Absence of Landau’s Diamagnetism in Two Dimensions

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Abstract. – The statistical weight \(W\) as a function of energy \(E\) for quasielectrons with mass \(m^*\) subject to a fixed magnetic field \(B\) is

\[
W/\text{A} = \left(\frac{m^*}{\pi \hbar^2}\right) E + \left(\frac{eB}{\pi^2 \hbar}\right) \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \cdot \sin \left(\frac{2\pi \nu E}{\hbar \omega_c}\right),
\]

where \(A\) is the sample area, and \(\omega_c \equiv eB/m^*\) the cyclotron frequency. Significantly, there is no Landau’s term proportional to \(B^2\) in 2D. This leads to the conclusion that the 2D electron system is always paramagnetic, but shows a magnetic oscillation.

Landau [1], Sondheimer and Wilson [2,3] discussed the de Haas-van Alphen (dHvA) oscillation [4] of a three-dimensional (3D) system of quasifree electrons. Nakamura [5] calculated the statistical weight \(W\) associated with the Landau states, and treated the dHvA oscillation. We extend Nakamura’s theory to a 2D system.

Let us consider a dilute system of electrons, each with effective mass \(m^*\), moving in a plane. Applying a magnetic field \(B\) perpendicular to the plane, each electron will be in a Landau state of energy

\[
E = (N_L + 1/2) \hbar \omega_c, \quad N_L = 0, 1, \cdots,
\]

where \(\omega_c = eB/m^*\) is the cyclotron frequency. The degeneracy of the Landau level (LL) is

\[
\frac{eBA}{2\pi \hbar}, \quad A = \text{sample area}.
\]

We introduce kinetic momenta

\[
\Pi_x = p_x + eA_x, \quad \Pi_y = p_y + eA_y,
\]

in terms of which the Hamiltonian \(\mathcal{H}\) for the electron is

\[
\mathcal{H} = \frac{1}{2m^*}(\Pi_x^2 + \Pi_y^2) = \frac{1}{2m^*} \Pi^2.
\]
After simple calculations, we obtain
\[ dx \, d\Pi_x \, dy \, d\Pi_y = dx \, dp_x \, dy \, dp_y. \]

We can then represent quantum states by quasi phase-space elements \( dx \, d\Pi_x \, dy \, d\Pi_y \). The Hamiltonian \( \mathcal{H} \) in eq. (3) does not depend on the position \((x, y)\). Assuming large normalization lengths \((L_1, L_2)\), we can represent the Landau states by concentric shells of the phase space having statistical weight \( 2\pi \Pi \Delta \Pi \cdot L_1 L_2 (2\pi \hbar)^{-2} = eBA/2\pi \hbar \), with \( A = L_1 L_2 \) and \( \hbar \omega_c = \Delta (\Pi^2/2m^*) = \Pi \Delta \Pi/m^* \). Hence, the LL degeneracy is given by eq. (2). Figure 1 represents a typical Landau state in the \( \Pi_x-\Pi_y \) space.

As the field \( B \) is raised the separation \( \hbar \omega_c \) increases, and the quantum states are collected (or bunched) together. As a result of bunching, the density of states \( N(\varepsilon) \) should change periodically since the Landau levels are spaced equally.

The electrons obey the Fermi-Dirac statistics. Considering a system of quasifree electrons, we define the Helmholtz free energy \( \mathcal{F} \) by
\[ \mathcal{F} = N\mu - 2k_B T \sum_i \ln \left[ 1 + e^{(\mu-E_i)/k_B T} \right], \]
where \( \mu \) is the chemical potential, and the factor 2 arises from spin degeneracy. The chemical potential \( \mu \) is determined from the condition
\[ \frac{\partial \mathcal{F}}{\partial \mu} = 0. \]

The magnetization \( \mathcal{M} \) for the system can be found from
\[ \mathcal{M} = -\frac{1}{A} \frac{\partial \mathcal{F}}{\partial B}. \]
Equation (5) is equivalent to the usual condition that the number of electrons $N$ can be obtained in terms of the Fermi distribution function $F$:

$$N = 2 \sum_i F(E_i), \quad F(E) \equiv \left[ e^{\beta(E-\mu)} + 1 \right]^{-1}. \quad (7)$$

The LL $E_i$ is characterized by the Landau oscillator quantum number $(N_L)_i$.

Let us introduce the density of states $dW/dE \equiv N(E)$ such that $N(E)dE =$ number of states having energy between $E$ and $E + dE$. We write eq. (4) in the form

$$F = N\mu - 2k_B T \cdot \int_0^\infty dE dW \ln \left[ 1 + e^{(\mu - E)/k_B T} \right] = N\mu - 2 \int_0^\infty dEW(E)F(E). \quad (8)$$

The statistical weight $W$ is the total number of states having energies less than the Landau energy $(N_L + 1/2) \hbar \omega_c$ in eq. (4). This $W$ from fig. 1 is

$$W = \frac{L_1L_2}{(2\pi \hbar)^2} 2\pi \Delta \Pi \cdot 2 \sum_{N_L=0}^\infty \Theta [E - (N_L + 1/2) \hbar \omega_c], \quad (9)$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}. \quad (10)$$

We introduce a dimensionless variable $\varepsilon \equiv 2\pi E/\hbar \omega_c$, and rewrite eq. (9) as

$$W(E) = C\hbar \omega_c \cdot 2 \sum_{N_L=0}^\infty \Theta[\varepsilon - (2N_L + 1)\pi], \quad (10)$$

with $C = 2\pi m^* A(2\pi \hbar)^{-2}$. We assume a high Fermi-degeneracy such that $\mu \simeq \varepsilon_F \gg \hbar \omega_c$. The sum in eq. (10) can be computed using Poisson’s summation formula [6]

$$\sum_{n=-\infty}^\infty f(2\pi n) = \frac{1}{2\pi} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty d\tau f(\tau)e^{-im\tau}, \quad (11)$$

where $\sum_{n=-\infty}^\infty f(2\pi n + t), \ 0 \leq t < 2\pi$ is by assumption uniformly convergent. We write the sum in eq. (10) as

$$2 \sum_{n=0}^\infty \Theta[\varepsilon - (2n + 1)\pi] = \Theta(\varepsilon - \pi) + \phi(\varepsilon; 0), \quad (12)$$

$$\phi(\varepsilon; x) = \sum_{n=-\infty}^\infty \Theta(\varepsilon - \pi - 2\pi|n + x|).$$

Note that $\phi(\varepsilon; x)$ is periodic in $x$. After the Fourier expansion, we set $x = 0$ to obtain eq. (12). By taking the real part (Re) of eq. (12) and using eq. (11), we obtain

$$\text{Re} \{ \text{Eq. (12)} \} = \frac{1}{\pi} \int_0^\infty d\tau \Theta(\varepsilon - \tau) + \frac{2}{\pi} \sum_{\nu=1}^\infty (-1)^\nu \int_0^\infty d\tau \Theta(\varepsilon - \tau) \cos \nu \tau, \quad (13)$$
where we assumed $\varepsilon \equiv 2\pi E/\hbar \omega_c \gg 1$, and neglected $\pi$ against $\varepsilon$. The integral in the first term in eq. (13) yields $\varepsilon$. The integral in the second term can be evaluated by integration by parts, and using $d\Theta/dy = \delta(y)$. We obtain

$$\int_0^\infty d\tau \Theta(\varepsilon - \tau) \cos \nu \tau = \frac{1}{\nu} \sin \nu \varepsilon.$$

Hence,

$$\text{Re}\{ \text{Eq. (12)} \} = \frac{1}{\pi} \varepsilon + \frac{2}{\pi} \sum_{\nu=1}^\infty (-1)^\nu \frac{\varepsilon}{\nu} \sin \nu \varepsilon. \quad (14)$$

Using eqs. (10) and (14), we obtain

$$W(E) = W_0 + W_{osc}, \quad (15)$$

$$W_0 = C \hbar \omega_c \frac{\varepsilon}{\pi} = \frac{A m^* E}{\pi \hbar^2}, \quad (16)$$

$$W_{osc} = C \hbar \omega_c \frac{2}{\pi} \sum_{\nu=1}^\infty (-1)^\nu \frac{\varepsilon}{\nu} \sin \left( \frac{2\pi \nu E}{\hbar \omega_c} \right). \quad (17)$$

The oscillatory term $W_{osc}$ contains an infinite sum with respect to $\nu$, but only the first term, $\nu = 1$, is important in practice as we see later. This term $W_{osc}$ can generate magnetic oscillations. There is no term proportional to $B^2$ generating the Landau diamagnetism. This is unexpected. We briefly discuss the difference between 2D and 3D systems.

In 3D, the LL $E$ is given by

$$E = \left( N_L + \frac{1}{2} \right) \hbar \omega_c + \frac{p_z^2}{2m^*}. \quad (18)$$

The energy $E$ is continuous in the bulk limit. The statistical weight $W'$ is the total number of states having energies less than $E$. The allowed values of $p_z$ are distributed over the range in which $|p_z|$ does not exceed $\sqrt{2m^*\hbar^2/\varepsilon - (2N_L + 1)} \frac{\pi}{2}$. We obtain

$$W'(E) = C' \left( \frac{\hbar \omega_c}{\pi} \right)^{3/2} \left( \frac{2}{\sqrt{2\pi}} \right) \sum_{N_L=0}^\infty \sqrt{\varepsilon - (2N_L + 1) \frac{\pi}{2}}, \quad (19)$$

where $C' \equiv V (2\pi m^*)^{3/2}/(2\pi \hbar)^3$, $\varepsilon \equiv 2\pi E/\hbar \omega_c$, and $V \equiv L_1 L_2 L_3$ is the sample volume. We proceed similarly, and obtain

$$W'(E) = W'_0 + W'_L + W'_{osc}, \quad (19)$$

$$W'_0 = C' \frac{4}{3\sqrt{\pi}} \frac{E^{3/2}}{\varepsilon^{3/2}}, \quad (20)$$

$$W'_L = -C' \frac{1}{24 \sqrt{\pi}} \frac{E^{1/2}}{(\hbar \omega_c)^2}, \quad (21)$$

$$W'_{osc} = C' \frac{1}{\sqrt{2}} \left( \frac{\hbar \omega_c}{\pi} \right)^{3/2} \sum_{\nu=1}^\infty \frac{(-1)^\nu}{\nu^{3/2}} \sin \left( \frac{2\pi \nu E}{\hbar \omega_c} - \frac{\pi}{4} \right). \quad (22)$$
In detail, we write the sum in eq. (18) as

\[ 2 \sum_{n=0}^{\infty} \sqrt{\varepsilon - (2n + 1)\pi} = (\varepsilon - \pi)^{1/2} + \psi(\varepsilon; 0), \]

\[ \psi(\varepsilon; x) \equiv \sum_{n=-\infty}^{\infty} (\varepsilon - \pi - 2\pi |n + x|)^{1/2}. \]

Since \( \psi(\varepsilon; x) \) is periodic in \( x \), we can use it for the Fourier expansion of eq. (23), and then set \( x = 0 \). By taking the real part (Re) of eq. (23) and using eq. (11), we obtain

\[ \text{Re} \{ \text{Eq. (23)} \} = \frac{1}{\pi} \int_{0}^{\varepsilon} d\tau (\varepsilon - \tau)^{1/2} + \frac{2}{\pi} \sum_{\nu=1}^{\infty} (-1)^{\nu} \int_{0}^{\varepsilon} d\tau (\varepsilon - \tau)^{1/2} \cos \nu \tau, \]

where we neglected \( \pi \) against \( \varepsilon \). The integral in the first term in eq. (24) yields \((2/3)\varepsilon^{3/2}\), leading to \( W_0' \) in eq. (20). The integral in the second term can be written after integrating by part, and changing the variable \((\nu\varepsilon - \nu\tau = t)\) as

\[ \frac{1}{2\nu^{3/2}} \left[ \sin \nu \varepsilon \int_{0}^{\nu \varepsilon} dt \frac{\cos t}{\sqrt{t}} - \cos \nu \varepsilon \int_{0}^{\nu \varepsilon} dt \frac{\sin t}{\sqrt{t}} \right]. \]

We now use asymptotic expansions for \( \nu \varepsilon = x \gg 1 \):

\[ \int_{0}^{x} dt \frac{\sin t}{\sqrt{t}} \sim \sqrt{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} - \ldots \]

\[ \int_{0}^{x} dt \frac{\cos t}{\sqrt{t}} \sim \sqrt{\frac{\pi}{2}} \frac{\sin x}{\sqrt{x}} - \ldots. \]

The second terms in the expansion lead to \( W_L' \) in eq. (21), where we used the identity

\[ \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu^2} = \frac{\pi^2}{12}. \]

The first terms lead to the oscillatory term \( W_{osc}' \) in eq. (22).

The term \( W_0' \), which is independent of \( B \), gives the weight equal to that for a free-electron system with no field. The term \( W_L' \) proportional to \( B^2 \) is negative (diamagnetic), and can generate a Landau diamagnetic moment.

The energy \( E \) of the 3D system is continuous, and hence the system is manageable or soft. In contrast, the energy \( E \) of the 2D system is discrete, and hence the system is less manageable. This explains the absence of Landau diamagnetism for the 2D system.

The statistical weight \( W_{osc} \) in eq. (17) has a sine term. Hence, the density of states, \( N = dW/dE \), has an oscillatory part of the form

\[ \sin \left( \frac{2\pi E}{\hbar \omega_c} \right), \quad E \equiv \frac{\Pi^2}{2m^*}. \]

If the density of states oscillates violently in the drop of the Fermi distribution function:

\[ F(E) \equiv [e^{\beta(E-\mu)} + 1]^{-1}, \]

the delta-function replacement formula

\[ -\frac{dF}{dE} = \delta(E - \mu), \]
cannot be used. The width of $-dF/dE$ is of the order $k_B T$. The critical temperature $T_c$ below which oscillations can be observed is $k_B T_c \sim \hbar \omega_c$. For $T < T_c$, we may proceed as follows. Let us consider the integral

$$I = \int_0^\infty dE \frac{F(E) \sin (2\pi E/\hbar \omega_c)}{e^{\beta (E-\mu)} + 1}.$$ 

We introduce a new variable $\zeta \equiv \beta (E-\mu)$, and extend the lower limit to $-\infty$ (low-temperature limit) so that

$$\int_0^\infty dE \cdots \frac{1}{e^{\beta (E-\mu)} + 1} = \frac{1}{\beta} \int_{-\infty}^\infty d\zeta \cdots \frac{1}{e^{\beta \zeta} + 1}.$$ 

With the help of the standard integral

$$\int_{-\infty}^\infty d\zeta \frac{e^{ia\zeta}}{e^{\zeta} + 1} = \frac{\pi}{a \sinh a},$$

we obtain

$$I = -\frac{\pi k_B T \cos (2\pi \varepsilon_F/\hbar \omega_c)}{\sinh (2\pi^2 k_B T m^*/\hbar e B)}.$$ 

(25)

For very low fields the oscillation number in the range $k_B T$ becomes great, and hence the sinusoidal contribution must cancel out. This effect is represented by the factor

$$[\sinh (2\pi^2 k_B T m^*/\hbar e B)]^{-1}.$$ 

We calculate the free energy indicated in eq. (8) using the statistical weight $W$ in eq. (15), and obtain

$$F = N\mu - 2A \frac{m^*}{\pi \hbar^2} \varepsilon_F + 2A \frac{e B}{\pi \hbar} k_B T \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu} \frac{\cos (2\pi \nu \varepsilon_F/\hbar \omega_c)}{\sinh (2\pi^2 \nu k_B T m^*/\hbar e B)}.$$ 

(26)

where we used the integration formula (25), and took the low-temperature limit except for the oscillatory terms. The magnetization $M$ can be obtained using eq. (26).

So far, we have not considered the Pauli spin magnetization [7]

$$M_{\text{Pauli}} = 2\mu_B^2 B N_0 / A = 2n\mu_B^2 B / \varepsilon_F,$$

with $\mu_B$ (Bohr magneton) $= e\hbar/2m$ and $n$ (electron number density) $= m^* \varepsilon_F / \pi \hbar^2$. Using eqs. (6), (7) and (26), we obtain the total magnetization $M_{\text{tot}} = M_{\text{Pauli}} + M_{\text{osc}}$:

$$M_{\text{tot}} = 2n\mu_B \frac{\mu_B^2 B}{\varepsilon_F} \left[ 1 + \frac{k_B T \mu_B}{m^* \sinh (2\pi^2 \mu_B T m^*/\hbar e B)} \right].$$ 

(27)

The $B$-dependence of $F$ is contained in the last term in eq. (26). The linear $B$-dependence of the multiplication factor is much stronger than the $B$-dependence of the alternating series. Therefore, the contribution from the $B$-derivative of the series is neglected. The variation of the statistical weight $W$ is periodic in $B^{-1}$, but it is far from sinusoidal. Only the first oscillatory term, $\nu = 1$, is important and kept in eq. (27) since $\sinh (2\pi^2 \mu_B T m^*/\hbar e B) \gg 1$.

The width of $dF/dE$ is finite for a finite $T$. In this $E$-range, many oscillations can occur if the field $B$ is made low. We assumed this condition to obtain eq. (27). The magnetic susceptibility $\chi$ is defined by the ratio $\chi = M/B$.

In conclusion, the 2D system is intrinsically paramagnetic since the Landau’s diamagnetic term is absent, but the system exhibits a dHvA oscillation.
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