Computing weight one modular forms over $\mathbb{C}$ and $\mathbb{F}_p$.

Kevin Buzzard

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Abstract

We report on a systematic computation of weight one cuspidal eigenforms for the group $\Gamma_1(N)$ in characteristic zero and in characteristic $p > 2$. Perhaps the most surprising result was the existence of a mod 199 weight 1 cusp form of level 82 which does not lift to characteristic zero.

Introduction

It is nowadays relatively easy to compute spaces of classical cusp forms of weights two or more, thanks to programs by William Stein written for the computer algebra packages Magma [BCP97] and SAGE [S+12]. On the other hand, there seems to be relatively little published regarding explicit computations of weight one cusp forms. Computations have been done by Buhler (Buh78) and Frey and his coworkers (Fre94), and there is a beautiful paper of Serre (Ser77) which explains several tricks for computing with weight one forms of prime level, but as far as we know there is nothing systematic in the literature. In this paper we report on a fairly systematic computation of weight 1 forms that we did using MAGMA about ten years ago now. The characteristic zero code we wrote has since been incorporated as part of MAGMA, but the methods work just as well in characteristic $p$; indeed for a given level $N$ we can compute mod $p$ for all primes $p \nmid N$ at once (we do not even attempt to define mod $p$ modular forms of level $N$ if $p | N$). We apologise for our laziness in writing up these results. Our methods and calculations have in the mean time been greatly extended by George Schaeffer and forthcoming work of his will make our example of a non-liftable mod 199 form of level 82 look rather puny.

The methods basically go back to Buhler’s thesis [Buh78], and the main idea is very simple: we cannot compute in weight 1 directly using modular symbols, but if we choose a non-zero form $f$ of weight $k \geq 1$ then multiplication by $f$ takes us from weight 1 to weight $k + 1 \geq 2$ where we can compute using modular symbols, and then we divide by $f$ again to produce the space of weight 1 forms with possible poles where $f$ vanishes. Repeating this idea for lots of choices of $f$ and intersecting the resulting spaces will often enable us to compute the space.
of holomorphic weight 1 forms rigorously. We explain the details in the next section.

Recent developments by Khare and Wintenberger on Serre’s conjecture give another approach for computing weight 1 forms in characteristic zero: instead of working on the automorphic side one can compute on the Galois side. The work of Khare and Wintenberger implies that there is a bijection between the set of weight 1 new eigenforms over \( \mathbb{C} \) and the set of continuous odd irreducible representations \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \) (see Théorème 4.1 of [DS74] for one direction and Corollary 10.2 of [KW09] for the other). Say \( \rho_f \) is the Galois representation attached to the form \( f \); then the level of \( f \) is the conductor of \( \rho_f \). Let us consider for a moment an arbitrary irreducible representation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \). The projective image of \( \rho \) in \( \text{PGL}_2(\mathbb{C}) \) is a finite subgroup of \( \text{PGL}_2(\mathbb{C}) \) and is hence either cyclic, dihedral, or isomorphic to \( A_4, S_4 \) or \( A_5 \), and in fact the cyclic case cannot occur because \( \rho \) is irreducible. If \( f \) is a characteristic zero eigenform then we say that the type of \( f \) is dihedral, \( A_4 \), \( S_4 \) or \( A_5 \) according to the projective image of \( \rho_f \). Because of the Khare–Wintenberger work, one approach for computing weight 1 modular forms of level \( N \) in characteristic zero would be to list all the finite extensions of \( \mathbb{Q} \) which could possibly show up as the kernel of the projective representation attached to a level \( N \) form, and then reverse-engineer the situation (carefully analysing liftings and ramification, which would not be much fun) to produce the forms themselves. Listing the extensions is just about possible: one can use class field theory to deal with the dihedral cases, and the \( A_4, S_4 \) and \( A_5 \) calculations can be done because the projective representation attached to a level \( N \) form is unramified outside \( N \), so to compute in level \( N \) one just has to list all \( A_4, S_4 \) and \( A_5 \) extensions unramified outside \( N \) (which is feasible nowadays for small \( N \) and indeed there are now tables of such things, see for example [JR] for an online resource). However lifting the projective representation to a representation can be troublesome to do in practice. Furthermore, this approach does not work in characteristic \( p \): the problem is if \( k \) is a finite subfield of \( \mathbb{F}_p \) then \( \text{PGL}_2(k) \) is a finite subgroup of \( \text{PGL}_2(\mathbb{F}_p) \), and this gives us infinitely many more extensions which must be checked for; hence the method breaks down. In fact looking for mod \( p \) Galois representations with large image is quite hard, it seems, and perhaps it is best to do the calculations on the automorphic side and use known theorems to deduce results on the Galois side. For example, as a consequence of our search at level 82 we proved the following result:

**Theorem 1.** (a) There is a mod 199 weight 1 cusp form of level 82 which is not the reduction of a characteristic zero form.

(b) There is a number field \( M \), Galois over \( \mathbb{Q} \), unramified outside 2 and 41, with Galois group \( \text{PGL}_2(\mathbb{Z}/199\mathbb{Z}) \).

We prove this result in the final section.

When we initially embarked upon this computation, the kinds of things we wanted to know were the following:

- What are the ten (or so) smallest integers \( N \) for which the space of weight 1
cusp forms of level $N$ is non-zero?

- What is the smallest $N$ for which there exists a weight 1 level $N$ eigenform whose associated Galois representation has projective image isomorphic to $A_4$? To $S_4$?

- Give some examples of pairs $(N, p)$ consisting of an integer $N$ and a prime $p$ for which there is a mod $p$ weight 1 eigenform of level $N$ which does not lift to a characteristic zero eigenform of level $N$ (Mestre had already given some examples when $p = 2$).

We found it hard to extract the answers to these questions from the literature, so we answered them ourselves with a systematic computation of weight one forms in characteristic zero and $p > 2$. We looked in the range $1 \leq N \leq 200$ in characteristic zero, and $1 \leq N \leq 82$ in characteristic $p$ (adopting a “quit while you’re ahead” policy). Both of these bounds are very modest and even ten years ago, when we actually did the calculations, it would have been possible to go further. George Schaeffer has since stepped up to the plate and his forthcoming PhD thesis pushes these calculations much further.

We did not find any characteristic zero weight 1 modular forms with associated Galois representations having projective image isomorphic to $A_5$ so we do not know what the smallest conductor of an $A_5$-representation is; as far as we know the smallest known conductor is Buhler’s example, which has conductor 800.

The outline of this paper is as follows. We explain our algorithm in section 1, summarise the characteristic zero results in section 2, and the characteristic $p$ results in section 3.

## 1 Computing weight one cusp forms.

We remind the reader of some definitions. If $N \geq 5$ is an integer, then there is a smooth affine curve $Y_1(N)$ over $\mathbb{Z}[1/N]$ parameterising elliptic curves over $\mathbb{Z}[1/N]$-schemes equipped with a point of exact order $N$. The fibres of $Y_1(N) \to \text{Spec}(\mathbb{Z}[1/N])$ are geometrically irreducible. If $E_1(N)$ denotes the universal elliptic curve over $Y_1(N)$ then the pushforward of $\Omega^1_{E_1(N)/Y_1(N)}$ is a sheaf $\omega$ on $Y_1(N)$. There is a canonical compactification $X_1(N)$ of $Y_1(N)$, obtained by adding cusps, and this curve is smooth and proper over $\mathbb{Z}[1/N]$. Furthermore the sheaf $\omega$ extends in a natural way to $X_1(N)$. If $R$ is any $\mathbb{Z}[1/N]$-algebra then we denote by $X_1(N)_R$ the pullback of $X_1(N)$ to $R$. We write $M_k(N; R)$ for $H^0(X_1(N)_R, \omega^{\otimes k})$ and refer to this space as the level $N$ weight $k$ modular forms defined over $R$. We write $S_k(N; R)$ for the sub-$R$-module of this $R$-module consisting of sections which vanish at every cusp.

If $K$ is a field where $N$ is invertible then the $K$-dimension of $M_k(N; K)$ is 0 if $k < 0$, it is 1 if $k = 0$, and can be easily computed if $k \geq 2$ using the Riemann–Roch formula. If however $k = 1$ then the Riemann–Roch theorem unfortunately only tells us the dimension of the subspace of Eisenstein series in $M_k(N; K)$. 

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Similarly for \( k \neq 1 \) the dimension of \( S_k(N; K) \) is also easily computed, but for \( k = 1 \), even the dimensions of these spaces seem to lie deeper in the theory.

A weight one cusp form of level \( N \) is a section of \( \omega \) which vanishes at every cusp, and is hence a section of \( \omega \otimes C^{-1} \) where \( C \) is the sheaf associated to the divisor of cusps. One can compute the degree of \( \omega \) on \( X_1(N) \) without too much trouble: for example \( \omega^\otimes 12 \) descends to a degree 1 sheaf on the \( j \)-line \( X_0(1) \) and hence the degree of \( \omega \) on \( X_1(N) \) is \( [\text{SL}_2(\mathbb{Z}) : \Gamma_1(N)]/24 \). Similarly the number of cusps on \( X_1(N) \) is well-known to be \( \frac{1}{2} \sum_{0 < d | N} \phi(d) \phi(N/d) \), the sum being over the positive divisors of \( N \). Hence the degree of \( \omega \otimes C^{-1} \) is easily computed in practice for small \( N \).

Lemma 2. Let \( K \) be a field in which the positive integer \( N \) is invertible. Then there are no non-zero weight 1 cusp forms of level \( N \) over \( K \), if \( 5 \leq N \leq 22 \), \( 24 \leq N \leq 28 \), \( N = 30 \) or \( N = 36 \).

Proof. Indeed, the degree of \( \omega \otimes C^{-1} \) is less than zero in these cases.

If \( N \leq 4 \) then there are theoretical issues with the approach we have adopted, because \( X_1(N)_K \) is only a coarse moduli space, and there is no sheaf \( \omega \) on \( X_1(N)_K \) (there is a problem at one of the cusps when \( N = 4 \), and problems at elliptic points when \( N \leq 3 \)). On the other hand, one can still give a rigorous definition of a modular form of level \( N \) for \( N \leq 4 \) (using the theory of algebraic stacks, for example, or the classical definition as functions on the upper half plane if \( K = \mathbb{C} \)) and one easily checks, using any of these definitions, that a cusp form of level \( N \) is also naturally a cusp form of level \( Nt \) for any positive integer \( t \). Because 1,2,3 and 4 all divide 12, and there are no non-zero cusp forms of level 12 by the above lemma, we conclude

Corollary 3. Let \( K \) be a field where the positive integer \( N \) is invertible. There are no non-zero cusp forms of level \( N \) over \( K \) for any \( N < 23 \).

The same argument shows that the dimension of the space of weight 1 cusp forms of level 23 is at most 1, because the degree of \( \omega \otimes C^{-1} \) on \( X_1(23) \) is 0. Moreover, there will be a non-zero level 23 weight 1 cusp form if and only if this sheaf is isomorphic to the structure sheaf on \( X_1(23) \). Conversely, there is indeed a non-zero level 23 weight 1 cusp form in characteristic zero: namely the form \( \eta(q) \eta(q^{23}) \), where \( \eta = q^{1/24} \prod_{n \geq 1} (1 - q^n) \). The mod \( p \) reduction of this form is a mod \( p \) cusp form for any \( p \neq 23 \), and this proves that the dimension of the level 23 forms is 1 in characteristic zero and in characteristic \( p \neq 23 \).

We have now solved the problem of computing weight one level \( N \) cusp forms for \( N \leq 28 \), but of course such tricks only work for small levels, and for \( N \geq 29 \) we used a computer to continue our investigations. Our strategy was as follows. Let \( K \) be an algebraically closed field where \( N \geq 29 \) is invertible. Let \( S_k(N; K) \) denote the weight \( k \) cusp forms of level \( N \) defined over \( K \). We wish
to compute $S := S_1(N; K)$. First let us choose a form $0 \neq f \in M_k(N; K)$ for some $k \geq 1$ that we can compute the $q$-expansion of to arbitrary precision (for example $f$ can be a form of weight at least 2, or a weight 1 Eisenstein series or theta series). Then $f.S := \{fh : h \in S\}$ is a subspace of $S_{k+1}(N; K)$, which is a space that we can compute as $k + 1 \geq 2$. We compute a basis of $q$-expansions for $S_{k+1}(N; K)$, and then divide each $q$-expansion by $f$, giving us an explicit finite-dimensional space of $q$-expansions which contains $S$. Repeating this for many choices of $f$ and continually taking intersections will typically cut this space down, but after a while its dimension will stabilise. Let $V$ be the space of $q$-expansions so obtained; this is now our candidate for $S$. We know for sure that it contains $S$. In fact, if we could somehow choose forms $f_1$ and $f_2$ as above, which were guaranteed to have no zeros in common on $X_1(N)_K$, then we would know for sure that our space really was $S$. However, we know of no efficient way of testing to see whether two given forms share a zero on $X_1(N)$, especially in characteristic $p$. Note that in [Fre94] they work over the complexes and do make a careful choice of $f_1$ and $f_2$, which they could prove had no common zero; our approach is more haphazard.

So far, we have a “candidate space” of $q$-expansions, which we know includes $S$ and this gives us a reasonable upper bound for the dimension of $S$. To get a good lower bound, because we were only really interested in the case of $N$ at most 200 or so, we wrote a program which counts dihedral representations. More precisely, what our program does is the following. For a given $N$ it counts the number of representation $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{C})$ which are continuous, odd, irreducible, induced from a character of a quadratic extension of $\mathbb{Q}$, and have conductor $N$. This computation is a finite one because if $M$ is a quadratic extension of $\mathbb{Q}$ and $\psi : \text{Gal}(\overline{M}/M) \rightarrow \mathbb{C}^\times$ is a continuous 1-dimensional representation then the conductor of $\text{Ind}(\psi)$ is the absolute value of $\text{disc}(M)|c(\psi)|$, where $c(\psi)$ is the conductor of $\psi$ (an ideal of the integers of $M$), and $|c(\psi)|$ is its norm. Hence there are only finitely many possibilities for $M$ and, for each possibility, one can use class field theory to enumerate the characters $\text{Gal}(\overline{M}/M) \rightarrow \mathbb{C}^\times$ of conductor $I$, for $I$ any ideal of $\mathcal{O}_M$. This approach gives a lower bound for the dimension of the space of newforms in $S$, and if we repeat the computation for divisors of $N$, then we get a lower bound for the dimension of $S$.

We have explained how to get both upper and lower bounds for the dimension of a space $S$ of characteristic zero weight one cusp forms. If these bounds coincide, which of course they often do in practice in the range we considered, then we have computed the dimension of $S$, we have also proved that the Galois representations associated to all eigenforms of level $N$ and conductor $\chi$ are induced from characters of index two subgroups, and furthermore, because both methods we have sketched are constructive, we now have two ways of actually computing the $q$-expansions of a basis for $S$ to as many terms as we like, within reason – an automorphic method and a Galois method.

If however we run the algorithm above, and the lower bound it produces is still strictly less than our upper bound, then we guess that there are some non-dihedral forms at level $N$ that are not contributing to our lower bound. What we
now need is a way of rigorously proving that these formal $q$-expansions really do correspond to holomorphic weight 1 forms rather than forms with poles. We do this as follows. Choose a form $h$ in our vector space $V$. We are now suspecting that $h$ is holomorphic; what we know is that $h = g/f$ for some non-zero weight $k$ form $f$ and weight $k+1$ form $g$. Let $D$ denote the divisor of zeros of $f$. Then $h$ is a meromorphic section of $\omega$, with divisor of poles bounded by $D$. In particular we have a bound on the degree of the divisor of poles of $h^2$, which is now meromorphic and weight 2. Now here's the trick. If we can find a holomorphic weight 2 form $\phi$ of level $N$ whose $q$-expansion is the same as that of $h^2$ up to order $q^{M+1}$, where $M$ is a large integer, then we have proved that $h^2$ is holomorphic; for $h^2 - \phi$ is a weight 2 form which is a holomorphic section of $\omega^{\otimes 2} \otimes (2D) \cong \omega^{\otimes (2+2k)}$ and yet it has a zero of order at least $M$ at $\infty$, so as long as $M$ is greater than the degree of $\omega^{\otimes (2+2k)}$ the form $h^2 - \phi$ must be identically zero, and in particular $h$ must be holomorphic. If we can prove that a basis for $V$ consists of holomorphic forms, then we have proved $V = S$. If this algorithm fails then we have really proved $V \neq S$ and we go back to choosing forms $f$ as above and dividing out. Eventually in practice the process terminates, at least for $N \leq 200$ or so on architecture that is now ten years old. As mentioned before, George Schaeffer has taken all of this much further now.

In practice we do not quite do what is suggested above. Firstly, instead of working with the full space of forms of level $N$ we fix a Dirichlet character of level $N$ and work with forms of level $N$ and this character. This gives us a huge computational saving because it cuts down the dimension of all the spaces we are working with by a factor of (very) approximately $N$. It does introduce some thorny issues at primes dividing $\phi(N)$, where the diamond operators may not be semisimple, but these can be dealt with by simply gritting one's teeth and ignoring diamond operators whose order divides $p$ in this case. In fact the issues became sufficiently thorny here for $p = 2$ that we decided to leave $p = 2$ alone and restrict to the case $p > 2$.

Secondly, in fact we do not work over a field at all; we do the entire calculation on the integral level, working over $\mathbf{Z}[\zeta_n]$ for $\zeta_n$ a primitive $n$th root of unity, $n$ chosen sufficiently large that all the relevant Dirichlet characters showing up in the computation have order dividing $n$. In characteristic zero we only need to compute with one Dirichlet character per Galois conjugacy class; but a characteristic zero conjugacy class can break into several conjugacy classes mod $p$ and so we need to reduce things not modulo $p$ but modulo the prime ideals of $\mathbf{Z}[\zeta_n]$. If $\chi$ and $\alpha$ are $\mathbf{Z}[\zeta_n]$-valued Dirichlet characters of level $N$, and we are trying to compute in level $N$, weight 1 and character $\chi$, then we choose $f \in S_k(N, \alpha; \mathbf{Z}[\zeta_n])$ and let $L$ denote the lattice $S_{k+1}(N, \alpha \chi; \mathbf{Z}[\zeta_n])/f$. We run through many choices of $f$ and, instead of intersecting vector spaces, we intersect lattices. When computing an intersection of two lattices $L_1$ and $L_2$ arising in the above way, one computes not just the intersection but also the size of the torsion subgroup of $(L_1 + L_2)/L_1$; if any prime number divides the order of this torsion subgroup then the intersection of $L_1$ and $L_2$ is bigger in characteristic $p$ than in characteristic zero. The torsion subgroup is a $\mathbf{Z}[\zeta_n]$-module and we compute the primes above $p$ in its support; the reduction of $\chi$ modulo
these prime ideals are the mod $p$ characters where there may be more mod $p$
forms than characteristic zero forms. Note that in practice the order of the
torsion subgroup of $(L_1 + L_2)/L_1$ can be so big that it is unfactorable, but this
does not matter because we simply collect all the orders of these torsion groups
and continually compute their greatest common divisor. What often happens
in practice is that we manage to prove that the intersection is no bigger in char-
acteristic $p$ than in characteristic zero for all odd $p \nmid N$; then we have proved
that all characteristic $p$ forms lift to characteristic zero.

Occasionally however we may run into a suspect prime number $p$ which shows
up in a lot of the torsion orders; we can then compute the $q$-expansion of our
candidate non-liftable form and square it and look in weight 2 in characteristic $p$
as explained above; if we can find the $q$-expansion of the square in weight 2 to
sufficiently high precision then we have constructed a non-liftable form.

These tricks, put together, always worked in the region in which we d id
computations, which was $N \leq 200$ in characteristic 0 and $N \leq 82$ in character-
istic $p > 2$. We stopped at $N = 200$ in characteristic zero because by then we
had seen $A_4$ and $S_4$ extensions, but we still felt a long way from $N = 800$ (for
which there is a known $A_5$ example) – there is no reason why one should not
be able to proceed further nowadays however. In characteristic $p$ we were run-
ning into problems of factoring very large integers when computing the torsion
subgroups of the quotients above, so we stopped at $N = 82$ because of a very
interesting example that we found there (see section 3).

2 Characteristic zero results.

We ran our calculations in characteristic 0 for all $N \leq 200$. Of course, the
dimension of $S_1(N, \chi; C)$ was usually zero.

2.1 Small level.

In characteristic zero there is a good theory of oldforms and newforms, and we
firstly list the dimensions of all the non-zero spaces $S_1^{\text{new}}(N, \chi; C)$ for $N \leq 60$.
Note that if $\chi_1$ and $\chi_2$ are Galois conjugate characters then the associated spaces
$S_1^{\text{new}}(N, \chi_1; C)$ and $S_1^{\text{new}}(N, \chi_2; C)$ are also Galois conjugate in a precise sense,
and in particular have the same dimension, so we only list characters u p to Galois
conjugacy. The (lousy) notation we use for characters is as follow s: if the prime
factorization of $N$ is $p^e q^f \ldots$, then a Dirichlet character $\chi$ of level $N$ can be
written as a product of Dirichlet characters $\chi_p$, $\chi_q \ldots$ of levels $p^e$, $q^f \ldots$. By
$p_a$, we mean a character $\chi_p$ of level $p^a$ and order $a$, and by $p_a q_b \ldots$ we mean
the product of such characters. This notation will not always specify the Galois
conjugacy class of a character uniquely (which is why it’s lousy), but it does in
the cases below apart from the case $N = 56$, where we need to add that the
character $2_2$ is the unique even character of level 8 and order 2 (thus making
the product $2_2 7_2$ odd).
One can now deduce, for example, that the dimension of $S_1(52; \mathbb{C})$ is two, because the character in the table above has a non-trivial Galois conjugate, and both the character and its conjugate contribute 1 to the dimension. All of these forms are of dihedral type and hence are easily explained in terms of ray class groups. For example, the two newforms at level 47 are explained by the fact that the class group of $L = \mathbb{Q}(\sqrt{-47})$ is cyclic of order 5, and if $H$ denotes the Hilbert class field of $L$ then the four non-trivial characters of $\text{Gal}(H/L)$ can all be induced up to give 2-dimensional Galois representations of $\text{Gal}(H/\mathbb{Q})$ of conductor 47 (one gets two isomorphism classes of 2-dimensional representations). In particular, the two newforms of level 47 are defined over $\mathbb{Q}(\sqrt{5})$ and are Galois conjugates. As another example, one checks that if $P$ is a prime above 13 in $\mathbb{Q}(i)$ then the corresponding ray class field of conductor $P$ has degree 3 over $\mathbb{Q}(i)$, and the corresponding order 3 character of the absolute Galois group of $\mathbb{Q}(i)$ can be induced up to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ giving a 2-dimensional representation of conductor 52 which is readily checked to be irreducible and odd, and is the representation corresponding to the level 52 form in the table above.

One can of course also explain all the forms in the table above using theta series: for example the first form in the list has level 23 and quadratic character; the corresponding normalised newform $f$ can be written down explicitly: $2f = \sum_{m,n} q^{m^2 + mn + 6n^2} - q^{2m^2 + mn + 3n^2}$. There is also another well-known formula for $f$, namely $f = \eta(q)\eta(q^{23})$, where $\eta(q) = q^{1/24}\prod_n(1 - q^n)$. For other examples one can see [Ser77].

### 2.2 Non-dihedral examples.

Our algorithm computed lower bounds for spaces of forms by counting dihedral representations, and hence our methods make it easy to spot when one has discovered a form which is not of dihedral type. To work out what is going on with these forms one needs to do both local and global calculations; the more pedantic amongst us might at this point like to choose algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ for all primes $\ell$, and also embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ (for all $\ell$) and into $\mathbb{C}$: this makes life slightly easier in terms of notation.

Notation: if $\ell$ is a prime then $D_\ell$ denotes the absolute Galois group of $\mathbb{Q}_\ell$, and...
and \( I_\ell \) is its inertia subgroup.

The smallest level where there is a non-dihedral form is \( N = 124 = 2^2 \times 31 \). In fact at level 124 there are four non-dihedral newforms (and no other newforms, although there is a 3-dimensional space of oldforms coming from level 31). Let \( \chi \) be a level 124 Dirichlet character with order 2 at 2 and order 3 at 31; then \( S_1(124, \chi; \mathbb{C}) \) has dimension 2, as does \( S_1(124, \chi^c; \mathbb{C}) \) where \( \chi^c \) denotes the complex conjugate of \( \chi \). What are the Galois representations attached to the corresponding eigenforms? Well, let \( f, g \) denote the two normalised eigenforms in \( S_1(124, \chi; \mathbb{C}) \).

**Lemma 4.** The projective image of the Galois representations associated to \( f \) and \( g \) are isomorphic to \( A_4 \); furthermore, in both cases the number field cut out by this projective representation is the splitting field \( K \) of \( x^3 + 7x^2 - 2x + 14 \).

**Proof.** We use the following strategy. We first construct two odd Galois representations to \( \text{GL}_2(\mathbb{C}) \) of conductor 124 and determinant \( \chi \), whose projective images both cut out \( K \); such representations must come from weight 1 forms of level 124 and hence they must be the representations associated to \( f \) and \( g \). The lemma is hence reduced to the construction of these two Galois representations, the heart of the matter being controlling the conductor. The strategy for doing such things is already in [Fre94]; a convenient reference is the Theorem (due to Tate) in §1 of [Kim94], which states that if we have a projective representation \( \overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \) and for each ramified prime \( \ell \) with associated decomposition group \( D_\ell \) we choose a lifting of \( \overline{\rho} : D_\ell \to \text{GL}_2(\mathbb{C}) \), then there is a global lift \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \) of \( \overline{\rho} \) such that if \( I_\ell \) is the inertia subgroup of \( D_\ell \) then \( \rho|_{I_\ell} \cong \rho_\ell|_{I_\ell} \). In particular the conductor of \( \rho \) is the product of the conductors of the \( \rho_\ell \). This result reduces the computation of conductors of global lifts to a local calculation, which we now do.

The splitting field \( K \) has degree 12 over \( \mathbb{Q} \). Let \( \overline{\tau} : \text{Gal}(K/\mathbb{Q}) \to \text{PGL}_2(\mathbb{C}) \) be an injection. We will lift \( \overline{\tau} \) to \( \rho \) with conductor 124. One easily checks using a computer algebra package that \( K \) is unramified outside 2 and 31. The decomposition group at 2 is cyclic of order 2 and the completion of \( K \) at a prime above 2 is isomorphic to \( \mathbb{Q}_2(\sqrt{3}) \). The decomposition group at 31 is cyclic of order 3 and cuts out a ramified degree 3 extension \( K_{31} \) of \( \mathbb{Q}_{31} \). The projective representation on the decomposition group at 2 lifts to a reducible representation \( 1 \oplus \overline{\tau} \), with \( \overline{\tau} \) the order 2 local character which cuts out \( \mathbb{Q}_2(\sqrt{3}) \). This local representation has conductor 4. Similarly the projective representation at 31 lifts to the reducible representation \( 1 \oplus \sigma \), with \( \sigma \) an order 3 character with kernel corresponding to \( K_{31} \). This local representation has conductor 31. Tate’s theorem now implies that there is a global lift \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{C}) \) of \( \overline{\tau} \) with conductor 124, which furthermore on \( I_2 \) looks like \( 1 \oplus \overline{\tau} \). We know that \( \rho \) is not induced from an index 2 subgroup of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) (if it were then the projective image of \( \rho \) would be dihedral), and hence \( \rho' := \rho \otimes \chi_4 \not\cong \rho \), where \( \chi_4 \) is the conductor 4 Dirichlet character. A local calculation at 2 shows that \( \rho' \) also has conductor 4 (note that \( \mathbb{Q}_2(\sqrt{\alpha}) = \mathbb{Q}_2(\sqrt{-1}) \)). Furthermore, \( \rho \) and \( \rho' \) are odd (because \( K \) is not totally real) and hence both modular, so correspond to two distinct newforms of level 124; these forms must be \( f \) and \( g \). \( \square \)
Note that the strategy of the above proof easily generalises to other levels, assuming one can find the relevant number field, which we could do in every case that we tried simply by looking through tables of number fields of small degree unramified outside a given set of primes. The next level where we see non-dihedral forms is $N = 133 = 7 \cdot 19$, with character $\chi$ of order 2 at 7 and 3 at 19; again this determines $\chi$ up to conjugation. The weight 1 forms and the corresponding representations were originally discovered by Tate and some of his students in the 1970s; see the concluding remarks of [Tat76] for some more historical information about the calculations (which were done by hand).

Again the dimension of the space of weight 1 forms of level $N$ and character $\chi$ is 2, both forms are of $A_4$ type and the corresponding $A_4$-extension of $\mathbb{Q}$ is the splitting field of $x^4 + 3x^2 - 7x + 4$. This can be proved using the same techniques as the preceding lemma; the splitting field is unramified outside 7 and 19, the decomposition group at 7 is cyclic of order 2 cutting out $\mathbb{Q}_7(\sqrt{7})$ and the decomposition group at 19 is cyclic of order 3; the global quadratic character one twists by to get the second form is the one with conductor 7.

The next non-dihedral newform occurs at level $148 = 2^2 \times 37$, and it is our first form of type $S_4$. If $\chi$ is a Dirichlet character of level 148 which is trivial at 2 and has order 4 at 37 (there are two such characters, and they are Galois conjugate) then the dimension of $S_1(148, \chi; \mathbb{C})$ is 1. One checks using similar techniques that the $S_4$-extension of $\mathbb{Q}$ cut out by the projective Galois representation must be the splitting field of $x^4 - x^3 + 5x^2 - 7x + 12$. Indeed, the splitting field of this polynomial has Galois group $S_4$, is unramified outside 2 and 37, the decomposition group at 2 is isomorphic to $S_3$ with inertia the order 3 subgroup, and the decomposition and inertia groups at 37 are both cyclic of order 4. Note that any inclusion $S_3 \to PGL_2(\mathbb{C})$ lifts to an inclusion $S_3 \to GL_2(\mathbb{C})$; the induced map $\rho_2 : D_2 \to GL_2(\mathbb{C})$ has conductor 4 because it is the sum of two order three tame characters on inertia.

The author confesses that he was initially slightly surprised to see this latter extension show up again in the next section.

### 3 Unliftable mod $p$ weight 1 forms.

Of perhaps more interest are the mod $p$ forms of level $N$ that do not lift to characteristic 0 forms of level $N$. We first note the following subtlety: there is a mod 3 form of level 52 with character of order 2 at 2 and trivial at 13, and this mod 3 form does not lift to a characteristic zero cusp form with order 2 character. But this is not surprising – indeed this phenomenon can happen in weight $k \geq 2$ as well, and in weight 2 it first happens at $N = 13$; this was the reason that Serre’s initial predictions about the character of the form giving rise to a modular representation needed a slight modification. The weight 1 mod 3 form of level 52 in fact lifts to a level 52 form with character of order 6, and also to an Eisenstein series of level 52 with order 2 character; on the Galois side what is happening is that the weight 1 level 52 cusp form of dihedral type

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\[\text{We thank Chandan Singh Dalawat for pointing out this reference to us on MathOverflow.}\]
mentioned in the previous section has associated Galois representation which is irreducible and induced from an index 2 subgroup, but the mod 3 reduction of this representation is reducible. The phenomenon of not being able to lift characters in an arbitrary manner was deemed “uninteresting” and we did not explicitly search for it (it happens again mod 5 at level 77 and many more times afterwards; note in particular that it can happen mod \( p \) for \( p \geq 5 \)).

We now restrict our attention to mod \( p \) weight 1 eigenforms of level \( N \) which do not lift to characteristic zero eigenforms of level \( N \). We did an exhaustive search for such examples with \( p > 2 \). The first example we found was at level 74, where there is a mod 3 form with character trivial at 2 and of order 4 at 37. The associated mod 3 Galois representation was checked to be irreducible and have solvable image, and this confused the author for a while, because he was under the impression that the only obstruction to lifting weight 1 forms was lifting the image of Galois. This notion is indeed vaguely true, but what is happening here is that the mod 3 form of level 74 lifts to no form of level 74 but to the form of type \( S_4 \) and level \( 148 = 2 \times 74 \) which we described in the previous section. Why has the conductor gone up? It is for the following reason: there is a 2-dimensional mod 3 representation of \( D_2 \) whose image is the subgroup \( \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \) of \( GL_2(\mathbb{F}_3) \) and such that the image of inertia has order 3. The conductor of this mod 3 representation is \( 2^1 \); however all lifts of this representation to \( GL_2(\mathbb{C}) \) have conductor at least \( 2^2 \). This example was nice to find, not least because it gave the author confidence that his programs were working, as well as emphasizing just how miraculous this whole theory is – these subtleties of conductors dropping show up on both the automorphic side and the Galois side.

What we now realised we really wanted was a mod \( p \) form which did not lift to any weight 1 form at all. Fortunately we soon found it – it was at level \( N = 82 = 2 \times 41 \), and to our surprise was a mod 199 form (note that 199 is prime) whose associated Galois representation had rather large image. We finish this note by explaining what we found here.

Let \( F_{199^2} \) denote the field with \( 199^2 \) elements. Fix a root \( \tau \) of \( X^2 + 127X + 1 \); changing \( \tau \) will just change everything below by the non-trivial field automorphism of \( F_{199^2} \). One can check that the multiplicative order of \( \tau \) is 40.

Let \( \chi \) be the group homomorphism \( (\mathbb{Z}/82\mathbb{Z})^\times \to F_{199^2}^\times \) which sends 47 \( \in (\mathbb{Z}/82\mathbb{Z})^\times \) (note that 47 is a generator of the cyclic group \( (\mathbb{Z}/82\mathbb{Z})^\times \)) to \( \tau \). Our programs showed that the space of mod 199 weight 1 cusp forms of level \( N \) and character \( \chi \) was 1-dimensional. Let \( f \) denote this weight 1 eigenform. The reader who wants to join in at home will need to know the first few terms in the \( q \)-expansion of \( f \):
\[ f = q + (18\tau + 85)q^2 + (183\tau + 55)q^3 + (120\tau + 135)q^4 + (171\tau + 45)q^5 \\
+ (187\tau + 187)q^6 + (140\tau + 128)q^7 + (194\tau + 161)q^8 + (84\tau + 141)q^9 \\
+ (151\tau + 150)q^{10} + (106\tau + 4)q^{11} + (127\tau + 191)q^{12} + (112\tau + 92)q^{13} \\
+ (27\tau + 2)q^{14} + (146\tau + 37)q^{15} + (172\tau + 44)q^{16} + (192\tau + 4)q^{17} \\
+ (137\tau + 125)q^{18} + (189\tau + 117)q^{19} + O(q^{20}) \]

This is enough to determine \( f \) uniquely. For one can formally multiply this
power series by a weight 1 Eisenstein series of level 82 and character \( \chi^{-1} \) and
the resulting \( q \)-expansion (which at this point we know up to \( O(q^{20}) \)) is the
\( q \)-expansion of a level 82 mod 199 weight 2 form with trivial character which
turns out to be determined uniquely by the first 19 coefficients of its \( q \)-expansion.
The \( q \)-expansion of this unique weight 2 form can be worked as far as one wants
within reason; we computed it to \( O(q^{20000}) \) in just a few minutes. Dividing
through by the Eisenstein series again gives us the \( q \)-expansion of \( f \) to as much
precision as one wants.

Computing the \( q \)-expansion of \( f \) to high precision gives us, for free, plenty of
facts about the mod 199 Galois representation associated to \( f \). Note first that
there is a mod 199 Galois representation attached to \( f \); one can multiply \( f \) by
the mod 199 Hasse invariant \( A \) to get a mod 199 form \( Af \) of weight 199 which
is an eigenvector for all Hecke operators away from 199; the smallest Hecke-
stable subspace containing \( Af \) is 2-dimensional and consists of two eigenvectors
\( f_1 \) and \( f_2 \); the \( T_\ell \)-eigenvalues of \( f_1 \) and \( f_2 \) all coincide for \( \ell \neq 199 \), and the
\( T_{199} \)-eigenvalues of \( f_1 \) and \( f_2 \) are the two roots of \( X^2 - a_{199}X + \chi(199) \), which
are distinct. Both \( f_1 \) and \( f_2 \) lift to characteristic zero forms which are ordinary
at 199, and the associated mod 199 Galois representations are isomorphic; this
is the mod 199 Galois representation \( \rho_f \) associated to \( f \).

It follows from the computation that the \( q \)-expansion of \( f \) has all coefficients
in \( \mathbf{F}_{199^2} \). Write \( f = \sum_{n \geq 1} a_n q^n \), with \( a_n \in \mathbf{F}_{199^2} \). Explicit computation
shows that \( a_2 \) and \( a_{41} \) are non-zero. The Brauer group of a finite field
is trivial and hence there is an associated semisimple Galois representation
\( \rho_f : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \text{GL}_2(\mathbf{F}_{199^2}) \). Standard facts about the mod \( p \) Galois representations associated to modular forms now imply that \( \rho_f \) is unramified outside \( \{2, 41\} \). Indeed, for \( \ell \neq 199 \) this is standard, and for \( \ell = 199 \) we use the fact that
the mod 199 representations of \( D_{199} \) attached to \( f_1 \) and \( f_2 \) are upper triangular
with unramified characters on the diagonal, and furthermore the unramified character
showing up as the subspace in \( \rho_{f_1} \) is the character that shows up as
the quotient in \( \rho_{f_2} \); hence \( \rho_f \) restricted to \( D_{199} \) is a direct sum of two unrami-
fied characters, \( \rho_f(\text{Frob}_{199}) \) is semisimple, and the characteristic polynomial of
\( \rho_f(\text{Frob}_{199}) \) is \( X^2 - a_{199}X + \chi(199) \), just as the characteristic polynomial of
\( \rho_f(\text{Frob}_{\ell}) \) is \( X^2 - a_\ell X + \chi(\ell) \) for all other primes \( \ell \neq \{2, 41\} \).

Standard mod \( p \) local-global results apply to \( f_1 \) and \( f_2 \), because they have
weight bigger than 1 and lift to characteristic zero. Furthermore there are
no weight 1 mod 199 forms of level 41 and character \( \chi \), hence \( \rho_f \) must be
ramified at 2 and at 41 (we are in a situation here where level-lowering results are known; indeed they are known for $f_1$ and $f_2$, which suffices because we shall show below that $\rho_f$ is absolutely irreducible). The kernel of $\rho_f$ hence corresponds to a number field $L$ ramified only at 2 and 41 such that $\text{Gal}(L/\mathbb{Q})$ is isomorphic to the image of $\rho_f$, which is a subgroup of $\text{GL}_2(\mathbb{F}_{199^2})$. The local representation at 2 is Steinberg, which means that inertia at 2 has order 199; the local representation at 41 is principal series corresponding to one unramified and one tamely ramified (but ramified) character, which implies that inertia at 41 has order 40. In particular $L$ is tamely ramified at both 2 and 41.

The only natural question left is: what is the image of $\rho_f$, or equivalently what is $\text{Gal}(L/\mathbb{Q})$? For several years we assumed that the image would contain $\text{SL}_2(\mathbb{F}_{199^2})$, but it was only in 2009 that we actually tried to sit down and compute it, and we discovered that in fact the image is smaller.

**Proposition 5.** The image $X$ of $\rho_f$ is, after conjugation in $\text{GL}_2(\mathbb{F}_{199^2})$ if necessary, contained in $Z \cdot \text{GL}_2(\mathbb{F}_{199^2})$, with $Z$ the subgroup of scalar matrices in $\text{GL}_2(\mathbb{F}_{199^2})$. Furthermore the quotient of $X$ by $X \cap Z$ is $\text{PGL}_2(\mathbb{F}_{199})$.

**Corollary 6.** (i) There is a number field $M$, Galois over $\mathbb{Q}$, unramified outside 2 and 41, tamely ramified at 2 and 41, and with $\text{Gal}(M/\mathbb{Q}) = \text{PGL}_2(\mathbb{F}_{199})$.

(ii) The weight 1 form $f$ does not lift to any weight 1 eigenform of characteristic zero.

**Proof.** (of corollary) (i) $M$ is the kernel of $\overline{\rho}_f$. (ii) There is no finite subgroup of $\text{GL}_2(\mathbb{C})$ with a subquotient isomorphic to the simple group $\text{PSL}_2(\mathbb{F}_{199})$ and so $\rho_f$ does not lift to $\text{GL}_2(\mathbb{C})$; hence neither does $f$. □

**Remark 7.** The proof of the proposition involves a lot of machine computation and in fact, the proof we present here involves far more machine computation than all of the other computations of the paper put together. Computing all the forms in characteristic zero and characteristic $p$ mentioned in this paper only took a day or two in 2002; proving the above proposition rigorously took several weeks of machine time (on a standard desktop running Debian linux and magma) in 2009.

**Proof.** (of Proposition) Let $\overline{\rho}_f$ denote the projective representation associated to $\rho_f$, so $\overline{\rho}_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{PGL}_2(\mathbb{F}_{199^2})$. It suffices to prove that the image of $\overline{\rho}_f$ is the subgroup $\text{PGL}_2(F_{199})$ of $\text{PGL}_2(F_{199^2})$. We now adopt a rather brute force approach. We have defined $X$ to be the image of $\rho_f$; we now let $\overline{X}$ denote the image of $\overline{\rho}_f$ in $\text{PGL}_2(F_{199^2})$. The finite subgroups of $\text{PGL}_2(F_p)$ for $p$ a prime were classified by Dickson (see [Dic58], sections 255 and 260); they are as follows. They are either conjugate in $\text{PGL}_2(F_p)$ to a subgroup of the upper-triangular matrices, are dihedral of order prime to $p$, are isomorphic to $A_4$, $S_4$ or $A_5$, or are conjugate to $\text{PSL}_2(k)$ or $\text{PGL}_2(k)$ for some finite subfield $k \subset \mathbb{F}_p$. We will rule out all but one possibility for $\overline{X}$; we know of no other way of proving the result. Ruling out $\text{PSL}_2(F_{199^2})$ will not be much fun at all.

We start by observing that looking at the $q$-expansion of $f$ gives upper and lower bounds for the size of $\overline{X}$. First, the size of $\overline{X}$ is bounded above

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by the size of $\text{PGL}_2(\mathbb{F}_{199^2})$, and this means that $\overline{X}$ cannot be conjugate to $\text{PSL}_2(k)$ or $\text{PGL}_2(k)$ for any finite field $k$ of size at least $199^2$. In fact because $\det(\rho_f) = \text{Im}(\chi) = \mu_{40} \subset \mathbb{F}_{199^2}$ consists of squares in $\mathbb{F}_{199^2}$, the image of $\rho_f$ is contained within $\mu_{80} \text{SL}_2(\mathbb{F}_{199^2})$ (where here $\mu_{80}$ is the cyclic group of 80th roots of unity considered as a subgroup of the scalars in $\text{GL}_2(\mathbb{F}_{199^2})$), which rules out the case $\overline{X} = \text{PGL}_2(\mathbb{F}_{199^2})$ as well.

On the other hand we can compute the semisimple conjugacy classes of the first 1500 unramified primes by explicitly computing the $q$-expansion of $f$ to high precision; this only takes a few minutes and shows that $\overline{X}$ has at least 199 elements (the point being that $a_\ell^2/\chi(\ell)$ takes on all 199 values of $\mathbb{F}_{199}$ as $\ell$ varies over the first 1500 unramified primes; we do not need to worry about whether Frobenii actually are semisimple, because if $q \in X \subseteq \text{GL}_2(\mathbb{F}_{199^2})$ is non-semisimple then its semisimplification is $g_{199^2} \in X$). This means that $\overline{X}$ cannot be isomorphic to $A_4$, $S_4$ or $A_5$. Next we observe that $\overline{X}$ cannot be conjugate to a subgroup of the upper-triangular matrices. For if it were, the semisimple representation $\rho_f$ would be the sum of two characters each having conductor a divisor of 82 and in particular one could deduce that if $\ell_1$ and $\ell_2$ were unramified primes which were congruent mod 82 then $\alpha_{\ell_1}$ and $\alpha_{\ell_2}$ would be equal; however this is not the case, as $a_7 \neq a_{89}$. As a consequence we deduce that $\rho_f$ is absolutely irreducible.

We are left with the following possibilities: $\overline{X}$ can be dihedral of order prime to 199, or conjugate to $\text{PSL}_2(k)$ for $k$ of size 199 or $199^2$, or conjugate to $\text{PGL}_2(\mathbb{F}_{199})$. We next rule out the dihedral case; if $\overline{X}$ were dihedral then $\rho_f$ restricted to an index two subgroup would be the sum of two characters, and hence $\rho_f$ would be induced from an index 2 subgroup corresponding to a quadratic extension of $\mathbb{Q}$ unramified outside 2 and 41. There are only seven such extensions, namely $\mathbb{Q}((\sqrt{D}))$ for $D \in \{2, 41, 82, -1, -2, -41, -82\}$, and for each one it is easy to find a prime $\ell \notin \{2, 41\}$ which is inert in the extension and such that $a_\ell \neq 0$ (indeed the smallest prime $\ell$ such that $a_\ell = 0$ is $\ell = 193$, but there is an inert prime $p \leq 13$ in each of these quadratic extensions). However all such Frobenius elements would have trace zero if $X$ were dihedral.

We next rule out the case $\overline{X} = \text{PSL}_2(\mathbb{F}_{199})$. The only method we know requires some serious computations. We have our weight 1 form $f = \sum_{n} a_n q^n$. Let $\psi : (\mathbb{Z}/41\mathbb{Z})^\times \to \mathbb{F}_{199^2}$ be an arbitrary group homomorphism; there are 40 choices for $\psi$. Extend $\psi$ to a map $\mathbb{Z} \to \mathbb{F}_{199^2}$ by defining $\psi(n) = 0$ if $41|n$.

We claim that the $q$-expansion $f_\psi = \sum_{n \geq 1} \psi(n) a_n q^n$ is a mod 199 modular form of level $2 \times 41^2$. For $\psi$ the trivial character this follows because $f_\psi(q) = f(q) - a_{41} f(q^{41})$. For non-trivial $\psi$ this follows because the weight 199 forms $f_1$ and $f_2$ lift to characteristic zero forms which are ramified principal series at 41 (with one tamely ramified and one unramified character), and the twists of the lifts by the Teichmuller lift of $\psi$ hence have level $2 \times 41^2$ and their $q$-expansions differ mod 199 only at coefficients $q^n$ with $199|n$; the difference of the twists of the lifts then reduces to a mod 199 form which has $q$-expansion which is a power series in $q^{199}$ and is hence the 199th power of a weight 1 form (by part (3) of the main theorem of [Kat77] whose Galois conjugate is $f_\psi$. 

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Hence the form $g := \sum_{t \geq 0} a_{1+41t} q^{t+41t}$ is a mod 199 weight 1 modular form of level $2 \times 41^2 = 3362$, it being a linear combination of the $f_\omega$. Note that this form is not an eigenform for the diamond operators, which is what makes the computation non-trivial. We know $a_{1+41t} \in \mathbb{F}_{199}$ for all $t \geq 0$; we claim that in fact $a_{1+41t} \in \mathbb{F}_{199}$ for all $t \geq 0$. This can be verified with the following horrendous computation, which took nearly a month to do; if $a_{1+41t} \not\in \mathbb{F}_{199}$ for some $t$ then $g$ would not be equal to its Galois conjugate $\overline{g}$, and hence $g - \overline{g}$ would be a non-zero mod 199 modular form of level 3362. However, one can check on a computer (given a month) that $a_{1+41t} \in \mathbb{F}_{199}$ for $0 \leq t \leq 8610$ and hence $g - \overline{g}$ has a zero at the cusp infinity of order at least 353011. However the degree of $\omega$ on $X_1(3362)$ is 353010 and hence $g - \overline{g}$ must be identically zero. The reason one needs to go so far is that one has to work with the whole of $X_1(3362)$ rather than the far smaller $X_0(3362)$, because $g$ is not an eigenform for the diamond operators.

The conclusion so far is that $a_{1+41t} \in \mathbb{F}_{199}$ for all $t \geq 0$. In particular, if $\ell \equiv 1 \mod 41$ is prime then the trace of $\rho_f(\text{Frob}_\ell)$ is in $\mathbb{F}_{199}$. Set $K = \mathbb{Q}(\zeta_{41})$; then $K$ is the field cut out by $\det(\rho_f)$ and in particular $K \subseteq L$, the field cut out by $\rho_f$. We regard $\rho_f$ as a representation of $\text{Gal}(L/\mathbb{Q})$ and we now consider its restriction to $\text{Gal}(L/K)$. The Frobenius conjugacy classes corresponding to primes of $K$ lying above rational primes $\ell \equiv 1 \mod 41$ cover $\text{Gal}(L/K)$, and the Brauer group of a finite field is trivial; we deduce that the restriction of $\rho_f$ to $\text{Gal}(L/K)$ can be conjugated within $\text{GL}_2(\mathbb{F}_{199})$ so that it lands within the subgroup $\text{GL}_2(\mathbb{F}_{199})$. In fact because $K$ is the field cut out by $\det(\rho_f)$ we see that $Y := \rho_f(\text{Gal}(L/K))$ lands in $\text{SL}_2(\mathbb{F}_{199})$ and in particular the size of $\rho_f(\text{Gal}(L/K))$ is at most the size of $\text{SL}_2(\mathbb{F}_{199})$ and hence is most $199^3 + 199$. Hence the size of $X = \rho_f(\text{Gal}(L/\mathbb{Q}))$ is at most $40(199^3 + 199)$, which is hence an upper bound for $X$. This upper bound is far smaller than the size of $\text{PSL}_2(\mathbb{F}_{199})$ which can hence be ruled out.

We finally have to distinguish between the two remaining possibilities for $X$, namely $X = \text{PSL}_2(\mathbb{F}_{199})$ and $X = \text{PGL}_2(\mathbb{F}_{199})$. Because $X$ can be no larger than $\text{PGL}_2(\mathbb{F}_{199})$ we do know that (after conjugation if necessary) $X \subseteq Z.\text{GL}_2(\mathbb{F}_{199})$, with $Z$ the scalars in $\text{GL}_2(\mathbb{F}_{199})$. Furthermore $\det(X) = \mu_{40} \subset \mathbb{F}_{199}^\times$ and hence $X \cap Z \subseteq \mu_{80}$. One checks that if $\ell$ is the prime 661 then $\rho_f(\text{Frob}_\ell)$ has semisimplification a scalar matrix with order 40 and hence $\mu_{40} \subseteq X \cap Z$. Furthermore the normal index 40 subgroup $Y$ of $X$ is contained within $\text{SL}_2(\mathbb{F}_{199})$ and hence $Y = Y \cap Z$ is a normal subgroup of $X$ of index at most 40 and hence a normal subgroup of $\text{PSL}_2(\mathbb{F}_{199})$ of index at most 40. But $\text{PSL}_2(\mathbb{F}_{199})$ is simple and hence $Y = \text{PSL}_2(\mathbb{F}_{199})$. This means that $Y$ is either $\text{SL}_2(\mathbb{F}_{199})$ or an index 2 subgroup – but $\text{SL}_2(\mathbb{F}_{199})$ is a perfect group and hence $Y = \text{SL}_2(\mathbb{F}_{199})$, and so $\mu_{40} \text{SL}_2(\mathbb{F}_{199}) \subseteq X$. Because we know $Y$ has index 40 in $X$ we deduce that $\mu_{40} \text{SL}_2(\mathbb{F}_{199})$ is an index 2 subgroup of $X$. If $\overline{X} = \text{PSL}_2(\mathbb{F}_{199})$ then this forces $X = \mu_{80} \text{SL}_2(\mathbb{F}_{199})$, but this cannot be the case because the eigenvalues $\alpha$ and $\beta$ of $\rho_f(\text{Frob}_3)$ have the following property: if $\delta \in \mu_{80} \subset \mathbb{F}_{199}$ satisfies $\delta^2 = \alpha\beta$ then $(\alpha + \beta)/\delta \not\in \mathbb{F}_{199}$. Hence $\overline{X} = \text{PGL}_2(\mathbb{F}_{199})$. □
We finish by remarking that the corresponding $\text{PGL}_2(F_{199})$-extension of $\mathbb{Q}$ contains a quadratic field $J$, corresponding to the subgroup $\text{PSL}_2(F_{199})$. It is easy to establish what this subextension is, as it is unramified outside 2 and 41 and in fact also unramified at 2, because $L/\mathbb{Q}$ is tamely ramified at 2, and hence it must be $\mathbb{Q}(\sqrt{41})$. In particular we deduce the existence of a Galois extension of $\mathbb{Q}(\sqrt{41})$, unramified outside 2 and 41, with Galois group $\text{PSL}_2(F_{199})$.

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Kevin Buzzard: buzzard@imperial.ac.uk.