Hitting Time Distributions for Denumerable Birth and Death Processes *

Yu Gong and Yong-Hua Mao†

School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, People’s Republic of China

Abstract

We proved the explicit formulas in Laplace transform of the hitting times for the birth and death processes on a denumerable state space with \( \infty \) the exit or entrance boundary. This extends the well known Keilson’s theorem from finite state space to infinite state space. We also apply these formulas to the fastest strong stationary time for strongly ergodic birth and death processes, and obtain the explicit convergence rate in separation.

Keywords and phrases: birth and death process, eigenvalues, hitting time, strong ergodicity, strong stationary time, exit/entrance boundary, separation.

AMS 2000 Subject classification: 60J27, 60J35, 37A30, 47A75

1 Introduction

In this paper, we will study the passage time between any two states of an irreducible birth and death process on the nonnegative integers \( \{0, 1, 2, \ldots \} \). A well-known theorem states that the passage time from state 0 to state \( d(\leq \infty) \) is distributed as a sum of \( d \) independent exponential random variables with distinct rates. These rates are just the non-zero eigenvalues of the associated generator for the process absorbed at state \( d \). This is a well-known theorem usually attributed to Keilson([13]), and it may be traced back at least as far as Karlin and McGregor ([13]). See Diaconis and Miclo [6] for historical comments.

Very recently, Fill [10] gave a first stochastic proof for the result via the duality. An excellent application of this theorem is to the distribution of the fastest strong stationary time for an ergodic birth and death process on \( \{0, 1, \ldots, d\} \). And it is also the starting point of studying separation cut-off for birth and death processes in [7]. By the similar

---

*Research supported in part by Program for New Century Excellent Talents in University (NCET), 973 Project(No 2006CB805901), NSFC(No 10721091)

†Corresponding author: maoyh@bnu.edu.cn
method, Fill proved an analogue result for the upward skip-free processes. Diaconis and Miclo presented another probabilistic proof for it, by using the “differential operators” for birth and death processes.

Consider a continuous-time birth and death process \((X_t)_{t \geq 0}\) with generator \(Q = (q_{ij})\) on \(\mathbb{Z}_+\). The \((q_{ij})\) is as follows

\[
q_{ij} = \begin{cases} 
  b_i, & \text{for } j = i + 1, i \geq 0; \\
  a_i, & \text{for } j = i - 1, i \geq 1; \\
  -(a_i + b_i), & \text{for } j = i \geq 1; \\
  -b_0, & \text{for } j = i = 0; \\
  0, & \text{for other } j \neq i.
\end{cases} \quad (1.1)
\]

Here \(a_i(i \geq 1), b_i(i \geq 0)\) be two sequences of positive numbers.

Let \(T_{i,n} = \inf \{t \geq 0 : X_t = n | X_0 = i\}\) be the hitting time of the state \(n\) starting from the state \(i\). The well known theorem of Keilson is the following.

**Theorem 1.1.** Let \(\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}\) be all (positive) \(n\) eigenvalues of \(-Q^{(n)}\), where

\[
Q^{(n)} = \begin{pmatrix} 
-b_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\
 a_1 & -(a_1 + b_1) & b_1 & 0 & \cdots & 0 & 0 \\
 0 & a_2 & -(a_2 + b_2) & b_2 & \cdots & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & a_{n-1} \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & -(a_{n-1} + b_{n-1})
\end{pmatrix}. \quad (1.2)
\]

Then \(T_{0,n}\) is distributed as a sum of \(n\) independent exponential random variables with rate parameters \(\{\lambda_1^{(n)}, \cdots, \lambda_n^{(n)}\}\). That is

\[
\mathbb{E}e^{-sT_{0,n}} = \prod_{\nu=1}^{n} \frac{\lambda_{\nu}^{(n)}}{s + \lambda_{\nu}^{(n)}}, s \geq 0. \quad (1.3)
\]

We will investigate the distribution of the hitting time \(T_{i,n}\) for the birth and death process \(X_t\) on the nonnegative integers. This includes four cases:

**Case I:** \(0 \leq i < n < \infty\);

**Case II:** \(0 \leq n < i \leq N < \infty\), where \(N\) is a reflecting state;

**Case III:** \(0 \leq i < n = \infty\);

**Case IV:** \(0 \leq n < i < \infty\).

Cases I&II are really easy consequence of Theorem 1.1 since the distributions are actually involved in finite states. This will be done in the next section. Indeed, for Case I, by using the property of the birth and death process and the strong Markov property, we can obtain the explicit formula for any \(0 \leq i < n < \infty\) from that of \(T_{0,n}\) in Theorem 1.1. See Corollary 2.1 below. For Case II, since \(N\) is a reflecting state, we can get the distribution of \(T_{i,n}(0 \leq n < i \leq N)\) from Case I via turning left-side to right, that is, we can take the mapping on the state space: \(j \to j' : j' = N - j\). See Corollary 2.2 below in Section 2.
When we deal with the birth and death process on \( \mathbb{Z}_+ \), we will face the classification of the state \( \infty \) at infinity concerning uniqueness, due to Feller [8]. See also [2, Chapter 8] for more details. According to [8], there are four types of the \( \infty \) boundary: regular, exit, entrance and natural boundaries.

Define \( \mu_0 = 1, \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, i \geq 1 \),

\[
\mu = \sum_{i=1}^{\infty} \mu_i. = \sum_{i=1}^{\infty} \mu_i.
\]

Let us also define \( R = \sum_{i=0}^{\infty} \frac{1}{\mu_i b_i} \sum_{j=0}^{i} \mu_j, S = \sum_{k=0}^{\infty} \frac{1}{\mu_k b_k} \sum_{i=k+1}^{\infty} \mu_i. \)

The \( \infty \) boundary is called exit if \( R < \infty, S = \infty \); entrance if \( R = \infty, S < \infty \). What we will do in this paper is to give distributions of \( T_{i,\infty} \) for the birth and death process with the exit boundary and that of \( T_{i,n}(i > n) \) for the entrance boundary.

The another difficulty for infinite birth and death processes is obviously about all the eigenvalues or the spectrum of the generator. By the spectral theory established in [16] and [17], we can eventually overcome the difficulty. Briefly speaking, we give distributions of \( T_{0,\infty} \) (the life time) for the minimal birth and death process corresponding to \( Q \) when \( \infty \) is the exit boundary. We will use a procedure of approximation with \( n \to \infty \) to derive the distribution of \( T_{0,\infty} \) from that of \( T_{0,n} \) in Theorem 1.1. To deal with the eigenvalues for birth and death processes in infinite state spaces, we should utilize the powerful theory of Dirichlet form. Dirichlet form helps one obtain the variational formulas for eigenvalues, and more importantly provide the approximation procedure. The similar situation appears when \( \infty \) is the entrance boundary, from a view point of [13] on the duality method. The duality method was used successfully in [5] to study the estimation of the principal eigenvalue for birth and death processes.

To end this section, we mention the Dirichlet form concerning birth and death processes. Let

\[
D(f) = \sum_{i=0}^{\infty} \mu_i b_i (f_i - f_{i+1})^2,
\]

and \( \mathcal{D}^{\max}(D) = \{ f \in L^2(\mu) : D(f) < \infty \} \). Then it is proven in [5] Proposition 1.3] that \( (D, \mathcal{D}^{\max}(D)) \) is regular if and only if

\[
\sum_{i=0}^{\infty} \left[ \frac{1}{\mu_i b_i} + \mu_i \right] = \infty.
\]

In other words, the Dirichlet form corresponding to \( Q \) is unique iff (1.6) holds.

We remark that when \( \infty \) is the exit or entrance boundary, the Dirichlet form is unique. Indeed, from [2, Section 8.1], we know that the equivalence condition for the exit boundary is \( R < \infty, \mu = \infty, \sum_{i=0}^{\infty} 1/\mu_i b_i < \infty \); the equivalence condition for the entrance boundary is \( S < \infty, \mu < \infty, \sum_{i=0}^{\infty} 1/\mu_i b_i = \infty \). Thus in any case, (1.6) holds. For regular boundary \( (R < \infty, S < \infty) \), the problem is that the Dirichlet form is not unique. We need to develop new technique other than that used in this paper. For the natural boundary \( (R < \infty, S = \infty) \), the situation is different. Although the Dirichlet form is unique, we
will face the difficulty of the essential spectrum problem. So the formula must be totally different from that in this paper.

The rest of the paper is organized as follows. In Section 2, we derive the distributions of the hitting times $T_{i,n}$ for finite birth and death process from Theorem 1.1. In Section 3, we give the distribution of the life time for the birth and death processes with $\infty$ the exit boundary, starting from any $i \geq 0$. In Section 4, we give the distributions of $T_{i,n}(i \geq n)$ for the birth and death processes with $\infty$ the entrance boundary. And finally in Section 5, the distribution of the (fastest) strong stationary time is derived and we also study the convergence in separation for the process.

2 The finite state space

Let’s first solve Case I from Theorem 1.1.

Corollary 2.1. For $0 \leq i < n < \infty$,

$$\mathbb{E}e^{-sT_{i,n}} = \prod_{\nu=1}^{n} \frac{\lambda_{\nu}^{(n)}}{s + \lambda_{\nu}^{(n)}}, \quad s \geq 0. \quad (2.1)$$

In particular,

$$\mathbb{E}T_{0,n} = \sum_{1 \leq \nu < n} \frac{1}{\lambda_{\nu}^{(n)}}, \quad \mathbb{E}T_{i,n} = \sum_{1 \leq \nu < n} \frac{1}{\lambda_{\nu}^{(n)}} - \sum_{1 \leq \nu < i} \frac{1}{\lambda_{\nu}^{(i)}}. \quad (2.2)$$

Proof. Since $T_{0,n} = T_{0,i} + T_{i,n}$ by the property of the birth and death process and $T_{0,i}, T_{i,n}$ are independent by the strong Markov property, the corollary follows immediately from Theorem 1.1.

$(2.2)$ follows from $(2.1)$ by a standard method to derive the moments from the Laplace transform.

We remark that $(2.2)$ can be called the eigentime identity for absorbed birth and death processes. Cf. [17]. It is different from that in [1] Chapter 3], where the eigentime identity for the ergodic finite Markov chain is involving in the average hitting time. In [16], this kind of eigentime identity for the continuous-time Markov chain on countable state space was studied. See Section 5 below.

For $0 \leq n < N < \infty$, let $\hat{\lambda}_{n,1}^{(N)} < \hat{\lambda}_{n,2}^{(N)} < \cdots < \hat{\lambda}_{n,N-n}^{(N)}$ be the positive eigenvalues of $-\hat{Q}_{n}^{(N)}$, where

$$\hat{Q}_{n}^{(N)} := \begin{pmatrix}
-(a_{n+1} + b_{n+1}) & b_{n+1} & 0 & 0 & \cdots & 0 & 0 \\
 a_{n+2} & -(a_{n+2} + b_{n+2}) & b_{n+2} & 0 & \cdots & 0 & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & a_{N} & -a_{N}
\end{pmatrix}. \quad (2.3)$$

This is the generator of the birth and death process on $\{n, \cdots, N\}$ with absorbing state $n$ and reflecting state $N$. 

4
As we mentioned in the last section, by taking the mapping on the state space: \( j \rightarrow j' = N - j \), we can easily obtain the following results from Theorem 1.1 and Corollary 2.1.

**Corollary 2.2.** For \( 0 \leq n < i \leq N < \infty \),

\[
E e^{-sT_{i,n}} = \prod_{\nu=1}^{N-n} \frac{\hat{\lambda}_{n,\nu}^{(N)}}{s + \hat{\lambda}_{n,\nu}^{(N)}}, \quad s \geq 0.
\]

In particular

\[
E e^{-sT_{N,n}} = \prod_{\nu=1}^{N-n} \frac{\hat{\lambda}_{n,\nu}^{(N)}}{s + \hat{\lambda}_{n,\nu}^{(N)}}, \quad s \geq 0
\]

and

\[
\sum_{\nu=1}^{N-n} \frac{1}{\hat{\lambda}_{n,\nu}^{(N)}} = E T_{N,n} = \sum_{j=n}^{N} \frac{1}{\pi_j b_j} \sum_{i=j+1}^{N} \pi_i.
\]

**Proof.** The second equality in (2.4) can be found on [Page 264].

3. \( \infty \) is the exit boundary

In this section, we will derive the life time distribution for the minimal birth and death process when \( Q \)-processes are not unique. Let’s recall the facts about the uniqueness of birth and death process, see for example [2, 3]. And then we will study the spectral theory for the minimal birth and death processes. The spectral theory helps us pass from finite states to infinite states, especially we will establish what are the limits of the eigenvalues for finite birth and death processes when the states go up to the infinity.

When \( R < \infty \), the corresponding \( Q \)-processes are not unique, for details see [2, Chapter 8] or [3, Chapter 4]. Let \((X_t, t \geq 0)\) be the corresponding continuous-time Markov chain with the minimal \( Q \)-function \( P(t) = (p_{ij}(t) : i, j \in E)\), that is,

\[
p_{ij}(t) = \mathbb{P}_i[X_t = j, t < \zeta]
\]

with \( \zeta = \lim_{n \to \infty} \xi_n \) the life time, where \( \xi_n \) be the successive jumps:

\[
\xi_0 = 0, \quad \xi_n = \inf \left\{ t : t > \xi_{n-1}, X_t \neq X_{\xi_{n-1}} \right\}, \quad n \geq 1.
\]

**Proposition 3.1.** (i) When staring from the state 0, the life time \( \zeta = \lim_{n \to \infty} T_{0,n} \) a.s.

(ii) \( \mathbb{P}_i[\zeta = \infty] = 1 \) or 0 for any \( i \in E \) and \( E_0 \zeta = R \).

**Proof.** (i) Let \( X_0 = 0 \). Note that \( \zeta \) is the (first) time that the process jumps infinite times, for any \( n \), before \( T_{0,n} \) the process jumps only finite times, so that \( \zeta \geq T_{0,n} \) for any \( n \). Thus \( \zeta \geq \lim_{n \to \infty} T_{0,n} \) a.s. Conversely, since the birth and death process jumps once to two nearest neighbors, then \( \xi_n \leq T_{0,n} \) for any \( n \). Thus \( \zeta = \lim_{n \to \infty} \xi_n \leq \lim_{n \to \infty} T_{0,n} \) a.s.

(ii) See [2, Chapter 8] or [3, Chapter 4].

\( \square \)
Denote by $L^2(\mu)$ the usual (real) Hilbert space on $E = \{0, 1, 2, \cdots\}$. Then it is well known that $Q^{(n)}, Q, P(t)$ are self-adjoint operators on $L^2(\mu)$. For a self-adjoint operator $A$ on $L^2(\mu)$, denote $\sigma(A), \sigma_{\text{ess}}(A)$ respectively the spectrum and the essential spectrum of $A$. Here, the essential spectrum consists of continuous spectrum and eigenvalues with infinite multiplicity. When $\sigma_{\text{ess}}(Q) = \emptyset$, denote by $\lambda_1 < \lambda_2 < \cdots$ all the eigenvalues of $-Q$. Actually, all the eigenvalues under consideration in this paper are of one multiplicity, for this see Theorem 3.4 below.

The following result in [17] is our start point of the spectral theory for the minimal birth and death process.

**Theorem 3.2.** If $R < \infty, S = \infty$ (the exit boundary), then $P(t)$ is a Hilbert-Schmidt operator for any $t > 0$. So that $\sigma_{\text{ess}}(Q) = \emptyset$ and

$$\sum_{n \geq 1} \lambda_n^{-1} = R.$$ 

To get the distribution of the life time, we need the following minimax principle for eigenvalues, which is a variant of classical Courant-Fischer theorem for symmetric matrices. See for example [12] p.149.

**Proposition 3.3.** Assume that $\sigma_{\text{ess}}(Q) = \emptyset$. Let $\lambda_\nu(\nu \geq 1), \lambda_\nu^{(n)}(1 \leq \nu < n)$ be eigenvalues for $-Q^{(n)}$ in (1.1) and $-Q$ in (1.2) respectively. Then for $\nu \geq 1$

$$\lambda_\nu = \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : \mu(f^2) = 1, \mu(f f_i) = 0, 0 \leq i < \nu \}.$$ 

and for $1 \leq \nu < n$

$$\lambda_\nu^{(n)} = \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f|_{[n, \infty)} = 0, \mu(f^2) = 1, \mu(f f_i) = 0, 0 \leq i < \nu \}.$$ 

**Proof.** a) We prove the assertion for $\lambda_\nu$ first. Let $e_\nu$ be the corresponding eigenfunction for $\lambda_\nu$, then

$$\lambda_\nu = \inf \{ D(f) : \mu(f^2) = 1, \mu(f e_j) = 1, 1 \leq j < \nu \}.$$ 

Since $f = \sum_{j=1}^{\infty} \mu(f e_j) e_j$ and $-Q e_j = \lambda_j e_j$, then

$$D(f) = \mu((-Q f) f) = \sum_{j=1}^{\infty} \lambda_j \mu(f e_j)^2,$$

so that for $f_i \in L^2(\mu)(1 \leq i < \nu)$,

$$\inf \{ D(f) : \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}$$

$$= \inf \left\{ \sum_{j=1}^{\infty} \lambda_j \mu(f e_j)^2 : \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \right\}$$

$$\leq \inf \left\{ \sum_{j=1}^{\nu} \lambda_j \langle f, e_j \rangle^2 : \mu(f^2) = 1, \mu(f f_i) = 0(1 \leq i < \nu), \mu(f e_j) = 0(j > \nu) \right\}$$

$$\leq \sup \left\{ \sum_{j=1}^{\nu} \lambda_j \langle f, e_j \rangle^2 : \mu(f^2) = 1, \mu(f e_j) = 0, j > \nu \right\}$$

$$= \lambda_\nu.$$
Therefore,

\[ \lambda_\nu \geq \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \} . \]

If we choose \( f_i = e_i, 1 \leq i < \nu \), the above equality holds.

b) For \( f|_{[n, \infty)} = 0 \),

\[ D(f) = \sum_{i=0}^{n-2} \mu_ib_i(f_i - f_{i+1})^2 + \mu_{n-1}b_{n-1}f_{n-1}^2 =: D^{(n)}(f), \quad (3.3) \]

then it’s easy to check that \( D^{(n)}(f) \) is Dirichlet form for \( Q^{(n)} \) in \((1.2)\). The rest of the proof is the same as above. \( \square \)

Let

\[ \mathcal{K} = \{ f : f \text{ has finite support} \} . \quad (3.4) \]

Define

\[ D(f) = \sum_{i=0}^{\infty} \mu_i b_i(f_i - f_{i+1})^2 \]

with the minimal domain \( \mathcal{D}(D) \) consisting of the functions in the closure of \( \mathcal{K} \) with respect to the norm \( || \cdot ||_D : ||f||_D = \mu(f^2) + D(f) \). In this paper, we deal with the minimal Dirichlet form or the minimal processes, cf. [3, Proposition 6.59].

**Theorem 3.4.** Assume that \( R < \infty, S = \infty, \) then

\[ \forall \nu \geq 1, \lambda_\nu^{(n)} \downarrow \lambda_\nu. \]

Moreover, all eigenvalues \( \lambda_\nu \) are distinct (each of one multiplicity).

**Proof.** a) We use Proposition 3.3 to prove the monotonicity for \( \lambda_\nu^{(n)} \). For any fixed \( 1 \leq \nu < n \) and \( f_1, \ldots, f_{\nu-1} \in L^2(\mu) \), if \( f \) is such that \( f|_{[n, \infty)} = 0, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \), then \( f|_{[n+1, \infty)} = 0, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \), so that

\[ \min \{ D(f) : f|_{[n, \infty)} = 0, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \} \]

\[ \geq \min \{ D(f) : f|_{[n+1, \infty)} = 0, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \} \]

and \( \lambda_\nu^{(n)} \geq \lambda_\nu^{(n+1)} \) for \( 1 \leq \nu < n \). This proves the monotonicity. Thus the limit

\[ \lim_{n \to \infty} \lambda_\nu^{(n)} =: \lambda_\nu \]

exists for any \( \nu \geq 1 \).

b) As pointed in Section 1, when \( R < \infty, \mu = \infty, (D, \mathcal{D}(D)) \) is a regular Dirichlet form. That is, let

\[ \mathcal{D}_{max}(D) = \{ f \in L^2(\mu) : D(f) < \infty \} , \]

and \( (D, \mathcal{D}_{max}(D)) \) be the maximum Dirichlet form, then \( \mathcal{D}(D) = \mathcal{D}_{max}(D) \). Therefore, when \( R < \infty, \mu = \infty \) it follows from Proposition 3.3 that

\[ \lambda_\nu = \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f \in \mathcal{K}, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \} . \quad (3.5) \]

7
c) On one hand, we have \( \lambda_\nu \leq \hat{\lambda}_\nu \). Indeed, it follows from (3.5) and monotonicity that for \( \nu \geq 1 \),

\[
\lambda_\nu = \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f \in \mathcal{H}, \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}
\]

\[
= \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : \exists n \geq 0, f|_{[n, \infty)} = 0; \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}
\]

\[
\leq \inf_{n \geq 1} \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f|_{[n, \infty)} = 0, \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}
\]

\[
\leq \lim_{n \to \infty} \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f|_{[n, \infty)} = 0, \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}
\]

\[
= \lim_{n \to \infty} \lambda_\nu(n) = \hat{\lambda}_\nu.
\]

On the other hand, from Corollary 2.2 we have

\[
\mathbb{E} T_{0, n} = \sum_{1 \leq \nu < n} \frac{1}{\lambda_\nu(n)} = \sum_{1 \leq \nu < \infty} \frac{1}{\lambda_\nu(n)} I_{[\nu < n]},
\]

then it follows from monotone convergence theorem that

\[
R = \mathbb{E}_0 \zeta = \sum_{1 \leq \nu < \infty} \frac{1}{\lambda_\nu}.
\]

But we already know from Theorem 3.2 that

\[
R = \sum_{1 \leq \nu < \infty} \frac{1}{\lambda_\nu} < \infty.
\]

Since \( \lambda_\nu \leq \hat{\lambda}_\nu(\nu \geq 1) \), it must hold that \( \lambda_\nu = \hat{\lambda}_\nu \) for any \( \nu \geq 1 \).

d) Next we will prove that all eigenvalues \( \{ \lambda_\nu, \nu \geq 1 \} \) are distinct. For this we only need to prove that the eigenspace for any \( \lambda_\nu \) is of one dimension. Indeed, let \( -\lambda \) be an eigenvalue and \( g \) the corresponding eigenfunction. From \( Qg(i) = -\lambda g_i, i \geq 0, \) we have

\[
b_0(g_1 - g_0) = -\lambda g_0, a_i(g_{i-1} - g_i) + b_i(g_{i+1} - g_i) = -\lambda g_i, i \geq 1.
\]

(3.8)

Since \( \mu_i b_i = \mu_{i+1} a_{i+1} (i \geq 0) \), it follows from (3.8) that

\[
g_{k+1} = \frac{\lambda}{\mu_k b_k} \sum_{i=0}^k \mu_i g_i + g_k, k \geq 0.
\]

(cf.[5]) This means that eigenfunction \( g \) is determined uniquely once \( g_0 \) is given.

Remark 3.5. For the first eigenvalue \( \nu = 1 \), it was proved by Chen M.-F.(2009) without the assumption \( \sigma_{\text{ess}}(Q) = \emptyset \) in case that \( \lambda_1 \) is defined by the classical Poincaré variational formula:

\[
\lambda_1 = \inf \left\{ \sum_{i=0}^{\infty} \mu_i b_i(f_i - f_{i+1})^2 : f \in \mathcal{H}, \sum_{i=0}^{\infty} \mu_i f_i^2 = 1 \right\}.
\]
Theorem 3.6. Assume $R < \infty, S = \infty$. Let $\zeta$ be the life time for the minimal process, then

$$
\mathbb{E}_0 e^{-s\zeta} = \prod_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{s + \lambda_{\nu}}, \ s \geq 0.
$$

And for any $i \geq 0$, let $T_{i,\infty} = \lim_{n \to \infty} T_{i,n}$, then

$$
\mathbb{E}_i e^{-s\zeta} = \mathbb{E} e^{-sT_{i,\infty}} = \prod_{\nu=1}^{\infty} \frac{\lambda_{\nu}^{(i)}}{s + \lambda_{\nu}^{(i)}}, \ s \geq 0.
$$

Proof. The assertions follow from the monotone convergence theorem and Proposition 3.1, Corollary 2.1, Theorem 3.4. \qed

For the exit boundary, the distributions of $T_{i,n}$ for $0 \leq i, n \leq \infty$ are all known. When $0 \leq i < n < \infty$, the distribution is given by Corollary 2.1, while $0 \leq i < n = \infty$, the distribution is given by Theorem 3.6.

4 \quad \infty \text{ is the entrance boundary}

In this section we will deal with Case IV for the birth and death process with $\infty$ the entrance boundary, i.e. $R = \infty, S < \infty$, and the corresponding $Q$-process is unique.

Since $S < \infty, \mu < \infty$. Let $\pi_i = \mu_i/\mu$, then $\pi = (\pi_i, i \geq 0)$ is a probability measure on $E$, so that the process is reversible with respect to $\pi$. Now we will consider the spectral theory for operators on Hilbert space $L^2(\pi)$.

For $n \geq 0$, let

$$
\hat{Q}_n = \begin{pmatrix}
-(a_{n+1} + b_{n+1}) & b_{n+1} & 0 & \cdots & \cdots & \cdots \\
a_{n+2} & -(a_{n+2} + b_{n+2}) & b_{n+2} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
$$

be the generator of the birth and death process absorbed at state $n$.

Let $\tilde{\pi}^{(n)} = (\pi_i : i > n)$ and $\tilde{E}_n = \{n+1, n+2, \cdots \}$. It is easy to check that $\hat{Q}_n$ is symmetric with respect to $\tilde{\pi}(n)$ and then $\hat{Q}_n$ is a self-adjoint operator in $L^2(\tilde{E}_n, \tilde{\pi}^{(n)})$. When $\sigma_{\text{ess}}(\hat{Q}_n) = \emptyset$, denote by $\tilde{\lambda}_{n,1} < \tilde{\lambda}_{n,2} < \cdots$ all the positive eigenvalues of $-\hat{Q}_n$, as we know from Theorem 3.4 that each eigenvalue is of one multiplicity. When $n = 0$, the subscript 0 is dropped.

Theorem 4.1. For the birth and death process with $\infty$ the entrance boundary, i.e. $R = \infty, S < \infty$, then $\sigma_{\text{ess}}(\hat{Q}_n) = \emptyset$, and for any $n \geq 0$

$$
S_n := \sum_{j=n}^{\infty} \frac{1}{\pi_j b_j} \sum_{i=j+1}^{\infty} \pi_i = \sum_{\nu \geq 1} \tilde{\lambda}_{n,\nu}^{-1} < \infty.
$$

9
Proof. It follows from [16, Theorem 1.4] that $\sigma_{\text{ess}}(Q) = \emptyset$ and $\sum_{\nu \geq 1} \hat{\lambda}_{\nu}^{-1} < \infty$, where $Q$ is defined by [14, Theorem 1.4] and $\{\lambda_{\nu} : \nu \geq 0\}$ is spectrum of $-Q$ in $L^2(\mu)$ with $\lambda_0 = 0$. But since $Q$ and $\hat{Q}_n$ differ only from a finite states, their essential spectrum is same (see for example [14, Theorem 5.35 on page 244]).

Now we prove the identity in (4.2). Let for $i, j > n$

$$p^{(n)}_{ij}(t) = \mathbb{P}_i[X_t = j, t < T_{i,n}], g^{(n)}_{ij} = \int_0^\infty p^{(n)}_{ij}(t)dt.$$ 

By a similar method as in the proof of [17, Theorem 1.4], we can get that

$$\sum_{\nu \geq 1} \hat{\lambda}_{n,\nu}^{-1} = \sum_{i > n} g^{(n)}_{ii} = \sum_{i > n} \frac{1}{(a_i + b_i)p^1_i[\tau^+_{i} = \infty]}.$$ 

Here $\tau^+_{i} = \inf \{ t \geq 1 : X_t = i \}$ is the return time. Note that once $[\tau^+_{i} = \infty|X_0 = i]$ happens, it must first jump to state $i - 1$, otherwise it can be back to state $i$ in finite time almost surely since the original $Q$-process is ergodic. Next when it comes to state $i - 1$, it must arrive to state $n$ before it arrives state $i$. Thus we have

$$p^1_i[\tau^+_{i} = \infty] = \frac{a_i}{a_i + b_i}p^1_{i-1}[T_{i-1,n} < \tau^+_{i}].$$ 

and by a standard martingale method and letting $s_i = \sum_{j < i}(\pi_j b_j)^{-1}$ be the scale function (see for example [21, Theorem 4 in §7.11]), we obtain

$$p^1_{i-1}[T_{i-1,n} < \tau^+_{i}] = \frac{s_i - s_{i-1}}{s_i - s_{n}} = \frac{(\mu_{i-1} b_{i-1})^{-1}}{\sum_{n \leq j < i}(\pi_j b_j)^{-1}} = \frac{(\mu_i a_i)^{-1}}{\sum_{n \leq j < i}(\pi_j b_j)^{-1}}.$$ 

Therefore

$$\sum_{\nu \geq 1} \hat{\lambda}_{n,\nu}^{-1} = \sum_{i > n} \pi_i \sum_{n \leq j < i} \frac{1}{\pi_j b_j} = \sum_{j = n}^{\infty} \frac{1}{\pi_j b_j} \sum_{i = j + 1}^{\infty} \pi_i.$$

As in the last section, we also need the minimax principle for eigenvalues of $\hat{Q}_n$.

**Proposition 4.2.** For $0 \leq n < N < \infty$, let

$$D^{(N)}_n(f) = \mu(-f\hat{Q}^{(N)}_n f) = \sum_{i = n+1}^{N-1} \mu_i b_i (f_i - f_{i+1})^2 + \mu_n b_n f_{n+1}^2.$$ 

Then for $1 \leq \nu < N - n$

$$\hat{\lambda}^{(N)}_{n,\nu} = \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D^{(N)}_n(f) : \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}.$$ 

**Proof.** The proof is direct and is omitted. \qed

**Proposition 4.3.** Assume that $\sigma_{\text{ess}}(Q) = \emptyset$. Let $\hat{\lambda}_{n,\nu}$ be eigenvalues for $-\hat{Q}^{(n)}$ in (4.4). Then for $\nu \geq 1$

$$\hat{\lambda}_{n,\nu} = \max_{f_1, \ldots, f_{\nu-1} \in L^2(\mu)} \min \{ D(f) : f_{[0,n]} = 0, \mu(f^2) = 1, \mu(f f_i) = 0, 1 \leq i < \nu \}.$$ 

(4.4)
Proof. Since \( D(f) = \sum_{i=n+1}^{\infty} \mu_i b_i (f_i - f_{i+1})^2 + \mu_n b_n f^2_{n+1} = \mu(-f\hat{Q}_n f) \). The rest of proof is similar to that of Proposition 3.3

We need the approximation procedure when \( N \to \infty \). For this purpose, we need to do more.

Fix \( n \geq 0 \) and define

\[
\tilde{\mathcal{K}} = \{ f \in L^\infty : f \neq 0 \} \subset \{ n+1, \cdots, N \} \text{ for some } N. 
\]

and \( \tilde{\mathcal{K}}_L = \{ g := cf + d : f \in \tilde{\mathcal{K}}, c, d \in \mathbb{R} \} \). Define

\[
D(f) = \sum_{i=0}^{\infty} \mu_i b_i (f_i - f_{i+1})^2
\]

with the domain \( D(D) \) consisting of the functions in the closure of \( \tilde{\mathcal{K}}_L \) with respect to the norm \( || \cdot ||_D : ||f||_D = \mu(f^2) + D(f) \).

Since \( \infty \) is the entrance boundary, the Dirichlet form is unique as explained in Section 1, thus \( D(D) = D^\text{max}(D) = \{ f \in L^2(\mu) : D(f) < \infty \} \) and we can rewrite (4.4) as

\[
\hat{\lambda}_{n,\nu} = \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \left\{ D(f) : f \in \tilde{\mathcal{K}}_L, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \right\}. 
\]

This leads to

**Theorem 4.4.** Assume that \( R = \infty, S < \infty \), then

\[
\forall \nu \geq 1, \hat{\lambda}_{n,\nu} \downarrow \hat{\lambda}_{n,\nu} \text{ as } N \to \infty. 
\]

Proof. (a) Let \( f \in \tilde{\mathcal{K}}_L \), assume that \( f_{[N, \infty)} = c \), then

\[
D(f) = \sum_{i=0}^{\infty} \mu_i b_i (f_i - f_{i+1})^2 = \sum_{i=n+1}^{\infty} \mu_i b_i (f_i - f_{i+1})^2 + \mu_n b_n f^2_{n+1}
\]

\[
= \sum_{i=n+1}^{N-1} \mu_i b_i (f_i - f_{i+1})^2 + \mu_n b_n f^2_{n+1} = D_n^{(N)}(f). 
\]

Here in \( D_n^{(N)}(f) \), \( f \) is viewed as a function on \( \{ n+1, \cdots, N \} \). From this, we can easily deduce the monotonicity of \( \hat{\lambda}_{n,\nu}^{(N)} \) in \( N \).

(b) By Theorem 9.11 in [1] and Proposition 4.3 above, we have that for \( \nu \geq 1 \),

\[
\hat{\lambda}_{n,\nu} = \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \left\{ D(f) : f \in \tilde{\mathcal{K}}_L, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \right\}
\]

\[
= \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \left\{ D_n^{(N)}(f) : \exists \, N > n, f_{[N, \infty)} = \text{constant}, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \right\}
\]

\[
\leq \lim_{N \to \infty} \max_{f_1, \cdots, f_{\nu-1} \in L^2(\mu)} \min \left\{ D_n^{(N)}(f) : f_{[N, \infty)} = \text{constant}, \mu(f^2) = 1, \mu(ff_i) = 0, 1 \leq i < \nu \right\}
\]

\[
= \lim_{N \to \infty} \hat{\lambda}_{n,\nu} = \alpha_{n,\nu}. 
\]
(c) It follows from (2.4) in Corollary 2.2 that

$$\sum_{j=n}^{N} \frac{1}{\pi_j b_j} \sum_{i=j+1}^{N} \pi_i = \sum_{\nu \geq 1} \frac{1}{\lambda_{n,\nu}^{(N)}},$$

from which by letting $N \to \infty$ we have

$$\sum_{j=n}^{\infty} \frac{1}{\pi_j b_j} \sum_{i=j+1}^{\infty} \pi_i = \sum_{\nu \geq 1} \frac{1}{\alpha_{n,\nu}}.$$

But for any $\nu \geq 1$, $\hat{\lambda}_{n,\nu} \leq \alpha_{n,\nu}$ and it follows from Theorem 4.1 that

$$\sum_{j=n}^{\infty} \frac{1}{\pi_j b_j} \sum_{i=j+1}^{\infty} \pi_i = \sum_{\nu \geq 1} \frac{1}{\lambda_{n,\nu}}.$$

Thus it must hold that $\hat{\lambda}_{n,\nu} = \alpha_{n,\nu} = \lim_{N \to \infty} \hat{\lambda}_{n,\nu}^{(N)}$ for any $\nu \geq 1$.

**Theorem 4.5.** Assume the birth and death process is such that $\infty$ the entrance boundary. For $n \geq 1$, we have

$$\mathbb{E}e^{-sT_{n,0}} = \prod_{\nu=1}^{\infty} \frac{\hat{\lambda}_\nu}{s + \hat{\lambda}_\nu}, s \geq 0. \quad (4.6)$$

Let $T_{\infty,0} = \lim_{n \to \infty} T_{n,0}$, then

$$\mathbb{E}e^{-sT_{\infty,0}} = \prod_{\nu=1}^{\infty} \frac{\hat{\lambda}_\nu}{s + \hat{\lambda}_\nu}, s \geq 0. \quad (4.7)$$

**Proof.**

a) By using the monotone convergence theorem, we can get (4.6) from Theorem 4.4 and Corollary 2.2 immediately.

b) To pass from (4.6) to (4.7), we only need to show that $\lim_{n \to \infty} \hat{\lambda}_{n,1} = \infty$, so that $\hat{\lambda}_{n,\nu}$ tend to infinity uniformly in $\nu \geq 1$ as $n \to \infty$. This is a consequence of [16, Corollary 1.1] and [18, Theorem 3.4].

For the entrance boundary, the distributions of $T_{i,n}$ for $0 \leq i, n \leq \infty$ are all known. When $0 \leq i < n < \infty$, the distribution of $T_{i,n}$ is given by Corollary 2.1, while $0 \leq i < n = \infty$, $T_{i,\infty} = \infty$ a.s. since the life time $\zeta = \infty$ a.s. When $0 \leq n < i \leq \infty$, the distribution of $T_{i,n}$ is given by Theorem 4.5.

5 Application to the fastest strong stationary time

In this section, we apply the theorems to the strong stationary time. We give the distribution to the fastest strong stationary times, and then study the exponential convergence in separation for the birth and death process.
First of all, we would like to recall some facts about the strong ergodicity for the birth and death process.

Let $T$ be the average hitting time: $T = \sum_{i,j} \pi_i \pi_j E T_{i,j}$. (cf. [1, Chapter 3]) From [2, Chapter 8], we can eventually calculate out that (cf. [16])

$$T = \sum_{k=0}^{\infty} \frac{1}{\pi_k b_k} \sum_{i=0}^{k} \pi_i \sum_{i=k+1}^{\infty} \pi_i.$$

In the following theorem, we summary the facts of strong ergodicity.

**Theorem 5.1.** Assume that the process is unique (i.e. $R = \infty$). The following statements are equivalent.

(i) $S < \infty$.

(ii) The process is strongly ergodic.

(iii) $T < \infty$.

(iv) $\sigma_{\text{ess}}(Q) = \emptyset$ and $\sum_{\nu \geq 1} \lambda_{\nu}^{-1} < \infty$.

Furthermore, when $\sigma_{\text{ess}}(Q) = \emptyset$, then

$$\sum_{\nu \geq 1} \lambda_{\nu}^{-1} = T.$$

**Proof.** The equivalence of (i) and (ii) was proved in [22] and the other assertions were prove in [16].

Now we study the distribution of the strong stationary time for the strongly ergodic birth and death process. A strong stationary time (SST) is a (minimal) randomized stopping time $\tau$ for $X_t$ such that $X_{\tau}$ has the distribution $\pi$ and it independent of $\tau$.

With the aid of Theorem 3.6 and the duality established in [9], we can obtain the following result, which extends the result in [10] to the denumerable case.

**Theorem 5.2.** For a strongly ergodic birth and death process (i.e. $R = \infty, S < \infty$), the SST $\tau$ has distribution

$$E_0 e^{-s \tau} = \prod_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{s + \lambda_{\nu}},$$

where $\{\lambda_{\nu} : \nu \geq 1\}$ are the positive eigenvalues of $-Q$ for the strongly ergodic birth and death process.

**Proof.** We follow the argument in [9, Section 3.3] with a minor modification. Let

$$H_i = \sum_{j \leq i} \pi_j,$$

where $\pi_j = \mu_j / \mu$ is as before. Then $0 < 1 / \mu = \pi_0 \leq H_i \leq 1$.

Define the dual $Q^*$-birth and death process with parameters $(a^*_i, b^*_i)$ given by

$$a^*_i = \frac{H_{i-1} b_i}{H_i}, \quad b^*_i = \frac{H_{i+1} a_{i+1}}{H_i}.$$

(5.1)
Also
\[ \mu^*_i = \frac{b_0 \cdots b_{i-1}^*}{a_1^* \cdots a_i^*} = \frac{b_0}{\mu_i b_i} H_i^2. \]

Note that \( H_i \geq H_0 \), we have
\[ \mu^* = \sum_{i=0}^{\infty} \mu^*_i = \sum_{i=0}^{\infty} \frac{b_0 H_i^2}{\mu_i b_i H_0^2} \geq \sum_{i=0}^{\infty} \frac{b_0 H_i^2}{\mu_i b_i} = \sum_{i=0}^{\infty} \frac{b_0}{\mu_i b_i} \sum_{j \leq i} \mu_j = b_0 R = \infty, \]
and as \( 1/\mu \leq H_j \leq 1, \mu_i b_i = \mu_{i+1} a_{i+1} \),
\[ R^* = \sum_{i=0}^{\infty} \frac{1}{\mu_i^* b_i} \sum_{j \leq i} \mu_j^* = \sum_{i=0}^{\infty} \frac{\mu_{i+1}}{b_0 H_i H_{i+1}} \sum_{j \leq i} \frac{b_0 H_j^2}{\mu_j b_j} \]
\[ = \sum_{j=0}^{\infty} \frac{H_j^2}{\mu_j b_j} \sum_{i \geq j} \frac{\mu_{i+1}}{H_i H_{i+1}} \leq \mu^{-2} S < \infty. \]

Thus the \( Q^* \)-process (minimal process) is with \( \infty \) the exit boundary and \( \tau \) has the same distribution as the life time \( \zeta^* \) (the time attaining \( \infty \)) for \( Q^* \)-process. Applying Theorem 3.6, we have
\[ \mathbb{E}_0 e^{-s \tau} = \mathbb{E}_0 e^{-s \zeta^*} = \prod_{\nu=1}^{\infty} \frac{\lambda^*_\nu}{s + \lambda^*_\nu}, \]
where \( \{\lambda^*_\nu, \nu \geq 1\} \) is spectrum of \(-Q^*\). To complete the proof, it suffices to prove that \( \lambda^*_\nu = \lambda_\nu \) for \( \nu \geq 1 \).

Actually, let the link matrix \( \Lambda = (\Lambda_{ij}) \) as \( \Lambda_{ij} = 1_{[j \leq i]} \pi_j / H_i \), then \( \Lambda Q = Q^* \Lambda \). This means that \( Q \) has the same spectrum in \( L^2(\mu) \) as \( Q^* \) in \( L^2(\mu^*) \). But as proved in Theorems 3.2 and 5.1, the spectra are all eigenvalues, so that \( \lambda^*_\nu = \lambda_\nu \) for \( \nu \geq 1 \).

Theorem 5.2 enables one to obtain the uniform convergence in separation. Actually we will prove that the uniform convergence in separation is equivalent to the strong ergodicity for the birth and death process.

Let \( p_{ij}(t) \) be the transition function for the birth and death process with stationary distribution \( \pi \), define the separation:
\[ s_i(t) = \sup_j \left( 1 - \frac{p_{ij}(t)}{\pi_j} \right), \forall i \geq 0, t \geq 0. \]

For the elementary properties of separation, see [9]. For example, we have the following relation for total variance distance and separation:
\[ ||p_i(t) - \pi||_{\text{Var}} := \sum_j |p_{ij}(t) - \pi_j| \leq s_i(t). \quad (5.2) \]

The main theorem in [9] (see also [10 Proposition 1]) says
\[ s_i(t) \leq \mathbb{P}_i[\tau > t], \forall 0 \leq t < \infty. \quad (5.3) \]
This leads to study the distribution of \( \tau \) starting from any state other than 0.
Proposition 5.3. Let $\tau$ be the SST in Theorem 5.2 and $X_t$ be the $Q$-process and $X^*_t$ be the minimal $Q^*$-process with $Q^*$ given by (5.1). Then for $t \geq 0$

$$P_0[\tau \leq t] = P_0[\zeta^* \leq t],$$

and for $i \geq 1$

$$P_i[\tau \leq t] = \frac{1}{\pi_i} \left\{ H_i P_i[\zeta^* \leq t] - H_{i-1} P_{i-1}[\zeta^* \leq t] \right\}. \tag{5.5}$$

Consequently, for $i \geq 1$, $E_i \tau \leq E_{i-1} \zeta^* \leq E_0 \zeta^*$ and

$$\sup_{i \geq 0} E_i \tau = E_0 \tau. \tag{5.6}$$

Proof. The proof of (5.4) can be found in [9]. We will prove (5.5). Let $m$ and $m^*$ be the distribution of $X_0$ and $X^*_0$ respectively, it follow from [9, Proposition 4] that if $m = m^* \Lambda$, then

$$P_m[\tau \leq t] = P_{m^*}[\zeta^* \leq t]. \tag{5.7}$$

For any $i \geq 1$, let $m^* = \delta_i$ the Dirac measure, then $m_k = \frac{\pi_k}{H_i} \delta_{[0\leq k \leq i]}$ and equality (5.7) implies that

$$\sum_{0 \leq k \leq i} \frac{\pi_k}{H_i} P_k[\tau \leq t] = P_i[\zeta^* \leq t] \quad \text{or} \quad \sum_{0 \leq k \leq i} \pi_k P_k[\tau \leq t] = H_i P_i[\zeta^* \leq t].$$

Thus

$$P_i[\tau \leq t] = \frac{1}{\pi_i} \left\{ H_i P_i[\zeta^* \leq t] - H_{i-1} P_{i-1}[\zeta^* \leq t] \right\}.$$ 

Since $P_i[\zeta^* \leq t]$ increases in $i \geq 0$, we have

$$P_i[\tau \leq t] \geq \frac{1}{\pi_i} \left\{ H_i P_{i-1}[\zeta^* \leq t] - H_{i-1} P_{i-1}[\zeta^* \leq t] \right\} = P_{i-1}[\zeta^* \leq t] \geq P_0[\zeta^* \leq t].$$

Thus

$$P_i[\tau > t] \leq P_{i-1}[\zeta^* > t] \leq P_0[\zeta^* > t],$$

which implies

$$E_i \tau = \int_0^\infty P_i[\tau > t] dt \leq \int_0^\infty P_{i-1}[\zeta^* > t] dt = E_{i-1} \zeta^* \leq E_0 \zeta^*. \tag{5.6}$$

Corollary 5.4. We have for any $\ell \geq 0$

$$E_0 \tau^\ell \leq (E_0 \tau)^\ell / \ell! \tag{5.8}$$

and for any $\lambda < 1 / E_0 \tau$

$$E_0 e^{\lambda \tau} \leq (1 - \lambda E_0 \tau)^{-1}. \tag{5.9}$$
Proof. Since from (5.5), $\tau$ and $\zeta^*$ have the same distribution under $\mathbb{E}_0$, we need only prove (5.8) with $\tau$ replaced by $\zeta^*$. It follows from [17, Lemma 4.1] that for any $\ell \geq 0$
\[ \mathbb{E}_0[(\zeta^*)^\ell] \leq (\mathbb{E}_0\zeta^*)^\ell / \ell! \]
and for any $\lambda < 1 / \mathbb{E}_0\tau$
\[ \mathbb{E}_0 e^{\lambda \tau} \leq (1 - \lambda \mathbb{E}_0\tau)^{-1}. \]

**Theorem 5.5.** Let
\[ S(t) = \sup_i s_i(t), \]
and $\tau$ be the SST in Theorem 5.2. Then the following statements are equivalent.

(i) The process is strongly ergodic.

(ii) $\tau < \infty$ a.s.

(iii) $\mathbb{E}_0\tau < \infty$.

(iv) $\lim_{t \to \infty} S(t) = 0$.

Proof. The equivalence of (i)-(iii) follows from Theorems 5.2 and 4.1 and Proposition 5.3. We will prove the implication of (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i).

(a) Suppose that $\sup_i \mathbb{E}_i\tau < \infty$. By duality, we have $\mathbb{E}_0\zeta^* < \infty$, it follows from (5.3) and Markov inequality that
\[ S(t) \leq \sup_i \mathbb{P}_i[\tau > t] \leq \frac{\sup_i \mathbb{E}_i\tau}{t} \leq \frac{\mathbb{E}_0\tau}{t} \to 0, \quad t \to \infty. \]

(b) Suppose $\lim_{t \to \infty} S(t) = 0$. It follows from the inequality (5.2) that
\[ \sup_i ||p_i(t) - \pi||_{\text{var}} \leq S(t) \to 0, \quad t \to \infty, \]
which is strong ergodicity. \qed

We remark that under any assumption of (1) − (4) above, letting
\[ \beta = \sup \{ \epsilon : \exists C < \infty, S(t) \leq Ce^{-\epsilon t}, \forall t \geq 0 \} \]
be the optimal convergence rate in separation, then it follows easily from Corollary 5.4 and Theorem 5.1 that
\[ \beta \geq \frac{1}{\mathbb{E}_0\tau} = \left( \sum_{\nu=1}^{\infty} \lambda_{\nu}^{-1} \right)^{-1} = \left( \sum_{k=0}^{\infty} \frac{1}{\pi_k b_k} \sum_{i=0}^{k} \pi_i \sum_{i=k+1}^{\infty} \pi_i \right)^{-1}. \]

Acknowledgement The authors would thank Professors Mu-Fa Chen and Feng-Yu Wang for their valuable suggestions.
References

[1] Aldous, D.J., Fill, J.A. Reversible Markov chains and random walks on graphs. URL www.berkeley.edu/users/aldous/book.html, 2003.

[2] Anderson, W. Continuous-time Markov chains. New York, Springer-Verlag, 1991.

[3] Chen, M.F. From Markov chains to non-equilibrium particle systems, Second edition. Singapore, Word Scientific, 2004.

[4] Chen, M.F. Eigenvalues, inequalities, and ergodic theory. New York, Springer, 2005

[5] Chen, M.-F. Speed of stability for birth and death processes. preprint, 2009.

[6] Diaconis, P., Miclo, L. On times to quasi-stationarity for birth and death processes. J. Theor. Probab., 22(2009), 558-586.

[7] Diaconis, P., Saloff-Coste, L. Separation cut-offs for birth and death chains. Ann. Appl. Probab., 16(2006), 2098-2122.

[8] Feller, W. The birth and death processes as diffusion processes. J. Math. Pures Appl., 38(1959), 301-345.

[9] Fill, J.A. Strong stationary duality for continuous-time Markov chains Part I: Theory. J. Theoret. Probab., 5(1992), 45-70.

[10] Fill, J.A. The passage time distribution for a birth-and-death chain: strong stationary duality gives a first stochastic proof. J. Theor. Probab., 22(2009), 543-557

[11] Fill, J.A. On hitting times and fastest strong stationary times for skip-free and more general chains. J. Theor. Probab., 22(2009), 587-600

[12] Horn, R.A., Johnson, C.R. Topics in matrix analysis (Vol.II). Cambridge University Press.

[13] Karlin, S., McGregor, J. Coincidence properties of birth and death processes. Pac. J. Math., 9 (1959), 1109-1140.

[14] Kato, T., Perturbation theory for linear operators. Springer-Verlag, 1966.

[15] Keilson, J. Markov chain models—rarity and exponentiality. Applied Mathematical Sciences, vol. 28. Springer, New York, 1979.

[16] Mao, Y.-H. Eigentime identity for continuous-time ergodic Markov chains. J. Appl. Probab., 41(2004), 1071-1080.

[17] Mao, Y.-H. Eigentime identity for transient Markov chains. J. Math. Anal. Appl., 315(2006), 415-424.

[18] Mao, Y.-H. On empty essential spectrum for Markov processes in dimension one. Acta Math. Sin. (Engl. Ser.), 22 (2006), 807-812.
[19] Mao, Y.-H. *Convergence rates in strong ergodicity for Markov processes*. Stoch. Proc. Appl., 116(2006), 1964-1976.

[20] Matthews, P. *Strong stationary times and eigenvalues*. J. Appl. Probab., 29(1992), 228-233.

[21] Wang Z.K., Yang X.Q. *Birth and death processes and Markov processes*, Science Press/Springer-Verlag, 1992.

[22] Zhang, H. J., Lin, X., Hou, Z. T. *Polynomial uniform convergence for standard transition functions*. Chinese Ann. Math. (Ser. A), 21(2000), 351-355. (in Chinese)

[23] Zhang, Y. H. *Strong ergodicity for single birth processes*. J. Appl. Prob., 38(2001), 270-277.