Multiple integral representation for the trigonometric SOS model with domain wall boundaries

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Abstract

Using the dynamical Yang-Baxter algebra we derive a functional equation for the partition function of the trigonometric SOS model with domain wall boundary conditions. The solution of the equation is given in terms of a multiple contour integral.

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1 Introduction

The study of two-dimensional lattice models in Statistical Mechanics advanced dramatically with the advent of Baxter’s concept of commuting transfer matrices [1]. This method introduced the concept of integrability in Statistical Mechanics and paved the way for the development of a variety of exact methods exploring the aforementioned commutativity. As examples of those methods we have Baxter’s $T − Q$ relation [1], the algebraic Bethe ansatz [2], the inversion trick [3], etc. Also a variety of models can be tackled by those same techniques such as vertex models, solid-on-solid models and hard square models.

Nevertheless, the implementation of those methods depends drastically on the boundary conditions chosen and the case of domain wall boundaries deserves special attention. This kind of boundary condition for the six vertex model was introduced by Korepin in [4] who also obtained a recurrence relation determining the model partition function. This recurrence relation was later on solved by Izergin in terms of a determinant [5]. Moreover, the exact solution of this model raised the issue of the sensitivity of the six vertex model bulk properties with respect to the boundary conditions in the thermodynamical limit [6].
A natural question that emerges in this scenario is how this sensitivity with respect to boundary conditions extends to the eight vertex model. In that case, although the partition function has been evaluated in \[7\–9\], the lack of manageable expressions have eluded the analysis of the thermodynamical limit except for a particular value of the anisotropy parameter \[10\].

Motivated by this scenario and keeping in mind the relation between Baxter’s eight vertex model and solid-on-solid models, also referred to as SOS models, here we demonstrate that the algebraic-functional method introduced in \[11\–12\] can also be used in that case. Using that method we derive a multiple integral formula for the partition function of the trigonometric SOS model with domain wall boundaries.

This paper is organised as follows. In the Sec. \[2\] we briefly describe SOS models in Statistical Mechanics with emphasis on the case of domain wall boundary conditions. In the Sec. \[3\] we present the dynamical Yang-Baxter algebra and demonstrate how it can be explored in order to obtain a functional equation for the partition function of the SOS model with domain wall boundaries. This functional equation is analysed in Sec. \[4\] and the solution of the equation is given in Sec. \[5\] as a multiple contour integral. Technical details are presented in App. \[A\] through App. \[F\].

2 Solid-on-solid (SOS) models

We consider a two-dimesional lattice formed by retangular cells juxtaposed as in Fig. 1. For a lattice with \(N + 1\) rows and \(M + 1\) columns we have \(M \times N\) retangular cells and we associate the Boltzmann weight \(w_{ij}(l_{i,j}^{l_{i+1,j}} l_{i+1,j}^{l_{i+1,j+1}} l_{i+1,j+1}^{l_{i,j+1}})\) to the cell enclosed by the cartesian coordinates \((i, j)\), \((i, j + 1)\), \((i + 1, j)\) and \((i + 1, j + 1)\). Each retangular cell is simply referred as \textit{face} and its configuration is characterised by the set of variables...
\[
\begin{pmatrix}
    w_{ij} & \left( \begin{array}{cc}
    l_{i+1,j} & l_{i+1,j+1} \\
    l_{i,j} & l_{i,j+1}
    \end{array} \right)
\end{pmatrix}
\]
The operator $T$ is commonly denominated (face) transfer matrix and it plays an important role in establishing the integrability of two-dimensional lattice models \[13\]. We proceed by defining a second transfer matrix $T'$ similarly to (2.4) but with Boltzmann weights $w'$ instead of $w$. Next we look for conditions on $w$ and $w'$ such that the transfer matrices $T$ and $T'$ form a commutative family, i.e. $[T, T'] = 0$. This requirement leads to the following relation

$$
\sum_{l_0} w_u \left( \frac{1}{l_2} \frac{1}{l_0} \right) w_{u+v} \left( \frac{1}{l_1} \frac{1}{l_2} \frac{1}{l_0} \right) w_u \left( \frac{1}{l_6} \frac{1}{l_5} \right) = \sum_{l_0} w_u \left( \frac{1}{l_2} \frac{1}{l_3} \right) w_{u+v} \left( \frac{1}{l_1} \frac{1}{l_2} \frac{1}{l_3} \right) w_u \left( \frac{1}{l_0} \frac{1}{l_5} \right) \, ,
$$

(2.5)

where $u$ and $v$ are complex variables parameterising the manifold where the transfer matrices form a commutative family. For a detailed derivation of (2.5) we refer to \[14\]. The Eq. (2.5) is usually referred to as Yang-Baxter relation \[15\], or simply Hexagon identity, and the local equivalence transformation described by (2.5) is depicted in Fig. 4.

Moreover, if we also consider periodic boundary conditions in the vertical direction, i.e. $l_{N+1,j} = l_{1,j}$, the partition function (2.1) becomes simply

$$
Z = \text{Tr} \left( T^N \right) \, ,
$$

(2.6)

and its evaluation is translated into an eigenvalue problem \[16\],[17\]. Throughout this paper we shall consider a different class of boundary conditions where (2.6) does not apply, although the bulk model is still governed by statistical weights satisfying (2.5).

**The trigonometric SOS model.** The variables $\{l_{i,j}, l_{i,j+1}, l_{i+1,j}, l_{i+1,j+1}\}$ depicted in Fig. 2 are also called height functions and they characterise the configuration of the associated face. The degree of freedom $l_{i,j}$ is also referred to as the colour of the face at the position $(i,j)$. Furthermore, we can also impose restrictions on $l_{i,j}$ such that only certain configurations of colours are allowed in the statistical sum (2.1).

In what follows we will be dealing with a lattice formed by coloured faces where each adjacent face can not have the same colour. As it was remarked by Baxter in \[18\], this system can be thought as a system of particles interacting through an infinitely repulsive force between nearest neighbors of the same type. For the trigonometric SOS model we will have $l_{i,j} = \theta + \bar{l}_{i,j} \gamma$, where $\bar{l}_{i,j}$ is an integer variable while $\theta$ and $\gamma$ are
complex numbers. In the Fig. 5 the height function $l_{i,j}$ characterising the colour of the face is projected into the center of the $(i,j)$ face. Interestingly enough, it was remarked by Lenard\footnote{See Note added in proof of \cite{19}.} an equivalence between this colouring of faces and the configurations of a six vertex model. This equivalence is depicted in Fig. 5 where basically one removes the outer edges of the four-faces set and places arrows on the internal edges according to a certain rule. This rule is as follows: each face of a four-faces set is visited in the anticlockwise direction. If the colour changes by $+\gamma$ when intersecting an edge, this edge receives an arrow pointing inwards; and if the colour changes by $-\gamma$ we place an arrow pointing outwards on that edge. For a particular class of statistical weights this model is also called \textit{Three-colouring model} \cite{18,20}. In that case the height function $l_{i,j}$ is conveniently labeled by an element of the ring $\mathbb{Z}_3$. More precisely, the $\mathbb{Z}_3$ structure labeling the colours of the faces is unveiled by considering the remainder after division of $\hat{l}_{i,j}$ by 3.

\textbf{The dynamical Yang-Baxter equation.} In \cite{21,23} it was demonstrated that the Boltzmann weights of a face model satisfying (2.5) are encoded in the solutions of the dynamical Yang-Baxter equation. This dynamical version of the quantum Yang-Baxter equation was proposed by Felder in \cite{21} as the quantised form of a modified classical Yang-Baxter equation. In that case this modified classical Yang-Baxter equation arises as the compatibility condition for the Knizhnik-Zamolodchikov-Bernard equations \cite{24,25}. Previous to that, the dynamical Yang-Baxter equation had appeared in connection to the Liouville string field theory in \cite{26}.

Now let $V = v_+ \oplus v_-$ be a two-dimensional complex vector space and consider the operator $R(\lambda, \theta) \in \text{End}(V \otimes V)$ with $\lambda, \theta \in \mathbb{C}$. The dynamical Yang-Baxter equation for the trigonometric SOS model then reads

$$R_{12}(\lambda_1 - \lambda_2, \theta - \gamma \hat{h}_3)R_{13}(\lambda_1 - \lambda_3, \theta)R_{23}(\lambda_2 - \lambda_3, \theta - \gamma \hat{h}_1) = R_{23}(\lambda_2 - \lambda_3, \theta)R_{13}(\lambda_1 - \lambda_3, \theta - \gamma \hat{h}_2)R_{12}(\lambda_1 - \lambda_2, \theta)$$

where $\hat{h} = \text{diag}(1,-1)$. The Eq. (2.7) is defined in $\text{End}(V_1 \otimes V_2 \otimes V_3)$ and the action of $R_{12}(\lambda, \theta - \gamma \hat{h}_3)$ on $v_1 \otimes v_2 \otimes v_3$ is understood as

$$[R(\lambda, \theta - \gamma \hat{h}) v_1 \otimes v_2] \otimes v_3,$$

keeping in mind that $\hat{h}_3 v_3 = \h v_3$. In other words, $\h$ is simply the eigenvalue of $\hat{h}$ on the
corresponding subspace. The explicit trigonometric solution of (2.7) is given by

$$R(\lambda, \theta) = \begin{pmatrix}
a_+(\lambda, \theta) & 0 & 0 & 0 \\
0 & b_+(\lambda, \theta) & c_+(\lambda, \theta) & 0 \\
0 & 0 & c_-(\lambda, \theta) & b_-(\lambda, \theta) \\
0 & 0 & 0 & a_-(\lambda, \theta)
\end{pmatrix}$$  \hspace{1cm} (2.9)

with non-null entries

$$a_\pm(\lambda, \theta) = \sinh(\lambda + \gamma)$$
$$b_\pm(\lambda, \theta) = \sinh(\lambda)\frac{\sinh(\theta \mp \gamma)}{\sinh(\theta)}$$
$$c_\pm(\lambda, \theta) = \sinh(\gamma)\frac{\sinh(\theta \mp \lambda)}{\sinh(\theta)}.$$  \hspace{1cm} (2.10)

The solution described by (2.9) and (2.10) consists of a particular limit of the elliptic solution found in [21, 22]. Also it is important to remark here that such solutions are in correspondence with Baxter’s eight-vertex model after a vertex-face transformation [27, 28]. Moreover, the dynamical $R$-matrix (2.9) satisfies the ice rule

$$\left[ R_{ab}(\lambda, \theta), \hat{h}_a + \hat{h}_b \right] = 0$$  \hspace{1cm} (2.11)

which plays an important role in establishing an algebra associated to dynamical $R$-matrices.

Now we turn our attention to the relation between solutions of the dynamical Yang-Baxter equation (2.7) and the statistical weights of a SOS model satisfying (2.5). At this stage the observations of Lenard [19], the vertex-face transformation introduced by Baxter [28] and Felder’s dynamical Yang-Baxter equation [22] converge to the same point. Following [22] we thus have

$$w_\lambda(\theta - \gamma \theta + \gamma) = a_+(\lambda, \theta)$$
$$w_\lambda(\theta - \gamma \theta + \gamma) = b_+(\lambda, \theta)$$
$$w_\lambda(\theta - \gamma \theta + \gamma) = c_+(\lambda, \theta)$$
$$w_\lambda(\theta + \gamma \theta - \gamma) = a_-(\lambda, \theta)$$
$$w_\lambda(\theta + \gamma \theta - \gamma) = b_-(\lambda, \theta)$$
$$w_\lambda(\theta + \gamma \theta - \gamma) = c_-(\lambda, \theta).$$  \hspace{1cm} (2.12)

This association is also depicted in Fig. 5.

The dynamical monodromy matrix. Still following [22, 23] we define an inhomogeneous monodromy matrix $T_a(\lambda, \theta)$ formed by the ordered product of dynamical $R$-matrices. More precisely, this dynamical monodromy matrix reads

$$T_a(\lambda, \theta) = \prod_{i=1}^L R_{ai}(\lambda - \mu_i, \theta_i)$$  \hspace{1cm} (2.13)
with \( \theta_i = \theta - \gamma \sum_{k=i+1}^{L} \hat{h}_k \), and it should not be confused with the face monodromy matrix defined in (2.2).

The dynamical monodromy matrix (2.13) is an operator living in the tensor product space \( \mathbb{V}_a \otimes \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \). The space \( \mathbb{V}_a \) will be refereed to as auxiliar space while the tensor product \( \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \) will be called quantum space. In this way \( T_a(\lambda, \theta) \) can be regarded as a matrix on the auxiliar space whose entries are matrices living in the quantum space. Here we shall restrict ourselves to the monodromy matrix built out of the \( R \)-matrix (2.9). Therefore, it can be conveniently denoted as

\[
T_a(\lambda, \theta) = \begin{pmatrix} A(\lambda, \theta) & B(\lambda, \theta) \\ C(\lambda, \theta) & D(\lambda, \theta) \end{pmatrix},
\]

(2.14)

Due to the dynamical Yang-Baxter equation (2.7) and the ice rule (2.11), one can demonstrate that the monodromy matrix (2.13) satisfies the algebraic relation

\[
R_{ab}(\lambda_1 - \lambda_2, \theta - \gamma \hat{h}) T_a(\lambda_1, \theta) T_b(\lambda_2, \theta - \gamma \hat{h}) = T_b(\lambda_2, \theta) T_a(\lambda_1, \theta - \gamma \hat{h}) R_{ab}(\lambda_1 - \lambda_2, \theta)
\]

(2.15)

where \( H = \sum_{k=1}^{L} \hat{h}_k \). With the help of the definition of \( \hat{h} \), we can identify \( H \) as the Cartan element of the \( \mathfrak{su}(2) \) algebra on the tensor product space \( \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \). Furthermore, in the limit \( \theta \to \infty \) we can immediately see that (2.9) becomes the standard \( R \)-matrix invariant under the quantum affine algebra \( U_q[\hat{\mathfrak{su}}(2)] \). This observation extends to the monodromy matrix (2.13) and the algebra (2.15), which become respectively the standard trigonometric six vertex model monodromy matrix and Yang-Baxter algebra [29,2].

**Domain wall boundary conditions.** We shall now consider the trigonometric SOS model on the lattice described in Fig. 1 with \( N = M = L + 1 \) and special boundary conditions. As for the boundary conditions, we set \( l_{1,j} = l_{j,1} = \theta + (L + 1 - j)\gamma \) and \( l_{L+1,j} = l_{j,L+1} = \theta + (j - 1)\gamma \). This special boundary conditions is illustrated in Fig. 6 together with the corresponding structure of vertices. In the vertices language we can immediately recognize this special boundary conditions as the case of domain wall boundaries introduced by Korepin in [4]. This observation if of fundamental importance allowing us to express the model partition function in terms of the entries of the monodromy matrix (2.14) analogously to the case of the standard six vertex model [4]. The diagrammatical interpretation of (2.13) is given in [22][23] and for a discussion on the construction of the partition function (2.1) in terms of the components (2.14) we refer to [30]. For the trigonometric SOS model considered here we only have to keep in mind that the entries of the dynamical monodromy matrix (2.14) also depends on the colour variable \( \theta \) governed by the height functions \( l_{j,1} \). In this way the partition function (2.1) for the trigonometric SOS model with domain wall boundaries can be written as

\[
Z_\theta = \langle \bar{0} \prod_{j=1}^{L} B(\lambda_j, \theta + j\gamma) | 0 \rangle
\]

(2.16)
\[ |0\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\bar{0}\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.17} \]

### 3 Dynamical Yang-Baxter algebra and functional relations

The dynamical Yang-Baxter algebra (2.15) encodes commutation rules for the entries of the dynamical monodromy matrix (2.13). In contrast to the standard Yang-Baxter algebra, the relation (2.15) not only contains the generators \( A(\lambda, \theta), B(\lambda, \theta), C(\lambda, \theta) \) and \( D(\lambda, \theta) \), but also the \( \mathfrak{su}(2) \) Cartan generator \( H \). This indicates that the algebra defined by (2.15) needs to be complemented.

**The \( \mathfrak{su}(2) \) Cartan generator.** The definition of \( H \) together with (2.13) and (2.14) allow us to directly compute the commutators between the \( \mathfrak{su}(2) \) Cartan generator \( H \) and the entries of the monodromy matrix (2.14). In the limit \( \theta \to \infty \) this analysis has been performed in [2, 4] and here we find that there are no significant modifications for arbitrary \( \theta \). Nevertheless, this analysis can also be found in App. A and we have

\[
\begin{align*}
[A(\lambda, \theta), H] &= 0 \\
[B(\lambda, \theta), H] &= 2B(\lambda, \theta) \\
[C(\lambda, \theta), H] &= -2C(\lambda, \theta) \\
[D(\lambda, \theta), H] &= 0. 
\end{align*} \tag{3.1}
\]

From the \( R \)-matrix (2.9) we can see that the generator \( H \) will appear in (2.15) only as \( e^{\pm \gamma H} \). Thus for convenience we define the operator \( K = q^H \) with \( q = e^\gamma \), in such a way that the commutation rules (3.1) imply the following relations:

\[
\begin{align*}
B(\lambda, \theta)K &= q^2 KB(\lambda, \theta) \\
C(\lambda, \theta)K &= q^{-2} KC(\lambda, \theta) \\
[A(\lambda, \theta), K] &= 0 \\
[D(\lambda, \theta), K] &= 0. 
\end{align*} \tag{3.2}
\]

The commutation rules (3.2) in addition to (2.15) form an extended Yang-Baxter algebra with generators \( A(\lambda, \theta), B(\lambda, \theta), C(\lambda, \theta), D(\lambda, \theta) \) and \( K^\pm \). For our purposes here
we only need to extract a few commutation relations from (2.15). Namely,
\[
B(\lambda_1, \theta)B(\lambda_2, \theta + \gamma) = B(\lambda_2, \theta)B(\lambda_1, \theta + \gamma)
\]
\[
A(\lambda_1, \theta + \gamma)B(\lambda_2, \theta) = \frac{s(\lambda_2 - \lambda_1 + \gamma)}{s(\lambda_2 - \lambda_1)} \frac{s(\theta + \gamma)}{s(\theta + 2\gamma)} B(\lambda_2, \theta + \gamma)A(\lambda_1, \theta + 2\gamma)
\]
\[
- \frac{s(\theta + \gamma - \lambda_2 + \lambda_1)}{s(\lambda_2 - \lambda_1)} \frac{s(\gamma)}{s(\theta + 2\gamma)} B(\lambda_1, \theta + \gamma)A(\lambda_2, \theta + 2\gamma)
\]
\[
D(\lambda_1, \theta - \gamma)B(\lambda_2, \theta) =
\]
\[
s(\lambda_1 - \lambda_2 + \gamma)B(\lambda_2, \theta - \gamma)D(\lambda_1, \theta)[tqK^{-1} - t^{-1}q^{-1}K][tq^2K^{-1} - t^{-1}q^{-2}K]^{-1}
\]
\[
- \frac{s(\gamma)}{s(\lambda_1 - \lambda_2)} B(\lambda_1, \theta - \gamma)D(\lambda_2, \theta)[tq\bar{x}_1\bar{x}_2^{-1}K^{-1} - t^{-1}q^{-1}\bar{x}_1\bar{x}_2K][tq^2K^{-1} - t^{-1}q^{-2}K]^{-1}
\]
\[
C(\lambda_1, \theta + \gamma)B(\lambda_2, \theta) =
\]
\[
\frac{s(\theta)}{s(\theta + \gamma)} B(\lambda_2, \theta + \gamma)C(\lambda_1, \theta + 2\gamma)[tqK^{-1} - t^{-1}q^{-1}K][tq^2K^{-1} - t^{-1}q^{-2}K]^{-1}
\]
\[
+ \frac{s(\gamma)}{s(\lambda_1 - \lambda_2)} \frac{s(\theta + \gamma - \lambda_1 + \lambda_2)}{s(\lambda_1 - \lambda_2)} A(\lambda_2, \theta + \gamma)D(\lambda_1, \theta)[tqK^{-1} - t^{-1}q^{-1}K][tq^2K^{-1} - t^{-1}q^{-2}K]^{-1}
\]
\[
- \frac{s(\gamma)}{s(\lambda_1 - \lambda_2)} A(\lambda_1, \theta + \gamma)D(\lambda_2, \theta)[tq\bar{x}_1\bar{x}_2^{-1}K^{-1} - t^{-1}q^{-1}\bar{x}_1\bar{x}_2K][tq^2K^{-1} - t^{-1}q^{-2}K]^{-1}.
\]

(3.3)

In the above commutation rules we have employed the notation \(s(\lambda) = \sinh(\lambda)\), \(t = e^{\theta}\) and \(\bar{x}_i = e^{\lambda_i}\). The algebra formed by the relations (3.2) and (3.3) will be one of the main ingredients in the derivation of a functional relation determining the partition function (2.16).

In order to proceed we will also need to consider the action of the generators \(A(\lambda, \theta)\), \(B(\lambda, \theta)\), \(C(\lambda, \theta)\), \(D(\lambda, \theta)\) and \(K\) on the states \(|0\rangle\) and \(|\bar{0}\rangle\) defined in (2.17). Those states are the \(\mathfrak{su}(2)\) highest and lowest weight states respectively, and from (2.9), (2.13) and (2.14) we readily obtain the relations
\[
K^\pm |0\rangle = q^{\pm L} |0\rangle
\]
\[
A(\lambda, \theta) |0\rangle = \prod_{i=1}^{L} s(\lambda - \mu_i + \gamma) |0\rangle
\]
\[
D(\lambda, \theta) |0\rangle = \frac{s(\theta + \gamma)}{s(\theta - (L - 1)\gamma)} \prod_{i=1}^{L} s(\lambda - \mu_i) |0\rangle.
\]

(3.4)

Moreover, we also obtain the properties
\[
B(\lambda, \theta) |0\rangle = \dagger \quad C(\lambda, \theta) |0\rangle = 0
\]
\[
B(\lambda, \theta) |\bar{0}\rangle = 0 \quad C(\lambda, \theta) |\bar{0}\rangle = \dagger,
\]

(3.5)
where the symbols † and ‡ stand for non-null values. The relations (3.5) together with [3.1] support considering \( B(\lambda, \theta) \) and \( C(\lambda, \theta) \) as creation and annihilation operators respectively with respect to the pseudo-vacuum state \( |0\rangle \).

Altogether the relations (3.2)-(3.5) allow us to derive the following formula,

\[
C(\lambda_0, \theta + \gamma) = \prod_{i=1}^{n} B(\lambda_i, \theta + (i-1)\gamma) |0\rangle = \sum_{i=1}^{n} M_i \prod_{j=1}^{n-1} B(\lambda_j^{(i)}, \theta + j\gamma) |0\rangle \\
+ \sum_{j=2}^{n} \sum_{i=1}^{j-1} N_{ji} B(\lambda_0, \theta + \gamma) \prod_{k=1}^{n-2} B(\lambda_k^{(j)}, \theta + (k+1)\gamma) |0\rangle,
\]

(3.6)

where

\[
N_{ji} = s(\gamma) \frac{s(\theta + \gamma) s(\lambda_0 - \lambda_i + \theta + (2n-1-L)\gamma) s(\theta + (n-L)\gamma)}{s(\lambda_j - \lambda_0) s(\theta + n\gamma) s(\theta + (2n-1-L)\gamma) s(\theta + (n-L)\gamma)} \\
\times \prod_{l=1}^{L} s(\lambda_0 - \mu_l + \gamma) s(\lambda_0 - \mu_l) \prod_{k=1}^{n} \frac{s(\lambda_k - \lambda_j + \gamma) s(\lambda_k - \lambda_0 + \gamma)}{s(\lambda_k - \lambda_j) s(\lambda_k - \lambda_0)} \\
+ s(\gamma) \frac{s(\lambda_0 - \lambda_i + \theta + \gamma) s(\theta + (n-L)\gamma)}{s(\lambda_0 - \lambda_i) s(\theta + n\gamma) s(\theta + (2n-1-L)\gamma) s(\theta + (n-L)\gamma)} \\
\times \prod_{l=1}^{L} s(\lambda_0 - \mu_l + \gamma) s(\lambda_0 - \mu_l) \prod_{k=1}^{n} \frac{s(\lambda_k - \lambda_i + \gamma) s(\lambda_k - \lambda_j + \gamma)}{s(\lambda_k - \lambda_i) s(\lambda_k - \lambda_j)} \prod_{l=1}^{L} s(\lambda_0 - \mu_l + \gamma) s(\lambda_0 - \mu_l)
\]

(3.7)

\[
M_i = s(\gamma) \frac{s(\theta + \gamma) s(\lambda_0 - \lambda_i + \theta + (2n-1-L)\gamma) s(\theta + (n-L)\gamma)}{s(\lambda_j - \lambda_0) s(\theta + n\gamma) s(\theta + (2n-1-L)\gamma) s(\theta + (n-L)\gamma)} \\
\times \prod_{l=1}^{L} s(\lambda_0 - \mu_l + \gamma) s(\lambda_0 - \mu_l) \prod_{k=1}^{n} \frac{s(\lambda_k - \lambda_j + \gamma) s(\lambda_k - \lambda_0 + \gamma)}{s(\lambda_k - \lambda_j) s(\lambda_k - \lambda_0)}
\]

(3.8)
considering (3.6) and the definition (2.16) we immediately obtain the relation
under the light of the extended Yang-Baxter algebra formed by (2.15) and (3.2). Thus
the quantity
The relation (3.6) is valid for the product of an arbitrary
Functional Equation. The relation (3.6) is valid for the product of an arbitrary
number of operators \( B(\lambda_i, \theta) \), and following the approach devised in [11], we examine
the quantity
under the light of the extended Yang-Baxter algebra formed by (2.15) and (3.2). Thus
considering (3.6) and the definition (2.16) we immediately obtain the relation
\[
\langle \bar{0} | C(\lambda_0, \theta + \gamma) \prod_{i=1}^{L+1} B(\lambda_i, \theta + (i-1)\gamma) | 0 \rangle = \sum_{i} M_i Z_\theta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1}) \\
+ \sum_{j=2}^{L+1} \sum_{i=1}^{j-1} N_{ji} Z_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{L+1}) ,
\]
(3.11)
where the coefficients \( M_i \) and \( N_{ji} \) are the ones given by (3.8) and (3.9) with \( n = L + 1 \).
On the other hand, the highest weight representation theory of the \( \mathfrak{su}(2) \) algebra tells us
that the LHS of (3.11) vanishes. This property has been already discussed in [11] but for
completeness we also include it in the App. A. In this way we are left with the following
functional equation for the partition function (2.16),
\[
\sum_{i=1}^{L+1} M_i Z_\theta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1}) \\
+ \sum_{j=2}^{L+1} \sum_{i=1}^{j-1} N_{ji} Z_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{L+1}) = 0
\]
(3.12)
Some comments are in order at this stage. The partition function \( Z_\theta \) depends on two
sets of variables, \( \{ \lambda_i \} \) and \( \{ \mu_i \} \), as well as parameters \( \gamma \) and \( \theta \). Nevertheless, within this
approach we can see that the set of variables \( \{ \mu_i \} \) can also be regarded as parameters.

As expected, the expressions (3.6)-(3.9) recover the ones presented in [4,11] in the limit
\( \theta \to \infty \). At this stage we have gathered most of the ingredients required to obtain a
functional equation determining the partition function (2.16).

\[
\langle \bar{0} | C(\lambda_0, \theta + \gamma) \prod_{i=1}^{L+1} B(\lambda_i, \theta + (i-1)\gamma) | 0 \rangle = \sum_{i} M_i Z_\theta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1}) \\
+ \sum_{j=2}^{L+1} \sum_{i=1}^{j-1} N_{ji} Z_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{L+1}) ,
\]
(3.11)
The Eq. (3.12) is linear and homogeneous in $Z_\theta$ and this observation will have important consequences for the characterisation of the desired solution. The homogeneity of (3.12) tells us that if $Z_\theta$ is a solution, so is $\alpha Z_\theta$ where $\alpha$ is a constant. In fact $\alpha$ only needs to be independent of the variables $\{\lambda_i\}$. Therefore, the Eq. (3.12) can determine the partition function at most up to a constant and the full determination of the partition function will require that we are at least able to compute $Z_\theta$ for some particular value of the variables $\lambda_i$. Moreover, the linearity of (3.12) tells us that if $Z_\theta^{(1)}$ and $Z_\theta^{(2)}$ are two solutions of (3.12), the linear combination $Z_\theta^{(1)} + Z_\theta^{(2)}$ is also a solution. This fact suggests that classifying the classes of unique solutions of (3.12) is an important step in this framework. The Eq. (3.12) has the same structure of the functional equation derived in [11], however the coefficients $M_i$ and $N_{ji}$ are deformed by the dynamical parameter $\theta$. In what follows these issues will be discussed and the desired solution of (3.12) will be presented.

4 The partition function

In this section we shall consider extra properties expected for the partition function (2.16) which will enable us to select the appropriate solution of (3.12). These properties include the multivariate polynomial structure of the partition function $Z_\theta$, as well as its asymptotic behaviour. In order to avoid an overcrowded section we shall discuss the mentioned properties in the App. B and here we present only the required results.

Polynomial structure. The partition function $Z_\theta$ is of the form,

$$Z_\theta(\lambda_1, \ldots, \lambda_L) = \frac{\bar{Z}_\theta(x_1, \ldots, x_L)}{\prod_{i=1}^{L} x_i^{\frac{\mu_i}{2}}} ,$$

where $\bar{Z}_\theta(x_1, \ldots, x_L)$ is a polynomial of order $L$ in each variable $x_i = e^{2\lambda_i}$ separately. The same polynomial structure holds if we consider the variables $\{\mu_i\}$ instead of $\{\lambda_i\}$, although this property will not be required.

Asymptotic behaviour. In the limit $x_i \to \infty$, the function $\bar{Z}_\theta$ possesses the asymptotic behaviour

$$\bar{Z} \sim \frac{(q - q^{-1})^L}{2^{L^2}} \frac{[L]_q^2!}{\prod_{n=1}^{L} (1 - q^{2n+1})} (x_1 \ldots x_L)^L ,$$

where $[n]_q! = 1(1 + q)(1 + q + q^2) \ldots (1 + q + \cdots + q^n)$ denotes the $q$-factorial function and $u_i = e^{2\mu_i}$.

For the moment we leave the properties (4.1) and (4.2) at rest and proceed with a more careful examination of the functional equation (3.12). Firstly we notice that besides the set of variables $\{\lambda_1, \lambda_2, \ldots, \lambda_L\}$ required to characterise its solution, the Eq. (3.12)
also depends on variables $\lambda_0$ and $\lambda_{L+1}$. Thus the variables $\lambda_0$ and $\lambda_{L+1}$ can be fixed in order to fulfill our needs. In particular, those variables can be chosen in such a way that solving (3.12) under the conditions (4.1) and (4.2) becomes systematic and simple. For illustrative purposes, let us see how this approach would work in the case $L = 2$. In that case our functional equation reads

\[
M_1 Z_\theta(\lambda_2, \lambda_3) + M_2 Z_\theta(\lambda_1, \lambda_3) + M_3 Z_\theta(\lambda_1, \lambda_2) + N_{21} Z_\theta(\lambda_0, \lambda_3) + N_{31} Z_\theta(\lambda_0, \lambda_2) + N_{32} Z_\theta(\lambda_0, \lambda_1) = 0
\]  

(4.3)

and we set $\lambda_0 = \mu_1$ and $\lambda_3 = \mu_1 - \gamma$. By doing so we find

\[
M_1 \big|_{\lambda_0=\mu_1, \lambda_3=\mu_1-\gamma} = M_2 \big|_{\lambda_0=\mu_1, \lambda_3=\mu_1-\gamma} = 0
\]  

(4.4)

and we also define

\[
m_2 = M_3 \big|_{\lambda_0=\mu_1, \lambda_3=\mu_1-\gamma} = -\frac{s(\theta + \gamma)}{s(\theta + 3\gamma)} s(\gamma)^2 s(\mu_1 - \mu_2 + \gamma) s(\mu_2 - \mu_1 + \gamma)
\]

\[
m_1 = N_{31} \big|_{\lambda_0=\mu_1, \lambda_3=\mu_1-\gamma} = \frac{s(\theta + \gamma - \lambda_1 + \mu_1)}{s(\theta + 3\gamma)} s(\gamma)^2 s(\mu_2 - \mu_1 + \gamma) s(\lambda_1 - \mu_2 + \gamma)
\]

\[
\times \frac{s(\lambda_2 - \mu_1)}{s(\lambda_2 - \lambda_1 + \gamma)} \frac{s(\lambda_2 - \lambda_1)}{s(\lambda_2 - \lambda_1 - \gamma)}
\]

\[
m_2 = N_{32} \big|_{\lambda_0=\mu_1, \lambda_3=\mu_1-\gamma} = \frac{s(\theta + \gamma - \lambda_2 + \mu_1)}{s(\theta + 3\gamma)} s(\gamma)^2 s(\mu_2 - \mu_1 + \gamma) s(\lambda_2 - \mu_2 + \gamma)
\]

\[
\times \frac{s(\lambda_1 - \mu_1)}{s(\lambda_1 - \mu_1 + \gamma)} \frac{s(\lambda_1 - \lambda_2 + \gamma)}{s(\lambda_1 - \lambda_2)}
\]  

(4.5)

With the above specialisation of the variables $\lambda_0$ and $\lambda_3$, the Eq. (4.3) reduces to

\[
Z_\theta(\lambda_1, \lambda_2) = \frac{s(\theta + \gamma - \lambda_2 + \mu_1)}{s(\theta + \gamma)} \frac{s(\lambda_2 - \mu_2 + \gamma)}{s(\lambda_2 - \mu_2 + \gamma)} \frac{s(\lambda_1 - \mu_1)}{s(\lambda_1 - \mu_1 + \gamma)} \frac{s(\lambda_1 - \lambda_2)}{s(\lambda_1 - \lambda_2 + \gamma)} Z_\theta(\mu_1, \lambda_1)
\]

\[
+ \frac{s(\theta + \gamma - \lambda_1 + \mu_1)}{s(\theta + \gamma)} \frac{s(\lambda_1 - \mu_1 + \gamma)}{s(\lambda_1 - \mu_1 + \gamma)} \frac{s(\lambda_2 - \mu_1)}{s(\lambda_2 - \mu_1 + \gamma)} \frac{s(\lambda_2 - \lambda_1 + \gamma)}{s(\lambda_2 - \lambda_1)} Z_\theta(\mu_1, \lambda_2)
\]

\[
- \frac{N_{21}}{M_3} Z_\theta(\mu_1, \mu_1 - \gamma)
\]

(4.6)

In addition to that the Eq. (4.6) reduces to the following identity when $\lambda_2 = \mu_1$,

\[
\left[ \frac{N_{21}}{M_3} \right]_{\lambda_3=\mu_1-\gamma} Z_\theta(\mu_1, \mu_1 - \gamma) = Z_\theta(\mu_1, \lambda_1) - Z_\theta(\lambda_1, \mu_1) .
\]  

(4.7)

As demonstrated in the App. [D] the functional equation (3.12) admits only symmetric solutions, i.e. $Z_\theta(\lambda_1, \lambda_2) = Z_\theta(\lambda_2, \lambda_1)$, and consequently the RHS of (4.7) vanishes. As the quantity $\left[ \frac{N_{21}}{M_3} \right]_{\lambda_3=\mu_1-\gamma}$ is finite we can thus conclude that $Z_\theta(\mu_1, \mu_1 - \gamma) = 0$. This
property simplifies \((4.6)\) and we are left with
\[
s(\mu_1 - \mu_2 + \gamma)s(\theta + \gamma)Z_\theta(\lambda_1, \lambda_2) = \\
s(\theta + \gamma - \lambda_2 + \mu_1)s(\lambda_2 - \mu_2 + \gamma)\frac{s(\lambda_1 - \mu_1)}{s(\lambda_1 - \mu_1 + \gamma)} \frac{s(\lambda_1 - \lambda_2 + \gamma)}{s(\lambda_1 - \lambda_2)} Z_\theta(\mu_1, \lambda_1) + \\
s(\theta + \gamma - \lambda_1 + \mu_1)s(\lambda_1 - \mu_2 + \gamma)\frac{s(\lambda_2 - \mu_1)}{s(\lambda_2 - \mu_1 + \gamma)} \frac{s(\lambda_2 - \lambda_1 + \gamma)}{s(\lambda_2 - \lambda_1)} Z_\theta(\mu_1, \lambda_2) .
\]
\[(4.8)\]

The vanishing condition of \(Z_\theta\) above unveiled have a special appeal since we are interested in the polynomial solution of \((4.3)\) with order dictated by \((4.1)\). For instance, they allow us to write
\[
Z_\theta(\lambda_1, \lambda_2)|_{\lambda_1=\mu_1} = s(\lambda_2 - \mu_1 + \gamma)V(\lambda_2)
\]
where now \(V(\lambda_2)\) needs to be a polynomial of the same order as \(Z_\theta\) with \(L = 1\) in order to satisfy \((4.1)\). The expression \((4.9)\) can now be replaced in \((4.8)\) yielding the following relation,
\[
Z_\theta(\lambda_1, \lambda_2|\mu_1, \mu_2) = \\
s(\theta + \gamma - \lambda_2 + \mu_1) s(\lambda_2 - \mu_2 + \gamma) \frac{s(\lambda_1 - \mu_1)}{s(\mu_1 - \mu_2 + \gamma)} \frac{s(\lambda_1 - \lambda_2 + \gamma)}{s(\lambda_1 - \lambda_2)} V(\lambda_1) + \\
s(\theta + \gamma - \lambda_1 + \mu_1) s(\lambda_1 - \mu_2 + \gamma) \frac{s(\lambda_2 - \mu_1)}{s(\mu_1 - \mu_2 + \gamma)} \frac{s(\lambda_2 - \lambda_1 + \gamma)}{s(\lambda_2 - \lambda_1)} V(\lambda_2) ,
\]
\[(4.10)\]
which can be substituted back into the original equation \((4.3)\). This step leave us with an equation involving the functions \(V(\lambda_0), V(\lambda_1), V(\lambda_2)\) and \(V(\lambda_3)\) for arbitrary values of those variables. Moreover, by setting \(\lambda_3 = \mu_1\) we are left with the equation
\[
K_1V(\lambda_2) + K_2V(\lambda_1) + L_{21}V(\lambda_0) = 0 ,
\]
\[(4.11)\]
where the coefficients \(K_1, K_2\) and \(L_{21}\) coincide respectively with \(M_1, M_2\) and \(N_{21}\) obtained from \((3.8)\) and \((3.9)\) with \(L = 1\) \((n = 2)\), \(\theta \rightarrow \theta + \gamma\) and \(\mu_1 \rightarrow \mu_2\). So the function \(V(\lambda)\) obeys the same equation as \(Z_{\theta+\gamma}(\lambda|\mu_2)\).

We have now reached an important stage of this approach. Let us suppose that the solution of \((3.12)\) with polynomial structure \((4.1)\) is unique. In fact, the uniqueness of the polynomial solutions is demonstrated in the App. \[F\] Since \(V(\lambda)\) and \(Z_\theta(\lambda)\) are polynomials of the same order, that is to say \(V(\lambda) = \Omega_2Z_{\theta+\gamma}(\lambda)\) where \(\Omega_2\) does not depend on \(\lambda\). The solution of \((3.12)\) for \(L = 1\) can be found in the App. \[F\] and here we shall only make use of the solution. In this way the results so far can be gathered and from \((4.10)\) and \((F.5)\) we immediately obtain an explicit solution for \(Z_\theta(\lambda_1, \lambda_2)\). The solution is then given by
\[
Z_\theta(\lambda_1, \lambda_2) = F_{12} + F_{21}
\]
\[(4.12)\]
where

\[ F_{ij} = \Omega_2 \frac{s(\theta + \gamma - \lambda_i + \mu_1)}{s(\theta + \gamma)} \frac{s(\theta + 2\gamma - \lambda_j + \mu_2)}{s(\theta + 2\gamma)} \frac{s(\lambda_i - \mu_2 + \gamma)}{s(\mu_1 - \mu_2 + \gamma)} \frac{s(\lambda_j - \mu_1)}{s(\lambda_j - \lambda_i)} \cdot \tag{4.13} \]

The constant factor \( \Omega_2 \) can be fixed by the asymptotic behaviour (4.2) and we find \( \Omega_2 = s(\gamma)^2 s(\mu_1 - \mu_2 + \gamma) \). Thus we have completely determined the partition function (2.16) for \( L = 2 \) using solely the polynomial structure (4.1) and the asymptotic behaviour (4.2), in addition to the functional equation (3.12). In what follows we shall consider the general \( L \) case.

**Special Zeroes and Symmetry.** The first step in order to consider the case with arbitrary values of \( L \) is to obtain an analogous of the relation (4.9) which can be obtained by uncovering special zeroes of our partition function. These zeroes have been unveiled in App. C, and in addition to that we shall also make use of the symmetry property discussed in App. D. Thus taking into account (C.13) and (D.5) we can write

\[ Z(\theta, \ldots, \lambda_i, \ldots, \lambda_j, \ldots) = \sum_{j=1}^{L} \prod_{k=1}^{L} s(\lambda_k - \mu_1 + \gamma) \frac{\bar{m}_j}{m_L} V(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_L), \tag{4.14} \]

We proceed by setting \( \lambda_0 = \mu_1 \) and \( \lambda_{L+1} = \mu_1 - \gamma \) in the functional equation (3.12). By doing so we obtain the expression

\[ Z(\theta, \ldots, \lambda) = \sum_{j=1}^{L} \prod_{k=1}^{L} s(\lambda_k - \mu_1 + \gamma) \frac{\bar{m}_j}{m_L} V(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_L), \tag{4.15} \]

where

\[ m_L = \frac{s(\theta + \gamma)}{s(\theta + (L + 1)\gamma)} (-1)^{L+1} s(\gamma)^2 \prod_{j=2}^{L} s(\mu_1 - \mu_j + \gamma) s(\mu_j - \mu_1 + \gamma) \]

\[ \bar{m}_j = \frac{s(\theta + \gamma - \lambda_j + \mu_1)}{s(\theta + (L + 1)\gamma)} (-1)^{L} s(\gamma)^2 \prod_{k=2}^{L} s(\mu_k - \mu_1 + \gamma) s(\lambda_j - \mu_k + \gamma) \]

\[ \times \prod_{k=1}^{L} s(\lambda_k - \mu_1 + \gamma) s(\lambda_k - \mu_1 + \gamma). \tag{4.16} \]

It is important to remark here that we have also considered (4.14) and the symmetry property \( Z(\theta, \ldots, \lambda, \ldots, \lambda, \ldots) = Z(\theta, \ldots, \lambda, \ldots, \lambda, \ldots) \) discussed in the App. D in order to obtain (4.15).
The relation (4.15) can now be substituted back into the Eq. (3.12) and considering $\lambda_{L+1} = \mu_1$ we obtain

\[
\sum_{i=1}^{L} K_i V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_L) + \sum_{j=2}^{L} \sum_{i=1}^{j-1} L_{ji} V(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_L) = 0.
\]

(4.17)

In their turn the coefficients $K_i$ and $L_{ji}$ appearing in (4.17) correspond respectively to the coefficients $M_i$ and $N_{ji}$ given in (3.8) and (3.9) with $L \to L - 1$ ($n = L$), $\theta \to \theta + \gamma$ and $\mu_i \to \mu_{i+1}$. Moreover, the compatibility between (4.14) and (4.1) tells us that the function $V$ is a multivariate polynomial of the same order as the partition function $Z_\theta$ for a lattice with dimensions $(L - 1) \times (L - 1)$. Thus the Eq. (4.17), together with the uniqueness property discussed in the App. E, implies in

\[
V(\lambda_1, \ldots, \lambda_n) = \Omega_n Z_{\theta+\gamma}(\lambda_1, \ldots, \lambda_n).
\]

(4.18)

In this way the relation (4.15) can be iterated using the results obtained in the App. F as initial condition.

By carrying on with this procedure we obtain the following expression for our partition function:

\[
Z_\theta(\lambda_1, \ldots, \lambda_L) = \sum_{\{i_1, \ldots, i_L\} \in S_L} F_{i_1 \ldots i_L}
\]

(4.19)

where

\[
F_{i_1 \ldots i_L} = \frac{\Omega_L}{\prod_{k=2}^{L} s(\mu_1 - \mu_k + \gamma)} \prod_{n=1}^{L} s(\theta + n\gamma - \lambda_{i_n} + \mu_n) \prod_{n=1}^{L} \prod_{j=n+1}^{L} s(\lambda_{i_n} - \mu_j + \gamma) \prod_{j=1}^{n-1} s(\lambda_{i_n} - \mu_j)
\]

\[
\times \prod_{n=1}^{L-1} \prod_{m=n+1}^{L} s(\lambda_{i_m} - \lambda_{i_n} + \gamma) / s(\lambda_{i_m} - \lambda_{i_n}).
\]

(4.20)

Here $S_L$ denotes the permutation group of order $L$ and the asymptotic behaviour (4.2) implies in $\Omega_L = s(\gamma) \prod_{k=2}^{L} s(\mu_1 - \mu_k + \gamma)$.

5 Multiple integral representation

The expression (4.19, 4.20) can be converted into a multiple contour integral similarly to the expression recently found in [30] for the $U_q[\widehat{su}(2)]$ vertex model. As a matter of fact, multiple contour integrals seems to fit naturally into the algebraic-functional framework presented here. We start by noticing that the solution of (3.12) for $L = 1$ given in (F.5) can be rewritten as

\[
Z_\theta(\lambda) = \frac{s(\gamma)}{2\pi i} \oint \frac{1}{s(w - \lambda)} \frac{s(\theta + \gamma - w + \mu_1)}{s(\theta + \gamma)} dw,
\]

(5.1)
where the integration contour contains the pole at $w = \lambda$. Now we look to the Eq. (4.15) considering (4.18) and keeping in mind that for $L = 1$ we have (5.1). This suggests that the iteration procedure described by (4.15) can be mimicked by Cauchy’s residue formula. It turns out that when we look to (4.15) searching for solutions as contour integrals, we find a factorised formula for the integrand. In this way we end up with the following expression for our partition function,

$$
Z_{\theta}(\lambda_1, \ldots, \lambda_L) =
\left[ \frac{s(\gamma)}{2\pi i} \right]^L \oint \cdots \oint \prod_{i=1}^L \prod_{j=i+1}^L \frac{s(w_j - w_i + \gamma)s(w_j - w_i)}{s(\theta + j\gamma - w_j + \mu_j) \times s(\theta + j\gamma)} \times \\
\prod_{i=1}^L \prod_{j=1}^{i-1} s(\mu_j - w_i) \prod_{j=i+1}^L s(w_i - \mu_j + \gamma) \, dw_1 \ldots dw_L,
$$

(5.2)

where the integration countours enclose the poles at $w_i = \lambda_j$. As expected the expression (5.2) coincides with (4.19)-(4.20) when evaluated using Cauchy’s residue formula. Moreover, in the limit $\theta \to \infty$ the formula (5.2) reduces to the one obtained in [30] after a relabelling of the variables $\mu_j$. This relabelling does not affect the solution $Z_{\theta}(\lambda_1, \ldots, \lambda_L)$ since this partition function is invariant under the exchange of variables $\mu_i \leftrightarrow \mu_j$ as discussed in [4].

6 Concluding remarks

The main result of this paper is the integral representation (5.2) obtained for the partition function of the trigonometric SOS model with domain wall boundaries. This integral formula has been obtained by solving a functional equation derived from the dynamical Yang-Baxter algebra. This approach has been proposed in [11, 12] and here we also present a more robust formulation of that method.

In contrast to the case considered in [11, 12], where the $\mathfrak{su}(2)$ algebra only appears in the final stages of the derivation of (3.12), here it plays an important role from the very beginning. For instance, the derivation of (3.6) requires the repeated use of the relations (3.2) and (3.4).

It is important to remark here that the elliptic version of this same partition function has been considered previously in [7–9, 31]. In particular, the work [8] discusses the lack of a single determinant expression for this partition function generalising the Izergin-Korepin determinant. For the three-colouring model case, a functional equation for this partition function was obtained in [32, 33] though a connection with the functional equation presented here is not obvious at the moment. It is also worth remarking that the trigonometric SOS model with one reflecting end, and the remaining boundaries of domain wall type has been considered in [34, 35]. In that case the dynamical Yang-Baxter algebra also plays an important role, though it is only responsible for a few out of six conditions determining uniquely the model partition function. The approach considered here makes use of only three conditions and it would be interesting to extend it to the case considered in [34, 35].
Moreover, it has been recently discussed in [36] the usefulness of such integral formulas for computing correlation functions for the case of domain wall boundaries which makes the representation (5.2) more attractive. The generalisation of our results for the elliptic case is under investigation and we hope to report on that in a future publication.

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A Dynamical Yang-Baxter algebra vs. \( \mathfrak{su}(2) \)

The analysis performed here will follow the same lines as the one presented in [12]. We shall consider the \( \mathfrak{su}(2) \) generators \( E, F \) and \( H \) satisfying the relations

\[
[E, F] = H \quad [H, E] = 2E \quad [H, F] = -2F ,
\]

(A.1)

and compute their commutation relations with the generators of the dynamical Yang-Baxter algebra \( A(\lambda, \theta), B(\lambda, \theta), C(\lambda, \theta) \) and \( D(\lambda, \theta) \). In fact we will only need their commutation rules with the Cartan generator \( H \) whose fundamental representation on the quantum space is given by

\[
H = \sum_{i=1}^{L} \hat{h}_i .
\]

(A.2)

Here \( \hat{h}_i \) consists of the Pauli matrix

\[
\hat{h} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(A.3)

acting non-trivially on the space \( \mathbb{V}_i \) of the tensor product \( \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \).

The ice rule (2.11) can be rewritten as

\[
[\mathcal{R}_{aj}(\lambda, \theta_j), \hat{h}_j] = -[\mathcal{R}_{aj}(\lambda, \theta_j), \hat{h}_a] \]

which immediately lead us to the relation

\[
[A(\lambda, \theta), H] = 0 \quad [B(\lambda, \theta), H] = 2B(\lambda, \theta) \quad [C(\lambda, \theta), H] = -2C(\lambda, \theta) \quad [D(\lambda, \theta), H] = 0 ,
\]

(A.5)
which allows us to exploit the representation theory of the \( \mathfrak{su}(2) \) algebra in order to gain insight into the dynamical Yang-Baxter algebra generators.

For instance, the \( \mathfrak{su}(2) \) highest and lowest weight states \( |0\rangle \) and \( |\bar{0}\rangle \) defined in (2.17) obey the relations \( H |0\rangle = L |0\rangle \) and \( H |\bar{0}\rangle = -L |\bar{0}\rangle \). These properties together with (A.5) allow us to obtain the relation

\[
H \prod_{i=1}^{n} B(\lambda_i, \theta + (i - 1)\gamma) |0\rangle = (L - 2n) \prod_{i=1}^{n} B(\lambda_i, \theta + (i - 1)\gamma) |0\rangle
\]

(A.6)

which is valid for any number \( n \) of operators \( B(\lambda, \theta) \). Now the relation (A.6) put us in position to use the \( \mathfrak{su}(2) \) representation theory to draw conclusions about the generator \( B(\lambda, \theta) \). For the case \( n = L \) the expression (A.6) tells us that

\[
\prod_{i=1}^{L} B(\lambda_i, \theta + (i - 1)\gamma) |0\rangle \sim |\bar{0}\rangle
\]

(A.7)

and from (3.5) we immediately have that

\[
\prod_{i=1}^{L+1} B(\lambda_i, \theta + (i - 1)\gamma) |0\rangle = 0
\]

(A.8)

The property (A.8) is an important ingredient for the derivation of the functional equation (3.12).

**B Polynomial structure and asymptotic behaviour**

In order to analyse the dependence of \( Z_{\theta} \) with the set of variables \( \{\lambda_i\} \) we first consider the following change of variables:

\[
x_i = e^{2\lambda_i}, \quad u_i = e^{2\mu_i}, \quad q = e^{\gamma}, \quad t = e^{\theta}.
\]

In terms of the above defined variables, the \( \mathcal{R} \)-matrix given by (2.9) and (2.10) can be written as

\[
\mathcal{R} = \frac{1}{8q\bar{x}} (xU + V),
\]

(B.1)

where

\[
U = (3q^2 + 1) 1 \otimes 1 + (q^2 - 1)H \otimes H + \frac{(q^2 - 1)(t^2 + 1)}{(t^2 - 1)} 1 \otimes H - \frac{(q^2 - 1)(t^2 + 1)}{(t^2 - 1)} H \otimes 1
\]

\[
+ \frac{4(1 - q^2)}{(t^2 - 1)} E \otimes F + \frac{4t^2(q^2 - 1)}{(t^2 - 1)} F \otimes E
\]

\[
V = -(3 + q^2) 1 \otimes 1 + (q^2 - 1)H \otimes H - \frac{(q^2 - 1)(t^2 + 1)}{(t^2 - 1)} 1 \otimes H + \frac{(q^2 - 1)(t^2 + 1)}{(t^2 - 1)} H \otimes 1
\]

\[
+ \frac{4t^2(q^2 - 1)}{(t^2 - 1)} E \otimes F + \frac{4(1 - q^2)}{(t^2 - 1)} F \otimes E.
\]

(B.2)
In (B.3) the generators E, F and H are the \( su(2) \) generators satisfying (A.1), and considering (B.2), (2.13) and (2.14) we readily obtain the expansion

\[
B(\lambda_i, \theta) = \frac{1}{x_i^L} \left[ f_L^{(i)} x_i^L + f_{L-1}^{(i)} x_i^{L-1} + \cdots + f_0^{(i)} \right].
\]  

(B.4)

Now looking to the product \( \prod_{j=1}^L B(\lambda_j, \theta + j\gamma) \) appearing in the definition (2.16), we can conclude that

\[
Z(\theta)(\lambda_1, \ldots, \lambda_L) = \frac{\bar{Z}(x_1, \ldots, x_L)}{\prod_{i=1}^L x_i^L},
\]

(B.5)

where \( \bar{Z}(x_1, \ldots, x_L) \) is a polynomial of order \( L \) in each variable \( x_i \).

Also from (B.2) we can see that in the limit \( x \to \infty \) only the operator \( U \) contributes for the partition function \( Z(\theta) \). In (B.3) the operator \( U \) is written in terms of \( su(2) \) generators which allows us to follow the same analysis of [11]. Without significant modifications we find that

\[
\bar{Z} \sim \frac{(q - q^{-1})^L}{2^{L^2}} \frac{[L]_q!}{(x_1 \ldots x_L)^L} \prod_{n=1}^L (1 - q^{2n} t^2) \bar{u}_n^L\text{ as } x_i \to \infty.
\]

(B.6)

Here the \( q \)-factorial function is defined as

\[
[n]_q! = 1(1 + q)(1 + q + q^2) \ldots (1 + q + \cdots + q^n).
\]

(B.7)

C Special Zeroes

One important ingredient for solving the Eq. (3.12) under the conditions (4.1) and (4.2) is the localisation of some special zeroes of our partition function. Since we are interested in the polynomial solution of (3.12), those zeroes will play an important role in the characterisation of our solution. We shall start by looking to particular values of \( L \) for illustrative purposes and then we proceed to the general case.

• \( L = 2 \):

We set \( \lambda_3 = \mu_1 \) and \( \lambda_2 = \mu_1 - \gamma \) in such a way that the functions \( M_2, M_3, N_{21}, N_{31} \) and \( N_{32} \) vanish. For these particular values of \( \lambda_3 \) and \( \lambda_2 \) we are thus left with

\[
M_1|_{\lambda_2, \lambda_3} Z(\mu_1 - \gamma, \mu_1) = 0.
\]

(C.1)

Since \( M_1|_{\lambda_2, \lambda_3} \) is finite we can conclude that \( Z(\mu_1 - \gamma, \mu_1) = 0 \).

• \( L = 3 \):
By setting $\lambda_4 = \mu_1$ and $\lambda_3 = \mu_1 - \gamma$ we obtain

$$
\sum_{i=0}^{2} P_i \, Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_i) = 0
$$

(C.2)

where

$$
P_0 = N_{21}|_{\lambda_0, \lambda_4} \quad , \quad P_1 = M_2|_{\lambda_3, \lambda_4} \quad \text{and} \quad P_2 = M_1|_{\lambda_3, \lambda_4} .
$$

(C.3)

In terms of the variables $x_i$, the functions $P_i$ are rational functions and thus $\exists \lambda_i : P_i = 0$. Besides the above specialisation of the variables $\lambda_4$ and $\lambda_3$, we also choose $\lambda_i | P_i = 0$ for $i = 1, 2$. Thus we are left with

$$
P_0|_{\lambda_1, \lambda_2} \, Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_0) = 0 ,
$$

(C.4)

and since $P_0|_{\lambda_1, \lambda_2}$ is finite we can conclude that $Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_0) = 0$.

- $L = 4$:

For the case $L \geq 4$ this analysis becomes a bit more involved. We start by setting $\lambda_5 = \mu_1$ and $\lambda_4 = \mu_1 - \gamma$ similarly to the previous cases. Under this specialisation the Eq. (3.12) reduces to

$$
M_1|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_2, \lambda_3, \mu_1 - \gamma, \mu_1) + M_2|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_1, \lambda_3, \mu_1 - \gamma, \mu_1) + M_3|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_1, \lambda_2, \mu_1 - \gamma, \mu_1) + N_{21}|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_0, \lambda_3, \mu_1 - \gamma, \mu_1) + N_{31}|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_0, \lambda_2, \mu_1 - \gamma, \mu_1) + N_{32}|_{\lambda_4, \lambda_5} \, Z_\theta(\lambda_0, \lambda_1, \mu_1 - \gamma, \mu_1) = 0 .
$$

(C.5)

Next we set $\lambda_0 = \mu_1$ and $\lambda_1 = \mu_1 - \gamma$. The Eq. (C.5) does not suffer significant simplifications and we then proceed by setting $\lambda_3 = \mu_1$ using the following properties:

$$
\lim_{\lambda_3 \to \mu_1} \frac{M_1}{N_{31}}|_{\lambda_0, \lambda_1, \lambda_4, \lambda_5} = -1
$$

(C.6)

By doing so we end up with the relation

$$
M_3|_{\lambda_0, \lambda_1, \lambda_3, \lambda_4, \lambda_5} \, Z_\theta(\mu_1 - \gamma, \lambda_2, \mu_1 - \gamma, \mu_1) = N_{21}|_{\lambda_0, \lambda_1, \lambda_3, \lambda_4, \lambda_5} \, Z_\theta(\mu_1, \mu_1 - \gamma, \mu_1) ,
$$

(C.7)

which is further simplified to

$$
N_{21}|_{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} \, Z_\theta(\mu_1, \mu_1 - \gamma, \mu_1) = 0
$$

(C.8)

with $\lambda_2 = \mu_1 - \gamma$. The quantity $N_{21}|_{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}$ is finite which allow us to conclude that

$$
Z_\theta(\mu_1, \mu_1 - \gamma, \mu_1) = 0 .
$$

(C.9)
Now we move backwards considering the consequences of (C.9) to the previous equations. From (C.8) and (C.9) we have that
\[ Z_\theta(\mu_1 - \gamma, \lambda_2, \mu_1 - \gamma, \mu_1) = 0 , \tag{C.10} \]
which can be reintroduced in (C.5) with the above mentioned specialisation of $\lambda_0$ and $\lambda_1$. This yields the following expression
\[ Z_\theta(\lambda_2, \lambda_3, \mu_1 - \gamma, \mu_1) = - \frac{N_{21}}{M_1} \bigg|_{\lambda_0, \lambda_1, \lambda_4, \lambda_5} Z_\theta(\mu_1, \lambda_3, \mu_1 - \gamma, \mu_1) - \frac{N_{31}}{M_1} \bigg|_{\lambda_0, \lambda_1, \lambda_4, \lambda_5} Z_\theta(\mu_1, \lambda_2, \mu_1 - \gamma, \mu_1) . \tag{C.11} \]
Now we replace (C.11) back into (C.5) to obtain the expression
\[ \sum_{i=0}^{3} Q_i Z_\theta(\mu_1, \lambda_i, \mu_1 - \gamma, \mu_1) = 0 . \tag{C.12} \]
The explicit form of $Q_i$ is not enlightening and shall not be presented here. Nevertheless, using similar arguments as for the cases $L = 2, 3$ we can conclude that $Z_\theta(\mu_1, \lambda, \mu_1 - \gamma, \lambda_1) = 0$. Thus from (C.11) we obtain the vanishing condition
\[ Z_\theta(\lambda_2, \lambda_3, \mu_1 - \gamma, \mu_1) = 0 . \tag{C.13} \]

**General $L$:**

For arbitrary values of $L$ we initially set $\lambda_{L+1} = \mu_1$ and $\lambda_L = \mu_1 - \gamma$ in the functional equation (3.12), followed by the specialisation $\lambda_0 = \mu_1$ and $\lambda_1 = \mu_1 - \gamma$. We collect the results at each one of the steps and then start fixing the variables $\lambda_{L-1} = \mu_1$, $\lambda_{L-2} = \mu_1 - \gamma$, $\lambda_{L-3} = \mu_1$ and so on until we exhaust all the variables. Then the consistency condition of each step with the previous ones allow us to conclude that
\[ Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_3, \ldots, \lambda_L) = 0 . \tag{C.14} \]
Together with the symmetry property discussed in App. D the relation (C.14) plays an important role for solving (3.12).

**D  $Z_\theta$ as a symmetric function**

In this appendix we intend to show that Eq. (3.12) admits only analytic solutions which are symmetric under the exchange of variables $\lambda_i \leftrightarrow \lambda_j$. This is an expected property of our partition function (2.16) due to the commutation relations (3.3). Nevertheless, we shall demonstrate that this property is not an extra input required to solve Eq. (3.12).
We start by integrating the Eq. (3.12) over the contour \( C_j \) containing only the variable \( \lambda_j \). For a given \( j \), the coefficients \( M_j \) and \( N_{kl} \) \((k, l \neq j)\) do not contain poles when \( \lambda_0 \to \lambda_j \). Moreover, we also have the following identities between the coefficients

\[
\begin{align*}
\lim_{\lambda_0 \to \lambda_j} s(\lambda_0 - \lambda_j) M_k &= -\lim_{\lambda_0 \to \lambda_j} s(\lambda_0 - \lambda_j) N_{jk} & k < j \\
\lim_{\lambda_0 \to \lambda_j} s(\lambda_0 - \lambda_j) M_k &= -\lim_{\lambda_0 \to \lambda_j} s(\lambda_0 - \lambda_j) N_{kj} & k > j
\end{align*}
\] (D.1)

for \( j = 1, \ldots, L \). Thus after the integration of (3.12) over the contour \( C_j \), we are left with the relation

\[
\sum_{i=1}^{L} \hat{M}_i = [Z_\theta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1}) - Z_\theta(\lambda_j, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1})] = Z_\theta(\lambda_j, \lambda_1, \ldots, \lambda_L) - Z_\theta(\lambda_1, \ldots, \lambda_L)
\] (D.2)

where

\[
\hat{M}_k = \lim_{\lambda_0 \to \lambda_j} s(\lambda_0 - \lambda_j) M_k .
\] (D.3)

Two observations are important at this stage. Firstly, the relation (D.2) is valid for \( j = 1, \ldots, L \) and thus it provides us with a total of \( L \) equations. Secondly, we notice that the RHS of (D.2) does not depend on the variable \( \lambda_{L+1} \). In fact this variable can be adjusted, together with the results obtained for the \((j-1)\)-th equation, in order to show that the RHS of (D.2) vanishes. Thus the relation (D.2) implies in

\[
Z_\theta(\lambda_1, \ldots, \lambda_L) = Z_\theta(\lambda_j, \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_L)
\] (D.4)

for \( j = 1, \ldots, L \) and consequently we have the desired symmetry relation

\[
Z_\theta(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L) = Z_\theta(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_L) .
\] (D.5)

### E Uniqueness

In this appendix we prove the uniqueness of the multivariate polynomial solution of the Eq. (3.12). In order to start we first need to introduce the modified coefficients

\[
\bar{M}_i = \prod_{j=1}^{L+1} x_j^{-\frac{2}{i}} M_i \quad \text{and} \quad \bar{N}_{ji} = \prod_{k=0}^{L+1} x_j^{-\frac{2}{i,k}} N_{ji} ,
\] (E.1)

with \( x_i = e^{2\lambda_i} \). In this way the Eq. (3.12) is given by

\[
\sum_{i=1}^{L+1} \bar{M}_i \bar{Z}_\theta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L+1}) + \sum_{j=2}^{L+1} \sum_{i=1}^{j-1} \bar{N}_{ji} \bar{Z}_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{L+1}) = 0 ,
\] (E.2)
in terms of $\bar{Z}_\theta$ which is a polynomial of order $L$ in each variable $x_i$ according to (4.1).

We shall now explore the linearity of the Eq. (E.2). More precisely that means the following: if $\bar{Z}_1$ and $\bar{Z}_2$ are two multivariate polynomials of type (4.1) satisfying (E.2), then

$$\bar{Z} = \alpha \bar{Z}_1 - \beta \bar{Z}_2$$

is also a solution for any constants $\alpha$ and $\beta$. Polynomials are characterised by the location of their zeroes and we can express $\bar{Z}$, $\bar{Z}_1$ and $\bar{Z}_2$ as

$$\bar{Z} \sim \prod_{i=1}^{L} (x - r_i) \quad \bar{Z}_1 \sim \prod_{i=1}^{L} (x - s_i) \quad \bar{Z}_2 \sim \prod_{i=1}^{L} (x - t_i)$$

(E.4)

where $x$ can represent any of the variables $x_i$. Since the constants $\alpha$ and $\beta$ in (E.3) are arbitrary they can always be fine tuned in order to ensure that $\bar{Z}$ is also of order $L$.

Next we set $x = r_j$ in (E.3) and from (E.4) we obtain

$$\alpha \prod_{i=1}^{L} (r_j - s_i) \sim \beta \prod_{i=1}^{L} (r_j - t_i).$$

(E.5)

The relation (E.5) allows us to make important conclusions. For instance, if we assume that $\{r_i\} \neq \{s_i\}$ then (E.5) implies that $\{s_i\} = \{t_i\}$ since $\alpha$ and $\beta$ can always be adjusted to compensate an overall factor. This implies that $\bar{Z}_1$ and $\bar{Z}_2$ are proportional to each other and so is $\bar{Z}$ due to (E.3). This consequence clearly contradicts the initial assumption $\{r_i\} \neq \{s_i\}$. The remaining option is allowing $\{r_i\} = \{s_i\}$ and thus $\bar{Z}$ and $\bar{Z}_1$ only differ by a constant. This fact together with (E.3) tell us that $\bar{Z}_2$ is also proportional to $\bar{Z}_1$. In summary this analysis shows that if we have two polynomials of the same order solving (E.2), they are essentially the same polynomial. This proves the uniqueness of the polynomial solution of (3.12).

**F Solution for $L = 1$**

Here we shall present the solution of the Eq. (3.12) for the case $L = 1$ which is of fundamental importance in order to derive the solution for general $L$. For the case $L = 1$ the Eq. (3.12) reads

$$M_1 Z_\theta(\lambda_2) + M_2 Z_\theta(\lambda_1) + N_{21} Z_\theta(\lambda_0) = 0.$$  

(F.1)

At first look the condition $\lambda_i = \lambda$ does not seem helpful in finding the solution of (F.1). However, a closer look reveals that the coefficients $M_1$, $M_2$ and $N_{21}$ contain poles when the variables $\lambda_i$ coincide. As we shall see this fact will be of fundamental importance.

We can compute the limit $\lambda_i = \lambda$ of the Eq. (F.1) using L’Hopital’s rule and we are left with the following second order differential equation

$$P_0 \ddot{Z}_\theta + P_1 \frac{d\dot{Z}_\theta}{dx} + P_2 \frac{dz^2}{dx^2} = 0,$$

(F.2)
given in terms of variables $x = e^{2\lambda}$ and $u = e^{2\mu_1}$. The coefficients in (F.2) are given by

\[ P_0 = (-4q^2 + 2q^4 t^2 + 2q^6 t^2)x + (2q^2 + 2q^4 - 4q^6 t^2)u \]
\[ P_1 = (-4q^4 t^2 + 2q^6 t^4 + 2q^8 t^4)u^2 + (4q^2 - 4q^8 t^4)xu + (-2q^2 - 2q^4 + 4q^6 t^2)x^2 \]
\[ P_2 = (1 + q^2 - 4q^4 t^2 + q^6 t^4 + q^8 t^4)xu^2 + (-4q^2 - q^2 t^2 + 5q^4 t^2 + 5q^6 t^2 - q^8 t^2 - 4q^8 t^4)ux^2 \]
\[ + (q^2 + q^4 - 4q^6 t^2 + q^8 t^4 + q^{10} t^4)x^3, \]

and by standard methods we find the general solution

\[ \tilde{Z}_\theta(x) = C_1(x - q^2 t^2 u) + C_2(x - q^2 t^2 u) \int e^{-\int_{x}^{x'} \frac{P_1(x')}{P_2(x')} \, dx'} \, dx, \]  

(F.4)

where $C_1$ and $C_2$ are two arbitrary integration constants. Now the polynomial structure (4.1) asks for $C_2 = 0$, while the asymptotic behaviour (4.2) implies in $C_1 = \frac{(q - q^{-1})}{2} (1 - t^2 q^3)^{-1} u_1^{-\frac{3}{2}}$. Thus our partition function for $L = 1$ is given by

\[ Z_\theta(\lambda) = s(\gamma) \frac{s(\theta + \gamma - \lambda + \mu_1)}{s(\theta + \gamma)}. \]  

(F.5)

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