A NUMBER THEORETIC CHARACTERIZATION OF $E$-SMOOTH AND (FRS) MORPHISMS: ESTIMATES ON THE NUMBER OF $\mathbb{Z}/p^k\mathbb{Z}$-POINTS

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Abstract. We provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of flat morphisms between smooth varieties whose fibers have rational singularities, termed (FRS) morphisms. For each individual fiber, the estimates were known by work of Avni and Aizenbud, but we render them uniform over all fibers. The proof technique for individual fibers is based on Hironaka’s resolution of singularities and Denef’s formula, but breaks down in the uniform case. Instead, we use recent results from the theory of motivic integration. Our estimates are moreover equivalent to the (FRS) property, just like in the absolute case by Avni and Aizenbud. In addition, we define new classes of morphisms, called $E$-smooth morphisms ($E \in \mathbb{N}$), which refine the (FRS) property, and use the methods we developed to provide uniform number-theoretic estimates as above for their fibers. Similar estimates are given for fibers of $\varepsilon$-jet flat morphisms, improving previous results by the last two authors.

1. Introduction

1.1. Overview. Let $\varphi : X \to Y$ be an algebraic morphism between smooth $K$-varieties, where $K$ is a number field. In this paper we give uniform arithmetic and analytic equivalent characterizations to the (FRS) property of $\varphi$, namely to the property of being flat with reduced fibers of rational singularities (see Theorem A). These results can be viewed as a common uniform improvement of the following two theorems:

1. [AA18, Theorem A], where bounds were given on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points of reduced local complete intersection schemes which have rational singularities (see also Theorem 1.3).
2. [AA16, Theorem 3.4], where pushforward of smooth measures with respect to $\varphi$ over non-Archimedean local fields were shown to have bounded density if and only if $\varphi$ is an (FRS) morphism (see also Theorem 1.3).

In order to prove our uniform characterizations of the (FRS) property, it seems natural to try and adapt the algebro-geometric proof of [AA18, Theorem A] to the relative case. This fails to work because of unsatisfactory behavior of resolution of singularities in families, with respect to taking points over $\mathbb{Z}$, $\mathbb{Z}/p^k\mathbb{Z}$ and $\mathbb{Z}_p$ (see Section 1.4.1). Instead, we prove a model theoretic result of independent interest about approximating suprema of a certain sub-class of motivic functions, which we call formally non-negative functions (see Theorem B). Using Theorem B and by analyzing the jets of $\varphi$, we prove Theorem A. Theorem [further strengthens CCH18, Theorem 2.1.3] in the case of formally non-negative functions. Finally, we provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of $E$-smooth morphisms, a new notion we introduce which refines the (FRS) property ($E \in \mathbb{N}$). Uniform estimates are also provided for fibers of $\varepsilon$-jet flat morphisms, achieving optimal bounds (c.f. [GH, Theorem 8.18]). See Section 2.1.1 and Theorems 4.11 and 4.12 for these notions and results.
1.2. Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the absolute case. Let $X$ be a finite type $\mathbb{Z}$-scheme. The study of the quantity $\#X(\mathbb{Z}/n\mathbb{Z})$, and its asymptotic behavior in $n \in \mathbb{N}$, is a long standing problem in number theory. When $n = p$ is prime, the asymptotic behavior is understood by the Lang-Weil estimates [LW54], and in particular, the family \[ \left\{ \frac{\#X(\mathbb{Z}/p^i\mathbb{Z})}{p^{i\dim_X}} \right\}_{p} \] is uniformly bounded.

Moving to the case where $n = p^k$ is a prime power (which suffices, by the Chinese remainder theorem), one can observe the following; if $X$ is smooth as a $\mathbb{Z}$-scheme, then an application of Hensel’s lemma shows that \[ \left\{ \frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim_X}} \right\}_{p,k} \] is uniformly bounded in both $p$ and $k$. On the other hand, taking the non-reduced scheme $X = \text{Spec} \mathbb{Z}[x]/(x^2)$, we see that $\#X(\mathbb{Z}/p^2\mathbb{Z}) = p^k$, which is not uniformly bounded. The following natural question arises.

**Question 1.1.** Is there a necessary and sufficient condition on $X$ such that \[ \left\{ \frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim_X}} \right\}_{p,k} \] is uniformly bounded?

In [AA18], Aizenbud and Avni, relying on results of Mustață [Mus01] and Denef [Den87], gave such a necessary and sufficient condition in the case where $X_\mathbb{Q}$ is a local complete intersection.

**Definition 1.2.** Let $K$ be a field of characteristic 0. A $K$-scheme of finite type $X$ has **rational singularities** if it is normal and for every resolution of singularities $\pi: \tilde{X} \to X$, one has $R^i\pi_*(O_{\tilde{X}}) = 0$ for $i \geq 1$.

**Theorem 1.3** (see [AA18, Theorem A] and [Gla19]). Let $X$ be a finite type $\mathbb{Z}$-scheme such that $X_\mathbb{Q}$ is equidimensional and a local complete intersection. Then the following are equivalent:

1. $X_\mathbb{Q}$ has rational singularities (and, in particular, $X_\mathbb{Q}$ is reduced).
2. There exists $C > 0$ such that for every prime $p$ and every $k \in \mathbb{N}$ one has $\frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim_X \mathbb{Q}}} < C$.
3. There exists $C > 0$ such that for every prime $p$ and every $k \in \mathbb{N}$ one has $\left| \frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim_X \mathbb{Q}}} - \frac{\#X(\mathbb{Z}/p^i\mathbb{Z})}{p^{i\dim_X \mathbb{Q}}} \right| < Cp^{1}$.

Motivated

1.3. Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the relative case. Let $X$ and $Y$ be smooth finite type $\mathbb{Z}$-schemes and let $\varphi: X \to Y$ be a dominant morphism. Our goal in this paper is to treat the relative analogue of Question 1.1.

**Question 1.4.** Is there a necessary and sufficient condition on $\varphi$ such that the size of each fiber of $\varphi: X(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{Z}/p^k\mathbb{Z})$, normalized by $p^{\dim_X \mathbb{Q} - \dim_Y \mathbb{Q}}$, is uniformly bounded when varying $p, k$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$?
Then the following are equivalent:

Theorem A

Definition 1.5. Let X and Y be smooth K-varieties, where K is a field with \( \text{char}(K) = 0 \).
We say that a morphism \( \varphi : X \to Y \) is (FRS) if it is flat and if every fiber of \( \varphi \) has rational singularities.

1.4. Main results. We are now ready to state the main result of this paper.

Theorem A (See Theorem [4.7] for a more general version). Let \( \varphi : X \to Y \) be a dominant morphism between finite type \( \mathbb{Z} \)-schemes X and Y, with \( X_Q, Y_Q \) smooth and geometrically irreducible. Then the following are equivalent:

1. \( \varphi_Q : X_Q \to Y_Q \) is (FRS).
2. There exists \( C_1 > 0 \) such that for every prime \( p, k \in \mathbb{N} \) and \( y \in Y(\mathbb{Z}/p^k\mathbb{Z}) \) one has
   \[
   \#\varphi^{-1}(y) \cdot p^{k(\dim X_Q - \dim Y_Q)} < C_1.
   \]
3. There exists \( C_2 > 0 \) such that for every prime \( p, k \in \mathbb{N} \) and \( y \in Y(\mathbb{Z}/p^k\mathbb{Z}) \) one has
   \[
   \left| \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_Q - \dim Y_Q)}} - \frac{\#\varphi^{-1}(y)}{p^{(\dim X_Q - \dim Y_Q)}} \right| < C_2 p^{-1},
   \]
   where \( \overline{Y} \) is the image of \( Y \) under the reduction \( Y(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{F}_p) \).
4. There exists \( C_3 > 0 \) such that the following hold for every prime \( p \). Let \( \mu_X(\mathbb{Z}_p) \) and \( \mu_Y(\mathbb{Z}_p) \) be the canonical measures on \( X(\mathbb{Z}_p) \) and \( Y(\mathbb{Z}_p) \) (see Lemma-Definition [4.2]). Then the pushforward measure \( \varphi_* \mu_X(\mathbb{Z}_p) \) has continuous density \( f_p \) with respect to \( \mu_Y(\mathbb{Z}_p) \), and \( \|f_p\|_\infty < C_3 \).

Using a jet-scheme characterization of rational singularities by Mustaţă [Mus01, Mus02], it can be shown that a morphism \( \varphi : X \to Y \) between smooth schemes is (FRS) if and only if for each \( k \in \mathbb{N} \), every non-empty fiber of the corresponding k-th jet map \( J_k(\varphi) : J_k(X) \to J_k(Y) \) is of dimension \( \dim J_k(X) - \dim J_k(Y) \) (i.e. \( J_k(\varphi) \) is flat) and has a singular locus of codimension at least 1 (see Subsection 2.1.1 and Lemma 2.4). Based on this characterization, it is natural to define two variations of the (FRS) property.

- A morphism \( \varphi \) is \( \varepsilon \)-jet flat, for \( \varepsilon \in \mathbb{R}_{>0} \), if the fibers of \( J_k(\varphi) \) are of dimension at most \( \dim J_k(X) - \varepsilon \dim J_k(Y) \), for all \( k \in \mathbb{N} \) (see [GH] Definition 3.22).
- A morphism \( \varphi \) is called E-smooth if it is 1-jet flat, and each of the fibers of \( J_k(\varphi) \) has singular locus of codimension at least \( E \).

In Section 4.3 using methods similar to the proof of Theorem A, we provide uniform estimates on the fibers of E-smooth and \( \varepsilon \)-jet flat morphisms (see Theorems 4.11 and 4.12). In particular, uniform estimates are given on fibers of flat morphisms whose fibers have terminal or log-canonical singularities.

1.4.1. Main difficulties in the proof of Theorem A. The proof of Theorem 4.3 in [AA18] proceeds by (locally) embedding \( X \) as a complete intersection in \( \mathbb{A}^N \) and choosing an embedded resolution of singularities for the pair \( (X_Q, \mathbb{A}^N_Q) \), also called a log-resolution, whose existence follows
from [Hir64]. For large \( p \), one can then use Denef’s formula [Den87, Theorem 3.1], to relate 
\(# X(\mathbb{Z}/p^k \mathbb{Z}) \) to \( \{ #E_i(F_p) \}_i \) and numerical data associated to the choice of resolution, where 
\( \{ E_i \}_i \) is a collection of constructible subsets built out of the prime divisors \( \{ E_i \}_i \) appearing in 
such a resolution. Combined with the Lang-Weil estimates for the \( E_i \)‘s, this yields estimates for 
\(# X(\mathbb{Z}/p^k \mathbb{Z}) \). To finally achieve the bounds of Theorem 1.3 one needs the reductions modulo \( p \) 
of the \( E_i \)’s to be of the expected dimensions over \( F_p \). This can always be done if the prime \( p \) is large enough (small primes are treated separately in [Gla19]).

If \( \varphi : X_{\mathbb{Q}} \to Y_{\mathbb{Q}} \) is (FRS), its fibers are local complete intersections with rational singularities, and one may try to mimic the strategy for Theorem 1.3. The weak point is that this only seems to work for each fiber separately, but does not give the desired uniformity in the choice of fiber. One can try to make this naive fiber-wise strategy more uniform by choosing some simultaneous resolutions of singularities. This can be done by breaking \( Y \) into constructible subsets, with resolutions over generic points of the pieces. However, such finite partition of \( Y \) into constructible sets does not behave well at all with respect to taking points over the rings \( \mathbb{Z}, \mathbb{Z}/p^k \mathbb{Z}, \) or \( \mathbb{Z}_p \). In fact, as far as we can see, the approach with resolutions of singularities in families is hard to adapt to the family situation of Theorem A.

To avoid these difficulties, we use the motivic nature of \( \mathbb{Z}/p^k \mathbb{Z} \)-point count of the fibers of \( \varphi \), that is, we use insights from motivic integration and uniform \( p \)-adic integration. Let \( r_k : Y(\mathbb{Z}_p) \to Y(\mathbb{Z}/p^k \mathbb{Z}) \) be the reduction map. Write \( d := \dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}} \). For each prime \( p \), each \( y \in Y(\mathbb{Z}_p) \) and each integer \( k \geq 1 \) we set

\[
g_p(y, k) = \frac{\# \varphi^{-1}(r_k(y))}{p^kd} \quad \text{and} \quad \tilde{h}_p(y, k) := g_p(y, k) - g_p(y, 1),
\]

as in the left-hand side of Items (2) and (3) of Theorem A. The collections of functions \( \{ g_p \}_p, \{ \tilde{h}_p \}_p \)
are examples of motivic functions, namely in a uniform \( p \)-adic sense as in [CGH18], but closely related to genuine motivic constructible functions from [CL08]. We use motivic integration to extract information on \( \{ g_p \}_p \) and \( \{ \tilde{h}_p \}_p \), which in turn allows us to prove Theorem A.

The proofs of the number-theoretic estimates for \( \varepsilon \)-jet flat and \( E \)-smooth morphisms (Theorems 1.11 and 1.12) share similar difficulties with the proof of Theorem A. Theorem 1.12 improves previous bounds for \( \varepsilon \)-jet flat morphisms: the bounds given in [VZnG08, Corollary 2.9] on \( g_p(y, k) \) are uniform in \( k \), but not in \( p \) and \( y \) (see Remark 2.8 for the relation of \( \varepsilon \)-jet flatness to the log canonical threshold), and the bounds given in [GH, Theorem 8.18] are uniform in \( p, y, k \), but are not optimal.

1.4.2. Model-theoretic results. We denote by \( \mathrm{Loc} \) the collection of all non-Archimedean local fields, by \( \mathrm{Loc}_0 \) the collection of all \( F \in \mathrm{Loc} \) of characteristic zero, and by \( \mathrm{Loc}_\geq \) the collection of all \( F \in \mathrm{Loc} \) with large enough residual characteristic, where ‘large enough’ changes according to our needs.

Let \( \mathcal{L}_{\text{DP}} \) denote the Denef-Pas language. This is a first order language with three sorts to account for a valued field \( F \), a residue field \( k_F \) and a value group which we identify with \( \mathbb{Z} \). An \( \mathcal{L}_{\text{DP}} \)-definable set \( X = \{ X_F \}_{F \in \mathrm{Loc}_\geq} \) is a collection of subsets \( X_F \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3} \) which is uniformly defined using an \( \mathcal{L}_{\text{DP}} \)-formula. Given \( \mathcal{L}_{\text{DP}} \)-definable sets \( X \) and \( Y \), a collection of functions \( \{ f : X_F \to Y_F \}_{F \in \mathrm{Loc}_\geq} \) is called an \( (\mathcal{L}_{\text{DP}}) \)-definable function if its graph is definable.
Given a definable set $X = \{X_F\}_{F \in \text{Loc}_\gg}$, the ring of motivic functions $\mathcal{C}(X)$ is a certain natural class of functions whose building blocks are the definable functions, and is closed under integration. Built on a natural notion of positivity, we define the semi-ring of formally non-negative functions $\mathcal{C}_+(X) \subset \mathcal{C}(X)$. As an example, the collection $\{\varphi_* \mu_F\}_{F \in \text{Loc}_\gg}$ of pushforwards of Haar measures $\mu_F$ on $\mathcal{O}_F^n$ under any polynomial map $\varphi$, as well as $\{g_p\}_p$ above are formally non-negative motivic functions. The classes $\mathcal{C}_+(X)$ and $\mathcal{C}(X)$ above are uniform $p$-adic specializations of more genuinely motivic functions defined in [CL08, CL10], but they go by similar methods and theories. See Subsection 2.2 for further details on motivic functions.

As a key step towards proving Theorem A, we show the following strengthening of [CGH18, Theorem 2.1.3] for the class of formally non-negative motivic functions:

**Theorem B** (Theorem 3.1). Let $f$ be in $\mathcal{C}_+(X \times W)$, where $X$ and $W$ are $\mathcal{L}_{\text{DP}}$-definable sets. Then there exists a constant $C > 0$, and a function $G \in \mathcal{C}_+(X)$ such that for any $F \in \text{Loc}_\gg$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on $W_F$, we have

$$\sup_{w \in W_F} f_F(x, w) \leq G(x) \leq C \cdot \sup_{w \in W_F} f_F(x, w).$$

The approximation of suprema given in Theorem B is best possible for the class of formally non-negative motivic functions $\mathcal{C}_+(X \times W)$, in the sense that one cannot choose $C$ to be a universal constant (see Proposition 3.6). In [CGH18, Theorem 2.1.3], a similar approximation result is shown (for motivic functions in $\mathcal{C}(X \times W)$ and in $\mathcal{C}_{\exp}(X \times W)$), but where the constant $C$ is replaced by $q_C F$, with $q_F$ the number of elements in the residue field $k_F$ of $F$, and where instead of $\sup f_F$ one approximates $\sup |f_F|^2$. For more details on the optimality of these approximation results, see the discussion in Subsection 3.1.

### 1.4.3. Sketch of proof of Theorem A

To prove Theorem A, we show $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$. The implications $(3) \Rightarrow (2) \Rightarrow (4)$ are rather easy and the implication $(4) \Rightarrow (1)$ essentially follows from an equivalent analytic characterization of the (FRS) property due to Aizenbud-Avni (see Theorem 4.3). The challenging part of the proof is to show $(1) \Rightarrow (3)$. Small primes are dealt using Theorem 4.3 and using basic properties of the canonical measure (Lemma-Definition 4.2). Thus we may consider only large enough primes $p$. Let us sketch the main strategy of the proof of $(1) \Rightarrow (2)$, for large $p$, which has similar difficulties to $(1) \Rightarrow (3)$.

(a) We use Theorem 4.3 to show that

$$\sup_{y, k} g_p(y, k) < C(p)$$

for some constant $C(p)$ depending on $p$.

(b) Item (a) and the fact that $g$ is formally non-negative as a motivic function (see Definition 2.10), allow us to utilize Theorem B to approximate (for each $k$ and $p$)

$$\sup_{y \in Y(\mathbb{Z}_p)} g_p(y, k)$$

by $G_p(k)$ for a single motivic function $\{G_p : \mathbb{Z}_{\geq 1} \to \mathbb{R}\}_p$.

(c) We use results from [CGH18] on approximate suprema of constructible Presburger functions to deduce that

$$\sup_{k \in \mathbb{Z}_{\geq 1}} G_p(k)$$
can be approximated by \( \sum_{l \in L} G_p(l) \) for some finite subset \( L \subseteq \mathbb{Z}_{\geq 1} \), with \( L \) independent of \( p \).

(d) To deal with \( G_p(l) \) for \( l \in L \), we use a transfer principle for boundedness of motivic functions from [CCH16] (see Theorem 2.14 below) to reduce to a question about the \( F_p \)-fibers of the \( (l-1) \)-th jet of \( \varphi \). We then combine Lang-Weil type arguments on the jets of \( \varphi \), together with a jet-scheme interpretation of the (FRS) property (Proposition 2.3), to deduce that \( G_p(l) < C \) for \( p \gg 1 \), \( l \in L \) and some constant \( C > 0 \) independent of \( p \).

This shows (1) \( \Rightarrow \) (2). To prove (1) \( \Rightarrow \) (3), we approximate \( \tilde{h}_p \) with a motivic function \( h_p \), which unlike \( h_p \), is formally non-negative. We then apply similar steps as above (with a few extra complications) to \( h_p \).

1.5. Further discussion. The (FRS) property was first introduced and studied in [AA16 AA18], where a very useful analytic interpretation was given as follows. Given a morphism \( \varphi : X \to Y \) between smooth \( \mathbb{Q} \)-varieties, the (FRS) property of \( \varphi \) is characterized by the property that for every \( F \in \text{Loc}_0 \) and every smooth, compactly supported measure \( \mu_X(F) \) on \( X(F) \), the push-forward measure \( \varphi_* (\mu_X) \) on \( Y(F) \) has continuous density (see Theorem 4.3 or [AA16 Theorem 3.4]). Our number theoretic characterization (Theorem A) can be seen as a refinement of this analytic characterization.

These characterizations allow one to use algebro-geometric tools to solve various problems in analysis, probability and group theory. For a motivating example, let \( \varphi : X \to \mathbb{G} \) be a semisimple algebraic \( \mathbb{Q} \)-group and let \( \varphi^\text{comm} : \mathbb{G}^{2t} \to \mathbb{G} \) be the map \((g_1, \ldots, g_{2t}) \mapsto [g_1, g_2] \cdots [g_{2t-1}, g_{2t}] \), corresponding to the product of \( t \) commutator maps. Using the above characterizations and a theorem of Frobenius, one has:

\[
(*) \quad \varphi^\text{comm} \text{ is (FRS) } \Rightarrow \# \{N \text{-dimensional irreducible } \mathbb{C} \text{-representations of } \mathbb{G}(\mathbb{Z}_p) \} = O(N^{2t-2}).
\]

Aizenbud and Avni showed in [AA16 AA18], that \( \varphi^\text{comm} \) is (FRS) for every \( \mathbb{G} \) as above, which via (*), confirmed a conjecture of Larsen-Lubotzky [LL08] about representation growth of compact \( p \)-adic and arithmetic groups. These bounds were improved in [Bud15 Kap GH].

The above situation can be generalized as follows. Let \( \varphi : X \to \mathbb{G} \) be a dominant morphism from a smooth \( \mathbb{Q} \)-variety \( X \) to a connected algebraic group \( \mathbb{G} \). We define the self-convolution \( \varphi * \varphi : X \times X \to \mathbb{G} \) of \( \varphi \) by \( \varphi * \varphi(x_1, x_2) = \varphi(x_1) \varphi(x_2) \), and write \( \varphi^{*t} : X^t \to \mathbb{G} \) for the \( t \)-th convolution power of \( \varphi \). Similarly to the usual convolution operation in analysis, this algebraic convolution operation has a smoothing effect on morphisms; In [GH19 GH21], it was shown that \( \varphi^{*t} : X^t \to \mathbb{G} \) has increasingly better singularity properties as \( t \) grows, and eventually, \( \varphi^{*t} \) becomes (FRS) for every \( t \geq t_0 \), for some \( t_0 \in \mathbb{N} \).

Moving to the probabilistic picture, let \( \mu_X(\mathbb{Z}_p) \) and \( \mu_{\mathbb{G}(\mathbb{Z}_p)} \) be the canonical measures on \( X(\mathbb{Z}_p) \) and \( \mathbb{G}(\mathbb{Z}_p) \), normalized to have total mass 1. One can then study the collection of random walks on \( \mathbb{G}(\mathbb{Z}_p) \), induced by the pushforward measures \( \{ \varphi_* \mu_X(\mathbb{Z}_p) \}_{p \text{ primes}} \), by analyzing the convergence rate of their self-convolutions \( \{ \varphi^{*t} \mu_X(\mathbb{Z}_p) \}^{*t} \) to \( \mu_{\mathbb{G}(\mathbb{Z}_p)} \), in the \( L^q \)-norm (\( q \geq 1 \)). This rate of convergence can be measured by the notion of \( L^q \)-mixing time (see e.g. [LP17 Chapter 4]). Note that the analytic convolution operation commutes with the algebraic convolution defined above, so that \( \{ \varphi^{*t} \mu_X(\mathbb{Z}_p) \}^{*t} = \{ \varphi^{*t} \} \mu_X(\mathbb{Z}_p) \). This makes Theorem A the connecting link between the algebraic and the probabilistic pictures above.
Explicitly, let us denote by $t_{alg}$ the minimal $t \in \mathbb{N}$ such that $\varphi^t$ is (FRS) and has geometrically irreducible fibers, and call it the *algebraic mixing time of* $\varphi$. Then Theorem A and its general form Theorem 4.7 imply that the algebraic mixing time of $\varphi$ is equal to the uniform (in $p \gg 1$) $L^\infty$-mixing time of the random walks on $\{\mathbb{Z}_p\}^n_p$ induced by $\{\varphi^t|_{\mathbb{Z}_p}\}_p$ (see GH Definition 9.2]). This philosophy was implemented in [GH], which motivated this work. There, the authors analyzed the singularity properties of word maps on semi-simple algebraic groups, using purely algebraic techniques, and obtained probabilistic results on word measures. In particular, Theorem A completes the proof of [GH] Theorems G and 9.3(2)].

1.6. Conventions.

- Throughout the paper, we use $K, K', K''$ to denote number fields and $O_K, O_{K'}, O_{K''}$ for their rings of integers. Similarly, local fields and their rings of integers are denoted by $F, F', F''$ and $O_F, O_{F'}, O_{F''}$, respectively.
- Given a local ring $A$, a morphism $\varphi : X \to Y$ of schemes $X$ and $Y$, and given $y \in Y(A)$ (i.e. a morphism $\text{Spec}(A) \to Y$), we denote by $X_{y,\varphi} := \text{Spec}(A) \times_Y X$ the scheme theoretic fiber over $y$, and simply by $\varphi^{-1}(y) \subseteq X(A)$ the set theoretic fiber of the induced map $\varphi : X(A) \to Y(A)$. Note that if $y \in Y$ is a schematic point, then it can be viewed as $y \in Y(\kappa(y))$, where $\kappa(y)$ is the residue field of $y$, so that $X_{y,\varphi} := \text{Spec}(\kappa(y)) \times_Y X$.
- Given a $K$-morphism $\varphi : X \to Y$ between $K$-varieties $X$ and $Y$, we denote by $X^\text{sm}$ (resp. $X^\text{sing}$) the smooth (resp. non-smooth) locus of $X$. We denote by $X^{\text{sm},\varphi}$ (resp. $X^{\text{sing},\varphi}$) the smooth (resp. non-smooth) locus of $\varphi$ in $X$.
- We denote the base change of an $S$-scheme $X$ with respect to $S' \to S$ by $X_{S'}$.

Acknowledgement. The author R.C. was partially supported by the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) with ERC Grant Agreement nr. 615722 MOTMELSUM, KU Leuven IF C14/17/083, and Labex CEMPI (ANR-11-LABX-0007-01). The author I.G. was partially supported by ISF grant 249/17, BSF grant 2018201 and by a Minerva foundation grant.

The authors wish to thank Rami Aizenbud, Nir Avni, Jan Denef and Julien Sebag for many useful discussions.

2. Preliminaries

2.1. Jet schemes and singularities. For a thorough discussion of jet schemes see [CLNS18] Chapter 3 and [EM09].

Definition 2.1 (cf. [CLNS18] Section 3.2)). Let $S$ be a scheme and let $X$ be a scheme over $S$.

1. For each $k \in \mathbb{N}$, we define the *$k$-th jet scheme* of $X$, denoted $J_k(X/S)$ as the $S$-scheme representing the functor

$$J_k(X/S) : W \mapsto \text{Hom}_{S\text{-schemes}}(W \times_{\text{Spec}Z} \text{Spec}(Z[t]/(t^{k+1})), X),$$

where $W$ is an $S$-scheme. We write $J_k(X)$ if the scheme $S$ is understood.

2. Given an $S$-morphism $\varphi : X \to Y$ and an $S$-scheme $W$, the composition with $\varphi$ induces a map $J_k(X/S)(W) \to J_k(Y/S)(W)$, which yields a morphism

$$J_k(\varphi) : J_k(X/S) \to J_k(Y/S),$$

called the *$k$-th jet of* $\varphi$. 

(3) For any $k_1 \geq k_2 \in \mathbb{N}$ the reduction map $\mathbb{Z}[t]/(t^{k_1+1}) \to \mathbb{Z}[t]/(t^{k_2+1})$ induces a natural collection of morphisms $\pi_{k_2,X}^{k_1} : J_{k_1}(X/S) \to J_{k_2}(X/S)$ which are called truncation maps. Note that the collection $\{\pi_k(\varphi) : J_k(X/S) \to J_k(Y/S)\}_{k \in \mathbb{N}}$ commutes with $\{\pi_m^m\}_{m \geq n}$.

(4) The natural map $\mathbb{Z} \to \mathbb{Z}[t]/(t^{m+1})$ induces a zero section $s_{m,X} : X \hookrightarrow J_m(X)$. We sometimes write $\pi_m^m$ and $s_m$ instead of $\pi_m^m$ and $s_{m,X}$, when $X$ is clear.

In the rest of this subsection, we assume $S = \text{Spec} K$. Mustaţă gave the following interpretation of rational singularities in terms of jet schemes:

**Theorem 2.2** ([Mus01]). Let $X$ be a geometrically irreducible, local complete intersection $K$-variety, with $\text{char}(K) = 0$. Then $J_k(X)$ is geometrically irreducible for all $k \geq 1$ if and only if $X$ has rational singularities.

Using Theorem 2.2 one can obtain a similar characterization of (FRS) morphisms:

**Proposition 2.3** ([GHI Corollary 3.12] and [Ish09]). Let $X$ and $Y$ be smooth, geometrically irreducible $K$-varieties, and let $\varphi : X \to Y$ be a $K$-morphism.

(1) Assume $\text{char}(K) = 0$. Then the morphism $\varphi$ is (FRS) if and only if $J_k(\varphi)$ is flat, with locally integral fibers for each $k \in \mathbb{N}$.

(2) The morphism $\varphi$ is smooth if and only if $J_k(\varphi)$ is smooth for each $k \in \mathbb{N}$.

**Remark 2.4.** Let $k$ be a natural number, and $K$ a field with $\text{char}(K) = 0$ or $\text{char}(K) \gg 1$ (in terms of $k$). Then the jet scheme $J_k(X)$ of an affine $K$-scheme $X \subseteq \mathbb{A}^n$ has a simple description: write $X = \text{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_l)$. Then

$$J_k(X) = \text{Spec} K[x_1, \ldots, x_n, x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(k)}, \ldots, x_n^{(k)}/\{(f_j)^{(l)}\}_{j=1, u=0},$$

where $f_j^{(l)}$ is the $u$-th formal derivative of $f_j$. For example, if $f = x_1 x_2^2$ then $f^{(1)} = x_1^{(1)} x_2^2 + 2x_1 x_2 x_2^{(1)}$. Similarly, $J_k(\varphi) = (\varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$ for a morphism $\varphi : X \to Y$ of affine $K$-schemes.

The next proposition will be useful in Section 4.

**Proposition 2.5.** Let $k \in \mathbb{N}$ and let $\varphi : X \to Y$ be a $K$-morphism as in Proposition 2.3 with $\text{char}(K) = 0$ or $\text{char}(K) \gg 1$ (in terms of $k$). Then $J_k(X)^{\text{sm}, J_k(\varphi)} = J_k(X^{\text{sm}, \varphi})$.

**Proof.** It follows from Proposition 2.3(2) that $J_k(X^{\text{sm}, \varphi}) \subseteq J_k(X)^{\text{sm}, J_k(\varphi)}$, so it is left to show the other inclusion. We may assume that $X$ and $Y$ are affine, and that $Y$ admits an étale map $\psi : Y \to \mathbb{A}^n_K$. We may further assume that $Y = \mathbb{A}^n_K$. Indeed, we have:

$$J_k(X)^{\text{sm}, J_k(\varphi)} = J_k(X)^{\text{sm}, J_k(\psi \circ \varphi)}$$

and

$$J_k(X^{\text{sm}, \psi \circ \varphi}) = J_k(X^{\text{sm}, \varphi}).$$

By Remark 2.4 we can write $X = \text{Spec} K[x_1, \ldots, x_{n+l}]/(f_1, \ldots, f_l)$, and

$$J_k(X) = \text{Spec} K[x_1, \ldots, x_{n+l}, x_1^{(k)}, \ldots, x_{n+l}^{(k)}/\{(f_j)^{(l)}\}_{j=1, u=0}.$$

Moreover $J_k(\varphi) = (\varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$ where $\varphi = (f_1^{(1)}, \ldots, f_{l+m}^{(1)}) : X \to \mathbb{A}^m_K$. Write $F_{u(l+m)+j} := f_j^{(u)}$ and $X_{u(n+l)+j} := x_{u+j}^{(u)}$, and let $\varpi := (a_1, \ldots, a_{(n+l)}) \in J_k(X)$. Then $J_k(\varphi)$ is smooth at $\varpi$ if and only if the matrix $M = \left(\frac{\partial F_{u(l+m)+j}}{\partial X_{u(n+l)+j}}\right)_{i=1, j=1}^{(n+l)(l+m)(k+1)}$ is of full rank $(l+m)(k+1)$. Note that $M$
has the shape

\[ M = \begin{pmatrix} M_{00} & M_{01} & \ldots & M_{0k} \\ 0 & M_{11} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{kk} \end{pmatrix}, \]

where \( M_{u_1u_2} = \left( \frac{\partial f(u_2)}{\partial x_i}(u_1) \right)^{(n+l),(l+m)} \) for \( 0 \leq u_1 \leq u_2 \leq k \). If \( M \) is of full rank, then also \( M_{00} \neq \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix} \) and all \( M_{0k} \neq \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix} \) must be of full rank, which in turn implies that \( \varphi \) is smooth at \( a \), and the proposition follows. \( \square \)

**Remark 2.6.** The case \( Y = \mathbb{A}^1 \) of Proposition 2.4 has essentially been proven in [EMY03] (proof of Theorem 3.3) and [Mus01, Proposition 4.12] (see also [Ish18, p.222]). Proposition 2.5 also relates to [Mus01, Questions 4.10 and 4.11], as follows. Given a local complete intersection variety \( X \), it can be written, locally, as a fiber \( \tilde{X}_{0,\varphi} \) of a flat morphism \( \varphi : \tilde{X} \to \mathbb{A}^m \), with \( \tilde{X} \) smooth. If we assume that \( J_k(\varphi) \) is flat for all \( k \), then Proposition 2.5 combined with [Har77, III, Theorem 10.2] implies that \( (\pi_{0,\tilde{X}_{0,\varphi}}^m)^{-1}(\tilde{X}_{0,\varphi})^{sm} = J_k(\tilde{X}_{0,\varphi})^{sm} \) for all \( k \), which gives a positive answer to [Mus01, Question 4.11] in this case. If \( J_k(\varphi) \) is not flat, one can still effectively describe its smooth locus, but it is harder to describe the smooth locus of its fibers.

### 2.1.1. \( E \)-smooth and \( \varepsilon \)-jet flat morphisms

We next introduce several properties of morphisms between smooth varieties: \( \varepsilon \)-flatness, \( \varepsilon \)-jet flatness, and \( E \)-smoothness. The first two notions were first introduced in [GH], whereas the \( E \)-smoothness notion is new.

**Definition 2.7.** Let \( X \) and \( Y \) be smooth, geometrically irreducible \( K \)-varieties, and let \( \varphi : X \to Y \) be a \( K \)-morphism, let \( E \geq 1 \) be an integer and let \( \varepsilon \in \mathbb{R}_{>0} \). Then:

1. \( \varphi \) is called \( \varepsilon \)-flat if for every \( x \in X \) we have \( \dim X_{\varphi(x),\varphi} \leq \dim X - \varepsilon \dim Y \).
2. \( \varphi \) is called \( \varepsilon \)-jet flat (resp. jet-flat) if \( J_k(\varphi) \) is \( \varepsilon \)-flat (resp. flat) for every \( k \in \mathbb{N} \).
3. A jet-flat morphism \( \varphi \) is called \( E \)-smooth if for all \( k \in \mathbb{Z}_{\geq 0} \) and all \( \tilde{x} \in J_k(X) \), the set \( (J_k(X),J_k(\varphi)(\tilde{x}),J_k(\varphi))^{\text{sing}} \) is of codimension at least \( E \) in \( J_k(X),J_k(\varphi)(\tilde{x}),J_k(\varphi) \).

**Remark 2.8.**

1. By [Mus02], a morphism \( \varphi \) as in Definition 2.7 is \( \varepsilon \)-jet flat if and only if \( \text{lct}(X,X_{\varphi(x),\varphi}) \geq \varepsilon \dim Y \) for all \( x \in X \), where \( \text{lct}(X,X_{\varphi(x),\varphi}) \) is the log-canonical threshold of the pair \( (X,X_{\varphi(x),\varphi}) \).
2. In addition, it follows from [Mus01, EM04] (see [GH, Corollary 3.12]) that if \( \varphi \) is a normal morphism, then it is jet-flat if and only if it is flat and has fibers with log-canonical singularities.

\( \varepsilon \)-flatness is a quantitative way to measure how close a morphism between smooth varieties is to being flat. Similarly, \( \varepsilon \)-jet flatness measures how close a morphism is to being jet-flat, which is very close to being an \( (FRS) \)-morphism. On the other hand, the starting point of \( E \)-smoothness is when \( \varphi \) is jet-flat, and the larger \( E \) is, the better the singularities of \( \varphi \) are. This is illustrated in the next lemma:

**Lemma 2.9.** Let \( \varphi : X \to Y \) be \( K \)-morphism between smooth, geometrically irreducible \( K \)-varieties.
Let Loc be the collection of all non-Archimedean local fields \( F \) of positive characteristic. For \( F \) of characteristic zero (resp. Loc of characteristic \( k \approx \mathbb{F}_p \)), \( \varphi \) is flat with fibers of terminal singularities if and only if \( \varphi \) is flat with normal fibers for each \( k \in \mathbb{N} \). In particular, in the situation of (1) and (2), \( \varphi \) is always jet-flat, and thus the fibers of \( J_k(\varphi) \) are local complete intersection, and hence Cohen-Macaulay. Serre's criterion for normality and reducedness [Gro67, Proposition 5.8.5, Theorem 5.8.6] and [Mus01, Proposition 1.4] now imply Items (1) and (2).

2.2. Motivic functions. In this subsection we recall the definition and some properties of motivic functions. In order to prove Theorem A, we encode the collection of motivic functions as was defined and studied in [CL08, Section 5.3]. In order to fully exploit the advantages of the motivic realm, we introduce the class of formally non-negative motivic functions, which is the specialization to local fields of [CL08, Section 5.3].

Throughout this subsection, we fix a number field \( k \). We use the (three-sorted) Denef-Pas language, denoted

\[ \mathcal{L}_{DP} = (\mathcal{L}_{Val}, \mathcal{L}_{Res}, \mathcal{L}_{Pres}, \text{val}, \text{ac}), \]

where:

1. The valued field sort \( \text{VF} \) is endowed with the language of rings \( \mathcal{L}_{Val} \), with coefficients in \( \mathcal{O}_K \).
2. The residue field sort \( \text{RF} \) is endowed with the language of rings \( \mathcal{L}_{Res} \).
3. The value group sort \( \text{VG} \) (which we just call \( \mathbb{Z} \)), is endowed with the Presburger language \( \mathcal{L}_{Pres} = (+, -, \leq, \{\equiv \text{mod } n \}_{n>0}, 0, 1) \) of ordered abelian groups along with constants 0, 1 and a family of relations \( \{\equiv \text{mod } n \}_{n>0} \) of congruences modulo \( n \).
4. \( \text{val} : \text{VF}\setminus\{0\} \to \mathbb{Z} \) and \( \text{ac} : \text{VF} \to \text{RF} \) are two function symbols.

Let Loc be the collection of all non-Archimedean local fields \( F \) with a ring homomorphism \( \mathcal{O}_K : F \to F \). We denote by Loc0 (resp. Loc+) the collection of all \( F \in \text{Loc} \) of characteristic zero (resp. positive characteristic). For \( F \in \text{Loc} \), we denote by \( \mathcal{O}_F \) its ring of integer, by \( k_F \) its residue field, and by \( q_F \) the number of elements in \( k_F \). Let us denote by \( \mathcal{O}_{k_F} \approx \mathbb{F}_p \) (resp. \( \mathcal{O}_{k_F} \approx \mathbb{F}_p \)), for the collection of \( F \in \text{Loc} \) (resp. Loc0, Loc+) with large enough residual characteristic (depending on some given data).

Given \( F \in \text{Loc} \) (and a chosen uniformizer \( \varpi_F \) of \( \mathcal{O}_F \)), we can interpret \( \text{val} \) and \( \text{ac} \) as the valuation map \( \text{val} : F^\times \to \mathbb{Z} \) and the angular component map \( \text{ac} : F \to k_F \), where \( \text{ac}(0) = 0 \) and \( \text{ac}(x) = x \cdot \varpi_F^{-\text{val}(x)} \) mod \( \varpi_F \mathcal{O}_F \) for \( x \neq 0 \). Hence, any formula \( \phi \) in \( \mathcal{L}_{DP} \) with \( n_1 \) free VF-variables, \( n_2 \) free RF-variables and \( n_3 \) free \( \mathbb{Z} \)-variables, yields a subset \( \phi(F) \subset F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3} \), a collection \( X = (X_F)_{F \in \text{Loc}} \) with \( X_F = \phi(F) \) is called an \( \mathcal{L}_{DP} \)-definable set. Given \( \mathcal{L}_{DP} \)-definable sets \( X \) and \( Y \), an \( \mathcal{L}_{DP} \)-definable function is a collection \( f = (f_F : X_F \to Y_F)_{F \in \text{Loc}} \) of functions whose collection of graphs is a definable set. We will often say “definable” instead of “\( \mathcal{L}_{DP} \)-definable”.

1Our notation for Loc+ is slightly more restrictive than the one used in [CGH18]. Here Loc+ consists of Loc0, Loc+ while in [CGH14], it consisted of Loc0 ∪ Loc+.\(^1\)
Definition 2.10 (See [CGH14, Subsections 4.2.4-4.2.5]). Let $X$ be an $\mathcal{L}_{\text{DP}}$-definable set. A collection $f = (f_F)_{F \in \text{Loc}_{\gg}}$ of functions $f_F : X_F \to \mathbb{R}$ is called a Presburger constructible function, if it can be written as

$$f_F(x) = \sum_{i=1}^{N_i} q^{\alpha_i,F(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{k=1}^{N_3} \frac{1}{1-q^{a_{ij,k,f}}},$$

where $N_1, N_2, N_3$ and $a_{ij}$ are non-zero integers, and $\alpha_i, \beta_{ij} : X \to \mathbb{Z}$ are definable functions. Given $f$ as above, set $\tilde{f}_F : X_F \times \mathbb{R}^{>1} \to \mathbb{R}$ by

$$\tilde{f}_F(x,s) := \sum_{i=1}^{N_1} s^{\alpha_i,F(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{k=1}^{N_3} \frac{1}{1-s^{a_{ij,k,f}}}.$$

We say that $f$ is formally non-negative if $\tilde{f}_F$ takes non-negative values for every $F \in \text{Loc}_{\gg}$. We denote by $\mathcal{P}(X)$ the ring of Presburger constructible functions on $X$, and by $\mathcal{P}_+(X)$ the sub-semiring of formally non-negative functions.

Definition 2.11. Let $X$ be an $\mathcal{L}_{\text{DP}}$-definable set. A collection $h = (h_F)_{F \in \text{Loc}_{\gg}}$ of functions $h_F : X_F \to \mathbb{R}$ is called a motivic function, if it can be written as:

$$h_F(x) = \sum_{i=1}^{N} \# Y_{i,F,x} \cdot f_i F(x),$$

where:

- $Y_{i,F,x} = \{ \xi \in k_F^r : (x, \xi) \in Y_{i,F} \}$ is the fiber over $x \in X_F$ of a definable set $Y_i \subseteq X \times \mathbb{R}^{r_i}$ with $r_i \in \mathbb{N}$.
- Each $f_i$ is a Presburger constructible function.

If furthermore every $f_i$ is formally non-negative, then we call $h$ a formally non-negative motivic function. We denote by $\mathcal{C}(X)$ the ring of motivic functions on $X$, and by $\mathcal{C}_+(X)$ the sub-semiring of formally non-negative motivic functions.

The classes $\mathcal{C}(X)$ and $\mathcal{C}_+(X)$ defined above are the specialization to local fields of more abstract classes of motivic functions defined in [CL08, Section 5] (e.g., see the discussion in [CGH14, Section 4.2]). In [CL08, Theorem 10.1.1], it is shown that these more general classes are preserved under a formal integration operation, and in [CL10, Section 9] it is shown that this formal integration operation commutes with usual $p$-adic integration under specialization. This implies the following theorem:

Theorem 2.12. Let $X$ be an $\mathcal{L}_{\text{DP}}$-definable set, and let $f$ be in $\mathcal{C}_+(X \times VF^m)$. Assume that for every $F \in \text{Loc}_{\gg}$ and every $x \in X_F$, the function $y \mapsto f_F(x,y)$ belongs to $L^1(F^m)$. Then there exists $g$ in $\mathcal{C}_+(X)$ such that

$$g_F(x) = \int_{y \in F^m} f_F(x,y) \, dy.$$

Remark 2.13. In [CGH14, Theorem 4.3.1] it was shown that the class of motivic functions is preserved under integration in the following stronger sense, namely, that given $f$ in $\mathcal{C}(X \times VF^m)$, one can find $g \in \mathcal{C}(X)$ such that for every $F \in \text{Loc}_{\gg}$ and $x \in X_F$, if $y \mapsto f_F(x,y)$ belongs to $L^1(F^m)$ then (2.1) holds. This stronger statement relies on an interpolation theorem [CGH14, Theorem 4.3.3] for functions in $\mathcal{C}(X)$. It would be interesting to prove a similar interpolation result for the class of formally non-negative motivic functions. This will imply the stronger formulation of Theorem 2.12 as in [CGH14, Theorem 4.3.1].
Finally, we need the following transfer result between $\text{Loc}_{0,\gg}$ and $\text{Loc}_{+,\gg}$.

**Theorem 2.14** (Transfer principle for bounds, [CGH16, Theorem 3.1]). Let $X$ be an $L_{\text{DP}}$-definable set, and let $H, G \in C(X)$ be motivic functions. Then the following holds for $F \in \text{Loc}_{\gg}$: if

$$|H_F(x)| \leq |G_F(x)|,$$

for each $x \in X_F$, then also

$$|H_{F'}(x)| \leq |G_{F'}(x)|,$$

for every $F' \in \text{Loc}$ with the same residue field as $F$, and each $x \in X_{F'}$.

3. An improvement of the approximation of suprema

The main goal of this section is to show the following improvement of [CGH18, Theorem 2.1.3] on approximate suprema. This improvement is made possible by placing ourselves in the special case of formally non-negative motivic functions and is not possible in the more general situation of [CGH18].

**Theorem 3.1** (Improved approximation of suprema). Let $f$ be in $C_+(X \times W)$, where $X$ and $W$ are definable sets. Then there exist a constant $C > 0$, and a function $G \in C_+(X)$ such that for any $F \in \text{Loc}_{\gg}$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on $W_F$, we have

$$\sup_{w \in W_F} f_F(x, w) \leq G_F(x) \leq C \cdot \sup_{w \in W_F} f_F(x, w).$$

The following lemma is immediate:

**Lemma 3.2.** Let $\{f_i\}_{i=1}^N$ be in $C_+(X \times W)$ and set $f = \sum_{i=1}^N f_i$. Then for $F \in \text{Loc}_{\gg}$, one has:

$$\frac{1}{N} \sum_{i=1}^N \sup_{w \in W_F} f_iF(x, w) \leq \sup_{w \in W_F} f_F(x, w) \leq \sum_{i=1}^N \sup_{w \in W_F} f_iF(x, w).$$

Let $f$ be in $C_+(X \times W)$. By Definition 2.11, we can write $f(x, w) = \sum_{i=1}^N \#Y_{i,x,w} \cdot g_i(x, w)$, where $g_i \in P_+(X \times W)$ and $Y_i \subseteq X \times W \times RF^*$. Lemma 3.2 thus implies the following:

**Corollary 3.3.** Let $f$ be in $C_+(X \times W)$, where $X$ and $W$ are definable sets.

1. Let $X \times W = \bigsqcup_{i=1}^M C_i$ be a definable partition and set $f_i(x, w) = f(x, w) \cdot 1_{C_i}$. Then it is enough to prove Theorem 3.1 for each $f_i$.

2. It is enough to prove Theorem 3.1 for $f$ of the form $f = \#Y_{x,w} \cdot g(x, w)$ where $g \in P_+(X \times W)$.

**Remark 3.4.** The key case of Theorem 3.1 is when neither $X$ nor $W$ involve valued field variables. The reduction to this case needs to be done with care. Naively, one can use quantifier elimination to eliminate the valued field variables, but this is problematic since it mixes the valued field variables of $X$ and $W$, making it hard to take supremum over the variables of $W$. In order to elude this problem, we will apply cell decomposition iteratively, first taking care of the $W$ variables and then taking care of the $X$ variables.
Proof of Theorem 3.1. Let \( f(x, w) = \#Y_{x, w} \cdot g(x, w) \) for some \( g \in \mathcal{P}_+(X \times W) \) and \( Y \subseteq X \times W \times \text{RF}^{r'} \). Without loss of generality, we may assume that \( X = \text{VF}^{n_1} \times \text{RF}^{n_2} \times \text{VG}^{n_3} \) and \( W = \text{VF}^{m_1} \times \text{RF}^{m_2} \times \text{VG}^{m_3} \) for some \( n_i \geq 0 \) and \( m_i \geq 0 \). We will first reduce to the case where there are no valued field variables, using the following claim.

Claim 1. We may assume that \( X = \text{RF}^{m_2} \times \text{VG}^{m_3} \) and \( W = \text{RF}^{m_2} \times \text{VG}^{m_3} \).

Proof of Claim 1. We first get rid of the valued field variables VF\(^{m_1}\) of \( W \). Without loss of generality we may assume that \( W = \text{VF}^{m_1} \). By induction, we may further assume that \( m_1 = 1 \). By [CL08] Theorem 7.2.1 there exists a definable surjection \( \lambda : X \times W \to C \subseteq X \times \text{RF}^{s} \times \text{Z}^{r'} \) over \( X \) as well as \( \psi \in C_{+}(C) \) such that \( f = \psi \circ \lambda \). Note that

\[
\sup_{w \in W_{F}} f_{F}(x, w) = \sup_{w \in W_{F}} \psi_{F} \circ \lambda_{F}(x, w) = \sup_{(\xi, k) \in k_{F}^{3} \times \text{Z}^{r'}} \psi_{F}(x, \xi, k),
\]

up to extending \( \psi \) by zero outside \( C \). We may therefore assume that \( W = \text{RF}^{m_2} \times \text{VG}^{m_3} \). We next get rid of the valued field variables \( \text{VF}^{m_1} \) of \( X \), denoted \( y := y_1, ..., y_{m_1} \). Write \( x = (y, \eta, t) \in X \) and \( w = (\xi, s) \in W \), with RF-variables \( \eta, \xi \) and VG-variables \( t, s \). By Definition 2.11 \( f \) is determined by a finite collection \( \alpha_i, \beta_{ij} : X \times W \to \text{Z} \) of definable functions, and by a definable set \( Y \subseteq X \times W \times \text{RF}^{r'} \). By quantifier elimination in the valued field variables [Pas89] Theorem 4.1, there exist finitely many polynomials \( g_1, ..., g_l \in \text{Z}[y_1, ..., y_{m_1}] \) such that the graphs of the functions in \( \{\alpha_i, \beta_{ij}\} \) can be defined by formulas of the form

\[
\bigvee_{i=1}^{L} \chi_i(\xi, \eta, ac(g_1(y)), ..., ac(g_l(y))) \land \theta_i(t, s, t', val(g_1(y)), ..., val(g_l(y))),
\]

and the subset \( Y \) can be defined by a formula of the form

\[
\bigvee_{i=1}^{L'} \tilde{\chi}_i(\xi, \eta, \xi', ac(g_1(y)), ..., ac(g_l(y))) \land \tilde{\theta}_i(t, s, val(g_1(y)), ..., val(g_l(y))),
\]

where \( \chi_i \) and \( \tilde{\chi}_i \) are \( \mathcal{L}_{\text{Res}} \)-formulas, \( \theta_i \) and \( \tilde{\theta}_i \) are \( \mathcal{L}_{\text{Pres}} \)-formulas, \( t' \) is in \( \text{Z} \) and \( \xi' \) is in \( \text{RF}^{r} \). We now set \( \lambda' : X \times W \to \text{RF}^{s} \times \text{Z}^{r'} \times W \) by \( \lambda'(x, w) = (\rho(x), w) \) with

\[
\rho(x) = \rho(y, \eta, t) := (\eta, ac(g_1(y)), ..., ac(g_l(y)), t, val(g_1(y)), ..., val(g_l(y))).
\]

Let \( C' \) be the image of \( \lambda' \). Note we may find definable functions \( \tilde{\alpha}_i, \tilde{\beta}_{ij} : C' \to \text{Z} \) and a definable subset \( \tilde{Y} \subseteq C' \times \text{RF}^{r'} \) such that \( \alpha_i = \tilde{\alpha}_i \circ \lambda' \), \( \beta_{ij} = \tilde{\beta}_{ij} \circ \lambda' \) and \( Y = (\lambda' \times \text{Id})^{-1}(\tilde{Y}) \). Using this new definable data, we construct \( \psi' \in C_{+}(C') \) such that \( f = \psi' \circ \lambda' \) and again we have

\[
\sup_{w \in W_{F}} f_{F}(x, w) = \sup_{w \in W_{F}} \psi'_F \circ \lambda'_F(x, w) = \sup_{w \in W_{F}} \psi'_F(\rho(x), w).
\]

Hence we have reduced to the case where \( X = \text{RF}^{m_2} \times \text{VG}^{m_3} \). This finishes the proof of Claim 1.

Claim 2. We may assume that \( X = \text{RF}^{m_2} \times \text{VG}^{m_3} \) and \( W = \text{RF}^{m_2} \).

Proof of Claim 2. Write \( x = (\eta, t) \) and \( w = (\xi, s) \) for the variables of \( X = \text{RF}^{m_2} \times \text{VG}^{m_3} \) and \( W = \text{RF}^{m_2} \times \text{VG}^{m_3} \). We would like to get rid of the value group variables \( \text{VG}^{m_3} \) of \( W \). Using the (model theoretic) orthogonality of the sorts VG and RF, there is a definable partition of \( X \times W \), such that each definable part \( A \) is a box \( A_1 \times A_2 \) with \( A_1 \subseteq \text{RF}^{m_2} \times \text{RF}^{m_2} \) and \( A_2 \subseteq \text{VG}^{m_3} \times \text{VG}^{m_3} \), and such that on each \( A \), \( f \) has the form

\[
f_{F}|_{A_{F}}(\eta, t, \xi, s) = \#Y_{\eta, \xi} \cdot H_{F}(t, s),
\]
for some $H \in \mathcal{P}_+(V^n_3 \times V^n_3)$ and $Y \subseteq RF^{m_2} \times RF^{m_2} \times RF'$. By Corollary 3.3 and by our assumption on $A$, we may assume $f_F = \#Y_{\eta, \xi} \cdot H_F(t, s)$. Note that for each $F \in \text{Loc}_{\geq}$ and each $(\eta, t, \xi) \in X_F \times k^{m_2}_F$ one has

$$\sup_{s \in \mathbb{Z}^{m_3}} f_F(\eta, t, \xi, s) = \#Y_{\eta, \xi} \cdot \sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s),$$

In order to approximate $\sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s)$, it is enough to consider the case where $m_3 = 1$ and proceed by induction on $m_3$. Using Presburger cell decomposition and rectilinearization (see [Chu03 Theorems 1 and 3]) we may assume that $H$ is in $\mathcal{P}_+(B)$ for $B \subseteq V^n_3 \times \mathbb{N}$ with $B_t := \{ s \in \mathbb{N} : (t, s) \in B \}$ is either a finite set for each $t \in \mathbb{Z}^{n_3}$, or $B_t = \mathbb{N}$, and moreover, $H$ is of the form

$$H_F(t, s) = \sum_{i=1}^{N} c_i F(t) s^{a_i} q^{b_i s},$$

with $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ and $c_i$ in $\mathcal{P}(V^n_3)$. Denote by $T$ the image of projection of $B$ to $V^n_3$. We repeat a part of the argument of the proof of [CGH15 Theorem 2.1.3]. Namely, by [CGH15 Lemmas 2.2.3 and 2.2.4], there exist $m, l \in \mathbb{N}_{\geq 1}$ and finitely many definable functions $h_1, ..., h_l : T \rightarrow \mathbb{N}$ with $h_j(t) \in B_t$ such that for each $t \in T$ for which $s \mapsto H_F(t, s)$ is bounded on $B_t$, one has

$$\sup_{s \in B_t} H_F(t, s) \leq m \cdot \max_{1 \leq j \leq l} H_F(t, h_j(t)).$$

In particular, setting $\tilde{H}(t) := m \cdot \sum_{j=1}^{l} H(t, h_j(t)) \in \mathcal{P}_+(T)$ we get:

$$\sup_{s \in B_t} H_F(t, s) < \tilde{H}(t) < m \cdot l \cdot \sup_{s \in B_t} H_F(t, s).$$

This finishes the proof of Claim 2.

\[\Box\]

**Claim 3.** We may assume that $X = RF^{n_2}$ and $W = RF^{m_2}$.

**Proof.** This follows directly by Claim 2 Corollary 3.3 and using the orthogonality of the sorts $V_G$ and $R_F$.

\[\Box\]

To continue the proof of Theorem 3.1, we may thus assume that $X = RF^{n_2}$ and $W = RF^{m_2}$. We may assume, again using Corollary 3.3 that $f$ is of the form $f(x, w) = f(\eta, \xi) = u \cdot \#Y_{\eta, \xi}$, with $\xi$ the coordinate on $W$, and $\eta$ on $X$ and $u = \{ u_F \}_{F \in \text{Loc}_{\geq}}$ is a motivic number. In particular, for each $\eta \in X_F$:

$$\sup_{w \in W_F} f_F(x, w) = \sup_{\xi \in \mathbb{Z}^{m_2}_F} f_F(\eta, \xi) = u_F \cdot \sup_{\xi \in \mathbb{Z}^{m_2}_F} \#Y_{\eta, \xi}.$$

By a definable variant of the Lang-Weil estimates (see [CvdDM92 Main Theorem]), there exists a definable partition $X \times W = \bigsqcup_{i=0}^{M} A_i$ and constants $C' > 0$, $d_i \in \mathbb{N}$ and $l_{i1}, l_{i2} \in \mathbb{Z}_{\geq 1}$, such that for each $1 \leq i \leq M$ and each $F \in \text{Loc}_{\geq}$:

$$A_{i,F} := \{ (\eta, \xi) \in X_F \times W_F : \#Y_{\eta, \xi} \leq \frac{l_{i1} d_i^{d_i^2}}{l_{i2}^{d_i^2}} \}.$$

Denote by $Z_j$ the projection of $A_i$ to $X$. For each subset $I \subseteq \{1, \ldots, M\}$, let $Z_I := \bigcap_{i \in I} Z_i \bigcup_{j \in I^c} Z_j$, with $Z_{\emptyset} := X \setminus \bigsqcup_{j=1}^{M} Z_j$. Then $X = \bigsqcup_{I} Z_I$ is a definable partition, and thus we may assume that
Theorem 3.1 shows that the family $C$ for each $1 \leq P$.

Remark 3.5

Let us take zero for the supremum of the empty set. Since $\{uF \cdot \sum_{i}2^il_i \cdot q_i^d\}_{F \in \text{Loc}_d}$ clearly lies in $C_+(X)$, this finishes the proof of Theorem 3.1. $\square$

3.1. Optimality of the bounds and further remarks. Let $X$ and $W$ be $\mathcal{L}_{dp}$-definable sets. Given a subclass $\mathcal{F} \subseteq \mathcal{C}(X \times W)$ of motivic functions, one can ask whether for any $f \in \mathcal{F}$, the function $\{\sup_{w \in W} f_F(x, w)\}_{F \in \text{Loc}_d}$ can be approximated by a motivic function in $\mathcal{C}(X)$ up to a constant $C$ in up to four increasing levels of approximation:

1. With $C$ depending on $F$ and $f$.
2. With $C$ depending on $f$ and independent of $F$.
3. With $C$ a universal constant, that is, uniform over all $f \in \mathcal{F}$ and $F \in \text{Loc}_d$.
4. With $C = 1 + C'q_F^{-1/2}$ for some $C'$ depending on $f$ and independent of $F$.

If the class $\mathcal{F}$ satisfies one of the Items (i) above, we say that $\mathcal{F}$ admits an approximation of suprema of type (i), or $\mathcal{F}$ is of type (i). Note that if $\mathcal{F}$ is of type (4) then it is also of type (3), as $C'q_F^{-1/2} < 2$ for $F \in \text{Loc}_d$. Similarly, type (i) is stronger than type (j) for $j < i$.

Remark 3.5

- The class $\mathcal{C}(X \times W)$ is not of type (1) (and thus of any type). Indeed, take $X = \mathbb{Z}^2$, $W = \{1, 2\} \subseteq \mathbb{Z}$ and define $f \in \mathcal{C}(X \times W)$ by $f(x, y, 1) = x^2 - y$ and $f(x, y, 2) = y - x^2$. Then $\sup_{w} f(x, y, w) = \max(x^2 - y, y - x^2)$ cannot be approximated by a motivic function on $\mathcal{C}(X)$, up to a constant depending on $F$ and $f$.

- The class $\mathcal{C}^\text{weak}_+(X \times W) := \{f \in \mathcal{C}(X \times W) : f_F \geq 0 \forall F \in \text{Loc}_d\}$ is of type (1), with $C = q_F^{C_0}$ for some $C_0 > 0$ depending only on $f$. This is a special case treated in the proof of [CGH18, Theorem 2.1.3]. One may wonder whether the class $\mathcal{C}^\text{weak}_+(X \times W)$ is of type (2).

Theorem 3.1 shows that the family $\mathcal{C}_+(X \times W)$, which is strictly contained in $\mathcal{C}^\text{weak}_+(X \times W)$, is of type (2). This is the best possible approximation, as already detected by the subclass $\mathcal{P}_+(X \times W) \subseteq \mathcal{C}_+(X \times W)$.

Proposition 3.6. The families $\mathcal{P}_+(X \times W)$ and $\mathcal{C}_+(X \times W)$ are not of type (3).

Proof. Let $X = \mathbb{Z}^\geq_1$, $W = \{1, \ldots, m\} \subseteq \mathbb{Z}$. Let $p_1 = 2 < p_2 = 3 < \cdots < p_{m+1}$ be the first $m$ prime numbers, and take $f(x_1, \ldots, x_m, w) = x_w^{p_m}$. Then for any $\epsilon > 0$ and any $g \in \mathcal{C}(X)$ with $\sup_{1 \leq w \leq m} f_F(x, w) \leq g_F(x)$ for $F \in \text{Loc}_d$, one cannot have

$$(*) \quad g_F(x) \leq (m - \epsilon) \cdot \sup_{1 \leq w \leq m} f_F(x, w)$$

for each $F \in \text{Loc}_d$ and $x \in X_F$. In fact, $\sum_{j=1}^{m} x_j^{p_j}$ is an optimal approximation (with constant $m$).
Here is a rough sketch. We assume, towards contradiction, the existence of $g \in C(X)$ satisfying $(\ast)$. Using Presburger cell decomposition \cite[Theorem 1]{Chu03}, we can decompose $X$ into cells $X = \bigsqcup_{i=1}^{N} C_i$, such that on each $C_i$, the definable Presburger functions appearing in $g$ are linear. We may find a large cell of the form

$$C = \{(x_1, \ldots, x_m) \in \mathbb{Z}_{\geq 1}^m : x_j \geq \alpha_j(x_{j+1}, \ldots, x_m) \land x_j = c_j \bmod r_j \},$$

for some affine functions $\alpha_j$, and integers $0 \leq c_j \leq r_j$. The cell $C$ is isomorphic to $\mathbb{Z}_{\geq 1}^m$ by an affine change of coordinates $\varphi : \mathbb{Z}_{\geq 1}^m \to C$, after which $g_F \circ \varphi$ has the form

$$\sum_{i=1}^{M} c_i(F) \cdot q_i^{a_1 e_1 + \ldots + a_m e_m} \cdot \prod_{j=1}^{m} e_j^{b_{ij}},$$

for $\{(a_{i1}, \ldots, a_{im}, b_{i1}, \ldots, b_{im})\}_i$ mutually different tuples of integers, where $b_{ij} \geq 0$. Since $\frac{g_F(x_1, \ldots, x_m)}{x_1^{p_1} + \ldots + x_m^{p_m}}$ is bounded from above and below by constants, it follows that all $a_{ij} \leq 0$. We can therefore write $g$ as:

$$g_F(x_1, \ldots, x_m) = P_F(x_1, \ldots, x_m) + E_F(x_1, \ldots, x_m),$$

where $P$ is a polynomial with coefficients in $F$ and $E$ consists of all the terms of $(\ast\ast)$ with $a_{ij} < 0$ for some $j \in \{1, \ldots, m\}$. Let us write $P_F(x_1, \ldots, x_m) = \sum_{i=1}^{m} d_j x_j^{p_j} + Q_F$ for some polynomial $Q$, which consists of monomials disjoint from $\{x_j^{p_j}\}_{j=1}^{m}$, and coefficients $d_j$ depending on $F$. Now the idea is to use the fact that the cell $C$ “sees” many asymptotic directions in $\mathbb{Z}_{\geq 1}^m$, to deduce:

1. For $F \in \text{Loc}_\gg$, the coefficients $d_j$ are bounded from below by constants arbitrary close to 1.

2. In a certain region $\tilde{C}$ in $C$, $E_F$ is negligible with respect to $P_F$, so that $g_F \sim P_F$.

The assumption that the $p_i$’s are prime numbers, guarantees that in certain asymptotic directions in $\tilde{C}$, $Q_F$ is negligible with respect to $\sum_{j=1}^{m} d_j x_j^{p_j}$, so that $g_F \sim \sum_{j=1}^{m} d_j x_j^{p_j}$.

Items (1) and (2) contradict our assumption on $g$. Note that without the assumption on the $p_i$’s, one can get tighter approximations than $\sum_{j=1}^{m} x_j^{p_j}$. For example, $\frac{4}{3} (x_1^2 - x_1 x_2 + x_2^2)$ gives a tighter upper bound for $\max(x_1^2, x_2^2)$, than $x_1^2 + x_2^2$, since $\frac{4}{3} (x_1^2 - x_1 x_2 + x_2^2) \leq \frac{4}{3} \max(x_1^2, x_2^2)$. \hfill $\square$

In \cite[Theorem 2.1.3]{CGH18}, an approximation of suprema result is proven for a more general class $C^{sp}(X \times W)$ of motivic exponential functions, which involves additive characters, and which is furthermore built out of functions which are definable in the generalized Denef-Pas language. Due to this larger generality, the approximation shown in \cite[Theorem 2.1.3]{CGH18} is a bit weaker than type (1) above (in \cite[Theorem 2.1.3]{CGH18}, one approximates $\sup |f|^2$ instead of $\sup f$). This is unavoidable, as already seen in Remark 3.6.

Remark 3.7. One can weaken the definition of approximation as follows. For a function $f \in C_+(X \times W)$, assume there exist motivic functions $\{g_i\}_{i=1}^{m} \in C_+(X)$, with $m \in \mathbb{N}$ such that

$$\max_{1 \leq i \leq m} \{g_i(x)\} \leq \sup_{w \in W_F} f_F(x, w) \leq C \cdot \max_{1 \leq i \leq m} \{g_i(x)\},$$

where $C$ is as in types (1)-(4) above. Using this weaker form of approximation, we expect $C_+(X \times W)$ to be of weakened type (4).

One may also weaken (3) by letting the constant $C$ depend on the number of variables running over $X \times W$, and wonder whether $C_+(X \times W)$ is of type (3) when weakened in this sense.
4. Number theoretic characterization of the (FRS) property

In this section we use Theorem 3.1 to prove a more general form of Theorem A for Loc\(\rightarrow\), providing a full number theoretic characterization of (FRS) morphisms (Theorem 4.4)

Throughout this section, \(K\) will be a fixed number field.

4.1. An analytic characterization of the (FRS) property. Given an \(\mathcal{O}_F\)-morphism \(\varphi : X \to Y\), we denote the natural maps \(X(\mathcal{O}_F/\mathfrak{m}_F^k) \to Y(\mathcal{O}_F/\mathfrak{m}_F^k)\) by \(\varphi\), therefore \(\varphi^{-1}(\mathfrak{y})\) is a finite set in \(X(\mathcal{O}_F/\mathfrak{m}_F^k)\), for any \(\mathfrak{y} \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)\). We denote by \(r_k : Y(\mathcal{O}_F) \to Y(\mathcal{O}_F/\mathfrak{m}_F^k)\) and by \(r_k^i : Y(\mathcal{O}_F/\mathfrak{m}_F^k) \to Y(\mathcal{O}_F/\mathfrak{m}_F^{k+1})\) the natural reduction maps for \(k \geq 1\).

Definition 4.1. Let \(Y\) be a smooth \(F\)-variety, with \(F \in \text{Loc}\). A measure \(\mu\) on \(Y(F)\) is called:

1. Smooth if for any \(y \in Y(F)\) there exists an analytic neighborhood \(U \subseteq Y(F)\) and an analytic diffeomorphism \(\psi : U \to \mathcal{O}_F^{\dim Y}\) such that \(\psi_*\mu\) is a Haar measure on \(\mathcal{O}_F^{\dim Y}\).
2. Schwartz if it is compactly supported and smooth.

Lemma 4.2 (cf. [Ser81, Wei82, Oes82]). Let \(F\) be in \(\text{Loc}\), and let \(Y\) be a finite type \(\mathcal{O}_F\)-scheme such that \(Y \times_{\text{Spec}\mathcal{O}_F} \text{Spec}F\) is smooth, of pure dimension \(d\). Then there is a unique Schwartz measure \(\mu_Y(\mathcal{O}_F)\) on \(Y(\mathcal{O}_F)\), and there exists \(k_0 \in \mathbb{N}\), such that for every \(k \geq k_0\) and every \(\bar{y} \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)\), one has

\[
\mu_Y(\mathcal{O}_F)(r_k^{-1}(\bar{y})) = q_F^{-kd}.
\]

The measure \(\mu_Y(\mathcal{O}_F)\) is referred to as the canonical measure on \(Y(\mathcal{O}_F)\). In the special case when \(Y\) is smooth over \(\mathcal{O}_F\), then (4.1) holds for every \(k \geq 1\).

Proof. If \(Y\) is affine, then the existence and uniqueness of \(\mu_Y(\mathcal{O}_F)\) follows from [Oes82, Lemma 3], building on [Ser81, Theorem 9]. In general, let \(Y = \bigcup_{i=1}^N U_i\) be an open affine cover by \(\mathcal{O}_F\)-subschemas \(U_i\). Then \(Y(\mathcal{O}_F) = \bigcup_{i=1}^N U_i(\mathcal{O}_F)\). Note that

\[
\mu_{U_i(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F) \cap U_j(\mathcal{O}_F)} = \mu_{U_j(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F) \cap U_j(\mathcal{O}_F)}
\]

by uniqueness, so we can glue them together to form \(\mu_Y(\mathcal{O}_F)\). If furthermore \(Y\) is smooth over \(\mathcal{O}_F\), then by applying Hensel’s lemma to (4.1) we can choose \(k_0 = 1\) (see also [Wei82, Theorem 2.25]).

In [AA16], Aizenbud and Avni gave an analytic characterization of the (FRS) property:

Theorem 4.3 ([AA16, Theorem 3.4]). Let \(\varphi : X \to Y\) be a map between smooth \(K\)-varieties. Then the following are equivalent:

1. \(\varphi\) is (FRS).
2. For any \(F \in \text{Loc}_0\) and any Schwartz measure \(\mu\) on \(X(F)\), the measure \(\varphi_*(\mu)\) has continuous density.
3. For any \(x \in X(\overline{K})\) and any finite extension \(K'/K\) with \(x \in X(K')\), there exists \(F \in \text{Loc}_0\) containing \(K'\), and a non-negative Schwartz measure \(\mu\) on \(X(F)\) that does not vanish at \(x\) such that \(\varphi_*(\mu)\) has continuous density.

The next result shows the above characterization extends to local fields of large positive characteristic.
Corollary 4.4. Let \( \varphi : X \to Y \) be a map between smooth \( K \)-varieties. Then \( \varphi \) is (FRS) if and only if for every \( F \in \text{Loc}_{\mathbb{F}_q} \), the measure \( \varphi_*(\mu_X(\mathcal{O}_F)) \) has bounded density with respect to \( \mu_Y(\mathcal{O}_F) \).

Proof. Without loss of generality, we may assume that \( Y \) is affine. By choosing an \( \mathcal{O}_K \)-model of \( Y \), we may identify it as an \( \mathcal{L}_{\text{Dp}} \)-definable set. Assume \( \varphi \) is (FRS). For each \( F \in \text{Loc}_{\mathbb{F}_q} \), write \( \tau_F := \varphi_*(\mu_X(\mathcal{O}_F)) \). By \cite[Theorem 3.4(2)]{AA16}, we can write \( \tau_F = f_F \cdot \mu_Y(\mathcal{O}_F) \) and \( f_F \) is continuous, for each \( F \in \text{Loc}_{\mathbb{F}_q} \). Moreover, locally, \( f \) can be written as an integral of a motivic function \( G \in C_\mathbb{A}(Y \times \text{VF}^\dim X - \dim Y) \), over \( \text{VF}^\dim X - \dim Y \). By \cite[Theorem 4.4.1]{GH14}, it follows that \( G_F(\cdot, \cdot) \) is integrable, for each \( F \in \text{Loc}_{\mathbb{F}_q} \) and \( y \in Y(\mathcal{O}_F) \). By Theorem \ref{thm:lang-weis-r}, we can choose \( f \) to be in \( C_\mathbb{A}(Y) \).

By \cite[Appendix B, Theorem 14.6]{ST16} (or more generally, by \cite[Theorem 2.1.2]{CGH18}) and since \( Y(\mathcal{O}_F) \) is compact, there exists \( a \in \mathbb{Z} \), such that for each \( F \in \text{Loc}_{\mathbb{F}_q} \) and each \( y \in Y(\mathcal{O}_F) \), one has \( f_F(y) < q_F^a \). By Theorem \ref{thm:lang-weis-r} we thus have \( f_F(y) < q_F^a \) for each \( F \in \text{Loc}_{\mathbb{F}_q} \) and each \( y \in Y(\mathcal{O}_F) \), as required. The other direction follows from Theorem \ref{thm:lang-weis-r} combined with \cite[Lemma 3.15]{GH19}, as in the proof of \cite[Proposition 3.16]{GH19}.

\[ \square \]

4.2. A number-theoretic characterization of the (FRS) Property. We now recall the Lang-Weil estimates, and set the required notation to state the main theorem.

Definition 4.5.

(1) For a finite type \( \mathbb{F}_q \)-scheme \( Z \), we denote by \( C_Z \) the number of its top-dimensional geometrically irreducible components which are defined over \( \mathbb{F}_q \).

(2) Let \( \varphi : X \to Y \) be a morphism between finite type \( \mathbb{Z} \)-schemes \( X \) and \( Y \), and let \( y \in Y(\mathbb{F}_q) \).

Then we write \( C_{X,y} := C_{X,\mathbb{F}_q} \) and \( C_{\varphi,y} := C(\varphi)_y \).

Theorem 4.6 (The Lang-Weil estimates \cite{LW51}). For every \( M \in \mathbb{N} \), there exists \( C(M) > 0 \), such that for every prime power \( q \), and any finite type \( \mathbb{F}_q \)-scheme \( X \) of complexity at most \( M \) (see e.g. \cite[Definition 7.7]{GH19}), one has

\[ \left| \#X(\mathbb{F}_q)/q^{\dim X} - C_X \right| < C(M)q^{-\frac{1}{2}}. \]

Let \( X,Y \) be finite type \( \mathcal{O}_K \)-schemes, with \( X_K, Y_K \) smooth and geometrically irreducible, and let \( \varphi : X \to Y \) be a dominant morphism. Let \( \mu_X(\mathcal{O}_F) \) and \( \mu_Y(\mathcal{O}_F) \) be the canonical measures on \( X(\mathcal{O}_F) \) and \( Y(\mathcal{O}_F) \) for \( F \in \text{Loc} \). Since \( \varphi \) is dominant, it follows that \( \tau_F := \varphi_*(\mu_X(\mathcal{O}_F)) \) is absolutely continuous with respect to \( \mu_Y(\mathcal{O}_F) \), and thus has an \( L^1 \)-density (see e.g. \cite[Corollary 3.6]{AA16}), so that \( \tau_F = f_F(y) \cdot \mu_Y(\mathcal{O}_F) \). When \( Y \) is affine, the collection \( f = \{ f_F : Y(\mathcal{O}_F) \to \mathbb{C})_{F \in \text{Loc}_{\mathbb{F}_q}} \) can be chosen to be formally non-negative. Indeed, as in the proof of Corollary \ref{thm:lang-weis-r} locally, \( f_F \) can be written as an integral of a motivic function \( G \in C_\mathbb{A}(Y \times \text{VF}^\dim X - \dim Y) \), over \( \text{VF}^\dim X - \dim Y \). Note there is an open affine sub scheme \( U \) of \( Y \), such that \( \varphi_K \) is smooth over \( U_K \). Then \( G_F(y, \cdot) \) is integrable for every \( y \in U(Y) \) and \( F \in \text{Loc}_{\mathbb{F}_q} \). By Theorem \ref{thm:lang-weis-r} it follows that \( f|_U \) is formally non-negative. Since \( U(Y) \) is dense in \( Y(F) \) for \( F \in \text{Loc}_{\mathbb{F}_q} \), by extending \( f|_U \) by 0 we get a collection of densities on \( \{ Y(\mathcal{O}_F) \}_{F \in \text{Loc}_{\mathbb{F}_q}} \) which is formally non-negative.

For \( F \in \text{Loc}_{\mathbb{F}_q} \), define a function \( g_F \) for \( y \in \mathcal{O}_F \) and \( k \in \mathbb{Z}_{\geq 1} \) by

\[ g_F(y,k) = \frac{1}{\mu_Y(\mathcal{O}_F)(B(y,k))} \int_{\bar{y} \in B(y,k)} f_F(\bar{y}) \mu_Y(\mathcal{O}_F), \]

where \( B(y,k) = r_k^{-1}(r_k(y)) \). By Theorem \ref{thm:lang-weis-r} it follows that \( \{ g_F : Y(\mathcal{O}_F) \times \mathbb{Z}_{\geq 1} \to \mathbb{C})_{F \in \text{Loc}_{\mathbb{F}_q}} \) is a formally non-negative motivic function.
For every $F \in \text{Loc}_{\gg}$, every $y \in Y(\mathcal{O}_F)$ and every $k \in \mathbb{Z}_{\geq 1}$, we have

\[(4.2) \quad g_F(y, k) = \frac{\varphi_*(\mu_X(\mathcal{O}_F))(B(y, k))}{\mu_Y(\mathcal{O}_F)(B(y, k))} = \frac{\#\varphi^{-1}(r_k(y))}{q_F^{k(\dim X_K - \dim Y_K)}},\]

where the last equality follows from Lemma-Definition 4.2 and the fact that $Y$ is smooth over $\mathcal{O}_F$ for $F \in \text{Loc}_{\gg}$. Set

\[h_F(y, k) = \frac{\#(\varphi^{-1}(r_k(y)) \cap \{X^{\text{sing}}_{\varphi}(k_F)\})}{q_F^{k(\dim X_K - \dim Y_K)}}.\]

The asymptotics of the functions $h$ and $g$, in $q_F$ and $k$, measure how wild are the singularities of $\varphi$. For example, if $\varphi_K$ is smooth, then $h_F(y, k) \equiv 0$ and $g_F(y, k) < C$ for $F \in \text{Loc}_{\gg}$ and some constant $C$. On the other hand, if $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is the map $x \mapsto x^m$, then $g(0, k) = h(0, k) = q_F^{-k\left\lfloor \frac{m}{k} \right\rfloor}$.

Furthermore, the motivic function $\{h_F\}_{F \in \text{Loc}_{\gg}}$ is formally non-negative (Proposition 4.9). This is used to prove our main theorem, which we state now.

**Theorem 4.7.** Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes $X$ and $Y$, with $X_K, Y_K$ smooth and geometrically irreducible. Then the following are equivalent:

1. $\varphi_K : X_K \to Y_K$ is (FRS).
2. There exists $C_1 > 0$, such that for each $F \in \text{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y' \in Y(\mathcal{O}_F)$:
   \[h_F(y', k) < C_1 q_F^{-1}.\]
3. There exists $C_2 > 0$ such that for each $F \in \text{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/m_F^k)$:
   \[
   \left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_k(y))}{q_F^{\dim X_K - \dim Y_K}} \right| < C_2 q_F^{-1}.
   \]
4. There exists $C_3 > 0$ such that for each $F \in \text{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/m_F^k)$:
   \[
   \left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - C_{\varphi, q_F, r_k}(y) \right| < C_3 q_F^{-\frac{1}{2}}.
   \]
5. There exists $C_4 > 0$ such that for each $F \in \text{Loc}_{\gg}$, $\varphi_*(\mu_X(\mathcal{O}_F))$ has continuous density $f_F$ with respect to $\mu_Y(\mathcal{O}_F)$, and for each $y' \in Y(\mathcal{O}_F)$, one has:
   \[|f_F(y') - C_{\varphi, q_F, r_k}(y')| < C_4 q_F^{-\frac{1}{2}}.\]

Before we prove Theorem 4.7, we first show it implies Theorem A.

**Proof of Theorem 4.7.** We prove $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$. To prove $(3) \Rightarrow (2)$, we first treat large primes using implication $(3) \Rightarrow (4)$ of Theorem 4.7 and then treat small primes using the Lang-Weil estimates. Implication $(4) \Rightarrow (1)$ follows from Theorem 1.3.

Let us assume that Condition (2) holds. By Lemma-Definition 4.2 and by Condition (2), there exists $C_1 > 0$, such that for every prime $p$, every $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ and every $k \geq k_0$, one has

\[(4.3) \quad \frac{\varphi_*\mu_X(\mathbb{Z}_p)(r_k^{-1}(y))}{\mu_Y(\mathbb{Z}_p)(r_k^{-1}(y))} = \frac{\mu_X(\mathbb{Z}_p)(\varphi^{-1}(r_k^{-1}(y)))}{p^{\dim X_\mathbb{Q}}} = \frac{\mu_X(\mathbb{Z}_p)(\varphi^{-1}(1))}{p^{\dim X_\mathbb{Q}}} = \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_\mathbb{Q} - \dim Y_\mathbb{Q})}} < C_1,
\]

where $\mu_X(\mathbb{Z}_p)$ and $\mu_Y(\mathbb{Z}_p)$ are the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)$. Let $f_p$ be the density of $\varphi_*\mu_X(\mathbb{Z}_p)$ with respect to $\mu_Y(\mathbb{Z}_p)$. Combining (4.3) with Lebesgue’s differentiation theorem, we
get for almost all \( y' \in Y(\mathbb{Z}_p) \):

\[
 f_p(y') = \lim_{k \to \infty} \frac{\varphi_* \mu_X(\mathbb{Z}_p)(r_k^{-1}(r_k(y')))}{\mu_Y(\mathbb{Z}_p)(r_k^{-1}(r_k(y')))} < C_1,
\]

which implies Condition (4).

It is left to prove (1) \( \Rightarrow \) (3). The case of large primes follows from the implication (1) \( \Rightarrow \) (3) of Theorem \ref{thm:main}. It is left to prove (3) for a fixed prime \( p \). By Theorem \ref{thm:nonvanishing} we have \( f_p < C(p) \) for some \( C(p) > 0 \). Using \ref{thm:nonvanishing}, we deduce that

\[
 (4.4) \quad \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_K - \dim Y_K)}} < C(p),
\]

for every \( k \geq k_0 \) and \( y \in Y(\mathbb{Z}/p^k \mathbb{Z}) \). For \( y \in Y(\mathbb{Z}/p^k \mathbb{Z}) \) with \( k < k_0 \) we can take the trivial bound \( \#\varphi^{-1}(y) \leq \sum_{i=1}^{k_0} \#X(\mathbb{Z}/p^i \mathbb{Z}) \) to deduce \ref{thm:nonvanishing} for every \( k \in \mathbb{N} \). Using the triangle inequality, and by applying the trivial upper bound \( \#\varphi^{-1}(\tilde{y}) < \#X(\mathbb{F}_p) \) for \( \tilde{y} \in Y(\mathbb{F}_p) \), we deduce (3).

\[ \square \]

Remark 4.8. One can easily adapt the proof of Theorem A above to prove a more general statement where the collection \( \{ \mathbb{Q}_p \}_p \) is replaced with all completions \( \mathcal{K}_p \) \( \mathbb{Q} \) of a fixed number field \( \mathcal{K} \). On the other hand, Theorem \ref{thm:main} is definitely not true for all \( F \in \text{Loc} \) (e.g. take \( \varphi(x) = 3x \), and consider unramified extensions of \( \mathbb{Q}_3 \)).

We now move to the proof of Theorem \ref{thm:main}. We start with the easier implications, and deal with the more challenging implication (1) \( \Rightarrow \) (2) in Subsection 4.2.1.

Proof of (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (1) of Theorem \ref{thm:main}. Implication (2) \( \Rightarrow \) (3): assume that \( h_F(y', k) < C_1 q_p^{-1} \) for each \( F \in \text{Loc}_{\infty} \), each \( k \in \mathbb{N} \) and each \( y' \in Y(O_F) \). Set \( y := r_k(y') \) and note that:

\[
 \left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_k(y))}{q_F^{k(\dim X_K - \dim Y_K)}} \right| = \left| \frac{\#\varphi^{-1}(\varphi_0)(y)}{q_F^{k(\dim X_K - \dim Y_K)}} + h_F(y', k) - \frac{\#\varphi^{-1}(\varphi_0)(r_k(y))}{q_F^{k(\dim X_K - \dim Y_K)}} - h_F(y', 1) \right|
\]

\[
= |h_F(y', k) - h_F(y', 1)| \leq 2C_1 q^{-1}.
\]

where the second equality follows from Hensel’s lemma and the inequality follows from our assumption on \( h \). Since \( r_k \) is surjective for \( F \in \text{Loc}_{\infty} \), this finishes the proof.

Implication (3) \( \Rightarrow \) (4): let us first prove that \( \varphi_K \) is flat, assuming Condition (3). It is enough to show that \( \varphi_{\mathcal{K}} \) is flat for infinitely many prime numbers \( p \). Let \( p \) be a prime large enough such that \( \dim X_K = \dim X_{\mathbb{F}_p}, \dim Y_K = \dim Y_{\mathbb{F}_p} \) and such that Condition (3) holds for \( F = \mathbb{F}_q((t)) \) for any \( q \) which is a power of \( p \). Note there are infinitely many primes \( p \) such that \( \mathbb{F}_p \) is a residue field of \( O_K \) for some prime of \( O_K \). Let \( x \in X(\mathbb{F}_q) \) for such \( q \) and let \( \bar{x} \in J_k(X)(\mathbb{F}_q) \simeq X(\mathbb{F}_q)[(t^{k+1})] \) be the image of \( x \) under the zero section embedding \( X(\mathbb{F}_q) \hookrightarrow J_k(X)(\mathbb{F}_q) \), so that \( r_k^{1}(\varphi(\bar{x})) = \varphi(x) \).

Then by Condition (3), we have for any \( k \in \mathbb{N} \):

\[
 (4.5) \quad \frac{\#\varphi^{-1}(\varphi(\bar{x}))}{q^{(k+1)(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(\varphi(x))}{q^{(k+1)(\dim X_K - \dim Y_K)}} < C_2 \cdot q^{-1}.
\]

By choosing \( q \) to be a suitable power of \( p \) we may assume \( C_{\varphi,q,\varphi(x)} \geq 1 \). Notice that \( \#\varphi^{-1}(\varphi(\bar{x})) = \#J_k(\varphi_{\mathcal{K}}(\varphi(x))) \) (\( \mathbb{F}_q \)). Since \( \dim J_k(\mathcal{K}) \varphi(x)_{\varphi_0} ) \geq (k + 1) \cdot \dim X_K \) \( \varphi(x)_{\varphi_0} \) and since \( C_{\varphi,q,\varphi(x)} \geq 1 \), we have by \ref{eq:dim_difference} and by the Lang-Weil estimates that

\[
 \dim X_{\varphi(x),\varphi_0} = \dim X_K - \dim Y_K = \dim X_{\mathbb{F}_q} - \dim Y_{\mathbb{F}_q}.
\]

By miracle flatness, we are done.
To prove Condition (4), by the triangle inequality, it is enough to find $C'_3$ such that for each $F \in \text{Loc}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/m_F^k)$:

$$\left| \frac{\#\varphi^{-1}(r_k^F(y))}{q_F^{\dim X_F - \dim Y_F}} - C_{\varphi, q_F, r_1^F}(y) \right| < C'_3q_F^{-\frac{1}{2}}.$$

This follows from the fact that $\varphi_K$ is flat, via a relative variant of the Lang-Weil estimates (see e.g. [GH, Theorem 8.4]).

Implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1): let $f_F$ be the density of $\varphi_*\mu_{X(\mathcal{O}_F)}$ with respect to $\mu_{Y(\mathcal{O}_F)}$. By Lebesgue’s differentiation theorem and Condition (4), for almost every $y' \in Y(\mathcal{O}_F)$, we have:

$$\left| f_F(y') - C_{\varphi, q_F, r_1}(y') \right| = \lim_{k \to \infty} \frac{\mu_X(\mathcal{O}_F)(\varphi^{-1}(B(y', k)))}{\mu_{Y(\mathcal{O}_F)}(B(y', k))} - C_{\varphi, q_F, r_1}(y')$$

(4.6)

$$= \lim_{k \to \infty} \frac{\#\varphi^{-1}(r_k(y'))}{q_F^{\dim X_F - \dim Y_F}} - C_{\varphi, q_F, r_1}(y') < C'_3q_F^{-\frac{1}{2}}.$$

(4.7)

This also shows that $f_F$ is essentially bounded for $F \in \text{Loc}_{\geq 0}$. By Corollary 4.4 and by Theorem 4.3 it follows that $f_F$ can be chosen to be continuous, so that (4.7) holds for all $y' \in Y(\mathcal{O}_F)$. This implies Condition (5), which implies Condition (1) using the same Corollary 4.4.

4.2.1. Proof of the implication (1) $\Rightarrow$ (2). In this section we will prove the remaining implication of Theorem 4.7, namely (1) $\Rightarrow$ (2). We first observe the following:

**Proposition 4.9.** Assume that $Y$ is affine. Then the motivic function $h$ is formally non-negative.

**Proof.** We first prove the special case with $X$ affine. Assume that $X \subseteq \mathbb{A}^m$ is the zero locus of $g_1, \ldots, g_t \in \mathcal{O}_K[x_1, \ldots, x_m]$. Since $X$ and $Y$ are affine, the map $\varphi = (f_1, \ldots, f_n) : X \to Y \subseteq \mathbb{A}^n$ is a polynomial map, thus with $f_i \in \mathcal{O}_K[x_1, \ldots, x_m]$. Given $y \in Y(\mathcal{O}_F)$, set:

$$S_{y,k,X} := \left\{ x \in \mathcal{O}_F^m : \min_{i,j} \{\text{val}(g_i(x)), \text{val}(f_j(x) - y_j)\} \geq k \right\}.$$

Now, for any $y \in Y(\mathcal{O}_F)$, we have

$$\#\varphi^{-1}(r_k(y)) = q_F^k \int_{\mathcal{O}_F^m} 1_{S_{y,k,X}} |dx_1 \wedge \ldots \wedge dx_m|.$$

Moreover,

$$\# \left( \varphi^{-1}(r_k(y)) \cap (r_1)^{-1}(X^{\text{sing}, \varphi}(k_F)) \right) = q_F^k \int_{\mathcal{O}_F^m} 1_{W_{y,k,X}} |dx_1 \wedge \ldots \wedge dx_m|,$$

where

$$W_{y,k,X} := \{ x \in S_{y,k,X} : r_1(x) \in X^{\text{sing}, \varphi}(k_F) \}.$$

Since $1_{W_{y,k,X}}$ is formally non-negative, we get by Theorem 2.12 that $h$ is formally non-negative as well. Now let $X = \bigcup_{i=1}^N U_i$ be a cover by smooth open affine subschemes $U_i$. For each $i$ and $F \in \text{Loc}_{\geq 0}$, write $V_i := U_i(\mathcal{O}_F) \setminus \bigcup_{j=1}^{i-1} U_j(\mathcal{O}_F)$ and note that

$$\# \left( \varphi^{-1}(r_k(y)) \cap (r_1)^{-1}(X^{\text{sing}, \varphi}(k_F)) \right) = \sum_{i=1}^N \# \left( (\varphi|_{U_i})^{-1}(r_k(y)) \cap (r_1)^{-1}(U_i^{\text{sing}, \varphi}(k_F) \cap r_1(V_i)) \right)$$

$$= \sum_{i=1}^N q_F^k \int_{\mathcal{O}_F^m} 1_{W_{y,k,X}} |dx_1 \wedge \ldots \wedge dx_m|,$$
where
\[W_{y,k,i} := \{ x \in S_{y,k,U_i} : r_1(x) \in U^{\text{sing},\varphi}(k_F) \cap r_1(V_i) \} .\]
This finishes the proof of Proposition 4.9.

We need one more lemma which we state in the generality of $E$-smooth morphisms, and which will further be used in the next section.

**Lemma 4.10.** Let $E \geq 1$ be an integer, let $\varphi$ be as in Theorem 4.7 and assume that $\varphi_K : X_K \to Y_K$ is $E$-smooth. Then, for each $k \in \mathbb{N}$ there exists a constant $C(k) > 0$ such that for each $F \in \text{Loc}_{+,\mathbb{Z}}$, one has
\[\sup_{y \in Y(O_F)} h_F(y,k) < C(k) \cdot q_F^{-E} .\]

**Proof.** Using Theorems 2.14 and 3.1 it is enough to prove the lemma for $F$ lying in $\text{Loc}_{+,\mathbb{Z}}$. By Proposition 2.5 we have
\[J_k(X_{kF}^{\text{sm},\varphi_{kF}}) = J_k(X_{kF})^{\text{sm},J_k(\varphi_{kF})} ,\]
for $F \in \text{Loc}_{+,\mathbb{Z}}$. Let $Z_{\bar{y}} := J_k(X_{kF})_{\bar{y},J_k(\varphi_{kF})}$ be a non-empty fiber of $J_k(\varphi_{kF})$ over $\bar{y} \in J_k(Y)(k_F)$. Since $J_k(\varphi_{kF})$ is flat and by (4.8), we have
\[Z_{\bar{y}}^{\text{sing}} = Z_{\bar{y}} \cap J_k(X_{kF})^{\text{sing},J_k(\varphi_{kF})} = Z_{\bar{y}} \cap J_k(X_{kF}^{\text{sing},\varphi_{kF}})^{-1} .\]
The $E$-smoothness of $\varphi_K$ implies that the right hand side is of codimension at least $E$ in $Z_{\bar{y}}$. By the definition of $h$, by the fact that all fibers of $J_k(\varphi_{kF})$ are of bounded complexity (for a fixed $k$) and using the Lang-Weil estimates, the lemma follows.

**Proof of the implication** $(1) \Rightarrow (2)$. We may assume that $Y$ is affine. Theorems 3.1 and Proposition 4.9 imply that there exist a constant $C_0 > 0$ and a motivic function $H$ in $C_+(\mathbb{Z}_{\geq 1})$ such that
\[\sup_{y \in Y(O_F)} h_F(y,k) < H_F(k) < C_0 \cdot \sup_{y \in Y(O_F)} h_F(y,k) .\]
It is thus enough to show that $\sup_k H_F(k) < C_1 \cdot q_F^{-1}$ for some constant $C_1$ which is independent of $F$.

By Corollary 4.4 by (4.8) and since $h_F \leq g_F$, we deduce that the function $(y,k) \mapsto h_F(y,k)$ is bounded for each $F \in \text{Loc}_{\mathbb{Z}}$. By (4.10) also $k \mapsto H_F(k)$ is bounded for each $F \in \text{Loc}_{\mathbb{Z}}$. As in the proof of Claim 2 of Theorem 3.1 it follows that there exist a finite set $L$ of $\mathbb{Z}_{\geq 1}$ and a constant $C'_0 > 0$ such that
\[\sup_k H_F(k) \leq C'_0 L \sum_{k \in L} H_F(k) .\]
Using (4.10), (4.11), Lemmas 2.9 and 4.1 and by setting $C_1 := C_0 C'_0 \cdot \sum_{k \in L} C(k)$, we obtain
\[\sup_k H_F(k) \leq C'_0 L \sum_{k \in L} H_F(k) \leq C'_0 C_0 \sum_{k \in L} \sup_{y \in Y(O_F)} h_F(y,k) < C_1 q_F^{-1} ,\]
for each $F \in \text{Loc}_{\mathbb{Z}}$. This finishes the proof of $(1) \Rightarrow (2)$.

4.3. **Number-theoretic estimates for $E$-smooth and $\varepsilon$-jet-flat morphisms.** In this subsection we use the improved approximation of suprema (Theorem 3.1), similarly as in Subsection 4.2.1 to provide uniform estimates for $E$-smooth morphisms and $\varepsilon$-jet flat morphisms, improving [GH] Theorem 8.18. We start by giving a characterization of $E$-smooth morphisms.
Theorem 4.11. Let $E \geq 1$ be an integer, and let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes $X$ and $Y$, with $X_K, Y_K$ smooth and geometrically irreducible. Then the following are equivalent:

1. $\varphi_K : X_K \to Y_K$ is $E$-smooth.
2. There exists $C_1 > 0$ such that for each $F \in \text{Loc}_\geq$, $k \in \mathbb{Z}_{\geq 1}$ and $y' \in Y(\mathcal{O}_F)$:
   $$h_F(y', k) < C_1q_F^{-E}.$$
3. There exists $C_2 > 0$ such that for each $F \in \text{Loc}_\geq$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/m_F^k)$:
   $$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(d\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{d\dim X_K - \dim Y_K}} \right| < C_2q_F^{-E}.$$ 

In particular, when $E = 2$, the conditions above are further equivalent to $\varphi_K : X_K \to Y_K$ being flat with fibers of terminal singularities (see Lemma 2.9).

Proof. The proof of $(1) \Rightarrow (2)$ is identical to the proof of $(1) \Rightarrow (2)$ in Theorem 4.7, where the only exception is the inequality $\sup_{y \in Y(\mathcal{O}_F)} h_F(y, k) < C_1q_F^{-E}$ for $F \in \text{Loc}_\geq$ which is similarly obtained using Lemma 4.10. $(2) \Rightarrow (3)$ is similar as in Theorem 4.7.

$(3) \Rightarrow (1)$: recall that condition $(3)$ implies that $\varphi_K$ is jet-flat and that

$$|h_F(y', k) - h_F(y', 1)| \leq C_2q_F^{-E},$$

for all $F \in \text{Loc}_\geq$, $k \in \mathbb{Z}_{\geq 1}$ and $y' \in Y(\mathcal{O}_F)$. Write $W_{y'} := (X_{kF})_{r_1(y'), \varphi_{kF}}$. We claim that $(W_{y'})^{\text{sing}}$ is of codimension at least $E + 1$ in $W_{y'}$ for all $F \in \text{Loc}_\geq$ and $y' \in Y(\mathcal{O}_F)$. Indeed, assume $(W_{y'})^{\text{sing}}$ is of codimension $r$ in $W_{y'}$ with $r \leq E$. Identifying $r_1(y')$ with $\tilde{y} := s_1(r_1(y')) = (r_1(y'), 0) \in J_1(Y)(k_F)$ under the zero section embedding $s_1 : Y \hookrightarrow J_1(Y)$, and using (4.9) one has

$$\left(J_1(X_{kF})_{\tilde{y}, J_1(\varphi_{kF})}\right)^{\text{sing}} = J_1(W_{y'}) \cap (\pi_{0, X_{kF}})^{-1}(X_{kF}^{\text{sing}, \varphi_{kF}}) = (\pi_{0, W_{y'}})^{-1}(W_{y'}^{\text{sing}}).$$

Since the dimension of the Zariski tangent space of a variety $Z$ at a singular point is larger than $\dim Z$, we have

$$\dim \left(J_1(X_{kF})_{\tilde{y}, J_1(\varphi_{kF})}\right)^{\text{sing}} \geq \dim(W_{y'})^{\text{sing}} + \dim X_K - \dim Y_K + 1$$
$$\geq \dim W_{y'} - r + \dim X_K - \dim Y_K + 1$$
$$\geq \dim J_1(X_{kF})_{\tilde{y}, J_1(\varphi_{kF})} - r + 1.$$ 

Hence $(J_1(X_{kF})_{\tilde{y}, J_1(\varphi_{kF})})^{\text{sing}}$ is of codimension at most $r - 1$. By replacing $F$ with a finite extension, and using the Lang-Weil estimates, one can find $C_3 > 0$ such that

$$h_F(y', 1) < C_3q_F^{-r}$$
$$h_F(y', 2) > \frac{1}{2}q_F^{-r+1}.$$ 

But this contradicts Condition (3). Therefore $h_F(y', 1) < C_3q_F^{-(E+1)}$ for all $F \in \text{Loc}_\geq$ and $y' \in Y(\mathcal{O}_F)$. But then by Condition (3), we deduce that $h_F(y', k) < C_3q_F^{-E}$ which implies that $\varphi_K$ is $E$-smooth. \hfill \Box

Finally, a number-theoretic estimate can be given to $\varepsilon$-jet-flat morphisms, sharpening the estimate in [GH, Theorem 8.18].
Theorem 4.12 (cf. [GH] Theorem 8.18). Let \( \varphi : X \to Y \) be a dominant morphism between finite type \( \mathcal{O}_K \)-schemes \( X \) and \( Y \), with \( X_K, Y_K \) smooth and geometrically irreducible and let \( 0 < \varepsilon \leq 1 \). Then the following are equivalent:

1. \( \varphi_K : X_K \to Y_K \) is \( \varepsilon \)-jet flat.
2. There exist \( C, M > 0 \) such that for each \( F \in \text{Loc}_\gg \), \( k \in \mathbb{Z}_{\geq 1} \) and \( y \in Y(\mathcal{O}_F/m_F^k) \), one has

\[
\frac{\#\varphi^{-1}(y)}{k^{(\dim X_K - \dim Y_K)}} < C \cdot k^M \cdot q_F^{(1-\varepsilon)\dim Y_K}.
\]

In particular, when \( \varepsilon = 1 \) and assuming \( \varphi_K \) has normal fibers, the conditions above are further equivalent to \( \varphi_K \) being flat with fibers of log-canonical singularities (Remark [2.5]).

The proof of (2) \( \Rightarrow \) (1) of Theorem 4.12 follows from [GH, Theorem 8.18]. In order to prove (1) \( \Rightarrow \) (2), we prove an auxiliary lemma.

Lemma 4.13. Let \( g \in C_+(\mathbb{Z}_{\geq 1}) \) be a formally non-negative motivic function such that for every \( \delta > 0 \) and \( k \in \mathbb{Z}_{\geq 1} \) we have (varying over \( F \in \text{Loc}_\gg \))

\[
\lim_{q_F \to \infty} q_F^{-\delta} g_F(k) = 0.
\]

Then there exist \( M \in \mathbb{N} \) and \( C > 0 \) such that \( g_F(k) < Ck^M \) for every \( k \in \mathbb{Z}_{\geq 1} \) and field \( F \in \text{Loc}_\gg \).

Proof. Since \( g \) is formally non-negative, we may write \( g_F = \sum \#Y_{F,i} f_{F,i} \) for \( f_{F,i} \in \mathcal{P}_+(\mathbb{Z}_{\geq 1}) \) formally non-negative and \( Y_{F,i} \subseteq \mathbb{Z}_{\geq 1} \times \mathbb{R}^{r_i} \). It is enough to show the claim for a single summand \( g_F = \#Y_{F} f_{F} \). Using Presburger cell decomposition and the orthogonality of RF and VG, we have a finite partition \( \mathbb{Z}_{\geq 1} = \bigcup A_i \) and we may write \( g_F(k)|_A = \sum \#Y_{F} c_i(q_F) q_F^{a_i k b_i} \) on each cell \( A_i \), where \( a_i \in \mathbb{Q}, b_i \in \mathbb{N} \), \( \{(a_i, b_i)\}_{i=1}^N \) are mutually different, and \( c_i(q) \) are rational functions in \( q \).

First assume our cell \( A \) is finite, in which case it is enough to prove the claim for a fixed \( k = k_0 \). Using [CvdDM92, Main Theorem], we have non-negative constants \( d, C_1 \) and \( C_2 \) such that

\[
(\dagger) \quad \#Y_F < C_2 q_F^d \quad \text{for all } F \in \text{Loc}_\gg, \quad \text{and} \quad C_1 q_F^d < \#Y_F < C_2 q_F^d
\]

for infinitely many fields \( F \in \text{Loc}_\gg \) (with infinitely many residual characteristics).

Therefore, for every \( \delta > 0 \) and infinitely many fields \( F \in \text{Loc}_\gg \), we have

\[
(\Delta) \quad \lim_{q_F \to \infty} q_F^{-\delta} \left( C_1 q_F^d \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} \right) \leq \lim_{q_F \to \infty} q_F^{-\delta} g_F(k_0) = 0,
\]

and thus \( \deg_{q} \left( C_2 q_F^d \sum q_F^{a_i k_0} c_i(q) k_0^{b_i} \right) \leq 0 \) as a rational function in \( q \). The claim now follows since there exists \( C_3 > 0 \) such that for every \( F \in \text{Loc}_\gg \) with \( q_F \) large enough,

\[
g_F(k_0) = \#Y_F \sum c_i(q_F) q_F^{a_i k_0 k_0^{b_i}} < C_2 q_F^d \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} < C_3.
\]

Now, assume our cell \( A \) is infinite and set \( a = \max\{a_i\} \). Using (\( \Delta \)) with a general \( k \) instead of a fixed \( k_0 \), we must have \( a \leq 0 \), as otherwise for every \( k \) large enough \( R(q) = C_1 q^d \sum q^{a_i k} c_i(q) k^{b_i} \) is a non-zero rational function in \( q \) whose degree is positive, and therefore \( \lim_{q_F \to \infty} q_F^{-\delta} R(q_F) \neq 0 \) for some \( \delta > 0 \).

Set \( H_F(k) = \sum_{i:a_i=0} \#Y_F c_i(q_F) k^{b_i} \) and \( E_F(k) = \sum_{i:a_i<0} \#Y_F c_i(q_F) q_F^{a_i k} k^{b_i} \), then we have

\[
g_F(k) = H_F(k) + E_F(k) \leq |H_F(k)| + |E_F(k)|.
\]
Using (1), we may find a constant $C'$ such that $|E_F(k)| < C'$ for every $k$ large enough and $F \in \text{Loc}_0$. It is therefore left to take care of $H_F(k)$. We may assume $A = \mathbb{Z}_{\geq 1}$.

We prove by induction on the number of summands $N$ that if $H_F = \sum_{i=1}^N \#Y_F c_i(q_F) k^{b_i}$ is a function satisfying $\lim_{q_F \to \infty} q_F^d H_F(k) = 0$ for every $k$ large enough and $\delta > 0$, then there exists a constant $C'' > 0$ such that for every $F \in \text{Loc}_0$ we have $|\#Y_F c_i(q_F)| < C''$ for all $1 \leq i \leq N$.

For $N = 1$ the claim follows by (1) as before by showing $|\#Y_F c(q_F)|$ is bounded by a rational function of non-positive $q$-degree. To prove the claim for $N > 1$, consider the functions

$$\tilde{H}_{j,F}(k) = H_F(2k) - 2^{b_j} H_F(k) = \sum_{i=1}^N \#Y_F (2^{b_j} - 2^{b_i}) c_i(q_F) k^{b_i}.$$ 

Each $\tilde{H}_{j,F}(k)$ has $N-1$ summands and satisfies the induction hypothesis since $H_F(k)$ and $H_F(2k)$ do, and therefore the proof by induction is concluded. Using the triangle inequality, we can now find a bound for $H_F(k)$ as required, proving the lemma.

**Proof of Theorem 4.12.** The proof of (2) $\Rightarrow$ (1) follows from [GH, Theorem 8.18]. It is left to prove (1) $\Rightarrow$ (2). By Theorem 4.1 we have exist $G \in C_+ (\mathbb{Z}_{\geq 1})$ and $C' > 1$ such that

$$\sup_{y \in Y(\mathcal{O}_F)} g_F(y,k) < G_F(k) < C' \sup_{y \in Y(\mathcal{O}_F)} g_F(y,k).$$

By [GH, Theorem 8.18] and Theorem 2.14 for each $0 < \epsilon' < \epsilon$ we have $g_F(y,k) < q_F^{k((1-\epsilon') \dim Y_k)}$ for all $F \in \text{Loc}_0$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F)$. Therefore, for every $0 < \epsilon' < \epsilon$ we have

$$G'_F(k) := G_F(k) q_F^{-k((1-\epsilon) \dim Y_k)} < C' q_F^{-k((1-\epsilon') \dim Y_k)} q_F^{k((1-\epsilon') \dim Y_k)} = C' q_F^{k((\epsilon - \epsilon') \dim Y_k)}$$

for every $F$ with residual characteristic large enough (which may depend on $\epsilon'$). For any fixed $k$ and $\delta$, choose $\epsilon'$ such that $k((\epsilon - \epsilon') \dim Y_k) < \delta$. Using Lemma 4.12 on $G'_F(k)$ the claim follows. \hfill $\Box$

**Remark 4.14.** To conclude the paper, we note that a possible deeper understanding of the estimates in Theorems 4.11 and 4.12 may come from the results on exponential sums in [CMN19] and may be related to the motivic oscillation index $\text{moi}(\varphi)$ of $\varphi$. The motivic oscillation index controls the decay rate of the Fourier transform of $\varphi_*(\mu_{\mathcal{O}_F})$ (see [CMN19, Proposition 3.11]). In the non-(FRS) case, optimal bounds on the decay rate were given in [CMN19, Theorem 1.5], proving a conjecture of Igusa on exponential sums [Igu78]. Here it can also be shown that $\text{moi}(\varphi)$ controls the explosion rate of the density of the pushforward measure $\varphi_*(\mu_{\mathcal{O}_F})$ near a critical point (see e.g. [GH, Theorem 8.18]). The (FRS) case of Igusa’s conjecture is open (see the discussion in [CMN19, Section 3.4]), and a potential connection between Theorems 4.11 and the $\text{moi}(\varphi)$ could be interesting in that regard.

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\footnote{Note we may assume $\epsilon$ is a rational number by Remark 2.8}  

\footnote{For the definition in the case that $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^1$ is a polynomial, see [CMN19, Section 3.4].}
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