First and second sound in a uniform Bose gas

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(January 9, 2022)

Abstract

We have recently derived two-fluid hydrodynamic equations for a trapped weakly-interacting Bose gas. In this paper, we use these equations to discuss first and second sound in a uniform Bose gas. These results are shown to agree with the predictions of the usual two-fluid equations of Landau when the thermodynamic functions are evaluated for a weakly-interacting gas. In a uniform gas, second sound mainly corresponds to an oscillation of the superfluid (the condensate) and is the low frequency continuation of the Goldstone-Bogoliubov symmetry-breaking mode.

PACS numbers: 03.75.F, 67.40.Db
I. INTRODUCTION

One of the most spectacular features \( ^4 \text{He} \) exhibited by superfluid \( ^4 \text{He} \) is the existence of two hydrodynamic sound modes, first and second sound. As first pointed out by Tisza \([2]\), the motion of a Bose condensate as a separate degree of freedom results in a two fluid hydrodynamics describing the superfluid and normal fluid components \([3]\). In recent work, the authors \([4]\) gave a microscopic derivation of the two-fluid hydrodynamic equations of motion for a trapped weakly-interacting Bose-condensed gas. In contrast to a Bose-condensed liquid like superfluid \( ^4 \text{He} \), the superfluid in a gas corresponds directly to the condensate atoms and the normal fluid corresponds to the non-condensate (or excited) atoms. In the present paper, we use these two-fluid equations to discuss the first and second sound modes of a uniform Bose-condensed gas. We find that at temperatures close to \( T_{\text{BEC}} \), first (second) sound mainly corresponds to an oscillation of the non-condensate (condensate) atoms. We also confirm \([5,6]\) that it is the second sound mode in a uniform gas which is the low frequency hydrodynamic analogue of the collisionless Bogoliubov-Popov Goldstone mode \([7]\).

These results for a uniform gas are of interest for comparison with the hydrodynamic oscillations of the condensate and non-condensate in a non-uniform trapped Bose gas \([4,8]\). They also may be of direct interest in connection with recent studies at MIT \([8]\) of the propagation of pulses along the \( z \)-axis of a cigar-shaped trap. The axial trap spring constant is so small that the condensate along the \( z \)-axis can be treated as effectively uniform (to a first approximation) in such propagation studies.

We recall that Ref. \([4]\) (ZGN) is based on: (a) a time-dependent Hartree-Fock-Popov equation of motion for the condensate wavefunction \( \Phi(\mathbf{r}, t) \); and (b) a set of hydrodynamic equations for the fluctuations of the thermal cloud (non-condensate) based on a kinetic equation which includes the effect on the atoms of the time-dependent self-consistent Hartree-Fock field. The analysis of ZGN uses the local equilibrium solution of the kinetic equation and thus does not include any hydrodynamic damping, such as Kirkpatrick and Dorfman \([10]\) consider. However it should be emphasized that a local equilibrium description is crucially dependent on collisions between the atoms and thus the hydrodynamic equations are only valid for low frequency phenomena (\( \omega \ll 1/\tau_c \), where \( \tau_c \) is the mean time between collisions of atoms in the thermal cloud).

In Section \([4]\), we solve the linearized hydrodynamic two-fluid equations for the coupled superfluid and normal fluid velocity fluctuations derived in ZGN. We exhibit the first and second sound normal modes valid at intermediate temperatures, defined as the temperature regime below \( T_{\text{BEC}} \) where the interaction energy of an atom is much less than the thermal kinetic energy (i.e., \( gn_0 \ll k_B T \); here, \( n_0 \) is the gas density and \( g = 4\pi a \hbar^2/m \) is the interaction parameter). The analysis of ZGN is built on a mean-field approximation for the equilibrium properties. As discussed in Section \([11]\), this simple theory is not valid close to the superfluid transition, where it gives rise to spurious discontinuities in the condensate density. In Section \([IV]\), we discuss the relation between our two-fluid equations written in terms of velocity fluctuations and the standard Landau formulation given in terms of density and entropy fluctuations \([11,12,13]\). All previous discussions \([12,7,5]\) of hydrodynamic modes in a dilute Bose gas have used the latter formulation.
II. COUPLED EQUATIONS FOR SUPERFLUID AND NORMAL FLUID VELOCITIES

When there is no trapping potential, the non-condensate density \( \tilde{n}_0 \) and condensate density \( n_{c0} \) do not depend on position. In this case, one can reduce the linearized 2-fluid equations given by Eqs. (12), (15) and (16) of ZGN to two coupled equations for the normal and superfluid local velocities

\[
\left( m \partial^2 - gn_{c0} \nabla (\nabla \cdot \delta v_N) + 2gn_0 \nabla (\nabla \cdot \delta v_S) \right) = 0 \quad (1a)
\]

\[
\frac{\partial^2 \delta v_N}{\partial t^2} = \left( \frac{5 \tilde{P}_0}{3 \tilde{n}_0} + 2g\tilde{n}_0 \right) \nabla (\nabla \cdot \delta v_N) + 2gn_{c0} \nabla (\nabla \cdot \delta v_S) \quad (1b)
\]

We emphasize that these equations are only valid at finite temperatures such that \( gn_0 \ll k_B T \). In deriving these equations, we have assumed that the contribution from the first term of Eq.(13) of ZGN is negligible in the long-wavelength limit of interest. These equations can be solved to give the low frequency hydrodynamic normal modes of a uniform Bose-condensed gas, as will be discussed. We defer discussion of the equilibrium quantities \( n_{c0}, \tilde{n}_0 \) and the kinetic contribution to the pressure \( \tilde{P}_0 \) which appear in (1a) and (1b) to Section III.

Introducing the velocity potentials \( \delta v_S \equiv \nabla \phi_S, \delta v_N \equiv \nabla \phi_N \), it is easy to see that (1a) and (1b) have plane-wave solutions \( \phi_{S,N}(r,t) = \phi_{S,N} e^{i(kr - \omega t)} \) satisfying

\[
\left[ \omega^2 - \frac{gn_{c0} k^2}{m} \right] \phi_S - \left( \frac{2g\tilde{n}_0 k^2}{m} \right) \phi_N = 0
\]

\[
- \left( \frac{2gn_{c0} k^2}{m} \right) \phi_S + \left[ \omega^2 - \left( \frac{5 \tilde{P}_0}{3 \tilde{n}_0} + \frac{2g\tilde{n}_0}{m} \right) k^2 \right] \phi_N = 0 . \quad (2)
\]

The zeros of the secular determinant of this coupled set of equations give two phonon solutions \( \omega^2_\pm = u^2_\pm k^2 \), where the velocities are the solution of

\[
u^4 - \nu^2 \left( \frac{5 \tilde{P}_0}{3 \tilde{n}_0} + \frac{2g\tilde{n}_0}{m} + \frac{gn_{c0}}{m} \right) + \frac{gn_{c0}}{m} \left( \frac{5 \tilde{P}_0}{3 \tilde{n}_0} - \frac{2g\tilde{n}_0}{m} \right) = 0 . \quad (3)
\]

Expanding to second order in the explicit dependence on \( g \), the sound velocities are given by

\[
\nu^2_+ = \frac{5 \tilde{P}_0}{3 \tilde{n}_0} + \frac{2g\tilde{n}_0}{m} + \frac{gn_{c0}}{m} + \epsilon \quad (4a)
\]

\[
\nu^2_- = \frac{gn_{c0}}{m} - \frac{gn_{c0}}{m} \epsilon , \quad (4b)
\]

where \( \epsilon \equiv 4g\tilde{n}_0 / \frac{5 \tilde{P}_0}{3 \tilde{n}_0} \ll 1 \) is the expansion parameter. We note (see Section III) that the ratio \( \frac{\tilde{P}_0}{\tilde{n}_0} = k_B T \left( \frac{g_{52}(z_0)}{g_{32}(z_0)} \right) \) depends weakly on \( g \). The \( \omega_+ \) mode in (1a) clearly corresponds to first sound. Using \( \omega^2_+ = u^2_+ k^2 \) in (2), one finds to leading order in \( g \) that
\[
\frac{\dot{\phi}_N}{\phi_S} \simeq \frac{2}{\epsilon} \gg 1.
\]

That is to say, the \(\omega_+\) first sound mode corresponds to an \textit{in-phase} oscillation in which the non-condensate velocity amplitude is much larger than that of the condensate. The \(\omega_-\) mode in (4b) is the second sound mode. Using \(\omega^2 = u^2 k^2\) in (2), one finds to leading order in \(g\) that

\[
-\frac{\dot{\phi}_S}{\phi_N} \simeq \frac{2}{\epsilon} \tilde{n}_0 \gg 1.
\]

Thus at finite temperatures where (1a) and (1b) are valid, second sound in a uniform weakly-interacting gas is seen to be an \textit{out-of-phase} oscillation, in which the condensate velocity amplitude is much larger than that of the non-condensate (a similar result was obtained many years ago in Ref. [11]).

**III. EQUILIBRIUM PROPERTIES IN THE POPOV APPROXIMATION**

The ZGN derivation of the coupled hydrodynamic equations for the two velocity fields given in (1a) and (1b) is built on a self-consistent Hartree-Fock description of the equilibrium properties. One of the earliest discussions of this mean-field theory was given by Popov [7] and it has become the standard approximation in recent studies of trapped Bose gases. Referring to ZGN, we recall that the equilibrium equation for the condensate yields the equilibrium chemical potential

\[
\mu_0 = 2g\tilde{n}_0 + gn_c.
\]

This parameter enters in the determination of the equilibrium excited-atom density given by

\[
\tilde{n}_0(T, n_0) = \frac{1}{\Lambda^3} g_{3/2}(z_0),
\]

where \((n_0 \equiv n_c + \tilde{n}_0)\)

\[
z_0 = e^{\beta(\mu_0 - 2gn_0)} = e^{-\beta gn_c}
\]

is the equilibrium fugacity and \(\Lambda = \sqrt{2\pi \hbar^2 / mk_BT}\) is the thermal de Broglie wavelength. The associated excited-atom \textit{kinetic} pressure is

\[
\tilde{P}_0(T, n_0) = \frac{1}{\beta \Lambda^3} g_{5/2}(z_0).
\]

We note that these results are equivalent to the simple “toy model” studied in Ref. [12].

Eqs. (3) and (6) must be solved self-consistently to determine \(n_c\) and \(\tilde{n}_0\) for a given total density \(n_0\). Condensation occurs when the density reaches the critical density \(n_{cr} = g_{3/2}(1)/\Lambda^3\). For \(n_0 < n_{cr}\), the condensate density is zero and (3) with \(\tilde{n}_0 = n_0\) determines the equilibrium fugacity. In Fig. 1, we show the equilibrium densities as a function of volume for
a fixed temperature. The parameter \( \gamma_{cr} \equiv \beta g n_{cr} \) is used to characterize the strength of the interaction. We see that the present level of approximation leads to a discontinuous change in the densities at the transition point \( [13] \). Moreover, below the critical volume \( v_{cr} = 1/n_{cr} \), \( \tilde{n}_0 \) decreases as a result of the interactions with the condensate, in contrast to the ideal gas behaviour which has the non-condensate maintaining a constant density of \( n_{cr} \). Fig. 2 gives the total pressure defined as \[ P = \tilde{P}_0 + \frac{1}{2} g(n_0^2 + 2n_0\tilde{n}_0 - \tilde{n}_0^2), \tag{11} \]

normalized by the critical pressure \( \tilde{P}_{cr} = g_{5/2}(1)/\beta \Lambda^3 \) of the ideal gas. The second term in (11) is the explicit interaction contribution, but it should be noted that \( \tilde{P}_0 \) also depends on interactions as a result of its dependence on \( z_0 \). The discontinuous behaviour of the non-condensate density leads to an analogous discontinuity in the pressure \( [13] \). In Figs. 3 and 4 we show the corresponding behaviour as a function of \( T \). It is of interest to note that for a trapped Bose gas, the use of these equilibrium properties in the Thomas-Fermi approximation leads to a similar discontinuous behaviour of the equilibrium condensate density, but now as a function of the radial distance from the center of the trap \( [14] \).

It is clear that the properties of the weakly-interacting gas are nonanalytic functions of the interaction strength \( g \) at the transition point within the mean-field Popov approximation described by (7)–(10). However, one should not take these features in the BEC critical region seriously. The simple mean-field Popov approximation for interactions is well known \( [15,16] \) not to be valid very close to the transition and the predicted discontinuities exhibited in Figs. 1–4 (characteristic of a first-order transition) are indicative of the limitations of the present simple theory. A correct treatment of this region would require a renormalization group (RG) analysis \( [17] \) which is outside the scope of the present paper.

For later purposes, we note that the kinetic pressure \( \tilde{P}_0 \) in (11) can be calculated by expanding the fugacity as \( z_0 \approx 1 - \beta g n_0 + \cdots \), which yields (using the identity \( z \partial g_n(z)/\partial z = g_{n-1}(z) \))

\[ \tilde{P}_0 \approx \tilde{P}_{cr} - gn_{cr} \tilde{n}_{cr}, \tag{12} \]

where \( \tilde{P}_{cr} \) and \( n_{cr} \) are the critical pressure and density of the ideal Bose gas introduced earlier. However, a similar perturbative expansion of the non-condensate density \( \tilde{n}_0 \) in (8) is not possible since the derivative of \( g_{3/2}(z) \) diverges at \( z = 1 \). Indeed, it is this non-perturbative dependence on \( g \) which leads to the discontinuities shown in Figs. 1–4.

**IV. RELATION TO STANDARD TWO-FLUID EQUATIONS**

First and second sound in a uniform Bose-condensed gas have been previously discussed in the literature \( [5,7,11] \). These earlier treatments start with the usual two-fluid equations of Landau \( [3] \). We recall that these linearized equations are (see ch.7 of Ref. \( [1] \))

\[ \frac{\partial \delta n}{\partial t} = -\nabla \cdot \delta j \]

\[ m \frac{\partial \delta v_s}{\partial t} = -\nabla \delta \mu \]
\[
\begin{align*}
\frac{m}{\partial t} \delta j &= -\nabla \delta P \\
\frac{\partial \delta s}{\partial t} &= -\nabla \cdot (s_0 \delta \mathbf{v}_N) \ , \quad (13)
\end{align*}
\]

where

\[
\begin{align*}
\delta n(r, t) &= \delta \tilde{n}(r, t) + \delta n_c(r, t) \\
\delta j(r, t) &= \tilde{n}_0 \delta \mathbf{v}_N + n_c \delta \mathbf{v}_S \ .
\end{align*}
\]

\(P\) and \(s\) are the pressure and entropy density, respectively. ZGN proved that the two-fluid equations which lead to (1a) and (1b) are in fact equivalent to the two-fluid equations in (13) when the thermodynamic functions in the latter are evaluated for the present model of a weakly-interacting Bose gas. Using the thermodynamic relation \([4]\),

\[
\rho \frac{\partial \mu}{\partial \rho} = \frac{\partial P}{\partial T} - s_0 \frac{\partial T}{\partial \rho} ,
\]

one can reduce the equations in (13) to

\[
\begin{align*}
m \frac{\partial^2 \delta n}{\partial t^2} &= \nabla^2 \delta P \\
m \frac{\partial^2 \delta s}{\partial t^2} &= \frac{\rho s}{\rho_N} s_0^2 \nabla^2 \delta T .
\end{align*}
\]

(15)

Solving this closed set of equations in terms of the variables \(\delta n\) and \(\delta s\), one finds two normal mode solutions \(\omega^2 \equiv u^2 k^2\), where \(u^2\) is given by the solution of the quadratic equation \([3]\)

\[
u^4 - u^2 \left[ \frac{\partial P}{\partial \rho} \bigg|_T + \frac{T}{c_v} \left( \frac{1}{\rho} \frac{\partial P}{\partial T} \bigg|_\rho \right)^2 + \frac{\rho s}{\rho_N} \frac{T s_0^2}{c_v} \frac{\partial P}{\partial \rho} \bigg|_T \right] = 0 .
\]

(16)

In this equation, \(c_v\) is the specific heat per unit mass and derivatives of the pressure have been expressed in terms of the independent thermodynamic variables \(T\) and \(\rho\). Although not immediately apparent, the coefficients in (16) are in fact consistent with those appearing in (3).

The problem is thus reduced to evaluation of the various equilibrium thermodynamic functions and derivatives which appear in (16). For the entropy per unit mass we have the expression \([4]\)

\[
\rho_0 s_0 T = \frac{5}{2} \tilde{P}_0 + g \tilde{n}_0 n_c ,
\]

(17)

from which we obtain

\[
\rho_0 c_v = \frac{3}{2} \rho_0 s_0 + g \left( \frac{3}{2} \tilde{n}_0 + n_c \right) \frac{\partial \tilde{n}}{\partial T} \bigg|_\rho .
\]

(18)

From the equation of state (11), we find that

\[
\frac{\partial P}{\partial \rho} \bigg|_T = \frac{gn_0}{m} \left( 1 + \frac{\partial \tilde{n}}{\partial n} \bigg|_T \right)
\]

(19)
and
\[
\frac{\partial P}{\partial T} \bigg|_\rho = \rho_0 \bar{s}_0 + g n_0 \frac{\partial \tilde{n}}{\partial T} \bigg|_\rho .
\] (20)

These quantities have been calculated previously in the limit that the interaction parameter \(g\) is regarded as small [4, 5, 18, 19]. In this situation, \(\tilde{P}_0\) in (10) is approximated by (12). An additional approximation is typically made whereby \(\tilde{n}_0\) is simply replaced by the ideal gas expression \(\tilde{n}_0\), in which case \(n_{c0} = n_0 - n_{cr}\). To the same level of approximation, one finds \(\frac{\partial \tilde{n}}{\partial n} \bigg|_T = 0\) and \(\frac{\partial \tilde{n}}{\partial T} \bigg|_\rho = 3n_{cr}/2T\). With these replacements, we also note that the expressions for the pressure and the entropy and energy densities given in ZGN reduce precisely to those of Refs. [18] and [19].

Using these results to calculate the thermodynamic quantities in (16), the first and second sound velocities are found (after some algebra) to be given by
\[
u_+^2 = \frac{5 k_B T g_{5/2}(1)}{3 m} g_{3/2}(1) + \frac{2 g n_{cr}}{m} \frac{5 g n_{c0}}{m} ,
\]

(21a)
\[
u_-^2 = \frac{g n_{c0}}{m} ,
\]
(21b)
keeping terms to first order in \(g\). The leading order terms in (21a) and (21b) were obtained from (16) by this method by Popov (see the last paragraph of Ref. [4]) as well as by Lee and Yang [11]. Precisely the same results follow from (4) to first order in \(g\) when (12) is again used for the kinetic pressure \(\tilde{P}_0\) and \(\tilde{n}_0\) is replaced by \(n_{cr}\). However, the results given by (3) are more general than those in (21), which only keep the leading order corrections to the properties of a non-interacting gas. As we discussed above, the analysis leading to (21) ignores any interaction-correction to the non-condensate density \(\tilde{n}\), which as can be seen from Fig. 1, becomes significant as the density increases beyond \(n_{cr}\).

As we emphasized in the beginning of Section II, the analysis of ZGN assumes that \(g n_0 \ll k_B T\) and thus the results are not really valid at low temperatures. To discuss the low temperature region would require a generalization of our work which is based on a quasiparticle spectrum exhibiting phonon-like behavior at long wavelengths (such a kinetic equation has been derived in Ref. [10]). The pioneering work of Lee and Yang [11] did include an analysis of both the low temperature and high temperature regions. At low temperatures, they found that the first and second sound modes avoid becoming degenerate by hybridizing and an interchange of the physical meaning of these two modes occurs as a result of this hybridization. While the sound velocities given by (3) are not really valid at low temperatures, Fig. 5 shows that our results do lead to this expected hybridization of first and second sound in a dilute gas.

V. CONCLUDING REMARKS

Recently two-fluid hydrodynamic equations were derived [3] for a trapped, weakly-interacting Bose gas. These are given in terms of coupled equations for the superfluid and normal fluid velocity fluctuations. In order to obtain more physical insight into these
hydrodynamic equations, we have given in the present paper a detailed analysis for a uniform Bose gas. In this case, it has been proven that the hydrodynamic equations of Ref. [4] are formally equivalent to the usual Landau two-fluid equations. As the present paper shows, this formal equivalence is somewhat hidden in explicit calculations of the first and second sound velocities. However, as discussed in Section IV, our results do reduce (to first order in the interaction $g$) to those found in earlier studies [11,7,5] based on the Landau formulation.

In superfluid $^4$He, one evaluates the equilibrium thermodynamic parameters in (16) using the phonon-roton excitation spectrum. As is well known [1,3], in superfluid $^4$He, first sound corresponds to an in-phase oscillation in which $v_N = v_S$. In contrast, second sound corresponds to an out-of-phase oscillation in which $\rho_n v_N = -\rho_S v_S$. The difference between second sound in a dilute gas at finite temperatures and in a liquid is a result of the dominance of the kinetic energy over the interaction energy for atoms in a gas. In both cases, however, we note that the second sound frequency goes to zero (becomes soft) at the superfluid transition. The mode does not exist above $T_{BEC}$. Moreover (4b) shows that second sound crucially depends on the interaction $g$. It would be absent if we had set $g = 0$ in (1a) and (1b).

As we have noted, second sound in a dilute gas largely involves an oscillation of the condensate atoms (superfluid density) and is a soft mode which vanishes in the normal phase. We recall that at finite temperatures [7], the generalization of the $T = 0$ Bogoliubov phonon gives a velocity formally identical to the first term in (4b). Thus we conclude that in a weakly interacting Bose-condensed gas at finite temperatures, second sound is the low frequency (hydrodynamic regime) continuation of the high frequency (collisionless or mean-field regime) Bogoliubov-Goldstone mode. This was first suggested in Refs. [5–7]. The situation is quite different in superfluid $^4$He, where the collisionless phonon spectrum is the continuation of hydrodynamic first sound [3] and there is no high-frequency analogue of the second sound branch.

**ACKNOWLEDGMENTS**

We thank W. Ketterle for a copy of Ref. [9] before publication and useful remarks. This work was supported by research grants from NSERC of Canada.
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**FIGURE CAPTIONS**

Fig.1: Density vs. volume per particle for a fixed temperature $T$. The chain curve corresponds to the non-condensate, the solid curve to the condensate. $\gamma_{cr}$ is the value of $gn_0/k_BT$ at the critical density $n_{cr} = g_{3/2}(1)/\Lambda^3$.

Fig.2: Pressure isotherms: the solid line is the total pressure according to (11), the chain curve is $\tilde{P}_0$ and the dashed curve corresponds to the usual approximation $P \simeq \tilde{P}_{cr} + \frac{1}{2}g(n^2 + n_{cr}^2)$.

Fig.3: As in Fig. 1, but as a function of $T$ for a fixed density $n_0$. Here, $\gamma_{cr} = gn_0/k_BT_{BEC}$.

Fig.4: Normalized pressure as a function of $T$ for a fixed density $n_0$. The solid curve corresponds to (11) and the chain curve is $\tilde{P}_0$. The dashed curve below $T = T_{BEC}$ is the ideal gas result $\tilde{P}_0/\tilde{P}_{cr} = (T/T_{BEC})^{5/2}$.

Fig.5: Squares of the first and second sound velocities (normalized by the first sound velocity of the ideal gas at $T = T_{BEC}$) vs. $T/T_{BEC}$. The value of $\gamma_{cr}$ has been increased to more clearly reveal the anti-crossing behavior at low temperatures. As discussed in Section IV, the low temperature results only indicate the qualitative behavior.
$\gamma_{cr} = 0.1$
\[ \gamma_{cr} = 0.1 \]
$\gamma_{cr} = 0.1$
\[ \gamma_{cr} = 0.2 \]

- **First Sound**
- **Second Sound**

The graph shows the relation between \( \frac{u^2}{u_{cr}^2} \) and \( \frac{T}{T_{BEC}} \).