ABOUT A CONJECTURE OF LIEB-SOLOVEJ

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Abstract. Very recently, E. H. Lieb and J. P. Solovej stated a conjecture about the constant of embedding between two Bergman spaces of the upper-half plane. A question in relation with a Werhl-type entropy inequality for the affine \(AX + B\) group. More precisely, that for any holomorphic function \(F\) on the upper-half plane \(\Pi^+\),

\[
\int_{\Pi^+} |F(x+iy)|^{2s} y^{2s-2} dxdy \leq \frac{\pi^{1-s}}{(2s-1)^{22s-2}} \left( \int_{\Pi^+} |F(x + iy)|^{2} dxdy \right)^s
\]

for \(s \geq 1\), and the constant \(\frac{\pi^{1-s}}{(2s-1)^{22s-2}}\) is sharp. We prove differently that the above holds whenever \(s\) is an integer and we prove that it holds when \(s \to \infty\). We also prove that when restricted to powers of the Bergman kernel, the conjecture holds. We next study the case where \(s\) is close to 1. Hereafter, we transfer the conjecture to the unit disc where we show that the conjecture holds when restricted to analytic monomials. Finally, we overview the bounds we obtain in our attempts to prove the conjecture.

1. Introduction

Let us denote the upper-half plane by \(\Pi^+ := \{ x + iy \in \mathbb{C} : y > 0 \}\). Let \(\nu > -1\) and \(1 \leq p < \infty\). We denote by \(A^p_\nu(\Pi^+)\) the weighted Bergman space consisting of holomorphic functions \(F\) on \(\Pi^+\) such that

\[
||F||_{A^p_\nu} := \left( \int_{\Pi^+} |F(x + iy)|^p y^\nu dx dy \right)^{\frac{1}{p}} < \infty.
\]

In \([1]\), in relation with a Werhl-type inequality for the affine \(AX + B\) group, E. H. Lieb and J. P. Solovej formulated the following conjecture. Conjecture: Let \(s \geq 1\). Then for any \(F \in A^2(\Pi^+) = A^2_0(\Pi^+)\), the following inequality holds

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy \leq \frac{\pi^{1-s}}{(2s-1)^{22s-2}} \left( \int_{\Pi^+} |F(x + iy)|^{2} dxdy \right)^s.
\]

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with equality if $F$ is proportional to $(z - z_0)^{-2}$.

The above inequality just translates the embedding of $A^2(\Pi^+)$ into $A^2_{2s-2}(\Pi^+)$ which is pretty easy to establish if one does not pay attention to the constant. Theorem 3 in [11] says that the inequality (1.1) holds when $s$ is an integer. The conjecture is then for the non-integer values of $s$.

We do not know yet how to prove this conjecture in general but provide a different proof in the case where $s$ is an integer and prove that in the general case, the constant in (1.1) is sharp. More precisely, we prove the following.

**Theorem 1.1.** The inequality (1.1) holds for all positive integers $s$.

In fact, Bayart, Brevig, Haimi, Ortega-Cerdà and Perfekt [4] also proved the same theorem and even settled the conjecture for $s = n + \frac{1}{2} \ (n = 1, 2, 3, \ldots)$. Their setting is the unit disc and we shall state their results precisely in Section 4, where we transfer the conjecture from the upper half-plane to the unit disc. Analogous conjectures on contractive inequalities for Hardy spaces were studied in [7]. Earlier sharp inequalities were obtained in [8].

In the sequel, for short, we adopt the following notation

$$C_s = \frac{\pi^{1-s}}{(2s-1)2^{2s-2}}.$$

We next prove that the above result holds whenever $s \to \infty$.

**Proposition 1.2.** The conjecture of Lieb-Solovej is asymptotically true, in the sense that

$$\lim_{s \to \infty} \max_{F \in A^2(\Pi^+), F \not\equiv 0} \frac{\int_{\Pi^+} |F(x+iy)|^{2s}y^{2s-2}dxdy}{\left(\int_{\Pi^+} |F(x+iy)|^{2}dxdy\right)^s} = 1. \tag{1.2}$$

We also obtain that when restricted only to powers of the Bergman kernel, the conjecture holds.

**Theorem 1.3.**

$$\int_{\Pi^+} \frac{y^{2s-2}}{|x+i(y+1)|^{2r}}dxdy \leq C_s \left(\int_{\Pi^+} \frac{dxdy}{|x+i(y+1)|^{2r}}\right)^s. \tag{1.3}$$

Moreover, equality holds in (1.3) if and only if $r = 2$.

A direct consequence of Theorem 1.3 is the following.

**Corollary 1.4.** May the Lieb-Solovej conjecture be true, the Bergman kernel functions $\frac{1}{\pi}(z - z_0)^{2} \ (z_0 \in \Pi^+)$ will be maximizing functions of this extremum problem.
We finally provide an equivalent form for the Lieb-Solovej conjecture for \( s \) close to 1.

**Theorem 1.5.** Let \( F \) be a holomorphic function in \( \Pi^+ \) such that

\[
\int_{\Pi^+} |F(x + iy)|^2 dx dy = 1.
\]

The following two assertions are equivalent.

1. the Lieb-Solovej conjecture is valid for \( s \) close to 1, i.e. there exists \( s_0 > 1 \) such that for every \( s \in (1, s_0) \), we have

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy \leq \frac{\pi^{1-s}}{(2s-1)2^{2s-2}}.
\]

2. \[
\int_{\Pi^+} \log \left[ \frac{1}{2\sqrt{\pi}|F(x + iy)|y} \right] |F(x + iy)|^2 dx dy \geq 1
\]

Unfortunately, we only succeeded in proving the following partial result.

**Theorem 1.6.** Let \( F \in A^2(\Pi^+) \) such that \( \|F\|_{A^2(\Pi^+)} = 1 \). We have

\[
\int_{\Pi^+} \log \left[ \frac{1}{2\sqrt{\pi}|F(x + iy)|y} \right] |F(x + iy)|^2 dx dy \geq \frac{\log 3}{2} \approx 0.5493.
\]

The plan of the paper is as follows. In Section 2, we prove estimates for the standard weighted Bergman kernels from which we deduce a related estimate (Corollary 2.3). There, we prove Theorem 1.1 and Proposition 1.2. In Section 3, we prove Theorem 1.3, Theorem 1.4 and Theorem 1.5. In Section 4, we transfer the conjecture and our results to the unit disc and we show that the conjecture holds when restricted to analytic monomials. Finally, in Section 5, we overview the bounds we obtain in our attempts to prove the Lieb-Solovej conjecture.

## 2. The cases where \( s \) is an integer and \( s \to \infty \).

### 2.1. Estimates for the standard weighted Bergman kernels.

For \( \nu > 0 \), we recall the expression of the standard weighted Bergman kernel \( B_\nu(z, w) \) of \( \Pi^+ \) (cf. e.g. [5]):

\[
B_\nu(z, w) = \frac{2^{\nu-1}\nu}{\pi} \left( \frac{z - \bar{w}}{i} \right)^{-\nu} (z, w \in \Pi^+).
\]

In particular, this kernel has the following reproducing property:

\[
F(z) = \int_{\Pi^+} B_\nu(z, u + iv)F(u + iv)\nu^{\nu-1}du dv \quad (F \in A^2(\Pi^+)).
\]
We shall need the following result from [3]. We give its proof for completeness.

**Proposition 2.1.** Let \( r > 0, \ t > -1 \) with \( 2r - t > 2 \). Then for \( x + iy \in \Pi^+ \),

\[
\int_{\Pi^+} \frac{v^r du \ dv}{|x + iy - u + iv|^{2r}} = \frac{4\pi \Gamma(1 + t) \Gamma(2r - t - 2)}{2^{2r} \Gamma(r)^2 y^{2r - t - 2}}.
\]

**Proof.** We first integrate with respect to \( u \). We have

\[
I_{y,v} := \int_{-\infty}^{\infty} \frac{du}{|x + iy - u + iv|^{2r}} = \int_{-\infty}^{\infty} \frac{du}{(x - u)^2 + (y + v)^2}.
\]

We apply the change of variable \( u \mapsto s = \frac{x-u}{y+v}, \ du = -(y+v)ds \); we get:

\[
I_{y,v} = \frac{c_r}{(y+v)^{2r-1}},
\]

where

\[
c_r := \int_{-\infty}^{\infty} \frac{1}{(s^2 + 1)^r} ds.
\]

We next integrate with respect to \( v \); applying the change of variable \( v \mapsto \tau = \frac{\tau}{y}, \ dv = y d\tau \), we obtain:

\[
\int_{0}^{\infty} \frac{v^r dv}{(y + v)^{2r-1}} = d_{r,t} \frac{1}{y^{2r - t - 2}},
\]

where

\[
d_{r,t} := \int_{0}^{\infty} \frac{\tau^r d\tau}{(1 + \tau)^{2r-1}}.
\]

We have shown that

\[
\int_{\Pi^+} \frac{v^r du \ dv}{|x + iy - u + iv|^{2r}} = c_r d_{r,t} \frac{1}{y^{2r - t - 2}}.
\]

To conclude the proof, we need the following lemma.

**Lemma 2.2.** We have the following equalities.

\[
c_r = \frac{\sqrt{\pi} \Gamma \left( r - \frac{1}{2} \right)}{\Gamma(r)}
\]

and

\[
d_{r,t} = \frac{\Gamma(t + 1) \Gamma(2r - t - 2)}{\Gamma(2r - 1)}.
\]
Proof of the lemma. We have:
\[ c_r = 2 \int_0^\infty \frac{1}{(s^2 + 1)^r} ds. \]
Applying the change of variable \( s = \sqrt{\sigma} \), \( ds = \frac{d\sigma}{2\sqrt{\sigma}} \), we get:
\[ c_r = \int_0^\infty \frac{\sigma^{-\frac{1}{2}}}{(\sigma + 1)^r} d\sigma. \]
We next apply the following well known formula:
\[ B(x, y) = \int_0^\infty \frac{t^{x-1}}{(t+1)^{x+y}} dt \quad (x > 0, \ y > 0), \]
where \( B(\cdot, \cdot) \) denotes the beta function. We obtain:
\[ c_r = B \left( \frac{1}{2}, r - \frac{1}{2} \right) = \frac{\Gamma(\frac{1}{2})\Gamma(r - \frac{1}{2})}{\Gamma(r)} = \frac{\sqrt{\pi}\Gamma(r - \frac{1}{2})}{\Gamma(r)} \]
and
\[ d_{r,t} = B(t + 1, 2r - t - 2) = \frac{\Gamma(t + 1)\Gamma(2r - t - 2)}{\Gamma(2r - 1)}. \]
It follows from this lemma that
\[ c_r d_{r,t} = \frac{\sqrt{\pi}\Gamma(r - \frac{1}{2})\Gamma(t + 1)\Gamma(2r - t - 2)}{\Gamma(r)\Gamma(2r - 1)}. \]
We finally apply the duplication formula (cf. e.g. [2]):
\[ \Gamma(x)\Gamma(x + \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2x)}{2^{2x-1}} \quad (x > 0). \]
For \( x = r - \frac{1}{2} \), we get:
\[ \frac{\Gamma(r - \frac{1}{2})}{\Gamma(2r - 1)} = \frac{\sqrt{\pi}}{2^{2r-2}\Gamma(r)}. \]
We then conclude that
\[ c_r d_{r,t} = \frac{4\pi\Gamma(1 + t)\Gamma(2r - t - 2)}{2^{2r}(\Gamma(r))^2}. \]
The result follows.

From the reproducing property (2.2) of the Bergman kernel and the previous proposition, we deduce the following estimate.
Corollary 2.3. For every $F \in A^2(\Pi^+)$, we have

$$\sup_{x+iy \in \Pi^+} |F(x + iy)| y \leq \frac{1}{2\sqrt{\pi}} \|F\|_{A^2(\Pi^+)}.$$  

This estimate is sharp as equality holds for $F_0(z) = \frac{1}{(z+i)^2}$.

Proof. From (2.1) and (2.2) in the particular case $\nu = 1$, it follows that

$$|F(x + iy)| \leq \frac{1}{\pi} \int_{\Pi^+} \frac{1}{|x + iy - u + iv|^2} |F(u + iv)| dudv.$$  

Applying the Schwarz inequality implies

$$|F(x + iy)| \leq \frac{1}{\pi} \left( \int_{\Pi^+} \frac{1}{|x + iy - u + iv|^4} dudv \right)^{\frac{1}{2}} \|F\|_{A^2(\Pi^+)}.$$  

Applying the previous proposition with $t = 0$ and $r = 2$, we obtain:

$$(2.4) \quad \int_{\Pi^+} \frac{1}{|x + iy - u + iv|^4} dudv = \frac{\pi}{4y^2}.$$  

So

$$\sup_{x+iy \in \Pi^+} |F(x + iy)| y \leq \frac{1}{2\sqrt{\pi}} \|F\|_{A^2(\Pi^+)}.$$  

For $F_0(x + iy) = \frac{1}{(z+i)^2}$, we have

$$\sup_{x+iy \in \Pi^+} |F_0(x + iy)| y \geq |F(i)| \times 1 = \frac{1}{4} = \frac{1}{2\sqrt{\pi}} \|F_0\|_{A^2(\Pi^+)},$$  

where the latter equality follows from (2.4). This proves that the estimate is sharp. \qed

2.2. Proof of Theorem 1.1.

Remark 2.4. The conjecture of Lieb-Solovej [11] can be written in the form of the following extremum problem. For all $s > 1$,

$$(2.5) \quad \max_{F \in A^2(\Pi^+), F \neq 0} \frac{\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy}{\left( \int_{\Pi^+} |F(x + iy)|^{2s} dxdy \right)^{\frac{s}{2}}} = \frac{\pi^{1-s}}{(2s-1)^{2s-2}}.$$  

May this conjecture be true, Theorem 1.3 says that the Bergman kernel functions $\frac{1}{\pi}(z - \bar{z}_0)^2$ ($z_0 \in \Pi^+$) will be maximizing functions of this problem.

Suggestion. Prove that the maximum is attained (easy). Show next that the maximum functions are 'unique' up to a multiplicative constant and are equal to Bergman kernel functions.
Lieb and Solovej have proved the conjecture when $s$ is an integer greater than 1. To this aim, they used representation theory. We provide a direct proof below based on Fourier-Laplace representation of functions in $A^2(\Pi^+)$. 

**Proof of Theorem 1.1.** We start this proof by recalling the following Paley-Wiener theorem for Bergman spaces (see [9, Theorem 1]).

**Proposition 2.5.** Let $F$ be a holomorphic function on $\Pi^+$, and let $\alpha > -1$. Then the following assertions are equivalent.

1. $F$ belongs to the weighted Bergman space $A^2_\alpha(\Pi^+)$. 
2. There exists a function $f : (0, \infty) \rightarrow \mathbb{C}$ satisfying the estimate 
   \[
   \int_0^\infty \frac{|f(t)|^2}{t^{\alpha + 1}} dt < \infty,
   \]
   such that 
   \[
   F(z) = \int_0^\infty f(t)e^{itz} dt \quad (z \in \Pi^+).
   \]
   In this case, 
   \[
   ||F||^2_{A^2_\alpha(\Pi^+)} = \frac{2\pi \Gamma(\alpha + 1)}{2^{\alpha + 1}} \int_0^\infty |f(t)|^2 \frac{dt}{t^{\alpha + 1}},
   \]
   where $\Gamma$ is the usual gamma function.

The following lemma is the key of our proof.

**Lemma 2.6.** For any $u > 0$, if $I_n$ is the integral defined by 

\[
I_n(u) := \int_{A_n} \left( u - \sum_{j=1}^n t_j \right) \left( \prod_{j=1}^n t_j \right) dt_1 \ldots t_n
\]

where $A_n := \{(t_1, \ldots, t_n) \in (0, \infty)^n : u - \sum_{j=1}^n t_j > 0\}$, then $I_n$ converges and 

\[
I_n(u) = \frac{u^{2n+1}}{\Gamma(2n+2)}.
\]

**Proof.** For $k = 0, \ldots, n-1$, define $I_{n-k}$ by 

\[
I_{n-k}(u) := \int_{A_{n,k}} \left( u - \sum_{j=1}^{n-k} t_j \right)^{2k+1} \left( \prod_{j=1}^{n-k} t_j \right) dt_1 \ldots t_{n-k}
\]

where $A_{n,k} := \{(t_1, \ldots, t_{n-k}) \in (0, \infty)^n : u - \sum_{j=1}^{n-k} t_j > 0\}$.

We observe that 

\[
I_1(u) = \int_{(0,\infty) \cap (0,u)} (u-t)^{2n-1} t dt = u^{2n+1}B(2n,2)
\]

where $B(\cdot, \cdot)$ is the usual beta function.
The lemma follows from the estimate of $I_1$ and the following relation.

\[ I_{n-k+1}(u) = B(2k, 2)I_{n-k}(u), \quad k = 0, \ldots, n - 1. \]

To prove (2.6), we recall that

\[ I_{n-k+1}(u) = \int_{A_{n,k+1}} \left( u - \sum_{j=1}^{n-k+1} t_j \right)^{2k-1} \prod_{j=1}^{n-k+1} t_j dt_1 \ldots t_{n-k+1}. \]

Put $t_{n-k+1} = \left( u - \sum_{j=1}^{n-k} t_j \right) t$. Then we obtain

\[ I_{n-k+1}(u) = \int_{A_{n,k}} \left( u - \sum_{j=1}^{n-k} t_j \right)^{2k+1} \prod_{j=1}^{n-k} t_j \left( \int_0^1 (1 - t)^{2k-1} td\right) dt_1 \ldots t_{n-k} = B(2k, 2)I_{n-k}(u). \]

\[ \square \]

Let us now prove Theorem 1, i.e. the conjecture in the case of integer exponents. Let $s = n > 1$ be an integer. Let $F \in A^2(\Pi^+)$. We recall with Proposition 2.5 that

\[ F(z) = \int_0^\infty e^{izt} f(t) dt, \quad z \in \Pi^+ \]
with

\[ \int_{\Pi^+} |F(z)|^2 dV(z) = \pi \int_0^\infty |f(t)|^2 t dt. \]

We observe that $\|F\|_{A^2_n} = \|F^n\|_{A^2_{2n-2}}$. We can write

\[ F^n(z) = \int_{(0, \infty)^n} e^{iz(t_1+t_2+\ldots+t_n)} f(t_1) \ldots f(t_n) dt_1 \ldots dt_n \]
\[ = \int_0^\infty e^{izu} g(u) du \]
where

\[ g(u) = \int_{A_{n-1}} f(u - \sum_{j=2}^{n} t_j) f(t_2) \ldots f(t_n) dt_2 \ldots dt_n, \]
\[ A_{n-1} = \{(t_2, \ldots, t_n) \in (0, \infty)^{n-1} : u - \sum_{j=2}^{n} t_j > 0\}. \]

By Proposition 2.5, we only need to estimate

\[ \frac{2\pi \Gamma(2n-1)}{2^{2n-1}} \int_0^\infty \frac{|g(u)|^2}{u^{2n-1}} du. \]

Using Hölder’s inequality, we easily obtain

\[ |g(u)|^2 \leq M_n(u) \times L_n(u) \]
where
\[ M_n(u) := \int_{A_{n-1}} \left| f(u - \sum_{j=2}^{n} t_j) \right|^2 \times \left| f(t_2) \right|^2 \times \ldots \times \left| f(t_n) \right|^2 \frac{dt_2 \ldots dt_n}{t_2 \ldots t_n} \]
and using Lemma 2.6,
\[ L_n(u) := \int_{A_{n-1}} \left( u - \sum_{j=2}^{n} t_j \right) t_2 \times \ldots \times t_n dt_2 \ldots dt_n = \frac{u^{2n-1}}{\Gamma(2n)}. \]

Therefore,
\[ \int_{\Pi^+} |F(x + iy)|^{2n} y^{2n-2} dxdy = \frac{2 \pi \Gamma(2n-1)}{2^{2n-1}} \int_0^\infty \frac{|g(u)|^2}{u^{2n-1}} du \leq \frac{2 \pi \Gamma(2n-1)}{2^{2n-1}} \frac{\pi^{-n}}{\Gamma(2n)} \left( \int_0^\infty \frac{|f(t)|^2}{t} dt \right)^n \]
\[ = \frac{\pi^{1-n}}{2^{2n-2} (2n-1)} \left( \int_{\Pi^+} |F(x + iy)|^2 dxdy \right)^n. \]

The proof of the estimate (2.5) in the case where \( s \geq 1 \) is an integer is complete. \( \square \)

We next prove Proposition 1.2, which says that the conjecture holds for \( s \to \infty \). We state it in the following more precise form.

**Proposition 2.7.** For \( s \geq 1 \), define
\[ \Phi(s) := \frac{\max_{F \in A^2(\Pi^+), F \neq 0} \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy}{\int_{\Pi^+} |F(x + iy)|^2 dxdy} \]
\[ \frac{\pi^{1-s}}{(2s-1)2^{2s-2}}. \]

Then the following hold.

(a) For any \( n \leq s \leq n + 1 \) where \( n = 1, 2, \ldots \), it holds that
\[ \frac{2s - 1}{2n + 1} \leq \Phi(s) \leq \frac{2s - 1}{2n - 1}. \]

In particular, we have the following.

(b) The conjecture of Lieb-Solovej is asymptotically true, in the sense that
\[ \lim_{s \to \infty} \frac{\max_{F \in A^2(\Pi^+), F \neq 0} \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy}{\int_{\Pi^+} |F(x + iy)|^2 dxdy} = 1. \]
Proof. We note that (2.8) follows from (2.7). Hence we only prove the latter. In fact, since by Theorem 2.5, $\Phi(n) = 1$ for any $n = 1, 2, \cdots$, it suffices to show that

$$\frac{2s - 1}{2n + 1} \Phi(n + 1) \leq \Phi(s) \leq \frac{2s - 1}{2n - 1} \Phi(n)$$

for any $n \leq s \leq n + 1$ where $n = 1, 2, \cdots$.

Using the pointwise estimate in Corollary 2.3, we first obtain

$$\int_{\Pi^+} |F(x + iy)|^{2(n+1)} y^{2(n+1)-2} dxdy$$

$$= \int_{\Pi^+} |F(x + iy)|^{2s} (|F(x + iy)|y)^{2n+2-2s} y^{2s-2} dxdy$$

$$\leq \frac{\pi^{s-n-1}}{2^{2n+2-2s}} \left( \int_{\Pi^+} |F(x + iy)|^{2s} dxdy \right)^{n+1-s} \times \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy.$$

It follows that

$$\Phi(n + 1) \leq \frac{\pi^{s-n-1}}{2^{2n+2-2s}} \times \frac{(2n + 1)2^{2n}}{\pi - n} \Phi(s)$$

$$= \frac{2n + 1}{2s - 1} \Phi(s).$$

Similarly, one obtains that

$$\Phi(s) \leq \frac{2s - 1}{2n - 1} \Phi(n).$$

Hence,

$$\frac{2s - 1}{2n + 1} \Phi(n + 1) \leq \Phi(s) \leq \frac{2s - 1}{2n - 1} \Phi(n)$$

from which (2.7) follows. \qed

3. The first test of the Lieb-Solovej conjecture.

Consequences of the conjecture

3.1. The test on the powers of the Bergman kernel function.

Proof of Theorem 1.3. Following Proposition 2.1 and Corollary 2.5, we must test out the Lieb-Solovej conjecture on the powers of the Bergman kernel function.

Question. Let $r, s > 1$. Prove or disprove the following estimate

$$\int_{\Pi^+} \frac{y^{2s-2}}{|x + i(y + 1)|^{2r_s}} dxdy \leq C_s \left( \int_{\Pi^+} \frac{dxdy}{|x + i(y + 1)|^{2r}} \right)^s.$$

Are there values of $r > 1$ for which equality holds in (3.1)?
By Proposition 2.1, the estimate (3.1) is equivalent to the following inequality for the Gamma function

\[
(3.2) \frac{\Gamma(2s)\Gamma(2s(r-1))}{(\Gamma(rs))^2} \leq \left( \frac{\Gamma(2(r-1))}{(\Gamma(r))^2} \right)^s,
\]

which is an equality when \( r = 2 \).

It follows from Theorem 2.7 that this inequality is true when \( s \) is a positive integer. We record the following corollary.

**Corollary 3.1.** For all integers \( n = 2, 3, \ldots \) and real numbers \( r > 1 \), the following estimate holds.

\[
\frac{\Gamma(2n)\Gamma(2n(r-1))}{(\Gamma(nr))^2} \leq \left( \frac{\Gamma(2(r-1))}{(\Gamma(r))^2} \right)^n.
\]

The test is indeed positive according to Theorem 1.3. This result is induced by the following theorem.

**Theorem 3.2.** The estimate (3.2) is valid for all \( r, s > 1 \). This estimate is an equality if and only if \( r = 2 \).

**Proof.** Without loss of generality, we assume that \( r \neq 2 \). Taking the logarithm, we must prove the following estimate

\[
(3.3) \log \Gamma(2s) - s \log \Gamma(2) + \log \Gamma(2s(r-1)) - s \log \Gamma(2(r-1))
\]

\[
-2[\log \Gamma(rs) - s \log \Gamma(r)] \leq 0 \quad (r, s > 1).
\]

In fact, it suffices to show that for every \( s > 1 \), the \( C^\infty \) function \( g = g_s \) defined on \((0, \infty)\) by

\[
g(u) = \log \Gamma(us) - s \log \Gamma(u)
\]

is concave. We prove that for every \( s > 1 \), this function \( g \) is strictly concave. This will imply that the inequality (3.3) (and equivalently, the inequality (3.2)) is strict except for \( r = 2 \). We adopt the usual notation

\[
\psi(x) = (\log \Gamma)'(x).
\]

We have

\[
g'(u) = s\psi(us) - s\psi(u)
\]

and

\[
g''(u) = s^2\psi'(us) - s\psi'(u) = s[\psi'(us) - \psi'(u)].
\]

To obtain that \( g''(u) < 0 \), it is enough to prove that \( s\psi'(us) - \psi'(u) < 0 \quad (u > 0) \). This reduces to the following lemma.

**Lemma 3.3.** The positive function \( h \), defined on \((0, \infty)\) by \( h(t) := tw\psi'(t) \), is strictly decreasing.
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Proof of the Lemma. We have

\[ h'(t) = \psi'(t) + t\psi''(t). \]

To simplify the notation, we call \( \eta \) this derivative function. We shall prove that \( \eta(x) < 0 \) for every \( x > 0 \). We recall the asymptotic expansions as \( x \to \infty \):

\[ \psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right) \]

and

\[ \psi''(x) \sim -\frac{1}{x^2} - \frac{1}{x^3} + O\left(\frac{1}{x^4}\right). \]

This implies that \( \eta(x) = -\frac{1}{2x^2} + O\left(\frac{1}{x^3}\right) \). So \( \eta(\infty) = \lim_{x \to \infty} \eta(x) = 0 \). We also recall the following formulas (cf. e.g. [1]):

\[ \psi'(x + 1) = \psi'(x) - \frac{1}{x^2} \]

and

\[ \psi''(x + 1) = \psi''(x) + \frac{2}{x^2}. \]

This yields

\[ \eta(x + 1) - \eta(x) = \psi''(x) = \psi''(x) + \frac{1}{x^2} + \frac{2}{x^3}. \]

From the formula (cf. e.g. [1])

\[ \psi(x) + \gamma = \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \]

where \( \gamma \) denotes the Euler constant, we deduce that

\[ \psi'(t) = \int_0^\infty \frac{te^{-xt}}{1 - e^{-t}} dt \]

and

\[ \psi''(t) = -\int_0^\infty \frac{t^2 e^{-xt}}{1 - e^{-t}} dt. \]

We obtain

\[ \eta(x + 1) - \eta(x) = -\int_0^\infty \frac{t^2 e^{-xt}}{1 - e^{-t}} dt + \int_0^\infty te^{-xt} dt + \int_0^\infty t^2 e^{-xt} dt \]

\[ = \int_0^\infty \left( -\frac{te^t}{e^t - 1} + 1 + t \right) te^{-xt} dt \]

\[ = \int_0^\infty \left( 1 - \frac{t}{e^t - 1} \right) te^{-xt} dt > 0, \]

since \( 1 - \frac{t}{e^t - 1} > 0 \) for every positive \( t \). Next we get \( g(x + 1) > g(x) \) for every \( x > 0 \). This gives

\[ \eta(x) < \eta(x + n) \quad (n = 1, 2, \ldots). \]
Letting $n$ tend to $\infty$, we conclude that $h'(x) = \eta(x) < \eta(\infty) = 0$ for every $x > 0$. The proof is complete.

The proof of the inequality (3.2) is then complete. This finishes the proof of Theorem 3.2.

3.2. Some consequences of the Lieb-Solovej conjecture. In this subsection, we assume that the Lieb-Solovej conjecture is true and we shall first draw as a consequence the following dual estimate.

**Corollary 3.4.** Assume that the Lieb-Sobolev conjecture is true. Let $s > 1$. Then the identity operator is bounded from $A^{\frac{2s}{2s-1}(\Pi^+)}$ to $A^2(\Pi^+)$ with operator norm $C_s^{\frac{2}{2s}}$.

**Proof.** For all $F \in A^2(\Pi^+)$ and $G \in A^{\frac{2s}{2s-1}(\Pi^+)}$, an application of the Hölder inequality gives

$$\left| \int_{\Pi^+} F(x + iy)\overline{G(x + iy)} dxdy \right|$$

$$= \left| \int_{\Pi^+} F(x + iy)y^{\frac{2s-2}{2s-1}}\overline{G(x + iy)y^{-\frac{2s-2}{2s-1}}} dxdy \right|$$

$$\leq \left( \int_{\Pi^+} |F(x + iy)|^{2s}y^{2s-2}dxdy \right)^{\frac{1}{2s}} \left( \int_{\Pi^+} |G(x + iy)|^{\frac{2s}{2s-1}}y^{-\frac{2s-2}{2s-1}}dxdy \right)^{\frac{2s-1}{2s}}.$$

We deduce from the conjecture that

$$\sup_{F \in A^2(\Pi^+), \|F\|_{A^2(\Pi^+)} = 1} \left| \int_{\Pi^+} F(x + iy)\overline{G(x + iy)} dxdy \right|$$

$$\leq C_s^{\frac{1}{2s}} \left( \int_{\Pi^+} |G(x + iy)|^{\frac{2s}{2s-1}}y^{-\frac{2s-2}{2s-1}}dxdy \right)^{\frac{2s-1}{2s}}.$$

We recall that the dual of the (Hilbert-)Bergman space $A^2(\Pi^+)$ is $A^2(\Pi^+)$ with respect to the duality pairing

$$< F, G > = \int_{\Pi^+} F(x + iy)\overline{G(x + iy)} dxdy.$$

We conclude that for every $G \in A^{\frac{2s}{2s-1}(\Pi^+)}$, we have

$$\|G\|_{A^2(\Pi^+)} = \left( \int_{A^2(\Pi^+)} |G(x + iy)|^2 dxdy \right)^{\frac{1}{2}}.$$
So the identity operator is bounded from $A^{\frac{2s}{2s-1}}(\Pi^+)$ to $A^2(\Pi^+)$ with operator norm $C_s^{\frac{1}{s}}$. □

We next prove Theorem 1.4. Let $F \in A^2(\Pi^+)$ such that $\|F\|_{A^2(\Pi^+)} = 1$. Assume that (1) is true, i.e. equivalently

$$\left(\int_{\Pi^+} |F(x + iy)| y^{2s-2} |F(x + iy)|^2 dxdy \right)^{\frac{1}{2s-2}} \leq \frac{1}{2\sqrt{\pi}(2s-1)^{\frac{1}{2s-2}}}.$$  

The measure $|F(x + iy)|^2 dxdy$ is a probability measure on $\Pi^+$. Letting $s$ tend to 1, we obtain (cf. e.g. [12], page 71, exercise 5):

$$\exp \left( \int_{\Pi^+} \log |F(x + iy)| y |F(x + iy)|^2 dxdy \right) \leq \frac{1}{2e\sqrt{\pi}}.$$  

Taking the logarithm of both sides, we obtain the estimate (1.5). In other words, (2) is true.

Conversely, assume that (2) is true. Taking the logarithm of both sides of (1.4), we obtain the following equivalent form

$$\varphi(s) := \log \left( \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy \right) - (1 - s) \log \pi + \log(2s - 1) + (2s - 2) \log 2 \leq 0.$$  

Since $\varphi(1) = 0$, it suffices to prove that $\varphi'(1) \leq 0$. For $s \geq 1$, we have

$$\varphi'(s) = \frac{4}{\pi} \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy + \log \pi + \frac{2}{2s-1} + 2 \log 2$$

$$= \frac{4}{\pi} \int_{\Pi^+} \frac{|F(x + iy)|^{2s} y^{2s-2} dxdy}{\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dxdy} + \log \pi + \frac{2}{2s-1} + 2 \log 2$$

In particular,

$$\varphi'(1) = 2 \int_{\Pi^+} \log(|F(x + iy)| y |F(x + iy)|^2 dxdy + \log \pi + 1 + 2 \log 2.$$  

The required estimate $\varphi'(1) \leq 0$ is equivalent to

$$\int_{\Pi^+} \log(2\sqrt{\pi} |F(x + iy)| y |F(x + iy)|^2 dxdy \leq -1,$$

which reduces to (2). In other words, (1) is true.
We finally prove Theorem 1.5, which provides a less precise estimate than the estimate (1.5). The measure $|F(x + iy)|^2 \, dx \, dy$ is a probability measure on $\Pi^+$. The function $\varphi(t) := \log \frac{1}{2\sqrt{\pi} t}$ is a convex function on $(0, \infty)$. An application of Jensen’s inequality (cf. e.g. [12], Theorem 3.3) gives

$$\int_{\Pi^+} \log \left[ \frac{1}{2\sqrt{\pi}|F(x + iy)|y} \right] |F(x + iy)|^2 \, dx \, dy \geq \log \frac{1}{2\sqrt{\pi} \int_{\Pi^+} |F(x + iy)|y \, |F(x + iy)|^2 \, dx \, dy}.$$ Now, by the Schwarz inequality, we obtain

$$\int_{\Pi^+} |F(x + iy)|y \, |F(x + iy)|^2 \, dx \, dy \leq \left( \int_{\Pi^+} |F(x + iy)|^2 \, |F(x + iy)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \left( \frac{\pi^{-1}}{3 \times 2^2} \right)^{\frac{1}{2}} = \frac{1}{2\sqrt{3\pi}},$$

since the conjecture is true for $s = 2$. We conclude that

$$\int_{\Pi^+} \log \left[ \frac{1}{2\sqrt{\pi}|F(x + iy)|y} \right] |F(x + iy)|^2 \, dx \, dy \geq \log \sqrt{3} = \frac{\log 3}{2}.$$  

4. The Lieb-Solovej conjecture on the unit disc

4.1. The statement of the conjecture. We denote by $\mathbb{D}$ the unit disc in the complex plane and by $dm$ the Lebesgue area measure in the complex plane. Via a transfer principle from the upper half-plane $\Pi^+$ to the unit disc $\mathbb{D}$, the Lieb-Solovej conjecture takes the following form on $\mathbb{D}$.

**Conjecture.** Let $s > 1$. For every $G \in A^2(\mathbb{D})$, we have

$$\int_{\mathbb{D}} |G(z)|^{2s} (1 - |z|^2)^{2s-2} \, dm(z) \leq \frac{\pi^{1-s}}{2s - 1} \left( \int_{\mathbb{D}} |G(z)|^2 \, dm(z) \right)^s,$$

with equality when $G(z)$ is a Bergman kernel function $\frac{1}{\pi} (1 - z \cdot \overline{z_0})^{-2}$, $z_0 \in \mathbb{D}$, of the unit disc $\mathbb{D}$.

**Proof.** We recall the Lieb-Solovej conjecture on the upper half-plane $\Pi^+$.

$$\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} \, dx \, dy \leq \frac{\pi^{1-s}}{(2s - 1)^{2s-2}} \left( \int_{\Pi^+} |F(x + iy)|^2 \, dx \, dy \right)^s \quad (F \in A^2(\Pi^+)).$$
We apply the linear fractional transformation $\Phi$ from $D$ to $\Pi^+$:

$$z = x + iy = \Phi(w) := \frac{i + w}{1 - w}.$$ 

Then $y = \Im z = \Im m\left(\frac{i + w}{1 - w}\right) = \frac{1 - |w|^2}{1 - |w|^2}$ and $\Phi'(w) = \frac{2i}{(1 - w)^2}$. So the estimate (4.1) takes the form

$$\int_D |F(\Phi(w))|^{2s} \left(\frac{1 - |w|^2}{|1 - w|^2}\right)^{2s-2} \frac{4}{|1 - w|^4} dm(w)$$

$$= 2^{-2s+2} \int_D |F(\Phi(w))|^{2s} \left(\frac{2}{|1 - w|^2}\right)^{2s} (1 - |w|^2)^{2s-2} dm(w)$$

$$= 2^{-2s+2} \int_D |F(\Phi(w))\Phi'(w)|^{2s} (1 - |w|^2)^{2s-2} dm(w)$$

$$\leq \frac{\pi^{1-s}}{(2s - 1)^{2s-2}} \int_D |F(\Phi(w))\Phi'(w)|^2 dm(w).$$

Without loss of generality, we take $G = (F \circ \Phi)\Phi'$: the result follows.

As mentioned in the introduction, Bayart, Brevig, Haimi, Ortega-Cerdà and Perfekt [4] proved differently the Lieb-Solovej conjecture for $s = 2, 3, 4, \cdots$ and they even settled it for $s = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$. More precisely, let $\alpha > 1$ and $0 < p < \infty$, and define the Bergman $B^p_\alpha(D)$ as the space of holomorphic $f$ on the unit disc $D$ whose norm

$$\|f\|_{B^p_\alpha(D)} := \left(\int_D |f(w)|^p |\alpha - 1| (1 - |w|^2)^{\alpha - 2} \frac{dm(z)}{\pi}\right)^{\frac{1}{p}}$$

is finite. For $\alpha_0 = \frac{1 + \sqrt{17}}{4}$, these authors prove the following theorem.

**Theorem 4.1.** [4, Theorem 1] Let $\alpha \geq \alpha_0$ and $0 < p < \infty$. For every $f \in B^p_\alpha(D)$,

(4.2) $$\|f\|_{B^p_\alpha(D)} \leq \|f\|_{B^p_\alpha(D)}.$$ 

Moreover, if $\alpha > \alpha_0$, equality holds in (4.2) if and only if there exist a complex constant $C$ and a point $\xi$ in $D$ such that $f(w) = \frac{C}{(1 - \xi \cdot w)^{\frac{4}{p}}}$.

From this theorem, they deduce the following corollary, which settles the Lieb-Solovej conjecture for $s = 2, 3, 4, \cdots$ and $s = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$.

**Corollary 4.2.** [4, Corollary 2] Let $f \in B^2_2(D)$. Then

$$\|f\|_{B^2_2(D)} \geq \|f\|_{B^3_3(D)} \geq \|f\|_{B^4_4(D)} \geq \cdots.$$
For the proof of Theorem 4.1, Bayart et al. solve a minimization problem in a Sobolev space $W^{1,2}(\mathbb{D})$.

### 4.2. A second test of the conjecture.

We must test out the Lieb-Solovej conjecture for the unit disc on the analytical monomials $z^n$ ($n = 1, 2, \cdots$).

**Question.** Let $s > 1$ and $n = 1, 2, \cdots$ Does the following estimate hold?

$$
(4.3) \quad \int_D |z^n|^{2s} (1 - |z|^2)^{2s-2} dm(z) \leq \frac{\pi^{1-s}}{2s-1} \left( \int_D |z^n|^2 dm(z) \right)^s.
$$

We move to the polar coordinates. The right hand side of (4.3) is equal to

$$
\frac{\pi^{1-s}}{2s-1} \left(2\pi \int_0^1 r^{2n+1} dr \right)^s = \frac{\pi^{1-s}}{2s-1} \left( \frac{\pi}{n+1} \right)^s = \frac{\pi}{(2s-1)(n+1)^s}.
$$

The left hand side is equal to

$$
L := 2\pi \int_0^1 r^{2ns}(1 - r^2)^{2s-2}rdr.
$$

We apply the change of variable $r^2 = \rho$, $2rdr = d\rho$. This gives

$$
L = \pi \int_0^1 \rho^{ns}(1 - \rho)^{2s-2} d\rho = \pi B(ns+1, 2s-1) = \pi \frac{\Gamma(ns+1)\Gamma(2s-1)}{\Gamma((n+2)s)}.
$$

Here, $B(\cdot, \cdot)$ denotes the Euler Beta function.

So the estimate (4.3) is equivalent to the following inequality for the Gamma function:

$$
(4.4) \quad \frac{\Gamma(ns+1)\Gamma(2s)}{\Gamma((n+2)s)} \leq \frac{1}{(n+1)^s} \quad (s > 1, n = 1, 2, \cdots).
$$

It appears that the inequality (4.4) (and hence the estimate (4.3)) is true.

**Theorem 4.3.** The estimate (4.3) is valid and strict for all $s > 1$ and $n = 1, 2, \cdots$.

**Proof.** Taking the logarithm of both sides of the inequality (4.4), we wish to prove that given $n = 1, 2, \cdots$, we have

$$
\varphi(s) := \log n + \log s + \log \Gamma(ns) + \log(2s) - \log \Gamma((n+2)s) + s \log(n+1) < 0
$$

for every $s > 1$. Since $\varphi(1) = 0$, it suffices to show that the derivative $\varphi'(s)$ of the function $\varphi(s)$ is negative. We have

$$
\varphi'(s) = \frac{1}{s} + \frac{n\Gamma'(ns)}{\Gamma(ns)} + \frac{2\Gamma'(2s)}{\Gamma(2s)} - \frac{(n+2)\Gamma'((n+2)s)}{\Gamma((n+2)s)} + \log(n+1).
$$
We conclude that (4.5) follows. The proof of the theorem is complete. □

We recall the following well-known identity (cf. e.g. [1]):

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} + \log x - \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-xt} dt \quad (x > 0).$$

So

$$\varphi'(s) = \frac{1}{s} + n \left\{ -\frac{1}{ns} + \log(ns) - \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-nst} dt \right\}$$

$$+ n \left\{ -\frac{1}{(n+2)s} + \log((n+2)s) - \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-(n+2)st} dt \right\}$$

$$- (n+2) \left\{ -\frac{1}{(2n+2)s} + \log(2n+2) - \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-(2n+2)st} dt \right\}$$

$$+ \log(n+1)$$

$$= n \log(ns) + 2 \log(2) - (n+2) \log((n+2)s) + \log(n+1)$$

$$- \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) \left\{ n e^{-nst} + 2e^{-2st} - (n+2)e^{-(n+2)st} \right\} dt$$

$$= n \log n + 2 \log 2 - (n+2) \log(n+2) + \log(n+1)$$

$$+ \frac{1}{s} \int_0^\infty \left( \frac{d}{dt} \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) \right) \left\{ e^{-nst} + e^{-2st} - e^{-(n+2)st} \right\} dt.$$

An integration by parts gives

$$\varphi'(s) = n \log(n+2) + \log(n+1)$$

$$+ \frac{1}{2s} - \frac{1}{s} \int_0^\infty \left\{ d \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) \right\} \left\{ e^{-nst} + e^{-2st} - e^{-(n+2)st} \right\} dt.$$

We next show that \(\varphi'(s) < 0\) \((s > 1) \text{ and } n = 1, 2, \ldots\). It is easy to check that \(e^{-nst} + e^{-2st} - e^{-(n+2)st} > 0\) and \(\frac{d}{dt} \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) > 0\). It suffices to prove that

\begin{equation}
(4.5) \quad n \log(n+2) + \log(n+1) + \frac{1}{2} < 0 \quad (n = 1, 2, \ldots).
\end{equation}

To this aim, we consider the function

$$h(y) := y \log y + 2 \log 2 - (y+2) \log(y+2) + \log(y+1) + \frac{1}{2} \quad (y \geq 1).$$

Its first order derivative is equal to

$$h'(y) = \log y - \log(y+2) + \frac{1}{y+1}$$

and its second order derivative is equal to

$$h''(y) = \frac{1}{y} - \frac{1}{y+2} - \frac{1}{(y+1)^2} = \frac{y^2 + 2y + 2}{y(y+2)(y+1)^2}.$$

Clearly, \(h''(y) > 0\) for every \(y \geq 1\), so that \(h'(y)\) increases from \(h'(1) = -\log 3 + \frac{1}{2}\) to \(\lim_{y \to \infty} h'(y) = 0\). In particular, \(h'(y) < 0\) for every \(y \geq 1\).

We conclude that \(h(y) < h(1) = 3 \log 2 - 3 \log 3 + \frac{1}{2} < 0\). The estimate (4.5) follows. The proof of the theorem is complete. □
4.3. Consequences of the conjecture for \( s \) close to 1 on the unit disc. Via the same transfer principle, the conjecture (1.5) for \( s \) close to 1 on the upper half-plane \( \Pi^+ \) takes the following form on the unit disc \( \mathbb{D} \):

\[
\int_\mathbb{D} \log \left[ \frac{1}{\sqrt{\pi} |G(z)|(1-|z|^2)} \right] |G(z)|^2 dm(z) \geq 1
\]

for every \( G \in A^2(\mathbb{D}) \) such that \( \| F \|_{A^2(\mathbb{D})} = 1 \).

Theorem 3.6 implies the following result on the unit disc.

**Corollary 4.4.** For every \( G \in A^2(\mathbb{D}) \) such that \( \| G \|_{A^2(\mathbb{D})} = 1 \), we have

\[
\int_\mathbb{D} \log \left[ \frac{1}{\sqrt{\pi} |G(z)|(1-|z|^2)} \right] |G(z)|^2 dm(z) \geq \frac{\log 3}{2}.
\]

5. The obtained bounds

5.1. A preliminary bound. The following corollary is a direct consequence of Corollary 2.3.

**Corollary 5.1.** For every \( F \in A^2(\Pi^+) \), we have

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy \leq \frac{\pi^{1-s}}{2^{2s-2}} \left( \int_{\Pi^+} |F(x + iy)|^2 dx dy \right)^s.
\]

5.2. An unsuccessful attempt via Minkowski’s integral inequality. A less successful attempt is via Minkowski’s integral inequality. We obtain the following result.

**Proposition 5.2.** Given \( s > 1 \) and \( F \in A^2(\Pi^+) \), for every \( \nu > 1 \), we have

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy \leq C_s \frac{\nu^s \Gamma(2s) \Gamma((\nu-1)s)}{\Gamma \left( \frac{(\nu+1)s}{2} \right)^2} \left( \int_{\Pi^+} |F(x + iy)|^2 dx dy \right)^s.
\]

In particular, for \( \nu = 3 \), we obtain

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy \leq 3^s C_s \left( \int_{\Pi^+} |F(x + iy)|^2 dx dy \right)^s.
\]

**Proof.** For every \( \nu > 1 \), we again rely on the reproducing property of the weighted Bergman kernel, we have

\[
[F(x + iy)]^2 = \frac{2^{\nu-1} \nu}{\pi} \int_{\Pi^+} \frac{v^{\nu-1}}{(x-u + i(y+v))^{\nu+1}} [F(u + iv)]^2 dudv.
\]
When we apply Minkowski’s integral inequality and Proposition 2.1, we obtain
\[
\left( \int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy \right)^{\frac{1}{s}}
\leq \frac{2^{\nu-1}}{\pi} \int_{\Pi^+} \left( \int_{\Pi^+} |x-u+i(y+v)|^{\nu+1} |F(u+iv)|^2 \frac{dudv}{s} y^{2s-2} dx dy \right)^{\frac{1}{s}}
\leq \frac{2^{\nu-1}}{\pi} \int_{\Pi^+} \left[ \int_{\Pi^+} |x-u+i(y+v)|^{\nu+1} |F(u+iv)|^2 \frac{dudv}{s} y^{2s-2} dx dy \right]^\frac{1}{s} \frac{y^{2s-2}}{s}
\leq \frac{2^{\nu-1}}{\pi} \int_{\Pi^+} \left[ \int_{\Pi^+} \frac{2^{\nu-1}}{2^{(\nu+1)s}} \frac{y^{2s-2}}{s} (\frac{\nu}{\nu+1})^s \frac{dudv}{s} y^{2s-2} dx dy \right]^\frac{1}{s} |F(u+iv)|^2 \frac{dudv}{s}
\leq \frac{\nu}{4\pi} \left( \frac{4\pi \Gamma(2s-1) \Gamma((\nu-1)s)}{\Gamma((\nu+1)s)^2} \right)^\frac{1}{s} \int_{\Pi^+} |F(u+iv)|^2 \frac{dudv}{s}.
\]
The result follows easily. □

**Remark 5.3.** Comparing the bounds of Corollary 5.1 and Proposition 5.2, namely \(2s-1\) and \(\inf_{\nu > 1} \frac{\nu \Gamma(2s) \Gamma((\nu-1)s)}{(\Gamma((\nu+1)s)^2)}\). It looks surprising that the first bound may be smaller than or equal to the second at least for \(1 < s < 2\). This is implied by the following proposition.

**Proposition 5.4.** For all \(1 < s < 2\), the following inequality holds
\[
2s-1 \leq \inf_{\nu > 1} \frac{\nu^s \Gamma(2s) \Gamma((\nu-1)s)}{(\Gamma((\nu+1)s)^2)}.
\]

**Proof.** In view of the convexity of the function \(\log \Gamma\), we have
\[
\frac{\Gamma(2s) \Gamma((\nu-1)s)}{(\Gamma((\nu+1)s)^2)} \geq 1 \quad (s > 1).
\]
Moreover, the inequality \(2s-1 \leq \nu^s\) holds if and only if \(\nu \geq (2s-1)^\frac{1}{s}\). In this case, we have
\[
2s-1 \leq \frac{\nu^s \Gamma(2s) \Gamma((\nu-1)s)}{(\Gamma((\nu+1)s)^2)}.
\]
Let us now suppose that $1 < \nu \leq (2s - 1)^{\frac{1}{s}}$. We recall the following identity (cf. e.g. [1], [10]):

$$\Gamma(x) = e^{-\gamma x} \frac{1}{x} \prod_{k=1}^{\infty} \frac{e^{\frac{x}{k}}}{1 + \frac{x}{k}}.$$  

This implies that

$$\Gamma(2s)\Gamma((\nu - 1)s) = \left(\frac{(\nu+1)s}{2}\right)^2 \frac{1}{1 - \frac{1}{\nu^2}} \prod_{k=1}^{\infty} \frac{(1 + \frac{1}{\nu k})^2}{(1 + \frac{2s}{k})/1 + \frac{(\nu-1)s}{k}}.$$  

We check easily that for all $\nu, s > 1$ and $k = 1, 2, \cdots$, we have

$$\frac{(1 + \frac{1}{\nu k})^2}{(1 + \frac{2s}{k})/1 + \frac{(\nu-1)s}{k}} \geq 1$$

and hence

$$\prod_{k=1}^{\infty} \frac{(1 + \frac{1}{\nu k})^2}{(1 + \frac{2s}{k})/1 + \frac{(\nu-1)s}{k}} \geq 1.$$  

It then suffices to show the inequality $2s - 1 \leq \frac{2\nu^s}{\nu^2 - 1}$ for all $1 < \nu \leq (2s - 1)^{\frac{1}{s}}$. We study the function

$$\varphi(\nu) = (2s - 1)(\nu^2 - 1) - 2\nu^s.$$  

Then $\varphi'(\nu) = 2(2s - 1)\nu - 2s\nu^{s-1}$ and $\varphi''(\nu) = 2(2s - 1) - 2s(s - 1)\nu^{s-2}$. The following equivalence holds

$$\varphi''(\nu) = 0 \Leftrightarrow \nu^{s-2} = \frac{2s - 1}{s(2s - 1)}.$$  

We show easily that $\frac{2s - 1}{s(2s - 1)} > 1$ if and only if $1 < s < \frac{3 + \sqrt{17}}{2}$. Assuming that $1 < s < 2$, since $\nu = \left(\frac{2s - 1}{s(2s - 1)}\right)^{\frac{1}{s-2}} < 1$ and $\lim_{\nu \to \infty} \varphi''(\nu) = 2(2s - 1) > 0$, we obtain that $\varphi''(\nu) > 0$ for all $\nu > 1$. So $\varphi'(s)$ increases from the positive value $2s - 2$ on the interval $[1, \infty)$; in particular, $\varphi'(s) > 0$ for all $1 < \nu < \infty$. Finally, we conclude that the function $\varphi(\nu)$ increases from the value $-2$ to $\infty$ on the interval $[1, \infty)$. We have

$$\varphi \left( (2s - 1)^{\frac{1}{s}} \right) = (2s - 1)^{\frac{2}{s} + 1} - 3(2s - 1) < 0$$

provided that $1 < s < 2$ (study the function $\psi(s) = \log(2s - 1) - \frac{s}{2} \log 3$). This proves that for all $1 < \nu < (2s - 1)^{\frac{1}{s}}$, we have $\varphi(\nu) < 0$ or
equivalently \(2s - 1 \leq \frac{2s^2}{p-1}\). This completes the proof of the proposition. \(\Box\)

5.3. **An improved bound via the complex interpolation method.**

A classical example of interpolation via the complex method concerns \(L^p\) spaces with a change of measures. We state it in our setting of the upper half-plane \(\Pi^+\).

**Theorem 5.5.** \([8,13]\] Let \(1 \leq p_0, p_1 \leq \infty\). Given two positive measurable functions (weights) \(\omega_0, \omega_1\) on \((0, \infty)\), then for every \(\theta \in (0, 1)\), we have

\[
[L^{p_0}(\Pi^+, \omega_0(y)dx dy), L^{p_1}(\Pi^+, \omega_1(y)dx dy)]_\theta = L^p(\Pi^+, \omega(y)dx dy)
\]

with equal norms, provided that

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]

\[
\frac{1}{\omega^p} = \frac{1 - \theta}{\omega_0^p} + \frac{\theta}{\omega_1^p}.
\]

We obtain the following theorem.

**Theorem 5.6.** We suppose that \(s > 1\) is not an integer. Let \(n\) be the integer such that \(n < s < n + \frac{1}{2}\) (resp. \(n + \frac{1}{2} < s < n + 1\)). Then for every \(F \in A^2(\Pi^+)\), we have the estimate

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy
\]

\[
\leq \frac{\pi^{1-s}}{(2n - 1)^{-2s+2n+1}(2n)^{2(s-n)}2^{2s-2}} \left( \int_{\Pi^+} |F(x + iy)|^2 dx dy \right)^s
\]

(resp.

\[
\int_{\Pi^+} |F(x + iy)|^{2s} y^{2s-2} dx dy
\]

\[
\leq \frac{\pi^{1-s}}{(2n - 1)^{-2s+2n+1}(2n)^{2(s-n)}2^{2s-1}} \left( \int_{\Pi^+} |F(x + iy)|^2 dx dy \right)^s
\]

Proof. We provide the proof for \(n < s < n + \frac{1}{2}\). In Theorem 5.5, we take \(p_0 = 2n, p_1 = 2n + 1, p = 2s, \omega_0(y) = y^{2n-2}, \omega_1(y) = y^{2n-1}\). Let \(\theta \in (0, 1)\) be defined by

\[
\frac{1}{s} = \frac{1 - \theta}{n} + \frac{\theta}{n + \frac{1}{2}}.
\]

We check easily that in the notations of Theorem 5.5, we have \(\omega(y) = y^{2s-2}\). We can then apply Theorem 5.5. First, we have trivially \([A^2(\Pi^+), A^2(\Pi^+)]_\theta = A^2(\Pi^+)\). Next, if we consider the identity operator \(i(F) = F\), then
by Corollary \[4.2\] \(i\) is bounded from \(A^2(\Pi^+)\) to \(L^{2n}(\Pi^+, y^{2n-2}dxdy)\) with operator norm \(\left(\frac{-\pi^{1-n}}{(2n-1)2^{2n-2}}\right)^{\frac{1}{2n}}\) and is bounded from \(A^2(\Pi^+)\) to \(L^{2n+1}(\Pi^+, y^{2n-1}dxdy)\) with operator norm \(\left(\frac{-\pi^{1-n}}{(2n)2^{2n-1}}\right)^{\frac{1}{2n+1}}\), we obtain that

\[
\left(\int_{\Pi^+} |F(x+iy)|^{2s} y^{2s-2}dxdy\right)^{\frac{1}{2s}} \leq \left(\left(\frac{-\pi^{1-n}}{(2n-1)2^{2n-2}}\right)^{\frac{1}{2n}}\right)^{1-\theta} \left(\left(\frac{-\pi^{1-n}}{(2n)2^{2n-1}}\right)^{\frac{1}{2n+1}}\right)^{\theta} \left(\int_{\Pi^+} |F(x+iy)|^{2}dxdy\right)^{\frac{1}{2}}.
\]

We deduce from \((5.1)\) that \(\theta = (1 - \frac{n}{s})(2n + 1)\) and \(1 - \theta = -2n + \frac{n}{s}(2n + 1)\). Replacing in \((5.2)\) gives the announced estimate. \(\square\)

**Remark 5.7.**

(1) The estimate in Theorem 5.6 clearly improves the one in Corollary 5.1.

(2) Corollary 5.6 also implies that the conjecture of Lieb-Solovej is asymptotically true, as proved in Proposition 2.7.

We next show that the bound in Theorem 5.6 is greater than the bound in the Lieb-Solovej conjecture when \(s\) is not an integer. It also makes more precise the constant \(\Phi(s)\) in Proposition 2.7. We recall that the Lieb-Solovej conjecture says that \(\Phi(s) = 1\). In fact, Theorem 5.6 provides a more accurate upper bound of \(\Phi(s)\) which lies in \((1, \frac{2s-1}{2n-1})\) when \(s \in (n, n+1)\); explicitly,

\[
\Phi(s) \leq \frac{2s - 1}{(2n - 1)^{-2s+2n+1}(2n)^{2(s-n)}}.
\]

These two assertions are consequences of the following elementary lemma.

**Lemma 5.8.** For \(s \in (n, n+1)\), the following double inequality holds

\[
\frac{1}{2s - 1} < \frac{1}{(2n - 1)^{-2s+2n+1}(2n)^{2(s-n)}} < \frac{1}{2n - 1}.
\]

**Proof.** The second inequality is obvious. The second inequality follows easily from the strict concavity of the function \(t \mapsto \log(2t - 1)\). \(\square\)

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