OPERATOR INEQUALITIES I. MODELS AND ERGODICITY

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Abstract. We discuss when an operator, subject to an inequality in hereditary form, admits a unitarily equivalent functional model of Agler type in the reproducing kernel Hilbert space associated to the inequality. To the contrary to the previous work, the kernel need not be of Nevanlinna-Pick type. We derive some consequences concerning the ergodic behavior of the operator.

1. Introduction

Let \( \alpha(t) = \sum \alpha_n t^n \) be an analytic function defined on the unit disc \( D := \{ |t| < 1 \} \), with \( \alpha_n \in \mathbb{R} \) for every \( n \geq 0 \) and \( \alpha_0 = 1 \). Let \( T \) be a bounded linear operator on a Hilbert space \( H \). We define

\[
\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^*nT^n,
\]

where the series is assumed to converge in the strong operator topology in \( L(H) \). Note that when \( \alpha \) is just a polynomial we do not have any convergence problem. For example, if \( \alpha(t) = 1 - t \) then the RHS of (1.1) is \( I - T^*T \), and therefore \( T \) is a contraction if and only if \( \alpha(T^*, T) \geq 0 \). In the 1960’s Nagy and Foiaş developed a beautiful spectral theory for contractions (see [30]). In this theory the size of the defect operator \( D := (I - T^*T)^{1/2} \) plays a central role. If \( D \) is of finite rank or \( D \) is Hilbert-Schmidt, much more consequences of the Nagy-Foiaş theory are available.

Hence, it is natural to try to develop a kind of spectral theory for operators \( T \) satisfying

\[
\alpha(T^*, T) \geq 0
\]

(1.2)

for more general functions \( \alpha \). There are many works in this setting when

\[
k(t) = \sum_{n=0}^{\infty} k_n t^n := 1/\alpha(t)
\]

(1.3)

gives a positive definite kernel \( k(z, w) := k(\overline{w}z) \) defining a reproducing kernel Hilbert space \( \mathcal{H}_k \) of analytic functions. For instance, in the Nagy-Foiaş case one obtains a Hardy space \( H^2 \). In his landmark paper [4], Agler has shown that if \( T \) has spectrum \( \sigma(T) \subset \mathbb{D} \), then it is natural to model the operator \( T \) by operators of the form \( B_k \otimes I_R \), where \( B_k \) is the backward shift on the space \( \mathcal{H}_k(\mathbb{D}) \) and \( I_R \) is the identity on some auxiliary Hilbert space \( R \). More generally, when \( \sigma(T) \subset \mathbb{D} \), it has been found that instead of \( B_k \otimes I_R \) one...
should consider operators of the form \((B_k \otimes I_R) \oplus U\), where \(U\) is an isometry or a unitary operator.

The study of this type of problem for tuples of commuting operators has received a lot of attention and there are many outstanding papers such as [25, 6, 7, 9]. One of the strongest results in this direction is contained in the recent papers by Bickel, Hartz and McCarthy [10] and by Clouâtre and Hartz [11]. It is stated for spherically symmetric tuples. For the case of a single operator, it can be formulated as follows.

**Theorem A** ([11, Theorem 1.3]). Let \(\alpha\) be an analytic function with \(\alpha_0 = 1\) and \(\alpha_n \leq 0\) for all \(n \geq 1\). Suppose that \(k\), given by (1.3), has radius of convergence 1, \(k_n > 0\) for every \(n \geq 0\) and

\[
\lim_{n \to \infty} \frac{k_n}{k_{n+1}} = 1.
\]

Then \(B_k\) is bounded on \(H_k(\overline{D})\), and for any Hilbert space operator \(T\) the following statements are equivalent.

(i) \(T\) satisfies \(\alpha(T^*, T) \geq 0\).

(ii) \(T\) is unitarily equivalent to a part of \((B_k \otimes I_R) \oplus S\), where \(S\) is an isometry on \(W\).

(Here \(R\) and \(W\) are auxiliary Hilbert spaces.)

In [10] the existence of a functional calculus in this context is also obtained. The formulation by Bickel, Clouâtre, Hartz and McCarthy speaks about a unitary operator \(U \in L(W)\), instead of isometry. It is easy to see that their formulation is equivalent to the version given above.

We denote by \(A_W\) the Wiener algebra of all analytic functions with summable Taylor coefficients.

In the present paper, we only focus on the single operator case. It will be always assumed that the functions \(\alpha\) and \(k\) are analytic in \(D\) and inverse to each other (see (1.3)). Their Taylor coefficients are real and normalized to \(\alpha_0 = k_0 = 1\). Moreover, we assume that \(\alpha\) is in \(A_W\) and that \(k_n > 0\) for all \(n \geq 0\). (See the discussion at Subsection 2.2).

Theorem A concerns the Nevanlinna-Pick case, when \(\alpha_n \leq 0\) for all \(n \geq 1\) (that is, \(k\) is a the Nevanlinna-Pick kernel).

Theorem A leads us to consider the following question.

**Question 1.1.** When an operator \(T \in L(H)\) satisfying \(\alpha(T^*, T) \geq 0\) can be modelled by a part of an operator of the form \((B_k \otimes I_R) \oplus S\), where \(S\) is an isometry?

The following result gives us a new positive answer.

**Theorem 1.2.** Suppose that \(k \in A_W\) does not vanish on \(\overline{D}\) and has positive Taylor coefficients satisfying

\[
\frac{k_j k_{n-j}}{k_n} \leq \tau_j \quad (\forall n \geq 2j)
\]

\[
\sum_{j=0}^{\infty} \tau_j < \infty.
\]
Then $B_k$ is bounded on $H_k$, and an operator $T \in L(H)$ is a part of $B_k \otimes I_R$ (for some Hilbert space $R$) if and only if

(i) both

$$\sum_{n=0}^{\infty} |\alpha_n| T^{*n} T^n$$

and

$$\sum_{n=0}^{\infty} k_n T^{*n} T^n$$

converge in the strong operator topology in $L(H)$; and

(ii) $\alpha(T^*, T) \geq 0$.

This theorem shows that above representation of $T$ exists in many cases when $k$ is not a Nevanlinna-Pick kernel and so Theorem A does not apply. For instance, given an integer $n \geq 2$, there are examples of functions $k$ satisfying the hypotheses of Theorem 1.2 with whatever prescribed signs of the coefficients $\alpha_2, \ldots, \alpha_n$. Note that $\alpha_1 = -k_1$ is always negative. Moreover, in Theorem 1.2 the quotients $k_n/k_{n+1}$ need not converge. See Examples 5.1 and 5.2.

The techniques employed in the proof of Theorem 1.2 are also different. We use, basically, a combination of Müller’s arguments in [24] and techniques based on Banach algebras.

The above theorems open a question of describing invariant subspaces of $B_k \otimes I_R$ and of constructing a functional model of operators under the study. We do not address this question here. In the recent work [12], Clouâtre, Hartz and Schiillo establish a Beurling–Lax–Halmos theorem for reproducing kernel Hilbert spaces in this Nevanlinna-Pick context. See also [27, 14, 28, 29].

Whenever (1.2) holds, we put

$$D := (\alpha(T^*, T))^{1/2},$$

where the non-negative square root is taken. This operator will play the same role as the defect operator in the Nagy-Foiaş theory. We denote by $\mathfrak{D}$ the closure of the range of $D$.

The operator

$$V_D : H \to H_k \otimes \mathfrak{D}, \quad V_D x(z) = D(I_H - zT)^{-1} x, \quad x \in H$$

will also be very important for us.

In the Nevanlinna-Pick case considered by Bickel, Clouâtre, Hartz and McCarthy, an easy computation shows that $V_D$ is a contraction (see Theorem 5.1). This is done without imposing extra conditions, such as (1.4). In particular, we do not use the existence of the model given in Theorem A. We just need the negativeness of $\alpha_n$ for $n \geq 1$.

Whenever the answer to Question 1.1 is affirmative, the operator $V_D$ is a contraction, and we can give an explicit model for $T$ (that is, give $R$ and $S$ explicitly). In particular, the following theorem can be seen as an addendum to Theorem A and Theorem 1.2.

**Theorem 1.3.** Suppose that the answer to Question 1.1 is affirmative. Then $V_D$ is a contraction, and hence we can define

$$W = (I_H - V_D^* V_D)^{1/2}, \quad W = \text{clos Ran } W.$$
Moreover, $S : \mathcal{W} \to \mathcal{W}$ given by $SWx := WTx$ is an isometry and the operator 

$$(V_D, W) : H \to (\mathcal{H}_k \otimes \mathcal{D}) \oplus \mathcal{W}, \quad (V_D, W)h = (V_D h, W h)$$

provides a model of $T$, in the sense that $(V_D, W)$ is isometric and 

$$((B_k \otimes I_\mathcal{D}) \oplus S)(V_D, W) = (V_D, W)T.$$ 

Since $\text{Ran} D$ is dense in $\mathcal{D}$ and $\text{Ran} W$ is dense in $\mathcal{W}$, this model is minimal in the sense given in Definition 3.5 (see Remark 3.6).

We distinguish the following two cases.

**Definition 1.4.** Let $\alpha \in \mathcal{A}_W$ be an analytic function with real Taylor coefficients, $\alpha_0 = 1$, such that $\alpha(t) \neq 0$ for $t \in \mathbb{D}$. If $\alpha(1) = 0$, we speak about the critical case, and if $\alpha(1) > 0$, we refer to it as the subcritical case. (Note that $\alpha(1)$ cannot be negative.)

In Section 2 we introduce two families of operators in $L(H)$ depending on a fixed analytic function $\alpha$: $\text{Adm}^w_\alpha$ and $C^w_\alpha$. Essentially, $\text{Adm}^w_\alpha$ is the family of operators $T$ for which we can define $\alpha(T^*, T)$ and $C^w_\alpha$ means that $\alpha(T^*, T) \geq 0$. We obtain some interesting properties for these families and characterize the membership of backward and forward weighted shifts to them.

In [8] the last two authors developed a kind of spectral theory, in the Nagy-Foiaş spirit, for operators $T$ satisfying $\alpha(T^*, T) \geq 0$ for suitable functions $\alpha$ assuming that the convergence in (1.1) is in norm. Since now we are assuming that the convergence of (1.1) is in the strong operator topology, our new hypothesis is weaker. This is the reason for the superscript notation “w” in $\text{Adm}^w_\alpha$ and $C^w_\alpha$. It will make us easier the referencing of some results there and how we improve them now.

In Section 3, we prove that the minimal model is unique in the critical case. In the subcritical case, it is not unique in general. However, in this case there always exists a model for which $V = V_D$ and $W$ is absent. See Theorem 3.7. This section also contains the proof of Theorem 1.3.

The proof of Theorem 1.2 is given in Section 4.

In Section 5 we study the scope of condition (1.5). There we present examples satisfying the hypothesis of Theorem 1.2 but were Theorem A does not apply.

In Section 6 we consider only functions $\alpha$ of the form $\alpha(t) = (1 - t)^a$ for some $a > 0$. In this case, we say that $T$ is an $a$-contraction when $\alpha(T^*, T) \geq 0$. Note that when $0 < a < 1$, we can apply Theorem A to obtain a model for $T$. With the help of this model we obtain some ergodic properties for $a$-contractions with $0 < a < 1$. For example, we have the following result.

**Theorem 1.5.** Let $T$ be an $a$-contraction with $0 < a < 1$ and let $b > 1 - a$. Then $T$ is quadratically $(C, b)$-bounded.
That $T$ is quadratically $(C,b)$-bounded means that
\[
\sup_{n \geq 0} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^{b}(n-j) \|T^{j}x\|^{2} \leq C \|x\|^{2} \quad (\forall x \in H)
\]
for some constant $C > 0$. Here $k^{b}(n)$ denotes the $n$-th Taylor coefficient of the function $(1 - t)^{-b}$. These are called Cesàro numbers. In that section we include definitions and basic properties in ergodic theory. We also give in Theorem 6.13 a characterization in ergodic theoretic terms for when the isometry $S$ appears in the model of an $a$-contraction $T$ (with $0 < a < 1$).

In the forthcoming paper Operator Inequalities II. Models up to similarity and Inclusions, we will study in more detail the $a$-contractions. There we obtain models up to similarity (instead of unitarily equivalent models), but also in this case we can discuss its usual consequences: completeness of eigenvectors, similarity to a normal operator, and so on.

2. Preliminaries on classes defined by operator inequalities

In this section we introduce the classes $\text{Adm}_{\alpha}^{w}$ and $\text{C}_{\alpha}^{w}$ associated to an analytic function $\alpha$, consisting on operators in $L(H)$. After studying them, we analyse why the conditions we will impose through the rest of the paper on the functions $\alpha$ and $k = 1/\alpha$ are natural. Finally, at the end of the section we discuss the membership of weighted shifts to the classes $\text{Adm}_{\alpha}^{w}$ and $\text{C}_{\alpha}^{w}$.

2.1. The classes $\text{Adm}_{\alpha}^{w}$ and $\text{C}_{\alpha}^{w}$. Associated to any analytic function $\alpha$ we introduce some families of operators in $L(H)$. Before entering into the definitions and basic properties of these families, let us mention the following well known result that will be used repeatedly.

**Lemma 2.1** (see [21, Problem 120]). If an increasing sequence $\{A_{n}\}$ of selfadjoint Hilbert space operators satisfies $A_{n} \leq CI$ for all $n$, where $C$ is a constant, then $\{A_{n}\}$ converges in the strong operator topology.

**Definition 2.2.** Let
\[
\text{Adm}_{\alpha}^{w} := \left\{ T \in L(H) : \sum_{n=0}^{\infty} |\alpha_{n}| \|T^{n}x\|^{2} < \infty \text{ for every } x \in H \right\}.
\]

Note that this class of operators is not affected if we change the signs of some coefficients $\alpha_{n}$'s.

**Notation 2.3.** In the next proposition and through this paper, we will use the following standard notation. If $X$ and $Y$ are two quantities (typically non-negative), then $X \lesssim Y$ (or $Y \gtrsim X$) will mean that $X \leq CY$ for some absolute constant $C > 0$. If the constant $C$ depends on some parameter $p$, then we write $X \lesssim_{p} Y$. We put $X \asymp Y$ when both $X \lesssim Y$ and $Y \lesssim X$.

**Proposition 2.4.** The following statements are equivalent.
(i) $T \in \text{Adm}_w^w$.
(ii) $\sum_{n=0}^{\infty} |\alpha_n| \|T^nx\|^2 \lesssim \|x\|^2$ for every $x \in H$.
(iii) The series $\sum_{n=0}^{\infty} |\alpha_n| T^*nT^n$ converges in the strong operator topology in $L(H)$.

Proof. Suppose that (i) is true. Note that for every $x, y \in H$ and $M > N$ we have

$$\left| \sum_{n=N+1}^{M} |\alpha_n| \langle T^n x, T^n y \rangle \right| \leq \sum_{n=N+1}^{M} |\alpha_n| \|T^n x\| \|T^n y\|$$

$$\leq \frac{1}{2} \left\{ \sum_{n=N+1}^{M} |\alpha_n| \|T^n x\|^2 + \sum_{n=N+1}^{M} |\alpha_n| \|T^n y\|^2 \right\} \to 0$$

as $N$ and $M$ go to infinity. Therefore

$$\sum_{n=0}^{\infty} |\alpha_n| \langle T^n x, T^n y \rangle \quad \text{converges (in } \mathbb{C} \text{), for every } x, y \in H.$$ Put

$$A_N := \sum_{n=0}^{N} |\alpha_n| T^*nT^n \in L(H)$$

for every non-negative integer $N$. Fix $x \in H$. By (2.2) we know that $\langle A_N x, y \rangle$ converges for every $y \in H$. This means that the sequence $\{A_N x\} \subset H$ is weakly convergent. Then $\sup_N \|A_N x\| < \infty$ for any $x \in H$ and therefore $\sup_N \|A_N\| < \infty$. Hence (ii) follows with absolute constant $\sup_N \|A_N\|$.

Now suppose we have (ii). This means that the operators $A_N$ given by (2.3) are uniformly bounded from above. So we can apply Lemma 2.1 to obtain (iii).

Finally, it is immediate that (iii) implies (i). This completes the proof. 

$\square$

Corollary 2.5. If $T \in \text{Adm}_w^w$ then the series

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^*nT^n$$

converges in the strong operator topology in $L(H)$.

Proof. Let $T \in \text{Adm}_w^w$. Using the equivalence between (i) and (iii) in Proposition 2.4, we know that the series $\sum |\alpha_n| T^*nT^n$ converges in SOT. Put

$$\alpha_n^+ := \begin{cases} \alpha_n & \text{if } \alpha_n \geq 0 \\ 0 & \text{if } \alpha_n < 0 \end{cases}, \quad \alpha_n^- := \begin{cases} 0 & \text{if } \alpha_n \geq 0 \\ -\alpha_n & \text{if } \alpha_n < 0 \end{cases}.$$ Hence

$$\sum_{n=0}^{N} \alpha_n T^*nT^n = \sum_{n=0}^{N} \alpha_n^+ T^*nT^n - \sum_{n=0}^{N} \alpha_n^- T^*nT^n.$$ It is immediate (using again Lemma 2.1) that both summands on the right hand side of (2.4) converge in SOT, and therefore the corollary follows. 

$\square$
This corollary allows us to introduce the following class of operators in $L(H)$, also depending on $\alpha$.

**Definition 2.6.** Let

$$C_w^\alpha := \{ T \in \text{Adm}_w^\alpha : \alpha(T^*, T) \geq 0 \}.$$ 

Sometimes, by abuse of notation, we simply write $\alpha(T^*, T) \geq 0$ instead of $T \in C_w^\alpha$. In particular, this means that $T \in \text{Adm}_w^\alpha$.

Recall that we denote by $A_W$ the Wiener algebra of analytic functions with summable Taylor coefficients.

**Proposition 2.7.** Let $T \in C_w^\alpha$. If $\alpha \notin A_W$, then $\sigma(T) \subset D$.

**Proof.** Let $T \in C_w^\alpha$, where $\alpha \notin A_W$ (that is, $\sum |\alpha_n| = \infty$). By Proposition 2.4 we know that there exists a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} |\alpha_n| \|T^n x\|^2 \leq C$$

for every $x \in H$ with $\|x\| = 1$.

Suppose that $T$ has spectral radius $\rho(T) \geq 1$. Let $\lambda$ be any point of $\sigma(T)$ such that $|\lambda| = \rho(T)$. Then $\lambda$ belongs to the boundary of the spectrum of $T$ and therefore it belongs to the approximate point spectrum. Put $R := |\lambda|^2 = \rho(T)^2 \geq 1$. Fix an integer $N$ sufficiently large so that

$$\sum_{n=0}^{N} |\alpha_n| > C + 1.$$ 

Next, choose a unit approximate eigenvector $h \in H$ corresponding to $\lambda$ so that

$$\|T^n h\|^2 - |\lambda|^{2n} \leq \left( \sum_{n=0}^{N} |\alpha_n| \right)^{-1}, \quad (m = 0, 1, \ldots, N).$$

Then

$$\sum_{n=0}^{N} |\alpha_n| R^n - \sum_{n=0}^{N} |\alpha_n| \|T^n h\|^2 \leq \sum_{n=0}^{N} |\alpha_n| \left| R^n - \|T^n h\|^2 \right| < 1,$$

and therefore

$$\sum_{n=0}^{N} |\alpha_n| \|T^n h\|^2 \geq \sum_{n=0}^{N} |\alpha_n| R^n - 1 \geq \sum_{n=0}^{N} |\alpha_n| - 1 > C.$$

But this contradicts (2.5). Hence $\rho(T)$ must be strictly less that 1, that is, $\sigma(T) \subset D$, as we wanted to prove. \qed

The next result follows immediately imitating the above proof. We denote by $r(\alpha)$ the radius of convergence of $\alpha$.

**Proposition 2.8.** If $T \in C_w^\alpha$, then $\rho(T)^2 \leq r(\alpha)$.
2.2. Analysis of the hypothesis on $\alpha$ and $k$. Let us study now the hypothesis we will impose through the rest of the paper on the functions $\alpha$ and $k = 1/\alpha$. First of all, the positivity of the coefficients $k_n$ is assumed in order to guarantee that we can obtain a reproducing kernel Hilbert space $H_k$ of analytic functions. Then, obviously, all the coefficients $\alpha_n$ are real. Normalizing, we can assume that $\alpha_0 = k_0 = 1$. Since we want to get rid of the condition $\sigma(T) \subset \mathbb{D}$, Proposition 2.7 shows that we need to assume that $\alpha \in A_W$. Finally, note that in Theorem 1.2, which is our new source of examples with respect to Theorem A, we need that $\alpha \notin A_W$. However, this assumption excludes automatically the critical case (when $\alpha(1) = 0$). Therefore, it is natural to just make the assumption that $k$ is analytic in $\mathbb{D}$, so we can still consider both cases: critical and subcritical. Observe that this assumption is the same as saying that $\alpha \notin A_W$ does not vanish on $\mathbb{D}$.

In our forthcoming paper Operator Inequalities II. Models up to similarity and Inclusions, we can get rid of the assumption that $\alpha$ does not vanish on $\mathbb{D}$ and just impose that $\alpha$ does not vanish on the interval $(0, 1)$. This assumption is in the spirit of our previous work [8].

2.3. The weighted shifts $B_\omega$ and $F_\omega$. Let $\omega$ be an analytic function with positive Taylor coefficients. This function gives a positive definite kernel $\omega(z, w) := \omega(\bar{w}z)$. We denote by $H_\omega$ the corresponding reproducing kernel Hilbert space. That is,

$$\langle f, \omega(\cdot, w) \rangle = f(w) \quad (\forall f \in H_\omega).$$

Obviously, the set of polynomials $e_n(t) := t^n$ for $n \geq 0$, is an orthogonal basis on $H_\omega$, and

$$\|e_n\|^2_{H_\omega} = \omega_n.$$

The backward and forward shifts $B_\omega$ and $F_\omega$ on $H_\omega$ are defined by

$$(2.7) \quad B_\omega f(t) := \frac{f(t) - f(0)}{t} \quad \text{and} \quad F_\omega f(t) := tf(t) \quad (\forall f \in H_\omega),$$

or equivalently

$$(2.8) \quad B_\omega e_n := \begin{cases} e_{n-1}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases} \quad \text{and} \quad F_\omega e_n := e_{n+1} \quad (\forall n \geq 0).$$

It is immediate that

$$\|B_\omega\|^2 = \sup_{n \geq 0} \frac{\omega_n}{\omega_{n+1}} \quad \text{and} \quad \|F_\omega\|^2 = \sup_{n \geq 0} \frac{\omega_{n+1}}{\omega_n}.$$

Therefore, in order to get the boundedness of both $B_\omega$ and $F_\omega$, we need to impose on the Taylor coefficients of $\omega$ the condition

$$(2.9) \quad C_1 \leq \frac{\omega_n}{\omega_{n+1}} \leq C_2 \quad (\forall n \geq 0)$$

for some positive constants $C_1, C_2$. 
Notation 2.9. Let us mention here a convenient notation that will be used in Section 6. We will deal with functions of the form $(1-t)^{-s}$ for real numbers $s$. Note that for $s > 0$ we have the reproducing kernel Hilbert space $H_{(1-t)^{-s}}$. We will just denote this space by $H_s$, emphasizing the exponent $s$. In the same way we use $B_s$ and $F_s$.

Lemma 2.10. Let $\kappa$ be an analytic function with positive Taylor coefficients satisfying (2.9). Let $T$ be one of the operators $B_\kappa$ or $F_\kappa$. Then:

(i) $T \in \text{Adm}_a$ if and only if

$$(2.10) \quad \sup_{m \geq 0} \left\{ \sum_{n=0}^{\infty} |\alpha_n| \frac{\|T^n e_m\|^2}{\|e_m\|^2} \right\} < \infty.$$

(ii) Suppose that $T \in \text{Adm}_a$. Then $T \in C_a$ if and only if

$$(2.11) \quad \sum_{n=0}^{\infty} \alpha_n \|T^n e_m\|^2 \geq 0 \quad (\forall m \geq 0).$$

Proof. (i) Let $T \in \text{Adm}_a$. By Proposition 2.4 (ii) we have

$$\sum_{n=0}^{\infty} |\alpha_n| \|T^n f\|^2 \lesssim \|f\|^2,$$

for every function $f \in H_\kappa$. Taking the vectors of the basis $f = e_m$ we obtain (2.10). Conversely, let us assume now (2.10). Fix a function $f \in H_\kappa$. Then

$$T^n f = \sum_{m=0}^{\infty} f_m T^n e_m \quad (\forall n \geq 0),$$

where the series are orthogonal. Therefore

$$\sum_{n=0}^{\infty} |\alpha_n| \|T^n f\|^2 = \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \|T^n e_m\|^2$$

$$= \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 \frac{\|T^n e_m\|^2}{\|e_m\|^2}$$

$$= \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 \sum_{n=0}^{\infty} |\alpha_n|^2 \frac{\|T^n e_m\|^2}{\|e_m\|^2}$$

$$\lesssim \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 < \infty,$$

where (2.10) allows us to justify the change of the summation indexes in the last equality.

(ii) Let $T \in \text{Adm}_a$. If $T \in C_a$, then obviously (2.11) follows. Conversely, if we have (2.11), then the last equality in (2.12) implies that $T \in C_a$. □

Particularizing this lemma for $B_\kappa$ and $F_\kappa$ separately, we immediately get the next two results.
Theorem 2.11. Let $\varphi$ be an analytic function with positive Taylor coefficients satisfying (2.9). Let $\beta$ be the analytic function whose Taylor coefficients are $\beta_n = |\alpha_n|$. Put $\gamma(t) = \beta(t) \varphi(t)$. Then:

(i) $B_{\varphi} \in \text{Adm}_\alpha^w$ if and only if
$$\sup_{m \geq 0} \left\{ \frac{\gamma_m}{\varphi_m} \right\} < \infty.$$

(ii) Suppose that $B_{\varphi} \in \text{Adm}_\alpha^w$. Then $B_{\varphi} \in C^w_{\alpha}$ if and only if all the Taylor coefficients of $\alpha(t) \varphi(t)$ are non-negative.

Before restating Lemma 2.10 for the forward shift $F_{\varphi}$, we need to introduce some terminology.

Notation 2.12. We denote by $\nabla$ the backward shift acting on one-sided sequences $\Lambda = \{\Lambda_m\}_{m \geq 0}$; that is, $\nabla\Lambda$ is the sequence whose $m$-th term is $\nabla\Lambda_m = \Lambda_{m+1}$, for every $m \geq 0$.

In general, if $\beta$ is an analytic function, we denote by $\beta(\nabla)\Lambda$ the sequence whose $m$-th term is given by
$$\beta(\nabla)\Lambda_m := \sum_{n=0}^{\infty} \beta_n \Lambda_{m+n}$$
whenever this series converges for every $m \geq 0$.

Theorem 2.13. Let $\varphi$ be an analytic function with positive Taylor coefficients satisfying (2.9). Let $\beta$ be the analytic function whose Taylor coefficients are $\beta_n = |\alpha_n|$. Then:

(i) $F_{\varphi} \in \text{Adm}_\alpha^w$ if and only if
$$\sup_{m \geq 0} \left\{ \frac{\alpha(\nabla)\varphi_m}{\varphi_m} \right\} < \infty.$$

(ii) Suppose that $F_{\varphi} \in \text{Adm}_\alpha^w$. Then $F_{\varphi} \in C_{\alpha}^w$ if and only if $\alpha(\nabla)\varphi_m \geq 0$ for every $m \geq 0$.

3. Proof of Theorem 1.3 and uniqueness of the model.

In this section we prove Theorem 1.3 and a result about the uniqueness of the model (see Theorem 3.7).

Let us start by proving that the operator $V_D$ given in (1.7) is a contraction in the Nevanlinna-Pick case.

Theorem 3.1. Let $\alpha_0 = 1$ and $\alpha_n \leq 0$ for $n \geq 1$. If $T \in L(H)$ satisfies $\alpha(T^*T) \geq 0$, then the operator $V_D$ given in (1.7) is a contraction.

Proof. Recall that $D^2 = \alpha(T^*T)$. Therefore
$$\|Dx\|^2 = \sum_{m=0}^{\infty} \alpha_m \|T^m x\|^2$$
for every $x \in H$. Hence
\[
\|DT^n x\|^2 = \sum_{m=0}^{\infty} \alpha_m \|T^{m+n} x\|^2
\]
for every $x \in H$ and every non-negative integer $n$. Fix a positive integer $N$. Then
\[
\sum_{n=0}^{N} k_n \|DT^n x\|^2 = \sum_{n=0}^{N} k_n \sum_{m=0}^{\infty} \alpha_m \|T^{m+n} x\|^2 \leq \sum_{j=0}^{\infty} \sum_{n+m=j} \tau_j \|T^j x\|^2,
\]
where in (*) we can rearrange the series because it converges absolutely (just use Proposition 2.4 (ii)). Since $\alpha k = 1$ we get
\[
\tau_0 = 1 \quad \text{and} \quad \tau_1 = \cdots = \tau_N = 0.
\]
Moreover,
\[
\tau_{N+i} = k_0 \alpha_{N+i} + \cdots + k_N \alpha_i < 0
\]
for every $i \geq 1$, because all the $\alpha_j$’s above are negative and the $k_j$’s are positive. Therefore
\[
\sum_{n=0}^{N} k_n \|DT^n x\|^2 \leq \|x\|^2
\]
for every $N$ and hence the series $\sum k_n \|DT^n x\|^2$ converges for every $x \in H$. This gives
\[
\|VD x\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 \leq \|x\|^2,
\]
as we wanted to prove. $\Box$

The following result is a simple and well-known fact.

**Proposition 3.2.** Let $T \in L(H)$ and let $\mathcal{R}$ be a Hilbert space. A bounded transform $V : H \to \mathcal{H}_\mathcal{R} \otimes \mathcal{R}$ satisfies
\[
VT = (B_\mathcal{R} \otimes I_{\mathcal{R}}) V
\]
if and only if there is a bounded linear operator $C : H \to \mathcal{R}$ such that $V = V_C$, where
\[
V_C x(z) = C(I - zT)^{-1} x, \quad x \in H.
\]

**Proof.** Let us suppose first that $V = V_C$ for some bounded linear operator $C : H \to \mathcal{R}$. That is,
\[
V_C x(z) = \sum_{n=0}^{\infty} C T^n x z^n.
\]
Recall that $B_\mathcal{R} \otimes I_{\mathcal{R}}$ is the operator on $\mathcal{H}_\mathcal{R} \otimes \mathcal{R}$ that sends
\[
\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_{n+1} z^n
\]
(where $a_n \in \mathcal{R}$). Therefore
\[
V_C T x(z) = \sum_{n=0}^{\infty} C T^{n+1} x z^n = (B_\mathcal{R} \otimes I_{\mathcal{R}}) V_C x(z).
\]
Conversely, suppose now that \( VT = (B_{\kappa} \otimes I_{\mathbb{R}})V \). Put
\[
V(x) := \sum_{n=0}^{\infty} a_n(x)z^n, \quad x \in H.
\]
Then
\[
\sum_{n=0}^{\infty} a_n(Tx)z^n = VTx = (B_{\kappa} \otimes I_{\mathbb{R}})Vx = \sum_{n=0}^{\infty} a_{n+1}(x)z^n.
\]
Therefore \( a_{n+1}(x) = a_n(Tx) \) and the statement follows using that \( a_0 : H \to \mathbb{R} \) must be a bounded linear operator (and then put \( C := a_0 \)).

**Proposition 3.3.** Let \( C : H \to \mathbb{R} \) be a bounded operator and let \( T \in C_{\alpha}^w \). Then there exists a bounded operator \( W : H \to \mathcal{W} \) such that the operator \((V_C, W)\) is isometric and transforms \( T \) into a part of the operator \((B_{\kappa} \otimes I_{\mathbb{R}}) \oplus S\), where \( S \in L(\mathcal{W}) \) is an isometry, if and only if the following conditions hold.

(i) \( V_C : H \to \mathcal{H}_k \otimes \mathbb{R} \) is a contraction.

(ii) The following identity holds:
\[
\|x\|^2 - \|V_Cx\|^2 = \|Tx\|^2 - \|V_CTx\|^2, \quad \forall x \in H.
\]

**Proof.** Let us suppose first the existence of such operator \( W \). Since \((V_C, W)\) is an isometry, (i) holds. Notice that (ii) is equivalent to proving that \( \|Wx\|^2 = \|WTx\|^2 \) for every \( x \in H \). But this is also immediate since \( SWx = WTx \) and \( S \) is an isometry.

Conversely, suppose now that (i) and (ii) are true. By (i), we can put \( W := (I - V_C^*V_C)^{1/2} \) and \( W := \text{Ran } W \). Using (ii) we have
\[
\|Wx\|^2 = \|x\|^2 - \|V_Cx\|^2 = \|Tx\|^2 - \|V_CTx\|^2 = \|WTx\|^2.
\]
We define
\[
S(Wx) := WTx,
\]
for every \( x \in H \). Note that \( S \) is well defined, since \( \|SWx\| = \|Wx\| \) by \((3.1)\). Since \( WH \) is dense in \( \mathcal{W} \), \( S \) can be extended to an isometry on \( \mathcal{W} \). By \((3.1)\) we know that \((V_C, W)\) is an isometry and it is immediate that
\[
(B_{\kappa} \otimes I_{D})V_C = V_CT \quad \text{and} \quad SW = WT.
\]
This completes the converse implication.

**Proposition 3.4.** Let \( T \in C_{\alpha}^w \). Assume that \( C : H \to \mathbb{R} \) and \( W : H \to \mathcal{W} \) are any bounded operators, such that \((V_C, W)\) is isometric and transforms \( T \) into a part of \((B_{\kappa} \otimes I_{\mathbb{R}}) \oplus S\), where \( S \in B(\mathcal{W}) \) is an isometry. Then
\[
\|Dx\|^2 = \|Cx\|^2 + \alpha(1)\|Wx\|^2, \quad \forall x \in H.
\]

**Proof.** Since \((V_C, W)\) is isometric, we have
\[
\|x\|^2 = \|V_Cx\|^2 + \|Wx\|^2 = \sum_{n=0}^{\infty} k_n \|CT^nx\|^2 + \|Wx\|^2.
\]
for every $x \in H$. Substituting $x$ by $T^j x$ above and multiplying by $\alpha_j$, we obtain that
\[
\alpha_j \|T^j x\|^2 = \sum_{n=0}^{\infty} \alpha_j k_n \|CT^{n+j} x\|^2 + \alpha_j \|WT^j x\|^2
\]
\[
= \sum_{n=0}^{\infty} \alpha_j k_n \|CT^{n+j} x\|^2 + \alpha_j \|W x\|^2,
\]
where we have used that $\|W x\|^2 = \|WT x\|^2$. Therefore
\[
\|D x\|^2 = \sum_{j=0}^{\infty} \alpha_j \|T^j x\|^2 = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \alpha_j k_n \|CT^{j+n} x\|^2 + \left( \sum_{j=0}^{\infty} \alpha_j \right) \|W x\|^2
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{j+n=m} \alpha_j k_n \right) \|CT^{n} x\|^2 + \alpha(1) \|W x\|^2.
\]
Since $\alpha k = 1$, the only non-vanishing summand in the last series above is for $m = 0$ and we obtain (3.2). Finally, note that the rearrangement in $(\ast)$ is correct as
\[
\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} |\alpha_j| k_n \|CT^{n+j} x\|^2 \leq \sum_{j=0}^{\infty} |\alpha_j| \|T^j x\|^2 < \infty,
\]
where we have used (3.3) and that $T \in C^\omega$.

By a minimal model we understand the following.

**Definition 3.5.** Let $\mathcal{L}$ be an invariant subspace of $(B_k \otimes R) \oplus S$, where $S \in L(W)$. We will say that the corresponding model operator
\[
\left( (B_k \otimes R) \oplus S \right)|\mathcal{L}
\]
is minimal if the following two conditions hold.

(i) $\mathcal{L}$ is not contained in $(\mathcal{H}_k \otimes \mathcal{R}') \oplus \mathcal{W}$ for any $\mathcal{R}' \subset \neq \mathcal{R}$.

(ii) The vectors $P_W \ell, \ell \in \mathcal{L}$ are dense in $\mathcal{W}$, where $P_W : (\mathcal{H}_k \otimes \mathcal{R}) \oplus \mathcal{W} \to \mathcal{W}$ is the orthogonal projection onto the second direct summand.

**Remark 3.6.** Suppose that the answer to Question 1.1 is affirmative. Then $T$ is unitarily equivalent to $\left( (B_k \otimes I_R) \oplus S \right)|\mathcal{L}$, where $\mathcal{L} = \text{Ran} (V_C, W)$ (see Proposition 3.2). Note that this model is minimal if and only if
(a) $\text{Ran} C = \mathcal{R}$; and
(b) $\text{Ran} W = \mathcal{W}$.

Indeed, in this case, it is easy to see that (a) is equivalent to (i), and (b) is equivalent to (ii) in the previous definition.

**Theorem 3.7.** Suppose that the answer to Question 1.1 is affirmative.
In the critical case the minimal model is unique and is obtained by taking $V = V_D$. More precisely, the pair of transforms $(V_D, W_0)$, where $W_0 = (I - V_D^2V_D) : H \to \mathcal{W}_0$, and $\mathcal{W}_0 := \text{Ran} W_0$ gives rise to a minimal model, and any minimal model is provided by $(V_C, W)$, where $C = vD$, $W = wW_0 : H \to W$ and $v, w$ are unitary isomorphisms.

(ii) In the subcritical case the minimal model is not unique (in general). However, there always exists a model for which $V = V_D$ and $W$ is absent (that is, $W = 0$).

**Proof.** (i) In the critical case (i.e., $\alpha(1) = 0$), (3.2) gives
$$\|Dx\| = \|Cx\| \quad (\forall x \in H),$$
so there exists a unitary operator $v$ such that $C = vD$. This implies the statement.

(ii) Suppose we are in the subcritical case (i.e., $\alpha(1) \neq 0$). Changing $x$ by $T^n x$ in (3.2) we obtain
$$\|DT^n x\|^2 = \|CT^n x\|^2 + \alpha(1) \|W x\|^2,$$
where we have used that $\|WTx\| = \|Wx\|$. Therefore
$$\|V_D x\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 = \sum_{n=0}^{\infty} k_n \|CT^n x\|^2 + k(1) \alpha(1) \|W x\|^2$$
$$= \|V_C x\|^2 + \|W x\|^2 = \|x\|^2,$$
so $V_D : H \to \mathcal{H}_k \otimes \mathcal{R}$ is an isometry and (ii) follows.

**Proof of Theorem 1.3.** It is an immediate consequence of Theorem 3.7 and Proposition 3.3 (i) that $V_D$ is a contraction. Finally, for proving that $(V_D, W)$ gives model, we just need to use the same argument employed in the reciprocal implication of Proposition 3.3.

As mentioned right after Theorem 1.3, the model obtained is minimal, since $\text{Ran} \, D$ is dense in $\mathcal{D}$, and $\text{Ran} \, W$ is dense in $\mathcal{W}$. The space $\mathcal{L}$ is just the closure of $\text{Ran}(V_D, W)$ in $(\mathcal{H}_k \otimes \mathcal{D}) \otimes \mathcal{W}$. (See Remark 3.6)

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. For that, we need some results concerning Banach algebras.

For any sequence of weights $\omega = \{\omega_n\}_{n=0}^{\infty}$ put
$$\ell^\infty(\omega) := \left\{ f(t) = \sum_{n=0}^{\infty} f_n t^n : \sup_{n \geq 0} |f_n| \omega_n < \infty \right\}.$$  
In general, it is an algebra of formal power series with respect to their formal multiplication.

**Proposition 4.1** (see [26]), $\ell^\infty(\omega)$ is a Banach algebra if and only if
$$\sup_{n \geq 0} \sum_{j=0}^{n} \frac{\omega_n}{\omega_j \omega_{n-j}} < \infty.$$ (4.1)
**Theorem 4.2.** Let $\omega_n > 0$ and $\omega_n^{1/n} \to 1$. If

\begin{equation}
\frac{\omega_n}{\omega_j \omega_{n-j}} \leq \tau_j \quad (\forall n \geq 2j)
\end{equation}

and \( \sum_{j=0}^\infty \tau_j < \infty \),

then the following is true.

(i) $\ell^\infty(\omega)$ is a Banach algebra.

(ii) If $f \in \ell^\infty(\omega)$ does not vanish on $\mathbb{D}$, then $1/f \in \ell^\infty(\omega)$.

We remark that instead of the condition (4.2), it suffices to require only a weaker condition

\begin{equation}
\lim_{m \to \infty} \sup_{n \geq 0} \sum_{m \leq j \leq n/2} \frac{\omega_n}{\omega_j \omega_{n-j}} = 0.
\end{equation}

This can be deduced from Theorem 3.4.1 and Lemma 3.6.3 from the paper [17] by El-Fallah, Nikolski and Zarrabi.

**Proof of Theorem 4.2.** (i) It is immediate that (4.2) implies (4.1). Hence the statement follows using Proposition 4.1.

(ii) Put $g := 1/f$. Suppose that $g \not\in \ell^\infty(\omega)$. This means that

\[ \sup_{n \geq 0} |g_n| \omega_n = \infty. \]

Hence, it is clear that there exists a sequence \( \{\rho^0_n\} \) in $[0,1]$ such that $\rho^0_n \to 0$ (slowly) and

\begin{equation}
\sup_{n \geq 0} |g_n| \omega_n \rho^0_n = \infty.
\end{equation}

**Claim.** There exists a sequence \( \{\rho_n\} \) with

\begin{equation}
\rho^0_n \leq \rho_n \leq 1 \quad \text{and} \quad \rho_n \to 0
\end{equation}

such that $\tilde{\omega}_n := \rho_n \omega_n$ defines a Banach algebra $\ell^\infty(\tilde{\omega})$.

Indeed, since $\sum \tau_j < \infty$, there exists a sequence of positive numbers \( \{c_j\} \) such that $c_j \not\to \infty$ and $\sum c_j \tau_j < \infty$. Then, obviously, any sequence \( \{\gamma_j\} \) such that

\begin{equation}
0 < \gamma_j \leq c_j
\end{equation}

satisfies that $\sum \gamma_j \tau_j < \infty$. Consider the sequence \( \{\rho_n\} \) given by

\begin{equation}
1 \quad 1 \quad \ldots \quad 1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \ldots \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{3} \quad \ldots \quad \frac{1}{3} \quad \ldots
\end{equation}

where $1/j$ is repeated $R_j$ times. Taking the blocks $R_j$ large enough (understanding that first we take $R_1$ large enough, then $R_2$ large enough, and so on), we can obviously guarantee (4.5) and also $\gamma_j := 1/\rho_j$ satisfies (4.6). Therefore, for $\tilde{\omega}_n := \rho_n \omega_n$ we have

\[ \frac{\tilde{\omega}_n}{\omega_j \tilde{\omega}_{n-j}} = \frac{\omega_n}{\omega_j \omega_{n-j}} \rho_n \leq \frac{\omega_n}{\omega_j \omega_{n-j}} \frac{1}{\rho_j \rho_{n-j}} \leq \tau_j \gamma_j \quad (\forall n \geq 2j) \]
and
\[ \sum_{j=0}^{\infty} \tau_j \gamma_j < \infty. \]

Hence, (i) gives that \( \ell^\infty(\tilde{\omega}) \) is a Banach algebra, and the proof of the Claim is complete.

Now fix \( \tilde{\omega}_n \) as in the Claim. Since
\[ f_n \tilde{\omega}_n = (f_n \omega_n) \rho_n \to 0, \]
we have \( f \in \ell^\infty(\tilde{\omega}) \) (the closure of the polynomials in \( \ell^\infty(\tilde{\omega}) \)). Then, using Gelfand theory, we get \( g \in \ell^\infty(\tilde{\omega}) \). In particular, \( g \in \ell^\infty(\omega) \), which means that
\[ \sup_{n \geq 0} |g_n| \tilde{\omega}_n < \infty. \]

But this contradicts (4.4). Therefore, the assumption \( g = 1/f \notin \ell^\infty(\omega) \) is false, as we wanted to prove. \( \square \)

**Proof of Theorem 1.2.** Note that statements (i) and (ii) of Theorem 1.2 can be rewritten as
\[ (4.8) \quad T \in C^w \alpha \cap \text{Adm}^w k. \]

Also, by Theorem 4.2 (i), \( \ell^\infty(\omega) \) is an algebra for \( \omega_n := 1/k_n \).

Suppose that \( T \) is a part of \( B_k \otimes I_R \). Then we just need to prove that \( B_k \) satisfies (4.8).

By Theorem 2.11 (i), we know that \( B_k \in \text{Adm}^w_k \) if and only if
\[ \sum_{j=0}^{m} k_j k_{m-j} \lesssim k_m, \]
which follows from Theorem 2.12 (i) and Proposition 4.1.

Now let us see that \( B_k \in C^w_\alpha \). By Theorem 4.2 (ii), \( \alpha = 1/k \) belongs to \( \ell^\infty(\omega) \), and therefore \( |\alpha_n| \lesssim k_n \). Then, since \( B_k \in \text{Adm}^w_k \), we obtain that \( B_k \in \text{Adm}^w_k \). Hence, Theorem 2.11 (ii) gives that \( B_k \in C^w_\alpha \) (because \( \alpha k = 1 \) has non-negative Taylor coefficients).

Conversely, let us assume now (4.8). We want to prove that \( T \) is a part of \( B_k \otimes I_R \). We adapt the argument of [24, Theorem 2.2].

Consider the operator
\[ V : H \to H \otimes \mathcal{D}, \quad Vx := \{Dx, DTx, DT^2x, \ldots\}. \]

Then obviously
\[ (B_k \otimes I_\mathcal{D})V = VT. \]

Moreover,
\[ \|Vx\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 = \sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} \alpha_m \|T^{n+m} x\|^2 \]
\[ = \sum_{j=0}^{\infty} \left( \sum_{n+m=j} k_n \alpha_m \right) \|T^j x\|^2 = \|x\|^2, \]
where we have used that $\sum_{n+m=j} k_n \alpha_m$ is equal to 1 if $j = 0$ and is equal to 0 if $j \geq 1$. The re-arrangement of the series is correct, since using (4.8) we have

$$\sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} |\alpha_m| \|T^{n+m}x\|^2 \lesssim \sum_{n=0}^{\infty} k_n \|T^n x\|^2 \lesssim \|x\|^2$$

and the series converges absolutely.

Hence $V$ is an isometry. Joined to (4.9), this proves that $T$ is a part of $B_k \otimes I_D$. □

5. Analysis of condition (1.5)

In this section we study the scope of condition (1.5) with examples where Theorem A does not apply.

Given an analytic function $f(t) = \sum f_n t^n$, we denote by $[f]_N$ its truncated polynomial of degree $N$, that is,

$$[f]_N := f_0 + f_1 t + \ldots + f_N t^N.$$

Example 5.1. Let $\sigma_2, \ldots, \sigma_N$ be an arbitrary sequence of signs (that is, a sequence of numbers $\pm 1$). We assert that there are functions $\alpha, k$ meeting all the hypotheses of Theorem 1.2 such that $\text{sign}(\alpha_n) = \sigma_n$, for $n = 2, \ldots, N$. Indeed, let $\tilde{\alpha}$ be a polynomial of degree $N$ such that $\tilde{\alpha}_0 = 1, \tilde{\alpha}_1 < 0$, and for $n = 2, \ldots, N$, let $\tilde{\alpha}_n < 0$ whenever $\sigma_n = -1$ and $\tilde{\alpha}_n = 0$ whenever $\sigma_n = 1$. We also impose that neither $\tilde{\alpha}$ nor the polynomial $\hat{k} := [1/\tilde{\alpha}]_N$ vanish on $D$. The existence of such $\tilde{\alpha}$ is obvious. The formula

$$\tilde{k}_n = \sum_{s \geq 1}^{n} \tilde{\alpha}_{n_1} \ldots \tilde{\alpha}_{n_s}$$

shows that all the coefficients of $\tilde{k}$ are positive. Now perturb the coefficients $\tilde{\alpha}_j$ that are equal to zero, obtaining a new function $\hat{\alpha}$, whose Taylor coefficients are

$$\hat{\alpha}_j := \begin{cases} \varepsilon & \text{if } \sigma_j = 1 \\ \tilde{\alpha}_j & \text{otherwise} \end{cases}.$$

By continuity, if $\varepsilon > 0$ is small enough, we can guarantee that the polynomial $\hat{k} = [1/\hat{\alpha}]_N$ also have positive Taylor coefficients, and we can also guarantee that $\hat{k}$ (which is a slight perturbation on $\tilde{k}$) does not vanish on $D$.

Finally, take as $k$ any function in $A_W$ whose first Taylor coefficients are

$$k_0 = \hat{k}_0 = 1, \quad k_1 = \hat{k}_1, \quad \ldots, \quad k_N = \hat{k}_N,$$

and such that

$$\frac{k_{n-j}}{k_n} \leq C \quad (\forall n \geq 2j),$$

for some constant $C$. For instance, one can put $k_n = An^{-b}$ for $n \geq N$, with $A > 0$ (small) and $b > 1$. It can be achieved that $k \in A_W$ does not vanish on $D$.

Then obviously $k$ satisfies the hypotheses of Theorem 1.2 (taking $\tau_j := Ck_j$) and $\alpha := 1/k$ in $A_W$ has the desired pattern of signs.
Finally, it is important to note that $\alpha_1 = -k_1$ is always negative.

**Example 5.2.** In the above example, with $k_n = An^{-b}$ for $n \geq N$, note that $k_n/k_{n+1} \to 1$. There are also examples of functions meeting the hypotheses of Theorem 1.2 such that the quotients $k_n/k_{n+1}$ do not converge. Indeed, take any function $k$ meeting these hypotheses and define $\hat{k}$ by $\hat{k}_0 = 1$ and $\hat{k}_n = g_n k_n$, $n \geq 1$, where $0 < \varepsilon < g_n < \varepsilon'$ for all $n$ and $g_n/g_{n+1}$ do not have limit. If $\varepsilon'$ is small enough, then $\hat{k}$ does not vanish on $\mathbb{D}$ and therefore the function $\hat{k}$ satisfies all the hypotheses of Theorem 1.2 but there is no limit of $\hat{k}_{n+1}/\hat{k}_n$.

6. Ergodic properties of $a$-contractions

In this section we focus only on functions $\alpha$ of the form

$$\alpha(t) = (1 - t)^a$$

for some $a > 0$. Note that indeed these type of functions satisfy all the hypotheses we impose in Subsection 2.2. Moreover, here we are in the critical case. We will study the particular properties of the classes $\text{Adm}_w^a$ and $C_w^a$ for these functions $\alpha$.

When $0 < a < 1$ we can apply Theorem A to obtain a model for operators $T \in L(H)$ satisfying $\alpha(T^*, T) \geq 0$. With the help of this model, we will derive some ergodic properties for $T$.

6.1. $a$-contractions. In order to emphasize the dependence on the exponent $a$ in (6.1), when $T \in C_w^a$ we will say that $T$ is an $a$-contraction, and instead of $\text{Adm}_w^a$ we will use the notation $\text{Adm}_w$. For example, the 1-contractions are just the contractions in $L(H)$.

Recall that the weighted Dirichlet space $\mathcal{D}_s$, with $s \in \mathbb{R}$, consists of all the analytic functions $f$ in $\mathbb{D}$ with finite norm

$$\|f\|_{\mathcal{D}_s} := \left( \sum_{n=0}^{\infty} (n+1)^s |f_n|^2 \right)^{1/2}.$$ 

Since $k(t) = 1/\alpha(t) = (1 - t)^{-a}$, it is easy to see that its Taylor coefficients $k_n$ have the asymptotic behaviour

$$(6.2) \quad k_n \asymp (n+1)^{a-1}.$$

In Proposition 6.3 we will see more properties of these coefficients.

Therefore, the norms in $\mathcal{H}_k$ and $\mathcal{D}_{a-1}$ are equivalent.

As an immediate consequence of Theorem 2.11 we obtain the following result. (See Notation 2.9)

**Theorem 6.1.** Let $a$ and $s$ be positive numbers. Then the following is true.

(i) $B_s \in \text{Adm}_w^a$.

(ii) $B_s$ is an $a$-contraction if and only if $a \leq s$. 
6.2. Ergodic properties. In this subsection we will study some ergodic properties of \(a\)-contractions. We start by introducing the Cesàro numbers. We consider the general setting of \(a \in \mathbb{C}\) to sate its definition and basic properties since they are not affected at all by this generality. After that, \(a\) is considered a real parameter, which is the situation we are dealing with.

**Definition 6.2.** Let \(a\) be a complex number. For any non-negative integer \(n\), we denote by \(k^a(n)\) the \(n\)-th Taylor coefficient of the function \((1 - t)^{-a}\); that is,

\[
(1 - t)^{-a} = \sum_{n=0}^{\infty} k^a(n) t^n \quad (|t| < 1).
\]

Therefore,

\[
k^a(n) = (-1)^n \left( \frac{a}{n} \right) = \begin{cases} 
    a(a+1) \cdots (a+n-1) & \text{if } n \geq 1 \\
    1 & \text{if } n = 0 
\end{cases}.
\]

These numbers are called *Cesàro numbers*. See [31, Vol. I, p.77].

Here we list some basic properties of the Cesàro numbers.

**Proposition 6.3.**

(i) Let \(m\) be a non-negative integer. Then

\[
k^{-m}(n) = 0 \quad \forall n \geq m + 1.
\]

Moreover, if \(m < a < m + 1\), then

\[
\text{sign } k^{-a}(n) = \begin{cases} 
    (-1)^n & \text{if } 0 \leq n \leq m \\
    (-1)^{m+1} & \text{if } n \geq m + 1
\end{cases}.
\]

(ii) If \(a < 0\), then

\[
\sum_{n=0}^{\infty} k^a(n) = 0.
\]

(iii) If \(0 < a \leq b\), then

\[
0 \leq k^a(n) \leq k^b(n) \quad \forall n \geq 0.
\]

(iv) As a function of \(n\), \(k^a\) is increasing for \(a > 1\), decreasing for \(0 < a < 1\), and \(k^1(n) = 1\) for every \(n \geq 0\).

(v) If \(a \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}\), then

\[
k^a(n) = \frac{\Gamma(n + a)}{\Gamma(a) \Gamma(n + 1)} = \left( \begin{array}{c} n + a - 1 \\ a - 1 \end{array} \right) \quad \forall n \geq 0,
\]

where \(\Gamma\) is the Euler’s Gamma function. Therefore

\[
k^a(n) = \frac{n^{a-1}}{\Gamma(a)} (1 + O(1/n)) \quad \text{as } n \to \infty.
\]
Moreover, if \(0 < a \leq 1\), then
\[
\frac{(n+1)^a-1}{\Gamma(a)} \leq k^a(n) \leq \frac{n^{a-1}}{\Gamma(a)} \quad \forall n \geq 1.
\]

Proof. (i), (ii) and (iii) are immediate. For (iv) see [31, Thm. III.1.17]. For (v) see see [31, Vol.I, p.77 (1.18)] and [19, Eq.(1)]. The last inequality of (v) follows from the Gautschi inequality (see [20, Eq.(7)]).

**Definition 6.4.** Let \(a\) be a non-negative (real) number. For any bounded linear operator \(T\) on a Banach space \(X\), we call the family of operators \(\{M^a_T(n)\}_{n \geq 0}\) given by
\[
M^a_T(n) := \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j)T^j,
\]
the Cesàro mean of order \(a\) of \(T\). When this family of operators is uniformly bounded, that is,
\[
\sup_{n \geq 0} \|M^a_T(n)\| < \infty,
\]
we say that \(T\) is \((C,a)\)-bounded.

Note that \((C,0)\)-boundedness is just power boundedness and \((C,1)\)-boundedness is the Cesàro boundedness. Also note that in this definition all the weights \(k^a(j)\)'s are non-negative because \(a \geq 0\).

It is well-known that if \(0 \leq a < b\), then \((C,a)\)-boundedness implies \((C,b)\)-boundedness. The converse is not true in general. For example, the Assani matrix
\[
T = \begin{pmatrix}
-1 & 2 \\
0 & -1
\end{pmatrix}
\]
is \((C,1)\)-bounded, but since
\[
T^n = \begin{pmatrix}
(-1)^n & (-1)^{n+1}2n \\
0 & (-1)^n
\end{pmatrix},
\]
it is not power bounded (see [18, Section 4.7]).

Properties, characterization through functional calculus and ergodic results for \((C,a)\)-bounded operators can be found in [3, 5, 13, 15, 16, 18, 22] and references therein.

**Definition 6.5.** If the sequence of operators \(\{M^a_T(n)\}_{n \geq 0}\) given in Definition 6.4 converges in the strong operator topology, we say that \(T\) is \((C,a)\)-mean ergodic.

If \(T\) is \((C,1)\)-mean ergodic, it is conventional just to say that \(T\) is mean ergodic.

In [23], Luo and Hou introduced a new definition of boundedness: a bounded linear operator \(T\) on a Banach space \(X\) is said to be absolutely Cesàro bounded if
\[
\sup_{n \geq 0} \frac{1}{n+1} \sum_{j=0}^{n} \|T^j x\| \lesssim \|x\|
\]
for every $x \in X$. This definition has been extended recently by the first author and Bonilla in [1]: $T$ is said to be absolutely $(C, a)$-Cesàro bounded for some $a > 0$ if

$$\sup_{n \geq 0} \frac{1}{k^a+1(n)} \sum_{j=0}^{n} k^a(n-j) \|T^j x\| \lesssim \|x\|$$

for every $x \in X$. Note that for $a = 1$ the definition of Luo and Hou is recovered.

We observe that the following relation holds.

$$\text{Power bounded } \Rightarrow \text{Absolutely } (C, a)\text{-bounded} \Rightarrow (C, a)\text{-bounded} \Rightarrow \|T^n\| = O(n^a).$$

The following extension of the above definitions will be important for us.

**Definition 6.6.** Let $a > 0$ and $p \geq 1$. We say that a bounded linear operator $T$ on a Banach space $X$ is $(C, a, p)$-bounded if

$$\sup_{n \geq 0} \frac{1}{k^a+1(n)} \sum_{j=0}^{n} k^a(n-j) \|T^j x\|^p \lesssim \|x\|^p,$$

for all $x \in X$.

Note that for $p = 1$ this definition is just the absolutely $(C, a)$-boundedness. We will use the term quadratically $(C, a)$-bounded instead of $(C, a, 2)$-bounded.

Using the asymptotic equivalence $k^a(n) \asymp (n+1)^a-1$, it is easy to see that $T$ is $(C, a, p)$-bounded if and only if

$$\sup_{n \geq 0} \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^a-1 \|T^j x\|^p \lesssim \|x\|^p \quad (\forall x \in X).$$

The following result will be essential for the proof of Theorem 1.5.

**Lemma 6.7.** The following holds.

(i) If $T$ is $(C, a, p)$-bounded, then any part of $T$ is also $(C, a, p)$-bounded.

(ii) If $T_1$ and $T_2$ are $(C, a, p)$-bounded, then the direct sum $T_1 + T_2$ is also $(C, a, p)$-bounded.

(iii) Let $T$ be a bounded linear operator on a Hilbert space. If $T$ is quadratically $(C, a)$-bounded, then $T \otimes I_R$ is also quadratically $(C, a)$-bounded, where $I_R$ is the identity operator on some Hilbert space $R$.

**Proof.** (i) and (ii) are immediate. For (iii) note that if $d = \dim R \leq \infty$, then the orthogonal sum of $d$ copies of $T$ is clearly quadratically $(C, a)$-bounded (by the Pythagoras Theorem).

The following result is very useful and its proof is simple.

**Lemma 6.8.** Let $0 \leq a < b$. Then $(C, a, p)$-boundedness implies $(C, b, p)$-boundedness.
Theorem 6.9. Let $a > 0$ and $1 \leq q < p$. If $T$ is $(C, a, p)$-bounded, then it is also $(C, b, q)$-bounded for each $b > qa/p$.

In particular, we obtain that $(C, a, p)$-boundedness implies $(C, a, q)$-boundedness.

For the proof of this Theorem (and later results), it is convenient to recall that if $r > -1$, then

$$m \sum_{j=1}^{m} j^{r} \lesssim m^{r+1} \quad (\forall m \geq 1).$$

Proof of Theorem 6.9. Let

$$s := \frac{p}{p - q}, \quad s' := \frac{p}{q}, \quad \gamma := \frac{q(a - 1)}{p(b - 1)}.$$

Note that $s$ and $s'$ are positive and satisfy $1/s + 1/s' = 1$. Since

$$(b - 1)(1 - \gamma)s = \frac{pb - qa}{p - q} - 1 > -1 \quad \text{and} \quad (b - 1)s' = a - 1,$$

using Hölder’s inequality and (6.4) it follows that

$$\frac{1}{(n + 1)^{a}} \sum_{j=0}^{n} (n + 1 - j)^{b - 1} \|T^{j}x\|^{q}$$

$$\leq \frac{1}{(n + 1)^{a}} \left( \sum_{j=0}^{n} (n + 1 - j)^{(b - 1)(1 - \gamma)s} \left( \sum_{j=0}^{n} (n + 1 - j)^{(b - 1)s'} \|T^{j}x\|^{q's'} \right)^{1/s'} \right)^{q/p}$$

$$\lesssim (n + 1)^{-qa/p} \left( \sum_{j=0}^{n} (n + 1 - j)^{a - 1} \|T^{j}x\|^{p} \right)^{q/p}$$

$$= \left( \frac{1}{(n + 1)^{a}} \sum_{j=0}^{n} (n + 1 - j)^{a - 1} \|T^{j}x\|^{p} \right)^{q/p}$$

for every $x \in X$ and every non-negative integer $n$. Hence the statement follows using (6.3). \hfill \Box

Lemma 6.10. Let $a > 0$ and $p \geq 1$. Then every isometry $S$ is $(C, a, p)$-bounded.

Proof. This is immediate, since indeed

$$\frac{1}{k^{\alpha + 1}(n)} \sum_{j=0}^{n} k^{\alpha}(n - j) \|S^{j}x\|^{p} = \frac{1}{k^{\alpha + 1}(n)} \left( \sum_{j=0}^{n} k^{\alpha}(n - j) \right) \|x\|^{p} = \|x\|^{p}$$

for every $x \in X$. \hfill \Box

Lemma 6.11. Let $0 < s < 1$ and let $a > 0$. Then $B_{s}$ is quadratically $(C, a)$-bounded if and only if $1 - s < a$. Moreover, for $1 - s < a$ we have

$$\lim_{n \to \infty} \frac{1}{k^{\alpha + 1}(n)} \sum_{j=0}^{n} k^{\alpha}(n - j) \|B_{s}^{j}x\|^{2} = 0 \quad (\forall x \in H).$$
Proof. Suppose that $a = 1 - s$. Then

\[
\frac{1}{(n + 1)^a} \sum_{j=0}^{n} (n + 1 - j)^{a-1} \|B_s^j f_n\|^2 \geq \frac{1}{(n + 1)^{1-s}} \sum_{j=0}^{n} (n + 1 - j)^{-s(n + 1 - j)^{a-1}} \]

(6.7)

\[
= \frac{1}{(n + 1)^{1-s}} \sum_{j=1}^{n+1} j^{-1} \geq \log(n + 2) \|f_n\|^2
\]

for every $n$. Therefore $B_s$ is not quadratically $(C,1-s)$-bounded, and by Lemma 6.8 we obtain that $B_s$ is not quadratically $(C,a)$-bounded for $a < 1 - s$.

Let us assume now that $1 - s < a \leq 1$ and fix $x \in H$. Write $x$ in the form $x = \sum x_m f_m$, where $x_m \in \mathbb{C}$. Then

\[
\|B_s^j x\|^2 = \sum_{m=j}^{\infty} k^s(m - j)|x_m|^2 \lesssim \sum_{m=j}^{\infty} (m + 1 - j)^{s-1} |x_m|^2,
\]

for every $j \geq 0$. Hence

\[
\frac{1}{(n + 1)^a} \sum_{j=0}^{n} (n + 1 - j)^{a-1} \|B_s^j x\|^2 \lesssim \frac{1}{(n + 1)^a} \sum_{j=0}^{n} (n + 1 - j)^{a-1} \sum_{m=j}^{\infty} (m + 1 - j)^{s-1} |x_m|^2
\]

\[
= \frac{1}{(n + 1)^a} \sum_{m=0}^{n} |x_m|^2 \sum_{j=0}^{m} (n + 1 - j)^{a-1} (m + 1 - j)^{s-1}
\]

\[
+ \frac{1}{(n + 1)^a} \sum_{m=n+1}^{\infty} |x_m|^2 \sum_{j=0}^{n} (n + 1 - j)^{a-1} (m + 1 - j)^{s-1}
\]

\[=: (I) + (II).
\]

In (I), note that since $1 - s < a \leq 1$, and $m \leq n$, we have

\[
\sum_{j=0}^{m} (n + 1 - j)^{a-1} (m + 1 - j)^{s-1} = \sum_{j=1}^{m+1} (n - m + j)^{a-1} j^{s-1}
\]

(6.8)

\[
\leq \sum_{j=1}^{m+1} j^{a-1} j^{s-1} = \sum_{j=1}^{m+1} j^{a+s-2} \lesssim (m + 1)^{a+s-1},
\]

where in the last estimate we used (6.4). Analogously, in (II) we have

\[
\sum_{j=0}^{n} (n + 1 - j)^{a-1} (m + 1 - j)^{s-1} \lesssim (n + 1)^{a+s-1},
\]
just by interchanging the roles of $m$ and $n$, and the roles of $a$ and $s$ in (6.8). Therefore

$$(I) \lesssim \frac{1}{(n+1)^a} \sum_{m=0}^{n} |x_m|^2 (m+1)^{a+s-1} = \left\{ \sum_{m=0}^{\lfloor \sqrt{n} \rfloor} + \sum_{m=\lfloor \sqrt{n} \rfloor+1}^{n} \right\} |x_m|^2 \frac{(m+1)^{a+s-1}}{(n+1)^a}$$

$$\lesssim \frac{\|x\|^2}{\sqrt{n^a}} + \sum_{m=\lfloor \sqrt{n} \rfloor+1}^{n} |x_m|^2 (m+1)^{a+s-1} \to 0 \quad (\text{as } n \to \infty),$$

and

$$(II) \lesssim \frac{1}{(n+1)^a} \sum_{m=n+1}^{\infty} |x_m|^2 (n+1)^{a+s-1} = \sum_{m=n+1}^{\infty} |x_m|^2 (n+1)^{s-1} \to 0 \quad (\text{as } n \to \infty).$$

Hence (6.6) follows when $1-s < a \leq 1$. Finally, suppose that $1 < a$. Then

$$\frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \|B_s^j x\|^2 \leq \frac{1}{n+1} \sum_{j=0}^{n} \|B_s^j x\|^2 \to 0 \quad (\text{as } n \to \infty),$$

since this is the case $a = 1$ in (6.6) (already proved). Note that (6.6) implies quadratically $(C,a)$-boundedness, so the proof is complete. \hfill \Box

This Lemma allows us to prove the following more general result.

**Theorem 6.12.** Let $0 < s < 1$ and $1 \leq q \leq 2$. Then $B_s$ is $(C,b,q)$-bounded if and only if $b > q(1-s)/2$. Moreover, for $b > q(1-s)/2$ we have

$$\lim_{n \to \infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b (n-j) \|B_s^j x\|^q = 0 \quad (\forall x \in H).$$

**Proof.** Note that $q = 2$ is precisely Lemma 6.11. So we assume that $1 \leq q < 2$. If $b = q(1-s)/2$, taking $x = f_n$, we get, as in (6.7), that

$$\frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b (n-j) \|B_s^j f_n\|^q \gtrsim \log(n+2) \|f_n\|^2$$

for every $n$. Therefore $B_s$ is not $(C,q(1-s)/2,q)$-bounded, and by Lemma 6.8 we get that $B_s$ is not $(C,b,q)$-bounded for $b < q(1-s)/2$.

Now suppose that $b > q(1-s)/2$. Then $b = qa/2$ for some $a > 1-s$. Using Hölder’s inequality as in the proof of Theorem 6.9 we obtain

$$\frac{1}{(n+1)^b} \sum_{j=0}^{n} (n+1-j)^{b-1} \|B_s^j x\|^q \lesssim \left( \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \|B_s^j x\|^2 \right)^{q/2} \to 0, \quad (n \to \infty),$$

by Lemma 6.11. Hence (6.9) follows. \hfill \Box

Here we state and prove a more general statement than Theorem 1.5.
Proof of Theorem 1.5. Let \( T \in C^w_a \) with \( 0 < a < 1 \) and let \( b > 1 - a \). By Theorem A and Theorem 3.7 (i), \( T \) is unitarily equivalent to a part of \((B_a \otimes I_D) \oplus S\). Hence, by Lemma 6.7 (i) and (i'), it is enough to prove that \((B_a \otimes I_D) \oplus S\) is quadratically \((C, b)\)-bounded. But this is immediate using Lemma 6.7 (ii) and (iii), and Lemmas 6.10 and 6.11. \(\square\)

**Theorem 6.13.** Let \( T \) be an \( a \)-contraction with \( 0 < a < 1 \) and let \( b > 1 - a \). Then the following statements are equivalent.

(i) The isometry \( S \) does not appear in the model of \( T \).

(ii) For every \( x \in H \),

\[
\exists \lim_{n \to \infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \| T^j x \|^2 = 0.
\]

(iii) For every \( x \in H \),

\[
\liminf_{n \to \infty} \| T^n x \| = 0.
\]

In the same spirit of Lemma 6.7 we have the following result.

**Lemma 6.14.** The following holds.

(i) If \( T \) satisfies (6.10), then any part of \( T \) also satisfies (6.10).

(ii) If \( T_1 \) and \( T_2 \) satisfy (6.10), then the direct sum \( T_1 \oplus T_2 \) also satisfies (6.10).

(iii) Let \( T \) be a bounded linear operator on a Hilbert space. If \( T \) satisfies (6.10), then the operator \( T \otimes I_R \) also satisfies (6.10), where \( I_R \) is the identity operator on some Hilbert space \( R \).

**Proof.** (i) and (ii) are immediate. For (iii) we use the same argument as in Lemma 6.7 (iii) and a simple application of Lebesgue’s Dominated Convergence Theorem. \(\square\)

**Proof of Theorem 6.13** As in the proof of Theorem 1.5 we have that \( T \) is unitarily equivalent to

\[
(B_a \otimes I_D) \oplus S | L,
\]

where \( L \) is a subspace of \((H_a \otimes D) \oplus W \) invariant by \((B_a \otimes I_D) \oplus S\).

Let us prove the circle of implications (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

Suppose that (i) is true. That is, \( T \) is unitarily equivalent to

\[
(B_a \otimes I_D) | L,
\]

where \( L \) is a subspace of \( H_a \otimes D \) invariant by \( B_a \otimes I_D \). Then (ii) follows using Lemmas 6.11 and 6.14.

Suppose now that

\[
\liminf_{n \to \infty} \| T^n x \| > 0
\]

for some \( x \in H \). Then, obviously, \( \| T^n x \| > \varepsilon > 0 \) for every \( n \geq 0 \). Hence for this vector \( x \) (6.10) does not hold. Therefore we have proved that (ii) \(\Rightarrow\) (iii).
Finally, suppose that the isometry $S$ appears in the model. Then for some vector $\ell = (\ell_1, \ell_2) \in \mathcal{L}$, its second component $\ell_2 \in \mathcal{W}$ is not 0. Therefore

$$
\lim_{n \to \infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \|((B_a \otimes I_2) \oplus S)^j \ell\|^2
$$

$$
= \lim_{n \to \infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \|((B_a \otimes I_2)^j \ell_1 + \oplus S^j \ell_2\|^2
$$

$$
= \lim_{n \to \infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \|S^j \ell_2\|^2.
$$

The first limit in the last equality is 0 because of Lemma 6.11 and Lemma 6.14 (iii), and the second limit is $\|\ell_2\|^2 \neq 0$ because of (6.3). Hence we obtain that (iii) $\Rightarrow$ (i). \qed

**Remark 6.15.**

(i) If $T$ is an operator in $C^w_a$ with $0 < a < 1$, by Theorem 1.5 and Theorem 6.9 it follows that $T$ is absolutely $(C, b, q)$-bounded for all $b > \frac{q(1-a)}{2}$.

(ii) Since $C^w_a$ is just the set of all contractions on $H$, $T \in C^w_a$ iff $T^* \in C^w_a$. On the contrary, if $a \in (0, 1)$, then there is an operator $T \in C^w_a$ such that $T^* \notin C^w_a$. An easy way to see it is to observe that $B_a^*$ is a forward weighted shift such that $\|B_a^n f_0\| \to \infty$ as $n$ goes to $\infty$. So $B_a \in C^w_a$, whereas its adjoint cannot belong to $C^w_a$, because such that $B_a^*$ is not quadratically $(C, b)$-bounded for any $b$ (see Lemma 6.11).

(iii) Let $T$ be an operator in $C^w_a$ with $0 < a < 1$. Using Theorem 1.5 (i) and Theorem 6.9 (with $p = 2$ and $q = 1$) we obtain that $T$ is $(C, b, 1)$-bounded for every $b > (1-a)/2$.

By [1] Corollary 3.1, we obtain that $T$ is $(C, b)$-mean ergodic, that is, there exists

$$
P_b x := \lim_{n \to \infty} M_T^b(n) x, \quad x \in H.
$$

Therefore, by [2] Theorem 3.3, we have

$$
H = \text{Ker}(I - T) \oplus \text{Ran}(I - T).
$$

In fact,

$$
\text{Ker}(I - T) = \text{Ran} P_b \quad \text{and} \quad \overline{\text{Ran}(I - T)} = \text{Ker} P_b.
$$

Also note that

$$
M_T^b(n) x = x \quad \text{for} \quad x \in \text{Ker}(I - T), \quad \text{and} \quad \lim_{n \to \infty} M_T^b(n) x = 0 \quad \text{for} \quad x \in \overline{\text{Ran}(I - T)}.
$$

Let now $0 < \gamma < 1$, by [2] Proposition 4.8 and Remark 4.9 it also follows that

$$
\text{Ker}(I - T) = \text{Ker}(I - T)\gamma, \quad \overline{\text{Ran}(I - T)} = \overline{\text{Ran}(I - T)\gamma},
$$

with $\text{Ran}(I - T) \subseteq \text{Ran}(I - T)\gamma$. Furthermore if $\gamma < 1 - b$, for $x \in \overline{\text{Ran}(I - T)}$,

$$
x \in \text{Ran}(I - T)\gamma \iff \sum_{n=1}^{\infty} \frac{1}{n^{1-\gamma}} T^n x \quad \text{converges},
$$
see [2, Theorem 9.2]. for $x \in \text{Ran}(I - T)^\gamma$.

(iv) By [1, Theorem 3.1], if $T$ is an operator in $C_w^{a}$ with $0 < a < 1$ and $b > (1 - a)/2$, then

$$\lim_{n \to \infty} \|M_T^{b}(n + 1) - M_T^{b}(n)\| = 0.$$ 

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