On Linearized Nordström Supergravity in Eleven and Ten Dimensional Superspaces (2)

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ABSTRACT

We present aspects of the component description of linearized Nordström Supergravity in eleven and ten dimensions. The presentation includes low order component fields in the supermultiplet, the supersymmetry variations of the scalar graviton and gravitino trace, their supercovariantized field strengths, and the supersymmetry commutator algebra of these theories.

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1 Introduction

A mathematically consistent (however requiring restrictions on the allowed general coordinate transformations) and simplified version of gravitation is provided by a variant \[1,2\] that may be called “Nordström Gravity.” In a previous work \[8\], we initiated a program of investigating whether one can construct Nordström Supergravity extensions in eleven and ten dimensional spacetimes of this simplified limit of gravitation at the linearized level.

One pointed motivation for our efforts has been the recent progress \[6,7\] in the derivation of M-Theory corrections to 11D Supergravity. A series of procedures connecting the corrections to a 3D, $\mathcal{N} = 2$ Chern-Simons theory \[3,4,5\] (used in a role roughly analogous to world-sheet $\sigma$-model $\beta$-function calculations for string corrections) has been successfully demonstrated. Though the works in \[6,7\] have presented a method of deriving these corrections beyond the supergravity limit, these solely treat purely bosonic M-Theory corrections, with no equivalent results describing fermionic corrections. One traditional way of accomplishing this is to embed the purely bosonic results into a superspace formulation. This impels us to a renewed interest in 11D supergravity in superspace.

The goal we are pursuing is to find a Salam-Strathdee superspace \[9\], as modified by Wess & Zumino \[10,11\], such that superspace Bianchi identities do not imply equations of motion for the component fields contained within the superspace description of Nordström SG. In particular, we are not currently investigating the prospect of writing action formulae for such supermultiplets of fields. While actions are the “gold standard,” it is useful to recall (as done below), this is not the first time the off-shell structure of a supergravity theory has been explored, without the additional exploration of an action principle.

This distinguishes our work from efforts taken by other. For example, there is a substantial literature that uses the concept of “pure spinors” \[12,13,14,15,16,17,18\] where the endpoint of action principles (see especially \[17,18\]) has been presented. While classically such approaches appear to work, there are troubling signs \[19,20,21,22\] that more needs to be done to justify complete acceptance at the level of quantum theory.

Perhaps an intuitive way to argue is in order to achieve quantization, it should be implemented in terms of variables that are basically free. The non-minimal pure spinor goes some way to giving a free resolution, but the resulting space of fields does not have a well-defined trace. The work of \[21\] offered a “fix” but if in the end the prescription still is not an integral over free variables with no other qualifications, then a proper quantization is still likely to be impeded.

Therefore, we adopt the rather more cautious approach by raising the query of whether it is possible to follow the pathway established by Wess & Zumino \[10,11\] wherein a “simple” Salam-Strathdee superspace is used as a starting point to building, in our case, Nordström SG in which Bianchi identities do not force equations of motion.

While it is often overlooked, the first off-shell description of 4D, $\mathcal{N} = 1$ supergravity was actually carried out by Breitenlohner \[23\] who took an approach equivalent to starting with the component fields of the Wess-Zumino gauge 4D, $\mathcal{N} = 1$ vector supermultiplet $(v_{\alpha}, \lambda_{\beta}, d)$ together with their

\footnote{In this previous work, a substantial citation review is undertaken and interested readers are encouraged to familiarize themselves with the literature via this means.}
familiar SUSY transformation laws,
\[
D_a v^b = (\gamma^b)_{a}^{c} \lambda_c ,
\]
\[
D_a \lambda_b = -i \frac{1}{4} ([\gamma^b, \gamma^d])_{ab} (\partial_c v_d - \partial_d v_c) + (\gamma^5)_{ab} d ,
\]
\[
D_a d = i (\gamma^5 \gamma^b)_{a}^{b} \partial_c \lambda_b ,
\]
followed by choosing as the gauge group the space time translations, SUSY generators, and the spin angular momentum generators as well as allowing additional internal symmetries. For the space time translations, this requires a series of replacements of the fields according to:
\[
v^b \rightarrow h^b_{\bar{c}} , \quad \lambda_b \rightarrow \psi^b_{\bar{c}} , \quad d \rightarrow A_{\bar{c}} ,
\]
(in the notation in \[23\] \(A_{\bar{a}} = B^5_{\bar{a}}\)) while for the SUSY generators, the replacements occur according to:
\[
v^b \rightarrow \chi^b_{\bar{c}} , \quad \lambda_b \rightarrow \phi^b_{\bar{c}} , \quad d \rightarrow \chi^5_{\bar{c}} ,
\]
and finally for the spin angular momentum generator, a replacement of
\[
v^b \rightarrow \omega^b_{\bar{c} \bar{d}} , \quad \lambda_b \rightarrow \chi^b_{\bar{c} \bar{d}} , \quad d \rightarrow D_{\bar{c} \bar{d}} ,
\]
was used. However, to be more exact, Breitenlohner also allowed for more symmetries like chirality to be included. Because the vector supermultiplet was off-shell (up to WZ gauge transformations) the resulting supergravity theory was off-shell and included a redundant set of auxiliary component fields, i.e. this is not an irreducible description of supergravity. But as seen from (1.2) the supergravity fields were all present and together with the remaining component fields a complete superspace geometry can be constructed.

In our approach to Nordström SG, the analog of the Wess-Zumino gauge 4D, \(N = 1\) vector supermultiplet is played by a scalar superfield in any of the 11D or 10D superspaces to be studied. This scalar superfield guarantees off-shell supersymmetry. However, like the approach taken by Breitenlohner, the resulting theory is expected to be reducible. Also like this earlier approach, the question of an action principle is not addressed.

The structure of the remainder of this work looks as follows.

In chapter two, a review of 4D, \(N = 1\) supergravity in superspace is given. This proves a detailed description of how to extract component results from the superspace geometry. Using the foundation in \textit{Superspace} \[24\], the general formalism for obtaining component level results is reviewed. In this context the composition rules for the parameter of spacetime translations, parameters of SUSY transformations, and Lorentz transformations are presented relating these to supergeometrical quantities is given. Next the SUSY transformation rules for the frame field, gravitino field, and spin connection relating these to supergeometrical quantities are given. Finally, the “supercovariantized” field strengths of the frame field, gravitino field, and spin connection relating these to supergeometrical quantities are given. and related to supergeometrical quantities. This chapter ends with the linearization of these results.

Chapter three uses the technology of the second chapter to present component level of the Nordström SG for 11D, \(N = 1\) superspace, 10D, \(N = 2A\) superspace, 10D, \(N = 2B\) superspace, and
10D, \( \mathcal{N} = 1 \) superspace. Component level descriptions of the local SUSY commutator algebras are provided. Linearized curvatures and torsions supertensors are presented and the supersymmetry variations of the linearized Nordström “scalar” graviton, the linearized spin-1/2 Nordström gravitino, and the spin connection are obtained.

Chapter four is a short chapter in comparison to the two that precede it. In 4D, \( \mathcal{N} = 1 \) supergravity [25,26], the concept of the “chiral compensator” was introduced some time ago. We demonstrate evidence that such a compensator exist for the 10D, \( \mathcal{N} = 2 \)B superspace. This is unique among supergravity theories in eleven and ten dimensions. However, we also present evidence that though such a chiral superfields appear to consistently exist, the linearized Nordström superspace is such that a chiral superfield of this type cannot be used as a compensator.

The fifth chapter is devoted to our conclusions and a summary.
2 Superspace Perspective On Component Results

In our previous paper [8], we restricted our focus solely to superfield considerations in eleven and ten dimensions. However given those result, the technology developed in Superspace [24] allows a presentation of some of the component results. In particular, the equations indicated in section (5.6) in this book can be applied to the case of eleven and ten dimensions. This is true even though the sole focus of the book is the case of 4D, \( \mathcal{N} = 1 \) supersymmetry. Nonetheless, the discussion in the book can be easily modified for use in 11D and 10D superspace theories. The relevant equations were designated as (5.6.13), (5.6.16) - (5.6.18), (5.6.21), (5.6.22) - (5.6.24), (5.6.28), (5.6.33), and (5.6.34). For the convenience of the reader, we bring these results all together in the text to follow. After this chapter, these are all going to be appropriately modified for the cases of 11D, \( \mathcal{N} = 1 \), 10D, \( \mathcal{N} = 2A \), 10D, \( \mathcal{N} = 2B \), and 10D, \( \mathcal{N} = 1 \) superspaces, respectively.

2.1 Recollection of 4D, \( \mathcal{N} = 1 \) Component/Superspace Results

In the context of 4D, \( \mathcal{N} = 1 \) superspace supergravity, we may distinguish among three types of symmetries:

(a.) space time translations with generator \( iK_{GC}(\xi^m) \), dependent on local parameters \( \xi^m(x) \),
(b.) SUSY transformations with generator \( iK_Q(\epsilon^2) \) dependent on local parameters \( \epsilon^2(x) \), and
(c.) tangent space transformations with generator \( iK_{TS}(\lambda^i) \) depend -ent on local parameters \( \lambda^i(x) \).

The tangent space transformations act as “internal angular momentum,” chirality, etc. on all “flat indices” associated with the superspace quantities.

The commutator algebra of two SUSY transformations generated by \( iK_Q(\epsilon_1^2) \), and \( iK_Q(\epsilon_2^2) \), respectively takes the form

\[
\left[ iK_Q(\epsilon_1), iK_Q(\epsilon_2) \right] = iK_{GC}(\xi^m) + iK_Q(\epsilon) + iK_{TS}(\lambda^i), \tag{2.1}
\]

where the parameters \( \xi^m \), \( \epsilon^2 \), and \( \lambda^i \) on the RHS of this equation are quadratic in \( \epsilon_1 \) and \( \epsilon_2 \), dependent on linear and quadratic terms in the gravitino, and linear terms in the superspace torsions and curvature supertensors according to:

\[
\xi^m = - \left[ (\epsilon_1^\alpha \epsilon_2^\beta + \epsilon_1^\beta \epsilon_2^\alpha)T_{\frac{\alpha}{\beta}}^\xi + \epsilon_1^\alpha \epsilon_2^\beta T_{\frac{\alpha}{\beta}}^e + \epsilon_1^\alpha \epsilon_2^\beta T_{\frac{\alpha}{\beta}}^c \right] e^m_{\xi}, \tag{2.2}
\]

\[
\epsilon^2 = - \left[ (\epsilon_1^\alpha \epsilon_2^\beta + \epsilon_1^\beta \epsilon_2^\alpha)(T_{\frac{\alpha}{\beta}}^\delta + T_{\frac{\alpha}{\beta}}^\xi) + \epsilon_1^\alpha \epsilon_2^\beta (T_{\frac{\alpha}{\beta}}^\delta - T_{\frac{\alpha}{\beta}}^\xi) + \epsilon_1^\alpha \epsilon_2^\beta (T_{\frac{\alpha}{\beta}}^\delta + T_{\frac{\alpha}{\beta}}^\xi) \right], \tag{2.3}
\]

\[
\lambda^i = - \left[ (\epsilon_1^\alpha \epsilon_2^\beta + \epsilon_1^\beta \epsilon_2^\alpha)(R_{\frac{\alpha}{\beta}}^i + T_{\frac{\alpha}{\beta}}^\phi_{\xi} + T_{\frac{\alpha}{\beta}}^\phi_{\delta}) + \epsilon_1^\alpha \epsilon_2^\beta (R_{\frac{\alpha}{\beta}}^i + T_{\frac{\alpha}{\beta}}^\phi_{\xi} + T_{\frac{\alpha}{\beta}}^\phi_{\delta}) + \epsilon_1^\alpha \epsilon_2^\beta (R_{\frac{\alpha}{\beta}}^i + T_{\frac{\alpha}{\beta}}^\phi_{\xi} + T_{\frac{\alpha}{\beta}}^\phi_{\delta}) \right]. \tag{2.4}
\]

The supersymmetry variations of the inverse frame field \( e_{\frac{m}{a}}(x) \), gravitino \( \psi_{\frac{\delta}{a}}(x) \), and connection fields for the tangent space symmetries \( \phi_{\frac{\alpha}{a}}^i(x) \) take the forms below and are expressed in terms
dependent on linear and quadratic in the gravitino, and linear in the superspace torsions and curvature supertensors.

\[
\delta Q e^{m}_{\alpha} = - \left[ \epsilon^\beta e^{d}_{\beta \alpha} + \epsilon^\delta e^{d}_{\delta \alpha} + (\epsilon^\delta \tilde{\psi}_a^\gamma + \epsilon^\delta \tilde{\bar{\psi}}_a^\gamma) T^{d}_{\beta \gamma} + \epsilon^\delta \tilde{\bar{\psi}}_a^\gamma T^{d}_{\beta \gamma} + \epsilon^\tilde{\psi} a^\delta T^{d}_{\gamma \bar{\beta}} + \epsilon^\tilde{\bar{\psi}} a^\delta T^{d}_{\gamma \bar{\beta}} \right] e^{m}_{d}, \tag{2.5}
\]

\[
\delta Q \psi_a^\delta = D_a^\delta - \epsilon^\beta (T^{\delta}_{\beta \alpha} + T^{\delta}_{\beta \alpha} \tilde{\psi}_a^\beta - \tilde{\bar{\psi}} a^\gamma (T^{\delta}_{\gamma \beta} + T^{\delta}_{\gamma \beta} \xi^\delta) - \epsilon^\delta \psi_a^\beta T^{\delta}_{\gamma \bar{\beta}} + \epsilon^\delta \tilde{\bar{\psi}} a^\gamma (T^{\delta}_{\gamma \bar{\beta}} + T^{\delta}_{\gamma \bar{\beta}} \xi^\delta) - \epsilon^\delta \psi_a^\beta (T^{\delta}_{\gamma \beta} + T^{\delta}_{\gamma \beta} \xi^\delta) - \epsilon^\delta \tilde{\bar{\psi}} a^\gamma (T^{\delta}_{\gamma \bar{\beta}} + T^{\delta}_{\gamma \bar{\beta}} \xi^\delta), \tag{2.6}
\]

\[
\delta Q \phi_a^i = - \epsilon^\beta (R^i_{\beta \alpha} + T^i_{\beta \alpha} \xi^i) - \tilde{\bar{\psi}} a^\gamma \phi_a^{i}, \tag{2.7}
\]

The supersymmetry covariantized versions of the torsions, gravitino field strength and field strengths associated respective with the inverse frame field $e^m_{\alpha}(x)$, gravitino $\psi_a^\delta(x)$, and connection fields for the tangent space symmetries $\phi_a^i(x)$ take the forms below and are expressed in terms dependent on linear and quadratic in the gravitino, and linear in the superspace torsions and curvature supertensors.

\[
T^{c}_{ab} = t^{c}_{ab} + \psi_a^\delta T^{c}_{\delta b} + \tilde{\bar{\psi}} a^\gamma T^{c}_{\gamma b} + \psi_a^\delta \tilde{\bar{\psi}} a^\beta T^{c}_{\beta b} + \psi_a^\delta \psi_a^\gamma T^{c}_{\gamma b} + \tilde{\bar{\psi}} a^\beta \tilde{\bar{\psi}} a^\gamma T^{c}_{\beta b}, \tag{2.8}
\]

\[
T^{\gamma}_{ab} = t^{\gamma}_{ab} + \psi_a^\delta T^{\gamma}_{\delta b} + \tilde{\bar{\psi}} a^\gamma T^{\gamma}_{\gamma b} + \psi_a^\delta \tilde{\bar{\psi}} a^\beta T^{\gamma}_{\beta b} + \psi_a^\delta \psi_a^\gamma T^{\gamma}_{\gamma b} + \tilde{\bar{\psi}} a^\beta \tilde{\bar{\psi}} a^\gamma T^{\gamma}_{\beta b}, \tag{2.9}
\]

\[
R^i_{ab} = r^i_{ab} + \psi_a^\delta R^i_{\delta b} + \tilde{\bar{\psi}} a^\gamma R^i_{\gamma b} + \psi_a^\delta \tilde{\bar{\psi}} a^\beta R^i_{\beta b} + \psi_a^\delta \psi_a^\gamma R^i_{\gamma b} + \tilde{\bar{\psi}} a^\beta \tilde{\bar{\psi}} a^\gamma R^i_{\beta b}. \tag{2.10}
\]

In the linearized limit of these theories, not all of the terms in (2.2) - (2.10) appear. Instead these equations take the forms

\[
\xi^m = - \left[ \epsilon^\alpha \epsilon^\beta a^\delta + \epsilon^\beta a^\gamma \epsilon^\alpha \right] T^{c}_{\alpha \beta} + \epsilon^\alpha \epsilon^\beta a^\gamma T^{c}_{\alpha \beta} + \epsilon^\alpha \epsilon^\beta a^\delta T^{c}_{\alpha \beta} \right] e^{m}_{c}, \tag{2.11}
\]

\[
\epsilon^\delta = - \left[ (\epsilon^\alpha \epsilon^\beta a^\gamma + \epsilon^\beta a^\gamma \epsilon^\alpha ) (T^{\delta}_{\alpha \beta} + T^{\delta}_{\alpha \beta} \xi^\delta) + \epsilon^\alpha \epsilon^\beta a^\gamma (T^{\delta}_{\alpha \beta} + T^{\delta}_{\alpha \beta} \xi^\delta) + \epsilon^\alpha \epsilon^\beta a^\gamma (T^{\delta}_{\alpha \beta} + T^{\delta}_{\alpha \beta} \xi^\delta) \right] \epsilon^\delta, \tag{2.12}
\]

\[
\chi^i = - \left[ (\epsilon^\alpha \epsilon^\beta a^\gamma + \epsilon^\beta a^\gamma \epsilon^\alpha ) (R^{i}_{\alpha \beta} + T^{i}_{\alpha \beta} \xi^i) + \epsilon^\alpha \epsilon^\beta a^\gamma (R^{i}_{\alpha \beta} + T^{i}_{\alpha \beta} \xi^i) + \epsilon^\alpha \epsilon^\beta a^\gamma (R^{i}_{\alpha \beta} + T^{i}_{\alpha \beta} \xi^i) \right] \epsilon^\delta, \tag{2.13}
\]

\[
\delta Q e^{m}_{\alpha} = - \left[ \epsilon^\delta T^{d}_{\beta \alpha} + \epsilon^\delta T^{d}_{\beta \alpha} + (\epsilon^\delta \psi_a^\gamma + \epsilon^\delta \tilde{\bar{\psi}} a^\gamma) T^{d}_{\beta \gamma} + \epsilon^\delta \psi_a^\gamma T^{d}_{\beta \gamma} + \epsilon^\delta \tilde{\bar{\psi}} a^\gamma T^{d}_{\beta \gamma} \right] e^{m}_{d}, \tag{2.14}
\]

\[
\delta Q \psi_a^\delta = D_a^\delta - \epsilon^\delta T^{\delta}_{\beta \alpha} - \epsilon^\delta T^{\delta}_{\beta \alpha} \right] e^{\tilde{\bar{\psi}} a^\delta}, \tag{2.15}
\]

\[
\delta Q \phi_a^i = - \epsilon^\delta R^i_{\beta \alpha} - \epsilon^\delta R^i_{\beta \alpha} \right] \epsilon^\delta, \tag{2.16}
\]

\[
T^{\gamma}_{ab} = t^{\gamma}_{ab}, \tag{2.17}
\]

\[
T^{c}_{ab} = t^{c}_{ab}. \tag{2.18}
\]
\[ R^i_{ab} = r^i_{ab} \]  \hspace{1cm} (2.19)

The terms on the RHS of the final three equation correspond to the non-supercovariantized versions of the respective torsions, gravitino field strength and connection field strengths.
3 Higher Dimensional Component Considerations

In the following four subsections, we will appropriately adapt these results to the cases of eleven and ten dimensional formulations appropriate for Nordström supergravity in those contexts. There are four steps:

(a.) define a Nordström SG linearized superspace supercovariant derivative in terms of a scalar prepotential leading to component fields,
(b.) express the geometrical tensors of each respective superspace in terms of the component field presented in the previous part,
(c.) express the ‘composition rules’ of the parameters of general coordinate, local Lorentz, and local SUSY transformations, and
(c.) write the component level SUSY transformation laws

that we undertake in each of the four cases of 11D, \( N = 1 \), 10D, \( N = 1 \), 10D, \( N = 2A \), and 10D, \( N = 2B \), theories.

3.1 Adaptation To 11D, \( \mathcal{N} = 1 \) Component/Superspace Results: Step 1

In the case of the 11D N(ordström)-SG covariant derivatives we define
\[
\nabla_\alpha = D_\alpha + \frac{1}{2}\Psi D_\alpha + \frac{1}{10}(\gamma^d_\alpha)^{\beta}(D_\beta \Psi)\mathcal{M}_{de}, \tag{3.1}
\]
\[
\nabla_a = \partial_a + \Psi \partial_a + i\frac{1}{4}(\gamma_a)^{\alpha\beta}(D_\alpha \Psi)D_\beta - (\partial_\Psi \mathcal{M}_a^\xi)\mathcal{M}^\xi_{\xi}. \tag{3.2}
\]

and “split” the spatial 11D N-SG covariant derivative into two parts
\[
\nabla_a\bigg| = D_a + \psi_a^\gamma \nabla_\gamma. \tag{3.3}
\]

On taking the \( \theta \to 0 \) limit the latter terms allows an identification with the gravitino and the leading term in this limit yields a component-level linearized gravitationally covariant derivative operator given by
\[
D_a = e_a + \phi_a^\xi \mathcal{M}_\xi = \partial_a + \Psi \partial_a + \phi_a^\xi \mathcal{M}_\xi. \tag{3.4}
\]

By comparison of the LHS to the RHS of (3.4), we see that a linearized frame field \( e_a^m = (1 + \Psi)\delta_a^m \) emerges to describe a scalar graviton. Finally, comparison of the coefficient of the Lorentz generator \( \mathcal{M}_\xi \) as it appears in the latter two forms of (3.4) informs us the spin connection is given by
\[
\phi_a^{de} = -\frac{1}{2}\delta_a^{[d}(\partial^e \Psi) \tag{3.5}
\]

Comparing the result in (3.2) with the one in (3.3) a component gravitino is identified via
\[
\psi_a^\gamma = i\frac{1}{4}(\gamma_a)^{\gamma\delta}(D_\delta \Psi) \tag{3.6}
\]

However, as this expression contains an explicit \( \gamma \)-matrix we see that it really defines the non-conformal spin-\(\frac{1}{2} \) part of the gravitino to be
\[
\psi_\beta \equiv (\gamma^a)^{\beta\gamma} \psi_a^\gamma. \tag{3.7}
\]
This is to be expected. As a Nordström type theory only contains a scalar graviton, it follows only the \(\gamma\)-trace of the gravitino can occur. So then we have

\[
D_\beta \Psi = i \frac{4}{11} (\gamma^a)_{\beta\gamma} \psi_a^\gamma \equiv i \frac{4}{11} \psi_\beta ,
\]

in the \(\theta \to 0\) limit.

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:

\[
K = C^{\gamma\delta}(D_\gamma D_\delta \Psi) , \quad K_{[3]} = (\gamma_{[3]})^{\gamma\delta}(D_\gamma D_\delta \Psi) , \quad K_{[4]} = (\gamma_{[4]})^{\gamma\delta}(D_\gamma D_\delta \Psi) ,
\]

or in other words,

\[
\frac{1}{2} D_{[\gamma} D_{\delta]} \Psi = \frac{1}{32} \left\{ C_{\gamma\delta} K - \frac{1}{3!} (\gamma_{[3]})_{\gamma\delta} K_{[3]} + \frac{1}{4!} (\gamma_{[4]})_{\gamma\delta} K_{[4]} \right\} .
\]

We emphasize that the component fields (the \(K\)'s) are defined by the \(\theta \to 0\) limit of these equations. The results in (3.9) and (3.10) follow as results from a Fierz identity

\[
\delta_{\gamma\delta}^{\alpha\beta} = \frac{1}{16} \left\{ C_{\gamma\delta} C^{\alpha\beta} - \frac{1}{3!} (\gamma_{[3]})_{\gamma\delta} (\gamma_{[3]})^{\alpha\beta} + \frac{1}{4!} (\gamma_{[4]})_{\gamma\delta} (\gamma_{[4]})^{\alpha\beta} \right\} ,
\]

valid for 11D spinors.

3.2 Adaptation To 11D, \(\mathcal{N} = 1\) Component/Superspace Results: Step 2

Torsions:

\[
T_{\alpha\beta}^\xi = i (\gamma^\xi)_{\alpha\beta} ,
\]

\[
T_{\alpha\beta}^{\gamma} = i \frac{3}{110} (\gamma_{[2]})_{\alpha\beta} (\gamma_{[2]})^{\gamma\delta} \psi_\delta ,
\]

\[
T_{ab}^\xi = i \frac{3}{11} \left[ \delta_{ab}^{\xi\delta} \delta_{\alpha}^{\beta} + \frac{3}{5} (\gamma_{[2]})_{\alpha}^{\xi \beta} \right] \psi_\beta ,
\]

\[
T_{ab}^{\gamma} = i \frac{1}{128} \left[ - (\gamma_{[2]})_{\alpha}^{\gamma} K + \frac{1}{2} (\gamma_{[2]})_{\alpha}^{\gamma} K_{[2]} - \frac{1}{3!} (\gamma_{[3]})_{\alpha}^{\gamma} K_{[3]} + \frac{1}{3!} (\gamma_{[3]})_{\alpha}^{\gamma} K_{[3]}^{[3]} - \frac{1}{4!} (\gamma_{[4]})_{\alpha}^{\gamma} K_{[4]} + \frac{1}{8} \left[ \delta_{ab}^{\xi\delta} \delta_{\alpha}^{\beta} + 3 (\gamma_{[2]})_{\alpha}^{\gamma} \right] (\partial_{ab} \Psi) \right] ,
\]

\[
T_{ab}^{\xi} = 0 ,
\]

\[
T_{ab}^{\gamma} = \frac{1}{11} (\gamma_{[2]})^{\gamma\delta} (\partial_{[a} \psi_{b]} \) .
\]

Curvatures:

\[
R_{\alpha\beta}^{\gamma} = \frac{1}{80} \left[ (\gamma_{[2]})_{\alpha\beta} K + (\gamma_{[1]})_{\alpha\beta} K_{[1]}^{[2]} - \frac{1}{3!} (\gamma_{[3]})_{\alpha\beta} K_{[3]} + \frac{1}{2} (\gamma_{[2]})_{\alpha\beta} K_{[2]}^{[2]} - \frac{1}{5!} 4! (\gamma_{[5]})_{\alpha\beta} K_{[5]} \right] ,
\]

\[
R_{ab}^{\gamma} = i \frac{3}{11} \left[ \delta_{ab}^{\xi\delta} (\partial_{[a} \psi_{b]} + \frac{1}{2} (\gamma_{[2]})_{\alpha}^{\gamma} \right] (\partial_{ab} \Psi) \right] ,
\]

\[
R_{ab}^{\xi} = - (\partial_{ab} \partial_{[a} \psi_{b]}) \delta_{ab}^{\xi\delta} .
\]
3.3 Adaptation To 11D, $\mathcal{N} = 1$ Component/Superspace Results: Step 3

Parameter Composition Rules:

\[
\xi_m = -i\epsilon_1^\alpha e^\beta_2 (\gamma^\xi)_{\alpha\beta} \delta^m_\xi (1 + \Psi),
\]

(3.21)

\[
\lambda^\alpha = -\frac{1}{80} \epsilon_1^\alpha e_2^\beta \left[ (\gamma^d)_{\alpha\beta} K + (\gamma_1)_{\alpha\beta} K^{1|d} - \frac{1}{3!} (\gamma^d\Psi)_{\alpha\beta} K^{[2]d} - \frac{1}{2} (\gamma_2)_{\alpha\beta} K^{2|d} \right] + \frac{1}{5!} \epsilon_1^\alpha e_2^\beta (\gamma^d)_{\alpha\beta} (\delta^2 \Psi),
\]

(3.22)

\[
e^\delta = i \frac{1}{11} \epsilon_1^\alpha e_2^\beta \left[ (\gamma^1)_{\alpha\beta} (\gamma_1)_{\delta} - \frac{3}{10} (\gamma^2)_{\alpha\beta} (\gamma_2)_{\delta} \right] \psi^\delta.
\]

(3.23)

3.4 Adaptation To 11D, $\mathcal{N} = 1$ Component/Superspace Results: Step 4

SUSY transformation laws:

\[
\delta Q e^m_\alpha = -i \frac{4}{11} e^\beta \left[ \delta^d_\alpha \delta^\gamma_\beta + \frac{1}{5} (\gamma^d_\alpha)_{\beta} \right] \delta^m_\xi \psi^\gamma,
\]

(3.24)

\[
\delta Q \psi^\delta_\alpha = (1 + \Psi) \partial^\delta_\alpha \psi^\delta - \epsilon^\gamma (\partial^\gamma_\alpha \psi^\delta) \mathcal{M}^\gamma_\xi
\]

\[-i \frac{1}{128} e^\beta \left[ -(\gamma^2_\alpha)_{\beta} \frac{1}{2} (\gamma^2_\alpha)_{\beta} K^{2|d} - \frac{1}{3!} (\gamma^2_\alpha)_{\beta} K^{[2]} + \frac{1}{3!} (\gamma^3_\alpha)_{\beta} K^{3|d} \right] \psi^\delta - \frac{1}{8} e^\beta \left[ \delta^\xi_\alpha \psi^\delta + 3 (\gamma^2_\alpha)_{\beta} \psi^\delta \right] (\partial^\xi_\alpha \psi^\delta),
\]

(3.25)

\[
\delta Q \phi^d_\alpha = -i \frac{4}{11} e^\beta \left[ \delta^d_\alpha (\partial^\beta \psi^\delta) + \frac{1}{5} (\gamma^d_\alpha)_{\beta} (\partial^\beta \psi^\delta) \right].
\]

(3.26)

In the remaining subsections of the chapter, the steps described for the case of the 11D, $\mathcal{N} = 1$ theory above will be repeated, essentially line by line, in each of the cases for 10D, $\mathcal{N} = 1$, 10D, $\mathcal{N} = 2\lambda$, and 10D, $\mathcal{N} = 2B$ superspaces. This will imply a certain repetitive nature to the respective presentation. There will only be slight various in explicit details. We are able to minimize this very slightly by noting the result in (3.4) applies universally in all three cases. So we will not explicitly rewrite it nor its resultant implications several more times.

3.5 Adaptation To 10D, $\mathcal{N} = 1$ Component/Superspace Results: Step 1

In the case of 10D $\mathcal{N} = 1$ N-SG covariant derivatives we define

\[
\nabla_\alpha = D_\alpha + \frac{1}{2} \Psi D_\alpha + \frac{1}{10} (\sigma^a_\alpha)_{\beta} (D_\beta \Psi) \mathcal{M}_{ab} \psi^a \psi^b,
\]

(3.27)

\[
\nabla_\alpha = \partial_\alpha + \Psi \partial_\alpha - i \frac{2}{5} (\sigma^a_\alpha)_{\beta} (D_\beta \Psi) D_\beta - (\partial^\gamma_\alpha \psi^\delta) \mathcal{M}^{\gamma\delta}_\xi.
\]

(3.28)

and “split” the spatial 10D $\mathcal{N} = 1$ N-SG covariant derivative into two parts

\[
\nabla_\alpha = D_\alpha + \psi^\gamma \nabla_\gamma.
\]

(3.29)

Comparing the result (3.28) in with the one in (3.29) a component gravitino is identified via

\[
\psi^\gamma = -i \frac{2}{5} (\sigma^a_\alpha)_{\beta} (D_\delta \Psi).
\]

(3.30)
However, as this expression contains an explicit $\sigma$-matrix we see that it defines the non-conformal spin-$\frac{1}{2}$ part of the gravitino to be
\[
\psi_\beta \equiv (\sigma^a)_\beta_\gamma \psi_a^\gamma.
\]
and it follows only the “$\sigma$-trace” of the gravitino can occur. So then we have
\[
D_\beta \Psi = i \frac{1}{4} (\sigma^a)_\beta_\gamma \psi_a^\gamma \equiv i \frac{1}{4} \psi_\beta ,
\]
in the $\theta \to 0$ limit.

The complete specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. We take advantage of the 10D Fierz identity
\[
\delta [\gamma_\alpha \delta_\beta] = \frac{1}{48} (\sigma^{[3]})_\gamma_\delta (\gamma^{[3]})^{\alpha_\beta} ,
\]
valid for 10D spinors, so we may define a bosonic field:
\[
G_{[3]} = (\sigma_{[3]})^{\gamma_\delta} (D_\gamma D_\delta \Psi) ,
\]
or in other words,
\[
\frac{1}{2} D_{[\gamma} D_{\delta]} \Psi = \frac{1}{16 \times 3!} (\sigma^{[3]})_{\gamma_\delta} G_{[3]} .
\]
We emphasize that the component field (the $G$) is defined by the $\theta \to 0$ limit of these equations.

### 3.6 Adaptation To 10D, $\mathcal{N} = 1$ Component/Superspace Results: Step 2

#### Torsions:

\[
\begin{align*}
T_{\alpha \beta}^\varepsilon &= i (\sigma^\varepsilon)_{\alpha_\beta} , \\
T_{\alpha}^\gamma &= 0 , \\
T_{\alpha b}^\varepsilon &= i \frac{3}{20} \left[ \delta_{\varepsilon}^2 \varepsilon_\alpha^\delta + (\sigma^\varepsilon)_{\alpha}^\delta \right] \psi_\delta , \\
T_{\alpha b}^\gamma &= \frac{1}{80} \left[ - (\sigma^{[2]})_{\gamma_\beta} G_{[2]} + \frac{1}{3} (\sigma_{[2]}^\gamma)_{\alpha} G^{[3]} \right] - \frac{3}{10} \left[ \delta_{\varepsilon}^2 \varepsilon_\alpha^\gamma - (\sigma^\varepsilon)_{\alpha}^\gamma \right] (\partial_\beta \Psi) , \\
T_{ab}^\gamma &= 0 , \\
T_{ab}^\varepsilon &= - \frac{1}{10} (\sigma_{[2]})^{\gamma_\delta} (\partial_{[2]} \psi_\delta) .
\end{align*}
\]

#### Curvatures:

\[
\begin{align*}
R_{\alpha \beta de} &= - i \frac{6}{5} (\sigma^{[d])_{\alpha_\beta} (\partial^{[e]} \Psi) - \frac{1}{40} \left[ \frac{1}{3!} (\sigma^{de[3]})_{\alpha_\beta} G_{[3]} + (\sigma_{[1]})_{\alpha_\beta} G_{[1])de} \right] , \\
R_{ab de} &= i \frac{1}{4} \left[ \delta_{[2]} (\partial_{[2]} \psi_\alpha) + \frac{1}{5} (\sigma^{de})_{\alpha}^\gamma (\partial_\gamma \psi_\beta) \right] , \\
R_{ab de} &= - (\partial_{[2]} G_{[2]} \psi_{[d]} )^{\delta_\varepsilon_\delta} .
\end{align*}
\]

### 3.7 Adaptation To 10D, $\mathcal{N} = 1$ Component/Superspace Results: Step 3

#### Parameter Composition Rules:

\[
\begin{align*}
\xi^m &= - i \epsilon_1^\alpha \epsilon_2^\beta (\sigma^\varepsilon)^{\alpha_\beta} \delta^\varepsilon_\varepsilon^m (1 + \Psi) , \\
\lambda^{de} &= \frac{1}{40} \epsilon_1^\alpha \epsilon_2^\beta \left[ \frac{1}{3!} (\sigma^{de[3]})_{\alpha_\beta} G_{[3]} + (\sigma_{[1]})_{\alpha_\beta} G_{[1])de} \right] + i \frac{17}{10} \epsilon_1^\alpha \epsilon_2^\beta (\sigma^{d])_{\alpha_\beta} (\partial^{[e]} \Psi) , \\
\varepsilon^d &= - i \frac{1}{10} \epsilon_1^\alpha \epsilon_2^\beta (\sigma^\varepsilon)^{\alpha_\beta} (\sigma^e)^{\beta_\delta} (\partial^{[e]} \Psi) ,
\end{align*}
\]
3.8 Adaptation To 10D, $\mathcal{N} = 1$ Component/Superspace Results: Step 4

SUSY transformation laws:

$$\delta Q e^m_a = -i \frac{1}{4} \epsilon^\beta \left[ \delta_a^d \delta_\beta^\gamma + \frac{1}{5} (\sigma^d_a)_\beta^\gamma \right] \delta_a^m \psi_\gamma \ ,$$  
(3.48)

$$\delta Q \psi_\alpha^\beta = (1 + \Psi) \partial_\alpha \epsilon^\beta - \epsilon^\beta (\partial_\alpha \psi) \mathcal{M}_\alpha^\epsilon$$

$$- i \frac{1}{80} \epsilon^\beta \left[ - (\sigma^2)_\alpha^\beta G_{2|2} + \frac{1}{3} (\sigma_{2|3})_\beta^\delta G_{3|2} \right] + \frac{3}{10} \epsilon^\beta \left[ \delta_a^\alpha \delta_a^\beta - \frac{1}{5} (\sigma^a_a)_\beta^\gamma \right] (\partial_\alpha \psi) \ ,$$  
(3.49)

$$\delta Q \phi_{d^e}^a = - i \frac{1}{4} \epsilon^\beta \left[ \delta_a^d (\partial_\alpha \psi) + \frac{1}{5} (\sigma^d_e)_\beta^\gamma (\partial_\alpha \psi) \right] .$$  
(3.50)

3.9 Adaptation To 10D, $\mathcal{N} = 2A$ Component/Superspace Results: Step 1

In the case of 10D $\mathcal{N} = 2A$ N-SG covariant derivatives we define

$$\nabla_\alpha = D_\alpha + \frac{1}{2} \Psi D_\alpha + \frac{1}{10} (\sigma^{ab})_\alpha^\beta (D_\beta \psi) \mathcal{M}_{ab} \ ,$$  
(3.51)

$$\nabla_\alpha = D_\alpha + \frac{1}{2} \Psi D_\alpha + \frac{1}{10} (\sigma^{ab})_\alpha^\beta (D_\beta \psi) \mathcal{M}_{ab} \ ,$$  
(3.52)

$$\nabla_a = \partial_a + \Psi \partial_a - \frac{1}{5} (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) D_\gamma - \frac{1}{5} (\sigma_a^\beta) (D_\beta \psi) D_\gamma - (\partial_a \psi) \mathcal{M}_a^\epsilon \ ,$$  
(3.53)

and “split” the spatial 10D $\mathcal{N} = 2A$ N-SG covariant derivative into three parts

$$\nabla_a \mid = D_a + \psi_a^\gamma \nabla_\gamma + \psi_a^\gamma \nabla_\gamma .$$  
(3.54)

On taking the $\theta \rightarrow 0$ limit the latter terms allow an identification with the component gravitinos are identified via

$$\psi_a^\gamma = - i \frac{1}{5} (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) \ , \ \psi_a^\gamma = - i \frac{1}{5} (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) \ .$$  
(3.55)

However, as this expression contains an explicit $\sigma$-matrix we see that it really defines the non-conformal spin-$\frac{1}{2}$ part of the gravitino to be

$$\psi_\beta \equiv (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) \ , \ \psi_\beta \equiv (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) .$$  
(3.56)

It follows only the “$\sigma$-trace” of the gravitino can occur. So then we have

$$D_\beta \psi = i \frac{1}{2} (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) \equiv i \frac{1}{2} \psi_\beta \ , \ \ D_\beta \psi = i \frac{1}{2} (\sigma_a^\beta) \gamma^\delta (D_\beta \psi) \equiv i \frac{1}{2} \psi_\beta \ ,$$  
(3.57)

in the $\theta \rightarrow 0$ limit.

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:

$$G_{[3]} = (\sigma_{[3]})^{\gamma^\delta} (D_\gamma D_\delta \psi) \ , \ H_{[3]} = (\sigma_{[3]})^{\gamma^\delta} (D_\gamma D_\delta \psi) \ ,$$  
(3.58)

$$N = C^{\gamma^\delta} (D_\gamma D_\delta \psi) \ , \ N_{[2]} = (\sigma_{[2]})^{\gamma^\delta} (D_\gamma D_\delta \psi) \ , \ N_{[4]} = (\sigma_{[4]})^{\gamma^\delta} (D_\gamma D_\delta \psi) \ ,$$  
(3.59)

or in other words,

$$\frac{1}{2} D_{[\gamma} D_{\delta]} \psi = \frac{1}{16 \times 3!} (\sigma_{[3]})^{\gamma^\delta} G_{[3]} \ , \ \frac{1}{2} D_{[\gamma} D_{\delta]} \psi = \frac{1}{16 \times 3!} (\sigma_{[3]})^{\gamma^\delta} H_{[3]} \ ,$$  
(3.60)

and

$$D_\gamma D_\delta \psi = \frac{1}{16} \left[ C_{\gamma^\delta} N + \frac{1}{2!} (\sigma_{[2]})^{\gamma^\delta} N_{[2]} + \frac{1}{4!} (\sigma_{[4]})^{\gamma^\delta} N_{[4]} \right] .$$  
(3.61)

We emphasize that the component fields (the $G$’s, $H$’s and $N$’s) are defined by the $\theta \rightarrow 0$ limit of these equations.
3.10 Adaptation To 10D, $N = 2A$ Component/Superspace Results: Step 2

Torsions:
\[
T_{\alpha \beta}^{\gamma} = i(\sigma^{\gamma}_{\alpha \beta}) ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = -\frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = i(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = -\frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = i(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = 0 ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{4}\left[\delta_{\alpha \beta}^{\gamma} + \frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{4}\left[\delta_{\alpha \beta}^{\gamma} + \frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{5}\left[\delta_{\alpha \beta}^{\gamma} + (\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{80}\left[-\frac{1}{2}(\sigma^{[2]})_{\alpha \beta}G_{[2]|} + \frac{1}{3!}(\sigma^{[3]})_{\alpha \beta}G^{[3]} - \frac{1}{5}\left[\delta_{\alpha \beta}^{\gamma} + (\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}\right]_{\gamma_{\delta}}^{\delta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{5}\left[\delta_{\alpha \beta}^{\gamma} + (\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{80}\left[(\sigma^{[2]})_{\alpha \beta}^\gamma N + (\sigma^{[1]})_{\alpha \beta}^\gamma N_{[1]} - \frac{1}{2}(\sigma^{[2]})_{\alpha \beta}^\gamma N^{[2]} - \frac{1}{3!}(\sigma^{[3]})_{\alpha \beta}^\gamma N_{[3]}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = \frac{1}{5}\left[\delta_{\alpha \beta}^{\gamma} + (\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}\right]_{\gamma_{\delta}}^{\delta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = -\frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\[
T_{\alpha \beta}^{\gamma} = -\frac{1}{10}(\sigma^{\gamma}_{\alpha \beta})_{\alpha \beta}(\sigma_{\alpha})^{\gamma}_{\beta} ,
\]
\]

Curvatures:
\[
R_{\alpha \beta}^{\gamma \delta} = -\frac{6}{5}(\sigma^{\gamma \delta}_{\alpha \beta})(\partial_{\alpha \beta}^{\gamma \delta}) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\gamma \delta}_{\alpha \beta})_{\alpha \beta}G_{[3]} + (\sigma^{[1]})_{\alpha \beta}G^{[1]}_{\alpha \beta\gamma \delta}\right] ,
\]
\[
R_{\alpha \beta}^{\gamma \delta} = -\frac{6}{5}(\sigma^{\gamma \delta}_{\alpha \beta})(\partial_{\alpha \beta}^{\gamma \delta}) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\gamma \delta}_{\alpha \beta})_{\alpha \beta}H_{[3]} + (\sigma^{[1]})_{\alpha \beta}H^{[1]}_{\alpha \beta\gamma \delta}\right] ,
\]
\[
R_{\alpha \beta}^{\gamma \delta} = \frac{1}{40}\left[(\sigma^{\gamma \delta}_{\alpha \beta})_{\alpha \beta}N_{\gamma \delta} - C_{\alpha \beta}N_{\gamma \delta} + \frac{1}{2}(\sigma^{\gamma \delta}_{\alpha \beta})_{\alpha \beta}N_{\gamma \delta}\right] ,
\]
3.11 Adaptation To 10D, \( \mathcal{N} = 2A \) Component/Superspace Results: Step 3

Parameter Composition Rules:

\[
\xi_m = - i \left[ \epsilon_1^\alpha \epsilon_2^\beta (\sigma^2)_{\alpha\beta} + \epsilon_1^\alpha \epsilon_2^\beta (\sigma^2)_{\dot{\alpha}\dot{\beta}} \right] \delta_e^m (1 + \Psi) ,
\]

\[
\chi^{de} = - \frac{1}{40} (\epsilon_1^\alpha \epsilon_2^\beta + \epsilon_1^\dot{\alpha} \epsilon_2^\dot{\beta}) \left[ (\sigma^0)^{de}_{\alpha\beta} \mathcal{N} - C_{\alpha\beta}^d \mathcal{N}^{de} + \frac{1}{2} (\sigma^0)^{de[2]}_{\alpha\beta} \mathcal{N}^{[2]} \right] \\
+ \frac{1}{2} (\delta^d_{\alpha\beta} \mathcal{N}^{de} + \frac{1}{4!} e^{de[4]} (\sigma^0)^{[4]}_{\alpha\beta} \mathcal{N}^{[4]}) \\
+ \frac{1}{2} (\delta^d_{\dot{\alpha}\dot{\beta}} (\sigma^0)^{de} + \frac{1}{4!} e^{de[4]} (\sigma^0)^{[4]}_{\dot{\alpha}\dot{\beta} \mathcal{N}^{[4]}} \right] ,
\]

\[
\epsilon^\delta = - i \frac{1}{4} (\epsilon_1^\alpha \epsilon_2^\beta + \epsilon_1^\dot{\alpha} \epsilon_2^\dot{\beta}) \left[ \delta^\delta_{\alpha} \delta^e_{\beta} + \frac{1}{10} (\sigma^2)^{\delta}_{\alpha} (\sigma^2)^{e}_{\beta} \right] \psi^\delta \\
- \frac{1}{5} \epsilon_1^\alpha \epsilon_2^\beta (\sigma^2)_{\alpha\beta} (\sigma^2)^{\delta e} \psi^\delta .
\]

3.12 Adaptation To 10D, \( \mathcal{N} = 2A \) Component/Superspace Results: Step 4

SUSY transformation laws:

\[
\delta_Q e^m_a = - i \frac{1}{2} \epsilon^{|d|} \left[ \delta^m_{a\gamma} + \frac{1}{5} (\sigma^0_a)^{|\dot{\alpha}|} \right] \delta^e_{a\psi^\gamma} - i \frac{1}{2} \epsilon^{|\dot{d}|} \left[ \delta^m_{a\dot{\gamma}} + \frac{1}{5} (\sigma^0_a)^{|\dot{\beta}|} \right] \delta^e_{a\psi^\gamma} ,
\]

\[
\delta_Q \psi^\delta = (1 + \Psi) \partial_a^\delta - \epsilon^\delta (\partial_a^\delta \psi^\delta) M^a \epsilon \\
- i \frac{1}{80} \epsilon^{|d|} \left[ - \frac{1}{2} (\sigma^0)^{|d|}_{a\beta} G_a^{[2]} + \frac{1}{3!} (\sigma^0)^{|d|}_{a\beta} G^{[3]} \right] + \frac{2}{5} \epsilon^{|\dot{d}|} \left[ \delta^d_{a\delta} - \sigma^0_a \right] \psi^\delta ,
\]

\[
\delta_Q \phi^{de} = - i \frac{1}{2} \epsilon^{|d|} \left[ \delta^d_{a\beta} (\partial_a^\beta \psi^\gamma) + \frac{1}{5} (\sigma^0)^{|d|}_{a\beta} (\partial_a^\beta \psi^\gamma) \right] - i \frac{1}{2} \epsilon^{|\dot{d}|} \left[ \delta^d_{a\beta} (\partial_a^\beta \psi^\gamma) + \frac{1}{5} (\sigma^0)^{|\dot{d}|}_{a\beta} (\partial_a^\beta \psi^\gamma) \right] .
\]

3.13 Adaptation To 10D, \( \mathcal{N} = 2B \) Component/Superspace Results: Step 1

In the case of 10D \( \mathcal{N} = 2B \) N-SG covariant derivatives we define

\[
\nabla_\alpha = D_\alpha + \frac{1}{2} \Psi D_\alpha + \frac{1}{10} (\sigma^{ab})_{\alpha}^\beta (D_\beta \Psi) M^{ab} ,
\]

\[
\bar{\nabla}_\alpha = \bar{D}_\alpha + \frac{1}{2} \Psi \bar{D}_\alpha + \frac{1}{10} (\sigma^{ab})_{\alpha}^\beta (\bar{D}_\beta \Psi) M^{ab} ,
\]

\( \nabla_\alpha = (D_\alpha + \frac{1}{2} \Psi D_\alpha) + \frac{1}{10} (\sigma^{ab})_{\alpha}^\beta (D_\beta \Psi) M^{ab} ,
\]

\( \bar{\nabla}_\alpha = (\bar{D}_\alpha + \frac{1}{2} \Psi \bar{D}_\alpha) + \frac{1}{10} (\sigma^{ab})_{\alpha}^\beta (\bar{D}_\beta \Psi) M^{ab} .
\]
\[ \nabla_a = \partial_a + \frac{1}{2} \Psi \partial_a + \frac{1}{2} \bar{\Psi} \partial_a - i \, \frac{1}{32} (\sigma_2)^{\alpha \beta} (D_\alpha \bar{\Psi}) \bar{D}_\beta - i \, \frac{1}{32} (\sigma_2)^{\alpha \beta} (\bar{D}_\alpha \Psi) D_\beta - i \, \frac{27}{160} (\sigma_2)^{\alpha \beta} (D_\alpha \bar{\Psi}) \bar{D}_\beta - i \, \frac{27}{160} (\sigma_2)^{\alpha \beta} (\bar{D}_\alpha \Psi) D_\beta - \frac{1}{2} (\partial_a \bar{\Psi}) M_{a \epsilon} - \frac{1}{2} (\partial_a \Psi) M_{a \epsilon} , \]

and “split” the spatial 10D \( N = 2 \) B-N-SG covariant derivative into three parts

\[ \nabla_a = D_a + \bar{\psi}_a^\beta \nabla_\beta + \bar{\psi}_a^\gamma \nabla_\gamma \ . \] (3.95)

On taking the \( \theta \to 0 \) limit the latter terms allows an identification with the gravitino and the leading term in this limit yields a component-level linearized gravitationally covariant derivative operator given by

\[ D_a = e_a + \phi_a^\epsilon M_\epsilon = \partial_a + \frac{1}{2} (\Psi + \bar{\Psi}) \partial_a + \phi_a^\epsilon M_\epsilon \ . \] (3.96)

Comparison of the LHS to the RHS of (3.96), we see that a linearized frame field \( e_a = (1 + \frac{1}{2} (\Psi + \bar{\Psi})) \delta_a \) emerges to describe a scalar graviton. Finally, comparison of the coefficient of the Lorentz generator \( M_\epsilon \) as it appears in the latter two forms of (3.96) informs us the spin connection is given by

\[ \phi_a^{\beta \epsilon} = - \frac{1}{4} \delta_a^{\beta \epsilon} (\partial_\epsilon (\Psi + \bar{\Psi})) \ . \] (3.97)

Comparing the result (3.94) in with the one in (3.95) the component gravitinos are identified via

\[ \psi_a^\gamma = - i \, \frac{1}{160} (\sigma_2)^{\gamma \delta} (\bar{D}_\delta (5 \Psi + 27 \bar{\Psi})) \ , \] (3.98)
\[ \bar{\psi}_a^\gamma = - i \, \frac{1}{160} (\sigma_2)^{\gamma \delta} (D_\delta (5 \Psi + 27 \bar{\Psi})) \ , \] (3.99)

which are equivalent to

\[ \bar{D}_a (5 \Psi + 27 \bar{\Psi}) = i 16 (\sigma_2)^\alpha_\gamma \psi_a^\gamma \ , \ \ D_a (5 \Psi + 27 \bar{\Psi}) = i 16 (\sigma_2)^\alpha_\gamma \bar{\psi}_a^\gamma \ . \] (3.100)

However, as this expression contains an explicit \( \sigma \)-matrix we see that it really defines the non-conformal spin-\( \frac{1}{2} \) part of the gravitino to be

\[ \psi_\beta \equiv (\sigma_2)^{\beta_\gamma} \psi_a^\gamma \ , \ \bar{\psi}_\beta \equiv - (\sigma_2)^{\beta_\gamma} \bar{\psi}_a^\gamma \ . \] (3.101)

Since the results in (3.100) are under-constrained, we are allowed to introduce a fermionic auxiliary field \( \lambda_\alpha \) and its complex conjugate \( \bar{\lambda}_\alpha \). So then we have

\[ \bar{D}_a \Psi = i \frac{1}{2} (\sigma_2)^{\alpha_\gamma} \psi_a^\gamma - 27 \bar{\lambda}_\alpha \equiv i \frac{1}{2} \psi_\alpha - 27 \bar{\lambda}_\alpha \ , \] (3.102)
\[ D_a \bar{\Psi} = i \frac{1}{2} (\sigma_2)^{\alpha_\gamma} \psi_a^\gamma + 5 \bar{\lambda}_\alpha \equiv i \frac{1}{2} \psi_a + 5 \bar{\lambda}_\alpha \ , \] (3.103)
\[ D_a \bar{\Psi} = i \frac{1}{2} (\sigma_2)^{\alpha_\gamma} \bar{\psi}_a^\gamma - 27 \lambda_\alpha \equiv - i \frac{1}{2} \psi_\alpha - 27 \lambda_\alpha \ , \] (3.104)
\[ D_a \Psi = i \frac{1}{2} (\sigma_2)^{\alpha_\gamma} \bar{\psi}_a^\gamma + 5 \lambda_\alpha \equiv - i \frac{1}{2} \bar{\psi}_a + 5 \lambda_\alpha \ . \] (3.105)
in the $\theta \to 0$ limit. Also observe that
\[
D_\alpha(\bar{\Psi} - \Psi) = 32\lambda_\alpha , \quad D_\alpha(\Psi - \bar{\Psi}) = 32\lambda_\alpha . \tag{3.106}
\]

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:
\[
\begin{align*}
U_3 &= (\sigma_3)^{\gamma\delta}(D_\gamma D_\delta \Psi) , \\
\bar{U}_3 &= - (\sigma_3)^{\gamma\delta}(\bar{D}_\gamma \bar{D}_\delta \bar{\Psi}) , \\
X_3 &= (\sigma_3)^{\gamma\delta}(D_\gamma D_\delta \bar{\Psi}) , \\
\bar{X}_3 &= - (\sigma_3)^{\gamma\delta}(\bar{D}_\gamma \bar{D}_\delta \bar{\Psi}) , \\
Y_3 &= (\sigma_3)^{\gamma\delta}(D_\gamma D_\delta \psi) , \\
\bar{Y}_3 &= - (\sigma_3)^{\gamma\delta}(\bar{D}_\gamma \bar{D}_\delta \bar{\psi}) .
\end{align*}
\tag{3.107}
\]

Ue emphasize that the component fields (the $U$'s, $X$'s and $Y$'s) are defined by the $\theta \to 0$ limit of these equations.

### 3.14 Adaptation To 10D, $\mathcal{N} = 2B$ Component/Superspace Results: Step 2

#### Torsions:
\[
\begin{align*}
T_{\alpha\beta}^\xi &= 0 , \\
T_{\alpha\beta}^\gamma &= -i\frac{1}{5}(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + 2(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\lambda_\delta , \\
T_{\alpha\gamma}^\gamma &= 0 , \\
T_{\alpha\beta}^\xi &= 0 , \\
T_{\alpha\beta}^\gamma &= 0 , \\
T_{\alpha\beta}^\gamma &= i\frac{1}{5}(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + 2(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\lambda_\delta , \\
T_{\alpha\beta}^{\xi} &= i(\sigma_5)^{\alpha\beta} , \\
T_{\alpha\beta}^{\gamma} &= -i\frac{1}{240}(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{8}[((\sigma_5)^{\gamma\delta}(\sigma_5)^{\gamma\delta} - \frac{1}{30}(\sigma_5)^{\gamma\delta}(\sigma_5)^{\gamma\delta})\lambda_\delta] , \\
T_{\alpha\beta}^{\gamma} &= i\frac{1}{240}(\sigma_5)^{\alpha\beta}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{8}[((\sigma_5)^{\gamma\delta}(\sigma_5)^{\gamma\delta} - \frac{1}{30}(\sigma_5)^{\gamma\delta}(\sigma_5)^{\gamma\delta})\lambda_\delta] , \\
T_{\alpha\beta}^{\xi} &= -i\frac{1}{5}[2\delta_2^{\nu}\delta_2^\gamma + (\sigma_5^{\nu})^\gamma\bar{\psi}_\gamma + [-11\delta_2^{\nu}\delta_2^\gamma + (\sigma_5^{\nu})^\gamma]\lambda_\gamma , \\
T_{\alpha\beta}^{\gamma} &= \frac{1}{64}[ -31\delta_2^{\nu}\delta_2^\gamma + 15(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) + \frac{1}{320}[27\delta_2^{\nu}\delta_2^\gamma + 53(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) , \\
T_{\alpha\beta}^{\gamma} &= -i\frac{1}{2560}[\frac{1}{2}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{320}[27\delta_2^{\nu}\delta_2^\gamma + 53(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) , \\
T_{\alpha\beta}^{\gamma} &= -i\frac{1}{2560}[\frac{1}{2}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{320}[27\delta_2^{\nu}\delta_2^\gamma + 53(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) , \\
T_{\alpha\beta}^{\gamma} &= -i\frac{1}{2560}[\frac{1}{2}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{320}[27\delta_2^{\nu}\delta_2^\gamma + 53(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) , \\
T_{\alpha\beta}^{\gamma} &= -i\frac{1}{2560}[\frac{1}{2}(\sigma_5)^{\gamma\delta}\bar{\psi}_\delta + \frac{1}{320}[27\delta_2^{\nu}\delta_2^\gamma + 53(\sigma_5^{\nu})^\gamma ](\bar{D}_\gamma \bar{\psi}) .
\end{align*}
\tag{3.108}
\]
\[ T_{ab}^\varphi = 0 \]  
\[ T_{ab}^\gamma = -\frac{1}{10} (\sigma_{[2]}^\gamma)(\partial_2 \bar{\psi}) \]  
\[ T_{ab}^\gamma = \frac{1}{10} (\sigma_{[2]}^\gamma)(\partial_2 \bar{\psi}) \]  

Curvatures:

\[ R_{\alpha\beta}^{\text{de}} = \frac{1}{40} \left[ \frac{\Gamma^3}{3!}(\sigma^{\text{de}[3]}_{\alpha\beta}U_{[3]} - (\sigma_{[1]}^\alpha\beta U_{[1]}^{\text{de}}) \right] \]  
\[ R_{\alpha\beta}^{\text{de}} = -\frac{1}{10} \left[ \frac{\Gamma^3}{3!}(\sigma^{\text{de}[3]}_{\alpha\beta}U_{[3]} - (\sigma_{[1]}^\alpha\beta U_{[1]}^{\text{de}}) \right] \]  
\[ R_{\alpha\beta}^{\text{de}} = -\frac{3}{5} (\sigma^{\text{de}}_{\alpha\beta}(\partial^\varphi(\Psi + \overline{\Psi})) - i \frac{1}{10} (\sigma^{\text{de}[2]}_{\alpha\beta}(\partial^\varphi(\Psi + \overline{\Psi})) \]  
\[ R_{\alpha\beta}^{\text{de}} = \frac{1}{10} \left[ \frac{\Gamma^3}{3!}(\sigma^{\text{de}[3]}_{\alpha\beta}U_{[3]} - (\sigma_{[1]}^\alpha\beta U_{[1]}^{\text{de}}) \right] \]  

3.15 Adaptation To 10D, \( N = 2B \) Component/Superspace Results: Step 3

Parameter Composition Rules:

\[ \xi^m = -i(e_1^{\alpha}\bar{e}_2^\beta + e_1^\beta\bar{e}_2^\alpha)(\sigma^\epsilon_{\alpha\beta}\delta_\epsilon^m(1 + \frac{1}{2}(\Psi + \overline{\Psi})) \]  
\[ \lambda^{\text{de}} = -(e_1^{\alpha}\bar{e}_2^\beta + e_1^\beta\bar{e}_2^\alpha) \left[ i - \frac{17}{20}(\sigma^{\text{de}}_{\alpha\beta}(\partial^\varphi(\Psi + \overline{\Psi})) - i \frac{1}{10} (\sigma^{\text{de}[2]}_{\alpha\beta}(\partial^\varphi(\Psi + \overline{\Psi}) \]  
\[ \xi^m - \frac{1}{40} e_1^{\alpha}\bar{e}_2^\beta \left[ \frac{\Gamma^3}{3!}(\sigma^{\text{de}[3]}_{\alpha\beta}U_{[3]} - (\sigma_{[1]}^\alpha\beta U_{[1]}^{\text{de}}) \right] + \frac{1}{40} \bar{e}_1^\alpha e_2^\beta \left[ \frac{\Gamma^3}{3!}(\sigma^{\text{de}[3]}_{\alpha\beta}U_{[3]} - (\sigma_{[1]}^\alpha\beta U_{[1]}^{\text{de}}) \right] \right] \]  
\[ e^\delta = -(e_1^{\alpha}\bar{e}_2^\beta + e_1^\beta\bar{e}_2^\alpha) \left[ i \frac{1}{10} \left( (\sigma^\delta_{\alpha\beta}(\sigma^{[6]})) + \frac{1}{24}(\sigma^{[6]}_{\alpha\beta}(\sigma^{[6]})) \delta^\epsilon \right) \psi_\epsilon \right. \]  
\[ + \frac{1}{8} \left( (\sigma^{[3]}_{\alpha\beta}(\sigma^{[3]})) + \frac{1}{30}(\sigma^{[5]}_{\alpha\beta}(\sigma^{[5]})) \delta^\epsilon \right) \lambda_\epsilon \]  
\[ - e_1^{\alpha}\bar{e}_2^\beta \left[ - i \frac{1}{5}(\sigma^\delta_{\alpha\beta}(\sigma^{[1]})) \delta^\epsilon \psi_\epsilon + 2(\sigma^\delta_{\alpha\beta}(\sigma^{[1]})) \delta^\epsilon \lambda_\epsilon \right] \]
\[\delta Q e_{a m} = - \epsilon^{\beta} \left[ - i \frac{1}{2} \left[ \delta_a^{d} \delta_{\beta}^{\gamma} + \frac{1}{5} (\sigma_a^{d})_{\beta}^{\gamma} \right] \bar{\psi}_\gamma + \left[ - 11 \delta_a^{d} \delta_{\beta}^{\gamma} + (\sigma_a^{d})_{\beta}^{\gamma} \right] \lambda_\gamma \right] \delta_q^{m} \]

\[\delta Q \psi_a^{\delta} = \left( 1 + \frac{1}{2} (\Psi + \bar{\Psi}) \right) \partial_a^{\delta} - \frac{1}{2} \epsilon^{\beta} (\partial_a (\Psi + \bar{\Psi})) M_a^{\delta} \]

\[\delta Q \phi_a^{de} = i \frac{1}{2} \epsilon^{\beta} \left[ \delta_a^{[d} (\partial_{e]} \bar{\psi}_\beta) + \frac{1}{5} (\sigma_a^{de})^{\gamma}_{\beta} (\partial_a^{\gamma} \bar{\psi}_\gamma) \right] - \epsilon^{\beta} \left[ - 11 \delta_a^{[d} (\partial_{e]} \lambda_\beta) + (\sigma_a^{de})^{\gamma}_{\beta} (\partial_a^{\gamma} \lambda_\gamma) \right] \]

\[\delta Q \phi_a^{de} = i \frac{1}{2} \epsilon^{\beta} \left[ \delta_a^{[d} (\partial_{e]} \psi_\beta) + \frac{1}{5} (\sigma_a^{de})^{\gamma}_{\beta} (\partial_a^{\gamma} \psi_\gamma) \right] - \epsilon^{\beta} \left[ - 11 \delta_a^{[d} (\partial_{e]} \bar{\lambda}_\beta) + (\sigma_a^{de})^{\gamma}_{\beta} (\partial_a^{\gamma} \bar{\lambda}_\gamma) \right] . \]
4 10D, $\mathcal{N} = 2B$ Chiral Compensator Considerations

In the limits where all supergravity fields are set to zero, four sets of super algebras emerge. These take the forms:

(a.) 11D, $\mathcal{N} = 1$,

$$\{ D_\alpha , D_\beta \} = i (\gamma^a)_{\alpha\beta} \partial_a , \ [D_\alpha , \partial_b] = 0 , \ [\partial_a , \partial_b] = 0$$ (4.1)

(b.) 10D, $\mathcal{N} = 1$,

$$\{ D_\alpha , D_\beta \} = i (\sigma^a)_{\alpha\beta} \partial_a , \ [D_\alpha , \partial_b] = 0 , \ [\partial_a , \partial_b] = 0$$ (4.2)

(c.) 10D, $\mathcal{N} = 2A$,

$$\{ D_\alpha , D_\beta \} = i (\sigma^a)_{\alpha\beta} \partial_a , \ \{ D_\alpha , D_{\dot{\beta}} \} = i (\sigma^a)_{\dot{\alpha}\dot{\beta}} \partial_a , \ \{ D_\alpha , D_{\dot{\beta}} \} = 0 ,$$ (4.3)

$$[D_\alpha , \partial_a] = 0 , \quad [D_{\dot{\alpha}} , \partial_a] = 0 , \quad [\partial_a , \partial_b] = 0 ,$$

(d.) 10D, $\mathcal{N} = 2B$,

$$\{ D_\alpha , D_\beta \} = 0 , \ \{ \bar{D}_\alpha , \bar{D}_\beta \} = 0 , \ \{ D_\alpha , \bar{D}_\beta \} = i (\sigma^a)_{\alpha\beta} \partial_a ,$$ (4.4)

$$[D_\alpha , \partial_a] = 0 , \quad [\bar{D}_\alpha , \partial_a] = 0 , \quad [\partial_a , \partial_b] = 0 ,$$

We next introduce a complex superfield denoted by $\Omega_d$ into each of these $d$-dimensional superspaces and seek to probe the implications of impose a first order differential equation imposed on this superfield that utilizes any of the spinorial derivatives above.

For either the 11D, $\mathcal{N} = 1$ or 10D, $\mathcal{N} = 1$ superspaces we have

$$D_\beta \Omega_d = 0 \rightarrow D_\alpha D_\beta \Omega_d = 0 \rightarrow \{ D_\alpha , D_\beta \} \Omega_d = 0 \rightarrow \partial_a \Omega_d = 0 \quad ,$$ (4.5)

and by analogy for the 10D, $\mathcal{N} = 2A$ superspace we find

$$D_\beta \Omega_d = 0 \rightarrow D_\alpha D_\beta \Omega_d = 0 \rightarrow \{ D_\alpha , D_\beta \} \Omega_d = 0 \rightarrow \partial_a \Omega_d = 0 \quad ,$$

$$D_\beta \Omega_d = 0 \rightarrow D_{\dot{\alpha}} D_\beta \Omega_d = 0 \rightarrow \{ D_{\dot{\alpha}} , D_\beta \} \Omega_d = 0 \rightarrow \partial_a \Omega_d = 0 \quad ,$$ (4.6)

Thus, from (4.5) and (4.6) we find the superfield $\Omega_d$ in each of these $d$-dimensional superspaces must be a constant. However, upon repeating these considerations for the 10D, $\mathcal{N} = 2B$ superspace we find

$$D_\beta \Omega_d = 0 \rightarrow D_\alpha D_\beta \Omega_d = 0 \rightarrow \{ D_\alpha , D_\beta \} \Omega_d = 0 \rightarrow 0 = 0 \quad ,$$ (4.7)

$$\bar{D}_\beta \Omega_d = 0 \rightarrow \bar{D}_\alpha \bar{D}_\beta \Omega_d = 0 \rightarrow \{ \bar{D}_\alpha , \bar{D}_\beta \} \Omega_d = 0 \rightarrow 0 = 0 \quad ,$$

which shows that the superfield $\Omega_d$ in this case can be a non-trivial representation of the translation operator.

The differential equation

$$\bar{D}_\beta \Omega_d = 0 \quad ,$$ (4.8)
in the context of four dimensions implies that \( \Omega_d \) is a “chiral superfield.” On the other hand the differential equation
\[
D \beta \Omega_d = 0 ,
\]
(4.9)
in the context of four dimensions implies that \( \Omega_d \) is a “anti-chiral superfield.” While it is not possible to simultaneously impose both conditions because a chiral superfield is the complex conjugate of an anti-chiral one, either one or the other can be imposed. This also means that neither the chiral nor the anti-chiral condition can be applied to a real superfield.

Let us return to the results shown (3.105) by focusing only on the equations that contain \( \lambda_{\alpha} \)
\[
\begin{align*}
D_\alpha \Psi &= \frac{i}{2} (\sigma^a)_{\alpha \gamma} \bar{\psi}_a^\gamma + 5 \lambda_{\alpha} \equiv -\frac{i}{2} \bar{\psi}_\alpha + 5 \lambda_{\alpha} , \\
D_\alpha \bar{\Psi} &= \frac{i}{2} (\sigma^a)_{\alpha \gamma} \bar{\psi}_a^\gamma - 27 \lambda_{\alpha} \equiv -\frac{i}{2} \bar{\psi}_\alpha - 27 \lambda_{\alpha} ,
\end{align*}
\]
(4.10)
since the remaining equations can be obtain by complex conjugation. In all the other cases we have explored, there is no spinor field such as \( \lambda_{\alpha} \). Taking the difference of the two equations that appear in (4.10), we may obtain
\[
i \frac{1}{32} D_\alpha (\Psi - \bar{\Psi}) = i \lambda_{\alpha} .
\]
(4.11)
However, the quantity \( i (\Psi - \bar{\Psi}) \) is a real superfield. The requirement that \( \lambda_{\alpha} = 0 \) is equivalent to the imposition of an anti-chirality condition on a real superfield and this condition possesses no non-trivial solution.

The inability to introduce such a chiral superfield distinguishes the type 2B theory from the other higher dimensional constructions we have considered. At first order in the \( \theta \)-expansion of \( \Psi \) both the spin-1/2 portion of the gravitino \( \bar{\psi}_a^\gamma \) and a separate spin-1/2 auxiliary spinor \( \lambda_{\alpha} \) must exist.
5 Conclusion & Possible Future Directions

In this work, we have presented the forms of the superspace torsions and curvature supertensors that are consistent with Nordström supergravity in eleven and ten dimensional superspaces. For the superspaces in 11D, \( \mathcal{N} = 1 \), 10D, \( \mathcal{N} = 1 \), 10D, \( \mathcal{N} = 2A \), and 10D, \( \mathcal{N} = 2B \), these results are found in the sets of equations given as (3.12) - (3.20), (3.36) - (3.44), (3.62) - (3.85), and (3.110) - (3.134), respectively. To our knowledge, these presentations initiate new results for the superspace torsions and curvature supertensors in these domains.

The use of the superfield \( \Psi \) in all cases guarantees all of these theories are “off-shell” supersymmetric without the need to impose some equations of motion for the fulfillment of a local supersymmetry algebra. The fact that \( \Psi \) used in each case does not satisfy any a priori superdifferential constraint implies the closure. Unfortunately, this same fact also implies that each of the descriptions we have provided is not an irreducible one. Exploring the possibility of imposing further superdifferential constraints to obtain one or more irreducible representations is the work for the future.

The work completed in this paper also suggests two new pathways to explore elements of 11D, \( \mathcal{N} = 1 \) supergeometry.

(a.)

In the works of \([27, 28]\) on the basis of the study of solutions to the 11D superspace Bianchi identities up to engineering dimension one, forms for the superspace torsions and curvature supertensors were proposed. Upon comparing particularly the results in the first of these references to the result derived in the current work as seen in (3.12) - (3.20), apparent concurrence is found. In the work of \([27]\), we can use the definition

\[
\nabla_\alpha J_\beta = C_{\alpha\beta}\mathcal{S} + \frac{1}{2}(\gamma^2)_{\alpha\beta}v_2 + \frac{1}{3!}(\gamma^3)_{\alpha\beta}t_2 + \frac{1}{4!}(\gamma^4)_{\alpha\beta}U_3 + \frac{1}{5!}(\gamma^5)_{\alpha\beta}Z_5 .
\]

In this former work, we must set the 11D “on-shell” superfield \( W_{abcd} \) to zero to make comparisons. When this is done, then by a change of notation where

\[
\psi_\alpha \rightarrow J_\alpha , \quad K_2 \rightarrow v_2 , \quad K_2 \rightarrow t_2 , \quad K_3 \rightarrow U_3 , \quad K_4 \rightarrow V_4 , \quad K_5 \rightarrow Z_5 ,
\]

we then look at (5.1) in contrast to the form of (3.8) and (3.9) in this work. We find in the Nordström limit,

\[
v_2 = \partial_2 \Psi , \quad t_2 = 0 , \quad Z_5 = 0 ,
\]

and thus there is significant overlap. In particular, the results in (5.3) tell us something interesting about the \( J_\alpha \) tensor. We can decompose it into two parts

\[
J_\alpha = J^{(T)}_\alpha + D_\alpha \Psi
\]

which is equivalent to the usual decomposition of a gauge field into its transverse and longitudinal parts. Upon setting the \( J^{(T)}_\alpha = 0 \), one recovers the Nordström theory.

\(\text{in comparison to these older works we have “rescaled” } t_2, U_3, V_4, \text{ and } Z_5 \text{ relative to the original definitions.}\)
There is a further feature noted in the work of [28] that also is indicated as a direction to include in this new pathway of exploration for 11D superspace supergravity.

While the notation of superconformal symmetry is not presently understood in a number of approaches to the study of 11D supergravity, the superspace approach in [28] is indicative of a specific further modification. In particular, by the introduction of a scaling transformation of the supervielbein, it was found that a modification of the spinor-spinor-vector component of the supertorsion that is given by the expression

$$ T_{\alpha\beta}^\epsilon = i(\gamma^\epsilon)_{\alpha\beta} + i(\gamma[^2])_{\alpha\beta} \mathcal{X}[^2]_{\alpha\beta}^\epsilon + i(\gamma[^5])_{\alpha\beta} \tilde{\mathcal{X}}[^5]_{\alpha\beta}^\epsilon $$

(5.5)
is consistent with the superspace scale transformations if and only if the “$\mathcal{X}$-tensor” and “$\tilde{\mathcal{X}}$-tensor” satisfy the conditions,

$$ \mathcal{X}[^8]_{\alpha\beta\gamma}^\epsilon = 0 \ , \ \epsilon[^8]_{\alpha\beta\gamma} \mathcal{X}[^8]_{\alpha\beta\gamma} = 0 \ , \ \mathcal{X}[^5]_{[4]}_{\alpha\beta\gamma}^\epsilon = 0 \ , \ \epsilon[^5]_{[a]bcedf} \tilde{\mathcal{X}}[^5]_{abcedf} = 0 \ . $$

(5.6)

A detailed and careful study of the 11D superspace supergravity Bianchi identities with the modifications in the current work as well as the works of [27,28] is indicated to assess the form of any equations of motion that emerges in the presence of retaining the on-shell field strength.

(b.) While the pathway for future investigation described above depends on the study of 11D supergravity supercovariant tensors and their Bianchi identities, the “Breitenlohner Approach” suggests a second pathway.

The 4D, $\mathcal{N} = 1$ Wess-Zumino gauge vector supermultiplet in (1.1) (or alternately the component level 4D, $\mathcal{N} = 1$ supermultiplet) arises in a very interesting way related to the 4D, $\mathcal{N} = 1$ real pseudoscalar superfield $V$. The components fields in $V$ may be expressed as an expansion in terms of the fermionic superspace D-operators followed by taking the limit as $\theta^a$ goes to zero. See equation (4.3.4a) in [24] and the equivalent expressions using the Majorana superspace coordinates associated with the superspace relevant to the component results in (1.1) take the forms

$$ C = V \ | \ , \ \chi_a = D_a V \ | \ , \ M = C^{ab} D_a D_b V \ | \ , \ N = i(\gamma^5)^{ab} D_a D_b V \ | \ , \\
 v_a = (\gamma^5 \gamma_a)^{ab} D_a D_b V \ | \ , \ \lambda^a = \epsilon^{a\beta\gamma} D_b D_c D_d V \ | \ , \ d = \epsilon^{a\beta\gamma} D_a D_b D_c D_d V \ | \ , $$

(5.7)

where $\epsilon^{a\beta\gamma}$ is the Levi-Civita tensor defined over the Majorana spinor indices. Also we have made adaptations in the notation that are appropriate for Majorana basis conventions in 4D. The results in (5.7) make clear there are eight bosons and eight fermions contained in this superfield. It is also clear there is a component level gauge 1-form $v_a$ that occurs at the quadratic order in the $\theta$-expansion of $V$.

Now let us consider the situation of an 11D, $\mathcal{N} = 1$ scalar superfield $V^{(11)}$ analogous to $V$. There are some differences of course. For example, in $V^{(11)}$ there are 2,147,483,648 bosonic component fields and 2,147,483,648 fermionic component fields. In the 11D, $\mathcal{N} = 1$ superspace, the quadratic order spinor supercovariant derivatives are in the $\{1\}$, $\{165\}$, and $\{330\}$ representations of the 11D Lorentz group whose explicit forms are

$$ \Delta^{(1)} = C^{\alpha\beta} D_\alpha D_\beta \ , \ \Delta^{(165)}_{abc} = (\gamma^{abc})^{\alpha\beta} D_\alpha D_\beta \ , \ \Delta^{(330)}_{abcd} = (\gamma^{abcd})^{\alpha\beta} D_\alpha D_\beta \ . $$

(5.8)
The implication of the existence of these operators is there is no component field at quadratic order in $V^{(11)}$ that occurs in the \{11\} representation of the 11D Lorentz group. This should be contrasted with the situation in 4D superspace where the operator $(\gamma^5\gamma_\alpha)^{ab}D_aD_b$ is in the \{4\} representation of the 4D Lorentz group.

However, at quartic utilizing the 11D spinorial derivatives we can define a superfield by the equation
\[ v^{(11)}_a = \frac{1}{3!} \left[ \Delta^{(165)abcd}\Delta^{(330)\ alpha\beta} V^{(11)} \right], \tag{5.9} \]
that is the analog to one of the equations seen to occur in (5.7). The “Breitenlohner Approach” can be followed by defining an operator valued supergravity co-vector $SG_a$ through the equation
\[ SG_a = \frac{1}{3!} \left[ \Delta^{(165)cde}\Delta^{(330)\alpha\beta\gamma} V^{(11)} \right] \partial_d + \frac{1}{3!} \left[ \Delta^{(165)cde}\Delta^{(330)\alpha\beta\gamma} V^{(11)} \right] D_\beta + \frac{1}{2}\cdots \tag{5.10} \]
and above $\partial_d$, $D_\beta$, $\mathcal{M}_{\hat{k}\hat{l}}$ denote respectively the 11D partial derivative operator, the 11D spinor superspace derivative, and the 11D Lorentz generators. There are other difference also. For the 4D superfield $V$ need only to be expanded to quartic order in $\theta^a$. In the case of the $V^{(11)}$ the $\theta^a$-expansion goes out to the order of $\theta$ raised to the thirty second power.

If no obstructions occur, this will describe 11D SG in superspace just as the expressions in (1.1) - (1.4) did for 4D, $N = 1$ superspace. It is possible the superfields $V^{(11)}h$, $V^{(11)}\beta$, and $V^{(11)}k\ell$ can be expressed in terms of more fundamental superfields as is the case in 4D, $N = 2$ superspace supergravity [29]. Moreover, were 11D superspace supergravity to follow the pattern of its lower dimensional “relatives,” the conformal part of the 11D graviton will be contained in the first term and the conformal part of the 11D gravitino will be contained in the second term of (5.10).

In any case, this approach would put a maximum limit on the number of component fields required in $SG_a$. We simply count the number of free indices on $V^{(11)}h$, $V^{(11)}\beta$, and $V^{(11)}k\ell$ to find $11 + 32 + 55 = 98$ and multiply by the number of component fields in $V^{(11)}$ to arrive at 210,453,397,504 bosonic component fields and 210,453,397,504 fermionic component fields. In fact depending on the size of the null space (and thus the gauge transformations of $V^{(11)}$) of the condition
\[ \left[ \Delta^{(165)cde}\Delta^{(330)\alpha\beta\gamma} V^{(11)} \right] \partial_d \left[ \delta V^{(11)} \right] = 0 , \tag{5.11} \]
the numbers could even be considerably less. There is also an argument that can be made to estimate the lower bound on the number of component fields involved.

In the works of [30,31] an algorithm was presented that, given a theory that possesses a number of 1D supercharges, determines the size of the smallest irreducible 1D SUSY representation. The theory in 11D, when reduced to 1D, corresponds to a 1D theory with 32 supercharges. For 1D, $N = 32$ supersymmetry the smallest off-shell representations determined by the algorithm possess 32,768 bosonic fields and 32,768 fermionic fields. Once more multiplying by 98 we are led to a lower bound of 3,211,264 bosonic components and 3,211,264 fermionic components. While 3.2 million component fields may seem a large number, it is far less than one percent of 210 billion.

Perhaps now the stage is set for us to (and very roughly paraphrasing Hilbert - reach beyond the level of Göttingen’s children) understand off-shell eleven dimensional supergravity supergeometry.
for M-Theory... and as well (with appropriate modifications), for ten dimensional supergravity supergeometries for heterotic and superstrings.

“Every boy in the streets of Göttingen understands more about four dimensional geometry than Einstein. Yet, in spite of that, Einstein did the work and not the mathematicians.”

- David Hilbert

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