NEW ZERO-FREE REGIONS FOR THE DERIVATIVES OF THE RIEMANN ZETA FUNCTION

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Abstract. The main aim of this paper is twofold. First we generalize, in a novel way, most of the known non-vanishing results for \( \zeta^{(k)}(s) \) by establishing the existence of an infinite sequence of regions in the right half-plane where these derivatives cannot have any zeros; and then, in the rare regions of the complex plane that do contain zeros of \( \zeta^{(k)}(s) \) (named “critical strips” in analogy with the classical case of \( \zeta(s) \)), we describe a unexpected phenomenon, which – especially for the hitherto-neglected high derivatives \( \zeta^{(k)}(s) \) – implies great regularities in their zero distributions. In particular, we prove sharp estimates for the number of zeros in each of these new critical strips, and we explain how they converge, in a very precise, periodic fashion, to their central, “critical” lines, as \( k \) increases. This not only shows that the zeros of \( \zeta^{(k)}(s) \) are not randomly scattered to the right of the line \( \sigma = \frac{1}{2} \), but that, in many respects, their two-dimensional distribution eventually becomes much simpler and more predictable than the one-dimensional behavior of the zeros of \( \zeta(s) \) on the line \( \sigma = \frac{1}{2} \).

1. Introduction

In this paper we investigate the distribution of zeros of higher derivatives of the Riemann zeta function. In order to put our main results in perspective, we first give a summary of some of the main results and conjectures in this area.

Let \( s = \sigma + it \). For all \( k \in \mathbb{N} \) the \( k \)-th derivative of the Riemann zeta function \( \zeta^{(k)}(s) \) is
\[
\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}, \quad \text{for } \sigma > 1, \tag{1}
\]
and can be extended to a meromorphic function on \( \mathbb{C} \), with a single pole (of order \( k \)) at the point \( s = 1 \). However, unlike \( \zeta(s) \) itself, the functions \( \zeta^{(k)}(s) \) have neither Euler products nor functional equations. Their non-trivial zeros do not lie on a line, but appear to be distributed (seemingly at random) to the right of the critical line \( \sigma = \frac{1}{2} \). In 1934 Speiser [7] was the first to show that the Riemann Hypothesis (denoted RH everywhere below) implies that \( \zeta'(s) \) has no zeros to the left of the critical line \( \sigma = \frac{1}{2} \). Unfortunately, for higher derivatives this particular property does not stay true. But, in 1974, Levinson and Montgomery [4] showed (again, assuming the RH) that the number of zeros of \( \zeta^{(k)}(s) \) in the left half-plane is always finite. More recently Yıldırım [14] proved that both \( \zeta''(s) \) and \( \zeta'''(s) \) have exactly one pair of non-trivial zeros with \( \sigma < 0 \), namely \( \zeta''(s) \) has a zero at approximately \(-0.35508433021 \pm 3.590839324398i\). He also showed that the RH implies that neither \( \zeta''(s) \) nor \( \zeta'''(s) \) have any zeros \( \rho \) with \( 0 < \Re(\rho) < \frac{1}{2} \).

2000 Mathematics Subject Classification. 11M26 (Primary), 11M06.

Key words and phrases. Riemann zeta-function, zeros, derivatives.
In regions to the right of the critical line, i.e. for $\sigma \geq \frac{1}{2}$, the total number of zeros of $\zeta^{(k)}(s)$ does not differ by much from the number of zeros of $\zeta(s)$. In fact, if we let $N(T)$ and $N_k(T)$ denote the number of such zeros $\rho$ with $0 \leq \Im(\rho) \leq T$ of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively, then according to Berndt [1]

$$N_k(T) = N(T) - \frac{T}{2\pi} \log 2 + O(\log T),$$  \hspace{1cm} (2)

where, by the classical Riemann-von Mangoldt formula (see Landau [3]),

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Most non-trivial zeros of $\zeta^{(k)}(s)$ are located close to the line $s = \frac{1}{2} + it$. Soundararajan [6] showed that, for $k = 1$, a positive portion of the zeros $\rho$ of $\zeta'(s)$ satisfies $\Re(\rho) < \frac{1}{2} + c/\log T$. Nevertheless many of the zeros of $\zeta^{(k)}(s)$ lie further to the right, even though their real parts can be bounded from above. For $k \geq 3$ such upper bounds were given by Spira [8] in 1965. They were improved by Verma and Kaur [12] (see Table 1).

**Table 1. Lower real bounds for zero-free regions in the right half-plane.**

| Author             | $\zeta$ | $\zeta'$ | $\zeta''$ | $\zeta^{(k)}$ for $k \geq 3$          |
|--------------------|---------|----------|-----------|----------------------------------------|
| Hadamard [2], de la Vallée-Poussin [11] | 1       | 3        | \(\zeta k + 2\) | \(\zeta^{(k)} k + 2\) |
| Titchmarsh [10]    |         |          |           |                                        |
| Spira [8]          |         |          |           |                                        |
| Verma & Kaur [12]  |         |          |           |                                        |
| Skorokhodov [5]    |         |          |           |                                        |

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In this paper we explicate some new, unexpected properties of the location of zeros of $\zeta^{(k)}(s)$ with $\frac{1}{2} \leq \Re(s) < 1.13588k + 2$. In particular we prove the existence of zero-free regions for $\zeta^{(k)}$ and show that the zeros exhibit a fascinating vertical periodicity between these zero-free regions, which we call critical strips in analogy to the critical strip of zeta. This enables us to give exact formulas for their number, while also proving that all zeros of $\zeta^{(k)}(s)$ inside them are simple. Figure 2 illustrates these phenomena for $\zeta^{(38)}(s)$.

**Figure 2.** Zeros of $\zeta^{(38)}(s)$ in $\mathbb{C}$, with zero-free regions (characterized by the dominance of $Q^{38}_M(s)$ for $M = 2$ and 3).

### 2. Statement of Main Results

In order to state our results precisely, we need to introduce some notation and definitions. Let $Q^k_n(s) := (\log n)^k/n^s$ denote the $n$-th term of the Dirichlet series (1) for $\zeta^{(k)}(s)$. All the previously known zero-free regions for $\zeta^{(k)}(s)$ have been obtained by finding solutions to

$$|\zeta^{(k)}(s)| = \left| \sum_{n=2}^{\infty} Q^k_n(s) \right| \geq Q^k_2(\sigma) - \sum_{n=3}^{\infty} Q^k_n(\sigma) > 0,$$

or some variation thereof (see [5, 10, 12]); that is, by finding the regions of the complex plane where the first non-zero term $Q^k_2(s)$ dominates all the other terms of the expansion (1) of $\zeta^{(k)}(s)$ (in the sense that $Q^k_2(s)$ is greater in modulus than the rest of the terms combined). Evidently, these conditions imply $\zeta^{(k)}(s) \neq 0$. However, $Q^k_2(s)$ is not always the dominant term; any other term can take this role. This is clear from the fact that $|Q^k_n(s)| = Q^k_n(\sigma)$ viewed as a function of $n$ has its global maximum at $n = e^{k/\sigma}$. Using this argument one can
show the existence of regions where $Q_n^k(s)$, $n \geq 2$ is the dominant term of $Q_k(s)$, which then provides us with a new zero-free region of $\zeta^{(k)}(s)$, for each $n$, for every sufficiently large $k$.

By $Q_n^k(s)$ we denote the term of $Q_k(s)$ which has the largest modulus. As we prove in Lemma 7, one important property is that if we fix some such $M$, then the moduli of the terms of $Q_k(s)$ grow, and on the right side of $Q_k(s)$ decrease, always in monotone fashion. Since no term $Q_k^M(s)$ can attain dominance on a line where its absolute value is equal to that of another term (and by the aforesaid property this can only happen when $Q_k^M(\sigma) = Q_{M+1}^k(\sigma)$ or $Q_k^M(\sigma) = Q_{M-1}^k(\sigma)$), it is reasonable to expect that the zeros of $\zeta^{(k)}(s)$ will be located close to these lines. For this purpose, we define

$$q_M := \log \left( \log \frac{M}{\log M+1} \right),$$

so that $Q_k^M(\sigma) = Q_{M+1}^k(\sigma)$ precisely at the line $\sigma = kq_M$. In particular, we have:

$q_2 \approx 1.13588, \quad q_3 \approx 0.808484, \quad q_4 \approx 0.668855$.

Note that $q_2$ is the constant of [12] that appears in Table 1.

In the above notation, our first main result can be stated as follows.

**Theorem 1.** (a) (The case of $M = 3$) Let $k \in \mathbb{N}$. We have $\zeta^k(s) \neq 0$ for

$$q_3k + 4 \log 3 \leq \sigma \leq q_2k - 2.$$

(b) (The case of $M > 3$) If $M \in \mathbb{N}$ and $M \geq 3$, then $\zeta^k(s) \neq 0$ for

$$q_M k + (M + 1) \log 3 \leq \sigma \leq q_{M-1} k - M \log 3.$$

For $k \geq 3$ and $M \geq 2$ we define the critical strip $S_k^M$ of $\zeta^{(k)}(s)$ as the region between the lines $\sigma = q_M k - (M + 1) \log 3$ and $\sigma = q_M k + (M + 1) \log 3$, as long as $q_{M+1} k + (M + 2) \log 3 < q_M k - (M + 1) \log 3$. The critical line of $S_k^M$ is given by $\sigma = q_M k$. A way to visualize the critical strips $S_k^M$ is to consider their location in the $\sigma k$-plane (see Figure 3). In this graphical representation, the wedge-shaped regions correspond to the zero-free regions, i.e. the regions of dominance of the terms $\frac{\log M}{M^k}$ (for $M = 2$ this is Verma and Kaur [12], for $M \geq 3$ it is new), while the critical strips $S_k^M$ are the narrow regions centered around the lines $\sigma = q_M k$ that separate the wedges. The tips of the wedges are at

$$k_M = \frac{(2M + 1) \log 3}{q_{M-1} - q_M},$$

which means that the first critical strips $S_1^k$ can be observed for all $k \geq 14$, the second $S_2^k$ for all $k \geq 41$, and the third $S_3^k$ for all $k \geq 87$.

It is an interesting corollary to Theorem 1 that, for the number $c(k)$ of critical strips of $\zeta^{(k)}(s)$ inside the region $1/2 \leq \sigma < q_2 k + 2$, we have

$$\frac{\sqrt{k}}{3 \log k} < c(k) < \frac{2\sqrt{k}}{\log k}.$$
If we also consider the imaginary parts of \( Q^k_M(q_Mk + it) = Q^k_{M+1}(q_Mk + it) \), then we obtain the solutions:
\[
t = (2j + 1) \frac{\pi}{\log(M + 1) - \log(M)}
\]
for \( j \in \mathbb{Z} \), showing that the precise location of the zeros \( \rho \) inside \( S^k_M \) should be close to
\[
k \cdot q_M + \frac{(2j + 1)\pi}{\log\left(\frac{M+1}{M}\right)} \cdot i
\]
for some \( j \in \mathbb{N} \). This suggests existence of an amazing vertical periodicity (in the limit) of the zeros of \( \zeta^{(k)}(s) \) at the critical lines, with the periods \( \frac{\pi}{\log(M+1) - \log(M)} \).

Although it is virtually impossible to give exact location of every transcendental zero in a given critical strip (and describe the way it approaches the limiting values with growing \( k \)), we are at least able to separate the zeros by horizontal line segments, whose imaginary parts lie between the values for \( t \) in (5). That is, we first establish that \( \zeta^{(k)}(s) \neq 0 \), for
\[
s = \sigma + \frac{2\pi j}{\log(M + 1) - \log M},
\]
where \( q_Mk - (M + 1)\log 3 \leq \sigma \leq q_Mk + (M + 1)\log 3 \), and then (with the help of Rouché’s theorem) we show that between every two consecutive lines that horizontally partition the critical strip \( S^k_M \) this way there is exactly one zero of \( \zeta^{(k)}(s) \). In other words:
Theorem 2. (a) Let \( j \in \mathbb{N} \). Then each rectangular region \( R \subset S_M^k \), consisting of all \( s = \sigma + it \) with
\[
q_M k - (M + 1) \log 3 < \sigma < q_M k + (M + 1) \log 3
\]
and
\[
\frac{2\pi j}{\log(M + 1) - \log(M)} < t < \frac{2\pi (j + 1)}{\log(M + 1) - \log(M)}
\]
contains exactly one zero of \( \zeta^{(k)}(s) \).

(b) For all \( M \geq 2 \) and \( k \in \mathbb{N} \), all the zeros of \( \zeta^{(k)}(s) \) inside \( S_M^k \) are simple.

Clearly, Theorem 2(a) can be converted into an exact formula for the number of zeros of \( \zeta^{(k)}(s) \) (for carefully chosen values of \( T \)) inside any given critical strip.

Corollary 3. Let \( N_M^k(T) \) denote the number of zeros \( \rho \) of \( \zeta^{(k)}(s) \) (with \( \Im(\rho) \leq T \)) inside the critical strip \( S_M^k \). Then, for all \( j \geq 1 \),
\[
N_M^k \left( \frac{2\pi j}{\log(M + 1) - \log(M)} \right) = j.
\]

Remark. As an immediate consequence of this result we have: For all \( k \geq 3 \), and all \( T > 0 \),
\[
N_M^k(T) = \frac{\log(M + 1) - \log(M)}{\pi} T + O(1).
\]
This implies that, for any given \( k \geq 3 \), the total number of zeros contained within all the critical strips is \( O(T) \), so always \( o(N_k(T)) \).

Finally, noticing – as we have in our last remark – that the important formula (5), that describes the vertical quasi-periodicity of zeros of \( \zeta^{(k)}(s) \), only contains \( M \), and is independent of \( k \), we realize that, with growing \( k \), the critical strips \( \{S_M^k\}_{k=2}^\infty \) can undergo a shift in one direction only: to the right, and with the length of the shift very close to \( q_M \) for each increment of \( k \). In other words, from Theorem 2 we can see that all zeros of \( \zeta^{(k)}(s) \) contained in a given critical strip \( S_M^k \) will keep shifting (almost) linearly, and with a (almost) fixed shift, the period, \( q_M \), to the right, as \( k \) grows to infinity.

A simple consequence of this observation is the following:

Conjecture 4. For all \( k \in \mathbb{N} \) there is a one-to-one correspondence between the non-trivial zeros of \( \zeta^{(k)}(s) \) and \( \zeta^{(k+1)}(s) \), where the zeros of \( \zeta^{(k+1)}(s) \) always stay to the right of the corresponding zeros of \( \zeta^{(k)}(s) \).

Remark. Spira [8] had already noticed that the zeros of \( \zeta'(s) \) and \( \zeta''(s) \) seem to come in pairs, where the zero of \( \zeta''(s) \) was always located to the right of the zero of \( \zeta'(s) \). With the help of extensive computations Skorokhodov [8] observed this behavior for higher derivatives. We conjecture not only the one-to-one correspondence, but also (for every fixed \( M \) – the existence of a quasi-lattice of zeros of \( \zeta^{(k)}(s) \), created as \( k = 1, 2, 3, \ldots \).

Remark. The zero-free regions obtained in Theorem 4 easily generalize to a large class of Dirichlet series. Since, in our proofs of the zero-free regions for \( \zeta^{(k)}(s) \), we only consider the absolute values of its coefficients, it follows that if \( L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \), and \( |a_M| \geq |a_n| \) for some \( M \geq 3 \) and all \( n \geq 2 \), then \( L^k(s) \neq 0 \) for \( q_M k + cM \leq \sigma \leq q_M k - c(M - 1) \), for some
computable constant $c > 0$. Our other results are slightly harder to extend and state in full generality. We relegate those investigations to a future project.

3. Two Auxiliary Lemmas

We consider the $\sigma k$–plane interpretation of Theorem 1. In general, the wedges in Figure 3 are the sets containing all points $(k, \sigma)$ that satisfy

$$q_M k + b_1 < \sigma < q_{M-1} k + b_2,$$

for some $M \in \mathbb{N}$ and constants $b_1$ and $b_2$. This implies that if $q_M k + b_1 \leq \sigma \leq q_{M-1} k + b_2$, then also

$$k \geq \frac{b_1 - b_2}{q_{M-1} - q_M},$$

with equality holding precisely when $k = k_M$, a point where the tip of the wedge is located. This fact will be of importance in the proof of (4) (see Corollary 6). The growth properties of $q_n$ play an important role in understanding the critical strips $S^k_m$:

**Lemma 5.** For all $n \geq 3$ we have

$$\frac{1}{\log n} \leq q_{n-1} \leq \frac{1}{\log(n-1)}.$$

**Proof.** In order to prove the lower bound, we write:

$$\alpha_{n-1} := \frac{\log(n-1)}{\log n} = 1 + \frac{\log(n-1) - \log n}{\log n},$$

$$\beta_{n-1} := \log(\alpha_{n-1}) = \log \left(1 + \frac{\log(n-1)}{\log n}\right) < \frac{\log(n-1)}{\log n},$$

the last inequality holds because $\log(1 + x) < x$, for all $x > -1$. The desired lower bound follows from $q_{n-1} = \beta_{n-1}/\log((n-1)/n)$. In order to prove the upper bound, we write:

$$q_{n-1} := \log \left(\log \left(\frac{\log(n-1)}{\log n}\right)\right) = \log \left(1 - \frac{\log(n-1)}{\log n}\right) = \log \left(\frac{1 - \log(n-1)}{\log n}\right)$$

$$= \frac{1}{\log n} + \frac{1}{2} \left(\frac{\log(n-1)}{\log n}\right)^2 + \frac{1}{3} \left(\frac{\log(n-1)}{\log n}\right)^3 + \cdots < \frac{1}{\log n} + \frac{2}{3(\log n)^2} < \frac{1}{\log(n-1)},$$

for all $n \geq 3$, giving us the desired bound. □

**Corollary 6.** For all $k$, the number $c(k)$ of critical strips of $\zeta(k)(s)$ is bounded by

$$\frac{\sqrt{k}}{3 \log k} < c(k) < \frac{2\sqrt{k}}{\log k}.$$  

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Proof. We count the number of wedges given by \( q_m^k (m + 1) \log 3 \leq \sigma \leq q_m^{k-1} k - m \log 3 \), which is equal to the number \( c(k) \) of critical strips \( S_m^k \). The tips of the wedges are at

\[
k_m = \frac{(2m + 1) \log 3}{q_m - q_{m-1}} \geq \frac{(2m + 1) \log 3}{\log(m-2)} - \frac{1}{\log m} = \frac{(2m + 1) \log 3}{\frac{\log(m-2)}{\log m} \cdot \log m} > \frac{1}{2} \log 3 \cdot m^2 (\log m)^2.
\]

Since every \( m \) that satisfies the above inequality corresponds to a unique critical strip \( S_m^k \), it follows that inverting this relationship will give us the wanted upper bound \([4]\) on \( c(k) \). For the intrinsically more interesting lower bounds, we similarly have:

\[
k_m = \frac{(2m + 1) \log 3}{q_m - q_{m-1}} \leq \frac{(2m + 1) \log 3}{\log(m-2)} - \frac{1}{\log m} - \frac{3}{2m(\log m)^2} < 6 \log 3 \cdot m^2 (\log m)^2,
\]

from which, by the same inversion, we obtain the desired result. \( \square \)

For a fixed \( M \geq 3 \) and \( k \in \mathbb{N} \), with the help of the above lemma we can now zoom in on the lines \( \sigma = q_M k \) (the expected critical lines), and prove that in their vicinity one has a monotonically (in the modulus) growth of terms of the Dirichlet series \( 1 \):

**Lemma 7.** Let \( M \) be fixed, let \( 2 \leq n < M - 1 \), and \( Q^k_n(\sigma) = \log \frac{x^n}{n} \) be the \( n \)-th term of \( 1 \). Then at the line \( \sigma = q_M k \), either \( Q^k_M(\sigma) \) or \( Q^k_{M-1}(\sigma) \) is the term with the largest modulus.

**Proof.** First: \( 0 = Q^k_M(\sigma) < Q^k_M(\sigma) < Q^k_{M-1}(\sigma) \), since for all \( k > 1 \), by Lemma \([5]\) we clearly have

\[
\frac{\log 2}{\log 3} < \left( \frac{2}{3} \right)^{q_M k} \iff Q^k_M(q_M k) = \left( \frac{\log 2}{2^{q_M}} \right)^k < \left( \frac{\log 3}{3^{q_M}} \right)^k = Q^k_{M} \]

Moreover, for \( \sigma = q_M k \), the function \( z(x) = \log^k x \) has a single maximum, precisely at:

\[
0 = z'(x) = \left( \frac{\log x}{x^{q_M}} \right)^k = k \left( \frac{\log x}{x^{q_M}} \right) \left( \frac{\log x}{x^{q_M}} \right) \left( \frac{\log x}{x^{q_M}} \right) = k \left( \frac{x^{q_M - 1} - q_M (\log x) x^{q_M - 1}}{x^{2q_M}} \right) \left( \frac{\log x}{x^{q_M}} \right) \left( \frac{\log x}{x^{q_M}} \right)
\]

i.e. at an \( x \) for which we have

\[
x^{q_M - 1} - q_M (\log x) x^{q_M - 1} = 0 \iff q_M = \frac{1}{\log x} \iff x = \exp(q_M^{-1}),
\]

which, by Lemma \([5]\) implies \( M - 1 < x \), and that proves the result. \( \square \)

### 4. Proof of Theorem \([4]\)

Now we are ready to prove our first main result. We will show that \( \zeta^{(k)}(s) \) has no zeros if \((k, \sigma)\) in the \( \sigma k \)-plane lies in one of the wedges given by an inequality of the form

\[
q_M k + b_1 \leq \sigma \leq q_{M-1} k + b_2
\]
for suitably chosen \( b_1, b_2 \in \mathbb{R} \). We choose \( b_1, b_2 \) such that these wedges are the regions where \( Q^k_M(s) = \log^k M \) is the dominant term (in the modulus) of \( \zeta^{(k)}(s) \). Everywhere hereafter we write \( H^k_M(s) \) for the “head” and \( T^k_M(s) \) for the “tail” of the series \( \zeta^{(k)}(s) \) split by \( Q^k_M(s) \):

\[
H^k_M(s) := \sum_{n=2}^{M-1} Q^k_n(s) = \sum_{n=2}^{M-1} \frac{\log^k n}{n^s} \quad \text{and} \quad T^k_M(s) := \sum_{n=M+1}^{\infty} Q^k_n(s) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^s}.
\]

Our goal will be to show that

\[
|\zeta^{(k)}(s)| \geq Q^k_M(\sigma) - H^k_M(\sigma) - T^k_M(\sigma) = Q^k_M(\sigma) \left( 1 - \frac{H^k_M(\sigma)}{Q^k_M(\sigma)} - \frac{T^k_M(\sigma)}{Q^k_M(\sigma)} \right) > 0
\]

for our choice of \( b_1 \) and \( b_2 \), keeping in mind that

\[
\frac{Q^M_{M+1}(q_Mk + b_1)}{Q^M_M}(q_Mk + b_1) = \left( \frac{M}{M+1} \right)^{b_1} \quad \text{and} \quad \frac{Q^{M-1}_M(q_{M-1}k + b_2)}{Q^M_M}(q_{M-1}k + b_2) = \left( \frac{M}{M-1} \right)^{b_2},
\]

as one can easily verify.

**The Tails.** We first find an upper bound for the tails \( T^k_M(\sigma) \).

**Lemma 8.** Fix some integer \( M \geq 2 \), and assume \( k - 1 < (\sigma - 1) \log M \). Then

\[
T^k_M(\sigma) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^\sigma} \leq \int_M^{\infty} \frac{\log^k x}{x^\sigma} dx < Q^k_M(\sigma) R^k_M(\sigma). \tag{7}
\]

where

\[
R^k_M(\sigma) = \frac{M}{\sigma - 1} \left( 1 + \frac{k}{(\sigma - 1) \log M - k + 1} \right).
\]

**Proof.** For \( k \in \mathbb{Z} \), the integral in (7) can be written in a closed form. Applying recursively the general formula (for all \( b, -a \neq -1 \)):

\[
\int M^{(a)} \log x^{a} dx = -\frac{(\log x)^a}{(a-1)x^{a-1}} + \frac{a}{a-1} \int M^{(a)} \log x^{a-1} dx,
\]

we obtain

\[
\int_M^{\infty} \frac{\log^k x}{x^\sigma} dx = \frac{\log^k M}{M^\sigma} \frac{M}{\sigma - 1} \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{\log^{-r} M}{(\sigma - 1)^r}
\]

\[
\leq Q^k_M(\sigma) \frac{M}{\sigma - 1} \left( 1 + \sum_{r=1}^{k} \frac{k(k-1)^{r-1}}{(\sigma - 1)^r} \frac{1}{(\log M)^r} \right)
\]

\[
< Q^k_M(\sigma) \frac{M}{\sigma - 1} \left( 1 + \frac{k}{(\sigma - 1) \log M} \sum_{r=0}^{\infty} \left( \frac{k}{(\sigma - 1) \log M} \right)^r \right)
\]

\[
= Q^k_M(\sigma) \frac{M}{\sigma - 1} \left( 1 + \frac{k}{(\sigma - 1) \log M - k + 1} \right),
\]

where the convergence of the geometric series is implied by \( k - 1 < (\sigma - 1) \log M \). \( \square \)

It is clear why estimating \( R^k_M(\sigma) \) will be vital for the proofs of our theorems. We note:
Lemma 9. If \(a_1 k + b_1 \leq \sigma\), and \(k \geq k_M\), then
\[
R^k_M(\sigma) \leq R^k_M(a_1 k + b_1) \leq R^{k_M}_M(a_1 k_M + b_1),
\tag{8}
\]
as long as the following two conditions are satisfied:
\[
a_1 > \frac{1}{\log M} \quad \text{and} \quad (a_1 \log M - 1)k_M + 1 + (b_1 - 1) \log M > 0,
\]
and in the case of \(b_1 < 1 - 1/\log M\) also:
\[
k_M \geq \frac{1}{a_1 \log M} \left( -(b_1 - 1) \log M - 1 + \sqrt{|(b_1 - 1) \log M + 1|} \right).
\]

Proof. The left-hand inequality of (8) is evident from the fact that \(R^k_M(\sigma)\) is decreasing when viewed as a function of \(\sigma\) alone. The right-hand inequality of (8) is equivalent to saying that \(R^k_M(\sigma)\) is decreasing as a function of \(k\). To see why this is the case, just notice that if we rewrite \(R^k_M(q_M k + b_1)\) in the form
\[
y(k) = \frac{1}{(c+1)k + d - 1} \frac{(c+1)k + d}{c k + d},
\]
where \(c := a_1 \log M - 1 > 0\) and \(d := 1 + (b_1 - 1) \log M\), then clearly
\[
y'(k) = -\frac{c(1 + c)^2 k^2 + 2cdk(1 + c) + (1 + cd)}{(c + 1)k + d - 1)^2 (ck + d)^2},
\]
from which it is easy to see that \(y'(k)\) can change sign only if \(d < 0\) (otherwise it remains nonpositive). However, the condition \(d < 0\) translates to \(b_1 < 1 - 1/\log M\), in which case one requires \(k_M \geq z_0\), where
\[
z_0 := -\frac{d}{1+c} + \frac{1}{1+c} \sqrt{|d|/c}
\]
is the right zero of the numerator of the above expression for \(y'(k)\). \(\square\)

The way the estimate for \(T^k_M(\sigma)\) from Lemma 8 will be used in the proof of Theorem 10 is via the separation:
\[
T^k_M(\sigma) = Q^k_M(\sigma) + T^k_{M+1}(\sigma) \leq Q^k_M(\sigma)(1 + R^k_{M+1}(\sigma)) \leq Q^k_M(q_M k + b_1)(1 + R^k_{M+1}(q_M + b_1)),
\]
where \(R^k_{M+1}(q_M + b_1)\) will converge because \(q_M > \frac{1}{\log(M+1)}\), by Lemma 5.

Also, a corollary of Lemma 8 and Lemma 9 is a proof of the result of [12]. We include it here because it exemplifies several of the important ideas and illustrates some of the key workings of our general method, being the special case of \(M = 2\) (representing the dominance of the term \(Q^k_2(\sigma)\)).

Theorem 10 ([12]). For all \(\sigma \geq q_2 k + 2\) we have \(\zeta^k(s) \neq 0\).

Proof. First write
\[
|\zeta^k(s)| \geq \frac{2^\sigma}{\log k^2} - T^k_2(\sigma) \geq Q^k_2(\sigma) \left(1 - \frac{Q^k_3}{Q^k_2}(\sigma) - \frac{Q^k_4}{Q^k_2}(\sigma) \left(1 + R^k_5(\sigma)\right) \right).
\]
By Lemma 9 we have \( R^k_1(q) \leq R^k_1(q_2k + 2) < 0.68 \), for \( k \geq 3 \). Furthermore,
\[
\frac{Q^k_3}{Q^k_2}(\sigma) = \frac{2^4}{\log^2 2} \cdot \log^2 \frac{4}{\sigma} < 2^4 \cdot \frac{\log^2 2}{4} = 2^{k-\sigma} \leq 2^{k-q_2k+2} \leq 2^{3(1-q_2)-2} \leq 0.19.
\]

The quotient \( \frac{Q^k_3}{Q^k_2}(\sigma) \) is decreasing in \( \sigma \), and hence \( \frac{Q^k_3}{Q^k_2}(\sigma) \leq \frac{Q^k_3}{Q^k_2}(q_2k + 2) = \frac{4}{9} \). So we obtain
\[
1 - \frac{Q^k_3}{Q^k_2}(\sigma) - \frac{Q^k_4}{Q^k_3}(\sigma) (1 + R^k_4(\sigma)) \geq 1 - \frac{4}{9} - 0.19(1 + 0.68) > 0,
\]
which establishes the result. \( \square \)

Since Theorem 1 (a) deals with the next case of \( M = 3 \) (corresponding to the dominance of the term \( Q_3^k(\sigma) \)), and only a little bit of extra effort is needed to cover it, we give a proof of it right now. The result states: \( \zeta^k(s) \neq 0 \), for
\[
q_3k + 4 \log 3 \leq \sigma \leq q_2k - 2.
\]

**Proof of Theorem 1 (a).** Separating the dominant term \( Q_3^k(\sigma) \), we get
\[
|\zeta^{(k)}(s)| \geq \frac{Q^k_3}{Q^k_2}(\sigma) - Q^k_2(\sigma) - T^k_3(\sigma) \geq \frac{Q^k_3}{Q^k_2}(\sigma) \left( 1 - \frac{Q^k_2}{Q^k_3}(\sigma) - \frac{Q^k_4}{Q^k_3}(\sigma) (1 + R^k_4(\sigma)) \right).
\]
Therefore we only need to show that
\[
1 - \frac{Q^k_2}{Q^k_3}(\sigma) - \frac{Q^k_4}{Q^k_3}(\sigma) (1 + R^k_4(\sigma)) > 0.
\]

From Lemma 9
\[
R^k_4(\sigma) \leq R^k_4(q_3k + 4) \leq R^k_4(q_3k + 4) < 0.72, \quad \text{for } q_3k + 4 \log 3 \leq \sigma \text{ and } k \geq k_3 = \frac{4 \log 3 + 2}{q_2 - q_3}.
\]
Also, \( \frac{Q^k_3}{Q^k_2}(\sigma) \leq \frac{Q^k_3}{Q^k_2}(q_3k + 4 \log 3) < 0.29 \) and \( \frac{Q^k_3}{Q^k_2}(q_2k - 2) < 0.29 \). Hence
\[
1 - \frac{Q^k_2}{Q^k_3}(\sigma) - \frac{Q^k_4}{Q^k_3}(\sigma) (1 + R^k_4(\sigma)) > 1 - 0.45 - 0.29(1 + 0.72) > 0,
\]
as desired. \( \square \)

Theorem 1 (b) deals with the dominance of the general term \( Q^k_3(\sigma) \), and consequently requires knowledge of the behavior of the sum of all the terms preceding it.

**The Heads.** We rewrite the heads of the series (1) in the following form:
\[
H^k_M(\sigma) = Q^k_M(\sigma) \left( \frac{Q^k_{M-1}}{Q^k_M}(\sigma) + \frac{Q^k_{M-2}}{Q^k_M}(\sigma) + \ldots + \frac{Q^k_2}{Q^k_M}(\sigma) \right)
= Q^k_M(\sigma) \left( \frac{Q^k_{M-1}}{Q^k_M}(\sigma) \left( 1 + \frac{Q^k_{M-2}}{Q^k_{M-1}}(\sigma) \left( 1 + \cdots \left( 1 + \frac{Q^k_2}{Q^k_3}(\sigma) \right) \cdots \right) \right) \right),
\]
and we will find upper bounds for all the above quotients \( \frac{Q^k_{n-1}}{Q^k_n}(\sigma) \) of consecutive terms.

Here, notice that \( \frac{Q^k_{n-1}}{Q^k_n}(\sigma) \) is always increasing with \( \sigma \). For \( 2 \leq n \leq M \), in the wedges (see Figure 3) delimited by \( q_Mk + b_2 \leq \sigma \leq q_{M-1}k + b_2 \), this yields
\[
\frac{Q^k_{n-1}}{Q^k_n}(\sigma) \leq \frac{Q^k_{n-1}}{Q^k_n}(q_{M-1}k + b_2) \leq \left( \frac{n}{n-1} \right)^{b_2},
\]
the second inequality holds because we have \( q_{M-1} < q_n \) for all \( n < M - 1 \), while the equality holds due to the fact that \( \sigma = q_{n-1}k \) is the solution of \( Q_n^k(\sigma) = Q_{n-1}^k(\sigma) \). Therefore

\[
\frac{Q_{n-1}^k(q_{n-2}k + b_2)}{Q_n^k(q_{n-2}^k)} = \left( \frac{n}{n-1} \right)^{b_2} \left( \frac{n-1}{n} \right)^{-b_2}.
\]

Thus, in order for \( H_M^k(\sigma) \) to stay bounded, we must choose \( b_2 < 0 \). It is not difficult to see that a choice of \( b_2 \) as a linear function of \( M \) (i.e. \( -b_2 \) of the form \( cM + d \), with \( d \geq c \)) is likely to work, since:

**Lemma 11.** Let \( c, d \in \mathbb{R} \) and \( 0 < c \leq d \). Then \( y(M) = \left( \frac{M}{M+1} \right)^{cM+d} \) is monotonously increasing with the asymptote \( 1/e^c \).

**Proof.** As \( \lim_{M \to \infty} \left( 1 + \frac{1}{M} \right)^{cM} = e^c \) and \( \lim_{M \to \infty} \left( \frac{M}{M+1} \right)^d = 1 \), for all fixed \( d > 0 \), we evidently have \( \lim_{M \to \infty} \left( \frac{M}{M+1} \right)^{cM+d} = 1/e^c \); and the function \( y(M) \) is monotonically increasing because, for all \( d > c > 0 \), we have

\[
y'(M) = \left( \frac{M}{M+1} \right)^{cM+d} \left( -c \log \left( \frac{M+1}{M} \right) + \frac{cM + d}{M} - \frac{cM + d}{M+1} \right) > 0,
\]

proving the result. \( \square \)

Due to a technical nature of our arguments, the proof of Theorem I(b) will be divided into two cases. The first case: \( M \geq 11 \), will be handled in full generality, while the second case: \( 4 \leq M \leq 10 \), will have to be treated separately for each value of \( M \).

**Lemma 12.** If \( 11 \leq M \), then \( \zeta^{(k)}(\sigma) \neq 0 \), for

\[
q_Mk + (M + 1) \log 3 \leq \sigma \leq q_{M-1}k - M \log 3.
\]

**Proof.** In a way similar to the approach we took in part (a) of the theorem, here we write

\[
\left| \zeta^{(k)}(s) \right| \geq Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma) \geq Q_M^k(\sigma) \left( 1 - \frac{H_M^k(\sigma)}{Q_M^k} - \frac{Q_{M+1}^k(\sigma)}{Q_M^k} \right). \]

By Lemma 9 for \( q_Mk + (M + 1) \log 3 \leq \sigma \), \( k \geq k_M = \frac{(M+1)\log 3}{q_{M-1}k-M} \), and \( M \geq 11 \), we have

\[
R_M^k(\sigma) \leq R_M^k(q_Mk + (M + 1) \log 3) \leq R_M^{kM}(q_Mk_M + (M + 1) \log 3) < \frac{1}{2}.
\]

On the other hand, from Lemma 11 we obtain the nice upper bound \( \frac{Q_{M+1}^k}{Q_M^k}(\sigma) < \frac{1}{3} \) for \( q_Mk + (M + 1) \log 3 \leq \sigma \) (this is precisely where the choice of the constant \( \log 3 \) is necessary); while, from Lemma 11 we also get:

\[
\frac{H_M^k(\sigma)}{Q_M^k} \geq \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} - 1 = \frac{1}{2}.
\]
Thus, for \(11 \leq M\) and \(q_M k + (M + 1) \log 3 \leq \sigma \leq q_{M-1} k + M \log 3\), we have

\[
1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{Q_{M+1}^k}{Q_M^k}(\sigma) \left(1 + R_M^k(\sigma)\right) > 1 - \frac{1}{2} - \frac{1}{3} \left(1 + \frac{1}{2}\right) = 0,
\]

which proves the lemma. \(\square\)

**Lemma 13.** If \(4 \leq M \leq 10\), and \(q_M k + (M + 1) \log 3 \leq \sigma \leq q_{M-1} k + M \log 3\), then \(\zeta^{(k)}(\sigma) \neq 0\).

**Proof.** Unfortunately, for \(4 \leq M \leq 10\), we have to consider each \(M\) separately. We proceed as above, but since here the estimate \(\frac{Q_{M+1}^k}{Q_M^k}(\sigma) < \frac{1}{3}\) is not quite sharp enough to give us sufficiently good results, we will need to list upper bounds for \(R_M^k\) and \(T_M^k\) for individual \(M\). We have (keeping, as above, \(\sigma_M := q_M k_M + (M + 1) \log 3\)):

| \(M\) | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|----|
| \(R_M^k(\sigma_M)\) | < 0.6 | < 0.57 | < 0.55 | < 0.53 | < 0.52 | < 0.51 | < 0.51 |
| \(Q_M^k(\sigma_M)\) | < 0.3 | < 0.31 | < 0.31 | < 0.31 | < 0.32 | < 0.32 | < 0.32 |
| \(T_M^k(\sigma_M)\) | < 0.47 | < 0.47 | < 0.48 | < 0.48 | < 0.48 | < 0.48 | < 0.48 |

As it is now easy to verify, for each \(M\) that satisfies \(4 \leq M \leq 10\), and for all \(\sigma\) in the range \(q_M k + (M + 1) \log 3 \leq \sigma \leq q_{M-1} k + M \log 3\), we again conclude:

\[
1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{T_M^k}{Q_M^k}(\sigma_M) > 1 - \frac{1}{2} - \frac{1}{3} (1 + \frac{1}{2}) = 0,
\]

which proves the lemma. \(\square\)

Combining Lemma 12 and Lemma 13 gives us a proof of Theorem 1(b).

**Remark.** The zero-free regions we have given are not the largest possible. For example, if one considered the lines \(\sigma = \frac{1}{2} (q_M + q_{M+1}) k + \frac{1}{2} \log 3\) through the centers of the wedges and searched for the lowest \(k\) for which there were no zeros on those lines, then one would obtain the following values for \(k_M\) (which are lower than the values we have for the tips of the wedge-shaped regions):

| \(M\) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|----|
| \(k_M\) on line | 14 | 41 | 87 | 154 | 247 | 368 | 519 | 703 |
| \(k_M\) at the tip | 20 | 71 | 151 | 269 | 429 | 638 | 898 | 1214 |

5. **Proof of Theorem 2**

In this last section we describe a counting technique that will allow us to obtain very precise (for many heights even exact) estimates for the number of zeros of \(\zeta^{(k)}(s)\) in all critical strips \(S_M^k\). It differs from the classical quantitative methods (notably Berndt’s, in 1, for \(N_k(T)\)) in its use of Rouché’s theorem. Because of the property of the quasi-periodicity of the zeros of \(\zeta^{(k)}(s)\) inside \(S_M^k\) we are able to count the zeros by individual separation. In order for our approach to work, we first find horizontal, periodically-spaced zero-free line segments within
the critical strips (in Lemma 14). Then we show that there is always exactly one zero of \( \zeta^{(k)}(s) \) in the rectangles \( R_j \) (for \( j \in \mathbb{N} \)) that are delimited by the vertical edges of two of the neighbouring zero-free regions and two of those horizontal zero-free lines (see Figure 4).

As already mentioned above, in the critical strips \( S^k_M \), which are located between the two zero-free regions, where the expansion of \( \zeta^{(k)}(s) \) is dominated by the terms \( Q^k_M(s) \) and \( Q^k_{M+1}(s) \) respectively, one can obtain values of the imaginary parts \( t \) of expected zeros by solving the equation \( Q^k_M(\sigma + it) = Q^k_{M+1}(\sigma + it) \), an act of balancing the real and imaginary parts of two largest terms, and then choosing the horizontal lines of separation exactly halfway between them, thus managing to avoid even the most irregular of zeros inside \( S^k_M \). That is exactly what we do below.

A nice consequence of this method is that all the zeros of \( \zeta^{(k)}(s) \) inside \( S^k_M \) are simple.

**Lemma 14.** Let \( M \geq 2 \) and \( k \in \mathbb{N} \). If \( s \in S^k_M \), then \( \zeta^{(k)}(s) \neq 0 \) for

\[
s = \sigma + i \cdot \frac{2\pi j}{\log(M + 1) - \log M},
\]

**Figure 4.** The curve \( \gamma \) is the boundary of the rectangle \( R_j \). The point \( \bullet \) represents a zero of \( Q^k_M(s) + Q^k_{M+1}(s) \) on the critical line \( \sigma = q_M k \).

**Proof.** For the sake of simplicity we consider the slightly wider rectangles \( R_j \), with the vertical boundaries: \( q_M k - (M + 1) \log 3 \leq \sigma \leq q_M k + (M + 1) \log 3 \), composition of which contains \( S^k_M \). In the center of the critical strip \( S^k_M \), that is on the critical line \( \sigma = q_M k \) we have \( |Q^k_M(s)| = |Q^k_{M+1}(s)| \). We consider the line segments in \( S^k_M \) with

\[
t = \frac{2\pi j}{\log(M + 1) - \log M}, \quad \text{where } j \in \mathbb{Z},
\]
so that \( Q^k_M(q_M k + it) = Q^k_{M+1}(q_M k + it) \). We set \( s = \sigma + it \), with \( t \) as above, and consider the real and imaginary parts of the expression

\[
\zeta^{(k)}(s) = \frac{1}{\sqrt{2}} \left( Q^k_M(\sigma) + Q^k_{M+1}(\sigma) - H^k_M(\sigma) - T^k_{M+1}(\sigma) \right) - \frac{1}{\sqrt{2}} \left( Q^k_M(\sigma) - Q^k_{M+1}(\sigma) \right)
\]

With \(|\Im(Q^k_M(s))| \leq Q^k_n(\sigma)\) and \(|\Re(Q^k_M(s))| \leq Q^k_n(\sigma)\) we obtain

\[
|\Re(\zeta^{(k)}(s))| \geq |\cos(t \log M)Q^k_M(\sigma) + \cos(t \log(M + 1))Q^k_{M+1}(\sigma)| - H^k_M(\sigma) - T^k_{M+1}(\sigma),
\]

\[
|\Im(\zeta^{(k)}(s))| \geq |\sin(t \log M)Q^k_M(\sigma) + \sin(t \log(M + 1))Q^k_{M+1}(\sigma)| - H^k_M(\sigma) - T^k_{M+1}(\sigma).
\]

Now, if \( t = 0 \), the situation is trivial, while if \( t \neq 0 \), then we either have \(|\sin(\tau)| \geq \sin(\pi/2) = 1/\sqrt{2} \) or \(|\cos(\tau)| \geq \cos(\pi/2) = 1/\sqrt{2} \). Therefore, because obviously \(|\zeta^{(k)}(s)| \geq |\Re(\zeta^{(k)}(s))|\) and \(|\zeta^{(k)}(s)| \geq |\Im(\zeta^{(k)}(s))|\) and by our choice of \( t \), we obtain:

\[
|\zeta^{(k)}(s)| \geq \frac{1}{\sqrt{2}} \left( Q^k_M(\sigma) + Q^k_{M+1}(\sigma) - H^k_M(\sigma) - T^k_{M+1}(\sigma) \right)
\]

\[
= Q^k_M(\sigma) \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} Q^k_{M+1}(\sigma) - H^k_M(\sigma) - Q^k_{M+1}(\sigma) \right) - Q^k_{M+1}(\sigma) \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} Q^k_{M+1}(\sigma) - T^k_{M+1}(\sigma) \right).
\]

Recall that \( Q^k_{M+2}(\sigma) \) and \( T^k_{M+2}(\sigma) \) are decreasing in \( \sigma \). From the proofs of Lemma 12 and Lemma 13 we know that

\[
\frac{1}{\sqrt{2}} - \frac{Q^k_{M+2}(\sigma)}{Q^k_{M+1}(\sigma)} \geq \frac{1}{\sqrt{2}} - \frac{Q^k_{M+2}(\sigma)}{Q^k_{M+1}(\sigma)} \left( 1 + R_{M+2}(\sigma) \right)
\]

\[
\geq \frac{1}{\sqrt{2}} - \frac{Q^k_{M+2}(\sigma)}{Q^k_{M+1}(\sigma)} (q_{M+1} k + (M + 2) \log 3) \left( 1 + R_{M+2}(q_{M+1} k + (M + 2) \log 3) \right) > 0.
\]

Similarly, since \( H^k_M(\sigma) \) is decreasing in \( \sigma \), we obtain

\[
\frac{1}{\sqrt{2}} - \frac{H^k_M(\sigma)}{Q^k_M(\sigma)} \geq \frac{1}{\sqrt{2}} - \frac{H^k_M(\sigma)}{Q^k_M(\sigma)} (q_{M-1} k - M \log 3) \geq \frac{1}{\sqrt{2}} - \frac{1}{2} > 0,
\]

which concludes the proof of the lemma.

\( \square \)

**Proof of Theorem 3** Let \( j \in \mathbb{N} \). Now that the non-vanishing of \( \zeta^{(k)}(s) \) along the horizontal division lines has been established (and those vertical lines, by definition, cannot contain any zeros), we can integrate along the entire boundary \( \gamma \) of each of the rectangles \( R_j \) described above (see Figure 4). Moreover, Theorem 1 and Lemma 14 assert that we have

\[
| \left( Q^k_M(s) + Q^k_{M+1}(s) \right) - \zeta^{(k)}(s) | \leq H^k_M(s) + T^k_{M+1}(s) \leq | Q^k_M(s) + Q^k_{M+1}(s) |,
\]

along the vertical and horizontal parts of the boundary curve \( \gamma \), respectively. Thus, along the entire boundary of \( R_j \), the function \( \zeta^{(k)}(s) \) is closely approximated by the function \( Q^k_M(s) + Q^k_{M+1}(s) \); and therefore, by Rouché’s Theorem, it has to have the same number of
The number of zeros for $\zeta^{(k)}(s)$ inside $S_k^M$, and the sharp formula for $N_k^M(T)$, as given in Corollary 3.

\[ s = q_M k + i \cdot \frac{(2j + 1)\pi}{\log(M + 1) - \log M}. \]

This proves both the simplicity of all zeros of $\zeta^{(k)}(s)$ inside $S_k^M$, and the sharp formula for $N_k^M(T)$, as given in Corollary 3. \[ \Box \]

6. Acknowledgments

This research was supported in part by a New Faculty Grant from UNC Greensboro. Most of the work on the paper was conducted while Thomas Binder was a visiting researcher at UNC Greensboro, in Fall 2008. The authors would like to thank Prof. Garry J. Tee from the University of Auckland for several helpful comments and many useful remarks. All computations and plots were done with the computer algebra system Sage \[9\].

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