On time-dependent orbital complexity in gravitational N-body simulations

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ABSTRACT
We implement an efficient method to quantify time-dependent orbital complexity in gravitational N-body simulations. The technique, which we name DWaTIM, is based on a discrete wavelet transform of velocity orbital time-series. The wavelet power spectrum is used to measure trends in complexity continuously in time. We apply the method to the test cases N = 3 Pythagorean configuration and a perturbed N = 5 Caledonian configuration. The method recovers the well-known time-dependent complexity of the dynamics in these small-N problems. We then apply the technique to an equal-mass collisional N = 256 body simulation run through core-collapse. We find that a majority of stars evolve on relatively complex orbits up to the time when the first hard binary forms, whereas after core-collapse, less complex orbits are found on the whole as a result of expanding mass shells.

Key words: methods: N-body simulations – methods: numerical – stars: kinematics.

1 INTRODUCTION

Computer simulations of the gravitational N-body problem aim to solve the set of 3N second-order ordinary differential equations

\[ F_i = \frac{m_i}{r_i} = -\sum_{l=1 \neq i}^{N} \frac{m_l (r_i - r_l)}{|r_i - r_l|^3}, \quad (i = 1, \ldots, N). \tag{1} \]

Here G is the gravitational constant, \( F_i \) and \( \dot{r}_i \) denote the force and the acceleration exerted by the \( N-1 \) particles \( l \) on particle \( i \) of mass \( m_i \) at position \( r_i \). A system with \( N > 2 \) is in general chaotic, i.e. the solution to equation (1) is known to be sensitive to the initial conditions. Miller (1964) was the first to show that individual orbits calculated from neighbouring initial configurations diverge on a short time-scale proportional to \( 4\xi/N \), where \( \xi \) is the mean time between two subsequent close encounters of a particle. Even when computed with double-precision arithmetic, the differences between the two integrations become comparable to the characteristic length- and velocity-scales of the cluster within only a few crossing times. Thus one may argue that the positions and velocities obtained from an N-body simulation are a fair rendition of the problem at hand in a statistical sense only (Aarseth & Lecar 1975; Quinlan & Tremaine 1992). Goodman, Heggie & Hut (1993) identify three mechanisms responsible for the exponential growth of the distance between nearby trajectories. Aside from numerical errors, \(^1\) they point out that the exponential instability of the solutions also arises through a fluctuating mean gravitational field, or through inherently chaotic orbits in a static field. However, for systems in dynamical equilibrium and with near-spherical symmetry (such as e.g. globular clusters), they conclude that the principal mechanism responsible for chaos is the cumulative effect of near-neighbour interactions, i.e. two-body encounters.

The exponential divergence of the N-body problem of equation (1) has been studied by several authors for systems with up to \( N = 10^5 \) particles (Kandrup & Smith 1991; Kandrup & Sideris 2001; Hemsendorf & Merritt 2002). The debate whether Miller’s instability is formally caused by chaos is still ongoing (see e.g. Kandrup & Sideris 2003; Helmi & Gómez 2007). The assumed chaoticity of such systems may be evaluated by computing the variational equations associated with equation (1) (Miller 1971) and by retrieving indicators of chaos, such as e.g. Lyapunov characteristic numbers (Benettin, Galgani & Strelcyn 1976). Other indicators of chaos commonly found in the literature are e.g. the relative Lyapunov indicator (Sándor, Érdi & Efthymiopoulos 2000), the mean exponential divergence of nearby orbits (Cincotta & Simó 2000) or the small-alignment index (Skokos 2001). All these indicators accurately quantify the exponential divergence of nearby trajectories.

In this work, we propose to investigate time-dependent orbital complexity in an N-body simulation. We define orbital complexity to be a measure of the richness and non-triviality of the frequency spectrum of an orbit at a given time \( t \). The possible connection between orbital chaos and orbital complexity has been pointed out by Kandrup, Eckstein & Bradley (1997). In that work, the authors find a strong correlation between their measure of orbital complexity and short-time Lyapunov exponents.

\(^1\) A round-off error at one time-step can be considered as a change in the initial conditions for the next time-step.
system governed by equation (1) that is complementary to the classical indicators of chaos mentioned above. For instance, the concept of complexity is exploited by Sideris & Kandrup (2002) to study the continuum limit in the case of large-N simulations. It is used to compare the orbital behaviour between particles evolving in smooth potentials and bodies orbiting in the corresponding frozen N-body configurations. The notion of complexity here allows us to contrast the discreteness effects of the N-body configuration to the orbital evolution obtained in the smooth case. However, whereas a global measure of orbital complexity has been implemented in several works (see e.g. Kandrup et al. 1997), a method to measure the impact of instantaneous changes in orbital complexity has not. A trajectory computed from equation (1) can show multiple, qualitatively distinct regimes in time. For instance, an orbit may display arcs of relatively smooth motion such as e.g. an unperturbed parabolic or hyperbolic orbit. At other times, the body may be gravitationally bound in a binary system with a particle of approximately the same mass. Likewise, the body may be temporarily trapped in a complicated higher order resonance, orbiting about a massive central body. All these states of motion can be identified in time by a suitable measure of complexity. The goal of this paper is precisely to discuss such time-resolved complexity by introducing a dedicated tool for complexity evaluation. The issue of formally relating complexity to chaos will not be addressed here.

Classical spectral methods and Fourier transform based techniques (see Laskar 1993; Carpintero & Aguilar 1998; Valluri & Merritt 1998) are of short execution time and able to provide an indicator of complexity obtained from the DWaT, namely the discrete coefficients as there are samples in the time-series, i.e. ∫−∞+∞S(t)Ψ(t−τa)dt.

Here (S, Ψa,τ) denotes the wavelet coefficients and Ψ is the complex conjugate of Ψ. Equation (3) can be inverted to reconstruct the original time-series. The CWT is known to produce a large amount of wavelet coefficients which implies considerable CPU execution time (see e.g. Daubechies 1992). In addition, the information the CWT displays at closely spaced scales or at closely spaced time points is highly correlated and thus unnecessary redundant.

For these reasons we instead compute a DWaT. The DWaT offers a highly efficient wavelet representation, which can be implemented with a simple recursive filter scheme (Daubechies 1992; Mallat 1999). Unlike the numerical CWT implementation which easily produces more than 10⁵ coefficients for a single orbital time-series of Q = 8192 data points, the DWaT only produces as many coefficients as there are samples in the time-series, i.e. Q. This property of the DWaT of avoiding redundant wavelet coefficients serves in defining a proper measure of complexity, as we will show in Section 2.2. For a given choice of the mother wavelet function Ψ(t) and for the discrete set of parameters aj = 2j and τj,k = 2k(j, k ∈ Z), the wavelet family

Ψj,k(t) = |aj|−1/2Ψ(t−τj,kaj),

defines an orthonormal basis of L²(R). The time-series f(tq) is sampled at Q = 2J (J ∈ N) constant time intervals of size Δ = τj+1 − τj. The discrete wavelet expansion then reads (see e.g. Rosso et al. 2006, equation 36)

f(t) = ∑j=12J−1∑k=1Q−1(f(tq), Ψj,k)discreteΨj,k(t)Δ−1.

For simplicity we set Δ = 1 throughout Section 2. Here f(t) is the reconstructed signal and J = log₂Q is the number of scales over which the time-series f(tq) is analysed. The DWaT coefficients (f(tq), Ψj,k)discrete can be understood as a representation of the wavelet power spectrum (or energy) at scale j and time τq, associated with the time-series f(tq). They represent the local residual errors between successive signal approximations at scales j and j − 1. In what follows, we set

Pj(k) ≡ (f, Ψj,k)discrete.

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for a more convenient notation of the DWaT coefficients. As mentioned earlier, there are $Q$ coefficients $P_j(k)$ and the number of coefficients computed for resolution level $j$ is $2^{j-1}$. The frequency band over which the $P_j(k)$ are computed is limited by the frequency $1/(Q \Delta)$ (scale $j = 1$) in the low-frequency domain and by the Shannon–Nyquist critical frequency $f_{\text{cr}} = 1/(2 \Delta)$ in the high-frequency domain (scale $j = J$; see also Press et al. 2002 sections 12.1 and 13.10). We refer to Samar et al. (1999) for further details about the wavelet representation.

We use bi-orthogonal cubic spline functions as mother wavelets (Cohen, Daubechies & Feauveau 1992, case $\{N, \tilde{N}\} = \{3, 9\}$ in their table 6.1),

\[ \Psi(t) = \sum_j g_j \Theta_3(2t - j), \]  

(7)

where the $g_j$ values are known as basic spline coefficients and

\[ \Theta_3(t) = \begin{cases} 
(t + 1)^2/2, & -1 \leq t \leq 0, \\
-(t - 1/2)^2 + 3/4, & 0 \leq t \leq 1, \\
(t - 2)^2/2, & 1 \leq t \leq 2, \\
0, & \text{otherwise}.
\]  

(8)

This choice is motivated by three arguments. First and most importantly, spline functions provide an excellent time–frequency localization when compared to other mother wavelet candidates (Unser 1999; Ahuja, Lertrattanapanich & Bose 2005). Instantaneous changes in the dynamics are accurately singled out by the DWaT. We further stress this point in Section 2.3. Secondly, the use of splines is computationally inexpensive (Thévenaz, Blue & Unser 2000) and provides further desirable properties, such as e.g. compact support and smoothness. (For an exhaustive discussion on spline interpolation, see Unser 1999; Thévenaz et al. 2000.) Finally, the use of a bi-orthogonal spline mother wavelet also implies reduced border effects, an undesired artefact of the wavelet transform algorithm (see Section 2.1.2 below).

We analyse the velocity time-series

\[ f(t_i) = v_{\alpha}(t_i) \quad (\alpha = x, y, z), \]  

(9)

i.e. the $v_x$, $v_y$, and $v_z$ velocity components of each particle $i$ ($i = 1, \ldots, N$). We do not use the information available on the positions of the bodies since there may be important differences in magnitude between the beginning and the end of the time-series. Positional information is then likely to produce pronounced DWaT border effects (see Section 2.1.2).

### 2.1.2 Border effects

Border effects are an artefact of the wavelet transform algorithm which enforces cyclical boundary conditions on the data vector (Lo Presti & Olmo 1996; Press et al. 2002, section 13.10). Such border effects depend on the values held by the two end points of the data vector, and may become important when both ends of the data set differ greatly. In the following, we aim to quantify the extent to which the diagnostics become erroneous due to these edge effects.

Fig. 1(a) shows the two sinusoids

\[ y_1 = \cos \left( \frac{2\pi}{8192} \sum_{q=1}^{Q} \frac{3\pi}{2} \right), \]

\[ y_2 = \cos \left( \frac{2\pi}{8192} \sum_{q=1}^{Q} \frac{15/2}{2} \right), \]  

(10)

sampled at $Q = 8192$ intervals ($q = 1, \ldots, Q$). The integration time is commensurate to the periodicity of signal $y_1$. This is not the case for signal $y_2$. Figs 1(b) and (c) show their respective DWaT scalograms. The scalogram is a grey-shaded representation of the DWaT coefficients $P_j(k)$. The darkest shade is for the largest values $|P_j(k)|$; white means $P_j(k) = 0$. For both sinuoids a maximum intensity is obtained for the scale that represents the base period $\Pi_1$ of the respective signal. These are scales $j = 6$ at period $\Pi_j = 2^{-j/2} Q \Delta = 2^{-5} \times 8192 = 256$ for $y_1$ and $j = 4$ for $y_2$. The region in which the DWaT of $y_1$ suffers from border effects is known as the cone of influence (Moortel, Munday & Hood 2004). The cone of influence is clearly visible at both edges in Fig. 1(c). On the left-hand edge of Fig. 1(c), for example, the DWaT gives an artificially high excited mode at scale $j = 5$ up to $t = 1024$. The same artefact is also found at scales of higher frequencies, although the magnitude of the effect then diminishes and is barely visible on the left-hand side of Fig. 1(c). In addition, scale $j = 4$ is wrongly excited in the intervals $0 \leq t \leq 3072$ and $7168 \leq t < 8192$. The size of the cone of influence depends on the choice of the mother wavelet and is especially significant when the spectrum of the signal contains low frequencies, when the period compares to the overall duration of the time-series (Moortel et al. 2004).

Fig. 1(d) shows a situation where the border effects have been reduced by extending the integration time to twice the original interval. Doubling the time interval of analysis implies an increased

\[ \Delta t \]
2.2 Measures of complexity

Our goal is to obtain from the DWaT a quantitative estimate of the time-dependent complexity of the frequency spectrum of a trajectory. Exploiting some notions of information theory, we here present the DWaTIM as an efficient indicator for complexity. In what follows, we provide a succinct overview of the concept. We follow closely the approach of Rosso et al. (2006) and Martin, Plastino & Rosso (2006). We refer the reader to these works for a more extended discussion.

2.2.1 Discrete wavelet transform information measure

For a chosen time window of size \( \kappa \Delta \) (where \( \kappa \) is an arbitrary integer) we compute the wavelet energy at each resolution level \( j \),

\[
E_j = \sum_{k=1}^{2^j-1} |P_j(k)|^2,
\]

and the total wavelet energy,

\[
E = \sum_j E_j,
\]

(11)

(12)

to obtain the so-called relative wavelet energy \( p_j = E_j/E \) at scale \( j \). The DWaT then provides a probability distribution\( \mathcal{P} = \{ p_j \}, \quad (j = 1, \ldots, J), \)

\[
\text{which weights the base frequency } j \text{ in the reconstruction of the original signal } v_w(t_x). \text{ By definition we have } \sum_j p_j = 1.
\]

The amount of disorder present at time \( t \) in an orbital time-series can be quantified by determining the information needed to describe the orbit at that time (Shannon 1948). An information measure (hereafter IM) can be seen as a quantity that describes the characteristics of the time-scale probability distribution \( \mathcal{P} \) of equation (13) (Rosso et al. 2006). The IM gives the amount of information required per time unit to specify the state of the system up to a given accuracy. The IM we use in this work is the Shannon entropy (Shannon 1948; Quian Quiroga, Rosso & Basar 1999; Sello 2003)

\[
W_S[\mathcal{P}] = -K \sum_j p_j \log_2(p_j)
\]

(14)

where \( K = 1 \) is an arbitrary numerical constant. A minimum of information entropy \( \min(W_S[\mathcal{P}]) = 0 \) is obtained if \( p_j = 1 \) for some scale \( j \) and 0 for all the remaining scales. This situation only occurs for the ordered dynamics of a periodic orbit with a single base frequency. Likewise, the state of highest complexity is obtained for the case of a white noise signal. For such an orbit, the entire band of base frequencies is sampled by the DWaT in equal proportions, and \( \mathcal{P} \) is characterized by the uniform probability distribution

\[
\mathcal{P}_u = \left\{ \frac{1}{J}, \ldots, \frac{1}{J} \right\},
\]

(15)

In this case the IM is max \( (W_S[\mathcal{P}] = W_S[\mathcal{P}_u] = \log_2 J \). For a given velocity component \( v_\alpha (\alpha = x, y, z; \ i = 1, \ldots, N) \) we define the DWaTIM to be the measure

\[
H_{\alpha i} \equiv \left( \frac{W_S[\mathcal{P}_i]}{W_S[\mathcal{P}_1]} \right)_{a_i},
\]

(16)

i.e. the Shannon entropy for velocity component \( \alpha \) of particle \( i \) normalized to the interval \([0, 1]\). The generalized DWaTIM of particle \( i \) is then obtained by taking the arithmetic mean over the 3 components,

\[
\Upsilon_i = \frac{H_{\alpha i} + H_{\beta i} + H_{\gamma i}}{3}.
\]

(17)

Finally, the overall DWaTIM \( \Upsilon_{\text{tot}} \) of the system is computed by averaging over all particles \( i \),

\[
\Upsilon_{\text{tot}} = \frac{\sum_{i=1}^{N} \Upsilon_i}{N}.
\]

(18)

In what follows we will argue that an increase in DWaTIM correlates with an increase in the complexity of the underlying orbital dynamics at that instant.

2.2.2 Complexity of a sinusoid

The temporal evolution of the DWaTIM indicator is computed by subdividing the input signal in non-overlapping time windows of size \( \kappa \Delta \). We use \( \kappa = 2 \). Fig. 2 shows our complexity analysis for the sinusoids \( y_1 \) and \( y_2 \) of Section 2.1.2. The DWaTIM of \( y_1 \), \( H_{y_1} \) is shown by the lower thick solid line. As expected for a stationary signal with a unique base frequency, \( H_{y_1} \approx 6.13 \times 10^{-4} \); the line

\[
\text{Figure 2. Complexity measure for the sinusoids } y_1 \text{ and } y_2 \text{ of Fig. 1. The thick solid line confounded with the } x \text{-axis shows the DWaTIM } H_{y_1}\text{ as obtained for } y_1 \text{ from Fig. 1(b). The upper solid line shows the DWaTIM } H_{y_2}\text{ obtained for } y_2 \text{ from Fig. 1(d) in which border effects where reduced by time-series extension. The dotted line indicates the } H_{y_1}\text{ curve obtained for the case where border effects have not been treated (see Fig. 1c).}
\]

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complexity values of up to the first 25 per cent of the diagnostic the effects are dramatic: we find border effects for that case still influence the result at all times.) For effects have been reduced by time-series extension. (Note that the obtained from the DWaT of Fig. 1(d), i.e. for the case where border shown by the upper solid line in Fig. 2. The line depicts the DWaTIM Hy = 2 is

\[
\text{Hy} = \sin \left( 2\pi \frac{32}{8192} q + \pi \right) + \sin \left( 2\pi \frac{128}{8192} q + \frac{5\pi}{4} \right). \tag{22}
\]

\[
\text{yc} = \sin \left( 2\pi \frac{32}{8192} q \right) + \sin \left( 2\pi \frac{128}{8192} q + \frac{\pi}{4} \right) + \sin \left( 2\pi \frac{612/3}{8192} q + \frac{\pi}{8} \right) + \sin \left( 2\pi \frac{8192/17}{8192} q + \frac{\pi}{17} \right). \tag{23}
\]

\[
\text{ye} = \text{yc} + G_3(q). \tag{24}
\]

\[
\text{ye} = G_2(q). \tag{25}
\]

Figure 3. Analysis of the time-dependent complexity of the toy model defined by equation (19). From top to bottom: (a) time-series, (b) DWaT, (c) DWaTIM Hyy.
DWaT scalogram of Fig. 3(b) illustrates the power spectrum of the base frequencies $1/\Omega_j$ as a function of time. The variation of complexity with time of signal $y_I$ is clearly identifiable, as the increase in complexity at $t = 4096$ is well recovered. The modes represented by scales $j = 8$ and 9 show a response within $4096 < t \leq 8192$. The related DWaTIM complexity measure $H_y$ is plotted in Fig. 3(c). Within $4096 < t \lesssim 4200$ the complexity rises from $H_y \simeq 6 \times 10^{-4}$ to an average value of $H_y \simeq 0.16$ for $t > 4200$. We note that in the time interval $4096 < t \lesssim 4200$, $H_y$ rises progressively. The size of this transient interval corresponds to about one-half of the period of $y_A$ which is 256 units. This is the longest of the two periods of the Fourier components involved in the transition $y_A \rightarrow y_B$. This sets a limit on the time-localization of the transition. In Section 3.2 we will discuss in detail the latency of transitions in the dynamics measured by the DWaTIM in an application to small-$N$ problems.

We also remark that border effects within $0 \leq t \lesssim 200$ affect the measure for less than 3 per cent of the signal.

Fig. 4 shows the results for the second toy model $y_B$ of equation (20). The related time-series is shown in Fig. 4(a) and the results for the complexity analysis are represented in Fig. 4(b) and (c). The time-dependent subtleties of the frequency spectrum are depicted by the DWaT in Fig. 4(b). For instance, the drop in complexity at $t \approx 3072$, where we remove the two incommensurate base frequencies of regime $y_C$ and switch to regime $y_B$, is recovered. The modes $j = 10$ and 11, excited within $1024 \lesssim t \lesssim 3072$ (the $j = 11$ response during that interval is hardly visible in Fig. 4b) do not show a response any more within $3072 \lesssim t \lesssim 4096$. The increased complexity between $y_C$ and its noisy counterpart $y_D$ is also apparent; scales 7 to 13 show a more irregular pattern within $4096 \lesssim t \lesssim 6144$ than within $1024 \lesssim t \lesssim 3072$. Finally, for the white noise regime $y_B$ within $6144 \lesssim t \lesssim 8192$, the DWaTIM response decreases. All the modes $j = 1, \ldots, 13$ are on average less excited than e.g. during regime $y_D$, and the DWaT intensity is spread over these scales in an almost uniform manner. This shows that the DWaT correctly identifies this regime as noise.

We remark that border effects may explain the feature seen at scale $j = 6$ within $0 \leq t \lesssim 1536$, namely that the magnitude of that mode increases and decreases until it stabilizes at the end of that interval. Therefore border effects may here persist for up to $1536/8192 \simeq 18$ per cent of the duration of the signal. This is still less than the conservative estimate of 25 per cent we have given in Section 2.2.2. Fig. 4(c) shows the DWaTIM $H_y$. Apart from the interval $0 < t \lesssim 1536$ where the border effects spoil the result, the DWaTIM provides a consistent overall diagnostic of time-dependent complexity. Each transition between the regimes $y_A$ and $y_B$, through regimes $y_C$, $y_B$ and $y_D$, is detected by the DWaTIM. For instance, the transition at $t = 3072$ from $y_C$ to $y_B$ corresponds to a drop of about 0.05 in average $H_y$ magnitude around that instant. We recover a DWaTIM value for regime $y_B$ of $H_y \approx 0.19$ within $3072 \lesssim t \lesssim 4096$, i.e. of about $3 \times 10^{-2}$ units larger than in toy model 1. This 19 per cent difference is due to the response of the three low-frequency scales $j = 1, 2$ and 3 in Fig. 4; at these times; in toy model 1 these modes do not show any response (see Fig. 3b within $4096 \lesssim t \lesssim 8192$). The performance of the DWaTIM to provide an absolute measure of complexity for a given dynamical regime depends on the time-series subjected to analysis and on the number of scales included in the DWaTIM computation. In Section 3.2 we investigate further this effect for the case of $N$-body orbits. Finally, we illustrate the behaviour of the DWaTIM in the white noise limit of $y_B$. As mentioned earlier, the total wavelet energy within $6144 \lesssim t \lesssim 8192$ (see grey-shade intensities in Fig. 4b) is reduced with respect to the interval $4096 \lesssim t \lesssim 6144$, pointing at the difficulty of the DWaT to single out privileged frequencies with a high probability during $6144 \lesssim t \lesssim 8192$. In this limit, the DWaTIM asymptotically reaches its maximum value of 1 (for the case of a perfect white noise signal and an infinite-length time-series; see Section 2.2.1).

3 FEW-BODY ENCOUNTERS

To validate the technique for the case of orbital dynamics, we present three applications to small $N$ problems ($N = 2, 3$ and 5). All individual orbits were computed using the STARBED software environment (Portegies Zwart et al. 2001). Integrations are performed using individual time-steps (Aarseth 1985) and a fourth-order Hermite predictor–corrector scheme (Makino & Aarseth 1992). time-series are constructed by Hermite interpolation at evenly spaced time intervals and by projecting the orbit of each star on the three orthogonal axes $x, y$ and $z$ in the centre of mass coordinate system. Standard $N$-body units are used throughout (Heggie & Mathieu 1986).

3.1 Binary motion: $N = 2$

We study independently the motion of two unperturbed, equal-mass binaries with respective eccentricities $e_1 = 0.05$ and $e_2 = 0.95$. Results for the $v_x$-component analysis of body #1 of each binary are presented in the left- and right-hand panels of Fig. 5, respectively. Fig. 5(a) shows the velocity time-series $v_x$ sampled at 8192 regular time intervals. The binaries are integrated over 16 $N$-body time units, so the sampling interval $\Delta$ is $16/8192 \approx 2^{-9}$. The period of both binaries is 0.5, allowing a sampling rate of 256 data points per revolution. We note that a consistent choice of $\Delta$ is of major importance to the method. If an orbit contains frequency components that

\[\text{The measure should give } 0 \text{ within } 0 < t \lesssim 1024. \text{ However, it is still true that the signal } y_A \text{ is quantified as less complex than } y_B \text{ and that the transition from } y_A \text{ to } y_B \text{ is picked up easily.}\]
Figure 5. Complexity measures of two unperturbed binaries with eccentricity $e = 0.05$ (left-hand panels) and $e = 0.95$ (right-hand panels), respectively. For each binary, we show the $x$-component result of body #1. From top to bottom: (a) velocity time-series, (b) DWaT, (d) DWaTIM $H_{x1}$. The solid horizontal lines in (c) show the average over the $x$, $y$, and $z$-components for 100 binaries with $e$ and $\Pi_j$ randomly chosen within $0 \leq e < 1$ and $2^{-8} \leq \Pi_j \leq 2$, respectively. Dashed lines are one standard deviation.

We also computed the average complexity of the $v_x$, $v_y$, and $v_z$ DWaTIM measures for a total of 100 different binaries. For each binary, the eccentricity and the periodicity was randomly chosen in the intervals $0 \leq e < 1$ and $2^{-8} \leq \Pi_j \leq 2$, respectively. The corresponding results for the DWaTIM are shown by the horizontal lines (in blue) on Fig. 5(c). The solid central line displays the mean DWaTIM; the interval delimited by the upper and the lower dashed line indicates the standard deviation. The average DWaTIM is $\approx 0.22$. This value can be seen as indicative for the complexity of unperturbed binaries with a periodicity $\Pi_j$ comprised within the DWaT bandwidth.

3.2 The Pythagorean problem: $N = 3$

The well-studied Pythagorean configuration (Burrau 1913) is a classic example of long-term complex behaviour (see e.g. Aarseth et al. 1994). The initial conditions consist of three particles at rest, placed at the vertices of a Pythagorean triangle. The initial conditions are given in Table 1 and the configuration is depicted in Fig. 6.

Orbital integrations were performed over 1500 time units by repeatedly rerunning the initial configuration with a reduced per-step integration error until convergence of the result was reached. In this way, the final trajectories showed a total absolute energy error $|E_{\text{final}} - E_{\text{initial}}|$ of less than $10^{-10}$. The analysis is restrained to the first 1024 time units of integration, during which we found by visual inspection that the system spent a comparable amount of time in trivial and in more complicated states. Once more, the time-series comprise $Q = 8192$ data points. Results of the respective time-series analysis for particle #1 and for the entire Pythagorean

Table 1. Pythagorean problem: initial conditions.

| Body | Mass | $x$ | $y$ | $z$ | $v_x$ | $v_y$ | $v_z$ |
|------|------|-----|-----|-----|-------|-------|-------|
| #1   | 3    | 1   | 3   | 0   | 0     | 0     | 0     |
| #2   | 4    | -2  | -1  | 0   | 0     | 0     | 0     |
| #3   | 5    | 1   | -1  | 0   | 0     | 0     | 0     |
problem are shown in the left- and right-hand panels of Fig. 7, respectively.

The $v_y$-coordinate of body #1 is shown in the left-hand panel of Fig. 7(a). In the right-hand panel of Fig. 7(a) we illustrate the dynamics of the whole Pythagorean problem by showing the position $x$ of all the 3 bodies. Here we see that the system is characterized by a highly complicated interplay of the three particles, including a sequence of intermittent binary formation and disruption. In the time interval $300 \lesssim t \lesssim 540$ for instance, the single body #1 strides away to large distance on a smooth trajectory (upper dashed line in the right-hand panel of Fig. 7(a) during that interval). Due to the recoil, a binary, formed of bodies #2 and #3, leaves in the opposite direction. The motion of the binary stars about their common centre of mass gives the thick line seen in the lower part of the right-hand panel of Fig. 7(a) within $300 \lesssim t \lesssim 540$.

The DWaT scalogram of body #1 and the average DWaT of the Pythagorean problem (the average magnitude of base frequency $1/\Pi_1$ at integration time $t$), computed by averaging over the 3 particles and over the $x$- and $y$-components, are shown in the left- and right-hand panels of Fig. 7(b), respectively. The DWaTIM for body #1, $\Upsilon_{1,1}$, and the overall DWaTIM of the Pythagorean problem, $\Upsilon_{\text{tot}}$ (see Section 2.2.1), are shown in Fig. 7(c). The dotted line depicts the DWaTIM as computed by taking into account all the scales $j = 1, \ldots, 13$. The solid line highlights the dramatic improvement of the diagnostic, obtained when ignoring the first 4 scales ($j = 1, \ldots, 4$) in the computation. As mentioned earlier (see Section 2.2.2), both complexity measures cease to yield reliable results in the limit where the orbit is resolved over less than eight complete oscillations (i.e. dynamical times). This motivated our decision to exclude the low-frequency DWaT scales in a general manner. We take the conservative choice of not including the $j = 4$ scale (i.e. the periodicities $\Pi_1$ corresponding to $1/8$ of the full signal). For the remainder of this work, we only compute and discuss this improved version of the DWaTIM.

The performance of the DWaTIM can be studied with respect to two criteria: (1) time-localization, i.e. the ability of the measure to identify accurately the instants $t$ where qualitative changes in the orbital dynamics occur and (2) continuous quantification of complexity, i.e. the ability of the measure to provide a consistent evaluation of complexity in an uninterrupted manner. In order to discuss (1), we examine the time-series of body #1 of the left-hand panel of Fig. 7(a). Let us consider the transitions from a regime where body #1 approaches bodies #2 and #3 on a comparatively smooth trajectory to the more complex regime where it undergoes multiple close encounters with the latter two bodies. By visual inspection of the left- and right-hand panels in Fig. 7(a), we identify...
such transitions to occur at e.g. \( t \approx 200 \), \( t \approx 530 \) and \( t \approx 800 \). In each of these cases, we can see in Fig. 7(c) that the measure \( H_1 \) shows the corresponding transition. For example, the transition at \( t \approx 530 \) is identified by the DWaTIM by an increase of about 0.7. The instantaneous close encounter among the three particles at \( t \approx 800 \) is also singled out. Here we find a local peak of \( H_1 \approx 0.8 \). Likewise, the instant \( t \approx 875 \) at which body \#1 forms a binary system with one of the remaining bodies is also recovered; the complexity shows an instantaneous peak at that instant and then gradually decreases from \( H_1 \approx 0.6 \) at \( t \approx 875 \) to a mean value of \( H_1 \approx 0.25 \) at \( t = 1024 \). The accuracy with which the epoch of a transition is resolved depends on the frequency bands involved in the transition. As mentioned earlier, the DWaTIM resolution in time increases by a factor of 2 when shifting from scale \( j \) to \( j + 1 \). High-frequency transitions can therefore be localized more accurately in time than low-frequency transitions. The maximum latency of the DWaTIM is obtained by studying the particular case of a transition involving the two lower scales \( j = 5 \) and 6. The maximum latency then corresponds to the resolution at scale \( j = 6 \), i.e. 1024/2^5 = 32 time units. This sets an upper limit to the error on the DWaTIM performance in time-localization. In conclusion, qualitative changes of orbital dynamics can be accurately localized in time by the DWaTIM indicator.

Concerning (2) the continuous quantification of complexity, we argue that, broadly speaking, the \( H_1 \) of the left-hand panel of Fig. 7(c) reproduces the complexity of the frequency spectrum obtained by the DWaT in Fig. 7(b) in a consistent manner. The relative complexity of the time-series is quantified continuously in time with respect to the two limiting cases of a single base frequency sinusoid with \( H_1 = 0 \) and a white noise signal of \( H_1 = 1 \). Excluding the low-frequency scales \( j = 1, \ldots, 4 \) has also some bearings on the reliability of the diagnostics. For instance, when these frequencies are included, the DWaTIM produces inconsistent results in the interval \( 300 \lesssim t \lesssim 540 \) (see dash–dotted line in Fig. 7c). The complexity in the case of unperturbed motion of body \#1 is then estimated to be higher than for the case where the body is bound in a binary (compare to the dash–dotted line in the interval \( 875 \lesssim t \lesssim 1024 \)).

We also investigated the effects of analysing the motion of body \#1 through a different time-series on the \( H_1 \) complexity diagnostic. This additional consistency check allows us to measure DWaT border effects for the case of the Pythagorean problem and to examine the extent to which the DWaTIM is able to provide an absolute measure of complexity. To compare two time-series we constructed a new one by taking the first half of the \( t \) sequence for the original \( Q = 8192 \) case and \( t = 512 \). This gave us a new time-series \( v_{i1}(t_q) \) of \( Q' = 4096 \) data points. The DWaT analysis was then performed over a different frequency domain than for the original \( Q = 8192 \) case (i.e. \( t = 1024 \)) case. The time-series \( v_{i1}(t_q) \) had also a different end value than its parent time-series \( v_{i1}(t_q) \), i.e. \( v_{i1}(t_q) \neq v_{i1}(t_q) \). The amplitude of DWaT border effects were therefore different than for the \( Q = 8192 \) case (see Section 2.1.2). We computed the DWaTIM \( H_1 \) and compared it with \( H_1 \) shown in Fig. 7(c) in the interval \( 0 \leq t \leq 512 \). Except for values right at the edge of Fig. 7(c), the results were nearly identical. In particular, the rms residual between the two measures was \( \approx 0.095 \) within \( 0 \leq t \leq 128 \) (corresponding to the first 25 per cent of the signal \( v_{i1} \)) and \( \approx 10^{-3} \) within \( 128 < t \leq 512 \).

We obtain similar results for the time-series of the other two bodies \#2 and \#3, and for the \( x \)-component. For that reason we omitted to display those results. The overall DWaTIM \( \Upsilon_{\text{tot}} \) of the Pythagorean problem (shown in Fig. 7c, right-hand panel) gives a time-resolved insight on the global orbital evolution obtained by integrating the initial conditions of Table 1. Once more, the features seen in the curve of \( \Upsilon_{\text{tot}} \) match those in real space of Fig. 7(a), right-hand side panel. We further discuss the utility of this general measure in Section 3.3.

### 3.3 Perturbed \( N = 5 \) Caledonian configuration

Next we perturb a configuration of the planar Caledonian symmetric four-body problem (hereafter CP; Széll et al. 2004). Initial conditions are shown in Fig. 8 and in Table 2. The CP is set up as described in fig. 1 of Széll et al. (2004). We specialize to the case where all bodies have masses equal to 1. For this configuration, the unperturbed CP consists of two stable binaries evolving in the \( xy \) plane around their common centre of mass. We now perturb the CP configuration with a fifth body of mass \( m = 1.5 \) (body \#5 in Table 2). The perturber approaches the \( xy \) plane as it moves towards the centre of mass of the two binaries on a time-scale of \( \approx 8\pi/5 \). At that moment the distance between body \#5 and the centre of mass is \( \approx 0.15 \), i.e. about the separation of the two binaries. Therefore the CP is strongly perturbed from \( t \approx 5 \) onwards. The subsequent evolution is characterized by repeated interactions including the formation of hierarchical systems with several particles orbiting around a hard central binary.

The outcome of the complexity analysis is summarized in Fig. 9. Fig. 9(a) gives the \( x \)-position time-series of the five particles; panels (b) and (c) show the average DWaT and \( \Upsilon_{\text{tot}} \) (similarly to the right-hand panels of Figs 7a–c). Fig. 9(a) gives an overview of the motion for the entire integration time of 64 time units. (Note that the motion of particle \#5, plotted with a dotted line in Fig. 9a, is difficult to disentangle and is hardly visible.) The intricate gravitational interplay among the five particles is difficult to follow by visual inspection: the individual evolution of the 5 orbits will not be discussed in detail. We focus on the evolution of the system as a whole. The dynamics up to \( t \approx 43 \) may be described by roughly two qualitatively different states of motion. The first one is sometimes referred to as

![Figure 8. Caledonian five-body problem: initial configuration.](image-url)
‘hierarchical interplay’ (see e.g. Gemmeke et al. 2008). It is characterized by a central binary and three particles orbiting at large radii. A clear example of this can be seen in Fig. 9 in the time interval $23 \lesssim t \lesssim 36$. (The hard central binary forms at $t \approx 23$ as indicated by the vertical arrow). The second state of motion is what we call ‘democratic interplay’. This regime is characterized by multiple close encounters among the five bodies, each body contributing an approximately equal amount to the overall complexity of the system. Examples of democratic interplay are the time intervals $5 \lesssim t \lesssim 10$ and $14 \lesssim t \lesssim 23$. At $t \approx 43$, the dynamics of the perturbed CP changes dramatically. After a close encounter around that time, all the particles stride away to larger distance. One particle immediately escapes the system on a nearly rectilinear trajectory (see upper dashed line in Fig. 9a at that instant). The remaining bodies stay close to each other until $t \approx 46$, when they are subjected to a further close encounter and another body is ejected (see lower solid line). The subsequent motion is relatively smooth. One particle remains (dashed line) and orbits around a hard binary (indicated by the horizontal arrow).

The overall complexity of the perturbed CP is shown by the DWaTIM in Fig. 9(c). Let us consider two time intervals, prior to and after $t = 43$. When $t \leq 43$ the DWaTIM $T_{tot}$ is roughly constant and it is interesting to observe that the local minima seen in that time interval can be found during the more quiescent hierarchical regimes, such as e.g. at $t \approx 32$. The two peaks observed at $t \approx 43$ and $t \approx 46$ arise from the two independent high-energy encounters that we have described earlier taking place at those times. When the final binary forms and the system dissolves the motion becomes unmistakably less complex and consequently $T_{tot}$ has a lower value of $\approx 0.2$ on average for all time $t \gtrsim 48$.

### 4 Equal Mass $N = 256$ Plummer Sphere

We apply the DWaTIM technique to a self-gravitating spherical polytrope of index $n = 5$ (Plummer 1911; Binney & Tremaine 1987, section 4.4.3). The phase-space distribution function $F$ of this system is a power law of $\varepsilon$,

$$F(\varepsilon) = \frac{24\sqrt{2}}{7\pi^3} \frac{R^2}{G^2 M^4} (-\varepsilon)^{7/2}.$$  \hspace{1cm} (26)

where $\varepsilon = (1/2) v^2 + \Phi(r) < 0$ is the mechanical energy per unit mass, $v$ the three-dimensional velocity, $\Phi$ the potential which is a function of the radius $r$ only. Note that $F = 0$ by construction whenever $\varepsilon > 0$ so that only mass elements bound by gravity are considered. Given a value of the gravitational constant $G$, the two free parameters $M$ and $R$ define the total system mass and a reference unit of length, respectively. Integrating $F$ over all velocities at constant radius yields the mass density $\rho$ at that radius, $\rho(r) = \int_0^{\sqrt{-2\varepsilon(r)}} 4\pi F(\varepsilon) v^2 dv = \frac{3}{4\pi R^3} \left(1 + \left[\frac{r}{R}\right]^2\right)^{-7/2}$, \hspace{1cm} (27)

where we have used $d^3v = 4\pi v^2 dv$ valid for an isotropic velocity field and where we have substituted for $\Phi(r)$ by solving Poisson’s equation. Note that the length $R$ defines the radius of a uniform-density core ($\rho[r \ll R] \approx$ constant). An $N = 256$ particle representation of equation (27) is obtained by random-sampling the mass density to assign three-dimensional positions. All bodies have mass $m_i = M/N$, where $i = 1, \ldots, N$. The particles’ energy is attributed similarly from equation (26) from which we compute the square velocity $v^2 = 2(\varepsilon - \Phi(r))$ as in the standard method of Aarseth, Hénon & Wielen (1974). In those circumstances the total kinetic and gravitational energies satisfy the virial theorem of equilibrium systems.

The nominal dynamical time is $t_d = 2R/\sigma$ where $\sigma^2$ is the mean squared velocity averaged over mass up to the half-mass radius. We found $\sigma^2 = 0.392 \ldots GM/R \approx \pi GM/(8R)$. The two-body relaxation time is conveniently defined as $t_i = N t_d / \ln 0.4N$ (see Binney & Tremaine 1987, Fig. 10).

Figure 9. Overall complexity of the Caledonian problem (see also Table 2 and Fig. 8). From top to bottom: (a) position time-series of the five particles, (b) averaged DWaT, (c) overall IM $T_{tot}$.

Figure 10. From bottom to top: core radius $r_c$, 20, 30, 40, 50, 60, 70, 80 and 90 per cent Lagrangian radii of the $N = 256$ equal-mass Plummer sphere. The solid vertical line at $t = 113$ marks the formation of the first hard binary in the system. The dashed vertical lines indicate the onset and the end of a notable core expansion within $165 \lesssim t \lesssim 180$. 

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of the mass, or 33 bodies). Fig. 10 displays the time-evolution for the initial configuration (see e.g. von Hoerner 1960; Casertano & Hut 1985). From equation (27) we find a quantity which monitors the rise of the central density. The density-averaged core radius in each time interval (the circles on the figure indicate where the core radius $r_c$ decreases on the mean. At that time, the first hard binary of binding energy $E < -100 kT$ forms. That event is marked with a vertical full line in Fig. 10. Note that $r_c$ along with the 20 and 30 per cent Lagrange radii increase significantly from $t \approx 165$ onwards. This phase of rapid expansion indicates the onset of post-collapse evolution for that simulation. The minimum value of $r_c$ occurs at $t \approx 165$ units and encloses 3 per cent of the total mass (eight bodies). We compute a dynamical time $t_{d\Sigma} \approx 0.05$ for the core at that time, a factor of $\approx 70$ smaller than the value computed from the initial conditions. An orbit confined to the core is adequately sampled with five points or more and hence we set $\Delta = 0.01$ for the complexity analysis. The time resolution of any features seen in the diagnostic of complexity is therefore $2\Delta = 0.02$ units, and any binary formed through dynamical evolution is well sampled provided its binding energy $E > -2.75 kT$.

4.1 Individual orbits

To illustrate the differences in orbital complexity a star may show between the initial time and the moment of core expansion, we graph in Fig. 11 the $xy$ projections of two individual orbits during two windows of four time units, the first running from $t = 0$ to 4 (top panels), the second running in the interval $t = 160$ to 164 (bottom panels). We picked two stars that happened to orbit within the core of the cluster at $t = 0$. The orbit starts off smooth and regular, but traces a much more intricate pattern later on.
Figure 12. Time evolution of six individual orbits in an $N = 256$ equal-mass Plummer sphere model. Each figure shows the complexity diagnostic obtained with respect to the $x$-coordinate of particle $i$, and comprises the velocity time-series $v_x$ (solid line in red, plotted against the left-hand axis) and the DWaTIM $H_x$ (solid blue line in the upper inset, plotted against the 0–1 scale shown on the right-hand axis).

These trends can be identified in a graph of the DWaTIM for these and four other orbits as displayed in Fig. 12. The figure shows $v_x$ versus time in the main frames and the DWaTIM as the top inset frame for each case. Note the change of scales: the DWaTIM is plotted against the scale shown at the right-hand side of each figure. Focusing on the top two panels in Fig. 12, we can identify the more complex phases around $160 \lesssim t \lesssim 164$ of the orbits displayed in Fig. 11(b) as local peaks in the DWaTIM during this time interval. It is clear that stars set on regular orbits initially can show more complex behaviour at later times. The opposite is also possible, as we illustrate in Fig. 12 with four more orbits also orbiting in and out of the core region. These and many others not shown here are typical of the wide variety of DWaTIM spectra: some orbits show rapid fluctuations in $v_x$ and yield a rather broad band of base frequencies (especially particles #6 and #182). Other trajectories have a narrower spectrum of frequencies (e.g. particle #128).

4.2 Global behaviour of the Plummer sphere

Fig. 13 shows as a function of time the cumulative number of stars whose DWaTIM $\Upsilon_i (i = 1, \ldots, N)$ exceeds a given threshold. The uppermost horizontal line on the figure denotes the total number of stars. The broken curves are for (from top to bottom) $\Upsilon_i > 0.2, 0.4, 0.5, 0.6$ and 0.7. Once more, the vertical solid and dashed lines indicate the time when a hard binary first formed and the interval of post-collapse core expansion, respectively (cf. also Fig. 10). Broadly speaking, the orbital complexity decreases on the mean with time. The uppermost lines showing $\Upsilon_i > 0.2$ and $\Upsilon_i > 0.4$ decrease
monotonically save for small localized fluctuations. In all the cases displayed, a pronounced drop in complexity is seen throughout the phase of core expansion, at times $160 \lesssim t \lesssim 185$. For instance, at the beginning of the expansion, about 110 stars have $\Upsilon_t > 0.4$ whereas at $t \approx 180$ only 75 stars reach that level. The situation is similar for higher threshold curves. After the formation of a hard binary but prior to post-collapse expansion, i.e. in the interval $113 \lesssim t \lesssim 165$, the complexity levels off or increases slightly with time. For example, the number of stars with $\Upsilon_t > 0.5$ increases from about 40 to approximately 55 stars in that interval. Hence, orbits that are already relatively complex at the time of binary formation yield a DWaTIM of even larger amplitude up to core-collapse and the onset of the expansion phase. This trend is also found in the curve $\Upsilon_t > 0.6$ and to a lesser extent in the $\Upsilon_t > 0.7$ curve. It may be important to note that the global results of Fig. 13 are not in contradiction with the apparent trend of increasing complexity depicted by the two stars of Fig. 11. Stars #6 and #182 of that figure are both part of the 8 stars that happen to reside in the core around $t \approx 165$. At these times the stars have a high DWaTIM with values of $\Upsilon_6$ and $\Upsilon_{182} > 0.6$ (see also the two upper panels of Fig. 12). Stars #6 and #182 therefore contribute to the local peak of the $\Upsilon_t > 0.6$ curve observed in Fig. 13 around $t \approx 160$. The snapshots given in Fig. 11(b) do not reflect a progressive increase in orbital complexity of the Plummer sphere within $0 < t \lesssim 164$. The enhanced two-body scatter attributable to the newly formed binary can be directly measured by the DWaT. In conclusion, both that event and the onset of post-collapse expansion can be singled out in Fig. 13 as a local minimum and a local maximum, respectively, in runs of the cumulative $\Upsilon_t$ (as exemplified e.g. by the curve of $\Upsilon_t > 0.5$).

A scalogram of the DWaT averaged over all $N = 256$ Plummer sphere particles is shown in Fig. 14. The dark shade illustrates base frequencies $1/\Pi_j$ of high amplitude, white means zero amplitude (see also Sections 3.2 and 3.3). Fig. 14(a) brushes a global picture for all scales $j = 1, \ldots, 15$, whereas Fig. 14(b) depicts only the high-frequency scales $j = 11, 12$ and 13. Some modes are growing in intensity and are then fading away after $t \approx 165$ units when the core starts to expand. A particularly good example is the scale $j = 6$ which reaches progressively higher amplitude in the interval $0 \lesssim t \lesssim 160$ before fading away during the expansion phase of the inner volume of the sphere. The onset of expansion triggers high-frequency modes (cf. Fig. 14b) which reflect the evolution towards more anisotropic radial orbits. Radial anisotropy implies more eccentric motion relative to the centre of mass and an enhanced spectrum of frequencies, cf. Fig. 5.

An illustration of the global measure of complexity $\Upsilon_{\text{tot}} = \sum \Upsilon_i/N$ is shown in Fig. 15. The solid and dashed vertical lines denote the same transitional phenomena as in all preceding figures. On the whole, the curve of the DWaTIM depicts the same global decrease in complexity as observed in Fig. 13. A close inspection of Fig. 15 suggests three different regimes in the evolution of the DWaTIM indicator $\Upsilon_{\text{tot}}$.

(i) Regime 1 takes place within $0 < t \lesssim 165$. In the interval $0 \lesssim t \lesssim 20$, we observe a rapid drop in $\Upsilon_{\text{tot}}$ which has no clear origin in a physical phenomenon (binary formation, core-collapse, etc.). The equilibrium cluster has a dynamical time $t_d$, whereas Fig. 14(b) depicts only the high-frequency scales $j = 11, 12$ and 13. Some modes are growing in intensity and are then fading away after $t \approx 165$ units when the core starts to expand. A particularly good example is the scale $j = 6$ which reaches progressively higher amplitude in the interval $0 \lesssim t \lesssim 160$ before fading away during the expansion phase of the inner volume of the sphere. The onset of expansion triggers high-frequency modes (cf. Fig. 14b) which reflect the evolution towards more anisotropic radial orbits. Radial anisotropy implies more eccentric motion relative to the centre of mass and an enhanced spectrum of frequencies, cf. Fig. 5.

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Figure 15. Overall DWaTIM complexity $\Upsilon_{\text{tot}}$ for an $N = 256$ equal-mass Plummer sphere. The curve is the average of $\Upsilon_t$ over all $N$ particles and over the three velocity time-series in each case (the $x$-, $y$- and $z$-components). The solid vertical line at $t = 113$ together with the rightmost arrow mark the formation of the first hard (stable) $E'/kT < -10$ binary in the system. The leftmost arrow marks the time when the first soft (unstable) $E'/kT < -10$ binary formed. The dashed vertical lines within $165 \lesssim t \lesssim 180$ bracket the period of expansion of the inner volume.

**5 DISCUSSION**

We presented a method to compute the time-dependent orbital complexity in $N$-body simulations. The gravitational $N$-body problem is described by the $3N$ second-order ordinary differential equations of equation (1). We extract a DWaTIM from the velocity time-series of the individual particles of the $N$-body simulation. We apply the technique to several few-body problems and to a larger $N = 256$ particle simulation. The method captures the time-dependent changes in the dynamics of three- and five-body systems and furthermore quantifies orbital complexity continuously in time. For example, we recovered and quantified the dynamically more complex phases of the well-studied Pythagorean problem (see Section 3.2) as well as the complex dynamics of a perturbed Caledonian configuration (see Section 3.3). We also applied the method to a set of $N = 256$ equal-mass Plummer spheres (see Section 3.3). We found that, on a global scale, orbital complexity decreases during the evolution. The occurrence of core-collapse and the subsequent core expansion causes a considerable drop in overall DWaTIM complexity $\Upsilon_{\text{tot}}$. Furthermore, we observed that the complexity of individual orbits with a DWaTIM $\Upsilon_t \gtrsim 0.5$ is a well-defined gauge of complexity. We obtain reliable diagnostics of complexity whenever the orbit is integrated as mother wavelets. This DWaT implementation avoids redundant wavelet coefficients. This helps in obtaining a proper measure of complexity for the two limiting cases of a unique base frequency sinusoid (DWaTIM = 0) and of a white noise signal (DWaTIM = 1), and thus to recover a well-defined gauge of complexity. We obtain a slower but constant decline sets in. The DWaTIM $\Upsilon_{\text{tot}}$ decreases from $\approx 0.38$ at $t \approx 20$ to $\Upsilon_{\text{tot}} \approx 0.35$ at $t \approx 165$; the formation of soft and hard binaries therefore has little impact on the global complexity of the system. That being said, it is difficult to disentangle the apparent trend of a drop in complexity during regime 1 because, first of all, the rapid drop early on is attributable to border effects; and secondly, the time-scale over which the trend becomes significant is comparable to the two-body relaxation time for that system. The data in Fig. 15 may be best understood in the light of Fig. 13. In that figure stars of low complexity show a decrease of $\Upsilon_t$ within $0 < t \lesssim 110$ (see the two upper curves $\Upsilon_t > 0.2$ and $>0.4$) whereas the number of stars on orbits giving a diagnostics of high-complexity remains approximately constant. This argues against border effects reaching beyond $t \approx 20$. The constant evolution towards more orbits of low-complexity diagnostics drive the trend of $\Upsilon_t$ to observed in Fig. 15, a statement that the system suffers from collisional effects when stars are shifted to higher energy long-period orbits by two-body collisions. Inspection of the rapid evolution of the core-radius comforts this view.

(ii) Regime 2 begins around $t \approx 165$ when the system enters post-collapse and the central volume expands systematically. This triggers a sudden drop in the DWaTIM from $\approx 0.35$ to $\approx 0.27$ at $t \approx 180$.

(iii) Regime 3 begins (loosely speaking) at $t \approx 180$ units when the run of $\Upsilon_{\text{tot}}$ resumes a slow decrease on the average. This contrast in DWaTIM orbital complexity between the pre- and post-collapse evolution has been seen in a set of ten $N = 256$ test simulations we performed with different random seeds. An attempt to obtain ensemble averaged results for this set of simulations proved fruitless owing to large scatter e.g. in the core-collapse time between two individual runs.

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