2m-Weak amenability of group algebras

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Abstract. A common fixed point property for semigroups is applied to show that the group algebra $L^1(G)$ of a locally compact group $G$ is $2m$-weakly amenable for each integer $m \geq 1$.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $X$ a Banach $\mathcal{A}$-bimodule. A linear mapping $D: \mathcal{A} \to X$ is called a derivation if it satisfies $D(ab) = aD(b) + D(a)b \ (a, b \in \mathcal{A})$. Given any $x \in X$, the mapping $a \mapsto ax - xa \ (a \in \mathcal{A})$ is a continuous derivation, called an inner derivation.

If $X$ is a Banach $\mathcal{A}$-bimodule, then the dual space $X^*$ of $X$ is naturally a Banach $\mathcal{A}$-bimodule with the $\mathcal{A}$-module actions defined by

$$
\langle x, af \rangle = \langle xa, f \rangle \quad \langle x, fa \rangle = \langle ax, f \rangle \quad (a \in \mathcal{A}, f \in X^*, x \in X).
$$

Note that the Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule with the product giving the module actions. So $\mathcal{A}^{(n)}$, the $n$-th dual space of $\mathcal{A}$, is naturally a Banach $\mathcal{A}$-bimodule in the above sense for each $n \in \mathbb{N}$. The Banach algebra $\mathcal{A}$ is called $n$-weakly amenable if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. If $\mathcal{A}$ is $n$-weakly amenable for each $n \in \mathbb{N}$ then it is called permanently weakly amenable.

Let $G$ be a locally compact group. The integral of a function $f$ on a measurable subset $K$ of $G$ against a fixed left Haar measure is denoted by $\int_K f dx$. Two functions on $G$ are regarded identical if they are equal to each other almost everywhere with respect to the left Haar measure. The group algebra $L^1(G)$ is the Banach algebra consisting of all absolutely integrable functions on $G$ (with respect to the left Haar measure), equipped with the convolution product and the usual $L^1$ norm

$$
\|f\|_1 := \int_G |f(t)| dt.
$$

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When $G$ is discrete, $L^1(G)$ is $\ell^1(G)$ consisting of all absolutely summable functions on $G$.

B.E. Johnson showed in [12] that $L^1(G)$ is always 1-weakly amenable for any locally compact group $G$. It was shown further in [5] that $L^1(G)$ is in fact $n$-weakly amenable for all odd numbers $n$. Whether this is still true for even numbers $n$ was left open in [5]. For a free group $G$, Johnson proved later in [13] that $\ell^1(G)$ is indeed $2m$-weakly amenable for any $m \in \mathbb{N}$. The problem has been resolved affirmatively for general locally compact group $G$ in [3] and in [19] independently, using a theory established in [20].

In this note we present a short proof to the $n$-weak amenability of $L^1(G)$ for even numbers $n$. Our proof is based on a common fixed point property for semigroups. In Section 2 we study this fixed point property. For the general theory concerning amenability and fixed point properties of locally compact groups we refer the reader to [8, 21]. The proof to the main result will be given in Section 3.

2. Common fixed points for semigroups

Let $S$ be a (discrete) semigroup. The space of all bounded complex valued functions on $S$ is denoted by $\ell^\infty(S)$. It is a Banach space with the uniform supremum norm. In fact $\ell^\infty(S) = (\ell^1(S))^*$, the dual space of $\ell^1(S)$. For each $s \in S$ and each $f \in \ell^\infty(S)$ let $\ell_s f$ be the left translate of $f$ by $s$, that is $\ell_s f(t) = f(st)$ ($t \in S$) (the right translate $r_s f$ is defined similarly). A function $f \in \ell^\infty(S)$ is called weakly almost periodic if its left orbit $\mathcal{LO}(f) = \{\ell_s f : s \in S\}$ is relatively compact in the weak topology of $\ell^\infty(S)$. The space of all weakly almost periodic functions on $S$ is denoted by $WAP(S)$, which is a closed subspace of $\ell^\infty(S)$ containing the constant function and invariant under the left and right translations. A linear functional $m \in WAP(S)^*$ is a mean on $WAP(S)$ if $\|m\| = m(1) = 1$. A mean $m$ on $WAP(S)$ is a left invariant mean (abbreviated LIM) if $m(\ell_s f) = m(f)$ for all $s \in S$ and all $f \in WAP(S)$. If $S$ is a group, it is well known that $WAP(S)$ always has a LIM [8].

Let $X$ be a Banach space and $C$ a nonempty subset of $X$. A mapping $T: C \to C$ is called nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. When $X$ is a separable locally convex topological space whose topology is determined by a family $Q$ of seminorms on $X$, we will denote it by $(X, Q)$ to highlight the topology $Q$.

Let $C$ be a subset of a locally convex topological vector space $(X, Q)$. We say that $\mathcal{S} = \{T_s : s \in S\}$ is a representation of $S$ on $C$ if for each $s \in S$, $T_s$ is a mapping from $C$ into $C$ and $T_{st}(x) =$
$T_s(T_t x)$ (for $s, t \in S$ and $x \in C$). The representation is called continuous if each $T_s$ ($s \in S$) is $Q$-$Q$ continuous; It is called equicontinuous if for each neighborhood $\mathcal{N}$ of 0 there is a neighborhood $\mathcal{O}$ of 0 such that $T_s(x) - T_s(y) \in \mathcal{N}$ whenever $x, y \in C$, $x - y \in \mathcal{O}$ and $s \in S$. The representation is called affine if $C$ is convex and each $T_s$ ($s \in S$) is an affine mapping, that is $T_s(ax + by) = aT_s(x) + bT_s(y)$ for all constants $a, b \geq 0$ with $a + b = 1$, $s \in S$ and $x, y \in C$. We say that $x \in C$ is a common fixed point for (the representation of) $S$ if $T_s(x) = x$ for all $s \in S$.

The following fixed point theorem was proved in [14].

**Theorem 1.** Let $S$ be a discrete semigroup and $\mathcal{G}$ an equicontinuous affine representation of $S$ on a weakly compact convex subset $C$ of a separated locally convex space $X$. If $WAP(S)$ has a left invariant mean then $C$ contains a common fixed point for $S$.

Let $B$ be a nonempty bounded subset of a Banach space $X$. By definition the Chebyshev radius of $B$ in $X$ is

$$r_B = \inf\{r \geq 0 : \exists x \in X \sup_{b \in B} \|x - b\| \leq r\}.$$  

Clearly we have $0 \leq r_B < \infty$ and

$$\sup_{b \in B} \|x - b\| \geq r_B \quad \text{for each } x \in X.$$  

The *Chebyshev center* of $B$ in $X$ is defined to be

$$C_B = \{x \in X : \sup_{b \in B} \|x - b\| \leq r_B\}.$$  

Chebyshev center has been extensively used in the field of fixed point theory (see [6, 7]). Some asymptotic version of it has been employed to study fixed point properties of semigroups [16, 17, 18].

We now recall that a Banach space $X$ is *L-embedded* if the image of $X$ under the canonical embedding into its bidual $X^{**}$, still denoted by $X$, is an $\ell_1$ summand in $X^{**}$, that is if there is a subspace $X_s$ of $X^{**}$ such that $X^{**} = X \oplus_1 X_s$, where $\oplus_1$ denotes the $\ell_1$ direct sum. The class of L-embedded Banach spaces includes all $L^1(\Sigma, \mu)$ (the space of all absolutely integrable functions on a measure space $(\Sigma, \mu)$), preduals of von Neumann algebras, dual spaces of M-embedded Banach spaces and the Hardy space $H_1$. In particular, given a locally compact group $G$, the space $L^1(G)$ is L-embedded. So are its even duals $L^1(G)^{(2m)}$ ($m \in \mathbb{N}$). We refer to [9] for more details of the theory concerning this type of Banach spaces. We also refer to [11, 2, 10] for the study of fixed points of various mappings in an L-embedded Banach space. In [11], as an application of a fixed point theorem, a surprising short solution to
the well-known derivation problem was given. The problem was first settled by V. Losert in [20].

We now give a common fixed point theorem for semigroups, which will provide the major machinery for our proof to the main result.

**Theorem 2.** Let $S$ be a discrete semigroup and $\mathcal{S}$ a representation of $S$ on an $L$-embedded Banach space $X$ as nonexpansive affine mappings. Suppose that $\text{WAP}(S)$ has a LIM and suppose that there is a nonempty bounded set $B \subset X$ such that $B \subseteq T_s(B)$ for all $s \in S$, then $X$ contains a common fixed point for $S$.

**Proof.** We use the idea of [1] to show that there is a nonempty weakly compact convex set in $X$ that is $S$-invariant. We first regard $B$ as a subset of $X^{**}$. Let $r_B$ be the Chebyshev radius and $C$ the Chebyshev center of $B$ in $X^{**}$. Then $C$ is nonempty, weak* compact and convex. In fact, for each $r > r_B$,

$$C_r := \{ x \in X^{**} : \sup_{b \in B} \| x - b \| \leq r \}$$

is nonempty by the definition of $r_B$. Note that $C_r = \cap_{b \in B} B[b, r]$, where $B[b, r]$ denotes the closed ball in $X^{**}$ centered at $b$ with radius $r$. The set $C_r$ is convex and weak* compact since each $B[b, r]$ is. The collection $\{ C_r : r > r_B \}$ is decreasing as $r$ decreases. Thus $C = \cap_{r > r_B} C_r$ is nonempty and is still weak* compact and convex. By the $L$-embeddedness of $X$ there is a subspace $X_s$ of $X^{**}$ such that $X^{**} = X \oplus_1 X_s$. Let $x \in C$. then there are $c \in X$ and $\xi \in X_s$ such that $x = c + \xi$. For each $b \in B$, $\| x - b \| = \| c - b \| + \| \xi \|$. So

$$r_B \geq \sup_{b \in B} \| x - b \| = \sup_{b \in B} \| c - b \| + \| \xi \| \geq r_B + \| \xi \|.$$  

The last inequality is due to (2.1). Therefore, we must have $\xi = 0$. This shows that $C \subset X$. The weak* compactness of $C$ (in $X^{**}$) is the same as the weak compactness of it (in $X$). So $C$ is a nonempty, weakly compact and convex subset of $X$.

Now for $s \in S$, $b \in B$ and $x \in C$ we have

$$\| T_s(x) - T_s(b) \| \leq \| x - b \| \leq r_B$$

since $T_s$ is nonexpansive. This implies that $\| T_s(x) - a \| \leq r_B$ for $a \in \overline{T_s(B)}$ ($s \in S$, $x \in C$). In particular, this holds for all $a \in B$ since $B \subseteq \overline{T_s(B)}$. Thus $T_s(x) \in C$ whenever $x \in C$ and $s \in S$, showing that $C$ is $S$-invariant. Note that a nonexpansive representation of $S$ is indeed equicontinuous. By Theorem 1 there is a common fixed point for $S$ in $C$. The proof is complete. \[\square\]
Theorem [1] has been extended to the general semitopological semigroup setting in [15]. A more general version of Theorem [2] and some discussion on when there is a set $B$ such that $T_s(B) = B$ for all $s \in S$ can also be found there.

3. 2m-weak amenability of $L^1(G)$

Let $X$ be a Banach space. Denote the space of all bounded linear operators on $X$ by $B(X)$. The space $B(X)$ is a Banach algebra with the operator norm topology and the composition product. So is $B(X) \times B(X)^{op}$ with the product topology and coordinatewise operations, where $B(X)^{op}$ is the algebra formed by reversing the order of the product in $B(X)$. The strong operator topology (or briefly so-topology) on $B(X) \times B(X)^{op}$ is the topology induced by the family of seminorms $\{p_x : x \in X\}$, where

$$p_x(S, T) = \max\{\|S(x)\|, \|T(x)\|\} \quad (S, T \in B(X))$$

(see [4] page 327).

Given a locally compact group $G$, let $M(G)$ be the space of all bounded complex valued regular Borel measures on $G$. With the convolution product of measures and with the norm induced by the total variation, $M(G)$ is a Banach algebra containing $L^1(G)$ as a closed ideal. In fact, $M(G)$ is the multiplier algebra of $L^1(G)$, and as the multiplier algebra of $L^1(G)$, $M(G)$ is a subalgebra of $B(L(G)) \times B(L(G))^{op}$ with each $\mu \in M(G)$ being identified with (the double multiplier) $(\ell_\mu, r_\mu) \in B(L(G)) \times B(L(G))^{op}$, where $\ell_\mu$ and $r_\mu$ denote, respectively, the left multiplier operator and the right multiplier operator on $L^1(G)$ implemented by $\mu$. We refer to [4] for the standard theory about multipliers and multiplier algebras.

It is well-known that $lin\{\delta_t : t \in G\}$, the linear space generated by the point measures $\delta_t$ ($t \in G$), is dense in $M(G)$ in the so-topology [4] Proposition 3.3.41(i)]. In particular, for each $h \in L^1(G)$ there is a net $(u_a) \subset lin\{\delta_t : t \in G\}$ such that $\|(u_a - h) \ast a\|_1 \to 0$ and $\|a \ast (u_a - h)\|_1 \to 0$ for all $a \in L^1(G)$.

Recall that if $\mathcal{A}$ is a Banach algebra, then its bidual $\mathcal{A}^{**}$ is a Banach algebra equipped with the Arens product $\Box$ defined

$$\langle f, u \Box v \rangle = \langle v \cdot f, u \rangle, \quad v \cdot f \in \mathcal{A}^* : \langle a, v \cdot f \rangle = \langle fa, v \rangle$$

for $u, v \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. If $X$ is a Banach $\mathcal{A}$-bimodule, then its bidual $X^{**}$ is naturally a Banach $\mathcal{A}^{**}$-bimodule with the module actions given by

$$\langle F, u \cdot M \rangle = \langle M \cdot F, u \rangle, \quad M \cdot F \in A^* : \langle a, M \cdot F \rangle = \langle F \cdot a, M \rangle$$
and
\[ \langle F, M \cdot u \rangle = \langle u \cdot F, M \rangle, \quad u \cdot F \in X^* : \langle x, u \cdot F \rangle = \langle F \cdot x, u \rangle, \]
\[ F \cdot x \in A^* : \langle a, F \cdot x \rangle = \langle x \cdot a, F \rangle \]
for \( u \in A^{**}, M \in X^{**}, F \in X^*, x \in X \) and \( a \in A \). In particular, for any integer \( m \in \mathbb{N} \), \( A^{(2m)} \) is a Banach \( A^{**} \)-bimodule.

A Banach \( A \)-bimodule \( X \) is called \textit{neo-unital} if \( X = AXA \), that is every element \( x \in X \) may be written in the form \( x = ayb \) for some \( a, b \in A \) and \( y \in X \). If \( A \) has a bounded approximate identity \( (e_\alpha) \) and \( X \) is a neo-unital Banach \( A \)-bimodule, then we may extend the \( A \) bimodule actions on \( X \) to \( M(A) \), the multiplier algebra of \( A \). The extension is defined as follows.

\[ \mu x = \lim_\alpha (\mu e_\alpha)x = (\mu a)yb, \quad x\mu = \lim_\alpha x(e_\alpha\mu) = ay(b\mu) \]

for \( \mu \in M(A) \) and \( x = ayb \in X \). Here we note that \( \mu a, b\mu \in A \) since \( A \) is (always) an ideal of \( M(A) \). These operations make \( X \) a unital Banach \( M(A) \)-bimodule. In this case a continuous derivation \( D: A \to X^* \) may be extended to a continuous derivation from \( M(A) \) to \( X^* \) by defining

\[ D(\mu) = \text{wk}^* \lim_\alpha D(\mu e_\alpha) \quad (\mu \in M(A)). \]

Moreover this extended \( D \) is \textit{so}-weak* continuous. In fact, if \( \mu_\alpha \to \mu \) in \( M(A) \) in the \textit{so}-topology and \( x = ayb \in X \) for \( a, b \in A \) and \( y \in X \), then

\[ \lim_\alpha \langle x, D(\mu_\alpha) \rangle = \lim_\alpha \langle ay, D(b\mu_\alpha) \rangle - \lim_\alpha \langle \mu_\alpha ay, D(b) \rangle = \langle ay, D(b\mu) \rangle - \langle \mu ay, D(b) \rangle = \langle x, D(\mu) \rangle. \]

We refer to the seminar paper [11] and the monograph [4] for more details of the above extensions.

We now can prove the main result of the paper.

**Theorem 3.** Let \( G \) be a locally compact group. Then the group algebra \( L^1(G) \) is \( 2m \)-weakly amenable for each \( m \in \mathbb{N} \).

**Proof.** Denote \( A = L^1(G), X = A^{(2m)} \) and \( Y = A^{(2m-1)} \). Then, as we have indicated, \( X \) is a Banach \( A^{**} \)-bimodule. Let \( (e_\alpha) \) be a bounded approximate identity of \( A \) and let \( E \) be a weak* cluster point of \( (e_\alpha) \) in \( A^{**} \). Then \( Ea = aE = a \) for all \( a \in A \). We have the \( A \)-bimodule decomposition \( X = X_1 \oplus X_2 \oplus X_3 \), where

\[ X_1 = \ell_E \circ r_E(X), \quad X_2 = (I - r_E)(X), \quad X_3 = (I - \ell_E) \circ r_E(X). \]
Here $I$ denotes the identity operator, $\ell_E$ is the left multiplication by $E$ and $r_E$ the right multiplication by $E$. It is readily seen that

\[ X_2 = (AY)^\perp \cong (Y/AY)^*, \quad X_1 \oplus X_3 = r_E(X) \cong (AY)^* \]

as Banach $\mathcal{A}$-bimodules. Similarly, in $(AY)^*$

\[ (I - \ell_E)((AY)^*) = (AY\mathcal{A})^\perp \cong (AY/AY\mathcal{A})^* \]

and

\[ \ell_E((AY)^*) \cong (AY\mathcal{A})^*. \]

as Banach $\mathcal{A}$-bimodules. We have

\[ X_3 \cong (AY/AY\mathcal{A})^* \quad \text{and} \quad X_1 \cong (AY\mathcal{A})^*. \]

Let $D: \mathcal{A} \to X$ be a continuous derivation. Then $D = D_1 + D_2 + D_3$, where

\[ D_1 = \ell_E \circ r_E \circ D : \mathcal{A} \to X_1, \quad D_2 = (I - r_E) \circ D : \mathcal{A} \to X_2, \]
\[ D_3 = (I - \ell_E) \circ r_E \circ D : \mathcal{A} \to X_3. \]

Since $\ell_E$ and $r_E$ are $\mathcal{A}$-bimodule morphisms, $D_1$, $D_2$ and $D_3$ are continuous derivations. Note that the left $\mathcal{A}$-module action on $Y/AY$ and the right $\mathcal{A}$-module action on $A/AY\mathcal{A}$ are trivial. From [11, Proposition 1.5], $D_2$ and $D_3$ are inner. We now show that $D_1$ is also inner. Then $D$ must be inner.

Since $AY\mathcal{A}$ is neo-unital, we may extend $D_1$ to a continuous derivation from $M(G)$, the multiplier of $\mathcal{A}$, to $X_1$. So we may consider $\Delta: G \to X_1 \subset X$ defined by

\[ \Delta(t) = D_1(\delta_t) \cdot \delta_{t^{-1}} \quad (t \in G). \]

It is readily seen that

\[ \Delta(ts) = \delta_t \cdot \Delta(s) \cdot \delta_{t^{-1}} + \Delta(t) \quad (t, s \in G). \]

Let $B = \Delta(G)$. Then $B$ is a nonempty bounded subset of $X$. For each $t \in G$, let $T_t$ be the self mapping on $X$ defined by

\[ T_t(x) = \delta_t \cdot x \cdot \delta_{t^{-1}} + \Delta(t) \quad (x \in X). \]

Using (3.1) one may check that $\mathcal{G} = \{T_t : t \in G\}$ defines a representation of $G$ on $X$ which is clearly nonexpansive and affine. Moreover, $T_t(\Delta(s)) = \Delta(ts)$ $(t, s \in G)$ and $T_e = I$. Since $G$ is a group, the above implies $T_t(B) = B$ for each $t \in G$. Here $G$ is regarded as a discrete group.

Since $WAP(G)$ has a LIM and $X$ is L-embedded, by Theorem [2] there is $\xi \in X$ such that

\[ \delta_t \cdot \xi \cdot \delta_{t^{-1}} + \Delta(t) = \xi \quad \text{for all} \ t \in G. \]
So $D_1(\delta_t) = \xi \cdot \delta_t - \delta_t \cdot \xi = \text{ad}_\xi(\delta_t)$ \hspace{1em} (t \in G). \hspace{1em} Let x = \ell_E \circ r_E(-\xi). \hspace{1em} Then x \in X_1. \hspace{1em} Also D_1(\delta_t) \in X_1. \hspace{1em} For any ayb \in AY.A \hspace{1em} with a,b \in A \hspace{1em} and \hspace{1em} y \in Y, \hspace{1em} we \hspace{1em} have

\begin{align*}
\langle ayb, D_1(\delta_t) \rangle &= \langle ayb \cdot \delta_t - \delta_t \cdot ayb, -\xi \rangle = \langle E(ayb \cdot \delta_t - \delta_t \cdot ayb)E, -\xi \rangle \\
&= \langle ayb \cdot \delta_t - \delta_t \cdot ayb, x \rangle = \langle ayb, \text{ad}_x(\delta_t) \rangle \hspace{1em} t \in G.
\end{align*}

So it is true that $D_1(\delta_t) = \text{ad}_x(\delta_t)$ for all $t \in G$. \hspace{1em} From what we have shown before stating the current theorem, both $D_1$ and $\text{ad}_x$, as continuous derivations from $M(G)$ into the dual of a neo-unital $A$-bimodule, are so-weak* continuous. \hspace{1em} Since $\text{lin}(\delta_t : t \in G)$ is dense in $M(G)$ in the so-topology, we finally have

$$D_1(f) = \text{ad}_x(f) \hspace{1em} (f \in A = L^1(G)),$$

therefore $D_1$ is inner. \hspace{1em} The proof is complete. \hfill \Box

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