Running primordial perturbations: Inflationary Dynamics and Observational Constraints

Richard Easther
Department of Physics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Benedict Bahr-Kalus and David Parkinson
Korea Astronomy and Space Science Institute, Yuseong-gu, Daedeok-daero 776, Daejeon 34055, Korea

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Inflationary cosmology proposes that the early Universe undergoes accelerated expansion, driven, in simple scenarios, by a single scalar field, or inflaton. The form of the inflaton potential determines the initial spectra of density perturbations and gravitational waves. We show that constraints on the duration of inflation together with the BICEP3/Keck bounds on the gravitational wave background imply that higher derivatives of the potential are nontrivial with a confidence of 99%. Such terms contribute to the scale-dependence, or running, of the density perturbation spectrum. We clarify the “universality classes” of inflation in this limit showing that a very small gravitational wave background can be correlated with a larger running. If pending experiments do not observe a gravitational wave background the running will be at the threshold of detectability if inflation is well-described at third-order in the slow roll expansion.

Now forty years old, inflation is the de facto description of the very early Universe. The clear consequences of generic inflationary models are well-verified: the Universe is spatially flat, almost homogeneous and isotropic and Gaussian, adiabatic perturbations induce large scale correlations in the polarization and temperature of the microwave background. The one ambiguous observable is the primordial gravitational wave background. Constraints have steadily tightened and the latest BICEP3/Keck data permits an amplitude of at most 4% of the density perturbations. A gravitational wave background is often viewed as the “smoking gun” of inflation since known alternatives do not generate a detectable signal but this is also true of many inflationary models. Moreover, algebraically simple slow-roll scenarios with large gravitational-wave signals must be “protected” by near-symmetries: such models can be proposed but nature need not employ them.

The amplitudes of the density and gravitational wave perturbations (expressed via their ratio, $r$), depend on the potential $V$ and its slope $V'$. The spectral index of the density perturbations $n_s$ further involves the second derivative, $V''$. Given a single field slow-roll prior, $n_s$ and $r$ are inputs for the inflationary inverse problem: the reconstruction of the potential from observational data.

We show that the latest BICEP3/Keck data implies that all viable implementations of slow-roll inflation with only $V$, $V'$ and $V''$ as free parameters produce more than 65 e-folds of inflation after astrophysically relevant perturbations leave the horizon, with 99% confidence. Without exotic post-inflationary physics, this is inconsistent with long-standing constraints so inflation can only terminate appropriately if higher derivatives are nontrivial or the potential is discontinuous.

A nontrivial $V'''$ modifies the dynamics relative to that derived with only $V'$ and $V''$. For any $n_s$ and $r$ one can fix $V'''$ to yield a specified amount of inflation. However, this leads to scale dependence in $V''$, contributing to the running of the spectral index, $\alpha_s = dn_s/d\ln k$ where $k$ is the comoving wavenumber. Experiments now under development are sensitive to $r \gtrsim 10^{-4}$, and the latest BICEP3/Keck data permits an amplitude of at most 4% of the density perturbations. A gravitational wave background is often viewed as the “smoking gun” of inflation since known alternatives do not generate a detectable signal but this is also true of many inflationary models. Moreover, algebraically simple slow-roll scenarios with large gravitational-wave signals must be “protected” by near-symmetries: such models can be proposed but nature need not employ them.

The analysis rests on the well-studied Hubble Slow Roll expansion. The full dynamical system has apparent attractors in the $(n_s,r)$ plane, issues with convergence and truncation, and does not account for initial, transient, field velocities.

The key finding is that all two-parameter single-field inflationary models are excluded with high confidence. The analysis rests on the well-studied Hubble Slow Roll expansion. The full dynamical system has apparent attractors in the $(n_s,r)$ plane, issues with convergence and truncation, and does not account for initial, transient, field velocities.

Given the measured value of $n_s$, we further show that tight constraints on $r$ imply a nontrivial running if the dynamics are treated at next-order in slow-roll. This formal linkage between the running and the duration of inflation is well known and we clarify the understanding of inflationary universality classes in this limit. Correlated expectations for $r$ and $\alpha_s$ depend on the truncated slow-roll hierarchy but three-parameter slow-roll is now the simplest feasible scenario. Excitingly, this linkage between $r$ and $\alpha_s$ presents a feasible target for future astrophysical measurements.
Two Parameter Slow-Roll Models: Single-field inflationary scenarios are governed by the Einstein-Klein-Gordon equations,

\[ H^2 = \frac{1}{3M_p^2} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right), \]

\[ \dot{H} = -\frac{1}{2M_p^2} \dot{\phi}^2, \]

\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0, \]

where the symbols have their usual meanings and we use the reduced Planck mass, \( M_p \). During the accelerated phase \( \phi \) evolves monotonically and is thus a “clock”. Eq. 2 can be rearranged to show that \( dH/d\phi \) is proportional to \(-d\phi/dt\), so

\[ V(\phi) = \frac{3M_p^2}{2} H(\phi)^2 - M_p^2 H'(\phi)^2. \]

For the purposes of parameter counting, we assume that the Potential and Hubble Slow Roll formulations are interchangeable. The Hubble Slow Roll hierarchy [21] provides a more succinct account of the dynamics,

\[ \epsilon(\phi) \equiv 2M_p^2 \left[ \frac{H'(\phi)}{H(\phi)} \right]^2, \]

\[ \ell \lambda_H = (2M_p)^{\ell} \frac{(H')^{\ell-1} d^{\ell+1}H}{d\phi^{\ell+1}}, \ell \geq 1. \]

with the convention that \( \eta = \frac{1}{2} \lambda_H \) and \( \xi = \frac{1}{2} \lambda_H \). The number of e-folds that will elapse before inflation ends is \( N = -\ln(a/a_{end}) \), where \( a_{end} \) is the scale factor as inflation completes. Noting \( H = \dot{a}/a \),

\[ \frac{dN}{d\phi} = \frac{1}{M_p} \frac{1}{\sqrt{2\epsilon}}, \]

the “flow equations” are

\[ \frac{d\epsilon}{dN} = 2\epsilon(\eta - \epsilon), \]

\[ \frac{d\eta}{dN} = -\epsilon \eta + \xi, \]

\[ \frac{d\ell \lambda_H}{dN} = [(\ell - 1)\eta - \epsilon] \times \ell \lambda_H + (\ell + 1)\lambda_H, \]

where \( N \) is now the independent variable. Accelerated expansion occurs when \( \dot{a} > 0 \) or equivalently \( \epsilon < 1 \). If \( \ell \lambda_H = 0 \) for all \( \ell \geq M \) at some \( \phi \) the system remains closed as it evolves [21] [31] [32], with \( M \) nontrivial slow-roll parameters. The amplitude of the potential is a further free parameter but scales out of the dynamics.

The foregoing treatment is exact but key observables are expressed in the slow-roll approximation, or

\[ n_s = 1 + 2\eta - 4\epsilon - 2(1+C)\epsilon^2 - \frac{1}{2}(3-C)\xi, \]

\[ r = 16\epsilon[1 + 2(C(\epsilon - \eta))], \]

FIG. 1. Likelihood contours in \( n_s \) and \( \ln r \) from the BK15 (blue) and BK18 (green) datasets, in combination with PlanckTTTEEE+lowE+lensing+BAO results [7, 9]. Shaded regions denote the 66% and 95% posteriors. Black contours indicate the number of e-folds \( N \) that take place after the pivot leaves the horizon with a two-term slow-roll hierarchy.

FIG. 2. The posterior for \( N \) with a two-term slow-roll hierarchy (as in Fig. 1) with a uniform prior on \( \tau \); for a logarithmic prior the distribution is roughly constant at larger \( N \).

\[ \alpha_s = -\frac{1}{1+\epsilon} \frac{d\phi}{dN} \frac{dn_s}{d\phi}, \]

where \( C = -2 + \ln 2 + \gamma, \ C = 4(\gamma + \ln 2) - 5, \) and \( \gamma \) is the Euler-Mascheroni constant. Finally, \( dN/d\ln k = -1/(1-\epsilon) \) is the rate at which modes leave the horizon and \( \epsilon \to 0 \) in the de Sitter limit where \( H \) is constant.

A two-term hierarchy maps \( n_s \) and \( r \) to an inflationary trajectory. Fig. 1 shows the constraints on \( n_s \) and \( \ln r \) derived from the BK15 [7] and BK18 [9] datasets (published in 2018 and 2021, respectively), together with Planck and Baryon Acoustic Oscillation data overlaid with the duration of inflation computed with two slow-roll terms. Fig. 2 shows the marginalised distributions for \( N \); BK18 yields \( P(N < 65) \approx 0.0024 \). Provided the post-
We write \( n_s - 1 = -a/N \), where \( a \) is a constant a little larger than unity. Dropping higher order terms and accounting for the difference between \( \eta \) and \( \eta^* \) we can set this equal to Eq. (11) or \( n_s \approx 1 - 6\epsilon V + 2\eta V \) to find a differential equation for \( \epsilon_V(N) \) (e.g. [30]). In the low \( r \) limit the solution has the form \( \epsilon_V \sim 1/(AN^3) \) where \( A \) is a large constant. Physically, this ensures that \( \eta V \) and \( \epsilon V \) are tightly correlated even when \( \epsilon V \ll \eta V \). However if \( r \lesssim |n_s - 1|^2 \) it would seem that \( \epsilon_V \) cannot be self-consistently ignored, since it contributes to the scale dependence of \( \eta V \) via

\[
\frac{d\eta V}{dN} \approx M_p^2 \left( \frac{V'}{V} \right)^2 \frac{V''}{V} \left( \frac{V'}{V} \right)^2 \,
\] (15)

and the second term can be far smaller than the first.

This regime corresponds to the Low-\( \epsilon \) limit of the Hubble Slow Roll hierarchy, and with three terms

\[
\frac{d\eta}{dN} \approx \xi, \quad \frac{d\xi}{dN} \approx \xi \eta.
\] (16)

These equations can be solved [24], showing

\[
\xi(N) = \frac{\eta(N)^2 - \eta_i^2}{2} + \xi_*,
\] (17)

where the star subscript denotes a value at the pivot.

To a good approximation \( \eta(N) = n_s - \xi \Delta N \) for astrophysically relevant modes, where \( \Delta N \) is the number of e-folds after the pivot leaves the horizon; the full solution for \( \eta(N) \) in this limit is the “2-parameter, Low-\( \epsilon \)” model of Ref. [24]. In particular, the relationship \( \xi \sim |n_s - 1|^2 \sim 1/N^2 \) is supplemented by an additive constant in the Low-\( \epsilon \) limit. Physically, this yields a near-inflexion point in the potential, where both \( \epsilon \) and \( n_s - 1 \) are necessarily very small.

Future Prospects: Recalling that \( r \sim (V'/V)^2 \), we can identify three regimes; \( V' > V'' \), \( V' \sim V'' \) and \( V' \ll V'' \) (with \( M_p = 1 \)). The first requires \( r \gtrsim 0.01 \) and is close to being ruled out; the second is eliminated if \( r \lesssim 10^{-4} \), a threshold which will be within reach by 2030 [19, 20].

If a primordial gravitational wave background is not detected in the coming decade, any viable single-field model will satisfy \( V' \ll V'' \) and is thus squarely inside the Low-\( \epsilon \) regime. Fig. 3 shows the likely values of \( \alpha_s \) on the \( n_s - \ln r \) plane for three different choices of the total number of e-folds. If \( r \lesssim 10^{-4} \) then \( \alpha_s < -10^{-3} \) for any self-consistent three-parameter scenario.

Fig. 3 shows the individual and combined limits on \( |\alpha_s| \) expected from CHIME [35] and SPHEREx [36], together with CMB-S4 [19]. Each experiment measures \( \alpha_s \) with an accuracy of, at best, \( 5 \times 10^{-3} \) but their combined sensitivity is similar to the expected running if \( r \lesssim 10^{-4} \). All these experiments aim to provide results by 2030. Consequently, if the early Universe passed through an inflationary universe is not dominated by matter whose stiffness exceeds that of radiation, \( N < 65 \) is a generic bound on the amount of inflation after the pivot leaves the horizon [16]. Subject to this proviso on the equation of state, all inflationary models described by the first two slow-roll parameters are now excluded.

This advance arises from tightening constraints on both \( n_s \) and \( r \). A spectral index of less than 0.95 was consistent with the full WMAP dataset [33] and inflation ends “on time” for smaller \( n_s \) without additional curvature in the potential. Consequently, better measurements of \( n_s \) combine with tighter bounds on the polarization to yield this result. Note too that this analysis implicitly assumes a “ski-run” inflationary potential with a smooth approach to regular expansion. Scenarios in which inflation abruptly terminates also require additional parameters, albeit outside the Hubble Slow Roll expansion.

Running and the End of Inflation We now extend the Hubble Slow Roll expansion to third order, so that \( \xi \) is non-zero. This can increase the scale-dependence of \( \eta \), as \( \alpha_s \approx -2\xi \) when \( \epsilon \) is small. Fig. 3 overlots the \( n_s \) and \( r \) constraints with contours showing the running resulting from choosing \( \xi \) such that \( N = 55 \) when the pivot leaves the horizon. The running is generically larger than in “standard” inflationary models [17] but still well inside recent constraints; e.g. \( d\eta/dN \approx -0.006 \pm 0.013 \) [6].

This adds nuance to statements that \( n_s - 1 \sim -1/N \) and \( \alpha_s \sim 1/N^2 \) which hold empirically for many simple models [17]. These expectations have been formalised in the Potential Slow Roll expansion [19, 28, 30], leading to what are sometimes referred to as “universality classes” [29]. In this framework \( \epsilon_V = M_p^2 (V'/V)^2/2 \), \( \eta_V = M_p^2 V''/V \) and \( \xi_V = M_p^4 V'''/V^2 \) and

\[
\frac{d\epsilon_V}{dN} \approx M_p^2 \left( \frac{V'}{V} \right)^2 \frac{V''}{V} \left( \frac{V'}{V} \right)^2 ,
\] (14)
accelerated phase the simplest currently viable inflationary models suggest that we can hope to have evidence that either $r$ or $\alpha_s$ is non-zero in a decade from now.

Discussion We have updated the priors on scalar field inflation using the latest data: models specified by only $V'$ and $V''$ at the pivot do not lead to a self-consistent inflationary era, at a 99% confidence level. There is a clear correlation between small $r$ and large $\alpha_s$ at third order in slow-roll. That said, it does not hold generically; even in slow-roll, if $3\lambda_H$ is nontrivial $\xi$ and the running can be small at the pivot. Moreover there are further counterexamples which cannot be easily described within the Hubble Slow Roll hierarchy, e.g. multifield models, and those with discontinuous or modulated potentials.

The current observational roadmap will investigate the range $10^{-4} \lesssim r \lesssim 10^{-2}$ and $|\alpha_s| \gtrsim 10^{-3}$ in the coming decade. Without a detection of the gravitational wave background there will be real pressure on the relationship between $\alpha_s$, $r$ and $N$ highlighted here. Consequently, even a null result will significantly constrain what is now the simplest viable inflationary model in terms of parameter count and qualitative complexity.

This analysis also illuminates inflationary universality classes for very small $r$, which arise from treating expressions for $n_s$ as differential relationships. Conversely, the Hubble Slow Roll parameters are akin to Taylor coefficients and the “flow equations” describe their running [21]. The first two terms set $\sqrt{\mathcal{F}}$ and $|n_s - 1|$ but near an extremum of $V(\phi)$ (or $H(\phi)$, since $V' = 0$ implies $H' = 0$) $r \ll 1$ and $V''$ is the next-to-leading order term.

A scenario in which $V'$ is very small and $V''$ is significant is most naturally an inflexion-point model. Interestingly, hilltop potentials of the form $V \sim V_0 - V_2 \phi^2 - V_4 \phi^4$ struggle to generate low values of $r$, given present constraints on $n_s$ [37]. In addition, for fixed $n_s$ there is an inverse correlation between $\alpha_s$ and $N$, depending on the overall inflationary scale [16] [24] and the possibly complicated and nonlinear physics of the post-inflationary universe [38] [44]. This overall discussion could be further sharpened by adopting a Bayesian model comparison framework [0] [45] [46], and these considerations illuminate the viable forms of the inflationary potential. Note too that if the inflationary patch of the potential is small it is more likely that models will be sensitive to the initial spatial configuration of the inflaton [47] [49].

In summary, all inflationary models fully described at second order in the Hubble slow roll expansion are now excluded by observational data with high confidence. This comes twenty years after the first nontrivial limits on inflationary models were delivered by WMAP [3] [50], and marks a significant advance in the ability to constrain inflation. Moreover, three-parameter slow-roll models, now the simplest scenarios (in terms of parameter count and

FIG. 4. The running $\alpha_s \times 10^3$ is plotted in the $n_s - \ln r$ plane for $N = 45, 55$ and 60, assuming a three-parameter slow-roll hierarchy. When $r \lesssim 10^{-4}$ we see that $\alpha_s < -10^{-3}$ for all values of $n_s$ consistent with presently available data.

FIG. 5. Forecast $n_s$-$\alpha_s$ constraints with CMB-S4, SPHEREx [S], CHIME [C]. The best combination promises to measure $\alpha_s$ to about $2.2 \times 10^{-3}$ at 95% confidence [31].

qualitative complexity), exhibit a correlation between the gravitational wave amplitude and the running. This will be testable over the coming decade, and either a verification or a null result would represent major progress.

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