ON PLURIPOTENTIAL THEORY ASSOCIATED TO QUATERNIONIC 
m-SUBHARMONIC FUNCTIONS

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ABSTRACT. Many aspects of pluripotential theory are generalized to quaternionic m-subharmonic functions. We introduce quaternionic version of notions of the m-Hessian operator, m-subharmonic functions, m-Hessian measure, m-capacity, the relative m-extremal function and the m-Lelong number, and show various propositions for them, based on d₀ and d₁ operators, the quaternionic counterpart of ∂ and ∂̅, and quaternionic closed positive currents. The definition of quaternionic m-Hessian operator can be extended to locally bounded quaternionic m-subharmonic functions and the corresponding convergence theorem is proved. The comparison principle and the quasicontinuity of quaternionic m-subharmonic functions are established. We also find the fundamental solution of the quaternionic m-Hessian operator.

1. Introduction

Pluripotential theory provides fine properties of plurisubharmonic functions, their Monge-Ampère measure and solutions to the complex Monge-Ampère equation \((dd^cu)^n = fβ^n\), where \(β\) is the fundamental Kähler form on \(\mathbb{C}^n\). Notably the Monge-Ampère operator \((dd^cu)^n\) is well defined for some non-smooth plurisubharmonic functions, e.g. continuous or locally bounded plurisubharmonic functions. This theory is a powerful tool in complex analysis of several variables, and was generalized to m-subharmonic functions, their Hessian measure and the complex m-Hessian equation \((dd^cu)^m ∧ β^{n−m} = fβ^n\). Pluripotential theory for m-subharmonic functions developed rapidly in last two decades, and there are vast literatures (cf. [1, 2, 8, 10, 12, 13, 15, 17, 19, 20, 22, 23, 25, 26, 30] and references therein).

On the quaternionic space, Alesker [3] introduced notions of quaternionic plurisubharmonic functions and quaternionic Monge-Ampère operator, proved a quaternionic version of the Chern-Levine-Nirenberg estimate and extended the quaternionic Monge-Ampère operator to continuous quaternionic plurisubharmonic functions. He also [6] used the Baston operator \(Δ\) to express the quaternionic Monge-Ampère operator by using methods of complex geometry. Then Wan-Wang [30] introduced the first-order differential operators \(d₀\) and \(d₁\) acting on the quaternionic version of differential forms and the notion of the closedness of a quaternionic positive current, motivated by 0-Cauchy-Fueter complex in quaternionic analysis [34]. The behavior of \(d₀\), \(d₁\) and \(Δ = d₀d₁\) is very similar to ∂, ∂̅ and ∂∂̅ in several complex variables, and many results in the complex pluripotential theory have been also extended to the quaternionic case (cf. [1, 2, 8, 11, 27, 28, 29, 31, 32, 33, 35] and references therein). Some aspects of quaternionic pluripotential theory has been generalized to the Heisenberg group [35]. The purpose of this paper is to generalize pluripotential theory to quaternionic m-subharmonic functions.

Key words and phrases. The quaternionic m-Hessian operator; quaternionic m-subharmonic function; quaternionic m-Hessian measure; quaternionic m-capacity; the comparison principle; quasicontinuity; the relative m-extremal function.

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The paper is organized as follows. In Section 2, a quaternionic version of Garding inequality is given by applying Garding’s theory of hyperbolic polynomials to symmetric function of eigenvalues of a quaternionic hyperhermitian matrix. In Section 3, we briefly recall positive forms, the first-order differential operators \( d_0 \) and \( d_1 \) and \( \Delta = d_0 d_1 \) and their various propositions. The quaternionic \( m \)-Hessian operator is introduced and can be written as \( (\Delta u)^m \wedge \beta_n^{n-m} \), where \( \beta_n \) is the fundamental form on \( \mathbb{H}^n \). In Section 4, we give the definition of nonsmooth quaternionic \( m \)-subharmonic function in terms of positive currents, which coincides with that for smooth ones, and prove basic properties of quaternionic \( m \)-subharmonic functions. In Section 5, for continuous quaternionic \( m \)-subharmonic functions, the locally uniform estimate, i.e. the Chern-Levine-Nirenberg estimate, the existence of \( m \)-Hessian measure and the comparison principle are established. We study the relative \( m \)-extremal function and quaternionic \( m \)-capacity in Section 6, and establish the quasicontinuity of quaternionic \( m \)-subharmonic functions, the extension of quaternionic \( m \)-Hessian operator to locally bounded quaternionic \( m \)-subharmonic functions and the corresponding convergence theorem (the Bedford-Taylor theory) in Section 7. In Section 8 we find the fundamental solution of the \( m \)-Hessian operator and define the \( m \)-Lelong number for a quaternionic \( m \)-subharmonic function.

We use the Sadullaev-Abdullaev approach [25] [26] to \( m \)-subharmonic functions and the complex \( m \)-Hessian operator, i.e. based on an integral estimate for \( \int_\Omega (\Delta u)^m \wedge \beta_n^{n-m} \) on a domain \( \Omega \). While in the classical approach (e.g. [18]), ones usually only use local estimate by using a cut-off function, e.g. in the proof of the Chern-Levine-Nirenberg estimate. We established such integral estimate by using a Stokes-type formula instead of Stokes formula, since our forms are not differential forms. The advantage of this approach is that we can quite quickly to establish necessary estimates and various results.

2. Hyperbolicity of symmetric functions of eigenvalues of a quaternionic hyperhermitian matrix

2.1. Quaternionic hyperhermitian matrix. An \( n \times n \) quaternionic matrix \( A = (a_{ij}) \) is called hyperhermitian if \( A^* = A \), i.e., \( a_{ij} = \overline{a}_{ji} \) for all \( i, j \). Denote by \( \mathcal{A}^n \) the space of all quaternionic hyperhermitian \( n \times n \) matrices, by \( GL_{\mathbb{H}}(n) \) the set of all invertible quaternionic \( (n \times n) \)-matrices, and by \( U_{\mathbb{H}}(n) \) the set of all unitary quaternionic \( (n \times n) \)-matrices, i.e. \( U_{\mathbb{H}}(n) = \{ M \in GL_{\mathbb{H}}(n), M^* M = M M^* = I_n \} \). Let us recall the definition of the Moore determinant [7] for \( M = (M_{ij}) \in \mathcal{A}^n \). Write a permutation \( \sigma \) of \((1, \ldots, n)\) as a product of disjoint cycles as

\[
\sigma = (n_{n_1} \ldots n_{n_l})(n_{21} \ldots n_{2l}) \cdots (n_{r1} \ldots n_{rl}),
\]

where for each \( i \), we have \( n_{i1} < n_{ij} \) for all \( j > 1 \), and \( n_{11} > \cdots > n_{r1} \). Then

\[
\det M = \sum_{\sigma \in S_n} \text{sgn} \sigma M_{n_{11}n_{12}} \cdots M_{n_{11}n_{11}} M_{n_{21}n_{22}} \cdots M_{n_{r1}n_{r1}}.
\]

Consider the homogeneous polynomial \( \det(s_1 M_1 + \cdots + s_n M_n) \) in real variables \( s_1, \ldots, s_n \) of degree \( n \). The coefficient of the monomial \( s_1 \cdots s_n \) divided by \( n! \) is called the mixed determinant of the hyperhermitian matrices \( M_1, \ldots, M_n \), and is denoted by \( \det(M_1, \ldots, M_n) \).

Proposition 2.1. (1) [3] Claim 1.1.4, 1.1.7] For a hyperhermitian \((n \times n)\)-matrix \( M \), there exits a unitary \( U \) such that \( U^* MU \) is diagonal and real.

(2) [3] Theorem 1.1.9] for any quaternionic hyperhermitian \((n \times n)\)-matrix \( M \) and for any quaternionic \((n \times n)\)-matrix \( C \), we have \( \det(C^* MC) = \det(M) \det(C^* C) \).
The mixed determinant is symmetric with respect to all variables, and linear with respect to each of them. In particular, \( \det(A, \ldots, A) = \det(A) \).

### 2.2. Hyperbolic polynomials

Recall Garding’s theory of hyperbolic polynomials [14]. Let \( P \) be a homogeneous polynomial of degree \( m \) in variables \( x \in \mathbb{R}^N \). We say that \( P \) is hyperbolic at \( a \in \mathbb{R}^N \) if the equation \( P(sa + x) = 0 \) has \( m \) real zeros for every \( x \in \mathbb{R}^N \). The completely polarized form of the polynomial \( P \) is given by

\[
M(x^1, \ldots, x^m) = \frac{1}{m!} \prod_k \left( \sum_i x^k_i \frac{\partial}{\partial x^k_i} \right) P(x),
\]

where \( x^k = (x^k_1, \ldots, x^k_N), x = (x_1, \ldots, x_N) \in \mathbb{R}^N \).

Let \( C(P,a) \) be the set of all \( x \in \mathbb{R}^N \) such that \( P(sa + x) \neq 0 \) when \( s \geq 0 \). If we factorize it as \( P(sa + x) = P(a) \prod_{i=1}^m (s + \mu_k(a, x)) \), for fixed \( x \in \mathbb{R}^N \), then \( x \in C(P,a) \) is equivalent to require

\[
\text{Proposition 2.2. Suppose a homogeneous polynomial } P \text{ on } \mathbb{R}^N \text{ of degree } m > 1 \text{ is hyperbolic at } a \in \mathbb{R}^N. \text{ Then (1) } [14] \text{ Lemma 1 } |Q| = \sum_{k=1}^N a_k \frac{\partial P}{\partial x^k} \text{ is hyperbolic at } a.
\]

(2) [14] Theorem 2] The function \( h \) defined in (2.3) is positive, homogeneous and concave, i.e. \( h(a, sx) = sh(a, x) \) for \( s \geq 0 \) and \( h(a, x + y) \geq h(a, x) + h(a, y) \). In particular, \( C = C(P,a) \) is convex. Further, \( P \) is hyperbolic at any \( b \in C \) and \( C(P,b) = C(P,a) \).

(3) [14] Theorem 3] \( \partial C = LP \) and \( x \) belongs to \( LP \) if and only if \( \mu_1(a, x) = \cdots = \mu_m(a, x) = 0 \).

### 2.3. The hyperbolicity of symmetric functions of eigenvalues of a quaternionic hyperhermitian matrix

Now we apply the above theory of hyperbolic polynomials to symmetric functions of eigenvalues of a quaternionic hyperhermitian matrix. An element \( x = (x_{ij}) \in \mathbb{H}^n \) is 1-1 correspondent to a point \( (x_{12}, \ldots, x_{(n-1)n}, x_{11}, \ldots, x_{nn}) \) in \( \mathbb{H}^m [n \times (n-1)] \times \mathbb{R}^n \). So we can identify \( \mathbb{H}^m \) with \( \mathbb{R}^N \) for \( N = 2n^2 - n \).

**Proposition 2.4.** \( P(x) = \det x \) is hyperbolic at \( I \) on \( \mathbb{H}^n \), where \( I \) is the identity matrix in \( \mathbb{H}^n \).

**Proof.** By definition (2.1) of the Moore determinant, we can write \( \det x = Q_1(x) + iQ_2(x) + jQ_3(x) + kQ_4(x) \) for some real polynomials \( Q_1, \ldots, Q_4 \) of degree \( n \). On the other hand by Proposition 2.1 (1), we have \( \det x = \prod_k \lambda_k(x) \in \mathbb{R} \) with \( \lambda_k(x) \) for \( k = 1, \ldots, n \) to be eigenvalues of the hyperhermitian matrix \( x \), which are all real. We see that \( \det x = Q_1(x) \). So \( P(x) = \det x \) is a real polynomial of degree \( n \). It follows from Proposition 2.1 (2) that there exists a unitary matrix \( U \) such that \( x = U \mathrm{diag}(\lambda_1, \ldots, \lambda_n)U^* \), and so

\[
P(sI + x) = \det(sI + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)) = \prod_{i=1}^n (s + \lambda_k).
\]
Therefore \( P(sI + x) \) has exactly \( n \) real zeros, i.e. \( P(x) = \det x \) is hyperbolic at \( I \). \( \square \)

For \( A \in \mathcal{H}^n \), let \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \) be eigenvalues of \( A \) and write \( \lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)) \) as a vector in \( \mathbb{R}^n \). Set

\[
\mathcal{H}_m(A) := S_m(\lambda(A)),
\]

where

\[
S_m(\lambda) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \lambda_{j_1} \cdots \lambda_{j_m},
\]

for \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), \( m = 1, \ldots, n \). The function \( \mathcal{H}_m \) is determined by

\[
\text{det}(sI + A) = \prod_{k=1}^{n} (s + \lambda_k(A)) = \sum_{m=0}^{n} \sum_{1 \leq j_1 < \cdots < j_m \leq n} \lambda_{j_1} \cdots \lambda_{j_m}(A)s^{n-m} = \sum_{m=0}^{n} \mathcal{H}_m(A)s^{n-m}
\]

for \( s \in \mathbb{R} \), by definition.

**Proposition 2.5.** \( \mathcal{H}_m(A) \) is a polynomial of order \( m \) on \( \mathcal{H}^n \) and is hyperbolic at \( I \) for \( m = 1, \ldots, n \).

**Proof.** By Proposition 2.4 \( \text{det}(A) = \mathcal{H}_n \) is hyperbolic at \( I \), i.e. \( \text{det}(A + sI) \) has \( n \) real zeros. If we take \( Q(sI + A) = \frac{d}{ds} \text{det}(sI + A) \), the equation \( Q(sI + A) = 0 \) has \( (n-1) \) real zeros separating those of the equation \( \text{det}(sI + A) = 0 \) by Rolle’s theorem (cf. [14, Lemma 1]). Thus

\[
Q(A) = Q(sI + A)|_{s=0} = \left. \frac{d}{ds} \right|_{s=0} \text{det}(A + sI) = \mathcal{H}_{n-1}(A)
\]

by (2.8), and it is hyperbolic at \( I \). The result follows by repeating this procedure. \( \square \)

Set

\[
\Gamma_m := \{ A \in \mathcal{H}^n : \mathcal{H}_m(sI + A) > 0 \text{ for any } s \geq 0 \}.
\]

By definition, \( \mathcal{H}_m(sI + A) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} (s + \lambda_{i_1}) \cdots (s + \lambda_{i_m}) > 0 \) for large \( s \). Then by the continuity of \( \mathcal{H}_m \), we see that \( \mathcal{H}_m(sI + A) \neq 0 \) for any \( s \geq 0 \) if and only if \( \mathcal{H}_m(sI + A) > 0 \) for any \( s \geq 0 \), and so \( C(\mathcal{H}_m, I) = \Gamma_m \) by definition of the cone \( C(\mathcal{H}_m, I) \).

**Corollary 2.1.** We have

\[
\Gamma_m = \{ \mathcal{H}_1(A) > 0 \} \cap \cdots \{ \mathcal{H}_m(A) > 0 \}.
\]

**Proof.** It follows from (2.6) that

\[
\mathcal{H}_m(sI + A) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} (s + \lambda_{i_1}) \cdots (s + \lambda_{i_m}) = \sum_{p=0}^{m} \binom{n-p}{m-p} \mathcal{H}_p(A)s^{m-p}.
\]

Since \( \mathcal{H}_m \) is hyperbolic at \( I \), for given \( A \in \Gamma_m \), there exist \( m \) positive number \( \mu_1, \ldots, \mu_m \) such that

\[
\mathcal{H}_m(sI + A) = \left( \frac{n}{m} \right) \prod_{j=1}^{m} (s + \mu_j) = \left( \frac{n}{m} \right) \sum_{p=0}^{m} \left( \sum_{1 \leq i_1 < \cdots < i_p \leq m} \mu_{i_1} \cdots \mu_{i_p} \right) s^{m-p}.
\]

So \( \mathcal{H}_p(A) = \binom{n}{m-p}^{-1} \sum_{1 \leq i_1 < \cdots < i_p \leq m} \mu_{i_1} \cdots \mu_{i_p} > 0 \) for \( p = 1, \ldots, m \). \( \square \)
Corollary 2.2. If $A_1, \ldots, A_m \in \Gamma_m$, then
\begin{equation}
\binom{n}{m} \det(A_1, \ldots, A_m, I, \ldots, I) \geq \mathcal{H}_m(A_1) \ldots \mathcal{H}_m(A_m) \frac{1}{H}.
\end{equation}

Proof. Apply Proposition 2.3 to $P = \mathcal{H}_m$ to get
\[ M(A_1, \ldots, A_m) \geq \mathcal{H}_m^\frac{1}{H}(A_1) \ldots \mathcal{H}_m^\frac{1}{H}(A_m), \]
where $M$ is the completely polarized form of $\mathcal{H}_m$. Recall that the completely polarized form $M$ of a hyperbolic polynomial $P$ is a polynomial uniquely determined by being linear in each argument, invariant under permutations and satisfying $M(x, \ldots, x) = P(x)$ (32). But $\det(A_1, \ldots, A_m, I, \ldots, I)$ is linear in $A_1, \ldots, A_m$, and invariant under permutations, and $\det(A, A, I, \ldots, I) = \mathcal{H}_m(A)/(\binom{n}{m})$ (cf. (3.16)). Therefore,
\begin{equation}
M(A_1, \ldots, A_m) = \binom{n}{m} \det(A_1, \ldots, A_m, I, \ldots, I).
\end{equation}
The result follows. \hfill \Box

3. The quaternionic $m$-Hessian operator

Alesker introduced the quaternionic Monge-Ampère operator in [3]. For a point $q = (q_0, \ldots, q_{n-1}) \in \mathbb{H}^n$, write $q_l = x_{4l} + x_{4l+1}i + x_{4l+2}j + x_{4l+3}k$, $l = 0, \ldots, n - 1$. The Cauchy-Fueter operator is
\begin{equation}
d\partial u = \partial x_{4l} + i\partial x_{4l+1} + j\partial x_{4l+2} + k\partial x_{4l+3},
\end{equation}
and its conjugate $\overline{d\partial u} = \partial x_{4l} - i\partial x_{4l+1} - j\partial x_{4l+2} - k\partial x_{4l+3}$. For a $C^2$ function $u$, the quaternionic Monge-Ampère operator on $\mathbb{H}^n$ is defined as the Moore determinant of its quaternionic Hessian
\begin{equation}
\det \left( \frac{\partial^2 u}{\partial q_l \partial q_k} \right),
\end{equation}
while the quaternionic $m$-Hessian operator $\mathcal{H}_m(u)$ is defined as
\begin{equation}
\mathcal{H}_m(u) := \mathcal{H}_m \left( \frac{\partial^2 u}{\partial q_l \partial q_k} \right).
\end{equation}

Let us recall that two first-order differential operator $d_0$ and $d_1$, introduced in [31], act on the quaternionic version of differential form. The behavior of $d_0$ and $d_1$ and $\Delta = d_0d_1$ is very similar to $\partial q_l$ and $\overline{\partial q_l}$ in several complex variables. This formulation of the quaternionic $m$-Hessian operator is fundamental here in the sense that we can use Stokes-type formula, etc.

3.1. Positive forms. Fix a basis $\{\omega^0, \omega^1, \ldots, \omega^{2n-1}\}$ of $\mathbb{C}^{2n}$. Let $\wedge^k \mathbb{C}^{2n}$ be the complex exterior algebra generated by $\mathbb{C}^{2n}$, $0 \leq k \leq n$. Recall the embedding $\tau : M_{2r}(p, r) \to M_{2p}(2p, 2r)$ as follows, where $M_{2r}(p, r)$ is the space of all $p \times r$-matrices over field $\mathbb{F}$. For a quaternionic $(p \times r)$-matrice $M$, write $M = a + bj$ for some complex matrices $a, b \in M_{2r}(p, r)$. Then
\begin{equation}
\tau(M) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\end{equation}
(cf. [33]). We will notations in [33], as the relabelling of those in [31], which have advantages in the proof of some properties of quaternionic linear algebra.
For $M \in M_C(2n, 2n)$, define its $C$-linear action on $\mathbb{C}^n$ as \[ M \cdot \omega^A = \sum_{B=0}^{2n-1} M_{AB} \omega^B, \]
and the induced action on $\wedge^2 \mathbb{C}^n$ as $M \cdot (\omega^A \wedge \cdots \wedge \omega^B) = M \omega^A \wedge \cdots \wedge M \omega^B$. For $M \in M_\mathbb{R}(n, n)$, defines its induced $C$-linear action on $\mathbb{C}^n$ as $M \cdot \omega^A = \tau(M) \omega^A$, and so on $\wedge^2 \mathbb{C}^n$. Then for $M \in U_\mathbb{R}(n)$, $M \cdot \beta_n = \beta_n$ and $M \cdot \Omega_{2n} = \Omega_{2n}$, where

\[ \beta_n = \sum_{l=0}^{n-1} \omega^I \wedge \omega^{n+l}, \quad \beta_n^\omega = \wedge^n \beta_n = n! \Omega_{2n}, \]

where $\Omega_{2n} := \omega^0 \wedge \omega^n \cdots \wedge \omega^{n-1} \wedge \omega^{2n-1}$.

There exists a real linear action $\rho(j)$ on $\mathbb{C}^n$\[ \rho(j) : \mathbb{C}^n \to \mathbb{C}^n, \quad \rho(j)(z) = zJ, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \]

An element $\omega$ of $\wedge^2 \mathbb{C}^n$ is called real if $\rho(j)\omega = \omega$. Denote by $\wedge^2 \mathbb{C}^n$ the subspace of all real elements in $\wedge^2 \mathbb{C}^n$, which is the counterpart of $(k, k)$-forms in complex analysis.

An element $\omega$ of $\wedge^n \mathbb{C}^n$ is called positive if $\omega = \kappa \Omega_{2n}$ for some non-negative number $\kappa$. An element $\omega \in \wedge^2 \mathbb{C}^n$ is said to be elementary strongly positive if there exist linearly independent right $\mathbb{H}$-linear mappings $\eta_j : \mathbb{H}^n \to \mathbb{H}$, $j = 1, \ldots, k$, such that

\[ \omega = \eta_1^j \tilde{\omega}^0 \wedge \eta_2^k \tilde{\omega}^1 \wedge \cdots \wedge \eta_k^\omega \wedge \eta_k^\omega, \]

where $\{\tilde{\omega}^0, \tilde{\omega}^1\}$ is a basis of $\mathbb{C}^2$ and $\eta_j^\ast : \mathbb{C}^2 \to \mathbb{C}^2$ is the induced $C$-linear pulling back transformation of $\eta_j$. An element $\omega \in \wedge^2 \mathbb{C}^n$ is called strongly positive if it belongs to the convex cone $SP^2 \mathbb{C}^n$ in $\wedge^2 \mathbb{C}^n$ generated by elementary strongly positive elements. An 2-k-element $\omega$ is said to be positive if for any elementary strongly positive element $\eta \in SP^2 \mathbb{C}^n$, $\omega \wedge \eta$ is positive. By definition, $\beta_n$ is a strongly positive 2-form, and $\beta_n^\omega$ is a positive 2n-form.

**Proposition 3.1.** \[ \text{(1) For a complex skew symmetric matrix } M = (M_{AB}) \in M_C(2n, 2n), \text{ the 2-form } \omega = \sum_{A,B=0}^{2n-1} M_{AB} \omega^A \wedge \omega^B \text{ is real if and only if there exists a hyperhermitian } n \times n \text{-matrix } M = (M_{jk}), \text{ such that } M = \tau(M)J. \]

(2) When $\omega$ is real, there exists a quaternionic unitary matrix $E \in U_\mathbb{H}(n)$ such that

\[ \tau(E)M \tau(E) = \begin{pmatrix} 0 & V \\ -V & 0 \end{pmatrix}, \quad \text{where} \quad V = \text{diag}(\nu_0, \ldots, \nu_{n-1}), \]

for some real numbers $\nu_0, \ldots, \nu_{n-1}$. Namely, we can normalize $\omega$ as $\omega = 2 \sum_{l=0}^{n-1} \nu_l \tilde{\omega}^l \wedge \tilde{\omega}^{l+n}$ with $\tilde{\omega}^A = E^\ast \omega^A$. In particular, $\omega$ is (strongly) positive if and only if each $\nu_l \geq 0$ ($>0$).

**Proposition 3.2.** \[ \text{Lemma } 3.3 \text{ For } \eta \in \wedge^2 \mathbb{C}^n \text{ with } \|\eta\| \leq 1, \beta_k^{\pm \epsilon} \text{ is positive 2k-form for some sufficiently small absolute constant } \epsilon > 0. \]

### 3.2. $d_0, d_1$ formulation of the quaternionic $m$-Hessian operator
We express the quaternionic $m$-Hessian operator in terms of $d_1, d_1$. Let $\Omega$ be a domain in $\mathbb{H}^n$. Denoted by $D^p(\Omega)$ the set of all $C^\infty_0(\Omega)$ functions valued in $\wedge^p \mathbb{C}^n$. $F \in D^k(\Omega)$ is called a (strongly) positive form if for any $q \in \Omega$, $F(q)$ is a (strongly) positive element. Define $d_0, d_1 : C^1(\Omega, \wedge^p \mathbb{C}^n) \to C(\Omega, \wedge^{p+1} \mathbb{C}^n)$ by

\[ d_\alpha F = \sum_{I} \sum_{A=0}^{2n-1} \nabla_A f I \omega^A \wedge \omega^{I}, \]
for $F = \sum f_I \omega^I \in C^1(\Omega, \wedge^p \mathbb{C}^{2n})$, where the multi-index $I = (i_1 \ldots i_p)$, $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$, and the first-order differential operators $\nabla_{A \alpha} (A = 0, \ldots, 2n - 1, \alpha = 0, 1)$ are

$$
\begin{pmatrix}
\nabla_{00} & \nabla_{01} \\
\vdots & \vdots \\
\nabla_{0n} & \nabla_{11} \\
\vdots & \vdots \\
\nabla_{n0} & \nabla_{n1} \\
\vdots & \vdots \\
\nabla_{(n+1)0} & \nabla_{(n+1)1}
\end{pmatrix} =
\begin{pmatrix}
\partial_{x_0} + i \partial_{x_1} & -\partial_{x_2} - i \partial_{x_3} \\
\vdots & \vdots \\
\partial_{x_4} + i \partial_{x_4} & -\partial_{x_{4+2}} - i \partial_{x_{4+3}} \\
\vdots & \vdots \\
\partial_{x_2} - i \partial_{x_3} & \partial_{x_0} - i \partial_{x_1} \\
\vdots & \vdots \\
-\partial_{x_{4+2}} - i \partial_{x_{4+3}} & \partial_{x_{4i}} - i \partial_{x_{4i+1}} \\
\vdots & \vdots 
\end{pmatrix}.
$$

(3.9)

**Proposition 3.3.** [31] Proposition 2.2] (1) $d_0 d_1 = -d_1 d_0$;
(2) $d_0^2 = d_1^2 = 0$;
(3) For $F \in C^1(\Omega, \wedge^p \mathbb{C}^{2n}), G \in C^1(\Omega, \wedge^q \mathbb{C}^{2n})$, we have

$$d_\alpha (F \wedge G) = d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G, \quad \alpha = 0, 1.$$  

The following nice identity will be frequently used.

**Proposition 3.4.** [31] Proposition 2.3] For $u_1, \ldots, u_n \in C^2$,

$$\Delta u_1 \wedge \Delta u_2 \wedge \cdots \wedge \Delta u_n = d_0 (d_1 u_1 \wedge \Delta u_2 \wedge \cdots \wedge \Delta u_n) = -d_1 (d_0 u_1 \wedge \Delta u_2 \wedge \cdots \wedge \Delta u_n) = d_0 d_1 (u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_n) = \Delta (u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_n).$$

(3.10)

Define

$$\int_{\Omega} F = \int_{\Omega} f dV,$$

if $F = f \Omega_{2n} \in L^1(\Omega, \wedge^{2n} \mathbb{C}^{2n})$, where $dV$ is the Lebesgue measure.

**Lemma 3.1.** [31] Lemma 3.2] (Stokes-type formula) Assume that $T = \sum A_T \omega^A$ is a $C^1(2n - 1)$-form in $\Omega$, where $\omega^A = \omega^A |_{\Omega_{2n}} := (-1)^{A-1} \omega^0 \wedge \cdots \wedge \omega^{A-1} \wedge \omega^{A+1} \wedge \cdots \wedge \omega^{2n-1}$. Then for a $C^1$ function $h$, we have

$$\int_{\Omega} h d_\alpha T = -\int_{\Omega} d_\alpha h \wedge T + \int_{\partial \Omega} \sum_{A=0}^{2n-1} h T_A \tau(n)_{A \alpha} dS, \quad \alpha = 0, 1,$$

(3.11)

where $n := (n_0, n_1, \ldots, n_{4n-1})$ is the unit outer normal vector to $\partial \Omega$, $dS$ denotes the surface measure of $\partial \Omega$, and $\tau(n)$ is a complex $(2n) \times 2$-matrix by definition [3.14] of $\tau$. In particular, if $h = 0$ on $\partial \Omega$, [3.14] has no boundary term.

Recall the Baston operator $\Delta u := d_0 d_1 u$ for a real $C^2$ function $u$.

**Proposition 3.5.** [31] Theorem 1.3] Let $u_1, \ldots, u_n$ be real $C^2$ functions on $\mathbb{H}^n$. Then we have

$$\Delta u_1 \wedge \cdots \wedge \Delta u_n = n! \det(A_1, A_2, \ldots, A_n) \Omega_{2n},$$

(3.12)

where $A_j = \left( \frac{\partial^2 u_j}{\partial u_1 \partial u_n}(q) \right)$.

**Proposition 3.6.**

$$\Delta u)^m \wedge H_{n-m} = m!(n-m)! H_m(u) \Omega_{2n}.$$

(3.13)
Proof. Apply Proposition 6.3 to \( u_1 = \cdots = u_m = u \) and \( u_{m+1} = \cdots = u_n = \|q\|^2 \) to get
\[
8^{n-m} (\Delta u)^m \land \beta_n^{n-m} = n! \det(A, \ldots, A, 8I, \ldots, 8I) \Omega_{2n},
\]
where \( A = \left( \frac{\partial^2 u}{\partial q_i \partial q_k} (q) \right), \) and
\[
\Delta \|q\|^2 = d_0 d_1 \|q\|^2 = 8 \beta_n.
\]
By definition, the coefficient of the monomial \( s_1 \ldots s_n \) of \( \det(s_1 A + \cdots + s_m A + 8 s_{m+1} I + \cdots + 8 s_n I) \) divided by \( n! \) is the \( \det(A, \ldots, A, 8I, \ldots, 8I). \) On the other hand, we can find a quaternionic unitary matrix \( U \in \mathbb{U}(n) \) such that \( U^* A U = \text{diag}(\lambda_1, \ldots, \lambda_n). \) Now apply Proposition 2.1 to get
\[
\det \left( \sum_{j=1}^{m} s_j A + 8 \sum_{j=m+1}^{n} s_j I \right) = \det \left( U^* \sum_{j=1}^{m} s_j A + 8 \sum_{j=m+1}^{n} s_j I \right) U
\]
\[
= \det \left( \sum_{j=1}^{m} s_j \text{diag}(\lambda_1, \ldots, \lambda_n) + 8 \sum_{j=m+1}^{n} s_j I \right)
\]
\[
= \prod_{p=1}^{n} \left( \lambda_p \sum_{j=1}^{m} s_j + 8 \sum_{j=m+1}^{n} s_j \right),
\]
whose coefficient of \( s_1 \ldots s_n \) is \( 8^{n-m} m!(n-m)! \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} \lambda_i_1 \ldots \lambda_i_m. \) Therefore
\[
(\Delta u)^m \land \beta_n^{n-m} = m!(n-m)! \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} \lambda_i_1 \ldots \lambda_i_m \Omega_{2n}.
\]
The result follows. \( \square \)

We also need the following elementary strong positivity (cf., e.g. [35] Proposition 4.2).

**Proposition 3.7.** For any \( C^1 \) real function \( u, d_0 u \land d_1 u \) is elementary strongly positive if \( \text{grad} \, u \neq 0. \)

### 4. Quaternionic \( m \)-subharmonic functions

#### 4.1. Smooth quaternionic \( m \)-subharmonic function.** A real \( C^2 \) functions \( u \) is said to be **quaternionic \( m \)-subharmonic** on \( \Omega \subset \mathbb{H}^n \) if
\[
\left( \frac{\partial^2 u}{\partial q_i \partial q_k} \right) (q) \in \mathbf{T}_m
\]
for any \( q \in \Omega. \) It follow from Corollary 2.1 and Proposition 3.6 that it is equivalent to require
\[
(\Delta u)^k \land \beta_n^{n-k} \geq 0, \quad \text{for } k = 1, 2, \ldots, m.
\]

**Proposition 4.1.** If \( u_1, \ldots, u_k \) are \( C^2 \) quaternionic \( m \)-subharmonic functions, \( 1 \leq k \leq m, \) then \( \Delta u_1 \land \cdots \land \Delta u_k \land \beta_n^{n-m} \geq 0. \)

**Proof.** Since \( u_1, \ldots, u_m \in \text{QSH}_m(\Omega) \cap C^2(\Omega), \) \( A_1 = \left( \frac{\partial^2 u_1}{\partial q_i \partial q_k} \right), \ldots, A_m = \left( \frac{\partial^2 u_m}{\partial q_i \partial q_k} \right) \in \mathbf{T}_m. \) Then we have
\[
\binom{n}{m} \det(A_1, \ldots, A_m, I, \ldots, I) \geq \mathcal{H}_m(A_1)^{\frac{m}{2}} \cdots \mathcal{H}_m(A_m)^{\frac{m}{2}} \geq 0
\]
by Garding’s inequality in Corollary 2.2. Then, by Proposition 3.5 we get
\[
\Delta u_1 \land \cdots \land \Delta u_m \land \beta_n^{n-m} = n! \det(A_1, \ldots, A_m, I, \ldots, I) \Omega_{2n} \geq 0.
\]
4.2. Closed positive currents. To define nonsmooth quaternionic $m$-subharmonic functions, we need to use currents. An element of the dual space $(\mathcal{D}^{2n-p}(\Omega)^\prime)$ is called a $p$-current. Obviously $2n$-currents are just distributions on $\Omega$. A $2k$-current $T$ is said to be positive if we have $T(\eta) \geq 0$ for any strongly positive form $\eta \in \mathcal{D}^{2n-2k}(\Omega)$. Let $\psi$ be a $p$-form whose coefficients are locally integrable in $\Omega$. One can associate with $\psi$ the $p$-current $T_\psi$ defined by $T_\psi(\varphi) = \int_\Omega \psi \wedge \varphi$ for $\varphi \in \mathcal{D}^{2n-p}(\Omega)$.

Now for a $p$-current $F$, we define the $(p+1)$-current $d_\alpha F$ as

$$(d_\alpha F)(\eta) := -F(d_\alpha \eta), \quad \alpha = 0, 1,$$

for any test form $\eta \in \mathcal{D}^{2n-p-1}(\Omega)$. We say a form (or a current) $F$ is closed if $d_0 F = d_1 F = 0$.

If a $p$-current $T$ has a continuous extension to the space of $(2n-p)$-forms with continuous coefficients, it is called a $p$-current of order zero or of measure type. A $p$-current $T$ is of measure type if and only if for any neighborhood $G \Subset \Omega$, there exists a constant $K_G$ such that $|T(\alpha)| \leq K_G ||\alpha||_G$, where $||\alpha||_G = \sum_{|\alpha| = 2n-p} \max_{x \in G} |\alpha_I(q)|$ for $\alpha = \sum_{|\alpha| = 2n-p} \alpha_I w^I$. Here the summation $\sum'$ is taken over increasing indices of length $2n-p$.

Denote by $\mathcal{M}^p(\Omega)$ the set of all $p$-currents of measure type, and it is identified with $\wedge^p$-valued Radon measures on $\Omega$. A sequence of currents $T_j \in \mathcal{M}^p(\Omega)$ weakly * converges to $T$ if $T_j(\alpha) \rightarrow T(\alpha)$ for any $(2n-p)$-forms with continuous coefficients. A family of currents $T_k \in \mathcal{M}^p(\Omega)$ is weakly * compact (or locally uniformly bounded) if and only if for any domain $G \Subset \Omega$ there is a constant $K_G$ depending only on $G$ such that

$$(4.4) \quad |T_k(\alpha)| \leq K_G ||\alpha||_G.$$ 

4.3. Non-smooth quaternionic $m$-subharmonic functions. A $[-\infty, \infty]$-valued upper semicontinuous function $u \in L_{\text{loc}}^1(\Omega)$ is called quaternionic $m$-subharmonic, if for any $C^4$ quaternionic $m$-subharmonic functions $v_1, \ldots, v_{m-1}$ on $\Omega$, the current $\Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n$ defined by

$$(4.5) \quad \Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n(\omega) = \int u \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n \wedge \Delta \omega,$$ 

for any $\omega \in C_0^\infty(\Omega)$, is nonnegative. The set of quaternionic $m$-subharmonic functions on $\Omega$ is denoted by $\text{QSH}_m(\Omega)$.

Proposition 4.2. A function $u \in C^2(\Omega)$ is quaternionic $m$-subharmonic in the above sense if and only if $(4.1)$ holds for any $q \in \Omega$.

Proof. For a function $u \in C^4(\Omega)$,

$$(4.6) \quad \int_\Omega u \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n \wedge \Delta \omega = \int_\Omega \omega \Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n,$$

by applying Stokes-type formula $(3.11)$ twice, since integrands vanish on the boundary. By continuity, $(4.6)$ is nonnegative for any nonnegative $\omega$ if and only if $\Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}^n$ is positive at each $q \in \Omega$. So in this case, the definition $(4.5)$ is equivalent to require $v_1, \ldots, v_{m-1}$ only to be quadratic $\text{QSH}_m$ polynomials.

For $k < m$, it is sufficient to prove that

$$(4.3) \quad \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_{n-m}^n \wedge \omega \geq 0,$$

for any elementary strongly positive $2(m-k)$-element $\omega = \eta_1^I \bar{\omega}^0 \wedge \eta_1^I \bar{\omega}^1 \wedge \cdots \wedge \eta_{m-k}^I \bar{\omega}^0 \wedge \eta_{m-k}^I \bar{\omega}^1,$

where $\eta_j : \mathbb{H}^n \rightarrow \mathbb{H}, j = 1, \ldots, m-k$, are linearly independent right $\mathbb{H}$-linear mappings and \{\{$\omega^0, \omega^1$\}\} is a basis of $\mathbb{C}^2.$ Since $\Delta ||q_0||^2 = 8 \bar{\omega}^0 \wedge \bar{\omega}^1$ and $\eta_j^I(\Delta ||q_0||^2) = \Delta(||\eta_j(q)||^2)$. So $(4.3)$ is proved by $\eta_j(q) \in \text{QPSH} \subset \text{QSH}_m(\Omega)$ and the case $k = m$ in $(4.1)$. \qed
\textbf{Sufficiency.} By Proposition \[1.1\] \(\Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_n^{-m}\) in \([4.5]\) is a positive form if the positivity in \([4.1]\) holds for \(u\).

\textbf{Necessity.} We prove it by induction on dimension \(n\) of the space and the number \(m\). Suppose that we have proved the result for dimension less than \(n\) and \(m-1\) on dimension \(n\). Now by rotation if necessary, we can assume that \(\frac{\partial^2 u}{\partial q_1 \partial q_2}(q_0)\) is diagonalized with eigenvalues \(\lambda_1(q_0) \leq \cdots \leq \lambda_n(q_0)\). Hence \(\lambda_n(q_0) \geq 0\) and

\begin{equation}
(4.7) \quad \mathcal{H}_m(u)(q_0) = \lambda_n(q_0) \sum_{1 \leq j_1 < \cdots < j_m \leq n-1} \lambda_{j_1}(q_0) \cdots \lambda_{j_m}(q_0) + \sum_{1 \leq j_1 < \cdots < j_m \leq n-1} \lambda_{j_1}(q_0) \cdots \lambda_{j_m}(q_0).
\end{equation}

If we take \(\Delta v_{m-1} = \omega^{n-1} \wedge \omega^{2n-1}\), i.e. \(v_{m-1} = |q_n|^2\). Then the positivity of \(\Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_n^{-m}\) at point \(q_0\) implies that

\[ \Delta' u(q_0) \wedge \Delta' v_1(q_0) \wedge \cdots \wedge \Delta' v_{m-1}(q_0) \wedge \beta_n^{-m} \]

is a positive element on \(\mathbb{H}^{n-1}\), where \(\Delta'\) is the Baston operator on \(\mathbb{H}^n\). By the assumption of induction for dimension \(n-1\), we see that \([\frac{\partial^2 u}{\partial q_k \partial q_h}(q_0)]\) belongs to \(\mathbb{T}_{m-1}\). Thus, the second sum in \((4.7)\) is non-negative. The first sum in \((4.7)\) is also non negative by the assumption of induction for \(m-1\) in dimension \(n\). \(\square\)

\textbf{Proposition 4.3.} Let \(\Omega\) be a domain in \(\mathbb{H}^n\). Then,

1. The standard approximation \(u_\varepsilon = u * \chi_\varepsilon\) is also a \(QSH_m\) function, and satisfies \(u_\varepsilon \downarrow u\) as \(\varepsilon \downarrow 0\).
2. \(QPSH = QSH_n \subset \cdots \subset QSH_1 = SH\).
3. \(au + bv \in QSH_m(\Omega)\) for any \(a, b \geq 0\).
4. If \(\gamma(t)\) is a convex increasing function on \(\mathbb{R}\) and \(u \in QSH_m\), then \(\gamma \circ u \in QSH_m\).
5. The limit of a uniformly converging or decreasing sequence of \(QSH_m\) functions is an \(QSH_m\) function.
6. The maximum of a finite number of \(QSH_m\) functions is a \(QSH_m\) function; for an arbitrary locally uniformly bounded family \(\{u_\alpha\} \subset QSH_m\), the regularization \(u^*(q)\) of the supremum \(u(q) = \sup_\alpha u_\alpha(q)\) is also a \(QSH_m\) function.
7. If \(D\) is an open subset of \(\Omega\), \(u \in QSH_m(\Omega)\), \(v \in QSH_m(D)\) and \(\lim sup_{q \to q_0} v(q) \leq u(q_0)\) for all \(q_0 \in \partial D \cap \Omega\), then the function defined by

\[ \phi = \begin{cases} u, & \text{on } \Omega \setminus D, \\ \max\{u, v\}, & \text{on } D, \end{cases} \]

belongs to \(QSH_m(\Omega)\).

\textbf{Proof.} Because there is no characterization of \(m\)-subharmonicity by the submean value inequality, the proof is different from that for plurisubharmonic functions.

(1) For any \(C^4(\Omega) \cap QSH_m(\Omega)\) functions \(v_1, \ldots, v_{m-1}\) and nonnegative function \(\omega \in C_0^\infty(\Omega)\), it is direct to see that if \(\varepsilon > 0\) small,

\begin{equation}
(4.9) \quad \int_\Omega \Delta u_\varepsilon(x) \wedge \Delta v_1(x) \wedge \cdots \wedge \Delta v_{m-1}(x) \wedge \beta_n^{-m} \wedge \omega(x)
\end{equation}

\begin{equation}
= \int_{B(0, \varepsilon)} \chi_\varepsilon(y) dV(y) \int_\Omega u(z) \Delta v_1(z + y) \wedge \cdots \wedge \Delta v_{m-1}(z + y) \wedge \beta_n^{-m} \wedge \Delta \omega(z + y) \geq 0,
\end{equation}

by \((4.5)\) for \(u\) with \(\omega(\cdot)\) replaced by \(\omega(\cdot + y)\) and \(v_j\) replaced by \(v_j(\cdot + y)\). Thus \(u_\varepsilon\) is \(QSH_m\).

For \(v_1, \ldots, v_{m-1} \in C^4(\Omega) \cap QSH_m(\Omega)\), denote

\begin{equation}
(4.10) \quad \alpha := \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_n^{-m}.
\end{equation}
Then the linear operator $A_\alpha$ defined by
$$A_\alpha(u) \cdot \Omega_{2n} = \Delta u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}$$
is a differential operator of the second order with $C^2$ coefficients, whose symbol $\sigma(A_\alpha)(\xi;q)$ at point $q$ and direction $0 \neq \xi \in \mathbb{R}^{4n}$ is given by
$$\sigma(A_\alpha)(\xi;q)\Omega_{2n} = d_0|\xi|^2 \wedge d_1|\xi|^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} \wedge \beta_{n-m} \geq 0$$
where $\omega_j = \Delta v_j(q)$, and $d_0|\xi|^2 \wedge d_1|\xi|^2$ is elementary strongly positive by Proposition 3.7. Without loss of generality, we may assume the it is strictly positive, i.e. $A_\alpha$ is a uniform elliptic operator. Otherwise, we replaced $v_j(q)$ by $v_j(q) + \varepsilon|q|^2$. It is also an operator of divergence form, which can be proved by $A_\alpha(u) \cdot \Omega_{2n} = d_0(d_1u \wedge \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m})$ by Proposition 3.3.

Now the positivity of (4.5) is equivalent to $A_\alpha u \geq 0$ in the sense of distributions, i.e. $u$ is $A_\alpha$-subharmonic. It is well known $A_\alpha$-subharmonicity can be characterized as the maximum principle, i.e. for every domain $G \subset \Omega$, if $v \in C(\overline{G})$ satisfies $A_\alpha v = 0$ and $u \leq v$ on $\partial G$, then $u \leq v$ in $G$.

All other properties can be proved by using this characterization and well known corresponding properties for $A_\alpha$-subharmonic functions (cf. e.g. [10]), since $A_\alpha$ is an elliptic differential operator of the second order with $C^2$ coefficients and of divergence form.

For example, for $u \in QSH_m(\Omega)$ and $v \in QSH_m(D)$, they are $A_\alpha$-subharmonic on $\Omega$ and $D$, respectively. Then the function $\phi$ in (4.8) is also $A_\alpha$-subharmonic on $\Omega$ for any $\alpha := \Delta v_1 \wedge \cdots \wedge \Delta v_{m-1} \wedge \beta_{n-m}$ with $v_1, \ldots, v_{m-1} \in C^4(\Omega) \cap QSH_m(\Omega)$. Thus (4.6) is nonnegative for any nonnegative $\omega$.

If $A_\alpha$ is not uniformly elliptic, we use $A_{\alpha_\epsilon}$, where $\alpha_\epsilon$ is the $\alpha$ in (4.10) with $v_j(q)$ replaced by $v_j(q) + \varepsilon|q|^2$. Since $A_{\alpha_\epsilon}$ is uniformly elliptic, $A_{\alpha_\epsilon}\phi \geq 0$ in the sense of distributions. Then $A_\alpha\phi \geq 0$ by letting $\varepsilon \to 0$. Thus, $\phi$ belongs to $QSH_m(\Omega)$ by definition. 

**Remark 4.1.** (1) In the definition of $QSH_m$, we require $v_j \in C^4$ instead of the usual condition $v_j \in C^2$ in order to make $A_\alpha$ of $C^2$ coefficients.

(2) In the complex case, the proof of these properties were only sketched in [25], as far as I know, by using integral representation formula of solutions to the operator $A_\alpha$. But there is also the degenerate problem there.

A set $E \subset \Omega$ is said to be quaternionic $m$-polar in $\Omega$, if there exists a function $u \in QSH_m(\Omega)$ such that $u \not= -\infty$ and $u|_E \equiv -\infty$.

5. Quaternionic $m$-Hessian measure and the comparison principle

5.1. Quaternionic $m$-Hessian measure. We need the following coarea formula.

**Proposition 5.1.** [21 Theorem 1.2.4] For a measurable nonnegative function $\Phi$ on an open subset $\Omega$ of $\mathbb{R}^N$ and $f \in C^{0,1}(\Omega)$, we have

$$\int_{\Phi(x) \leq \Phi(0)} |\text{grad } f(x)| dV(x) = \int_0^\infty d\int_{\Omega \cap \{|f| \leq s\}} \Phi(x) dS(x),$$

where $dS$ is the $(N-1)$-dimension Hausdorff measure $dH^{N-1}$, which equals to the surface measure if the surface is smooth.

A domain $\Omega$ is called $m$-hyperconvex if there exists a continuous function $\rho \in QSH_m(\Omega)$ such that $\rho < 0$ in $\Omega$ and $\lim_{q \to \partial \Omega} \rho(q) = 0$, i.e. $\{\rho(q) < c\}$ is relatively compact in $\Omega$ for any $c < 0$. It is called strongly $m$-hyperconvex if $\rho \in QSH_m(G)$ for some open set $G \supset \Omega$. We need the following key
integral estimate. See Sadullaev-Abdullaev [24] Theorem 16.2 for plurisubharmonic functions and [25] for \( m \)-subharmonic functions on a ball.

**Theorem 5.1.** Let \( \Omega = \{ \varphi < 0 \} \) be a \( m \)-hyperconvex domain with \( \varphi \in C^2(\Omega) \), \( \sigma = \min_\Omega \varphi \). For \( u_1 \cdots u_k \in \text{QSH}_m(\Omega) \cap C(\Omega) \), \( k = 0, \ldots, m \), and any \( \sigma < r < 0 \),

\[
\int_{\sigma}^{r} dt \int_{e^t}^{\sigma} (\varphi^{n-k}) \wedge \Delta u_1 \wedge \cdots \wedge \Delta u_k \leq (M - M') \int_{e^t}^{r} (\varphi^{n-k+1}) \wedge \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1},
\]

where \( M = \max_{e^t \leq \Omega} \{ u_1, \ldots, u_k \} \), \( M' = \min_{e^t} \{ u_1, \ldots, u_k \} \). In particular, if \( u_k|_{q = r} = 0 \), we have

\[
\int_{\sigma}^{r} dt \int_{e^t}^{\sigma} (\varphi^{n-k}) \wedge \Delta u_1 \wedge \cdots \wedge \Delta u_k = - \int_{e^t}^{r} u_k(\varphi^{n-k+1}) \wedge \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1}.
\]

We first prove the result under the \( C^2 \) assumption.

**Lemma 5.1.** Theorem 5.1 holds for \( u \in \text{QSH}_m(B) \cap C^2(\Omega) \).

**Proof.** Note that \( n = \text{grad} \varphi / |\text{grad} \varphi| \) and so \( \tau(n)_{A_0} = \nabla A_0 \varphi / |\text{grad} \varphi| \). Denote \( \Theta := \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1} \).

Apply Proposition 3.3 Stokes-type formula (3.11) and the coarea formula (5.1) to get

\[
\int_{\sigma}^{r} dt \int_{e^t}^{\sigma} (\varphi^{n-k}) \wedge \Delta u_1 \wedge \cdots \wedge \Delta u_k = \int_{\sigma}^{r} dt \int_{e^t}^{\sigma} d_0 (d_1 u_k \wedge (\varphi^{n-k}) \wedge (\Delta u)^{k-1})
\]

\[
= \int_{\sigma}^{r} dt \int_{e^t}^{\sigma} \sum_{A=0}^{2n-1} (d_1 u_k \wedge \Theta \wedge (\varphi^{n-k})_{A}) \nabla A_0 \varphi dV
\]

\[
= \int_{e^t}^{r} \sum_{A=0}^{2n-1} (d_1 u_k \wedge \Theta \wedge (\varphi^{n-k})_{A}) \nabla A_0 \varphi dV
\]

\[
= \int_{e^t}^{r} d_1 u_k \wedge \Theta \wedge (\varphi^{n-k}) - \int_{e^t}^{r} u_k \wedge (\varphi^{n-k+1}) = I_1 + I_2.
\]

In the forth identity, we have used

\[
\int_{\sigma}^{r} dt \int_{e^t}^{\sigma} d_0 (d_1 u_k \wedge \Theta \wedge (\varphi^{n-k})_{A}) \wedge (\Omega_{2n} = d_0 \varphi \wedge d_1 u_k \wedge \Theta \wedge (\varphi^{n-k}),
\]

since \( d_0 \varphi = \sum_{A=0}^{2n-1} \nabla A_0 \varphi \omega^A \). But

\[
\sum_{A=0}^{2n-1} \tau(n)_{A_1} (d_0 \varphi \wedge \Theta \wedge (\varphi^{n-k})_A \wedge (\Omega_{2n} = d_0 \varphi \wedge d_1 u_k \wedge \Delta u_1 \wedge \cdots \Delta u_{k-1} \wedge (\varphi^{n-k} / |\text{grad} \varphi|)
\]

is nonnegative by using Proposition 3.7 and 4.1. So we have

\[
I_1 \leq -M \int_{e^t}^{r} \sum_{A=0}^{2n-1} (d_0 \varphi \wedge \Theta \wedge (\varphi^{n-k})_A \tau(n)_{A_1} dS = M \int_{e^t}^{r} \Theta \wedge (\varphi^{n-k},
\]

and

\[
I_2 \leq -M' \int_{e^t}^{r} \Theta \wedge (\varphi^{n-k+1}.
\]
The estimate follows. If $u_k|_{|q|=r} = 0$, we get $I_1 = 0$. \hfill \square

Applying (5.2) to the ball $B = B(0,1)$ with $\varrho(q) = |q|^2 - 1$ repeatedly, we get
\[
\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} \int_{|q|^2 \leq t_k} \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k} \leq (M - M')^k \int_{|q|^2 \leq 1} \beta_n^m = C(M - M')^k,
\]
for $k = 0, 1, \ldots, m$. On the other hand, for a fixed $0 < r < 1$, the left hand side above can be estimated from below as
\[
\int_0^1 dt_1 \cdots \int_0^{t_k} \int_{|q|^2 \leq r} \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k} \leq \frac{(1-r)^k}{k!} \int_{|q|^2 \leq r} \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}.
\]
So we get
\[
\int_{|q|^2 \leq r} \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k} \leq \frac{Ck!(M - M')^k}{(1-r)^k},
\]
which implies the local Chern-Levine-Nirenberg estimate for $QSH_m \cap C^2$ functions.

**Corollary 5.1.** In the function class $L_M = \{ u \in QSH_m(\Omega) \cap C^2(\Omega) : |u| \leq M \}$, the integrals $\int_K \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}$ are uniformly bounded for any compact subset $K$, $k = 1, \ldots, m$.

**Theorem 5.2.** For $u_1, \ldots, u_m \in QSH_m(\Omega) \cap C(\Omega)$, the recurrence relation
\[
(5.5) \quad \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k} = \int u_k \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1} \wedge \beta_n^{m-k} \wedge \Delta \omega, \quad k = 1, \ldots, m,
\]
for $\omega \in \mathcal{D}^{2m-2k}(\Omega)$, defines a closed positive current.

Moreover, the following weak * convergence of currents of measure type holds for the standard approximations $u_j^t \downarrow u_j$ ($j = 1, 2, \ldots, k$) as $t \to \infty$,
\[
(5.6) \quad \Delta u_1^t \wedge \cdots \wedge \Delta u_k^t \wedge \beta_n^{m-k} \to \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}.
\]

**Proof.** The closedness follows from definition. For $k = 1$, the left hand side of (5.5) is the Laplace operator. The result holds.

Suppose that the result holds for $k - 1$. Then $\Delta u_1 \wedge \cdots \wedge \Delta u_{k-1} \wedge \beta_n^{m-k}$ is a closed positive current of measure type. Thus the right hand side of (5.5) is well defined, and defines a linear continuous functional on $\mathcal{D}^{2m-2k}(\Omega)$.

To show the positivity of this current, note that the standard approximations $u_j^t$ locally uniformly converges to $u_j$. Thus, for a strongly positive form $\omega \in \mathcal{D}^{2m-2k}(\Omega)$, by the convergence (5.6) of currents of measure type for $k - 1$, we have
\[
\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}(\omega) = \int u_k \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1} \wedge \beta_n^{m-k} \wedge \Delta \omega
\]
\[
= \lim_{t \to \infty} \int u_k \Delta u_1^t \wedge \cdots \wedge \Delta u_{k-1}^t \wedge \beta_n^{m-k} \wedge \Delta \omega
\]
\[
= \lim_{s \to \infty} \lim_{t \to \infty} \int u_k^s \Delta u_1^t \wedge \cdots \wedge \Delta u_{k-1}^t \wedge \beta_n^{m-k} \wedge \Delta \omega,
\]
which is nonnegative since
\[
\int u_k^s \Delta u_1^t \wedge \cdots \wedge \Delta u_{k-1}^t \wedge \beta_n^{m-k} \wedge \Delta \omega = \int \Delta u_k^s \wedge \Delta u_1^t \wedge \cdots \wedge \Delta u_{k-1}^t \wedge \beta_n^{m-k} \wedge \omega \geq 0,
\]
Proposition 5.3.

4.1. So note that if

Proof. 

∆ positive currents

Proposition 5.2.

Theorem 5.3.

The comparison principle.

We need the following proposition to prove this theorem.
Proposition 5.5. Let $\Omega$ be a bounded domain with smooth boundary, and let $u, v \in C^2(\Omega) \cap QSH_m(\Omega)$. If $u = v$ on $\partial \Omega$ and $u \leq v$ in $\Omega$, then
\begin{equation}
\int_{\Omega} (\Delta u)^m \wedge \beta_n^{m-m} \geq \int_{\Omega} (\Delta v)^m \wedge \beta_n^{m-m}.
\end{equation}

Proof. We can choose a defining function $\varrho$ of $\Omega$ with $|\nabla \varrho| = 1$. Then
\begin{align}
\int_{\Omega} (\Delta u)^m \wedge \beta_n^{m-m} - \int_{\Omega} (\Delta v)^m \wedge \beta_n^{m-m} &= \int_{\Omega} \sum_{p=1}^{m} (\Delta v)^{p-1} \wedge \Delta (u - v) \wedge (\Delta u)^{n-p} \wedge \beta_n^{m-m} \\
&= \sum_{p=1}^{m} \int_{\partial \Omega} d0 \left[ d1 \ (u - v) \wedge (\Delta v)^{p-1} \wedge (\Delta u)^{n-p} \wedge \beta_n^{m-m} \right] \\
&= \sum_{p=1}^{m} \sum_{A=0}^{2n-1} \int_{\partial \Omega} [d1 \ (u - v) \wedge (\Delta v)^{p-1} \wedge (\Delta u)^{n-p} \wedge \beta_n^{m-m}]_A \cdot \nabla A_0 \varrho \, dS
\end{align}

by using Stokes formula (5.1). Note that we have
\begin{equation}
\sum_{A=0}^{2n-1} [d1 \ (u - v) \wedge (\Delta v)^{p-1} \wedge (\Delta u)^{n-p} \wedge \beta_n^{m-m}]_A \cdot \nabla A_0 \varrho \, dS
\end{equation}
as in (5.10). Since $u = v$ on $\partial \Omega$ and $u \leq v$ in $\Omega$, then for a point $q \in \partial \Omega$ with $\nabla (u - v)(q) \neq 0$, we can write $u - v = h \varrho$ in a neighborhood of $q$ for some positive smooth function $h$. Consequently, we have $\nabla (u - v)(q) = h(q) \nabla \varrho$, and so $\nabla A_1 (u - v)(q) = h(q) \nabla A_1 \varrho(q)$ on $\partial \Omega$. Thus,
\begin{align}
s0 \varrho(q) \wedge d1 \ (u - v)(q) = h(q) s0 \varrho(q) \wedge d1 \varrho(q) \quad \text{on the boundary},
\end{align}
which is elementary strongly positive by Proposition 5.7. Since $(\Delta v)^{p-1} \wedge (\Delta u)^{n-p} \wedge \beta_n^{m-m}$ is also positive by Proposition 4.1, we find that the right hand of (5.11) is a positive 2-form by definition. So the integrant in the right hand of (5.10) on $\partial \Omega$ is nonnegative if $\nabla (v - u)(q) \neq 0$. While if $\nabla (v - u)(q) = 0$, the integrant at $q$ in (5.10) vanishes. Therefore the difference in (5.10) is nonnegative. \hfill \Box

Proof of Theorem 5.5. At first, we assume that $u, v \in QSH_m(\Omega) \cap C^2(\Omega)$. Let $G_\eta := \{u < v - \eta\}$. Then $G := \{u < v\} = \bigcup_{\eta > 0} G_\eta$ and by Sard’s theorem, $G_\eta$ are open sets with smooth boundaries for almost all $\eta > 0$. For such $\eta$, we have
\begin{align}
\int_{G_\eta} (\Delta u)^m \wedge \beta_n^{m-m} \geq \int_{G_\eta} (\Delta v)^m \wedge \beta_n^{m-m}
\end{align}
by Proposition 5.6, (5.8) follows by taking limit $\eta \to 0$.

Now if $u, v \in QSH_m(\Omega) \cap C(\Omega)$, consider the standard approximations $u_j \downarrow u, \ v_j \downarrow v$ by smooth $QSH_m$ functions. Denote $G_p := \{q \in G; u < v - 1/p\}$ and $G_{j,k,p} := \{q \in G; u_j < v_k - 1/p\}$.

For any open set $G' \subseteq G$ we can choose positive integers $p_0$ and $p_1$ such that $G' \subseteq G_{p_0} \subseteq G_{p_1} \subseteq G$. Since $u_j, v_j$ converge locally uniformly in $G$, there exist $k_0$ such that $G' \subseteq G_{j,k,p_0} \subseteq G_{p_1} \subseteq G$ for all $j, k > k_0$. Then
\begin{align}
\int_{G_{j,k,p_0}} (\Delta u_j)^m \wedge \beta_n^{m-m} \geq \int_{G_{j,k,p_0}} (\Delta v_k)^m \wedge \beta_n^{m-m}
\end{align}
Corollary 5.2. Let $E \in \Omega$. The
which contradicts to the assumption $\langle \Delta u \rangle^m \land \beta_n^{\alpha - m}$. By convergence of currents of measure type, we get
$$\int_G (\Delta u)^k \land \beta_n^{\alpha - m} \geq \int_{G'} (\Delta v)^k \land \beta_n^{\alpha - m}.$$ The result follows since the $G' \subseteq G$ is arbitrarily chosen. \hfill $\square$

**Proposition 5.6.** Let $\Omega$ be a bounded domain with smooth boundary, and let $u, v \in C(\Omega) \cap QSH_m(\Omega)$. Suppose that $(\Delta u)^m \land \beta_n^{\alpha - m} \leq (\Delta v)^m \land \beta_n^{\alpha - m}$ on $\Omega$, and $\lim_{q \in \Omega} (u(q) - v(q)) \geq 0$. Then $u \geq v$ in $\Omega$.

**Proof.** Assume that $v(q_0) - u(q_0) = \eta > 0$ at some point $q_0 \in \Omega$. Thus the open set $G := \{D : u(q) < v(q) - \eta/4\}$ is not empty. Then
$$G_1 := \{D : u(q) < v(q) - \eta/2 + \varepsilon|q - q_0|^2\} \subseteq G,$$ and contains $q_0$ for sufficiently small $\varepsilon > 0$. By applying the comparison principle in Theorem 5.3 and Proposition 5.3, we get
$$\int_{G_1} (\Delta u)^m \land \beta_n^{\alpha - m} \geq \int_{G_1} (\Delta v + \varepsilon|q - q_0|^2)^m \land \beta_n^{\alpha - m} \geq \int_{G_1} (\Delta v)^m \land \beta_n^{\alpha - m} + (8\varepsilon)^m \int_{G_1} \beta_n^n$$
which contradicts to the assumption $(\Delta u)^m \land \beta_n^{\alpha - m} \leq (\Delta v)^m \land \beta_n^{\alpha - m}$. \hfill $\square$

We also need the following proposition for several functions.

**Corollary 5.2.** Let $\Omega$ be a bounded domain and let $u_j, v_j \in C(\Omega) \cap QSH_m(\Omega)$. If $u_j = v_j$ outside a compact subset of $\Omega$, then
$$\int_{\Omega} \Delta u_1 \land \cdots \land \Delta u_m \land \beta_n^{\alpha - m} = \int_{\Omega} \Delta v_1 \land \cdots \land \Delta v_m \land \beta_n^{\alpha - m}. \tag{5.12}$$

**Proof.** If the domain has smooth boundary and $u_j, v_j \in C^2(\Omega) \cap QSH_m(\Omega)$, this identity is obtained as in (5.11) by applying
$$\Delta u_1 \land \cdots \land \Delta u_m - \Delta v_1 \land \cdots \land \Delta v_m = \sum_{p=1}^m \Delta v_1 \land \cdots \land \Delta v_p \land \Delta (u_p - v_p) \land \Delta u_{p+1} \land \cdots,$$
since there is no boundary term in this case. The general case easily follows from approximation. \hfill $\square$

6. QUATERNIONIC RELATIVE m-EXTREMAL FUNCTION AND QUATERNIONIC m-CAPACITY

For a domain $\Omega$ in $\mathbb{H}^n$ and $E \subseteq \Omega$, let
$$U(E, \Omega) := \{u \in QSH_m(\Omega), u|_{\partial \Omega} \leq 0, u|_E \leq -1\},$$ and
$$\omega(q, E, \Omega) := \sup\{u(q) : u \in U(E, \Omega)\},$$ whose upper semicontinuous regularization $\omega^*(q, E, \Omega)$ is called a relative $m$-extremal function of the set $E$ in $\Omega$. The $P_m$-capacity is defined as
$$P_m(E, \Omega) := -\int_{\Omega} \omega^*(q, E, \Omega) \beta_n^n.$$ The relative extremal function has the following simple properties:
(1) (monotonicity) if \( E_1 \subseteq E_2 \), then \( \omega^*(q, E_1, \Omega) \geq \omega^*(q, E_2, \Omega) \); if \( E \subseteq D_1 \subseteq D_2 \), then \( \omega^*(q, E, D_1) \geq \omega^*(q, E, D_2) \) for \( q \in D_1 \).

(2) \( \omega^*(q, E, \Omega) \equiv 0 \) if and only if \( E \) is m-polar in \( \Omega \). The proof is the same as the complex case [18].

(3) Let \( \Omega = \{ q < 0 \} \) be m-hyperconvex. If \( E \Subset \Omega \), then \( \omega^*(q, E, \Omega) \to 0 \) as \( q \to \partial \Omega \).

Note that \( M_\Omega \in \mathcal{U}(E, \Omega) \) for a suitable \( M > 0 \) since \( E \subseteq \Omega \). Then \( 0 \leq \omega^*(q, E, \Omega) \leq M_\Omega \) on \( \Omega \). We must have \( \omega^*(q, E, \Omega) \to 0 \) as \( q \to \partial \Omega \).

(4) Let \( \Omega = \{ q < 0 \} \) be a strongly m-hyperconvex. If \( E \Subset \Omega \), then the relative m-extremal function \( \omega^*(q, E, \Omega) \) admits a quaternionic m-subharmonic extension to a neighborhood of the closure \( \overline{\Omega} \).

By \( \omega^*(q, E, \Omega) \geq M_\Omega \) on \( \Omega \) as above, the quaternionic m-subharmonic function

\[
 w(q) = \begin{cases} 
 \omega^*(q, E, \Omega), & q \in \Omega, \\
 M_\Omega, & q \notin \Omega,
\end{cases}
\]

gives an extension to a neighborhood of \( \overline{\Omega} \).

A point \( q_0 \in K \) is called an m-regular point of the compact set \( K \Subset \Omega \) if \( \omega^*(q_0, K, \Omega) = -1 \). A compact set \( K \Subset \Omega \) is called m-regular in \( \Omega \) if each point of \( K \) is m-regular. A function \( u \in QSH_m(\Omega) \) is called maximal if it satisfies the maximum principle in the class \( QSH_m(\Omega) \), i.e., for any \( D \Subset \Omega \), if \( v \in QSH_m(D) \) and \( \lim_{q \to \partial D}(u(q) - v(q)) \geq 0 \), then \( u \geq v \) in \( D \).

Since a quaternionic m-subharmonic function is subharmonic by Proposition [4,3] (2), a regular compact set of the classical potential theory is m-regular. In general, an m-regular compact set is always m'-regular if \( m' > m \). Therefore, for any compact subset \( K \) of an open set \( U \), there exists an m-regular compact set \( E \) such that \( K \Subset E \Subset U \).

**Proposition 6.1.** Let \( K \) be an m-regular compact subset of of an m-hyperconvex domain \( \Omega \). Then, (1) relative m-extremal function \( \omega^*(q, K, \Omega) \) is maximal in \( \Omega \setminus K \); (2) \( \omega^*(\cdot, K, \Omega) \in C(\Omega) \); (3)

\[
 (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_{n-m} = 0 \quad \text{on} \quad \Omega \setminus K.
\]

**Proof.** (1) Suppose that \( \omega^*(\cdot, K, \Omega) \) is not maximal. Then there exists a domain \( G \Subset D \setminus K \) and a function \( v \in QSH_m(G) \) such that \( \lim_{q \to \partial G}(u(q) - v(q)) \geq 0 \), but \( v(q_0) > \omega^*(q_0, K, \Omega) \) at some point \( q_0 \in G \). Since \( \omega^*(q, K, \Omega)|_{K} = -1 \), the function

\[
 w(q) = \begin{cases} 
 \max(v(q), \omega^*(q, K, \Omega)), & q \in G, \\
 \omega^*(q, K, \Omega), & q \notin G,
\end{cases}
\]

belongs to \( w \in \mathcal{U}(K, \Omega) \) by definition (6.1), and so \( w \leq \omega^*(\cdot, K, \Omega) \). This contradicts to \( w(q_0) = v(q_0) > \omega^*(q_0, K, \Omega) \).

(2) Consider \( \Omega_j := \{ q \in \Omega; \omega^*(q, K, \Omega) < -1/j \} \) for positive integers \( j \). Then \( \Omega_j \subset \Omega_{j+1} \) and \( \Omega_j \Subset \Omega \) since \( \Omega \) is m-hyperconvex. Fixed a \( j_0 \), the relative m-extremal function can be approximated on \( \overline{\Omega}_{j_0} \) by smooth \( QSH_m \) functions \( v_t \downarrow \omega^*(\cdot, E, \Omega) \). Applying Hartogs’ Lemma for subharmonic functions twice to this sequence, we see that there exists \( t_0 \) such that for \( t > t_0 \), we have \( v_t \leq 0 \) on \( \overline{\Omega}_{j_0} \) and simultaneously, \( v_t \leq -1 + 1/j_0 \) on \( K \). Then the function

\[
 \tilde{w}(q) = \begin{cases} 
 \max(v_t(q) - 1/j_0, \omega^*(q, K, \Omega)), & q \in \Omega_{j_0}, \\
 \omega^*(q, K, \Omega), & q \notin \Omega_{j_0},
\end{cases}
\]

belongs to \( \mathcal{U}(K, \Omega) \), and so

\[
 \omega^*(q, K, \Omega) - 1/j_0 \leq v_t(q) - 1/j_0 \leq \tilde{w}(q) \leq \omega^*(q, K, \Omega)
\]
for \( q \in \overline{\Omega}_v \). Consequently, \( v_t \) converges uniformly to \( \omega^*(\cdot, K, \Omega) \) on compact subsets of \( \Omega \). So it is continuous.

(3) Suppose \( (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{m-m} \) does not vanish on \( \Omega \setminus K \). There exists a ball \( B(q_0, r) \) where \( (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{m-m} \neq 0 \). Let \( v(q) \) be the Bremermann-Perron solution to the generalized Dirichlet problem \( (\Delta v)^m \wedge \beta_n^{m-m} = 0 \) on the ball with continuous boundary value \( \omega^*(\cdot, K, \Omega)|_{\partial B(q_0, r)} \). Such a solution exits, and is unique and continuous. The proof is exactly as in the complex case \([10]\). We omit details. It is is maximal by construction, i.e. \( v \geq \omega^*(\cdot, K, \Omega) \) on \( B(q_0, r) \). But \( v \neq \omega^*(q, K, \Omega) \), since \( (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{m-m} \neq 0 \) on \( B(q_0, r) \). Therefore, \( v(q') > u(q') \) for some \( q' \in B(q_0, r) \). But

\[
\begin{align*}
\omega(q) = \begin{cases} \omega(q, E, \Omega), & q \in \Omega \setminus B(q_0, r) , \\ \max\{v(q), \omega^*(q, E, \Omega)\}, & q \in B(q_0, r), \\ \end{cases}
\end{align*}
\]

belongs to \( \mathcal{U}(K, \Omega) \). Then \( w(q') > u(q') \) contradicts to the maximality of \( \omega^*(q, K, \Omega) \) in (1).

\[
\square
\]

6.1. **Quaternionic \( m \)-capacity.** See \([25]\) Section 3 for complex \( m \)-capacity. Given a compact set \( K \) in a domain \( \Omega \subset \mathbb{H}^n \), let

\[
\mathcal{U}^*(K, \Omega) = \left\{ u \in QSH_m(\Omega) \cap C(\Omega), u|_K \leq -1, \lim_{q \to \partial \Omega} u(q) \geq 0 \right\}.
\]

The quaternionic \( m \)-capacity of the condenser \((K, \Omega)\) is defined as

\[
C_m(K) = \inf \left\{ \int_\Omega (\Delta u)^m \wedge \beta_n^{m-m} : u \in \mathcal{U}^*(K, \Omega) \right\}
\]

and the quaternionic \( m \)-capacity of an open set \( U \subset \Omega \) is

\[
C_m(U) = \sup \{ C_m(K) : K \subset U \}.
\]

The **exterior \( m \)-capacity** of a set \( E \subset \Omega \) is defined as

\[
C_m^*(E) = \sup \{ C_m(U) : \text{open } U \supset E \}
\]

\( m \)-capacity is obviously monotonic by definition.

**Proposition 6.2.** Let \( \Omega \) be a \( m \)-hyperconvex domain in \( \mathbb{H}^n \). Then,

1. For any \( m \)-regular compact set \( K \subset \Omega \),

\[
C_m(K) = \int_K (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{m-m}.
\]

2. For any compact subset \( K \subset \Omega \), \( C_m(K) = \inf \{ C_m(E) : \Omega \supset E \supset K \text{ and } E \text{ is an } m \text{-regular compact set} \} \). In particular, \( C_m^*(K) = C_m(K) \).

3. If \( K \) is an \( m \)-regular compact subset, then

\[
C_m(K) = \sup \left\{ \int_K \Delta u_1 \wedge \cdots \wedge \Delta u_m \wedge \beta_n^{m-m} : u_j \in QSH_m(\Omega) \cap C(\Omega), -1 \leq u_j < 0 \right\},
\]

4. Suppose that \( \Omega \) is strongly \( m \)-hyperconvex. If \( U \subset \Omega \) is an open set, then

\[
C_m(U) = \sup \left\{ \int_U (\Delta u)^m \wedge \beta_n^{m-m} : u \in QSH_m(\Omega) \cap C(\Omega), -1 \leq u < 0 \right\}
\]

\[
= \sup \left\{ \int_U (\Delta u)^m \wedge \beta_n^{m-m} : u \in QSH_m(\Omega) \cap C^\infty(\Omega), -1 \leq u < 0 \right\}
\]

5. The exterior capacity is monotonic, i.e. if \( E_1 \subset E_2 \), then \( C_m^*(E_1) \subset C_m^*(E_2) \), and countably subadditive, i.e. \( C_m^*(\cup_j E_j) \leq \sum_j C_m^*(E_j) \).
If \( U_1 \subset U_2 \subset \ldots \) are open subsets of \( \Omega \), then \( C_m \left( \bigcup_{j=1}^\infty U_j, \Omega \right) = \lim_{j \to \infty} C_m(U_j, \Omega) \).

If \( E \subset D \subset \Omega \), then \( C_m^*(E, D) \leq C_m^*(E, \Omega) \).

**Proof.**

(1) For \( u \in U^*(K, \Omega) \) and any \( 0 < \varepsilon < 1 \), consider the open set

\[
O := \{ q \in \Omega; u(q) < (1 - \varepsilon)\omega^*(q, K, \Omega) - \varepsilon/2 \} \subset \Omega.
\]

Note that \( O \ni K \). Then, we have

\[
(1 - \varepsilon)^m \int_K (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{-m} = (1 - \varepsilon)^m \int_O (\Delta \omega^*(q, K, \Omega))^m \wedge \beta_n^{-m} \leq \int_O (\Delta u)^m \wedge \beta_n^{-m} \leq \int_\Omega (\Delta u)^m \wedge \beta_n^{-m}
\]

by the comparison principle and \([6.4]\). Letting \( \varepsilon \to 0 \), we see that the infimum on the right hand side of \([6.4]\) is attained by the relative \( m \)-extremal function \( \omega^*(q, K, \Omega) \).

(2) \( C_m(K) \leq C_m(E) \) by monotonicity. Conversely, for any \( 0 < \varepsilon < 1 \), choose \( u \in U^*(K, \Omega) \) such that

\[
\int_\Omega (\Delta u)^m \wedge \beta_n^{-m} < C_m(K) + \varepsilon.
\]

Since \( \{ q \in \Omega; u(q) < -1 + \varepsilon \} \) is a neighborhood of the compact set \( K \), there exists an \( m \)-regular compact set \( E \) such that \( K \subset E \subset U \). Consider

\[
O := \{ q \in \Omega; u(q) < (1 - 2\varepsilon)\omega^*(q, E, \Omega) \}.
\]

Then, \( E \subset O \ni \{ q \in \Omega; u(q) < -1 + \varepsilon \} \), and so

\[
C_m(E) = \int_E (\Delta \omega^*(q, E, \Omega))^m \wedge \beta_n^{-m} \leq \int_O (\Delta \omega^*(q, E, \Omega))^m \wedge \beta_n^{-m} \leq \frac{1}{(1 - 2\varepsilon)^m} \int_\Omega (\Delta u)^m \wedge \beta_n^{-m} \leq \frac{C_m(K) + \varepsilon}{(1 - 2\varepsilon)^m},
\]

by using \([6.5]\) for the \( m \)-regular compact subset \( E \) and the comparison principle. The result follows by letting \( \varepsilon \to 0 \).

(3) \( C_m(K) \) is less that or equal to the right hand side of \([6.6]\) by using \([6.5]\). On the other hand, for any \( u_j \in QSH_m(\Omega) \cap C(\Omega) \) with \( -1 \leq u_j < 0 \), consider

\[
v_j(q) := \max \left\{ (1 + \varepsilon)\omega^*(q, K, \Omega), \frac{u_j(q) - \varepsilon/2}{1 + \varepsilon/2} \right\}.
\]

Then, \( v_j \in QSH_m(\Omega) \cap C(\Omega) \) with \( -1 \leq v_j < 0 \), \( \lim_{q \to \partial \Omega} v_j(q) = 0 \), and \( v_j \equiv (1 + \varepsilon)\omega^*(\cdot, K, \Omega) \) near the boundary. We get

\[
(1 + \varepsilon)^m \int_\Omega (\Delta \omega^*)^m \wedge \beta_n^{-m} = \int_\Omega \Delta v_1 \wedge \cdots \wedge \Delta u_m \wedge \beta_n^{-m} \geq \frac{1}{(1 + \varepsilon/2)^m} \int_K \Delta u_1 \wedge \cdots \wedge \Delta u_m \wedge \beta_n^{-m},
\]

by using Corollary \([6.2]\) and \( v_j \equiv (u_j - \varepsilon/2)/(1 + \varepsilon/2) \) in a neighborhood of \( K \). Letting \( \varepsilon \to 0 \), we get the another direction of inequality, since \( (\Delta \omega^*)^m \wedge \beta_n^{-m} = 0 \) on \( \Omega \setminus K \).

(4) For any \( u \in QSH_m(\Omega) \cap C(\Omega) \) with \( -1 \leq u < 0 \), we have \( C_m(U) \geq C_m(K) \geq \int_K (\Delta u)^m \wedge \beta_n^{-m} \) by (3). Then \( C_m(U) \geq \int_U (\Delta u)^m \wedge \beta_n^{-m} \) since \( K \) can be arbitrarily chosen. Thus \( C_m(U) \) is larger than or equal to the right hand side of \([6.7]\).

Since \( \Omega \) is a strongly \( m \)-hyperconvex domain, the relative \( m \)-extremal function \( \omega^*(q, E, \Omega) \) admits a quaternionic \( m \)-subharmonic extension to a neighborhood of the closure \( \Omega \), and so it can be approximated.
in a neighborhood $U$ of $\overline{\Omega}$ by $QSH_m \cap C^\infty$ functions $v_j \downarrow \omega^*(q, K, \Omega)$. Hence,

$$C_m(K) = \int_K (\Delta u^*(q, K, \Omega))^m \wedge \beta_n^{-m} = \int_{\Omega} (\Delta u^*(q, K, \Omega))^m \wedge \beta_n^{-m} \leq \lim_{\varepsilon \to \infty} \int_{\Omega} (\Delta v_j)^m \wedge \beta_n^{-m} \leq \lim_{\varepsilon \to \infty} (1 + \varepsilon)^m \int_{\Omega} (\Delta w_j)^m \wedge \beta_n^{-m}$$

if we denote $w_j = (v_j - \varepsilon)/(1 + \varepsilon)$. Here $-1 \leq w_j < 0$ if $j$ is large. So $C_m(K)$ is controlled by the right hand side of (6.7) multiplying $(1 + \varepsilon)^m$. The result follows by letting $\varepsilon \to 0$.

(5) The monoticity of $C_m^*(E)$ follows from the monotonicity of $C_m(K)$ for compact sets $K$. If $E_j$'s are open sets, then

$$C_m(\cup_j E_j) = \sup \left\{ \int_{\cup_j E_j} (\Delta u)^m \wedge \beta_n^{-m} : u \in QSH_m(\Omega) \cap C(\Omega), -1 \leq u < 0 \right\}$$

$$\leq \sup \left\{ \sum_j \int_{E_j} (\Delta u)^m \wedge \beta_n^{-m} : u \in QSH_m(\Omega) \cap C(\Omega), -1 \leq u < 0 \right\} \leq \sum_j C_m(E_j).$$

In general, we find an open set $U_j \supset E_j$ such that $C_m(U_j) - C_m^*(E_j) \leq \varepsilon/2^j$. Then

$$\sum_j C_m^*(E_j) \geq \sum_j C_m(U_j) - \varepsilon \geq C_m(\cup_j U_j) - \varepsilon \geq C_m(\cup_j E_j) - \varepsilon.$$

We get the result by letting $\varepsilon \to 0$. \hfill \Box

(6) It is obvious by definition.

By (4) and (5), we get a useful estimate: for a strongly $m$-hyperconvex domain $\Omega$, there exists a neighborhood $\Omega' \supset \overline{\Omega}$ such that

$$\int_U (\Delta u_1) \wedge \cdots \wedge (\Delta u_m) \wedge \beta_n^{-m} \leq C_m(U)$$

for any $u_j \in QSH_m(\Omega) \cap C(\Omega)$ with $-1 \leq u_j < 0$ on $\Omega$ and $|u_j| \leq 1$ on $\Omega'$.

**Proposition 6.3.** If $E \subset B(0, r)$, $r < 1$, then

$$C_m^*(E, B) \leq \frac{m!\mathcal{P}_m(E, B)}{(1 - r^2)^m}.$$

**Proof.** It is sufficient to prove (6.9) for $m$-regular compact set $E$. Apply Theorem 5.1 for $q(q) = |q|^2 - 1$, $\Omega = B$ and $u = \omega = \omega(q, E, B)$ repeatedly to get

$$\int_0^1 dt_1 \cdots \int_0^{t_m-1} dt_m \int_{|q|^2 \leq t_m} (\Delta \omega)^m \wedge \beta_n^{-m} \leq \int_0^1 dt_1 \int_{|q|^2 \leq t_1} \Delta \omega \wedge \beta_n^{-1} = - \int_B \omega \beta_n^{-1} = \mathcal{P}_m(E, \Omega).$$

On the other hand,

$$\int_0^1 dt_1 \cdots \int_0^{t_m-1} dt_m \int_{|q|^2 \leq t_m} (\Delta \omega)^m \wedge \beta_n^{-m} \geq \int_{r^2}^1 dt_1 \cdots \int_{r^2}^{t_m-1} dt_m \int_{|q|^2 \leq r^2} (\Delta \omega)^m \wedge \beta_n^{-m} = \frac{(1 - r^2)^m}{m!} \int_{|q|^2 \leq r^2} (\Delta \omega)^m \wedge \beta_n^{-m}.$$

The estimate follows. \hfill \Box
7. The quasicontinuity of quaternionic m-subharmonic functions and the Bedford-Taylor theory

**Lemma 7.1.** [32] Corollary 3.1 If \( u, v \in C^2(\Omega) \) and let \( \alpha \) be a positive \((2n-2)\)-form. Then

\[
(7.1) \quad \left| \int_\Omega d_0 u \wedge d_1 v \wedge \alpha \right|^2 \leq \int_\Omega d_0 u \wedge d_1 u \wedge \alpha \cdot \int_\Omega d_0 v \wedge d_1 v \wedge \alpha.
\]

**Theorem 7.1.** Any bounded quaternionic m-subharmonic function is continuous almost everywhere with respect to m-capacity, i.e., given \( u \in QSH_m(\Omega) \) and any \( \epsilon > 0 \), there exists an open set \( U \subset \Omega \) such that \( C_m(U, \omega) < \epsilon \) and \( u \) is continuous on \( \Omega \setminus U \).

**Proof.** Firstly, we establish an integral inequality for \( QSH_m \) functions on \( B \). Let \( \mathcal{L} \) be the class of smooth \( QSH_m \) functions \( u \) on the ball \( B(0,1+\delta) \) for \( \delta > 0 \), such that \( |u| \leq 1 \). Consider functions \( v, u, u_1, \ldots, u_m \in \mathcal{L} \) such that \( \varphi_0 = v - u \geq 0 \) in \( B \) and \( \varphi_0 = \text{const} \) on the sphere \( S = \partial B \). Then if we denote \( \Theta := d_1 u_1 \wedge \Delta u_2 \wedge \cdots \Delta u_m \wedge \beta_n^{-m} \), we get

\[
(7.2) \quad \int_B \varphi_0 \Delta u_1 \wedge \cdots \Delta u_m \wedge \beta_n^{-m} = \int_B \varphi_0 d_0 \Theta = \sum_{A=0}^{2n-1} \int_B \varphi_0 \Theta_A \tau(n) A_0 dS - \int_B d_0 \varphi_0 \wedge \Theta
\]

by using Stokes-type formula \((3.11)\) to functions in \( \mathcal{L} \), where \( C \) is an absolute constant independent of \( u_1, \ldots, u_m \in \mathcal{L} \) by Corollary 5.1. Applying Lemma 7.1 to \( u = \varphi, v = u_1 \) and closed positive form \( \alpha = \Delta u_2 \wedge \cdots \Delta u_m \wedge \beta_n^{-m} \), and using Stokes-type formula \((3.11)\) twice, we get

\[
(7.3) \quad \left| \int_B d_0 \varphi_0 \wedge d_1 u_1 \wedge \alpha \right|^2 \leq C \left( \int_B d_0 u_1 \wedge d_1 u_1 \wedge \alpha \right) \left( \int_B d_0 \varphi_0 \wedge d_1 \varphi_0 \wedge \alpha \right)
\]

where \( \varphi_0^+ = \frac{u+v}{2} \in \mathcal{L} \). The second inequality follows from locally uniform estimate in Proposition 5.4 \( \varphi_0|_s = \|\varphi_0\|_s \) and

\[
\int_B \Delta \varphi_0 \wedge \alpha \leq \int_B (u + v) \wedge \alpha \leq 2C,
\]

while the last inequality in \((7.3)\) follows from the fact \( \varphi_0 \geq 0 \) and \( \Delta v \wedge \alpha \geq 0 \).
Applying this procedure repeatedly, we obtain the inequality
\[
(7.4) \quad \int_B \varphi_0 \Delta u_1 \wedge \cdots \wedge \Delta u_m \wedge \beta_n^{m-m} \leq \gamma \left( \|\varphi_0\|s + \int_B \varphi_0 (\Delta \varphi_0^+)^m \wedge \beta_n^{m-m} \right),
\]
for some absolute constants \(\gamma, \kappa > 0\).

Since the capacity is countably subadditive, it suffices to prove the theorem for the unit ball \(B \subset \Omega\) and show that for any \(\epsilon > 0\) there exists an open set \(U \subset B'\) such that \(C_m(U \cap B', B) < \epsilon\) and \(u\) is continuous in \(B' \setminus U\), where \(B' = B(0, \frac{1}{2})\). Assume \(-1 \leq u \leq 0\). If replace \(u\) by \(\max\{u(q), v(q)\}\) with \(v(q) = 2(|q|^2 - \frac{2}{3})\), then \(v(q)|_{\partial B} = \frac{1}{2} > 0 > u(q)\), i.e. \(u \equiv v\) in a neighborhood of the sphere \(S = \partial B\). Let \(u_p \downarrow u\), \(v_p \downarrow v\) be the standard approximations. Note that \(u_p \equiv v_p\) in a neighborhood of \(S\) for \(p > p_0\). We can assume the sequence \(\int_B u_p (\Delta u_p)^m \wedge \beta_n^{m-m}\) has a limit by passing to subsequence if necessary, since it is bounded by Proposition 5.2. For a fixed \(\sigma > 0\), consider \(U_{p,N}(\sigma) := \{q \in B' : u_p(q) - u_{p+N}(q) > \sigma\}\), then we have \(U_{p,N}(\sigma) \subset U_{p,N+1}(\sigma)\), and \(\bigcup_{N=1}^{\infty} U_{p,N} = U_p(\sigma) := \{q \in B' : u_p(q) - u(q) > \sigma\}\). Then we have
\[
(7.5) \quad C_m \left( \bigcup_{N=1}^{\infty} U_{p,N}(\sigma) \right) = C_m(U_p(\sigma)) = \lim_{N \to \infty} C_m(U_{p,N}(\sigma))
\]
by Proposition 6.2 (6).

Denote \(\varphi_{p,N} := u_p - u_{p+N}\). Since the open set \(U_{p,N}(\sigma) \subset B' \subset B\), it follows from (6.3) that
\[
(7.6) \quad C_m(U_{p,N}(\sigma)) = \sup \left\{ \int_{U_{p,N}(\sigma)} (\Delta u)^m \wedge \beta_n^{m-m} : u \in \mathcal{L} \right\} \leq \sup \left\{ \frac{1}{\sigma} \int_{U_{p,N}(\sigma)} \varphi_{p,N}(\Delta u)^m \wedge \beta_n^{m-m} : u \in \mathcal{L} \right\} \leq \frac{\gamma}{\sigma} \left( \|u_p - v\|s + \int_B \varphi_{p,N}(\Delta \varphi_{p,N}^+)^m \wedge \beta_n^{m-m} \right),
\]
by the estimate (7.4), where \(\varphi_{p,N}^+ := (u_p + u_{p+N})/2\). Note that
\[
(7.7) \quad (\Delta \varphi_{p,N}^+)^m \wedge \beta_n^{m-m} = 2^{-m}(\Delta u_p + \Delta u_{p+N})^m \wedge \beta_n^{m} = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} (\Delta u_p)^k \wedge (\Delta u_{p+N})^{m-k} \wedge \beta_n^{m}. \]

It is sufficient to prove \(\int_B (u_p - u_{p+N}))(\Delta u_p)^k \wedge (\Delta u_{p+N})^{m-k} \wedge \beta_n^m\) tends to 0 uniformly as \(N \to \infty\) and then \(p \to \infty\).

For any closed \(C^2\) smooth \(2(n-1)\)-form \(\alpha\), i.e., \(d_0 \alpha = 0, d_1 \alpha = 0\), such that \(\Delta u_{p+N} \wedge \alpha \geq 0\), we have
\[
\int_B u_p \Delta u_{p+N} \wedge \alpha = \sum_{A=0}^{2n-1} \int_S u_p (d_1 u_{p+N} \wedge \alpha) A \tau(n) A_0dS + \int_B d_1 u_{p+N} \wedge d_0 u_p \wedge \alpha + \sum_{A=0}^{2n-1} \int_S u_{p+N} (d_0 u_p \wedge \alpha) A \tau(n) A_1 dS + \int_B u_{p+N} \Delta u_p \wedge \alpha \leq A_{p,N} + \int_B u_p \Delta u_p \wedge \alpha,
\]
by using Stokes-type formula (3.11) and $d_0 d_1 = -d_1 d_0$, where

$$A_{p,N} := \sum_{A=0}^{2n-1} \int_S [v_p(d_1 v_p + \alpha)A\tau(n)A_0 + v_p d_0 v_p + \alpha)A\tau(n)A_1] dS,$$

(7.8) since $u_p = v_p$ in a neighborhood of $S$ for $p > p_0$. Similarly,

$$\int_B u_{p,N} \Delta u_p \wedge \alpha = B_{p,N} + \int_B u_p \Delta u_{p,N} \wedge \alpha \geq B_{p,N} + \int_B u_{p,N} \Delta u_{p,N} \wedge \alpha,$$

(7.9) by $u_p \geq u_{p,N}$ and $\Delta u_p \wedge \alpha \geq 0$, where

$$B_{p,N} := \sum_{A=0}^{2n-1} \int_S [v_p(d_1 v_p + \alpha)A\tau(n)A_0 + v_p d_0 v_p + \alpha)A\tau(n)A_1] dS.$$

Repeating this procedure, finally we get

$$\int_B (u_p - u_{p,N})(\Delta u_p)^k \wedge (\Delta u_{p,N})^{m-k} \wedge \beta_n^m \leq \sigma(v_p, N) + \int_B u_p(\Delta u_p)^m \wedge \beta_n^m - \int_B u_{p,N}(\Delta u_{p,N})^m \wedge \beta_n^m$$

(7.10) where $\sigma(v_p, N)$ is the sum of terms of type $A_{p,N}$ and $B_{p,N}$ above. Because the sequence $\{v_p\}$ converges in the $C^2(B)$, we have

$$A_{p,N} \to \sum_{A=0}^{2n-1} \int_S [v_p(d_1 v_p + \alpha)A\tau(n)A_0 + v_p d_0 v_p + \alpha)A\tau(n)A_1] dS$$

(7.11)

$$= \int_B (d_0 v_p \wedge d_1 v_p + v_p d_0 d_1 v_p) \wedge \alpha + \int_B (d_1 v_p \wedge d_0 v_p + v_p d_1 d_0 v_p) \wedge \alpha = 0,$$

as $N \to \infty$, by using Stokes-type formula (3.11) again. Similarly $B_{p,N} \to 0$ as $N \to \infty$. Since the sequence $\int_B u_p(\Delta u_p)^m \wedge \beta_n^m$ has a limit as $p \to \infty$, the right hand side of (7.10) tends to 0. Hence

$$\lim_{p \to \infty} C_m(U_p(\sigma)) = \lim_{p \to \infty} \lim_{N \to \infty} (U_{p,N}(\sigma)) = 0.$$

Now for fixed $\epsilon > 0$, there exist $p_j > 0$ such that if we denote $U_{p_j} := U_p(1/j)$ for $\sigma = 1/j$, we have $C_m(U_{p_j}) \leq \frac{\epsilon}{2^j}$. Since $u_p(q) - u_p(q) < \frac{1}{j}$ for $p > p_j$ outside the set $U_{p,j}$, then we see that $u_p$ convergence to $u$ uniformly outside the open set $U = \cup_{j=1}^{\infty} U_{p,j}$. Since $u_p \in C^\infty(B)$, $u$ is continuous outside $U$, and

$$C_m(U) = C_m \left( \bigcup_{j=1}^{\infty} U_{p,j} \right) \leq \sum_{j=1}^{\infty} C_m(U_{p,j}) \leq \epsilon.$$

The theorem is proved.

**Proposition 7.1.** Let $u_1, \ldots, u_m \in QSH_m(\Omega) \cap L^\infty_m(\Omega)$. Then, (1) the recurrence relation

$$\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}(\omega) = \int u_k \Delta u_1 \wedge \cdots \wedge \Delta u_{k-1} \wedge \beta_n^{m-k} \wedge \Delta \omega$$

(7.12) for $\omega \in \mathcal{D}^{2n-2k}(\Omega), k = 1, \ldots, m$, defines a closed positive $2(n-m+k)$-current.

(2) The following convergence of closed positive currents (of measure type) holds for the standard approximations $u_i^t \downarrow u_i, i = 1, \ldots, m$, as $t \to \infty$:

$$\Delta u_1^t \wedge \cdots \wedge \Delta u_k^t \wedge \beta_n^{m-k} \to \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge \beta_n^{m-k}, \quad k = 1, \ldots, m.$$
Proof. Let us prove the theorem by induction on $k$. The case $k = 1$ is obvious.

Assume that it holds for $k - 1$. Then for a fixed strongly positive form $\omega \in D^{2m-2k}(\Omega)$, we have

\[
(7.14) \quad \int u_k^* \Delta u_1^i \land \cdots \land \Delta u_{k-1}^i \land \beta_n^{m-n} \land \Delta \omega = \int \Delta u_1^i \land \cdots \land \Delta u_{k-1}^i \land \Delta u_k^i \land \beta_n^{m-n} \land \omega \geq 0,
\]

by Proposition [4.1] for smooth $QS\mathit{H}_m$, which yields the limits $\int u_k^* \Delta u_1^i \land \cdots \land \Delta u_{k-1}^i \land \beta_n^{m-n} \land \Delta \omega \geq 0$ as $t \rightarrow \infty$. If let $s \rightarrow \infty$, we find that $\int u_k \Delta u_1 \land \cdots \land \Delta u_{k-1} \land \beta_n^{m-n} \land \Delta \omega \geq 0$. Hence, the current $\Delta u_k \land \Delta u_1 \land \cdots \land \Delta u_{k-1} \land \beta_n^{m-n}$ is positive. It is closed by definition.

To prove (2), note that if the convergence

\[
(7.15) \quad E := u_1^i \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} - u_1 \Delta u_2 \land \cdots \land \Delta u_k \land \beta_n^{m-n} \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty,
\]

is valid for $k$, then (7.13) is valid for $k$, since

\[
\int \Delta u_1^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \omega = \int u_1^i \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \Delta \omega,
\]

for $\omega \in D^{2m-2k}(\Omega)$. So it suffices to prove (7.15) for $k$, provided that (7.13) is valid for $k - 1$.

By the quasicontinuity in Theorem [7.1], for a fixed $\varepsilon > 0$, we can find an open $U \subset \Omega$ such that $C_m(U) < \varepsilon$ and $u_1 \in C(\Omega \setminus U)$. Let $\bar{u} \in C(\Omega)$ satisfy $u_1 \equiv \bar{u}$ on $\Omega \setminus U$ and $\|\bar{u}\| \leq \|u\|_\Omega$. Denote $E_\omega := \text{supp} \omega$. Then,

\[
|E \land \omega| \leq \left| \int_{E_\omega} (u_1^i - u_1) \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \omega \right| + \left| \int_{\Omega} u_1 (\Delta u_2^i \land \cdots \land \Delta u_k^i - \Delta u_2 \land \cdots \land \Delta u_k) \land \beta_n^{m-n} \land \omega \right| \leq \left| \int_{E_\omega \setminus U} (u_1^i - u_1) \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \omega \right| + \left| \int_{E_\omega \cap U} (u_1^i - u_1) \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \omega \right| + \left| \int_{E_\omega \cap U} (\bar{u} - u_1) (\Delta u_2^i \land \cdots \land \Delta u_k^i - \Delta u_2 \land \cdots \land \Delta u_k) \land \beta_n^{m-n} \land \omega \right| + \left| \int_{\Omega} \bar{u} (\Delta u_2^i \land \cdots \land \Delta u_k^i - \Delta u_2 \land \cdots \land \Delta u_k) \land \beta_n^{m-n} \land \omega \right|.
\]

The integral over the sets $E_\omega \setminus U$ on the right hand side tends to zero as $t \rightarrow \infty$ since $u_1^i \rightarrow u_1$ uniformly in $E_\omega \setminus U$, while the forth integral over $\Omega$ tends to zero because

\[
\lim_{t \rightarrow +\infty} \int_{E_\omega \cap U} \Delta u_2^i \land \cdots \land \Delta u_k^i \land \beta_n^{m-n} \land \omega = \Delta u_2 \land \cdots \land \Delta u_k \land \beta_n^{m-n},
\]

as currents of measure type by the assumption of induction, and $\bar{u}$ continuous on $\Omega$. The second and third integrals reduces to estimating integrals of the type

\[
\int \Delta v_2 \land \cdots \land \Delta v_k \land \beta_n^{m-n} \land \omega,
\]

where $v_2, \ldots, v_k \in Q\mathit{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, which are small because the capacity $C_m(U) < \varepsilon$ is small.

At last, a positive current is a current of measure type by Proposition 3.4 in [31]. $\Box$
8. The fundamental solution of the quaternionic \(m\)-Hessian operator and the \(m\)-Lelong number

**Proposition 8.1.** Let \(\kappa_m = \frac{2m}{m} - 1\). Then the function \(K_m(q) := \frac{-1}{|q - a|^{2m}}\) is QSH\(_m\) and is the fundamental solution to the quaternionic \(m\)-Hessian operator \(H_m\), i.e.

\[
H_m(K_m) = C_{m,n}d_a
\]

where \(C_{m,n} = \frac{8^n n! \pi^{2n} \kappa_m}{(2n)! m! (n-m)!}\).

**Proof.** Without loss of generality, we may assume that \(a = 0\). Denote \(K_{m,\epsilon} := \frac{-1}{|q - \epsilon a|^{2m}}\). Then,

\[
d_1K_{m,\epsilon} = \frac{\kappa_m d_1|q|^2}{(|q|^2 + \epsilon)^{\kappa_m+1}},
\]

and

\[
\Delta K_{m,\epsilon} = d_0 \left( \frac{\kappa_m d_1|q|^2}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right) = -\frac{\kappa_m (\kappa_m + 1)}{(|q|^2 + \epsilon)^{\kappa_m+2}} d_0|q|^2 \wedge d_1|q|^2 + \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} =: A + B.
\]

Hence,

\[
(\Delta K_{m,\epsilon})^p \wedge \beta_n^{n-p} = (pA \wedge B^{p-1} + B^p) \wedge \beta_n^{n-p}, \quad p = 1, \ldots, m,
\]

by \(\omega \wedge \omega = 0\) for any 1-form \(\omega\). Now apply

\[
d_0|q|^2 \wedge d_1|q|^2 = 4 \sum_{l=0}^{n-1} |q|^2 \omega^l \wedge \omega^{n+l} + \sum_{|j-k| \neq n} a_{jk} \omega^j \wedge \omega^k
\]

(cf. [33 (3.12)]) to \(\Delta K_{m,\epsilon}\) to get

\[
(\Delta K_{m,\epsilon})^p \wedge \beta_n^{n-p} = \left[ -4p \kappa_m (\kappa_m + 1) \sum_{l=0}^{n-1} |q|^2 \omega^l \wedge \omega^{n+l} \wedge \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^{p-1} + \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^p \right] \wedge \beta_n^{n-p}
\]

\[
= -4p \kappa_m (\kappa_m + 1) \sum_{l=0}^{n-1} |q|^2 \omega^l \wedge \omega^{n+l} \wedge \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^{p-1} + \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^p \wedge \beta_n^{n-p}
\]

\[
= -4p \kappa_m (\kappa_m + 1) \sum_{l=0}^{n-1} |q|^2 \omega^l \wedge \omega^{n+l} \wedge \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^{p-1} + \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{\kappa_m+1}} \right)^p \wedge \beta_n^{n-p}
\]

by \(-p(\kappa_m + 1) + 2n = 2n(1 - p/m) \geq 0\). Thus, \(K_{m,\epsilon} \in QSH_m\) by definition, and so is \(K_m \in QSH_m\) by \(K_{m,\epsilon} \downarrow K_m\). In particular,

\[
(\Delta K_{m,\epsilon})^{m} \wedge \beta_n^{n-m} = \epsilon \left( \frac{8 \kappa_m \beta_n}{(|q|^2 + \epsilon)^{2n+1}} \right)^m \Omega_{2n}.
\]

Letting \(\epsilon \to 0\), we get

\[
(\Delta K_m)^m \wedge \beta_n^{n-m} = 0 \quad \text{on} \quad \mathbb{H}^n \setminus \{0\}.
\]

For any \(\varphi \in C_0^\infty(\mathbb{R}^{4n})\), by rescaling \(q = q' \epsilon^\frac{1}{2}\), we get

\[
\lim_{\epsilon \to 0} \epsilon \int_{\mathbb{R}^{4n}} \frac{\varphi(q' \epsilon^\frac{1}{2})}{(|q'|^2 + \epsilon)^{2n+1}} \varphi(q) dV(q) = \lim_{\epsilon \to 0} \epsilon \int_{\mathbb{R}^{4n}} \frac{\varphi(q' \epsilon^\frac{1}{2})}{(|q'|^2 + 1)^{2n+1}} dV(q) = \frac{S_{4n}}{4n} \varphi(0),
\]
by
\[
\int_{S^4} \frac{1}{(|q|^2 + 1)^{2n+1}} dV(q) = \lim_{R \to \infty} S_{4n} \int_0^R \frac{r^{4n-1}}{(1 + r^2)^{2n+1}} dr = \lim_{R \to \infty} S_{4n} \int_0^{\arctan R} \frac{\tan^{4n-1} \theta}{\sec^{4n} \theta} d\theta \\
= \lim_{R \to \infty} S_{4n} \int_0^{\arctan R} \sin^{4n-1} d\sin \theta = \lim_{R \to \infty} S_{4n} \int_0^{\arctan \frac{R}{\sqrt{1 + R^2}}} t^{4n-1} dt \\
= \lim_{R \to \infty} S_{4n} \frac{1}{4n} \frac{R^{4n}}{(1 + R^2)^{2n}} = \frac{S_{4n}}{4n},
\]
where \( S_{4n} = 4n \pi^{2n} / (2n)! \). Thus (8.1) follows.

\[\square\]

**Proposition 8.2.** Suppose that \( \Omega \subseteq \mathbb{H}^n \) is a domain and \( B(a, R) \in \Omega \) for some \( R > 0 \). For \( u \in QSH_m(\Omega) \) and \( 0 < r < R \), denote
\[
\sigma(a, r) = \int_{B(a, r)} \Delta u \wedge \beta_{n-1}.
\]
Then, \( \frac{\sigma(a, r)}{r^{2n-1}} \) is an increasing function of \( r \) for \( 0 < r < R \), and
\[
(8.7)
\]
exists and is nonnegative. It is called the \( m \)-Lelong number of \( u \) at \( a \).

**Proof.** : For \( 0 < r_1 < r_2 < R \), consider
\[
v_u(r_1, r_2) := \int_{r_1 < |q| \leq r_2} \Delta u \wedge (\Delta K_m)^{m-1} \wedge \beta_{n-m}.
\]
Since \( K_m \in QSH_m \), the integrant in (8.7) is a nonnegative measure on \( B(a, R) \). Without loss of generality, we may assume that \( a = 0 \). Firstly, assume \( u \in QSH_m(B(0, R)) \cap C^\infty(B(0, R)) \). Then we have
\[
v_u(r_1, r_2) = \int_{r_1 < |q| \leq r_2} d_0 \left( d_1 K_m \wedge \Delta u \wedge (\Delta K_m)^{m-2} \wedge \beta_{n-m} \right)
\]
\[
= \frac{\kappa_m}{r_2^{2(\kappa_m + 1)}} \int_{|q| = r_2} (d_1 |q|^2 \wedge \Delta u \wedge (\Delta K_m)^{m-2} \wedge \beta_{n-m})_A \tau(n)_{A_0} dS
\]
\[
- \frac{\kappa_m}{r_1^{2(\kappa_m + 1)}} \int_{|q| = r_1} (d_1 |q|^2 \wedge \Delta u \wedge (\Delta K_m) \wedge \beta_{n}^{m-1})_A \tau(n)_{A_0} dS
\]
\[
= (8\kappa_m)^{m-1} \left( \frac{\sigma(a, r_2)}{r_2^{4n-m+1}} - \frac{\sigma(a, r_1)}{r_1^{4n-m+1}} \right) > 0,
\]
by using Stokes theorem, (8.2) for \( \epsilon = 0 \), and
\[
\frac{\kappa_m}{r_2^{2(\kappa_m + 1)}} \int_{|q| = r} (d_1 |q|^2 \Delta u \wedge (\Delta K_m)^{m-2} \wedge \beta_{n-m})_A \tau(n)_{A_0} dS
\]
\[
= \frac{8\kappa_m}{r_2^{2(\kappa_m + 1)}} \int_{|q| \leq r} \Delta u \wedge (\Delta K_m)^{m-2} \wedge \beta_{n-m+1} = \cdots = \left( \frac{8\kappa_m}{r_2^{2(\kappa_m + 1)}} \right)^{m-1} \int_{|q| \leq r} \Delta u \wedge \beta_{n}^{m-1}
\]
Now using the convergence of \( u \ast \chi_\epsilon \downarrow u \) and \( \lim_{\epsilon \to 0} \Delta (u \ast \chi_\epsilon) \wedge \beta_{n-m} \to \Delta u \wedge \beta_{n-m} \) as currents of measure type, we get the result.
\[\square\]
The proof given here also simplifies the proof of the existence of the Lelong number for a plurisubharmonic function in [31].

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