Volume and Boundary Face Area of a Regular Tetrahedron in a Constant Curvature Space

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An example of the volume and boundary face area of a curved polyhedron for the case of regular spherical and hyperbolic tetrahedron is discussed. An exact formula is explicitly derived as a function of the scalar curvature and the edge length. This work can be used in loop quantum gravity and Regge calculus in the context of a non-vanishing cosmological constant.

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I. INTRODUCTION

In geometry, the calculation of volume and boundary face area of a curved polyhedron (geodesic polyhedron\(^1\)) is one of the most difficult problems. In the case of spherical and hyperbolic tetrahedra, a lot of efforts has been made by mathematicians for calculating the volume and boundary face area: the volume formula are discussed by N. Lobachevsky and L. Schlafli in refs [1] for an orthoscheme tetrahedron, by G. Martin in ref [2] for a regular hyperbolic tetrahedron and by several authors in refs [3–9] for an arbitrary hyperbolic and spherical tetrahedron. All these results are based on the Schlafli differential equation where a unit sectional curvature was taken and they are given by a combination of dilogarithmic or Lobachevsky functions in terms of the dihedral angles. In the present paper, the volume and boundary face area of a regular spherical and hyperbolic tetrahedron are explicitly recalculated in terms of the curvature radius \( r = \sqrt{\frac{R}{|R|}} \) and the edge length \( a \). We directly perform the integration over the area and volume elements to end up with simple formula for the boundary face area and volume of a regular tetrahedron in a space of a constant scalar curvature \( R \). This can be done by using the projection map to the Cayley-Klein-Hilbert coordinates system (CKHcs) which maps a regular geodesic tetrahedron \( T(a) \) of an edge length \( a \) in the manifold of a constant curvature \( R \) to a regular Euclidean tetrahedron \( T(a_0) \) of an edge length \( a_0 \) in the CKHcs. Then, one can express the area and volume measure elements in terms of their Euclidean ones. A comparison between the regular Euclidean, spherical and hyperbolic tetrahedron is studied and their implications are discussed. In physics, a direct application of the volume and boundary face area of a regular tetrahedron is essentially in loop quantum gravity (LQG) and Regge calculus. In LQG, the Euclidean tetrahedron interpretation of a 4-valent intertwiner state was shown in ref [10]. The main important feature of the formula which we are looking for is to find another possible correspondence between the 4-valent intertwiner state with a constant curvature regular tetrahedra shapes; this can be achieved by inverting the resulted functions. Thus, one can obtain the scalar curvature measure for a regular tetrahedron shape which allows us to know what kind of space in which the 4-valent intertwiner state can be represented by a regular tetrahedron [11]. It is worth mentioning that the idea supporting this new correspondence in the context of LQG with a non-vanishing cosmological constant was initiated in refs [11–14]. In the context of Regge calculus, the use of a constant curvature triangulation of spacetime was suggested in ref [15–17] and it can be useful for constructing a quantum gravity version with a non-vanishing cosmological

\(^1\)Geodesic polyhedron is the convex region enclosed by the intersection of geodesic surfaces. A geodesic surface is a surface with vanishing extrinsic curvature and the intersection of two such surfaces is necessarily a geodesic curve.
constant. The paper is organized as follows: In section II, the volume and boundary face area of a geodesic polyhedron in general curved space are discussed. In section III, we give general integration formula of the volume and area for constant curvature spaces. In section IV, an exact formula for regular spherical and hyperbolic tetrahedra is explicitly derived as a function of the curvature radius and the edge length. Finally, in section V we draw our conclusions.

II. VOLUME AND BOUNDARY FACE AREA OF A POLYHEDRON IN A GENERAL CURVED SPACE

For any n-dimensional Riemannian manifold \(M\) equipped with an arbitrary metric \(g\) and a coordinates chart \(\{U \subset M, \tilde{x}\}\), one has to find another coordinates chart system \(\{U \subset M, x\}\), such that the straight lines in the second are geodesics of the manifold \(M\). In other words, it maps the geodesic curves of the manifold in the first coordinates system to the straight line in the second one. Such a coordinates system denoted by CKHcs (Cayley-Klein-Hilbert coordinates system)\(^2\) is very useful to calculate the volume and boundary face area of a geodesic polyhedron (i.e. every geodesic polygons and polyhedrons in the manifold maps to Euclidean polygons and polyhedrons in the CKHcs respectively).

Finding such coordinates system is not an easy task for general metric spaces because it depends on the geometry itself and one has to solve a differential equation to find the CKHcs. If we denote by \(\varphi\) the coordinates transformation between the first and the CKHcs:

\[
x^A = \varphi^A(\tilde{x}) = \tilde{1}, \quad A = 1, n,
\]

one can define the CKHcs by coordinates transformation that satisfying the following differential equation (See Appendix A):

\[
\tilde{\nabla}_V \tilde{\nabla}_V \varphi^A(\tilde{x}) = 0,
\]

where

\[
\tilde{\nabla}_V V = 0,
\]

Eq. (2) holds for any vector field \(V\) tangent to geodesic curves and \(\tilde{\nabla}_V\) stands for the covariant directional derivative along the vector field \(V\) in the coordinates system \(\{U, \tilde{x}\}\).

By knowing the metric in the first coordinates system, one can determine the corresponding Christoffel symbols \(\tilde{\Gamma}'\)'s and then solve the differential equation (2) to get the ideal frame

\(^2\)It is usually known as the Klein projection.
for calculating the volume of a geodesic polyhedron $Pol$ and its boundary face area $\partial Pol_f$ in an arbitrary $n$-dimensional Riemannian space:

$$\int_{Pol \subset U \subset M} dV_{Riem} = \int_{x(Pol) \subset x(U) \subset \mathbb{R}^n} \sqrt{|\det(g(x))|} \ dV_{Euc}, \quad (4)$$

$$\int_{\partial Pol_f \subset U \subset M} dA^Riem = \int_{x(\partial Pol_f) \subset x(U) \subset \mathbb{R}^n} \sqrt{|\det(g(x)|_{\partial Pol_f})} \ dA^Euc_f, \quad (5)$$

where $dA^Euc_f$ and $dV^Euc_{f}$ are the Euclidean face area and volume measures of a geodesic polyhedron respectively, $g(x)$ is the metric in the CKHcs, $g(x)|_{\partial Pol_f}$ is the induced metric in the geodesic surface $\partial Pol_f$.

![Figure 1. The Cayley-Klein-Hilbert coordinates system (CKHcs).](image)

**III. VOLUME AND BOUNDARY FACE AREA OF A POLYHEDRON IN A 3D-CONSTANT CURVATURE SPACE**

Let $\Sigma$ be a $3$-sphere or $3$-hyperbolic metric space. The metric of the $S^3$ and $H^3$ can be combined in a unified expression and induced from the Euclidean $Euc^4$ and the Minkowski $Mink^4$ spaces respectively by using a compact form $\epsilon$ such that:

$$\epsilon = \begin{cases} 1 & \text{for } S^3 \subset Euc^4 \\ i & \text{for } H^3 \subset Mink^4 \end{cases} \quad (6)$$

Let us consider the cartesian coordinates chart for the two spaces $Euc^4$ and $Mink^4$

$$X : M \longrightarrow \mathbb{R}^3 \times \epsilon \mathbb{R} \\
\quad m \quad \mapsto \quad X^A(m) = (x^1, x^2, x^3, \epsilon x^4), \quad (7)$$

where

$$\epsilon \mathbb{R} = \begin{cases} \mathbb{R} & \text{for } Euc^4 \\ i\mathbb{R} = Im(\mathbb{C}) & \text{for } Mink^4 \end{cases} \quad (8)$$
Basically, the metric of the \( Euc^4 \) and \( Mink^4 \) in this coordinates system is written as:

\[
 ds^2 = \delta_{AB}dX^A dX^B = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \epsilon^2(dx^4)^2, \tag{9}
\]

In the spherical coordinates \( \{\tilde{x}\} = \{\rho, \psi, \theta, \varphi\} \) one has:

\[
\begin{aligned}
\rho &= \sqrt{\delta_{AB} X^A X^B} \\
\psi &= \arctan \left( \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} / x^4 \right) \\
\theta &= \arctan \left( \sqrt{(x^1)^2 + (x^2)^2} / x^3 \right) \\
\varphi &= \arctan \left( x^3 / x^4 \right)
\end{aligned}
\]

\[
\begin{aligned}
X^1 &= \frac{\rho}{\epsilon} \cos(\varphi) \sin(\theta) \sin(\epsilon \psi) \\
X^2 &= \frac{\rho}{\epsilon} \sin(\varphi) \sin(\theta) \sin(\epsilon \psi) \\
X^3 &= \frac{\rho}{\epsilon} \cos(\theta) \sin(\epsilon \psi) \\
X^4 &= \rho \cos(\epsilon \psi)
\end{aligned}, \tag{10}
\]

\[
 ds^2 = \epsilon^2 d\rho^2 + \rho^2 \left[ d\psi^2 + \epsilon^2 \sin^2(\epsilon \psi) \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right], \tag{11}
\]

Now, we define the 3d- metric spaces \( S^3_\epsilon \) and \( H^3_\epsilon \) as hyper-surfaces embedded in \( Euc^4 \) and \( Mink^4 \) respectively as:

\[
 X^2 = \delta_{AB} X^A X^B = (\epsilon r)^2, \tag{12}
\]

where \( r \) is a positive real number known as the radius of curvature. Geodesics can be obtained by the intersection of \( S^3_\epsilon \) (or \( H^3_\epsilon \)) surface with two distinct 3d- hypersurfaces through the centre of the \( S^3_\epsilon \) (or \( H^3_\epsilon \)):

\[
\begin{aligned}
\delta_{AB}X^A X^B = (\epsilon r)^2 \\
a_A X^A = 0 \\
b_A X^A = 0
\end{aligned}, \tag{13}
\]

Where \( a_A \) and \( b_A \) are two non-collinear vectors of \( \mathbb{R}^3 \times \epsilon \mathbb{R} \). After dividing Eq. (13) by \( \cos(\epsilon \psi) \), the geodesics satisfy:

\[
\begin{aligned}
a_1 \cos(\varphi) \sin(\theta) \tan(\epsilon \psi) + a_2 \sin(\varphi) \sin(\theta) \tan(\epsilon \psi) + a_3 \cos(\theta) \tan(\epsilon \psi) + a_4 &= 0 \\
b_1 \cos(\varphi) \sin(\theta) \tan(\epsilon \psi) + b_2 \sin(\varphi) \sin(\theta) \tan(\epsilon \psi) + b_3 \cos(\theta) \tan(\epsilon \psi) + b_4 &= 0
\end{aligned}, \tag{14}
\]

where \( \psi \neq \frac{\pi}{2} \) is used in the case of the 3-sphere \( S^3_\epsilon \). Therefore, we can get from the geodesic equations (14), the coordinates transformation to the CKHcs \( \{\tilde{x}\} = \{x, y, z\} \) that satisfying the differential equation condition (2) for both spherical and hyperbolic cases:

1. For the spherical case \( S^3_\epsilon \) (\( \epsilon = 1 \Rightarrow R = \frac{\rho}{\epsilon} \)), the coordinates transformation to the CKHcs and its inverse read:

\[
\begin{aligned}
\varphi_{S^3_\epsilon}^{-1} : \tilde{x}(U_{S^3_\epsilon} \subset S^3_\epsilon) &\to x(U_{S^3_\epsilon} \subset S^3_\epsilon) \\
(\psi, \theta, \varphi) &\mapsto (x, y, z) \\
\varphi_{S^3_\epsilon} : x(U_{S^3_\epsilon} \subset S^3_\epsilon) &\to \tilde{x}(U_{S^3_\epsilon} \subset S^3_\epsilon) \\
(x, y, z) &\mapsto (\psi, \theta, \varphi)
\end{aligned}, \tag{15}
\]
and are defined by
\[
\begin{align*}
\mathbf{x} &= r \cos(\varphi) \sin(\theta) \tan(\psi) \\
y &= r \sin(\varphi) \sin(\theta) \tan(\psi) \\
z &= r \cos(\theta) \tan(\psi)
\end{align*}
\]
\[
\begin{align*}
\psi &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{r}\right) \\
\theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
\varphi &= \arctan\left(\frac{y}{x}\right)
\end{align*}
\]
\begin{equation}
(16)
\end{equation}

Notice that \( U^{S^3_i} \subset S^3_i \) is the top half 3-sphere divided by the hyper-surface of the equation \( \psi = \frac{\pi}{2} \):
\[
\tilde{x}(U^{S^3_i}) = \{(\psi, \theta, \varphi) \mid \psi \in [0, \frac{\pi}{2}], \theta \in [0, \pi], \varphi \in [0, 2\pi]\},
\end{equation}
\begin{equation}
(17)
\end{equation}

2. For the hyperbolic case \( S^3_r \) (\( \epsilon = i \Rightarrow R = \frac{\epsilon}{r^2} \)), the coordinates transformation to the CKHcs and its inverse read:
\[
\begin{align*}
\varphi_{H^3_r} : \tilde{x}(U^{H^3_r} \subset H^3_r) &\rightarrow [-r, r]^3 \\
(\psi, \theta, \varphi) &\mapsto (x, y, z)
\end{align*}
\]
\[
\begin{align*}
\varphi_{H^3_r}^{-1} : [-r, r]^3 &\rightarrow \tilde{x}(U^{H^3_r} \subset H^3_r) \\
(x, y, z) &\mapsto (\psi, \theta, \varphi)
\end{align*}
\]
\begin{equation}
(18)
\end{equation}

and are defined by
\[
\begin{align*}
\mathbf{x} &= r \cos(\varphi) \sin(\theta) \tanh(\psi) \\
y &= r \sin(\varphi) \sin(\theta) \tanh(\psi) \\
z &= r \cos(\theta) \tanh(\psi)
\end{align*}
\]
\[
\begin{align*}
\psi &= \text{arctanh}\left(\frac{\sqrt{x^2 + y^2 + z^2}}{r}\right) \\
\theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
\varphi &= \arctan\left(\frac{y}{x}\right)
\end{align*}
\]
\begin{equation}
(19)
\end{equation}

Notice that, in order to get an isomorphism between the two coordinates systems, we have to take the cubic interval \([-r, r]^3\) since \( \tanh(\psi) \) is bounded by the interval \([-1, 1]\). Moreover, we have also considered the region \( U^{H^3_r} \subset H^3_r \) as the top sheet of the 3d-spherical hyperboloid \( H^3_r \).

By using the compact form \((6)\), one can unify the transformation between the two coordinates charts for both spherical and hyperbolic cases:
\[
\begin{align*}
\mathbf{x} &= \epsilon \cos(\varphi) \sin(\theta) \tan(\psi) \\
y &= \epsilon \sin(\varphi) \sin(\theta) \tan(\psi) \\
z &= \epsilon \cos(\theta) \tan(\psi)
\end{align*}
\]
\[
\begin{align*}
\psi &= \epsilon \text{arctan}\left(\frac{\sqrt{x^2 + y^2 + z^2}}{r}\right) \\
\theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
\varphi &= \arctan\left(\frac{y}{x}\right)
\end{align*}
\]
\begin{equation}
(20)
\end{equation}

The metric in the 3-sphere \( S^3_r \) and 3-hyperbolic \( H^3_r \) spaces is:
\[
ds^2 = r^2 \left[ d\psi^2 + \epsilon^2 \sin^2(\epsilon \psi) \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right],
\end{equation}
\begin{equation}
(21)
\end{equation}

Using the differential form chain rule, one can write:
\[
d\psi = \frac{\epsilon^2}{(\epsilon^2 r^2 + |\mathbf{z}|^2)|\mathbf{x}|} dx + \frac{\epsilon^2}{(\epsilon^2 r^2 + |\mathbf{z}|^2)|\mathbf{x}|} dy + \frac{\epsilon^2}{(\epsilon^2 r^2 + |\mathbf{z}|^2)|\mathbf{x}|} dz,
\end{equation}
\begin{equation}
(22)
\end{equation}

\(^{3}\)Knowing that the biggest possible spherical tetrahedron is the half of 3-sphere \( S^3_i \).
\[ d\theta = \frac{x}{|\vec{r}|^2 \sqrt{x^2 + y^2}} \, dx + \frac{y}{|\vec{r}|^2 \sqrt{x^2 + y^2}} \, dy - \frac{\sqrt{x^2 + y^2}}{|\vec{r}|^2} \, dz, \quad (23) \]

\[ d\varphi = -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy, \quad (24) \]

Thus, the metric in the CKHcs becomes:

\[ ds^2 = g_{AB} dx^A dx^B = -\left( \sum_{A=1}^3 x^A dx^B \right)^2 + \sum_{A=1}^3 \frac{dx^A}{\epsilon^2 r^2 + |\vec{r}|^2}, \quad (25) \]

The components of the metric elements read:

\[ g_{AB} = \begin{bmatrix}
\epsilon^2 r^2 (\epsilon^2 r^2 + y^2 + z^2) & -\epsilon^2 r^2 xy & -\epsilon^2 r^2 xz \\
-\epsilon^2 r^2 xy & (\epsilon^2 r^2 + y^2 + z^2) & \epsilon^2 r^2 (\epsilon^2 r^2 + y^2 + z^2) \\
-\epsilon^2 r^2 xz & \epsilon^2 r^2 (\epsilon^2 r^2 + y^2 + z^2) & (\epsilon^2 r^2 + y^2 + z^2)
\end{bmatrix}, \quad (26) \]

and the Jacobian \( J(\vec{r}) \)

\[ J(\vec{r}) = \sqrt{|\det (g(\vec{r}))|} = \frac{r^4}{(\epsilon^2 r^2 + |\vec{r}|^2)^2}, \quad (27) \]

Finally, we can determine the volume of a geodesic polyhedron \( Pol \) and its boundary face area \( \partial Pol_f \):

1. For a spherical polyhedron \( (R = \frac{6}{r^2}) \)

\[ \int_{\partial Pol_f \subset S^3 \subset \mathbb{R}^3} dA^S_f = \int_{x(\partial Pol_f) \subset \mathbb{R}^3} dA^Euc \sqrt{|\det (g(x)|^{S^3}_{\partial Pol_f})|}, \quad (28) \]

\[ \int_{Pol \subset S^3 \subset \mathbb{R}^3} dV^S = \int_{x(Pol) \subset \mathbb{R}^3} dV^Euc \frac{r^4}{(r^2 + |\vec{r}|^2)^2}, \quad (29) \]

2. For a hyperbolic polyhedron \( (R = \frac{6}{r^2}) \)

\[ \int_{\partial Pol_f \subset H^3 \subset \mathbb{R}^3} dA^H_f = \int_{x(\partial Pol_f) \subset \mathbb{R}^3} dA^Euc \sqrt{|\det (g(x)|^{H^3}_{\partial Pol_f})|}, \quad (30) \]

\[ \int_{Pol \subset H^3 \subset \mathbb{R}^3} dV^H = \int_{x(Pol) \subset \mathbb{R}^3} dV^Euc \frac{r^4}{(-r^2 + |\vec{r}|^2)^2}, \quad (31) \]

The induced Jacobian \( \sqrt{|\det (g(x)|^{S^3}_{\partial Pol_f})|} \) and \( \sqrt{|\det (g(x)|^{H^3}_{\partial Pol_f})|} \) for both spherical and hyperbolic respectively can be determined after restricting the metric in the boundary surface area \( \partial Pol_f \).
IV. APPLICATION: REGULAR TETRAHEDRON IN A CONSTANT CURVATURE SPACE

Let $T(a)$ be a regular geodesic tetrahedron with an edge length $a$ embedded in a constant curvature 3d-space $\Sigma$, and $\left\{ \vec{A}_f \right\}_{f=T(a)}$ be normal area vectors of $T(a)$. In what follows, we will calculate the volume of a geodesic regular tetrahedron $T(a)$ and its boundary face area $\partial T(a)_f$ in 3d-sphere $S^3_r$ and Hyperbolic $H^3_r$ manifolds:

$$A^\Sigma_f(r, a) = \int_{x(\partial T(a)_f) \subset \mathbb{R}^3} dA^\text{Euc} \sqrt{\det(g(x)|\partial T(a)_f)} ,$$  \hspace{1cm} (32)

$$V^\Sigma(r, a) = \int_{x(T(a)) \subset \mathbb{R}^3} dV^\text{Euc} \frac{r^4}{(c^2 r^2 + |\vec{z}|^2)^2} ,$$  \hspace{1cm} (33)

FIG. 2. A regular tetrahedron $T(a_0)$ in $\mathbb{R}^3$ (CKHcs).

The ignorance of how this new coordinates system CKHcs can map an Euclidean length to spherical and hyperbolic length measures, one has to be careful in choosing the location of the tetrahedron $T(a)$. From our choice in Fig. 2, it obvious to see that the image of a regular geodesic tetrahedron $T(a)$ of an edge length $a$ in the manifold is an Euclidean regular tetrahedron $T(a_0)$ of a different edge length $a_0$ in the CKHcs:

$$x(T(a)) = T(a_0) ,$$  \hspace{1cm} (34)

Our objective is to have an expression for the starting Euclidean length $a_0$ in terms of the geodesic length $a$. In order to determine how this coordinates system measure the length different from the original one, we have to consider two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ in the CKHcs where the corresponding geodesic line between them is parameterized by:

$$\begin{align*}
y &= \alpha x + \beta \\
z &= \gamma x + \delta ,
\end{align*}$$  \hspace{1cm} (35)
where
\[
\alpha = \frac{y_2 - y_1}{x_2 - x_1}, \quad \beta = \frac{x_2y_1 - x_1y_2}{x_2 - x_1},
\]
\[
\gamma = \frac{z_2 - z_1}{x_2 - x_1}, \quad \delta = \frac{x_2z_1 - x_1z_2}{x_2 - x_1}.
\]

The geodesic length between \(M_1\) and \(M_2\) is:
\[
d(M_1M_2) = \epsilon r \arctan \left( \frac{(\alpha^2 + \gamma^2 + 1) x + \alpha \beta + \gamma \delta}{\sqrt{\epsilon^2 r^2 + \beta^2 + \delta^2 + (\alpha^2 + \gamma^2) \epsilon^2 r^2 + \alpha^2 \delta^2 + \gamma^2 \beta^2 - 2\alpha \beta \gamma \delta}} \right) \bigg|_{x_1}^{x_2},
\]

Since \(d(M_1M_2)\) depends strongly on the ending points, a special care has to be done in the location of the Euclidean regular tetrahedron in the CKHcs as it is shown in Fig. 2.

One can check that:
\[
a = 2\epsilon r \arctan \left( \frac{1}{2} \frac{a_0}{\sqrt{\epsilon^2 r^2 + \frac{a_0^2}{8}}} \right), \quad \text{(39)}
\]

In order to obtain a geodesic edge length \(a\), one has to solve Eq. (39) for the unknown \(a_0\) and get:
\[
a_0 = \frac{2 \epsilon r \tan \left( \frac{a}{2\epsilon r} \right)}{\sqrt{1 - \frac{1}{2} \tan^2 \left( \frac{a}{2\epsilon r} \right)}}, \quad \text{(40)}
\]

1. For the spherical case \(S^3_\epsilon\) (\(\epsilon = 1 \Rightarrow R = \frac{6}{\epsilon^2}\)) , one has:
\[
a = 2r \arctan \left( \frac{1}{2} \frac{a_0}{\sqrt{r^2 + \frac{a_0^2}{8}}} \right), \quad \text{(41)}
\]

In this case, one can check that the regular tetrahedron has a maximal edge \(a_{max}\) (for \(a_0 \to \infty\)) given by:
\[
a_{max} = 2 \arctan \left( \sqrt{2} \right) r, \quad \text{(42)}
\]

2. For the hyperbolic case \(S^3_\epsilon\) (\(\epsilon = i \Rightarrow R = \frac{-\hat{6}}{\epsilon^2}\)) , one has:
\[
a = 2r \arctanh \left( \frac{1}{2} \frac{a_0}{\sqrt{r^2 - \frac{a_0^2}{8}}} \right), \quad \text{(43)}
\]

Due to the compactness property (see Eq. (18)) of the coordinates chart, the initial value of the Euclidean length \(a_0\) must be bounded \(a_0 < \frac{2}{3} \sqrt{6} r\). However, \(a\) has no upper bound.
IV.1. Boundary area of a regular tetrahedron in $S^3_r$ and $H^3_r$

The faces area of a geodesic regular tetrahedron of an edge length $a$ are all equal \( A^E_f (r,a) = A^E (r,a) \), \( \forall f = 1,4 \). In fact, the geodesic surface of the $S^3_r$ and $H^3_r$ are portions of the great 2-dimensional spheres $S^2$ and hyperbolic $H^2_r$ respectively. Accordingly, we expect to obtain the same area expression of the spherical and hyperbolic trigonometry. Due to the symmetric property of the constant curvature spaces, we restrict ourselves to perform the integration over one of the faces $P = \partial \Sigma$ in general very hard to evaluate. To do so, one has to make a series expansion of the Jacobian $J$.

The boundary face area is:

\[
A^\Sigma (r,a) = \int_{P_1 P_2 P_3 \subset \mathbb{R}^3} dA^E_{\text{vec}} \frac{e^{2r^2} \sqrt{e^{2r^2} + \frac{a_0^2}{24}}}{(e^{2r^2} + x^2 + y^2 + \frac{a_0^2}{24})^{3/2}}, \tag{45}
\]

with

\[
dA^E_{f} = \frac{1}{2} \sum_{i,j,k=1}^3 \epsilon_{ijk} A^i_f dx^j \wedge dx^k, \tag{46}
\]

where $A^i_f$ is the $i^{th}$ component of the normal area vector $\vec{A}_f$. The integral in Eq. (45) is in general very hard to evaluate. To do so, one has to make a series expansion of the Jacobian $J(\vec{x})$ given in (27) with respect to the coordinates variables $\{\vec{x}\}$ and then easily perform the integration over one of the faces $P_1 P_2 P_3$, we get the following expression:

\[
A^\Sigma (r,a) = \frac{\sqrt{3}}{4} a^2 \{1 + \frac{1}{8} \left( \frac{a}{cr} \right)^2 + \frac{1}{60} \left( \frac{a}{cr} \right)^4 + \frac{583}{241920} \left( \frac{a}{cr} \right)^6 + \frac{227}{604800} \left( \frac{a}{cr} \right)^8 + \frac{23}{36960} \left( \frac{a}{cr} \right)^{10} + \frac{1418693}{130767436800} \left( \frac{a}{cr} \right)^{12} + \mathcal{O} \left( \left( \frac{a}{cr} \right)^{14} \right) \}, \tag{47}
\]

Using the symmetry of the triangle faces of a regular tetrahedron, the exact formula of the boundary face area reads:

\[
A^\Sigma (r,a (a_0)) = 2 \int_0^{a_0} dx \int_{-\sqrt{3}x + \sqrt{3a_0}}^{-\sqrt{3}x - \sqrt{3a_0}} dy \frac{e^{2r^2} \sqrt{e^{2r^2} + \frac{a_0^2}{24}}}{(e^{2r^2} + x^2 + y^2 + \frac{a_0^2}{24})^{3/2}}, \tag{48}
\]

Straightforward but tedious calculations (See Appendix B) give the following analytical expression of the boundary face area $A^\Sigma (r,a)$ of a regular spherical and hyperbolic tetrahedron with an edge length $a$ in the curved space $\Sigma$ of a constant curvature $R = \frac{6}{cr^2}$:

\[
A^\Sigma (r,a) = e^{2r^2} \left( 3 \arccos \left( \frac{\cos \left( \frac{a}{cr} \right)}{\cos \left( \frac{a}{cr} \right) + 1} \right) - \pi \right), \tag{49}
\]

It is easy to check that the expansion of the resulted formula (49) in terms of the $\frac{a}{cr}$ variable is exactly the one in Eq. (47) and thus ensuring the correctness of the integration.
1. For the spherical case $S^3_r (\epsilon = 1 \Rightarrow R = \frac{6}{\sqrt{2r}})$, one has:

$$A^{S_3^r} (r, a) = r^2 \left( 3 \arccos \left( \frac{\cos \left( \frac{a}{r} \right)}{\cos \left( \frac{a}{r} \right) + 1} \right) - \pi \right), \quad (50)$$

As it is expected, it is the familiar expression of the regular spherical triangle embedded in the 2-sphere $S^2_r$ where the dihedral angle is defined by $\Theta = \arccos \left( \frac{\cos \left( \frac{a}{r} \right)}{\cos \left( \frac{a}{r} \right) + 1} \right)$ which is the cosine rule formula for spherical trigonometry. We can check that the boundary area $A^{S_3^r}$ for the maximal edge length $a_{\text{max}}$ in Eq. (42) corresponds to an upper bound $A^{S_3^r}_{\text{max}} = \pi r^2$. The boundary area of a regular spherical tetrahedron is always greater than the boundary area of a regular Euclidean one.

2. For the hyperbolic case $S^3_i (\epsilon = i \Rightarrow R = \frac{-6}{\sqrt{2r}})$, one has:

$$A^{H_3^i} (r, a) = r^2 \left( \pi - 3 \arccos \left( \frac{\cosh \left( \frac{a}{r} \right)}{\cosh \left( \frac{a}{r} \right) + 1} \right) \right), \quad (51)$$

As it is expected, it is the familiar expression of the regular hyperbolic triangle embedded in the 2-hyperbolic $H^2_r$ where the dihedral angle is defined by $\Theta = \arccos \left( \frac{\cosh \left( \frac{a}{r} \right)}{\cosh \left( \frac{a}{r} \right) + 1} \right)$ which is the cosine rule formula for hyperbolic trigonometry. Notice that in this case, there is no upper bound and for a given pair $(r, a)$. The boundary area of a regular hyperbolic tetrahedron is always smaller than the boundary area of a regular Euclidean one.

3. For the Euclidean case $Euc^3 (R = 0)$, one has:

$$A^{Euc^3} (r, a) = \lim_{r \to \infty} A^\Sigma (r, a) = \frac{\sqrt{3}}{4} a^2, \quad (52)$$

The Euclidean limit is well-defined.

FIG. 3.: Function surface of the boundary face area for spherical (green), Euclidean (blue) and hyperbolic (red) regular tetrahedra.
IV.2. Volume of a regular tetrahedron in $S^3_r$ and $H^3_r$

The volume $V^\Sigma$ of a regular spherical and hyperbolic tetrahedron is:

$$V^\Sigma(r, a(a_0)) = \int_{T(a_0) \subset \mathbb{R}^3} dV^Euc \frac{r^4}{(\epsilon^2 r^2 + |\vec{x}|^2)^\frac{3}{2}}, \quad (53)$$

Since the integration is very hard to deal with, it is better to make again a series expansion of the Jacobian $J(\vec{x})$ given in (27) in terms of the coordinates variables $\{\vec{x}\}$ and then easily perform the integration to end up with:

$$V^\Sigma(r, a(a_0)) = \sqrt{2} \frac{\sqrt{3}}{12} a^3 \left( 1 + \frac{23}{80} \frac{a}{\epsilon r} + \frac{3727}{53760} \frac{a^4}{(\epsilon r)^6} + \frac{124637}{7741440} \frac{a^6}{(\epsilon r)^8} + \frac{20283401}{5449973760} \frac{a^8}{(\epsilon r)^{10}} + \frac{14700653069}{17003918131200} \frac{a^{10}}{(\epsilon r)^{12}} + O(a^{14}) \right), \quad (54)$$

Using the symmetry of the regular tetrahedron, the exact expression of the volume of a regular spherical and hyperbolic tetrahedron is:

$$V^\Sigma(r, a(a_0)) = 2 \int_{-\sqrt{6}a_0}^{\sqrt{6}a_0} dz \int_0^{\frac{\pi}{2} a(z)} dx \int_{-\sqrt{3}a(z)}^{-\sqrt{3}a(z)} dy \frac{r^4}{(\epsilon^2 r^2 + |\vec{x}|^2)^\frac{3}{2}} \quad (55),$$

where

$$a(z) = \frac{-\sqrt{6}}{2} z + \frac{3a_0}{4}. \quad (56)$$

Which can be rewritten in the following integral form (See Appendix C) as:

$$V^\Sigma(r, a) = 12 \epsilon^3 r^3 \int_0^{\tan(\frac{\pi}{2} z)} dt \frac{t \arctan(t)}{(3 - t^2) \sqrt{2 - t^2}} \quad (57).$$

Notice that this integral has no analytic formula (we can carry the integration by using numerical methods) and can be expressed in terms of some special functions like the dilogarithm $Li_2(z)$, the Clausen of order 2 $Cl_2(\varphi)$ or the digamma $\Psi(x)$. It is easy to check that the expansion of the resulted formula (57) in terms of the $\frac{a}{\epsilon r}$ variable is exactly the one in Eq. (54) and thus ensuring the correctness of the integration.

1. For the spherical case $S^3_r$ ($\epsilon = 1 \Rightarrow R = \frac{6}{r}$), one has:

$$V^{S^3_r}(r, a) = 12 \epsilon^3 r^3 \int_0^{\tan(\frac{\pi}{2} z)} dt \frac{t \arctan(t)}{(3 - t^2) \sqrt{2 - t^2}} \quad (58).$$

The volume for a maximal edge length $V^{S^3_r}(r, a_{max})$ (as it is expected) is half of the 3-dimensional cubic hyperarea of 3-sphere of radius $r$:

$$V^{S^3_r}(r, a_{max}) = \pi^2 r^3 = \frac{1}{2} \text{Area} (S^3_r \subset \mathbb{R}^4), \quad (59)$$

Notice that for a given pair $(r, a)$ the volume of a regular spherical tetrahedron is always greater than the regular Euclidean one.
2. For the hyperbolic case $S^3_\epsilon (\epsilon = i \Rightarrow R = \frac{\pi}{r^2})$, one has:

$$V^{H^2} (r, a) = 12 \, r^3 \int_0^{\tanh(\frac{\pi}{2})} dt \frac{t \ \text{arctanh}(t)}{(3 + t^2) \sqrt{2 + t^2}},$$

(60)

has an upper bound:

$$\lim_{a \to \infty} V^{H^3} (r, a) = 1.0149416064096536250 \, r^3,$$

(61)

$$= \text{Im} \left[ \text{Li}_2 \left( e^{i\pi} \right) \right] r^3 = \frac{\sqrt{6}}{3} \left( \frac{1}{3} - \frac{2}{3} \pi^2 \right) r^3 = C_{12} \left( \frac{\pi}{3} \right) r^3,$$

(62)

Notice that for a given pair $(r, a)$ the volume of a regular hyperbolic tetrahedron is always smaller than the regular Euclidean one.

3. For the Euclidean case $\text{Euc}^3 (R = 0)$, one has:

$$V^{\text{Euc}^3} (r, a) = \lim_{r \to \infty} V^{\Sigma} (r, a) = \frac{\sqrt{2}}{12} a^3,$$

(63)

The Euclidean limit is well-defined.

FIG. 4. Function surface of regular tetrahedron volume for spherical (green), Euclidean (blue) and hyperbolic (red) cases.

IV.3. The volume-area ratio function

We define the volume-area ratio function $VRA^\Sigma$ for a regular geodesic tetrahedron as:

$$VRA^\Sigma (r, a) = \frac{V^{\Sigma} (r, a)}{(A^{\Sigma} (r, a))^2},$$

(64)

It is obvious that the $VRA^\Sigma$ for a regular Euclidean tetrahedron is a constant:

$$VRA^{\text{Euc}^3} = \lim_{r \to \infty} VRA(r, a) = \frac{\sqrt{2}}{12 \left( \frac{\sqrt{3}}{4} \right)^2} = 0.4136,$$

(65)
Corollary IV.0.1 according to the useful inequality
\[ V_{\text{RA}}^{H^3}(r, a) \leq V_{\text{RA}}^{Euc^3}(r, a) \leq V_{\text{RA}}^{S^3}(r, a), \]

the \( V_{\text{RA}}^{\Sigma} \) function allows us to know what kind of geometry inside the regular geodesic tetrahedron: (see Fig. 5)

\[
\begin{align*}
V_{\text{RA}}^{\Sigma}(r, a) > 0.4136 & \quad S^3_r \\
V_{\text{RA}}^{\Sigma}(r, a) = 0.4136 & \quad Euc^3 \\
V_{\text{RA}}^{\Sigma}(r, a) < 0.4136 & \quad H^3_r
\end{align*}
\]

FIG. 5. The volume-area ratio function for spherical (green), Euclidean (blue) and hyperbolic (red) cases.

IV.4. The volume function in terms of scalar curvature and area

From the area formula (49), one can express the edge length \( a \) by:
\[
a(A, R) = \left( \pi - \arccos \left( \frac{\pi}{6} + \frac{A}{3r^2} \right) \frac{1}{\sin\left(\frac{\pi}{6} + \frac{A}{3r^2} \right)} \right) cr,
\]

substitute it in Eq. (57) to get a volume function in terms of the 3d- Ricci scalar curvature and boundary face area of a regular tetrahedron:

\[
V^{\Sigma} = V^{\Sigma}(R, a(R, A)) = V^{\Sigma}(R, A),
\]

Corollary IV.0.2 the volume of a regular geodesic tetrahedron for a fixed boundary area satisfies the following inequality

\[
\text{For any } R_1, R_2 \in \mathbb{R} \text{ if } R_1 < R_2 \text{ then } V^{\Sigma}(R_1, A) < V^{\Sigma}(R_2, A),
\]

this results from the fact that the function \( V^{\Sigma} \) increases with respect to \( R \) for a fixed area norm \( A \) (see Fig. 6).
V. CONCLUSION

In this paper, we explicitly derived the boundary face area and volume of a regular spherical and hyperbolic tetrahedron in terms of the curvature radius (or the scalar curvature) and the edge length. We have directly performed the integration over the area and volume elements by using the Cayley-Klein-Hilbert coordinates system (CKHcs) to end up with simple formula given in Eqs. (49,57). A comparison between the Euclidean, spherical and hyperbolic cases is studied and their implications are discussed. It is shown that the volume function of a regular geodesic tetrahedron for a fixed boundary face area is a strictly increasing in the scalar curvature interval.

Appendix A: Proof of the relation (2)

The geodesics in the CKHcs \{U \subset M, \vec{x}\} are straight lines, one has:

\[ \vec{x}^A = 0, \]  

(A1)

The condition

\[ \Gamma^A_{BC}(x)\vec{x}^B \vec{x}^C = 0, \]  

(A2)

must be hold, which implies:

\[ \Gamma^A_{BC}(x) \frac{\partial \varphi^B(\vec{x})}{\partial \vec{x}^J} \frac{\partial \varphi^C(\vec{x})}{\partial \vec{x}^J} \vec{x}^I \vec{x}^I = 0, \]  

(A3)

Under the transformation (1), the Christoffel symbols transform as:

\[ \Gamma^A_{BC}(x) = \frac{\partial \varphi^J}{\partial x^B} \frac{\partial \varphi^K}{\partial x^C} \frac{\partial \varphi^A}{\partial x^J} \Gamma^J_{JK}(\vec{x}) - \frac{\partial \varphi^J}{\partial x^B} \frac{\partial \varphi^K}{\partial x^C} \frac{\partial^2 \varphi^A}{\partial \vec{x}^J \partial \vec{x}^K}, \]  

(A4)

By substituting it in Eq. (A3), one can obtain the transformation condition Eq. (2) to the ideal CKHcs frame.
Appendix B: Proof of the area formula

The boundary face area \((P_1 P_2 P_3)\) of a regular spherical and hyperbolic tetrahedron of an edge length \(a\) is given by an integral form in Eq. (48). For simplicity, we drop the triangle face \(P_1 P_2 P_3\) to \(\Pi(P_1 P_2 P_3)\) in the \(XY\)-plane (since the area of a fixed triangle is the same wherever its location inside the constant curvature manifold). In this case, the induced Jacobian can be written as:

\[
\sqrt{|\text{det}(g(x)|_{\Pi(P_1 P_2 P_3)}|) = \frac{e^3 r^3}{(e^2 r^2 + x^2 + y^2)^{3/2}}, \tag{B1}\]

The boundary face area is given by:

\[
A^\Sigma (r, a) = 2 \int_0^{a_0} dx \int_{-\frac{\sqrt{3}a_0}{r}}^{\frac{\sqrt{3}a_0}{r}} dy \frac{e^3 r^3}{(e^2 r^2 + x^2 + y^2)^{3/2}}, \tag{B2}\]

where one can check the starting Euclidean length \(a_0\) in this case is given by:

\[
a_0 = 2er \frac{\tan(\frac{a}{2r})}{\sqrt{1 - \frac{1}{3} \tan^2(\frac{a}{2r})}}, \tag{B3}\]

Performing the Integral over \(y\) variable, one get:

\[
\int_{-\frac{\sqrt{3}a_0}{r}}^{\frac{\sqrt{3}a_0}{r}} dy \frac{e^3 r^3}{(e^2 r^2 + x^2 + y^2)^{3/2}} = \frac{e^3 r^3(-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})}{(e^2 r^2 + x^2)(e^2 r^2 + x^2 + (\sqrt{3}x + \frac{\sqrt{3}a_0}{3})^2)} + \frac{e^3 r^3 \frac{\sqrt{3}a_0}{6}}{(e^2 r^2 + x^2)\sqrt{e^2 r^2 + x^2 + \frac{a_0^2}{12}}}. \tag{B4}\]

Let us preform the second integral over the \(x\) variable. By integrating each term separately, one has:

\[
t_1(x) = \int_0^{a_0} dx \frac{e^3 r^3(-\sqrt{3}x + \frac{\sqrt{3}a_0}{3})}{(e^2 r^2 + x^2)\sqrt{e^2 r^2 + x^2 + (\sqrt{3}x + \frac{\sqrt{3}a_0}{3})^2}} = e^2 r^2 \arctan(\frac{F(a_0, r; x)}{G(a_0, r; x)}), \tag{B5}\]

where

\[
F(a_0, r; x) = -\frac{\sqrt{3}}{3} \sqrt{e^2 r^2 + 4x^2 - 2xa_0 + \frac{a_0^2}{3}}(e^2 r^2 + \frac{a_0^2}{9})(-e^2 r^2 + \frac{a_0 x}{3}) - a_0(\frac{5e^2 r^2}{9} + \frac{a_0^2}{27})(e^2 r^2 + x^2) + \frac{a_0^4 x}{81} + r^4 x + 2a_0^2 e^2 r^2 x - 9, \tag{B6}\]

and

\[
G(a_0, r; x) = \frac{er \sqrt{3}}{3} \sqrt{e^2 r^2 + 4x^2 - 2xa_0 + \frac{a_0^2}{3}}(e^2 r^2 + \frac{a_0^2}{9})(x + \frac{a_0}{3}) + \frac{2a_0^2 e^2 r^2 x^2}{27} + \frac{e^5 r^5}{3} - \frac{4a_0^2 e^2 r^3}{27} - \frac{a_0^4 e r}{81} + \frac{4e^3 r^3 x^2}{3}. \tag{B7}\]
Now, let us focus on the second integral over the $x$ variable. By integrating each term separately, one has:

$$t_2(x) = \int_0^{\frac{\alpha}{a_0}} dx \frac{3\sqrt{\alpha_0}}{\sqrt{(e^2r^2 + x^2)^3}} = \epsilon^2 r^2 \arctan \left( \frac{a_0 x}{r\sqrt{12e^2r^2 + 12x^2 + a_0^2}} \right),$$

(B8)

Adding the terms together, we obtain:

$$A^\Sigma (r, a) = 2(t_1(x) + t_2(x)) |_{x=0}^{x=a_0/2} = 2\epsilon^2 r^2 \arctan \left( \frac{9a_0^3 \sqrt{3a_0} - \sqrt{3a_0} \sqrt{9e^2r^2 + 3a_0^2} + 18\epsilon^2 r^2}{3a_0^3 - 63a_0^3 \epsilon^2 r^2 - 216a_0 r^4 - \sqrt{3} \sqrt{9e^2 r^2 + 3a_0^2 (18a_0^3 \epsilon^2 r^2 + 144 r^4 - a_0^4)} \right) \quad (B9)$$

When we replace $a_0$ given in Eq. (B3), we get the area function formula of Eq. (49).

Appendix C: Proof of the volume formula

The volume of a regular spherical and hyperbolic tetrahedron of an edge length $a$ is given by an integral form in Eq. (55). Using the integration by shell method (taking the sum of parallel triangles of constant $z$). Performing the integral over the $y$ variable, one get:

$$\int_{\frac{-\sqrt{3x + \sqrt{3a(z)}}}{\alpha}}^{\frac{\sqrt{3a(z)}}{\alpha}} dy \frac{r^4}{(e^2r^2 + x^2 + y^2 + \alpha(z)^2)^2} = \frac{32\sqrt{3} r^4 (-3x + \alpha(z))}{32x^2 - 16\alpha(z)x + 8e^2r^2 + 3\alpha(z)^2 \left( 24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2 \right)} + \frac{24\sqrt{6} r^4 \arctan \left( \frac{2\sqrt{3} (3x + \alpha(z))}{\sqrt{24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2}} \right)}{24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2} + \frac{48\sqrt{3} r^4 \alpha(z)}{(24x^2 + 24\epsilon^2 r^2 + 3\alpha(z)^2) \left( 24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2 \right)} + \frac{24\sqrt{6} r^4 \arctan \left( \frac{\sqrt{2} \alpha(z)}{\sqrt{24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2}} \right)}{24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2} \quad (C1)$$

Now, let us focus on the second integral over the $x$ variable. By integrating each term separately, one has:

$$t_1 (x) = \int dx \frac{32\sqrt{3} r^4 (-3x + \alpha(z))}{32x^2 - 16\alpha(z)x + 8e^2r^2 + 3\alpha(z)^2 \left( 24x^2 + 24\epsilon^2 r^2 + \alpha(z)^2 \right)} = -6\sqrt{3} r^4 \ln \left( \frac{32x^2 - 16\alpha(z)x + 8e^2r^2 + 3\alpha(z)^2}{72\epsilon^2 r^2 + 11\alpha(z)^2} \right) + \frac{8\sqrt{3} r^4 \alpha(z) \arctan \left( \frac{8x - 2\alpha(z)}{\sqrt{16\epsilon^2 r^2 + 2\alpha(z)^2}} \right)}{72\epsilon^2 r^2 + 11\alpha(z)^2} \sqrt{164\epsilon^2 r^2 + 6\alpha(z)^2} + \frac{48\sqrt{3} r^4 \alpha(z) \arctan \left( \frac{12x}{\sqrt{144\epsilon^2 r^2 + 6\alpha(z)^2}} \right)}{72\epsilon^2 r^2 + 11\alpha(z)^2} \sqrt{144\epsilon^2 r^2 + 6\alpha(z)^2} \quad (C2)$$
Adding all four terms together, we obtain:

\[ T_2 (x) = \int dx \frac{24\sqrt{6} \, r^4 \arctan \left( \frac{2\sqrt{3}(-3x+\alpha(z))}{\sqrt{24x^2+24x^2r^2+\alpha(z)^2}} \right)}{(24x^2+24r^2r^2+\alpha(z)^2)} = \]

\[ 48\sqrt{6} \, r^4 \sqrt{8r^2 + \alpha(z)^2} \arctan \left( \frac{\sqrt{2}(4x-\alpha(z))}{\sqrt{8r^2 + \alpha(z)^2}} \right) - 6\sqrt{3} \, r^4 \ln \left( 24x^2 + 24r^2r^2 + \alpha(z)^2 \right) \]

\[ + \frac{6\sqrt{3} \, r^4 \ln \left( 96x^2 - 48\alpha(z)x + 24r^2r^2 + 9\alpha(z)^2 \right)}{72r^2r^2 + 11\alpha(z)^2} - \frac{24\sqrt{2} \arctan \left( \frac{\sqrt{2}\sqrt{4x^2}}{\sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2}} \right)}{72r^2r^2 + 11\alpha(z)^2} \sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2} \]

\[ + \frac{24\sqrt{6} \, r^4 \arctan \left( \frac{\sqrt{2}(4x-\alpha(z))}{\sqrt{24x^2+24r^2r^2+\alpha(z)^2}} \right)}{24r^2r^2 + \alpha(z)^2} \sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2}, \quad (C3) \]

\[ T_3 (x) = \int dx \frac{48\sqrt{3} \, r^4 \alpha(z)}{(24x^2 + 24r^2r^2 + 3\alpha(z)^2)(24x^2 + 24r^2r^2 + \alpha(z)^2)} = \]

\[ \frac{6\sqrt{2} \, r^4 \arctan \left( \frac{2\sqrt{2} \, x}{\sqrt{24x^2 + \alpha(z)^2 + \alpha(z)^2}} \right)}{\alpha(z)\sqrt{24r^2r^2 + \alpha(z)^2}} - \frac{2\sqrt{3} \, r^4 \arctan \left( \frac{2\sqrt{3} \, x}{\sqrt{8x^2 + \alpha(z)^2}} \right)}{\alpha(z)\sqrt{8x^2r^2 + \alpha(z)^2}}, \quad (C4) \]

\[ T_4 (x) = \int dx \frac{24\sqrt{6} \, r^4 \arctan \left( \frac{\sqrt{2}\alpha(z)}{\sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2}} \right)}{(24x^2 + 24r^2r^2 + \alpha(z)^2)^{3/2}} = \]

\[ 6\sqrt{6} \, r^4 \sqrt{8r^2 + \alpha(z)^2} \arctan \left( \frac{2\sqrt{2} \, x}{\sqrt{8x^2r^2 + \alpha(z)^2}} \right) + \frac{24\sqrt{6} \, r^4 x \arctan \left( \frac{\sqrt{2}\alpha(z)}{\sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2}} \right)}{24x^2r^2 + \alpha(z)^2} \]

\[ - \frac{6\sqrt{2} \, r^4 \arctan \left( \frac{2\sqrt{2} \, x}{\sqrt{24x^2 + 24r^2r^2 + \alpha(z)^2}} \right)}{\alpha(z)\sqrt{24x^2r^2 + \alpha(z)^2}} \], \quad (C5) \]

Adding all four terms together, we obtain:

\[ 2(T_1 (x) + T_2 (x) + T_3 (x) + T_4 (x)) \bigg|^{x=\alpha(z)/2}_{x=0} = 24\sqrt{6} \, r^4 \alpha(z) \arctan \left( \frac{\sqrt{2}\alpha(z)}{\sqrt{8x^2 + \alpha(z)^2}} \right) \]

\[ = \frac{\sqrt{2}\alpha(z)}{\sqrt{8x^2 + \alpha(z)^2}}, \quad (C6) \]

Making the following change of variable in the third integral over \( z:\)

\[ t = \frac{\sqrt{2}\alpha(z)}{\sqrt{8x^2 + \alpha(z)^2}}, \quad (C7) \]

When we replace \( a_0 \) given in Eq. (40), we get the volume function formula of Eq. (57).

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