Abstract

We present $N = 2$ supersymmetry transformations, both in $N = 1$, $D = 4$ Minkowski and anti-de Sitter superspaces, for higher superspin massless theories. It is noted that the existence of dual versions of massless supermultiplets with arbitrary superspin $s$ may provide a basis for understanding duality in $N = 1$, $D = 4$ superstring theory. We further conjecture that the $N = 1$, $D = 4$ supergravity pre-potential together with all higher superspin $s$ pre-potentials are components of an $N = 1$, $D = 4$ superstring pre-potential.
1. Recently, the linearized Lagrangian gauge theory formulations for arbitrary higher superspin massless multiplets have been constructed in $N = 1, D = 4$ Minkowski superspace \cite{1, 2} (see also \cite{3}) and then extended to the case of anti-de Sitter (AdS) supersymmetry \cite{4}. These models are realized in terms of real unconstrained superfield pre-potentials (being the higher spin analogues of the gravitational axial vector superfield \cite{5}) along with complex constrained ones: longitudinal linear superfields or transversal linear superfields (non-trivial analogues of the chiral compensator in the old minimal supergravity \cite{6} and the complex linear compensator in the nonminimal supergravity \cite{7}, respectively). Transversal and longitudinal linear superfields are known to be native inhabitants of the AdS superspace \cite{8}. Combining massless multiplets of all superspins in the AdS superspace, one probably results in a theory \cite{9} possessing an infinite-dimensional symmetry algebra whose finite-dimensional subalgebra includes the $N = 2$ AdS superalgebra $osp(4,2)$. In this respect, it would be of interest to analyze possible realizations of $N = 2$ supersymmetry for higher superspin massless multiplets.

A major motivation to pursue the approach of references \cite{1, 2} is to develop a new tool with which to analyze superstring theory. It should be clearly stated that the theories developed thus far \cite{1, 2, 4} are not linearized $D = 4$ superstrings but are a more general class of higher superspin theories. There is no impediment to adding together arbitrary numbers of these linearized actions with arbitrary values of $s$ and taking the $s \to \infty$ limit. However, with the proper choice of multiplicities and spectrum $N = 1, D = 4$, superstrings should emerge as special cases.

2. The formulations of Refs. \cite{1, 2} in $N = 1, D = 4$ Minkowski superspace (with standard covariant derivatives $D_{\alpha} = (\partial_{\alpha}, \bar{D}_{\dot{\alpha}})$) involve so-called transversal and longitudinal linear superfields. A complex tensor superfield $\Gamma(k, l)$ subject to the constraint

$$
\bar{D}^{\alpha} \Gamma_{\alpha(k)\dot{\alpha}(l)} = 0 \quad , \quad l > 0 \quad ,
$$

$$
\bar{D}^{2} \Gamma_{\alpha(k)} = 0 \quad , \quad l = 0 \quad ,
$$

(1)
is said to be transversal linear. A longitudinal linear superfield $G(k, l)$ is defined to satisfy the constraint

$$
\bar{D}_{\dot{\alpha}} G_{\alpha(k)\dot{\alpha}(l)} = 0 \quad ,
$$

(2)
(the symmetrization over all dotted indices is assumed). The above constraints imply that $\Gamma(k, l)$ and $G(k, l)$ are linear in the usual sense

$$
\bar{D}^{2} \Gamma(k, l) = \bar{D}^{2} G(k, l) = 0 \quad .
$$

(3)

4 Throughout the paper we consider only Lorentz tensors symmetric in their undotted indices and separately in their dotted ones. For a tensor of type $(k, l)$ with $k$ undotted and $l$ dotted indices we use the shorthand notations $\Psi(k, l) \equiv \Psi_{\alpha(k)\dot{\alpha}(l)} \equiv \Psi_{\alpha_1 \ldots \alpha_k \dot{\alpha}_1 \ldots \dot{\alpha}_l} = \Psi_{(\alpha_1 \ldots \alpha_k)(\dot{\alpha}_1 \ldots \dot{\alpha}_l)}$. Following Ref. \cite{10} we assume that the indices, which are denoted by one and the same letter, should be symmetrized separately with respect to upper and lower indices; after the symmetrization, the maximal possible number of the upper and lower indices denoted by the same letter are to be contracted. In particular $\phi_{\alpha(k)} \psi_{\alpha(l)} \equiv \phi_{(\alpha_1 \ldots \alpha_k}(\psi_{\alpha_{k+1} \ldots \alpha_l)})$. Given two tensors of the same type, their contraction is denoted by $f \cdot g \equiv f^{\alpha(k)\dot{\alpha}(l)} g_{\alpha(k)\dot{\alpha}(l)}$. 


The constraints (4) and (5) can be resolved in terms of unconstrained superfields

\[ \Gamma_{\alpha(k)\hat{\alpha}(l)} = \hat{D}^\hat{\alpha} \Phi_{\alpha(k)\hat{\alpha}(l+1)}, \]
\[ G_{\alpha(k)\hat{\alpha}(l)} = \hat{D}_{\alpha} \Psi_{\alpha(k)\hat{\alpha}(l-1)}. \]

Here the superfields \( \Phi \) and \( \Psi \) are defined modulo arbitrary shifts of the form

\[ \delta \Phi(k, l + 1) = \gamma(k, l + 1), \]
\[ \delta \Psi(k, l - 1) = g(k, l - 1), \]

involving a transversal linear superfield \( \gamma(k, l + 1) \) and a longitudinal linear superfield \( g(k, l - 1) \). It follows that any transversal linear superfield \( \Gamma(k, l) \), which appears in a supersymmetric field theory, can be equivalently replaced by an unconstrained one, via rule (4), that introduces the additional gauge invariance (6). The gauge parameter, which is a transversal linear superfield, can also be re-expressed in terms of an unconstrained superfield introduced via the same rule, and on and on. The number of dotted indices on the superfields increases indefinitely in this process, thus providing us with a gauge structure of infinite stage reducibility. Any longitudinal linear superfield \( g(k, l) \) can be replaced by an unconstrained one in correspondence with the rule (5). Similarly, the longitudinal linear gauge parameter in Eq. (7) can be re-expressed in terms of an unconstrained one and so on. This results in a gauge structure of \( l - 1 \) stage reducibility.

Two formulations for the massless multiplet of a half-integer superspin \( s + 1/2 \) \((s = 1, 2, \ldots)\) which were called in Ref. [1] transversal and longitudinal, contain the following dynamical variables respectively:

\[ V_{\perp}^{s+1/2} = \{ H(s, s), \Gamma(s - 1, s - 1), \bar{\Gamma}(s - 1, s - 1) \}, \]
\[ V_{\parallel}^{s+1/2} = \{ H(s, s), G(s - 1, s - 1), \bar{G}(s - 1, s - 1) \}. \]

Here \( H(s, s) \) is real, \( \Gamma(s - 1, s - 1) \) transversal linear and \( G(s - 1, s - 1) \) longitudinal linear tensor superfields. The case \( s = 1 \) corresponds to the supergravity multiplet investigated in detail in literature (see Ref. [3] for a review). Two formulations of Ref. [2] for the massless multiplet of an integer superspin \( s \), \((s = 1, 2, \ldots)\) longitudinal and transversal, contain the following dynamical variables respectively:

\[ V_{s}^{\perp} = \{ H'(s - 1, s - 1), \Gamma'(s, s), \bar{\Gamma}'(s, s) \}, \]
\[ V_{s}^{\parallel} = \{ H'(s - 1, s - 1), G'(s, s), \bar{G}'(s, s) \}. \]

Here \( H'(s - 1, s - 1) \) is real, \( \Gamma'(s, s) \) transversal linear and \( G'(s, s) \) longitudinal linear tensor superfields. The case \( s = 1 \) corresponds to the gravitino multiplet [2, 12, 13, 14] (see Ref. [3] for a review).

In the transversal half-integer-superspin formulation, the action functional reads

\[ S_{s+1/2}^{\perp} = \left(-\frac{1}{2}\right)^s \int d^8 z \left\{ \frac{1}{8} H^{\alpha(s)\hat{\alpha}(s)} D^\beta \bar{D}^\gamma D_\beta D_\gamma H_{\alpha(s)\hat{\alpha}(s)} \right\}. \]
In the longitudinal formulation, the action takes the form

\[ S_{s+1/2}^\parallel = \left( -\frac{1}{2} \right)^s \int d^8 z \left\{ \frac{1}{8} H^{\alpha(s)}D^\beta D^2 D^\beta H^{\alpha(s)} + \frac{s}{2s+1} D^\alpha D^\alpha g + \left( \frac{s+1}{s} \right) V + \text{c.c.} \right\} \]

(12)

The gauge transformations for the superfields \( H(s,s), \Gamma(s-1,s-1) \) and \( G(s-1,s-1) \) are given in the form

\[ \delta_g H(s,s) = g(s,s) + \bar{g}(s,s) \]

(14)

\[ \delta_g \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{s}{2s+1} \bar{D}^\alpha D^\alpha \bar{g}_{\alpha(s)\dot{\alpha}(s)} \]

(15)

\[ \delta_g G_{\alpha(s-1)\dot{\alpha}(s-1)} = \frac{s}{2s+1} D^\alpha \bar{D}^\alpha g_{\alpha(s)\dot{\alpha}(s)} + is\bar{\alpha}^{\dot{\alpha}} g_{\alpha(s)\dot{\alpha}(s)} \]

(16)

with a longitudinal linear parameter \( g(s,s) \). The action \( S_{s+1/2}^\perp \) is invariant under the gauge transformations (14) and (13), and \( S_{s+1/2}^\parallel \) is invariant under the gauge transformations (14) and (16). The actions (12) and (13) are dually equivalent [1].

The massless multiplet of integer superspin \( s \) is described in the longitudinal formulation by the action functional

\[ S_s^\parallel = \left( -\frac{1}{2} \right)^s \int d^8 z \left\{ \frac{1}{8} H^{\alpha(s-1)}D^\beta D^2 D^\beta H'_{\alpha(s-1)} + \frac{s}{2s+1} D^\alpha D^\alpha g + \left( \frac{s+1}{s} \right) V + \text{c.c.} \right\} \]

(17)

while the action in the transversal formulation takes the form

\[ S_s^\perp = -\left( -\frac{1}{2} \right)^s \int d^8 z \left\{ -\frac{1}{8} H^{\alpha(s-1)}D^\beta D^2 D^\beta H'_{\alpha(s-1)} + \frac{1}{8} \left( \frac{s^2}{(s+1)(2s+1)} \right) \left[ [D^\alpha, \bar{D}^\dot{\alpha}] H^{\alpha(s-1)} + \text{c.c.} \right] \right\} \]
The gauge freedom for the superfields $H'(s - 1, s - 1)$, $G'(s, s)$ and $\Gamma'(s, s)$ reads

$$\delta_\gamma H'(s - 1, s - 1) = \gamma(s - 1, s - 1) + \bar{\gamma}(s - 1, s - 1),$$  \hspace{1cm} (19)

$$\delta_\gamma G'_{\alpha(s)\bar{\alpha}(s)} = \frac{1}{2} \bar{\partial}_{\bar{\alpha}} D_\alpha \bar{\gamma}_{\alpha(s-1)\bar{\alpha}(s-1)},$$  \hspace{1cm} (20)

$$\delta_\gamma \Gamma'_{\alpha(s)\bar{\alpha}(s)} = \frac{1}{2} D_\alpha \bar{\partial}_{\bar{\alpha}} \gamma_{\alpha(s-1)\bar{\alpha}(s-1)} - i s \partial_{\alpha\bar{\alpha}} \gamma_{\alpha(s-1)\bar{\alpha}(s-1)},$$  \hspace{1cm} (21)

with a transversal linear parameter $\gamma(s - 1, s - 1)$. So $S_s^\parallel$ is invariant under the gauge transformations (19) and (20), and $S_s^\perp$ is invariant under the gauge transformations (19) and (21). The actions (14) and (15) are dually equivalent [2].

The appearance of these dual equivalent descriptions is very noteworthy. It extends a property of superspace supergravity theory. There it is known that duality transformations on the superspace are related to changes of the Weyl compensating multiplet [13]. In 1985 [16], it was proposed that this same type of duality transformation must exist for dual descriptions of the $D = 10$ heterotic string low-energy effective action [1] and it was conjectured that this was likely the case for the entire theory. Presently duality is one of the most intensively studied topics in superstring theory having led to many new insights into the non-perturbative aspects of the theory. Although the original conjectures were for $D = 10$ theories, the fact that these dual actions described above exist for arbitrary values of $s$ provides additional support for the now widely accepted belief in the presence of such dual versions, at least for $N = 1$, $D = 4$ superstrings.

The first lines in (14) and (15) together with the existence of the duality transformations discussed above lead to a simple interpretation. The pre-potentials $H(s, s)$ and $H'(s - 1, s - 1)$ are likely to be the superspin $s$ contributions to a generalization of the conventional $N = 1$, $D = 4$ conformal supergravity pre-potential. For obvious reasons this generalization may be called “the $N = 1$, $D = 4$ conformal superstring pre-potential.” It seems likely that the $N = 1$, $D = 4$ conformal superstring pre-potential should be the null-string [17] limit of conventional superstring theory. Finally within an ultimate formulation of the $N = 1$, $D = 4$ superstring, the gauge variations in (14) and (15) should emerge as the different superspin $s$ components of a “$N = 1$, $D = 4$ conformal superstring group” that would constitute the “stringy” extension for the case of and (8) and (9) in the case of $s = 1$. The role of the compensating superfields $\Gamma(s - 1, s - 1)$ or $G(s - 1, s - 1)$ would be to break down this to the “$N = 1$, $D = 4$ Poincaré superstring group.”

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5This proposal was the first suggestion of string-string duality in the literature.
3. Let us turn to the analysis of $N = 2$ supersymmetry transformations in the model described by the action $S^\perp_{s+1/2} + S^\parallel_s$. It follows from considerations of dimension that such transformations cannot be expressed, in manifestly $N = 1$ supersymmetric form, via the linear superfields $\Gamma(s-1,s-1)$ and $G'(s,s)$ entering the action functionals, but only in terms of their unconstrained potentials $\Phi(s-1,s)$ and $\Psi'(s,s-1)$ introduced according to the rules (4) and (5). Accordingly, one then readily arrives at the following supersymmetry transformations

$$\delta H_\alpha(s)\dot{\alpha}(s) = -4i\bar{\xi}_\alpha\Psi'_\alpha(s)\dot{\alpha}(s-1) + \text{c.c.},$$

$$\delta \Phi_\alpha(s-1)\dot{\alpha}(s) = \frac{i(s-1)}{2(s+1)}\xi_\alpha D_\alpha D^\beta H'_{\beta\alpha(s-2)\dot{\alpha}(s-1)} - \frac{is}{2(s+1)}\xi_\beta D_\alpha D^\delta H'_{\alpha\beta(s-1)\dot{\alpha}(s-1)}$$

$$+ \frac{is}{s+1}\xi_\alpha D_\beta \Psi'(s) - i\xi_\beta D_\beta \bar{\Psi}'_\alpha(s-1)\dot{\alpha}(s)$$

$$+ \frac{is}{s-1}\xi_\alpha D_\beta \bar{\Phi}'_\beta(s-2)\dot{\alpha}(s);$$

$$\delta H'_\alpha(s-1)\dot{\alpha}(s-1) = -4i\bar{\xi}_\alpha\Phi_\alpha(s-1)\dot{\alpha}(s) + \text{c.c.},$$

$$\delta \Psi'_\alpha(s)\dot{\alpha}(s-1) = \frac{i}{2}\xi_\alpha \bar{D}_\alpha H_\beta\alpha\dot{\alpha}(s-1)\dot{\alpha}(s) - \frac{i}{2}\xi_\beta \bar{D}_\alpha D^\beta H_\alpha\dot{\alpha}(s)$$

$$+ \frac{is-1}{s-1}\xi_\alpha \bar{D}_\beta \Phi'(s-1)\dot{\alpha}(s-1) + i\xi_\beta D_\beta \bar{\Phi}'_\alpha(s-1)\dot{\alpha}(s-1)$$

$$+ i\xi_\beta D_\alpha \bar{\Phi}'_\beta(s-1)\dot{\alpha}(s-1),$$

(22)

where $\xi_\alpha$ is a constant spinor parameter. These transformations leave the total action $S^\perp_{s+1/2} + S^\parallel_s$ invariant.

We have obtained $N = 2$ ($s, s + 1/2$) multiplets described in terms of a special $N = 1$ superfield realization: the longitudinal formulation for superspin $s$ and the transversal formulation for superspin $s + 1/2$. It can be seen that there exist three other $a$ priori combinations to describe $N = 2$ multiplets from $N = 1$ superspin-$s$ and superspin-$(s + 1/2)$ multiplets. However none seem to allow the construction of an $N = 2$ multiplet. As for the possibility of an $N = 2$ multiplet constructed from $(s-1/2,s)$, all four choices do not lead to non-trivial supersymmetry transformations. In some more detail, the only possibility allowing non-trivial mixing of complex potentials via each other, under hypothetical supersymmetry transformations, appears to be the same: the longitudinal formulation for integer superspin $s$ and the transversal formulation for half-integer superspin $s-1/2$. Here the dynamical superfields are: $H(s-1,s-1)$, $\Phi(s-2,s-1)$ (superspin $s-1/2$) and $H'(s-1,s-1)$, $\Psi'(s,s-1)$ (superspin $s$). This case differs from that previously considered only by superfield tensor structure. When the highest superspin was half-integer, the tensor structures coincided for complex superfields, but differed by two indices for the real superfields. When the highest superspin is integer, the tensor structures coincide for the real superfields and differ by two indices for the complex ones. This fact restricts the number of independent candidate structures for $N = 2$ supersymmetric transformation laws. Modulo purely gauge variations, the general ansatz for such a transformation reads

$$\delta H_\alpha(s-1)\dot{\alpha}(s-1) = \alpha_1\xi_\alpha\Psi'_\alpha(s)\dot{\alpha}(s-1) + \alpha_2\xi_\beta D_\alpha H'_{\beta\alpha(s-2)\dot{\alpha}(s-1)},$$

5
Here the superscript ‘transversal formulations. For this purpose one should eliminate the gauge degrees of freedom of the dynamical superfields. As a result, one finds

\[ \delta \Phi_{\alpha(s-2)\dot{\alpha}(s-1)} = i\beta_1 \bar{\epsilon}_{\dot{\alpha}} \partial_{\alpha} H'_{\alpha(s-2)\dot{\alpha}(s-2)} \beta \]

\[ + i\beta_2 \bar{\epsilon}_{\dot{\alpha}} \partial_{\alpha} H'_{\alpha(s-1)\dot{\alpha}(s-1)} + \beta_3 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha H'_{\alpha(s-1)\dot{\alpha}(s-1)} \]

\[ + \beta_4 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha \Psi'_{\alpha(s)\dot{\alpha}(s-1)} + \beta_5 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha \Psi'_{\alpha(s-1)\dot{\alpha}(s)} \]\n
\[ \delta H''_{\alpha(s-1)\dot{\alpha}(s-1)} = \gamma_1 \delta_{\alpha} \Phi_{\alpha(s-2)\dot{\alpha}(s-1)} + \gamma_2 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha H_{\alpha(s-1)\dot{\alpha}(s-1)} \]

\[ + \gamma_3 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha H_{\alpha(s-1)\dot{\alpha}(s-1)} + \gamma_4 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha \Psi'_{\alpha(s)\dot{\alpha}(s-1)} + \gamma_5 \bar{\epsilon}_{\dot{\alpha}} \bar{D}^\alpha \Psi'_{\alpha(s-1)\dot{\alpha}(s-2)} \]

(23)

where \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i \) are arbitrary coefficients. Surprisingly enough, direct calculations show that the requirement of invariance of the action \( S_{s-1/2}^\perp + S_{s}^\parallel \) under this transformation can be satisfied only when all the coefficients vanish.

To clearly illustrate the difference between the considered \( N = 2 \) supermultiplets, let us investigate the off-shell superspin content of the theories (a similar analysis for the multiplet of \( N = 2 \) supergravity was given many years ago for the linearized theory \([14, 18]\) and later the full non-linear theory \([19]\). That can be done by decomposing unconstrained and linear superfields onto irreducible components according to the rules given in \([20]\).

We will not reproduce here the decompositions of the dynamical superfields onto irreducible ones, but only describe the superspin contents:

\[ H(s, s) = (s + \frac{1}{2})^r \oplus s \oplus (s - \frac{1}{2}) \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(24)

\[ \Gamma(k, l) = \frac{k + l + 1}{2} \oplus \frac{k + l}{2} \oplus \ldots \oplus \frac{k - l - 1}{2} \oplus \frac{k - l}{2} \]  

(25)

\[ \Gamma(k, k) = \frac{2k + 1}{2} \oplus k \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(26)

\[ G(k, l) = \frac{k + l}{2} \oplus \frac{k + l - 1}{2} \oplus \ldots \oplus \frac{k - l + 1}{2} \oplus \frac{k - l}{2} \]  

(27)

Here the superscript ‘\( r \)’ is used for real representations. It is seen that \( G(k, l) \) and \( \Gamma(k, l - 1) \) describe equivalent representations.

Now, we are in position to deduce the superspin content in the longitudinal and transversal formulations. For this purpose one should eliminate the gauge degrees of freedom of the dynamical superfields. As a result, one finds

\[ S_{s+1/2}^\perp : (s + \frac{1}{2})^r \oplus (s - \frac{1}{2}) \oplus (s - 1) \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(28)

\[ S_{s+1/2}^\parallel : (s + \frac{1}{2})^r \oplus (s - 1) \oplus (s - \frac{3}{2}) \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(29)

\[ S_s^\parallel : s \oplus (s - \frac{1}{2})^r \oplus (s - 1) \oplus (s - \frac{3}{2}) \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(30)
\[ S_s^\perp : (s + \frac{1}{2}) \oplus s \oplus (s - \frac{1}{2})^r \oplus (s - 1) \oplus (s - \frac{3}{2}) \oplus \ldots \oplus \frac{1}{2} \oplus 0 \]  

(31)

It is now obvious that only pairing the formulations \( S_s^\perp_{s+1/2} \) and \( S_s^\parallel \) allows the possibility to join the \( N = 1 \) supermultiplets, entering the actions, into \( N = 2 \) supermultiplets which off-shell should have the following general structure (see, e.g., [11])

\[
(\lambda + \frac{1}{2} r) \oplus \lambda \oplus (\lambda - \frac{1}{2} r) \text{ or } (\frac{1}{2} r) \oplus 0.
\]

4. Now we turn to the \( N = 1, D = 4 \) AdS superspace defined by the algebra of covariant derivatives \( D_A = (D_a, D_\alpha, \bar{D}^\dot{\alpha}) \)

\[
\{D_\alpha, \bar{D}^\dot{\alpha}\} = -2i \epsilon_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad [D_\alpha, D_\beta] = -2 \bar{\mu} \epsilon_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad [D_\alpha, \bar{D}^\dot{\beta}] = i \bar{\mu} \epsilon_{\alpha\beta} \bar{D}^\dot{\beta},
\]

(32)

and conjugates to (33). Here \( M \) and \( \bar{M} \) are the Lorentz generators, and \( \mu \) the torsion (the square-root of the constant curvature) of the AdS superspace. The AdS analogues of the constraints (1) and (2) read [8]

\[
\bar{D}^\dot{\alpha} \Gamma_{\alpha(k)\dot{\alpha}(l)} = 0 \iff (\bar{D}^2 - 2(l + 2)\bar{\mu}) \Gamma(k, l) = 0, \quad l > 0,
\]

(34)

\[
\bar{D}^\dot{\alpha} G_{\alpha(k)\dot{\alpha}(l)} = 0 \iff (\bar{D}^2 + 2l\bar{\mu}) G(k, l) = 0,
\]

(35)

and imply some specific features for transversal and longitudinal superfields. First, the corresponding subspaces in the space of tensor superfields of type \((k, l)\) have empty intersection and supplement each other. This can be proved by checking the properties of the projectors \( \mathcal{P}_\perp \) and \( \mathcal{P}_\parallel \) on the subspaces of transversal and longitudinal linear superfields respectively

\[
\mathcal{P}_\perp = \frac{\bar{D}^2 + 2l\bar{\mu}}{4(l + 1)\bar{\mu}}, \quad \mathcal{P}_\parallel = -\frac{\bar{D}^2 - 2(l + 2)\bar{\mu}}{4(l + 1)\bar{\mu}},
\]

(36)

Second, the equivalence of the spaces of superfields \( G(k, l) \) and \( \Gamma(k, l - 1) \), we have mentioned after Eq. (27), becomes evident in the AdS superspace from the explicit form of the intertwining operators:

\[
G_{\alpha(k)\dot{\alpha}(l)} = \bar{D}^\dot{\alpha} \Gamma_{\alpha(k)\dot{\alpha}(l-1)} \iff \Gamma_{\alpha(k)\dot{\alpha}(l-1)} = -\frac{1}{2(l + 1)\bar{\mu}} \bar{D}^\dot{\alpha} G_{\alpha(k)\dot{\alpha}(l)}.
\]

(37)

The latter relations allow the potentials \( \Phi(k, l + 1) \) and \( \Psi(k, l - 1) \), which are defined in complete analogy with Eqs. (4), (5), to be expressed in terms of their strengths \( \Gamma(k, l) \) and \( G(k, l) \) modulo purely gauge parts.

It was shown in Ref. [4] that each of the formulations [12], [13], [17] and [18] has its counterpart in the AdS superspace. The corresponding actions are obtained from the flat ones by replacing the flat derivatives \( D \)'s with the AdS covariant ones.
$D$'s along with adding $\mu$-dependent terms. Similarly to the flat superspace, we now consider the $N = 2$ supersymmetry of the action $S_{s+1/2}^1[H, \Gamma] + S_s^0[H', G']$. For the second supersymmetry to be covariant with respect to the $N = 1$ AdS superalgebra, it is convenient to pass from constant spinor parameter $\xi_{\alpha}$ to a Killing spinor superfield $\varepsilon_{\alpha}$ defined by

$$D_{\alpha} \varepsilon_{\alpha} = \bar{D}_{\dot{\alpha}} \varepsilon_{\dot{\alpha}} = 0 \quad , \quad \mu D^\alpha \varepsilon_{\alpha} = \bar{\mu} \bar{D}_{\dot{\alpha}} \bar{\varepsilon}_{\dot{\alpha}} \quad .$$

The description in terms of $\varepsilon_{\alpha}$ and its conjugate is equivalent in the AdS superspace to the use of a real linear superfield $\varepsilon$ subject to the constraints

$$(\bar{D}^2 - 4\mu)\varepsilon = D_\alpha \bar{D}_{\dot{\alpha}} \varepsilon_{\dot{\alpha}} = 0 \quad \varepsilon = \bar{\varepsilon} \quad .$$

The superfields $\varepsilon_{\alpha}$ and $\varepsilon$ are connected as follows

$$D^\alpha \varepsilon_{\alpha} = 2\bar{\mu} \varepsilon \quad D_{\alpha} \varepsilon = 2\varepsilon_{\alpha} \quad .$$

The $N = 2$ supersymmetry transformations in the AdS superspace differs in form from the flat ones (22) only by $\mu$-dependent terms:

$$\delta_\varepsilon H_{\alpha(s)\dot{\alpha}(s)} = -4i\bar{\varepsilon}_{\dot{\alpha}} \bar{\Psi}'_{(s)} + \text{c.c.} \quad ,$$

$$\delta_\varepsilon \Phi_{\alpha(s-1)\dot{\alpha}(s)} = \frac{i(s-1)}{2(s+1)} \varepsilon_\alpha \bar{D}_{\dot{\alpha}} D^\beta H_{\beta \dot{\alpha}(s-2)\dot{\alpha}(s-1)} - \frac{is}{2(s+1)} \varepsilon_\alpha \bar{D}_{\dot{\alpha}} D_\beta H'_{\beta \alpha(s-1)\dot{\alpha}(s)}$$

$$+ 2is\bar{\mu} \varepsilon_{\dot{\alpha}} H'_{\alpha(s-1)\dot{\alpha}(s-1)} + \frac{is}{s+1} \varepsilon_\alpha \bar{D}_{\dot{\alpha}} \Psi_{\alpha(s)} \dot{\alpha}(s-1)$$

$$+ \frac{is}{s} \varepsilon_\alpha \bar{D}_{\dot{\alpha}} \bar{\Psi}'_{\beta\alpha(s-2)\dot{\alpha}(s)} \quad ,$$

$$\delta_\varepsilon H'_{\alpha(s-1)\dot{\alpha}(s)} = -4i\varepsilon_{\dot{\alpha}} \Phi_{\dot{\alpha}(s-1)\dot{\alpha}(s)} + \text{c.c.} \quad ,$$

$$\delta_\varepsilon \Psi'_{\alpha(s-1)\dot{\alpha}(s)} = \frac{i}{2} \varepsilon^\beta \bar{D}^{\dot{\alpha}} D_\alpha H_{\beta \alpha(s-1)\dot{\alpha}(s)} - \frac{i}{2} \varepsilon_\beta \bar{D}^{\dot{\alpha}} D^\beta H_{\alpha(s)\dot{\alpha}(s)}$$

$$+ 2is\bar{\mu} \varepsilon_{\dot{\alpha}} H_{\alpha(s)\dot{\alpha}(s)} + \frac{is}{s} \varepsilon_\alpha \bar{D}^{\dot{\alpha}} \Phi_{\alpha(s)\dot{\alpha}(s)}$$

$$+ \frac{is}{s} \varepsilon_\alpha \bar{D}^{\dot{\alpha}} \bar{\Phi}_{\alpha(s)\dot{\alpha}(s)} \quad .$$

The transformations (41) leave the action $S_{s+1/2}^1 + S_s^0$ in the AdS superspace invariant and reduce in the flat limit ($\mu \to 0$) to (22). It is a property of the AdS superspace that the transformation laws can always be expressed in terms of the strengths $\Gamma(s-1, s-1)$ and $G'(s, s)$:

$$\delta_\varepsilon H(s, s) = 2i\varepsilon \left( G'(s, s) - G'(s, s) \right) \quad ,$$

$$\delta_\varepsilon \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = -\frac{i}{2} (\bar{D}^2 + 2(s-1)\mu) \varepsilon \bar{D}^{\delta\beta} H'_{\beta \alpha(s-1)\dot{\alpha}(s-1)}$$

$$+ 2is\bar{\mu} \varepsilon^\delta \bar{D}^{\dot{\alpha}} H'_{\dot{\alpha}(s-1)\dot{\alpha}(s-1)} - \frac{2is}{s+1} \varepsilon^\delta \bar{D}^{\dot{\alpha}} G'_{\alpha(s)} \dot{\alpha}(s)$$

$$- \frac{is}{s+1} \varepsilon \bar{D}^{\dot{\alpha}} \varepsilon \bar{D}^\alpha G'_{\alpha(s)\dot{\alpha}(s)} \quad .$$
\[ \delta_\varepsilon H'(s-1, s-1) = 2i\varepsilon \left( \Gamma(s-1, s-1) - \Gamma(s-1, s-1) \right) , \]
\[ \delta_\varepsilon G'_{(a(s)\dot{a}(s))} = -\frac{i}{2}(\mathcal{D}^2 - 2(s + 2)\mu)\varepsilon^\beta \mathcal{D}_\beta H_{\alpha(s)\dot{a}(s)} + 2is\bar{\mu}\bar{\mathcal{D}}_{\dot{\alpha}}\varepsilon^\beta H_{\alpha(s)\dot{a}(s-1)\dot{\dot{a}}} - 2i\varepsilon_\alpha \mathcal{D}_{\dot{\alpha}}\Gamma_{\alpha(s-1)\dot{a}(s-1)} - i\mathcal{D}_{\dot{\alpha}}\varepsilon \mathcal{D}_{\alpha} \Gamma_{\alpha(s-1)\dot{a}(s-1)} . \]  

(42)

Here the derivatives are assumed to act on all the objects placed to their right. A new feature of the \( N = 2 \) supersymmetry variations is that the covariant derivatives enter some expressions in higher powers. As a consequence, we are able to add appropriate variations proportional to the equations of motion of the actions \( S^\perp_{s+1/2} \) and \( S^\perp_s \). Such a possibility did not arise for the previous expression \([11]\) of the second supersymmetry nor for the supersymmetry between the component actions for arbitrary spin fields \([21]\). In fact, to come to the latter form of the transformation laws \([12]\) one has first to express the potentials via their strengths (modulo purely gauge parts), in accordance with \([17]\), and then to add some purely gauge variations as well as certain terms proportional to the equations of motion.

The transformations turn out to form a closed algebra off the mass-shell! It is an instructive exercise to derive the relation

\[ [\delta_\varepsilon, \delta_\varepsilon'] = \delta_K + \delta_g + \delta_\gamma . \]  

(43)

Here \( \delta_K \) denotes an AdS transformation acting on a tensor superfield \( U \) by the law

\[ \delta_K U = \mathcal{K} U \quad , \quad \mathcal{K} = -\frac{1}{2}k^{\dot{a}\dot{a}} \mathcal{D}_{\dot{a}\dot{a}} + (k^\alpha \mathcal{D}_\alpha + k^{\alpha(2)} M_{\alpha(2)} + \text{c.c.}) = \bar{\mathcal{K}} \]  

(44)

with the parameters constrained by

\[ k_\alpha = \frac{i}{8} \mathcal{D}^{\dot{\alpha}} k_{\alpha\dot{a}} \quad , \quad k_{\alpha(2)} = \mathcal{D}_\alpha k_\alpha , \]
\[ \mathcal{D}_{\dot{\alpha}} k_{\alpha\dot{a}} = \mathcal{D}_\alpha k_{\alpha\dot{a}} = 0 \quad , \quad \mathcal{D}^{\alpha} \mathcal{D}^{\dot{\alpha}} k_{\alpha\dot{a}} = \mathcal{D}^{\dot{\alpha}} \mathcal{D}^{\alpha} k_{\alpha\dot{a}} = 0 . \]  

(45)

and chosen in the case at hand as follows

\[ k_{\alpha\dot{a}} = 16i(\varepsilon_\alpha \varepsilon'_\dot{\alpha} - \varepsilon'_\alpha \varepsilon_\dot{\alpha}) . \]  

(46)

\( \delta_g \) and \( \delta_\gamma \) denote special \( \varepsilon \)-dependent gauge transformations \([14, 16]\) and \([19, 21]\)

\[ g_{\alpha\dot{a}} = (\mathcal{D}^2 - 2(s + 2)\mu)(\varepsilon^{\beta} \varepsilon' - \varepsilon'^\beta \varepsilon) \mathcal{D}_\beta H_{\alpha(s)\dot{a}(s)} - 4s\bar{\mu} \bar{\mathcal{D}}_{\dot{\alpha}}(\varepsilon^{\beta} \varepsilon' - \varepsilon'^\beta \varepsilon) H_{\alpha(s)\dot{a}(s-1)\dot{\dot{a}}} \]
\[ \gamma_{\alpha(s-1)\dot{a}(s-1)} = (\mathcal{D}^2 + 2(s - 1)\mu)(\varepsilon^{\beta} \varepsilon' - \varepsilon'^\beta \varepsilon) \mathcal{D}_\beta H_{\alpha(s-1)\dot{a}(s-1)} - 4s\bar{\mu} \bar{\mathcal{D}}_{\dot{\alpha}}(\varepsilon\varepsilon' - \varepsilon'\varepsilon) H_{\alpha(s-1)\dot{a}(s-1)} \]  

(47)

Eq. \([15]\) defines a Killing supervector \( \mathcal{K} \) of the AdS superspace, \([\mathcal{K}, \mathcal{D}_\alpha] = 0 \). The set of all Killing supervectors is known to form the Grassmann shell of the superalgebra
Remarkably, the union of the transformations (12) and the \( N = 1 \) AdS ones (14) provides a realization of the \( N = 2 \) supersymmetry algebra, \( osp(4, 2) \). The point is that the spinor Killing superfield \( \varepsilon_\alpha \) defined by Eq. (38) contains among its components two independent constant parameters: a spinor and a scalar. The latter proves to correspond to \( O(2) \)-rotations which enter into \( osp(4, 2) \).

Eqs. (41), (42) allow us to obtain new representations for \( N = 2 \) supersymmetry in flat superspace. The point is that there are two different possibilities for carrying out the limit to flat superspace: (I) One can keep \( \varepsilon \) fixed in the limit \( \mu \to 0 \); (II) \( \varepsilon \) goes to infinity in the limit \( \mu \to 0 \) such that \( \bar{\mu}\varepsilon \) remains finite and non-zero. In the former case, the flat-superspace limit of Eqs. (39), (40) reads

\[
\varepsilon_\alpha = \frac{1}{2} D_\alpha \varepsilon , \quad \bar{D}^2 \varepsilon = D_\alpha \bar{D}\bar{\alpha} \varepsilon = 0 , \quad \varepsilon = \bar{\varepsilon} .
\]

and \( \varepsilon \) has the explicit form

\[
\varepsilon = f + 2\theta^\alpha \xi_\alpha + 2\bar{\theta}^\dot{\alpha} \xi^{\dot{\alpha}} , \quad f = \text{const} , \quad \xi_\alpha = \text{const} .
\]

Then (12) turns into

\[
\begin{align*}
\delta_\varepsilon H(s, s) &= 2i\varepsilon \left( G'(s, s) - \bar{G}'(s, s) \right) , \\
\delta_\varepsilon \Gamma_\alpha(s-1)\dot{\alpha}(s-1) &= -\frac{i}{2} \bar{D}^2 \varepsilon D_\beta H'_\alpha(s-1)\dot{\alpha}(s-1) - \frac{2is}{s+1} \varepsilon D_\beta \bar{G}'_\alpha(s)\dot{\alpha}(s) \\
&\quad - \frac{is}{s+1} \bar{D}\bar{\alpha} \varepsilon D_\beta \bar{G}'_\alpha(s)\dot{\alpha}(s) , \\
\delta_\varepsilon H'(s-1, s-1) &= 2i\varepsilon \left( \Gamma(s-1, s-1) - \bar{\Gamma}(s-1, s-1) \right) , \\
\delta_\varepsilon G'_\alpha(s)\dot{\alpha}(s) &= -\frac{i}{2} \bar{D}^2 \varepsilon D_\beta H_\alpha(s)\dot{\alpha}(s) - 2i\varepsilon D_\alpha \bar{D}\bar{\alpha} \Gamma_\alpha(s-1)\dot{\alpha}(s-1) \\
&\quad - i\bar{D}_\alpha \varepsilon D_\beta \bar{D}_\dot{\alpha} \Gamma_\alpha(s-1)\dot{\alpha}(s-1) .
\end{align*}
\]

These expressions describe \( N = 2 \) supersymmetry transformations (22) but written now in terms of \( \Gamma \) and \( G \). Due to the explicit dependence of \( \varepsilon \) on \( \theta \), the transformations obtained do not have a manifestly \( N = 1 \) supersymmetric form; however, commuting (50) with an \( N = 1 \) super-translation results in a purely gauge shift. Similarly, the parameter \( f \) in (49) generates purely gauge transformations. Now, let us consider the second possibility of reduction to the flat superspace. Here \( \varepsilon_\alpha \) is well-defined in the limit \( \mu = \bar{\mu} \to 0 \), satisfies the equations

\[
D_\alpha \varepsilon_\alpha = \bar{D}\bar{\alpha} \varepsilon_\alpha = 0 , \quad D^\alpha \varepsilon_\alpha = \bar{D}\bar{\alpha} \varepsilon^{\dot{\alpha}} .
\]

and therefore has the following form

\[
\varepsilon_\alpha = \xi_\alpha + \theta_\alpha c , \quad \xi_\alpha = \text{const} , \quad c = \text{const} .
\]
Under this contraction only the transformation law (41) possesses a nonsingular flat-superspace limit which reduces to the replacement $\mathcal{D}_A \to \mathcal{D}_A, \mu \to 0$. Choosing $c = 0$ we then recover the exact result in Eq. (22). For $\xi_\alpha = 0$, on the other hand, our transformation law will describe internal $O(2)$-rotations of the dynamical superfields in Minkowski superspace!

5. Let us comment on the $N = 2$ supersymmetry between the formulations $S_{s+1/2}^\parallel [H, G]$ and $S_{s}^\perp [H', \Gamma']$ dually equivalent to $S_{s+1/2}^\parallel [H, \Gamma]$ and $S_{s}^\perp [H', G']$ respectively. It turns out that the use of duality transformations does allow us to obtain some symmetry, with a Killing spinor parameter, of the action $S_{s+1/2}^\parallel + S_{s}^\perp$ in the AdS superspace, but this symmetry will have a nonstandard form. In more detail, the duality of the actions $S_{s+1/2}^\perp + S_{s}^\parallel$ was established in [4] with the aid of the auxiliary action

$$S_{\mathcal{V}}[H, \Gamma, G, V_\parallel] = S_{s+1/2}^\parallel [H, \Gamma] + \left( -\frac{1}{2} \right)^s \int d^8 z E^{-1} \left\{ V_\parallel \cdot \bar{V}_\parallel - \frac{1}{2} G \cdot V_\parallel + c.c. \right\} .$$

(53)

Here $d^8 z E^{-1}$ is the invariant measure of the AdS superspace and both the auxiliary superfield $V_\parallel$ and dynamical superfield $G$ are longitudinal linear. Using the equation of motion for $G$, $V_\parallel$ vanishes and the action $S_{\mathcal{V}}$ reduces to $S_{s+1/2}^\parallel$. The superfields $\Gamma$ and $V_\parallel$ can be expressed in terms of $G$ and $H$ using their own equations of motion, that leads to the formulation $S_{s+1/2}^\parallel$. Now, we are to look for transformation laws of superfields $G$ and $V_\parallel$ for which the variation of $S_{\mathcal{V}}$ equals to that of $S_{s+1/2}^\parallel$. First we put $\delta V_\parallel = 0$. Then we find an equation on $\delta G$

$$\delta S_{\mathcal{V}} - \delta S_{s+1/2}^\parallel = \left( -\frac{1}{2} \right)^s \int d^8 z E^{-1} (2\bar{\Gamma} - \frac{2}{s} \delta G) \cdot V_\parallel = 0 .$$

(54)

If the quantity in the parentheses is transversal linear, the integrand will be a total derivative. This uniquely requires

$$\delta G = s\bar{\mathcal{P}}_\parallel \delta \bar{\Gamma} ,$$

(55)

where the projector $\mathcal{P}_\parallel$ was defined in Eq. (36). The variation $\delta \Gamma'$ of the formulation $S_{s}^\parallel [H, \Gamma']$ can be determined analogously, and we obtain a symmetry between the auxiliary actions for superspins $s + 1/2$ and $s$. However, after eliminating $\Gamma$, $G'$ and the auxiliary superfields, with the use of their equations of motion, we will obtain rather subtle expressions for the transformation laws of the rest superfields which involve higher derivatives, $1/\mu$-factors and do not admit flat-superspace limit.

Finally, let us end by noting that the investigation of the existence of these massless arbitrary superspin-$s N = 2, D = 4$ multiplets is not an idle concern from the view of the $N = 1, D = 4$ superstring. It has long been known that massive representations of $N = 1, D = 4$ supersymmetry are equivalent to $N = 2$ representations. Thus, our confirmation that the $N = 2$ representations exist for arbitrary superspins gives a positive indication for the existence of massive $N = 1, D = 4$ arbitrary
superspin-$s$ multiplets. Since all $N = 1, D = 4$, $s > 1$ supermultiplets must be massive in the case of the superstring, our present work is seen to be consistent with our conjecture that the pre-potentials $H(s, s)$ and $H'(s - 1, s - 1)$ can be consistently interpreted as different components of a superstring theory.

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**References**

[1] S.M. Kuzenko, V.V. Postnikov and A.G. Sibiryakov, JETP Lett. 57 (1993) 534.

[2] S.M. Kuzenko and A.G. Sibiryakov, JETP Lett. 57 (1993) 539.

[3] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*. Institute of Physics Publishing, Bristol and Philadelphia, 1995.

[4] S.M. Kuzenko and A.G. Sibiryakov, Phys. At. Nucl. 57 (1994) 1257.

[5] V.I. Ogievetsky and E. Sokatchev, Nucl. Phys. B124 (1977) 309.

[6] W. Siegel and S.J. Gates, Jr., Nucl. Phys. B147 (1979) 77.

[7] S.J. Gates and W. Siegel, Nucl. Phys. B163 (1980) 519.

[8] E.A. Ivanov and A.S. Sorin, J. Phys. A13 (1980) 1159.

[9] S.M. Kuzenko and A.G. Sibiryakov, *Towards a unified theory of massless superfields of all superspins*, in preparation.

[10] M.A. Vasiliev, Fortschr. Phys. 35 (1987) 741; Nucl. Phys. B 307 (1988) 319.

[11] S.J. Gates, Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace*. Benjamin–Cummings, Reading, MA, 1983.

[12] V.I. Ogievetsky and E. Sokatchev, J. Phys. A10 (1977) 2021.

[13] E.S. Fradkin and M.A. Vasiliev, Lett. Nuov. Cim. 25 (1979) 79; B. de Wit and J.W. van Holten, Nucl. Phys. B155 (1979) 530.

[14] S.J. Gates, Jr. and W. Siegel, Nucl. Phys. B164 (1980) 484; S.J. Gates, Jr. and V. A. Kostelecky, Nucl. Phys. B248 (1984) 570.
[15] S.J. Gates, Jr., M. Roček and W. Siegel, Nucl. Phys. B198 (1982) 113.

[16] S. J. Gates, Jr. and H. Nishino, Phys. Lett. 173B (1986) 46.

[17] A. Schild, Phys. Rev. D16 (1977) 1722; A. Garcia, J.F. Plebanski and I. Robinson, Gen. Relativity Grav.8 (1977) 841; J. Sachel Phys. Rev. D21 (1980) 2182; P. Budinich, Commun. Math. Phys. 107 (1986) 455; F. Lizzi, B. Rai, G. Sparano and A. Srivastava, Phys. Lett. 182B (1986) 326; J. Gamboa, C. Ramirez and M. Ruiz-Altaba, Nucl. Phys. B338 (1990) 143; idem., Phys. Lett. 231B (1989) 57; idem., Phys. Lett. B225 (1989) 335-339; F. Lizzi and G. Sparano, Phys. Lett. 232B (1989) 311; J. Barcelos-Neto, C. Ramirez and M. Ruiz-Altaba, Z. Phys. C47:241-246, 1990; A. A. Zheltukhin, Proceedings of the Protvino HEP Workshop (1989) 77; idem., Phys. Lett. B233 (1989) 112; idem., Yad. Fiz. 51 (1990) 1504; U. Lindstrom, B. Sundborg and G. Theodoridis; Phys. Lett. B253 (1991) 319.

[18] S.J. Gates, Jr., in Superspace and Supergravity, eds., Hawking and Roček, Cambridge Univ. Press (1981), pp. 219-235.

[19] S.J. Gates, Jr., A. Karlhede, M. Roček, U. Lindstrom, Nucl. Phys. B243 (1984) 221; idem. Class. Quant. Grav. 1 (1984) 227; J. M. F. Labastida, M. Roček, E. Sanchez-Velasco and Peter Wills, Phys. Lett. 151B (1985) 111.

[20] V. I. Ogievetsky and E. Sokatchev, Nucl. Phys. B99 (1975) 96; S. J. Gates, Jr. and W. Siegel, Nucl. Phys. B189 (1981) 295.

[21] T. Curtright, Phys. Lett. B85 (1979) 219.