Notes on Lattice-Reduction-Aided MMSE Equalization

Robert F.H. Fischer

Abstract—Over the last years, novel low-complexity approaches to the equalization of MIMO channels have gained much attention. Thereby, methods based on lattice basis reduction are of special interest, as they achieve the optimum diversity order. In this paper, a tutorial overview on LRA equalization optimized according to the MMSE criterion is given. It is proven that applying the zero-forcing BLAST algorithm to a suitably augmented channel matrix—the inverse of the square root of the correlation matrix of the data symbols times the noise variance as its lower part—indeed is already widely used but lacks a formal proof. It turns out that it is more important to take the correlations of the data correctly into account than what type of lattice reduction actually is used.

I. INTRODUCTION

The joint reception of signals transmitted in parallel—either considering multi-antenna systems or multi-user scenarios—will become even more important over the next years. When designing transmission systems for such multiple-input/multiple-output (MIMO) channels, the interference among the individual signals has to be dealt with by means of equalization.

During the last decade, numerous techniques known from intersymbol-interference channels—e.g., linear equalization, decision-feedback equalization (DFE), also known as successive interference cancellation (SIC) and also used in the Bell Laboratories space-time (BLAST) approach), maximum-likelihood detection, cf. [5, Table E.1]—have been transferred to the MIMO setting. However, novel approaches based on lattice basis reduction, e.g., [21], [17], are of special interest. Using these lattice-reduction-aided (LRA) techniques, low-complexity equalization achieving the optimum diversity behavior [15] is enabled.

In this paper, a tutorial overview on LRA equalization optimized according to the minimum mean-squared error (MMSE) criterion is given. It is shown that it is more important to take the inherently introduced correlations of the data symbols correctly into account, than which lattice reduction approach actually is used. The main result of the paper is to establish a connection of the V-BLAST algorithm to the successive MMSE estimation of correlated data starting from basic principles. It is proven that applying (zero-forcing) BLAST to a suitably augmented channel matrix—having the inverse of the square root of the correlation matrix of the data symbols times the noise variance as its lower part—indeed results in the optimum solution. To the best knowledge of the authors, a formal proof for this fact yet has not being presented in literature. However, it has been used widely without taking care of its validity.

The paper is organized as follows: in Sec. III the channel model is introduced and conventional equalization techniques are briefly reviewed. Lattice-reduction-aided equalization is addressed in Sec. III and its MMSE DFE version is analyzed in detail in Sec. IV. Concluding remarks follow in Sec. V and in the Appendix fundamentals on estimation are compiled.

II. CHANNEL MODEL AND EQUALIZATION

We consider uncoded multiple-antenna transmission over flat-fading channels where joint equalization at the receiver side is possible. The input/output relation is given by the usual equation:

\[ y = Ha + n. \]  

(1)

This model has either to be understood in complex base-band notation or as its real-valued model with doubled dimensionality [9]. As all subsequent discussions can either be applied to the complex or the real model, we do not distinguish both approaches in the sequel. In each case, the channel matrix is expected to be of dimension \( N \times N \). The differences between both views are discussed if required. For successive schemes, due to the larger degree of freedom, usually the real-valued model has some advantages [4].

Each component \( a_m \) of \( a \) is independently drawn from a zero-mean one-dimensional \( M \)-ary ASK constellation \( \mathcal{A} = \{ \pm 1/2, \pm 3/2, \ldots, \pm (M - 1)/2 \} \) or an \( M \)-ary QAM constellation, with an \( M \)-ary ASK per quadrature component. The correlation matrix of the data vector hence reads \( \Phi_{aa} \equiv E\{aa^H\} = \sigma_a^2 I \), with variance \( \sigma_a^2 \equiv E\{|a|^2\} \). The noise is assumed to be spatially white with variance \( \sigma_n^2 \) per component, i.e., \( \Phi_{nn} \equiv E\{nn^H\} = \sigma_n^2 I \).

A. Linear Equalization

The interference between the parallel data streams can be eliminated by means of equalization, i.e., via \( r = H_R y \) a decision vector is generated. Having \( r \), individual threshold decision can be performed.

Using linear equalization (LE), optimized according to the zero-forcing (ZF) criterion, the receive matrix reads

\[ H_R^{(LE,ZF)} = \left( H^H H \right)^{-1} H^H, \]  

(2)

Notation: \( A^T \): transpose of matrix \( A \); \( A^H \): Hermitian (i.e., conjugate) transpose; \( A^{-1} \): inverse of the Hermitian transpose of a square matrix \( A \); \( I \): identity matrix; matrices are denoted by uppercase letters, vectors by lower case letters. E{·}: expectation.
i.e., the receive matrix is given by the Moore-Penrose left pseudo inverse of $H$. Already in [9] it has been observed that the minimum mean-squared error solution is obtained by using the augmented matrix ($\xi$ is the inverse signal-to-noise ratio)

$$\hat{H} = \left[ \frac{H}{\sqrt{\xi}} \right]_{(N_R+N_T) \times N_T} \bar{y} = \left[ \begin{array}{c} y \\ 0 \end{array} \right]_{(N_R+N_T)}$$

(3)
in the ZF solution and feeding $\bar{y}$ into the resulting receive matrix rather than $y$. Subsequently, all quantities corresponding to the augmented channel model are marked by a horizontal bar.

B. Decision-Feedback Equalization

Some gains over linear equalization can be achieved by using sorted decision-feedback equalization, also known as BLAST or SIC. The required matrices for ZF DFE are obtained by performing a sorted QR-type decomposition such that

$$HP = QL,$$

(4)

where $P$ is a permutation matrix (a single one in each row and column), $Q$ is unitary and $L$ is lower triangular. From these quantities, the feedforward matrix $F$ and the lower triangular, unit main diagonal feedback matrix $B$ are calculated as $F \triangleq \text{diag}(L)^{-1}Q^H$ and $B \triangleq \text{diag}(L)^{-1}L$, respectively.

Again, the MMSE solution is obtained by plugging the augmented channel matrix into [3], cf. [13].

III. LATITUDE-REDUCTION-AIDED EQUALIZATION

Unfortunately, using linear equalization or DFE, only a diversity order of $N_R - N_T + 1$ (for the complex-valued model) is possible. Lattice-reduction-aided equalization schemes, e.g., [21], [17], have proven to require only low complexity, nevertheless being able to achieve the full diversity order $N_R$ of the MIMO channel [15]. The idea is to choose a “more suited” representation of the lattice spanned by the columns of the channel matrix $H$; equalization is done with respect to the new basis, which is desired to be close to orthogonal. At the very end, the change of basis is reversed.

A. Lattice-Reduction-Aided Linear Equalization

For performing LRA equalization, in the first step lattice basis reduction, e.g., by using the LLL algorithm [12] (or some complex-valued version thereof, e.g., [6]), is performed to obtain

$$H = CZ,$$

(5)

where $Z \in \mathbb{Z}_{N_T \times N_T}$ is an integer unimodular matrix, i.e., has only integer coefficients and $|\det(Z)| = 1$. The reduced channel matrix $C$ is usually required to have columns close to orthogonal and of small norms (depending on the definition of “reduced”). Using (5), the receive signal is given by $y = CZa + n$.

In the second step, only $C$ is treated and the signal $z \triangleq Za$, which is taken from a translate of the integer lattice $(\mathbb{Z}N_T = \mathbb{Z}^{N_T})$ and hence can be obtained by individual threshold decision per component, is to be estimated. This transformed data vector has zero mean, $\mu_z = E[z] = E\{Za\} = ZE\{a\} = Z0 = 0$, but is correlated with covariance matrix

$$\Phi_{zz} = E\{zz^H\} = E\{Zaa^HZ^H\} = \sigma_z^2 ZZ^H.$$  

(6)

Third, the change of basis is reversed via $Z^{-1}$.

1) LRA ZF Linear Equalization: Applying LRA ZF linear equalization the correlations are ignored and the receive matrix is simply the left pseudo inverse of the reduced channel matrix

$$H^{(LRA, ZF)}_R = (\check{C}^H C)^{-1} C^H.$$  

(7)

2) LRA MMSE Linear Equalization: As in the conventional case, the MMSE solution may be obtained by applying all operations to the augmented channel model, cf. [20]. Hence, in the first step $\tilde{H}$ is fed into the lattice basis reduction, resulting in (note: $Z$ usually differs from the ZF case)

$$\tilde{H} = \check{C}Z.$$  

(8)

Using the definition of $\tilde{H}$, the reduced augmented matrix can be written as

$$\tilde{C} = \left[ \frac{H}{\sqrt{\xi}} \right] \check{C}^{-1} = \left[ \frac{HZZ^{-1}}{\sqrt{\xi}Z^{-1}} \right] \triangleq \left[ \begin{array}{c} A \\ Z^{-1} \end{array} \right]$$

(9)

with the obvious definitions of $C$ and $A$. The receive matrix (with respect to $\bar{y}$) is then given by

$$H^{(LRA, MMSE)}_R = (\check{C}^H C)^{-1} C^H = (\check{C}^H C + \check{Z}^{-1} Z^{-1})^{-1} \left[ \check{C}^H \sqrt{\xi}Z^{-1} \right],$$

(10)

or with respect to $y$, when deleting the last $N_T$ columns

$$H^{(LRA, MMSE)}_R = \left( \check{C}^H C + \check{Z}^{-1} Z^{-1} \right)^{-1} C^H = Z \left( H^H H + \check{I} \right)^{-1} C^H.$$  

(11)

This receive matrix takes the correlations of the data perfectly into account. To see this, note that from the basic literature on estimation, e.g., [13] Theorem 2.6.1], the optimum MMSE linear estimator is given by

$$\left( C^H \Phi_{nn}^{-1} C + \Phi_{zz}^{-1} \right)^{-1} C^H \Phi_{nn}^{-1},$$

(13)

which, since white channel noise was assumed and $\Phi_{zz} = \sigma_z^2 ZZ^H$, exactly gives the receive matrix (11). The covariance matrix of the resulting minimum mean-squared error $e$ is given by

$$\Phi_{ee} = \left( C^H \Phi_{nn}^{-1} C + \Phi_{zz}^{-1} \right)^{-1} \sigma_z^2 \left( C^H C + \check{Z}^{-1} Z^{-1} \right)^{-1}.$$  

(14)
B. Lattice-Reduction-Aided ZF DFE

In order to enhance performance, linear equalization can be replaced by DFE, resulting in lattice-reduction-aided DFE, cf. Fig. 1. As in the classical case, for performing DFE, the (sorted) QR-type factorization of the respective channel matrix is required. For LRA ZF DFE, the factorization has the form

\[ CP = QL. \]  

(15)

Feedforward and feedback matrices \( F \) and \( B \) are calculated as explained above. In the feedback loop, the components of \( z \) are detected in an optimized order described by the permutation matrix \( P \). After reestablishing the original ordering, an estimate of the original data vector \( a \) is generated via the inverse of the integer unimodular matrix \( Z \).

IV. LATTICE-REDUCTION-AIDED MMSE DFE

The optimization of the LRA DFE according to the MMSE criterion is not as straightforward as in the ZF case. This is due to the correlation of the data symbols \( z_k \) to be estimated in an optimum succession within the DFE loop. Up to now, in the literature this fact has not been treated in detail; usually simply the ZF solution with respect to the augmented channel matrix has been used, e.g., [20], [13]. We first review the straightforward application of the BLAST algorithm [8] to the augmented channel model and then compare these results to those obtained from the theory of optimum estimation of correlated Gaussian random variables.

A. Lattice Reduction

As in the LRA MMSE linear case, we stick to the augmented channel model \( H \) and consider the lattice reduction according to [5] and [9]. Assume for simplicity of notation, that the columns of \( C \) are sorted according to the optimum decision order, i.e., we replace \( C \) implicitly by \( CP \), thereby anticipating the permutation matrix \( P \) to be determined during the calculation of the required matrices. Thereby, the optimization criterion is—as proposed in the V-BLAST system—the noise enhancement encountered in the feedforward processing. For the MMSE solution this criterion is identical to looking at the minimum main diagonal element of the error covariance matrix.

B. V-BLAST Algorithm

We first simply perform the (MMSE) V-BLAST algorithm with respect to the augmented channel matrix \( \tilde{C} \). Assuming that \( l \) \((l = 0, \ldots, N_T - 1)\) symbols are already known, the BLAST approach is to simply delete the \( l \) first columns (due to the assumed sorting) of \( \tilde{C} \) and proceed with the residual augmented channel matrix \( \tilde{C}[\cdot] \).

1) Feedforward Matrix: Having deleted the first \( l \) columns, the potential feedforward matrix (with respect to the augmented channel model) for estimating the remaining \( N_T - l \) symbols reads

\[
\tilde{F}_k^{(l)} = \begin{bmatrix}
\tilde{f}_k^{(l)} \\
\tilde{f}_k^{(l)} \\
\vdots \\
\tilde{f}_k^{(l)}
\end{bmatrix} = \begin{bmatrix}
C_H^{[\cdot]} & C_H^{[\cdot]}
\end{bmatrix}^{-1} \begin{bmatrix}
C_H^{[\cdot]}
\end{bmatrix}
\]

(16)

In each step the row \( \tilde{f}_k^{(l)} \), corresponding to the symbol \( z_k \) which can be detected most reliably, is appended to the entire feedforward matrix \( \tilde{F} \). The feedforward matrix for the non-augmented, original channel is obtained from \( \tilde{F} \) by deleting the last \( N_T \) columns.

2) Optimum Sorting: In the BLAST algorithm, usually the norms of the row of the feedforward matrix are considered as sorting criterion [8]. These are proportional to the noise enhancement and hence determine the error rate. Using (16), these row norms are given by the diagonal elements of

\[
\tilde{F}_k^{(l)} (\tilde{F}_k^{(l)})^H = \begin{bmatrix}
C_H^{[\cdot]} & C_H^{[\cdot]}
\end{bmatrix}^{-1} \begin{bmatrix}
C_H^{[\cdot]}
\end{bmatrix}
\]

(17)

If \( \tilde{C} \) has already been sorted optimally, the upper left diagonal element will be the smallest. Otherwise, the first row of \( \tilde{F} \) and that with the smallest norm are exchanged; this exchange is also recorded in the permutation matrix \( P \). After \( N_T \) iterations the entire feedforward matrix \( \tilde{F} \) and the optimum processing order, represented by the permutation matrix \( P \) are known.

3) Feedback Matrix: Knowing \( \tilde{F} \) and \( P \), the feedback matrix \( B \) can be calculated. It is well-known [7] that the approaches of a) canceling before applying the feedforward matrix (as usually proposed in the BLAST context) and b) canceling at the output of the feedforward matrix (as is preferred in the DFE context) are equivalent. Here, we consider the latter strategy, cf. also Fig. 1.

Since it is optimum to cancel all known interference, the feedback matrix calculates to

\[
B = \begin{bmatrix}
\tilde{b}_1 \\
\vdots \\
\tilde{b}_{N_T}
\end{bmatrix} = \tilde{F} \tilde{C} \hat{P} = \tilde{F} \begin{bmatrix}
\tilde{C} \\
\hat{A}
\end{bmatrix} \hat{P}.
\]

(18)

As in each step \( \tilde{F}_k^{(l)} \tilde{C}_k^{(l)} = I \) holds, i.e., the remaining symbols are equalized and the already canceled are ignored, it is easy to see that \( \hat{B} \) is a lower triangular matrix with unit main diagonal.

---

4Given a matrix \( M \), let \( M^{(l)} \) denote the matrix obtained from \( M \) by deleting the first \( l \) columns, and \( M^{(l)}_\cdot \) denote the matrix composed of the first \( l \) columns of \( M \), i.e., \( M = [M^{(l)}_\cdot \ M^{(l)}_\cdot] \). Please distinguish that from the notation \( .(\cdot) \), which indicates a quantity present in step \( l \) (counting from zeros to \( N_T - 1 \)) of the algorithm.

---

Fig. 1. Lattice-reduction-aided DFE.
Moreover, by construction, the rows of $\bar{F}$ are orthogonal; via a diagonal gain matrix $G$ we can write $\bar{F} = GQ^{H}$, where $Q$ is an $(N_{R} + N_{T}) \times N_{T}$ matrix with orthonormal columns. In summary, using the lower triangular matrix $L \triangleq G^{-1} B$, can be written in the form

$$\bar{C} P = \bar{Q} L,$$  \hspace{1cm} (19)

i.e., applying the BLAST algorithm a sorted QR-type (QL) factorization of the reduced augmented channel matrix $\bar{C}$ is inherently performed.

In more detail, the $(l + 1)$th row of the feedback filter is given by

$$b_{l+1} = \tilde{f}_{l}^{(f)} \bar{C} P,$$  \hspace{1cm} (20)

which, using (15), is the first row of the matrix

$$\bar{M}^{(f)} = \left( C_{l}^{H} C_{l} + A_{l}^{H} A_{l} \right)^{-1} \left[ C_{l}^{H} A_{l} \right] \left\{ \begin{array}{c} C \end{array} \right\} P.$$  \hspace{1cm} (21)

Writing $\bar{C} P = \left[ C_{1} ; C_{2} \right]$, with $C_{2} = C_{l}$ and $A_{2} = A_{l}$, we can write

$$\bar{M}^{(f)} = \left( C_{2}^{H} C_{2} + A_{2}^{H} A_{2} \right)^{-1} \left[ C_{2}^{H} A_{2} \right] \left[ C_{1} ; C_{2} \right]$$

$$= \left( \left( C_{2}^{H} C_{2} + A_{2}^{H} A_{2} \right)^{-1} \left( C_{2}^{H} A_{2} \right) \right) \left\{ \begin{array}{c} C \end{array} \right\} P,$$  \hspace{1cm} (22)

C. Optimum Estimation of Correlated Data

We now turn to the situation of deriving the required matrices directly from the theory of minimum mean-squared estimation and the properties of correlated random vectors when parts of the variables are already known. Looking at the optimum linear estimator (33), summarized in the Appendix, feedforward and feedback matrices can immediately be given by identifying the respective quantities suitably.

However, from (37) it can be deduced that the optimal processing depends on the mean and covariance matrix of the vector of not yet detected symbols. These quantities, however, depend on the previous decisions when performing DFE. In turn, optimum filtering and the optimum processing order potentially may depend on the actual decisions made so far within the DFE. In the following we show, that this is actually not the case. All required matrices can be calculated in advance and the influence of previous decisions is taken into account via the feedback matrix in an optimum way.

1) Feedforward Matrix: Again assume that the first $l$ symbols $z_{k}$ (contained in the vector $z_{1}$) have already been detected. We can partition the vector $z$, mean vector and correlation matrix of this vector in the form

$$z = \left[ \begin{array}{c} z_{1} \\ z_{2} \end{array} \right], \quad \mu_{z} = \left[ \begin{array}{c} \mu_{1} \\ \mu_{2} \end{array} \right], \quad \Phi_{zz} = \left[ \begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right].$$  \hspace{1cm} (23)

Under white noise, the feedforward matrix—filtering the receive vector $y$ (non-augmented model) for obtaining estimates of the remaining $N_{T} - l$ symbols—is given by (cf. (37))

$$F^{(l)} = \left[ \begin{array}{c} \tilde{f}_{1}^{(f)} \\ \tilde{f}_{N_{T}-l}^{(f)} \end{array} \right] = \left( C_{l}^{H} C_{l} \right)^{-1} = \left( C_{l}^{H} C_{l} \right)^{-1} = \left( C_{l}^{H} C_{l} + \sigma_{n}^{2} \Phi_{22}^{-1} z_{l} \right)^{-1} = \left( C_{l}^{H} C_{l} + \sigma_{n}^{2} \Phi_{22}^{-1} z_{l} \right)^{-1} C_{l}^{H} C_{l}.$$  \hspace{1cm} (24)

The conditioned covariance matrix $\Phi_{zz}^{-1}$ can be written as follows. Since from (2) $\sqrt{Z}^{-1} = \bar{A} \triangleq [A_{1} A_{2}]$, we have on the one hand

$$\Phi_{zz}^{-1} = \frac{1}{\sigma_{n}} Z^{-H} Z^{-1} = \frac{1}{\sigma_{n}} A_{l}^{H} A_{l}$$

$$= \frac{1}{\sigma_{n}} \left[ A_{l}^{H} A_{l} + A_{l}^{H} A_{l} \right].$$  \hspace{1cm} (25)

On the other hand, with the partitioning (23) and using (10) Page 472, Eq. (7.7.5)), we can write (elements marked by * are irrelevant)

$$\Phi_{zz}^{-1} = \left( \left( \Phi_{22} - \Phi_{21} \Phi_{11}^{-1} \Phi_{12} \right) - I \right)^{-1}.$$  \hspace{1cm} (26)

A comparison of (25) and (26) reveals that for all $l$, we have

$$\sigma_{n}^{2} \left( \Phi_{22} - \Phi_{21} \Phi_{11}^{-1} \Phi_{12} \right) = \sigma_{n}^{2} \left( \Phi_{22} - \Phi_{21} \Phi_{11}^{-1} \Phi_{12} \right)^{-1}.$$  \hspace{1cm} (27)

Hence, the optimum feedforward matrix calculates to

$$F^{(l)} = \left( C_{l}^{H} C_{l} + A_{l}^{H} A_{l} \right)^{-1} C_{l}^{H}.$$  \hspace{1cm} (28)

2) Optimum Sorting: According to the general theory of estimation (Eqs. (38) and (41)), given $z_{l}$ and applying the optimum linear estimator (feedforward processing), the correlation matrix of the error with respect to the remaining, not yet known symbols $z_{k}$ is given as

$$\Phi_{ee} = \left( C_{l}^{H} C_{l} \right)^{-1} \left( \Phi_{22}^{-1} z_{l} \right),$$

$$= \sigma_{n}^{2} \left( C_{l}^{H} C_{l} \right)^{-1} \left( \Phi_{22}^{-1} z_{l} \right)^{-1} = \sigma_{n}^{2} \left( \Phi_{22}^{-1} z_{l} \right)^{-1}.$$  \hspace{1cm} (29)

The next symbol to be detected is the one, for which the corresponding main diagonal element of $\Phi_{ee}$ is minimum. Assuming the channel matrix has been accordingly rearranged, the upper left main diagonal element is the smallest and only the first row of the feedforward matrix is used to produce a decision symbol. Otherwise, the respective rows are exchanged which is kept track of in the permutation matrix $P$.

3) Feedback Matrix: From (37) and using (40), (41), the feedback filter follows immediately, too. The influence of the already detected symbols has additionally to be canceled from the receive vector $y$. This is done by remodulating the vector $z_{1}$ of decisions via $C_{l}$, containing the first $l$ columns of $C$. Moreover, the mean of $z_{2}$ given $z_{1}$ has to be taken into account (starting from $\mu_{z} = 0$). The task of the feedback filter is hence twofold: to cancel the known interference and at the same time to predict the not yet decided symbols from the known ones.

With the goal to have the cancellation point at the output of the feedforward matrix, the feedback filter, when already $l$ symbols are known, calculates to

$$F^{(l)} = \left[ \left( C_{l}^{H} C_{l} \right)^{-1} + \Phi_{22}^{-1} z_{l} \right]^{-1} C_{l}^{H}.$$  \hspace{1cm} (30)
and with the above abbreviations (partitioning of \( \hat{C} \)), after straightforward manipulations, we arrive at
\[
\begin{align*}
&= \left( C_1^H C_2 + A_2^H A_2 \right)^{-1} C_1^H C_2 - I) \Phi_{21} \Phi_{11}^T + \left( C_2^H C_2 + A_2^H A_2 \right)^{-1} C_1^H C_1 \\
&= \left( C_1^H C_2 + A_2^H A_2 \right)^{-1} \left( C_1^H C_1 - A_2^H A_2 \Phi_{21} \Phi_{11}^T \right).
\end{align*}
\]
\[ (31) \]
From (25), the correlation matrix is given as
\[
\Phi_{zz} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}
\]
\[ (32) \]
Again using (10) Eq. (7.7.5), we have
\[
\begin{align*}
\Phi_{11} &= \left( O - V U^{-1} V^H \right)^{-1} \\
\Phi_{21} &= \left( V^H O^{-1} V - U \right)^{-1} V^H O^{-1}
\end{align*}
\]
\[ (33) \quad (34) \]
and together with \( A_2^H A_2 \Phi_{21} \), we arrive at
\[
\begin{align*}
A_2^H A_2 \Phi_{21} &= U \left( V^H O^{-1} V - U \right)^{-1} V^H O^{-1} \left( O - V U^{-1} V^H \right) \\
&= U \left( V^H O^{-1} V - U \right)^{-1} \left( U - V^H O^{-1} V \right) U^{-1} V^H \\
&= -V^H.
\end{align*}
\]
\[ (35) \]
In summary, the feedback matrix, when \( l \) symbols are already known, is given by
\[
M^{(l)} = \left( C_1^H C_2 + A_2^H A_2 \right)^{-1} \left( C_2^H C_1 + A_2^H A_1 \right).
\]
\[ (36) \]
Assuming that the symbols \( z_k \) are in the optimum ordering, as for the feedforward matrix, since only a single next symbol (the currently best) is decided, only the first row of the matrix \( M^{(l)} \) is actually used. Note that the respective row of the feedback matrix \( B \) is obtained from that row by appending a single one and then \( N_T - l - 1 \) trailing zeros.

D. Comparison and Discussion

From the above derivations it is immediate that both perspectives on LRA MMSE DFE lead to the same result. A comparison of (14)—here deleting the last \( N_R \) columns to return from the augmented to the original channel model—and (25) reveals that for both cases the feedforward matrices are identical.

The sorting is based on (17) and (29), respectively. As the feedforward processing is identical, this also holds for the error variances or the norms of the filter vectors, proportional to these variances and hence the same decision orders result.

Finally, the feedback filters are also identical; this is revealed by comparing (22) and (26).

Hence, the straightforward application of the V-BLAST algorithm for sorted QR decomposition to the extended channel matrix indeed results in the optimum solution to LRA MMSE DFE. The “trick” behind this lies in the lower part of the augmented matrix. Whereas for classical DFE the (scaled) identity matrix is present, in case of LRA the inverse of the square root of the correlation matrix of the vector \( z \) to be estimated is present (cf. (25) and (26)). As shown, deleting columns and calculating the feedback matrix on this reduced channel matrix has the same effect as updating the correlation matrix of the residual symbols.

The above derivation also reveals that in case of MMSE DFE for correlated symbols the feedback matrix fulfills two tasks: the cancellation of the interference of already detected symbols and some kind of prediction of the still unknown information symbols from the symbols up to now known. In case of white data symbols, only cancellation is required.

Numerical simulations reveal that it is more important to take the correlations of the data symbols correctly into account than using a specific type of lattice reduction. Conducting lattice reduction on the original channel matrix, and using the resulting matrices \( C \) and \( Z \) to create an augmented matrix on which the QR decomposition is done, performs only marginally worse than starting rightaway with the augmented matrix. However, using the LLL on the original channel requires less complexity and is independent of the current SNR.

As the above derivation is valid for any channel model and any correlation of the data, we can conclude that when performing MMSE DFE for correlated symbols, optimum feedforward and feedback matrices and the optimum sorting can be calculated via the V-BLAST algorithm. Thereby, the algorithm simply has to work on an augmented channel matrix, which has the inverse of the square root of the correlation matrix of the data symbols times the noise variance as its lower part. In other words, all required matrices are obtained by performing a sorted QR-type decomposition of this augmented channel matrix. However, with regard to computational complexity this procedure is far from optimum as the algorithm has to work on a matrix of approximately doubled number of rows. Fortunately, the efficient “fast V-BLAST algorithm” proposed in (11) Table III can simply be modified to take correlated data (correlation introduced via a matrix \( Z \)) into account. Here, only the computation of \( R \) and \( Q \) according to (11) Eqs. (26) and (28) has to be modified. Using the initializations (notation from (11)) \( R_0 = \alpha Z^H Z \) and \( Q_0 = (1/\alpha)(Z^H Z)^{-1} \), this algorithm efficiently delivers the same results as the ZF BLAST algorithm applied to the augmented channel matrix.

LRA equalization for MIMO channels can be viewed as the counterpart to partial-response signaling (PRS) [11], [2] for intersymbol interference channels, see [5]. In both cases an integer polynomial/matrix is split from the actual channel transfer function/matrix and only the residual system is considered. Equalization is done towards the target polynomial/unimodular matrix. The non-whiteness of the data sequence to be detected has to be taken into account for MMSE equalization of PRS (e.g., [2] Appendix A); the same is true in LRA schemes. However, in contrast to PRS, which is usually employed to achieve some desired transmitter side characteristics (spectral zeros at DC or Nyquist frequency), the use of LRA enables
full diversity of the MIMO transmission system and hence is the key to significantly improve error performance of uncoded transmission.

V. SUMMARY AND CONCLUSIONS

Lattice-reduction-aided equalization optimized according to the MMSE criterion of MIMO channels has been studied. For the first time it has been proven that applying the zero-forcing BLAST algorithm to a suitably augmented channel matrix—having the inverse of the square root of the correlation matrix of the data symbols times the noise variance as its lower part—indeed results in the optimum solution. It is more important to take the correlations of the data correctly into account than what specific type of lattice reduction actually is used.

Finally it should be noted that taking the uplink/downlink duality into account, instead of employing receiver-side equalization, MMSE LRA precoding can be performed. The given results can immediately be transferred to this transmitter-side technique, which is of great importance in the multi-user downlink.

APPENDIX

SOME FUNDAMENTALS OF ESTIMATION THEORY

In this appendix, for convenience, two important properties on minimum mean-squared error estimation of correlated and non-zero mean random variables are reviewed from the literature.

First, we consider a vector $x$ with (possibly) non-zero mean $\mu_x$ and covariance matrix $\Phi_{xx}$. This vector is observed through the matrix $H$ and disturbed by (zero-mean) Gaussian noise $n$ with covariance matrix $\Phi_{nn}$. Hence, the observation $y = Hx + n$ is present. The optimum linear estimator for this setting is given by, e.g., [4] Page 68

$$
\hat{x} = \left( H^H \Phi_{nn}^{-1} H + \Phi_{xx}^{-1} \right)^{-1} H^H \Phi_{nn}^{-1} y - \left( H^H \Phi_{nn}^{-1} H + \Phi_{xx}^{-1} \right)^{-1} H^H \Phi_{nn}^{-1} H - I \right) \mu_x.
$$

(37)

The covariance matrix of the resulting estimation error can be written as

$$
\Phi_{ee} = \left( \Phi_{xx}^{-1} + H^H \Phi_{nn}^{-1} H \right)^{-1}.
$$

(38)

Second, assume a multivariate Gaussian distribution (random vector $w$) of dimension $Q$ with mean $\mu_w$ and covariance matrix $\Phi_{ww}$. Let the random vector, the mean vector, and the covariance matrix be partitioned according to

$$
w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \mu_w = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Phi_{ww} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}
$$

(39)

where the dimensions of the upper and left parts are $q$, e.g., $\dim(w_1) = \dim(\mu_1) = q$, $\dim(\Phi_{11}) = q \times q$, etc.

Having already knowledge on the first $q$ components of the random vector $w$—i.e., the vector $w_1$—the mean and the covariance matrix for the residual $Q-q$ variables (vector $w_2$), conditioned on the knowledge $w_1$, calculate to

$$
\mu_{2|w_1} = \mu_2 + \Phi_{21} \Phi_{11}^{-1} (w_1 - \mu_1)
$$

(40)

$$
\Phi_{22|w_1} = \Phi_{22} - \Phi_{21} \Phi_{11}^{-1} \Phi_{12}.
$$

(41)

Note that the new covariance matrix is the Schur complement of $\Phi_{11}$ in $\Phi$; it does not depend on the actual value of $w_1$. Note additionally that both quantities can be obtained in one step or successively in $q$ steps, each time assuming additional knowledge of a single symbol.

REFERENCES

[1] J. Benesty, Y. Huang, J. Chen. A Fast Recursive Algorithm for Optimum Sequential Signal Detection in a BLAST System. IEEE Transactions on Signal Processing, Vol. 51, No. 7, pp. 1722–1730, July 2003.

[2] J.M. Ciofﬁ, G.P. Dudevoir, M.V. Eyubo˘ glu, and G.D. Forney. MMSE Decision-Feedback Equalizers and Coding—Part I: Equalization Results, Part II: Coding Results. IEEE Transactions on Communications, Vol. 43, No. 10, pp. 2582–2604, Oct. 1995.

[3] R.F.H. Fischer. Precoding and Signal Shaping for Digital Transmission, John Wiley & Sons, New York, 2002.

[4] R.F.H. Fischer, C. Windpassinger. Real- vs. Complex-Valued Equalization in V-BLAST Systems. Electronics Letters, Vol. 39, No. 5, pp. 470–471, Mar. 2003.

[5] R.F.H. Fischer, C. Siegl. On the Relation between Lattice-Reduction-Aided Equalization and Partial-Response Signalling. International Zurich Seminar (IZS), pp. 34–37, Zurich, Switzerland, Feb. 2006.

[6] Y.H. Gan, C. Ling. W.H. Mow. Complex Lattice Reduction Algorithm for Low-Complexity Full-Diversity MIMO Detection. IEEE Transactions on Signal Processing, Vol. 57, No. 7, pp. 2701–2710, July 2009.

[7] G. Ginis, J.M. Ciofﬁ. On the relation between V-BLAST and the GDFE. IEEE Communications Letters, Vol. 5, No. 9, pp. 364–366, Sept. 2001.

[8] G.D. Golden, G.J. Foschini, R.A. Valenzuela, P.W. Wolniansky. Detection Algorithm and Initial Laboratory Results Using V-BLAST Space-Time Communication Architecture. Electronics Letters, Vol. 35, No. 1, pp. 14–15, Jan. 1999.

[9] B. Hassibi. An Efﬁcient Square-Root Algorithm for BLAST. IEEE International Conference on Acoustics, Speech, and Signal Processing pp. 737–740, Istanbul, Turkey, June 2000.

[10] R.A. Horn, C.R. Johnson. Matrix Analysis. Cambridge University Press, Cambridge, UK, 1985.

[11] J. Huber. Trettlscodeierung. Springer Verlag, Berlin, Heidelberg, 1992. (in German).

[12] A.K. Lenstra, H.W. Lenstra, L. Lovász. Factoring polynomials with rational coefﬁcients, Mathematische Annalen, Vol. 261, No. 4, pp. 515–534, 1982.

[13] A.D. Murugan, H. El Gamal, M.O. Damen, G. Caire. A Unified Framework for Tree Search Decoding: Rediscovering the Sequential Decoder. IEEE Transactions on Information Theory, Vol. 53, No. 3, pp. 933–953, Mar. 2006.

[14] A.H. Sayed. Fundamentals of Adaptive Filtering, John Wiley & Sons, New York, 2003.

[15] M. Taherzadeh, A. Mobasher, A.K. Khandani. LLL Reduction Achieves Near-Maximum-Likelihood Detection and Precoding for MIMO Systems using Lattice Reduction. IEEE Information Theory Workshop, pp. 345–348, Paris, France, Mar./Apr. 2003.

[16] C. Windpassinger. Detection and Precoding for Multiple Input Multiple Output Channels. Dissertation, Erlangen, June 2004.

[17] C. Windpassinger, R.F.H. Fischer. Low-Complexity Near-Maximum-Likelihood Detection and Precoding for MIMO Systems using Lattice Reduction. IEEE Information Theory Workshop, pp. 345–348, Paris, France, Mar./Apr. 2003.

[18] C. Windpassinger, Detection and Precoding for Multiple Input Multiple Output Channels. Dissertation, Erlangen, June 2004.

[19] C. Windpassinger, R.F.H. Fischer, J.B. Huber. Lattice-Reduction-Aided Broadcast Precoding. IEEE Transactions on Communications, Vol. 52, No. 12, pp. 2057–2060, Dec. 2004.

[20] D. Wübben, R. Böhnke, V. Kühn, K.D. Kammermeyer. Near-Maximum-Likelihood Detection of MIMO Systems using MMSE-Based Lattice Reduction. IEEE International Conference on Communications, pp. 798–802, Paris, France, June 2004.

[21] H. Yao, G.W. Wornell. Lattice-Reduction-Aided Detectors for MIMO Communication Systems. IEEE Global Communications Conference, Taipei, Taiwan, Nov. 2002.