Claw Finding Algorithms Using Quantum Walk

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Abstract

The claw finding problem has been studied in terms of query complexity as one of the problems closely connected to cryptography. For given two functions, \( f \) and \( g \), as an oracle which have domains of size \( N \) and \( M (N \leq M) \), respectively, and the same range, the goal of the problem is to find \( x \) and \( y \) such that \( f(x) = g(y) \). This problem has been considered in both quantum and classical settings in terms of query complexity. This paper describes an optimal algorithm using quantum walk that solves this problem. Our algorithm can be slightly modified to solve a more general problem of finding a tuple consisting of elements in the two function domains that has prespecified property. Our algorithm can also be generalized to find a claw of \( k \) functions for any constant integer \( k > 1 \), where the domains of the functions may have different size. Keywords: quantum computing, query complexity, oracle computation

1 Introduction

The most significant discovery in quantum computation would be Shor’s polynomial-time quantum algorithms for factoring integers and computing discrete logarithms \[15\], both of which are believed to be hard to solve in classical settings and are thus used in arguments for the security of the widely used cryptosystems. Another significant discovery is Grover’s quantum algorithm for the problem of searching an unstructured set \[11\], i.e, the problem of searching for \( i \in \{0, 1, \ldots, N - 1\} \) such that \( f(i) = 1 \) for a hidden Boolean function \( f \); it has yielded a variety of generalizations \[4, 12, 2, 16, 13\]. Grover’s algorithm and its generalizations assume the oracle computation model, in which a problem instance is given as a black box (called an oracle) and any algorithm needs to make queries to the black box to get sufficient information on the instance. In the case of searching an unstructured set, any algorithm needs to make queries of the form “what is the value of function \( f \) for input \( i \)” to the given oracle. In the oracle computation model, the efficiency of an algorithm is usually measured by the number of queries the algorithm needs to make, i.e., the query complexity of the algorithm. The query complexity of a problem means the query complexity of the algorithm that solves the problem with fewest queries.

One of the earliest applications of Grover’s algorithm was the bounded-error algorithm of Brassard, Høyer and Tapp \[5\]; it addressed the collision problem in a cryptographic context, i.e., finding pair \( (x, y) \) such that \( f(x) = f(y) \), in a given 2-to-1 function \( f \) of domain size \( N \). Their quantum algorithm requires \( O(N^{1/3}) \) queries, whereas any bounded-error classical algorithm needs \( \Theta(N^{1/2}) \) queries. Subsequently, Aaronson and Shi \[1\] proved the matching lower bound. Brassard et al. \[5\] considered two more related problems: the element distinctness problem and the claw finding problem. These problems are also important in a cryptographic sense. Furthermore, studying these problems has deepened our understanding of the power of quantum computation.

The element distinctness problem is to decide whether or not \( N \) integers given as an oracle are all distinct. Buhrman et al. \[8\] gave a bounded-error algorithm for the problem, which makes \( O(N^{3/4}) \) queries (strictly

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speaking, they assumed a comparison oracle, which returns just the result of comparing function values for two specified inputs, and, in this case, the query complexity is $O(N^{5/4} \log N)$. Subsequently, Ambainis [2] gave an improved upper bound $O(N^{2/3})$ by introducing a new framework of quantum walk (his quantum walk algorithm was reviewed from a slightly more general point of view in [14, 10], and a much more general framework was given by Szegedy [16]). This upper bound matches the lower bound proved by Aaronson and Shi [1].

The claw finding problem is defined as follows. Given two functions $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ as an oracle, decide whether or not there exists at least one pair $(x, y) \in X \times Y$, called a claw, such that $f(x) = g(y)$, and find a claw if it exists, where $X$ and $Y$ are domains of size $N$ and $M$ ($N \leq M$), respectively. By claw finding $(N, M)$, we mean this problem.

After Brassard et al. [5] considered a special case of the claw finding problem, Buhrman et al. [7] gave a quantum algorithm that requires $O(N^{1/2}M^{1/4})$ queries for $N \leq M < N^2$ and $O(M^{1/2})$ queries for $M \geq N^2$ (strictly speaking, they assumed a comparison oracle, which returns just the result of comparing function values for two inputs). They also proved that any algorithm requires $\Omega(M^{1/2})$ queries by reducing the search problem over an unstructured set to the claw finding problem. Thus, while their bounds of the query complexity are tight when $M \geq N^2$, there is still a big gap when $N \leq M < N^2$. Furthermore, they considered the case of $k$ functions, i.e., the $k$-claw finding problem defined as follows: given $k$ functions $f_i : X_i := \{1, \ldots, N_i\} \rightarrow Z$ ($i \in \{1, \ldots, k\}$) as an oracle, where $k > 1$ is any constant integer, and $N_i \leq N_j$ if $i < j$, decide whether or not there exists at least one $k$-claw, i.e., a tuple $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that $f_i(x_i) = f_j(x_j)$ for any $i, j \in \{1, \ldots, k\}$, and find a $k$-claw if it exists. A generalization of their algorithm works well for the $k$-claw finding problem; its query complexity is $O(N^{1-1/2^k})$ if $N_i = N$ for all $i \in \{1, \ldots, k\}$. It is shown in [14] that the quantum-walk algorithm in [2] for the element distinctness problem is general enough to be applied with slight modification to the $k$-claw finding problem; this yields query complexity $O((\sum_{i=1}^k N_i)^{1/2^k})$ if the promise is assumed that there is at most one solution, and, with random reduction, query complexity $\tilde{O}((\sum_{i=1}^k N_i)^{1/2^k})$ for the problem without the single-solution promise. Zhang [17] generalized the quantum-walk algorithm in [2] to solve the claw finding problem with the single-solution promise by making $O((NM)^{1/3})$ queries for $N \leq M < N^2$ and $O(M^{1/2})$ for $M \geq N^2$. This upper bound is optimal, since the matching lower bound $\Omega((NM)^{1/3})$ was proved in the paper by reducing the collision problem to the claw finding problem. Zhang also showed that the algorithm can be generalized to solve a more general problem of finding a tuple consisting of elements in the domains of given $k$ functions with the single-solution promise. To solve the problems without the promise, we usually use a randomly reduction to the problem with the single-solution promise, which is known to increase the query complexity by at most a log factor as pointed out in [14] (if the problem has certain robust properties, there is a random reduction that increases the query complexity by a constant multiplicative factor, e.g., [2]).

This paper gives an optimal quantum algorithm that directly (i.e., without using such a random reduction) solves the claw finding problem without the single-solution promise. The query complexity of our algorithm is as follows:

$$Q(\text{claw finding} (N, M)) = \begin{cases} O((NM)^{1/3}) & (N \leq M < N^2) \\ O(M^{1/2}) & (M \geq N^2), \end{cases}$$

where $Q(P)$ means the number of queries required to solve problem $P$ with one-sided bounded error (i.e., with the one-sided error probability bounded by a certain constant, say, 1/3). The optimality is guaranteed by the lower bounds given in [7, 17]. Our algorithm can be modified to solve a more general problem of finding a tuple $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in X^p \times Y^q$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for any $i \neq j$, and $(f(x_1), \ldots, f(x_p), g(y_1), \ldots, g(y_q)) \in R$, for given $R \subseteq Z^{p+q}$, where $p$ and $q$ are positive constant integers. We call this problem $(p, q)$-subset finding problem and denote it by $(p, q) - \text{subset finding} (N, M))$. Thus, claw finding $(N, M)$ is a special case of $(p, q) - \text{subset finding} (N, M))$ with $p = q = 1$ and equality relation $R$. The query complexity is

$$Q((p, q) - \text{subset finding} (N, M)) = \begin{cases} O((NM^q)^{1/(p+q+1)}) & (N \leq M < N^{1+1/q}) \\ O(M^{q/(1+q)}) & (M \geq N^{1+1/q}), \end{cases}$$
Our claw finding algorithm first finds subsets \( \tilde{X} \subseteq X \) and \( \tilde{Y} \subseteq Y \) of size \( O(1) \) such that there is a claw in \( \tilde{X} \times \tilde{Y} \), by using binary and 4-ary searches over \( X \) and \( Y \); in order to decide which branch we should proceed at each visited node in the search trees, we use a subroutine that \textit{decides}, with one-sided bounded error, whether or not there exists a claw of two functions \( f \) and \( g \). The algorithm then searches \( \tilde{X} \times \tilde{Y} \) for a claw by making classical queries. If we na"ively repeated the bounded-error subroutine \( O(\log M) \) times at each visited node to guarantee bounded error as a whole, a “\( \log \)” factor would be multiplied to the total query complexity. Instead, at the node of depth \( s \) in the search trees, we repeat the subroutine \( O(s) \) times to amplify success probability. This achieves bounded error as a whole, while pushing up the query complexity by just a constant multiplicative factor. This binary search technique can be used to solve other problems such as the search version of the element distinctness problem, with the quantum walk algorithm for the problems in [16]. (Høyer et al. [12] introduced an error reduction technique with a similar flavor; however, their technique is used in an algorithmic context different from ours: their error reduction is performed at each recursion level while ours is sequentially used at each step of the search tree.)

The subroutine is developed around the Szegedy’s quantum walk framework [16] over a Markov chain on the graph categorical product of two Johnson graphs, which correspond to the two functions (with an idea similar to the one used in [9]). The \textit{Johnson graph} \( J(n,k) \) is a connected regular graph with \( \binom{n}{k} \) vertices such that every vertex is a subset of size \( k \) of \( [n] \); two vertices are adjacent if and only if the symmetric difference of their corresponding subsets has size 2. For two functions \( f \) and \( g \) with domains \( X \) and \( Y \) such that \( |X| \leq |Y| \), the subroutine applies Szegedy’s quantum walk to the graph categorical product of two Johnson graphs \( J_f = J(|X|,(|X||Y|)^{1/3}) \) and \( J_g = J(|Y|,(|X||Y|)^{1/3}) \) if \( |Y| \leq |X|^2 \), and \( J_f = J(|X|,|X|) \) and \( J_g = J(|Y|,|X|) \) otherwise.

Our algorithm can be generalized to the \( k \)-claw finding problem. For the \( k \)-claw finding problem \( \text{k-claw}_{\text{finding}}(N_1, \ldots, N_k) \) against the \( k \) functions with domain sizes \( N_i \) \( (i = 1, \ldots, k) \), respectively,

\[
Q(\text{k-claw}_{\text{finding}}(N_1, \ldots, N_k)) = \begin{cases} O\left(\left(\prod_{i=1}^{k} N_i\right)^{1/3}\right) & \text{if } \prod_{i=2}^{k} N_i = O(N_1^k), \\ O\left(\sqrt{\prod_{i=2}^{k} N_i/N_1^{k-2}}\right) & \text{otherwise}. \end{cases}
\]

Our algorithms can work with slight modification even against a comparison oracle (i.e., against an oracle that, for a given pair of inputs \((x_i, x_j) \in X_i \times X_j\), only decides which is the larger of two function values \( f_i(x_i) \) and \( f_j(x_j) \)); the query complexity increases by a multiplicative factor of \( \log N_1 \) for the \( k \)-function case (\( \log N \) for the two-function case).

### Related works

Recently, Magniez et al. [13] developed a new quantum walk over a Markov chain. One of the advantages of their quantum walk over Szegedy’s quantum walk is that their quantum walk can \textit{find} a marked vertex if there is at least one marked vertex, which would simplify our algorithm. Interestingly, our algorithm shows Szegedy’s quantum walk together with carefully adjusted binary search can find a solution in some interesting problems such as the claw finding problem and the element distinctness problem with the same order of query complexity.

## 2 Preliminaries

This section defines problems and introduces some useful techniques. We denote the set of positive integers by \( \mathbb{Z}^* \), the set of \( \{i \mid j \leq i \leq k \text{ for } i, j, k \in \mathbb{Z}^* \} \) by \( [j,k] \), and \([1,k]\) by \([k]\) for short.

**Problem 1 (Claw Finding Problem)** Given two functions \( f : X := [N] \rightarrow Z \) and \( g : Y := [M] \rightarrow Z \) as an oracle for \( N \leq M \), where \( Z = |[Z]| \), find a pair \((x,y) \in X \times Y\) such that \( f(x) = g(y) \) if such a pair exists.
Actually, $Z$ is allowed to be any totally ordered set, but we adopt the above definition for simplicity.

In a quantum setting, the two functions are given as quantum oracle $O_{f,g}$ which is defined as $O_{f,g} : |p,z,w⟩ \rightarrow |p,z \oplus P(p) \pmod{|Z|}, w⟩$, where $p \in X \cup Y$, $z \in Z$, $w$ is work space, $P(p)$ is defined as $f(p)$ if $p \in X$ and $g(p)$ if $p \in Y$ (note that it easy to know whether $p$ is in $X$ or $Y$ by using one more bit to represent $p$). This kind of oracle, which returns the value of the function(s), is called a standard oracle.

Another type of oracle is called the comparison oracle, which, for given two inputs, only decides which is the larger of the two function values corresponding to the inputs. More formally, comparison oracle $O_{f,g}$ is defined as $O_{f,g} : |p,q,b,w⟩ \rightarrow |p,q,b \oplus |P(p) \leq Q(q)|, w⟩$, where $p,q \in X \cup Y$, $b \in \{0,1\}$, $w$ and $P$ are defined as in the standard oracle, $Q$ is defined in the same way as $P$, and $|P(p) \leq Q(q)|$ is the predicate such that its value is 1 if and only if $P(p) \leq Q(q)$.

It is obvious that, if we are given a standard oracle, we can realize a comparison oracle by issuing $O(1)$ queries to the standard oracle. Thus, upper bounds for a comparison oracle are those for a standard oracle, and lower bounds for a standard oracle are those for a comparison oracle, if we ignore constant multiplicative factors.

Buhrman et al. [7] generalized the claw finding problem to a $k$-function case.

**Problem 2 (k-Claw Finding Problem)** Given $k$ functions $f_i : X_i := [N_i] \rightarrow Z$ ($i \in [k]$) as an oracle, where $N_i \leq N_j$ if $i < j$, and $Z := [|Z|]$, find a $k$-claw, i.e., a $k$-tuple $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that $f_i(x_i) = f_j(x_j)$ for any $i,j \in [k]$, if it exists.

Standard and comparison oracles are defined in almost the same way as in the two-function case, except that inputs $p$ and $q$ belong to one of $X_i$’s, respectively, for $i \in [k]$.

The next theorem describes Szegedy’s framework, which we use to prove our upper bounds.

**Theorem 1 [16]** Let $M$ be a symmetric Markov chain with state set $V$ and transition matrix $P$ and let $\delta_M$ be the spectral gap of $P$, i.e., $1 - \max_{\lambda_i} |\lambda_i|$ for the eigenvalues $\lambda_i$’s of $P$. For a certain subset $V' \subseteq V$ with the promise that $|V'|$ is either 0 or at least $\varepsilon|V|$ for $0 < \varepsilon < 1$, any element in $V'$ is marked. For $T = O(1/\sqrt{\varepsilon \delta_M})$, the next quantum algorithm decides whether $|V'|$ is 0 (“false”) or at least $\varepsilon|V|$ (“true”) with one-sided bounded error with cost $O(C_U + (C_F + C_W)/\sqrt{\delta_M \varepsilon})$, where $C = \sum_i |c_i⟩⟨c_i|$ for $|c_i⟩ = \sum_j |P_{i,j}|0⟩|j⟩$ and $R = \sum_j |r_j⟩⟨r_j|$ for $|r_j⟩ = \sum_i |P_{j,i}|0⟩|i⟩$:

1. Prepare $|0⟩$ in a one-qubit register $R_0$, and prepare a uniform superposition $|φ_0⟩ := \frac{1}{\sqrt{|V|}} \sum_{i,j \in V, P_{i,j} \neq 0} |i⟩|j⟩$ in a register $R_1$ with cost at most $C_U$, where $r$ is the number of adjacent states (of any state) in $M$.
2. Apply the Hadamard operator to $R_0$.
3. For randomly and uniformly chosen $1 \leq t \leq T$, apply the next operation $W$ $t$ times to $R_1$ if the content of $R_0$ is “1.”
   3.1 To any $|i⟩|j⟩$, perform the next steps: (i) Check if $i \in V'$ with cost at most $C_F$, (ii) If $i \notin V'$, apply diffusion operator $2C - I$ with cost at most $C_W$.
   3.2 To any $|i⟩|j⟩$, perform the next steps: (i) Check if $j \in V'$ with cost at most $C_F$, (ii) If $j \notin V'$, apply diffusion operator $2R - I$ with cost at most $C_W$.
4. Apply the Hadamard operator to $R_0$, and measure registers $R_0$ and $R_1$ with respect to the computational basis.
5. If the result of measuring $R_0$ is 1 or a marked element is found by measuring $R_1$, output “true”; otherwise output “false.”

### 3 Claw Detection

In this section, we describe “claw-detection” algorithms that detect the existence of a claw. The claw-detection algorithms will be used as subroutines in the “claw-search” algorithms presented in the next section that find a
claw.

Before presenting the claw-detection algorithm, we introduce some notions. The Johnson graph $J(n, k)$ is a connected regular graph with $\binom{n}{k}$ vertices such that every vertex is a subset of size $k$ of $[n]$; two vertices are adjacent if and only if the symmetric difference of their corresponding subsets has size 2. The graph categorical product $G = (V_G, E_G)$ of two graphs $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$, denoted by $G = G_1 \times G_2$, is a graph having vertex set $V_G = V_{G_1} \times V_{G_2}$ such that $((v_1, v_2), (v'_1, v'_2)) \in E_G$ if and only if $(v_1, v'_1) \in E_{G_1}$ and $(v_2, v'_2) \in E_{G_2}$.

The next two propositions are useful in analyzing the claw-detection algorithms we will describe.

**Proposition 2** For Markov chains $M, M_1, \ldots, M_k$, the spectral gap $\delta$ of $M$ is the minimum of those $\delta_1, \ldots, \delta_k$ of $M_1, \ldots, M_k$, i.e., $\delta = \min_i [\delta_i]$ if the underlying graph of $M$ is the graph categorical product of those of $M_1, \ldots, M_k$.

The eigenvalues of the Markov chain on $J(n, k)$ are $\frac{(k-j)(n-k-j) - j}{k(n-k)}$ for $j \in [0, k]$ [6] pages 255–256, from which the next proposition follows.

**Proposition 3** The Markov chain on Johnson graph $J(n, k)$ has spectral gap $\delta = \Omega(\frac{1}{k})$, if $2 \leq k \leq n/2$.

We will first describe a claw-detection algorithm against a comparison oracle, from which we can almost trivially obtain a claw-detection algorithm against a standard oracle. Let Claw_Detect denote the algorithm. To construct Claw_Detect, we apply Theorem II on the graph categorical product of two Johnson graphs $J_f = J(\lfloor X \rfloor, l)$ and $J_g = J(\lceil Y \rceil, m)$ for the domains $X$ and $Y$ of functions $f$ and $g$, respectively, where $l$ and $m$ ($l \leq m$) are integers fixed later.

More precisely, let $F$ and $G$ be any vertices of $J_f$ and $J_g$, respectively, i.e., any $l$-element subset and $m$-element subset of $X$ and $Y$, respectively. Then $(F, G)$ is a vertex in $J_f \times J_g$. Similarly, for any edges $(F, F')$ and $(G, G')$ of $J_f$ and $J_g$, respectively, $((F, G), (F', G'))$ is an edge connecting two vertices $(F, G)$ and $(F', G')$ in $J_f \times J_g$. We next define “marked vertices” as follows. Vertex $(F, G)$ is marked if there is a pair of $(x, y) \in F \times G$ such that $f(x) = g(y)$. To check if $(F, G)$ is marked or not, we just sort all elements in $F \cup G$ on their function values. Although we have to sort all elements in the initial vertex, we have only to change a small part of the sorted list we have already had when moving to an adjacent vertex. For every vertex $(F, G)$, we maintain a representation $L_{F,G}$ of the sorted list of all elements in $F \cup G$ on their function values, and we identify $(F, G, L_{F,G})$ as a vertex of $J_f \times J_g$. Here, we want to guarantee that $L_{F,G}$ is uniquely determined for any pair $(F, G)$ in order to avoid undesirable quantum interference; we have just to introduce some appropriate rules that break ties, i.e., the situation where there are multiple elements in $F \cup G$ that have the same function value.

As the state $|\phi_0\rangle$ in Theorem II we prepare

$$|\phi_0\rangle = \frac{1}{\sqrt{\binom{N}{l}} \binom{M}{m} \binom{(N-l)m}{M-m} \prod_{(F',G') \neq (F,G) \text{ s.t. } f(F') = f(F), g(G') = g(G) \neq m} |F, G, L_{F,G}, F', G', L_{F',G'}\rangle,$$

in register $R_1$. The number $1 \leq t \leq \frac{1}{\sqrt{\delta \epsilon}}$ of repeating $W$ is chosen randomly and uniformly for some constant $c$, $\delta := \Omega(1/m)$ and $\epsilon := lm/(NM)$.

We next describe the implementation of operation $W$. Since diffusion operator $2C - I$ depends on $L_{F,G}$’s, it cannot be performed without queries to the oracle. We thus divide operator $2C - I$ into a few steps. For every unmarked vertex $(F, G, L_{F,G})$, we first transform $|F, G, L_{F,G}, F', G', L_{F',G'}\rangle$ into $|F, G, L_{F,G}, F', G', L_{F,G}\rangle$ with queries to the oracle. We then perform a diffusion operator on the registers where the contents “$F$, $G$” and “$F'$, $G'$” are stored, to obtain a superposition of $|F, G, L_{F,G}, F'', G'', L_{F,G}\rangle$ over all $(F'', G'')$ adjacent to $(F, G)$. Finally, we transform $|F, G, L_{F,G}, F'', G'', L_{F,G}\rangle$ into $|F, G, L_{F,G}, F'', G', L_{F,G}\rangle$. Operator $2R - I$ can be implemented in a similar way.
Lemma 4 Let $Q_2(\text{claw}_{\text{detect}}(N, M))$ be the number of queries needed to decide whether there is a claw or not for functions $f : X := [N] \to Z$ and $g : Y := [M] \to Z$ given as a comparison oracle. Then,

$$Q_2(\text{claw}_{\text{detect}}(N, M)) = \begin{cases} O((NM)^{1/3} \log N) & (N \leq M < N^2) \\ O(M^{1/2} \log N) & (M \geq N^2). \end{cases}$$

Proof We will estimate $C_U$, $C_F$ and $C_W$ for ClawDetect, and then apply Theorem 1.

To generate $|\phi_0\rangle$, we first prepare the uniform superposition of $|F, G\rangle|F', G'\rangle$ over all $F, F', G, G'$ such that $(F, F')$ and $(G, G')$ are edges of $J_f$ and $J_g$, respectively. Obviously, this requires no queries. We then compute $L_{FG}$ and $L_{F', G'}$ for each basis state by issuing $O((l + m) \log(l + m))$ queries to oracle $O_{fg}$. Thus, $C_U = O((l + m) \log(l + m))$.

We can check if there is a pair of $(x, y) \in F \times G$ such that $f(x) = g(y)$ by looking through $L_{FG}$ (without any queries). Thus, $C_F = 0$.

For every unmarked $(F, G, L_{FG})$, step (a).ii of operation $W$ transforms $|F, G, L_{FG}\rangle|F', G', L_{F', G'}\rangle$ into a superposition over all $|F, G, L_{FG}\rangle|F'', G'', L_{F'', G''}\rangle$ such that $|F \triangle F''| = |G \triangle G''| = 2$. This is realized by insertion and deletion of $O(1)$ elements to/from the sorted list of $O(l + m)$ elements, and diffusion operators without queries. Each insertion or deletion can be performed with $O(\log(l + m))$ queries by using binary search. Similarly, step (b).ii of operation $W$ needs $O(\log(l + m))$ queries (without any queries). Thus, we have $C_W = O(\log(l + m))$.

We set $\epsilon = \frac{1}{N^2} \times \frac{m}{\sqrt{m}}$, since the probability that a state is marked is minimized when only one claw exists for $f$ and $g$, in which case the probability is $\frac{1}{N^2} \times \frac{m}{\sqrt{m}}$. Since, from Proposition 3 the spectral gaps of the Markov chains on $J(N, l)$ and $J(M, m)$ are $\Omega(\frac{1}{N})$ and $\Omega(\frac{1}{\sqrt{m}})$, respectively, the spectral gap of the Markov chain on $J(N, l) \times J(M, m)$ is $\Omega(\min\{\frac{1}{N^2}, \frac{1}{\sqrt{m}}\})$ due to $l \leq m$ and Proposition 2.

From Theorem 1 the total number of queries is $Q_2(\text{claw}_{\text{detect}}(N, M)) = O((l + m) \log(l + m) + \log(l + m) \sqrt{mNM/(lm)}) = O((l + m) \log(l + m) + \sqrt{NM/l\log(l + m)}).

When $N \leq M < N^2$, we set $l = m = \Theta((NM)^{1/3})$, which satisfies condition $l \leq N$. The total number of queries is $Q_2(\text{claw}_{\text{detect}}(N, M)) = O((NM)^{1/3} \log N)$. When $M \geq N^2$, we set $l = m = N$, implying that $Q_2(\text{claw}_{\text{detect}}(N, M)) = O(M^{1/2} \log N)$. \hfill \Box

The standard oracle case can be handled by using almost the same approach.

Corollary 5 Let $Q_2(\text{claw}_{\text{detect}}(N, M))$ be the number of queries needed to decide whether there is a claw or not for functions $f : X = [N] \to Z$ and $g : Y = [M] \to Z$ given as a standard oracle. Then,

$$Q_2(\text{claw}_{\text{detect}}(N, M)) = \begin{cases} O((NM)^{1/3}) & (N \leq M < N^2) \\ O(M^{1/2}) & (M \geq N^2). \end{cases}$$

The claw-detection algorithm against a standard oracle can easily be modified in order to solve the more general problem of detecting a tuple $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in X^p \times Y^q$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for any $i \neq j$, and $(f(x_1), \ldots, f(x_p), g(y_1), \ldots, g(y_q)) \in R$, for given $R \subseteq Z^{p+q}$, where $p$ and $q$ are any constant positive integers. A modification is made to the part of the algorithm that decides whether a vertex of the underlying graph is marked or not; the modification can be made without changing the number of queries. The query complexity can be analyzed by using almost the same approach as used in claw detection with $\epsilon = \frac{N^p}{l^{p-1}} \times \frac{N^q}{m^{q-1}} \times (M^r)^{1/3} / \sqrt{mNM}$; the query complexity is $O(N^p M^q / \sqrt{mNM})$ for $N \leq M < N^{1+1/4}$ and $O(M^{1+1/4})$ for $M \geq N^{1+1/4}$. The problem of finding such a tuple can also be solved with the same order of complexity as above by using the algorithm for detecting it as a subroutine.

Our algorithm for detecting a claw can easily be generalized to the case of $k$ functions of domains of size $N_1, \ldots, N_k$, respectively. More concretely, we apply Theorem 1 to the Markov chain on the graph categorical.
product of the $k$ Johnson graphs, each of which corresponds to one of the $k$ functions. We denote this “$k$-claw detection” algorithm by $k$-Claw\_Detect in the next section.

**Lemma 6** For any positive integer $k > 1$, let $Q_2(k\text{-claw}_{\text{detect}}(N_1, \ldots, N_k))$ be the number of queries needed to decide whether there is a $k$-claw or not for functions $f_i : X_i := [N_i] \to Z$ ($i \in [k]$) given as a comparison oracle, where $N_i \leq N_j$ if $i < j$. If $k$ is constant,

$$Q_2(k\text{-claw}_{\text{detect}}(N_1, \ldots, N_k)) = \begin{cases} O\left(\left(\prod_{i=1}^k N_i\right)^{\frac{1}{k+1}} \log N_1\right) & \text{if } \prod_{i=2}^k N_i = O(N_1^k), \\ O\left(\sqrt[2k+2]{\prod_{i=2}^k N_i} \frac{1}{\log N_1}\right) & \text{otherwise}. \end{cases}$$

**Proof (Sketch).** In a way similar to the case of two functions, we apply Theorem 1 on the graph categorical product of $k$ Johnson graphs $J_k := J([X_i], l_i)$ ($i \in [k]$) for the domains $X_i$'s of functions $f_i$'s, where $l_i$'s are integers fixed later such that $l_i \leq l_j$ for $i < j$.

To generate $|\psi_0\rangle$, we first prepare the uniform superposition of $|F_1, \ldots, F_k\rangle|F'_1, \ldots, F'_k\rangle$ over all $F_i$ and $F'_i$ such that $(F_i, F'_i)$ is an edge of $J_k$ for every $i$. This requires no queries. As in the case of two functions, define $L_{F_1, \ldots, F_k}$ for any $F_1, \ldots, F_k$ as a representation of the sorted list of all elements in $\bigcup_{i=1}^k F_i$ so that it can be uniquely determined for each tuple $(F_1, \ldots, F_k)$. We then compute $L_{F_1, \ldots, F_k}$ and $L_{F'_1, \ldots, F'_k}$ for each basis state by issuing $O((l_1 + \cdots + l_k)(l_1 + \cdots + l_k))$ queries to the oracle. Thus, $C_U = O((l_1 + \cdots + l_k)(l_1 + \cdots + l_k))$. $C_F$ and $C_W$ can be estimated as $O(\log l_1 + \cdots + l_k)$, respectively, in a way similar to the case of two functions. We set $\epsilon$ to $\prod_{i=1}^k l_i/N_i$ and $\delta$ to $\min(1/l_i) = 1/l_k$.

When $\prod_{i=2}^k N_i = O(N_1^k)$, we set $l_i := \Theta\left(\prod_{i=1}^k N_i\right)^{\frac{1}{2k+2}}$ for every $i \in [k]$, which satisfies condition $l_i \leq N_i \leq N_i$ for every $i \in [k]$. When $\prod_{i=2}^k N_i = \Omega(N_1^k)$, we set $l_i := \Theta(N_1)$ for every $i \in [k]$. Against a standard oracle, we obtain a similar result.

**Corollary 7** For any positive integer $k > 1$, let $Q_2(k\text{-claw}_{\text{detect}}(N_1, \ldots, N_k))$ be the number of queries needed to decide whether there is a $k$-claw or not for functions $f_i : X_i := [N_i] \to Z$ ($i \in [k]$) given as a standard oracle, where $N_i \leq N_j$ if $i < j$. If $k$ is constant,

$$Q_2(k\text{-claw}_{\text{detect}}(N_1, \ldots, N_k)) = \begin{cases} O\left(\left(\prod_{i=1}^k N_i\right)^{\frac{1}{2k+2}}\right) & \text{if } \prod_{i=2}^k N_i = O(N_1^k), \\ O\left(\sqrt[2k+2]{\prod_{i=2}^k N_i} \frac{1}{\log N_1}\right) & \text{otherwise}. \end{cases}$$

### 4 Claw Finding

We now describe an algorithm, Claw\_Search, that finds a claw. The algorithm consists of three stages. In the first stage, we find an $O(N)$-sized subset $Y'$ of $Y$ such that there is a claw in $X \times Y'$, by performing binary search over $Y$ with Claw\_Detect. In the second stage, we perform 4-ary search over $X$ and $Y'$ with Claw\_Detect to find $O(1)$-sized subsets $X''$ and $Y''$ of $X'$ and $Y'$, respectively, such that there is a claw in $X'' \times Y''$. In the final stage, we search $X'' \times Y''$ for a claw by issuing classical queries. To keep the error rate moderate, say, at most $1/3$, Claw\_Detect is repeated $O(s)$ times against the same pair of domains at the $s$th node of the search tree at each stage. This pushes up the query complexity by only a constant multiplicative factor.

Figure 1 precisely describes Claw\_Search. Steps 2, 3 and 4 in the figure correspond to the first, second and final stages, respectively.
Algorithm Claw\_Search

**Input:** Integers $M$ and $N$ such that $M \geq N$; Comparison oracle $O_{f,g}$ for functions $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, respectively, such that $X := [N]$ and $Y := [M]$.

**Output:** Claw pair $(x, y) \in X \times Y$ such that $f(x) = g(y)$ if such a pair exists; otherwise $(-1, -1)$.

1. Set $\tilde{X} := X$ and $\tilde{Y} := Y$.
2. Set $s := 1$, and repeat the next steps until $u_f - l_f \leq |\tilde{X}|$, where $u_f$ and $l_f$ are the largest and smallest values, respectively, in $\tilde{Y}$.
   
   2.1 Set $\Xi_Y := \{(l_f, m_f - 1), (m_f, u_f]\}$, where $m_f := [(l_f + u_f)/2]$.
   
   2.2 For every $\tilde{Y}' \in \Xi_Y$, do the following.
   
      If all $\tilde{Y}' \in \Xi_Y$ are examined, output $(-1, -1)$ and halt.
   
      2.2.1 Apply Claw\_Detect $(s + 2)$ times to $f$ and $g$ restricted to domains $\tilde{X}$ and $\tilde{Y}$, respectively.
      
      2.2.2 If at least one of the $(s + 2)$ results is “true,” set $\tilde{Y} := \tilde{Y}'$, and break (leave (b)).
   
   2.3 Set $s := s + 1$.

3. Set $s := 1$, and repeat the next steps until $u_D - l_D \leq c$ for every $D \in \{\tilde{X}, \tilde{Y}\}$ and some constant $c$, say, 100, where $u_D$ and $l_D$ are the largest and smallest values, respectively, in $D$.

3.1 For every $D \in \{\tilde{X}, \tilde{Y}\}$, set $\Xi_D := \{(l_D, u_D]\}$ if $u_D - l_D \leq c$, and otherwise, set $\Xi_D := \{(l_D, m_D - 1), (m_D, u_D]\}$ where $m_D := [(l_D + u_D)/2]$.

3.2 For every pair $(\tilde{X}', \tilde{Y}') \in \tilde{X} \times \Xi_F$, do the following.

   If all the pairs are examined, output $(-1, -1)$ and halt.

3.2.1 Apply Claw\_Detect $(s + 3)$ times to $f$ and $g$ restricted to domains $\tilde{X}'$ and $\tilde{Y}'$, respectively.

3.2.2 If at least one of the $(s + 3)$ results is “true,” set $\tilde{X} := \tilde{X}'$ and $\tilde{Y} := \tilde{Y}'$, and break (leave (b)).

3.3 Set $s := s + 1$.

4. Classically search $\tilde{X} \times \tilde{Y}$ for a claw.

5. Output claw $(x, y) \in \tilde{X} \times \tilde{Y}$ if it exists; otherwise output $(-1, -1)$.

![Figure 1: Algorithm Claw\_Search](image-url)

**Theorem 8** Let $Q_2(\text{claw}\_\text{finding}(N, M))$ be the number of queries needed to locate a claw if it exists for functions $f : X = [N] \rightarrow Z$ and $g : Y = [M] \rightarrow Z$ given as a comparison oracle. Then,

$$Q_2(\text{claw}\_\text{finding}(N, M)) = \begin{cases} O\left((NM)^{1/3} \log N\right) & N \leq M < N^2 \\ O\left(M^{1/2} \log N\right) & M \geq N^2. \end{cases}$$

**Proof** We will analyze Claw\_Search in Fig. [1].

When there is no claw, Claw\_Search always outputs the correct answer. Suppose that there is a claw. The algorithm may output a wrong answer if at least one of the following two cases happens. In case (1), one of $O(\log M/N)$ runs of step 2.(b) errs; in case (2), one of $O(\log N)$ runs of step 3.(b) errs.

Without loss of generality, the error probability of Claw\_Detect can be assumed to be at most $1/3$. The error probability of each single run of step 2.(b) is at most $\frac{2}{3^{m+2}}$. The error probability of each run of step 2.(b) is at most $\frac{2}{3^{m+2}} < \frac{1}{3^{m+2}}$. The error probability of case (1) is thus at most $\sum_{s=1}^{\log M/N} \frac{1}{3^{m+2}} < \frac{1}{6}$. The error probability of case (2) is also at most $\sum_{s=1}^{\log N/1} \frac{1}{3^{m+1}} < \frac{1}{6}$ by similar calculation. Therefore, the overall error probability is at most $1/6 + 1/6 = 1/3$. 

We next estimate the number of queries. If \( N \leq M < N^2 \), the size of \( \tilde{X} \) is always at most quadratically different from that of \( X \). Thus, the \( s \)th repetition of step 2 requires \( O(s(NM/2^s)^{1/3} \log N) \) queries by Lemma 4. Similarly, the \( s \)th repetition of step 3 requires \( O(s(N/2^s)^{2/3} \log N) \) queries by Lemma 4.

The total number of queries is

\[
O\left( \sum_{s=1}^{[\log(M/N)]} \left( s \left( \frac{M}{2^s} \right)^{1/3} \log N \right) \right) + \sum_{s=1}^{[\log N]} \left( s(N/2^s)^{2/3} \log N \right) = O\left((NM)^{1/3} \log N \right).
\]

If \( M \geq N^2 \), the \( s \)th repetition of step 2 requires \( O(s((NM/2^s)^{1/3} + (M/2^s)^{1/2}) \log N) \) by Lemma 4. Thus, similar calculation gives \( O(M^{1/2} \log N) \) queries. \( \square \)

We can easily obtain the standard-oracle version of the above theorem by using Corollary 5 instead of Lemma 4.

**Corollary 9** Let \( Q_2(\text{claw}_{\text{finding}}(N, M)) \) be the number of queries needed to locate a claw if it exists for functions \( f : X := [N] \rightarrow Z \) and \( g : Y := [M] \rightarrow Z \) given as a standard oracle. Then,

\[
Q_2(\text{claw}_{\text{finding}}(N, M)) = \begin{cases} 
O\left((NM)^{1/3}\right) & N \leq M < N^2 \\
O(M^{1/2}) & M \geq N^2.
\end{cases}
\]

Similarly, we can find a \( k \)-claw by using \( k \)-Claw\_Detect as a subroutine. First, we find \( O(N_1) \)-sized subset \( X_i' \) of \( X_i \) for every \( i \in [2,k] \) such that there is a claw in \( X_1 \times X_2' \times \cdots \times X_k' \), by performing \( 2^{k-1} \)-ary search over \( X_i' \)'s for all \( i \in [2,k] \) with \( k \)-Claw\_Detect. Let \( X_i' := X_1 \). We then perform \( 2^{k-1} \)-ary search over \( X_i' \) for all \( i \in [k] \) with \( k \)-Claw\_Detect to find \( O(1) \)-sized subset \( X_i'' \) of \( X_i' \) for every \( i \in [k] \) such that there is a claw in \( X_1'' \times \cdots \times X_k'' \). Finally, we search \( X_1'' \times \cdots \times X_k'' \) for a claw by issuing classical queries. A more precise description of the algorithm, \( k \)-Claw\_Search, is given in Fig. 2.

**Theorem 10** For any positive integer \( k > 1 \), let \( Q_2(\text{claw}_{\text{finding}}(N_1, \ldots, N_k)) \) be the number of queries needed to locate a \( k \)-claw if it exists for \( k \) functions \( f_i : X_i := [N_i] \rightarrow Z \) (\( i \in [k] \)) given as a comparison oracle, where \( N_i \leq N_j \) if \( i < j \). If \( k \) is constant,

\[
Q_2(\text{claw}_{\text{finding}}(N_1, \ldots, N_k)) = \begin{cases} 
O\left( \left( \prod_{i=1}^{k} N_i \right)^{1/k} \log N_1 \right) & \text{if } \prod_{i=2}^{k} N_i = O(N_1^k), \\
O\left( \sqrt{\prod_{i=2}^{k} N_i/N_1^{k-2}} \log N_1 \right) & \text{otherwise}.
\end{cases}
\]

We can easily obtain the standard-oracle version of the above theorem by using Corollary 7 instead of Lemma 6.

**Corollary 11** For any positive integer \( k > 1 \), let \( Q_2(\text{claw}_{\text{finding}}(N_1, \ldots, N_k)) \) be the number of queries needed to locate a \( k \)-claw if it exists for \( k \) functions \( f_i : X_i := [N_i] \rightarrow Z \) (\( i \in [k] \)) given as a standard oracle, where \( N_i \leq N_j \) if \( i < j \). If \( k \) is constant,

\[
Q_2(\text{claw}_{\text{finding}}(N_1, \ldots, N_k)) = \begin{cases} 
O\left( \left( \prod_{i=1}^{k} N_i \right)^{1/k} \right) & \text{if } \prod_{i=2}^{k} N_i = O(N_1^k), \\
O\left( \sqrt{\prod_{i=2}^{k} N_i/N_1^{k-2}} \right) & \text{otherwise}.
\end{cases}
\]
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Algorithm $k$-Claw\_Search

**Input:** $k$ integers $N_1, \ldots, N_k$ such that $N_i \leq N_j$ if $i < j$.
Comparison oracle $O_{f_1, \ldots, f_k}$ for functions $f_i : X_i \rightarrow Z$ such that $X_i := [N_i]$ for every $i \in [k]$.

**Output:** $k$-claw $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that $f_i(x_i) = f_j(x_j)$ for every $i, j \in [k]$ if it exists; otherwise $(-1, \ldots, -1)$.

1. Set $\tilde{X}_i := X_i$ for every $i \in [k]$.

2. Set $s := 1$, and repeat the next steps until $u_i - l_i \leq |\tilde{X}_1|$ for all $i \in [2,k]$, where $u_i$ and $l_i$ are the largest and smallest values, respectively, in $\tilde{X}_i$.

   2.1 For every $i \in [2,k]$, set $\Xi_i := \{(u, u_i)\}$ if $u_i - l_i \leq |\tilde{X}_1|$, and otherwise, set $\Xi_i := \{(l_i, m_i - 1), (m_i, u_i)\}$ where $m_i := \lfloor (l_i + u_i)/2 \rfloor$.

   2.2 For every tuple $(\tilde{X}_1', \tilde{X}_2', \ldots, \tilde{X}_k') \in (\tilde{X}_1) \times \Xi_2 \times \cdots \times \Xi_k$, do the following.
      
      If all the tuples are examined, output $(-1, \ldots, -1)$ and halt.

      2.2.1 Apply $k$-Claw\_Detect $(s + 1) + \lceil \log_2 2^{k-1} \rceil$ times to the $k$ functions $f_i$ restricted to domains $\tilde{X}_i'$, respectively, for every $i \in [k]$.

      2.2.2 If at least one of the $(s + 1) + \lceil \log_2 2^{k-1} \rceil$ results is “true,” set $\tilde{X}_i := \tilde{X}_i'$ for every $i \in [2,k]$, and break (leave (b)).

   2.3 Set $s := s + 1$.

3. Set $s := 1$, and repeat the next steps until $u_i - l_i \leq c$ for all $i \in [k]$ and some constant $c$, say, 100, where $u_i$ and $l_i$ are the largest and smallest values, respectively, in $\tilde{X}_i$.

   3.1 For every $i \in [k]$, set $\Xi_i := \{(u, u_i)\}$ if $u_i - l_i \leq c$, and otherwise, set $\Xi_i := \{(l_i, m_i - 1), (m_i, u_i)\}$ where $m_i := \lfloor (l_i + u_i)/2 \rfloor$.

   3.2 For every tuple $(\tilde{X}_1', \tilde{X}_2', \ldots, \tilde{X}_k') \in \Xi_1 \times \cdots \times \Xi_k$, do the following.
      
      If all the tuples are examined, output $(-1, \ldots, -1)$ and halt.

      3.2.1 Apply $k$-Claw\_Detect $(s + 1) + \lceil \log_2 2^k \rceil$ times to the $k$ functions $f_i$ restricted to domains $\tilde{X}_i'$ for every $i \in [k]$.

      3.2.2 If at least one of the $(s + 1) + \lceil \log_2 2^k \rceil$ results is “true,” set $\tilde{X}_i := \tilde{X}_i'$ for every $i \in [k]$, and break (leave (b)).

   3.3 Set $s := s + 1$.

4. Classically search $\tilde{X}_1 \times \cdots \times \tilde{X}_k$ for a $k$-claw.

5. Output $k$-claw $(x_1, \ldots, x_k) \in X_1' \times \cdots \times X_k'$ if it exists; otherwise output $(-1, \ldots, -1)$.

Figure 2: Algorithm $k$-Claw\_Search