Abstract

We show that the poset of $SL(n)$-orbit closures in the product of two partial flag varieties is a lattice if the action of $SL(n)$ is spherical.

Keywords: Spherical double cones, partial flag varieties, ladder posets.
MSC: 06A07, 14M15

1 Introduction

Let $G$ be a connected reductive algebraic group. A normal algebraic variety $X$ is called a spherical $G$-variety if there exists an algebraic action $G \times X \to X$ such that the restriction of the action to a Borel subgroup $B$ of $G$ has an open orbit in $X$. In this case, we say that the action is spherical.

Let $P_1, \ldots, P_k \subset G$ be a list of parabolic subgroups containing the same Borel subgroup $B$ and let $X$ denote the product variety $X = G/P_1 \times \cdots \times G/P_k$. Then $X$ is a smooth, hence normal, $G$-variety via the diagonal action. The study of functions on an affine cone over $X$ is important for understanding the decompositions of tensor products of representations of $G$, see [15, 10]. In particular, determining when the diagonal action of $G$ on $X$ is spherical is important for understanding the multiplicity-free representations of $G$. In his ground breaking article [6], Littelmann initiated the classification problem and gave a list of all possible pairs of maximal parabolic subgroups $(P_1, P_2)$ such that $G/P_1 \times G/P_2$ is a spherical $G$-variety. In [7], for group $G = SL(n)$ and in [8] for $G = Sp(2n)$, Magyar, Weyman, and Zelevinski classified the parabolic subgroups $P_1, \ldots, P_k$ such that the product $X = G/P_1 \times \cdots \times G/P_k$ is a spherical $G$-variety. According to [7], if $X$ is a spherical $G$-variety, then the number of factors is at most 3, and $k = 3$ occurs in only special cases. Therefore, the gist of the problem lies in the case $k = 2$. This case is settled in full detail by Stembridge. In [12], for a semisimple complex algebraic group $G$, Stembridge listed all pairs of parabolic subgroups $(P_1, P_2)$ such that $G/P_1 \times G/P_2$ is a spherical $G$-variety.
For motivational purposes, we will mention some recent related developments. Let $K$ be a connected reductive subgroup of $G$ and let $P$ be a parabolic subgroup of $G$. One of the major open problems in the classification of spherical actions is the following: What are the possible triplets $(G, K, P)$ such that $G/P$ is a spherical $K$-variety? When $K$ is a Levi subgroup of a parabolic subgroup $Q$, this question is equivalent to asking when $G/Q \times G/P$ is a spherical $G$-variety via diagonal action; it has a known solution as we mentioned earlier. For an explanation of this equivalence, see [1, Lemma 5.4]. In [1], Avdeev and Petukhov gave a complete answer to the above question in the case $G = SL(n)$. If we assume that $K$ is a symmetric subgroup of $G$, then our initial question is equivalent to asking when $G/P \times K/B_K$ has an open $K$-orbit via its diagonal action. Here, $B_K$ is a Borel subgroup of $K$. In this case, the answer is recorded in [4]. See also the related work of Pruijssen [14]. Finally, let us mention another extreme situation where the answer is known: $G$ is an exceptional simple group, $P$ is a maximal parabolic subgroup, and $K$ is a maximal reductive subgroup of $G$, see [9].

We go back to the products of flag varieties and let $P$ and $Q$ be two parabolic subgroups from $G$. From now on we will call a product variety of the form $G/P \times G/Q$ a double flag variety. If the diagonal action of $G$ on a double flag variety $X = G/P \times G/Q$ is spherical, then we will call $X$ a spherical double flag variety for $G$. As it is shown by Littelmann in his previously mentioned article, the problem of deciding if a double flag variety is spherical or not is closely related to a study of the invariants of a maximal unipotent subgroup in the coordinate ring of an affine cone over $X$. In turn, this study is closely related to the combinatorics of the $G$-orbits in $X$. In this regard, our goal in this note is to prove the following result on the poset of inclusion relationships between the $G$-orbit closures in a spherical double flag variety.

**Theorem 1.1.** Let $G$ denote the special linear group $SL(n+1)$. If $X$ is a spherical double flag variety for $G$, then the poset of $G$-orbit closures in $X$ is a lattice.

In fact, we have a precise description of the possible lattices in Theorem 1.1. It turns out that the Hasse diagram of such a lattice look like a “ladder”, or the lattice is a chain, see Theorem 3.2.

The structure of our paper is as follows. In the next section we set up our notation and review some basic facts about the double-cosets of parabolic subgroups. In Subsection 2.2, we show that the inclusion poset of $G$-orbit closures in $G/P \times G/Q$ is isomorphic to the inclusion poset of $P$-orbit closures in $G/Q$. In Subsection 2.4 we review the concept of tight Bruhat order due to Stembridge. We use the information gained from this subsection in our analysis of the cases that are considered in the subsequent Section 3, where we prove our main result.
2 Preliminaries

2.1

For simplicity, let us assume that $G$ is a semisimple simply-connected complex algebraic group and let $B$ be a Borel subgroup of $G$. Let $T$ be a maximal torus of $G$ that is contained in $B$. The unipotent radical of $B$ is denoted by $U$, so that $B = UT$. We denote by $\Phi$ the root system corresponding to the pair $(G, T)$ and we denote by $\Delta$ the subset of simple roots corresponding to $B$. A parabolic subgroup $P$ of $G$ is said to be standard with respect to $B$ if the inclusion $B \subseteq P$ holds true. In this case, $P$ is uniquely determined by a subset $I \subseteq \Delta$.

The Weyl group of $(G, T)$, that is $N_G(T)/T$, is denoted by $W$ and we use the letter $R$ to denote the set of simple reflections $s_\alpha \in W$, where $\alpha \in \Delta$. We will allow ourselves to be confused by using the letter $I$ (and $J$) to denote a subset of $\Delta$ or the corresponding subset of simple reflections in $R$. The length of an element $w \in W$, denoted by $\ell(w)$, is the minimal number of Coxeter generators $s_i \in R$ that is needed for the equality $w = s_1 \cdots s_k$ to hold true. In this case, the product $s_1 \cdots s_k$ is called a reduced expression for $w$. Note that when $W$ is the symmetric group of permutations $S_{n+1}$, the length of a permutation $w = w_1 \cdots w_{n+1} \in S_{n+1}$ is equal to the number of pairs $(i, j)$ with $1 \leq i < j \leq n + 1$ and $w_i > w_j$.

The Bruhat-Chevalley order on $W$ can be defined by declaring $v \leq w$ ($w, v \in W$) if a reduced expression of $v$ is obtained from a reduced expression $s_1 \cdots s_k$ of $w$ by deleting some of the simple reflections $s_i$.

Let $X(T) := \text{Hom}(T, \mathbb{G}_m)$ denote the group of characters of the maximal torus $T$. Let $\{\omega_1, \ldots, \omega_r\}$ denote the set of fundamental weights corresponding to the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. By our assumptions on $G$, we have $\omega_i \in X(T)$ for every $i \in \{1, \ldots, r\}$. The Weyl group $W$ acts on the weight lattice, that is $X(T)$. Let $E$ denote the real vector space that is spanned by the fundamental weights, so that $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The action of $W$ on the weight lattice extends to give a linear action on $E$. A vector $\theta$ from $E$ is called a dominant vector if it is of the form $\theta = a_1 \omega_1 + \cdots + a_r \omega_r$, where $a_i$'s are nonnegative real numbers. Let $W(\omega_i)$ ($i \in \{1, \ldots, r\}$) denote the isotropy group of $\omega_i$ in $W$. Then $W(\omega_i)$ ($i \in \{1, \ldots, r\}$) is a parabolic subgroup of $W$, and furthermore, the subgroup of $G$ that is generated by $B$ and $W(\omega_i)$ is a maximal parabolic subgroup.

2.2

Let $G$ act on two irreducible varieties $X_1$ and $X_2$, and let $x_i \in X_i$, $i = 1, 2$ be two points in general positions. If $G_i \subseteq G$ denotes the stabilizer subgroup of $x_i$ in $G$, then $\text{Stab}_G(x_1 \times x_2)$ coincides with the stabilizer in $G_1$ of a point in general position from $G/G_2$ (or, equivalently, with the stabilizer in $G_2$ of a point in general position from $G/G_1$), see [10].

Let $P_1$ and $P_2$ be two parabolic subgroups of $G$. By applying the idea from the previous paragraph to the double flag variety $X := G/P_1 \times G/P_2$, where $B \subset P_1 \cap P_2$, we notice that
the study of $G$-orbits in $X$ reduces to the study of $P_1$-orbits in the flag variety $G/P_2$. But more is true; this correspondence between $G$-orbits and $P_1$-orbits respects the inclusions of their closures in Zariski topology.

**Lemma 2.1.** The poset of $G$-orbit closures in $X$ is isomorphic to the poset of $P_2$-orbit closures in $G/P_1$.

**Proof.** Let $X$ denote $G/P_1 \times G/P_2$. The canonical projection $\pi: X \to G/P_2$ is $G$-equivariant and it turns $X$ into a homogeneous fiber bundle over $G/P_2$ with fiber $G/P_1$ at every point $gP_2$ ($g \in G$) of the base $G/P_2$. To distinguish it from the other fibers, let us denote by $Y$ the fiber $G/P_1$ at the 'origin' $eP_2$ of $G/P_2$. Then any $G$-orbit in $X$ meets $Y$. Note also that if $g \cdot y \in Y$ for some $g \in G$ and $y \in X$, then $g \in P_2$. There are two useful consequences of this observation: 1) $Y$ is a $P_2$-variety; 2) any $G$-orbit $O$ meeting $Y$, actually meets $Y$ along a $P_2$-orbit. Therefore, the map $O \mapsto O \cap Y$ gives a bijection between the set of all $G$-orbits in $X$ and the set of all $P_2$-orbits in $Y$.

Since $G$ and $P_2$ are connected algebraic groups, the Zariski closures of their orbits are irreducible. Furthermore, the boundaries of the orbit closures are unions of orbits of smaller dimensions. At the same time, $Y$ is closed in $X$, therefore, the extension of the orbit-correspondence map,

$$
O \mapsto O \cap Y,
$$

(1)

gives a poset isomorphism between the inclusion orders on the Zariski closures of $G$-orbits in $X$ and the Zariski closures of $P_2$-orbits in $Y$. This finishes the proof of our lemma. □

**Remark 2.2.** By looking at the $(P_1, P_2)$-double cosets in $G$, as far as the combinatorics of orbit closures is concerned, we see that there is no real difference between the study of $P_1$-orbits in $G/P_2$ and the study of $P_2$-orbits in $G/P_1$.

2.3

We preserve our assumptions/notation from the previous subsections; $P_1$ and $P_2$ are two standard parabolic subgroups with respect to $B$. If $I$ and $J$ are the subsets of $R$ (or, of $\Delta$) that determine $P_1$ and $P_2$, respectively, then we will write $P_I$ (resp. $P_J$) in place of $P_1$ (resp. $P_2$). The Weyl groups of $P_I$ and $P_J$ will be denoted by $W_I$ and $W_J$, respectively. In this subsection, we will present some well-known facts regarding the set of $(W_I, W_J)$-double cosets in $W$, denoted by $W_I \backslash W/W_J$.

First of all, the set $W_I \backslash W/W_J$ is in a bijection with the set of $P_I$-orbits in $G/P_J$, see [2, Section 21.16]. For $w \in W$, we denote by $[w]$ the double coset $W_I w W_J$. Let

$$
\pi: W \to W_I \backslash W/W_2
$$

denote the canonical projection onto the set of $(W_1, W_2)$-double cosets. It turns out that the preimage in $W$ of every double coset in $W_I \backslash W/W_2$ is an interval with respect to Bruhat-Chevalley order, hence it has a unique maximal and a unique minimal element, see [3].
Moreover, if \([w], [w'] \in W_1W/W_2\) are two double cosets, \(w_1\) and \(w_2\) are the maximal elements of \([w]\) and \([w']\), respectively, then \(w \leq w'\) if and only if \(w_1 \leq w_2\), see [5]. It follows that \(W_1W/W_2\) has a natural combinatorial partial ordering defined by

\[ [w] \leq [w'] \iff w \leq w' \iff w_1 \leq w_2 \]

where \([w], [w'] \in W_1W/W_2\) and \(w_1\) and \(w_2\) are the maximal elements, \(w_1 \in [w]\) and \(w_2 \in [w']\). This partial order is geometric in the following sense; if \(O_1\) and \(O_2\) are two \(P_I\)-orbits in \(G/P_J\) with the corresponding double cosets \([w_1]\) and \([w_2]\), respectively, then \(O_1 \subseteq \overline{O_2}\) if and only if \(w_1 \leq w_2\). The bar on \(O_2\) stands for the Zariski closure in \(G/P_J\).

Now let \([w]\) be a double coset from \(I'W \cap W'J\) represented by an element \(w \in W\) such that \(\ell(w) \leq \ell(v)\) for every \(v \in [w]\). It turns out that the set of all such minimal length double coset representatives is given by \(I'W \cap W'J\), where \(I'W\) stands for the set of minimal length coset representatives for \(W_1W/W\). We denote \(I'W \cap W'J\) by \(X_{I',J}^-\). Set \(H = I \cap wJw^{-1}\). Then \(uw \in W_J^J\) for \(u \in W_I\) if and only if \(u\) is a minimal length coset representative for \(W_IW_H\). In particular, every element of \(W_IwW_J\) has a unique expression of the form \(uwwv\) with \(u \in W_I\) is a minimal length coset representative of \(W_IW_H\), \(v \in W_J\) and \(\ell(uwwv) = \ell(u) + \ell(w) + \ell(v)\).

Another characterization of the sets \(X_{I',J}^-\) is as follows. For \(w \in W\), the right ascent set is defined as

\[ \text{Asc}_R(w) = \{ s \in R : \ell(ws) > \ell(w) \} \]

The right descent set, \(\text{Des}_R(w)\) is the complement \(R - \text{Asc}_R(w)\). Similarly, the left ascent set of \(w\) is

\[ \text{Asc}_L(w) = \{ s \in R : \ell(sw) > \ell(w) \} \quad (= \text{Asc}_R(w^{-1})) \]

Then

\[ X_{I',J}^- = \{ w \in W : I \subseteq \text{Asc}_L(w) \text{ and } J \subseteq \text{Asc}_R(w) \} \tag{2} \]
\[ = \{ w \in W : I^c \supseteq \text{Des}_R(w^{-1}) \text{ and } J^c \supseteq \text{Des}_R(w) \} \tag{3} \]

For our purposes we need the distinguished set of maximal length representatives for each double coset. It is given by

\[ X_{I',J}^+ = \{ w \in W : I \subseteq \text{Des}_R(w^{-1}) \text{ and } J \subseteq \text{Des}_R(w) \} \tag{4} \]
\[ = \{ w \in W : I^c \supseteq \text{Asc}_R(w^{-1}) \text{ and } J^c \supseteq \text{Asc}_R(w) \} \tag{5} \]

For a proof of this characterization of \(X_{I',J}^+\), see [3, Theorem 1.2(i)].

**Remark 2.3.** The Bruhat-Chevalley orders on \(X_{I',J}^-\) and \(X_{I',J}^+\) are isomorphic.
2.4

We mentioned in the introductory section that Littelmann classified the pairs of parabolic subgroups \((P_I, P_J)\) corresponding to fundamental dominant weights such that the diagonal \(G\) action on \(G/P_I \times G/P_J\) is spherical. Said differently, we know all pairs \((I, J)\) of subsets of \(R\) such that

- \(|I| = |J| = |R| - 1\), and
- \(G/P_I \times G/P_J\) is a spherical double flag variety for \(G\).

In particular, under the maximality assumption of the subsets \(I\) and \(J\), the poset of \(G\)-orbit closures is a chain, see [6, Proposition 3.2]. In the light of our Lemma 2.1, this is equivalent to the statement that with respect to the Bruhat-Chevalley order, the set \(X^+_{I,J}\) is a chain.

We mention also that the classification of Littelmann is extended by Stembridge to cover all pairs of subsets \((I, J)\) in \(R\) such that \(G/P_I \times G/P_J\) is a spherical double flag variety for \(G\). See Corollaries 1.3.A – 1.3.D, 1.3.E6, 1.3.E7, and 1.3.{E8,F4,G2} in [12].

**Remark 2.4.** 1. We call a spherical double flag variety \(G/P_I \times G/P_J\) trivial if one of the factors is isomorphic to a point, that is \(P_I = G\) or \(P_J = G\). In the cases of E8,F4, and G2 all of the spherical double flag varieties are trivial.

2. In the cases of A–D, E6, and E7, if \(G/P_I \times G/P_J\) is a spherical double flag variety for \(G\), then at least one of the subsets \(I\) and \(J\) is maximal, that is to say, of cardinality \(|R| - 1\). Without loss of generality we always choose \(I\) to be the maximal one.

2.5

In this subsection we will review the useful concept of “tight Bruhat order.” We maintain our notation from the previous subsections.

One way to define the Bruhat-Chevalley order on \(W\) is to use the reflection representation of \(W\) as the group of isometries of \(E\). Let \(\langle \ , \rangle\) denote the \(W\)-invariant inner product on \(E\), and let \(\theta \in E\) be a vector such that \(\langle \theta, \beta \rangle \geq 0\) for all \(\beta \in \Phi^+\). Such a vector is called dominant. It is indeed dominant in the sense of Subsection 2.1.

It is well known that the stabilizer of a dominant vector is a parabolic subgroup \(W_J \subset W\), where \(J = \{s_\alpha \in R : \langle \theta, \alpha \rangle = 0\}\). Thus, as a set, the minimal length coset representatives \(W^J \subset W\) of the quotient \(W/W_J\) can be identified with the orbit \(W\theta\). Following Stembridge, we are going to call the orbit map \(w \mapsto w \cdot \theta\) the evaluation.

A proof of the following result can be found in [11].
**Proposition 2.5.** Let \( \theta \in E \) be a dominant vector with stabilizer \( W_J \). The evaluation map induces a poset isomorphism between the Bruhat-Chevalley order on \( W^J \) and the orbit \( W\theta \) with partial order \( \leq_B \) defined by the transitive closure of the following relations:

\[
\mu \leq_B s_\beta(\mu) \quad \text{for all } \beta \in \Phi^+ \text{ such that } \langle \mu, \beta \rangle > 0.
\]

Let \( I \) be a subset of \( R \) and let \( \Phi_I \subset \Phi \) denote the root subsystem corresponding to the parabolic subgroup \( W_I \). We denote by \( \Phi_I^+ \) the intersection \( \Phi^+ \cap \Phi_I \). If \( \theta \) is a dominant vector and its stabilizer subgroup is \( W_J \) with \( J \subset R \), then we define

\[
(W\theta)_I := \{ \mu \in W\theta : \langle \mu, \beta \rangle \geq 0 \text{ for all } \beta \in \Phi_I^+ \}. \tag{6}
\]

A proof of the following result can be found in [13, Proposition 1.5].

**Proposition 2.6.** Let \( I, J \subset R \) be two sets of Coxeter generators for \( W \) and let \( \theta \in E \) be a dominant vector with stabilizer \( W_J \). Then the evaluation map induces a poset isomorphism between the (restriction of) Bruhat-Chevalley order on \( X_{I,J}^- \) and \( (W\theta)_I \) with partial order defined by the transitive closure of the relations

\[
\mu \leq_B s_\beta(\mu) \quad \text{for all } \beta \in \Phi^+ \text{ such that } s_\beta(\mu) \in (W\theta)_I \text{ and } \langle \mu, \beta \rangle > 0.
\]

Now we come to the definition of a critical notion for our proof. There is a natural partial ordering on the roots defined by

\[
\nu \leq \mu \iff \mu - \nu \in \mathbb{R}^+ \Phi^+ . \tag{7}
\]

It turns out, when the interpretation of Bruhat-Chevalley ordering as given in Proposition 2.5 is used, there is a natural order reversing implication:

\[
\mu \leq_B \nu \implies \nu \leq \mu . \tag{8}
\]

If the converse implication also holds, then the poset \( W\theta \) is called tight. More precisely, a subposet \( (M, \leq_B) \) of the Bruhat-Chevalley order on \( (W\theta, \leq_B) \) is called tight if

\[
\mu \leq_B \nu \iff \nu \leq \mu
\]

for all \( \nu, \mu \) in \( M \subset E \).

In the light of our Remark 2.4 part 3, we assume that \( I \subset R \) is a maximal subset of the form \( I = R - \{s\} \) for some \( s \in R \). Also, we assume that there exists a dominant \( \theta \in E \) such that \( W_J \) is its stabilizer subgroup. Now, by [13, Theorem 2.3], we see that if \( W^J \) is tight, then \( X_{I,J}^- = X_{R-\{s\},J}^- \) is a chain. The list of tight quotients is also given in [13]; \( (W^J, \leq_B) \) is tight if and only if \( W \) is of at most rank 2, or \( J = R \), or one of the following holds:
• $W \cong A_n$ and $J^c = \{s_j\} \ (1 \leq j \leq n)$ or $J^c = \{s_j, s_{j+1}\} \ (1 \leq j \leq n-1)$,

• $W \cong B_n$ and $J^c = \{s_1\}, \{s_2\}, \{s_n\}$, or $J^c = \{s_1, s_2\}$,

• $W \cong D_n$ and $J^c = \{s_1\}, \{s_2\}$ or $J^c = \{s_n\}$,

• $W \cong E_6$ and $J^c = \{s_1\}$ or $J^c = \{s_6\}$,

• $W \cong E_7$ and $J^c = \{s_7\}$,

• $W \cong F_4$ and $J^c = \{s_1\}$ or $J^c = \{s_4\}$, or

• $W \cong H_3$ and $J^c = \{s_1\}$ or $J^c = \{s_3\}$.

Therefore, in these cases (when $I$ is maximal and $J$ is as in this list) we know that $X^-_{I,J} = X^-_{R-(s),J}$ is a chain. We finish our preliminaries section by listing the remaining cases under the assumption that $I$ is of the form $R - \{s\}$ for some $s \in R$.

• $W \cong A_n$
  1. $I^c \in \{\{s_2\}, \{s_{n-1}\}\}$ and $J^c = \{s_p, s_q\}$ with $1 < p < p + 1 < q < n$;
  2. $I^c \in \{\{s_1\}, \{s_n\}\}$ and $|J^c| \geq 2$ (but $J^c \neq \{s_j, s_{j+1}\} \ (1 \leq j \leq n - 1)$);
  3. $I^c \in \{\{s_2\}, \ldots, \{s_{n-1}\}\}$, and $J^c = \{s_1, s_j\}$ or $J^c = \{s_j, s_n\}$ with $2 < j < n - 1$.

• $W \cong C_n$
  1. $I^c = \{s_n\}$ and $|J^c| = 1$;
  2. $I^c = J^c = \{s_1\}$.

• $W \cong D_n \ (n \geq 4)$
  1. $I^c = \{s_n\}$ and $J^c = \{s_i, s_{i+1}\}$ with $1 \leq i \leq n$ and $1 \leq l \leq 2$;
  2. $I^c \in \{\{s_1\}, \{s_2\}\}$, and $J^c \subseteq \{s_1, s_2, s_n\}$ or $J^c \subseteq \{s_{n-1}, s_n\}$ or $J^c = \{s_{n-2}\}$;
  3. $(n = 4 \text{ case only}) \ I^c = \{s_1\}$ and $J^c = \{s_2, s_3\}$ or $I^c = \{s_2\}$ and $J^c = \{s_1, s_3\}$.

• $W \cong E_6$
  1. $I^c \in \{\{s_1\}, \{s_6\}\}$ and $J^c = \{s_1, s_6\}$.
3 Proof of the main result

The Weyl group of $(SL(n+1),T)$, where $T$ is the maximal torus of diagonal matrices is isomorphic to the symmetric group $S_{n+1}$. A set of Coxeter generators $R \subset S_{n+1}$ is given by the set

$$R = \{s_i = (i, i+1) : i = 1, \ldots, n\},$$

where $(i, i+1)$ is the simple transposition that interchanges $i$ and $i+1$ and leaves everything else fixed. For easing our notation, whenever it is clear from the context, we will denote the simple transposition $s_i$ by its index $i$.

In the light of Lemma 2.1, Subsection 2.3, and Subsection 2.5, to prove our main result Theorem 1.1, it will suffice to analyze the Bruhat-Chevalley order on the set of distinguished double coset representatives, $X^+_{I,J}$. We will do this analysis on a case-by-case basis for

1. $I^c \in \{\{2\}, \{n-1\}\}$ and $J^c = \{p, q\}$ $(1 < p < p + 1 < q < n)$;
2. $I^c \in \{\{1\}, \{n\}\}$ and $|J^c| \geq 2$ (but $J^c \neq \{j, j+1\}$ $(1 \leq j \leq n-1)$);
3. $I^c \in \{\{2\}, \ldots, \{n-1\}\}$, and $J^c = \{1, j\}$ or $J^c = \{j, n\}$ with $2 < j < n-1$.

3.1 Case 1.

We start with a general remark which we will use in the sequel.

**Remark 3.1.** Let $w \in S_{n+1}$ be a permutation whose one-line notation ends with the decreasing string $k k-1 \ldots 2 1$. In this case, any element in the upper interval $[w, w_0] \subset S_{n+1}$ has the same ending. In other words, if $w' \in [w, w_0]$, then the last $k$ entries of $w'$ are exactly $k, k-1, \ldots, 1$ in this order. Similarly, if $w$ begins with the decreasing string $n+1 \ldots k$ for some $k \in \{1, \ldots, n+1\}$, then any element in the upper interval $[w, w_0] \subset S_{n+1}$ has the same beginning. So, essentially, these elements form an upper interval of $S_{n+1}$, which is isomorphic to $S_{n+1-k}$. In a similar way, if we consider the set of permutations that starts with the string $1 2 \ldots k$, then we obtain a lower interval that is isomorphic to $S_{n+1-k}$ in $S_{n+1}$.

Now we proceed to give our proof, starting with the sub-case $I^c = \{2\}$.

Let $w = w_1 \ldots w_{n+1}$ be an element, in one-line notation, from $X^+_{I,J}$. Recall that

$$X^+_{I,J} = \{w \in W : I^c \supseteq \text{Asc}_R(w^{-1}) \text{ and } J^c \supseteq \text{Asc}_R(w)\}.$$

The meaning of $I^c = \{2\} \supseteq \text{Asc}_R(w^{-1})$ is that either $\text{Asc}_R(w^{-1}) = \emptyset$, in which case $w$ is equal to $w_0$, the longest permutation, or, $\text{Asc}_R(w^{-1}) = \{2\}$ hence 2 comes before 3 in $w$ and there are no other consecutive pairs $(a, a+1)$ such that $a$ comes before $a+1$ in $w$. Note also that $\text{Asc}_R(w)$ cannot be empty unless $X^+_{I,J} = \{w_0\}$.

We continue with the assumption that $w \neq w_0$. Suppose $J^c = \{p, q\}$ for $1 < p < p + 1 < q < n$. We are going to write $L_1$ for the segment $w_1 \ldots w_p$, $L_2$ for the segment $w_{p+1} \ldots w_q$,
and $L_3$ for the segment $w_{q+1} \ldots w_{n+1}$. By our assumptions, all three of these segments are decreasing sequences. In particular, since 2 comes before 3 in $w$, 2 cannot appear in $L_3$. In fact, 2 and 3 cannot appear in the same segment.

First, we assume that $p = 2$. Since any element of $X^+_{I,J}$ has descents (at least) at the positions $J = \{1, 2, 3, 4, \ldots, q, \ldots, n+1\}$, the bottom element $\tau_0$ is either of the form

$$\tau_0 = 2 \ 1 \ | \ n + 1 \ n \ldots n - q + 3 \ n - q + 2 \ n - q + 1 \ldots 3, \quad (9)$$

or it is of the form

$$\tau_0 = n + 1 \ n \ldots n - q + 4 \ 2 \ 1 \ | \ n - q + 3 \ n - q + 2 \ldots 3. \quad (10)$$

The bars between numbers indicate the possible positions of ascents. Note that the number of inversions of the former permutation is $1 + \binom{n-1}{2}$, and the rank of the latter is

$$f_n(q) := \left( \sum_{i=1}^{q-2} n + 1 - i \right) + 1 + \left( \sum_{i=q+1}^{n} n + 1 - i \right)$$

$$= \binom{n+1}{2} + 1 - (n + 1 - q) - (n + 1 - (q - 1)),$$

which is always greater than the former. Therefore, the minimal element $\tau_0$ of $X^+_{I,J}$ starts with 2 1 (as in 9).

This element has a single ascent at the 2-nd position. We will analyze the covers of $\tau_0$. Since an upward covering in Bruhat-Chevalley order is obtained by moving a larger number to the front, $n + 1$ of $L_2$ moves into $L_1$ and accordingly either 2 or 1 from $L_1$ moves into $L_2$.

Recall that each double coset $W_I z W_J$ is an interval of $W$ in Bruhat-Chevalley order and $X^+_{I,J}$ consists of maximal elements of these intervals (see [3, Theorem 1.2(ii)]). It follows from this critical observation that, to obtain a covering of $\tau_0$, 1 has to move, and it becomes the last entry of $L_2$. In other words, the permutation

$$\tau_1 = n + 1 \ 2 \ | \ n \ldots n - q + 3 \ 1 \ | \ n - q + 2 \ n - q + 1 \ldots 3$$

is the unique element in $X^+_{I,J}$ that covers $\tau_0$.

Next, we analyze the covers of $\tau_1$; it has only two possible coverings which are obtained as follows: 1) 2 moves into $L_2$ and $n$ moves into $L_1$, 2) 1 moves into $L_3$ and $n - q + 2$ moves into $L_2$. The resulting elements are

$$\tau_2 = n + 1 \ n \ | \ n - 1 \ldots n - q + 3 \ 2 \ 1 \ | \ n - q + 2 \ n - q + 1 \ldots 3,$$

$$\tau_3 = n + 1 \ 2 \ | \ n \ldots n - q + 3 \ n - q + 2 \ | \ n - q + 1 \ldots 3 \ 1.$$
It is not difficult to see that each of these two elements are covered by the same element, namely
\[ \tau_4 = n + 1 \ n \ | \ n - 1 \ldots n - q + 3 \ n - q + 2 \ | \ n - q + 1 \ldots 3 \ 1. \]

Observe that, in \( \tau_4 \) the only entry that can be moved is 2 and this is possible only if the inequality \( q \leq n - 1 \) holds. This agrees with our assumption on \( q \). Therefore, there exists a unique cover of \( \tau_4 \), which is \( w_0 \). Note that all that is said above is independent of \( n \) as long as \( p = 2 \) and \( 3 < q < n \). Hence, our poset is as in Figure 1.

Finally, we look at the case for \( p > 2 \). The only difference between this and \( p = 2 \) case is that the first \( p - 2 \) terms of the elements of \( X^+_{I,J} \) all start with \( n + 1 \ n \ n - 2 \ldots n - p \).

By using Remark 3.1 and induction, we reduce this case to the case of \( p = 2 \). Therefore, our poset \( X^+_{I,J} \) is isomorphic to the one in Figure 1.

We proceed with the second sub-case of Case 1; we assume that \( I^c = \{n - 1\} \) and \( J = \{p, q\} \) with \( 2 \leq p < p + 1 < q \leq n - 1 \). As in the previous sub-case, for an element \( w \in X^+_{I,J} \) these conditions imply that \( w \) is of the form \( w = L_1|L_2|L_3 \), where \( L_i, i = 1, 2, 3 \) are decreasing sequences of lengths \( p, q - p \) and \( n + 1 - q \), respectively, and the number \( n - 1 \) appears before \( n \) in \( w \). It follows that the smallest element of \( X^+_{I,J} \) is of the form
\[
\tau_0 = w_1 \ldots w_p | w_{p+1} \ldots w_q | w_{q+1} \ldots w_{n+1} = n - 1 \ n - 2 \ldots n - q \ | \ n + 1 \ n \ n - q - 1 \ n - q - 2 \ldots 1
\]
Then arguing exactly as in the previous case one sees that the poset under consideration is also of the form Figure 1.

3.2 Case 2.

We start with the sub-case \( I^c = \{1\} \), and we let generously \( J^c \) be any proper subset \( J^c \subset \{1, \ldots, n\} \). Let \( w = w_1 \ldots w_{n+1} \) be an element from \( X^+_{I,J} \) and let \( v_1 \ldots v_{n+1} \) denote the
inverse, $w^{-1}$ of $w$. Since $\text{Asc}_R(w^{-1}) \subseteq \{1\}$, we have either $w = w^{-1} = w_0$, or
\[ v_1 < v_2 > v_3 > \cdots > v_{n+1}. \] (11)
Let $V'$ denote the set of permutations whose entries satisfy the inequalities in (12) and set
\[ V := V' \cup \{w_0\}. \]
Then $V$ has $n + 1$ elements, and furthermore, $(V, \leq)$ is a chain. But in Bruhat-Chevalley order we have
\[ u \leq v \iff u^{-1} \leq v^{-1} \] for every $u, v \in S_{n+1}$.
Therefore, $V^{-1} := \{v^{-1} : v \in V\}$ is a chain also. It follows that, as a subposet of $V^{-1}$, $X^{+}_{I,J}$ is a chain as well. This finishes the proof of the first part of Case 2.

Next, we assume that $I^c = \{n\}$ and let $w = w_1 \ldots w_{n+1} \in X^{+}_{I,J}$. If $w^{-1} = v_1 \ldots v_{n+1}$ denotes the inverse of $w$, then, as before, we have either $w = w^{-1} = w_0$, or
\[ v_1 > v_2 > \cdots > v_n < v_{n+1}. \] (12)
By arguing as in the previous paragraph we see that $X^{+}_{I,J}$ is a chain in this case as well, and hence, the proof of Case 2 is finished.

### 3.3 Case 3.

Now, we proceed with the proof of Case 3 but since we have symmetry, we will consider the case of $I^c = \{i\}$ with $2 \leq i \leq n - 1$ and $J^c = \{1, j\}$ with $2 < j < n - 1$ only. Let us note also that as the number $i \in I^c$ grows up to $\lfloor \frac{n+1}{2} \rfloor$ we get more freedom to position $i$ and $i + 1$ in an element $w \in X^{+}_{I,J}$; this makes $X^{+}_{I,J}$ grow taller as a poset. Now we are ready to present the structure of our poset in detail.

A generic element $w = w_1 \ldots w_{n+1}$ from $X^{+}_{I,J}$ is viewed as a concatenation of three segments, $w = L_1 L_2 L_3$ where $L_1 = w_1$, $L_2 = w_2 \ldots w_j$, and $L_3 = w_{j+1} \ldots w_{n+1}$. The possible ascents are at the 1-st and at the $j$-th positions. At the same time, if $w \neq w_0$, then we have $w^{-1} \neq w_0$, therefore, $w^{-1}$ has an ascent at the $i$-th position. This means that $i$ comes before $i + 1$ in $w$ and there are no other pairs $(a, a + 1)$ such that $a$ comes before $a + 1$ in $w$. Therefore, $i$ and $i + 1$ are always contained in distinct segments except for $w = w_0$. In particular, $i$ appears either in $L_1$ or in $L_2$.

We proceed to determine the smallest element $\tau_0$ of $X^{+}_{I,J}$. Let us write $\tau_0$ in the form $\tau_0 = L_1 L_2 L_3$ as in the previous paragraph and let $k$ be the number in $L_1$. We observe that if $k \neq n + 1$, then we have $k = i$. Indeed, if we assume otherwise that $k \neq i$ and that $k \neq n + 1$, then we find that $k + 1$ comes after $k$ in $\tau_0$; this is a contradiction. As a consequence of this observation we see that $\tau_0$ starts either with $n + 1$ or with $i$. On the other hand, if $k = n + 1$, then by interchanging $k$ with the first entry of $L_2$ we obtain another element in $X^{+}_{I,J}$ and this new element is smaller than $\tau_0$ in Bruhat-Chevalley order. This is a contradiction as well. Therefore, in $\tau_0$, we have $i$ as the first entry. Now there are two easy cases;
1) $j \leq i$ and $\tau_0$ is of the form

$$\tau_0 = i \ i - 1 \ldots i - j + 1 \mid n + 1 \ n \ldots i + 1 \ i - j \ i - j - 1 \ldots 1.$$  \hspace{1cm} (13)

2) $j > i$ and $\tau_0$ is of the form

$$\tau_0 = i \ n + 1 \ n \ldots n + 2 - (j - i) \ i - 1 \ i - 2 \ldots 1 \mid n + 1 - (j - i) \ n - (j - i) \ldots i + 1.$$  \hspace{1cm} (14)

Note that the vertical bar is between the $j$-th and the $j + 1$-st positions.

We proceed with some observations regarding how the posets climb up in the Bruhat-Chevalley order on $X_{I,J}^+$, starting with $\tau_0$’s as in (13) and (14). First of all, if $\tau_0$ is as in (13), then to get a covering relation, there is only one possible interchange, namely, moving $i - j + 1 \in L_2$ into $L_3$. In this case, to maintain the descents, the number that is replaced by $i - j + 1$ has to be $n + 1$, which goes into the first entry of $L_2$. In other words, the unique $w \in X_{I,J}^+$ that covers $\tau_0$ is

$$w = i \ n + 1 \ i - 1 \ldots i - j + 2 \mid n \ldots i + 1 \ i - j + 1 \ i - j \ i - j - 1 \ldots 1.$$  \hspace{1cm} (15)

It is easy to verify that there are exactly two elements that covers $w$;

$$w_{(2)} = n + 1 \ i \ i - 1 \ldots i - j + 2 \mid n \ldots i + 1 \ i - j + 1 \ i - j \ i - j - 1 \ldots 1$$  \hspace{1cm} (16)

and

$$w^{(2)} = i \ n + 1 \ n \ i - 1 \ldots i - j + 3 \mid n - 1 \ldots i + 1 \ i - j + 2 \ i - j + 1 \ldots 1.$$  \hspace{1cm} (17)

By Remark 3.1 we see that all elements that lie above $w_{(2)}$ in $X_{I,J}^+$ start with $n + 1$. Also, since there is no ascent at the 1-st position for such elements, the resulting upper interval $[w_{(2)}, w_0]$ in $X_{I,J}^+$ is isomorphic to a double coset poset in $S_{n+1}$ with $I^c = \{i\}$ and $J^c = \{j\}$, hence it is a chain.

There are two covers of $w^{(2)}$; one of them, $w_{(3)}$, is an element of the interval $[w', w_0]$ (hence $w_{(3)}$ covers $w'$ as well). The other cover of $w^{(2)}$ is

$$w_{(3)} = i \ n + 1 \ n \ n - 1 \ i - 1 \ldots i - j + 4 \mid n - 1 \ldots i + 1 \ i - j + 3 \ i - j + 2 \ldots 1.$$  \hspace{1cm} (18)

Now the pattern is clear; $w^{(3)}$ has exactly two covers one of which lies in $[w_{(3)}, w_0]$ and the other $w^{(4)}$ has a similar structure. Therefore, the bottom portion of the resulting poset is a ‘ladder’, as depicted in Figure 2, and the chains $w^{(p)}$ and $w_{(p)}$, $p \geq 3$ climb up to meet for the first time either at $w_0$, or at

$$w^{(m+1)} = w_{(m+1)} = n + 1 \ n \ldots n + 1 - (j - 2) \ i \mid n + 1 - (j - 3) \ldots i \ldots 2 \ldots 1.$$  \hspace{1cm} (19)

In the latter case, of course, $w_{0}$ is the unique cover of $w^{(m+1)} = w_{(m+1)}$ and it is easy to check from (19) that this happens if and only if $n + 1 - (j - 1) > i$. In both of these cases, the height of our poset does not exceed $j$.  

13
Figure 2: The Bruhat-Chevalley order on $X_{I,J}^+$ for $I^c = \{i\}$, $J^c = \{1, j\}$, where $2 < j \leq i$.

Now we look at the covers of $\tau_0$ in the case of (14). In this case, there are exactly two covers of $\tau_0$:

$$w^{(1)} = n + 1 \ i \ n \ldots n + 2 - (j - i) \ i - 1 \ i - 2 \ldots 1 \ | \ n + 1 - (j - i) \ n - (j - i) \ldots i + 1$$

$$w^{(2)} = n + 1 \ i \ n \ldots n + 2 - (j - i) \ n + 1 - (j - i) \ i - 1 \ i - 2 \ldots 2 \ | \ n - (j - i) \ldots i + 1 \ 1.$$
Figure 3: The Bruhat-Chevalley order on $X_{I,J}^+$ for $I^c = \{i\}$, $J^c = \{1, j\}$, where $2 \leq i < j$.

**Theorem 3.2.** Let $G$ denote $SL(n+1)$ and let $P_I$ and $P_J$ be two standard parabolic subgroups of $G$. If $G/P_I \times G/P_J$ is a spherical double flag variety, then the inclusion poset $(Z, \subseteq)$ of $G$-orbit closures is either a chain or one of the “ladder lattices” as depicted in Figure 4. More precisely, we have

1. if $|I^c| = |J^c| = 1$, then $Z$ is isomorphic to a chain;
2. if $|I^c| = 1$ and $J^c = \{s_j, s_{j+1}\}$ ($1 \leq j \leq n - 1$), then $Z$ is isomorphic to a chain;
3. if $I^c \in \{\{s_2\}, \{s_{n-1}\}\}$ and $J^c = \{s_p, s_q\}$ ($1 < p < p + 1 < q < n$), then the Hasse diagram of $Z$ is as in Figure 1;
4. if $I^c \in \{\{s_1\}, \{s_n\}\}$ and $|J^c| \geq 2$ (but $J^c \neq \{s_j, s_{j+1}\}$ ($1 \leq j \leq n - 1$)), then $Z$ is isomorphic to a chain;
5. if $I^c \in \{\{s_2\}, \ldots, \{s_{n-1}\}\}$, and $J^c = \{s_1, s_j\}$ or $J^c = \{s_j, s_n\}$ with $2 < j < n - 1$, then
   (a) the Hasse diagram of $Z$ is as in (A) in Figure 4 for $2 < j \leq i$ and $i + j - 2 < n$;
   (b) the Hasse diagram of $Z$ is as in (B) in Figure 4 for $2 < j \leq i$ and $i + j - 2 \geq n$;
   (c) the Hasse diagram of $Z$ is as in (C) in Figure 4 for $j > i \geq 2$ and $i + j - 2 < n$;
   (d) the Hasse diagram of $Z$ is as in (D) in Figure 4 for $j > i \geq 2$ and $i + j - 2 \geq n$.

**Acknowledgements.** We are grateful to John Stembridge for several reasons, including for his Maple codes and for answering our questions about his work. We thank Bill Graham for bringing this problem to our attention. We thank Roman Avdeev for his comments on the first version of this manuscript. Finally, we thank the referee for very careful reading of our paper and for the constructive suggestions.
Figure 4: The ladder posets.
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