Geometric construction of voting methods that protect voters’ first choices

Alex Small

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Abstract

We consider the possibility of designing an election method that eliminates the incentives for a voter to rank any other candidate equal to or ahead of his or her sincere favorite. We refer to these methods as satisfying the “Strong Favorite Betrayal Criterion” (SFBC). Methods satisfying our strategic criteria can be classified into four categories, according to their geometrical properties. We prove that two categories of methods are highly restricted and closely related to positional methods (point systems) that give equal points to a voter’s first and second choices. The third category is tightly restricted, but if criteria are relaxed slightly a variety of interesting methods can be identified. Finally, we show that methods in the fourth category are largely irrelevant to public elections. Interestingly, most of these methods for satisfying the SFBC do so only “weakly,” in that these methods make no meaningful distinction between the first and second place on the ballot. However, when we relax our conditions and allow (but do not require) equal rankings for first place, a wider range of voting methods are possible, and these methods do indeed make meaningful distinctions between first and second place.

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This is a draft. I would appreciate feedback. Public discussion of this draft can be undertaken at http://votingmath.blogspot.com.

Voting theorists have known since the work of Gibbard and Satterthwaite that voting systems using ranked ballots will give incentives for insincere voting[1, 2]. Nonetheless, many people defend various voting reform proposals (e.g. Instant Runoff Voting) by claiming that their proposal will solve the “lesser of two evils” problem, and allow
voters to support their sincere favorite candidate [3, 4]. It is easy to show that this claim is false for Instant Runoff Voting, as there will still be cases in which voters have an incentive to insincerely rank a “lesser evil” in first place. Still, the claims are common, suggesting that there is public interest in the design of voting methods that eliminate the incentive to list a “lesser evil” in first place. It is therefore worth exploring the extent to which incentives for manipulation can be reduced, with particular attention to incentives regarding first place rankings. We can analyze this by considering whether voting methods satisfy a criterion called the “Favorite Betrayal Criterion” (FBC), which relates to the incentives voters face when deciding which candidate to list in first place [5, 6]. The FBC can be stated as:

**Definition 1** A voting method satisfies the **Favorite Betrayal Criterion** (FBC) if there do not exist situations where a voter is only able to obtain a more preferred outcome (i.e. the election of a candidate that he or she prefers to the current winner) by insincerely listing another candidate ahead of his or her sincere favorite.

A variety of methods are known to satisfy this statement of the FBC[5, 6]. One well-known example is Approval Voting[7]. However, most of these methods satisfy the FBC by allowing a voter to rank one or more candidates equal to his or her sincere favorite. This is, in some sense, a “weak” way of protecting one’s favorite candidate. We will instead explore a stronger version of the FBC, in which it is further stipulated that a voter never has an incentive to rank another candidate equal to his or her favorite:

**Definition 2** A voting method satisfies the **Strong Favorite Betrayal Criterion** (SFBC) if there do not exist situations where a voter is only able to obtain a more preferred outcome (i.e. the election of a candidate that he or she prefers to the current winner) by insincerely listing another candidate ahead of or equal to his or her sincere favorite.

Note that neither definition requires that there never be situations in which a more preferred outcome can be obtained by insincerely voting another candidate ahead of one’s favorite. Such situations can still exist with an FBC-compliant method. However, the voter should be able to do at least as well by indicating some other insincere ordering
that leaves his or her favorite(s) at the top of the list while insincerely changing the relative rankings of the other candidates.

To make this concrete, consider a simple example of a method that satisfies FBC and SFBC: antiplurality voting. In this method, voters rank the candidates on their ballots, each candidate receives one point for each ballot on which he or she is not ranked last, and the candidate with the most points wins. A voter has no disincentive to rank his or her sincere favorite in first place, so it complies with SFBC. However, it is still a manipulable method, in accordance with the Gibbard-Satterthwaite Theorem, since a voter may have an incentive to insincerely rank some other candidate in last place, if there is a close race between that candidate and the voter’s favorite.

Also, antiplurality voting illustrates our point about the possibility of obtaining a more preferred outcome either by demoting one’s favorite OR by indicating some other ranking that leaves one’s favorite candidate(s) at the top of the list. For instance, suppose that in an antiplurality election with 4 candidates the top contenders are a voter’s second and third choices. (Call these respective candidates $c_2$ and $c_3$, for convenience.) That voter has a clear strategic incentive to insincerely rank $c_3$ in last place. Which candidate he or she ranks in first place on his or her ballot is irrelevant, as all candidates except $c_3$ will receive one point each. In this case, ranking $c_2$ in first place will have the same effect as ranking $c_2$ in second place and leaving his or her sincere favorite (called $c_1$ for convenience) in first place, as long as $c_3$ is ranked last. Similar SFBC-compliant methods can be designed in which the first and second choices each receive one point, and candidates ranked lower receive specified fractions of a point.

Note, however, that antiplurality voting only “weakly” satisfies the intent of the SFBC, since the first place designation given to the favorite is purely ceremonial. There is no practical distinction between ranking a candidate in first place or second place. Still, while there is no difference between first and second place for determining the outcome, there might be political or social significance attached to the first place votes, e.g. as a measure of party strength.

The issue that we address here is whether other SFBC-compliant methods exist, and the nature of such voting methods. We will use geometric techniques to show that SFBC-compliant methods can be classified into 4 categories. Two of the categories are highly restricted and have some features similar to anti-plurality voting. A third category is heavily restricted, but if the SFBC condition is relaxed to FBC
it includes a variety of recently-proposed methods [5, 6]. We will show that the fourth category is largely irrelevant to public elections.

Finally, before beginning our formal analysis, a note on voter incentives: One could argue that our analysis is irrelevant because individuals have almost no incentive to vote insincerely if elections decided by a single vote are rare. A trivial response is that almost no incentive is not the same as no incentive. Our more serious response is that in practice one can consider strategic choices facing campaigners and activists seeking to influence voters. Should they advise a faction of voters with similar preferences to vote sincerely or strategically? If enough voters heed the advice of a campaigner, especially when using a complicated ranked method that makes strategic incentives opaque to the non-expert, then the decisions of an individual performing strategic calculations can indeed influence the course of an election.

2 Ballot Design

Voting theorists have proposed a wide range of election methods using a wide range of ballot types. Here we restrict our attention to methods which can be conducted with ballots on which each voter assigns a rank to each candidate, and no other information is indicated on the ballot for the purpose of determining the election outcome. We make no assumptions about whether more than one candidate can be assigned the same rank, or whether a voter can leave some ranks unused. For instance, one could imagine a method in which a candidate receives 5 points if assigned the first rank, 4 points if assigned the second rank, 3 points if assigned the third rank, and so on down to the sixth rank (for which a candidate receives zero points). In this method, a voter might assign a favorite candidate the first rank, a compromise candidate the second rank, and all remaining candidates in the last rank (or not list the remaining candidates, formally equivalent to listing them in the sixth rank). A rank has still been assigned to every candidate, even if some ranks were not used and other ranks were used multiple times.

Likewise, we do not exclude from consideration methods in which the number of ranks is less than the number of candidates. For instance, in Approval Voting every candidate is either approved or disapproved (equivalent to ranking every candidate in either the first or the second slot on a ballot) and the winner is the candidate ranked
“approved” on the greatest number of ballots[7]. As another example, in the “plurality” or “first past the post” elections commonly used in the United States, only one candidate receives a vote, implicitly equivalent to a first rank, and all of the other candidates are implicitly ranked in the second position.

3 Criteria

For the sake of simplicity, we impose several criteria on the election methods under consideration.

1. Anonymity: The outcome doesn’t depend upon which voter submits which ballot, but only on how many voters cast ballots of each type. Formally, the outcome is function of \(\{n_k\}\), where \(n_k\) is the number of voters submitting ballots that indicate a given preference order \(P_k\).

2. Neutrality: All candidates are treated equally, so that the level of support required for victory is the same for all candidates. Formally, if every voter changes his or her ballot in such a way that the winning candidate \(c_1\) is swapped with a candidate \(c_2\) (while all other preferences are left unchanged), then \(c_2\) should win. Conversely, if every voter changes his or her ballot in such a way that the winning candidate \(c_1\) retains the same ranking but losing candidates \(c_2\) and \(c_3\) are swapped (while all other preferences are left unchanged), then \(c_1\) should still win.

3. No Turnout Quota: The number of voters participating in an election is irrelevant to the result. Only the fraction of the electorate casting ballots of a particular type is relevant.

4. Linearity: The conditions for a candidate to win can be expressed by a series simple inequalities that are linear in the tallies of the ballot types. Formally, the inequalities that must be satisfied for candidate \(c_i\) to win involve conditions of the form:

\[
\sum_k u_{ijk} n_k > 0
\]

where \(\{u_{ijk}\}\) are constant coefficients (some of which may be negative), \(i\) refers to the candidate, \(j\) indexes the condition being checked, and \(k\) indexes the preference orders.
Note that because of the No Turnout Quota criterion the right hand sides of the inequalities can be set to zero without loss of generality: Suppose the right hand side of an inequality were some constant $\alpha$. Because the numbers of voters with each preference order will sum to $n_V$ (the total number of voters), we can write the righthand side as $\alpha \cdot \left( \sum_k n_k \right) / n_V$, and then rewrite the inequality as $\sum_k u_{ijk} \cdot n_k - \alpha \cdot n_k / n_V = \sum_k (u_{ijk} - \alpha / n_V) \cdot n_k > 0$. With a suitable redefinition of the coefficients, we can write the inequality with a zero on the righthand side.

5. Decisiveness: There is always a single winning candidate, except in the case of ties. Formally, a necessary condition for a tie is that at least one of the expressions being evaluated (i.e. an expression of the form $\sum_k u_{ijk} \cdot n_k$) to determine the outcome is equal to zero.

Our Anonymity and Neutrality criteria are violated in some notable cases. For instance, American Presidential elections violate our formulation of the Anonymity criterion, since additional supporters only help a candidate if those supporters live in states that the candidate needs to win. Also, a recall election might be considered in violation of Neutrality if, for instance, an incumbent with 49.9% support is removed for lacking a majority, and succeeded by somebody who wins 35% support in a 3-way race. Nonetheless, the Anonymity and Neutrality criteria are satisfied in most public elections held in democratic societies. We impose them in this work to gain considerable simplicity without undue loss of generality.

The Linearity criterion, while not always explicitly discussed, is satisfied in every seriously proposed election method that we are aware of. Still, one could invent hypothetical election methods utilizing inequalities that are nonlinear in ballot tallies, so we must impose Linearity explicitly. Fortunately, it is a very weak criterion: In our analysis we will examine criteria satisfied by boundaries and their normal vectors. We expect that if one wanted to analyze nonlinear methods, most of the results would carry over because they impose constraints on the orientation of boundaries.

The No Turnout Quota criterion eliminates from our analysis any election rules in which the result depends on the absolute number of ballots cast, e.g. elections in which a quorum of eligible citizens must support a result for it to be valid. In such situations, abstention may become a viable strategy for voters, a complication that we neglect.
here. Due to our Decisiveness criterion, the only elections without a valid winner will be elections that end in ties, and the set of tied elections will be of lower dimension than the set of possible elections (since a linear expression must be exactly equal to zero for a tie to occur).

4 Outline

The rest of this paper proceeds as follows: In the next section we will outline the geometrical formalism that we use to describe election methods. Our formalism is closely related to that used by Saari[8], representing the set of ballots cast as a point in the unit simplex. In this formalism, an election method partitions the simplex into regions, each region corresponding to victory by a particular candidate. The components of the normal vectors to the boundaries are related to the coefficients \( \{ u_{ijk} \} \) introduced in our definition of the Linearity criterion. Using reasoning similar to Saari’s proof of the Gibbard-Satterthwaite Theorem, we will show that the boundaries between these regions determine the strategic incentives confronting a voter. We will then derive requirements that each boundary (and its normal vector) must satisfy in order for a method to comply with the SFBC.

Based on the geometric properties of the boundaries, we will classify SFBC-compliant election methods into 4 distinct categories. The most important result in this work concerns methods in which each boundary satisfies a different. These methods correspond to certain positional methods (elections in which a candidate receive points according to the number of voters assigning him each rank). In the second category, some of the boundaries satisfy multiple conditions, and we will prove that methods in this category can be thought of as a hybrid of runoffs and positional methods, akin to the Bucklin method[9]. In the third category, all of the boundaries satisfy multiple conditions, and we will give examples of methods in this category. Finally, in the fourth category every boundary satisfies every possible condition, and we will argue that methods in this category are uninteresting for public elections.
5 Formalism

5.1 Vector Notation for Electorates

Because of our Anonymity criterion, the outcome of the election can be determined if we know the number of voters indicating each preference order \( P_i \). In addition, as discussed above, our No Voter Turnout Quota criterion implies that the total number of voters \( n_V \) participating is irrelevant to the election outcome. It then follows that a listing of the fraction of voters indicating each preference order \( P \) contains sufficient information to specify the election outcome. We will therefore follow the lead of Saari and describe the electorate with what Saari refers to as a normalized profile vector \( p \), where each component \( p_k = n_k/n_V \) indicates the fraction of the voters who cast a ballot indicating a preference \( P_k \). (This definition for the components also defines the basis that we will use for working in this vector space.)

The dimensionality \( d \) of the space that these vectors reside in depends on the number of candidates \( n_c \) and whether we allow voters to cast ballots where some candidates are ranked equal. In the case where equal rankings are disallowed (recall that the Strong FBC rules out the need for ballots listing multiple candidates in first place), the dimensionality of the space is \( n_c! \).

In any case, profile vectors will lie in the unit simplex \( Si(d) \), defined as the set of all vectors in \( d \)-dimensional space such that all of their components (in our chosen basis) are non-negative and add up to unity. It is convenient to express this summation of components to unity as a condition on the inner product \( (p, I) = \sum_k p_k \cdot 1 : \)

\[
(p, I) = 1 \quad (2)
\]

where \( I \) is a vector for which all of the components are 1. The vector \( I \) is significant in this work in part because the relation \( (p, I) = 1 \) enables us to simplify certain results, and also because \( I/d \) lies at a symmetry point (the center of the simplex).

We can also formulate our victory conditions in terms of inner products, and re-write Eq. (1) as:

\[
\sum_k u_{ijk} \cdot n_k = \sum_k u_{ijk} \cdot p_k \cdot n_V = (p, u_{ij}) \cdot n_V > 0 \quad (3)
\]

where \( u_{ij} \) is a vector of coefficients defining the \( j \)th condition for candidate \( c_i \) to win. Because \( n_V > 0 \), Eq. (3) can just as easily be written as \( (p, u_{ij}) > 0 \).
5.2 Boundaries

An election method can be considered as a procedure for dividing the simplex into \( n_c \) regions, each region corresponding to the election of one of the \( n_c \) candidates. Geometrically, we can say that an election method specifies the boundaries between regions and assigns a winner to each of the \( n_c \) regions. In this view, the Neutrality criterion specifies the symmetry of the boundaries.

Consider the boundary between a region where candidate \( c_i \) wins and a region where candidate \( c_j \) wins. Mathematically, this boundary can be specified piecewise in terms of its normal vector \( \mathbf{N}_{ij} \). The vector \( \mathbf{N}_{ij} \) points outward from the boundary and into the region where candidate \( c_i \) wins, while \(-\mathbf{N}_{ij}\) points outward from the boundary and into the region where \( c_j \) wins. A basic fact of analytical geometry is that all of the points in a flat surface will have the same projection onto the normal vector. In our formalism, this means that \((\mathbf{p}, \mathbf{N}_{ij})\) is constant along the boundary. We can set the constant to whatever value we like and still have a geometrically valid definition of a boundary. However, in keeping with Eq. (3) we will assume that the inner product is zero, along a boundary, i.e.

\[
(\mathbf{p}, \mathbf{N}_{ij}) = 0 \tag{4}
\]

The choice of zero for the right hand side corresponds to the notion that boundaries represent tied results, in which case two weighted ballot counts exactly cancel. Also, there is no loss of generality here, due to our assumption that \((\mathbf{p}, \mathbf{I}) = 1\). Suppose that one wanted to define boundaries so that \((\mathbf{p}, \mathbf{N}_{ij})\) is equal to some constant value \( a \). We can rewrite our new victory condition \((\mathbf{p}, \mathbf{N}_{ij}) = a\) as \((\mathbf{p}, \mathbf{N}_{ij}) = (\mathbf{p}, a\mathbf{I})\), or \((\mathbf{p}, \mathbf{N}_{ij} - a\mathbf{I}) = 0\), which amounts to a redefinition of the normal vector in Eq. (4).

Note that because the boundaries may be defined piecewise (giving rise to kinks or folds in profile space), satisfaction of Eq. (4) may not be a sufficient condition for a profile to lie on the \( i - j \) boundary. The piecewise specification of the boundaries is the subject of the next section.

The Neutrality condition imposes certain requirements on the boundaries. For instance, suppose a profile vector \( \mathbf{p} \) lies on the \( i - j \) boundary defined by the normal vector \( \mathbf{v} \), i.e. \((\mathbf{p}, \mathbf{v}) = 0\) and \( \mathbf{v} \) points from the boundary into the region where \( c_i \) wins. If every voter were to swap candidates \( c_i \) and \( c_j \) on his or her ballot, then the profile vector should
still be somewhere on the $i - j$ boundary (although perhaps not on a portion with normal vector $v$, if the boundary is defined piecewise). Voters swapping candidates $c_i$ and $c_j$ on their ballots corresponds to swapping components of the profile vector $p$. This is a linear operation, and so its effect on the profile vector $p$ can be represented by a matrix $S_{i,j}$ (which we will call a "symmetry operator") acting on $p$. There should hence be some other normal vector $v'$ (possibly equal to $-v$) that defines some portion of the (possibly piecewise-defined) $i - j$ boundary such that $(S_{i,j}p, v') = 0$.

It is easy to show that the operator $S_{i,j}$ is unitary or self-adjoint (in the language of linear algebra), and this enables us to relate the inner normal vectors $v'$ and $v$. The fact that $S_{i,j}$ is unitary implies that:

$$(S_{i,j}p, v') = (p, S_{i,j}v') = 0 \quad (5)$$

The condition $(p, v) = 0$ therefore implies that $(p, S_{i,j}v') = 0$, and hence $v$ and $S_{i,j}v'$ must be linearly dependent, i.e. they must be proportional to each other by some constant scalar factor. The magnitude of the factor is irrelevant for our purposes, and whether it is positive or negative depends on how we define the orientations of $v$ and $v'$ relative to the $i - j$ boundary. However, in the special case where $v$ and $v'$ are also linearly dependent (i.e. the electorate obtained by swapping $c_i$ and $c_j$ on every ballot lies on the same portion of the piecewise defined boundary) it must be the case that $v' = -v$. In that special case, one of the vectors ($v$ or $v'$ defines the inner normal $N_{ij}$ pointing toward the region where $c_i$ wins, and the other vector defines the inner normal $N_{ji}$ pointing from the same boundary toward the region where $c_j$ wins.

A similar line of reasoning shows that if a vector $v$ is normal to some portion of the $i - j$ boundary, then $S_{j,k}v$ should be normal to some portion of the $i - k$ boundary. We thus see that as a consequence of the Neutrality criterion the boundaries between victory regions are related by symmetry operations involving the exchange of candidate names.

### 5.3 Election Methods as Sequential Procedures

Having established the vector notation for defining election methods according to the vectors normal to boundaries between victory regions, let us now consider how boundaries might be defined piecewise. Election methods can generally be specified as procedure defined by a series
of if-then-else statements regarding the numbers of voters submitting
different ballot types. For instance, the simple (and SFBC-compliant)
method of antiplurality voting could be formally specified as:

IF The number of voters listing \( c_1 \) in last place
is less than the number of voters listing \( c_2 \) in
last place,
AND The number of voters listing \( c_1 \) in last place
is less than the number of voters listing \( c_3 \) in
last place,
AND ... 
THEN Candidate \( c_1 \) wins.
ELSE
IF The number of voters listing \( c_2 \) in last place
is less than the number of voters listing \( c_1 \) in
last place,
AND The number of voters listing \( c_2 \) in last place
is less than the number of voters listing \( c_3 \) in
last place,
AND ... 
THEN Candidate \( c_2 \) wins.
ELSE
etc.

If the conditions are expressed as sets of linear inequalities, we get:

IF \((p, u_{11}) > 0\)
AND \((p, u_{12}) > 0\)
AND ... 
THEN Candidate \( c_1 \) wins.
ELSE
IF \((p, u_{21}) > 0\)
AND \((p, u_{32}) > 0\)
AND ... 
THEN Candidate \( c_2 \) wins.
ELSE
etc.

where the vectors \( \{u_i\} \) are chosen so that \((p, u_{11})\) is (for this particular example) the difference between the number of last place votes received by candidate \( c_2 \) and candidate \( c_1 \), and so forth.

Notice that in the text description of the voting method the conditions for candidate \( c_2 \) to win could be obtained by swapping the labels 1 and 2 for the candidates, in accordance with our Neutrality crite-
rion. If the conditions are translated into sets of linear inequalities, then (as discussed above) the vectors $u_{ij}$ defining those conditions are related by the symmetry operator $S_{1,2}$. In general, once we define a set of sufficient conditions under which some candidate $c_i$ wins, we can invoke Neutrality to define conditions for any other candidate to win.

We will refer to a set of conditions related by symmetry operations as a “stage” of an election method.

**Definition 3** Given a set of linear inequalities such that the simultaneous satisfaction of those inequalities is a sufficient for a particular candidate $c_i$ to win (and the vectors used to define the associated linear inequalities), we can generate a stage of victory conditions by applying swap operators $S_{j,k}$ to those vectors for all $j$ and $k$. Cases where $i = j$ or $i = k$ will produce conditions for some other candidate to win. Cases where $j \neq i$ and $k \neq i$ will produce alternative conditions for $c_i$ to win.

The popular Instant Runoff method is an example of a method where a single stage will have multiple conditions for the same candidate to win. In Instant Runoff, a candidate can win if he or she survives successive eliminations to become one of the 2 final candidates considered, and receives majority support over the other finalist. For that method, there would be multiple scenarios under which the candidate could win, corresponding to different opponents in the final round.

Whether a stage of conditions has $n_c$ conditions (1 per candidate) or some multiple of $n_c$ (i.e. multiple ways for each candidate to win) depends on the vectors defining the primary set of inequalities used to generate the stage. If we have a series of inequalities defining a sufficient set of conditions for $c_i$ to win, and if applying any swap operator $S_{j,k}$ ($j$, $k \neq i$) to any of the vectors associated with that set of conditions generates another vector of that set of conditions, then the only swap operation that changes the set of inequalities is one that changes the name of the winner, and so there are only $n_c$ conditions in that stage (one per candidate). Otherwise, the number of conditions in that stage will be a multiple of $n_c$. This fact will prove to be useful later.

The sets of conditions for different candidates to win in a stage must be mutually exclusive, so that a stage produces a unique winner. However, the conditions need not be exhaustive, i.e. a stage could
select no winner. Numerous election methods can be expressed as procedures with stages that sometimes yield no winner. For instance, if a method elects a candidate listed in first place by a majority of the voters, and uses some alternative procedure when there is no majority favorite, that method will be one in which the first stage does not always yield a winner. When that is the case, our Decisiveness criterion requires that the first stage of the method be augmented by a series of subsequent stages so that the method yields a unique winner in every case except ties, which will correspond to a lower-dimensional subset of the unit simplex that the electorate vectors reside in.

Also, although a stage of victory conditions can be generated by applying swap operations to a single set of conditions for a particular candidate to win, it may be that not all of the vectors defining that primary set of conditions are necessary to specify the election method. Some of the vectors in that primary set used to generate the stage may also be related to each other by swap operations. For instance, if a set of conditions for a candidate to win consists of 5 linear inequalities, it may be that the vectors defining some of those inequalities can be derived from each other by swap operations. It will be useful later on in this work to consider the minimal number of vectors needed to specify all of the inequalities in a stage of an election method. We therefore introduce this definition:

**Definition 4** A Minimal Set of Generators $M$ for a stage $s$ of an election method is a set of vectors $\{v_i\}$ such that:

- Every vector defining every inequality in that stage can be obtained from a vector in $M$ by a suitable sequence of swap operations.

- No vector in $M$ can be obtained from any other vector in $M$ by any sequence of swap operations.

The number of vectors in $M$ is said to be the number of generators for $s$.

Note that the minimal set $M$ for a given stage $s$ is not unique. If the same swap operator were applied to every vector in $M$, we would still have another minimal set of generators for the same stage $s$.

Finally, it is worth noting that when boundaries are specified piece-wise the same vector may define different boundaries under different circumstances. Consider, for instance, 3 candidates in an election conducted with Instant Runoff. Suppose that no candidate is the first
choice of a majority but candidate $c_1$ is the first choice of the greatest number of voters. Suppose also that candidate $c_1$ can defeat candidate $c_2$ in a one-to-one contest, but he or she cannot defeat $c_3$. A necessary condition in determining whether $c_1$ or $c_3$ wins is whether $c_2$ has more first place votes than $c_3$ or fewer first place votes than $c_3$, and so the vector $v$ used to specify that linear inequality is the vector normal to a portion of the $1-3$ boundary. However, if $c_1$ could defeat $c_3$ in a one-to-one contest but could not defeat $c_2$, then that same vector $v$ would be normal to a portion of the $1-2$ boundary. We therefore see that some of the vectors defining our linear inequalities may be normal to multiple boundaries, depending on the circumstances.

6 Voter Incentives and Geometry

Whether or not a voter has an incentive to vote insincerely depends on the boundaries between the victory regions for different candidates. Suppose, for instance, that the condition for candidate $c_i$ to win instead of $c_j$ is $(p, N_{ij}) > 0$ (with the case $(p, N_{ij}) > 0$ corresponding to a victory for candidate $c_j$). The inner product $(p, N_{ij})$ is in essence a weighted sum over the ballot types of the number of each type of ballot submitted, with the ballot numbers corresponding to the components of the vector $p$ and the weighting factors corresponding to the components of the normal vector $N_{ij}$. A voter therefore has an incentive to submit whichever ballot type will receive the greatest weight (positive or negative, depending on their preference). Voters who prefer $c_i$ to $c_j$ will want to cast a ballot of a type corresponding to the largest positive component of $N_{ij}$, and voters who prefer $c_j$ to $c_i$ will want to cast a ballot of a type corresponding to the largest negative component of $N_{ij}$.

This line of reasoning was used by Saari in his proof of the Gibbard-Satterthwaite Theorem [8], where he showed that the theorem requires all of the components of the normal vectors to be either the same positive number of the same negative number. He then went on to show that election methods corresponding to boundaries defined by such normal vectors inevitably lead to paradoxes. Here, we will show that when this line of reasoning is applied to the SFBC we can obtain somewhat less stringent conditions on the normal vectors.
6.1 SFBC-Compliant Methods

Consider a voter whose sincere favorite candidate is $c_1$, and who prefers some candidate $c_i$ ($i$ may or may not be equal to 1) to some other candidate $c_j$ ($j \neq 1$). If the election is a close race between $c_i$ and $c_j$, then the condition determining the election outcome will be whether $(p, N_{ij})$ is positive ($c_i$ wins) or negative ($c_j$ wins). As argued above, this voter will want to cast a ballot of a type that corresponds to the largest component(s) of $N_{ij}$. In order for the method to comply with the SFBC, that ballot type must list $c_1$ in first place. A similar analysis holds for any other voter who prefers $c_i$ to $c_j$, and so the vector $N_{ij}$ must have at least $n_c - 1$ elements with the same maximum (largest positive) value. Of those maximum elements, at least one must correspond to a preference order listing each of the candidates other than $c_j$ in first place. Otherwise, there will be voters with an incentive to list some candidate other than their favorite in first place.

A similar analysis shows that the normal vector $N_{ij}$ must have at least $n_c - 1$ minimum (largest negative) elements, and of those at least one must correspond to a preference order listing each of the candidates other than $c_i$ in first place. This leads us to our first significant result:

**Theorem 1** If a voting method complies with the SFBC, then any normal vector being used to define the $i - j$ boundary must satisfy the following conditions:

1. At least $n_c - 1$ components must have the same maximum (largest positive) value, and for each candidate other than $c_j$ there must be at least one component of $N_{ij}$ corresponding to a preference order with that candidate in first place.

2. At least $n_c - 1$ components must have the same minimum (largest negative) value, and for each candidate other than $c_i$ there must be at least one component of $N_{ij}$ corresponding to a preference order with that candidate in first place.

Also, our Neutrality condition implies that the normals to different boundaries can be obtained by the application of swap operators, which would exchange candidates in preference orders and correspondingly swap components of the normals. Moreover, if a vector satisfies the conditions for $N_{ij}$ in an SFBC-compliant method, then multiplying it by $-1$ exchanges the maximum (largest positive) and minimum (largest negative) components. Because the conditions for normals are
stated in terms of minimum and maximum components, it follows that multiplying a suitable \(N_{ij}\) vector by \(-1\) gives a suitable \(N_{ji}\) vector. 

Note that the requirements laid out in Theorem 1 are necessary but not sufficient for a method to satisfy SFBC. It is not enough for a method to define victory conditions in terms of vectors satisfying the requirements of the theorem. The conditions must also be arranged in such an order that non-satisfaction of a condition only leads to certain results. When an election result changes because an inequality of the form \((p, v) > 0\) is no longer satisfied, and when the vector \(v\) satisfies the requirements for the normal \(N_{12}\), the only possible outcome should be that the election result is now the victory of candidate \(c_2\) instead of \(c_1\).

As was remarked above, when boundaries are defined piecewise, the same vector may define multiple boundaries, depending on the situation. This does not necessarily pose a problem for the normal vectors specifying a SFBC-compliant method. A normal vector \(v\) could satisfy the necessary conditions for the normal vectors \(N_{ij}\) and \(N_{ik}\) \((j \neq k)\). In that case, for any candidate other than \(c_i\) there must be at least one element of \(v\) that has the minimum (largest negative) value and corresponds to a preference order with that candidate in first place. Also, for any candidate other than \(c_j\) there must be at least one element of \(v\) that has the maximum (largest positive) value and corresponds to a preference order with that candidate in first place, and for any candidate other than \(c_k\) there must be at least one element of \(v\) that has the maximum (positive) value and corresponds to a preference order with that candidate in first place. The second pair of conditions imply that for every candidate there must be at least one element of \(v\) with the maximum (largest positive) value and corresponding to a preference order with that candidate in first place. This implies that \(v\) satisfies the conditions for any normal vector \(N_{ix}\) \((x \neq i)\) in an SFBC-compliant method. This leads to the following useful result:

**Theorem 2** If a vector \(v\) satisfies the necessary conditions for the normal vectors \(N_{ij}\) and \(N_{ik}\) \((j \neq k)\) in an SFBC-compliant method, \(v\) satisfies the necessary conditions for any normal vector \(N_{ix}\) \((x \neq i)\) in an SFBC-compliant method.

Analogous reasoning leads to the following result:

**Theorem 3** If a vector \(v\) satisfies the necessary conditions for the normal vectors \(N_{ji}\) and \(N_{ki}\) \((j \neq k)\) in an SFBC-compliant method,
v satisfies the necessary conditions for any normal vector N_{xi} (x \neq i) in an SFBC-compliant method.

Also, the same considerations lead to one more result concerning normal vectors satisfying multiple conditions:

**Theorem 4** If a vector v satisfies the necessary conditions for the normal vectors N_{ij} and N_{kl} (i \neq k and j \neq l) in an SFBC-compliant method, v satisfies the necessary conditions for any normal vector N_{xi} (for all admissible values of x and y) in an SFBC-compliant method.

We therefore see that a vector normal to a boundary in an SFBC-compliant method can fall into 3 categories, defined as follows:

**Definition 5** We will refer to 3 different types of normal vectors for the boundaries between victory regions.

1. A **Type 1 vector** only satisfies the requirements necessary for the boundary between 2 particular victory regions.
2. A **Type 2 vector** satisfies the requirements for all of the boundaries to the victory region for a single candidate c_i.
3. A **Type 3 vector** satisfies the requirements for all of the boundaries between all of the victory regions defined by this election method.

Note that while we have given these results in terms of compliance with SFBC, the theorems given here also apply to FBC-compliant methods if weakened slightly to include the possibility of ranking two candidates together in first place.

### 6.2 Geometric Classification of SFBC-Compliant Methods

Having considered the requirements that individual normal vectors must satisfy for an election method to satisfy the SFBC, let us now consider how we can use these requirements to classify the stages that comprise an election procedure. Each stage will consist of a series of linear inequalities defined by vectors that are in turn related to each other by swap operators. Each vector may satisfy the requirements for a single boundary (case 1 above), multiple boundaries on the same region (case 2 above), or all possible boundaries (case 3 above). Given those possibilities, the following taxonomy will be useful for the results that follow:
Definition 6 The stages of an election method compliant with the SFBC shall be referred to as coming in 4 different types.

1 A stage of an SFBC-compliant election method will be referred to as a **Type 1 stage** if each vector defining a victory condition is of Category 1, i.e. it only satisfies the requirements for a single boundary in an SFBC-compliant method. So, if a vector defining a condition for candidate \( c_i \) to win satisfies the requirements for the normal vector \( N_{ij} \) it will not satisfy the requirements for the normal vector \( N_{ik} \) (\( k \neq j \)).

1b A stage of an SFBC-compliant election method will be referred to as a **Type 1b stage** if some of the vectors satisfy a single condition (as in Type 1 stages) and other vectors are of category 2, i.e. they satisfy requirements for multiple boundaries of the same region (i.e. some vectors are of Category 2, i.e. they satisfy the requirements for \( N_{ij} \) and \( N_{ik} \)) in an SFBC-compliant method.

2 A stage of an SFBC-compliant election method will be referred to as a **Type 2 stage** if all of the vectors defining a condition for a candidate to win are of Category 2, i.e. they satisfy the requirements for multiple boundaries of the same region. For these types of stages, any vector defining a condition for candidate \( c_i \) to win will satisfy the requirements for \( N_{ij} \) and \( N_{ik} \) (for all \( k \neq j \)).

3 A stage of an SFBC-compliant election method will be referred to as a **Type 3 stage** if at least one of the vectors defining a victory condition is of Category 3, i.e. it satisfies the necessary requirements for all possible boundaries.

This taxonomy is somewhat arbitrary, but the types are non-overlapping and are related to key results proved below. In what follows we will prove that Type 1 stages are heavily restricted and are related to the ’positional methods’ that Saari has studied extensively [8]. We will then prove that Type 1b stages are also heavily restricted, and are again related to positional methods. Type 2 stages are somewhat more varied and cannot be easily categorized, but we will prove that they are nonetheless subject to significant restrictions. We will also show that Type 2 stages occur in methods that have been studied by others, if SFBC is relaxed to FBC [5, 6].

We will not devote much attention to Type 3 stages, as it is difficult to conceive of a socially desirable election method that makes use of
Type 3 stages. We are not aware of any methods that use Type 3 stages, and for the case of 3 candidates Type 3 stages turn out to be rather tricky to construct in a manner that avoids multiple winners \((i.e.\) it is tricky to construct a Type 3 stage in which the conditions for different candidates to win are mutually exclusive). The reason for these complications in the case of 3 candidates is that if there are 3 candidates and 3! = 6 preference orders \((i.e.\) 6 components to a normal vector) then 3 of those components must have the same maximum value and 3 must have the same minimum value. This tightly constrains the number of possible orientations for the vectors, and so it turns out to be necessary to construct several conditions of the form \(g_1 \leq (p, v) \leq g_2\) (where \(g_1\) and \(g_2\) are constants) to ensure that the victory regions thus defined are non-overlapping. When these conditions are constructed and examined, it is difficult to interpret them in terms of considerations usually used in the construction of election method \((e.g.\) majority support, point totals, or being one of the top 2 candidates to make a runoff).

The constraints are less significant in the case of 4 or more candidates, but generally election method designers prefer methods that can be stated in terms of simple criteria that can be just as easily described regardless of the number of candidates. If the conditions defining an election method cease to make sense when the number of candidates is reduced to 3, the election method is unlikely to be of interest. Moreover, even ignoring the issue of the number of candidates, in elections using Type 3 stages it is possible that two voters with the same favorite candidate may have completely opposite effects on that candidate’s election prospects, because of the way that vectors in Type 3 methods have both a maximum and a minimum component corresponding to preferences with the same candidate in first place. While it is often the case that casting a ballot with a particular preference order \((e.g.\) a preference order listing one’s sincere favorite in first place) will be sub-optimal \((hence the interest in analyzing strategic incentives)\), methods with Type 3 stages are more pathological than most other methods \((including methods that fail to satisfy FBC or SFBC)\). For this reason, we will not analyze Type 3 stages in any detail here, and the remainder of this work will focus on the other Types of SFBC-compliant stages.
7 Type 1 Stages

We will prove two main results here: First, we will prove that in order to comply with SFBC a Type 1 stage must always return a result, i.e. there can be no cases where a Type 1 method fails to return a result because the conditions checked indicate that $c_1$ beats $c_2$ ($(p, N_{12}) > 0$), $c_2$ beats $c_3$ ($(p, N_{23}) > 0$), and $c_3$ beats $c_1$ ($(p, N_{31}) > 0$). Such failures to return a result are related to the familiar cyclic paradox described by Condorcet[8]. Second, we will prove that methods of that sort are point systems, in which candidates are assigned points according to the ranks given by voters, and the candidate with the most points wins.

7.1 Type 1 Stages and Paradoxes

We will begin by supposing that some stage of an election method, numbered $s$ in the sequence of stages, is a Type 1 stage, and we will assume that this stage does not always return a result. If the vectors normal to the boundaries defined in stage $s$ are denoted $N_{ij}^s$, then we can find a cyclic region of the unit simplex where $(p, N_{12}^s) < 0$, $(p, N_{23}^s) < 0$, and $(p, N_{31}^s) < 0$. In this case, we need to check the subsequent stage $s+1$. We will make no assumptions about whether stage $s+1$ is of Type 1 or some other type. We will suppose, without loss of generality, that stage $s+1$ selects candidate $c_3$ as the winner, implying that $(p, N_{32}^{s+1}) > 0$ and $(p, N_{31}^{s+1}) > 0$. As long as one of the vectors $N_{12}^s$, $N_{23}^s$, and $N_{31}^s$ cannot be written as a linear combination of the other two (a condition assured by our assumption that $(p, N_{12}^s) < 0$, $(p, N_{23}^s) < 0$, and $(p, N_{31}^s) < 0$), it is possible to find a point where the boundaries defined in stage $s$ intersect. Our neutrality criterion then assures that it is possible for this intersection to coincide with a region of profile space where stage $s+1$ selects candidate $c_3$.

Now consider the hypersurface consisting of the intersection of the boundaries defined by the normal vectors $N_{12}^s$, $N_{23}^s$, and $N_{31}^s$. We specifically consider a portion of this hypersurface lying in a region where stage $s+1$ selects candidate $c_3$. (Below we will address the issue of whether this intersection hypersurface lies inside the unit simplex.) Assume initially that the profile vector is on the hypersurface defined by the intersection of those boundaries. We can then displace the profile vector slightly away from that intersection, so that we are in a region where stage $s$ selects candidate $c_1$. Subsequently, we can
change \( p \) slightly so that it crosses the \( 1 - 2 \) boundary defined in stage \( s \) \((\text{without crossing the } 2 - 3 \text{ boundary!})\) and we now have a situation in which stage \( s \) yields no winner and so \( c_3 \) wins. However, this is a violation of SFBC, because the outcome changed from \( c_1 \) to \( c_3 \) by crossing a boundary that does not satisfy necessary conditions for the \( 1 - 3 \) boundary in SFBC-compliant methods.

In order to prove that this is a violation of SFBC, however, we must address whether this intersection of planes occurred within the unit simplex \( i.e. \) did it occur for a valid profile\?). A paradox that only occurs when the profile vector \( p \) is outside the unit simplex \( e.g. \) some components of \( p \) are negative) is of no consequence for election methods, since we cannot have a negative number of voters submitting a ballot of a particular type. We will address this by considering what happens when the normal vectors are changed in a way that shifts the intersection out of the unit simplex.

Suppose initially that the boundaries all meet in the center of the simplex at the point \( I/d \) discussed above. Our analysis above, in which we displace the profile slightly from the intersection, clearly applies. Suppose that we then start shifting boundaries around by changing the right-hand side of each condition, so that the victory condition is now \( (p, N_{ij}^s) > \delta_{ij}^s \). This will move each boundary around in a direction parallel to its normal vector. However, we have seen previously that changing the normal vector another vector proportional to \( I \). Adding to a normal some vector proportional to \( I \) does not affect compliance with SFBC, because all of the elements of a normal vector change by the same amount, so our conditions regarding maximum and minimum elements are still met. We also change the profile so \( p \) so that its position relative to the intersection is unchanged.

Even if we can translate the cyclic region entirely outside the unit simplex, however, there is a second cyclic region, that intersects the original cyclic region only along a surface of dimension 2 less than the dimension of the boundary. This second cyclic region cannot be reached by crossing only a single boundary, as it involves a different intransitive relationship, one in which each pairwise comparison has been reversed. Translating the first cyclic region outside the simplex does not translate the second cyclic region outside, and because the cyclic regions extend infinitely far out from their intersection point, it is impossible to translate both regions outside of the simplex.

It therefore follows that SFBC violations are inevitable if a condition \( (p, N_{ij}^s) < 0 \) (note the ”less than” symbol in the inequality)
causes a stage to return no winner in a case where none of the previous stages of the method returned a winner, and the normal vector $N_{ij}$ only meets the requirements for a single type of boundary in an SFBC-compliant method, i.e. it is a Category 1 normal vector. This gives us the following theorem:

**Theorem 5** In a voting method that complies with SFBC, if one of the normal vectors defining an inequality only satisfies the requirements for a single boundary, non-satisfaction of that inequality cannot cause the stage to return no winner (unless there is a tie and none of the subsequent stages return a winner).

This may seem like a surprising result, because if a stage contains an inequality then there should be cases where that inequality is not satisfied; otherwise there would be no reason to include that inequality. However, the non-satisfaction of an inequality need not cause the entire stage to return no winner. It could be that non-satisfaction of a particular inequality just means that the stage returns some other winner. Suppose we take a particular victory condition in a stage and examine the vectors defining the associated inequalities. Normal vectors can be sorted into the three categories (enumerated at the end of Section 6.1, depending on whether they satisfy the requirements for only a single boundary (category 1) while other vectors may satisfy the requirements for multiple boundaries (categories 2 and 3, depending on how many requirements are satisfied). Theorem 5 tells us that if we examine all of the possible victory conditions for the various candidates in a given stage, in at least one of those conditions all of the inequalities involving vectors of the first category will be satisfied. Whether or not the stage returns a winner then depends on the satisfaction of inequalities involving vectors of the second and third categories.

One consequence of Theorem 5 is that if all of the vectors defining a stage of conditions are of the first category (i.e. each vector only satisfies the requirements for a single boundary in an SFBC-compliant method), then there can be no case where the non-satisfaction of any of the inequalities in that stage causes the stage to return no winner. This leads to the following Theorem:

**Theorem 6** If an SFBC-compliant election method includes a Type 1 stage, that stage must be the last stage of the method.
7.2 Type 1 stages must return winners

Proving that a Type 1 stage of an election method must be the final stage does not address the issue of whether a type 1 stage must have a winner. One could consider the possibility of an election method in which the final stage is a Type 1 stage that only gives paradoxes (no winner) in situations where some previous stage has already yielded a winner. We will now prove that such paradoxes lead to violations of the SFBC, because constructing a method to avoid such paradoxes imposes requirements on the earlier stages that are incompatible with the SFBC.

Suppose that stage \( s \) of an SFBC-compliant method is of Type 1. Theorem 6 says that it must be the last stage of the election method. Stage \( s \) is only utilized if all of the previous stages returned no winner. A necessary condition for a previous stage to return no winner is for a number of inequalities (involving vectors of the second or third categories defined above) to be unsatisfied. We could therefore generate a number of new stages that involve all of the conditions and inequalities in \( s \), augmented with inequalities indicating that other stages returned no winners. Those inequalities would involve vectors of the second and third categories, so these new new stages will be Type 1b or Type 3.

There would be a great number of possible ways to reach stage \( s \) (depending on which inequalities were unsatisfied in the previous stages) so there would be a great many new stages formed by augmenting \( s \). However, these new stages could be inserted anywhere in our specification of the procedure without changing the election result. We could thus place one of these augmented stages at the beginning of the procedure.

If there are cases where stage \( s \) has a paradox like that of Condorcet (i.e. \((p, N_{12}) < 0, (p, N_{23}) < 0, \text{ and } (p, N_{31}) < 0\)) then there will be cases where the first stage of the election method fails to return a winner because of an unsatisfied inequality involving a vector of the first category. According to Theorem 6, this leads to a violation of SFBC. This leads to the following Theorem:

**Theorem 7** A Type 1 stage in an election method satisfying SFBC must always return a winner (except in the case of ties). There can be no Condorcet-type paradoxes in a Type 1 stage of an election method satisfying SFBC.

Finally, we can show that the SFBC imposes an additional con-
straint on Type 1 stages. One could suppose that a Type 1 stage sometimes returns multiple winners (i.e., conditions for different candidates to win are not mutually exclusive) but that these multiple victories only happen in situations where a previous stage has returned a winner, avoiding the difficulty of multiple winners. We will use reasoning analogous to that of Theorem 7 to show that a Type 1 stage must always return a unique winner (except in the case of ties, which are a lower-dimensional subset of the unit simplex).

Suppose that we generate new stages by augmenting the conditions of stage \( s \) with linear inequalities that are satisfied when other stages return no winners. The inequalities added to the augmented stages will again involve vectors of the second or third categories, so the augmented stages are Type 1b or Type 3. There will again be a great number of augmented stages, reflecting the different ways that stages prior to \( s \) could fail to return a winner. However, we must also augment the conditions of stage \( s \) with additional inequalities specifying that none of the other candidates win in stage \( s \). These inequalities can be obtained by taking inequalities from other conditions of stage \( s \) and multiplying the vectors by \(-1\). The non-satisfaction of one of these inequalities (involving normal vectors of Category 1) will cause the stage to return no winner, in violation of Theorem 5. This gives our final theorem of this section:

**Theorem 8** A stage of Type 1 in an election method satisfying SFBC must always return a unique winner, except in the case of ties.

### 7.3 Generators of Type 1 Stages must not produce paradoxes

We will prove this result by constructing stages that have multiple generators, some of them with paradoxes and showing that they lead to problems that can only be solved by introducing an additional single-generator rule that decides the outcome in all cases. A generator defines a series of relationships (possibly intransitive) between candidates of the form \( c_i \) beats \( c_j \). Suppose that a method has at least 2 generators, and we call 2 of these \( A \) and \( B \). We have 3 possibilities: Both generators always generate transitive relations, both can generate intransitive relations, and one always generates a transitive relation while the other can generate an intransitive relation. Additionally, we consider the possibility that the stage has more than one
condition for each candidate to win, as well as the possibility that the stage has exactly one condition for each candidate to win.

Now consider stages in which each condition has inequalities produced from different generators. Suppose that all of the conditions for \( c_1, c_2, \) and \( c_3 \) to win have inequalities of the form \((p, N_{12}^A)\), \((p, N_{23}^A)\), and \((p, N_{31}^A)\), and each is supplemented by an inequality generated from \( B \). (These inequalities might also be supplemented by inequalities from other generators, but in what follows we get sufficient conditions on \( A \) and \( B \).) If \( A \) sometimes gives a paradox then this stage can return no winner. It therefore follows that \( A \) must always return a winner, and the only candidate who can win this stage is the candidate elected according to \( A \). The inequalities generated by \( B \) are either superfluous or else lead to paradoxes.

We are left with the possibility of producing a Type 1 stage by combining inequalities from different generators in each condition. Considering only swaps of candidates \( c_1, c_2, \) and \( c_3 \), we get 6 possible conditions from \( A \) and \( B \). We list the associated sets of inequalities below:

| Condition 1.1 | Condition 2.1 | Condition 3.1 |
|---------------|---------------|---------------|
| \((p, N_{12}^A) > 0\) | \((p, N_{12}^A) > 0\) | \((p, N_{12}^A) > 0\) |
| \((p, N_{23}^B) > 0\) | \((p, N_{23}^B) > 0\) | \((p, N_{23}^B) > 0\) |

| Condition 1.2 | Condition 2.2 | Condition 3.2 |
|---------------|---------------|---------------|
| \((p, N_{13}^A) > 0\) | \((p, N_{13}^A) > 0\) | \((p, N_{13}^A) > 0\) |
| \((p, N_{21}^B) > 0\) | \((p, N_{21}^B) > 0\) | \((p, N_{21}^B) > 0\) |

Suppose, without loss of generality, that \( A \) generates a transitive relationship \( c_1 \succ c_2 \succ c_3 \) and \( B \) generates an intransitive relationship \( c_1 \succ c_2, c_2 \succ c_3, \) and \( c_3 \succ c_1 \). The stage described above gives:

| Condition 1.1 | Condition 2.1 | Condition 3.1 |
|---------------|---------------|---------------|
| \(c_1 \succ c_2\) | \(c_2 \succ c_3\) | \(c_3 \prec c_1\) |
| \(c_1 < c_3\) | \(c_2 < c_1\) | \(c_3 < c_2\) |

| Condition 1.2 | Condition 2.2 | Condition 3.2 |
|---------------|---------------|---------------|
| \(c_1 \succ c_2\) | \(c_2 \succ c_3\) | \(c_3 \succ c_1\) |
| \(c_1 < c_3\) | \(c_2 < c_1\) | \(c_3 < c_2\) |

In this case, \( c_1 \) satisfies Condition 1.2 and the inequalities generated by \( B \) are thus superfluous. If the cycle generated by \( B \) were reversed, \( c_1 \) would still win, but by Condition 1.1.

Finally, suppose that \( A \) and \( B \) give opposite paradoxes with all

\[ ^1 \text{We are using the notation } c_i \succ c_j \text{ and } c_j \prec c_i \text{ to mean “} c_i \text{ is preferred to } c_j \text{”. The notation } c_i \sim c_j \text{ will mean that neither candidate is preferred to the other.} \]
relationships reversed. We get the following table of outcomes:

| Condition 1.1 | Condition 2.1 | Condition 3.1 |
|---------------|---------------|---------------|
| $c_1 \succ c_2$ | $c_2 \succ c_3$ | $c_3 \succ c_1$ |
| $c_1 \succ c_3$ | $c_2 \succ c_1$ | $c_3 \succ c_2$ |

| Condition 1.2 | Condition 2.2 | Condition 3.2 |
|---------------|---------------|---------------|
| $c_1 \prec c_2$ | $c_2 \prec c_3$ | $c_3 \prec c_1$ |
| $c_1 \prec c_3$ | $c_2 \prec c_1$ | $c_3 \prec c_2$ |

The only way to decide this contest is to introduce an additional generator $C$ and augment the conditions with more inequalities. The outcome is then decided by $C$ in this case, and in order to avoid paradoxes, $C$ must also return the same winner as this stage would return without $C$, in cases where $A$ and $B$ together return a unique winner. It therefore follows that in all cases this stage must return the result that $C$ would return, and so "$A$ and $B$ are superfluous.

We are therefore left to conclude that when constructing a Type 1 stage with multiple generators, none of the generators can give a paradox.

### 7.4 Type 1 Stages are Point Systems

Given that a Type 1 stage must always return a unique winner (except in the case of ties) and hence must always be the last stage in an election method satisfying SFBC, and that the generators cannot give rise to paradoxes, we now ask what sorts of election rules are defined by generators of Type 1 stages. We will prove below that such election rules are equivalent to point systems, in which a candidate receives points based on the position assigned on a ballot, irrespective of how other candidates are ranked on that ballot, a candidate’s points from all of the ballots are summed, and the candidate with the most points wins. When the election method requires voters to submit a complete ranking of candidates, with all ranks used and no equal rankings, these methods are commonly called ”positional methods” [8]. One obvious SFBC-compliant election method is anti-plurality voting. Any other positional methods that assigns equal (and maximum) points to first and second choice candidates on a ballot would also comply with SFBC, as there is no disincentive for a voter to list his or her sincere favorite in first place: If there is a close race between two candidates, and neither of those candidates is a voter’s favorite, that voter can list his or her sincere favorite in first place and the more preferred of the front-runners in second place, because the second place candidate on
the ballot will receive the maximum point total.

However, positional methods are not the only possible Type 1 stages. In Section 2, we included in our analysis ballots in which some positions may be left empty and other positions may be assigned to multiple candidates, and we remarked that Approval Voting is one such method. Approval Voting does not satisfy SFBC, because there are times when a voter has an incentive to approve multiple candidates, giving equal points to his or her favorite as well as a compromise candidate. However, a modified form of Approval Voting, with 3 ranks on the ballot and equal points for the first and second places would satisfy our phrasing of SFBC, albeit on a technicality. Likewise, we could modify Range Voting, an SFBC-compliant technique in which voters assign points to candidates within some specified range (e.g. 0 to 5). In a modified SFBC-compliant form of Range Voting, there would be two top positions on the ballot, with equal points but one would be recorded as a voter’s true favorite. This illustrates yet again that while SFBC can be satisfied in principle, it is difficult to design an SFBC-compliant method that makes a meaningful distinction between first and second place.

The fact that a Type 1 stage must always return a winner means that the stage will never give a paradox, which in turn implies that there is always a transitive relationship among the candidates. If \((\mathbf{p}, \mathbf{N}_{12}^s) = (\mathbf{p}, \mathbf{N}_{23}^s) = 0\), i.e. ties between \(c_1\) and \(c_2\) as well as \(c_2\) and \(c_3\), then \((\mathbf{p}, \mathbf{N}_{13}^s) = 0\), i.e. there is also a tie between \(c_1\) and \(c_3\). More generally, if \((\mathbf{p}, \mathbf{N}_{12}^s) = (\mathbf{p}, \mathbf{N}_{23}^s) = ... = (\mathbf{p}, \mathbf{N}_{nc-1,nc}^s) = 0\) implies that all other inner products \((\mathbf{p}, \mathbf{N}_{ij}^s)\) are also zero, then the \(n_c(n_c-1)/2\) normal vectors \(\{\mathbf{N}_{ij}^s\}\) are not linearly independent. This implies that of all the normal vectors to the boundaries, at most \(n_c-1\) of them are linearly independent. We can show that no fewer than \(n_c-1\) normal vectors are linearly independent by considering ties between \(c_1\) and \(c_2\), \(c_2\) and \(c_3\), etc. all the way to \(c_{n_c-2}\) and \(c_{n_c-1}\). These conditions will mean that many pairs of candidates are tied with each other (e.g. \(c_1\) and \(c_{n_c-1}\) are tied) but they do not guarantee a tie between \(c_{n_c-1}\) and \(c_{n_c}\).

Due to the Neutrality criterion, in a stage with exactly 1 victory condition per candidate and \(n_c-1\) inequalities per candidate, we also have that each boundary normal \(\mathbf{N}_{ij}^s\) must be unchanged by swaps of any candidates other than \(c_i\) and \(c_j\). It therefore follows that \((\mathbf{p}, \mathbf{N}_{ij}^s)\) depends only on how many voters list \(c_i\) and \(c_j\) in each spot on the ballot. We could therefore write \((\mathbf{p}, \mathbf{N}_{ij}^s) = T_i - T_j\) where \(T_i\) and \(T_j\)
are point totals assigned to each candidate according to how many voters list that candidate in a particular spot on the ballot. There are \( n_c \) point totals to calculate, and if we calculate the differences between all possible pairs of point totals we get \( n_c - 1 \) linearly independent quantities, which correspond to the \( n_c - 1 \) linearly independent normal vectors that can be found.

Moreover, if a point system is to satisfy SFBC, it must award equal points to a voter’s first and second choices, so that the voter can give the maximum possible points to a contender in a close race without having to demote his/her sincere favorite from the top place on the ballot. However, as we have discussed above, this is only a technical case of compliance with SFBC, since the method makes no practical distinction between first and second place. Hence, Type 1 stages only satisfy “Strong” FBC in a very weak sense.

The only remaining question to ask is whether we could use two or more different Type 1 generators, each giving rise to a different point system, to produce a Type 1 stage with multiple conditions for a candidate to win. However, in that case, a condition would be a mix of inequalities related to different point systems. In general, there will be cases in which candidates \( c_1 \) and \( c_2 \) both beat \( c_3 \) in two different point systems generated by vectors \( A \) and \( B \), but \( A \) selects \( c_1 \) over \( c_2 \), while \( B \) selects \( c_2 \) over \( c_1 \). Using the same table of outcomes as above, we get:

| Condition 1.1          | Condition 2.1          | Condition 3.1          |
|------------------------|------------------------|------------------------|
| \( c_1 \succ c_2 \)   | \( c_2 \succ c_3 \)   | \( c_3 \prec c_1 \)   |
| \( c_1 \succ c_3 \)   | \( c_2 \succ c_1 \)   | \( c_3 \prec c_2 \)   |

| Condition 1.2          | Condition 2.2          | Condition 3.2          |
|------------------------|------------------------|------------------------|
| \( c_1 \prec c_2 \)   | \( c_2 \succ c_3 \)   | \( c_3 \prec c_1 \)   |
| \( c_1 \prec c_3 \)   | \( c_2 \prec c_1 \)   | \( c_3 \prec c_2 \)   |

In this case, a third generator is needed to generate another point system to decide between \( c_1 \) and \( c_2 \), and this third generator must, as before, always give the same results as the method generated by \( A \) and \( B \), rendering those two generators redundant with the third generator. We can thus conclude that a Type 1 stage can only have a single generator, which generates a point system.

We therefore get this theorem, which is our primary result in this paper:

**Theorem 9** Any Type 1 stage that satisfies SFBC, Neutrality, Anonymity, Linearity, No Turnout Quota, and Decisiveness is a point system in
which candidates receive points based on how many voters list the candidate in each spot on the ballot and the candidate with the most points wins. In these point systems, the candidates listed in the first and second places on the ballot must receive equal points.

If the method requires a strict ranking (i.e. no ties and no unused ballot spots) of all \( n_c \) candidates, then the method is a Positional Method of the sort studied by Saari.

8 Type 1b Stages

We now consider Type 1b stages. Although a stage of an SFBC-compliant method can never fail to return a winner due to the non-satisfaction of an inequality specified by a Type 1 vector, a Type 1 stage (based on a point system) could be augmented by additional inequalities if these inequalities involve Type 2 vectors. Because the underlying Type 1 stage would always return a winner, the only way the stage would fail to return a winner is through the non-satisfaction of an inequality specified by a Type 2 vector.

However, we must be careful here: Suppose that candidate \( c_1 \) satisfies all of the Type 2 inequalities required for victory, and all but one of the Type 1 inequalities required for victory. The remaining inequality is not satisfied because the profile is on a boundary: \( (p, N_{s12}^*) = 0 \).

The transitivity of the relationships defined by the Type 1 inequalities implies that \( c_2 \) also satisfies all of the Type 1 inequalities required for victory in stage \( s \), except that \( (p, N_{s21}^*) = 0 \).

In this case, changing the profile to change the sign of \( (p, N_{s12}^*) \) should change the outcome from \( c_1 \) to \( c_2 \) without requiring the use of a subsequent stage of conditions. This only works if, whenever the profile is on the \( 1-2 \) boundary defined by the Type 1 inequalities (i.e. \( c_1 \) and \( c_2 \) have equal points), and \( c_1 \) satisfies a Type 2 inequality, \( c_2 \) also satisfies the analogous Type 2 inequality. Because the Type 1 inequalities are expressed in terms of point totals, the Type 2 inequalities must then also be expressed in terms of point totals. In other words, when \( c_1 \) and \( c_2 \) have equal points, and have more points than any other candidate, both candidates must either satisfy the Type 2 inequalities or both candidates must not satisfy the Type 2 inequalities.

It then follows that the Type 2 inequalities can only depend on the total number of points that a candidate receives, rather than a
comparison of the points received by 2 different candidates. In order to satisfy the No Turnout Quota criterion, the criterion must be a threshold of support points proportional to the number of voters participating, rather than some fixed number of points independent of the number of voters. We then get the following result:

**Theorem 10** Any Type 1b stage that satisfies SFBC, Neutrality, Anonymity, Linearity, No Turnout Quota, and Decisiveness is a point system in which candidates receive points based on how many voters list the candidate in each spot on the ballot and the candidate with the most points wins if the number of points received by that candidate exceeds a threshold that is proportional to the number of voters. In these point systems, the candidates listed in the first and second places on the ballot must receive equal points.

Consider an example of a Type 1b voting rule that satisfies SFBC: Voters rank candidates, and a candidate receives 1 point from each voter who ranks that candidate in first or second place. If the points received exceed a quota (e.g. at least 75% of the voters give that candidate a point). Otherwise, the winner is the candidate ranked in last place by the fewest voters.

Another example would be a method in which ballots have 4 places, and the option to list no candidates in some of the places. The winner is the candidate ranked in first or second place by the greatest number of voters, if that candidate is ranked in those places by a majority of the voters. Otherwise, the candidate ranked in first, second, or third place by the greatest number of voters is the winner. While this method does not make a meaningful distinction between first and second place, it does not obligate the voter to put a candidate in second place. There will be situations in which a voter has a strategic incentive to rank a candidate in second place (usually because his or her favorite is not a contender, but some other less-preferred candidates are contenders), but there will also be situations in which a voter has no such incentive (usually because his or her favorite is a contender). In the later case, the voter is able to make a meaningful distinction between first place and the next most highly-ranked candidate on the ballot.
9 Type 2 Stages

General results are harder to find for Type 2 stages, in which all of the inequalities are defined by Type 2 vectors, i.e. the vectors defining the conditions for candidate $c_1$ to win satisfy the conditions for the normals to the $1-2$, $2-3$, etc. boundaries. However, we can show that if we restrict our attention to 3-candidate elections in which voters must submit strict and complete rankings (i.e. all candidates are ranked and no two candidates are ranked equal to each other) then such methods must be non-monotonic. We will use the following definition of monotonicity, taken from Saari [8]:

**Definition 7** A voting method is **monotonic** if when $c_j$ is chosen with some profile $p$, and the only voters to change preferences change them to give $c_j$ a higher ranking (while preserving the relative rankings of all other candidates) then $c_j$ is still elected with the new profile $p'$.

Let us examine the implications of this requirement for a normal vector $v$ that satisfies the requirements of SFBC for $N_{1,2}$ and $N_{1,3}$ (i.e. $v_1$ is a Type 2 vector). Suppose that candidate $c_3$ is elected and a voter has submitted a ballot with the preference $c_1 \succ c_2 \succ c_3$. If that voter changes his ballot to read $c_1 \succ c_3 \succ c_2$ then $c_3$ should still win rather than $c_1$. This implies that the component of $v_1$ corresponding to $c_1 \succ c_2 \succ c_3$ is greater than or equal to the component corresponding to $c_1 \succ c_3 \succ c_2$. However, this same vector also satisfies the conditions for the $1-2$ boundary. Suppose that we now have a situation where $c_2$ wins (under the same conditions, i.e. the outcome is determined by the sign of the inner product $(p, v_1)$) and a voter then changes his ballot from reading $c_1 \succ c_3 \succ c_2$ to $c_1 \succ c_2 \succ c_3$. In this case, $c_2$ should still win rather than $c_1$, and so the component of $v_1$ corresponding to $c_1 \succ c_3 \succ c_2$ should be greater than or equal to the component corresponding to $c_1 \succ c_3 \succ c_2$. We therefore conclude that the components corresponding to $c_1 \succ c_2 \succ c_3$ and $c_1 \succ c_3 \succ c_2$ must be equal.

SFBC also implies that these components must be the largest components of the vector, so that voters whose favorite is $c_1$ do not have to list their favorite below first place in order to submit a ballot that gives $c_1$ the maximum benefit. For simplicity, we will assume that these components are both $+1$. Furthermore, SFBC implies that 2 other components must also be equal to $+1$, with one of those components corresponding to a ballot that lists $c_2$ in first place, and the
other corresponding to a ballot that lists $c_3$ in first place. We will consider the implications of monotonicity to determine which components should be $+1$.

Suppose that $c_1$ wins in a situation where the outcome is determined by the sign of the inner product $(p, v_1)$. If a voter changes his or her ballot from $c_2 \succ c_3 \succ c_1$ to $c_2 \succ c_1 \succ c_3$, the candidate $c_1$ should still win, implying that the component corresponding to $c_2 \succ c_1 \succ c_3$ must be larger than the component corresponding to $c_2 \succ c_3 \succ c_1$, and hence must be equal to $+1$. A similar analysis implies that the component corresponding to $c_3 \succ c_1 \succ c_2$ must also be $+1$. The other components, corresponding to $c_2 \succ c_3 \succ c_1$ and $c_3 \succ c_2 \succ c_1$ must be equal to some negative number $-m$. We hence see that in the case of SFBC-compliant and monotonic election methods with only 3 candidates and strict rankings, all of the Type 2 normal vectors are of the form:

$$v_1 = (+1, +1, +1, -m, -m, +1)$$

We are working in the basis defined by Saari [8]:

A similar form can be obtained for a Type 2 vector $v_2$ or $v_3$ that expresses a condition for $c_2$ or $c_3$ to win. This sort of condition could be expressed as “$c_1$ wins if the number of voters listing $c_1$ in first or second place is greater than $m$ times the number of voters listing $c_1$ in last place.” It thus follows that if an SFBC-compliant method is monotonic, and if there are only 3 candidates and the ballots require voters to give complete rankings of the candidates without equal rankings (in any position) then there is only one possible form for a Type 2 condition.

### Table 1: Basis for SFBC-compliant methods with 3 candidates

| Preference | Vector |
|------------|--------|
| $c_1 \succ c_2 \succ c_3$ | $(1, 0, 0, 0, 0, 0)$ |
| $c_1 \succ c_3 \succ c_2$ | $(0, 1, 0, 0, 0, 0)$ |
| $c_3 \succ c_1 \succ c_2$ | $(0, 0, 1, 0, 0, 0)$ |
| $c_3 \succ c_2 \succ c_1$ | $(0, 0, 0, 1, 0, 0)$ |
| $c_2 \succ c_3 \succ c_1$ | $(0, 0, 0, 0, 1, 0)$ |
| $c_2 \succ c_1 \succ c_3$ | $(0, 0, 0, 0, 0, 1)$ |
The problem is that in this case the Type 2 conditions for \( c_1 \) to win and for \( c_2 \) to win are not mutually exclusive. This is easy to show by example. Suppose that all of the voters list \( c_3 \) in last place. It then follows that the number of voters listing \( c_1 \) in first or second place is greater than \( m \) times the number listing \( c_1 \) in last place (since that number is zero), and the same holds true for the number listing \( c_2 \) in first or second place. This Type 2 stage can therefore not decide between \( c_1 \) and \( c_2 \), violating our decisiveness condition. We therefore get the following result:

**Theorem 11** If an SFBC-compliant method is monotonic and satisfies Anonymity, Neutrality, Linearity, Decisiveness, and No Turnout Quota, it cannot have a Type 2 stage if there are 3 candidates and the ballots require voters to rank all candidates without any equal rankings.

10 Ties and SFBC

So far, we have largely neglected ties except to note their existence and equate them to boundaries between victory regions in profile space. However, we have not considered the rules used to break ties. Interestingly, in the case of tie-breaking procedures, it is possible to satisfy Strong FBC rather than just a weak form. Suppose that we have some SFBC-compliant method, and we supplement it with a rule that when there is a 2-way tie (i.e., two boundaries intersect) the winner is whichever of the 2 candidates is ranked above the other by the most voters. There is no incentive to list another candidate ahead of one’s favorite in this tie-breaking procedure, and there is no incentive to list another candidate ahead of one’s favorite in any other part of the voting procedure (due to SFBC-compliance). However, the tie-breaking procedure truly satisfies SFBC in a strong sense. Notably, pairwise comparisons work in ties, but not in general multi-candidate elections, because ties are 2-way elections, whereas a general multi-candidate election is susceptible to the Condorcet paradox. Unfortunately, this stronger form of compliance with SFBC only happens in an exceedingly rare case.

11 FBC vs. SFBC

Interestingly, while we can show that election methods are highly restricted if we insist that no voter ever have an incentive to rank an-
other candidate equal to his or her sincere favorite, a much wider variety of election methods are possible if we relax SFBC to FBC, and consider methods in which a voter sometimes has an incentive to rank another candidate equal to his or her sincere favorite, but never has an incentive to rank another candidate above his or her sincere favorite. Interestingly, some of these methods actually make meaningful distinctions between first and second place, unlike methods that satisfy our very strict formulation of SFBC as discussed above.

We will illustrate this point with three examples:

11.1 Range Voting

In Range Voting[5, 6], each voter assigns each candidate points on some scale (typically 0 to some upper bound), and the candidate with the most points wins. In the case where the upper bound is 1, Range Voting is equivalent to Approval Voting. A voter may have an incentive to assign the maximum score to some candidate other than his or her sincere favorite (if that candidate is in a close race with a less-preferred candidate) but there is never a disincentive to give the top score to the sincere first choice. This is a point system, just like the Type 1 SFBC-compliant methods, and is a Type 1 FBC-compliant method.

11.2 Majority Choice Approval

A very simple example of a Type 1b FBC-compliant method is Majority Choice Approval Voting. In this simple method, a voter can rate each candidate as “Preferred”, “Approved”, or “Disapproved.” The candidate who is rated “Preferred” by the greatest number of voters wins, provided that he or she is rated thus by a majority of voters. Otherwise, the candidate with the greatest combined “Preferred” and “Approved” ratings wins. The comparisons of vote totals are all expressed by Type 1 vectors, and the requirement that the total exceed a majority threshold is expressed by a Type 2 vector, giving a Type 1b method.

11.3 Majority Defeat Disqualification Approval

Another FBC-compliant method is Majority Defeat Disqualification Approval (MDDA), first studied by Kevin Venzke[5, 6]. In MDDA
Table 2: Ballot counts being compared to determine if \( c_1 \) dominates \( c_2 \).

| Does not prefer \( c_2 \) to \( c_1 \) (vector component = +1) | Prefers \( c_2 \) to \( c_1 \) (vector component = −1) |
|---------------------------------------------------------------|--------------------------------------------------|
| \( c_1 \succ c_2 \succ c_3 \) | \( c_2 \succ c_1 \succ c_3 \) |
| \( c_1 \succ c_3 \succ c_2 \) | \( c_2 \succ c_3 \succ c_1 \) |
| \( c_3 \succ c_1 \succ c_2 \) | \( c_3 \succ c_2 \succ c_1 \) |
| \( c_1 \sim c_3 \succ c_2 \) | \( c_2 \sim c_3 \succ c_1 \) |
| \( c_1 \sim c_2 \succ c_3 \) | \( c_2 \sim c_3 \succ c_1 \) |
| \( c_3 \succ c_1 \sim c_2 \) | |

and related methods, voters rank as many candidates as they wish, with equal rankings allowed; all unranked candidates are treated as being ranked equal to each other in last place. All ranked candidates are said to be approved. A candidate \( c_i \) is said to be dominated by a candidate \( c_j \) if a majority of voters rank \( c_j \) above \( c_i \). If one candidate dominates all other candidates, that candidate wins. Because equal rankings are allowed, it is possible that there will be no majority favoring \( c_i \) over \( c_j \) and no majority favoring \( c_j \) over \( c_i \), in which case neither candidate is dominated by the other. If there are multiple undominated candidates, or no un-dominated candidates, some other method must be used.

In MDDA and related methods based on the concept of majority dominance, the outcome is determined by whether or not a candidate is dominated by another candidate. Let us define a vector \( \mathbf{d}_{12} \) such that \((\mathbf{p}, \mathbf{d}_{12}) > 0\) if \( c_1 \) is not dominated by \( c_2 \). The elements of the \( \mathbf{d}_{12} \) are summarized in Table 2.

For each candidate, there is at least one preference that corresponds to a maximum (largest positive) element (+1) of \( \mathbf{d}_{12} \). There are also minimum (largest negative) elements corresponding to preferences that list \( c_2 \) and \( c_3 \) in first place. This is a type 2 vector. For a stage in which the outcome is determined only by comparing candidates and eliminating dominated candidates, all of the vectors are Type 2, and we thus see that a wider range of Type 2 methods are possible if we relax SFBC to FBC.

If there is more than 1 undominated candidate, the outcome is decided by a stage that elects the undominated candidate ranked last by the fewest people. In this case, Type 2 vectors (to determine who is
not dominated) are combined with Type 1 vectors (comparing points based on last place rankings) to give a Type 1b stage.

12 Least Favorite Promotion

Interestingly, we can formulate an analogous criterion for voters who are less interested in supporting their favorite and more interested in avoiding the accidental election of their least favorite due to strategic manipulations. This fear may be well-warranted in some cases: In a 3-candidate election with an SFBC-compliant method, promoting the least favorite to second place may be a viable strategy for electing the voter’s favorite over his or her second choice, but such strategies always come at a risk.

If we were to pursue the same approach as used above for identifying SFBC-compliant methods, and call the new criterion “Least Favorite Promotion”, we would find that a major portion of the methods are point systems of some sort or another. Some types of point systems (e.g. Range Voting, Approval Voting, positional methods that give equal points to first and second choices and zero points to last and second-last choices) would satisfy both SFBC and also Least Favorite Promotion.

13 Conclusions

In conclusion, we have shown that the Strong Favorite Betrayal Criterion is exceedingly difficult to satisfy. Two of the four geometrical categories of methods are restricted to point systems. The Type 2 methods have more variety, but the most interesting methods are only found if SFBC is relaxed to allow for the possibility of sometimes ranking another candidate equal to one’s favorite, e.g. Majority Defeat Disqualification Approval. When SFBC is replaced with FBC, the resulting methods actually make meaningful distinctions between first and second place, satisfying the “spirit” but not the “letter” of SFBC. The most significant practical consequence of these results is that election reformers who want to be free from disincentives against supporting their sincere favorite above all others must accept systems in which voters sometimes rank another candidate equal to their favorite (either explicitly, in FBC, or implicitly in point systems that give equal points to first and second place).
Acknowledgements: I thank all of the members of the Election Methods Mailing List for useful discussions of this problem, especially Forest Simmons and Warren D. Smith. I first learned of the Favorite Betrayal Criterion from reading the website ElectionMethods.org, where Mike Ossipoff and Russ Paielli noted that it is satisfied by Approval Voting, but not by any other methods that they had analyzed. (The site content, alas, has been changed since then, and the articles on FBC can no longer be found there.)

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