Can observed randomness be certified to be fully intrinsic?

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Randomness comes in two qualitatively different forms. Apparent randomness can result both from ignorance or lack of control of degrees of freedom in the system. In contrast, intrinsic randomness should not be ascribable to any such cause. While classical systems only possess the first kind of randomness, quantum systems are believed to exhibit some intrinsic randomness. In general, any observed random process includes both forms of randomness. In this work, we provide quantum processes in which all the observed randomness is fully intrinsic. These results are derived under minimal assumptions: the validity of the no-signalling principle and an arbitrary (but not absolute) lack of freedom of choice. The observed randomness tends to a perfect random bit when increasing the number of parties, thus defining an explicit process attaining full randomness amplification.

Physical theories aim at providing the best possible predictions for the phenomena occurring in nature. Consequently, on observing a probabilistic process, a natural question arises: how much -if any- of that is intrinsically unpredictable?

Consider an experimental setup in which a variable takes different values with different probabilities. This variable has *observed randomness* that can be easily estimated from the measured statistics. In general, we can distinguish two qualitatively different forms of randomness contributing to the observed randomness of a process. The first is the *apparent randomness*, which appears as a consequence of imperfections of the system, such as lack of knowledge and control of all the relevant degrees of freedom. Clearly, an improvement on our control of the setup reduces this form of randomness. The second form of randomness is termed *intrinsic randomness* and refers to the component of observed randomness that cannot be ascribed to imperfections. It is this second form of randomness that should be considered truly random, as any improvement on our control of the setup leaves it unchanged.

The quantitative contribution of each form of randomness to the observed randomness depends on the physical theory used to describe the process. In classical theories, for instance, all observed randomness is apparent, as it is always possible to explain any random classical process as the probabilistic mixture of deterministic classical processes. Moving to the quantum domain, the axioms of quantum theory state that measurements on quantum particles yield intrinsically random outcomes. Yet, the fact that a theory makes predictions only in terms of probabilities does not necessarily imply the existence of intrinsic randomness. It may simply reflect some limitations of the formalism, in the sense that a better, more complete theory could restore determinism. However, the non-local correlations observed when measuring entangled particles allow one to assess the randomness of a process independent of the full quantum formalism. Under only two assumptions, (i) the impossibility of instantaneous communication, - known as the no-signaling principle - and (ii) that the measurement settings in a Bell test can be chosen at random - known as freedom of choice - non-local quantum correlations necessarily imply intrinsic randomness. This is because such correlations cannot be described as the probabilistic mixture of deterministic processes.

Up to now, in all Bell tests, the intrinsic randomness revealed by quantum non-locality (under said assumptions) is also mixed with apparent randomness, resulting from the non-completeness of quantum theory. In this work, we ask the following fundamental question: is there any quantum process that is as intrinsically random as it is observed to be? We answer this question in the affirmative by providing a family of quantum processes whose intrinsic randomness can be computed analytically for arbitrary system sizes and also demonstrating that this is strictly equal to the observed randomness.

Our results are related to recent attempts to prove the completeness of quantum physics. In Colbeck and Renner claimed that no no-signalling theory can have a better predictive power than quantum theory. However, the proof, which is based on the quantum violation of the chained Bell inequality, assumes that the settings in the inequality can be chosen freely. This introduces a circularity in the argument, as the free process needed in the proof is already assumed to be complete. A possible way out to break this circularity is to consider protocols for randomness amplification. There, the intrinsic randomness of a quantum process could be proven using a source of imperfect randomness. In fact, the protocol for full randomness amplification given in provides a
Bell test in which a measured variable has an intrinsic randomness that tends to be equal to the observed randomness in the limit of an infinite number of parties. Our results provide finite-size Bell setups in which observed and intrinsic randomness are strictly equal using arbitrarily small randomness for the choice of measurements. In this sense, they represent the strongest proof of completeness of a quantum process.

Preliminaries. — Suppose that a Bell test is performed repeatedly among $N$ parties and the resulting statistics is given by $P_{\text{obs}}(a|x)$, where $a = (a_1, \ldots, a_N)$ and $x = (x_1, \ldots, x_N)$ are the string of outcomes and measurement inputs of the parties involved. Let $g$ be a function acting on the measurement results $a$. As previously explained, there are different physically relevant notions of randomness.

First, the observed randomness of $g$ for measurements $x$ is the randomness computed directly from the statistics. Operationally, this may be defined as the optimal probability of guessing the outcome of $g$ for input $x$,

$$G_{\text{obs}}(g, x, P_{\text{obs}}) = \max_{k \in \text{Im}(g)} P_{\text{obs}}(g(a) = k|x).$$

(1)

where $\text{Im}(g)$ is the image of function $g$.

Moving to the definition of the intrinsic randomness, one should consider all possible preparations of the observed statistics in terms of no-signalling probability distributions. In our context, a particular preparation reads

$$P_{\text{obs}}(a|x) = \sum_e p(e|x)P_e^{\text{ex}}(a|x)$$

(2)

where the $P_e^{\text{ex}}$ are extremal points of the no-signaling set [9]. The terms $p(e|x)$ may depend on $x$, which accounts for possible correlations between the preparation $e$ and the measurement settings $x$, given that the choice of measurements are not assumed to be free. Hence, we define the intrinsic randomness of a function $g$ by optimizing over all possible non-signalling preparations of $P_{\text{obs}}$ so as to minimize the randomness of $g$. In other words,

$$G_{\text{intr}}(g, x, P_{\text{obs}}) = \max_{\{p(e|x),P_e^{\text{ex}}\}} \sum_e p(e|x)G_{\text{obs}}(g(x, P_e^{\text{ex}})$$

subject to:

$$\sum_e p(e|x)P_e^{\text{ex}}(a|x) = P_{\text{obs}}(a|x)$$

(3)

$$p(x|e) \geq \delta \quad \text{with} \quad \delta > 0; \quad \forall x, e$$

(4)

where $G_{\text{obs}}(g, x, P_e^{\text{ex}}) = \max_k P_e^{\text{ex}}(g(a) = k|x)$ is also the intrinsic randomness of $P_e^{\text{ex}}$, since intrinsic and observed randomness must coincide for extremal points of the non-signalling set. Note that condition $p(x|e) \geq \delta > 0$ allows for an arbitrary (but not absolute) relaxation of the freedom of choice assumption by allowing for arbitrary (yet not complete) correlations between the preparation and the measurement settings. Physically, this condition ensures that all measurement combinations appear for all possible preparations $e$. An example of a source of randomness fulfilling this condition is a Santha-Vazirani source [10]. Note however that our definition allows sources more general than the Santha-Vazirani sources.

From a cryptographic point of view, the observed randomness is the one perceived by the parties performing the Bell test, whereas the intrinsic randomness is that perceived by a non-signalling eavesdropper possessing knowledge of the preparation of the observed correlations and with the ability to arbitrarily (yet not fully) bias the choice of the measurement settings.

In general, $G_{\text{obs}}$ is strictly larger than $G_{\text{intr}}$, as the set of non-signalling correlations is larger than the quantum. The results in [11, 12] provide a Bell test in which $G_{\text{intr}}$ approaches $G_{\text{obs}}$ (and to 1/2 in the limit of an infinite number of measurements and assuming free choices, that is, $p(x|e)$ in (2) is independent of $e$). The results in [8] allow some relaxation of this last condition. The results in [8] arbitrarily relaxed the free-choice condition and give a Bell test in which $G_{\text{intr}}$ tends to $G_{\text{obs}}$ (and both tend to 1/2 in the limit of an infinite number of parties). Here, we provide a significantly stronger proof, as we allow the same level of relaxation on free choices and provide Bell tests in which $G_{\text{intr}} = G_{\text{obs}}$ for any number of parties. Moreover, a perfect random bit is obtained in the limit of an infinite number of parties.

Scenario. — Our scenario consists of $N$ parties where each performs two measurements of two outcomes. In what follows, we adopt a spin-like notation and label the outputs by $+1$. Then, any non-signalling probability distribution can be written as (for simplicity we give the expression for three parties, but it easily generalizes to an arbitrary number)

$$P(a_1, a_2, a_3|x_1, x_2, x_3) =$$

$$\frac{1}{8} \left( 1 + a_1 A_1^{(x_1)} + a_2 A_2^{(x_2)} + a_3 A_3^{(x_3)} + a_1 a_2 A_1^{(x_1)} A_2^{(x_2)} + a_1 a_3 A_1^{(x_1)} A_3^{(x_3)} + a_2 a_3 A_2^{(x_2)} A_3^{(x_3)} + a_1 a_2 a_3 A_1^{(x_1)} A_2^{(x_2)} A_3^{(x_3)} \right),$$

(5)

where $A_i^{(x_i)}$ denotes the outputs of measurement $x_i$ by each party $i$. In this scenario, we consider Mermin Bell inequalities, whose Bell operator reads

$$M_N = \frac{1}{2} M_{N-1}(A_N^{(0)} + A_N^{(1)}) + \frac{1}{2} M_{N-1}(A_N^{(0)} - A_N^{(1)}),$$

(6)

where $M_2$ is the Clauser-Horne-Shimony-Holt operator and $M_{N-1}$ is obtained from $M_{N-1}$ after swapping $A_i^{(0)} \leftrightarrow A_i^{(1)}$. We study probability distributions that give the maximal non-signalling violation of the Mermin inequalities and focus our analysis on a function $f$ that maps
the $N$ measurement results into one bit as follows:

\[ f(a) = \begin{cases} +1 & n_-(a) = (4j + 2); \text{ with } j \in \{0, 1, 2, \ldots\} \\ -1 & \text{otherwise} \end{cases} \]  

(7)

where $n_-(a)$ denotes the number of results in $a$ that are equal to $-1$.

Results.— Our goal in what follows is to quantify the intrinsic randomness of the bit defined by $f(a)$ for those distributions maximally violating the Mermin inequality for odd $N$. We first prove the following

Lemma 1. Let $P_M(a|x)$ be an $N$-partite (odd $N$) non-signalling probability distribution maximally violating the corresponding Mermin inequality. Then, for any input $x$ appearing in the inequality

\[ P_M(f(a) = h_N(x) \geq 1/2, \text{ with } h_N = \sqrt{2} \cos \left( \frac{\pi(N + 4)}{4} \right). \]  

(8)

Note that, as $N$ is odd, $h_N = \pm 1$. Operationally, the Lemma implies that, for all points maximally violating the Mermin inequality, the bit defined by $f$ is biased towards the same value $h_N$. Since the proof of the Lemma for arbitrary odd $N$ is convoluted, we give the explicit proof for $N = 3$ here, which already conveys the main ingredients of the general proof, and relegate the generalization to the Appendix.

Proof for three parties.— With some abuse of notation, the tripartite Mermin inequality may be expressed as,

\[ M_3 = \langle x_1x_2x_3 \rangle = (A^{(1)}_{x_1}A^{(2)}_{x_2}A^{(3)}_{x_3}) \]  

where $\langle x_1x_2x_3 \rangle = (A^{(1)}_{x_1}A^{(2)}_{x_2}A^{(3)}_{x_3})$ and similar for the other terms. The maximal non-signalling violation assigns $M_3 = 4$ which can only occur when the first three correlators in (8) take their maximum value of $+1$ and the last takes its minimum of $-1$.

Take any input combination appearing in the inequality (8), say, $x_3 = (0, 0, 1)$. Maximal violation of $M_3$ imposes the following conditions:

1. $\langle 001 \rangle = 1$. This further implies $\langle 0 \rangle_1 = \langle 01 \rangle_{23}$, $\langle 0 \rangle_2 = \langle 01 \rangle_{13}$ and $\langle 1 \rangle_3 = \langle 00 \rangle_{12}$.

2. $\langle 010 \rangle = 1$ implying $\langle 0 \rangle_1 = \langle 10 \rangle_{23}$, $\langle 1 \rangle_2 = \langle 00 \rangle_{13}$ and $\langle 0 \rangle_3 = \langle 01 \rangle_{12}$.

3. $\langle 100 \rangle = 1$ implying $\langle 1 \rangle_1 = \langle 00 \rangle_{23}$, $\langle 0 \rangle_2 = \langle 10 \rangle_{13}$ and $\langle 0 \rangle_3 = \langle 10 \rangle_{12}$.

4. $\langle 111 \rangle = -1$ implying $\langle 1 \rangle_1 = -\langle 11 \rangle_{23}$, $\langle 1 \rangle_2 = -\langle 11 \rangle_{13}$ and $\langle 1 \rangle_3 = -\langle 11 \rangle_{12}$

Imposing these relations on (8) for input $x_m = (0, 0, 1)$ one gets

\[ P_M(a_1, a_2, a_3|0, 0, 1) = \frac{1}{8} (a_1a_2a_3 + (a_1 + a_2a_3)|0\rangle_1 + (a_2 + a_1a_3)|0\rangle_2 + (a_3 + a_1a_2)|1\rangle_3) \]  

(10)

Using all these constraints and the definition of the function $f$, Eq. (5) can be expressed as

\[ P_M(f(a) = +1|x_m) = P_M(1, -1, -1|x_m) + P_M(-1, -1, -1|x_m) \]  

\[ = \frac{1}{4} (3 - \langle 0 \rangle_1 - \langle 0 \rangle_2 - \langle 0 \rangle_3) \]  

(11)

Proving that $P(f(a) = +1|x_m) \geq 1/2$ then amounts to showing that $\langle 0 \rangle_1 + \langle 0 \rangle_2 + \langle 0 \rangle_3 \leq 1$. This form is very convenient since it reminds one of a positivity condition of probabilities.

We then consider the input combination $x_m$ such that all the bits in $x_m$ are different from those in $x_m$. We call this the swapped input, which in the previous case is $x_m = (1, 1, 0)$. Note that this is not an input appearing in the Mermin inequality. However, using the previous constraints derived for distributions $P_M$ maximally violating the inequality, one has

\[ P_M(a_1, a_2, a_3|1, 1, 0) = \frac{1}{8} ((1 + a_1)(1) + a_2(1) + a_3(0)) + a_1a_2(1)_{12} \]  

\[ + a_1a_3(1)_{13} + a_2a_3(1)_{23} + a_1a_2a_3(1)_{123}) \]  

(12)

where the second equality results from the relations $\langle 1 \rangle_{12} = \langle 1 \rangle_{13}$, $\langle 1 \rangle_{12} = \langle 0 \rangle_{12}$ and $\langle 0 \rangle_{23} = \langle 0 \rangle_{13}$.

It can be easily verified that summing the two positivity conditions $P_M(1, 1, -1|x_m) \geq 0$ and $P_M(-1, -1, 1|x_m) \geq 0$ gives the result we seek, namely $1 - \langle 0 \rangle_1 - \langle 0 \rangle_2 - \langle 0 \rangle_3 \geq 0$, which completes the proof. \(\square\)

Using the previous Lemma, it is rather easy to prove the following

Theorem 1. Let $P_M(a|x)$ be an $N$-partite (odd $N$) non-signalling probability distribution maximally violating the corresponding Mermin inequality. Then the intrinsic and the observed randomness of the function $f$ are equal for any input $x$ appearing in the Mermin inequality:

\[ G_{\text{int}}(f, x, P_M) = G_{\text{obs}}(f, x, P_M) \]  

where

\[ G_{\text{obs}}(f, x, P_M) = \max_{k \in \{+1, -1 \}} P_{\text{obs}}(f(a) = k|x) \]
Proof of Theorem 1. — Since $P_{\text{obs}}$ maximally and algebraically violates the Mermin inequality, all the extremal distributions $P_{e}^{\text{ex}}$ appearing in its decomposition must also necessarily lead to the maximal violation of the Mermin inequality (see Appendix for details). Hence, the randomness of $f$ in these distributions as well satisfies Eqn. (8) of Lemma 1. Using this, we find,

$$G_{\text{obs}}(f, x, P_{e}^{\text{ex}}) = \max_{k \in \{+1, -1\}} P_{e}^{\text{ex}}(f(a) = k|x)$$

$$= |P_{e}^{\text{ex}}(f(a) = h_{N}|x) - 1/2| + 1/2$$

$$= P_{e}^{\text{ex}}(f(a) = h_{N}|x),$$

for every $e$. Therefore,

$$G_{\text{int}}(f, x, P_{\text{obs}}) = \max_{\{p(e|x), P_{e}^{\text{ex}}\}} \sum_{e} p(e|x) G_{\text{obs}}(f, x, P_{e}^{\text{ex}})$$

$$= \max_{\{p(e|x), P_{e}^{\text{ex}}\}} \sum_{e} p(e|x) P_{e}^{\text{ex}}(f(a) = h_{N}|x)$$

$$= P_{\text{obs}}(f(a) = h_{N}|x),$$

(13)

where the last equality follows from the constraint $\sum_{e} p(e|x) P_{e}(a|x) = P_{\text{obs}}(a|x)$. On the other hand, the observed randomness for $f$ is, $G_{\text{obs}}(f, x, P_{\text{obs}}) = P_{\text{obs}}(f(a) = h_{N}|x)$. □

The previous technical results are valid for any non-signalling distribution maximally violating the Mermin inequality. For odd $N$ this maximal violation can be attained by a unique quantum distribution, denoted by $P_{\text{ghz}}(a|x)$, resulting from measurements on a Greenberger-Horne-Zeilinger (GHZ) state. When applying Theorem 1 to this distribution, one gets

Main result: Let $P_{\text{ghz}}(a|x)$ be the $N$-partite (odd $N$) quantum probability probability distribution attaining the maximal violation of the Mermin inequality. The intrinsic and observed randomness of $f$ for a Mermin input satisfy

$$G_{\text{int/obs}}(f, x, P_{\text{ghz}}) = \frac{1}{2} + \frac{1}{2^{(N+1)/2}}$$

(15)

This follows straightforwardly from Theorem 1 since $P_{\text{ghz}}(a|x) = 1/2^{N-1}$ for outcomes $a$ with an even number of results equal to $-1$ and for those measurements appearing in the Mermin inequality.

It is important to remark that $f(a|x_{m})$ approaches a perfect random bit exponentially with the number of parties. In fact, this bit defines a process in which full randomness amplification takes place. Yet, it is not a complete protocol as, contrary to the existing proposal in [8], no estimation part is provided.

Discussion — We have identified a quantum process whose observed randomness can be proven to be fully intrinsic. In other words, for the considered process, quantum theory gives predictions as accurate as any non-signalling theory, possibly supra-quantum, can give. Our results hold under the minimal assumptions: the validity of the no-signaling principle and an arbitrary (but not complete) relaxation of the freedom of choice. The latter is subtle and much attention in recent years has focused on relaxing it in Bell experiments [13–17].

Our work raises several questions. Our main motivation here has been to understand the ultimate limits allowed by quantum theory on intrinsic randomness and, thus, we have worked in a noise-less regime. It is interesting to consider how would our results have to be modified to encompass scenarios including noise and hence amenable to experiments. The presence of noise modifies our results from two different viewpoints. First, noise is due to lack of control of the setup and, thus, a source of apparent randomness, which immediately implies a gap between intrinsic and observed randomness.

Second, in a noisy situation, it is impossible to arbitrarily relax the freedom of choice assumption, quantified by $\delta$ in Eq. 13. In fact, there is a tradeoff between the amount of relaxation of this condition and the violation needed to certify the presence of any intrinsic randomness. The reason is that, for a sufficiently small value of $\delta$, any correlations not attaining the maximal non-signalling violation of a Bell inequality can be reproduced using purely deterministic local strategies. It seems natural, in a practical context, to extend the definition of intrinsic randomness by considering bounded relaxations of the freedom of choice assumption and non-maximal violations of Bell inequalities. These investigations would constitute the strongest tests on the completeness of quantum predictions, given that they would rely on significantly more relaxed assumptions than any other quantum experiment performed to date.

From a purely theoretical perspective, our results certify a maximum of one bit of randomness for any system size. It would be interesting to extend these analytical results to certify randomness that scale with the number of parties. This could for instance be accomplished with functions of increasing outcomes. In a related context, it would also be interesting to explore whether similar results are possible in a bipartite scenario or, on the contrary, whether an asymptotic number of parties is necessary for full randomness amplification.

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Here we prove the principal theorem of the main text. It is basically a generalization of the the proof for $N = 3$. We would like to prove that the function $f$ defined in the main text, satisfies the property:

$$P(f(a) = h_N|x_m| \geq 1/2$$

for any $N$-partite distribution (odd $N$) that maximally violates the Mermin inequality. As in the tripartite case, in order to prove the result we (I) express condition (10) in terms of some correlators and (II) use positivity conditions from the swapped input to prove the inequality.

An $N$-partite no-signalling probability distribution $P(a|x)$ with inputs $x \in \{0,1\}^N$ and outputs $a \in \{+1,-1\}^N$ can be parameterized in terms of correlators as,

$$P(a|x) = \frac{1}{2^N} \left( 1 + \sum_{i=1}^{N} a_i(x_i) + \sum_{i<j} a_i a_j \langle x_i x_j \rangle + \sum_{i<j<k} a_i a_j a_k \langle x_i x_j x_k \rangle + \cdots + a_1 a_2 \cdots a_N \langle x_1 x_2 \cdots x_N \rangle \right),$$

Restricting $P(a|x)$ to those maximally violating the $N$-partite Mermin inequality is equivalent to requiring all correlators of input strings of odd parity to take their extremal values. Namely, we have,

$$\langle x_1 x_2 \cdots x_N \rangle = (-1)^{\left(1+\sum_{i=1}^{N} x_i\right)/2},$$

for all $N$-point correlators satisfying $\sum_{i=0}^{N} x_i = 1 \mod 2$. For instance, $(0,0,\ldots,1) = 1$ and similarly for all permutations. Also, $(0,0,\ldots,0,1,1,1) = -1$ as well as for all permutations, etc. In the following we will use the notation $\langle \cdot \rangle_k$ to denote a $k$-point correlator. The input combination used to extract randomness is a generalization of the tripartite case and denoted by $x_m = (0,0,\ldots,0,1)$. The corresponding $N$-point correlator satisfies $(0,0,\ldots,0,1) = 1$ for all $N$. The latter implies two useful relations:
1. Half the total outcomes vanish. In particular these are the terms for which the product of outcomes is $-1$ i.e. $P(\prod_{i=1}^{N} a_i = -1|x_m) = 0$.

2. $\langle \rangle_{N-k} = \langle \rangle_k$ for all $1 \leq k \leq (N-1)/2$ where the correlators $\langle \rangle_{N-k}$ and $\langle \rangle_k$ are complementary in the input $x_m$.

One can use these in Eqn. [17] to express $P(a|x_m)$ in terms of only the first $(N - 1)/2$-point correlators as,

$$P(a|x_m) = \frac{1}{2^{N-1}} \left( 1 + \sum a_i \langle x_i \rangle + \sum a_i a_j \langle x_i x_j \rangle + \cdots + \sum a_i a_j \cdots a_p \langle x_i x_j \cdots x_p \rangle (N-1)/2 \right).$$

where $a_1 \cdot a_2 \cdot a_3 \ldots a_N = +1$ since $P(a|x_m) = 0$ when $a_1 \cdot a_2 \cdot a_3 \ldots a_N = -1$.

### Expressing the inequality in terms of correlators

As mentioned, our first goal is to express Eq. (16) as a function of some correlators. Let us recall the function we use in our main theorem,

$$f(a) = \begin{cases} +1 & n_-(a) = (4j + 2); \text{ with } j \in \{0, 1, 2, \ldots\} \\ -1 & \text{otherwise} \end{cases}$$

where $n_-(a)$ denotes the number of results in $a$ that are equal to $-1$.

It turns out that the quantity (Eq. (16)) we would like to calculate, namely, $P(f(a) = h_N|x_m) - 1/2$ can be equivalently expressed as $h_N \cdot (P(f(a) = +1|x_m) - 1/2)$. The latter form is convenient since the function only takes value $+1$ for all $N$.

We proceed to express the latter in terms of correlators (as in the proof for three parties in the main text),

$$(h_N \cdot P(f(a) = +1|x_m) - 1/2) = 2^{-(N-1)} \alpha' \cdot c,$$

where

$$\alpha' = h_N \cdot (\alpha_0 - 2^{N-2}, \alpha_1, \alpha_2, \ldots, \alpha_{(N-1)/2})$$

$$c = \left( 1, \sum_{S^1} \langle \rangle_1, \sum_{S^2} \langle \rangle_2, \ldots, \sum_{S^{(N-1)/2}} \langle \rangle_{(N-1)/2} \right)$$

Note that, since the function $f$ symmetric under permutations, the vector $c$ consists of the different sums of all $k$-point correlators, denoted by $S^k$, where $k$ ranges from 0 to $(N - 1)/2$ because of Eq. (18). The vector $\alpha'$ is the vector of coefficients for each sum of correlators. Our next goal is to compute this vector.

Recall that function $f$ is such that $f(a) = +1$ if $n_-(a) = 4j + 2$ for any $j \in \mathbb{N} \cup \{0\}$. By inspection, the explicit values of $\alpha_i$ can be written as

$$\alpha_i = \sum_{r=0}^{i} (-1)^r \binom{i}{r} \sum_{j \geq 0} \binom{n-i}{4j+2-r}.$$  

For example, $\alpha_0 = \sum_{j \geq 0} \binom{n}{4j+2}$ as one would expect since $\alpha_0$ simply counts the total number of terms $P(a|x_m)$ being summed to obtain $P(f(a) = +1|x_m)$.

Making use of the closed formula $\sum_{j \geq 0} \binom{n}{4j+2} = \frac{1}{2} \sum_{k \geq 0} \omega^{-ka}(1 + \omega^k)^n$, where $\omega = e^{i2\pi/r}$ is the $r$th root of unity, we can simplify the second sum appearing in Eq. (22). Finally we recall that the phase $h_N$ was defined (in the main text) to be $h_N = \sqrt{2} \cos (N + 4)\pi/4$. Putting all this together and performing the first sum in Eq. (22) gives us,

$$\alpha'_i = 2^{N-3} \left( -2 \cos \frac{(N-2i)\pi}{4} \cos \frac{(N+4)\pi}{4} \right)$$

Notice that the term in the parenthesis is a phase taking values in the set $\{+1, -1\}$ since $N$ is odd while the amplitude is independent of $N$. Thus, we can simplify Eqn. (23) for even and odd values of $i$ as,
\[\alpha'_i = \begin{cases} 2^{(N-3)/2}(-1)^{\frac{N-1}{2}} & i \text{ odd} \\ 2^{(N-3)/2}(-1)^{\frac{i}{2}} & i \text{ even} \end{cases} \quad (24)\]

Thus, to prove that \( f \) possesses the property \( h_N \cdot (P(f(a) = +1|x_m) - 1/2) \geq 0 \) necessary to proving the main theorem is equivalent to proving

\[\alpha' \cdot c \geq 0, \quad (25)\]

for \( c \) as defined in Eqn. (21) and for the values of \( \alpha' \) given by Eqn. (24). This is the task of the following section, where we show that it follows from positivity constraints on \( P(a|x) \).

proving the inequality from positivity constraints

We show that positivity conditions derived from the swapped input \( \bar{x}_m = (1,1,\ldots,1,0) \) may be used to show \( \alpha' \cdot c \geq 0 \). Notice that the components of \( \bar{x}_m \) and \( x_m \) are opposite, i.e. \( \{\bar{x}_m\}_i = \{x_m\}_i \oplus 1 \) for all \( i \). In the following we will repeatedly use the Mermin conditions of Eqn. (17).

We start by summing the positivity conditions \( P(+ + + + -|\bar{x}_m) \geq 0 \) and \( P(- - - - +|\bar{x}_m) \geq 0 \). Using Eqn. (17), one can easily see that upon summing, all \( k \)-point correlators for odd \( k \) are cancelled out since these are multiplied by coefficients (products of \( a,\bar{a} \)) that appear with opposite signs in the two positivity expressions. In contrast, \( k \)-point correlators for even \( k \) add up since they are multiplied by coefficients that appear with the same sign in the two expressions. For example, \( N \) being odd, the full correlator always cancels out while the \( (N-1) \)-point correlators always appear.

This leaves us with an expression containing only the even-body correlators,

\[1 + \sum_{i<j} a_i a_j \langle x_i x_j \rangle + \sum_{i<j<k<l} a_i a_j a_k a_l \langle x_i x_j x_k x_l \rangle + \cdots + \sum a_i \ldots a_p \langle x_i \ldots x_p \rangle \geq 0. \quad (26)\]

Note once again, that this inequality is derived from the so-called swapped input \( \bar{x}_m \). We aim to cast it in a form that can be compared directly with Eqn. (21), which comes from the chosen Mermin input \( x_m \). To this end, we need to convert Eqn. (26) to an expression of the form,

\[(\beta_0, \beta_1, \ldots, \beta_{(N-1)/2}): \left(1, \sum \langle \cdot \rangle_1, \ldots, \sum \langle \cdot \rangle_{(N-1)/2} \right) \geq 0 \quad (27)\]

We first highlight the similarities and differences between the two preceding expressions, namely, the one we have i.e. Eqn. (26) and the one we want, i.e. Eqn. (27). Each contains \((N-1)/2\) distinct classes of terms. However the former contains only even \( k \)-point correlators for \( k = 2 \) to \((N-1)\) while the latter contains all terms from \( k = 1 \) to \((N-1)/2\). Thus, terms of Eq. (26) must be mapped to ones in Eqn. (27). Moreover, since the point of making this mapping is to finally compare with Eqn. (21), we also note that the correlators appearing in Eqn. (26) are locally swapped relative to those appearing in Eqn. (24). Thus, our mapping must also convert correlators of the swapped input into those corresponding to the chosen input.

We demonstrate next that one may indeed transform the inequality (26) into the inequality (27) satisfying both the demands above. To this end, all the even \( k \)-point correlators (for \( k \geq \frac{N-1}{2} \)) appearing in Eqn. (26) are mapped to odd \( (N-k) \)-point correlators in Eqn. (27). Likewise, all the even \( k \)-point correlators (for \( k < \frac{N-1}{2} \)) of the swapped input appearing in Eqn. (26) are mapped to the corresponding \( k \)-point correlators of the chosen input in Eqn. (27).

These mappings make systematic use of the Mermin conditions Eqn. (17) and are made explicit in the following section.

Even-point correlators

Consider a \( 2k \)-point correlator where \( 2k \leq (N-1)/2 \). The correlators are of two forms and we show how they are transformed in each case:
\begin{itemize}
  \item \langle 11 \ldots 1 \rangle_{2k} \text{. We would like to map this to the correlator } \langle 00 \ldots 0 \rangle_{2k} \text{ appearing in } \mathbf{x}_m. \text{ We achieve the mapping by completing each to the corresponding Mermin full-correlators } \langle 11 \ldots 110 \ldots 0 \rangle_N = (-1)^k \text{ and } \langle 00 \ldots 010 \ldots 0 \rangle_N = (-1)^N = 1. \text{ From the signs, we have the relation, } \langle 11 \ldots 1 \rangle_{2k} = (-1)^k \langle 00 \ldots 0 \rangle_{2k} \text{.}
  
  \item \langle 11 \ldots 10 \rangle_{2k} \text{, which we would like to map to } \langle 00 \ldots 01 \rangle_{2k}. \text{ Using the same ideas we get } \langle 11 \ldots 10110 \ldots 0 \rangle_N = (-1)^k \text{ and } \langle 00 \ldots 0110 \ldots 0 \rangle_N = (-1)^{N-2k} = -1. \text{ Thus, giving us the relation } \langle 11 \ldots 10 \rangle_{2k} = (-1)^{k+1} \langle 00 \ldots 01 \rangle_{2k}.
\end{itemize}

By inspection one can write the relationship

\[ \langle a_1 a_2 \ldots a_{2k} (x_1 x_2 \ldots x_{2k}) \rangle = (-1)^k \langle x_1 x_2 \ldots x_{2k} \rangle \]

for correlators of either form discussed above on multiplying with their corresponding coefficients. Since we have finally converted to the desired correlators of the chosen input \( \bar{x} \), we can read off \( \beta_i \) as the corresponding phase. Thus, \( \beta_i = (-1)^{i/2} \) for even \( i \).

**Odd-point correlators**

Consider now a 2\( k \)-point correlator where \( 2k \geq (N - 1)/2 \). The correlators are again of two forms and may be transformed to the required \( (N - 2k) \)-point correlators in each case. The only difference from before is that the two correlators are now complementary to each other in the swapped input. Since the details are similar, we simply state the final result \( \beta_i = (-1)^{(N-i)/2} \) for odd \( i \).

The final expression thus reads,

\[ \beta_i = \begin{cases} (-1)^{N-i} & \text{if odd} \\ (-1)^{i/2} & \text{if even} \end{cases} \]  

(28)

Thus, the values of \( \beta \) given in Eqs. (28) exactly match the ones for \( \alpha'_i \) (up to the constant factor) given in Eqn. (24). Together with the correlators matching those in \( \mathbf{c} \), it proves that \( f \) satisfies the required \( \alpha' \cdot \mathbf{c} \geq 0 \) and hence the full result.

**Proof that all distributions in decomposition maximally violate the Mermin inequality**

We end by proving the claim made in the main text that if an observed probability distribution \( P_{\text{obs}}(\mathbf{a}|\mathbf{x}) \) violates maximally and algebraically the corresponding Mermin inequality, all the no-signaling components \( P_{\text{ex}}(\mathbf{a}|\mathbf{x}) \) present in its preparation must also algebraically violate the inequality.

We recall that the decomposition appears in the definition of intrinsic randomness given by,

\[ G_{\text{int}}(g, \mathbf{x}, P_{\text{obs}}) = \max_{\{p(e|\mathbf{x}), P_{\text{ex}}\}} \sum_e p(e|\mathbf{x})G_{\text{obs}}(g, \mathbf{x}, P_{\text{ex}}) \]

subject to:

\[ \sum_e p(e|\mathbf{x})P_{\text{ex}}(\mathbf{a}|\mathbf{x}) = P_{\text{obs}}(\mathbf{a}|\mathbf{x}) \]

\[ p(e|\mathbf{x}) \geq \delta \text{ with } \delta > 0 \exists \mathbf{x}, e \]  

(29)

Since \( P_{\text{obs}} \) algebraically violates the Mermin inequality, this definition imposes stringent conditions on the correlators of \( P_{\text{obs}} \) satisfying the Mermin condition (17), namely that,

\[ \langle x_1 \ldots x_N \rangle_{P_{\text{obs}}} = \pm 1 = \sum_e p(e|x_1, \ldots, x_N) \langle x_1 \ldots x_N \rangle_{P_{\text{ex}}} \]  

(31)
where by normalization $\sum_e p(e|x_1, \ldots, x_N) = +1$ and $-1 \leq \langle x_1 \ldots x_N \rangle_p_{\text{ex}} \leq +1$. Note that condition $p(x|e) \geq \delta$ for all $x, e$ for $\delta > 0$ can be inverted using the Bayes’ rule to obtain $p(e|x) > 0$ for all $x, e$. Now is clear by convexity that the condition $p(x|e) \geq \delta$ (denying absolute relaxation of freedom of choice) implies that all the correlator $\langle x_1 \ldots x_N \rangle_p_{\text{ex}}$ appearing in the Mermin inequality must also necessarily satisfy $\langle x_1 \ldots x_N \rangle_p_{\text{ex}} = \pm 1$ for all $e$ thus maximally violating the Mermin inequality. In fact it is also clear that this constraint on $p(x|e)$ is strictly necessary to ensure that the decomposition correlations satisfy maximal Mermin violation. To see this, suppose $p(x|e_0) = 0$, then the corresponding $\langle x_1 \ldots x_N \rangle_p_{e_0}$ is fully unconstrained while satisfying Eq. (31).

[1] Arthur T. Benjamin, Bob Chen, and Kimberly Kindred. Sums of evenly spaced binomial coefficients. Mathematics Magazine, 83:370373, 2010.