HECKE ALGEBRAS ASSOCIATED TO \(\Lambda\)-ADIC MODULAR FORMS

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Abstract. We show that if an Eisenstein component of the \(p\)-adic Hecke algebra associated to modular forms is Gorenstein, then it is necessary that the plus-part of a certain ideal class group is trivial. We also show that this condition is sufficient whenever a conjecture of Sharifi holds.

1. Introduction

In this paper, we study whether the Eisenstein component of the \(p\)-adic Hecke algebra associated to modular forms is Gorenstein, and how this relates to the theory of cyclotomic fields. We will first prepare some notation in order to state our results. See section 1.4 for some remarks on the notation.

1.1. Notation. Let \(p \geq 5\) be a prime and \(N\) an integer such that \(p \nmid \phi(N)\) and \(p \nmid N\). Let \(\theta: (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times_p\) be an even character and let \(\chi = \omega^{-1} \theta\), where \(\omega: (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times\) denotes the Teichmüller character. We assume that \(\theta\) satisfies the same conditions as in [S] and [K-F] – namely that 1) \(\theta\) is primitive, 2) if \(\chi|_{\mathbb{Z}/p\mathbb{Z}^\times} = 1\), then \(\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times} (p) \neq 1\), 3) if \(N = 1\), then \(\theta \neq \omega^2\).

If \(\phi: G \to \overline{\mathbb{Q}}^\times_p\) is a character of a group \(G\), let \(\mathbb{Z}_p[\phi]\) denote the \(\mathbb{Z}_p\)-algebra generated by the values of \(\phi\), on which \(G\) acts through \(\phi\). If \(M\) is a \(\mathbb{Z}_p[G]\)-module, denote by \(M_{\phi}\) the \(\phi\)-eigenspace:

\[
M_{\phi} = M \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\phi].
\]

Let

\[
H' = \lim_{\leftarrow} H^1(\mathbb{X} \times_{\mathbb{Z}} (Np^r), \mathbb{Z}_p)^{ord}_{\theta}
\]

and

\[
\hat{H}' = \lim_{\leftarrow} H^1(\mathbb{Y} \times_{\mathbb{Z}} (Np^r), \mathbb{Z}_p)^{ord}_{\theta},
\]

where \(ord\) denotes the ordinary part for the dual Hecke operator \(T^*(p)\), and the subscript refers to the eigenspace for the diamond operators.

Let \(h'\) (resp. \(\hat{h}'\)) be the algebra of dual Hecke operators acting on \(H'\) (resp. \(\hat{H}'\)). Let \(I\) (resp. \(\mathcal{I}\)) be the Eisenstein ideal of \(h'\) (resp. \(\hat{h}'\)). Let \(\mathcal{H}\) denote the Eisenstein component \(\mathcal{H} = \mathcal{H}^\prime_m\) the localization at the unique maximal ideal \(m\) containing \(I\). We can define the Eisenstein component \(h\) of \(h'\) analogously. Let \(\hat{H} = \hat{H}' \otimes_{\hat{h}'} \mathcal{H}\) and \(H = H' \otimes_{h'} \mathcal{H}\), the Eisenstein components.

Let \(G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). For a \(G_Q\)-module \(M\), let \(M^+\) and \(M^-\) for the eigenspaces of complex conjugation.

Let \(Q_\infty = \mathbb{Q}(\zeta_{Np^{\infty}})\); let \(M\) be the maximal abelian \(p\)-extension of \(Q_\infty\) unramified outside \(Np\) and let \(L\) be the maximal abelian \(p\)-extension of \(Q_\infty\) unramified everywhere. Let \(X = \text{Gal}(M/Q_\infty)\) and \(X = \text{Gal}(L/Q_\infty)\).
Let \( \Lambda = \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\), the Iwasawa algebra.

1.2. **Statement of Results.** Many authors have studied the Gorenstein property of Hecke algebras, and its relationship to arithmetic (cf. [Kur], for example). Ohta has the following theorem.

**Theorem 1.1.** (Ohta, [O2]) Suppose that \( \mathcal{X}_\beta^+ = 0 \). Then \( \mathcal{H} \) is Gorenstein.

Skinner and Wiles have obtained similar results using different methods [S-W]. This is a sufficient condition for \( \mathcal{H} \) to be Gorenstein. The main result of this paper gives a necessary condition that is conjecturally also sufficient.

**Theorem 1.2.** Suppose that \( \mathcal{X}^- \neq 0 \). Then

1) If \( \mathcal{H} \) is Gorenstein, then \( \mathcal{X}_\beta^+ = 0 \).
2) If Sharifi’s conjecture is true, then \( \mathcal{X}_\beta^+ = 0 \) implies \( \mathcal{H} \) is Gorenstein.

**Remarks 1.3.** Sharifi’s conjecture is stated in ([S], 4.12). It says that the map \( \Upsilon \), described in section 2.4 below, is an isomorphism. In fact, for the proof of theorem 1.2 (2), we only need that it is a surjection. For partial results on Sharifi’s conjecture, see [K-F].

Note that \( \mathcal{X}_\beta^+ \) is a quotient of \( \mathcal{X}_\beta^+ \), which appears in Ohta’s result. If Sharifi’s conjecture is true, then this is a stronger result.

The assumption \( \mathcal{X}^- \neq 0 \) is just to make the result interesting: if \( \mathcal{X}^- = 0 \), then \( \mathcal{H} \cong \Lambda \).

From Ohta’s and Skinner-Wiles’s results, one may hope that \( \mathcal{H} \) is always Gorenstein, but our result shows that this is not true.

**Corollary 1.4.** The ring \( \mathcal{H} \) is not always Gorenstein.

**Proof.** We need only find \( p, N \) and \( \theta \) satisfying the assumptions of [1.1] and such that \( \mathcal{X}_\beta^+ \neq 0 \) and \( \mathcal{X}^- \neq 0 \). To find an example, we first note that, by Iwasawa theory (cf. section 3), the condition \( \mathcal{X}_\beta^+ \neq 0 \) is implied by the condition that \( \mathcal{X}_\beta^+ \) is not cyclic as a \( \Lambda \)-module. By the theory of Iwasawa adjoints (cf. section 4), \( \mathcal{X}_\beta^+ \) is not cyclic if \( \mathcal{X}^- \neq 0 \) is not cyclic.

One way to search for examples is to find an odd primitive character \( \chi \) of conductor \( pN \) (with \( p \nmid \varphi(N) \)) and of order two such that \( \mathcal{X}^- = \mathcal{X}^- \) is not cyclic (and hence not zero).

To find such a character, we look at tables of class groups of imaginary quadratic fields ([B]) and find that, for \( p = 5 \) and \( N = 350267 \), the \( p \)-Sylow subgroup of the class group of \( \mathbb{Q}(\sqrt{-Np}) \) is not cyclic. Let \( \chi : G_\mathbb{Q} \to \{\pm 1\} \) be the character associated to \( \mathbb{Q}(\sqrt{-Np}) \); since \( \mathbb{Q}(\sqrt{-Np}) \) embeds into \( \mathbb{Q}(\zeta_N) \), this factors through \( (\mathbb{Z}/Np\mathbb{Z})^\times \). This is the required \( \chi \).

We also have applications to the theory of cyclotomic fields, as in [O2].

**Corollary 1.5.** Assume Sharifi’s conjecture, and that \( \mathcal{X}_\beta^+ = 0 \). Then the following are true:

1) \( \mathcal{X}^- \) is cyclic.
2) \( I \) is a complete intersection.
3) \( I \) is principal.
4) \( I \) is principal.

Proof. Item 1 follows from \( X'_\theta = 0 \) via the Iwasawa Main Conjecture and the theory of Iwasawa adjoints (cf. section 4). Now, from our assumptions, we have that \( \mathcal{H} \) is Gorenstein. The other items now follow from ([O2], corollary 4.2.13).

Remark 1.6. In the case \( N = 1 \), the assertion \( X^+ = 0 \) is called Vandiver’s conjecture. The previous corollary says that if Vandiver’s and Sharifi’s conjectures are true, then the structures of the Hecke algebras and Iwasawa modules are very simple.

1.3. Outline of the proof. The idea of the proof comes from Kato and Fukaya’s work on Sharifi’s conjectures [K-F]. They consider the Drinfeld-Manin modification \( \tilde{H}_{DM} = \mathfrak{h} \otimes_R \tilde{H} \) of \( \tilde{H} \) and some subquotients:

\[
\mathcal{R} = \tilde{H}_{DM}/\mathcal{H}, \quad \mathcal{P} = H^-/IH^- , \quad \mathcal{Q} = (H/\mathcal{H})/\mathcal{P}.
\]

Using Otha’s theory of \( \Lambda \)-adic Eichler-Shimura cohomology, one can show that \( \tilde{H}^- \cong \text{Hom}_\Lambda(\mathcal{H}, \Lambda) \), and thus that \( \mathcal{H} \) is Gorenstein if and only if \( \tilde{H}^- \) is free of rank one as an \( \mathcal{H} \)-module.

We construct a commutative diagram:

\[
\begin{array}{ccc}
X \otimes X & \xrightarrow{\Theta \otimes \Upsilon} & \text{Hom}(\mathcal{R}, \mathcal{Q}) \otimes \text{Hom}(\mathcal{Q}, \mathcal{P}) \\
\downarrow \nu' \otimes 1 & & \downarrow \\
\Lambda/\xi \otimes X & \xrightarrow{\tau} & \mathcal{P}
\end{array}
\]

The unlabeled maps are the natural ones, and \( \xi \in \Lambda \) denotes the \( p \)-adic Riemann zeta function. In section 2, we describe the top horizontal map, and we show that \( \mathcal{H} \) is Gorenstein if and only if the clockwise map \( X \otimes X \rightarrow \mathcal{P} \) is surjective. In section 3, we describe the map \( \nu' \) and show that the diagram is commutative. It is known that if \( \mathcal{H} \) is Gorenstein, then \( \Upsilon \) is an isomorphism. Since \( X'^- \neq 0 \), we get that if \( \mathcal{H} \) is Gorenstein, then \( \nu' \) is surjective. In section 4, we recall the theory of Iwasawa adjoints, and use this to show that \( \nu' \) is surjective if and only if \( X'^+ = 0 \). This completes the proof of the first statement.

In Sharifi’s paper [S], he formulates conjectures that imply that \( \Upsilon \) is an isomorphism even without the hypothesis that \( \mathcal{H} \) is Gorenstein (cf. [K-F]). If \( \Upsilon \) is surjective and \( X'^+ = 0 \), then \( \nu' \) is also surjective, so the clockwise map \( X \otimes X \rightarrow \mathcal{P} \) is surjective, and so \( \mathcal{H} \) is Gorenstein.

1.4. Remarks on the Notation. There are many small choices of notational convention in this area of study, and it seems that every author has a different convention. The major result that is used in this paper comes from [K-F], so we align to their notation.

We use the standard model for \( X_1(M) \): as the moduli space for \( (E, \mathbb{Z}/M\mathbb{Z} \subset E) \) elliptic curves with a subgroup isomorphic to \( \mathbb{Z}/M\mathbb{Z} \). Sharifi and Ohta use the model that is a moduli space for \( (E, \mathbb{Z}/M\mathbb{Z}(1) \subset E) \), which effects the Galois action (cf. proposition 2.1). In the notation of [K-F], we use the model \( X_1(M) \), where Ohta and Sharifi use the model \( X'_1(M) \).

We use the algebra of dual Hecke operators \( T^*(a) \), but the eigenspace with respect to the usual diamond operators \( (a) \). In effect, this means that our Hecke
algebra $\mathcal{H}$ would be denoted by $\mathcal{H}_{\Lambda}^{(+,-)}$ in [S]. We hope that, with this dictionary, the reader may now easily translate between the various papers mentioned herewith.

Finally, throughout the paper, we use notation such as $X_\theta^+$, which is redundant since $X_\theta^+ = X_\theta$. However, we find it helpful, since the plus-minus parts often determine the most important behaviour, and it can be difficult to remember the parity of the characters. We hope that this does not cause any confusion.

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2. Galois Actions

In this section, we recall some results on $H/IH$. These results are mainly due to many authors – Ohta, Mazur-Wiles, Sharifi, Fukaya-Kato. See ([K-F], section 6) for an excellent exposition, including proofs. The purpose of this section is to construct the clockwise map in the commutative diagram (*), and show that it is surjective if $\mathcal{H}$ is Gorenstein.

2.1. Let $\mathcal{L} = \text{Frac}(\Lambda)$. For a $\Lambda$ module $M$, we use $M_\mathcal{L} = M \otimes_\Lambda \mathcal{L}$. It is known that $H^+$ (resp. $H^-$) is a free $\mathcal{H}$-module (resp. $\mathcal{H}_\mathcal{L}$-module) of rank 1. Choosing bases allows us to decompose the Galois representation $G_\mathcal{L} \to \text{Aut}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}}) = GL_2(\mathcal{H}_\mathcal{L})$ into 4 homomorphisms:

$$a(\sigma) \in \text{Hom}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}}^-, H_{\mathcal{H}_\mathcal{L}}^-), \quad b(\sigma) \in \text{Hom}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}}^+, H_{\mathcal{H}_\mathcal{L}}^-),$$

$$c(\sigma) \in \text{Hom}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}}^-, H_{\mathcal{H}_\mathcal{L}}^+), \quad d(\sigma) \in \text{Hom}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}}^+, H_{\mathcal{H}_\mathcal{L}}^+).$$

We may consider $a(\sigma), b(\sigma)$ etc. as elements of $\mathcal{H}_\mathcal{L}$. Let $B$ (resp. $C$) be the $\mathcal{H}$-submodule of $\mathcal{H}_\mathcal{L}$ generated by the $b(\sigma)$ (resp. $c(\sigma)$). It has been shown by Ohta that $BC = I$. It is known that $B$ (resp. $C$) is generated by $\{b(\tau) \mid \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty)\}$ (resp. $\{c(\tau) \mid \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty)\}$) (cf. [S], Prop 4.2).

2.2. As in ([O2], section 4.2), we also have a commutative diagram of $\mathcal{H}$-modules with exact rows:

$$\begin{array}{c}
0 \longrightarrow H^+ \longrightarrow H \longrightarrow H^- \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow H^+ \longrightarrow \bar{H}_D \longrightarrow \bar{H}_D^- \longrightarrow 0
\end{array}$$

The cokernel of the middle vertical arrow is, by definition, $\mathcal{R}$. By (loc. cit.), we have $\mathcal{R} \cong \mathcal{H}/I \cong \Lambda/\xi$. This is a torsion $\Lambda$-module, so tensoring $\mathcal{L}$ makes the all the vertical arrows into isomorphisms. So we see that $(\bar{H}_D^-)_{\mathcal{H}_\mathcal{L}}$ is a free $\mathcal{H}_\mathcal{L}$-module of rank 1 – let $e^- \in (\bar{H}_D^-)_{\mathcal{H}_\mathcal{L}}$ be a basis element. Let $e^+ \in H^+$ be a basis as an $\mathcal{H}$ module, and, by abuse of notation, let $e^+$ also denote its image in $H^+_{\mathcal{H}_\mathcal{L}}$.

By (loc. cit.), the bottom exact sequence splits as $\mathcal{H}$-modules, so $\bar{H}_D = H^+ \oplus \bar{H}_D^-$. We can decompose the Galois representation $G_\mathcal{L} \to \text{Aut}_{\mathcal{H}}(\bar{H}_D)$ (similarly to $G_\mathcal{L} \to \text{Aut}_{\mathcal{H}_\mathcal{L}}(H_{\mathcal{H}_\mathcal{L}})$):

$$\tilde{a}(\sigma) \in \text{Hom}_{\mathcal{H}}(\bar{H}_D, \bar{H}_D^-), \quad \tilde{b}(\sigma) \in \text{Hom}_{\mathcal{H}}(H^+, \bar{H}_D^-),$$
Proposition 2.3. The image of $\Phi$ is generated by $B$ of section 2.1.

Since everything is rank 1 after tensoring $L$, we may consider $\tilde{a}(\sigma), \tilde{b}(\sigma)$, etc., as elements of $\mathfrak{h}_L$—to avoid confusion, let us write $\tilde{b}_\sigma$ for the element of $\mathfrak{h}_L$ such that $\tilde{b}(\sigma)(e^+) = b_\sigma e^-$ (similarly, $\tilde{c}(\sigma)(e^-) = c_\sigma e^+$, etc.). We let $B$ (resp. $\tilde{C}$) be the $\mathfrak{h}$-submodule of $\mathfrak{h}_L$ generated by the elements $\tilde{b}_\sigma$ (resp. the elements $\tilde{c}_\sigma$).

Note that, since $H \otimes L = H_{DM} \otimes L$ and $H^- \otimes L = H_{DM}^- \otimes L$, we have $B\tilde{C} = BC$. Thus $B\tilde{C} = I$.

2.3. Let $\{0, \infty\} \in \hat{H}_{DM}$ denote the class corresponding to the homology class of the path from $0$ to $\infty$. The quotient $R$ is free of rank 1 over $\mathfrak{h}/I$ with canonical basis $\{0, \infty\}$ (Remark 4.2).

2.4. The action of $G_Q$ on $R$, $\mathcal{P}$, $Q$ is known, and is summarized by the following proposition.

**Proposition 2.1.** $\mathcal{P}$ is a $G_Q$-submodule of $H/IH$; $\sigma \in G_Q$ acts as $\kappa(\sigma)^{-1}$ on $\mathcal{P}$ and $R$ and as $\kappa(\sigma)^{-1}$ on $Q$.

Sharifi has defined a map

$$\Upsilon : X \to \text{Hom}_B(Q, \mathcal{P})$$

where $\Upsilon(\sigma)$ is induced by the map $x \mapsto (\sigma - 1)x$, for $x \in H/IH$: since the action of $X$ on $\mathcal{P}$ is trivial, this factors through $Q$; since the action of $X$ on $Q$ is trivial, we have $(\sigma - 1)x \in \mathcal{P}$ for all $x \in H/IH$.

We can similarly define a map

$$\Theta : X \to \text{Hom}_B(R, Q)$$

where $\Theta(\tau)$ is induced by the map $r \mapsto ((\tau - 1)r \mod I) \mod \mathcal{P}$, for $r \in \hat{H}_{DM}$; since $X$ acts trivially on $Q$, we see that this factors through $R$; since $X$ acts trivially on $R$, we see that $(\tau - 1)r \in H$ for any $r \in \hat{H}_{DM}$.

We have a natural map $\text{Hom}(R, Q) \otimes \text{Hom}(Q, \mathcal{P}) \to \text{Hom}(R, \mathcal{P})$, given by composition. There is then a natural map $\text{Hom}(R, \mathcal{P}) \to \mathcal{P}$ given by evaluation at $\{0, \infty\}$, the canonical basis of $R$.

We obtain the clockwise map $\Phi : X \otimes X \to \mathcal{P}$ in the commutative diagram $(*):$ it is given by $\Phi(\tau \otimes \sigma) = \Upsilon(\sigma) \circ \Theta(\tau)(\{0, \infty\})$.

2.5. We have the following result about the image of $\Phi$.

**Lemma 2.2.** The image of $\Phi$ is $I\{0, \infty\}$, where $I$ is the Eisenstein ideal.

**Proof.** Let $C(\tau) : H^- \to H^+$ denote the restriction of $\tau - 1$ to $H^-$ followed by the natural projection to $H^+$. Similarly, let $B(\sigma) : H^+ \to H^-$ denote the correct component of $\sigma - 1$. Note that, since, by the Eichler-Shimura theory, $\text{Hom}_A(\mathfrak{h}, \Lambda) \cong H^-$, we have $\text{Hom}_B(H^-, H^-) \cong \mathfrak{h}$. In particular, the composition $B(\sigma)C(\tau)$ acts trivially on $\mathfrak{h}$.

Note further that the maps $C(\tau)$ and $B(\sigma)$ are the same as the maps $c(\tau)$ and $b(\sigma)$ of section 2.1.

We then have that the $\mathfrak{h}$-module generated by $\{B(\sigma)C(\tau) \mid \sigma, \tau \in X\}$ is $I$. The image of $\Phi$ is generated by $B(\sigma)C(\tau)\{0, \infty\}$, and so it is equal to $I\{0, \infty\}$.

This leads to the main result of the section.

**Proposition 2.3.** $\mathfrak{h}$ is Gorenstein if and only if the map $\Phi$ is surjective.

The proof follows from the next two lemmas.
Lemma 2.4. If $\mathcal{H}$ is Gorenstein, then $\hat{H}_{DM}$ is free of rank 1 with basis $\{0, \infty\}$. In particular, $\mathcal{P}$ is generated by the image of $I\{0, \infty\}$.

Proof. The assumption $\mathcal{H}$ is Gorenstein means that we have $\text{Hom}_{\Lambda}(\mathcal{H}, \Lambda) \cong \mathcal{H}$ as $\mathcal{H}$-modules. We know from Ohta’s Eichler-Shimura theory that $\hat{H}/H = \hat{H}_{DM}$ is a free $h$-module of rank 1.

Now, since $\hat{H}/H$ is generated by $\{0, \infty\}$, as an $\mathcal{H}$-module and $\hat{H}$ is free of rank one, we see that $\hat{H}$ is also generated by $\{0, \infty\}$.

We have the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^- & \rightarrow & \hat{H}_{DM} & \rightarrow & \mathcal{P} & \rightarrow & 0 \\
0 & \rightarrow & I & \rightarrow & h & \rightarrow & h/I & \rightarrow & 0
\end{array}
$$

where the centre and rightmost vertical arrows are the isomorphisms given by $1 \mapsto \{0, \infty\}$. We see that $H^- = I\{0, \infty\}$. \qed

Lemma 2.5. If $\mathcal{P}$ is generated by $I\{0, \infty\}$, then $\mathcal{H}$ is Gorenstein.

Proof. By Nakayama’s lemma, $H^-$ is generated by $I\{0, \infty\}$. Then, in the diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^- & \rightarrow & \hat{H}^- & \rightarrow & \hat{H}^- / H^- & \rightarrow & 0 \\
0 & \rightarrow & I & \rightarrow & \mathcal{H} & \rightarrow & \mathcal{H}/I & \rightarrow & 0
\end{array}
$$

where the centre and rightmost arrows are given by $1 \mapsto \{0, \infty\}$, the leftmost and rightmost arrows are surjective. We see that the $\hat{H}^- \cong \text{Hom}_{\Lambda}(\mathcal{H}, \Lambda)$ is generated by one element, and so $\mathcal{H}$ is Gorenstein. \qed

3. Relation to cyclotomic units

In this section, we recall some definitions from the theory of cyclotomic fields. We apply these ideas to construct the map $\nu'$ in the diagram (*). We then show how a result of Fukaya and Kato implies that (*) is commutative.

3.1. Let $U$ be the group of principal local units and $E$ be the group of principal global units with respect to the tower $\mathbb{Q}(\mu_{p^r})$, as in [W], section 13). Recall that $U/E \cong \text{Gal}(M/L)$, where $M$ and $L$ are as in [11]. We get an exact sequence:

$$(3.1) \quad 0 \rightarrow E \rightarrow U \rightarrow X \rightarrow X \rightarrow 0$$

3.2. The Hilbert symbol

$$(\ , )_r : U \times U \rightarrow \mu_{p^r}$$

given by

$$(a, b)_r = \frac{\sigma_{b_r}(a_1^{1/p^r})}{a_1^{1/p^r}},$$

where $\sigma_{b_r}$ is the image of $b_r$ under the Artin map. We also have

$$[\ , ]_r : E \times X \rightarrow \mu_{p^r}.$$
given by

\[ [u, \sigma]_r = \frac{\sigma(u_1)^{1/p^r}}{u_1^{1/p^r}} \]

Choosing a compatible system of \( p^r \)-th roots of unity, we can and do identify \( \mu_{p^r} \) with \( \mathbb{Z}/p^r\mathbb{Z} \).

We have a Kummer pairing maps

\[ (\ , \ )_{Kum} : U \times U \to \Lambda, \quad [\ , \ ]_{Kum} : E \times \mathfrak{X} \to \Lambda \]

given by

\[ (a, b)_{Kum} = \lim_{\leftarrow} \sum_{c \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} (\sigma_c(a), b)_n \langle c^{-1} \rangle, \]

\[ [u, \sigma]_{Kum} = \lim_{\leftarrow} \sum_{c \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} [\sigma_c(u), \sigma]_n \langle c^{-1} \rangle, \]

where \( \sigma_c \) is defined by

\[ \sigma_c(\zeta_{Np^n}) = \zeta_{Np^n}^c. \]

3.3. commutivity In particular, we have a map

\[ \nu : \mathfrak{X} \to \Lambda \]

given by \( \nu(\sigma) = [(1 - \zeta_{Np^n}), \sigma]_{Kum} \), the pairing with cyclotomic units. The map \( \nu' : \mathfrak{X} \to \Lambda/\xi \) in the diagram (*) is the composition of \( \nu \) with the natural map \( \Lambda \to \Lambda/\xi \).

**Proposition 3.2.** The composite

\[ \mathfrak{X} \otimes X \xrightarrow{\nu' \otimes 1} \Lambda/\xi \otimes X \xrightarrow{\gamma} P \]

coincides with the map \( \Phi \).

This proposition is exactly saying that the diagram (*) is commutative. This is a restatement of the following theorem of Fukaya and Kato ([Fukaya-Kato], section 9).

**Theorem 3.3.** Let \( E \) denote the natural extension of \( R \) by \( Q \), given by their structure as subquotients of \( H_{DM} \). Then the class of \( E \) is given by cyclotomic units.

To explain how proposition 3.2 follows from this, we must explain what is meant by ‘is given by cyclotomic units’.

The modules \( R \) and \( Q \) are both free of rank one as \( \Lambda/\xi \)-modules, with canonical bases. By proposition 2.1, we have canonical isomorphisms of \( \Lambda/(\langle \xi \rangle[G_Q]) \)-modules:

\[ R = \Lambda/\xi(-1), \quad Q = \Lambda^\# /\xi, \]

where \( \Lambda^\# \) denotes \( \Lambda \) with \( \sigma \in G_Q \) acting by \( \langle a^{-1} \rangle \) where \( \sigma(\zeta_{Np^r}) = \zeta_{Np^r}^a \) for any \( r \).

Thus group of extension class of \( R \) by \( Q \) coincides with the group

\[ \text{Ext}^1_{\Lambda/(\langle \xi \rangle[G_Q])}(\Lambda/\xi(-1), \Lambda^\#/\xi) = H^1(\mathbb{Z}[1/Np], \Lambda^\#/\xi(1)). \]

As an element of this Galois cohomology, the class \( E \) is given by the map \( \Theta \) of section 2.4.

We have a natural map

\[ H^1(\mathbb{Z}[1/Np], \Lambda^\#(1)) \xrightarrow{j} H^1(\mathbb{Z}[1/Np], \Lambda^\#/\xi(1)). \]

The map \( \nu \) gives an element of \( H^1(\mathbb{Z}[1/Np], \Lambda^\#(1)) \), and theorem 3.3 says exactly that \( j(\nu) = \nu' = E \). Thus \( \nu' = \Theta \), whence proposition 3.2.
4. Iwasawa Adjunction

In this section, we recall the theory of Iwasawa adjoints in terms of Ext.

4.1. We recall the interpretation of $\mathcal{X}$ and $X$ in terms of Galois cohomology. Let $G_r = \text{Gal}(K/Q(\zeta_{Np^r}))$ and $G_\infty = \text{Gal}(K/Q_\infty)$, where $K$ is the maximal Galois extension of $Q(\zeta_{Np})$ that is unramified outside $Np$, and $Q_\infty$ is as in [1.1]. Let $H^q_\infty = \lim\limits_{\leftarrow r} H^q(G_r, Z_p)$. We have

$$\mathcal{X} = H^1(G_\infty, Q_p/Z_p)^\vee,$$

where $(-)^\vee$ denotes the Pontryagin dual, and an exact sequence

$$0 \to X(-1) \to H^2_\infty \to \bigoplus_v Z_p \to Z_p \to 0.$$

Thus for any non-trivial character $\phi$, $X(-1)_\phi \cong (H^2_\infty)_\phi$.

4.2. For a $\Lambda$-module $M$, denote by $E^i(M)$ the group $\text{Ext}^i_\Lambda(M, \Lambda)$. The group $E^1(M)$ is called the Iwasawa adjoint of $M$. We recall some generalities.

**Proposition 4.1.** For any finitely generated $\Lambda$-module $M$, let $T_0(M)$ be the maximal finite submodule of $M$, and let $T_1(M)$ be the $\Lambda$-torsion submodule of $M$. We have the following for any finitely generated $\Lambda$-module $M$:

1) The module $E^3(M)$ is $\Lambda$-torsion.
2) If $M$ is $\Lambda$-torsion, then $T_0(E^2(M)) = 0$.
3) $E^1(M/T_0(M)) = E^1(M)$, and $E^1(M) = 0$ if and only if $M/T_0(M)$ is free.
4) $E^2(M) = T_0(M)^\vee$.
5) $E^1(E^1(M)) = T_1(M)/T_0(M)$.
6) $E^1(M)$ is finite if and only if $T_1(M)$ is finite.

**Proof.** See ([N-S-W], section 5.4).

This is relevant to our situation because of the following proposition.

**Proposition 4.2.** There is a spectral sequence of finitely generated $\Lambda$-modules

$$E^{p,q}_2 = E^p(H^q(G_\infty, Q_p/Z_p)^\vee) \Rightarrow H^{p+q}_\infty.$$

**Proof.** This is a special case of ([N], theorem 1).

**Corollary 4.3.** We have $X_{\chi^{-1}} = E^1(\mathcal{X}_\theta^+)$ and $E^1(X_{\chi^{-1}}) = \mathcal{X}_\theta^+$.

**Proof.** In the spectral sequence of proposition 4.2, we have $E^{0,0}_2 = E^{0,2}_2 = 0$ ($E^{2,0}_2 = 0$ is in ([N], Lemma 5), and $E^{0,2}_2 = 0$ because $H^2(G_\infty, Q_p/Z_p) = 0$ by the abelian case of Leopoldt’s conjecture). This gives an isomorphism $H^2_\infty \cong E^1(H^1(G_\infty, Q_p/Z_p)^\vee) = E^1(\mathcal{X})$. Thus,

$$X_{\chi^{-1}} = X_{\chi^{-1}}(-1) = X(-1)_{\theta^{-1}} \cong (H^2_\infty)_{\theta^{-1}} \cong E^1(\mathcal{X})_{\theta^{-1}} = E^1(\mathcal{X}_\theta^+).$$

We now have $E^1(X_{\chi^{-1}}) = E^1(E^1(\mathcal{X}_\theta^+)) = T_1(\mathcal{X}_\theta^+)/T_0(\mathcal{X}_\theta^+)$ using proposition 4.1 item 6. The result now follows from the fact that $\mathcal{X}^+$ is torsion and has no non-zero finite submodules.
4.3. For the remainder of the section, we consider the restriction of $\nu$ to $X_{X-1}$, which we also denote by $\nu : X_{X-1} \rightarrow \Lambda$. Let $\xi_{X-1} \in \Lambda$ be the characteristic power series of $X_{X-1}$. For more on this, see [W]. We require some lemmas.

**Lemma 4.4.** There exists a natural commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & U_{X-1} & \rightarrow & X_{X-1} & \rightarrow & \Lambda \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \xi_{X-1} & \rightarrow & \Lambda & \rightarrow & \Lambda/\xi_{X-1} & \rightarrow & 0
\end{array}
$$

Thus the map $\nu$ induces a map $\nu : X_{X-1} \rightarrow \Lambda/\xi_{X-1}$.

**Lemma 4.5.** There is a natural exact sequence:

$$
0 \rightarrow E^+_0/C^+_0 \rightarrow \Lambda/\xi_{X-1} \rightarrow E^+_1(\nu) \rightarrow X^+_0 \rightarrow X^+_1 \rightarrow 0,
$$

where $E^+_0$ is the group of global units and $C^+_0$ is the cyclotomic units.

**Corollary 4.6.** The following are equivalent:

1) $X^+_0$ is finite.
2) $E^+_0/C^+_0 = 0$.
3) $E^+_1(\nu)$ is injective.

**Proof.** By the Iwasawa main conjecture, the characteristic ideals of $X^+$ and $E^+_0/C^+$ are equal. Thus $X^+_0$ is finite if and only if $E^+_0/C^+_0$ is finite. But $E^+_0/C^+_0$ injects into $\Lambda/\xi$, which has no nonzero finite submodules, so $E^+_0/C^+_0$ is finite if and only if $E^+_0/C^+_0 = 0$. This gives the equivalence of 1) and 2). The equivalence 2) and 3) is immediate from lemma 4.5. \( \square \)

We leave the proofs of lemmas 4.4 and 4.5 to the next section. Using these lemmas, we get the following result.

**Proposition 4.7.** 1) We have $E^1(\ker(\nu)) = \ker(E^1(\nu))$.

2) Assume $X^+_0$ is finite. Then we have $\ker(E^1(\nu)) = \ker(\nu)$.

**Proof.** For ease of notation, we let $B \subset \Lambda/\xi_{X-1}$ be the image of $\nu$, $K = \ker(\nu)$, and $C = \ker(\nu)$.

Applying the Iwasawa adjoint to the sequence

$$
0 \rightarrow K \rightarrow X_{X-1} \rightarrow B \rightarrow 0
$$

we get

$$
0 \rightarrow E^1(B) \rightarrow E^1(X_{X-1}) \rightarrow E^1(K) \rightarrow 0
$$

noting that $E^0(K) = E^2(B) = 0$.

Applying Iwasawa adjoint to the sequence

$$
0 \rightarrow B \rightarrow \Lambda/\xi_{X-1} \rightarrow C \rightarrow 0
$$

we get

$$
0 \rightarrow E^1(C) \rightarrow E^1(\Lambda/\xi_{X-1}) \rightarrow E^1(B) \rightarrow E^2(C) \rightarrow 0,
$$
noting that $E_0(\mathcal{B}) = E_2(\mathcal{B}) = 0$.

1) Since the composition $E_1(\Lambda/\xi_{\chi^{-1}}) \to E_1(\mathcal{B}) \to E_1(X_\phi^-)$ is $E_1(\nu)$, we see that $\ker(E_1(\nu)) = E_1(C)$.

2) By corollary 1.6 and part 1), we see $E_1(C) = 0$. By proposition 1.1, this implies that $C$ is finite and $E_2(C) = C'$. Putting this together, we get a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \to & \Lambda/\xi_{\chi^{-1}} & \xrightarrow{E_1(\nu)} & X_\phi^+ & \to & 0 \\
& \downarrow & & & \downarrow & & \\
0 & \to & E_1(\mathcal{B}) & \xrightarrow{E_1(\nu)} & E_1(X_{\chi^{-1}}) & \to & E_1(K) \to 0
\end{array}
\]

which implies that $E_1(K)$ is a quotient of $X_\phi^+$, and is thus finite. By 1.6, we have that $T_1(K)$ is finite, but $K \subset X_{\chi^{-1}}$ is torsion, so $K$ is finite. But $X_{\chi^{-1}} = E_1(X_\phi^+)$, so $X_{\chi^{-1}}$ has no finite submodules – thus $K = 0$.

Finally, the sequence

\[
0 \to X_{\chi^{-1}} \xrightarrow{\nu} \Lambda/\xi_{\chi^{-1}} \xrightarrow{\gamma} C \to 0
\]

yields

\[
0 \to E_1(\Lambda/\xi_{\chi^{-1}}) \xrightarrow{E_1(\nu)} E_1(X_{\chi^{-1}}) \xrightarrow{E_2(\nu)} E_2(C) \to 0
\]

so $C' = E_2(C) = \operatorname{coker}(E_1(\nu)) = X_\phi^+$. □

4.4. We now explain how proposition 4.7 implies our main result, theorem 1.2.

**Theorem 4.8.** If $\mathfrak{H}$ is Gorenstein, then $X_\phi^+ = 0$.

**Proof.** Recall the diagram

\[
\begin{array}{ccccccc}
\mathfrak{X} \otimes X & \xrightarrow{\Theta \otimes \gamma} & \operatorname{Hom}(\mathcal{R}, \mathcal{Q}) \otimes \operatorname{Hom}(\mathcal{Q}, \mathcal{P}) \\
& \nu' \otimes 1 \downarrow & & \downarrow \gamma & & \\
\Lambda/\xi \otimes X & \xrightarrow{\gamma} & X & \xrightarrow{\nu} & \mathcal{P}.
\end{array}
\]

This is commutative by proposition 3.2. Since $\mathfrak{H}$ is Gorenstein, proposition 2.3 implies that the map $\mathfrak{X} \otimes X \to \mathcal{P}$ is surjective, and $\gamma$ is an isomorphism (O3, cf. (S, Prop 4.10)). Thus $\nu'$ is surjective.

Now, by proposition 3.2, $\nu' = \Theta$. Since the character of $\operatorname{Hom}(\mathcal{R}, \mathcal{Q})$ is $\chi^{-1}$ and $\Theta$ is Galois-equivariant, we see that $\nu'$ factors through $\mathfrak{X}_{\chi^{-1}}$. By Nakayama’s lemma, $\nu' : \mathfrak{X}_{\chi^{-1}} \to \Lambda/\xi$ is surjective if and only if $\nu : \mathfrak{X}_{\chi^{-1}} \to \Lambda$ is surjective. If $\nu$ is surjective, then $\nu'$ is surjective.

We now have that $\nu'$ is surjective. Then by proposition 4.7 part 1), we have that $E_1(\nu)$ is injective, and so $X_\phi^+$ is finite by corollary 1.6. We may now apply proposition 4.7 part 2), which implies that $\operatorname{coker}(E_1(\nu)) = 0$. But $\operatorname{coker}(E_1(\nu)) = X_\phi^+$ by lemma 4.5.

We also have the partial converse.

**Theorem 4.9.** If $X_\phi^+ = 0$, then $\nu'$ is surjective. If, in addition, $\gamma$ is surjective, then $\Phi$ is surjective and, in particular, $\mathfrak{H}$ is Gorenstein.
Proof. It is immediate from lemma 4.5 and proposition 4.7 that $X_\mathfrak{p}^{+} = 0$ implies that $\mathfrak{p}$ is surjective. From the assumption $p \mid B_{1,1}$ we see that $\xi_{X-1}$ is not a unit. By Nakayama’s lemma, we have that $\nu$ is surjective, and thus that $\nu'$ is surjective. The second statement follows from the commutative diagram (*) and proposition 2.3.

Remark 4.10. Sharifi’s conjectures imply that $\Upsilon$ is an isomorphism, and so this completes the proof of 1.2.

5. Colman power series and explicit reciprocity

The purpose of this section is to prove lemmas 4.4 and 4.5. The method of proof is to show that the map $E_1(\nu)$ coincides with the composite

$$\Lambda/\xi_{X-1} \to U_0^+/C_0^+ \to X_0^+$$

where $C$ is the groups of cyclotomic units, $\Lambda/\xi_{X-1} \to U^+/C^+$ is the isomorphism of Iwasawa’s theorem defined by Colman power series, and $U^+/C^+ \to X^+$ is the natural map from the sequence 3.1. We first review the theory of Colman power series and the relation to explicit reciprocity.

5.1. The Colman power series map $\text{col} : U \to \Lambda$ is defined by

$$\text{col}(u) = L(f_u),$$

where $f_u$ is the power series satisfying $f_u(1 - \zeta_{X^{-1}}) = u_n$, and

$$L(f_u)(T) = \frac{1}{p} \log \left( \frac{f(T)^p}{f(1 - (1 - T)^p)} \right).$$

We use the standard notations $D = (1 - T)d/dT$ and $\omega = 1 - (1 - T)^p$, and we use $f * g$ for the multiplication in $\Lambda$—in terms of power series this is induced by $(1 - T) * (1 - T) = (1 - T)$—and $t$ for the Tate twist $t(1 - T) = (1 - T)^{-1}$. The following lemma is proved in [P-R], but the situation in that paper is quite general, and so it may be difficult to see how the results there apply here. We produce a proof here for the convenience of the reader.

Lemma 5.1. The following diagram is commutative:

$$\begin{array}{ccc}
U \times U & \xrightarrow{(\cdot, \cdot)_{\text{sum}}} & \Lambda \\
\downarrow \text{col, col} & & \downarrow \\
\Lambda \times \Lambda & \xrightarrow{1 \times D, t} & \Lambda \times \Lambda \xrightarrow{*} \Lambda
\end{array}$$

Proof. Let $a, b \in U$, and let $f = f_a$ and $g = g_b$. Recall that $g_{a,b}(T) = g_b(1 - (1 - T)^c)$. In Coleman’s paper [C], he shows that

$$(a, b)_r = \frac{1}{p^r} \sum_{\xi \in \mu_{p^r}} \text{Log}(f)(1 - \xi) \text{Log}(g)(1 - \xi) \mod p^r \mathbb{Z}_p.$$}

Let $\text{Log}(f)(T) = \sum x_n (1 - T)^n$ and $\text{Log}(g)(T) = \sum y_m (1 - T)^m$. Note that

$$\text{Log}(f)(T) = \sum_{p \mid n} x_n (1 - T)^n$$

and

$$\text{Log}(g_{c, -1})(T) = \sum c^{-1} m y_m (1 - T)^m.$$
Coleman’s formula then gives
\[(a, \sigma_{c-1}b)_r = \frac{1}{p^r} \sum_{\xi \in \mu_{p^r}} \sum_{\{n, m : p \nmid n\}} c^{-1}my_{m}x_{n}(\xi)^{c-1}m+n \mod p^r\]

Now, since \(\frac{1}{p^r} \sum_{\xi \in \mu_{p^r}} \xi^j = 1\) if \(p^r \mid j\) and 0 otherwise, we see
\[(a, \sigma_{c-1}b)_r = \sum_{\{n, m : p^r \mid c-1m+n , p \nmid n\}} c^{-1}my_{m}x_{n} \mod p^r.\]

On the other hand, we compute
\[tD\xi(f) * \xi(g)(T) = \sum_{n,m : p \nmid n,m} nx_{n}y_{m}(1 - T)^{-m-1} \mod (p^r, \omega_r)\]
\[= \sum_{c \in \mathbb{Z}/p^r\mathbb{Z}} \left( \sum_{m = nc \mod p^r} nx_{n}y_{m} \right) (1 - T)^c \mod (p^r, \omega_r)\]
\[= \sum_{c \in \mathbb{Z}/p^r\mathbb{Z}} (a, \sigma_{c-1}b)_r(1 - T)^c \mod (p^r, \omega_r)\]

□

5.2. We complete the proof of lemmas 4.4 and 4.5

Proof. (Of lemma 4.4) The diagram is obtained by restricting one of the \(U\) factors in lemma 5.1 to the cyclotomic units.

Proof. (Of lemma 4.5) From lemma 4.4, we have
\[0 \rightarrow U_{\chi^{-1}} \rightarrow X_{\chi^{-1}} \rightarrow X_{\chi^{-1}} \rightarrow 0\]
\[\Delta \rightarrow \Lambda \stackrel{\chi_{-1}}{\rightarrow} \Lambda \rightarrow \Lambda/\chi_{-1} \rightarrow 0\]

When we apply Iwasawa adjoint, the non-zero part of the resulting complex is:
\[E^0(X_{\chi^{-1}}) \rightarrow E^0(U_{\chi^{-1}}) \rightarrow E^1(X_{\chi^{-1}}) \rightarrow E^1(X_{\chi^{-1}})\]
\[\Delta \stackrel{\chi_{-1}}{\rightarrow} \Lambda \rightarrow \Lambda/\chi_{-1} \rightarrow 0\]

We have \(E^0(X_{\chi^{-1}}) = E^0_{\theta} + \), and \(E^0(U_{\chi^{-1}}) \cong U_{\theta}^+\), by the Colman power series. So the diagram is:
\[E^0_{\theta} \rightarrow U^+_\theta \rightarrow X^+_\theta \rightarrow X^+_\theta\]
\[\Delta \stackrel{\chi_{-1}}{\rightarrow} \Lambda \rightarrow \Lambda/\chi_{-1} \rightarrow 0\]

In the proof of Iwasawa’s theorem, one shows that the Coleman power series maps \(C^+_\theta\) isomorphically on to \(\chi_{-1} \Lambda\). We can now read off the desired exact sequence.

□
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