RELATIVELY COMPACT SETS IN THE REDUCED C*-ALGEBRAS OF COXETER GROUPS

GERO FENDLER

Abstract. We characterize relatively norm compact sets in the regular C*-algebra of finitely generated Coxeter groups using a geometrically defined positive semigroup acting on the algebra.

1. Introduction

Let \((X, d)\) be a compact metric space, \(x_0 \in X\). In \(C(X)\), the continuous complex valued functions on \(X\), consider the convex, balanced and closed set

\[ K = \{ f : |f(x) - f(y)| \leq d(x, y), f(x_0) = 0 \}. \]

The Arzela-Ascoli theorem shows that \(K\) is relatively compact. On the other hand this theorem can be thought to compare any relatively compact set against this special set.

In the non-commutative context this has been made precise by Antonescu and Christensen [1] as follows:

Let \(A\) be a unital, separable C*-algebra and \(S\) the set of its states endowed with the \(w^*\)-topology.

**Definition 1.** \(K \subset A\) is called a metric set if it is convex, balanced norm compact and separates the states of \(A\).

**Lemma 2** ([1]). If \(K \subset A\) is a metric set then

\[ d(\varphi, \psi) := \sup_{x \in K} |\varphi(x) - \psi(x)|, \quad \varphi, \psi \in S \]

defines a metric on \(S\), which generates the \(w^*\)-topology.

Their general non-commutative Arzela-Ascoli Theorem reads as follows

**Theorem 3** ([1]). Let \(A\) be a unital C*-algebra \(K \subset A\) a metric set then \(H \subset A\) is relatively compact if and only if \(H\) is bounded and for all \(\epsilon > 0\) exists \(N > 0\) such that

\[ H \subset B_\epsilon(0) + NK + CId, \]

where \(B_\epsilon(0) \subset A\) is the ball of radius \(\epsilon\) around 0.

Our aim here is to give an example of some such set \(K\) in the reduced C*-algebra \(A = C^*_r(G)\) of a finitely generated Coxeter group \(G\).

Let \(G, S\) be a Coxeter group and \(l\) the length function associated to the generating set \(S\). (For the readers convenience in the next two sections we recall some notions and assertions related to the regular C*-algebra of Coxeter groups.)

**Theorem 4.**

\[ K = \{ \lambda(f) : \| \lambda(f) \| \leq 1 \text{ and } \| \lambda(l \cdot f) \| \leq 1 \} \]
is relatively compact in \(C^*_r(G)\).

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The proof of this theorem is given in our last section.

Since the set $\mathcal{K}$ in $C^*_\lambda(G)$ separates the states, is convex and balanced an application of the theorem of Antonescu and Christensen characterizes relatively compact subsets of $C^*_\lambda(G)$ as follows:

**Corollary 1.** A set $\mathcal{H} \subset C^*_\lambda(G)$ is relatively compact if and only if it is bounded and for all $\epsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\mathcal{H} \subset m\mathcal{K} + C\lambda(\delta(\epsilon)) + B_\epsilon(0),$$

where $B_\epsilon(0) \subset C^*_\lambda(G)$ is the ball of radius $\epsilon$ and center 0.

2. Coxeter group

**Definition 5.** A pair $(G, S)$ is a Coxeter group if, $S$ is a finite generating subset of the group $G$ with the following presentation:

$$s^2 = e, \quad s \in S,$$

$$(st)^m(s,t) = e, \quad s, t \in S, \quad s \neq t,$$

where $m(s,t) \in \{2, 3, 4, \ldots, \infty\}$.

A specific tool for working with Coxeter groups is their geometric representation. Let $V = \bigoplus_{s \in S} \mathbb{R} \alpha_s$ be an abstract real vector space with basis $\{\alpha_s : s \in S\}$. Define a bilinear form on it: $B(\alpha_s, \alpha_t) = \begin{cases} 1 & s = t, \\ -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq \infty, \\ -1 & m(s,t) = \infty. \end{cases}$ For $s \in S$ define a reflection by $\sigma_s : \xi \rightarrow \xi - 2B(\alpha_s, \xi)\alpha_s$. Then

$\circ: V = \mathbb{R}\alpha_s \oplus H_s$, where $H_s = \{\xi : B(\alpha_s, \xi) = 0\}$ is stabilized point wise by $\sigma_s$ and $\sigma_s\alpha_s = -\alpha_s$.

$\circ: s \mapsto \sigma_s$ extends multiplicatively to a representation $\sigma : G \rightarrow \text{Gl}(V)$ of the Coxeter group.

$\circ: \sigma$ is faithful and $\sigma(G)$ a discrete subgroup of $\text{Gl}(V)$.

We dualise the representation $\sigma$ to obtain the adjoint representation

$$\sigma^*(g)f(\xi) = f(\sigma(g^{-1})\xi), \quad f \in V^*, \xi \in V$$

For $s \in S$ let $Z_s$ be the hyperplane $Z_s = \{f \in V^* : f(\alpha_s) = 0\}$, and $A_s$ the half-space $A_s = \{f \in V^* : f(\alpha_s) > 0\}$; define a family of Hyperplanes in $V^* \mathcal{H} = \cup_{g \in G} gZ_s$.

Denote $C = \cap_{s \in S} A_s$ the intersection of the halfspaces, its closure $D = \overline{C} \setminus \{0\}$ is called the fundamental chamber usually considered as a subset of the union of its translates $U = \cup_{g \in G} gD$, the Tit’s cone.

The following facts hold true:

(i): $C$ is a simplicial cone, its faces are the sets $Z_s \cap D$.

(ii): $U$ is a convex cone, $D$ a fundamental domain for the action of $G$ on it.

(iii): a closed line segment $[u, c] \subset U$ meets only finitely many members of $\mathcal{H}$.

(iv: Moreover, for any $c \in C$:

$\text{card}([Z \in \mathcal{H} : gc, c] \cap Z \neq \emptyset)) = l(g),$

where $l(g) = \inf\{k : g = s_1 \ldots s_k, \ s_i \in S\}$ denotes the usual length with respect to the generating set $S$. This construction due to Tits was used by Bożejko, M. and Januszkiewicz, T. and Spatzier, R. J., we recall their short proof of their theorem

**Theorem 6** ([3]). For $t > 0$ $\varphi_t : g \mapsto e^{-tl(g)}$

is a positive definite function on $G$. 

Relatively compact sets in the reduced $C^*$-algebras of Coxeter groups

Proof.

$$l(g^{-1} h) = \text{card}\{ Z \in \mathcal{H} : \{hc, gc\} \cap Z \neq \emptyset\} = \sum_{Z \in \mathcal{H}} \left| \chi_h(Z) - \chi_g(Z)^2 \right|,$$
where $c \in C$ is arbitrary and $\chi_h$ is the characteristic function of $N^h = \{ Z \in \mathcal{H} : \{hc, c\} \cap Z \neq \emptyset\}$. Hence $l(.)$ is negative definite and, by a theorem of Schoenberg [9] (we only need the part already due to Schur [10]), $e^{-l(.)}$ is positive definite, see e.g. [2] Theorem 7.8.

3. Regular representation

For functions $f, h : G \to \mathbb{C}$ their convolution is defined by:

$$f \ast h(y) = \sum_{x \in G} f(x)h(x^{-1} y).$$

For summable $f : G \to \mathbb{C}$ we denote $\lambda(f) : l^2(G) \to l^2(G)$ the associated convolution operator $\lambda(f)h = f \ast h$. The regular (or reduced) $C^*$-algebra $C^*_r(G)$ is the just the operator norm closure of $\{\lambda(f) : f \in l^1(G)\}$. Denote for $g \in G \delta_g$ the point mass one in $g \in G$ then $\lambda(\delta_g)$ is just left translation by $g^{-1}$ on $l^2(G)$ and we are just dealing with the integrated version of the left regular representation. Since for $A \in C^*_r(G)$ there is a unique $f = A \delta_e \in l^2$ we abuse notation to denote $A = \lambda(f)$. The Tits cone with its division by the hyperplanes can be seen as a subset of a cubical building. This allows to estimate certain convolution operator norms. The first example of such an estimation was given for the free group on two generators by U. Haagerup [6] and accordingly such inequalities are called Haagerup inequality.

Theorem 7. A Coxeter group is a group of rapid decay: there is $C > 0$ and $k \in \mathbb{N}$ such that

$$\|\lambda(f)\| \leq C \left( \sum_g |f(g)|^2 (1 + l(g))^{2k} \right)^{\frac{1}{2}}.$$

A consequence of this theorem is the following lemmata due to Haagerup [6, 7]. For the readers convenience we recall their proofs.

Lemma 8. If $\varphi : G \to G$ is such that $\sup_g |\varphi(g)|(1 + l(g))^k < \infty$, then for all $\lambda(f) \in C^*_r(G)$

$$\|\lambda(\varphi \cdot f)\| \leq C \sup_g |\varphi(g)|(1 + l(g))^k \|\lambda(f)\|.$$

Here $C$ and $k$ are the constants in the Haagerup inequality.

Proof. From $\lambda(f)\delta_e = f$ we have $\sum_g |f(g)|^2 \frac{1}{2} = \| f \|_2 \leq \|\lambda(f)\|$ and by the Haagerup inequality:

$$\|\lambda(\varphi \cdot f)\|^2 \leq C \sum_g |\varphi(g)f(g)|^2 (1 + l(g))^{2k} \leq \sum_g |f(g)|^2 C \sup_g |\varphi(g)|^2 (1 + l(g))^{2k}.$$

Lemma 9. There is a sequence of finitely supported functions $(\psi_m)$ such that for $\lambda(f) \in C^*_r(G)$:

- $\lambda(\psi_m \cdot f) \to \lambda(f)$, as $m \to \infty$
- $\|\lambda(\psi_m \cdot f)\| \leq 3\|\lambda(f)\|$
Proof. Since, by theorem 6, the functions $\varphi_t$ are positive definite, they define contractive (i.e. norm non-increasing) multipliers on the regular $C^*$-algebra. Let

$$\varphi_{n,t} = \begin{cases} e^{-tl(g)} & \text{if } l(g) \leq n \\ 0 & \text{else} \end{cases}$$

Then

$$\|\lambda(\varphi_{n,t} \cdot f) - \lambda(f)\| \leq \|\lambda(\varphi_{n,t} \cdot f) - \lambda(\varphi_t \cdot f)\| + \|\lambda(\varphi_t \cdot f) - \lambda(f)\| \leq C \sup_{l>n} e^{-tl}(1 + l)^k \|\lambda(f)\| + \|\lambda(\varphi_t \cdot f) - \lambda(f)\|$$

Since $\sup_{l>n} e^{-tl}(1 + l)^k \to 0$ as $n \to \infty$ we can extract the $\psi_m$ from the $\varphi_{n,t}$. □

4. Relatively compact sets

First we notice that the positive definite functions $\varphi_t : g \mapsto e^{-tl(g)}$ define a $C_0$-semigroup of multipliers on $C^*_\lambda(G)$ given by $M_t : C^*_\lambda(G) \to C^*_\lambda(G)$, $\lambda(f) \mapsto \lambda(\varphi_t \cdot f)$.

**Lemma 10.** $M : t \mapsto M_t$ is a $C_0$-semigroup of contractions on $C^*_\lambda(G)$.

**Proof.** Since $\varphi_t$ is positive definite

$$\| M_t \| = \varphi_t(e) = 1.$$ 

For finitely supported $f$ everything is elementary and now an approximation proves the assertion. □

**Lemma 11.** The generator $D$ of the semigroup $M_t$ is given by

$$D(\lambda(f)) = -\lambda(l \cdot f)$$

$$\text{Dom}(D) = \{ \lambda(f) : \lambda(l \cdot f) \in C^*_\lambda(G) \}$$

**Proof.** We have

$$\lambda(\varphi_t \cdot \delta_g) = e^{-tl(g)} \lambda(\delta_g),$$

hence the assertion is clear for finitely supported $f = \sum g f(g) \delta_g$.

Now as a generator of a $C_0$-contraction semi group the operator $D$ has a closed graph. But if $\lambda(f)$ and $\lambda(l \cdot f) \in C^*_\lambda(G)$, then for the finitely supported $\psi_m$ as above:

$$\lambda(\psi_m \cdot f) \to \lambda(f)$$

and

$$\lambda(\psi_m \cdot l \cdot f) \to \lambda(l \cdot f).$$

□

**Proof of Theorem 4.** We shall show that for $\epsilon > 0$ there exists a finite dimensional bounded set

$$\tilde{K}_\epsilon \subset C^*_\lambda(G)$$

such that for all $f \in K$

$$\text{dist}(f, \tilde{K}_\epsilon) \leq \epsilon.$$ 

(this show that $K$ is totally bounded)

We have for $f \in K$:

$$\lambda(\varphi_t \cdot f) - \lambda(f) = M_t(\lambda(f)) - \lambda(f) = \int_0^t D(M_s(\lambda(f))) \, ds$$
RELATIVELY COMPACT SETS IN THE REDUCED C*-ALGEBRAS OF COXETER GROUPS

Hence
\[ \| \lambda(\varphi_t \cdot f) - \lambda(f) \| \leq t \sup_{s < t} \| \lambda(l e^{-st} \cdot f) \| \leq t \| \lambda(t \cdot f) \| \leq t. \]

and
\[ \| \lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f) \| \leq C \sup_{l > n} e^{-tl} (1 + l)^k \| \lambda(f) \| \leq C \sup_{l > n} e^{-tl} (1 + l)^k \]

taking first \( t \) small and then \( n \) large we have an approximation to \( \lambda(f) \) by certain \( \lambda(\varphi_{n,t} \cdot f) \) up to \( \epsilon \) uniformly in \( \lambda(f) \in K \). Further for this \( n \)
\[ \| \lambda(\varphi_{n,t} \cdot f) \| \leq \| \lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f) \| + \| \lambda(\varphi_t \cdot f) \| \leq C (\sup_{l > n} e^{-tl} (1 + l)^k + 1) \| \lambda(f) \| \leq C (\sup_{l > n} e^{-tl} (1 + l)^k + 1) \]

So these \( \lambda(\varphi_{n,t} \cdot f) \) are from a bounded set and all have their support in words of length at most \( n \). The functions with support in this finite set give rise to a finite dimensional subspace of \( C^*_\lambda(G) \).

\[ \square \]

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NuHAG, Faculty of Mathematics, University of Vienna, Nordbergstasse 15, 1090 Vienna, Austria

E-mail address: gero.fendler<at>univie.ac.at