Asymptotic analysis of non-autonomous discrete hungry integrable systems

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Discrete hungry integrable systems have an interesting application in the computation of eigenvalues of totally non-negative (TN) matrices. In the case where the system’s variables are restricted to be all positive, they or their combinations converge to TN matrix eigenvalues as the discrete-time variable goes to infinity. To accelerate the convergence, arbitrary parameters are introduced that act as implicit shifts of origin without loss of the positivity of the variables in matrix similarity transformations. Discrete hungry integrable systems with the arbitrary parameters depending on independent variables are called non-autonomous. In this study, we show the convergence of the solution to non-autonomous discrete hungry integrable systems to matrix eigenvalues without assuming positivity of the variables. Thus, we present convergence theorems, which can contribute to computing eigenvalues of general matrices, not just TN matrices. To this end, we show determinant solutions to two non-autonomous discrete hungry integrable systems. We also clarify a Bäcklund transformation, which relates these two non-autonomous discrete hungry integrable systems.

Keywords: asymptotic behaviour; determinant solution; discrete hungry Lotka–Volterra system; discrete hungry Toda equation.

1. Introduction

The discrete hungry Toda (dhToda) equations and the discrete hungry Lotka–Volterra (dhLV) systems are two representative discrete hungry integrable systems. While the famous discrete Toda (dToda) equation was considered first in studies of discrete integrable systems, the discrete Lotka–Volterra (dLV) system was extended first to a hungry scheme. There is not an exact definition of integrability of the discrete
integrable systems; however, the dhLV systems which are just the discrete Bogoyavlensky lattices satisfy a discrete analogue of Liouville integrability according to Suris [1, 2].

The dLV system is a time-discretization of the integrable Lotka–Volterra (LV) system, which describes the population of each species in predator–prey relationships at any time. For \((2m − 1)\) species, the LV system assumes that the 1st, 2nd, \ldots, \((2m − 2)\)th species prey only on the 2nd, 3rd, \ldots, \((2m − 1)\)th species, respectively. There is no predator for the first species and no prey for the \((2m − 1)\)th species.

The two extended LV systems, where each species may prey on one or more species, are called hungry LV (hLV) systems [3–5]. Time-discretizations of these systems are known as dhLV systems, where the simple dLV system is a special case of the dhLV systems. Hama et al. [6] distinguished the two systems, which differ in whether the majority of species survives over the passage of time, by designating the dhLV I and dhLV II systems.

One of the dhToda equations first appeared in the study of box and ball systems [7]. In particular, it is derived from an inverse ultra-discretization of the ultra-discrete integrable system that describes a numbered box and ball system. Since this dhToda equation can be transformed to the dhLV I system using the Bäcklund transformation [8], it is sometimes called the dhToda I equation. The other, known as the dhToda II equation, was defined as the corresponding dhToda equation to the dhLV II system.

These four systems are all applicable to computing eigenvalues of totally non-negative (TN) matrices, where TN matrices are square matrices with all minors positive [9–11]. Under the initial settings given from entries of the TN matrices, as the discrete-time variable goes to infinity, some of the variables of the discrete hungry integrable systems or their combinations converge to TN eigenvalues, and the others converge to 0.

Fukuda et al. [10] and Sumikura et al. [11] introduced shifts of origin into algorithms based on the dhToda I and dhToda II equations, respectively, to accelerate the rate of convergence. That is, to design effective TN eigenvalue algorithms, they included arbitrary parameters, leading to the so-called non-autonomous dhToda I and dhToda II equations, respectively. The non-autonomous dhToda I and dhToda II variables given from the TN matrices are all positive at the initial discrete time. The positivity of the non-autonomous dhToda I and dhToda II variables at any discrete time then holds under a suitable choice of arbitrary parameters. Fukuda et al. [10] and Sumikura et al. [11] thus focused on the case where the non-autonomous dhToda I and dhToda II variables are always positive. It is to be noted here that the non-autonomous dhToda I and dhToda II equations can generate similarity transformations involving implicit shifts of general matrices, not just TN matrices. In fact, the convergence to matrix eigenvalues was shown not to be restricted to the case where the autonomous dhToda II variables are all positive [12]. Though a few algorithms specialized in computing eigenvalues of TN matrices have been designed, no algorithm specialized, for example, in the case of the so-called M-matrices [13], to the best of our knowledge, have been found yet. Many of the other non-symmetric matrices have not been even theoretically classified. Generally, eigenvalues of non-symmetric matrices are thus computed using the versatile QR algorithm [14, 15].

Despite the potential property of the dhToda II equation, the convergence in the non-autonomous dhToda II equation has not been discussed without requiring the positivity of the non-autonomous dhToda II variables. The positivity of the non-autonomous dhToda I and dhToda II variables contribute to the derivation of the convergence theorems in Fukuda et al. [10] and Sumikura et al. [11], so the theorems are not easily generalized. In this article, we therefore observe the asymptotic behaviour of the non-autonomous dhToda I variables from the viewpoint of studies of discrete integrable systems as the discrete-time variable goes to infinity. That is, we give determinant expressions of the non-autonomous dhToda I variables, and then present a generalized convergence theorem for the non-autonomous dhToda I variables without
assuming the positivity. We also clarify determinant expressions and asymptotic convergence of dhLV₁ variables.

This article focuses only on the non-autonomous dhToda₁ equation and the dhLV₁ system, between which we obtain a Bäcklund transformation. For simplicity and following some conventions, we hereinafter refer to these as the non-autonomous dhToda equation and the dhLV system, respectively.

The remainder of this article is organized as follows. In Section 2, we associate an infinite sequence with respect to two discrete-time variables with matrix eigenvalues, and then examine properties of extended Hankel determinants expressed using elements of the infinite sequence. In Section 3, by observing extended Hadamard polynomials based on the extended Hankel determinants, we show the determinant solution to the non-autonomous dhToda equation. In Section 4, we also clarify an eigenvalue problem associated with the non-autonomous dhToda equation. In Section 5, by symmetrizing the extended Hadamard polynomials, we derive the determinant solution to the dhLV system and a Bäcklund transformation between the non-autonomous dhToda equation and the dhLV system. In Section 6, by asymptotically expanding the determinant solutions to the non-autonomous dhToda equation and the dhLV system, we complete the convergence theorems for the non-autonomous dhToda and dhLV variables. We also illustrate an application of the non-autonomous dhToda equation to computing eigenvalues of non-positive matrices with non-positive entries. Finally, in Section 7, we present our conclusion.

2. Moments and extended Hankel determinants with the two discrete-time variables

In this section, we first introduce an infinite complex sequence \( \{f_s(t)\}_{s,t=0}^{\infty} \) with respect to two types of discrete-time variables \( s \) and \( t \), and then give an expression of the general term of the infinite sequence \( \{f_s(t)\}_{s,t=0}^{\infty} \). Next, we define the extended Hankel determinants associated with the infinite sequence \( \{f_s(t)\}_{s,t=0}^{\infty} \), and finally present an expansion of the extended Hankel determinants \( H_k^{(s,t)} \) in terms of the general term of the infinite sequence \( \{f_s(t)\}_{s,t=0}^{\infty} \). Solutions to several discrete integrable systems are known to be expressed using determinants associated with infinite sequences. Elements of the infinite sequence are often defined by certain inner products such as expected values of powers of differences between random variables and their statistical or probabilistic averages. Thus, the elements of the infinite sequences are sometimes called moments, hereinafter denoted by \( f_s(t) \).

For arbitrary distinct constants \( \lambda_1, \lambda_2, \ldots, \lambda_m \), let us introduce an \( m \)-degree polynomial with respect to \( z \in \mathbb{C} \),

\[
p(z) := (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m). \tag{2.1}
\]

The polynomial \( p(z) \) can then be regarded as the characteristic polynomial of an \( m \)-by-\( m \) matrices with distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \). Obviously, the polynomial \( p(z) \) can be expanded as

\[
p(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \tag{2.2}
\]

where \( a_1, a_2, \ldots, a_m \) are complex constants.
Let us consider the case where the moment sequence \( \{f_s^{(i)}\}_{s,i=0}^\infty \) satisfies linear equations with respect to the coefficients \( a_1, a_2, \ldots, a_m \) appearing in (2.2),

\[
f_{s+Mm} + \sum_{i=1}^m a_i f_{i+(m-i)M}^{(i)} = 0, \quad s, t = 0, 1, \ldots
\]

(2.3)

Moreover, let us assume that \( f_s^{(i)} \) are transformed to \( f_s^{(i+1)} \) as

\[
f_s^{(i+1)} = f_{s+M}^{(i)} - \mu^{(i)} f_s^{(i)}, \quad s, t = 0, 1, \ldots
\]

(2.4)

where \( \mu^{(i)} \) are arbitrary complex. Obviously, if \( f_0^{(0)}, f_1^{(0)}, \ldots, f_{M-1}^{(0)} \) are given, then any element of the moment sequence \( \{f_s^{(i)}\}_{s,i=0}^\infty \) is uniquely fixed. The following proposition gives an expression concerning general terms of the moments \( f_s^{(i)} \) satisfying the linear dependences (2.3) and (2.4).

**Proposition 2.1** Let us assume that the moments \( f_s^{(i)} \) satisfy the linear dependences (2.3) and (2.4). Then, the moments \( f_s^{(i)} \) can be expressed using \( \lambda_1, \lambda_2, \ldots, \lambda_m \) as

\[
f_s^{(i)} = \sum_{p=1}^m c_p^{(s \mod M)} \lambda_p^{\frac{s}{M}} \rho_p^{(i)} \lambda_p^{\frac{i}{M}}, \quad s, t = 0, 1, \ldots
\]

(2.5)

where \( c_1^{(i)}, c_2^{(i)}, \ldots, c_m^{(i)} \) are constants given using \( f_0^{(0)}, f_1^{(0)}, \ldots, f_{M-1}^{(0)} \) by

\[
\begin{pmatrix}
  c_1^{(i)} \\
  c_2^{(i)} \\
  \vdots \\
  c_m^{(i)}
\end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_m^{-1} \end{pmatrix}^{-1} \begin{pmatrix} f_0^{(i)} \\ f_1^{(i)} \\ \vdots \\ f_{M-1}^{(i)} \end{pmatrix}, \quad j = 0, 1, \ldots, M - 1
\]

(2.6)

and \( \rho_1^{(i)}, \rho_2^{(i)}, \ldots, \rho_m^{(i)} \) are constants such that

\[
\rho_1^{(i)} := 1, \quad \rho_2^{(i)} := 1, \quad \ldots, \quad \rho_m^{(i)} := 1,
\]

(2.7)

\[
\rho_i^{(i)} = (\lambda_i - \mu^{(0)})(\lambda_i - \mu^{(1)}) \cdots (\lambda_i - \mu^{(i-1)}), \quad i = 1, 2, \ldots, m, \quad t = 1, 2, \ldots
\]

(2.8)

**Proof.** Using (2.5), we can rewrite the left-hand side of the linear dependence (2.3) as

\[
\sum_{p=1}^m c_p^{(s+Mm \mod M)} \rho_p^{(i)} \lambda_p^{\frac{s+Mm}{M}} + \sum_{i=1}^m a_i \left( \sum_{p=1}^m c_p^{(s+(m-i)M \mod M)} \rho_p^{(i)} \lambda_p^{\frac{s+(m-i)M}{M}} \right)
\]

\[
= \sum_{p=1}^m c_p^{(s \mod M)} \rho_p^{(i)} \lambda_p^{\frac{s}{M}} + \sum_{i=1}^m a_i \left( \sum_{p=1}^m c_p^{(s \mod M)} \rho_p^{(i)} \lambda_p^{\frac{s}{M}} \lambda_p^{\frac{i}{M}} \right)
\]

\[
= \sum_{p=1}^m c_p^{(s \mod M)} \rho_p^{(i)} \lambda_p^{\frac{s}{M}} \left( \lambda_p^m + \sum_{i=1}^m a_i \lambda_p^{m-i} \right).
\]

(2.9)
From (2.2), it is obvious that

$$\lambda_p^m + \sum_{i=1}^{m} a_i \lambda_p^{m-i} = p(\lambda_p).$$  \hspace{1cm} (2.10)

Combining (2.9) and (2.10) with \(p(\lambda_p) = 0\), we thus see that the moments \(f_s^{(i)}\) in (2.5) satisfy the linear dependence (2.3). Setting \(s = 0, 1, \ldots, m - 1\) in (2.5) with \(i = 0\) and considering \(\rho_p^{(0)} := 1\), we derive

$$\begin{pmatrix}
    f_s^{(0)} \\
    f_{s+M}^{(0)} \\
    \vdots \\
    f_{s+(m-1)M}^{(0)}
\end{pmatrix} = 
\begin{pmatrix}
    1 & 1 & \cdots & 1 \\
    \lambda_1 & \lambda_2 & \cdots & \lambda_m \\
    \vdots & \vdots & \ddots & \vdots \\
    \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_m^{-1}
\end{pmatrix}
\begin{pmatrix}
    c_1^{(i)} \\
    c_2^{(i)} \\
    \vdots \\
    c_m^{(i)}
\end{pmatrix}, \quad j = 0, 1, \ldots, M - 1. \hspace{1cm} (2.11)
$$

On the right-hand side of (2.11), the coefficient matrix involving distinct \(\lambda_1, \lambda_2, \ldots, \lambda_m\) is non-singular because the coefficient matrix is the Vandermonde matrix. Therefore, by considering the inverse of the coefficient matrix in (2.11), we have (2.6). From (2.4) and (2.5), we easily obtain

$$\rho_i^{(t+1)} = (\lambda_i - \mu^{(i)}) \rho_i^{(t)}, \quad i = 1, 2, \ldots, m,$$  \hspace{1cm} (2.12)

which immediately leads to (2.7).

Proposition 2.1 claims that the moments \(f_s^{(i)}\) have \((M + 1)m\) arbitrary constants \(\lambda_1, \lambda_2, \ldots, \lambda_m\) and \(c_1^{(j)}, c_2^{(j)}, \ldots, c_m^{(j)}\) for \(j = 0, 1, \ldots, M - 1\). The general term of the moments sequence \(\{f_s^{(i)}\}_{s,t=0}^{\infty}\) involving \(\lambda_1, \lambda_2, \ldots, \lambda_m\) and \(c_1^{(j)}, c_2^{(j)}, \ldots, c_m^{(j)}\) varies from previously used because the moment sequence \(\{f_s^{(i)}\}_{s,t=0,1,\ldots}^{\infty}\) in this article is an extension of one in our previous article [12]. Moreover, for \(k = 1, 2, \ldots, m + 1\), let us consider determinants of square matrices of order \(k\) involving the moments \(f_s^{(i)}\),

$$H_k^{(s,t)} := 
\begin{vmatrix}
    f_s^{(i)} & f_{s+M}^{(i)} & \cdots & f_{s+k-1}^{(i)} \\
    f_{s+M}^{(i)} & f_{s+M+1}^{(i)} & \cdots & f_{s+M+k-1}^{(i)} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{s+(k-1)M}^{(i)} & f_{s+(k-1)M+1}^{(i)} & \cdots & f_{s+(k-1)M+k-1}^{(i)}
\end{vmatrix}, \quad s, t = 0, 1, \ldots, \hspace{1cm} (2.13)$$

where \(H_1^{(s,t)} := 0\) and \(H_0^{(s,t)} := 1\). Since \(H_k^{(s,t)}\) with \(M = 1\) are just standard Hankel determinants appearing in solution expressions of the dToda equation and the dLV system [20], we can regard \(H_k^{(s,t)}\) as an extension of the Hankel determinants. Hereinafter, \(H_k^{(s,t)}\) refers to extended Hankel determinants.

Considering the linear dependence (2.3), we derive the following proposition for the extended Hankel determinants \(H_k^{(s,t)}\) with the case \(k = m + 1\).

PROPOSITION 2.2 It holds that

$$H_{m+1}^{(s,t)} = 0, \quad s, t = 0, 1, \ldots,$$  \hspace{1cm} (2.14)
Proof. Multiplying the 1st, 2nd, \ldots, mth rows in the extended Hankel determinants $H_{m+1}^{(s,j)}$ by $a_m$, $a_{m-1}$, \ldots, $a_1$, respectively, and adding these to the $(m + 1)$th row, we obtain

$$H_{m+1}^{(s,j)} = \begin{vmatrix} f_s^{(i)} & f_{s+1}^{(i)} & \cdots & f_{s+m}^{(i)} \\ f_{s+1}^{(i)} & f_{s+2}^{(i)} & \cdots & f_{s+m+1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+m}^{(i)} & f_{s+m+1}^{(i)} & \cdots & f_{s+2m}^{(i)} \\ \sum_{i=0}^{m} a_{s+(m-i)M} f_{s+(m-i)M}^{(i)} & \sum_{i=0}^{m} a_{s+(m-i)M+1} f_{s+(m-i)M+1}^{(i)} & \cdots & \sum_{i=0}^{m} a_{s+(m-i)M+m} f_{s+(m-i)M+m}^{(i)} \end{vmatrix},$$

where $a_0 := 1$. Since it is obvious from (2.3) that the $(m + 1, 1), (m + 1, 2), \ldots, (m + 1, m + 1)$ entries become 0, we thus have (2.14). 

With the help of Proposition 2.1, we also obtain the following proposition concerning expansions of the extended Hankel determinants $H_{k}^{(s,j)}$.

**Proposition 2.3** Let us assume the moments $f_{s}^{(i)}$ satisfy the linear dependences (2.3) and (2.4). Then, for $\ell = 0, 1, \ldots$ and $j = 0, 1, \ldots, M - 1$, the extended Hankel determinants $H_{k}^{(s,j)}$ can be expanded as,

$$H_{k}^{(M+j,s)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} K_{i_1 i_2 \ldots i_k}^{(j)} (\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} )^{\ell} \rho_{i_1}^{(j)} \rho_{i_2}^{(j)} \cdots \rho_{i_k}^{(j)},$$

where

$$K_{i_1 i_2 \ldots i_k}^{(j)} := \begin{vmatrix} z_{i_1}^{(j)} & z_{i_2}^{(j)} & \cdots & z_{i_k}^{(j)} \\ z_{i_1}^{(j+1)} & z_{i_2}^{(j+1)} & \cdots & z_{i_k}^{(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_1}^{(j+k-1)} & z_{i_2}^{(j+k-1)} & \cdots & z_{i_k}^{(j+k-1)} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{k-1} & \lambda_{i_2}^{k-1} & \cdots & \lambda_{i_k}^{k-1} \end{vmatrix},$$

$$C_{p}^{(s) \equiv c_{p \pmod{M}} \lambda_{p}} = \begin{vmatrix} \lambda_{p} \end{vmatrix}, \quad p = 1, 2, \ldots, m, \quad s = 0, 1, \ldots$$

**Proof.** Combining Proposition 2.1 with (2.13), we can express the extended Hankel determinants $H_{k}^{(M+j,s)}$ as

$$H_{k}^{(M+j,s)} = \begin{vmatrix} \sum_{p=1}^{m} c_{p}^{(j)} \rho_{p}^{(j)} \lambda_{p}^{\ell} & \sum_{p=1}^{m} c_{p}^{(j+1)} \rho_{p}^{(j)} \lambda_{p}^{\ell} & \cdots & \sum_{p=1}^{m} c_{p}^{(j+k-1)} \rho_{p}^{(j)} \lambda_{p}^{\ell} \\ \sum_{p=1}^{m} c_{p}^{(j)} \rho_{p}^{(j)} \lambda_{p}^{\ell+1} & \sum_{p=1}^{m} c_{p}^{(j+1)} \rho_{p}^{(j)} \lambda_{p}^{\ell+1} & \cdots & \sum_{p=1}^{m} c_{p}^{(j+k-1)} \rho_{p}^{(j)} \lambda_{p}^{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{p=1}^{m} c_{p}^{(j)} \rho_{p}^{(j)} \lambda_{p}^{\ell+k-1} & \sum_{p=1}^{m} c_{p}^{(j+1)} \rho_{p}^{(j)} \lambda_{p}^{\ell+k-1} & \cdots & \sum_{p=1}^{m} c_{p}^{(j+k-1)} \rho_{p}^{(j)} \lambda_{p}^{\ell+k-1} \end{vmatrix}.$$
The extended Hankel determinants $H_{k}^{(M+j,t)}$ can be rewritten as

$$H_{k}^{(M+j,t)} = \det \left\{ \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_m \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_m^{k-1}
\end{array} \right\} \left( \begin{array}{cccc}
\rho_1^{(i)} & \lambda_1^{(i)} \\
\rho_2^{(i)} & \lambda_2^{(i)} \\
\vdots & \vdots \\
\rho_m^{(i)} & \lambda_m^{(i)}
\end{array} \right) \times \left( \begin{array}{cccc}
-\tilde{c}_1^{(j)} & -\tilde{c}_1^{(j+1)} & \cdots & -\tilde{c}_1^{(j+k-1)} \\
-\tilde{c}_2^{(j)} & -\tilde{c}_2^{(j+1)} & \cdots & -\tilde{c}_2^{(j+k-1)} \\
\vdots & \vdots & & \vdots \\
-\tilde{c}_m^{(j)} & -\tilde{c}_m^{(j+1)} & \cdots & -\tilde{c}_m^{(j+k-1)}
\end{array} \right) \right\}. \quad (2.18)$$

By applying the Cauchy–Binet formula to the right-hand side and using the elementary transformations of the determinants, we immediately obtain

$$H_{k}^{(M+j,t)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} (\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k})^{\ell} \rho_{i_1}^{(i_1)} \rho_{i_2}^{(i_2)} \cdots \rho_{i_k}^{(i_k)} \times \left( \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_k} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{i_1}^{k-1} & \lambda_{i_2}^{k-1} & \cdots & \lambda_{i_k}^{k-1}
\end{array} \right) \left( \begin{array}{cccc}
\tilde{c}_1^{(j)} & \tilde{c}_1^{(j+1)} & \cdots & \tilde{c}_1^{(j+k-1)} \\
\tilde{c}_2^{(j)} & \tilde{c}_2^{(j+1)} & \cdots & \tilde{c}_2^{(j+k-1)} \\
\vdots & \vdots & & \vdots \\
\tilde{c}_m^{(j)} & \tilde{c}_m^{(j+1)} & \cdots & \tilde{c}_m^{(j+k-1)}
\end{array} \right) \right\}. \quad (2.19)$$

Thus, we have (2.15). □

3. Extended Hadamard polynomials and the non-autonomous dhToda equation

In this section, we define extended Hadamard polynomials associated with the extended Hankel determinants $H_{k}^{(s,t)}$. We then relate roots of the extended Hadamard polynomials to matrix eigenvalues, and find identities for the extended Hadamard polynomials involving the extended Hankel determinants $H_{k}^{(s,t)}$. Through deriving the non-autonomous dhToda equation from the polynomial identities, we finally clarify the solution to the non-autonomous dhToda equation in terms of the extended Hankel determinants $H_{k}^{(s,t)}$.

For $k = 1, 2, \ldots, m$, under the assumption $H_{k}^{(s,t)} \neq 0$, let us define polynomials of degree $k$ with respect to $z \in \mathbb{C}$ as

$$\mathcal{H}_{k}^{(s,t)}(z) := \frac{H_{k}^{(s,t)}(z)}{H_{k}^{(s,t)}}, \quad s, t = 0, 1, \ldots, \quad (3.1)$$

where $H_{k}^{(s,t)}(z)$ are polynomials of degree $k$ with respect to $z$ given by

$$H_{k}^{(s,t)}(z) := \begin{vmatrix}
f_{s}^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k-1}^{(t)} & 1 \\
f_{s+1}^{(t)} & f_{s+1}^{(t)} & \cdots & f_{s+k}^{(t)} & z \\
\vdots & \vdots & & \vdots & \vdots \\
f_{s+kM}^{(t)} & f_{s+kM}^{(t)} & \cdots & f_{s+kM+k-1}^{(t)} & z^k
\end{vmatrix} \quad (3.2)$$
It is obvious that \( \mathcal{H}^{(s,t)}_k(z) \) are monic polynomials of degree \( k \). Since the polynomials \( \mathcal{H}^{(s,t)}_k(z) \) with \( M = 1 \) and without \( t \) are the simple Hadamard polynomials given by Henrici [17], hereinafter, we refer to \( \mathcal{H}^{(s,t)}_k(z) \) as the extended Hadamard polynomials. The following proposition relates the extended Hadamard polynomials \( \mathcal{H}^{(s,t)}_k(z) \) with \( k = m \) to the characteristic polynomial \( p(z) \) of \( m \)-by-\( m \) matrices.

**Proposition 3.1** The extended Hadamard polynomials \( \mathcal{H}^{(s,t)}_m(z) \) coincide with the characteristic polynomial of matrices having eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \), namely,

\[
\mathcal{H}^{(s,t)}_m(z) = p(z), \quad s, t = 0, 1, \ldots
\]

**Proof.** Setting \( f^{(s,t)}_k := (f^{(s)}_k, f^{(s)}_{k+1}, \ldots, f^{(s)}_{k+(k-1)M})^\top \) and \( z_t := (1, z, z^2, \ldots, z^{k-1})^\top \), we can express the polynomials \( H^{(s,t)}_m(z) \) as

\[
H^{(s,t)}_m(z) = \begin{vmatrix}
  f^{(s,t)}_{m+1} & f^{(s+1,t)}_{m+1} & \cdots & f^{(s+m-1,t)}_{m+1} & z_{m+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(s,t)}_m & f^{(s+1,t)}_m & \cdots & f^{(s+m-1,t)}_m & z_m \\
  f^{(s,t)}_{m-1} & f^{(s+1,t)}_{m-1} & \cdots & f^{(s+m-1,t)}_{m-1} & z_{m-1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(s,t)}_1 & f^{(s+1,t)}_1 & \cdots & f^{(s+m-1,t)}_1 & z_1 \\
  f^{(s,t)}_0 & f^{(s+1,t)}_0 & \cdots & f^{(s+m-1,t)}_0 & z_0 
\end{vmatrix}. \tag{3.4}
\]

Multiplying the 1st, 2nd, \ldots, \( m \)th rows on the right-hand side of (3.4) by \( a_m, a_{m-1}, \ldots, a_1 \), respectively, and adding these to the \( (m+1) \)th row, we obtain

\[
H^{(s,t)}_m(z) = \begin{vmatrix}
  f^{(s,t)}_{m+1} & f^{(s+1,t)}_{m+1} & \cdots & f^{(s+m-1,t)}_{m+1} & z_{m+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(s,t)}_m & f^{(s+1,t)}_m & \cdots & f^{(s+m-1,t)}_m & z_m \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(s,t)}_1 & f^{(s+1,t)}_1 & \cdots & f^{(s+m-1,t)}_1 & z_1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f^{(s,t)}_0 & f^{(s+1,t)}_0 & \cdots & f^{(s+m-1,t)}_0 & z_0 
\end{vmatrix}. \tag{3.5}
\]

From the linear dependence (2.3), it is obvious that the \((m+1, 1), (m+1, 2), \ldots, (m+1, m)\) entries become 0. From (2.1) and (2.2), we see that the \( (m+1, m+1) \) entry is just the characteristic polynomial \( p(z) \) of matrices with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \). Thus, we can rewrite the polynomials \( H^{(s,t)}_m(z) \) using the extended Hankel determinants \( H^{(s,t)}_m(z) \) and the characteristic polynomial \( p(z) \) as \( H^{(s,t)}_m(z) = p(z)H^{(s,t)}_m(z) \). Combining this with the definition of the extended Hadamard polynomials \( \mathcal{H}^{(s,t)}_m(z) = H^{(s,t)}_m(z)H^{(s,t)}_m(z) \), we therefore have (3.3).

\[\square\]

Proposition 3.1 implies that \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are roots of the extended Hadamard polynomial \( \mathcal{H}^{(s,t)}_m(z) \), namely, \( \mathcal{H}^{(s,t)}_m(\lambda_i) = 0 \) for \( i = 1, 2, \ldots, m \). The following proposition gives discrete-time evolutions with respect to \( s \) for the extended Hadamard polynomials \( \mathcal{H}^{(s,t)}_k(z) \).

**Proposition 3.2** ([12, Proposition 3.1]) Let us assume that the moment sequence \( \{f^{(s)}_i\}_{i=0}^\infty \) makes the extended Hankel determinants \( H^{(s,t)}_k(z) \) be all non-zero. Then, for \( s, t = 0, 1, \ldots \), the extended Hadamard polynomials \( \mathcal{H}^{(s,t)}_k(z) \) satisfy

\[
\begin{align*}
\mathcal{H}^{(s+1,t)}_k(z) &= \mathcal{H}^{(s,t)}_k(z) + q^{(s,t)}_k \mathcal{H}^{(s,t)}_{k-1}(z), \quad k = 1, 2, \ldots, m, \\
\mathcal{H}^{(s,t)}_k(z) &= \mathcal{H}^{(s+1,t)}_k(z) + e^{(s,t)}_k \mathcal{H}^{(s,t)}_{k-1}(z), \quad k = 0, 1, \ldots, m,
\end{align*}
\]

where \( q^{(s,t)}_k \) and \( e^{(s,t)}_k \) are given using the extended Hankel determinants \( H^{(s,t)}_k(z) \) as,

\[
\begin{align*}
q^{(s,t)}_k &:= \frac{H^{(s,t)}_{k-1}H^{(s+1,t)}_k}{H^{(s+M,t)}_kH^{(s,t)}_{k-1}}, \quad k = 1, 2, \ldots, m, \\
e^{(s,t)}_k &:= \frac{H^{(s,t)}_{k+1}H^{(s+1,t)}_{k-1}}{H^{(s,t)}_kH^{(s+1,t)}_{k-1}}, \quad k = 0, 1, \ldots, m.
\end{align*}
\]
We can regard the identities (3.5) in Proposition 3.2 as the recursion formula generating the discrete-time evolutions of the extended Hadamard polynomials from $\mathcal{H}^{(s,t)}_k(z)$ to $\mathcal{H}^{(s+M,t)}_k(z)$. According to the theory of orthogonal polynomials [16], we can recognize the identities (3.5) with $M = 1$ as the Christoffel transformations for the simple Hadamard polynomials. Since the identities (3.6) generate inverse discrete-time evolutions for the identities (3.5), we see that (3.6) with $M = 1$ are just the Geronimus transformations for the simple Hadamard polynomials. Considering the Christoffel and Geronimus transformations simultaneously, we derive the simple discrete Toda equation [18]. Extending $M$ from 1 to arbitrary integers, we can obtain the dToda equation [12].

Focusing on identities for the minors of the polynomials $H^{(s,t)}_k(z)$, we obtain the following proposition for discrete-time evolutions concerning the extended Hadamard polynomials $\mathcal{H}^{(s,t)}_k(z)$ with respect to not $s$ but $t$.

**Proposition 3.3** Let us assume that the moment sequence $\{s_j\}_{s_j=0}^{\infty}$ makes the extended Hankel determinants $H^{(s,t)}_k(z)$ be all non-zero. Then, for $s, t = 0, 1, \ldots$, the extended Hadamard polynomials $\mathcal{H}^{(s,t)}_k(z)$ satisfy

$$
(z - \mu^{(t)})\mathcal{H}^{(s,t+1)}_{k-1}(z) = \mathcal{H}^{(s,t)}_k(z) + Q^{(s,t)}_k \mathcal{H}^{(s,t)}_{k-1}(z), \quad k = 1, 2, \ldots, m, \quad (3.9)
$$

where $Q^{(s,t)}_k$ are given using the extended Hankel determinants $H^{(s,t)}_k(z)$ as

$$
Q^{(s,t)}_k := \frac{H^{(s,t)}_k H^{(s,t+1)}_{k-1}}{H^{(s,t)}_{k-1} H^{(s,t+1)}_k}, \quad k = 1, 2, \ldots, m. \quad (3.10)
$$

**Proof.** Let $H^{(s,t)}_k(z) \left[ \begin{array}{c} i_1, i_2, \ldots \\ j_1, j_2, \ldots \end{array} \right]$ denote determinants obtained by deleting $i_1, i_2, \ldots$ rows and $j_1, j_2, \ldots$ columns of the polynomials $H^{(s,t)}_k(z)$. Then, it follows that

\[
H^{(s,t)}_k(z) \left[ \begin{array}{c} k + 1 \\ k + 1 \end{array} \right] = \begin{vmatrix} f^{(s,t)}_k & f^{(s+1,t)}_{k+1} & \cdots & f^{(s+k-1,t+1)}_{k-1} \\ f^{(s+1,t)}_k & f^{(s+2,t+1)}_{k+1} & \cdots & f^{(s+k,t+2)}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(s+k-1,t+1)}_k & f^{(s+k,t+2)}_{k+1} & \cdots & f^{(s+k,t)}_{k-1} \end{vmatrix} = H^{(s,t)}_k(z),
\]

\[
H^{(s,t)}_k(z) \left[ \begin{array}{c} 1 \\ k \end{array} \right] = \begin{vmatrix} f^{(s,t)}_k & f^{(s+1,t)}_{k+1} & \cdots & f^{(s+k,t+1)}_{k-1} \\ f^{(s+1,t)}_k & f^{(s+2,t+1)}_{k+1} & \cdots & f^{(s+k-1,t+2)}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(s+k,t+1)}_k & f^{(s+k-1,t+2)}_{k+1} & \cdots & f^{(s+k-1,t)}_{k-1} \end{vmatrix} = H^{(s,t)}_k(z)(z - \mu^{(t)})z_k = (z - \mu^{(t)})H^{(s,t+1)}_{k-1}(z),
\]

\[
H^{(s,t)}_k(z) \left[ \begin{array}{c} 1 \\ k \end{array} \right] = \begin{vmatrix} f^{(s,t)}_k & f^{(s+1,t)}_{k+1} & \cdots & f^{(s+k-1,t+1)}_{k-1} \\ f^{(s+1,t)}_k & f^{(s+2,t+1)}_{k+1} & \cdots & f^{(s+k-2,t+2)}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(s+k-1,t+1)}_k & f^{(s+k-2,t+2)}_{k+1} & \cdots & f^{(s+k-1,t)}_{k-1} \end{vmatrix} = H^{(s,t)}_{k+1}(z),
\]

\[
H^{(s,t)}_k(z) \left[ \begin{array}{c} 1 \\ k+1 \end{array} \right] = \begin{vmatrix} f^{(s,t)}_k & f^{(s+1,t)}_{k+1} & \cdots & f^{(s+k-1,t+1)}_{k-1} \\ f^{(s+1,t)}_k & f^{(s+2,t+1)}_{k+1} & \cdots & f^{(s+k-2,t+2)}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(s+k-1,t+1)}_k & f^{(s+k-2,t+2)}_{k+1} & \cdots & f^{(s+k,t)}_{k-1} \end{vmatrix} = H^{(s,t+1)}_k(z),
\]

Thus, we can rewrite Jacobi’s identity [19] as

$$
(z - \mu^{(t)})H^{(s,t)}_k H^{(s,t+1)}_{k-1}(z) = H^{(s,t+1)}_{k-1} H^{(s,t)}_k(z) + H^{(s,t+1)}_k H^{(s,t)}_{k-1}(z).
$$
Dividing both sides by $H_k^{(s,t)} H_{k-1}^{(s+1,t)}$ and recalling that $\mathcal{H}_k^{(s,t)}(z) = H_k^{(s,t)}(z) / H_k^{(s,t)}$, we therefore have (3.9) and (3.10).

With the help of Proposition 3.3, we have the following theorem that gives the non-autonomous dhToda equation involving the parameter $\mu^{(t)}$.

**Theorem 3.4** Let us assume that the moment sequence $\{f^{(t)}_j\}_{j=0}^{\infty}$ makes the extended Hankel determinants $H_k^{(s,t)}$ be all non-zero. Then, the variables $q_k^{(s,t)}$, $e_k^{(s,t)}$ and $Q_k^{(s,t)}$ satisfy, for $s, t = 0, 1, \ldots$,

$$
\begin{align*}
q_k^{(s,t+1)} + Q_k^{(s,t)} &= q_k^{(s,t)} + Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m - 1, \\
q_k^{(s,t+1)} Q_k^{(s,t)} &= q_k^{(s,t)} Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m,
\end{align*}
$$

(3.11)

and

$$
\begin{align*}
e_k^{(s,t+1)} + Q_k^{(s,t)} &= e_k^{(s,t+1)} + Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m, \\
e_k^{(s,t+1)} Q_k^{(s,t)} &= e_k^{(s,t+1)} Q_k^{(s+1,t)}, & k = 1, 2, \ldots, m - 1,
\end{align*}
$$

(3.12)

where $Q_1^{(s,t)} = q_1^{(s,t)} - \mu^{(t)}$.

**Proof.** Replacing $t$ with $t + 1$ in (3.5), multiplying $(z - \mu^{(t)})$ on both sides and using (3.9) in the first and second terms on the right-hand side, we obtain

$$
z(z - \mu^{(t)}) \mathcal{H}_k^{(s+1,t+1)} = \mathcal{H}_k^{(s,t)} + (q_k^{(s,t+1)} + Q_k^{(s,t+1)}) \mathcal{H}_k^{(s,t)} + q_k^{(s,t+1)} Q_k^{(s,t)} \mathcal{H}_k^{(s,t)}.
$$

(3.13)

Furthermore, by replacing $s$ with $s + M$ in (3.9), multiplying $z$ on both sides and using (3.5) in the first and second terms on the right-hand side, we derive

$$
z(z - \mu^{(t)}) \mathcal{H}_k^{(s+M,t+1)} = \mathcal{H}_k^{(s,t)} + (q_k^{(s,t+1)} + Q_k^{(s+M,t)}) \mathcal{H}_k^{(s,t)} + q_k^{(s,t+1)} Q_k^{(s+M,t)} \mathcal{H}_k^{(s,t)}.
$$

(3.14)

Observing (3.13) and (3.14), we have

$$
(q_k^{(s,t+1)} + Q_k^{(s,t)} \mathcal{H}_k^{(s,t)}) + q_k^{(s,t+1)} Q_k^{(s,t)} \mathcal{H}_k^{(s,t)} = (q_k^{(s,t+1)} + Q_k^{(s+M,t)}) \mathcal{H}_k^{(s,t)} + q_k^{(s,t+1)} Q_k^{(s+M,t)} \mathcal{H}_k^{(s,t)}.
$$

(3.15)

Since $\mathcal{H}_k^{(s,t)}(z)$ and $\mathcal{H}_k^{(s,0)}(z)$ are monic polynomials of degree $k$ and $k - 1$, respectively, it is obvious that they are linearly independent. Combining this with $\mathcal{H}_k^{(s,t)}(z) \neq 0$ for any $s$ and $t$, we thus obtain (3.11). Similarly, by using (3.6) in the first and second terms on the right-hand side in (3.9), we obtain

$$
(z - \mu^{(t)}) \mathcal{H}_k^{(s+1,t+1)} = \mathcal{H}_k^{(s+1,t)}(z) + (e_k^{(s,t)} + Q_k^{(s,t)} \mathcal{H}_k^{(s+1,t)} + e_k^{(s+1,t)} Q_k^{(s+1,t)} \mathcal{H}_k^{(s+1,t)}).
$$

(3.16)

The replacements $t$ with $t + 1$ in (3.6) and $s$ with $s + 1$ in (3.9), respectively, immediately lead to

$$
\mathcal{H}_k^{(s+1,t+1)} = \mathcal{H}_k^{(s+1,t+1)}(z) + e_k^{(s,t)} \mathcal{H}_k^{(s+1,t+1)}(z),
$$

(3.17)

$$
(z - \mu^{(t)}) \mathcal{H}_k^{(s+1,t+1)} = \mathcal{H}_k^{(s+1,t)}(z) + Q_k^{(s+1,t)} \mathcal{H}_k^{(s+1,t)}(z).
$$

(3.18)
By multiplying \((z - \mu^{(l)})\) on both sides in (3.17) and using (3.18) in the first and second terms on the right-hand side, we derive

\[
(z - \mu^{(l)}) \mathcal{H}_k^{(s,l+1)}(z) = \mathcal{H}_k^{(s,l)}(z) + (e_k^{(s,l+1)} + Q_k^{(s,l+1)} \mathcal{H}_k^{(s,l+1)}(z) + e_k^{(s,l+1)} Q_k^{(s,l+1)} \mathcal{H}_k^{(s,l+1)}(z). \tag{3.19}
\]

Combining (3.16) with (3.19), we have (3.12). Since \(Q_1^{(s,l)}\) are expressed as (3.10), we easily see \(Q_1^{(s,l)} = H_1^{(s,l+1)}/H_1^{(s,l)} = f_s^{(l+1)}/f_s^{(l)}.\) Combining this with the linear dependence (2.4) and \(q_1^{(s,l)} = H_1^{(s+M,l)}/H_1^{(s,l)} = f_{s+M}^{(l)} / f_s^{(l)},\) we have \(Q_1^{(s,l)} = q_1^{(s,l)} - \mu^{(l)}.\)

It is easy to check that (3.11) with \(s = 0\) and (3.12) with \(s = 0, 1, \ldots, M - 1\) are mathematically equivalent to the non-autonomous dhToda equation appearing in Sumikura et al. [11]. We thus conclude that the solution to the non-autonomous dhToda equation is given using the extended Hankel determinants \(H_1^{(s,l)}\) as (3.7) with \(s = 0\) and (3.8) with \(s = 0, 1, \ldots, M - 1.\)

From the viewpoint of numerical stability, we employ the auxiliary variables

\[
D_k^{(0,l)} := \frac{Q_k^{(0,l+1)}}{Q_k^{(0,l)}},
\tag{3.20}
\]

\[
F_k^{(s,l)} := \frac{Q_k^{(s,l+1)}}{Q_k^{(s,l)}}, \quad s = 0, 1, \ldots, M - 1. \tag{3.21}
\]

With the help of (3.20) and (3.21), we can reduce subtractions in the non-autonomous dhToda equation (3.11) with \(s = 0\) and (3.12) with \(s = 0, 1, \ldots, M - 1\) via the differential form proposed by Sumikura et al. [11]. Combining (3.7) and (3.10) with (3.20), we obtain the determinant expression of the auxiliary variables \(D_k^{(0,l)},\)

\[
D_k^{(0,l)} = \frac{H_k^{(0,l+1)}}{H_k^{(0,l)}} \left( \frac{H_k^{(l+1)}}{H_k^{(l)}} - \frac{H_k^{(0,l+1)}}{H_k^{(0,l)}} \right). \tag{3.22}
\]

Similarly, from (3.8), (3.10) and (3.21), it follows that

\[
F_k^{(s,l)} = \frac{H_k^{(s,l+1)}}{H_k^{(s,l)}} \left( \frac{H_k^{(s,l+1)}}{H_k^{(s,l)}} - \frac{H_k^{(s+1,l+1)}}{H_k^{(s+1,l+1)}} \right). \tag{3.23}
\]

4. The related eigenvalue problem

In this section, by examining the extended Hadamard polynomials \(\mathcal{H}_k^{(s,l)}(z)\), we describe the eigenvalue problem of Hessenberg matrices associated with the non-autonomous dhToda equation.

For \(s, t = 0, 1, \ldots,\) let us introduce \(m\)-by-\(m\) bidiagonal matrices involving the non-autonomous dhToda variables \(e_1^{(s,t)}, \ldots, e_{m-1}^{(s,t)}\) and \(q_1^{(s,t)}, \ldots, q_m^{(s,t)}\),

\[
L^{(s,t)} := \begin{pmatrix}
1 \\
e_1^{(s,t)} & 1 \\
& \ddots & \ddots \\
& & e_{m-1}^{(s,t)} & 1
\end{pmatrix}, \quad R^{(s,t)} := \begin{pmatrix}
q_1^{(s,t)} & 1 \\
& q_2^{(s,t)} & \ddots \\
& & \ddots & 1 \\
& & & q_m^{(s,t)}
\end{pmatrix}. \tag{4.1}
\]
From the viewpoint of the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$, we then obtain a proposition concerning the eigenpairs of $A^{(s,t)} := L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)}$.

**Proposition 4.1** Eigenvalues of $A^{(s,t)}$ coincide with the roots $\lambda_i$ of the polynomial $p(z)$. Moreover, eigenvectors corresponding to $\lambda_i$ are $\mathcal{H}_k^{(s,t)}(\lambda_i) := (\mathcal{H}_0^{(s,t)}(\lambda_i), \mathcal{H}_1^{(s,t)}(\lambda_i), \ldots, \mathcal{H}_{m-1}^{(s,t)}(\lambda_i))^T$. Namely,

$$A^{(s,t)} \mathcal{H}_k^{(s,t)}(\lambda_i) = \lambda_i \mathcal{H}_k^{(s,t)}(\lambda_i), \quad i = 1, 2, \ldots, m. \quad (4.2)$$

**Proof.** From the identities (3.5) with $z = \lambda_i$, it follows that

$$\begin{align*}
&\begin{cases}
\lambda_i \mathcal{H}_0^{(s+M,t)}(\lambda_i) = \mathcal{H}_0^{(s,t)}(\lambda_i) + q_{1,t}^{(s,t)} \mathcal{H}_1^{(s,t)}(\lambda_i), \\
\lambda_i \mathcal{H}_1^{(s+M,t)}(\lambda_i) = \mathcal{H}_2^{(s,t)}(\lambda_i) + q_{2,t}^{(s,t)} \mathcal{H}_1^{(s,t)}(\lambda_i), \\
\vdots \\
\lambda_i \mathcal{H}_{m-1}^{(s+M,t)}(\lambda_i) = \mathcal{H}_m^{(s,t)}(\lambda_i) + q_m^{(s,t)} \mathcal{H}_{m-1}^{(s,t)}(\lambda_i).
\end{cases} 
\end{align*} \quad (4.3)$$

Since $\mathcal{H}_m^{(s,t)}(\lambda_i) = 0$, we can express (4.3) using the upper bidiagonal matrix $R^{(s,t)}$ as

$$\lambda_i \mathcal{H}_0^{(s+M,t)}(\lambda_i) = R^{(s,t)} \mathcal{H}_0^{(s+1,t)}(\lambda_i), \quad i = 1, 2, \ldots, m. \quad (4.4)$$

Similarly, the identities (3.6) with $z = \lambda_i$ lead to

$$\mathcal{H}_k^{(s,t)}(\lambda_i) = L^{(s,t)} \mathcal{H}_k^{(s+1,t)}(\lambda_i), \quad i = 1, 2, \ldots, m. \quad (4.5)$$

Multiplying both sides of (4.4) by $L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)}$ from the left, we obtain

$$\lambda_i L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} \mathcal{H}_k^{(s+M,t)}(\lambda_i) = A^{(s,t)} \mathcal{H}_k^{(s,t)}(\lambda_i), \quad i = 1, 2, \ldots, m. \quad (4.6)$$

Using (4.5) repeatedly, we can rewrite the left-hand side of (4.6) as $\lambda_i \mathcal{H}_k^{(s,t)}(\lambda_i)$. Thus, we have (4.2). \qed

With the help of (4.4) and (4.5) in the proof of Proposition 4.1, we derive a proposition for the two types of similarity transformations of $A^{(s,t)}$.

**Proposition 4.2** The similarity transformations for $A^{(s,t)}$ with respect to $s$ are given as

$$A^{(s+M,t)} = R^{(s,t)} A^{(s,t)} (R^{(s,t)})^{-1}, \quad (4.7)$$

$$A^{(s+1,t)} = (L^{(s,t)})^{-1} A^{(s,t)} L^{(s,t)}. \quad (4.8)$$

**Proof.** For $s, t = 0, 1, \ldots$, by multiplying both sides of (4.5) by $R^{(s,t)}$ from the left and considering (4.4) and (4.5) on the left-hand side, we obtain

$$\lambda_i L^{(s+M,t)} \mathcal{H}_k^{(s+M+1,t)}(\lambda_i) = R^{(s,t)} L^{(s,t)} \mathcal{H}_k^{(s+1,t)}(\lambda_i), \quad i = 1, 2, \ldots, m. \quad (4.9)$$

Using (4.4) again, we can rewrite the left-hand side of (4.9) as $L^{(s+M,t)} R^{(s+1,t)} \mathcal{H}_k^{(s+1,t)}(\lambda_i)$. Recalling that $\mathcal{H}_k^{(s+1,t)}(\lambda_i)$ are eigenvector corresponding to $\lambda_i$, we see that $\mathcal{H}_k^{(s+1,t)}(\lambda_i) \neq 0$. Thus, it follows that

$$L^{(s+M,t)} R^{(s+1,t)} = R^{(s,t)} L^{(s,t)}, \quad s, t = 0, 1, \ldots. \quad (4.10)$$
Using (4.10) repeatedly, we can gradually transform $A^{(s+M,t)}$ as follows

$$A^{(s+M,t)} = L^{(s+M,t)} L^{(s+M+1,t)} \cdots L^{(s+2M-1,t)} R^{(s+M,t)}$$

$$= L^{(s+M,t)} L^{(s+M+1,t)} \cdots L^{(s+2M-2,t)} R^{(s+M-1,t)} L^{(s+M-1,t)}$$

$$\vdots$$

$$= R^{(s,t)} L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)}$$

$$= R^{(s,t)} L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)} (R^{(s,t)})^{-1}.$$

Therefore, by noting that $L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)} = A^{(s,t)}$, we have (4.7).

Similarly, by using (4.10) for $A^{(s+1,t)}$, we obtain

$$A^{(s+1,t)} = (L^{(s,t)})^{-1} L^{(s+1,t)} L^{(s+2,t)} \cdots L^{(s+M-1,t)} R^{(s,t)} L^{(s,t)}.$$  \hspace{1cm} (4.11)

Applying (4.11) to $A^{(s,t)} = L^{(s,t)} L^{(s+1,t)} \cdots L^{(s+M-1,t)} R^{(s,t)}$, we thus derive (4.8).

Now, let us introduce $m$-by-$m$ upper bidiagonal matrices involving $Q_1^{(s,t)}$, $Q_2^{(s,t)}$, \ldots, $Q_m^{(s,t)}$, not $q_1^{(s,t)}$, $q_2^{(s,t)}$, \ldots, $q_m^{(s,t)}$, like so

$$R^{(s,t)} := \begin{pmatrix}
Q_1^{(s,t)} & 1 \\
Q_2^{(s,t)} & \ddots \\
& \ddots & 1 \\
& & Q_m^{(s,t)}
\end{pmatrix}. \hspace{1cm} (4.12)$$

Considering Proposition 3.3 with $z = \lambda_i$, we obtain

$$R^{(s,t)} H^{(s,t)}(\lambda_i) = (\lambda_i - \mu^{(t)}) H^{(s,t+1)}(\lambda_i).$$  \hspace{1cm} (4.13)

Equation (4.13) with (4.4) and (4.5) gives relationships of $R^{(s,t)}$ to $L^{(s,t)}$ and $R^{(s,t)}$.

**Theorem 4.3** The similarity transformation for $A^{(s,t)}$ with respect to $t$ is given by

$$A^{(s,t+1)} = R^{(s,t)} A^{(s,t)} (R^{(s,t)})^{-1}. \hspace{1cm} (4.14)$$

**Proof.** By substituting (4.13) and (4.5) into $(\lambda_i - \mu^{(t)}) H^{(s,t+1)}(\lambda_i)$ in this order, we obtain $(\lambda_i - \mu^{(t)}) H^{(s,t+1)}(\lambda_i) = R^{(s,t)} L^{(s,t)} H^{(s,t+1)}(\lambda_i)$. By substituting them in the reverse order, we obtain $(\lambda_i - \mu^{(t)}) H^{(s,t+1)}(\lambda_i) = L^{(s,t+1)} R^{(s,t+1)} H^{(s,t+1)}(\lambda_i)$. Since $H^{(s,t+1)}(\lambda_i) \neq 0$, we thus have

$$R^{(s,t)} L^{(s,t)} = L^{(s,t+1)} R^{(s,t+1)}. \hspace{1cm} (4.15)$$

Similarly, it follows from (4.4) with (4.13), that $R^{(s,M,t)} H^{(s,t)}(\lambda_i) = \lambda_i (\lambda_i - \mu^{(t)}) H^{(s,M,t+1)}(\lambda_i) = R^{(s,t+1)} R^{(s,t)} H^{(s,t)}(\lambda_i)$, which immediately leads to

$$R^{(s,t+1)} R^{(s,t)} = R^{(s,M,t)} R^{(s,t)}. \hspace{1cm} (4.16)$$
Applying (4.16) to $A^{(s,t+1)} = L^{(s,t+1)}L^{(s,t+1)}\ldots L^{(s+M-1,t+1)}R^{(s,t+1)}$, we derive
\[
A^{(s,t+1)} = L^{(s,t)}L^{(s,t)}\ldots L^{(s+M-1,t)}R^{(s,t)}(R^{(s,t)})^{-1}.
\]
Along a similar line as a part of the proof of Proposition 4.2, we can rewrite $A^{(s,t+1)}$ using (4.15) repeatedly as
\[
A^{(s,t+1)} = \mathcal{H}^{(s,t)}L^{(s+1,t)}\ldots L^{(s+M-1,t)}R^{(s,t)}(R^{(s,t)})^{-1},
\]
which implies (4.14). □

Proposition 4.2 and Theorem 4.3 imply that the eigenvalues of the Hessenberg matrices $A^{(s,t)}$ do not vary for any $s$ and $t$. We here describe how to apply the evolution from $t$ to $t+1$ in the non-autonomous dhToda equation to generate similarity transformations of $A^{(s,0)} := L(0,0)L(1,0)\ldots L(M-1,0)R(0,0)$. The sequences $\{q_k^{(0)}\}_{k=1}^m, \{e_k^{(0)}\}_{k=1}^{m-1}$ satisfying (4.15) as well as those of $\{q_k^{(1)}\}_{k=1}^m, \{e_k^{(1)}\}_{k=1}^{m-1}$ as follows:

1. Compute $Q_1^{(0)} = q_1^{(0)} - \mu_1^{(0)}$.
2. Compute $Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(M,0)}$ recursively using the first equation of (3.12) with $k = 1$.
3. Compute $q_{1}^{(0,1)}$ using the second equation of (3.11) with $k = 1$.
4. Compute $Q_1^{(2,0)}$ using the first equation in (3.11) with $k = 1$.
5. Compute $e_{1}^{(0,1)}$ recursively using the second equation of (3.12) with $k = 1$.
6. Compute $Q_2^{(0,1)}$ recursively using the first equation of (3.12) with $k = 1$.
7. Compute $e_{1}^{(1,1)}, e_{1}^{(2,1)}, \ldots, e_{1}^{(M-1,1)}$ and $Q_2^{(1,0)}, Q_2^{(2,0)}, \ldots, Q_2^{(M,0)}$ as per Steps 5–6.
8. Compute the sequence $\{Q_k^{(0)}\}_{k=1}^m, \{Q_k^{(1)}\}_{k=3}^m, \ldots, \{Q_k^{(M,0)}\}_{k=3}^m$ and $\{q_k^{(1)}\}_{k=2}^m, \{e_k^{(1)}\}_{k=2}^m, \{e_k^{(1,1)}\}_{k=2}^m$ as per Steps 2–7.
9. Compute $q_{m}^{(0,1)}$ using the second equation of (3.11) with $k = m$.

Figure 1 shows the discrete-time evolutions with respect to $s$ and $t$ in the non-autonomous dhToda equation. Similarly to the above procedure for deriving the sequences in Sub(0) and Main(1) from those in Main(0), we can recursively generate the sequences in Sub(1) and Main(t + 1) from those in Main(t) for $t = 1, 2, \ldots$. Thus, we can obtain $A^{(0,1)}, A^{(0,2)}, \ldots$, whose eigenvalues are all equal to $A^{(0,0)}$ by computing the sequences in Main(t + 1) from those in Main(0) using the non-autonomous dhToda equation.

5. The discrete hungry Lotka–Volterra system

In this section, we first reconsider Propositions 3.2 and 3.3 by symmetrizing with the extended Hadamard polynomials $\mathcal{H}_k^{(s,t)}(z)$. Next, we derive the dhLV system by observing the resulting propositions for the symmetric extended Hadamard polynomials. We then describe the solution to the dhLV system in terms of the extended Hankel determinants $H_k^{(s,t)}$.

Let us define the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_k^{(s,t)}(z)$ by
\[
\tilde{\mathcal{H}}_k^{(s,t)}(z) := z^s \mathcal{H}_k^{(s+j)}(z^{M+1}), \quad j = 0, 1, \ldots, M, \quad k = 0, 1, \ldots, m,
\]
Fig. 1. Discrete-time evolutions in the non-autonomous dhToda equation.

where \( \tilde{H}^{(s,t)}_0(z) := 0 \), \( \tilde{H}^{(s,t)}_1(z) := 0 \), \ldots , \( \tilde{H}^{(s,t)}_{-M}(z) := 0 \). Also, the \( \tilde{H}^{(s,t)}_k(z) \) are monic polynomials of degree \( k \). Proposition 3.2 then gives three-term recurrence relations with respect to the symmetric extended Hadamard polynomials \( \tilde{H}^{(s,t)}_{k}(z) \).

**Proposition 5.1** For \( t = 0, 1, \ldots \), the symmetric extended Hadamard polynomials \( \tilde{H}^{(s,t)}_{k}(z) \) satisfy

\[
  z \tilde{H}^{(s,t)}_{k-1}(z) = \tilde{H}^{(s,t)}_{k}(z) + v^{(s,t)}_{k-M} \tilde{H}^{(s,t)}_{k-M-1}(z), \quad k = 1, 2, \ldots , (M + 1)m + M, \tag{5.2}
\]

where \( v^{(s,t)}_{(M+1)k-M} := q^{(s,t)}_k \) and \( v^{(s,t)}_{(M+1)k-M+j+1} := e^{(s+j,t)}_k \) for \( j = 0, 1, \ldots , M - 1 \).

**Proof.** Replacing \( z \) with \( z^{M+1} \) in the three-term recurrence relations (3.5) and (3.6), we obtain

\[
  z^{M+1} \tilde{H}^{(s+1,t)}_{k-1}(z^{M+1}) = \tilde{H}^{(s+1,t)}_{k}(z^{M+1}) + q^{(s,t)}_{k-M} \tilde{H}^{(s+1,t)}_{k-M-1}(z^{M+1}), \tag{5.3}
\]

\[
  \tilde{H}^{(s,t)}_{k}(z^{M+1}) = \tilde{H}^{(s+1,t)}_{k}(z^{M+1}) + e^{(s,t)}_{k-M} \tilde{H}^{(s+1,t)}_{k-M-1}(z^{M+1}). \tag{5.4}
\]

Multiplying both sides of (5.4) with \( s \mapsto s, s+1, \ldots , s+M-1 \) by \( z, z^2, \ldots , z^M \), respectively, we derive, for \( j = 0, 1, \ldots , M - 1 \),

\[
  z^{j+1} \tilde{H}^{(s+j,t)}_{k}(z^{M+1}) = z^{j+1} \tilde{H}^{(s+j+1,t)}_{k}(z^{M+1}) + e^{(s+j,t)}_{k} z^{j} \tilde{H}^{(s+j+1,t)}_{k-M-1}(z^{M+1}). \tag{5.5}
\]
Using $v_{(M+1)k-M}^{(x,t)} = q_k^{(x,t)}$, $v_{(M+1)(k-M+j+1)}^{(x,t)} = \tilde{c}_k^{(x+j,t)}$, and (5.1), we can rewrite (5.3) and (5.5), respectively, as

\begin{align}
&z \tilde{\mathcal{H}}_{(M+1)(k-1)+M}^{(x,t)}(z) = \tilde{\mathcal{H}}_{(M+1)k}^{(x,t)}(z) + v_{(M+1)k-M}^{(x,t)} \tilde{\mathcal{H}}_{(M+1)(k-1)}^{(x,t)}(z), \\
&z \tilde{\mathcal{H}}_{(M+1)k+j}^{(x,t)}(z) = \tilde{\mathcal{H}}_{(M+1)k+j+1}^{(x,t)}(z) + v_{(M+1)k-M+j+1}^{(x,t)} \tilde{\mathcal{H}}_{(M+1)(k-1)+j+1}^{(x,t)}(z).
\end{align}

Equations (5.6) and (5.7) are just identities in the cases where $k$ is replaced with $(M+1)k$ and $(M+1)k+j+1$ in (5.2), respectively.

Proposition 3.3 also yields the time evolution from $t$ to $t+1$ in the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_{k}^{(x,t)}(z)$.

**Proposition 5.2** For $t = 0, 1, \ldots$, the symmetric extended Hadamard polynomials $\tilde{\mathcal{H}}_{k}^{(x,t)}(z)$ satisfy

\begin{equation}
(z^{M+1} - \mu^{(t)}) \tilde{\mathcal{H}}_{k-1}^{(x,t+1)}(z) = \tilde{\mathcal{H}}_{k-1}^{(x,t)}(z) + V_{k}^{(x,t)} \tilde{\mathcal{H}}_{k}^{(x,t)}(z), \quad k = 1, 2, \ldots, (M+1)m,
\end{equation}

where $V_{(M+1)(k-1)+j+1} = Q_{k}^{(x+j,t)}$ for $j = 0, 1, \ldots, M$.

**Proof.** Replacing $s$ and $z$ with $s+j$ and $z^{M+1}$, respectively, in the three-term recurrence relation (3.9) of Proposition 3.3, and multiplying both sides with $z^{j}$, we obtain, for $j = 0, 1, \ldots, M$ and $k = 1, 2, \ldots, m$,

\begin{equation}
z^{j} (z^{M+1} - \mu^{(t)}) \tilde{\mathcal{H}}_{k-1}^{(x+j,t+1)}(z^{M+1}) = z^{j} \tilde{\mathcal{H}}_{k-1}^{(x+j,t)}(z^{M+1}) + Q_{k}^{(x+j,t)} \tilde{\mathcal{H}}_{k}^{(x+j,t)}(z^{M+1}).
\end{equation}

Considering $Q_{k}^{(x+j,t)} = V_{k}^{(x,t)} \tilde{\mathcal{H}}_{k}^{(x,t)}(z)$ and (5.1), and rewriting $(M+1)(k-1)+j+1$ as $k$, we have (5.8).

Propositions 5.1 and 5.2 lead to relationships of the variables $v_{k}^{(x,t)}$ and $V_{k}^{(x,t)}$.

**Lemma 5.1** For $s, t = 0, 1, \ldots$, the variables $v_{k}^{(x,t)}$ and $V_{k}^{(x,t)}$ satisfy

\begin{align}
v_{k}^{(x,t+1)} + V_{k}^{(x,t+1)} &= v_{k+M+1}^{(x,t)} + V_{k+M}^{(x,t)}, \quad k = 0, 1, \ldots, (M+1)(m-1), \\
v_{k}^{(x,t)} V_{k}^{(x,t)} &= v_{k+M}^{(x,t)} V_{k+M}^{(x,t)}, \quad k = 1, 2, \ldots, (M+1)m-M.
\end{align}

**Proof.** Letting $t \rightarrow t+1$ in (5.2), multiplying both sides by $(z^{M+1} - \mu^{(t)})$ and using (5.8) in the first and second terms on the right-hand side, we have

\begin{equation}
z(z^{M+1} - \mu^{(t)}) \tilde{\mathcal{H}}_{k-1}^{(x,t+1)}(z) = \tilde{\mathcal{H}}_{k-1}^{(x,t)}(z) + (v_{k+M+1}^{(x,t+1)} + V_{k+M}^{(x,t+1)}) \tilde{\mathcal{H}}_{k}^{(x,t)}(z) + v_{k}^{(x,t+1)} V_{k}^{(x,t)} \tilde{\mathcal{H}}_{k-M-1}^{(x,t)}(z).
\end{equation}

On the other hand, by multiplying (5.8) by $z$ and using (5.2) in the first and second terms on the right-hand side, we have

\begin{equation}
z(z^{M+1} - \mu^{(t)}) \tilde{\mathcal{H}}_{k-1}^{(x,t+1)}(z) = \tilde{\mathcal{H}}_{k-1}^{(x,t)}(z) + (v_{k+1}^{(x,t)} + V_{k}^{(x,t)}) \tilde{\mathcal{H}}_{k}^{(x,t)}(z) + v_{k+M}^{(x,t+1)} V_{k+M}^{(x,t)} \tilde{\mathcal{H}}_{k-M-1}^{(x,t)}(z).
\end{equation}

Thus, from (5.12) and (5.13), it follows that

\begin{equation}
(v_{k-M}^{(x,t+1)} + V_{k+1}^{(x,t+1)} - v_{k+1}^{(x,t)} - V_{k}^{(x,t)}) \tilde{\mathcal{H}}_{k}^{(x,t)}(z) + (v_{k+M}^{(x,t+1)} V_{k+M}^{(x,t)} - v_{k-M}^{(x,t)} V_{k}^{(x,t)}) \tilde{\mathcal{H}}_{k-M-1}^{(x,t)}(z) = 0.
\end{equation}
Since $\mathcal{H}_{0}^{(s,t)}(z)$, $\mathcal{H}_{1}^{(s,t)}(z)$, \ldots, $\mathcal{H}_{(M+1)m+M}^{(s,t)}(z)$ are monic polynomials of distinct degrees, they are linearly independent. The symmetric extended Hadamard polynomials $\mathcal{H}_{k}^{(s,t)}(z)$ and $\mathcal{H}_{k-M-1}^{(s,t)}(z)$ are thus linearly independent. Therefore, we derive

\begin{align*}
v_{k-M}^{(s,t)} + V_{k+1}^{(s,t)} &= v_{k+1}^{(s,t)} + V_{k}^{(s,t)}, \quad k = M, M + 1, \ldots, (M + 1)(m - 1), \\
v_{k-M}^{(s,t)} V_{k-M}^{(s,t)} &= v_{k-M}^{(s,t)} V_{k}^{(s,t)}, \quad k = M + 1, M + 2, \ldots, (M + 1)m,
\end{align*}

which are equivalent to (5.10) and (5.11), respectively. \hfill \Box

Since $v_{k}^{(s,t)}$ are aliases for $e_{k}^{(s,t)}$ and $V_{k}^{(s,t)}$ are also aliases for $Q_{k}^{(s,t)}$, the pair of (5.10) and (5.11) is just another expression of (3.11) and (3.12).

Now, let us introduce an infinite sequence $\{\kappa^{(t)}\}_{t=0}^{\infty}$ given using the infinite sequence $\{\mu^{(t)}\}_{t=0}^{\infty}$ as

\begin{equation}
\kappa^{(t)} := \frac{M+1}{\sqrt{\mu^{(t)}}, \quad t = 0, 1, \ldots}.
\end{equation}

Moreover, let $u_{k}^{(s,t)}$ be new variables defined by

\begin{equation}
u_{k}^{(s,t)} := v_{k}^{(s,t)} \frac{\mathcal{H}_{k+1}^{(s,t)}(\kappa^{(t)})}{\mathcal{H}_{k}^{(s,t)}(\kappa^{(t)})},
\end{equation}

Then, we obtain relationships of the variables $v_{k}^{(s,t)}$ and $V_{k}^{(s,t)}$ to the new variables $u_{k}^{(s,t)}$.

**Theorem 5.3** The variables $v_{k}^{(t)}$ and $V_{k}^{(t)}$ are expressed using the variables $u_{k}^{(t)}$ and the parameters $\kappa^{(t)}$ as

\begin{align*}
v_{k}^{(s,t)} &= u_{k}^{(s,t)} \prod_{\ell=1}^{M}(\kappa^{(t)} - u_{k-\ell}^{(s,t)}), \quad k = -M + 1, -M + 2, \ldots, (M + 1)m, \\
V_{k}^{(s,t)} &= -\prod_{\ell=0}^{M}(\kappa^{(t)} - u_{k-\ell}^{(s,t)}), \quad k = 1, 2, \ldots, (M + 1)m.
\end{align*}

Moreover, it holds that

\begin{equation}
M_{k}^{(s,t)} \prod_{\ell=1}^{M}(\kappa^{(t)} - u_{k-\ell}^{(s,t+1)}) = u_{k}^{(s,t)} \prod_{\ell=1}^{M}(\kappa^{(t)} - u_{k+\ell}^{(s,t)}).
\end{equation}

**Proof.** Dividing both sides of the three-term recurrence relation (5.2) with $z = \kappa^{(t)}$ in Proposition 5.1 by $\mathcal{H}_{k-1}^{(s,t)}(\kappa^{(t)})$, we derive

\begin{equation}
\frac{\mathcal{H}_{k}^{(s,t)}(\kappa^{(t)})}{\mathcal{H}_{k-1}^{(s,t)}(\kappa^{(t)})} = \kappa^{(t)} - u_{k-M}^{(s,t)}.
\end{equation}

Replacing $k$ with $k + M - \ell$ in (5.19), multiplying the both sides from $\ell = 1$ to $M$ and combining the results with (5.15), we have (5.16). From the three-term recurrence relation (5.8) with $z = \kappa^{(t)}$ in
Proposition 5.2, we see that \( V_k^{(s,t)} = -\tilde{\mathcal{H}}_k^{(s,t)}(\kappa(t)) / \mathcal{H}_k^{(s,t)}(\kappa(t)) \). Combining this with (5.19), we thus obtain (5.17). Moreover, from (5.16) and (5.17) with (5.11) in Lemma 5.1, we find that

\[
V_k^{(s,t+1)} = u_k^{(s,t)} \prod_{\ell=0}^{M-1} (\kappa(t) - u_{k+M-\ell}^{(s,t)}).
\]

(5.20)

Since it is obvious that \( \prod_{\ell=0}^{M-1} (\kappa(t) - u_{k+M-\ell}^{(s,t)}) = \prod_{\ell=1}^{M} (\kappa(t) - u_{k+\ell}^{(s,t)}) \), we therefore have (5.18). We can easily check that (5.16) and (5.17) give (5.10).

**Theorem 5.4** Let us assume that the moment sequence obtained from Proposition 5.2, we see that \( V_k^{(s,t)} = -\tilde{\mathcal{H}}_k^{(s,t)}(\kappa(t)) / \mathcal{H}_k^{(s,t)}(\kappa(t)) \). Combining this with (5.19), we thus obtain (5.17). Moreover, from (5.16) and (5.17) with (5.11) in Lemma 5.1, we find that

\[
V_k^{(s,t+1)} = u_k^{(s,t)} \prod_{\ell=0}^{M-1} (\kappa(t) - u_{k+M-\ell}^{(s,t)}).
\]

(5.20)

Since it is obvious that \( \prod_{\ell=0}^{M-1} (\kappa(t) - u_{k+M-\ell}^{(s,t)}) = \prod_{\ell=1}^{M} (\kappa(t) - u_{k+\ell}^{(s,t)}) \), we therefore have (5.18). We can easily check that (5.16) and (5.17) give (5.10).

**Theorem 5.3** is essentially equivalent to the dhLV system

\[
\begin{align*}
U_k^{(s,t+1)} \prod_{\ell=1}^{M} (1 + \delta(t+1) U_{k-\ell}^{(s,t+1)}) &= U_k^{(s,t)} \prod_{\ell=1}^{M} (1 + \delta(t) U_{k+\ell}^{(s,t)}), \quad k = 1, 2, \ldots, (M + 1)m - M, \\
U_{j-M}^{(s,t)} &= 0, \quad U_{(M+1)m+j-M}^{(s,t)} = 0, \quad j = 1, 2, \ldots, M, \\
i = 0, 1, \ldots.
\end{align*}
\]

(5.21)

This is because the dhLV system is actually derived by replacing \( u_k^{(s,t)} \) and \( (\kappa(t)M + 1) \) with \( U_k^{(s,t)} / (\kappa(t)M + 1) \) and \(-1/\delta(t)\), respectively, in (5.18). Considering that \( \epsilon_0 = 0 \) and \( \epsilon_1 = 0 \) for \( s, t = 0, 1, \ldots \), we also obtain the boundary conditions \( U_1^{(s,t)} = 0 \), \( U_2^{(s,t)} = 0 \), \ldots, \( U_0^{(s,t)} = 0 \) and \( U_{(M+1)m-M+1}^{(s,t)} = 0 \), \( U_{(M+1)m-M+2}^{(s,t)} = 0 \), \ldots, \( U_{(M+1)m}^{(s,t)} = 0 \).

Here, let \( \tilde{H}_k^{(s,t)} := (\delta(t) \delta(s) \cdots \delta(t-s) - 1)^k \tilde{H}_k^{(s,t)} \), namely,

\[
\tilde{H}_k^{(s,t)} = \begin{vmatrix}
\tilde{J}_s^{(t)} & \tilde{J}_s^{(t)} & \cdots & \tilde{J}_s^{(t)} \\
\tilde{J}_{s+M}^{(t)} & \tilde{J}_{s+M}^{(t)} & \cdots & \tilde{J}_{s+M}^{(t)} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{J}_{s+(k-1)M}^{(t)} & \tilde{J}_{s+(k-1)M+1}^{(t)} & \cdots & \tilde{J}_{s+(k-1)M+k-1}^{(t)}
\end{vmatrix}, \quad s, t = 0, 1, \ldots,
\]

(5.22)

where \( \tilde{J}_s^{(t)} := \delta(t) \delta(s) \cdots \delta(t-s) - 1 \). The following theorem then gives the determinant solution to the dhLV system (5.21).

**Theorem 5.4** Let us assume that the moment sequence \( \{f_s^{(t)}\}_{s,t=0}^{\infty} \) makes \( H_k^{(s,t)} \) be all non-zero. Then, the dhLV variables \( U_k^{(s,t)} \) can be expressed as

\[
U_k^{(s,t)} = \frac{\tilde{H}_k^{(s,t+1)}}{\tilde{H}_k^{(s,t)}}, \quad k = 0, 1, \ldots, m - 1,
\]

(5.23)

\[
U_k^{(s,t)} = \frac{\tilde{H}_k^{(s+1,t+1)}}{\tilde{H}_k^{(s,t)}}, \quad k = 0, 1, \ldots, m - 1.
\]

(5.24)

**Proof.** From (5.15), it is obvious that \( u_k^{(s,t)} = v_k^{(s,t)} / \tilde{\mathcal{H}}_k^{(s,t)}(\kappa(t)) \). Using (5.1), we can rewrite this as

\[
u_k^{(s,t)} = q_k^{(s,t)} \frac{\mathcal{H}_k^{(s,t)}(\mu(t))}{\mathcal{H}_k^{(s+M,t)}(\mu(t))}, \quad k = 0, 1, \ldots, m - 1.
\]

(5.25)
Theorem 5.4 shows the solution to the dhLV system in terms of the determinants $\tilde{H}_k^{(s,t)} = (\delta(0) \delta(1) \cdots \delta(t-1))^k H_k^{(s,t)}$ associated with the moment sequence $U_{s,t}^{(x)}$ involving two types of discrete-time variables $s$ and $t$. It is worth noting here that the dhLV variables $U_k^{(x,t)}$ and the non-autonomous dhToda variables $q_k^{(x,t)}$, $e_k^{(x,t)}$, and $Q_k^{(x,t)}$ can both be expressed using the extended Hankel determinants $H_k^{(s,t)}$. The following theorem thus gives relationships of the dhLV variables $U_k^{(x,t)}$ to the non-autonomous dhToda variables $q_k^{(x,t)}$, $e_k^{(x,t)}$, and $Q_k^{(x,t)}$.

**Theorem 5.5** For $k = 1, 2, \ldots, m$, the non-autonomous dhToda variables $q_k^{(x,t)}$, $e_k^{(x,t)}$, and $Q_k^{(x,t)}$ and the dhLV variables $U_k^{(x,t)}$ satisfy

\[ q_k^{(x,t)} = U_k^{(x,t)}(M+1)(k-1)+1 \prod_{\ell=1}^M (1 + \delta(\ell) U_k^{(x,t)}(M+1)(k-1)+1-\ell), \]

\[ e_k^{(x,t)} = U_k^{(x,t)}(M+1)(k-1)+j+1 \prod_{\ell=1}^M (1 + \delta(\ell) U_k^{(x,t)}(M+1)(k-1)+j+1-\ell), \quad j = 1, 2, \ldots, M, \]

\[ Q_k^{(x,t)} = \frac{1}{\delta(t)} \prod_{\ell=0}^M (1 + \delta(\ell) U_k^{(x,t)}(M+1)(k-1)+j+1-\ell), \quad j = 0, 1, \ldots, M. \]
6.1 Let us assume that the moment sequence \( q^{(s)}_k, u^{(s)}_k = U_k^{(s)}/(k^{(s)})^M \) and \( \delta^{(s)} = -1/(k^{(s)})^{M+1} \) in (5.16) with \( k = (M+1)(k-1)+1 \) of Theorem 5.3, we easily obtain (5.28). Similarly, since \( e^{(s+j-1)}_k = x^{(s_j)}_k \) and \( Q^{(s+j)}_k = V^{(s)}_{(M+1)(k-1)+j+1} \), we have (5.29) and (5.30), respectively.

Equations (5.28)–(5.30) in Theorem 5.5 can be regarded as the Bäcklund transformation between the non-autonomous dhToda equation and the dhLV system. The Bäcklund transformations in Fukuda et al. [8] and Hama et al. [6] relate the other autonomous Toda equation to two dhLV systems as \( \delta \to \infty \). The Bäcklund transformation (5.28)–(5.30) differs from the already known ones in that the dhLV system with arbitrary \( \delta^{(s)} \) is related to the non-autonomous dhToda equation. Another difference is that we obtained the transformation by focusing on the determinant solutions, not the associated LR transformations. See also [9] concerning a Lax representation for the dhLV system.

6. Asymptotic behaviour of the determinantal solution to the discrete hungry systems

In this section, we first give asymptotic expansions as \( t \to \infty \) of the extended Hankel determinants \( H_k^{(s)} \), and then present asymptotic analysis as \( t \to \infty \) for solutions to the non-autonomous dhToda equation and the dhLV system.

Reconsidering Proposition 2.3, we can easily derive asymptotic expansions as \( t \to \infty \) of the extended Hankel determinants \( H_k^{(s)} \).

**Lemma 6.1** Let us assume that \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are constants such that \( |\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}| \). Moreover, let us assume that there are constants \( q_1, q_2, \ldots, q_m \in [0, 1) \) such that \( q_k > |\lambda_{k+1} - \mu^{(t)}|/|\lambda_k - \mu^{(t)}| \) for \( k = 1, 2, \ldots, m-1 \) and \( t = 0, 1, \ldots \). Then, for \( k = 1, 2, \ldots, m \), it holds that

\[
H_k^{(M+j)} = K_{1,2,\ldots,k}^{(j)} \rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)} (1 + O(q_k)), \quad t \to \infty.
\] (6.1)

**Proof.** Observing the expansion of the extended Hankel determinants \( H_k^{(M+j)} \) appearing in Proposition 2.3, we easily see that the dominant term as \( t \to \infty \) is \( K_{1,2,\ldots,k}^{(j)} \rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)} \) if it holds,

\[
|\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}|, \quad t = 0, 1, \ldots
\] (6.2)

Thus, we immediately have (6.1).

Using Lemma 6.1, we obtain asymptotic behaviour as \( t \to \infty \) of the non-autonomous dhToda variables \( q_k^{(s)}, e_k^{(s)} \) and \( Q_k^{(s)} \).

**Theorem 6.1** Let us assume that the moment sequence \( \{f^{(s)}_k\}_{k=0}^{\infty} \) satisfies \( H_k^{(s)} \neq 0 \) and the infinite sequence \( \{\mu^{(t)}\} \) converges to some constant \( \mu^* \) as \( t \to \infty \). Moreover, let \( \lambda_1, \lambda_2, \ldots, \lambda_m \) be constants such that \( |\lambda_1 - \mu^{(t)}| > |\lambda_2 - \mu^{(t)}| > \cdots > |\lambda_m - \mu^{(t)}| \). Then, it holds that

\[
\lim_{t \to \infty} q_k^{(s)} = \lambda_k, \quad k = 1, 2, \ldots, m,
\] (6.3)
\[
\lim_{t \to \infty} e_k^{(s)} = 0, \quad k = 1, 2, \ldots, m-1,
\] (6.4)
\[
\lim_{t \to \infty} Q_k^{(s)} = \lambda_k - \mu^*, \quad k = 1, 2, \ldots, m.
\] (6.5)
Proof. Applying Lemma 6.1 to the non-autonomous dhToda variables \( q^{(s,j)}_k = H^{(s,j)}_{k-1} H^{(j+M,j)}_k / (H^{(j,j)}_k H^{(j+M,j)}_k) \), we immediately obtain (6.3). Using Lemma 6.1 and replacing \( s \) with \( M\ell + j \) in (3.8), we can rewrite
\[
e_k^{(M+j+\ell)} = H^{(M+j+\ell)}_{k+1} H^{(M+j+1,\ell)}_k / (H^{(M+j+1,\ell)}_k H^{(M+j,\ell)}_k)
\] as
\[
e_k^{(M+j,\ell)} = \frac{K^{(j)}_{1,2,\ldots,k} K^{(j+1)}_{1,2,\ldots,k-1}}{K^{(j)}_{1,2,\ldots,k} K^{(j+1)}_{1,2,\ldots,k}} \cdot \frac{\lambda_k + \rho_k^{(t)} / \rho_k^{(0)}}{\lambda_k} \cdot (1 + O(\epsilon_k^{(t)}))(1 + O(\epsilon_k^{(t-1)})) / (1 + O(\epsilon_k^{(t)}))(1 + O(\epsilon_k^{(t-1)})).
\] (6.6)
Recalling \( \rho_k^{(t)} = (\lambda_i - \mu^{(0)})(\lambda_i - \mu^{(1)}) \cdots (\lambda_i - \mu^{(t-1)}) \) in (2.7), we see that
\[
\frac{(\rho_1^{(0)} \rho_2^{(0)} \cdots \rho_k^{(0)})(\rho_1^{(t)} \rho_2^{(t)} \cdots \rho_k^{(t)})}{(\rho_1^{(0)} \rho_2^{(0)} \cdots \rho_k^{(0)})^2} = \frac{(\lambda_k+\mu^{(0)})(\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)}) \cdots (\lambda_k - \mu^{(t-1)})}.
\] (6.7)
Thus, by combining (6.7) with (6.6), we have (6.4) under the assumption (6.2). Similarly, by considering Lemma 6.1 with (6.7) in the non-autonomous dhToda variables \( Q^{(s,j)}_k = H^{(s,j)}_{k-1} H^{(j+1,1)}_k / (H^{(j,j)}_k H^{(j+1,1)}_{k-1}) \), we derive
\[
\lim_{t \to \infty} Q^{(s,j)}_k = \lim_{t \to \infty} \frac{\rho_k^{(t+1)}}{\rho_k^{(0)}} \cdot \frac{(1 + O(\epsilon_k^{(t+1)}))(1 + O(\epsilon_k^{(t-1)}))}{(1 + O(\epsilon_k^{(t)}))(1 + O(\epsilon_k^{(t-1)}))} = \lim_{t \to \infty} (\lambda_k - \mu^{(0)}) \cdot \frac{(1 + O(\epsilon_k^{(t+1)}))(1 + O(\epsilon_k^{(t-1)}))}{(1 + O(\epsilon_k^{(t)}))(1 + O(\epsilon_k^{(t-1)}))},
\]
which leads to (6.5). □

The following theorem shows asymptotic convergence of the auxiliary variables \( D^{(0,t)}_k, F^{(0,t)}_k, F^{(1,t)}_k, \ldots, F^{(M-1,t)}_k \) in the differential non-autonomous dhToda equation.

**Theorem 6.2** As \( t \to \infty \), the auxiliary non-autonomous dhToda variables \( D^{(0,t)}_k = H^{(M,t+1)}_{k-1} / H^{(M+1,t+1)}_{k-1} \) and \( F^{(s,t)}_k = H^{(s+1,t)}_{k-1} / H^{(s+1,t+1)}_{k-1} \) converge to \(-\mu^s + \lambda_k - \mu^s\) in the differentiable non-autonomous dhToda equation.

**Proof.** From Lemma 6.1 with \( \rho_k^{(t)} = (\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)}) \cdots (\lambda_k - \mu^{(t-1)}) \), it follows that
\[
\lim_{t \to \infty} D^{(0,t)}_k = \lim_{t \to \infty} \left[ \frac{\lambda_k \rho_k^{(t+1)} K^{(0)}_{1,2,\ldots,k}}{K^{(0)}_{1,2,\ldots,k-1}} \cdot \frac{1 + O(\epsilon_k^{(t+1)})}{1 + O(\epsilon_k^{(t-1)})} \right] \times \left[ \frac{1}{\lambda_k} \cdot \frac{K^{(0)}_{1,2,\ldots,k}}{K^{(0)}_{1,2,\ldots,k-1}} \cdot \frac{1 + O(\epsilon_k^{(t-1)})}{1 + O(\epsilon_k^{(t-1)})} - \frac{1}{\rho_k^{(0)}} \cdot \frac{K^{(0)}_{1,2,\ldots,k} K^{(0)}_{1,2,\ldots,k-1}}{1 + O(\epsilon_k^{(t+1)})} \right]
\]
\[
= \lim_{t \to \infty} \left( \frac{\rho_k^{(t+1)}}{\rho_k^{(0)}} \cdot \frac{(1 + O(\epsilon_k^{(t+1)}))(1 + O(\epsilon_k^{(t-1)}))}{(1 + O(\epsilon_k^{(t)}))(1 + O(\epsilon_k^{(t-1)})))} \right) = -\mu^s,
\]
\[
\lim_{t \to \infty} F^{(s,t)}_k = \lim_{t \to \infty} \left[ \rho_k^{(t+1)} \frac{K^{(s)}_1 \cdot K^{(s)}_{1,2,\ldots,k} \cdot \lambda_k - \mu^s}{K^{(s)}_{1,2,\ldots,k-1}} \cdot \frac{1 + O(q_k^{(s)})}{1 + O(q_k^{(t+1)})} \right]
\]

\[
= \lim_{t \to \infty} \left[ \frac{\rho_k^{(t+1)} \cdot (1 + O(q_k^{(s)}))(1 + O(q_k^{(t+1)}))}{\lambda_k - \mu^s} \right]
\]

\[
= \lambda_k - \mu^s, \quad s = 0, 1, \ldots, M - 1,
\]

\[
\lim_{t \to \infty} F^{(M-1,t)}_k = \lim_{t \to \infty} \left[ \rho_k^{(t+1)} \frac{K^{(M-1)}_1 \cdot K^{(M-1)}_{1,2,\ldots,k} \cdot \lambda_k - \mu^s}{K^{(M-1)}_{1,2,\ldots,k-1}} \cdot \frac{1 + O(q_k^{(s)})}{1 + O(q_k^{(t+1)})} \right]
\]

\[
= \lim_{t \to \infty} \left[ \frac{\rho_k^{(t+1)} \cdot (1 + O(q_k^{(s)}))(1 + O(q_k^{(t+1)}))}{\lambda_k - \mu^s} \right]
\]

\[
= \lambda_k - \mu^s.
\]

Combining Theorem 5.4 with Lemma 6.1, we can identify the asymptotic convergence of the non-autonomous dhLV variables \(U_k^{(s,t)}\) as \(t \to \infty\).

**Theorem 6.3** Let us assume that the moment sequence \( \{f_i^{(t)}\}_{t=0}^{\infty} \) satisfies \(H_k^{(s,t)} \neq 0\), and \(\lambda_1, \lambda_2, \ldots, \lambda_m\) are constants such that \(|\lambda_1 - \mu^{(0)}| > |\lambda_2 - \mu^{(0)}| > \cdots > |\lambda_m - \mu^{(0)}|\). Moreover, let us assume that there are constants \(q_1, q_2, \ldots, q_{M-1} \in [0, 1)\) such that \(q_k > |\lambda_{k+1} - \mu^{(0)}|/|\lambda_k - \mu^{(0)}|\) for \(k = 1, 2, \ldots, m - 1\) and \(t = 0, 1, \ldots\). Then, asymptotic behaviour as \(t \to \infty\) of the dhLV variables \(U_k^{(t)}\) is given by

\[
\lim_{t \to \infty} U_k^{(t)}_{(M+1)k+1} = \lambda_k, \quad k = 0, 1, \ldots, m - 1,
\]

\[
\lim_{t \to \infty} U_k^{(t)}_{(M+1)(k+1)+j} = 0, \quad j = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, m.
\]

**Proof.** Noting that \(\tilde{H}_k^{(s,t)} = (\delta^{(0)} \delta^{(1)} \cdots \delta^{(t-1)})^k H_k^{(s,t)}\) in Theorem 5.4, we see that \(U_k^{(s,t)}_{(M+1)k+1} = H_k^{(s,M+1)} / (H_k^{(s,1)} H_k^{(s,t+1)} H_k^{(s,t+1)} / H_k^{(s,1)} H_k^{(s,t+1)})).\) Applying Lemma 6.1, we derive

\[
U_k^{(t)}_{(M+1)k+1} = \frac{\lambda_1 \lambda_2 \cdots \lambda_{k+1}}{\lambda_1 \lambda_2 \cdots \lambda_k} \cdot \frac{1 + O(q_k^{(s)})}{1 + O(q_k^{(t+1)})}.
\]
which immediately leads to (6.8). Let us consider the case \( j' + j < M \), where \( j' \) are integers such that \( s = M \ell + j' \). Using Theorem 5.4 and Lemma 6.1, as \( t \to \infty \), we can also rewrite \( U^{(j,j')}_{(M+1)(k-1)+j+1} = H^{(s+j-1,0)}_{k+1} H^{(s+j+1,1)}_{k} \) for \( j = 1, 2, \ldots, M \) as

\[
U^{(M+j,j')}_{(M+1)(k-1)+j+1} = \frac{1}{\delta(t)} \cdot \frac{K^{(j,j')}_{1,2,\ldots,k+1} K^{(j,j')}_{1,2,\ldots,k-1}}{K^{(j,j')}_{1,2,\ldots,k} K^{(j,j')}_{1,2,\ldots,k}} \cdot \frac{\lambda_{k+1} \ell}{\lambda_{k} (\lambda_{k} - \mu^{(t)})} \times \frac{(\lambda_{k+1} - \mu^{(0)}) (\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_{k} - \mu^{(0)}) (\lambda_{k} - \mu^{(1)}) \cdots (\lambda_{k} - \mu^{(t-1)})} \times \frac{(1 + O(\varrho_{k+1}^t)) (1 + O(\varrho_{k+1}^{t+1}))}{(1 + O(\varrho_{k}^t))(1 + O(\varrho_{k}^{t+1}))}, \quad t \to \infty.
\]

(6.11)

Considering \( \rho_k^{(t)} = (\lambda_k - \mu^{(0)})(\lambda_k - \mu^{(1)}) \cdots (\lambda_k - \mu^{(t-1)}) \) in (6.11), we obtain, for \( j = 1, 2, \ldots, M \),

\[
U^{(M+j,j')}_{(M+1)(k-1)+j+1} = \frac{1}{\delta(t)} \cdot \frac{K^{(j,j')}_{1,2,\ldots,k+1} K^{(j,j')}_{1,2,\ldots,k-1}}{K^{(j,j')}_{1,2,\ldots,k} K^{(j,j')}_{1,2,\ldots,k}} \cdot \frac{\lambda_{k+1} \ell}{\lambda_{k} (\lambda_{k} - \mu^{(t)})} \times \frac{(\lambda_{k+1} - \mu^{(0)}) (\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_{k} - \mu^{(0)}) (\lambda_{k} - \mu^{(1)}) \cdots (\lambda_{k} - \mu^{(t-1)})} \times \frac{(1 + O(\varrho_{k+1}^t)) (1 + O(\varrho_{k+1}^{t+1}))}{(1 + O(\varrho_{k}^t))(1 + O(\varrho_{k}^{t+1}))}, \quad t \to \infty.
\]

(6.12)

which leads to (6.9). Similarly, in the cases where \( j' + j = M \) and \( j' + j > M \), from Theorem 5.4 and Lemma 6.1, it respectively follows that

\[
U^{(M+j,j')}_{(M+1)(k-1)+j+1} = \frac{1}{\delta(t)} \cdot \frac{K^{(M-1)}_{1,2,\ldots,k+1} K^{(M-1)}_{1,2,\ldots,k-1}}{K^{(M-1)}_{1,2,\ldots,k} K^{(M-1)}_{1,2,\ldots,k}} \cdot \frac{\lambda_{k+1} \ell}{\lambda_{k} (\lambda_{k} - \mu^{(t)})} \times \frac{(\lambda_{k+1} - \mu^{(0)}) (\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_{k} - \mu^{(0)}) (\lambda_{k} - \mu^{(1)}) \cdots (\lambda_{k} - \mu^{(t-1)})} \times \frac{(1 + O(\varrho_{k+1}^t)) (1 + O(\varrho_{k+1}^{t+1}))}{(1 + O(\varrho_{k}^t))(1 + O(\varrho_{k}^{t+1}))}, \quad t \to \infty, \quad j' + j = M
\]

(6.13)

and

\[
U^{(M+j,j')}_{(M+1)(k-1)+j+1} = \frac{1}{\delta(t)} \cdot \frac{K^{(j,j'-M)}_{1,2,\ldots,k+1} K^{(j,j'-M)}_{1,2,\ldots,k-1}}{K^{(j,j'-M)}_{1,2,\ldots,k} K^{(j,j'-M)}_{1,2,\ldots,k}} \cdot \frac{\lambda_{k+1} \ell}{\lambda_{k} (\lambda_{k} - \mu^{(t)})} \times \frac{(\lambda_{k+1} - \mu^{(0)}) (\lambda_{k+1} - \mu^{(1)}) \cdots (\lambda_{k+1} - \mu^{(t-1)})}{(\lambda_{k} - \mu^{(0)}) (\lambda_{k} - \mu^{(1)}) \cdots (\lambda_{k} - \mu^{(t-1)})} \times \frac{(1 + O(\varrho_{k+1}^t)) (1 + O(\varrho_{k+1}^{t+1}))}{(1 + O(\varrho_{k}^t))(1 + O(\varrho_{k}^{t+1}))}, \quad t \to \infty, \quad j' + j > M,
\]

(6.14)

which imply (6.9). \( \square \)
According to Proposition 4.1, \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are eigenvalues of the Hessenberg matrix \( A^{(0,0)} = L^{(0,0)} L^{(1,0)} \cdots L^{(M-1,0)} R^{(0,0)} \) involving \( \{q_k^{(0,0)}\}_{k=1}^m, \{e_k^{(0,0)}\}_{k=1}^{m-1}, \{e_k^{(1,0)}\}_{k=1}^{m-1}, \ldots, \{e_k^{(M-1,0)}\}_{k=1}^{m-1} \). Theorems 6.1 and 6.3 thus suggest that the non-autonomous dhToda equation with initial settings \( \{q_k^{(0,0)}\}_{k=1}^m, \{e_k^{(0,0)}\}_{k=1}^{m-1}, \{e_k^{(1,0)}\}_{k=1}^{m-1}, \ldots, \{e_k^{(M-1,0)}\}_{k=1}^{m-1} \) and the dhLV system with initial settings \( \{U_{j-M}^{(0)}\}_{j=1}^M \) and \( \{U_{(M+1)m+j-M}^{(0)}\}_{j=1}^M \) are both applicable to computing eigenvalues of the Hessenberg matrix \( A^{(0,0)} \). This observation differs from Sumikura et al. [11] in that \( \{q_k^{(0,0)}\}_{k=1}^m, \{e_k^{(0,0)}\}_{k=1}^{m-1}, \{e_k^{(1,0)}\}_{k=1}^{m-1}, \ldots, \{e_k^{(M-1,0)}\}_{k=1}^{m-1} \) are not restricted to be positive and so \( A^{(0,0)} \) is not restricted to be TN. In other words, algorithms based on the non-autonomous dhToda equation and the dhLV system can compute the eigenvalues of even the Hessenberg matrix \( A^{(0,0)} \) involving some negative \( q \)'s and \( e \)'s values.

Here, we give examples to demonstrate asymptotic convergence without requiring positivity of some of the non-autonomous dhToda variables. We used a computer with Mac OS Sierra (ver. 10.12.6) and CPU: 3.1 GHz Intel Core i7 CPU. We employed floating-point arithmetic in the numerical computing software MATLAB [ver. 9.1.0.441655 (R2016b)]. As the first example, let us consider the case where \( M = 3, m = 3, q_1^{(0,0)} = -6, q_2^{(0,0)} = -6, q_3^{(0,0)} = -0.5, e_1^{(0,0)} = 2, e_2^{(0,0)} = -6, e_1^{(1,0)} = 1, e_2^{(1,0)} = 5, e_1^{(2,0)} = -4 \) and \( e_2^{(2,0)} = -6.5 \) in the non-autonomous dhToda equation. Moreover, let \( \mu^{(0)} = F_3^{(2,0)} \) where \( \mu^{(0)} = 0.1 \). Then the non-autonomous dhToda equation gives similarity transformations of the Hessenberg matrix

\[
A^{(0,0)} = L^{(0,0)} L^{(1,0)} L^{(2,0)} R^{(0,0)} = -\begin{pmatrix} 6 & -1 & 0 \\ -6 & 7 & -1 \\ -12 & -43 & 8 \end{pmatrix}.
\]

It is remarkable that \( \hat{A}^{(0,0)} := -A^{(0,0)} \) is not a TN matrix but an M-matrix [13]. Square matrices are M-matrices if they form \( \alpha I - B \) for non-negative matrices \( B \) and real numbers \( \alpha \) that are larger than
the spectrum radius of $B$. Computed eigenvalues of the $M$-matrix $\tilde{A}^{(0,0)}$ using the MATLAB function \texttt{eig} were $\tilde{\lambda}_1 = 14.3089310318622$, $\tilde{\lambda}_2 = 6.49746173237448$ and $\tilde{\lambda}_3 = 0.193607235763316$. Figure 2 shows the convergence of $q_1^{(0,0)}$, $q_2^{(0,0)}$ and $q_3^{(0,0)}$ to $-\tilde{\lambda}_1$, $-\tilde{\lambda}_2$ and $-\tilde{\lambda}_3$, respectively, as $t$ grows larger. Since $q_1^{(0,0)} = -14.3089310318622$, $q_2^{(0,50)} = -6.49746173237441$ and $q_3^{(0,50)} = -0.193607235763316$, the relative differences from $-\tilde{\lambda}_1$, $-\tilde{\lambda}_2$ and $-\tilde{\lambda}_3$ are considerably small.

As the second example, let us consider the case where $M = 2$, $m = 20$ and $q_1^{(0,0)} = 51$, $q_2^{(0,0)} = 6$, $q_3^{(0,0)} = -4$, $q_4^{(0,0)} = 65$, $q_5^{(0,0)} = 28$, $q_6^{(0,0)} = 67$, $q_7^{(0,0)} = -9$, $q_8^{(0,0)} = 68$, $q_9^{(0,0)} = 15$, $q_{10}^{(0,0)} = 91$, $q_{11}^{(0,0)} = 93$, $q_{12}^{(0,0)} = 13$, $q_{13}^{(0,0)} = 17$, $q_{14}^{(0,0)} = 60$, $q_{15}^{(0,0)} = 92$, $q_{16}^{(0,0)} = 95$, $q_{17}^{(0,0)} = 97$, $q_{18}^{(0,0)} = 10$, $q_{19}^{(0,0)} = 12$, $q_{20}^{(0,0)} = 80$, $e_1^{(0,0)} = 6$, $e_2^{(0,0)} = 10$, $e_3^{(0,0)} = 10$, $e_4^{(0,0)} = 8$, $e_5^{(0,0)} = 4$, $e_6^{(0,0)} = 2$, $e_7^{(0,0)} = 5$, $e_8^{(0,0)} = 7$, $e_9^{(0,0)} = 7$, $e_{10}^{(0,0)} = 2$, $e_{11}^{(0,0)} = 5$, $e_{12}^{(0,0)} = 7$, $e_{13}^{(0,0)} = 9$, $e_{14}^{(0,0)} = 4$, $e_{15}^{(0,0)} = 4$, $e_{16}^{(0,0)} = 5$, $e_{17}^{(0,0)} = 5$, $e_{18}^{(0,0)} = 7$, $e_{19}^{(0,0)} = 3$, $e_{20}^{(0,0)} = 3$, $e_{21}^{(0,0)} = 8$, $e_{22}^{(0,0)} = 9$, $e_{23}^{(0,0)} = 7$, $e_{24}^{(0,0)} = 6$, $e_{25}^{(0,0)} = 6$, $e_{26}^{(0,0)} = 7$, $e_{27}^{(0,0)} = 7$, $e_{28}^{(0,0)} = 7$, $e_{29}^{(0,0)} = 7$, $e_{30}^{(0,0)} = 7$, $e_{31}^{(0,0)} = 7$, $e_{32}^{(0,0)} = 7$, $e_{33}^{(0,0)} = 7$, $e_{34}^{(0,0)} = 7$, $e_{35}^{(0,0)} = 7$, $e_{36}^{(0,0)} = 7$, $e_{37}^{(0,0)} = 7$, $e_{38}^{(0,0)} = 7$, $e_{39}^{(0,0)} = 7$, $e_{40}^{(0,0)} = 7$, $e_{41}^{(0,0)} = 7$, $e_{42}^{(0,0)} = 7$, $e_{43}^{(0,0)} = 7$, $e_{44}^{(0,0)} = 7$, $e_{45}^{(0,0)} = 7$, $e_{46}^{(0,0)} = 7$, $e_{47}^{(0,0)} = 7$, $e_{48}^{(0,0)} = 7$, $e_{49}^{(0,0)} = 7$, $e_{50}^{(0,0)} = 7$. Table 1 implies that the computed eigenvalues of $A^{(0,0)}$ and the exact characteristic polynomial $p(z)$ are very close. More precisely, the relative differences from the computed eigenvalues of $A^{(0,0)}$ and the exact characteristic polynomial $p(z)$ imply that $p(z)$ is a polynomial with roots that are computed eigenvalues. Since the entries of $A^{(0,0)}$ are all integer, we can easily obtain the exact characteristic polynomial $p(z).$

As the Hessenberg matrix $A^{(0,0)}$, while the Matlab function \texttt{eig} returns eighteen real and two complex values, the dhToda equations (3.11) and (3.12) generate only real values. Two eigenvalues $\lambda_8, \lambda_9$ computed using \texttt{eig} and the dhToda equation (3.11) and (3.12) are (0.790759327837088 + 0.0066041099069751i) $\times 10^{2}$, (0.790759327837088 − 0.0066041099069751i) $\times 10^{2}$ and 0.784971872967535 $\times 10^{2}$, respectively. To dig into the numerical differences arising in the both, we compare twenty coefficients of the characteristic polynomial of $A^{(0,0)}$, namely, $a_1, a_2, \ldots, a_{20}$ appearing in $p(z) = z^{20} + a_1 z^{19} + \cdots + a_{19} z + a_{20}$, with polynomials whose roots are computed eigenvalues. Since the entries of $A^{(0,0)}$ are all integer, we can easily obtain the exact characteristic polynomial $p(z).$

Table 1 describes the relative differences of coefficients between characteristic polynomials constructed from the computed eigenvalues of $A^{(0,0)}$ and the exact characteristic polynomial $p(z)$ Table 1 implies that eigenvalues computed using the non-autonomous dhToda equations (3.11) and (3.12) are more desirable as roots of the characteristic polynomial $p(z)$ than those using the MATLAB function \texttt{eig}.  

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Table 1. The relative errors of 20 coefficients of characteristic polynomials constructed from computed eigenvalues

|   | MATLAB function `eig` | Non-autonomous dhToda equation |
|---|------------------------|--------------------------------|
| a₁ | 1.35572038168877 × 10⁻¹⁵ | 9.68371701206269 × 10⁻¹⁶ |
| a₂ | 2.58316111183205 × 10⁻¹⁵ | 1.84511507988004 × 10⁻¹⁵ |
| a₃ | 3.33101839723298 × 10⁻¹⁵ | 2.75171084988812 × 10⁻¹⁵ |
| a₄ | 3.18458284995627 × 10⁻¹⁵ | 4.19024059204772 × 10⁻¹⁵ |
| a₅ | 5.80923282114099 × 10⁻¹⁵ | 5.26883907033718 × 10⁻¹⁵ |
| a₆ | 1.73380629987787 × 10⁻¹⁴ | 6.31810770294479 × 10⁻¹⁵ |
| a₇ | 3.07245234358733 × 10⁻¹⁴ | 7.84005080777456 × 10⁻¹⁵ |
| a₈ | 8.88567301021745 × 10⁻¹⁴ | 8.6443301898983 × 10⁻¹⁵ |
| a₉ | 4.46778162861198 × 10⁻¹³ | 1.01654581999318 × 10⁻¹⁴ |
| a₁₀ | 1.29430877053763 × 10⁻¹² | 1.14301513133542 × 10⁻¹⁴ |
| a₁₁ | 3.88111776521558 × 10⁻¹² | 1.31146289724997 × 10⁻¹⁴ |
| a₁₂ | 5.8910922120296 × 10⁻¹¹ | 1.4425339763311 × 10⁻¹⁴ |
| a₁₃ | 1.46516743249783 × 10⁻¹¹ | 1.52813054086397 × 10⁻¹⁴ |
| a₁₄ | 1.94263069463187 × 10⁻⁹ | 1.52150651877853 × 10⁻¹⁴ |
| a₁₅ | 1.20924727500864 × 10⁻⁸ | 5.06725071542775 × 10⁻¹⁵ |
| a₁₆ | 2.22334352436917 × 10⁻⁸ | 2.73319889922195 × 10⁻¹⁴ |
| a₁₇ | 1.76219331323578 × 10⁻⁸ | 2.42739416243640 × 10⁻¹⁴ |
| a₁₈ | 5.48654356151481 × 10⁻⁸ | 2.34568661355177 × 10⁻¹⁴ |
| a₁₉ | 4.59169583992666 × 10⁻⁷ | 2.11800970433668 × 10⁻¹⁴ |
| a₂₀ | 1.50853899504407 × 10⁻⁷ | 2.42924318356651 × 10⁻¹⁴ |

7. Conclusion

In this article, we first examined the properties of the extended Hankel determinants associated with the moment sequence \( \{ f_s^{(t)} \}_{s,t=0}^{\infty} \) with respect to two types of discrete-time variables \( s \) and \( t \). Observing the extended Hadamard polynomials based on the extended Hankel determinants, we expressed the solution to the non-autonomous dhToda equation using the extended Hankel determinants. We thus clarified the eigenvalue problem of Hessenberg matrices associated with the non-autonomous dhToda equation. We also gave the determinant solution to the dhLV system in terms of the extended Hankel determinants. Finally, we showed the asymptotic convergence to eigenvalues of Hessenberg matrices associated with the non-autonomous dhToda equation and dhLV system as the discrete-time variable \( t \) goes to infinity.

It is worth noting that, through this article, the non-autonomous dhToda and dhLV variables are not restricted to be all positive. The Hessenberg matrices in this article thus differ from those restricted to be TN in Sumikura et al. [11]. The discrete-time variable \( t \) and an auxiliary discrete-time variable \( s \) enabled us to get a deeper understanding of the non-autonomous dhToda equation and dhLV system. The auxiliary discrete-time variable \( s \) played key roles in completing asymptotic analysis of the autonomous dhToda equation and the dhLV system. Though the discrete system involving both \( s \) and \( t \) in Theorem 3.4 is an auxiliary system in this article, examining integrable properties of this discrete system is so interesting as a future work.
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