Comparison of the operation of the equations of the surface harmonics method and the finite difference method in the test problem

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Abstract. Currently, one of the main methods of neutron-physical calculation of the reactor is the homogenization method, in which after obtaining effective small-group characteristics of cells, the heterogeneous core, in fact, turns into a piecewise homogeneous one. To find the distribution of neutrons in such a zone, the diffusion equation is solved by finite-difference (or nodal) methods. One of the methods justifying this approach is the surface harmonics method (SHM), which in the initial period of its development acted as a justification and refinement of the homogenization method. In the simplest versions of the SHM, the resulting finite-difference equations are reduced to a form similar to the finite-difference approximation of the diffusion equation. It is interesting to compare in the simplest cases the advantages and disadvantages of a simple finite-difference approximation of the diffusion equation and the finite-difference equations obtained using SHM. An analytical comparison is made using the example of two-dimensional geometry. To do this, the paper briefly describes how to obtain equations in SHM.

1. Introduction
Currently, one of the main methods of neutron-physical calculation of the reactor is the homogenization method, in which after obtaining effective small-group characteristics of cells, the heterogeneous core, in fact, turns into a piecewise homogeneous one. To find the distribution of neutrons in such a zone, the diffusion equation is solved by finite-difference (or nodal) methods.

One of the methods justifying this approach is the surface harmonics method (SHM), which in the initial period of its development acted as a justification and refinement of the homogenization method (see, for example, [1], [2]). The main principles of the method laid down in it are described in [3], [4] and more modern [5], [6].

In the simplest versions of the SHM, the resulting finite-difference equations are reduced to a form similar to the finite-difference approximation of the diffusion equation. It is interesting to compare in the simplest cases the advantages and disadvantages of a simple finite-difference approximation of the diffusion equation and the finite-difference equations obtained using SHM. An analytical comparison is made using the example of two-dimensional geometry. To do this, the paper briefly describes how to obtain equations in SHM.

2. Equations of SHM
In SHM the distribution of neutrons in a cell is constructed as a linear combination of some trial functions meeting the neutron transport equation inside the cell. As the boundary conditions for test
functions we use zero odd angular moments of neutron distribution along all faces, besides for the \( n \)-th face \((n=1,\ldots,N)\) of the cell, on which we set the unit \( l \)-th odd angular moment \((l=1,3,\ldots,L)\) with the \( g \)-th neutron spectrum \( \Theta^{gl}_{n}(r_s,E)\) \((\int_{\Gamma}^{r_1} \Theta^{gl}_{n}(r_s,E)dSdE=1,\ g=1,2,\ldots,G)\). Thus, for each cell we have \( N \frac{L+1}{2} G \) trial function (we will denote \( f_{nngl}(r_s,E,\Omega)\) or, for short \( f_{nngl}(\omega)\)), and the distribution of neutrons in the cell is written as \( \Phi(\omega) = \sum_{n=1}^{N} \sum_{g=1}^{G} \sum_{l=1}^{L} A_{nngl} f_{nngl}(\omega) \), (the cell index is omitted). These trial functions are usually used in the response matrix method. However, work \cite{7}, for example, showed that in SHM it is always necessary to arrange trial functions according to the scheme of setting odd moments on the cell faces (or by importance, in the simplest case-by symmetry, as recommended in previous works on SHM). For two-dimensional geometry with not necessarily square cells, the neutron flow patterns are shown in Figure 1. The distribution of neutrons in the cell after ordering the test functions can be written in matrix-vector form

\[
\Phi(\omega) = \varphi^T(\omega)\mathbf{I} + \psi_x^T(\omega)\mathbf{J}_x + \psi_y^T(\omega)\mathbf{J}_y + \xi^T(\omega)\mathbf{P}
\]  

(1)

In this expression, the vectors \( \mathbf{I} \), \( \mathbf{J} \), \( \mathbf{P} \) consist of the amplitudes of test functions (first ordered by the index of groups, then by the index of angular moments), «\( T \)» - is the transpose sign.

Figure 1. Neutron Flow (\( g \)-th group, \( l \)-th moment) for different test functions.

Ased on the desire to reduce the discrepancy of the neutron transfer equation when substituting representation (1) in it Error! Reference source not found., the conditions for cross-linking neutron distributions at the cell edges are derived in the SHM. The crosslinking conditions in the SHM are the equality of odd angular moments (in the traditional case only of the normal projection of the neutron current) and the equality of some combinations of even moments on each face of the cell. Usually, the neutron flux density is cross-linked. In General, linear combinations of two consecutive even angular moments should be cross-linked together. The vector of such combinations on the cell surface will be denoted by \( \Phi(r_s,E)\). The conditions for crosslinking on the \( n \)-th face with coefficients before the amplitude vectors of trial functions will include coefficients-matrices of the type

\[
\mathbf{f}^{n}_{n} = \int_{\Gamma}^{r_1} \int_{0}^{\infty} \Theta^{n}_{n}(r_s,E) \tilde{f}^{n}_{n}(r_s,E)dSdE \quad \text{dimension} \quad \frac{L+1}{2} G \times \frac{L+1}{2} G . \quad \text{The matrix} \quad \Theta^{n}_{n}(r_s,E) \quad \text{has} \quad \frac{L+1}{2} G \quad \text{rows and} \quad \frac{L+1}{2} \quad \text{columns (in the simplest case for} \quad L=1 \quad \text{it is a column vector, and the matrix} \quad \tilde{f}^{n}_{n}(r_s,E) \]


contains the above linear combinations of two consecutive even angular moments and consists of \( \frac{L+1}{2} \) rows and \( \frac{L+1}{2} \cdot G \) columns (for \( L = 1 \) it is a row vector).

For example, let’s write down the crosslinking conditions on the right side of the cell shown in Figure 1. For clarity, we will number the faces separately by \( x \) and \( y \). The right and upper faces will be numbered as index 1, left and lower as index 2. Cells on the \( x \)-axis will be numbered with the index \( i \), on the \( y \)-axis-j, the upper index of the test functions indicates on which face the even moments of the trial functions are calculated. For \( x \)-symmetric cells, the values of flow densities on faces 1 and 2 will be the same by module (similar for \( y \)), but for generality, we will write equations for non-symmetric cells:

\[
\begin{align*}
I_{i,j}^{x} + J_{i,j}^{x} + P_{i,j}^{x} &= -I_{i+1,j}^{x} + J_{i+1,j}^{x} - P_{i+1,j}^{x} \\
\Phi_{i,j}^{x} + \psi_{i,j}^{x} J_{i,j}^{x} + \xi_{i,j}^{x} P_{i,j}^{x} &= \Phi_{i+1,j}^{x} J_{i+1,j}^{x} - \psi_{i+1,j}^{x} J_{i+1,j}^{x} + \xi_{i+1,j}^{x} P_{i+1,j}^{x}
\end{align*}
\]

(2)

(3)

If we exclude the amplitude \( J_{i,j}^{x} \) from these equations (2) and (3), rectangular cells are considered symmetric in \( x \) and \( y \) (for non-symmetric cells, the equations are derived in [8]), enter the usual variable \( \Phi_{i,j} = \bar{\Phi}_{i,j} J_{i,j}^{x} \) - vector of the average cell group flow densities (even moments of neutron distribution), and as an additional variable enter (for symmetry), \( f_{i,j}^{x} = \psi_{i,j}^{x} P_{i,j}^{x}, \) where \( \psi_{i,j}^{x} = (\psi_{i,j}^{x} + \psi_{i,j}^{y}) / 2 \), we get:

\[
\begin{align*}
I_{i,j}^{x} + J_{i,j}^{x} + P_{i,j}^{x} &= (\psi_{i,j}^{x} + \psi_{i,j+1}^{y})^{-1} (R_{i+1,j}^{x} \Phi_{i+1,j} + r_{i+1,j}^{x} f_{i+1,j}^{x} - R_{i,j}^{x} \Phi_{i,j} - r_{i,j}^{x} f_{i,j}^{x}) \\
R_{i,j}^{x} &= (\Phi_{i,j}^{x} - \psi_{i,j}^{x}) \bar{\Phi}_{i,j}^{-1} \cdot r_{i,j}^{x} = (\xi_{i,j}^{x} - \psi_{i,j}^{x}) \Phi_{i,j}^{-1}
\end{align*}
\]

(4)

Expression (4) is nothing more than a generalization of Fick’s law for the current (odd angular moments of neutrons between cells \( i \) and \( i+1 \). On the right side in parentheses it contains the difference in the average flux density (the even moments of the distribution) of neutrons in these cells. Expression \( (\psi_{i,j}^{x} + \psi_{i,j+1}^{y})^{-1} \) is a combination of diffusion coefficients (complete matrices) in cells \( i \) and \( i+1 \).

A similar (4) Error! Reference source not found. expression can be obtained by crosslinking odd and even moments of neutron distribution on the face \( x_{2} \):

\[
\begin{align*}
I_{i,j}^{x} - J_{i,j}^{x} + P_{i,j}^{x} &= (\psi_{i,j}^{x} + \psi_{i,j+1}^{y})^{-1} (R_{i+1,j}^{x} \Phi_{i+1,j} + r_{i+1,j}^{x} f_{i+1,j}^{x} - R_{i,j}^{x} \Phi_{i,j} - r_{i,j}^{x} f_{i,j}^{x}) \\
R_{i,j}^{x} &= (\Phi_{i,j}^{x} - \psi_{i,j}^{x}) \bar{\Phi}_{i,j}^{-1} \cdot r_{i,j}^{x} = (\xi_{i,j}^{x} - \psi_{i,j}^{x}) \Phi_{i,j}^{-1}
\end{align*}
\]

(5)

Note that the sum of equations (4) Error! Reference source not found. and (5) can be used for calculating \( J_{i,j}^{x} \) when \( \Phi_{i,j} \) and \( f_{i,j} \) are calculated.

The same kind of expression can be obtained for the currents (odd moments of distribution) of neutrons flowing into the cell from the \( y \)-faces.

\[
\begin{align*}
I_{i,j}^{y} + J_{i,j}^{y} - P_{i,j}^{y} &= (\psi_{i,j}^{y} + \psi_{i,j+1}^{x})^{-1} (R_{i+1,j}^{y} \Phi_{i+1,j} + r_{i+1,j}^{y} f_{i+1,j}^{y} - R_{i,j}^{y} \Phi_{i,j} + r_{i,j}^{y} f_{i,j}^{y}) \\
R_{i,j}^{y} &= (\Phi_{i,j}^{y} - \psi_{i,j}^{y}) \bar{\Phi}_{i,j}^{-1} \cdot r_{i,j}^{y} = (\xi_{i,j}^{y} - \psi_{i,j}^{y}) \Phi_{i,j}^{-1}
\end{align*}
\]

(6)

(7)

These four above obtained equations (4) Error! Reference source not found. – (7) can be written for each cell (and there are four unknown vectors in the cell), so we get a closed system of algebraic equations (taking into account the edge conditions). The main variable in this system of equations is \( \Phi_{i,j} \). To find it, it is convenient to use the equations obtained by the sum (4) Error! Reference source
not found. – (7), the final equation is similar to a finite-difference approximation of the diffusion equation:

\begin{equation}
(\psi_{i,j}^x + \psi_{i,j}^y)^{-1}(R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) + (\psi_{i,j}^x + \psi_{i,j}^y)^{-1}(R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) +
+ (\psi_{i,j}^x + \psi_{i,j}^y)^{-1}(R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) + (\psi_{i,j}^x + \psi_{i,j}^y)^{-1}(R_{i,j}^y \Phi_{i,j} - R_{i,j}^x \Phi_{i,j}) - \Sigma_{i,j} \Phi_{i,j} + q_{i,j} = 0
\end{equation}

In equation (8) Error! Reference source not found. matrix $\Sigma_{i,j} = 4(\Phi_{i,j})^{-1}$ is introduced, which in its physical sense is a matrix of neutron absorption macro cross-sections averaged over a cell and neutron transitions from group to group due to scattering and nuclear fission. Summand $Q_{i,j}$ is associated with additional trial functions:

\begin{equation}
q_{i,j} = (\psi_{i,j}^x + \psi_{i,j}^y)^{-1} (r_{i,j}^x f_{i,j} - r_{i,j}^y f_{i,j}) + (\psi_{i,j}^x + \psi_{i,j}^y)^{-1} (r_{i,j}^x f_{i,j} - r_{i,j}^y f_{i,j}) -
- (\psi_{i,j}^x + \psi_{i,j}^y)^{-1} (r_{i,j}^x f_{i,j} - r_{i,j}^y f_{i,j}) - (\psi_{i,j}^y + \psi_{i,j}^y)^{-1} (r_{i,j}^y f_{i,j} - r_{i,j}^y f_{i,j}) - \sigma_{i,j} f_{i,j} + Q_{i,j} = 0
\end{equation}

where $\sigma_{i,j} = 4(\Phi_{i,j})^{-1}$,

\begin{equation}
Q_{i,j} = (\psi_{i,j}^x + \psi_{i,j}^y)^{-1} (R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) + (\psi_{i,j}^y + \psi_{i,j}^y)^{-1} (R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) -
- (\psi_{i,j}^x + \psi_{i,j}^y)^{-1} (R_{i,j}^x \Phi_{i,j} - R_{i,j}^y \Phi_{i,j}) - (\psi_{i,j}^y + \psi_{i,j}^y)^{-1} (R_{i,j}^y \Phi_{i,j} - R_{i,j}^y \Phi_{i,j})
\end{equation}

For a homogeneous lattice consisting of rectangular symmetric (and identical cells), equations (8) Error! Reference source not found., (10) are written as:

\begin{equation}
\frac{(\psi^x)^{-1}}{2} R^x(\Phi_{i,j} + \Phi_{i,j} - 2\Phi_{i,j}) + \frac{(\psi^y)^{-1}}{2} R^y(\Phi_{i,j} + \Phi_{i,j} - 2\Phi_{i,j}) - \Sigma_{i,j} \Phi_{i,j} +
+ \frac{(\psi^x)^{-1}}{2} R^x(\Phi_{i,j} + \Phi_{i,j} - 2\Phi_{i,j}) + \frac{(\psi^y)^{-1}}{2} R^y(\Phi_{i,j} + \Phi_{i,j} - 2\Phi_{i,j}) - \sigma_{i,j} f_{i,j} + Q_{i,j} = 0
\end{equation}

In the lowest approximation of SHM, the amplitudes of additional test functions are assumed to be zero $f_{i,j} = 0$, and the equation (12) for $\Phi_{i,j}$ is written as:
which is similar to a finite-difference approximation of the diffusion equation.

3. Limit transition to one-dimensional geometry

Now let's move on to one-dimensional geometry. Suppose, for example, that the neutron flux density is the same on the \(y\)-axis. Then the system (12), (13) is reduced to the form

\[
\frac{(\psi^*)^{-1}}{2} \left[ R' (\Phi_{i+1,j} + \Phi_{i-1,j} - 2\Phi_{i,j}) + (\psi^*)^{-1} \right] - \Sigma_{i,j} \Phi_{i,j} = 0,
\]

which is exactly the kind that is obtained in SHM in one-dimensional geometry (see, for example, [9], [10]). Equation (21) is also similar to the finite-difference approximation of the one-dimensional diffusion equation \(D \Delta \Phi(x) - \Sigma \Phi(x) = 0\), which in the finite-difference method for the one-group case on a uniform lattice is written as (\(b\) is the lattice pitch on the \(x\)-axis):

\[
D \left( \frac{\Phi_{i+1,j} + \Phi_{i-1,j} - 2\Phi_{i,j}}{b^2} \right) - \Sigma_{i,j} \Phi_{i,j} = 0.
\]

For the case where the real neutron flux density behaves as \(A \exp(-x/L)\), where \(L\) is the length of the neutron diffusion, this approximate equation will give an approximate value of the diffusion length \(L_\ast\), which can be found by substituting the solution in the equation (22):
\[(\exp(-b/L) + \exp(b/L) - 2) - b^2/L^2 = 0 \quad \text{or} \quad Sh^2 \frac{b}{2L} = \frac{b^2}{4L^2}. \]  

(23)

For \( z = \frac{b}{2L} \ll 1 \) \( Shz \approx z(1 + z^2/6), \) \( Sh^2z \approx z^2(1 + z^2/3) \) therefore, we have

\[ \frac{L^2}{L^2} \approx 1 + \frac{b^2}{12L^2}. \]  

(24)

Note that equation (21) for this case gives \( L^2 = \tilde{L}^2 \) for any cell size.

In the following sections of this paper, we will consider and compare the constructed finite-difference schemes with the traditional simplest finite-difference approximations used to solve the two-dimensional diffusion equation. Comparison is carried out on the example of one-velocity problems in homogeneous non-multiplying media for the case when the real density of the neutron flux behaves as \( A \exp(-x/L), \) where \( L \) - is the length of the neutron diffusion. Thus, the test problem is that we are trying to get the correct neutron distribution proportional in a certain area of space by solving the diffusion equation using a finite-difference method on a two-dimensional lattice that does not "successfully" take into account the one-dimensional nature of the neutron distribution (see figure 2).

**Figure 2.** the Finite difference grid is rotated by an angle \( \alpha \) relative to the \( x \text{-axis} \) (cell size by \( x = b \), cell size by \( y = a \)).

For generality, we use a rectangular lattice, not a square one.

4. **A simple finite-difference method for solving the diffusion equation**

Two-dimensional diffusion equation

\[ D \frac{d^2 \Phi(x,y)}{dx^2} + D \frac{d^2 \Phi(x,y)}{dy^2} - \Sigma \Phi(x,y) = 0 \]  

(25)

is approximated in a rectangular grid by the following equations
\[
\frac{\Phi_{i-1,j} - 2\Phi_{i,j} + \Phi_{i+1,j}}{a^2} + \frac{\Phi_{i,j-1} - 2\Phi_{i,j} + \Phi_{i,j+1}}{b^2} - \frac{\Phi_{i,j}}{L^2} = 0
\]  

(26)

The solution of a system of algebraic equations will probably give an approximate value of the diffusion length \( L \neq L_0 \). To find it, substitute an approximate solution \( \Phi_i = A \exp(-x/L) \) in (26). Denoting \( x_* = \frac{b \cos \alpha}{2L_0}, \ y_* = \frac{a \sin \alpha}{2L_0} \), we get:

\[
\frac{Sh^2(y_*)}{a^2} + \frac{Sh^2(x_*)}{b^2} = \frac{1}{4L_0^2}
\]

(27)

For small \( z \), \( Sh^2(z) \approx z^2 (1 + z^2 / 3) \). Substituting this decomposition in (27), we have

\[
\frac{x_*^2}{b^2} (1 + \frac{x_*^2}{3}) + \frac{y_*^2}{a^2} (1 + \frac{y_*^2}{3}) \approx \frac{1}{4L_0^2}
\]

or, opening designations for \( x_* \) и \( y_* \),

\[
\frac{L_0^2}{L^2} \approx 1 + \frac{b^2 \cos^4 \alpha}{12L_0^2} + \frac{a^2 \sin^4 \alpha}{12L_0^2}
\]

(28)

For \( \alpha = 0 \) or \( \alpha = \pi / 2 \) we have the ratio obtained (24) for a one-dimensional geometry.

5. **Surface Harmonics Method**

5.1 **Lowest approximation (one unknown per cell)**

In this case, we have equation (14), which, after using the solution type \( \Phi(x, y) = A \exp(-x/L) \) we write in a form similar to the equation (27):

\[
\frac{b^2 (\psi^s - \psi^r) - (\psi^r - \psi^s)}{8L_0^2} \frac{Sh^2 x_*}{b^2} + \frac{a^2 (\psi^r) - (\psi^r - \psi^s)}{8L_0^2} \frac{Sh^2 y_*}{a^2} = \frac{1}{4L_0^2}
\]

(29)

We see that in comparison with (27) in equation (29) Laplacians have correction factors\( \frac{b^2 (\psi^s - \psi^r)}{8L_0^2}, \frac{a^2 (\psi^r - \psi^s)}{8L_0^2} \). To calculate these multipliers, you will need an explicit form of trial functions. When using (20) and normalizing (on the edge) the incoming neutron currents at 1 n/sec, expressions can be written for the trial functions \( x = \frac{b}{2L}, \ y = \frac{a}{2L} \)

\[
\phi_x^s = \frac{Chx}{2yDShx}, \ \phi_y^s = \frac{Chy}{2xDShy}, \ \phi_x^r = \bar{\phi}_x^r = \frac{2}{ab\Sigma_a} = \frac{1}{2xyD}, \ \phi_y^r = \frac{Shx}{2yDChx}, \ \psi^s = \frac{Shx}{2xDChx}, \ \psi^r = \frac{Shy}{2xDChy}.
\]

(30)

Then, up to the quadratic terms
\[
\frac{b^2}{8L^2}(\psi^y - \psi^x) = \frac{xCh x}{2Sh x} + \frac{x^2}{2Sh x^2} \approx \frac{1+1/2}{2} + \frac{1}{2(1+x^2/6)} = 1
\]  

Similarly, \[
\frac{a^2}{8L^2}(\psi^y - \psi^x) \approx 1 \quad \text{(or } (\psi^y)^{-1}(\psi^x - \psi^y) \approx \frac{8L^2}{a^2} \text{) with accuracy up to quadratic terms. It turns out that the lowest SHM approximation, if applicable in the medium of a one-velocity diffusion approximation (and homogeneous cells), is approximately of the same order of accuracy as the standard finite-difference scheme (for small \( \frac{b^2}{L^2} \)).}

5.2 SHM equations with two unknowns per cell

After substituting in (12), (13) the form of the solution \( \Phi_{ij} = Ae^{\frac{x}{L}}, f_{ij} = Be^{\frac{x}{L}} \), we have:

\[
\left[ \frac{(\psi^y)^{-1}}{2} - \frac{r^y 4Sh^2x}{2} - \frac{r^y 4Sh^2y}{2} - \Sigma_{i,j} \right] A + \left[ \frac{(\psi^y)^{-1}}{2} - \frac{r^y 4Sh^2x}{2} - \frac{r^y 4Sh^2y}{2} - \sigma_{i,j} \right] B = 0 
\]

\[
\left[ \frac{(\psi^y)^{-1}}{2} - \frac{r^y 4Sh^2x}{2} - \frac{r^y 4Sh^2y}{2} \right] A + \left[ \frac{(\psi^y)^{-1}}{2} - \frac{r^y 4Sh^2x}{2} - \frac{r^y 4Sh^2y}{2} \sigma_{i,j} \right] B = 0 
\]

The coefficients for unknown \( A \) and \( B \) are obtained up to the second-order smallness terms (quadratic terms). Using that:

\[
(\psi^y)^{-1}(\psi^x - \psi^y) \approx \frac{2}{x^2}, \quad (\psi^y)^{-1}(\psi^x - \psi^y) \approx \frac{2}{y^2},
\]

\[
(\psi^y)^{-1} (\xi^y - \psi^y) \approx -\frac{2}{3} \left(1 + \frac{19x^2}{120} \right), \quad (\psi^y)^{-1} (\xi^y - \psi^y) \approx -\frac{2}{3} \left(1 + \frac{19y^2}{120} \right)
\]

equation (33) is converted to the form

\[
\left( \begin{array}{c}
\frac{4Sh^2x}{x^2} - \frac{4Sh^2y}{y^2} \\
\end{array} \right) A = \left( \begin{array}{c}
\frac{4}{3} Sh^2 x + \frac{4Sh^2y}{3} + 4 \\
\end{array} \right) B
\]

or:

\[
\frac{B}{A} \approx \frac{L^2 \cos^2 \alpha}{L^2} \left( 1 - \frac{y^2}{3} \right) - \frac{L^2 \sin^2 \alpha}{L^2} \left( 1 - \frac{x^2}{3} \right)
\]

Substituting (35) in equation (32), we get:

\[
\frac{L^2}{L^2} \approx 1 + \left( \frac{b^2}{12L^2} + \frac{a^2}{12L^2} \right) \cos^2 \alpha \sin^2 \alpha
\]

From (36) it can be seen that for \( \alpha = 0 \) and \( \alpha = \pi / 2 \) the solution of the system of equations SHM (12), (13) gives the correct solution \( L^2 = L^2 \) regardless of the selected grid sizes on the \( x \) and \( y \) axes. For intermediate lattice positions, the result is still better than in the lowest approximation of the SHM. This result can be shown graphically if you set a certain ratio between the dimensions \( a \) and \( b \) (for example, \( a = b \) or \( a = b/2 \) and \( a = 2b \)). Then you can draw (see Figure 3) the resulting coefficient before
in the expressions (36) (SHM equations with two unknowns per cell) and (28) (the lowest approximation of the SHM, close to the standard finite difference scheme for small \( \frac{b^2}{L^2} \)).

![Graph](image.png)

**Figure 3.** Coefficient \( k \) in the ratio \( \frac{L^2}{L^2} \approx 1 + k \frac{b^2}{L^2} \) for finite-difference equations.

This coefficient is shown in Figure 3 for different ratios \( a \) and \( b \) (the coefficient for \( b^2/L^2 \) is shown).

6. Conclusion
From the figure, we see that the best result with the correct limit transition to one-dimensional geometry is shown by the SHM equations with two unknowns per cell.

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