Quantum Networks for Generating Arbitrary Quantum States

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Abstract: Quantum protocols often require the generation of specific quantum states. We describe a quantum algorithm for generating any prescribed quantum state. For an important subclass of states, including pure symmetric states, this algorithm is efficient.

1 Introduction

Many results in quantum information theory require the generation of specific quantum states, such as EPR pairs, or the implementation of specific quantum measurements, such as a von Neumann measurement in a Fourier transformed basis. Some states and measurements can be efficiently implemented using standard quantum computational primitives such as preparing a qubit in the state $|0\rangle$ and applying a sequence of quantum gates (from a finite set). EPR pairs can be prepared from the state $|00\rangle$ using a Hadamard gate and a controlled-NOT gate. A von Neumann measurement in the Fourier basis can be efficiently realized by applying an inverse quantum Fourier transform and performing a von Neumann measurement in the standard computational basis, i.e. $\{|0\rangle, |1\rangle\}$. However many states and basis changes cannot be efficiently realized. This paper focusses on the generation of quantum states. For example, in [HMPEPC98], their improved frequency standard experiment requires the preparation of specific symmetric states on $n$ qubits, where $n$ is a parameter (number of ions). The algorithm we describe here will efficiently prepare the required symmetric state. This short paper will focus on the algorithm for generating the state, and will ignore issues related to errors and decoherence (for which the theory of fault-tolerant error-correction, or other stabilization methods, will apply). We do not have space to elaborate on the details of precision, but simple calculations that require $O(\log(\frac{1}{\epsilon}))$ extra space and $\text{polylog}(\frac{1}{\epsilon})$ elementary operations allow us to generate any state with fidelity at least $1 - \epsilon$.

Suppose we want to generate the state $|\Psi\rangle = \sum_{x \in \{0,1\}^n} e^{i\gamma_x} \alpha_x |x\rangle$. In practice it suffices to generate a state $|\tilde{\Psi}\rangle$ satisfying $|\langle \tilde{\Psi} | \Psi \rangle|^2 > 1 - \epsilon$ for a given small real number $\epsilon > 0$. In this case, it suffices to approximate each $\gamma_x$ and $\alpha_x$ to accuracy $\text{poly}(\epsilon)$ (i.e. $O(\log(\frac{1}{\epsilon}))$ bits of accuracy). Here we have factored out the phase in each term, and so the $\alpha_x$ are all non-negative real values. Note that if we can prepare the state $|\Psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$, then we can approximate $|\Psi\rangle$ arbitrarily well by introducing appropriate phase factors using methods discussed in [CEMM98]. We will therefore focus on a method for generating states $|\Psi\rangle$ with non-negative real amplitudes.

2 The algorithm

In order to create the $n$-qubit state $|\Psi\rangle$, we will implement in sequence $n$ controlled rotations, with the $k$th rotation controlled by the state of the previous $k - 1$ qubits for $k > 0$.

We will first define these controlled rotations, and then in the next section we will describe how we would implement them.

Will extend the definition of $\alpha_x$ to $x \in \{0,1\}^j$ for $1 \leq j < n$. Suppose we had a copy of $|\Psi\rangle$, and we measured the leftmost $j$ qubits in the computational basis. Let $\alpha_x$ be the non-negative real number so that $\alpha_x^2$ equals the probability the measurement result is $x$. Then $(\alpha_{x_1x_2\ldots x_{k-1}0}/\alpha_{x_1x_2\ldots x_{k-1}1})^2$ gives the conditional probability that the $k$th qubit is $|0\rangle$, conditioned on the state of the first state of the $k - 1$ qubits being $|x_1x_2\ldots x_{k-1}\rangle$. Define a controlled rotation $c - U^\Psi_{x_1x_2\ldots x_{k-1}0}$ by:

$$|x_1\rangle |x_2\rangle \ldots |x_{k-1}\rangle |0\rangle \xrightarrow{c - U^\Psi_{x_1x_2\ldots x_{k-1}0}} |x_1\rangle |x_2\rangle \ldots |x_{k-1}\rangle \left(\frac{\alpha_{x_1x_2\ldots x_{k-1}0}}{\alpha_{x_1x_2\ldots x_{k-1}1}} |0\rangle + \frac{\alpha_{x_1x_2\ldots x_{k-1}1}}{\alpha_{x_1x_2\ldots x_{k-1}1}} |1\rangle\right)$$
As shown in Figure 1, the algorithm for generating the $n$-qubit state $|\psi\rangle$ is a sequence of $n$ such controlled rotations. It is easy to show by induction that after the first $j$ controlled rotations are applied we have produced the state

$$\sum_{x_1x_2...x_j \in \{0,1\}^j} \alpha_{x_1x_2...x_j} |x_1x_2...x_j\rangle$$

and therefore after all $n$ controlled rotations we have

$$c - U^\Psi_{x_1x_2...x_{n-1}0} \left( c - U^\Psi_{x_1x_2...x_{n-2}0} \left( ... c - U^\Psi_{x_10} (U^\Psi_{0}|0\rangle |0\rangle) |0\rangle \right) |0\rangle \right) = |\Psi\rangle.$$

Fig. 1. Network to generate $|\Psi\rangle$

3 Details

In this section we show how to implement the controlled-$U^\Psi_{x_1x_2...x_{k-1}0}$ with arbitrary precision. First assume we have a quantum register $|\Psi\rangle$ which encodes some “classical” description of the state $\Psi$. The state $|\Psi\rangle$ must contain enough information to allow the probabilities $\alpha_x^2$ (or a related quantity, such as the $\omega_x$ we define below) to be computed. We also use an ancilla register of $O(\log(1/\epsilon))$ qubits initialized to the state $|0\rangle$. Then we define operators $U_k$ for each $1 \leq k \leq n$ as follows:

$$|\Psi\rangle|0\rangle|x_1\rangle ... |x_{k-1}\rangle \xrightarrow{U_k} |\Psi\rangle|\omega_k\rangle|x_1\rangle ... |x_{k-1}\rangle$$

where $\omega_k$ satisfies

$$\cos^2(2\pi\omega_k) = \left(\frac{\alpha_{x_1x_2...x_{k-1}0}/\alpha_{x_1x_2...x_{k-1}}}{\alpha_{x_1x_2...x_{k-1}0}}\right)^2 + O(poly(\epsilon)).$$

A simple application of the techniques in [CEMM98] allow us to approximate (arbitrarily well) the transformation:

$$c - S_\omega : \ |\omega\rangle|0\rangle \longrightarrow |\omega\rangle e^{2\pi i \omega}|0\rangle \ , \ |\omega\rangle|1\rangle \longrightarrow |\omega\rangle e^{-2\pi i \omega}|1\rangle.$$

Also, define $V = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$. With these components, a network implementing $c - U^\Psi_{x_1x_2...x_{k-1}0}$ is shown in Figure 2. Here we assumed that the algorithm works for a general family of states with classical descriptions $|\Psi\rangle$. If we are only interested in producing a specific state $|\Psi\rangle$ the network can be simplified by removing the register containing $|\Psi\rangle$ and simplifying each $U_k$ to work only for that specific $|\Psi\rangle$ (in the same way that one can simplify a circuit for adding variable inputs $x$ and $y$ to one that adds a fixed input 5 to variable input $y$).

4 Efficiency: an example

The overall efficiency of our algorithm depends on how efficiently we can implement $U_k$; in other words, how efficiently we can compute the conditional probabilities $(\alpha_{x_1x_2...x_{k-1}0}/\alpha_{x_1x_2...x_{k-1}})^2$ or equivalently $(\alpha_{x_1x_2...x_{k}0}/\alpha_{x_1x_2...x_{k}})^2$.
One example for which this is easy is the symmetric states. The symmetric state $|S_r⟩$ is defined to be an equally-weighted superposition of the computational basis states $|x⟩$ that have Hamming weight $H(x) = r$ ($H(x)$ is the number of bits of $x$ that equal 1). That is,

$$|S_r⟩ = \frac{1}{\sqrt{\binom{n}{r}}} \sum_{H(x) = r} |x⟩.$$

The conditional probability $\left( \frac{\alpha_{x_1 x_2 \ldots x_{k-1} 1}}{\alpha_{x_1 x_2 \ldots x_{k-1}}} \right)^2$ is easily computed to be

$$\frac{r - H(x_1 x_2 \ldots x_{k-1})}{n - k}$$

for $1 \leq k < n$, and to be $k - H(x_1 x_2 \ldots x_{k-1})$ for $k = n$. The Hamming weight can be efficiently computed as shown in [KM01]. Then we simply need to reversibly compute the $\omega_k$ satisfying $\cos^2(2\pi \omega_k) = \left( \frac{\alpha_{x_1 x_2 \ldots x_{k-1} 1}}{\alpha_{x_1 x_2 \ldots x_{k-1}}} \right)^2 + \text{poly}(\epsilon)$.

Another example for which we can efficiently implement $U_k$ is for more general symmetric pure states

$$\sum_{j=0}^{n} \beta_j |S_j⟩$$

where we are given the $\beta_j$ values (as required in [HMPEPC98]).

This technique will not allow us to generate efficiently all quantum states, but it will work for any family of states where for some reordering of the qubits we can efficiently compute the conditional probabilities $\left( \frac{\alpha_{x_1 x_2 \ldots x_{k-1} 1}}{\alpha_{x_1 x_2 \ldots x_{k}}} \right)^2$.

References

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