Exceptional points, phase rigidity and nonlinear Schrödinger equation

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Abstract

In order to describe theoretically the present-day high-resolution experimental data at high level density, obtained in different fields of physics, different extensions of standard quantum mechanics are considered in literature. Among others, nonlinear terms are introduced in the Schrödinger equation or a non-Hermitian part is added to the Hamiltonian. In the last case, the system is considered to be open, and the coupling to the environment is taken into account from the very beginning. The control of the system properties by varying a certain parameter provides us information on the system that is, in many cases, counterintuitive and points to the same phenomena in different systems in spite of specific differences between them. In our paper, we use a schematic model for the non-Hermitian operator in order to simulate the main features of small open quantum systems. At low level density, the system is described well by standard Hermitian quantum physics while fundamental differences appear at high level density due to the non-Hermiticity of the Hamiltonian which cannot be neglected under this condition. The results obtained and discussed are the following. (i) At high level density, the dynamics of the system is determined by singular, so-called exceptional points (EPs) at which two eigenvalues of the non-Hermitian operator $\mathcal{H}$ coalesce; (ii) nonlinear terms appear in the Schrödinger equation in the neighborhood of EPs; (iii) in approaching an EP, the phases of the eigenfunctions of $\mathcal{H}$ are not rigid; (iv) the wavefunctions of two coalescing states at the EP are mixed in a finite parameter range around the EP; (v) the EPs cause dynamical phase transitions at high level density; (vi) interesting new information on open quantum systems can be obtained when not only “loss” (as usually) is taken into account but also “gain”. These results, characteristic for an open quantum system, are fundamentally different from those of a closed quantum system described by a Hermitian operator. The transition from a closed system at low level density to an open one at high level density occurs smoothly.
I. INTRODUCTION

The basic features of quantum mechanics are worked out about 90 years ago: the Schrödinger equation is linear and allows superpositions of quantum states to be solutions of the Schrödinger equation; the Hamiltonian describing the system is Hermitian, its eigenvalues $E^B_i$ are real and its eigenfunctions $\Phi^B_i$ are normalized according to $\langle \Phi_i | \Phi_j \rangle = \delta_{i,j}$. The system described in this manner is closed since its coupling to an environment is not involved in the theory. The finite lifetime of most states of a (small) system is calculated without taking into account any feedback from the environment onto the system. This theory is proven experimentally during multi-year studies performed on different systems at low level density.

For the last years, not only the resolution of most experimental devices has increased considerably but also calculations with higher accuracy have become possible. As a result, the standard quantum theory has shown its limit to describe successfully experimental results. Counterintuitive results are obtained in different experiments. An example is the observation of an unexpected regularity of the measured transmission phases (so-called phase lapses) in mesoscopic systems [1] which could not explained in the framework of Hermitian quantum physics in spite of much effort [2, 3]. They are explainable however by considering the feedback from the environment onto the system [4]. Another example is the experimental observation and theoretical description of a dynamical phase transition (DPT) in the spin swapping operation [5, 6]. While Fermi’s golden rule holds below the DPT, it is violated above it. In a new experimental paper [7], the formation of a protected sub-band for conduction in quantum point contacts under extreme biasing is found, see also [8]. This sub-band is a collective robust mode of non-equilibrium transport that is immune to local heating. It has potential practical implications for nanoscale devices made of quantum point contacts and quantum dots.

In order to improve the theoretical description of these limiting cases, in some papers the coupling of the system to an environment is taken into account explicitly. Mostly, this is done by replacing the Hermitian Hamilton operator, or part of it, by a non-Hermitian one, see e.g. the reviews [9, 10] and the book [11]. In other papers, nonlinearities are added to the Schrödinger equation. An example is the review [12] where the role of nonlinear Fano resonances in theoretical and experimental studies of light propagation in photonic devices
and charge transport through quantum dots (nanostructures) is reviewed. By this means, the description of experimental results could be improved considerably in all cases.

A non-Hermitian Hamiltonian in the Schrödinger equation appears when the system is considered to be open, i.e. to be embedded into an environment, and the coupling between the system and its environment is taken into account from the very beginning. A natural environment is the continuum of scattering wavefunctions to which the states of the system are coupled and into which they decay. It can be changed by external fields, however never be deleted. The finite lifetime of the states of the system is calculated directly from the non-Hermitian part of the Hamiltonian $H$ [9, 10]. The feedback from the environment onto the system is involved in the non-Hermitian Hamiltonian $H$ and therefore also in its eigenvalues $\mathcal{E}_i$ and eigenfunctions $\Phi_i$. The basic assumption of this description is supported experimentally by the recent observation that remote states are coupled through the continuum [13].

Meanwhile there are many calculations performed with a non-Hermitian Hamiltonian. As usual, the behavior of the system is controlled by means of varying a certain parameter. The restriction of the parameter dependence of the Hamiltonian $H$ to its explicitly non-Hermitian part (by neglecting the parameter dependence of its real Hermitian part) allows us to receive a quick overview on the spectroscopic redistribution processes occurring in the system under the influence of the coupling to the environment, see e.g. [4, 14, 15]. Most interesting is the appearance of unexpected collective coherent phenomena in different systems. They are similar to the phenomenon of Dicke superradiance [16] which is known in optics for many years. It has been shown, moreover, that the reorganization of the spectrum of the system under the influence of the coupling to the environment at a critical value of the control parameter occurs globally over the whole energy range of the spectrum [14]. It takes place by a cooperative action of all states and the length scale diverges as well as the degree of non-Hermiticity of the Hamiltonian. It has been shown further that the reordering of the spectrum corresponds, indeed, to a second-order phase transition [14], justifying the notation dynamical phase transition. The states below and beyond the DPT are non-analytically connected. This method is shown to describe also phase transitions in, e.g., biological systems [17].

The calculation of the eigenvalues $\mathcal{E}_i$ and eigenfunctions $\Phi_i$ of the non-Hermitian Hamiltonian $H$ hits upon some mathematically non-trivial problems due to the existence of singular
points in the continuum. At these points, two eigenvalues coalesce and the two corresponding eigenfunctions are linearly dependent \[10\]. The geometric phase of these points differs from the Berry phase of a diabolic point by a factor 2. These singular points, called usually *exceptional points* (EPs), are well-known in mathematics \[18\]. Their meaning for the dynamics of open quantum systems and the behavior of the two eigenfunctions at an EP is however studied only recently. Numerical results for the eigenvalues and eigenfunctions of \( \mathcal{H} \) under the influence of an EP in a concrete system are obtained, e.g., for atoms \[19, 20\], for the transmission through quantum dots \[21–24\] and for charge transport in molecular networks \[25\]. In the early papers, the EPs are called mostly *branch points in the complex plane* or *double poles of the S matrix*. Phase transitions in open quantum systems which are associated with the formation of long-lived and short-lived states according to \[14\], are related to EPs first in \[26\]. More recent results can be found in the review \[10\]. The drawback of all these studies is the unsolved question how different EPs influence one another at high level density and how they are related to a DPT.

The eigenfunctions of a non-Hermitian operator \( \mathcal{H} \) are biorthogonal according to \( \langle \Phi_i^* | \mathcal{H} | \Phi_j \rangle = \mathcal{E}_i \delta_{ij} \) (where \( \mathcal{E}_i \) is a complex eigenvalue of \( \mathcal{H} \)). They have to be normalized therefore by means of \( \langle \Phi_i^* | \Phi_j \rangle \) which is a complex number (in difference to the norm \( \langle \Phi_i | \Phi_j \rangle \) which is a real number). In order to guarantee a smooth transition from the description of an open quantum system to an almost (and eventually really) closed one, the eigenfunctions of \( \mathcal{H} \) should be normalized according to \( \langle \Phi_i^* | \Phi_j \rangle = \delta_{ij} \). This is possible only by the additional requirement \( \text{Im}(\langle \Phi_i^* | \Phi_j \rangle) = 0 \). This condition implies that the relation between the phases of the two states \( i \) and \( j \) is, generally, not rigid: far from an EP, the two wavefunctions are (almost) orthogonal to one another in (nearly) the same manner as the eigenfunctions of a Hermitian operator while they become linearly dependent in approaching an EP \[10\] such that the biorthogonality of them cannot be neglected. This is quantitatively expressed by the phase rigidity \( r_i \equiv \langle \Phi_i^* | \Phi_j \rangle / \langle \Phi_i | \Phi_j \rangle \) which is reduced in approaching an EP, \( r_i \to 0 \). Here, the environment can put its information into the system by aligning states of the system with states of the environment, i.e. by enhancing their decay width.

The phase rigidity of the eigenfunctions and its reduction near to the singular EP is the most interesting value when a realistic quantum system is described by a Schrödinger equation with non-Hermitian Hamiltonian. Since the environment is able to change the spectroscopic properties of the system only if \( r_i < 1 \), an EP may influence strongly the
dynamics of an open quantum system. This is in contrast to a closed system described by a Hermitian operator and rigid phases \( r_i = 1 \) of its eigenfunctions. In \([21, 22]\), the correlation between non-rigid phases of the eigenfunctions \( \Phi_i \) of the non-Hermitian Hamiltonian in the neighborhood of an EP and the transmission through a quantum dot is demonstrated in calculations for a special quantum dot. The enhancement is a collective effect caused by \( r_i < 1 \) for many levels \( i \) in a certain finite parameter range. It has been shown further \([10]\) that the Schrödinger equation of the system contains nonlinear terms when \( r_i < 1 \), i.e. in the neighborhood of EPs. In contrast to the usual calculations, it is therefore not necessary to introduce nonlinear terms into the Schrödinger equation by hand. They are part and parcel of the non-Hermitian quantum physics, and appear only in the vicinity of EPs (where \( r_i < 1 \)).

Recently, PT symmetric non-Hermitian Hamiltonians are studied the eigenvalues of which are real in a broad parameter range \([27]\) (here P and T stand for the parity and time operators, respectively). PT symmetry breaking causes the eigenvalues of the Hamiltonian to become complex as shown theoretically \([27]\) as well as experimentally \([28–30]\). The meaning of EPs and their relation to PT-symmetry breaking is studied, e.g. \([31–34]\). Less studied is the question whether or not PT symmetry breaking can be considered to be a DPT in the sense described above.

It is the aim of the present paper to study the meaning of the mathematical non-trivial properties of non-Hermitian operators for the physics of open quantum systems. The mathematical properties are the existence of singular points (EPs), the reduced phase rigidity \( (r_i) \) in their vicinity and the appearance of nonlinear terms in the Schrödinger equation due to \( r_i < 1 \). The physical observable effects are DPTs known to appear at high level density, where the ranges of influence of different EPs overlap. To find an answer to the question how these different EPs cause a DPT in a physical system is the main interest of the present paper. Most of these studies have to be performed numerically, what is an expression of the well-known fact that the states below and beyond a phase transition are non-analytically connected. We restrict ourselves, in our paper, to these coherent (collective) effects induced into an open quantum system by its embedding into an environment, which are very robust. Besides these effects exist, of course, also other effects arising from the coupling between system and environment, which are of decoherent nature. We will not consider them in the present paper since they do not cause any global new features of the system such as e.g. a
DPT, in which we are interested.

In our calculations we use a schematic model to simulate typical features of open quantum systems induced coherently by the environment. The results obtained are generic. The basic formalism used by us, is worked out in nuclear physics many years ago \[35\] where it is, however, never used without introducing statistical assumptions for the individual states. In contrast, we consider the individual eigenvalues $E_i$ and eigenfunctions $\Phi_i$ of the non-Hermitian Hamiltonian $\mathcal{H}$. In particular, we are interested in the influence of the singular EPs onto these values. The eigenvalues show level repulsion and width bifurcation at a certain critical parameter value in the usual manner. The eigenfunctions characterize the parameter range over which the influence of the EPs can be seen and how different EPs influence one another.

First, we consider a two-level system with real, complex and imaginary coupling coefficients between system and environment. Then the crossing of three states is obtained analytically and their relevance for physical processes is discussed shortly. In the following section we consider numerically the influence of a nearby state onto an EP. All numerical studies are performed for open quantum systems as well as for those with PT symmetry. The obtained results are discussed in Sect. \[VI\]. In the first part (Sect. \[VI A\]) we discuss the relation between reduced phase rigidity near to an EP and the appearance of nonlinear terms in the Schrödinger equation. In the second part (Sect. \[VI B\]), we address the problem of the relation between EPs and DPTs in systems with more than two states. We are able to justify the restriction of the parameter dependence of the Hamiltonian $\mathcal{H}$ to its non-Hermitian part in order to describe the most important global spectroscopic redistribution processes, i.e. the main features of the DPT. In the last part (Sect. \[VI C\]), we discuss the results for PT symmetric systems and their relation to those obtained in open quantum systems. Some conclusions are drawn in the last section.
II. CROSSING OF TWO STATES IN AN OPEN QUANTUM SYSTEM WITH SYMMETRIC NON-HERMITIAN HAMILTONIAN

A. Basic equations, Hamiltonian near an exceptional point

In an open quantum system, the discrete states described by a Hermitian Hamiltonian $H^B$, are embedded into the continuum of scattering wavefunctions, which exists always and can not be deleted. Due to this fact the discrete states turn into resonance states the lifetime of which is usually finite. The Hamiltonian $H$ of the system which is embedded into the environment, is non-Hermitian. Its eigenvalues are complex and provide not only the energies of the states but also their lifetimes (being inverse proportional to the widths).

The non-Hermitian Hamiltonian of an open quantum system reads \[10\]

$$
H = H^B + V_{BC} G^{(+)} C V_{CB}
$$

where $V_{BC}$ and $V_{CB}$ stand for the interaction between system and environment and $G^{(+)}_C$ is the Green function in the environment. The so-called internal (first-order) interaction between two discrete states $i$ and $j$ is involved in $H^B$ while their external (second-order) interaction via the common environment is described by the last term of (1). Generally, the coupling matrix elements that determine the external interaction of two states consist of the principal value integral

$$
\text{Re} \langle \Phi^B_i | H | \Phi^B_j \rangle - E_i^B \delta_{ij} = \frac{1}{2\pi} \mathcal{P} \int_{E_c}^{E'_c} dE' \frac{\gamma^0_i \gamma^0_j}{E - E'}
$$

which is real, and the residuum

$$
\text{Im} \langle \Phi^B_i | H | \Phi^B_j \rangle = -\frac{1}{2} \gamma^0_i \gamma^0_j
$$

which is imaginary \[10\]. Here, the $\Phi^B_i$ and $E_i^B$ are the eigenfunctions and (discrete) eigenvalues, respectively, of the Hermitian Hamiltonian $H^B$ which describes the states in the subspace of discrete states without any interaction of the states via the environment. The $\gamma^0_{ic} \equiv \sqrt{2\pi} \langle \Phi^B_i | V | \xi^E_c \rangle$ are the (energy-dependent) coupling matrix elements between the discrete states $i$ of the system and the environment of scattering wavefunctions $\xi^E_c$. The $\gamma^0_{kc}$ have to be calculated for every state $i$ and for every channel $c$ (for details see \[10\]). When $i = j$, (2) and (3) give the selfenergy of the state $i$. The coupling matrix elements (2) and (3) (by adding $E_i^B \delta_{ij}$ in the first case) are often simulated by complex values $\omega_{ij}$, e.g. \[36, 37\].
In order to study the interaction of two states via the common environment it is convenient to start from two resonance states (instead of two discrete states). Let us consider, for example, the symmetric $2 \times 2$ matrix

$$\mathcal{H}^{(2)} = \begin{pmatrix} \varepsilon_1 \equiv e_1 + i\frac{\gamma_1}{2} & \omega_{12} \\ \omega_{21} & \varepsilon_2 \equiv e_2 + i\frac{\gamma_2}{2} \end{pmatrix}$$

with $\gamma_i \leq 0$. The diagonal elements of (4) are the two complex eigenvalues $\varepsilon_i$ ($i = 1, 2$) of a non-Hermitian operator $\mathcal{H}^0$. That means, the $e_i$ and $\gamma_i$ denote the energies and widths, respectively, of the two states when $\omega_{ij} = 0$. The $\omega_{12} = \omega_{21} \equiv \omega$ stand for the coupling matrix elements of the two states via the common environment which are, generally, complex due to (2) and (3). The selfenergy of the states is assumed to be included into the $\varepsilon_i$. The Hamiltonian $\mathcal{H}^{(2)}$ allows to consider the properties of the system near to and at an EP because here the distance between the two states, that coalesce at the EP, relative to one another is much smaller than that relative to the other states of the system. Note that the coupling matrix elements $\gamma^0_{kc}$ have the dimension of square root of energy while the widths $\gamma_k$ of the individual eigenstates have, of course, the dimension of energy.

**B. Eigenvalues of $\mathcal{H}^{(2)}$**

The eigenvalues of $\mathcal{H}^{(2)}$ are

$$\mathcal{E}_{i,j} \equiv E_{i,j} + \frac{i}{2} \Gamma_{i,j} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm Z ; \quad Z \equiv \frac{1}{2} \sqrt{\left(\varepsilon_1 - \varepsilon_2\right)^2 + 4 \omega^2}$$

where $E_i$ and $\Gamma_i$ stand for the energy and width, respectively, of the eigenstate $i$. When the energy detuning of the two levels is varied, different behaviors of the eigenvalues (5) will be observed which depend on the coupling strength $\omega$ between the states and their environment. Generally, resonance states with nonvanishing widths $\Gamma_i$ repel each other in energy according to $\text{Re}(Z)$ while the widths bifurcate according to $\text{Im}(Z)$. The transition from level repulsion to width bifurcation is studied numerically in e.g. [38]. The two states cross when $Z = 0$. This crossing point is an EP according to the definition of Kato [18]. Here, the two eigenvalues coalesce, $\mathcal{E}_1 = \mathcal{E}_2$.

According to [5], two interacting discrete states (with $\gamma_1 = \gamma_2 = 0$ and $e_1 \neq e_2$) avoid always crossing since $\omega \equiv \omega_0$ and $\varepsilon_1 - \varepsilon_2$ are real in this case and the condition $Z = 0$ can
not be fulfilled,
\[(e_1 - e_2)^2 + 4 \omega_0^2 > 0 \, . \tag{6}\]

In this case, the EP can be found only by analytical continuation into the continuum. This situation is called usually avoided crossing of discrete states. It holds also for narrow resonance states if \(Z = 0\) cannot be fulfilled due to the small widths of the two states. The physical meaning of this result is very well known since many years: the avoided crossing of two discrete states at a certain critical parameter value \([39, 40]\) means that the two states are exchanged at this point, including their populations (population transfer).

When \(\omega = i \omega_0\) is imaginary, \(Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 + i(e_1 - e_2)(\gamma_1 - \gamma_2) - 4\omega_0^2} \, \tag{7}\)
is complex. The condition \(Z = 0\) can be fulfilled only when \((e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 = 4\omega_0^2\) and \((e_1 - e_2)(\gamma_1 - \gamma_2) = 0\), i.e. when \(\gamma_1 = \gamma_2\) (or when \(e_1 = e_2\)). In this case
\[\begin{align*}
(e_1 - e_2)^2 - 4\omega_0^2 = 0 & \quad \rightarrow \quad e_1 - e_2 = \pm 2\omega_0 \, , \quad (8)
\end{align*}\]
and two EPs appear. It holds further
\[\begin{align*}
(e_1 - e_2)^2 > 4\omega_0^2 & \quad \rightarrow \quad Z \in \mathbb{R} \, \tag{9} \\
(e_1 - e_2)^2 < 4\omega_0^2 & \quad \rightarrow \quad Z \in \mathbb{I} \, \tag{10}
\end{align*}\]
independent of the parameter dependence of the \(e_i\). In the first case, the eigenvalues \(\mathcal{E}_i = E_i + i/2 \Gamma_i\) differ from the original values \(\varepsilon_i = e_i + i/2 \gamma_i\) by a contribution to the energies and in the second case by a contribution to the widths. The width bifurcation starts in the very neighborhood of one of the EPs and becomes maximum in the middle between the two EPs. This happens at the crossing point \(e_1 = e_2\) where \(\Delta \Gamma/2 \equiv |\Gamma_1/2 - \Gamma_2/2| = 4\omega_0\). A similar situation appears when \(\gamma_1 \approx \gamma_2\), see numerical results in Sect. [D]. The physical meaning of this result is completely different from that discussed above for discrete and narrow resonance states. It means that different time scales may appear without any enhancement of the coupling strength to the continuum (for details see [41]).

The cross section can be calculated by means of the \(S\) matrix \(\sigma(E) \propto |1 - S(E)|^2\). For an isolated resonance, it gives the well-known Breit-Wigner line shape according to
\[S = 1 + i \frac{\Gamma_1}{E - E_1 - \frac{1}{2} \Gamma_1} \, \tag{11}\]
where $E$ is the energy and $E_1$ and $\Gamma_1$ are defined in Eq. (5). This expression can be rewritten as

$$S = \frac{E - E_1 + \frac{i}{2} \Gamma_1}{E - E_1 - \frac{i}{2} \Gamma_1}$$

(12)

which is explicitly unitary when the energy dependence of the $E_i$ and $\Gamma_i$ is taken into account [10]. Extending the problem to that of two closely neighboring resonance states that are coupled to one common continuum of scattering wavefunctions the unitary representation (12) of the $S$ matrix reads (up to a background term) [10]

$$S = \frac{(E - E_1 + \frac{i}{2} \Gamma_1) (E - E_2 + \frac{i}{2} \Gamma_2)}{(E - E_1 - \frac{i}{2} \Gamma_1) (E - E_2 - \frac{i}{2} \Gamma_2)}.$$

(13)

In this expression, the influence of an EP onto the cross section is contained in the eigenvalues $\mathcal{E}_i = E_i + i/2 \Gamma_i$ of $\mathcal{H}^{(2)}$. Reliable results can be obtained therefore also when an EP is approached and the $S$ matrix has a double pole at the parameter value corresponding to the EP. Here, the line shape of the two overlapping resonances is described by

$$S = 1 + 2i \frac{\Gamma_d}{E - E_d - \frac{i}{2} \Gamma_d} - \frac{\Gamma_d^2}{(E - E_d - \frac{i}{2} \Gamma_d)^2}$$

(14)

by rewriting (13), where $E_1 = E_2 \equiv E_d$ and $\Gamma_1 = \Gamma_2 \equiv \Gamma_d$. It deviates from the Breit-Wigner line shape of an isolated resonance due to interferences between the two resonances. The first term of (14) is linear (with the factor 2 in front) while the second one is quadratic. As a result, two peaks with asymmetric line shape appear in the cross section (for a numerical example see Fig. 9 in [43]).

C. Eigenfunctions of $\mathcal{H}^{(2)}$

The eigenfunctions of a non-Hermitian $\mathcal{H}$ must fulfill the conditions $\mathcal{H} |\Phi_i\rangle = \mathcal{E}_i |\Phi_i\rangle$ and $\langle \Psi_i| \mathcal{H} = \mathcal{E}_i \langle \Psi_i|$ where $\mathcal{E}_i$ is an eigenvalue of $\mathcal{H}$ and the vectors $|\Phi_i\rangle$ and $\langle \Psi_i|$ denote its right and left eigenfunctions, respectively. When $\mathcal{H}$ is a Hermitian operator, the $\mathcal{E}_i$ are real, from which we arrive at the well-known relation $\langle \Psi_i| = |\Phi_i\rangle$, and the eigenfunctions can be normalized by using the expression $\langle \Phi_i| \Phi_j\rangle$. For the symmetric non-Hermitian Hamiltonian $\mathcal{H}^{(2)}$, however, we have $\langle \Psi_i| = \langle \Phi_i^\dagger|$. This means, the eigenfunctions are biorthogonal and have to be normalized by means of $\langle \Phi_i^\dagger| \Phi_j\rangle$. This is, generally, a complex value, in contrast to the real value $\langle \Phi_i| \Phi_j\rangle$ of the Hermitian case. To smoothly describe the transition from a
closed system with discrete states, to a weakly open one with narrow resonance states, we normalize the $\Phi_i$ according to [10]

$$\langle \Phi_i^* | \Phi_j \rangle = \delta_{ij} \quad (15)$$

(for details see sections 2.2 and 2.3 of [10]). It follows

$$\langle \Phi_i | \Phi_i \rangle = \text{Re} \left( \langle \Phi_i | \Phi_i \rangle \right) ; \quad A_i \equiv \langle \Phi_i | \Phi_i \rangle \geq 1 \quad (16)$$

and

$$\langle \Phi_i | \Phi_{j \neq i} \rangle = i \text{Im} \left( \langle \Phi_i | \Phi_{j \neq i} \rangle \right) = -\langle \Phi_{j \neq i} | \Phi_i \rangle \quad (17)$$

At an EP $A_i \to \infty$ and $|B_j^i| \to \infty$. The $\Phi_i$ contain (like the $E_i$) global features that are caused by many-body forces induced by the coupling $\omega_{ik}$ of the states $i$ and $k \neq i$ via the environment. They contain moreover the self-energy contributions of the states $i$ due to their coupling to the environment.

At the EP, the eigenfunctions $\Phi_i^{\text{cr}}$ of $\mathcal{H}^{(2)}$ of the two crossing states are linearly dependent from one another,

$$\Phi_1^{\text{cr}} \rightarrow \pm i \Phi_2^{\text{cr}} ; \quad \Phi_2^{\text{cr}} \rightarrow \mp i \Phi_1^{\text{cr}} \quad (18)$$

according to analytical as well as numerical and experimental studies, see Appendix of [41], section 2.5 of [10] and Figs. 4 and 5 in [33]. That means, the wavefunction $\Phi_1$ of the state 1 jumps, at the EP, via the wavefunction $\Phi_1 \pm i \Phi_2$ of a chiral state to $\pm i \Phi_2$ [44].

The Schrödinger equation with the non-Hermitian operator $\mathcal{H}^{(2)}$ is equivalent to a Schrödinger equation with $\mathcal{H}^0$ and source term [45]

$$(\mathcal{H}^0 - \varepsilon_i) |\Phi_i\rangle = - \begin{pmatrix} 0 & \omega_{ij} \\ \omega_{ji} & 0 \end{pmatrix} |\Phi_j\rangle \equiv W|\Phi_j\rangle \quad (19)$$

Due to the source term, two states are coupled via the common environment of scattering wavefunctions into which the system is embedded, $\omega_{ij} = \omega_{ji} \equiv \omega$. The Schrödinger equation (19) with source term can be rewritten in the following manner [45],

$$(\mathcal{H}^0 - \varepsilon_i) |\Phi_i\rangle = \sum_{k=1,2} \langle \Phi_k | W | \Phi_i \rangle \sum_{m=1,2} \langle \Phi_k | \Phi_m \rangle |\Phi_m\rangle \quad (20)$$
According to the biorthogonality relations (16) and (17) of the eigenfunctions of $\mathcal{H}^{(2)}$, (20) is a nonlinear equation. Most important part of the nonlinear contributions is contained in

$$\langle \mathcal{H}^0 - \varepsilon_n \rangle |\Phi_n\rangle = \langle \Phi_n | W | \Phi_n \rangle |\Phi_n\rangle^2 |\Phi_n\rangle.$$  

(21)

The nonlinear source term vanishes far from an EP since $\langle \Phi_k | \Phi_k \rangle \to 1$ and $\langle \Phi_k | \Phi_{l \neq k} \rangle = -\langle \Phi_{l \neq k} | \Phi_k \rangle \to 0$ as follows from the normalization (15). Thus, the Schrödinger equation with source term is linear far from an EP, as usually assumed. It is however nonlinear in the neighborhood of an EP.

The biorthogonality of the eigenfunctions $\Phi_k$ of the non-Hermitian operator $\mathcal{H}^{(2)}$ is determined quantitatively by the ratio

$$r_k \equiv \frac{\langle \Phi_k^* | \Phi_k \rangle}{\langle \Phi_k | \Phi_k \rangle} = A_k^{-1}.$$  

(22)

Usually $r_k \approx 1$ for decaying states which are well separated from other decaying states (according to the fact that Hermitian quantum physics is a good approach at low level density). The situation changes however completely when an EP is approached:

(i) When two levels are distant from one another, their eigenfunctions are (almost) orthogonal, $\langle \Phi_k^* | \Phi_k \rangle \approx \langle \Phi_k | \Phi_k \rangle \equiv A_k \approx 1$.

(ii) When two levels cross at the EP, their eigenfunctions are linearly dependent according to (18) and $\langle \Phi_k | \Phi_k \rangle \equiv A_k \to \infty$.

These two relations show that the phases of the two eigenfunctions relative to one another change dramatically when the crossing point (EP) is approached. We call $r_k$, defined by (22), the phase rigidity of the eigenfunction $\Phi_k$. Generally $1 \geq r_k \geq 0$. The non-rigidity $r_k$ of the phases of the eigenfunctions of $\mathcal{H}^{(2)}$ follows directly from the fact that $\langle \Phi_k^* | \Phi_k \rangle$ is a complex number (in difference to the norm $\langle \Phi_k | \Phi_k \rangle$ which is a real number) such that the normalization condition (15) can be fulfilled only by the additional postulation $\text{Im} \langle \Phi_k^* | \Phi_k \rangle = 0$ (what corresponds to a rotation).

When $r_k < 1$, an analytical expression for the eigenfunctions as function of a certain control parameter can, generally, not be obtained. The non-rigidity $r_k < 1$ of the phases of the eigenfunctions of $\mathcal{H}^{(2)}$ in the neighborhood of EPs is the most important difference between the non-Hermitian quantum physics and the Hermitian one. Mathematically, it causes nonlinear effects in quantum systems in a natural manner, as shown above. Physically,
it gives the possibility that one of the states of the system aligns at (or near to) the EP with the common environment and receives, by this, a large width. This alignment is nothing but a quantitative measure of the influence of the environment onto the spectroscopic properties of the system \[10\].

It is meaningful to represent the eigenfunctions $\Phi_i$ of $\mathcal{H}^{(2)}$ in the set of basic wavefunctions $\Phi_i^0$ of $\mathcal{H}^0$

$$\Phi_i = \sum_{j=1}^{N} b_{ij} \Phi_j^0 ; \quad b_{ij} = |b_{ij}| e^{i \theta_{ij}} .$$

Also the $b_{ij}$ are normalized according to the biorthogonality relations of the wavefunctions $\{\Phi_i\}$. The angle $\theta_{ij}$ can be determined from $\tan(\theta_{ij}) = \text{Im}(b_{ij})/\text{Re}(b_{ij})$.

It should be remarked here that the eigenfunctions $\Phi_k$ of $\mathcal{H}^{(2)}$ represent only the localized part of the resonance wavefunction inside the system. The wavefunction of the resonance state $k$ in the whole function space of discrete and scattering states contains additionally a “tail” due to the coupling to the scattering wavefunctions, see \[10\].

**D. Numerical results for two states**

In our calculations, the mixing coefficients $b_{ij}$, defined in \[23\], of the wavefunctions of the two states due to their avoided crossing are not calculated. The fact that the two wavefunctions are mixed in a finite parameter range around the critical value of their avoid crossing \[45\] can be simulated by assuming a Gaussian distribution \[36, 37, 46\]

$$\omega_{i \neq j} = \omega e^{-(e_i - e_j)^2}$$

for the coupling coefficients. The results reproduce very well those obtained numerically exact in \[45\] for two levels and real coupling $\omega \[36, 37, 46\]$. Further, the selfenergies of the states are assumed, in our calculations, to be included into the $\varepsilon_i$.

Let us first consider the $2 \times 2$ matrix \[4\] with $e_1 = 1 - \frac{a}{2}$, $e_2 = a$ and with $\gamma_i \ (i = 1, 2)$ and $\omega_{12} = \omega_{21} \equiv \omega$ independent of $a$. For illustration, we show in Fig. \[1\] the eigenvalue trajectories $E_i$ and $\Gamma_i/2$ and in Fig. \[2\] the mixing coefficients $b_{ij} = |b_{ij}| e^{i \theta_{ij}}$ of the eigenfunctions of $\mathcal{H}^{(2)}$ in the neighborhood of an EP. The calculations are performed with real, complex and imaginary coupling coefficients $\omega$, see upper, middle and lower row, respectively, of Figs. \[1\] and \[2\]
The upper (middle) rows of Figs. 1 and 2 with real (complex) \( \omega \) and \( \gamma_1 \neq \gamma_2 \) show an EP at the critical parameter value \( a = a_{cr} \). Here the eigenvalue trajectories cross and \( |b_{ij}| \to \infty \). The lower row is calculated with imaginary \( \omega \) and \( \gamma_1 = \gamma_2 \). Here, two EPs appear that are related to one another. Also here \( |b_{ij}| \to \infty \) at every EP.

The critical parameter range has a finite extension at both sides of the EPs. When \( \omega \) is imaginary, the critical parameter range includes both EPs and their vicinity. Between the two EPs the eigenfunctions are strongly mixed (1:1) with one another. Beyond the critical parameter region, the eigenvalues trajectories \( E_i(a) \) approach the trajectories \( \varepsilon_j(a) \) after exchange of \( i \) and \( j \).

The main difference of the eigenvalue trajectories with real to those with imaginary coupling coefficients \( \omega \) are related to the relations (6) to (10) obtained analytically and discussed in Sect. 11B. For real and complex \( \omega \) and \( \gamma_1 \neq \gamma_2 \), the results show one EP (when the condition \( Z = 0 \) is fulfilled, see Fig. 1 upper and middle rows). This EP is isolated from other EPs, generally, at low level density. In the case of imaginary \( \omega \) and \( \gamma_1 \approx \gamma_2 \), however, two related EPs appear (Fig. 1 lower row). Between these two EPs, the widths \( \Gamma_i \) bifurcate: the width of one of the two states increases by varying \( a \) although the coupling strength \( \omega \) between system and environment remains constant.

Interesting are also the phases of the eigenfunctions in the neighborhood of an EP. As can be seen from Fig. 2 (right panel) the phases of all components of the eigenfunctions jump at the EP either by \(-\pi/4\) or by \(+\pi/4\). That means the phases of both eigenfunctions jump in the same direction by the same amount. Thus, there is a phase jump of \(-\pi/2\) (or \(+\pi/2\)) when one of the eigenfunctions passes into the other one at the EP. This result is in agreement with (18). It holds true for real as well as for imaginary \( \omega \) as can be seen from Fig. 2.

The position of an isolated EP can always be found by varying a third parameter. For example, with \( e_1 = 1 - \frac{a}{2} + r \cos \theta; \ e_2 = a + r \sin \theta \) one EP appears in any case in the parameter range \( 0 \leq \theta \leq \pi \). The results obtained in the neighborhood of and at this EP show the same characteristic features as those in Figs. 1 and 2 around the crossing point (EP) of the eigenvalue trajectories, the eigenfunctions are mixed and \( |b_{ij}| \to \infty \) at the EP. The phase jumps are of the same type as those shown in Fig. 2 confirming the relation (18) between the two eigenfunctions at the EP also by these calculations.

Figs. 1 and 2 show further that an EP has a strong influence onto its neighborhood. This
FIG. 1: Energies $E_i$ (left panel) and widths $\Gamma_i/2$ (right panel) of $N = 2$ states coupled to a common channel as a function of $a$. Parameters: $e_1 = 1 - 0.5 a$; $e_2 = a$; $\gamma_1/2 = -0.5$ (a,b); -0.5505 (c,d); -0.6 (e,f); $\gamma_2/2 = -0.6$; $\omega = 0.05$ (a,b); 0.025 (1+i) (c,d); 0.05 i (e,f). The dashed lines in (a, c, e) show $e_i(a)$.

Fact is important from a physical point of view since it determines the dynamics of an open quantum system:

(i) The wavefunctions of the two crossing states are mixed and the phases of the wavefunctions of the two states relative to one another depend on the parameter $a$ in a finite parameter range in the neighborhood of the EP. The reduction of the phase rigidity $r_k$ (corresponding to [22]) allows one of the states to align to the states of the environment, i.e. to receive a large width, while the other state almost decouples from the environment. Thus, the spectroscopic properties of the states will be changed near to an EP under the influence of the environment.

(ii) When the interaction of the two states via the environment is imaginary and the
FIG. 2: Mixing coefficients $b_{ij} = |b_{ij}|e^{i\theta_{ij}}$ of $N = 2$ states coupled to a common channel as a function of $a$. The parameters are the same as in Fig. 1.

widths of both states are similar to one another ($\gamma_1 \approx \gamma_2$), width bifurcation occurs between the two EPs according to (8) and (10) without any enhancement of the coupling strength to the environment. The phases jump at the two EPs in different directions and the eigenvalues approach the original values only beyond the two EPs.

While the mathematical properties of the eigenvalues of $\mathcal{H}^{(2)}$ are studied in many papers for isolated EPs, their influence onto the vicinity of the EPs and onto the eigenfunctions is investigated in only a few papers, see e.g. the review [10]. As can be seen from Fig. 2 the areas of influence of different EPs overlap at high level density and new characteristic features arise in the system which can not be described by relating them to a certain number of isolated EPs.
III. CROSSING OF TWO STATES IN PT SYMMETRIC SYSTEMS WITH NON-
SYMMETRIC NON-HERMITIAN HAMILTONIAN

A. Basic equations, PT-symmetric Hamiltonian

An example for a non-symmetric non-Hermitian Hamiltonian is the Hamiltonian used
for the description of a PT symmetric system. As has been shown in [47–50], the quantum
mechanical Schrödinger equation and the optical wave equation in PT symmetric optical lat-
tices are formally equivalent. Complex PT symmetric structures can be realized by involving
symmetric index guiding and an antisymmetric gain/loss profile.

The main difference of these optical systems to open quantum systems consists in the
asymmetry of gain and loss in the first case while the states of an open quantum system can
only decay (Im(ε<sub>1,2</sub>) < 0 and Im(ℰ<sub>1,2</sub>) < 0 for both states). Thus, the modes involved in the
non-Hermitian Hamiltonian in optics appear in complex conjugate pairs while this is not
the case in an open quantum system. As a consequence, the Hamiltonian for PT symmetric
structures in optical lattices may have real eigenvalues in a large parameter range [32].

The 2 × 2 non-Hermitian Hamiltonian may be written, in this case, as [28, 29, 31]

\[
H_{PT} = \begin{pmatrix} e^{-i\frac{\gamma}{2}} & w \\ w^* & e^{i\frac{\gamma}{2}} \end{pmatrix},
\]

(25)

where \( e \) stands for the energy of the two modes, \( \pm \gamma \) describes gain and loss, respectively, and
the coupling coefficients \( w \) stand for the coupling of the two modes via the lattice. When
the PT symmetric optical lattices are studied with vanishing gain, the Hamiltonian reads

\[
H'_{PT} = \begin{pmatrix} e^{-i\frac{\gamma}{2}} & w \\ w^* & e \end{pmatrix}.
\]

(26)

In realistic systems, \( w \) in (25) and (26) is mostly real (or almost real).

B. Eigenvalues of the PT-symmetric Hamiltonian

The eigenvalues of the Hamiltonian (25) differ from (5),

\[
ℰ_{PT}^{\pm} = e \pm \frac{1}{2} \sqrt{4|w|^2 - \gamma^2} \equiv e \pm Z_{PT}.
\]

(27)
A similar expression is derived in [28, 29]. Since $e$ and $\gamma$ are real, the $\mathcal{E}_{\pm}^{PT}$ are real when $4|w|^2 > \gamma^2$. Under this condition, the two levels repel each other in energy what is characteristic of discrete interacting states. When the interaction $w$ is fixed, the level repulsion decreases with increasing $\gamma$. When $4|w|^2 = \gamma^2$ the two states cross. Here, $\mathcal{E}_{\pm}^{PT} = e$ and $\gamma = \pm \sqrt{4|w|^2}$. With further increasing $\gamma$ and $4|w|^2 < \gamma^2$ ($w$ fixed for illustration), width bifurcation (PT symmetry breaking) occurs and $\mathcal{E}_{\pm}^{PT} = e \pm \frac{1}{2} \sqrt{\gamma^2 - 4|w|^2}$.

These relations are in accordance with (6) to (10) for open quantum systems. Since $|w|$ is real, two EPs exist according to

$$4|w|^2 = (\pm \gamma)^2. \quad (28)$$

Further

$$\gamma^2 < 4|w|^2 \rightarrow Z_{PT} \in \Re \quad (29)$$

$$\gamma^2 > 4|w|^2 \rightarrow Z_{PT} \in \Im \quad (30)$$

independent of the parameter dependence of $\gamma$ and of the ratio Re($w$)/Im($w$).

In the case of the Hamiltonian (26), the eigenvalues read

$$\mathcal{E}_{\pm}^{PT} = e - i \frac{\gamma}{4} \pm \frac{1}{2} \sqrt{4|w|^2 - \gamma^2} = e - i \frac{\gamma}{4} \pm Z_{PT}. \quad (31)$$

We have level repulsion as long as $4|w|^2 > \frac{\gamma^2}{4}$. While level repulsion decreases with increasing $\gamma$, the loss increases with increasing $\gamma$. At the crossing point, $\mathcal{E}_{\pm}^{PT} = e - i \frac{\gamma}{4}$. With further increasing $\gamma$ and $4|w|^2 \ll \frac{\gamma^2}{4}$

$$\mathcal{E}_{\pm}^{PT} \rightarrow e - i \frac{\gamma}{4} \pm i \frac{\gamma}{4} = \begin{cases} e \\ e - i \frac{\gamma}{4} \end{cases}. \quad (32)$$

The two modes (32) behave differently. While loss in one of them is large, it is almost zero in the other one. Thus, only one of the modes effectively survives. Equation (32) corresponds to high transparency at large $\gamma$.

Further, two EPs exist according to

$$4|w|^2 = (\pm \gamma/2)^2 \quad (33)$$

and

$$\frac{\gamma^2}{4} < 4|w|^2 \rightarrow Z_{PT}' \in \Re \quad (34)$$

$$\frac{\gamma^2}{4} > 4|w|^2 \rightarrow Z_{PT}' \in \Im. \quad (35)$$
In analogy to (28) up to (30) these relations are independent of the parameter dependence of $\gamma$ and of the ratio $\text{Re}(w)/\text{Im}(w)$.

Thus, the difference between the eigenvalues $\mathcal{E}_i$ of $\mathcal{H}^{(2)}$ of an open quantum system and the eigenvalues of the Hamiltonian of a PT symmetric system consists, above all, in the fact that the $\mathcal{E}_i$ depend on the ratio $\text{Re}(\omega)/\text{Im}(\omega)$ while the $\mathcal{E}_i^{PT}$ and $\mathcal{E}_i'^{PT}$ are independent of $\text{Re}(w)/\text{Im}(w)$.

There exist however similarities between the two cases. Interesting is the comparison of the eigenvalues $\mathcal{E}_i$ of $\mathcal{H}^{(2)}$ obtained for imaginary non-diagonal matrix elements $\omega$, with the eigenvalues of (25) or (26) for real $w$. In both cases, there are two EPs. In the first case, the energies $E_i$ are constant and the widths $\Gamma_i$ bifurcate between the two EPs. This situation is characteristic of an open quantum system at high level density with complex (almost imaginary) $\omega$, see Eqs. (8) to (10). In the second case however the difference $|E_1 - E_2|$ in the energies increases (level repulsion) while the widths $\Gamma_i$ of both states are equal in the parameter range between the two EPs, see (28) to (30) and (33) to (35), respectively. Between the two EPs, level repulsion causes the two levels to be distant from one another and $w$ is expected to be (almost) real according to (2) and (3). Formally, the role of energy and width is exchanged in the two cases.

It should be underlined here that the non-Hermitian Hamiltonian describing an open quantum system may also have real eigenvalues if certain conditions are fulfilled. Such a case is studied already more than 80 years ago \cite{51}, later in atomic physics \cite{19, 20, 52, 53} and also in other systems such as double quantum dots \cite{10, 24}. The so-called bound states in the continuum are caused by width bifurcation due to which the width of the long-lived resonance state may approach zero. This mechanism is different from that considered here since it creates real eigenvalues of the non-Hermitian Hamiltonian only at a few special parameter values.

C. Eigenfunctions of the PT-symmetric Hamiltonian

The eigenfunctions of the two $2 \times 2$ Hamiltonians (25) and (26) show the same characteristic features as those of the Hamiltonian (4). The eigenmodes can be normalized, generally, according to (15) where $\Phi_i^{PT}$ ($\Phi_i'^{PT}$) denotes the right eigenmode. Far from an EP, the eigenfunctions $\Phi_i^{PT}$ ($\Phi_i'^{PT}$) are orthogonal to one another. The orthogonality is lost
in approaching the crossing point of the eigenvalue trajectories. Here, the modes show some skewness according to (16). As in the case of open quantum systems, the phase rigidity \( r_\lambda \) can be defined according to (22). It varies between 1 and 0 and is a quantitative measure for the skewness of the modes. Thus, the phases of the eigenmodes of the non-Hermitian Hamiltonians (25) and (26) are not rigid, and spectroscopic redistribution processes may occur under the influence of the environment (lattice).

The eigenfunctions \( \Phi_i^{PT} \) of \( \mathcal{H}_{PT} \) (and \( \Phi_i'^{PT} \) of \( \mathcal{H}'_{PT} \)) can be represented in a set of basic wavefunctions in full analogy to the representation of the eigenfunctions \( \Phi_i \) of \( \mathcal{H}^{(2)} \) in (23). They contain valuable information on the mixing of the wavefunctions under the influence of the non-diagonal coupling matrix elements \( w \) and \( w^* \) in (25) and (26), respectively, as well as its relation to EPs.

D. Numerical results for a PT-symmetric system

In realistic systems, the non-diagonal matrix elements \( w \) of the non-Hermitian Hamiltonians (25) and (26) are real (or almost real) as follows from the level repulsion occurring between the two EPs (see above, Sect. III B). Nevertheless, we did some calculations also for complex and imaginary \( w \) (results are not shown).

According to (25) and (26), the energies \( e_i \) and widths \( \gamma_i \) of the two states are the same. We choose \( e_1 = e_2 = e \) independent of the parameter \( a \) in the considered region and \( \gamma_i \) (gain and loss) to be parameter dependent.

In Fig. 3, the eigenvalues \( E^{PT} \) and \( E'^{PT} \) of (25), left panel, and (26), right panel, are shown. They do not depend on the ratio \( \text{Re}(w)/\text{Im}(w) \). The corresponding eigenfunctions shown in the lower part of Fig. 3 are calculated with real \( w \). In contrast to the eigenvalues, they depend on the ratio \( \text{Re}(w)/\text{Im}(w) \).

As can be seen from the results, the EPs cause PT symmetry breaking. The level repulsion appearing between the two EPs is accompanied by a complete (1:1) mixing of the eigenfunctions. The mixing vanishes only far from the EP (Figs. 3e and f). Hence, PT symmetry will be broken in a finite vicinity of an EP.

This result is in full analogy to the results shown in Figs. 1e,f and 2e for open quantum systems with imaginary \( \omega \) where width bifurcation is accompanied by a complete mixing of the eigenfunctions between the two EPs and the mixing vanishes only far from the EPs.
FIG. 3: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of the eigenfunctions $\Phi_i$ of $N = 2$ states coupled to a common channel as a function of $a$. Parameters: $e = 0.5$; $w = 0.05$; $\gamma_1/2 = -0.05 a$; and $\gamma_2 = -\gamma_1$ (left panel); $\gamma_2 = 0$ (right panel). In order to illustrate the symmetry properties, the results are shown for positive as well as for negative values $a$. The dashed lines in (a, b) show $e$.

Further numerical studies showed that also the phases of the eigenfunctions jump always by $\pi/4$ at the EPs (not shown in Fig. 3).

We state therefore the following. The results of Fig. 3 obtained from calculations for PT symmetric systems with real $w$ are formally similar to those received for open quantum systems with imaginary coupling coefficients $\omega$ (lower row in Figs. 1 and 2). In the two cases, the role of energy and width is formally exchanged.

In order to receive a better understanding of the role of PT symmetry in Fig. 3, we performed another calculation with slightly different energies $e_i$ of the two states. The
results shown in Fig. 4 are very similar to those in Fig. 3. The differences are of the same type as those obtained in corresponding calculations for open quantum systems with $\omega = 0.05i$, see Fig. 1 (left panel) in [38] with $\gamma_1 = \gamma_2$ and Fig. 2 (left panel) in [38] with $\gamma_1 \approx \gamma_2$, respectively.

IV. CROSSING OF $N = 3$ STATES IN AN OPEN QUANTUM SYSTEM

We consider a system consisting of three levels coupled to one common continuum of scattering wavefunctions. The Hamiltonian reads

$$\mathcal{H}^{(3)} = \begin{pmatrix}
\epsilon_1 = \epsilon_1 + \frac{i\omega_1}{2} & \omega_{12} & \omega_{13} \\
\omega_{21} & \epsilon_2 = \epsilon_2 + \frac{i\omega_2}{2} & 0 \\
\omega_{31} & 0 & \epsilon_3 = \epsilon_3 + \frac{i\omega_3}{2}
\end{pmatrix}$$

(36)
where $\omega_{23} = \omega_{32} = 0$ is assumed by using the doorway picture $[38]$. For simplicity, we assume that all coupling coefficients are equal to one another, $\omega_{ij} = \omega$. They may be real or complex.

According to $[54]$, the eigenvalues can be determined from the polynomial

$$\lambda^3 + R\lambda^2 + S\lambda + T = 0 \quad (37)$$

with

$$R = -(\epsilon_1 + \epsilon_2 + \epsilon_3)$$
$$S = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 - 2\omega^2$$
$$T = \omega^2\epsilon_2 + \omega^2\epsilon_3 - \epsilon_1\epsilon_2\epsilon_3 \quad (38)$$

Eq. (37) can be transformed to

$$y^3 + py + q = 0 \quad (39)$$

with

$$y = \lambda + \frac{R}{3} \quad (40)$$
$$p = \frac{3S - R^2}{3} \quad (41)$$
$$q = \frac{2R^3}{27} - \frac{RS}{3} + T \quad (42)$$

Using (41), (42) and

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} \quad (43)$$
$$v = -\frac{p}{3u} \quad (44)$$

the three eigenvalues read

$$\lambda_1 = \frac{u+v}{2} - \frac{R}{3} \quad (45)$$
$$\lambda_2 = -\frac{u+v}{2} - \frac{R}{3} + \frac{u-v}{2}i\sqrt{3} \quad (46)$$
$$\lambda_3 = -\frac{u+v}{2} - \frac{R}{3} - \frac{u-v}{2}i\sqrt{3} \quad (47)$$
When \( y = 0 \) and \( q = 0 \) the three eigenvalues cross according to (40) and (39). Here

\[
\lambda_{1,2,3} = -\frac{R}{3} = \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}
\]  

(48)

where the definition (38) of \( R \) is used. Further,

\[
u = \sqrt{\frac{p}{3}} \quad \text{and} \quad v = -\sqrt{\frac{p}{3}} = -u .
\]  

(49)

Equation (48) shows that the crossing point is determined by the distance \( R \) between the three complex energies \( \epsilon_i \) of the states. The way on which the crossing point is approached plays therefore an important role. When approaching the crossing point by means of \( u = -v \rightarrow 0 \), the eigenvalues (45) to (47) at the crossing point read

\[
\lambda_1 = -\frac{R}{3}
\]  

(50)

\[
\lambda_2 \rightarrow -\frac{R}{3} + iu\sqrt{3}
\]  

(51)

\[
\lambda_3 \rightarrow -\frac{R}{3} - iu\sqrt{3}
\]  

(52)

The two eigenvalues \( \lambda_2 \) and \( \lambda_3 \) form an EP when \( u \rightarrow 0 \). The eigenvalue \( \lambda_1 \) is not influenced by this condition. It plays the role of an observer state that can be exchanged at the EP with one of the other two states, but does not participate in the spectroscopic redistribution processes taking place at the EP. When however \( u = 0 \) and the EP is approached by varying another parameter (being independent of \( u \)), all three states participate in the spectroscopic redistribution processes.

The conclusion of this analytical study is the following. According to the parameter varied in approaching the EP, two different types of EPs exist when the eigenvalues of three states cross. The differences between these two types of EPs consist in the following:

- when the crossing point is approached by \( u \rightarrow 0 \), two states show the signatures of an EP while the third state is an observer state that, generally, may exchange with one of the other states.

- when the crossing point is approached by keeping constant \( u = 0 \) and varying another independent parameter, the three states form together a common EP, at which all three states participate in the spectroscopic redistribution processes.
In any case, the spectroscopic redistribution caused by the crossing of three states is a nonlinear process according to (37).

Similar results are obtained when a PT symmetric system with the Hamiltonian $H_{PT}$ or $H'_{PT}$ is considered. In (36) and the following expressions, $\omega$ has to be replaced by $|w|$ and gain ($\gamma_i > 0$) has to be taken into account additionally to loss ($\gamma_i < 0$).

The detailed study of the crossing point of all three eigenvalues of the Hamiltonian (36) is of high mathematical interest. For example, it allows us to determine the geometric phase related to the respective crossing point. As a result it is, in any case, different from that of an EP for crossing of two levels. This holds true even in the case when the third state is an observer state since this state may exchange with each of the two other states. The geometric phase depends therefore on the number $N$ of crossing (or nearly crossing) states.

However, an EP is a point in the continuum (being of measure zero) and its influence onto physical observables can be seen only in its vicinity. The same holds true for the crossing point of three states. Any third state will cross with one of the two states in the neighborhood of the EP whose wavefunctions differ from those of the original states (because they are mixed with one another near to the EP). Thus, all three states are influenced by the crossing points of two states, the EPs. For each of them, the corresponding geometric phase is well defined: it differs from the Berry phase of a diabolic point by a factor 2, see [10]. While the properties of isolated EPs are studied experimentally as well as theoretically in many papers, the influence of a nearby state onto an EP is less understood, up to now. In any case, the areas of influence of different EPs overlap and nonlinear terms will appear not only at the EPs themselves but, above all, in a finite vicinity of them. Analytical studies are restricted due to these nonlinearities. When combined with numerical studies they are expected to give reliable results, see the next section.

V. INFLUENCE OF A NEARBY STATE ONTO AN EXCEPTIONAL POINT

A. Open quantum system with three nearby states

The results of numerical calculations shown in Figs. 5 to 8 are performed with the non-Hermitian Hamiltonian (36) for an open quantum system and with (24) for the coupling coefficients $\omega$. We assume that only the energies $e_i$ of the states depend on a parameter $a$
FIG. 5: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of the eigenfunctions $\Phi_i$ of $N = 2$ (left panel a,c,e) and $N = 3$ (right panel b,d,f) states of an open system coupled to a common channel by $\omega = 0.01$ as a function of $a$. Parameters: $e_1 = 1 - 1/2 \ a; \ e_2 = a; \ e_3 = -1/3 + 3/2 \ a$ (b,d,f); $\gamma_1/2 = \gamma_2/2 = -0.495; \ \gamma_3/2 = -0.485$ (b,d,f). The dashed lines in (a,b) show $e_i(a)$.

while the widths $\gamma_i$ are constant in the considered parameter range. In all calculations the position of the EP is determined independently by means of the $2 \times 2$ Hamiltonian (4). We show the eigenvalues $\mathcal{E}_i = E_i + i/2 \ \Gamma_i$ and the mixing coefficients $|b_{ij}|$ of the eigenfunctions $\Phi_i$ of (36) for real coupling coefficient $\omega$ (Fig. 5) as well as for imaginary $\omega$ (Fig. 6) and compare them with those of (4) in the neighborhood of the EP.

In the left panel of Fig. 5 (real $\omega$), we see the typical avoided crossing of two levels with an exchange of the wavefunctions around the critical parameter value $a = a_{cr} = 2/3$ (see Sect. II D). The right panel of Fig. 5 shows the crossing of the two levels with a third
FIG. 6: The same as Fig. 5 but $\omega = 0.01i$ and $\gamma_3/2 = -0.4853$ (b,d,f).

one. Here, two intersections can be seen: the first one at $a = a_1 = 0.65775 < a_{cr}$ and the second one at $a = a_2 = 0.675 > a_{cr}$. At the two intersections $|b_{ij}| > 1$ what is a clear hint to an EP. The eigenfunctions are mixed not only in the parameter range between the two intersections but also in a comparable large parameter range beyond them. As can be seen from the eigenvalue pictures Fig. 5.b,d the third state interacts with the two other ones and exchanges with them.

Fig. 6 shows the results obtained with imaginary $\omega$. In the left panel, we see two EPs at $a = a_1 = 0.6539 < a_{cr}$ and $a = a_2 = 0.6774 > a_{cr}$, respectively. The appearance of two EPs and the width bifurcation between them is characteristic for two levels having the same (or similar) widths $\gamma_i$ (see Sect. III D). In the right panel of Fig. 6 a third level crosses the energy of the two states in the parameter range in which the widths of the two states
FIG. 7: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of $N = 3$ states of an open system coupled to one common channel as a function of $s$ with $e_3 = s - 1/3 + 3/2a$. $\omega = 0.01$; $a = a_{cr} = 2/3$ (left); $a = a_1 = 0.6539 < a_{cr}$ (mid); $a = a_2 = 0.675 > a_{cr}$ (right). The $e_1, e_2, \gamma_1, \gamma_2, \gamma_3$ are the same as in Fig. 5. The dashed lines in (a,b,c) show $e_i(s)$. The bifurcation. Width bifurcation occurs now with participation of all three states (Fig. 6.d). Altogether, the width bifurcation is stronger than in the two-level case. At $a = a_{cr}$, the difference between the largest and smallest values of $\Gamma_i/2$ is 0.02 in the two-level case and 0.03 in the three-level case. Here, the widths of the three levels appear in two groups: the largest width is much larger than the two other widths which, on their part, differ by a comparable small value from one another. This result is in agreement with the analytical result that the widths bifurcate at every eigenvalue crossing.

As in the case of real coupling coefficients $\omega$ (Fig. 5), the eigenfunctions obtained with imaginary $\omega$ (Fig. 6) are mixed strongly in the parameter range between the two EPs as well as beyond this range. The mixing is symmetrical around the critical parameter value
FIG. 8: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of $N = 3$ states of an open system coupled to one common channel as a function of $s$ with $e_3 = s - 1/3 + 3/2 a$. $\omega = 0.01i$; $a = a_{cr} = 2/3$ (left); $a = a_1 = 0.6539 < a_{cr}$ (mid); $a = a_2 = 0.6774 > a_{cr}$ (right). The $e_1, e_2, \gamma_1, \gamma_2, \gamma_3$ are the same as in Fig. 6. The dashed lines in (a,b,c) show $e_i(s)$. $a = a_{cr}$ in both cases and the large $|b_{ij}|$ point clearly to EPs.

In order to see more clearly the influence of a third level onto an avoided level crossing or an EP, we keep fixed all the parameters used in Figs. [5] and [6] with the exception of the energy $e_3$ of the third state. We choose $e_3 = s - 1/3 + 3/2 a$ and trace the eigenvalues and eigenfunctions of (36) as a function of $s$. At the value $a = a_{cr} = 2/3$, the three levels with energies $e_i$ cross (see Figs. [5b] and [6b]).

We show the results for different $a$ in Figs. [7] (real $\omega$) and [8] (imaginary $\omega$). The energies $E_i$ show level repulsion when $\omega$ is real while the widths $\Gamma_i$ of all three states bifurcate when $\omega$ is imaginary. The eigenfunctions $\Phi_i$ are mixed strongly nearby the intersection points and the EPs, respectively. Some mixing remains at large values of $|s|$. The eigenvalue and
eigenfunction pictures are completely symmetric around $s = 0$ when $a = a_{cr}$ while this is not the case when $a \neq a_{cr}$, neither for real nor for imaginary $\omega$. These last cases are, of course, more realistic than the symmetric one and will appear in physical systems. Comparing the results for different $a$, one sees the sensitive parameter dependence of the results in the critical region.

All the results shown in Figs. 5 to 8 are obtained for the case that the widths of two states are equal, $\gamma_1 = \gamma_2$. Very similar results are obtained when $\gamma_1 \approx \gamma_2$. Further, the results of calculations with complex $\omega$ (not shown) show characteristic features of the calculations with real as well as of those with imaginary $\omega$ (compare Figs. 1 and 2).

B. PT symmetric system with three nearby states

In Figs. 9 and 10 we show the influence of a nearby state onto an EP when the system is PT symmetric and the coupling coefficient $w$ is real. Instead of width bifurcation occurring in the case of an open quantum system and imaginary coupling coefficient $\omega$, we see now the separation of the states from one another in energy. However, the third state has a large influence on the eigenvalue trajectories as well as on the parameter range in which the eigenfunctions of $H$ are mixed.

In Fig. 9 the widths $\gamma_i$ of the states 2 and 3 increase with $a$, while that of state 1 decreases with $a$. Due to this, there are altogether four EPs, at $a \approx -4; -2; +2; +4$. At $a \approx -4$, the widths $\Gamma_i$ of the two states with $\gamma_i \propto a$ separate from one another while their energies $E_i$ coalesce. The opposite happens at $a \approx -2$: the width $\Gamma_i$ of the state with $\gamma_i \propto -a$ coalesces with that of one of the two states with $\gamma_i \propto a$ while the two states separate in energy at these critical $a$ value. The figure is symmetrical around $a = 0$. The eigenfunctions of $H$ are mixed in a parameter range, which is much larger than the range of mixing without the third state.

In the left panel of Fig. 9 the influence of the third state onto the eigenvalues and eigenfunctions of $H$ is shown by varying its distance $s$ to the two crossing states. The calculation is performed with $a = 0$ and the results are symmetrical around $s = 0$.

In Fig. 10 we show results for the case that the width of the third state is independent of the parameter $a$. It is $\gamma_1/2 = -\gamma_2/2 = -0.05a$, $\gamma_3/2 = 0.05$ (left panel) and $\gamma_1/2 = -0.05a; \gamma_2/2 = \gamma_3/2 = 0$ (right panel). The figures are similar to one another. In both
FIG. 9: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of $N = 3$ states of a PT symmetric system coupled to one common channel as a function of $a$ (left panel) and $s$ (right panel). The parameters are $e_1 = e_2 = 0.5$ and $e_3 = 0.487$ (left panel), $e_3 = 0.487 + s$; $a = 0$ (right panel); $\gamma_1/2 = -0.05a$; $\gamma_2/2 = 0.05a$; $\gamma_3/2 = 0.05a$; $w = 0.05$. The dashed lines in (a,b) show $e_i(a)$. In these cases, we have two EPs. The wavefunctions are strongly mixed between the two EPs as well as in a finite parameter range beyond them.
FIG. 10: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of $N = 3$ states of a PT symmetric system coupled to one common channel as a function of $a$. The parameters are $e_1 = e_2 = e_3 = 0.5$ and $\gamma_1/2 = -0.05a$; $\gamma_2/2 = 0.05a$; $\gamma_3/2 = 0.05$ (left panel); $\gamma_1/2 = -0.05a$; $\gamma_2/2 = \gamma_3/2 = 0$ (right panel); $w = 0.05$.

VI. DISCUSSION OF THE RESULTS

A. Nonlinear Schrödinger equation and reduced phase rigidity of the wavefunctions nearby an EP

In textbooks of quantum mechanics it is claimed that the Schrödinger equation is linear, that the Hamiltonian describing a quantum system is Hermitian and has real eigenvalues that provide us the energies $E_i^B$ of the states, that the wavefunctions of the states are
normalized according $\langle \Phi^B_i | \Phi^B_j \rangle = \delta_{ij}$ and that the phases of the wavefunctions relative to one another are rigid. These statements are proven by the successful description of very many experimental data obtained by studying quantum systems at low level density during the about 80 years quantum mechanics. They should be kept in force in any extension of quantum mechanics.

The eigenvalues of a Hermitian operator are real, and the finite lifetime of the states is calculated mostly by assuming that particles may tunnel from the system to the environment of scattering states. By this, the feedback from the environment onto the system is neglected in calculating the finite lifetime of the states of a decaying quantum system. The numerical results obtained theoretically do agree usually well with the experimental values under the condition that the states are isolated from one another. In such a case, the interaction of two states via the environment is a second-order process, see Eq. (1), and is usually small in comparison with the direct (first-order) interaction of two states contained in $H^B$ of (1).

The situation changes at high level density. Here, it is no longer justified to neglect the feedback from the environment on the system. Instead, the system has to be considered to be open, and its coupling to the environment of scattering states has explicitly to be taken into account from the very beginning. As a consequence, the Hamiltonian (1) of the open system is non-Hermitian. The eigenvalues of $\mathcal{H}$ are complex and provide not only the energies $E_i$ of the states but also their lifetimes being inverse proportional to the widths $\Gamma_i$, see Sect. II A. Also the eigenfunctions $\Phi_i$ are complex. They are biorthogonal and normalized by means of the complex value $\langle \Phi^*_i | \Phi_j \rangle$.

From a mathematical point of view, the description of an open quantum system results in some nontrivial problems. Most important is the existence of singular points, the EPs, in the environment (continuum of scattering wavefunctions), see Sects. III B III C. Here, not only two eigenvalues coalesce but also the two related eigenfunctions are linearly dependent from one another according to (18).

From a physical point of view, any extension of quantum physics should, with decreasing level density, pass smoothly into the standard Hermitian quantum physics. Hence, $\langle \Phi^*_i | \Phi_j \rangle$ should be normalized to the value $\delta_{ij}$. This normalization results in the fact that the phases of every two states are, generally, not rigid relatively to one another because $\langle \Phi^*_i | \Phi_j \rangle$ is a complex number and may become real only by the additional condition $\text{Im} \langle \Phi^*_i | \Phi_j \rangle = 0$, see the discussion around (22). Approaching an EP, the phase rigidity $r_i \equiv \frac{\langle \Phi^*_i | \Phi_j \rangle}{\langle \Phi_i | \Phi_j \rangle} \to 0$,
the environment is able to put information into the system and the second-order term of
the non-Hermitian Hamiltonian $\mathcal{H}$, Eq. (1), may become important, even the dominant
one. As a result, a DPT may take place under the influence of the environment, see the
following section VI B. Further, due to the biorthogonality of the eigenfunctions $\Phi_i$ of the
non-Hermitian Hamiltonian $\mathcal{H}$ and $r_i \leq 1$, the Schrödinger equation is, generally, nonlinear,
see (21). The nonlinear contributions vanish far from EPs. They can, however, not be
neglected near to them.

All the numerical results shown in the present paper for the eigenvalues and, above all, for
the eigenfunctions of two states (Figs. 1 to 4) illustrate the role of reduced phase rigidity and
non-linear contributions in the Schrödinger equation describing an open quantum system.
In all cases, the influence of the EPs occurs in a finite parameter range around them.

Some years ago, calculations for the transmission through quantum dots have been per-
formed in the tight-binding approach [21, 22]. These calculations for realistic systems have
demonstrated the correlation between reduced phase rigidity, on the one hand, and en-
hanced transmission, on the other hand, in systems with a small as well with a large number
of states. The results of the present paper show that these results are generic.

B. Crossing of three states and dynamical phase transitions

In the present paper, we studied the influence of a nearby state onto an EP. First we
considered the problem analytically for three different eigenvalues (Sect. IV). As a result,
the eigenvalues of all three states may coalesce in one point either by keeping intact an EP
(with all its signatures) formed by two of the states or by forming a new common singular
point. In any case, the geometric phase differs from that of an EP since the third state
may exchange with the other two states. A similar result for the geometric phase has been
obtained in [55]. We found further that it depends on the manner the crossing point is
approached which of the two possibilities will be realized.

However, these results cannot give any answer to the question how different EPs influence
one another. The problem is that every crossing point of eigenvalue trajectories is a point
in the continuum and therefore of measure zero. More important than the crossing points
themselves is therefore their influence onto the system properties in some finite parameter
range around an EP. All our calculations (Figs. 5 to 10) have shown generic features similar
to those discussed for two crossing states. The areas of influence of different EPs overlap and amplify, collectively, their impact onto physical values. The eigenfunctions $\Phi_i$ are mixed in the basic wavefunctions $\Phi_0^j$ (according to Eq. (23)) in a finite critical parameter range. The range is larger than in the 2-level case. Also the width bifurcation (when the $\omega$ are imaginary) is larger in the 3-level case than in the 2-level case (Fig. 6). It occurs stepwise and may explain, in this manner, the hierarchical trapping of resonance states found many years ago [56].

In the systems with three states, the various states lose their individual character in some critical parameter range, over which the distinct regions of influence of the various EPs overlap. The significance of the individual EPs is therefore lost, so that more than two states are unable to coalesce at a single point (recall that an EP is a point in the continuum and its influence on physical observables can be seen only in its vicinity). While in the former case of an isolated EP the two states are exchanged at the EP (in accordance with the predictions of Eq. (18)), in the latter case (with overlapping areas of influence of EPs), the states are no longer directly related to the original ones (at low level density). Instead a spectroscopic redistribution, caused by several EPs, takes place that is aimed at achieving a dynamical stabilization of the system by accumulation as much coupling strength between system and environment as possible onto just one specific state (in the one-channel case) of the localized system. At the same time, the remaining states decouple strongly from the environment. The phase rigidity of the eigenfunctions is found to be reduced over a relatively large parameter range, where also significant nonlinearities appear. These effects persist when the number of interacting states is increased beyond three. It should be underlined here that this process of stabilization of the system occurs with the participation of all states and that, in the two-level case, it corresponds to nothing but width bifurcation.

The common feature of the DPTs is the presence of EPs. This can be seen directly in the two-level case involving just a single EP. In the many-level case, the role of the EPs is somewhat hidden because the areas influenced by different EPs overlap and the positions of the EPs cannot be determined analytically. The numerical calculations show, however, clearly that they are responsible for the spectroscopic reordering processes which finally cause the DPT. This can be seen in Figs. 5 and 6 in which the eigenvalues and eigenfunctions of $\mathcal{H}$ are compared for the three-level case with those for the much simpler two-level case. According to these results, the non-Hermitian Hamilton operator (11) of the open quantum
system can be approximated, in the region of the DPT, quite well by

\[ \mathcal{H} = H^B - i\alpha VV^+ \]  

(53)

where \( VV^+ \) stands for the residuum (3) (and the principal value integral (2) is neglected due to its small value at high level density where the DPT occurs). Many calculations are performed, indeed, by using the approximation (53) and tuning the parameter \( \alpha \). The Hamiltonian (53) describes the main features of the spectroscopic reordering in the critical parameter region when starting from a certain distribution of the levels (and allowing for the non-avoided level crossings in the critical range of the parameter \( \alpha \) where the widths bifurcate). Although not directly seen from its structure, (53) relates these features to the existence of EPs as discussed above. This holds true not only for the two-level case, involving just a single EP, but also for the many-level problem, in which one must deal with several EPs whose areas of influence overlap with one another. In both cases, the common feature exhibited by these open systems is that their quantum dynamics exhibits non-analytically connected states on either side of the transition.

Summarizing, our results obtained for \( N > 2 \) states allow us to state that the basic features of a DPT can be seen already at a relatively small number of nearby states. They are caused by several EPs (the crossing points of two resonance states) the areas of influence of which overlap. Fermi’s golden rule holds below the DPT but is violated beyond it.

C. Role of PT symmetry in systems with gain and loss

During last about 15 years, the properties of non-Hermitian Hamiltonians with PT symmetry are studied in many papers, see the review [27]. The interest in this topic arises from the fact that the Hamiltonian of a PT symmetric system may have real eigenvalues in a large parameter range although it is non-Hermitian. After showing the equivalence of the quantum mechanical Schrödinger equation and the optical wave equation in PT symmetric optical lattices [47-50], complex PT symmetric structures have been realized by involving symmetric index guiding and an antisymmetric gain/loss profile [28-30]. The experimental results have shown PT symmetry breaking and, at the same time, its importance for providing new devices with interesting properties. The relation between PT symmetry breaking and a DPT according to Sect. is discussed in, e.g., [31]. Both phenomena are of the
FIG. 11: Energies $E_i$ (top), widths $\Gamma_i/2$ (mid) and mixing coefficients $|b_{ij}|$ (bottom) of $N = 2$ states of a quantum system with gain and loss which is coupled to one common channel, as a function of $a$. The parameters are $e_1 = 1 - a/2; e_2 = a/2$ and $\gamma_1/2 = -0.05; \gamma_2/2 = 0.05; \omega = 0.05$ (left panel); $\gamma_1/2 = -0.05; \gamma_2/2 = 0.0205; \omega = 0.05(1 + i)/\sqrt{2}$ (right panel) same type.

The main difference of these optical systems to open quantum systems consists in the asymmetry of gain and loss in the first case while the states of an open quantum system can only decay ($\text{Im}(\varepsilon_{1,2}) \leq 0$ and $\text{Im}(\mathcal{E}_{1,2}) \leq 0$). Thus, the modes involved in the non-Hermitian Hamiltonian in optics appear in complex conjugate pairs while this is not the case in a usual open quantum system. However, open quantum systems with gain and loss can easily be produced, even without any PT symmetry.

It is interesting to compare Fig. 3 with Figs. 1e, f and 2e. The last calculations are
performed for open quantum systems with imaginary non-diagonal matrix elements $\omega$ while the results of Fig. 3 are received for a PT symmetric system with real non-diagonal matrix elements $w$. The eigenvalues and eigenfunctions are very similar to one another if, formally, energy and width are exchanged. In both cases, we have two EPs. While in the first case the widths of the states bifurcate between the two EPs, the energies bifurcate in the second case in the corresponding parameter range. Contrary to these results, the energies coalesce in the first case and the widths coincide in the second case between the two EPs. However, the widths vanish and the eigenvalues are real in the whole parameter range between the two EPs only in Fig. 3.c.

In order to receive more information on the role of PT symmetry in quantum systems with gain and loss, we perform calculations with the Hamiltonian (4) but different signs for the two $\gamma_i$ (and $\omega = \omega_{12} = \omega_{21}$). In this case, the eigenvalues $\mathcal{E}_{i,j} \equiv E_{i,j} + \frac{1}{2} \Gamma_{i,j}$ are given by (5) with

$$Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 - \frac{1}{4} (\gamma_1 - \gamma_2)^2 + i (e_1 - e_2)(\gamma_1 - \gamma_2) + 4\omega^2}.$$  (54)

According to the condition $Z = 0$ for the appearance of an EP, we have one EP at the crossing point $a = a_{cr}$ of the two $e_i$ trajectories (where $e_1(a) = e_2(a)$), if $\gamma_1 = -\gamma_2$ is parameter independent and $\omega = |\gamma_i/2|$ is real. There is however no EP when $\omega$ is imaginary. If $\omega$ is complex and the widths $\gamma_i$ of the two states have different signs, there is also one EP. We show the corresponding numerical results with one EP in Fig. 11.

We underline here that the results of Fig. 11 are obtained by using the Hamiltonian (4) for a system with parameter independent loss and gain. As usual, the EP appears at the crossing point of the energy trajectories if $\omega$ is real. The system is not PT symmetric and shows the characteristic features of an open quantum system. A balance between gain and loss will provide PT symmetry, is however not necessary. Systems of this type will surely allow many different applications.

**VII. CONCLUSIONS**

Concluding we state the following. The present-day high resolution experimental studies require a description of quantum systems by taking into account their embedding into the continuum of scattering wavefunctions. This natural environment exists always. It can be
changed by means of external fields, however it cannot be deleted. This basic assumption of the description of open quantum systems which is used in the present paper, is proven experimentally: in semiconductor nanostructures (quantum point contacts) \[13\], the interaction between two remote quantum states is essentially mediated by the continuum. In the case of remote states, \(\text{Re}(\omega) \gg \text{Im}(\omega)\) according to (2) and (3), and the interaction \(\omega\) causes the well-known avoided level crossing phenomenon, which is found experimentally, indeed.

The coupling of the open quantum system to the environment causes the Hamiltonian to be non-Hermitian whose eigenvalues \(E_i\) are complex. They provide not only the energies \(E_i = \text{Re}(E_i)\) of the states of the system but also their lifetimes which are inverse proportional to the widths \(\Gamma_i/2 = \text{Im}(E_i)\). The feedback of the environment onto the system is involved in the non-Hermitian Hamiltonian \(\mathcal{H}\) as well as in its eigenvalues \(E_i\) and eigenfunctions \(\Phi_i\). At low level density or (and) weak coupling between system and environment, the feedback can be neglected, to a good approximation, and the non-Hermitian Hamiltonian \(\mathcal{H}\), Eq. (1), passes smoothly into the standard Hermitian Hamiltonian \(H^B\).

The coupling of the system to the environment of scattering wavefunctions entails some non-trivial mathematical problems. First of all, there are singular (exceptional) points in the continuum which influence strongly the dynamics of open quantum systems. They cause nonlinear terms in the Schrödinger equation (Sect. VI A) and non-rigid phases of the wavefunctions (Sect. VI A), as well as DPTs in open quantum systems beyond which Fermi’s golden rule is violated (Sect. VI B), and PT symmetry breaking in systems with balanced loss and gain (Sect. VI C). While nonlinear terms are included by hand in many present-day calculations for concrete systems, and DPTs and PT symmetry breaking are observed experimentally in different systems, the basic features of these phenomena are considered, for the first time, in the present paper.

According to the results of our schematic-model calculations presented in the present paper, the basic features of open quantum systems described by a non-Hermitian Hamiltonian are the following.

(i) Width bifurcation occurs due to \(\text{Im}(\omega)\). It is large in the case of closely spaced states of an open quantum system. Here, EPs may appear in the continuum as shown analytically as well as numerically in Sect. II. They cause nonlinear effects and are of crucial relevance for the dynamics of open quantum systems. It is necessary therefore to prove experimentally their existence by tracing the eigenvalues of \(\mathcal{H}\) as function of a certain parameter in a realistic
(ii) Information on the influence of EPs onto their vicinity in a realistic open quantum system can be reached best by studying the eigenfunctions of $\mathcal{H}$. Of special interest are the phases of the wavefunctions of the two crossing states relative to one another when the EP is approached. According to theory, the phases are not rigid (Sect. II C), and the phase rigidity $r_i \to 0$ at the EP. This result expresses the most characteristic feature of non-Hermitian quantum physics when applied to the description of realistic systems. It shows the mechanism (alignment of a state of the system to the scattering states of the environment and receiving, by this, a large width) due to which it is possible for the environment to put information into the system. This mechanism is strengthened when the system has more than two states. It is aimed at the stabilization of the system by accumulating almost the whole coupling strength between system and environment onto one state (in the one-channel case).

(iii) The influence of a nearby state onto an EP and its vicinity shows that the wavefunctions of all three states are strongly mixed with one another in a finite parameter range around the EP. The eigenvalues of the three eigenstates of $\mathcal{H}$ do not cross exactly in one point. By this, the EPs cause DPTs in realistic open quantum systems which can be described well by the simple non-Hermitian Hamiltonian (53).

(iv) The states of open many-body quantum systems are assumed, usually, to decay into the environment of scattering states. This fact corresponds to a loss of particles into the environment. Our understanding of open quantum systems will be enriched by studying systems with inclusion of gain. Of special interest for applications are systems in which both, loss and gain, are parameter independent, see Sect. VI C.

Experimental studies are necessary to prove these basic features of open quantum systems. The results will allow us, on the one hand, to receive a better understanding of DPTs and, on the other hand, to design new devices with desired properties.

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