Color-kinematics duality and dimensional reduction for graviton emission in Regge limit

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Abstract

In this talk we review the work in [1, 2, 3] where we have studied the applicability of the color-kinematics duality to the scattering of two distinguishable scalar matter particles with one gluon emission in QCD, or one graviton emission in Einstein gravity. We have shown that the duality works well in the Regge limit under two different extensions of the gauge theory: the introduction of a new scalar contact interaction and the relaxation of the distinguishability of the scalars. Both modifications correspond to theories obtained by dimensional reduction from higher-dimensional pure gauge theories.

1 Introduction

The study of the relation between gravity and gauge theories both in the strong-weak aspect of AdS/CFT [4] and in a weak-weak set up [5] is an active field of research at present. Early results were the Kawai-Lewellen-Tye (KLT) relations [6] or the further studies in [7] which strengthened the idea that, at the level of scattering amplitudes, gravity should in some sense correspond to the square of gauge theory. More recently, Bern, Carrasco and one of the current authors (BCJ), showed that there exists an underlying duality between color and kinematics in gauge theory [8] which generates gravity amplitudes by replacing the color factors in the gauge-theory side with kinematic numerator functions depending on particle momenta and states, giving a double-copy

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representation of gravity amplitudes. This is expected to also work at loop level \[9, 10, 12, 13, 14, 15\].

The duality is known to work at tree level in pure (super)-Yang-Mills theories in various dimensions \[8, 10, 11\]. The treatment of general matter states and interactions in the color-kinematics duality has not been understood yet. In \[2\] three of us studied the duality in the context of inelastic amplitudes involving scalar particles in multi-Regge kinematics (\[16, 17, 18\]). A simple extension of the BCJ duality to the scattering of two scalar particles with gluon emission in scalar QCD only correctly retrieves the square of two Lipatov’s QCD emission vertices \[19, 20, 21\]. The terms responsible for the cancellation of simultaneous divergences in overlapping channels \[22, 23, 24, 25, 18, 1\], as required by unitarity \[26\] were not reproduced correctly. In \[3\] this problem was approached with two different modifications: first, we considered the scattering of two distinguishable scalars in the adjoint representation in Yang-Mills theory introducing a quartic matter self-coupling (characteristic of the bosonic sector of \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory). Second, we repeated the calculations in \[2\] with identical adjoint scalars. In both cases the duality reproduced the correct gravitational amplitude in the Regge limit computed in \[1\].

2 Scalar matter and Color-kinematics duality

We focus on the scattering of two scalars with momenta \(p_1, p_2\) producing two scalars with momenta \(p_3, p_4\) and an emitted gluon (in QCD) or graviton (in gravity) with momentum \(p_5\). The gauge-theory amplitude is written as a sum over 15 channels,

\[
\mathcal{A}_5 = g^3 \sum_{i=1}^{15} \frac{c_i n_i}{d_i},
\]

where \(c_i\) are the color factors:

\[
\begin{align*}
    c_1 &= f^{a_5 a b} f^{b a c} f^{c a_2 a_1}, & c_2 &= f^{a_5 a b} f^{b a_3 c} f^{c a_2 a_1}, \\
    c_3 &= f^{a_5 a b} f^{b a_3 c} f^{c a_3 a_1}, & c_4 &= f^{a_5 a b} f^{b a_2 c} f^{c a_3 a_1}, \\
    c_5 &= f^{a_5 a b} f^{b a_1 c} f^{c a_3 a_1}, & c_6 &= f^{a_5 a b} f^{b a_2 c} f^{c a_2 a_4}, \\
    c_7 &= f^{a_5 a b} f^{b a_2 c} f^{c a_3 a_1}, & c_8 &= f^{a_5 a_4 c} f^{c a_1 b} f^{b a_3 a_1}, \\
    c_9 &= f^{a_5 a b} f^{b a_2 c} f^{c a_4 a_1}, & c_{10} &= f^{a_5 a b} f^{b a_3 c} f^{c a_2 a_1}, \\
    c_{11} &= f^{a_5 a b} f^{b a_4 c} f^{c a_3 a_1}, & c_{12} &= f^{a_5 a_2 b} f^{b a_3 c} f^{c a_4 a_1}, \\
    c_{13} &= f^{a_5 a b} f^{b a_4 c} f^{c a_2 a_1}, & c_{14} &= f^{a_5 a_2 b} f^{b a_3 c} f^{c a_3 a_1}, \\
    c_{15} &= f^{a_5 a b} f^{b a_3 c} f^{c a_4 a_1},
\end{align*}
\]

where \(f^{abc}\) are structure constants and the denominators \(d_i = \prod_{\alpha} s_{a_i}\) correspond to the product of the kinematic invariants associated with the internal
The graviton emission effective vertex $M$ by three of us more recently in [1] is

$$j_1 \equiv c_{12} - c_9 + c_{15}, \quad j_2 \equiv c_{11} - c_7 + c_{14}, \quad j_3 \equiv -c_4 + c_5 + c_3,$$

$$j_4 \equiv c_1 - c_2 - c_3, \quad j_5 \equiv -c_{10} + c_6 - c_{14}, \quad j_6 \equiv -c_{13} + c_8 - c_{15}, \quad j_7 \equiv c_4 - c_{10} + c_{13}, \quad j_8 \equiv c_8 + c_7 - c_2, \quad j_9 \equiv c_6 + c_9 - c_1. \quad (3)$$

The numerators $n_i$ obtained from the Feynman rules in general do not satisfy the Jacobi-like identities $\pm n_i \pm n_j \pm n_k = 0$, corresponding to $j_\alpha$ with $c_i \rightarrow n_i$. A generalized gauge transformation, adding zero to the original amplitude in the form

$$A_5 = \sum_{i=1}^{15} \frac{c_i n_i}{d_i} + \sum_{\alpha=1}^{9} \gamma_\alpha j_\alpha = \sum_{i=1}^{15} \frac{c_i n'_i}{d_i}. \quad (4)$$

The numerators $n'_i$ are obtained by collecting the coefficients of each color factor $c_i$ and multiplying by corresponding denominator: $n'_i = d_i \partial_{c_i} A_5$. The parameters $\gamma_\alpha$ are chosen such that $j_\alpha\big|_{c_i \rightarrow n'_i} = 0$. These new numerators are used to construct the gravitational amplitude using the BCJ double-copy prescription

$$-iM = \left(\frac{\alpha}{2}\right)^3 \sum_{i=1}^{15} \frac{n'_i n'_i}{d_i}, \quad (5)$$

where $\alpha$ is the gravitational coupling constant. We express the momenta as

$$p_3 = -p_1 + k_1, \quad p_4 = -p_2 - k_2, \quad p_5 = -k_1 + k_2, \quad (6)$$

where $k_1^\mu = \alpha_1 p_1^\mu + \beta_1 p_2^\mu + k_{1,\perp}^\mu$ and $k_2^\mu = \alpha_2 p_1^\mu + \beta_2 p_2^\mu + k_{2,\perp}^\mu$, with $k_{i,\perp}$ being orthogonal to $p_1$ and $p_2$. In this way we have

$$p_5^\mu = (\alpha_2 - \alpha_1) p_1^\mu + (\beta_2 - \beta_1) p_2^\mu + k_{2,\perp}^\mu - k_{1,\perp}^\mu. \quad (7)$$

Multi-Regge kinematics is defined in terms of Sudakov parameters as $1 \gg \alpha_1 \gg \alpha_2$ and $1 \gg |\beta_2| \gg |\beta_1|$. The gravitational amplitude then reads

$$-iM = -iA_{kk} M^{\mu\nu} \epsilon_{\mu\nu}(p_5), \quad (8)$$

where $\epsilon_{\mu\nu}(p_5)$ is the graviton polarization tensor and $[\parallel]$

$$M^{\mu\nu} = (k_1 + k_2)^\mu (k_1 + k_2)^\nu + A_{k1} \left[(k_1 + k_2)^\mu p_1^\nu + p_1^\mu (k_1 + k_2)^\nu\right] + A_{k2} \left[(k_1 + k_2)^\mu p_2^\nu + p_2^\mu (k_1 + k_2)^\nu\right] + A_{11} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + A_{12} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + A_{22} p_1^\mu p_2^\nu. \quad (9)$$

The graviton emission effective vertex $M^{\mu\nu}$ calculated by Lipatov in [23] and by three of us more recently in [1] is

$$M^{\mu\nu} = \Omega^\mu \Omega^\nu - N^\mu N^\nu. \quad (10)$$
where the vertex for the coupling of two reggeized gluons and one on-shell is

\[
\Omega^\mu \simeq \left( \alpha_1 - \frac{2\beta_1}{\beta_2} \right) \mu^\mu + \left( \beta_2 + \frac{2\alpha_2}{\alpha_1} \right) q^\mu - (k_1 + k_2)_T^\mu, \tag{11}
\]

and the term \( \mathcal{N}^\mu \mathcal{N}^\nu \) removes simultaneous poles in \( \alpha_1 = 0 \) and \( \beta_2 = 0 \) with

\[
\mathcal{N}^\mu \simeq -2i\sqrt{\beta_1 \alpha_2} \left( \frac{\mu^\mu}{\beta_2} + \frac{q^\nu}{\alpha_1} \right). \tag{12}
\]

In [2] three of us studied the scattering of two distinguishable scalars \( \Phi \) and \( \Phi' \) showing that the obtained color-kinematics solution only reproduces the QCD-like part \( (\Omega^\mu \Omega^\nu) \) in the gravity side. This problem with the incorrect \( \mathcal{N}^\mu \mathcal{N}^\nu \) term was solved in [3] by embedding the Yang-Mills + 2 scalar theory into the bosonic sector of \( \mathcal{N} = 2 \) super-Yang-Mills theory (with the scalars transforming in the adjoint representation and introducing the matter self-coupling for the two scalars \( \Delta \mathcal{L} = \frac{g^2}{2} \text{Tr} \left( \Phi, \Phi'^2 \right) \)). This adds four more diagrams without \( t \)-channel poles. Now the numerators are

\[
\begin{align*}
n_1' &= (p_1 + p_2)^2 \left[ - (\gamma_9 - \gamma_4)(p_3 + p_5)^2 - 2p_3 \cdot \epsilon(p_5) \right], \\
n_2' &= (p_1 + p_2)^2 \left[ - (\gamma_4 + \gamma_8)(p_4 + p_5)^2 + 2p_4 \cdot \epsilon(p_5) \right], \\
n_3' &= (\gamma_3 - \gamma_4)(p_1 + p_2)^2(p_3 + p_4)^2, \\
n_4' &= (p_3 + p_4)^2 \left( (\gamma_7 - \gamma_3)(p_1 + p_5)^2 + 2p_1 \cdot \epsilon(p_5) \right), \\
n_5' &= -(p_3 + p_4)^2 \left[ - \gamma_3(p_2 + p_5)^2 + 2p_2 \cdot \epsilon(p_5) \right], \\
n_6' &= -(p_3 + p_5)^2 \left[ - (\gamma_5 + \gamma_9)(p_2 + p_4)^2 + (p_2 - p_4) \cdot \epsilon(p_5) \right] \\
&\quad - 2(p_2 - p_4) \cdot (p_1 - p_3 - p_5)(p_3 \cdot \epsilon(p_5)), \\
n_7' &= -(p_4 + p_5)^2 \left[ (\gamma_2 - \gamma_8)(p_1 + p_3)^2 + (p_4 - p_1) \cdot \epsilon(p_5) \right] \\
&\quad - 2(p_3 - p_1) \cdot (p_2 - p_4 - p_5)(p_3 \cdot \epsilon(p_5)), \\
n_8' &= (p_2 + p_3)^2 \left( \gamma_6 + \gamma_8)(p_4 + p_5)^2 + 2p_4 \cdot \epsilon(p_5) \right), \\
n_9' &= -(p_1 + p_4)^2 \left[ (\gamma_1 - \gamma_9)(p_3 + p_5)^2 + 2p_3 \cdot \epsilon(p_5) \right], \\
n_{10}' &= -(p_1 + p_5)^2 \left[ (\gamma_5 + \gamma_7)(p_2 + p_4)^2 + (p_2 - p_4) \cdot \epsilon(p_5) \right] \\
&\quad - 2(p_2 - p_4) \cdot (-p_1 + p_3 - p_5)(p_1 \cdot \epsilon(p_5)), \\
n_{11}' &= -(p_2 + p_5)^2 \left[ - \gamma_2(p_1 + p_3)^2 + (p_3 - p_1) \cdot \epsilon(p_5) \right] \\
&\quad - 2(p_3 - p_1) \cdot (-p_2 + p_4 - p_5)(p_2 \cdot \epsilon(p_5)), \\
n_{12}' &= (p_1 + p_4)^2 \left[ \gamma_1(p_2 + p_5)^2 + 2p_2 \cdot \epsilon(p_5) \right], \\
n_{13}' &= -(p_2 + p_3)^2 \left[ (\gamma_6 - \gamma_7)(p_1 + p_5)^2 + 2p_1 \cdot \epsilon(p_5) \right],
\end{align*}
\]
There are four independent $\gamma$ variables which we take to be $\gamma_{1,3,6,7}$ and write

$$\gamma_2 = \frac{(p_2 + 2p_3 + p_4) \cdot \epsilon(p_5)}{s\beta_1} - \gamma_1 \frac{1 + \beta_2}{\beta_1} - \gamma_3 \frac{-1 + \alpha_1 - \alpha_2 + \beta_1 - \beta_2}{\beta_1},$$

$$\gamma_4 = \frac{2(p_3 + p_4) \cdot \epsilon(p_5)}{s} + \gamma_3 (1 - \alpha_1 + \alpha_2 - \beta_1 + \beta_2) + \gamma_7 (\beta_1 - \beta_2), \quad (14)$$

$$\gamma_5 = \frac{(-p_2 + p_4) \cdot \epsilon(p_5)}{s\alpha_2} - \gamma_3 \frac{1 - \alpha_1 + \alpha_2 - \beta_1 + \beta_2}{\alpha_2} + \gamma_6 \frac{1 - \alpha_1}{\alpha_2} - \gamma_7 \frac{\beta_1 - \beta_2}{\alpha_2},$$

$$\gamma_8 = \frac{2(p_2 + p_3) \cdot \epsilon(p_5)}{s(\alpha_1 + \beta_1)} - \gamma_1 \frac{1 + \beta_2}{\alpha_1 + \beta_1} + \gamma_6 \frac{1 - \alpha_1}{\alpha_1 + \beta_1} - \gamma_7 \frac{\beta_1 - \beta_2}{\alpha_1 + \beta_1},$$

$$\gamma_9 = \frac{-2(p_2 + p_3) \cdot \epsilon(p_5)}{s(\alpha_2 + \beta_2)} + \gamma_1 \frac{1 + \beta_2}{\alpha_2 + \beta_2} - \gamma_6 \frac{1 - \alpha_1}{\alpha_2 + \beta_2}.$$

Applying the BCJ prescription we construct the gravitational amplitude which in multi-Regge kinematics limit has the coefficients

$$A_{11} \approx \alpha_1^2 - \frac{4\alpha_1 \beta_1}{\beta_2} + \frac{4\beta_1^2}{\beta_2^2} + \frac{4\alpha_2 \beta_1}{\beta_2^2} + \ldots,$$

$$A_{22} \approx \beta_2^2 + \frac{4\alpha_2 \beta_1}{\alpha_1} + \frac{4\alpha_2 \beta_1}{\alpha_1^2} + \frac{4\beta_2^2}{\alpha_1^2} + \ldots,$$

$$A_{12} \approx \alpha_1 \beta_2 - 2\beta_1 + 2\alpha_2 + \ldots, \quad (15)$$

$$A_{k1} \approx -\alpha_1 + \frac{2\beta_1}{\beta_2} + \ldots,$$

$$A_{k2} \approx -\beta_2 + \frac{2\alpha_2}{\alpha_1} + \ldots,$$

which correctly reproduce the full form of Lipatov’s effective graviton emission vertex.

A second method to avoid the problem found in [2] is to consider a single adjoint scalar minimally coupled to a nonabelian gauge field and compute the scattering amplitude with indistinguishable scalars. The number of Feynman diagrams is increased and resolving the four-point vertices in terms of trivalent ones we obtain the numerators

$$n_1 = -(p_3 + p_5) \cdot (p_2 - p_1) \cdot \epsilon(p_5) - 2(p_2 - p_1) \cdot (-p_3 + p_4 - p_5) [p_3 \cdot \epsilon(p_5)],$$

$$n_2 = -(p_4 + p_5) \cdot (p_2 - p_1) \cdot \epsilon(p_5) - 2(p_2 - p_1) \cdot (p_3 - p_4 - p_5) [p_4 \cdot \epsilon(p_5)],$$
In this contribution we have summarized the work in [1, 2, 3] related to the use of the color-kinematics duality to the scattering of two distinguishable scalar matter particles with gluon emission, or graviton emission. In [2] it was shown that in transferring the BCJ double-copy prescription to the scattering of minimally coupled distinguishable scalars an important part of the gravitational amplitude in multi-Regge kinematics was not correctly reproduced.

In [3] we have studied two extensions of the theory for which the BCJ prescription generates the correct the Regge limit of [24, 1]. In one of them a contact interaction between the two scalar particles is introduced, while in the other we give up the distinguishability of the scalars. For both cases we obtain valid gravity amplitudes from the BCJ double-copy prescription in the Regge limit.
Both cases can be thought of as originating from the bosonic sector of $D = 4$ $\mathcal{N} = 2$ super-Yang-Mills theory, keeping either both scalars, or only one scalar. They can be interpreted as coming from subsectors of $\mathcal{N} = 4$ super-Yang-Mills theory, for which double-copy prescription is proven to give valid gravity tree-level amplitudes [10]. An important observation is that the $D = 4$ Yang-Mills + scalar theories studied in [9] are via dimensional reduction directly related to pure Yang-Mills theory in $D = 6$ and $D = 5$ dimensions, respectively. Indeed the new interaction term can be obtained by dimensionally reducing $D = 6$ pure Yang-Mills to $D = 4$, where the gauge field along the extra two dimensions are interpreted as two scalars, $\Phi \equiv A_4$, $\Phi' \equiv A_5$. We find that the successful application of color-kinematics duality in [3] stems from its validity in higher-dimensional Yang-Mills theory and gravity [8, 9, 10, 12, 15] (other practical examples of dimensional reduction can be found in [27]).

Nevertheless, the inclusion of general matter states and interactions in the color-kinematics and double-copy formalism is still an open problem. In particular, it would be interesting to study how to relate tree amplitudes in Yang-Mills theory with minimally-coupled fermions and scalars to that of Einstein gravity with the same matter content. Embedding the gauge and gravity theories into their respective higher-dimensional versions is probably the correct path to follow. The results here discussed with the help of the multi-Regge limit are a first step towards understanding this general matter case.

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