Coefficient of restitution of colliding viscoelastic spheres

Rosa Ramírez, Thorsten Pöschel, Nikolai V. Brilliantov, and Thomas Schwager

1Dpto. de Física, F.C.F.M., Universidad de Chile, Casilla 487-3, Santiago, Chile
2Humboldt-Universität zu Berlin, Institut für Physik, Invalidenstr. 110, D-10115 Berlin, Germany
3Moscow State University, Physics Department, Moscow 119899, Russia

We perform a dimension analysis for colliding viscoelastic spheres to show that the coefficient of normal restitution \( \epsilon \) depends on the impact velocity \( g \) as \( \epsilon = 1 - \gamma_1 g^{1/5} + \gamma_2 g^{2/5} + \ldots \), in accordance with recent findings. We develop a simple theory to find explicit expressions for coefficients \( \gamma_1 \) and \( \gamma_2 \). Using these and few next expansion coefficients for \( \epsilon(g) \) we construct a Padé-approximation for this function which may be used for a wide range of impact velocities where the concept of the viscoelastic collision is valid. The obtained expression reproduces quite accurately the existing experimental dependence \( \epsilon(g) \) for ice particles.

I. INTRODUCTION

The change of relative velocity of inelastically colliding particles can be characterized by the coefficient of restitution \( \epsilon \). The normal component of the relative velocity after a collision \( g' = \vec{v}'_{12} \cdot \hat{e} \) follows from that before the collision \( g = \vec{v}_{12} \cdot \hat{e} \) via

\[
g' = -\epsilon g
\]

where \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}'_1, \vec{v}'_2 \) are respectively the velocities before and after the collision, while the unit vector \( \hat{e} \equiv \vec{r}_{12}/|\vec{r}_{12}| \) gives the direction of the inter-particle vector \( \vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \) at the instant of the collision.

From experiments as well as from theory it is well known that the coefficient of normal restitution \( \epsilon \) is not constant but it depends sensitively on the impact velocity \( g \). Although most of the results in the field of granular gases have been derived neglecting this dependence but using a velocity-independent coefficient of restitution (e.g. [12–18]) it has been shown that the impact-velocity dependence of the coefficient of restitution has serious consequences for various problems in granular gas dynamics [19–24].

The equation of motion for inelastically colliding 3D-spheres reads, therefore,

\[
\ddot{\xi} + \frac{g}{m_{eff}} \left( \xi^{3/2} + \frac{3}{2} A \sqrt{\xi} \xi' \right) = 0 , \tag{5}
\]

with \( \xi(0) = 0 \), \( \dot{\xi}(0) = g \), and with \( m_{eff} \equiv m_1 m_2/(m_1 + m_2) \) \((m_1, m_2)\) the masses of the colliding particles. To obtain the dependence of the restitution coefficient on the impact velocity for 3D-spheres the equation of motion (5) was solved numerically [10,24] and analytically [32], where the velocity-dependent restitution coefficient has been obtained as a series in powers of \( g^{1/5} \):

\[
\epsilon = 1 - C_1 \left( \frac{3}{2} A \right)^{2/5} \left( \frac{g}{m_{eff}} \right)^{1/5} + C_2 \left( \frac{3}{2} A \right)^{4/5} \left( \frac{g}{m_{eff}} \right)^{2/5} + \ldots \tag{6}
\]

The first coefficients \( C_1 = 1.15344 \) and \( C_2 = 0.79826 \) were evaluated analytically and then confirmed by numerical simulations [32].

Although in [13] a general method of derivation of all coefficients of the expansion (6) has been proposed, to obtain these, extensive calculations have to be performed. This approach does not provide closed-form expressions
II. THE DIMENSIONAL ANALYSIS

To perform the general dimensional analysis we adopt the following form for the elastic and dissipative forces:

\[ F_{\text{el}} = m^\text{eff} D_1 \xi^\alpha \]
\[ F_{\text{diss}} = m^\text{eff} D_2 \xi^\gamma \hat{\xi}^\beta. \]

This general form (at least for small \( \xi \) and \( \hat{\xi} \)) follows from the fact that both elastic and dissipative forces vanish at \( \xi = 0 \) and \( \hat{\xi} = 0 \), respectively. With these notations the equation of motion for colliding particles reads

\[ \ddot{\xi} + D_1 \xi^\alpha + D_2 \xi^\gamma \hat{\xi}^\beta = 0, \]
with \( \xi(0) = 0, \hat{\xi}(0) = g \),

where \( g \) has already been introduced. Now we choose as the characteristic length \( \xi_0 \) of the problem the maximal compression for the elastic case. It may be found from the condition that the initial kinetic energy \( m^\text{eff} g^2 / 2 \) equals the maximal elastic energy \( m^\text{eff} D_1 \xi_0^{\alpha+1}/(\alpha+1) \), which yields

\[ \xi_0 = \left( \frac{\alpha+1}{2 D_1} \right)^{\frac{\alpha+1}{\alpha+1}} g^{\frac{\alpha}{\alpha+1}}. \]

Choosing then the characteristic time of the problem as \( \tau_0 = \xi_0 \hat{\xi}_0 \), we construct new dimensionless variables

\[ \hat{\xi} = \xi/\xi_0, \quad \hat{\xi} = \hat{\xi}/g, \quad \hat{\xi} = \frac{g^2}{\xi_0} \xi, \]

and recast the equation of motion into dimensionless form:

\[ \ddot{\hat{\xi}} + \delta(g) \hat{\xi} + \frac{1+\alpha}{2} \hat{\xi}^\alpha = 0 \quad \text{with} \]
\[ \hat{\xi}(0) = 0, \quad \hat{\xi}(0) = 1 \]
\[ \hat{\xi}(\tau_e) = 0, \quad \hat{\xi}(\tau_e) = -\epsilon \]

In the last Eq. (10) we supplemented the pre-collisional initial conditions at \( \tau = 0 \) with the after-collisional conditions at \( \tau = \tau_e \) (\( \tau_e \) is the dimensionless time and \( \tau \) is the dimensionless duration of the collision). These follow just from the definition of the restitution coefficient. We point out that all dependence on the initial impact velocity on any quantity of the problem, including \( \epsilon \) (this is just the dimensionless after-collisional velocity) comes only through the constant \( \delta \), which reads

\[ \delta(g) = D_2 \left( \frac{1+\alpha}{2 D_1} \right)^{\frac{\alpha+1}{\alpha+1}} g^{\frac{2(\gamma+\alpha)}{\gamma+\alpha}-\beta}. \]

Hence, \( \epsilon(g) = \epsilon(\delta(g)) \). A similar result for \( \epsilon \rightarrow 0 \), \( \beta = 1 \) and \( \alpha = 3/2 \) has been obtained in [33].

If we assume that the restitution coefficient does not depend on the impact velocity \( g \) then follows

\[ 2(\gamma - \alpha) + \beta (1 + \alpha) = 0. \]

For a linear dependence of the dissipative force on the velocity, i.e. for \( \beta = 1 \) (this seems to be the most realistic for small \( \xi \)) one obtains a constant restitution coefficient for

- the linear elastic force, \( F_{\text{el}} \sim \xi \), i.e. \( \alpha = 1 \). The condition (12) implies \( \gamma = 0 \), and thus the linear dissipative force \( F_{\text{diss}} \sim \hat{\xi} \).
- the Hertz law for 3D-spheres \( \alpha = 3/2 \), therefore \( \gamma = \frac{1}{4} \) and \( F_{\text{diss}} \sim \hat{\xi}^{1/4} \) provides a constant restitution coefficient.

We now ask the question: What kind of \( \epsilon(g) \) dependence corresponds to the forces which act during collisions of viscoelastic particles? It may be generally shown [25,26,37] that the relation

\[ F_{\text{diss}} = A \hat{\xi} \frac{\partial}{\partial \xi} F_{\text{el}}(\xi). \]

between the dissipative and elastic forces with the dissipative constant \( A \) given in eq. (10) holds, provided three conditions are met [38]:

(i) The elastic part of the stress tensor depends linearly on the deformation tensor [28].

(ii) The dissipative part of the stress tensor depends linearly on the deformation rate tensor [28].

(iii) The conditions of quasistatic motion are provided, i.e. \( g \ll c, \tau_{\text{vis}} \ll \tau_e \) [23,24] (here \( c \) is the speed of sound in the material of particles, \( \tau_{\text{vis}} \) is relaxation time of viscous processes in its bulk).
From this follows that $\beta = 1$, $\gamma = \alpha - 1$, and thus the constant restitution coefficient may be observed only for collisions of cubic particles with surfaces normal to the direction of collision. We wish to emphasize that this conclusion comes from the general analysis of viscoelastic collisions.

Let us discuss now collisions between spheres with elastic and dissipative forces as given by (2) and (3), respectively. For these we have $m^{\text{eff}}_1 = \rho$, $\alpha = 3/2$ and $m^{\text{eff}}_2 = \frac{3}{2} A \rho$, $\gamma = 1/2$, $\beta = 1$ which yields the functional dependence for $\delta(g)$ and $\epsilon(g)$ respectively:

$$\delta = \frac{3}{2} \left( \frac{5}{4} \right)^{3/5} \frac{\sqrt{\rho}}{m^{\text{eff}}} \frac{g^{1/5}}{\xi^{2/5}}$$

$$\epsilon = \epsilon \left( A \left( \frac{\rho}{m^{\text{eff}}} \right)^{2/5} g^{1/5} \right)$$

(skipping the prefactor of $\delta(g)$ in the last equation) in accordance with (1) as found previously.

III. THE RESTITUTION COEFFICIENT FOR SPHERES

Using $\frac{d\xi}{d\xi} = \xi \frac{d\xi}{d\xi}$ it is convenient to write the equation of motion for a collision in the form

$$\frac{d}{d\xi} \left( \frac{1}{2} \xi^2 + \frac{1}{2} \xi^{5/2} \right) = -\delta \xi \xi^{1/2} = \frac{dE(\xi)}{d\xi}$$

$$\dot{\xi}(0) = 0; \quad \ddot{\xi}(0) = 1$$

where we introduce the mechanical energy

$$E \equiv \frac{1}{2} \xi^2 + \frac{1}{2} \xi^{5/2}.$$  (17)

To find the energy losses in the first stage of the collision, which starts with zero compression and ends in the turning point with maximal compression $\tilde{\xi}_0$

$$\int_{\tilde{\xi}_0}^{\tilde{\xi}_0} \frac{dE}{d\xi} d\xi = -\delta \int_{\tilde{\xi}_0}^{\tilde{\xi}_0} \xi^{1/2} d\xi$$

one needs to know the dependence of the compression rate $\dot{\xi}$ as a function of the compression $\xi$.

For the case of elastic collisions, the maximal compression is $\tilde{\xi}_0 = 1$, according to the definition of our dimensionless variables. Hence, the dependence $\dot{\xi}(\xi)$ follows from the conservation of energy:

$$\dot{\xi}(\xi) \approx \sqrt{1 - \xi^{5/2}}.$$  (19)

The velocity $\dot{\xi}$ vanishes at the turning point $\tilde{\xi} = 1$. For inelastic collisions the maximal compression $\tilde{\xi}_0$ is smaller than 1, therefore, one can write an approximation relation for the inelastic case:

$$\dot{\xi}(\xi) \approx \sqrt{1 - (\xi/\tilde{\xi}_0)^{5/2}}$$  (20)

which also gives vanishing velocity $\dot{\xi}$ at the turning point $\tilde{\xi}_0$. Integration in Eq. (18) may be performed yielding

$$\frac{1}{2} \tilde{\xi}_0^{5/2} - \frac{1}{2} = -\delta d \tilde{\xi}_0^{3/2}$$  (21)

where we take into account that $E(\tilde{\xi}_0) = \frac{1}{2} \tilde{\xi}_0^{5/2}$, $E(0) = \frac{1}{2} \tilde{\xi}(0) = 1$ and introduce a constant

$$d \equiv \int_{0}^{1} x^{1/2} \sqrt{1 - x^{5/2}} = \frac{1}{5} \Gamma \left( \frac{5}{4} \right).$$  (22)

Consider now the inverse collision, which is defined as a collision which starts with velocity $\epsilon$ and ends with velocity $g$. According to the concept of the inverse collision introduced in [23] (which is a useful auxiliary model), it is characterized by a negative damping (the energy “is pumped” into the system during the collision). The maximal compression $\tilde{\xi}_0$ is the same in both collisions, the direct and the inverse.

Rescaling equation of motion for the inverse collision in the very same way as for the direct collision yields

$$\frac{dE(\tilde{\xi})}{d\tilde{\xi}} = +\delta \tilde{\xi}^{1/2}$$

$$\dot{\tilde{\xi}}(0) = 0, \quad \ddot{\tilde{\xi}}(0) = \epsilon.$$  (23)

This suggests the following approximative relation for $\tilde{\xi}(\tilde{\xi})$ during the inverse collision:

$$\dot{\tilde{\xi}}(\tilde{\xi}) \approx \epsilon \sqrt{1 - (\tilde{\xi}/\tilde{\xi}_0)^{5/2}},$$  (24)

with the additional prefactor $\epsilon$, which is the initial velocity in the inverse collision.

Integration of the energy gain for the first stage of the inverse collision (which equals up to its sign the energy loss in the second stage of the direct collision [22]) may be performed just in the same way as for the direct collision, yielding the result

$$\frac{1}{2} \tilde{\xi}_0^{5/2} - \frac{\epsilon^2}{2} = +\epsilon \delta d \tilde{\xi}_0^{3/2},$$  (25)

where we again use $E(\tilde{\xi}_0) = \frac{1}{2} \tilde{\xi}_0^{5/2}$ and $E(0) = \frac{1}{2} \epsilon^2$. Multiplying Eq. (21) by $\epsilon$ and summing it up with Eq. (23) we obtain a simple approximative relation between the restitution coefficient and the (dimensionless) maximal compression:

$$\epsilon = \tilde{\xi}_0^{5/2}$$  (26)

Substituting this into Eq. (21) we arrive at an equation for the restitution coefficient.
\[
\epsilon + 2\delta d e^{3/5} = 1. 
\] (27)

The formal solution to this equation may be written as a continuum fraction (which does not diverge in the limit \( g \to \infty \)):

\[
\epsilon^{-1} = 1 + 2\delta d \left( 1 + 2\delta (1 + \cdots )^{2/5} \cdots \right)^{2/5}. 
\] (28)

Another way of representation of the restitution coefficient \( \epsilon \) is a series expansion in terms of \( \delta \). For practical applications it is convenient to return to dimensional units. We define the characteristic velocity \( g^* \) such that

\[
\delta \equiv \frac{1}{2d} \left( \frac{g}{g^*} \right)^{1/5},
\] (29)

with \( d \) being defined in Eq. (22). Using the definition \( g^* \) together with Eq. (22) we find for the characteristic velocity

\[
(g^*)^{-1/5} = \frac{\sqrt{7}}{2^{1/5} 5^{2/5}} \frac{\Gamma (3/5)}{\Gamma (21/10)} \left( \frac{3}{2} A \right) \left( \frac{\rho_{\text{mell}}}{m} \right)^{2/5}. 
\] (30)

Evaluating the numerical prefactor finally yields

\[
(g^*)^{-1/5} = 1.15344 \left( \frac{3}{2} A \right) \left( \frac{\rho_{\text{mell}}}{m} \right)^{2/5}. 
\] (31)

Note that the numerical constant 1.15344 has to be equal to \( C_1 \) in Eq. (3).

With this new notation the restitution coefficient reads

\[
\epsilon = 1 - a_1 \left( \frac{g}{g^*} \right)^{1/5} + a_2 \left( \frac{g}{g^*} \right)^{2/5} - a_3 \left( \frac{g}{g^*} \right)^{3/5} + a_4 \left( \frac{g}{g^*} \right)^{4/5} + \cdots ,
\] (32)

with \( a_1 = 1, a_2 = 3/5 \) (which are exact values), \( a_3 = 6/25 = 0.24, a_4 = 7/125 = 0.056, \cdots \) (which deviate from the correct ones, see below). Comparing (32) with (3) we conclude that our simple approximative theory reproduces exactly the coefficients \( C_1 \) and \( C_2 \), which were found before using extensive analysis (22).

We also performed rigorous but elaborated calculations according to the general scheme of (22) to find the exact coefficients (details are given in the Appendix)

\[
a_3 = 0.315119, \quad a_4 = 0.161167 , \quad \text{or, respectively,}
\]

\[
C_3 = -0.483582, \quad C_4 = 0.285279 . 
\] (34)

Hence, we observe that while the first two coefficients \( a_1 = 1 \) and \( a_2 = 3/5 \) are correctly obtained from the approximative theory, the next approximated coefficients \( a_3, a_4 \) differ from the exact ones.

For practical applications such as Molecular Dynamics simulations, however, the expansion (32) is of limited value, due to its divergence for high impact velocities, \( g \to \infty \). According to the velocity distribution function there is a certain probability that the relative velocity \( g \) of colliding particles exceeds the limit of applicability of (22). Therefore, we use the obtained coefficients to construct a Padé-approximation for \( \epsilon (g) \) which reveals the correct limits of the boundary conditions, \( \epsilon (0) = 1 \) and \( \epsilon (\infty) = 0 \). Since the dependence \( \epsilon (g) \) is expected to be a smooth, monotonically decreasing function, we choose a “1-4” Padé-approximation

\[
\epsilon = \frac{1 + d_1 \left( \frac{g}{g^*} \right)^{1/5}}{1 + d_2 \left( \frac{g}{g^*} \right)^{2/5} + d_3 \left( \frac{g}{g^*} \right)^{3/5} + d_4 \left( \frac{g}{g^*} \right)^{4/5} + d_5 \left( \frac{g}{g^*} \right)^{5/5}}. 
\] (35)

Standard analysis yields the coefficients \( d_k \) in terms of the coefficients \( a_k \) (see Table I)

| \( d_0 \) | \( d_1 \) | \( d_2 \) | \( d_3 \) | \( d_4 \) | \( d_5 \) |
|--------|--------|--------|--------|--------|--------|
| \( a_1 - 2a_2 - a_2^2 + 3a_2 - 1 \) | \( 1 - a_2 + a_3 - 2a_4 + (a_2 - 1)(3a_2 - 2a_1) \) | \( a_1 - 2a_2 \) | \( a_3 + a_2^2 (a_2 - 1) - a_4 (a_2 + 1) \) | \( a_4 (a_2 - 1) + (a_3 - a_2)(a_2^2 - 2a_1) \) | \( 2(a_3 - a_2)(a_1 - a_2a_3) - (a_1 - a_2)^2 - a_2 (a_3 - a_2^2) \) |

\[
\text{Table I. The coefficients of the Padé formula (33) as derived from the coefficients } a_k .
\]

Using the characteristic velocity \( g^* = 0.32 \text{ cm/s} \) for ice as a fitting parameter we could reproduce fairly well the experimental dependence of the restitution coefficient of ice as a function of the impact velocity \( g \) in the whole range of \( g \) (Fig. 1). The discrepancy with the experimental data at small \( g \) follows from the fact that the extrapolation expression, \( \epsilon = 0.32/g^{0.234} \) used in (3) has an unphysical divergence at \( g \to 0 \) and does not imply the fail of the theory for this region. The scattering of the experimental data presented in (3) is large for small impact velocity according to experimental complications, hence the fit formula of (3) cannot be expected to be accurate.
enough for too small velocities. Moreover in the region of very small velocity possibly other than viscolelastic interactions influence the collision behavior, e.g. adhesion.

FIG. 1. Dependence of the normal restitution coefficient on the impact velocity for ice particles. The solid line — experimental data of [5], dashed line — the Padé-approximation (32) with the constants given in the Table and with the characteristic velocity for ice $g^* = 0.32 \text{cm/s}$.

IV. CONCLUSION

We developed a dimensional analysis for the inelastic collision of spherical particles. We could show that for 3D-spheres the functional form for $\epsilon(g)$ agrees with that derived previously [32] using a much more complicated approach. Using a simple approximative theory we found a continuum-fraction representation for $\epsilon(g)$ and obtained explicit expressions for the coefficients of the series expansion of the restitution coefficient in terms of the impact velocity. The first two coefficients in this series coincide with that found previously by numerical evaluation. We report also a few next coefficients which we have derived within the general approach of a previous study [32]. Using the first four coefficients of this series expansion we constructed a Padé-approximation for $\epsilon(g)$. It reproduces fairly well the experimental data for colliding ice particles. The obtained relation for the restitution coefficient may be used for a wide range of the impact velocities, provided that the energy loss during a collision is attributed to viscoelasticity and that all the other dissipative processes (plastic deformation, fragmentation of particles) may be ignored.

ACKNOWLEDGMENTS

The authors want to thank W. Ebeling and L. Schimansky-Geier for discussion. The work was supported by Deutsche Forschungsgemeinschaft through grant Po 472/3-2 and by FONDECYT Chile, through project 02960021.

V. APPENDIX

The general method of derivation of the expansion coefficients $C_k$ has been given in [32]. Here, we briefly sketch the main lines of derivation, and provide some details for the particular case of $C_5$ and $C_6$. Since the method of derivation is based on the collection of terms with different dependence on the initial velocity $g$, it is convenient to use a scaling, somewhat different from that used before for the dimensional analysis. Namely, we rescale the time as $t' = (\rho/m_{\text{eff}})^{2/5} g^{-1/5} t$ and the length as $x = (\rho/m_{\text{eff}})^{2/5} / 2\xi$ to recast Eq. (3) into the form

$$x'' + \alpha g^{-1/5} x' \sqrt{x} + g^{-2/5} x^{3/2} = 0,$$

(36)

with $\alpha \equiv \frac{3}{4} A (\rho/m_{\text{eff}})^{2/5}$ and using all the notations introduced previously. The initial conditions for the rescaled Eq. (36) now read $x(0) = 0$ and $x'(0) = g^{4/5}$. For simplicity of notations we will keep in what follows $t$ for the rescaled time. As it was shown in [32], the trajectory may be expanded in terms of $\sqrt{t}$ as

$$x(t) = b_1 t^{1/2} + b_2 t + b_3 t^{3/2} + b_4 t^2 + b_5 t^{5/2} + b_6 t^3 + b_7 t^{7/2} + \ldots .$$

(37)

Clearly, both, $b_1$ and $b_3$ should be zero to avoid divergence of velocity and acceleration at $t = 0$. At the same time $b_2 = g^{4/5}$ and $b_4 = 0$ due to the equation of motion at vanishing compression. This yields

$$x(t) = g^{4/5} t + b_3 t^{5/2} + b_6 t^3 + b_7 t^{7/2} + \ldots .$$

(38)

From (38) one obtains $x'(t)$ and $x''(t)$ which are to be substituted into the equation of motion (36). One also needs $\sqrt{x}$ and $x^{3/2}$, the expansions for these in terms of $\sqrt{t}$ read

$$\sqrt{x} = g^{2/5} t^{1/2} + \frac{b_5}{2g^{2/5}} t^{1/2} + \frac{b_6}{2g^{2/5}} t^{5/2} + \frac{b_7}{2g^{2/5}} t^3 + \ldots .$$

(39)

and

$$x^{3/2} = g^{6/5} t^{3/2} + \frac{3}{2} g^{2/5} b_5 t^{3/2} + \frac{3}{2} g^{2/5} b_6 t^{7/2} + \ldots .$$

(40)

Inserting the expansions for $x'(t)$, $x''(t)$, $\sqrt{x}$ and $x^{3/2}$ into (36) and collecting the orders of $t$ we obtain

$$0 = \left( \frac{15}{4} b_5 + \alpha g^{1/5} \right) t^{1/2} + 6 b_6 t + \left( \frac{35}{4} b_7 + 1 \right) t^{3/2}$$

$$+ \left( 12 b_8 + 3 \alpha g^{1/5} b_5 \right) t^2 + \left( \frac{63}{4} b_9 + \frac{7}{2} \alpha g^{1/5} b_6 \right) t^{5/2} .$$

(41)

This suggests the coefficients:
so that the solution for the trajectory finally reads

\[
x(t) = g^{4/5}t - \frac{4}{15}ag^{5/2} - \frac{4}{35}g^{4/5}t^{7/2} + \frac{1}{15}a^2g^{6/5}t^4 + \ldots
\]

In order to get the higher orders, which is conceptionally simple but requires extensive calculus, we wrote a program [11], which by formula manipulations performs exactly the steps we described above and which is able to find the trajectory up to any desired order.

Generally, it is convenient to write the solution as a series

\[
x(t) = g^{4/5}\left(x_0(t) + ag^{1/5}x_1(t) + a^2g^{2/5}x_2(t) + \ldots\right).
\]

Here \(x_0(t)\) is a “zero-order” trajectory, which refers to the case of undamped collision, the “first-order” trajectory, \(x_1(t)\), accounts for damping in linear (with respect to \(a\)) approximation, the “second-order” trajectory, \(x_2(t)\), corresponds to the next approximation \(\sim a^2\), etc. Here we give our results for these “n-order” trajectories up to \(n = 3\), obtained using the above mentioned program up to the order \(t^{11}\):

\[
\begin{align*}
x_0 &= t - \frac{4}{35}t^{7/2} + \frac{1}{175}t^6 - \frac{22}{104125}t^{17/2} + \frac{52}{8017625}t^{11}, \\
x_1 &= -\frac{4}{15}t^{5/2} + \frac{3}{70}t^5 - \frac{713}{238875}t^{15/2} + \frac{6126}{42639187}t^{10}, \\
x_2 &= \frac{1}{15}t^4 + \frac{937}{75075}t^{13/2} + \frac{871}{808500}t^9, \\
x_3 &= -\frac{38}{2475}t^{11/2} + \frac{43943}{13513500}t^8 - \frac{1184627}{3594591000}t^{21/2},
\end{align*}
\]

To proceed we need to find the maximal compression \(x_{\text{max}}\), which is reached at time \(t_{\text{max}}\). The point of maximal compression is a turning point of the trajectory, where the velocity is zero. Therefore the condition

\[
x_{\text{max}}'(t_{\text{max}}) = 0
\]

holds at this point. With the above expression for the trajectory Eqs. [18-49], the last Eq. [40] is an equation to determine \(t_{\text{max}}\), which may be then used to find \(x_{\text{max}}\). This equation, however, is a high-order algebraic equation for \(\Gamma_{\text{max}}\), which is not generally solvable. On the other hand, for the undamped collision \(t_{\text{max}}\) equals one-half of the collision duration \(t_c/2\) and both quantities of interest are known [23]:

\[
\begin{align*}
t_0 &= \frac{t_c}{2} + \frac{3}{5}\left(\frac{\Gamma_{\text{max}}}{\Gamma}\right) - \frac{4}{35}\frac{\Gamma_{\text{max}}^2}{\Gamma}\left(\frac{\Gamma_{\text{max}}}{\Gamma}\right) - \frac{4}{35}\frac{\Gamma_{\text{max}}^3}{\Gamma^2}\left(\frac{\Gamma_{\text{max}}}{\Gamma}\right) + \ldots \\
x_0 &= \left(\frac{5}{4}\right)^{2/5}.
\end{align*}
\]

For a viscoelastic collision \(t_{\text{max}}\) certainly differs from \(t_c/2\), so that \(t_{\text{max}} = \frac{t_c}{2} + \delta t\). If the dissipation parameter \(\alpha\) is not large, the deviation \(\delta t\) is presumably small, therefore we expand \(x'(t_{\text{max}}) = x'\left(\frac{t_c}{2} + \delta t\right)\) in terms of \(\delta t\):

\[
g^{-4/5}x'(t_{\text{max}}) = \left[\frac{x_0'\left(\frac{t_c}{2}\right) + \delta tx_0''\left(\frac{t_c}{2}\right) + \delta t^2x_0'''\left(\frac{t_c}{2}\right) + \ldots}{2}\right]
\]

\[
+ ag^{1/5}\left[\frac{x_1'\left(\frac{t_c}{2}\right) + \delta tx_1''\left(\frac{t_c}{2}\right) + \delta t^2x_1'''\left(\frac{t_c}{2}\right) + \ldots}{2}\right]
\]

\[
+ a^2g^{2/5}\left[\frac{x_2'\left(\frac{t_c}{2}\right) + \delta tx_2''\left(\frac{t_c}{2}\right) + \delta t^2x_2'''\left(\frac{t_c}{2}\right) + \ldots}{2}\right]
\]

\[
+ a^3g^{3/5}\left[\frac{x_3'\left(\frac{t_c}{2}\right) + \delta tx_3''\left(\frac{t_c}{2}\right) + \delta t^2x_3'''\left(\frac{t_c}{2}\right) + \ldots}{2}\right] = 0,
\]

where we use representation [18] for the trajectory. The deviation \(\delta t\), vanishes at \(\alpha = 0\) and, thus, suggests the expansion in terms of \(\alpha\):

\[
\delta t = \tau_1\alpha + \tau_2\alpha^2 + \tau_3\alpha^3 + \ldots
\]

Substituting \(\delta t\), given by Eq. [24], into [23] and collecting terms of the same order of \(\alpha\) yields

\[
Y_0 + \alpha Y_1 + \alpha^2 Y_2 + \alpha^3 Y_3 + \cdots = 0,
\]

with the abbreviations

\[
Y_0 = x_0'\left(\frac{t_c}{2}\right),
\]

\[
Y_1 = \tau_1x_0''\left(\frac{t_c}{2}\right) + g^{1/5}x_1'\left(\frac{t_c}{2}\right),
\]

\[
Y_2 = \tau_2x_0''\left(\frac{t_c}{2}\right) + \frac{\tau_2^2}{2}x_0''\left(\frac{t_c}{2}\right) + g^{1/5}\tau_1x_1'\left(\frac{t_c}{2}\right) + g^{2/5}x_2'\left(\frac{t_c}{2}\right)
\]}
\[ Y_3 = \tau_3x''_0 \left( \frac{t^0}{2} \right) + \tau_2x''_0 \left( \frac{t^0}{2} \right) + \frac{x^3}{6}x''''_0 \left( \frac{t^0}{2} \right) + g^{1/5}\tau_2x'_1 \left( \frac{t^0}{2} \right) + g^{1/5}\tau_1^2x''_1 \left( \frac{t^0}{2} \right) + g^{2/5}\tau_1x''_2 \left( \frac{t^0}{2} \right) + g^{3/5}x'_3 \left( \frac{t^0}{2} \right). \]

The conditions \( Y_k = 0 \) for \( k = 0, \ldots, 3 \) together with Eq. (58) allows to express the constants \( \tau_1, \tau_2, \tau_3 \), etc., in terms of functions \( x_1(t), x_2(t), x_3(t) \), etc., and their time derivatives taken at time \( \left( t^0_c/2 \right) \):

\[
\tau_1 = -g^{1/5} \frac{x'_1 \left( \frac{t^0}{2} \right)}{x''_0 \left( \frac{t^0}{2} \right)},
\tau_2 = g^{2/5} \left[ -\frac{x^2 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right)}{2} + \frac{x'_1 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right)}{x''_0 \left( \frac{t^0}{2} \right)} - \frac{x'_2 \left( \frac{t^0}{2} \right)}{x''_0 \left( \frac{t^0}{2} \right)} \right].
\]

We do not write the expression for \( \tau_3 \), since due to the special properties of the problem, i.e. due to the fact that \( x'_0 \left( t^0_c/2 \right) = 0 \) the value \( \tau_3 \) is not needed for calculation of \( \epsilon \) up to fourth order of \( \alpha \). The functions \( x_1(t), x_2(t), x_3(t) \) are known and given by Eqs. (49), so that the constants \( \tau_1 \) and \( \tau_2 \) may be found explicitly.

Writing the maximal compression as

\[ x_{\text{max}} = g^{4/5} \left[ x_0 \left( \frac{t^0}{2} + \delta t \right) + \alpha g^{1/5}x_1 \left( \frac{t^0}{2} + \delta t \right) + \alpha^2 g^{2/5}x_2 \left( \frac{t^0}{2} + \delta t \right) + \alpha^3 g^{3/5}x_3 \left( \frac{t^0}{2} + \delta t \right) \right], \]

and expanding this in terms of \( \delta t \), using then representation of \( \delta t \) as \( \delta t = \alpha \tau_1 + \alpha^2 \tau_2 + \cdots \), with \( \tau_1, \tau_2 \) from (56) and collecting terms of the same order of \( \alpha \) we obtain

\[ x_{\text{max}} = g^{4/5} \left( y_0 + \alpha g^{1/5}y_1 + \alpha^2 g^{2/5}y_2 + \alpha^3 g^{3/5}y_3 \right), \]

where \( y_0, \ldots, y_3 \) are pure numbers:

\[
y_0 = x_0 \left( \frac{t^0}{2} \right) = 1.093362
\]

\[
y_1 = x_1 \left( \frac{t^0}{2} \right) = -0.504455
\]

\[
y_2 = \left[ x_2 \left( \frac{t^0}{2} \right) - \frac{1}{2} x'_1 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right) \right] = 0.260542
\]

\[
y_3 = \left[ x_3 \left( \frac{t^0}{2} \right) - \frac{1}{2} x'_1 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right) + x'_2 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right) - \frac{1}{2} x''_1 \left( \frac{t^0}{2} \right) x''_0 \left( \frac{t^0}{2} \right) \right] = -0.136769
\]

and where we use expressions (49) for \( x_1(t), x_2(t) \) and \( x_3(t) \).

To calculate the coefficient of restitution one has to use the concept of inverse collision, as was introduced in (32) and discussed in previous chapters of the present study. One obtains the solution of this inverse collision by replacing \( g \to \epsilon g \) for the initial velocity and \( \alpha \to -\alpha \) for the dissipative coefficient. In particular, this applies to the maximal compression of the inverse collision \( x_{\text{max}}^{\text{inv}} = x_{\text{max}}(g \to \epsilon g, \alpha \to -\alpha) \). For consistency one has to require the maximum compressions for direct and inverse collision to be equal, i.e.

\[ x_{\text{max}}^{\text{inv}} = x_{\text{max}} \]

or using (32),

\[ \epsilon^{1/5} g^{1/2} \left( y_0 - \alpha \epsilon^{1/5} g^{1/2} y_1 + \alpha^2 \epsilon^{2/5} g^{2/2} y_2 - \alpha^3 \epsilon^{3/5} g^{3/2} y_3 + \cdots \right) = g^{1/2} \left( y_0 + \alpha g^{1/5} y_1 + \alpha^2 g^{2/5} y_2 + \alpha^3 g^{3/5} y_3 + \cdots \right). \]

Eq. (64) is in fact an algebraic equation for \( \epsilon^{1/5} \), which may not be generally solved. For this reason we write \( \epsilon \) as an expansion of \( \alpha g^{1/5} \), which is the only combination in which both parameters appear
\[ \epsilon = 1 + C_1 \alpha g^{1/5} + C_2 \left( \alpha g^{1/5} \right)^2 + C_3 \left( \alpha g^{1/5} \right)^3 + C_4 \left( \alpha g^{1/5} \right)^4 + \ldots \]  

and substitute (65) into (64). Collecting orders we find

\[ \left[ -\frac{4}{5} y_0 C_1 + 2 y_1 \right] \alpha g^{1/5} + \left[ \left( -\frac{4}{5} C_2 + \frac{2}{25} C_1 \right) y_0 + y_1 C_1 \right] \alpha^2 g^{2/5} + \\
\left[ -\frac{4}{5} C_3 + \frac{4}{25} C_1 C_2 - \frac{4}{125} C_1^2 \right] y_0 + y_1 C_2 - \frac{6}{5} y_2 C_1 + 2 y_3 \right] \alpha^3 g^{3/5} + \\
\left\{ -\frac{4}{5} C_4 + \frac{2}{25} \left( C_2^2 + 2 C_1 C_3 \right) - \frac{12}{125} C_1^2 C_2 + \frac{11}{125} C_1^3 \right\} y_0 + y_1 C_3 + \left( -\frac{6}{5} C_2 - \frac{3}{25} C_1 \right) y_2 + \frac{7}{5} y_3 C_1 \right) \alpha^4 g^{4/5} = 0. \]

The last Eq. (65) yields the final result for the coefficients:

\[ C_1 = \frac{5 y_1}{2 y_2} = -1.153449 \]
\[ C_2 = \frac{15}{4} \left( \frac{y_1}{y_2} \right)^2 = \frac{3}{5} C_1^2 = 0.798267 \]
\[ C_3 = \frac{95}{16} \frac{y_1}{y_2}^3 - \frac{15}{4} \frac{y_2 y_1}{y_0} + \frac{5}{2} \frac{y_3}{y_0} = -0.483582 \]
\[ C_4 = \frac{315}{32} \left( \frac{y_1}{y_2} \right)^4 - \frac{105}{8} \frac{y_2 y_1}{y_0^2} + \frac{35}{4} \frac{y_3 y_1}{y_0} = 0.285279 \]

Using \((g^*)^{-1/5} = C_1 \alpha\), we obtain for coefficients \(a_k\) in expansion (32):

\[ a_1 = 1 \]
\[ a_2 = C_2/C_1^2 = 3/5 \]
\[ a_3 = C_3/C_1^3 = 0.315119 \]
\[ a_4 = C_4/C_1^4 = 0.161167. \]

Note that although the general method given in Appendix allows to evaluate up to a desired precision \(a_k\), in principle, coefficients \(C_k\), it does not provide the closed form expression for \(C_1\) as the simple approximate approach given in the main text does.

[1] R. M. Brach, J. Appl. Mech. 56, 133 (1989).
[2] S. Wall, W. John, H. C. Wang, and S. L. Goren, Aerosol Sci. Tech. 12, 926 (1990).
[3] W. Goldsmith, Impact: The Theory and Physical Behavior of Colliding Solids, Edward Arnold (London, 1960).
[4] P. F. Luckham, Pow. Tech. 58, 75 (1989).
[5] F. G. Bridges, A. Hatzes, and D. N. C. Lin, Nature 309, 333 (1984).
[6] E. Hodgkinson, Report of the 4th Meeting of the British Association for the Advancement of Science, London (1835).
[7] C. V. Raman, Phys. Rev. 12, 442 (1918).
[8] S. F. Foerster, M. Y. Louge, H. Chang, and Kh. Allia, Phys. Fluids 6, 1108 (1994).
[9] S. Hatzes, F. G. Bridges, and D. N. C. Lin, Mon. Not. R. Astr. Soc. 231, 1191 (1988).
[10] S. Luding, E. Clément, A. Blumen, J. Rajchenbach, and J. Duran, Phys. Rev. E 50, 4113 (1994).
[11] S. Luding, E. Clément, J. Rajchenbach, and J. Duran, Europhys. Lett. 36, 247 (1996).
[12] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. 70, 1619 (1993).
[13] Y. Wu, H. Li, and L. P. Kadanoff, Phys. Rev. Lett. 74, 1268 (1995).
[14] T. Zhou and L. P. Kadanoff, Phys. Rev. E 54, 623 (1996).
[15] T. P. C. van Noije, M. H. Ernst, and R. Brito, Phys. Rev. E 57, 4891 (1998).
[16] J. A. C. Orza, R. Brito, T. P. C. van Noije, and M. H. Ernst, Int. J. Mod. Phys. C 8, 953 (1997).
[17] T. C. P. van Noije, M. H. Ernst, R. Brito, and J. A. G. Orza, Phys. Rev. Lett. 79, 411 (1997).
[18] J. J. Brey and D. Cubero, Phys. Rev. E 57, 2019 (1998).
[19] H. Salo, J. Lukkari, and J. Hanninen, Earth, Moon, and Planets 43, 33 (1988).
[20] K. A. Hämeen-Anttila and J. Lukkari, Astrophys. Space. Sci. 71, 475 (1980).
[21] F. Spahn, U. Schwarz, and J. Kurths, Phys. Rev. Lett. 78, 1596 (1997).
[22] T. Pöschel and T. Schwager, Phys. Rev. Lett. 80, 5708 (1998).
[23] N. V. Brilliantov and T. Pöschel, cond-mat/9803387.
[24] F. Spahn, J.-M. Hertzsch, and N. V. Brilliantov, Chaos, Solitons and Fractals, 5, 1945 (1995).
[25] N. V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel, Phys. Rev. E 53, 5382 (1996).
[26] J.-M. Hertzsch, F. Spahn, and N. V. Brilliantov, J.Phys.II (France), 5, 1725 (1995).
[27] H. Hertz, J. f. reine un. angewandte Math. 92, 156 (1862).
[28] L. D. Landau and E. M. Lifschitz, Theory of Elasticity, Oxford University Press (Oxford, 1965).
[29] G. Kuwabara and K. Kono, Jpn. J. Appl. Phys. 26, 1230 (1987).
[30] W. A. M. Morgado and I. Oppenheim, Phys. Rev. E 55, 1940 (1997).
[31] W. A. M. Morgado and I. Oppenheim, Physica A 246, 547 (1997).
[32] T. Schwager and T. Pöschel, Phys. Rev. E 57, 650 (1998).
As usual for collision (e.g. [3,28]) the two-particle problem is reduced to scattering problem of a single-particle with an effective mass $m^{\text{eff}}$.

Derivation of the dissipative force given in [25,26] for colliding spheres may be straightforwardly generalized to obtain the relation (13) (or (A17) in [25,26]) for colliding bodies of any shape, provided that displacement field in the bulk of the material of bodies in contact is a one-valued function of the compression (see also [38]).

Obviously, the coefficients of the Padé-approximation may be chosen up to an arbitrary factor to multiply numerator and denominator; we chose it to have unity as a leading term for both of these.

Note that, in difference to the calculations in the main part of the article the quantities $x$, $x'$ and $x''$ do have units, namely $(m/\text{sec})^{4/5}$. The rescaled time is dimensionless. The purpose of this scaling was only to simplify the dependence of the problem on the material parameters, it was necessary to keep the explicit dependence of the problem on the initial velocity.

The maple-program is available at:

\[ \text{http://www.summa.physik.hu-berlin.de/~kies/papers/DimAnalysis/epsilon_simple.txt} \]