Abstract. Let $X$ be a union of symplectic manifolds whose dimensions tend to infinity and let $M$ be a manifold with a closed 2-form $\omega$. We use Tischler’s elementary method of constructing symplectic embeddings in $\mathbb{C}P^{\infty}$ to prove a lifting property for the map that pulls back the limiting symplectic form on $X$ along an embedding of $M$. As an application, we show that, under suitable assumptions, the inclusion $\text{Emb}_{\omega}(M, X) \subset \text{Map}_{\omega}(M, X)$ is a weak homotopy equivalence. For example, each connected component of the space of symplectic embeddings of a connected symplectic manifold $M$ with integral symplectic form in $\mathbb{C}P^{\infty}$ (with the standard symplectic form) is weakly equivalent to $(S^1)^{b_1(M)} \times \mathbb{C}P^{\infty}. \text{We also show that, if } b_2(M) < \infty, \text{any compact family of closed 2-forms on } M \text{not intersecting the trivial cohomology class can be obtained by pulling back a family of standard forms on } (\mathbb{C}P^{\infty})^{b_2(M)} \text{along a family of embeddings.}$

1. Introduction

Whitney’s embedding theorem states that any manifold $M$ can be embedded in $\mathbb{R}^N$ when $N$ is sufficiently large. This is a basic property of smooth manifolds which is often useful in differential topology. Although the embedding is by no means unique, it is easy to see that as $N$ increases, the connectivity of the space of embeddings of $M$ in $\mathbb{R}^N$ (with the weak Whitney topology [Hi, Section 2.1]) tends to infinity. Thus, the space of embeddings of $M$ in $\mathbb{R}^{\infty} = \bigcup_n \mathbb{R}^n$ (defined as the union of the spaces of embeddings in $\mathbb{R}^n$) is contractible. This result says that the embedding of $M$ in $\mathbb{R}^{\infty}$ is “homotopy unique” and leads, for instance, in the case when $M$ is closed, to a model for the classifying space of the group of diffeomorphisms of $M$ as the space of submanifolds of $\mathbb{R}^{\infty}$ which are diffeomorphic to $M$.

It is natural to consider the problem of finding an analogous “universal” space for embedding a manifold $M$ with the added structure given by a closed 2-form $\omega$. It is unreasonable to expect the existence of $(X, \omega_X)$ playing the role of $\mathbb{R}^{\infty}$ in the previous paragraph. Indeed, taking $M = S^2$ we see that $X$ would need to have an uncountable second homotopy group and hence could not be the union of a sequence of (second countable, Hausdorff) manifolds.

However, if we impose suitable restrictions on the cohomology class of $\omega$, universal spaces do exist. Tischler proved in [Ti] that if $\omega$ is integral (i.e. if $[\omega]$ is in the image of the map $H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{R})$) and $M$ is closed, then the role of $\mathbb{R}^N$ is played by $(\mathbb{C}P^{N}, \omega_{FS})$ where $\omega_{FS}$ denotes the standard Fubini-Study form on complex projective space. In [Po], Popov showed how to extend Tischler’s argument to the case when $M$ is not compact and he also proved that any manifold with an exact 2-form $\omega$ embeds in some $\mathbb{R}^{2N}$ with the standard symplectic form. Popov and Tischler’s results also follow from Gromov’s h-principle for symplectic embeddings (see [Gr], 3.4.2 (B), p. 335] and [EM, Theorem 12.1.1]) but they are much more elementary.
The aim of this paper is to refine Tischler and Popov’s arguments so as to obtain a parametric and relative version of their results, and derive some consequences concerning the topology of embedding spaces. In order to state our main theorem we will first establish some notation that will be fixed for the remainder of the paper. Let \((X_n, \omega_n)\) be a sequence of symplectic manifolds and \(i_n: X_n \to X_{n+1}\) be closed embeddings of positive codimension preserving the symplectic forms. Given a manifold \(M\) together with a closed 2-form \(\omega\), we give the space \(\text{Emb}(M, X_n)\) of embeddings of \(M\) in \(X_n\) the weak Whitney topology and write

\[
\text{Emb}_{\omega}(M, X_n) = \{ \phi \in \text{Emb}(M, X_n): \phi^* \omega_n = \omega \}
\]

for the subspace of embeddings preserving the 2-form, and

\[
\text{Emb}_{[\omega]}(M, X_n) = \{ \phi \in \text{Emb}(M, X_n): \phi^* [\omega_n] = [\omega] \}
\]

for the subspace of embeddings preserving the cohomology class of the 2-form. Note that the latter is a union of components of the space of embeddings. We write \(\text{Emb}_{\omega}(M, X)\) for the union (or colimit) of the spaces \(\text{Emb}_{\omega}(M, X_n)\). Recall that a subset of the union is closed if its intersection with each of the sets \(\text{Emb}_{\omega}(M, X_n)\) is closed. The space \(\text{Emb}_{[\omega]}(M, X)\) is defined and topologized analogously. We give the space \(\Omega^2(M)\) of 2-forms on \(M\) the subspace topology determined by the weak Whitney topology on \(C^\infty(M, \Lambda^2(TM))\).

The following is our refinement of Tischler and Popov’s theorems (which correspond to the cases when \(A\) is a one point space, \(U\) is the empty set and \(X_n = \mathbb{CP}^n\) or \(\mathbb{R}^{2n}\)).

**Theorem 1.1.** Let \(M\) be a manifold and \(\omega\) a closed 2-form on \(M\). Let \(\pi: \text{Emb}_{[\omega]}(M, X) \to [\omega] \subset \Omega^2(M)\) be the map given by pulling back the 2-forms \(\omega_n\).

Let \(A\) be a compact subset of a smooth manifold and suppose given subspaces \(B \subset U \subset A\) with \(U\) open and \(B\) closed. Then for each pair of maps \(f, g\) making the solid diagram commute

there exists a lift \(h\) of \(g\) which is homotopic to \(f\) relative to \(B\).

Note that the space \(A\) in the previous statement can be, for instance, an arbitrary finite cell complex. In order to prove this theorem we will use the fact that a continuous family of exact 2-forms on a (not necessarily compact) manifold \(M\) can be obtained by differentiating a continuous family of 1-forms. Since we have not been able to find a suitable reference in the literature we have included a construction of a continuous deRham anti-differential in an appendix (see Theorem A.1).

Theorem 1.1 implies that the map \(\pi\) gives rise to a long exact sequence of homotopy groups and hence implies that the inclusion of each fiber of \(\pi\)

\[
\text{Emb}_{\omega}(M, X) \hookrightarrow \text{Emb}_{[\omega]}(M, X)
\]

is a weak homotopy equivalence (see Corollary 3.3). Using basic differential and algebraic topology we can then identify the weak homotopy type of the space \(\text{Emb}_{\omega}(M, X)\) in several examples - see section 3. For instance, in the case when \(X_n = \mathbb{CP}^n\) with the standard form, we
will see in 3.6.2 below that for \( \omega \) an integral 2-form on a connected manifold \( M \), the number of connected components of \( \text{Emb}_\omega(M,X) \) is in one-to-one correspondence with \( \text{Ext}(H_1(M),\mathbb{Z}) \) and each connected component is weakly homotopy equivalent to \( (S^1)^{b_1(M)} \times \mathbb{C}P^\infty \). This result also appears (essentially, in the case when \( M \) is closed) in the work of Gal and Kedra [GK, Theorem 3.2, Corollary 3.7], where it is deduced from Gromov’s parametric \( h \)-principle for symplectic immersions. As Gal and Kedra point out [GK, Remark 3.3], these results concerning the weak homotopy type of spaces of embeddings should also follow easily from the parametric version of the \( h \)-principle for symplectic embeddings. Unfortunately it seems that even a complete statement of the parametric \( h \)-principle for symplectic embeddings has not yet appeared in the literature.

Finally we would like to mention another application of Tischler’s method appearing below in 3.6.4. This says that the standard family of 2-forms on \( (\mathbb{C}P^\infty)^r \) given by taking \( \mathbb{R} \)-linear combinations of the pullbacks of the Fubini-Study forms on \( \mathbb{C}P^\infty \) via the projections is versal (i.e. all families can be obtained from it but not uniquely) among families of closed 2-forms not meeting the trivial cohomology class on manifolds with \( b_2 = r \). We hope this may prove useful in the study of the topology of the space of symplectic forms on a compact manifold.

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**2. Proof of the main theorem**

Let \((N,\omega)\) be a symplectic manifold of dimension \(2n\). By a Darboux chart for \( N \) we will mean a symplectomorphism \( \varphi: U \to V \) where \( U \subset N \) is open and contractible and \( V \) is an open subset of \( \mathbb{R}^{2n} \) with the standard symplectic form.

Recall that \((X_n,\omega_n)\) denotes a sequence of symplectic manifolds, and we assume there are symplectic closed embeddings \( i_n: X_n \to X_{n+1} \) of positive codimension. In the examples we have in mind, the \( X_n \) are either compact or \( \mathbb{R}^{2n} \).

The idea of the proof of the Theorems of Tischler and Popov in the cases when \( X_n = \mathbb{C}P^n \) (respectively \( \mathbb{R}^{2n} \)) is the following. Given a manifold \( M \) and an integral (respectively exact) closed 2-form \( \omega \) on \( M \), it is easy (by topological arguments in the case of \( \mathbb{C}P^n \) and trivially in the case of \( \mathbb{R}^{2n} \)) to produce a smooth map \( f: M \to X_n \) for some \( n \) such that \( f^*[\omega_n] = [\omega] \). One can then write \( f^*\omega_n - \omega = \sum_k dh_k \wedge dt_k \) for some smooth functions \( h_k, t_k: M \to \mathbb{R} \). The proof proceeds by inductively adding the pairs of functions \( (h_k, t_k) \) as extra coordinates, starting with a map of the form \((f, h_1, t_1): M \to X_{n+1} \) (in Darboux coordinates on \( X_{n+1} \)) which pulls back \( \omega_{n+1} \) to \( f^*\omega_n + dh_1 \wedge dt_1 \). For this to work it is important that the support of the functions \( h_k, t_k \) and (in the case of \( \mathbb{C}P^n \)) their magnitude be suitably constrained. In the end one gets a symplectic map \( g: M \to X_N \) for some \( N > n \), which is necessarily an immersion. Finally one applies Moser’s lemma (requiring careful estimates in the non-compact case) and the density of embeddings to turn the immersion \( g \) into an embedding.

Our proof will start with an embedding instead of an arbitrary map and use the fact that the perturbations made to the initial map will stay within embeddings. This will allow us to skip the application of Moser’s lemma, but will otherwise consist of obtaining parametric versions of the other steps in the proof and making use of the symplectic neighborhood theorem to handle the case of general symplectic manifolds \( X_n \). This extra generality leads, in our view, to a considerable simplification of the proofs given in [Ti] [P].
Unless otherwise specified, the topology we give spaces of smooth functions is the weak Whitney topology. We’ll indicate the strong Whitney topology [Hi, Section 2.1] on functions by the subscript $s$. For instance $C^s_x(M)$ denotes the space of smooth functions on $M$ with the strong topology. For $X$ and $Y$ topological spaces $C(X,Y)$ denotes the space of continuous maps from $X$ to $Y$ with the compact-open topology and $C_s(X,Y)$ denotes the space of continuous maps with the strong topology (for which a basis of open sets is given by $\{f \in C(X,Y) : \text{Graph}(f) \subset U}\}$ for $U$ an arbitrary open subset of $X \times Y$).

**Lemma 2.1.** Let $A$ be a compact subset of a smooth manifold and $B \subset A$ a closed subset. Let $M$ be a manifold of dimension $n$, and $\eta: A \to \Omega^1(M)$ be a continuous map. Assume $\{W_\alpha\}_{\alpha \in I}$ is a locally finite open cover of $A \times M$ such that, for each $\alpha$, there is a coordinate neighborhood $U$ of $M$ so that $W_\alpha \subset A \times U$. Then there exist continuous functions $h_\alpha^r, t_\alpha^r: A \to C^\infty(M)$ for $r = 1, \ldots, n$ and $\alpha \in I$, such that

(i) For each $r = 1, \ldots, n$ and $\alpha \in I$, the functions $(z, x) \mapsto h_\alpha^r(z)(x)$ and $(z, x) \mapsto t_\alpha^r(z)(x)$ are supported on $W_\alpha$.

(ii) If $\eta(z) = 0$ for all $z$ in a neighborhood of $B$, then $h_\alpha^r(z) = t_\alpha^r(z) = 0$ for all $z \in B$.

(iii) $d\eta(z) = \sum_{\alpha \in I} \sum_{r=1}^n dh_\alpha^r(z) \wedge dt_\alpha^r(z)$.

**Proof.** Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{W_\alpha\}$ of $A \times M$ (this makes sense because $A$ can be regarded as a closed subset of a smooth manifold). We can write $\rho_\alpha \eta = \sum_{r=1}^n h_\alpha^r(z) ds_\alpha^r$ where $s_\alpha^r$ are local coordinates on some coordinate neighborhood $U$ so that $W_\alpha \subset A \times U$ and $h_\alpha^r: A \to C^\infty(M)$ are continuous functions such that the support of $(z, x) \mapsto h_\alpha^r(z)(x)$ is contained in the support of $\rho_\alpha$.

Let $\phi_\alpha: A \times M \to \mathbb{R}$ be a smooth cut-off function supported on $W_\alpha$ which is equal to 1 on an open set containing the support of $\rho_\alpha$. Suppose $U$ is a neighborhood of $B$ where $\eta$ vanishes and let $\psi: A \to [0, 1]$ be a continuous function which is supported on $U$ and equal to 1 on $B$. Setting $t_\alpha^r(z) = (1 - \psi(z))\phi_\alpha(z, \cdot)s_\alpha^r$ we obtain the required expression: $d\eta(z) = \sum_\alpha d(\rho_\alpha \eta(z)) = \sum_\alpha \sum_{r=1}^n dh_\alpha^r(z) \wedge dt_\alpha^r(z)$. \qed

We will need the fact that the functions produced in the previous lemma can be made arbitrarily small. The next lemma will ensure this can be arranged. We note that the proof of this point given in [Ti, Lemma 2 (3),(4)] is mistaken.

**Lemma 2.2.** (cf. [EM, 12.1.5]) Let $D^2_s$ denote the disk of radius $s$ in $\mathbb{R}^2$. Given $r, R > 0$, there exists a smooth symplectic map $D^2_R \to D^2_s$ sending the origin to itself.

**Proof.** A map with the required properties can be obtained by composing a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ of the form $z \mapsto \frac{z^{N+1}}{N+1}$ for a suitable $N \in \mathbb{N}$ with translations of $\mathbb{C}$ on both sides. \qed

**Proof of Theorem 1.1.** We will use capital letters to denote adjoint maps. For instance $F: A \times M \to X$ denotes the map $(z, x) \mapsto f(z)(x)$. Since $A$ is compact and $\text{Emb}_{[\omega]}(M, X)$ has the colimit topology, there exists $N$ such that the image of $F$ is contained in $X_N$.

Let $n = \dim M$, let $k$ be the dimension of a smooth manifold containing $A$, let $d_j = \dim X_j$, and set $C = n(n + k + 1)$. Let $U_N = \{U_{N, \beta}\}_{\beta \in J}$ be a locally finite cover of $X_N$ by Darboux coordinate neighborhoods and $\phi_{N, \beta}: U_{N, \beta} \to \mathbb{R}^{2d_N}$ be corresponding Darboux charts.

By the symplectic neighborhood theorem we can pick successive extensions of the Darboux coordinates on $X_N$ to the manifolds $X_j$ which we denote

$$\phi_{j, \beta}: U_{j, \beta} \to \mathbb{R}^{2d_j}, \quad j = N + 1, \ldots, N + C$$
The families $\mathcal{U}_j = \{U_{j,\beta}\}_{\beta \in J}$ are open covers of $X_N$ in $X_j$ satisfying $\mathcal{U}_j \cap X_{j'} = \mathcal{U}_{j'}$ for $j' < j$. We will use the charts $\phi_{j,\beta}$ to construct a map $H: A \times M \to X_{N+C}$ whose adjoint will lift $g$.

Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be a locally finite refinement of the cover $F^{-1}(U_W)$ with the property that each element of $\mathcal{W}$ has compact closure and is contained in $A \times U$ for some coordinate chart $U$ on $M$. We'll denote the refinement function by $I: \mathcal{W} \to \mathcal{W}$. According to [Mu, Lemma 2.7] we can (by further refining $\mathcal{W}$ if necessary) partition the indexing set $I$ of $\mathcal{W}$ as

$$I = I_0 \coprod \ldots \coprod I_{n+k}$$

so that if $\alpha, \alpha' \in I_i$ then $W_\alpha \cap W_{\alpha'} = \emptyset$.

By Theorem [A.1] there is a continuous map $\eta: A \to \Omega^1(M)$ which vanishes on $U$ and satisfies,

$$d(\eta(z)) = g(z) - f(z)^\ast \omega_N$$

for all $z \in A$. Lemma [2.1] then provides continuous functions

$$h_{\alpha}^r, t_{\alpha}^r: A \to C^\infty(M), \quad r = 1, \ldots, n; \ \alpha \in I$$

such that

1. $S_{\alpha} = \cup_{r=1}^{n} \text{Supp}(H_{\alpha}^r \cup \text{Supp}(T_{\alpha}^r))$ is contained in $W_\alpha$,
2. $h_{\alpha}^n(z) = t_{\alpha}^n(z) = 0$ for $z \in B$,
3. $d\eta(z) = \sum_{r,\alpha} dh_{\alpha}^r(z) \wedge dt_{\alpha}^r(z)$.

We are now in a position to complete the proof by constructing a homotopy

$$k: A \times [0, C] \to \text{Emb}_{[\omega]}(M, X)$$

such that

1. $k(z, 0) = f(z)$ for all $z \in A$,
2. $k(z, t) = f(z)$ for all $z \in B$,
3. $\pi k(z, C) = g(z)$, for all $z \in A$.

For each $m \in \{0, \ldots, n+k\}$ and $r \in \{1, \ldots, n\}$ define $h_{m}^r = \sum_{\alpha \in I_m} h_{\alpha}^r$ and $t_{m}^r = \sum_{\alpha \in I_m} t_{\alpha}^r$. Then

$$d\eta(z) = \sum_{r=1}^{n} \sum_{m=0}^{n+k} dh_{m}^r(z) \wedge dt_{m}^r(z).$$

Note that the maps

$$(H_{m}^r, T_{m}^r): A \times M \to \mathbb{R}^2$$

are supported on a disjoint union of compact subsets of $A \times M$ indexed by the set $I_{m}$. Thus, if $\mathcal{N}$ is any neighborhood of $0$ in $C_s(A \times M, \mathbb{R}^2)$, Lemma [2.2] allows us to construct from $h_{m}^r, t_{m}^r$ maps

$$(H_{m}^r, T_{m}^r): A \times M \to \mathbb{R}^2$$

so that

1. $H_{m}^r, T_{m}^r \in \mathcal{N}$,
2. $\text{Supp}(H_{m}^r, T_{m}^r) \subset \text{Supp}(H_{m}^r, T_{m}^r) \subseteq \cup_{\alpha \in I_{m}} S_{\alpha}$,
3. $(H_{m}^r, T_{m}^r) \in C(A, C^\infty(M, \mathbb{R}^2))$,
4. $dh_{m}^r(z) \wedge dt_{m}^r(z) = dH_{m}^r(z) \wedge dT_{m}^r(z)$ for all $z \in A$. 

The homotopy \( k : A \times [0, C] \to C^\infty(M, X) \) is constructed inductively as follows. We pick the neighborhood \( N \) of 0 in \( C_s(A \times M, \mathbb{R}^2) \) so small that the map

\[
k|_{A \times [0,1]} : A \times [0, 1] \to C^\infty(M, X_{N+1})
\]
given by the formula

\[
k(z, s)(x) = \begin{cases} 
\phi_{N+1, \psi(a)}(\phi_N, \psi(a))(F(z, x)), s\overline{H}_0(z, x), sT\overline{I}(z, x), 0, \ldots, 0) & \text{if } (z, x) \in \cup_{\alpha \in I_0} S_{\alpha} \\
F(z, x) & \text{otherwise.}
\end{cases}
\]
is well defined and satisfies

1. \( k(z, s) \in \text{Emb}_{[\omega]}(M, X_{N+1}) \) for all \( z \in A \) and \( s \in [0, 1] \),
2. \( k(z, s)(S_{\gamma}) \subset \overline{U}_{N+1, \psi(\gamma)} \) for all \( z \in A, s \in [0, 1] \) and \( \gamma \in I \).

This is possible, because (1) and (2) can be enforced by imposing a finite number of bounds on the functions \( \overline{H}_0, T_0 \) on the compact set \( S_{\alpha} \) for each \( \alpha \in I_0 \).

Then setting \( g_1(z) = k(z, 1) \) we have

\[
g_1(z)^*\omega_{N+1} = f(z)^*\omega_N + dh_0(z) \wedge dt_0(z).
\]

Since \( G_1(S_{\gamma}) \subset \overline{U}_{N+1, \psi(\gamma)} \), we can proceed with the construction of the homotopy in the same way. Namely we obtain \( g_2 = k(z, 2) : A \to \text{Emb}_{[\omega]}(M, X_{N+2}) \) with \( g_2(z)^*\omega_{N+2} = g_1(z)^*\omega_{N+1} + dh_0(z) \wedge dt_0(z) \) and \( G_2(S_{\gamma}) \subset \overline{U}_{N+2, \psi(\gamma)} \) for all \( \gamma \in I \), etc. Setting \( h(z) = k(z, C) \) we’ll have \( h(z)^*\omega_{N+C} = f(z)^*\omega_N + dh(z) = g(z) \) as required.

**Remark 2.3.** The construction given in the previous proof shows that the lift \( h \) can be chosen so that \( H \) approximates \( F \) arbitrarily in \( C_s(A \times M, X_{N+C}) \).

### 3. Applications

A map \( \pi : E \to X \) is said to have the **weak homotopy lifting property** with respect to a space \( B \) if given a commutative diagram

\[
\begin{array}{ccc}
B \times 0 & \xrightarrow{k} & E \\
\downarrow & & \downarrow \pi \\
B \times [0, 1] & \xrightarrow{H} & X
\end{array}
\]

there exists a homotopy \( \tilde{H} : B \times [0, 1] \to E \) such that \( \pi \tilde{H} = H \) and \( \tilde{H}_0 \) is homotopic to \( k \) as maps over \( \pi \) (i.e. \( \tilde{H}_0 \) is vertically homotopic to \( k \)). Clearly this is equivalent to the usual homotopy lifting property for homotopies \( H \) which are initially constant (i.e. which, for some \( \epsilon > 0 \), satisfy \( H(b, t) = H(b, 0) \) for all \( b \in B \) and \( t \leq \epsilon \)). See for instance [St 13.1.3] for more on the weak homotopy lifting property.

**Corollary 3.1.** The map \( \pi : \text{Emb}_{[\omega]}(M, X) \to \{[\omega] \subset \Omega^2(M) \} \) given by pulling back the forms \( \omega_n \) has the weak homotopy lifting property with respect to compact subsets of smooth manifolds.

**Proof.** Let \( B \) be a compact subset of a smooth manifold. Given a homotopy \( H : B \times [0, 1] \to \{[\omega] \} \) which is constant in the interval \( [0, \epsilon] \) and a lift \( k : B \to \text{Emb}_{[\omega]}(M, X) \) of \( H_0 \), apply Theorem [1.1] with \( A = B \times [0, 1], U = B \times [0, \epsilon], B = B \times 0, g = H \) and \( f(a, t) = k(a) \). \( \square \)
Remark 3.2. When $M$ is compact, a version of Moser’s trick can be used to show that the map $\pi$ in Corollary 3.1 has the homotopy lifting property with respect to compact subsets of manifolds for smooth homotopies (i.e. maps $H: B \times [0,1] \to [\omega]$ whose adjoints lie in $C(B, C^\infty([0,1] \times M, \Lambda^2(T^*M)))$).

Corollary 3.3. Let $M$ be a manifold and $\omega \in \Omega^2(M)$ be a closed 2-form. Then the inclusion
\[ \text{Emb}_\omega(M, X) \to \text{Emb}_{[\omega]}(M, X) \]
is a weak homotopy equivalence.

Proof. The usual construction of the long exact sequence of homotopy groups in a fibration goes through for maps satisfying the weak homotopy lifting property with respect to closed disks. Since $[\omega]$ is convex, the result follows from Corollary 3.1. □

In order to compare the homotopy type of the space of embeddings with the homotopy type of the space of continuous functions it is convenient to make the following additional assumption on the symplectic embeddings $i_n: X_n \to X_{n+1}$.

Assumption 3.4. The connectivity of the symplectic embeddings $i_n: X_n \to X_{n+1}$ goes to $\infty$ with $n$ (i.e. for any given $N$, there exists $m$ so that $i_{n*}: \pi_k(X_n, *) \to \pi_k(X_{n+1}, *)$ is an isomorphism for all basepoints, all $k \leq N$ and $n \geq m$).

Recall that for $X, Y$ topological spaces, $C(X, Y)$ denotes the space of continuous maps with the compact open topology. Under Assumption 3.4 it follows from Whitehead’s Theorem (see for instance [AGP Theorem 5.1.32]) that the canonical map
\[ \text{colim}_n C(M, X_n) \to C(M, X) \]
is a weak equivalence for any manifold $M$. Moreover the cohomology classes $[\omega_n] \in H^2(X_n; \mathbb{R})$ determine a unique element $[\omega_X] \in H^2(X; \mathbb{R})$ pulling back to $[\omega_n]$ under the inclusions.

Corollary 3.5. Let $M$ be a manifold, $\omega$ be a closed 2-form on $M$ and assume the symplectic embeddings $i_n: X_n \to X_{n+1}$ satisfy Assumption 3.4. Let
\[ C_{[\omega]}(M, X) = \{ f \in C(M, X): f^*[\omega_X] = [\omega] \}. \]
Then the inclusion
\[ \text{Emb}_\omega(M, X) \to C_{[\omega]}(M, X) \]
is a weak homotopy equivalence.

Proof. First recall that a map $\phi: X \to Y$ is a weak homotopy equivalence, if and only if for all $k \geq 0$ and all commutative squares
\[ \begin{array}{ccc}
S^{k-1} & \xrightarrow{f} & X \\
\downarrow h & & \downarrow \phi \\
D^k & \xrightarrow{g} & Y
\end{array} \]
(where $D^k$ denotes the closed unit ball in $\mathbb{R}^k$) there exists a lift $h$ making the upper triangle commute and the lower triangle commute up to homotopy relative to $S^{k-1}$ (see for instance [Ma Lemma in section 9.6]).
In view of Corollary 3.3 it suffices to show that each of the inclusions \( i_1: \text{Emb}_{[\omega]}(M, X) \to C^\infty_{[\omega]}(M, X) \) and \( i_2: C^\infty_{[\omega]}(M, X) \hookrightarrow C_{[\omega]}(M, X) \) are weak equivalences (where spaces of smooth maps to \( X \) mean the union of the corresponding spaces of smooth maps to the manifolds \( X_n \)).

In the case of \( i_2 \), since the map \( (\ref{1}) \) is a weak equivalence, it is enough to show that the inclusions \( i_2: C^\infty(M, X_n) \hookrightarrow C(M, X_n) \) are weak equivalences for all \( n \). But the required lifts in \( (\ref{2}) \) exist by the density of smooth maps in continuous maps in the strong topology (cf. \[HH\] Theorem 2.6).

In order to prove that the map \( i_1 \) satisfies the lifting conditions \( (\ref{2}) \), note that compactness of \( D^k \) implies that the image of \( g \) is contained in \( C^\infty(M, X_n) \) for some \( n \). As long as the dimension of \( X_n \) is large enough, the density of embeddings in smooth maps in the strong topology (cf. \[HH\] Theorem 2.13) allows us to construct the required lift. The reader is referred to \[AG\] Sections 3.2 and 3.3 for more details of these arguments. \( \square \)

### 3.6. Examples.

#### 3.6.1. Embeddings in \( \mathbb{R}^\infty \).

Let \( X_n = \mathbb{R}^{2n} \), let \( \omega_n \) denote the standard symplectic form on \( X_n \) and \( i_n: \mathbb{R}^{2n} \to \mathbb{R}^{2n+2} \) denote the canonical inclusions. Assumption 3.3 is obviously satisfied. In this case, Corollary 3.3 says that for \((M, \omega)\) a manifold with an exact 2-form, the space \( \text{Emb}_{\omega}(M, \mathbb{R}^\infty) \) is contractible. For instance if \( \omega = 0 \), this says that the space of Lagrangian embeddings of \( M \) in \( \mathbb{R}^\infty \) is contractible. Since the group \( \text{Diff}_{\omega}(M) \) of diffeomorphisms of \( M \) which preserve \( \omega \) acts freely on \( \text{Emb}_{\omega}(M, \mathbb{R}^\infty) \), we see that, when \( M \) is compact, the space of submanifolds of \( \mathbb{R}^\infty \) which are diffeomorphic to \((M, \omega)\) provides a model for the classifying space \( B \text{Diff}_{\omega}(M) \).

#### 3.6.2. Embeddings in \( \mathbb{C}P^{\infty} \).

Let \( X_n = \mathbb{C}P^n \), with \( \omega_n \) the standard Fubini-Study form, and \( i_n: \mathbb{C}P^n \to \mathbb{C}P^{n+1} \) denote the canonical inclusions. Assumption 3.3 is satisfied, so if \((M, \omega)\) is a manifold with a closed 2-form, Corollary 3.3 says that the inclusion

\[ \text{Emb}_{\omega}(M, \mathbb{C}P^{\infty}) \hookrightarrow C_{[\omega]}(M, \mathbb{C}P^{\infty}) \]

is a weak homotopy equivalence (cf. also \[GK\] Theorem 3.2 where this result is proved for \( M \) compact).

Since \( \mathbb{C}P^{\infty} \) is an Eilenberg-Maclane space \( K(\mathbb{Z}, 2) \), the space \( C(M, \mathbb{C}P^{\infty}) \) is weakly equivalent to a product of Eilenberg-Maclane spaces

\[ C(M, \mathbb{C}P^{\infty}) = H^2(M; \mathbb{Z}) \times K(H^1(M; \mathbb{Z}), 1) \times K(H^0(M; \mathbb{Z}), 2). \]

This is a standard computation but since we were unable to find a suitable reference we’ll sketch an argument: \( \mathbb{C}P^{\infty} \) is weakly equivalent to a topological abelian group \( X \) (see for instance \[AGP\] Corollary 6.4.23), hence \( C(M, \mathbb{C}P^{\infty}) \) is weakly equivalent to \( C(M, X) \). The singular complex \( \text{Sing}(C(M, X)) \) is a simplicial abelian group, and so, by the Dold-Kan correspondence and elementary properties of chain complexes of abelian groups, is weakly equivalent to a simplicial abelian group \( G = \prod_{k \geq 0} G_k \) with each \( G_k \) a simplicial abelian group with only one simplex in degrees less than \( k \) and \( \pi_i(G_k) = 0 \) for \( i \neq k \). The geometric realization \( |G| \) of \( G \) is weakly equivalent to \( C(M, \mathbb{C}P^{\infty}) \) and also to \( \prod_k |G_k| = \prod_k K(\pi_k(G_k), k) \). The computation of the homotopy groups of \( C(M, \mathbb{C}P^{\infty}) \) in terms of the cohomology of \( M \) follows from the fact that \( \mathbb{C}P^{\infty} \) classifies degree 2 cohomology (cf. \[GK\] Lemma 3.6).

If \( \omega \) is an integral form, then the space \( C_{[\omega]}(M, \mathbb{C}P^{\infty}) \) is the union of the components corresponding to the class \( [\omega] \in H^2(M; \mathbb{R}) \), so

\[ \text{Emb}_{\omega}(M, \mathbb{C}P^{\infty}) \cong \text{Ext}(H_1(M), \mathbb{Z}) \times (S^1)^{\beta_1(M)} \times (\mathbb{C}P^{\infty})^{\beta_0(M)} \]
where $\beta_0(M)$ denotes the number of connected components of $M$ and $\beta_1(M)$ the first Betti number (which may be infinite).

In particular, if $(M, \omega)$ is simply connected (and integral), $\text{Emb}_\omega(M, \mathbb{C}P^\infty) \cong \mathbb{C}P^\infty$ with the weak equivalence induced by evaluation at a given point in $M$. Once we fix a base point in $\mathbb{C}P^\infty$ the space $\text{Emb}_{\omega, *}(M, \mathbb{C}P^\infty)$ of pointed embeddings is contractible and so, for $M$ compact, the quotient by the space $\text{Diff}_{\omega}(M, *)$ of diffeomorphisms preserving $\omega$ and fixing the basepoint provides a model for the classifying space $B\text{Diff}_{\omega}(M, *)$.

3.6.3. Embeddings in products of copies of $\mathbb{C}P^\infty$. Let $M$ be a manifold with finite second Betti number and $\omega$ a closed 2-form on $M$. Let $r = r(|\omega|)$ be the least positive integer such that $[\omega]$ can be written as a real linear combination of $r$ elements in $H^2(M; \mathbb{Z})$. We can pick $\alpha_i \in H^2(M; \mathbb{Z})$ and $\lambda_i \in \mathbb{R} \setminus \{0\}$ for $i = 1, \ldots, r$ such that

$$[\omega] = \lambda_1 \alpha_1 + \ldots + \lambda_r \alpha_r.$$  

Consider the sequence of symplectic manifolds $X_n = (\mathbb{C}P^n)^r$ with symplectic form $\omega_n = \sum_{i=1}^r \lambda_i \omega_i$ where $\omega_i$ denotes the pullback of the standard Fubini-Study form via the $i$-th projection. Assumption 3.4 is satisfied and therefore, by Corollary 3.5, the classes $\lambda_i$ determine a connected component of $\text{Emb}_\omega(M, (\mathbb{C}P^\infty)^r)$. As in the previous example, the weak homotopy type of this connected component is $(S^1)^{\beta_1(M)r} \times (\mathbb{C}P^\infty)^{\beta_0(M)r}$.

3.6.4. Realization of families of closed forms by embeddings. Let $M$ be a manifold with finite second Betti number $r = \beta_2(M)$. Let $\alpha_1, \ldots, \alpha_r \in H^2(M; \mathbb{Z})$ be classes spanning $H^2(M; \mathbb{R})$. As before, the $r$-tuple $(\alpha_i)$ determines a component $\text{Emb}_{(\alpha_i)}(M, (\mathbb{C}P^\infty)^r)$ of the space of embeddings.

The argument used in the proof of Theorem 1.1 allows us to realize a family of closed 2-forms on $M$ with non-vanishing cohomology class by a family of embeddings in $(\mathbb{C}P^\infty)^r$ as follows.

Let $\Omega^2_{\text{cl}}(M)$ and $\Omega^2_{\text{ex}}(M)$ be the spaces of closed and exact two forms on $M$, respectively, and let $\pi: \Omega^2_{\text{cl}}(M) \to H^2(M; \mathbb{R})$ denote the obvious projection map (which is continuous because $\beta_2(M)$ is finite). The choice of the $r$-tuple $(\alpha_i)$ above gives an identification $\mathbb{R}^r \to H^2(M; \mathbb{R})$ which we will omit from the notation. For $\lambda \in \mathbb{R}^r$ we’ll write $\omega_\lambda$ for the closed 2-form on $(\mathbb{C}P^n)^r$ given by

$$\omega_\lambda = \sum_{i=1}^r \lambda_i \omega_i$$

where $\omega_i$ is the pullback of the standard Fubini-Study form by the projection on the $i$-th component.

Let $A$ be a compact subspace of a smooth manifold and let $g: A \to (\Omega^2_{\text{cl}}(M) \setminus \Omega^2_{\text{ex}}(M))$ be a continuous map. We’ll show there exists a map $f: A \to \text{Emb}_{(\alpha_i)}(M, (\mathbb{C}P^\infty)^r)$ such that

$$f(z)^* (\omega_{\pi(g(z))}) = g(z).$$

The previous equation says that the (nonzero) linear combinations of the pullbacks of the Fubini-Study forms on the factors give a versal family of closed 2-forms with non-vanishing cohomology class. Note that the non-vanishing condition is automatic for families of symplectic forms on compact manifolds.

The reason the proof of Theorem 1.1 applies even though the classes $\omega_\lambda$ are not necessarily symplectic is the following. Setting $X_n = (\mathbb{C}P^n)^r$ and letting $i_n: X_n \to X_{n+1}$ denote the standard inclusions, we can consider the standard forms $\omega_n = \omega_{(1, \ldots, 1)}$ on $X_n$. Using the
notation of the proof of Theorem 1.1 if we pick Darboux charts on $X_N$ which are $r$-fold Cartesian products of Darboux charts on $\mathbb{C}P^N$, we can likewise pick symplectic neighborhoods of the Darboux coordinate neighborhoods on $X_j$ which are $r$-fold Cartesian products. Let $\lambda_i: A \to \mathbb{R}$ denote the components of the map $\pi \circ g: A \to H^2(M; \mathbb{R}) \cong \mathbb{R}^r$. Since $A$ is compact, the functions

$$\mu_i(z) = \frac{\lambda_i(z)}{\sum_{i=1}^r \lambda_i(z)^2}$$

are bounded. Using the product splitting of the symplectic tubular neighborhoods, the inductive construction of the homotopy $k: A \times [0, C] \to \text{Emb}_{(\alpha_i)}(M, X)$ can then be altered as follows: in the first step replace the expression $(F(z, x), s\mathcal{H}_0^i(z, x), sT^i_0(z, x), 0, \ldots, 0)$ by

$$(F(z, x), s\mu_1(z)\mathcal{H}_0^i(z, x), s\mu_1(z)T^i_0(z, x), s\mu_2(z)\mathcal{H}_0^i(z, x), \ldots, s\mu_r(z)T^i_0(z, x)).$$

As long as $T^i_0, T^i_l \in C(A, C^\infty(M, \mathbb{R}^2))$ are chosen to lie in a sufficiently small neighborhood $N$ of 0 in $C_n(A \times M, \mathbb{R}^2)$, the boundedness of the functions $\mu_i$ ensures that the homotopy is well defined and the construction of $k$ may proceed.

3.6.5. Embeddings in $BU$. Let $X_n = \text{Gr}_n(\mathbb{C}^{2n})$ be the Grassmann manifold of $n$-dimensional complex subspaces of $\mathbb{C}^{2n}$. The manifolds $X_n$ admit canonical Plücker embeddings in $\mathbb{C}P^{2n}$ which are compatible with the standard inclusions $i_n: X_n \to X_{n+1}$ and $\mathbb{C}P^{2n} \subset \mathbb{C}P^{2(n+1)}$ (for a suitable ordering of the coordinates in projective space). The restriction of the Fubini-Study forms on the projective spaces give rise to canonical symplectic forms $\omega_n$ on $X_n$ such that $i_n^*\omega_{n+1} = \omega_n$. The form $\omega_n$ is integral and its cohomology class generates $H^2(X_n; \mathbb{Z}) \cong \mathbb{Z}$.

The colimit $X$ of the inclusions $i_n$ is the classifying space for stable complex bundles usually denoted $BU$. Assumption 3.4 is satisfied, so given a manifold $M$ and a closed 2-form $\omega$ on $M$, Corollary 3.5 identifies the weak homotopy type of $\text{Emb}_w(M, BU)$ with that of $C[\omega](M, BU)$. The homotopy groups of this space can be computed in terms of (complex, representable) $K$-theory of $M$ since $BU$ classifies reduced complex $K$-theory.

In more detail, there is a map $c: BU \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots$ classifying the stable Chern classes which is an equivalence on rational cohomology. The composite of $c$ with the projection onto $K(\mathbb{Z}, 2) \cong \mathbb{C}P^\infty$ can be identified with the inclusion of $BU$ in $\mathbb{C}P^\infty$ determined by the Plücker embeddings. Therefore the set of connected components of $C[\omega](M, BU)$ is

$$\pi_0(C[\omega](M, BU)) = \{\alpha \in \tilde{K}^0(M): c_1(\alpha) = [\omega]\}.$$

For this set to be non-empty, $[\omega]$ must of course be an integral class and, since $c$ is a rational equivalence, the set will be non-empty whenever $[\omega]$ is sufficiently divisible in $H^2(M, \mathbb{Z})$.

Since $BU$ has a group-like multiplication, all the connected components of $C[\omega](M, BU)$ are weakly equivalent and the homotopy groups agree with those of the connected component corresponding to the constant map from $M$ to $BU$. Hence for each connected component $\alpha$ we have

$$\pi_i(\text{Emb}_w(M, BU)_\alpha) = \begin{cases} K^1(M) & \text{if } i \text{ is odd}, \\ \tilde{K}^0(M) & \text{if } i > 0 \text{ is even}. \end{cases}$$

Appendix A. A continuous deRham anti-differential

Let $M$ be a (second countable, Hausdorff) smooth manifold without boundary. We give spaces of differential forms on $M$ the weak Whitney topology. The aim of this appendix is to prove the following result which was used in the proof of Theorem 1.1.
Theorem A.1. For each $k \geq 1$, there exists a continuous linear right inverse for the deRham differential $d: \Omega^{k-1}(M) \to d(\Omega^{k-1}(M))$.

The idea of the proof is suggested in [MS] p.96 (although the reference there to an inductive procedure seems misleading). A similar formula for the anti-differential appears in the PhD Thesis of Ioan Marcut [Ma, Section 3.4.4] under the added assumption that the manifold $M$ is of finite type (the details are spelled out for $k = 2$). As pointed out in [Ma], the existence of a smooth family of primitives for a smooth family of exact forms on an arbitrary manifold is stated as [GLSW] Lemma, p. 617 and a sketch proof is also given there.

One difference between the argument we give and the one described in [Ma, GLSW] is that we make use of the Čech-deRham complex for cohomology with compact supports instead of the usual Čech-deRham complex. The proof will consist of using explicit quasi-isomorphisms (cf. [BT, I.9]) between the deRham complex on $M$ and the usual Čech-deRham complex. The proof will consist of using explicit quasi-isomorphisms (cf. [BT, I.9]) between the deRham complex on $M$, the Čech-deRham double complex with compact supports and a complex for Čech homology with real coefficients to translate the problem of finding an anti-differential to the combinatorial problem of finding a bounding chain in the Čech complex.

A.2. The Čech-deRham double complex with compact supports. Suppose the dimension of the manifold $M$ is $n$ and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a cover of $M$ such that the sets $\overline{U_\alpha}$ are compact, every non-empty intersection of open subsets in $\mathcal{U}$ is diffeomorphic to $\mathbb{R}^n$, and every intersection of $n+2$ elements of $\mathcal{U}$ is empty (one can, for instance, cover $M$ by the open stars of vertices in a smooth triangulation). We pick a total ordering of the indexing set $I$ and consider the Čech-deRham double complex\footnote{Our notation below is meant to suggest Čech homology with coefficients in a cosheaf (cf. [Br, VI.4]) even though the Čech complexes we use are a completed version where direct sums are replaced with products.} for compactly supported cohomology given by

$$C^{-p,q} = \hat{C}_p(\mathcal{U}; \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(U_{\alpha_0}\cdots_{\alpha_p})$$

with $p, q \in \{0, \ldots, n\}$. We’ll write $\omega = (\omega_{\alpha_0\cdots\alpha_p})$ for an element of $C^{-p,q}$ and use the convention $\omega_{\beta_0\cdots\beta_p} = 0$ if $\beta_i = \beta_j$ for some $i \neq j$. Moreover, if $\sigma$ is a permutation of $\{\alpha_0, \ldots, \alpha_p\}$ with $\alpha_0 < \cdots < \alpha_p$, we set $\lambda_{\sigma(\alpha_0)\cdots\sigma(\alpha_p)} = (-1)^{sgn(\sigma)}\lambda_{\alpha_0\cdots\alpha_p}$.

The horizontal and vertical differentials $d^h: C^{-p,q} \to C^{-(p-1),q}$ and $d^v: C^{-p,q} \to C^{-p,q+1}$ are given by the formulas

$$d^h(\omega)_{\alpha_0\cdots\alpha_p} = \sum_{\alpha \in I} \omega_{\alpha\alpha_0\cdots\alpha_p}, \quad (d^v(\omega))_{\alpha_0\cdots\alpha_p} = (-1)^pd\omega_{\alpha_0\cdots\alpha_p}.$$

We can augment the complex $C^{*,*}$ in the horizontal direction by setting $C^{1,*} = \Omega^*(M)$ and defining $d^v = -d$ on $C^{1,*}$, and $d^h: C^0,q \to C^{1,q}$ by the formula

$$d^h(\omega_\alpha) = \sum_{\alpha \in I} \omega_\alpha.$$

We can also augment the complex $C^{*,*}$ in the vertical direction by setting

$$C^{-p,n+1} = \hat{C}_p(\mathcal{U}; \mathbb{R}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathbb{R}$$
with horizontal differential $d^h$ still given by (11), and $d^v : C^{-p,n} \to C^{-p,n+1}$ given by

$$d^v(\omega_{\alpha_0 \cdots \alpha_p}) = (-1)^p \left( \int_{U_{\alpha_0 \cdots \alpha_p}} \omega_{\alpha_0 \cdots \alpha_p} \right).$$

**A.3. Explicit contractions of rows and columns.** Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to the cover $U$ and define $K : C^{-p,q} \to C^{-(p+1),q}$, for $0 \le q \le n$ and $-n \le -p \le 1$ by

$$K\omega_{\alpha_0 \cdots \alpha_p} = \sum_{i=0}^{p+1} (-1)^i \rho_{\alpha_i} \omega_{\alpha_0 \cdots \alpha_i \cdots \alpha_{p+1}}. \tag{6}$$

One easily checks (cf. [BT, (8.7) p.95]) for the case of the Čech-deRham complex with not necessarily compact supports) that for each $q \in \{0, \ldots, n\}$, the operator $\mathbb{R}$ is a cochain contraction (i.e. $d^hK + Kd^h = \text{id}$) of the complexes $(C^{*,q}, d^h)$ with $-n \le * \le 1$, for each $q$ such that $0 \le q \le n$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $(n-1)$-coordinates and $e(t)dt \in \Omega^1_c(\mathbb{R})$ be a 1-form with integral 1. There are operators

$$\pi_* : \Omega^k_c(\mathbb{R}^n) \to \Omega^{k-1}_c(\mathbb{R}^{n-1}), \quad \text{and} \quad e_* : \Omega^{k-1}_c(\mathbb{R}^{n-1}) \to \Omega^k_c(\mathbb{R}^n)$$

given respectively by integration along the fiber of $\pi$ and exterior product with $e(x_n)dx_n$. Multiplying by $(-1)^{*-1}$ the operator $K$ in [BT, Proposition I.4.6, p. 38], we obtain operators $Q : \Omega^*_c(\mathbb{R}^n) \to \Omega^{*-1}_c(\mathbb{R}^n)$ such that $dQ + Qd = 1 - e_*\pi_*$. The cochain complex $\Omega^*_c(\mathbb{R}^n)$ can be augmented by $\mathbb{R}$ in degree $n + 1$ by setting $d : \Omega^*_c(\mathbb{R}^n) \to \mathbb{R}$ to be $d\omega = \int_{\mathbb{R}^n} \omega = (\pi_*)^n \omega$, and it is then easy to check that $L : \Omega^*_c(\mathbb{R}^n) \to \Omega^{*-1}_c(\mathbb{R}^n)$ defined by

$$L = \begin{cases} \sum_{j=0}^{n-1} (e_*)_j Q(\pi_*)^j & \text{if } * \le n, \\ (e_*)^n & \text{if } * = n + 1, \end{cases} \tag{7}$$

(we have identified $\mathbb{R} = \Omega^{n+1}_c(\mathbb{R}^n)$ with $\Omega^0_c(\mathbb{R}^0)$) is a cochain contraction of the augmented complex $\Omega^*_c(\mathbb{R}^n)$. Taking the product of these cochain contractions we obtain cochain contractions (still denoted by $L$) of the complexes $(C^{-p,*}, d^p)$ with $0 \le * \le n + 1$, for each $p$ such that $0 \le p \le n$.

**Figure 1.** The augmented Čech-deRham double complex.
Let $C^* = \text{Tot}(C^{*,*})$ denote the total cochain complex associated to the double complex $C^{p,q}$ with $-n \leq p \leq 0, 0 \leq q \leq n$. Thus $C^k = \bigoplus_{i \leq 0} C^{i,k-i}$, and the differential in $C^*$ is given by $D = d^h + d^v$. We write
\[
\omega^{(m)} \in C^{m,k+m} \text{ for the components of } \omega \in C^k.
\]

The cochain contractions of the rows and columns of the augmented double complex imply that we have quasi-isomorphisms
\[
\hat{C}^* (U; \mathbb{R}) \xrightarrow{I} C^* \xrightarrow{S} \Omega^*(M)
\]
given in degree $k$ by
\[
I(\omega) = \left( \int_{\mathbb{R}^n} \omega^{(n-k)}_{\alpha_0, \ldots, \alpha_{n-k}} \right) \quad \text{and} \quad S(\omega) = \sum_{\alpha \in I} \omega^{(0)}_{\alpha}.
\]

We will use the following lemma to lift a cochain $x$ along $S$ or $I$ given we already have a lift of $dx$. In the statement below, the $N$-th column (with the negative of the vertical differential) is to be regarded as an augmentation of the double complex to its left. We leave the proof to the reader.

**Lemma A.4.** Let $(C^{p,q})_{p,q \in \mathbb{Z}}$ be a double cochain complex bounded on the right (i.e. $C^{p,q} = 0$ for $p > N$) with horizontal and vertical differentials $d^h, d^v$. Assume $K: C^{p,*} \to C^{p-1,*}$ satisfy $d^h K + K d^v = \text{id}$.

Let $x \in C^{N,q}$ and suppose that $\alpha_i \in C^{N-i-1,q+i+1}$ for $i \geq 0$ are such that
\[
(1) \quad d^h \alpha_{i+1} + d^v \alpha_i = 0 \text{ for } i \geq 0,
\]
\[
(2) \quad d^h(\alpha_0) = -d^v x.
\]

Then defining
\[
\beta_i = \sum_{j=1}^{i} (-1)^{j-1} (Kd^v)^{j-1} K\alpha_{i-j} + (-1)^i (Kd^v)^i Kx \in C^{N-i-1,q+i} \text{ for } i \geq 0
\]
we have
\[
(1) \quad d^h \beta_{i+1} + d^v \beta_i = \alpha_i \text{ for } i \geq 0,
\]
\[
(2) \quad d^h(\beta_0) = x.
\]

**A.5. Proof of the theorem.** We note that the cochain contractions $K$, $L$ defined in (6) and (7) are continuous when we give $\Omega^k(M)$ the weak Whitney topology, $\hat{C}_p(U; \mathbb{R})$ the product topology and $\hat{C}_p(U; \Omega^k)$ the product topology of the weak (or strong, in fact) Whitney topologies. Moreover the differentials $d^h$ and $d^v$ in the augmented double complex are also continuous with respect to these topologies.

**Proof of Theorem A.7.** Let $\omega \in \Omega^k(M)$ be an exact form. By Lemma A.4 (where we take $x = \omega$ and $\alpha_i = 0$),
\[
(8) \quad \gamma = \sum_{i=0}^{n-k} \gamma^{(i)} = \sum_{i=0}^{n-k} (-1)^i (Kd^v)^i K\omega \in C^k
\]
is a cocycle in $C^*$, such that $S(\gamma) = \omega$. Since $S$ and $I$ are quasi-isomorphisms, we have that $I(\gamma) \in \hat{C}_{n-k}(U; \mathbb{R})$ is a coboundary.

We pick linear maps
\[
T: \hat{C}_{p-1}(U; R) \to \hat{C}_p(U; \mathbb{R})
\]
such that $d^h T d^h = d^h$ and each component of $T(c) \in \check{C}_p(U; \mathbb{R}) = \prod_{j_0 < \cdots < j_p} \mathbb{R}$ depends only on finitely many components of $c \in \prod_{j_0 < \cdots < j_{p-1}} \mathbb{R}$ (so $T$ is continuous for the product topologies). This is possible because the complex $\check{C}_*(U; \mathbb{R})$ is the $\mathbb{R}$-dual of a chain complex

$$\cdots \leftarrow \oplus_{j_0 < \cdots < j_p} \mathbb{R} \xrightarrow{\partial} \oplus_{j_0 < \cdots < j_{p-1}} \mathbb{R} \leftarrow \cdots$$

It suffices to pick any map $t$ such that $\partial t \partial = \partial$ and take $T = t^\vee$.

We can now apply Lemma [A, J] (with the horizontal and vertical axis interchanged) to $x = TI(\gamma)$ and $\alpha_i = (-1)^{n-k-1} \gamma_{(n-k-i)}$. It tells us that

$$\delta = \sum_{i=0}^{n-k+1} \delta^{(i)} = \sum_{i=0}^{n-k+1-i} \left( \sum_{j=1}^{n-k+1-i} (-1)^{j-1} (Ld^h)^{j-1} L \gamma^{(i+j-1)} + (-1)^i (Ld^h)^{n-k+1-i} \right)$$

is a cochain in $C^{k-1}$ with $D\delta = \gamma$. The sought after primitive of $\omega$ is therefore $S(\delta)$ which is explicitly given by the following formula in terms of the differentials in the extended double complex and the contraction operators for rows and columns

$$d^h \left( \sum_{j=1}^{n-k+1} (-1)^{j-1} (Ld^h)^{j-1} \gamma^{(j-1)} + (-1)^{n-k} (Ld^h)^{n-k+1} \right)$$

where the $\gamma^{(i)}$ are given by (8). This completes the proof. \qed

Remark A.6. (i) It is clear from the formula (9) for the anti-derivative that it takes smooth families of exact forms to smooth families of forms.

(ii) Suppose $M$ is a smooth manifold with boundary. Using a collaring, it is easy to obtain a continuous anti-differential on $\Omega^*(M)$ from the one constructed in the proof of Theorem [A, J] on $\Omega^*(\text{int } M)$.

(iii) The proof of Theorem [A, J] also gives a continuous linear inverse for $d$: $\Omega^k_c(M) \rightarrow d(\Omega^k_c(M))$. It suffices to replace the direct products in (8) and (9) by direct sums. The existence of the splittings $T$ is immediate in this case.

(iv) A continuous right inverse for the deRham differential does not in general exist if we consider the strong Whitney topology on spaces of forms (it suffices to consider the case when $M = \mathbb{R}$).

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