Cylindrically symmetric non-aligned Einstein–Maxwell solutions with rotation and pseudorotation of the types $O$ and $N$

Héctor Vargas Rodríguez
Department of physics, C.U.C.E.I, Guadalajara University, México.
E-mail: hv_8@yahoo.com, hvargas@udgphys.intranets.com

Abstract. A new solution to the Einstein–Maxwell field equations is presented describing a cylindrically symmetric homogeneous cosmology. The solution is conformally flat, it possesses seven Killing vectors of which the timelike one is rotating and one of the spacelike, pseudorotating. Our solution also admits a Kerr-Schild form. It is alternatively produced by different electromagnetic sources some of which represent constant null electromagnetic fields, while the others, a circularly polarized plane electromagnetic wave (seemingly, a unique situation in general relativity). The concrete electromagnetic four-potentials are found from the assumption that they are proportional to the Killing covectors. The general solution is obtained for timelike and null geodesics. Finally, we find that this space-time admits closed timelike non-geodesic lines.

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1. Introduction

A stationary space-time is that which admits a temporal Killing vector $\xi_{[1]} = \partial_t$, besides it is said to be cylindrically symmetric if it is axisymmetric about an infinite axis and translationally invariant along that axis (here the $z$ axis); i.e. the space-time also admits the spatial Killing vectors $\xi_{[2]} = \partial_z$, $\xi_{[3]} = \partial_\varphi$. These three Killing vectors form an Abelian group $G_3$. We suppose that both $\xi_{[1]}$ and $\xi_{[2]}$ are not hypersurface–orthogonal, 

$$\ast(\xi_{[1]} \wedge d\xi_{[1]}) \neq 0, \quad \ast(\xi_{[2]} \wedge d\xi_{[2]}) \neq 0;$$

of course this means that the temporal Killing vector $\xi_{[1]}$ is rotating, however since $\xi_{[2]}$ is a spatial vector, we shall say that it is ‘pseudorotating’ and that the corresponding space-time is ‘pseudostationary’ in this sense. Hence the line element becomes (we suppose that in rotation and pseudorotation one and the same function, $f$, is present)

$$ds^2 = e^{2\alpha}(dt + f d\varphi)^2 - e^{2\beta} d\rho^2 - e^{2\delta} d\varphi^2 - e^{2\alpha}(dz + f d\varphi)^2,$$
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which depends only on \( \rho \); although below we first shall use Cartesian-like coordinates, later the cylindrical-like coordinates are to be revisited. One can find in the literature examples of cylindrically symmetric Einstein-Maxwell fields with rotation, like that of Wilson; or with pseudorotation, like that of Chitre, see for example Kramer et al [5]; but there are no examples possessing the both properties at once.

In this paper we obtain an exact solution of the self-consistent Einstein–Maxwell equations having the both properties, rotation and pseudorotation as well. Here a nonstandard approach in solving the Einstein–Maxwell equations is followed: Once the solution to Einstein’s equations is known, one proceeds to find all its Killing vectors. But then we propose the ansatz that at least one of the Killing (co)vectors is proportional to the exact electromagnetic four-potential (co)vector of this problem. It is revealed that four of the seven Killing vectors in fact satisfy the sourceless Einstein–Maxwell equations. The ansatz is inspired by a conjecture formulated by Horský and Mitskievich, see [3],[4]. This conjecture is a generalization of the fact that the timelike Killing covectors of the Schwarzschild and Kerr solutions coincide (up to a constant factor) with the electromagnetic four-potentials of the Reissner–Nordström and Kerr–Newman fields respectively. The conjecture states that the electromagnetic four-potential covector of a stationary self-consistent Einstein–Maxwell field is simultaneously proportional (up to a constant factor) to the Killing covector of the corresponding vacuum space-time when the parameter connected with the electromagnetic field of the self consistent problem is set equal to zero, this parameter coinciding with the aforementioned constant factor (see Cataldo et al [1]).

In Section 2, we obtain the solution in both Cartesian-like and cylindrical-like coordinates, a natural tetrad basis being introduced for either form of the metric, and the coordinate transformation connecting the both forms are explicitly given.

In Section 3, the properties of our solution are outlined, i.e. that it is of the Petrov type O, moreover it possesses an isometry group \( G_7 \), in addition it accepts a Kerr-Schild form; besides, its electromagnetic sources alternatively correspond to two different kinds: either constant orthogonal electric and magnetic fields, or a left circularly polarized plane electromagnetic wave [in our space-time, these two distintic alternatives lead to one and the same stress-energy tensor (5); this situation seems to be a unique in the electromagnetic theory]. In all these cases the electromagnetic fields are non aligned, i.e. they posses other symmetries than that of the metric field (which naturally coincides with the symmetry of the electromagnetic stress-energy tensor). Also the motion of test particles and photons in this space-time is studied. Finally, it is shown that closed timelike lines exist in this space-time.

In Section 4, not only the concluding remarks are given, but an exact superposition of our solution and the ‘anti-Mach’ Oszváth-Schücking solution is also presented. These both results (our solution and its superposition with the ‘anti-Mach’ one) are counter-examples to some widely accepted views ‘prohibiting’ such types of exact solutions.

We denote derivative with respect to the coordinate \( x \) by a prime, and use the units in which \( c = 1 \). Four-dimensional indices are Greek ones, the space-time signature
being \((+-+-)\), the Ricci tensor \(R_{\mu\nu} := R^\lambda_{\mu\nu\lambda}\). The symbol \(*\) denotes the Hodge star as well as the Levi-Civita dual conjugation (when it is written under or over a pair of corresponding tensor indices).

2. A stationary and pseudostationary solution

Consider the line element (2),

\[ ds^2 = e^{2\alpha}(dt + f dy)^2 - e^{2\beta}dx^2 - e^{2\delta}dy^2 - e^{2\alpha}(dz + f dy)^2, \]

where \(\alpha, \beta, \delta\) and \(f\) are functions of \(x\) only. We introduce the natural orthonormalized basis related to this metric, its covariant and contravariant forms being respectively

\[ \theta^{(0)} = e^\alpha(dt + f dy), \quad \theta^{(1)} = e^\beta dx, \quad \theta^{(2)} = e^\delta dy, \quad \theta^{(3)} = e^\alpha(dz + f dy), \]

and

\[ X_0 = e^{-\alpha} \partial_t, \quad X_1 = e^{-\beta} \partial_x, \quad X_2 = e^{-\delta} (\partial_y - f \partial_t - f \partial_z), \quad X_3 = e^{-\alpha} \partial_z. \]

Making use of Cartan’s structure equations, it is easy to calculate the Riemann tensor components and then those of the Ricci curvature:

\[ R_{(0)(0)} = - \left( \alpha' e^{2\alpha} - \beta' e^{-\beta} - \gamma' e^{-\gamma} - \delta' e^{-\delta} - \right) \frac{1}{2} f'^2 e^{2(\alpha - \beta - \delta)}, \]

\[ R_{(1)(1)} = 2 \left( \alpha' e^{\alpha - \beta} + \gamma' e^{\gamma - \delta} + \right) e^{-\beta}, \]

\[ R_{(2)(2)} = \left( \delta' e^{\beta + \delta} - \right) e^{-\beta}, \]

\[ R_{(3)(3)} = \left( \alpha' e^{2\alpha - \beta + \delta} - \right) e^{-2\alpha - \beta - \delta}, \]

\[ R_{(0)(2)} = - R_{(2)(0)} = - \frac{1}{2} \left( f' e^{4\alpha - \beta - \delta} \right) e^{-3\alpha - \beta}, \]

\[ R_{(0)(3)} = \frac{1}{2} f'^2 e^{2(\alpha - \beta - \delta)}, \]

(all other Ricci components are equal to zero);

\[ \frac{1}{2} R = - \left( \alpha' e^{2\alpha - \beta + \delta} - \right) e^{-2\alpha - \beta - \delta} - \left( \alpha' e^{\alpha - \beta} - \right) e^{-\alpha - \beta} - \left( \delta' e^{\alpha - \beta + \delta} \right) e^{-\alpha - \beta - \delta}. \]

Then, the Einstein tensor components are

\[ G_{(0)(0)} = \left( \alpha' e^{\alpha - \beta} \right) e^{-\alpha - \beta} + \left( \delta' e^{\delta - \beta} \right) e^{-\delta - \beta} + \alpha' \delta' e^{-2\beta} - \frac{1}{2} f'^2 e^{2(\alpha - \beta - \delta)}, \]

\[ G_{(1)(1)} = -\alpha' (\alpha' + 2 \beta') e^{-2\beta}, \]

\[ G_{(2)(2)} = -2 \left( \alpha' e^{\alpha - \beta} \right) e^{-\alpha - \beta} - \alpha'^2 e^{-2\beta}, \]

\[ G_{(3)(3)} = - \left( \alpha' e^{\alpha - \beta} \right) e^{-\alpha - \beta} - \left( \delta' e^{\alpha - \beta + \delta} \right) e^{-\alpha - \beta - \delta} - \frac{1}{2} f'^2 e^{2(\alpha - \beta - \delta)}, \]

\[ G_{(0)(3)} = R_{(0)(3)}, \quad G_{(0)(2)} = -G_{(2)(0)} = R_{(0)(2)}. \]
We shall consider the gravitational field with electromagnetic sources, so that the scalar curvature vanishes, \( R = 0 \). Assuming that \( G_{(1)(1)} = G_{(2)(2)} = G_{(0)(2)} = -G_{(2)(3)} = 0 \), one sees immediately that \( G_{(1)(1)} = G_{(2)(2)} = 0 \) are satisfied if \( \alpha = 0 \). Using the freedom on the choice of the \( x \) coordinate we put \( \beta = 0 \). To satisfy \( R = 0 \), we have two choices, the first one, \( \delta = 0 \), takes us to Cartesian-like coordinates, and the second one, \( \delta = \ln(x) \), to cylindrical-like coordinates, hence reinterpreting the coordinates \( t, x, y, z \) as \( \tilde{t}, \rho, \phi, z \). The both result in the same solution. Finally, \( G_{(0)(2)} = -G_{(2)(3)} = 0 \) is fulfilled in the first case if \( f(x) = Cx \), and in the second case if \( f(\rho) = \frac{C}{2}\rho^2 \). Thus the solution in Cartesian-like coordinates is

\[
 ds^2 = (dt + Cxdy)^2 - dx^2 - dy^2 - (dz + Cxdy)^2, \tag{3}
\]

with the natural tetrad

\[
 \theta^{(0)} = dt + Cxdy, \quad \theta^{(1)} = dx, \quad \theta^{(2)} = dy, \quad \theta^{(3)} = dz + Cxdy.
\]

In cylindrical-like coordinates,

\[
 ds^2 = \left( d\tilde{t} + \frac{C}{2}\rho^2d\varphi \right)^2 - d\rho^2 - \rho^2d\varphi^2 - \left( d\tilde{z} + \frac{C}{2}\rho^2d\varphi \right)^2, \tag{4}
\]

with

\[
 \tilde{\theta}^{(0)} = d\tilde{t} + \frac{C}{2}\rho^2d\varphi, \quad \tilde{\theta}^{(1)} = d\rho, \quad \tilde{\theta}^{(2)} = \rho d\varphi, \quad \tilde{\theta}^{(3)} = d\tilde{z} + \frac{C}{2}\rho^2d\varphi.
\]

The both forms are related via the transformation

\[
 t = \tilde{t} - \frac{C}{2}xy, \quad z = \tilde{z} - \frac{C}{2}xy, \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi.
\]

In the both cases the electromagnetic energy-momentum tensor (EEMT) is

\[
 T = -\frac{1}{\kappa}G = \frac{C^2}{2\kappa} (\tilde{\theta}^{(0)} - \tilde{\theta}^{(3)}) \otimes (\tilde{\theta}^{(0)} - \tilde{\theta}^{(3)}) = \frac{C^2}{2\kappa} (dt - dz) \otimes (dt - dz) =
\]

\[
 = \frac{C^2}{2\kappa} (\tilde{\theta}^{(0)} - \tilde{\theta}^{(3)}) \otimes (\tilde{\theta}^{(0)} - \tilde{\theta}^{(3)}) = \frac{C^2}{2\kappa} (d\tilde{t} - d\tilde{z}) \otimes (d\tilde{t} - d\tilde{z}), \tag{5}
\]

\( \kappa \) being the gravitational constant. The nature of the electromagnetic sources is discussed in the subsection 3.2. Here it is worth being mentioned that (5) has the canonical structure for a null electromagnetic field (cf. Synge [10]).

### 3. Properties of the new solution

#### 3.1. Geometry

**3.1.1. Petrov type.** A calculation immediately shows that in the space-time (3), (4) the Weyl tensor vanishes, thus it is of the Petrov type \( O \). Of course this means that the metric is conformally flat and that the (Ricci-flat) Riemann tensor could be expressed exclusively in terms of the sources

\[
 R_{(\alpha)(\beta)(\gamma)(\delta)} = \kappa \left( T_{(\alpha)(\gamma)}\eta(\delta)(\beta) - T_{(\beta)(\gamma)}\eta(\delta)(\alpha) \right), \tag{6}
\]
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3.1.2. Killing vectors. Our solution admits isometry group $G_7$, the corresponding Killing vectors in contravariant form using Cartesian-like coordinates are

$$
\xi[1] = \partial_t, \\
\xi[2] = \partial_z, \\
\xi[3] = \frac{C}{2}(y^2 - x^2)(\partial_t + \partial_z) - y\partial_x + x\partial_y, \\
\xi[4] = \partial_y, \\
\xi[5] = Cy(\partial_t + \partial_z) - \partial_x, \\
\xi[6] = Cx\sin[C(t - z)](\partial_t + \partial_z) - \cos[C(t - z)]\partial_x - \sin[C(t - z)]\partial_y, \\
\xi[7] = -Cx\cos[C(t - z)](\partial_t + \partial_z) - \sin[C(t - z)]\partial_x + \cos[C(t - z)]\partial_y,
$$

and the covariant ones,

$$
\xi[1] = \partial_t + Cx\partial_y, \\
\xi[2] = -\partial_z - Cx\partial_y, \\
\xi[3] = \frac{C}{2}(x^2 + y^2)(\partial_t - \partial_z) + y\partial_x - x\partial_y, \\
\xi[4] = Cx(\partial_t - \partial_z) - \partial_y, \\
\xi[5] = Cy(\partial_t - \partial_z) + \partial_x, \\
\xi[6] = \cos[C(t - z)]dx + \sin[C(t - z)]dy, \\
\xi[7] = \sin[C(t - z)]dx - \cos[C(t - z)]dy.
$$

The corresponding contravariant forms in cylindrical-like coordinates are

$$
\xi[1] = \partial_{\tilde{t}}, \\
\xi[2] = \partial_{\tilde{z}}, \\
\xi[3] = \partial_{\varphi}, \\
\xi[4] = \frac{C}{2}\rho\cos(\varphi)(\partial_{\tilde{t}} + \partial_{\tilde{z}}) + \sin(\varphi)\partial_{\rho} - \frac{1}{\rho}\cos(\varphi)\partial_{\varphi}, \\
\xi[5] = \frac{C}{2}\rho\sin(\varphi)(\partial_{\tilde{t}} + \partial_{\tilde{z}}) - \cos(\varphi)\partial_{\rho} + \frac{1}{\rho}\sin(\varphi)\partial_{\varphi}, \\
\xi[6] = -\sin[C(\tilde{t} - \tilde{z}) - \varphi]\left(\frac{1}{2}C\rho(\partial_{\tilde{t}} + \partial_{\tilde{z}}) + \frac{1}{\rho}\partial_{\varphi}\right) - \cos[C(\tilde{t} - \tilde{z}) - \varphi]\partial_{\rho}, \\
\xi[7] = -\cos[C(\tilde{t} - \tilde{z}) - \varphi]\left(\frac{1}{2}C\rho(\partial_{\tilde{t}} + \partial_{\tilde{z}}) - \frac{1}{\rho}\partial_{\varphi}\right) - \sin[C(\tilde{t} - \tilde{z}) - \varphi]\partial_{\rho},
$$
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and the covariant forms,
\[ \xi_{[1]} = d\tilde{t} + \frac{C}{2} \rho^2 d\varphi, \]
\[ \xi_{[2]} = -d\tilde{z} - \frac{C}{2} \rho^2 d\varphi, \]
\[ \xi_{[3]} = \frac{C}{2} \rho^2 (d\tilde{t} - d\tilde{z}) - \rho^2 d\varphi, \]
\[ \xi_{[4]} = C \rho \cos(\varphi)(d\tilde{t} - d\tilde{z}) - \sin(\varphi) d\rho - \rho \cos(\varphi) d\varphi, \]
\[ \xi_{[5]} = C \rho \sin(\varphi)(d\tilde{t} - d\tilde{z}) + \cos(\varphi) d\rho - \rho \sin(\varphi) d\varphi, \]
\[ \xi_{[6]} = \cos[C(\tilde{t} - \tilde{z}) - \varphi] d\rho + \rho \sin[C(\tilde{t} - \tilde{z}) - \varphi] d\varphi, \]
\[ \xi_{[7]} = \sin[C(\tilde{t} - \tilde{z}) - \varphi] d\rho - \rho \cos[C(\tilde{t} - \tilde{z}) - \varphi] d\varphi. \]

Contravariant Killing vectors have the following non trivial commutators:
\[ [\xi_{[1]}, \xi_{[6]}] = -C \xi_{[7]}, \quad [\xi_{[3]}, \xi_{[5]}] = -C \xi_{[4]}, \]
\[ [\xi_{[1]}, \xi_{[7]}] = C \xi_{[6]}, \quad [\xi_{[3]}, \xi_{[6]}] = C \xi_{[7]}, \]
\[ [\xi_{[2]}, \xi_{[6]}] = C \xi_{[7]}, \quad [\xi_{[3]}, \xi_{[7]}] = -C \xi_{[6]}, \]
\[ [\xi_{[2]}, \xi_{[7]}] = -C \xi_{[6]}, \quad [\xi_{[4]}, \xi_{[5]}] = C(\xi_{[1]} + \xi_{[2]}), \]
\[ [\xi_{[3]}, \xi_{[4]}] = -C \xi_{[5]}, \quad [\xi_{[6]}, \xi_{[7]}] = C(\xi_{[1]} + \xi_{[2]}). \]

Only \( \xi_{[1]} \) is timelike and all others are space-like,
\[ \xi_{[1]} \cdot \xi_{[1]} = 1, \quad \xi_{[3]} \cdot \xi_{[3]} = -\rho^2, \]
\[ \xi_{[2]} \cdot \xi_{[2]} = \xi_{[4]} \cdot \xi_{[4]} = \xi_{[5]} \cdot \xi_{[5]} = \xi_{[6]} \cdot \xi_{[6]} = \xi_{[7]} \cdot \xi_{[7]} = -1, \]

Finally we could see that \( \xi_{[1]} \) is rotating and \( \xi_{[2]} \) is pseudorotating,
\[ \omega = \frac{1}{2} * (\xi_{[1]} \wedge d\xi_{[1]}) = \frac{C}{2} \tilde{\theta}^{(3)}, \quad \varpi = \frac{1}{2} * (\xi_{[2]} \wedge d\xi_{[2]}) = \frac{C'}{2} \tilde{\theta}^{(0)}, \]

here \( \omega \) being the usual space-time angular velocity which is directed along the negative \( z \) axis, \( \varpi \) is the space-time pseudo-angular-velocity (introduced as an analogue of \( \omega \)), it is directed along the \( t \) axis; for the both, it could be seen that if \( C \) changes its sign, they change their directions.

It is interesting that \( ds^2 \) can be written exclusively in terms of the Killing vectors, for example (4) is equivalent to
\[ ds^2 = \xi_{[1]} \xi_{[1]} - \xi_{[2]} \xi_{[2]} - \xi_{[6]} \xi_{[6]} - \xi_{[7]} \xi_{[7]} \] (7)
3.1.3. Kerr-Schild form of the metric

If the metric (4) is written as
\[ ds^2 = d\tilde{t}^2 - d\rho^2 - d\tilde{z}^2 + \rho^2 d[C(\tilde{t} - \tilde{z}) - \varphi]d\varphi \]
and one performs the change
\[ \varphi = \tilde{\varphi} + \frac{C}{2}(\tilde{t} - \tilde{z}), \]
one arrives to the Kerr–Schild form
\[ ds^2 = d\tilde{t}^2 - d\rho^2 - C^2 4 \rho^2 (d\tilde{t} - d\tilde{z})^2. \] (8)

It is easily checked that \( d\tilde{t} - d\tilde{z} \) is a null (co)vector from the viewpoint of the both Minkowski and the stationary and pseudostationary metric (8), and that this vector is geodesic. Contrary to the Kerr congruence this congruence does not rotate; this may suggest that not all stationary space-times correspond to null rotating congruences or that the pseudorotation somehow compensates the rotation of the null congruence.

3.2. The electromagnetic field

Modifying the conjecture proposed by Horský and Mitskievich, we shall now look for electromagnetic four-potentials proportional to the Killing covectors (below their Cartesian forms being used),
\[ A = k\xi \] (9)
(the electromagnetic field tensor is its exterior derivative, \( F = dA \)) and check if they would satisfy the vacuum Maxwell equations, in our space-time,
\[ (\sqrt{-g}F^{\mu\nu})_{,\nu} = 0, \]
and lead to one and the same form of the EEMT (5). This will put a constraint on the proportionality constant \( k \). It is found that four of these Killings \( \xi[4], \xi[5], \xi[6], \xi[7] \), and the linear superposition \( \xi[4] \pm \xi[5] \), fulfill the above requirements. So we have five cases. [In fact, \( \xi[3] \) satisfies the vacuum Maxwell equations, but it yields the EEMT different from (5), thus this candidate of the four-potential describes a merely test electromagnetic field in our space-time. However it could serve as the first step in a further generalization of our solution.]

In order to describe the electric and magnetic field vectors we shall introduce a reference frame described by the monad (see Mitskievich [9]),
\[ \tau = \theta^{(0)} = dt + Cxdy. \] (10)
This reference frame is rotating
\[ \omega = \frac{1}{2} * (\tau \wedge d\tau) = \frac{C}{2} \theta^{(3)}. \] (11)
but has neither acceleration
\[ G = - * (\tau \wedge *d\tau) = 0, \] (12)
nor expansion and shear since the rate-of-strain tensor vanishes
\[ D_{\mu\nu} = \frac{1}{2} \mathcal{L}_\tau b_{\mu\nu} = 0, \quad (13) \]
here [b_{\mu\nu}] = g_{\mu\nu} - n_\mu n_\nu is the three metric in the local subspace orthogonal to the monad
and \( \mathcal{L}_\tau \) denotes the Lie derivative with respect to the monad which coincides with \( \xi^{[1]} \).

With respect to this reference frame we split the electromagnetic field tensor in the electric and magnetic (co)vectors
\[ E = * (\tau \wedge * F), \quad B = * (\tau \wedge F). \quad (14) \]
the electromagnetic momentum density (Poynting covector, \( c = 1 \)) being (cf. [9])
\[ S = \frac{1}{4\pi} * (E \wedge \tau \wedge B). \quad (15) \]
In all the five cases below it is found that the Poynting covector is
\[ S = -\frac{C^2}{2\pi} \theta^{(3)}, \quad (16) \]
as it could be seen from the EEMT; it is directed along the positive \( z \) axis and does not depend on the sign of \( C \). Also for the all five cases below
\[ F^{(\mu)(\nu)} F^{(\mu)(\nu)} = 0, \quad F^{(\mu)(\nu)} F^{(\mu)(\nu)} * = 0. \]
Since these both electromagnetic field invariants vanish, we come to the pure null type electromagnetic field.

The first three cases, \( \xi^{[4]}, \xi^{[5]} \) and \( \xi^{[4]} \pm \xi^{[5]} \) correspond to a constant orthogonal electric and magnetic fields, while the last two cases, \( \xi^{[6]} \) and \( \xi^{[7]} \), correspond to a left circularly polarized (positive helicity) plane electromagnetic wave propagating in the direction of the positive \( z \) axis. Hence the plane electromagnetic wave has its spin angular momentum in an opposite direction to that of the angular velocity of the space-time. If \( C \) changes its sign, then the plane electromagnetic wave acquires negative helicity and the relative situation continues to be as before.

In the following subsections, the reader will see that the alternative variants of electromagnetic field are always non aligned with the space-time geometry, though the (one and the same) stress-energy tensor does exactly correspond to the space-time symmetry.

3.2.1. The fourth Killing vector case
The electromagnetic four-potential and field tensor are
\[ A^{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)} (dt - dz), \quad F^{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)} \wedge \left( \theta^{(0)} - \theta^{(3)} \right). \]
with the corresponding electric and magnetic covectors
\[ E^{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)}, \quad B^{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(2)}. \]
3.2.2. The fifth Killing vector case

\[ A_{[5]} = \sqrt{\frac{2\pi}{\kappa}} C y (dt - dz), \quad F_{[5]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(2)} \land \left( \theta^{(0)} - \theta^{(3)} \right). \]

\[ E_{[5]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(2)}, \quad B_{[5]} = -\sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)}. \]

3.2.3. A linear superposition of the fourth and fifth Killing vector cases

\[ A_{[4,5]} = \sqrt{\frac{2\pi}{\kappa}} C (x \pm y) (dt - dz), \]

\[ F_{[4,5]} = \sqrt{\frac{2\pi}{\kappa}} C (\theta^{(1)} \pm \theta^{(2)}) \land \left( \theta^{(0)} - \theta^{(3)} \right). \]

\[ E_{[4,5]} = \sqrt{\frac{2\pi}{\kappa}} C (\theta^{(1)} \pm \theta^{(2)}), \quad B_{[4,5]} = \sqrt{\frac{2\pi}{\kappa}} C (\theta^{(2)} \mp \theta^{(1)}). \]

3.2.4. The sixth Killing vector case

\[ A_{[6]} = \sqrt{\frac{2\pi}{\kappa}} \left\{ \cos[C(t - z)] dx + \sin[C(t - z)] dy \right\}, \]

\[ F_{[6]} = \sqrt{\frac{2\pi}{\kappa}} C \left\{ \sin[C(t - z)] \theta^{(1)} - \cos[C(t - z)] \theta^{(2)} \right\} \land \left( \theta^{(0)} - \theta^{(3)} \right), \]

\[ E_{[6]} = \sqrt{\frac{2\pi}{\kappa}} C \left\{ \sin[C(t - z)] \theta^{(1)} - \cos[C(t - z)] \theta^{(2)} \right\}, \]

\[ B_{[6]} = -\sqrt{\frac{2\pi}{\kappa}} C \left\{ \cos[C(t - z)] \theta^{(1)} + \sin[C(t - z)] \theta^{(2)} \right\}. \]

3.2.5. The seventh Killing vector case  This case could be obtained from the later, since if one changes, for example, \( z \) by \( z + \pi/(2C) \) in \( A_{[6]} \), it becomes \( A_{[7]} \). We come to the electromagnetic four-potential

\[ A_{[7]} = \sqrt{\frac{2\pi}{\kappa}} \left\{ \sin[C(t - z)] dx - \cos[C(t - z)] dy \right\} \]

and the field tensor

\[ F_{[7]} = \sqrt{\frac{2\pi}{\kappa}} C \left( \theta^{(0)} - \theta^{(3)} \right) \land \left\{ \cos[C(t - z)] \theta^{(1)} + \sin[C(t - z)] \theta^{(2)} \right\}, \]

while the corresponding electric and magnetic covectors are

\[ E_{[7]} = -\sqrt{\frac{2\pi}{\kappa}} C \left\{ \cos[C(t - z)] \theta^{(1)} + \sin[C(t - z)] \theta^{(2)} \right\}, \]

\[ B_{[7]} = \sqrt{\frac{2\pi}{\kappa}} C \left\{ \sin[C(t - z)] \theta^{(1)} - \cos[C(t - z)] \theta^{(2)} \right\}. \]

When \( C > 0 \), their contravariant forms are interpreted as pertaining to a left circularly polarized electromagnetic wave (a wave with positive helicity).
3.3. Timelike and null geodesics

From the geodesic line equation
\[ \frac{d}{d\lambda} \left( g^{\mu\nu} \frac{dx^\nu}{d\lambda} \right) = \frac{1}{2} g^{\alpha\beta,\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}, \]
and the metric (3), the first integrals follow
\[ \mu = 0 : \quad \frac{dt}{d\lambda} + Cx \frac{dy}{d\lambda} = K_0, \]
\[ \mu = 2 : \quad Cx \left( \frac{dt}{d\lambda} - \frac{dz}{d\lambda} \right) - \frac{dy}{d\lambda} = K_2, \]
\[ \mu = 3 : \quad \frac{dz}{d\lambda} + Cx \frac{dy}{d\lambda} = K_3. \]

Alongside with the equation
\[ \mu = 1 : \quad \frac{d^2x}{d\lambda^2} + C \left( \frac{dt}{d\lambda} - \frac{dz}{d\lambda} \right) \frac{dy}{d\lambda} = 0. \]

We obtain the solution to the above equations, and then insert it in \( ds^2 \),
\[ \left( \frac{ds}{d\lambda} \right)^2 = \left( \frac{dt}{d\lambda} + Cx \frac{dy}{d\lambda} \right)^2 - \left( \frac{dx}{d\lambda} \right)^2 - \left( \frac{dz}{d\lambda} + Cx \frac{dy}{d\lambda} \right)^2 - \left( \frac{dy}{d\lambda} \right)^2 = \eta, \]
since it is already a first integral of the equations of motion. Here we take \( \eta = 1 \) for timelike geodesics (which describe the motion of test massive particles) or \( \eta = 0 \) for null geodesics (which describe the motion of test photons), so that timelike and null geodesics have the following parametric expressions:
\[
\begin{align*}
    t(\lambda) & = \left[ \eta + (K_0 - K_3)^2 \right]^{\frac{1}{2}} \lambda - \frac{K_0^2 - K_3^2 - \eta}{4C(K_0 - K_3)^2} \sin[2C(K_0 - K_3)\lambda], \\
    x(\lambda) & = \sqrt{\frac{K_0^2 - K_3^2 - \eta}{C^2(K_0 - K_3)^2}} \cos[C(K_0 - K_3)\lambda], \\
    y(\lambda) & = \sqrt{\frac{K_0^2 - K_3^2 - \eta}{C^2(K_0 - K_3)^2}} \sin[C(K_0 - K_3)\lambda], \\
    z(\lambda) & = \left[ \eta - (K_0 - K_3)^2 \right]^{\frac{1}{2}} \lambda - \frac{K_0^2 - K_3^2 - \eta}{4C(K_0 - K_3)^2} \sin[2C(K_0 - K_3)\lambda]; \\
\end{align*}
\]

Here \( K_0 \) is always positive since it is usually interpreted as energy per unit rest mass, in fact \( K_0 \geq \sqrt{\eta + K_3^2} \). For test massive particles the canonical parameter \( \lambda \) is interpreted as the proper time along the particle’s world line. Some of the integration constants were put equal to zero by performing a translation in \( \lambda \), or by choosing the origin of the coordinates.

For test massive particles, \( \eta = 1 \), we could see that they could be at rest on the \( z \)-axis, or moving along it, but if a test massive particle is outside the \( z \)-axis, it revolves around the \( z \)-axis against the space-time rotation while this particle is travelling along the \( z \)-axis (when \( C > 0 \)).

For test photons, \( \eta = 0 \), we could see that they can travel only in the negative \( z \)-axis direction, and they revolve against the space-time rotation (when \( C > 0 \)).
3.4. The closed-circuit journey along a timelike line

In another form, the metric \( ds^2 \) reads

\[
\left( d\tilde{t} - d\tilde{z} \right) \left( d\tilde{t} + d\tilde{z} + C\rho^2 d\varphi \right) - d\rho^2 - \rho^2 d\phi^2.
\]

We divide the closed-circuit journey into two parts, the both corresponding to \( d\rho = 0 \):

\[
(A) \quad \left. \frac{d\tilde{t}}{ds} \right|_A = -\alpha < 0, \quad \left. \frac{d\tilde{z}}{ds} \right|_A = \beta > 0, \quad \left. \frac{d\varphi}{ds} \right|_A = \gamma \neq 0,
\]

and

\[
(B) \quad \left. \frac{d\tilde{t}}{ds} \right|_B = a > 0, \quad \left. \frac{d\tilde{z}}{ds} \right|_B = -b < 0, \quad \left. \frac{d\varphi}{ds} \right|_B = 0.
\]

At any stage, the constants \( \alpha, \beta, \gamma, a \) and \( b \) are subject to the constraint

\[
\left( \frac{d\tilde{t}}{ds} - \frac{d\tilde{z}}{ds} \right) \left( \frac{d\tilde{t}}{ds} + \frac{d\tilde{z}}{ds} + C\rho^2 d\varphi \right) - \rho^2 \left( \frac{d\varphi}{ds} \right)^2 = 1.
\]

All this guarantees that the parameter \( s \) grows monotonically along the whole circuit, while the time coordinate changes its behaviour from diminishing in \( A \) to growth in \( B \). In fact, in \( A \) we have

\[
(\alpha + \beta)(\alpha - \beta - C\rho^2\gamma) - \rho^2\gamma^2 = 1
\]

and in \( B \), simply \( a^2 - b^2 = 1 \). The last relation is equivalent to \( a = \cosh \psi, \ b = \sinh \psi \).

Let us also suppose that

\[
\int_A d\tilde{t} + \int_B d\tilde{t} = 0, \quad \int_A d\tilde{z} + \int_B d\tilde{z} = 0, \quad \int_A d\varphi = -2\pi
\]

respectively,

\[
-\alpha s_A + a s_B = 0, \quad \beta s_A - b s_B = 0, \quad \gamma s_A = -2\pi,
\]

\( s_A \) and \( s_B \) being the proper-time durations of the two parts of journey, the minus sign in the \( \varphi \) integral guarantees that \( s_A > 0 \).

When \( s_A = s_B \), then \( \alpha = a \) and \( \beta = b \), and (17) yields

\[
\gamma = -C(\alpha + \beta) = -Ce^\psi.
\]

It is interesting to observe that the sign of \( \gamma \) is automatically the opposite to that of \( C \) (the azimuthal motion follows the space-time rotation, and this motion is clearly non-geodesic).

From the last integral in (18), we found the proper-time durations

\[
s_A = s_B = \frac{2\pi}{C} e^{-\psi}.
\]
Cylindrically symmetric non-aligned Einstein–Maxwell solutions

4. Concluding remarks

One could find in Kramer *et al* [5], in table 33.5, a conclusion that conformally flat solutions with non-aligned null electromagnetic fields do not exist. The solution presented above is however a counterexample. This conclusion in [5] is based on the theorem 7.4 (in the same book), if the electromagnetic null field is supposed to be only a test one (as this usually occurs in the case when the four-potential is simultaneously, up to a constant factor, a Killing vector of the space-time under consideration). But we have shown that here we have a self-consistent solution of the Einstein-Maxwell equations. See also in [5] the theorem 28.7 which should correspond to principally non-existing solutions according to the same table 33.5.

The abundance of isometries in our solution suggests to consider it as describing an electromagnetic homogeneous cosmological model. This model is stationary (and pseudostationary), and its curvature (the Riemann tensor) completely reduces to the electromagnetic stress-energy tensor [see (6)]: one may say that there is no free (intrinsic) gravitational field in this space-time. The overall simplicity of this solution permits to easily perform a complete integration of the geodesic equation in its space-time in the general case. What is even stranger, there are several equally good candidates for the concrete electromagnetic fields as sources of its space-time geometry (the reader would willingly accept the ‘candidates’ mutually related by a dual conjugation or dual rotation which do not change the electromagnetic stress-energy tensor at all, but here we have so different fields as, for example, a plane monochromatic electromagnetic wave with the frequency $C$, and a pair of time-independent electric and magnetic fields, a zero-frequency wave, if one would wish to call it so). It is worth mentioning that the four-potentials of these fields are also the Killing vectors of this space-time. We have also seen that four of the seven Killing vectors can be used in a very simple representation of our solution (7).

Like some other space-times, our solution permits existence of closed timelike (non-geodesic) paths (*cf.* this property of the Gödel cosmological solution [2]).

It is easy to see that the Oszváth–Schücking ‘anti-Mach’ solution (10.13), written in non-null coordinates, in [5] permits an exact superposition with our solution. In the Kerr–Schild form, this superposition is simply

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2 - (H + \frac{C^2}{4\rho^2})(dt - dz)^2,$$

with

$$H(t, \rho, \varphi, z) = A\rho^2 \cos(2B(t - z) + 2\varphi).$$

Our expressions for the electromagnetic field and the Ricci tensor do not change their forms in the new orthonormal basis, the Petrov type is now $N$. The possibility of such superposition is based on the fact that the both fields, that of the Oszváth–Schücking wave and our electromagnetic field with its Poynting vector, propagate in one and the same direction with the fundamental velocity $c = 1$, thus without any
interaction on the Minkowski background of the Kerr–Schild metric (cf [6,7]); the same property of no interaction takes place for any objects (both particles and fields, not only electromagnetic ones) with the same type of propagation.

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