Optimal two weight codes from trace codes over a chain ring *

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Abstract: In this paper, we construct an infinite family of two-weight codes for the homogeneous metric over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + \cdots + u^{k-1}\mathbb{F}_2$, where $u^k = 0$. These codes are defined as trace codes. They have the algebraic structure of abelian codes. Their homogeneous weight distribution is computed by using character sums. In particular, we give a necessary and sufficient condition of optimality for the Gray image codes by using the Griesmer bound. We have shown that if $k > 3$, these images are projective. Their support structure is determined. An application to secret sharing schemes is given.

Key words: Two-weight codes; Homogeneous distance; Trace codes

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1 Introduction

Two-weight codes form a class of combinatorial codes which are closely related to combinatorial designs, finite geometry, and graph theory. Information on them can be found in [3, 4]. It is worth mentioning that many contributions about few weights codes have been done in [5, 6, 8]. In [11, 12], the authors have constructed an infinite family of binary and \( p \)-ary two-weight codes from trace codes over \( \mathbb{F}_2 + u\mathbb{F}_2 \) and \( \mathbb{F}_p + u\mathbb{F}_p \), respectively.

In the present paper, following this trend, we use trace codes over the larger ring \( R \) defined in the abstract. Note that \( R = \mathbb{F}_2 + u\mathbb{F}_2 \) when \( k = 2 \). Although most of previous works on two-weight codes were done on cyclic codes, using cyclotomy [2], the codes we construct here are provably abelian but perhaps not cyclic. Their coordinate places are indexed by the group of units of an algebraic extension of a finite ring. Their weight distribution is determined by using exponential character sums. After Gray mapping, we obtain an infinite family of binary abelian two-weight codes. In special cases, they are shown to be optimal for given length and dimension by application of the Griesmer bound [9], and they meet that bound with equality when \( k = 2 \).

The manuscript is organized as follows. Basic notations and definitions are provided in Section 2. Section 3 shows that the codes and their binary images are abelian. Main results in this paper, together with some examples, are presented in Section 4. Section 5 gives a necessary and sufficient condition for their binary images to be optimal. Section 6 determines the minimum distance of the dual codes. The support structure of binary images and an application to secret sharing schemes are given in Section 8.

2 Preliminaries

We consider the chain ring \( \mathbb{F}_2 + u\mathbb{F}_2 + \cdots + u^{k-1}\mathbb{F}_2 \), denoted by \( R \), with \( u^k = 0 \), and \( k \geq 2 \), a positive integer. Given an integer \( m \), we can construct the ring extension of \( R \) of degree \( m \) denoted by \( \mathcal{R} = \mathbb{F}_{2^m} + u\mathbb{F}_{2^m} + \cdots + u^{k-1}\mathbb{F}_{2^m} \). There is a Frobenius operator \( F \) which maps \( a_0 + a_1u + \cdots + a_{k-1}u^{k-1} \) to \( a_0^2 + a_1^2u + \cdots + a_{k-1}^2u^{k-1} \). The
Trace function, denoted by $Tr$ is then defined as

$$Tr = \sum_{j=0}^{m-1} F^j.$$ 

It is immediate to check that

$$Tr(a_0 + a_1u + \cdots + a_ku^{k-1}) = tr(a_0) + tr(a_1)u + \cdots + tr(a_{k-1})u^{k-1},$$

for all $a_i \in \mathbb{F}_{2^n}$, $i = 0, 1, \ldots, k-1$. Here $tr()$ denotes the standard trace of $\mathbb{F}_{2^n}$ down to $\mathbb{F}_2$.

For convenience, let $M$ denote the maximal ideal of $R$, i.e., $M = \{a_1u + a_2u^2 + \cdots + a_{k-1}u^{k-1} : a_i \in \mathbb{F}_{2^n}, i = 1, \ldots, k-1\}$. The group of units in $\mathcal{R}$, denoted by $\mathcal{R}^*$, is $\{a_0 + a_1u + \cdots + a_{k-1}u^{k-1} : a_0 \in \mathbb{F}_{2^n}, a_i \in \mathbb{F}_{2^n}, i = 1, 2, \ldots, k-1\}$, and $|\mathcal{R}^*| = 2^{(k-1)m}(2^m - 1)$. It is obvious that $\mathcal{R}^*$ is not a cyclic group and that $\mathcal{R} = \mathcal{R}^* \cup M$.

A linear code $C$ over $R$ of length $n$ is an $R$-submodule of $R^n$. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ are two elements of $R^n$, their standard inner product is defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, where the operation is performed in $R$. The dual code of $C$ is denoted by $C^\perp$ and defined as $C^\perp = \{y \in R^n | \langle x, y \rangle = 0, \forall x \in C\}$.

For any $a = a_0 + a_1u + \cdots + a_{k-1}u^{k-1} \in R$, let $I = \{a_0, a_1, \ldots, a_{k-2}\}$, a set with $k - 1$ elements. Now we define the Gray map $\Phi : R \to \mathbb{F}_2^{k-1}$, $\Phi(a) = (A_0, A_1, \ldots, A_{k-1})$, where $A_0 = a_{k-1}, A_j = (a_{k-1} + a_{i_1} + a_{i_2} + \cdots + a_{i_t})_{a_{i_t} \in I}, 1 \leq t \leq j \leq k-1,$ and $a_{i_1} \neq a_{i_2}$ if $i_1 \neq i_2$. For instance, when $k = 2$, then $\Phi(a_0 + a_1u) = (a_1, a_0 + a_0)$. As additional examples, for $k = 4$, we have $\Phi(a_0 + a_1u + a_2u^2 + a_3u^3) = (A_0, A_1, A_2, A_3)$, where $A_0 = a_3, A_1 = (a_3 + a_0, a_3 + a_1, a_3 + a_2), A_2 = (a_3 + a_0 + a_1, a_3 + a_0 + a_2, a_3 + a_1 + a_2), A_3 = (a_3 + a_0 + a_1 + a_2)$. It is easy to extend this Gray map to $R^n$, and we also know from [13] that $\Phi$ is injective and linear.

For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_2^n, d_H(x, y) = |\{i : x_i \neq y_i\}|$ is called the Hamming distance between $x$ and $y$ and $w_H(x) = d_H(x, 0)$, the Hamming weight of $x$. The Hamming weight of a codeword $c = (c_1, c_2, \ldots, c_n)$ of $\mathbb{F}_2^n$ can also be equivalently defined as $w_H(c) = \sum_{i=1}^{n} w_H(c_i)$, where $w_H(c_i)$ equals to 0 if and only if $c_i$ is a zero element.

For $x \in R$, we define the homogeneous weight of $x$ as follows:

$$w_{\text{hom}}(x) = \begin{cases} 
0, & \text{if } x = 0, \\
2^{k-1}, & \text{if } x \in \langle u^{k-1} \rangle \backslash \{0\}, \\
2^{k-2}, & \text{if } x \in R \backslash \langle u^{k-1} \rangle.
\end{cases}$$
The homogeneous weight of a codeword $c = (c_1, c_2, \ldots, c_n)$ of $R^n$ is defined as $w_{\text{hom}}(c) = \sum_{i=1}^{n} w_{\text{hom}}(c_i)$. For any $x, y \in R$, the homogeneous distance $d_{\text{hom}}$ is given by $d_{\text{hom}}(x, y) = w_{\text{hom}}(x - y)$. As was observed in [13], $\Phi$ is a distance preserving isometry from $(R^n, d_{\text{hom}})$ to $(\mathbb{F}_2^{2k-1}n, d_H)$, where $d_{\text{hom}}$ and $d_H$ denote the homogeneous and Hamming distance in $R^n$ and $\mathbb{F}_2^{2k-1}n$, respectively. This means if $C$ is a linear code over $R$ with parameters $(n, 2t, d)$, then $\Phi(C)$ is a binary linear code of parameters $[2^{k-1}n, t, d]$.

Given a finite abelian group $G$, a code over $R$ is said to be abelian if it is an ideal of the group ring $R[G]$. In other words, the coordinates of $C$ are indexed by elements of $G$ and $G$ acts regularly on this set. In the special case when $G$ is cyclic, the code is a cyclic code in the usual sense [10].

3 Symmetry

For $a \in \mathcal{R}$, we define the vector $Ev(a)$ by the following evaluation map $Ev(a) = (Tr(ax))_{x \in \mathcal{R}^*}$. Define the code $C_m$ by the formula $C_m = \{Ev(a) | a \in \mathcal{R}\}$. Thus $C_m$ is a code of length $|\mathcal{R}^*|$ over $R$.

**Proposition 3.1** The group of units $\mathcal{R}^*$ acts regularly on the coordinates of $C_m$.

**Proof.** For any $v', u' \in \mathcal{R}^*$ the change of variables $x \mapsto (u'/v')x$ permutes the coordinates of $C_m$, and maps $v'$ to $u'$. Such a permutation is unique, given $v', u'$.

The code $C_m$ is thus an abelian code with respect to the group $\mathcal{R}^*$. In other words, it is an ideal of the group ring $R[\mathcal{R}^*]$. As observed in the previous section, $\mathcal{R}^*$ is not a cyclic group, and thus $C_m$ may be not cyclic. The next result shows that its binary image is also abelian.

**Proposition 3.2** A finite group of size $2^{k-1}|\mathcal{R}^*|$ acts regularly on the coordinates of $\Phi(C_m)$.

**Proof.** It is similar to the proof in [11], and we omit it here.
4 The homogeneous weight enumerator

In order to determine the homogeneous weight enumerator of the code $C_m$, we first recall the following classic lemmas.

**Lemma 4.1** [10, (6) p.412] If $y = (y_1, y_2, \cdots, y_n) \in \mathbb{F}_2^n$, then $2w_H(y) = n - \sum_{i=1}^{n} (-1)^{y_i}$.

**Lemma 4.2** [10, Lemma 9 p.143] If $z \in \mathbb{F}_2^*$, then $\sum_{x \in \mathbb{F}_2^m} (-1)^{\text{tr}(zx)} = 0$.

We are now ready to discuss the homogeneous weight of the codewords in the above abelian codes.

**Theorem 4.3** For $a \in \mathbb{R}$, the homogeneous weight of the codewords of $C_m$ is given below.

(a) If $a = 0$, then $w_{\text{hom}}(Ev(a)) = 0$;

(b) If $a \in M \setminus \{0\}$, then
   - if $a = \alpha u^{k-1}$, where $\alpha \in \mathbb{F}_2^*$, then $w_{\text{hom}}(Ev(a)) = 2^{k-2}|\mathbb{R}^*| + 2^{(k-1)(m+1)-1}$,
   - if $a \in M \setminus \{\alpha u^{k-1} : \alpha \in \mathbb{F}_2^*\}$, then $w_{\text{hom}}(Ev(a)) = 2^{k-2}|\mathbb{R}^*|$;

(c) If $a \in \mathbb{R}^*$, then $w_{\text{hom}}(Ev(a)) = 2^{k-2}|\mathbb{R}^*|$.

**Proof.** (a) If $a = 0$, then $Ev(a) = (0, 0, \cdots, 0)$. So $w_{\text{hom}}(Ev(a)) = 0$.

(b) Let $a = a_1u + a_2u^2 + \cdots + a_{k-1}u^{k-1} \in M \setminus \{0\}$, $x = x_0 + x_1u + \cdots + x_{k-1}u^{k-1} \in \mathbb{R}^*$. So we have

\[
ax = a_1x_0u + (a_1x_1 + a_2x_0)u^2 + \cdots + (a_1x_{k-2} + a_2x_{k-3} + \cdots + a_{k-1}x_0)u^{k-1}
\]

\[
= B_1u + B_2u^2 + \cdots + B_{k-1}u^{k-1},
\]

and

\[
\text{Tr}(ax) = \text{tr}(B_1)u + \text{tr}(B_2)u^2 + \cdots + \text{tr}(B_{k-1})u^{k-1}.
\]

Taking Gray map yields

\[
\Phi(\text{Ev}(a)) = (\text{tr}(B_{k-1}), \text{tr}(B_{k-2}) + \text{tr}(B_1), \cdots, \text{tr}(B_{k-1}) + \text{tr}(B_1) + \cdots + \text{tr}(B_{k-2})).
\]
Using Lemma 4.1, we have
\[ 2^{k-1}|\mathcal{R}^*| - 2w_{hom}(Ev(a)) = \sum_{x_1, x_2, \ldots, x_{k-1} \in \mathbb{F}_2^m} \sum_{x_0 \in \mathbb{F}_2^m} (-1)^{\text{tr}(B_{k-1})} \]
\[ + \sum_{x_1, x_2, \ldots, x_{k-1} \in \mathbb{F}_2^m} \sum_{x_0 \in \mathbb{F}_2^m} (-1)^{\text{tr}(B_{k-1}) + \text{tr}(B_1) + \ldots} \]
\[ + \sum_{x_1, x_2, \ldots, x_{k-1} \in \mathbb{F}_2^m} \sum_{x_0 \in \mathbb{F}_2^m} (-1)^{\text{tr}(B_{k-1}) + \text{tr}(B_1) + \ldots + \text{tr}(B_{k-2})}. \]

By using Lemma 4.2, we know that the RHS is not equal to zero if and only if \( a_{k-1} \neq 0 \), and \( a_i = 0 \), \( i = 1, 2, \ldots, k-2 \). So according to the above equation, we have

\[ w_{hom}(Ev(a)) = \begin{cases} 2^{k-2}|\mathcal{R}^*| + 2^{(k-1)(m+1)-1}, & \text{if } a = \alpha u^{k-1}, \\ 2^{k-2}|\mathcal{R}^*|, & \text{if } a \in M \setminus \{\alpha u^{k-1} : \alpha \in \mathbb{F}_2\}. \end{cases} \]

(c) Let \( a = a_0 + a_1 u + \cdots + a_{k-1} u^{k-1} \in \mathcal{R}^*, \ x = x_0 + x_1 u + \cdots + x_{k-1} u^{k-1} \in \mathcal{R}^* \). Then we have

\[ ax = a_0 x_0 + (a_0 x_1 + a_1 x_0) u + \cdots + (a_0 x_{k-1} + a_1 x_{k-2} + \cdots + a_{k-1} x_0) u^{k-1} \]

and

\[ \text{Tr}(ax) = \text{tr}(a_0 x_0) + \text{tr}(a_0 x_1 + a_1 x_0) u + \cdots + \text{tr}(a_0 x_{k-1} + a_1 x_{k-2} + \cdots + a_{k-1} x_0) u^{k-1} \]
\[ =: D_0 + D_1 u + \cdots + D_{k-1} u^{k-1}. \]

We can easily check that

\[ \sum_{x_1, x_2, \ldots, x_{k-1} \in \mathbb{F}_2^m} \sum_{x_0 \in \mathbb{F}_2^m} (-1)^{D_{k-1}} = 0. \]

Next we can use Lemmas 4.1 and 4.2, so we obtain

\[ 2^{k-1}|\mathcal{R}^*| - 2w_{hom}(Ev(a)) = 0. \]

Thus, \( w_{hom}(Ev(a)) = 2^{k-2}|\mathcal{R}^*| \). \( \square \)

According to Theorem 4.3, we have constructed a binary code of length \( n = 2^{(k-1)m+k-1}(2^m - 1) \), dimension \( km \), with two weights \( \omega_1 \) and \( \omega_2 \ (\omega_1 < \omega_2) \) of values

\[ \omega_1 = 2^{m(k-1)+k-2}(2^m - 1), \ \omega_2 = 2^{(m+1)k-2}, \]

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with respective frequencies \( f_1, f_2 \) given by

\[
f_1 = 2^{km} - 2^m, \quad f_2 = 2^m - 1.
\]

**Remark:** These values are reminiscent of the family \( SU1 \) of [3]. However, identifying the weights would yield, in the notation of [3],

\[
\ell =: (m + 1)k - 1 \\
k =: (m + 1)k - m - 1,
\]

which is not consistent with the second frequency \( A_{w_2} \) or the dimension \( \ell > mk \).

**Example 4.4.** Let \( m = 2, k = 2 \). Then we obtain a binary code of parameters \([24, 4, 12]\). The weights are \( \{12, 16\} \), of frequencies 12 and 3.

**Example 4.5.** Let \( m = 2, k = 3 \). Then we obtain a binary code of parameters \([192, 6, 96]\). The weights are \( \{96, 128\} \), of frequencies 60 and 3.

## 5 Optimality

Recall the Griesmer bound on the parameters of an \([n, K, d]\) binary code:

\[
\sum_{i=0}^{K-1} \left\lceil \frac{d}{2^i} \right\rceil \leq n.
\]

**Theorem 5.1** If the code \( \Phi(C_m) \) is defined above for given length and dimension, then \( \Phi(C_m) \) is optimal if and only if \( m \geq \max\{k - 1, \lfloor \frac{2^{k-1} - k}{k-1} \rfloor + 1\} \).

**Proof.** Recall that the parameters of \( \phi(C_m) \) are \([n, K, d] = [2^{m(k-1)+k-1}(2^m - 1), km, d]\), where \( d = 2^{m(k-1)+k-2}(2^m - 1) \). We claim that \( \sum_{i=0}^{K-1} \left\lceil \frac{d+1}{2^i} \right\rceil > n \), violating the Griesmer bound. Indeed, depending on the range of \( i \), two expressions for the inner ceiling function may occur,

- If \( 0 \leq i < (m + 1)k - m - 2 \), then \( \left\lceil \frac{d+1}{2^i} \right\rceil = 2^{(m+1)k-m-2-i}(2^m - 1) + 1 \);
- If \( i \geq (m + 1)k - m - 2 \), then \( \left\lceil \frac{d+1}{2^i} \right\rceil = 2^{(m+1)k-2-i} \).
Thus
\[\sum_{i=0}^{k-1} \left\lfloor \frac{d+1}{2^i} \right\rfloor = \sum_{i=0}^{(m+1)k-m-3} \left\lfloor \frac{d+1}{2^i} \right\rfloor + \sum_{i=(m+1)k-m-2}^{km-1} \left\lfloor \frac{d+1}{2^i} \right\rfloor = \sum_{i=0}^{(m+1)k-m-3} (2^{(m+1)k-m-2-i}(2^m - 1) + 1) + \sum_{i=(m+1)k-m-2}^{km-1} (2^{(m+1)k-2-i}) = (2^{(m+1)k-m-1} - 2)(2^m - 1) + (m + 1)k - m - 2 + 2m^1 - 2^{k-1}.
\]

We want to guarantee that \((m + 1)k - m - 2 \leq km - 1\), i.e., \(m \geq k - 1\). Then if the code \(\Phi(C_m)\) is optimal, we need \(\sum_{i=0}^{k-1} \left\lfloor \frac{d+1}{2^i} \right\rfloor - n > 0\), i.e., \(m(k - 1) + k - 2^{k-1} > 0\), so we get \(m \geq \left\lceil \frac{2^{k-1}}{k-1} \right\rceil + 1\). In all, we have \(m \geq \max\{k - 1, \left\lceil \frac{2^{k-1}}{k-1} \right\rceil + 1\}\), and thus the theorem is proved. 

\[
\square
\]

6 The dual code

We compute the dual homogeneous distance of \(C_m\). A property of the trace that we need is that it is nondegenerate.

**Lemma 6.1** If for all \(a \in \mathcal{R}\), we have that \(Tr(ax) = 0\), then \(x = 0\).

**Proof.** Let \(a = a_0 + a_1 u + \cdots + a_{k-1} u^{k-1}\) and \(x = x_0 + x_1 u + \cdots + x_{k-1} u^{k-1}\), where \(x\) is a fixed element of \(R\) and \(a_i, x_i \in \mathbb{F}_{2^n}, i = 0, 1, \ldots, k - 1\). Then

\[ax = a_0 x_0 + (a_0 x_1 + a_1 x_0) u + \cdots + (a_0 x_{k-1} + a_1 x_{k-2} + \cdots + a_{k-1} x_0) u^{k-1} =: E_0 + E_1 u + \cdots + E_{k-1} u^{k-1}.
\]

Thus \(Tr(ax) = 0\) is equivalent to \(tr(E_i) = 0, i = 0, 1, \ldots, k - 1\). Using the nondegenerate character of \(tr()\), we first have \(tr(E_0) = 0\), then we get \(x_0 = 0\). Next we take \(x_0 = 0\) into \(tr(E_1) = 0\), and we have \(x_1 = 0\). Continuing this way, we finally get \(x_i = 0, i = 0, 1, \ldots, k - 1\), so \(x = 0\). This completes the proof. 

Next, we give the dual homogeneous distance of the two-homogeneous-weight codes \(C_m\).

**Theorem 6.2** For all \(m \geq 2\), the dual homogeneous distance \(d'_{\text{hom}}\) of \(C_m\) is \(2^{k-1}\).

**Proof.** First, we check that \(d'_{\text{hom}} \geq 2^{k-1}\). We just need to show that \(C_m^{\perp}\) does not contain a codeword that only has a word of homogeneous weight \(2^{k-2}\). If not, we
assume that there is a codeword of $C_m^\perp$ that has a word $\gamma = \gamma_0 + \gamma_1 u + \cdots + \gamma_{k-1} u^{k-1} \in \mathcal{R}\backslash\langle u^{k-1} \rangle$ at some $x \in \mathcal{R}^*$, so we know that $\gamma$ at least exist a coefficient $\gamma_j \neq 0$, where $j = 0, 1, \ldots, k-2$. Let $a = a_0 + a_1 u + \cdots + a_{k-1} u^{k-1} \in \mathcal{R}$, $x = x_0 + x_1 u + \cdots + x_{k-1} u^{k-1} \in \mathcal{R}^*$. Then we have $\gamma Tr(ax) = 0$, which gives $k$ equations, according to the first equation and Lemma 6.1, we have $tr(\gamma_0 x_0) = 0$, $\gamma_0 x_0 = 0$, but $x_0 \neq 0$, so $\gamma_0 = 0$. Applying the same technique to the next equations, we get $\gamma_1 = \gamma_2 = \cdots = \gamma_{k-2} = 0$, a contradiction.

Next, we prove that there exist a codeword of $C_m^\perp$ that has homogeneous weight $2^{k-1}$. First, we claim that $C_m^\perp$ does not contain a codeword that only has a word of homogeneous weight $2^{k-1}$. We can use a similar approach as above to prove it. Then we assume that there exist a codeword of $C_m^\perp$ have two values $\alpha = \alpha_0 + \alpha_1 u + \cdots + \alpha_{k-1} u^{k-1}, \beta = \beta_0 + \beta_1 u + \cdots + \beta_{k-1} u^{k-1} \in \mathcal{R}\backslash\langle u^{k-1} \rangle$ at some $x, y \in \mathcal{R}^*$, where $x = x_0 + x_1 u + \cdots + x_{k-1} u^{k-1}, y = y_0 + y_1 u + \cdots + y_{k-1} u^{k-1} \in \mathcal{R}^*$. Without loss of generality, we let $\alpha_0, \beta_0 \neq 0$. Then we get $\alpha Tr(ax) + \beta Tr(ay) = 0$, we just consider the constant term, i.e., $tr(\alpha_0 x_0 + \beta_0 y_0)) = 0$. According to Lemma 6.1, we have $\alpha_0 x_0 + \beta_0 y_0$, and due to $x_0, y_0 \in \mathbb{F}_2^m$, so we can let $\alpha_0 = x_0^{-1} \neq 0, \beta_0 = -y_0^{-1} \neq 0$, so the assumption is correct, and such $\alpha, \beta$ exist.

7 Connections with strongly regular graphs

Let us consider the strongly regular graph (SRG) built on the cosets of the dual code. Denote $r, s$ the two non restricted values of that graph. It is well-known (see [2] Th. 9.8.1 (iii)]) that they are related to the two weights $\omega_1 < \omega_2$ by the formulas

$$r = N - 2\omega_1, s = N - 2\omega_2,$$

where $N = 2^{k-1}n$ is the length of the binary image code. After using the values of $\omega_1, \omega_2$ of the preceding section, we find

$$r = 0, s = -2^{(k-1)(m+1)}.$$

The quadratic equation for $r, s$ shows that the parameters of the SRG satisfy, by [2], Thm 9.1.3.(ii)] the relations

$$\lambda = K + r + s + rs, \mu = K + rs,$$

where $K = N$ is the degree of the graph. Thus we obtain a graph with

$$\lambda = K + s = N + s, \mu = K = N.$$
But the coincidence $\mu = K$ suffices to prove that this graph is multipartite complete \[2, \S9.1.4\]. As a two-class association scheme it is imprimitive, since its complement graph is disconnected.

8 Application to secret sharing schemes

8.1 Support structure

Let $q$ be a prime power, and $n$ an integer. Let $\mathbb{F}_q$ denote the finite field of order $q$. The support $s(x)$ of a vector $x$ in $\mathbb{F}_q^n$ is defined as the set of indices where it is nonzero. We say that a vector $x$ covers a vector $y$ if $s(x)$ contains $s(y)$. A minimal codeword of a linear code $C$ is a nonzero codeword that does not cover any other nonzero codeword. In general determining the minimal codewords of a given linear code is a difficult task. However, there is a numerical condition, derived in \[1\], bearing on the weights of the code, that is easy to check.

**Lemma 8.1** (Ashikmin-Barg) Denote by $w_0$ and $w_\infty$ the minimum and maximum nonzero weights, respectively. If

$$\frac{w_0}{w_\infty} > \frac{q - 1}{q},$$

then every nonzero codeword of $C$ is minimal.

We can infer from there the support structure for the codes of this paper.

**Proposition 8.2** All the nonzero codewords of $\Phi(C_m)$, for $m \geq 2$, are minimal.

**Proof.** By the preceding lemma with $w_0 = \omega_1$, and $w_\infty = \omega_2$. Rewriting the inequality of the lemma as $2\omega_1 > \omega_2$, we obtain successively

$$2\omega_1 - \omega_2 = 2^{m(k-1)+k-1}(2^m - 1) - 2^{(m+1)k-2} = 2^{(m+1)k-2}(1 - 2^{1-m}) > 0.$$ 

Hence the proposition is proved. \qed

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8.2 Massey’s scheme

A secret sharing scheme (SSS) is a protocol involving a dealer and $U$ users. The dealer owns a very important secret (concretely an element of a large finite field) and is willing to reveal it to a large subset of the users but not to each one. To that end, he distributes some part of the secret, the so-called shares, (concretely an element of the same large finite field) to the users, one share per user. To recover the secret, the users need to collude together, in so-called coalitions, to combine their shares. Massey’s scheme is a construction of such a scheme where a code $C$ of length $n$ over $\mathbb{F}_p$ gives rise to a SSS with $U = n - 1$. In essence, the secret is carried by the first coordinate of a codeword, and the coalitions correspond to supports of codewords in the dual code with a one in that coordinate. The members of a coalition can compute the secret by using this codeword as a linear relation to combine their shares. See [14] for a detailed explanation of the mechanism of that scheme. Now, the coalition structure is related to the support structure of $C$. Indeed, the minimal coalitions are in one-to-one correspondence with the minimal codewords in the dual with a one in the secret-carrying coordinate. It would be interesting to know the dual Hamming distance (not only the dual homogeneous distance), as this would impact the SSS democratic or dictatorial character [7]. We leave this as an open problem to the diligent reader.

9 Conclusion

In this paper, we have studied a family of trace codes over $\mathbb{F}_2 + u\mathbb{F}_2 \cdots + u^{k-1}\mathbb{F}_2$, which includes [11] as a special case. These codes are provably abelian, but not visibly cyclic. Using a character sum approach, we have been able to determine their homogeneous weight distribution, and we have obtained a family of abelian binary two-weight codes by the Gray map. These codes are shown to be optimal under some condition by using the Griesmer bound. An application to secret sharing schemes is given. Determining the dual Hamming distance of the considered codes is a challenging open problem.
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