Conserved charges and soliton solutions in affine Toda theory

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Abstract

We study the conserved charges of affine Toda field theories by making use of the conformally invariant extension of these theories. We compute the values of all charges for the single soliton solutions, and show that these are related to eigenvectors of the Cartan matrix of the finite-dimensional Lie algebra underlying the theory.
1. Introduction

Affine Toda field theory is a scalar field theory constructed from an underlying affine Lie algebra \( \hat{\mathfrak{g}} \), with many properties of the theory reflecting those of the algebra. Thus, for example, there are infinitely many conserved charges in the theory [1–4], with spins equal to the exponents of \( \hat{\mathfrak{g}} \), and by constructing S-matrix elements for the theory it has been observed that the values of these charges form eigenvectors of the Cartan matrix of the finite-dimensional Lie algebra \( \mathfrak{g} \) associated to \( \hat{\mathfrak{g}} \) [5–10]. The allowed three-point couplings were also seen to be related to \( \mathfrak{g} \) [5,6,11–13], and in [14] Dorey gave a characterization of these couplings that explained many of the above results. In a separate development, an explanation was given of the Lie-algebraic nature of the classical mass spectrum and couplings [15,16] starting from the Lagrangian of the theory.

It was subsequently proposed by Hollowood [17] to study the theory with an imaginary value of the coupling constant, for which there are infinitely many degenerate vacuum states with soliton solutions interpolating between them. Such solutions were constructed first for the \( A_n^{(1)} \) theories, and although they are not real they nevertheless have real energy. Further soliton solutions were found for other Lie algebras in [18-24], with the most general solution being given in [18,19], and the energy and momentum were found to be real in these cases also. The masses of the single soliton solutions were seen to form the \textit{left} Perron-Frobenius eigenvector of the Cartan matrix of \( \mathfrak{g} \) [18], while the masses of the fundamental excitations formed the \textit{right} Perron-Frobenius eigenvector. These results provide evidence for a conjectured duality between affine Toda field theories, in which the solitons of one theory correspond to the fundamental excitations of another.

In this paper we extend the calculation of Olive, Turok and Underwood [18] by computing the classical values of all conserved charges for the single soliton solutions. We shall see that these are all real, apart from a possible overall phase factor, and that the values of the charge of spin-\( N \) form the left eigenvector of the Cartan matrix of \( \mathfrak{g} \) with eigenvalue \( 2(1 - \cos \pi N/h) \), where \( h \) is the Coxeter number of \( \mathfrak{g} \). This provides further evidence for the duality of affine Toda theories.

The conserved charges are also the subject of interesting work by Niedermaier [25,26], in which expressions for the charges are obtained by the inverse scattering method for the case \( \hat{\mathfrak{g}} = A_n^{(1)} \). It would be worthwhile to extend this
approach to arbitrary affine algebras and to relate it to that given here.

2. Untwisted affine Lie algebras and the affine Toda equation

We first review those concepts from the theory of untwisted affine Lie algebras that will be needed in this paper [27].

2.1 Affine Lie algebras

Let \( g \) be a finite-dimensional simple Lie algebra of rank \( r \), with simple roots \( \alpha_1, \ldots, \alpha_r \) and Cartan matrix \( K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2 \). The so-called extended Cartan matrix is constructed similarly from an extended set of roots, obtained by adjoining to the set of simple roots an extra root \(-\psi\), where \( \psi \) is the highest root of \( g \). This extra root will be denoted \( \alpha_0 \), and the extended Cartan matrix will again be written as \( K_{ij} \) but with \( i \) and \( j \) taking values from 0 to \( r \). This matrix will be the Cartan matrix of the untwisted affine Lie algebra \( \hat{g} \) corresponding to \( g \). In the Chevalley basis \( \hat{g} \) is generated by elements \( e_i, f_i, h_i \), for \( i = 0, \ldots, r \), satisfying the commutation relations

\[
[h_i, h_j] = 0, \\
[h_i, e_j] = K_{ji}e_j, \\
[h_i, f_j] = -K_{ji}f_j, \\
[e_i, f_j] = \delta_{ij}h_i
\] (2.1)

and subject to the Serre relations

\[
(ad e_i)^{1-K_{ji}}e_j = 0 \quad (2.2)
\]

and

\[
(ad f_i)^{1-K_{ji}}f_j = 0. \quad (2.3)
\]

With respect to this basis there is a natural grading of the algebra, called the principal grading, such that the elements \( e_i \) have grade 1 and the \( f_i \) grade \(-1\).† The elements of grade > 0 generate a subalgebra \( \hat{n}_+ \) and those of grade < 0 a subalgebra \( \hat{n}_- \). We shall assume that it is legitimate to exponentiate these

† It is often useful to add to this algebra an extra element called the derivation, the adjoint action of which produces this grading, but we shall not do this here.
subalgebras where necessary, thereby obtaining groups $\hat{N}_\pm$, and we shall write $\hat{T}$ for the group obtained by exponentiating elements of grade zero.

The fact that the highest root of $\mathfrak{g}$ has an expansion in terms of simple roots implies the existence of a relation $\sum_0^r n_i \alpha_i = 0$, where the $n_i$ are positive integers and $n_0 = 1$. From this it follows that the Cartan matrix of $\hat{\mathfrak{g}}$ has a left-eigenvector with eigenvalue zero,

$$\sum_{i=0}^r n_i K_{ij} = 0. \quad (2.4)$$

By considering the Lie algebra $\check{\mathfrak{g}}$ whose roots $\check{\alpha} = 2\alpha/\alpha^2$ are the co-roots of $\mathfrak{g}$ it follows similarly that

$$\sum_{j=0}^r K_{ij} m_j = 0, \quad (2.5)$$

with $m_j$ positive integers and $m_0 = 1$. The Coxeter number $h$ and dual Coxeter number $\check{h}$ of $\mathfrak{g}$ are defined by $h = \sum_{i=0}^r n_i$ and $\check{h} = \sum_{i=0}^r m_i$.

A consequence of eqn (2.1) is that $\hat{\mathfrak{g}}$ contains a central element $x = \sum_0^r m_i h_i$, with $[x, e_i] = [x, f_i] = 0$. Let $\check{E}_1$ and $\check{E}_{-1}$ be linear combinations of the $e_i$ and $f_i$ respectively such that

$$[\check{E}_1, \check{E}_{-1}] = x; \quad (2.6)$$

this condition does not determine $\check{E}_\pm$ uniquely, but a convenient choice is $\check{E}_1 = \sum_i \sqrt{m_i} e_i$ and $\check{E}_{-1} = \sum_i \sqrt{m_i} f_i$. Most of our results, however, will be independent of any particular choice. Consider now the set of non-trivial elements of $\hat{\mathfrak{g}}$ that commute with $\check{E}_\pm$ modulo the central element $x$. Each such element can be chosen to have definite principal grade, and it can be shown [27] that an element $\check{E}_N$ of grade $N$ exists precisely when $N$ is equal to an exponent of $\mathfrak{g}$ modulo the Coxeter number. Furthermore, with a suitable choice of normalisation we have

$$[\check{E}_N, \check{E}_M] = x N \delta_{N+M,0}, \quad (2.7)$$

so the elements $\check{E}_N$ and $x$ span an infinite-dimensional Heisenberg subalgebra of $\hat{\mathfrak{g}}$.

### 2.2 The affine Toda equations

Affine Toda field theory is formulated in terms of a field $\phi$ taking values in the grade zero subalgebra $\hat{\mathfrak{g}}_0$ of $\hat{\mathfrak{g}}$, spanned by $h_0, \ldots, h_r$. The equation of motion for
\( \phi \) is
\[
\beta \partial_{\mu} \partial^{\mu} \phi + 4m^2 [e^{\beta \phi} \hat{E}_1 e^{-\beta \phi}, \hat{E}_{-1}] = 0, \tag{2.8}
\]
which is independent of any particular choice that is made for \( \hat{E}_{\pm 1} \) satisfying eqn (2.6). It should be remarked that this equation differs slightly from the conventional form of the affine Toda equation, but this can be recovered from eqn (2.8) by expanding \( \phi \) in the form \( \phi = \sum_{i} \phi_i h_i + \phi_x x \). The equations for \( \phi_i \) are
\[
\beta \partial_{\mu} \partial^{\mu} \phi_i + 4m^2 m_i \left( e^{\beta \sum_{j=1}^{r} K_{ij} \phi_j} - e^{\beta \sum_{j=1}^{r} K_{0j} \phi_j} \right) = 0, \tag{2.9}
\]
which are just the usual affine Toda equations, while \( \phi_x \) is determined non-locally in terms of the \( \phi_i \) by the equation
\[
\beta \partial_{\mu} \partial^{\mu} \phi_x + 4m^2 e^{\beta \sum_{j=1}^{r} K_{0j} \phi_j} = 0. \tag{2.10}
\]
Eqns (2.9) and (2.10) are the conformal affine Toda equations [28], with the field corresponding to the derivation set to zero.

It will be useful to introduce light-cone coordinates \( x^\pm = x \pm t \), with respect to which the equation of motion (2.8) becomes
\[
-\beta \partial_{+} \partial_{-} \phi + m^2 [e^{\beta \phi} \hat{E}_1 e^{-\beta \phi}, \hat{E}_{-1}] = 0. \tag{2.11}
\]
This can be written as a zero curvature condition,
\[
[\partial_{+} + A_{+}, \partial_{-} + A_{-}] = 0, \tag{2.12}
\]
with
\[
A_{+} = me^{\beta \phi} \hat{E}_1 e^{-\beta \phi} \tag{2.13}
\]
and
\[
A_{-} = m \hat{E}_{-1} - \beta \partial_{-} \phi. \tag{2.14}
\]
The standard procedure used to demonstrate the integrability of this system is to make a gauge transformation \( A_{\mu} \rightarrow A_{\mu}^\omega = \omega^{-1} \partial_{\mu} \omega + \omega^{-1} A_{\mu} \omega \) such that \( A_{\mu}^\omega \) lies entirely in the subspace of \( \hat{g} \) spanned by the \( \hat{E}_N \) with \( N \geq -1 \). This is possible with \( \omega \) having the form \( \omega = \exp\{\omega_1 + \omega_2 + \ldots\} \), where \( \omega_i \) is a local function of \( \partial_{-} \phi \) with grade \( i \). The equation of motion then implies that \( A_{+}^\omega \) lies in the span
of the $\hat{E}_N$ with $N \geq 2$, from which it follows that $A^- \omega$ and $A^+ \omega$ commute. In this
gauge the equation of motion takes the form

$$\partial_+ A^- \omega = \partial_- A^+ \omega,$$

which implies immediately the existence of an infinite number of conserved currents. These can conveniently be expressed by taking the commutator of eqn (2.15) with $\hat{E}_N$, giving an infinite set of conserved currents taking values in the centre

$$\partial_+ [A^- \omega, \hat{E}_N] = \partial_- [A^+ \omega, \hat{E}_N].$$

Unfortunately this procedure for constructing conserved charges suffers from a
problem in the present context. The gauge transformation $A_- \to A^- \omega$ is not
uniquely determined, since there is a freedom to multiply $\omega$ on the right by the
exponential of any linear combination of elements of the Heisenberg subalgebra
with grade $\geq 2$. This changes the conserved currents by a total derivative, and
although this would not normally have any effect on the conserved charges it does
here. The reason for this is that even though the fields $\phi_i$ and their derivatives can
be taken to vanish at infinity, the same is not true for $\phi_x$, because this field must
satisfy eqn (2.10). For the vacuum solution $\phi_i = 0$, for example, we find $\phi_x = m^2 x^+ x^- / \beta$, and in fact the boundary terms coming from integrating derivatives
of $\phi_x$ will be crucial in obtaining the conserved charges of soliton solutions.

There is, however, another form for the conserved currents, which does not suffer
from these disadvantages. This alternative form was first given by Wilson [1]
for the unextended affine Toda theory (2.9), and in that context the relation to
the construction outlined above is somewhat obscure. In the extended theory
we are considering, however, the connection between these two approaches follows
immediately from an attempt to construct commuting flows for the equation (2.8).

### 2.3 Commuting flows in extended affine Toda theory

In the construction of commuting flows it is convenient to regard $x^-$ as the spatial
coordinate and to consider $x^+$ as the time direction. We introduce an infinite
collection of additional time variables $t_{-N}$ and define the evolution of $\phi$ with
respect to these by the zero curvature conditions

$$ \left[ \frac{\partial}{\partial x^-} + A_-, \frac{\partial}{\partial t_{-N}} + A_- N \right] = 0$$

(2.17)
for some suitable $A_{-N}$. A natural choice would be to take $A_{-N}$ equal to $m^N \sum_{n \leq 0} (\omega \hat{E}_{-N} \omega^{-1})_n$, with $\omega$ being the group element introduced earlier and with $(\omega \hat{E}_{-N} \omega^{-1})_n$ denoting the terms of grade $n$ in $(\omega \hat{E}_{-N} \omega^{-1})$. This generalises $A_-$, in that $A_{-1} = A_-$, and in the non-extended theory the analogue of this construction leads to an infinite set of commuting flows. In the extended theory, however, the evolution of the additional field $\phi_x$ depends explicitly on the choice of $\omega$, and the evolution operators $\partial_{-N}$ do not commute either amongst themselves or with $\partial_+$ when acting on $\phi_x$. In this case a better choice is to take instead

$$A_{-N} = -m^N \sum_{n>0} (\omega \hat{E}_{-N} \omega^{-1})_n,$$

which leads to the same evolution equations in the non-extended theory but does not suffer from the above difficulties.

In order for the evolution equation (2.17) to be consistent, it is essential that $\partial_{-N} \phi$ as determined by it should have grade zero. This follows from the fact that $A_{-N}$ can be written in two distinct ways,

$$A_{-N} = -m^N \sum_{n>0} (\omega \hat{E}_{-N} \omega^{-1})_n$$

and

$$A_{-N} = -m^N (\omega \hat{E}_{-N} \omega^{-1}) + m^N \sum_{n \leq 0} (\omega \hat{E}_{-N} \omega^{-1})_n.\tag{2.20}$$

From these we obtain the following two expressions for $\partial_{-N} \partial_{-} \phi$,

$$\beta \frac{\partial}{\partial t_{-N}} \partial_{-} \phi = m^{N+1} [\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]\tag{2.21}$$

and

$$= m^N [A_{-N}, \hat{E}_{-N}] - m^N \partial_{-} (\omega \hat{E}_{-N} \omega^{-1})_0,\tag{2.22}$$

and from the first of these we see that, as claimed above, the evolution $\partial_{-N}$ does not depend on the particular choice of $\omega$ that we make, because the quantity $[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]$ is unchanged under the group of transformations that leaves the form of $A_{-N}$ invariant,

$$[\hat{E}_{-1}, (\omega e^{b_N E_N}) \hat{E}_{-N} (e^{-b_N E_N} \omega^{-1})] = [\hat{E}_{-1}, \omega (\hat{E}_{-N} + Nb_N x) \omega^{-1}]$$

$$= [\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})].\tag{2.23}$$

To verify the commutativity of the flows, we consider the action of the evolution operators on $A_-$. Thus, for example, $[\partial_{-M}, \partial_{-N}]A_-$ is given by

$$[\partial_{-M}, \partial_{-N}]A_- = [\partial_+ + A_-, \partial_{-M}A_{-N} - \partial_{-N}A_{-M} + [A_{-M}, A_{-N}]]\tag{2.24}$$
and this can be seen to vanish when we insert the explicit expressions for $A_{-N}$ and $A_{-M}$ from eqn (2.19) and make use of the fact that $A^\omega_{-N}$ and $A^\omega_{-M}$ are forced, as a consequence of eqn (2.17), to lie in the Heisenberg subalgebra of $\hat{g}$. The vanishing of $[\partial_+, \partial_{-N}]$ is essentially similar except that in this case we need to evaluate $[\partial_{-N}\phi, A_+]$, while from eqns (2.21, 2.22) we know only $\partial_{-N}\partial_{-}\phi$. From eqn (2.22), however, we see that

$$\partial_{-} \left\{ \beta \frac{\partial}{\partial t_{-N}} \phi + m^N (\omega \hat{E}_{-N} \omega^{-1})_0 \right\}$$

(2.25)

will vanish in any commutator, so that within a commutator it will be consistent to set $\beta \partial_{-N}\phi$ equal to $-m^N (\omega \hat{E}_{-N} \omega^{-1})_0$. Provided we make this choice, the flows $\partial_{+}$ and $\partial_{-N}$ will commute.

Comparing the two expressions (2.21) and (2.22) we see that†

$$m[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1] = [A^\omega_{-}, \hat{E}_{-N}] - \partial_{-} (\omega \hat{E}_{-N} \omega^{-1})_0.$$  

(2.26)

Thus $m[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]$ is a conserved density, since it differs from $[A^\omega_{-}, \hat{E}_{-N}]$ by a derivative term. This is the form for the conserved densities given by Wilson [1]. It should be remarked that only the terms proportional to the central element $x$ in eqn (2.26) are of interest, because $[A^\omega_{-}, \hat{E}_{-N}]$ lies in the centre of $\hat{g}$ and so all components of $[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]$ other than that proportional to $x$ are given by derivatives of local fields. The corresponding charges therefore vanish.

It is interesting to note that the field $\phi_x$ does not enter into the expression $[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]$ for the conserved densities [4]. To see this we exploit the fact that these quantities are independent of $\omega$ to make the following convenient choice. We first find $\omega_0$ depending only on $\phi_1, \ldots, \phi_r$ such that $A^{\omega_0}_{-}$ takes the form

$$A^{\omega_0}_{-} = m\hat{E}_{-1} - \beta \partial_{-}\phi_x x + \sum_{N>0} a_N \hat{E}_N;$$

(2.27)

$\omega_0$ is just the gauge transformation that would be used in the non-conformal affine Toda field theory to transform $A_{-}$ and $A_{+}$ into an abelian subalgebra of $g$. We

† We note that eqn (2.26) could have been obtained immediately, without introducing the time variable $t_{-N}$, by evaluating directly the expression $\partial_{-} (\omega \hat{E}_{-N} \omega^{-1})_0$ and making use of the fact that $A^\omega_{-}$ lies in the Heisenberg subalgebra.

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can now make a further transformation by \( \exp (\phi x \hat{E}_1) \) to remove the central term from eqn (2.27), but the argument given above implies that this has no effect on \([\hat{E}_{-1}, (\omega \hat{E}_{-N}\omega^{-1})_1]\) and so the conserved densities are indeed independent of \( \phi X \).

In addition to the conserved densities \([\hat{E}_{-1}, (\omega \hat{E}_{-N}\omega^{-1})_1]\) of negative spin, there is another set of conserved densities with positive spin. These are obtained by first making a gauge transformation on the potentials \( A_+ \) and \( A_- \) to bring them into the form

\[
A_+ = m\hat{E}_1 + \beta\phi \tag{2.28}
\]

and

\[
A_- = me^{-\beta\phi} \hat{E}_{-1} e^{\beta\phi}. \tag{2.29}
\]

We then make a further gauge transformation \( A_\mu \rightarrow A'_\mu \) such that \( A'_\mu \) lies in the subspace of \( \hat{g} \) spanned by the \( \hat{E}_N \) with \( N \leq 1 \). Conserved densities of positive spin can then be constructed by a procedure entirely analogous to that given above. From here on we shall treat only the negative spin densities, but equivalent results will hold for those of positive spin.

### 3. The conserved charges of soliton solutions

In order to calculate the values of the conserved charges for soliton solutions it is necessary to obtain explicit expressions for \( \partial_{-N}\phi \); the conserved charges will then follow from eqns (2.21) and (2.22).

#### 3.1 The Leznov-Saveliev solution and the conserved charges

We shall use the method of Leznov and Saveliev [29] to solve the Toda equations. This is closely related to the approach of Date et al [30]; the relation between these methods is explained in reference [31]. The starting point is the zero curvature condition

\[
[\partial_- + A_-, \partial_+ + A_+] = 0, \tag{3.1}
\]

which implies the existence of some monodromy matrix \( T \) such that

\[
A_+ = T^{-1} \partial_+ T
\]

\[
A_- = T^{-1} \partial_- T \tag{3.2}
\]
Following Leznov and Saveliev we now consider two different Borel decompositions of $T$,

$$T = UHV^{-1}, \quad U \in \hat{N}_+, \ H \in \hat{T}, \ V \in \hat{N}_-, \quad (3.3)$$

and

$$T = \tilde{V} \tilde{H} \tilde{U}^{-1}, \quad \tilde{V} \in N_-, \ \tilde{H} \in T, \ \tilde{U} \in N_+. \quad (3.4)$$

We shall write $H = e^h$ and $\tilde{H} = e^{\tilde{h}}$. From the equation for $A_+$ we obtain

$$\partial_+ \tilde{V} = 0 \quad (3.5)$$
$$\partial_+ \tilde{h} = 0 \quad (3.6)$$
$$\tilde{U} \partial_+ \tilde{U}^{-1} = me^{\beta \phi} \hat{E}_1 e^{-\beta \phi} \quad (3.7)$$
$$H^{-1} U^{-1} \partial_+ U H = me^{\beta \phi} \hat{E}_1 e^{-\beta \phi}, \quad (3.8)$$

and from $A_-$ we have

$$\partial_- U = 0 \quad (3.9)$$
$$\partial_- h + \beta \partial_- \phi = 0 \quad (3.10)$$
$$\tilde{H}^{-1} \tilde{V} \tilde{H}^{-1} \partial_- \tilde{V} \tilde{H} = m \hat{E}_{-1} \quad (3.11)$$
$$\tilde{U} \partial_- \tilde{U}^{-1} + \tilde{U} (m \hat{E}_{-1} + \partial_- \tilde{h}) \tilde{U}^{-1} = m \hat{E}_{-1} - \beta \partial_- \phi. \quad (3.12)$$

A convenient way to obtain solutions to the affine Toda equations is to consider the group element

$$g = e^{-\beta \phi V^{-1} \tilde{U}}$$
$$= e^{-\beta \phi H^{-1} U^{-1} \tilde{V} \tilde{H}}. \quad (3.13)$$

From eqns (3.5–3.12) it follows that

$$\partial_+ gg^{-1} = -m \hat{E}_1 - \partial_+ (\beta \phi + h) \quad (3.14)$$
$$g^{-1} \partial_- g = m \hat{E}_{-1} + \partial_- \tilde{h}. \quad (3.14)$$

In addition we have $\partial_- (\partial_+ gg^{-1}) = \partial_+ (g^{-1} \partial_- g) = 0$, so that $g$ can be thought of as a constrained WZW field [32]. The general solution to the affine Toda equations is obtained by solving (3.14) to express $g$ in terms of the free fields $\tilde{h}$ and $h + \beta \phi$ and then recovering $\phi$ by taking the matrix elements of $g$ between general highest weight states. The soliton solutions are precisely those with $\tilde{h} = 0$ and $h = -\beta \phi$ [18], in which case

$$g(x^+, x^-) = e^{-m \hat{E}_1 x^+} g(0) e^{m \hat{E}_{-1} x^-}, \quad (3.15)$$
where the group element $g(0)$ is a constant of integration. Then, if $|\Lambda\rangle$ is any highest weight state, soliton solutions are given by

$$
\langle \Lambda | e^{-\beta \phi} | \Lambda \rangle = \langle \Lambda | e^{-m \hat{E}_1 x^+} g(0) e^{m \hat{E}_{-1} x^-} | \Lambda \rangle.
$$

(3.16)

Furthermore, when $\tilde{h}$ and $h + \beta \phi$ vanish, a candidate for the group element $\omega$ is $\tilde{U}$, since from eqns (3.12) and (3.7) we see that

$$
A_{\tilde{U}} = m \hat{E}_{-1}, \quad A_+ = 0.
$$

(3.17)

In order to find the conserved charges of soliton solutions we need to evaluate the term in the current $[\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1]$ that is proportional to the central element $x$. To do this we can take the expectation of this current in the highest weight state $|\Lambda_0\rangle$ of the vacuum representation of $\hat{g}$ at level one. From eqns (2.26) and (3.17), with $\omega = \tilde{U}$, this is given by

$$
m \langle \Lambda_0 | [\hat{E}_{-1}, (\tilde{U} \hat{E}_{-N} \tilde{U}^{-1})_1] | \Lambda_0 \rangle = -\partial_- \langle \Lambda_0 | (\tilde{U} \hat{E}_{-N} \tilde{U}^{-1}) | \Lambda_0 \rangle.
$$

(3.18)

The group element $\tilde{U}^{-1}$ acts as the identity on highest weight states, and from the definition of $g$ in eqn (3.13) we see that

$$
\langle \Lambda_0 | \tilde{U} \hat{E}_{-N} | \Lambda_0 \rangle = \frac{\langle \Lambda_0 | g \hat{E}_{-N} | \Lambda_0 \rangle}{\langle \Lambda_0 | g | \Lambda_0 \rangle}.
$$

(3.19)

The general expression for the conserved charge is therefore

$$
Q_{-N} \equiv \int_{-\infty}^{\infty} dx^- m^N \langle \Lambda_0 | [\hat{E}_{-1}, (\omega \hat{E}_{-N} \omega^{-1})_1] | \Lambda_0 \rangle
$$

$$
= \left[ m^N \frac{\langle \Lambda_0 | g \hat{E}_{-N} | \Lambda_0 \rangle}{\langle \Lambda_0 | g | \Lambda_0 \rangle} \right]_{x^- = -\infty}^{x^- = \infty}.
$$

(3.20)

Just as for the energy-momentum tensor, the only contribution for soliton solutions is a surface term.

### 3.2 Conserved charges and the Cartan matrix

For the single soliton solutions, eqn (3.20) is particularly simple to evaluate. These solutions are characterised by $g(0)$ having the form $g(0) = \exp\{Q \hat{F}(\alpha, \rho)\}$ [18],



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where $Q$ is a constant and $\hat{F}(\alpha, \rho)$ is an element of $\hat{\mathfrak{g}}$ that diagonalises the adjoint action of the Heisenberg subalgebra,

$$[\hat{E}_N, \hat{F}(\alpha, \rho)] = (\alpha, E_{\nu}) \rho^N \hat{F}(\alpha, \rho). \quad (3.21)$$

Here $\rho$ is a constant, $\alpha$ is a root of $\mathfrak{g}$, and $E_{\nu}$ is the element of $\mathfrak{g}$ of which $\hat{E}_N$ is the abelianization; $\nu$, which is equal to $N$ mod $h$, is an exponent of $\mathfrak{g}$. It can be shown that the $E_{\nu}$ span a Cartan subalgebra of $\mathfrak{g}$ [33], and $(\alpha, E_{\nu})$ is the natural pairing between roots and elements of this Cartan subalgebra. These results follow from the isomorphism of the affine Lie algebra $\hat{\mathfrak{g}}$ to the central extension of the loop algebra of $\mathfrak{g}$ [27]. Using these results the single soliton solution can be written in the form

$$\langle \Lambda | e^{-\beta \phi} | \Lambda \rangle = \langle \Lambda | e^{-x m^2 x^+ x^-} \exp \left\{ Q(x^+, x^-) \hat{F}(\alpha, \rho) \right\} | \Lambda \rangle, \quad (3.22)$$

where

$$Q(x^+, x^-) = Q e^{-m x^- (\alpha, E_{h-1}) \rho^{-1} - m x^+ (\alpha, E_1) \rho}. \quad (3.23)$$

For static solitons, which is all that we shall consider here, the parameter $\rho$ must satisfy the condition

$$(\alpha, E_1) \rho = (\alpha, E_{h-1}) \rho^{-1}; \quad (3.24)$$

$Q(x^+, x^-)$ will then be independent of $t$, and the only time-dependence in $\phi$ will be in the field $\phi_x$.

To evaluate explicitly the values of the charges for these static solutions we write

$$\langle \Lambda_0 | g(x^+, x^-) \hat{E}_{-N} \Lambda_0 \rangle = -(\alpha, E_{h-\nu}) \rho^{-N} Q(x^+, x^-) \times$$

$$\langle \Lambda_0 | e^{-x m^2 x^+ x^-} \hat{F}(\alpha, \rho) \exp \left\{ Q(x^+, x^-) \hat{F}(\alpha, \rho) \right\} | \Lambda_0 \rangle, \quad (3.25)$$

using the commutation relations of $\hat{F}$ with the Heisenberg subalgebra. The value of the charge $Q_{-N}$ is then

$$Q_{-N} = -(\alpha, E_{h-\nu}) \left( \frac{m}{\rho} \right)^{N} \times \left[ \frac{Q(x^+, x^-) \langle \Lambda_0 | \hat{F}(\alpha, \rho) \exp \left\{ Q(x^+, x^-) \hat{F}(\alpha, \rho) \right\} | \Lambda_0 \rangle}{\langle \Lambda_0 | \exp \left\{ Q(x^+, x^-) \hat{F}(\alpha, \rho) \right\} | \Lambda_0 \rangle} \right]_{x^- = -\infty}^{\infty} \quad (3.26)$$
Since the highest non-vanishing power of $\hat{F}(\alpha, \rho)$ in a representation of level $x$ is $2x/\alpha^2$ [19], it is easy to see by expanding the exponentials in this expression as a power series that

$$Q_{-N} = \pm \frac{2}{\alpha^2} (\alpha, E_{h_{\nu}}) \left( \frac{m}{\rho} \right)^N$$

(3.27)

for a soliton at rest, where the sign depends on whether $(\alpha, E_{h_{-1}})\rho^{-1}$ is positive or negative.

Alternatively we could use eqn (3.26) to derive the relation

$$Q_{-N} = \frac{(\alpha, E_{h_{-\nu}})}{(\alpha, E_{h_{-1}})} \left( \frac{m}{\rho} \right)^{N-1} Q_{-1}$$

(3.28)

and then find the values of $Q_{-N}$ by using the fact that the values of the charge $Q_{-1}$ are known [18] to be proportional to the entries in the left Perron-Frobenius eigenvector of the Cartan matrix of $g$.

To complete the evaluation of the conserved charges we need to know the quantities $(\alpha, E_{\nu})$, which can be calculated [15,16] using the fact that the elements $E_{\nu}$ are eigenvectors of a Coxeter element $\sigma$ of the Weyl group $W$ of $g$ [33]. Let us review briefly this procedure, which relates eigenvectors of the Coxeter element to those of the Cartan matrix [34,35]. The easiest way to do this is to make use of the isomorphism between the Cartan subalgebra $H$ of $g$ and its dual $H^*$, spanned by the roots of $g$ [16]. Thus, to each element $E_{\nu}$ there corresponds an element $q_{\nu}$ such that

$$(E_{\nu}, h) = (q_{\nu}, h) \quad \forall \ h \in H,$$

(3.29)

where $(E_{\nu}, h)$ denotes the natural inner product on $H$ and $(q_{\nu}, h)$ the pairing between elements of $H^*$ and $H$. By means of this isomorphism $H^*$ inherits an inner product from that on $H$, and using this we can write the relation between $E_{\nu}$ and $q_{\nu}$ as

$$(\alpha, E_{\nu}) = (\alpha, q_{\nu}).$$

(3.30)

We shall therefore look for the eigenvectors $q_{\nu}$ of $\sigma$ acting on the root space. We label the points of the Dynkin diagram of $g$ alternately black and white, and write $\sigma = \sigma_B \sigma_W$, where $\sigma_B$ (respectively $\sigma_W$) is a product of Weyl reflections in all of the black (white) simple roots. Writing the Cartan matrix of $g$ as $K = 2 - C$, 

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so that $C$ is the incidence matrix of the Dynkin diagram of $g$ and takes the block form

$$C = \begin{pmatrix} 0 & C_{wb} \\ C_{bw} & 0 \end{pmatrix},$$

(3.31)

it is elementary to show that

$$\sigma_W(\alpha_w) = -\alpha_w$$

$$\sigma_B(\alpha_w) = \alpha_w + \sum_{b \in B} C_{wb} \alpha_b$$

(3.32)

and

$$\sigma_B(\alpha_b) = -\alpha_b$$

$$\sigma_W(\alpha_b) = \alpha_b + \sum_{w \in W} C_{bw} \alpha_w.$$  

(3.33)

Suppose now that $(x^\nu_w, x^\nu_b)$ is a left eigenvector of $C$ with eigenvalue $2 \cos \pi \nu / h$,

$$\sum_b x^\nu_b C_{bw} = 2 \cos(\pi \nu / h) x^\nu_w, \quad \sum_w x^\nu_w C_{wb} = 2 \cos(\pi \nu / h) x^\nu_b.$$  

(3.34)

Then $(x^\nu_w, -x^\nu_b)$ is also an eigenvector of $C$, with eigenvalue $2 \cos \pi (h - \nu) / h$, so we can take $(x^{h-\nu}_w, x^{h-\nu}_b)$ to be proportional to $(x^\nu_w, -x^\nu_b)$. The constant of proportionality can always be chosen to be 1, except for the case $\nu = h/2$ when it can be either $\pm 1$. For simplicity we shall not treat the case $\nu = h/2$ here, although the analysis is similar. The right eigenvectors of $C$ are given by $(y^\nu_w, y^\nu_b) = (x^\nu_w \alpha_w^2, x^\nu_b \alpha_b^2)$, and we will choose the normalisation

$$\sum_w x^\nu_w y^\mu_w + \sum_b x^\nu_b y^\mu_b = h \delta_{\nu, \mu}.$$  

(3.35)

The eigenvectors of the Coxeter element $\sigma$ can now be written in terms of those of the Cartan matrix. The eigenvector with eigenvalue $\exp(2\pi i \nu / h)$ can be chosen to be

$$\exp\left(\frac{-\pi i \nu}{2h}\right) \sum_w x^\nu_w \alpha_w + \exp\left(\frac{\pi i \nu}{2h}\right) \sum_b x^\nu_b \alpha_b,$$  

(3.36)

and given the normalization $(E_\nu, E_\mu) = h \delta_{\nu+\mu, h}$ it is easy to see that the correctly normalized $q_\nu$ are given in terms of these eigenvectors by

$$q_\nu = e^{\pi i / 4} \sin \pi \nu / h \left\{ \exp\left(\frac{-\pi i \nu}{2h}\right) \sum_w x^\nu_w \alpha_w + \exp\left(\frac{\pi i \nu}{2h}\right) \sum_b x^\nu_b \alpha_b \right\},$$  

(3.37)
from which we obtain

\[
(\alpha_w, q_\nu) = \exp(\pi i\nu/2h - \pi i/4) y_w' \tag{3.38}
\]

and

\[
(\alpha_b, q_\nu) = \exp(-\pi i\nu/2h + 3\pi i/4) y_b'. \tag{3.39}
\]

We now have all the information we need to compute the charges of the solitons. There are \(r\) species of solitons, given by taking \(\alpha\) equal to \(\alpha_w\) and \(-\alpha_b\) in \(\hat{F}(\alpha, \rho)\) [18]. Let us consider the case \(\alpha = \alpha_w\) first. We have

\[
(\alpha_w, q_1) = \exp(\pi i/2h - \pi i/4) y_w^1 \tag{3.40}
\]

and

\[
(\alpha_w, q_{h-1}) = \exp(-\pi i/2h + \pi i/4) y_w^1, \tag{3.41}
\]

from which it follows

\[
\rho = \mp \exp(\pi i/4 - \pi i/2h), \tag{3.42}
\]

where the minus sign corresponds to a soliton and the plus to an antisoliton. We have also

\[
(\alpha_w, q_{h-\nu}) = \exp(\pi i/4 - \pi i\nu/2h) y_w^\nu, \tag{3.43}
\]

so from eqn (3.27) we obtain

\[
Q_{-N} = 2(-1)^N m^N \exp(-(N - 1)\pi i/4 + (N - \nu)\pi i/2h) x_w^\nu \tag{3.44}
\]

for a soliton.

In the case \(\alpha = -\alpha_b\), we have

\[
\rho = \mp \exp(\pi i/4 + \pi i/2h) \tag{3.45}
\]

and

\[
(-\alpha_b, q_{h-\nu}) = \exp(\pi i/4 + \pi i\nu/2h) y_b^\nu, \tag{3.46}
\]

so

\[
Q_{-N} = 2(-1)^N m^N \exp(-(N - 1)\pi i/4 - (N - \nu)\pi i/2h) x_b^\nu \tag{3.47}
\]

for a soliton solution.
Since $N - \nu$ is equal to 0 (mod $h$), it follows that

$$\left( \exp\left((N - \nu)\pi i/2h\right)x_u^\nu, \exp\left(-(N - \nu)\pi i/2h\right)x_u^\nu \right)$$

is a left eigenvector of the matrix $C$ with eigenvalue $2 \cos(N\pi/h)$. Thus we arrive finally at the result that the values of the conserved charge $Q_{-N}$ for soliton solutions form a left eigenvector of the Cartan matrix of $\mathfrak{g}$, with eigenvalue $2(1 - \cos N\pi/h)$. By a suitable choice of phase all of these values can be taken to be real. The antisoliton solutions can be treated similarly; these are related to the solitons by

$$Q_{-N}(\text{antisolon}) = (-1)^{N-1}Q_{-N}(\text{soliton}),$$

as one might expect.

**Conclusions**

We have shown how to calculate all conserved charges for single soliton solutions in an affine Toda theory based on an arbitrary untwisted affine Lie algebra $\hat{\mathfrak{g}}$. The values of these charges form left eigenvectors of the Cartan matrix of the finite-dimensional Lie algebra $\mathfrak{g}$ corresponding to $\hat{\mathfrak{g}}$. Our results provide further evidence for the existence of a unitary theory contained within an affine Toda theory with imaginary coupling and for the duality of such a theory with an affine Toda theory associated to the algebra $\hat{\mathfrak{g}}$. It would be interesting to calculate quantum corrections [36,37] to the results we have obtained, although this is likely to be much more difficult than the classical calculation we have presented.

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