ON $M$–TERMS APPROXIMATIONS BESOV CLASSES IN LORENTZ SPACES

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Abstract. In this paper we consider Lorentz space with a mixed norm of periodic functions of many variables. We obtain the exact estimation of the best $M$-term approximations of Nikol’ski’s, Besov’s classes in the Lorentz space with the mixed norm.

Keywords: Lorentz space, and Besov’s class, and approximation
MSC: 41A10 and 41A25

1. Introduction

Let $\bar{x} = (x_1, ..., x_m) \in \mathbb{T}^m = [0, 2\pi]^m$ and $\theta_j, p_j \in [1, +\infty)$, $j = 1, ..., m$. Let $L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m)$ denotes the space of Lebesgue measurable functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have $2\pi$–period with respect to each variable such that

$$\|f\|_{\bar{p}, \bar{\theta}} = \|...\|_{p_1, \theta_1} ... \|_{p_m, \theta_m} < +\infty,$$

where

$$\|g\|_{p, \theta} = \left\{ \frac{2\pi}{\int_0^{2\pi} (g^*(t))^{\theta} t_k^{\theta_k - 1} dt} \right\}^\frac{1}{\theta},$$

where $g^*$ a non-increasing rearrangement of the function $|g|$ (see. [1]).

It is known that if $\theta_j = p_j$, $j = 1, ..., m$, then $L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m) = L_{\bar{p}}(\mathbb{T}^m)$ the Lebesgue measurable space of functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have $2\pi$–period with respect to each variable with the norm

$$\|f\|_{\bar{p}} = \left[ \int_0^{2\pi} \cdots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{\frac{1}{p_1}} \cdots \right]^{\frac{1}{p_m - 1}} \int_0^{2\pi} dx_m < +\infty,$$

where $\bar{p} = (p_1, ..., p_m)$, $1 \leq p_j < +\infty$, $j = 1, ..., m$ (see [2],p. 128).

Any function $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$ can be expanded to the Fourier series

$$\sum_{\rho \in \mathbb{Z}^m} a_\rho(f) e^{i\langle \rho, \bar{x} \rangle},$$

where $a_\rho(f)$ Fourier coefficients of $f \in L_1(\mathbb{T}^m)$ with respect to multiple trigonometric system $\{e^{i\langle \rho, \bar{x} \rangle}\}_{\rho \in \mathbb{Z}^m}$, and $\mathbb{Z}^m$ is the space of points in $\mathbb{R}^m$ with integer coordinates.

For a function $f \in L(\mathbb{T}^m)$ and a number $s \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ let us introduce the notation

$$\delta_0(f, \bar{x}) = a_0(f), \quad \delta_s(f, \bar{x}) = \sum_{\rho \in \rho(s)} a_\rho(f) e^{i\langle \rho, \bar{x} \rangle},$$

where $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$,

$$\rho(s) = \left\{ \bar{k} = (k_1, ..., k_m) \in \mathbb{Z}^m : \left[ 2^{s-1} \right] \leq \max_{j=1,...,m} |k_j| < 2^s \right\},$$

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where \([a]\) is the integer part of the number \(a\).

Let us consider Nikol’skii, Besov classes (see [2], [3]). Let \(1 < p_j < +\infty, 1 < \theta_j < +\infty, j = 1, ..., m, 1 \leq \tau \leq \infty, \) and \(r > 0\)

\[
H_{p,\theta}^r = \left\{ f \in L_{p,\theta}(\mathbb{T}^m) : \sup_{s \in \mathbb{Z}_+} 2^{sr} \| \delta_s(f) \|_{p,\theta} \leq 1 \right\}.
\]

\[
B_{p,\theta,\tau}^r = \left\{ f \in L_{p,\theta}(\mathbb{T}^m) : \left( \sum_{s \in \mathbb{Z}_+} 2^{s\tau r} \| \delta_s(f) \|_{p,\theta}^r \right)^{\frac{1}{r}} \leq 1 \right\}.
\]

It is known that for \(1 \leq \tau \leq \infty\) the following holds

\[ B_{p,\theta,1}^r \subset B_{p,\theta,\tau}^r \subset B_{p,\theta,\infty}^r = H_{p,\theta}^r. \]

Let \(f \in L_{p,\theta}(\mathbb{T}^m)\) and \(\tilde{k}^{(j)}_{j=1}^M\) be a system of vectors \(\tilde{k}^{(j)} = (k_1^{(j)}, ..., k_m^{(j)})\) with integer coordinates. Consider the quantity

\[
e_M(f)_{p,\theta} = \inf_{\tilde{k}^{(j)}_{j=1}^M} \left\| f - \sum_{j=1}^M b_j e(\tilde{k}^{(j)}_{j=1}^M) \right\|_{p,\theta},
\]

where \(b_j\) are arbitrary numbers. The quantity \(e_M(f)_{p,\theta}\) is called the best \(M - \)term approximation of a function \(f \in L_{p,\theta}(\mathbb{T}^m)\). For a given class \(F \subset L_{p,\theta}(\mathbb{T}^m)\) let

\[
e_M(F)_{p,\theta} = \sup_{f \in F} e_M(f)_{p,\theta}.
\]

The best \(M - \)term approximation was defined by S.B.S. Stechkin [4]. Estimations of \(M - \)term approximations of different classes were provided by R.S. Ismagilov [5], E.S. Belinsky [6], V.E. Maiorov [7], B.S. Kashin [8], R. DeVore [9], V.N. Temlyakov [10], A.S. Romanyuk [11], Dinh Dung [12], Wang Heping and Sun Yongsheng [13], L. Q. Duan and V. E. Maiorov [7], B. S. Kashin [8], R. DeVore [9], V. N. Temlyakov [10], A. S. Romanyuk [11], Dinh Dung [12], Wang Heping and Sun Yongsheng [13], L. Q. Duan and G. S. Fang [14], W. Sickel and M. Hansen [15], S. A. Stasyuk [16], [17] and others (see bibliography in [18], [19], [20]).

For the case \(p_1 = ... = p_m = p\) and \(q_1 = ... = q_m = \theta_1 = ... = \theta_1 = q\) R.A. DeVore and V.N. Temlyakov [20] proved the following theorem.

**Theorem A.** Let \(1 \leq p, q, \tau \leq \infty\) and \(r(p, q) = m\left(\frac{1}{p} - \frac{1}{q}\right)_+\) if \(1 \leq p \leq q \leq 2, \) or \(1 \leq q \leq p < \infty\) and \(r(p, q) = \max\left\{\frac{m}{p}, \frac{m}{\tau}\right\}\) in other cases. Then for \(r > r(p, q)\) the following holds

\[
e_M(B_{p,\tau}^r) \lesssim M^{-\frac{a_+}{r}} + \left(\frac{1}{r} - \max\left\{\frac{1}{p}, \frac{1}{\tau}\right\}\right)_+,
\]

where \(a_+ = \max\{a; 0\}\).

Moreover, in the case of \(m\left(\frac{1}{p} - \frac{1}{q}\right)_+ < r < \frac{m}{p}, 1 < p < 2 < q < \infty\) S.A. Stasyuk [16] proved \(e_M(B_{p,\tau}^r) \lesssim M^{-\frac{a_+}{r}} + \left(\frac{1}{r} - \max\left\{\frac{1}{p}, \frac{1}{\tau}\right\}\right)_+\). In the case \(r = \frac{m}{p}\) obtained \(e_M(B_{p,\tau}^r) \asymp M^{-\frac{a_+}{r}}(\log M)^{1 - \frac{a_+}{r}}\) (see [17]).

The main goal of the present paper is to find the order of the quantity \(e_M(F)_{q,\theta}\) for the class \(F = B_{p,\theta,\tau}^r\).

The notation \(A(y) \asymp B(y)\) means that there exist positive constants \(C_1, C_2\) such that \(C_1 A(y) \leq B(y) \leq C_2 A(y)\). If \(A \leq C_2 B\) or \(A \geq C_2 B\), then we write \(A \ll B\) or \(A \gg B\).
2. Auxiliary results

To prove the main results the following auxiliary propositions are used.

**Theorem B ([21])**. Let \( p \in (1, \infty) \). Then there exist positive numbers \( C_1(p), C_2(p) \) such that for any function \( f \in L_p(T^n) \) the following inequality holds:

\[
\|f\|_p < \left\| \left( \sum_{s=0}^\infty |\delta_s(f)|^2 \right)^{\frac{1}{2}} \right\|_p < \|f\|_p.
\]

**Theorem C ([22])**. Let \( \tilde{n} = (n_1, \ldots, n_m) \), \( n_j \in \mathbb{N}, j = 1, \ldots, m \) and

\[
T_{\tilde{n}}(\bar{x}) = \sum_{|\kappa| \leq \tilde{n}, j=1,\ldots,m} c_{\kappa} e^{i\langle \kappa, \bar{x} \rangle}.
\]

Then for \( 1 \leq p_j < q_j < \infty, 1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty, j = 1, \ldots, m \) the following inequality holds

\[
\|T_{\tilde{n}}\|_{\bar{q},\theta^{(2)}} < \left( \prod_{j=1}^m n_j^{\frac{1}{p_j} - \frac{1}{q_j}} \right) \|T_{\tilde{n}}\|_{\tilde{p},\bar{\theta}^{(1)}}.
\]

Let \( \Omega_M \) be a set containing no more than \( M \) vectors \( \bar{k} = (k_1, \ldots, k_m) \) with integer coordinates, and \( P(\Omega_M, \bar{x}) \) be any trigonometric polynomial, which consists of harmonics with “indices” in \( \Omega_M \).

**Lemma 1** (see [18]). Let \( 2 < q_j < +\infty, j = 1, \ldots, m \). Then for any trigonometric polynomial \( P(\Omega_N) \) and for any natural number \( M < N \) there exists trigonometric polynomial \( P(\Omega_M) \), such that the following estimation holds

\[
\|P(\Omega_N) - P(\Omega_M)\|_q < \left( NM^{-1} \right)^{\frac{1}{2}} \|P(\Omega_N)\|_2,
\]

and, moreover, \( \Omega_M \subset \Omega_N \).

3. Main results

Let us prove the main results.

**Theorem 1.**. Let \( \bar{p} = (p_1, \ldots, p_m), \bar{q} = (q_1, \ldots, q_m), \bar{\theta}^{(1)} = (\theta_1^{(1)}, \ldots, \theta_m^{(1)}), \bar{\theta}^{(2)} = (\theta_1^{(2)}, \ldots, \theta_m^{(2)}), 1 < p_j < q_j < \infty, 1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty, j = 1, \ldots, m, 1 \leq \tau \leq \infty \).

1. If \( \sum_{j=1}^m \left( \frac{1}{p_j} - \frac{1}{q_j} \right) < r < \sum_{j=1}^m \frac{1}{p_j} \), then

\[
e_M \left( B^r_{\bar{p},\bar{\theta}^{(1)},\tau} \right)_{\bar{q},\bar{\theta}^{(2)}} \asymp M^{-(2 \sum_{j=1}^m \frac{1}{q_j} - 1) \left( r - \sum_{j=1}^m \left( \frac{1}{p_j} - \frac{1}{q_j} \right) \right)}.
\]

2. If \( r = \sum_{j=1}^m \frac{1}{p_j} \), then

\[
e_M \left( B^r_{\bar{p},\bar{\theta}^{(1)},\tau} \right)_{\bar{q},\bar{\theta}^{(2)}} \asymp M^{-\frac{1}{2} (\log_2 M)^{1 - \frac{1}{2}}},
\]

for \( M > 1 \).

3. If \( r > \sum_{j=1}^m \frac{1}{p_j} \), then

\[
e_M \left( B^r_{\bar{p},\bar{\theta}^{(1)},\tau} \right)_{\bar{q},\bar{\theta}^{(2)}} \asymp M^{-\frac{1}{2} \left( r + \sum_{j=1}^m \left( \frac{1}{2} - \frac{1}{p_j} \right) \right)}.
\]
**Proof.** Firstly, we are going to consider the upper bound in the first item. Taking into account the inclusion $B^r_{\bar{p},\bar{\theta}(1)} \subset H^r_{\bar{p},\bar{\theta}(1)}$, $1 \leq \tau < +\infty$, it suffices to prove it for the class $H^r_{\bar{p},\bar{\theta}(1)}$.

Let $1 < p_j \leq 2 < q_j < \infty$, $j = 1, \ldots, m$, and $N$ be the set of natural numbers. For a number $M \in N$ choose a natural number $n$ such that $2^nm < M \leq 2^{(n+1)m}$. For a function $f \in H^r_{\bar{p},\bar{\theta}(1)}$, it is known that

$$
\|\delta_s(f)\|_{\bar{p},\bar{\theta}(1)} \leq 2^{-sr}, \quad 1 < p_j < \infty, \quad j = 1, \ldots, m.
$$

We will seek an approximation polynomial $P(\Omega_M, \bar{x})$ in the form

$$
P(\Omega_M, \bar{x}) = \sum_{s=0}^{n-1} \delta_s(f, \bar{x}) + \sum_{n \leq s < \alpha n} P(\Omega_n, \bar{x}),
$$

(1)

where the polynomials $P(\Omega_n, \bar{x})$ will be constructed for each $\delta_s(f, \bar{x})$ in accordance with Lemma 1, and the number $\alpha > 1$ will be chosen during the construction.

Let $\sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^{m} \frac{1}{p_j}$. Suppose

$$
N_s = \left[ 2^{nm} 2^{s(\frac{1}{j_1} - \frac{1}{q_j})} 2^{\alpha n (\frac{1}{j_1} - \frac{1}{q_j})} + 1 \right],
$$

where $[y]$ integer part of the number $y$.

Now we are going to show that polynomials (1) have no more than $M$ harmonics (in terms of order). By definition of the number $N_s$, we have

$$
\sum_{s=0}^{n-1} 2\{k = (k_1, \ldots, k_m) : [2^{s-1}] \leq \max_{j=1,\ldots,m} |k_j| < 2^s \} + \sum_{n \leq s < \alpha n} N_s \leq C2^{nm} + \sum_{n \leq s < \alpha n} \left( 2^{nm} 2^{s(\frac{1}{j_1} - \frac{1}{q_j})} 2^{\alpha n (\frac{1}{j_1} - \frac{1}{q_j})} + 1 \right) \ll 2^{nm} + (\alpha - 1)n \ll 2^{nm} \times M,
$$

where $\sharp A$ denotes the number of elements of the set $A$.

Next, by the property of the norm we have

$$
\|f - P(\Omega_M)\|_{\bar{q},\bar{\theta}(2)} \leq \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_n)) \right\|_{\bar{q},\bar{\theta}(2)} + \left\| \sum_{n \leq s < \alpha n} \delta_s(f) \right\|_{\bar{q},\bar{\theta}(2)} = J_1(n) + J_2(n).
$$

(2)

Let us estimate $J_2(n)$. Applying the inequality of different metrics for trigonometric polynomials (see Theorem C), we can obtained

$$
J_2(n) \leq \sum_{n \leq s < \alpha n} \|\delta_s(f)\|_{\bar{q},\bar{\theta}(2)} \ll \sum_{n \leq s < \alpha n} 2^{s(\frac{1}{j_1} - \frac{1}{q_j})} \|\delta_s(f)\|_{\bar{p},\bar{\theta}(1)}.
$$

Therefore, taking into account $f \in H^r_{\bar{p},\bar{\theta}(1)}$ and $\sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{q_j}) < r$, we get

$$
J_2(n) \ll \sum_{n \leq s < \alpha n} 2^{s(r - \frac{m}{j_1} - \frac{1}{q_j})} \ll 2^{\alpha n (r - \frac{m}{j_1} - \frac{1}{q_j})}.
$$

(3)
Let us estimate $J_1(n)$. Using the property of the quasi-norm, Lemma 1 and the inequality of different metrics (see Theorem C), we get

$$J_1(n) = \left\| \sum_{n \leq s < \alpha n} \left( \delta_s(f) - P(\Omega_N) \right) \right\|_{q(\bar{g})^{(2)}} << \sum_{n \leq s < \alpha n} \left\| \delta_s(f) - P(\Omega_N) \right\|_{q(\bar{g})^{(2)}} << \sum_{n \leq s < \alpha n} (N_s^{\frac{1}{2s}}) \left\| \delta_s(f) \right\|_2 << \sum_{n \leq s < \alpha n} (N_s^{\frac{1}{2s}})^{\frac{1}{2}} \left\| \delta_s(f) \right\|_{\bar{g}^2(1)} << \sum_{n \leq s < \alpha n} N_s^{-\frac{s}{2}} \frac{1}{2} \sum_{j=1}^{m} \frac{1}{r_j} 2^{-sr} << 2^{-\alpha m} \frac{1}{2} \sum_{j=1}^{m} \frac{1}{r_j} \frac{1}{2} \sum_{n \leq s < \alpha n} \left( \frac{1}{r_j} - \frac{1}{q_j} \right) \sum_{j=1}^{m} \frac{1}{r_j} 2^{-sr} \equiv 2^{-\alpha m} \frac{1}{2} \sum_{j=1}^{m} \frac{1}{r_j} \frac{1}{2} \sum_{n \leq s < \alpha n} \left( \frac{1}{r_j} - \frac{1}{q_j} \right) \sum_{j=1}^{m} \frac{1}{r_j} 2^{-sr}.$$  

(4)

Suppose $\alpha = m(2 \sum \frac{1}{q_j})^{-1}$. Then from the inequality (4), we get

$$J_1(n) \leq C2^{-\alpha m} \left( \sum \frac{1}{q_j} \right)^{-1} (r - \sum \frac{1}{r_j} - \sum \frac{1}{q_j}) \leq M^{-2} \left( \sum \frac{1}{q_j} \right)^{-1} (r - \sum \frac{1}{r_j} - \sum \frac{1}{q_j}).$$

(5)

For $\alpha = m(2 \sum \frac{1}{q_j})^{-1}$, using the inequality (3) and taking into account $2^{nm} \propto M$ we obtain

$$J_2(n) \equiv M^{-2} \left( \sum \frac{1}{q_j} \right)^{-1} (r - \sum \frac{1}{r_j} - \sum \frac{1}{q_j}).$$

(6)

By (5) and (6), we get from the inequality (2) the following

$$\| f - P(\Omega_M) \|_{q(\bar{g})^{(2)}} << M^{-2} \left( \sum \frac{1}{q_j} \right)^{-1} (r - \sum \frac{1}{r_j} - \sum \frac{1}{q_j})$$

for any function $f \in H^{r}_{\bar{g}^{2}(1)}$ in the case of $\sum \frac{1}{r_j} - \sum \frac{1}{q_j} < r < \sum \frac{1}{r_j}$.

From the inclusion $B^{r}_{\bar{g}^{2}(1)} \subset H^{r}_{\bar{g}^{2}(1)}$ and the definition of $M$ – term approximation, it follows that

$$e_M \left( B^{r}_{\bar{g}^{2}(1)} \right) \equiv M^{-2} \left( \sum \frac{1}{q_j} \right)^{-1} (r - \sum \frac{1}{r_j} - \sum \frac{1}{q_j})$$

in the case of $\sum \frac{1}{r_j} - \sum \frac{1}{q_j} < r < \sum \frac{1}{r_j}$.

Let us consider the lower bound. We will use the well-known formula ([23], p.25)

$$e_M(f)_{q(\bar{g})^{(2)}} = \inf_{B_M} \sup_{P \in L^{+}(\|P\|_{q(\bar{g})^{(2)}} \leq 1)} \left| \int_{\Omega_M} f(\bar{x}) P(\bar{x}) d\bar{x} \right|,$$

(7)

where $\bar{q} = (q_1', ..., q_m')$, $\bar{g}^{(2)} = (g_1^{(2)}', ..., g_m^{(2)}')$, $\frac{1}{q_j} + \frac{1}{g_j} = 1$, $\frac{1}{q_j} + \frac{1}{\bar{g}_j^{(2)}} = 1 + \frac{1}{\bar{g}_j^{(2)}}$, $\bar{q}_j = 1$, $j = 1, ..., m$, and $L_M^{+}$ is the set of functions that are orthogonal to the subspace of trigonometric polynomials with harmonics in the set $\Omega_M$. 
Consider the function
\[ F_{q,n}(\vec{x}) = \sum_{j=1}^{\max_{j=1,...,m} |k_j| \leq 2} e^{i\langle k,\vec{x} \rangle}. \]

Let \( \Omega_M \) be a set of \( M \) vectors with integer coordinates. Suppose
\[ g(\vec{x}) = F_{q,n}(\vec{x}) - \sum_{k \in \Omega_M} e^{i\langle k,\vec{x} \rangle}, \]
where the sum \( \sum_{k \in \Omega_M} \) contains those terms in the function \( F_{q,n}(\vec{x}) \) with indices only in \( \Omega_M \).

By the inequality
\[ \left\| \sum_{j=1}^{\max_{j=1,...,m} |k_j| \leq 2} e^{i\langle k,\vec{x} \rangle} \right\|_{\vec{p}, \vec{q}(1)} << 2 \sum_{j=1}^{\max_{j=1,...,m} |k_j| \leq 2} \left( \frac{1}{p_j} \right) \]
and the Perseval’s equality for \( 1 < q_j' < 2, j = 1, ..., m \), we obtain
\[ \|g\|_{\vec{p}, \vec{q}(2')} \leq \|F_{q,n}\|_{\vec{p}, \vec{q}(2')} + \left( 2\pi \sum_{j=1}^{\max_{j=1,...,m} |k_j| \leq 2} \left( \frac{1}{p_j} \right) \right) \|\sum_{k \in \Omega_M} e^{i\langle k,\vec{x} \rangle}\|_2 << \left( \frac{2nm}{2^m + M^{\frac{1}{2}}} \right) \left( \frac{2nm}{2^m + M^{\frac{1}{2}}} \right). \]

Now we consider the function
\[ P_1(\vec{x}) = C_2^{-\frac{nm}{2}} g(\vec{x}). \]

Then (9) implies follows that the function \( P_1 \) satisfies the assumptions of the formula (7) for some constant \( C_2 > 0 \).

Consider the function
\[ f_n(\vec{x}) = C_3 2^{-nm(2 \sum_{j=1}^{m} \frac{1}{\eta_j})^{-1}(r - \sum_{j=1}^{m} (\frac{1}{\eta_j} - 1))} F_{q,n}(\vec{x}). \]

By the inequality (8), we get
\[ \sum_{s=0}^{\infty} 2^{sr} \|\delta_s(f_n)\|_{\vec{p}, \vec{q}(1)} << \]
\[ \sum_{s=0}^{\infty} 2^{sr} \|\delta_s(F_{q,n})\|_{\vec{p}, \vec{q}(1)} << \]
\[ \sum_{s=0}^{\infty} 2^{sr} \|\delta_s(F_{q,n})\|_{\vec{p}, \vec{q}(1)} \leq C_3. \]

Hence, the function \( C_3^{-1} f_n \in B_{\vec{p}, \vec{q}(1)}^\epsilon \).

For the functions (10) and (11), we have by the formula (7), the following
\[ \epsilon_M(\vec{p}, \vec{q}(2)) >> \inf_{\Omega_M} \left\{ \left\| \int_{\Omega_M} f_n(\vec{x}) P_1(\vec{x}) d\vec{x} \right\| \right\} >> \]
\[ >> 2^{-nm(2 \sum_{j=1}^{m} \frac{1}{\eta_j})^{-1}(r - \sum_{j=1}^{m} (\frac{1}{\eta_j} - 1))} 2^{-\frac{nm}{2^m}} \left( \|F_{q,n}\|_2^2 - M \right) >> \]
Therefore by applying Theorem B and by the norm property we obtain

\[
>> 2^{-nm(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1}(r-\sum_{j=1}^{m} \frac{1}{p_j}-1)} - \frac{nm(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1}}{2} \leq C^2.
\]

Hence, it follows from (12) by the inclusion \(B_{\hat{p},\hat{\theta}(1),1}^r \subset B_{\hat{p},\hat{\theta}(1),r}^r\) that

\[
e_M(f_n)_{\hat{q},\hat{\theta}(2)} >> 2^{-nm(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1}(r-\sum_{j=1}^{m} \frac{1}{p_j}-1)} - \frac{nm(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1}}{2}
\]
in the case of \(\sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^{m} \frac{1}{p_j}\).

So we have proved the first item.

Now we consider the case \(r = \sum_{j=1}^{m} \frac{1}{p_j}\). Let \(f \in B_{\hat{p},\hat{\theta}(1),r}^r\). Suppose

\[
\alpha = m(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1} \quad \text{and} \quad N_s = \left[2^{nm} n^{\frac{1}{r} - 1} \||\delta_s(f_n)||_{\hat{p},\hat{\theta}(1)} 2^{sr} \right] + 1.
\]

Then, by definition of the numbers \(N_s\) and by H"older’s inequality, we obtain

\[
\sum_{s=0}^{n-1} \#(p(s)) + \sum_{n \leq s < \alpha n} N_s << 2^{nm} + (\alpha - 1)n + 2^{nm} n^{\frac{1}{r} - 1} \sum_{n \leq s < \alpha n} ||\delta_s(f_n)||_{\hat{p},\hat{\theta}(1)} 2^{sr} << \]

\[
<< 2^{nm} + (\alpha - 1)n + 2^{nm} n^{\frac{1}{r} - 1}((\alpha - 1)n)^{\frac{1}{r}} \left(\sum_{s=0}^{\infty} ||\delta_s(f_n)||_{\hat{p},\hat{\theta}(1)} 2^{srr} \right)^{\frac{1}{r}} << \]

\[
<< 2^{nm} \sim M.
\]

To estimate \(J_1(n)\) let \(\beta = \max\{q_1, ..., q_m\}\). Then \(\beta > 2\) and \(L_\beta(\mathbb{T}^m) \subset L_{\hat{q},\hat{\theta}(2)}(\mathbb{T}^m)\).

Therefore by applying Theorem B and by the norm property we obtain

\[
J_1(n) = \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\hat{q},\hat{\theta}(2)} ||P(\Omega_{N_s})||_{\beta} \leq C \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\beta} << \]

\[
<< \left\| \left( \sum_{n \leq s < \alpha n} ||\delta_s(f) - P(\Omega_{N_s})||^2 \right)^{\frac{1}{2}} \right\|_{\beta} \leq \left( \sum_{n \leq s < \alpha n} ||\delta_s(f) - P(\Omega_{N_s})||^2 \right)^{\frac{1}{2}}.
\]

It implies by Lemma 1 and by the inequality of different metrics (see Theorem C) that

\[
J_1(n) << \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{nm} ||\delta_s(f)||^2 \right)^{\frac{1}{2}} << \]

\[
<< \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{nm} \sum_{j=1}^{m} \frac{1}{p_j} \||\delta_s(f)||^2 \right)^{\frac{1}{2}}.
\]
Next, since \( r = \sum_{j=1}^{m} \frac{1}{p_j} \), we have by definition of the numbers \( N_s \) and using Holders inequality, the following

\[
J_1(n) \ll \left( \sum_{n \in s < \infty} N_{s}^{-1} 2^{sm} 2^{s} \sum_{j=1}^{n} \left( \frac{1}{p_j} - \frac{1}{p_{j}} \right) \| \delta_s(f) \|_{\bar{p}, \bar{q}(1)}^{2} \right)^{\frac{1}{2}} \ll \ll (2^{-nm} n^{1-\frac{1}{2}})^{\frac{1}{2}} \left( \sum_{n \in s < \infty} 2^{sr} \| \delta_s(f) \|_{\bar{p}, \bar{q}(1)}^{2} \right)^{\frac{1}{2}} \ll \ll (2^{-nm} n^{1-\frac{1}{4}})^{\frac{1}{2}} \left( \sum_{n \in s < \infty} 2^{sr\theta} \| \delta_s(f) \|_{\bar{p}, \bar{q}(1)}^{r} \right)^{\frac{1}{r}} (\alpha - 1)^{\frac{1}{2r}} = C 2^{-nm} n^{1-\frac{1}{4}} \times M^{-\frac{1}{2}} (\log M)^{-\frac{1}{4}}.
\]

Thus

\[
J_1(n) \ll M^{-\frac{1}{4}} (\log M)^{1-\frac{1}{4}}
\] (13)

in the case of \( r = \sum_{j=1}^{m} \frac{1}{p_j} \).

For the estimation of \( J_2(n) \) we apply Holders inequality and taking into account that \( r = \sum_{j=1}^{m} \frac{1}{p_j} \) and \( \alpha = m(2 \sum_{j=1}^{m} \frac{1}{q_j})^{-1} \), we obtain

\[
J_2(n) \ll \sum_{n \in s < \infty} 2^{sr} \sum_{j=1}^{m} \left( \frac{1}{p_j} - \frac{1}{p_{j}} \right) \| \delta_s(f) \|_{\bar{p}, \bar{q}(1)}^{r} \ll \ll \left( \sum_{s=0}^{\infty} 2^{sr\theta} \| \delta_s(f) \|_{\bar{p}, \bar{q}(1)}^{r} \right)^{\frac{1}{r}} \left( \sum_{n \in s < \infty} 2^{-sr\theta} \sum_{j=1}^{m} \frac{1}{q_j} \right)^{\frac{1}{r}} \ll \ll 2^{-nm} \sum_{j=1}^{m} \frac{1}{q_j} = C2^{-nm} \times M^{-\frac{1}{2}},
\] (14)

where \( \tau' = \frac{r}{r-1} \).

By (13) and (14) the inequality (2) implies that

\[
\| f - P(\Omega_M) \|_{\bar{q}, \bar{q}(2)} \ll M^{-\frac{1}{4}} (\log M)^{1-\frac{1}{4}}
\]

in the case \( r = \sum_{j=1}^{m} \frac{1}{p_j} \). It proves the upper bound estimation in the second item.

Let \( r > \sum_{j=1}^{m} \frac{1}{p_j} \). Suppose

\[
N_s = \left[ 2^{-n(r - \sum_{j=1}^{m} \frac{1}{p_j} - 1)} - s(r - \sum_{j=1}^{m} \frac{1}{p_j}) \right] + 1.
\]

Then

\[
\sum_{s=0}^{n-1} \# \rho(s) + \sum_{n \in s < \infty} N_s \ll \ll 2^{nm} + (\alpha - 1)n + 2^{n(r - \sum_{j=1}^{m} \frac{1}{p_j} - 1)} \sum_{n \in s < \infty} 2^{-s(r - \sum_{j=1}^{m} \frac{1}{p_j})} \ll \ll 2^{nm} + (\alpha - 1)n + 2^{n(r - \sum_{j=1}^{m} \frac{1}{p_j} - 1)} \sum_{n \in s < \infty} 2^{-s(r - \sum_{j=1}^{m} \frac{1}{p_j})} \ll \ll 2^{nm} + (\alpha - 1)n + 2^{n(r - \sum_{j=1}^{m} \frac{1}{p_j} - 1)} \sum_{n \in s < \infty} 2^{-s(r - \sum_{j=1}^{m} \frac{1}{p_j})}.
\]
If $f \in H^r_{\bar{p}, \bar{g}(1)}$, then, by definition of the numbers $N_s$ and $r > \sum_{j=1}^m \frac{1}{p_j}$ we obtain

$$J_1(n) \ll \left( \sum_{n \leq s < an} N_s^{-1} 2^{sn/2} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) \| \delta_s(f) \|_{\bar{p}, \bar{g}(1)}^2 \right)^{\frac{1}{2}} \ll \frac{1}{2} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})) \left( \sum_{n \leq s < an} 2^{s(r + \sum_{j=1}^m \frac{1}{p_j})} \| \delta_s(f) \|_{\bar{p}, \bar{g}(1)}^2 \right)^{\frac{1}{2}} \ll 2^{-n(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}.$$  

Thus,

$$J_1(n) \ll M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \tag{15}$$

in the case of $r > \sum_{j=1}^m \frac{1}{p_j}$.

To estimate $J_2(n)$, we suppose $\alpha = (r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})) (r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))^{-1}$ and get

$$J_2(n) \ll \sum_{n \alpha \leq s < \infty} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) \| \delta_s(f) \|_{\bar{p}, \bar{g}(1)}^2 \ll \sum_{n \alpha \leq s < \infty} 2^{-s(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \ll 2^{-n(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \ll M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \tag{16}$$

for a function $f \in H^r_{\bar{p}, \bar{g}(1)}$. By (15) and (14), it follows from (2) that

$$\| f - P(\Omega_M) \|_{\bar{p}, \bar{g}(2)} \ll M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

for any function $f \in H^r_{\bar{p}, \bar{g}(1)}$ in the case of $r > \sum_{j=1}^m \frac{1}{p_j}$.

From $B^r_{\bar{p}, \bar{g}(1), \tau} \subset H^r_{\bar{p}, \bar{g}(1)}$ it follows that

$$\varepsilon_M \left( B^r_{\bar{p}, \bar{g}(1), \tau} \right)_{\bar{q}, \bar{g}(2)} \ll M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

in the case of $r > \sum_{j=1}^m \frac{1}{p_j}$. It proves the upper bound estimation in the item 3.

Let us consider the lower bound estimation in the case $r = \sum_{j=1}^m \frac{1}{p_j}$. Consider the function

$$g_1(\bar{x}) = \sum_{s=1}^{n} \sum_{k \in \rho(s)} \prod_{j=1}^m k_j^{-1} \cos k_j x_j. \quad \tag{17}$$

Then

$$\delta_s(g_1, \bar{x}) = \sum_{k \in \rho(s)} \prod_{j=1}^m k_j^{-1} \cos k_j x_j.$$

\section*{On $M -$ Terms Approximations Besov Classes in Lorentz Spaces}

$<< 2^{nm} + (\alpha - 1)n << 2^{nm} << M$. 
It is known that for a function \( d_s(\bar{x}) = \sum_{k \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j \) the following relation holds
\[
\|d_s\|_{\tilde{p}, \tilde{q}(1)} \asymp 2^{s} \sum_{j=1}^{m} (1 - \frac{1}{p_j}), \quad 1 < p_j, \theta_j^{(1)} < +\infty, \quad j = 1, ..., m.
\]

Therefore by the inequality of distinct metrics (see Theorem C) and by the Marcinkiewicz theorem on multipliers, we have
\[
\|\delta_s(g_1)\|_{\tilde{p}, \tilde{q}(1)} \ll 2^{-s m} \|d_s\|_{\tilde{p}, \tilde{q}(1)} \lesssim C 2^{-s} \sum_{j=1}^{m} \frac{1}{p_j}.
\]

Hence, since \( r = \sum_{j=1}^{m} \frac{1}{p_j} \) we obtain
\[
\left( \sum_{s=0}^{\infty} 2^{s r} \|\delta_s(g_1)\|_{r, \tilde{q}(1)}^r \right)^{\frac{1}{r}} \leq C_1 n^{\frac{1}{r}}.
\]

Therefore the function \( f_1(\bar{x}) = C_1 n^{-\frac{1}{r}} g_1(\bar{x}) \) belongs to the class \( B_{\tilde{p}, \tilde{q}(1)}, \tau \), \( 1 < p_j < +\infty, \quad j = 1, ..., m \).

Now, we are going to construct a function \( P_1 \), which satisfies the conditions of the formula (7). Let
\[
v_1(\bar{x}) = \sum_{s=1}^{n} \sum_{k \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j
\]
and \( \Omega_M \) be an arbitrary set of vectors \( \bar{k} = (k_1, ..., k_m) \) in \( M \) with integer coordinates. Consider the function
\[
u_1(\bar{x}) = \sum_{s=1}^{n} \sum_{k \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j.
\]

Suppose \( w_1(\bar{x}) = v_1(\bar{x}) - u_1(\bar{x}) \). Then, since \( 1 < q_j' = \frac{q_j}{q_j - 1} < 2, \quad j = 1, ..., m \), we obtain, by the Perseval’s equality, the following
\[
\|w_1\|_{q, \tilde{q}(2)} \leq \|v_1\|_{q, \tilde{q}(2)}' + \|u_1\|_2 \leq \|v_1\|_{q, \tilde{q}(2)}' + C M^{\frac{1}{q}}.
\]

By the property of quasi-norm and the estimation of the norm of the Dirichlet kernel in the Lorentz space, we have
\[
\|v_1\|_{q, \tilde{q}(2)}' \ll \sum_{s=1}^{n} \|\delta_s(g_1)\|_{q, \tilde{q}(2)}' \ll \sum_{s=1}^{n} 2^{s} \sum_{j=1}^{m} (1 - \frac{1}{q_j}) < 2^{s} \sum_{j=1}^{m} (1 - \frac{1}{q_j}) = C 2^{s} \sum_{j=1}^{m} \frac{1}{p_j}.
\]

Therefore, taking into account \( \frac{1}{q_j} < \frac{1}{2}, \quad j = 1, ..., m \), we get
\[
\|w_1\|_{q, \tilde{q}(2)}' \ll (\frac{2^{nm}}{M^{\frac{1}{q}}} + M^{\frac{1}{q}}) \leq C_2 2^{\frac{nm}{q}}.
\]

Hence the function
\[
P_1(\bar{x}) = C_2^{-1} 2^{-\frac{nm}{q}} w_1(\bar{x})
\]
satisfies the conditions of the formula (7). Then, by substituting the functions \( f_1 \) and \( P_1 \) into (7) and by orthogonally of a trigonometric system, we obtain
\[
e_M(f_1)_{\tilde{q}, \tilde{\theta}(2)} >> \sum_{n_1 \leq s < n} \sum_{k \in \rho(s)} \prod_{j=1}^{m} k_j^{-\frac{1}{2}} 2^{-\frac{nm}{2}} n^{-\frac{1}{2}} >>
\]
\[
\geq C(\ln 2)^m \sum_{n_1 \leq s < n} 2^{-\frac{nm}{2}} n^{-\frac{1}{2}} = C(\ln 2)^m 2^{-\frac{nm}{2}} n^{-\frac{1}{2}} (n - n_1) \geq
\]
\[
>> (\ln 2)^m 2^{-\frac{nm}{2}} n^{1-\frac{1}{2}} \geq M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{2}},
\]
where \( n_1 \) is a natural number such that \( n_1 < \frac{n}{2} \).

So, for the function \( f_1 \in B_{\tilde{p}, \tilde{\theta}(1), \tau} \) it has been proved that
\[
e_M(f_1)_{\tilde{q}, \tilde{\theta}(2)} >> M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{2}}
\]
in the case of \( r = \sum_{j=1}^{m} \frac{1}{p_j} \). Hence
\[
e_M\left( B_{\tilde{p}, \tilde{\theta}(1), \tau}^{r} \right)_{\tilde{q}, \tilde{\theta}(2)} >> M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{2}}
\]
in the case of \( r = \sum_{j=1}^{m} \frac{1}{p_j} \). It proves the lower bound estimation in the second item.

Let us prove the lower bound estimation for the case \( r > \sum_{j=1}^{m} \frac{1}{p_j} \). Since in this case an upper bound estimation of the quantity \( e_M\left( B_{\tilde{p}, \tilde{\theta}(1), \tau}^{r} \right)_{\tilde{q}, \tilde{\theta}(2)} \) does not depend on \( \tau \) and \( B_{\tilde{p}, \tilde{\theta}(1), 1}^{r} \subset B_{\tilde{p}, \tilde{\theta}(1), \tau}^{r}, 1 < \tau < +\infty \), it suffices to prove the lower bound estimation for \( B_{\tilde{p}, \tilde{\theta}(1), 1}^{r} \).

For a number \( M \in \mathbb{N} \), we choose a natural number \( n \) such that \( 2^{nm} < M \leq 2^{n+1} \) and \( 2M \leq \sharp \rho(n) \), where \( \sharp \rho(n) \) denotes the number of elements in the set \( \rho(n) \).

Consider the following function
\[
f_3(\vec{x}) = n^{-1} \sum_{s=1}^{n} 2^{-s} \sum_{j=1}^{m} \prod_{k \in \rho(s)} k_j^{-\frac{1}{2}} \cos k_j x_j.
\]
Then
\[
\| \delta_s(f_3) \|_{\tilde{p}, \tilde{\theta}(1)} << 2^{-sr} n^{-1}.
\]
Hence
\[
\sum_{s=0}^{\infty} 2^{sr} \| \delta_s(f_3) \|_{\tilde{p}, \tilde{\theta}(1)} \leq C_3
\]
i.e. the function \( C_3^{-1} f_3 \in B_{\tilde{p}, \tilde{\theta}(1), 1}^{r} \).

Next, consider the functions
\[
v_3(\vec{x}) = \sum_{s=1}^{n} \sum_{k \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j,
\]
\[
u_3(\vec{x}) = \sum_{s=1}^{n} \sum_{k \in \rho(s) \cap \Omega_M} \prod_{j=1}^{m} \cos k_j x_j.
\]
Suppose \( w_3(\bar{x}) = v_3(\bar{x}) - u_3(\bar{x}) \). By the Perseval’s equality,
\[
\|u_3\|_2 \leq M^{\frac{1}{2}},
\]
\[
\|v_3\|_2 = 2^{(\alpha - 1)m}.
\]
From these relations, we obtain, by the properties of the norm, the following
\[
\|w_3\|_2 \leq \|v_3\|_2 + \|u_3\|_2 \leq C_4 2^{nm}.
\]
Therefore the function \( P_3(\bar{x}) = C_4^{-1}2^{-\frac{m}{r}}w_3(\bar{x}) \) satisfies the conditions of the formula (7).

Since \( 2 < q_j \) \( j = 1, ..., m \), we have \( e_M(f_3)_2 \leq C e_M(f_3)_{\bar{q}, \bar{q}(2)} \). Now, by the formula (7), we get
\[
e_M(f_3)_{\bar{q}, \bar{q}(2)} \gg e_M(f_3)_{2} \gg n^{-1}2^{-nm} \sum_{s=1}^{n} 2^{s} \sum_{j=1}^{m} k_j^{-\frac{r}{m}} \gg
\]
\[
\gg n^{-1}2^{-nm} \sum_{s=1}^{n} 2^{s} \sum_{j=1}^{m} (1 - \frac{1}{pj}) 2^{s(m-r)} =
\]
\[
= Cn^{-1}2^{-nm} \sum_{s=1}^{n} 2^{-s(r - \sum_{j=1}^{m} \frac{1}{pj})} \gg 2^{n(r + \sum_{j=1}^{m} (\frac{1}{r} - \frac{1}{pj}))}.
\]
It follows from the relation \( 2^{nm} \propto M \) that
\[
e_M(f_3)_{\bar{q}, \bar{q}(2)} \gg M^{-\frac{1}{2} \left( r + \sum_{j=1}^{m} (\frac{1}{r} - \frac{1}{pj}) \right)}
\]
in the case \( r > \sum_{j=1}^{m} \frac{1}{pj} \) for the function \( C_3^{-1}f_3 \in B_{\bar{p}, \bar{q}(1)}^{r} \). Hence
\[
e_M(B_{\bar{p}, \bar{q}(1)}^{r})_{\bar{q}, \bar{q}(2)} \gg M^{-\frac{1}{2} \left( r + \sum_{j=1}^{m} (\frac{1}{r} - \frac{1}{pj}) \right)}
\]
and therefore
\[
e_M(B_{\bar{p}, \bar{q}(1)}^{r})_{\bar{q}, \bar{q}(2)} \gg M^{-\frac{1}{2} \left( r + \sum_{j=1}^{m} (\frac{1}{r} - \frac{1}{pj}) \right)}
\]
in the case \( r > \sum_{j=1}^{m} \frac{1}{pj} \). So Theorem 1 has been proved.

**Theorem 2.** Let \( \bar{p} = (p_1, ..., p_m), \bar{q} = (q_1, ..., q_m) \), \( 1 < p_j < q_j \leq 2, 1 < q_j^{(1)}, q_j^{(2)} < \infty \), \( j = 1, ..., m \), \( 1 \leq r \leq +\infty \).

If \( r > \sum_{j=1}^{m} \left( \frac{1}{p_j} - \frac{1}{q_j} \right) \), then
\[
e_M(B_{\bar{p}, \bar{q}(1)}^{r})_{\bar{q}, \bar{q}(2)} \propto M^{-\frac{1}{2} \left( r - \sum_{j=1}^{m} (\frac{1}{r} - \frac{1}{q_j}) \right)}.
\]

**Proof.** For a number \( M \in \mathbb{N} \) choose a natural number \( n \) such that \( M \propto 2^{nm} \). By the inequality of distinct metrics and by Holder’s inequality, we have
\[
\|f - \sum_{s=0}^{n} \delta_s(f)\|_{\bar{q}, \bar{q}(2)} \leq \sum_{s=n}^{\infty} \|\delta_s(f)\|_{\bar{q}, \bar{q}(2)} \leq
\]
\[
\leq \left[ \sum_{s=0}^{\infty} 2^{sr} \|\delta_s(f)\|_{\bar{q}, \bar{q}(2)}^r \right]^\frac{1}{r} \leq \left[ \sum_{s=n}^{\infty} 2^{sr} \right]^\frac{1}{r} \ll
\]
By Theorem 1, for \( f \in B^r_{\bar{p}, \bar{q}(1), \tau} \), \( \frac{1}{r} + \frac{1}{\tau} = 1 \). Therefore,

\[
e_{M}(f)_{\bar{q}, \bar{q}(2)} \leq \| f - \sum_{s=0}^{n} \delta_{s}(f) \|_{\bar{q}, \bar{q}(2)} \leq M \left( \frac{1}{r} \sum_{j=1}^{m} (\frac{1}{r_j} - \frac{1}{q_j}) \right) \]

for \( n \in B^r_{\bar{p}, \bar{q}(1), \tau} \). Let \( n \in B^r_{\bar{p}, \bar{q}(1), \tau} \).

Hence

\[
e_{M} \left( B^r_{\bar{p}, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \leq M \left( \frac{1}{r} \sum_{j=1}^{m} (\frac{1}{r_j} - \frac{1}{q_j}) \right).
\]

It proves the upper bound estimation.

For the lower bound estimation, let us consider the function

\[
f_0(\bar{x}) = n^{-r} + \sum_{j=1}^{m} \left( \frac{1}{p_j} - 1 \right) V_n(\bar{x}),
\]

where \( V_n(\bar{x}) \) is a Valle-Poisson sum with multiplicity.

Next, following the proof in [19] (pp. 46-47) and applying Theorem B, we obtain the lower bound estimation of the quantity \( e_{M} \left( B^r_{\bar{p}, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \).

**Theorem 3.** Let \( \bar{p} = (p_1, \ldots, p_m) \), \( \bar{q} = (q_1, \ldots, q_m) \), \( 2 \leq p_j < q_j < \infty \), \( 1 < \theta_j(1), \theta_j(2) < \infty \), \( j = 1, \ldots, m \), \( 1 \leq \tau \leq +\infty \). If \( r > \frac{m}{2} \), then

\[
e_{M} \left( B^r_{\bar{p}, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \asymp M^{-\frac{r}{m}}.
\]

**Proof.** By the inclusion \( B^r_{\bar{p}, \bar{q}(1), \tau} \subset B^r_{2, \bar{q}(1), \tau} \subset H^r_{2, \bar{q}(2), \tau} \), we have

\[
e_{M} \left( B^r_{\bar{p}, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \leq e_{M} \left( B^r_{2, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \leq e_{M} \left( H^r_{2, \bar{q}(2), \tau} \right)_{\bar{q}, \bar{q}(2)}.
\]

By Theorem 1,

\[
e_{M} \left( H^r_{2, \bar{q}(2), \tau} \right)_{\bar{q}, \bar{q}(2)} \leq M^{-\frac{r}{m}}.
\]

for \( p_j = 2, j = 1, \ldots, m \). Hence

\[
e_{M} \left( B^r_{\bar{p}, \bar{q}(1), \tau} \right)_{\bar{q}, \bar{q}(2)} \leq M^{-\frac{r}{m}}.
\]

it proves the upper bound estimation.

Let us consider the lower bound estimation. Consider Rudin-Shapiros polynomial (see [24], p.155) of the type

\[
R_s(x) = \sum_{s=2^s-1}^{2^s} \varepsilon_k e^{ikx}, \quad x \in [0, 2\pi], \quad \varepsilon_k = \pm 1.
\]

it is known that \( \| R_s \|_{\infty} = \max_{x \in [0,2\pi]} | R_s(x) | \leq 2^{\frac{r}{2}} \) (see [24], p. 155). For a given number \( M \) choose a number \( n \) such that \( M \asymp 2^{nm} \). Now consider the function

\[
f_0(\bar{x}) = 2^{-n(\frac{m}{2} + \tau)} \sum_{s=1}^{n} \prod_{j=1}^{m} R_s(x_j)
\]

Then, by the continuity, \( f_0 \in L_{\bar{p}, \bar{q}(1)}(\mathbb{T}^m) \) and

\[
\sum_{s=0}^{\infty} 2^{s\tau} \| \delta_s(f_0) \|_{\bar{p}, \bar{q}(1)} \tau = 2^{n(\frac{m}{2} + \tau)} \sum_{s=1}^{n} 2^{s\tau} \| \prod_{j=1}^{m} R_s(x_j) \|_{\bar{p}, \bar{q}(1)} \leq
\]
\[ \leq 2^{-n\left(\frac{2}{p} + \tau\right)} \sum_{s=1}^{n} 2^{s\left(\frac{2}{p} + \tau\right)} \leq C_0. \]

Hence, the function \( C_0^{-1} f_0 \in B_{\overline{p}, \overline{\theta}(1), \tau}^{r}. \) Now construct a function \( P(\bar{x}), \) which would satisfy the conditions in the formula (7). Suppose

\[
v_0(\bar{x}) = \sum_{s=1}^{n} m \prod_{j=1}^{s} R_s(x_j), \quad u_0(\bar{x}) = \sum_{s=1}^{n} m \prod_{j=1}^{s} R_s(x_j),
\]

where the sign * means that the polynomial \( u_0(\bar{x}) \) contains only those harmonics of \( v_0 \) which have indices in \( \Omega_M. \) Suppose \( w_0(\bar{x}) = v_0(\bar{x}) - u_0(\bar{x}). \) Then, since \( 1 < q_j' = \frac{q_j}{q_j - 1} < 2, \quad j = 1, ..., m, \) and by the Percevals equality, we have

\[ \|w_0\|_{q, \overline{\theta}(3)}' \leq \|w_0\|_2 \leq C_1 2^{\frac{nm}{r}}. \]

Therefore, for the function \( P_0(\bar{x}) = C_1^{-1} 2^{-\frac{nm}{r}} w_0(\bar{x}), \) the inequality holds \( \|P_0\|_{q, \overline{\theta}(3)}' \leq 1. \) Now using the formula (7), we obtain

\[
e_M \left( B_{\overline{p}, \overline{\theta}(1), \tau}^{r} \right)_{\overline{q}, \overline{\theta}(2)}' \gg e_M(f_0)_{\overline{q}, \overline{\theta}(2)}' \gg 2^{-n\left(\frac{2}{p} + \tau\right)} 2^{-\frac{nm}{r}} (2^{nm} - M) \gg
\[
\gg 2^{-n(m+r)} 2^{nm} \gg M^{-\frac{1}{m}}.
\]

So

\[
e_M \left( B_{\overline{p}, \overline{\theta}(1), \tau}^{r} \right)_{\overline{q}, \overline{\theta}(2)}' \gg M^{-\frac{1}{m}}.
\]

It proves Theorem 3.

**Corollary.** Let \( 1 < p \leq 2 < q < \infty, 1 \leq \tau \leq \infty \) and \( r = \frac{m}{p}. \) Then

\[
e_M \left( B_{\overline{p}, \overline{\theta}(1), \tau}^{r} \right) q \gg M^{-\frac{1}{2}} (\log M)^{1 - \frac{1}{p}}.
\]

The proof follows from the second item of Theorem 2.1 if \( p_j = \theta_j^{(1)} = p, \quad q_j = \theta_j^{(2)} = q, \quad j = 1, ..., m. \)

**Remark.** In the case \( p_j = \theta_j^{(1)} = p, \quad q_j = \theta_j^{(2)} = q, \quad j = 1, ..., m \) and \( r > m(\frac{1}{p} - \frac{1}{q}), \) the results of R.A. DeVore and V.N. Temlyakov [20] follow from Theorem 1 - 3. If \( 1 < p \leq 2 < q < \infty \) and \( m(\frac{1}{p} - \frac{1}{q}) < r \leq \frac{m}{p}, \) the results of S.A. Stasyuk [16], [17] follow from the first and second items of Theorem 1.

The cases \( p_j = \theta_j^{(1)}, \quad q_j = \theta_j^{(2)}, \quad j = 1, ..., m. \) of Theorem 1 - 3 were announced in [25] and in of Theorem 1 the first item proved [26].

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