On an arithmetic inequality on $\mathbb{P}^1_{\mathbb{Q}}$

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Abstract

We establish an inequality comparing the height and the $\chi$-arithmetic volume of toric metrized divisors on $\mathbb{P}^1_{\mathbb{Q}}$. This gives a partial answer to a question of Burgos, Moriwaki, Philippon and Sombra ([5, remark 5.13]).

In [5, remark 5.13] the authors ask if the following inequality

$$h_D(X) \leq \hat{\text{vol}}_{\chi}(X, D)$$

holds for any toric DSP metrized $\mathbb{R}$-divisor $D$ on $X = \mathbb{P}^1_K$, where $K$ is a global field, $h_D(X)$ is the height of $X$ and $\hat{\text{vol}}_{\chi}(X, D)$ is $\chi$-arithmetic volume with respect to $D$.

In this note we give an affirmative answer to this question when $K = \mathbb{Q}$, $D$ is nef and $D$ is a toric DSP metrized divisor such that the metric on all non-archimedean places is the canonical metric (see theorem (0.2)).

Let $L$ be a line bundle on $\mathbb{P}^1(\mathbb{C})$. A metric $\|\cdot\|$ on $L$ is semipositive if it is the uniform limit of a sequence of semipositive smooth metrics. The metric $\|\cdot\|$ is DSP if it is the quotient of two semipositive ones.

We denote by $M_{\mathbb{Q}}$ the set of places of $\mathbb{Q}$. For any $v \in M_{\mathbb{Q}}$, we denote by $\mathbb{P}^1_{\mathbb{Q}}$ the $v$-adic analytification of $\mathbb{P}^1_{\mathbb{Q}}$. Similarly a line bundle $L$ on $\mathbb{P}^1_{\mathbb{Q}}$ defines a collection of analytic line bundles $\{L^v\}_{v \in M_{\mathbb{Q}}}$, see [5, §3] for more details.

**Definition 0.1.** A metrized divisor on $\mathbb{P}^1_{\mathbb{Q}}$ is a pair $\mathcal{D} = (D, (\|\cdot\|_v)_{v \in M_{\mathbb{Q}}})$ formed by a divisor $D$ with $\|\cdot\|_\infty$ is a continuous hermitian metric on $\mathcal{O}(D)_{\infty}$ and $\|\cdot\|_v$ is the canonical metric of $\mathcal{O}(D)_v$ for $v$ a non-archimedean place. We say that $\mathcal{D}$ is smooth or semipositive if so is the metric $\|\cdot\|_\infty$. We say that $\mathcal{D}$ is a DSP divisor if it is the difference of two semipositive divisors. The Green function of $\mathcal{D}$ is the function $g_{\mathcal{D}} : \mathbb{P}^1(\mathbb{C}) \setminus |D| \to \mathbb{R}$ given by

$$g_{\mathcal{D}}(p) = -\log \|s_D(p)\|_\infty,$$

where $s_D$ is the canonical section of $\mathcal{O}(D)$.

Let $\mathcal{D}$ be a metrized DSP divisor on $\mathbb{P}^1_{\mathbb{Q}}$ as in (0.1). We suppose that $\mathcal{D}$ is toric, see [5 §4]. This means that $D$ is a toric divisor and $\|\cdot\|_\infty$ is invariant under the action of $S^1$ the compact torus of $\mathbb{P}^1(\mathbb{C})$ (see [5, definition 4.12] and [5, proposition 4.16]). In the sequel, we assume that $\mathcal{D}$ satisfies these hypothesis and $D$ is nef.

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Theorem 0.2. Under the previous hypothesis, we have

\[ h_{\overline{D}}(\mathbb{P}^1_q) \leq \overline{\text{vol}}(\mathbb{P}^1_q, \overline{D}). \]

In order to prove this theorem, we assume first that \( \overline{D} \) is smooth. By definition, \( g_{\overline{D}} \) is a smooth weight of \( \| \cdot \|_\infty \). We denote by \( P_{\overline{D}} \) the equilibrium weight of \( g_{\overline{D}} \) (see the appendix) instead of \( P_{\overline{D}} \) and by \( \| \cdot \|_{\overline{P}} \) the hermitian metric defined by \( P_{\overline{D}} \) and we denote by \( \overline{D}_P \) the metrized divisor \( D \) endowed with the metric \( \| \cdot \|_P \) on the archimedean place and with the canonical metric on all non-archimedean places.

Claim 0.3. \( \overline{D}_P \) is a semipositive toric divisor.

Proof. By definition \( P_{\overline{D}} \) is a psh weight on \( \mathcal{O}(D) \) and we know that \( \| \cdot \|_P \) is a continuous metric (see for instance [2, §1.4, before (1.8)]). Then the Chern current \( c_1((\mathcal{O}(D), P\| \cdot \|)) \) is semipositive. By [3] theorem 4.6.1, \( \| \cdot \|_P \) is a semipositive metric.

Let \( g \) be a psh weight function on \( \mathcal{O}(D) \) with \( g \leq g_{\overline{D}} \). Let \( \theta \in \mathbb{S}^1 \). We set \( g_\theta \) the function given by \( g_\theta(z) = g(\theta \cdot z) \) for any \( z \in \mathbb{P}^1(\mathbb{C}) \). Then \( g_\theta \) is clearly a psh weight on \( \mathcal{O}(D) \). We have \( g_\theta(z) = g(\theta \cdot z) \leq g_{\overline{D}}(\theta \cdot z) = g_{\overline{D}}(z), \forall z \in \mathbb{P}^1(\mathbb{C}) \). Then, \( g_\theta(z) \leq P_{\overline{D}}(z), \forall z \in \mathbb{P}^1(\mathbb{C}) \). Therefore, \( P_{\overline{D}}(\theta \cdot z) \leq P_{\overline{D}}(z), \forall \theta \in \mathbb{S}^1, \forall z \in \mathbb{P}^1(\mathbb{C}) \). We conclude that

\[ P_{\overline{D}}(\theta \cdot z) = P_{\overline{D}}(z) \quad \forall \theta \in \mathbb{S}^1, \forall z \in \mathbb{P}^1(\mathbb{C}). \]

Which means that \( \| \cdot \|_P \) is an invariant metric. We conclude that \( \overline{D}_P \) is a semipositive toric divisor on \( \mathbb{P}^1_q \).

Recall that if \( \overline{D} := (D, (\| \cdot \|_{\overline{D}})_{\nu \in \mathcal{M}_D}) \) is a smooth metrized divisor as in (0.1), then by [4] proposition 3.2.2, we have

\[ h_{\overline{D}}(\mathbb{P}^1_q) - h_{\overline{D}}(\mathbb{P}^1_q) = - \int_X (g_{\overline{D}} - g_{\overline{D}})(c_1(\mathcal{O}(D), \| \cdot \|) + c_1(\mathcal{O}(D), \| \cdot \|')). \]

By [7], one can extend this equality to the case of DSP divisor \( \overline{D}_P \), and we have

\[ h_{\overline{D}}(\mathbb{P}^1_q) - h_{\overline{D}}(\mathbb{P}^1_q) = - \int_X (g_{\overline{D}} - g_{\overline{D}})(c_1(\mathcal{O}(D), \| \cdot \|) + c_1(\mathcal{O}(D), \| \cdot \|')). \]

where \( c_1(\mathcal{O}(D), \| \cdot \|') \) is the first Chern current of \( (\mathcal{O}(D), \| \cdot \|') \).

Since \( \overline{D}_P \) is semipositive, then the previous equality gives

\[ h_{\overline{D}}(\mathbb{P}^1_q) - h_{\overline{D}}(\mathbb{P}^1_q) = - \int_X (g_{\overline{D}} - g_{\overline{D}})(c_1(\mathcal{O}(D), \| \cdot \|) + c_1(\mathcal{O}(D), \| \cdot \|)). \]

From (6), we have

\[ h_{\overline{D}}(\mathbb{P}^1_q) \leq h_{\overline{D}}(\mathbb{P}^1_q). \tag{1} \]

Since \( \overline{D}_P \) is a semipositive toric divisor, then by [5] corollary 5.8

\[ h_{\overline{D}_P}(\mathbb{P}^1_q) = \overline{\text{vol}}(\mathbb{P}^1_q, \overline{D}_P), \tag{2} \]

and by [5] theorem 5.6, we have

\[ \overline{\text{vol}}(\mathbb{P}^1_q, \overline{D}_P) = 2 \int_{\Delta_D} \vartheta_{\overline{D}_P} d\text{vol}_{\Delta}, \]

where \( \vartheta_{\overline{D}_P} \) is the roof function associated to \( \overline{D}_P \) (see [5] definition 4.17).
Claim 0.4. We have,\[ \vartheta_{\mathcal{D}} = \vartheta_{\mathcal{D}}, \]
on \Delta_D.

Proof. This is an easy consequence of the combination of [5] proposition 5.1 (1) and [3] proposition 2.8. Indeed, by [3] proposition 2.8 we have supp_{\mathcal{D}} \parallel s \parallel_{k \mathcal{D}} = \sup_{\mathcal{D}} \parallel s \parallel_{k \mathcal{D}}^1,\]
for any \( s \in H^0(\mathbb{P}^1, \mathcal{O}(kD)) \) and \( k \in \mathbb{N}^*. \) But, we know that \( \sup_{\mathcal{D}} \parallel s_m \parallel = \exp(-k\vartheta_{\mathcal{D}}(m/k)) \) where \( s_m \) is the global section of \( \mathcal{O}(kD) \) corresponding to \( m \in k\Delta_D \cap \mathbb{Z} \) (see for instance [5] proposition 5.1)). By continuity and density arguments we deduce the equality of the claim.

By [5] theorem 5.6 and the claim (0.4) we have,\[ \hat{\text{vol}}_{\chi}(\mathbb{P}_Q^1, \mathcal{D}) = \hat{\text{vol}}_{\chi}(\mathbb{P}_Q^1, \overline{\mathcal{D}}). \]

Then from (1), (2) and (3) we conclude that\[ h_{\mathcal{D}}(\mathbb{P}_Q^1) \leq \hat{\text{vol}}_{\chi}(\mathbb{P}_Q^1, \mathcal{D}). \]

This ends the proof of the theorem (0.2).

1 Appendix

Let \( X \) be compact manifold of dimension \( n \) and \( L \) an ample holomorphic line bundle on \( X \). Let \( \phi \) be a weight of a continuous hermitian metric \( e^{-\phi} \) on \( L \). When \( \phi \) is smooth we define the Monge-Ampère operator as\[ \text{MA}(\phi) := (dd^c \phi)^n. \]

The equilibrium weight of \( \phi \) is defined as: \[ P_X \phi := \sup \{ \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}. \]

where * denotes upper semicontinuous regularization. When \( \phi \) is smooth then \( P_X \phi = \sup \{ \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}. \) It is known that \( P_X \phi \) is a psh weight and the metric \( e^{-P_X \phi} \) is continuous (see for instance [2] §1.4, before (1.8)). By the theory of Bedford-Taylor, the Monge-Ampère operator can be extended to

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3Here semipositive, means that the associated first Chern form is semipositive
locally bounded psh weights \( \phi \) (see [1]).

By [3] proposition 2.10 we have

\[
\int_X (P_X \phi - \phi) \text{MA}(P_X \phi) = 0. \tag{5}
\]

When \( \dim(X) = 1 \), we have

\[
\int_X (\phi - P_X \phi)(\text{MA}(\phi) + \text{MA}(P_X \phi)) \leq 0 \tag{6}
\]

Indeed,

\[
\frac{1}{2} \int_X (\phi - P_X \phi)(\text{MA}(\phi) + \text{MA}(P_X \phi)) = \frac{1}{2} \int_X (\phi - P_X \phi)(dd^c \phi - dd^c P_X \phi) \quad \text{by [5]}
\]

\[
= -\int_X d(\phi - P_X \phi) \wedge d^c (\phi - P_X \phi)
\]

\[
\leq 0.
\]

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