FREE LIMITS OF THOMPSON’S GROUP $F$

AZER AKHMEDOV, MELANIE STEIN, AND JENNIFER TABACK

Abstract. We produce a sequence of markings $S_k$ of Thompson’s group $F$ within the space $G_n$ of all marked $n$-generator groups so that the sequence $(F, S_k)$ converges to the free group on $n$ generators, for $n \geq 3$. In addition, we give presentations for the limits of some other natural (convergent) sequences of markings to consider on $F$ within $G_3$, including $(F, \{x_0, x_1, x_n\})$ and $(F, \{x_0, x_1, x_0^3\})$.

1. Introduction

Sela defined the notion of a limit group in conjunction with his solution to the problem of Tarski which asks whether all free groups of rank at least 2 have the same elementary theory \cite{S1, S2}. Limit groups arise in Sela’s analysis of equations in free groups, and he shows that they coincide with the class of finitely generated, fully residually free groups \cite{Se1}. Work of Sela, along with Kharlampovich and Myasnikov \cite{S1, KM1, KM2} shows that limit groups can be constructed recursively from building blocks consisting of free, surface and free abelian groups by taking a finite sequence of free products and amalgamations over $\mathbb{Z}$. Alternately, a group is a limit group in the sense of Sela if and only if it is an iterated generalized double, defined below in Section 4.

Define a marked group $(G, S)$ to be a group $G$ with a fixed and ordered set of generators $S = \{g_1, g_2, \ldots, g_n\}$, and let $G_n$ be the set of all groups marked by $n$ elements up to isomorphism of marked groups. A marked group $(G, S)$ is equipped with a canonical epimorphism from the free group on $|S|$ letters to $G$. The space $G_n$ admits a topology in which two marked groups $(G_1, S_1)$ and $(G_2, S_2)$ are at distance at most $e^{-R}$ if they have same
relations of length at most $R$. With respect to this topology, the limit

groups of Sela, equivalently the class of all finitely generated fully residually

free groups, arise naturally as limits of marked free groups.

This topological approach towards marked groups opens the notion of limit
groups to include limits of other, non free, groups within $G_n$, for a fixed $n$.

This topology was defined in [G] by Grigorchuk, and an earlier equivalent

construction was presented in [C]; Champetier and Guirardel study this
topology in [CG], and Guyot and Stalder in [SL GS] investigate limits of

marked copies of Baumslag-Solitar groups.

Just as the class of finitely generated fully residually free groups arise nat-
urally as limits of marked free groups, one can extend the definition of

“fully residually” to non-free classes of groups in such a way that these

groups arise naturally as limits of marked sequences of other, non-free,
groups. More specifically, a finitely generated group $G$ is defined to be

fully residually $\mathcal{P}$ (where we view $\mathcal{P}$ as a property of groups, e.g. free or

finite) if for any finite collection $\{w_1, w_2, \ldots, w_k\}$ of elements of $G$, there

is a surjective homomorphism $\phi$ from $G$ to a $\mathcal{P}$ group so that the images

$\{\phi(w_1), \phi(w_2), \ldots, \phi(w_k)\}$ of the original set of elements are all nontrivial.

In the case $\mathcal{P}=\text{free}$, we can omit the requirement that $\phi$ be surjective, as

any subgroup of a free group is itself free. Just as for the special case of

fully residually free groups, if a group $G$ is fully residually $\mathcal{P}$, there is a se-

quence of markings of the $\mathcal{P}$ group so that the sequence of marked groups

converges to $G$. With this definition, any group which is fully residually

Thompson arises as a limit of marked copies of Thompson’s group $F$. One

aim of this paper is to show that the free group $F_k$ for $k \geq 3$ is fully

residually Thompson.

The goal of this paper is to analyze several sequences of markings of Thomp-
son’s group $F$. We begin in Section 3 by partially answering a question of

Sapir, and show in Corollary 3.9 that there exist sequences of marked copies

of Thompson’s group $F$ which converge to the free group $F_k$ within $G_k$ for

any $k \geq 3$, that is, we show that the free group is fully residually Thomp-

son. Brin [B] has recently shown that there is a sequence of markings of $F$
in $G_2$ which converges to the free group $F_2$. This is related to the notion of

a group exhibiting $k$-free-like behavior, defined by Olshanskii and Sapir in

[OS], motivated by work in [BNP].

A group $G$ is said to be $k$-free-like for $k \geq 2$ if there exists a sequence of
generating sets $Z_i$ for $i \geq 1$, each with $k$ elements, such that the Cayley

graph $\Gamma(G, Z_i)$ satisfies no relation of length less than $i$, and the Cheeger

constant of this graph is uniformly (in $i$) bounded away from zero. The

Cheeger constant is the infimum over all subsets $A$ of the group of the ratio
of the size of the boundary of $A$ to the size of $A$, with respect to a fixed generating set. If one can show that $F$ is uniformly nonamenable, it will follow from Corollary 3.9 that $F$ exhibits $k$-free-like behavior.

In Section 4 we investigate the limits of several sequences of marked copies of $F$ within $G_3$ where the markings are chosen to be very “natural”, for example the sequence $\{x_0, x_1, x_n\}$, where the generators are taken from the standard infinite presentation for $F$. We generalize our concrete examples to sequences of markings where the third element in the triple is simply subject to certain conditions on its support. We note that in each case, there is an amalgamated product (or HNN extension) of copies of Thompson’s group $F$, or subgroups of $F$, over a maximal abelian subgroup which is a generalized double over the limit group obtained.

The problem of determining all sequences of markings of $F$ of the form $\{x_0, x_1, g_n\}$, where $g_n \in F$, which converge in $G_3$, and the presentation of any resulting limit groups, is extremely interesting to the authors. In his thesis, Zarzycki considers this problem as well; he has obtained preliminary results which state that limit of a sequence of marked copies of $F$ using markings of the form $\{x_0, x_1, g_n\}$ can never be a central HNN-extension [Z1]. His methods are significantly different from the techniques presented here.

2. Preliminaries

In this section we present brief background material on several topics used in this paper.

2.1. A brief introduction to Thompson’s group $F$. Thompson’s group $F$ can be viewed as the group of piecewise-linear orientation-preserving homeomorphisms of the unit interval, subject to two conditions:

1. the coordinates of all breakpoints lie in the set of dyadic rational numbers, and
2. the slopes of all linear pieces are powers of 2.

Group elements can be viewed uniquely in this way if we require that the slope change at all given breakpoints. Group multiplication then simply corresponds to function composition.

This group is commonly studied via a standard infinite presentation $\mathcal{P}$:

$$\mathcal{P} = \langle x_0, x_1, x_2, \cdots | x_i^{-1} x_j x_i = x_{j+1} \text{ for } i < j \rangle.$$
It is clear from the above presentation that \( x_0 \) and \( x_1 \) are sufficient to generate the group, and we thus obtain the standard finite presentation \( \mathcal{F} \):

\[
\mathcal{F} = \langle x_0, x_1 | [x_1x_0^{-1}, x_0^{-1}x_1x_0], [x_1x_0^{-1}, x_0^{-2}x_1x_0^2] \rangle.
\]

The generators \( x_0 \), \( x_1 \) and \( x_n \) are depicted as homeomorphisms of the interval in Figure 1. Notice that the support of \( x_n \) is exactly the interval \([1 - \frac{1}{2^n}, 1]\). As a consequence, any element with support contained in \([0, 1 - \frac{1}{2^n}]\) will commute with \( x_n \), as elements of \( F \) with disjoint support always commute. Analyzing the relators in the finite presentation \( \mathcal{F} \), it is not hard to see that the support of \( x_1x_0^{-1} \) is \([0, \frac{3}{4}]\), while the supports of \( x_2 = x_0^{-1}x_1x_0 \) and \( x_3 = x_0^{-2}x_1x_0^2 \) are contained in \([\frac{3}{4}, 1]\), hence these elements commute.

![Figure 1. The generators \( x_0 \) and \( x_n \) of \( F \) as homeomorphisms of \([0, 1]\).](image)

With respect to the infinite presentation \( \mathcal{P} \), elements of \( F \) have a standard (infinite) normal form, given by

\[
x^{r_1}_{i_1}x^{r_2}_{i_2} \cdots x^{r_k}_{i_k}x^{-s_l}_{j_l} \cdots x^{-s_2}_{j_2}x^{-s_1}_{j_1},
\]

where \( r_i, s_i > 0 \), \( 0 \leq i_1 < i_2 \cdots < i_k \) and \( 0 \leq j_1 < j_2 \cdots < j_l \). This normal form is unique with the additional condition that when both \( x_i \) and \( x_i^{-1} \) occur, so does \( x_{i+1} \) or \( x_{i+1}^{-1} \), as discussed by Brown and Geoghegan in [BG]. We will always mean unique normal form when we refer to a word \( w \) in normal form. A positive word (resp. negative word) has unique normal form containing only positive (resp. negative) exponents.

For a thorough introduction to Thompson’s group \( F \) we refer the reader to [CFP].

2.2. Girth and limit groups. The girth of a group \( G \) with respect to a finite generating set \( S \) is defined to be the length of the shortest relator
satisfied in \((G, S)\). The girth of a finitely generated group \(G\) is the supremum of \(\text{girth}(G, S)\) over all finite generating sets \(S\) for \(G\). It is proven in [AK] that within the class of non-cyclic groups, those which are finitely generated and hyperbolic (or one-relator or linear) have infinite girth if and only if they are not virtually solvable.

We will use the notion of girth together with the following proposition of Stalder, which provides an algebraic formulation of convergence of a series of marked groups in this topology, to exhibit free limits of marked copies of Thompson’s group \(F\). If \((G, S)\) is a marked group and \(|S| = k\), we say that \(S\) is a marking of \(G\) of length \(k\).

**Proposition 2.1 ([S]).** Let \((\Gamma_n, S_n)\) be a sequence of marked groups on \(k\) generators. The following are equivalent:

1. \((\Gamma_n, S_n)\) is convergent in \(G_k\);
2. for all \(w \in F_k\) we have either \(w = 1\) in \(\Gamma_n\) for \(n\) large enough, or \(w \neq 1\) in \(\Gamma_n\) for \(n\) large enough, where \(F_k\) is the free group on \(k\) letters.

In the sections below, we will be interested not only in whether particular sequences of marked groups converge, but in the presentation of the limit group of such a convergent subsequence. The following proposition, while a restatement of the definition of convergence of a sequence of marked groups, states explicitly how we characterize this limit group.

**Proposition 2.2.** Let \((\Gamma_n, S_n)\) be a sequence of marked groups on \(k\) generators. Then \((\Gamma_n, S_n)\) converges to \((\Gamma, S)\) in \(G_k\) if for all words \(w \in F_k\)

1. if \(w = 1\) in \(\Gamma\), then \(w = 1\) in \(\Gamma_n\) for sufficiently large \(n\).
2. if \(w = 1\) in \(\Gamma_n\) for infinitely many \(n\), then \(w = 1\) in \(\Gamma\).

In the case that the limit group is a free group, the first property holds trivially, so we need only check the second. Combining the notion of girth with the above proposition yields a straightforward characterization of when a sequence of markings of a fixed group converges to a free group. Namely:

**Proposition 2.3.** If a group \(G\) has a sequence of markings \(S_n\) of length \(k\) so that the girth of \((G, S_n)\) goes to infinity with \(n\), then the sequence \((G, S_n)\) converges to \(F_k\) in \(G_k\).

3. Free limits of Thompson’s group \(F\)

The goal of this section is to prove that there is a sequence \(\{S_n\}\) of markings of \(F\) so that \((F, S_n)\) converges to the free group of rank \(k\) in \(G_k\), for all
$k \geq 3$. The methods used to prove this theorem yield an additional result. Namely they produce a different sequence of markings of $F$ so that the resulting sequence of marked groups converges to the free product $F * F_l$, the amalgamation of Thompson’s group $F$ with a free group of rank $l$ for any $l \geq 1$.

The main result is proven by exhibiting a sequence of markings of $F$ of a fixed length so that the girth of the group with respect to these markings approaches infinity. As a corollary we obtain that the same sequence of markings on $F$ converges to the free group of the appropriate rank. As the finiteness of the girth of a group is closely related to whether the group satisfies a law, we first recall the result of Brin and Squier [BS] (and reproven by Abért [A] and later by Esyp [E]) that Thompson’s group $F$ satisfies no law.

First define $W_{m,k}$ to be the set of all nontrivial reduced words in the free group $F(a, b_1, b_2, \cdots, b_{k-1})$ of rank $k$ of length at most $m$. The reason for using both $a$ and $b_j$ to denote the generators of the free group will be clear below.

**Proposition 3.1.** Fix $m \in \mathbb{N}$ and let $k \geq 2$. There exist $u_1, u_2, \cdots, u_k \in F$ so that for any $w \in W_{m,k}$ the word $w(u_1, u_2, \cdots, u_k) \in F$ is nontrivial.

**Proof.** Suppose that there are $l$ words $w_1, \cdots, w_l$ in $W_{m,k}$. As $F$ satisfies no group law, for each $i$ with $1 \leq i \leq l$ we can find $u_{i,1}^l, u_{i,2}^l, \cdots, u_{i,k}^l \in F$ so that $w_i(u_{i,1}^l, u_{i,2}^l, \cdots, u_{i,k}^l)$ is nontrivial. For any two dyadic rationals $a$ and $b$ with $a < b$, there is an isomorphism $\phi_{[a, b]}$ from $F$ to the isomorphic copy of $F$ supported on the interval $[a, b]$, which we denote $F_{[a, b]}$, as detailed in [BST].

Choosing dyadic rationals $0 = a_0 < a_1 < a_2 < \cdots < a_i < a_i = 1$, let $I_j = [a_j, a_{j+1}]$ for $1 \leq j < i$ and define $\phi_{I_j} : F \to F_{I_j}$ to be the corresponding isomorphisms. Let $u_{j,r} = \phi_{I_j}(u_{j,r})$ for $1 \leq r \leq k$ and $1 \leq j \leq l$. It is clear that the support of $u_{j,r}$ lies in the interval $I_j$.

For $1 \leq r \leq k$, define $w_r(x) = w_{j,r}(x)$ for $x \in I_j$. If $w \in W_{m,k}$, then $w(u_1, u_2, \cdots, u_k)$ must be nontrivial on at least one interval $I_j$ by construction. \hfill \Box

The next proposition shows that for any word $w = w(a,b_1,b_2,\cdots,b_{k-1}) \in W_{m,k}$, we can always find $u_1, u_2, \cdots, u_{k-1} \in F$ so that $w(x_0, u_1, u_2, \cdots, u_{k-1})$ is nontrivial. We first state a basic lemma regarding the construction of elements of $F$ with certain proscribed values. A proof of this lemma can be found in [CFP].

**Lemma 3.2 ([CFP], Lemma 4.2).** If $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ are partitions of $[0, 1]$ consisting
of dyadic rational numbers, then there exists \( f \in F \) so that \( f(x_i) = y_i \) for \( i = 0, 1, 2, \ldots, n \). Furthermore, if \( x_{i-1} = y_{i-1} \) and \( x_i = y_i \) for some \( i \) with \( 1 \leq i \leq n \), then \( f \) can be taken to be the identity on the interval \([x_{i-1}, x_i]\).

**Proposition 3.3.** Fix \( m \in \mathbb{N} \), \( 0 < \epsilon < \frac{1}{2} \), and \( k \geq 2 \). For any \( w = w(a, b_1, b_2, \ldots, b_{k-1}) \in W_{m,k} \), there exist elements \( u_1, u_2, \ldots, u_{k-1} \in F \) so that for \( 1 \leq i \leq k-1 \), we have \( w(x_0, u_1, u_2, \ldots, u_{k-1})|_{(0, \epsilon)} \neq 1 \) and \( \text{Support}(u_i) \subseteq [0, \epsilon) \).

**Proof.** As \( m \) is fixed, define \( N_m = \{-m, -(m-1), \ldots, 0, 1, 2, \ldots, m\} \). Choose \( I_0 \subset (0, \epsilon) \) to be a small closed interval whose endpoints lie in \( \mathbb{Z}^{\left\lfloor \frac{1}{2} \right\rfloor} \) with the property that the collection of intervals \( \{I_i = x_0^i(I_0)\} \) is pairwise disjoint for \( i \in N_m \), and all \( I_i \subset (0, \epsilon) \). Given \( w \in W_{m,k} \) we will construct \( u_1, u_2, \ldots, u_{k-1} \in F \) supported in \( \bigcup_{i \in N_m} I_i \) so that \( w(x_0, u_1, u_2, \ldots, u_{k-1}) \) is nontrivial.

More specifically, choosing \( x \) in the interior of \( I_0 \), we will define a sequence of elements \( u_1, u_2, \ldots, u_{k-1} \in F \) so that \( w(x_0, u_1, u_2, \ldots, u_{k-1})(x) \neq x \).

Each element \( u_j \) will be constructed so that

1. \( u_j \) preserves each interval, though not point-wise, that is, \( u_j(I_i) = I_i \) for \( i \in N_m \), and
2. \( u_j \) is the identity off the union of these intervals, that is, for any \( y \notin \bigcup_{i \in N_m} I_i \), we have \( u_j(y) = y \).

The construction will be accomplished by selecting, for each \( j \) and each \( i \), a (possibly empty) pair of sequences of points \( \alpha_1 < \alpha_2 < \cdots < \alpha_r \) and \( \beta_1 < \beta_2 < \cdots < \beta_r \) in the interior of \( I_i \), and applying Lemma 3.2 to define \( u_j|_{I_i} \).

To begin the construction, choose \( x \in \mathbb{Z}^{\left\lfloor \frac{1}{2} \right\rfloor} \) in the interior of \( I_0 \). Let

\[ w = a^{e_1}B_1a^{e_2}B_2 \cdots a^{e_r}B_r \]

be a word in the free group \( F(a, b_1, b_2, \ldots, b_{k-1}) \) where each \( B_j \) is a word in \( b_1^{\pm 1}, b_2^{\pm 1}, \ldots, b_{k-1}^{\pm 1} \). We allow \( e_1 = 0 \) and \( B_r = 1 \), but \( e_l \neq 0 \) for \( 1 < l \leq r \) and \( B_l \) is not the trivial word for \( 1 \leq l < r \).

Let \( B_s = b_{s,1}b_{s,2} \ldots b_{s,q_s} \) for \( b_{s,j} \in \{b_1^{\pm 1}, b_2^{\pm 1}, \ldots, b_{k-1}^{\pm 1}\} \). All points that are chosen in the construction below are assumed to lie both in the interior of an interval \( I_j \) and in \( \mathbb{Z}^{\left\lfloor \frac{1}{2} \right\rfloor} \). First choose an increasing sequence of dyadic rationals in the interior of \( I_0 \) beginning with \( x \) (which we relabel in keeping with our indexing scheme),

\[ x = y_r = y_r^{q_r+1} < y_r^{q_r} < y_r^{q_{r-1}} < \cdots < y_r^1 = x_r. \]
Note that the length of this sequence is one more than the length of $B_r$. Moving through the word $w$ from right to left, we now consider $a^{e_r}$, which indicates which interval we use to choose the next sequence of points. Recalling that $x_0$ will be substituted for $a$, this next sequence will be chosen in the interior of $x_0^{e_r}(I_0)$, beginning with the image of the final point in the previous sequence under $x_0^{e_r}$ and with length $|B_{r-1}| + 1$. More precisely, let $y_{r-1} = x_0^{e_r}(x_r) \in x_0^{e_r}(I_0) = I_j$ for $I_j \neq I_0$. Next, chose an increasing sequence of dyadic rationals in the interior of $I_j$,

$$y_{r-1} = y_{r-1}^{q_{r-1}+1} < y_{r-1}^{q_{r-1}} < \cdots < y_{r-1}^1 = x_{r-1}.$$ 

Continue in this way through the word $w$, constructing increasing sequences of points in the various intervals $I_j$. We remark that for each index $s$ the sequence constructed consists of more than one point, with the possible exception of $s = r$, that is, the initial sequence constructed. When $s \neq r$, we know that $B_s \neq 1$, hence $y_s < x_s$. Notice that it is possible for more than one such sequence to be chosen within a single interval $I_j$. If it is the case that two sequences of the form $\{y_j^i\}$ and $\{y_m^j\}$ for fixed $l$ and $m$ where $l < m$ both are chosen in $I_j$, then by construction, $y_m^j < y_l^j$ for any superscripts $s$ and $p$. In fact, we claim this inequality is strict. To see this, note that since $m > 1$, $e_m \neq 0$, and hence $m - 1 \neq l$. In particular, $l + 1 \neq r$, and hence $y_{l+1} < x_{l+1}$. Therefore, $y_m^s < y_{l+1} < x_{l+1} \leq y_l^p$, and so $y_m^s < y_l^p$, as desired.

Now for each pair of points $y_s^{i+1} < y_s^i$, we obtain a pair $(z, u_n(z))$ for some $n \in \{1, 2, \ldots, k-1\}$ as follows. If $b_{s,i} = b_n$, let $u_n(y_s^{i+1}) = y_s^i$, and if $b_{s,i} = b_n^{-1}$, let $u_n(y_s^i) = y_s^{i+1}$. We claim that if $(z_1, u_n(z_1))$ and $(z_2, u_n(z_2))$ are two such pairs, corresponding to two letters $b_1^{l_1}$ and $b_2^{l_2}$ in the word $w$, for some $n \in \{1, 2, \ldots, k-1\}$ and $\epsilon_1, \epsilon_2 \in \{+1, -1\}$ where $z_1 \leq z_2$, then in fact $z_1 \neq z_2$ and $u_n(z_1) < u_n(z_2)$.

To see this, first consider the case that both $b_1^{l_1}$ and $b_2^{l_2}$ occur within the same subword $B_j$. If $\epsilon_1 = \epsilon_2$, the claim is clear. If $\epsilon_1 \neq \epsilon_2$, since $B_j$ is freely reduced, $b_1^{l_1}$ and $b_2^{l_2}$ are not adjacent in the word. Hence, $b_1^{l_1} = b_{j,i}$ and $b_2^{l_2} = b_{j,l}$ with $l + 1 < i$, and $y_i^{j+1} < y_i^j < y_1^{j+1} < y_1^j$. Hence, regardless of the values of $\epsilon_1$ and $\epsilon_2$, since we have the setwise equalities $\{z_1, u_n(z_1)\} = \{y_i^{j+1}, y_i^j\}$ and $\{z_2, u_n(z_2)\} = \{y_1^{j+1}, y_1^j\}$, we see that $z_1 \neq z_2$ and $u_n(z_1) < u_n(z_2)$.

On the other hand, suppose $b_1^{l_1}$ is in the subword $B_j$ and $b_2^{l_2}$ is in the subword $B_k$ with $k \neq j$. Then if $z_1 \in I_i$ and $z_2 \in I_l$ with $i \neq l$, since $I_l \cap I_i = \emptyset$, the claim is clearly true. So suppose both $z_1$ and $z_2$ are in $I_i$. Then since $z_1 \leq z_2$, $k < j$. Then $\{z_1, u_n(z_1)\} = \{y_i^j, y_i^{j+1}\}$ and
\{z_2, u_n(z_2)\} = \{y_k^r, y_k^{s+1}\} for some superscripts \(r\) and \(s\). Then as previously established, \(z_1 < z_2\) and \(u_n(z_1) < u_n(z_2)\).

It follows from the last claim that for a given \(n \in \{1, 2, \ldots, k-1\}\), the collection of pairs \(\{(z_\alpha, u_n(z_\alpha))\}\) defined above satisfy the hypothesis Lemma 3.2. Namely, the domain points \(\{z_\alpha\}\) are all distinct, and if \(z_\alpha_1 < z_\alpha_2\) then \(u_n(z_\alpha_1) < u_n(z_\alpha_2)\). Therefore, we can apply Lemma 3.2 for any pair \(j\) and \(n\) to define \(u_n|_{I_j}\). If, for a given pair, no points have been chosen in \(I_j\) to be domain and range points for \(u_n\), then simply define \(u_n|_{I_j}\) to be the identity. Also, for points \(y\) outside all intervals \(I_j\), define \(u_n(y) = y\) for any \(n\). If it is the case that \(\exp(a, w)\), the sum of the exponents of all instances of \(a\) in the word \(w\), is zero then \(w(x_0, u_1, u_2, \cdots, u_{k-1})(x) \in I_0\), and hence by construction, \(w(x_0, u_1, u_2, \cdots, u_{k-1})(x) > x\). If \(\exp(a, w) \neq 0\), then \(w(x_0, u_1, u_2, \cdots, u_{k-1})(x) \notin I_0\), and hence \(w(x_0, u_1, u_2, \cdots, u_{k-1})(x) \neq x\). In either case, \(w(x_0, u_1, u_2, \cdots, u_{k-1})\) is nontrivial.

Note that since the elements \(u_1, u_2, \cdots, u_{k-1}\) constructed in Proposition 3.3 have support in a small interval close to zero, whereas \(x_1\) has support \([\frac{1}{2}, 1]\), the above proof is easily extended to obtain the following corollary.

**Corollary 3.4.** Fix \(m \in \mathbb{N}\), \(0 < \epsilon < \frac{1}{2}\), and let \(k \geq 2\). For any word \(w\) in the free group \(F(a, c, b_1, b_2, \cdots, b_{k-1})\) of rank \(k + 1 \geq 3\), which is reduced, has length at most \(m\) and contains at least one \(a\) or \(b_j^{-1}\) for some \(1 \leq j \leq k-1\), there exist \(u_1, u_2, \cdots, u_{k-1} \in F\) so that \(\text{Support}(u_i) \subseteq [0, \epsilon)\) for \(1 \leq i \leq k-1\) and \(w(x_0, x_1, u_1, u_2, \cdots, u_{k-1})|_{[0, \epsilon)}\) is nontrivial.

To prove this corollary, simply delete all occurrences of the letter \(c\) (corresponding to the generator \(x_1\) of \(F\)) and carry through the same construction of the elements \(u_i\) with the resulting word. In Corollary 3.3 the rank of the free group considered is at least 3. The above corollary would hold with \(x_1\) replaced by any element whose support was contained in \([\frac{1}{2}, 1]\).

We next extend Proposition 3.3 to a collection of words in the free group \(F(a, b_1, b_2, \cdots, b_{k-1})\) of length at most \(m\). The proof of this proposition is analogous to that of Proposition 3.1.

**Proposition 3.5.** Fix \(m \in \mathbb{N}\), \(k \geq 2\), \(0 < \epsilon < \frac{1}{2}\) and let \(w_1, w_2, \cdots, w_q \in \mathcal{W}_{m,k}\). There exists \(u_1, u_2, \cdots, u_{k-1} \in F\) with \(\text{Support}(u_j) \subseteq [0, \epsilon)\) for \(1 \leq j \leq k-1\) and \(w_i(x_0, u_1, u_2, \cdots, u_{k-1})|_{[0, \epsilon)} \neq 1\) for all \(1 \leq i \leq q\).

**Proof.** Choose the interval \(I_0\) as in the proof of Proposition 3.3 so that it is contained in \((0, \epsilon)\) and its translates under \(x_0^i\) are pairwise disjoint for \(i \in \mathcal{N}_m = \{-m, \cdots, m\}\) and all contained in \((0, \epsilon)\). Choose \(q\) pairwise disjoint subintervals \(J_1, J_2, \cdots, J_q\) of \(I_0\), all having endpoints which
are dyadic rationals, with the same properties as $I_0$, that is, the translates of each $J_i$ are pairwise disjoint under the above list of powers of $x_0$. Following the proof of Proposition 3.3, for each $i$ with $1 \leq i \leq q$ define elements $u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k-1}$ supported in $\bigcup_{s \in N_m} x_s^i(J_i)$ so that $w_i(x_0, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k-1})$ is nontrivial.

For $1 \leq j \leq k - 1$, define $u_j(x) = u_{i,j}(x)$ for $x \in \bigcup_{s \in N_m} x_s^i(J_i)$ over all $1 \leq l \leq q$, and $u_j(x) = x$ for all other $x \in [0,1]$. By construction, each $u_j$ is supported in $[0,\epsilon)$ for $1 \leq j \leq k - 1$. Moreover, as homeomorphisms of $[0,1]$ we have $w_r(x_0, u_1, u_2, \ldots, u_{k-1})|_{[0,\epsilon)} \neq 1$ for $1 \leq r \leq q$.

The proof of Proposition 3.3 extends naturally to the following corollary, analogous to Corollary 3.4.

**Corollary 3.6.** Fix $m \in \mathbb{N}$, $0 < \epsilon < \frac{1}{2}$, and $k \geq 2$. Let $w_1, w_2, \ldots, w_q \in F(a,c,b_1,b_2,\ldots,b_{k-1})$ all be words of length at most $m$, each containing at least one occurrence of $b_j$ or $b_j^{-1}$ for $1 \leq j \leq k - 1$. Then there exist $u_1, u_2, \ldots, u_{k-1} \in F$ with $\text{Support}(u_j) \subseteq [0,\epsilon)$ for $1 \leq j \leq k - 1$ and $w_i(x_0, x_1, u_1, u_2, \ldots, u_{k-1})|_{[0,\epsilon)} \neq 1$ for $1 \leq i \leq q$.

Corollary 3.6 yields our first result about possible limit groups arising from marked copies of Thompson’s group $F$.

**Corollary 3.7.** For any $k \geq 2$, there is a sequence of marked groups

$$G_m = \langle a, c, b_1, \ldots, b_{k-1} | R_1 = [ca^{-1}, a^{-1}ca],$$

$$R_2 = [ca^{-1}, a^{-2}ca^2], b_i = w_{i,m}(a, b) \text{ for } 1 \leq i \leq k - 1 \rangle$$

each isomorphic to Thompson’s group $F$, which converge to the marked group $G = \langle a, c, b_1, \ldots, b_{k-1} | R_1 = [ca^{-1}, a^{-1}ca], R_2 = [ca^{-1}, a^{-2}ca^2] \rangle$, the amalgamation of Thompson’s group $F$ and $F_{k-1}$, the free group of rank $k - 1 \geq 1$.

**Proof.** For a fixed value of $m$, Corollary 3.6 with $k \geq 2$ yields the elements $u_{1,m}, u_{2,m}, \ldots, u_{k-1,m}$ so that for any word $w$ in the free group of rank $k+1$ whose length is at most $m$, we know that $w(x_0, x_1, u_{1,m}, u_{2,m}, \ldots, u_{k-1,m})$ is nontrivial by construction. For each $i$, write $u_{i,m}$ as a word in $x_0$ and $x_1$. Replacing $x_0$ with $a$ and $x_1$ with $b$ yields the word $w_{i,m}$ in the presentation of $G_m$.

Let $w = w(a,c,b_1,\ldots,b_{k-1})$ be a word of length $r$ in $F_{k+1}$. If $w$ contains only the letters $a$ and $c$, then $w$ is trivial in $G_m$ for some $m$ if and only if it is trivial in $G_m$ for every $m$. On the other hand, if $w$ contains $b_i$ for any $i$, then for any $m > r$, by construction, $w(x_0, x_1, u_{1,m}, \ldots, u_{k-1,m})$ is not trivial in $G_m$. It then follows from Proposition 2.2 that the marked groups $G_m$ converge to the marked group $G$ given above. \qed
Theorem 3.8. Thompson’s group $F$ has infinite girth. Moreover, for any $l \geq 3$ there is a sequence of generating sets $S_{l,n}$ of length $l$ for $F$ so that the girth of $(F, S_{l,n})$ approaches infinity as $n$ approaches infinity.

Proof. Fix $l \in \mathbb{N}$ with $l \geq 3$. To prove the theorem we exhibit a family of generating sets of length $l$ for $F$ so that the girth with respect to these generating sets approaches infinity. This proves the second statement in the theorem, which includes the first statement in the theorem.

Choose $m \in \mathbb{N}$, and fix $\epsilon > 0$ so that $2^m \epsilon < \frac{1}{2}$. This choice is made initially so that later in the argument, the supports of certain elements are disjoint from $\text{supp}(x_1) = \left[\frac{1}{2}, 1\right]$.

Using Proposition 3.5 with $k = l - 1$ we can construct a set of elements $u_1, u_2, \ldots, u_{l-1} \in F$ so that for all $w(x_0, x_1, x_2, \ldots, x_{l-2}) \in W_{m^2 + l - 1}$ we have $w(x_0, u_1, u_2, \ldots, u_{l-2})$ nontrivial, and additionally, as a homeomorphism $w(x_0, u_1, u_2, \ldots, u_{l-2}^m)$ is nontrivial on the interval $(0, \epsilon)$.

Consider the following generating set of length $l$ for $F$:

$$S_{l,m} = \{\alpha = x_0, \beta = x_0 u_1, \gamma_1 = u_1, \ldots, \gamma_{l-2} = u_{l-2}\}.$$

Any word $w$ in the generators in $S_{l,m}$ and their formal inverses can be rewritten in the generators $x_0, x_1$ and $u_1, u_2, \ldots, u_{l-2}$ to form a nonempty reduced word, at the cost of increasing word length by a factor of $m + 2$. Namely,

$$w(\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_{l-2}) = w_1(x_0, x_1, u_1, u_2, \ldots, u_{l-2})$$

and if the length of $w$ is at most $m$, then the length of $w_1$ is at most $m^2 + 2m$. Moreover, $\alpha, \beta$ and the $\gamma_i$ have been chosen so that after replacement by $x_0, x_1$ and the $u_i$, not all the letters cancel. Namely, if all the generators canceled when $\beta$ was replaced by $x_0 u_1^m x_1$ then the word $w$ in the generators $S_{l,m}$ would have had length greater than $m$, contradicting the fact that it had length at most $m$.

Similarly, if we remove all instances of the letter $x_1$ from $w_1$ and reduce the resulting word, we obtain a new word $w_2(x_0, u_1, u_2, \ldots, u_{l-2})$ of length at most $m^2 + m$ and at least $1$. We note that $w_2$ is nontrivial, as we assumed that all words $w(x_0, u_1, u_2, \ldots, u_{l-2})$ of length at most $m^2 + m$ were nontrivial in $F$. As $\text{supp}(u_i) \subset [0, \epsilon)$ by construction and $\text{supp}(x_1) \subset [\frac{1}{2}, 1]$, these supports are disjoint. Since there are at most $m$ occurrences of the generator $x_0$ in $w_1$ and $2^m \epsilon < \frac{1}{2}$, for any $x \in (0, \epsilon)$, we see that $w_2(x) = w_1(x) \neq x$. As $w(x) = w_1(x)$, we have shown that $w(\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_{l-2})$ is nontrivial. Hence, the girth of $(F, S_{l,m})$ is at least
Therefore, the girth of \((F, S_{l,m})\) approaches infinity as \(m\) approaches infinity.

\[
\square
\]

As a direct consequence of Proposition 2.3, we obtain the following corollary.

**Corollary 3.9.** For each \(l \in \mathbb{N}, l \geq 3\), the sequence of marked groups
\[
G_m = (F, \{x_0, x_0 u_{1,m}^m x_1, u_{1,m}, \ldots, u_{l-2,m}\})
\]
converges to the free group \((F_l, \{a_1, a_2, \ldots, a_l\})\), where \(u_{1,m}, \ldots, u_{l-2,m}\) are the elements constructed above in the proof of Proposition 2.3 with \(k = l-1\).

## 4. Non-free limits of \(F\) within \(G_3\)

The results in this section are motivated by considering several natural sequences of markings of \(F\) within \(G_3\) of the form \(\{x_0, x_1, g_n\}\) for some \(g_n \in F\). In particular, we consider the cases \(g_n = x_n\), the \((n+1)\)-st generator in the infinite presentation for \(F\), and \(g_n = x_0^n\). However, the convergence of the resulting sequence of marked groups relies less on the actual elements chosen and more on their supports. Hence we are able to state convergence results for more general sequences of markings of \(F\).

As a corollary of Theorem 4.3 we see that \((F, \{x_0, x_1, x_n\})\) is convergent in \(G_3\) and obtain a presentation for the resulting limit group; in Theorem 4.5 we prove that \((F, \{x_0, x_1, x_0^n\})\) is convergent in \(G_3\) as well and give a presentation of the limit group. For consistency in the notation of the marking, we identify \(a = x_0, b = x_1\) and \(c = g_n\), the additional generator in the marking.

There are various characterizations of limit groups of sequences of marked finitely generated free groups in the literature. For instance, Kharlampovich and Myasnikov [KM2] prove that a group is a limit group (in the sense of Sela) if and only if it is a subgroup of an iterated extension of centralizers of a free group. Alternatively, the class of limit groups can be characterized by iterating the generalized double construction, which is defined below, as proven by Champetier and Guirardel in the following theorem, derived from work of Sela.

**Theorem 4.1** ([CG], Theorem 4.6). A group is a limit group if and only if it is an iterated generalized double.

To make this precise, we define the notion of a generalized double over a limit group \(L\) which is the limit of marked copies of free groups.
**Definition 4.2.** A generalized double over a limit group $L$ is a group $G = A \ast_C B$ (or $G = A \ast_C C$) such that both vertex groups $A$ and $B$ are finitely generated and

1. $C$ is a nontrivial abelian group whose images under both embeddings are maximal abelian in the vertex groups
2. there is an epimorphism $\varphi : G \to L$ which is one-to-one in restriction to each vertex group.

While there is no analogous characterization for limits of non-free groups, we note that in each of our theorems below there is an amalgamated product (or HNN extension) of copies of Thompson’s group $F$, or subgroups of $F$, over a maximal abelian subgroup which is a generalized double over the limit group obtained. Thus the same structure emerges in our limit groups as appears in the case of limits of marked free groups.

The first example of a convergent sequence of marked copies of $F$, given in Theorem 4.3, is a generalization of the natural sequence of markings $\{x_0, x_1, x_n\}$ of $F$ in $G_3$, that is, $a = x_0$, $b = x_1$ and $c = x_n$.

**Theorem 4.3.** Let

\[ G_n = \langle a, b, c | R_1 = [ba^{-1}, a^{-1}ba], R_2 = [ba^{-1}, a^{-2}ba^2], c^{-1}w_n \rangle, \]

where $w_n$ is a word in $a$ and $b$, such that viewed as an element of $F$ where $a = x_0$ and $b = x_1$, $w_n$ has support in $[t_n, 1]$, where $\lim_{n \to \infty} t_n = 1$, and $w_n$ maps $[\frac{3t_n^2}{4}, 1]$ linearly to $[\frac{3t_n^2}{4}, 1]$. Then the sequence $(G_n, \{a, b, c\})$ converges to $(G, \{a, b, c\})$, where

\[ G = \langle a, b, c | R_1, R_2, R_3 = [ca^{-1}, a^{-1}ca], R_4 = [ca^{-1}, a^{-2}ca^2], \]

and $[ba^{-1}, a^i ca^{-i}]$ for all $i \in \mathbb{Z}$.

**Proof.** Note first that $R_3$ and $R_4$ are true in $G_n$ for any $n$ for the following reason. Since $w_n$ maps $[\frac{3t_n^2}{4}, 1]$ linearly to $[\frac{3t_n^2}{4}, 1]$, the support of $ca^{-1}$ is $[0, \frac{1}{2}t_n]$. As the support of $a^{-i}ca^i$ is $a^{-i}(\text{Supp}(c))$ and the support of $c$ is contained in $[t_n, 1]$, we see that $a^{-i}[t_n, 1] \subset [\frac{1+t_n}{2}, 1]$ for $i \geq 1$. As the supports of these two elements are disjoint, they must commute and we obtain the relations $R_3$ and $R_4$.

Next, for a relator of the form $[ba^{-1}, a^i ca^{-i}]$, choose $N$ (and there are infinitely many such choices) so that $a^i[t_N, 1] \subseteq [3/4, 1]$. Then for any $n \geq N$, we see that $a^i ca^{-i}$, as a homeomorphism in $G_n$, has support in $[3/4, 1]$ and hence commutes in $G_n$ with $ba^{-1}$, whose support lies in $[0, 3/4]$. Therefore the relation $[ba^{-1}, a^i ca^{-i}]$ holds in $G_n$ for all $n \geq N$. 

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We follow Proposition 2.2 and first prove that if \( w \) is trivial in \( G \) then \( w \) is trivial in \( G_n \) for sufficiently large \( n \). Let \( w = w(a, b, c) \) be a word in \( a, b, c \) and their inverses which is the identity in \( G \). Then \( w \) may be expressed as a product of conjugates of finitely many relators of \( G \). But, for any finite set of relators, there is some \( N \) so that the relators all hold in \( G_n \) for \( n \geq N \), and thus \( w \) is trivial in \( G_n \) for all \( n \geq N \).

Now suppose \( w \) is a word in the letters \( a, b, c \) and their inverses which is the identity in \( G_n \) for infinitely many \( n \). First, using a finite number of relations if the form \([ba^{-1}, a'ca^{-i}]\), we may move all occurrences of \( c \) to the left of all occurrences of \( b \) with the penalty of increasing the number of occurrences of the generator \( a \) in the word. Namely, introducing pairs of the form \( a^{-i}a^i \) to the left or right of instances of the generator \( c \) within the word we can create terms of the form \( a'^{-i}a_i \) adjacent to terms of the form \( ba^{-1} \); as these expressions commute via the given relator, we can rewrite \( w \) in this way as \( w_1w_2 \), where \( w_1 \) is a word in the generators \( a \) and \( c \) and \( w_2 \) is a word in the generators \( a \) and \( b \). Choose \( M \) large enough so that the finite collection of relators used in this process all hold in \( G_n \) for \( n > M \), and thus we can rewrite \( w \) in \( G_n \) as well.

In both \( G_n \) and in \( G \), the elements \( a \) and \( c \) satisfy the relators \( R_3 \) and \( R_4 \). The subgroup \((a, c|R_3, R_4) \cong F \) and so we can rewrite \( w_1 \) in an infinite normal form using \( c_0 = a, c_1 = c, \ldots, c_{1+i} = a^{-i}ca^i, \ldots \) for \( i \geq 1 \). Similarly, we can rewrite \( w_2 \) using the relators \( R_1 \) and \( R_2 \) in the standard infinite normal form in the letters \( x_0 = a, x_1 = b, x_2, \ldots, x_{1+i} = a^{-1}ba^i, \ldots \) for \( i \geq 1 \). Suppose that \( w_1 = c_0^+w_1'c_0^- \), where \( w_1' \) has infinite normal form in \( c_i^\pm \) for \( i \geq 1 \), and \( w_2 = x_0^+w_2'x_0^- \), where \( w_2' \) has infinite normal form in \( x_i^\pm \) for \( i \geq 1 \). Combine the terms \( c_0^{-i} \) and \( x_0^i \), and use the relators \( R_1, R_2, R_3, \) and \( R_4 \), to move this combined power of \( x_0 \) completely to the left or right of the expression, depending on the sign of the exponent. As a result of the application of these relators, indices of other terms may be increased, and the final and initial exponents \( \epsilon \) and \( \beta \) may change as well. Without loss of generality we then assume that there is an \( M \in \mathbb{N} \) so that for \( n > M \) we can write \( w = w_1w_2 \) in both \( G \) and \( G_n \), where \( w_1 = c_0^+w_1', w_2 = w_2'x_0^- \), and \( w_1' \) has infinite normal form in \( c_i^\pm \), for \( i \geq 1 \), and \( w_2' \) has infinite normal form in \( x_i^\pm \), for \( i \geq 1 \).

Now suppose that a fixed word \( w \) is the identity in \( G_n \) for infinitely many \( n \). Then \( w \) is certainly the identity for infinitely many \( n > M \), where we can write \( w = w_1w_2 = c_0^+w_1'w_2'x_0^- \) in the form above in \( G_n \) for all \( n > M \). For any such \( n \), in \( G_n \), the word \( w \) represents a particular homeomorphism. As a homeomorphism, \( c_1 \) has support in \([t_n, 1]\), so \( c_n \) has support contained in \([t_n, 1]\) as well for all \( n \geq 2 \). Thus \( w_1' \) must also have support in \([t_n, 1]\).
Similarly, as the support of $x_1$ is $[\frac{1}{2}, 1]$ and the support of $x_n$ is contained in $[\frac{1}{2}, 1]$ for all $n \geq 2$, we see that $w'_2$ has support in $[\frac{1}{2}, 1]$. But then the slope of $w_1 w_2$ near zero will be $2^{\beta - \varepsilon}$; as this homeomorphism is the identity in $G_n$, we must have $\varepsilon = \beta$.

For each of the infinitely many $n > M$ for which $w$ is the identity in $G_n$, recalling that $x_0$ and $c_0$ are both equal to the generator $a$, we may conjugate $w$ by $a^\varepsilon$ to obtain the word $w'_1 w'_2$, which must also be the identity in $G_n$. We claim that $w'_2$ must in fact be the empty word. For if not, then thinking of $w'_2$ as a homeomorphism in $G_n$, there is some $x \in (0, 1)$ which is not fixed by $w'_2$. However, for sufficiently large $n$, $w'_2(x)$ will be outside of the support of $w'_1$, and hence $w'_1 w'_2$ will not fix $x$ in $G_n$ for such large $n$. But if $w'_2$ is the empty word, then $w'_1$ must be the empty word as well. But this means that in fact, for all $n > M$, the original $w$ can be written as $c_0^\varepsilon x_0^{-\varepsilon}$ in $G$, and hence can be transformed to the identity in $G$. \qed

Using the notation in the definition of the generalized double over a limit group, we remark that when $A = B = F$ and $C = \mathbb{Z}$, where both inclusions of $C$ in $A$ and $B$ map $C$ to the subgroup generated by $x_0$, then $A \ast_C B$ is a generalized double over the limit group $G$ obtained in Theorem 4.3. We note that as the support of $x_0$ is the entire interval $[0, 1]$, the subgroup generated by $x_0$ is a maximal abelian subgroup of $F$.

In the previous example, the additional generator $c$ in $G_n$ had support in a small neighborhood of 1 for large $n$. Alternatively, if we choose a sequence of additional generators to have supports in arbitrarily small intervals, close to zero but not including zero, we obtain the following convergent sequence of marked copies of $F$.

**Theorem 4.4.** Choose $g_n \in F$ to have support in $[r_n, s_n] \subseteq [0, 1]$, where $r_n < s_n < 2r_n$ and $\lim_{n \to \infty} r_n = 0$, and choose a word $w_n$ in $a^{\pm 1}, b^{\pm 1}$ so that $w_n(x_0, x_1) = g_n$. Let

$$G_n = \langle a, b, c \mid R_1 = [ba^{-1}, a^{-1}ba], R_2 = [ba^{-1}, a^{-2}ba^2], c^{-1}w_n(a, b) \rangle.$$  

Then the sequence $(G_n, \{a, b, c\})$ converges to $(G, \{a, b, c\})$, where $G = \langle a, b, c \mid R_1, R_2, [a^i ca^{-i}, c], [a^i ba^{-i}, c] \text{ for every } i \in \mathbb{Z} \rangle$.

**Proof.** First, suppose $w = 1$ in $G$. Then $w$ can be transformed to the empty word using only a finite number of relations of $G$. There exists some $M$ so that $s_n < 1/2$ for all $n \geq M$, and then it follows that $x_0([r_n, s_n])$ is disjoint from $[r_n, s_n]$. Therefore, for any $n \geq M$, the relations of the form $[a^i ca^{-i}, c]$ hold in $G_n$. On the other hand, $[a^i ba^{-i}, c] = 1$ holds in $G_n$ for sufficiently large $n$, so we may choose $N$ so that all of these relations hold in $G_n$ for $n \geq N$, and thus $w$ is trivial in $G_n$ for $n \geq N$. \hfill \qed
Now suppose \( w = 1 \) in \( G_n \) for infinitely many \( n \). By inserting pairs of the form \( a^{-1}a^i \) adjacent to certain instances of the generator \( c \) in the word \( w \) as in the proof of Theorem 4.4 and using the fourth relation given in the presentation for \( G \), the word \( w \) can be rewritten in the form \( w_1w_2 \) where \( w_1 \) is a word in the generators \( a \) and \( c \), and \( w_2 \) is a word in the generators \( a \) and \( b \).

As above, let \( c_0 = c \) and \( c_i = a^i c a^{-i} \) for \( i \geq 1 \). Then using the third type of relation in the presentation above for \( G \), the word \( w_1 \) can be rewritten as a word in the \( c_i^{\pm 1} \), at the expense of a power of \( a \) at the right, which we shift into \( w_2 \). So \( w_1 \) is a word in the \( c_i \), and \( w_2 \) is a word in \( a \) and \( b \). For every \( n \), as a homeomorphism in \( G_n \), the element \( w_1 \) has slope 1 near zero and near 1, so \( w_2 \) must be supported in \([\epsilon, 1 - \epsilon]\) for some \( \epsilon > 0 \). But for sufficiently large \( n \), we know that \( w_1 \) is supported in \([0, \epsilon]\) as a homeomorphism in \( G_n \). Therefore, it follows that \( w_2 \) must be the identity, and thus can be reduced to the empty word using \( R_1 \) and \( R_2 \). Therefore, \( w_1 \) is the identity in \( G_n \) for infinitely many \( n \). But in both \( G_n \) and in \( G \), the element \( c_i \) commutes with \( c_j \) for every \( i \) and \( j \). Since the supports of these elements are disjoint, \( \exp_i(w_1) \), the net exponent of all occurrences of \( c_i \), must be zero for every \( i \). Thus \( w_1 \) can be transformed to the empty word in \( G \) as well as \( G_n \). □

Using the notation in the definition of the generalized double over a limit group, taking \( A = F \), \( B = \mathbb{Z} \wr \mathbb{Z} = \langle a, c | [a^{-1}ca^i, a^{-j}ca^j] \rangle \) for all \( i, j \in \mathbb{Z} \) and \( C = \mathbb{Z} \), where the inclusions of \( C \) in \( A \) maps \( C \) to the subgroup generated by \( x_0 \), and the inclusion of \( C \) in \( B \) maps \( C \) to the subgroup generated by \( a \), the group \( A \rtimes_C B \) is a generalized double over the limit group \( G \) obtained in Theorem 4.3.

In the list of motivating “natural” sequences of markings, it remains to consider the case where the additional generator \( g_n \) of \( G_n \) is taken to be \( x_0^n \).

**Theorem 4.5.** Let

\[
G_n = \langle a, b, c | R_1 = [ba^{-1}, a^{-1}ba], R_2 = [ba^{-1}, a^{-2}ba^2], c^{-1}a^n \rangle.
\]

Then the sequence \( (G_n, \{a, b, c\}) \) converges to \( (G, \{a, b, c\}) \), where

\[
G = \langle a, b, c | R_1, R_2, [c, a], [(a^{-j}ba^j)a^{-1}, c^{-i}(a^{-k}ba^k)c^j], i \geq 1, j \geq 0, k \geq 0 \rangle.
\]

**Proof.** Note first that in both \( G \) and \( G_n \), \( a \) and \( b \) generate a subgroup isomorphic to \( F \), and that the relator \( [c, a] \) holds in \( G_n \) for all \( n \) as \( c = a^n \).

As above, we identify \( x_0 \) with \( a \), \( x_1 \) with \( b \), and recall that \( x_{i+1} = a^{-1}ba^i \); making these substitutions into the final relator in the presentation of \( G \) with \( c = a^n \), we see that this relator claims that \( x_{ni+k+1} \) commutes with \( x_{j+1}x_0^{-1} \). As the support of \( x_{ni+k+1} \) is easily seen to be \([1 - \frac{1}{2n+k+1}, 1]\) and
the support of \( x_{i+1}^{-1} x_0^{-1} \) is \([0, 1 - \frac{1}{2^{i+1}}]\), a relator of this form is satisfied in \(G_n\) as long as \(j + 2 < n i + k + 1\), that is, \(n \geq \frac{j-k+1}{i}\).

We note for later use that the last two types of relators together, in \(G\), yield relators of the form

\[
x_{j+1}(c^{-i}x_{k+2}e^j) = (c^{-i}x_{k+1}e^j)x_{j+1},
\]

for \(j \geq 0, i \geq 1, k \geq 0\). Furthermore, if \(n \geq (j - k + 1)/i\), then the relation holds in \(G_n\) as well.

Suppose \(w\) is a word in \(a, b, c\) and their inverses which is the identity in \(G\). Then it may be expressed as a product of conjugates of finitely many relators of \(G\). But, for any finite set of relators, there is some \(M\) so that the relators all hold in \(G_n\) for \(n \geq M\), so \(w\) is also the identity in \(G_n\) for all \(n \geq M\).

Given any word \(w\) in \(a, b, c\) and their inverses, \(w\) can be expressed, in both \(G\) and \(G_n\) for any \(n\), as

\[
c^{m_1}w_1c^{n_2}w_2 \cdots c^{n_k}w_k,
\]

where for each \(i\), \(w_i\) is a word in \(a^\pm 1\) and \(b^\pm 1\) and \(n_j \neq 0\) for \(2 \leq j \leq k\). As usual, using the notation \(x_0 = a, x_1 = b, x_{1+i} = a^{-i}ba^i\), we may assume for each \(i\) that \(w_i\) is a word in the standard infinite normal form for \(F\). Next, since \(x_0\) commutes with \(c\) in both \(G\) and \(G_n\), and then also using relators involving just the \(x_j\), we may assume (changing the \(w_i\)’s without renaming) that \(w\) is of the form

\[
x_0^a c^{n_1}w_1c^{n_2}w_2 \cdots c^{n_k}w_kx_0^{-b},
\]

where \(a\) and \(b\) are positive integers and \(w_i\) is a word in infinite normal form without \(x_0^\pm 1\).

Now suppose that \(w\) is the identity in \(G_n\) for infinitely many \(n\). Then it follows that the total exponent sum of \(x_0\) must be zero for those indices \(n\), in other words, \((a - b) + (n_1 + n_2 + \cdots + n_k)n = 0\) for each of those indices \(n\). Therefore \(a = b\) and \(n_1 + n_2 + \cdots + n_k = 0\). But \(w = id\) in \(G_n\) if and only if \(x_0^{-a}w_0^n = id\), so we know that

\[
w' = c^{n_1}w_1c^{n_2}w_2 \cdots c^{n_k}w_k,
\]

where \(w_i\) is a word in infinite normal form without \(x_0^\pm 1\), is the identity in \(G_n\) for infinitely many \(n\). Moreover, conjugating if necessary, we may assume that \(n_i \neq 0\) (for \(i \neq 1\)) and \(w_i\) is not the empty word, for all \(i\). Now since \(n_1 + n_2 + \cdots + n_k = 0\), we may rewrite \(w'\) as:

\[
w' = (c^{n_1}w_1c^{-n_1})(c^{n_1+n_2}w_2c^{-(n_1+n_2)}) \cdots (c^{n_1+\cdots+n_{k-1}}w_{k-1}c^{-(n_1+\cdots+n_{k-1})})(w_k).
\]
We claim that the word $w'$ must also be the identity in $G$. For if not, suppose that amongst the words of this form which are the identity in $G_n$ for infinitely many $n$ and are not the identity in $G$, the word $w'$ has $k$ minimal. Next, let $a = \max(n_1, n_1 + n_2, \ldots, n_1 + n_2 + \cdots + n_{k-1})$, and since $w'$ is the identity (in $G_n$ or $G$) if and only if $c^{-a}w'c^a$ is the identity, we may assume $w'$ is of the form:

$$(c^{-m_1}w_1c^{m_1})(c^{-m_2}w_2c^{m_2}) \cdots (c^{-m_k}w_kc^{m_k}),$$

where $m_i \geq 0$ for all $i$, and at least one $m_i = 0$. For each value of $i$ where $m_i = 0$, the subword $(c^{-m_i}w_ic^{m_i}) = w_i = p_iq_i$, where $p_i$ (resp. $q_i$) is a positive word (resp. a negative word) in normal form involving no $x_0^\pm 1$. Thus, in $G$, using finitely many relators of the form

$$x_j(c^{-i}x_{k+1}c^i) = (c^{-i}x_kc^i)x_j$$

for $j \geq 1, i \geq 1$, we can move $p_i$ to the left and $q_i$ to the right of the expression, until eventually $w$ can be written (reindexing) as $p(l)c^{-i}w_ic^{m_i}q$, where $l < k$ and $p$ (respectively $q$) is a positive (respectively negative) word in infinite normal form. Since this can be done in $G$ using only finitely many relations, for sufficiently large $n$ it can be done in $G_n$ as well, so we may assume it can be done in $G_n$ for infinitely many $n$. Now notice that by choosing the minimal value of the index $n$ to be perhaps even larger, we can ensure that once we replace $c$ by $x_0^n$ in $G_n$, the product $\Pi_{i=1}^l c^{-m_i}w_ic^{m_i}$, when written in the standard infinite normal form, involves only the generators $x_j$ with $j$ much larger that the subscripts of the generators in $p$ or in $q$. But since $w$ is the identity for infinitely many $n$, it follows that $p = q^{-1}$. Hence, $\Pi_{i=1}^l c^{-m_i}w_ic^{m_i}$, with $l < k$, is the identity in $G_n$ for infinitely many $n$, but is not the identity in $G$, which is a contradiction, as we assumed that $k$ was minimal. □

Using the notation in the definition of the generalized double over a limit group, we remark that when $A = F$, and $C = \mathbb{Z}$, where both inclusions of $C$ in $A$ map $C$ to the subgroup generated by $x_0$, then $A * C$ is a generalized double over the limit group $G$ obtained in Theorem 4.5.

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