Remote state preparation using non-maximally entangled states

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We present a scheme in which any pure qubit \( |\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \) could be remotely prepared by using minimum classical bits and the previously shared non-maximally entangled states, on condition that the receiver holds the knowledge of \( \theta \). Several methods are available to check the trade-off between the necessary entanglement resource and the achievable fidelity.

I. INTRODUCTION

The elementary resources in quantum information theory are quantum entanglement and classical communication. By means of them, one unknown quantum state (“qubit”) could be transmitted from a sender (“Alice”) to a receiver (“Bob”), i.e., the process of teleportation [1, 2], which indicates that people have found a new way to broadcast information and shows better prospect than the traditional technique [3]. Similar to teleportation, the remote state preparation (RSP) is assumed that Alice completely knows the state to be prepared by Bob, who will know part of the knowledge on this state at most (in many situations he even knows nothing about this state). The essential concern for teleportation and RSP is the trade-off between entanglement and classical communication. It is clear that two bits of forward classical communication and one bit of entangled teleported qubit are both necessary and sufficient during the process of teleportation. However when it turns to the RSP, whether the amount of both quantum and classical resources could be reduced and how the trade-off between entanglement and classical communication will change has been checked by many authors. For instance, Pati [4] has shown that a qubit chosen from equatorial or polar great circles on a Bloch sphere can be remotely prepared with one classical bit from Alice to Bob if they share one bit of entanglement, which implies that the lower bound of classical communication [5] is possibly reached. Many other techniques [6–11] about faithful RSP have been constructed, including both exact and asymptotical methods.

Unlike the conventional disposal, recently, Ye et al. [12] proposed a new scheme in which non-maximally entangled state plays the role of quantum channel, instead of EPR [3] singlet. They showed that any pure quantum state can be faithfully prepared by using finite classical bits and any previously shared non-maximally entangled state. An explicit procedure is given by [13]. The scheme [12] of many ensembles of states remotely prepared by using minimum classical bits and previously shared entangled state, including all the ensembles in two-dimensional case, has been also established.

In this paper we study a RSP protocol, in which a series of non-maximally entangled states are employed as the quantum channel, each of which will correspond to one area where the transmitted state lies. In section II we describe this scheme and demonstrate that the prior fidelity expected can be achieved, provided that enough number of entangled states is supplied and Bob knows the content of \( \theta \). In section III we provide several techniques to reduce the entanglement resource for the deterministic fidelity, we try to find out the lowest expense. We compare the present work with several former techniques, in order to represent different characteristics of RSP protocols in section IV. Finally, we present our conclusion and some open problems.

II. EXPLICIT SCHEME

The protocol is characterized as follows. A pure qubit state and its orthonormal state have the form

\[
|\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle, \quad (1)
\]

\[
|\tilde{\phi}\rangle = \sin \theta |0\rangle - \cos \theta e^{i\varphi} |1\rangle. \quad (2)
\]

Here, two real parameters are valued in \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( 0 \leq \varphi \leq 2\pi \), which define the qubit \( |\phi\rangle \) as a point on the Bloch sphere [3]. Alice plans to transmit this state \( |\phi\rangle \) to Bob who has the knowledge of \( \theta \). Here we define that \( A_n = \frac{\pi}{2} \arcsin[(2\varphi - 1)^n] \), \( n = 0, 1, 2, ..., \text{and} \varphi \in [\frac{1}{2}, 1) \) which is the expected fidelity with which Alice transmits qubit \( |\phi\rangle \) to Bob. When \( \theta \in [\frac{\pi}{4} - A_n, \frac{\pi}{4} - A_{n+1}] \), the prior-entangled state shared by Alice and Bob is assumed like this:

\[
|\Psi_{AB}\rangle = |0\rangle |0\rangle + \tan(\frac{\pi}{4} - A_{n+1}) |1\rangle |1\rangle. \quad (3)
\]

Notice that we don’t normalize the above state for convenience and the same reason is applicable to all following cases. As the first step, Alice performs a unitary operation

\[
U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-i\varphi} \\
\frac{1}{\sqrt{2}} e^{-i\varphi} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

then Alice measures her particle with basis \{\( |0\rangle, |1\rangle \)\} and broadcast 1 bit to inform Bob about the result of her measurement. After receiving the information Bob will

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do nothing if he gets 0 or \( \sigma_z \) if he gets 1. Therefore he could always get such state
\[
|\psi\rangle = |0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle. \tag{4}
\]

Second, Bob performs the CNOT gate on \( |\psi\rangle \) and an ancilla state \( |\psi_{anc}\rangle = |0\rangle + y|1\rangle \)
\[
U_{\text{CNOT}} |\psi_B\rangle |\psi_{anc}\rangle = (|0\rangle |0\rangle + y|0\rangle |1\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle |1\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle |0\rangle)_{B,\text{anc}},
\]
where \( y = a + i\sqrt{\frac{2a\cot(2A_{n+1})}{\tan(2\theta)}} - a^2 - 1, \) \( a \) is some constant which keeps
\[\frac{2a\cot(2A_{n+1})}{\tan(2\theta)} - a^2 - 1 \geq 0 \text{ [14].} \]
The reduced density matrix of \( B \) then becomes
\[
\rho_B = [|0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle]
\[ [|0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle + y|0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle + y^* (|0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)e^{i\varphi}|1\rangle].
\]

Using \( \{|\phi\rangle, |\bar{\phi}\rangle\}, \) the basis \( \{|0\rangle, |1\rangle\} \) can be reexpressed as
\[
|0\rangle = \cos \theta |\phi\rangle + \sin \theta |\bar{\phi}\rangle, \tag{5}
\]
\[
|1\rangle = e^{-i\varphi}(\sin \theta |\phi\rangle - \cos \theta |\bar{\phi}\rangle). \tag{6}
\]

So \( \rho_B \) is written as
\[
\rho_B = C_0 |\phi\rangle \langle \phi | + C_1 |\bar{\phi}\rangle \langle \bar{\phi}| + C_2 |\phi\rangle \langle \bar{\phi}| + C_3 |\bar{\phi}\rangle \langle \phi|,
\]
where
\[
C_2 = C_3^* \]
\[= (\cos \theta + y \tan\left(\frac{\pi}{4} - A_{n+1}\right) \sin \theta)
\]
\[= (\sin \theta - y^* \tan\left(\frac{\pi}{4} - A_{n+1}\right) \cos \theta)
\]
\[= (y \cos \theta + \tan\left(\frac{\pi}{4} - A_{n+1}\right) \sin \theta)
\]
\[= (y^* \sin \theta - \tan\left(\frac{\pi}{4} - A_{n+1}\right) \cos \theta)
\]
\[= 0.
\]

Hence, we get
\[
\rho_B = C_0 |\phi\rangle \langle \phi | + C_1 |\bar{\phi}\rangle \langle \bar{\phi}|. \tag{7}
\]

From the above equation, one can read off the fidelity of \( |\phi\rangle \)
\[
F(|\phi\rangle \langle \phi|) = \frac{C_0}{C_0 + C_1} = \frac{1}{1 + \chi}. \tag{8}
\]

where
\[
\chi = \frac{C_1}{C_0} = \frac{\cos 2\theta - \sin 2A_{n+1}}{\cos 2\theta + \sin 2A_{n+1}}. \tag{9}
\]

According to \( \theta \in \left[\frac{\pi}{4} - A_n, \frac{\pi}{4} - A_{n+1}\right], \) we find
\[
\chi_{\text{MIN}} = 0,
\]
\[
\chi_{\text{MAX}} = q^{-1} - 1.
\]

It can be easily found that \( F(|\phi\rangle \langle \phi|) \in [q, 1], \) which implies that \( q \) is the minimum fidelity with which Bob gets state \( |\phi\rangle \).

Until now the parameter \( \theta \) is confined in some smaller region. Since \( \theta \in \left[\frac{\pi}{4} - A_n, \frac{\pi}{4} - A_{n+1}\right] \) and \( A_n = \frac{1}{2}\arcsin(2q - 1)n, \) \( n = 0, 1, 2, \ldots, \) we can see that \( A_n \) will become smaller as \( n \) goes up and finally
\[
\lim_{n \to \infty} A_n = 0.
\]

If all regions are combined (note that \( A_0 = \frac{\pi}{4} \))
\[
\left[\frac{\pi}{4} - A_0, \frac{\pi}{4} - A_1\right] \cup \left[\frac{\pi}{4} - A_1, \frac{\pi}{4} - A_2\right] \cup \ldots \left[\frac{\pi}{4} - A_n, \frac{\pi}{4} - A_{n+1}\right] \cup \ldots = [0, \frac{\pi}{4}], \tag{10}
\]

the whole region of \([0, \frac{\pi}{4}]\) is covered. Now, if we own sufficient non-maximally entangled states \( |\Psi_{AB}\rangle = |0\rangle |0\rangle + \tan\left(\frac{\pi}{4} - A_{n+1}\right)|1\rangle |1\rangle, n = 0, 1, 2, \ldots, \) the protocol for \( \theta \in [0, \frac{\pi}{4}] \) is completed.

On the other hand, we can deal with the region \( \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \) in a similar method. First, \( \theta \) is divided into many small regions, i.e., \([\frac{\pi}{4} + A_{n+1}, \frac{\pi}{4} + A_n], n = 0, 1, 2, \ldots \). Then, on each small region, a non-maximally entangled state is provided in the following form
\[
|\Psi_{AB}\rangle = |0\rangle |0\rangle + \tan\left(\frac{\pi}{4} + A_{n+1}\right)|1\rangle |1\rangle. \tag{11}
\]

Subsequently, the procedure is entirely the same as that of region \( \theta \in [0, \frac{\pi}{4}] \), except that \( y \) will be redefined as \( y = a + i\sqrt{-\frac{2a\cot(2A_{n+1})}{\tan(2\theta)}} - a^2 - 1. \) Again \( a \) is some constant which keeps \( -\frac{2a\cot(2A_{n+1})}{\tan(2\theta)} - a^2 - 1 \geq 0 \). After performing all steps, we get
\[
\chi = \frac{C_1}{C_0} = \frac{\cos 2\theta + \sin 2A_{n+1}}{\cos 2\theta - \sin 2A_{n+1}}. \tag{12}
\]

According to \( \theta \in [\frac{\pi}{4} + A_{n+1}, \frac{\pi}{4} + A_n], n = 0, 1, 2, \ldots, \) we also get \( \chi \in [0, q^{-1} - 1], \) which induces \( F(|\phi\rangle \langle \phi|) \in [q, 1]. \)

Therefore \( q \) also denotes the minimum fidelity on each region. Since this protocol can be carried out on any region \([\frac{\pi}{4} + A_{n+1}, \frac{\pi}{4} + A_n], n = 0, 1, 2, \ldots, \) we have completed the scheme for region \([\frac{\pi}{4}, \frac{\pi}{2}]\). Combined with the conclusion on region \([0, \frac{\pi}{4}]\), the explicit protocol is feasible on the whole region \([0, \frac{\pi}{2}]\).

In the above protocol, the total classical cost we need is 1 bit. A certain number of non-maximally entangled states is required for distinct regions of \( \theta \). The
sufficient number is easily imaginable, e.g., random astronomical number. However, it is unclear what the necessary number is. Obviously, the smaller this number is, the better this protocol will become. From the above protocol we need two non-maximally entangled states \(|0\rangle |0\rangle + \tan(\frac{\pi}{4} - A_n)|1\rangle |1\rangle\) for region \(\theta \in [\frac{\pi}{4} - A_n, \frac{\pi}{4} + A_n]\) and \(|0\rangle |0\rangle + \tan(\frac{\pi}{4} + A_{n+1})|1\rangle |1\rangle\) for region \(\theta \in [\frac{\pi}{4} + A_{n+1}, \frac{\pi}{4} + A_n]\). A glancing observation will lead to the result that these two states are interconvertible by jointly local operation \((\sigma_x)_{A}(\sigma_x)_{B}\) for any \(n = 0, 1, 2, \ldots\), which implies that the family of non-maximally entangled states for the region \([0, \frac{\pi}{4}]\) or the other family for the region \([\frac{\pi}{4}, \frac{\pi}{2}]\) will be enough for the whole region \(\theta \in [0, \frac{\pi}{4}]\). This result immediately decreases the original number of entangled states to its half. From this brief process we infer that the necessary entanglement resource can be reduced. We analyze this parameter for the explicit protocol above in appendix A, where the main result is that we have to supply more and more shared entangled states with increasing fidelity \(g\) and approximation accuracy around the central point \(\theta = \frac{\pi}{4}\).

III. REDUCTION OF SHARED ENTANGLEMENT

We will show that it is possible to carry out the above protocol with high fidelity, provided 1 qubit and some small number of entangled states are available. According to the above argument, it is necessary that

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} \arcsin((2q - 1)^n) = 0. 
\]

Therefore \(n\) must be very large as the minimum fidelity \(q\) gradually tends to one, which says that \(q\) has direct dependence to the number of shared entangled states. The proposition from appendix A indicates that, e.g., at least 38 non-maximally entangled states are necessary to carry out the above protocol under the condition \(q = 0.95, A_{38} \simeq 0.01\). While \(q\) is required to be larger, \(N\) will increase very fast. This conclusion also shows that we need much more entanglement resource when \(A_n\) is smaller, so we try to improve the qualification of the above protocol by focusing on the small region near central point \(\theta = \frac{\pi}{4}\).

IMPROVED PROTOCOL I

The technique in section II supposes that Alice and Bob share the non-maximally entangled state \(|\Psi_{AB}\rangle = |0\rangle |0\rangle + \tan(\frac{\pi}{4} - A_n)|1\rangle |1\rangle\) when \(\theta \in [\frac{\pi}{4} - A_n, \frac{\pi}{4} + A_n] \cup [\frac{\pi}{4} + A_n, \frac{\pi}{4} + A_{n+1}]\), \(n = 0, 1, 2 \ldots, N\), whose combination will be \(\theta \in [0, \frac{\pi}{4} - A_N] \cup [\frac{\pi}{4} + A_N, \frac{\pi}{4}]\). Here we deal with the small central region \(\theta \in [\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N]\) by using one maximally entangled state shared by Alice and Bob:

\[
|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle). 
\]

First Alice performs the following unitary operation on her particle

\[
U = \begin{pmatrix} \cos \theta & \sin \theta e^{i \phi} \\ \sin \theta e^{-i \phi} & - \cos \theta \end{pmatrix},
\]

then she measures this particle with basis \(|0\rangle, |1\rangle\) and broadcast 1 bit to Bob, who will do nothing if he gets 0 or \(\sigma_z\) if he gets 1. Thus Bob will get the following states both in 50% probability:

\[
|\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i \phi} |1\rangle, \\
|\phi'\rangle = \sin \theta |0\rangle + \cos \theta e^{i \phi} |1\rangle.
\]

So if Bob receives 0 the protocol comes true, while he gets state \(|\phi'\rangle\), we observe that \(\theta \in [\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N]\) and if \(A_N\) is very small, the two qubits \(|\phi\rangle\) and \(|\phi'\rangle\) will be very close to each other. Hence, the fidelity is

\[
F(|\phi\rangle, |\phi'\rangle) \equiv ||\langle \phi | \phi \rangle| = \sin 2\theta,
\]

so we obtain

\[
F \geq F_{\min} = \sin\left(\frac{\pi}{2} \pm 2A_N\right) = \sqrt{1 - (2q - 1)^2N}.
\]

The connection between \(N\) and \(q\) is described in figure 1 and figure 2, while several key points \(N + 1\) in the following table for there is one added maximally-entangled state. This protocol is a kind of \textit{approximate} substitution, which indeed requires that qubit \(|\phi'\rangle\) replaces \(|\phi\rangle\) with a high fidelity. We also give another scheme based on the above discussion in appendix B, which is a kind of probabilistic exact protocol [15]. Combined with the technique in section II on region \([0, \frac{\pi}{4} - A_N] \cup [\frac{\pi}{4} + A_N, \frac{\pi}{4}]\), the whole protocol is now completed.

![FIG. 1: trade-off between \(N\) and \(q\) in \([0.5, 0.9]\). The three curves from downside to upside represent \(F_{\min} = 0.99, 0.999, 0.9999\), whose precision is gradually improved. \(N\) increases very slowly.](image-url)
FIG. 2: trade-off between $N$ and $q \in [0.9, 0.99]$. The three curves from downside to upside represent $F_{\text{min}} = 0.99, 0.999, 0.9999$. We find $N$ here increases less than that of Appendix A.

**IMPROVED PROTOCOL II**

We take an ulterior step to decrease the entanglement resource. The present idea is based on the technique in appendix A, which leaves out the small symmetry region $\left[\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N\right]$. We define $C_n = \tan(\frac{\pi}{4} - A_n)$, thus $N$ necessary entangled states are $\left|\Psi_{AB}\right\rangle = |0\rangle|0\rangle + C_{l_k}|1\rangle|1\rangle, l_k \in [1, M]$.

(19)

Evidently, each section contains several entangled states. Our aim is to replace $|\Psi_k\rangle$ by

$$|\Phi_k\rangle = |0\rangle|0\rangle + B_k|1\rangle|1\rangle, k \in [1, M].$$

(20)

$B_k$ is a positive constant. We introduce a POVM measurement

$$M_{k0} = \begin{pmatrix} \sqrt{\frac{B_k^2 + 1}{C_{l_k}^2 + 1}} C_{l_k} & 0 \\ 0 & \sqrt{\frac{B_k^2 + 1}{C_{l_k}^2 + 1}} B_k \end{pmatrix},$$

$$M_{k1} = \sqrt{I - M_{k0}^2},$$

(21)

where $P_k \in [\frac{1}{B_k^2 + 1}, 1], C_{l_k} \in [\sqrt{P_k B_k^2 + P_k - 1}, B_k]$. After performing measurement $M_{k0}$, therefore $|\Phi_k\rangle$ can be transformed into $|\Psi_k\rangle$. The probability Alice carries out $M_{k0}$ is

$$P(M_{k0}) = \langle \Phi_k | M_{k0}^\dagger M_{k0} | \Phi_k \rangle = P_k.$$  

(22)

The above argument implies that Alice can decrease the necessary number of entangled states with probability $P_k$, by substituting one state $|\Phi_k\rangle$ for each section $|\Psi_k\rangle$.

Here we give a concrete procedure to show how entanglement resource is reduced. According to the result in appendix A, at least $N = 194$ entangled states are required so that the small region $\left[\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N\right]$ can be left out with $q = 0.99$. For convenience we suppose $P_k = 0.99$ for $k \in [1, M]$. The necessary condition to be satisfied is $C_{l_k} \in [\sqrt{\frac{P_k B_k^2 + P_k}{B_k^2 + 1}} - 1, B_k]$, thus all $f_k$ belonging to this region will lead to the fact that $|\Psi_k\rangle = |0\rangle|0\rangle + C_{l_k}|1\rangle|1\rangle$. First we set $B_M = \tan(\frac{\pi}{4} - A_{194})$, i.e., $l_M = 194$. In order to get $l_{M-1}$ we calculate

$$\sqrt{0.99 \times \tan(\frac{\pi}{4} - A_{194})^2 + 0.99 - 1} \leq C_{l_k} \leq \tan(\frac{\pi}{4} - A_{194}),$$

(23)

which leads to $l_{M-2} = 159$. The technique for rest region is analogous to the above procedure. At last the total number of entangled states $|\Psi_k\rangle = |0\rangle|0\rangle + \tan(\frac{\pi}{4} - A_k)|1\rangle|1\rangle$ is $M = 50 : k = 194, 173, 158, 146, 136, 128, 121, 115, 109, 104, 99, 94, 90, 86, 82, 78, 75, 72, 69, 66, 63, 60, 57, 52, 1 \leq t \leq 27$. Notice that $P_k \geq \frac{1}{B_k^2 + 1}$ for all $k$ above. If lower probability is allowed, e.g., $P_k = 0.98$ for $k \in [1, M]$, similar technique says that only $M = 29$ entangled states are required.

Now we summarize the whole protocol. First Alice and Bob share $M$ quantum channels $|\Phi_k\rangle = |0\rangle|0\rangle + B_k|1\rangle|1\rangle, k \in [1, M]$. By local POVM measurement $\{M_{k0}, M_{k1}\}$, Alice can transform each $|\Phi_k\rangle$ into corresponding string $|\Psi_k\rangle = |0\rangle|0\rangle + C_{l_k}|1\rangle|1\rangle, f_k \in [l_{k-1}, l_k], k \in [1, M]$, with probability $P_k$. Here we define $l_0 = 1$ and $l_M = N$. $N$ is determined by $q$ and the approximation accuracy of region $\left[\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N\right]$. Next step follows the technique in section II, since we have got $|\Psi_{AB}\rangle = |0\rangle|0\rangle + \tan(\frac{\pi}{4} - A_k)|1\rangle|1\rangle, k \in [1, N]$. The assumption Bob knows $\theta$ assists Bob by distinguishing which channel is in use, i.e., state $|\Phi_k\rangle$ corresponds to $\theta \in \left[\frac{\pi}{4} - A_{l_k-1}, \frac{\pi}{4} - A_{l_k-1}\right] \cup \left[\frac{\pi}{4} + A_{l_k-1}, \frac{\pi}{4} + A_{l_k-1}\right]$ when $1 \leq k \leq M - 1$, and $\theta \in \left[\frac{\pi}{4} - A_{l_{M-1}}, \frac{\pi}{4} - A_{l_{M-1}}\right] \cup \left[\frac{\pi}{4} + A_{M}, \frac{\pi}{4} + A_{l_{M-1}}\right]$ when $k = M$. Finally Bob will get the expected state $|\psi\rangle$ with a minimum fidelity $q \times P_k$.

Hitherto we construct protocol II based on the demonstration in appendix A. However, it is completely feasible to adopt the technique in protocol I and appendix B for the disposal of region $\left[\frac{\pi}{4} - A_N, \frac{\pi}{4} + A_N\right]$. Combined with appendix B, a probabilistic exact protocol with higher efficiency is practicable. If more entanglement resource is available, we can improve the success probability farther.
IV. MORE ARGUMENT ABOUT RSP

All techniques above describe a sort of RSP protocol with a decided fidelity, which requires the expense of certain number of entangled states and one bit of classical communication. An apparent deficiency in this protocol is that the receiver needs to know the content of \( \theta \), and the entanglement resource required may be large. The technique based on the dark states [16, 17] provided an explicit scheme in which Bob knows \( \theta \) or \( \varphi \), under this condition Alice could transmit any qubit \( |\psi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \) to Bob with the expense of one maximally-entangled state and one bit of classical communication. However the transformation it requires in the case Bob knows \( \theta \) is not unitary but Hermitian, therefore it is not possible to carry out such protocol on a quantum computer [3]. The probabilistic exact protocol in [18] could transmit any polar state with relatively lower fidelity from one sender to different receivers, where the expense is one deliberate entangled state and one cbit.

Another idea from [12] has given a faithful scheme in which many ensembles of states can be remotely prepared by using minimum classical bits and previously shared entangled state, especially they have found all the ensembles in two-dimensional case. It seems this is a better protocol for it needs only one shared entangled state and one cbit, furthermore this is a faithful scheme. Here we do some simple analysis on this scheme in two-dimensional case. As described in [12], the ensemble that can be remotely prepared must be in the form

\[
\{ v |\Phi\rangle = v (\alpha_0 |0\rangle + \alpha_1 e^{i\omega} |1\rangle) \mid \alpha_0, \alpha_1 > 0, \alpha_0^2 + \alpha_1^2 = 1, \forall \omega \}
\]

by a previously shared entangled state

\[
|\Psi_{AB}\rangle = \alpha_0 |0\rangle |0\rangle + \alpha_1 |1\rangle |1\rangle.
\]

(26)

Here, \( \alpha_0, \alpha_1 \) and \( \omega \) are known to Alice and \( v \) is a unitary operator in two-dimensional Hilbert space. Therefore we suppose

\[
v = \begin{pmatrix}
\cos \gamma & -e^{i\delta} \sin \gamma \\
e^{i\beta} \sin \gamma & e^{i(\beta+\delta)} \cos \gamma
\end{pmatrix},
\]

where \( \beta, \gamma, \delta \) are real parameters. This operation is done by Bob so the parameters should be independent of \( |\phi\rangle \).

Since all the ensembles have been found, we can infer

\[
v |\Phi\rangle = A e^{-i\alpha} |\phi\rangle = A e^{-i\alpha} (\cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle), \quad (28)
\]

where \( A \) and \( \alpha \) are any real number. Obviously \( A = \pm 1 \), and if \( A = -1 \) we can set \( \gamma \to \gamma + \pi \). Thus

\[
v |\Phi\rangle = e^{-i\alpha} (\cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle). \quad (29)
\]

Some simple algebra will lead to

\[
\alpha_0 = \sqrt{\cos^2 \gamma - \cos 2\gamma \sin^2 \theta + \frac{1}{2} \sin 2\gamma \sin 2\theta \cos(\varphi - \beta)}
\]

\[
\alpha_1 = \sqrt{\cos^2 \gamma - \cos 2\gamma \cos^2 \theta - \frac{1}{2} \sin 2\gamma \sin 2\theta \cos(\varphi - \beta)}
\]

where we have employed the assumption that \( \alpha_0, \alpha_1 > 0, \alpha_0^2 + \alpha_1^2 = 1 \). A simple observation shows that both \( \alpha_0 \) and \( \alpha_1 \) must be related to \( \theta \) or \( \varphi \) under the assumption that \( \beta \) and \( \gamma \) are constant. Therefore the shared entangled state \( |\Psi_{AB}\rangle = \alpha_0 |0\rangle |0\rangle + \alpha_1 |1\rangle |1\rangle \) can’t be constant, i.e., it is a variable which transforms with the change of \( \theta \). That is to say, infinite amount of entangled states are required to perform this protocol, for there are infinite number of \( \theta \) during the region \( [0, \pi/2] \). However, it is possible that \( \alpha_0 \) and \( \alpha_1 \) become constant, provided \( \beta \) and \( \gamma \) are related to \( \theta \) or \( \varphi \). We simply rewrite the expression of \( \alpha_0 \) and \( \alpha_1 \) by employing the two ends \( \theta = 0, \pi/2 \) to get

\[
\alpha_0 = \alpha_1 = |\cos \gamma| = |\sin \gamma| = \frac{1}{\sqrt{2}},
\]

\[
\beta = \varphi \pm \frac{\pi}{2}, \omega = \pi - \delta - 2\theta. \quad (30)
\]

Therefore Bob has to know \( \varphi \), in addition the shared state \( |\Psi_{AB}\rangle \) has become an EPR singlet, which breach the origin thought in [12]. It is a trivial scheme similar to that in [17]. To say the least, \( \alpha_0 \) and \( \alpha_1 \) will still be connected with \( \theta \) or \( \varphi \), while \( \beta \) and \( \gamma \) are merely related to \( \theta \). It is because that the term \( \frac{1}{2} \sin 2\gamma \sin 2\theta \cos(\varphi - \beta) \) will not disappear until \( \sin 2\gamma = 0 \), under which the shared entangled state remains \( |\Psi_{AB}\rangle = \cos \theta |0\rangle |0\rangle + \sin \theta |1\rangle |1\rangle \) or \( |\Psi_{AB}\rangle = \sin \theta |0\rangle |0\rangle + \cos \theta |1\rangle |1\rangle \). From these reasons we infer that the protocol in [12] always requires one entangled state for one corresponding \( \theta \), otherwise it will be a trivial scheme. Therefore infinite entanglement resource is required for all transmitted qubits. Furthermore, the condition Bob holds the knowledge of \( \theta \) will not help decrease the necessary shared entanglement. In fact, the protocol in this paper is an effective method in economizing entanglement, by the help that Bob knows \( \theta \).

One latest technique in [13] is an exactly faithful RSP protocol, which requires finite cbits and one entangled state in \( d \)-dimensional Hilbert space. This technique requires relatively more classical communication than other techniques, however it needs only one arbitrary entangled state to transmit any one qubit, which is established on the transformation of original entangled state. The idea in improved protocol II is based on this scheme. If two forward cbits are allowed, the scheme in this paper may become another form like this. The quantum channel shared by Alice and Bob is one Greenberger-Horne-Zeilinger (GHZ) state

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle |0\rangle + |1\rangle |1\rangle |1\rangle). \quad (31)
\]

Here, Alice and Bob have two and one particles respectively. First Alice performs a local unitary operation on one of her particles

\[
U = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

Then Alice measures this particle with basis \( \{ |0\rangle, |1\rangle \} \) and broadcast 1 bit to inform Bob about the result \( k \) of
her measurement. The corresponding operations done by Alice and Bob are respectively

\[ U_{A0} = \begin{pmatrix} 1 & 0 \\ 0 & -e^{i\varphi} \end{pmatrix}, \quad U_{B0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad k = 0 \] (32)

and

\[ U_{A1} = \begin{pmatrix} 0 & 1 \\ e^{i\varphi} & 0 \end{pmatrix}, \quad U_{B1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = 1. \] (33)

Thus they will always share

\[ |\psi\rangle = \cos \theta |0\rangle |0\rangle + \sin \theta e^{i\varphi} |1\rangle |1\rangle. \] (34)

Second, Alice performs a Hadamard gate and broadcast 1 bit to Bob, who does nothing if he gets 0 or \( \sigma_z \) if he gets 1. Now Bob gets the expected state \( |\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \). This protocol is also a typical application which employs the idea of entanglement transformation.

In general, all kinds of RSP protocols try to find out the best trade-off between the classical communication and entanglement resource under different preconditions. As we have compared various RSP protocols, the technique advanced in this paper widens the traditional restriction, i.e., the receiver Bob owns the content of parameter \( \theta \) of a general qubit \( |\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \). This is indeed a generalization of Pati’s RSP protocol [4] by loosing the range of \( \theta \). Moreover, we also provide several techniques to decrease the necessary entanglement resource effectively. Since the lower bound of the entanglement consumption remains to be found, we may expect there should be better idea to reduce the necessary resource, following the technique in this paper. On the other hand, we also employ the non-maximally entangled state as the quantum channel like that in [12, 13]. The non-maximally state is more flexible and it expands our choice in the aspect of quantum connection. However, the generally faithful RSP protocol is still hard to establish if the condition on quantum and classical resources is restricted. Although the asymptotic techniques, e.g., [6, 7, 15] have successfully transmitted arbitrary state at a cost of 1 bit and 1 ebit per qubit sent, their preconditions required are plenty of shared entanglement resource and classical communication.

V. CONCLUSIONS

We have given an explicit protocol for performing the RSP protocol, using minimum classical bits and certain number of non-maximally entangled states as the quantum channel. The trade-off between the necessary entanglement resource and the achievable fidelity is discussed in detail by several different techniques. The evaluation of this protocol should be focused on how far we can reduce entanglement resource, however the optimal choice is hard to make out despite the above discussion. One useful finding in this paper is that the condition Bob knows \( \theta \) will be helpful to transmit the qubit with lower necessary entanglement. We may consider, that there is some latent connection between the necessary resource and how far the receiver owns the knowledge of the qubit sent, during an exact RSP process. The idea of entanglement transformation is a good method which may lead to a better effect later.

APPENDIX A: RESOURCE FOR THE EXPLICIT PROTOCOL

It is defined that \( A_n = \frac{\pi}{4} \arcsin (2q - 1)^n \), \( n = 0, 1, 2, \ldots, N \) and \( q \in \left[ \frac{1}{2}, 1 \right] \). The task we face is how to make \( A_N \) as small as possible, so that the region \( \left[ \frac{\pi}{4} - A_N, \frac{\pi}{4} \right] \) can be ignored, in other words this region has been concentrated on the point \( \theta = \frac{\pi}{4} - A_N \). In this case qubit \( |\phi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \) will be very close to the polar state \( \cos \frac{\pi}{4} |0\rangle + \sin \frac{\pi}{4} e^{i\varphi} |1\rangle \) during the region \( \left[ \frac{\pi}{4} - A_N, \frac{\pi}{4} \right] \). The second half is similarly treated with, i.e., as \( A_N \) is very small, region \( \left[ \frac{\pi}{4} + A_N, \frac{\pi}{4} \right] \) will be concentrated on the point \( \theta = \frac{\pi}{4} + A_N \). The estimation procedure is below.

Firstly, to the first half we suppose

\[ \frac{\sin \left( \frac{\pi}{4} - A_N \right)}{\sin \frac{\pi}{4}} = 1 - 10^{-m}, \quad m = 2, 3, 4, 5, 6, \ldots, \]

i.e., \( A_N = \frac{\pi}{4} - \arcsin \left( \frac{1 - 10^{-m}}{\sqrt{2}} \right) \), which also keeps

\[ \frac{\cos \left( \frac{\pi}{4} - A_N \right)}{\cos \frac{\pi}{4}} < 1 + 10^{-m}, \quad m = 2, 3, 4, 5, 6, \ldots, \]

consequently the qubit \( \cos \left( \frac{\pi}{4} - A_N \right) |0\rangle + \sin \left( \frac{\pi}{4} - A_N \right) e^{i\varphi} |1\rangle \) could approximately substitute the qubit \( \cos \theta |0\rangle + \sin \theta e^{i\varphi} |1\rangle \) with high fidelity, if \( m \) is large enough. Secondly, the above procedure also applies to the second half. Since we already have

\[ \frac{\sin \left( \frac{\pi}{4} + A_N \right)}{\cos \frac{\pi}{4}} < 1 + 10^{-m}, \quad m = 2, 3, 4, 5, 6, \ldots, \]

\[ \frac{\cos \left( \frac{\pi}{4} + A_N \right)}{\sin \frac{\pi}{4}} = 1 - 10^{-m}, \quad m = 2, 3, 4, 5, 6, \ldots, \]

it implies that qubit \( \cos \left( \frac{\pi}{4} + A_N \right) |0\rangle + \sin \left( \frac{\pi}{4} + A_N \right) e^{i\varphi} |1\rangle \) is also close to the polar qubit \( \cos \frac{\pi}{4} |0\rangle + \sin \frac{\pi}{4} e^{i\varphi} |1\rangle \). According to the above argument we describe the relationship between \( N \) and \( q \) in Figure 3 where

\[ N = \frac{\log (2 + 10^{-m} - 10^{-2m})}{\log (2q - 1)}. \]

Here \( \log (x) \) denotes logarithms to base 2. Notice that \( q \) is the minimum fidelity for Bob to get the state \( |\phi\rangle \) during the region \( \left[ 0, \frac{\pi}{4} - A_N \right] \cup \left[ \frac{\pi}{4} + A_N, \frac{\pi}{4} \right] \). Several necessary numbers \( N \) of non-maximally entangled states are explicitly provided in the following table.
\[ F(\theta \in [\pi/4 - A_N, \pi/4 + A_N])_{\text{min}} = \frac{1}{(2q - 1)^N + 1}. \]

or another form

\[ N = \frac{\log(1/F_{\text{min}} - 1)}{\log(2q - 1)}. \]

This function is described in Figure 4. From it we find that \( N \) increases slowly in a majority of region while \( N \) will become very large as \( q \) tends to 1, which is similar to the situation in improved protocol I. Here we also need only \( N + 1 \) entangled states and several key point \( N + 1 \) is made out in the table below. The fidelity in Figure 4 is lower than that of improved protocol I. However it is noticeable that this protocol is a probabilistic exact protocol, i.e., combined with the argument in section II Bob could get the explicit qubit \( |\phi\rangle \) during the whole region \( \theta \in [0, \pi/2] \) with a high fidelity. Therefore such protocol is more assuring than improved protocol I and it can also be combined with the explicit scheme in section II to get a better qualification.

\[ \begin{array}{cccccc}
 m & A_N & q = 0.90 & q = 0.95 & q = 0.98 & q = 0.99 \\
 2 & 9.95066 \times 10^{-4} & 17.55 & 37.18 & 95.95 & 193.89 \\
 4 & 9.99950 \times 10^{-9} & 38.17 & 80.84 & 208.64 & 421.59 \\
 6 & 9.99999 \times 10^{-12} & 58.81 & 124.55 & 321.45 & 649.54 \\
\end{array} \]

APPENDIX B: ANOTHER IMPROVED PROTOCOL

We deal with the region \( \theta \in [\pi/4 - A_N, \pi/4 + A_N] \) in another way which is based on correct protocol one, until Bob gets states \( |\phi\rangle \) and \( |\phi'\rangle \) both in 50\%. If he gets \( |\phi'\rangle \) then Bob carries out one POVM measurement

\[ M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \tan^2 \theta \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - \tan^4 \theta} \end{pmatrix}, \]

if \( \theta \in [\pi/4 - A_N, \pi/4] \) or

\[ M_0' = \begin{pmatrix} \cot^2 \theta & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1' = \begin{pmatrix} \sqrt{1 - \cot^4 \theta} & 0 \\ 0 & 0 \end{pmatrix}, \]

if \( \theta \in [\pi/4 + A_N, \pi/4 + A_N] \). The probabilities with which Bob performs \( M_0 \) and \( M_0' \) are

\[ P(M_0) = \langle \phi' | M_0 M_0^\dagger | \phi' \rangle = \tan^2 \theta, \]

\[ P(M_0') = \langle \phi' | M_0' M_0'^\dagger M_0 M_0'^\dagger | \phi' \rangle = \cot^2 \theta. \]

Evidently both \( M_0 \) and \( M_0' \) will transform state \( |\phi'\rangle \) into \( |\phi\rangle \). As \( A_N \) is a small amount, we may infer Bob will do \( M_0 \) or \( M_0' \) with a high probability. On the other hand Bob always gets the expected state \( |\phi\rangle \) with probability 50\%. Therefore the fidelity is

\[ F(\theta \in [\pi/4 - A_N, \pi/4]) = \frac{1 + \tan^2 \theta}{2} = \frac{\sec^2 \theta}{2}, \]

\[ F(\theta \in [\pi/4 + A_N]) = \frac{1 + \cot^2 \theta}{2} = \frac{\csc^2 \theta}{2}, \]

which implies

\[ F(\theta \in [\pi/4 - A_N, \pi/4 + A_N])_{\text{min}} = \frac{1}{(2q - 1)^N + 1}. \]

\[ F_{\text{min}} q = 0.90 | q = 0.95 | q = 0.98 | q = 0.99 \\
 0.97 | 16.58 | 33.99 | 86.15 | 173.06 \\
 0.99 | 21.59 | 44.61 | 113.57 | 228.45 \\
\]

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y = tan 2θ cot 2A_{n+1} + i \sqrt{2} - \frac{(tan 2θ cot 2A_{n+1})^2 - 1}{2}, in which θ = 0, π/2 can be consistently defined.

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