Multiplier systems for Siegel modular groups

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Abstract
Deligne proved in [De] (s. also [Hi], 7.1) that the weights of Siegel modular forms on any congruence subgroup of the Siegel modular group of genus \( g > 1 \) must be integral or half integral. Actually he proved that for a system \( v(M) \) of complex numbers of absolute value 1

\[
v(M) \det(CZ + D)^r \quad (r \in \mathbb{R})
\]

can be an automorphy factor only if \( 2r \) is integral. We give a different proof for this. It uses Mennicke's result that subgroups of finite index of the Siegel modular group are congruence subgroups and some techniques from the paper [BMS] of Bass-Milnor-Serre.

Introduction
We fix a natural number \( g \) (which later will be 2). We denote by \( E = E^{(g)} \) the \( g \times g \)-unit matrix and by

\[
I = I^{(g)} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}
\]

the standard alternating matrix. The symplectic group \( \text{Sp}(g, \mathbb{R}) \) consists of all \( M \in \text{GL}(2g, \mathbb{R}) \) with the property \( M'M = I \). Here \( M' \) denotes the transposed matrix of \( M \). We consider the usual action \( MZ = (AZ + B)(CZ + D)^{-1} \) of the real symplectic group \( \text{Sp}(g, \mathbb{R}) \) on the Siegel upper half plane. The function

\[
J(M, Z) = \det(CZ + D)
\]

has no zeros on the half plane. Since the half plane is convex, there exists a continuous choice \( L(M, Z) = \arg J(M, Z) \) of the argument. We normalize it such that it is the principal value for \( Z = iE \) where \( E \) denotes the unit matrix. Recall that the principal value \( \text{Arg}(a) \) is defined such that it is in the interval \( (-\pi, \pi] \). So we have

\[
L(M, iE) = \text{Arg}(J(M, iE)) \in (-\pi, \pi].
\]
We consider
\[ w(M, N) := \frac{1}{2\pi} \left( (L(MN, Z) - L(M, NZ) - L(N, Z)) \right). \]

Obviously,
\[ e^{2\pi i w(M, N)} = 1. \]

Hence \( w(M, N) \) is a constant (independent of \( Z \)),
\[ w(M, N) \in \mathbb{Z}. \]

**Remark.** The function \( w : \text{Sp}(n, \mathbb{R}) \times \text{Sp}(n, \mathbb{R}) \to \mathbb{Z} \) is a cocycle in the following sense:
\[ w(M_1M_2, M_3) + w(M_1, M_2) = w(M_1, M_2M_3) + w(M_2, M_3), \]
\[ w(E, M) = w(M, E) = 0. \]

The computation of \( w(M, N) \) in genus 1 is easy for the following reason. From the definition we have
\[ 2\pi w(M, N) = \text{Arg}\left( (c\alpha + d\gamma)i + c\beta + d\gamma \right) - \text{arg}(cN(i) + d) - \text{Arg}(\gamma i + \delta) \]
for
\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]
where \( \text{arg}(cN(i) + d) \) is obtained from the principal value of \( \text{arg}(ci + d) \) through continuous continuation. But \( cz + d \) for \( z \) in the upper half plane never crosses the real axis. Hence the result of the continuation is the principal value too. So all three arguments in the definition of \( w(M, N) \) are the principal values (in genus 1). This makes it easy to compute \( w \). We rely on tables for the values of \( w \) which have been derived by Petersson and reproduced by Maass [Ma1], Theorem 16.

**0.1 Lemma.** Let \( M = \begin{pmatrix} \ast & \ast \\ m_1 & m_2 \end{pmatrix} \), \( S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be two real matrices with determinant 1 and \( m'_1, m'_2 \) the second row of the matrix \( MS \). Then
\[ 4w(M, S) = \begin{cases} 
\text{sgn} c + \text{sgn} m_1 - \text{sgn} m'_1 - \text{sgn}(m_1m'_1) & \text{if } m_1m'_1 \neq 0, \\
(1 - \text{sgn} c)(1 - \text{sgn} m_1) & \text{if } cm_1 \neq 0, m'_1 = 0, \\
(1 + \text{sgn} c)(1 - \text{sgn} m_2) & \text{if } cm'_1 \neq 0, m_1 = 0, \\
(1 - \text{sgn} a)(1 + \text{sgn} m_1) & \text{if } m_1m'_1 \neq 0, c = 0, \\
(1 - \text{sgn} a)(1 - \text{sgn} m_2) & \text{if } c = m_1 = m'_1 = 0.
\end{cases} \]

**Corollary.** Assume that \( m_1m'_1 \neq 0 \) and that \( m_1m'_1 > 0 \) or \( m_1c < 0 \). Then \( w(M, S) = 0 \).

We give an example.
0.2 Lemma. We have

\[ w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = w\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 0. \]

We denote by \( \Gamma_g[q] \) the principal congruence subgroup level \( q \). This is the kernel of the natural homomorphism \( \text{Sp}(g, \mathbb{Z}) \rightarrow \text{Sp}(g, \mathbb{Z}/q\mathbb{Z}) \).

1. Some special values of the cocycle

We give some examples for values of \( w \) in genus \( g > 1 \).

1.1 Lemma. One has

\[ w\left(\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, M\right) = 0. \]

The proof is trivial and can be omitted. \( \square \)

1.2 Lemma. Let \( g = 2 \) and

\[ P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

We have

\[ w(P, M) = w(M, P) = \begin{cases} 0 & \text{if } \text{Im} \det(iC + D) < 0, \\ -1 & \text{if } \text{Im} \det(iC + D) > 0. \end{cases} \]

Proof. Let \( z := \det(Ci + D) \). One computes

\[ 2\pi w(P, M) = 2\pi w(M, P) = \text{Arg}(-z) - \text{Arg}(z) - \text{Arg}(-1). \] \( \square \)

1.3 Definition. The Siegel parabolic group consists of all symplectic matrices of the form

\[ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}. \]

The two Klingen parabolic groups in the case \( g = 2 \) consist of all symplectic matrices of the form

\[ \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \text{ resp. } \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_4 & b_3 & b_4 \\ 0 & 0 & d_1 & 0 \end{pmatrix}. \]
There is a character on the Siegel parabolic group
\[ \varepsilon \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(D). \]

For an element \( M \) of the Siegel parabolic group, the expression \( \det(CZ + D) = \det(D) \) is independent of \( Z \). Hence
\[ L(M, Z) = 0 \text{ if } \varepsilon(M) > 0. \]

An immediate consequence is the following lemma.

**1.4 Lemma.** For two elements \( M, N \) of the Siegel parabolic group we have \( w(M, N) = 0 \) if \( \varepsilon(M) > 0. \)

**1.5 Lemma.** Let \( g = 2 \) and let \( M \) be a Klingen parabolic matrix and \( N \) a Siegel parabolic matrix with \( \varepsilon(N) > 0. \) Then \( w(M, N) = 0. \)

**Proof.** Since \( L(N, iE) = 1 \), we have to show that the arguments of \( J(MN, iE) \) and \( L(M, N(iE)) \) are the same. Both determinants are equal. But the argument of the first is the principal part and that of the second is defined by continuation from the argument of \( J(M, iE) \). Hence it is sufficient to show that the principal part of the argument of \( L(M, Z) \) is continuous. This is the case if \( \Im J(M, Z) \) is always \( \geq 0 \) or always \( < 0 \). Actually, for the first Klingen parabolic group
\[ \Im J(M, Z) = c_1d_4 \Im z_0 \quad \text{where} \quad Z = \begin{pmatrix} z_0 & * \\ * & * \end{pmatrix}. \]

The argument for the second Klingen parabolic group is the same. This proves the lemma.

**1.6 Lemma.** Let \( g = 2 \) and let
\[ M = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}. \]

Then
\[ w(I, M) = \begin{cases} 0 & \text{if } \tr(S) \geq 0, \\ -1 & \text{else.} \end{cases} \]

**Proof.** From the definition we have
\[ 2\pi w(I, M) = \Arg \det(iE + S) - \Arg \det(E) - \arg \det(iE + S). \]

The third argument is defined through continuation along \( \det(iE + tS) \), beginning from \( t = 0 \) to \( t = 1. \) For \( t = 0 \) we have to take the principal value which is \( \pi. \) The imaginary part of \( \det(iE + tS) \) equals \( \trt(S). \) In the case \( \tr(S) \geq 0 \) we keep the principal value. But if it is negative we make a jump by \( -2\pi. \)
§2. Multipliers

1.7 Lemma. Let \( g = 2 \) and let

\[ M = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix}. \]

Then

\[ w(M, I) = \begin{cases} -1 & \text{if } \text{tr}(S) \geq 0, \\ 0 & \text{else}. \end{cases} \]

Proof. Let \( z = \det(iS + E) \). One computes \( 2\pi w(M, I) = \text{Arg}(-z) - \pi - \text{Arg}(z) \).
This depends on the imaginary part of \( z \) which is \( \text{tr}(S) \). \( \square \)

2. Multipliers

2.1 Definition. Let \( \Gamma \subset \text{Sp}(g, \mathbb{R}) \) be an arbitrary subgroup and let \( r \) be a real number. A system \( v(M), M \in \Gamma \), of complex numbers of absolute value 1 is called a multiplier system of weight \( r \) if

\[ v(MN) \equiv v(M)v(N)\sigma(M, N) \]

where

\[ \sigma(M, N) = \sigma_r(M, N) := e^{2\pi i r w(M, N)}. \]

The elliptic modular group \( \text{Sp}(1, \mathbb{Z}) = \text{SL}(2, \mathbb{Z}) \) admits multipliers for every real \( r \). One can construct them by means of the discriminant function \( \Delta \). This is a modular form without zeros. Hence we can choose a holomorphic power \( f(z) = \Delta(z)^{r/12} \). This can be used to construct a multiplier.

Maass [Ma2] proved that the full Siegel modular group of genus \( g > 1 \) admits only multipliers for integral \( r \) and their values can be only \( \pm 1 \). As a consequence (s. [Ch]), for every multiplier system on a subgroup \( \Gamma \) of finite index of the modular group the weight \( r \) is rational and the values of \( v \) are contained in a finite subgroup of \( S^1 \).

Let \( \Gamma_{g,\theta} \) be the theta group of degree \( g \). It consists of all integral symplectic matrices such that \( AB' \) and \( CD' \) have even diagonal. The function

\[ \vartheta(Z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i n' Z n} \]

is a modular form of weight 1/2 on the theta group. It can be used to construct a multiplier system of weight 1/2.

The result of Deligne states:
2.2 Theorem. Let $g > 1$ and let $\Gamma \subset \text{Sp}(g, \mathbb{Z})$ be any subgroup of finite index of the Siegel modular group. Multiplier systems of weight $r$ can only exist if $2r$ is integral.

It is sufficient to prove this in the case $g = 2$. So we assume from now on $g = 2$.

We assume that a natural number $q'$ is given and that $v$ is a multiplier system of weight $r$ on $\Gamma_2[q']$.

For any $L \in \text{Sp}(2, \mathbb{Z})$ we can consider a conjugate multiplier system $[\text{FB}]$ that is defined by

$$\tilde{v}(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)}.$$

It is easy to check that this is a multiplier system. The quotient of two multiplier systems of the same weight is a homomorphism, as we know into a finite group. Since every subgroup of finite index of the Siegel modular group is a congruence subgroup, we obtain $\tilde{v}(M) = v(M)$ on some subgroup $\Gamma_2[q] \subset \Gamma_2[q']$ (where $q$ may depend on $L$).

2.3 Lemma. For given $L$ in the full modular group there exists a multiple $q$ of $q'$ such that such that

$$v(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)}$$

for each $M \in \Gamma_2[q]$.

This will be used for several matrices, in particular for $M = I$.

2.4 Proposition. There exists a multiple $q$ of $q'$ such that the following holds. Let $U$ be an element from the subgroup that is generated by the matrices $(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix})$. Let $M$ be a matrix from $\Gamma_2[q]$ of the form

$$M = \begin{pmatrix} U' & * \\ 0 & U^{-1} \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} E & 0 \\ * & E \end{pmatrix}.$$

Then $v(M) = 1$.

Proof. The matrices of the first type build a finitely generated group. The number of generators is independent on $q$. It is enough to prove $v(M) = 1$ for the generators, since $w(M, N) = 0$ for all $M, N$ in this group. We also have $v(M)^n = v(M^n)$. Since the values of $v$ are contained in a finite group, we find an $n$ such that $v(M^n) = 1$ for all of the generators.

The second case is more difficult. Due to Lemma 2.3 it is sufficient to prove $\sigma(IMI^{-1}, I) = \sigma(I, M)$ for translation matrices $M$. This follows from the Lemmas 1.6 and 1.7. \qed
3. Embedded subgroups

We have to consider three embeddings of \( SL(2, \mathbb{Z}) \) into \( Sp(2, \mathbb{Z}) \), namely

\[
\iota_1, \iota_2, \iota_3 : SL(2) \longrightarrow Sp(2),
\]

\[
\iota_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cccc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad \iota_2 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{array} \right),
\]

\[
\iota_3(M) = \left( \begin{array}{cc} M & 0 \\ 0 & M^{-1} \end{array} \right) = \left( \begin{array}{cccc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{array} \right).
\]

We have \( w(\iota_3(M), \iota_3(N)) = 0 \). Hence \( M \mapsto v(\iota_3(M)) \) is a homomorphism into a finite group. Its kernel is a subgroup of finite index in \( SL(2, \mathbb{Z}) \). We will show that it is in fact a congruence subgroup.

Let \( P \) as in Lemma 1.2. We have

\[
P \iota_1(M) P^{-1} = \iota_2(M).
\]

From Lemma 1.2 follows \( w(\iota_2(M), P) = w(P, \iota_1(M)) \). Hence we obtain from Lemma 2.3 the following result.

3.1 Lemma. \( We \ have \)

\[
v(\iota_1(M)) = v(\iota_2(M))
\]

for \( M \in \Gamma_1[q] \).

For sake of simplicity we write

\[
v(M) = v(\iota_1(M)) = v(\iota_2(M)).
\]

This is a multiplier system in genus 1. We have

\[
w(M, N) = w(\iota_\nu(M), \iota_\nu(N)), \quad \text{for} \quad \nu = 1, 2.
\]

3.2 Lemma. \( The \ value \ v(M), M \in \Gamma_1[q], \ depends \ only \ on \ the \ second \ row \ of \ M. \)

Proof. When \( M, N \) have the same second row, then \( \left( \begin{array}{c} 1_x \\ 0_1 \end{array} \right) M = N \). We know \( w(\left( \begin{array}{c} 1_x \\ 0_1 \end{array} \right), M) = 0 \) and \( v(\left( \begin{array}{c} 1_x \\ 0_1 \end{array} \right)) = 1 \) (Proposition 2.4). \( \square \)
3.3 Lemma. Assume that $v$ is a multiplier system of weight $r$ on $\Gamma_2[q']$. There exists a multiple $q$ of $q'$ such that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1[q]$ we have

$$v\begin{pmatrix} d_1 & -c_1 & 0 & 0 \\ -b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix} \cdot v\begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1^2 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1^2 c_2 & 0 & y \end{pmatrix}$$

where

$$y = d_1 - b_1 c_1 d_2 + c_1 c_2 b_1 b_2 d_1.$$

Proof. The proof depends on a certain relation which occurs in [BMS] during the proof of Lemma 13.3. We reproduce it here. We set

$$H_1 = \begin{pmatrix} d_1 & -c_1 & 0 & 0 \\ -b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1^2 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1^2 c_2 & 0 & y \end{pmatrix}.$$

We consider the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ ac_2 & c_1 c_2 & 1 & 0 \\ c_1 c_2 & 0 & 0 & 1 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_1 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & -ad_1^2 b_2 & b_1 b_2 d_1 \\ 0 & 1 & b_1 b_2 d_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
We are going to compute \( w(R_2, H_3) \). A direct computation gives

\[
\text{Im } J(R_2 H_3, iE) = c_2 (1 + c_1^2).
\]

Next we treat \( J(R_2, H_3(iE)) \). Here the argument has to be defined by continuation from the principal value of the argument of \( J(R_2, iE) \). We can do this along the straight line from \( iE \) to \( H_3(iE) \). The points on this line are of the form \((i \tau, 0)\) where \( \tau \) is in the upper half plane. One computes

\[
J(R_2, \begin{pmatrix} i & 0 \\ 0 & \tau \end{pmatrix}) = \det \begin{pmatrix} 1 + ac_2i & c_1 c_2 \tau \\ c_1 c_2 i & 1 \end{pmatrix}.
\]

The real part is \( 1 + c_1^2 c_2^2 \text{Im } \tau \) which is positive. Hence the principal value of the argument is continuous along the line. So we see

\[
L(R_2, H_3(iE)) \in (-\pi, \pi]
\]

Finally we compute

\[
\text{Im } J(H_3, iE) = c_1^2 c_2.
\]

Now we see that the imaginary part of \( J(R_2 H_3, iE) \) and \( J(H_3, iE) \) have the same sign (namely the sign of \( c_2 \)). Hence their arguments are both contained in \((0, \pi)\) or in \((-\pi, 0)\). This means that \( 2\pi w(R_2, H_3) \) is contained in \((0, \pi) - (-\pi, \pi) = (0, \pi)\) or in \((-\pi, 0) - (-\pi, \pi) = (-\pi, 0)\). This is \((-2\pi, 2\pi)\) in both cases. We obtain

\[
w(R_2, H_3) = 0.
\]

The case \( c_1 c_2 = 0 \) is easy and can be omitted.

From Lemma 1.5 we can take \( w(H_2, R_1 R_3 R_4) = 0 \). For trivial reason one has \( w(H_1, H_2 R_1 R_3 R_4) = 0 \). Now we evaluate

\[
v(R_2 H_3) = v(H_1 H_2 R_1 R_3 R_4).
\]

The left hand side is

\[
v(R_2)v(H_3)\sigma(R_2, H_3) = v(R_2)v(H_3).
\]

But \( v(R_2) = 1 \) (Proposition 2.4). Hence the left hand side is just \( v(H_3) \). The right hand side is

\[
v(H_1)v(H_2 R_1 R_3 R_4)\sigma(H_1, H_2 R_1 R_3 R_4) = v(H_1)v(H_2 R_1 R_3 R_4).
\]

Similarly we see

\[
v(H_2 R_1 R_2 R_3) = v(H_2)v(R_1 R_2 R_3)\sigma(H_2, R_1 R_3 R_4) = v(H_2)v(R_1 R_3 R_4).
\]

From Proposition 2.4 we know \( v(R_1 R_3 R_4) = 1 \). Hence we get \( v(H_3) = v(H_1)v(H_2) \). \( \square \)
4. Mennicke symbol

We have seen that \( v(\iota_1(M)) = v(\iota_2(M)) \) depends only on the second row of \( M \in \Gamma_1[q] \). Hence we can define

\[
\begin{bmatrix} c \\ d \end{bmatrix} = v \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}^{-1}.
\]

We also can define

\[
\begin{bmatrix} b \\ a \end{bmatrix} = v \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.
\]

It is clear that this does not depend on the choice of \( c, d \).

4.1 Proposition. For a suitable multiple \( q > 2 \) of \( q' \) the bracket \( \begin{bmatrix} b \\ a \end{bmatrix} \) is a Mennicke symbol. This means that it is a function on the set of all coprime \( (a, b) \) with the property \( a \equiv 1 \mod q \) and \( b \equiv 0 \mod q \) such that the following properties hold.

MS1 It is invariant under the transformations \( (a, b) \mapsto (a + xb, b) \) and \( (a, b) \mapsto (a, b + qay) \) for integral \( x, y \).

MS2 It satisfies the rule

\[
\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = v \begin{pmatrix} b_1 \\ a \end{pmatrix} \begin{pmatrix} b_2 \\ a \end{pmatrix}.
\]

**Proof of MS1.** We notice that \( w \) is trivial on the image of \( \iota_3 \). Hence \( v \) is a character on this group. The invariance under \( (a, b) \mapsto (a + xb, b) \) follows from the equation

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & qy \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b + qay \\ * & * \end{pmatrix}.
\]

To prove the invariance under \( (a, b) \mapsto (a + xb, b) \), we consider

\[
\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a + xb & b \\ * & * \end{pmatrix}.
\]

Due to Lemma 2.3 we can assume that \( v(\iota_3(M)) \) is invariant under conjugation with \( \iota_3(\begin{pmatrix} 10 \\ 11 \end{pmatrix}) \). This proves MS1.

**Proof of MS2.** The proof of MS2 needs two Lemmas which we now have to formulate and prove now. We make use of

\[
v(\iota_\nu(M^{-1})) = v(\iota_\nu(M))^{-1}.
\]
This is true since in genus 1 one has \( w(M, M^{-1}) = 0 \). (This is a general rule for \( c \neq 0 \) but also for \( c = 0 \) and \( a > 0 \). But in our case \( c = 0 \) implies \( a = 1 \) since we assume \( q > 2 \).) This relation implies

\[
\left\{ \begin{array}{c} c \\ d \end{array} \right\} = \left\{ \begin{array}{c} -c \\ a \end{array} \right\}^{-1}.
\]

From Lemma 3.3 we get after the replacement, \( c_2 \mapsto -c_2 \) the following general rule (compare Lemma 13.3 in [BMS]).

**4.2 Lemma.** Let \( a - 1 \equiv c_1 \equiv c_2 \equiv 0 \mod q \) and let \( a, c_1 \) and \( a, c_2 \) be coprime. Then

\[
\left[ \begin{array}{cc} c_1 \\ a \end{array} \right] \left\{ \begin{array}{c} c_2 \\ a \end{array} \right\} = \left\{ \begin{array}{c} c_1^2 c_2 \\ a \end{array} \right\}.
\]

We need also the following simple lemma.

**4.3 Lemma.** We have

\[
\left\{ \begin{array}{c} 1 - a \\ a \end{array} \right\} = 1
\]

for \( a \equiv 1 \mod q \).

**Proof.** We use

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 - a & a - 1 \\ 1 - a & a \end{pmatrix}.
\]

\( \square \)

We insert un Lemma 4.2 now \( c_2 = 1 - a \) to obtain the following formula.

\[
\left[ \begin{array}{c} c \\ a \end{array} \right] = \left\{ \begin{array}{c} c^2(1 - a) \\ a \end{array} \right\}.
\]

Before we continue, we mention that \( \left\{ \right\} \) is not a Mennicke symbol. It does not satisfy MS1.

**4.4 Lemma.** We have

\[
\left\{ \begin{array}{c} c \\ d \end{array} \right\} = \left\{ \begin{array}{c} c \\ d + yc \end{array} \right\}
\]

and

\[
\left\{ \begin{array}{c} c + xqd \\ d \end{array} \right\} = \left\{ \begin{array}{c} c \\ d \end{array} \right\} e^{2\pi i rs}.
\]

where

\[
s = w \left( \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix} \right).
\]
Proof. The first relation can be derived from
\[
\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d + cy \end{pmatrix}.
\]
To derive the second one we consider the relation
\[
\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c + dxq & d \end{pmatrix}.
\]
It shows
\[
\left\{ c + dxq \atop d \right\} = \left\{ c \atop d \right\} e^{2\pi i rs}.
\]
The $w$-value $s$ is usually not zero. □

Proof of Proposition 4.1 (MS2) continued. Now we use
\[
\begin{pmatrix} * & * \\ c^2 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c^2 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c^2 - ac^2 & a \end{pmatrix}.
\]
From the corollary of the table of Maass in the introduction we get
\[
w\left( \begin{pmatrix} * & * \\ c^2 & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -c^2 & 1 \end{pmatrix} \right) = 0
\]
and hence
\[
\left\{ c^2(1 - a) \atop a \right\} = \left\{ c^2 \atop a \right\}.
\]
So we obtain
\[
\begin{bmatrix} c \\ a \end{bmatrix} = \left\{ c^2 \atop a \right\}
\]
and moreover
\[
\begin{bmatrix} c_1c_2 \\ a \end{bmatrix} = \left\{ c_1^2c_2^2 \atop a \right\} = \begin{bmatrix} c_1 \end{bmatrix} \begin{bmatrix} c_2^2 \\ a \end{bmatrix} = \begin{bmatrix} c_1 \\ a \end{bmatrix} \begin{bmatrix} c_2 \end{bmatrix}.
\]
This finishes the proof of Proposition 4.1. □

The main result about Mennicke symbols is that they are trivial [BMS], Theorem 3.6. Hence we obtain now the important result.

4.5 Proposition. The multiplier system $v$ is identically one on all
\[
\begin{pmatrix} M & 0 \\ 0 & M'^{-1} \end{pmatrix} \quad \text{for} \; M \in \Gamma_1[q].
\]
From Lemma 4.2 follows now
\[ \{ \frac{c^2}{d} \} = 1 \quad \text{and} \quad \{ \frac{c_1 c_2}{d} \} = \{ \frac{c_1 c_2^2}{d} \} \]
for \( c \equiv c_1 \equiv c_2 \equiv 0 \mod q \) and \( d \equiv 1 \mod q \). This can be generalized. We have to consider the Kronecker symbol \( (\frac{c}{d}) \). For its definition and properties we refer to [Di]. We will need it only for \( c \neq 0 \) and for odd \( d \). We collect some properties (always assuming this condition)
\[ \left( \frac{c_1 c_2}{d} \right) = \left( \frac{c_1}{d} \right) \left( \frac{c_2}{d} \right), \quad \left( \frac{c}{d_1 d_2} \right) = \left( \frac{c}{d_1} \right) \left( \frac{c}{d_2} \right). \]
Assume \( d > 0 \) or \( c_1 c_2 > 0 \). Then
\[ \left( \frac{c_1}{d} \right) = \left( \frac{c_2}{d} \right) \quad \text{if} \quad c_1 \equiv c_2 \mod d. \]
Also the relation
\[ \left( \frac{c}{d_1} \right) = \left( \frac{c}{d_2} \right) \quad \text{if} \quad \begin{cases} d_1 \equiv d_2 \mod c \quad \text{and} \quad c \equiv 0 \mod 4, \\ d_1 \equiv d_2 \mod 4c \quad \text{and} \quad c \equiv 2 \mod 4 \end{cases} \]
is valid. Finally we mention
\[ \left( \frac{c}{-1} \right) = \begin{cases} 1 & \text{for } c > 0, \\ -1 & \text{for } c < 0. \end{cases} \]
Since one of the rules demands \( c \equiv 0 \mod 4 \), we will from now on assume that \( q \equiv 0 \mod 4 \).

4.6 Proposition. Let \( q \) be a suitable multiple of \( q' \) and let
\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1[q], \quad \left( \frac{c}{d} \right) = 1. \]
Then \( v(M) = 1 \).

Proof. We use the invariance under \((c, d) \mapsto (c, d+xc)\). We can apply Dirichlet’s prime number theorem and therefore assume that \( d = p \) is a (positive) prime. But then the Kronecker symbol is the usual Legendre symbol. Since \( d \equiv 1 \mod q \) we have \( (\frac{q}{d}) = 1 \). This implies \( (\frac{c/p}{d}) = 1 \). Since \( d \) is a prime, we get a solution of \( c/q = x^2 + dy \) or \( c = qx^2 + dxy \). Now use
\[ \begin{pmatrix} * & * \\ qx^2 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qy & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}. \]
In the case \( c > 0 \) the \( w \)-value is zero. This follows from the corollary in the table of Maass in the introduction. In the case \( c < 0 \) we must have \( y < 0 \) and
again from this corollary follows that the $w$-value is zero. (In the notation of the table the sign distribution of $(m_1, c, m'_1)$ is $(+, *, +)$ or $(+, -, *)$.) Now we get

$$v(M) = v\left( \begin{array}{cc} * & * \\ c & d \end{array} \right) = v\left( \begin{array}{cc} * & * \\ qx^2 & d \end{array} \right) = \{qx^2\}. \quad \{x^2q\} = \{x^2q^3\} = \{q(xq)^2\} = \{q\} = \{1\} = 1. \quad \square$$

\textbf{4.7 Lemma.} Assume that the matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is contained in \(\Gamma_1[q]\) and has the following properties. All entries are positive and \(dq < c(q-1)\). Then

$$v(M) = e^{-2\pi ir} \quad \text{if} \quad \left( \frac{c}{d} \right) = -1.$$

\textbf{Proof.} We consider

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & -q \\ q & 1+q \end{array} \right) = \left( \begin{array}{cc} * & * \\ c- qc + dq & -cq + d + dq \end{array} \right).$$

Clearly \(\left( \frac{c}{1+q} \right) = 1\). We also claim

$$\left( \frac{c - cq + dq}{-cq + d + dq} \right) = 1.$$

To prove this, we observe

$$\left( \frac{c - cq + dq}{-cq + d + dq} \right) = \left( \frac{c - qc + dq}{d - c} \right) = \left( \frac{c - qc + dq}{-1} \right) \left( \frac{c - qc + dq}{c - d} \right).$$

Now we use \(c - qc + dq < 0\). It follows \(c - d > 0\). Hence we get

$$= -\left( \frac{c}{c - d} \right) = -\left( \frac{c}{-d} \right) = -\left( \frac{c}{d} \right) = -(-1) = 1.$$

Now we have proved

$$v\left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & -q \\ q & 1+q \end{array} \right) \right) = 1.$$

The left hand side equals

$$v\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \exp \left\{ 2\pi i w \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \left( \begin{array}{cc} 1 & -q \\ q & 1+q \end{array} \right) \right) \right\} = 1.$$
From Maass’ table in the introduction follows that the \( w \)-value is 1. (The sign distribution of \((m_1, c, m'_1)\) is \((+, +, -)\).) This proves Lemma 4.7. \(\square\) There exist two coprime natural numbers \(c, d\) such that \(c \equiv 0 \mod q\) and \(d \equiv 1 \mod q\) and such that \(\left(\frac{c}{d}\right) = -1\). We also can assume \(dq < c(q - 1)\). The pair \((c, d)\) is the second row of a matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1[q]\). We want to compute \(v(M)\). Since we can add a multiple of the second row to the first one, we can assume that \(a\) and \(b\) are also positive. From Lemma 4.7 we know \(v(M) = e^{-2\pi ir}\). Now we consider

\[
v(M^2) = v(M)^2 e^{2\pi ir w(M, M)}.
\]

Since all entries from \(M\) are positive, we have \(w(M, M) = 0\). So we get

\[
v(M^2) = e^{-4\pi ir}.
\]

We compute \(\left(\frac{c}{d}\right)\) for the matrix

\[
N = M^2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.
\]

We get

\[
\left(\frac{c}{d}\right) = \left(\frac{c(a + d)}{cb + d^2}\right) = \left(\frac{c}{cb + d^2}\right)\left(\frac{a + d}{cb + d^2}\right).
\]

We have

\[
\left(\frac{c}{cb + d^2}\right) = \left(\frac{c}{d^2}\right) = 1
\]

and

\[
\left(\frac{a + d}{cb + d^2}\right) = \left(\frac{a + d}{d(a + d) - 1}\right).
\]

Since \(a + d \equiv 2 \mod 4\) we only can change the denominator mod \(4(a + d)\). Since \(d \equiv 1 \mod 4\) we see

\[
\left(\frac{a + d}{d(a + d) - 1}\right) = \left(\frac{a + d}{a + d - 1}\right) = \left(\frac{1}{a + d - 1}\right) = 1.
\]

This shows \(v(N) = 1\) and we get the relation

\[
e^{-4\pi ir} = 1
\]

which implies that \(2r\) is integral. This finishes the proof of the main result.
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