INITIAL SOFT L-FUZZY PREPROXIMITIES

YOUNG SUN KIM\textsuperscript{a} AND YONG CHAN KIM\textsuperscript{b,∗}

Abstract. In this paper, we introduce the notions of soft $L$-fuzzy preproximities in complete residuated lattices. We prove the existence of initial soft $L$-fuzzy preproximities. From this fact, we define subspaces and product spaces for soft $L$-fuzzy preproximity spaces. Moreover, we give their examples.

1. Introduction

Hájek [5] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [6,7-9]. Recently, Molodtsov [11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,4]. Pawlak’s rough set [12,13] can be viewed as a special case of soft rough sets [4]. The topological structures of soft sets have been developed by many researchers [2,7-9,15-17].

Čimoka et.al [3] introduced $L$-fuzzy syntopogenous structures as fundamentals and application to $L$-fuzzy topologies, $L$-fuzzy proximities and $L$-fuzzy uniformities in a complete residuated lattice. Kim [7] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where $L$ is a complete residuated lattice. Kim [7-9] introduced the soft topological structures, soft $L$-fuzzy quasi-uniformities and soft $L$-fuzzy topogenous orders in complete residuated lattices.

In this paper, we prove the existence of initial soft $L$-fuzzy preproximities. From this fact, we define subspaces and product spaces for soft $L$-fuzzy preproximity spaces. Moreover, we give their examples.
2. Preliminaries

**Definition 2.1 ([5,6])**. An algebra \((L, \land, \lor, \circ, \to, 0, 1)\) is called a *complete residuated lattice* if it satisfies the following conditions:

(C1) \(L = (L, \leq, \lor, \land, 1, 0)\) is a complete lattice with the greatest element 1 and the least element 0;

(C2) \((L, \circ, 1)\) is a commutative monoid;

(C3) \(x \circ y \leq z\) iff \(x \leq y \to z\) for \(x, y, z \in L\).

In this paper, we assume that \((L, \leq, \circ, \to, \oplus^*)\) is a complete residuated lattice with an order reversing involution \(^*\) which is defined by \(x \oplus y = (x^* \circ y^*)^*\) and \(x^* = x \to 0\).

**Lemma 2.2 ([5,6])**. For each \(x, y, z, x_i, y_i, w \in L\), we have the following properties.

1. \(1 \to x = x, 0 \circ x = 0\),

2. If \(y \leq z\), then \(x \circ y \leq x \circ z, x \oplus y \leq x \oplus z, x \to y \leq x \to z\) and \(z \to x \leq y \to x\),

3. \(x \circ y \leq x \land y \leq x \lor y \leq x \oplus y\),

4. \((\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*\),

5. \(x \circ (\bigvee_i y_i) = \bigvee_i (x \circ y_i)\),

6. \(x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)\),

7. \(x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)\),

8. \((\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)\),

9. \(x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)\),

10. \((\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)\),

11. \((x \circ y) \to z = x \to (y \to z) = y \to (x \to z)\),

12. \(x \circ (x \to y) \leq y\) and \(x \to y \leq (y \to z) \to (x \to z)\),

13. \((x \to y) \circ (z \to w) \leq (x \circ z) \to (y \circ w)\),

14. \((x \to y) \circ (z \to w) \leq \bigvee \circ (x \oplus w)\),

15. \(x \to y \leq (x \circ z) \to (y \circ z)\) and \((x \to y) \circ (y \to z) \leq x \to z\),

16. \(x \circ y \circ (z \circ w) \leq (x \circ z) \oplus (y \circ w)\),

17. \(x \to y = y^* \to x^*\).

**Definition 2.3 ([7-9])**. Let \(X\) be an initial universe of objects and \(E\) the set of parameters (attributes) in \(X\). A pair \((F, A)\) is called a *fuzzy soft set* over \(X\), where \(A \subset E\) and \(F : A \to L^X\) is a mapping. We denote \(S(X, A)\) as the family of all fuzzy soft sets under the parameter \(A\).
**Definition 2.4** ([7-9]). Let \((F, A)\) and \((G, A)\) be two fuzzy soft sets over a common universe \(X\).

1. \((F, A)\) is a fuzzy soft subset of \((G, A)\), denoted by \((F, A) \subseteq (G, A)\) if \(F(\epsilon) \subseteq G(\epsilon)\), for each \(\epsilon \in A\).
2. \((F, A) \land (G, A) = (F \land G, A)\) if \((F \land G)(\epsilon) = F(\epsilon) \land G(\epsilon)\) for each \(\epsilon \in A\).
3. \((F, A) \lor (G, A) = (F \lor G, A)\) if \((F \lor G)(\epsilon) = F(\epsilon) \lor G(\epsilon)\) for each \(\epsilon \in A\).
4. \((F, A) \circ (G, A) = (F \circ G, A)\) if \((F \circ G)(\epsilon) = F(\epsilon) \circ G(\epsilon)\) for each \(\epsilon \in A\).
5. \((F, A)^* = (F^*, A)\) if \(F^*(\epsilon) = (F(\epsilon))^*\) for each \(\epsilon \in A\).
6. \((F, A) \oplus (G, A) = (F \oplus G, A)\) if \((F \oplus G)(\epsilon) = (F(\epsilon) \oplus G(\epsilon))^*\) for each \(\epsilon \in A\).

**Definition 2.5** ([8, 9]). Let \(S(X, A)\) and \(S(Y, B)\) be the families of all fuzzy soft sets over \(X\) and \(Y\), respectively. The mapping \(f_\phi : S(X, A) \rightarrow S(Y, B)\) is a soft mapping where \(f : X \rightarrow Y\) and \(\phi : A \rightarrow B\) are mappings.

1. The image of \((F, A) \in S(X, A)\) under the mapping \(f_\phi\) is denoted by \(f_\phi((F, A)) = (f_\phi(F), B)\) where
   
   \[
   f_\phi(F)(b)(y) = \begin{cases} 
   \bigvee_{a \in \phi^{-1} \{b\}} (f_\phi(F)(a))(y), & \text{if } \phi^{-1} \{b\} \neq \emptyset, \\
   0, & \text{otherwise}.
   \end{cases}
   \]

2. The inverse image of \((G, B) \in S(Y, B)\) under the mapping \(f_\phi\) is denoted by \(f_\phi^{-1}((G, B)) = (f_\phi^{-1}(G), A)\) where
   
   \[
   f_\phi^{-1}(G)(a)(x) = f_\phi^{-1}(G(\phi(a)))(x), \ \forall a \in A, x \in X.
   \]

3. The soft mapping \(f_\phi : S(X, A) \rightarrow S(Y, B)\) is called injective (resp. surjective, bijective) if \(f\) and \(\phi\) are both injective (resp. surjective, bijective).

**Lemma 2.6** ([8, 9]). Let \(f_\phi : S(X, A) \rightarrow S(Y, B)\) be a soft mapping. Then we have the following properties. For \((F, A), (F_i, A) \in S(X, A)\) and \((G, B), (G_i, B) \in S(Y, B),\)

1. \((G, B) \geq f_\phi(f_\phi^{-1}((G, B)))\) with equality if \(f\) is surjective,
2. \((F, A) \leq f_\phi^{-1}(f_\phi((F, A)))\) with equality if \(f\) is injective,
3. \(f_\phi^{-1}(\bigvee_{i \in I} (G_i, B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i, B)),\)
4. \(f_\phi^{-1}(\bigwedge_{i \in I} (G_i, B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i, B)),\)
5. \(f_\phi(\bigvee_{i \in I} (F_i, A)) = \bigvee_{i \in I} f_\phi((F_i, A)),\)
6. \(f_\phi(\bigwedge_{i \in I} (F_i, A)) \leq \bigwedge_{i \in I} f_\phi((F_i, A))\) with equality if \(f\) is injective,
7. \(f_\phi^{-1}((G_1, B) \circ (G_2, B)) = f_\phi^{-1}((G_1, B)) \circ f_\phi^{-1}((G_2, B)),\)
8. \(f_\phi^{-1}((G_1, B) \oplus (G_2, B)) = f_\phi^{-1}((G_1, B)) \oplus f_\phi^{-1}((G_2, B)),\)
(9) \( f_\delta((F_1, A) \odot (F_2, A)) \leq f_\delta((F_1, A)) \odot f_\delta((F_2, A)) \) with equality if \( f \) is injective.
(10) \( f_\delta((F_1, A) \oplus (F_2, A)) \leq f_\delta((F_1, A)) \oplus f_\delta((F_2, A)) \).

**Definition 2.7.** A function \( \delta : L^X \times L^X \to L \) is called a soft L-fuzzy pre-proximity on \( X \) if it satisfies the following conditions:

(SP1) \( \delta((1_X, A), (0_X, A)) = 0 \) and \( \delta((0_X, A), (1_X, A)) = 0 \).

(SP2) If \( (F, A) \leq (F_1, A) \) and \( (G, A) \leq (G_1, A) \), then \( \delta((F, A), (G, A)) \leq \delta((F_1, A), (G_1, A)) \).

(SP3) If \( \delta((F, A), (G, A)) \neq 1 \), then \( (F, A) \leq (G, A)^* \).

(SP4)

\( \delta((F_1, A) \odot (F_2, A), (H_1, A) \oplus (H_2, A)) \leq \delta((F_1, A), (H_1, A)) \odot \delta((F_2, A), (H_2, A)). \)

The triple \((X, A, \delta)\) is said to be a soft L-fuzzy pre-proximity space.

A soft L-fuzzy pre-proximity space is called a soft L-fuzzy quasi-proximity if (SQ)

\( \delta((F, A), (G, A)) \geq \bigwedge_{(H,A)\in S(X,A)} \{ \delta((F, A), (H, A)) \oplus \delta((H, A)^*, (G, A)) \} \).

A soft L-fuzzy pre-proximity space is called perfect if

(P) \( \delta(\bigvee_{i \in I}(F_i, A), (G, A)) \leq \bigcup_{i \in I} \delta((F_i, A), (G, A)). \)

Let \((X, A, \delta_1)\) and \((X, A, \delta_2)\) be soft L-fuzzy pre-proximity spaces. We say that \( \delta_1 \) is finer than \( \delta_2 \) (\( \delta_2 \) is coarser than \( \delta_1 \)) if \( \delta_1((F, A), (G, A)) \leq \delta_2((F, A), (G, A)) \) for all \((F, A), (G, A) \in S(X, A)\).

Let \((X, A, \delta_X)\) and \((Y, B, \delta_Y)\) be soft L-fuzzy pre-proximity spaces and \( f_\delta : X \to Y \) be a soft map. Then \( f \) is called a fuzzy proximity soft map if \( \forall (F, A), (G, A) \in S(X, A), \delta_X((F, A), (G, A)) \leq \delta_X(f_\delta((F, A)), (f_\delta((G, A)))). \)

**Remark 2.8.**

(1) If a complete residuated lattice \((L, \leq, \odot, \oplus, *)\) is a completely distributive lattice \((L, \leq, \land, \lor, *)\) with a strong negation * with \( \odot = \land \) and \( \oplus = \lor \), the above definition coincide with that in the sense [3].

(2) Let \((X, A, \delta)\) be a soft L-fuzzy pre-proximity space. By (SP4), we have

\[ \delta((\bigodot_{i=1}^p(F_i, A), \bigoplus_{k=1}^p(G_k, A)) \leq \bigwedge_{\sigma \in K} (\bigoplus_{i=1}^p \delta((F_i, A), (G_{\sigma(i)}), A))) \]

where \( K = \{ \sigma \mid \sigma : \{1, 2, ..., p\} \to \{1, 2, ..., p\} \} \) is a bijective function.

(3) Let \( L \) be an idempotent complete residuated lattice, that is, \( x \odot x = x \), for each \( x \in L \). Since \( (F, A) \odot (F, A) = (F, A) \) and \( (G, A) \oplus (G, A) = (G, A) \), then

\( \delta((F, A), (G, A)) \leq \delta((F, A), (G, A)) \oplus \delta((F, A), (G, A)) \) and

\( \delta((F_1, A) \odot (F_2, A), (G, A)) \leq \delta((F_1, A), (G, A)) \oplus \delta((F_2, A), (G, A)). \)
3. Initial soft $L$-fuzzy Preproximities

**Theorem 3.1.** Let $\{(X_i, A_i, \delta_i) \mid i \in \Gamma\}$ be a family of soft $L$-fuzzy pre-proximity spaces. Let $X$ be a set and, for each $i \in \Gamma$, $f_i : X \to X_i$ and $\phi_i : A \to A_i$ mappings. Define the function $\delta : S(X, A) \times S(X, A) \to L$ on $X$ by

$$
\delta((F, A), (G, A)) = \bigwedge \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{i \in \Gamma} \delta((f_j_\phi_i)((F_j, A)), (f_j_\phi_i((G_{\sigma(j)}, A))) \right) \right\},
$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^{p}(F_j, A)\}$, $\{(G_k, A) \mid (G, A) = \bigoplus_{k=1}^{p}(G_{\sigma(j)}, A)\}$ and

$$K = \{\sigma \mid \sigma : \{1, \ldots, p\} \to \{1, \ldots, p\} \text{ is a bijective function}\}.$$

Then:

1. $\delta$ is the coarsest soft $L$-fuzzy pre-proximity on $X$ which all $(f_i)_{\phi_i}, i \in \Gamma$, are fuzzy proximity soft maps.

2. If $\{(X_i, A_i, \delta_i) \mid i \in \Gamma\}$ is a family of soft $L$-fuzzy quasi-proximity spaces, $\delta$ is a soft $L$-fuzzy quasi-proximity on $X$.

3. A map $f_\phi : (Y, B, \delta_0) \to (X, A, \delta)$ is a fuzzy proximity soft map iff each $(f_i)_{\phi_i} \circ f_\phi : (Y, B, \delta_0) \to (X_i, A_i, \delta_i)$ is a fuzzy proximity soft map.

**Proof.** (1) First, we will show that $\delta$ is a soft $L$-fuzzy pre-proximity on $X$.

(SP1) Since $\delta((F, A), (0_X, A)) \leq \delta((f_i)_{\phi_i}, ((F, A)), (0_X, A)) = 0$ for all $(F, A) \in S(X, A)$, it is clear.

(SP2) It follows from the definition of $\delta$.

(SP3) We will show that if $(F, A) \not\leq (G, A)^*$, then $\delta((F, A), (G, A)) = 1$.

Let $(F, A) \not\leq (G, A)^*$. Then, for every two finite families $\{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^{p}(F_j, A)\}$ and $\{(G_k, A) \mid (G, A) = \bigoplus_{k=1}^{p}(G_{\sigma(j)}, A)\}$ and $\sigma \in K$, there exist $j_0, \sigma(j_0)$, $x_0$ such that $(F_{j_0}, A)(x_0) \not\leq (G_{\sigma(j_0)}, A)(x_0)^*$. It follows that, for all $i \in \Gamma$,

$$(f_i)_\phi((F_{j_0}, A))(x_0) \not\leq (f_i)_\phi((G_{\sigma(j_0)}, A))(x_0)^*.$$ 

Since $\delta_i$ is a soft $L$-fuzzy pre-proximity on $X_i$, for each $i \in \Gamma$, by (SP3),

$$\delta_i((f_i)_\phi((F_{j_0}, A)), (f_i)_\phi((G_{\sigma(j_0)}, A))) = 1.$$ 

So, $\bigwedge_{i \in \Gamma} \delta_i((f_i)_\phi((F_{j_0}, A)), (f_i)_\phi((G_{\sigma(j_0)}, A))) = 1$. By Lemma 2.2(3), it follows

$$\bigoplus_{j=1}^{p} \left( \bigwedge_{i \in \Gamma} \delta_i((f_i)_\phi((F_j, A)), (f_i)_\phi((G_{\sigma(j)}, A))) \right) = 1,$$
for every two finite families \(\{(F_j, A) \mid (F, A) = \circ_{j=1}^{p}(F_j, A)\}\) and \(\{(G_k, A) \mid (G, A) = \bigoplus_{k=1}^{p}(G_k, A)\}\) and \(\sigma \in K\). Hence \(\delta((F, A), (G, A)) = 1\).

(4) Suppose there exist \((F_1, A), (G_i, A) \in S(X, A)\) such that
\[
\delta((F_1, A) \circ (F_2, A), (G_1, A) \oplus (G_2, A)) \\
\geq \delta((F_1, A), (G_1, A)) \oplus \delta((F_2, A), (G_2, A)).
\]
By the definition of \(\delta((F_1, A), (G_1, A))\) and Lemma 2.2(6), there exist two finite families \(\{(F_{i_j}, A) \mid (F_1, A) = \circ_{j=1}^{p}(F_{i_j}, A)\}\) and \(\{(G_{1_{\sigma(j)}}, A) \mid (G_1, A) = \bigoplus_{j=1}^{p}(G_{1_{\sigma(j)}}, A)\}\) with a bijective function \(\sigma\), we have
\[
\delta((F_1, A) \circ (F_2, A), (G_1, A) \oplus (G_2, A)) \\
\geq \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{\ell \in \Gamma} \delta_i((f_i)_\phi(((F_1)_\sigma, A)), (f_i)_\phi(((G_{1_{\sigma(j)}}, A))) \right) \right\} \oplus \delta((F_2, A), (G_2, A))
\]
Again, by the definition of \(\delta((F_2, A), (G_2, A))\) and Lemma 2.2(6), there exist two finite families \(\{(F_{2_{k_j}}, A) \mid (F_2, A) = \circ_{k=1}^{q}(F_{2_{k_j}}, A)\}\) and \(\{(G_{2_{\epsilon(k)}}, A) \mid (F_2, A) = \bigoplus_{k=1}^{q}(G_{2_{\epsilon(k)}}, A)\}\) with a bijective function \(\epsilon\), we have
\[
\delta((F_1, A) \circ (F_2, A), (G_1, A) \oplus (G_2, A)) \\
\geq \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{\ell \in \Gamma} \delta_i((f_i)_\phi(((F_1)_\sigma, A)), (f_i)_\phi(((G_{1_{\sigma(j)}}, A))) \right) \right\} \oplus \delta((F_2, A), (G_2, A))
\]
By Lemma 2.2(6), for each \(j, \sigma(j)\) and \(k, \epsilon(k)\), there exist \(i_j, i_k \in \Gamma\) such that
\[
\delta((F_1, A) \circ (F_2, A), (G_1, A) \oplus (G_2, A)) \\
\geq \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{\ell \in \Gamma} \delta_i((f_i)_\phi(((F_1)_\sigma, A)), (f_i)_\phi(((G_{1_{\sigma(j)}}, A))) \right) \right\} \oplus \delta((F_2, A), (G_2, A))
\]
On the other hand, since
\[
(F_1, A) \circ (F_2, A) = \left( \bigcirc_{j=1}^{p}(F_{i_j}, A) \right) \circ \left( \bigcirc_{k=1}^{q}(F_{2_{k_j}}, A) \right),
\]
\[
(G_1, A) \oplus (G_2, A) = \left( \bigoplus_{j=1}^{p}(G_{1_{\sigma(j)}}, A) \right) \oplus \left( \bigoplus_{k=1}^{q}(G_{2_{\epsilon(k)}}, A) \right),
\]
for a bijective function \(\sigma \cup \epsilon\), we have
\[
\delta((F_1, A) \circ (F_2, A), (G_1, A) \oplus (G_2, A)) \\
\leq \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{\ell \in \Gamma} \delta_i((f_i)_\phi(((F_1)_\sigma, A)), (f_i)_\phi(((G_{1_{\sigma(j)}}, A))) \right) \right\} \oplus \delta((F_2, A), (G_2, A))
\]
It is a contradiction. Hence the condition (4) holds.

Second, from the definition of \(\delta\), for two families \(\{(F, A) \mid (F, A) = (F, A)\}\) and \(\{(G, A) \mid (G, A) = (G, A)\}\), since
\[
\delta((F, A), (G, A)) \leq \bigwedge_{\ell \in \Gamma} \delta_i((f_i)_\phi(((F, A)), (f_i)_\phi(((G, A)))) \\
\leq \delta((f_i)_\phi(((F, A)), (f_i)_\phi(((G, A))))
\]
We will show that all two finite families \( (f_i) \) \( : (X, A, \delta) \rightarrow (X_i, A_i, \delta_i) \) is a fuzzy proximity soft map.

If all \( (f_i) \) \( : (X, A, \delta_0) \rightarrow (X_i, A_i, \delta_i) \) are fuzzy proximity soft maps, then, for all two finite families \( \{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^{p}(F_j, A)\} \) and \( \{(G_k, A) \mid (G, A) = \bigoplus_{k=1}^{q}(G_k, A)\} \) and \( \sigma \in K \),

\[
\delta((F, A), (G, A)) = \bigwedge_{(H, A) \in S(X, A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.
\]

Thus, \( \delta_0((F, A), (G, A)) \leq \delta((F, A), (G, A)) \) for each \( (F, A), (G, A) \in S(X, A) \).

(2) Let \( \{(X_i, A_i, \delta_i) \mid i \in \Gamma\} \) be a family of soft \( L \)-fuzzy quasi-proximity spaces.

We will show that \( \delta \) is an soft \( L \)-fuzzy quasi-proximity on \( X \).

Suppose there exist \( (F, A), (G, A) \in S(X, A) \) such that

\[
\delta((F, A), (G, A)) \not\geq \bigwedge_{(H, A) \in S(X, A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.
\]

By the definition of \( \delta \), there are finite families \( \{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^{p}(F_j, A)\} \) and \( \{(G_k, A) \mid (G, A) = \bigoplus_{k=1}^{q}(G_k, A)\} \) and a bijective function \( \sigma \) such that

\[
\bigoplus_{j=1}^{p}\left\{\bigwedge_{i \in \Gamma} \delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\right\}
\not\geq \bigwedge_{(H, A) \in S(X, A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.
\]

It follows that for any \( j, \sigma(j) \), there exists an \( i_j \in \Gamma \) such that

\[
\bigoplus_{j=1}^{p}\left\{\delta_i((f_i)_{\phi_i}((F_j, A)), (f_i)_{\phi_i}((G_{\sigma(j)}, A)))\right\}
\not\geq \bigwedge_{(H, A) \in S(X, A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.
\]

Since \( \delta_{i_j} \) is a soft \( L \)-fuzzy quasi-proximity on \( X_{i_j} \), by (SQ), there exists \( (H_{i_j}, A_{i_j}) \in S(X_{i_j}, A_{i_j}) \) such that

\[
(\bigoplus_{j=1}^{p}\left\{\delta_i((f_i)_{\phi_i}((F_j, A)), (H_{i_j}, A_{i_j}))(H_{i_j}, A_{i_j}))\right\}
\not\geq \bigwedge_{(H, A) \in S(X, A)} \{\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A))\}.
\]

On the other hand, put \( (H, A) = \bigoplus_{j=1}^{p}(f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j}, A_{i_j})) \). Since

\[
(f_{i_j})_{\phi_{i_j}}((f_{i_j})^{-1}_{\phi_{i_j}}((H_{i_j}, A_{i_j}))) \leq (H_{i_j}, A_{i_j}),
\]

for the identity \( \sigma(j) = j \), then

\[
\delta((F, A), (H, A)) \leq \bigoplus_{k=1}^{q}\delta_i((f_i)_{\phi_i}((F_j, A)), (f_{i_j})_{\phi_{i_j}}((H_{i_j}, A_{i_j})))
\leq \bigoplus_{k=1}^{q}\delta_i((f_i)_{\phi_i}((F_j, A)), (H_{i_j}, A_{i_j}))).
\]
Let \[ \delta \] be defined as in (2.8), and let \( (H^*, A), (G, A) \). Hence, from Remark 2.8(3) and Theorem 3.1, we obtain the following corollary.

Each two finite families \( (\delta \phi_i, A), (\delta \phi_j, A) \) is a fuzzy proximity soft map. Let \( (F, A), (H, A) \). From Remark 2.8(2), we have

\[
\delta((H^*, A), (G, A)) \leq \bigoplus_{j=1}^{p} \delta_{ij} ((f_{ij})_{\phi_{ij}}^1 ((H_{ij}, A_{ij}))^*, (f_{ij})_{\phi_{ij}} ((G_{\sigma(j)}, A)))
\]

It implies

\[
\delta((F, A), (H, A)) \oplus \delta((H^*, A), (G, A)) \leq \bigoplus_{j=1}^{p} \delta_{ij} ((f_{ij})_{\phi_{ij}} ((F_{ij}, A)), (H_{ij}, A_{ij})) \oplus \bigoplus_{j=1}^{p} \delta_{ij} ((H_{ij}, A_{ij})^*, (f_{ij})_{\phi_{ij}} ((G_{\sigma(j)}, A))).
\]

It is a contradiction for (B). Thus, the result follows.

(3) Necessity of the composition condition is clear since the composition of fuzzy proximity soft maps is a fuzzy proximity soft map.

Each two finite families \( \{(F_j, A) | f_{((F, A))} = \bigoplus_{k=1}^{p} (F_k, A) \} \) and \( \{(G_k, A) | f_{((G, A))} = \bigoplus_{k=1}^{p} (G_k, A) \} \) and each \( \sigma \in K \), we have

\[
\delta(f_{((F, A))}, f_{((G, A))}) = \bigwedge \left\{ \bigoplus_{j=1}^{p} \left( \bigwedge_{j=1}^{p} \delta_{ij} ((f_{ij})_{\phi_{ij}}((F_{ij}, A)), (f_{ij})_{\phi_{ij}}((G_{\sigma(j)}, A))) \right) \right\}.
\]

It follows

\[
(F, A) \leq f_{((F, A))}^{-1}(f_{((F, A))}) = \bigoplus_{j=1}^{p} f_{((F_j, A))}^{-1}(f_{((F_j, A))}) \quad \text{and} \quad (G, A) \leq \bigoplus_{k=1}^{p} f_{((G_k, A))}^{-1}(f_{((G_k, A))}).
\]

Since \( (f_{ij})_{\phi_{ij}} \circ f_{((F, A))} \) is a fuzzy proximity soft map and for any \( j, \sigma(j) \),

\[
(f_{ij})_{\phi_{ij}}(f_{((F, A))}^{-1}((F_{ij}, A))) \leq (f_{ij})_{\phi_{ij}}((F_{ij}, A)),
\]

\[
\delta_0(f_{((F, A))}^{-1}((F_{ij}, A)), f_{((G_{\sigma(j)}, A))}^{-1}((G_{\sigma(j)}, A))) \leq \delta_0((f_{ij})_{\phi_{ij}}((F_{ij}, A)), (f_{ij})_{\phi_{ij}}((G_{\sigma(j)}, A))).
\]

Since \( (F, A) \leq \bigoplus_{j=1}^{p} f_{((F_j, A))}^{-1}((F_j, A)) \), we have, for all \( j, \sigma(j) \) and \( i \in \Gamma \),

\[
\delta_0((F, A), (G, A)) \leq \bigwedge_{\sigma \in K} \left\{ \bigoplus_{j=1}^{p} \delta_0(f_{((F, A))}^{-1}((F_{ij}, A)), f_{((G_{\sigma(j)}, A))}^{-1}((G_{\sigma(j)}, A))) \right\}
\]

(by Remark 2.8(2))

\[
\leq \bigwedge_{\sigma \in K} \left\{ \bigoplus_{j=1}^{p} \bigwedge_{i \in \Gamma} \delta_0((f_{ij})_{\phi_{ij}}((F_{ij}, A)), (f_{ij})_{\phi_{ij}}((G_{\sigma(j)}, A))) \right\}
\]

Hence \( \delta_0((F, A), (G, A)) \leq \delta(f_{((F, A))}, f_{((G, A))}) \).

From Remark 2.8(3) and Theorem 3.1, we obtain the following corollary.

**Corollary 3.2.** Let \((L, \bigvee, \leq)\) be an idempotent complete residuated lattice. Let \( \{(X_i, A_i, \delta_i) | i \in \Gamma \} \) be a family of soft \( L\)-fuzzy pre-proximity spaces. Let \( X \) be a
set and, for each $i \in \Gamma$, $f_i : X \to X_i$ a mapping. Define the function $\delta : S(X, A) \times S(X, A) \to L$ on $X$ by

$$\delta((F, A), (G, A)) = \bigwedge \left\{ \bigwedge_{i \in \Gamma} \delta_i \left( \bigwedge_{j=1}^p \delta_i((f_i)_\phi((F_j, A)), (f_i)_\phi((G_{\sigma(j)}, A)))) \right) \right\},$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^p (F_j, A)\}$ and $\{(G_{\sigma(j)}, A) \mid (G, A) = \bigoplus_{j=1}^p (G_{\sigma(j)}, A)\}$. Then $\delta$ is the coarsest soft $L$-fuzzy pre-proximity on $X$ which for each $i \in \Gamma$, $(f_i)_\phi$ is a fuzzy proximity soft map.

Let $SPROX$ be a category with object $(X, A, \delta_X)$ where $\delta_X$ is a soft $L$-fuzzy pre-proximity with a morphism $f_\phi : (X, A, \delta_X) \to (Y, B, \delta_Y)$ is a fuzzy proximity soft map. Let $SET$ be a category with object $(X, f)$ where $X$ is a set with a morphism $f : X \to Y$ is a function.

**Theorem 3.3.** The forgetful functor $U : SPROX \to SET$ defined by $U(X, A, \delta) = X$ and $U(f) = f$ is topological.

**Proof.** From Theorem 3.1, every $U$-structured source $(f_i : X \to U(X_i, A_i, \delta_i))_{i \in \Gamma}$ has a unique $U$-initial lift $(f_i : (X, A, \delta) \to (X_i, \delta_i))_{i \in \Gamma}$ where $\delta$ in Theorem 3.1.

**Corollary 3.4.** Let $(Y, B, \delta_Y)$ be a soft $L$-fuzzy pre-proximity space. Let $X$ be a set, $f : X \to Y$ and $\phi : A \to B$ mappings. Define the function $\delta : S(X, A) \times S(X, A) \to L$ on $X$ by

$$\delta((F, A), (G, A)) = \bigwedge \left\{ \bigwedge_{\sigma \in \mathcal{K}} \left\{ \bigoplus_{j=1}^p \left( \delta_Y(f_\phi((F_j, A)), f_\phi((G_{\sigma(j)}, A))) \right) \right\} \right\},$$

where the first $\bigwedge$ is taken over all two finite families $\{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^p (F_j, A)\}$, $\{(G_{\sigma(j)}, A) \mid (G, A) = \bigoplus_{j=1}^p (G_{\sigma(j)}, A)\}$ and $A = \{\sigma \mid \sigma : \{1, \ldots, p\} \to \{1, \ldots, p\} \text{ is a bijective function}\}$. Then $\delta$ is the coarsest soft $L$-fuzzy pre-proximity on $X$ which $f_\phi$ is a fuzzy proximity soft map such that $\delta((F, A), (G, A)) = \delta_Y(f_\phi((F, A)), f_\phi((G, A)))$.

**Proof.** From Theorem 3.1 and the definition of $\delta((F, A), (G, A))$, we only show:

$$\delta((F, A), (G, A)) \leq \delta_Y(f_\phi((F, A)), f_\phi((G, A))).$$

Suppose $\delta((F, A), (G, A)) \not\leq \delta_Y(f_\phi((F, A)), f_\phi((G, A)))$. Then there exist two finite families $\{(F_j, A) \mid (F, A) = \bigoplus_{j=1}^p (F_j, A)\}$, $\{(G_{\sigma(j)}, A) \mid (G, A) = \bigoplus_{j=1}^p (G_{\sigma(j)}, A)\}$.
and \( \sigma \in K \) such that
\[
\oplus_{j=1}^{p} \left( \delta_{Y}(f_{\phi}((F_{j}, A)), f_{\phi}((G_{\sigma(j)}, A))) \right) \geq \delta_{Y}(f_{\phi}((F, A)), f_{\phi}((G, A))).
\]
On the other hand, since \( \bigcirc_{j=1}^{p} f_{\phi}((F_{j}, A)) \geq f_{\phi}(\bigcirc_{j=1}^{p} (F_{j}, A)) \) and \( \oplus_{j=1}^{p} f_{\phi}((G_{\sigma(j)}, A)) \geq f_{\phi}(\bigoplus_{j=1}^{p} (G_{\sigma(j)}, A)) \) from Lemma 2.6(9,10), we have
\[
\oplus_{j=1}^{p} \left( \delta_{Y}(f_{\phi}((F_{j}, A)), f_{\phi}((G_{\sigma(j)}, A))) \right) \geq \delta_{Y}(\bigcirc_{j=1}^{p} f_{\phi}((F_{j}, A)), \bigoplus_{j=1}^{p} f_{\phi}((G_{\sigma(j)}, A))) \geq \delta_{Y}(f_{\phi}((F, A)), f_{\phi}((G, A))).
\]
It is a contradiction. Hence the result holds.

**Definition 3.5.** Let \((X, A, \delta_{X})\) be a soft \(L\)-fuzzy pre-proximity space, \(Z \subseteq X\) and \(C \subseteq A\). The pair \((Z, C, \delta)\) is said to be a subspace of \((X, A, \delta_{X})\) if it is endowed with the initial soft \(L\)-fuzzy pre-proximity with respect to \((Z, i, \delta_{X})\) where \(i\) is the inclusion function. From Corollary 3.8, we define the function \(\delta : L^{2} \times L^{Z} \to L\) on \(A\) by
\[
\delta((F, A), (G, A)) = \delta_{X}(i_{1}((F, A)), i_{2}((G, A))).
\]

**Definition 3.6.** Let \(X = \prod_{i \in \Gamma} X_{i}\) be the product of the sets from family \(\{(X_{i}, A_{i}, \delta_{i})| i \in \Gamma\}\) of soft \(L\)-fuzzy pre-proximity spaces. The initial soft \(L\)-fuzzy pre-proximity \(\delta = \bigotimes \delta_{i}\) on \(X\) with respect to the family \(\{\pi_{i} : X \to (X_{i}, A_{i}, \delta_{i})| i \in \Gamma\}\) of all projection maps is called the product soft \(L\)-fuzzy pre-proximity of \(\{\delta_{i} | i \in \Gamma\}\), and \((X, \prod_{i \in \Gamma} A_{i}, \bigotimes \delta_{i})\) is called the product soft \(L\)-fuzzy pre-proximity space.

**Example 3.7.** Let \(U = \{h_{i} | i = \{1, ..., 6\}\}\) with \(h_{i}=\text{house}\) and \(E = \{e, b, w, c, i\}\) with \(e=\text{expensive}, b=\text{beautiful}, w=\text{wooden}, c=\text{creative}, i=\text{in the green surroundings}\). Define a binary operation \(\odot\) on \([0, 1]\) by
\[
x \odot y = \max\{0, x + y - 1\}, \quad x \to y = \min\{1 - x + y, 1\}
\]
\[
x \oplus y = \min\{1, x + y\}, \quad x^{*} = 1 - x
\]
Then \([0, 1], \wedge, \to, 0, 1\) is a complete residuated lattice (ref. [5, 6]). Let \(A = \{b, c, i\} \subseteq E\) and \(X = \{h^{1}, h^{4}, h^{5}, h^{6}\}\). Put \((H, A)\) be a fuzzy soft set as follow:

| \((H, A)\) | \(h^{1}\) | \(h^{4}\) | \(h^{5}\) | \(h^{6}\) |
|------------|--------|--------|--------|--------|
| \(b\)     | 0.5    | 0.6    | 0.2    | 0.6    |
| \(c\)     | 0.1    | 0.5    | 0.5    | 0.6    |
| \(i\)     | 0.4    | 0.6    | 0.6    | 0.5    |


\[
(H, A) \odot (H, A) \quad h^{1} \quad h^{4} \quad h^{5} \quad h^{6} \quad h^{1} \quad h^{4} \quad h^{5} \quad h^{6}
\]

- \(b\): 0.0 0.2 0.0 0.2
- \(c\): 0.0 0.0 0.0 0.2
- \(i\): 0.0 0.2 0.2 0.0
\[(H^*, A) \times_h^i h^1 h^4 h^5 h^6 (H^*, A) \oplus_h^i (H^*, A) h^1 h^4 h^5 h^6\]

\[
b = \begin{pmatrix} 0.5 & 0.4 & 0.8 & 0.4 \\ 0.9 & 0.5 & 0.5 & 0.4 \end{pmatrix} \quad c = \begin{pmatrix} 1.0 & 0.8 & 1.0 & 0.8 \end{pmatrix} \quad i = \begin{pmatrix} 0.6 & 0.4 & 0.4 & 0.5 \end{pmatrix} \]

\[(K, A) \times_h^i h^1 h^4 h^5 h^6 (H, A) \oplus (K, A) h^1 h^4 h^5 h^6\]

\[
b = \begin{pmatrix} 0.6 & 0.5 & 0.4 & 0.6 \end{pmatrix} \quad c = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.6 \end{pmatrix} \quad i = \begin{pmatrix} 0.5 & 0.3 & 0.3 & 0.7 \end{pmatrix} \]

(1) We define soft L-fuzzy preproximities \(\delta_1, \delta_2 : S(X, A) \times S(X, A) \rightarrow L\) as

\[
\delta_1((F, A), (G, A)) = \begin{cases} 0, & \text{if } (F, A) = (0_X, A) \text{ or } (G, A) = (0_X, A) \\ 0.4, & \text{if } (F, A) \leq (H, A) \leq (G, A)^*, \\
& (F, A) \not\leq (H, A) \circ (H, A) \\ 0.7, & \text{if } (0_X, A) \neq (F, A) \leq (H, A) \circ (H, A) \\
& \leq (G, A)^*, (H, A) \not\leq (G, A)^*, \\ 1, & \text{otherwise}, \end{cases}
\]

\[
\delta_2((F, A), (G, A)) = \begin{cases} 0, & \text{if } (F, A) = (0_X, A) \text{ or } (G, A) = (0_X, A) \\ 0.5, & \text{if } (F, A) \leq (K, A) \leq (G, A)^*, \\ 1, & \text{otherwise}, \end{cases}
\]

But \(\delta_i^i\) for \(i = 1, 2\), is not a soft L-fuzzy quasi-proximity because

\[
1 = \bigwedge_{(F, A) \in S(X, A)} \delta_1((H, A) \circ (H, A), (F, A)) \oplus \delta_1((F^*, A), (H, A)^* \circ (H, A)^*)) \\
\not\leq \delta_1((H, A) \circ (H, A), (H, A)^* \circ (H, A)^*)) = 0.7.
\]

\[
1 = \bigwedge_{(F, A) \in S(X, A)} \delta_1((H, A) \circ (H, A), (F, A)) \oplus \delta_1((F^*, A), (H, A)^* \circ (H, A)^*)) \\
\not\leq \delta_1((H, A) \circ (H, A), (H, A)^* \circ (H, A)^*)) = 0.7.
\]

(2) By Theorem 3.1, let \(f_1 = f_2 : X \rightarrow S\) and \(\phi_1 = \phi_2 : A \rightarrow A\) be identity maps. We obtain the coarsest soft L-fuzzy preproximity \(\delta : S(X, A) \times S(X, A) \rightarrow L\) which is finer than \(\delta_i, i = 1, 2\), as follows

\[
\delta((F, A), (G, A)) = \begin{cases} 0, & \text{if } (F, A) = (0_X, A) \text{ or } (G, A) = (0_X, A) \\ 0.4, & \text{if } (F, A) \leq (H, A) \leq (G, A)^*, \\
& (F, A) \not\leq (H, A) \circ (H, A) \\ 0.5, & \text{if } (F, A) \leq (K, A) \leq (G, A)^*, \\
& (F, A) \not\leq (H, A) \circ (K, A) \\ 0.7, & \text{if } (0_X, A) \neq (F, A) \leq (H, A) \circ (H, A) \\
& \leq (G, A)^*, (H, A) \not\leq (G, A)^*, \\ 0.9, & \text{if } (0_X, A) \neq (F, A) \leq (H, A) \circ (K, A) \\
& \leq (G, A)^*, (H, A) \not\leq (K, A)^*, \\ 1, & \text{otherwise}. \end{cases}
\]
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*Department of Applied Mathematics, Pai Chai University, Dae Jeon, 35345, Korea
Email address: yskim@pcu.ac.kr*

*Department of Mathematics, Gangneung-Wonju National Gangneung 25457, Korea
Email address: yck@gwnu.ac.kr*