Viability of vector-tensor theories of gravity

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Abstract. We present a detailed study of the viability of general vector-tensor theories of gravity in the presence of an arbitrary temporal background vector field. We find that there are six different classes of theories which are indistinguishable from General Relativity by means of local gravity experiments. We study the propagation speeds of scalar, vector and tensor perturbations and obtain the conditions for classical stability of those models. We compute the energy density of the different modes and find the conditions for the absence of ghosts in the quantum theory. We conclude that the only theories which can pass all the viability conditions for arbitrary values of the background vector field are not only those of the pure Maxwell type, but also Maxwell theories supplemented with a (Lorentz type) gauge fixing term.
1. Introduction

Although the interest in alternative gravity theories has been present over the years \cite{1, 2}, it has been recently renewed by the unexpected observations made in cosmological and astrophysical contexts. Indeed, the difficulties found by the Standard Model (SM) of elementary particles and General Relativity (GR) in order to explain the nature of dark matter and dark energy suggest either the need for new physics beyond the SM or the modification of GR at very large scales.

Among such modifications, we find the so called vector-tensor theories of gravity, proposed in the early 70’s \cite{3, 4}, and where, in addition to the metric tensor, a dynamical vector field is introduced in such a way that it only couples to matter through gravitational interactions.

After some initial proposals, these theories were practically forgotten due to consistency problems and only particular classes of models with fixed norm vector or potential terms have been considered in the context of violations of Lorentz invariance (see \cite{5} and references therein). However, very recently, it has been shown \cite{6, 7} that unconstrained vector-tensor theories without potential terms are excellent candidates for dark energy.

The main three difficulties found in this type of theories were the following \cite{8, 9}:

a) Inconsistencies with local gravity tests: in addition to modifications in the static parametrized post-Newtonian (PPN) parameters ($\gamma, \beta$), these theories typically predict preferred frame effects, parametrized by ($\alpha_1, \alpha_2$). Such effects are produced by the motion with respect to the privileged frame selected by the presence of the background vector field. The increasing observational precision in Earth and Solar System experiments seems to exclude this type of effects \cite{2}.

b) Classical instabilities: they are generated by the presence of modes with negative propagation speed squared. This gradient instabilities give rise to exponentially growing perturbations which make the theory unphysical.

c) Quantum instabilities (ghosts): they typically appear when the energy of some perturbation modes becomes negative. In such a case it has been shown that the vacuum state of the corresponding quantum theory is unstable \cite{10}.

It has been also argued \cite{11} that the presence of faster than light propagation modes could give rise to causality inconsistencies although this conclusion is far from clear (see for instance \cite{12}) and even some authors claim that consistency requires the presence of such modes \cite{9}. For that reason in this work we will not include it as a viability requirement.

Although it is usually claimed that vector-tensor theories are plagued by the mentioned consistency problems, the actual situation is that none of them has been studied in detail, and they are only based on qualitative arguments which, in many cases, turn out to be inappropriate. In this work we present a detailed analysis of the viability of unconstrained vector-tensor theories and obtain the precise conditions in order to satisfy the above requirements.
2. Local gravity constraints

Without any other restriction, apart from having second order linear equations of motion, the most general action for a vector-tensor theory without potential terms can be written as follows [2]:

\[ S[g_{\mu\nu}, A_{\mu}] = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} R + \omega RA_{\mu}A^{\mu} + \sigma R_{\mu\nu}A^{\mu}A^{\nu} + \tau \nabla_{\mu}A_{\nu}\nabla^{\mu}A^{\nu} + \varepsilon F_{\mu\nu}F^{\mu\nu} \right] \]  

(1)

with \( \omega, \sigma, \tau, \varepsilon \) dimensionless parameters and \( F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \). Notice that the term including the Ricci tensor can be rewritten as \( \nabla_{\mu}A_{\nu}\nabla^{\mu}A^{\nu} - \nabla_{\mu}A^{\nu}\nabla_{\nu}A^{\mu} \). These two terms do not appear independently in the most general action because they can be recasted in the terms written in (1).

Since GR agrees with Solar System experiments with high precision, it would be desirable that a viable vector-tensor theory had the same set of PPN parameters as GR, i.e. \( \gamma = \beta = 1 \) and \( \alpha_1 = \alpha_2 = 0 \). Notice that the rest of PPN parameters vanishes identically for models described by (1). The PPN parameters for general vector-tensor theories have been calculated in [2] assuming the existence of a constant background time-like vector field:

\[ A_{\mu} = (A, 0, 0, 0) \]  

(2)

and can be found in the Appendix. When we impose that such parameters agree with those of GR for any value of \( A \), we obtain two sets of compatible models, according to their behavior in flat space-time, both with \( \omega = 0 \):

- **Gauge non-invariant models.** These models have \( \tau \neq 0 \) and the corresponding term \( \nabla_{\mu}A_{\nu}\nabla^{\mu}A^{\nu} \) breaks the \( U(1) \) gauge invariance in Minkowski space-time. The three possibilities we obtain in this case are:  
  i) \( \sigma = -3\tau = -6\varepsilon \), ii) \( \sigma = -2\tau = -2\varepsilon \), iii) \( \sigma = \tau \).

- **Gauge invariant models.** In this case we have \( \tau = 0 \) and the only term remaining in Minkowski space-time is the gauge invariant one. The possibilities in this case are \( \sigma = m\varepsilon \) with \( m = 0, -2, -4 \).

Notice that except for the \( \sigma = \tau = 0 \) case (which is nothing but GR plus Maxwell electromagnetism), all the considered cases break gauge invariance in general space-times. Therefore, there are six different classes of models which are indistinguishable from GR by means of Solar System experiments and, therefore, do not spoil the current bounds on the PPN parameters. To our knowledge best, none of these models (apart from Maxwell’s theory) had been considered previously in the literature.

We will also use the present constraints on the variation of the Newton’s constant which are given by \( \dot{G}/G \lesssim 10^{-13} \text{ yr}^{-1} \) [2].

3. Classical and quantum stability

To study the existence of unstable classical modes and ghosts we shall perform perturbations around a Minkowski background. In addition, we will also consider perturbations around the constant background vector field introduced in (2). This is possible because Minkowski space-time is an admissible solution of the theory in the case in which the vector field takes a constant value, as that introduced above. For this constant background, the vector field breaks Lorentz invariance and we have a preferred frame defined as that in which the vector field has only temporal component. In this background we decompose the perturbations in Fourier modes and solve the equations for the corresponding amplitudes. That way, we obtain the dispersion relation, which provides us with the propagation speed of the modes, that is required to be real in order not to have exponentially growing perturbations. As we explained above, here we shall not care about superluminal propagation of the modes, although it could be easily imposed at some point. Notice that although we are assuming constant background, in practice, this background could evolve on cosmological timescales. The effects of such evolution could have important cosmological consequences [6, 7].

On the other hand, we define the energy for the modes as [13]:

$$\rho = \langle T^{(2)}_{00} - \frac{1}{8\pi G} G^{(2)}_{00} \rangle$$

(3)

where $T^{(2)}_{\mu\nu}$ and $G^{(2)}_{\mu\nu}$ are the energy-momentum tensor of the vector field and the Einstein’s tensor calculated up to quadratic terms in the perturbations and $\langle \cdot \cdot \cdot \rangle$ denotes an average over spatial regions. Then, we insert the solutions obtained for the perturbations into this expression and study under which conditions they are positive.

Notice also that the preferred frame respects the invariance under spatial rotations and therefore, in order to simplify the analysis we can perform the usual split of the perturbations into spin-0 (scalar), spin-1 (vector) and spin-2 (tensor). For simplicity, the longitudinal gauge has been chosen in the calculations below, although we have checked that the final results for the mode frequencies and energy densities defined in (3) do not depend on the gauge choice.

The scalar perturbations of the vector field can be written as $S_\mu = (S_0, \bar{\nabla} S)$ and the perturbed metric in the longitudinal gauge is:

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\psi)\delta_{ij}dx^i dx^j$$

(4)

For vector perturbations we have $V_\mu = (0, \bar{v})$ and the metric is as follows:

$$ds^2 = dt^2 + 2\bar{F} \cdot d\bar{x} dt - \delta_{ij} dx^i dx^j$$

(5)

with $\nabla \cdot \bar{v} = \nabla \cdot \bar{F} = 0$. Although the vector field does not generate tensor perturbations, we still can have effects by the presence of the background vector field as we shall see later. Let us first consider the scalar and vector perturbations.
3.1. Gauge non-invariant models

3.1.1. Model I: $\sigma = -3\tau = -6\varepsilon$ In this model, the Fourier components of the gravitational potentials relate to those of the vector field as follows: \[ \phi_k = \frac{4\varepsilon A}{1 - 8\varepsilon A^2} \left[ 3\dot{S}_k - S_{0k} \right], \] \[ \psi_k = -\frac{4\varepsilon A}{1 - 8\varepsilon A^2} \left[ (1 - 32\varepsilon A^2)\dot{S}_k + S_{0k} \right], \] whereas $S_{0k}$ and $S_k$ are related to each other by means of: \[ S_{0k} = \frac{1}{k^2(1 + 8\varepsilon A^2)} \left[ 48\varepsilon A^2 k^2 \frac{dS_k}{dt} \right. \right. \] \[ \left. \left. - (1 - 8\varepsilon A^2)(1 - 16\varepsilon A^2)(1 - 64\varepsilon A^2) \frac{d^3S_k}{dt^3} \right] \right]. \] Thus, the problem is solved once we know the solution of $S_k$, which happens to satisfy the following fourth order equation: \[ \frac{d^4S_k}{dt^4} + \frac{k^2}{(1 - 16\varepsilon A^2)(1 - 64\varepsilon A^2)} \left[ 2(1 - 40\varepsilon A^2) \frac{d^2S_k}{dt^2} + k^2 S_k \right] = 0. \] This equation yields two independent modes: \[ S_k = C_1 e^{-i\omega_1 t} + C_2 e^{-i\omega_2 t}, \] with their respective dispersion relations: \[ \omega_1^2 = \frac{k^2}{1 - 64\varepsilon A^2}, \] \[ \omega_2^2 = \frac{k^2}{1 - 16\varepsilon A^2}. \] Then, in order to have stable solutions we need to satisfy the following condition: \[ 64\varepsilon A^2 < 1 \] because, in that case, both modes have real propagation speeds.

On the other hand, the energy density evaluated over the solutions for each mode is given by: \[ \rho_{\omega_1}^{(s)} = -384\varepsilon^2 A^2 k^4 \frac{(1 - 128\varepsilon A^2)}{(1 - 64\varepsilon A^2)^2} |C_1|^2 \] \[ \rho_{\omega_2}^{(1)} = 384\varepsilon^2 A^2 k^4 \frac{(1 - 32\varepsilon A^2)}{(1 + 8\varepsilon A^2)^2} |C_2|^2 \] These energies are both positive under the following constraint: \[ 1 < 128\varepsilon A^2 < 4 \] Now, by combining (13) and (16) we find the following viability condition for the scalar modes: \[ 1 < 128\varepsilon A^2 < 2 \] ‡ Hereafter we will measure the field in units of $\sqrt{4\pi G}$. 

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Concerning vector perturbations, we obtain the following relation between the amplitudes:

$$\vec{F}_k = -\frac{16\varepsilon A^2}{1 - 16\varepsilon A^2} \vec{v}_k$$

(18)

and both evolve as plane waves with the following dispersion relation:

$$\omega^2 = \frac{k^2}{1 - 16\varepsilon A^2}$$

(19)

which leads to the stability condition:

$$16\varepsilon A^2 < 1$$

(20)

Finally, the energy density associated to the vector perturbations is given by:

$$\rho^{(v)} = 8\varepsilon \left( 1 - \frac{32\varepsilon A^2}{(1 - 16\varepsilon)^2} k^2 |\vec{v}_0|^2 \right)$$

(21)

From this expression we see that $\varepsilon$ must be positive and we have to satisfy:

$$32\varepsilon < 1$$

(22)

in order not to have vector modes with negative energy.

In this model, according to the definition in the Appendix, $G = 1$ and the constraints on the variation of $G$ do not set any limit on the possible variation of $A$.

3.1.2. Model II: $\sigma = -2\tau = -2\varepsilon$ In this case, $\phi$ and $\psi$ are given in terms of the perturbation of the vector field as follows:

$$\phi_k = \frac{3\tau A}{1 - 3\tau A^2} \dot{S}_k,$$

(23)

$$\psi_k = -3\tau A \frac{1 - 6\tau A}{1 - 3\tau A^2} \dot{S}_k.$$  

(24)

On the other hand, the perturbation $S_{0k}$ can be expressed as:

$$S_{0k} = -\frac{1}{2k^2(1 - 3\tau A^2)} \left[ (1 - 6\tau A^2)(1 - 15\tau A^2) \frac{d^3 S_k}{dt^3} ight.$$  

$$- k^2 (1 + 3\tau A^2) \frac{d S_k}{dt} \right].$$

(25)

Then, all the perturbations are given in terms of $S_k$, for which we can obtain the following fourth-order differential equation:

$$\frac{d^4 S_k}{dt^4} + 2k^2 \frac{1 - 3\tau A^2}{(1 - 6\tau A^2)(1 - 15\tau A^2)} \left[ 2(1 - 9\tau A^2) \frac{d^2 S_k}{dt^2} + k^2 S_k \right] = 0.$$  

(26)

The solution of this equation is a superposition of two independent modes:

$$S_k = C_+ e^{-i\omega_+ t} + C_- e^{-i\omega_- t},$$

(27)

with their respective dispersion relations:

$$\omega_{\pm}^2 = \frac{k^2}{(1 - 6\tau A^2)(1 - 15\tau A^2)} \left[ 1 - 12\tau A^2 + 27\tau^2 A^4 ight.$$  

$$\pm 3|\tau|A^2 \sqrt{(1 - 3\tau A^2)(5 - 27\tau A^2)} \right].$$

(28)
Then, we have two modes with two different speeds of propagation which depend on $\tau A^2$ as it is shown in Fig. 1. The $\omega_-$-mode has real propagation speed for $\tau A^2 < \frac{1}{6}$, whereas for the $\omega_+$-mode to have real propagation speed we need to satisfy either $\tau A^2 < \frac{1}{15}$ or $\tau A^2 > \frac{1}{3}$. Therefore, the necessary condition in order not to have instabilities is $\tau A^2 < \frac{1}{15}$. Finally, the degeneracy disappears for $\tau A^2 = 0$ when both propagation speeds are 1, recovering thus the usual result.

The energy density corresponding to each mode can be expressed as:

$$\rho_{\pm}^{(s)} = \tau f_{\pm}(\tau A^2)k^4 |C_{\pm}|^2$$

where $f_{\pm}(\tau A^2)$ are the functions plotted in Fig. 2. Notice that $f$ and $\tau$ must have the same sign for the energy density to be positive. We find the following condition in order to have positive energy density for both modes:

$$\tau A^2 \in (0.033, 0.105) \cup (0.383, 0.5)$$

For this model, the vector perturbation on the metric relates to that of the vector field by means of:

$$\vec{F}_k = -\frac{6\tau A}{1 - 6\tau A^2} \vec{v}_k$$

which evolve as:

$$\vec{v}_k = \vec{v}_{0k} e^{-i\omega v t}$$

where $\vec{v}_{0k} \cdot \vec{k} = 0$ and

$$\omega_v^2 = \frac{1 - 3\tau A^2}{1 - 6\tau A^2} k^2$$

From this expression we obtain that for $\tau A^2 < \frac{1}{6}$ the propagation speed is real.

The energy density corresponding to the vector perturbations is:

$$\rho^{(v)} = 6\tau k^2 \frac{1 - 12\tau A^2 + 18\tau A^4}{(1 - 6\tau A^2)^2} |\vec{v}_{0k}|^2$$
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Figure 2. This plot shows the functions $f_\pm(\tau A^2)$ which determine the sign of the energy density for the scalar modes in the model II: blue for $f_+$ and dashed-red for $f_-$.  

One can easily verify that this expression is positive if:

$$\tau A^2 \in [0, c_-] \cup [c_+, \infty)$$

with $c_\pm = \frac{1}{3}(1 \pm \frac{1}{\sqrt{2}})$. Note that $\tau > 0$ is a necessary condition and that the singular value $\tau A^2 = \frac{1}{6}$ is not contained in the interval.

In this case, $G = 1 + 6\tau A^2$. The present constraints on the variation of the Newton’s constant will translate into a limit on the possible variation of $A$ which will depend on the present cosmological value of $A$.

3.1.3. Model III: $\sigma = \tau$ The perturbations of this model propagate at the speed of light so there are no classically unstable modes. However, the scalar perturbations are not just plane waves, but they have also a growing mode:

$$S_{0k} = [-i k \lambda(D_0 + i k D)t + D_0] e^{-i k t}$$

$$S_k = [\lambda(D_0 + i k D)t + D] e^{-i k t}$$

with

$$\lambda = \frac{\epsilon + 2\tau(2\epsilon + \tau)A^2}{\epsilon + \tau + 2\tau(2\epsilon + \tau)A^2}$$

In principle, this solution can give rise to both positive or negative energies depending on the value of the amplitudes. However, there exists a way to get this difficulty around which is motivated by the fact that the action corresponding to this model can be rewritten as that of Maxwell’s electromagnetism in the Gupta-Bleuler formalism, that is:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2} \left( \nabla_\mu A^\mu \right)^2 \right]$$

Notice that this action is the physically relevant action in the covariant formalism, since the energy of the modes and all the observables are calculated from (38) (see for
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In order to get rid of the negative energy modes, it is necessary to restrict the Hilbert space of the theory, by imposing the (Lorentz) condition $\langle \phi | \partial_\mu A^\mu | \phi \rangle = 0$, which determines the physical states $|\phi\rangle$. As shown in [7], this is equivalent to remove the growing mode i.e. $D_0 = -ikD$ so that perturbations of the vector field propagate as pure plane waves. Then, using this condition we obtain that, as in the electromagnetic case, in the restricted Hilbert space, the energy of the scalar modes is identically zero. A detailed treatment of the quantization for this model is performed in [7].

The solution of the vector perturbation of the vector field is:

$$\vec{v}_k = \vec{v}_{0k} e^{-ikt} \quad (39)$$

Moreover, the vector perturbation of the metric vanishes.

The energy density associated to the vector perturbations is:

$$\rho^{(v)} = (2\varepsilon + \tau)k^2 |\vec{v}_{0k}|^2 \quad (40)$$

which is positive if $2\varepsilon + \tau > 0$.

In this model we have again $G = 1$ and no constraints on the variation of $A$ can be established.

3.2. Gauge invariant models: $\tau = 0$, $\sigma = m\varepsilon$, $m = 0, -2, -4$

As the case $m = 0$ is nothing but Einstein’s gravity plus electromagnetism (which can be easily seen to satisfy all the viability conditions), we shall focus just on the cases $m = -2, -4$. In such cases, we can obtain the following relations between the perturbations:

$$\dot{S}_k = \frac{1 + (m + 3)m\varepsilon A^2}{m\varepsilon A(1 + 2m\varepsilon A^2)} \psi_k,$$

$$S_{0k} = \frac{1 - m\varepsilon A^2(3 - 2m^2\varepsilon A^2)}{m\varepsilon A(1 + 2m\varepsilon A^2)} \psi_k,$$

$$\phi_k = \psi_k - 2m\varepsilon A S_k. \quad (41)$$

Therefore, all the perturbations can be immediately obtained once we know the solution of $\psi_k$ which happens to evolve as:

$$\psi_k = C_k e^{-\omega_s t}, \quad (42)$$

with:

$$\omega_s^2 = \frac{13 + 2(4 + m)m\varepsilon A^2}{3 - 1 + 2m\varepsilon A^2} k^2. \quad (43)$$

Then, the classical stability of these modes is guaranteed in the range:

$$\varepsilon A^2 \in \left( -\infty, -\frac{1}{2m} \right) \cup \left( -\frac{3}{2(4 + m)m}, \infty \right). \quad (44)$$

Notice that the second interval vanishes for the model with $m = -4$. 

When we insert the corresponding solutions into the energy density we obtain:

$$
\rho^{(s)} = \frac{3 + 4m\varepsilon A^2 (3 + (4 + m)m\varepsilon A^2)}{2\pi G(1 + 2m\varepsilon A^2)} k^2 |C_k|^2.
$$

(45)

This energy density is positive for:

$$
\varepsilon A^2 \in (-\infty, a_+) \cup \left(-\frac{1}{2m}, a_-\right),
$$

(46)

with:

$$
a_\pm = -\frac{1}{2m \left(1 \pm \sqrt{-\frac{1+m}{3}}\right)}.
$$

(47)

Notice that $a_-$ is $\infty$ for the model $m = -4$.

On the other hand, the vector perturbations of the field evolve as plane waves $\vec{v}_k = \vec{v}_0 e^{-i\omega_{vt}}$ with the following dispersion relation:

$$
\omega_{\vec{v}}^2 = \frac{1 + (2 + \frac{1}{2}m)m\varepsilon A^2}{1 + 2m\varepsilon A^2} k^2,
$$

(48)

which imposes the condition:

$$
\varepsilon A^2 \in \left(-\infty, -\frac{1}{2m}\right) \cup \left(-\frac{1}{(2 + \frac{1}{2}m)m}, \infty\right)
$$

(49)

in order to have real propagation speed. Notice that for $m = -4$ the second interval vanishes.

Besides, the vector perturbation of the metric relates to $\vec{v}$ by means of:

$$
\vec{F}_k = \frac{2m\varepsilon A}{1 + 2m\varepsilon A^2} \vec{v}_k.
$$

(50)

These solutions for the vector perturbations lead to the following expression for the energy density:

$$
\rho^{(v)} = \varepsilon \frac{1 + 4m\varepsilon A^2 (1 + (1 + \frac{1}{2}m)m\varepsilon A^2)}{(1 + 2m\varepsilon A^2)} k^2 |\vec{v}_0|^2
$$

(51)

The requirement for this energy to be positive is:

$$
\varepsilon A^2 \in (0, b_+) \cup \left(-\frac{1}{2m}, b_-\right),
$$

(52)

where

$$
b_\pm = -\frac{1}{m(2 \pm \sqrt{-m})}
$$

(53)

Note also that for $m = -4$, $b_-$ becomes $+\infty$.

In this case we get: $G = 1 - \varepsilon m(4 + m)A^2$, so the only model with $G \neq 1$ is that with $m = -2$. 


Table 1: In this table we summarize the conditions obtained in order to have both classical and quantum stability for the models with the same set of PPN parameters as GR studied in this work. The \( m = 0 \) gauge invariant model satisfies all the viability conditions.

4. Gravitational Waves

At first glance, one may think that as the vector field does not generate tensor modes at first order, gravitational waves will not be affected. However, the presence of a constant value of the vector field in the background can modify the speed of propagation of tensor perturbations. For the general vector-tensor action \((1)\) we have the following dispersion relation:

\[
\omega_i^2 = \frac{k^2}{\sqrt{1 + 2(\sigma - \tau)A^2}}
\]  

for both \( \oplus \) and \( \otimes \) polarizations. Therefore, the speed of gravitational waves is modified in the presence of the vector field, recovering the usual value for \( A = 0 \). Thus, if \( 2(\sigma - \tau)A^2 > -1 \) we do not have unstable modes. In particular, the constraint \( \sigma - \tau \geq 0 \) is a sufficient condition (although not necessary) which is independent of the background vector field.

On the other hand, the energy density associated to the tensor perturbations is also modified by the presence of the background vector field:

\[
\rho^{(t)} = \frac{k^2}{32\pi G} \frac{1 + 4(\sigma - \tau)A^2}{1 + 2(\sigma - \tau)A^2} \left( |C_{\oplus}|^2 + |C_{\otimes}|^2 \right)
\]

where \( C_{\oplus}, C_{\otimes} \) are the amplitudes of the corresponding graviton polarizations.

Then, in order not to have modes with negative energy density we need either \( 2(\sigma - \tau)A^2 < -1 \) or \( 4(\sigma - \tau)A^2 > -1 \). These conditions combined with the classical stability condition lead to the constraint \( 4(\sigma - \tau)A^2 > -1 \). For models I and II this condition reads \( 2\varepsilon A^2 < 1 \) and \( 12\varepsilon A^2 < 1 \) respectively. On the other hand, Model III has \( \sigma = \tau \) so gravitational waves are unaffected. Finally, for the gauge invariant models we obtain \( \varepsilon A^2 < \frac{1}{4m} \).
We have studied the viability conditions for unconstrained vector-tensor theories of gravity. In Table I we summarize the different conditions obtained for the different models analyzed. If we concentrate only on classical stability and local gravity constraints, we find that there are different types of vector-tensor theories which can be made viable for certain values of the coefficients. Thus for example: Model I with $\epsilon < 0$ or Model II with $\tau < 0$ for arbitrary values of $A$.

However if we also impose that the models are free from ghosts, we have found that only theories of the Maxwell type or Maxwell plus a gauge-fixing term can be made compatible with all the consistency conditions for arbitrary $A$. Notice that on small (sub-Hubble) scales, such vector-tensor theories are indistinguishable from General Relativity, however it has been shown (see [6, 7] for different examples) that in a cosmological scenario, the presence of the (dynamical) background vector field could have important consequences. Indeed in [7] it has been shown that the electromagnetic theory with a gauge fixing term can explain in a natural way the existence and the smallness of the cosmological constant, solving in this way the problem of establishing what is the fundamental nature of dark energy. Present and forthcoming astrophysical and cosmological observations could thus help us determining what is the true theory for the gravitational interaction.

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§ Notice that we have only considered models with exactly the same PPN parameters as GR, if we relax this condition and we only demand compatibility with current experimental limits on PPN parameters then the range of viable models could be much larger.
Appendix: PPN parameters for vector-tensor theories of gravity.

The PPN parameters for the vector-tensor theory \( \Pi \) are given by \[2\]:

\[
\begin{align*}
\gamma &= \frac{1 + 4\omega A^2 (1 + 2\frac{2\omega + \sigma - \tau}{2\omega + \tau})}{1 - 4\omega A^2 (1 - \frac{8\omega}{2\omega + \tau})} \\
\beta &= \frac{1}{4}(3 + \gamma) + \frac{1}{2}\Theta \left[ 1 + \frac{\gamma(\gamma - 2)}{G} \right] \\
\alpha_1 &= 4(1 - \gamma) \left[ 1 + (2\varepsilon + \tau)\Delta \right] + 16\omega A^2 \Delta a \\
\alpha_2 &= 3(1 - \gamma) \left[ 1 + \frac{2}{3}(2\varepsilon + \tau)\Delta \right] + 8\omega A^2 \Delta a - 2 \frac{bA^2}{G} \\
\alpha_3 &= \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0
\end{align*}
\]

with:

\[
\begin{align*}
\Theta &= \frac{(1 - 4\omega A^2)(2\varepsilon + \sigma - 2\omega)}{(1 - 4\omega A^2)(2\varepsilon + \tau) + 32\omega^2 A^2} \\
\Delta &= \frac{1}{2A^2(\sigma - \tau)^2 - (2\varepsilon + \tau) [1 - 4A^2(\omega + \sigma - \tau)]} \\
a &= (2\varepsilon + \tau)(1 - 3\gamma) + 2(\sigma - \tau)(1 - 2\gamma) \\
b &= \begin{cases} 
(2\omega + \sigma - \tau) \left[ (2\gamma - 1)(\gamma + 1) + \Theta(\gamma - 2) \right] & \tau \neq 0 \\
-2(\gamma - 1)^2(2\omega + \sigma) \left( 1 - \frac{2\omega + \sigma}{\tau} \right) & \tau = 0 
\end{cases}
\end{align*}
\]

Moreover, it is possible to define an effective Newton’s constant given by:

\[
G_{\text{eff}} \equiv G \left[ \frac{1}{2}(\gamma + 1) + 6\omega A^2(\gamma - 1) \\
- 2A^2(\sigma - \tau)(1 + \Theta) \right]^{-1}.
\]

The above expressions are obtained assuming \( G_{\text{eff}} = 1 \).