A PRIMITIVE ASSOCIATED TO THE CANTOR-BENDIXSON DERIVATIVE ON THE REAL LINE

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Abstract. We consider the class of compact countable subsets of the real numbers \( \mathbb{R} \). By using an appropriate partition, up to homeomorphism, of this class we give a detailed proof of a result shown by S. Mazurkiewicz and W. Sierpinski related to the cardinality of this partition. Furthermore, for any compact subset of \( \mathbb{R} \), we show the existence of a “primitive” related to its Cantor-Bendixson derivative.

1. Introduction

The earliest ideas of limit point and derived set in the space of the real numbers were both introduced and investigated by Georg Cantor since 1872 (see also [1, 2, 3, 4, 6]) to analyze the convergence set of a trigonometric series. These two concepts have been generalized to the case of any arbitrary topological space. Thus, let \( X \) be a topological space and let \( A \) be a subset of \( X \), we write \( A' \) to denote the derived set of \( A \), that is, the set of all limit points of \( A \). The next definition extends the process of taking the derivative of a set for any ordinal number.

Definition 1.1 (Cantor-Bendixson’s derivative). Let \( A \) be a subset of a topological space. For a given ordinal number \( \alpha \), we define, using Transfinite Recursion, the \( \alpha \)-th derivative of \( A \), written \( A^{(\alpha)} \), as follows:

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• $A^{(0)} = A$,
• $A^{(β+1)} = (A^{(β)})'$, for all ordinal $β$,
• $A^{(λ)} = \bigcap_{γ < λ} A^{(γ)}$, for all limit ordinal $λ \neq 0$.

In this paper, we are initially concerned with the Cantor-Bendixson derivative of compact countable subsets of the real numbers, where a countable set is either a finite set or a countably infinite set. Thus, we consider the set

$$K = \{K \subset \mathbb{R} : K \text{ is compact and countable}\}. \quad (1.1)$$

Moreover, for all $K_1, K_2 \in K$, we define the relation

$$K_1 \sim K_2 \iff \text{there exists } f : K_1 \hookrightarrow K_2 \text{ continuous and bijective}. \quad (1.2)$$

It is not hard to see that $\sim$ is an equivalence relation on the set $K$ and since the elements of $K$ are compact sets, we have that for all $K_1, K_2 \in K$

$$K_1 \sim K_2 \iff \text{there exists } f : K_1 \hookrightarrow K_2 \text{ homeomorphism}. \quad (1.3)$$

Therefore, there is a partition of the set $K$, and we denote by

$$\mathcal{K} = \mathcal{K}/\sim \quad (1.4)$$

the set of all equivalence classes of $K$.

In 1920, S. Mazurkiewicz and W. Sierpinski [7] showed that the cardinality of $\mathcal{K}$ is $\aleph_1$. In Section 2, we show in detail that for any countable ordinal number $α$, and for any $p \in \omega$, there is a set $K \in K$ such that $K^{(α)}$ has exactly $p$ elements. This last fact was first briefly mentioned by Cantor in [3]. The results shown in Section 2 allow us to prove, in Theorem 3.4, that the cardinality of $\mathcal{K}$ is greater than or equal to $\aleph_1$. On the other hand, the cardinality of $\mathcal{K}$ is smaller than or equal to $\aleph_1$ as a consequence of Theorem 3.3.

Section 3 considers Cantor-Bendixson’s characteristic, denoted by $\mathcal{CB}$. First, we show that for any element $K \in K$ with $\mathcal{CB}(K) = (α, p)$, we get $p = 0$ if and only if $K = \emptyset$. Moreover, we use Lemma 3.6 to prove Theorem 3.3 where the injectivity of function $\tilde{\mathcal{CB}}$, defined in (3.12), is shown. These two last results were first mentioned in [7]; however, for the sake of completeness, we include here their detailed proofs. Finally, Theorem 3.5 shows that for any compact subset of the reals, there exists a primitive-like set connected with its Cantor-Bendixson derivative.

We recall that if $F$ is a closed subset of $\mathbb{R}$, then $(F^{(α)})_{α \in \mathbb{R}}$ is a decreasing family of closed subsets of the real line. Furthermore, if $K \in \mathcal{K}$, then $(K^{(α)})_{α \in \mathbb{R}}$ is a decreasing family of elements of $K$. 
We denote by $\text{OR}$, the class of all ordinal numbers. Moreover, $\omega$ is used to designate the set of all natural numbers and $\Omega$ represents the set of all countable ordinal numbers. In addition, the cardinality of a set $B$ is denoted by $|B|$.

2. A family of elements in $\mathcal{K}$ having a Cantor-Bendixson’s derivative with any given finite number of elements

First, we remark that any finite subset of $\mathbb{R}$ is an element of $\mathcal{K}$ with empty derived set. Thus, a set of this kind satisfies the property that its Cantor-Bendixson’s derivative is empty for all ordinal number greater than or equal to 1. The following theorem let us find some elements belonging to $\mathcal{K}$ not satisfying this last property. The main idea of the next result was given in [3], for completeness, we present below its proof in detail.

**Theorem 2.1.** For any countable ordinal number $\alpha \in \Omega$, and for all $a, b \in \mathbb{R}$ such that $a < b$, there is a set $K \in \mathcal{K}$ such that $K \subset (a, b]$ and $K^{(\alpha)} = \{b\}$.

**Proof.** We will use Transfinite Induction.

(a) First, we consider the case $\alpha = 0$. For any $a, b \in \mathbb{R}$ such that $a < b$, the result follows by taking the set $K = \{b\} \in \mathcal{K}$.

(b) Now, we suppose that for a given countable ordinal number $\alpha \in \Omega$, and for all $c, d \in \mathbb{R}$ such that $c < d$, there is a set $\tilde{K} \in \mathcal{K}$ such that $\tilde{K} \subset (c, d]$ and $\tilde{K}^{(\alpha)} = \{d\}$. Let $a, b \in \mathbb{R}$ be such that $a < b$. We take a strictly increasing sequence, $(x_n)_{n\in\omega}$, in $(a, b]$ such that $x_n \to b$ as $n \to +\infty$. Defining $x_{-1} := a$ and applying the hypothesis to the real numbers $x_{m-1} < x_m$, $m \in \omega$, it follows that there exists a sequence of sets $(K_m)_{m\in\omega}$ such that for all $m \in \omega$, $K_m \in \mathcal{K}$, $K_m \subset (x_{m-1}, x_m]$ and $K_m^{(\alpha)} = \{x_m\}$. Now, we define the set

$$K := \bigcup_{m \in \omega} K_m \cup \{b\}. \quad (2.1)$$

The set $K$, given in (2.1), satisfies the following properties:

- $K \subset (a, b]$, since $K_m \subset (x_{m-1}, x_m] \subset (a, b]$, for all $m \in \omega$.
- $K$ is countable, since it is the countable union of countable sets.
- $K$ is compact. In fact, given $(A_i)_{i \in I}$ an open cover of $K$, there is a $j \in I$ such that $b \in A_j$. Since $A_j$ is an open set and $(x_n)_{n\in\omega}$ is a strictly increasing sequence that converges to $b$, there exists $N_1 \in \omega$ such that $K_n \subset A_j$ for all $n \in \omega$ with $n > N_1$. On the other hand, the set $C := \bigcup_{n=0}^{N_1} K_n$ is compact, since it is the finite union of compact sets. Thus, $C$ has a finite open subcover $(A_i)_{i \in J}$. Then, $(A_i)_{i \in J \cup \{j\}}$ is a finite open subcover of $K$. 

For all ordinal number $\beta$ with $\beta \leq \alpha$,

$$K^{(\beta)} = \bigcup_{m \in \omega} K^{(\beta)}_m \cup \{b\}. \quad (2.2)$$

Last expression is obtained by using Transfinite Induction on $\beta$. In fact, the case $\beta = 0$ is immediate from (2.1). Now, we suppose that for a given ordinal number $\beta < \alpha$, (2.2) holds. Since $\beta + 1 \leq \alpha$, we have that $K^{(\alpha)}_m \subset K^{(\alpha)} \subset K^{(\beta+1)}$ for all $m \in \omega$. Moreover, since $x_m \in K^{(\alpha)}_m \subset K^{(\beta+1)}$, for all $m \in \omega$, and $x_m \to b$ as $m \to +\infty$, we see that $b \in K^{(\beta+1)}$. Therefore,

$$\bigcup_{m \in \omega} K^{(\beta+1)}_m \cup \{b\} \subset K^{(\beta+1)}. \quad (2.3)$$

In order to prove the other inclusion, let $x \in K^{(\beta+1)}$. Using the induction hypothesis, we see that

$$K^{(\beta+1)} \subset K^{(\beta)} = \bigcup_{m \in \omega} K^{(\beta)}_m \cup \{b\}.$$ 

Therefore, either $x = b$ or $x \in K^{(\beta)}_m$ for some $m \in \omega$. If $x = b$, then there is nothing else to prove. If $x \neq b$, there exists $M \in \omega$ such that

$$x \in K^{(\beta)}_M \subset K_M \subset (x_{M-1}, x_M].$$

We claim that $x \in K^{(\beta+1)}_M$. To prove the last assertion, we suppose, by contradiction, that $x \notin K^{(\beta+1)}_M$. Thus, $x$ is an isolated point of $K^{(\beta)}_M$. However, we know that $\{x_M\} = K^{(\alpha)}_M \subset K^{(\beta+1)}_M$. Then, $x \neq x_M$. Thus, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset (x_{M-1}, x_M)$ and

$$(x - \epsilon, x + \epsilon) \cap K^{(\beta)}_M = \{x\}.$$ 

Moreover, since $(x - \epsilon, x + \epsilon) \subset (x_{M-1}, x_M)$, we conclude that for all $m \in \omega \setminus \{M\}$,

$$(x - \epsilon, x + \epsilon) \cap K^{(\beta)}_m = \emptyset.$$ 

Hence,

$$\{x\} = (x - \epsilon, x + \epsilon) \cap \left( \bigcup_{m \in \omega} K^{(\beta)}_m \cup \{b\} \right)$$

$$= (x - \epsilon, x + \epsilon) \cap K^{(\beta)},$$

where in the last equality we have used the assumption that (2.2) holds for $\beta$. Even so, this last expression is a contradiction with the
fact that $x \in K^{(\beta+1)}$. Then, $x \in K^{(\beta+1)}_M$. Thus,
\[ K^{(\beta+1)} = \bigcup_{m \in \omega} K^{(\beta+1)}_m \cup \{ b \}. \quad (2.4) \]

Using (2.3) and (2.4), we get
\[ K^{(\beta+1)} = \bigcup_{m \in \omega} K^{(\beta+1)}_m \cup \{ b \}. \]

Finally, let $\gamma \neq 0$ be a limit ordinal such that $\gamma \leq \alpha$ and suppose that
\[ K^{(\delta)} = \bigcup_{m \in \omega} K^{(\delta)}_m \cup \{ b \}, \quad (2.5) \]

for all ordinal number $\delta$ such that $\delta < \gamma$. Following a similar procedure to the one performed above to obtain (2.3), we have that
\[ \bigcup_{m \in \omega} K^{(\gamma)}_m \cup \{ b \} \subset K^{(\gamma)}. \quad (2.6) \]

To obtain the other inclusion, let $x \in K^{(\gamma)}$. Using the induction hypothesis (2.5), we see that
\[ K^{(\gamma)} := \bigcap_{\delta < \gamma} K^{(\delta)} = \bigcap_{\delta < \gamma} \left( \bigcup_{m \in \omega} K^{(\delta)}_m \cup \{ b \} \right). \]

Then, either $x = b$ or for all ordinal number $\delta$ such that $\delta < \gamma$, there exists $m \in \omega$ such that $x \in K^{(\delta)}_m$. If $x = b$, then there is nothing left to prove. If $x \neq b$, there exists $M \in \omega$ such that $x \in K^{(0)}_M = K_M \subset (x_{M-1}, x_M]$. We claim now that for all ordinal number $\delta$ such that $\delta < \gamma$, $x \in K^{(\delta)}_M$. In fact, we suppose, by contradiction, that there is an ordinal number $\delta_0$ with $\delta_0 < \gamma$ and such that $x \notin K^{(\delta_0)}_M$. However, we know that there exists $m_0 \in \omega$ with $m_0 \neq M$ such that $x \in K^{(\delta_0)}_{m_0} \subset K_{m_0} \subset (x_{m_0-1}, x_{m_0}]$. Since $m_0 \neq M$, we get $(x_{m_0-1}, x_{m_0}] \cap (x_{M-1}, x_M] = \emptyset$, which is a contradiction with the fact that $x \in (x_{m_0-1}, x_{m_0}] \cap (x_{M-1}, x_M]$. Therefore,
\[ x \in \bigcap_{\delta < \gamma} K^{(\delta)}_M =: K^{(\gamma)}_M \subset \bigcup_{m \in \omega} K^{(\gamma)}_m. \]

Then,
\[ K^{(\gamma)} \subset \bigcup_{m \in \omega} K^{(\gamma)}_m \cup \{ b \}. \quad (2.7) \]

By (2.6) and (2.7), we have that
\[ K^{(\gamma)} = \bigcup_{m \in \omega} K^{(\gamma)}_m \cup \{ b \}. \]

Hence, (2.2) holds for all ordinal number $\beta$ such that $\beta \leq \alpha$. 
Applying now (2.2) to the ordinal number \( \alpha \), and since \( K^{(\alpha)}_m = \{ x_m \} \), for all \( m \in \omega \), we conclude that
\[
K^{(\alpha)} = \bigcup_{m \in \omega} K^{(\alpha)}_m \uplus \{ b \} \\
= \bigcup_{m \in \omega} \{ x_m \} \uplus \{ b \} \\
= \{ x_m : m \in \omega \} \uplus \{ b \}.
\]

Therefore,
\[
K^{(\alpha + 1)} = (K^{(\alpha)})' = \{ b \}.
\]

(c) Finally, let \( \lambda \neq 0 \) be a countable limit ordinal number. We suppose that for all ordinal number \( \rho \) such that \( \rho < \lambda \) and for all \( c, d \in \mathbb{R} \) such that \( c < d \), there is a set \( \widetilde{K} \in \mathcal{K} \) such that \( \widetilde{K} \subset (c, d] \) and \( \widetilde{K}^{(\rho)} = \{ d \} \). Since \( \lambda \) is a countable limit ordinal number, there exits a strictly increasing sequence \( (\rho_n)_{n \in \omega} \) in \( \Omega \) such that \( \rho_n < \lambda \), for all \( n \in \omega \), and \( \sup\{ \rho_n : n \in \omega \} = \lambda \). Let \( a, b \in \mathbb{R} \) be such that \( a < b \). We take a strictly increasing sequence, \( (x_n)_{n \in \omega} \), in \( (a, b] \) such that \( x_n \to b \) as \( n \to +\infty \).

Defining again \( x_{-1} = a \) and applying the hypothesis to the real numbers \( x_{m-1} < x_m \), and the ordinal number \( \rho_m, m \in \omega \), it follows that there exists a sequence of sets \( (K_m)_{m \in \omega} \) such that for all \( m \in \omega \), \( K_m \in \mathcal{K} \), \( K_m \subset (x_{m-1}, x_m] \) and \( K_m^{(\rho_m)} = \{ x_m \} \). We also define, as in the previous case, the set
\[
K := \bigcup_{m \in \omega} K_m \uplus \{ b \}.
\]

It can be shown, similarly to the case (b) above, that the set \( K \), defined in (2.8), satisfies the following properties:

- \( K \subset (a, b] \).
- \( K \) is countable.
- \( K \) is compact.
- For all ordinal number \( \rho \) with \( \rho \leq \lambda \),
\[
K^{(\rho)} = \bigcup_{m \in \omega} K_m^{(\rho)} \uplus \{ b \}.
\]

Last expression is obtained by using Transfinite Induction on \( \rho \). In fact, the case \( \rho = 0 \) is immediate from (2.8). Now, we suppose that for a given ordinal number \( \rho < \lambda \), (2.9) holds. Since \( \lambda \) is a limit ordinal, we have that \( \rho + 1 < \lambda \), and then there exists \( N \in \omega \) such that \( \rho + 1 < \rho_m \) for all \( m \in \omega \) with \( m > N \). Therefore,
\[
x_m \in K_m^{(\rho_m)} \subset K_m^{(\rho+1)} \subset K^{(\rho+1)} \text{ for all } m \in \omega \text{ with } m > N,
\]
since $x_m \to b$ as $m \to +\infty$, we see that $b \in K^{(\rho+1)}$. Then,
\[
\bigcup_{m \in \omega} K^{(\rho+1)}_m \uplus \{b\} \subset K^{(\rho+1)}. \tag{2.10}
\]
In order to prove the other inclusion, let $x \in K^{(\rho+1)}$. Using the induction hypothesis, we see that
\[K^{(\rho+1)} \subset K^{(\rho)} = \bigcup_{m \in \omega} K^{(\rho)}_m \uplus \{b\}.
\]
Therefore, either $x = b$ or $x \in K^{(\rho)}_m$ for some $m \in \omega$. If $x = b$, then there is nothing else to prove. If $x \neq b$, there exists $M \in \omega$ such that
\[x \in K^{(\rho)}_M \subset K_M \subset (x_{M-1}, x_M].
\]
Since $K^{(\rho M+1)}_M = \emptyset$, we have that $\rho < \rho_M + 1$, that is $\rho \leq \rho_M$. We claim that $x \in K^{(\rho+1)}_M$. To prove the last assertion, we suppose, by contradiction, that $x \notin K^{(\rho+1)}_M$. Thus, $x$ is an isolated point of $K^{(\rho)}_M$. However, we know that $K_M \cap K_{M+1} = \emptyset$, then $x \notin K_{M+1}$. Hence, $x \notin K^{(\rho)}_{M+1}$. Thus, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset (x_{M-1}, x_{M+1})$, $(x - \epsilon, x + \epsilon) \cap K^{(\rho)}_M = \emptyset$ and
\[(x - \epsilon, x + \epsilon) \cap K^{(\rho)}_M = \{x\},
\]
where in the second expression above we have used the fact that $K^{(\rho)}_{M+1}$ is a closed subset of $\mathbb{R}$. Moreover, since $(x - \epsilon, x + \epsilon) \subset (x_{M-1}, x_{M+1})$, we conclude that for all $m \in \omega \setminus \{M\}$,
\[(x - \epsilon, x + \epsilon) \cap K^{(\rho)}_m = \emptyset.
\]
Hence,
\[
\{x\} = (x - \epsilon, x + \epsilon) \cap \left( \bigcup_{m \in \omega} K^{(\rho)}_m \uplus \{b\} \right)
\]
\[= (x - \epsilon, x + \epsilon) \cap K^{(\rho)},
\]
where in the last equality we have used the assumption that (2.9) holds for $\rho$. Nevertheless, this last expression is a contradiction with the fact that $x \in K^{(\rho+1)}$. Then, $x \in K^{(\rho+1)}_M$. Thus,
\[K^{(\rho+1)} \subset \bigcup_{m \in \omega} K^{(\rho+1)}_m \uplus \{b\}. \tag{2.11}
\]
Using (2.10) and (2.11), we get
\[K^{(\rho+1)} = \bigcup_{m \in \omega} K^{(\rho+1)}_m \uplus \{b\}.
\]
Finally, let $\gamma \neq 0$ be a limit ordinal such that $\gamma \leq \lambda$ and suppose that

$$K^{(\delta)} = \bigcup_{m \in \omega} K^m_m \cup \{b\}, \quad (2.12)$$

for all ordinal number $\delta$ such that $\delta < \gamma$. We have, using (2.12), that

$$\bigcup_{m \in \omega} K^m_m \cup \{b\} = \bigcup_{m \in \omega} \left( \bigcap_{\delta < \gamma} K^m_m \cup \{b\} \right) \subset \bigcap_{\delta < \gamma} \left( \bigcup_{m \in \omega} K^m_m \cup \{b\} \right) = \bigcap_{\delta < \gamma} K^{(\delta)} = K^{(\gamma)}. \quad (2.13)$$

To get the other inclusion, we can follow a similar procedure to the one performed above to obtain (2.7). Thus, we have that

$$K^{(\gamma)} \subset \bigcup_{m \in \omega} K^m_m \cup \{b\}. \quad (2.14)$$

By (2.13) and (2.14), we obtain

$$K^{(\gamma)} = \bigcup_{m \in \omega} K^m_m \cup \{b\}. \quad (2.15)$$

Consequently, (2.9) holds for all ordinal number $\rho$ such that $\rho \leq \lambda$. Furthermore, since for all $m \in \omega$, $\rho_m + 1 < \lambda$, it follows that for all $m \in \omega$

$$K^{(\lambda)}_m \subset K^{(\rho_m + 1)}_m = (K^{(\rho_m)}_m)' = (\{x_m\}') = \emptyset. \quad (2.16)$$

Therefore,

$$K^{(\lambda)} = \bigcup_{m \in \omega} K^{(\lambda)}_m \cup \{b\} = \{b\}. \quad (2.17)$$

From (a), (b) and (c), the theorem is proved.

The next lemma will be used in the proof of Corollary 2.1 below.

**Lemma 2.1.** Suppose that $n \in \omega$. Let $F_1, F_2, \ldots, F_n$ be closed subsets of $\mathbb{R}$. Then, for all ordinal number $\alpha \in \text{OR}$, we have that

$$\left( \bigcup_{k=1}^n F_k \right)^{(\alpha)} = \bigcup_{k=1}^n F_k^{(\alpha)}. \quad (2.18)$$
Proof. The general case, \( n \in \omega \), is a consequence of the result for \( n = 2 \) and the Principle of Finite Induction. Thus, we suppose that \( n = 2 \). We will now use Transfinite Induction.

(a) If \( \alpha = 0 \), then there is nothing else to prove.
(b) We now suppose that for a given ordinal number \( \alpha \in \text{OR} \), \((F_1 \cup F_2)^{(\alpha)} = F_1^{(\alpha)} \cup F_2^{(\alpha)}\). Therefore,
\[(F_1 \cup F_2)^{(\alpha+1)} = (F_1 \cup F_2)^{(\alpha)} = (F_1^{(\alpha)} \cup F_2^{(\alpha)})^\prime = F_1^{(\alpha+1)} \cup F_2^{(\alpha+1)},\]
where in the last equation we have used the fact that the derived set of a finite union of subsets of a metric space equals the union of their derived sets.
(c) Finally, let \( \lambda \neq 0 \) be a limit ordinal number. We suppose that for all \( \beta \in \text{OR} \) such that \( \beta < \lambda \), \((F_1 \cup F_2)^{(\beta)} = F_1^{(\beta)} \cup F_2^{(\beta)}\). Then,
\[F_1^{(\lambda)} \cup F_2^{(\lambda)} = \bigcap_{\beta<\lambda} F_1^{(\beta)} \cup \bigcap_{\beta<\lambda} F_2^{(\beta)} \subset \bigcap_{\beta<\lambda} (F_1^{(\beta)} \cup F_2^{(\beta)}) \]
\[= \bigcap_{\beta<\lambda} (F_1 \cup F_2)^{(\beta)} \]
\[= (F_1 \cup F_2)^{(\lambda)}.\]
In order to prove the other inclusion, we take \( x \in (F_1 \cup F_2)^{(\lambda)} \). We suppose, for the sake of contradiction, that \( x \notin F_1^{(\lambda)} \) and \( x \notin F_2^{(\lambda)} \). Thus, there exist \( \beta_1, \beta_2 \in \text{OR} \), with \( \beta_1 < \lambda \) and \( \beta_2 < \lambda \), such that \( x \notin F_1^{(\beta_1)} \) and \( x \notin F_2^{(\beta_2)} \). If \( \beta_1 \leq \beta_2 \), then \( F_1^{(\beta_2)} \subset F_1^{(\beta_1)} \). Hence, \( x \notin F_1^{(\beta_2)} \cup F_2^{(\beta_2)} = (F_1 \cup F_2)^{(\beta_2)} \), which contradicts the fact that \( x \in (F_1 \cup F_2)^{(\lambda)} = \bigcap_{\beta<\lambda} (F_1 \cup F_2)^{(\beta)} \). The proof of the other case, \( \beta_2 < \beta_1 \), is similar. Therefore,
\[(F_1 \cup F_2)^{(\lambda)} = F_1^{(\lambda)} \cup F_2^{(\lambda)}.\]
Consequently, the lemma is proved. □

The following result is a generalization of Theorem 2.1.

Corollary 2.1. Given any countable ordinal number \( \alpha \) and given any \( p \in \omega \), there exists \( K \in \mathcal{K} \) such that \( |K^{(\alpha)}| = p \).

Proof. Let \( \alpha \in \Omega \). If \( p = 0 \), we take \( K = \emptyset \). If \( p \in \omega \setminus \{0\} \), it is enough to apply Theorem 2.1 to a collection of \( p \) pairwise disjoint intervals. Thus, for all \( k \in \{1, \ldots, p\} \), there exists \( K_k \in \mathcal{K} \), such that \( K_k^{(\alpha)} \) has only one
element, and $K_i \cap K_j = \emptyset$ for $i, j \in \{1, \ldots, p\}$ with $i \neq j$. We now define

$$K := \biguplus_{k=1}^{p} K_k.$$ 

Hence, $K \in \mathcal{K}$ and, using Lemma 2.1, we get

$$K^{(\alpha)} = \biguplus_{k=1}^{p} K_k^{(\alpha)}.$$ 

Therefore, $K^{(\alpha)}$ has exactly $p$ elements. \qed

**Remark 2.1.** Even though the proofs of (2.2) and (2.9) are similar, it is worth mentioning that they are not identical. In fact, to prove (2.2) we have that $\alpha \in \Omega$ and for all $m \in \omega$, $K_m^{(\alpha)} = \{x_m\}$. On the other hand, to obtain (2.9) we consider $\lambda \neq 0$ a countable limit ordinal and a strictly increasing sequence $(\rho_m)_{m \in \omega}$ in $\Omega$, with $\sup\{\rho_m : m \in \omega\} = \lambda$, such that for all $m \in \omega$, $\rho_m < \lambda$ and $K_{\rho_m}^{(\alpha)} = \{x_m\}$, where $\rho_m$ depends on $m$. In addition, we point out that the process developed to obtain (2.13) can also be used to get (2.3), (2.6) and (2.10).

### 3. Some results concerning Cantor-Bendixson’s derivative

It is a well-known fact that, for all $K \in \mathcal{K}$, $(K^{(\alpha)})_{\alpha \in \text{OR}}$ is a decreasing family of elements of $\mathcal{K}$. The following two results were first proved by G. Cantor in [5] and they imply that for all $K \in \mathcal{K}$, $(K^{(\alpha)})_{\alpha \in \text{OR}}$ is in fact a strictly decreasing family of sets in $\mathcal{K}$ up to a countable ordinal number and such that all of its subsequent derivative sets are empty.

**Lemma 3.1.** If $K \in \mathcal{K}$ and $K \neq \emptyset$, then $K' \neq K$.

The above lemma implies the following theorem.

**Theorem 3.1.** If $K \in \mathcal{K}$, then there exists a countable ordinal number $\beta$ such that $K^{(\beta)}$ is finite.

Since $\Omega$ is a well-ordered set, by the previous theorem, we see that for all $K \in \mathcal{K}$, there exists the smallest countable ordinal number $\alpha$ such that $K^{(\alpha)}$ is finite. We can now give the next definition.

**Definition 3.1** (Cantor-Bendixson’s characteristic). Let $K \in \mathcal{K}$. We say that $(\alpha, p) \in \Omega \times \omega$ is the Cantor-Bendixson characteristic of $K$ if $\alpha$ is the smallest countable ordinal number such that $K^{(\alpha)}$ is finite and $|K^{(\alpha)}| = p$. In this case, we write $\text{CB}(K) = (\alpha, p)$.
By Theorem 2.1 for all countable ordinal number $\alpha$, there exists a set $K \in \mathcal{K}$ having Cantor-Bendixson’s characteristic $(\alpha, 1)$. Furthermore, by Corollary 2.1, we have that for all $p \in \omega \setminus \{0\}$ and for all $\alpha \in \Omega$, there exists $K \in \mathcal{K}$ such that $\mathcal{CB}(K) = (\alpha, p)$. In addition, we obviously see that $\mathcal{CB}(\varnothing) = (0, 0)$. Moreover, we have the next result concerning the empty set.

**Proposition 3.1.** Let $K \in \mathcal{K}$ be such that $\mathcal{CB}(K) = (\alpha, p) \in \Omega \times \omega$. Then, $p = 0$ if and only if $K = \varnothing$.

**Proof.** If $K = \varnothing$, then $\mathcal{CB}(K) = (0, 0)$, and thus the result holds. Now, we suppose that $K \neq \varnothing$. We consider three cases.

- If $\alpha = 0$, then $K = K^{(0)}$ is finite. Since $K \neq \varnothing$, we have that $|K^{(0)}| \neq 0$. Hence, $p \neq 0$.
- We suppose now that $\alpha$ is a nonzero limit ordinal. Then, for all $\beta \in \Omega$ such that $\beta < \alpha$, $K^{(\beta)}$ is infinite. Therefore, $(K^{(\beta)})_{\beta < \alpha}$ is a decreasing nested family of nonempty compact subsets of $\mathbb{R}$. By using the Cantor Intersection Theorem, we obtain
  \[ K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)} \neq \varnothing. \]
  Then, $|K^{(\alpha)}| \neq 0$, and so $p \neq 0$.
- Finally, we assume that $\alpha$ is a successor ordinal. Thus, there exists an ordinal $\beta \in \Omega$ such that $\beta + 1 = \alpha$. Since $\beta < \alpha$, it follows that $K^{(\beta)}$ is infinite. Then,
  \[ K^{(\alpha)} = K^{(\beta+1)} = (K^{(\beta)})' \neq \varnothing. \]
  Therefore, $|K^{(\alpha)}| \neq 0$. Hence, $p \neq 0$. \qed

3.1. **Partition of $\mathcal{K}$.** In this subsection, we show some general results concerning the equivalence relation $\sim$ defined on the set $\mathcal{K}$ by (1.2).

**Proposition 3.2.** Let $K_1, K_2 \in \mathcal{K}$ be such that $K_1 \sim K_2$. Then, $K_1' \sim K_2'$.

More precisely, if $f$ is a homeomorphism of $K_1$ onto $K_2$, then $f|_{K_1'}$ is also a homeomorphism of $K_1'$ onto $K_2'$.

**Proof.** Since the image of a limit point, under a homeomorphism, is also a limit point, we see that $f(K_1') = K_2'$. Hence, $f|_{K_1'} : K_1' \mapsto K_2'$ is a homeomorphism. Therefore, $K_1' \sim K_2'$. \qed

By using Transfinite Induction, we get the following result.
Corollary 3.1. Let $K_1, K_2 \in \mathcal{K}$ be such that $K_1 \sim K_2$, and let $\alpha$ be any ordinal number. Then, $K_1^{(\alpha)} \sim K_2^{(\alpha)}$. More precisely, if $f$ is a homeomorphism of $K_1$ onto $K_2$, then $f|_{K_1^{(\alpha)}}$ is also a homeomorphism of $K_1^{(\alpha)}$ onto $K_2^{(\alpha)}$.

It follows from the last corollary that if $K_1, K_2 \in \mathcal{K}$, $K_1 \sim K_2$ and \( CB(K_1) = (\alpha, p) \in \Omega \times \omega \), then there exists a bijective function of $K_1^{(\alpha)}$ onto $K_2^{(\alpha)}$. Therefore, $|K_2^{(\alpha)}| = |K_1^{(\alpha)}| = p$. Hence, $CB(K_2) = (\alpha, p)$. This last result about the Cantor-Bendixson characteristic, which was given by S. Mazurkiewicz and W. Sierpinski in [7], is expressed in the following theorem.

Theorem 3.2. If $K_1, K_2 \in \mathcal{K}$ and $K_1 \sim K_2$, then $CB(K_1) = CB(K_2)$.

The above theorem shows that the Cantor-Bendixson characteristic is preserved for equivalent elements of $\mathcal{K}$, i.e., given $K \in \mathcal{K}$, we have that $CB(K_1) = CB(K)$, for all $K_1 \in [K]$, where $[K]$ denotes the equivalence class of $K$. The reciprocal of Theorem 3.2, which was likewise given by S. Mazurkiewicz and W. Sierpinski in [7], is also true, and for completeness we give a more explicit proof of this fact in Theorem 3.3 below. In the following, we consider any ordinal number as a topological space with the order topology. Lemmas 3.2 to 3.6 will be used in the proof of Theorem 3.3.

Lemma 3.2. Let $K \in \mathcal{K}$ be such that $CB(K) = (1, 1)$. Then, there exists a homeomorphism of $K$ onto $\omega + 1$.

Proof. There is an $x \in \mathbb{R}$ such that $K' = \{x\}$. The set $K \setminus K'$ is infinite and countable. Therefore, there exists a bijective function $g$ of $K \setminus K'$ onto $\omega$. Now, we define $f : K \mapsto \omega + 1$

\[ z \mapsto f(z) = \begin{cases} g(z), & \text{if } z \neq x, \\ \omega, & \text{if } z = x. \end{cases} \]

We see that $f$ is a bijective function. Furthermore, since $\omega + 1$ is a compact topological space, $(\omega + 1)' = \{\omega\}$, $f$ is an injective function, and $f(K') = f(\{x\}) = \{\omega\}$, we have that $f$ is a continuous function. Moreover, since $\omega + 1$ is a Hausdorff space, it follows that $f$ is in fact a homeomorphism. \[\square\]

Lemma 3.3. Let $\alpha$ be a countable ordinal number such that $\alpha > 1$. Suppose that for all ordinal number $\beta$ such that $0 < \beta < \alpha$ and for all $\tilde{K} \in \mathcal{K}$ such that $CB(\tilde{K}) = (\beta, p) \in \tilde{\Omega} \times (\omega \setminus \{0\})$, there exists a homeomorphism $\tilde{f}$ of $\tilde{K}$ onto $\omega^\beta \cdot p + 1$. Then, for all $K \in \mathcal{K}$ such that $CB(K) = (\alpha, 1)$, there exists a homeomorphism of $K$ onto $\omega^\alpha + 1$. 
Proof. Let \( K \in \mathcal{K} \) be such that \( \text{CB}(K) = (\alpha, 1) \). Then, there exists an \( x \in K \) such that \( K^{(\alpha)} = \{x\} \). We have that \( x \in K^{(\alpha)} \subset K'' \). Thus, \( x \) is a limit point of \( K' \). Hence, there exists a strictly increasing or strictly decreasing sequence \( (x_n)_{n \in \omega} \) in \( K' \) such that it converges to \( x \). We suppose that \( (x_n)_{n \in \omega} \) is an strictly increasing sequence in \( K' \), the other case is similar.

We claim that for all \( n \in \omega \), we can take \( r_n > 0 \) such that \( x_n < x - r_n < x_{n+1} \) and \( x - r_n, x + r_n \notin K \). In fact, if we suppose the contrary, then there exists \( l \in \omega \) such that

\[
[x_l - x, x_{l+1} - x] \subset \{r \in \mathbb{R} : x - r \in K \text{ or } x + r \in K\}.
\]

However, the set on the right-hand side of the last inclusion is countable, which is a contradiction. Hence, the claim is proved. We remark that the sequence \( (r_n)_{n \in \omega} \) converges to 0 as \( n \) goes to infinity. We now define the sets

\[
K_0 = K \cap ((-\infty, x - r_0] \cup [x + r_0, +\infty)),
K_k = K \cap ([x - r_{k-1}, x - r_k] \cup [x + r_k, x + r_{k-1}]), \quad k \in \omega \setminus \{0\}. \tag{3.1}
\]

We see that for all \( k \in \omega \), \( x_k \in K_k \). In addition, the sequence of sets \( (K_k)_{k \in \omega} \) satisfies the following properties.

- \( K_k \subset K \), for all \( k \in \omega \).
- \( K_k \in \mathcal{K} \), for all \( k \in \omega \), since they are countable closed subsets of \( K \).
- \( x_k \in K'_k \neq \varnothing \), for all \( k \in \omega \). In fact, let \( \varepsilon > 0 \). First, we consider the case \( k \in \omega \setminus \{0\} \). We now take \( \bar{\varepsilon} := \min\{\varepsilon, x_k - x + r_{k-1}, x - r_k - x_k\} > 0 \). Since \( x_k \in K' \), there exists \( z \in [(x_k - \bar{\varepsilon}, x_k + \bar{\varepsilon}) \setminus \{x_k\}] \cap K \). Thus, \( z \in [(x_k - \varepsilon, x_k + \varepsilon) \setminus \{x_k\}] \cap K_k \). Hence, \( x_k \in K'_k \). For the case \( k = 0 \), by taking \( \bar{\varepsilon} := \min\{\varepsilon, x - r_0 - x_0\} > 0 \), and proceeding in a similar way as in the previous case, we see that \( x_0 \in K'_0 \).

- \( (K_k)_{k \in \omega} \) is a pairwise disjoint sequence in \( \mathcal{K} \).
- \( \bigcup_{k \in \omega} K_k \cup \{x\} = K \). The fact that \( \bigcup_{k \in \omega} K_k \cup \{x\} \subset K \) follows directly from (3.1). In order to prove the reverse inclusion, we take \( z \in K \). If \( z = x \), there is nothing else to show. Now, we suppose that \( z \neq x \). Since \( r_n \to 0 \) as \( n \to +\infty \), we can choose the smallest natural number \( N \in \omega \) such that \( r_N < |x - z| \). Thus, \( z \in K_N \).

Moreover, from (3.1) we see that for all \( k \in \omega \), \( x \notin K_k^{(\alpha)} \subset \{x\} \). Therefore, for all \( k \in \omega \), \( K_k^{(\alpha)} = \varnothing \). Thus, for all \( k \in \omega \), \( \text{CB}(K_k) = (\beta_k, p_k) \in \Omega \times \omega \) implies that \( 0 < \beta_k < \alpha \). We remark that for all \( k \in \omega \), \( K_k \neq \varnothing \) implies that \( p_k \in \omega \setminus \{0\} \). Using the hypothesis, we conclude that for all \( k \in \omega \), there exists a homeomorphism \( f_k \) of \( K_k \) onto \( \omega^{\beta_k} \cdot p_k + 1 \). We now define the
where

\[ f(b) = \begin{cases} f_0(z), & \text{if } z \in K_0, \\ \sum_{j=0}^{k-1} \omega^{\beta_j} \cdot p_j + 1 + f_k(z), & \text{if } z \in K_k, k \in \omega \setminus \{0\}, \\ \tau, & \text{if } z = x, \end{cases} \]

Thus, there exists an ordinal number \( \mu := \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k := \sup \left\{ \sum_{k=0}^{n} \omega^{\beta_k} \cdot p_k : n \in \omega \right\} \).

(a) First, we remark that \( f \) is an injective function. In fact, let \( u, v \in K \) be such that \( f(u) = f(v) \). If \( u = x \) and \( v \in K_0 \), for some \( q \in \omega \), then

\[ f(v) \leq \sum_{k=0}^{q} \omega^{\beta_k} \cdot p_k < \tau = f(u), \]

which is a contradiction. Thus, there exists \( r \in \omega \) such that \( u \in K_r \). We suppose, by contradiction, that \( q \neq r \). Without loss of generality, we may assume that \( q < r \). Then,

\[ f(v) \leq \sum_{k=0}^{q} \omega^{\beta_k} \cdot p_k \leq \sum_{k=0}^{r-1} \omega^{\beta_k} \cdot p_k < \sum_{k=0}^{r-1} \omega^{\beta_k} \cdot p_k + 1 + f_r(u) = f(u), \]

which is not possible. Hence, \( q = r \). Thus,

\[ \sum_{k=0}^{q-1} \omega^{\beta_k} \cdot p_k + 1 + f_q(u) = f(u) = f(v) = \sum_{k=0}^{q-1} \omega^{\beta_k} \cdot p_k + 1 + f_q(v), \]

implies that \( f_q(u) = f_q(v) \). Using the fact that \( f_q \) is an injective function, it follows that \( u = v \).

(b) We will now show that \( f \) is onto. In fact, let \( \gamma \leq \tau \). If \( \gamma = \tau \), we have that \( f(x) = \tau = \gamma \). If \( \gamma < \tau \), we take \( M := \min\{n \in \omega : \gamma \leq \sum_{k=0}^{n} \omega^{\beta_k} \cdot p_k\} \). In case \( M = 0 \), \( \gamma \leq \omega^{\beta_0} \cdot p_0 \). Since, \( f_0 \) is onto, there exists \( z \in K_0 \subset K \) such that \( f(z) = f_0(z) = \gamma \). We now assume that \( M \in \omega \setminus \{0\} \). Then,

\[ \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 \leq \gamma \leq \sum_{k=0}^{M} \omega^{\beta_k} \cdot p_k. \]

Thus, there exists an ordinal number \( \mu \) such that

\[ \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 + \mu = \gamma \leq \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + \omega^{\beta_M} \cdot p_M. \]

Then, \( \mu \leq \omega^{\beta_M} \cdot p_M \). Since \( f_M \) is onto, there exists \( z \in K_M \subset K \) such that \( f_M(z) = \mu \). So, \( f(z) = \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 + f_M(z) = \gamma \).
Moreover, for all \( k \in \omega \), \( f|_{K_k} \) equals an ordinal number, i.e. a constant function, plus a continuous function. Thus, for all \( k \in \omega \), \( f|_{K_k} \) is a continuous function. In addition, since \( (K_k)_{k \in \omega} \) is a pairwise disjoint sequence of open subsets in \( K \), it follows that \( f \) is a continuous function at any element of \( \bigcup_{k \in \omega} K_k \). Furthermore, \( f \) is also continuous at the point \( x \in K \). If fact, let \( \mu \) be an ordinal number such that \( \mu < \tau \). There exists \( m \in \omega \) such that \( \mu < \sum_{j=0}^{m} \omega^{\beta_j} \cdot p_j \). We claim that

\[
\left( x - r_m, x + r_m \right) \cap K \subset \left( \mu, \tau + 1 \right). \quad (3.2)
\]

Let \( y \in \left( x - r_m, x + r_m \right) \cap K \). If \( y = x \), then \( f(y) = f(x) = \tau \in (\mu, \tau + 1) \). We now suppose that \( y \neq x \). Then, there is \( i \in \omega \) such that \( y \in K_i \).

Since \( (r_n)_{n \in \omega} \) is a strictly decreasing sequence of positive numbers, we conclude that \( i > m \). Then,

\[
f(y) = \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + f_i(y) \geq \sum_{j=0}^{m} \omega^{\beta_j} \cdot p_j > \mu. \quad (3.3)
\]

Moreover,

\[
f(y) = \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + f_i(y) \leq \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + \omega^{\beta_i} \cdot p_i
\]

\[
= \sum_{j=0}^{i} \omega^{\beta_j} \cdot p_j \leq \tau < \tau + 1. \quad (3.4)
\]

From (3.3) and (3.4), we see that \( f(y) \in (\mu, \tau + 1) \). Thus, (3.2) follows. Hence, \( f \) is continuous at the point \( x \).

By (a) and (b), \( f \) is a bijective function. In addition, by (c), \( f \) is a continuous function of \( K \) onto \( \tau + 1 \).

We will now prove that \( \tau = \omega^{\alpha} \). In order to get this, let \( \tilde{\alpha} := \sup\{ \beta_k : k \in \omega \} \in \text{OR} \). We see that \( \tilde{\alpha} \leq \alpha \).

(i) First, we consider the case \( \tilde{\alpha} < \alpha \). Then, \( \tilde{\alpha} + 1 \leq \alpha \). Thus, for all \( k \in \omega \), \( K_k^{(\tilde{\alpha} + 1)} = \emptyset \). Using Transfinite Induction, and proceeding as in the proof of (2.2), we get

\[
K^{(\tilde{\alpha} + 1)} = \bigcup_{k \in \omega} K_k^{(\tilde{\alpha} + 1)} \cup \{ x \} = \{ x \}.
\]

Then, \( \tilde{\alpha} + 1 = \alpha \). Since for all \( k \in \omega \), \( \omega^{\beta_k} \cdot p_k \leq \omega^{\tilde{\alpha}} \cdot p_k \), we see that

\[
\tau = \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \leq \omega^{\tilde{\alpha}} \cdot \left( \sum_{k \in \omega} p_k \right) = \omega^{\tilde{\alpha}} \cdot \omega = \omega^{\tilde{\alpha} + 1} = \omega^{\alpha}. \quad (3.5)
\]

On the other hand, we claim that

\[
|\{ n \in \omega : \beta_n = \tilde{\alpha} \}| = \aleph_0. \quad (3.6)
\]
In order to prove (3.6), we first suppose, by contradiction, that for all \( n \in \omega \), \( \beta_n < \tilde{\alpha} \). Thus, for all \( n \in \omega \), \( \beta_n + 1 \leq \tilde{\alpha} \), and we get 
\[
K_n^{(\tilde{\alpha})} \subset K_n^{(\beta_n + 1)} = \emptyset.
\]
Moreover, we see that \( K^{(\tilde{\alpha})} = \bigcup_{k \in \omega} K_k^{(\tilde{\alpha})} \cup \{ x \} = \{ x \} \). Then, \( \tilde{\alpha} = \alpha \), which is a contradiction. Hence, there exists at least one \( n \in \omega \) such that \( \beta_n = \tilde{\alpha} \). We now suppose, again by contradiction, that the set \( \{ n \in \omega : \beta_n = \tilde{\alpha} \} \neq \emptyset \) is finite. Let \( N := \max\{ n \in \omega : \beta_n = \tilde{\alpha} \} \in \omega \). We have that for all \( k \in \omega \) such that \( k > N \), \( \beta_k < \tilde{\alpha} \). Then,
\[
K^{(\tilde{\alpha})} = \bigcup_{k \in \omega} K_k^{(\tilde{\alpha})} \cup \{ x \} = \bigcup_{k=0}^{N} K_k^{(\tilde{\alpha})} \cup \{ x \}.
\]
It follows that, \( K^{(\tilde{\alpha})} \) is a finite set. Hence, \( K^{(\alpha)} = K^{(\tilde{\alpha} + 1)} = \emptyset \), which is a contradiction with the fact that \( K^{(\alpha)} = \{ x \} \). Therefore, (3.6) is proved. We now define, for all \( n \in \omega \),
\[
m_n := |\{ k \in \omega : k \leq n \text{ and } \beta_k = \tilde{\alpha} \}| \in \omega.
\]
Then, for all \( n \in \omega \), we have that
\[
\sum_{k=0}^{n} \omega^{\beta_k} \cdot p_k \geq \omega^{\tilde{\alpha}} \cdot m_n.
\]
For this reason,
\[
\tau = \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \geq \omega^{\tilde{\alpha}} \cdot \sup\{ m_n : n \in \omega \}
\]
\[
= \omega^{\tilde{\alpha}} \cdot \omega = \omega^{\tilde{\alpha}+1} = \omega^\alpha.
\]
Using (3.5) and (3.7), we conclude that \( \tau = \omega^\alpha \).

(ii) We now consider the case \( \tilde{\alpha} = \alpha \). We claim that for all \( k \in \omega \), \( \beta_k < \tilde{\alpha} \). In fact, if there exists \( l \in \omega \) such that \( \beta_l = \tilde{\alpha} \), then
\[
K_l^{(\beta_l)} \cup \{ x \} \subset \bigcup_{i \in \omega} K_i^{(\beta_l)} \cup \{ x \} = K^{(\beta_l)} = K^{(\alpha)} = \{ x \},
\]
contradicting the fact that \( |K_l^{(\beta_l)}| = p_l > 0 \). We now remark that \( \alpha \) is a limit ordinal. In order to prove the last assertion, we suppose, for the sake of contradiction, that \( \alpha \) is a successor ordinal. Then, there exists an ordinal number \( \lambda \) such that \( \alpha = \lambda + 1 \). Thus, for all \( k \in \omega \), \( \beta_k \leq \lambda < \alpha = \tilde{\alpha} \), which is a contradiction with the definition of \( \tilde{\alpha} \). On the other hand, since for all \( k \in \omega \), \( \omega^{\beta_k} \leq \omega^{\beta_k} \cdot p_k \leq \tau \), it follows that
\[
\omega^\alpha = \omega^{\tilde{\alpha}} = \sup\{ \omega^{\beta_k} : k \in \omega \} \leq \tau.
\]
We now define, for all $n \in \omega$,
\[
\beta_{k_n} := \max \{\beta_k : k = 0, 1, \ldots, n\}, \\
p_{k_n} := \max \{p_k : k = 0, 1, \ldots, n\}.
\]
Then, for all $n \in \omega$, we see that
\[
\sum_{k=0}^{n} \omega^{\beta_k} \cdot p_k \leq \omega^{\beta_{k_n}} \cdot p_{k_n} \cdot n < \omega^{\beta_{k_n}+1} \leq \omega^\alpha,
\]
where in the last inequality we have used the fact that $\beta_{k_n} < \beta_{k_n} + 1 \leq \alpha$. In consequence,
\[
\tau = \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \leq \omega^\alpha. \quad (3.9)
\]
Equations (3.8) and (3.9) imply that $\tau = \omega^\alpha$.

Therefore, $f$ is a bijective and continuous function of $K$ onto $\tau + 1 = \omega^\alpha + 1$. In addition, since $\omega^\alpha + 1$ is a Hausdorff space, we conclude that $f$ is a homeomorphism of $K$ onto $\omega^\alpha + 1$. $\square$

**Lemma 3.4.** Suppose that $K$ and $F$ are closed subsets of $\mathbb{R}$ such that $K \cap F = K \cap \overset{\circ}{F}$, where $\overset{\circ}{F}$ is the set of all interior points of $F$. Then, for all $\alpha \in \text{OR}$, we have that
\[
(K \cap F)^{(\alpha)} = K^{(\alpha)} \cap F. \quad (3.10)
\]

**Proof.** We proceed by Transfinite Induction.

- The case $\alpha = 0$ is immediate.
- We now suppose that the result is true for $\alpha \in \text{OR}$. Then,
\[
(K \cap F)^{(\alpha+1)} = ((K \cap F)^{(\alpha)})' = (K^{(\alpha)} \cap F)' \subset (K^{(\alpha)})' \cap F' \subset K^{(\alpha+1)} \cap F,
\]
where in the last expression we have used the induction hypothesis and the fact that $F$ is closed. In order to prove the reverse inclusion, let $x \in K^{(\alpha+1)} \cap F$. Since $K$ is closed, $x \in K \cap F = K \cap \overset{\circ}{F}$. Thus, there exists $r > 0$ such that $(x - r, x + r) \subset F$. Let $\varepsilon > 0$. We now take $\varepsilon := \min \{\varepsilon, r\} > 0$. Then,
\[
\emptyset \neq (\{x - \varepsilon, x + \varepsilon\} \cap \{x\}) \cap K^{(\alpha)} = (\{x - \varepsilon, x + \varepsilon\} \cap \{x\}) \cap K^{(\alpha)} \cap F \subset (\{x - \varepsilon, x + \varepsilon\} \cap \{x\}) \cap (K \cap F)^{(\alpha)}.
\]
Hence, $x \in (K \cap F)^{(\alpha+1)}$. Therefore, $(K \cap F)^{(\alpha+1)} = K^{(\alpha+1)} \cap F$.
- Finally, let $\lambda \neq 0$ be a limit ordinal number. We suppose that for all $\beta \in \text{OR}$ such that $\beta < \lambda$, $(K \cap F)^{(\beta)} = K^{(\beta)} \cap F$. Then,
\[
(K \cap F)^{(\lambda)} = \bigcap_{\beta < \lambda} (K \cap F)^{(\beta)} = \bigcap_{\beta < \lambda} (K^{(\beta)} \cap F) = \bigcap_{\beta < \lambda} K^{(\beta)} \cap F = K^{(\lambda)} \cap F.
\]
This concludes the proof. \hfill \Box

**Lemma 3.5.** Let $\alpha$ be a countable ordinal number such that $\alpha > 0$. Let $p \in \omega \setminus \{0\}$. Suppose that for all $\tilde{K} \in K$ such that $\text{CB}(\tilde{K}) = (\alpha, 1)$, there exists a homeomorphism of $\tilde{K}$ onto $\omega^\alpha + 1$. Then, for all $K \in K$ such that $\text{CB}(K) = (\alpha, p)$, there exists a homeomorphism of $K$ onto $\omega^\alpha \cdot p + 1$.

**Proof.** Let $K \in K$ be such that $\text{CB}(K) = (\alpha, p) \in \Omega \times \omega$. We write $K^{(\alpha)} = \{x_1, x_2, \ldots, x_p\}$, where $x_i < x_j$, for all $i, j \in I := \{1, \ldots, p\}$ with $i < j$. We see that for all $k \in \{1, \ldots, p - 1\}$, there exists $z_k \in (x_k, x_{k+1})$ such that $z_k \notin K$. We now consider the sets
\[
K_1 = K \cap (-\infty, z_1],
K_k = K \cap [z_{k-1}, z_k], \quad k \in \{2, \ldots, p - 1\},
K_p = K \cap [z_{p-1}, +\infty).
\]
(3.11)

Proceeding as in the proof of Lemma 3.3, it is possible to show that the finite family $(K_k)_{k \in I}$ satisfies the following properties:

- $K_k \subset K$, for all $k \in I$.
- $K_k \in K$, for all $k \in I$.
- $x_k \in K_k' \neq \emptyset$, for all $k \in I$.
- $(K_k)_{k \in I}$ is a pairwise disjoint finite sequence in $K$.
- $\biguplus_{k \in I} K_k = K$.

By using Lemma 3.4, we have that for all $k \in I$, $K_k^{(\alpha)} = \{x_k\}$. Therefore, for all $k \in I$, $\text{CB}(K_k) = (\alpha, 1)$. Thus, for all $k \in I$, there exists a homeomorphism $f_k$ of $K_k$ onto $\omega^\alpha + 1$. We now define the function $f$ given by
\[
f : K \mapsto \tau + 1
z \mapsto f(z) = \begin{cases} f_1(z), & \text{if } z \in K_1, \\ \sum_{j=1}^{k-1} \omega^\alpha + 1 + f_k(z), & \text{if } z \in K_k, \text{ for some } k \in I \setminus \{1\}, \end{cases}
\]
where
\[
\tau := \sum_{j=1}^{p} \omega^\alpha = \omega^\alpha \cdot \sum_{j=1}^{p} 1 = \omega^\alpha \cdot p.
\]
Proceeding in a similar fashion as in the items (a), (b) and (c) in the proof of Lemma 3.3, we obtain that $f$ is a homeomorphism of $K$ onto $\omega^\alpha \cdot p + 1$. \hfill \Box

**Lemma 3.6.** Let $\alpha$ be a countable ordinal number such that $\alpha > 0$. Let $p \in \omega \setminus \{0\}$. Then, for all $K \in K$ such that $\text{CB}(K) = (\alpha, p)$, there exists a homeomorphism of $K$ onto $\omega^\alpha \cdot p + 1$. 

Proof. We will use Strong Transfinite Induction. By Lemmas 3.2 and 3.5 the result holds for \( \alpha = 1 \). We now consider \( \alpha \in \Omega \) such that \( \alpha > 1 \), and we suppose that the result is true for all ordinal number \( \beta \) such that \( 0 < \beta < \alpha \). Lemmas 3.3 and 3.5 imply the result for \( \alpha \). Hence, the lemma is proved. \( \square \)

Next result contains the reciprocal of Theorem 3.2.

**Theorem 3.3.** If \( K_1, K_2 \in \mathcal{K} \) and \( \text{CB}(K_1) = \text{CB}(K_2) \), then \( K_1 \sim K_2 \).

**Proof.** If \( \text{CB}(K_1) = \text{CB}(K_2) = (0, p) \in \Omega \times \omega \), we get \( |K_1| = |K_2| = p \). Then, \( K_1 \sim K_2 \).

We now suppose that \( \text{CB}(K_1) = \text{CB}(K_2) = (\alpha, p) \), with \( \alpha > 0 \). By Proposition 3.1, \( p \in \omega \setminus \{0\} \). By Lemma 3.6, there exist two homeomorphisms, \( g \) of \( K_1 \) onto \( \omega \cdot p + 1 \) and \( h \) of \( K_2 \) onto \( \omega \cdot p + 1 \). Therefore, \( f = h^{-1} \circ g: K_1 \mapsto K_2 \) is a homeomorphism of \( K_1 \) onto \( K_2 \). Hence, \( K_1 \sim K_2 \). \( \square \)

Theorems 3.2 and 3.3 fully characterize the partition of \( \mathcal{K} \) by the Cantor-Bendixson characteristic.

### 3.2. Cardinality of the set \( \mathcal{K} \).

Combining the previous results we obtain the cardinality of \( \mathcal{K} \).

**Theorem 3.4.** The set \( \mathcal{K} \), given by (1.4), has cardinality \( \aleph_1 \).

**Proof.** We define the function

\[
\widehat{\text{CB}}: \mathcal{K} \mapsto \left( \Omega \times (\omega \setminus \{0\}) \right) \cup (0, 0)
\]

\[
[K] \mapsto \text{CB}([K]) = \text{CB}(K) = (\alpha, p).
\] (3.12)

By Theorem 3.2 and Proposition 3.1, we see that \( \widehat{\text{CB}} \) is well-defined. Moreover, Corollary 2.1 implies that \( \widehat{\text{CB}} \) is a surjective function. Furthermore, by Theorem 3.3, \( \widehat{\text{CB}} \) is an injective function. Then,

\[
|\mathcal{K}| = \left| \left( \Omega \times (\omega \setminus \{0\}) \right) \cup (0, 0) \right| = |\Omega \times \omega| = |\Omega| = \aleph_1.
\] \( \square \)

Last theorem shows that

\[
\aleph_0 < \aleph_1 = |\mathcal{K}| \leq 2^{\aleph_0} = c,
\]

where \( c \) is the cardinality of \( \mathbb{R} \).

### 3.3. A “primitive” related to the Cantor-Bendixson derivative of compact subsets of the real line.

We end this paper with a last theorem that we can view as a generalization of Theorem 2.1 and Corollary 2.1 given in Section 2. The next result shows that for any compact subset of the reals, there is a primitive-like set associated to its Cantor-Bendixson derivative.
Theorem 3.5. Suppose that \( \alpha \in \Omega \). Let \( F \) be a compact subset of \( \mathbb{R} \). Then, there exists a compact set \( \mathcal{F} \subset \mathbb{R} \) such that \( \mathcal{F}(\alpha) = F \).

Proof. If \( \alpha = 0 \), we define \( \mathcal{F} = F \) and the result holds. From now on, we suppose that \( \alpha > 0 \). There are two cases. First, if \( F \) is perfect, i.e. \( F = F' \), we can take \( \mathcal{F} = F \), and the result follows.

We now assume that \( F \neq F' \). Since \( F \setminus F' \) is the set of all isolated points of \( F \), we have that \( F \setminus F' \neq \emptyset \) is countable. Hence, \( F \setminus F' = \{x_n : n \in I\} \), where \( \emptyset \neq I \subset \omega \), and \( x_n \neq x_m \), for all \( n, m \in I \) with \( n \neq m \). Furthermore, for all \( n \in I \), there exists \( r_n \in (0, \frac{1}{n+1}) \) such that \( (x_n - r_n, x_n + r_n) \cap F = \{x_n\} \).

By Theorem 2.1 we see that for all \( n \in I \), there exits \( K_n \in \mathcal{K} \) such that \( K_n \subset (x_n - r_n, x_n] \) and \( K_n^{(\alpha)} = \{x_n\} \). Since \( ((x_n - r_n, x_n])_{n \in I} \) is a pairwise disjoint sequence of intervals, we see that \( (K_n)_{n \in I} \) is a pairwise disjoint sequence in \( \mathcal{K} \). We now define the set \( \mathcal{F} \subset \mathbb{R} \) given by

\[
\mathcal{F} := \bigcup_{n \in I} K_n \cup F. \tag{3.13}
\]

Claim 1. \( \mathcal{F} \) is a compact subset of \( \mathbb{R} \).

In fact, let \( (z_k)_{k \in \omega} \) be a sequence in \( \mathcal{F} \) such that \( z_k \to z \in \mathbb{R} \) when \( k \to +\infty \). There are three cases.

(i) If \( \{k \in \omega : z_k \in F\} \) is infinite, there exists a subsequence \( (z_{\phi(k)})_{k \in \omega} \) in \( F \), where \( \phi : \omega \to \omega \) is a strictly increasing function. Since \( F \) is closed, we conclude that \( z \in F \subset \mathcal{F} \).

(ii) We now suppose that there exists \( m \in I \) such that \( \{k \in \omega : z_k \in K_m\} \) is infinite. Similarly as in the previous case, we obtain that \( z \in K_m \subset \mathcal{F} \).

(iii) Finally, we assume that for all \( n \in I \), \( \{k \in \omega : z_k \in K_n\} \) is a finite set and \( \{k \in \omega : z_k \in F\} \) is also finite. Thus, there exists a subsequence \( (z_{\psi(k)})_{k \in \omega} \), where \( \psi : \omega \to \omega \) is a strictly increasing function, and there is also a strictly increasing function \( \sigma : \omega \to I \) such that for all \( k \in \omega \)

\[
z_{\psi(k)} \in K_{\sigma(k)} \subset (x_{\sigma(k)} - r_{\sigma(k)}, x_{\sigma(k)}]. \tag{3.14}
\]

In order to prove the last assertion, we see that there exists \( n_0 \in I \) such that \( \{k \in \omega : z_k \in K_{n_0}\} \neq \emptyset \). Then, there is \( k_0 \in \omega \) with \( z_{k_0} \in K_{n_0} \). We thus define \( \psi(0) := k_0 \) and \( \sigma(0) := n_0 \). We now get \( n_1 \in I \) with \( n_1 > n_0 \) and such that \( \{k \in \omega : z_k \in K_{n_1}, k > k_0\} \neq \emptyset \). So, there exists \( k_1 \in \omega \) with \( k_1 > k_0 \) and such that \( z_{k_1} \in K_{n_1} \). We define \( \psi(1) := k_1 \) and \( \sigma(1) := n_1 \). By continuing this process, functions \( \psi \) and \( \sigma \) are recursively obtained. From (3.14), we have that for all \( k \in \omega \), \( |x_{\sigma(k)} - z_{\psi(k)}| < r_{\sigma(k)} < \frac{1}{\sigma(k)+1} \). As \( (z_{\psi(k)})_{k \in \omega} \) converges to
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It follows that \((x_{\sigma(k)})_{k \in \omega}\) also converges to \(z\). Since, the elements of the last sequence belong to \(F\), and \(F\) is closed, we conclude that \(z \in F \subset \mathcal{F}\).

From (i), (ii) and (iii), \(\mathcal{F}\) is a closed subset of \(\mathbb{R}\). Moreover, since \(F\) is bounded, there exist \(a, b \in \mathbb{R}\), with \(a < b\), such that \(F \subset [a, b]\). Then, \(\mathcal{F} \subset [a-1, b]\), i.e., \(\mathcal{F}\) is bounded. Hence, \(\mathcal{F}\) is a compact subset of \(\mathbb{R}\).

**Claim 2.** \(\mathcal{F}^{(\alpha)} = F\).

Actually, we will show the following more general result: for all countable ordinal number \(\beta \in \Omega\) such that \(\beta \leq \alpha\)

\[
\mathcal{F}^{(\beta)} = \bigcup_{n \in I} K_n^{(\beta)} \cup F. \tag{3.15}
\]

In order to prove (3.15), we proceed by Transfinite Induction as in Theorem 2.1.

(a) If \(\beta = 0\), then the result holds immediately.

(b) We now suppose that (3.15) is true for a given \(\beta \in \Omega\) such that \(\beta < \alpha\). We note that for all \(n \in I\), \(K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}\). Then,

\[
\bigcup_{n \in I} K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}.
\]

Furthermore, by the induction hypothesis, \(F \subset \mathcal{F}^{(\beta)}\). Then, \(F' \subset \mathcal{F}^{(\beta+1)}\). Moreover,

\[
F \setminus F' = \bigcup_{n \in I} \{x_n\} = \bigcup_{n \in I} K_n^{(\alpha)} \subset \bigcup_{n \in I} K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}.
\]

Hence,

\[
\bigcup_{n \in I} K_n^{(\beta+1)} \cup F \subset \mathcal{F}^{(\beta+1)}. \tag{3.16}
\]

In order to show the reverse inclusion, we take \(x \in \mathcal{F}^{(\beta+1)}\). Using the induction hypothesis, we see that

\[
x \in \mathcal{F}^{(\beta+1)} = (\mathcal{F}^{(\beta)})' = \left(\bigcup_{n \in I} K_n^{(\beta)} \cup F\right)' = \left(\bigcup_{n \in I} K_n^{(\beta)}\right)' \cup F'.
\]

Using now Claim 1, we have that \(\mathcal{F}\) is closed. Then,

\[
x \in \mathcal{F}^{(\beta+1)} \subset \mathcal{F}^{(\beta)} = \bigcup_{n \in I} K_n^{(\beta)} \cup F.
\]

If \(x \in F\), there is nothing left to show. On the other hand, if \(x \notin F\), there exists \(m \in I\) such that \(x \in K_m^{(\beta)} \subset (x_m - r_m, x_m]\). We now assume, by contradiction, that \(x \notin K_m^{(\beta+1)}\). Then, \(x\) is an isolated point of \(K_m^{(\beta)}\).
Since $x \neq x_m \in F$, there is $0 < \varepsilon < \min\{x - x_m + r_m, x_m - x\}$ such that
\[(x - \varepsilon, x + \varepsilon) \cap K_m^{(\beta)} = \{x\}.
\]
Moreover, as $(x - \varepsilon, x + \varepsilon) \subset (x_m - r_m, x_m)$, we conclude that for all $n \in I$ with $n \neq m$,
\[(x - \varepsilon, x + \varepsilon) \cap K_n^{(\beta)} = \emptyset.
\]
Then,
\[(x - \varepsilon, x + \varepsilon) \cap \bigcup_{n \in I} K_n^{(\beta)} = \{x\}.
\]
Therefore, $x$ is an isolated point of $\bigcup_{n \in I} K_n^{(\beta)}$. Since $x \notin F$, and $F$ is closed, we see that $x \notin F'$. Hence, $x \in \left(\bigcup_{n \in I} K_n^{(\beta)}\right)'$, which is contradictory. In consequence,
\[x \in K_m^{(\beta+1)} \subset \bigcup_{n \in I} K_n^{(\beta+1)}.
\]
Thus, summarizing, we can conclude that
\[F^{(\beta+1)} \subset \bigcup_{n \in I} K_n^{(\beta+1)} \cup F. \tag{3.17}
\]
From (3.16) and (3.17), we get
\[F^{(\beta+1)} = \bigcup_{n \in I} K_n^{(\beta+1)} \cup F. \tag{3.18}
\]
(c) Finally, let $\gamma \neq 0$ be a limit ordinal such that $\gamma \leq \alpha$ and we assume that for all ordinal number $\delta$ such that $\delta < \gamma$,
\[F^{(\delta)} = \bigcup_{n \in I} K_n^{(\delta)} \cup F. \tag{3.18}
\]
Using (3.18), we obtain
\[
\bigcup_{n \in I} K_n^{(\gamma)} \cup F = \bigcup_{n \in I} \left(\bigcap_{\delta < \gamma} K_n^{(\delta)}\right) \cup F \\
\subset \bigcap_{\delta < \gamma} \left(\bigcup_{n \in I} K_n^{(\delta)}\right) \cup F \\
= \bigcap_{\delta < \gamma} \left(\bigcup_{n \in I} K_n^{(\delta)} \cup F\right) \\
= \bigcap_{\delta < \gamma} F^{(\delta)} \\
= F^{(\gamma)}. \tag{3.19}
\]
In order to show the other inclusion, we take \( x \in \mathcal{F}(\gamma) \). Using the induction hypothesis (3.18), we see that

\[
\mathcal{F}(\gamma) = \bigcap_{\delta < \gamma} \mathcal{F}(\delta) = \bigcap_{\delta < \gamma} \left( \bigcup_{n \in I} K_n^{(\delta)} \cup F \right).
\]

Then, either \( x \in F \) or for all ordinal number \( \delta \) such that \( \delta < \gamma \), there exists \( n \in I \) such that \( x \in K_n^{(\delta)} \). If \( x \notin F \), there is \( N \in I \) such that \( x \in K_N^{(0)} = K_N \). We now assume, to get a contradiction, that there is an ordinal number \( \delta_0 \) with \( \delta_0 < \gamma \) and such that \( x \notin K_N^{(\delta_0)} \). Since there is \( l \in I \) with \( l \neq N \) such that \( x \in K_l^{(\delta_0)} \subset K_l \), we obtain a contradiction with the fact that \( K_l \cap K_N = \emptyset \). Hence, for all ordinal number \( \delta \) such that \( \delta < \gamma \), \( x \in K_N^{(\delta)} \). In consequence,

\[
x \in \bigcap_{\delta < \gamma} K_N^{(\delta)} = K_N^{(\gamma)} \subset \bigcup_{n \in I} K_n^{(\gamma)}.
\]

Thus,

\[
\mathcal{F}(\gamma) \subset \bigcup_{n \in I} K_n^{(\gamma)} \cup F. \tag{3.20}
\]

From (3.19) and (3.20), we have that

\[
\mathcal{F}(\gamma) = \bigcup_{n \in I} K_n^{(\gamma)} \cup F.
\]

By (a), (b) and (c), we obtain (3.15) for all countable ordinal number \( \beta \) such that \( \beta \leq \alpha \). Finally, using (3.15) with \( \alpha \), we get

\[
\mathcal{F}(\alpha) = \bigcup_{n \in I} K_n^{(\alpha)} \cup F = \bigcup_{n \in I} \{ x_n \} \cup F = F,
\]

which finishes the proof. \( \square \)

References

[1] G. Cantor, Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen, Math. Ann., 5 (1872), pp. 123–132.
[2] ———, Ueber unendliche, lineare Punktmannichfaltigkeiten I, Math. Ann., 15 (1879), pp. 1–7.
[3] ———, Ueber unendliche, lineare Punktmannichfaltigkeiten II, Math. Ann., 17 (1880), pp. 355–358.
[4] ———, Ueber unendliche, lineare Punktmannichfaltigkeiten III, Math. Ann., 20 (1882), pp. 113–121.
[5] ———, Sur divers théorèmes de la théorie des ensembles de points situés dans un espace continu à n dimensions, Acta Math., 2 (1883), pp. 409–414.
[6] ———, Ueber unendliche, lineare Punktmannichfaltigkeiten IV, Math. Ann., 21 (1883), pp. 51–58.
[7] S. Mazurkiewicz and W. Sierpinski, Contribution à la topologie des ensembles dénombrables, Fund. Math., 1 (1920), pp. 17–27.