ON SMOOTHABLE SURGERY FOR 4-MANIFOLDS

QAYUM KHAN

Abstract. Under certain homological hypotheses on a compact 4-manifold, we prove exactness of the topological surgery sequence at the stably smoothable normal invariants. The main examples are the class of finite connected sums of 4-manifolds with certain product geometries. Most of these compact manifolds have non-vanishing second mod 2 homology and have fundamental groups of exponential growth, which are not known to be tractable by Freedman-Quinn topological surgery. Necessarily, the +-construction of certain non-smoothable homotopy equivalences requires surgery on topologically embedded 2-spheres and is not attacked here by transversality and cobordism.

Contents

1. Introduction 1
2. Smoothing normal bordisms 5
3. Homotopy self-equivalences 6
4. Smoothable surgery for 4-manifolds 9
References 17

1. Introduction

1.1. Objectives. The main theorem of this paper is a limited form of the surgery exact sequence for compact 4-manifolds (Theorem 1.1). Corollaries include exactness at the smooth normal invariants of the 4-torus $T^4$ (Example 1.4) and the real projective 4-space $\mathbb{R}P^4$ (Corollary 1.7). C.T.C. Wall proved an even more limited form of the surgery exact sequence [Wa99, Theorem 16.6] and remarked that his techniques do not apply to $T^4$ and $\mathbb{R}P^4$. Although our new hypotheses depend on the $L$-theory assembly map, we provide a remedy along essentially the same lines.

1.2. Results. Let $(X, \partial X)$ be a based, compact, connected, topological 4-manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(\tau X) : \pi \to \mathbb{Z}^\times$. The reader is referred to Section 1.4 for an explanation of surgical language.

If $X$ has a preferred smooth structure, consider the following surgery sequence.

\begin{equation}
\mathcal{S}_{\text{DIFF}}(X) \xrightarrow{\eta} \mathcal{N}_{\text{DIFF}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega)
\end{equation}

Otherwise, let $\text{TOP0}$ refer to manifolds with the same smoothing invariant as $X$.

\begin{equation}
\mathcal{S}_{\text{TOP0}}(X) \xrightarrow{\eta} \mathcal{N}_{\text{TOP0}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega)
\end{equation}

The first examples consists of orientable 4-manifolds $X$ with torsion-free, infinite fundamental groups, mostly of exponential growth. These include the 4-torus $T^4$ and connected sums of certain aspherical 4-manifolds of non-positive curvature.
Corollary (4.3). Let $\pi$ be a free product of groups of the form
\[
\pi = \star_{i=1}^{n} \Lambda_i
\]
for some $n > 0$, where each $\Lambda_i$ is a torsion-free lattice in either $\text{Isom}(\mathbb{E}^{m_i})$ or $\text{Isom}(\mathbb{H}^{m_i})$ or $\text{Isom}(\mathbb{C}^{m_i})$ for some $m_i > 0$. Suppose the orientation character $\omega$ is trivial. Then the surgery sequences (1.0.1) and (1.0.2) are exact.

The second examples consist of a generalization $X$ of non-aspherical, orientable, simply-connected 4-manifolds. These include the outcome of smooth surgery on the core circle of the mapping torus of an orientation-preserving self-diffeomorphism of a 3-dimensional lens space $L(p, q)$. The fundamental groups have torsion.

Corollary (4.6). Let $\pi$ be a free product of groups of the form
\[
\pi = \star_{i=1}^{n} \Omega_i
\]
for some $n > 0$, where each $\Omega_i$ is an odd-torsion group. (Necessarily $\omega$ is trivial.) Then the surgery sequences (1.0.1) and (1.0.2) are exact.

The third examples consist of non-aspherical, non-orientable 4-manifolds $X$ whose connected summands are non-orientable with fundamental group of order two. These include the real projective 4-space $\mathbb{RP}^4$.

Corollary (4.7). Suppose $X$ is a DIFF 4-manifold of the form
\[
X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)
\]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is either $S^2 \times \mathbb{RP}^2$ or $S^2 \times \mathbb{RP}^2$ or $\# S^1(n\mathbb{RP}^4)$ for some $1 \leq n \leq 4$. Then the surgery sequences (1.0.1) and (1.0.2) are exact.

The fourth examples consist of orientable 4-manifolds $X$ whose connected summands are mostly aspherical 3-manifold bundles over the circle. The important non-aspherical examples include $\# n(S^3 \times S^1)$ with free fundamental group. The aspherical examples are composed of fibers of a specific type of Haken 3-manifolds.

Corollary (4.8). Suppose $X$ is a TOP 4-manifold of the form
\[
X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)
\]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is the total space of a fiber bundle
\[H_i \to X_i \to S^1.\]
Here, we suppose $H_i$ is a compact, connected 3-manifold such that:

1. $H_i$ is $S^3$ or $D^3$, or
2. $H_i$ is irreducible with non-zero first Betti number.

Moreover, if $H_i$ is non-orientable, we assume that the quotient group $H_i(H_i; \mathbb{Z})_{\alpha_i}$ of coinvariants is 2-torsionfree, where $\alpha_i : H_i \to H_i$ is the monodromy homeomorphism. Then the surgery sequence (1.0.2) is exact.

Finally, the fifth examples consist of possibly non-orientable 4-manifolds $X$ with torsion-free fundamental group. The connected summands are surface bundles over surfaces, most of which are aspherical with fundamental groups of exponential growth. The aspherical, non-orientable examples of subexponential growth include simple torus bundles $T^2 \times Kl$ over the Klein bottle, excluded from Corollary 4.3.
Corollary 1.9. Suppose $X$ is a TOP 4-manifold of the form
\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is the total space of a fiber bundle
\[ \Sigma_i^f \longrightarrow X_i \longrightarrow \Sigma_i^b. \]
Here, we suppose the fiber and base are compact, connected 2-manifolds, $\Sigma_i^f \neq \mathbb{RP}^2$, and $\Sigma_i^b$ has positive genus. Moreover, if $X_i$ is non-orientable, we assume that the fiber $\Sigma_i^f$ is orientable and that the monodromy action of $\pi_1(\Sigma_i^b)$ of the base preserves any orientation on the fiber (i.e. the bundle is simple). Then the surgery sequence (1.0.2) is exact.

1.3. Techniques. Our methods employ various bits of geometric topology: topological transversality in all dimensions (Freedman-Quinn [FQ90]), and the analysis of smooth normal invariants of the Novikov pinching trick, which is used to construct homotopy self-equivalences of 4-manifolds (Cochran-Habegger, Wall [CH90]). Our hypotheses are algebraic-topological in nature and come from the surgery characteristic class formulas of Sullivan-Wall [Wal99] and from the assembly map components of Taylor-Williams [TW79], as well as control of $\pi_2$ in non-orientable cases.

Jonathan Hillman has successfully employed these now standard techniques to classify 4-manifolds, up to s-cobordism, in the homotopy type of certain surface bundles over surfaces [Hil02, Ch. 6]. Along the same lines, our abundant families of 4-manifold examples also have fundamental groups of exponential growth, and so, too, are currently inaccessible by topological surgery [FQ90, FT95, KQ00].

The reader should be aware that the topological transversality used in Section 3 produces 5-dimensional TOP normal bordisms $W \to X \times \Delta^1$ which may not be smoothable, although the boundary $\partial W = \partial_+ W \cup \partial_- W$ is smoothable. In particular, $W$ may not admit a TOP handlebody structure relative to $\partial_- W$. Hence $W$ may not be the trace of surgeries on topologically embedded 2-spheres in $X$. Therefore, in general, $W$ cannot be produced by Freedman-Quinn surgery theory, which has been developed only for a certain class of fundamental groups $\pi_1(X)$ of subexponential growth. In this way, topological cobordism is superior to surgery.

1.4. Language. For any group $\pi$, we shall write $\text{Wh}_0(\pi) := \tilde{K}_0(\mathbb{Z}[\pi])$ for the projective class group and $\text{Wh}_1(\pi) := \tilde{K}_1(\mathbb{Z}[\pi])/\langle \langle \rangle \rangle$ for the Whitehead group.

Let CAT be either the manifold category TOP or PL = DIFF in dimensions $< 7$. Suppose $(X, \partial X)$ is a based, compact, connected CAT 4-manifold. Let us briefly introduce some basic notation used throughout this paper. The fundamental group $\pi = \pi_1(X)$ depends on a choice of basepoint; a basepoint is essential if $X$ is non-orientable. The orientation character $\omega = w_1(X) : \pi \to \mathbb{Z}^\times$ is a homomorphism that assigns $+1$ or $-1$ to a loop $\lambda : S^1 \to X$ if the pullback bundle $\lambda^*(\tau_X)$ is orientable or non-orientable. Recall that any finitely presented group $\pi$ and arbitrary orientation character $\omega$ can be realized on some closed, smooth 4-manifold $X$ by a straightforward surgical construction. A choice of generator $[X] \in H_4(X, \partial X; \mathbb{Z}^\omega)$ is called a twisted orientation class.

Let us introduce the terms in the surgery sequence investigated in Section 3. The simple structure set $S_{\text{CAT}}(X)$ consists of CAT $s$-bordism classes in $\mathbb{R}^\infty$ of simple homotopy equivalences $h : Y \to X$ such that $\partial h : \partial Y \to \partial X$ is the identity. Here, simple means that the torsion of the acyclic $\mathbb{Z}[\pi]$-module complex
Cone($\tilde{h}$) is zero, for some preferred finite homotopy CW-structures on $Y$ and $X$. Indeed, any compact topological manifold $(X, \partial X)$ has a canonical simple homotopy type, obtained by cleanly embedding $X$ into euclidean space \cite[Thm. III.4.1]{KS77}. Therefore, for any abelian group $A$ and $n > 1$, pulling back the inverse of the Hurewicz isomorphism induces a bijection from $[X/\partial X, K(A, n)]_0$ to $H^n(X, \partial X; A)$. This identification shall be used implicitly throughout the paper.

Denote $G_n$ as the topological monoid of homotopy self-equivalences $S^{n-1} \to S^n$, and $G := \text{colim}_n G_n$ as the direct limit of $\{ G_n \to G_{n+1} \}$. The \textbf{normal invariant set} $\mathcal{N}_{\text{CAT}}(X) \cong [X/\partial X, G/\text{CAT}]_0$ consists of CAT normal bordism classes in $\mathbb{R}^\infty$ of degree one. CAT normal maps $f : M \to X$ such that $\partial f : \partial X \to \partial X$ is the identity; we suppress the normal data and define $X/\partial X = X \sqcup \text{pt}$. Denote $f : X/\partial X \to G/\text{CAT}$ as the associated homotopy class of based maps. Indeed, transversality in the TOP category holds for all dimensions and codimensions \cite{KS77, PQ90}. The normal invariants map $\eta : \mathcal{S}_{\text{CAT}}(X) \to \mathcal{N}_{\text{CAT}}(X)$ is a forgetful map. The \textbf{surgery obstruction group} $L^1_4(\mathbb{Z}[[\pi]])$ consists of Witt classes of non-singular quadratic forms over the group ring $\mathbb{Z}[[\pi]]$ with involution ($g \mapsto \omega(g)g^{-1}$). The surgery obstruction map $\sigma^1_4 : \mathcal{N}_{\text{CAT}}(X) \to L^1_4(\mathbb{Z}[[\pi]])$ vanishes on the image of $\eta$. The basepoint of the former two sets is the identity map $1_X : X \to X$, and the basepoint of the latter set is the Witt class 0.

1.5. \textbf{Invariants}. The unique homotopy class of classifying maps $u : X \to B\pi$ of the universal cover induces homomorphisms

$$u_0 : H_0(X; \mathbb{Z}^\omega) \to H_0(\pi; \mathbb{Z}^\omega)$$

$$u_2 : H_2(X; \mathbb{Z}_2) \to H_2(\pi; \mathbb{Z}_2).$$

Next, recall that the manifold $X$ has a second Wu class

$$v_2(X) \in H^2(X; \mathbb{Z}_2) = \text{Hom}(H_2(X; \mathbb{Z}_2), \mathbb{Z}_2)$$

defined for all $a \in H^2(X, \partial X; \mathbb{Z}_2)$ by $\langle v_2(X), a \cap [X] \rangle = \langle a \cup a, [X] \rangle$. This un-oriented cobordism characteristic class is uniquely determined from the Stiefel-Whitney classes of the tangent microbundle $\tau_X$ by the formula

$$v_2(X) = w_1(X) \cup w_1(X) + w_2(X).$$

Observe that $v_2(X)$ vanishes if $X$ is a TOP Spin-manifold.

Finally, let us introduce the relevant surgery characteristic classes. Observe

$$H_0(\pi; \mathbb{Z}^\omega) = \mathbb{Z}/\langle \omega(g) - 1 \mid g \in \pi \rangle = \begin{cases} \mathbb{Z} & \text{if } \omega = 1 \\ \mathbb{Z}_2 & \text{if } \omega \neq 1. \end{cases}$$

The 0th component of the 2-local assembly map $A_{\ast}$ \cite{TW79} has an integral lift

$$I_0 : H_0(\pi; \mathbb{Z}^\omega) \to L^1_4(\mathbb{Z}[[\pi]]).$$

The image $I_0(1)$ equals the Witt class of the $E_8$ quadratic form \cite[Rmk. 3.7]{Dav05}. The 2nd component of the 2-local assembly map $A_{\ast}$ \cite{TW79} has an integral lift

$$\kappa_2 : H_2(\pi; \mathbb{Z}_2) \to L^1_4(\mathbb{Z}[[\pi]]).$$

Let $f : M \to X$ be a degree one, TOP normal map. According to René Thom \cite{Th65a}, every homology class in $H_2(X, \partial X; \mathbb{Z}_2)$ is represented by $g_*[\Sigma]$ for some
compact, possibly non-orientable surface $\Sigma$ and TOP immersion $g : (\Sigma, \partial \Sigma) \to (X, \partial X)$. The codimension two Kervaire-Arf invariant

$$\ker(f) : H_2(X, \partial X; \mathbb{Z}_2) \to \mathbb{Z}_2$$

assigns to each two-dimensional homology class $g_*[\Sigma]$ the Arf invariant of the degree one, normal map $g^*(f) : f^*(\Sigma) \to \Sigma$. The element $\ker(f) \in H^2(X, \partial X; \mathbb{Z}_2)$ is invariant under TOP normal bordism of $f$; it may not vanish for homotopy equivalences. If $M$ and $X$ are oriented, then there is a signature invariant

$$\text{sign}(f) := (\text{sign}(M) - \text{sign}(X))/8 \in H_0(X; \mathbb{Z}),$$

which does vanish for homotopy equivalences. For any compact topological manifold $X$, the Kirby-Siebenmann invariant $\text{ks}(X) \in H^4(X, \partial X; \mathbb{Z}_2)$ is the sole obstruction to the existence of a DIFF structure on $X \times \mathbb{R}$ or equivalently on $X \# r(S^2 \times S^2)$ for some $r \geq 0$. Furthermore, the image of $\text{ks}(X) \cap [X]$ in $\mathbb{Z}_2$ under the augmentation map $X \to \text{pt}$ is an invariant of unoriented TOP cobordism [FQ90, §10.2B]. Define

$$\text{ks}(f) := f_*(\text{ks}(M) \cap [M]) - (\text{ks}(X) \cap [X]) \in H_0(X; \mathbb{Z}_2).$$

In Section 4, we shall use Sullivan’s surgery characteristic class formulas as geometrically identified in dimension four by J.F. Davis [Dav05, Prop. 3.6]:

\begin{equation}
(1.0.4) \hat{f}^*(k_2) \cap [X] = \ker(f) \cap [X] \in H_2(X; \mathbb{Z}_2)
\end{equation}

\begin{equation}
(1.0.5) \hat{f}^*(\ell_4) \cap [X] = \begin{cases}
\text{sign}(f) \in H_0(X; \mathbb{Z}) & \text{if } \omega = 1 \\
\ker(f) + (\ker(f)^2 \cap [X]) \in H_0(X; \mathbb{Z}_2) & \text{if } \omega \neq 1.
\end{cases}
\end{equation}

Herein is used the 5th stage Postnikov tower [KS77, Wal99]

$$k_2 + \ell_4 : G/TOP^5 \to K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4).$$

The two expressions in (1.0.5) agree modulo two [KS77, Annex 3, Thm. 15.1]:

\begin{equation}
(1.0.6) \text{ks}(f) = \left(\text{red}_2(\hat{f})^*(\ell_4) - (\hat{f})^*(k_2)^2\right) \cap [X] \in H_0(X; \mathbb{Z}_2).
\end{equation}

2. Smoothing normal bordisms

Let $(X, \partial X)$ be a based, compact, connected, DIFF 4-manifold. We start with group-theoretic criteria on the existence and uniqueness of smoothing the topological normal bordisms relative $\partial X$ from the identity map on $X$ to itself.

**Proposition 2.1.** With respect to the Whitney sum $H$-space structures on the CAT normal invariants, there are exact sequences of abelian groups:

\begin{align*}
0 & \to \text{Tor}_1(H_0(\pi; \mathbb{Z}_2), \mathbb{Z}_2) \to N_{\text{DIFF}}(X) \xrightarrow{\text{red}_{\text{TOP}}} N_{\text{TOP}}(X) \xrightarrow{\ker} H_0(\pi; \mathbb{Z}_2) \to 0 \\
0 & \to \text{Tor}_1(H_4(\pi; \mathbb{Z}^\omega), \mathbb{Z}_2) \to N_{\text{DIFF}}(X \times \Delta^1) \xrightarrow{\text{red}_{\text{TOP}}} N_{\text{TOP}}(X \times \Delta^1) \xrightarrow{\ker} H_4(\pi; \mathbb{Z}^\omega) \otimes \mathbb{Z}_2 \to 0.
\end{align*}

**Proof.** Since $X$ is a CAT manifold, by CAT transversality and Cerf’s result that PL/O is 6-connected [KS77, FQ90], we can identify the based sets

\begin{align*}
N_{\text{DIFF}}(X) &= [X/\partial X, G/\text{PL}]_0 \\
N_{\text{TOP}}(X) &= [X/\partial X, G/\text{TOP}]_0 \\
N_{\text{DIFF}}(X \times \Delta^1) &= [S^1 \wedge (X/\partial X), G/\text{PL}]_0 \\
N_{\text{TOP}}(X \times \Delta^1) &= [S^1 \wedge (X/\partial X), G/\text{TOP}]_0.
\end{align*}

Furthermore, each right-hand set is an abelian group with respect to the $H$-space structure on $G/\text{CAT}$ given by Whitney sum of CAT microbundles.
For any based space $Z$ with the homotopy type of a CW-complex, there is the Siebenmann-Morita exact sequence of abelian groups \cite[Annex 3, Thm. 15.1]{KS77}:

$$0 \longrightarrow \text{Cok} \left( \text{red}^{(3)}_2 \right) \longrightarrow \left[ Z, G/\text{PL} \right]_0 \overset{\text{red}_{\text{TOP}}}{\longrightarrow} \left[ Z, G/\text{TOP} \right]_0 \overset{\text{Im} \left( \text{red}^{(4)}_2 \right)}{\longrightarrow} \text{Im} \left( \text{red}^{(4)}_2 + \text{Sq}^2 \right) \longrightarrow 0.$$ 

Here, the stable cohomology operations

$$\text{red}^{(n)}_2 : H^n(Z; \mathbb{Z}) \longrightarrow H^n(Z; \mathbb{Z}_2)$$

$$\text{Sq}^2 : H^2(Z; \mathbb{Z}_2) \rightarrow H^4(Z; \mathbb{Z}_2)$$

are reduction modulo two and the second Steenrod square. The homomorphism $k_s$ is given by the formula $k_s(a, b) = \text{red}^{(4)}_2(a) - \text{Sq}^2(b)$, as stated in \cite[3.0.1]{10.2307/1970799}, which follows from Sullivan’s determination \cite[3.0.1]{10.2307/1970799} below.

Suppose $Z = X/\partial X$. By Poincaré duality and the universal coefficient sequence, there are isomorphisms

$$\text{Cok} \left( \text{red}^{(3)}_2 \right) \cong \text{Cok} \left( \text{red}_2 : H_1(X; \mathbb{Z}^\omega) \rightarrow H_1(X; \mathbb{Z}_2) \right) \cong \text{Tor}_1(H_0(X; \mathbb{Z}^\omega), \mathbb{Z}_2)$$

$$\text{Im} \left( \text{red}^{(4)}_2 \right) \cong \text{Im} \left( \text{red}_2 : H_1(X; \mathbb{Z}^\omega) \rightarrow H_0(X; \mathbb{Z}_2) \right) = H_0(X; \mathbb{Z}_2).$$

Therefore we obtain the exact sequence for the normal invariants of $X$.

Suppose $Z = S^1 \wedge (X/\partial X)$. By the suspension isomorphism $\Sigma$, Poincaré duality, and the universal coefficient sequence, there are isomorphisms

$$\text{Cok} \left( \text{red}^{(3)}_2 \right) \cong \text{Cok} \left( \text{red}^{(2)}_2 : H^2(X, \partial X; \mathbb{Z}) \rightarrow H^2(X, \partial X; \mathbb{Z}_2) \right)$$

$$\cong \text{Cok} \left( \text{red}_2 : H^2(X; \mathbb{Z}^\omega) \rightarrow H_2(X; \mathbb{Z}_2) \right) \cong \text{Tor}_1(H_1(X; \mathbb{Z}^\omega), \mathbb{Z}_2).$$

Note, since the cohomology operations $\text{red}^{(4)}_2$ and $\text{Sq}^2$ are stable, that

$$(\Sigma^{-1} \circ k_s)(\Sigma a, \Sigma b) = \text{red}^{(3)}_2(a) - \text{Sq}^2(b) = \text{red}^{(3)}_2(a)$$

for all $a \in H^3(X, \partial X; \mathbb{Z})$ and $b \in H^1(X, \partial X; \mathbb{Z}_2)$. Then, by Poincaré duality and the universal coefficient sequence, we have

$$\text{Im}(k_s) \cong \text{Im} \left( \text{red}^{(3)}_2 : H^3(X, \partial X; \mathbb{Z}) \rightarrow H^3(X, \partial X; \mathbb{Z}_2) \right)$$

$$\cong \text{Im} \left( \text{red}_2 : H^1(X; \mathbb{Z}^\omega) \rightarrow H_1(X; \mathbb{Z}_2) \right) = H_1(\pi; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Therefore we obtain the exact sequence for the normal invariants of $X \times \Delta^1$. \hfill \Box

3. Homotopy self-equivalences

Recall Sullivan’s determination \cite{10.2307/1970799, Wa99}.

\[(3.0.1) \quad k_2 + 2\ell_4 : G/\text{PL} \overset{\cong}{\longrightarrow} K(\mathbb{Z}_2, 2) \times \delta(\text{Sq}^2) K(\mathbb{Z}, 4).\]

The homomorphism $\delta : H^4(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) \rightarrow H^5(K(\mathbb{Z}_2, 2); \mathbb{Z})$ is the Bockstein associated to the coefficient exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$, and the element $\text{Sq}^2$ is the 2nd Steenrod square. The cohomology classes $k_2$ and $2\ell_4$ map to a generator of the base $K(\mathbb{Z}_2, 2)$ and of the fiber $K(\mathbb{Z}, 4)$. Moreover, the cohomology class $2\ell_4$ of $G/\text{PL}$ is the pullback of the cohomology class $\ell_4$ of $G/\text{TOP}$ under the forgetful map $\text{red}_{\text{TOP}} : G/\text{PL} \rightarrow G/\text{TOP}$. The above homotopy equivalence gives

$$\text{red}_2(2\ell_4) = (k_2)^2 \in H^4(G/\text{PL}; \mathbb{Z}_2);$$
compare \cite{MM79} Theorem 4.32, Footnote. There exists a symmetric $L$-theory twisted orientation class $[X]_L \in H_4(X, \partial X; L^-)$ fitting into a commutative diagram

\[
\begin{array}{cccc}
N_{PL}(X) & \xrightarrow{\sigma_*} & L_4^h(Z[\pi]^-) \\
[X/\partial X, G/PL]_0 & \xrightarrow{\text{red}_\text{TOP}} & \tilde{H}_4(B\pi_+ \wedge G/PL, \omega) & H_0(\pi; \mathbb{Z}^\omega) \oplus H_2(\pi; \mathbb{Z}_2) \\
[X/\partial X, G/TOP]_0 & \xrightarrow{\text{red}_\text{TOP}} & H_4(B\pi_+ \wedge G/TOP; MSTOP^-) & H_4(B\pi; G/TOP^-) \\
H^0(X, \partial X; G/TOP) & \xrightarrow{\cap([X]_L)} & H_4(X; G/TOP^-) \\
\end{array}
\]

due to Sullivan-Wall \cite{Wal99} Thm. 13B.3 and Quinn-Ranicki \cite{Ran92} Thm. 18.5.

Here, the identification $N_{PL}(X) = [X/\partial X, G/PL]_0$ only makes sense if $ks(X) = 0$. It follows that the image $\hat{\sigma}(g) \in H_4(\pi; G/TOP^-)$, through the scalar product $\text{act}\ |_{\tilde{H}_4(B\pi_+ \wedge G/PL, \omega)}$ of a normal invariant $g : X/\partial X \to G/PL$ consists of two characteristic classes:

\[
\hat{\sigma}(g) = u_0(g^*(\ell_4) \cap [X]) \oplus u_2(g^*(k_2) \cap [X]),
\]

which are determined by the TOP manifold-theoretic invariants in Subsection 3.3.

We caution the reader that $\ell_4 \notin H^4(G/PL; \mathbb{Z})$; the notation $2\ell_4$ is purely formal.

**Definition 3.1.** Let $(X, \partial X)$ be any based, compact, connected TOP 4-manifold. Define the **stably smoothable** subsets

\[
\begin{align*}
N_{\text{TOP}^0}(X) & := \{ f \in N_{\text{TOP}}(X) \mid ks(f) = 0 \} \\
S_{\text{TOP}^0}(X) & := \{ h \in S_{\text{TOP}}(X) \mid ks(h) = 0 \}.
\end{align*}
\]

Recall $X$ has fundamental group $\pi$ and orientation character $\omega$.

**Hypothesis 3.2.** Let $X$ be orientable. Suppose that the homomorphism

\[
k_2 : H_2(\pi; \mathbb{Z}_2) \to L_4^h(Z[\pi]^-)
\]

is injective on the subgroup $u_2(\text{Ker} v_2(X))$.

**Hypothesis 3.3.** Let $X$ be non-orientable such that $\pi$ contains an orientation-reversing element of finite order, and if CAT = DIFF, then suppose that orientation-reversing element has order two. Suppose that $k_2$ is injective on all $H_2(\pi; \mathbb{Z}_2)$, and suppose that $\text{Ker}(u_2) \subseteq \text{Ker}(v_2)$.

**Hypothesis 3.4.** Let $X$ be non-orientable such that there exists an epimorphism $\pi^\omega \to \mathbb{Z}^-$. Suppose that $k_2$ is injective on the subgroup $u_2(\text{Ker} v_2(X))$.

**Proposition 3.5.** Let $f : M \to X$ be a degree one, normal map of compact, connected TOP 4-manifolds such that $\partial f = 1_{\beta X}$. Suppose Hypothesis 3.2 or 3.3 or 3.4. If $\sigma_+(f) = 0$ and $\text{Ker}(f) = 0$, then $f$ is TOP normally bordant to a homotopy self-equivalence $h : X \to X$ relative to $\partial X$.

\footnote{$L^h(1) = G/TOP$ is a module spectrum over the ring spectrum $L^- = MSTOP$ via Brown representation. Refer to \cite{Ran92} Rmk. B9 \cite{Wal99} Thm. 9.8 on the level of homotopy groups or to Sullivan’s original method of proof in his thesis \cite{Sul90}.}
Proof. Since $\kappa(f) = 0$, there is a (formal) based map $g : X/\partial X \to G/\text{PL}$ such that $\text{red}_{\text{TOP}} \circ g = \hat{f} : X/\partial X \to G/\text{TOP}$.

So $g$ has vanishing surgery obstruction:

$$0 = \sigma_4(f) = (I_0 + \kappa_2)(\hat{\sigma}(g)) \in \tilde{L}_4^h(\mathbb{Z}[\pi^\omega]).$$

Suppose $X$ is orientable; that is, $\omega = 1$. Then the inclusion $1^+ \to \pi^\omega$ is retractive and induces a split monomorphism $L_4^h(\mathbb{Z}[1]) \to \tilde{L}_4^h(\mathbb{Z}[\pi])$ with cokernel defined as $\tilde{L}_4^h(\mathbb{Z}[\pi])$. So the above sum of maps is direct:

$$0 = (I_0 \oplus \kappa_2)(\hat{\sigma}(g)) \in \tilde{L}_4^h(\mathbb{Z}[1]) \oplus \tilde{L}_4^h(\mathbb{Z}[\pi]).$$

Then both the signature and the square of the Kervaire-Arf invariant vanish ($1,0,0$):

$$0 = g^*([2]) = g^*(k_2)^2.$$ 

So $(\hat{f})^*(k_2) \cap [X] \in \text{Ker} \ o_2(X)$. Therefore, since $\kappa_2$ is injective on the subgroup $u_2(\text{Ker} \ o_2(X))$, we have $(\hat{f})^*(k_2) \cap [X] \in \text{Ker}(u_2)$. So, by the Hopf exact sequence

$$
\pi_2(X) \oplus \mathbb{Z}_2 \xrightarrow{\text{Hur}} H_2(X; \mathbb{Z}_2) \xrightarrow{u_2} H_2(\pi; \mathbb{Z}_2) \to 0,
$$

there exists $\alpha \in \pi_2(X)$ such that our homology class is spherical:

$$(\hat{f})^*(k_2) \cap [X] = (\text{red}_2 \circ \text{Hur})(\alpha).$$

Suppose $X$ is non-orientable; that is, $\omega \neq 1$. Let $x \in \pi$ be an orientation-reversing element: $\omega(x) = -1$. First, consider the case that $x$ has finite order. By taking an odd order power, we may assume that $x$ has order $2^N$ for some $N > 0$. Then the map induced by $1^+ \to \pi^\omega$ has a factorization through $(C_{2^N})^-:

L_4^h(\mathbb{Z}[1]) \to L_4^h(\mathbb{Z}[C_{2^N}]^-) \xrightarrow{x^{2^N}} L_4^h(\mathbb{Z}[\pi^\omega]).

The abelian group in the middle is zero by [Wal76, Theorem 3.4.5, Remark]. Then, since $I_0 : H_0(\pi; \mathbb{Z}) \to L_4^h(\mathbb{Z}[\pi^\omega])$ factors through $L_4^h(\mathbb{Z}[1]) \cong \mathbb{Z}$, generated by the Witt class $[E_8]$, we must have $I_0 = 0$. So

$$0 = \sigma_4(f) = \kappa_2(\hat{\sigma}(f)) \in L_4^h(\mathbb{Z}[\pi^\omega]),$$

and since $\kappa_2$ is injective on all $H_2(\pi; \mathbb{Z}_2)$, we have

$$0 = \hat{\sigma}(f) = u_2 \left( (\hat{f})^*(k_2) \cap [X] \right).$$

Then $(\hat{f})^*(k_2) \cap [X] \in \text{Ker}(u_2) \subseteq \text{Ker}(v_2)$ by hypothesis, and the class is spherical.

Next, consider the case there are no orientation-reversing elements of finite order. Then, by hypothesis, there is an epimorphism $p : \pi^\omega \to \mathbb{Z}^-$, which is split by a monomorphism with image generated by some orientation-reversing infinite cyclic element $y \in \pi$. Define $\overline{L}_4^h(\mathbb{Z}[\pi^\omega])$ as the kernel of $p_*$. Then $\hat{f}$ induces a direct sum decomposition

$$L_4^h(\mathbb{Z}[\pi^\omega]) = L_4^h(\mathbb{Z}[\mathbb{Z}^-]) \oplus \overline{L}_4^h(\mathbb{Z}[\pi^\omega]).$$

The abelian group of the non-orientable Laurent extension in the middle is isomorphic to $\mathbb{Z}_2$, generated by the Witt class $[E_8]$, according to the quadratic version of [MR90, Theorem 4.1] with orientation $u = -1$. Then the map $I_0 : H_0(\pi; \mathbb{Z}) \to L_4^h(\mathbb{Z}[\pi^\omega])$ factors through the summand $L_4^h(\mathbb{Z}[\mathbb{Z}^-])$ by an isomorphism; functorially, $\kappa_2$ has zero projection onto that factor. So the sum of maps is direct, similar to the oriented case:

$$0 = (I_0 \oplus \kappa_2)(\hat{\sigma}(g)) \in L_4^h(\mathbb{Z}[\mathbb{Z}^-]) \oplus \overline{L}_4^h(\mathbb{Z}[\pi^\omega]).$$
A similar argument, using the smooth normal invariant $g$, shows that

$$0 = (\hat{f})^*(\ell_4) = (\hat{f})^*(k_2)^2.$$ 

Hence $(\hat{f})^*(k_2) \cap [X] \in \text{Ker} v_2(X)$. Since $\kappa_2$ is injective on $u_2(\text{Ker} v_2(X))$, we also have $(\hat{f})^*(k_2) \cap [X] \in \text{Ker}(u_2)$, thus the class is spherical.

Let us return to the general case of $X$ without any condition on orientability. For any $\alpha \in \pi_2(X)$, there is a homotopy operation, called the Novikov pinch map, defined by the homotopy self-equivalence

$$h : X \xrightarrow{\text{pinch}} X \vee S^4 \xrightarrow{1_x \vee \Sigma \eta} X \vee S^3 \xrightarrow{1_x \vee \eta} X \vee S^2 \xrightarrow{1_x \vee \alpha} X.$$ 

Here, $\eta : S^3 \to S^2$ and $\Sigma \eta : S^4 \to S^3$ are the complex Hopf map and its suspension that generate the stable homotopy groups $\pi^s_4$ and $\pi^s_5$.

For the normal invariant of the self-equivalence $h : X \to X$ associated to our particular $\alpha$, there is a formula in the simply-connected case due to Cochran and Habegger [CH90 Thm. 5.1] and generalized to the non-simply connected case by Kirby and Taylor [KT01 Thm. 18, Remarks]:

$$\hat{(h)}^*(k_2) = \left(1 + \left\langle v_2(X), (\hat{f})^*(k_2) \cap [X] \right\rangle\right) \cdot (\hat{f})^*(k_2)$$

$$\hat{(h)}^*(\ell_4) = 0 = (\hat{f})^*(\ell_4).$$

Here, we have used $(\hat{f})^*(k_2) \cap [X] \in \text{Ker} v_2(X)$ and, if $X$ is non-orientable, $\text{ks}(f) = 0$ in (1.0.5). Therefore $f : M \to X$ is TOP normally bordant to the homotopy self-equivalence $h : X \to X$ relative to the identity $\partial X \to \partial X$ on the boundary. \qed

4. Smoothable surgery for 4-manifolds

Terry Wall asked if the smooth surgery sequence is exact at the normal invariants for the 4-torus $T^4$ and real projective 4-space $\mathbb{RP}^4$; see the remark after [Wal99 Thm. 16.6]. The latter case of $\mathbb{RP}^4$ was affirmed implicitly in the work of Cappell and Shaneson [CS70]. The main theorem of this section affirms the former case of $T^4$ and extends their circle sum technique for $\mathbb{RP}^4$ to a broader class of non-orientable 4-manifolds, using the assembly map and smoothing theory.

**Theorem 4.1.** Let $(X, \partial X)$ be a based, compact, connected, CAT 4-manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(X) : \pi \to \mathbb{Z}^\times$.

1. Suppose Hypothesis 3.2 or 3.3 Then the surgery sequence of based sets is exact at the smooth normal invariants:

$$S^*_\text{DIFF}(X) \xrightarrow{\eta} N^*_\text{DIFF}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^{\omega}).$$

2. Suppose Hypothesis 3.2 or 3.3 or 3.4 Then the surgery sequence of based sets is exact at the stably smoothable normal invariants:

$$S^*_\text{TOP}(X) \xrightarrow{\eta} N^*_\text{TOP}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^{\omega}).$$

The above theorem generalizes a statement of Wall [Wal99 Theorem 16.6] proven correctly by Cochran and Habegger [CH90] for closed, oriented DIFF 4-manifolds.

**Corollary 4.2** (Wall). Suppose the orientation character $\omega$ is trivial and the group homology vanishes: $H_2(\pi; \mathbb{Z}_2) = 0$. Then the surgery sequence (4.1.1) is exact.
A fundamental result from geometric group theory is that any torsion-free, finitely presented group $\Gamma$ is of the form $\Gamma = \star_{i=1}^{n} \Gamma_i$ for some $n \geq 0$, where each $\Gamma_i$ is either $\mathbb{Z}$ or a one-ended, finitely presented group. Geometric examples of such $\Gamma_i$ are torsion-free lattices of any rank. The Borel/Novikov Conjecture (i.e. Integral Novikov Conjecture) would imply that $\kappa_2$ is injective for all finitely generated, torsion-free groups $\pi$ and all $\omega$ \cite{Dav05}. At the moment, we have:

**Corollary 4.3.** Let $\pi$ be a free product of groups of the form

$$\pi = \star_{i=1}^{n} \Lambda_i$$

for some $n > 0$, where each $\Lambda_i$ is a torsion-free lattice in either $\text{Isom}(\mathbb{E}^{m_i})$ or $\text{Isom}(\mathbb{H}^{m_i})$ or $\text{Isom}(\mathbb{CH}^{m_i})$ for some $m_i > 0$. Suppose the orientation character $\omega$ is trivial. Then the surgery sequences \eqref{4.1.1} and \eqref{4.1.2} are exact.

**Example 4.4.** Besides stabilization with connected summands of $S^2 \times S^2$, the preceding corollary includes the orientable manifolds $X = T^4 = \coprod 4(S^1)$ and $X = \#n(S^1 \times S^3)$ and $X = \#n(T^2 \times S^2)$ for all $n > 0$. Also included are the compact, connected, orientable 4-manifolds $X$ whose interiors $X - \partial X$ admit a complete hyperbolic metric. In addition, the corollary applies to the total space of any orientable fiber bundle $S^2 \to X \to \Sigma$ for some compact, connected, orientable 2-manifold $\Sigma$ of positive genus.

**Remark 4.5.** Many surgical theorems on TOP 4-manifolds require $\pi$ to have subexponential growth \cite{FT95, KQ00} in order to find topologically embedded Whitney discs. Currently, the Topological Surgery Conjecture remains open for the more general class of discrete, amenable groups. In our case, observe that all crystallographic groups $1 \to \mathbb{Z}^n \to \pi \to \text{finite} \to 1$ have subexponential growth for all $m > 0$. On the other hand, observe that all torsion-free lattices $\pi$ in $\text{Isom}(\mathbb{H}^{m})$ and all free groups $\pi = F_n$ have exponential growth if and only if $m, n > 1$.

Indeed, taking all $\mathbb{E}^{m_i} = \mathbb{R}$, we obtain the finite-rank free groups $\pi = F_n$. Thus we partially strengthen a theorem of Krushkal and Lee \cite{KL02} if $X$ is a compact, connected, oriented TOP 4-manifold with fundamental group $F_n$. They only required $X$ to be a finite Poincaré complex of dimension 4 ($\partial X = \emptyset$) but insisted that the intersection form over $\mathbb{Z}[\pi]$ of their degree one, TOP normal maps $f : M \to X$ be tensored up from the simply-connected case $\mathbb{Z}[1]$. Now, our shortcoming is that exactness is not proven at $\mathcal{N}_{\text{TOP}}(X)$. This is because self-equivalences do not represent the homotopy equivalences with $k_{sp} \neq 0$, such as the well-known non-smoothable homotopy equivalences $\# \mathbb{C}P^2 \to \mathbb{C}P^2$ and $\# \mathbb{R}P^4 \to \mathbb{R}P^4$.

Consider examples of infinite groups with odd torsion and trivial orientation. The original case $n = 1$ below was observed by S. Cappell \cite[Thm. 5]{Cap76}. Observe that the free products below have exponential growth if and only if $n > 1$.

**Corollary 4.6.** Let $\pi$ be a free product of groups of the form

$$\pi = \star_{i=1}^{n} O_i$$

for some $n > 0$, where each $O_i$ is an odd-torsion group. (Necessarily $\omega$ is trivial.) Then the surgery sequences \eqref{4.1.1} and \eqref{4.1.2} are exact.

Consider non-orientable 4-manifolds $X$ whose fundamental group $\pi = \star n(C_2)$ is infinite and has 2-torsion. We denote $S^2 \times \mathbb{R}P^2$ the total space of the 2-sphere bundle classified by the unique homotopy class of non-nullhomotopic map $\mathbb{R}P^2 \to BSO(3)$. 

This total space was denoted as the sphere bundle $S(\gamma \oplus \gamma \oplus \mathbb{R})$ in the classification of \[HKT94\], where $\gamma$ is the canonical line bundle over $\mathbb{R}P^2$.

**Corollary 4.7.** Suppose $X$ is a DIFF 4-manifold of the form
\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is either $S^2 \times \mathbb{R}P^2$ or $S^2 \times \mathbb{R}P^2$ or $\# S^n(\mathbb{R}P^4)$ for some $1 \leq n \leq 4$. Then the surgery sequences (4.1.1) and (4.1.2) are exact.

In symplectic topology, the circle sum $M \#_S N$ is defined as $(M - E) \cup_{\partial E} (N - E)$, where $E$ is the total space of a 3-plane bundle over $S^1$ with given embeddings in the 4-manifolds $M$ and $N$. The preceding corollary takes circle sums along the order-two generator $\pi_1(P_j)$; the normal sphere bundle $\partial E = S^2 \times \mathbb{R}P^1$ is non-orientable. Observe that all the free products $\pi$ in Corollary 4.7 have exponential growth if and only if $n > 2$.

Next, consider non-orientable 4-manifolds $X$ whose fundamental groups $\pi$ are infinite and torsion-free. Interesting examples have $\pi$ in Waldhausen’s class $\text{Cl}$ of groups with vanishing Whitehead groups $\text{Wh}(\pi)$ [Wal78, §19], such as Haken 3-manifold bundles over the circle.

**Corollary 4.8.** Suppose $X$ is a TOP 4-manifold of the form
\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is the total space of a fiber bundle
\[ H_i \to X_i \to S^1. \]

Here, we suppose $H_i$ is a compact, connected 3-manifold such that:
1. $H_i$ is $S^3$ or $D^3$, or
2. $H_i$ is irreducible with non-zero first Betti number.

Moreover, if $H_i$ is non-orientable, we assume that the quotient group $H_i/\mathbb{Z}_{\alpha_i}$ of coinvariants is 2-torsionfree, where $\alpha_i : H_i \to H_i$ is the monodromy homeomorphism. Then the surgery sequence (4.1.2) is exact.

Finally, consider certain surface bundles over surfaces, which have fundamental group in the same class $\text{Cl}$. Let $K_I = \mathbb{R}P^2 \# \mathbb{R}P^2$ be the Klein bottle, whose fundamental group $\pi = \mathbb{Z}^+ \times \mathbb{Z}^-$ has the indicated orientation. Observe that any non-orientable, compact surface of positive genus admits a collapse map onto $K_I$.

**Corollary 4.9.** Suppose $X$ is a TOP 4-manifold of the form
\[ X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2) \]
for some $n > 0$ and $r \geq 0$, and each summand $X_i$ is the total space of a fiber bundle
\[ \Sigma_i^f \to X_i \to \Sigma_i^b. \]

Here, we suppose the fiber and base are compact, connected 2-manifolds, $\Sigma_i^f \neq \mathbb{R}P^2$, and $\Sigma_i^b$ has positive genus. Moreover, if $X_i$ is non-orientable, we assume that the fiber $\Sigma_i^f$ is orientable and that the monodromy action of $\pi_1(\Sigma_i^b)$ of the base preserves any orientation on the fiber (i.e. the bundle is simple). Then the surgery sequence (4.1.2) is exact.
4.1. Proofs in the orientable case.

Proof of Theorem 4.1 for orientable $X$. Suppose $X$ satisfies Hypothesis 3.2. Let $f: M \to X$ be a degree one, normal map of compact, connected, oriented TOP 4-manifolds such that: $\partial f = 1_{\partial X}$ on the boundary, $f$ has vanishing surgery obstruction $\sigma_*(f) = 0$, and $f$ has vanishing Kirby-Siebenmann stable PL triangulation obstruction $\kappa_4(f) = 0$.

Then, by Proposition 3.5, $f$ is TOP normally bordant to a homotopy self-equivalence $h : X \to X$ relative to $\partial X$. Thus exactness is proven at $N_{\text{TOP}}(X)$.

Note, since $X$ is orientable [10.3], that

\[ \text{Tor}_1(H_0(\pi; \mathbb{Z}^n), \mathbb{Z}_2) = \text{Tor}_1(\mathbb{Z}, \mathbb{Z}_2) = 0. \]

Then, by Proposition 2.1, red$\_{\text{TOP}}$ induces an isomorphism from $N_{\text{DIFF}}(X)$ to $N_{\text{TOP}}(X)$. Thus exactness is proven at $N_{\text{DIFF}}(X)$. \qed

Proof of Corollary 4.2. The result follows immediately from Theorem 4.1 since

\[ \kappa_2 : H_2(\pi; \mathbb{Z}_2) = 0 \longrightarrow L_4^h(\mathbb{Z}[\pi]) \]

is automatically injective. \qed

Proof of Corollary 4.3. By Theorem 4.1, it suffices to show $\kappa_2$ is injective by induction on $n$. Suppose $n = 0$. Then it is automatically injective:

\[ \kappa_2 : H_2(1; \mathbb{Z}_2) = 0 \longrightarrow L_4^h(\mathbb{Z}[1]) = \mathbb{Z}. \]

Let $\Lambda$ be a torsion-free lattice in either $\text{Isom}(\mathbb{E}^m)$ or $\text{Isom}(\mathbb{H}^m)$ or $\text{Isom}(\mathbb{C} \mathbb{H}^m)$. Since isometric quotients of the homogeneous4 spaces $\mathbb{E}^m$ or $\mathbb{H}^m$ or $\mathbb{C} \mathbb{H}^m$ have uniformly bounded curvature matrix (hence $A$-regular), by [FJ98, Proposition 0.10], the connective (integral) assembly map is split injective:

\[ A_\Lambda(1) : H_4(\Lambda; \mathbb{G}/\text{TOP}) = H_0(\Lambda; \mathbb{Z}) \oplus H_2(\Lambda; \mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\Lambda]). \]

The decomposition of the domain follows from the Atiyah-Hirzebruch spectral sequence for the connective spectrum $\mathbb{G}/\text{TOP} = \mathbb{L}(1)$. Therefore the integral lift of the 2-local component is injective:

\[ \kappa_2 : H_2(\Lambda; \mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\Lambda]). \]

Suppose for some $n > 0$ that $\kappa_2$ is injective for $\pi_n = \bigstar_{i=1}^{n-1} \Lambda_i$. Then $\Lambda_n$ is a torsion-free lattice in either $\text{Isom}(\mathbb{E}^m)$ or $\text{Isom}(\mathbb{H}^m)$. Write $\pi := \pi_n * \Lambda_n$. By the Mayer-Vietoris sequence in $K$-theory [Wal78], and since

\[ \text{Wh}_1(1) = \text{Wh}_0(1) = 0 = \text{Nil}_0(\mathbb{Z}[1]; \mathbb{Z}[\pi_n - 1], \mathbb{Z}[\Lambda_n - 1]), \]

note $\text{Wh}_1(\pi) = \text{Wh}_1(\pi_n) \oplus \text{Wh}_1(\Lambda_n)$. Also, since the trivial group 1 is square-root closed in the torsion-free groups $\pi$ and $\Lambda_n$, we have

\[ \text{UNil}_4^h(\mathbb{Z}[1]; \mathbb{Z}[\pi_n - 1], \mathbb{Z}[\Lambda_n - 1]) = 0 \]

by [Cap74] Corollary 4, which was proven in [Cap76] Lemmas II.7.8.9. So

\[
\begin{align*}
L_4^h(\mathbb{Z}[\pi]) &= \tilde{L}_4^h(\mathbb{Z}[\pi_n]) \oplus L_4^h(\mathbb{Z}[1]) \oplus \tilde{L}_4^h(\mathbb{Z}[\Lambda_n]) \\
H_2(\pi; \mathbb{Z}_2) &= H_2(\pi_n; \mathbb{Z}_2) \oplus H_2(1; \mathbb{Z}_2) \oplus H_2(\Lambda_n; \mathbb{Z}_2)
\end{align*}
\]

4Homogeneous: the full group of isometries acts transitively on the riemannian manifold.
by the Mayer-Vietoris sequences in $L$-theory \cite[Theorem 5(ii)]{Cap74} and group homology \cite[§VII.9]{Bro94}. Therefore, since $\kappa_2$ factors through the summand $L^4_1(\mathbb{Z}[\pi])$, we conclude that

$$\kappa_2 = \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix} : H_2(\pi; \mathbb{Z}_2) \to L^4_1(\mathbb{Z}[\pi])$$

is injective. The corollary is proven for $n$ factors, thus completing the induction. \hfill \Box

Proof of Corollary 4.6. Since each $O_i$ is odd-torsion, a transfer argument \cite{Bro94} shows that $H_2(O_i; \mathbb{Z}_2) = 0$. Then, by the Mayer-Vietoris sequence in group homology \cite[§VII.9]{Bro94} and induction, we conclude $H_2(\star_{i=1}^n O_i; \mathbb{Z}_2) = 0$. Therefore $\kappa_2$ is automatically injective. \hfill \Box

### 4.2. Proofs in the non-orientable case.

Proof of Theorem 4.1 for non-orientable $X$. Suppose $X$ satisfies Hypothesis 3.3. Let $f : M \to X$ be a degree one, TOP normal map such that $\sigma_*(f) = 0$ and $\text{ks}(f) = 0$. Then, by Proposition 3.5, $f$ is TOP normally bordant to a homotopy self-equivalence $h : X \to X$ relative to $\partial X$. Thus exactness is proven at $N_{\text{TOP}}(X)$.

Further suppose the non-orientable 4-manifold $X$ is smooth. Since in this case we assume that $\pi$ has an orientation-reversing element of order two, by \cite[Theorem 3.1]{CS76}, there exists a closed DIFF 4-manifold $X_1$ and a simple homotopy equivalence $h_1 : X_1 \to X$ such that $\eta_{\text{DIFF}}(h_1) \neq 0$ and $\eta_{\text{TOP}}(h_1) = 0$. The above argument of Proof 4.1 in the orientable case shows for non-orientable $X$ that the kernel of $N_{\text{DIFF}}(X) \to N_{\text{TOP}}(X)$ is cyclic of order two. Note

$$\eta_{\text{DIFF}}(h \circ h_1) = \eta_{\text{DIFF}}(h) + h_* \eta_{\text{DIFF}}(h_1) \neq \eta_{\text{DIFF}}(h)$$

$$\eta_{\text{TOP}}(h \circ h_1) = \eta_{\text{TOP}}(h) + h_* \eta_{\text{TOP}}(h_1) = \eta_{\text{TOP}}(h)$$

by the surgery sum formula given in Proposition 4.10 and \cite[Proposition 4.3]{Ran80}. Therefore $f$ is DIFF normally bordant to either the simple homotopy equivalence $h : X \to X$ or $h \circ h_1 : X_1 \to X$ relative to $\partial X$. Thus exactness is proven at $N_{\text{DIFF}}(X)$.

Suppose $X$ satisfies Hypothesis 3.4. Let $f : M \to X$ be a degree one, TOP normal map such that $\sigma_*(f) = 0$ and $\text{ks}(f) = 0$. Then, by Proposition 3.5, $f$ is TOP normally bordant to a homotopy self-equivalence $h : X \to X$ relative to $\partial X$. Thus exactness is proven at $N_{\text{TOP}}(X)$. \hfill \Box

The following formula generalizes an analogous result of J. Shaneson \cite[Prop. 2.2]{Sha70}, which was stated in the smooth case.

**Proposition 4.10.** Suppose $M, N, X$ are compact PL manifolds. Let $f : M \to N$ be a degree one, PL normal map such that $\partial f : \partial M \to \partial N$ is the identity map. Let $h : N \to X$ be a homotopy equivalence such that $\partial h : \partial N \to \partial X$ is the identity. Then there is a sum formula for PL normal invariants:

$$\tilde{h} \circ \tilde{f} = \eta(h) + h_* (\tilde{f}) \in [X/\partial X, G/\text{PL}]_0.$$  

**Proof.** Any element of the abelian group $[X/\partial X, G/\text{PL}]_0$ is the stable equivalence class of a pair $(\xi, t)$, where $\xi$ is a PL fiber bundle over $X/\partial X$ with fiber $(\mathbb{R}^n, 0)$ for some $n$, and $t : \xi \to e^n = (X/\partial X) \times (\mathbb{R}^n, 0)$ is a fiber homotopy equivalence of the absolute fiber $\mathbb{R}^n - \{0\} \simeq S^{n-1}$. The abelian group structure on $[X/\partial X, G/\text{PL}]_0$ is the $\pi_0$ of the Whitney sum $H$-space structure on the $\Delta$-set $\text{Map}_0(X/\partial X, G/\text{PL})$ defined rigorously in \cite[Proposition 2.3]{Ron96}. 

Let $\nu_M$ be the PL normal $(\mathbb{R}^n, 0)$-bundle of the unique isotopy class of embedding $M \hookrightarrow S^{n+\dim(M)}$, where $n > \dim(M) + 1$. For a certain stable fiber homotopy trivialization $s$ induced by the embedding of $M$ and the normal map $f$, the normal invariant of the degree one, PL normal map $(f, \xi) : M \rightarrow N$ is defined by

$\hat{(f, \xi)} = (\xi - \nu_N, s)$.

For the homotopy equivalence $h : N \rightarrow X$ with homotopy inverse $h : X \rightarrow N$ and any PL bundle $\chi$ over $N$, define the pushforward bundle $h^* (\chi)$ over $X$.

Let $r$ be the stable fiber homotopy trivialization associated to the degree one, PL normal map $(h, h^*(\nu_N))$. Then note

$\hat{h} \circ f = (h^*(\xi) - \nu_X, r + h^*(s))$

$= (h^*(\nu_N) - \nu_X, r + (h^*(\xi) - h^*(\nu_X), h^*(s)))$

$= \eta(h) + h^* (\hat{f})$.

Here, the addition is the Whitney sum of stable PL bundles with fiber $(\mathbb{R}^n, 0)$ equipped with stable fiber homotopy trivializations.

**Proposition 4.11** (López de Medrano). The following map is an isomorphism:

$\kappa_2 : H_2(C_2; \mathbb{Z}_2) \rightarrow L_4^h (\mathbb{Z} [C_2]^-)$.

Note that the source and target of $\kappa_2$ are isomorphic to $\mathbb{Z}_2$ [Wal99, Thm. 13A.1].

**Proof.** Observe that the connective assembly map

$A_\pi(1) : H_\oplus (\pi; G/\text{TOP}^\omega) \rightarrow L_\oplus^h (\mathbb{Z} [\pi]^-)$

is a homomorphism of $L^*(\mathbb{Z})$-modules. Then, by action of the symmetric complex $\sigma^*(\mathbb{C}P^2) \in L^4(\mathbb{Z})$, there is a commutative diagram

$H_2(C_2; \mathbb{Z}_2) \xrightarrow{\kappa_2} L_4^h (\mathbb{Z} [C_2]^-)$

$\xrightarrow{\sigma^*(\mathbb{C}P^2)} \cong L_4^h (\mathbb{Z} [C_2]^-)$

where the vertical maps are isomorphisms by decorated periodicity [Su96]. So it is equivalent to show that $\kappa_2^{(8)}$ is non-trivial.

Consider the commutative diagram

$\begin{align*}
\mathcal{N}_{PL}(\mathbb{R}P^8) & \xrightarrow{\sigma_*} \mathcal{N}_{TOP}(\mathbb{R}P^8) \\
\text{red}_{\text{TOP}} \downarrow & \downarrow \text{proj} \\
\mathcal{N}_{TOP}(\mathbb{R}P^8) & \xrightarrow{\kappa_2^{(8)}} H_2(C_2; \mathbb{Z}_2) \\
\text{transv} \downarrow & \downarrow \text{proj} \\
H^0(\mathbb{R}P^8; G/\text{TOP}) & \xrightarrow{\cap [\mathbb{R}P^8]_1} H_8(\mathbb{R}P^8; G/\text{TOP}^-)
\end{align*}$
The PL surgery obstruction map $\sigma_*$ for $\mathbb{R}P^8$ was shown to be non-trivial in [LM71, Theorem IV.3.3] and given by a codimension two Kervaire-Arf invariant. So the map $\kappa_2^{(8)}$ is non-trivial. Therefore $\kappa_2$ is an isomorphism. □

**Proof of Corollary 4.7.** We proceed by induction on the number $n > 0$ of free $C_2$ factors in $\pi$ to show that $\kappa_2 : H_2(\pi; \mathbb{Z}_2) \to L^h_4(\mathbb{Z}[\pi^\omega])$ is injective.

Suppose $n = 1$. Then

$$\kappa_2 : H_2(C_2; \mathbb{Z}_2) \to L^h_4(\mathbb{Z}[C_2]^-)$$

is an isomorphism by Proposition 4.11.

Suppose the inductive hypothesis is true for $n > 0$. Write $\Lambda = \star(n - 1)(C_2)$

$$\pi = (\pi_n)^{\omega_n} \ltimes (C_2)^-.$$ 

By the Mayer-Vietoris sequence in group homology [Bro94, §VII.9], we have

$$H_2(\pi; \mathbb{Z}_2) = H_2(\pi_n; \mathbb{Z}_2) \oplus H_2(C_2; \mathbb{Z}_2).$$

By the Mayer-Vietoris sequence in $L^h_4$-theory [Cap74, Thm. 5(ii)], using the Mayer-Vietoris sequence in $K$-theory [Wal78] for $h$-decorations, we have

$$L^h_4(\mathbb{Z}[\pi]) = L^h_4(\mathbb{Z}[\pi_n]) \oplus L^h_4(\mathbb{Z}[C_2]) \oplus \text{Unil}^h_4(\mathbb{Z}; \mathbb{Z}[\pi_n - 1]^\omega, \mathbb{Z}^-).$$

Since $\kappa_2$ is natural in groups with orientation character, we have

$$\kappa_2 = \begin{pmatrix} \kappa_n & 0 \\ 0 & \kappa_2 \end{pmatrix} : H_2(\pi; \mathbb{Z}_2) \to L^h_4(\mathbb{Z}[\pi])$$

Therefore, by induction, we obtain that $\kappa_2$ is injective for the free product $\pi^\omega$.

Let $i > 0$. If $X_i = S^2 \times \mathbb{R}P^2$ or $X_i = S^2 \times \mathbb{R}P^2$, then a Leray-Serre spectral sequence argument shows that

$$\text{Ker}(u_2) = Z_2[S^2] \subseteq Z_2[S^2] \oplus Z_2[\mathbb{R}P^2] = \text{Ker}(v_2).$$

If $X_i = P_j \# S_i P_k$, then a Mayer-Vietoris and Poincaré duality argument shows that

$$\text{Ker}(u_2) = Z_2([\mathbb{R}P^2_j] + [\mathbb{R}P^2_k]) = \text{Ker}(v_2).$$

Hence Theorem 4.1 applies for both sets of $X_i$. □

### 4.3. Proofs in both cases of orientability.

**Proof of Corollary 4.8.** Write $\Lambda_i$ as the fundamental group and $\omega_i$ as the orientation character of $X_i$. Then the connective assembly map

$$A_{\Lambda_1}(1) : H_4(B\Lambda_1; G/\text{TOP}^{\omega_1}) \to L^h_4(\mathbb{Z}[\Lambda_1])$$

is an isomorphism, as follows. Note $\Lambda_1 = \pi_1(X_1) = \pi_1(H_1) \times \mathbb{Z}$.

Suppose $H_1$ has type (1). Then, by [Wal99, Theorem 13A.8], the map $A_{\Lambda_1}(1)$ is an isomorphism in dimension 4, given by signature (mod 2 if $\omega_1 \neq 1$).

Suppose $H_1$ has type (2). Then, by [Rou00a, Theorem 1.1(1)] if $\partial H_1$ is non-empty and by [Rou00a, Theorem 1.2] if $\partial H_1$ is empty, the connective assembly map $A_{\pi_1(H_1)}(1)$ is an isomorphism in dimensions 4 and 5. Since $\pi_1(H_1)$, hence $\Lambda_1$, is a member of Waldhausen’s class $\text{Cl}$ [Wal78, Prop. 19.5(6,8)], we obtain $\text{Wh}_*(\Lambda_1) = 0$ by [Wal78, Proposition 19.3]. So, by the Ranicki-Shaneson sequence in $L^h_4$-theory [Ran73, Thm. 5.2], and by the five-lemma, we obtain that the connective assembly map $A_{\Lambda_1}(1)$ is an isomorphism in dimension 4.
Therefore, for both types, the integral lift $\kappa_2$ of the 2-local component of $A_{\Lambda_i}(1)$ is injective. So, by the inductive Mayer-Vietoris argument of Corollary 4.3 we conclude that $\kappa_2$ is injective for the free product $\pi = \star_{i=1}^n \Lambda_i$.

If $X$ is orientable, then $X$ satisfies Hypothesis 3.2. Otherwise, suppose $X$ is non-orientable. Then consider all $X_i$ which are non-orientable. If $H_i$ is orientable, then the monodromy homeomorphism $\alpha_i : H_i \to H_i$ must reverse orientation. So there is a lift $\pi_1(X_i) \to \pi_1(S^1) \xrightarrow{1} \mathbb{Z}$ of the orientation character. Otherwise, if $H_i$ is non-orientable, then $H_i(X_i) = H_i(H_i)(\alpha_i) \times \mathbb{Z}$ by the Wang sequence and is 2-torsionfree by hypothesis. So there is a lift $\pi_1(X_i) \to H_i(X_i) \to \mathbb{Z}$ of the orientation character. Hence there is an epimorphism $(\Lambda_i)^{\omega_i} \to \mathbb{Z}^-$. Thus there is an epimorphism $\pi^{\omega} \to \mathbb{Z}^-$. So $X$ satisfies Hypothesis 3.4. Therefore, in both cases of orientability of $X$, Theorem 4.1 is applicable.

**Proof of Corollary 4.2** Write $\Lambda_i$ as the fundamental group and $\omega_i$ as the orientation character of $X_i$.

Suppose $\Sigma_i^f = S^2$. Since $\pi_1(X_i) = \pi_1(\Sigma_i^b)$ is the fundamental group of an aspherical, compact surface, by the proof of a result of J. Hillman [Hil91, Lemma 8], the connective assembly map $A_{\Lambda_i}(1)$ is an isomorphism in dimension 4.

Suppose $\Sigma_i^f \neq S^2$. Since $\Sigma_i^f$ and $\Sigma_i^b$ are aspherical, $X_i$ is aspherical. By a result of J. Hillman [Hil91, Lemma 6], the connective assembly map $A_{\Lambda_i}(1)$ is an isomorphism in dimension 4.

Indeed, in both cases, the Mayer-Vietoris argument extends to fiber bundles where the surfaces are aspherical, compact, and connected, which are possibly non-orientable and with non-empty boundary (see [CHS06, Thm. 2.4] for detail).

Then the integral lift of the 2-local component of $A_{\Lambda_i}(1)$ is injective:

$$\kappa_2 : H_2(\Lambda_i; \mathbb{Z}) \to L_4^b(\mathbb{Z}[\Lambda_i]^{\omega_i}).$$

So, by the Mayer-Vietoris argument of Corollary 4.3, we conclude that $\kappa_2$ is injective for the free product $\pi = \star_{i=1}^n \Lambda_i$.

If $X$ is orientable, then $X$ satisfies Hypothesis 3.2. Otherwise, suppose $X$ is non-orientable. Then consider all $X_i$ which are non-orientable. By hypothesis, the fiber $\Sigma_i^f$ is orientable and the monodromy action of $\pi_1(\Sigma_i^b)$ on $H_2(\Sigma_i^f; \mathbb{Z})$, induced by the bundle $\Sigma_i^f \to X_i \to \Sigma_i^b$, is trivial. We must have that the surface $\Sigma_i^b$ is non-orientable. So, since $\Sigma_i^b$ is the connected sum of a compact orientable surface and non-zero copies of Klein bottles $Kl$, by collapsing to any $Kl$-summand, there is a lift $\pi_1(X_i) \to \pi_1(\Sigma_i^b) \to \pi_1(Kl) \to \mathbb{Z}$ of the orientation character. Hence there is an epimorphism $(\Lambda_i)^{\omega_i} \to \mathbb{Z}^-$. Thus there is an epimorphism $\pi^{\omega} \to \mathbb{Z}^-$. So $X$ satisfies Hypothesis 3.4. Therefore, in both cases of orientability of $X$, Theorem 4.1 is applicable.

**Acknowledgments.** The author is grateful to his doctoral advisor, Jim Davis, for discussions on the assembly map in relation to smooth 4-manifolds [Dav05]. The bulk of this paper is a certain portion of the author’s thesis [Kha06], with various improvements from conversations with Chris Connell, Ian Hambleton, Chuck Livingston, Andrew Ranicki, John Ratcliffe, and Julius Shaneson. Finally, the author would like to thank his pre-doctoral advisor, Professor Louis H. Kauffman, for the years of encouragement and geometric intuition instilled by him.
ON SMOOTHABLE SURGERY FOR 4-MANIFOLDS

References

[Bro94] Kenneth S. Brown. Cohomology of groups, volume 87 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[Cap74] Sylvain E. Cappell. Unitary nilpotent groups and Hermitian $K$-theory. Bull. Amer. Math. Soc., 80:1117–1122, 1974.

[Cap76] Sylvain E. Cappell. A splitting theorem for manifolds. Invent. Math., 33(2):69–170, 1976.

[CH90] Tim D. Cochran and Nathan Habegger. On the homotopy theory of simply connected four manifolds. Topology, 29(4):419–440, 1990.

[CHS06] Alberto Cavicchioli, Friedrich Hegenbarth, and Fulvia Spaggiari. Manifolds with poly-surface fundamental groups. Monatsh. Math., 148(3):181–193, 2006.

[Cap74] Sylvain E. Cappell and Julius L. Shaneson. Some new four-manifolds. Ann. of Math. (2), 104(1):61–72, 1976.

[Dav05] James F. Davis. The Borel/Novikov conjectures and stable diffeomorphisms of 4-manifolds. In Geometry and topology of manifolds, volume 47 of Fields Inst. Commun., pages 63–76. Amer. Math. Soc., Providence, RI, 2005.

[FJ98] F. T. Farrell and L. E. Jones. Rigidity for aspherical manifolds with $\pi_1 \subset \text{GL}_m(\mathbb{R})$. Asian J. Math., 2(2):215–262, 1998.

[FQ90] Michael H. Freedman and Frank Quinn. Topology of 4-manifolds, volume 39 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1990.

[FT95] Michael H. Freedman and Peter Teichner. 4-manifold topology. I. Subexponential groups. Invent. Math., 122(3):509–529, 1995.

[Hill02] J. A. Hillman. Four-manifolds, geometries and knots, volume 5 of Geometry & Topology Monographs. Geometry & Topology Publications, Coventry, 2002.

[HKT94] Ian Hambleton, Matthias Kreck, and Peter Teichner. Nonorientable 4-manifolds with fundamental group of order 2. Trans. Amer. Math. Soc., 344(2):649–665, 1994.

[Kha06] Qayum Khan. On connected sums of real projective spaces. Indiana University, Ph.D. dissertation, 2006. Thesis advisor: James F. Davis.

[KL02] Vyacheslav S. Krushkal and Ronnie Lee. Surgery on closed 4-manifolds with free fundamental group. Math. Proc. Cambridge Philos. Soc., 133(2):305–310, 2002.

[KQ00] Vyacheslav S. Krushkal and Frank Quinn. Subexponential groups in 4-manifold topology. I. Subexponential groups. Invent. Math., 122(3):509–529, 1995.

[Hil91] Jonathan A. Hillman. On 4-manifolds homotopy equivalent to surface bundles over surfaces. Topology Appl., 40(3):275–286, 1991.

[Hil92] J. A. Hillman. Four-manifolds, geometries and knots, volume 5 of Geometry & Topology Monographs. Geometry & Topology Publications, Coventry, 2002.

[KT01] Vyacheslav S. Krushkal and Ronnie Lee. Surgery on closed 4-manifolds with free fundamental group. Math. Proc. Cambridge Philos. Soc., 133(2):305–310, 2002.

[KS77] Robion C. Kirby and Laurence C. Siebenmann. Foundational essays on topological manifolds, smoothings, and triangulations. Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[KT01] Robion C. Kirby and Laurence R. Taylor. A survey of 4-manifolds through the eyes of surgery. In Surveys on surgery theory, Vol. 2, volume 149 of Ann. of Math. Stud., pages 387–421. Princeton Univ. Press, Princeton, NJ, 2001.

[LdM71] S. López de Medrano. Involutions on manifolds. Springer-Verlag, New York, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 59.

[MM79] Ib Madsen and R. James Milgram. The classifying spaces for surgery and cobordism of manifolds, volume 92 of Annals of Mathematics Studies. Princeton University Press, Princeton, N.J., 1979.

[MR90] R. J. Milgram and A. A. Ranicki. The $L$-theory of Laurent extensions and genus 0 function fields. J. Reine Angew. Math., 406:121–166, 1990.

[Ran73] A. A. Ranicki. Algebraic $L$-theory. III. Twisted Laurent extensions. In Algebraic $K$-theory, III: Hermitian $K$-theory and geometric application (Proc. Conf. St. Res. Center, Battelle Memorial Inst., 1972), pages 412–463. Lecture Notes in Mathematics, Vol. 343. Springer, Berlin, 1973.

[Ran80] Andrew Ranicki. The algebraic theory of surgery. II. Applications to topology. Proc. London Math. Soc. (3), 40(2):193–283, 1980.

[Ran92] A. A. Ranicki. Algebraic $L$-theory and topological manifolds, volume 102 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1992.
[Rou96] C. P. Rourke. The Hauptvermutung according to Casson and Sullivan. In *The Hauptvermutung book*, volume 1 of *K-Monogr. Math.*, pages 129–164. Kluwer Acad. Publ., Dordrecht, 1996.

[Rou00a] Sayed K. Roushon. L-theory of 3-manifolds with nonvanishing first Betti number. *Internat. Math. Res. Notices*, (3):107–113, 2000.

[Rou00b] Sayed K. Roushon. Vanishing structure set of Haken 3-manifolds. *Math. Ann.*, 318(3):609–620, 2000.

[Sha70] Julius L. Shaneson. Non-simply-connected surgery and some results in low dimensional topology. *Comment. Math. Helv.*, 45:333–352, 1970.

[Sul96] D. P. Sullivan. Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric Topology Seminar Notes. In *The Hauptvermutung book*, volume 1 of *K-Monogr. Math.*, pages 69–103. Kluwer Acad. Publ., Dordrecht, 1996.

[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[TW79] Laurence Taylor and Bruce Williams. Surgery spaces: formulae and structure. In *Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978)*, volume 741 of *Lecture Notes in Math.*, pages 170–195. Springer, Berlin, 1979.

[Wal76] C. T. C. Wall. Classification of Hermitian Forms. VI. Group rings. *Ann. of Math. (2)*, 103(1):1–80, 1976.

[Wal78] Friedhelm Waldhausen. Algebraic K-theory of generalized free products. I, II, III, IV. *Ann. of Math. (2)*, 108(1,2):135–256, 1978.

[Wal99] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

Department of Mathematics, Vanderbilt University, Nashville, TN 37240 U.S.A.

E-mail address: qayum.khan@vanderbilt.edu