BOHR PHENOMENON FOR $K$-QUASICONFORMAL HARMONIC MAPPINGS AND LOGARITHMIC POWER SERIES

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Abstract. In this article, we establish the Bohr inequalities for the sense-preserving $K$-quasiconformal harmonic mappings defined in the unit disk $\mathbb{D}$ involving classes of Ma-Minda starlike and convex univalent functions, usually denoted by $S^*(\psi)$ and $C(\psi)$ respectively, and for $\log(f(z)/z)$ where $f$ belongs to the Ma-Minda classes or satisfies certain differential subordination. We also estimate Logarithmic coefficient’s bounds for the functions in $C(\psi)$ for the case $\psi(\mathbb{D})$ be convex.

2010 AMS Subject Classification. 30B10, 30C45, 30C50, 30C80, 30C62, 31A05
Keywords and Phrases. Bohr radius, Radius problems, Harmonic mappings, Starlike and Convex functions, Logarithmic coefficients.

1. Introduction

Let $\mathcal{H}$ be the class of complex valued harmonic functions $h$ (which satisfy the Laplacian equation $\Delta h = 4h_{zz} = 0$) defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then we can write $h = f + \overline{g}$, where $f$ and $g$ are analytic and satisfies $h(0) = g(0)$. We say that $h$ is sense-preserving in $\mathbb{D}$ whenever the Jacobian $J_h := |f'|^2 - |g'|^2 > 0$. A sense-preserving homeomorphism defined in $\mathbb{D}$ which is also harmonic is called $K$-quasiconformal, $K \in [1, \infty)$ if the (second complex) dilatation $w_h := g'/f'$ satisfies $|w_h(z)| \leq k$, $k = (K-1)/(K+1) \in [0, 1)$.

In 1914, H. Bohr [7] proved a power series inequality which is also known as the classical Bohr inequality.

**Theorem 1.1** (Bohr’s Theorem, [7]). Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be an analytic function in $\mathbb{D}$ and $|g(z)| < 1$ for all $z \in \mathbb{D}$, then

$$\sum_{m=0}^{\infty} |a_m||z|^m \leq 1, \text{ for } |z| \leq \frac{1}{3}. \quad (1.1)$$

K. Gangania thanks to University Grant Commission, New-Delhi, India for providing Junior Research Fellowship under UGC-Ref. No.:1051/(CSIR-UGC NET JUNE 2017).
functions. g has Bohr phenomenon if for any r ∈ 0, 1 such that |z| = r ≤ r₀, where d(f(0), ∂f(Ω)) denotes the Euclidean distance between f(0) and the boundary of domain f (Ω). When h in S(h) is an important subclass of univalent functions denoted by S∗ that is symmetric about real axis. Note that the concept of subordination for analytic functions can be adopted for harmonic functions without any change, see [9]. Now let us consider the class A which consists of analytic function with power series of the form f(z) = z + ∑∞ m=2 aₘzᵐ, and it’s important subclass of univalent functions denoted by S. Further, the classes of Ma-Minda starlike and convex function [13], respectively are defined as:

\[ S^*(ψ) := \left\{ f ∈ A : \frac{zf''(z)}{f'(z)} < ψ(z) \right\} \]

\[ C(ψ) := \left\{ f ∈ A : 1 + \frac{zf''(z)}{f'(z)} < ψ(z) \right\}, \]

where ψ is analytic and univalent with \( \Re \psi(z) > 0, ψ'(0) > 0, ψ(0) = 1 \) and \( ψ(Ω) \) is symmetric about real axis. Note that ψ ∈ P, the class of normalized Carathéodory functions. Also when \( ψ(z) = (1 + z)/(1 - z), S^*(ψ) \) and \( C(ψ) \) reduces to the standard classes \( S^* \) and \( C \) of univalent starlike and convex functions.

In view of Muhanna [14], the class S(f) of functions g subordinate to f has Bohr phenomenon if for any g(z) = ∑∞ m=0 bₘzᵐ ∈ S(f), there exist an r₀ ∈ (0, 1] such that

\[ \sum_{m=1}^{∞} |bₘ|rᵐ ≤ d(f(0), ∂f(Ω)) \]  \( \text{(1.2)} \)

holds for |z| = r ≤ r₀, where d(f(0), ∂f(Ω)) denotes the Euclidean distance between f(0) and the boundary of domain f (Ω). When h in S(h) is an important subclass of univalent functions denoted by S.
harmonic mapping, several Bohr inequalities have been investigated with certain assumptions on the analytic part $f$. Bhowmik and Das [3] assumed that $h$ is a sense-preserving $K$-quasiconformal harmonic mapping, where the analytic part $f$ is univalent of convex univalent function, and their result states as follows:

**Theorem 1.2.** [3, Theorem 1] Suppose that $h(z) = f(z) + g(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $f$ is univalent and $h_1(z) = f_1(z) + g_1(z) = \sum_{m=0}^{\infty} c_m z^m + \sum_{m=1}^{\infty} d_m z^m \in S(h)$. Then

$$\sum_{m=1}^{\infty} (|c_m| + |d_m|)r^m \leq d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r \leq (5K + 1 - \sqrt{8K(3K + 1)})/(K + 1)$. This result is sharp for the function $p(z) = z/(1 - z)^2 + k z/(1 - z)^2$, where $k = (K - 1)/(K + 1)$. Moreover, if we take $f$ to be convex univalent then the result holds for $r \leq r_0 = (K + 1)/(5K + 1)$ with sharpness for the function $q(z) = z/(1 + z) + k z/(1 - z)$.

Previously, several improved classical Bohr type inequality were discussed in [4] for the sense-preserving $K$-quasiconformal harmonic mappings such that $f$ satisfy $|f(z)| < 1$ and it’s applications to the corresponding analytic functions by taking $k = 0$ were shown. We state here one among them:

**Theorem 1.3.** [4, Theorem 2.9] Let $h(z) = f(z) + g(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $|f(z)| < 1$ and $0 \leq a = |a_0| < 1$. Then the following inequality holds

$$|f(z)| + \sum_{m=1}^{\infty} (|a_m| + |b_m|)r^m \leq 1,$$

for all $a \geq \alpha_k$ and $|z| = r \leq r_{a,k}$ (the radius is sharp), where

$$\alpha_k = \frac{\sqrt{k^2 + 12k + 12} - (2k + 3)}{k + 1} \quad \text{and} \quad r_{a,k} = \frac{B_{a,k} - (k + 2)(1 + a)}{2a^2(k + 1) + 2ak},$$

where $B_{a,k} = \sqrt{a^2(k^2 + 8k + 8) + 2a(k^2 + 6k + 4) + (k + 2)^2}$.

Motivated by the above results and noticing the role of sharp coefficient bounds for functions in a given class to establish such inequalities, see [3, 19, 20, 22, 23], and observing that sharp coefficients bounds are not available in general for the class $S^*(\psi)$ and $C(\psi)$. It is promising to discuss the case where analytic part $f$ of a sense-preserving $K$-quasiconformal harmonic mapping belongs to the general class $S^*(\psi)$ and $C(\psi)$. More precisely,

**Definition 1.4.** Let $f \in S^*(\psi)$ (or $C(\psi)$), and $h(z) = f(z) + g(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m$ be a sense-preserving $K$-quasiconformal harmonic
mapping defined in \( \mathbb{D} \). Then
\[
S_\psi(h) := \{ h_1(z) : h_1(z) \prec h(z) \},
\]
where
\[
h_1(z) = f_1(z) + g_1(z) = z + \sum_{m=2}^{\infty} c_m z^m + \sum_{m=1}^{\infty} d_m z^m.
\]

Similarly, in the literature of coefficient’s problems, much of the attention has been on the inequalities related to the logarithmic coefficients, see [26], due to it’s important role in settling the Bieberbach conjecture, we refer to see the recent articles [1, 28, 29] and their references, which are defined by the following logarithmic power series
\[
\log \left( \frac{f(z)}{z} \right) = 2 \sum_{m=1}^{\infty} \gamma_m z^m,
\]
where \( f \in S \). Recently, Adegani et. al [1, Theorem 1, Sec 2] settled the problem of sharp bounds for \( \gamma_m \) for the functions in the class \( S^*(\psi) \), but obtained only three initial logarithmic coefficient’s bounds for the class \( C(\psi) \) in [1, Theorem 2, Sec 2]. Therefore, we consider the problem of Bohr inequality similar to (1.1) for \( \log(f(z)/z) \), where \( f \) either belongs to the classes \( S^*(\psi) \) and \( C(\psi) \) or satisfies certain standard differential subordination[27]. The reason why we consider the Bohr inequality of classical type for the series (1.3) instead of (1.2) follows with the observations discussed by Bhowmik and Das: The quantity \( d(g, \partial \Omega) \), where \( \Omega \) is the image of \( \mathbb{D} \) under \( g(z) := \log(f(z)/z) \), can be arbitrarily small positive number for \( f \in S \) which is observed with the help of univalent polynomials \( f_n(z) = z + (z^2/n) \) for each \( n \geq 2 \) such that the image of \( \log(f_n(z)/z) \) does not include the point \( \log(1 + (1/n)) \).

While proving our result related to Bohr inequality for the logarithmic power series, we surprisingly get estimates for the logarithmic coefficient’s bounds for the class \( C(\psi) \) for the case \( \psi(\mathbb{D}) \) being convex, see Theorem 3.4. Note that till date obtaining sharp bounds even for a particular class of convex functions is an open problem.

2. Bohr phenomenon for \( K \)-quasiconformal mappings

For convenience, let us consider the functions \( f_n \in S^*(\psi) \), \( n \in \{0, 1, 2, \cdots \} \) defined in the unit disk \( \mathbb{D} \) as
\[
\frac{zf_n'(z)}{f_n(z)} = \psi(z^{n+1}).
\]
In case when the coefficients of \( f_n \) in its power series expansion are positive, we denote \( f_n \) by \( \hat{f}_n \). Since sharp bounds for the Taylor coefficients in general for the functions in the class \( S^*(\psi) \) are yet not known. Therefore, we need the following result, obtained in [19, Lemma 2.1, Sec 2].

**Lemma 2.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) be analytic in \( \mathbb{D} \) and \( g \prec f \), then
\[
\sum_{k=N}^{\infty} |b_k|^k \leq \sum_{n=N}^{\infty} |a_n|^n.
\]
for $|z| = r \leq \frac{1}{3}$ and $N \in \mathbb{N}$.

which was further used to prove \cite{19} Corollary 2.1, Sec 2, and there replacing $g(z)$ by $g(z)/M$ for $M > 0$ we easily get:

**Lemma 2.2.** Let the analytic functions $f$, $g$ and $h$ satisfy $g(z) = M \phi(z)f(\omega(z))$ in $\mathbb{D}$, where $\omega$ is the Schwarz function. Assume $|\phi(z)| \leq \tau$ for $|z| < \tau \leq 1$.

Then

$$\sum_{k=N}^{\infty} |b_k| r^k \leq \tau M \sum_{n=N}^{\infty} |a_n| r^n, \quad \text{for} \quad 0 \leq |z| = r \leq \frac{\tau}{3},$$

where $M > 0$ and $N \in \mathbb{N}$.

The Lemma 2.2 generalizes the result of Alkhaleefah et al. \cite{4, Theorem 2.1}. For other inequalities of this kind, see \cite{17}.

Now the following result is a generalization of Theorem 1.2 in the sense that $f$ be a Ma-Minda univalent starlike functions, while its convex analogue is presented in Theorem 2.3.

**Theorem 2.1.** Let $h(z) = f(z) + g(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $f \in S^*(\psi)$ and $h_1(z) = f_1(z) + g_1(z) = z + \sum_{m=2}^{\infty} c_m z^m + \sum_{m=1}^{\infty} d_m z^m \in S_\psi(h)$. Then

$$\sum_{m=1}^{\infty} (|c_m| + |d_m|) r^m \leq d(f(0), \partial f(\mathbb{D}))$$

(2.2)

holds for $|z| \leq r^* = \min\{r_0, \frac{1}{3}\}$, where $r_0$ is the unique root in $(0, 1)$ of

$$\frac{2K}{K+1} \hat{f}_0(r) = -f_0(-1).$$

The result is sharp for the function $p(z) = f_0(z) + k f_0(z)$, where $\hat{f}_0 = f_0$ as defined in (2.1) when $r^* = r_0$.

**Proof.** Clearly, $f'(z) \neq 0$ as $f \in S^*(\psi)$. From the dilatation $w_h(z) = g'/f'$, we have $|w_f| \leq k = (K-1)/(K+1) < 1$. We consider the case $w_h$ being non constant, and the case $w_f(z) = cz$, $|c| = 1$ can be handled on similar lines. Thus, by Maximum-Modulus principle there exist $\phi : \mathbb{D} \to \mathbb{D}$ such that

$$g'(z) = k \phi(z)f'(z).$$

(2.3)

Now in Lemma 2.2 setting $\tau = 1 = N$ and $w(z) = z$ with $p(z) = \sum_{n=0}^{\infty} p_n z^n$ and $q(z) = \sum_{n=0}^{\infty} q_n z^n$ such that $p(z) = M \phi(z)q(z)$ for some $M > 0$, $\phi : \mathbb{D} \to \mathbb{D}$, we have for $|z| = r \leq 1/3$

$$\sum_{n=0}^{\infty} |q_n| r^n \leq M \sum_{n=0}^{\infty} |p_n| r^n,$$

which on applying to (2.3) and Lemma 2.1 with $N = 1$ on $f(z)/z < f_0(z)/z$, and then integrating from 0 to $r$ finally yield

$$\sum_{m=1}^{\infty} |b_m| r^m \leq k \sum_{m=1}^{\infty} |a_m| r^m \leq k \hat{f}_0(r)$$

(2.4)
for $|z| = r \leq 1/3$. Thus from (2.4), we have for $|z| = r \leq 1/3$

$$
\sum_{m=1}^{\infty} |a_m|r^m + \sum_{m=1}^{\infty} |b_m|r^m \leq (1 + k) \sum_{m=1}^{\infty} |a_m|r^m. \quad (2.5)
$$

Now let us consider the function

$$
T(r) := (1 + k)\hat{f}_0(r) + f_0(-1), \quad 0 \leq r \leq 1.
$$

Then it is a continuous function of $r$ and $T'(r) \geq 0$. Also $T(0) = f_0(-1) < 0$ and $T(1) > 0$. Therefore, $T(r)$ has a unique root, say $r_0$ in $(0, 1)$. Hence by (2.5)

$$
\sum_{m=1}^{\infty} |a_m|r^m + \sum_{m=1}^{\infty} |b_m|r^m \leq d(f(0), \partial f(D))
$$

holds for $|z| = r \leq \min\{r_0, 1/3\}$, where $r_0$ is the root of $T(r)$. Now it remains to prove that

$$
\sum_{m=1}^{\infty} (|c_m| + |d_m|)r^m \leq \sum_{m=1}^{\infty} |a_m|r^m + \sum_{m=1}^{\infty} |b_m|r^m
$$

is true for $|z| \leq r^*$ which in fact holds, since $f_1 < f$ and $g_1 < g$ [30, P. 164, Sec 2] with the application of Lemma 2.1 yields $\sum_{m=1}^{\infty} |c_m|r^m \leq \sum_{m=1}^{\infty} |a_m|r^m$ and $\sum_{m=1}^{\infty} |d_m|r^m \leq \sum_{m=1}^{\infty} |b_m|r^m$ for $r \leq 1/3$. Sharpness of the result follows for the function $p(z)$ as defined in the hypothesis by direct computation for $|z| = r \leq r^*$ such that equality holds in (2.2). \hfill \Box

In the next result, we establish the Bohr-Rogosinski phenomenon for the the class $S_{\psi}(h)$ following the proof of Theorem 2.1. Note that if we take $N \to 1$ and $n \to \infty$ in the result below, we trace back Theorem 2.1.

**Corollary 2.2.** Let $h(z) = f(z) + \overline{g(z)} = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $f \in S^*(\psi)$ and $h_1(z) = f_1(z) + \overline{g_1(z)} = z + \sum_{m=2}^{\infty} c_m z^m + \sum_{m=1}^{\infty} d_m z^m \in S_{\psi}(h)$. Then

$$
|f(z^n)| + \sum_{m=N}^{\infty} (|c_m| + |d_m|)r^m \leq d(f(0), \partial f(D)) \quad (2.6)
$$

holds for $|z| \leq r^* = \min\{r_0, 1/3\}$, where $n \in \mathbb{N}$ and $r_0$ is the unique root in $(0, 1)$ of

$$
(K + 1)(\hat{f}_0(r^n) + f_0(-1)) + 2K(\hat{f}_0(r) - S_N(\hat{f}_0)) = 0,
$$

where $S_N(f) = \sum_{m=1}^{N-1} a_m r^m$. The result is sharp for the function $p(z) = f_0(z) + kr_0(z)$, where $\hat{f}_0 = f_0$ as defined in (2.1) when $r^* = r_0$.

**Proof.** Since $|f(z^n)| \leq \hat{f}_0(r^n)$ for $|z| = r$. Therefore, applying Lemma 2.1 on (2.3), and Lemma 2.1 on $f(z)/z \prec f_0(z)/z$, and then integrating from 0 to $r$, we deduce that

$$
|f(z^n)| + \sum_{m=N}^{\infty} (|c_m| + |d_m|)r^m
$$
\[ \leq |\hat{f}_0(r^n)| + (1 + k) \sum_{m=N}^{\infty} |a_m|r^m \]
\[ \leq -f_0(-1) \leq d(f(0), \partial f(\mathbb{D})) \]

holds for \(|z| = r \leq \{r_0, 1/3\}\), where \(r_0 \in (0, 1)\) is the unique positive root of the equation

\[ T_N(r) := \hat{f}_0(r^n) + f_0(-1) + (1 + k) \sum_{m=N}^{\infty} |a_m|r^m = 0. \]

Existence of the root \(r_0\) follows, since \(T_N(r)\) is continuous function of \(r\) and \(T_N'(r) \geq 0\) with \(T_N(0) = f_0(-1)\) and \(T_N(1) > 0\). \(\square\)

**Corollary 2.3.** [23, Theorem 5.1] [20, Theorem 3.1] Let \(f \in S^*(\psi)\) and \(f_1(z) = \sum_{m=1}^{\infty} c_m z^m \in S_\psi(f)\). Then

\[ \sum_{m=1}^{\infty} |c_m|r^m \leq d(f(0), \partial f(\mathbb{D})) \]

holds for \(|z| \leq r^* = \min\{r_0, 3/4\}\), where \(r_0\) is the unique root in \((0, 1)\) of

\[ \hat{f}_0(r) = -f_0(-1). \]

The result is sharp for the function \(\hat{f}_0 = f_0\) as defined in (2.1) when \(r^* = r_0\).

**Corollary 2.4.** Let \(f \in S^*\) in Theorem 2.1. Then the inequality (2.2) holds for \(|z| = r \leq r_0 < 1/3\), where

\[ r_0 = \frac{5K + 1 - \sqrt{8K(3K + 1)}}{(K + 1)}. \]

The radius is sharp.

**Corollary 2.5.** Let \(\psi(z) = \frac{1+Dz}{1+Ez}\) in Theorem 2.1, where \(-1 \leq E < D \leq 1\). Then the inequality (2.2) holds for \(|z| = r \leq \min\{r_0, 1/3\}\), where \(r_0\) is the unique root of the equation

\[ \frac{2K}{K + 1}(r + \sum_{m=2}^{\infty} \prod_{t=0}^{m-2} \frac{|E - D + Et|}{t + 1} r^m) - (1 - E)^{\frac{D-E}{3}} = 0. \]

Further assume that \(f_0 = \hat{f}_0\) as defined in (2.1), and

(i) If \(E \neq 0\) and \(3(1-E)^{\frac{D-E}{3}} \leq (1 + E)^{\frac{D-E}{3}}\);

(ii) If \(E = 0\) and \(D \geq \frac{3}{4} \log 3\).

Then the radius \(r_0\) is sharp.

**Corollary 2.6.** Let \(\psi(z) = \frac{1+(1-2\alpha)z}{1-z}\) in Theorem 2.1, where \(0 \leq \alpha \leq 1/2\). Then the inequality (2.2) holds for \(|z| = r < r_0 < 1/3\), where \(r_0\) is the unique root of the equation

\[ K^{2(1-\alpha)+1}r - (K + 1)(1 - r)^{2(1-\alpha)} = 0. \]

The radius is sharp.
Remark 2.1. Substituting $\alpha = 0$ in Corollary 2.6, we retrace Corollary 2.4. One should also note that if we take $K = 1$ and $\alpha = 0$ in Corollary 2.4, then the radius $r_0 = 3 - 2\sqrt{2}$, which is equal to the Bohr radius for the class $S^*$. Moreover, thinking in the direction of inclusion relationship for a class $\mathcal{M}$ of starlike functions such that $\mathcal{M} \subset S^*(\alpha)$ for some $\alpha$, then we observe that Corollary 2.6 provides a lower bound for the radius $r_0$ in Theorem 2.1 for the case of $\mathcal{M}$ in place of $S^*(\psi)$ in general.

Now recall that a function $f \in S$ which has the property that for every circular arc $\Gamma$ contained in $\mathbb{D}$ with center $\xi \in \mathbb{D}$, the image arc $f(\Gamma)$ is a starlike arc with respect to $f(\xi)$, is called Goodman uniformly starlike ($\equiv \text{UST}$). Similarly the class of uniformly convex function ($\equiv \text{UCV}$) is also defined. Motivated by this class recently Darus introduced the class:

Let $f \in A$. Then $f \in k - \text{UCST}(\alpha)$ if and only if

$$
\Re\left\{ \frac{(zf'(z))'}{f(z)} \right\} > \left| k \left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( zf'(z) \right)' - 1 \right|, 
$$

$k \geq 0, 0 \leq \alpha \leq 1$. Now following the Remark 2.1 and the inclusion relation given in [8, Theorem 2.1], Corollary 2.6 gives:

Example 2.7. Let us replace $S^*(\psi)$ by $k - \text{UCST}(\alpha)$ in Theorem 2.1. Then the inequality (2.2) holds for $|z| = r_0 \leq 1/3$, where $r_0$ is the unique root of the equation

$$
\frac{2Kr}{(1 - r)^{2(1 - \delta)}} - \frac{K + 1}{4(1 - \delta)} = 0,
$$

where

$$
\delta = \frac{(2\gamma - \beta) + \sqrt{(2\gamma - \beta)^2 + 8\beta}}{4}
$$

and $0 \leq \delta < 1, \beta = \frac{1 + nk}{1 + k}$ and $\gamma = \frac{1}{1 + k}$ and further satisfy

$$(1 - \alpha)k^2 - (1 + \alpha)k - 2 \geq 0.
$$

In view of the Remark 2.1, note that $k - \text{UCST}(\alpha) \subset S^*(\delta)$, and therefore the radius $r_0$ can be further improve.

Now we conclude this section with the following result which is convex analogue of Theorem 2.1.

Theorem 2.8. Let $h(z) = f(z) + g(z) = z + \sum_{m=2}^{\infty}a_mz^m + \sum_{m=1}^{\infty}b_mz^m$ be a sense-preserving $K$-quasiconformal harmonic mapping defined in $\mathbb{D}$ such that $f \in C(\psi)$ and $h_1(z) = f_1(z) + g_1(z) = z + \sum_{m=2}^{\infty}c_mz^m + \sum_{m=1}^{\infty}d_mz^m \in S_\psi(h)$. Then

$$
\sum_{m=1}^{\infty}(|c_m| + |d_m|)r_0^m \leq d(f(0), \partial f(\mathbb{D})),
$$

holds for $|z| \leq r^* = \min\{r_0, \frac{1}{3}\}$, where $r_0$ is the unique root in $(0, 1)$ of

$$
\frac{2K}{K + 1} f_0(r) = -f_0(-1).
$$
The result is sharp for the function $p(z) = f_0(z) + kf_0(z)$ when $r^* = r_0$, where $f_0 = f_0$ is the solution of

$$1 + \frac{zf''(z)}{f'(z)} = \psi(z).$$

**Proof.** It follows from Theorem 2.1 by suitably using Alexander’s relation between starlike and convex functions. 

**Corollary 2.9.** Let $f \in C$ in Theorem 2.8. Then the inequality (2.2) holds for $|z| = r \leq r_0 < 1/3$, where

$$r_0 = \frac{(K + 1)}{(5K + 1)}.$$

The radius is sharp.

The above corollary was obatined by Liu and Ponnusamy [24, Theorem 1], (also see [11, Theorems 1.1 and 1.3]), where the analytic part $f$ was not normalized.

### 3. Bohr phenomenon for Logarithmic power series

Let us recall that logarithmic coefficients $\gamma_m$ of the functions $f \in S^*(\psi)$ are defined as

$$\log \left( \frac{f(z)}{z} \right) = 2 \sum_{m=1}^{\infty} \gamma_m z^m. \quad (3.1)$$

In the following theorem, we generalize and provide a simple proof of the result [5, Theorem 4, Sec 3].

**Theorem 3.1.** Let $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ and $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in S^*(\psi)$, where $\psi(\mathbb{D})$ is convex. Then the logarithmic coefficient’s of $f$ satisfies

$$2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 1, \text{ for } |z| = r \leq 1 - \frac{1}{\exp \left( \frac{1}{|B_1|} \right)}.$$

The result is sharp for the case when the function $f$ defined as $zf'(z)/f(z) = \psi(z)$ is an extremal for the logarithmic coefficient’s bounds.

**Proof.** Recall the result [11, Theorem 1, Sec 2] which says: If $f \in S^*(\psi)$, then $\gamma_m$ as defined in (3.1) satisfies

$$|\gamma_m| \leq \frac{|B_1|}{2m}, \quad (3.2)$$

whenever $\psi(\mathbb{D})$ is convex. Hence

$$2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 2 \sum_{m=1}^{\infty} \frac{|B_1|}{2m} r^m = |B_1| \sum_{m=1}^{\infty} \frac{1}{m} r^m.$$
Since the sum of the power series \( \sum_{m=1}^{\infty} \frac{x^m}{m} = -\log(1-x) \) for \( x \in (-1, 1] \).
Therefore, we have
\[
2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq |B_1| \log \left( \frac{1}{1-r} \right) \leq 1,
\]
which holds whenever \( r \leq 1 - 1/(\exp(\frac{1}{|B_1|})) \).

Now we apply the above theorem to some well-known and recently introduced classes:

**Corollary 3.2.** Let \( g \in A \) and \( f \in S^*(\psi) \), where \( \psi(\mathbb{D}) \) is convex. Then
\[
2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 1, \quad \text{for} \quad |z| = r \leq r_0 = 1 - \frac{1}{\exp(\frac{1}{|B_1|})},
\]
follows for each one of the following cases:

(i) \( B_1 = D - E \) when \( \psi(z) = \frac{|z| + D}{1 + Ez} \), where \(-1 \leq E < D \leq 1\).

(ii) \( B_1 = 2(1 - \alpha) \) when \( \psi(z) = \frac{1+1(1-2\alpha z)}{1-z} \), where \( 0 \leq \alpha < 1 \).

(iii) \( B_1 = 2\eta \) when \( \psi(z) = \left( \frac{1+z}{z} \right)^\eta \), where \( 0 < \eta \leq 1 \).

(iv) \( B_1 = (5\sqrt{2} - 6)/(2\sqrt{2}) \) when \( \psi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}} \).

(v) \( B_1 = b^z/a \) when \( \psi(z) = (b(1 + z))^{1/a} \), where \( a \geq 1 \) and \( b \geq 1/2 \).

(vi) \( B_1 = 1 - \alpha \) when \( \psi(z) = \alpha + (1 - \alpha)e^z \), where \( 0 \leq \alpha < 1 \).

(vii) \( B_1 = 1/2 \) when \( \psi(z) = \frac{2}{1+e^{-z}} \).

The result is sharp for the case when the function \( f \) defined as \( zf'(z)/f(z) = \psi(z) \) is an extremal for the logarithmic coefficient’s bounds.

**Remark 3.1.** In Theorem 3.1, let \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in S^*(\psi) \), where \( \psi(\mathbb{D}) \) is starlike with respect to 1 but convex. Then the logarithmic coefficients of \( f \) satisfies, see [I] Theorem 1, Sec 2,\]
\[
|\gamma_m| \leq \frac{|B_1|}{2}, \quad n \in \mathbb{N},
\]
and following the Proof of Theorem 3.1, we deduce that
\[
2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 1 \quad \text{for} \quad |z| = r \leq (1 + |B_1|)^{-1}.
\]

Note that all sharp logarithmic coefficient’s bounds for the functions in the class \( C(\psi) \) of normalized convex function are not available till now, see [I] Theorem 1, Sec 2]. But using the technique of differential subordination, we overcome this difficulty to achieve the Bohr radius for logarithmic power series for the Ma-Minda classes of convex functions in the following result.

**Theorem 3.3.** Let \( \phi \) be convex in \( \mathbb{D} \) and suppose \( \psi \) be the convex solution of the differential equation
\[
\psi(z) + \frac{z\psi'(z)}{\psi(z)} = \phi(z).
\]
If \( f \in \mathcal{C}(\phi) \) with \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \). Then the logarithmic coefficients of \( f \) satisfies
\[
2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 1, \tag{3.4}
\]
for
\[
|z| = r \leq 1 - \frac{1}{\exp \left( \frac{2}{|B_1|} \right)}.
\]
The result is sharp for the case when the function \( f \) defined as \( 1 + zf''(z)/f'(z) = \phi(z) \) is an extremal for the logarithmic coefficient’s bounds.

**Proof.** Let us define \( p(z) := zf'(z)/f(z) \). Since \( f \in \mathcal{C}(\phi) \), therefore we have \( 1 + zf''(z)/f'(z) \prec \phi(z) \), which can be equivalently written as
\[
p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z). \tag{3.5}
\]
Since \( \Re \phi(z) > 0 \) and \( \phi \) is convex in \( \mathbb{D} \), therefore using [27, Theorem 3.2d, p. 86] the solution \( \psi \) of the differential equation \( \psi'(z) = h(z) \) is analytic in \( \mathbb{D} \) with \( \Re \psi(z) > 0 \) and has the following integral form given by
\[
\psi(z) := h(z) \left( \int_0^z \frac{h(t)}{t} \, dt \right)^{-1} = 1 + \frac{B_1}{2} z + \frac{B_1^2 + 4B_2}{12} z^2 + \cdots, \tag{3.6}
\]
where
\[
h(z) = z \exp \int_0^z \frac{\phi(t) - 1}{t} \, dt.
\]
Since \( \Re \psi(z) > 0 \) and \( p \) satisfies the subordination \( \psi \prec \psi \), therefore using [27, Lemma 3.2e, p. 89] we conclude that \( \psi \) is univalent and \( p \prec \psi \), where \( \psi \) is the best dominant. Thus we have obtained that \( f \in \mathcal{C}(\phi) \) implies \( zf''(z)/f'(z) \prec \psi(z) \) and \( \psi \) is the best dominant, which is a univalent Carathéodory function. Now applying Theorem 3.1, the desired radius is achieved. \( \square \)

**Remark 3.2.** If \( f \in \mathcal{C} \). Then the logarithmic coefficients of \( f \) satisfies
\[
2 \sum_{m=1}^{\infty} |\gamma_m| r^m \leq 1,
\]
for
\[
|z| = r \leq 1 - 1/e.
\]
and the radius is sharp for the function \( f(z) = z/(1 - z) \). The result was obtained by Bhowmik and Das in [5, Remark on Page no 741].

Now we estimate the Logarithmic coefficient’s bounds for the functions in the class \( \mathcal{C}(\psi) \) for the case \( \psi(\mathbb{D}) \) being convex, which generalize the result [11, Theorem 2, Sec 2] where sharp two initial Logarithmic coefficient’s bounds and estimate for the third bound \( (\gamma_m, \text{for } m = 1, 2 \text{ and } 3) \) were discussed.
Theorem 3.4. (Convex analogue of [1, Theorem 1, Sec 2]) Let $\phi$ be convex in $\mathbb{D}$ and suppose $\psi$ be the solution of the differential equation
\[
\psi(z) + \frac{z\psi'(z)}{\psi(z)} = \phi(z).
\]
If $f \in \mathcal{C}(\phi)$ with $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then the logarithmic coefficients of $f$ satisfies the inequalities:

1. If $\psi$ is convex. Then
   \[
   |\gamma_m| \leq \frac{|B_1|}{4m}, \quad m \in \mathbb{N},
   \]
   \[
   \sum_{m=1}^{n} |\gamma_m|^2 \leq \frac{1}{4} \sum_{m=1}^{n} \frac{|c_m|^2}{m^2}, \quad n \in \mathbb{N},
   \]
   and
   \[
   \sum_{m=1}^{\infty} |\gamma_m|^2 \leq \frac{1}{4} \sum_{m=1}^{\infty} \frac{|c_m|^2}{m^2},
   \]
   where $c_m$ are the Taylor series coefficients of $\psi$ (see Eq. (3.6)).

2. If $\psi$ is starlike with respect to $1$. Then
   \[
   |\gamma_m| \leq \frac{|B_1|}{4}, \quad m \in \mathbb{D}.
   \]

The inequalities (3.7), (3.8) and (3.9) are sharp for the case when the function $f$ defined as $zf'(z)/f(z) = \psi(z)$ serves as an extremal for the logarithmic coefficient’s bounds.

Proof. Following the Proof of Theorem 3.3 and then applying the result [1, Theorem 1, Sec 2] completes the proof. $\square$

In our next result, we show the application to the Janowski class, which covers many well-known classes. Here $\mathcal{C}[D, E] := \mathcal{C}((1 + Dz)/(1 + Ez)).$

Let us first consider the confluent and Gaussian hypergeometric functions, respectively as follows:
\[
q(z) = \begin{cases} 
2F_1 \left(1 - \frac{D}{E}, 1, 2; \frac{Ez}{1 + Ez}\right), & \text{if } E \neq 0; \\
1F_1 (1, 2; -Dz), & \text{if } E = 0.
\end{cases}
\]
(3.10)

We note that for $1 + D/E \geq 0$ and $-1 \leq E < 0$,
\[
\min_{|z|=r} \Re \psi(z) = \psi(-r) = \frac{1}{q(-r)} > 0, \text{ where } \psi(z) = 1/q(z).
\]

Corollary 3.5. Let $f$ belongs to $\mathcal{C}[D, E]$, where $-1 \leq E < D \leq 1$, and in addition $q$ as defined in (3.10) be convex. Then the logarithmic coefficients of $f$ satisfies (3.4) for
\[
|z| = r \leq \begin{cases} 
1 - \frac{1}{\exp \left(\frac{1}{\sigma - r}\right)}, & \text{if } E, D \neq 0; \\
1 - \frac{1}{\exp \left(\frac{2}{|E|}\right)}, & \text{if } E \neq 0, D = 0,
\end{cases}
\]
Proof. In Theorem 3.3 put $\phi(z) = (1 + Dz)/(1 + Ez)$. Then we have $\psi(z) := 1/q(z)$, where

$$q(z) = \begin{cases} \int_0^1 \left(1 + \frac{Etz}{1+Ez}\right)^{\frac{B-E}{E}} dt, & \text{if } E \neq 0; \\ \int_0^1 e^{D(t-1)z} dt, & \text{if } E = 0, \end{cases}$$

which with some computation can be written explicitly as

$$\psi(z) = \begin{cases} \frac{Dz(1+Ez)^{\frac{B-E}{E}}}{1+(1+Ez)^{\frac{B-E}{E}}} = 1 + \frac{D-E}{2} z + \frac{D^2-6DE+5E^2}{12} z^2 + \cdots, & \text{if } E, D \neq 0; \\ \frac{Ez}{(1+Ez)\log(1+Ez)} = 1 - \frac{E}{2} z + \frac{5E^2}{12} z^2 - \cdots, & \text{if } E \neq 0, D = 0; \\ \frac{Dze^{Dz}}{e^{Dz}-1}, & \text{if } E = 0, \end{cases}$$

and satisfies the differential Eq. (3.3). Hence, the result follows from Theorem 3.1. □

Now we have the result for the class of convex functions of order $\alpha$ using Corollary 3.5:

Corollary 3.6. Let $f$ belongs to $C[1-2\alpha, -1]$, where $0 \leq \alpha < 1$. Then the logarithmic coefficients of $f$ satisfies (3.11) for

$$|z| = r \leq 1 - \exp(1/(\alpha - 1)), \text{ when } \alpha \neq 1/2$$

and

$$|z| = r \leq 1 - 1/\exp(2), \text{ when } \alpha = 1/2.$$

Corollary 3.7. Let $f$ belongs to $C[D, 0]$. Then the logarithmic coefficients of $f$ satisfies (3.11) for

$$|z| = r \leq 1 - \exp(-\frac{2}{D}),$$

Proof. From the proof of Corollary 3.5 we obtain that

$$\psi(z) = Dze^{Dz}/(e^{Dz}-1) = 1 + \frac{D}{2} z + \frac{D^2}{2} z^2 - \cdots,$$

when $\phi(z) = 1+Dz$. Now with a little computation, we find that the function $l(z) = ze^z/(e^z-1)$ is convex univalent in $D$. Therefore, the function $\psi(z) = l(Dz)$ is also convex in $D$ for each fixed $0 < D \leq 1$. □

Theorem 3.8. Let $\phi$ be convex in $D$ with $\Re\phi(z) > 0$, and suppose $f \in A$ satisfies the differential subordination

$$zf'(z) + z \left(\frac{zf'(z)}{f(z)}\right)' < \phi(z) = 1 + B_1z + B_2z^2 + \cdots. \quad (3.11)$$

Then the logarithmic coefficients of $f$ satisfies (3.11) for

$$|z| = r \leq 1 - \frac{1}{\exp\left(\frac{2}{|B_1|}\right)}.$$

The result is sharp for the case when the function $f$ defined as $zf'(z)/f(z) = \psi(z)$ is an extremal for the logarithmic coefficient’s bounds.
Proof. Let \( p(z) = zf'(z)/f(z) \). Then the subordination (3.11) can equivalently be written as: \( p(z) + zp'(z) \prec \phi(z) \). A simple calculation show that the analytic function
\[
\psi(z) := (1/z) \int_0^z \phi(t)dt = 1 + \frac{B_1}{2} z + \frac{B_2}{3} z^2 + \cdots
\]
satisfies
\[
\psi(z) + z\psi'(z) = \phi(z).
\]
Now from the Hallenbeck and Ruscheweyh result [27, Theorem 3.1b, p. 71], we have \( p \prec \psi \), where \( \psi \) is the best dominant and also convex. Further, since \( \Re\phi(z) > 0 \), using the integral operator [27, Theorem 4.2a, p. 202] preserving functions with positive real part, we see that \( \psi \) is a Carathéodory function. Thus we have
\[
zf'(z)/f(z) \prec \psi(z).
\]
Now applying Theorem 3.1, the result follows. \( \square \)

Corollary 3.9. Suppose \( f \in A \) satisfies the differential subordination
\[
\frac{zf'(z)}{f(z)} + 2z\left(\frac{zf'(z)}{f(z)}\right)' \prec \frac{1+z}{1-z}.
\]
Then the logarithmic coefficients of \( f \) satisfies (3.4) for \( |z| = r \leq (e - 1)/e \approx 0.632121 \),

Corollary 3.10. Suppose \( f \in A \) satisfies the differential subordination
\[
\frac{zf'(z)}{f(z)} + z\left(\frac{zf'(z)}{f(z)}\right)' \prec \exp(z).
\]
Then the logarithmic coefficients of \( f \) satisfies (3.4) for \( |z| = r \leq (e^2 - 1)/e^2 \approx 0.864665 \),

Theorem 3.11. Let \( \phi \) be convex in \( D \) with \( \Re\phi(z) > 0 \), and suppose \( f \in A \) satisfies the differential subordination
\[
\frac{zf'(z)}{f(z)} \left(2z\left(\frac{zf'(z)}{f(z)}\right)\right)' \prec \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, B_1 \neq 0.
\]
(3.12)
Then the logarithmic coefficients of \( f \) satisfies (3.4) for
\[
|z| = r \leq 1 - \frac{1}{\exp(1/|B_1|)}.
\]
The result is sharp for the case when the function \( f \) defined as \( zf'(z)/f(z) = \sqrt{\psi(z)} \) is an extremal for the logarithmic coefficient’s bounds.

Proof. Let \( p(z) = zf'(z)/f(z) \). Then the subordination (3.12) can be equivalently written as:
\[
p^2(z) + 2zp(z)p'(z) \prec \phi(z),
\]
which using the change of variable $P(z) = p^2(z)$ becomes

$$P(z) + zP'(z) \prec \phi(z).$$

Now proceeding as in Theorem 3.8 we see that $p(z) \prec \sqrt{\psi(z)}$ and $\sqrt{\psi(z)}$ is the best dominant. Further, since $\Re \phi(z) > 0$, using [27] Theorem 4.2a, p. 202], we see that $\psi$ is a Carathéodory function. Thus we have

$$\frac{zf'(z)}{f(z)} \prec \sqrt{\psi}(z) = 1 + \frac{B_1}{4}z + \left(\frac{B_2}{6} - \frac{B_1^2}{32}\right)z^2 + \cdots,$$

where $\psi(z) = \frac{1}{z} \int_0^z \phi(t)dt$. Now if we let $q(z) = 1 + z\psi''(z)/\psi'(z)$. Then

$$q(0) = 1 \quad \text{and} \quad \psi'(z)(q(z) + 1) = \phi'(z),$$

which further by logarithmic differentiation gives

$$\Psi(r, s; z) := q(z) + \frac{zq'(z)}{q(z) + 1} = 1 + \frac{z\phi''(z)}{\phi'(z)}. \quad (3.13)$$

Since the function $\Psi$ satisfy the admissible conditions [27] 2.3-11, p. 35], using [27] Theorem 2.3i (i), p. 35] in (3.13), we see that $\psi$ is convex. Let

$$h(z) = (\psi(z) - 1)/\psi'(0) \in A.$$

Now it is easy to conclude that $h \in S^*$, hence univalent. Finally, note that

$$|\arg \sqrt{\psi}(z)| = \frac{1}{2}|\arg \psi(z)| \leq \frac{\pi}{4},$$

which implies $\Re \sqrt{\psi(z)} > 0$. Thus we see that $\sqrt{\psi(z)}$ is a Carathéodory univalent convex function. Now applying Theorem 3.11 the result follows. □

**Corollary 3.12.** Suppose $f \in A$ satisfies the differential subordination

$$zf'(z) - f(z) \left(zf'(z) - f(z) + 2z \left(zf'(z) - f(z)\right)\right) \prec 1 + \frac{2z(\alpha - 1)}{1 + z}, \quad \alpha \in [0, 1).$$

Then the logarithmic coefficients of $f$ satisfies (3.4) for $|z| = r \leq 1 - \exp\left(\frac{2}{\alpha - 1}\right)$.

**Corollary 3.13.** Suppose $f \in A$ satisfies the differential subordination

$$zf'(z) - f(z) \left(zf'(z) - f(z) + 2z \left(zf'(z) - f(z)\right)\right) \prec 1 + az, \quad \alpha \in (0, 1].$$

Then the logarithmic coefficients of $f$ satisfies (3.4) for $|z| = r \leq 1 - 1/\exp(4/\alpha)$.  

**Conflict of interest**

The authors declare that they have no conflict of interest.
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