GLOBAL WELL-POSEDNESS FOR THE
HOMOGENEOUS LANDAU EQUATION

MARIA GUALDANI AND NESTOR GUILLEN

Abstract. Global well-posedness and exponential decay to equilibrium are proved for the homogeneous Landau equation from kinetic theory. The initial distribution is only assumed to be bounded and decaying sufficiently fast at infinity. In particular, discontinuous initial configurations that might be far from equilibrium are covered. Despite the lack of a comparison principle for the equation, the proof of existence relies on barrier arguments and parabolic regularity theory. Uniqueness and decay to equilibrium are then obtained through weighted integral inequalities. Although the focus is on the spatially homogeneous case with Coulomb potential, the methods introduced here may be applied elsewhere in nonlinear kinetic theory.

1. Introduction

In this note we deal with the Cauchy problem for the homogeneous Landau equation, which consists in finding a non-negative function \( f(v,t) \) with \( f : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R} \), taking a prescribed initial condition \( f_{\text{in}}(v) \) and solving for positive times the nonlinear evolution equation

\[
\partial_t f(v,t) = Q(f,f),
\]  

where \( Q(f,f) \) denotes a quadratic integro-differential operator known as the Landau collisional operator, and commonly it takes the form (\( \gamma \in \mathbb{R} \))

\[
Q(f,f) = \frac{\partial}{\partial v_j} \int_{\mathbb{R}^3} \left( f(w) \frac{\partial f}{\partial v_i}(v) - f(v) \frac{\partial f}{\partial w_i}(w) \right) \left( \delta_{ij} - \frac{(v_i - w_i)(v_j - w_j)}{|v-w|^2} \right) |v-w|^\gamma + dw.
\]

This equation has been considered for different values of the parameter \( \gamma \). Our main result deals specifically with the case \( \gamma = -3 \) which in the physical interpretation corresponds to particles interacting by a Coulombic force. We show that if a given \( f_{\text{in}} \) is bounded and decays fast enough at infinity then there is a unique solution to (1.1) starting from \( f_{\text{in}} \) and which becomes smooth for all positive times.

This means there is no finite time breakdown for (1.1) even for initial conditions far from equilibrium. Breakdown had only been ruled out for interactions corresponding to the so called hard spheres. The question of breakdown for the homogeneous Landau equation for the Coulomb case is discussed further by Villani in [21, Chapter 5, Section 1.3].

1.1. Main result. Although the methods in this manuscript work in greater generality (in particular, dimensions other than 3), we shall focus on the case of a Coulomb interaction \( \gamma = -3 \) in \( \mathbb{R}^3 \). The spatially inhomogeneous case will be studied in forthcoming work. Further background, strategy of the proof and overall notation are given in the sections below.

Date: Monday 13th May, 2013.
Consider the Cauchy problem for (1.1)
\[
\begin{aligned}
\partial_t f &= Q(f, f) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\
f_{|t=0} &= f_{\text{in}} \quad \text{in } \mathbb{R}^3,
\end{aligned}
\] (1.3)
where \(Q(f, f)\) is given by (1.6) with \(\Psi(v) = (8\pi|v|)^{-1}\).

**Definition 1.1.** Let \(f_{\text{in}} \in L^1(\mathbb{R}^3)\) be non-negative. A non-negative function \(f : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}\) is said to be a weak solution of (1.3) if \(f \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^3))\) and

1. \(f\) belongs to \(L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^3))\) and \(L^2(\mathbb{R}^+, \dot{H}^1_{\text{loc}}(\mathbb{R}^3))\).
2. For any smooth test function \(\phi : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}\) we have
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} f \partial_t \phi \, dvdt - \int_{\mathbb{R}^3 \times \mathbb{R}^+} (\nabla \phi, A[f] \nabla f - f \nabla a[f]) \, dvdt = \int_{\mathbb{R}^3} f_{\text{in}} \phi_0 \, dv.
\]

The one condition required of the initial distribution \(f_{\text{in}}\) is that it lies below two Maxwellians with the same mass and energy, and whose centers of mass are not too close together.

**Assumption on \(f_{\text{in}}\):** For some \(m > 0, \tau \in \mathbb{S}^2\) and \(R \geq 4\) the initial distribution \(f_{\text{in}}\) satisfies the bound
\[
f_{\text{in}}(v) \leq \frac{m}{(2\pi)^{3/2}} \min \left\{ e^{-|v-R|^2/2}, e^{-|v+R|^2/2} \right\}, \quad v \in \mathbb{R}^3.
\] (1.4)

The main result is the global well-posedness and decay to equilibrium of (1.3) for such \(f_{\text{in}}\).

**Theorem 1.2.** Assume that the initial condition \(f_{\text{in}}\) satisfies the pointwise bound (1.4). Then there is a unique weak solution of (1.3) such that
\[
f(v, t) \leq \frac{m}{(2\pi)^{3/2}} \min \left\{ e^{-|v-R|^2/2}, e^{-|v+R|^2/2} \right\}, \quad v \in \mathbb{R}^3, \quad t \geq 0.
\]
This solution becomes infinitely differentiable in both \(v\) and \(t\) for all positive times. Moreover, \(f_t = f(\cdot, t)\) converges exponentially in \(L^1\) to the unique equilibrium distribution \(\mathcal{M}^{f_{\text{in}}}\) with the same mass, momentum and energy as \(f_{\text{in}}\). Specifically, there is a \(\lambda = \lambda(f_{\text{in}}) > 0\) such that
\[
\|f_t - \mathcal{M}^{f_{\text{in}}}\|_{L^1(\mathbb{R}^3)} \leq e^{-\lambda t} \sqrt{2H(f_{\text{in}} \mid \mathcal{M}^{f_{\text{in}}} \forall t > 0,}
\]
where \(H(f \mid g)\) denotes the relative entropy of \(f\) with respect to \(g\) (see Section 5).

**Remark 1.3.** The decay assumption for Theorem 1.2 is restrictive, however, it is only needed for the existence part of the theorem, the uniqueness proved in Section 5 requires only polynomial decay of the solutions. Thus, it is likely that Theorem 1.2 can be extended to initial data that decays polynomially by combining compactness properties observed by Lions [17] with the integral estimates from Section 5.

1.2. **Physical model.** The nonlinear evolution equation (1.1) arises in kinetic theory, notably plasma physics. It was derived by Landau [15] and it serves as a formal approximation to the Boltzmann equation in the spatially inhomogeneous setting
\[
\partial_t f + v \cdot \nabla_x f = Q_B(f, f)
\]
where \(Q_B\) denotes the Boltzmann collisional operator. In this setting the function \(f(x, v, t)\) is represents the density of particles with position \(x\) and velocity \(v\) at time \(t\). Landau’s original
intent in deriving this approximation was to make sense of the Boltzmann collisional operator, which always diverges when considering an interaction by a Coulombic force.

However, by neglecting all types of collision other than grazing ones Landau gave a formal justification that $f$ essentially solves the equation

$$
\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad x, v \in \mathbb{R}^3, \; t > 0,
$$

(1.5)

for a bilinear operator $Q(f, f)$ obtained from $Q_B$. This operator $Q(f, f)$ has been known since as the Landau collisional operator. The quadratic operator $Q$ acts only with respect to the $v$ variable, typically it is as in (1.2). In general, it has the form

$$
Q(f, f)(v) = \text{div} \left( \int_{\mathbb{R}^3} \Psi(v - w) \Pi(v - w) (f(w) \nabla_v f(v) - f(v) \nabla_w f(w)) \, dw \right).
$$

(1.6)

Here $\Pi(v)$ denotes the projection in the orthogonal complement to $v$,

$$
\Pi(v) := I - \hat{v} \otimes \hat{v}.
$$

and $\Psi(v)$ is a scalar valued function determined from the original Boltzmann kernel describing how particles interact. Note that in writing $Q(f, f)$ we have switched from the index notation in (1.2) to vector notation. If the interaction between particles at a distance $r$ is proportional to $1/r^s$, $\Psi$ takes a form which is typically written as

$$
\Psi(v) = |v|^{2+\gamma}, \quad \gamma = (s - 5)/(s - 1).
$$

In particular, for a Coulombic force ($s = 2$) we have $\gamma = -3$ and $\Psi(v) = c|v|^{-1}$ for some physical constant $c$.

In the kinetic literature, $\gamma \geq 0$ is referred as the hard-potentials case and $\gamma < 0$ as the case of soft-potentials. Although the case of hard potentials does not include the physically important Coulomb interaction, the equation is much better understood in this case (see next section).

1.3. **Mathematics background.** Even though we do not expect (1.1) to satisfy a comparison principle, it is quite natural to approach its analysis from the point of view of nonlinear parabolic equations. To see why this must be so, it suffices to note that any smooth solution to (1.1) with (1.6) solves a parabolic equation. One only needs to take the common factor $\nabla f_v(v)$ and $f(v)$ out of the integral in (1.6), which can be done given enough smoothness (see below).

The result equation is semi-linear except for the fact that the diffusion coefficients depend non-locally on the solution itself. It may be written in either divergence or non-divergence form,

$$
\partial_t f = \text{div} \left( A[f] \nabla f - f \nabla a[f] \right) = \text{Tr}(A[f] D^2 f) - f a[f],
$$

where $a[f] := \text{Tr}A[f]$ and for $\Psi$ as in (1.6) the matrix $A[f]$ is given by

$$
A[f](v) := \int_{\mathbb{R}^3} \Pi(w) \Psi(w) f(v - w) \, dw,
$$

(1.7)

in other words, $A[f]$ is obtained by convolving $f$ with the matrix kernel $w \to \Pi(w) \Psi(w)$.

Although $f$ solves a parabolic equation, the non-local dependence of the coefficients on the solution prevents the full equation from satisfying a comparison principle: if $f^{(1)}$ touches $f^{(2)}$ from below, it does not follow that $Q(f^{(1)}, f^{(1)}) \leq Q(f^{(2)}, f^{(2)})$ at the contact point. Specifically, in this situation one cannot expect an inequality such as

$$
\text{Tr}(A[f^{(1)}] D^2 f^{(1)}) \leq \text{Tr}(A[f^{(2)}] D^2 f^{(2)})
$$
since when \( f^{(1)} \leq f^{(2)} \) one only has \( A[f^{(1)}] \neq A[f^{(2)}] \) at every point except when \( f^{(1)} \equiv f^{(2)} \) (a manifestation of the non-locality). There is not even a maximum principle, since at a maximum point for \( f \) we only obtain \( \partial_t f \leq -f \Delta a[f] \), which does not rule out finite time blow up of the maximum of \( f \). However, if one can control (for instance, in \( L^\infty \)) the size of \( a[f] \) and the ellipticity (i.e. eigenvalues) of the matrix \( A[f] \), then the parabolic equation will provide higher regularity for \( f \) via a bootstrapping effect. Thus, the problem of regularity estimates for (1.1) becomes a question of bounding \( \|f\|_\infty \).

In the particular setting of hard potentials (\( \gamma \geq 0 \)) Desvillettes and Villani overcome this particular difficulty, allowing them to develop a rather complete theory of existence, uniqueness and convergence to equilibrium [7, 8]. They observe that higher moment bounds should be propagated by the equation. Exploiting this observation they control the Hölder norms of \( A[f] \) and \( a[f] \) over time, using the higher moments in the pointwise bound

\[
\sup_v |A[f](v)| \leq C \int_{\mathbb{R}^3} f(w)(1 + |w|^{2+\gamma}) \, dw.
\]

They also control the lowest eigenvalue of \( A[f] \) over time, which allows them to use parabolic theory as hinted at above. In this way they obtain the existence of weak solutions as well as their instantaneous regularization. On the other hand, uniqueness of weak solutions was achieved via Gronwall’s lemma. This required several moment estimates for the solutions constructed and a delicate calculation involving a weighted \( L^2 \) norm of the difference of two solutions.

The decay to equilibrium is done in [8]. It is shown there are constants \( \lambda, c_0 > 0 \) only depending on the initial mass, first and second moments, such that

\[
\|f(\cdot, t) - \mathcal{M}\|_{L^1(\mathbb{R}^3)} \leq c_0 e^{-\lambda t} \quad \forall t > 0,
\]

where \( \mathcal{M} \) is a Maxwellian, that is an equilibrium distribution. They key step of their argument is showing that, for hard potentials, the entropy production can be used to control the Fisher information of \( f \), this controls the relative entropy of \( f \) with respect to \( \mathcal{M} \) thanks to the logarithmic Sobolev inequality. Once this is shown, exponential convergence to equilibrium in \( L^1 \) follows via a Csiszár-Kullback inequality (see discussion at the end of the next section).

Analyzing the soft potential case has proved more difficult. Villani [20] introduced the so called \( H \)-solutions in this case, which enjoy (weak) a priori bounds in a weighted Sobolev space. However, the issue of their uniqueness or regularity (no finite time break down occurs) has remained open, even for smooth initial data (see [21, Chapter 1, Chapter 5] for further discussion).

A completely different approach based on perturbation theory is developed by Guo in [11], where the spatially inhomogeneous Landau equation is considered in \( \mathbb{R}^3 \). The potential is given by the Coulomb interaction, and \( f \) is periodic in the spatial variable \( x \). It is shown that if the initial data is close enough in a certain high Sobolev norm (containing derivatives of order 8) then a unique global solution exists. Moreover, as remarked in [11], this approach also extends to the case of potentials where \( \gamma \) might even take values below \(-3\).

From a different perspective, Krieger-Strain [13] consider an isotropic version of (1.1)

\[
\partial_t f = \{\Delta^{-1} f\} \Delta f + \alpha f^2 \quad \alpha \in (0, 1 - \varepsilon),
\]

and show existence of smooth global solutions starting from very general initial data.
1.4. Main difficulties and general strategy. In the case of soft potentials \( \psi(v) = c_\gamma |v|^\gamma + 2 \), \( \gamma < 0 \), note that \( \Delta a[f] \) is given by convolving \( f \) with a kernel which is not locally bounded. As such, the obvious conserved integral quantities (mass, moments) do not suffice to control \( \|\Delta a[f]\|_\infty \). This is particularly bad for the Coulomb interaction \( \gamma = -3 \), where \( a[f] \) is the Newtonian potential of \( f \), so \( \Delta a[f] = -f \) and the non-divergence form of the equation becomes
\[
\partial_t f = \text{Tr}(A[f]D^2 f) + f^2.
\]
This is reminiscent of the well studied semi-linear heat equation
\[
\partial_t f = \Delta f + f^2,
\]
which is known to blow up in finite time, and suggests (1.1) may also break down for finite \( t \). However, as mentioned in the previous section, this has only been ruled out when the initial condition is a small and smooth perturbation from equilibrium [11].

However, unlike the semilinear heat equation, the Landau equation preserves the \( L^1 \) norm, while blow up for the semilinear heat equation is known to happen for every \( L^p \) norm. Another notable difference is that compared to the semilinear heat equation the Landau equation admits a richer class of equilibrium solution: every Maxwellian \( M \) solves
\[
Q(M, M) = 0
\]
which holds in particular, for those with arbitrarily large mass.

Our strategy is partly inspired by work on a different non-local equation which also does not enjoy a comparison principle. The non-local porous medium equation is given by
\[
\partial_t u + \text{div}(u \nabla (-\Delta)^{-s} u) = 0,
\]
and it falls in the class of active scalar equations. It may also be written as
\[
\partial_t u + u(-\Delta)^{1-s} u + (\nabla u, \nabla (-\Delta)^{-s} u) = 0.
\]
Here the non-locality is in the diffusion operator and the drift and not on the diffusion coefficients. Although this equation does not enjoy a comparison principle, Caffarelli and Vazquez [5] identify very specific functions, that they call “true supersolutions”, with the property that any solution that starts below them stays below them for all later times. This fact allows the construction of global weak solutions for fairly general initial data. Such solutions have been shown to be Hölder continuous [4]. Uniqueness, however, is only known in one dimension [3].

Remark 1.4. Broadly speaking, barrier arguments have not been very useful in obtaining upper bounds on solutions to kinetic equations. They have, however, been useful in deriving lower pointwise bounds. Indeed, by neglecting the \( f^2 \) term in the equation one obtains
\[
\partial_t f \geq \text{Tr}(A[f]D^2 f),
\]
so any solution \( f \) is a supersolution of a linear parabolic equation for which barriers are available provided bounds for the eigenvalues of \( A[f] \) are available.

In vague terms, the key observation in this work (see Section 2) is that a solution to the Landau equation can only touch a Maxwellian from below in a very specific way: if a solution \( f \) is such that \( f(\cdot, t) \leq M(\cdot) \) up to time \( t_0 \) and \( f(v_0, t_0) = M(v_0) \) for some \( v_0 \), then it turns out that either \( f(\cdot, t_0) \) is uniformly close to \( M(\cdot) \), or \( v_0 \) is really close to the center of mass of \( M \). This observation also holds for linear approximations to the Landau equation (see Lemma 2.4).
This contact analysis suggests one may think of Maxwellians as “partial barriers” that can be used to trap a solution below them for all times. This can be done in even greater generality than what is needed here by invoking the machinery of viscosity solutions (Lemma 3.2). Furthermore, we believe this method could be used elsewhere, even in the spatially inhomogeneous setting.

**Remark 1.5.** The validity of upper Gaussian (i.e. Maxwellian) bounds is an important question in kinetic theory. For the Boltzmann equation with hard potentials, where a priori estimates are available, upper Maxwellian bounds were obtained by Gamba, Panferov and Villani [9]. In the present work, the emphasis has been on showing global existence of solutions for soft potentials, where a priori estimates were not previously available. A question left open by our method is the global well-posedness for initial distributions with finite mass, energy and entropy, but which might not be bounded and have polynomial spatial decay.

The propagation of upper Maxwellian bounds for the linear approximations (Lemma 4.5) is combined with the effects of parabolic regularization in Section 4. This permits the construction of global solutions via an iteration scheme.

Since the comparison principle fails, one must look elsewhere for methods to show uniqueness. A Gronwall lemma for the $L^2$ distance between two solutions is obtained in Lemma 5.5. This shows uniqueness in the class of solutions which are bounded and have enough spatial decay. The proof of the lemma is based on the bilinear structure of the interaction operator and the application of weighted singular integral estimates (see Proposition 5.4). Finally, exponential decay to equilibrium is proved by arguing as in the proof of the logarithmic Sobolev inequality from the $H^1 \rightarrow L^{2*}$ Sobolev embedding, except here we rely on the entropy production functional directly and not on the Fisher information. In this sense, this last estimate can be regarded as a weighted inequality in its own right.

1.5. **Outline.** The rest of the paper is organized as follows. An overview of the notation is done at the end of this section. In Section 2 the contact analysis hinted at above is carried out in detail. Section 3 reviews the viscosity solution machinery which is used to prove a general barrier lemma. These tools are used in Section 4 to construct a global solution to (1.3) through an iteration scheme, by using the propagated Maxwellian bounds and parabolic regularity theory. In Section 5 weighted integral estimates are applied to prove both the uniqueness component of Theorem 1.2 and the exponential decay to equilibrium.

1.6. **Notation.** Universal constants will be denoted by $c, c_0, c_1, C_0, C_1, C$. Vectors in $\mathbb{R}^3$ will be denoted by $v, w, x, y$ and so on, the inner product between $v$ and $w$ will be written $(v, w)$. We will also write $\langle v \rangle = 1 + |v|$. $B_R(v_0)$ denotes closed ball of radius $R$ centered at $v_0$, if $v_0 = 0$ we simply write $B_R$. The set of unit vectors in $\mathbb{R}^n$ is $S^{n-1}$, and $\mathbb{R}_+$ as is usual will denote the open interval $(0, +\infty)$.

The identity matrix will be denoted by $I$, the trace of a matrix $X$ will be denoted $\text{Tr}(X)$. We will denote by $f, f_1, \hat{f}, \text{etc}$ both functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$, in the latter case, we will often use the subscript notation $f_t$ to denote the function restricted to the fixed time $t$.

The initial distribution for the Cauchy problem will always be denoted by $f_{in}$.

**Norms.** Given any $p \in [1, \infty)$ and $a \in \mathbb{R}$ we introduce the weighted $L^p$ norm

$$
\|g\|_{L^p_a(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} |g(v)|^p \langle v \rangle^a \, dv \right)^{1/p}.
$$
Likewise, we shall write $\|f\|_{L^\infty(R^3)} := \|f(\cdot)^a\|_{L^\infty(R^3)}$. Given $\alpha, \beta \in (0, 1)$ we also use the standard notation for the space-time Hölder semi-norm $[\phi]_{C^{\alpha, \beta}}$ as well as $C^{k,l}$ and $C^{k,l}_{\text{loc}}$.

Maxwellians and conserved quantities. A particular type of distribution that will play an important role in all what follows is the Maxwellian.

$$M(v) := \frac{m}{(2\pi)^{3/2}T^3} e^{-|v-v_0|^2/(2T^2)}.$$  

In particular, we will use $M_0$ for the standard (normalized) Maxwellian,

$$M_0(v) := \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$  

If $\xi \in \mathbb{R}^3$ then we will write $M_\xi(v) := M_0(v - \xi)$. For a given $f$ we will denote its mass, momentum, energy and entropy respectively by

$$M(f) := \int_{\mathbb{R}^3} f \, dv, \quad P(f) := \int_{\mathbb{R}^3} f v \, dv, \quad E(f) := \int_{\mathbb{R}^3} f |v|^2 \, dv, \quad H(f) := \int_{\mathbb{R}^3} f \log(f) \, dv.$$  

Moreover, given any $f$, $M^f$ will denote the unique Maxwellian given by

$$M(f) = M(M^f), \quad P(f) = P(M^f), \quad E(f) = E(M^f).$$  

2. Ellipticity and contact analysis

This section deals with properties of smooth solutions of the linear parabolic equation

$$\partial_t g = Q(f, g)$$

for some non-negative smooth function $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. In particular, $f$ is a smooth solution of the Landau equation if and only if $g = f$ is itself a solution of this linear equation.

The main observation (Lemma 2.4) is that a smooth $g$ solving the above equation cannot touch a Maxwellian $M_\xi$ too far away from its center of mass $\xi$. Applying this in particular to $f$ solving (1.3) (in which case $g$ above agrees with $f$) yields a necessary condition for a smooth solution of (1.3) to touch a Maxwellian from below. The proof of this estimate follows a standard practice from the theory of viscosity solutions, and it consists in simply evaluating the Hessian and time derivative of $g$ at the contact point, and then using the equation to compare them with the derivatives of $M_\xi$ (which is also a solution).

2.1. Ellipticity estimates. The following propositions will guarantee that the matrix $A[f]$ is uniformly positive in compact subsets of $\mathbb{R}^3$, and its smallest eigenvalue decays like $|v|^{-1}$, the bounds depending only on the mass, energy and entropy of $f$. The arguments are very much in the spirit of the analysis done for hard potentials in [7, Section 4].

Proposition 2.1. Let $f \in L^1(\mathbb{R}^3)$ be non-negative. If there are constants $r \geq 1, \delta > 0$ and $\mu \in [0,1]$ such that

$$S \subset B_r, \quad |S| \geq (1-\mu)|B_r| \Rightarrow \int_S f(v) \, dv \geq \delta,$$

then the following lower pointwise bound holds

$$A[f](v) \geq \frac{\delta r \sqrt{\mu}}{16\pi (r + |v|)^2} \quad \forall \, v \in \mathbb{R}^3.$$
Proof. Let us recall two elementary geometric facts. The first is that if \( v \in \mathbb{R}^3 \) and \( S \subset B_r \) then \( w \in S \Rightarrow |v - w| \leq r + |v| \), so that

\[
\int_S \frac{f(w)}{8\pi v - w} \, dw \geq \frac{1}{8\pi(r + |v|)} \int_S f(w) \, dw.
\]

The second is that for any cone of the form \((v, \xi, \theta)\)

\[ K_v(\xi, \theta) := \{ w \in \mathbb{R}^3 \mid -\theta \leq \angle(\xi, v - w) \leq \theta \}, \]

the following inequality holds

\[ (\Pi(v - w)\xi, \xi) \geq \sin(\theta)|\xi|^2, \quad \forall w \notin K_v(\xi, \theta). \]

Indeed, if the lines through \( \xi \) and \( v - w \) form an angle \( \angle(\xi, v - w) \in (0, \pi/2) \), then

\[ (\Pi(v - w)\xi, \xi) = \sin(\theta)|\xi|^2. \]

Now, \((A[f](v)\xi, \xi)\) can be bounded as follows: if \( v \in \mathbb{R}^3, \xi \in S^2, \theta \in (0, \pi/2)\)

\[
(A[f](v)\xi, \xi) \geq \left( \int_{B_r \setminus K_v(\xi, \theta)} \frac{f(w)}{8\pi|v - w|} (\Pi(v - w)\xi, \xi) \, dw \right) \sin(\theta)|\xi|^2.
\]

Using the first observation above with \( S = B_r \setminus K_v(\xi, \theta) \) yields

\[
(A[f](v)\xi, \xi) \geq \left( \frac{1}{8\pi(r + |v|)} \int_S f(w - v) \, dw \right) \sin(\theta)|\xi|^2.
\]

Note \( B_r \cap K_v(\xi, \theta) \) is contained in a cylinder of height \( 2r \) and base \( \tan(\theta)(r + |v|) \), so

\[
|B_r \cap K_v(\xi, \theta)| \leq 2r\pi \tan(\theta)^2(r + |v|) = \frac{3\tan(\theta)^2}{2r^2} (r + |v|)^2 |B_r|.
\]

Let \( \theta > 0 \) be taken so that

\[
\tan(\theta)^2 = \frac{2r^2}{3(r + |v|)^2} \mu.
\]

In which case \(|S| \geq (1 - \mu)|B_r|\) and then (by the assumption of the lemma) the integral over \( S \) is at least \( \delta \). This proves the bound

\[ (A[f](v)\xi, \xi) \geq \frac{\delta \sin(\theta)}{8\pi(r + |v|)} |\xi|^2. \]

On the other hand, the way \( \theta \) was picked guarantees that (recall that \( r \geq 1, \mu \in (0, 1) \))

\[
\sin(\theta) = \frac{\tan(\theta)}{\sqrt{1 + (\tan(\theta))^2}} = \frac{2r\sqrt{\mu}}{\sqrt{9(r + |v|)^2 + 4r\mu}} \geq \frac{2r\sqrt{\mu}}{\sqrt{13(r + |v|)^2}}.
\]

Using that \( 2/\sqrt{13} \geq 1/2 \) and substituting the bound for \( \sin(\theta) \) in the bound for \( A[f] \) we obtain

\[ (A[f](v)\xi, \xi) \geq \frac{\delta r \sqrt{\mu}}{16\pi(r + |v|)^2} |\xi|^2. \]

This proves the estimate. \( \square \)
Proposition 2.2. Let $f$ be a non-negative function in $L^1(\mathbb{R}^d)$ with positive mass $M$ and finite energy $E$ and entropy $H$. Then for any $S \subset B_\rho$ with $|B_\rho \setminus S| \leq \mu |B_\rho|$ we have

$$\int_S f(v) \, dv \geq \frac{M}{2},$$

where $R$ and $\mu$ are determined by

$$\rho := 2\sqrt{\frac{E}{M}}, \quad \mu := \sqrt{\frac{M^{d+1}}{E^d}} \frac{e^{-8\hat{H}}}{2^{d+3}|B_1|},$$

(2.1)

where $\hat{H} = \int_{\mathbb{R}^3} f |\log(f(v))| \, dv$. In particular, $\rho$ and $\mu$ are controlled by $M$, $E$ and $H$.

Proof. Observe that for any $\rho > 0$

$$\int_{B_\rho^c} f(v) \, dv \leq \frac{1}{\rho^2} \int_{B_\rho^c} f(v)^2 \, dv = \frac{E}{\rho^2}.$$  

From here it follows that

$$\int_{B_\rho} f(v) \, dv \geq M - \frac{E}{\rho^2}. \quad (2.2)$$  

On the other hand, given any $S \subset B_\rho$, and setting $\hat{H} = \int_{\mathbb{R}^3} f |\log(f(v))| \, dv$,

$$\int_{B_\rho \setminus S} f(v) \, dv = \int_{B_\rho \setminus S} f(v) \mathbf{1}_{\{f \leq e^{2\hat{H}/\epsilon}\}} \, dv + \int_{B_\rho \setminus S} f(v) \mathbf{1}_{\{f > e^{2\hat{H}/\epsilon}\}} \, dv,$$

$$\leq |B_\rho \setminus S| e^{2\hat{H}/\epsilon} + \int_{B_\rho \setminus S} f(v) \log f(v) \frac{2\hat{H}}{\epsilon} \, dv,$$

$$\leq |B_\rho \setminus S| e^{2\hat{H}/\epsilon} + \epsilon/2.$$  

Then, if $\eta(\epsilon) := \frac{\epsilon}{2e^{2\hat{H}/\epsilon}}$ the above inequalities yield

$$|B_\rho \setminus S| \leq \eta(\epsilon) \Rightarrow \int_{B_\rho \setminus S} f(v) \, dv \geq \int_{B_\rho} f(v) \, dv - \epsilon. \quad (2.3)$$

Finally, to prove the proposition, observe that if $\rho$ is as in (2.1) then (2.2) yields

$$\int_{B_\rho} f(v) \, dv \geq \frac{3}{4} M.$$  

Moreover, a basic calculation shows that if $\rho$ and $\mu$ are as in (2.1) then also

$$\mu |B_\rho| \leq \eta(\frac{1}{4} M).$$

The last two inequalities combined with (2.3) prove the proposition. \hfill \Box

Corollary 2.3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a non-negative function with finite positive mass $M$, finite energy $E$ and finite entropy $H$. Then with $\rho$ and $\mu$ as in (2.1)

$$A[f](v) \geq \frac{M \rho \sqrt{\rho}}{32\pi (\rho + |v|)^2} \mathbb{I} \quad \forall \ v \in \mathbb{R}^3.$$
2.2. **Contact of classical solutions with Maxwellians.** Although elementary, the following observation serves as motivation for the proof of Theorem 1.2.

**Lemma 2.4.** Let $g : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative $C^{2,1}$ function solving
\[
\partial_t g = Q(f, g) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+
\]
for some function $f(v, t)$. Suppose there is $\xi \in \mathbb{R}^3$ and $m > 0$ for which the following holds,

1. There is $(v_0, t_0) \in \mathbb{R}^3 \times (0, T)$ at which $g$ touches $m\mathcal{M}_\xi$ from below, namely
\[
g \leq m\mathcal{M}_\xi \quad \text{in } \mathbb{R}^3 \times [0, t_0],
\]
\[
g = m\mathcal{M}_\xi \quad \text{at } (v_0, t_0).
\]

2. The function $f$ is such that $f_{10} \leq \mathcal{M}_\xi$ and $f_{10} \neq \mathcal{M}_\xi$ in a set of positive measure, and
\[
A[m\mathcal{M}_\xi - f(\cdot, t_0)] \geq \frac{C}{1 + |v - \xi|}a[m\mathcal{M}_\xi - f(\cdot, t_0)]I \quad \forall v \in \mathbb{R}^3,
\]
for some $C > 0$.

Then, the contact point $v_0$ must lie in $B_{R_0}(\xi)$, where $R_0 = 4(1 + C)^{-1}$.

**Proof.** The fact that $g$ is touching $m\mathcal{M}_\xi$ from below and from below at $(v_0, t_0)$ implies that
\[
\partial_t g(v_0, t_0) \geq 0,
\]
\[
mD^2\mathcal{M}_\xi(v_0, t_0) \geq D^2 g(v_0, t_0).
\]
Combining these two inequalities with the equation $\partial_t g = Q(f, g)$ it follows that at $(v_0, t_0)$
\[
0 = \partial_t g - \text{Tr}(A[f]D^2 g) - gf \geq -m\text{Tr}(A[f]D^2 \mathcal{M}_\xi) - m^2 \mathcal{M}_\xi^2.
\]
Adding and subtracting $m\text{Tr}(A[m\mathcal{M}_\xi - f]D^2 \mathcal{M}_\xi)$ it follows that
\[
0 \geq m\text{Tr}(A[m\mathcal{M}_\xi - v]D^2 \mathcal{M}_\xi),
\]
where the cancellations followed from $Q(\mathcal{M}_\xi, \mathcal{M}_\xi) = 0$. To analyze this inequality, recall that
\[
D^2 \mathcal{M}_\xi(v) = (-I + (v - \xi) \otimes (v - \xi))\mathcal{M}_\xi(v)
\]
Substituting,
\[
0 \geq \text{Tr}[A[m\mathcal{M}_\xi - f]D^2 \mathcal{M}_\xi] = \text{Tr}[A[m\mathcal{M}_\xi - f](-I + (v - \xi) \otimes (v - \xi))]\mathcal{M}_\xi
\]
\[
= \left(\left(A[m\mathcal{M}_\xi - f](v - \xi), v - \xi\right) - \text{Tr}[A[m\mathcal{M}_\xi - f]]\right)\mathcal{M}_\xi
\]
Since $\mathcal{M}_\xi > 0$, it follows that the quantity inside the parenthesis is non-positive. Then, the inequality in the second assumption yields
\[
0 \geq (A[m\mathcal{M}_\xi - f](v - \xi), (v - \xi)) - a[m\mathcal{M}_\xi - f],
\]
\[
\geq a[m\mathcal{M}_\xi - f]\left(C(1 + |v - \xi|)^{-1}|v - \xi|^2 - 1\right).
\]
Since $\mathcal{M}_\xi > f$ in some small ball, $a[m\mathcal{M}_0 - u] > 0$. In this case, it follows that
\[
C|v - \xi|^2 \leq (1 + |v - \xi|)
\]
from where it follows that $|v - \xi| \leq 4(1 + C)^{-1}$, and the lemma is proved.

\[\square\]
Figure 1. If $f$ solves the linear Landau equation or the full Landau equation then it cannot “touch” a given Maxwellian from below in an arbitrary way. The closer the point of contact is to the maximum of the Maxwellian, the closer that $f$ must be to the Maxwellian everywhere.

The above lemma shows there is some interest in finding sufficient conditions for a function $f$ to satisfy, for some $m, \xi$ and $C$, the bounds

$$
\begin{align*}
A[mM_\xi - f] &\geq \frac{C}{1 + |v - \xi|} a[mM_\xi - f] I \quad \forall \ v \in \mathbb{R}^3, \\
f &\leq mM_\xi \quad \text{in} \ \mathbb{R}^3, \quad \text{with } |\{mM_\xi - f > 0\}| > 0.
\end{align*}
$$

(2.4)

In fact, an equivalent and more convenient formulation of Lemma 2.4 is the following.

**Proposition 2.5.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a non-negative function, $C, m > 0$ and $\xi \in \mathbb{R}^3$ such that (2.4) is satisfied. Then, for $R_0 := 4(1 + C)^{-1}$ the following holds

$$
Q(f, mM_\xi)(v) = m\text{Tr}(A[f]D^2M_\xi) + mM_\xi f < 0 \quad \text{in} \ \mathbb{R}^3 \setminus B_{R_0}(\xi).
$$

As it turns out, (2.4) holds if $f \geq 0$ and lies below two Maxwellians. Namely,

$$
0 \leq f(v) \leq \min\{mM_{R\tau}(v), mM_{-R\tau}(v)\}, \quad \forall \ v \in \mathbb{R}^3.
$$

(2.5)

**Lemma 2.6.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, satisfies (2.5) for some $R \geq 2$, $\tau \in S^2$ and $m > 0$. Then, (2.4) holds with the same $m$, $C = 1/50$ and $\xi = R\tau$.

**Proof.** The proof consists in obtaining a lower bound for $A[mM_{R\tau} - f]$ and an upper bound for $a[mM_{R\tau} - f]$, we begin with the latter. Observe that $a[mM_{R\tau} - f] \leq a[mM_{R\tau}]$ due to the positivity of $f$. Moreover,

$$
a[mM_{R\tau}](v) = a[mM_0](v - R\tau) \quad \forall \ R, \tau.
$$

Recall Newton’s formula for the Newtonian potential of a radial function (see [16, Theorem 9.7]). It says that if $g$ is a radial function $g(v) = h(|v|)$ then

$$
a[g](v) = \frac{1}{3|v|} \int_0^{|v|} h(t)t^2 \ dt + \frac{1}{3} \int_{|v|}^{+\infty} h(t)t \ dt,
$$

Let $h(t) = (2\pi)^{-3/2}e^{-t^2/2}$ so that $M_0(v) = h(|v|)$, then

$$
a[M_0] = \frac{1}{3(2\pi)^{3/2}|v|} \int_0^{|v|} e^{-t^2/2}t^2 \ dt + \frac{1}{3(2\pi)^{3/2}} \int_{|v|}^{+\infty} e^{-t^2/2}t \ dt.$$


By a straightforward estimate, it follows that

\[
a[M_0] \leq \begin{cases} 
\frac{1}{3\pi|v|} & \text{if } |v| \geq 1, \\
\frac{1}{3\pi} & \text{if } |v| < 1.
\end{cases}
\]

Therefore,

\[
a[M_0] \leq \frac{1}{\pi} \frac{1}{1 + |v|}, \quad \forall \, v \in \mathbb{R}^3.
\]

Translating by \(R\tau\) leads to the upper estimate

\[
a[mM_{R\tau} - f] \leq \frac{1}{\pi} \frac{m}{1 + |v - R\tau|}, \quad \forall \, v \in \mathbb{R}^3.
\] (2.6)

For the lower bound, since by assumption \(f \leq mM_{-R\tau}\), we have

\[
mM_{-R\tau}(v) - f(v) \geq mM_{-R\tau}(v) - mM_{-R\tau}(v) = me^{-\frac{1}{2}|v-R\tau|^2} - me^{-\frac{1}{2}|v+R\tau|^2}.
\]

Take \(R \geq 2\) and \(|v-R\tau| \leq 1\) so that \(v \cdot \tau \geq 1\). This guarantees that

\[
e^{-\frac{1}{2}|v-R\tau|^2} - e^{-\frac{1}{2}|v+R\tau|^2} \geq \frac{1}{2}e^{-\frac{1}{2}|v-R\tau|^2},
\]

\[
\Rightarrow mM_{R\tau}(v) - f(v) \geq \frac{m}{16}e^{-\frac{1}{2}|v-R\tau|^2}.
\]

In particular, this implies that

\[
mM_{R\tau}(v) - f(v) \geq \frac{1}{4}m \quad \forall \, v \in B_1(R\tau).
\]

Proposition 2.1 with \(r = 1, \mu = \frac{3}{4}, \delta = \frac{1}{4}|B_1| = \pi/3\), gives

\[
A[\chi_{B_1(R\tau)}](v) \geq \frac{\pi\sqrt{3}}{6\pi} \frac{1}{16\pi(1 + |v|)^2} = \frac{1}{32\sqrt{3}(1 + |v|)^2}I,
\]

and this becomes

\[
A[mM_{R\tau} - f] \geq \frac{m}{4}A[\chi_{B_1(R\tau)}] \geq \frac{m}{240} \frac{1}{(1 + |v-R\tau|)^2}I. \quad (2.7)
\]

Putting inequalities (2.6) and (2.7) together leads to

\[
A[mM_{R\tau} - f] \geq \frac{1}{50} \frac{a[mM_{R\tau} - f]}{1 + |v-R\tau|}I,
\]

and the lemma is proved.

\[
\square
\]

**Corollary 2.7.** Suppose \(f\) satisfies (2.5) for some \(m > 0\), \(R \geq 4\) and \(\tau \in \mathbb{S}^2\), then

\[
Q(f, mM_{R\tau})(v) < 0 \quad \forall \, v \in \mathbb{R}^3 \setminus B_R(R\tau).
\]

**Proof.** By Lemma 2.6, the function \(f\) satisfies (2.4) with \(C = 1/50\), \(\xi = R\tau\) and the same \(m\). Since \(R_0 < 4 \leq R\) we always have \(\mathbb{R}^3 \setminus B_R(R\tau) \subset \mathbb{R}^3 \setminus B_{R_0}(R\tau)\) in which case the Corollary follows from Proposition 2.5.

\[
\square
Figure 2. If a solution $f$ lies below two Maxwellians whose centers are far apart then it cannot ever touch their minimum from below.

3. TRAPPING THROUGH PARTIAL BARRIERS

The one lemma in this section involves the comparison principle for parabolic equations in all of $\mathbb{R}^3$. In particular, it is observed that the simultaneous use of several barriers helps can be used to overcome fact that the equation (1.1) does not have a comparison principle. When only one barrier is needed, it reduces to the usual comparison principle.

Remark 3.1. It is worth noting that the lemma is stated in greater generality than required for the proof of Theorem 1.2. The initial proof of Theorem 1.2 involved several barriers for different parabolic equations, and the lemma was proved to handle such a situation. In the end everything worked out for a single barrier (namely, the minimum of two Maxwellians, see Section 4). However, the idea of using barriers for different equations in different regions is of independent interest (in particular, it might be needed for the spatially inhomogeneous case), and the proof is not much made much more complicated by this feature.

The proof of the lemma is through the method of doubling of variables, which is a staple of the theory of viscosity solutions both for Hamilton-Jacobi equations and second order parabolic equations. See the guide by Crandall, Ishii and Lions [6, Sections 3 and 5] for a detailed and general presentation of this theory.

In the interest of making the proof as accessible as possible, the lemma is stated stated and proved only for smooth functions. A regularization procedure in Section 4 will give enough regularity. On the other hand, the lemma will still hold for viscosity solutions, by invoking the Crandall-Jensen-Ishii Lemma (see [12] and also [6, Appendix]), and the proof of the existence theorem could have been done within the framework of viscosity solutions.

Lemma 3.2. Let $f : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$, and assume that

1. $f$ is a $C^{2;1}_{loc}$ function which is bounded in $\mathbb{R}^3 \times [0, T]$ for each $T > 0$.
2. There is a finite open cover $\{D_i\}_i$ of $\mathbb{R}^3$.
3. For each $i$ there is a $F_i: \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ bounded in the $C^{2;1}$ norm and such that

$F := \min_j F_j \equiv F_i$ in $D_i \times \mathbb{R}_+$ $\forall$ $i$.  


(4) For each $i$ there is an elliptic operator $L_i$

$$L_i \phi := \text{Tr}(A_i(v,t)D^2 \phi) + b_i(v,t) \cdot \nabla \phi + c_i(v,t) \phi,$$

where $A_i(v,t)$ is a symmetric matrix with non-negative eigenvalues and

$$|A_i|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}_+)} \leq C$$

for some $(\alpha, \beta, \epsilon)$. For any $\epsilon > 0$ there is an elliptic operator

$$L \phi := \text{Tr}(L \phi) + b \phi + c \phi,$$

where $L$ is a symmetric matrix with non-negative eigenvalues and

$$|L|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}_+)} \leq C$$

with a constant $C$ that is independent of $i$.

(5) The functions $f, F_i$ satisfy the following inequalities in the viscosity sense

$$\begin{align*}
\partial_t f & \leq L_i f & \text{in } D_i \times \mathbb{R}_+, \\
\partial_t F_i & \geq L_i F_i
\end{align*}$$

(3.1)

(6) $f(v,0) < F_i(v,0)$ for each $i$.

Then, it follows that $f \leq F_i$ in $\mathbb{R}^3 \times \mathbb{R}_+$ for each $i$.

*Proof.* Arguing by contradiction, suppose there are $T, \delta > 0$ such that

$$f(x_0,t_0) - F(x_0,t_0) \geq 2\delta$$

for some $(x_0,t_0) \in \mathbb{R}^3 \times (0,T]$.

As in the theory of viscosity solutions, let us argue by doubling the number of spatial variables. The argument has two parts.

**Part 1: Picking test functions.** For any $\alpha, \beta, \epsilon > 0$ (determined below) define

$$\Phi(x,y,t) := f(x,t) - F(y,t) - \frac{\alpha}{2}|x-y|^2 + \beta(t-T) - \epsilon(|x|^2 + |y|^2),$$

which is a continuous function defined in $\mathbb{R}^6 \times \mathbb{R}_+$. Note that as long as

$$\epsilon \leq \frac{\delta}{8|x_0|^2}, \quad \beta \leq \frac{\delta}{4T},$$

the following inequality will hold

$$\Phi(x_0,x_0,t_0) \geq 2\delta + \beta(t_0 - T) - 2\epsilon|x_0|^2 \geq \delta > 0.$$  

On the other hand, $\Phi$ is bounded since $f, F$ are bounded in $\mathbb{R}^3 \times [0,T]$. It follows that

$$0 < \delta \leq \sup_{\mathbb{R}^6 \times [0,T]} \Phi(x,y,t) < +\infty.$$  

By picking $\alpha$ sufficiently large, we can guarantee that $\Phi < 0$ outside a large compact set contained in $\mathbb{R}^6 \times (0,T]$. It follows there exists a number $\bar{t} = \bar{t}(\alpha, \beta, \epsilon) \in (0,T)$ such that

$$\sup_{\mathbb{R}^6 \times [0,\bar{t}]} \Phi(x,y,t) = 0$$

and the first supremum is achieved at some $(\bar{x}, \bar{y}, \bar{t})$, where $\bar{x} = \bar{x}(\alpha, \beta, \epsilon), \bar{y} = \bar{y}(\alpha, \beta, \epsilon)$. These will be the points where the differential inequalities in assumption (5), as such it will be convenient to estimate $|\bar{x} - \bar{y}|$.

Indeed, since $\Phi(\bar{x}, \bar{x}, \bar{t}) \leq \Phi(\bar{x}, \bar{y}, \bar{t})$ it follows that

$$\Phi(\bar{x}, \bar{y}, \bar{t}) = f(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t}) - \frac{\alpha}{2}|\bar{x} - \bar{y}|^2 + \beta(\bar{t} - T) - \epsilon(|\bar{x}|^2 + |\bar{y}|^2),$$

$$\geq \Phi(\bar{x}, \bar{x}, \bar{t}),$$

$$= f(\bar{x}, \bar{t}) - F(\bar{x}, \bar{t}) + \beta(\bar{t} - T) - 2\epsilon|\bar{x}|^2.$$
Therefore
\[ \frac{\alpha}{2}|\bar{x} - \bar{y}|^2 + \epsilon|\bar{y}|^2 - \epsilon|x|^2 \leq F(\bar{x}, \tilde{t}) - F(\bar{y}, \tilde{t}). \]
The same argument starting this time from \( \Phi(\bar{y}, \bar{y}, \tilde{t}) \leq \Phi(\bar{x}, \bar{y}, \tilde{t}) \) also leads to
\[ \frac{\alpha}{2}|\bar{x} - \bar{y}|^2 + \epsilon|\bar{x}|^2 - \epsilon|\bar{y}|^2 \leq f(\bar{x}, \tilde{t}) - f(\bar{y}, \tilde{t}). \]
Summing the two inequalities we conclude that
\[ \frac{\alpha}{2}|\bar{x} - \bar{y}|^2 \leq 2\|f\|_{\infty} + 2\|F\|_{\infty}. \]
Using that \( \Phi(\bar{x}, \bar{y}, \tilde{t}) = 0 \) and using that \( \tilde{t} - T \leq 0 \) we obtain the further bound
\[ \epsilon(|\bar{x}|^2 + |\bar{y}|^2) \leq f(\bar{x}, \tilde{t}) - F(\bar{y}, \tilde{t}) \]
In conclusion, the bounds above imply that
\[
\begin{aligned}
\lim_{\epsilon \to 0} \epsilon(|\bar{x}|^2 + |\bar{y}|^2) &= 0 \text{ for each fixed } \alpha, \\
\lim_{\alpha \to \infty} |\bar{x} - \bar{y}| &= 0 \text{ uniformly in } \epsilon, \beta.
\end{aligned}
\tag{3.2}
\]

**Part 2: Invoking the differential inequalities.** Assumption (3) yields that \( F(\bar{y}, \tilde{t}) = F_i(\bar{y}, \tilde{t}) \) for some \( i \) such that \( \bar{y} \in D_i \). By taking \( \alpha \) large enough we can guarantee that \( \bar{x}, \bar{y} \) both lie in some \( D_i \). Since \( F(\bar{y}, t) = F_i(\bar{y}, \tilde{t}) \) in \( D_i \times \mathbb{R}_+ \), it follows that
\[
\Phi_i(x, y, t) := f(x, t) - F_i(y, t) - \frac{\alpha}{2}|x - y|^2 + \beta(t - T) - \epsilon(|x|^2 + |y|^2)
\]
achieves a local maximum at \((\bar{x}, \bar{y}, \tilde{t})\) which is equal to zero. This means that \( \partial_x \Phi_i \geq 0 \) and that the Hessian of \( \Phi \) is positive at \((\bar{x}, \bar{y}, \tilde{t})\). Given the form of \( \Phi \), this leads to the inequalities
\[
\beta \leq \partial_t F_i(\bar{y}, \tilde{t}), \quad f(\bar{x}, \tilde{t}) - F_i(\bar{y}, \tilde{t}) \leq \frac{\alpha}{2}|\bar{x} - \bar{y}|^2 + \epsilon(|\bar{x}|^2 + |\bar{y}|^2),
\]
and
\[
\begin{pmatrix}
D^2f(\bar{x}, \tilde{t}) & 0 \\
0 & -D^2F_i(\bar{y}, \tilde{t})
\end{pmatrix} \leq \begin{pmatrix}
(\alpha + \epsilon)I & -\alpha \\
-\alpha & (\alpha + \epsilon)I
\end{pmatrix}
\]
where we used that \( \beta(\tilde{t} - T) \leq 0 \) to get the second inequality. The third inequality is understood in the usual matrix sense.

As it is standard in the viscosity solution literature, the above inequalities will imply an upper bound on \( L_i f(\bar{x}, \tilde{t}) - L_i F_i(\bar{y}, \tilde{t}) \). Recalling this argument, observe that for nonnegative symmetric matrices \( M, N, C \) and \( D \) there is the elementary identity
\[
\text{Tr}(CM - DN) = \text{Tr} \left[ \begin{pmatrix}
C & C^{1/2}D^{1/2} \\
C^{1/2}D^{1/2} & D
\end{pmatrix} \begin{pmatrix}
M & 0 \\
0 & -N
\end{pmatrix} \right].
\]
We may apply this identity to \( D^2f(\bar{x}, \tilde{t}), \ D^2F_i(\bar{y}, \tilde{t}), \ A_i(\bar{x}, \tilde{t}) \) and \( A_i(\bar{y}, \tilde{t}) \). Then, using the matrix inequality and simplifying the resulting expression leads to
\[
\begin{aligned}
\text{Tr} \left[ A_i(\bar{x}, \tilde{t})D^2f(\bar{x}, \tilde{t}) - A_i(\bar{y}, \tilde{t})D^2F_i(\bar{y}, \tilde{t}) \right] &\leq \alpha \text{Tr} \left[ (A_i(\bar{x}, \tilde{t})^{1/2} - A_i(\bar{y}, \tilde{t})^{1/2})^2 \right] + \epsilon \text{Tr} [A_i(\bar{x}, \tilde{t}) + A_i(\bar{y}, \tilde{t})].
\end{aligned}
\]
Furthermore, the regularity of $A_i(x, t)$ in space implies that that
\[
\text{Tr} \left[(A_i(x, \bar{t}))^{1/2} - A_i(y, \bar{t})^{1/2}\right]^2 \leq C|x - \bar{y}|,
\]
\[
\text{Tr}[A_i(x, \bar{t}) + A_i(y, \bar{t})] \leq C.
\]
where $C$ is as in assumption (4). Thus
\[
\text{Tr}(A_i(x, \bar{t})D^2 f(x, \bar{t}) - A_i(y, \bar{t})D^2 F_i(y, \bar{t})) \leq C(\alpha|x - \bar{y}| + \epsilon).
\]
Finally, since $\bar{x}, \bar{y} \in D_i$ the differential inequalities in assumption (5) say that
\[
\partial_t f(x, \bar{t}) - \partial_t F_i(y, \bar{t}) < L_i f(x, \bar{t}) - L_i F_i(y, \bar{t}).
\]
Expanding the operator $L_i$ and canceling the common terms leads to
\[
L_i f(x, \bar{t}) - L_i F_i(y, \bar{t}) = \text{Tr}(A_i(x, \bar{t})D^2 f(x, \bar{t}) - A_i(y, \bar{t})D^2 F_i(y, \bar{t}))
+ c_i(x, \bar{t})f(x, \bar{t}) - c_i(y, \bar{t})F_i(y, \bar{t}),
\leq C(\alpha|x - \bar{y}| + \epsilon) + (c_i(x, \bar{t}) - c_i(y, \bar{t}))f(x, \bar{t})
+ c_i(y, \bar{t})(f(x, \bar{t}) - F_i(y, \bar{t})
\]
Again by assumption (4), the right hand side of the last inequality is no larger than
\[
C(\alpha|x - \bar{y}| + \epsilon) + C|x - \bar{y}|f(x, \bar{t}) + C(f(x, \bar{t}) - F_i(y, \bar{t}))_+,
\]
Given the upper bound on $(f(x, \bar{t}) - F_i(y, \bar{t}))$, we conclude that
\[
L_i f(x, \bar{t}) - L_i F_i(y, \bar{t}) \leq C \left[(\alpha + \|f\|_\infty)|x - \bar{y}| + \epsilon + \epsilon(|x|^2 + |\bar{y}|^2)\right]
\]
Combining the last two inequalities it follows that
\[
\beta < C \left[(\alpha + \|f\|_\infty)|x - \bar{y}| + \frac{\alpha}{2}|x - \bar{y}|^2 + \epsilon + \epsilon(|x|^2 + |\bar{y}|^2)\right]
\]
Due to (3.2), by fixing $\beta$, taking $\epsilon$ small enough and $\alpha$ large enough we arrive at a contradiction. \qed

4. Existence of global solutions

In this section we prove the existence of component of Theorem 1.2. As before $f_{in}$ will denote a fixed initial distribution satisfying (1.4) for some $m > 0$, $\tau \in S^2$ and $R \geq 4$.

The proof of existence can be summarized as follows: a sequence of functions $f^{(k)}(v, t)$ is defined by $f^{(0)}(v, t) \equiv f_{in}(v)$ and letting for each $k \in \mathbb{N}$,
\[
\partial_t f^{(k)} = Q(f^{(k-1)}, f^{(k)}), \quad f^{(k)}_0 = f_{in},
\]
that a unique $f^{(k)}$ exists for each given $f^{(k-1)}$ follows from the theory of linear parabolic equations in divergence form, and this is done in Lemma 4.2. The contact analysis for solutions of linear equations and Maxwellians done in Section 2 is then applied here. In fact, its only purpose is to show that if $f^{(k)}$ satisfies (2.5) then so does $f^{(k+1)}$ (see Lemma 4.5), this follows specifically from Corollary 2.7 and Lemma 3.2. Since all the $f^{(k)}$ satisfy (2.5), the ellipticity of $A[f_k]$ is controlled locally but uniformly in $k$ using the estimates from Section 2.

Lastly, the regularity theory for parabolic equations in divergence form is invoked to control the derivatives of $f^{(k)}$ in compact subsets of space and time, this gives enough compactness to extract a converging subsequence, which will be a classical solution to (1.3). The well-posedness and regularity of weak solutions for linear parabolic equations is a well known subject and a
thorough presentation of the theory can be found in the classical treatise by Ladyženskaja, Solonnikov and Ural’ceva [14, Chapters III-V].

4.1. **Linear theory.** We will work with functions in the space \( \mathcal{X} \) given by

\[
\mathcal{X} := \left\{ f : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}, \ f \geq 0, \quad \| f_0 \|_{L^1(\mathbb{R}^3)} = \| f_0 \|_{L^1(\mathbb{R}^3)}, \ \text{and} \ t \to \| f_t \|_{L^\infty(\mathbb{R}^3)} \in L^\infty_{\text{loc}}([0, \infty)) \right\} \quad (4.2)
\]

Any \( f \in \mathcal{X} \) defines a linear elliptic operator through \( Q(\cdot, \cdot) \). Specifically, if \( \phi \in C^2(\mathbb{R}^3) \),

\[
\phi \to Q(f, \phi) = \text{div}(A[f] \nabla \phi - \phi \nabla a[f]) = \text{Tr}(A[f] D^2 \phi) + f \phi.
\]

Note that since \( f_t \in L^1 \cap L^\infty \) for all \( t \) this (degenerate parabolic) operator can be written both in divergence and non-divergence forms.

Given \( f \in \mathcal{X} \), we shall consider the linear Cauchy problem,

\[
\begin{aligned}
\partial_t g &= Q(f, g) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\
g &= f \quad \text{in } \mathbb{R}^3 \times \{0\}.
\end{aligned}
\]

(4.3)

A solution being understood in the following weak sense.

**Definition 4.1.** For any \( f \in \mathcal{X} \), a weak solution to the Cauchy problem (4.3) is a non-negative function \( g : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R} \) such that

1. For all \( t > 0 \) \( \int_{\mathbb{R}^3} g_t \, dv = \int_{\mathbb{R}^3} f_0 \, dv \). In particular, \( g \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^3)) \)
2. \( g \in L^2(\mathbb{R}^+; H^1_{\text{loc}}(\mathbb{R}^3)) \).
3. For any smooth test function \( \phi : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R} \) the following holds

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} g \partial_t \phi \, dvdt + \int_{\mathbb{R}^3 \times \mathbb{R}^+} (\nabla \phi, A[f] \nabla g - g \nabla a[f]) \, dvdt = \int_{\mathbb{R}^3} f_0 \phi_0 \, dv.
\]

**Lemma 4.2.** (Linear well posedness) For any \( f \in \mathcal{X} \), there is a unique weak solution to (4.3) in the sense of Definition 4.1. Moreover, this solution belongs to \( \mathcal{X} \) too.

**Proof.** Existence. Given \( f \in \mathcal{X} \) and \( \epsilon > 0 \) define

\[
L_\epsilon \phi := Q(f_\epsilon, \phi) + \epsilon \Delta \phi.
\]

where \( f_\epsilon \) is a mollification of \( f \). Then for each \( \epsilon > 0 \) the operator \( L_\epsilon \) is a uniformly elliptic operator and its coefficients are Hölder continuous in \( \mathbb{R}^3 \times [0, T] \) for every \( T > 0 \). Then, [14, Theorem 5.1, page 320] says there is a unique classical solution \( g^\epsilon \) of the problem

\[
\begin{aligned}
\partial_t g^\epsilon &= L_\epsilon g^\epsilon \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\
g^\epsilon &= f \quad \text{in } \mathbb{R}^3 \times \{0\}.
\end{aligned}
\]

Multiplying both sides of the equation by \( g^\epsilon \) and integrating leads to the energy inequality

\[
\int_{\mathbb{R}^3} (g^\epsilon_T)^2 \, dv + \int_0^T \int_{\mathbb{R}^3} (A[f_\epsilon] \nabla g^\epsilon_t, \nabla g^\epsilon_t) + \epsilon |\nabla g^\epsilon_t|^2 \, dvdt \leq \int_{\mathbb{R}^3} (g^\epsilon_0)^2 \, dv + \int_0^T \int_{\mathbb{R}^3} f(t) (g^\epsilon_t)^2 \, dvdt,
\]

which in turn implies the bound,

\[
\| g^\epsilon_T \|_2^2 + \int_0^T \int_{\mathbb{R}^3} (A[f_\epsilon] \nabla g^\epsilon_t, \nabla g^\epsilon_t) \, dvdt \leq \| g^\epsilon_0 \|_2^2 + \int_0^T \| f(t) \|_{L^\infty} \| g^\epsilon_t \|_2^2 \, dvdt.
\]
By Gronwall’s inequality, and the fact that $g_0^t = f_0$, it follows that
\[
\|g_0^t\|^2_{L^2} \leq e^{\int_0^T \|f_t\|_{L^\infty} \, dt} \|f_0\|^2_{L^2}, \quad \forall \ t \in (0, T).
\]
Using this with the previous energy inequality yields
\[
\|g_T^t\|^2_{L^2} + \int_0^T \int_{\mathbb{R}^3} (A[f_t] \nabla g_t^t, \nabla g_t^t) \, dv \, dt \leq \left(1 + e^{\int_0^T \|f_t\|_{L^\infty} \, dt} \int_0^T \|f_t\|_{L^\infty} \, dt \right) \|f_0\|^2_{L^2}.
\]
Thus, as $\epsilon \to 0$ there is a subsequence of $g^\epsilon$ converging weakly to a function $g$ such that
\[
\|g_T^t\|^2_{L^2} + \int_0^T \int_{\mathbb{R}^3} (A[f] \nabla g_t, \nabla g_t) \, dv \, dt \leq \left(1 + e^{\int_0^T \|f_t\|_{L^\infty} \, dt} \int_0^T \|f_t\|_{L^\infty} \, dt \right) \|f_0\|^2_{L^2}.
\]
Furthermore, this function is a weak solution of the linear Cauchy problem 4.3. To show that $g \in \mathcal{X}$ it remains to show that $\|g_t\|_{L^\infty(\mathbb{R}^3)}$ is bounded in every bounded interval. This follows at once from the bound
\[
g_t^\epsilon(v, t) \leq e^{\int_0^T \|f_t\|_{L^\infty(\mathbb{R}^3)} \, dt} \|g^\epsilon(v, 0)\|_{L^\infty} = e^{\int_0^T \|f_t\|_{L^\infty(\mathbb{R}^3)} \, dt} \|f_0\|_{L^\infty}
\]
which can be proved from a standard application of the comparison principle.

Uniqueness. If $g^{(1)}$ and $g^{(2)}$ are weak solutions of (4.3), then $\tilde{g} = g^{(1)} - g^{(2)}$ solves
\[
\begin{cases}
\partial_t \tilde{g} = Q(f, \tilde{g}) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\
\tilde{g} = 0 & \text{in } \mathbb{R}^3 \times \{0\}.
\end{cases}
\]
Using this equation it follows that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \tilde{g}_t^2 \, dv \leq \int_{\mathbb{R}^3} 2(\tilde{g}_t \nabla \tilde{g}_t, \nabla [f_t]) \, dv = \int_{\mathbb{R}^3} f_t \tilde{g}_t^2 \, dv \leq \|f_t\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \tilde{g}_t^2 \, dv
\]
Then, Gronwall’s inequality yields for any given $T > 0$ the estimate
\[
\sup_{0 \leq t \leq T} \|g_t^{(1)} - g_t^{(2)}\|_{L^2(\mathbb{R}^3)} \leq e^{MT} \|g_0^{(1)} - g_0^{(2)}\|_{L^2(\mathbb{R}^3)},
\]
where $M = \sup_{(0, T)} \|f_t\|_{L^\infty} < \infty$. Since $g_0^{(1)} = g_0^{(2)}$ we conclude that $g_t^{(1)} = g_t^{(2)}$ for $t > 0$. \hfill \Box

Remark 4.3. In light of Lemma 4.2 one may properly talk about the operator
\[
T : \mathcal{X} \to \mathcal{X}.
\]
where $Tf$ is defined as the unique weak solution of (4.3).

To complement the well-posedness for weak solutions, let us recall the local regularity estimates for solutions to parabolic equations.

Theorem 4.4. Suppose $g : Q \to \mathbb{R}$ is a weak solution of the equation
\[
\partial_t g = \text{div} (B \nabla g + gb),
\]
where $Q \subset \mathbb{R}^3 \times \mathbb{R}_+$ is a space-time cylinder of parabolic diameter $R$, $b \in L^\infty(Q; \mathbb{R}^3)$ is a bounded vector field, and $B$ is a diffusion matrix for which there are $\lambda, \Lambda > 0$ such that
\[
\lambda \mathbb{I} \leq B(v, t) \leq \Lambda \mathbb{I} \text{ a.e. in } Q.
\]
Then, the following two estimates hold
(1) (De Giorgi-Nash-Moser estimates) \( \exists \alpha \in (0,1), C > 0 \) determined by \( \lambda, \Lambda \) such that
\[
[\phi]_{C^{\alpha, \alpha/2}(Q_{1/2})} \leq \frac{C}{R^n} \left( \|\phi\|_{L^\infty(Q)} + R^2 \|b\|_{L^\infty(Q)} \right). \tag{4.4}
\]

(2) (Schauder estimates) If \( B, b \in C^{\beta, \beta/2}(Q) \), then there is a finite \( C \) such that
\[
[D^2 g]_{C^{\beta, \beta/2}(Q_{1/2})} + [\partial_t g]_{C^{\beta, \beta/2}(Q_{1/2})} \leq C \left( \lambda, \Lambda, R, \|B\|_{C^{\beta, \beta/2}(Q)}, \|b\|_{C^{\beta, \beta/2}(Q)}, \|g\|_{L^\infty(Q)} \right). \tag{4.5}
\]

The first part of this theorem can be found in [14, Chapter III, Theorem 10.1, page 204], for the second part, see [14, Chapter IV, Theorem 10.1, page 351].

4.2. Trapping and a priori estimates.

Lemma 4.5. (Trapping is preserved) Suppose \( f \) is as in (4.2) and that \( m > 0, R \geq 4, \tau \in S^2 \) are such that \( f_t \) satisfies (2.5) for all \( t \), then \( (Tf)_t \) also satisfies (2.5) for all \( t \) with the same \( m, R \) and \( \tau \).

Proof. The proof will consist in checking that the five assumptions listed in Lemma 3.2 are satisfied. First, Lemma 4.2 guarantees that \( Tf \) satisfies Assumption (1). Define
\[
D_1 := \mathbb{R}^3 \setminus B_R(R\tau), \quad D_2 := \mathbb{R}^3 \setminus B_R(-R\tau).
\]
It is clear that \( \{D_i\}_{i=1,2} \) is an open cover of \( \mathbb{R}^3 \) and that it has a positive Lebesgue number, so Assumption (2) holds. Define also
\[
F_i(v, t) := m\mathcal{M}_{R\tau}(v), \quad mF_2(v, t) := \mathcal{M}_{-R\tau}(v) \quad \forall \ (v, t) \in \mathbb{R}^3 \times \mathbb{R}_+.
\]
Setting \( F(v, t) = \min_{i=1,2} F_i(v, t) \) it is clear that
\[
F(v, t) < F_i(v, t) \text{ if } v \notin D_i.
\]
Thus assumption (3) is also met. On the other hand, the linear elliptic operator given by \( Q(f, \cdot) \) certainly has bounded coefficients so it complies with assumption (4). Observe that,
\[
f(v, t) \leq m\mathcal{M}_{\pm R\tau}(v), \quad \forall \ (v, t) \in \mathbb{R}^3 \times \mathbb{R}_+,
\]
in which case Corollary 2.7 guarantees that
\[
\partial_t F = 0 > Q(f, F_i) \text{ in } D_i \times \mathbb{R}_+,
\]
moreover \( \partial_t Tf = Q(f, Tf) \) by construction, so that assumption (5) holds as well. Finally, \( Tf = f \) at \( t = 0 \) and by assumption \( f \leq F(v, 0) \) which gives assumption (6). In conclusion, we may apply Lemma 3.2 to \( Tf \) and conclude that
\[
Tf(v, t) \leq F(v, t) \quad \forall \ (v, t) \in \mathbb{R}^3 \times \mathbb{R}_+.
\]

The next proposition shows that if \( f \) lies below two Maxwellians, then the linear elliptic operator given by \( Q(f, \cdot) \) is uniformly elliptic in compact subsets of \( \mathbb{R}^3 \).

Proposition 4.6. Let \( f \in L^1(\mathbb{R}^3) \) satisfy (2.5) for some \( m > 0, R \geq 4, \tau \in S^2 \). Then,
Additionally, if

\( Q \)

is any space time cylinder

\( \) 

existence of global solutions.

4.3. Corollary 4.7.

\( \| \nabla A[f] \|_{C^{\alpha}(\mathbb{R}^3)} \leq C(m, R, \alpha). \)

Proof. The upper bound in (4.6) follows exactly as in the proof of (2.6), using that

\( \| A[f](v) \| \leq a[f](v) \| \leq m a[\mathcal{M}_{\pm Rv}](v). \)

On the other hand, the lower bound in (4.6) follows from Corollary 2.3. Just note that the constants in Corollary 2.3 are given by \( M(f), E(f), H(f) \) and \( \tilde{H}(f) \), the last three being uniformly controlled by \( m \) and \( R \) since \( f \) satisfies (2.5). This proves the first part of the lemma.

The estimate in the second part follows by standard real analysis. Since \( f \) satisfies (2.5),

\( \| f \|_{L^p(\mathbb{R}^3)} \leq m \| \mathcal{M}_{Rv} \|_{L^p(\mathbb{R}^3)} + m \| \mathcal{M}_{-Rv} \|_{L^p(\mathbb{R}^3)} = 2m \| \mathcal{M}_0 \|_{L^p(\mathbb{R}^3)} < +\infty, \)

for every \( p \in [1, +\infty] \). Then, Calderón-Zygmund estimates [19] applied to the kernels \( \partial_{kl} K_{ij} \)

\( K_{kl,ij}(y) := \frac{\partial^2}{\partial y_k \partial y_j} \left( \frac{1}{8\pi |y|} \left( \delta_{ij} - \frac{y_i y_j}{|y|^2} \right) \right), \)

give the bound (for some dimensional constant \( C_3 \))

\( \| D^2 A[f] \|_{L^p(\mathbb{R}^3)} \leq 2C_3 m \| \mathcal{M}_0 \|_{L^p(\mathbb{R}^3)}, 1 < p < \infty. \)

Likewise, Young’s inequality can be used to show that \( \| A[f] \|_{L^p(\mathbb{R}^3)} \) is bounded for every \( p \in [1, +\infty] \). Then, using Morrey’s inequality for each \( \alpha \in (0, 1) \) we get the second estimate. \( \square \)

A combination of Proposition 4.6, Lemma 4.5 and Theorem 4.4 gives uniform bounds for \( Tf \).

Corollary 4.7. Let \( f \in \mathcal{X} \) be such that \( f_t \) satisfies (2.5) for all \( t \) for some \( m, R \geq 4, \tau \in \mathbb{S}^2 \). Then for any space time cylinder \( Q \) and any \( \alpha \in (0, 1) \) we have the bound

\( \| Tf \|_{C^{\alpha,\alpha/2}(Q_{1/2})} \leq C(Q, m, M(f), \alpha) \| Tf \|_{L^\infty(Q)}. \)

Additionally, if \( f \in C^{\alpha,\alpha/2}(Q) \) we have the estimates

\( [\partial_t Tf]_{C^{\alpha,\alpha/2}(Q_{1/2})} + [D^2 Tf]_{C^{\alpha,\alpha/2}(Q_{1/2})} \leq C(Q, m, M(f), \alpha, \| Tf \|_{L^\infty(Q)}, \| f \|_{C^{\alpha,\alpha/2}(Q)}). \)

4.3. Existence of global solutions. Now, proving the existence of a global solution to the nonlinear problem is straightforward.

Proof of Theorem 1.2. Define the sequence of approximating solutions \( \{ u_k \}_k \) by letting

\( f_0(v, t) := f_{in}(v), \quad f_{k+1}(v, t) := Tf_k(v, t) \quad \forall k \in \mathbb{N}. \)

Lemma 4.5 guarantees that \( 0 \leq f_k(v, t) \leq m \mathcal{M}_{\pm Rv}(v) \) in \( \mathbb{R}^3 \times \mathbb{R}_+ \) for every \( k \). In particular, each \( f_k \) with \( k > 1 \) solves a linear parabolic equation with Hölder continuous coefficients.

Therefore, Corollary 4.7 can be applied to each \( f_k \), giving uniform bounds for \( f_k \) in each space time cylinder \( Q \). Then, a Cantor diagonalization argument shows there is a subsequence, which we still call \( f_k \), such that \( f_k, \nabla f_k, D^2 f_k \) and \( \partial_t f_k \) converge locally uniformly in compact subsets
of $\mathbb{R}^3 \times \mathbb{R}$ to $f, \nabla f, D^2 f$ and $\partial_t f$ respectively, for some $f$. In particular, $f$ is $C^2_{\text{loc}}$ in the spatial variable, $C^1_{\text{loc}}$ in the temporal variable and satisfies the pointwise bound $f(v, t) \leq mM_{\pm R_T}(v)$.

On the other hand, recall that, the weak form of the equation solved by each $f_k$ says that for any smooth test function $\phi$ with compact support

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} f_k \partial_t \phi \, dv \, dt + \int_{\mathbb{R}^3} (\nabla \phi, A[f_{k-1}]) \nabla f_k - f_k \nabla a[f_{k-1}] \, dv \, dt = \int_{\mathbb{R}^3} \phi(v, 0)f_{\text{in}} \, dv.$$

The strong convergence of the $f_k$ and its derivatives guarantees that taking $k \to +\infty$ the above yields (again, for any test function $\phi$)

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} f \partial_t \phi \, dv \, dt + \int_{\mathbb{R}^3} (\nabla \phi, A[f]) \nabla f - f \nabla a[f] \, dv \, dt = \int_{\mathbb{R}^3} \phi(v, 0)f_{\text{in}} \, dv.$$

This shows $f$ is a global classical solution of the Landau equation with initial data $f_{\text{in}}$. □

**Remark 4.8.** Since $f$ is differentiable for positive times, the coefficients $A[f]$ will be at least twice differentiable in space and differentiable in time. Standard parabolic Schauder estimates imply higher regularity for $f$, then bootstrapping implies that $f \in C^\infty_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$.

5. **Integral estimates, uniqueness and decay to equilibrium**

In this section the rest of Theorem 1.2 is proved, namely, uniqueness of weak solutions and their decay to equilibrium, always assuming that the initial distribution $f_{\text{in}}$ satisfies (1.4).

5.1. **Uniqueness.** Uniqueness is proved for solutions $f(v, t)$ whose spatial gradients have enough decay at infinity. The next lemma shows that as long as the distribution itself decays fast enough then we can have as much decay for the gradient as needed. This guarantees that the uniqueness and stability lemma proved afterwards covers the solutions built in Section 4.

**Lemma 5.1.** Let $f$ be a weak solution of (1.3) in the sense of Definition 1.1. Then, for any $p > 3$ and any $a > 0$

$$\int_0^T \| \nabla f_t \|^2_{L^p(\mathbb{R}^3)} \, dt \leq C \left( \int_0^T \| f_t \|^2_{L^{p+4}_a(\mathbb{R}^3)} \, dt + \| f_{\text{in}} \|^2_{L^{p+2}_a(\mathbb{R}^3)} \right),$$

for some $C$ completely determined by $M(f_{\text{in}}), E(f_{\text{in}}), H(f_{\text{in}}), a, p$ and $\sup_{t \in [0,T]} \| f_t \|_{L^p(\mathbb{R}^3)}$.

**Proof.** Let $\phi \in \text{Lip}_a(\mathbb{R}^3)$ and $\phi \geq 0$. Then, using $\phi^2 f$ as a test function leads to

$$\frac{d}{dt} \int_{\mathbb{R}^3} \phi^2 f^2 \, dv + \int_{\mathbb{R}^3} (A[f] \nabla f, \nabla (\phi^2 f)) \, dv = \int_{\mathbb{R}^3} (f \nabla a[f], \nabla (\phi^2 f)) \, dv.$$

Expand the integrand on the left,

$$(A[f] \nabla f, \nabla (\phi^2 f)) = (A[f] \nabla f, \phi^2 \nabla f) + 2(A[f] \phi \nabla f, f \nabla \phi),$$

and note that “completing the square” leads to the identity

$$(A[f] \nabla f, \nabla (\phi^2 f)) = (A[f] \nabla (\phi f), \nabla (\phi f)) - f^2 (A[f] \nabla \phi, \nabla \phi).$$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \phi^2 f^2 \, dv + \int_{\mathbb{R}^3} (A[f] \nabla (\phi f), \nabla (\phi f)) \, dv = \int_{\mathbb{R}^3} (f \nabla a[f], \nabla (\phi^2 f)) + f^2 (A[f] \nabla \phi, \nabla \phi) \, dv.$$
For the other integral, by parts and \(-\Delta a[f] = f\) yields
\[
\int_{\mathbb{R}^3} (f \nabla a[f], \nabla (\phi^2 f)) \, dv = \int_{\mathbb{R}^3} \phi^2 f^3 \, dv - \int_{\mathbb{R}^3} \phi^2 f (\nabla f, \nabla a[f]) \, dv,
\]
where \(\Psi[\phi, f] := (\phi \nabla \phi, \nabla a[f]) + (A[f] \nabla \phi, \nabla \phi)\). Integrating for \(t \in (0, T)\) and rearranging,
\[
\int_0^T \int_{\mathbb{R}^3} (A[f] \nabla (\phi f), \nabla (\phi f)) \, dv dt \leq \int_0^T \int_{\mathbb{R}^3} \phi^2 f^3 + \Psi[\phi, f] f^2 \, dv dt + \int_{\mathbb{R}^3} \phi^2 f^2 \, dv. \quad (5.1)
\]
To obtain the estimate, for each \(k \in \mathbb{N}\) let \(\phi_k : \mathbb{R}^3 \to \mathbb{R}\) be such that
\[
0 \leq \phi_k \leq 1 \text{ in } \mathbb{R}^3, \quad \|\nabla \phi_k\|_{L^\infty(\mathbb{R}^3)} \leq 1,
\]
\[
\phi_k \equiv 1 \text{ in } R_k, \quad \phi_k \equiv 0 \text{ in } \hat{R}_k,
\]
where \(R_k := \{v \mid k \leq |v| \leq k + 1\}\), \(\hat{R}_k := \{v \mid k - 1 \leq |v| \leq k + 2\}\). Then,
\[
\int_0^T \int_{R_k} (A[f] \nabla f, \nabla f) \, dv dt \leq \int_0^T \int_{\hat{R}_k} f^3 + |\nabla a[f]| f^2 + |A[f]| f^2 \, dv dt + \int_{\hat{R}_k} f^2 \, dv.
\]
Moreover,
\[
\int_0^T \int_{R_k} (A[f] \nabla f, \nabla f) \, dv dt \leq \int_0^T \int_{R_k} f^3 + C_1 f^2 \, dv dt + \int_{\hat{R}_k} f^2 \, dv,
\]
where \(C_1 = C_1(f) := \|A[f]\|_{L^\infty(\mathbb{R}^3)} + \|\nabla a[f]\|_{L^\infty(\mathbb{R}^3)}\). Next, Corollary 2.3 implies that
\[
(A[f] \nabla f, \nabla f) \geq \delta \frac{\rho}{(\rho + |v|)^2} |\nabla f|.
\]
For some \(\rho\) and \(\delta\) determined by \(M(f_{\text{in}}), E(f_{\text{in}})\) and \(H(f_{\text{in}})\). Observe that
\[
\rho + t \leq \max\{1, (1 + \rho)/2\} (1 + t) \forall \, t \geq 1,
\]
and thus, with \(c_1 := \delta \max\{1, (1 + \rho)/2\}\)
\[
(A[f] \nabla f, \nabla f) \geq c_1 \langle v \rangle^{-2} |\nabla f| \text{ in } B_1^c.
\]
Let \(a > 0\), multiplying both sides of the above equation by \(\langle v \rangle^{a+2}\) and integrating over \(R_k\),
\[
c_1 \int_0^T \int_{R_k} |\nabla f|^2 \langle v \rangle^a \, dv dt \leq \int_0^T \int_{R_k} (A[f] \nabla f, \nabla f) \langle v \rangle^{a+2} \, dv dt.
\]
Then, using that \(\langle v \rangle \leq 2(k + 1)\) in \(\hat{R}_k\) for every \(k > 1\), it follows that
\[
c_1 \int_0^T \int_{R_k} |\nabla f|^2 \langle v \rangle^a \, dv dt \leq 2^{a+2}(k + 1)^{2+a} \left( \int_0^T \int_{R_k} f^3 + C_1 f^2 \, dv dt + \int_{\hat{R}_k} f^2 \, dv \right) .
\]
By the same token, \( \frac{1}{5}(k+1) \leq \langle v \rangle \) in \( \hat{R}_k \), thus
\[
\int_0^T \int_{\hat{R}_k} |\nabla f|^2 \langle v \rangle^a \, dv \, dt \leq 6^{a+2} \left( \int_0^T \int_{\hat{R}_k} (f^3 + C_1 f^2) \langle v \rangle^{2+a} \, dv \, dt + \int_{\hat{R}_k} f_{\text{in}}^2 \langle v \rangle^{2+a} \, dv \right).
\]
Adding up these bounds for each \( k > 1 \) leads to a bound over all space, by noting that
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla f|^2 \langle v \rangle^a \, dv \, dt \leq \int_0^T \int_{\mathbb{R}^3} |\nabla f|^2 \langle v \rangle^a \, dv \, dt + \sum_{k>1} \int_0^T \int_{\hat{R}_k} |\nabla f|^2 \langle v \rangle^a \, dv \, dt,
\]
so that
\[
c_0 \int_0^T \int_{\mathbb{R}^3} |\nabla f|^2 \langle v \rangle^a \, dv \, dt \leq \int_0^T \int_{\mathbb{R}^3} (f^2 + C_1 f^3) \langle v \rangle^{a+2} \, dv \, dt + \int_{\mathbb{R}^3} f_{\text{in}}^2 \langle v \rangle^{a+2} \, dv \, dt.
\]
In other words, 
\[
c_1 \int_0^T \|\nabla f_t\|^2_{L^2(\mathbb{R}^3)} \, dt \leq 6^{a+2} \left( \int_0^T \|f_t\|^2_{L^2_{a+2}(\mathbb{R}^3)} \, dt + C_1 \int_0^T \|f_{\text{in}}\|^3_{L^3_{a+2}(\mathbb{R}^3)} \, dt + \|f_{\text{in}}\|^2_{L^2_{a+2}(\mathbb{R}^3)} \right).
\]
Since \( c_1 \) depends only on \( M(f_{\text{in}}), E(f_{\text{in}}), H(f_{\text{in}}) \) and \( C_1(f) \leq C_p \|f\|_{L^p(\mathbb{R}^3)} \) for any \( p > 3 \), the lemma is proved.

The uniqueness proof (Lemma 5.5) will involve certain weighted integral inequalities. There is a broad literature on weighted \( L^p \) bounds for singular integrals, we refer the reader to Sawyer-Wheeden [18] for a complete discussion of the subject. The estimates needed involves the sub-additive operator
\[
I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-\alpha}} \, dy, \quad \alpha \in (0, d),
\]
in the cases when \( d = 3 \) and \( \alpha = 1, 2 \).

**Remark 5.2.** For a given \( f : \mathbb{R}^3 \to \mathbb{R} \), the following pointwise bounds hold
\[
|f(x)| \leq cI_1(|\nabla f|)(x),
\]
\[
|A[f](x)| \leq cI_2(f)(x),
\]
\[
|\nabla A[f](x)| \leq cI_1(f)(x).
\]

**Theorem 5.3.** [18, Section 1] Let \( w_1(x), w_2(x) \) be non-negative functions satisfying a reverse doubling condition
\[
\int_{\delta B} w_i(x) \, dx \leq \epsilon \int_B w_i(x) \, dx, \quad i = 1, 2.
\]
Then, there is some positive \( C = C(p, q, w_1, w_2, d) \) such that for all functions \( f \) we have
\[
\left( \int_{\mathbb{R}^d} |I_\alpha f(x)|^q w_2(x) \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^d} |f(x)|^p w_1(x) \, dx \right)^{1/p}
\]
if and only if
\[
\sup_B \left\{ |B|^{(\alpha-d)/d} \left( \int_B w_2(x) \, dx \right)^{1/q} \left( \int_B w_1(x)^{1-p'} \, dx \right)^{1/p} \right\} < \infty.
\]
the supremum being over all balls \( B \subset \mathbb{R}^d \).

This theorem implies as a special case the following bounds.
Proposition 5.4. The following inequalities hold
\[
\|f\|_{L^2_4(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2_4(\mathbb{R}^3)},
\]
\[
\|f\|_{L^5_3(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^5_3(\mathbb{R}^3)},
\]
\[
\|\text{div}A[f]\|_{L^6(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}.
\]

Proof. According to Remark 5.2, each inequality follows from the estimate
\[
\|I_\alpha(f)\|_{L^3_a(\mathbb{R}^3)} \leq C\|f\|_{L^2_b(\mathbb{R}^3)}
\]
for \(a, q, p, b\) in a certain range. Namely, the proposition will be proved if \(\alpha, q, p, w_1(x) = \langle x \rangle^b\) and \(w_2(x) = \langle x \rangle^a\) satisfy the assumptions of Theorem 5.3. Checking that \(w_i\) satisfies the two conditions of the Theorem is tedious but straightforward, one only needs to take into account how far is \(B\) from the origin. We omit the details of the verification of the doubling property and focus on checking the second condition in detail.

Let \(B = B_r(y)\). If \(r \leq |y|/2\), then
\[
\frac{1}{2} \leq \frac{\langle x \rangle}{|y|} \leq 2, \quad \forall x \in B.
\]
Setting \(\gamma := (\alpha - 3)/3 + 1/q + 1/2\) we obtain,
\[
|B|^{(\alpha-3)/3} \left( \int_B \langle x \rangle^a \, dx \right)^{1/q} \left( \int_B \langle x \rangle^{-b} \, dx \right)^{1/2} \leq C_{a,q,b}|B|^\gamma |\langle y \rangle|^{a/b-2/q},
\]
On the other hand, if \(r \geq \frac{1}{2}|y|\), then \(B \subset B_{2r}(0)\) and \(|B_{2r}(0)| = 2^3|B|\), so
\[
\int_B \langle x \rangle^a \, dx \leq \int_{B_{2r}(0)} \langle x \rangle^a \, dx = C_a|B|(1 + |B|^{a/3}),
\]
\[
\int_B \langle x \rangle^{-b} \, dx \leq \int_{B_{2r}(0)} \langle x \rangle^{-b} \, dx = C_b|B|(1 + |B|^{-b/3}),
\]
as long as \(b \leq 0\) (remember we are only interested in the cases \(b = 0, -2\)). In conclusion,
\[
|B|^{(\alpha-3)/3} \left( \int_B \langle x \rangle^a \, dx \right)^{1/q} \left( \int_B \langle x \rangle^{-b} \, dx \right)^{1/2} \leq C_{a,q,b}|B|^\gamma (1 + |B|^{a/3})^{1/q}(1 + |B|^{-b/3})^{1/2},
\]
with the same \(\gamma\) as before. It follows that the assumptions of Theorem 5.3 hold as long as \(\gamma \geq 0, \ a/q - b/2 \leq 0, \ \gamma + a/(3q) - b/6 \leq 0\).

Then, it is easy to see that for the \(\alpha, q, a\) and \(b\) for the given values in the statement of the proposition satisfy the above inequalities, and the proposition is proved.

Having gone through the preliminaries, a Gronwall type estimate can now be proved.

Lemma 5.5. Let \(f^{(1)}\) and \(f^{(2)}\) both solve (1.1) with initial data as in Theorem 1.2 and such that
\[
\int_0^T \|f^{(1)}\|_{L^6_3(\mathbb{R}^3)}^2 \, dt, \quad \int_0^T \|\nabla f^{(2)}\|_{L^{10/3}_3(\mathbb{R}^3)}^2 \, dt < +\infty.
\]

(5.2)
Then, there is a $B(t) \in L^1(0, T)$ such that for $t \in (0, T)$,

$$
\frac{d}{dt} \|f^{(1)} - f^{(2)}\|_{L^2(\mathbb{R}^3)}^2 \leq B(t) \|f^{(1)} - f^{(2)}\|_{L^2(\mathbb{R}^3)}^2.
$$

In particular, if $f^{(1)} = f^{(2)}$ at $t = 0$ it follows that $f^{(1)} = f^{(2)}$ for all later times.

Proof. For the rest of the proof, let us write $u = f^{(1)} - f^{(2)}$ and $w = f^{(1)} + f^{(2)}$. Then $u$ solves

$$
\partial_t u = Q(f^{(1)}, f^{(1)}) - Q(f^{(2)}, f^{(2)}).
$$

This can be rewritten as

$$
\partial_t u = Q(f^{(1)}, u) + Q(u, f^{(2)}).
$$

Using this equation we compute the rate of change of $\|u\|_{L^2(\mathbb{R}^3)}$,

$$
\frac{d}{dt} \|u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} u(Q(f^{(1)}, u) + Q(u, f^{(2)})) \, dx,
$$

$$
= - \int_{\mathbb{R}^3} (A[f^{(1)}] \nabla u, \nabla u) + \int_{\mathbb{R}^3} u \text{div} \left(-u \nabla a[f^{(1)}] + A[u] \nabla f^{(2)} - f^{(2)} \nabla a[u]\right) \, dx.
$$

Expanding the divergence term and simplifying leads to

$$
\frac{d}{dt} \|u\|^2_{L^2(\mathbb{R}^3)} = - \int_{\mathbb{R}^3} (A[f^{(1)}] \nabla u, \nabla u) + \int_{\mathbb{R}^3} h \, dx,
$$

where $h = h^{(1)} + h^{(2)}$ and

$$
h^{(1)} := u \text{Tr}(A[u] D^2 f^{(2)}), \quad h^{(2)} := -u(\nabla u, \nabla a[f^{(1)})] + 2u^2 f^{(1)}.
$$

Let us see the lemma follows from the bound

$$
\int_{\mathbb{R}^3} h_t \, dx \leq C \left( \|f^{(1)}_t\|_{L^\infty_t(\mathbb{R}^3)} + \|\nabla f^{(2)}_t\|_{L^2_t(\mathbb{R}^3)} \right) \|\nabla u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}.
$$

Indeed, suppose that (5.3) holds for all $t \in (0, T)$ for some $C > 0$. Recall that, as shown in Lemma 5.1 there is a $c_1 = c_1(f^{(1)}_0)$ such that

$$
c_1 \|\nabla u_t\|^2_{L^2(\mathbb{R}^3)} \leq \int_{\mathbb{R}^3} (A[f^{(1)}_t] \nabla u_t, \nabla u_t) \, dx.
$$

It follows that

$$
\frac{d}{dt} \|u_t\|^2_{L^2(\mathbb{R}^3)} \leq -c_1 \|\nabla u_t\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} h_t \, dx, \quad t \in (0, T).
$$

Define

$$
\tilde{B}(t) = C \left( \|f^{(1)}_t\|_{L^\infty_t(\mathbb{R}^3)} + \|\nabla f^{(2)}_t\|_{L^2_t(\mathbb{R}^3)} \right), \quad t \in (0, T).
$$

Then, for each $\epsilon > 0$ and $t \in (0, T)$ we have

$$
\int_{\mathbb{R}^3} h_t \, dx \leq \epsilon \tilde{B}(t) \|\nabla u_t\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\epsilon} \tilde{B}(t) \|u_t\|^2_{L^2(\mathbb{R}^3)}.
$$

Taking $\epsilon = \frac{c_1}{2\tilde{B}(t)}$ with $c_1$ as above leads to

$$
\frac{d}{dt} \|u_t\|^2_{L^2(\mathbb{R}^3)} \leq \frac{c_1^2}{2} \|\nabla u_t\|^2_{L^2(\mathbb{R}^3)} + \frac{2}{c_1} \tilde{B}(t)^2 \|u_t\|^2_{L^2(\mathbb{R}^3)}.
$$

Therefore,

$$
\frac{d}{dt} \|u_t\|^2_{L^2(\mathbb{R}^3)} \leq \frac{2}{c_1} \tilde{B}(t)^2 \|u_t\|^2_{L^2(\mathbb{R}^3)}.
$$
This proves the lemma, since $B(t) := \dot{B}(t)^2$ is locally with respect to $t$ due to assumption (5.2). It only remains to show (5.3) holds. Note that

$$-\int_{\mathbb{R}^3} u(\nabla u, \nabla a[f^{(1)}]) \, dx = -\frac{1}{2} \int_{\mathbb{R}^3} (\nabla u^2, \nabla a[f^{(1)}]) \, dx,$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} u^2 \Delta a[f^{(1)}] \, dx.$$ 

Since $\Delta a[f^{(1)}] = -f^{(1)}$ it follows that

$$\int_{\mathbb{R}^3} h_t^{(2)} \, dx = \frac{5}{2} \int_{\mathbb{R}^3} u^2 f^{(1)} \, dx,$$

$$\leq \frac{5}{2} \|u\|_{L^2_4(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \|f^{(1)}\|_{L^1_\infty(\mathbb{R}^3)}.$$ 

Controlling $h^{(1)}$ will be a bit more delicate, observe that

$$\int_{\mathbb{R}^3} h_t^{(1)} \, dx = \int_{\mathbb{R}^3} u\text{Tr}(A[u]D^2 f^{(2)}) \, dx.$$ 

Integration by parts shows this integral is equal to

$$-\int_{\mathbb{R}^3} (\nabla u, A[u] \nabla f^{(2)}) \, dx - \int_{\mathbb{R}^3} u(\text{div}(A[u]), \nabla f^{(2)}) \, dx.$$ 

Note that

$$\left| \int_{\mathbb{R}^3} (\nabla u, A[u] \nabla f^{(2)}) \, dx \right| \leq \int_{\mathbb{R}^3} \langle x \rangle^{-1} |\nabla u| |A[u]| |\nabla f^{(2)}| \langle x \rangle \, dx.$$ 

A pointwise bound for $A[u]$ can be obtained through a standard interpolation argument, and the compactness enjoyed by $u_t$. Note that

$$|A[u](x)| \leq \int_{B_R(x)} \frac{2|u(y)|}{|x-y|} \, dy + \int_{B_R(x)^c} \frac{2|u(y)|}{|x-y|} \, dy,$$

$$\leq 4\sqrt{\pi R} \|u\|_{L^1(B_R)} + 2R^{-1}\|u\|_{L^1(B_R^c)}.$$ 

A compactness argument shows there is a $R_0 > 0$ depending only on $f^{(1)}$, $f^{(2)}$ and $T$ such that

$$\int_{B_{R_0}} |u_t| \, dx \geq \int_{B_{R_0}^c} |u_t| \, dx, \quad \text{for } t \in (0, T).$$ 

Then, taking $R = R_0$ above implies that $\|u_t\|_{L^1(B_R^c)} \leq \|u_t\|_{L^1(B_R)} \leq 2\pi R^{3/2}\|u_t\|_{L^2(B_R)}$, thus

$$|A[u_t](x)| \leq (4\sqrt{\pi R_0} + 2\pi R_0^{3/2})\|u_t\|_{L^2(\mathbb{R}^3)}, \quad \forall \, x \in \mathbb{R}^3.$$ 

In conclusion,

$$\left| \int_{\mathbb{R}^3} (\nabla u, A[u] \nabla f^{(2)}) \, dx \right| \leq C(R_0)\|u_t\|_{L^2(\mathbb{R}^3)} \|\nabla u_t\|_{L^2_4(\mathbb{R}^3)} \|\nabla f_t^{(2)}\|_{L^2_2(\mathbb{R}^3)}.$$ 

On the other hand, we may write

$$\int_{\mathbb{R}^3} u(\text{div}(A[u]), \nabla f^{(2)}) \, dx = \int_{\mathbb{R}^3} \langle v \rangle^{-5/3} |u| |\text{div}A[u]| |\langle v \rangle|^{5/3} \nabla f^{(2)}| \, dx.$$ 

Then, Hölder inequality for three functions and exponents 3, 6 and 2 yields,

$$\left| \int_{\mathbb{R}^3} u(\text{div}(A[u]), \nabla f^{(2)}) \, dx \right| \leq \|u_t\|_{L^3_\infty(\mathbb{R}^3)} \|\text{div}A[u_t]\|_{L^6(\mathbb{R}^3)} \|\nabla f_t^{(2)}\|_{L^{10/3}_2(\mathbb{R}^3)}.$$
This, by Proposition 5.4, implies
\[ \left| \int_{\mathbb{R}^3} u(\text{div}(A[u]), \nabla f^{(2)}) \, dx \right| \leq C \| \nabla u_t \|_{L^2_\omega(\mathbb{R}^3)} \| u_t \|_{L^2(\mathbb{R}^3)} \| \nabla f_t^{(2)} \|_{L^2_{10/3}(\mathbb{R}^3)}. \]

The bounds we have obtained for \( h^{(1)} \) and \( h^{(2)} \) yield
\[ \int_{\mathbb{R}^3} h_t \, dx \leq C \left( \| f_t^{(1)} \|_{L^\infty_\omega(\mathbb{R}^3)} + \| \nabla f_t^{(2)} \|_{L^2_{10/3}(\mathbb{R}^3)} \right) \| \nabla u_t \|_{L^2_\omega(\mathbb{R}^3)} \| u_t \|_{L^2(\mathbb{R}^3)}, \]
which finishes the proof. \( \square \)

5.2. Exponential decay to equilibrium. As it is well known (see [8, 21] for further references and discussion), decay to equilibrium follows from any bound for the relative entropy between \( f_t \) and \( M f_t \) and the entropy production at time \( t \). Traditionally, this would follow from a bound on the Fisher information of the distribution \( f_t \) in terms of the entropy production. However, for soft potentials the lowest eigenvalue of \( A[f] \) decays at infinity like \( |v|^{-2} \), making it difficult to bound the Fisher information in terms of the entropy production.

Here, instead, the approach taken does not invoke directly the Stam-Gross logarithmic Sobolev inequality but instead imitates its proof based on the Sobolev embedding itself (see [2]) to relate the entropy production with the relative entropy.

Remark 5.6. Ideally, one should seek a bound relating entropy to the entropy functional, understood as a trilinear functional (since \( Q(f, f) \) is already bilinear). Recent work of Beckner [1] goes in that direction, although the bounds derived in [1] are bilinear and isotropic. Here, an inequality guaranteeing exponential decay is proved through elementary methods, at the price that the rate obtained is non-explicit (and thus, far from optimal).

The argument below combines a quantitative bound which holds when \( f_t \) is close enough to equilibrium in \( L^2(\mathbb{R}^3) \), and an implicit bound provided \( f_t \) is far from equilibrium. These are respectively the statements of the next two lemmas.

Lemma 5.7. Let \( f \) and \( \lambda_0 \) be such that \( \int_{\mathbb{R}^3} f \, dv = 1 \) and
\[ \int_{\mathbb{R}^3} f^2 \, dv \leq (2 - \lambda_0) \int_{\mathbb{R}^3} f \log(f) \, dv + 1 + 3\lambda_0, \]
then the following inequality holds
\[ \lambda_0 H(f \mid M f) \leq 4 \int_{\mathbb{R}^3} \left( A[f] \nabla \sqrt{f}, \nabla \sqrt{f} \right) \, dv - \int_{\mathbb{R}^3} f^2 \, dv. \]

Proof. Jensen’s inequality says that
\[ 2 \int_{\mathbb{R}^3} f \log(f) \, dv \leq \log \int_{\mathbb{R}^3} f^2 \, dv \]
On the other hand, the integral inequality (see [20], or [10] for another proof) says that
\[ \int_{\mathbb{R}^3} f^2 \, dv \leq 4 \int_{\mathbb{R}^3} \left( A[f] \nabla \sqrt{f}, \nabla \sqrt{f} \right) \, dv \]
Then
\[ 2 \int_{\mathbb{R}^3} f \log(f) \, dv + 1 \leq 4 \int_{\mathbb{R}^3} \left( A[f] \nabla \sqrt{f}, \nabla \sqrt{f} \right) \, dv. \]
Subtracting $\|f\|_{L^2}^2$ from both sides,
\[2 \int_{\mathbb{R}^3} f \log(f) \, dv + 1 - \int_{\mathbb{R}^3} f^2 \, dv \leq 4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv - \int_{\mathbb{R}^3} f^2 \, dv.\]
This leads to
\[
\lambda_0 \left( \int_{\mathbb{R}^3} f \log(f) \, dv - 3 \right) \leq 2 \int_{\mathbb{R}^3} f \log(f) \, dv + 1 - \int_{\mathbb{R}^3} f^2 \, dv
\]
\[
\square
\]

**Lemma 5.8.** Let $f$ be a distribution such that $M(f) = 1$, $\|f - \mathcal{M}f\|_{L^2} \geq 1$ and $E(f), H(f)$ and $\|f\|_{L^2(\mathbb{R}^3)}$ are all finite. Then, there is some $\lambda_1 = \lambda_1(E(f), H(f), \|f\|_{L^2(\mathbb{R}^3)})$ such that
\[
\lambda_1 H(f \mid \mathcal{M}f) \leq 4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv - \int_{\mathbb{R}^3} f^2 \, dv
\]

**Proof.** If such a $\lambda_1$ did not exist there would be some $C_0$ and a sequence $f_k$ with
\[
M(f_k) = 1, \quad \text{and } E(f_k), H(f_k), \|f_k\|_{L^2(\mathbb{R}^3)} \leq C_0,
\]
such that
\[
\|f_k - \mathcal{M}f_k\|_{L^2} \geq 1,
\]
and yet
\[
0 \leq 4 \int_{\mathbb{R}^3} (A[f_k] \nabla \sqrt{f_k}, \nabla \sqrt{f_k}) \, dv - \int_{\mathbb{R}^3} f_k^2 \, dv \leq 2^{-k} H(f_k \mid \mathcal{M}f_k).
\]
This shows first that
\[
\int_{\mathbb{R}^3} (A[f_k] \nabla \sqrt{f_k}, \nabla \sqrt{f_k}) \, dv
\]
is bounded in $k$. Moreover, by the lower bound for $A[f]$ from Corollary 2.3, it follows that $\nabla \sqrt{f_k}$ is bounded in $L^2_{loc}(\mathbb{R}^3)$. Passing to a subsequence, we may assume without loss of generality that for some $f \in L^2(\mathbb{R}^3)$
\[
f_k \to f, \quad \sqrt{f_k} \to \sqrt{f} \text{ in } L^2(\mathbb{R}^3), \quad A[f_k] \to f \text{ in } L^\infty(\mathbb{R}^3).
\]
It follows that
\[
4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv - \int_{\mathbb{R}^3} f^2 \, dv = 0,
\]
it is well known that in this case $f = \mathcal{M}f$. This contradicts the fact that $f_k \to f$ in $L^2(\mathbb{R}^3)$ and $\|f_k - \mathcal{M}f_k\|_{L^2} \geq 1$, and the lemma is proved. \(\square\)

With the previous lemmas in hand, obtaining convergence to equilibrium is straightforward.

**Lemma 5.9.** Let $f$ be the unique solution to (1.3) with $f_{in}$ satisfying (1.4), then
\[
\|f_t - \mathcal{M}f_{in}\|_{L^1(\mathbb{R}^3)} \leq e^{-\lambda t} \sqrt{2H(f_{in} \mid \mathcal{M}f_{in})}, \quad \lambda = \lambda(f_{in})
\]

**Proof.** Due to the symmetries of the equation, it suffices to consider the case where $f_{in}$ is normalized, that is
\[
\int_{\mathbb{R}^3} f_{in} \, dx = 1, \quad \int_{\mathbb{R}^3} f_{in} v \, dx = 0, \quad \int_{\mathbb{R}^3} f_{in} |v|^2 \, dx = 3.
\]
Recall that in this case, the normalized Maxwellian $\mathcal{M}f_{\text{in}} = \mathcal{M}_0$ has entropy
\[
H(\mathcal{M}_0) = \int_{\mathbb{R}^3} \mathcal{M}_0 \log(\mathcal{M}_0) \, dx = 3.
\]
A standard computation gives,
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \log(f) \, dv = \int_{\mathbb{R}^3} Q(f,f) \log(f) \, dv,
\]
\[
= -\int_{\mathbb{R}^3} \left( A[f] \nabla f - f \nabla a[f], \frac{\nabla f}{f} \right) \, dv.
\]
Note that
\[
\left( A[f] \nabla f - f \nabla a[f], \frac{\nabla f}{f} \right) = \left( A[f] \frac{\nabla f}{\sqrt{f}}, \frac{\nabla f}{\sqrt{f}} \right) - (\nabla a[f], \nabla f)
\]
\[
= \left( 2A[f] \nabla \sqrt{f}, 2\nabla \sqrt{f} \right) - (\nabla a[f], \nabla f),
\]
where we used that $\frac{1}{\sqrt{f}} \nabla f = 2 \nabla \sqrt{f}$. Then,
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \log(f) \, dv = -4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv + \int_{\mathbb{R}^3} (\nabla a[f], \nabla f) \, dv,
\]
\[
= -4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv - \int_{\mathbb{R}^3} f \Delta a[f] \, dv.
\]
Integrating by parts in the last integral, and recalling that $-\Delta a[f] = f$ it follows that
\[
\frac{d}{dt} H(f_t | \mathcal{M}_0) = \frac{d}{dt} \int_{\mathbb{R}^3} f \log(f) \, dv = -4 \int_{\mathbb{R}^3} (A[f] \nabla \sqrt{f}, \nabla \sqrt{f}) \, dv + \int_{\mathbb{R}^3} f^2 \, dv
\]
For those $t$ such that $\|f_t - \mathcal{M}_0\|_{L^2(\mathbb{R}^3)} \leq 1$, Lemma 5.7 says that
\[
\frac{d}{dt} H(f_t | \mathcal{M}_0) \leq -\lambda_0 H(f_t, \mathcal{M}_0).
\]
Otherwise, $\|f_t - \mathcal{M}_0\|_{L^2(\mathbb{R}^3)} \geq 1$ in which case Lemma 5.8 yields
\[
\frac{d}{dt} H(f_t | \mathcal{M}_0) \leq -\lambda_1 H(f_t, \mathcal{M}_0).
\]
In either case,
\[
\frac{d}{dt} H(f | \mathcal{M}_0) \leq -2\lambda H(f | \mathcal{M}_0) \forall t > 0,
\]
where $\lambda(f_{\text{in}}) := \frac{1}{\pi^2} \min\{\lambda_0, \lambda_1\}$. Therefore, $H(f | \mathcal{M}_0) \leq e^{-2\lambda t} H(f_{\text{in}} | \mathcal{M}_0)$ and the lemma follows from the Csiszár-Kullback-Pinsker inequality (see [22, Section 2], also compare with the argument in [8, Section 4]),
\[
\|f - \mathcal{M}^f\|_{L^1} \leq \sqrt{2H(f | \mathcal{M}^f)}.
\]
\[\square\]

Acknowledgments: M.P. Gualdani is supported by NSF-DMS 1109682. N. Guillen is supported by NSF-DMS 1201413. The authors would like to thank the hospitality of MSRI during the program Free Boundary Problems, Theory and Applications in the Spring of 2011, where this work was started.
References

[1] William Beckner. Embedding estimates and fractional smoothness. International Mathematics Research Notices, 2012.
[2] William Beckner and Michael Pearson. On sharp sobolev embedding and the logarithmic sobolev inequality. Bulletin of the London Mathematical Society, 30(1):80–84, 1998.
[3] Piotr Biler, Grzegorz Karch, and Régis Monneau. Nonlinear diffusion of dislocation density and self-similar solutions. Communications in Mathematical Physics, 294(1):145–168, 2010.
[4] Luis Caffarelli, Fernando Soria, and Juan Luis Vazquez. Regularity of solutions of the fractional porous medium flow. arXiv preprint arXiv:1201.6048, 2012.
[5] Luis Caffarelli and Juan Luis Vazquez. Nonlinear porous medium flow with fractional potential pressure. Archive for rational mechanics and analysis, 202(2):537–565, 2011.
[6] Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. Users guide to viscosity solutions of second order partial differential equations. Bulletin of the American Mathematical Society, 27(1):1–67, 1992.
[7] Laurent Desvillettes and Cédric Villani. On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. Comm. Partial Differential Equations, 25(1-2):179–259, 2000.
[8] Laurent Desvillettes and Cédric Villani. On the spatially homogeneous Landau equation for hard potentials. II. H- theorem and applications. Comm. Partial Differential Equations, 25(1-2):261–298, 2000.
[9] IM Gamba, V Panferov, and C Villani. Upper maxwellian bounds for the spatially homogeneous boltzmann equation. Archive for rational mechanics and analysis, 194(1):253–282, 2009.
[10] Philip T Gressman, Joachim Krieger, and Robert M Strain. A non-local inequality and global existence. Advances in Mathematics, 230(2):642–648, 2012.
[11] Y. Guo. The landau equation in a periodic box. Communications in mathematical physics, 231(3):391–434, 2002.
[12] Robert Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. Archive for Rational Mechanics and Analysis, 101(1):1–27, 1988.
[13] Joachim Krieger and Robert M. Strain. Global solutions to a non-local diffusion equation with quadratic non-linearity. Comm. Partial Differential Equations, 37(4):647–689, 2012.
[14] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
[15] LD Landau. Die kinetische gleichung fuer den fall coulombscher wechselwirkung. Phys. Z. Sowjet, 10:154, 1936.
[16] Elliott H Lieb and Michael Loss. Analysis, volume 14 of graduate studies in mathematics. American Mathematical Society, Providence, RI, 4, 2001.
[17] PL Lions. On boltzmann and landau equations. Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences, 346(1679):191–204, 1994.
[18] Eric Sawyer and Richard L Wheeden. Weighted inequalities for fractional integrals on euclidean and homogeneous spaces. American Journal of Mathematics, 114(4):813–874, 1992.
[19] Elias M. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[20] C. Villani. On a new class of weak solutions to the spatially homogeneous boltzmann and landau equations. Archive for rational mechanics and analysis, 143(3):273–307, 1998.
[21] C. Villani. A review of mathematical topics in collisional kinetic theory. Handbook of mathematical fluid dynamics, 1:71–74, 2002.
[22] Cedric Villani. Trend to equilibrium for dissipative equations, functional inequalities and mass transportation. Contemporary Mathematics, 353:95–109, 2004.

Maria Pia Gualdani, Department of Mathematics, George Washington University, 2115 G Street NW Washington DC, 20052, USA
E-mail address: gualdani@gwu.edu

Nestor Guillen, Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA
E-mail address: nestor@math.ucla.edu.