\[ P^{(N,K)}(q) = \sum_{r=0}^{NK} b_{2r} q^{2r} \]

As shown in \[3\], the Poincare polynomial appears as a particular \(t \to 0\) limit of the superconformal index taken with \(q = y/t\) held fixed. Taking this limit in the fixed-point sum (4.5) yields eqn (4.8) of \[3\],

\[ P^{(N,K)}(\frac{y}{t}) = \sum_{\vec{Y}: ||\vec{Y}||=K}^{N} \prod_{j=1}^{N} (\frac{y}{t})^{2N|Y_j| - 2j \ell(Y_j)} \tag{4.9} \]

Where \(|Y_j|\) and \(\ell(Y_j)\) are the weight and length of the partition corresponding to the Young Tableau \(Y_j\) respectively. This agrees with the standard formula for the Poincare polynomial due to Nakajima \[19\] and yields a closed formula for the degeneracy \(N^{(n,n)}\) of the half BPS multiplet \(S(n,n)\) as the number of \(N\)-coloured partitions of total weight \(K\) satisfying a particular linear constraint,

\[ N^{(n,n)} = \left| \left\{ \bar{Y} : ||\bar{Y}|| = K, \sum_{j=1}^{N} j \ell(Y_j) = n \right\} \right| \]

This set of states was also considered in \[10\] and was found to correctly account for the chiral primaries of the \((2,0)\) theory of type \(A_{N-1}\) in DLCQ.

The second set of protected states form the multiplets, \(SS(r/2, d-1, d-1)\) for integer \(r \geq 2\) (together with \(S(d, d-1)\) for \(r = 1\) and \(S(d, d)\) for \(r = 0\)) which are in one to one correspondence with the holomorphic functions on the instanton moduli space. For each irreducible representation \(R\) of the global symmetry, the corresponding degeneracy \(N^{(j,d-1,d-1)}_R\) can be extracted from the Hilbert series;

\[ HS(t, x, Z) := Tr_{H^{0,0}}(t^{R} x^{L} \prod_{i} z_{i}) \]

Here \(H^{0,0}\) denotes the space of holomorphic functions on \(M_{K,N}\). More precisely, if we expand the Hilbert series in characters \(\chi_{R}\) of the irreducible representations \(R\) of the global symmetry group,

\[ HS(t, x, Z) = \sum_{R} HS_{R}(t) \chi_{R}(x, Z) \tag{4.10} \]
then we have,
\[
\HS_R(t) = N_{R}^{(d,d)} + N_{R}^{(d,d-1)} t + \sum_{r=0}^{\infty} N_{R}^{(r/2,d-1,d-1)} t^{2r} \quad (4.11)
\]

The Hilbert series itself is the coefficient of the highest power of \( y \) in the expansion of the superconformal index \([3]\) and can be extracted by taking a \( y \to \infty \) limit of the latter. Taking the limit in equation (4.5), we obtain the explicit formula,
\[
\HS(t, x, Z) = \sum_{\vec{Y}} \prod_{i,j=1}^{N} \prod_{s \in Y_i} \text{PE} \left[ t^{g_{ij}(s)-1} x f_{ij}(s) \frac{z_i}{z_j} (1 + t^2) \right], \quad (4.12)
\]

where the sum over \( N \)-vectors \( \vec{Y} \) whose components are Young Tableaux is the same as in (4.5). This result agrees with the existing formulae for the Hilbert series of instanton moduli space \([24]\). The degeneracies of each protected multiplet are then uniquely determined by comparing equation (4.12) with (4.10, 4.11).

To make this explicit one needs to expand the fixed point formula for the Hilbert series in terms of characters of \( SU(2) \times SU(N) \) as we did for the case \( k = 1, N = 2 \) above.

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