Early and late stage profiles for a new chemotaxis model with density-dependent jump probability and quorum-sensing mechanisms

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Abstract

In this paper, we derive a new chemotaxis model with degenerate diffusion and density-dependent chemotactic sensitivity, and we provide a more realistic description of cell migration process for its early and late stages. Different from the existing studies focusing on the case of non-degenerate diffusion, the new model with degenerate diffusion causes us some essential difficulty on the boundedness estimates and the propagation behavior of its compact support. In the presence of logistic damping, for the early stage before tumour cells spread to the whole body, we first estimate the expanding speed of tumour region as $O(t^\beta)$ for $0 < \beta < \frac{1}{2}$. Then, for the late stage of cell migration, we further prove that the asymptotic profile of the original system is just its corresponding steady state. The global convergence of the original weak solution to the steady state with exponential rate $O(e^{-ct})$ for some $c > 0$ is also obtained.

1 Introduction

The motion of cells moving towards the higher concentration of a chemical signal is called chemotaxis. For example, bacteria moves toward the highest concentration of food molecules to find food. A well-known chemotaxis model was initially proposed by Keller and Segel \[8\] in 1971, subsequently, a number of variations of the Keller-Segel system were proposed and have been extensively studied during the past four decades, for example, see the survey papers \[1,7\] and the references therein. Especially, chemotaxis

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models also appear in medical mathematics. Many factors effect the migration mechanisms of tumour cells. For example, the extracellular matrix (ECM), to which the tumour cell to be attached, inhibits the cell polarizes and elongates to migrate. ECM-degrading enzymes (MDE) cleave ECM fibers into smaller chemotactic fragments to facilitate cell-migration [4]. In [3], Chaplain and Anderson introduced a model for tumour invasion mechanism, which describes tumour invasion phenomenon in accounting for the role of chemotactic ECM fragments named ECM*, produced by a biological reaction between ECM and MDE. In these models, the cancer cell random motility is assumed to be a constant, which leads to linear isotropic diffusion. However, in realistic situation, it is emphasized in that migration of the cancer cells through the ECM fibers should rather be regarded like movement in a porous medium with degenerate diffusion from a physical point of view [22]. Compared with the classical tumour invasion model with linear diffusion, the mathematical analysis of the nonlinear diffusion system has to cope with considerable additional challenges and is much less understood. Several chemotaxis models with nonlinear diffusion have been recently proposed and analyzed, e.g. [9, 22, 26, 27], where the nonlinear diffusions in these studies were still assumed to be non-degenerate. For tumour angiogenesis model and relevant mathematical analysis with or without degenerate diffusion, we refer to [11, 12, 29, 30, 31, 34] and the references therein.

Biological experiments suggest that no cell migration (in particular no diffusivity) occurs in regions where the tissue is absent [36]. In order to account for this biological feature, we extend Chaplain and Anderson’s model [3] to a new one with density-dependent jump probability and quorum-sensing mechanisms of tumour cells as follows, which is concerned with the competition between the following several biological mechanisms: degenerate diffusion, density-dependent chemotaxis, and logistic growth. That is,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta (q(u)u) - \nabla \cdot (\phi(u)q(u)u\nabla v) + \mu u^p(1 - ru), \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + wz, \quad x \in \Omega, \ t > 0, \\
\frac{\partial w}{\partial t} &= -wz, \quad x \in \Omega, \ t > 0, \\
\frac{\partial z}{\partial t} &= \Delta z - z + u, \quad x \in \Omega, \ t > 0.
\end{align*}
\]

The detailed derivation of the model (1) will be carried out in the last part of the next section. Here, \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary. The four variables \(u, w, z\) and \(v\) represent the cancer cell density, ECM concentration, the MDE concentration and the ECM* concentration, respectively. \(q(u)\) denotes the jump probability of a cell depending on the population pressure at its present location, which is increasing with respect to \(u\) with \(q(0) = 0, q(1) = 1\), namely, the jump probability is 1 when the cell
density exceeds maximum and it is zero when the cell density is zero, and $f(u) = \mu u^\delta(1 - ru)$ is the logistic growth term, where $\mu > 0$ and $r > 0$ are the proliferation rate and reciprocal of carrying capacity, respectively, $\delta \geq 1$ is a constant. $\phi(u)$ is the density-dependent chemotactic functions responding to quorum-sensing mechanisms, satisfying $|\phi(s)| \leq 1$ and $|\phi'(s)| \leq 1$. While $\phi(u)$ can be sign-changing representing the phenomenon that some chemicals have been shown to elicit both attractive and repellent responses [13, 19]. Moreover, some reasonable structure conditions on $\phi(s)$, and $q(s)$ are also required in discussing the existence of solutions, which we leave in Section 2 after the formulation of this model.

A recent interesting work related to the chemotaxis model mentioned above is [5], in which they considered the following chemotaxis system with linear diffusion

$$\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u\nabla v), \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + wz, \quad x \in \Omega, \ t > 0, \\
\frac{\partial w}{\partial t} &= -wz, \quad x \in \Omega, \ t > 0, \\
\frac{\partial z}{\partial t} &= \Delta z - z + u, \quad x \in \Omega, \ t > 0.
\end{align*}$$

It is proved the existence of global solutions and the asymptotic behaviors of global solutions as time goes to infinity by using the properties of the Neumann heat semigroup $e^{t\Delta}$ in $\Omega$.

Compared to the linear cases, the chemotactic system with degenerate diffusion and chemotactic sensitivity is more complex and challenging. Since the first equation of (1) is degenerate at any point where $u(x, t) = 0$, there is no classical solution in general. The spatial derivatives of $u$ may not exist in classical sense, and may even do not belong to the class of locally integrable generalized functions, that is, there might hold $u \notin W^{2,1}_{\text{loc}}$.

As we all know, for random walk equations, $u(x, t) > 0$ for $t > 0$ and any $x \in \mathbb{R}^N$, thus a linear diffusion process predicts a non-zero number of the tumour cells for arbitrarily large displacements at arbitrarily small time, namely, the underlying propagation speed is infinite [20]. This means that these models are valid for large time and describe the dynamics of tumour cells when they spread to all parts of body. Since most cancers develop at one anatomical site as primary tumour and then go on to metastasis, it is vital to study the mechanism of tumour cell migration at the early stage of cancer development when tumour progression begins and proceed to yield a cancerous mass [32].

In this paper, we provide a more realistic description of cell migration process for early and late stages. It is worth to mention that our stability results of the model (1) give a certain estimate for the expanding speed of tumour region before cancer cells spread to the whole body. We prove that there
exist $t_0$ and two families of monotone increasing open sets \( \{A_1(t)\}_{t \geq 0}, \{A_2(t)\}_{t \in (0, t_0)} \) such that

\[
A_1(t) \subset \text{supp } u(\cdot, t) \subset A_2(t) \subset \Omega, \quad t \in (0, t_0),
\]

\( \partial A_1(t) \) and \( \partial A_2(t) \) have finite derivatives with respect to \( t \), namely, \( \{A_1(t)\}_{t \geq 0} \) and \( \{A_2(t)\}_{t \in (0, t_0)} \) both expand at finite speeds. This indicates the finite speed propagation property of our chemotaxis model. As shown late in Remarks 2.1 and 2.2 in the porous media diffusion case, we estimate that, in the early stage the expanding speed of tumour region is somehow like the algebraic rate of \((1 + t)^{\beta}\) for some \( \beta \in (0, \frac{1}{2}) \).

In contrast with the well known linear cases, the degenerate diffusion is endowed with an interesting feature of slow diffusion, that is, the compact support of solutions propagates at a finite speed. The slow diffusion feature has some advantages and accuracy for describing specified biological processes in the point of view of the physical reality, and it also leads to more challenges in the mathematical studies. For example, in order to investigate the asymptotic behavior of solutions, one must appropriately describe the propagation behavior of its support, which is more likely to be a compact subset of the prescribed domain for some time interval if the initial data is given so. We mention that the Neumann heat semigroup theory and functional transform methods have been proved to be effective in studying the global boundedness and large time behavior for the linear diffusion equations, but they are all inapplicable in the degenerate diffusion case due to the nonlinearity. We establish the global existence of bounded weak solutions to this model by energy estimate technique and methods based on Moser-type iteration. Then we prove that, as the late stage of the tumour migration, the original weak solution time-asymptotically converges to its steady state, even if the initial perturbation is large, namely, the global stability of the steady state. The adopted approach is the technical compactness analysis with the help of the comparison principle deduced by the approximate Hohmgren’s approach and two kinds of lower solutions showing the expanding support and the exponentially convergence. The one is a self similar weak lower solution of Barenblatt type and the other kind is an ODE solution.

This paper is organized as follows. In Section 2, we derive the models based on realistic biological assumptions, which incorporate density-dependent jump probability and quorum-sensing mechanisms and leads to the form of equation given by (1). We leave the global existence of weak solutions and their regularity to the corresponding chemotaxis system into Section 3 as preliminaries. Section 4 is devoted to the study of compact support property of the tumour cells at early stage and the large time behavior at late stage, showing the exponentially convergence of solutions.
2 Main results and formulations of new chemotaxis model

In this section, we first state our main results on the study of expanding compact support of the tumour cells at early stage and the asymptotic behavior at late stage. We leave the detailed derivation on the new chemotaxis model \((1)\) with density-dependent jump probability and quorum-sensing mechanisms in the second part of this section.

2.1 Main results

We estimate the upper bound and lower bound for expanding speed of tumour cell region at early stage (before the tumour cells spread to the whole body) and show the exponentially convergence of solutions for large time. The derivation of this new chemotaxis model with degenerate diffusion is presented at the end of this section.

We consider the following system \((3)\) with degenerate diffusion

\[
egin{align*}
\psi_t &= \Delta(q(u))u - \nabla \cdot (\phi(u)q(u)u\nabla \nu) + \mu u^\delta(1 - u), \\
\nu_t &= \Delta v + w_z, \\
w_t &= -w_z, \\
z_t &= \Delta z - z + u, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial \nu}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \\
u(x, 0) &= u_0(x), \\
v(x, 0) &= v_0(x), \\
w(x, 0) &= w_0(x), \\
z(x, 0) &= z_0(x),
\end{align*}
\]

\(x \in \Omega, \ t > 0, \) \(x \in \partial \Omega, \ t > 0, \) \(x \in \Omega, \)

where \(\delta \geq 1, \mu > 0, u_0, v_0, w_0, z_0\) are nonnegative functions, \(\nu\) is the unit outer normal vector, and \(q(u) \geq 0\) with \(q(0) = 0\). Here and after, the IBVP \((3)\) will be our main target equations.

Since degenerate diffusion equations may not have classical solutions in general, we need to formulate the following definition of generalized solutions for the initial boundary value problem \((3)\).

**Definition 2.1** Let \(T \in (0, \infty)\). A quadruple \((u, v, w, z)\) is said to be a weak solution to the problem \((3)\) in \(Q_T = \Omega \times (0, T)\) if

1. \(u \in L^\infty(Q_T), v \in L^\infty(Q_T) \cap L^2((0, T); L^2(\Omega)),\) and \(q(u)u_t \in L^2((0, T); L^2(\Omega));\)
2. \(v \in L^\infty(Q_T) \cap L^2((0, T); W^{2,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega));\)
3. \(w \in L^\infty(Q_T), w_t \in L^2((0, T); L^2(\Omega));\)
4. \(z \in L^\infty(Q_T) \cap L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega));\)
5. the identities

\[
\int_0^T \int_{\Omega} u_t \psi_t dxdt + \int_\Omega u_0(x) \psi(x, 0) dx = \int_0^T \int_{\Omega} \nabla q(u) u \cdot \nabla \psi dxdt
\]
cells exist only in finite part of the body at the early stage. The description of cell migration process are as follows. First, we show that the tumour description of (3) with the initial data

\[ u \geq \delta > 0 \]

which is ill-posed if \( 0 < \delta \). Note that for constant initial data \((u_0, v_0, w_0, z_0)\) hold for all \( v, \psi, \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \) with \( \psi(x, T) = 0 \) for \( x \in \Omega \);

(6) \( (v, w, z) \) is a weak solution of (3) in \( Q_T \) for any \( T \in (0, \infty) \), then we call it a global weak solution.

A quadruple \((u, v, w, z)\) is said to be a globally bounded weak solution to the problem (3) if there exists a constant \( C \) such that

\[
\sup_{t \in \mathbb{R}^+} \left\{ ||u||_{L^\infty(\Omega)} + ||v||_{W^{1,\infty}(\Omega)} + ||w||_{L^\infty(\Omega)} + ||z||_{W^{1,\infty}(\Omega)} \right\} \leq C.
\]

Throughout this paper we assume that \( q(u) = u^{m-1} \) with \( m > 1 \), \( |\psi(s)| \leq 1 \) and \( |\psi'(s)| \leq 1 \), and the initial data satisfy \( u_0 \in C^0(\overline{\Omega}) \), \( v_0 \in W^{2,\infty}(\Omega) \), \( w_0 \in C^{2,\theta}(\overline{\Omega}) \), \( \theta \in (0, 1) \), \( \frac{\partial u_0}{\partial \nu} = 0 \) on \( \partial \Omega \), \( z_0 \in C^0(\overline{\Omega}) \). Here we note that for constant initial data \((u_0, v_0, w_0, z_0)\), the first equation of (3) is reduced to

\[
u'(t) = \mu \Psi^\delta (1 - u), \quad u(0) = u_0,
\]

which is ill-posed if \( 0 < \delta < 1 \). Therefore, we only consider the case \( \delta \geq 1 \).

As preliminaries, we leave the global existence and regularity results into Section 3. Our main results concerned with the description of cell migration process are as follows. First, we show that the tumour cells exist only in finite part of the body at the early stage.

**Theorem 2.1** (Early stage profile - upper bound) Let \((u, v, w, z)\) be a globally bounded weak solution of (3) with the initial data

\[
supp u_0 \subset \overline{B}_{r_0}(x_0) \subset \Omega,
\]

for some \( r_0 > 0 \) and \( x_0 \in \Omega \). Then there exists a time \( t_1 > 0 \) and a family of monotone increasing open sets \( \{A(t)\}_{t \in (0, t_1)} \) such that

\[
supp u(\cdot, t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0, t_1),
\]

and \( \partial A(t) \) has a finite derivative with respect to \( t \). More precisely, we can choose

\[
A(t) = \{ x \in \Omega; |x - x_0|^2 < \eta(t + \tau) \}, \quad t \in (0, t_1),
\]

with some appropriate \( \eta, \tau > 0 \).
Remark 2.1  As a typical finite propagating model, the Barenblatt solution of the porous medium equation is

\[ B(x, t) = (1 + t)^{-k} \left[ \left( 1 - \frac{k(m-1)}{2mn} \right) \frac{|x|^2}{(1 + t)^{2k/n}} \right]^{\frac{1}{m-1}} \] (4)

with \( k = 1/(m - 1 + 2/n) < n/2 \) for \( m > 1 \), and its support is expanding at the rate \((1 + t)^{k/n}\). Here we have proved the tumour cells are located within a ball expanding at the rate \((1 + t)^{1/2}\).

Next, we show the propagating property of the tumour cells at the early stage.

Theorem 2.2 (Early stage profile - lower bound)  Let \((u, v, w, z)\) be a globally bounded weak solution of (3). Assume that \(1 \leq \delta < m, \Omega\) is convex and \(u_0 \neq 0\). Then there exists a time \(t_0 > 0\) such that the support of \(u\) expands to the whole \(\Omega\) when \(t \geq t_0\). Precisely speaking, there exist a family of monotone increasing open sets \(\{A(t)\}_{t > 0}\) (we can choose \(A(t) = \{x \in \Omega; |x - x_0|^2 < \eta(1 + t)^\beta\}\) with sufficiently small \(\beta, \eta > 0\)) such that

\[ A(t) \subset \text{supp } u(\cdot, t), \quad t > 0, \]

and \(A(t) = \Omega\) for \(t \geq t_0\), \(\partial A(t)\) has a finite derivative with respect to \(t\).

Remark 2.2  For this chemotaxis system, we proved that the tumour cells will expand to the whole body when the time \(t\) increases. Compared with the porous medium equation, whose Barenblatt solution \(B(x, t)\) in (4) is expanding at the rate \((1 + t)^{2k/n}\), the tumour cells of (3) migrate to at least a ball expanding at the rate \((1 + t)^\beta\). Here in the proof we have selected \(\beta > 0\) sufficiently small, which means the support is expanding with a much slower rate.

Under the hypotheses of Theorem 2.1 and Theorem 2.2, we see that there exist \(t_0\) and two family of monotone increasing open sets \(\{A_1(t)\}_{t > 0}, \{A_2(t)\}_{t \in (0, t_0)}\) such that

\[ A_1(t) \subset \text{supp } u(\cdot, t) \subset \overline{A_2(t)} \subset \Omega, \quad t \in (0, t_0), \]

\(\partial A_1(t)\) and \(\partial A_2(t)\) have finite derivatives with respect to \(t\), which means that \(\{A_1(t)\}_{t > 0}\) and \(\{A_2(t)\}_{t \in (0, t_0)}\) both expand at finite speeds. This indicates immediately the finite speed propagation property of this chemotaxis model, though we have not proved it directly.

After the tumour cells spread to the whole body, we can investigate the large time behavior. We show that the solution converges to its steady state exponentially.

Theorem 2.3 (Late stage profile)  Let \((u, v, w, z)\) be a globally bounded weak solution of (3). Assume that the hypothesis in Theorem 2.2 is valid. Then there exist \(C\) and \(c > 0\) such that

\[ ||u(\cdot, t) - 1||_{L^\infty(\Omega)} + ||w(\cdot, t)||_{W^{1,\infty}(\Omega)} + ||v(\cdot, t) - (\overline{v_0} + \overline{w_0})||_{W^{2,\infty}(\Omega)} + ||z(\cdot, t) - 1||_{L^\infty(\Omega)} \leq Ce^{-ct}, \]

for all \(t > 0\), where \(\overline{v_0} = \frac{1}{|\Omega|} \int_\Omega v_0(x)dx\) and \(\overline{w_0} = \frac{1}{|\Omega|} \int_\Omega w_0(x)dx\).
The main difficulty lies in proving the expanding property of the support of the first component. We first prove the comparison principle by the approximate Hohmgren’s approach, and then construct two kinds of lower solutions. The one is a self similar weak lower solution with much slower expanding support and slightly faster decaying maximum compared with the Barenblatt solution to the porous medium equation, the other kind is an ODE solution. After showing the expanding property, we formulate several upper and lower solutions that converge to steady state exponentially by utilizing the exponential decay of other components.

2.2 Derivation of the new chemotaxis model

We extend the derivation of the classical taxis models in [20]. The derivation of the model begins with a master equation for a continuous-time and discrete-space random walk

$$\frac{\partial u_i}{\partial t} = T_{i-1}^+ u_{i-1} + T_{i+1}^- u_{i+1} - (T_{i}^+ + T_{i}^-) u_i,$$  \hspace{1cm} (5)

where $T_{i}^\pm(\cdot)$ denote the transitional-probabilities per unit time of a one-step jump to $i \pm 1$ and $u_i$ denotes the cell density at $i$.

Cancer cells can modify their migration mechanisms in response to different conditions [4]. There are two potentially important factors: (i) the effect of cell-density on the probability of cell movement; (ii) the effect of signal-mediated cell-density sensing mechanisms on movement [17].

For neighbor-based and gradient-based rules, Painter and Hillen [17] proposed volume filling approach, that is, the movement of cells is inhibited by the neighboring site where the cells are densely packed. The transitional probability then takes the form

$$T_{i}^\pm = q(u_{i\pm 1})(\alpha + \beta(\tau(v_{i\pm 1}) - \tau(v_i))),$$ \hspace{1cm} (6)

where $q(u)$ denotes the probability of a cell finding space at its neighboring location, constant $\alpha$ is the intrinsic dispersion coefficient, constant $\beta$ the coefficient signal detection, $v_i$ the signal concentration, and $\tau$ the mechanism of tactic responses in cell populations, such as chemotaxis, haptotaxis or phototaxis. Substituting (6) to the master equation (5), in the PDE limits they derive

$$\frac{\partial u}{\partial t} = \nabla \cdot (d_1 (q(u) - q'(u))\nabla u - \chi(v) q(u) u \nabla u)$$

where $d_1 = k\alpha$, $\chi(v) = 2k\beta \frac{\partial(\tau(v))}{\partial v}$, $k$ is a scaling constant. Note that $q(u)$ is a non-increasing function in this model, which says that the probability of a cell finding space at its neighboring site decreases in the cell density at that site.
Since a different combination of the above strategies may be necessary to reflect cell movement, we combine the local and gradient-based strategies and assume the transitional probability of the form

\[ T_i^\pm = q(u_i)\left(\alpha + \beta(\tau(v_{i+1}) - \tau(v_i))\right), \tag{7} \]

where \( q(u) \) represents the jump probability of a cell due to the population pressure at present site. At the microscopic level, a high cell density results in increased probability of a cell being “pushed” from departure site \([10, 15, 18]\), for example due to the pressure exerted by neighboring cells. We shall assume that only a finite number of cells, \( U_{\text{max}} \), can be accommodated at any site. We study the relative density \( \bar{u} = u_i/U_{\text{max}} \), (and drop the symbol \( \sim \) for simplicity). Moreover, the jump probability is 1 when the cell density exceeds \( U_{\text{max}} \) and it is zero when the cell density is zero. Thus we stipulate the following conditions on \( q \):

\[ q(0) = 0, \quad q(1) = 1 \quad \text{and} \quad q(u) \geq 0, \quad \text{for all } 0 \leq u \leq 1. \]

A natural choice for \( q(u) \) is

\[ q(u) = u^{m-1}, \quad m > 1, \tag{8} \]

which states that the probability of a jump leaving one site increases with the cell density at that site \([14, 21]\).

In most tissues, to control cell density at proper level, cells also secrete quorum-sensing molecule \( z \), then the concentration of the modules sensed by the cell will be an indicator of local density \([17]\). Cells sense the same gradient of chemical at the surface, but the “strength” signalled to the movement dynamics is modulated by quorum-sensing molecule \( z \) \([19]\). A more general choice of transitional-probabilities \( T_i^\pm(\cdot) \) can also be considered, namely

\[ T_i^\pm = q(u_i)(\alpha + \beta(z_i)(\tau(v_{i+1}) - \tau(v_i))), \tag{9} \]

where \( \beta(z) \) is a tactic function responding to quorum-sensing molecule \( z \). Substituting (9) into the Master Equation (5) gives:

\[
\frac{d}{dt} u_i = q_{i-1}(\alpha + \beta_{i-1}(\tau_i - \tau_{i-1}))u_{i-1} + q_{i+1}(\alpha + \beta_{i+1}(\tau_i - \tau_{i+1}))u_{i+1} \\
- q_i(\alpha + \beta(\tau_{i+1} - \tau_i))u_i - q_i(\alpha + \beta_i(\tau_{i-1} - \tau_i))u_i \\
= \alpha(q_{i-1}u_{i-1} + q_{i+1}u_{i+1} - 2q_iu_i) \\
+ \beta_{i-1}q_{i-1}(\tau_i - \tau_{i-1})u_{i-1} + \beta_{i+1}q_{i+1}(\tau_i - \tau_{i+1})u_{i+1} - \beta_iq_i(\tau_{i+1} + \tau_{i-1} - 2\tau_i)u_i \\
= \alpha(q_{i-1}u_{i-1} + q_{i+1}u_{i+1} - 2q_iu_i) \\
- \beta_{i+1}q_{i+1}u_{i+1}(\tau_{i+1} - \tau_i) + \beta_iq_iu_i(\tau_i - \tau_{i-1}) - (\beta_iq_iu_i(\tau_{i+1} - \tau_i) - \beta_{i-1}q_{i-1}u_{i-1}(\tau_i - \tau_{i-1}))
\]
we obtain for the cell density $u_i$.

We set $x = kh$, interpret $x$ as a continuous variable and extend the definition of $u_i$ accordingly. The transitional probabilities of jumping to a neighboring location depend on the spatial scale $h$. Thus we assume that $\mathcal{T}_h^\pm = \frac{k}{h^2} \mathcal{T}^\pm$ for some scaling constant $k$. Expanding the right-hand side with respect to $h$, we obtain for the cell density $u(x, t)$:

$$\frac{\partial u}{\partial t} = k\left(\alpha \frac{\partial^2 (q(u)u)}{\partial x^2} - 2 \frac{\partial}{\partial x}\left(\beta(z)q(u)\frac{\partial \tau}{\partial x}\right)\right) + O(h^2).$$

By taking the limit of $h \to 0$, we arrive at the following model:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 (q(u)u)}{\partial x^2} - \frac{\partial}{\partial x}\left(\beta(z)\chi(v)q(u)\frac{\partial v}{\partial x}\right),$$

where $D_u = k\alpha$, $\chi(v) = 2k \frac{d\tau(v)}{dv}$. The function $\chi(v)$ is commonly referred as the tactic sensitivity function. The simplest form is $\chi(v) = \chi_0$ with $\chi_0$ being a constant.

Apart from that, we consider a modification of the Verhulst logistic growth term to model organ size evolution introduced by Blumberg \cite{2} and Turner \cite{24}, which is called hyper-logistic function, accordingly:

$$f(u) = ru^\delta (1 - \mu u).$$

In the special case, the quorum sensing molecule $z = z(u)$ is not diffusing and a monotone increasing function of the cell density. Denote $\beta(z) = \beta(z(u)) := \phi(u)$. Assume that $z$ switches the response to chemotaxis concentration $v$ from attractant at low concentrations of $v$ to repellent at high concentrations, namely, $\beta$ is a sign-changing and non-increasing function. (e.g. $\beta(z) = 1 - z/z^*$) \cite{17, 6}. Including cell kinetics and signal dynamics, we derive the resulting model for the cell movement

$$\frac{\partial u}{\partial t} = \frac{D_u \Delta (q(u)u) - \chi_0 \nabla \cdot \left(\phi(u)q(u)u \nabla v\right) + \mu u^\delta (1 - ru)}{\text{dispersion chemotaxis proliferation}}.$$

Incorporating the kinetic equation of ECM and MDE, we arrive at a modified Chaplain and Lolas’ chemotaxis model, see \cite{3}, where we assume the constants $D_u, \chi_0, r = 1$ for simplification.

### 3  Preliminaries: Global existence, boundedness and regularity

As preliminaries, we prove the existence, boundedness and regularity of a global weak solution in this section. The main preliminary results are as follows.
Theorem 3.1 (Existence of globally bounded weak solutions) For $1 \leq n \leq 3$, the problem \((\text{3})\) admits a globally bounded weak solution \((u, v, w, z)\).

Theorem 3.2 (Regularity) Let \((u, v, w, z)\) be a globally bounded weak solution of \((\text{3})\). Then there exist \(\alpha \in (0, 1)\) and \(C(p) > 0\) such that

\[
\|u\|_{L^p(\Omega \times [t, t+1])} + \|v\|_{L^2(\Omega \times [t, t+1])} + \|w\|_{L^p(\Omega \times [t, t+1])} + \|z\|_{W^{2,1}_p(\Omega \times [t, t+1])} \leq C(p),
\]

for any \(p > 1\) and \(t \geq 1\).

We first use the artificial viscosity method to get smooth approximate solutions. Despite the absence of comparison principle, we can prove a special case compared with a lower solution, which is helpful for establishing the regularity estimates. By making use of the special structure of dispersion, we carry on the estimates on \(u^m\) in \(W^{1,2}(Q_T)\), instead of \(u\). These energy estimates ensure the global existence of weak solution.

Consider the following corresponding regularized problem

\[
\begin{cases}
    u_t = \nabla \cdot (m(a_e(u))^{m-1}\nabla u) - \nabla \cdot (u^m \phi(u) \nabla v) + \mu |u|^{\delta-1}u(1-u) + \varepsilon, \\
    v_t = \Delta v + wz, \\
    w_t = -wz, \\
    z_t = \Delta z - z + u, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \\
    (u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), z(x, 0) = z_0(x), \\
    x \in \Omega, t > 0, \\
    x \in \partial \Omega, t > 0, \\
    x \in \Omega,
\end{cases}
\]

where \(\varepsilon \in (0, 1)\), \(a_e \in C^\infty(\mathbb{R})\), \(a_e(s) = s + \varepsilon\) for \(s \geq 0\), \(a_e(s) = \varepsilon/2\) for \(s < -\varepsilon\), \(a_e\) is monotone increasing with \(0 \leq a_\varepsilon^\prime \leq 1\), and \(u_0, v_0, w_0, z_0\) are smooth approximations of \(u_0, v_0, w_0, z_0\), respectively, with

\[
\begin{align*}
    \varepsilon &\leq u_0 \leq u_0 + \varepsilon, \quad 0 \leq v_0 \leq v_0 + \varepsilon, \\
    0 &\leq w_0 \leq w_0 + \varepsilon, \quad 0 \leq z_0 \leq z_0 + \varepsilon, \\
    |\nabla u_0| &\leq 2|\nabla u_0|, \quad |\nabla v_0| \leq 2|\nabla v_0|, \\
    |\nabla w_0| &\leq 2|\nabla w_0|, \quad |\Delta w_0| \leq 2|\Delta w_0|, \quad |\nabla z_0| \leq 2|\nabla z_0|,
\end{align*}
\]

and \(\frac{\partial u_0}{\partial \nu} = 0\) on \(\partial \Omega\). The local existence of the regularized problem \((10)\) is trivial and we denote the unique solution by \((u_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon)\). Let \((0, T_{\max})\) be its maximal existence interval.

As usual, there is no comparison principle for the system, because the system is strongly coupled. However, we have the following lemma.
Lemma 3.1 There holds \( u_e \geq 0, \), \( v_e \geq 0, \), \( w_e \geq 0, \) and \( z_e \geq 0 \) for all \( x \in \Omega \) and \( t \in (0, T_{\text{max}}) \).

**Proof.** We denote \((u_e, v_e, w_e, z_e)\) by \((u, v, w, z)\) in this proof for the sake of simplicity. We argue by contradictions. Since \( u_{0e} \geq \varepsilon > 0 \), there exists \( t_0 \in (0, T_{\text{max}}) \) such that \( u > 0 \) for all \( x \in \Omega \) and \( t \in (0, t_0) \). \( u(x_0, t_0) = 0 \) for some \( x_0 \in \overline{\Omega} \) and \( u(x, t_0) \geq 0 \) for all \( x \in \Omega \).

Now we divide this proof into two parts. If \( x_0 \in \Omega \), then \( \nabla u(x_0, t_0) = 0 \) and

\[
\nabla \cdot (m_a(u))^m \nabla u = m_a(u)^m \Delta u + m(m - 1)a'_e(u)\nabla u + 2 \varepsilon \nabla (u^m - 1) = 0,
\]

which contradicts to \( \partial_t u(x_0, t_0) \leq 0 \).

If \( x_0 \in \partial \Omega \), then \( \partial_t u(x_0, t_0) = 0 \), \( \partial^2_t u(x_0, t_0) \geq 0 \) for any tangent vector \( t \), and the boundary condition shows that \( \partial^2_t u(x_0, t_0) = 0 \). We assert that \( \nabla u(x_0, t_0) \geq 0 \). In fact, if it were not true, Taylor expansion at \((x_0, t_0)\) shows that there would exist a point \( x' \in \Omega \) such that \( u(x', t_0) < 0 \). Therefore, we also have \( \nabla u(x_0, t_0) = 0 \) and the above equalities. Those contradictions complete the proof. ☐

Since \( u_e \geq 0 \), the first equation of (10) is equivalent to

\[
\frac{\partial u}{\partial t} = \Delta (u + \varepsilon)^m - \nabla \cdot (u^m \phi(u) \nabla v) + \mu u^\varepsilon (1 - u) + \varepsilon, \quad u \geq 0.
\]

Now we present some energy estimates independent of time \( t \) and the parameter \( \varepsilon \).

**Lemma 3.2** The first solution component \( u_e \) satisfies

\[
\sup_{t \in (0, T_{\text{max}})} \int_\Omega u_e(\cdot, t) dx \leq \max \left\{ \int_\Omega u_0 dx + |\Omega|, \left( \frac{2(C_1 + |\Omega|)}{\mu C_2} \right)^{1/(\varepsilon + 1)} \right\},
\]

where \( C_1 = \mu^2 |\Omega| \) and \( C_2 = 1/|\Omega|^{\varepsilon} \).

**Proof.** We denote \( u_e \) by \( u \) in this proof for the sake of simplicity. Since \( u \) is nonnegative and \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} = 0 \) on \( \partial \Omega \), integration of the first equation of (10) over \( \Omega \) yields

\[
\frac{d}{dt} \int_\Omega u dx \leq \mu \int_\Omega u^\varepsilon dx - \mu \int_\Omega u^{\varepsilon + 1} dx + |\Omega|,
\]

for all \( t \in (0, T_{\text{max}}) \). We note that

\[
\mu \int_\Omega u^\varepsilon dx \leq \frac{1}{2} \mu \int_\Omega u^{\varepsilon + 1} dx + C_1.
\]

and

\[
\int_\Omega u^{\varepsilon + 1} dx \geq C_2 \left( \int_\Omega u dx \right)^{\varepsilon + 1},
\]
where \( C_1 = \mu 2^\delta |\Omega| \) and \( C_2 = 1/|\Omega|^\delta \). Let \( y(t) = \int_\Omega u(\cdot, t)dx \) for \( t \in [0, T_{\text{max}}] \). We find
\[
y'(t) \leq C_1 + |\Omega| - \frac{\mu C_2}{2} y^{\delta + 1}(t).
\]
The comparison principle of ODE shows that
\[
y(t) \leq \max \left\{ y(0), \left( \frac{2(C_1 + |\Omega|)}{\mu C_2} \right)^{1/(\delta + 1)} \right\}
\]
for all \( t \in (0, T_{\text{max}}) \).

Here we recall some lemmas about the \( L^p-L^q \) type estimates for the components of the solution, and we refer the readers to [5] for details.

**Lemma 3.3** ([5]) Let \( p \geq 1 \) and
\[
\begin{cases}
q \in [1, \frac{np}{n-2p}], & p \leq \frac{n}{2}, \\
q \in [1, \infty], & p > \frac{n}{2}.
\end{cases}
\]
Then for any \( T \in (0, T_{\text{max}}] \), there exists a constant \( C_z(p, q) \) such that
\[
\sup_{t \in (0, T)} \| z_\varepsilon(\cdot, t) \|_{L^q(\Omega)} \leq C_z(p, q)(\| z_0 \|_{L^q(\Omega)} + \sup_{t \in (0, T)} \| u_\varepsilon(\cdot, t) \|_{L^p(\Omega)}).
\]

**Lemma 3.4** ([5]) Let \( q \geq 1 \) and
\[
\begin{cases}
r \in [1, \frac{nq}{n-q}], & q \leq n, \\
r \in [1, \infty], & q > n.
\end{cases}
\]
Then for any \( T \in (0, T_{\text{max}}] \), there exists a constant \( C_v(q, r) \) such that
\[
\sup_{t \in (0, T)} \| \nabla v_\varepsilon(\cdot, t) \|_{L^r(\Omega)} \leq C_v(q, r)(\| \nabla v_0 \|_{L^r(\Omega)} + \sup_{t \in (0, T)} \| z_\varepsilon(\cdot, t) \|_{L^q(\Omega)}).
\]

**Lemma 3.5** There holds
\[
\| w_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq \| w_0 \|_{L^\infty(\Omega)} + 1, \quad t \in (0, T_{\text{max}}),
\]
and
\[
\int_\Omega v_\varepsilon(x, t)dx \leq \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx + 2|\Omega|, \quad t \in (0, T_{\text{max}}).
\]

**Proof.** Since both \( w_\varepsilon \) and \( z_\varepsilon \) are nonnegative, it is clear from the third equation of (10) that
\[
|w_\varepsilon(x, t)| \leq w_{0\varepsilon}(x, t) \leq \| w_0 \|_{L^\infty(\Omega)} + 1.
\]
We add the third to the second equation of (10) and integrate over $\Omega$ to obtain
\[
\frac{d}{dt} \int_{\Omega} (v_\varepsilon + w_\varepsilon) dx = \int_{\Omega} \Delta v_\varepsilon dx = 0, \quad t \in (0, T_{\text{max}}).
\]
Thus,
\[
\int_{\Omega} (v_\varepsilon + w_\varepsilon) dx \leq \int_{\Omega} v_{0\varepsilon}(x) dx + \int_{\Omega} w_{0\varepsilon}(x) dx \leq \int_{\Omega} v_0(x) dx + \int_{\Omega} w_0(x) dx + 2|\Omega|,
\]
for all $t \in (0, T_{\text{max}})$. \hfill \Box

**Lemma 3.6** Let $1 \leq n \leq 3$. There exists a constant $C$ independent of $t$ and $\varepsilon$ such that
\[
\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C, \quad t \in (0, T_{\text{max}}).
\]
For any $r \geq 1$, there exists a constant $C(r)$ independent of $t$ and $\varepsilon$ such that
\[
\|\nabla v_\varepsilon\|_{L^r(\Omega)} \leq C(r), \quad t \in (0, T_{\text{max}}).
\]

**Proof.** According to Lemma 3.2, $\|u_\varepsilon\|_{L^1(\Omega)}$ is uniformly bounded. Since $n \leq 3$, we can apply Lemma 3.3 and 3.4 to complete this proof. \hfill \Box

The following Gagliardo-Nirenberg inequality (see [27, 33]) will be used in deriving the $L^p$ estimates of $u_\varepsilon$.

**Lemma 3.7** Let $0 < s \leq p \leq \frac{2p}{(n-2)+p}$. There exists a positive constant $C$ such that for all $u \in W^{1,2}(\Omega) \cap L^s(\Omega),$
\[
\|u\|_{L^p(\Omega)} \leq C(\|\nabla u\|_{L^2(\Omega)}^a \|u\|_{L^s(\Omega)}^{1-a} + \|u\|_{L^s(\Omega)}),
\]
is valid with $a = \frac{n(s-n/p)}{1-n/2p} \in (0, 1)$.

We present the following $L^p$ estimate of $u_\varepsilon$.

**Lemma 3.8** Let $1 \leq n \leq 3$. For any given $p \geq 1$, there exists a constant $C(p) > 0$ independent of $t$ and $\varepsilon$ such that
\[
\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C(p), \quad t \in (0, T_{\text{max}}).
\]

**Proof.** We denote $u_\varepsilon, v_\varepsilon$ by $u, v$ in this proof for the sake of simplicity. By a straightforward computation, testing the first equation in (10) by $u^r$ for $r > 0$ and integrating by parts we find that
\[
\frac{1}{r+1} \frac{d}{dt} \int_{\Omega} u^{r+1} dx + \int_{\Omega} \nabla(u \varepsilon)^m \cdot \nabla u dx 
\leq \int_{\Omega} u^m \phi(u) \nabla v \cdot \nabla u dx + \mu \int_{\Omega} u^{\delta r} dx - \mu \int_{\Omega} u^{\delta + r+1} dx + \int_{\Omega} u^r dx. \quad (11)
\]
We note that
\[
\mu \int_{\Omega} u^{\delta r} dx \leq \frac{1}{4} \mu \int_{\Omega} u^{\delta + r+1} dx + C_1, \quad (12)
\]
and
\[ \int_{\Omega} u' dx \leq \frac{1}{4} \mu \int_{\Omega} u^{\delta + \epsilon} dx + C_2. \]  
(13)

where \( C_1 \) and \( C_2 \) are constants independent of \( t \). Then by Young’s inequality, we see that
\[ \int_{\Omega} u^m \phi(u) \nabla \cdot \nabla u' dx \leq r \int_{\Omega} u^{m + r - 1} |\nabla u|^2 dx \]
\[ \leq \frac{mr}{2} \int_{\Omega} (u + \epsilon)^{m-1} u'^{-1} |\nabla u|^2 dx + \frac{r}{2m} \int_{\Omega} u^{m+r} |\nabla v|^2 dx \]
\[ \leq \frac{1}{2} \int_{\Omega} \nabla (u + \epsilon)^m \cdot \nabla u' dx + \frac{r}{2m} \int_{\Omega} u^{m+r} |\nabla v|^2 dx. \]  
(14)

We use Hölder’s inequality to see that
\[ \frac{r}{2m} \int_{\Omega} u^{m+r} |\nabla v|^2 dx \leq \frac{r}{2m} \left( \int_{\Omega} u^{m+r+k} dx \right)^{\frac{m+k}{m+r}} \left( \int_{\Omega} |\nabla v|^{\frac{2(m+k)}{m}} dx \right)^{\frac{m}{m+r}} \]
\[ \leq C_3 \left( \int_{\Omega} u^{m+r+k} dx \right)^{\frac{m+k}{m+r}} \]
where \( \kappa > 0 \) is a constant to be determined and \( C_3 \) is a constant depending on the \( L^{\frac{2(m+k)}{m}} (\Omega) \) norm of \( \nabla v \) which is uniformly bounded according to Lemma 3.6. Now we use the Gagliardo-Nirenberg inequality Lemma 3.7 to obtain
\[ \left( \int_{\Omega} u^{m+r+k} dx \right)^{\frac{m+k}{m+r+k}} = \| u^{\frac{m+k}{m+r+k}} \|_{L^{\frac{2(m+k)}{m}}(\Omega)} \]
\[ \leq C_4 \| \nabla u^{\frac{m}{m+k}} \|_{L^2(\Omega)}^{2} \| u^{\frac{m+k}{m}} \|_{L^{\frac{2(m+k)}{m}}(\Omega)}^{2(1-a)} + \| u^{\frac{m+k}{m}} \|_{L^{\frac{2(m+k)}{m}}(\Omega)}^{2(1-a)} \]
\[ \leq C_5 (1 + \| \nabla u^{\frac{m}{m+k}} \|_{L^2(\Omega)}^{2} + \| u^{\frac{m+k}{m}} \|_{L^{\frac{2(m+k)}{m}}(\Omega)}^{2(1-a)}), \]
where \( C_4 \) is a constant, \( C_5 \) depends on \( \| u \|_{L^1(\Omega)} \), and

\[ a = \frac{n(m+r)/2 - n(m+r)/(2(m+r+k))}{1 - n/2 + n(m+r)/2} \in (0, 1), \]
provided that \( \frac{2(m+r+k)}{m+r} < \frac{2a}{(n-2)} \). This can be done by taking \( \kappa > 0 \) and appropriately small. Therefore, we have
\[ \frac{r}{2m} \int_{\Omega} u^{m+r} |\nabla v|^2 dx \leq C_3 C_5 (1 + \| \nabla u^{\frac{m}{m+k}} \|_{L^2(\Omega)}^{2}) \]
\[ \leq \frac{2mr}{(m+r)^2} \| u^{\frac{m+k}{m}} \|_{L^2(\Omega)}^{2} + C_6 \]
\[ \leq \frac{mr}{2} \int_{\Omega} (u + \epsilon)^{m-1} u'^{-1} |\nabla u|^2 dx + C_6 \]
\[ \leq \frac{1}{2} \int_{\Omega} \nabla (u + \epsilon)^m \cdot \nabla u' dx + C_6. \]  
(15)
since $a \in (0, 1)$. Combining (12), (13), (14), (15) with (11), we infer that
\[
\frac{d}{dt} \int_\Omega u^{r+1} dx \leq -\frac{\mu(r+1)}{2} \int_\Omega u^{\delta+r+1} dx + (r+1)(C_1 + C_2 + C_6).
\]
According to
\[
\int_\Omega u^{\delta+r+1} dx \geq \frac{1}{|\Omega|^{\frac{1}{n}}} \left( \int_\Omega u^{r+1} dx \right)^{\frac{\delta+r+1}{r+1}},
\]
we obtain
\[
\frac{d}{dt} \int_\Omega u^{r+1} dx \leq (r+1)(C_1 + C_2 + C_6) - \frac{\mu(r+1)}{2|\Omega|^{\frac{1}{n}}} \left( \int_\Omega u^{r+1} dx \right)^{\frac{\delta+r+1}{r+1}}.
\]
By an ODE comparison,
\[
\int_\Omega u^{r+1} dx \leq \max \left\{ \int_\Omega (u_0 + 1)^{r+1} dx, \left( \frac{2(C_1 + C_2 + C_6)|\Omega|^{\frac{1}{n}}}{\mu} \right)^{\frac{\delta+r+1}{r+1}} \right\}
\]
for all $t \in (0, T)$.  

**Lemma 3.9** Let $1 \leq n \leq 3$. There exists a constant $C > 0$ independent of $T_{\max}$ and $\varepsilon$ such that
\[
\sup_{t \in (0, T_{\max})} \| \nabla v_\varepsilon \|_{L^\infty(\Omega)} \leq C.
\]

**Proof.** According to Lemma 3.8, $\| u_\varepsilon \|_{L^{r+1}(\Omega)}$ is uniformly bounded. We can apply Lemma 3.3 and Lemma 3.4 to obtain the boundedness of $\| \nabla v_\varepsilon \|_{L^\infty(\Omega)}$.  

We now employ the following Moser-type iteration to get the $L^\infty(\Omega)$ estimate of $u$.  

**Lemma 3.10** Let $1 \leq n \leq 3$. There exists a constant $C > 0$ independent of $T_{\max}$ and $\varepsilon$ such that
\[
\sup_{t \in (0, T_{\max})} \| u_\varepsilon \|_{L^\infty(\Omega)} \leq C.
\]

**Proof.** We denote $u_\varepsilon, v_\varepsilon$ by $u, v$ in this proof for the sake of simplicity. We test the first equation in (10) by $u^r$ for $r > 0$ and integrating by parts we find that
\[
\frac{1}{r + 1} \frac{d}{dt} \int_\Omega u^{r+1} dx + \int_\Omega \nabla (u + \varepsilon)^m \cdot \nabla u^r dx
\leq \int_\Omega u^m \phi(u) \nabla v \cdot \nabla u^r dx + \mu \int_\Omega u^{\delta+r} dx - \mu \int_\Omega u^{\delta+r+1} dx + \int_\Omega u^r dx.
\]
Similar to the proof of Lemma 3.8 using Young’s inequality we can estimate
\[
\mu \int_\Omega u^{\delta+r} dx \leq \frac{1}{4} \mu \int_\Omega u^{\delta+r+1} dx + 4^{\delta+r} \mu |\Omega|, \\
\int_\Omega u^r dx \leq \frac{1}{4} \mu \int_\Omega u^{\delta+r+1} dx + \left( \frac{4}{\mu} \right)^{\frac{1}{\delta+r}} |\Omega|,
\]

...
where according to Lemma $3.9$ $∥\nabla v∥_{L^\infty(\Omega)}$ is uniformly bounded. Now we apply the Gagliardo-Nirenberg inequality Lemma $3.7$ to obtain

$$\int_\Omega u^{m+r} \, dx = ||u^{\frac{mr}{m+r}}||_{L^2(\Omega)}^2 \leq C_0(\|\nabla u^{\frac{2a}{m+r}}||_{L^2(\Omega)}^2 ||u^{\frac{2a}{m+r}}||_{L^2(\Omega)}^{2(1-a)} + ||u^{\frac{m}{m+r}}||_{L^2(\Omega)}^2,$$

where $a = n/(n+2) \in (0, 1)$ and $C_0$ is the constant in the Gagliardo-Nirenberg inequality which is independent of $r$. Therefore, we have

$$\frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} \int_\Omega u^{m+r} \, dx \leq \frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} C_0(\|\nabla u^{\frac{2a}{m+r}}||_{L^2(\Omega)}^2 ||u^{\frac{2a}{m+r}}||_{L^2(\Omega)}^{2(1-a)} + ||u^{\frac{m}{m+r}}||_{L^2(\Omega)}^2)$$

$$\leq \frac{mr}{(m+r)^2} ||\nabla u^{\frac{m}{m+r}}||^2_{L^2(\Omega)} + \left( \frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} C_0 \right)^{\frac{1}{2}} \left( \frac{(m+r)^2}{mr} \right)^{\frac{1}{2}} ||u^{\frac{m}{m+r}}||^2_{L^2(\Omega)}$$

$$+ \frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} C_0 ||u^{\frac{m}{m+r}}||^2_{L^2(\Omega)} \leq \frac{1}{4} \int_\Omega \nabla (u + \varepsilon)^m \cdot \nabla u' \, dx + C_1(r) ||u^{\frac{m}{m+r}}||^2_{L^1(\Omega)},$$

where

$$C_1(r) = \left( \frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} C_0 \right)^{\frac{1}{2}} \left( \frac{(m+r)^2}{mr} \right)^{\frac{1}{2}} + \frac{r}{m} ||\nabla v||^2_{L^2(\Omega)} C_0.$$

Inserting the above estimates (17), (18) into (16) yields

$$\frac{d}{dt} \int_\Omega u^{r+1} \, dx + \int_\Omega u^{r+1} \, dx \leq C_1(r)(r+1)||u^{\frac{m}{m+r}}||^2_{L^1(\Omega)} + (r+1)(4^6 \mu \Omega) + \left( \frac{4}{\mu} \right)^{\frac{mr}{m+r}} \Omega)$$

$$+ \int_\Omega u^{r+1} \, dx - \frac{1}{2} \mu \int_\Omega u^{\delta+r+1} \, dx \leq C_1(r)(r+1)||u^{\frac{m}{m+r}}||^2_{L^1(\Omega)} + C_2(r),$$

where

$$C_2(r) = (r+1)\left( 4^6 \mu \Omega + \left( \frac{4}{\mu} \right)^{\frac{mr}{m+r}} \Omega \right) + \left( \frac{2}{\mu} \right)^{\frac{mr}{m+r}} \Omega.$$
Therefore, we can rewrite (22) into
\[ (20) \] into
\[ j \]
\[ \text{We can invoke Lemma 3.8 to find } C_0 \text{ such that} \]
\[ \sup_{t \in (0,T_{\max})} \| u \|_{L^\infty_t(\Omega)} \leq C_0. \]
From (19) and an ODE comparison, we have
\[ \sup_{t \in (0,T_{\max})} \| u \|_{L^{j+1}_t(\Omega)}^{j+1} \leq \max \left\{ \int_\Omega (u_0 + 1)^{j+1} dx, C_1(r_j)(r_j + 1) \cdot \sup_{t \in (0,T_{\max})} \| u \|_{L^{j+1}_t(\Omega)}^{2(r_j+1)} + C_2(r_j) \right\}. \] (20)
A simple analysis shows that \( C_1(r)(r+1) \leq a_1 b_1 \) and \( C_2(r) \leq a_2 b_2^r \) for some positive constants \( a_1, a_2 \) and \( b_1, b_2 \) that all are greater than 1 and independent of \( r \). Therefore, we can rewrite the above inequality (20) into
\[ \sup_{t \in (0,T_{\max})} \| u \|_{L^{j+1}_t(\Omega)}^{j+1} \leq \max \left\{ \int_\Omega (u_0 + 1)^{j+1} dx, a_1 b_1 r_j \cdot \sup_{t \in (0,T_{\max})} \| u \|_{L^{j+1}_t(\Omega)}^{2(r_j+1)} + a_2 b_2^r \right\}. \] (21)
Let
\[ M_j = \max \left\{ \sup_{t \in (0,T_{\max})} \int_\Omega u^{j+1} dx, 1 \right\}. \]
Since boundedness of \( u \) in \( L^\infty(\Omega) \) is evident in the case when \( M_j \leq \max\{ \int_\Omega (u_0 + 1)^{j+1} dx, 1 \} \) for infinitely many \( j \geq 1 \), we may assume that \( M_j \geq \max\{ \int_\Omega (u_0 + 1)^{j+1} dx, 1 \} \) and thus, according to (21), there holds
\[ M_j \leq a_1 r_j M_{j-1}^2 + a_2 b_2^r. \] (22)
We note that if \( M_{j-1}^2 \leq a_2 b_2^r \) for infinitely many \( j \geq 1 \), then
\[ M_{j-1}^{-1} \leq (a_2 b_2^r)^{-1} \leq a_2^{-1} b_2^{-r} \leq 2 b_2, \]
for \( j \) sufficiently large, which shows the boundedness of \( u \) in \( L^\infty(\Omega) \). Otherwise, \( M_{j-1}^2 \geq a_2 b_2^r \) except for a finite number of \( j \geq 1 \). Thus, there exists a \( j_0 \geq 1 \) such that
\[ M_{j-1}^2 \geq a_2 b_2^r, \quad j \geq j_0. \]
Therefore, we can rewrite (22) into
\[ M_j \leq 2 a_1 r_j M_{j-1}^2 \leq D^j M_{j-1}^2 \] (23)
for all \( j \geq j_0 \) with a constant \( D \) independent of \( j \), whence upon enlarge \( D \) if necessary we can achieve that (23) actually holds for all \( j \geq 1 \). By introduction, this yields

\[
M_j \leq D^{(j-i)/2 - 1} M_1^{2j-1} = D^{2j-1} M_1^{2j-1} \leq D^{2j+1} M_1^{2j-1}
\]

for all \( j \geq 1 \), and hence that

\[
M_j^{j+1} \leq D^{j/2} M_0^{j/2} \leq D^2 M_1,
\]

for all \( j \geq 1 \). This implies that \( u \) indeed belongs to \( L^\infty(\Omega \times (0, T_{\max})) \).

Now we turn to the regularity estimates.

**Lemma 3.11**  Let \( 1 \leq n \leq 3 \). Then there exists a constant \( C \) independent of \( t \) and \( \varepsilon \) such that

\[
\sup_{t \in (0, T_{\max})} (\|v\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} + \|\nabla v_0\|_{L^\infty(\Omega)}) \leq C.
\]

And the third solution component \( w_0 \) fulfills

\[
\|\nabla w_0\|_{L^\infty(\Omega)} \leq 2\|\nabla w_0\|_{L^\infty(\Omega)} + (\|w_0\|_{L^\infty(\Omega)} + 1) \sup_{t \in (0, T_{\max})} \|\nabla w_0\|_{L^\infty(\Omega)} t, \quad t \in (0, T_{\max}).
\]

**Proof.** According to Lemma 8.10, Lemma 8.3, Lemma 8.4 we see that \( \|u_0\|_{L^\infty(\Omega)}, \|z_0\|_{L^\infty(\Omega)}, \|\nabla v_0\|_{L^\infty(\Omega)} \) are uniformly bounded in \( (0, T_{\max}) \). The standard \( L^p - L^q \) type estimates also shows the boundedness of \( \|v_0\|_{L^\infty(\Omega)} \) and \( \|\nabla v_0\|_{L^\infty(\Omega)} \). We denote \( v, w, z \) by \( v, w, z \) in this proof for the sake of simplicity. Since both \( w \) and \( z \) are nonnegative according to the third and fourth equation in (10) and the initial data, we have

\[
w(x, t) = w_0(x) e^{-\frac{1}{\varepsilon} \int_0^t z(x, \tau) d\tau},
\]

\[
\nabla w(x, t) = \nabla w_0(x) e^{-\frac{1}{\varepsilon} \int_0^t z(x, \tau) d\tau} - w_0(x) e^{-\frac{1}{\varepsilon} \int_0^t z(x, \tau) d\tau} \int_0^t \nabla z(x, \tau) d\tau.
\]

Therefore,

\[
|\nabla w(x, t)| \leq |\nabla w_0(x, t)| + w_0(x) \sup_{t \in (0, T_{\max})} \|\nabla z\|_{L^\infty(\Omega)} t
\]

\[
\leq 2\|\nabla w_0\|_{L^\infty(\Omega)} + (\|w_0\|_{L^\infty(\Omega)} + 1) \sup_{t \in (0, T_{\max})} \|\nabla z\|_{L^\infty(\Omega)} t.
\]

This completes the proof.

**Lemma 3.12**  There exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( T \), such that

\[
\int_0^T \int_\Omega |\Delta v_0|^2 dx dt \leq C(1 + T^2), \quad T \in (0, T_{\max}).
\]
\textbf{Proof.} We denote \( v_\varepsilon, w_\varepsilon, z_\varepsilon \) by \( v, w, z \) in this proof for the sake of simplicity. Multiplying the second equation in (10) by \(-\Delta v\) and integrating over \( \Omega \) yields
\[
\int_\Omega \frac{\partial}{\partial t} |\nabla v|^2 \, dx + \int_\Omega |\Delta v|^2 \, dx = \int_\Omega \nabla v \cdot \nabla (wz) \, dx \leq C \left( \int_\Omega |\nabla w|^2 \, dx + 1 \right) \leq C(1 + t),
\]
since \( \nabla v, z \) and \( \nabla z \) are uniformly bounded in \( L^\infty(\Omega) \) according to Lemma 3.11. Integrating over \((0, T)\), we complete this proof. \( \square \)

\textbf{Lemma 3.13} There exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( T \), such that
\[
\int_0^T \int_\Omega |\nabla u_\varepsilon^m|^2 \, dx \, dt \leq C(1 + T), \quad T \in (0, T_{\text{max}}).
\]

\textbf{Proof.} We denote \( u_\varepsilon, v_\varepsilon \) by \( u, v \) in this proof for the sake of simplicity. We test the first equation in (10) by \((u + \varepsilon)^m\) and get
\[
\frac{1}{m + 1} \frac{d}{dt} \int_\Omega (u + \varepsilon)^{m+1} \, dx + \int_\Omega |\nabla(u + \varepsilon)^m|^2 \, dx
\leq \int_\Omega u^m \phi(u) \nabla v \cdot \nabla (u + \varepsilon)^m \, dx + \mu \int_\Omega u^\delta (u + \varepsilon)^m \, dx
\quad - \mu \int_\Omega u^{\delta+1} (u + \varepsilon)^m \, dx + \int_\Omega (u + \varepsilon)^m \, dx. \tag{24}
\]
According to Lemma 3.9 and Lemma 3.10, \( \nabla v \) and \( u \) are uniformly bounded. Thus,
\[
\int_\Omega u^m \phi(u) \nabla v \cdot \nabla (u + \varepsilon)^m \, dx \leq \frac{1}{2} \int_\Omega |\nabla(u + \varepsilon)^m|^2 \, dx + C_1,
\]
where \( C_1 \) is a constant independent of \( t \) and \( \varepsilon \). Integrating (24) on \((0, T)\) yields
\[
\int_\Omega (u + \varepsilon)^{m+1} \, dx + \int_0^T \int_\Omega |\nabla(u + \varepsilon)^m|^2 \, dx \leq \int_\Omega (u_{0\varepsilon} + \varepsilon)^{m+1} \, dx + CT. \tag{25}
\]
We note that
\[
|\nabla u^m| = m(u + \varepsilon)^{m-1}|\nabla u| \leq m(u + \varepsilon)^{m-1} |\nabla(u + \varepsilon)| = |\nabla(u + \varepsilon)^m|.
\]
This completes the proof. \( \square \)

\textbf{Lemma 3.14} There exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( T \), such that
\[
\int_0^T \int_\Omega \left( |\nabla u_\varepsilon^{m+1}|^2 \right) \, dx \, dt + \int_\Omega \left| \nabla u_\varepsilon^m \right|^2 \, dx \leq C(1 + T^2), \quad T \in (0, T_{\text{max}}).
\]
Moreover,
\[
\int_0^T \int_\Omega \left( |u_\varepsilon^{m+1}|^2 \right) \, dx \, dt \leq \frac{4m^2}{(m + 1)^2} ||u_{\varepsilon}||_{L^\infty(\Omega)} ||u_{\varepsilon}||_{L^m(\Omega)} \int_0^T \int_\Omega \left( |u_\varepsilon^{m+1}|^2 \right) \, dx \, dt \leq C(1 + T^2), \quad T \in (0, T_{\text{max}}).
\]
Proof. We denote $u_\varepsilon, v_\varepsilon$ by $u, v$ in this proof for the sake of simplicity. We multiply the first equation in (10) by $[(u + \varepsilon)^m]$, and then we have

$$
\int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + \int_\Omega \nabla(u + \varepsilon)^m \cdot \nabla[(u + \varepsilon)^m] \, dx \\
\leq \int_\Omega u^m \phi(u) \nabla v \cdot \nabla[(u + \varepsilon)^m] \, dx + \mu \int_\Omega u^\delta[(u + \varepsilon)^m] \, dx \\
- \mu \int_\Omega u^{\delta+1}[(u + \varepsilon)^m] \, ds + \int_\Omega |(u + \varepsilon)^m| \, dx. \tag{26}
$$

We note that $\|u\|_{L^\infty(\Omega)}$ is uniformly bounded and then

$$
\int_\Omega \mu \phi^\delta[(u + \varepsilon)^m] \, dx = \int_\Omega m u \phi^\delta(u + \varepsilon)^{m-1} u_t \, dx \leq \frac{1}{5} \int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + C_1, \\
\int_\Omega -\mu u^{\delta+1}[(u + \varepsilon)^m] \, dx = -\int_\Omega m u \phi^\delta(u + \varepsilon)^{m-1} u_t \, dx \leq \frac{1}{5} \int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + C_2, \\
\int_\Omega |(u + \varepsilon)^m| \, dx = \int_\Omega m(u + \varepsilon)^{m-1} u_t \, dx \leq \frac{1}{5} \int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + C_3,
$$

where $C_1, C_2, C_3$ are constants independent of $t$ and $\varepsilon$. We also have

$$
\int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx = \frac{4m}{(m+1)^2} \int_\Omega \left|\left((u + \varepsilon)^{m+1}\right)_t\right|^2 \, dx,
$$

and

$$
\int_\Omega \nabla(u + \varepsilon)^m \cdot \nabla[(u + \varepsilon)^m] \, dx = \frac{1}{2} \frac{\partial}{\partial t} \int_\Omega \left|\nabla(u + \varepsilon)^m\right|^2 \, dx.
$$

There holds

$$
\int_\Omega u^m \phi(u) \nabla v \cdot \nabla[(u + \varepsilon)^m] \, dx = -\int_\Omega [(u + \varepsilon)^m]_t \nabla \cdot (u^m \phi(u) \nabla v) \, dx \\
= -\int_\Omega m(u + \varepsilon)^{m-1} u_t \cdot (m u^{m-1} \phi(u) \nabla u + u^m \phi'(u) \nabla v + u^n \phi(u) \nabla v) \, dx \\
\leq \frac{1}{5} \int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + C_4 \int_\Omega (u + \varepsilon)^{2(m-1)}|\nabla u|^2 \, dx + C_5 \int_\Omega |\nabla v|^2 \, dx \\
\leq \frac{1}{5} \int_\Omega m(u + \varepsilon)^{m-1}|u_t|^2 \, dx + C_4 \int_\Omega |\nabla(u + \varepsilon)^m|^2 \, dx + C_5 \int_\Omega |\nabla v|^2 \, dx,
$$

where $C_4$ and $C_5$ are constants independent of $t$ and $\varepsilon$, since the uniform boundedness of $\|\nabla v\|_{L^\infty(\Omega)}$. Inserting the above inequalities into (26), and noticing the inequality (25) in the proof of Lemma 3.13 we find a constant $C$ independent of $t$ and $\varepsilon$ such that

$$
\int_0^T \int_\Omega \left|\left((u + \varepsilon)^{m+1}\right)_t\right|^2 \, dx dt + \int_\Omega \left|\nabla(u + \varepsilon)^m\right|^2 \, dx \leq \int_\Omega \left|\nabla(u_0 \varepsilon) + \varepsilon\right|^2 \, dx + C(1 + T^2) \leq C(1 + T^2).
$$
Clearly, we have
\[
\left(\frac{m+1}{4}\right)^2 u^{m-1} |u_t|^2 \leq \frac{(m+1)^2}{4} (u + \varepsilon)^{m-1} |u_t|^2 = \left(\frac{(u + \varepsilon)^{m+1}}{4}\right)^2,
\]
and
\[
|(u^m)_t|^2 \leq \frac{4m^2}{(m+1)^2} |u_{e}|^{m-1} \left(\frac{(u + \varepsilon)^{m+1}}{4}\right)^2 \leq \frac{4m^2}{(m+1)^2} |u_{e}|^{m-1} \left(\frac{(u + \varepsilon)^{m+1}}{4}\right)^2.
\]

The proof is completed. \(\square\)

**Proof of Theorem 3.7** According to the estimates, for any \(\varepsilon\), the approximation solution \((u_{e}, v_{e}, w_{e}, z_{e})\) exists globally. The regularity estimates of \(v_{e}, w_{e}, z_{e}\) are trivial. For any \(T \in (0, \infty)\), we see that \(u_{e}^{m} \in L^{\infty}(Q_{T}), \nabla u_{e}^{m} \in L^{2}(Q_{T}), \) and \(\partial_{t} u_{e}^{m} \in L^{2}(Q_{T})\), Thus, there exists a function \(\bar{u} \in W^{1,2}(Q_{T})\), such that \(u_{e}^{m}\) weakly in \(W^{1,2}(Q_{T})\) and strongly in \(L^{2}(Q_{T})\) converges to \(\bar{u}\). We denote \(u = \bar{u}^{1/m}\) since \(\bar{u} \geq 0\). Thus, \(u_{e}^{m}\) converges almost everywhere to \(u^{m}\), and \(u_{e}\) converges almost everywhere to \(u\). We can verify the integral identities in the definition of weak solutions. By taking a sequence of \(T \in (0, \infty)\) and the diagonal subsequence procedure, we can find the existence of a global weak solution. \(\square\)

Now we show the regularity of the globally bounded weak solution.

**Lemma 3.15** Let \((u, v, w, z)\) be a globally bounded weak solution of \((3)\) such that \(\|u(\cdot, t)\|_{L^{2}(\Omega)}\) is uniformly bounded with \(\gamma = \max\{1, n/3\}\). Then there exists a constant \(C\) such that
\[\sup_{t \in \mathbb{R}^{+}} \left\{ \|u\|_{L^{\infty}(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} + \|w\|_{L^{\infty}(\Omega)} + \|z\|_{W^{1,\infty}(\Omega)} \right\} \leq C.\]

**Proof.** Since \(\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2}\) is uniformly bounded, for any \(r \geq 1\) we can apply Lemma 3.3 and 3.4 to find a constant \(C(r)\) independent of \(t\) such that \(\|\nabla v(\cdot, t)\|_{L^{r}(\Omega)} \leq C(r)\) for all \(t > 0\). The estimates in the proof of Theorem 3.1 in section 3 can be carried on to complete this proof here. \(\square\)

In Lemma 3.11 we have proved \(\|\nabla w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C(1 + t)\) (same as \(\nabla w_{e}\)) for some constant \(C > 0\). However that is an estimate depending on time \(t\). Employing the method in the proof of Lemma 3.4 in next Section and iteration technique, we can prove the following uniform estimate.

**Lemma 3.16** Let \((u, v, w, z)\) be a globally bounded weak solution of \((3)\). Then for any \(p \geq 1\) there holds
\[\int_{\Omega} |\nabla w(\cdot, t)|^{p} dx \leq C(p), \quad t > 0,
\]
for some constant \(C(p)\) independent of time \(t\).

**Proof.** This proof proceeds along the idea of the arguments of Lemma 4.3 in \([23]\) and Lemma 4.1 in \([23]\). Since
\[w(x, t) = w_{0}(x)e^{-\int_{0}^{t} v(x, s) ds},\]
and
\[ \nabla w(x, t) = \nabla w_0(x)e^{-\int_0^t \zeta(x, s)ds} - w_0(x)e^{-\int_0^t \zeta(x, s)ds} \int_0^t \nabla z(x, s)ds. \]

We see that
\[ |\nabla w(x, t)|^2 \leq 2|\nabla w_0(x)|^2 e^{-2\int_0^t \zeta(x, s)ds} + 2|w_0(x)|^2 e^{-2\int_0^t \zeta(x, s)ds} \int_0^t |\nabla z(x, s)ds|^2. \]

And thus
\[
\int_\Omega |\nabla w(x, t)|^2 dx \leq C + C \int_\Omega e^{-2 \int_0^t \zeta(x, s)ds} \left( \int_0^t |\nabla z(x, s)ds|^2 \right) dx
\leq C - \frac{C}{2} \int_\Omega \nabla e^{-2 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t \nabla z(x, s)ds \right) dx
\leq C + \frac{C}{2} \int_\Omega e^{-2 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t \Delta z(x, s)ds \right) dx
\leq C + \frac{C}{2} \int_\Omega e^{-2 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t (z_t + z - u)ds \right) dx
\leq C + \frac{C}{2} \int_\Omega e^{-2 \int_0^t \zeta(x, s)ds} \cdot (z(x, t) + \int_0^t z(x, s)ds) dx
\leq C.
\]

Using the same method, we have
\[ |\nabla w(x, t)|^4 \leq 2^3|\nabla w_0(x)|^4 e^{-4\int_0^t \zeta(x, s)ds} + 2^3|w_0(x)|^4 e^{-4\int_0^t \zeta(x, s)ds} \int_0^t |\nabla z(x, s)ds|^4, \]

and
\[
\int_\Omega |\nabla w(x, t)|^4 dx \leq C + C \int_\Omega e^{-4 \int_0^t \zeta(x, s)ds} \left( \int_0^t |\nabla z(x, s)ds|^4 \right) dx
\leq C - \frac{C}{4} \int_\Omega \nabla e^{-4 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t |\nabla z(x, s)ds|^3 \right)^2 dx
\leq C + \frac{3C}{4} \int_\Omega e^{-4 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t |\nabla z(x, s)ds|^2 \right)^2 \cdot \left( \int_0^t \Delta z(x, s)ds dx \right)
\leq C + \frac{3C}{4} \int_\Omega e^{-4 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t |\nabla z(x, s)ds|^2 \right)^2 \cdot \left( \int_0^t (z_t + z - u)ds \right) dx
\leq C + \frac{3C}{4} \int_\Omega e^{-4 \int_0^t \zeta(x, s)ds} \cdot \left( \int_0^t |\nabla z(x, s)ds|^2 \right)^2 \cdot (z(x, t) + \int_0^t z(x, s)ds) dx
\leq C + \frac{3C}{4} \int_\Omega e^{-2 \int_0^t \zeta(x, s)ds} \cdot \left( z(x, t) + \int_0^t z(x, s)ds \right) dx
\leq C,
\]
according to the proof of the previous estimate on $\|\nabla w\|_{L^2(\Omega)}$ and the boundedness of $\|z\|_{L^\infty(\Omega)}$. Repeating this process for $\|\nabla w\|_{L^k(\Omega)}$ with $k = 6, 8, \ldots$, we complete this proof by iteration. □

Proof of Theorem 3.2 Lemma 3.16 shows the uniform boundedness of $\|\nabla w\|_{L^n(\Omega)}$. According to the third equation of (3), we see that $\|w_t\|_{L^\infty(\Omega)} = \|wz\|_{L^\infty(\Omega)} \leq C$. Therefore, $w \in W^{1,n+2}(\Omega \times (t, t+1))$ and its norm is uniformly bounded for any $t > 0$. Sobolev embedding theorem implies the existence of $\alpha \in (0, 1)$ and $C > 0$ such that

$$\|w\|_{C^\alpha(\Omega \times [t, t+1])} \leq C, \quad t > 0.$$ 

Since $u$ is uniformly bounded, the strong solution theory of parabolic equation applied to the fourth equation in (3) shows

$$\|z\|_{L^p(\Omega \times (t, t+1))} + \|\Delta z\|_{L^p(\Omega \times (t, t+1))} \leq C(p), \quad t > 0,$$

for some constant $C(p) > 0$. Taking $p > 1 + n/2$, we see that for some $\alpha \in (0, 1)$

$$\|z\|_{C^\alpha(\Omega \times [t, t+1])} \leq C, \quad t > 0.$$ 

Thus,

$$\|wz\|_{C^\alpha(\Omega \times [t, t+1])} \leq C, \quad t > 0.$$ 

This can also be deduced by

$$\|\nabla wz\|_{L^p(\Omega)} + \|wz_t\|_{L^p(\Omega \times (t, t+1))} \leq C, \quad t > 0,$$

with $p > n + 1$. Using bootstrap arguments involving the standard parabolic regularity theory, we can verify that

$$\|v\|_{C^{2,\alpha,1+a/2}(\Omega \times [t, t+1])} + \|z\|_{W^{2,1}_{p'}(\Omega \times (t, t+1))} \leq C(p).$$

The proof is completed. □

4 Propagating properties and large time behavior

This section is devoted to the study of the propagating properties of the tumour cells and the large time behavior of the weak solution $(u, v, w, z)$ to the problem (3). In contrast with the heat equation, it is known that the porous medium equation has the property of finite speed of propagation. Therefore, the first component $u$ may not have positive minimum for some time $t > 0$. We use the comparison principle together with two kinds of weak lower solutions, one is decaying but its support is expanding with finite speed of propagation, the other one is an increasing function of time $t$, to overcome the difficulty of degenerate dispersion.

We first present the following comparison principle of the first component.
Lemma 4.1  Let $T > 0$ and the function space

$$E = \{ u \in L^\infty(Q_T); u \geq 0, \nabla u_m \in L^2((0, T); L^2(\Omega)), u^{m-1}_t \in L^2((0, T); L^2(\Omega))\},$$

$u_1, u_2 \in E, \nabla v \in L^\infty(Q_T)$, and $u_1, u_2$ satisfy the following differential inequalities

$$\frac{\partial u_1}{\partial t} \geq \Delta u_1^m - \nabla \cdot (u_1^m \phi(u_1) \nabla v) + \mu u_1^\delta(1 - u_1),$$

$$\frac{\partial u_2}{\partial t} \leq \Delta u_2^m - \nabla \cdot (u_2^m \phi(u_2) \nabla v) + \mu u_2^\delta(1 - u_2), \quad x \in \Omega, t \in (0, T),$$

$$\frac{\partial u_1}{\partial \nu} \geq 0 \geq \frac{\partial u_2}{\partial \nu}, \quad x \in \partial\Omega, t \in (0, T),$$

$$u_1(x, 0) \geq u_2(x, 0) \geq 0, \quad x \in \Omega,$$

in the sense that the following inequalities

$$\int_0^T \int_\Omega u_1 \varphi dxdt + \int_\Omega u_{10}(x) \varphi(x, 0)d\Omega \leq \int_0^T \int_\Omega \nabla u_1^m \cdot \nabla \varphi dxdt$$

$$\quad - \int_0^T \int_\Omega \nabla u_1^m \phi(u_1) \nabla v \cdot \nabla \varphi dxdt - \int_0^T \int_\Omega (\mu u_1^\delta(1 - u_1)) \varphi dxdt,$$

$$\int_0^T \int_\Omega u_2 \varphi dxdt + \int_\Omega u_{20}(x) \varphi(x, 0)d\Omega \geq \int_0^T \int_\Omega \nabla u_2^m \cdot \nabla \varphi dxdt$$

$$\quad - \int_0^T \int_\Omega \nabla u_2^m \phi(u_2) \nabla v \cdot \nabla \varphi dxdt - \int_0^T \int_\Omega \mu u_2^\delta(1 - u_2) \varphi dxdt,$$

hold for some fixed $u_{10}, u_{20} \in L^2(\Omega)$ such that $u_{10} \geq u_{20} \geq 0$ on $\Omega$ and all test functions $0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega))$ with $\varphi(x, T) = 0$ on $\Omega$. Then $u_1(x, t) \geq u_2(x, t)$ almost everywhere in $Q_T$.

Proof. The following inequality

$$\int_0^T \int_\Omega (u_1 - u_2) \varphi dxdt \leq \int_0^T \int_\Omega \nabla (u_1^m - u_2^m) \cdot \nabla \varphi dxdt$$

$$\quad - \int_0^T \int_\Omega (u_1^m \phi(u_1) - u_2^m \phi(u_2)) \nabla v \cdot \nabla \varphi dxdt - \int_0^T \int_\Omega (\mu u_1^\delta(1 - u_1) - u_2^\delta(1 - u_2)) \varphi dxdt,$$

holds for all $0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega))$ with $\varphi(x, T) = 0$. Let

$$a(x, t) = \begin{cases} u_1^m - u_2^m, & u_1(x, t) \neq u_2(x, t), \\ u_1 - u_2, & u_1(x, t) = u_2(x, t) \end{cases},$$

$$b(x, t) = \begin{cases} (u_1^m \phi(u_1) - u_2^m \phi(u_2)) \nabla v, & u_1(x, t) \neq u_2(x, t), \\ (\mu u_1^\delta(1 - u_1) - u_2^\delta(1 - u_2)) \varphi, & u_1(x, t) = u_2(x, t). \end{cases}$$
and

\[ c(x,t) = \begin{cases} 
\frac{\mu(u_1^0(1-u_1) - u_2^0(1-u_2))}{u_1 - u_2}, & u_1(x,t) \neq u_2(x,t), \\
\mu\delta u_1^0 - \mu(\delta + 1)u_1^0, & u_1(x,t) = u_2(x,t).
\end{cases} \]

Since \( \nabla v, u_1, u_2 \) are bounded and \( \phi \) is smooth enough, there exists a constant \( C > 0 \) such that \( |b| \leq Ca \) and \( |c| \leq C \). Henceforth, a generic positive constant (possibly changing from line to line) is denoted by \( C \). However, \( c \) is not bounded by \( Ca \) and we have no estimate on \( \nabla c \) since we only assume that \( \delta \geq 1 \). Then for all \( 0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega)) \) with \( \varphi(x, T) = 0 \) on \( \Omega \) and \( \frac{\partial c}{\partial \nu} = 0 \) on \( \partial \Omega \times (0, T) \), there holds

\[
\int_0^T \int_{\Omega} (u_1 - u_2)(\varphi_t + a(x, t)\Delta \varphi + b(x, t) \cdot \nabla \varphi + c(x, t)\varphi) dx dt \leq 0.
\]

We employ the standard duality proof method or the approximate Hohmgren’s approach to complete this proof (see Theorem 6.5 in [25], Chapter 1.3 and 3.2 in [35]). For any smooth function \( \psi(x, t) \geq 0 \), we solve the inverse-time problem

\[
\begin{aligned}
\varphi_t + (\kappa + a_\varepsilon(x, t))\Delta \varphi + b(x, t) \cdot \nabla \varphi + c_\theta(x, t)\varphi + \psi &= 0, & (x, t) &\in Q_T, \\
\frac{\partial \varphi}{\partial \nu} &= 0, & (x, t) &\in \partial \Omega \times (0, T), \\
\varphi(x, T) &= 0, & x &\in \Omega,
\end{aligned}
\]

(27)

where \( \kappa > 0, \theta > 0, a_\varepsilon \) is a smooth approximation of \( a, a_\varepsilon \geq a \), and

\[
c_\theta(x, t) = \begin{cases} 
\frac{\mu(u_1^0(1-u_1) - u_2^0(1-u_2))}{u_1 - u_2}, & |u_1(x,t) - u_2(x,t)| \geq \theta, \\
0, & |u_1(x,t) - u_2(x,t)| < \theta.
\end{cases}
\]

This definition of \( c_\theta \) allows us to find a constant \( C(\theta) \) such that

\[
\frac{c_\theta^2}{a} \leq C(\theta).
\]

We may also need to replace \( b(x, t) \) and \( c_\theta(x, t) \) by their smooth approximation functions \( b_\varepsilon(x, t) \) and \( c_{\theta,\varepsilon}(x, t) \) respectively in (27). For the sake of simplicity we omit this procedure. Here we note that (27) is a standard parabolic problem as the initial data is imposed at the end time \( t = T \). Therefore, it has a smooth solution \( \varphi \geq 0 \). Maximum principle shows the boundedness of \( \varphi \). Then we get the estimate

\[
\begin{align*}
\iint_{Q_T} (u_1 - u_2)\psi dx dt &\geq -\iint_{Q_T} |u_1 - u_2||a - a_\varepsilon||\Delta \varphi| dx dt \\
&\quad - \kappa \iint_{Q_T} |u_1 - u_2||\Delta \varphi| dx dt - \iint_{Q_T} |u_1 - u_2||c - c_\theta|\varphi dx dt \\
&= -I_1 - I_2 - I_3.
\end{align*}
\]
Now we need the a priori estimate on \( a_e |\Delta \varphi|^2 \). We can assume that \( T \) is appropriately small, otherwise we can prove step by step on each time interval. We multiply the equation (27) by \( \eta(t) \Delta \varphi \) where \( 1/2 \leq \eta(t) \leq 1 \) is a smooth function with \( \eta'(t) \geq M > 0 \) for \( t \in (0, T) \). Since \( T \) is small, we can choose \( M \) appropriately large. Integrating over \( Q_T \) yields

\[
\int_{Q_T} \varphi \eta \Delta \varphi \, dx \, dt + \int_{Q_T} \eta (\kappa + a_e) (\Delta \varphi)^2 \, dx \, dt \\
\leq \int_{Q_T} \eta |b| \| \nabla \varphi \| \Delta \varphi \, dx \, dt + \int_{Q_T} \eta \psi \varphi \Delta \varphi \, dx \, dt + \int_{Q_T} \eta \psi \Delta \varphi \, dx \, dt \\
\leq \int_{Q_T} \eta |C| \| \nabla \varphi \| \Delta \varphi \, dx \, dt + \frac{1}{4} \int_{Q_T} \eta (\kappa + a_e) (\Delta \varphi)^2 \, dx \, dt \\
+ \int_{Q_T} \frac{\eta \kappa^2}{\kappa + a_e} \, dx \, dt + \int_{Q_T} \eta |\nabla \psi| |\nabla \varphi| \, dx \, dt \\
\leq \frac{1}{2} \int_{Q_T} \eta (\kappa + a_e) (\Delta \varphi)^2 \, dx \, dt + \int_{Q_T} \frac{\eta C^2 \kappa^2}{\kappa + a_e} |\nabla \varphi|^2 \, dx \, dt + \int_{Q_T} \eta C |\theta|^2 \, dx \, dt \\
+ \int_{Q_T} \eta |\nabla \psi|^2 \, dx \, dt + \int_{Q_T} \eta |\nabla \varphi|^2 \, dx \, dt.
\]

Using \( \varphi(x, T) = 0 \), we have

\[
\int_{Q_T} \varphi \eta \Delta \varphi \, dx \, dt = - \int_{Q_T} \eta \nabla \varphi \cdot \nabla \varphi \, dx \, dt = - \frac{1}{2} \int_{Q_T} \eta \frac{\partial}{\partial t} |\nabla \varphi|^2 \, dx \, dt \\
\geq \frac{1}{2} \int_{Q_T} \eta' \eta(t) |\nabla \varphi|^2 \, dx \, dt \geq \frac{M}{2} \int_{Q_T} |\nabla \varphi|^2 \, dx \, dt.
\]

Therefore,

\[
\int_{Q_T} |\nabla \varphi|^2 \, dx \, dt + \int_{Q_T} (\kappa + a_e) (\Delta \varphi)^2 \, dx \, dt \leq C(\theta). \tag{28}
\]

It follows that

\[
I_1 = \int_{Q_T} |u_1 - u_2|^2 |a - a_e| |\Delta \varphi| \, dx \, dt \\
\leq \left( \int_{Q_T} (\kappa + a_e) |\Delta \varphi|^2 \, dx \, dt \right)^{\frac{1}{2}} \cdot \left( \int_{Q_T} \frac{|a - a_e|^2}{\kappa + a_e} |u_1 - u_2|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
\leq C(\theta) \left( \int_{Q_T} \frac{|a - a_e|^2}{\kappa + a_e} \, dx \, dt \right)^{\frac{1}{2}} \\
\leq C(\theta)^{\frac{1}{2}} \left( \int_{Q_T} |a - a_e|^2 \, dx \, dt \right)^{\frac{1}{2}},
\]

which converges to zero if we let \( \varepsilon \to 0 \). For any fixed \( \gamma > 0 \), denote

\[
F_\gamma = \{(x, t) \in Q_T; |u_1 - u_2| \geq \gamma\},
\]
and
\[ G_{\gamma} = \{(x, t) \in Q_T; |u_1 - u_2| < \gamma\}. \]

Then there exists a constant \( C(\gamma) \) such that \( a(x, t) \geq C(\gamma) \) on \( F_{\gamma} \) and
\[
I_2 = \kappa \int_{Q_T} |u_1 - u_2| |\Delta \varphi| \,dx \,dt \\
\leq \kappa \int_{G_{\gamma}} |u_1 - u_2| |\Delta \varphi| \,dx \,dt + \kappa \int_{F_{\gamma}} |u_1 - u_2| |\Delta \varphi| \,dx \,dt \\
\leq \gamma \int_{G_{\gamma}} |\kappa| |\Delta \varphi| |dx \,dt + \frac{\kappa C}{C(\gamma)^{\frac{\gamma}{2}}} \int_{F_{\gamma}} a^2 |\Delta \varphi| \,dx \,dt \\
\leq C(\gamma) \left( \int_{Q_T} |\kappa| |\Delta \varphi|^2 \,dx \,dt \right)^{\frac{1}{2}} + \frac{\kappa C}{C(\gamma)^{\frac{\gamma}{2}}} \left( \int_{Q_T} a |\Delta \varphi|^2 \,dx \,dt \right)^{\frac{1}{2}} \\
\leq \gamma C(\theta) + \frac{\kappa C(\theta)}{C(\gamma)^{\frac{\gamma}{2}}},
\]
which converges to zero if we first let \( \kappa \to 0 \) and then let \( \gamma \to 0 \). We also have
\[
I_3 = \int_{Q_T} |u_1 - u_2| |c - c_0| |\varphi| \,dx \,dt \leq C \left( \int_{Q_T} |c - c_0|^2 \,dx \,dt \right)^{\frac{1}{2}},
\]
which converges to zero if we let \( \theta \to 0 \). Now we conclude that
\[
\int_{Q_T} (u_1 - u_2) \psi \,dx \,dt \geq 0
\]
for any given \( \psi \geq 0 \) and then \( u_1 \geq u_2 \) almost everywhere on \( Q_T \).

Here we recall some lemmas about the asymptotic behavior of solutions to evolutionary equations.

**Lemma 4.2** ([5]) Let \((u, v, w, z)\) be a global solution of (3). Then there exists a constant \( L \geq 0 \) such that
\[
\|v(\cdot, t) - L\|_{W^{1,\infty}(\Omega)} \to 0, \quad \text{as } t \to \infty.
\]
In particular,
\[
\|\nabla v(\cdot, t)\|_{L^{\infty}(\Omega)} \to 0, \quad \text{as } t \to \infty.
\]

**Lemma 4.3** ([28] Lemma 4.1) If \( z \) is a global classical solution of
\[
\begin{align*}
\frac{dz}{dt} &= \Delta z - z + u, & x \in \Omega, \ t > 0, \\
\frac{\partial z}{\partial \nu} &= 0, & x \in \partial \Omega, \ t > 0, \\
z(x, 0) &= z_0(x), & x \in \Omega,
\end{align*}
\]
where \( u(x, t) \geq 0 \) is given. Then there exist constants \( C_1 \) and \( C_2 > 0 \) only depend on \( \text{diam}\Omega \) and \( \sup_{T < t} \|u\|_{L^1(\Omega)} \) respectively, such that
\[
\int_0^t z(x, s) \,ds \geq C_1 \int_0^t \int_\Omega u(y, s) \,dy \,ds - C_2, \quad x \in \Omega, \ t > 0.
\]
Lemma 4.4 ([28] Lemma 4.3, [23] Lemma 4.1) If \((w, z)\) is a global solution of
\[
\begin{cases}
w_t = -wz, \\
z_t = \Delta z - z + u, & x \in \Omega, \; t > 0, \\
\frac{\partial z}{\partial n} = 0, & x \in \partial \Omega, \; t > 0, \\
w(x, 0) = w_0(x), \\
z(x, 0) = z_0(x), & x \in \Omega,
\end{cases}
\]
with \(u \geq 0\) on \(\Omega \times \mathbb{R}^+\) and \(\frac{\partial m}{\partial v} = 0\) on \(\partial \Omega\), then
\[
\int_\Omega |\nabla w(t, \cdot)|^2 \, dx \leq 2 \int_\Omega |\nabla w_0|^2 \, dx + \frac{|\Omega|}{2e} \|w_0\|_{L^\infty(\Omega)}^2 + \|w_0\|_{L^\infty(\Omega)}^2 \int_\Omega z(\cdot, t) \, dx
\]
for all \(t > 0\).

Now we construct a self similar weak lower solution with expanding support.

Lemma 4.5 Let \((u, v, w, z)\) be a globally bounded weak solution of \((3)\) with the first component initial data \(u_0 \geq 0, u_0 \neq 0\) and \(1 \leq \delta < m, \Omega\) is convex. Define a function
\[
g(x, t) = e(1 + t)^{-k} \left[ (\eta - \frac{|x - x_0|^2}{(1 + t)^\beta})^+ \right]^d, \quad x \in \Omega, \; t \geq 0,
\]
where \(d = 1/(m - 1)\), \(\beta \in (0, 1/2)\) is sufficiently small, \(\kappa = (1 - \beta)/(m - 1)\), \(x_0 \in \Omega\) such that \(\inf_{x \in B_r(x_0)} u_0(x) > 0\) for some \(r > 0, \varepsilon \in (0, 1/2), \eta > 0\). Then by appropriately selecting \(\beta, \varepsilon\) and \(\eta\), the function \(g(x, t)\) can be a weak lower solution of the first equation in \((3)\), that is,
\[
\begin{cases}
\frac{\partial g}{\partial t} \leq \Delta g^m - \nabla \cdot (g^m \phi(g) \nabla v) + \mu g^\delta (1 - g), & x \in \Omega, \; t \in (0, T), \\
\frac{\partial g}{\partial n} \leq 0, & x \in \partial \Omega, \; t \in (0, T), \\
0 \leq g(x, 0) \leq u_0(x), & x \in \Omega,
\end{cases}
\]
in the sense that the following inequality
\[
\int_0^T \int_\Omega g \varphi \, dx \, dt + \int_\Omega g(x, 0) \varphi(x, 0) \, dx \geq \int_0^T \int_\Omega \nabla g^m \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_\Omega g^m \phi(g) \nabla v \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_\Omega \mu g^\delta (1 - g) \varphi \, dx \, dt,
\]
holds for any \(T > 0\) and all test functions \(0 \leq \varphi \in L^2((0, T); W^{1,2}(\Omega)) \cap W^{1,2}((0, T); L^2(\Omega))\) with \(\varphi(x, T) = 0\) on \(\Omega\), and \(0 \leq g(x, 0) \leq u_0(x)\) on \(\Omega\). Therefore, \(u(x, t) \geq g(x, t)\) and there exist \(t_0 > 0\) and \(\varepsilon_0 \geq 0\) such that \(u(x, t) \geq \varepsilon_0\) for all \(x \in \Omega\) and \(t \geq t_0\).
We denote $C_\Omega$. In order to find a weak lower solution $u_0$, we let

$$h(x, t) = \left( \eta - \frac{|x - x_0|^2}{(1 + t)^\beta} \right)_+, \quad x \in \Omega, \quad t \geq 0,$$

and

$$A(t) = \left\{ x \in \Omega; \frac{|x - x_0|^2}{(1 + t)^\beta} < \eta \right\}, \quad t \geq 0.$$ 

Since $u_0 \geq 0$, $u_0 \not\equiv 0$ and $u_0 \in C(\overline{\Omega})$, we see that there exists $x_0 \in \Omega$ such that $u_0(x) \geq \varepsilon_1$ on $B_r(x_0)$ for some $r > 0$ and $\varepsilon_1 > 0$. Without loss of generality, we may assume that $B_r(x_0) \subset \Omega$, $x_0 = 0$ and $\varepsilon_1 \leq 1/2$. Straightforward computation shows that

$$g_r = -\kappa \varepsilon(1 + t)^{-1}h^d + \varepsilon(1 + t)^{-x}dh^{d-1} \frac{\beta|x|^2}{(1 + t)^{\beta+1}},$$

$$\nabla g^m = -\varepsilon^m(1 + t)^{-ms}mdh^{md-1} \frac{2x}{(1 + t)^\beta},$$

$$\Delta g^m = \varepsilon^m(1 + t)^{-ms}md(\varepsilon^m)h^{md-2} \frac{4|x|^2}{(1 + t)^{2\beta}} - \varepsilon^m(1 + t)^{-ms}mdh^{md-1} \frac{2n}{(1 + t)^\beta},$$

for all $x \in A(t)$ and $t > 0$. According to the definition of $g$, we see that $\frac{\partial g}{\partial n} \leq 0$ and $\frac{\partial g^m}{\partial n} \leq 0$ on $\partial \Omega$ since $\Omega$ is convex, and

$$g(x, 0) = \varepsilon[(\eta - |x|^2)_+]^d \leq \varepsilon_1 1_{B_r(x_0)} \leq u_0(x), \quad x \in \Omega,$$

provided that

$$\eta \leq \varepsilon^2, \quad \varepsilon_1 \leq \varepsilon_1. \quad (29)$$

In order to find a weak lower solution $g$, we only need to check the following differential inequality on $A(t)$

$$\frac{\partial g}{\partial t} \leq \Delta g^m - \nabla \cdot (g^m \phi(g) \nabla v) + \mu g^\phi(1 - g), \quad x \in A(t), \quad t > 0. \quad (30)$$

Since $g(x, t) \leq \varepsilon_1 \leq 1/2$, we see that $\mu g^\phi(1 - g) \geq \mu g^\phi/2$ for all $x \in \Omega$ and $t > 0$. Further,

$$|\nabla \cdot (g^m \phi(g) \nabla v)| \leq g^m |\phi(g)||\Delta v| + |mg^{m-1} \phi(g) + g^m \phi'(g)||\nabla g||\nabla v|$$

$$\quad \leq g^m ||\Delta v||_{L^\infty(\Omega \times \mathbb{R}^+)} + (m + 1)||\nabla g^m|| \cdot ||\nabla v||_{L^\infty(\Omega \times \mathbb{R}^+)}.$$

We denote $C_1 = ||\nabla v||_{L^\infty(\Omega \times \mathbb{R}^+)}$ and $C_2 = ||\Delta v||_{L^\infty(\Omega \times \mathbb{R}^+)}$ for convenience, since they are bounded according to Theorem 3.2. A sufficient condition of inequality (30) is

$$\varepsilon(1 + t)^{-x}dh^{d-1} \frac{\beta|x|^2}{(1 + t)^{\beta+1}} + \varepsilon^m(1 + t)^{-ms}mdh^{md-1} \frac{2n}{(1 + t)^\beta}.$$
Next, we construct another constant lower solution and thus

\[
\eta \quad \text{and then implies}
\]

\[
+ C_2 \varepsilon^m (1 + t)^{-\kappa \delta} h^{d-1} + (m + 1) C_1 \varepsilon^m (1 + t)^{-\kappa \delta} m \varepsilon h^{d-1} \frac{2|x|}{(1 + t)^\delta}
\]

\[
\leq \kappa \varepsilon (1 + t)^{-\kappa \delta} h^{d-1} + \varepsilon^m (1 + t)^{-\kappa \delta} m \varepsilon h^{d-1} - 4|x|^2
\]

\[
+ \frac{\mu}{2} \varepsilon^\delta (1 + t)^{-\kappa \delta} h^{d+1}, \quad x \in A(t), \; t > 0.
\]

(31)

As we have chosen \( d = 1/(m - 1) \) and \( \kappa = (1 - \beta)/(m - 1) \), we rewrite (31) into

\[
\frac{\varepsilon \beta}{m - 1} \frac{|x|^2}{(1 + t)^\delta} + 2n \frac{m}{m - 1} \varepsilon^m h
\]

\[
+ C_2 \varepsilon^m (1 + t)^{\beta} h^2 + 2(m + 1) \varepsilon^m \frac{m}{m - 1} h|x|
\]

\[
\leq \kappa \varepsilon h + \varepsilon^m \frac{m}{(m - 1)^2} \frac{4|x|^2}{(1 + t)^\delta}
\]

\[
+ \frac{\mu}{2} \varepsilon^\delta (1 + t)^{-\kappa \delta} h^{d+1}, \quad x \in A(t), \; t > 0.
\]

(32)

Let \( \varepsilon, \beta \) and \( \eta \) be chosen such that

\[
\begin{cases}
\varepsilon \beta \leq \frac{4 \varepsilon^m}{m - 1}, \\
2n \frac{m}{m - 1} \varepsilon^m \leq \frac{1}{4} \kappa \varepsilon, \\
2m C_1 \varepsilon^m |x| \leq \frac{\kappa \varepsilon}{4}, \\
C_2 \varepsilon^m h^{d+1-\delta} \leq \frac{\mu}{2} \varepsilon^\delta (1 + t)^{-\kappa \delta + \kappa + 1 - \beta}, \quad x \in A(t), \; t > 0.
\end{cases}
\]

(33)

Since \( 1 \leq \delta < m, \beta \in (0, 1/2), \kappa = (1 - \beta)/(m - 1) \geq 1/[2(m - 1)], \; h \leq 1/2 \) and \( |x| \leq \text{diam} \Omega \), we see that \( d + 1 - d\delta = d(m - \delta) > 0, \; -\kappa \delta + \kappa + 1 - \beta = (m - \delta) \kappa > 0 \). Thus, for (29) and (33), it suffices to choose \( \eta = t^2 \),

\[
\varepsilon = \min \left\{ \left( \frac{1}{8nm} \right)^{\frac{1}{m-1}}, \left( \frac{1}{8m(m - 1) \text{diam} \Omega} \right)^{\frac{1}{m-1}}, \left( \frac{\varepsilon_0}{r^2 d} \right)^{\frac{1}{m-1}}, \left( \frac{\mu}{2C_2} \right)^{\frac{1}{m-1}} \right\},
\]

and then \( \beta = 4 \varepsilon^{m-1} m/(m - 1) \).

Now, we find a weak lower solution with expanding support and comparison principle Lemma 4.1 implies

\[
u(x, t) \geq g(x, t) = \varepsilon (1 + t)^{-\kappa} \left[ \left( \eta - \frac{|x - x_0|^2}{(1 + t)^\delta} \right)^d \right], \quad x \in \Omega, \; t > 0.
\]

There exists a \( t_0 \) such that

\[
\eta - \frac{|x - x_0|^2}{(1 + t_0)^\delta} \geq \frac{\eta}{2}, \quad x \in \Omega,
\]

and thus

\[
u(x, t_0) \geq g(x, t_0) \geq \varepsilon (1 + t_0)^{-\kappa} \left( \frac{\eta}{2} \right)^d, \quad x \in \Omega.
\]

Next, we construct another constant lower solution

\[
u(x, t) \equiv \varepsilon_0, \quad x \in \Omega, \; t > t_0,
\]
with \(0 < \varepsilon_0 \leq \varepsilon(1 + t_0)^{-\eta/2} \leq 1/2\) to be determined. Clearly, \(\frac{\partial u}{\partial \nu} = 0\) on \(\partial \Omega\). We only need to check the following differential inequality

\[
0 \leq -\varepsilon_0^m \phi(\varepsilon_0) \Delta v(x, t) + \mu \varepsilon_0^\delta (1 - \varepsilon_0), \quad x \in \Omega, \ t > t_0,
\]

which is valid if we further let

\[
\varepsilon_0 \leq \left( \frac{\mu}{2\|\Delta v\|_{L^\infty(\Omega \times (t_0, +\infty))}} \right)^{1/m},
\]

since \(\delta < m\) and \(\Delta v\) is uniformly bounded according to Theorem 3.2. Applying the comparison principle Lemma 4.1 again, we find

\[
u(x, t) \geq \nu(x, t) \equiv \varepsilon_0, \quad x \in \Omega, \ t > t_0.
\]

This completes the proof.

Remark. It is interesting to compare the self similar weak lower solution \(g(x, t)\) in the proof of Lemma 4.5 to the Barenblatt solution of porous medium equation

\[
B(x, t) = (1 + t)^{-k} \left[ 1 - \frac{k(m - 1)}{2mn} \frac{|x|^2}{(1 + t)^{2k/n}} \right]^{\frac{1}{m - 1}},
\]

with \(k = 1/(m - 1 + 2/n)\). The Barenblatt solution \(B(x, t)\) is decaying at the rate \((1 + t)^{-1/(m - 1 + 2/n)}\) in \(L^\infty(\mathbb{R}^n)\) and the support is expanding at the rate \((1 + t)^{2k/n}\). While the self similar weak lower solution \(g(x, t)\) is decaying at the rate \((1 + t)^{-(1 - \beta)/(m - 1)}\) and its support is expanding at the rate \((1 + t)^\beta\). Here in the proof we have selected \(\beta > 0\) sufficiently small, which means the support of \(g\) is expanding with a much slower rate and the maximum of \(g\) is decaying at a slightly faster rate.

Proof of Theorem 2.2 This has been proved in Lemma 4.5.

After proving the support expanding property of the first equation in (3), which is a degenerate diffusion equation, we can deduce the following convergence properties of all components.

Lemma 4.6 Let \((u, v, w, z)\) be a globally bounded weak solution of (3) with the first component initial data \(u_0 \geq 0, u_0 \neq 0\) and \(1 \leq \delta < m\). Then there exist constants \(C_1, C_2 > 0\) and \(c_1, c_2 > 0\) independent of \(t\) such that

\[
\|w(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 e^{-c_1 t},
\]

and

\[
\|v(\cdot, t) - (\overline{v}_0 + \overline{w}_0)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} + \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 e^{-c_2 t},
\]

for all \(t > 0\), where \(\overline{f} = \int_{\Omega} f dx/|\Omega|\).
Proof. Applying Lemma 4.3, we see that
\[
\int_0^t z(x, t) ds \geq C \int_0^t \int_{\Omega} u(y, s) dy ds - C
\]
\[
\geq C \int_0^t \int_{\Omega} u(y, s) dy ds - C
\]
\[
\geq C|\Omega| e_0(t - t_0) - C
\]
\[
\geq c_1 t - C, \quad x \in \Omega, \ t > t_0,
\]
since \(u(x, t) \geq e_0\) for \(x \in \Omega\) and \(t > t_0\) according to Lemma 4.5. Therefore,
\[
w(x, t) = w_0(x) e^{-\int_0^t z(x, s) ds} \leq w_0(x) e^{-c_1 t + C} \leq C_1 e^{-c_1 t}, \quad x \in \Omega, \ t > t_0. \tag{34}
\]
This is also valid for \(t \in (0, t_0)\) upon enlarging \(C_1\) if necessary and hereafter we only need to prove this lemma for \(t > t_0\). We also have
\[
|\nabla w(x, t)| = |\nabla w_0(x)| e^{-\int_0^t z(x, s) ds} + w_0(x) e^{-\int_0^t z(x, s) ds} \int_0^t |\nabla z(x, s) ds|
\]
\[
\leq C e^{-c_1 t} + C e^{-c_1 t} t \leq C' e^{-c_1 t}, \quad x \in \Omega, \ t > t_0,
\]
with \(0 < c_1' < c_1\). We may write \(C_1'\) and \(c_1'\) as \(C_1\) and \(c_1\) for simplicity. Therefore,
\[
|\nabla (wz)(x, t)| \leq |z \nabla w(x, t)| + |w \nabla z(x, t)| \leq C e^{-c_1 t}, \quad x \in \Omega, \ t > t_0,
\]
It follows from the second equation in (3) that
\[
v(x, t) = e^{t \Delta} v_0 + \int_0^t e^((t-s) \Delta)(wz)(\cdot, s) ds, \quad t > 0,
\]
and
\[
\nabla v(x, t) = e^{t \Delta} \nabla v_0 + \int_0^t e^((t-s) \Delta) \nabla (wz)(\cdot, s) ds, \quad t > 0,
\]
Using the standard \(L^p - L^q\) type estimate for \(\Delta v\), we get
\[
||\Delta v(x, t)||_{L^\infty(\Omega)} \leq ||\nabla e^{t \Delta} |\nabla v_0||_{L^\infty(\Omega)} + \int_0^t ||\nabla e^{(t-s) \Delta} |\nabla (wz)(x, s)||_{L^\infty(\Omega)} ds
\]
\[
\leq C(1 + t^{-\frac{1}{2}}) e^{-a t} ||\nabla v_0||_{L^\infty(\Omega)}
\]
\[
+ C \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-a t} ||\nabla (wz)(\cdot, s)||_{L^\infty(\Omega)} ds
\]
\[
\leq C e^{-a t} + C \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-a t} e^{-c_1 s} d s
\]
\[
\leq C_2 e^{-c_1 t}, \quad x \in \Omega, \ t > t_0,
\]
where \( \lambda_1 > 0 \) is the first nonzero eigenvalue of \(-\Delta\) with homogeneous Neumann boundary condition. The \( L^\infty \) estimate of \( \nabla v \) can be deduced in a similar way. In the proof of Lemma 3.5 we have obtained
\[
\int_\Omega (v(x,t) + w(x,t))dx = \int_\Omega (v_0(x) + w_0(x))dx,
\]
which is the same as the estimate of \( v_\varepsilon + w_\varepsilon \). It follows from (34) that \( w(x,t) \) is decaying to zero exponentially. This implies that
\[
v(t) = \frac{1}{|\Omega|} \int_\Omega v(x,t)dx\]
is converging to \( v_0 + w_0 \) exponentially. A Poincaré type inequality shows
\[
\|v(x,t) - \nabla(t)\|_{L^\infty(\Omega)} \leq C\|\nabla v(x,t)\|_{L^\infty(\Omega)} \leq Ce^{-c't}.
\]
Therefore,
\[
\|v(x,t) - (v_0 + w_0)\|_{L^\infty(\Omega)} \leq \|v(x,t) - \nabla(t)\|_{L^\infty(\Omega)} + \|\nabla(t) - (v_0 + w_0)\|_{L^\infty(\Omega)}
\]
\[
\leq \|v(x,t) - \nabla(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}
\]
\[
\leq Ce^{-c't}, \quad x \in \Omega, \ t > t_0,
\]
The proof is completed. \( \square \)

**Lemma 4.7** For constants \( C, c > 0 \) and \( m > 1 \), the local solution \( g \) of the following ODE
\[
\begin{aligned}
g'(t) &= Ce^{-ct}g^m, \quad t > 0, \\
g(0) &= g_0 > 0,
\end{aligned}
\]
blooms up in finite time if \( c/C < (m-1)g_0^{m-1} \), while remains bounded if \( c/C > (m-1)g_0^{m-1} \).

**Proof.** There holds
\[
\frac{-1}{m-1}(\frac{1}{g^{m-1}})' = Ce^{-ct}, \quad t > 0.
\]
Integrating over \((0, t)\) shows
\[
\frac{1}{m-1}(\frac{1}{g^{m-1}} - \frac{1}{g_0^{m-1}}) = \frac{C}{c}(1 - e^{-ct}).
\]
A simple analysis completes this proof. \( \square \)

**Lemma 4.8** Let \((u, v, w, z)\) be a globally bounded weak solution of (3) with the first component initial data \( u_0 \geq 0, u_0 \neq 0 \) and \( 1 \leq \delta < m \). Then there exist constants \( C_3 > 0 \) and \( c_3 > 0 \) independent of \( t \) such that
\[
\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C_3 e^{-c_3 t},
\]
for all \( t > 0 \).
**Proof.** Lemma 4.5 implies that $u(x, t) \geq \varepsilon_0$ for $x \in \Omega$ and $t > t_0$. It suffices to prove this lemma for $t \geq t_1$ with some fixed $t_1 \geq t_0$ to be determined. We use upper and lower solution method to achieve this. Let $u_1(t)$ and $u_2(t)$ be one pair of the solutions of the following ODE

\[
\begin{align*}
  u'_1(t) &\geq u_1^m \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} + \mu u_1^\delta (1 - u_1), \\
  u'_2(t) &\leq -u_2^m \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} + \mu u_2^\delta (1 - u_2), \quad t > t_1, \\
  u_1(t_1) &\geq \|u(\cdot, t_1)\|_{L^\infty(\Omega)}, \\
  u_2(t_1) &\leq \varepsilon_0.
\end{align*}
\]

(35)

Lemma 4.1 shows that

\[u_1(t) \geq u(x, t) \geq u_2(t), \quad x \in \Omega, \ t > t_0.\]

We only need to find one pair of $(u_1, u_2)$ such that $u_1$ and $u_2$ both converge to 1 exponentially. A sufficient condition of (35) is

\[
\begin{align*}
  u'_1(t) &= C_2 e^{-c_2 t} u_1^m + \mu u_1^\delta (1 - u_1), \\
  u'_2(t) &= -C_2 e^{-c_2 t} u_2^m + \mu u_2^\delta (1 - u_2), \quad t > t_1, \\
  u_1(t_1) &= \|u(\cdot, t_1)\|_{L^\infty(\Omega)} + 1, \\
  u_2(t_1) &= \varepsilon_0,
\end{align*}
\]

(36)

since $\|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 e^{-c_2 t}$ according to Lemma 4.6. We note that we can choose $t_1$ sufficiently large such that

\[
\frac{c_2}{C_2 e^{-c_2 t_1}} > 2(m - 1) \left( \sup_{t > 0} \|u(\cdot, t)\|_{L^\infty(\Omega)} + 1 \right)^{m-1}.
\]

Lemma 4.7 implies that $u_1(t)$ is uniformly bounded by some constant $C$. And a simple ODE comparison shows that $u_1(t) > 1$ for all $t > t_1$. Therefore,

\[
\begin{align*}
  u'_1(t) &\leq C_m C_2 e^{-c_2 t} + \mu u_1^\delta (1 - u_1), \quad t > t_1, \\
  u_1(t_1) &= \|u(\cdot, t_1)\|_{L^\infty(\Omega)} + 1.
\end{align*}
\]

We see that $u_1(t)$ is an upper solution of $u(x, t)$ and an upper solution of $u_1(t)$ is $\overline{u}_1(t)$ such that

\[
\begin{align*}
  \overline{u}'_1(t) &= C_m C_2 e^{-c_2 t} + \mu u_1^\delta (1 - \overline{u}_1), \quad t > t_1, \\
  \overline{u}_1(t_1) &= \|u(\cdot, t_1)\|_{L^\infty(\Omega)} + 1,
\end{align*}
\]

(37)

which can be solved as

\[
\begin{align*}
  \overline{u}_1(t) &= 1 + e^{-\mu u_1^\delta (t-t_1)} (\|u(\cdot, t_1)\|_{L^\infty(\Omega)} + 1) + C_m C_2 \int_{t_1}^t e^{-\mu u_1^\delta (t-s)} e^{-c_2 s} ds - e^{-\mu u_1^\delta (t-t_1)} \\
  &\leq 1 + e^{-\mu u_1^\delta (t-t_1)} (\|u(\cdot, t_1)\|_{L^\infty(\Omega)} + 1) + C_m C_2 e^{-\min[\mu u_1^\delta (t-t_1)]/2}, \quad t > t_1.
\end{align*}
\]
On the other hand, the lower solution of $u(x, t)$ satisfies

$$
\begin{cases}
  u'_2(t) = -C_2 e^{-c_2 t} u_2^m + \mu u'_2(1 - u_2), & t > t_1, \\
  u_2(t_1) = \varepsilon_0.
\end{cases}
$$

We note that we can choose $t_1$ sufficiently large that

$$C_2 e^{-c_2 t_1} \geq \mu \varepsilon_0^m (1 - \varepsilon_0).$$

An ODE comparison shows that $\varepsilon_0 \leq u_2(t) < 1$ for all $t > t_1$ and

$$
\begin{cases}
  u'_2(t) \geq -C_2 e^{-c_2 t} + \mu \varepsilon_0(1 - u_2), & t > t_1, \\
  u_2(t_1) = \varepsilon_0.
\end{cases}
$$

We see that $u_2(t)$ is a lower solution of $u(x, t)$ and a lower solution of $u_2(t)$ is $u_2(t)$ such that

$$
\begin{cases}
  u'_2(t) = -C_2 e^{-c_2 t} + \mu \varepsilon_0(1 - u_2), & t > t_1, \\
  u_2(t_1) = \varepsilon_0.
\end{cases}
$$

This can also be solved as

$$
\begin{align*}
  u_2(t) &= 1 + e^{-\mu \varepsilon_0^m(t-t_1)} \varepsilon_0 - C_2 \int_{t_1}^t e^{-\mu \varepsilon_0^m(s-t)} e^{-c_2 s} ds - e^{-\mu \varepsilon_0^m(t-t_1)} \\
  &\geq 1 - e^{-\mu \varepsilon_0^m(t-t_1)} - C_2 C e^{-\min[\mu \varepsilon_0^m, c_2] t/2}, \quad t > t_1.
\end{align*}
$$

Thus, we conclude

$$u_2(t) \leq u(t) \leq u_1(t) \leq u(t) \leq u_1(t), \quad t > t_1,$$

and $u_2(t), u_1(t)$ converge to 1 exponentially.

**Lemma 4.9** Let $(u, v, w, z)$ be a globally bounded weak solution of (3) with the first component initial data $u_0 \geq 0$, $u_0 \neq 0$ and $1 \leq \delta < m$. Then there exist constants $C_4 > 0$ and $c_4 > 0$ independent of $t$ such that

$$
\|z(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C_4 e^{-c_4 t},
$$

for all $t > 0$.

**Proof.** From the fourth equation in (3), we have

$$
 z(x, t) = e^{(\Delta - 1) t} x_0 + \int_0^t e^{(t-s)(\Delta - 1)} u(\cdot, s) ds, \quad t > 0.
$$

We note that

$$
\int_0^t e^{(t-s)(\Delta - 1)} ds = 1 - e^{-t},
$$
which can be deduced by solving the ODE $z' = -z + 1$ with $z(0) = 0$. Therefore,
\[
\|z(x, t) - 1\|_{L^\infty(\Omega)} \leq \|e^{t(\Delta - 1)} z_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta - 1)} (u(\cdot, s) - 1)\|_{L^\infty(\Omega)} ds + e^{-t}
\leq C e^{-t} (\|z_0\|_{L^\infty(\Omega)} + 1) + C \int_0^t e^{-(t-s)} \|u(\cdot, s) - 1\|_{L^\infty(\Omega)} ds
\leq C e^{-t} + C C_3 \int_0^t e^{-(t-s)} e^{-c_3 s} ds 
\leq C_4 e^{-c_3 t}, \quad t > 0.
\]

The proof is completed. \(\square\)

**Proof of Theorem 2.3** This is proved by collecting Lemma 4.5, Lemma 4.6, Lemma 4.8 and Lemma 4.9.

Finally, we construct a self similar upper solution with expanding support to prove Theorem 2.1. We note that for constructing a weak upper solution for the heat equation, one should replace the cut-off composite function $(\cdot)_+$ by $(\cdot)_-$. But here for the degenerate porous medium type equation and the self similar function of the form $g = [(1 - |x|^2)^+]^q$ with $m d > 1$, we can check that $\nabla g^m$ is continuous and $\Delta g^m \in L^q(\Omega)$ for some $q > 1$. This shows that the differential inequality for an upper solution only need to be valid almost everywhere, without the possible Radon measures on the boundary of its support.

**Lemma 4.10** Let $(u, v, w, z)$ be a globally bounded weak solution of (3). We further assume that
\[
supp u_0 \subset \overline{B}_{r_0}(x_0) \subset \Omega,
\]
for some $r_0 > 0$ and $x_0 \in \Omega$. Define a function
\[
g(x, t) = e^{(\tau + t)^\sigma} \left[\eta - \frac{|x - x_0|^2}{(\tau + t)^\beta}\right]^d, \quad x \in \Omega, \ t \geq 0,
\]
where $d = 1/(m - 1)$, $\beta > 0$, $\sigma > 0$, $e > 0$, $\eta > 0$, $\tau \in (0, 1)$. Then by appropriately selecting $\beta$, $\sigma$, $e$, $\eta$ and $\tau$, the support of $g(x, t)$ is contained in $\Omega$ for $t \in (0, t_0)$ with some $t_0 > 0$ and the function $g(x, t)$ can be an upper solution of the first equation in (3) on $\Omega \times (0, t_0)$, that is,
\[
\begin{align*}
\frac{\partial g}{\partial t} &\geq \Delta g^m - \nabla \cdot (g^m \phi(g) \nabla v) + \mu g^\delta (1 - g), \quad x \in \Omega, \ t \in (0, t_0), \\
\frac{\partial g}{\partial v} &\geq 0, \quad x \in \partial \Omega, \ t \in (0, t_0), \\
g(x, 0) &\geq u_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
\]
in the sense that the following inequality
\[
\int_0^{t_0} \int_\Omega g \varphi dx dt + \int_\Omega g(x, 0) \varphi(x, 0) dx \leq \int_0^{t_0} \int_\Omega \nabla g^m \cdot \nabla \varphi dx dt
\]
Straightforward computation shows that
\[ g \text{ holds for all test functions } 0 \leq \varphi \in L^2((0, t_0); W^{1,2}(\Omega)) \cap W^{1,2}((0, t_0); L^2(\Omega)) \text{ with } \varphi(x, t_0) = 0 \text{ on } \Omega \text{ and } g(x, 0) \geq u_0(x) \geq 0 \text{ on } \Omega. \] Therefore, \( u(x, t) \leq g(x, t) \) and there exist a family of monotone increasing open sets \( \{A(t)\}_{t \in (0, t_0)} \) such that
\[ \text{supp } u(\cdot, t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0, t_0), \]
and \( \partial A(t) \) has a finite derivative with respect to \( t \).

**Proof.** For simplicity, we let
\[ h(x, t) = \left( \eta - \frac{|x - x_0|^2}{(\tau + t)^\beta} \right)_+, \quad x \in \Omega, \ t \geq 0, \]
and
\[ A(t) = \left\{ x \in \Omega; \frac{|x - x_0|^2}{(\tau + t)^\beta} < \eta \right\}, \quad t \geq 0. \]
Since \( u_0 \in C(\overline{\Omega}) \) and \( \text{supp } u_0 \subset \overline{B}_{r_0}(x_0) \subset \Omega \), we see that there exist \( r_1 > r_0 \) and \( \varepsilon_1 > 0 \) such that \( B_{r_1}(x_0) \subset \subset \Omega \) and \( u_0(x) \leq \varepsilon_1 \) for all \( x \in \Omega \). Without loss of generality, we may assume that \( x_0 = 0 \). Straightforward computation shows that
\[ g_t = \sigma \varepsilon (\tau + t)^\sigma - 1 \int h^d + \varepsilon (\tau + t)^\sigma dh^{d-1} \frac{2x}{(\tau + t)^\beta}, \]
\[ \nabla g^m = - \varepsilon^m (\tau + t)^{m-1} \frac{d}{d} \frac{2x}{(\tau + t)^\beta}, \]
\[ \Delta g^m = \varepsilon^m (\tau + t)^{m-1} \frac{d}{d} \frac{4|x|^2}{(\tau + t)^2} - \varepsilon^m (\tau + t)^{m-1} \frac{2n}{(\tau + t)^\beta}, \]
for all \( x \in A(t) \) and \( t > 0 \). Let \( \tau \in (0, 1) \) to be determined and
\[ r_2 = \frac{r_0 + \tau_1}{2}, \quad \eta = \frac{\tau_1^2}{\tau^\beta}, \quad t_0 = \min \left\{ \tau, \tau \left( \frac{r_1}{r_2} \right)^\beta - 1 \right\}. \tag{38} \]
According to the definition of \( g \), we see that \( A(0) = B_{r_2}(x_0) \), \( \text{supp } u_0 \subset \subset \overline{A}(0) \subset \Omega \), and \( A(t_0) \subset \subset B_{r_1}(x_0) \subset \subset \Omega \). Therefore, \( \frac{\partial g}{\partial \nu} = 0 \) and \( \frac{\partial g^m}{\partial \nu} = 0 \) on \( \partial \Omega \) for all \( t \in (0, t_0) \), and
\[ g(x, 0) = \varepsilon \tau^\sigma \left( \eta - \frac{|x - x_0|^2}{\tau^\beta} \right)_+^d \geq \varepsilon \tau^\sigma \left( \frac{r_2^2}{\tau^\beta} - \frac{r_0^2}{\tau^\beta} \right)^d \cdot 1_{B_{r_0}(x_0)} \geq \varepsilon_1 1_{B_{r_0}(x_0)} \geq u_0(x), \quad x \in \Omega, \]
provided that
\[ \varepsilon \tau^\sigma \left( \frac{r_2^2}{\tau^\beta} - \frac{r_0^2}{\tau^\beta} \right)^d \geq \varepsilon_1. \tag{39} \]
In order to find a weak lower solution \( g \), we only need to check the following differential inequality on \( A(t) \)

\[
\frac{\partial g}{\partial t} \geq \Delta g^m - \nabla \cdot (g^m \phi(g) \nabla v) + \mu g^\delta (1 - g), \quad x \in A(t), \ t \in (0, t_0).
\] (40)

Since \( 0 \leq g \leq \epsilon \eta^d \), we see that \( \mu g^\delta (1 - g) \leq \mu g^\delta \) for all \( x \in \Omega \) and \( t \geq 0 \). Further,

\[
|\nabla \cdot (g^m \phi(g) \nabla v)| \leq g^m |\phi(g)| |\Delta v| + |mg^{m-1} \phi(g) + g^m \phi'(g) \nabla g||\nabla v|
\]

\[
\leq g^m ||\Delta v||_{L^{\infty}(\Omega \times \mathbb{R}^+)} + (m + \epsilon \tau^\eta \eta^d) ||\nabla g^m|| ||\nabla v||_{L^{\infty}(\Omega \times \mathbb{R}^+)}.
\]

We denote \( C_1 = ||\nabla v||_{L^{\infty}(\Omega \times \mathbb{R}^+)} \) and \( C_2 = ||\Delta v||_{L^{\infty}(\Omega \times \mathbb{R}^+)} \) for convenience, since they are bounded according to Theorem 3.2. A sufficient condition of inequality (40) is

\[
\sigma \epsilon (\tau + t)^{\sigma - 1} h + \epsilon (\tau + t)^{\sigma} dh^{d-1} \frac{\beta |x|^2}{(\tau + t)^{\beta + 1}} + \epsilon^m (\tau + t)^{m \sigma} dh^m \frac{2n}{(\tau + t)^{\beta}}
\]

\[
\geq C_2 \epsilon m (\tau + t)^{m \sigma} h^m + (m + \epsilon \tau^\eta \eta^d) C_1 \epsilon m (\tau + t)^{m \sigma} dh^{m - 1} \frac{2|x|^2}{(\tau + t)^{\beta}} + \mu \epsilon^\delta (\tau + t)^{\delta \sigma} h^\delta, \quad x \in A(t), \ t \in (0, t_0).
\] (41)

As we have chosen \( d = 1/(m - 1) \), we rewrite (41) into

\[
\sigma \epsilon (\tau + t)^{\sigma - 1} h + \frac{\epsilon \beta}{m - 1} \frac{|x|^2}{(\tau + t)^{\beta + 1}} + 2n \frac{m}{m - 1} \epsilon^m (\tau + t)^{m \sigma} \frac{h}{(\tau + t)^{\beta}}
\]

\[
\geq C_2 \epsilon m (\tau + t)^{m \sigma} h^2 + 2(m + \epsilon \tau^\eta \eta^d) C_1 \epsilon m (\tau + t)^{m \sigma} dh \frac{|x|}{(\tau + t)^{\beta}}
\]

\[
+ \frac{m}{(m - 1)^2} \epsilon^m (\tau + t)^{m \sigma} \frac{4|x|^2}{(\tau + t)^{2 \beta}} + \mu \epsilon^\delta (\tau + t)^{\delta \sigma} h^\delta - d + 1, \quad x \in A(t), \ t \in (0, t_0).
\]

Let \( \epsilon, \beta, \sigma \) and \( \tau \) be chosen such that

\[
\begin{cases}
\frac{1}{2} \frac{\epsilon \beta}{m - 1} \frac{|x|^2}{(\tau + t)^{\beta + 1}} \geq \frac{m}{(m - 1)^2} \epsilon^m (\tau + t)^{m \sigma} \frac{4|x|^2}{(\tau + t)^{2 \beta}}, \\
\frac{1}{3} \sigma \epsilon (\tau + t)^{\sigma - 1} h \geq C_2 \epsilon^m (\tau + t)^{m \sigma} h^2, \\
\frac{1}{3} \sigma \epsilon (\tau + t)^{\sigma - 1} h \geq \mu \epsilon^\delta (\tau + t)^{\delta \sigma} h^{\delta - d + 1}, \\
\frac{1}{2} \frac{\epsilon \beta}{m - 1} \frac{|x|^2}{(\tau + t)^{\beta + 1}} + \frac{1}{3} \sigma \epsilon (\tau + t)^{\sigma - 1} h \geq 2(m + \epsilon \tau^\eta \eta^d) C_1 \epsilon^m (\tau + t)^{m \sigma} dh \frac{|x|}{(\tau + t)^{\beta}}, 
\end{cases}
\] (42)

We have the following estimate

\[
2(m + \epsilon \tau^\eta \eta^d) C_1 \epsilon^m (\tau + t)^{m \sigma} dh \frac{|x|}{(\tau + t)^{\beta}}
\]
We note that the solution in Lemma 4.5 is expanding with a much slower rate and decaying at a slightly faster rate. Here, 

\[ \sigma/3 \geq \mu \epsilon^{\delta-1}(t + \tau)^{(\delta-1)r+1} h^{\delta-1}, \quad x \in A(t), \ t \in (0, t_0). \]

Therefore, a sufficient condition of (42) is

\[
\frac{(m-1)\beta}{2\sigma/3} \geq 8m \epsilon^{m-1}(t + \tau)^{(m-1)r-\beta+1}, \\
2\sigma/3 \geq (C_2 + m + \epsilon \eta)^2 C_1^2 m \epsilon^{m-1}(t + \tau)^{(m-1)r+1} h, \\
\sigma/3 \geq \mu \epsilon^{\delta-1}(t + \tau)^{(\delta-1)r+1} h^{\delta-1},
\]

(43)

We note that \( \eta, \tau \) and \( t_0 \) satisfy the condition (38) and (39), and then \( h \leq \eta = r_2/\tau^\beta, \tau + t \leq \tau + t_0 \leq 2\tau, \)
\[ \epsilon \tau^{\sigma-d\beta}(r_2^2 - r_0^2)^d \geq \epsilon_1. \] For \( \tau \in (0, 1) \), we choose
\[ \epsilon = \frac{\epsilon_1}{\tau^{\sigma-d\beta}(r_2^2 - r_0^2)^d} := C_3 \tau^{d\beta-\sigma}. \]

Now, we only need to find \( \tau \in (0, 1) \) such that
\[
\begin{align*}
\frac{(m-1)\beta}{2\sigma/3} & \geq 8m C_3^{m-1} 2^{\max(0,(m-1)(r-\beta+1))}, \\
2\sigma/3 & \geq (C_2 + m + C_3 r_2^d) C_1^2 m C_3^{m-1} 2^{(m-1)r} r_2^2 h, \\
\sigma/3 & \geq \mu C_3^{\delta+1} 2^{\delta(r-d-1)} r_2^{d(\delta-1)},
\end{align*}
\]

This can be done by selecting \( \beta = 1, \sigma = 1, \) and \( \tau \in (0, 1) \) sufficiently small.

The comparison principle Lemma 4.1 implies that \( u(x, t) \leq g(x, t) \) for all \( x \in \Omega \) and \( t \in (0, t_0). \) Thus,
\[ \text{supp } u(\cdot, t) \subset \overline{A}(t) = \{ x \in \Omega; |x - x_0|^2 < \eta(t + \tau)\beta \}, \quad t \in (0, t_0), \]
and
\[ \partial A(t) = \{ x \in \Omega; |x - x_0| = \eta(t + \tau)\beta \}, \quad t \in (0, t_0), \]
which has finite derivative with respect to \( t. \)

**Remark.** Similar to the weak lower solution in Lemma 4.5, we compare the self similar weak upper solution \( g(x, t) \) to the Barenblatt solution of porous medium equation
\[ B(x, t) = (1 + t)^{-k} \left[ \left( 1 - \frac{k(m-1)}{2mn} \frac{|x|^2}{(1 + t)^{2k/n}} \right)^+ \right]^{\frac{1}{m-1}}, \]
with \( k = 1/(m-1 + 2/n). \) The Barenblatt solution \( B(x, t) \) is decaying at the rate \((1 + t)^{-1/(m-1+2/n)}\) in \( L^\infty(\mathbb{R}^n) \) and the support is expanding at the rate \((1 + t)^{2k/n} \). As we have shown the support of the lower solution in Lemma 4.5 is expanding with a much slower rate and decaying at a slightly faster rate. Here, the upper solution is increasing at the rate \((\tau + t)^{\sigma} \) and its support is expanding at the rate \((\tau + t)\beta \). The increasing of \( g(x, t) \) makes it possible to be an upper solution, which can be seen from the proof.

From the proof of Lemma 4.10, we can choose \( \beta > 0 \) to as small as we want. But we note that \( \text{supp } u_0 \subset \subset \text{supp } g(\cdot, 0) \) and if we choose a smaller \( \beta > 0 \), then the parameters \( \tau \) and \( t_0 \) are also smaller.
This shows if we let the upper solution expands slower, then it may only be an upper solution for a smaller time interval. Thus, the slower expanding upper solution $g(x,t)$ on a smaller time interval does not contradict to the possible feature that the solution $u(x,t)$ expands at a fixed rate since $\text{supp } u_0 \subset \subset \text{supp } g(\cdot,0)$ at the initial time.

**Proof of Theorem 2.7** This has been proved in Lemma 4.10.

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