Two-pion exchange and strong form-factors in covariant field theories

G. Ramalho$^1$, A. Arriaga$^1$ and M. T. Peña$^{1,2}$

$^1$Centro de Física Nuclear da Universidade de Lisboa, 1699 Lisboa Codex, Portugal
$^2$Departamento de Física, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal

(March 19, 2018)

Abstract

In this work improvements to the application of the Gross equation to nuclear systems are tested. In particular we evaluate the two pion exchange diagrams, including the crossed-box diagram, using models developed within the spectator-on-mass-shell covariant formalism. We found that the form factors used in these models induce spurious contributions that violate the unitary cut requirement. We tested then some alternative form-factors in order to preserve the unitarity condition. With this new choice, the difference between the exact and the spectator-on-mass-shell amplitudes is of the order of the one boson scalar exchange, supporting the idea that this difference may be parameterized by this type of terms.

I. INTRODUCTION

Electron scattering experiments off light nuclei at high momentum transfer (few GeV/c) are expected to be performed at facilities such as TJNAF. These experiments will provide new information on the intermediate and short range behavior of the nuclear interaction, establishing an important challenge to theoretical models. In fact, at these regimes, theoretical descriptions of the adequate hadronic degrees of freedom – nucleons, mesons and resonances – have to be based on both relativistic kinematics and dynamics.

Over the past years different relativistic formulations have been investigated for application to light nuclei, which can be classified in two major categories: Relativistic Hamiltonian Dynamics and Relativistic Field Theories. The first one, not considered in the present paper, assumes basically one of two alternative forms, the light-front form and the instant form. The latter is developed within a Schrödinger equation framework, uses variational Monte Carlo techniques and has been applied recently to the 3- and 4-nucleon systems. In this formalism relativistic covariance is achieved through the Poincaré Group algebra, and the resulting Hamiltonian consists of a relativistic kinematic energy operator and 2- and 3-body interactions. The 2-body interaction, containing asymptotic relativistic corrections, is parameterized in order to reproduce the NN data ($\chi^2 \simeq 1$ for $\sim 30$ parameters) below 350 MeV; it also comprises boost corrections up to the order of $(P/2M)^2$, where $P$ is the
total 3-momentum of the interacting pair and $M$ is the nucleon mass. Finally, the 3-body interaction is fitted to the triton binding energy. The generalization to heavier systems is in principle possible.

The second category methods are based on meson exchange diagrams. The non-perturbative character of the nuclear interaction definitely requires infinite sums of these amplitudes. As well known, infinite series can effectively be summed by means of integral equations, which assigns to the Bethe-Salpeter equation a predominant role in the relativistic description of the NN problem.

The Bethe-Salpeter equation is a manifestly covariant 4-dimensional integral equation. In principle, its interaction kernel should include all irreducible diagrams derived from a considered Lagrangian, which is clearly out of reach of the present calculations. This limitation determines inevitably the truncation of the kernel expansion and only a restrict set of appropriate meson exchange diagrams is kept.

Even with a truncated kernel the Bethe-Salpeter equation is, beyond ladder approximation, very hard to solve. This difficulty lead to the development of covariant dimensional reductions, imposing restrictions on the energy variable, from which 3-dimensional integral equations, known as Quasi-Potential (QP) equations, are obtained.

Among QP equations, the Gross equation considers only ladder and crossed-ladder diagrams, and is derived from the assumption that important cancellations between the two categories of diagrams occur. These cancellations mean that their sum is dominated by the contribution, to the ladder diagrams, of the mass pole of the heavier particle or, for equal interacting masses, by the combination of the corresponding mass poles. In the first case, the Gross prescription consists on replacing the sum of the ladder and crossed-ladder diagrams by the ladder diagrams with the massive particle on its mass shell. In the second case, a symmetrized combination of the ladder diagrams with one of the interacting particles on its mass shell is required. The amplitudes obtained contain ($\sim 13$) parameters (meson masses and coupling constants) fitted to the NN data ($\chi^2 \simeq 2 - 3$) below the pion production threshold.

For scalar interacting particles in the static limit (the limit when the mass of one of the interacting particles goes to infinity), the above cancellation is proved to be exact order by order $\mathcal{O}[5]$, meaning that in this case the Gross prescription is exact. When one of the masses, although finite, remains much larger than the other, the prescription is a very good approximation as shown by Maung and Gross. This result motivates the application of the Gross or spectator equation to the proton-nucleus scattering, such as $p-^{40}\text{Ca}$ scattering. For equal masses it may be shown that the approximation is still reasonably good.

In nuclear applications, however, the spinor structure of the nucleon is essential, and the vertex interaction includes a Dirac matrix structure. Moreover, for pion exchanges with pseudovector coupling (PV), the coupling supported by chiral symmetry arguments, the integral equations diverge, forcing the adoption of regularization schemes of some kind. The most common regularization procedure consists on the inclusion of form-factors, which could, in principle, be reinterpreted in terms of self-energy contributions of nucleons and mesons. Nevertheless, the determination from first principles of self-energies, due to radiative corrections, would require the resolution of coupled Dyson-Schwinger equations, again far beyond the present calculations capability. Furthermore, the inclusion of form-factors can also be justified as an effective way of representing the internal structure of the
hadrons and, as a consequence, phenomenological form-factors have to be considered.

In these situations, where Dirac particles exchange mesons with realistic couplings implying the transfer of some quantum numbers, the quality of the Gross prescription has not yet been studied systematically. Therefore its application in such circumstances has, at present, only heuristic and pragmatic motivations. Even so, the Gross prescription has been already applied to the 2- and 3-nucleon systems [11–13], which urges, on one hand, the study of its ability to approximate the exact solutions of the Bethe-Salpeter equation with ladder and crossed-ladder series (from here on we use exact in this sense) and, on the other, the search for ways to systematically improve it. This study is precisely the goal of the present work.

The analysis is focused on fourth-order diagrams with incoming and outgoing particles on-mass-shell, since the complexity of the calculations is substantial and the information obtained is already very rich and relevant. These amplitudes were considered before in references [14,15]: in the first one only the static limit was evaluated, in the second one the recent NN models were not considered yet, and the comparison between the Gross prescription and the exact calculation is presented only for one energy. In Sec. II a brief introduction to the Bethe-Salpeter equation and its 3-dimensional reductions is given. In Sec. III the box and crossed-box pion exchange amplitudes are analyzed in detail. In Sec. IV different form-factors are examined and, in particular, new form-factors are suggested aiming the resolution of problems inherent to the conventional ones. In Sec. V results of applications are discussed and, finally, in Sec. VI conclusions are presented.

II. BETHE-SALPETER EQUATION AND THREE DIMENSIONAL REDUCTIONS

Within a covariant field theory framework, the scattering amplitude $\mathcal{M}$ is given by the sum of all the possible interaction processes (represented by Feynman diagrams) derived from the Lagrangian under consideration. The scattering amplitude of the interaction between particles of masses $m$ and $M$ is determined by the Bethe-Salpeter (BS) equation

$$\mathcal{M}(p, p'; P) = \mathcal{V}(p, p'; P) + i \int \frac{d^4k}{(2\pi)^4} \mathcal{V}(p, k; P) G(k, P) \mathcal{M}(k, p'; P),$$

where $P$ is the total 4-momentum, and $p$ and $p'$ are respectively the initial and final momenta of one of the particles. For this equation we use the shorthand notation

$$\mathcal{M} = \mathcal{V} + \mathcal{V} G \mathcal{M},$$

keeping in mind that the homogeneous term includes a 4-dimensional integration. In the above equation $\mathcal{V}$ is the interaction kernel which involves, in principle, all irreducible diagrams, and $G$ is the 2-particle propagator given by

$$G(k, P) = G_1(k) G_2(P - k),$$

with the explicit form for Dirac particles.
\[ G_i(k) = \frac{-i}{m_i - \not{k} - i\varepsilon} = \frac{-i(m_i + \not{k})}{m_i^2 - k^2 - i\varepsilon}, \]  

(2.4)

with \( m_1 = M, m_2 = m \) and \( i = 1, 2 \).

As mentioned earlier, the exact solution of the BS equation for general couplings is not technically possible at present, and approximations are unavoidable. A common approximation consists on restricting \( \mathcal{V} \) to a sum of One Boson (or Meson) Exchange diagrams, and is usually called the Ladder Approximation. There are two main objections to this approximation: firstly the one body limit (the Klein-Gordon or Dirac equations with interaction terms) is not recovered when one of the masses goes to infinity \[4,5\]; secondly it has been shown that, in general, crossed-ladder diagrams have important contributions \[4,5\], and hence should not be neglected.

Alternative ways rely on the replacement of the BS equation by the system

\[
\mathcal{M} = K + Kg\mathcal{M} \tag{2.5}
\]

\[
K = \mathcal{V} + \mathcal{V}(G-g)K. \tag{2.6}
\]

which is perfectly equivalent to the BS equation. In this equations \( g \) is a different 2-particle propagator and \( K \) a new kernel whose iteration reads

\[
K = \mathcal{V} + \mathcal{V}(G-g)\mathcal{V} + \mathcal{V}(G-g)\mathcal{V}(G-g)\mathcal{V} + ... \tag{2.7}
\]

A Quasi-Potential equation is obtained through a covariant dimensional reduction of the Eq. \((2.5)\). Since there are no first principle rules to dictate how this reduction should be performed, a large ambiguity gives room for many options. Usually, different approximations comprise different appropriate choices to constraint the energy-component of the free 4-momentum in the propagator \( g \). If \( g \) is a good approximation to \( G \), then only the first term in Eq. \((2.7)\) can be kept and a simpler 3-dimensional, but still covariant, equation is obtained. If needed, and possible, the kernel \( K \) can be corrected by the inclusion of higher order terms, following Eq. \((2.7)\).

Examples of QP equations are the Blankenbecler-Sugar \[16\], the Equal-Time \[17\] and the Gross \[4\] equations, all satisfying the required one-body limit. The Gross equation, hereafter the only considered, results from the choice

\[
g(k,P) = -i2\pi \frac{\delta_+(k^2 - M^2)}{m^2 - (W - k_0)^2 - i\varepsilon} \Lambda_1(k)\Lambda_2(P - k) \tag{2.8}
\]

\[
= -i2\pi \frac{\delta(k_0 - E_k)}{2E_k \left[ \varepsilon_k^2 - (W - E_k)^2 - i\varepsilon \right]} \Lambda_1(k)\Lambda_2(P - k),
\]

where

\[
W = \sqrt{P^2} \tag{2.9}
\]

\[
E_k = \sqrt{M^2 + k^2} \tag{2.10}
\]
$e_k = \sqrt{m^2 + k^2}$  \hspace{1cm} (2.11)

$\delta_+(k^2 - M^2) = \delta(k^2 - M^2)\theta(k_0)$  \hspace{1cm} (2.12)

$\Lambda_i(k) = m_i + \not{k}$.  \hspace{1cm} (2.13)

The second line of Eq. (2.8) is valid only in the CM reference frame. As it can be seen from this equation, the Gross choice means that the particle with mass $M$ is placed on its mass shell in all intermediate states.

The principal motivation for the Gross equation comes from the cancellation theorem [4,5], that states that for scalar particles in the static limit ($M \to \infty$), the sum of all ladder and crossed-ladder diagrams is exactly given by the sum of ladder diagrams with the massive particle on its mass shell. This is shown to be a result of important cancellations between ladder and crossed-ladder amplitudes in all orders. In Fig. 1 this cancellation is illustrated for fourth-order diagrams. In all figures the cross on a line means that the corresponding particle is on-mass-shell; we will refer to the diagram with one of the particles on its mass-shell in the intermediate state as the Gross amplitude.

For identical interacting particles however, equal treatment is required and Eq. (2.8) has to be properly symmetrized in order to comply with the Pauli principle. This procedure has been implemented recently using a system of coupled equations for the half-on-mass-shell amplitudes [11]. Each of these equations can be written as follows:

$$M(p, p', P) = V(p, p', P)$$  \hspace{1cm} (2.14)

$$+i \int \frac{d^4k}{(2\pi)^4} V(p, k; P) g(k; P) M(k, p'; P);$$

here the propagator takes into account an adequate symmetrization of the particles in all intermediate states having the form:

$$g(k; P) = \frac{1}{2} (2\pi) \Lambda_1(k) G_2(P - k) \delta_+(M^2 - k^2)$$  \hspace{1cm} (2.15)

$$\frac{1}{2} (2\pi) G_1(k) \Lambda_2(P - k) \delta_+(M^2 - (P - k)^2)$$

where the equal mass condition ($m = M$) has been used. The 3-dimensional reduction is trivially obtained by performing the $k_0$ integration using the $\delta_+$ function.

III. BOX AND CROSSED-BOX AMPLITUDES

In this section we consider the crossed-box diagram and the contributions to the box which are not explicitly contained in the spectator formalism. The last ones, often denoted by subtracted-box contributions, are shown in Fig. 2. Together, crossed-box and subtracted-box diagrams are the lowest order corrections to the one boson exchange kernel of the Gross equations (see Eq. (2.6)). We calculated them exactly for a variety of kinematic conditions (energy and scattering angle) in order to assess their importance relatively to the Gross amplitude.

We started by deriving the general expressions for the box and crossed-box diagrams, the $\mathcal{M}_I$ and $\mathcal{M}_{II}$ amplitudes respectively, which are represented in Fig. 3 along with detailed
notation for the kinematic variables in the inner loop. In all diagrams particle 1 is represented by the lower line and particle 2 by the upper line.

These amplitudes result from an effective Lagrangian for spin-1/2 particles of equal mass $M$, interacting through the exchange of a meson of mass $\mu$:

$$\mathcal{L}_\sigma = \frac{1}{2} \bar{\psi} (i \gamma_\mu \partial^\mu - M) \psi + \frac{1}{2} (\partial^\mu \sigma \partial_\mu \sigma - \mu^2 \sigma^2) - g_\sigma \sigma \bar{\psi} \psi,$$

for scalar vertices, or

$$\mathcal{L}_\pi = \frac{1}{2} \bar{\psi} (i \gamma_\mu \partial^\mu - M) \psi + \frac{1}{2} (\partial^\mu \pi \cdot \partial_\mu \pi - \mu^2 \pi^2)$$

$$- ig_\pi \bar{\psi} \left[ \lambda \gamma^5 + (1 - \lambda) \frac{q^2}{2M} \gamma^5 \right] \tau \cdot \pi \psi,$$

for a isovector, pseudoscalar and pseudovector coupling admixture, with a mixing parameter $\lambda$ ($0 \leq \lambda \leq 1$).

Application of the Feynman rules gives for the amplitudes:

$$\mathcal{M}_I = i \Gamma \int \frac{d^4 k}{(2\pi)^4} H_I(p, k, p') [\xi^1_I(k) \otimes \xi^2_I(P - k)]$$

$$\times D_1(k) D_2(P - k) d_3(p - k) d_4(p' - k)$$

for the box amplitude, and

$$\mathcal{M}_{II} = i \Gamma \int \frac{d^4 k}{(2\pi)^4} H_{II}(p, k, p') [\xi^1_{II}(k) \otimes \xi^2_{II}(P - k)]$$

$$\times D_1(p' + p - k) D_2(P - k) d_3(p - k) d_4(p' - k),$$

for the crossed-box amplitude.

In the above equation $\Gamma$ includes the coupling constants and the isospin operators defined in the Appendix (Eqs. (A1) and (A8)). The propagator of spinor $i$, $i = 1, 2$, has been separated into a function $D_i$ (the denominator of the Dirac propagator following the second line of equation 2.4) which coincides with the propagator of a scalar particle with the same mass $M$

$$D_i(k) = \frac{1}{M^2 - k^2 - i\varepsilon_i},$$

and a function carrying the Dirac structure of the spin–1/2 particle which, together with the two interaction vertices, defines the Dirac structure functions $\xi^i_\alpha(k)$ ($i = 1, 2; \alpha = I, II$). Although the explicit form of $\xi^i_\alpha(k)$ depends on the interaction considered, it has the general structure

$$\xi^i_\alpha(k) = a^i_\alpha(k) + b^i_\alpha(k) \bar{k}.$$
The expressions for $a_i^\alpha(k)$ and $b_i^\alpha(k)$ are given in the Appendix. The required explicit expressions for spinors and phase conventions can be found in the work of F. Gross et al. [11].

The $d_i$ function stands for the propagator of meson $i$

$$d_i(q) = \frac{1}{\mu^2 - q^2 - i\varepsilon_i}.$$  \hspace{1cm} (3.7)

Finally, $H_\alpha$ are scalar functions involving the regularization mechanisms of the vertices, or form-factor functions. They will be discussed at length in Sec. IV.

We note that in the case of scalar particles, no regularization scheme is needed, and we may simply take $H_\alpha = 1$, together with $a_i^\alpha(k) = 1$ and $b_i^\alpha(k) = 0$, $(i = 1, 2; \alpha = I, II)$. This particular situation was considered by us as a test study, meant firstly to check partial analytical and numerical results.

Introducing explicitly the on-mass-shell energy variables $E_k$ defined by Eq. (2.10) and

$$\omega_q = \sqrt{\mu^2 + q^2},$$  \hspace{1cm} (3.8)

and the off-mass-shell energy-component $k_0$ we can write

$$\mathcal{M}_I = i\Gamma \int \frac{d^4k}{(2\pi)^4} H_I \left[ \xi^2_1 \otimes \xi^2_2 \right] h_I(p, k, p'; k_0)$$  \hspace{1cm} (3.9)

$$\mathcal{M}_{II} = i\Gamma \int \frac{d^4k}{(2\pi)^4} H_{II} \left[ \xi^2_{II1} \otimes \xi^2_{II2} \right] h_{II}(p, k, p'; k_0)$$  \hspace{1cm} (3.10)

where part of the integrand function arguments have been omitted for simplicity. The $h_I$ and $h_{II}$ functions are given by:

$$h_I(p, k, p'; k_0) =$$

$$\frac{1}{E_k^2 - k_0^2 - i\varepsilon_1} \frac{1}{E_k^2 - (W - k_0)^2 - i\varepsilon_2} \times$$

$$\frac{1}{\omega_{p-k}^2 - (E_p - k_0)^2 - i\varepsilon_3} \frac{1}{\omega_{p-k}^2 - (E_p - k_0)^2 - i\varepsilon_4}$$

and

$$h_{II}(p, k, p'; k_0) =$$

$$\frac{1}{E_{p+k}^2 - k_0^2 - i\varepsilon_1} \frac{1}{E_k^2 - (W - k_0)^2 - i\varepsilon_2} \times$$

$$\frac{1}{\omega_{p-k}^2 - (E_p - k_0)^2 - i\varepsilon_3} \frac{1}{\omega_{p-k}^2 - (E_p - k_0)^2 - i\varepsilon_4}.$$  \hspace{1cm} (3.11)

From the last two equations we can locate the $k_0$ poles, related with the physical singularities present to satisfy unitarity. If the form-factor functions $H_\alpha$ have no singularities, and since each propagator contains two poles near the real axis, the $h_\alpha$ functions will have eight physical complex singularities, four in the upper-half complex plane and four in the lower
one. The $k_0$ integration can be done using the Cauchy theorem by closing the integration contour, and for practical reasons we chose to close the contour integration in the lower-half complex plane. The box amplitude splits into the four following contributions

\[ \mathcal{M}_I = \mathcal{M}_I^+ + \mathcal{M}_I^- + \mathcal{M}_I^{10} + \mathcal{M}_I^{20} \]  

(3.13)

where

- $\mathcal{M}_I^+$ is the residue of the $k_0 = E_k$ pole, present in the propagator of particle 1, meaning that this particle is on-mass-shell with 4-momentum $(E_k, k)$ (positive energy). This amplitude is the Gross or spectator fourth-order amplitude.

- $\mathcal{M}_I^-$ is the residue of the $k_0 = W + E_k$ pole, present in the propagator of particle 2, meaning that this particle is on-mass-shell with 4-momentum $(-E_k, -k)$ (negative energy).

- $\mathcal{M}_I^{10}$ is the residue of the $k_0 = E_p + \omega_{k-p}$ pole, present in the propagator of the meson carrying momentum $p-k$, meaning that this meson is on-mass-shell with 4-momentum $(\omega_{p-k}, p - k)$ (positive energy).

- $\mathcal{M}_I^{20}$ is the residue of the $k_0 = E_p + \omega_{p'-k}$ pole, present in the propagator of the meson carrying momentum $k-p'$, meaning that this other meson is on-mass-shell with 4-momentum $(-\omega_{p'-k}, k - p')$ (negative energy).

Similarly, the crossed-box amplitude decomposes into

\[ \mathcal{M}_{II} = \mathcal{M}_{II}^+ + \mathcal{M}_{II}^- + \mathcal{M}_{II}^{10} + \mathcal{M}_{II}^{20} \]  

(3.14)

where

- $\mathcal{M}_{II}^+$ corresponds to the residue of the $k_0 = W + E_{p+p'-k}$ pole, meaning that particle 1 is on-mass-shell with 4-momentum $(-E_{p'+p-k}, p' + p - k)$ (negative energy).

- $\mathcal{M}_{II}^-$ corresponds to the residue of the $k_0 = W + E_k$ pole, meaning that particle 2 is on-mass-shell with 4-momentum $(-E_k, -k)$ (also negative energy).

- $\mathcal{M}_{II}^{10}$ and $\mathcal{M}_{II}^{20}$ have the same interpretation as $\mathcal{M}_I^{10}$ and $\mathcal{M}_I^{20}$ have for the box amplitude.

Given the above interpretation we can separate retardation effects from nucleon negative energy states contributions. The retardation effects are obtained by adding

\[ \mathcal{M}_I^0 = \mathcal{M}_I^{10} + \mathcal{M}_I^{20} \]  

(3.15)

from the box diagram to

\[ \mathcal{M}_{II}^0 = \mathcal{M}_{II}^{10} + \mathcal{M}_{II}^{20} \]  

(3.16)

from the crossed-box diagram. The nucleon negative energy states contributions are given by adding $\mathcal{M}_I^-$ from the box diagram to
\[ M_{II}^+ = M_{II}^0 + M_{II}^- \]  

from the crossed-box diagram.

It can be proved that if the \( H_\alpha \) functions have only real singularities, the sum of \( M_I^0 \), \( M_{II}^0 \) and \( M_{II}^- \) is real \[ \Box \]. This result guarantees in particular that the fourth-order amplitude satisfies unitarity: the only possible imaginary part comes from the two-nucleon unitarity cut. Still, the existence of singularities in the \( H_\alpha \) functions may yield extra terms to Eqs. (3.13) and (3.14), other than the physical on-mass-shell particle pole contributions discussed. We anticipate here that those contributions are in general not negligible. Our study will henceforth show that the inclusion of crossed-box and subtracted-box diagrams in the kernel allows us to be more discriminative in our choices for the short-range parameterization of the the NN interaction. We will be back to this discussion in the next sections.

Finally, we note that the usual procedure for antisymmetrizing the amplitudes has to be performed. This can be achieved by permuting the particles in the final state \[ \Box \], as usually. One obtains for the fourth-order amplitude:

\[
M^{(4)} = \frac{1}{2} (M_I + \delta \tilde{M}_I) + \frac{1}{2} (M_{II} + \delta \tilde{M}_{II})
\]  

where \( \delta = (-1)^I \) (\( I \) being the total isospin of the system) and \( \tilde{M} \) denote the permuted terms, obtained by application of the permutation operator to the final state particles.

Additionally, as discussed in reference \[ \Box \], the Pauli principle imposes an extra symmetrization on the spectator formalism amplitudes: this symmetrization is intrinsic to the intermediate states, where only one of the particles is on-mass-shell. As a consequence, the fourth-order diagrams resulting from the Gross equations are not only the diagrams with particle 1 on-mass-shell in the intermediate state, amplitude \( M_I^+ \), but also the diagrams where particle 2 is on-mass-shell in the intermediate state, amplitude \( M_I^{+}' \), as can be seen from Fig. \[ \Box \]:

\[
M^{(4)}_{\text{Gross}} = \frac{1}{4} M_I^+ + \delta \frac{1}{4} \tilde{M}_I^+ + \frac{1}{4} M_{II}^{+}' + \delta \frac{1}{4} \tilde{M}_{II}^{+}'.
\]  

The coefficients affecting the several sub-amplitudes force the two particles to be equally on-mass-shell in the intermediate state, which is sufficient for the total amplitude to satisfy the Pauli Principle.

**IV. STRONG FORM-FACTORS**

Nucleons are not elementary particles and consequently their hadronic structure has to be taken into account. Mathematically, such structure provides the necessary regularization of the integrals for the high order loops, needed to build in the analytical structure of a bound state wavefunction, or of a scattering transition matrix. Unfortunately, it is not yet possible to obtain the description of the nucleon structure directly from the QCD Lagrangian,
or even from effective models inspired on constituent quark structure. The compositeness of the nucleons is therefore currently described by means of form-factors which have to be phenomenologically fixed. Yet, it has been shown by Gross and Riska \[10\] that it is possible to describe consistently the electromagnetic interaction with nucleons, within a relativistic and manifestly covariant framework, with phenomenological strong form-factors for the meson-nucleon vertices, present in the current conservation relations following the Ward-Takahashi identities.

In non relativistic applications, analytic dipolar (or even higher power) functions of the 3-momentum are used as form-factors. In relativistic applications the covariance requirement motivates the use of form-factors of the type:

\[
f(q^2) = \frac{\Lambda^2}{\Lambda^2 - q^2}
\]

where \( q^2 = q_0^2 - \mathbf{q}^2 \) substitutes \( \mathbf{q}^2 \) in the non-relativistic form-factors.

The introduction of the strong form-factors can be interpreted as vertex-dressing, and we take them to have a factorized separable form, as done in reference \[11\]:

\[
F_m(p,p') = f_m(q^2)f_N(p^2)f_N(p'^2)
\]

where \( f_N \) and \( f_m \) stand for the nucleon and meson form-factors respectively, \( p \) and \( p' \) are the nucleon 4-momenta and \( q = p' - p \) the meson 4-momentum. As a consequence, the \( H_\alpha \) function, introduced in the Sec. [\[\]), includes from each vertex two nucleon form-factors \( f_N \) and one meson form-factor \( f_m \). The function \( H_\alpha \) is therefore a product of eight form-factor functions, since the form-factors of on-mass-shell nucleons are normalized to one.

Alternatively, the dressing can be associated with self-energy corrections to the Dirac and meson propagators (radiative corrections). Following these lines, a product of two strong form-factors \( f_m(q^2) \), each coming from a different vertex, can be incorporated in the renormalization of the meson propagator according to reference \[10\]:

\[
\frac{f_m^2(q^2)}{\mu^2 - q^2} = \frac{1}{\mu^2 - q^2 + \Pi_m(q^2)}
\]

being the meson self-energy becomes fixed by

\[
\Pi_m(q^2) = \left[ \frac{1}{f_m^2(q^2)} - 1 \right](\mu^2 - q^2).
\]

Analog equations for the nucleon Dirac propagator and self-energy can be written \[11\]:

\[
\frac{f_N^2(p^2)}{M - \not{p}} = \frac{1}{M - \not{p} + \Sigma(p)}
\]

being the nucleon self-energy becomes fixed by

\[
\Sigma(p) = \left[ \frac{1}{f_N^2(p^2)} - 1 \right](M - \not{p}).
\]
In the most recent works which applied the Gross equation [13] the form-factors were assumed to have the expression:

\begin{align}
 f_N(p^2) &= \frac{(\Lambda^2_N - M^2)^2}{(\Lambda^2_N - M^2)^2 + (M^2 - p^2)^2} \\
 f_m(q^2) &= \frac{(\Lambda^2_m - \mu^2)^2 + \Lambda^4_m}{(\Lambda^2_m - q^2)^2 + \Lambda^4_m}
\end{align}

(4.7) (4.8)

The two parameters \( \Lambda_N \) and \( \Lambda_m \) are the nucleon and meson cut-offs respectively and are fixed, along with the couplings and meson masses, through a fit to the nucleon-nucleon scattering data.

These choices imply that the effective nucleon propagator for those models is the product

\begin{align}
 f_N^2(p^2) &= \frac{1}{M - \hat{p}} \times \\
 &\times \left[ \frac{\tilde{\Lambda}_N^2}{M^2 + i\Lambda^2_N - p^2} \right]^2 \left[ \frac{\tilde{\Lambda}_N^2}{M^2 - i\Lambda^2_N - p^2} \right]^2
\end{align}

(4.9)

with \( \tilde{\Lambda}_N^2 = \Lambda^2_N - M^2 \). A similar expression can be written for the meson effective propagator. The first factor on the r.h.s. of Eq. (4.9) is the non renormalized propagator, while the second and third factors are effectively propagators of resonances of width \( \tilde{\Lambda}_N^2 \). Such widths have ad-hoc energy and 3-momentum dependencies, in particular they are constant in the entire energy and momentum range, with the consequence of inducing spurious imaginary contributions in the scattering amplitude. We can interpret these complex mass terms as inelasticity sources associated with nucleon or meson excitations, which nevertheless are non physical, namely because they open channels at energies below the pion production threshold.

As we will see in the next section, the spurious contributions may be small under some conditions, e.g. for light meson scalar exchange, but they have sizeable effects for a pion exchange with pseudovector coupling.

The pathology of the form-factor that we are discussing manifests only when we use it in the full Bethe-Salpeter equation, that is to say, when we treat the energy-component as a totally independent variable. For the Gross reduction, where these form-factors were used and calibrated by the data, the spurious components are not evident: the prescription for the \( k_0 \) variable constrains its range such that the non-physical complex singularities are irrelevant. In other words, the spectator equation is insensitive to the existence of the complex poles of the form-factors. Nevertheless, the study of the validity of the Gross prescription, as well as the introduction of possible required corrections to the kernel of the integral equation, need to incorporate crossed-box and subtracted-box diagrams which may be crucially affected by the form-factors poles.

In order to solve the problem we explored alternative form-factor functions. Since the form-factors are essentially needed to cut the 3-momentum range of the integrals, we made the choice of not allowing the regularization function to depend on the relative energy variable. However, in order to still satisfy covariance, the new form-factors were built by giving to the square of the 3-momentum \( \mathbf{k} \) a covariant form. More specifically, in an arbitrary frame where the total two nucleon momentum is \( \mathbf{P} \), \( \mathbf{k}^2 \) is replaced by the Lorentz invariant
\[ \bar{k}^2 = \left( \frac{P \cdot k}{P^2} \right)^2 - k^2, \]  

(4.10)

with \( \bar{k}^2 = k^2 \) in the CM frame. \( \bar{k} \) stands for the square root of \( \bar{k}^2 \). It becomes clear from this equation that the price to be paid for this new choice, which does not consider the relative energy dependence explicitly, is to have form-factors depending on the on-mass-shell momentum of the external particles, or, in other words, on the total momentum \( P \) of the 2 nucleon system. The complexities implied by this choice are only relevant, but with the possibility of being actually handled in practice, for the 3-nucleon applications, where the 2-nucleon subsystem has to be treated in the 3-nucleon system CM frame. They will also imply a new version of the work on the electromagnetic currents of reference [10], since in this reference the nucleon-nucleon interaction does not depend on the total momentum of the system.

Examples of this type of form-factor choices can be found, for instance, in the work of the Nijmegen [19] and Bonn [18] groups that used

\[ f_m(q^2) = e^{-\frac{q^2}{2\Lambda^2}} \]

(4.11)

and

\[ f_m(q^2) = \left[ \frac{\Lambda^2}{\Lambda^2 + q^2} \right]^n. \]

(4.12)

respectively, which were already written in terms of the Lorentz invariant \( q^2 \).

For our calculations we took two different models for the meson form-factor functions. Both choices satisfy the criterion of leaving out the \( k_0 \) dependence from those functions. The first choice corresponds to Eq. (4.12) with \( n = 1 \). For the second choice, we selected a functional form which would furthermore not alter the results of the calculations already done within the spectator formalism, preserving namely the description of the two nucleon scattering observables. Explicitly, we used

\[ f_m(p^2, p'^2, q^2) = \left. \frac{(\Lambda_m^2 - \mu^2)^2 + \Lambda_m^4}{(\Lambda_m^2 - q^2)^2 + \Lambda_m^4} \right|_{k_0=E_k} \]

(4.13)

\[ = \frac{(\Lambda_m^2 - \mu^2)^2 + \Lambda_m^4}{[\Lambda_m^2 + q^2 - (E_p - E_{p'})^2]^2 + \Lambda_m^4}. \]

The \( k_0 = E_k \) condition is necessary in order to keep the results for the \( \mathcal{M}_I^+ \) amplitude, that are part of the calculations in references [11,13]. This is the reason why in the denominator a "retardation-like" term \( E_p - E_{p'} \) appears, even though it may seem arbitrary at first thought. We may advance at this point that this term has very little importance: the results with and without it differ by at most 10%, with the exception of the region where the amplitudes vanish in the \( I = 1 \) channel.

A new version of the nucleon form-factors \( f_N \) is also required. We again made them dependent only on the relative 3-momentum which, together with the manifest covariance requirement, made them depending on the total 2 nucleon 4-momentum (Eq. (4.10)). Explicitly, we took the general nucleon-form-factor used in reference [13] calculated as for the \( \mathcal{M}_I^+ \) amplitude, which reads
\[ f_N(k^2; \tilde{p}^2) = \left[ \frac{\bar{\Lambda}_N^4}{\bar{\Lambda}_N^4 + (P - k)^2}_{k_0=E_k} \right]^{1/2} \left[ \frac{\bar{\Lambda}_N^4}{\bar{\Lambda}_N^4 + W^2(W - 2E_k)^2} \right]^{1/2}. \] (4.14)

In this equation \( W = 2E_{\tilde{p}} \) is the invariant mass, \( \tilde{p}^2 \) is defined from the 3-momentum of the initial and final particles and \( \bar{k}^2 \) is defined from the off-mass-shell particle 3-momentum.

The introduction of these new form-factor functions allowed us to eliminate the imaginary spurious contributions to the box and crossed-box diagrams, and successfully construct an amplitude satisfying the elastic cut unitarity requirement. This conclusion will be accounted for in detailed and quantitative terms in the next section.

V. RESULTS AND DISCUSSION

In order to describe the scattering process between two Dirac interacting particles we need to specify, in addition to the energy and scattering angle, the initial and final isospin and helicity (or spin) states. In general we have five independent helicity channels from a total of sixteen possible. We present here only the results for the matrix element corresponding to the \( ++ \rightarrow ++ \) transition, since similar conclusions can be drawn for the other channels.

If the form-factors have no singularities, we can use the decomposition of Eqs. (3.13) and (3.14) to compute the amplitudes. This is not the case, however, of the form-factors of Eqs. (4.7) and (4.8) whose singularity structure implies the inclusion of the corresponding residues.

A. Scalar exchange

As a reference calculation (an important check to our analytical calculations and numerical results) we considered the simplest case of scalar exchange between scalar identical particles of mass \( M \) (equal masses), where no form-factors are needed. We used the parameters

\[ \frac{g^2}{4\pi M^2} = 0.5 \]

\[ \mu = M/7. \]

The results are shown in Fig. 5 for three different kinetic energies in the lab reference frame. The curves refer to the box and crossed-box amplitudes, their sum and the spectator amplitude. For comparison we also include the result for the amplitude corresponding to the Blankenbecler-Sugar quasi-potential equation.

The figure clearly shows the cancellation between parts of the box and the crossed-box amplitudes, which has been shown previously to occur only in the static limit \[4,5\]. This conclusion is interesting, and in the meantime path integral techniques have been used to verify its validity for all orders \[6\].
Next we considered the exchange of a scalar meson between nucleons, which forces us to consider the \( I = 0 \) and \( I = 1 \) isospin NN channels. Form-factors of Eqs. (4.7) and (4.8) were included, and the parameters were taken from the models used in the three-nucleon calculations \([13,20]\), which read

\[
\frac{g^2}{4\pi} = 4.82 \\
\mu = m_\sigma = 498 \text{ MeV} \\
\Lambda_\sigma = \Lambda_m = 1141 \text{ MeV} \\
\Lambda_N = 1862 \text{ MeV}.
\]

We note that our calculation does not include the general off-shell couplings of the original model \([13]\).

The results for laboratory kinetic energy \( T_{\text{lab}} \) of 100, 200, and 300 MeV are shown in Figs. 6 and 7, for the \( I = 0 \) and \( I = 1 \) cases respectively. One concludes that the real part of the sum of the box and crossed-box amplitudes is well differs from the Gross amplitude, being the relative deviation of the order of 15\% for both isospin channels, at 100 MeV, and 30\% at 200 MeV. For 300 MeV these deviations become 50\% for \( I = 0 \) and 70\% for \( I = 1 \). This quantifies how the approximation deteriorates with increasing energy. It is interesting to add that in general the fourth order \( \sigma \) exchange contributions are 10\% of the second order (one \( \sigma \) exchange).

The quality of the Gross prescription is slightly improved for a lighter exchanged meson, as shown in Figs. 8 and 9, where the results were obtained with:

\[
\frac{g^2}{4\pi} = 1 \\
\mu = m_\pi = 138 \text{ MeV} \\
\Lambda_m = \Lambda_\pi = 2109 \text{ MeV}
\]

Since this exchange does not correspond to the pseudovector pion exchange, the value of the coupling constant was arbitrarily chosen by the convenience. This situation has been considered to study how the quality of the Gross prescription depends on the meson mass. From the same figures we can conclude also that while at 100 MeV the crossed-box amplitudes are very small, they become important at higher energies and cannot be neglected, as assumed by the ladder approximation. As for the retardation effects we can say that the contributions from the meson poles in the box and crossed-box tend to cancel each other. We may then conclude that the ladder approximation overestimates meson retardation effects.

Note that we showed only the real part of the fourth-order amplitudes. As discussed in the Sec. (II), the only imaginary contribution comes from the spectator amplitude \( \mathcal{M}_I^\pm \) when the form-factors have no singularities. This is not obviously the case of the form-factors of Eqs. (4.7) and (4.8), but fortunately the additional spurious residues contributions are small and mean less than a 2\% contribution. So in this case the unitarity violation is very small.
B. PV coupling

Next we evaluated the box and crossed-box amplitudes for the \( \pi \)NN coupling of Eq. (3.2) restricted to PV coupling \((\lambda = 0)\). We also used the form-factors introduced in Eqs. (4.7) and (4.8). The pion mass and cut-off parameters were given above and the coupling constant taken from the references \([13,20]\)

\[
g_\pi^2 = \frac{g_\pi^2}{4\pi} = 13.34.
\]

The allowed exchange of isospin introduces weighting factors for the box and crossed-box that are not the same for the two different isospin channels, resulting for the fourth-order amplitude

\[
\mathcal{M}^{(4)} = 3(\mathcal{M}_I + \mathcal{M}_{II}) - 2(\mathcal{M}_I - \mathcal{M}_{II})\tau_1 \cdot \tau_2.
\]  

The sign difference in the coefficients affecting the crossed-box contribution in the isoscalar and isovector parts, makes difficult the simultaneous representation of the exact amplitudes, in the two isospin channels, by the spectator-on-mass-shell approximation.

The real and imaginary parts of the amplitudes for \( T_{lab} = 100 \text{ MeV} \) are represented in Fig. 10. The first observation to be made is that their magnitudes, specially for the case of the imaginary part, are abnormally large: the OBE contributions corresponding to exchanged heavy mesons of the same model are \( \sim 100 \text{ GeV}^{-2} \). The second observation is that contrarily to the scalar coupling case the imaginary contribution of the crossed-box amplitude, that should be exactly zero, does not vanish, being two orders of magnitude larger than the box imaginary part. These conclusions are true for both isospin channels. As mentioned earlier, the reason for these results lies in the complex singularities present in the form-factors of (4.7) and (4.8), which clearly violate the unitarity condition.

Small changes on the cut-off parameters \( \Lambda_\pi \) modify the final results, but do not change the orders of magnitude obtained. The figures also show the dominance of the crossed-box amplitude justified by a cumbersome combination of the presence of \( \gamma^5 \) in the coupling and the \( f_\pi(q^2) \) form-factor. As can be seen from Fig. 11 this form-factor peaks in the region \( q^2 \geq 4M^2 \), where the negative-energy nucleon poles, defining the \( \mathcal{M}^+_I \) and \( \mathcal{M}^-_{II} \) contributions, occur: consequently these amplitudes are largely (and artificially) enhanced. Contrarily, in the box amplitude, the negative energy-state contribution included in \( \mathcal{M}^-_I \) is strongly suppressed by the nucleon \( f_N(k^2) \) form-factor, since one of the nucleons is very much off-mass-shell. Furthermore, since the region involved in \( \mathcal{M}^+_I \) (the Gross amplitude) is \( q^2 \leq 0 \), this amplitude is not crucially affected by the pronounced peak of the meson form-factor. The spectator equation solutions are then not affected by the spurious singularities for the form-factors, as announced in Sec. [V].

C. PV coupling – new form-factors

As a consequence of the above results, we tailored alternative choices for the meson and nucleon form-factors. We tested two different models: model A combines the pion form-factor of Eq. (4.13) with the nucleon form-factor of Eq. (4.14); in model B the pion form-factor of Eq. (4.12) with \( n = 1 \) is used together with the nucleon form-factor of Eq.
Since those form-factors have no singularities in the $k_0$ variable, the only terms to be considered in the amplitudes calculation are given by the decompositions of Eqs. (3.13) and (3.14).

As explained in Sec. IV, model A preserves the spectator amplitude $\mathcal{M}^+_I$. Nevertheless it changes crucially the remaining amplitudes relatively the models of reference [13,20]. Models A and B give in general very similar results as can be seen from Figs. 12 and 13, with the exception of the Gross amplitude in the $I = 1$ channel. Anyhow, since equal conclusions can be drawn from both models, we present only the final results for model A.

From Figs. 14 and 15 it is clear that the exact and Gross amplitudes do differ much less than in the model considered in the previous sub-section. They are now of the same order of magnitude, being the difference between them of the same order of OBE scalar exchange amplitude ($\sim 100 \text{GeV}^{-2}$), which gives the hope that the spectator-on-mass-shell prescription can be easily improved through the inclusion of OBE-type terms in the kernel. From the same figures we conclude that the box amplitude is dominant for the $I = 0$ channel, although the crossed-box amplitude is by no means negligible for the $I = 1$ channel.

Having found a way of preserving the unitary condition, we can now analyze relativistic contributions such as retardation, given by $\mathcal{M}^0_I$ (box) and $\mathcal{M}^{0\,\text{I}}_I$ (crossed-box), and nucleon negative-energy states, given by $\mathcal{M}^-_I$ (box) and $\mathcal{M}^{\text{II}}_{I\,\text{I}}$ (crossed-box).

In Figs. 16 and 17 we separately show those effects for the box amplitude, and in the Fig. 18 and 19 for the crossed-box amplitude. For the box amplitude the meson pole contributions are always important, and with opposite sign to the total amplitude. We conclude that in this case retardation effects cannot be neglected. The nucleon negative-energy state contributions are now very much suppressed relatively to the case of Sec. V B, and typically of the order of $\sim 15\%$. For the crossed-box amplitude, the meson poles and negative energy contributions are smaller than for the box, and have a partial important cancelation between them, giving rise to a net small result. This last cancellation decreases slowly with increasing energy.

VI. CONCLUSIONS

Recently, methods based on path-integral techniques allowed field-theory calculations, including ladder and crossed-ladder series, for scalar particles to be performed to all orders [8]. When the status of the calculations will reach the point of dealing with Dirac particles and general couplings, the comparison of their results with the ones obtained with different Quasi-Potential formalisms will clarify the unsettled point of selecting between different covariant formulations, for the description of nuclear systems.

In the meantime we decided to perform a study to quantify how well the amplitudes, calculated within the spectator-on-mass-shell formalism, represent the exact ones when Dirac particles interact through boson-exchange, in particular pion-exchange. This will eventually orient us towards possible required corrections, which may get increasingly important at higher energies, namely above the pion production threshold. We considered here two-pion exchange contributions not included in the spectator-on-mass-shell formalism. We aimed to check whether they can be included in a kernel restricted to have the OBE form, such that they can more easily be incorporated in current calculations and present-day computer codes.
The main conclusions of our work are:

1. The form-factors of the effective NN models calibrated with the spectator equations can not be used to investigate the quality of the spectator amplitudes, since the corresponding crossed-box and subtracted-box amplitudes, required for this study, strongly violate unitarity.

2. For the same reason the above NN models are not adequate for use in extensions of the kernel that include crossed-box and subtracted-box terms.

3. With appropriate choices of the form-factors, meaning no unitary violation, the difference between the full and Gross fourth-order amplitudes is of the order of magnitude of typical OBE scalar exchange amplitudes, raising the expectation that this difference may be parameterized by this type of exchange terms. The study of possible improvements of the spectator-on-mass-shell prescription, through the inclusion of OBE-type terms in the integral equation kernel, is planned for the near future.

4. There is a crucial (energy-dependent) interplay between retardation effects and nucleon negative-energy state contributions in the box and crossed-box diagrams, which does not allow the ladder approximation to work well as a representation of the exact Bethe-Salpeter series (ladder plus crossed-ladder). This is consistent with the findings of Nieuwenhuis et al. [6].

ACKNOWLEDGMENTS

The authors wish to thank Franz Gross for very helpful discussions and suggestions, and J. A. Tjon for his useful advice. They also thank A. Stadler for many discussions and for having explained the details of the NN models. This work was performed under the grants PRAXIS XXI 2/2.1/Fis/223/94 and PRAXIS XXI BD/9450/96.
REFERENCES

[1] B.D. Keister and W.N. Polyzou, Adv. Nucl. Phys. 20, 225 (1991).
[2] B. S. Pudliner, V. R. Pandharipande, J. Carlson and R. B. Wiringa, Phys. Rev. Lett. 74, 4396 (1995); B. S. Pudliner, V. R. Pandharipande, J. Carlson, Steven C. Pieper and R. B. Wiringa, Phys. Rev. C56, 1720 (1997); J. L. Forest, V. R. Pandharipande and A. Arriaga, [nucl-th/9805033](1998).
[3] Salpeter and Bethe, Phys. Rev. 84, 1232 (1951).
[4] F. Gross, Phys. Rev. 186, 1448 (1969).
[5] F. Gross, Relativistic Quantum Mechanics and Field Theory, Chap. 12, John Wiley & Sons (1993).
[6] T. Nieuwenhuis, J. A. Tjon and Y. A. Simonov Few-Body Systems Supp. 7, 286 (1994); T. Nieuwenhuis and J. A. Tjon, Phys. Rev. Lett. 77, 814 (1996).
[7] K. M. Maung and F. Gross Phys. Rev. C42, 1681 (1990).
[8] F. Gross, K. M. Maung, J. A. Tjon, L. W. Townsend and S. J. Wallace, Phys. Rev. C40, R10 (1989).
[9] G. Ramalho, “Two pion exchange in a manifest covariant formalism”, master thesis, unpublished (1998).
[10] F. Gross and D. O. Riska, Phys. Rev. C36, 1928 (1987).
[11] F. Gross, J. W. Van Orden and K. Holinde, Phys. Rev. C45, 2094 (1992).
[12] W. W. Buck and F. Gross, Phys. Rev. D20, 2361, (1979); J. P. Pinto, A. Amorim and F. D. Santos, Phys. Rev. C53, 2376 (1986).
[13] A. Stadler and F. Gross, Phys. Rev. Lett. 78, 26 (1997).
[14] F. Gross, Phys. Rev. C26, 2203 (1982)
[15] M. J. Zuilhof and J. A. Tjon, Phys. Rev. C26, 1277 (1982).
[16] R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
[17] S. J. Wallace and V. B. Mandelzweig, Nucl. Phys. A503, 673 (1989).
[18] R. Machleidt, Adv. Nucl. Phys. 19, 1189 (1989).
[19] T. A. Rijken, Annals of Physics 208, 253 (1991).
[20] A. Stadler, private communication (1997).

APPENDIX A: COUPLING PARAMETERS COEFFICIENTS

1. Scalar vertex

\[ \Gamma = g_4^4 \]  
\[ a^1_i = M \]  
\[ b^1_i = 1 \]  
\[ a^2_i = a^2_{II} = M \]  
\[ b^2_i = b^2_{II} = 1 \]  
\[ a^1_{II} = 3M \]  
\[ b^1_{II} = -1 \]  

18
2. Pseudovector-pseudoscalar mixture

\[ \Gamma = g_\pi^4 \left\{ \frac{3 - 2\tau_1 \cdot \tau_2}{3 + 2\tau_1 \cdot \tau_2} \right\} \text{Box Crossed-Box} \quad (A8) \]

\[ a_i^1 = \lambda^2 M + \lambda(1 - \lambda) \frac{M^2 + k^2}{M} + (1 - \lambda)^2 \frac{M^2 + 3k^2}{4M} \quad (A9) \]

\[ b_i^1 = \lambda^2 + 2\lambda(1 - \lambda) + (1 - \lambda)^2 \frac{3M^2 + (P - k)^2}{4M^2} \quad (A10) \]

\[ a_i^2 = a_{iI} = \lambda^2 M + \lambda(1 - \lambda) \frac{M^2 + (P - k)^2}{M} \]
\[ + (1 - \lambda)^2 \frac{M^2 + 3(P - k)^2}{4M} \quad (A11) \]

\[ b_i^2 = b_{iI} = -\lambda^2 + 2\lambda(1 - \lambda) \]
\[ + (1 - \lambda)^2 \frac{3M^2 + (P - k)^2}{4M^2} \quad (A12) \]

\[ a_{iI} = -\lambda^2 M + \lambda(1 - \lambda) \frac{3M^2 - (p' + p - k)^2}{M} \]
\[ + (1 - \lambda)^2 \frac{5M^2 - (p' + p - k)^2}{4M} \quad (A13) \]

\[ b_{iI}^1 = \lambda^2 + 2\lambda(1 - \lambda) \]
\[ + (1 - \lambda)^2 \frac{3M^2 + (p' + p - k)^2}{4M^2} \quad (A14) \]

3. Form-factor structure

Box amplitude:

\[ H_I(p, k, p') = \left[ f_N(k^2) \right]^2 \left[ f_N((P - k)^2) \right]^2 \]
\[ \left[ f_m((p - k)^2) \right]^2 \left[ f_m((p' - k)^2) \right]^2 \quad (A15) \]

Crossed-box amplitude:

\[ H_{II}(p, k, p') = \left[ f_N((p' + p - k)^2) \right]^2 \left[ f_N((P - k)^2) \right]^2 \]
\[ \left[ f_m((p - k)^2) \right]^2 \left[ f_m((p' - k)^2) \right]^2 \quad (A16) \]
FIGURES

FIG. 1. Fourth-order exact cancellation for scalar particles in the static limit. The thick line indicates the heavier particle, and the cross on the propagator line means that the particle is on-mass-shell. The diagram with the cross defines the Gross amplitude.

FIG. 2. Subtracted-box diagrammatic definition.

FIG. 3. Box ($\mathcal{M}_I$) and crossed-box ($\mathcal{M}_{II}$) diagrams and notation for the kinematical variables. The sum of the two diagrams defines the exact fourth-order amplitude.

FIG. 4. Symmetrized Gross amplitude for identical particles. The initial and final particles are on-mass-shell (cross symbol). The phase $\delta$ is 1 for scalar particles, and $(-1)^I$ for nucleons, where $I$ is the total isospin.

FIG. 5. Real part of the fourth-order amplitudes for equal scalar particles of mass $M$; the exchange scalar meson has a mass of $M/7$. The solid line corresponds to the exact result, the dashed line to the box amplitude, the long-dashed line to the crossed-box and the dotted line to the Gross amplitude.

FIG. 6. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 0$ channel; the exchange scalar meson has a mass of 498 MeV. The curves have the same meaning as in Fig. 5.

FIG. 7. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 1$ channel; the exchange scalar meson has a mass of 498 MeV. The curves have the same meaning as in Fig. 5.

FIG. 8. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 0$ channel; the exchange scalar meson has a mass of 138 MeV. The curves have the same meaning as in Fig. 5.

FIG. 9. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 1$ channel; the exchange scalar meson has a mass of 138 MeV. The curves have the same meaning as in Fig. 5.

FIG. 10. Real and imaginary parts of the fourth-order amplitudes for interacting nucleons in the isospin $I = 0, 1$ channels; the exchange pseudovector meson has a mass of 138 MeV. The curves have the same meaning as in Fig. 5 and correspond to a kinetic energy of 100 MeV. We can see a dominance of the crossed-box amplitude, which is almost coincident with the full result in some cases. Also the Box and Gross lines are almost coincident in some cases.
FIG. 11. Pion form-factor of references [13,20].

FIG. 12. Real part of the Gross amplitude for interacting nucleons in the isospin $I = 0,1$ channels; the exchange pseudovector meson has a mass of 138 MeV. Comparison between models A and B for three different kinetic energies.

FIG. 13. Real part of the box and crossed-box amplitudes for interacting nucleons in the isospin $I = 0,1$ channels; the exchange pseudovector meson has a mass of 138 MeV. Comparison between models A and B for three different kinetic energies.

FIG. 14. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 0$ channel; the exchange pseudovector meson has a mass of 138 MeV. The curves have the same meaning as in Fig. 5. Model A has been considered.

FIG. 15. Real part of the fourth-order amplitudes for interacting nucleons in the isospin $I = 1$ channel; the exchange pseudovector meson has a mass of 138 MeV. The curves have the same meaning as in Fig. 5. Model A has been considered.

FIG. 16. Retardation and nucleon negative-energy contributions for the box amplitude, in the $I = 0$ isospin channel, for a pseudovector exchanged meson of mass 138 MeV. Model A has been considered. The full line is the total box amplitude, the long-dashed line is the $M_{I}^{-}$ amplitude and the dashed line the $M_{I}^{0}$ amplitude.

FIG. 17. Retardation and nucleon negative-energy contributions for the box amplitude, in the $I = 1$ isospin channel, for a pseudovector exchanged meson of mass 138 MeV. Model A has been considered. The full line is the total box amplitude, the long-dashed line is the $M_{I}^{-}$ amplitude and the dashed line the $M_{I}^{0}$ amplitude.

FIG. 18. Retardation and nucleon negative-energy contributions for the crossed-box amplitude, in the $I = 0$ isospin channel, for a pseudovector exchanged meson of mass 138 MeV. Model A has been considered. The full line is the total crossed-box amplitude, the long-dashed line is the sum of the $M_{I}^{+}$ and $M_{I}^{-}$ amplitudes and the dashed line the $M_{I}^{0}$ amplitude.

FIG. 19. Retardation and nucleon negative-energy contributions for the crossed-box amplitude, in the $I = 1$ isospin channel, for a pseudovector exchanged meson of mass 138 MeV. Model A has been considered. The full line is the total crossed-box amplitude, the long-dashed line is the sum of the $M_{I}^{+}$ and $M_{I}^{-}$ amplitudes and the dashed line the $M_{I}^{0}$ amplitude.
$T_{\text{lab}} = 100$ MeV

$T_{\text{lab}} = 200$ MeV

$T_{\text{lab}} = 300$ MeV

$\cos \theta_{\text{cm}}$
\[ I = 0 \quad ++ \rightarrow ++ \]

\[ T_{\text{lab}} = 100 \text{ MeV} \]

\[ T_{\text{lab}} = 200 \text{ MeV} \]

\[ T_{\text{lab}} = 300 \text{ MeV} \]
\( I = 1 \quad ++ \rightarrow ++ \)

\[
\begin{align*}
& T_{\text{lab}} = 100 \text{ MeV} \\
& T_{\text{lab}} = 200 \text{ MeV} \\
& T_{\text{lab}} = 300 \text{ MeV}
\end{align*}
\]
$I = 0 \quad \leftrightarrow \quad ++ \rightarrow ++$

$T_{lab} = 100 \text{ MeV}$

$T_{lab} = 200 \text{ MeV}$

$T_{lab} = 300 \text{ MeV}$
\[ I = 1 \quad \rightarrow \quad ++ \rightarrow ++ \]

\[ T_{\text{lab}} = 100 \text{ MeV} \]

\[ T_{\text{lab}} = 200 \text{ MeV} \]

\[ T_{\text{lab}} = 300 \text{ MeV} \]
\[ T_{lab} = 100 \text{ MeV} \quad ++ \rightarrow ++ \]

![Diagram](image-url)
\[ f_\pi(q^2) \]
GROSS  \( \rightarrow \) ++

\[ I = 0 \]

\[ I = 1 \]

\[ \text{Re} [V^{(4)}](GeV^{-2}) \]

\[ \cos \theta_{cm} \]
$I = 0 \quad ++ \rightarrow ++ \quad \text{Model A}$

$T_{lab} = 100 \text{ MeV}$

$T_{lab} = 200 \text{ MeV}$

$T_{lab} = 300 \text{ MeV}$

$\cos \theta_{cm}$
$I = 1 \quad ++ \rightarrow ++ \quad \text{Model A}$

$T_{\text{lab}} = 100 \text{ MeV}$

$T_{\text{lab}} = 200 \text{ MeV}$

$T_{\text{lab}} = 300 \text{ MeV}$

$\cos \theta_{\text{cm}}$
$I = 0 \quad ++ \rightarrow ++ \quad \text{Model A}$

\begin{align*}
T_{lab} &= 100 \text{ MeV} \\
T_{lab} &= 200 \text{ MeV} \\
T_{lab} &= 300 \text{ MeV}
\end{align*}
$I = 1 \quad ++ \rightarrow ++ \quad \text{Model A}$

$T_{lab} = 100 \ \text{MeV}$

$T_{lab} = 200 \ \text{MeV}$

$T_{lab} = 300 \ \text{MeV}$
$I = 0 \quad ++ \rightarrow ++ \quad \text{Model A}$

$T_{\text{lab}} = 100 \text{ MeV}$

$T_{\text{lab}} = 200 \text{ MeV}$

$T_{\text{lab}} = 300 \text{ MeV}$
$I = 1 \quad ++ \rightarrow ++$  Model A

$T_{lab} = 100$ MeV

$T_{lab} = 200$ MeV

$T_{lab} = 300$ MeV

$\cos \theta_{cm}$