Asymptotics of PDE in random environment 
by paracontrolled calculus 

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Abstract 
We apply the paracontrolled calculus to study the asymptotic behavior of a certain quasilinear PDE with smeared mild noise, which originally appears as the space-time scaling limit of a particle system in random environment on one dimensional discrete lattice. We establish the convergence result and show a local in time well-posedness of the limit stochastic PDE with spatial white noise. It turns out that our limit stochastic PDE does not require any renormalization. We also show a comparison theorem for the limit equation. 

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1 Introduction 
In this paper, we study the asymptotic behavior of a solution of a certain quasilinear partial differential equation (PDE) with mild noise, which arises in the hydrodynamic scaling limit of a microscopic interacting particle system called zero-range process in a random environment. We apply the method of the paracontrolled calculus to show that the solution converges to that of a stochastic partial differential equation (SPDE) with spatial white noise. 

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1.1 PDE in Sinai’s random environment

Landim, Pacheco, Sethuraman and Xue [19] recently studied the hydrodynamic space-time scaling limit for the zero-range process on one dimensional discrete lattice \( \mathbb{Z} \) in a random environment of Sinai’s type and derived the following PDE with mild noise for the macroscopic density \( v = v(t, x) \) of particles on \( \mathbb{R} \):

\[
\partial_t v = \Delta \{ \varphi (v) \} - \nabla \{ \varphi (v) \hat{w}^\varepsilon (x) \}, \quad x \in \mathbb{R}.
\]

(1.1)

Here, \( \Delta = \partial^2_x \), \( \nabla = \partial_x \) and \( \varphi (v) \left( = \hat{E}^{\varepsilon v} [g(\eta_0)] \right) \) is a strictly increasing \( C^2 \)-function of \( v \geq 0 \) such that \( \varphi (0) = 0 \) and \( \varphi'(v) \geq c \) for some \( c > 0 \), where \( \nu \) is a certain product measure on \( \{0, 1, 2, \ldots \}^\mathbb{Z} \) with mean \( v \) associated with the jump rate \( g(k) \) of the zero-range process and \( \eta_0 \) denotes the particle number at 0; see \( \Phi (\alpha) \) in (3.8) of [18], p.30. The simplest example is \( \varphi (v) = v \) taking \( g(k) = k \). The noise is given by

\[
\hat{w}^\varepsilon (x) = \frac{1}{(a + b)\varepsilon} \left( w(x + a\varepsilon) - w(x - b\varepsilon) \right),
\]

where \( \{w(x)\}_{x \in \mathbb{R}} \) is a two-sided Brownian motion and \( a, b > 0 \); actually \( a = b = 1 \) in [19]. Therefore, \( \hat{w}^\varepsilon (x) \rightarrow \hat{w}(x) \) as \( \varepsilon \downarrow 0 \) and the limit \( \hat{w}(x) \) is the spatial white noise. Instead of this specific form of \( \hat{w}^\varepsilon (x) \), we may take general smeared noise \( \psi^\varepsilon \ast \hat{w}(x) \) of \( \hat{w}(x) \) defined in Subsection 1.3 or Section 5.

We consider the PDE (1.1) with mild noise in more general form, assuming \( \varphi (v) \) is defined for all \( v \in \mathbb{R} \) such that \( \varphi \in C^2 (\mathbb{R}) \) and \( \varphi' (v) \geq c > 0 \), by replacing the second \( \varphi (v) \) with \( -\chi(v) \) such that \( \chi \in C^1 (\mathbb{R}) \):

\[
\partial_t v = \Delta \{ \varphi (v) \} + \nabla \{ \chi (v) \hat{w}^\varepsilon (x) \}, \quad x \in \mathbb{R}.
\]

(1.3)

Our goal is to show the convergence of the solution \( v = v^\varepsilon \) of (1.3) to that of the following SPDE

\[
\partial_t v = \Delta \{ \varphi (v) \} + \nabla \{ \chi (v) \xi \}, \quad x \in \mathbb{R},
\]

(1.4)

with \( \xi(x) = \hat{w}(x) \). Roughly speaking, \( \hat{w} \in C^{\frac{-d}{2}-} ( := \cap_{\delta > 0} C^{\frac{-d}{2}-\delta} ) \) if the spatial dimension is \( d \) instead of 1 so that we expect \( \nabla \{ \chi (v) \hat{w} \} \in C^{\frac{-d}{2}-1-} \) (if \( v \) is reasonable) and by Schauder estimate, we would have \( v \in C^{\frac{-d}{2}-1+2-} = C^{\frac{-d}{2}-} \). Thus, one can guess that (1.4) has meaning only when \( d = 1 \).

We will show comparison theorems for (1.3) and (1.4), by which one can deal with \( v \) and \( \chi \) defined only for \( v \geq 0 \) if \( \chi \) satisfies \( \chi (0) = 0 \), see Corollary 1.2 and Lemma 6.1.

If we drop \( \nabla \) in the second term of (1.4), it is an equation known as the generalized parabolic Anderson model and studied in the framework of singular SPDEs, [11], [14], [15], [16], [4], [10]. Note that the equation (1.4), with both \( \varphi, \chi \) being linear and \( \nabla \) dropped, is originally called the parabolic Anderson model. The generalized parabolic Anderson model has meaning for the spatial dimension \( d \leq 3 \).

1.2 Integrated quasilinear stochastic PDE

We actually study, instead of (1.4), the equation

\[
\partial_t u = a(\nabla u) \Delta u + g(\nabla u) \cdot \xi,
\]

(1.5)
where $a(v) = \varphi'(v)$, $g(v) = \chi(v)$, $v \in \mathbb{R}$, and $\xi$ is the spatial white noise. If we set $v := \nabla u$, then we can recover the equation (1.4):

$$ \partial_t v = \nabla(a(v)\nabla v) + \nabla(g(v) \cdot \xi) = \Delta \{ \varphi(v) \} + \nabla \{ \chi(v) \cdot \xi \}. $$

In other words, (1.5) is an integrated form of (1.4). The relation between the equations (1.4) and (1.5) is similar to that of stochastic Burgers equation and KPZ equation.

We work with the equation (1.5) on $T = [0,1]$ with periodic boundary condition following the method of Bailleul, Debussche and Hofmanová [4], in which they studied the case $a = a(u), g = g(u)$ on $T^2$ instead of $a = a(\nabla u), g = g(\nabla u)$ on $T$ in our case. Roughly, the noise term behaves as $g(\nabla u) \cdot \xi \in C^{2,1} \times C^{1,1}$ in our case, while $g(u) \cdot \xi \in C^{1,1} \times C^{1,1}$ in [4].

The equation (1.4) considered on $T$ has a mass conservation law:

$$ \int_T v(t, x) dx = m $$

for all $t \geq 0$ with a constant $m \in \mathbb{R}$. This is caused by the conservation of particle number of the underlying microscopic system. Since $v = \nabla u \geq 0$ for the original model, $u$ should be increasing in $x$ and $u(t, 1) = u(t, 0) + m$ holds with some $m > 0$. Therefore, it is more natural to consider (1.5) under the modified periodic condition:

$$ u(t, x + n) = u(t, x) + nm, \quad n \in \mathbb{Z}, \quad x \in \mathbb{R}. \tag{1.6} $$

Indeed, to consider (1.5) under the condition (1.6), set $\bar{u}(t, x) := u(t, x) - mx$, then $\bar{u}(t, x)$ satisfies the usual periodic boundary condition $\bar{u}(t, x + 1) = \bar{u}(t, x)$ and the SPDE

$$ \partial_t \bar{u} = a(\nabla \bar{u}) \Delta \bar{u} + g(\nabla \bar{u} + m) \cdot \xi. $$

Therefore, instead of (1.6), we may consider (1.5) under the usual periodic boundary condition with $a(v), g(v)$ replaced by $a(v + m), g(v + m)$, respectively.

The equation (1.5) has a property that, if $u$ is a solution, then $u + c$ is also a solution for every $c \in \mathbb{R}$. In other words, (1.5) is an equation for the shape of $u$ and its graph is invariant under the vertical shift. Or, (1.5) is essentially an equation for the slope $\nabla u$ of $u$. This is close to the $\nabla \phi$-interface model [9], though its driving force is the space-time noise. This gives a clear difference from the equation considered in [4]. One can expect that the structure of invariant measures for the system on $\mathbb{R}$ would be quite different.

The equation (1.5) with $a(v) = \frac{1}{2}$ and $g(v) = v$ can be interpreted as the Kolmogorov equation associated with the stochastic differential equation (SDE)

$$ dX_t = b(X_t) dt + dW_t, \tag{1.7} $$

where $b(x) = \xi(x)$ and $W_t$ is a one dimensional Brownian motion, see [8]. The SDE (1.7) is studied by [6] for $b \in C^\beta$ with $\beta \in (-\frac{3}{2}, -\frac{1}{2})$ and this covers the one dimensional spatial white noise as in our paper. The process $X_t$ determined by the SDE (1.7) with $b = \xi$, the spatial white noise independent of $W$, is called the Brox diffusion, cf. [19].
1.3 Main results and structure of the paper

Let $C^\alpha, \alpha \in \mathbb{R}$ and $L^2_T, \alpha \in (0, 2)$ be the spatial and parabolic H"older spaces, respectively, explained in Subsection 2.1. Let $\xi$ be the spatial white noise on $\mathbb{T}$. We call $\xi^\varepsilon, \varepsilon > 0$, the smeared noise of $\xi$ if it is defined by $\xi^\varepsilon = \psi^\varepsilon \ast \xi$, where $\psi^\varepsilon(x) = \frac{1}{\varepsilon} \psi(\frac{x}{\varepsilon})$ and the mollifier $\psi$ is a measurable and integrable function on $\mathbb{R}$ with compact support satisfying $\int_{\mathbb{R}} \psi(x)dx = 1$, see Lemma 5.2. Note that the noise $\hat{\psi}^\varepsilon$ in (1.2) considered on $\mathbb{T}$ is a smeared noise of the spatial white noise $\xi = \hat{w}$ in this sense, see Remark 5.1.

Our main result is formulated as follows.

**Theorem 1.1.** Assume that $a \in C^3_b(\mathbb{R})$ satisfies $c \leq a(v) \leq C$ for some $c, C > 0$ and $g \in C^2_b(\mathbb{R})$. Let an initial value $u_0 \in C^\alpha$ with $\alpha \in (\frac{4}{3}, 2)$ be given. Then, there exists a random time $T > 0$ defined on the same probability space as $\xi$ such that the solutions $u^\varepsilon$ of the SPDE

$$\partial_t u^\varepsilon = a(\nabla u^\varepsilon) \Delta u^\varepsilon + g(\nabla u^\varepsilon) \cdot \xi^\varepsilon, \quad u^\varepsilon(0) = u_0,$$

with the smeared noise $\xi^\varepsilon$ of $\xi$ converge in probability in $L^3_T$ as $\varepsilon \downarrow 0$ to $u \in L^3_T$, which is a unique solution up to the time $T$ of the SPDE (1.5) on $\mathbb{T}$ defined in paracontrolled sense, that is, satisfying the condition (2.2) below, with $u(0) = u_0$. In particular, the limit $u$ is independent of the choice of the mollifier $\psi$.

In our case, we do not need any renormalization. This is due to Lemma 5.1 below, and different from the generalized parabolic Anderson models, see [4] and [10].

When we consider (1.5) under the modified periodic condition (1.6) with $m > 0$ assuming $\nabla u(0, x) \geq 0$, we can expect that $\nabla u(t, x) \geq 0$ holds for the solution of (1.5) for $t > 0$. We will show this for $v = \nabla u$ by considering (1.4) with smooth noise $\xi^\varepsilon$, see Lemma 6.1. Then, by Theorem 1.1, we can find a subsequence of $\varepsilon \downarrow 0$ along which $u^\varepsilon$ converge to $u$ almost surely and, taking the limit, we obtain the following corollary.

**Corollary 1.2.** Assume $\varphi \in C^4([0, \infty))$, $\chi \in C^5([0, \infty))$ and they satisfy the conditions $\chi(0) = 0$, $c \leq \varphi'(v) \leq C$ and $|\chi'(v)| \leq C \varphi'(v)$ for some $c, C > 0$. Then, for the solution $u(t)$ of the paracontrolled SPDE (1.5), if $\nabla u(0, x) \geq 0$ holds for all $x \in \mathbb{T}$, we have $\nabla u(t, x) \geq 0$ for all $0 \leq t \leq T$ and $x \in \mathbb{T}$.

In Section 2, we rewrite the SPDE (1.5) into a certain fixed point problem in a setting of the paracontrolled calculus and we solve it in Section 3, see Theorem 3.1. Theorem 1.1 follows from Theorem 3.1 and Lemma 5.2. Section 4 is devoted to the proof of a Lipschitz estimate on $\zeta$, which appears as a remainder term in the fixed point problem. In Section 5, we study the convergence in $C^{2\alpha - 3}$ of the resonant term $II(\nabla X, \xi)$, which is a quadratic function of the spatial white noise $\xi$, where $X = (-\Delta)^{-1}Q\xi$, $Q\xi = \xi - \xi(\mathbb{T})$ and $\alpha$ is as in Theorem 1.1. The reason why our equation does not require renormalization lies in the fact that this term contains the derivative $\nabla X$. This is different from [4] as pointed above and more close to the stochastic Burgers equation. In Section 6, we show a comparison theorem for the SPDE (1.4) with smooth noise and this leads to Corollary 1.2.
2 Reduction to fixed point problem

The main purpose of this section is to reduce the SPDE (1.5) driven by one dimensional spatial white noise $\xi$ on $\mathbb{T}$ to a fixed point problem for the map $\Phi$ defined by (2.16) by means of the paracontrolled calculus.

2.1 Function spaces, regularity exponents, paraproduct and variable $X$

Let us first introduce several function spaces. As in [11], [4], for regularity exponent $\alpha \in \mathbb{R}$, we denote by $C^\alpha := C^\alpha([0, T])$ the space of $C^\alpha$-continuous functions on $[0, T]$ endowed with the norm $\| \cdot \|_{C^\alpha}$. For $T > 0$, we write $C^\alpha_T := C([0, T], C^\alpha)$ for the space of $C^\alpha$-valued continuous functions on $[0, T]$ endowed with the supremum norm $\| \cdot \|_{C^\alpha_T}$ and write $C^\alpha_T L^\infty := C^\alpha([0, T], L^\infty)$ with $\alpha \in (0, 1)$ for the space of $\alpha$-Hölder continuous functions from $[0, T]$ to $L^\infty$ with the seminorm $\|f\|_{C^\alpha_T L^\infty} = \sup_{0 \leq s \neq t \leq T} \|f(t) - f(s)\|_{L^\infty}/|t - s|^\alpha$ for $f \in C^\alpha_T L^\infty$, where $L^\infty = L^\infty(\mathbb{T})$. For $\alpha \in (0, 2)$, we also define $\mathcal{L}^\alpha_T := C^\alpha_T \cap C^{\alpha/2}_T L^\infty$, equipped with the norm $\| \cdot \|_{\mathcal{L}^\alpha_T} = \| \cdot \|_{C^\alpha_T} + \| \cdot \|_{C^{\alpha/2}_T L^\infty}$.

In the following, we will use three regularity exponents $\alpha, \beta$ and $\gamma$. In particular, $\alpha$ is the exponent for the solution $u$ of (1.5) and $\beta$ is that for a certain function of its derivative $\nabla u$ (see $u'$ below) so that $\beta < \alpha - 1$. We expect $\alpha = \frac{3}{2} - \delta$ for every $\delta > 0$, since $\xi \in C^{-\frac{1}{2} - \delta}$ for one dimensional spatial white noise would imply $u \in C^{\frac{3}{2} - \delta}$ by Schauder estimate.

More precisely, we assume the following conditions for these three exponents. For $\alpha$, we assume that $\alpha \in (\frac{1}{2}, \frac{3}{2})$ and consider $\xi \in C^{\alpha - 2}$. For $\beta$, noting that $\frac{1}{2} < \alpha - 1$ for $\alpha$ in this interval, we take $\beta \in (\frac{1}{2}, \alpha - 1)$. Then, we have that $3 - 2\alpha < \frac{1}{3}$ and $2\alpha + \beta - 3 > 0$. The third regularity exponent $\gamma$ is taken as $\gamma \in (2\beta + 1, \alpha + \beta)$, which will be mainly used in Section 3. In particular, we have $\gamma + \beta - 2 > 0$. Throughout this paper, unless otherwise noted, we will assume $\alpha, \beta, \gamma$ satisfy the aforementioned conditions.

As in [4], we denote by $\Pi_{fg}(= f \ast g)$ in [11]) the paraproduct, by $\Pi_{fg}(= f \ast g)$ the resonant term and by $\tilde{\Pi}_{fg}(= f \ast g)$ the modified paraproduct, respectively. Then, for two distributions $f$ and $g$, the general product $fg$ can be (at least formally) written as $fg = \Pi_{fg} + \Pi_{f} g + \Pi_{g} f$, which is called the Littlewood-Paley decomposition. See Section 2.1 and (36) of [11] or p. 43 and p. 45 of [4] for precise definitions of these notions.

Define $X = (-\Delta)^{-1} Q\xi$, more precisely, a zero spatial mean solution of

\begin{equation}
-\Delta X = Q\xi,
\end{equation}

where $Q\xi := \xi - \xi(\mathbb{T})$ and $\xi(\mathbb{T}) \equiv \xi(0) = \int_{\mathbb{T}} \xi(x) dx$, see also (5.1) below. Consider $(u, u') \in \mathcal{L}^\beta_T \times \mathcal{L}^\beta_T$. We call $(u, u')$ is paraprocontrolled by $X$ if

\begin{equation}
(2.2) \quad u = \tilde{\Pi}_{u'} X + u^*, \quad u^* \in \mathcal{L}^\alpha_T
\end{equation}

holds with

\begin{equation}
(2.3) \quad \|(u, u')\|_{\alpha, \beta, \gamma} := \|u'\|_{\mathcal{L}^\beta_T} + \|u^*\|_{\mathcal{L}^\alpha_T} + \sup_{0 < t \leq T} t^{-\frac{\gamma}{2}} \|u^*(t)\|_{C^\gamma} < \infty.
\end{equation}
We denote by $C_{\alpha,\beta,\gamma}(X)$ the space of all functions $(u, u')$ paracontrolled by $X$. Actually, we introduce $(u, u')$ expecting $u$ to be the solution of (1.5) and $u' = \frac{g(\nabla u)}{m(\nabla u)}$.

### 2.2 Basic estimates in paracontrolled calculus

Before starting to transform the equation (1.5) into a fixed point problem, let us prepare some fundamental estimates in the paracontrolled calculus, which will be frequently used in the following.

Let us first summarize some known results for the Hölder-Besov space $C^\alpha$, which are mentioned in Lemma 2.1 and p. 62 of [11]. Throughout this paper, we write $a \lesssim b$ for two non-negative functions $a$ and $b$ (mostly norms of distributions or their products) if there exists a positive constant $C$ independent of the variables under consideration such that $a \lesssim Cb$.

**Lemma 2.1.** (i) For $\alpha \leq \beta$, we have $\| \cdot \|_{C^\alpha} \lesssim \| \cdot \|_{C^\beta}$. Moreover, $\| \cdot \|_{L^\infty} \lesssim \| \cdot \|_{C^\alpha}$ for $\alpha > 0$ and conversely $\| \cdot \|_{C^\alpha} \lesssim \| \cdot \|_{L^\infty}$ for $\alpha \leq 0$.

(ii) (Bony’s estimate) The following hold.

- For $\alpha > 0$ and $\beta \in \mathbb{R}$, $\| \Pi u \|_{C^\beta} \lesssim \| u \|_{L^\infty} \| v \|_{C^\beta}$.
- For $\alpha \neq 0$ and $\beta \in \mathbb{R}$, $\| \Pi u \|_{C^{(\alpha,0)+\beta}} \lesssim \| u \|_{C^\alpha} \| v \|_{C^\beta}$.
- For $\alpha + \beta > 0$, $\| \Pi(\Pi u,v) \|_{C^{\alpha+\beta}} \lesssim \| u \|_{C^\alpha} \| v \|_{C^\beta}$.

In particular, the product $uv$ is well-defined if and only if $\alpha + \beta > 0$, in this case $uv \in C^{\alpha+\beta}$ and

$$\| uv \|_{C^{\alpha+\beta}} \lesssim \| u \|_{C^\alpha} \| v \|_{C^\beta}$$

holds for $\alpha \beta \neq 0$.

The next is an important result for the commutator.

**Lemma 2.2.** (Lemma 2.4 [11], Commutator lemma) Let $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbb{R}$ satisfy the conditions of $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$. For any smooth functions $u, v, w$, define $C(u,v,w)$ by

$$C(u,v,w) := \Pi(\Pi u,v,w) - u\Pi(v,w).$$

Then, $C(u,v,w)$ is uniquely extended to a bounded trilinear operator from $C^\alpha \times C^\beta \times C^\gamma$ to $C^{\alpha+\beta+\gamma}$ and

$$\| C(u,v,w) \|_{C^{\alpha+\beta+\gamma}} \lesssim \| u \|_{C^\alpha} \| v \|_{C^\beta} \| w \|_{C^\gamma}.$$

Since $\nabla u$ is included in the coefficients of the SPDE (1.5), we need to study the regularity of $\nabla u$ whenever $u \in L^\alpha_T$. The next lemma shows that $\nabla u \in L^{\alpha-1}_T$ holds by an interpolation theorem.

**Lemma 2.3.** Let $\alpha \in (1,2)$. Then, for any $u \in L^\alpha_T$, we have $\nabla u \in L^{\alpha-1}_T$ and

$$(2.4) \quad \| \nabla u \|_{L^{\alpha-1}_T} \lesssim \| u \|_{L^\alpha_T}.$$
Proof. From the fact that the operator \( \nabla : C^{\alpha} \mapsto C^{\alpha-1} \) is continuous, see Proposition 2.3 of [17] or Lemma A.1 and its remark in [11], it follows that for all \( u \in C^{\alpha} \), \( \| \nabla u \|_{C^{\alpha-1}} \lesssim \| u \|_{C^\alpha} \). Then, we can easily show that \( \nabla u \in C_T^{\alpha-1} \cap C_T^{\alpha/2} \). More precisely, we have

\[
(2.5) \quad \| \nabla u \|_{C_T^{\alpha-1}} \lesssim \| u \|_{C_T^\alpha} \quad \text{and} \quad \| \nabla u \|_{C_T^{\alpha/2} \cap C_T^{-1}} \lesssim \| u \|_{C_T^{\alpha/2} L^\infty}.
\]

The first part is clear. Noting that \( u \in C_T^{\alpha/2} L^\infty \) and the continuous embedding of \( L^\infty \) in \( C^0 \) (see Lemma 2.1(i)), we have

\[
\| \nabla u(t) - \nabla u(s) \|_{C^{-1}} \lesssim \| u(t) - u(s) \|_{C^{\alpha}} \lesssim \| u(t) - u(s) \|_{L^\infty},
\]

which implies that \( \| \nabla u \|_{C_T^{\alpha/2} \cap C_T^{-1}} \lesssim \| u \|_{C_T^{\alpha/2} L^\infty} \) and in particular \( \nabla u \in C_T^{\alpha/2} \).

Now it is enough for us to show \( \| \nabla u \|_{C_T^{\alpha/2} L^\infty} \lesssim \| u \|_{L^\infty} \). Noting that \( B_{\infty,1}^\alpha \subset L^\infty \) and using Theorem 2.80 of [2] (see Remark 2.1 for details), we have that

\[
(2.6) \quad \| v \|_{L^\infty} \lesssim \| v \|_{B_{\infty,1}^\alpha} \lesssim \| v \|_{C_T^{\alpha}}^{\frac{1}{\alpha}} \| v \|_{C_T^{\alpha-1}}^{\frac{\alpha-1}{\alpha}}
\]

holds for any \( v \in C^{\alpha-1} \cap C^{-1} \).

Applying (2.6) to \( \nabla u(t) - \nabla u(s) \) and then using (2.5), we obtain that

\[
(2.7) \quad \| \nabla u \|_{C_T^{\alpha}}^{\frac{1}{\alpha}} \lesssim \sup_{0 \leq s < t \leq T} \frac{\| \nabla u(t) - \nabla u(s) \|_{C_T^{\alpha}}^{\frac{1}{\alpha}} \| \nabla u(t) - \nabla u(s) \|_{C_T^{\alpha-1}}^{\frac{\alpha-1}{\alpha}}}{| t - s |^{\frac{\alpha-1}{\alpha}}}
\]

\[
\lesssim \left( \| \nabla u \|_{C_T^{\alpha}} \right)^{\frac{1}{\alpha}} \| \nabla u \|_{C_T^{\alpha-1}}^{\frac{\alpha-1}{\alpha}}
\]

\[
\lesssim \left( \| u \|_{C_T^{\alpha} L^\infty} \right)^{\frac{1}{\alpha}} \| u \|_{C_T^{\alpha-1}}^{\frac{\alpha-1}{\alpha}}
\]

\[
\lesssim \| u \|_{C_T^{\alpha}}
\]

Consequently, combining (2.5) with (2.7), we obtain (2.4).

\[\square\]

Remark 2.1. To show (2.6), we utilize the fact \( B_{\infty,\infty}^\alpha = C^\alpha \) and the second part of Theorem 2.80 of [2], the interpolation inequality for general nonhomogeneous Besov spaces: there exists a constant \( C > 0 \) such that for \( \alpha_1 < \alpha_2, \, \theta \in (0, 1) \), and \( p \in [1, \infty] \),

\[
\| u \|_{B_{p,1}^{\alpha_1} \cap (1-\theta) \alpha_2} \leq \frac{C}{(\alpha_2 - \alpha_1)^{\theta(1-\theta)}} \| u \|_{B_{p,\infty}^{\alpha_1}} \| u \|_{B_{p,2}^{\alpha_2}}^{1-\theta},
\]

where \( B_{p,1}^{\alpha} \) and \( B_{p,\infty}^{\alpha} \) denote nonhomogeneous Besov spaces, see Definition 2.68 of [2] for details. This interpolation inequality is originally stated for the spaces on \( \mathbb{R} \) in [2]. However, by similar arguments, we can show the same inequality for the spaces on \( \mathbb{T} \), see pp. 62 of [11] for the definition of the general Besov spaces \( B_{p,q}^{\alpha}(\mathbb{T}^d) \) on the torus, and see pp. 62 and p. 63 of [11] and references therein for the relation between \( B_{p,q}^{\alpha}(\mathbb{R}^d) \) and \( B_{p,q}^{\alpha}(\mathbb{T}^d) \).
On the other hand, a simple interpolation theorem for the spaces $C^\alpha$ implies
\[ \|u\|_{C^{\alpha_1 + (1-\theta)\alpha_2}} \leq \|u\|_{C^{\alpha_1}}^{\theta} \|u\|_{C^{\alpha_2}}^{1-\theta}, \]
for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\theta \in (0, 1)$, see Proposition 2.1 of [17] or the first part of Theorem 2.80 of [2]. By using this inequality, due to a little gap between two spaces $L^\infty$ and $C^0$ as in Lemma 2.1-(i), we can show a weaker result than (2.4), that is, for all $\beta \in (\frac{1}{2}, \alpha - 1)$, $\|\nabla u\|_{L^\beta_T} \lesssim \|u\|_{C^\alpha}$ with implicit constants $C = C(\alpha, \beta)$, without relying on general nonhomogeneous Besov spaces.

We summarize estimates for the modified paraproduct $\Pi$. Note that the first estimate (2.8) in the following lemma is similar to that for $\Pi u v$ in Lemma 2.1-(ii), but, since the definition of the modified paraproduct involves time integral, the estimate is given by uniform norms in time.

**Lemma 2.4.** Let $\beta \in \mathbb{R}$. Then the following hold.
(i) If $\alpha \neq 0$, then for any $u \in C_T C^\alpha$ and $v \in C_T C^\beta$, we have
\[ \|\Pi u v\|_{C_T C^{(\alpha,0) + \beta}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta}. \]  
(ii) If $\alpha \in (0, 2)$, then for any $u \in L^2_T$ and $v \in C_T C^\beta$, we have
\[ \|\Pi u v - \Pi u v\|_{C_T C^{\alpha+\beta}} \lesssim \|u\|_{L^2_T} \|v\|_{C_T C^\beta}. \]

Lemma 2.4-(i) follows obviously from Lemma 2.7 of [13] and $\|u\|_{C_T L^\infty} \lesssim \|u\|_{C_T C^\alpha}$ for $\alpha > 0$. Lemma 2.4-(ii) is taken from Lemma 2.8 of [13] and the special case with $\alpha \in (0, 1)$ is proved in Lemma 5.1 of [11].

Combining Lemma 2.4-(ii) with Lemma 2.2, we easily have the following modified version of commutator lemma.

**Lemma 2.5.** Let $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$ satisfy the same conditions as Lemma 2.2. For any smooth functions $u, v, w$ on $[0, T] \times \mathbb{T}$, define $\hat{C}(u, v, w)$ by
\[ \hat{C}(u, v, w) := \Pi(\Pi u v, w) - u \Pi(\Pi u, v). \]

Then, $\hat{C}(u, v, w)$ is uniquely extended to a bounded trilinear operator from $L^2_T \times C_T C^\beta \times C_T C^\gamma$ to $C_T C^{\alpha+\beta+\gamma}$ and
\[ \|\hat{C}(u, v, w)\|_{C_T C^{\alpha+\beta+\gamma}} \lesssim \|u\|_{L^2_T} \|v\|_{C_T C^\beta} \|w\|_{C_T C^\gamma}. \]

**Proof.** The result is obvious from the inequalities
\[ \|C(u, v, w)\|_{C_T C^{\alpha+\beta+\gamma}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta} \|w\|_{C_T C^\gamma} \]
by Lemma 2.2 and
\[ \|\hat{C}(u, v, w) - C(u, v, w)\|_{C_T C^{\alpha+\beta+\gamma}} = \|\Pi(\Pi u v - \Pi u v, w)\|_{C_T C^{\alpha+\beta+\gamma}} \lesssim \|u\|_{L^2_T} \|v\|_{C_T C^\beta} \|w\|_{C_T C^\gamma} \]
by Lemma 2.1-(ii) and Lemma 2.4-(ii).
The next lemma gives estimates on the commutators $[\nabla, \Pi]$ and $[\Delta, \Pi]$ and follows from Lemma 2.4-(i), cf. Lemma 5.1 of [11].

**Lemma 2.6.** Let $T > 0$, $\alpha \in (0,1)\beta \in \mathbb{R}$ and let $u \in C_T C^\alpha$ and $v \in C_T C^\beta$. Then, the following hold.

\begin{align*}
(2.10) \quad & \| [\nabla, \Pi_u] v \|_{C_T C^{\alpha+\beta-1}} \lesssim \| u \|_{C_T C^\alpha} \| v \|_{C_T C^\beta}, \\
(2.11) \quad & \| [\Delta, \Pi_u] v \|_{C_T C^{\alpha+\beta-2}} \lesssim \| u \|_{C_T C^\alpha} \| v \|_{C_T C^\beta}.
\end{align*}

**Proof.** Since the proofs for (2.10) and (2.11) are essentially same, we only give the proof of (2.10), note that Lemma 2.3 is easily generalized to $\Delta u$. By the definition of $u v$, we have

$$[\nabla, \Pi_u] v = \nabla [\Pi_u v]$$

Therefore, noting $\alpha - 1 < 0$ and then applying Lemma 2.4-(i) together with the fact that $\| \nabla u \|_{C^{\alpha-1}} \lesssim \| u \|_{C^\alpha}$, we have

$$\| \Pi_{\nabla u} v \|_{C_T C^{\alpha+\beta-1}} \lesssim \| \nabla u \|_{C_T C^{\alpha-1}} \| v \|_{C_T C^\beta} \lesssim \| u \|_{C_T C^\alpha} \| v \|_{C_T C^\beta}$$

and this shows the conclusion. \hfill \square

We also need the associativity for the modified paraproduct $\Pi$. To show it, we first state the associative property for the paraproduct.

**Lemma 2.7.** (Lemma 2.6 [13]) If $\alpha > 0$ and $\beta \in \mathbb{R}$, then

$$\| \Pi_u (\Pi_v w) - \Pi_{uv} w \|_{C^{\alpha+\beta}} \lesssim \| u \|_{C^\alpha} \| v \|_{C^\beta} \| w \|_{C^\beta}$$

holds for all $u, v \in C^\alpha$ and $w \in C^\beta$.

Based on the above associative property, we can show that the associativity holds for the modified paraproduct $\Pi$.

**Lemma 2.8.** Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$, and let us define

$$R(u, v; w) = \Pi_u (\Pi_v w) - \Pi_{uv} w$$

for $u \in C_T C^\alpha$, $v \in L^\alpha_T$ and $w \in C^\beta$. Then we have

$$\| R(u, v; w) \|_{C_T C^{\alpha+\beta}} \lesssim \| u \|_{C_T C^\alpha} \| v \|_{L^\alpha_T} \| w \|_{C^\beta}.$$

**Proof.** Noting $\alpha \in (0,1)$ and then applying Lemmas 2.1, 2.7 and (2.9) in Lemma 2.4, we deduce that

$$\| R(u, v; w)(t) \|_{C^{\alpha+\beta}} \lesssim \| (\Pi_u (\Pi_v w) - \Pi_{uv} w)(t) \|_{C^{\alpha+\beta}} + \| (\Pi_u (\Pi_v w) - \Pi_{uv} w)(t) \|_{C^{\alpha+\beta}}$$

$$\lesssim \| u(t) \|_{C^\alpha} \| (\Pi_v w - \Pi_{uv} w)(t) \|_{C^{\alpha+\beta}} + \| u(t) \|_{C^\alpha} \| v(t) \|_{C^\beta} \| w \|_{C^\beta}$$

for all $t \in [0, T]$, which gives the result. \hfill \square
In the end, we give some estimates for nonlinear functions of $\nabla u$. These estimates are mainly used in Sections 3 and 4, where we will take $F = a, a', g$ or $g'$ and choose a proper exponent $\alpha$ according to the situation; recall that we assume that $a, g \in C^3_b(\mathbb{R})$.

**Lemma 2.9.** Let $F \in C^2_b(\mathbb{R})$ and $\alpha \in (1, 2)$. Then the following hold.

(i) For any $u \in C^\alpha$, we have

\[
\|F(\nabla u)\|_{C^{\alpha-1}} \lesssim \|F\|_{C^1}(1 + \|u\|_{C^\alpha}),
\]

and for any $u, v \in C^\alpha$, we have the local Lipschitz estimate

\[
\|F(\nabla u) - F(\nabla v)\|_{C^{\alpha-1}} \lesssim \|F\|_{C^2}(1 + \|u\|_{C^\alpha})\|u - v\|_{C^\alpha}.
\]

(ii) Let $0 < \beta \leq \alpha - 1$. Then, for any $u \in \mathcal{L}^\beta_T$, we have

\[
\|F(\nabla u)\|_{L^\beta_T} \lesssim T^{2-\beta-1} \|F\|_{C^1}(1 + \|u\|_{L^2_T}) + \|F(\nabla u(0))\|_{C^\beta}.
\]

For any $u, v \in L^2_T$ satisfying $u(0) = v(0)$, we have the local Lipschitz estimate

\[
\|F(\nabla u) - F(\nabla v)\|_{L^2_T} \lesssim T^{\alpha-\beta-1} \|F\|_{C^2}(1 + \|u\|_{L^2_T})\|u - v\|_{L^2_T}.
\]

**Proof.** Using Lemma 9-1 of [4], we have for any $u \in C^\alpha$

\[
\|F(\nabla u)\|_{C^{\alpha-1}} \lesssim \|F\|_{C^1}(1 + \|\nabla u\|_{C^{\alpha-1}}).
\]

Then, recalling the continuity of the operator $\nabla : C^\alpha \to C^{\alpha-1}$, we have the desired result (2.12). We can show (2.13) by analogous arguments.

As for the results in (ii), they can be shown by combining Lemma 9-2 of [4] and Lemma 2.3. □

### 2.3 Derivation of fixed point problem

Let $\alpha, \beta, \gamma$ be the exponents given in Subsection 2.1. Let us now reformulate the equation (1.5) into a fixed point problem for the map $\Phi$ on $C_{\alpha, \beta, \gamma}(X)$ defined as follows:

\[
\Phi(u, u') := (v, v')
\]

and

\[
v' = g(\nabla u) - \frac{(a(\nabla u) - a(\nabla u_0^T))u'}{a(\nabla u_0^T)},
\]

\[
L^0v = \Pi(\nabla u_0^T)\omega \xi + g'(\nabla u)\Pi(\nabla u', \xi) - a'(\nabla u)\Pi(\nabla u', \Pi u'\xi)
+ (a(\nabla u) - a(\nabla u_0^T))\Delta u' + \zeta,
\]

where $L^0 := \partial_t - a(\nabla u_0^T)\Delta$, $u_0^T := e^{t\Delta}u_0$ and $e^{t\Delta}$ denotes the semigroup generated by $\Delta$ on $T$. The term $\zeta = \zeta(u, u') \in C_T^0\mathbb{C}^{\alpha+\beta-2}$ is defined in (2.33) below, which is considered as a remainder term in our analysis. Moreover, we assume $v(0) = u_0$. Note that $u' = v'$
is equivalent to \( u' = \frac{g(\nabla u)}{a(\nabla u)} \). The sum of the second and third terms in the right hand side of (2.18) will be denoted by \( \varepsilon_1(u, u') \), and the fourth term by \( \varepsilon_2(u, u') \), respectively, and estimated in Lemma 3.4 below.

The key point in our analysis is to rewrite the SPDE (1.5) as in the form

\[
(2.19) \quad L^0 u = (a(\nabla u) - a(\nabla u_0^T)) \Delta u + g(\nabla u) \cdot \xi.
\]

Although the leading part in the above equation is still the term \( (a(\nabla u) - a(\nabla u_0^T)) \Delta u \), we can show that it is well-defined when \( u \) is paracontrolled by \( X \). To make it clear, we generalize Lemma 3 of [4] to the next result, which is important for our goal.

**Lemma 2.10.** For any \( f \in L_T^\beta \) and \( u_0 \in C^\alpha \), the following intertwining continuity estimate holds.

\[
\|L^0(\Pi_f X) - \Pi_{a(\nabla u_0^T)} f(-\Delta X)\|_{C^{\gamma-1}\alpha+\beta-2}_T \lesssim (1 + T^{-\frac{2-\alpha}{2}}\|u_0\|_{C^\alpha}) \|f\|_{L_T^\beta} \|X\|_{C^\alpha}.
\]

**Proof.** This estimate can be shown analogously to Proposition 12 of [4]. One of the key points in the proof of Proposition 12 of [4] is Lemma 11 of [4]. We remark that Lemma 11 of [4] can be easily generalized to Lemma 2.11 below and then we see that for any \( f \in C^\beta, g \in C^{\gamma-1} \) and \( v \in C^{\alpha-2} \), we have

\[
\|g f v - \Pi_{g f} v\|_{C^{\alpha+\beta-2}} \lesssim \|f\|_{C^\beta} \|g\|_{C^{\gamma-1}} \|v\|_{C^{\alpha-2}};
\]

recall that \( \alpha + \gamma > 3 \). On the other hand, by (2.12) for \( F = a \) in Lemma 2.9 and the Schauder estimate on \( u_0^T \), see (3.1) in Lemma 3.2 below, we observe that

\[
(2.20) \quad \|a(\nabla u_0^T)\|_{C^{\gamma-1}} \lesssim \|a\|_{C^1} (1 + \|u_0^T\|_{C^\gamma}) \lesssim \|a\|_{C^1} (1 + T^{-\frac{2-\alpha}{2}}\|u_0\|_{C^\alpha}).
\]

We can now complete the proof of this lemma by mimicking the proof of Proposition 12 of [4].

We generalize Lemma 11 of [4] to the next lemma, which is used in the proof of Lemma 2.10 above. Our proof is elementary and based only on Lemmas 2.1 and 2.7.

**Lemma 2.11.** Let \( \alpha, \beta > 0, \gamma < 0 \) satisfy the conditions \( \alpha \geq \beta \) and \( \alpha + \gamma > 0 \). Then for any \( u \in C^\alpha, v \in C^\beta \) and \( w \in C^\gamma \), we have

\[
\|u a w - \Pi_{a w} w\|_{C^{\beta+\gamma}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta} \|w\|_{C^\gamma}.
\]

**Proof.** This can be shown easily by combining Lemma 2.1 with Lemma 2.7. In fact, by the Littlewood-Paley decomposition, we have

\[
(2.21) \quad u a w - \Pi_{a w} w = u a (\Pi_{a w} w) - \Pi_{a w} w + \Pi_{a w} u + \Pi(u, \Pi_{a w} w).
\]

Noting that \( \alpha \geq \beta > 0 \) and then using Lemma 2.7, we obtain that

\[
\|\Pi_{a w} (\Pi_{a w} w) - \Pi_{a w} w\|_{C^{\beta+\gamma}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta} \|w\|_{C^\gamma}.
\]
Under our assumptions, we have better estimates for the last two terms of (2.21). Since $\beta > 0$, Lemma 2.1-(ii) gives that $\|\Pi_v w\|_{C^\gamma} \lesssim \|v\|_{C^\beta}\|w\|_{C^\gamma}$. Then using this estimate and Lemma 2.1-(ii) again, we easily have

$$\|v w\|_{C^{\alpha+\gamma}} + \|\Pi(u, v w)\|_{C^{\alpha+\gamma}} \lesssim \|u\|_{C^\alpha}\|v\|_{C^\beta}\|w\|_{C^\gamma},$$

where $\gamma < 0$ and $\alpha + \gamma > 0$ have been used. Consequently, we obtain the desired result by the above estimates together with the condition $\alpha \geq \beta$. \qed

Now let us deal with (2.19). Note that by the definition of $X$,

$$(2.22) \quad \|X\|_{C^\alpha} \lesssim \|\xi\|_{C^{\alpha-2}}$$

holds, see p. 40 of [11] for example. In particular, the estimate in Lemma 2.10 is given by $\|\xi\|_{C^{\alpha-2}}$ instead of $\|X\|_{C^\alpha}$, see also Remark 3.2 below.

To derive the fixed point problem for the map $\Phi$, in (2.19), we first expand

$$(2.23) \quad g(\nabla u) \cdot \xi = \Pi_g(\nabla u)\xi + \Pi g(\nabla u) + \Pi(g(\nabla u), \xi),$$

where, as in p. 40 of [11], by Lemma 2.7 (Taylor expansion) of [11], we rewrite the resonant term as follows:

$$\Pi(g(\nabla u), \xi) = g'(\nabla u)\Pi(\nabla u, \xi) + P_g(\nabla u, \xi),$$

where $P_g$ was denoted by $\Pi_g$ in Lemma 2.7 of [11]. Note that $P_g(\nabla u, \xi)$ has nice spatial regularity at each $t > 0$, it should be evaluated in some proper space weighted in time under some regularity assumption on $u^\beta$ because of the explosive temporal property as $t \downarrow 0$, see Lemma 3.4 for details.

By Lemma 2.5, $\tilde{C}(u', \nabla X, \xi) \in C_T C^{2\alpha+\beta-3}$ is a nice function. Therefore, showing that $\Pi(R_1, \xi) \in C_T C^{2\alpha+\beta-3}$ is also nice by the regularity of $R_1$ and $2\alpha + \beta - 3 > 0$, we have

$$(2.24) \quad \Pi(\nabla u, \xi) = \Pi(\tilde{C}(u', \nabla X, \xi) + \Pi(\nabla u^\beta, \xi) + \Pi(R_1, \xi)$$

where $\tilde{C}$ is the modified commutator defined in Lemma 2.5. Note that although $\Pi(\nabla u^\beta, \xi)$ has nice spatial regularity at each $t > 0$, it should be evaluated in some proper space weighted in time under some regularity assumption on $u^\beta$ because of the explosive temporal property as $t \downarrow 0$, see Lemma 3.4 for details.

By Lemma 2.5, $\tilde{C}(u', \nabla X, \xi) \in C_T C^{2\alpha+\beta-3}$ is a nice function. Therefore, showing that $\Pi(R_1, \xi) \in C_T C^{2\alpha+\beta-3}$ is also nice by the regularity of $R_1$ and $2\alpha + \beta - 3 > 0$, we have

$$\Pi(g(\nabla u), \xi) = g'(\nabla u)u'\Pi(\nabla X, \xi) + g'(\nabla u)\Pi(\nabla u^\beta, \xi) + A_1,$$

where

$$(2.25) \quad A_1 = P_g(\nabla u, \xi) + g'(\nabla u)\{\tilde{C}(u', \nabla X, \xi) + \Pi(R_1, \xi)\} \in C_T C^{2\alpha+\beta-3}.$$
Hence, plugging the above equation into (2.23), we have
\[(2.26)\quad g(\nabla u) \cdot \xi = \Pi_{g(\nabla u)} \xi + \Pi_{\xi} g(\nabla u) + g'(\nabla u) u' \Pi(\nabla X, \xi) + g'(\nabla u) \Pi(\nabla u^2, \xi) + A_1.\]

Next, for the first term in (2.19), by (2.2), we have
\[(a(\nabla u) - a(\nabla u_0^T)) \Delta u = (a(\nabla u) - a(\nabla u_0)) (\Delta \Pi_{u'} X + \Delta u^2).\]

Then, by the continuity result (2.11) on the commutator \([\Delta, \Pi]\) given in Lemma 2.6, we have
\[
\Delta \Pi_{u'} X = \Pi_{u'} \Delta X + R_2
\]
with \(R_2 := [\Delta, \Pi_{u'}] X \in C_T C^{\alpha+\beta-2}\) (note that \(\Pi_{u'} 1 = 0\) by definition), so that
\[
(a(\nabla u) - a(\nabla u_0^T)) \Delta u = -(a(\nabla u) - a(\nabla u_0^T)) \Pi_{u'} \xi + (a(\nabla u) - a(\nabla u_0^T)) (\Delta u^2 + R_2),
\]
which, by Lemma 2.8, is further rewritten as
\[(2.27)\quad (a(\nabla u) - a(\nabla u_0^T)) \Delta u = -\Pi(a(\nabla u) - a(\nabla u_0^T)) u' \xi - \Pi((a(\nabla u) - a(\nabla u_0^T)), \Pi_{u'} \xi) + (a(\nabla u) - a(\nabla u_0^T)) \Delta u^2 + A_2,
\]
where
\[(2.28)\quad A_2 = -R(a(\nabla u) - a(\nabla u_0^T), u'; \xi) - \Pi_{u'} \xi (a(\nabla u) - a(\nabla u_0^T)) + (a(\nabla u) - a(\nabla u_0^T)) R_2 \in C_T C^{\alpha+\beta-2};
\]
see Lemma 2.8 for the definition of the first term \(R\) in \(A_2\).

As we stated above, the term including \(u^2\) should be estimated in a weighted space with weight in time under the regularity assumption on \(u^t\). Hence we can not consider it as a remainder such as \(A_2\).

To derive the map \(\Phi\), we first note that the resonant term \(\Pi((a(\nabla u_0^T), \Pi_{u'} \xi)\) in (2.27) is well-defined for \(u_0 \in C^\alpha\). In fact, by Lemma 2.1-(ii), (2.20) and Lemma 2.4-(i), we have that
\[(2.29)\quad \|\Pi((a(\nabla u_0^T), \Pi_{u'} \xi)\|_{C_T C^{\alpha+\gamma-3}} \lesssim \|a(\nabla u_0^T)\|_{C^{\gamma-1}} \|\Pi_{u'} \xi\|_{C_T C^{\alpha-2}} \lesssim \|a\|_{C^1} (1 + T^{-\frac{3-\alpha}{2}} \|u_0\|_{C^\alpha}) \|u'\|_{L^2} \||\xi||_{C^{\alpha-2}};
\]
recall that for any \(\alpha + \gamma - 3 > 0\). For the resonant term \(\Pi((a(\nabla u), \Pi_{u'} \xi)\) in (2.27), by Lemma 2.7 of [11] (Taylor expansion) and (2.24),
\[
\Pi((a(\nabla u), \Pi_{u'} \xi) = a'((\nabla u) \Pi(\nabla u, \Pi_{u'} \xi) + P_a(\nabla u, \Pi_{u'} \xi)\]
\[= a'((\nabla u) \Pi(\Pi_{u'} \nabla X + \nabla u^2 + R_1, \Pi_{u'} \xi) + P_a(\nabla u, \Pi_{u'} \xi).
\]
Lemma 2.5 yields
\[
\Pi(\tilde{\Pi}\nu, \nabla X, \tilde{\Pi}\nu\xi) = u'\Pi(\nabla X, \tilde{\Pi}\nu\xi) + C(u', \nabla X, \tilde{\Pi}\nu\xi) = (u')^2\Pi(\nabla X, \xi) + u'\tilde{C}(u', \xi, \nabla X) + \tilde{C}(u', \nabla X, \tilde{\Pi}\nu\xi),
\]
where the last two commutators are in \(CT^{2\kappa+\beta-3}\). Thus,
\[
\Pi(a(\nabla u), \tilde{\Pi}\nu\xi) = a'\Pi(\nabla X, \xi) + a'(\nabla u)\Pi(\nabla u^\sharp, \tilde{\Pi}\nu\xi) + A_3,
\]
where
\[
(2.30) \quad A_3 = P_u(\nabla u, \tilde{\Pi}\nu\xi) + a'(\nabla u)\Pi(\nabla u^\sharp, \tilde{\Pi}\nu\xi)
\]
bilds to \(CT^{2\kappa+\beta-3}\), since \(\tilde{\Pi}\nu\xi \in CT^{\alpha-2}\) by Lemma 2.4(i).

Thus, noting (2.27), we have
\[
(2.31) \quad (a(\nabla u) - a(\nabla u_0^\sharp))\Delta u
= -\Pi(a(\nabla u) - a(\nabla u_0^\sharp), \tilde{\Pi}\nu\xi) + (a(\nabla u) - a(\nabla u_0^\sharp))\Delta u^\sharp + A_2,
\]
\[
- \{a'(\nabla u)(u')^2\Pi(\nabla X, \xi) + a'(\nabla u)\Pi(\nabla u^\sharp, \tilde{\Pi}\nu\xi) + A_3\}.
\]

Thus, assuming that \(\Pi(\nabla X, \xi)\) is nicely defined (see Lemma 5.2 below) and using (2.26), (2.31), we can rewrite the equation (2.19) and therefore (1.5) as
\[
(2.32) \quad \mathcal{L}^0 u = P_{\Pi g(\nabla u) - (a(\nabla u) - a(\nabla u_0^\sharp))\nabla u} + g'(\nabla u)\Pi(\nabla u^\sharp, \xi) - a'(\nabla u)\Pi(\nabla u^\sharp, \tilde{\Pi}\nu\xi)
\]
\[
+ (a(\nabla u) - a(\nabla u_0^\sharp))\Delta u^\sharp + \zeta,
\]
where
\[
(2.33) \quad \zeta = (u, u') = \Pi g(\nabla u) + g'(\nabla u)u'\Pi(\nabla X, \xi) + A_1 + \Pi(a(\nabla u_0^\sharp), \tilde{\Pi}\nu\xi) + A_2
\]
\[
- \{a'(\nabla u)(u')^2\Pi(\nabla X, \xi) + A_3\}.
\]

We have \(\zeta \in CT^{\alpha+\beta-2}\). This leads to (2.18) with (2.17), more precisely, (2.32) coincides with (2.18) if \(u = v\) and \(u' = v'\).

By the above analysis, it is clear that the remainder \(\zeta\) is described explicitly in terms of multilinear maps of \(\nabla u, u'\) or \(C^2\) functions of \(\nabla u\), whose \(C^{\alpha+\beta-2}\)-norm depends polynomially on the norm \(\|(u, u')\|_{\alpha, \beta, \gamma}\) and the data. Hereafter, the data mean \(\|u_0\|_{C^{\alpha}}, \|\xi\|_{C^{\alpha-2}}, \|\Pi(\nabla X, \xi)\|_{C^{\alpha-3}}\) and the norms relative to the coefficients \(a\) and \(g\).

Moreover, as a function defined on \(C_{\alpha, \beta, \gamma}(X)\), we will show that \(\zeta\) is locally Lipschitz continuous with a Lipschitz constant depending polynomially on \(\|(u, u')\|_{\alpha, \beta, \gamma}\) and the data, see Proposition 3.8 in Section 3 for details.

Throughout this paper, we denote by \(K_0\) a generic constant depending possibly on \(\|u_0\|_{C^{\alpha}}\) and the norms relative to \(a\) and \(g\), but not on \(X, \xi\) and \(\Pi(\nabla X, \xi)\). In addition, for simplicity, we denote by \(K(\|(u, u')\|_{\alpha, \beta, \gamma})\) a positive constant depending polynomially on both \(\|(u, u')\|_{\alpha, \beta, \gamma}\) and \(K_0\) of the above type, and by \(K(\|(u_1, u_1')\|_{\alpha, \beta, \gamma}, \|(u_2, u_2')\|_{\alpha, \beta, \gamma})\) a positive constant depending polynomially on \(\|(u_1, u_1')\|_{\alpha, \beta, \gamma}, \|(u_2, u_2')\|_{\alpha, \beta, \gamma}\) and \(K_0\) when we study the Lipschitz property. All of the constants may change from line to line.
3 Solving fixed point problem

In this section, we solve the fixed point problem for the map $\Phi$ on $C_{\alpha,\beta,\gamma}(X)$ defined by (2.16) and give the proof of Theorem 1.1.

3.1 Formulation of the result and proof of Theorem 1.1

To solve our equation (1.5) by the fixed point theorem, for $\lambda > 0$, we set

$$B_T(\lambda) := \{(u, u') \in C_{\alpha,\beta,\gamma}(X); \ u(0) = u_0, \ u'(0) = \frac{g(\nabla u_0)}{a(\nabla u_0)}, \ (u, u')_{\alpha,\beta,\gamma}, \ \lambda \},$$

where $u_0 \in C^{\alpha}, \alpha \in (\frac{4}{3}, \frac{3}{2})$, is given as in Theorem 1.1. Observe that the initial data $u(0)$ and $u'(0)$ are fixed in $B_T(\lambda)$.

The main theorem of this section is the following.

**Theorem 3.1.** (i) There exist a large enough $\lambda > 0$ and a small enough $T > 0$ such that the map $\Phi$ defined by (2.16) is contractive from $B_T(\lambda)$ into itself. In particular, $\Phi$ has a unique fixed point on $[0, T]$ for $T > 0$ small enough, which solves the paracontrolled SPDE (1.5) locally in time.

(ii) The map $\Phi$ depends continuously on the enhanced noise $\xi := (\xi, \Pi(\nabla X, \xi)) \in C^{\alpha-2} \times C^{2\alpha-3}$ and its contractivity on $B_T(\lambda)$ is locally uniform in $\xi$. In particular, the unique fixed point of $\Phi$ in $B_T(\lambda)$ inherits the continuity in $\xi$.

Once Theorem 3.1 is shown, one can prove Theorem 1.1.

**Proof of Theorem 3.1.** Let the initial value $u_0 \in C^{\alpha}$ of (1.5) be given. The fixed point $(u, u')$ of the map $\Phi$ satisfies (2.17) and (2.18) with $v = u$ and $v' = u'$ so that, by the arguments in Section 2, we see that $u$ is a solution of SPDE (1.5) paracontrolled by $X$. Conversely, if $u$ is a solution of (1.5) paracontrolled by $X$, it is a fixed point of $\Phi$. Therefore, Theorem 3.1-(i) shows the existence and uniqueness of a local solution of the SPDE (1.5).

For each enhanced noise $\xi = (\xi, \Pi(\nabla X, \xi)) \in C^{\alpha-2} \times C^{2\alpha-3}$, let us denote the fixed point by $J(\xi)$. Given a spatially mild noise $\eta$ such as the smeared noise $\xi^{\epsilon}$ in Theorem 1.1 or Lemma 5.2 on $T$, let us denote by $I(\eta)$ the unique solution of the well-posed equation (1.5) with the mild noise $\eta$ instead of $\xi$. We have that $J(\eta) := J(\eta, \Pi(\nabla Y, \eta))$ extends the map $I(\eta)$ in the sense that $J(\eta) = I(\eta)$ for any mild noise $\eta$, where $Y = (-\Delta)^{-1} Q \eta$. In particular, the convergence result in Theorem 1.1 is obtained by the continuity of the map $J$ shown in Theorem 3.1-(ii) and Lemma 5.2 in Section 5.

3.2 Lipschitz estimates and Schauder estimates

We will give the proof of Theorem 3.1 at the end of this section. Here, we prepare growth and Lipschitz estimates (Lemmas 3.4 and 3.5 and Proposition 3.8) and Schauder estimates (Lemmas 3.2, 3.3 and 3.6 and Corollary 3.7).

Let us first summarize some known results relative to the semigroup $P_t = e^{t\Delta}, t \geq 0$ generated by $\Delta$ on $T$, which will be frequently used in this section.
Lemma 3.2. Let $\alpha \in \mathbb{R}$ and $t \in (0, T]$. Then the following hold.
(i) For $\beta \geq 0$ and $u \in C^\alpha$,
\begin{equation}
\|P_t u\|_{C^{\alpha+\beta}} \lesssim_T t^{-\frac{\beta}{2}} \|u\|_{C^\alpha}, \quad t \in (0, T].
\end{equation}

Here and in the sequel, \(\lesssim_T\) means the implicit multiplicative constant in the right hand depends uniformly on $T$ for all $t \in (0, T]$. For $\beta \geq 0$,
\[\|P_t u\|_{C^\beta} \lesssim_T t^{-\frac{\beta}{2}} \|u\|_{L^\infty}, \quad t \in (0, T]\]
and conversely for $\beta < 0$
\begin{equation}
\|P_t u\|_{L^\infty} \lesssim_T t^{\frac{\beta}{2}} \|u\|_{C^\beta}, \quad t \in (0, T].
\end{equation}

(ii) Let $\text{Id}$ denote the identity operator and $\beta \in (0, 2)$. Then, we have
\begin{equation}
\|(P_t - \text{Id}) u\|_{C^\alpha} \lesssim_T t^{\frac{\alpha}{2}} \|u\|_{C^{\alpha+\beta}}, \quad t \in (0, T],
\end{equation}
\begin{equation}
\|(P_t - \text{Id}) u\|_{L^\infty} \lesssim_T t^{\frac{\beta}{2}} \|u\|_{C^\beta}, \quad t \in (0, T].
\end{equation}

(iii) For any $u \in C_T C^\alpha$, let us denote by $U(t)$ the convolution of $P_t$ and $u = u(\cdot)$, that is, $U(t) = \int_0^t P_{t-s} u(s) ds$.
- For all $\beta \in [0, 1)$, we have $t^\beta \|U(t)\|_{C^{\alpha+2}} \lesssim \sup_{s \in [0, t]} (s^\beta \|u(s)\|_{C^\alpha})$, $t \in [0, T]$.
- If $\alpha \in (-2, 0)$, then $\|U\|_{C_T^{\alpha+2}} \lesssim \|u\|_{C_T C^\alpha}$.

Remark 3.1. The results in Lemma 3.2 are mainly taken from A.7-A.9 of [11], Corollary 2.7 of [17] and the proof of Lemma 6 of [4]. For readers’ convenience, we give a little more explanation. (i) except (3.2) and (iii) are taken directly from Lemma A.7 and respectively Lemma A.9 of [11]. (3.2) comes from the proof of Lemma 6 of [4], see (10) and the beginning of p. 53 of [4] and references therein for details. (3.3) is taken from Corollary 2.7 of [17]. On the other hand, (3.4) can be obtained as a trivial modification of Lemma A.8 of [11], where $\beta \in (0, 1)$ is assumed, by noting the boundedness of the torus.

The next lemma is important for us to prove Lemma 3.6 below.

Lemma 3.3. Let us denote by $Q_t$ the semigroup generated by $\nabla (b \nabla \cdot)$ for some positive function $b \in C^1_b(\mathbb{T})$. Then the statements (i)-(iii) in Lemma 3.2 hold for $Q_t$ instead of $P_t$ where all regularity exponents are taken from $(-\infty, 2)$, with implicit multiplicative constants depending only on the $C^\alpha$-norm of $b$ and positive lower bound of $b$.

Proof. As shown in Theorem 4.15 and Corollary 4.23 of [1], the operator $L := \nabla (b \nabla \cdot)$ generates the analytic semigroup $\{Q_t = e^{tL}\}_{t \geq 0}$, whose fundamental solution $q_t(x,y)$ satisfies the Gaussian estimates
\[|q_t(x,y)| + |t^{\frac{1}{2}} \nabla_x q_t(x,y)| \lesssim_T h_t(x,y), \quad t \in (0, T],\]
with some constant $c > 0$, and with a heat kernel $h_t(x,y)$ of the Laplacian $\Delta$. In [3], Bailleul and Bernicot defined the Besov type norm
\[\|f\|_{C^\alpha_T} := \|e^{L} f\|_{L^\infty} + \sup_{0 < t \leq 1} t^{-\frac{\alpha}{2}} \|(tL) e^{tL} f\|_{L^\infty}, \quad \alpha \in (-\infty, 2),\]
for any nonnegative self-adjoint operator \( L \) on \( L^2(\mathbb{T}) \) which satisfies the Gaussian estimates as above. We can show that the norm \( \| \cdot \|_{C^\alpha_L} \) is equivalent to the standard Besov norm \( \| \cdot \|_{C^\alpha} \) for any \( \alpha \in (-\infty, 2) \), by a modification of Theorem 5.1 of [5], where the homogeneous Besov norms are considered. Since (iii) is a consequence of (i) and (ii), it is sufficient to show (i) and (ii) with \( C^0 \) replaced by \( C^\alpha_L \).

We show the estimate (3.1) for the norm \( \| \cdot \|_{C_L^T} \). Because of the semigroup property, it is sufficient to consider the case \( T = 1 \). Moreover, note that the estimate \( \|(tL)e^{tL}f\|_{L^\infty} \lesssim t^{\frac{\alpha}{2}} \|f\|_{C_L^T} \) holds for any \( 0 < t \leq 2 \), by the semigroup property again. Since \( \{e^{tL}\}_{0 < t \leq 1} \) are uniformly bounded in \( L^\infty \), we have

\[
s^{-\frac{\alpha + \beta}{2}}\|(sL)e^{(s+t)L}f\|_{L^\infty} \lesssim s^{-\frac{\alpha + \beta}{2}}\|(sL)e^{sL}f\|_{L^\infty} \lesssim s^{-\frac{\alpha}{2}}\|f\|_{C_L^T} \lesssim t^{-\frac{\alpha}{2}}\|f\|_{C_L^T}
\]

if \( t \leq s \leq 1 \), and have

\[
s^{-\frac{\alpha + \beta}{2}}\|(sL)e^{(s+t)L}f\|_{L^\infty} \lesssim s^{-\frac{\alpha}{2}}\|(sL)e^{tL}f\|_{L^\infty} \lesssim s^{1 - \frac{\alpha + \beta}{2}}t^{-1}\|f\|_{C_L^T} \lesssim t^{-\frac{\alpha}{2}}\|f\|_{C_L^T}
\]

if \( s \leq t \), where we used \( \alpha + \beta < 2 \). Hence we have

\[
\|e^{tL}f\|_{C_L^{\alpha + \beta}} = \|e^{(tL)}f\|_{L^\infty} + \sup_{0 < s \leq 1} s^{-\frac{\alpha + \beta}{2}}\|(sL)e^{(s+t)L}f\|_{L^\infty} \lesssim (1 + t^{-\frac{\alpha}{2}})\|f\|_{C_L^T}.
\]

The other estimates of (i) are obtained by similar arguments.

We show (3.3) for the norm \( \| \cdot \|_{C_L^T} \). As shown in Proposition A.3 of [3], the norm \( \| \cdot \|_{C_L^T} \) is equivalent to the one

\[
\|f\|_{C_L^{T,2}} := \|e^L f\|_{L^\infty} + \sup_{0 < t \leq 1} t^{-\frac{\alpha}{2}}\|(tL)^2 e^{tL} f\|_{L^\infty}.
\]

Therefore, by noting that

\[
(sL)e^{tL}(e^{tL} - Id)f = \int_s^{s+t} (sL)\partial_r(e^{rL}f)dr = \int_s^{s+t} s L^2 e^{rL}fdr,
\]

we have

\[
\|(sL)e^{tL}(e^{tL} - Id)f\|_{L^\infty} \lesssim \int_s^{s+t} s \cdot r^{-\frac{\alpha + \beta}{2} - 2}\|f\|_{C_{t,r}^{\alpha + \beta}} dr \\
\lesssim t \cdot s \cdot s^{-\frac{\alpha + \beta}{2} - 2}\|f\|_{C_{t,s}^{\alpha + \beta}} = ts^\frac{\beta}{2} \cdot s^\frac{\alpha}{2}\|f\|_{C_{t,s}^{\alpha + \beta}} \lesssim t^\frac{\beta}{2}s^2\|f\|_{C_{t,s}^{\alpha + \beta}}
\]

if \( t \leq s \leq 1 \), where we used \( \beta < 2 \). On the other hand, we have

\[
\|(sL)e^{sL}(e^{sL} - Id)f\|_{L^\infty} \lesssim \|(sL)e^{sL}f\|_{L^\infty} \lesssim s^{-\frac{\alpha + \beta}{2}}\|f\|_{C_L^{\alpha + \beta}} \lesssim t^\frac{\beta}{2}s^2\|f\|_{C_L^{\alpha + \beta}}
\]

if \( s \leq t \), where we used \( \beta > 0 \). Moreover, we have

\[
\|e^L(e^{tL} - Id)f\|_{L^\infty} = \int_1^{1+t} \partial_r e^{rL}f dr \|_{L^\infty} \lesssim \int_1^{1+t} r^{-\frac{\alpha + \beta}{2} - 1} dr \|f\|_{C_L^{\alpha + \beta}} \lesssim t\|f\|_{C_L^{\alpha + \beta}}.
\]

Therefore, we obtain

\[
\|(e^{tL} - Id)f\|_{C_L^T} = \|e^L(e^{tL} - Id)f\|_{L^\infty} + \sup_{0 < s \leq 1} s^{-\frac{\alpha}{2}}\|(sL)e^{sL}(e^{tL} - Id)f\|_{L^\infty} \\
\lesssim t^\frac{\beta}{2}\|f\|_{C_L^{\alpha + \beta}}.
\]

The other estimate (3.4) is obtained by a similar argument. \( \square \)
As we stated in Section 2, one task is to evaluate the terms including $u^\gamma$. It is formulated in the following lemma.

**Lemma 3.4.** For $u := (u, u') \in \mathcal{B}_T(\lambda)$, set

$$
\varepsilon_1(u) := \varepsilon_1(u, u') = g'(\nabla u)\Pi(\nabla u^\gamma, \xi) - a'((\nabla u)\Pi(\nabla u^\gamma, \Pi u^\gamma \xi)),
$$

$$
\varepsilon_2(u) := \varepsilon_2(u, u') = (a(\nabla u) - a(\nabla u_0))\Delta u^\gamma.
$$

Then we have the local growth estimates

$$
\sup_{0 < t \leq T} t^{\frac{2-\alpha}{2}} \|\varepsilon_1(u)(t)\|_{C^{2\alpha-3}} \lesssim K(\|u\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C_{\alpha-2}})\|\xi\|_{C_{\alpha-2}},
$$

$$
\sup_{0 < t \leq T} t^{\frac{2-\alpha}{2}} \|\varepsilon_2(u)(t)\|_{C^{7-2}} \lesssim T^{\frac{\alpha-\beta-1}{2}} K(\|u\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C_{\alpha-2}}) + K_0
$$

for two constants $K(\|u\|_{\alpha, \beta, \gamma})$ and $K_0$.

Moreover, we have the local Lipschitz estimates: for any $u_1 := (u_1, u_1')$ and $u_2 := (u_2, u_2') \in \mathcal{B}_T(\lambda)$,

$$
\sup_{0 < t \leq T} t^{\frac{2-\alpha}{2}} \|(\varepsilon_1(u_1) - \varepsilon_1(u_2))(t)\|_{C^{2\alpha-3}}
\lesssim K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}) \|u_1 - u_2\|_{\alpha, \beta, \gamma}(1 + \|\xi\|_{C_{\alpha-2}})^2 \|\xi\|_{C_{\alpha-2}},
$$

$$
\sup_{0 < t \leq T} t^{\frac{2-\alpha}{2}} \|(\varepsilon_2(u_1) - \varepsilon_2(u_2))(t)\|_{C^{7-2}}
\lesssim T^{\frac{\alpha-\beta-1}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}) \|u_1 - u_2\|_{\alpha, \beta, \gamma}(1 + \|\xi\|_{C_{\alpha-2}})^2
$$

hold for some constant $K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})$.

**Remark 3.2.** In the following in the proofs of Lemmas 3.4, 3.5, Corollary 3.7, Proposition 3.8 and Lemmas 4.1-4.4, we keep the norm $\|X\|_{C^\alpha}$ in estimates to make its origin clear, but eventually by (2.22), we bound it by $\|\xi\|_{C_{\alpha-2}}$.

**Proof of Lemma 3.4.** To prove this lemma, let us first note that for any $u \in \mathcal{B}_T(\lambda)$, we have

$$
\|u\|_{L^p_T} \lesssim \|u'\|_{L^p_T} \|X\|_{C^\alpha} + \|u^\gamma\|_{L^p_T} \leq (1 + \|X\|_{C^\alpha}) \|u\|_{\alpha, \beta, \gamma},
$$

which can be shown by Lemma 13 of [4], see also Lemma 2.10 of [13]. Indeed, the proof of Lemma 13 of [4] shows that $\|\Pi g\|_{L^p_T} \lesssim \|f\|_{L^p_T} \|g\|_{C^\alpha}$ for any $f \in L^p_T, \beta \in (0, 1)$ and $g \in C^\alpha, \alpha \in (0, 2)$. So we obtain (3.9) by recalling (2.2). By Lemma 2.4-(i), we easily have the weak estimate $\|u\|_{C_T C^\alpha} \lesssim (1 + \|X\|_{C^\alpha}) \|u\|_{\alpha, \beta, \gamma}$, which is enough for us to prove the estimates on $\varepsilon_1(u)$. But, for $\varepsilon_2(u)$ and Lemma 3.5 below, we have to use (3.9). So we state it here.

We now prove (3.5). Noting that $\alpha < \gamma$ and $0 < \alpha + \gamma - 3 < \beta$, we have

$$
\|\varepsilon_1(u)(t)\|_{C^{2\alpha-3}} \lesssim \|\varepsilon_1(u)(t)\|_{C^{\alpha+\gamma-3}}
\lesssim\|g'(\nabla u)(t)\|_{C^\beta} \|\Pi(\nabla u^\gamma, \xi)(t)\|_{C^{\alpha+\gamma-3}} + \|a'(\nabla u)(t)\|_{C^\beta} \|\Pi(\nabla u^\gamma, \Pi u^\gamma \xi)(t)\|_{C^{\alpha+\gamma-3}}
$$

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\[ \lesssim \|g'(\nabla u)(t)\|_{C^\beta} \|\nabla u^2(t)\|_{C^{\gamma-1}} \|\xi\|_{C^{\alpha-2}} \]

\[ + \|a'(\nabla u)(t)\|_{C^\beta} \|\nabla u^2(t)\|_{C^{\gamma-1}} \|u\|_{C_T^r} \|\xi\|_{C^{\alpha-2}}, \]

where we have used Lemma 2.1-(i) and (ii) for the first and second inequalities, respectively, and Lemma 2.1-(ii) and Lemma 2.4 for the third inequality.

Using (2.12) in Lemma 2.9 together with (3.9), we have

\[ (3.11) \quad \|g'(\nabla u)(t)\|_{C^\beta} + \|a'(\nabla u)(t)\|_{C^\beta} \lesssim (\|g'\|_{C^1} + \|a'\|_{C^1})(1 + \|u(t)\|_{C^{\beta+1}}) \]

\[ \lesssim (\|g'\|_{C^1} + \|a'\|_{C^1})(1 + (1 + \|X\|_{C^\alpha})\|u\|_{[\alpha,\beta,\gamma]}), \]

\[ \lesssim (\|g'\|_{C^1} + \|a'\|_{C^1})(1 + \|u\|_{[\alpha,\beta,\gamma]})(1 + \|X\|_{C^\alpha}). \]

Combining (3.10) with (3.11), we have

\[ (3.12) \quad \|\varepsilon_1(u)(t)\|_{C^{2\alpha-3}} \]

\[ \lesssim (\|g'\|_{C^1} + \|a'\|_{C^1})(1 + \|u\|_{[\alpha,\beta,\gamma]}^2)(1 + \|X\|_{C^\alpha})\|u^2(t)\|_{C^{\gamma}} \|\xi\|_{C^{\alpha-2}}, \]

and then by multiplying both sides of (3.12) by \( t^{\frac{2\alpha}{3}} \), we obtain (3.5) with the constant

\[ K([u]_{[\alpha,\beta,\gamma]})(1 + \|u\|_{[\alpha,\beta,\gamma]}^2\|u\|_{[\alpha,\beta,\gamma]}^2). \]

We remark all of the other constants \( K \) and \( K_0 \) used in this paper can be obtained by the similar way. So the details for them will be omitted in the sequel.

The local Lipschitz estimate (3.7) of \( \varepsilon_1 \) is proved as follows. By the multilinearity of the resonant and the modified paraproduct, we have

\[ (3.13) \quad \|([\varepsilon_1(u_1) - \varepsilon_1(u_2)](t)\|_{C^{2\alpha-3}} \]

\[ \lesssim (\|g'(\nabla u_1)\Pi(\nabla(u_1^2 - u_2^2),\xi) - a'(\nabla u_1)\Pi(\nabla(u_1^2 - u_2^2),\Pi u_1^2\xi)(t)\|_{C^{2\alpha-3}} \]

\[ + \|(g'(\nabla u_1) - g'(\nabla u_2))\Pi(\nabla(u_1^2,\xi) - (a'(\nabla u_1) - a'(\nabla u_2))\Pi(\nabla u_2^2,\Pi u_2^2\xi)(t)\|_{C^{2\alpha-3}} \]

\[ + \|(a'(\nabla u_2)\Pi(\nabla u_2^2,\Pi u_2 - u_2^2\xi)(t)\|_{C^{2\alpha-3}} \]

\[ =: I(t) + II(t) + III(t). \]

From (3.12), it follows that

\[ (3.14) \quad I(t) \lesssim K([u]_{[\alpha,\beta,\gamma]})(1 + \|X\|_{C^\alpha})\|u_1^2(t) - u_2^2(t)\|_{C^{\gamma}}(1 + \|X\|_{C^\alpha})\|\xi\|_{C^{\alpha-2}}, \quad t \in [0,T]. \]

To evaluate the second term \( II(t) \), we first note that

\[ (3.15) \quad \|([g'(\nabla u_1) - g'(\nabla u_2)](t))\|_{C^\beta} \]

\[ \lesssim (\|g'\|_{C^2} + \|a'\|_{C^2})(1 + \|X\|_{C^\alpha})\|u_1\|_{[\alpha,\beta,\gamma]})(1 + \|X\|_{C^\alpha})\|u_1 - u_2\|_{[\alpha,\beta,\gamma]} \]

\[ \lesssim K([u]_{[\alpha,\beta,\gamma]})\|u_1 - u_2\|_{[\alpha,\beta,\gamma]}(1 + \|X\|_{C^\alpha})^2, \quad t \in [0,T], \]

which follows from (2.13) and the similar arguments for (3.11). We remark that (3.11) and (3.15) also hold for \( g, a \) instead of \( g', a' \), which will be used in the sequel. Using this estimate and repeating essentially the same arguments for (3.12), we have

\[ (3.16) \quad II(t) \lesssim K([u]_{[\alpha,\beta,\gamma]})(u_2^2(t))\|u_1 - u_2\|_{[\alpha,\beta,\gamma]}(1 + \|X\|_{C^\alpha})^2\|\xi\|_{C^{\alpha-2}}. \]
for all $t \in [0, T]$. Moreover, thanks to (3.11), the similar arguments give that

$$III(t) \lesssim K(\|u_2\|_{\alpha, \beta, \gamma}) \|u_1 - u_2\|_{\alpha, \beta, \gamma} \|u^2_T(t)\|_{C^\gamma(1 + \|X\|_{C^\alpha})}\|\xi\|_{C^\alpha-2}, \ t \in [0, T].$$

Plugging this, (3.14) and (3.16) into (3.13), we obtain (3.7).

Let us now turn to the proofs of (3.6) and (3.8). Since $\beta > 0 > \gamma - 2$ and $\beta + \gamma - 2 > 3\beta - 1 > 0$, Lemma 2.1-(ii) implies for all $t \in [0, T]$,

$$\|\varepsilon_2(u)(t)\|_{C^{\gamma-2}} \lesssim \|\varepsilon_2(u)(t)\|_{C^{\beta+\gamma-2}} \lesssim \|a(\nabla u) - a(\nabla u_0^T)(t)\|_{C^\beta} \|\Delta u^2(t)\|_{C^{\gamma-2}} \lesssim \|a(\nabla u) - a(\nabla u_0^T)\|_{C^\beta} \|\varepsilon^2(u)\|_{C^\gamma}.$$ 

Then, we can obtain (3.6) if there exist two constants $K(\|u\|_{\alpha, \beta, \gamma})$ and $K_0$ such that

$$\|a(\nabla u) - a(\nabla u_0^T)\|_{C^\beta} \lesssim T^{\frac{\alpha-\beta-1}{2}} \{K(\|u\|_{\alpha, \beta, \gamma})(1 + \|X\|_{C^\alpha}) + K_0\}.$$ 

To obtain the important factor $T^{\frac{\alpha-\beta-1}{2}}$, we use Lemma 2.9-(ii) and show a stronger result: for any $u \in \mathcal{L}^\sigma_T$,

$$\|a(\nabla u) - a(\nabla u_0^T)\|_{L^\beta_T} \lesssim T^{\frac{\alpha-\beta-1}{2}} \{K(\|u\|_{\alpha, \beta, \gamma})(1 + \|X\|_{C^\alpha}) + K_0\},$$

which will be frequently used in the sequel.

From (2.15) and $\beta < \alpha - 1$, it follows that

$$\|a(\nabla u) - a(\nabla u_0)\|_{L^\beta_T} \lesssim T^{\frac{\alpha-\beta-1}{2}} \|a\|_{C^2(1 + \|u_0\|_{C^\alpha})} \|u - u_0\|_{L^\sigma_T} \lesssim T^{\frac{\alpha-\beta-1}{2}} \|a\|_{C^2(1 + \|u_0\|_{C^\alpha})}(\|u\|_{\alpha, \beta, \gamma} + \|u_0\|_{C^\alpha})(1 + \|X\|_{C^\alpha})$$

where the strong estimate (3.9) has been used for the last inequality. On the other hand, applying (2.12) and then using (3.3) in Lemma 3.2, we have

$$\|a(\nabla u_0^T) - a(\nabla u_0)\|_{C^\beta} \lesssim \|a\|_{C^2(1 + \|u_0\|_{C^\alpha})} \|u^T_0 - u_0\|_{C^\alpha} \lesssim \|a\|_{C^2(1 + \|u_0\|_{C^\alpha})} \|u_0\|_{C^\alpha}.$$ 

Combining this with (3.18), we obtain (3.17) and therefore complete the proof of (3.6).

Finally, let us show the local Lipschitz estimate (3.8) of $\varepsilon_2$. By repeating essentially the same arguments for (3.18), we have

$$\|a(\nabla u_1) - a(\nabla u_2)\|_{L^\beta_T} \lesssim T^{\frac{\alpha-\beta-1}{2}} \|a\|_{C^2(1 + \|u_1\|_{\alpha, \beta, \gamma})} \|u_1 - u_2\|_{\alpha, \beta, \gamma}(1 + \|X\|_{C^\alpha})^2.$$ 

The definition of $\varepsilon_2$ gives that

$$\|(\varepsilon_2(u_1) - \varepsilon_2(u_2))(t)\|_{C^{\gamma-2}} \lesssim \|a(\nabla u_1) - a(\nabla u_2)\|_{L^\beta_T} \|u^2_T(t)\|_{C^\gamma} + \|a(\nabla u_2) - a(\nabla u_0^T)\|_{L^\beta_T} \|u^2_T - u^2_0(t)\|_{C^\gamma}.$$
\[
\lesssim T^{\frac{\alpha - r - 1}{2}} K(\|u_1\|_{\alpha, \beta, r}, \|u_2\|_{\alpha, \beta, r})(1 + \|X\|_{C^\alpha})^2 \\
\times (\|u_1 - u_2\|_{\alpha, \beta, r} \|u'_1(t)\|_{C^r} + \|u'_2(t) - u''_2(t)\|_{C^r}),
\]
where we have used (3.19) and (3.17) with \( u = u_2 \) for the last inequality. Hence, we complete the proof of (3.8) and then the proof of this lemma.

The next lemma gives the local growth and local Lipschitz properties of the map \( \Phi(u, u') \) in \( u' \).

**Lemma 3.5.** For \( u_1 = (u_1, u'_1) \) and \( u_2 = (u_2, u'_2) \in B_T(\lambda) \), set \( v_1 := \Phi(u_1) = (v_1, v'_1) \) and \( v_2 := \Phi(u_2) = (v_2, v'_2) \). Then, we have the local growth estimate

\[
\|v'_1\|_{L^\beta_T} \lesssim T^{\frac{\alpha - r - 1}{2}} K(\|u_1\|_{\alpha, \beta, r}, \|u_2\|_{\alpha, \beta, r})(1 + \|\xi\|_{C^\alpha - 2}) + K_0.
\]

Moreover, the following local Lipschitz estimate also holds.

\[
\|v'_1 - v'_2\|_{L^\beta_T} \lesssim T^{\frac{\alpha - r - 1}{2}} K(\|u_1\|_{\alpha, \beta, r}, \|u_2\|_{\alpha, \beta, r})|u_1 - u_2|_{\alpha, \beta, r}(1 + \|\xi\|_{C^\alpha - 2})^2.
\]

**Proof.** The proof is similar to that of Lemma 5 of [4]. By (2.14) in Lemma 2.9 and (3.9), we have that

\[
\|g(\nabla u_1)\|_{L^\beta_T} \lesssim T^{\frac{\alpha - r - 1}{2}} |g|_{C^1}(1 + \|u_1\|_{L^\infty_T}) + |g|_{C^1}(1 + \|u_0\|_{C^\alpha})
\]

\[
\lesssim T^{\frac{\alpha - r - 1}{2}} |g|_{C^1}(1 + \|u_1\|_{\alpha, \beta, r})(1 + \|X\|_{C^\alpha}) + |g|_{C^1}(1 + \|u_0\|_{C^\alpha}).
\]

Recalling the assumption on \( a: a \in C^3_b(\mathbb{R}) \) and \( 0 < c \leq a(v) \leq C \), we have \( \frac{1}{a(\nabla u_0^T)} \in C^\beta \). More precisely, (2.12) and (3.1) in Lemma 3.2 yield that

\[
\left\| \frac{1}{a(\nabla u_0^T)} \right\|_{C^\beta} \lesssim c^{-2} |a(\nabla u_0^T)|_{C^\alpha} \lesssim c^{-2} |a|_{C^1}(1 + \|u_0^T\|_{C^\alpha})
\]

\[
\lesssim c^{-2} |a|_{C^1}(1 + \|u_0\|_{C^\alpha}).
\]

Then, by the definition of \( \Phi \) and Lemma 2.1, it follows that

\[
\|u'_1\|_{L^\beta_T} \lesssim \left\| \frac{1}{a(\nabla u_0^T)} \right\|_{C^\beta} \left\{ \|g(\nabla u_1)\|_{L^\beta_T} + \|a(\nabla u_1) - a(\nabla u_0^T)\|_{L^\beta_T} \|u'_1\|_{L^\beta_T} \right\},
\]

which gives (3.20) by (3.17) with \( u = u_1 \), (3.22) and (3.23).

Next, we give the proof of (3.21). It is enough for us to show the local Lipschitz estimates for \( g(\nabla u) \) and \( (a(\nabla u) - a(\nabla u_0^T))u' \), respectively. For the term \( g(\nabla u) \), thanks to (2.15) in Lemma 2.9, we see that the local Lipschitz estimate (3.19) holds also for \( g \) instead of \( a \). To deal with \( (a(\nabla u) - a(\nabla u_0^T))u' \), we set \( b_i = a(\nabla u_i) - a(\nabla u_0^T) \) for \( i = 1, 2 \). Then it is enough to estimate \( \|b_1 u'_1 - b_2 u'_2\|_{L^\beta_T} \). Since \( \|fg\|_{L^\beta_T} \lesssim \|f\|_{L^\beta_T} \|g\|_{L^\beta_T} \) holds for any \( f, g \in L^\beta_T \), we have

\[
\|b_1 u'_1 - b_2 u'_2\|_{L^\beta_T} \lesssim \|b_1\|_{L^\beta_T} \|u'_1 - u'_2\|_{L^\beta_T} + \|b_1 - b_2\|_{L^\beta_T} \|u'_2\|_{L^\beta_T}
\]

\[
\lesssim T^{\frac{\alpha - r - 1}{2}} K(\|u_1\|_{\alpha, \beta, r}, \|u_2\|_{\alpha, \beta, r})\|u_1 - u_2\|_{\alpha, \beta, r}(1 + \|X\|_{C^\alpha})^2,
\]

where we have used (3.17) and (3.19). Therefore, we have (3.21).
Now we turn to the study of the property of $v^\#$. To do it, according to our observation in Section 2, we give the following Schauder estimate as a preparation, which is a modification of Lemma 6 in [4]. We will take $b = a(\nabla u^\#_0)$ in the next lemma to show Corollary 3.7 for $v^\#$.

**Lemma 3.6. (Schauder estimate)** Let an initial value $f_0 \in C^\alpha$ and a function $b \in C^2_b(T)$, which is uniformly positive: $b \geq c > 0$, be given. Let $\phi_1 \in C((0,T], C^{\gamma-2})$ satisfying

\[
\sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_1(t)\|_{C^{\gamma-2}} < \infty,
\]

and $\phi_2 \in C((0,T], C^{\alpha+\beta-2})$ satisfying

\[
\sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_2(t)\|_{C^{\alpha+\beta-2}} < \infty,
\]

be given. Let $f$ be the solution of the parabolic equation

\[
\partial_t f - b \Delta f = \phi_1 + \phi_2, \quad f(0) = f_0.
\]

Then, choosing $T > 0$ small enough, we have

\[
\sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|f(t)\|_{C^{\gamma}} + \|f\|_{L^2}\]

\[
\lesssim \|f_0\|_{C^\alpha} + \sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_1(t)\|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_2(t)\|_{C^{\alpha+\beta-2}}
\]

with an implicit multiplicative positive constant in the right hand side depending only on the $C^\alpha$-norm of $b$ and the lower bound $c > 0$ of $b$.

**Proof.** Our proof is mainly inspired by that of Lemma 6 of [4] and to show this lemma, we use the semigroup approach similar to that used in Lemma 6 of [4]. By the relation $b \Delta f = \nabla (b \nabla f) - \nabla b \cdot \nabla f$, the parabolic equation (3.25) can be rewritten as

\[
\partial_t f - \nabla (b \nabla f) = -\nabla b \cdot \nabla f + \phi_1 + \phi_2, \quad f(0) = f_0.
\]

Let $Q_t$ denote the semigroup generated by $\partial_t - \nabla (b \nabla \cdot)$. Then the solution $f$ of (3.25) can be represented in the mild form as

\[
f(t) = Q_t f_0 + \int_0^t Q_{t-s} \phi_1(s) ds + \int_0^t Q_{t-s} \phi_2(s) ds - \int_0^t Q_{t-s} (\nabla b \cdot \nabla f(s)) ds
\]

\[
= I_0(t) + I_1(t) + I_2(t) + I_3(t).
\]

From now on, we divide the proof into three steps.

**Step 1.** We show that there exists a small enough time horizon $T > 0$ such that

\[
\sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|f(t)\|_{C^{\gamma}} \lesssim \|f_0\|_{C^\alpha} + \sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_1(t)\|_{C^{\gamma-2}}
\]

\[
+ T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < t \leq T} t^{\frac{\gamma-2}{2}} \|\phi_2(t)\|_{C^{\alpha+\beta-2}}.
\]
This will be used in both Step 2 and Step 3. From (i) and (iii) in Lemma 3.3, it follows easily that

\[(3.29) \quad t^{\frac{\gamma - \alpha}{2}} \| I_0(t) \|_{C^\gamma} \lesssim t \sup_{0 < s \leq t} \| f_0 \|_{C^\alpha} \quad \text{and} \quad t^{\frac{\gamma - \alpha}{2}} \| I_1(t) \|_{C^\gamma} \lesssim \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| \phi_1(s) \|_{C^{\gamma - 2}}.\]

Since \( \gamma > 0 > \alpha + \beta - 2 \), by (i) in Lemma 3.3, we see that

\[(3.30) \quad t^{\frac{\gamma - \alpha}{2}} \| I_2(t) \|_{C^\gamma} \leq t^{\frac{\gamma - \alpha}{2}} \int_0^t \| Q_{t-s} \phi_2(s) \|_{C^\gamma} ds \]

\[\lesssim t^{\frac{\gamma - \alpha}{2}} \int_0^t (t-s)^{-\frac{\gamma - (\alpha + \beta - 2)}{2}} \| \phi_2(s) \|_{C^{\alpha + \beta - 2}} ds \]

\[\lesssim t^{\frac{\gamma - \alpha}{2}} \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| \phi_2(s) \|_{C^{\alpha + \beta - 2}} \int_0^t (t-s)^{-1 + \frac{\alpha + \beta - \gamma}{2}} s^{\frac{\gamma - \alpha}{2}} ds \]

\[\lesssim T^{\frac{\alpha + \beta - \gamma}{2}} \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| \phi_2(s) \|_{C^{\alpha + \beta - 2}}, \quad t \leq T.\]

Here the fact that for any \( t > 0 \)

\[(3.31) \quad \int_0^t (t-s)^{p-1}s^{q-1} ds = t^{p+q-1} B(p, q), \quad p, q \in (0, 1),\]

and \( \alpha + \beta - \gamma, \gamma - \alpha \in (0, 1) \) have been used for the last line, where \( B(p, q) \) denotes the beta function with parameters \( p, q \).

To evaluate the third term \( I_3(t) \), we first note that for any \( 0 < t \leq T \)

\[(3.32) \quad \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| \nabla b \cdot \nabla f(s) \|_{C^{\alpha - 1}} \lesssim \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| \nabla b \|_{C^{\alpha - 1}} \| \nabla f(s) \|_{C^{\gamma - 1}} \]

\[\lesssim \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| f(s) \|_{C^\gamma},\]

where \( 1 < \alpha < \gamma \) and Lemma 2.1 have been used for the first inequality. Then, by analogous arguments for (3.30), we have

\[t^{\frac{\gamma - \alpha}{2}} \| I_3(t) \|_{C^\gamma} \lesssim t^{\frac{\gamma - \alpha}{2}} \int_0^t (t-s)^{-\frac{\gamma - \alpha - 1}{2}} \| \nabla b \cdot \nabla f(s) \|_{C^{\alpha - 1}} ds \]

\[\lesssim T^{1-\frac{\gamma - \alpha}{2}} \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| f(s) \|_{C^\gamma}, \quad t \in [0, T],\]

where we have used \( 1 - \gamma + \alpha \in (0, 1) \) and (3.31) for the last inequality. So, using the relation \( 1 - \gamma + \alpha > 0 \), we can choose a small enough \( T > 0 \) such that

\[(3.33) \quad t^{\frac{\gamma - \alpha}{2}} \| I_3(t) \|_{C^\gamma} \leq \frac{1}{2} \sup_{0 < s \leq t} s^{\frac{\gamma - \alpha}{2}} \| f(s) \|_{C^\gamma}, \quad t \in [0, T].\]

Consequently, by (3.29), (3.30), (3.33) and \( 1 - \gamma + \alpha > 0 \), we obtain (3.28).

**Step 2.** In this step, we derive the estimate on the norm \( \| f \|_{C^\gamma C^\alpha} \). By (i) with \( \beta = 0 \) in Lemma 3.3, it is easy to know that \( \| I_0(t) \|_{C^\alpha} \lesssim \| f_0 \|_{C^\alpha} \). Using (i) in Lemma 3.3 and the relation \( \alpha - \gamma + 2 \in (1, 2) \), \( \gamma - \alpha \in (0, 1) \), we have

\[(3.34) \quad \| I_1(t) \|_{C^\alpha} + \| I_2(t) \|_{C^\alpha}\]

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\[ \begin{align*}
&\leq \int_0^t \left( \|Q_t-s\phi_1(s)\|_{C^\alpha} + \|Q_t-s\phi_2(s)\|_{C^\alpha} \right) ds \\
&\lesssim \int_0^t \left( (t-s)^{-\frac{\alpha-4\beta+2}{2}} \|\phi_1(s)\|_{C^{\gamma-2}} + (t-s)^{-\frac{2\alpha}{2}} \|\phi_2(s)\|_{C^{\alpha+\beta-2}} \right) ds \\
&\leq \sup_{0<s\leq t} s^{\frac{-\alpha}{2}} \|\phi_1(s)\|_{C^{\gamma-2}} \int_0^t (t-s)^{-\frac{\alpha-4\beta+2}{2}} s^{\frac{-\alpha}{2}} ds \\
&\quad + \sup_{0<s\leq t} s^{\frac{-\alpha}{2}} \|\phi_2(s)\|_{C^{\alpha+\beta-2}} \int_0^t (t-s)^{-\frac{2\alpha}{2}} s^{\frac{-\alpha}{2}} ds \\
&\lesssim \sup_{0<s\leq t} s^{\frac{-\alpha}{2}} \|\phi_1(s)\|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0<s\leq t} t^{\frac{-\alpha}{2}} \|\phi_2(s)\|_{C^{\alpha+\beta-2}}, \quad t \leq T.
\end{align*} \]

Using (i) in Lemma 3.3, we similarly have

\[ \begin{align*}
(3.35) \quad \|J_3(t)\|_{C^\alpha} &\leq \int_0^t \|Q_t-s(\nabla b \cdot \nabla f(s))\|_{C^\alpha} ds \\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla b \cdot \nabla f(s)\|_{C^{\alpha-1}} ds \\
&\lesssim T^{\frac{1-\gamma+\alpha}{2}} \sup_{0<s\leq T} s^{\frac{-\alpha}{2}} \|f(s)\|_{C^{\gamma}}, \quad t \in [0,T].
\end{align*} \]

From the above estimates and (3.28), we obtain that

\[ \begin{align*}
(3.36) \quad \|f\|_{C^\alpha L^\infty} &\lesssim \|f_0\|_{C^\alpha} + \sup_{0<t\leq T} t^{\frac{-\alpha}{2}} \|\phi_1(t)\|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0<t\leq T} t^{\frac{-\alpha}{2}} \|\phi_2(t)\|_{C^{\alpha+\beta-2}}.
\end{align*} \]

**Step 3.** We devote to evaluating \( \|f\|_{C^\alpha L^\infty} \). Let \( 0 \leq s < t \leq T \). Then by (3.27), we have

\[ \begin{align*}
&\|f(t) - f(s)\|_{L^\infty} \\
\leq &\|\left( (Q_t - Q_s) f_0 \right)_{L^\infty} \|
\quad + \left\| \int_0^s (Q_{t-r} - Q_{s-r}) \phi_1(r) dr \right\|_{L^\infty} 
\quad + \left\| \int_s^t Q_{t-r} \phi_1(r) dr \right\|_{L^\infty} \\
\quad + \left\| \int_0^s (Q_{t-r} - Q_{s-r}) \phi_2(r) dr \right\|_{L^\infty} 
\quad + \left\| \int_s^t Q_{t-r} \phi_2(r) dr \right\|_{L^\infty} \\
\quad + \left\| \int_0^s (Q_{t-r} - Q_{s-r}) (\nabla b \cdot \nabla f(r)) dr \right\|_{L^\infty} 
\quad + \left\| \int_s^t Q_{t-r} (\nabla b \cdot \nabla f(r)) dr \right\|_{L^\infty} \\
= &J_0(s,t) + J_1(s,t) + J_2(s,t) + J_3(s,t) + J_4(s,t) + J_5(s,t) + J_6(s,t).
\end{align*} \]

By (ii) in Lemma 3.3 and the contractivity of the semigroup \( Q_1 \) on \( L^\infty \), we have

\[ J_0(s,t) \lesssim \|(Q_{t-s} - \text{Id})f_0\|_{L^\infty} \lesssim |t-s|^{\frac{\alpha}{2}} \|f_0\|_{C^\alpha}. \]

Thanks to (ii) in Lemma 3.3, the terms \( J_i(s,t), i = 1, 3, 5 \) can be estimated by repeating essentially the same arguments in **Step 2**. In fact, using (ii) in Lemma 3.3, we have

\[ \begin{align*}
J_1(s,t) + J_3(s,t) &\lesssim |t-s|^{\frac{\alpha}{2}} \int_0^s \left( \|Q_{s-r}\phi_1(r)\|_{C^\alpha} + \|Q_{s-r}\phi_2(r)\|_{C^\alpha} \right) dr, \\
J_5(s,t) &\lesssim |t-s|^{\frac{\alpha}{2}} \int_0^s \|Q_{s-r}(\nabla b \cdot \nabla f(r))\|_{C^\alpha} dr.
\end{align*} \]
Then, by the analogous arguments for (3.34) and (3.35), we have
\[
J_1(s,t) + J_3(s,t) \lesssim |t-s|^{\frac{3}{2}} \left( \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| \phi_1(r) \|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| \phi_2(r) \|_{C^{\alpha+\beta-2}} \right),
\]
\[
J_5(s,t) \lesssim |t-s|^{\frac{2}{3}} T^{\frac{\gamma+\alpha}{2}} \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| f(r) \|_{C^{\gamma}}.
\]
It is easier to evaluate \(J_2(s,t)\) and \(J_4(s,t)\) by noting that \(\gamma - 2 < 0\) and \(\alpha + \beta - 2 < 0\) and using (i) in Lemma 3.3. In fact, we have
\[
J_2(s,t) + J_4(s,t)
\]
\[
\lesssim \int_s^t \left( |t-r|^{-\frac{3}{2}} \| \phi_1(r) \|_{C^{\gamma-2}} + |t-r|^{\frac{\alpha+\beta-2}{2}} \| \phi_2(r) \|_{C^{\alpha+\beta-2}} \right) dr
\]
\[
\lesssim |t-s|^{\frac{3}{2}} \sup_{0 < r \leq t} r^{-\frac{\alpha}{2}} \| \phi_1(r) \|_{C^{\gamma-2}} \int_0^t (t-r)^{-1+\frac{\gamma}{2}} r^{-\frac{\alpha}{2}} dr
\]
\[
+ |t-s|^{\frac{3}{2}} \sup_{0 < r \leq t} r^{-\frac{\alpha}{2}} \| \phi_2(r) \|_{C^{\alpha+\beta-2}} \int_0^t (t-r)^{-1+\frac{\beta}{2}} r^{-\frac{\alpha}{2}} dr
\]
\[
\lesssim |t-s|^{\frac{3}{2}} \left( \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| \phi_1(r) \|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| \phi_2(r) \|_{C^{\alpha+\beta-2}} \right).
\]
For the term \(J_6(s,t)\), thanks to the contractivity of the semigroup \(Q_t\) on \(L^\infty\) and the fact \(C^{\alpha-1} \subset L^\infty\), by (3.32), we have
\[
J_6(s,t) \leq \int_s^t \| Q_{t-r}(\nabla b \cdot \nabla f(r)) \|_{L^\infty} dr
\]
\[
\lesssim \int_s^t \| \nabla b \cdot \nabla f(r) \|_{C^{\alpha-1}} dr
\]
\[
\lesssim |t-s|^{\frac{3}{2}} T^{1-\frac{\gamma}{2}} \sup_{0 < r \leq T} r^{-\frac{\alpha}{2}} \| f(r) \|_{C^{\gamma}}.
\]
Noting that \(1 - \frac{\gamma}{2} > 0\) and using the above estimates in this step together with (3.28), we have that
\[
(3.37) \quad \| f \|_{C^{\frac{\alpha}{2}}_{x} L^\infty} \lesssim \| f_0 \|_{C^{\alpha}} + \sup_{0 < t \leq T} t^{-\frac{\alpha}{2}} \| \phi_1(t) \|_{C^{\gamma-2}} + T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < r \leq T} t^{-\frac{\alpha}{2}} \| \phi_2(t) \|_{C^{\alpha+\beta-2}}
\]
holds for small enough \(T > 0\).

Consequently, we obtain the desired result (3.26) by (3.28), (3.36) and (3.37). Hence the proof is completed.

Applying Lemma 3.6 to the equation (3.38) below, which is essentially (2.18) in our fixed point problem, we have the following estimate for \(v^\phi := v - \Pi_{v} X\) for \(v\) determined by (2.18) and (2.17).

**Corollary 3.7.** Let \(\phi_1 \in C((0,T], C^{\gamma-2})\) and \(\phi_2 \in C((0,T], C^{\alpha+\beta-2})\) be functions as in Lemma 3.6. For given \(z_0 \in C^{\alpha}\) and \(z' \in L^\beta_T\), let \(z\) be the solution of the equation
\[
(3.38) \quad (\partial_t - a(\nabla u_0^T)) \Delta z = \Pi_{a(\nabla u_0^T)} z' \xi + \phi_1 + \phi_2, \quad z(0) = z_0.
\]
Then, \((z, z') \in C_{\alpha, \beta, \gamma}(X)\) and we have the estimate

\[
(3.39) \quad \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|z^\sharp(t)\|_{C^\gamma} + \|z^\sharp\|_{L^2_T}
\]

\[
\lesssim \|z_0\|_{C^\alpha} + \|z'(0)\|_{L^\infty} \|\xi\|_{C^{\alpha - 2}} + T^{\frac{\alpha + \beta - \gamma}{2}} (1 + \|u_0\|_{C^\alpha}) \|z'\|_{L^2_T} \|\xi\|_{C^{\alpha - 2}}
\]

\[
+ \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|\phi_1(t)\|_{C^{\gamma - 2}} + T^{\frac{\alpha + \beta - \gamma}{2}} \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|\phi_2(t)\|_{C^{\alpha + \beta - 2}}
\]

with a multiplicative positive constant in the right hand side depending only on the \(C^\alpha\)-norm of \(u_0\).

If \((y, y') \in C_{\alpha, \beta, \gamma}(X)\) is associated similarly to another set of data \(\psi_1, \psi_2, y_0\) and \(y'\) with \(y\) the solution of the equation

\[
(\partial_t - a(\nabla u_0^T) \Delta) y = \Pi_{a(\nabla u_0^T) y'_0} \xi + \psi_1 + \psi_2, \quad y(0) = y_0 \in C^\alpha,
\]

then we have the Lipschitz continuity bound

\[
(3.40) \quad \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|z^\sharp(t) - y^\sharp(t)\|_{C^\gamma} + \|z^\sharp - y^\sharp\|_{L^2_T}
\]

\[
\lesssim \|z_0 - y_0\|_{C^\alpha} + \|z'(0) - y'(0)\|_{L^\infty} \|\xi\|_{C^{\alpha - 2}} + T^{\frac{\alpha + \beta - \gamma}{2}} (1 + \|u_0\|_{C^\alpha}) \|z' - y'\|_{L^2_T} \|\xi\|_{C^{\alpha - 2}}
\]

\[
+ \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|\phi_1(t) - \psi_1(t)\|_{C^{\gamma - 2}} + T^{\frac{\alpha + \beta - \gamma}{2}} \sup_{0 < t \leq T} t^{-\frac{\gamma - \alpha}{2}} \|\phi_2(t) - \psi_2(t)\|_{C^{\alpha + \beta - 2}}
\]

with a multiplicative positive constant in the right hand side depending only on the \(C^\alpha\)-norm of \(u_0\).

**Proof.** Set

\[
z^\sharp = z - \Pi_{z'} X.
\]

Then, to prove the first part of this lemma, it is enough for us to show (3.39). Recalling that \(L^0 = \partial_t - a(\nabla u_0^T) \Delta\) and noting that

\[
L^0(\Pi_{z'} X) = \left\{L^0(\Pi_{z'} X) - \Pi_{a(\nabla u_0^T) z'} (-\Delta X)\right\} + \Pi_{a(\nabla u_0^T) z'} (-\Delta X),
\]

we have that

\[
(3.41) \quad L^0 z^\sharp = \phi_1 + \phi_2 - \left\{L^0(\Pi_{z'} X) - \Pi_{a(\nabla u_0^T) z'} (-\Delta X)\right\}
\]

with the initial condition \(z^\sharp(0) = z_0 - (\Pi_{z'} X)(0) = z_0 - \Pi_{z'}(0) X \in C^\alpha\). Note that \((\Pi_{z'} X)(0) = \Pi_{z'}(0) X\) because \(X\) is independent of time \(t\), see p. 45 of [4]. We write \(\Pi_{z'}(0) X\) for \((\Pi_{z'} X)(0)\) whenever \(z' \in L^2_T\) in this paper.

Since \(z' \in L^2_T\), by the intertwining continuity estimate in Lemma 2.10, we see that

\[
\|L^0(\Pi_{z'} X) - \Pi_{a(\nabla u_0^T) z'} (-\Delta X)\|_{C^\gamma} \lesssim \left(1 + T^{-\frac{\gamma - \alpha}{2}} \|u_0\|_{C^\alpha}\right) \|z'\|_{L^2_T} \|X\|_{C^\alpha},
\]

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which particularly implies that

\[(3.42)\]

\[
\sup_{0 < t \leq T} \left| \frac{t^{\gamma - \alpha}}{\Gamma(t)} \right| \leq K(\|u_1\|_{C_0^{\alpha + \beta - 2}}, u_2) \leq K(\|u_1\|_{C_0^{\alpha + \beta - 2}}, u_2),
\]

So, the condition (3.24) in Lemma 3.6 holds for \(f_2 - \\{L^0(\Pi_{z'}X) - \Pi_a(\nabla u_0')\}z'(\Delta X)\). Now, let \(b(x) = a(\nabla u_0')(x)\). By the assumption on \(a\), we know \(b\) satisfies the assumption of Lemma 3.6. Then, (3.39) is obtained by Lemma 3.6, (3.42) and noting that \(\|z(0)\|_{C_0} \leq \|z_0\|_{C_0} + \|z(0)\|_{L^\infty}\|X\|_{C_0}\).

Next, let us give the proof of (3.40), which is shown similarly to (3.39) with the help of Lemma 3.6. In fact, by the assumptions on \((z, z')\) and \((y, y')\), we deduce that

\[
\begin{align*}
(\partial_t - a(\nabla u_0')\Delta)(y - z) &= \Pi_a(\nabla u_0')(y'-z')\xi + (\psi_1 - \phi_1) + (\psi_2 - \phi_2) \\
\text{with } y(0) - z(0) &= y_0 - z_0 \in C_0.
\end{align*}
\]

Setting \(y^t = y - \Pi_{y'}X\) and then similarly to (3.41), we have that

\[
\begin{align*}
L^0(y^t - z^t) &= (\psi_1 - \phi_1) + (\psi_2 - \phi_2) - \{L^0(\Pi_{y'}X) - \Pi_a(\nabla u_0')(y'-\Delta X)\} \\
&\quad + \{L^0(\Pi_{z'}X) - \Pi_a(\nabla u_0')(z'-\Delta X)\} \\
&= (\psi_1 - \phi_1) + (\psi_2 - \phi_2) - \{L^0(\Pi_{y'}X) - \Pi_a(\nabla u_0')(y'-z')(\Delta X)\}
\end{align*}
\]

with the initial condition \((y^t - z^t)(0) = (y_0 - z_0) - \Pi_{(y'-z')}(0)X \in C_0\). Hence, the desired result (3.40) can be easily obtained by Lemma 3.6.

The next proposition shows the local growth and local Lipschitz continuity of the remainder term \(\zeta\) defined by (2.33). The proof is given in Section 4.

**Proposition 3.8.** For any \(u_1 = (u_1, u_1'), u_2 = (u_2, u_2') \in \mathcal{B}_T(\lambda),\) we have

\[(3.43)\]

\[
\sup_{0 < t \leq T} t^{\gamma - \alpha} |\zeta(u_1, u_2)(t)|_{C_0^{\alpha + \beta - 2}} \leq K(\|u_1\|_{C_0^{\alpha + \beta - 2}}, u_2)
\]

\[(3.44)\]

\[
\sup_{0 < t \leq T} t^{\gamma - \alpha} \left( |\zeta(u_1) - \zeta(u_2)|_{C_0^{\alpha + \beta - 2}} \right) \leq K(\|u_1\|_{C_0^{\alpha + \beta - 2}}, \|u_2\|_{C_0^{\alpha + \beta - 2}})
\]

where

\[
K_1(X, \xi) = (1 + \|\xi\|_{C_0^{\alpha - 2}})^2 \left( \|\Pi(\nabla X, \xi)\|_{C_0^{\alpha - 3}} \right)
\]

\[
K_2(X, \xi) = (1 + \|\xi\|_{C_0^{\alpha - 2}})^2 \left( (1 + \|\xi\|_{C_0^{\alpha - 2}})^2 + \|\Pi(\nabla X, \xi)\|_{C_0^{\alpha - 3}} \right)
\]

**3.3 Proof of Theorem 3.1**

Based on these preparations, we are at the position to give the proof of Theorem 3.1.

First, we give the proof of (i). Let us recall that for any \(u = (u, u') \in \mathcal{B}_T(\lambda),\) the map \(\Phi(u) = \Phi(u, u')\) is defined by (2.16). Using the notations \(\varepsilon_1(u, u')\) and \(\varepsilon_2(u, u')\) introduced in Lemma 3.4, we can rewrite \(L^0v\) in (2.18) as the following:

\[(3.45)\]

\[
L^0v = \Pi_a(\nabla u_0')\varepsilon_1 + \varepsilon_1(u, u') + \varepsilon_2(u, u') + \zeta,
\]

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with the initial value $v(0) = u_0 \in C^\alpha$.

We first show that $\Phi$ maps $B_T(\lambda)$ into itself, if we choose $\lambda > 0$ sufficiently large and $T > 0$ small enough. We have $v' \in \mathcal{L}_T^\beta$ for $v'$ determined by (2.17) by Lemma 3.5. Since $\beta < \alpha - 1$, Lemmas 2.1-(i) and 3.4 yield that

\begin{equation}
\sup_{0 < t \leq T} t^{\frac{-\alpha}{2}} \|\varepsilon_1(u, u')\|_{C^{\alpha+\beta-2}} \lesssim K(\|u\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C^{\alpha-2}})\|\xi\|_{C^{\alpha-2}}.
\end{equation}

Now, let $\phi_1 = \varepsilon_2(u, u')$ and $\phi_2 = \varepsilon_1(u, u') + \zeta$. By Lemma 3.4, (3.46) and (3.43), we see that $\phi_1$ and $\phi_2$ satisfy the assumptions in Corollary 3.7. In addition, according to the definition of $v'$, we have $v'(0) = v'(0) \in L^\infty$, by recalling that $u'(0) = \frac{g(\nabla u_0)}{a(\nabla u_0)}$. Applying Corollary 3.7 to (3.45), and then using Lemmas 3.4, 3.5 and (3.43), we have

\begin{equation}
\sup_{0 < t \leq T} t^{\frac{-\alpha}{2}} \|v(t)\|_{C^\gamma} + \|v^2\|_{L_T^\beta} 
\lesssim \|u_0\|_{C^\alpha} + \|v'(0)\|_{L^\infty} \|\xi\|_{C^{\alpha-2}} + T^{\frac{\alpha+\beta-\gamma}{2}}(1 + \|u_0\|_{C^\alpha})\|v'(0)\|_{L_T^\beta} \|\xi\|_{C^{\alpha-2}} 
\end{equation}

where $K(X, \xi)$ is the constant defined in Proposition 3.8, and the relation $0 < \alpha + \beta - \gamma < \alpha - \beta - 1$ has been used to obtain the order $\frac{\alpha+\beta-\gamma}{2}$ of $T$ in the last inequality.

Recall that $\|\Phi(u)\|_{\alpha, \beta, \gamma} = \|(v, v')\|_{\alpha, \beta, \gamma} = \|v'\|_{\mathcal{L}_T^\beta} + \|v^2\|_{L_T^\alpha} + \sup_{0 < t \leq T} t^{\frac{-\alpha}{2}} \|\varepsilon(t)\|_{C^\gamma}$. Therefore, thanks to Lemma 3.5 and (3.47), we obtain

\begin{equation}
\|\Phi(u)\|_{\alpha, \beta, \gamma} \lesssim T^{\frac{\alpha+\beta-\gamma}{2}} K(\|u\|_{\alpha, \beta, \gamma}) K_1(X, \xi) + K_0(1 + \|\xi\|_{C^{\alpha-2}}),
\end{equation}

where we have used the relation $\alpha + \beta - \gamma > 0$, and $\alpha - \beta - 1$ again. Consequently, noting that $\alpha + \beta - \gamma > 0$, we can choose sufficient large $\lambda > 0$ and small enough $T > 0$ such that $\Phi$ maps $B_T(\lambda)$ into itself.

Now we give the proof of the contractive property of $\Phi$ on $B_T(\lambda)$. We use the same notations introduced in Lemmas 3.4 and 3.5. Note that for any $u_1 := (u_1, u'_1) \in B_T(\lambda)$ and $u_2 := (u_2, u'_2) \in B_T(\lambda)$, we have $u_1(0) = u_2(0) = u_0$ and $u'_1(0) = u'_2(0)$. So, the definition of $\Phi$ implies that $v_1(0) = v_2(0) = u_0$ and $v'_1(0) = v'_2(0)$. Applying (3.40) with this fact and repeating essentially the same arguments for (3.47), we have

\begin{equation}
\sup_{0 < t \leq T} t^{\frac{-\alpha}{2}} \|v_1(t) - v_2(t)\|_{C^\gamma} + \|v'_1 - v'_2\|_{L_T^\alpha} 
\lesssim \sup_{0 < t \leq T} t^{\frac{-\alpha}{2}} \|\varepsilon_2(u_1) - \varepsilon_2(u_2)\|_{C^{\alpha-2}} + T^{\frac{\alpha+\beta-\gamma}{2}}(1 + \|u_0\|_{C^\alpha})\|v'_1 - v'_2\|_{L_T^\beta} \|\xi\|_{C^{\alpha-2}} 
\end{equation}

\begin{equation}
+ T^{\frac{\alpha+\beta-\gamma}{2}} \sup_{0 < t \leq T} \left( \|\varepsilon_1(u_1) - \varepsilon_1(u_2)\|_{C^{\alpha+\beta-2}} + \|\xi(u_1) - \xi(u_2)\|_{C^{\alpha+\beta-2}} \right) 
\lesssim T^{\frac{\alpha+\beta-\gamma}{2}} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma}) \|u_1 - u_2\|_{\alpha, \beta, \gamma} K_2(X, \xi),
\end{equation}

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where $\tilde{K}_2(X, \xi)$ is the constant defined in Proposition 3.8. Here, for the second inequality, we have used Lemma 3.4 for the terms involving $\varepsilon_1$ (note $\alpha + \beta - 2 < 2\alpha - 3$) and $\varepsilon_2$ (note $\alpha + \beta - \gamma < \alpha - \beta - 1$), Lemma 3.5 for $\|v_i - v_j\|_{e^T}$ and (3.44) in Proposition 3.8 for $\xi = \xi(u, u')$. Finally, by (3.21) in Lemma 3.5 again and (3.49), we have that

$$\|\Phi(u_1) - \Phi(u_2)\|_{\alpha, \beta, \gamma} \leq T^{2+\beta-\gamma} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})\|u_1 - u_2\|_{\alpha, \beta, \gamma} \tilde{K}_2(X, \xi),$$

which clearly implies that the map $\Phi$ inherits its contractivity on $B_T(\lambda)$ if $T > 0$ is chosen sufficient small. This concludes the proof of the assertion (i) of the theorem.

In the end, let us give the proof of (ii). Let $\lambda$ and $T$ be the chosen values in the proof of (i). We first show the contractive property of $\Phi$ is locally uniform in the enhanced noise $\xi = (\xi, \Pi(\nabla X, \xi)) \in C^{\alpha-2} \times C^{2\alpha-3}$. According to the definition of $\tilde{K}_2(X, \xi)$, we easily observe that $\tilde{K}_2(X, \xi)$ is bounded from above by an increasing polynomial $P$ of the norm of $\xi$. Then, for all $\xi \in B_\tau := \{\xi = (\xi, \Pi(\nabla X, \xi)) : \|\xi\|_{C^{\alpha-2}} + \Pi(\nabla X, \xi)\|_{C^{2\alpha-3}} \leq r\}$, $r > 0$, by (3.50), we have $\|\Phi(u_1) - \Phi(u_2)\|_{\alpha, \beta, \gamma}$ is controlled by $T^{2+\beta-\gamma} K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})\|u_1 - u_2\|_{\alpha, \beta, \gamma} P(r)$. Therefore, using the fact that $\|u_i\|_{\alpha, \beta, \gamma} \leq \lambda$, $i = 1, 2$ and $K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})$ does not depend on $X, \xi$ and $\Pi(\nabla X, \xi)$, we obviously see the map $\Phi$ is locally Lipschitz on $B_T(\lambda)$ for all $\xi \in B_\tau$ by making $T > 0$ small enough if necessary. The important observation is that $T$ can be chosen as the function of $r$ for all $\xi \in B_\tau$. Therefore, the desired result is proved.

Next, let us show the remaining part, that is, $\Phi$ depends continuously on $\xi$ and the unique fixed point of $\Phi$ in $B_T(\lambda)$ is also continuous in $\xi$. In order to do it, it is more convenient to take the pair $\tilde{u} = (u', u')$ as a variable instead of $u = (u, u')$ and regard $\Phi = \Phi(\tilde{u}, \xi)$ as a map from $C_{\alpha, \beta, \gamma} \times (C^{\alpha-2} \times C^{2\alpha-3})$ to $C_{\alpha, \beta, \gamma}$, where $C_{\alpha, \beta, \gamma} = \{\tilde{u} \equiv (u', u') \in L_T^{2, \beta} \times L_T^{2, \alpha} : \|\tilde{u}\|_{\alpha, \beta, \gamma} \leq \infty\}$. Note that the norm $\|\cdot\|_{\alpha, \beta, \gamma}$ in (2.3) is defined essentially for $\tilde{u}$ and all estimates we obtained in $u$ is the same in $\tilde{u}$. In particular, the space $C_{\alpha, \beta, \gamma}$ does not depend on the noise $\xi$ or $X$.

Then, thanks to the implicit function theorem, see Proposition C.1.1 of [7] and the locally uniform contractivity of $\Phi$ in $\tilde{u}$, it is enough for us to show the continuity of $\Phi$ in $(\tilde{u}, \xi)$. In fact, we can show that $\Phi$ is locally Lipschitz continuous in $(\tilde{u}, \xi)$. For any two elements $(\tilde{u}_1, \xi_1)$ and $(\tilde{u}_2, \xi_2)$, we see $\|\Phi(\tilde{u}_1, \xi_1) - \Phi(\tilde{u}_2, \xi_2)\|_{\alpha, \beta, \gamma}$ is bounded by

$$\|\Phi(\tilde{u}_1, \xi_1) - \Phi(\tilde{u}_2, \xi_1)\|_{\alpha, \beta, \gamma} + \|\Phi(\tilde{u}_2, \xi_1) - \Phi(\tilde{u}_2, \xi_2)\|_{\alpha, \beta, \gamma}.$$

According to (3.50), we only have to study the second term. However, thanks to the multilinear property of $\xi_1$ and $\xi_2$ in the variable $(X, \xi, \Pi(\nabla X, \xi))$, by analogous, but simpler arguments for (3.48), we have

$$\|\Phi(\tilde{u}_2, \xi_1) - \Phi(\tilde{u}_2, \xi_2)\|_{\alpha, \beta, \gamma} \leq T^{2+\beta-\gamma} K(\|\tilde{u}_2\|_{\alpha, \beta, \gamma})(1 + \|\xi_1\|_{C^{\alpha-2}} + \|\xi_2\|_{C^{\alpha-2}})\|\xi_1 - \xi_2\|_{C^{\alpha-2}}.$$

Indeed, the cubic power of $\|\xi_i\|_{C^{\alpha-2}}$, $i = 1, 2$, in the Lipschitz coefficient in the above estimate arises from the terms $P_a$ in $A_1$ and $P_a$ in $A_3$ of $\Phi$ as computed in Lemmas 4.2 and 4.4, respectively, later and this is reflected in $\tilde{K}_2(X, \xi)$ in Proposition 3.8, which involves the fourth power of $\|\xi\|_{C^{\alpha-2}}$. All other terms in $\Phi$ have lower orders as we can see from Lemmas 3.4, 3.5, Proposition 3.8 and (3.47). Consequently, the proof is completed. \(\square\)
4 Proof of Proposition 3.8

This section is devoted to the proof of Proposition 3.8. In order to do it, we study each term appeared in \( \zeta \) given in (2.33) separately and divide it into four lemmas.

Let us denote by \( A_0 \) all of the terms in \( \zeta \) except the three terms \( A_1, A_2 \) and \( A_3 \), that is,

\[
A_0 = A_0(u, u') = \Pi \xi g(\nabla u) + g' (\nabla u) u' \Pi (\nabla X, \xi) + \Pi(a(\nabla u'_0), \Pi u'_0) - a' (\nabla u) (u')^2 \Pi(\nabla X, \xi),
\]

so that \( \zeta = A_0 + A_1 + A_2 - A_3 \). We first show \( A_0 \) has the desired estimates.

**Lemma 4.1.** For any \( u_1 = (u_1, u'_1), u_2 = (u_2, u'_2) \in \mathcal{B}_T(\lambda) \), we have

\[
\sup_{0 < t \leq T} t^{-\frac{\alpha}{2}} \| A_0(u_1)(t) \|_{C^{\alpha + \beta - 2}} \lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}) (1 + \| X \|_{C^{\alpha - 2}}) (\| \xi \|_{C^{\alpha - 2}} + \| \Pi(\nabla X, \xi) \|_{C^{2\alpha - 3}}),
\]

\[
\sup_{0 < t \leq T} t^{-\frac{\alpha}{2}} \| (A_0(u_1) - A_0(u_2))(t) \|_{C^{\alpha + \beta - 2}} \lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}, \| u_2 \|_{\alpha, \beta, \gamma}) \| u_1 - u_2 \|_{\alpha, \beta, \gamma} \times (1 + \| \xi \|_{C^{\alpha - 2}})^2 (\| \xi \|_{C^{\alpha - 2}} + \| \Pi(\nabla X, \xi) \|_{C^{2\alpha - 3}}).
\]

**Proof.** We first give the proof of (4.1). Applying Lemma 2.1 together with (3.11) for \( g \) instead of \( g' \), we get

\[
\| \Pi \xi g(\nabla u_1)(t) \|_{C^{\alpha + \beta - 2}} \lesssim \| g(\nabla u_1)(t) \|_{C^{\alpha}} \| \xi \|_{C^{\alpha - 2}} \lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}) (1 + \| X \|_{C^{\alpha}}) \| \xi \|_{C^{\alpha - 2}},
\]

where we have used the relation \( 0 < \alpha + \beta - 2 < \beta \) for the first inequality.

Recall that \( 0 < 2\alpha + \beta - 3 < \beta \) and \( \Pi(\nabla X, \xi) \in C^{2\alpha - 3} \) is assumed. Applying now Lemma 2.1 and (3.11), we obtain

\[
\| (g'(\nabla u_1) u'_1 \Pi(\nabla X, \xi))(t) \|_{C^{2\alpha - 3}} \lesssim \| g'(\nabla u_1)(t) \|_{C^{\beta}} \| u'_1(t) \|_{C^{\beta}} \| \Pi(\nabla X, \xi) \|_{C^{2\alpha - 3}} \lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}) (1 + \| X \|_{C^{\alpha}}) \| \Pi(\nabla X, \xi) \|_{C^{2\alpha - 3}}.
\]

Thanks to (3.11), the similar arguments give that

\[
\| (a'(\nabla u_1)(u'_1)^2 \Pi(\nabla X, \xi))(t) \|_{C^{2\alpha - 3}} \lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}) (1 + \| X \|_{C^{\alpha}}) \| \Pi(\nabla X, \xi) \|_{C^{2\alpha - 3}}.
\]

By (2.29) and \( \gamma - \alpha > 0 \), we easily have

\[
\sup_{0 < t \leq T} t^{-\frac{\alpha}{2}} \| \Pi(a(\nabla u'_0), \Pi u'_0)(t) \|_{C^{\alpha + \beta - 2}} \lesssim \| a \|_{C^1} \| u'_0 \|_{C^{\alpha}} \| u_1 \|_{\alpha, \beta, \gamma} \| \xi \|_{C^{\alpha - 2}}.
\]

Thus, by the above estimates together with \( \gamma - \alpha > 0 \), we obtain the desired result (4.1); recall Remark 3.2 and the same applies hereafter.

Next, we give the proof of (4.2). For the term \( \Pi \xi g(\nabla u) \), by (3.15) for \( g \) and the bilinearity of the paraproduct, we have

\[
\| \Pi \xi g(\nabla u_1)(t) - \Pi \xi g(\nabla u_2)(t) \|_{C^{\alpha + \beta - 2}}
\]

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For any $t \in [0, T]$. Thus, from the above estimates, it follows that
\[
\|P(\nabla u_1(t) - g(\nabla u_2(t))\|_{C^\beta} \lesssim K(\|u_1\|_{C^\beta} + \|u_2\|_{C^\beta} + 1 + \|X\|_{C^\alpha})^2 \|\xi\|_{C^\alpha-2}.
\]

For the term $g'((\nabla u_1)u' \Pi(\nabla X, \xi)$, noting that $\Pi(\nabla X, \xi) \in C^{2\alpha - 3}$, $2\alpha - 3 + \beta > 0$ and using the estimate for the product $uv$ given in Lemma 2.1, we may give the estimate on
\[
\|g'((\nabla u_1)u' - g'((\nabla u_2)u'')(t)\|_{C^\beta}.
\]
But, this is bounded again by the estimate for the product in Lemma 2.1 as
\[
\lesssim K(\|u_1\|_{C^\beta} + \|u_2\|_{C^\beta} + 1 + \|X\|_{C^\alpha})^2 \|\xi\|_{C^\alpha-2},
\]
for all $t \in [0, T]$, where (3.11) and (3.15) have been used for the last inequality.

The term $a'(\nabla u)(u'' \Pi(\nabla X, \xi)$ is treated similarly, for example, by estimating as
\[
\|(u_1''(t) - u_2'')\|_{C^\beta} \lesssim (\|u_1''(t)\|_{C^\beta} + \|u_2''(t)\|_{C^\beta})(\|u_1''(t)\|_{C^\beta} + \|u_2''(t)\|_{C^\beta}) \lesssim (\|u_1\|_{C^\beta} + \|u_2\|_{C^\beta})(\|u_1 - u_2\|_{C^\beta})
\]
for all $t \in [0, T]$. Thus, from the above estimates, it follows that
\[
\|P(\nabla u_1(t) - g(\nabla u_2(t))(u_1''(t) - (u_1''(t) - u_2'')(t))\Pi(\nabla X, \xi)\|_{C^{2\alpha - 3}} \lesssim K(\|u_1\|_{C^\beta} + \|u_2\|_{C^\beta} + 1 + \|X\|_{C^\alpha})^2 \|\Pi(\nabla X, \xi)\|_{C^{2\alpha - 3}}.
\]

For the third term $\Pi(a(\nabla u_0 T), \nabla u' \xi)$, from (2.29) and $\alpha + \beta - 2 < 0 < \alpha + \gamma - 3$, we have
\[
\|\Pi(a(\nabla u_0 T), \nabla u_1\xi - \nabla u_2\xi)(t)\|_{C^{\alpha + \beta - 2}} \lesssim \|\Pi(a(\nabla u_0 T), \nabla u_1\xi - \nabla u_2\xi)(t)\|_{C^{\alpha + \gamma - 3}} \lesssim \|a\|_{C^1(1 + T^{-\gamma/\alpha}} \|u_0\|_{C^\alpha})^2 \|u_1 - u_2\|_{C^\beta} \|\xi\|_{C^\alpha-2}.
\]

Multiply the both sides by $t^{\gamma/\alpha}$ and take the supremum in $t \in (0, T]$. Then the factor $T^{-\gamma/\alpha}$ (which is large for small $T > 0$) is absorbed and we obtain the desired estimate for this term.

Consequently, the proof of (4.2) is completed by recalling $\alpha + \beta - 2 < 2\alpha - 3$. □

We proceed to handle the term $A_1$ defined by (2.25) with $R_1 = R_1(u', X) = [\nabla, \Pi u'] X$.

**Lemma 4.2.** For any $u_1 = (u_1, u_1')$, $u_2 = (u_2, u_2') \in \mathcal{B}_T(\lambda)$, we have
\[
\|A_1(u_1)\|_{C^{\alpha + \beta - 3}} \lesssim K(\|u_1\|_{C^{\alpha + \beta - 3}}(1 + \|\xi\|_{C^{\alpha - 2}})^2 \|\xi\|_{C^{\alpha - 2}},
\]
\[
\|A_1(u_1) - A_1(u_2)\|_{C^{\alpha + \beta - 3}} \lesssim K(\|u_1\|_{C^{\alpha + \beta - 3}}(1 + \|\xi\|_{C^{\alpha - 2}})^4.
\]

**Proof.** We start with the proof of (4.3). Noting that $2\alpha - 3 < 0$ and $3\alpha - 4 > 0$, by Lemma 2.7 of [11], we have $P_g : (\nabla u, \xi) \in C^{\alpha - 1} \times C^{\alpha - 2} \rightarrow P_g(\nabla u, \xi) \in C^{2(\alpha - 1) + \alpha - 2} = C^{3\alpha - 4}$ and
\[
\|P_g(\nabla u_1, \xi)\|_{C^{\alpha + \beta - 3}} \lesssim g(1 + \|\nabla u_1\|_{C^{\alpha + \beta - 1}}^2) \|\xi\|_{C^{\alpha - 2}}
\]

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\[ \lesssim \|g\|_{C^2}(1 + (1 + \|X\|_{C^\alpha})^2\|u_1\|_{\alpha,\beta,\gamma}^2)\|\xi\|_{C^{\alpha-2}} \]
\[ \lesssim \|g\|_{C^2}(1 + \|u_1\|_{\alpha,\beta,\gamma}^2)(1 + \|X\|_{C^\alpha})^2\|\xi\|_{C^{\alpha-2}}, \]

where we have used (3.9) for the second inequality. On the other hand, thanks to Lemmas 2.5 and 2.6 together with (3.11), we easily have that the \( C_T C^{2\alpha+\beta-3}\)-norm of the second term of \( A_1 \) is bounded by

\[ K(\|u_1\|_{\alpha,\beta,\gamma})(1 + \|X\|_{C^\alpha})\|X\|_{C^\alpha}\|\xi\|_{C^{\alpha-2}}. \]

Therefore, we obtain (4.3) by the above estimates.

From now on, we show (4.4). For the first term of \( A_1 \), by the Lipschitz estimate given in Lemma 2.7 of [11] and (3.9), we have

\[ (4.6) \]
\[ \|P_g(\nabla u_1, \xi) - P_g(\nabla u_2, \xi)\|_{C_T C^{3\alpha-4}} \]
\[ \lesssim \|g\|_{C^3}\left( 1 + (\|\nabla u_1\|_{C_T C^{\alpha-1}} + \|\nabla u_2\|_{C_T C^{\alpha-1}})^2 + \|\xi\|_{C^{\alpha-2}} \right) \|\nabla u_1 - \nabla u_2\|_{C_T C^{\alpha-1}} \]
\[ \lesssim \|g\|_{C^3}\left( 1 + (1 + \|X\|_{C^\alpha})^2(\|u_1\|_{\alpha,\beta,\gamma} + \|u_2\|_{\alpha,\beta,\gamma})^2 + \|\xi\|_{C^{\alpha-2}} \right)(1 + \|X\|_{C^\alpha})\|u_1 - u_2\|_{\alpha,\beta,\gamma}, \]

which gives the bound

\[ \|P_g(\nabla u_1, \xi) - P_g(\nabla u_2, \xi)\|_{C_T C^{3\alpha-4}} \]
\[ \lesssim K(\|u_1\|_{\alpha,\beta,\gamma}, \|u_2\|_{\alpha,\beta,\gamma})\|u_1 - u_2\|_{\alpha,\beta,\gamma}(1 + \|X\|_{C^\alpha})^3(1 + \|\xi\|_{C^{\alpha-2}}). \]

Thanks to the multilinear properties of \( \tilde{C} \), the resonant and \( R_1 \), we can easily show that

\[ \|g'(\nabla u_1)\{ \tilde{C}(u_1', \nabla X, \xi) + \Pi(R_1(u_1', X), \xi) \} \]
\[ - g'(\nabla u_2)\{ \tilde{C}(u_2', \nabla X, \xi) + \Pi(R_1(u_2', X), \xi) \} \|_{C_T C^{2\alpha+\beta-3}} \]
\[ \lesssim K(\|u_1\|_{\alpha,\beta,\gamma}, \|u_2\|_{\alpha,\beta,\gamma})\|u_1 - u_2\|_{\alpha,\beta,\gamma}(1 + \|X\|_{C^\alpha})^2\|X\|_{C^\alpha}\|\xi\|_{C^{\alpha-2}}. \]

For instance, by the linearity of \( \tilde{C} \) in \( u' \) and Lemma 2.5, we have

\[ \|\tilde{C}(u_1', \nabla X, \xi) - \tilde{C}(u_2', \nabla X, \xi)\|_{C_T C^{2\alpha+\beta-3}} \]
\[ \lesssim \|u_1 - u_2\|_{\alpha,\beta,\gamma}\|X\|_{C^\alpha}\|\xi\|_{C^{\alpha-2}}. \]

Then, thanks to (3.11) and (3.15), the usual arguments give the desired result for \( g'(\nabla u_1) \)
\( \tilde{C}(u_1', \nabla X, \xi) \). Consequently, the above estimates and the fact that \( 2\alpha + \beta - 3 < 3\alpha - 4 \)
yield (4.4).

For the reader’s convenience, we recall the definition of \( A_2 \) from (2.28):

\[ A_2 = A_2(u, u') = -R(a(\nabla u) - a(\nabla u_0^T), u'; \xi) - \Pi_{\nabla u'}(a(\nabla u) - a(\nabla u_0^T)) \]
\[ + (a(\nabla u) - a(\nabla u_0^T))R_2, \]

where \( R \) is defined in Lemma 2.8 and \( R_2 = R_2(u', X) = |\Delta, \Pi_{u'}|X \). The next lemma gives the estimates on \( A_2 \).
Lemma 4.3. For any \(u_1 = (u_{11}, u_{12}), u_2 = (u_{21}, u_{22}) \in \mathcal{B}_T(\lambda)\), we have

\[
\begin{align*}
(4.7) & \quad \|A_2(u_1)\|_{C_T^{2\alpha + \beta - 2}} \lesssim K(\|u_1\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C^{\alpha - 2}})\|\xi\|_{C^{\alpha - 2}}, \\
(4.8) & \quad \|A_2(u_1) - A_2(u_2)\|_{C_T^{2\alpha + \beta - 2}} \\
& \lesssim K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})\|u_1 - u_2\|_{\alpha, \beta, \gamma}(1 + \|\xi\|_{C^{\alpha - 2}})^2\|\xi\|_{C^{\alpha - 2}}.
\end{align*}
\]

Proof. Let us first give the proof of (4.7). For the first term in \(A_2\), using Lemma 2.8 together with (3.17), we have

\[
\begin{align*}
(4.9) & \quad \|R(a(\nabla u_1) - a(\nabla u_0^T), u_1')\|_{C_T^{2\alpha + \beta - 2}} \lesssim \|a(\nabla u_1) - a(\nabla u_0^T)\|_{C_T^\beta}\|u_1'\|_{L_T^\beta}\|\xi\|_{C^{\alpha - 2}} \\
& \lesssim K(\|u_1\|_{\alpha, \beta, \gamma})(1 + \|X\|_{C^{\alpha}})\|\xi\|_{C^{\alpha - 2}}.
\end{align*}
\]

By Lemmas 2.4 and 2.6 together with (3.17), the similar arguments yield the same local growth of the other two terms as above. So, we have (4.7).

Next, we show the local Lipschitz estimate (4.8). The proof is essentially same as that of (4.7), because of the multilinear properties of \(R\), the resonant, the modified paraproduct and \(R_2\). Here we only give the proof for the term \(R\) as an example. By the trilinearity of \(R\), the same arguments for (4.9) give that

\[
\begin{align*}
\|R(a(\nabla u_1) - a(\nabla u_0^T), u_1')\|_{C_T^{2\alpha + \beta - 2}} & \lesssim \|R(a(\nabla u_1) - a(\nabla u_0^T), u_1')\|_{C_T^{\alpha + \beta - 2}} + \|R(a(\nabla u_2) - a(\nabla u_0^T), u_1' - u_2')\|_{C_T^{\alpha + \beta - 2}} \\
& \lesssim K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})\|\xi\|_{C^{\alpha - 2}}.
\end{align*}
\]

where we have used (3.17) and (3.19).

Finally, let us give the estimates on \(A_3\). Recall the definition (2.30) of \(A_3\):

\[
A_3 = P_a(\nabla u, \Pi u'\xi) + a'(\nabla u)\{C(u', \nabla X, \Pi u'\xi) + \Pi(1, \Pi u'\xi) + u'C(u', \xi, \nabla X)\}.
\]

Lemma 4.4. For any \(u_1 = (u_{11}, u_{12}), u_2 = (u_{21}, u_{22}) \in \mathcal{B}_T(\lambda)\), we have

\[
\begin{align*}
(4.10) & \quad \|A_3(u_1)\|_{C_T^{2\alpha + \beta - 3}} \lesssim K(\|u_1\|_{\alpha, \beta, \gamma})(1 + \|\xi\|_{C^{\alpha - 2}})^2\|\xi\|_{C^{\alpha - 2}}, \\
& \quad \|A_3(u_1) - A_3(u_2)\|_{C_T^{2\alpha + \beta - 3}} \\
& \lesssim K(\|u_1\|_{\alpha, \beta, \gamma}, \|u_2\|_{\alpha, \beta, \gamma})\|u_1 - u_2\|_{\alpha, \beta, \gamma}(1 + \|\xi\|_{C^{\alpha - 2}})^4.
\end{align*}
\]

Proof. Let us first give the estimates on the term \(A_{3,1}(u, u') := a'(\nabla u)u'C(u', \xi, \nabla X)\). Noting that \(\beta > 2\alpha + \beta - 3 > 0\) and using Lemma 2.1-(ii) and Lemma 2.5, we have that

\[
(4.11) \quad \|u_1' C(u_2', \xi, \nabla X)\|_{C_T^{2\alpha + \beta - 3}} \lesssim \|u_1'\|_{L_T^\beta}\|u_2'\|_{L_T^\beta}\|X\|_{C^{\alpha}}\|\xi\|_{C^{\alpha - 2}}
\]

holds for \(u_1', u_2' \in L_T^\beta\). Using (4.11) with \(u_1' = u_2'\) and (3.11), we get the local growth estimate on \(A_{3,1}\), that is,

\[
\|A_{3,1}(u_1, u_1')\|_{C_T^{2\alpha + \beta - 3}} \lesssim K(\|u_1\|_{\alpha, \beta, \gamma})(1 + \|X\|_{C^{\alpha}})\|X\|_{C^{\alpha}}\|\xi\|_{C^{\alpha - 2}}.
\]
Moreover, noting the multilinear property of \( C \), by (4.11), (3.11) and (3.15), we immediately have the local Lipschitz estimate on \( A_{3,1} \), that is,
\[
\|A_{3,1}(u_1, u'_1) - A_{3,1}(u_2, u'_2)\|_{C_T C^{2\alpha + \beta - 3}} \\
\lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}, \| u_2 \|_{\alpha, \beta, \gamma}) \| u_1 - u_2 \|_{\alpha, \beta, \gamma} (1 + \| X \|_{C^\alpha})^2 \| X \|_{C^\alpha} \| \xi \|_{C^{\alpha - 2}}.
\]

Now comparing the terms in \( A_3 \) except \( A_{3,1} \) with \( A_1 \), see (2.25), we find the difference is that \( g' \) and \( \xi \) in \( A_1 \) are replaced by \( a' \) and \( \Pi_a \xi \) in \( A_3 \), respectively. So, noting \( \| \Pi_a \xi \|_{C_T C^{\alpha - 2}} \lesssim \| a' \|_{C_T} \| \xi \|_{C^{\alpha - 2}} \) from Lemma 2.4-(i), we can easily conclude the proof by mimicking that of Lemma 4.2. Here, we only explain the factor \((1 + \| \xi \|_{C^{\alpha - 2}})^{4}\) appearing in (4.10) in a little more detail. It comes from the estimate on \( P_a \) as we saw in the proof of (4.4). In fact, using the similar arguments for (4.6) and then (3.9), we easily have
\[
\| P_a(\nabla u_1, \Pi_a \xi) - P_a(\nabla u_2, \Pi_a \xi) \|_{C_T C^{\alpha - 4}} \\
\lesssim \| a \|_{C^3} \left( 1 + (\| \nabla u_1 \|_{C_T C^{\alpha - 1} \| + \| \nabla u_2 \|_{C_T C^{\alpha - 1}})^2 \right) \quad \times \left( \| \nabla u_1 - \nabla u_2 \|_{C_T C^{\alpha - 1}} + \| \Pi_a \xi \|_{C_T C^{\alpha - 2}} \right) \\
\lesssim \| a \|_{C^3} \left( 1 + (1 + \| X \|_{C^{\alpha}})^2 \right) \left( \| u_1 \|_{\alpha, \beta, \gamma} + \| u_2 \|_{\alpha, \beta, \gamma} \right)^2 \quad \times \left( \| u_1 - u_2 \|_{\alpha, \beta, \gamma} (1 + \| X \|_{C^\alpha} + \| \xi \|_{C^{\alpha - 2}}) \right) \\
\lesssim K(\| u_1 \|_{\alpha, \beta, \gamma}, \| u_2 \|_{\alpha, \beta, \gamma}) \| u_1 - u_2 \|_{\alpha, \beta, \gamma} \times (1 + \| X \|_{C^\alpha})^2 (1 + \| \xi \|_{C^{\alpha - 2}}) (1 + \| X \|_{C^\alpha} + \| \xi \|_{C^{\alpha - 2}}),
\]
which gives the desired result. \( \square \)

To conclude this section, let us give the proof of Proposition 3.8.

**Proof of Proposition 3.8.** Noting that \( \alpha + \beta - 2 < 2\alpha + \beta - 3 \) and \( \gamma - \alpha > 0 \), we obtain immediately Proposition 3.8 by Lemmas 4.1-4.4. \( \square \)

## 5 Convergence of the resonant term

Recall \( \xi \in C^{\alpha - 2} \) and \( \nabla X \in C^{\alpha - 1} \). Then we can define \( \Pi(\nabla X, \xi) \in C^{2\alpha - 3} \), which is denoted by \( \nabla X \circ \xi \) in Lemma 5.2, though their product is definable with less regularity: \( \nabla X \cdot \xi \in C^{\alpha - 2} \) as we will see in Lemma 5.3, note that \( 2\alpha - 3 > \alpha - 2 \).

We follow the arguments in Section 5.2 of [11] noting that they discuss two dimensional case taking \( T = [0, 2\pi] \), while we are in one dimension but consider \( \nabla X \) instead of \( X (= \vartheta) \). Recall \( T = [0, 1] \) in our case and set \( \hat{u}(k) = \int_T e^{-2\pi ikx} u(x) dx, k \in \mathbb{Z} \). Let \( \xi \) be the spatial white noise on \( T \). Then,
\[
E[\tilde{\xi}(k) \tilde{\xi}(k')] = 1_{k=-k'}, \quad k, k' \in \mathbb{Z},
\]
and \( \overline{\xi(k)} = \tilde{\xi}(-k) \). Note that the mean zero solution \( X \) of (2.1) is given by
\[
X = \int_0^\infty P_t Q \xi dt,
\]

(5.1)
where \( P_t = e^{t\Delta} \) and \( Q\xi = \xi - \xi(0), \) note \( \xi(0) = \xi(T). \)

The following expectation appears to compensate the 0th order term to define \( \nabla X \circ \xi \) in (5.2) (cf. Lemma 5.6 of [11]), but it vanishes in our case.

**Lemma 5.1.** For \( x \in \mathbb{T} \) and \( t > 0, \) we have

\[
E[\Pi(\nabla P_t Q \xi, \xi)(x)] = \sum_{k \in \mathbb{Z} \setminus \{0\}} 2\pi i ke^{-4\pi^2 k^2 t} = 0.
\]

**Proof.** Compared with Lemma 5.6 of [11] (they consider on \( \mathbb{T}^2 = [0, 2\pi]^2 \) so that they have \((2\pi)^{-2}\)), we have \( \nabla \) which yields \( 2\pi ik \) in Fourier mode. \( \square \)

We assume the following rather mild condition for the mollifier \( \psi \) to cover the noise in (1.2), see Remark 5.1 below. Let \( \psi \) be a measurable and integrable function on \( \mathbb{R}, \) which has a compact support and satisfies \( \int_{\mathbb{R}} \psi(x)dx = 1. \) We set \( \psi^\varepsilon(x) = \frac{1}{\varepsilon}\psi(\frac{x}{\varepsilon}) \) for \( \varepsilon > 0. \) Note that the support of \( \psi^\varepsilon \) is included in \( \mathbb{T} \left( = \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) \) for sufficiently small \( \varepsilon > 0 \) and \( \psi^\varepsilon(\xi(x)) = \int_{\mathbb{T}} \psi^\varepsilon(x-y)\xi(y)dy, \) \( x \in \mathbb{T} \) is well-defined (by considering \( \psi^\varepsilon \) periodically on \( \mathbb{R} \) if necessary). The compact support property is assumed for simplicity and can be removed. We call \( \xi^\varepsilon := \psi^\varepsilon \ast \xi \) the smeared noise of the spatial white noise \( \xi \) on \( \mathbb{T}. \)

**Lemma 5.2.** (cf. Lemma 5.8 of [11]) Set

\[
(5.2) \quad \nabla X \circ \xi = \int_0^\infty \Pi(\nabla P_t Q \xi, \xi)dt.
\]

Then we have

\[
(5.3) \quad E[\|\nabla X \circ \xi\|_{C^{2a-3}}^p] < \infty \quad \text{for all } \alpha < \frac{3}{2} \text{ and } p \geq 1.
\]

Moreover, for \( \psi \) satisfying the above condition, we set \( \xi^\varepsilon = \psi^\varepsilon \ast \xi \) for \( \varepsilon > 0 \) and \( \nabla X^\varepsilon = \int_0^\infty \nabla P_t Q \xi^\varepsilon dt \left( = \nabla (-\Delta)^{-1} Q \xi^\varepsilon \right). \) Then, we have

\[
(5.4) \quad \lim_{\varepsilon \downarrow 0} E[\|\nabla X \circ \xi - \Pi(\nabla X^\varepsilon, \xi^\varepsilon)\|_{C^{2a-3}}^p] = 0
\]

for all \( p \geq 1. \) We also have

\[
(5.5) \quad \begin{align*}
c^\varepsilon := & E[\nabla X^\varepsilon(x)\xi^\varepsilon(x)] = E[\Pi(\nabla X^\varepsilon, \xi^\varepsilon)(x)] \\
= & \int_0^\infty E[\Pi(\nabla P_t Q \xi^\varepsilon, \xi^\varepsilon)(x)]dt = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|\hat{\psi}(\varepsilon k)|^2}{4\pi^2 k^2} 2\pi ik \\
= & 0,
\end{align*}
\]

for \( x \in \mathbb{T} \) and \( \varepsilon > 0 \) such that \( \text{supp}(\psi) \subset \{|x| \leq \frac{1}{2\varepsilon}\}, \) where \( \hat{\psi}(y) := \int_{\mathbb{R}} e^{-2\pi i yx} \psi(x)dx, \) \( y \in \mathbb{R} \) is the Fourier transform on \( \mathbb{R}. \)

**Proof.** We first note that \( c^\varepsilon = 0 \) in (5.5) follows from the symmetry of \( |\hat{\psi}(\varepsilon k)|^2 = \hat{\psi}(\varepsilon k)\hat{\psi}(-\varepsilon k) \) in \( k \) due to the fact that \( \psi \) is real-valued.

To show (5.3) and (5.4), we divide the time integral on \( (0, \infty) \) in (5.2) and \( \nabla X^\varepsilon \) into those on \( (0, 1] \) and \( (1, \infty). \) Let us first show (5.4) for the contribution from the integral

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on \((1, \infty)\), (5.3) for this part is shown similarly. Noting that \(2\alpha - 3 < 0 < 2\alpha - 2\) for \(\alpha \in (\frac{3}{4}, \frac{3}{2})\), we have

\[
\begin{align*}
\| \int_1^\infty \{ \Pi(\nabla P_t Q \xi) - \Pi(\nabla P_t Q \xi^\varepsilon) \} dt \|_{C^{2\alpha - 3}} \\
\leq \int_1^\infty \| \Pi(\nabla P_t Q(\xi - \xi^\varepsilon)) \|_{C^{2\alpha - 2}} + \| \Pi(\nabla P_t Q \xi^\varepsilon, \xi - \xi^\varepsilon) \|_{C^{2\alpha - 2}} ) dt \\
\lesssim \int_1^\infty \{ \| \nabla P_t Q(\xi - \xi^\varepsilon) \|_{C^{\alpha}} \| \xi \|_{C^{\alpha - 2}} + \| \nabla P_t Q \xi^\varepsilon \|_{C^{\alpha}} \| \xi - \xi^\varepsilon \|_{C^{\alpha - 2}} \} dt \\
\lesssim \int_1^\infty \{ \| P_t Q(\xi - \xi^\varepsilon) \|_{C^{\alpha+1}} \| \xi \|_{C^{\alpha - 2}} + \| P_t Q \xi^\varepsilon \|_{C^{\alpha+1}} \| \xi - \xi^\varepsilon \|_{C^{\alpha - 2}} \} dt \\
\lesssim \| \xi - \xi^\varepsilon \|_{C^{\alpha - 2}} \{ \| \xi \|_{C^{\alpha - 2}} + \| \xi^\varepsilon \|_{C^{\alpha - 2}} \} \int_1^\infty t^{-\frac{3}{2}} dt,
\end{align*}
\]

where we have used Lemma 2.1-(i) (noting \(2\alpha - 2 > 0\)) for the second inequality, Lemma 3.2-(i) for the last inequality, and note that \(\int_1^\infty t^{-\frac{3}{2}} dt < \infty\). We then have the desired convergence for the part arising from the integral on \((1, \infty)\) by showing

\[
E[\| \xi - \xi^\varepsilon \|_{C^{\alpha - 2}}^{2p}] \leq E[\| \xi - \xi^\varepsilon \|_{C^{\alpha - 2}}^{2p}] \lesssim \| \xi \|_{C^{\alpha - 2}} \| \xi^\varepsilon \|_{C^{\alpha - 2}} \int_1^\infty t^{-\frac{3}{2}} dt < \infty\]

under our condition on \(\psi\). Indeed, we first compute

\[
E[\| \xi - \xi^\varepsilon \|_{L^{2p;2p}}^{2p}] = \sum_{j=-1}^{\infty} 2^{2pj(\alpha - 2)} \int \| \Delta_j (\xi - \xi^\varepsilon)(x) \|_{L^{2p}}^{2p} dx.
\]

Then, by Gaussian hypercontractivity (equivalence of moments, Lemma 4.6 of [11]),

\[
E[\| \Delta_j (\xi - \xi^\varepsilon)(x) \|_{L^{2p}}^{2p}] \leq C_p E[\| \Delta_j (\xi - \xi^\varepsilon)(x) \|_{L^{2p}}^{2p}]
\]

for some \(C_p > 0\). Here, with \(K_j := \hat{\rho}_j\) (inverse Fourier transform of dyadic partition \(\{\rho_j\}_{j=-1}^{\infty}\) of unity), we can rewrite as

\[
E[\Delta_j (\xi - \xi^\varepsilon)(x)^2] = E[\left( \int_T K_j(x - y)(\xi - \xi^\varepsilon)(y) dy \right)^2]
\]

\[
= \| K_j - \rho \|_{L^2(T)}^2
\]

\[
= \| \rho_j - \hat{\rho}_j \hat{\psi}^\varepsilon \|_{L^2(Z)}^2
\]

\[
= \sum_{k \in Z} \rho_j(k)^2 \{ 1 - 2 \text{Re} \hat{\psi}^\varepsilon(k) + |\hat{\psi}^\varepsilon(k)|^2 \}.
\]

We have used Plancherel identity for the third equality. However, by our condition on \(\psi\),

\[
|\hat{\psi}^\varepsilon(k)| = \frac{1}{\varepsilon} \int_T \psi\left( \frac{x}{\varepsilon} \right) e^{-2\pi ikx} dx
\]

\[
= \left| \int_R \psi(y) e^{-2\pi iky} dy \right| \leq \| \psi \|_{L^1(R)}
\]

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and this tends to 0 as $\varepsilon \downarrow 0$ for each $k$. Thus, since $\sum_{k \in \mathbb{Z}} \rho_j^2(k) \sim \int \rho^2(2^{-j}x)dx \sim 2^j$, we can show by Lebesgue’s convergence theorem that

$$E[\|\xi - \xi^\varepsilon\|^2_{L^{2p} \to L^{2p}}] \xrightarrow{\varepsilon \downarrow 0} 0,$$

if $2(\alpha - 2) + 1 < 0$, that is $\alpha < \frac{3}{2}$. Since we have continuous embedding $B^\alpha_{p,p} \subset B^\alpha_{\infty,\infty} = C^{\alpha - \frac{1}{2}}$ by Besov embedding theorem (Lemma A.2 of [11] or Lemma 8 of [12]), taking $p$ large, we see

$$E[\|\xi - \xi^\varepsilon\|^2_{C^{\alpha-2}}] \xrightarrow{\varepsilon \downarrow 0} 0,$$

if $\alpha < \frac{3}{2}$ and this implies (5.6).

For the integral on $(0, 1]$, since one can apply Gaussian hypercontractivity, Lebesgue’s convergence theorem and Besov embedding theorem as above, we only show (5.3) for this part:

$$E\left[\left\|\int_0^1 \Xi_t dt\right\|_{C^{2\alpha-3}}^p\right] < \infty,$$

where $\Xi_t = \Pi(\nabla P_t \mathcal{Q}\xi, \xi)$. To show this, noting that $\nabla$ yields $2\pi ik$ in Fourier mode, similarly to Lemma 5.8 of [11] and also as above, we have

$$\text{Var}(\Delta_m \Pi(\nabla P_t \xi, \xi)(x)) \leq \sum_{i,j,i',j'} \sum_{k_1 \neq k_2} \sum_{|i-j| \leq 1} \left[1_{m \leq i, m \leq j} \rho_m^2(k_1 + k_2) \rho_i(k_1) \rho_j(k_2) \rho_i(k_2) \rho_j(k_1) 4\pi^2 k_1^2 e^{-8\pi^2 k_1^2 t} + 1_{m \leq i} 1_{m \leq j'} \rho_m^2(k_1 + k_2) \rho_i(k_1) \rho_j(k_2) \rho_i(k_2) \rho_j(k_1) 4\pi^2 k_1 k_2 e^{-4\pi^2 k_1^2 t - 4\pi^2 k_2^2 t}\right].$$

Then, noting that $k_1 \in \text{supp}(\rho_i)$ and $k_2 \in \text{supp}(\rho_j)$ with $|i - j| \leq 1$ imply $k_1^2 \sim 2^{2i}$ and $|k_1k_2| \sim 2^{2i}$, we have

$$\text{Var}(\Delta_m \Pi(\nabla P_t \xi, \xi)(x)) \lesssim \sum_{i,j',j'} 1_{m \leq i} 1_{m \leq j'} \sum_{k_1,k_2} \text{supp}(\rho_m)(k_1 + k_2) 1_{\text{supp}(\rho_i)}(k_1) 1_{\text{supp}(\rho_j)}(k_2) 2^{2i} e^{-2tc2^{2i}} \lesssim \sum_{i,j' \geq 2m} 2^{2m} 2^{2i} e^{-2te2^{2i}} \lesssim \frac{2^m}{t^{3/2}} \sum_{i \geq 2m} e^{-tc2^{2i}} \lesssim \frac{2^m}{t^{3/2}} e^{-tc2^{2m}},$$

where we used that $\|k_1 \leq C2^i, \|k_2 \leq C2^m$ in the sum $\sum_{k_1,k_2}$ in the second line (instead of $\|k_1 \leq C2^{2i}, \|k_2 \leq C2^{2m}$ in two dimensional case) and $2^{3i} e^{-tc2^{2i}} \leq Ct^{-3/2}$ in the third line.

Thus, we obtain

$$E[\|\Xi_t\|_{B^{2\alpha-3}_{2p,2p}}] \lesssim \left( \sum_{m \geq -1} 2^{m(2\alpha-3)m2p} E[\|\Delta_m \Xi_t\|_{L^{2p}(\mathbb{T})}^{2p}] \right)^{1/2p} \lesssim t^{-3/4} \left( \sum_{m \geq -1} 2^{m(2\alpha-3)m2p} 22^{m2p} e^{-tc2^{2m}} \right)^{1/2p} \lesssim t^{-3/4} \left( \int_{-1}^{\infty} (2^{x})^{2p(2\alpha-\frac{3}{2})} e^{-tc(x^2)} dx \right)^{1/2p}.$$
Recall that

\[ E[|\|\Xi\|_{B^{2\alpha - 3}_{2}, 2}] \lesssim t^{-3/4} \left( t^{-p(2\alpha - \frac{3}{2})} \int_0^\infty y^{2p(2\alpha - \frac{3}{2}) - 1} e^{-cy^2} dy \right)^{1/2p}. \]

If \( \alpha > \frac{5}{4} \), the integral in the right hand side is finite for all large \( p \) and therefore

\[ E[|\|\Xi\|_{B^{2\alpha - 3}_{2}, 2}] \lesssim t^{-\frac{3}{2} - \frac{1}{2}(2\alpha - \frac{3}{2})} = t^{-\alpha + \frac{1}{2}} \] so that \( \int_0^1 E[|\|\Xi\|_{B^{2\alpha - 3}_{2}, 2}] dt < \infty \) for all \( \alpha < \frac{3}{2} \). \( \square \)

The constant \( c_e \) usually diverges as \( \varepsilon \downarrow 0 \), and in such case it is called the renormalization constant. However, since \( c_e = 0 \) in our case, our equation (1.5) does not require any renormalization.

**Remark 5.1.** (i) Let us consider the noise \( w^\varepsilon(x) \) introduced in (1.2) with \( \{w(x)\}_{x \in \mathbb{Z}} \). It has a representation: \( w^\varepsilon(x) = \psi^\varepsilon \ast w(x) \) with

\[ \psi(x) = \frac{1}{a + b} 1_{[-\alpha, b]}(x), \]

which satisfies the assumption of Lemma 5.2. Indeed, since \( \psi^\varepsilon(x) = \frac{1}{(a + b)e} 1_{[-\alpha, b]}(x) \), we see

\[ \psi^\varepsilon \ast w(x) = \int_T \psi^\varepsilon(x - y) w(dy) = \frac{1}{(a + b)e} \int_{x - b}^{x + a\varepsilon} \psi(dy) = \psi^\varepsilon(x). \]

(ii) At least heuristically, \( \xi = w(x) \) and \( \nabla X = \nabla (\Delta)^{-1} \omega = (-\nabla)^{-1} \omega = -w(x) \) is a periodic mean zero Brownian motion, that is

\[ w(x) = B(x) - xB(1) - \int_0^1 \{B(y) - yB(1)\} dy, \quad x \in \mathbb{T} \simeq [0, 1), \]

where \( B \) is a standard Brownian motion.

Finally, we note that the product \( \nabla X \cdot \xi \) can be defined directly for \( \xi = w(x) \) in a usual sense as a limit, but we see \( \nabla X \cdot \xi \in C^{\frac{3}{2} - \delta} \) for every \( \delta > 0 \), though \( \Pi(\nabla X, \xi) \in C^{2\alpha - 3}, \alpha < \frac{3}{2} \). In particular, we see again that we don’t need any renormalization.

**Lemma 5.3.** Let \( w(x) \) be as in Remark 5.1-(ii) and let \( \xi(x) = w(x) \). Then, \( \nabla X^\varepsilon(x) \cdot \xi^\varepsilon(x) \) converges to \( -\frac{1}{2} \nabla(w^2(x)) \) as \( \varepsilon \downarrow 0 \) in \( C^{\frac{3}{2} - \delta} \) for every \( \delta > 0 \).

**Proof.** Recall that

\[ \xi^\varepsilon(x) = \int_T \psi^\varepsilon(x - y) \xi(dy) = \int_T \psi^\varepsilon(x - y) \omega(dy) \]

\[ = \int_T \nabla_x \psi^\varepsilon(x - y) \omega(dy), \]

since \( -\nabla_y \psi^\varepsilon(x - y) = \nabla_x \psi^\varepsilon(x - y) \). On the other hand, let \( \{\varphi_0, \varphi_{k, \pm} \in C^\infty(\mathbb{T})\}_{k=1}^\infty \) be the eigenfunctions of \( -\Delta: -\Delta \varphi_{k, \pm}(x) = 4\pi^2 k^2 \varphi_{k, \pm}(x) \), where

\[ \varphi_{k, +}(x) = \sqrt{2} \sin 2\pi kx, \quad \varphi_{k, -}(x) = \sqrt{2} \cos 2\pi kx, \quad x \in \mathbb{T} \simeq [0, 1), \]

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and \( \varphi_0(x) \equiv 1 \). Then the equation 
\[- \Delta X^\varepsilon = Q \xi^\varepsilon \]
is solved as
\[
X^\varepsilon(x) = \sum_{k=1, \pm}^{\infty} \varphi_{k, \pm}(x) \int_{\mathbb{T}} \varphi_{k, \pm}(y) \xi^\varepsilon(y) dy.
\]
Therefore, by integration by parts,
\[
\nabla X^\varepsilon(x) = \sum_{k=1, \pm}^{\infty} \frac{\varphi'_{k, \pm}(x)}{4\pi^2 k^2} \int_{\mathbb{T}} \varphi_{k, \pm}(y) dy \int_{\mathbb{T}} \nabla \chi(y-z) w(z) dz
\]
\[
= - \int_{\mathbb{T}} \partial_y \partial_x Y_y(x) dy \int_{\mathbb{T}} \chi(y-z) w(z) dz
\]
\[
= - \int_{\mathbb{T}} \chi(x-z) w(z) dz,
\]
where
\[
Y_y(x) = \sum_{k=1, \pm}^{\infty} \frac{\varphi_{k, \pm}(x)}{4\pi^2 k^2} \varphi_{k, \pm}(y)
\]
is a (mean zero) solution of
\[- \Delta Y_y(x) = \delta_y(x)
\]
for each \( y \in \mathbb{T} \), that is, \( Y_y(x) \) is a Green function of \(-\Delta\). Note that \( \partial_y \partial_x Y_y(x) = \delta_y(x) \).

From these, we have
\[
\nabla X^\varepsilon(x) \cdot \xi^\varepsilon(x) = - \frac{1}{2} \nabla \left( \int_{\mathbb{T}} \chi(x-y) w(y) dy \right)^2.
\]
In particular, since \( \int_{\mathbb{T}} \chi(x-y) w(y) dy \) converges to \( w(x) \) in \( C^{\frac{1}{2} - \delta} \) as \( \varepsilon \downarrow 0 \) for every \( \delta > 0 \), we obtain the conclusion. \( \square \)

6 Comparison theorem for SPDE (1.4) with smooth noise

We show a comparison theorem for (1.4) on \( \mathbb{T} \) (or \( \mathbb{R} \)) with smooth \( \xi \).

**Lemma 6.1.** (1) Assume \( \chi \in C^1(\mathbb{R}) \) and satisfy \( |\chi'(v)| \leq C \varphi(v) \), \( v \in \mathbb{R} \) for some \( C > 0 \). We also assume \( \xi \in C^\infty(\mathbb{T}) \). Then, for two solutions \( v_1, v_2 \) of (1.4) on \( \mathbb{T} \), if \( v_1(0) \geq v_2(0) \) holds, we have \( v_1(t) \geq v_2(t) \) for all \( t \geq 0 \), where \( v_1 \geq v_2 \) means that \( v_1(x) \geq v_2(x) \) for all \( x \in \mathbb{T} \) for \( v_i = (v_i(x))_{x \in \mathbb{T}} \), \( i = 1, 2 \).

(2) In addition, assume \( \chi(0) = 0 \). Then, if \( v(0) \geq 0 \), we have \( v(t) \geq 0 \) for all \( t \geq 0 \).

**Proof.** The assertion (2) follows from (1), since \( v(t) \equiv 0 \) is a solution of (1.4) by noting \( \chi(0) = 0 \). To show (1), assume that \( v_1(s, x) \geq v_2(s, x) \) for all \( 0 \leq s \leq t \) and \( x \in \mathbb{T} \) and \( v_1(t, x_0) = v_2(t, x_0) \) at some \( t \geq 0 \) and \( x_0 \in \mathbb{T} \). Then, noting that the solutions \( v_1 \) and \( v_2 \) of (1.4) are smooth, we have
\[
\partial_t (v_1(t, x_0) - v_2(t, x_0))
\]
\[
= \Delta \{ \varphi(v_1(t, \cdot)) - \varphi(v_2(t, \cdot)) \}(x_0) + \nabla \{ (\chi(v_1(t, \cdot)) - \chi(v_2(t, \cdot))) \xi(\cdot) \}(x_0)
\]
\[ \lim_{\delta \to 0} \left[ \frac{1}{\delta^2} \sum_{\pm} \{ \varphi(v_1(t, x_0 \pm \delta)) - \varphi(v_2(t, x_0 \pm \delta)) \} \right. \\
\left. + \frac{1}{\delta} \left\{ (\chi(v_1(t, x_0 + \delta)) - \chi(v_2(t, x_0 + \delta))) \xi(x_0 + \delta) \right\} \right]. \]

However, since \(|\chi'(v)| \leq C\varphi'(v)\) and \(|\xi(x)| \leq M\) for some \(M > 0\), we see
\[
\frac{1}{\delta} \left| \chi(v_1(t, x_0 + \delta)) - \chi(v_2(t, x_0 + \delta)) \right| |\xi(x_0 + \delta)| \\
\leq \frac{M}{\delta} \int_{v_1(t, x_0 + \delta)}^{v_2(t, x_0 + \delta)} |\chi'(v)| dv \\
\leq \delta CM \cdot \frac{1}{\delta^2} \{ \varphi(v_1(t, x_0 + \delta)) - \varphi(v_2(t, x_0 + \delta)) \},
\]
which implies
\[ \partial_t (v_1(t, x_0) - v_2(t, x_0)) \geq 0. \]

This shows that \(v_2\) cannot exceed \(v_1\) at \(t\) and \(x_0\).

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