Approximation order of Kolmogorov diameters via $L^q$-spectra and applications to polyharmonic operators

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Abstract

We establish a connection between the $L^q$-spectrum of a Borel measure $\nu$ on the $m$-dimensional unit cube and the approximation order of Kolmogorov diameters of the unit sphere with respect to Sobolev norms in $L^p_\nu$. This leads to improvements of classical results of Borzov and Birman/Solomjak for a broad class of singular measures. As an application, we consider spectral asymptotics of polyharmonic operators and obtain improved upper bounds of the decay rate of their eigenvalues. For measures with non-trivial absolutely continuous parts as well as for self-similar measures the exact approximation orders are stated.

Keywords: Kolmogorov diameters/widths, polyharmonic operator; spectral asymptotics; $L^q$-spectrum; piecewise polynomial approximations; Kreĭn–Feller operator, Sobolev spaces, adaptive approximation algorithms.

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Let us start with some basic notations. Let \( \mathbb{R}^m \) denote the \( m \)-dimensional euclidean space, \( m \in \mathbb{N} \). For a multi-index \( k := (k_1, \ldots, k_m) \in \mathbb{N}^m \) we define \( |k| := \sum_{i=1}^{m} k_i \) and \( x_1 \cdots x_m := \prod_{i=1}^{m} x_i^k \) and for a bounded open subset \( \Omega \subset \mathbb{R}^m \) and \( p \geq 1 \), we let \( L^p(\Omega) \) denote the set of all \( p \)-integrable functions on \( \Omega \) with respect to the Lebesgue measure \( \Lambda \) restricted to \( \Omega \). The Sobolev space \( W^{k,p}_0(\Omega) \) (see e.g. \[30\] and \[25\]) is defined to be the set of all functions \( f \in L^p(\Omega) \) for which the weak derivatives up to order \( \ell \in \mathbb{N} \) lie in \( L^p(\Omega) \) and

\[
\|f\|_{W^{\ell,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\partial_\ell f\|_{L^p(\Omega)} < \infty,
\]

where we set \( \|f\|_{L^p(\Omega)} := \left( \int_\Omega |f|^p \, d\Lambda \right)^{1/p} \) with \( |\partial_\ell f| := \left( \sum_{|i| = \ell} |D_i^k f|^2 \right)^{1/2} \) and \( D_i^k f := \partial_i^{k_1} \partial_2^{k_2} \cdots \partial_m^{k_m} f \). We let \( W^{k,p}_{0,\ell}(\Omega) \) denote the completion of \( C^\infty(\Omega) \) with respect to \( \|\cdot\|_{W^{\ell,p}(\Omega)} \). Note that for each \( f \in W^{k,p}_{0,\ell}(\Omega) \) an equivalent norm is given by \( \|\cdot\|_{L^p(\Omega)} \). If \( \ell p/m > 1 \), then \( W^{k,p}_{0,\ell}(\Omega) \) is compactly embedded into \( (C(\overline{\Omega}), \|\cdot\|_{C(\overline{\Omega})}) \), with \( \|\cdot\|_{C(\overline{\Omega})} \) denoting the uniform norm, and therefore we always pick a continuous representative of \( W^{k,p}_{0,\ell}(\Omega) \). For the set of continuous function from \( A \subset \mathbb{R}^m \) to \( \mathbb{R} \) we write \( C(A) \).

For a normed vector space \((V, \|\cdot\|_{V})\) and a subset \( K \subset V \), the \textit{Kolmogorov n-diameter} (or \textit{n-widths}) of \( K \) in \( V \), \( n \in \mathbb{N} \), is given by

\[
d_n(K, V) := \inf \left\{ \sup_{x \in K} \inf_{y \in V_n} \|x - y\|_V : V_n \text{ is } n \text{-dimensional subspace of } V \right\}.
\]

If \( K \) is pre-compact, then the \( n \)-diameters converge to zero and one could say that the \( n \)-diameter \( d_n(K, V) \) measures the extend to which \( K \) can be approximated by \( n \)-dimensional subspaces of \( V \). We call the value

\[
\overline{\text{ord}}(K, V) := \limsup_{n \to \infty} \frac{\log(d_n(K, V))}{\log n}
\]

the \textit{upper approximation order} of \( K \) in \( V \). If the upper approximation order coincides with the \textit{lower approximation order} \( \underline{\text{ord}}(K, V) \) defined by replacing the limit superior with the limit inferior in the above definition, we call the common value \( \text{ord}(K, V) \) the \textit{approximation order}. See \[26\] for further details on this topic. For the \textit{unit sphere} in \( V \) we write \( \mathcal{S}V := \{f \in V : \|f\|_V = 1\} \). In the following we will concentrate on the particular choice \( V = L^p_q(Q) \) for a Borel measure \( \nu \) on the half-open unit cube.
\( Q := (0, 1)^m \) and \( K \in \{ W^f_p(Q), W^\ell_{q,p}(Q) \}, q \geq p > 1. \) Throughout, we will assume that
\( \varrho := q(\ell - m/p) > 0 \) and \( q \geq p > 1. \) (1.1)

Under this condition, using the Landau symbols, it has been shown in [2] that
\[
d_n\left(W^f_p, L^q_n\right) = O\left(n^{-\ell(\ell/m - 1/p + 1/q)}\right)
\]
(1.2) and in the case that \( \nu \) is a singular measure with respect to the Lebesgue measure, we know from [5] that even
\[
d_n\left(W^f_p, L^q_n\right) = o\left(n^{-\ell(\ell/m - 1/p + 1/q)}\right).
\]

In this paper we want to address the question to what extent these estimates can be effectively improved for arbitrary Borel measures on \( Q. \) We will see how our main result can be obtained from auxiliary measure-geometric quantities involving the \( L^q \)-spectrum of \( \nu \) combined with some ideas from [2] dealing with piecewise polynomial approximation in \( L^q \) of elements in \( W^f_p(Q) \) (see Section 3.1). For \( n \in \mathbb{N}, \) we set
\[
\mathcal{D}_n := \left\{ Q = \prod_{k=1}^m (l_k 2^{-n}, (l_k + 1)2^{-n}] : (l_k)_{k=1,...,m} \in \mathbb{Z}^m, \nu(Q) > 0 \right\}, \quad \mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n
\]
and the \( L^q \)-spectrum of \( \nu \) is given, for \( s \in \mathbb{R}, \) by
\[
\beta_\nu(s) := \limsup_{n \to \infty} \beta^\nu_n(s) \quad \text{with} \quad \beta^\nu_n(s) := \log \left( \sum_{C \in \mathcal{D}_n} \nu(C)^s \right) / \log (2^n).
\]
The \( L^q \)-spectrum has gained some high attention from various authors in recent years, e.g. [24, 29, 16]. Note that \( \beta_\nu \) is—as a limit superior of convex functions—itself a convex function and that \( \beta_\nu(0) \) is equal to the upper Minkowski dimension of \( \text{supp} \, \nu \) denoted by \( \dim_M \nu. \) Before stating our main result, we introduce the following key quantity
\[
s_b := \inf \{ s > 0 : \beta_\nu(s) - bs \leq 0 \} \quad \text{for} \quad b > 0.
\]

**Theorem 1.1.** Assuming (1.1), we have
\[
\overline{\text{ord}}\left(W^f_p, L^q\right) \leq -\frac{1}{q \cdot s_b}.
\]

**Remark 1.2.** Note that
\[
-1 \leq -\frac{\varrho}{q \dim_M \nu} - \frac{1}{q} \leq 1/p - 1/q - \frac{\ell}{m} < 0,
\]
and that \(-1/(q \cdot s_b) = -\ell/m + 1/p - 1/q\) if and only if \( \beta_\nu(s) = m(1-s) \) for some and hence for all \( s \in (0, 1). \) These claims follow readily from the convexity of \( \beta_\nu \) by observing that for all \( s \in [0, 1] \) we have \( \beta_\nu(s) \leq \beta_\nu(0)(1-s) \leq m(1-s). \)

If \( \nu \) has a non-trivial absolutely continuous part with respect to Lebesgue, then an application of Jensen inequality guarantees \( \beta_\nu(s) = m(1-s), \) for all \( s \in (0, 1). \) Hence,
we gain a new perspective on the estimate in (1.2) in terms of the $L^q$-spectrum. Namely, the intersection with the line through the origin with slope $\varrho$ and $s \mapsto m(1 - s)$ is given by $m/(m + \varrho)$, which leads to the general upper bound $-\ell/m + 1/p - 1/q$ as obtained in [2] (for an illustration of this observation see Fig. 1.1 on page 4). Consequently, whenever $\beta_\nu(s) < m(1 - s)$, for some $s \in (0, 1)$, Theorem 1.1 improves the classical result of [2] and [5, Theorem 5.1]. Indeed, a strict inequality occurs for many singular measures, for example if $\dim_M(\nu) < m$. Roughly speaking, the more nonuniform the mass of $\nu$ is distributed compared to the Lebesgue measure, the faster $(d_n(\mathcal{H}_{\ell}^p, L_2^q))$ decreases.

As a direct application of our result we consider polyharmonic operators, i.e. we restrict to the Hilbert spaces setting $p = q = 2$, $H^\ell := W_2^\ell(Q)$ and $H_0^\ell := W_{02}^\ell(Q)$: By Theorem 4.2, $d_{n-1}(\mathcal{H}_{\ell}^{0}, L_2^q)$ can be identified with the square root of the $n$-th eigenvalue of the associated polyharmonic operator with respect to $\nu$. This gives rise to improved upper bounds of the decay rate of their eigenvalues (Theorem 4.3 in Section 4.1). Further, using this connection for $\nu$ with non-trivial absolutely continuous part with respect to Lebesgue, we deduce from [3, Theorem 5.1] that $2s_\nu = m/\ell$ and moreover, for some explicit constant $c > 0$,

$$d_n(\mathcal{H}_{\ell}^{0}, L_2^q) \sim cn^{-\ell/m}.$$
In Section 4.2 we consider self-similar measures \( \nu \) under the open set condition (cf. Example 3.5 for definitions); it follows from \([21]\) for \( m = 1 \) and from Theorem 4.6 for \( m > 1 \) that

\[
\text{ord} \left( \mathcal{H}_0^\ell, L^2_\nu \right) = -\frac{1}{2s_\ell}.
\]

In Section 4.3 we finally consider polyharmonic operators for the special case \( \ell = m = 1 \). We will show that the associated spectral problem is equivalent to the spectral problem of the classical Krein–Feller operator (see e.g. \([15, 14, 22, 23]\)). Using the superadditivity established in Theorem 4.2 (see also \([16, \text{Theorem 1.1}]\)), we have equality in Theorem 1.1 for all finite Borel measure on \((0, 1)\), i.e.

\[
\text{ord} \left( \mathcal{H}_1^1, L^2_\nu \right) = \text{ord} \left( \mathcal{H}_0^1, L^2_\nu \right) = -\frac{1}{2s_1}.
\]

2. Optimal partitions

Fix \( a > 0 \) and a Borel probability measure \( \nu \) on \( Q \). Let \( \Upsilon_n \) denote the set of all finite partitions consisting of at most \( n \in \mathbb{N} \) half-open \( m \)-dimensional subcubes of \( Q \). As in \([2, 6]\), we introduce an auxiliary target quantity for the underlying optimisation problem given by the Kolmogorov \( n \)-diameter: For \( n \in \mathbb{N}, a > 0 \), and with \( \mathfrak{z}_a(Q) := \Lambda(Q)^a \nu(Q) \), \( Q \) half-open subcube, we let

\[
\gamma_{a,n} := \inf_{\Xi \in \Upsilon_n} \max_{Q \in \Xi} J_a(Q), \quad \text{(2.1)}
\]

and define the exponential growth rate of its reciprocal

\[
\alpha_a := \lim inf_{n \to \infty} \frac{\log(1/\gamma_{a,n})}{\log(n)}.
\]

Remark 2.1. The quantity \( \gamma_{a,n} \) naturally arises in the study of approximation order in \( L^2_\nu \) of functions in \( W^p_\nu(Q) \) by piecewise polynomial approximations (see for instance \([3]\) and Proposition 3.3) as well as in the study of the spectral behaviour of polyharmonic operators as defined in Section 4, see also \([3, 4]\). This common ground reveals a deep connection between these two aspects. It is also worth pointing out that the so-called quantization problem, that is the speed of approximation of a compactly supported Borel probability measure by finitely supported measures (see \([11]\) for an introduction), has also close links to the growth rate of \( n \gamma_{a,n} \). This will be subject of the forthcoming paper \([17]\).

We make use of the fact that the asymptotic optimisation problem in (2.1) can, as a result of Proposition 2.3, be transformed into the following counting problem. Motivated by \([16, 2, 18]\), we introduce follow quantities. Let \( \Pi \) denote the sets of all partitions of \( Q \) by half-open \( m \)-dimensional cubes. Then the exponential growth rate of

\[
\mathcal{N}_a(t) := \inf \left\{ \text{card}(P) : P \in \Pi : \max_{Q \in P} \mathfrak{z}_a(Q) < 1/t \right\}, \ t > 0,
\]
2.1 The $L^q$-spectrum and optimal partitions

given by

$$h_a := \limsup_{t \to \infty} \frac{\log N_a(t)}{\log t},$$

will be called the (upper) $v$-partition entropy with parameter $a$. Let us begin with the preparatory observation that for $a > 0$, the sequence $(\gamma_{a,2^{rn}})$ is either strictly decreasing or eventually constant zero.

**Lemma 2.2.** For all $n \in \mathbb{N}$ and $a > 0$, we have $\gamma_{a,2^{rn+1}} \leq \frac{1}{2} \gamma_{a,2^{rn}}$.

**Proof.** For $\Xi \in \mathcal{Y}_{2^{rn}}$ we can divide each $Q \in \Xi$ into $2^n$ disjoint, equally sized, half-open cubes. The new resulting partition denoted by $\Xi'$ satisfies $\text{card}(\Xi') \leq 2^{m(n+1)}$ and $\Lambda(Q')^{2^m} = \Lambda(Q)^m$, for all $Q' \subset Q \in \Xi$ with $Q' \in \Xi'$. This implies

$$\max_{Q \in \Xi'} 3_a(Q) \leq \frac{1}{2^{rn}} \max_{Q \in \Xi} 3_a(Q).$$

**Proposition 2.3.** For $a > 0$ we have $h_a = 1/\alpha_a$.

**Proof.** The proof follows along the same lines as the proof of the elementary Lemma [18, Lemma 2.2]. First note that for $0 < \varepsilon < 1$ we have $N_a(1/\varepsilon) = \inf \{n \in \mathbb{N} \mid \gamma_{a,n} < \varepsilon\}$. By Lemma 2.2 we have that $(\gamma_{a,2^{rn}})_n$ is a strictly decreasing null sequence or eventually constant zero. The latter case is immediate. For the first case the strict monotonicity gives as in [18, Lemma 2.2] for $B(\varepsilon) := \inf \{n \in \mathbb{N} \mid \gamma_{a,2^{rn}} < \varepsilon\}

$$\frac{1}{\alpha_a} = \limsup_{k \to \infty} -\log \gamma_{a,k} = \limsup_{n \to \infty} \frac{mn \log 2}{-\log \gamma_{a,2^{rn}}} = \limsup_{\varepsilon \to 0} \frac{B(\varepsilon) \log 2}{-\log \varepsilon} = h_a,$$

where the second equality follows by squeezing $2^{m(n-1)} < k \leq 2^{mn}$, and the last equality by noting that $2^{m(B(\varepsilon)-1)} \leq N_a(1/\varepsilon) \leq 2^{m(B(\varepsilon))}\). \hfill \Box$

2.1. The $L^q$-spectrum and optimal partitions

For $b > 0$ and $n \in \mathbb{N}$, by the monotonicity and continuity of $\beta^*_n$ there exists a unique number $s_{n,b} \in [0, 1]$ such that

$$\beta^*_n(s_{n,b}) = b \cdot s_{n,b}.$$

**Lemma 2.4.** For all $b > 0$, $s_b = \limsup_{n \to \infty} s_{n,b}$

and if $s_b > 0$, then $\beta_v(s_b) = b \cdot s_b$.

**Remark 2.5.** The assumption $s_b > 0$ cannot be dropped to guarantee the equality $\beta_v(s_b) = b \cdot s_b$. In fact, for the finite measure $\eta := \sum p_k \delta_{x_k}$ with $p_k := e^{-k}$ and $x_k := 1/k$ where $\delta_x$ denotes the Dirac measure on $x$, we have $\dim_m(\eta) = 1/2$,

$$\beta_v(s) = \begin{cases} 1/2, & s = 0, \\ 0, & s > 0 \end{cases}$$

and therefore $\beta_v(0) = 1/2 \neq b \cdot s_b = 0.$
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\[ s_b = \inf \{ s > 0 : \beta_r(s) - b \cdot s \leq 0 \} \]

and define \( s_* := \limsup_{n \to \infty} s_{n,b} \). Then for every \( \varepsilon > 0 \) and \( n \) large enough we have \( s_{n,b} \leq s_* + \varepsilon \) and consequently \( \beta'_n(s_* + \varepsilon) \leq \beta'_n(s_{n,b}) \). This implies

\[ \beta'_r(s_* + \varepsilon) \leq b \cdot s_*. \]

If \( s_* = 0 \), then \( \beta'_r(s) = 0 \) for all \( s > 0 \), which shows \( s_b = 0 \). Assuming \( s_* > 0 \), the continuity of \( \beta_r \) in \((0,1)\) gives \( \beta'_r(s_*) \leq b \cdot s_* \). Let \((n_k)_{k \in \mathbb{N}}\) be such that \( s_{n_k,b} = s_* \). Then for all \( \eta \in (0,s_*/2) \) and for \( k \) large we have \( s_{n_k,b} \geq s_* - \eta \), which gives \( \beta'_{n_k}(s_{n_k,b}) \leq \beta'_{n_k}(s_* - \eta) \). This implies

\[ b s_* \leq \limsup_k \beta'_{n_k}(s_* - \eta) \leq \beta'_r(s_* - \eta). \]

The continuity of \( \beta_r \) in \((0,1)\) gives \( b s_* = \beta'_r(s_*) \) and therefore \( s_b = s_* \).

\[ \square \]

The following lemma is the key to estimate \( h_a \) in terms of the \(L^q\)-spectrum \( \beta_r \). To this end, for a given threshold \( t \in (0, \log_k(\nu))] \), we will construct partitions by dyadic cubes as a function of \( t \) via an \textit{adaptive approximation algorithm} in the sense of \cite{8} (see also \cite{13}) as follows. We say \( Q \in D \) is \textit{bad}, if \( \log_k(\nu) \geq t \), otherwise we call \( Q \) \textit{good}. The goal is to construct a partition of \( Q \) with minimal cardinality, denoted by \( P_{a,t} \), consisting of elements of half-open dyadic cubes that are good. In the first step, we divide \( Q \) into \(2^d\) half open cubes of equal size and move good cubes among them to \( P_{a,t} \). Now, repeat this procedure with respect to each of the remaining bad cubes until no bad cubes are left. Since for each \( Q \in P_{a,t} \), we have \( \log_k(\nu) \leq \log_k(\nu') \), where \( \nu' \) denotes the unique dyadic cube such that \( \nu' \) is the predecessor of \( \nu \). This ensures that the procedure terminates after finitely many steps. The resulting finite partition \( P_{a,t} \) is optimal (in the sense of minimizing the cardinality) among all partitions \( P \) by half-open dyadic cubes fulfilling \( \max_{Q \in P} \log_k(\nu) < t \). This indicates that \( \text{card} P_{a,t} \) provides a good approximation of \( N_{a,t}(1/t) \). Now, the remaining task is to connect the asymptotic behaviour of \( \text{card}(P_{a,t}) \) with the \(L^q\)-spectrum \( \beta_r \). Motivated by ideas from large derivation theory and the thermodynamic formalism \cite{28} we are able to bound \( h_a \) from above by \( s_{am} \), namely, by comparing the cardinality of \( P_{a,t} \) and \( Q_{a,t} := \{ Q \in D : \log_k(\nu) \geq t \} \). This will be the key idea in the proof of Lemma 2.6.

\textbf{Lemma 2.6.} For all \( a > 0 \), we have

\[ h_a \leq s_{am}. \]

\textit{Proof.} Without loss of generality, we assume that \( \nu \) is a probability measure. For \( t \in (0,1) \),

\[ P_{a,t} = \left\{ C \in D : \log_k(\nu) < t \wedge \exists C' \in D_{\log_k(\nu)/m} \in C' \supset C \wedge \log_k(\nu') \geq t \right\} \]

is a partition of \( Q \) by dyadic cubes. With \( Q_{a,t} \) as defined above, we note that for \( C \in P_{a,t} \) there is exactly one \( C' \in Q_{a,t} \cap D_{\log_k(\nu)/m} \) with \( C \subset C' \) and for each \( C' \in Q_{a,t} \cap D_{\log_k(\nu)/m} \) there are at most \( 2^m \) elements of \( P_{a,t} \cap D_{\log_k(\nu)/m} \) which are subsets of \( C' \). Hence,

\[ \text{card} P_{a,t} \leq 2^m \text{card} Q_{a,t}. \]
By the definition of $s_{n,a,m}$, we have

$$\sum_{C \in D_k} \nu(C)^{s_{n,a,m}} = 2^{m s_{n,a,m}}.$$ 

For $s > s_{a,m}$ and $\varepsilon := (s - s_{a,m})/2$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have $s_{a,m} + \varepsilon > s_{k,a,m}$. This gives $s - \varepsilon = s_{a,m} + (s - s_{a,m})/2 > s_{k,a,m}$ and we obtain for all $0 < t < 1$,

$$t^s \text{card} P_{a,t} = \sum_{k=1}^{\infty} \sum_{C \in P_{a,t} \cap D_k} t^s \leq 2^m \sum_{k=1}^{K} \sum_{C \in Q_{a,t} \cap D_{k-1}} 1 \leq 2^m \sum_{k=1}^{K} \sum_{C \in Q_{a,t} \cap D_{k-1}} \frac{(3\alpha(C))^{s_{k-1,a,m} + \varepsilon}}{t^{s_{k-1,a,m} + \varepsilon}} \leq 2^m \sum_{k=1}^{K} \sum_{C \in Q_{a,t} \cap D_{k-1}} 1 + \sum_{k=K+1}^{\infty} 2^m \left(1 - a(k-1)\varepsilon\right) t^{s_{k-1,a,m} + \varepsilon} \leq 2^m \sum_{k=0}^{K-1} \sum_{C \in D_k} 1 + 2^m \left(1 + a\varepsilon\right) \sum_{k=K+1}^{\infty} 2^{-a(k-1)\varepsilon} < \infty.$$

This implies

$$\limsup_{t \downarrow 0} \frac{\log(\text{card} P_{a,t})}{-\log(t)} \leq s$$

and since $s > s_{a,m}$ was arbitrary,

$$h_a = \limsup_{t \downarrow 0} \frac{\log(N_a(1/t))}{-\log(t)} \leq \limsup_{t \downarrow 0} \frac{\log(\text{card} P_{a,t})}{-\log(t)} \leq s_{a,m}. \quad \square$$

Now, we are in the position state one of our core results needed in the proof of Theorem 1.1.

**Proposition 2.7.** *For all $a > 0$,*

$$\limsup_{n \to \infty} \frac{\log(\gamma_{a,n})}{\log(n)} = -\alpha_a = -\frac{1}{h_a} \leq -\frac{1}{s_{a,m}}.$$  

**Proof.** This follows immediately from Proposition 2.3 and Lemma 2.6.  

**Remark 2.8.** In the case $m = a = 1$ we have shown in [16] that even equality holds, i.e.

$$\limsup_{n \to \infty} \frac{\log(\gamma_{1,n})}{\log(n)} = -\frac{1}{s_1}. \quad \square$$
3. Approximation order

3.1. Piecewise polynomial approximations of functions of the Sobolev space in the metric of $L^q$

In this section we recall some results of [2, §3] and [4] which will be important for our applications to $n$-diameters and polyharmonic operators. Let $Q \subset Q$ denote a cube. As pointed out in the introduction our standing assumption $\ell p/m > 1$ ensures that $W^\ell_p(Q)$ is compactly embedded in $(C(\overline{Q}), \| \cdot \|_{C(\overline{Q})})$. In the case $\ell p/m \leq 1$ the situation becomes more involved; in general, we have no compact embedding from $W^\ell_p(Q)$ into $L^2_\nu(Q)$ (e.g. [12, 20, 19]). Further, without loss of generality we assume that $\nu$ is Borel probability measure on $Q$. For every $u \in W^\ell_p(Q)$, we associate a polynomial $r \in \mathbb{R}[x_1, \ldots, x_m]$ of degree at most $\ell - 1$ satisfying the conditions

$$\int_Q x^k r(x) \, d\Lambda(x) = \int_Q x^k u(x) \, d\Lambda(x) \text{ for all } |k| \leq \ell - 1. \tag{3.1}$$

By an application of Hilbert’s Projection Theorem with respect to $L^2(\nu)$, we have that $r$ is uniquely determined by (3.1) and set $P_Q u := r$. Note that $P_Q$ defines a linear projection operator which maps from $W^\ell_p(Q)$ to the finite-dimensional space of polynomials in $m$ variables of degree not exceeding $\ell - 1$ and we denote the dimension of this finite dimensional space of polynomials by $\kappa$.

We finish this section with two crucial observations which follow from [2].

**Lemma 3.1** ([2, Lemma 3.1]). Let $Q \subset Q$ be a cube. Then there exists $C_1 > 0$ independent of $Q$ such that for all $u \in W^\ell_p(Q)$

$$\|u - P_Q u\|_{C(\overline{Q})} \leq C_1 \Lambda(Q)^{\ell/m - 1/p} \|u\|_{L^\ell_p(Q)}.$$

**Definition 3.2.** Let $\Xi$ be a partition of $Q$ into half open cubes and we define $P(\Xi, \ell - 1)$ to be the space of piecewise-polynomial functions which restrict on each cube $Q \in \Xi$ to a polynomial of degree $\ell - 1$. We define

$$P_\Xi : W^\ell_p(Q) \to P(\Xi, \ell - 1)$$

$$u \mapsto \sum_{Q \in \Xi} 1_Q P_Q u,$$

where $1_Q$ denotes the characteristic function on the cube $Q$.

**Proposition 3.3.** For a finite Borel measure $\nu$ on $Q$ and $1 \leq p \leq q$, there exists $C_2 > 0$ such for all partitions $\Xi$ of $Q$ of half open cubes and every $u \in W^\ell_p(Q)$, we have

$$\|u - P_\Xi u\|_{L^\ell_Q(Q)} \leq C_2 \|u\|_{L^\ell_p(Q)} \left( \max_{Q \in \Xi} \frac{Z_{\Xi/m}(Q)}{\Lambda(Q)} \right)^{1/q}.$$ 

**Proof.** This follows from the proof of [2, Theorem 3.3] using Lemma 3.1. □
3.2 Approximation order of Kolmogorov $n$-diameters

In this section we will prove our main result.

Lemma 3.4. Under the assumption (1.1), there exists a constant $C_3 > 0$ depending only on $p, q, m, \ell$ such for all $n \in \mathbb{N}$ we have

$$d_n \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \leq C_3 \left( \frac{\gamma_{E/m,n}}{m} \right)^{1/q}.$$

Proof. For $n \in \mathbb{N}$ and $\Xi \in \Upsilon_n$ we have $\dim P(\Xi, \alpha) = \text{card}(\Xi) \leq n$, and therefore

$$d_n \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \leq d \text{card}(\Xi) \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right).$$

Hence, we obtain by Proposition 3.3

$$d \text{card}(\Xi) \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \leq \sup_{u \in \mathcal{S}_{W_p^\ell}} \inf_{y \in P(\Xi, \ell-1)} \| u - y \|_{L_q^\nu} \leq \sup_{u \in \mathcal{S}_{W_p^\ell}} \| u - P_{\Xi} u \|_{L_q^\nu} \leq C_2 \sup_{Q \in \Xi} \| u \|_{L_{q/p}(Q)} \left( \max_{Q \in \Xi} J_{q/p}(Q) \right)^{1/q} \leq C_2 \left( \max_{Q \in \Xi} J_{q/p}(Q) \right)^{1/q}.$$

Taking the infimum over all partitions with cardinality less than or equal to $n$ proves the lemma. \qed

We are now in the position to prove our main theorem.

Proof of Theorem 1.1. For $N \in \mathbb{N}$ and $n(N) := \lfloor N/k \rfloor$, Lemma 3.4 gives

$$d_N \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \leq d_{n(N)} \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \leq C_3 \left( \frac{\gamma_{E/m,n}}{m} \right)^{1/q}.$$

Using $\kappa n(N) \leq N$, we obtain

$$\log \left( d_N \left( \mathcal{S}_{W_p^\ell, L_q^\nu} \right) \right) \leq \frac{q \log(C_3) + \log \left( \frac{\gamma_{E/m,n}}{m} \right)}{q \log(\kappa n(N))} \leq \frac{q \log(C_3) + \log \left( \frac{\gamma_{E/m,n}}{m} \right)}{q \log \left( \frac{\gamma_{E/m,n}}{m} \right)} \leq - \frac{1}{q \cdot s_q}. \quad \Box$$

In the following example we consider self-similar measure under the open set condition. In this case the $L^\nu$-spectrum is well-known, allowing us to give a formula of $s_q$ only in terms of the probability weights and contraction ratios.
Example 3.5. For fixed \( n \in \mathbb{N} \) let \((T_1, \ldots, T_n)\) be a set of contracting similarities of \( \mathbb{R}^m \) with ratios \( r_1, \ldots, r_n \in (0, 1) \) that is for all \( x, y \in \mathbb{R}^d \) and \( i = 1, \ldots, n \)

\[
|T_i(x) - T_i(y)| = r_i |x - y|.
\]

Furthermore, we assume the open set condition (OSC) is fulfilled, i.e. there exists an open set \( O \subset \mathbb{R}^m \) such that

\[
T_i(O) \subset O \quad \text{and} \quad T_i(O) \cap T_j(O) = \emptyset, \quad i \neq j.
\]

Moreover, there exists a unique compact set \( K \) such that

\[
K = \bigcup_{i=1}^n T_i(K).
\]

Without loss of generality, we assume \( K \subset (0, 1)^m \). For \((p_1, \ldots, p_n) \in (0, 1)^n\) let \( \nu \) be the unique Borel measure with

\[
\nu = \sum_{i=1}^n p_i \nu \circ T_i^{-1}.
\]

The measure \( \nu \) is called self-similar measure with respect to the weights \((p_1, \ldots, p_n)\) and ratios \((r_1, \ldots, r_n)\) and we have \( \text{supp} \ \nu = K \). By [27, Theorem 16] the \( L^q \)-spectrum \( \beta_q \) on \( \mathbb{R}_{\geq 0} \) is given by the unique solution \( s \) of

\[
\sum_{i=1}^n p_i s_i r_i^{\beta_q(s)} = 1.
\]

Hence, under our standing assumptions \((p \leq q \quad \text{and} \quad p \ell/m > 1)\) and applying Theorem 1.1, we have

\[
\overline{\text{ord}} \left( \mathcal{W}^p_{\ell}, L^q_0 \right) \leq - \frac{1}{\frac{1}{p} - \frac{1}{q}},
\]

where \( s_\delta \) is the unique solution \( s \) of the equation \( \sum_{i=1}^n r_i^{\beta(s)} = 1 \). In particular, for the ‘geometric’ choice of the weights \( p_i := r_i^{\delta}, \quad i = 1, \ldots, n \), where \( \delta \in [0, m] \) is Hausdorff dimension \( \text{dim}_H (K) \) of \( K \) determined as the unique solution of \( \sum_{i=1}^n r_i^\delta = 1 \), we obtain

\[
s_\delta = \frac{\delta}{\delta + \delta}.
\]

Consequently, inserting \( \delta = \ell q - mq/p \) we get

\[
\overline{\text{ord}} \left( \mathcal{W}^p_{\ell}, L^q_0 \right) \leq \frac{\ell m}{\delta} \left( \frac{m}{\ell p} - 1 \right) - \frac{1}{q} \leq \frac{m}{\ell p - 1} \left( \frac{m}{\ell p - 1} - \frac{1}{q} \right) = \frac{m - 1}{p} + \frac{1}{q}.
\]

4. Application to polyharmonic operators

4.1. General setup and eigenvalue asymptotics

In this section let \( \nu \) be a finite Borel measure on \( \hat{Q} \) and we restrict to the Hilbert space setting \( H^p_0 \), respectively \( H^\ell \). First, let us define the polyharmonic operator as in
[3, 5, 31, 32]. We define the following quadratic forms

\[ J_v(u) := \int |u|^2 \, dv, \quad I_v(u) := \int_\Omega \sum_{|\alpha| \leq \ell} |D^\alpha u|^2 \, d\Lambda, \quad u \in H^\ell_0 \]

and let \( J_v(u, v) \) and \( I_v(u, v) \) denote the corresponding bilinear forms. Observe that \( I_v^{1/2} \) is an equivalent norm in \( H^\ell_0 \) (see [1, 6.30 Theorem]) and in virtue of our standing assumption \( 2\ell/m > 1 \), we obtain that \( H^\ell_0 \) is compactly embedded into \( C(\overline{Q}), \| \cdot \|_{C(\overline{Q})} \) (see e.g. [1, Theorem 6.3, Part II]). In particular, there is a constant \( C > 0 \) such that for all \( u \in H^\ell_0 \),

\[ J_v(u) \leq CI_v(u) \]

and by an application of the Cauchy-Schwarz inequality, for fixed \( u \in H^\ell_0 \), the map \( w \mapsto J_v(u, w) \) defines a bounded linear functional. By the Riesz Representation Theorem, we can define a bounded linear non-negative self-adjoint operator \( T_v \) mapping from \( \left( H^\ell_0, I_v (\cdot, \cdot) \right) \) to itself such that, for all \( u, w \in H^\ell_0 \),

\[ J_v(u, w) = I_v(T_v(u), w) \]

and

\[ \sqrt{I_v(T_v(u), T_v(u))} = \|J_v(u, \cdot)\| := \sup_{y \in H^\ell_0} \frac{|J_v(u, y)|}{I_v(y, y)^{1/2}} \leq C \sqrt{I_v(u, u)}. \]

To finally show that \( T_v \) is compact, let \( (u_m)_{m \in \mathbb{N}} \) be a bounded sequence in \( (H^\ell_0, I_v) \). Then, by the compact embedding of \( H^\ell_0 \) into \( C(\overline{Q}), \| \cdot \|_{C(\overline{Q})} \), there exists a subsequence \( (u_{m_k})_{k \in \mathbb{N}} \), which is a Cauchy in \( C(\overline{Q}), \| \cdot \|_{C(\overline{Q})} \). Hence, for all \( k, m \in \mathbb{N} \), we have

\[
\begin{align*}
I_v(T_v(u_m) - T_v(u_{m_k})) &= \int_\Omega (u_m - u_{m_k}) T_v(u_m - u_{m_k}) \, dv \\
&\leq \|u_k - u_m\|_{C(\overline{Q})} \sqrt{\int_\Omega T_v(u_m - u_{m_k})^2 \, dv} \sqrt{I_v(u_m - u_{m_k})} \\
&\leq \|u_k - u_m\|_{C(\overline{Q})} \sqrt{C(v(Q))} \sqrt{I_v(T_v(u_m - u_{m_k}))} \sqrt{I_v(u_m - u_{m_k})} \\
&\leq \|u_k - u_m\|_{C(\overline{Q})} \sqrt{C^2(v(Q))I_v(u_m - u_{m_k})},
\end{align*}
\]

taking into account that the sequence \( (u_m)_{m \in \mathbb{N}} \) is bounded with respect to \( I_v \), we deduce that \( (T_v(u_m))_{m \in \mathbb{N}} \) is a Cauchy sequence in the Hilbert space \( (H^\ell_0, I_v) \).

**Definition 4.1.** An element \( f \in H^\ell_0 \setminus \{0\} \) is called an eigenfunction of \( T_v \) with eigenvalue \( \lambda \), if for all \( g \in H^\ell_0 \), we have

\[ J_v(f, g) = \lambda I_v(f, g). \]
4.2 Application to self-similar measures

From the spectral theorem for self-adjoint compact operator we deduce that there is a decreasing sequence of non-negative eigenvalues \((\lambda_n^\nu)_{n \in \mathbb{N}}\) tending to 0. We are interested in the decay rate of the sequence \((\lambda_n^\nu)_{n \in \mathbb{N}}\). Note that \(\ker (T_\nu)\) can be quite large; for example in the case that \(\nu\) equals the Dirac measure \(\delta_{1/2}\) on \((0,1)\), we have \(\ker (T_\nu) = \{f \in H^0(0,1) \mid f(1/2) = 0\}\) and there is exactly one eigenvalue not equal to zero, namely \(\lambda = 1/4\) with (normalized) eigenfunction \(f_{1/4}(x) := \mathbb{1}_{(0,1/2)}2x + \mathbb{1}_{[1/2,1)}(1-2x)\).

As we will see, the growth rate of the eigenvalues is encoded by the \(L^q\)-spectrum. Using the variational principle it can be shown that the eigenvalues can be computed in terms of \(n\)-diameter (e.g. [9, Theorem 4.5]).

**Theorem 4.2.** Let \((\lambda_n^\nu) \downarrow 0\) be the decreasing sequence of eigenvalues of the polyharmonic operator \(T_\nu\) with respect to the Borel measure \(\nu\). Then we have

\[
\sqrt{\lambda_{n+1}^\nu} = d_n(\mathcal{S}H^0(L^2_\nu)).
\]

Combining Theorem 4.2, Theorem 1.1 and the fact \(d_n(\mathcal{S}H^0(L^2_\nu)) \leq d_n(\mathcal{S}H(L^2_\nu))\), we obtain the following upper bound.

**Theorem 4.3.** We have

\[
\limsup_{n \to \infty} \frac{\log(\lambda_n^\nu)}{\log(n)} \leq -\frac{1}{s_2(\nu)} - \frac{2\ell - m}{\dim_M(\nu)} + 1 \leq -\frac{2\ell - m}{m}.
\]

**Remark 4.4.** If \(\nu\) is a singular measure with respect to the Lebesgue measure, the result of [5] gives us

\[
\lambda_n^\nu = o\left(n^{-2s/m}\right).
\]

In the case \(\beta_s(s) < m(1-s)\), for some \(s \in (0,1)\), we can improve this estimate, in fact we have for every \(\varepsilon > 0\)

\[
\lambda_n^\nu = O\left(n^{-1/s+\varepsilon}\right).
\]

In general one cannot expect that \(\lambda_n^\nu \approx n^{-s}\) for some \(s > 0\) (see for example [? ] or [16]).

4.2 Application to self-similar measures

To treat self-similar measures \(\nu\) in more detail, we need the following characterisation of the eigenvalues of the linear self-adjoint compact operator \(T_\nu\) by the well-known max-min principle (see for example [10, Section 4], [26, Theorem 2.1, page 64]).

**Proposition 4.5.** For all \(i \in \mathbb{N}\), we have

\[
\lambda_i^\nu = \sup \left\{ \inf_{\mu \in C[0]} \{ J_\nu(\psi, \psi) : I_\nu(\psi, \psi) = 1 \} : G \subset H^0_0, \text{i-dimensional subspace} \right\}.
\]

**Proof.** Let \((e_j)_{j \in \mathbb{N}}\) be an orthonormal basis of eigenfunctions of \(T_\nu\) corresponding to the eigenvalues \((\lambda_j^\nu)_{j \in \mathbb{N}}\). Let \(G_i\) be an \(i\)-dimensional subspace of \(H^0_0\) and define \(E_i :=\)
span \( \{e_j : j \geq i\} \). Observe that \( \dim \left( H_0^i / E_i \right) = i - 1 \). Using the first and second isomorphism theorem,

\[
i - \dim(G_i \cap E_i) = \dim(G_i / G_i \cap E_i) = \dim((G_i + E_i) / G_i) \leq \dim \left( H_0^i / G_i \right) = i - 1.\]

Therefore, there exists \( u \in E_i \cap G_i \neq \{0\} \) with \( I_r(u, u) = 1 \) and we may write \( u = \sum_{k \geq i} c_k e_k \) with \( \sum_{k \geq i} c_k^2 = 1 \). Consequently,

\[
\inf_{\psi \in G_i \setminus \{0\}} \{ J_r(\psi, \psi) : I_r(\psi, \psi) = 1 \} \leq J_r(u, u) = I_r(T_r(u), u)
\]

\[
= I_r \left( \sum_{k \geq i} \lambda_k c_k e_k, \sum_{k \geq i} c_k e_k \right) \leq \lambda_i^r.
\]

Conversely, for \( G_i := \text{span} \{e_1, \ldots, e_i\} \), we have \( \inf_{\psi \in G_i \setminus \{0\}} \{ J_r(\psi, \psi) : I_r(\psi, \psi) = 1 \} = \lambda_1^r \).

The following theorem has been established in [21, Theorem 3.1] for dimension \( m = 1 \). We prove the corresponding result for arbitrary dimension \( m \in \mathbb{N} \) of the ambient space.

**Theorem 4.6.** Let \( \nu \) be a self-similar measure under OSC with feasible open set \( O \subset (0, 1)^m \) as defined in Example 3.5, \( \ell \in \mathbb{N} \), and assume \( \nu(\partial O) = 0 \). Then,

\[
\text{ord} \left( \mathcal{H}_0^\ell, L_\nu^2 \right) = \text{ord} \left( \mathcal{H}_0^\ell, L_\nu^2 \right) = \lim_{l \to \infty} \frac{\log(\lambda_1^r)}{\log(l)} = -\frac{1}{2d_0}.
\]

**Proof.** Here, we follow partially [20]. For \( t \) with \( 0 < t < \min \{r \vee (2\ell - m)/s \} \), let

\[
E_t := \left\{ \omega \in I^* : p_\omega r_\omega^{2\ell - m} < t \leq p_\omega r_\omega^{2\ell - m} \right\},
\]

where \( I^* = \bigcup_{j \geq 1} \{1, \ldots, n\}^j \) and \( p_\omega r_\omega^{2\ell - m} := \prod_{k \geq 1} \omega_k 2^{2\ell - m}, \omega = \omega_1 \cdot \ldots \cdot \omega_k \in I^*, k \in \mathbb{N} \). Then, by our assumption \( \nu(\partial O) = 0 \), for all \( \omega \in E_t \), it follows that \( \nu(\partial T_\omega(O)) = 0 \) and

\[
\nu \left( \bigcup_{\omega \in E_t} T_\omega(O) \right) = 1 \text{ and } T_\omega(O) \cap T_{\omega'}(O) = \emptyset \text{ for all } \omega, \omega' \in E_t, \text{ with } \omega \neq \omega'.
\]

Hence,

\[
\sum_{\omega \in E_t} r_\omega x_\omega r_\omega^{2\ell - m} = 1 \leq t^e \text{ card}(E_t).
\]

Fix \( a \in K \) and choose \( u_0 \in C_0^\infty(O) \) such that \( u_0(a) > 0 \). For \( \omega \in E_t \), we set

\[
u_{\omega}(x) := \begin{cases} u_0(T_{\omega}^{-1}(x)) & , x \in T_\omega(O) \\ 0 & , x \in [0, 1]^d \setminus T_\omega(O) \end{cases}.
\]

Then \( u_\omega \in C_0^\infty(T_\omega(O)) \). Since the supports of \( (u_\omega)_{\omega \in E_t} \) are disjoint, it follows that the \( (u_\omega)_{\omega \in E_t} \) are mutually orthogonal both in \( L_\nu^2 \) and in \( H_0^{\ell} \), and \( \text{span}(u_\omega : \omega \in E_t) \) is therefore a card(\(E_t\))-dimensional subspace of \( H_0^{\ell} \). Moreover, we have

\[
J_r(u_\omega) = p_\omega \int u_\omega^2 \, dv \text{ and } I_r(u_\omega) = r_\omega^{-2\ell} I_r(u_0).
\]
4.3 One-dimensional Kreĭn–Feller operators

We obtain
\[
\frac{J_f(u_0)}{I_f(u_0)} = R^{2^{l-m}} p_0 \frac{J_f(u_0)}{I_f(u_0)}.
\]

Now, for \( u = \sum_{c\in E_i} c_i u_0 \in H^l_0 \setminus \{0\} \) with \( c_i \in \mathbb{R} \), we have
\[
\frac{J_f(u_0)}{I_f(u_0)} = \frac{\sum_{c\in E_i} c_i^2 J_f(u_0)}{\sum_{c\in E_i} c_i^2 I_f(u_0)} = R \frac{\sum_{c\in E_i} c_i^2 p_0^{2^{l-m}} I_f(u_0)}{\sum_{c\in E_i} c_i^2 I_f(u_0)} \geq tR \min p_i 2^{l-m}.
\]

The min-max principle stated in Proposition 4.5 gives
\[
tR \min p_i 2^{l-m} \leq \lambda^y_{\text{card}(E_i)} \leq \lambda^y_{[l^{-1}]^c}.
\]

In particular, for \( t = l^{-1/\kappa} \) and \( l \in \mathbb{N} \) large,
\[
l^{-1/\kappa} R \min p_i 2^{l-m} \leq \lambda^y_i.
\]

Combining this with (3.2), gives
\[
-\frac{1}{s_0} \leq \liminf_{l \to \infty} \frac{\log \left( \frac{\lambda^y_i}{\lambda^{y_{[l^{-1}]}^c}} \right)}{\log(l)} \leq \limsup_{m \to \infty} \frac{\log \left( \frac{\lambda^y_i}{\lambda^{y_{[l^{-1}]}^c}} \right)}{\log(l)} \leq -\frac{1}{s_0}.
\]

4.3 One-dimensional Kreĭn–Feller operators

In this final section, we show that the spectral problem of the Kreĭn–Feller operator in dimension one is actually equivalent to the spectral problem of polyharmonic operator \( T_{\nu} \) (excluding the eigenvalue zero) for the case \( l = m = 1 \) and \( p = q = 2 \). Let us recall the general setting for the one-dimensional Kreĭn–Feller Operator with respect to the finite Borel measure \( \nu \) on \((0, 1)\). We set
\[
C_\nu([0, 1]) := \{ f \in C([0, 1]) \mid f \text{ is affine linear on the components of } [0, 1] \setminus \text{supp}(\nu) \}
\]
and \( \text{dom}(E_\nu) := H^1_0 \cap C_\nu([0, 1]) \) with Dirichlet form \( E_\nu(f, g) := \int_{(0,1)} \nabla f \nabla g \ d\Lambda \). Now, we consider the spectral problem of the classical Kreĭn–Feller operator considered in [15, 16, 22]. We call \( u \in \text{dom}(E_\nu) \) \( \{0\} \) an eigenfunction with eigenvalue \( \lambda \) if
\[
\int_{(0,1)} \nabla f \nabla u \ d\Lambda = \lambda \int f u \ dv,
\]
for all \( f \in \text{dom}(E_\nu) \). We need a decomposition
\[
[0, 1] \setminus \text{supp}(\nu) =: A_1 \cup A_2 \cup \bigcup_{i \in I}(a_i, b_i),
\]
where \( I \subseteq \mathbb{N}, A_1 := [0, d_1] \) if \( 0 \notin \text{supp}(\nu) \) otherwise \( A_1 = \emptyset, A_2 := (d_2, 1] \) if \( 1 \notin \text{supp}(\nu) \) otherwise \( A_2 = \emptyset, \) and the intervals \([0, c_1], (c_2, 1], (a_i, b_i), i \in I, \) are mutually disjoint.

The following Lemma will provide a map from \( H^1_0 \) to \( \text{dom}(E_\nu) \).
4.3 One-dimensional Krein–Feller operators

Lemma 4.7 ([16, Lemma 2.1]). The map \( \tau_\nu : H^1_0 \to \text{dom}(\mathcal{E}_\nu) \)

\[
\tau_\nu(f)(x) := \begin{cases} 
  f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i} (x - a_i), & x \in (a_i, b_i), \ i \in N, \\
  f(x), & x \in \text{supp}(\nu), \\
  \frac{f(c_1)}{1 - c_1} (1 - x), & x \in (c_2, 1], \ 1 \not\in \text{supp}(\nu),
\end{cases}
\]

is surjective, \( \tau_\nu(f) = f \) as elements of \( L^2_\nu \), and we have

\[
\nabla \tau_\nu(f)(x) := \begin{cases} 
  \frac{f(b_i) - f(a_i)}{b_i - a_i}, & x \in (a_i, b_i), \ i \in N, \\
  \nabla f(x), & x \in \text{supp}(\nu), \\
  \frac{f(c_1)}{1 - c_1}, & x \in [0, c_1), \ 0 \not\in \text{supp}(\nu), \\
  -\frac{f(c_2)}{1 - c_2}, & x \in (c_2, 1], \ 1 \not\in \text{supp}(\nu).
\end{cases}
\]

Lemma 4.8. We have \( \varphi \in \text{dom}(\mathcal{E}_\nu) \) is an eigenfunction of \((4.1)\) with eigenvalue \( \lambda \), if and only if, for all \( g \in H^1_0 \), we have

\[
\int_{[0,1]} \nabla \varphi \nabla g \ d\Lambda = \lambda \int \varphi g \ dv.
\]

Proof. Recall, \([0, 1] \setminus \text{supp}(\nu) = A_1 \cup A_2 \cup \bigcup_{i \in I} (a_i, b_i)\) and define \( c_i = \nabla \varphi \left( \frac{a_i + b_i}{2} \right) \). For simplicity we assume \( 0, 1 \in \text{supp}(\nu) \) and let \( g \in H^1_0 \) be, then we have

\[
\int_{[0,1]} \nabla \varphi \nabla g \ d\Lambda = 
\int_{\text{supp}(\nu)} \nabla \varphi \nabla g \ d\Lambda + \sum_{i \in I} \int_{(a_i, b_i)} c_i \nabla g \ d\Lambda \\
= \int_{\text{supp}(\nu)} \nabla \varphi \nabla g \ d\Lambda + \sum_{i \in I} c_i (g(b_i) - g(a_i)) \\
= \int_{\text{supp}(\nu)} \nabla \varphi \nabla g \ d\Lambda + \sum_{i \in I} c_i (b_i - a_i) \left( \frac{g(b_i) - g(a_i)}{b_i - a_i} \right) \\
= \int_{\text{supp}(\nu)} \nabla \varphi \nabla \tau(g) \ d\Lambda + \sum_{i \in I} \int_{(a_i, b_i)} \nabla \varphi \nabla \tau(g) \ d\Lambda \\
= \int_{(0,1)} \nabla \varphi \nabla \tau(g) \ d\Lambda = \lambda \int f g \ dv. \quad \Box
\]

Proposition 4.9. Let \( \varphi \) be an eigenfunction of \( T_\nu \) with eigenvalue \( \lambda > 0 \). Then \( \varphi \) is affine linear on the connected components of \([0, 1] \setminus \text{supp}(\nu)\).

Proof. Here we closely follow [7, Proposition 3.2]. Let \((a, b)\) be a component \([0, 1] \setminus \text{supp}(\nu)\). Then we have for all \( f \in H^1_0 \)

\[
\lambda \int_{[0,1]} \nabla f \nabla \varphi \ d\Lambda = \int f \varphi \ dv.
\]

For \( x_1, x_2 \in (a, b), x_1 < x_2 \) and for all \( \delta > 0 \) sufficiently small such that

\[
a < x_1 - \delta < x_1 < x_1 + \delta < x_2 - \delta < x_2 < x_2 + \delta < b,
\]

we have

\[
\lambda \int_{(x_1 - \delta, x_1 + \delta)} \nabla f \nabla \varphi \ d\Lambda = \int f(x) \varphi(x) \ dx \]

and

\[
\lambda \int_{(x_2 - \delta, x_2 + \delta)} \nabla f \nabla \varphi \ d\Lambda = \int f(x) \varphi(x) \ dx.
\]

Therefore, \( \varphi \) is affine linear on the connected components of \([0, 1] \setminus \text{supp}(\nu)\).
we have that
\[
g : x \mapsto \frac{x - (x_2 - \Delta)}{2\Delta} \mathbb{1}_{(x_2 - \Delta, x_2 + \Delta)}(x) + \frac{x + \Delta - x}{2\Delta} \mathbb{1}_{(x_1 - \Delta, x_1 + \Delta)}(x).
\]
defines an element in \(H^1_0\). Hence, we obtain
\[
0 = \int_{[0,1]} \nabla g \nabla \varphi \, d\lambda = \frac{1}{2\Delta} \int_{(x_1 - \Delta, x_1 + \Delta)} \nabla \varphi \, d\lambda - \frac{1}{2\Delta} \int_{(x_2 - \Delta, x_2 + \Delta)} \nabla \varphi \, d\lambda.
\]
The Lebesgue differentiation theorem forces \(\nabla \varphi(x) = c\) almost everywhere. For all \(x \in (a, b)\), we obtain
\[
\varphi(x) = \varphi(a) + \int_{[a,x]} \nabla \varphi \, d\lambda = \varphi(a) + c(x - a).
\]
The following proposition shows that the equivalence of the spectral problems. \(\square\)

**Proposition 4.10.** We have that \(\lambda > 0\) is an eigenvalue of \(T_v\) if and only if \(1/\lambda\) is an eigenvalue of \((4.1)\).

**Proof.** Let \(\varphi\) be an eigenfunction of \(T_v\) with eigenvalue \(\lambda > 0\). Using \(4.9\) it follows \(\varphi \in \text{dom}(E_{\nu})\) and by definition we have for all \(f \in \text{dom}(E_{\nu}) \subset H^1_0\)
\[
\lambda \int_{[0,1]} \nabla f \nabla \varphi \, d\lambda = \int f \varphi \, dv.
\]
Hence, \(\varphi\) is an eigenfunction of \((4.1)\) with eigenvalue \(1/\lambda\).

Reversely, let \(\varphi \in \text{dom}(E_{\nu})\) be an eigenfunction of \((4.1)\) with eigenvalue \(\lambda\). Then it follows \(\lambda > 0\) and by Lemma \(4.8\) we have for all \(f \in H^1_0\)
\[
\int_{[0,1]} \nabla f \nabla \varphi \, d\lambda = \lambda \int f \varphi \, dv,
\]
which shows that \(\varphi\) is an eigenfunction of \(T_v\) with eigenvalue \(1/\lambda\). \(\square\)

We end this section by using the above observation to give a short proof of the sub-/superadditivity of the eigenvalue counting function announced in the introduction.

Now, for \(d_0 = 0 < d_1 < \cdots < d_n < d_{n+1} = 1\) with \(\nu([d_k]) = 0\), we define the following closed subspace of \(H^1_0\) given by
\[
F := \{ u \in H^1_0 : u(d_i) = 0, i \in \{1, \ldots, n\}\}.
\]
Note that \(F\) can be identified with
\[
F \simeq H^1_0((d_0, d_1)) \times H^1_0((d_2, d_3)) \times \cdots \times H^1_0((d_n, d_{n+1})).
\]
Furthermore, let \(T_{k,v}\) denote the operator on \(H^1_0((d_k, d_{k+1}))\) with respect to the form \((f, g) \mapsto \int_{[d_k,d_{k+1}]} fg \, dv\) and let \(T_{F,v}\) be the operator on \(F\) with respect to the form \((f, g) \mapsto \int fg \, dv\). We define the eigenvalue counting function of \(S \in \{T_v, T_{k,v}, T_{F,v}\}\) by
\[
N(x,S) := \text{card}\{n \in \mathbb{N} : \lambda_n^S \geq x\}, \ x > 0.
\]
Then the following sub-/superadditivity holds true.
Proposition 4.11. For all $x \geq 0$, we have
\[ \sum_{k=0}^{n} N(x, T_{k,v}) = N(x, T_{F,v}) \leq \sum_{k=0}^{n} N(x, T_{k,v}) + n. \]

Proof. From max-min principle we deduce
\[ N(x, T_{F,v}) \leq N(x, T_{v}). \]
Moreover, we have $\dim \left( H_{0}^{1}/F \right) = n$. Hence, it follows from [32, Proposition 1]
\[ N(x, T_{v}) \leq N(x, T_{F,v}) + n. \]
It remains to show $N(x, T_{F,v}) = \sum_{k=0}^{n} N(x, T_{k,v})$. Let $f$ be an eigenfunction with eigenvalue $\lambda > 0$ of $T_{k,v}$ with $\nu((d_{k}, d_{k+1})) > 0$. Then define
\[ g(x) = \|_{(d_{k}, d_{k+1})} f, \]
it follows $g \in F$ and we have for all $h \in F$
\[ \lambda \int_{(0,1)} \nabla h \nabla g \, d\Lambda = \lambda \int_{(d_{k}, d_{k+1})} \nabla h \nabla g \, d\Lambda = \int_{(d_{k}, d_{k+1})} hg \, d\nu = \int_{(0,1)} hg \, d\nu. \]
This implies $N(x, T_{F,v}) \geq \sum_{k=0}^{n} N(x, T_{k,v})$. On the other hand, if $f$ is an eigenfunction with eigenvalue $\lambda > 0$ of $T_{F,v}$, then $g = \|_{(d_{k}, d_{k+1})} f$ is an eigenfunction with eigenvalue $\lambda$ of $T_{k,v}$ provided $g \neq 0$ and $\nu((d_{k}, d_{k+1})) > 0$. Hence, we obtain $N(x, T_{F,v}) = \sum_{k=0}^{n} N(x, T_{k,v})$. □

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