Excluding Zeno Behaviour in Event-Triggered Time-Delay Systems by Impulsive Controls

Kexue Zhang and Bahman Gharesifard

Abstract

In this paper, we study the problem of event-triggered control for stabilization of general nonlinear time-delay systems. Based on a Razumikhin-type input-to-state stability result for time-delay systems, we propose an event-triggered control algorithm to stabilize nonlinear time-delay systems. In order to exclude the Zeno behaviors, we combine a novel impulsive control mechanism with the proposed event-triggered strategy; in this sense, our proposed algorithm is a hybrid impulsive and event-triggered strategy. We then obtain sufficient conditions for the stabilization of the nonlinear control systems with time-delay by using Lyapunov method and Razumikhin technique. Numerical simulations are provided to show the effectiveness of our theoretical results.

Index Terms

Event-triggered control, Zeno behavior, time-delay system, impulsive control, Razumikhin technique.

I. INTRODUCTION

Event-triggered control strategies allow for updating the sequence of control inputs in an effective manner while still guaranteeing the underlying desired performance. There is a very large literature on this subject, which we are unable to survey here but refer to [1], [2] and references therein. Event-triggered control method has been widely applied to various control problems, e.g., consensus problems [3], distributed optimization protocols [4], fault detection [5], and sensor scheduling [6].

Kexue Zhang is with the Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta T2N 1N4, Canada (e-mail: kexue.zhang@ucalgary.ca).

Bahman Gharesifard is with the Department of Mathematics and Statistics, Queen’s University, Kingston, Ontario K7L 3N6, Canada (e-mail: bahman.gharesifard@queensu.ca).
Time-delay is ubiquitous in many practical systems and dynamical systems with time-delay present in many fields (see, e.g. [7]–[10], and many references therein). Due to the advantage of event-triggered control in efficiency improvements and the significance of time-delay systems in modeling the real-world phenomena, it is crucially needed to design event-triggered control strategies for time-delay systems. The past few years have witnessed a widespread increase of interests in this research area, especially in consensus problems of multi-agent systems which are normally described as linear time-delay systems. In [11], the authors studied leader-following consensus of multi-agent systems with time delay by employing event-triggered consensus protocols, and Zeno behavior (a phenomenon of infinite number of control updates over a finite time period) was successfully excluded from the this consensus problem. The idea of ruling out Zeno behavior introduced in [11] was then applied to various consensus problems with time-delay, such as observer-based consensus [12] and stochastic consensus [13]. The event-triggered consensus protocols considered in [14], [15] both require the explicit information of the time-delay since the updates of the control signal depend on the delayed state of each agent. However, the study of nonlinear time-delay systems is challenging and still to a large extent open. One main challenge in this area is to exclude Zeno behavior, which is the subject of our work.

In this paper, we study stabilization problem of nonlinear time-delay systems by applying the event-triggered control approach. Coupled with the event-triggered control mechanism, we use impulsive controls to help rule out Zeno behavior. The method of impulsive control has been proved to be powerful in the design and synthesis of control systems (see, e.g., [16]–[20] and the references therein). The main contributions of this research are as follows.

**Statement of contributions.** We use a Razumikhin-type input-to-state stability result for time-delay systems to derive an event-triggered control algorithm which enforces control input updates whenever the norm of a certain measurement error becomes large. This idea is a natural adaptation and generalization of the work [21] on nonlinear delay-free systems. We show that the proposed algorithm guarantees the asymptotic stability, however, when applied to stabilize a class of scalar linear systems with time-delay, as we demonstrate both analytically and numerically, it can naturally lead to Zeno behaviors. Therefore, the analysis of excluding Zeno behavior in [21] cannot be extended to the class of time-delayed control systems under study, which is the main challenge that we address here. In particular, to exclude Zeno behavior, we introduce an impulsive control mechanism alongside the event-triggered control strategy. Our “hybrid” control
algorithm works as follows: We first prescribe a threshold constant \( h > 0 \) which serves as a lower bound of each inter-execution time (the time between two successive control updates). If the measurement error becomes large enough at a time later than \( h \) units of time after the last control update, then we update the control input. Otherwise, the control signal will not be updated until \( h \) units of time after the previous control update, at which time we execute an impulsive control input and then update the feedback control signal. This newly proposed hybrid algorithm ensures that the inter-execution times are at least \( h \) units of time, which implies the exclusion of Zeno behavior in the control systems. By using Lyapunov function method and Razumikhin technique, we construct sufficient conditions on the impulse inputs and impulse moments to guarantee the asymptotic stability of the corresponding control systems under this hybrid control algorithm.

The rest of this paper is organized as follows. Sections II contains some mathematical preliminaries. In Section III, we derive an event-triggered control algorithm and show that in the presence of delay, even a linear control system can exhibit Zeno behavior. In Section IV, we propose a hybrid impulsive and event-triggered control algorithm to exclude such Zeno behaviors. Our idea of future research is summarized in Section V.

II. Preliminaries

Let \( \mathbb{N} \) denote the set of positive integers, \( \mathbb{R} \) the set of real numbers, \( \mathbb{R}^+ \) the set of nonnegative reals, and \( \mathbb{R}^n \) the \( n \)-dimensional real space equipped with the Euclidean norm denoted by \( \| \cdot \| \). For \( a, b \in \mathbb{R} \) with \( b > a \), let \( \mathcal{PC}([a, b], \mathbb{R}^n) \) denote the set of piecewise right continuous functions \( \varphi : [a, b] \to \mathbb{R}^n \), and \( \mathcal{PC}([a, \infty), \mathbb{R}^n) \) the set of functions \( \varphi : [a, \infty) \to \mathbb{R}^n \) satisfying

\[
\varphi|_{[a, b]} \in \mathcal{PC}([a, b], \mathbb{R}^n),
\]

for all \( b > a \), where \( \varphi|_{[a, b]} \) is a restriction of \( \varphi \) on interval \([a, b]\). Given \( \tau > 0 \), the linear space \( \mathcal{PC}([-\tau, 0], \mathbb{R}^n) \) is equipped with a norm defined by

\[
\| \varphi \|_\tau := \sup_{s \in [-\tau, 0]} \| \varphi(s) \|,
\]

for \( \varphi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n) \). For simplicity, we use \( \mathcal{PC} \) to represent \( \mathcal{PC}([-\tau, 0], \mathbb{R}^n) \). For \( x \in \mathcal{PC}([a, \infty), \mathbb{R}^n) \), define \( \Delta x \) as

\[
\Delta x(t) := x(t^+) - x(t^-),
\]

where \( x(t^+) \) and \( x(t^-) \) denote respectively the right- and left-hand limits of \( x \) at \( t \).
Consider the following time-delay control system:

\[
\begin{align*}
\dot{x}(t) &= f(t, x_t, u), \\
 x_{t_0} &= \varphi,
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the system state; \(u \in \mathcal{PC}([t_0, \infty), \mathbb{R}^m)\) represents the input; \(\varphi \in \mathcal{PC}\) is the initial function; \(f : \mathbb{R}^+ \times \mathcal{PC} \times \mathbb{R}^m \to \mathbb{R}^n\) satisfies \(f(t, 0, 0) = 0\) for all \(t \in \mathbb{R}^+\); \(x_t\) is defined as \(x_t(s) = x(t+s)\) for \(s \in [-\tau, 0]\), and \(\tau > 0\) is the maximum involved delay. Given \(u \in \mathcal{PC}([t_0, \infty), \mathbb{R}^m)\), define \(g(t, \phi) = f(t, \phi, u(t))\) and assume \(g\) satisfies all the necessary conditions in [22] so that, for any initial condition \(\varphi \in \mathcal{PC}\), system (1) has a unique solution \(x(t, t_0, \varphi)\) that exists in a maximal interval \([t_0 - \tau, t_0 + \Gamma]\), where \(0 < \Gamma \leq \infty\).

**Remark 1.** Throughout this paper, we assume the initial condition \(\varphi\), the continuous dynamics described by \(f\) and the system trajectory \(x\) are all piecewise right-continuous. This is a less conservative requirement than what has been considered in the community of time-delay systems which suppose the system trajectories are continuous (see, e.g., [9], [23], [24]). The main reason of making such less conservative assumption is that we will introduce impulsive control input to control system (1) that will bring in the discontinuities (due to the state jumps or impulses) to the system state. The fundamental theory for existence and uniqueness of solutions to impulsive time-delay systems established in [22] are applicable to time-delay control system (1), since system (1) is a special case of impulsive time-delay systems.

The notion of input-to-state stability (ISS), introduced by Sontag in [25], has been proved powerful in the analysis and controller design of dynamical systems, especially in the design of event-triggered controllers (see, e.g., [3], [4], [11], [21]). We introduce the following function classes before giving the formal ISS definition for system (1). A continuous function \(\alpha : \mathbb{R}^+ \to \mathbb{R}\) is said to be of class \(\mathcal{K}\) and we write \(\alpha \in \mathcal{K}\), if \(\alpha\) is strictly increasing and \(\alpha(0) = 0\). If \(\alpha\) is also unbounded, we say that \(\alpha\) is of class \(\mathcal{K}_\infty\) and we write \(\alpha \in \mathcal{K}_\infty\). A continuous function \(\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+\) is said to be of class \(\mathcal{KL}\) and we write \(\beta \in \mathcal{KL}\), if \(\beta(\cdot, t) \in \mathcal{K}\) for each \(t \in \mathbb{R}^+\) and \(\beta(s, t)\) decreases to 0 as \(t \to \infty\) for each \(s \in \mathbb{R}^+\). Now we are in the position to state the ISS definition for system (1).

**Definition 1.** System (1) is said to be input-to-state stable (ISS) if there exist functions \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}_\infty\) such that, for each initial condition \(\varphi \in \mathcal{PC}\) and input function \(u \in \mathcal{PC}([t_0, \infty), \mathbb{R}^m)\),
the corresponding solution to (1) exists globally and satisfies

\[ \|x(t)\| \leq \beta(\|\varphi\|_r, t - t_0) + \gamma \left( \sup_{s \in [t_0, t]} \|u(s)\| \right), \text{ for all } t \geq t_0. \]

Next, we present several concepts regarding Lyapunov functions and review a Razumikhin-type ISS result that will be used for the design of our event-triggered control mechanism. A function \( V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \) is said to be of class \( \nu_0 \) and we write \( V \in \nu_0 \), if, for each \( x \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^n) \), the composite function \( t \mapsto V(t, x(t)) \) is also in \( \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^n) \) and can be discontinuous at some \( t' \in \mathbb{R}^+ \) only when \( t' \) is a discontinuity point of \( x \). Given a function \( V \in \nu_0 \) and an input \( u \in \mathcal{PC}([t_0, \infty), \mathbb{R}^m) \), the upper right-hand derivative \( D^+V \) of the Lyapunov function candidate \( V \) with respect to system (1) is defined as follows:

\[ D^+V(t, \phi(0)) = \lim_{h \to 0^+} \frac{V(t + h, \phi(0) + hf(t, \phi, u)) - V(t, \phi(0))}{h} \]

for \( \phi \in \mathcal{PC} \). Reference [26] studied a more general form of system (1) with impulse effects. Here, we review a special case in which no impulses are considered and the corresponding ISS result is as follows, see also [27].

**Theorem 1.** Assume that there exist functions \( V \in \nu_0 \) and \( \alpha_1, \alpha_2, \chi \in \mathcal{K}_\infty \), and constants \( q > 1 \), \( c > 0 \) such that, for all \( t \in \mathbb{R}^+ \), \( x \in \mathbb{R}^n \) and \( \phi \in \mathcal{PC} \),

(i) \( \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \);

(ii) whenever \( qV(t, \phi(0)) \geq V(t + s, \phi(s)) \) for all \( s \in [-\tau, 0] \),

\[ D^+V(t, \phi(0)) \leq -cV(t, \phi(0)) + \chi(\|u\|). \]

Then system (1) is ISS.

It can be seen from the proof of Theorem 1 in [26] that the global asymptotic stability (GAS) of system (1) is guaranteed when \( u = 0 \). The Razumikhin-type condition (ii) in Theorem 1 plays an essential role in the event-triggered controller design, as we demonstrate in the next section.

**III. Event-Triggered Control Algorithm**

Consider system (1) with a feedback control input \( u \) as follows:

\[
\begin{align*}
\dot{x}(t) &= f(t, x_t, u), \\
\quad u(t) &= k(x(t)), \quad t \in [t_i, t_{i+1}) \\
\quad x_{t_0} &= \varphi,
\end{align*}
\] (2)
where \( u \in \mathbb{R}^m \) is the feedback control input and \( k : \mathbb{R}^n \to \mathbb{R}^m \) is the feedback control law. The time sequence \( \{t_i\}_{i=1}^{\infty} \) is implicitly defined by certain execution rule to be determined later based on the measurement of the system states, and each time instant \( t_i \) corresponds to a control input update \( u(t_i) \). To be more specific, the controller \( u \) samples the system states and updates its input signal both at each \( t_i \) while remaining constant between two successive control updates.

Let us define the state measurement error by
\[
e(t) = x(t_i) - x(t),
\]
for \( t \in [t_i, t_{i+1}) \) with \( i \in \mathbb{N} \), and then rewrite
\[
u(t) = k(x(t_i)) = k(e(t) + x(t)).
\]

Substituting (4) into system (2) gives the following closed-loop system:
\[
\begin{cases}
\dot{x}(t) = f(t, x, k(e + x)), \\
x_{i_0} = \varphi.
\end{cases}
\]

We make the following assumption on the control system (5).

**Assumption III.1.** There exist functions \( V \in \nu_0 \) and \( \alpha_1, \alpha_2, \chi \in \mathcal{K}_{\infty} \), and constants \( q > 1, c > 0 \) such that all the conditions of Theorem 1 hold for system (5) with input \( u \) replaced with \( e \).

It can be seen from Theorem 1 that assumption III.1 guarantees that the closed-loop system (5) is ISS with respect to the measurement error \( e \), and system (5) is GAS provided \( e = 0 \). In this paper, we design an execution rule to determine the time sequence \( \{t_i\}_{i=1}^{\infty} \) for the updates of the feedback control input \( u \) so that the closed-loop system (5) with the measurement error \( e \) is still GAS. To do so, we restrict \( e \) to satisfy
\[
\chi(||e||) \leq \sigma \alpha_1(||x||) \text{ for some } \sigma > 0,
\]
then the dynamics of \( V \) is bounded by
\[
D^+ V(t, x) \leq -c V(t, x) + \sigma \alpha_1(||x||) \leq -(c - \sigma) V(t, x)
\]
whenever \( q V(t, x(t)) \geq V(t + s, x(t + s)) \) for all \( s \in [-\tau, 0] \). This guarantees the control system (5) is GAS provided \( \sigma < c \). The updating of the control input \( u \) can be triggered by the execution rule (or event)
\[
\chi(||e||) = \sigma \alpha_1(||x||).
\]
The event times are the instants when the event happens, that is,

\[ t_{i+1} = \inf\{t \geq t_i \mid \chi(||e||) = \sigma \alpha_1(||x||)\}. \tag{9} \]

According to the control law in (2), the control input is updated at each \( t_i \) (the error \( e \) is set to zero simultaneously), remains constant until the next event time \( t_{i+1} \), and then the error \( e \) is reset to zero again. Therefore, the proposed event times in (9) insures the GAS of control system (2).

Since the event times in (9) are defined implicitly, it is essential to rule out the existence of Zeno behavior, which we define below for completeness.

**Definition 2** (Zeno Behavior). *If there exists \( T > 0 \) such that \( t_l \leq T \) for all \( l \in \mathbb{N} \), then system (2) is said to exhibit Zeno behavior.*

It worth mentioning that for control systems without time-delay, a well-known result in [21] says that if the functions \( f \) and \( k \) in (2) are Lipschitz continuous on compact sets, then it is possible to exclude Zeno behavior (some extra conditions are required for a definite exclusion, see [21] for more details). However, this is not true for event-triggered control systems with time-delay. For demonstration, let us study the following linear scalar control system with time-delay

\[
\begin{align*}
\dot{x}(t) &= b x(t - r) + u(t), \\
x_{t_0} &= \phi,
\end{align*}
\tag{10}
\]

where state \( x \in \mathbb{R} \), \( \phi(s) = 1 \) for \( s \in [-r, 0] \), \( r = 16 \), \( b = -0.1 \), and \( u(t) = kx(t) \) is the state feedback control with \( k = -0.2 \). By considering Lyapunov function \( V(x) = x^2 \), it can be derived from Theorem 1 (with \( e = 0 \) ) that system (10) is GAS. However, the system is unstable without the control input (i.e., \( u = 0 \), and see Fig. 1 for illustration).

Consider the event-triggered implementation of \( u \) in system (10), and then the closed-loop system can be rewritten in the form of (5):

\[
\begin{align*}
\dot{x}(t) &= b x(t - r) + kx(t) + ke(t), \\
x_{t_0} &= \phi,
\end{align*}
\tag{11}
\]

where \( e(t) = x(t_i) - x(t) \) for \( t \in [t_i, t_{i+1}) \) with \( i \in \mathbb{N} \), and the sequence of event times \( t_1, t_2, \ldots \) is to be determined according to (9). To derive the functions \( \alpha_1 \) and \( \chi \) in (9), we choose Lyapunov
function $V(x) = x^2$. Then condition (i) of Theorem 1 is satisfied with $a_1(|x|) = a_2(|x|) = x^2$. From the dynamics of system (11), it follows that

$$
\dot{V}(x(t)) = 2kx^2 + 2bxx(t - r) + 2kxe \\
\leq 2kx^2 + |b|(|e|x^2 + e^{-1}x^2(t - r)) + |k|(e|x^2 + e^{-1}e^2) \\
= (2k + e|b| + e|k|)V(x(t)) + e^{-1}|b|V(x(t - r)) + e^{-1}|k|e^2
$$

where $\varepsilon = \sqrt{q}$ and $\epsilon = \sigma/|k|$. Whenever

$$qV(x(t)) \geq V(x(t + s)),
$$

for all $s \in [-r, 0]$ with some $q > 1$, we have

$$
\dot{V}(x) = (2k + e|b| + e^{-1}q|b| + e|k|)V(x) + e^{-1}|k|e^2,
$$

then, condition (ii) of Theorem 1 holds with

$$
c = -(2k + e|b| + e^{-1}q|b| + e|k|) > 0 \text{ and } \chi(|e|) = e^{-1}|k|e^2.
$$

Therefore, assumption III.1 is satisfied for system (11). The event times defined by (9) are as follows:

$$t_{i+1} = \inf\{t \geq t_i | e^2 = \sigma_0 x^2\} \quad (12)
$$

where positive constant $\sigma$ satisfies $\sigma < c$. Using the fact $\epsilon = \sigma/|k|$, we can rewrite (12) as

$$t_{i+1} = \inf\{t \geq t_i | e^2 = \sigma_0 x^2\} \quad (13)
$$

and the condition $\sigma < c$ as

$$2k + e|b| + e^{-1}q|b| + e|k| + e^{-1}|k|\sigma_0 < 0 \quad (14)
$$

where $\sigma_0 = e^2$. Then,

$$k + \sqrt{q}|b| + \sqrt{\sigma_0}|k| < 0 \quad (15)
$$

implies (14), by the facts that $\varepsilon = \sqrt{q}$ and $\epsilon = \sigma/|k|$. According to our analysis of the nonlinear control system (5), if (15) holds, then closed-loop system (11) is GAS with the event times determined by (13).

We now choose $\sigma_0 = 0.36$ so that (15) is satisfied, and we will show there are infinitely many event times over the time interval $[0, 10]$. We first prove that $t_1 < 10$ and $0 < x(t_1) < 1 - t_1/10$. For $t \leq r$, we have $x(t - r) = x(t_0) = 1$ and

$$\dot{x} = -0.1x(t - r) - 0.2x(t_0) = -0.3 < -0.1,$$
for \( t \leq \min(t_1, r) \). Therefore, both \( x \) and \( e \) are positive, \( x \) is strictly decreasing, and \( e \) is strictly increasing for \( t \in (t_0, \min(t_1, r)) \). By (13), \( e^2(t_1) = \sigma_0 x^2(t_1) \), that is, \( e(t_1) = 0.6x(t_1) \) which implies \( x(t_1) = x(t_0)/1.6 \). We then can conclude that \( x \) strictly decreases from 1 at \( t_0 \) to \( x(t_0)/1.6 \) at \( t_1 \) with \( t_1 < 10 \) and the decreasing rate is smaller than \(-0.1\). Thus, \( 0 < x(t_1) < 1 - t_1/10 \). See Fig. 2 for a demonstration. Next, suppose \( t_m < 10 \) and \( 0 < x(t_m) < 1 - t_m/10 \) for some \( m \geq 1 \). From the dynamics of system (11), we have

\[
\dot{x} = -0.1 - 0.2x(t_m) < -0.1,
\]

for \( t \in [t_m, \min(t_{m+1}, r)] \). Then we derive that \( x \) strictly decreases from \( x(t_m) \) at \( t = t_m \) to \( x(t_{m+1}) = x(t_m)/1.6 \) at \( t = t_{m+1} \) with \( t_{m+1} < 10 \) and decreasing rate less than \(-0.1\). Hence, \( x(t_{m+1}) < 1 - t_{m+1}/10 \). Based on the above discussion, we conclude from mathematical induction that there are infinitely many event times on \([0, 10]\), that is, control system (11) exhibits Zeno behavior (see Fig. 3 for demonstration).

For this specific system, its trajectory intersects with the time axis, and \( \dot{x} \) is bounded. Therefore, it takes less and less time for \( e^2 \) to evolve from 0 to \( \sigma_0 x^2 \) as \( x \) getting closer and closer to 0. For linear scalar systems without time-delay, this property does not hold mainly because its trajectory does not go to zero in a finite time. Thus, the well-known result in [21] for delay-free
systems cannot be generalized to time-delay systems seamlessly in the sense of excluding Zeno behavior. The above example indicates that ruling out the existence of Zeno behavior cannot be guaranteed even for linear control systems with time-delay, though the sequence of event times determined by (9) assures the GAS of the event-triggered control system (11). Excluding the existence of Zeno behavior is the objective of Section IV.

IV. EXCLUDING ZENO BEHAVIOR VIA IMPULSES

As we demonstrated, a linear control system with time-delay can exhibit Zeno behavior under natural event-triggered control strategies. Our main objective is to show that one can still use event-triggered strategies, as long as they are paired with impulsive control signals, designed to exclude Zeno behaviors.

To proceed, let us define a sequence of event-time candidates

$$\tilde{t}_{i+1} = \inf\{t \geq t_i \mid \chi(||e||) = \sigma \alpha_1(||x||)\} \quad (16)$$

where the sequence of event times $t_1, t_2, t_3, \ldots$ is to be determined with a lower bound $h > 0$ of the inter-execution time $\inf_{i \in \mathbb{N}}\{\tilde{t}_{i+1} - t_i\}$ according to the execution rule displayed at Hybrid-EI below.

It can be seen from Hybrid-EI that the inter-execution times $\{t_{i+1} - t_i\}_{i \in \mathbb{N}}$ are lower bounded by $h$, that is, $t_{i+1} - t_i \geq h$ for all $i \in \mathbb{N}$. This excludes the existence of Zeno behavior. The
Hybrid Event-triggered/Impulsive Strategy (Hybrid-EI)

1. If \( \bar{t}_{i+1} - t_i > h \), then let \( t_{i+1} = \bar{t}_{i+1} \) and update the control input signal at \( t = t_{i+1} \).

2. If \( \bar{t}_{i+1} - t_i \leq h \), then activate an impulse (state jump) \( \Delta x(t) = I(t, x) \) at \( t = t_i + h \) where \( I : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) regulates the state jump. Let \( t_{i+1} = t_i + h \) and update the control input at \( t = t_{i+1} \) after the jump.

The closed-loop system can be written as an impulsive system:

\[
\begin{aligned}
\dot{x}(t) &= f(t, x, u), \\
\Delta x(t_{i+1}) &= I(t_{i+1}, x(t_{i+1})), \quad \text{if } t_{i+1} = t_i + h \\
x_{t_0} &= \varphi, \\
\end{aligned}
\]  

(17)

If \( t_{i+1} = t_i + h \), we call \( t_{i+1} \) an impulse time. We assume that \( x \) is right-continuous at each impulse time and system (17) satisfies all the necessary conditions in [22] so that for any initial condition \( \varphi \in \mathcal{PC} \), system (17) has a unique global solution \( x(t, t_0, \varphi) \). To ensure the asymptotic stability of system (17), the next theorem presents several sufficient conditions on the continuous dynamics of system (17), the impulses, the lower bound of inter-execution times, and the relation among them.

**Theorem 2.** Suppose that assumption III.1 holds with \( V \in \nu_0, q > 1 \) and \( c > 0 \). For some \( h > 0 \), the event times \( t_1, t_2, t_3, ... \) are defined according to Hybrid-EI with the event-time candidates given in (16) and \( \sigma < c \). If \( t_{i+1} = t_i + h \), we further assume that there exist positive constants \( \bar{c} \) and \( \rho < 1 \) such that

(i) for \( t \in [t_i, t_{i+1}) \), \( D^+ V(t, x) \leq \bar{c} V(t, x) \), whenever \( qV(t, x(t)) \geq V(t+s, x(t+s)) \), for all \( s \in [-\tau, 0] \):
(ii) \( V(t_{i+1}, x(t_{i+1})) \leq \rho V(t_{i+1}, x(t_{i+1})) \)
(iii) \( q > \frac{1}{\rho} > e^{\bar{c}h} \).

Then the closed-loop system (17) is GAS.

**Proof.** Condition (iii) implies that there exists a small enough \( \varepsilon > 0 \) such that \( \frac{1}{\rho} > \frac{1}{\rho + \varepsilon} > e^{\bar{c}h} \). We then can find a positive constant \( \lambda \) close to zero and \( \lambda \leq c - \sigma \) so that both

\[
q \geq e^{\lambda \tau} > \frac{1}{\rho} > e^{(\bar{c}+\lambda)h}
\]
and
\[ q > \frac{e^{tr}}{\rho + \varepsilon} > \frac{1}{\rho + \varepsilon} > e^{\left(c+1\right)h} \]
are satisfied. Let \( M = qe^{-tr} \), then we have \( M > 1 \) and \((\rho + \varepsilon)M > 1\). Let \( v(t) = V(t, x(t)) \) and define \( w(t) = e^{l(t-t_0)}v(t) \) for \( t \geq t_0 - \tau \). By induction, we will show that, for \( t \in [t_i, t_{i+1}) \),
\[
w(t) \leq \begin{cases} 
(\rho + \varepsilon)Ma_2(||\varphi||_r), & \text{if } t_{i+1} > t_i + h \\
Ma_2(||\varphi||_r), & \text{if } t_{i+1} = t_i + h
\end{cases}
\]
(18)
For \( s \in [-\tau, 0] \), we have
\[
w(t_0 + s) = e^{l(t_0+s-\tau)}v(t_0 + s) \leq v(t_0 + s) \leq a_2(||\varphi||_r) \]
\[ < (\rho + \varepsilon)Ma_2(||\varphi||_r). \]
Therefore, (18) is true on \([t_0 - \tau, t_0]\). We now prove (18) holds on \([t_0, t_1)\). To do this, we consider the following two cases.

**Case I:** \( t_1 > t_0 + h \). We will show that
\[
w(t) \leq (\rho + \varepsilon)Ma_2(||\varphi||_r) \]
(19)
is true for \( t \in [t_0, t_1) \). We do this by contradiction. Suppose (19) is not true on \([t_0, t_1)\), then there exists some \( t \in (t_0, t_1) \) so that
\[
w(t) > (\rho + \varepsilon)Ma_2(||\varphi||_r). \]
Define now
\[
t^* = \inf\{t \in (t_0, t_1) | w(t) > (\rho + \varepsilon)Ma_2(||\varphi||_r)\}. \]
By the continuity of \( w \), we conclude that
\[
w(t^*) = (\rho + \varepsilon)Ma_2(||\varphi||_r) \]
and
\[
w(t) \leq (\rho + \varepsilon)Ma_2(||\varphi||_r), \]
for all \( t \in [t_0, t^*] \). We next define
\[
t^{**} = \sup\{t \in [t_0, t^*) | w(t) \leq a_2(||\varphi||_r)\}. \]
Since \( w(t_0) \leq a_2(||\varphi||_r) \) and \( w(t^*) > a_2(||\varphi||_r) \), we conclude that
\[
w(t^{**}) = a_2(||\varphi||_r) \]
and that
\[ \alpha_2(\|\varphi\|_r) \leq w(t) \leq (\rho + \varepsilon)\alpha_2(\|\varphi\|_r), \]
for \( t \in [t^*, t^*'] \). Thus, for any \( t \in [t^*, t^*'] \), we have \( t + s \leq t^* \) for all \( s \in [-\tau, 0] \) and \( w(t + s) \leq (\rho + \varepsilon)\alpha_2(\|\varphi\|_r) \leq (\rho + \varepsilon)Mw(t) \) which implies
\[
v(t + s) = w(t + s)e^{-\lambda(t+s-t_0)} < (\rho + \varepsilon)Mw(t)e^{-\lambda(t+s-t_0)} < (\rho + \varepsilon)Me^{\lambda t}v(t) < qv(t) \tag{20}
\]
where we used the facts that \((\rho + \varepsilon)M > 1\), \( \rho + \varepsilon < 1 \) and \( M = qe^{\lambda t} \). We then can conclude from condition (ii) of Theorem 1 and (7) that for \( t \in [t^*, t^*'] \)
\[
D^+w(t) = \lambda e^{\lambda t}v(t) + e^{\lambda t}D^+v(t) = (\lambda - c + \sigma)w(t) \leq 0,
\]
which indicates that \( w(t) \) is nonincreasing on \([t^*, t^*']\) and then \( w(t^*) \geq w(t^*) \) which is a contradiction to the definitions of \( t^* \) and \( t^{**} \). Hence, (19) is true on \([t_0, t_1)\) for Case I.

**Case II: \( t_1 = t_0 + h \).** We will show that
\[
w(t) \leq \alpha_2(\|\varphi\|_r) \tag{21}
\]
holds on \([t_0, t_1)\) by contradiction. Assume that (21) does not hold, then there exists some \( t \in [t_0, t_1) \) such that \( w(t) > \alpha_2(\|\varphi\|_r) \); we define
\[
\bar{t} = \inf\{t \in [t_0, t_1) \mid w(t) > \alpha_2(\|\varphi\|_r)\}.
\]
It follows from the continuity of \( w \) on \([t_0, t_1)\) that
\[
w(\bar{t}) = \alpha_2(\|\varphi\|_r)
\]
and
\[
w(t) \leq \alpha_2(\|\varphi\|_r),
\]
for all \( t \in [t_0, \bar{t}] \). Since \( w(t_0) \leq \alpha_2(\|\varphi\|_r) \), there exists a \( t \in [t_0, \bar{t}) \) so that \( w(t) > \alpha_2(\|\varphi\|_r) \). Let
\[
\tilde{t} = \sup\{t \in [t_0, \bar{t}) \mid w(t) \leq \alpha_2(\|\varphi\|_r)\}.
\]
Then we have that \( w(\tilde{t}) = \alpha_2(\|\varphi\|_{\tau}) \) and
\[
\alpha_2(\|\varphi\|_{\tau}) \leq w(t) \leq M\alpha_2(\|\varphi\|_{\tau}),
\]
for all \( t \in [\tilde{t}, \bar{t}] \). Therefore, for \( t \in [\tilde{t}, \bar{t}] \), we have
\[
w(t + s) \leq M\alpha_2(\|\varphi\|_{\tau}) \leq Mw(t),
\]
for all \( s \in [-\tau, 0] \), which then implies
\[
v(t + s) = e^{-\lambda(t+s-t_0)}w(t + s) \leq e^{-\lambda(t+s-t_0)}Mw(t) = Me^{-\lambda s}v(t) \leq qv(t)
\]
Condition (ii) states that
\[
D^+w(t) = \lambda e^{\lambda(t-t_0)}v(t) + e^{\lambda(t-t_0)}D^+v(t) \leq \lambda w(t) + \bar{c}e^{\lambda(t-t_0)}v(t) = (\lambda + \bar{c})w(t)
\]
for all \( t \in [\tilde{t}, \bar{t}] \), then
\[
w(\bar{t}) \leq w(\tilde{t})e^{(\lambda + \bar{c})(\bar{t} - \tilde{t})} \leq w(\tilde{t})e^{(\lambda + \bar{c})h} = e^{(\lambda + \bar{c})h}\alpha_2(\|\varphi\|_{\tau}) < M\alpha_2(\|\varphi\|_{\tau}),
\]
which is a contradiction to the definition of \( \bar{t} \). Therefore, (21) is true on \([t_0, t_1]\) for this case. We hence conclude from the above two cases that (18) holds for \( t \in [t_0, t_1] \).

Now suppose (18) holds on \([t_0, t_m]\) where \( m \geq 1 \) and we next prove that (18) is still true for \( t \in [t_m, t_{m+1}] \). Similar to the above discussion, we consider two scenarios.

**Case I**: \( t_{m+1} > t_m + h \). We have \( w(t) \leq M\alpha_2(\|\varphi\|_{\tau}) \) for all \( t < t_m \) from (18), and will prove (19) holds on \([t_m, t_{m+1}]\). For \( t = t_m \), we have
\[
w(t_m) = \rho w(t_m^-) \leq \rho M\alpha_2(\|\varphi\|_{\tau}) < (\rho + \varepsilon)M\alpha_2(\|\varphi\|_{\tau}),
\]
that is, (19) is true for $t = t_m$. We next show (19) is true on $(t_m, t_{m+1})$ by a contradiction argument. Suppose (19) does not hold, then we can find a $t \in (t_m, t_{m+1})$ so that $w(t) > (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)$.

To proceed, we define

$$t_* = \inf\{t \in (t_m, t_{m+1}) \mid w(t) > (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)\}.$$  

Using the facts that  

$$w(t_m) < (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)$$  

and the continuity of $w$ on $(t_m, t_{m+1})$, we conclude  

$$w(t_*) = (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)$$  

and  

$$w(t) \leq (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)$$  

on $[t_m, t_*]$. We further define

$$t_{**} = \sup\{t \in [t_m, t_*) \mid w(t) \leq \rho M\alpha_2(\|\varphi\|_r)\}.$$  

Since $w(t_m) \leq \rho M\alpha_2(\|\varphi\|_r)$ and $w(t_*) > \rho M\alpha_2(\|\varphi\|_r)$, we conclude from the continuity of $w$ that  

$$w(t_{**}) = \rho M\alpha_2(\|\varphi\|_r)$$  

and  

$$\rho M\alpha_2(\|\varphi\|_r) \leq w(t) \leq (\rho + \varepsilon)M\alpha_2(\|\varphi\|_r)$$  

on $[t_{**}, t_*]$. For $s \in [-\tau, 0]$, we have $t + s \leq t_*$ when $t \in [t_{**}, t_*]$, then $w(t + s) \leq M\alpha_2(\|\varphi\|_r)$. Therefore,  

$$w(t + s) \leq \frac{1}{\rho} w(t),$$  

for $t \in [t_{**}, t_*]$, which implies  

$$v(t + s) = w(t + s)e^{-\lambda(t+s-t_0)} \leq \frac{1}{\rho} w(t)e^{-\lambda(t-t_0)}e^{-\lambda s} \leq \frac{e^{\lambda \tau}}{\rho} v(t) \leq qv(t)$$  

where we used the fact $\frac{e^{\lambda \tau}}{\rho} \leq q$. Similar to Case I, we can derive the contradiction: $w(t_*) \leq w(t_{**})$. Thus, we conclude that (19) is true on $[t_m, t_{m+1})$ for Case I’.
Case II: $t_{m+1} = t_m + h$. We will show (21) holds on $[t_m, t_{m+1})$. Based on the assumption of $w$ on $[t_0, t_m)$, we have $w(t) \leq M\alpha_2(\|\varphi\|_r)$ for $t \in [t_0, t_m)$. When $t = t_m$, it follows from condition (ii) that

$$w(t_m) \leq \rho w(t_m) \leq \rho M\alpha_2(\|\varphi\|_r) < M\alpha_2(\|\varphi\|_r)$$

which implies (21) is true at $t = t_m$. We next will prove (21) holds on $(t_m, t_{m+1})$ by contradiction. Suppose there exist a $t \in (t_m, t_{m+1})$ so that $w(t) > M\alpha_2(\|\varphi\|_r)$. We define

$$\hat{t} = \inf \{ t \in (t_m, t_{m+1}) \mid w(t) > M\alpha_2(\|\varphi\|_r) \}.$$

The continuity of $w$ yields that

$$w(t) = M\alpha_2(\|\varphi\|_r)$$

and

$$w(t) \leq M\alpha_2(\|\varphi\|_r),$$

for all $t \leq \hat{t}$. Let

$$\hat{t} = \sup \{ t \in [t_m, t) \mid w(t) \leq \rho M\alpha_2(\|\varphi\|_r) \}.$$

Then the facts that

$$w(t_m) \leq \rho M\alpha_2(\|\varphi\|_r)$$

and

$$w(t) > \rho M\alpha_2(\|\varphi\|_r)$$

imply that

$$w(\hat{t}) = \rho M\alpha_2(\|\varphi\|_r) \text{ and } w(t) \geq \rho M\alpha_2(\|\varphi\|_r)$$

on $[\hat{t}, t]$. Therefore, for $s \in [-\tau, 0]$ and $t \in [\hat{t}, t]$, it follows that

$$w(t + s) \leq M\alpha_2(\|\varphi\|_r) \leq \frac{1}{\rho} w(t) \leq Mw(t).$$

By an argument similar to the one in Case II, we have that

$$v(t + s) \leq qv(t)$$

for $s \in [-\tau, 0]$; therefore, (22) holds on $[\hat{t}, t]$ that implies

$$w(t) < M\alpha_2(\|\varphi\|_r),$$
which is a contradiction with the definition of $t$. Hence, (21) is true on $(t_m, t_{m+1})$. This completes the induction proof for all $t \geq t_0$. Therefore, $w(t) \leq M\alpha_2(\|\varphi\|_r)$ on $[t_0, \infty)$, as claimed and the global asymptotic stability of closed-loop system (17) hence follows.

Let us revisit the event-triggered control system (11). We incorporate the linear impulses $\Delta x = \beta x$ at $t = t_i + 1$ when $t_i + 1 = t_i + h$ and use the proposed hybrid strategy with Hybrid-EI. Here, the constants $\beta$ and $h$ are to be determined by using Theorem 2. The closed-loop system can then be written as a linear impulsive system:

$$\begin{align*}
\dot{x}(t) &= bx(t - r) + kx(t_i), \quad \text{for } t \in [t_i, t_{i+1}) \\
\Delta x(t_{i+1}) &= \beta x(t_{i+1}^-), \quad \text{if } t_{i+1} = t_i + h \\
x_{t_0} &= \phi
\end{align*}
$$

(23)

It is not hard to observe that assumption III.1 holds with the given parameters and by selecting the Lyapunov function as $V(x) = x^2$. Next, we show that conditions (i) and (ii) of Theorem 2 hold for system (23): When $t_{i+1} = t_i + h$, we can derive from the continuous dynamics of (23) that, whenever $qV(x(t)) \geq V(x(t + s))$ for all $s \in [-\tau, 0]$, we have that

$$\dot{V}(x(t)) = 2x[bx(t - r) + kx(t_i)]$$

$$\leq |b|([\varepsilon x^2 + \varepsilon^{-1}x^2(t - r)] + |k|[\varepsilon^2 x^2 + \varepsilon^{-1}x^2(t_i)])$$

$$= (\varepsilon|b| + \varepsilon|k|)x^2 + \varepsilon^{-1}|b|x^2(t - r) + \varepsilon^{-1}|k|x^2(t_i)$$

$$\leq [\varepsilon|b| + \varepsilon|k| + q(\varepsilon^{-1}|b| + \varepsilon^{-1}|k|)]V(x(t))$$

$$= \ddot{c}V(x(t)),
$$

(24)

where

$$\varepsilon = \sqrt{q/|b|}, \quad \varepsilon = \sqrt{q/|k|},$$

$$\ddot{c} = 2\sqrt{q(|b| + |k|)} \quad \text{and} \quad \tau = \max\{r, h\}.$$
Therefore, we conclude from Theorem 2 that if there exists a $q > 1$ such that both (15) and condition (iii) of Theorem 2 are satisfied, the closed-loop system (23) is globally asymptotically stable, and the lower bound of the inter-execution times is $h$. To demonstrate the effectiveness of the proposed control algorithm and Theorem 2, let $q = 3$, $h = 0.666$ and $\beta = -0.293$ so that both (15) and condition (iii) of Theorem 2 hold. Fig. 4 shows the stability of system (23), and a clear view of the impulse effects in system (23) is also provided within the figure. Actually, system (23) is globally exponentially stable since $\alpha_1(|x|) = \alpha_2(|x|) = V(x) = x^2$ in condition (i) of Theorem 1. As discussed for system (11), the event-triggered control inputs are updated more and more frequently when the state $x$ gets closer and closer to zero. This explains why the impulses are generally activated around the intersections between the trajectory $x$ and the time axis in Fig. 4. The reason for the existence of large inter-execution times is that it takes more time for $e^2$ to evolve from zero at each event time to $\sigma_0 x^2$ at the next event time if $x^2$ is fairly large and/or $|i|$ is relatively small.

V. Conclusions

We have studied the event-triggered control problem of general nonlinear time-delay systems. An event-triggered control algorithm has been proposed to stabilize the nonlinear systems with time-delay. To exclude Zeno behavior due to the presence of delay, we have incorporated
the impulsive control mechanism into the event-triggered control algorithm to guarantee the nonexistence of Zeno behavior. Future work includes applying our control algorithm to various related control problems, such as, consensus of multi-agent systems and distributed optimization, seeking parallel control algorithms based on the method of Lyapunov-Krasovskii functionals, and extending these algorithms to control of switching time-delay systems.

References

[1] Z.-P. Jiang and T.-F. Liu, A survey of recent results in quantized and event-based nonlinear control, International Journal of Automation and Computing, vol 12, no. 5, pp. 455-466, 2015.
[2] C. Nowzari, E. Garcia, and J. Cortés, Event-triggered communication and control of networked systems for multi-agent consensus, Automatica, vol. 105, pp. 1-27, 2019.
[3] G.S. Seyboth, D.V. Dimarogonas, and K.H. Johansson, Event-based broadcasting for multi-agent average consensus, Automatica, vol. 49, no. 1, pp. 245-252, 2013.
[4] S.S. Kia, J. Cortés, and S. Martínez, Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication, Automatica, vol. 55, pp. 254-264, 2015.
[5] A. Golabi, M. Davoodi, N. Meskin, and J. Mohammadpour, Event-triggered fault detection for discrete-time LPV systems with application to a laboratory tank system, vol. 32, no. 11, pp. 1591-1606, 2018.
[6] A.S. Leong, S. Dey, and D.E. Quevedo, Sensor scheduling in variance based event triggered estimation with packet drops, IEEE Transactions on Automatic Control, vol. 62, no. 4, pp. 1880-1895, 2017.
[7] M. Dhamala, V.K. Jirsa, and M. Ding, Enhancement of neural synchrony by time delay, Physical Review Letters, vol. 92, no. 7, 074104, 2004.
[8] T.B. Sheridan, Space teleoperation through time delay: review and prognosis, IEEE Transactions on Robotics and Automation, vol. 9, no. 5, pp. 592-606, 1993.
[9] J.-P. Richard, Time-delay systems: an overview of some recent advances and open problems, Automatica, vol. 39, no. 10, pp. 1667-1694, 2003.
[10] A. Waibel, T. Hanazawa, G. Hinton, K. Shikano, and K.J. Lang, Phoneme recognition using time-delay neural networks, IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 37, no. 3, pp. 328-339, 1989.
[11] W. Zhang and Z.-P. Jiang, Event-based leader-following consensus of multi-agent systems with input time delay, IEEE Transactions on Automatic Control, vol. 60, no. 5, pp. 1362-1367, 2015.
[12] D. Zhao and T. Dong, Reduced-order observer-based consensus for multi-agent systems with time delay and event triggered strategy, IEEE Access, vol. 5, pp. 1263-1271, 2017.
[13] X. Tan, J. Cao, X. Li, and A. Alsaedi, Leader-following mean square consensus of stochastic multi-agent systems with input delay via event-triggered control, IET Control Theory & Applications, vol. 12, no. 2, pp. 299-309, 2017.
[14] L. Li, D.W.C. Ho, and J. Lu, Event-based network consensus with communication delays, Nonlinear Dynamics, vol. 97, pp. 1847-1858, 2017.
[15] N. Mu, X. Liao, and T. Huang, Event-based consensus control for a linear directed multiagent system with time delay, IEEE Transactions on Circuits and Systems-II: Express Briefs, vol. 62, no. 3, pp. 281-285, 2015.
[16] A. Bensoussan and C.S. Tapiero, Impulsive control in management: Prospects and applications, Journal of Optimization Theory and Applications, vol. 37, no. 4, pp. 419-442, 1982.
[17] G.N. Silva and R.B. Vinter, Necessary conditions for optimal impulsive control problems, SIAM Journal on Control and Optimization, vol. 35, no. 6, pp. 1829-1846, 1997.

[18] I.-C. Morarescu, S. Martin, A. Girard, and A. Muller-Gueudin, Coordination in networks of linear impulsive agents, IEEE Transactions on Automatic Control, vol. 61, no. 9, pp. 2402-2415, 2016.

[19] O.N. Samsonyuk and S.A. Timoshin, Optimal control problems with states of bounded variation and hysteresis, Journal of Global Optimization, vol. 74, no. 3, pp 565-596, 2019.

[20] X. Liu and K. Zhang, Input-to-state stability of time-delay systems with delay-dependent impulses, IEEE Transactions on Automatic Control, 2019, DOI: 10.1109/TAC.2019.2930239

[21] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Transactions on Automatic Control, vol. 52, no. 2, pp. 1680-1685, 2007.

[22] G. Ballinger and X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynamics of Continuous, Discrete & Impulsive Systems, vol. 5, pp. 579-591, 1999.

[23] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[24] E. Fridman, Introduction to Time-Delay Systems: Analysis and Control, Birkhauser, Basel, 2014.

[25] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Transactions on Automatic Control, vol. 34, no. 4, pp. 435-443, 1989.

[26] W.H. Chen and W.X. Zheng, Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, Automatica, vol. 45, pp. 1481-1488, 2009.

[27] A.R. Teel, Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem, IEEE Transactions on Automatic Control, vol. 43, no. 7, pp. 960-964, 1998.