Determining the Equation of State of the Expanding Universe: Inverse Problem in Cosmology

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ABSTRACT

Even if the luminosity distance as a function of redshift is obtained accurately using, for example, Type Ia supernovae, the equation of state of the Universe cannot be determined uniquely but depends on one free parameter $\Omega_0 = k/(a_0^2 H_0^2)$, where $a_0$ and $H_0$ are the present scale factor and the Hubble parameter, respectively. This degeneracy might be resolved if, for example, the time variations of the redshift of quasars are measured as proposed recently by Loeb. Therefore the equation of state of the Universe (or the metric of the universe) might be determined without any theoretical assumption on the matter content of the Universe in future.

Key words: cosmology: theory – dark matter.

1 INTRODUCTION

To determine the structure and dynamics of an astrophysical system, the equation of state is usually necessary. For example, consider the structure of a spherical neutron star. If the pressure $p$ is known as a function of the density $\rho$, we can determine the gravitational mass $M$ and the radius $R$ of the star as a function of the central density $\rho_c$ by solving the Oppenheimer-Volkoff equation (Oppenheimer & Volkoff1939). This means that we can determine the mass-radius relation $M(R)$ theoretically in principle. However, the equation of state relevant to the neutron star is not established yet, although it may be determined by Quantum Chromo Dynamics in future. Therefore the mass-radius relation of the neutron star is not known well theoretically at present.

Observationally, however, the mass-radius relation of neutron stars may be determined in the near future. If gravitational waves from the coalescing binary neutron stars are detected by the LIGO/VIRGO/GEO/TAMA network which will be in operation around 2000, the mass of each neutron star as well as its radius may be determined by analyzing the waveform in the last three minutes of the binary inspiral (Thorne1995). In general the mass of each observed neutron star can be different so that there is a chance to determine $M(R)$ observationally. In this case, as shown by Lindblom (1991), the equation of state of the high density matter can be determined from $M(R)$. This is, in a sense, the inverse problem.

Now let us consider an isotropic and homogeneous universe which describes the Universe quite well in a global sense. The amount of the radiation $\rho_r(z)$ in the Universe is well known as a function of the redshift $z$ from the present temperature of the cosmic background radiation. The amount of baryons $\rho_b(z)$ has a constraint from the big bang nucleosynthesis. We know that dark matter should exist but we do not know its density $\rho_d(z)$ or pressure $p_d(z)$. A cosmological constant may exist but introducing a non-zero cosmological constant needs a fine-tuning of the vacuum energy, and at present we do not have any convincing explanation for why such an extremely small value of the cosmological constant (in Planck units) is required. Since we do not know the equation of state of the matter in the standard model well, we cannot determine the scale factor as a function of time theoretically. Moreover recently several authors have considered the more general equation of state for a dark component called x-matter or quintessence and have explored its cosmological implications (Turner & White1997; Chiba, Sugiyama & Nakamura1997; Chiba, Sugiyama & Nakamura1998; Caldwell, Dave & Steinhardt1998; Chiba & Nakamura1998). The situation is worse than the neutron star case; the equation of state in the expanding universe is almost unknown theoretically.

Observationally we have several quantities such as the luminosity distance, the angular diameter distance and number counts as a function of the redshift. Among these the luminosity distance $d_L(z)$ may be determined quite accurately by using Type Ia supernovae (Garnavich et al.1998; Perlmutter et al.1998; Riess et al.1998; Schmidt et al.1998). In the future the
afterglows of a certain class of gamma ray bursts might serve as a standard candle\cite{Cohen & Piran1997}. Therefore in this paper we assume that a quite accurate luminosity distance may be obtained and examine whether the equation of state of the expanding universe can be determined uniquely. Namely we discuss the inverse problem in cosmology. By “equation of state of the universe”, we mean the relation between the total energy density of cosmic matter and the total pressure. Nearly three decades ago, Weinberg studied the possibility of determining the metric from the observed luminosity distance, with negative conclusion \cite{Weinberg1970}. We shall argue that the improvement of observational techniques now enables us to determine the metric of the universe (or the equation of state of the universe) directly from the observational data.

In section 2 we show that the equation of state can be determined if the scale factor is given as a function of time. We also show that the scalar field potential can be determined similarly. In section 3 we discuss how to determine the metric of the universe from the luminosity distance as a function of the redshift \(z\).

### 2 MATTER FIELD IN TERMS OF SCALE FACTOR

Consider any given scale factor \(a(t)\) which is a monotonically increasing function of time. We denote the inverse function of \(a(t)\) as \(t(a)\). Then every function of \(t\) can be considered as a function of \(a\). For example the Hubble parameter \(H\) can be written as

\[
\frac{\dot{a}}{a} = \left(\frac{dt}{da}\right)^{-1} = H(a)
\]

which we call the equation of state of the Universe. The Einstein equations are

\[
\kappa^2 = \frac{8\pi G}{3} \rho + \frac{k}{a^2},
\]

\[
\dot{H}(a) + H(a)^2 = -\frac{\kappa^2}{6} (\rho + 3p),
\]

\[
\dot{\rho} = -3H(p + \rho),
\]

where \(\kappa^2 = 8\pi G\) and \(k = 0, -1\) and 1 for flat, open and closed universes, respectively. First, from Eq.(3) \(\rho\) as a function of \(a\) can be written as

\[
\kappa^2 \rho(a) = 3 \left( H(a)^2 + \frac{k}{a^2} \right).
\]

Then, from Eq.(4) \(p\) as a function of \(a\) can also be written as

\[
\kappa^2 p(a) = - \left( 2\dot{H}(a) + 3H(a)^2 + \frac{k}{a^2} \right).
\]

Since \(\rho(a)\) and \(p(a)\) are given, we can determine the equation of state \(p(\rho)\). Note that we can consider even the case where the weak energy condition \cite{Hawking & Ellis 1973} is violated: \(\rho + p < 0\) or \(\rho < 0\).

Next consider the minimally coupled scalar field \(\phi\) with the potential \(V(\phi)\), which is an example of the general energy momentum tensor described above. Then

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi),
\]

\[
p = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

\[
\ddot{\phi} + 3H \dot{\phi} = -V''.
\]

In this case the weak energy condition is satisfied. We assume \(\dot{\phi} \geq 0\) for simplicity though it is not necessary. From Eqs.(5) and (6) we immediately have,

\[
\kappa^2 V = \left( 3H^2 + \dot{H} + \frac{2k}{a^2} \right),
\]

\[
\kappa^2 \dot{\phi}^2 = \kappa^2 \left( \frac{d\phi}{da} H a \right)^2 = -2 \left( \dot{H} - \frac{k}{a^2} \right).
\]

From Eq.(11) \(V\) can be written as a function of \(a\). Also, from Eq.(9) \(\dot{\phi}\) can be written as a function of \(a\). Therefore, \(V\) can be written as a function of \(\phi\).
Determining the Equation of State of the Universe

3 DETERMINING THE METRIC FROM OBSERVATIONS

3.1 Matter field in terms of the luminosity distance

From observations we might know, for example, the luminosity distance as a function of redshift \(d_L(z)\) accurately by using Type Ia supernovae\(^1\). In this section we therefore will regard the Hubble parameter \(H(z)\) as a function of the redshift instead of as a function of the scale factor in the previous section. \(\rho(z)\) and \(p(z)\) are then given by

\[
\kappa^2 \rho(z) = 3 \left( H(z)^2 + (1 + z)^2 H_0^2 \Omega_{k0} \right),
\]

\[
\kappa^2 p(z) = -3 H(z)^2 + 2(1 + z) H(z) \frac{dH}{dz} - (1 + z)^2 H_0^2 \Omega_{k0},
\]

\[
\Omega_{k0} \equiv -\frac{k}{a_0^2 H_0^2},
\]

where \(a_0\) and \(H_0\) are the present scale factor and the Hubble parameter, respectively. It is apparent that in general the equation of state of the Universe depends on \(\Omega_{k0}\). To determine \(H(z)\) we will use the luminosity distance. Since all cosmological observations are made on the past light cone, the argument is similar for other distance indicators. The luminosity distance \(d_L(z)\) is defined by

\[
d_L(z) = a_0 (1 + z) f(\chi) \equiv (1 + z) r(z),
\]

\[
\chi = \frac{1}{a_0} \int_0^z \frac{du}{H(u)}
\]

Then \(H(z)\) can be written in terms of \(r(z)\) as

\[
H(z) = (dr/dz)^{-1} \sqrt{1 - r(z)^2 H_0^2 \Omega_{k0}}.
\]

Note here that the above formula is valid irrespective of the sign of \(k\). Since \(r(0) = 0\), \(H_0\) can be determined irrespective of \(\Omega_{k0}\). One may think that \(\Omega_{k0}\) can be determined only from \(r(z)\). However, this is not the case. To show this, we rewrite Eq.(18) as

\[
\frac{dr}{dz} = H(z)^{-1} \sqrt{1 - r(z)^2 H_0^2 \Omega_{k0}}.
\]

From the second derivative of Eq.(18), we have

\[
\frac{d^2 r}{dz^2}(0) = \frac{d^2}{dz^2} \left( \frac{1}{H} \right) - \frac{\Omega_{k0}}{H_0}.
\]

Eq.(20) shows that we cannot determine \(\Omega_{k0}\) without the knowledge of the second derivative of the Hubble parameter. This is due to the fact that as far as the expansion of the universe is concerned, the effect of the curvature is equivalent to “matter” with the equation of state \(p = -\rho/3\).

Now using \(r(z)\) we express \(\rho(z)\) and \(p(z)\) explicitly as

\[
\kappa^2 \rho(z) = 3 \left[ \frac{1}{(dr/dz)^2} + (1 + z)^2 \left( 1 - \frac{r^2}{(dr/dz)^2} \right) \right] H_0^2 \Omega_{k0},
\]

\[
\kappa^2 p(z) = -\frac{3}{(dr/dz)^2} + (1 + z) \frac{d}{dz} \left( \frac{1}{(dr/dz)^2} \right) - \left[ (1 + z)^2 - \frac{3 r^2}{(dr/dz)^2} \right] \left[ (1 + z) \frac{d}{dz} \left( \frac{r^2}{(dr/dz)^2} \right) \right] H_0^2 \Omega_{k0}.
\]

The above equations show that the equation of state of the Universe cannot be determined uniquely from the luminosity distance \(d_L(z) = (1 + z) r(z)\) but depends on one free parameter \(\Omega_{k0}\). This degeneracy was first pointed out by Weinberg\(^2\). This means that even if the luminosity distance is observed accurately as a function of \(z\), in order to determine the equation of the state of the Universe (or the metric) some assumption about \(\Omega_{k0}\) is needed. Intuitively, this can be understood as follows. Since the luminosity distance or other cosmological distance measure only gives information on the past light cone, it is not enough to infer what happens inside the light cone.

3.2 Breaking the degeneracy

To determine \(\Omega_{k0}\) we need other dynamical information. We here show that \(\Omega_{k0}\) can be determined if we use a new observational technique proposed by Loeb\(^3\). He pointed out that the time variation of cosmic redshift might be detectable through two observations of \(\sim 10^2\) quasars set a decade apart with the HIRES instrument of the Keck 10 meter telescope. The key point of his proposal is to use an existing spectroscopic technique, which was recently employed in planet searches. If a jupiter size planet exists around a certain G/F type star, the G/F type star is moving around the center of mass of the star-planet system with rotational velocity of the order of \(\sim 30m/s\). In principle, the wave length of an absorption line in the spectrum of...
the star is doppler shifted. By observing only one absorption line it is difficult to determine a period and an orbital velocity to obtain the mass and the orbital radius of the planet because the expected shift of the line is too small compared with the line width. However if one observes many absorption lines, the signal-to-noise ratio(S/N) will increase in proportion to \(\sqrt{N}\) where \(N\) is the number of lines observed. The same technique can be applied to cosmology in principle. Each line from a quasar has a large line width so that it is difficult to determine the time variation of the redshift. However by observing many quasars and many absorption lines it might be possible to determine one parameter \(\Omega_\text{ko}\). If such an observation is performed, the change of the redshift \(\Delta z\) can be obtained as

\[
\Delta z = [(1 + z)H_0 - H(z)] \Delta t \equiv g(z) \Delta t
\]

where \(\Delta t\) is the time interval of two observations. A spectroscopic velocity shift is then

\[
\Delta v = \frac{\Delta z}{1 + z} = \frac{g(z) \Delta t}{1 + z}^{-1}
\]

which is of order \(\sim 1\text{ms}^{-1}\) over \(\Delta t \sim 10^2\) years for a single quasar. Loeb showed that it is feasible to detect the cosmic signal (with signal-to-noise ratio of \(\sim 100\)) only over a decade with \(\sim 10^2\) quasars.

Since \(g(z)\) can be determined observationally, \(H(z)\) can be determined independently from Eq.(19) as

\[
H(z) = (1 + z)H_0 - g(z)
\]

In reality it may be hard to determine \(g(z)\) for various \(z\). However if \(g(z)\) is determined at a certain \(z_s\) quite accurately, we can determine \(\Omega_\text{ko}\) as

\[
\Omega_\text{ko} = r(z_s)^{-2}H_0^{-2} \left[ 1 - (dr(z_s)/dz)^2 \right] \left( 1 + z_s \right) H_0 - g(z_s)^2
\]

Even if \(g(z)\) is not determined quite accurately for any \(z\), if the luminosity distance \(d_L(z) = (1 + z)r(z)\) is obtained accurately, we may determine \(\Omega_\text{ko}\) by using a maximum likelihood test for many observations of \(g(z)\), since only one parameter \(\Omega_\text{ko}\) is to be determined.

### 3.3 Application to two component model

Finally, we note that Eq.(22) and Eq.(23) can be applied to the two component model such that \(\rho = \rho_M + \rho_X\), where \(\rho_M\) is dust matter and \(\rho_X\) refers to the “x-component”, and that we can determine, once \(\Omega_{M0}\) is known, the equation of state (or the effective potential) of the x-component. For example, in the case of a fluid we have

\[
\kappa^2 p_X(z) = 3 \left[ \frac{1}{(dr/dz)^2} + \left( 1 + z \right)^2 - \frac{r^2}{(dr/dz)^2} \right] H_0^2 \Omega_\text{ko} - H_0^2 \Omega_{M0}
\]

\[
\kappa^2 p_X(z) = -\frac{3}{(dr/dz)^2} + (1 + z) \frac{d}{dz} \left( \frac{1}{(dr/dz)^2} \right) - \left[ (1 + z)^2 - \frac{3r^2}{(dr/dz)^2} + (1 + z) \frac{d}{dz} \left( \frac{r^2}{(dr/dz)^2} \right) \right] H_0^2 \Omega_\text{ko}
\]

Note that the cosmological constant corresponds to the \(p_X = -\rho_X\) case. Alternatively, in the case of a scalar field we have

\[
\kappa^2 \phi^2 = \kappa^2 (1 + z)^2 \left[ 1 - r^2 H_0^2 \Omega_{M0} \right] \left( \frac{d\phi(z)}{dz} \right)^2
\]

\[
\kappa^2 \phi^2 = (1 + z) \frac{d}{dz} \left( \frac{1}{(dr/dz)^2} \right) + 2(1 + z)^2 - \left( 1 + z \right) \frac{d}{dz} \left( \frac{r^2}{(dr/dz)^2} \right) \right] H_0^2 \Omega_\text{ko} - 3H_0^2 \Omega_{M0}
\]

\[
\kappa^2 V(z) = \frac{3}{(dr/dz)^2} - \frac{1}{2} (1 + z) \frac{d}{dz} \left( \frac{1}{(dr/dz)^2} \right) + 2(1 + z)^2 - \frac{3r^2}{(dr/dz)^2} + \frac{1}{2} (1 + z) \frac{d}{dz} \left( \frac{r^2}{(dr/dz)^2} \right) \right] H_0^2 \Omega_\text{ko} - \frac{3}{2} H_0^2 \Omega_{M0}
\]

Once \(\Omega_{M0}\) is measured by other observations, we can determine unambiguously the equation of state or effective potential of a scalar field from the luminosity distance \(d_L(z) = (1 + z)r(z)\).

### 4 SUMMARY

We have explored the possibility of determining the equation of state (or the metric) directly from the observational data. We have shown that by combining the observation of the luminosity distance and the observation of the time variation of the cosmic redshift, the equation of state of the Universe (or the metric) might be directly determined unambiguously in future without any theoretical assumption on the matter content and the geometry of the Universe. Since there exists no convincing candidate for the equation of state of the x-component or the effective potential, it may be more promising to determine it directly from observational data.
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