Persistent Superconductor Currents in Holographic Lattices

Norihiro IIZUKA,†‡ and Kengo MAEDA§,

†Interdisciplinary Fundamental Physics Team, Interdisciplinary Theoretical Science Research Group, RIKEN, Wako 351-0198, JAPAN
‡Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, JAPAN
§Faculty of Engineering, Shibaura Institute of Technology, Saitama 330-8570, JAPAN

We consider a persistent superconductor current along the direction with no translational symmetry in a holographic gravity model. Incorporating a lattice structure into the model, we numerically construct novel solutions of hairy charged stationary black brane with momentum/rotation along the lattice direction. The lattice structure prevents the horizon from rotating, and the total momentum is only carried by matter fields outside the black brane horizon. This is consistent with the black hole rigidity theorem, and suggests that in dual field theory with lattices, superconductor currents are made up by “composite” fields, rather than “fractionalized” degrees of freedom. We also show that our solutions are consistent with the superfluid hydrodynamics.

- Introduction -

Over the past few years a considerable number of studies have been made in applying the idea of holography, or the gauge/gravity duality, to strongly coupled quantum systems. Famous examples of such are hairy black hole solutions which are dual to the boundary superconductor (see, e.g., [2] and references therein). However, most of the models so far considered do not adequately capture essential features of realistic condensed matter models in the sense that they admit translational symmetry, under which momentum is conserved. Only quite recently, there have appeared some attempts to construct holographic models with no translational symmetry [3–6], in which infinite conductivity was confirmed in superconducting states [7, 8]. In such holographic superconductors without translational symmetry, the persistent superconductor current is expected to have a holographic description by a stationary rotating black hole solution with momentum along the direction with no translational symmetry. On the other hand, according to black hole rigidity theorem [9,11], a stationary rotating black hole must have a symmetry along the direction of momentum conserved [28]. In this paper, we resolve the apparent conflict mentioned above by constructing novel black brane solutions with momentum along the direction of no translational invariance, which is dual to a superconducting state without dissipation.

In our solutions, the lattice structure prevents the horizon from rotating, and the total momentum is only carried by matter fields outside the black brane horizon. A key point is that we incorporate a lattice structure without inducing inhomogeneities and in doing so, our solutions have non-dissipating momentum along the lattice. To our best knowledge, this is the first example of a holographic gravity model that has—as a legitimate dual to a superconducting phase—no dissipation, and that is only made possible by taking into account the effects beyond the linear response theory.

- Helical lattices from Bianchi type VII0 -

In order to introduce the lattice effects in holography, we will make use of the Bianchi type VII geometry, characterized by the following three Killing vectors,

\[ \xi_1 = \partial_{x^2}, \quad \xi_2 = \partial_{x^3}, \quad \xi_3 = \partial_{x^1} - x^3 \partial_{x^2} + x^2 \partial_{x^3}, \quad (1) \]

which form the Lie algebra, \([\xi_i, \xi_j] = C^{k}_{ij} \xi_k\) with \(C^{1}_{23} = -C^{1}_{32} = -1, \quad C^{2}_{13} = -C^{3}_{12} = 1\) and the rest \(C^{i}_{jk} = 0\). Associated with them are the following three one-forms,

\[ \omega^1 = \cos(x^1) dx^2 + \sin(x^1) dx^3, \]
\[ \omega^2 = -\sin(x^1) dx^2 + \cos(x^1) dx^3, \quad \omega^3 = dx^1, \quad (2) \]

each of which is invariant under all the Killing vectors \(\xi_i\). Using these, in this paper we make the following metric ansatz;

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + e^{2v_3(r)} (\omega^1 - \Omega(r) dt)^2 + e^{2v_1(r)} (\omega^2)^2 + e^{2v_2(r)} (\omega^3)^2. \quad (3) \]

where \(f(r), \Omega(r), v_i(r)\), with \(i = 1, 2, 3\), are functions of the radial coordinate \(r\) only. Under more generic Bianchi type anisotropic metric ansatz, the near horizon geometries of static, i.e., \(\Omega(r) = 0\), black brane solutions, which admit homogeneous [29] but generic anisotropic metric ansatz, are classified and studied in [15,16]. With \(\Omega(r) \neq 0\), there is a flow of the geometry along \(x^1\), i.e., the black branes can “rotate” along the \(x^1\) direction.

Note that if \(v_1(r) = v_2(r)\), then, due to \((\omega^1)^2 + (\omega^2)^2 = (dx^2)^2 + (dx^3)^2\), we have translational invariance along \(x^1\). However, as long as \(v_1 \neq v_2\), there is no translational invariance along \(x^1\) direction, i.e., \(\partial_{x^1}\) is not a Killing vector of the geometry. Since \(x^1 \rightarrow x^1 + 2\pi n\) (with \(n\) integer) is a discrete symmetry, there is a “helical lattice” structure along \(x^1\) direction [17]. However the metric ansatz [3] is a homogeneous one and this homogeneity enables

norihiro.iizuka@riken.jp  iizuka@phys.sci.osaka-u.ac.jp
akihiro@phys.kindai.ac.jp
maeda302@sic.shibaura-it.ac.jp
us to reduce the Einstein equations to a tractable set of ordinary differential equations. Note that in the homogeneous model without angular momentum, a Drude peak and nonzero resistance are found in a normal state [17], implying that momentum is generically lost due to the umklapp scattering.

Before discussing our explicit model, we can, at this stage, argue generic nature of our geometry. Under the metric ansatz (3), we are concerned with black branes whose event horizon $H$ is given by $f(r) = 0$. Then, the tangent vector $l$ along the null geodesics of such an $H$ is

$$l = \partial_t + \Omega_h \partial_x , \quad \Omega_h \equiv \Omega|_{r=r_h} ,$$

where $r = r_h$ denotes the root of $f(r) = 0$, the horizon radius. Then, we have, on $H$

$$R_{\mu\nu}l^\mu l^\nu = -2\Omega_h^2 (\sinh(v_1 - v_2))^2 .$$

In order to satisfy the null energy condition, i.e., $R_{\mu\nu}l^\mu l^\nu \geq 0$, we need either $\Omega_h = 0$ or $v_1 = v_2$ at the horizon [30]. This implies that it is impossible to have a rotating horizon along $x^1$ with the lattice effects. This is a consequence of the black hole rigidity theorem, which claims that under the analyticity assumption, a stationary rotating black hole must be axisymmetric [9–11] [31].

- A Holographic Model -

The model we consider is five-dimensional action $S = \int d^5 x \sqrt{-g} \mathcal{L}$ where the Lagrangian has $U(1) \times U(1)$ gauge symmetry;

$$\mathcal{L} = R + \frac{12}{L^2} - \frac{1}{4} F^2 - \frac{1}{4} W^2 - |D\Phi|^2 - m^2 |\Phi|^2 ,$$

where $R$ is Einstein-Hilbert term, $L$ represents AdS scale and $A_\mu, B_\mu$ are one-form gauge potentials, and their field strengths are $F = dA, W = dB$. $\Phi$ is a complex scalar field, which is charged under only the gauge potential $A_\mu$, but is neutral to $B_\mu$. Covariant derivative acting on $\Phi$ is $D_\mu = \nabla_\mu - iq A_\mu$.

The one-form $A_\mu$ is to introduce a chemical potential. We make an ansatz for the other one-form $B$ to be proportional to type VII 0 Bianchi form, so that it induces holographic “helical lattice” effects. If we set $B = 0$, then it makes our model the same type as [18].

We solve the field equations with the metric ansatz (3) and

$$A_\mu dx^\mu = A_t(r) \omega^3 + A_t(r) dt ,$$

$$B_\mu dx^\mu = b(r) \omega^1 , \quad \Phi = \phi(r) ,$$

where we have set the phase of $\Phi$ to be zero by the gauge transformation, as easily checked from the equation of motion.

The equations of motion for the metric component $\xi$ are given by

$$v_1(r) - v_2(r) \text{ and } b \text{ are given by}$$

$$f \xi'' + \left( f' + f (v_1' + v_2' + v_3) \right) \xi' - 2 e^{-2v_3} \left( 1 - e^{2v_3} f^{-1} \Omega^2 \right) \sinh 2 \xi$$

$$= \frac{1}{2} e^{-2(v_2+v_3)} \left( 1 - e^{2v_3} f^{-1} \Omega^2 \right) b^2 - \frac{1}{2} e^{-2v_3} \Omega' b^2 ,$$

$$f b'' + \left( f' + f (v_2' + v_3 - v_1') \right) b'$$

$$- e^{-2(v_1-v_2)} \left( e^{-2v_3} - f^{-1} \Omega^2 \right) b = 0 ,$$

where $'$ is the derivative with respect to $r$. The gauge potential $b$ plays the role of “source term” for $\xi$ evolution [17]. It is then clear that nonzero $b$ only introduces the helical lattice effects, i.e., nonzero $\xi \neq 0$. There is no matter field other than $b$ which plays the role of source to induce the disparity between $\omega^1$ and $\omega^2$ for the metric ansatz (3) in our model (32). Therefore, we seek for the solution with $b(r) \neq 0$, and therefore $\xi(r) \neq 0$: One form $B$ with ansatz (8) is our source for holographic helical lattice effects.

- Near horizon analysis -

Let us analyze the near horizon of (10) in the finite temperature case. Near the horizon, the metric should have a single zero $f(r) \approx \kappa (r - r_h)$ ($\kappa > 0$). Furthermore let us assume that $v_1$ is at least $C^2$ and set

$$\lim_{r \to r_h} \Omega(r) = \Omega_h , \quad \lim_{r \to r_h} \xi(r) = \xi(r_h) .$$

Now suppose that $\Omega_h$ is nonzero for a moment. Then, as the r. h. s. of (5) must be non-negative, we must impose $\xi(r_h) = 0$, and thus reduce (10) to

$$\kappa^2 (r-r_h)^2 b'' + \kappa^2 (r-r_h)' + \Omega_h b \simeq 0 ,$$

in the $r \to r_h$ limit. This admits solutions

$$b(r) \propto (r-r_h)^{\eta} , \quad \eta \equiv \pm \frac{\Omega_h}{\kappa} .$$

Since both solutions are singular at the horizon, we have to choose their coefficients to vanish and thus $b(r) = b'(r) = 0 \text{ at } r \to r_h$ to obtain a smooth solution. However, this means that $b(r)$ must vanish identically in all the radius as $b(r)$ obeys the 2nd order differential equation (10), and as a consequence $\xi(r) = 0$. This itself does not cause any problem with the rigidity theorem, since it implies that either $\Omega_h$ or $\xi$ must vanish at the horizon. However since we are interested in constructing a holographic model with lattices, we are interested in a solution with $b(r) \neq 0$, and this forces us to choose

$$\lim_{r \to r_h} \Omega(r) = 0 .$$

Therefore we conclude that in our set-up, the lattice effects and smoothness condition force us to have flows (or “rotation”) only outside the horizon. Black brane horizon cannot be rotating. In addition, we choose for $A_t$ at the horizon $\lim_{r \to r_h} A_t(r) = 0$, as in the static case (32).

By the condition (14), $\Omega \simeq \Omega_h^t (r-r_h)$ near the horizon. Furthermore, from the explicit equations of motion in
terms of the component $f, \Omega, v_1, v_3, \xi, A_t, A_{x^1}, b, \phi$, one can show that there are the 9 free parameters:

$$
A_{x^1}(r_h), \quad A'_t(r_h), \quad \phi(r_h), \quad \xi(r_h), \quad v_1(r_h), \quad v_3(r_h), \quad \kappa, \quad \Omega'_h, \quad b(r_h). \quad (15)
$$

We will tune these nonzero parameters in such a way that we will have asymptotically AdS geometry with no non-normalizable modes, except for $b$ field [34]. So, we impose at $r \to \infty$,

$$
\lim_{r \to \infty} \xi(r) \equiv v_1(r) - v_2(r) = 0. \quad (16)
$$

Then the lattice effect disappears and we have translational invariance restored at UV.

The asymptotic behavior of $\Omega$ is given by

$$
\Omega \simeq \Omega_0 + \frac{\Omega_N}{r^4}. \quad (17)
$$

Then, we impose

$$
\lim_{r \to \infty} \Omega(r) = 0, \quad (i.e., \quad \Omega_0 = 0) \quad (18)
$$

because we are interested in a solution with no non-normalizable modes for $g_{\mu \nu}$.

Near the boundary $r \to \infty$, the scalar field behaves

$$
\phi \simeq C_+ r^{\lambda_+} + C_- r^{\lambda_-}, \quad \lambda_{\pm} = -2 \mp \sqrt{4 + m^2 L^2}. \quad (19)
$$

For numerics purpose, we will choose the value of mass as $m^2 L^2 = -\frac{15}{4}$. In this case $\phi \simeq C_+ r^{-\frac{5}{2}} + C_- r^{-\frac{3}{2}}$, and both are normalizable modes. As usual, we can impose the boundary condition that either $C_+$ or $C_-$ becomes zero [18]. In this paper, we shall impose $C_- = 0$. These conditions [16, 18], and $C_- = 0$ are achieved by tuning the parameters [15]. Under these conditions, following requirement of vanishing non-normalizable mode is satisfied,

$$
\lim_{r \to \infty} \left( f(r) - \frac{r^2}{L^2} \right) = O \left( \frac{1}{r^2} \right). \quad (20)
$$

**- Numerical Solution Interpolating IR and UV-**

A numerical solution is shown in FIG. 1. This is for the parameter choice,

$$
\begin{align*}
& r_h = 3.033, \quad \kappa = 1.821, \quad \phi(r_h) = 1, \quad \Omega'_h = 0.1, \\
& b(r_h) = 7.354, \quad A'_t(r_h) = 2.589, \quad A_{x^1}(r_h) = -0.7470, \\
& \xi(r_h) = -0.125, \quad v_1(r_h) = 0.8457, \quad v_3(r_h) = 1, \quad (21)
\end{align*}
$$

and

$$
q = 1, \quad L^2 = 2, \quad m^2 = -15/8. \quad (22)
$$

In our solution, asymptotically at $r \to \infty$, $e^{v_1} = e^{v_2} = e^{v_3} = r$ and the conditions [16, 18] are satisfied so that lattice disappears at the boundary and geometry approaches AdS metric.

The asymptotic behavior of $A_{x^1}$ is $A_{x^1} \simeq a_{x^0} + \frac{b_{x^0}}{r}$. According to the AdS/CFT dictionary, $\sqrt{2} a_{x^1}$ corresponds to the current in the dual field theory, while $-2\sqrt{2} \Omega_N$ in (17) corresponds to $\langle T'_{x^1} \rangle$ component of the expectation value of the energy momentum tensor $\langle T_{\mu \nu} \rangle$ on the boundary theory [35]. We numerically find that $a_{x^1N} \simeq 5.873$ and $\Omega_N \simeq 12.61$ under the boundary condition $\Omega_0 = 0$. Similarly, the asymptotic behavior of $b$ is $b \simeq b(\infty) + \frac{b_{x^0}}{r}$, and we numerically find $b(\infty) = 8.012$.

Since $\Omega(r) \to 0$ at the horizon, black branes are not “rotating.” However, as $\Omega(r) \neq 0$ between the horizon and infinity, our solution is not static. This may be viewed that the matter field outside the horizon is rotating. Note also that there is no ergosphere with respect to $\partial_r$ outside of the horizon, as $g_{tt}(r) < 0$ for all the range between horizon to infinity.

**- Superfluid Hydrodynamics -**

Given above solution, it would be interesting to change various input parameters and see if there is relationship between various solutions. In TABLE I, we show some of our solutions, including the one in FIG 1, where we vary the parameters such as: $\phi(r_h)$ (charged scalar field at the horizon), $T = \kappa/4\pi$ (temperature), $\mu \equiv A_t(\infty)$ (chemical potential), $b(\infty)$ and $\zeta \equiv A_{x^1}(\infty)/A_{x^1}(\infty)$ (superfluid fraction) from the previous choice [21], and obtained the boundary expectation values of $\langle T_{x^1} \rangle$ and $\langle j_{x^1} \rangle$ from the normalizable modes of $g_{x^1x^1}$ and $A_{x^1}$ at the boundary. In FIG. 2, we draw the 3D plot of $(T/\mu, b(\infty)/\mu, -\zeta)$ values of our solutions, by connecting vertex points which we obtained from the solutions. We keep the parameters in [22] the same.

One can show that in all of the solutions, including the ones in the TABLE I, the relation,

$$
\frac{\langle T_{x^1} \rangle}{\mu \langle j_{x^1} \rangle} = -1.000 \pm O(10^{-4}), \quad (23)
$$

holds in a very high precision, independent on the various parameter choices for $\phi(r_h)$, $T$, $\mu$, $b(\infty)$. In the
TABLE I: Boundary stress tensor \( \langle T_{x^1} \rangle \) and current \( \langle j_{x^1} \rangle \) for various choices of output parameters: charged scalar field at horizon \( \phi(r_h) \), temperature \( T = \kappa/4\pi r \), chemical potential \( \mu = A_1(\infty) \), non-normalizable source field \( b(\infty) \), and superfluid fraction \( \zeta = A_{x^1}(\infty)/A_1(\infty) \).

| \( \phi(r_h) \) | \( T \) | \( \mu \) | \( b(\infty) \) | \( -\zeta \) | \( \langle T_{x^1} \rangle \) | \( \langle j_{x^1} \rangle \) |
|---|---|---|---|---|---|---|
| 1 | 0.08138 | 4.325 | 5.927 | 0.5489 | -61.60 | 14.24 |
| 1 | 0.1450 | 4.295 | 5.812 | 0.2491 | -35.67 | 8.306 |
| 2/3 | 0.03570 | 4.071 | 4.955 | 0.7103 | -24.03 | 5.903 |
| 2/3 | 0.1059 | 3.919 | 7.057 | 0.5018 | -23.06 | 5.885 |
| 4/5 | 0.1513 | 4.003 | 7.048 | 0.2524 | -20.47 | 5.114 |

FIG. 2: 3D plot of dimensionless parameter \( (T/\mu, b(\infty)/\mu, -\zeta) \) relation, fixing \( \phi(r_h) \) values. Vertex points are obtained from numerical data. The upper surface is for fixing \( \phi(r_h) = 2/3 \), and the lower surface is for fixing \( \phi(r_h) = 1 \). One can clearly see the tendency that the surface exhibits a slope falling to the right. This implies that as we increase either \( T/\mu \), or \( b(\infty)/\mu \), or \( |\zeta| \), the condensate VEV \( \phi(r_h) \) decreases.

In this paper, we have numerically constructed stationary, non-static hairy black brane solutions, which have momentum/rotation along the direction of no translational invariance due to lattices without dissipation [35]. This is dual to the persistent superconductor current along the lattice direction, which is more realistic than previous holographic superconductor solutions. In the bulk, we consider the backreaction of \( A_{x^1} \) to the metric \( g_{x^1} \). This is because the DC conductivity diverges and therefore current can be large without external electric field. Note that in our solutions, there are no source term corresponding to the external electric fields.

One of the key features in constructing our solutions is the rigidity theorem, which forces us one of the below choices: 1. Holographic lattice effects survive near the horizon, but black brane cannot be rotating. Rotation is carried by the matter field surrounding black brane. 2. Black brane is rotating, but holographic lattice effects disappear. Note that in our metric ansatz, we have shown that in case of 2, lattice effects actually disappear in all of the radius due to the regularity assumption, and therefore, we are forced to choose 1. Actually we expect that this is a generic nature. We conjecture that on the gravitational action without any artificial sources, holographic lattices prevent black brane from rotating and momentum must be carried by the matter fields outside.

It is very interesting to ask what it implies in dual field theory that holographic lattices kill the rotation of black branes. This might suggest that in dual field theory, in the absence of the lattices, “fractionalized” [24, 25] degrees of freedom cannot have current along the lattice direction in the stationary limit. “Fractionalized” degrees of freedom are thought to be responsible for the non-Fermi liquid behavior, and actually they are degrees of freedom violating Luttinger theorem [26]. It is interesting to ask what is the implication of the rigidity theorem and how much it gives restriction on the dynamics of “fractionalized” degrees of freedom, in condensed matter physics.

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[28] This aspect may be explained by the fact that if an asymmetric black hole solution is rotating along the direction of no symmetry, it loses the angular momentum by the emission of gravitational or electromagnetic waves [14]. Indeed, no such an asymmetric rotating black hole solution has so far been found except a dissipating case where entropy increases [12, 13].

[29] Homogeneous space is the one where any two points are connected by the isometry of the space. In general this isometry is not translational symmetry.

[30] Near the horizon, the scalar field $\Delta$ behaves as $\Delta \sim 2^{-\nu} (\sinh (v_1 - v_2))^2 / f$. So, we can reach the same conclusion from the regularity condition of the curvature.

[31] This is because of the fact that for a stationary black hole, there must exist a null Killing vector $K$ on $H$. However if the black hole is rotating, then $\partial_\omega$ becomes by definition spacelike on $H$, and therefore there must exist another Killing vector that, combined together with $\partial_\omega$, provides $K$.

[32] If $b = 0$ and the black hole is rotating, then one can check that $\xi$ cannot have a regular solution at horizon.

[33] This condition can be derived from the non-divergence of $A_r A^r$ at the horizon.

[34] We do have a non-normalizable mode for $A_r$ and $\phi$, but not for metric nor charged scalar field. Furthermore note that the non-normalizable mode for $A_r$ is purely constant and can be gauged away, giving zero electric field in AdS/CFT.

[35] Note that we choose the convention that $16\pi G = 1$ and $L = \sqrt{2}$, and we use the stress tensor given in [19].

[36] See also [22, 23] for related works on holographic superfluid hydrodynamics.

[37] Precisely speaking, a scalar condensate should be read from the non-normalizable mode of $\phi(r)$ at the boundary, instead of horizon value. However we have seen that the latter becomes larger as the former becomes larger in all of our numerical solutions. Therefore we use $\phi(r_h)$ as “condensate parameter”.

[38] Our solutions are stationary non-rotating but not static. This property and our metric ansatz appear to be quite similar to the solution found in [24], besides the obvious difference of matter fields considered. However, it should be emphasised that the solution of [24] contains a cross-term of $dt$ and $\omega^2$ or $\omega^2$ and therefore its non-staticity is due to the matter flow in a direction of translational symmetry $\xi_1 = \partial_x$ and $\xi_2 = \partial_y$. In contrast, our solution, having the cross-term of $dt$ and $\omega^2$, contains the matter fields that flow in a direction of no translational symmetry, $\partial_z$, and thus is a novel black brane solution.