SEVERAL FORMULAS FOR BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. A generalized Stirling numbers of the second kind \( S_{a,b}(p, k) \), involved in the expansion 
\( (an + b)^p = \sum_{k=0}^{p} k! S_{a,b}(p, k) \binom{p}{k} \), where \( a \neq 0, b \) are complex numbers, have studied in [16]. In this paper, we show that Bernoulli polynomials \( B_p(x) \) can be written in terms of the numbers \( S_{1,x}(p, k) \), and then use the known results for \( S_{1,x}(p, k) \) to obtain several new explicit formulas, recurrences and generalized recurrences for Bernoulli numbers and polynomials.

1. Introduction

Bernoulli numbers have played a very important role in the development of mathematics over the last two centuries. Since they appeared in the 18th century, as part of coefficients in a formula for sums of powers of non-negative integers, they have been present in different fields of mathematics (see [9]).

Bernoulli polynomials \( B_p(x) \) can be defined by the generating function [2, p. 48]

\[
t e^x t = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi,
\]

or by the explicit formula [5, p. 367]

\[
B_p(x) = \sum_{j=0}^{p} \binom{p}{j} x^{p-j} B_j,
\]

where \( B_j \) is the \( j \)-th Bernoulli number \( (B_j = B_j(0)) \).

There are many works showing explicit formulas for Bernoulli numbers and polynomials (see the Gould’s paper [4]). Some of these works pursue links between the Bernoulli’s world with other kind of numbers, mainly Stirling numbers of the second kind and relatives [6, 7, 10, 11, 12, 15, 18].
In a previous work [16] we studied a generalization of Stirling numbers of the second kind, namely,

\[
S_{a,x} (p,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a (k-j) + x)^p,
\]

where \(a, x\) are arbitrary complex numbers, \(a \neq 0\). Plainly we have \(S_{1,0} (p,k) = S(p,k)\) (the standard Stirling numbers), and also \(S_{1,1} (p,k) = S(p+1,k+1)\). In this work we use some of the results for the generalized Stirling numbers (GSN, for short) of the type \(S_{1,x} (p,k)\), contained in [16], to obtain results involving Bernoulli numbers and polynomials: we obtain explicit formulas for Bernoulli numbers ((14), (41), (50), (52)), explicit formulas for the sum and the difference of two consecutive even Bernoulli numbers ((42), (43), (46), (47), (48), (57)), explicit formulas for sums of products of Stirling numbers of the first kind and Bernoulli numbers (Proposition 4, Proposition 5), among other results.

We summarize next some facts about the GSN \(S_{1,x} (p,k)\) (to be used in this work).

• Some values of the GSN \(S_{1,x} (p,k)\) are:

\[
\begin{align*}
S_{1,x} (p,0) & = x^p, \\
S_{1,x} (p,1) & = (x+1)^p - x^p, \\
S_{1,x} (p,2) & = \frac{1}{2} (x+2)^p - (x+1)^p + \frac{1}{2} x^p, \\
& \vdots \\
S_{1,x} (p,p) & = 1.
\end{align*}
\]

• The GSN \(S_{1,x} (p,k)\) can be written in terms of standard Stirling numbers as follows:

\[
S_{1,x} (p,k) = \frac{1}{k!} \sum_{j=0}^{p} \binom{p}{j} (x-m_1)^p-j \sum_{t=0}^{m_1} \binom{m_1}{t} (k+t)! S(j,k+t),
\]

where \(m_1\) is an arbitrary non-negative integer, and also as

\[
S_{1,x} (p,k) = \sum_{j=0}^{p} \binom{p}{j} (x-m_2)^p-j \sum_{t=0}^{m_2-1} (-1)^t s(m_2,m_2-t) S(j+m_2-t,k+m_2),
\]

where \(m_2\) is an arbitrary positive integer, and \(s(\cdot,\cdot)\) are the Stirling numbers of the first kind (with recurrence \(s(q+1,k) = s(q,k-1) + qs(q,k)\)).

• The GSN \(S_{1,x} (p,k)\) satisfy the identity:

\[
S_{1,x+1} (p,k) = S_{1,x} (p,k) + (k+1) S_{1,x} (p,k+1).
\]

• For \(0 \leq l \leq p_1 + p_2\), we have the following identity:

\[
S_{1,x} (p_1+p_2,l) = \sum_{m=0}^{p_2} S_{1,x} (p_2,m) S_{1,x+m} (p_1,l-m).
\]

• The GSN \(S_{1,x} (p,k)\) satisfy the recurrence:

\[
S_{1,x} (p,k) = S_{1,x} (p-1,k-1) + (k+x) S_{1,x} (p-1,k).
\]
In section 2 we show that Bernoulli polynomials \( B_p(x) \) can be written in terms of the GSN \( S_{1,x}(p, k) \), \( k = 0, 1, \ldots, p \), and give several one-parameter families of formulas for Bernoulli numbers and polynomials. In section 3 we obtain two generalized recurrences for Bernoulli polynomials (other interesting results on recurrences for Bernoulli numbers are contained in the works of M. Merca [13, 14]). In section 4 we deduce some corollaries from the recurrences obtained in section 3.

2. First results

The relation of Bernoulli numbers with Stirling numbers of the second kind is a well-known story, that dates back to Worpitzky [19] (see also [5, p. 560] and [8, p. 61]): we have

\[ B_p = \sum_{k=0}^{p} S(p, k) \frac{(-1)^k k!}{k + 1}. \]  

According to (3) (with \( m_1 = 0 \)), the GSN \( S_{1,x}(p, k) \) can be written in terms of standard Stirling numbers as

\[ S_{1,x}(p, k) = \sum_{j=0}^{p} \binom{p}{j} x^{p-j} S(j, k). \]

On the other hand, observe that from (8) and (9), we have that

\[ \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k!}{k + 1} = \sum_{j=0}^{p} \binom{p}{j} x^{p-j} \sum_{k=0}^{j} S(j, k) \frac{(-1)^k k!}{k + 1} = \sum_{j=0}^{p} \binom{p}{j} x^{p-j} B_j. \]

That is, the \( p \)-th Bernoulli polynomial \( B_p(x) \) can be written as

\[ B_p(x) = \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k!}{k + 1}. \]

We have to say that (10) is not a new formula. If we use (2) for \( S_{1,x}(p, k) \), expression (10) becomes formula (2.6) in [3, Vol. 8]. Thus, formula (10) is a “new way to write a known formula”. However, this new way to write the Bernoulli polynomials \( B_p(x) \) allows us to use some of the results in [16] for the GSN \( S_{1,x}(p, k) \), in order to obtain (new) results for Bernoulli polynomials \( B_p(x) \).

The case \( x = 0 \) of (10) is (8). The case \( x = 1 \) gives us (by using that \( B_p(1) = (-1)^p B_p \))

\[ B_p = (-1)^p \sum_{k=0}^{p} S(p + 1, k + 1) \frac{(-1)^k k!}{k + 1}. \]
From (3) and (4) we can write the following families of explicit formulas for the Bernoulli polynomials $B_p(x)$ involving arbitrary integer parameters $m_1$ and $m_2$.

\begin{equation}
B_p(x) = \sum_{k=0}^{p} \sum_{j=0}^{m_1} \binom{p}{j} (x - m_1)^{p-j} \sum_{t=0}^{m_1} \binom{m_1}{t} S(j, k + t) \frac{(-1)^k (k + t)!}{k + 1},
\end{equation}

\begin{equation}
= \sum_{k=0}^{p} \sum_{j=0}^{m_2 - 1} \binom{p}{j} (x - m_2)^{p-j} \times \sum_{t=0}^{m_2 - 1} (-1)^t s(m_2, m_2 - t) S(j + m_2 - t, k + m_2) \frac{(-1)^k k!}{k + 1},
\end{equation}

where $m_1 \geq 0$ and $m_2 > 0$. The simplest case of (12) (with $m_1 = 0$) is (10), and the simplest case of (13) (with $m_2 = 1$) is

\begin{equation}
B_p(x) = \sum_{k=0}^{p} \sum_{j=0}^{m_2 - 1} \binom{p}{j} (x - 1)^{p-j} S(j + 1, k + 1) \frac{(-1)^k k!}{k + 1}.
\end{equation}

From (12) and (13) we have at once the following families of formulas for Bernoulli numbers

\begin{equation}
B_p = \sum_{k=0}^{p} \sum_{j=0}^{m_1} \binom{p}{j} (-m_1)^{p-j} \sum_{t=0}^{m_1} \binom{m_1}{t} S(j, k + t) \frac{(-1)^k (k + t)!}{k + 1},
\end{equation}

\begin{equation}
= \sum_{k=0}^{p} \sum_{j=0}^{m_2 - 1} \binom{p}{j} (-m_2)^{p-j} \times \sum_{t=0}^{m_2 - 1} (-1)^t s(m_2, m_2 - t) S(j + m_2 - t, k + m_2) \frac{(-1)^k k!}{k + 1},
\end{equation}

\begin{equation}
= (-1)^p \sum_{k=0}^{p} \sum_{j=0}^{m_2 - 1} \binom{p}{j} (1 - m_1)^{p-j} \sum_{t=0}^{m_1} \binom{m_1}{t} S(j, k + t) \frac{(-1)^k (k + t)!}{k + 1},
\end{equation}

\begin{equation}
= (-1)^p \sum_{k=0}^{p} \sum_{j=0}^{m_1} \binom{p}{j} (1 - m_2)^{p-j} \times \sum_{t=0}^{m_2 - 1} (-1)^t s(m_2, m_2 - t) S(j + m_2 - t, k + m_2) \frac{(-1)^k k!}{k + 1},
\end{equation}

(where $m_1 \geq 0$ and $m_2 > 0$ are arbitrary integers), and the following expressions for the value of the Bernoulli polynomial $B_p(x)$ at $x = m \in \mathbb{N}$:

\begin{equation}
B_p(m) = \sum_{k=0}^{p} \sum_{t=0}^{m} \binom{m}{t} S(p, k + t) \frac{(-1)^k (k + t)!}{k + 1},
\end{equation}

\begin{equation}
= \sum_{k=0}^{p} \sum_{t=0}^{m-1} (-1)^t s(m, m - t) S(p + m - t, k + m) \frac{(-1)^k k!}{k + 1}.
\end{equation}

3. Generalized recurrences

We begin this section with a formula for $B_{p_1+p_2}(x)$ in terms of GSN.
Proposition 1. We have

\begin{equation}
B_{p_1+p_2} (x) = \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} S_{1,x} (p_2, k_2) S_{1,x+k_2} (p_1, k_1) \frac{(-1)^{k_1+k_2} (k_1 + k_2)!}{k_1 + k_2 + 1}.
\end{equation}

Proof. By using (10) and (6) we have

\begin{align*}
B_{p_1+p_2} (x) &= \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} \frac{(-1)^{k_1} k_1!}{k_1 + 1} \sum_{p_1}^{p_2} \frac{(-1)^{k_1} k_1!}{k_1 + 1} \\
&= \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} S_{1,x} (p_2, k_2) S_{1,x+k_2} (p_1, k_1 - k_2) \frac{(-1)^{k_1} k_1!}{k_1 + 1} \\
&= \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} S_{1,x} (p_2, k_2) S_{1,x+k_2} (p_1, k_1) \frac{(-1)^{k_1+k_2} (k_1 + k_2)!}{k_1 + k_2 + 1},
\end{align*}

as expected. \(\square\)

If we use the simplest case \(m_1 = 0\) of (3) to write the GSN \(S_{1,x} (p_2, l)\) and \(S_{1,x+l} (p_1, k)\) of the right-hand side of (17), in terms of standard Stirling numbers, we get

\begin{align*}
B_{p_1+p_2} (x) &= \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \left( \begin{array}{c}
p_1 \\
j_1
\end{array} \right) \left( \begin{array}{c}
p_2 \\
j_2
\end{array} \right) x^{p_2-j_2} (x + k_2) x^{p_1-j_1} S_{1,x} \left( \begin{array}{c}
 j_1 \\
k_1
\end{array} \right) S_{1,x+k_2} \left( \begin{array}{c}
 j_2 \\
k_2
\end{array} \right) \frac{(-1)^{k_1+k_2} (k_1 + k_2)!}{k_1 + k_2 + 1}.
\end{align*}

(18)

From (18) with \(p_1 = 1\) or \(p_2 = 1\) we obtain that

\begin{align*}
B_{p+1} (x) &= xB_p (x) + \sum_{k=0}^{p} \sum_{j=k}^{p} \left( \begin{array}{c}
p \\
j
\end{array} \right) (x + 1)^{p-j} S (j, k) \frac{(-1)^{k+1} (k + 1)!}{k + 2},
\end{align*}

(19)

\begin{align*}
B_{p+2} (x) &= xB_p (x) + \sum_{k=0}^{p} \sum_{j=k}^{p} \left( \begin{array}{c}
p \\
j
\end{array} \right) x^{p-j} S (j, k) \frac{(-1)^{k+1} k!}{(k + 1)(k + 2)},
\end{align*}

(20)

respectively. From (18) with \(p_2 = 2\), and (19), we see that

\begin{align*}
B_{p+2} (x) &= -x (x + 1) B_p (x) + (2x + 1) B_{p+1} (x) \\
&\quad + \sum_{k=0}^{p} \sum_{j=k}^{p} \left( \begin{array}{c}
p \\
j
\end{array} \right) (x + 2)^{p-j} S (j, k) \frac{(-1)^{k} (k + 2)!}{k + 3}
\end{align*}

(21)

Similarly, from (18) with \(p_1 = 2\), and (20), we obtain that

\begin{align*}
B_{p+2} (x) &= -x^2 B_p (x) + 2x B_{p+1} (x) \\
&\quad + \sum_{k=0}^{p} \sum_{j=k}^{p} \left( \begin{array}{c}
p \\
j
\end{array} \right) x^{p-j} S (j, k) \frac{(-1)^{k+1} k! (k - 1)}{(k + 1)(k + 2)(k + 3)}.
\end{align*}

(22)

Examples (19) and (21) are particular cases of the following general result.

Proposition 2. For non-negative integers \(p, q\) we have

\begin{align*}
\sum_{k=0}^{q} (-1)^k \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x + i) = \sum_{k=0}^{p} \frac{(-1)^k (k + q)!}{k + q + 1}.
\end{align*}

(23)
Proof. We proceed by induction on \( q \). The case \( q = 0 \) of (23) is (10). If we assume that (23) is true for \( q \in \mathbb{N} \), then

\[
\sum_{k=0}^{q+1} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q} (x+i) \]

\[
= \sum_{k=0}^{q+1} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \left( (x+q) \prod_{i=0}^{q-1} (x+i) \right) \]

\[
= \sum_{k=0}^{q+1} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x+i) \quad \text{for} \quad x = (x+q) \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \prod_{i=0}^{q-1} (x+i) \]

\[
= (x+q) \sum_{k=0}^{q} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x+i) \]

\[
+ \sum_{k=1}^{q+1} (-1)^k B_{p+k} (x) \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \prod_{i=0}^{q-1} (x+i) \]

\[
= (x+q) \sum_{k=0}^{q} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x+i) \]

\[
- \sum_{k=0}^{q} (-1)^k B_{p+1+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x+i) \]

\[
= (x+q) \sum_{k=0}^{p} S_{1,x+q} (p, k) \frac{(-1)^k (k+q)!}{k+q+1} \quad - \sum_{k=0}^{p+1} S_{1,x+q} (p+1, k) \frac{(-1)^k (k+q)!}{k+q+1}. \]

Now we use the recurrence (7) and formula (5) to write

\[
\sum_{k=0}^{q+1} (-1)^k B_{p+k} (x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q} (x+i) \]

\[
= (x+q) \sum_{k=0}^{q} S_{1,x+q} (p, k) \frac{(-1)^k (k+q)!}{k+q+1} \]

\[
- \sum_{k=0}^{p+1} (S_{1,x+q} (p, k-1) + kS_{1,x+q} (p, k)) \frac{(-1)^k (k+q)!}{k+q+1} \]

\[
= - \sum_{k=0}^{p+1} (S_{1,x+q} (p, k-1) + kS_{1,x+q} (p, k)) \frac{(-1)^k (k+q)!}{k+q+1} \]

\[
= - \sum_{k=1}^{p+1} S_{1,x+q+1} (p, k-1) \frac{(-1)^k (k+q)!}{k+q+1} \]

\[
= \sum_{k=0}^{p} S_{1,x+q+1} (p, k) \frac{(-1)^k (k+q+1)!}{k+q+2}, \]

as desired. \( \square \)

Examples (20) and (22) are particular cases of the following general result.
Proposition 3. For non-negative integers \( p, q \), we have

\[
(24) \quad \sum_{k=0}^{q} \binom{q}{k} (-x)^{q-k} B_{p+k}(x) = -\sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)},
\]

where \( R_{-1}(k) = -1 \), \( R_0(k) = 1 \), and for \( q \geq 2 \) the \((q-1)\)-th degree polynomials \( R_{q-1}(k) \) are defined recursively by

\[
R_{q-1}(k) = k (k+q+1) R_{q-2}(k) - (k+1)^2 R_{q-2}(k+1).
\]

Proof. We proceed by induction on \( q \). The case \( q = 0 \) of (24) is (10). Let us assume that (24) is true for \( q \in \mathbb{N} \). Then

\[
\sum_{k=0}^{q+1} \binom{q+1}{k} (-x)^{q+1-k} B_{p+k}(x)
\]

\[
= -x \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)} - \sum_{k=0}^{q+1} S_{1,x}(p+1, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)}
\]

\[
= x \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)} - (S_{1,x}(p, k-1) + (k+x) S_{1,x}(p, k)) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)}
\]

\[
= -\sum_{k=0}^{p} (S_{1,x}(p, k-1) + k S_{1,x}(p, k)) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)}
\]

\[
= \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k (k+1)! R_{q-1}(k+1)}{\prod_{i=1}^{q+1} (k+i+1)} - \sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)}
\]

\[
= -\sum_{k=0}^{p} S_{1,x}(p, k) \left( k (k+q+1) R_{q-1}(k) - (k+1)^2 R_{q-1}(k+1) \right) \frac{(-1)^k k!}{\prod_{i=1}^{q+2} (k+i)}
\]

\[
= -\sum_{k=0}^{p} S_{1,x}(p, k) \frac{(-1)^k k! R_q(k)}{\prod_{i=1}^{q+2} (k+i)},
\]

as expected. \(\square\)
Some of the polynomials \( R_p(y) \) are:

\[
R_0(y) = 1, \\
R_1(y) = y - 1, \\
R_2(y) = y(y - 5), \\
R_3(y) = (y - 1)(y^2 - 15y - 4), \\
R_4(y) = y(y^3 - 42y^2 + 119y + 42), \\
R_5(y) = (y - 1)(y^4 - 98y^3 + 659y^2 + 518y + 120), \\
R_6(y) = y(y^5 - 219y^4 + 3721y^3 - 6189y^2 - 7250y - 2160), \\
R_7(y) = (y - 1)\left(y^6 - 465y^5 + 15241y^4 - 57735y^3 + 518y^2 - 12096\right), \\
R_8(y) = y\left(y^7 - 968y^6 + 60082y^5 - 595760y^4 + 371569y^3 + 98906y^2 - 12096\right), \\
R_9(y) = (y - 1)\left(y^8 - 1980y^7 + 212994y^6 - 23438610y^5 + 149545929y^4 + 31139516y^3 + 15573360y^2 + 3024000\right), \\
R_{10}(y) = y\left(y^9 - 4017y^8 + 733902y^7 - 23438610y^6 + 149545929y^5 + 87456447y^4 - 427451752y^3 - 754107900y^2 - 117936000\right).
\]

4. Some consequences

In this section we consider some consequences of (23) and (24). We will use, without further comments, that for any non-negative integers \( q \) and \( k \), we have

\[
\left[ \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x + i) \right]_{x=0} = s(q, k),
\]

and

\[
\left[ \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x + i) \right]_{x=1} = s(q + 1, k + 1).
\]

We begin by noting that the case \( q = 0 \) of (23) is (10). The case \( p = 0 \) of (23) is

\[
\sum_{k=0}^{q} (-1)^k B_k(x) \frac{1}{k!} \frac{d^k}{dx^k} \prod_{i=0}^{q-1} (x + i) = \frac{q!}{q + 1}.
\]

In particular, by setting \( x = 0 \) and \( x = 1 \) in (27) we obtain that

\[
\sum_{k=0}^{q} (-1)^k s(q, k) B_k = \sum_{k=0}^{q} s(q + 1, k + 1) B_k = \frac{q!}{q + 1},
\]

(see [18], formula (2.7)). More generally, if we set \( x = 0 \) in (23) we get

\[
\sum_{k=0}^{q} (-1)^k s(q, k) B_{p+k} = \sum_{k=0}^{p} S_1(q, p, k) \frac{(-1)^k (k + q)!}{k + q + 1},
\]

(29)
which includes the identity $\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{k} = \frac{q!}{q+1}$ in (28) as the particular case $p = 0$. The cases $p = 1, 2$ of (29) are

$$\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{k+1} = -\frac{q!}{(q+1)(q+2)},$$

$$\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{k+2} = -\frac{q!(q-1)}{(q+1)(q+2)(q+3)},$$

respectively.

The following proposition gives more information about (29).

**Proposition 4.** We have

$$\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{p+k} = \sum_{k=0}^{p} S_{1,q}(p, k) \frac{(-1)^{k}(k+q)!}{k+q+1}$$

$$= -\frac{q!\mathcal{R}_{p-1}(q)}{\prod_{i=1}^{p+1} (q+i)}$$

where the $(p-1)$-th degree $q$-polynomials $\mathcal{R}_{p-1}(q)$ are defined in Proposition 3.

**Proof.** Identity (30) is (29). To prove (31), we proceed by induction on $p$. The case $p = 0$ is (28). If we assume (31) is true for a given $p \in \mathbb{N}$, then

$$-q!\mathcal{R}_{p}(q) = q!\left((q+p+2)\mathcal{R}_{p-1}(q) - (q+1)^{2}\mathcal{R}_{p-1}(q+1)\right)$$

$$= \frac{q!q\mathcal{R}_{p-1}(q)}{\prod_{i=1}^{p+1} (q+i)} + \frac{(q+1)!\mathcal{R}_{p-1}(q+1)}{\prod_{i=1}^{p+1} (q+1+i)}$$

$$= q\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{p+k} - \sum_{k=0}^{q+1} (-1)^{k} s(q+1, k) B_{p+k}$$

$$= q\sum_{k=0}^{q} (-1)^{k} s(q, k) B_{p+k} - \sum_{k=0}^{q+1} (-1)^{k} (s(q, k-1) + qs(q, k)) B_{p+k}$$

$$= -\sum_{k=0}^{q+1} (-1)^{k} s(q, k-1) B_{p+k}$$

$$= \sum_{k=0}^{q} (-1)^{k} s(q, k) B_{p+1+k},$$

as desired. \qed

Now we set $x = 1$ in (23) to get

$$(-1)^{p} \sum_{k=0}^{q} s(q+1, k+1) B_{p+k} = \sum_{k=0}^{p} S_{1,q+1}(p, k) \frac{(-1)^{k}(k+q)!}{k+q+1},$$

as desired.
which includes the identity \(\sum_{k=0}^{q} s(q+1,k+1)B_k = \frac{q!}{q+1}\) of (28) as the particular case \(p = 0\). The case \(q = 0\) of (32) is (11). The cases \(p = 1, 2\) of (32) are

\[
\sum_{k=0}^{q} s(q+1,k+1)B_{k+1} = -\frac{q!}{q+2},
\]

\[
\sum_{k=0}^{q} s(q+1,k+1)B_{k+2} = \frac{(q+1)!}{(q+2)(q+3)},
\]

respectively.

The following proposition gives more information about (32).

**Proposition 5.** For integers \(p > 1, q \geq 0\), we have

\[
(-1)^p \sum_{k=0}^{p+1} s(q+1,k+1)B_{p+k} = \sum_{k=0}^{p} S_{1,q+1}(p,k) \frac{(-1)^k (k+q)!}{k+q+1}
\]

\[
= \frac{(q+1)! R_{p-2}(q+1)}{\prod_{i=2}^{p+1} (q+i)},
\]

where the \((p-2)\)-th degree \(q\)-polynomials \(R_{p-2}(q+1)\) are defined in Proposition 3.

**Proof.** Identity (35) is (32). Let us prove (36). We proceed by induction on \(p\). The case \(p = 2\) of (36) is (34). If we assume (36) is true for a given integer \(p > 2\), then

\[
(-1)^{p+1} \frac{(q+1)! R_{p-1}(q+1)}{\prod_{i=2}^{p+2} (q+i)}
\]

\[
= (-1)^{p+1} \frac{(q+1)!}{\prod_{i=2}^{p+2} (q+i)} \left((q+1)(q+p+2)R_{p-2}(q+1) - (q+2)^2 R_{p-2}(q+2)\right)
\]

\[
= (-1)^{p+1} \frac{(q+1)!}{\prod_{i=2}^{p+2} (q+i)} \left((q+1)R_{p-2}(q+1) - (-1)^{p+1} (q+2) R_{p-2}(q+2)\right)
\]

\[
= - (q+1) \sum_{k=0}^{p} s(q+1,k+1)B_{p+k} + \sum_{k=0}^{q+1} s(q+2,k+1)B_{p+k}
\]

\[
= - (q+1) \sum_{k=0}^{p} s(q+1,k+1)B_{p+k}
\]

\[
+ \sum_{k=0}^{q+1} (s(q+1,k) + (q+1) s(q+1,k+1))B_{p+k}
\]

\[
= \sum_{k=0}^{q} s(q+1,k+1)B_{p+1+k},
\]

as desired. \(\square\)
For example, the polynomial $R_3(y) = (y - 1) (y^2 - 15y - 4)$ is involved in (31) with $p = 4$, and in (36) with $p = 5$. The corresponding results are

$$\sum_{k=0}^{q} (-1)^k s(q, k) B_{k+4} = -\frac{q! (q - 1) (q^2 - 15q - 4)}{(q + 1)(q + 2)(q + 3)(q + 4)(q + 5)},$$

and

$$\sum_{k=0}^{q} s(q + 1, k + 1) B_{k+5} = -\frac{(q + 1)!q (q^2 - 13q - 18)}{(q + 2)(q + 3)(q + 4)(q + 5)(q + 6)}.$$

We can write (30) and (35), by using (3) (with $m_1 = 0$) as

$$\sum_{k=0}^{q} (-1)^k s(q, k) B_{p+k} = \sum_{k=0}^{p} \sum_{j=k}^{p} \binom{p}{j} q^{p-j} S(j, k) \frac{(-1)^k (k + q)!}{k + q + 1},$$

and

$$\sum_{k=0}^{q} s(q + 1, k + 1) B_{p+k} = (-1)^p \sum_{k=0}^{p} \sum_{j=k}^{p} \binom{p}{j} (q + 1)^{p-j} S(j, k) \frac{(-1)^k (k + q)!}{k + q + 1},$$

respectively. By setting $q = 2$ in (37) and $q = 1$ in (38), we obtain, respectively

$$-B_{p+1} + B_{p+2} = \sum_{k=0}^{p} \sum_{j=k}^{p} \binom{p}{j} 2^{p-j} S(j, k) \frac{(-1)^k (k + 2)!}{k + 3},$$

and

$$B_p + B_{p+1} = (-1)^p \sum_{k=0}^{p} \sum_{j=k}^{p} \binom{p}{j} 2^{p-j} S(j, k) \frac{(-1)^k (k + 1)!}{k + 2}.$$

Replace $p$ by $2p - 1$ in (39) and in (40) to obtain (for $p > 1$)

$$B_{2p} = -\sum_{k=0}^{2p-1} \sum_{j=k}^{2p-1} \binom{2p-1}{j} 2^{2p-1-j} S(j, k) \frac{(-1)^k (k + 2)!}{k + 3},$$

and

$$B_{2p} + B_{2p+2} = \sum_{k=0}^{2p} \sum_{j=k}^{2p} \binom{2p}{j} 2^{2p-j} S(j, k) \frac{(-1)^k (k^2 + 5k + 17)(k + 1)!}{(k + 2)(k + 3)}.$$

The sum of (39) and (40), with $p$ replaced by $2p$, gives us the following formula for the sum of two consecutive even Bernoulli numbers (with $p > 0$)

$$B_{2p} + B_{2p+2} = \sum_{k=0}^{2p} \sum_{j=k}^{2p} \binom{2p}{j} 2^{2p-j} S(j, k) \frac{(-1)^k (k^2 + 5k + 7)(k + 1)!}{(k + 2)(k + 3)}.$$

Subtract (39) from (40), with $p$ replaced by $2p$, gives us the following formula for the difference of two consecutive even Bernoulli numbers (with $p > 0$)

$$B_{2p} - B_{2p+2} = \sum_{k=0}^{2p} \sum_{j=k}^{2p} \binom{2p}{j} 2^{2p-j} S(j, k) \frac{(-1)^{k+1} (k^2 + 3k + 1)(k + 1)!}{(k + 2)(k + 3)}.$$
If we set $q = 4$ in (37) and $q = 3$ in (38), we obtain, respectively

\[(44) \quad -6B_{p+1} + 11B_{p+2} - 6B_{p+3} + B_{p+4} = \frac{p \choose j} {4^{p-j} S(j,k)} \left[ \frac{(-1)^k (k+4)!}{k+5} \right].\]

\[(45) \quad 6B_{p+1} + 11B_{p+2} + 6B_{p+3} = (-1)^p \frac{p \choose j} {4^{p-j} S(j,k)} \left[ \frac{(-1)^k (k+3)!}{k+4} \right].\]

From (44) with $p$ replaced by $2p-1$, we obtain (for $p > 0$)

\[(46) \quad B_{2p} + B_{2p+2} = -\frac{1}{6} \sum_{k=0}^{2p-1} \sum_{j=k}^{2p-1} \binom{2p-1}{j} \frac{(-1)^k (k+4)!}{k+5}.\]

From (45) with $p$ replaced by $2p$, we obtain (for $p > 0$)

\[(47) \quad B_{2p} + B_{2p+2} = \frac{1}{6} \sum_{k=0}^{2p} \sum_{j=k}^{2p} \binom{2p}{j} \frac{(-1)^k (k+3)!}{k+4}.\]

The sum of (44) and (45), with $p$ replaced by $2p+1$, gives us (for $p > 0$)

\[(48) \quad B_{2p+2} - B_{2p+4} = \frac{1}{5} \sum_{k=0}^{2p+1} \sum_{j=k}^{2p+1} \binom{2p+1}{j} \frac{(-1)^k (k^2 + 7k + 11)(k+3)!}{(k+4)(k+5)}.\]

Now, let us consider some consequences of (24). The case $q = 0$ of (24) is (10). The case $p = 0$ of (24) is

\[(49) \quad \sum_{k=0}^{q} \binom{q}{k} (-x)^{q-k} B_k(x) = -\frac{R_{q-1}(0)}{(q+1)!},\]

which gives us the following formula for Bernoulli numbers in terms of the independent term of the polynomials $R_{q-1}(y)$ described in Proposition 3, namely,

\[(50) \quad B_q = \frac{R_{q-1}(0)}{(q+1)!}.\]

For example, the first 12 Bernoulli numbers, according to (50), are as follows (see the corresponding polynomials $R_{q-1}(y)$ at the end of Section 3)

\[
\begin{align*}
B_0 &= -\frac{R_{-1}(0)}{(0+1)!} = 1, \\
B_1 &= -\frac{R_{1}(0)}{(1+1)!} = -\frac{1}{2}, \\
B_2 &= -\frac{R_{2}(0)}{(2+1)!} = -\frac{1}{3} = \frac{1}{6}, \\
B_3 &= -\frac{R_{3}(0)}{(3+1)!} = 0, \\
B_4 &= -\frac{R_{4}(0)}{(4+1)!} = -\frac{4}{5!} = -\frac{1}{5}, \\
B_5 &= -\frac{R_{5}(0)}{(5+1)!} = 0, \\
B_6 &= -\frac{R_{5}(0)}{(6+1)!} = -\frac{120}{7!} = \frac{1}{42}, \\
B_7 &= -\frac{R_{7}(0)}{(7+1)!} = 0, \\
B_8 &= -\frac{R_{8}(0)}{(8+1)!} = -\frac{12096}{9!} = \frac{1}{30}, \\
B_9 &= -\frac{R_{9}(0)}{(9+1)!} = 0, \\
B_{10} &= -\frac{R_{10}(0)}{(10+1)!} = -\frac{3024000}{11!} = -\frac{5}{66}, \\
B_{11} &= -\frac{R_{11}(0)}{(11+1)!} = 0.
\end{align*}
\]

The case $x = 0$ of (24) is

\[(51) \quad B_{p+q} = -\sum_{k=0}^{p} S(p,k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k+i)}.\]
which is a generalization of (8) (case \(q = 0\)). From (51), we have the following formula for even Bernoulli numbers

\[
B_{2p} = - \sum_{k=0}^{p} S(p, k) \frac{(-1)^k k! R_{p-1}(k)}{\prod_{i=1}^{p+1} (k + i)}.
\]

Plainly, the non-negative integers \(p\) and \(q\) in (51) commute. That is, we have

\[
B_{p+q} = - \sum_{k=0}^{p} S(p, k) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k + i)} = - \sum_{k=0}^{q} S(q, k) \frac{(-1)^k k! R_{p-1}(k)}{\prod_{i=1}^{p+1} (k + i)}.
\]

For example, we have

\[
B_{p+2} = - \sum_{k=0}^{p} S(p, k) \frac{(-1)^k k! (k-1)}{(k+1)(k+2)(k+3)} = \frac{R_{p-1}(1)}{(p+2)!} - \frac{4R_{p-1}(2)}{(p+3)!}.
\]

The case \(x = 1\) of (24) is

\[
(-1)^{p+q} \sum_{k=0}^{q} \binom{q}{k} B_{p+k} = - \sum_{k=0}^{p} S(p+1, k+1) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k + i)},
\]

which includes (11) as the particular case \(q = 0\). Recall that

\[
(-1)^{p+q} \sum_{k=0}^{q} \binom{q}{k} B_{p+k} = \sum_{k=0}^{p} \binom{p}{k} B_{q+k},
\]

Formula (55) is the famous Carlitz identity [1] (see also [17] and references therein). That is, we can write (54) as

\[
(-1)^p \sum_{k=0}^{q} \binom{q}{k} B_{p+k} = - (-1)^q \sum_{k=0}^{p} S(p+1, k+1) \frac{(-1)^k k! R_{q-1}(k)}{\prod_{i=1}^{q+1} (k + i)} = - (-1)^p \sum_{k=0}^{q} S(q+1, k+1) \frac{(-1)^k k! R_{p-1}(k)}{\prod_{i=1}^{p+1} (k + i)} = (-1)^q \sum_{k=0}^{p} \binom{p}{k} B_{q+k}.
\]
For example, we have

\[
\sum_{k=0}^{q} \binom{q}{k} B_{k+2} = (-1)^q \left( \frac{R_{q-1}(0)}{(q+1)!} - \frac{3R_{q-1}(1)}{(q+2)!} + \frac{4R_{q-1}(2)}{(q+3)!} \right)
\]

\[
= -\sum_{k=0}^{q} S(q+1,k+1) \frac{(-1)^k k! (k-1)}{(k+1)(k+2)(k+3)}
\]

(56)

(57) \quad B_{2q} + B_{2q+2} = -\sum_{k=0}^{2q} S(2q+1,k+1) \frac{(-1)^k k! (k-1)}{(k+1)(k+2)(k+3)}

With \( q \) replaced by \( 2q \) we obtain from (56) that (for \( q > 0 \))

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REFERENCES

[1] L. Carlitz, Problem 795, Math. Mag., 44 (1971), 107.
[2] L. Comtet, Advanced Combinatorics, Reidel, 1974.
[3] H. W. Gould, Tables of Combinatorial Identities, Edited and compiled by Prof. Jocelyn Quaintance, 2010. Available from: https://math.wvu.edu/~hgould/.
[4] H. W. Gould, Explicit formulas for Bernoulli numbers, Amer. Math. Monthly, 79 (1972), 44-51.
[5] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics. A Foundation for Computer Science, 2\textsuperscript{nd} edition, Addison-Wesley, 1994.
[6] B. N. Guo, I. Mező and F. Qi, An explicit formula for Bernoulli polynomials in terms of \( r \)-Stirling numbers of the second kind, Rocky Mountain J. Math., 46 (2016), 1919–1923.
[7] B. N. Guo and F. Qi, An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind, J. Ana. Num. Theor., 3 (2015), 27–30.
[8] B. C. Kellner, Identities between polynomials related to Stirling and harmonic numbers, Integers, 14 (2014), 54–76.
[9] B. Mazur, Bernoulli Numbers and the Unity of Mathematics, Available from: http://people.math.harvard.edu/~mazur/papers/slides.Bartlett.pdf
[10] M. Merca, A new connection between \( r \)-Whitney numbers and Bernoulli polynomials, Integral Transforms Spec. Funct., 25 (2014), 937–942.
[11] M. Merca, A connection between Jacobi-Stirling numbers and Bernoulli polynomials, J. Number Theory, 151 (2015), 223–229.
[12] M. Merca, Connections between central factorial numbers and Bernoulli polynomials, Period. Math. Hungar., 73 (2016), 259–264.
[13] M. Merca, On lacunary recurrences with gaps of length four and eight for the Bernoulli numbers, Bull. Korean Math. Soc., 56 (2019), 491–499.
[14] M. Merca, Bernoulli numbers and symmetric functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Serie A, Matemáticas, 114 (2020), 29–36.
[15] I. Mező, A new formula for the Bernoulli polynomials, Results Math., 58 (2010), 329–335.
[16] C. Pita-Ruiz, Generalized stirling Numbers I, preprint, arXiv:1803.05953v1.
[17] C. Pita-Ruiz, Carlitz-Type and other Bernoulli Identities, J. Integer Seq., 19 (2016), 27 pp.
[18] F. A. Shiha, An explicit formula for Bernoulli polynomials with a $q$ parameter in terms of $r$-Whitney numbers, *J. Ana. Num. Theor.*, 6 (2018), 47–50.

[19] J. Worpitzky, Studien über die Bernoullischen und Eulerschen Zahlen, *J. Reine Angew. Math.*, 94 (1883), 203–232.

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