A NOTE ON $L^p_\nu(\nu, X, Y)$ SPACES OF VECTOR-VALUED FUNCTIONS WITH RESPECT TO VECTOR MEASURES

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Dedicated to the memory of Professor Guillermo Restrepo Sierra

Abstract. In this work we introduce the $L^p_\nu(\nu, X, Y)$ spaces for the case where $\nu$ is a vector measure and the functions are vector-valued. We establish fundamental properties for such spaces.

1. Introduction

$L^p$ spaces play a crucial role in modern mathematics. Their properties, among other things, have originated fundamental results in several fields such as differential equations, measure theory, and functional analysis. In this paper, we will pursue two aims. Firstly, to define $L^p_\nu(\nu, X, Y)$ spaces for $X$-valued functions, where $X$ and $Y$ are Banach spaces, and $\nu$ is a vector measure valued on $Y$. Secondly, to establish some properties as separability, density of simple functions, and the Hölder inequality for these spaces.

The pioneering works in geometry and the theory of the Banach spaces in infinite dimension have their theoretical origins in vector measures. An example of this is found in the theorem of Orlicz-Pettis, which can be considered as a link between the theory of vector measures and properties of the Banach spaces, because, it relates the convergence of series in Banach spaces with $\sigma$-additive vector measures.

A paramount question in the theory of vector measures is to determine the existence of Radon-Nikodym derivatives of a vector measure with respect to another. This condition has crucial consequences in the study of weak compactness in spaces of functions such as the Lebesgue-Bochner space $L^1(\mu, X)$, where $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Furthermore, it is essential in the Lebesgue-Bochner space duality theorem $L^p(\mu, X)$, where its respective dual is $L^q(\mu, X)$ if and only if $X'$ has the property of Radon-Nikodym for $p^{-1} + q^{-1} = 1$. For further applications of vector measures, such as in game
theory, control theory, stochastic differential equations, and statistics, just to mention a few, see for example [EO16] and [GL17].

The introduction of integrable scalar or vector-valued functions with respect to vector measures leads to define interesting $L^p$ spaces, which might be used in other areas of mathematics. For example, in [OB16] and [DM09], some properties of this sort of spaces for scalar functions and vector measures were extended yielding to important results in the harmonic analysis theory. This type of applications motivates us to explore more about these spaces and their connections with other areas.

For this purpose we will focus on studying $L^p$ spaces for strongly measurable functions but weakly integrable with respect to a vector measure and the injective tensor product. In particular, we will use the definition of a $\otimes$-integrable function given in [Ste11] and the characterization of the $\otimes$-integrable functions of order $p$, for $1 \leq p < \infty$.

Let $(\Omega, \mathcal{A})$ be a measurable space. We will say that a vector-valued function is $\otimes$-integrable of order $p$ if and only if $\|f\|^p$ is $\nu$-integrable, i.e., $\|f\|^p$ is $|y' \circ \nu|$-integrable for all $y' \in Y'$ and for each $E \in \mathcal{A}$ there exists an element of $Y$ denoted by $\int_E \|f\|^p d\nu$, such that

$$y' \left( \int_E \|f\|^p d\nu \right) = \int_E \|f\|^p d(y' \circ \nu), \quad \text{for each } y' \in Y'.$$

We will base our construction of the $L^p_w(\nu, X, Y)$ spaces on $\otimes$-integrable functions of order $p$, for $1 \leq p < \infty$, that only satisfy the first condition. We will determine their properties, and by means of an extension of the norm introduced in [CB08], we will prove that $L^p_w(\nu, X, Y)$ is a Banach space. We will proved that the spaces $L^p_w(\nu, X, Y)$ are separable showing that the set of simple functions is dense in $L^p_w(\nu, X, Y)$, and that when the $\sigma$-algebra is countably generated and $X$ is a separable Banach space, $L^p_w(\nu, X, Y)$ is separable. Additionally we will show that $L^p(\nu, X, Y)$ is a closed subspace of $L^p_w(\nu, X, Y)$ which is, a generalization of the proof by Stenfansson in [Ste93]. We point out the difficulty of dealing with these spaces. Since their duals might not coincide with $L^q_w$ versions. Even in the case of $X = \mathbb{R}$ the pathology remains. At the end of this note we will discuss this issue in more detail.

2. Preliminares

To start with, we will use the concept of $\nu$-integrability that was introduced by Lewis in [Lew70] for the case scalar valued-functions and vector measures.

**Definition 2.1.** Let $(\Omega, \mathcal{A})$ be a measurable space, $f$ be a scalar valued measurable function, $Y$ be a Banach space, $y' \in Y'$ with $Y'$ the dual of $Y$,
and $\nu : \mathcal{A} \to Y$ a $\sigma$-additive vector measure. A $\nu$-measurable function $f$ is $\nu$-integrable if

1. $f$ is $y' \circ \nu$-integrable for each $y' \in Y'$, i.e.,
$$\int_{\Omega} |f|d|y' \circ \nu| < \infty,$$

2. for every $E \in \mathcal{A}$ there exists an element of $Y$ denoted by $\int_{E} f \, d\nu$, such that
$$y'\left(\int_{E} f \, d\nu\right) = \int_{E} f \, d(y' \circ \nu),$$
for each $y' \in Y'$.

Using this definition, on 1975 Klůvánek and Knowles defined in [KG75] the $L^1(\nu)$ space as the space of the $\nu$-integrable functions that satisfy both conditions. Years later, G. Curbera in [GR75] and [G.P94] established additional properties of this space and Sanchez in [P01] extended these studies para $1 < p < \infty$.

Next, consider the following definition:

**Definition 2.2.** Let $1 < p < \infty$, and $\nu$ be a vector measure $\sigma$-additive. We will say that a scalar measurable function is $p$-integrable with respect to $\nu$ if $|f|^p$ is $\nu$-integrable.

Sánchez showed that with the norm
$$\|f\|_{L^p(\nu)} = \sup_{\|y'\| \leq 1} \left\{ \left( \int_{\Omega} |f|^p \, d|y' \circ \nu| \right)^{1/p} \right\},$$
$L^p(\nu)$ is a Banach space and the set of simple functions is dense in $L^p(\nu)$.

Moreover, in [P01], Sánchez illustrated by means of an example an interesting situation. If $p$ and $q$ are such that $p^{-1} + q^{-1} = 1$, then the spaces $L^q(\nu)$ and $(L^p(\nu))'$ are different.

In [Ste93], Stefánsson defined that a scalar valued measurable function in the usual sense is weakly integrable, if for all $y' \in Y'$, $f$ is $y' \circ \nu$-integrable and denotes $L^1_{\text{w}}(\nu)$ as the set of weakly integrable functions. Additionally, it showed that $L^1_{\text{w}}(\nu)$ is a Banach space with the norm
$$\|f\|_{L^1_{\text{w}}(\nu)} = \sup_{\|y'\| \leq 1} \left\{ \int_{\Omega} |f| \, d|y' \circ \nu| \right\},$$
which contains $L^1(\nu)$ as a closed subspace.

In [DM09], for $1 < p < \infty$ the space $L^p_{\text{w}}(\nu)$ is defined.
**Definition 2.3.** Let $1 < p < \infty$. The space $L^p_w(\nu)$ is defined as the space \((\|\nu\|\text{-equivalence classes of})\) of the measurable functions such that $|f|^p \in L^1_w(\nu)$.

It can be shown that with the norm \[
\|f\|_{L^p_w(\nu)} = \sup_{\|y'\| \leq 1} \left\{ \left( \int_{\Omega} |f|^p d|y' \circ \nu| \right)^{1/p} \right\},
\]
$L^p_w(\nu)$ is a Banach space. Additionally, the following relationships between these spaces hold.

$L^p(\nu) \subset L^p_w(\nu) \subset L^1_w(\nu)$.

**Remark 2.4.** Thanks to Rybakov’s theorem, we know that for every vector measure $\nu$ there exists a real measure of control $\mu$ that makes the vector measure $\nu$ to be $\mu$-continuous. Further, in [DM09] it is showed that for $p = \infty$, the space $L^\infty(\nu)$ of the measurable functions that are bounded $\nu$-c.t.p, coincide with the space $L^\infty(\mu)$. That is the reason that do not consider this case.

Following Stenfánsson in [Ste11], to extend the notion of these spaces to vector-valued and $\nu$-measurable functions. We will say that a vector-valued and $\nu$-measurable function $f$ is $\otimes$-integrable, if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ of $\Omega$ in $X$ such that

\[
\lim_{n \to \infty} \sup_{\|y'\| \leq 1} \left\{ \int_{\Omega} \|f - f_n\| d|y' \circ \nu| \right\} = 0.
\]

From this definition, we make the following observations.

(1) It can be shown from this definition that $(\int_{\Omega} f_n \otimes d\nu)_{n \in \mathbb{N}}$ is a Cauchy sequence on $X \hat{\otimes}_\gamma Y$.

(2) This integral is defined for each $E \in \mathcal{A}$ and the $\lim_{n \to \infty} \int_E f_n \otimes d\nu$ exists and is unique since $X \hat{\otimes}_\gamma Y$ is a Banach space.

(3) If $f$ is $\otimes$-integrable then $\int_E f \otimes d\nu$ is the vector such that

\[
\int_E f \otimes d\nu = \lim_{n \to \infty} \int_E f_n \otimes d\nu.
\]

(4) It can be shown that this limit does not depend on of the choice of $(f_n)_{n \in \mathbb{N}}$.

Let $f$ be a $\nu$-measurable function. In [Ste11], Stenfánsson defines the space $L^1(\nu, X, Y)$ as the vector space of all (||\nu||-equivalence classes of) functions $\otimes$-integrables equipped with the norm $\|

\|f\|_{L^1(\nu, X, Y)} = \sup_{\|y'\| \leq 1} \left\{ \int_{\Omega} \|f\| d|y' \circ \nu| \right\},
\]
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and shows that is a Banach space.

Furthermore, in [CB07] additional properties of $L^1(\nu, X, Y)$ are shown as the fact of being a Banach lattice, a separable space, and the density of the set of simple functions in $L^1(\nu, X, Y)$.

In [CB08], the spaces $L^p(\nu, X, Y)$ for $1 < p < \infty$ are defined as follows.

**Definition 2.5.** A $\nu$-measurable function $f : \Omega \to X$ is called $\otimes$-integrable of order $p$, if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ of $\Omega$ in $X$ such that

$$\lim_{{n \to \infty}} \sup_{\|y\| \leq 1} \left\{ \left( \int_\Omega \|f(w) - f_n(w)\|^p |d|\circ \nu \right)^{1/p} \right\} = 0.$$

As a consequence of this definition, it can be stated that if $f$ is $\otimes$-integrable of order $p$, then the expression

$$\|f\|_{L^p(\nu, X, Y)} = \sup_{\|v\| \leq 1} \left\{ \left( \int_\Omega \|f\|^p |d|\circ \nu \right)^{1/p} \right\},$$

is a norm, and hence $L^p(\nu, X, Y)$ is a Banach space.

Additionally, in [CB08], Chakraborty and Basu, considered some properties similar to those given by Stenfánsson in [Ste11]. One of them is that $f$ is $\otimes$-integrable of order $p$ if and only if $\|f\|^p$ is $\nu$-integrable, i.e., $\|f\|^p \in L^1(\nu)$. (See Theorem 1 in [CB08], page 90). Remarkably, this property allowed to show the existence of the dominated convergence theorem of order $p$.

In addition to the above, and as a generalization of Sánchez work in [P01], Chakraborty and Basu showed that $L^p(\nu, X, Y)$ is a Banach lattice, $L^p(\nu, X, Y) \subset L^1(\nu, X, Y)$ for $1 < p < \infty$, and exhibits a dual of the spaces $L^p(\nu, X, Y)$.

Furthermore, in [CB08], they generalized Sánchez’s spaces in the case of real-valued weakly integrable functions, considering weakly measurable functions defining the $w$- $L^p(\nu, X, Y)$ spaces and establishing several properties.

3. Results

In this section we will introduce the $L^p_w(\nu, X, Y)$ spaces for functions $\nu$-measurable and integrable with respect to $|y|\circ \nu$ for all $y \in Y'$. Additionally we will state properties of separability and density of simple functions. We remark that we are based on the Theorem 1 in [CB08] which states that $f$ is $\otimes$-integrable of order $p$ if and only if $\|f\| \in L^p(\nu)$.

**Definition 3.1.** Let $1 \leq p < \infty$. We define $L^p_w(\nu, X, Y)$ as the space of the functions $\nu$-measurable ($\|\nu\|$ equivalence classes) such that $\|f\|^p$ is $|y|\circ \nu$.
integrable, that is,
\[ \int_{\Omega} \| f \|^p |y' \circ \nu| < \infty, \]
for each \( y' \in Y' \).

If the expression
\[ \sup_{\| y' \| \leq 1} \left\{ \left( \int_{\Omega} \| f \|^p |y' \circ \nu| \right)^{1/p} \right\} < \infty, \]
then it can be shown that it is a norm that we will denote by \( \| f \|_{L^p(\nu,X,Y)} \). In addition, under this norm it can be proved that \( L^p(\nu,X,Y) \) is a Banach space, and the proof of this fact is similar to that given in [Ste11] by Stenfansson for the case \( p = 1 \).

**Observation 3.2.** Let be \( 1 < p < \infty \). Note that if \( X = \mathbb{R} \) then \( L^p(\nu,X,Y) = L^p(\nu) \). If \( Y = \mathbb{R} \) then \( L^p(\nu,X,Y) = L^p(\nu, X) \) where \( L^p(\nu, X) \) corresponds to Lebesgue-Bochner spaces.

We will show that \( L^p(\nu,X,Y) \) is a closed subspace of \( L^p(\nu,X,Y) \).

**Theorem 3.3.** \( L^p(\nu,X,Y) \) is a closed subspace of \( L^p(\nu,X,Y) \).

**Proof.** Let be \( f \in L^p(\nu,X,Y) \) and \( (f_m)_{m \in \mathbb{N}} \) be a sequence of \( \nu \)-measurable functions in \( L^p(\nu,X,Y) \) that converges to \( f \) in \( L^p(\nu,X,Y) \). Then \( (\| f_m \|^p)_{m \in \mathbb{N}} \) is a sequence of measurable functions in \( L^1(\nu) \). If we define
\[ \mu_m(E) = \int_E \| f_m \|^p d\nu, \quad \text{and} \quad \mu(E) = \int_E \| f \|^p d\nu, \]
then \( \mu_m \) and \( \mu \) are vector measures \( \sigma \)-additive and therefore
\[ \| \mu_m(E) - \mu(E) \| \leq \| \mu_m - \mu \| (E) \rightarrow 0, \]
as
\[ \| \mu_m - \mu \| (E) = \sup_{\| y' \| \leq 1} \left\{ \int_E \| f_m \|^p - \| f \|^p |y' \circ \nu| \right\} \]
\[ \leq \sup_{\| y' \| \leq 1} \left\{ \int_E \| f_m - f \|^p |y' \circ \nu| \right\} \]
\[ = \| f_m - f \|_{L^p(\nu,X,Y)} \rightarrow 0 \]
when \( m \rightarrow \infty \) and holds for all \( E \in \mathcal{A} \). And hence \( \| f \|^p \in L^1(\nu) \). In conclusion \( f \in L^p(\nu,X,Y) \). \( \square \)
Of the previously defined, we can also show that
\[ L^p(\nu, X, Y) \subset L^p_w(\nu, X, Y) \subset L^1(\nu, X, Y) \]
and
\[ L^p(\nu, X, Y) \subset L^1(\nu, X, Y) \subset L^1_w(\nu, X, Y). \]

We are going to state properties of density and separability for these spaces.

**Theorem 3.4.** Let \( 1 \leq p < \infty \). The set of simple functions is dense in \( L^p_w(\nu, X, Y) \).

**Proof.** Let \( f \in L^p_w(\nu, X, Y) \). As a consequence of the Pettis measurability theorem for vector measures, there exists a sequence of functions \((f_n)_{n \in \mathbb{N}}\) \( \nu \)-measurables that only assume countably many values such that \( \|f_n - f\| \leq 1/n, \nu \)-c.t.p. Hence
\[
\|f_n\|^p \leq 2^p \left( \|f\|^p + \frac{1}{n^p} \right),
\]
and
\[
\sup_{\|y\| \leq 1} \int_{\Omega} \|f_n\|^p |y' \circ \nu| < \infty,
\]
that means that \((f_n)_{n \in \mathbb{N}} \subset L^p_w(\nu, X, Y)\). We can show that
\[
\lim_{\|\nu\|(E) \to 0} \sup_{\|y\| \leq 1} \int_{\Omega} \|f_n \chi_E\|^p |y' \circ \nu| = 0
\]
for each \( E \in \mathcal{A} \). Indeed, writing
\[
f_n = \sum_{m=1}^{\infty} x_{nm} \chi_{E_{nm}},
\]
where \( E_{nk} \cap E_{nj} = \emptyset \) if \( k \neq j \) and \( x_{nm} \in X \), we obtain for each \( y' \in Y' \) the next:
\[
\int_{\Omega} \|f_n \chi_E\|^p |y' \circ \nu| = \int_{\Omega} \sum_{m=1}^{\infty} \|x_{nm}\|^p \chi_{E_{nm} \cap E} |y' \circ \nu|
\]
\[
= \sum_{m=1}^{\infty} \int_{\Omega} \|x_{nm}\|^p \chi_{E_{nm} \cap E} |y' \circ \nu|
\]
\[
= \sum_{m=1}^{\infty} \|x_{nm}\|^p |y' \circ \nu| (E_{nm} \cap E)
\]
\[
\leq \sum_{m=1}^{\infty} \|x_{nm}\|^p |y' \circ \nu| (E)
\]
Taking the supremo on \(\|y'\| \leq 1\), we obtain
\[
\sup_{\|y'\| \leq 1} \int_{\Omega} \|f_n\|^p d|y' \circ \nu| = \sup_{\|y'\| \leq 1} \sum_{m=1}^{\infty} \|x_{nm}\|^p |y' \circ \nu|(E) \\
= \sup_{\|y'\| \leq 1} |y' \circ \nu|(E) \sum_{m=1}^{\infty} \|x_{nm}\|^p \\
= \|\nu\|(E) \sum_{m=1}^{\infty} \|x_{nm}\|^p \rightarrow 0
\]
when \(\|\nu\|(E) \rightarrow 0\).

For each \(n\), we can choose a \(p_n\) large enough so that
\[
\sup_{\|y'\| \leq 1} \int_{\bigcup_{m \geq p_n+1} E_{nm}} \|f_n\|^p d|y' \circ \nu| \leq \frac{\|\nu\|(\Omega)}{n}
\]
If \(\phi_n = \sum_{m=1}^{p_n} x_{nm} \chi_{E_{nm}}\), for all \(y' \in Y'\) we have
\[
\int_{\Omega} \|f - \phi_n\|^p d|y' \circ \nu| \leq 2^p \left\{ \int_{\Omega} \|f - f_n\|^p d|y' \circ \nu| + \int_{\Omega} \|f_n - \phi_n\|^p d|y' \circ \nu| \right\} \\
\leq 2^p \left[ \frac{\|\nu\|(\Omega)}{n^p} + \int_{\bigcup_{m=p_n+1} E_{nm}} \sum_{m=1}^{\infty} x_{nm} \chi_{E_{nm}} \right]^p d|y' \circ \nu|
\]
Taking the supremum over all \(\|y'\| \leq 1\), we get
\[
\sup_{\|y'\| \leq 1} \int_{\Omega} \|f - \phi_n\|^p d|y' \circ \nu| \leq 2^p \left[ \frac{\|\nu\|(\Omega)}{n^p} + \int_{\bigcup_{m=p_n+1} E_{nm}} \|f_n\|^p d|y' \circ \nu| \right] \\
\leq 2^p \left[ \frac{\|\nu\|(\Omega)}{n^p} + \frac{\|\nu\|(\Omega)}{n} \right]
\]
Then
\[
\lim_{n \to \infty} \|f - \phi_n\|_{L^p_w(\nu,X,Y)} = 0,
\]
which completes the proof. \(\square\)

Next, we will show that the \(L^p_w(\nu,X,Y)\) spaces are separable.

**Theorem 3.5.** Let \((\Omega, \mathcal{A})\) be a measurable space and be \(p\) a real such that \(1 \leq p < \infty\). If \(X\) is a separable Banach space and the \(\sigma\)-algebra is countably generated, then the space \(L^p_w(\nu,X,Y)\) is separable.

**Proof.** Let \(D\) be a dense and countable subset of \(X\). Since \(\mathcal{A}\) is countably generated then there exits a countable set \(\mathcal{F} \subset \mathcal{A}\). We consider the collection of all finite sums \(\sum_{j=1}^{m} q_j \chi_{D_j}\) where \(q_j \in D\) and \(D_j \in \mathcal{F}\). This collection is
countable and is contained in the space $L^p_w(\nu, X, Y)$. In what follows, we will show that this collection is a dense subset.

Let $f \in L^p_w(\nu, X, Y)$ and $\epsilon > 0$. Since the set of simple functions is dense in $L^p_w(\nu, X, Y)$. There exists a simple function $g$ such that

$$\|f - g\|_{L^p_w(\nu, X, Y)} \leq \epsilon.$$ 

From now on, we assume that $g$ is written as $\sum_{j=1}^n \alpha_j \chi_{A_j}$ where each $\alpha_j \in X$ and every $A_j \in \mathcal{A}$. Our purpose here is to show that for this function $g$, there exist $q_j \in D$ and $D_j \in \mathcal{F}$ such that

$$\left\| g - \sum_{j=1}^n q_j \chi_{D_j} \right\|_{L^p_w(\nu, X, Y)} \leq \epsilon.$$ 

We begin by noting that since $X$ is separable for each $\alpha_j$ we can choose a $q_j$ such that

$$\|\alpha_j - q_j\| \leq \frac{\epsilon}{(2n\|\nu\|(\Omega))^{1/p}},$$

for $1 \leq j \leq n$. For $\mathcal{A}$ countably generated we have

$$\sup_{\|y\| \leq 1} \int_{\Omega} |\chi_{A_j} - \chi_{D_j}| d|y| \circ \nu \leq \frac{\epsilon^p}{2n\beta^p},$$

where $\beta = \sup_{1 \leq j \leq n} \|\alpha_j\|$. Then

$$\left\| \sum_{j=1}^n \alpha_j \chi_{A_j} - \sum_{j=1}^n q_j \chi_{D_j} \right\|_{L^p_w(\nu, X, Y)}^p = \sup_{\|y\| \leq 1} \int_{\Omega} \left\| \sum_{j=1}^n \alpha_j \chi_{A_j} - \sum_{j=1}^n q_j \chi_{D_j} \right\|^p d|y| \circ \nu$$

where $\beta = \sup_{1 \leq j \leq n} \|\alpha_j\|$. Then

$$\sum_{j=1}^n \alpha_j \chi_{A_j} - \sum_{j=1}^n q_j \chi_{D_j}$$

is contained in the space $L^p_w(\nu, X, Y)$. In what follows, we will show that this collection is a dense subset.
This shows that the space $L^p_w(\nu, X, Y)$ is separable. \qed

We now establish a Hölder inequality for our spaces.

**Proposition 3.6.** (Hölder inequality). Let be $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $(\Omega, \mathcal{A})$ be a measurable space, $X, Y$ Banach spaces and $\nu : \mathcal{A} \to Y$ a $\sigma$-additive vector measure. If $f \in L^p_w(\nu, X, Y)$ and $g \in L^q_w(\nu, X, Y)$ then

\[
\sup_{\|y\|_X \leq 1} \left\{ \int_{\Omega} \|f(w)x\| \|g(w)\| d|y'\circ \nu|(w) \right\} \leq \|f\|_{L^p_w(\nu, X, Y)} \|g\|_{L^q_w(\nu, X, Y)}.
\]

**Proof.** We will use Young’s inequality

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

In this case, let be

\[
a = \frac{\|f(w)\|}{\|f\|_{L^p_w(\nu, X, Y)}}, \quad b = \frac{\|g(w)\|}{\|g\|_{L^q_w(\nu, X, Y)}}.
\]

Then

\[
\sup_{\|y\|_X \leq 1} \left\{ \int_{\Omega} \frac{\|f(w)\| \|g(w)\|}{\|f\|_{L^p_w(\nu, X, Y)} \|g\|_{L^q_w(\nu, X, Y)}} d|y'\circ \nu|(w) \right\} \leq \frac{1}{p} + \frac{1}{q} = 1,
\]

so that

\[
\sup_{\|y\|_X \leq 1} \left\{ \int_{\Omega} \|f(w)\| \|g(w)\| d|y'\circ \nu|(w) \right\} \leq \|f\|_{L^p_w(\nu, X, Y)} \|g\|_{L^q_w(\nu, X, Y)}.
\]

\qed

If $X$ is a Banach algebra, the Hölder inequality is modified as follows:

**Proposition 3.7.** (Hölder Inequality). Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p_w(\nu, X, Y)$ and $g \in L^q_w(\nu, X, Y)$ then

\[
\sup_{\|y\|_X \leq 1} \left\{ \int_{\Omega} \|f(w)g(w)\| \|x\| \|y'\circ \nu|(w) \right\} \leq \|f\|_{L^p_w(\nu, X, Y)} \|g\|_{L^q_w(\nu, X, Y)}.
\]

In the above statement $f(w)g(w)$ denotes the multiplication of $f(w)$ and $g(w)$ in the algebra $X$.

Despite the existence of reasonable Hölder inequalities, the study of duals for these spaces exhibit some pathologies. Thus, we devote the last part
of this work to note some observations regarding the duality of the spaces $L^p_w(\nu, X, Y)$. To do so, we will begin by stating the following definitions taken from [KG75].

**Definition 3.8.** Let $Y$ be a Banach space. We say that $Y$ has the $B$-$P$ property if given a sequence $(y_n)_{n \in \mathbb{N}}$ such that \[ \sum_{n=1}^{\infty} |(y', y_n)| < \infty \] for each $y' \in Y'$ there exists an element $y \in Y$ with $y = \sum_{n=1}^{\infty} y_n$.

If $Y$ is a Banach space, then $Y$ has the B-P property if and only if $Y$ does not contain an isometric copy of $c_0$.

As a consequence of the above, we have the following theorem.

**Theorem 3.9.** Let $Y$ be a Banach space with the B-P property and $\nu : A \to Y$ a $\sigma$-additive vector measure. If $f$ is a real valued measurable function and $y' \circ \nu$-integrable for each $y' \in Y'$ then $f$ is $\nu$-integrable.

**Proof.** See [KG75], Theorem 1, Page 31. □

With this in mind, if $X = \mathbb{R}$ and $Y$ has the B-P property, then $L^p_w(\nu) = L^p(\nu)$ and thanks to the example by Sanchez in [P01] constructed in the case $Y = \ell^2$, one can show us that these spaces have some obstacles to characterize their duals. In addition, if $Y$ has the B-P property, we can conclude that $L^p(\nu, X, Y) = L^p_w(\nu, X, Y)$ since the previous theorem is applied to real valued measurable function $\|f\|^p$.

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**References**

[BD55] R.G. Bartle, N. Dunford and J. Schwartz. Weak compactness and vector measures. Canad. J. Math. 7 (1955), 289-305.

[OB16] O. Blasco. Fourier analysis for vector-measures on compact abelian groups. Racsam. 110:519–539, 2016.

[CS13] J.M. Calabuig, F. Galaz, E.M. Navarrete, E.A. Sánchez. Fourier Transform and Convolutions on $L^p$ of a Vector Measure on a Compact Hausdorff Abelian Group. J. Fourier Anal Appl. 19: 312–332, 2013.

[CB07] N. Chakraborty and S. Basu. On some properties of the space of tensor integrable functions. Analysis Mathematica. 33:1–16, 2007.

[CB08] N. Chakraborty and S. Basu. Spaces of $p$-tensor integrable functions and related Banach space properties. Real Analysis Exchange. 34(1):87–103, 2008.

[DM09] O. Delgado and P. J. Miana. Algebra structure for $L^p$ of a vector measure. Journal of Mathematical Analysis and Applications. (358):355–363, 2009.

[EO16] O. Edhan. Values of vector measure market games and their representations. Internat. J. Game Theory. (45): 411–433, 2016.
[G.P94] G.P. Curbera. *When $l^1$ of a vector measure is an AL-space*. Pacific Journal of Mathematics. 162(2), 1994.

[GR75] G.P. Curbera and W. Rickerl. *Vector measures, Integration and Applications*. In: Positivity (in Trends Math.), Birkhäuser, Basel, pp. 127–160 (2007).

[GL17] J. J. Grobler, C.C.A. Labuschagne. *The Itô integral for martingales in vector lattices*. Journal of Mathematical Analysis and Applications. (450): 1245–1274 (2017).

[KG75] I. Kluvánek and G. Knowles. *Vector measures and control systems*. North-Holland Amsterdam, 1975.

[Lew70] D. Lewis. *Integration with respect to vector measures*. J. Differential Equations. 33:157–165, 1970.

[Pó01] E. S. Pérez. *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces*. Illinois Journal of Mathematics. 45(3):907–923, 2001.

[Ste93] G. F. Stefánsson. *$l^1$ of a vector measure*. Le Matematiche. 48(48):219–234, 1993.

[Ste11] G. F. Stefánsson. *Integration in vector spaces*. Illinois Journal of Mathematics. 45(3):925–938, 2011.

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