Distribution Functions for Edge Eigenvalues in Orthogonal and Symplectic Ensembles: Painlevé Representations

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Abstract

We derive Painlevé-type expressions for the distribution of the $m$th largest eigenvalue in the Gaussian Orthogonal and Symplectic Ensembles in the edge scaling limit. The work of Johnstone and Soshnikov (see [7], [10]) implies the immediate relevance of our formulas for the $m$th largest eigenvalue of the appropriate Wishart distribution.

1 Introduction

The Gaussian $\beta$–ensembles are probability spaces on $n$-tuples of random variables $\{\lambda_1, \ldots, \lambda_N\}$, with joint density functions

$$P_{N,\beta}(\lambda_1, \ldots, \lambda_N) = P_{N,\beta}(\vec{\lambda}) = C_{N,\beta} \exp \left[ -\frac{1}{2} \beta \sum_{j=1}^{N} \lambda_j^2 \right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \quad (1.1)$$

The $C_{N,\beta}$ are normalization constants, and by setting $\beta = 1, 2, 4$ we recover the Gaussian Orthogonal Ensemble (GOE$_N$), Gaussian Unitary Ensemble (GUE$_N$), and Gaussian Symplectic Ensemble (GSE$_N$), respectively. We restrict ourselves to those three cases in this paper, and refer the reader to [3] for recent results on the general $\beta$ case. Originally the $\lambda_j$ are eigenvalues of randomly chosen matrices from corresponding matrix ensembles, so we will henceforth refer to them as eigenvalues. With the eigenvalues ordered so that $\lambda_j \geq \lambda_{j+1}$, define

$$\hat{\lambda}_m^{(N)} = \frac{\lambda_m - \sqrt{2N}}{2^{-1/2}N^{-1/6}}, \quad (1.2)$$

to be the rescaled $m$th eigenvalue measured from edge of spectrum. A standard result of Random Matrix Theory about the distribution of the largest eigenvalue
in the $\beta$–ensembles is that
\[ \hat{\lambda}_1^{(N)} \xrightarrow{D} \hat{\lambda}_1, \] (1.3)
whose law is given by the Tracy–Widom distributions.

Theorem 1.1 (Tracy, Widom [13], [14]).
\[
F_2(s) := P_{\text{GUE}}(\hat{\lambda}_1 \leq s) = \exp \left[ - \int_s^\infty (x - s) q^2(x) dx \right],
\] (1.4)
\[
F_1^2(s) := \left[ P_{\text{GOE}}(\hat{\lambda}_1 \leq s) \right]^2 = F_2 \cdot \exp \left[ - \int_s^\infty q(x) dx \right],
\] (1.5)
\[
F_4^2(\frac{s}{\sqrt{2}}) := \left[ P_{\text{GSE}}(\hat{\lambda}_1 \leq s) \right]^2 = F_2 \cdot \cosh^2 \left[ - \frac{1}{2} \int_s^\infty q(x) dx \right].
\] (1.6)
The function $q$ is the unique (see [6], [2]) solution to the Painlevé II equation
\[ q'' = x q + 2 q^3, \] (1.7)
such that $q(x) \sim \text{Ai}(x)$ as $x \to \infty$, where $\text{Ai}(x)$ is the solution to the Airy equation which decays like $\frac{1}{\sqrt{2\pi}} x^{-1/4} \exp \left( -\frac{2}{3} x^{3/2} \right)$ at $+\infty$. The density functions $f_\beta$ corresponding to the $F_\beta$ are graphed in Figure 1.

Let $F_2(s, m)$ denote the distribution for the $m^{th}$ largest eigenvalue in GUE. Tracy and Widom showed in [13] that if we define $F_2(s, 0) \equiv 0$, then
\[
F_2(s, m + 1) - F_2(s, m) = (-1)^m \frac{1}{m!} \frac{d^m}{d\lambda^m} D_2(s, \lambda) \bigg|_{\lambda=1}, \quad m \geq 0,
\] (1.8)

*The square root of 2 in the argument of $F_4$ reflects a normalization chosen in [13] to agree with Mehta’s original one. It can be removed by choosing a different normalization.
where
\[ D_2(s, \lambda) = \exp \left[ - \int_s^\infty (x - s) q^2(x, \lambda) dx \right], \quad (1.9) \]
and \( q(x, \lambda) \) is the solution to (1.7) such that \( q(x, \lambda) \sim \sqrt{\lambda} \text{Ai}(x) \) as \( x \to \infty \). An intermediate step leading to (1.9) is to first show that \( D_2(s, \lambda) \) can be expressed as a Fredholm determinant
\[ D_2(s, \lambda) = \det(I - \lambda K_{\text{Ai}}), \quad (1.10) \]
where \( K_{\text{Ai}} \) is the integral operator with kernel
\[ K_{\text{Ai}}(x, y) = \text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y) \quad (x - y). \quad (1.11) \]
In the \( \beta = 1, 4 \) cases a result similar to (1.10) holds with the difference that the operators in \( D_{\beta}(s, \lambda) \) have matrix–valued kernels (see e.g. [11]). In fact, the same combinatorial argument used to obtain the recurrence (1.8) in the \( \beta = 2 \) case also works for the \( \beta = 1, 4 \) cases, leading to
\[ F_{\beta}(s, m + 1) - F_{\beta}(s, m) = \frac{(-1)^m}{m!} \frac{d^m}{d\lambda^m} D_{\beta}^{1/2}(s, \lambda) \bigg|_{\lambda=1}, \quad m \geq 0, \beta = 1, 4, \quad (1.12) \]
where \( F_{\beta}(s, 0) \equiv 0 \). Given the similarity in the arguments up to this point and comparing (1.9) to (1.4), it is natural to conjecture that \( D_{\beta}(s, \lambda), \beta = 1, 4, \) can be obtained simply by replacing \( q(x) \) by \( q(x, \lambda) \) in (1.5) and (1.6).
However the following conjecture, which had long been in the literature, and whose verification is the content of Corollary (2.2), hints that this cannot be the case:

**Conjecture 1.2 (Baik, Rains [11]).** In the appropriate scaling limit, the distribution of the largest eigenvalue in GSE corresponds to that of the second largest in GOE. More generally, the joint distribution of every second eigenvalue in the GOE coincides with the joint distribution of all the eigenvalues in the GSE, with an appropriate number of eigenvalues.

Forrester and Rains subsequently proved (see [5]) the equivalence of alternate GOE eigenvalues and GSE eigenvalues at finite \( N \) ensemble level, lending weight to Conjecture (1.2). This so-called “interlacing property” between GOE and GSE had been noticed by Mehta and Dyson (see [8]). Conjecture (1.2) does not agree with the formulae we postulated for \( D_{\beta}(s, \lambda), \beta = 1, 4 \). Indeed, combining the two leads to incorrect relationships between derivatives of \( q(x, \lambda) \) evaluated at \( \lambda = 1 \). To be precise, the conjecture is true for \( D_1(s, \lambda) \) but it is false for \( D_4(s, \lambda) \). The correct forms for both \( D_{\beta}(s, \lambda), \beta = 1, 4 \) are given below in Theorem (2.1).

This work also extends that of Johnstone in [7] (see also [4]), since \( F_1(s, m) \) gives the asymptotic behavior of the \( m \)th largest eigenvalue of a \( p \) variate Wishart distribution on \( n \) degrees of freedom with identity covariance. This holds under very general conditions on the underlying distribution of matrix
entries by Soshnikov’s universality theorem (see [10] for a precise statement). In Table 1, we compare our distributions to finite $n$ and $p$ empirical Wishart distributions as in [7].

2 Statement of the Main Results

Theorem 2.1. The distributions for the $m$th largest eigenvalues in the GOE and GSE satisfy the recurrence

$$D_1(s, \lambda) = D_2(s, \tilde{\lambda}) \frac{\lambda - 1 - \cosh \mu(s, \tilde{\lambda}) + \sqrt{\lambda} \sinh \mu(s, \tilde{\lambda})}{\lambda - 2},$$

where

$$D_4(s, \lambda) = D_2(s, \lambda) \cosh^2 \left( \frac{\mu(s, \lambda)}{2} \right),$$

and $q(x, \lambda)$ is the solution to (1.7) such that $q(x, \lambda) \sim \sqrt{\lambda} \text{Ai}(x)$ as $x \to \infty$.

Corollary 2.2 (Interlacing property).

$$F_4(s, m) = F_1(s, 2m), \quad m \geq 1.$$  

In the next section we outline the proof of these theorems. In the last, we present an efficient numerical scheme to compute $F_{\beta}(s, m)$. We implemented this scheme using MATLAB, and compared the results to simulated Wishart distributions.

3 Sketch of the Proofs

3.1 Distribution for the Next Largest Eigenvalues and Finite $N$ Gaussian Ensembles

With the joint density function defined as in (1.1), let $J$ be an interval on the real line, and $\chi = \chi_J(x)$ its characteristic function. We denote by $\tilde{\chi} = 1 - \chi$ the characteristic function of the complement of $J$, and define $\tilde{\lambda} = 2\lambda - \lambda^2$. Furthermore, let $E_{\beta,N}(m, J)$ equal the probability that exactly the $m$ largest eigenvalues of a matrix chosen at random from a (finite $N$) $\beta$–ensemble lie in $J$. We also define

$$E_{\beta,N}^{(\lambda)}(J) = \int \cdots \int \tilde{\chi}(x_1) \cdots \tilde{\chi}(x_N) P_N(\lambda \beta, \ldots, \lambda \beta) d x_1 \cdots d x_N.$$
Figure 2: $10^4$ realizations of $10^3 \times 10^3$ GOE matrices; the solid curves are, from right to left, the theoretical limiting densities for the first through fourth largest eigenvalue.

For $\lambda = 1$ this is just $E_{\beta,n}(0,J)$, the probability that no eigenvalues lie in $J$. The following propositions are easy combinatorial facts that can be proved by induction (see e.g. [12]).

**Proposition 3.1.**

$$E_{\beta,N}^{(\lambda)}(J) = \sum_{k=0}^{N} (-\lambda)^k \binom{N}{k} \int \cdots \int P_{\beta}(x_1, \ldots, x_N) \, dx_1 \cdots dx_N. \quad (3.2)$$

**Proposition 3.2.**

$$E_{\beta,N}(m,J) = \left. \frac{(-1)^m}{m!} \frac{d^m}{d\lambda^m} E_{\beta,N}^{(\lambda)}(J) \right|_{\lambda=1}, \quad m \geq 0. \quad (3.3)$$

The next step is to find a useful expression for the multiple integral $E_{\beta,N}^{(\lambda)}(J)$. It turns out that through standard RMT techniques (see e.g. [15]), the integral can be expressed as the determinant of an operator on the Hilbert space $L^2(J) \times L^2(J)$. Let

$$D_{\beta,N}(s,\lambda) = \det(I - \lambda K_{\beta,N}) \quad , \quad \beta = 1,4, \quad (3.4)$$

for

$$K_{1,N} = \chi \begin{pmatrix} S + \psi \otimes \epsilon \varphi & SD - \psi \otimes \varphi \\ \epsilon S - \epsilon \psi \otimes \epsilon \varphi & S + \epsilon \varphi \otimes \psi \end{pmatrix} \chi. \quad (3.5)$$


Here $\epsilon$ is the integral operator with kernel $\epsilon(x-y) = \frac{1}{2} \text{sgn}(x-y)$, and $D$ denotes the differentiation operator $\frac{d}{dx}$. $S$ is the integral operator with kernel

$$S(x,y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y},$$

and the functions $\varphi$ and $\psi$ are

$$\varphi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_N(x), \quad (3.7)$$

$$\psi(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_{N-1}(x), \quad (3.8)$$

where

$$\varphi_N(x) = \frac{1}{\sqrt{2^N N! \sqrt{\pi}}} e^{-x^2/2} H_N(x), \quad (3.9)$$

and the $H_N(x)$ are the classical Hermite polynomials. This implies that the $\varphi_n(x)$ are orthonormal with respect to the Lebesgue measure on $\mathbb{R}$. Similarly, let

$$K_{4,N} = \frac{1}{2} \chi \left( \begin{array}{ccc} S + \psi \otimes \epsilon \varphi & SD - \psi \otimes \varphi \\ \epsilon S + \epsilon \psi \otimes \epsilon \varphi & S + \varphi \otimes \epsilon \psi \end{array} \right) \chi. \quad (3.10)$$

Following the same approach as in [15] and [14], we arrive at

$$E_{\beta,N}^{(\lambda)}(J) = D_{\beta,N}^{1/2}(s,\lambda). \quad (3.11)$$

### 3.2 Edge-Scaling

#### 3.2.1 The GOE case: reduction of the determinant

The above determinants are Fredholm determinants of operators on $L^2(J) \times L^2(J)$. Our first task will be to rewrite these determinants as those of operators on $L^2(J)$. This part follows exactly the proof in [13]. To begin, note that

$$[S,D] = \varphi \otimes \psi + \psi \otimes \varphi \quad (3.12)$$

so that (using the fact that $D \epsilon = \epsilon D = I$)

$$[\epsilon,S] = \epsilon S - S \epsilon = \epsilon S D \epsilon - \epsilon D S \epsilon = \epsilon [S,D] \epsilon = \epsilon \varphi \otimes \psi \epsilon + \epsilon \psi \otimes \varphi \epsilon = \epsilon \varphi \otimes \epsilon^t \psi + \epsilon \psi \otimes \epsilon^t \varphi = -\epsilon \varphi \otimes \epsilon^t \psi - \epsilon \psi \otimes \epsilon^t \varphi, \quad (3.13)$$

where the last equality follows from the fact that $\epsilon^t = -\epsilon$. We thus have
\[
D (\epsilon S + \epsilon \psi \otimes \epsilon \varphi) = S + \psi \otimes \epsilon \varphi, \\
D (\epsilon SD - \epsilon \psi \otimes \varphi) = SD - \psi \otimes \varphi.
\]

The expressions on the right side are the top entries of \(K_{1,N}\). Thus the first row of \(K_{1,N}\) is, as a vector,
\[
D (\epsilon S + \epsilon \psi \otimes \epsilon \varphi, \epsilon SD - \epsilon \psi \otimes \varphi).
\]

Now (3.13) implies that
\[
\epsilon S + \epsilon \psi \otimes \epsilon \varphi = S \epsilon - \epsilon \varphi \otimes \epsilon \psi.
\]
Similarly (3.12) gives
\[
\epsilon [S, D] = \epsilon \varphi \otimes \psi + \epsilon \psi \otimes \varphi,
\]
so that
\[
\epsilon SD - \epsilon \psi \otimes \varphi = \epsilon DS + \epsilon \varphi \otimes \psi = S + \epsilon \varphi \otimes \psi.
\]

Using these expressions we can rewrite the first row of \(K_{1,N}\) as
\[
D (S \epsilon - \epsilon \varphi \otimes \epsilon \psi, S + \epsilon \varphi \otimes \psi).
\]

Applying \(\epsilon\) to this expression shows the second row of \(K_{1,N}\) is given by
\[
(\epsilon S - \epsilon \psi \otimes \epsilon \varphi, S + \epsilon \varphi \otimes \psi)
\]
Now use (3.13) to show the second row of \(K_{1,N}\) is
\[
(S \epsilon - \epsilon \varphi \otimes \epsilon \psi, S + \epsilon \varphi \otimes \psi).
\]

Therefore,
\[
K_{1,N} = \chi \begin{pmatrix}
D (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) & D (S + \epsilon \varphi \otimes \psi) \\
S \epsilon - \epsilon + \epsilon \varphi \otimes \epsilon \psi & S + \epsilon \varphi \otimes \psi
\end{pmatrix} \chi
\]
\[
= \begin{pmatrix}
\chi D & 0 \\
0 & \chi
\end{pmatrix} \begin{pmatrix}
(S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \chi & (S + \epsilon \varphi \otimes \psi) \chi \\
(S \epsilon - \epsilon + \epsilon \varphi \otimes \epsilon \psi) \chi & (S + \epsilon \varphi \otimes \psi) \chi
\end{pmatrix}.
\]

Since \(K_{1,N}\) is of the form \(AB\), we can use the fact that \(\det(I - AB) = \det(I - BA)\) and deduce that \(D_{1,N}(s, \lambda)\) is unchanged if instead we take \(K_{1,N}\) to be
\[
K_{1,N} = \begin{pmatrix}
(S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \chi & (S + \epsilon \varphi \otimes \psi) \chi \\
(S \epsilon - \epsilon + \epsilon \varphi \otimes \epsilon \psi) \chi & (S + \epsilon \varphi \otimes \psi) \chi
\end{pmatrix} \begin{pmatrix}
\chi D & 0 \\
0 & \chi
\end{pmatrix}
\]
\[
= \begin{pmatrix}
(S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \chi D & (S + \epsilon \varphi \otimes \psi) \chi \\
(S \epsilon - \epsilon + \epsilon \varphi \otimes \epsilon \psi) \chi D & (S + \epsilon \varphi \otimes \psi) \chi
\end{pmatrix}.
\]
Therefore

\[ D_{1,N}(s,\lambda) = \det \begin{pmatrix} I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D & - (S + \epsilon \varphi \otimes \psi) \lambda \chi \\ - (S \epsilon - \epsilon + \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D & I - (S + \epsilon \varphi \otimes \psi) \lambda \chi \end{pmatrix}. \]  

(3.14)

Now we perform row and column operations on the matrix to simplify it, which do not change the Fredholm determinant. Justification of these operations is given in [14]. We start by subtracting row 1 from row 2 to get

\[ \begin{pmatrix} I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D & - (S + \epsilon \varphi \otimes \psi) \lambda \chi \\ I + \epsilon \lambda \chi D & I \end{pmatrix}. \]

Next, adding column 2 to column 1 yields

\[ \begin{pmatrix} I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D - (S + \epsilon \varphi \otimes \psi) \lambda \chi & - (S + \epsilon \varphi \otimes \psi) \lambda \chi \\ \lambda \chi D & \lambda \chi \end{pmatrix}. \]

Then right-multiply column 1 by \(-\epsilon \lambda \chi D\) and add it to column 1 to get

\[ \begin{pmatrix} I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D - (S + \epsilon \varphi \otimes \psi) \lambda \chi (\epsilon \lambda \chi D - I) & - (S + \epsilon \varphi \otimes \psi) \chi \\ 0 & 0 \end{pmatrix}. \]

Finally we multiply row 2 by \(S + \epsilon \varphi \otimes \psi\) and add it to row 1 to arrive at

\[ \det \begin{pmatrix} I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D + (S + \epsilon \varphi \otimes \psi) \lambda \chi (\epsilon \lambda \chi D - I) & 0 \\ 0 & I \end{pmatrix}. \]

Thus the determinant we want equals the determinant of

\[ I - (S \epsilon - \epsilon \varphi \otimes \epsilon \psi) \lambda \chi D + (S + \epsilon \varphi \otimes \psi) \lambda \chi (\epsilon \lambda \chi D - I). \]  

(3.15)

So we have reduced the problem from the computation of the Fredholm determinant of an operator on \(L^2(J) \times L^2(J)\), to that of an operator on \(L^2(J)\).

### 3.2.2 The GOE Case: differential equations

Next we want to write the operator in (3.15) in the form

\[ \left( I - K_{2,N} \right) \left( I - \sum_{i=1}^{L} \alpha_i \otimes \beta_i \right), \]  

(3.16)

where the \(\alpha_i\) and \(\beta_i\) are functions in \(L^2(J)\). In other words, we want to rewrite the determinant for the GOE case as a finite dimensional perturbation of the corresponding GUE determinant. The Fredholm determinant of the product is then the product of the determinants. The limiting form for the GUE part
is already known, and we can just focus on finding a limiting form for the
determinant of the finite dimensional piece. It is here that the proof must be
modified from that in [14]. A little simplification of (3.15) yields

\[ I - \lambda S \chi - \lambda (1 - \lambda) \epsilon \chi D - \lambda (\epsilon \varphi \otimes \chi \psi) - \lambda (\epsilon \varphi \otimes \psi) (1 - \lambda \chi) \epsilon \chi D. \]

Writing \[ \epsilon [\chi, D] + \chi \] for \[ \epsilon \chi D \] and simplifying \[ (1 - \lambda \chi) \chi \] to \[ (1 - \lambda) \chi \] gives

\[ I - \lambda S \chi - \lambda (1 - \lambda) \epsilon \chi D - \lambda (\epsilon \varphi \otimes \chi \psi) - \lambda (\epsilon \varphi \otimes \psi) (1 - \lambda) \epsilon [\chi, D]. \]

Define \[ \tilde{\lambda} = 2 \lambda - \lambda^2 \] and let \[ \sqrt{\tilde{\lambda}} \varphi \to \varphi, \] and \[ \sqrt{\tilde{\lambda}} \psi \to \psi \] so that \[ \tilde{\lambda} S \to S \] and \[ (3.15) \] goes to

\[ I - S \chi - (\epsilon \varphi \otimes \chi \psi) - \frac{\tilde{\lambda}}{\lambda} S \epsilon \chi D - \lambda (\epsilon \varphi \otimes \psi) (1 - \lambda \chi) \epsilon [\chi, D]. \]

Now we define \[ R := (I - S \chi)^{-1} S \chi = (I - S \chi)^{-1} - I \] (the resolvent of \[ S \chi \]), whose kernel we denote by \[ R(x, y) \], and \[ Q_\epsilon := (I - S \chi)^{-1} \epsilon \varphi \]. Then \[ (3.15) \] factors into

\[ A = (I - S \chi) B. \]

where \( B \) is

\[ I - (Q_\epsilon \otimes \chi \psi) - \frac{\lambda}{\tilde{\lambda}} (I + R) S \epsilon \chi D - \lambda (Q_\epsilon \otimes \psi) (1 - \lambda \chi) \epsilon [\chi, D], \quad \lambda \neq 1. \]

Hence

\[ D_{1, N}(s, \lambda) = D_{2, N}(s, \tilde{\lambda}) \det(B). \]

In order to find \( \det(B) \) we use the identity

\[ \epsilon [\chi, D] = \sum_{k=1}^{2m} (-1)^k \epsilon_k \otimes \delta_k, \quad (3.17) \]

where \( \epsilon_k \) and \( \delta_k \) are the functions \( \epsilon(x - a_k) \) and \( \delta(x - a_k) \) respectively, and the \( a_k \)
are the endpoints of the (disjoint) intervals considered, \( J = \cup_{k=1}^{m} (a_{2k-1}, a_{2k}). \)

We also make use of the fact that

\[ a \otimes b \cdot c \otimes d = (b \cdot c) \cdot a \otimes d \quad (3.18) \]
where \((\ldots)\) is the usual \(L^2\)–inner product. Therefore

\[
(Q \otimes \psi) (1 - \lambda \chi) \epsilon [\chi, D] = \sum_{k=1}^{2m} (-1)^k Q \otimes \psi \cdot (1 - \lambda \chi) \epsilon_k \otimes \delta_k
\]

\[
= \sum_{k=1}^{2m} (-1)^k (\psi, (1 - \lambda \chi) \epsilon_k) Q \otimes \delta_k.
\]

It follows that

\[
\frac{D_{1,N}(s, \lambda)}{D_{2,N}(s, \lambda)}
\]

equals the determinant of

\[
I - Q \otimes \chi \psi
\]

\[
- \frac{\lambda}{2\lambda} \sum_{k=1}^{2m} (-1)^k [(S + RS) (1 - \lambda \chi) \epsilon_k + (\psi, (1 - \lambda \chi) \epsilon_k) Q] \otimes \delta_k.
\]

We now specialize to the case of one interval \(J = (t, \infty)\), so \(m = 1\), \(a_1 = t\) and \(a_2 = \infty\). We write \(\epsilon_t = \epsilon_1\), and \(\epsilon_\infty = \epsilon_2\), and similarly for \(\delta_k\). Writing the terms in the summation and using the facts that

\[
\epsilon_\infty = -\frac{1}{2},
\]

and

\[
(1 - \lambda \chi) \epsilon_t = -\frac{1}{2} (1 - \lambda \chi) + (1 - \lambda \chi) \chi,
\]

then yields

\[
I - Q \otimes \chi \psi - \frac{\lambda}{2\lambda} [(S + RS) (1 - \lambda \chi) + (\psi, (1 - \lambda \chi) \chi) Q] \otimes (\delta_t - \delta_\infty)
\]

\[
+ \frac{\lambda}{\lambda} [(S + RS) (1 - \lambda \chi) \chi + (\psi, (1 - \lambda \chi) \chi) Q] \otimes \delta_t
\]

which, to simplify notation, we write as

\[
I - Q \otimes \chi \psi - \frac{\lambda}{2\lambda} [(S + RS) (1 - \lambda \chi) + a_{1,\lambda} Q] \otimes (\delta_t - \delta_\infty)
\]

\[
+ \frac{\lambda}{\lambda} [(S + RS) (1 - \lambda \chi) \chi + \tilde{a}_{1,\lambda} Q] \otimes \delta_t,
\]

where

\[
a_{1,\lambda} = (\psi, (1 - \lambda \chi)), \quad \tilde{a}_{1,\lambda} = (\psi, (1 - \lambda \chi) \chi).
\]

Now we can use the formula:

\[
\det \left( I - \sum_{i=1}^{L} \alpha_i \otimes \beta_i \right) = \det (\delta_{jk} - (\alpha_j, \beta_k)_{1 \leq j,k \leq L})
\]
In this case, \( L = 3 \), and
\[
\begin{align*}
\alpha_1 &= Q, \\
\alpha_2 &= \frac{\lambda}{\lambda} [(S + RS)(1 - \lambda \chi) + a_{1,\lambda} Q], \\
\alpha_3 &= -\frac{\lambda}{\lambda} [(S + RS)(1 - \lambda \chi) + \hat{a}_{1,\lambda} Q], \\
\beta_1 &= \chi \psi, \\
\beta_2 &= \delta_t - \delta_\infty, \\
\beta_3 &= \delta_t.
\end{align*}
\] (3.23)

In order to simplify the notation, define
\[
\begin{align*}
Q(x, \lambda, t) &= (I - S \chi)^{-1} \varphi, \\
P(x, \lambda, t) &= (I - S \chi)^{-1} \psi, \\
Q_\epsilon(x, \lambda, t) &= (I - S \chi)^{-1} \epsilon \varphi, \\
P_\epsilon(x, \lambda, t) &= (I - S \chi)^{-1} \epsilon \psi, \\
q_N(t) &= Q(t, \lambda, t), \\
p_N(t) &= P(t, \lambda, t), \\
q_\epsilon(t) &= Q_\epsilon(t, \lambda, t), \\
p_\epsilon(t) &= P_\epsilon(t, \lambda, t), \\
u_\epsilon &= (Q, \chi \epsilon \varphi) = (Q_\epsilon, \chi \varphi), \\
u_\epsilon &= (Q, \chi \epsilon \psi) = (P_\epsilon, \chi \psi), \\
v_\epsilon &= (P, \chi \epsilon \varphi) = (Q_\epsilon, \chi \varphi), \\
w_\epsilon &= (P, \chi \epsilon \psi) = (P_\epsilon, \chi \psi), \\
P_{1,\lambda} &= \int (1 - \lambda \chi) P \, dx, \\
\hat{P}_{1,\lambda} &= \int (1 - \lambda \chi) \chi P \, dx, \\
Q_{1,\lambda} &= \int (1 - \lambda \chi) Q \, dx, \\
\hat{Q}_{1,\lambda} &= \int (1 - \lambda \chi) \chi Q \, dx, \\
R_{1,\lambda} &= \int (1 - \lambda \chi) R(x, t) \, dx, \\
\hat{R}_{1,\lambda} &= \int (1 - \lambda \chi) \chi R(x, t) \, dx.
\end{align*}
\] (3.24)

Note that all quantities in (3.25) and (3.26) are functions of \( t \) alone. Furthermore, let
\[
\begin{align*}
c_\varphi &= \epsilon \varphi(\infty) = \frac{1}{2} \int_{-\infty}^\infty \varphi(x) \, dx, \\
c_\psi &= \epsilon \psi(\infty) = \frac{1}{2} \int_{-\infty}^\infty \psi(x) \, dx.
\end{align*}
\] (3.27)

From [14] we find
\[
\lim_{N \to \infty} c_\varphi = \sqrt{\frac{\lambda}{2}}, \quad \lim_{N \to \infty} c_\psi = 0,
\] (3.28)

and at \( t = \infty \),
\[
\begin{align*}
P_{1,\lambda}(\infty) &= 2 c_\varphi, \\
Q_{1,\lambda}(\infty) &= 2 c_\varphi, \\
R_{1,\lambda}(\infty) &= 0, \\
\hat{P}_{1,\lambda}(\infty) &= \hat{Q}_{1,\lambda}(\infty) = \hat{R}_{1,\lambda}(\infty) = 0.
\end{align*}
\]
Hence
\begin{align}
(\alpha_1, \beta_1) &= \tilde{\nu}_\epsilon, \quad (\alpha_1, \beta_2) = q_\epsilon - c_\varphi, \quad (\alpha_1, \beta_3) = q_\epsilon, \\
(\alpha_2, \beta_1) &= \frac{\lambda}{2\lambda} [\mathcal{P}_{1,\lambda} - a_{1,\lambda} (1 - \tilde{\nu}_\epsilon)], \\
(\alpha_2, \beta_2) &= \frac{\lambda}{2\lambda} [\mathcal{R}_{1,\lambda} + a_{1,\lambda} (q_\epsilon - c_\varphi)], \\
(\alpha_2, \beta_3) &= \frac{\lambda}{2\lambda} [\mathcal{R}_{1,\lambda} + a_{1,\lambda} q_\epsilon], \\
(\alpha_3, \beta_1) &= -\frac{\lambda}{\lambda} \left[ \tilde{\mathcal{P}}_{1,\lambda} - \tilde{a}_{1,\lambda} (1 - \tilde{\nu}_\epsilon) \right], \\
(\alpha_3, \beta_2) &= -\frac{\lambda}{\lambda} \left[ \tilde{\mathcal{R}}_{1,\lambda} + \tilde{a}_{1,\lambda} (q_\epsilon - c_\varphi) \right], \\
(\alpha_3, \beta_3) &= -\frac{\lambda}{\lambda} \left[ \tilde{\mathcal{R}}_{1,\lambda} + \tilde{a}_{1,\lambda} q_\epsilon \right].
\end{align}

As an illustration, let us do the computation that led to (3.31) in detail. As in [14], we use the facts that \( S = S \), and \( (S + S R^t) \chi = R \) which can be easily seen by writing \( R = \sum_{k=1}^{\infty} (S \chi)^k \). Furthermore we write \( R(x, t) \) to mean \( \lim_{y \to t} R(x, y) \). In general, since all evaluations are done by taking the limits from within \( J \), we can use the identity \( \chi \delta_k = \delta_k \) inside the inner products. Thus
\begin{align}
(\alpha_2, \beta_2) &= \frac{\lambda}{\lambda} \left[ ((S + R S) (1 - \lambda \chi), \delta_t - \delta_\infty) + a_{1,\lambda} (Q_\epsilon, \delta_t - \delta_\infty) \right] \\
&= \frac{\lambda}{\lambda} \left[ ((1 - \lambda \chi), (S + R^t S) \chi (\delta_t - \delta_\infty)) + a_{1,\lambda} (Q_\epsilon(t) - Q_\epsilon(\infty)) \right] \\
&= \frac{\lambda}{\lambda} \left[ ((1 - \lambda \chi), (S + R^t S) \chi (\delta_t - \delta_\infty)) + a_{1,\lambda} (q_\epsilon(t) - c_\varphi) \right] \\
&= \frac{\lambda}{\lambda} \left[ (1 - \lambda \chi), R(x, t) - R(x, \infty) \right] + a_{1,\lambda} (q_\epsilon(t) - c_\varphi) \\
&= \frac{\lambda}{\lambda} [\mathcal{R}_{1,\lambda}(t) - \mathcal{R}_{1,\lambda}(\infty) + a_{1,\lambda} (q_\epsilon - c_\varphi)] \\
&= \frac{\lambda}{\lambda} [\mathcal{R}_{1,\lambda}(t) + a_{1,\lambda} (q_\epsilon - c_\varphi)].
\end{align}

We want the limit of the determinant
\[
\det (\delta_{jk} - (\alpha_j, \beta_k))_{1 \leq j, k \leq L},
\]
as \( N \to \infty \). In order to get our hands on the limits of the individual terms involved in the determinant, we will find differential equations for them first as in [14].
Row operation on the matrix show that $a_{1,\lambda}$ and $\tilde{a}_{1,\lambda}$ fall out of the determinant; to see this add $\lambda a_{1,\lambda}/(2\lambda)$ times row 1 to row 2 and $\lambda \tilde{a}_{1,\lambda}/\lambda$ times row 1 to row 3. So we will not need to find differential equations for them. Our determinant is

$$
\det \begin{pmatrix}
1 - \bar{v}_\epsilon & -(q_\epsilon - c_\varphi) & -q_\epsilon \\
-\frac{\lambda \tilde{P}_{1,\lambda}}{2\lambda} & 1 - \frac{\lambda \tilde{R}_{1,\lambda}}{2\lambda} & -\frac{\lambda \tilde{R}_{1,\lambda}}{2\lambda} \\
\frac{\lambda \tilde{R}_{1,\lambda}}{\lambda} & -\frac{\lambda \tilde{R}_{1,\lambda}}{\lambda} & 1 + \frac{\lambda \tilde{R}_{1,\lambda}}{\lambda}
\end{pmatrix}.
$$

(3.37)

Proceeding as in [14] we find the following differential equations

$$
\frac{d}{dt} u_\epsilon = q_N q_\epsilon,
\frac{d}{dt} q_\epsilon = q_N - q_N \bar{v}_\epsilon - p_N u_\epsilon,
$$

(3.38)

$$
\frac{d}{dt} Q_{1,\lambda} = q_N (\lambda - \tilde{R}_{1,\lambda}),
\frac{d}{dt} P_{1,\lambda} = p_N (\lambda - \tilde{R}_{1,\lambda}),
$$

(3.39)

$$
\frac{d}{dt} R_{1,\lambda} = -p_N Q_{1,\lambda} - q_N P_{1,\lambda},
\frac{d}{dt} \tilde{R}_{1,\lambda} = -p_N \tilde{Q}_{1,\lambda} - q_N \tilde{P}_{1,\lambda},
$$

(3.40)

$$
\frac{d}{dt} \tilde{Q}_{1,\lambda} = q_N \left(\lambda - 1 - \tilde{R}_{1,\lambda}\right),
\frac{d}{dt} \tilde{P}_{1,\lambda} = p_N \left(\lambda - 1 - \tilde{R}_{1,\lambda}\right).
$$

(3.41)

Let us derive the first equation in (3.39) for example. From (3.37) (equation 2.17), we have

$$
\frac{\partial Q}{\partial t} = -R(x,t) q_N.
$$

Therefore

$$
\frac{\partial Q_{1,\lambda}}{\partial t} = \frac{d}{dt} \left[ \int_{\tilde{Q}_{1,\lambda}}^t Q(x,t) \, dx - (1 - \lambda) \int_{\tilde{Q}_{1,\lambda}}^t Q(x,t) \, dx \right]
= q_N + \int_{\tilde{Q}_{1,\lambda}}^t \frac{\partial Q}{\partial t} \, dx - (1 - \lambda) \left[ q_N + \int_{\tilde{Q}_{1,\lambda}}^t \frac{\partial Q}{\partial t} \, dx \right]
= q_N - q_N \int_{\tilde{Q}_{1,\lambda}}^t R(x,t) \, dx - (1 - \lambda) q_N + (1 - \lambda) q_N \int_{\tilde{Q}_{1,\lambda}}^t R(x,t) \, dx
= \lambda q_N - q_N \int_{\tilde{Q}_{1,\lambda}}^t (1 - \lambda) R(x,t) \, dx
= \lambda q_N - q_N \tilde{R}_{1,\lambda} = q_N (\lambda - \tilde{R}_{1,\lambda}).
$$

Now we change variable from $t$ to $s$ where $t = \tau(s) = 2 \sigma \sqrt{N} + \frac{q_N}{N^{1/6}}$. Then we take the limit $N \to \infty$, denoting the limits of $u_\epsilon$, $P_{1,\lambda}$, $Q_{1,\lambda}$, $R_{1,\lambda}$, $\tilde{P}_{1,\lambda}$, $\tilde{Q}_{1,\lambda}$, $\tilde{R}_{1,\lambda}$ and the common limit of $u_\epsilon$ and $\tilde{v}_\epsilon$ respectively by $\underline{u}_{1,\lambda}$, $\underline{Q}_{1,\lambda}$, $\underline{R}_{1,\lambda}$, $\underline{P}_{1,\lambda}$, $\underline{\tilde{Q}}_{1,\lambda}$, $\underline{\tilde{R}}_{1,\lambda}$ and $\underline{v}$. We eliminate $\underline{Q}_{1,\lambda}$ and $\underline{R}_{1,\lambda}$ by using the facts that $\underline{Q}_{1,\lambda} = \underline{P}_{1,\lambda} + \lambda \sqrt{2}$ and $\underline{Q}_{1,\lambda} = \underline{P}_{1,\lambda}$. These limits hold uniformly for bounded $s$ so we can interchange $\lim$ and $\frac{d}{ds}$. Also $\lim_{N \to \infty} N^{-1/6} q_N = \lim_{N \to \infty} N^{-1/6} p_N = q$, where $q$
is as in (1.9). We obtain the systems

\[ \frac{d}{ds} \mu = -\frac{1}{\sqrt{2}} q, \quad \frac{d}{ds} q = \frac{1}{\sqrt{2}} q \left( 1 - 2 \mu \right), \]  

(3.42)

\[ \frac{d}{ds} P_{1,\lambda} = -\frac{1}{\sqrt{2}} q \left( R_{1,\lambda} - \lambda \right), \quad \frac{d}{ds} R_{1,\lambda} = -\frac{1}{\sqrt{2}} q \left( 2 P_{1,\lambda} + \sqrt{2} \lambda \right), \]  

(3.43)

\[ \frac{d}{ds} \bar{P}_{1,\lambda} = \frac{1}{\sqrt{2}} \left( 1 - \lambda - \bar{R}_{1,\lambda} \right), \quad \frac{d}{ds} \bar{R}_{1,\lambda} = -q \sqrt{2} \bar{P}_{1,\lambda}. \]  

(3.44)

The change of variables \( q \rightarrow \mu = \int_{s}^{\infty} q(x) \, dx \) transforms these systems into

\[ \frac{d}{d\mu} \mu = \frac{1}{\sqrt{2}} q, \quad \frac{d}{d\mu} q = -\frac{1}{\sqrt{2}} \left( 1 - 2 \mu \right), \]  

(3.45)

\[ \frac{d}{d\mu} P_{1,\lambda} = \frac{1}{\sqrt{2}} \left( R_{1,\lambda} - \lambda \right), \quad \frac{d}{d\mu} R_{1,\lambda} = \frac{1}{\sqrt{2}} \left( 2 P_{1,\lambda} + \sqrt{2} \lambda \right), \]  

(3.46)

\[ \frac{d}{d\mu} \bar{P}_{1,\lambda} = -\frac{1}{\sqrt{2}} \left( 1 - \lambda - \bar{R}_{1,\lambda} \right), \quad \frac{d}{d\mu} \bar{R}_{1,\lambda} = \sqrt{2} \bar{P}_{1,\lambda}. \]  

(3.47)

Since \( \lim_{s \rightarrow \infty} \mu = 0 \), corresponding to the boundary values at \( t = \infty \) which we found earlier for \( P_{1,\lambda}, R_{1,\lambda}, \bar{P}_{1,\lambda}, \bar{R}_{1,\lambda} \), we now have initial values at \( \mu = 0 \).

Therefore

\[ P_{1,\lambda}(0) = \bar{P}_{1,\lambda}(0) = P_{1,\lambda}(0) = \bar{P}_{1,\lambda}(0) = 0. \]  

(3.48)

We use this to solve the systems and get

\[ q = \frac{\sqrt{\lambda} - 1}{2 \sqrt{2}} e^{\mu} + \frac{\sqrt{\lambda} + 1}{2 \sqrt{2}} e^{-\mu}, \]  

(3.49)

\[ \mu = \frac{\sqrt{\lambda} - 1}{4} e^{\mu} - \frac{\sqrt{\lambda} + 1}{4} e^{-\mu} + \frac{1}{2}, \]  

(3.50)

\[ P_{1,\lambda} = \frac{\sqrt{\lambda} - \lambda}{2 \sqrt{2}} e^{\mu} + \frac{\sqrt{\lambda} + \lambda}{2 \sqrt{2}} e^{-\mu} - \frac{\sqrt{\lambda}}{2}, \]  

(3.51)

\[ R_{1,\lambda} = \frac{\sqrt{\lambda} - \lambda}{2} e^{\mu} - \frac{\sqrt{\lambda} + \lambda}{2} e^{-\mu} + \lambda, \]  

(3.52)

\[ \bar{P}_{1,\lambda} = \frac{1}{2 \sqrt{2}} (e^{\mu} - e^{-\mu}), \quad \bar{R}_{1,\lambda} = \frac{1 - \lambda}{2} (e^{\mu} + e^{-\mu} - 2). \]  

(3.53)

Substituting these expressions into the determinant gives \( D_{1} \), namely

\[ D_{1}(s, \lambda) = D_{2}(s, \lambda) \frac{\lambda - 1 - \cosh \mu(s, \lambda) + \sqrt{\lambda} \sinh \mu(s, \lambda)}{\lambda - 2}, \]  

(3.54)

where \( D_{\beta} = \lim_{N \rightarrow \infty} D_{\beta,N} \).
3.3 The GSE Case

The GSE case is the easy one. All calculations in [15] and [14] go through essentially unchanged except for the trailing factor of $\lambda$. Therefore we will not reproduce them here.

3.4 Interlacing property

The following series of lemmas establish Corollary (2.2):

**Lemma 3.3.** Define

$$a_j = \frac{d^j}{d \lambda^j} \sqrt{\frac{\lambda}{2 - \lambda}} \bigg|_{\lambda=1}. \quad (3.55)$$

Then $a_j$ satisfies the following recursion

$$a_j = \begin{cases} 
1 & \text{if } j = 0, \\
(j-1)a_{j-1} & \text{for } j \geq 1, j \text{ even}, \\
ja_{j-1} & \text{for } j \geq 1, j \text{ odd}.
\end{cases} \quad (3.56)$$

**Proof.** Consider the expansion of the generating function $f(\lambda) = \sqrt{\frac{\lambda}{2-\lambda}}$ around $\lambda = 1$

$$f(\lambda) = \sum_{j \geq 0} \frac{a_j}{j!} (\lambda - 1)^j = \sum_{j \geq 0} b_j (\lambda - 1)^j$$

Since $a_j = j!b_j$, the statement of the lemma reduces to proving the following recurrence for the $b_j$

$$b_j = \begin{cases} 
1 & \text{if } j = 0, \\
\frac{j-1}{j} b_{j-1} & \text{for } j \geq 1, j \text{ even}, \\
b_{j-1} & \text{for } j \geq 1, j \text{ odd}.
\end{cases} \quad (3.57)$$

Let

$$f^{\text{even}}(\lambda) = \frac{1}{2} \left( \sqrt{\frac{\lambda}{2-\lambda}} + \sqrt{\frac{2-\lambda}{\lambda}} \right), \quad f^{\text{odd}}(\lambda) = \frac{1}{2} \left( \sqrt{\frac{\lambda}{2-\lambda}} - \sqrt{\frac{2-\lambda}{\lambda}} \right).$$

These are the even and odd parts of $f$ relative to the reflection $\lambda - 1 \to -(\lambda - 1)$ or $\lambda \to 2 - \lambda$. Recurrence (3.57) is equivalent to

$$\frac{d}{d \lambda} f^{\text{even}}(\lambda) = (\lambda - 1) \frac{d}{d \lambda} f^{\text{odd}}(\lambda)$$

which is easily shown to be true. \qed
Lemma 3.4. Define
\[ f(s, \lambda) = 1 - \sqrt{\frac{\lambda}{2 - \lambda}} \tanh \frac{\mu(s, \tilde{\lambda})}{2}, \] (3.58)
for \( \tilde{\lambda} = 2\lambda - \lambda^2 \). Then
\[ \frac{\partial^{2n}}{\partial \lambda^{2n}} f(s, \lambda) \bigg|_{\lambda=1} - \frac{1}{2n + 1} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} f(s, \lambda) \bigg|_{\lambda=1} = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ if } n \geq 1. \end{cases} \] (3.59)

Proof. The case \( n = 0 \) is readily checked. The main ingredient for the general case is Faà di Bruno’s formula
\[ \frac{d^n}{dt^n} g(h(t)) = \sum_{k_1! \cdots k_n!} \frac{n!}{k_1! \cdots k_n!} \left( \frac{d^k g}{dh^k}(h(t)) \right) \left( \frac{1}{1!} \frac{dh}{dt} \right)^{k_1} \cdots \left( \frac{1}{n!} \frac{dh}{dt^n} \right)^{k_n}, \] (3.60)
where \( k = \sum_{i=1}^n k_i \) and the above sum is over all partitions of \( n \), that is all values of \( k_1, \ldots, k_n \) such that \( \sum_{i=1}^n i k_i = n \). We apply Faà di Bruno’s formula to derivatives of the function \( \tanh \frac{\mu(s, \tilde{\lambda})}{2} \), which we treat as some function \( \tilde{\lambda}(\lambda) \).

Notice that for \( j \geq 1 \), \( \frac{\partial \tilde{\lambda}}{\partial \lambda} \bigg|_{\lambda=1} \) is nonzero only when \( j = 2 \), in which case it equals \(-2\). Hence, in (3.55), the only term that survives is the one corresponding to the partition all of whose parts equal 2. Thus we have
\[ \frac{\partial^{2n-k}}{\partial \lambda^{2n-k}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1} = \begin{cases} 0 & \text{if } k = 2j + 1, j \geq 0 \\ \frac{(-1)^{n-j}}{(n-j)!} \frac{\partial^{n-j}}{\partial \lambda^{n-j}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1} & \text{for } k = 2j, j \geq 0 \end{cases} \]
\[ \frac{\partial^{2n-k+1}}{\partial \lambda^{2n+1-k}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1} = \begin{cases} 0 & \text{if } k = 2j, j \geq 0 \\ \frac{(-1)^{n-j}}{(n-j)!} \frac{\partial^{n-j}}{\partial \lambda^{n-j}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1} & \text{for } k = 2j + 1, j \geq 0 \end{cases} \]

Therefore, recalling the definition of \( a_j \) in (3.55) and setting \( k = 2j \), we obtain
\[ \frac{\partial^{2n}}{\partial \lambda^{2n}} f(s, \lambda) \bigg|_{\lambda=1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{\partial^k}{\partial \lambda^k} \sqrt{\frac{\lambda}{2 - \lambda}} \frac{\partial^{2n-k}}{\partial \lambda^{2n-k}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1} \]
\[ = \sum_{j=0}^{n} \binom{2n}{2j} \frac{(-1)^{n-j}}{(2j)! (n-j)!} a_{2j} \frac{\partial^{n-j}}{\partial \lambda^{n-j}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \bigg|_{\lambda=1}. \]
Similarly, using $k = 2j + 1$ instead yields
\[
\frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} f(s, \lambda) \bigg|_{\lambda=1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\partial^k}{\partial \lambda^k} \sqrt{\frac{\lambda}{2 - \lambda}} \frac{\partial^{2n+1-k}}{\partial \lambda^{2n+1-k}} \tanh \frac{\mu(s, \lambda)}{2} \bigg|_{\lambda=1} = (2n + 1) \sum_{j=0}^{n} \frac{(2n)!}{(2j)!(n-j)!} \frac{a_{2j+1} (n-j)!}{2j+1} \frac{\partial^{n-j}}{\partial \lambda^{n-j}} \tanh \frac{\mu(s, \lambda)}{2} \bigg|_{\lambda=1} = (2n + 1) \frac{\partial^{2n}}{\partial \lambda^{2n}} f(s, \lambda) \bigg|_{\lambda=1},
\]

since $a_{2j+1}/(2j+1) = a_{2j}$. Rearranging this last equality leads to (3.55). \( \square \)

**Lemma 3.5.** Let $D_1(s, \lambda)$ and $D_4(s, \tilde{\lambda})$ be as in (2.11) and (2.12). Then
\[
D_1(s, \lambda) = D_4(s, \tilde{\lambda}) \left( 1 - \sqrt{\frac{\lambda}{2 - \lambda}} \tanh \frac{\mu(s, \tilde{\lambda})}{2} \right)^2.
\] (3.61)

**Proof.** Using the facts that $-1 - \cosh x = -2 \cosh^2 \frac{x}{2}, 1 = \cosh^2 x - \sinh^2 x$ and \(\sinh x = \sinh \frac{x}{2} \cosh \frac{x}{2}\) we get

\[
D_1(s, \lambda) = \frac{-2}{\lambda - 2} D_4(s, \tilde{\lambda}) + D_2(s, \tilde{\lambda}) \frac{\lambda + \sqrt{\lambda} \sinh \mu(s, \tilde{\lambda})}{\lambda - 2},
\]

\[
= \frac{-2}{\lambda - 2} D_4(s, \lambda) + D_2(s, \tilde{\lambda}) \frac{\lambda \cosh^2 \left( \frac{\mu(s, \lambda)}{2} \right) + \lambda \sinh^2 \left( \frac{\mu(s, \lambda)}{2} \right) + \sqrt{\lambda} \sinh \mu(s, \tilde{\lambda})}{\lambda - 2},
\]

\[
= D_4(s, \tilde{\lambda}) + \frac{D_4(s, \tilde{\lambda}) \lambda \sinh^2 \left( \frac{\mu(s, \lambda)}{2} \right) + \sqrt{\lambda} \sinh \mu(s, \tilde{\lambda})}{\cosh^2 \left( \frac{\mu(s, \lambda)}{2} \right) \lambda - 2},
\]

\[
= D_4(s, \tilde{\lambda}) \left( 1 - \frac{\lambda \sinh^2 \left( \frac{\mu(s, \lambda)}{2} \right) + 2 \sqrt{\lambda} \sinh \left( \frac{\mu(s, \lambda)}{2} \right) \cosh \left( \frac{\mu(s, \lambda)}{2} \right)}{(\lambda - 2) \cosh^2 \left( \frac{\mu(s, \lambda)}{2} \right)} \right),
\]

\[
= D_4(s, \tilde{\lambda}) \left( 1 - 2 \sqrt{\frac{\lambda}{2 - \lambda}} \tanh \left( \frac{\mu(s, \lambda)}{2} \right) + \frac{\lambda}{2 - \lambda} \tanh^2 \left( \frac{\mu(s, \lambda)}{2} \right) \right),
\]

\[
= D_4(s, \tilde{\lambda}) \left( 1 - \frac{\sqrt{\lambda}}{2 - \lambda} \tanh \left( \frac{\mu(s, \lambda)}{2} \right) \right)^2.
\]

\( \square \)

For notational convenience, define $d_1(s, \lambda) = D_{1/2}^1(s, \lambda), d_4(s, \lambda) = D_{4/2}^4(s, \lambda)$. Then

**Lemma 3.6.** For $n \geq 0$,
\[
\left[ -\frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} + \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} \right] d_1(s, \lambda) \bigg|_{\lambda=1} = (-1)^n \frac{\partial^n}{\partial \lambda^n} d_4(s, \lambda) \bigg|_{\lambda=1}.
\]
Proof. Let
\[ f(s, \lambda) = 1 - \sqrt{\frac{\lambda}{2 - \lambda}} \tanh \frac{\mu(s, \lambda)}{2} \]
by the previous lemma, we need to show that
\[
\left[ -\frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} + \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} \right] d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1} = \left( -\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} d_4(s, \tilde{\lambda}) \right) \bigg|_{\lambda=1}.
\]
Now formula (3.60) applied to \(d_4(s, \tilde{\lambda})\) gives
\[
\frac{\partial^k}{\partial \lambda^k} d_4(s, \tilde{\lambda}) \bigg|_{\lambda=1} = \begin{cases} 0 & \text{if } k = 2j + 1, j \geq 0, \\ \frac{(-1)^j j!}{\partial \lambda^j} d_4(s, \tilde{\lambda}) & \text{if } k = 2j, j \geq 0. \end{cases}
\]
Therefore
\[
-\frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1} = -\frac{1}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\partial^k}{\partial \lambda^k} d_4 \frac{\partial^{2n+1-k}}{\partial \lambda^{2n+1-k}} f \bigg|_{\lambda=1}
\]
\[
= -\sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j+1)! j! \partial \lambda^j} d_4 \frac{\partial^{2n-2j+1}}{\partial \lambda^{2n-2j+1}} f \bigg|_{\lambda=1}
\]
Similarly
\[
\frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1} = \frac{1}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} \frac{\partial^k}{\partial \lambda^k} d_4 \frac{\partial^{2n-k}}{\partial \lambda^{2n-k}} f \bigg|_{\lambda=1}
\]
\[
= \sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j)! j! \partial \lambda^j} d_4 \frac{\partial^{2n-2j}}{\partial \lambda^{2n-2j}} f \bigg|_{\lambda=1}
\]
Therefore
\[
\left[ -\frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} + \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \lambda^{2n}} \right] d_4(s, \tilde{\lambda}) f(s, \lambda) \bigg|_{\lambda=1}
\]
\[
= \sum_{j=0}^{n} \frac{(-1)^j}{(2n-2j)! j! \partial \lambda^j} d_4(s, \tilde{\lambda}) \left[ \frac{\partial^{2n-2j}}{\partial \lambda^{2n-2j}} f \right] - \frac{1}{2n-2j+1} \frac{\partial^{2n-2j+1}}{\partial \lambda^{2n-2j+1}} f \bigg|_{\lambda=1}
\]
Now Lemma 3.4 shows that the square bracket inside the summation is zero unless \(j = n\), in which case it is 1. The result follows. \(\square\)

Lemma 3.6 establishes the inductive step in the proof of Corollary 2.2.
4 Numerics

Let

\[ q_n(x) = \left. \frac{\partial^n}{\partial \lambda^n} q(x, \lambda) \right|_{\lambda=1}, \tag{4.1} \]

so that \( q_0 \) equals \( q \) from (1.7). In order to compute \( F_\beta(s, m) \) it is crucial to know \( q_n \) accurately. Asymptotic expansions for \( q_n \) at \(-\infty\) are given in [13]. We outline how to compute \( q_0 \) and \( q_1 \) as an illustration. From [13], we know that, as \( t \to +\infty \)

\[ q_0(-t/2) = \frac{1}{2} \sqrt{t} \left( 1 - \frac{1}{t^3} - \frac{73}{2t^6} - \frac{10657}{2t^9} - \frac{1391277}{8t^{12}} + O\left( \frac{1}{t^{15}} \right) \right), \]

\[ q_1(-t/2) = \frac{\exp \left( \frac{1}{3} \frac{t^{3/2}}{2} \right)}{2\sqrt{2\pi} t^{1/4}} \left( 1 + \frac{17}{24t^{3/2}} + \frac{1513}{2^7 3^2 t^3} + \frac{850193}{2^{10} 3^4 t^{9/2}} - \frac{407117521}{2^{15} 3^5 t^6} + O\left( \frac{1}{t^{15/2}} \right) \right). \]

Quantities needed to compute \( F_\beta(s, m), m = 1, 2 \), are not only \( q_0 \) but also integrals involving \( q_0 \), such as

\[ I_0 = \int_s^{\infty} (x-s) q_0^2(x) \, dx, \quad J_0 = \int_s^{\infty} q_0(x) \, dx. \tag{4.3} \]

Instead of computing these integrals afterwards, it is better to include them as variables in a system together with \( q_0 \), as suggested in [9]. Therefore all quantities needed are computed in one step, greatly reducing errors, and taking full advantage of the powerful numerical tools in MATLAB. Since

\[ I_0' = -\int_s^{\infty} q_0^2(x) \, dx, \quad I_0'' = q_0^2, \quad J_0' = -q_0, \tag{4.4} \]

the system closes, and can be concisely written

\[
\begin{pmatrix}
q_0 \\
q_0' \\
q_0'' \\
I_0 \\
I_0' \\
J_0
\end{pmatrix}
= \begin{pmatrix}
q_0 & q_0' & q_0'' & s q_0 + 2 q_0^3 \\
q_0' & I_0 & q_0'' & I_0' \\
q_0'' & I_0' & q_0 & J_0 \\
s q_0 + 2 q_0^3 & I_0 & q_0 & -q_0
\end{pmatrix}.
\tag{4.5}
\]

We first use the MATLAB built-in Runge–Kutta based ODE solver \texttt{ode45} to obtain a first approximation to the solution of (4.5) between \( x = 6 \), and \( x = -8 \), with an initial values obtained using the Airy function on the right hand side. Note that it is not possible to extend the range to the left due to the high instability of the solution a little after \(-8\); (This is where the transition region between the three different regimes in the so-called “connection problem” lies. We circumvent this limitation by patching up our solution with the asymptotic expansion to the left of \( x = -8 \).) The approximation obtained is then used as a trial solution in the MATLAB boundary value problem solver \texttt{bvp4c}, resulting in an accurate solution vector between \( x = 6 \) and \( x = -10 \).
Similarly, if we define
\[
I_1 = \int_{s}^{\infty} (x - s) q_0(x) q_1(x) \, dx, \quad J_1 = \int_{s}^{\infty} q_0(x) q_1(x) \, dx,
\]
(4.6)
then we have the first-order system
\[
\frac{d}{ds} \begin{pmatrix} q_1' \\ q_1' \\ I_1' \\ I_1' \\ J_1' \\ J_1' \end{pmatrix} = \begin{pmatrix} q_1' \\ q_1' \\ s q_1 + 6 q_0^2 q_1 \\ I_1' \\ q_0 q_1 \\ -q_0 q_1 \end{pmatrix},
\]
(4.7)
which can be implemented using \texttt{bvp4c} together with a “seed” solution obtained in the same way as for \( q_0 \). Work is in progress to provide publicly downloadable versions of the MATLAB routines.

Table 1 shows a comparison of percentiles of the \( F_1 \) distribution to corresponding percentiles of empirical Wishart distributions. Here \( \lambda_i \) denotes the \( i \)th largest eigenvalue in the Wishart Ensemble. The percentiles in the \( \lambda_i \) columns were obtained by finding the ordinates corresponding to the \( F_1 \)–percentiles listed in the first column, and computing the proportion of eigenvalues lying to the left of that ordinate in the empirical distributions for the \( \lambda_i \). The bold entries correspond to the levels of confidence most commonly used in statistical applications. The reader should compare this table to a similar one in [7].

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