NOTES ON NONREPETITIVE GRAPH COLOURING

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Abstract. A vertex colouring of a graph is nonrepetitive on paths if there is no path $v_1, v_2, \ldots, v_{2t}$ such that $v_i$ and $v_{t+i}$ receive the same colour for all $i = 1, 2, \ldots, t$. We determine the maximum density of a graph that admits a $k$-colouring that is nonrepetitive on paths. We prove that every graph has a subdivision that admits a 4-colouring that is nonrepetitive on paths. The best previous bound was 5. We also study colourings that are nonrepetitive on walks, and provide a conjecture that would imply that every graph with maximum degree $\Delta$ has a $f(\Delta)$-colouring that is non-repetitive on walks. We prove that every graph with treewidth $k$ and maximum degree $\Delta$ has a $O(k \Delta)$-colouring that is nonrepetitive on paths, and a $O(k \Delta^3)$-colouring that is nonrepetitive on walks.

1. Introduction

We consider simple, finite, undirected graphs $G$ with vertex set $V(G)$, edge set $E(G)$, and maximum degree $\Delta(G)$. Let $[t] := \{1, 2, \ldots, t\}$. A walk in $G$ is a sequence $v_1, v_2, \ldots, v_t$ of vertices of $G$, such that $v_i v_{i+1} \in E(G)$ for all $i \in [t-1]$. A $k$-colouring of $G$ is a function $f$ that assigns one of $k$ colours to each vertex of $G$. A walk $v_1, v_2, \ldots, v_{2t}$ is repetitively coloured by $f$ if $f(v_i) = f(v_{t+i})$ for all $i \in [t]$. A walk $v_1, v_2, \ldots, v_{2t}$ is boring if $v_i = v_{t+i}$ for all $i \in [t]$. Of course, a boring walk is repetitively coloured by every colouring. We say a colouring $f$ is nonrepetitive on walks (or walk-nonrepetitive) if the only walks that are repetitively coloured by $f$ are boring. Let $\sigma(G)$ denote the minimum $k$ such that $G$ has a $k$-colouring that is nonrepetitive on walks.

A walk $v_1, v_2, \ldots, v_t$ is a path if $v_i \neq v_j$ for all distinct $i, j \in [t]$. A colouring $f$ is nonrepetitive on paths (or path-nonrepetitive) if no path of $G$ is repetitively coloured by $f$. Let $\pi(G)$ denote the minimum $k$ such that $G$ has a $k$-colouring that is nonrepetitive on paths. Observe that a colouring that is path-nonrepetitive is proper, in the sense that adjacent vertices receive distinct colours. Moreover, a path-nonrepetitive colouring has no 2-coloured $P_4$ (a path on four vertices). A proper colouring with no 2-coloured $P_4$ is called a star colouring since each bichromatic subgraph is a star forest; see [1, 8, 17, 18, 22]. The star chromatic number $\chi_{st}(G)$ is the minimum number of colours in a
proper colouring of $G$ with no 2-coloured $P_4$. Thus

$\chi(G) \leq \chi_{st}(G) \leq \pi(G) \leq \sigma(G)$. 

Path-nonrepetitive colourings are widely studied \[2, 3, 4, 6, 9, 10, 12, 13, 19, 21\]; see the survey by Grytczuk \[20\]. Nonrepetitive edge colourings have also been considered \[4, 3\].

The seminal result in this field is by Thue \[24\], who in 1906 proved that the $n$-vertex path $P_n$ satisfies

$$\pi(P_n) = \begin{cases} 
  n & \text{if } n \leq 2, \\
  3 & \text{otherwise}.
\end{cases}$$

A result by Kündgen and Pelsmajer \[21\] (see Lemma 3.4) implies

$$\sigma(P_n) \leq 4.$$

Currie \[11\] proved that the $n$-vertex cycle $C_n$ satisfies

$$\pi(C_n) = \begin{cases} 
  4 & \text{if } n \in \{5, 7, 9, 10, 14, 17\}, \\
  3 & \text{otherwise}.
\end{cases}$$

Let $\pi(\Delta)$ and $\sigma(\Delta)$ denote the maximum of $\pi(G)$ and $\sigma(G)$, taken over all graphs $G$ with maximum degree $\Delta(G) \leq \Delta$. Now $\pi(2) = 4$ by \[2\] and \[1\]. In general, Alon et al. \[4\] proved that

$$\frac{\alpha \Delta^2}{\log \Delta} \leq \pi(\Delta) \leq \beta \Delta^2,$$

for some constants $\alpha$ and $\beta$. The upper bound was proved using the Lovász Local Lemma, and the lower bound is attained by a random graph.

In Section 2 we study whether $\sigma(\Delta)$ is finite, and provide a natural conjecture that would imply an affirmative answer.

In Section 3 we study path- and walk-nonrepetitive colourings of graphs of bounded treewidth. Kündgen and Pelsmajer \[21\] and Barát and Varjú \[6\] independently proved that graphs of bounded treewidth have bounded $\pi$. The best bound is due to Kündgen and Pelsmajer \[21\] who proved that $\pi(G) \leq 4^k$ for every graph $G$ with treewidth at most $k$. Whether there is a polynomial bound on $\pi$ for graphs of treewidth $k$ is an open question. We answer this problem in the affirmative under the additional assumption

Footnotes:

1. The nonrepetitive 3-colouring of $P_n$ by Thue \[24\] is obtained as follows. Given a nonrepetitive sequence over \{1, 2, 3\}, replace each 1 by the sequence 12312, replace each 2 by the sequence 131232, and replace each 3 by the sequence 1323132. Thue \[24\] proved that the new sequence is nonrepetitive. Thus arbitrarily long paths can be nonrepetitively 3-coloured.

2. The treewidth of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k + 2$ vertices. Treewidth is an important graph parameter, especially in structural graph theory and algorithmic graph theory; see the surveys \[7, 23\].
of bounded degree. In particular, we prove a $O(k\Delta)$ upper bound on $\pi$, and a $O(k\Delta^3)$ upper bound on $\sigma$.

In Section 4 we prove that every graph has a subdivision that admits a path-nonrepetitive 4-colouring; the best previous bound was 5.

In Section 5 we determine the maximum density of a graph that admits a path-nonrepetitive $k$-colouring, and prove bounds on the maximum density for walk-nonrepetitive $k$-colourings.

2. Is $\sigma(\Delta)$ bounded?

Consider the following elementary lower bound on $\sigma$, where $G^2$ is the square graph of $G$. That is, $V(G^2) = V(G)$, and $vw \in E(G^2)$ if and only if the distance between $v$ and $w$ in $G$ is at most 2. A proper colouring of $G^2$ is called a distance-2 colouring of $G$.

Lemma 2.1. Every walk-nonrepetitive colouring of a graph $G$ is distance-2. Thus $\sigma(G) \geq \chi(G^2) \geq \Delta(G) + 1$.

Proof. Consider a walk-nonrepetitive colouring of $G$. Adjacent vertices $v$ and $w$ receive distinct colours, as otherwise $v, w$ would be a repetitively coloured path. If $u, v, w$ is a path, and $u$ and $w$ receive the same colour, then the non-boring walk $u, v, w, v$ is repetitively coloured. Thus vertices at distance at most 2 receive distinct colours. Hence $\sigma(G) \geq \chi(G^2)$. In a distance-2 colouring, each vertex and its neighbours all receive distinct colours. Thus $\chi(G^2) \geq \Delta(G) + 1$. □

Hence $\Delta(G)$ is a lower bound on $\sigma(G)$. Whether high degree is the only obstruction for bounded $\sigma$ is an open problem.

Open Problem 2.2. Is there a function $f$ such that $\sigma(\Delta) \leq f(\Delta)$?

First we answer Open Problem 2.2 in the affirmative for $\Delta = 2$. The following lemma will be useful.

Lemma 2.3. Fix a distance-2 colouring of a graph $G$. If $W = (v_1, v_2, \ldots, v_{2t})$ is a repetitively coloured non-boring walk in $G$, then $v_i \neq v_{t+i}$ for all $i \in [t]$.

Proof. Suppose on the contrary that $v_i = v_{t+i}$ for some $i \in [t-1]$. Since $W$ is repetitively coloured, $c(v_{i+1}) = c(v_{t+i+1})$. Each neighbour of $v_i$ receives a distinct colour. Thus $v_{i+1} = v_{t+i+1}$. By induction, $v_j = v_{t+j}$ for all $j \in [i, t]$. By the same argument, $v_j = v_{t+j}$ for all $j \in [1, i]$. Thus $W$ is boring, which is the desired contradiction. □

Proposition 2.4. $\sigma(2) \leq 5$.

Proof. A result by Kündgen and Pelsmajer \[21\] implies that $\sigma(P_n) \leq 4$ (see Lemma 3.4). Thus it suffices to prove that $\sigma(C_n) \leq 5$. Fix a walk-nonrepetitive 4-colouring of the path $(v_1, v_2, \ldots, v_{2n-4})$. Thus for some $i \in [1, n-2]$, the vertices $v_i$ and $v_{n+i-2}$ receive distinct colours. Create a cycle $C_n$ from the sub-path $v_i, v_{i+1}, \ldots, v_{n+i-2}$ by adding one
vertex $x$ adjacent to $v_i$ and $v_{n+i-2}$. Colour $x$ with a fifth colour. Observe that since $v_i$ and $v_{n+i-2}$ receive distinct colours, the colouring of $C_n$ is distance-2. Suppose on the contrary that $C_n$ has a repetitively coloured walk $W = y_1, y_2, \ldots, y_{2t}$. If $x$ is not in $W$, then $W$ is a repetitively coloured walk in the starting path, which is a contradiction. Thus $x = y_i$ for some $i \in [t]$ (with loss of generality, by considering the reverse of $W$). Since $x$ is the only vertex receiving the fifth colour and $W$ is repetitive, $x = y_{t+i}$. By Lemma 2.3 $W$ is boring. Hence the 5-colouring of $C_n$ is walk-nonrepetitive. □

Below we propose a conjecture that would imply a positive answer to Open Problem 2.2. First consider the following lemma which is a slight generalisation of a result by Barát and Varjú [5]. A walk $v_1, v_2, \ldots, v_t$ has length $t$ and order $|\{v_i : 1 \leq i \leq t\}|$. That is, the order is the number of distinct vertices in the walk.

**Lemma 2.5.** Suppose that in some coloured graph, there is a repetitively coloured non-boring walk. Then there is a repetitively coloured non-boring walk of order $k$ and length at most $2k^2$.

**Proof.** Let $k$ be the minimum order of a repetitively coloured non-boring walk. Let $W = v_1, v_2, \ldots, v_{2t}$ be a repetitively coloured non-boring walk of order $k$ and with $t$ minimum. If $2t \leq 2k^2$, then we are done. Now assume that $t > k^2$. By the pigeonhole principle, there is a vertex $x$ that appears at least $k + 1$ times in $v_1, v_2, \ldots, v_t$. Thus there is a vertex $y$ that appears at least twice in the set $\{v_{t+i} : v_i = x, i \in [t]\}$. As illustrated in Figure 1 $W = AxByCA'yB'yC'$ for some walks $A, B, C, A', B', C'$ with $|A| = |A'|$, $|B| = |B'|$, and $|C| = |C'|$. Consider the walk $U := AxCA'yC'$. If $U$ is not boring, then it is a repetitively coloured non-boring walk of order at most $k$ and length less than $2t$, which contradicts the minimality of $W$. Otherwise $U$ is boring, implying $x = y$, $A = A'$, and $C = C'$. Thus $B \neq B'$ since $W$ is not boring, implying $xBxB'$ is a repetitively coloured non-boring walk of order at most $k$ and length less than $2t$, which contradicts the minimality of $W$. □

![Figure 1. Illustration for the proof of Lemma 2.5.](image)

We conjecture the following strengthening of Lemma 2.5.
**Conjecture 2.6.** Let $G$ be a graph. Consider a path-nonrepetitive distance-2 colouring of $G$ with $c$ colours, such that $G$ contains a repetitively coloured non-boring walk. Then $G$ contains a repetitively coloured non-boring walk of order $k$ and length at most $h(c) \cdot k$, for some function $h$ that only depends on $c$.

**Theorem 2.7.** If Conjecture 2.6 is true, then there is a function $f$ for which $\sigma(\Delta) \leq f(\Delta)$. That is, every graph $G$ has a walk-nonrepetitive colouring with $f(\Delta(G))$ colours.

Theorem 2.7 is proved using the Lovász Local Lemma [16].

**Lemma 2.8 ([16]).** Let $A = A_1 \cup A_2 \cup \cdots \cup A_r$ be a partition of a set of ‘bad’ events $A$. Suppose that there are sets of real numbers \{$p_i \in [0, 1) : i \in [r]\}$, \{${x_i \in [0, 1) : i \in [r]}$\}, and \{${D_{ij} \geq 0 : i, j \in [r]}$\} such that the following conditions are satisfied by every event $A \in A_i$:

- the probability $P(A) \leq p_i \leq x_i \prod_{j=1}^{r}(1 - x_j)^{D_{ij}}$, and
- $A$ is mutually independent of $A \setminus \{A \cup D_A\}$, for some $D_A \subseteq A$ with $|D_A \cap A_j| \leq D_{ij}$ for all $j \in [r]$.

Then

$$P \left( \bigwedge_{A \in A} \overline{A} \right) \geq \prod_{i=1}^{r}(1 - x_i)^{|A_i|} > 0.$$

That is, with positive probability, no event in $A$ occurs.

**Proof of Theorem 2.7.** Let $f_1$ be a path-nonrepetitive colouring of $G$ with $\pi(G)$ colours. Let $f_2$ be a distance-2 colouring of $G$ with $\chi(G^2)$ colours. Note that $\pi(G) \leq \beta \Delta^2$ for some constant $\beta$ by Equation (5), and $\chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta^2 + 1$ by a greedy colouring of $G^2$. Hence $f_1$ and $f_2$ together define a path-nonrepetitive distance-2 colouring of $G$.

The number of colours $\pi(G) \cdot \chi(G^2)$ is bounded by a function solely of $\Delta(G)$. Consider this initial colouring to be fixed. Let $c$ be a positive integer to be specified later. For each vertex $v$ of $G$, choose a third colour $f_3(v) \in [c]$ independently and randomly. Let $f$ be the colouring defined by $f(v) = (f_1(v), f_2(v), f_3(v))$ for all vertices $v$.

Let $h := h(\pi(G) \cdot \chi(G^2))$ from Conjecture 2.6. A non-boring walk $v_1, v_2, \ldots, v_{2t}$ of order $i$ is interesting if its length $2t \leq hi$, and $f_1(v_j) = f_1(v_{t+j})$ and $f_2(v_j) = f_2(v_{t+j})$ for all $j \in [t]$. For each interesting walk $W$, let $A_W$ be the event that $W$ is repetitively coloured by $f$. Let $A_i$ be the set of events $A_W$, where $W$ is an interesting walk of order $i$. Let $A = \bigcup_i A_i$.

We will apply Lemma 2.8 to prove that, with positive probability, no event $A_W$ occurs. This will imply that there exists a colouring $f_3$ such that no interesting walk is repetitively coloured by $f$. A non-boring non-interesting walk $v_1, v_2, \ldots, v_{2t}$ of order $i$ satisfies (a) $2t > hi$, or (b) $f_1(v_j) \neq f_1(v_{t+j})$ or $f_2(v_j) \neq f_2(v_{t+j})$ for some $j \in [t]$. In case (a), by the assumed truth of Conjecture 2.6, $W$ is not repetitively coloured by $f$. In
case (b), \( f(v_j) \neq f(v_{t+j}) \) and \( W \) is not repetitively coloured by \( f \). Thus no non-boring walk is repetitively coloured by \( f \), as desired.

Consider an interesting walk \( W = v_1, v_2, \ldots, v_{2t} \) of order \( t \).

We claim that \( v_{\ell} \neq v_{t+\ell} \) for all \( \ell \in [t] \). Suppose on the contrary that \( v_{\ell} = v_{t+\ell} \) for some \( \ell \in [t] \). Since \( W \) is not boring, \( v_j \neq v_{t+j} \) for some \( j \in [t] \). Thus \( v_j = v_{t+j} \) and \( v_{j+1} \neq v_{t+j+1} \) for some \( j \in [t] \) (where \( v_{t+j+1} \) means \( v_1 \)). Since \( W \) is interesting, \( f_2(v_{j+1}) = f_2(v_{t+j+1}) \), which is a contradiction since \( v_{j+1} \) and \( v_{t+j+1} \) have a common neighbour \( v_j \) (= \( v_{t+j} \)). Thus \( v_j \neq v_{t+j} \) for all \( j \in [t] \), as claimed.

This claim implies that for each of the \( i \) vertices \( x \) in \( W \), there is at least one other vertex \( y \) in \( W \), such that \( f_3(x) = f_3(y) \) must hold for \( W \) to be repetitively coloured. Hence at most \( c^{i/2} \) of the \( c^i \) possible colourings of \( W \) under \( f_3 \), lead to repetitive colourings of \( W \) under \( f \). Thus the probability \( P(\Delta W) \leq p_i := c^{-i/2} \), and Lemma 2.8 can be applied as long as

\[
(6) \quad c^{-i/2} \leq x_i \prod_j (1 - x_j)^{D_{ij}},
\]

Every vertex is in at most \( h j \Delta^{hj} \) interesting walks of order \( j \). Thus an interesting walk of order \( i \) shares a vertex with at most \( h i j \Delta^{hj} \) interesting walks of order \( j \). Thus we can take \( D_{ij} := h i j \Delta^{hj} \). Define \( x_i := (2\Delta^h)^{-i} \). Note that \( x_i \leq \frac{1}{2} \). So \( 1 - x_i \geq e^{-2x_i} \).

Thus to prove (6) it suffices to prove that

\[
\begin{align*}
  c^{-i/2} & \leq x_i \prod_j e^{-2x_j D_{ij}}, \\
  \iff \quad c^{-i/2} & \leq (2\Delta^h)^{-i} \prod_j e^{-2(2\Delta^h)^{-j} h i j \Delta^{hj}}, \\
  \iff \quad c^{-1/2} & \leq (2\Delta^h)^{-1} \prod_j e^{-2(2^{-1} h) j}, \\
  \iff \quad c^{-1/2} & \leq (2\Delta^h)^{-1} e^{-2h \sum j 2^{-j}}, \\
  \iff \quad c^{-1/2} & \leq (2\Delta^h)^{-1} e^{-4h}, \\
  \iff \quad c & \geq 4(4^4 e^4 \Delta)^{2h}.
\end{align*}
\]

Choose \( c \) to be the minimum integer that satisfies this inequality, and the lemma is applicable. We obtain a \( c \)-colouring \( f_3 \) of \( G \) such that \( f \) is nonrepetitive on walks. The number of colours in \( f \) is at most \( h \lfloor 4(4^4 e^4 \Delta)^{2h} \rfloor \), which is a function solely of \( \Delta \). \( \square \)

3. Trees and Treewidth

We start this section by considering walk-nonrepetitive colourings of trees.

**Theorem 3.1.** Let \( T \) be a tree. A colouring \( c \) of \( T \) is walk-nonrepetitive if and only if \( c \) is path-nonrepetitive and distance-2.
Proof. For every graph, every walk-nonrepetitive colouring is path-nonrepetitive (by definition) and distance-2 (by Lemma 2.1).

Now fix a path-nonrepetitive distance-2 colouring $c$ of $T$. Suppose on the contrary that $T$ has a repetitively coloured non-boring walk. Let $W = (v_1, v_2, \ldots, v_{2t})$ be a repetitively coloured non-boring walk in $T$ of minimum length. Some vertex is repeated in $W$, as otherwise $W$ would be a repetitively coloured path. By considering the reverse of $W$, without loss of generality, $v_i = v_j$ for some $i \in [1, t - 1]$ and $j \in [i + 2, 2t]$.

Choose $i$ and $j$ to minimise $j - i$. Thus $v_i$ is not in the sub-walk $(v_{i+1}, v_{i+2}, \ldots, v_{j-1})$. Since $T$ is a tree, $v_{i+1} = v_{j-1}$. Thus $i + 1 = j - 1$, as otherwise $j - i$ is not minimised. That is, $v_i = v_{i+2}$. Assuming $i \neq t - 1$, since $W$ is repetitively coloured, $c(v_{t+i}) = c(v_{t+i+2})$, which implies that $v_{t+i} = v_{t+i+2}$ because $c$ is a distance-2 colouring. Thus, even if $i = t - 1$, deleting the vertices $v_i, v_{i+1}, v_{t+i}, v_{t+i+1}$ from $W$, gives a walk $(v_1, v_2, \ldots, v_{i-1}, v_{i+2}, v_{t+i+2}, \ldots, v_{2t})$ that is also repetitively coloured. This contradicts the minimality of the length of $W$. □

Note that Theorem 3.1 implies that Conjecture 2.6 is vacuously true for trees.

Since every tree $T$ has a path-nonrepetitive 4-colouring \cite{21} and a distance-2 ($\Delta(T) + 1$)-colouring, Theorem 3.1 implies the following result, where the lower bound is Lemma 2.1

**Corollary 3.2.** Every tree $T$ satisfies $\Delta(T) + 1 \leq \sigma(T) \leq 4(\Delta(T) + 1)$.

In the remainder of this section we prove the following polynomial upper bounds on $\pi$ and $\sigma$ in terms of the treewidth and maximum degree of a graph.

**Theorem 3.3.** Every graph $G$ with treewidth $k$ and maximum degree $\Delta \geq 1$ satisfies $\pi(G) \leq ck\Delta$ and $\sigma(G) \leq ck\Delta^3$ for some constant $c$.

We prove Theorem 3.3 by a series of lemmas. The first is by Kündgen and Pelsmajer \cite{21}.

**Lemma 3.4 (\cite{21}).** Let $P^+$ be the pseudograph obtained from a path $P$ by adding a loop at each vertex. Then $\sigma(P^+) \leq 4$.

Now we introduce some definitions by Kündgen and Pelsmajer \cite{21}. A *levelling* of a graph $G$ is a function $\lambda : V(G) \rightarrow \mathbb{Z}$ such that $|\lambda(v) - \lambda(w)| \leq 1$ for every edge $vw \in E(G)$. Let $G_{\lambda=k}$ and $G_{\lambda>k}$ denote the subgraphs of $G$ respectively induced by $\{v \in V(G) : \lambda(v) = k\}$ and $\{v \in V(G) : \lambda(v) > k\}$. The *$k$-shadow* of a subgraph $H$ of $G$ is the set of vertices in $G_{\lambda=k}$ adjacent to some vertex in $H$. A levelling $\lambda$ is *shadow-complete* if the $k$-shadow of every component of $G_{\lambda>k}$ induces a clique. Kündgen and Pelsmajer \cite{21} proved the following lemma for repetitively coloured paths. We show that the same proof works for repetitively coloured walks.

\[3\]The 4-colouring in Lemma 3.4 is obtained as follows. Given a nonrepetitive sequence on $\{1, 2, 3\}$, insert the symbol 4 between consecutive block of length two. For example, from the sequence 123132123 we obtain 1243143241243.
Lemma 3.5. For every levelling \(\lambda\) of a graph \(G\), there is a 4-colouring of \(G\), such that every repetitively coloured walk \(v_1, v_2, \ldots, v_{2t}\) satisfies \(\lambda(v_j) = \lambda(v_{t+j})\) for all \(j \in [t]\).

Proof. The levelling \(\lambda\) can be thought of as a homomorphism from \(G\) into \(P^+\), for some path \(P\). By Lemma 3.4, \(P^+\) has a 4-colouring that is nonrepetitive on walks. Colour each vertex \(v\) of \(G\) by the colour assigned to \(\lambda(v)\) (thought of as a vertex of \(P^+\)). Suppose \(v_1, v_2, \ldots, v_{2t}\) is a repetitively coloured walk in \(G\). Thus \(\lambda(v_1), \lambda(v_2), \ldots, \lambda(v_{2t})\) is a repetitively coloured walk in \(P^+\). Since the 4-colouring of \(P^+\) is nonrepetitive on walks, \(\lambda(v_1), \lambda(v_2), \ldots, \lambda(v_{2t})\) is boring. That is, \(\lambda(v_j) = \lambda(v_{t+j})\) for all \(j \in [t]\). \(\square\)

Lemma 3.6 ([21]). If \(\lambda\) is a shadow-complete levelling of a graph \(G\), then

\[
\pi(G) \leq 4 \cdot \max_k \pi(G_{\lambda=k}).
\]

Now we generalise Lemma 3.6 for walks.

Lemma 3.7. If \(H\) is a subgraph of a graph \(G\), and \(\lambda\) is a shadow-complete levelling of \(G\), then

\[
\sigma(H) \leq 4 \chi(H^2) \cdot \max_k \sigma(G_{\lambda=k}) \leq 4(\Delta(H)^2 + 1) \cdot \max_k \sigma(G_{\lambda=k}).
\]

Proof. Let \(c_1\) be the 4-colouring of \(G\) from Lemma 3.5. Thus every repetitively coloured walk \(v_1, v_2, \ldots, v_{2t}\) satisfies \(\lambda(v_j) = \lambda(v_{t+j})\) for all \(j \in [t]\). Let \(c_2\) be an optimal walk-nonrepetitive colouring of each level \(G_{\lambda=k}\). Let \(c_3\) be a proper \(\chi(H^2)\)-colouring of \(H^2\). The second inequality in the lemma follows from the first since \(\chi(H^2) \leq \Delta(H)^2 + 1\). Let \(c(v) := (c_1(v), c_2(v), c_3(v))\) for each vertex \(v\) of \(H\). We claim that \(c\) is nonrepetitive on walks in \(H\).

Suppose on the contrary that \(W = v_1, \ldots, v_{2s}\) is a non-boring walk in \(H\) that is repetitively coloured by \(c\). Then \(W\) is repetitively coloured by each of \(c_1, c_2, \) and \(c_3\). Thus \(\lambda(v_i) = \lambda(v_{t+i})\) for all \(i \in [t]\) by Lemma 3.5. Let \(W_k\) be the sequence (allowing repetitions) of vertices \(v_i \in W\) such that \(\lambda(v_i) = k\). Since \(v_i \in W_k\) if and only if \(v_{t+i} \in W_k\), each sequence \(W_k\) is repetitively coloured. That is, if \(W_k = x_1, \ldots, x_{2s}\) then \(c(x_i) = c(x_{s+i})\) for all \(i \in [s]\).

Let \(k\) be the minimum level containing a vertex in \(W\). Let \(v_i\) and \(v_j\) be consecutive vertices in \(W_k\) with \(i < j\). If \(j = i + 1\) then \(v_iv_j\) is an edge of \(W\). Otherwise there is walk from \(v_i\) to \(v_j\) in \(G_{\lambda > k}\) (since \(k\) was chosen minimum), implying \(v_iv_j\) is an edge of \(G\) (since \(\lambda\) is shadow-complete). Thus \(W_k\) forms a walk in \(G_{\lambda=k}\) that is repetitively coloured by \(c_2\). Hence \(W_k\) is boring. In particular, some vertex \(v_i = v_{t+i}\) is in \(W_k\). Since \(W\) is not boring, \(v_j \neq v_{t+j}\) for some \(j \in [t]\). Without loss of generality, \(i < j\) and \(v_\ell = v_{t+\ell}\) for all \(\ell \in [i, j - 1]\). Thus \(v_j\) and \(v_{t+j}\) have a common neighbour \(v_{j-1} = v_{t+j-1}\) in \(H\), which implies that \(c_3(v_j) \neq c_3(v_{t+j})\). But \(c(v_j) = c(v_{t+j})\) since \(W\) is repetitively coloured, which is the desired contradiction. \(\square\)

Note that some dependence on \(\Delta(H)\) in Lemma 3.7 is unavoidable, since \(\sigma(H) \geq \chi(H^2) \geq \Delta(H) + 1\).
Lemma 3.8. Every tree $T$ satisfies $\Delta(T) + 1 \leq \sigma(T) \leq 4 \Delta(T)$.

Proof. Let $r$ be a leaf vertex of $T$. Let $\lambda(v)$ be the distance from $r$ to $v$ in $T$. Then $\lambda$ is a shadow-complete levelling of $T$ in which each level is an independent set. A greedy algorithm proves that $\chi(T^2) \leq \Delta(T) + 1$. Thus Lemma 3.7 implies that $\sigma(T) \leq 4 \Delta(T) + 4$. Observe that the proof of Lemma 3.7 only requires $c_3(v) \neq c_3(w)$ whenever $v$ and $w$ are in the same level and have a common parent. Since $r$ is a leaf, each vertex has at most $\Delta(T) - 1$ children. Thus a greedy algorithm produces a $\Delta(T)$-colouring with this property. Hence $\sigma(T) \leq 4 \Delta(T)$. □

A tree-partition of a graph $G$ is a partition of its vertices into sets (called bags) such that the graph obtained from $G$ by identifying the vertices in each bag is a forest (after deleting loops and replacing parallel edges by a single edge).

Lemma 3.9. Let $G$ be a graph with a tree-partition in which every bag has at most $\ell$ vertices. Then $G$ is a subgraph of a graph $G'$ that has a shadow-complete levelling in which each level satisfies

$$\pi(G'_{\lambda=k}) \leq \sigma(G'_{\lambda=k}) \leq \ell.$$ 

Proof. Let $G'$ be the graph obtained from $G$ by adding an edge between all pairs of nonadjacent vertices in a common bag. Let $F$ be the forest obtained from $G'$ by identifying the vertices in each bag. Root each component of $F$. Consider a vertex $v$ of $G'$ that is in the bag that corresponds to node $x$ of $F$. Let $\lambda(v)$ be the distance between $x$ and the root of the tree component of $F$ that contains $x$. Clearly $\lambda$ is a levelling of $G'$. The $k$-shadow of each connected component of $G'_{\lambda=k}$ is contained in a single bag, and thus induces a clique on at most $\ell$ vertices. Hence $\lambda$ is shadow-complete. By colouring the vertices within each bag with distinct colors, we have $\pi(G'_{\lambda=k}) \leq \sigma(G'_{\lambda=k}) \leq \ell$. □

Lemmas 3.6, 3.7 and 3.9 imply:

Lemma 3.10. If a graph $G$ has a tree-partition in which every bag has at most $\ell$ vertices, then $\pi(G) \leq 4\ell$ and $\sigma(G) \leq 4\ell(\Delta(G)^2 + 1)$.

Wood [25] proved that every graph with treewidth $k$ and maximum degree $\Delta \geq 1$ has a tree-partition in which every bag has at most $\frac{5}{2}(k+1)\left(\frac{1}{2}\Delta - 1\right)$ vertices. With Lemma 3.10 this proves the following quantitative version of Theorem 3.3.

Theorem 3.11. Every graph $G$ with treewidth $k$ and maximum degree $\Delta \geq 1$ satisfies $\pi(G) \leq 10(k+1)\left(\frac{1}{2}\Delta - 1\right)$ and $\sigma(G) \leq 10(k+1)\left(\frac{1}{2}\Delta - 1\right)(\Delta^2 + 1)$.

4The proof by Kündgen and Pelsmajer [21] that $\pi(G) \leq 4^k$ for graphs with treewidth at most $k$ can also be described using tree-partitions; cf. [15].

5The proof by Wood [25] is a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [14].
4. Subdivisions

The results of Thue [24] and Currie [11] imply that every path and every cycle has a subdivision $H$ with $\pi(H) = 3$. Brešar et al. [9] proved that every tree has a subdivision $H$ such that $\pi(H) = 3$. Which graphs have a subdivision $H$ with $\pi(H) = 3$ is an open problem [20]. Grytczuk [20] proved that every graph has a subdivision $H$ with $\pi(H) \leq 5$. Here we improve this bound as follows.

**Theorem 4.1.** Every graph $G$ has a subdivision $H$ with $\pi(H) \leq 4$.

**Proof.** Without loss of generality $G$ is connected. Say $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$. As illustrated in Figure 2 let $H$ be the subdivision of $G$ obtained by subdividing every edge $v_iv_j \in E(G)$ (with $i < j$) $j - i - 1$ times. The distance of every vertex in $H$ from $v_0$ defines a levelling of $H$ such that the endpoints of every edge are in consecutive levels. By Lemma 3.5 there is a 4-colouring of $H$, such that for every repetitively coloured path $x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$ in $H$, $x_j$ and $y_j$ have the same level for all $j \in [t]$. Hence there is some $j$ such that $x_{j-1}$ and $x_{j+1}$ are at the same level. Thus $x_j$ is an original vertex $v_i$ of $G$. Without loss of generality $x_{j-1}$ and $x_{j+1}$ are at level $i - 1$. There is only one original vertex at level $i$. Thus $y_j$, which is also at level $i$, is a division vertex. Now $y_j$ has two neighbours in $H$, which are at levels $i - 1$ and $i + 1$. Thus $y_{j-1}$ and $y_{j+1}$ are at levels $i - 1$ and $i + 1$, which contradicts the fact that $x_{j-1}$ and $x_{j+1}$ are both at level $i - 1$. Hence we have a 4-colouring of $H$ that is nonrepetitive on paths. □

![Figure 2. The subdivision $H$ with $G = K_6$.](image)

It is possible that every graph has a subdivision $H$ with $\pi(H) \leq 3$. If true, this would provide a striking generalisation of the result of Thue [24] discussed in Section 1.

5. Maximum Density

In this section we study the maximum number of edges in a nonrepetitively coloured graph.
Proposition 5.1. The maximum number of edges in an \( n \)-vertex graph \( G \) with \( \pi(G) \leq c \) is \( (c - 1)n - \binom{c}{2} \).

Proof. Say \( G \) is an \( n \)-vertex graph with \( \pi(G) \leq c \). Fix a \( c \)-colouring of \( G \) that is nonrepetitive on paths. Say there are \( n_i \) vertices in the \( i \)-th colour class. Every cycle receives at least three colours. Thus the subgraph induced by the vertices coloured \( i \) and \( j \) is a forest, and has at most \( n_i + n_j - 1 \) edges. Hence the number of edges in \( G \) is at most
\[
\sum_{1 \leq i < j \leq c} (n_i + n_j - 1) = \sum_{1 \leq i \leq c} (c - 1)n_i - \binom{c}{2} = (c - 1)n - \binom{c}{2}.
\]
This bound is attained by the graph consisting of a complete graph \( K_{c-1} \) completely connected to an independent set of \( n - (c - 1) \) vertices, which obviously has a \( c \)-colouring that is nonrepetitive on paths.

Now consider the maximum number of edges in a coloured graph that is nonrepetitive on walks. First note that the example in the proof of Proposition 5.1 is repetitive on walks. Since \( \sigma(G) \geq \Delta(G) + 1 \) and \( |E(G)| \leq \frac{1}{2}\Delta(G)|V(G)| \), we have the trivial upper bound,
\[
|E(G)| \leq \frac{1}{2}(\sigma(G) - 1)|V(G)|.
\]
This bound is tight for \( \sigma = 2 \) (matchings) and \( \sigma = 3 \) (cycles), but is not known to be tight for \( \sigma \geq 4 \).

We have the following lower bound.

Proposition 5.2. For all \( p \geq 1 \), there are infinitely many graphs \( G \) with \( \sigma(G) \leq 4p \) and
\[
|E(G)| \geq \frac{1}{8}(3\sigma(G) - 4)|V(G)| - \frac{1}{8}\sigma(G)^2.
\]

Proof. Let \( G \) be the lexicographic product of a path and \( K_p \); that is, \( G \) is the graph with a levelling \( \lambda \) in which each level induces \( K_p \), and every edge is present between consecutive levels. Let \( c_1 \) be the 4-colouring of \( G \) from Lemma 3.3. If \( v \) is the \( j \)-th vertex in its level, where \( j \in [p] \), then let \( c(v) := (c_1(v), j) \). The number of colours is \( 4p \). We claim that \( c \) is nonrepetitive on walks in \( G \). Suppose on the contrary that \( W = v_1, \ldots, v_{2t} \) is a non-boring walk in \( G \) that is repetitively coloured by \( c \). Then \( W \) is repetitively coloured by \( c_1 \). Thus \( \lambda(v_i) = \lambda(v_{i+t}) \) for all \( i \in [t] \) by Lemma 3.5. Since \( W \) is not boring, some \( v_i \neq v_{i+t} \). By construction, \( c(v_i) \neq c(v_{i+t}) \), which contradicts the assumption that \( W \) is repetitively coloured. Hence \( \sigma(G) \leq 4p \). Now we count the edges:
\[
|E(G)| = \frac{1}{2}(3p - 1)|V(G)| - p^2.
\]
As a lower bound, \( \sigma(G) \geq \Delta(G) + 1 = 3p \). Thus
\[
|E(G)| \geq \frac{1}{8}(3\sigma(G) - 4)|V(G)| - (\sigma(G)/3)^2.
\]

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