The microlocal spectrum condition, initial value formulations and background independence

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Abstract

We analyze the implications of the microlocal spectrum/Hadamard condition for states in a (linear) quantum field theory on a globally hyperbolic spacetime $M$ in the context of a (distributional) initial value formulation. More specifically, we work in a $3+1$-split $M \cong \mathbb{R} \times \Sigma$ and give a bound, independent of the spacetime metric, on the wave front sets of the initial data for a quasi-free Hadamard state in the quantum field theory defined by a normally hyperbolic differential operator $P$ acting in a vector bundle $E \to M$. This aims at a possible way to apply the concept of Hadamard states within approaches to quantum field theory/gravity relying on a Hamiltonian formulation, potentially without a (classical) background metric $g$.

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1 Introduction

Quantum field theory on curved spacetimes is nowadays a well-developed subject, which allows for the rigorous treatment of perturbative quantization of classical field theories on curved spacetimes (see [1] for a recent review), including scalar fields, Dirac fields, Yang-Mills fields, and even the treatment of perturbative quantum gravity in a locally covariant fashion [2]. At the basis of this approach are the linear(ized) field theories and their quantum theories, which are probably the most studied examples of quantum field theories. In the framework of algebraic quantum field theory, the concept of Hadamard states for linear quantum fields plays an important role (see

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e.g. [8,9]). These states replace the concept of vacuum state in a locally covariant manner by mimicking the short-distance behavior of the latter in purely spacetime geometric fashion. It is known that there are sufficiently many of these states on arbitrary globally hyperbolic spacetimes (cf. [7,9]). A particularly elegant characterization of Hadamard states is the so-called microlocal spectrum condition [10], which is a prescription for the wave front set of the associated two-point function(al), and can be interpreted as a remnant of the spectrum condition in quantum field theory on Minkowski space.

In this article, we analyze the relation between the microlocal spectrum condition and the initial value formulation of the quantum field theory with the aim to extract a condition that is manifestly independent of the spacetime metric. Such a condition could be used as a filter for physically interesting states in the matter sector of approaches to quantum gravity, where a (classical) spacetime metric is not available, e.g. loop quantum gravity. Furthermore, our analysis provides a first step to elucidate the structures that need to be present in a theory of quantum gravity coupled to matter, such that quantum field theory on curved spacetime can be extracted in a semi-classical limit. Potential candidates for a semi-classical analysis of loop quantum gravity in this regard are the reparametrizing models (see [11] for an overview). Notably, the concept of adiabatic vacua, which is related to that of Hadamard states [12,13], has already been applied in the framework of loop quantum cosmology, a loop quantization of symmetry reduced models, to treat cosmological perturbations [14,15].

The organization of the article is as follows:

The main part of the article is section 2. It deals with the quantum theory of linear field theories in the context locally covariant quantum field theory [6], and the important notion of Hadamard states, which are characterized by prescription for the wave front set of the two-point correlation function(al). In the first subsection 2.1 beside recalling important results about linear field theories on Lorentzian manifolds, we prove, building on work by Dimock [16], a theorem on the initial value problem for generalized wave equations with distributional initial data. In subsection 2.2 we discuss the microlocal spectrum/Hadamard condition, and prove the main theorem of this article: A bound on the wave front sets of the initial data for a quasi-free Hadamard state of a linear quantum field theory, which is independent of the spacetime metric.

In section 3 we discuss the wave front set bound for the initial data in view of dynamical aspects of the microlocal spectrum condition and available construction procedures for Hadamard states. Furthermore, we outline how the wave front set bound could be applied as an a priori condition for semi-classical states of quantum matter fields in background independent theories like loop quantum gravity.

Section 5 provides an appendix with some essential material from the theory of distributions and their wavefront sets.

Let us fix some notation:

Throughout the article, $(M,g)$, or $M$ for short, denotes spacetime, i.e. a globally hyperbolic, time-/space-oriented, Hausdorff, second-countable, $\sigma$-compact ($C^\infty$-) manifold ($\dim(M) = m < \infty$). The metric induced volume form on $M$ is $dV_g$. A Cauchy surface for $M$ is called $\Sigma$, i.e. $M \cong \mathbb{R} \times \Sigma$. The induced volume form on $\Sigma$ is $dA_g$. For the causal future/past of a subset $K \subset M$, we use the usual notation $J_+(K)$ ($J(K) := J_+(K) \cup J_-(K)$), $K \subset M$ indicates a compact subset. $E \xrightarrow{\pi} M$, or simply $E$, is a finite dimensional, (real) vector bundle over $M$ ($\text{rank}(E) = e$), and $E^*$ its dual. If we have two such vector bundle $E, E'$, we denote the exterior tensor product over $M \times M'$ by $E \boxtimes E'$, and the interior tensor product, for $M = M'$, over $M$ by $E \otimes E'$. If we do not specify a connection in the tangent bundles $TM, T\Sigma$, these are given resp. induced by the Levi-Civita connection of $g$.

The functional spaces, we frequently use, are:

1. The compactly supported, smooth functions on $M$ or $\Sigma$, and their distributional duals:

   $\mathcal{D}(M), \mathcal{D}(\Sigma)$ and $\mathcal{D}'(M), \mathcal{D}'(\Sigma)$.

2. The smooth functions on $M$ or $\Sigma$, and their distributional duals:

   $\mathcal{E}(M), \mathcal{E}(\Sigma)$ and $\mathcal{E}'(M), \mathcal{E}'(\Sigma)$.

3. The smooth function with “spacelike compact” support on $M$, and their dual:

   $\mathcal{E}_{sc}(M)$ and $\mathcal{E}_{sc}(M)$ (cf. [17]).

$\mathcal{E}_{sc}(M)$ are the smooth function on $M$ which are “spacelike compact”, i.e. if $f \in \mathcal{E}_{sc}(M)$ there exists a compact set $K \subset M$ s.t. $\text{supp}(f) \subset J(K)$.
4. The generalizations of these spaces to sections in a vector bundle $E$ over $M$ or its restriction $E_{\Sigma}$ to $\Sigma$:

$$\mathcal{D}(M, E), \mathcal{D}'(M, E^*)$$ etc. (cf. [17]).

5. Distribution spaces with specified wave front sets:

$$\mathcal{D}'_\Gamma, \mathcal{E}'_\Lambda,$$

where $\Gamma, \Lambda$ are conical subsets of $T^* M$ or $T^* \Sigma$ (see definition 5.14, cf. [18, 19]).

The embeddings of the type $\mathcal{D}(M, E) \hookrightarrow \mathcal{D}'(M, E)$ are understood by means of the volume form $dV_g$ and the fibre metric $g_E$, i.e.

$$\forall f' \in \mathcal{D}(M, E) : (f, f') := \int_M g_E(f, f') dV_g,$$

associates a unique distribution to every $f \in \mathcal{D}(M, E)$. All space will be equipped with one of their usual topologies. Thus, we refrain from restating the various definitions, and refer the interested reader to the appendix and references. Let us also issue a word of caution regarding the notion of continuity and sequential continuity: In general, sequentially continuous maps between locally convex topological vectors spaces are not necessarily continuous in the topological sense. Although, equality of the concepts holds for bornological topologies (cf. [20]), it may fail for non-bornological spaces like $\mathcal{D}'_\Gamma, \Gamma$ a closed, but not open cone (cf. [19]). In this article, we restrict ourselves to the simpler case of sequential continuity.

2 Linear quantum fields in curved spacetimes

We start this section by a brief outline of some essential facts for the understanding of linear quantum fields in curved spacetime and our analysis of the microlocal spectrum/Hadamard condition (cf. [6,16,17,21]). We conclude the first subsection 2.1 by proving that the distributional initial value problem for generalized wave equations can be considered well-posed. After this, we proceed to the discussion of the microlocal spectrum/Hadamard condition for the quantum theory, and prove the main theorem of the article.

2.1 The initial value formulation for generalized wave equations

Let us consider a spacetime $M$, and a vector bundle $E$ on $M$ equipped with a (non-degenerate) fibre metric $g_E$. The fibre metric $g_E$ provides an identification of $E$ and $E^*$, which we will use freely. Global hyperbolicity implies the existence of a $3 + 1$-split of spacetime, $M \cong \mathbb{R} \times S$ in the $C^\infty$-sense (cf. [22]), and we have a well-posed initial value problem (cf. [17]), with initial data in $\mathcal{D}(\Sigma, E_{\Sigma})$, for generalized wave equations

$$Pu = 0, \ u \in \mathcal{E}'(M, E),$$

where $P : \mathcal{E}'(M, E) \to \mathcal{E}'(M, E)$ is a formally self-adjoint, normally hyperbolic differential operator, i.e.

$$\int_M g_E(Pu, v) dV_g = \int_M g_E(u, Pv) dV_g,$$

and the principal symbol of $P$ is given by the spacetime metric $g$:

$$P = g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + a^k(x) \frac{\partial}{\partial x^k} + b(x)$$

in local coordinates $x = (x^1, ..., x^m)$ on $U \subset M$ subordinate to a local trivialization $E_{|U} \cong U \times \mathbb{R}^e$ with matrix valued coefficients $a, b : U \to \mathbb{R}^e$. Moreover, there exist unique advanced and retarded fundamental solutions

$$G^\pm : \mathcal{D}(M, E) \to \mathcal{E}_{\text{sc}}(M, E),$$

$$P \circ G^\pm = \text{id}_{\mathcal{D}(M, E)}, \ G^\pm \circ P_{\mathcal{D}(M, E)} = \text{id}_{\mathcal{D}(M, E)}$$

$$\forall f \in \mathcal{D}(M, E) : \text{supp}(G^\pm(f)) \subset J^M_e(\text{supp}(f)),$$
and we may write (cp. \[16\])

\[
f = (G' \circ (\nu^*)')(f_1) - (G' \circ (\nu^*)')(f_0), \quad f_0, f_1 \in \mathcal{D}'(\Sigma, E^*_\Sigma) \cong \mathcal{D}'(\Sigma, E_\Sigma) \subset \mathcal{D}'(\Sigma, E^*_\Sigma),
\]

(2.7)

where \( \iota : \Sigma \hookrightarrow M \) is the inclusion of the Cauchy surface, \( G' : \mathcal{E}'(M, E^*) \to \mathcal{D}'(M, E^*) \) is the (formal) adjoint of the causal propagator \( G = G^\pm = G^+ \), and \((\nu^*)', (\nu^*)' : \mathcal{D}'(\Sigma, E^*_\Sigma) \to \mathcal{E}'_{\text{sc}}(M, E^*) \) denote the adjaunts of the maps

\[
i^* : \mathcal{E}'(\Sigma, E_\Sigma), \quad u \mapsto u|_{\Sigma} = i^*u \tag{2.8}
\]

\[
u^* : \mathcal{E}'(\Sigma, E_\Sigma), \quad u \mapsto (\nabla_n u)|_{\Sigma} = \nu^*(\nabla_n u),
\]

where \( u \in \mathcal{E}(\Sigma, TM_\Sigma) \) denotes the timelike, future oriented, unit normal to \( \Sigma \), and \( \nabla \) is the unique \( P \)-compatible connection in \( E \) (cf. \[17,23\]). We notice that the adjaunts of the advanced and retarded fundamental solutions satisfy \( G^\pm = (G^\mp)' \), because of the formal self-adjointness of \( P \). From \[16,17\], we know that the restrictions

\[
G' \circ (\nu^*)', G' \circ (\nu^*)' : \mathcal{D}'(\Sigma, E^*_\Sigma) \subset \mathcal{D}'(\Sigma, E^*_\Sigma) \to \mathcal{E}'_{\text{sc}}(M, E^*) \subset \mathcal{D}'(M, E^*)
\]

(2.9)

are (sequentially) continuous maps, and one finds the identities

\[
i^* \circ G \circ (\nu^*)' = 0, \quad \nu^* \circ G \circ (\nu^*)' = -\text{id}_{\mathcal{D}'(\Sigma, E_\Sigma)}, \tag{2.10}
\]

\[
\nu^* \circ G \circ (\nu^*)' = \text{id}_{\mathcal{D}'(\Sigma, E_\Sigma)}, \quad \nu^* \circ G \circ (\nu^*)' = 0,
\]

(2.11)

and a short exact sequence of (sequentially) continuous maps

\[
0 \longrightarrow \mathcal{D}(M, E) \overset{P}{\longrightarrow} \mathcal{D}(M, E) \overset{E}{\longrightarrow} \mathcal{E}'_{\text{sc}}(M, E) \overset{P}{\longrightarrow} \mathcal{E}'_{\text{sc}}(M, E).
\]

(2.12)

Furthermore, it follows from the results of \[16\], and \(2.12\), that \(2.7\) can be utilized to construct solutions with distributional initial data \( u_0, u_1 \in \mathcal{D}'(\Sigma, E^*_\Sigma) \), i.e.

\[
u = (G' \circ (\nu^*)')(u_1) - (G' \circ (\nu^*)')(u_0) \in \mathcal{D}'(M, E^*). \tag{2.13}
\]

\[
\text{Remark 2.1:}
\]

Equation \(2.13\) admits an important refinement, because there exists a strong constraint on the wave front set of any distributional solution \( u \in \mathcal{D}'(M, E^*) \) to a linear partial differential equation \( Pu = 0 \) (see theorem \[5.22\]), cf. \[24\]):

\[
WF(u) \subset \text{Char } P. \tag{2.14}
\]

The conical subset \( \text{Char } P \subset T^*M \setminus \{0\} \), called the characteristic set of \( P \), is defined in theorem \[5.22\] of the appendix. The definition of the wave front set of a distribution can be found in the appendix (see definition \[5.3\]), as well. We conclude that \(2.13\) can be replace by

\[
u = (G' \circ (\nu^*)')(u_1) - (G' \circ (\nu^*)')(u_0) \in \mathcal{D}'_{\text{Char } P}(M, E^*). \tag{2.15}
\]

What is not achieved in \[16\], although one finds a contrary statement in \[8\], is an answer to the questions, which distributional solutions \( u \in \mathcal{D}'(M, E^*) \) arise in this way, and in which sense the initial value problem can be considered well-posed for initial data in \( \mathcal{D}'(\Sigma, E^*_\Sigma) \). A (partial) answer to these questions can be given in form of the following theorem.

\[
\text{Theorem 2.2 (The distributional initial value problem):}
\]

Let \( E \) be a vector bundle over a globally hyperbolic spacetime \( M \), equipped with a non-degenerate fibre metric \( g_\Sigma \). Furthermore, let \( P \) be a formally self-adjoint, normally hyperbolic operator acting in \( E \). Then, given \( u_0, u_1 \in \mathcal{D}'(\Sigma, E^*_\Sigma) \), there exists a unique, proper, distributional solution \( u \in \mathcal{D}'_{\text{Char } P}(M, E^*) \) to the equation \( Pu = 0 \), s.t. \( u|_{\Sigma} = u_0, (\nabla_n u)|_{\Sigma} = u_1 \). Here, a distributional solution \( u \) is called proper (cp. \[22\]), if it can be approximated by
a sequence of regular solutions \( \{u_j\}_{j=1}^{\infty} \subset \mathcal{E}(M,E^*) \) in the (weak) topology of \( \mathcal{D}'_{\text{Char} P}(M,E^*) \), i.e.

\[
\forall f \in \mathcal{D}(M,E) : (u,f) = \lim_{j \to \infty} (u_j,f),
\]

\[
\forall j : Pu_j = 0.
\]

Moreover, the map

\[
\mathcal{D}'(\Sigma,E^*_\Sigma) \otimes 2 \to \mathcal{D}'_{\text{Char} P}(M,E^*)
\]

sends \((u_0,u_1)\) to the solution \(u\), s.t. \(Pu = 0, \ u|_{\Sigma} = u_0, (\nabla nu)|_{\Sigma} = u_1\), is (sequentially) continuous.

Before we start the proof of this theorem, we state useful results concerning a generalized Green’s identity and the continuity of some of the maps introduced above.

**Lemma 2.3 (Green’s identity for normally hyperbolic differential operators, cf. [17]):**

Let \(P : \mathcal{E}(M,E) \to \mathcal{E}(M,E)\) be normally hyperbolic, and \(\nabla\) be the unique \(P\)-compatible connection in \(E^2\). Then, we have for every \(u \in \mathcal{E}(M,E^*)\) and \(f \in \mathcal{E}(M,E)\) the identity

\[
(u,Pf) - (P^*u,f) = \text{div}_g(W),
\]

where \(W \in \mathcal{E}(M,TM)\) is defined by

\[
g(W,X) = (\nabla_X u,f) - (u,\nabla_X f), \ \forall X \in \mathcal{E}(M,E).
\]

Here, \(\text{div}_g\) denotes the divergence operator associated with the Levi-Civita connection of \(g\).

This lemma and the following corollary are essential to prove uniqueness in theorem 2.2.

**Corollary 2.4 (Fresnel-Kirchhoff integral, cp. [16]):**

Assume that \(P\) is also formally self-adjoint. If \(u \in \mathcal{E}(M,E^*)\) is a solution to \(Pu = 0\), we have:

\[
\forall f \in \mathcal{D}(M,E) : \int_M (u,f)dV_g = -\int_{\Sigma} ((\nabla_n u,G(f)) - (u,\nabla_n G(f)))dA_g.
\]

**Proof:**

We integrate (2.18) with \(f\) replaced by \(G^\pm(f)\) in the domains \(J_{\pm}(\Sigma)\) with the common boundary \(\partial J_{\pm}(\Sigma)\), and apply Gauss’ theorem:

\[
\int_{J_{\pm}(\Sigma)} (u,f) = \mp \int_{\Sigma} ((\nabla_n u,G^\pm(f)) - (u,\nabla_n G^\pm(f)))dA_g.
\]

Adding the two expression gives the result.

Clearly, the formulas for the solution (2.7) and (2.13) mimic (2.20).

**Lemma 2.5:**

The maps \(\iota^*,\nu^* : \mathcal{E}_{ac}(M,E) \to \mathcal{D}(\Sigma,E^\Sigma)\) (see (2.8)) are sequentially continuous.

**Proof:**

For \(f \in \mathcal{E}_{ac}(M,E)\), take a converging sequence \(\{f_j\}_{j=1}^{\infty} \subset \mathcal{E}_{ac}(M,E)\), i.e. there exists a compact subset \(K \subset M\), s.t.

\[
\forall j : \text{supp}(f), \text{supp}(f_j) \subset J(K),
\]

\[
\forall k \in \mathbb{N}_0, K' \subset M \text{ cpt. : } \lim_{j \to \infty} ||f - f_j||_{C^k(K,E)} = 0.
\]

\(\nabla\) induces a connection in \(E^\ast\), which we denote by the same symbol.
where \( \|f\|_{C^k(K', E)} := \max_{n=1, \ldots, k} \sup_{x \in K'} \|\nabla^nf(x)\|_{g(x)} \) for \( f \in \mathcal{E}(M, E) \). We need to show that \( \lim_{j \to \infty} \iota^* f_j = \iota^* f \) and \( \lim_{j \to \infty} \nu^* f_j = \nu^* f \) in \( \mathcal{D}(\Sigma, E_{\Sigma}) \). Because \( M \) is globally hyperbolic and \( \Sigma \) is a Cauchy surface, we know that \( J(K) \cap \Sigma =: K'' \) is compact. Moreover by the definition of the maps in question, we have

\[
\text{supp}(\iota^* f), \text{supp}(\iota^* f_j), \text{supp}(\nu^* f), \text{supp}(\nu^* f_j) \subset K'' ,
\]  

(2.24)

and

\[
\|\iota^* f - \iota^* f_j\|_{C^k(K'', E_\Sigma)} \leq \|f - f_j\|_{C^k(K'', E)}
\]  

(2.25)

\[
\|\nu^* f - \nu^* f_j\|_{C^k(K'', E_\Sigma)} \leq \|f - f_j\|_{C^k(K'', E)}
\]  

(2.26)

for some compact subset \( K'' \subset M \), s.t. \( K'' \subset K''' \), and all \( k \in \mathbb{N}_0 \). This proves the statement.

Proposition 2.6:

The compositions of adjoint maps \( G' \circ (\iota^*)' \circ (\nu^*) : \mathcal{D}'(\Sigma, E_{\Sigma}) \to \mathcal{D}'_{\text{Char} P}(M, E^*) \) are sequentially continuous w.r.t. Hörmander's (pseudo-)topology on \( \mathcal{D}'_{\text{Char} P}(M, E^*) \) (see definition 5.14 & [18, 19]).

Proof:

For \( u \in \mathcal{D}'(\Sigma, E_{\Sigma}) \), take a converging sequence \( \{u_j\}_{j=1}^{\infty} \subset \mathcal{D}'(\Sigma, E_{\Sigma}) \), i.e.

\[
\forall f \in \mathcal{D}(\Sigma, E^*_\Sigma) : \lim_{j \to \infty} (u_j, f) = (u, f).
\]  

(2.27)

To show that \( \lim_{j \to \infty} (G' \circ (\iota^*)'(u_j) = (G' \circ (\iota^*)'(u) \) and \( \lim_{j \to \infty} (G' \circ (\nu^*)'(u_j) = (G' \circ (\nu^*)'(u) \), we use a characterization of convergence in \( \mathcal{D}'(M, E^*) \), \( \Gamma \subset T^*M \setminus \{0\} \) closed and conical, proven in [19].

Given a sequence \( \{u_j\}_{j=1}^{\infty} \subset \mathcal{D}'(M, E^*) \), s.t. \( \lim_{j \to \infty} (u_j, v) = \lambda_v \) exists for all \( v \in \mathcal{E}'_\Lambda(M, E) \), then \( \lim_{j \to \infty} u_j = u \in \mathcal{D}'(M, E^*) \) exists, s.t. \( (u, v) = \lambda_v \) for all \( v \in \mathcal{E}'_\Lambda(M, E) \).

Here, \( \Lambda \subset T^*M \setminus \{0\} \) is the complement of the inversion of \( \Gamma \):

\[
\Lambda := (\Gamma^\circ)^c = \{(x, k) \in T^*M \setminus \{0\} \mid (x, -k) \notin \Gamma\}.
\]  

(2.28)

Next, we observe that we have an extension \( G : \mathcal{E}'_\Lambda(M, E) \to \mathcal{E}_{\text{loc}}(M, E) \) in the sense of theorem 5.20. To achieve this, we use the fact that Schwartz' kernel theorem gives us a distribution \( K_G \in \mathcal{D}'(M \times M, E \boxtimes E^*) \), and check that the composition \( K_G \circ v \) for \( v \in \mathcal{E}'_\Lambda(M, E) \) is well-defined. The wave front set of \( K_G \) is well-known (cf. [10, 26]):

\[
\text{WF}(K_G) = \{(x, k; x', k') \in (\text{Char } P)^2 \mid (x, k) \sim_{H_P} (x', -k')\},
\]  

(2.29)

where \( (x, k) \sim_{H_P} (x', k') \) means that \( (x, k), (x', k') \in T^*M \) lie on the same integral curve of the Hamiltonian vector field \( H_P \) of the principal symbol of \( P \). Because \( P \) is normally hyperbolic, its principal symbol is given by the (inverse) metric \( \sigma_P(x, k) = g^{ij}(x)k_i k_j \). Thus, \( \text{Char } P = C^*M \setminus \{0\} \) is the co-light cone bundle without the zero section, and an integral curve of \( H_P \) joining \( (x, k) \) and \( (x', k') \) is a null geodesic strip in \( T^*M \), which projects to the null geodesic in \( M \) from \( x \) to \( x' \) with co-tangents \( k \in T^*_x M \) and \( k' \in T^*_{x'} M \) (cf. [10]). Clearly, the Hamiltonian flow in \( T^*M \) is in one-to-one correspondence with the null geodesics flow in \( TM \) via the metric \( g \). To apply theorem 5.20, we need to check that

\[
\text{WF}(v) \cap (-\text{WF}(K_G)_{M_2|M_1}) = \emptyset.
\]  

(2.30)

This is trivially satisfied, because \( -\text{WF}(K_G)_{M_2|M_1} = \{(x', k') \in T^*M \setminus \{0\} \mid (x, 0; x', k') \in \text{WF}(K_G)\} = \emptyset \) by 5.20. Theorem 5.20 gives us information on the wave front set of \( G(v) \), as well:

\[
\text{WF}(G(v)) \subset \underbrace{\text{WF}(K_G)_{M_2|M_1}}_{= \emptyset} \cup \underbrace{\text{WF}'(K_G) \circ \text{WF}(v)}_{= \emptyset} = \emptyset,
\]  

(2.31)

by the definition of \( \Lambda \) and 2.29. It follows that \( G(v) \in \mathcal{E}(M, E) \). What remains to be checked, is that \( G(v) \in \)

---

3See appendix 5.1 for the construction of the norms

4At coinciding point \( x = x' \), we have \( k = k' \neq 0 \).
\(E_{sc}(M, E)\). To see this, we notice that \(\text{supp}(v) \subseteq M\) is compact, because \(v \in E_{\lambda}(M, E)\), which implies:

\[
(G(v), f) = -(v, G(f)) = 0
\tag{2.32}
\]

for all \(f \in \mathcal{D}(M, E^\ast)\), s.t. \(\text{supp}(f) \subseteq (\text{supp}(v))^\circ\). Putting everything together, we find:

\[
\lim_{j \to \infty} (\mathcal{G}' \circ (\nu^\ast))'(u_j, v) = \lim_{j \to \infty} -((\nu^\ast)'(u_j), \mathcal{G}(v))_{E, E'} = \lim_{j \to \infty} -(u_j, \nu^\ast \mathcal{G}(v))_{E, E'} = -(u, \nu^\ast \mathcal{G}(v)) = ((\mathcal{G}' \circ (\nu^\ast))'(u, v) \quad \forall v \in E_{\lambda}(M, E).
\tag{2.33}
\]

The argument for \(\mathcal{G}' \circ (\nu^\ast)\) is analogous. 

\footnote{Theorem 2.2}

Now, we are in the position to prove theorem \footnote{Theorem 2.2}. 

\textbf{Proof:}

1. \textit{Existence:}

Given \(u_0, u_1 \in \mathcal{D}'(\Sigma, E^*_{\Sigma})\), we use equation \footnote{Equation 2.15} to define a solution \(u \in \mathcal{D}'_{\text{char}}P(M, E^*)\):

\[
u := (\mathcal{G}' \circ (\nu^\ast))'(u_1) - (\mathcal{G}' \circ (\nu^\ast))'(u_0).
\tag{2.34}
\]

We need to show that this solution satisfies \(\nu^*u = u_0\), \(\nu^*u = u_1\). To this end, we observe that the extended maps

\[
i^*, \nu^* : \mathcal{D}'_{\text{char}}P(M, E^*) \to \mathcal{D}'(\Sigma, E^*_{\Sigma})
\tag{2.35}
\]

are well-defined and (sequentially) continuous by virtue of theorem \footnote{Theorem 5.15} because the co-normal \(N_i\) of \(i : \Sigma \hookrightarrow M\) has empty intersection with the co-light cone bundle \(\text{Char} P = C^*M \setminus \{0\}\):

\[
N_i \cap \text{Char} P = \emptyset.
\tag{2.36}
\]

Since \(\mathcal{D}(\Sigma, E^*_{\Sigma})\) is (sequentially) dense in \(\mathcal{D}'(\Sigma, E^*_{\Sigma})\), we find sequences \(\{u_{0,j}\}, \{u_{1,j}\} \subset \mathcal{D}(M, E^*)\), s.t. \(\lim_{j \to \infty} u_{0,j} = u_0\) and \(\lim_{j \to \infty} u_{1,j} = u_1\). Using the continuity of the maps \footnote{Equation 2.35} and proposition \footnote{Proposition 2.6}, we may write:

\[
i^*u_j = i^* \mathcal{G}' \circ (i^\ast)'(u_{1,j}) - i^* \mathcal{G}' \circ (i^\ast)'(u_{0,j}) = \lim_{j \to \infty} i^* \mathcal{G}' \circ (i^\ast)'(u_{0,j})
\tag{2.37}
\]

where we used the identities \footnote{Equation 2.10} after the next-to-last line. The argument for \(\nu^*u = u_1\) is analogous.

2. \textit{Uniqueness:}

If we want prove uniqueness of the solution \footnote{Equation 2.34} among the proper solutions of \(Pu = 0\), we first need to check that \(u\) is indeed proper, and second, that any other proper solution \(u' \in \mathcal{D}'_{\text{char}}P(M, E^*)\) with \(i^*u' = u_0\), \(\nu^*u = u_1\) is identical to \(u\), i.e. \(u' \equiv u\).

To see that \(u\) is proper, we choose sequences \(\{u_{0,j}\}, \{u_{1,j}\} \subset \mathcal{D}(M, E^*)\), s.t. \(\lim_{j \to \infty} u_{0,j} = u_0\) and \(\lim_{j \to \infty} u_{1,j} = u_1\), as before. Then, we set

\[
\forall j : \quad u_j := (\mathcal{G}' \circ (\nu^\ast))'(u_{1,j}) - (\mathcal{G}' \circ (\nu^\ast))'(u_{0,j}) \in E_{sc}(M, E^*),
\tag{2.38}
\]
which is a sequence of smooth solutions, s.t. \( \lim_{j \to \infty} u_j = u \) in \( \mathcal{D}'_{\text{Char}} P(M, E^*) \), by proposition 2.6 and 2.9.

For the second statement, we observe that another solution \( u' \neq u \) would imply the existence of a non-trivial, proper solution with vanishing initial data, i.e.

\[
0 \neq u'' := u - u' \in \mathcal{D}'_{\text{Char}} P(M, E^*) : P u'' = 0, \quad \nu^* u'' = 0, \quad \nu^* u'' = 0. \tag{2.39}
\]

Thus, to conclude uniqueness, we need to show that the only proper solution with vanishing initial data is \( u'' \equiv 0 \). This can be done by an appeal to corollary 2.4. Assume we are given a proper solution \( u'' \) as in (2.39). Then, we choose a approximating sequence \( \{ u''_j \}_{j=1}^\infty \subset \mathcal{D}(M, E^*) \), \( \lim_{j \to \infty} u''_j = u'' \), \( \forall j : \ P u''_j = 0 \), and compute:

\[
\forall f \in \mathcal{D}(M, E) : \quad (u'', f) = \lim_{j \to \infty} (u''_j, f)
\]

\[
\text{Cor. 2.4} \quad \lim_{j \to \infty} \left( (\nu^* u''_j, \nu^* G(f)) - (\nu^* u''_j, \nu^* G(f)) \right) = 0,
\]

where we use the continuity of \( \nu^*, \nu^* \) in the last line. But this contradicts \( u'' \neq 0 \).

3. Continuous dependence on initial data:

This is precisely the content of proposition 2.6 \( \square \)

Remark 2.7:

Clearly, the statement of theorem 2.2 can be improved, if \( M \) is a linear manifold and \( P \) has constant coefficients, e.g. Minkowski space \( M \) and \( P \) is the d’Alembertian. Namely, every distributional solution \( u \in \mathcal{D}'_{\text{Char}} P(M, E^*) \) is then a proper solution by virtue of the existence of an approximate identity \( \{ \phi_\varepsilon \} \subset \mathcal{D}(M) \), \( \lim_{\varepsilon \to 0} \phi_\varepsilon = \delta_0 \), Hörmander’s density theorem (see [24], p.262-263) and the convolution identities:

\[
\phi_\varepsilon * u \in \mathcal{E}(M, E^*), \quad \lim_{\varepsilon \to 0} \phi_\varepsilon * u = u \text{ in } \mathcal{D}'_{\text{Char}} P(M, E^*),
\]

\[
P(\phi_\varepsilon * u) = \phi_\varepsilon * (Pu) = 0.
\]

Interestingly, (2.4) and (2.11) tell us that \( K_G \) is a proper, distributional (bi-)solution.

2.2 Quasifree states and the microlocal spectrum/Hadamard condition

We are now ready to turn our attention to the quantum theory associated with the classical setup of the previous subsection. From the exact sequence (2.12), the well-posedness of the Cauchy problem for \( P \) with initial data in \( \mathcal{D}(\Sigma, E\Sigma) \) and the identities (2.10), we know, that we have a pair of isomorphic linear, symplectic spaces representing the space of (smooth) solution \( \text{Sol}^\infty_0(P) \) with compactly supported (smooth) initial data

\[
\mathcal{D}(M, E)/\text{im} P|_{\mathcal{D}(M, E)} \cong \mathcal{D}(\Sigma, E\Sigma)^{\Sigma^2}. \tag{2.42}
\]

The symplectic structures are given by (cf. [17])

\[
\sigma^M([f], [f']) := \int_M g_E(G(f), f') dV_g
\]

\[
\text{Cor. 2.4} \quad \int_{\Sigma} (g_E(\nabla_n G(f), G(f')) - g_E(G(f), \nabla_n G(f'))) dA_g =: \sigma^\Sigma((f_0, f_1), (f'_0, f'_1)),
\]

for \( [f], [f'] \in \mathcal{D}(M, E)/\text{im} P|_{\mathcal{D}(M, E)} \) and \( (f_0, f_1), (f'_0, f'_1) \in \mathcal{D}(\Sigma, E\Sigma)^{\Sigma^2} \), which are identified by virtue of the isomorphism (2.42). The expressions are well-defined, because \( \forall f \in \mathcal{D}(M, E) : \ G(Pf) = 0 \text{ and } (\nabla_n G(f))|_{\Sigma} \), \( G(f)|_{\Sigma} \) defines the initial data for the solution \( G(f) \in \text{Sol}^\infty_0(P) \).

This allows us to consider the space \( \text{Sol}^\infty_0(P) \) as a symplectic space, with symplectic structure \( \sigma \), and it is well-known that we can associate a \( (C^*)\)-Weyl algebra \( W_P \) with it\(^8\). This algebra is generated by the Weyl elements

---

\(^8\)See [17] for a detailed exposition with an emphasis on local covariance [6] and functoriality of the construction. The are alternative algebraic structures, as well, e.g. the Resolvent algebra [24][28]. These could be worthwhile to consider, since the Weyl algebra only
\[ W(G(f)) \in \text{Sol}^\infty_\omega(P), \text{ subject to the CCR relations in Weyl form:} \]
\[ W(G(f))W(G(f')) = e^{-\frac{i}{2}\sigma(G(f),G(f'))}W(G(f + f')). \]  
(2.44)

A Hilbert space representation of the quantum system defined by the Weyl algebra \( \mathcal{W}_P \) is obtained by specifying an (algebraic) state \( \omega : \mathcal{W}_P \to \mathbb{C} \) and passing to the GNS representation \((\mathfrak{F}_\omega, \pi_\omega, \Omega_\omega)\) (see [29][30] for a detailed account on the algebraic formulation of quantum theory). An important class of states on \( \mathcal{W}_P \) is given by the (regular) quasi-free states, i.e. states \( \omega \), which are solely determined via their two-point function(al) (cf. [6]):
\[ \omega(W(G(f))) = e^{-\frac{i}{2}\omega_2(f,f)}, \]
(2.45)\[ \omega_2(f,f') := -\frac{\partial^2}{\partial t \partial s|_{t = s = 0}} \omega(W(G(tf))W(G(sf'))). \]

It is important for the following that this definition requires \( \omega_2 : \mathcal{D}(M,E)^{\times 2} \to \mathbb{C} \) to be a distributional (bi-)solution for \( P \), i.e. \( \forall f, f' \in \mathcal{D}(M,E) : \omega_2(Pf,f) = 0 = \omega_2(f,Pf') \). Among the quasi-free states are the physically important Hadamard states, which can be regarded as a replacement for the vacuum state of quantum field theory on Minkowski space, since they can be characterized as having a short-distance singularity structure analogous to that of the Minkowski vacuum (cf. [3], and [6] for important structural properties of the folium of Hadamard states).

In a seminal paper [10], Radzikowski showed that Hadamard states are equivalently characterized by a specific form of the wave front set of their two-point function(al) (cp. [24][20]):
\[ \text{WF}(\omega_2) = \text{WF}(K_G) \cap (C_{\ast} M \times C_{\ast} M) \]
(2.46)\[ = \{(x, k; x', k') \in (\text{Char } P)^{\times 2} \mid (x, k) \sim_{H_P} (x', -k'), k \text{ is future-directed}\}, \]
where \( C_{\ast} M \) are the future-/past-directed, co-light cone bundles of \( M \). Thus, the two-point function(al) of a Hadamard state has a wave front set resembling the spectral condition, i.e. positivity of the energy, of quantum field theory on Minkowski space in a microlocal fashion, which justifies the name microlocal spectrum condition for (2.46). What is even more remarkable, is the fact that the microlocal spectrum condition admits a generalization to allow for locally covariant treatment of interacting quantum field on curved spacetimes in a perturbative setting [4][3][31][41]. A crucial observation in this respect is the fact that a Hadamard state defines a Feynman propagator [10]:
\[ \omega_F := i\omega_2 - K_G, \]
(2.47)\[ \text{admits a very restricted set of dynamics (C\textsuperscript{\ast}-automorphism 1-parameter groups).} \]

\[ ^6 \text{Along the diagonal } \Delta_M \subset M \times M \text{ we have } \text{WF}(\omega_2)_{|\Delta_M} = \{(x, k; x - k) \in (T^\ast M)^{\times 2} \setminus \{0\} \mid k \in C_{\ast_{|x}} M \setminus \{0\}\}. \]
Theorem 2.8 (The microlocal spectrum of Hadamard initial data): Given a quasi-free Hadamard state \( \omega \). If the two-point function(al) \( \omega_2 \in \mathcal{D}'_{\text{Char}}(M \times M, E^* \boxtimes E^*) \) is proper (in the sense of theorem 2.2), the initial data (2.48) satisfies the bound

\[
\bigcup_{i,j=0}^1 \text{WF}(\omega_{2,ij}) = N_\Delta \setminus \{0\},
\]

where \( N_\Delta = \{(\Delta(x),(k,k')) \in T^* \Sigma^x \times \Sigma \mid k = -k'\} \) is the co-normal of the diagonal map \( \Delta : \Sigma \to \Sigma \times \Sigma \).

Before we start the proof, let us outline the rough idea and why (2.51) is plausible from the point of view of canonical quantization on \( \Sigma \), we will follow:

The wave front set of an initial datum \( u_{|\Sigma} \in \mathcal{D}'(\Sigma, E^*_\Sigma) \) for a (proper) solution \( u \in \mathcal{D}'(M, E^*) \) can be estimated by the tools of microlocal analysis from \( \text{WF}(u) \), because, on the one hand, \( u \) arises from \( u_{|\Sigma} \) by composition with the causal propagator \( K_G \) of \( P \) (see 2.14), and on the other hand, \( u_{|\Sigma} \) is the restriction of \( u \). Thus, the knowledge of \( \text{WF}(\omega_2) \) gives us a two-sided estimate on the wave front sets \( \text{WF}(\omega_{2,ij}) \), \( i,j = 0,1 \). Furthermore, since the \( K_G \) propagates singularities along the co-light cone bundle (see theorem 5.22), the initial data for a Hadamard state must contain enough singular directions to satisfy microlocal spectrum condition, which is the reason for (2.51). In view of canonical quantization, where

\[
\Phi(x), (\nabla_n \Phi)(x') \sim \delta^{m-1}(x,x'),
\]

and 2.68 below, this seems adequate.

**Proof:**

We prove the theorem by showing the inclusions \( \bigcup_{i,j=0}^1 \text{WF}(\omega_{2,ij}) \subset N_\Delta \setminus \{0\} \) and \( \bigcup_{i,j=0}^1 \text{WF}(\omega_{2,ij}) \supset N_\Delta \setminus \{0\} \).

1. \( \bigcup_{i,j=0}^1 \text{WF}(\omega_{2,ij}) \subset N_\Delta \setminus \{0\} \):

   Because \( \text{WF}(Pu) \subset \text{WF}(u) \) for any differential operator (see corollary 5.12), we have \( \text{WF}(\nu^*u) \subset \iota^* \text{WF}(u) \) for any \( u \in \mathcal{D}'_{\text{Char}}(M,E^*) \). Thus, if we show \( \text{WF}(\omega_{2,00}) \subset N_\Delta \setminus \{0\} \), the first inclusion follows. Using theorem 5.15, we find:

   \[
   \text{WF}(\omega_{2,00}) \subset (\iota^* \times \iota^*) \text{WF}(\omega_2) \]

   \[
   \text{WF}(\omega_{2,00}) \subset (\iota^* \times \iota^*) \text{WF}(\omega_2) \]

   \[
   = \{(x,k) \in (T^* \Sigma \setminus \{0\}) \times \Sigma) \mid (x,k) \sim_{H_p} (x',k'), k \in C^*_+|_{\iota(x)}M\}
   \]

   \[
   = \{(x,k) \in (T^* \Sigma \setminus \{0\}) \times \Sigma) \mid (x,k) \sim_{H_p} (x',k'), k \in C^*_+|_{\iota(x)}M\}
   \]

   \[
   = \{(x,k) \in (T^* \Sigma \setminus \{0\}) \times \Sigma) \mid (x,k) \sim_{H_p} (x',k'), k \in C^*_+|_{\iota(x)}M\}
   \]

   The last line follows, because:

   (a) \( (\iota(x),k) \sim_{H_p} (\iota(x'),k') \) requires \( \iota(x), \iota(x') \in \Sigma \) to lie on a common null geodesic or be equal. Thus, the only possibility is \( (x',k') = (x,k) \), since \( \Sigma \) is acausal (cf. 32).

   (b) \( d_{\iota x} : \text{Char} P_{\iota(x)} \cap C^-_{\iota(x)}M \to T_{\iota x} \Sigma \) is an isomorphism of conical sets for every \( x \in \Sigma \).

2. \( \bigcup_{i,j=0}^1 \text{WF}(\omega_{2,ij}) \supset N_\Delta \setminus \{0\} \):

   From equation 2.49, we find:

   \[
   \text{WF}(\omega_2) \subset \text{WF}((G' \circ (\nu^*)') \times (G' \circ (\nu^*)')\omega_{2,00}) \cup \text{WF}((G' \circ (\nu^*)') \times (G' \circ (\nu^*)')\omega_{2,01})
   \]

   \[
   \cup \text{WF}((G' \circ (\nu^*)') \times (G' \circ (\nu^*)')\omega_{2,10}) \cup \text{WF}((G' \circ (\nu^*)') \times (G' \circ (\nu^*)')\omega_{2,00}).
   \]

   Thus, we may derive the second inclusion, if we compute a bound on the wave front set of the individual contributions in 2.49. This can be done with the help of theorem 5.20 because

   \[
   ((G' \circ (\nu^*)') \times (G' \circ (\nu^*)')\omega_{2,00} = \omega_{2,00} \circ (\iota^* \times G) \times (\iota^* \circ G))
   \]

   \[
   = \omega_{2,00} \circ (\iota^* \times \text{id}_M^G) K^\otimes 2 \text{ etc.,}
   \]
2.2 Quasifree states and the microlocal spectrum/Hadamard condition

where the last line is interpreted as composition of distribution

\[ \mathcal{D}'(\Sigma \times \mathbb{R}) \times \mathcal{D}'(\Sigma \times \mathbb{R}) \times M \otimes \mathcal{E}(\Sigma) \otimes \mathcal{E}(\Sigma) \rightarrow \mathcal{D}'(M \otimes \mathbb{R}) \otimes \mathcal{E}(\Sigma) \otimes \mathcal{E}(\Sigma). \]  

(2.56)

Furthermore, it is important that we have (see corollary 5.12 & theorem 5.13):

\[ \text{WF}(\nu^* \otimes \text{id}_M K_G) \subset (\nu^* \otimes \text{id}_M K_G)(\mathcal{E}(\Sigma)) \subset 
\{(x, (d_{(x)}\nu^*) \kappa; x', k') \in (T^* \Sigma \setminus \{0\}) \times \text{Char} P \mid (\nu^* \otimes \text{id}_M K_G) \}
\]

(2.46)

By virtue of proposition 5.17, we can determine the wave front set of \(((\nu^* \otimes \text{id}_M K_G)K_G)^{\otimes 2}\):

\[ \text{WF}(\phi \otimes \nu^* \otimes \phi \otimes \nu^*) \subset \text{WF}(\phi \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^*) \]

(2.58)

Putting (2.57), (2.58) & (2.59) together (see also (5.32)), we find:

\[ \text{WF}(\phi \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^*) \subset \text{WF}(\phi \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^* \otimes \nu^*) \]

(2.60)

because \( \text{WF}(\nu^* \otimes \text{id}_M K_G)K_G \subseteq \{(x, \kappa) \in T^* \Sigma \setminus \{0\} \mid \kappa \in \text{WF}(\nu^* \otimes \text{id}_M K_G), \kappa \in \Sigma \} \). Thus, what remains to be computed, is the composition of wave front sets \( \text{WF}(\omega_{2,00}) \otimes \text{WF}(\nu^* \otimes \text{id}_M K_G)^{\otimes 2} \), which can be done by means of (2.57) (second line):

\[ \text{WF}(\omega_{2,00}) \circ \text{WF}(\nu^* \otimes \text{id}_M K_G)^{\otimes 2} \]

(2.61)

If we combine the rather complicated looking expression in the last line with the requirement (2.61) and microlocal spectrum condition (2.46), we realize that we have to require that the wave front sets \( \text{WF}(\omega_{2,00}) \) etc. contain elements \( (x', -(d_{(x')\nu^*}) \kappa; x', (d_{(x')\nu^*}) \kappa') \in (T^* \Sigma)^{\otimes 2} \setminus \{0\}, \kappa' \in C_{-|x'\nu^*}|x'\nu^*} \)

This is the case, because the relations

\[ \text{WF}(\omega_{2,00}) \circ \text{WF}(\nu^* \otimes \text{id}_M K_G)^{\otimes 2} \]

(2.62)

have to hold simultaneously, which can only be satisfied if \( \nu^*(x') = \nu^*(x'') \), implying \(-k'' = k'' \), since \( \Sigma \) is causal (cf. [12]). But, elements of the form \( (x', -(d_{(x')\nu^*}) \kappa; x', (d_{(x')\nu^*}) \kappa') \in (T^* \Sigma)^{\otimes 2} \setminus \{0\} \) are exactly those of the co-normal set \( N_\Delta \setminus \{0\}, \) and \( d_{(x')\nu^*} \) : \( \text{Char} P_{(x)} \cap C_{\pm |x|} M \rightarrow T^* \Sigma \) is an isomorphism of conical sets for every \( x \in \Sigma \). This implies the second inclusion.

To illustrate (2.51), we consider the ground state of Klein-Gordon field of mass \( m > 0 \) on an ultra-static spacetime, which is known to be Hadamard [7], and includes the important case of the vacuum state in Minkowski space.
Example 2.9:
Take an ultra-static spacetime \((M, g) = (\mathbb{R}, -dt^2) \times (\Sigma, h)\), where \((\Sigma, h)\) is a complete, d-dimensional, Riemannian manifold, e.g. \(\mathbb{R}^3\) with its standard metric. The analysis of Klein-Gordon operator \(\square_g + m^2 = \partial_t^2 - \Delta_h + m^2\) is conveniently phrased in terms of the strictly positive \((m > 0)\), elliptic operator \(D = -\Delta_h + m^2\) (cf. \[43\] for a detailed exposition), which is essentially self-adjoint together with all its natural powers on \(L^2(\Sigma, dV_h)\) with dense domain \(\mathcal{D}(\Sigma)\) \[44\]. In an abuse of notation, we denote the closure of \(D\) by the same letter. Then, the operator \(\sqrt{D}\) is a strictly positive, elliptic, self-adjoint, pseudo-differential operator, and we have the important property \(\[8, 45\]
\[
WF(\sqrt{D}u) = WF(u), \ u \in \mathcal{E}'(\Sigma).
\]
(2.63)
Moreover, \(\sqrt{D}\) admits a suitable (Borel) functional calculus \([8]\). Since \(\mathcal{D}(\Sigma)\) is nuclear, we can find a spectral resolution of the kernel \(K_D \in \mathcal{D}'(\Sigma \times \Sigma)\) of \(D\) as an integral operator in \(L^2(\Sigma, dV_h)\) \[46, 47\]:
\[
K_D = \int_{\sigma(\sqrt{D})} \omega^2 f_\omega \tilde{f}_\omega d\mu(\omega),
\]
(2.64)
where \(f_\omega \in \mathcal{E}(\Sigma), Df_\omega = \omega^2 f_\omega\), and \(\sigma(K) \ni \omega \geq m > 0\). This said, the two-point function(al) of the ground state for the quantum field theory of the Klein-Gordon field can be expressed as an integral operator in \(L^2(\Sigma, dV_h)\), as well:
\[
\omega_{2,\infty}(t, x; t', x') = \int_{\sigma(\sqrt{D})} \frac{e^{-i\omega(t-t')}}{2\omega} f_\omega(x) \tilde{f}_\omega(x') d\mu(\omega),
\]
(2.65)
which allows us to compute the initial data relative to \(\Sigma_0 = \{0\} \times \Sigma:\)
\[
\omega_{2,\infty,00}(x, x') = \int_{\sigma(\sqrt{D})} \frac{1}{2\omega} f_\omega(x) \tilde{f}_\omega(x') d\mu(\omega) = \frac{1}{2} \sqrt{D}^{-1} \frac{\delta(d)(x, x')}{\sqrt{h}(x)},
\]
(2.66)
\[
\omega_{2,\infty,01}(x, x') = -\omega_{2,\infty,10}(x, x') = \int_{\sigma(\sqrt{D})} f_\omega(x) \tilde{f}_\omega(x') d\mu(\omega) = -i \frac{\delta(d)(x, x')}{2\sqrt{h}(x)},
\]
\[
\omega_{2,\infty,11}(x, x') = \int_{\sigma(\sqrt{D})} \omega f_\omega(x) \tilde{f}_\omega(x') d\mu(\omega) = \frac{1}{2} \sqrt{D} \frac{\delta(d)(x, x')}{\sqrt{h}(x)}.
\]
This implies the for wave front sets
\[
WF(\omega_{2,\infty,00}) = WF(\omega_{2,\infty,01}) = WF(\omega_{2,\infty,10}) = WF(\omega_{2,\infty,11}) = WF(\sqrt{h} \delta^{(d)}),
\]
(2.67)
because of \((2.63)\) and the smoothness of \(\sqrt{h}\). Furthermore, we have in a local coordinate system \(U \subset \mathbb{R}^d\)
\[
(\sqrt{h} \delta^{(d)})(f e^{-ik(\cdot)}, g e^{-ik'(\cdot)}) = \int_U f(x) g(x) e^{-i(k+k')x} \sqrt{h}(x) dt^d x, \ f, g \in \mathcal{D}(U),
\]
(2.68)
and thus \(WF(\sqrt{h} \delta^{(d)}) = N_\Delta \setminus \{0\}\). This shows that \((2.51)\) holds and is maximally saturated.

3 Concluding remarks
In the previous section, we have shown that the initial data of a Hadamard state with proper two-point function(al) must satisfy the bound
\[
\bigcup_{i,j=0}^1 WF(\omega_{2,ij}) = N_\Delta \setminus \{0\},
\]
(3.1)
where \(N_\Delta = \{(\Delta(x), (k, k')) \in T^*\Sigma \times \Sigma | k = -k'\}\) is the co-normal of the diagonal map \(\Delta : \Sigma \to \Sigma \times \Sigma\). This bound on the wave front set of the initial data could be regarded as optimal in the following sense: If a the two-point
function(al) \( \omega_2 \) has only a single non-smooth initial datum, e.g. \( \text{WF}(\omega_{2,00}) \neq \emptyset \), the bound (2.51) is strict:

\[
\omega_2 \text{ satisfies the microlocal spectrum condition } \Rightarrow \text{WF}(\omega_{2,00}) = N_\Delta \setminus \{0\}.
\]  

(3.2)

But, as we will see below (3.10) (cp. also (2.49)), the requirement that only one initial datum is non-smooth is unstable w.r.t. the dynamics. Furthermore, the theorem tells us that the wave front sets of the initial data are already restricted in terms of the geometry of the Cauchy surface \( \Sigma \), only. There is no reference to the metric (or causal) structure of \( M \), besides the fact that \( \Sigma \) is Cauchy. Although, it is true that assigning initial data to a solution depends on the spacetime metric via the maps (2.8), choosing initial data does not depend on this structure. \( \xi \) is background independent in this sense. Thus, we have a condition that is applicable to settings, where no spacetime metric is available, e.g. loop quantum gravity.

Interestingly, the proof of theorem 2.8 shows that the form of the (primed!) wave front set \( \text{WF}'(\omega_2) \) required by the microlocal spectrum condition represents a minimal, conical, \( H_P \)-invariant (cf. theorem 3.22) subset of \( (C^* M \setminus \{0\})^{\times 2} \), s.t. the pullback of the restriction of \( \text{WF}(\omega_2) \) (unprimed!) to the diagonal \( \Delta_M \) in \( M \times M \) gives the full, non-zero co-normal of the diagonal \( \Delta_\Sigma \) in \( \Sigma \times \Sigma \), i.e. \( (i \times i)^* \text{WF}(\omega_2)|_{\Delta_M} = N_\Delta \setminus \{0\} \). The minimality of \( \text{WF}'(\omega_2) \) follows from the fact that a subset \( V \subset (T^* M)^{\times 2} \setminus \{0\} \) with these properties must contain \( d\delta_{\Sigma}(C^*_\Sigma M \setminus \{0\}) \), \( x \in M \), when restricted to the diagonal \( \Delta_M \).

On the other hand, the proof also shows that initial data subject to (3.1) does not uniquely correspond to a (bi-)solution with a wave front set satisfying the microlocal spectrum condition. For example, the initial data of the recently proposed S-J vacuum \([48]\) for the Klein-Gordon field of mass \( m > 0 \) on an ultra-static slab spacetime \((M, g) = (I_x, -dt^2) \times (\Sigma, h)\), satisfies, and even saturates, this bound, as well, but does not define a Hadamard state in general \([49, 50] \). Here, \( I_x = (-\tau, \tau), \tau > 0 \) and \((\Sigma, h)\) is a compact, d-dimensional, Riemannian manifold. To see that the initial data of the S-J vacuum respects (3.2), we argue in same way as in example 2.8.

Since \( \Sigma \) is assumed to be compact, the spectral measure \( \mu \) in (2.64) is supported in a countable set of points \( \{\omega_j\}_{j \in J} \), and the two-point function(al) of the S-J vacuum is given by (cf. [49]):

\[
\omega_{2,SJ}(t, x; t', x') = \sum_{j \in J} \frac{1}{2\omega_j(1 - \delta_j)}(e^{-i\omega_j t} + i\delta_j \sin(\omega_j t))(e^{i\omega_j t'} - i\delta_j \sin(\omega_j t'))f_{\omega_j}(x)f_{\omega_j}(x'),
\]

(3.3)

where \( 1 - \delta_j = \sqrt{\frac{1 - \sin(2\omega_j t)}{1 + \sin(2\omega_j t)}}, j \in J \). It is important for the following that \( 1 - \delta_j \) is strictly bounded away from zero and bounded above as a function of \( \omega_j \), because \( \omega_j \geq m > 0 \). The initial data relative to \( \Sigma_0 = \{0\} \times \Sigma \) takes form:

\[
\omega_{2,SJ,00}(x, x') = \sum_{j \in J} \frac{1}{2\omega_j(1 - \delta_j)}f_{\omega_j}(x)f_{\omega_j}(x') = \frac{1}{2}(\sqrt{D}(1 - \delta)(\sqrt{D}))^{-1}\delta^{(d)}(x, x'),
\]

(3.4)

\[
\omega_{2,SJ,01}(x, x') = -\omega_{2,SJ,10} = -\frac{i}{2} \sum_{j \in J} f_{\omega_j}(x)f_{\omega_j}(x') = -\frac{\delta^{(d)}(x, x')}{\sqrt{h(x)}},
\]

\[
\omega_{2,SJ,11}(x, x') = \frac{\omega_j(1 - \delta_j)}{2} = \frac{1}{2}(\sqrt{D}(1 - \delta)(\sqrt{D}))\delta^{(d)}(x, x') \sqrt{h(x)}.
\]

Here, we defined the elliptic, self-adjoint, pseudo-differential operator \( (1 - \delta)(\sqrt{D}) \) by the functional calculus of \( \sqrt{D} \) (see example 2.10). By a similar argument as above, we have:

\[
\text{WF}(\omega_{2,SJ,00}) = \text{WF}(\omega_{2,SJ,01}) = \text{WF}(\omega_{2,SJ,10}) = \text{WF}(\omega_{2,SJ,11}) = N_\Delta \setminus \{0\}.
\]

(3.5)

Summarizing, we expect that (3.1) does not capture the full dynamical content of the microlocal spectrum condition.

Let us phrase this in more physical terms: A Hadamard two-point function(al) \( \omega_2 \) has only singularities with positive/negative frequencies w.r.t. to its first/second argument, while the causal propagator kernel \( K_\Sigma \) has singularities with positive and negative frequencies equally contributing to both arguments. Nevertheless, the restriction of both distributions and their future normal derivatives to a Cauchy surface \( \Sigma \) gives rise to (3.1). But, the propagation of singularities is a problem of the dynamical law governing the identification of initial data and actual solutions.

---

7. \( \text{WF}(\omega_2) \) satisfies these conditions, too, which would correspond to choosing the opposite time orientation on \( M \).

8. We denote the diagonal map \( M \to M \times M \) also by \( \Delta \).
(see [242]), and the microlocal spectrum condition constrains the relevant state space subject to this evolution in a dynamical manner, which is only to a partial extent covered by [31]. In a sense, we may view this as an instance of Haag’s theorem, which tells us that kinematical and dynamical aspects are tightly entangled in quantum field theory. This point is further elucidated by the fact that Hadamard states are suitable to define locally covariant, renormalized Wick products, time-ordered product and a stress-energy tensor, which are related to (perturbative) dynamical questions.

Therefore, let us have a short look at the dynamical law defined by $P$ in terms of initial data. Given two Cauchy surfaces $\Sigma_1, \Sigma_2 \subseteq M$, the causal propagator $G$ can be used to define a canonical transformation (see (2.43))

\[
\mathcal{D}(\Sigma_1, E_{\Sigma_1})^{0,2} \xrightarrow{\alpha_{\Sigma_2,\Sigma_1}} \mathcal{D}(\Sigma_2, E_{\Sigma_2})^{0,2}
\]

\[
(f_0^{\Sigma_1}, f_1^{\Sigma_1}) \xrightarrow{\alpha_{\Sigma_2,\Sigma_1}} (f_0^{\Sigma_1}, f_1^{\Sigma_1}) := (f_0^{\Sigma_2}, f_1^{\Sigma_2})
\]

with

\[
f_0^{\Sigma_2} = \nu_{\Sigma_2}^* \left( (G' \circ (\nu_{\Sigma_1}^*)') (f_0^{\Sigma_1}) - (G' \circ (\nu_{\Sigma_1}^*)')(f_0^{\Sigma_1}) \right),
\]

\[
f_1^{\Sigma_2} = \nu_{\Sigma_2}^* \left( (G' \circ (\nu_{\Sigma_1}^*)') (f_1^{\Sigma_1}) - (G' \circ (\nu_{\Sigma_1}^*)')(f_1^{\Sigma_1}) \right).
\]

This induces a $*$-isomorphism, also denoted by $\alpha_{\Sigma_2,\Sigma_1}^G$, of the corresponding Weyl algebras by

\[
\mathcal{W}_{\Sigma_1} \xrightarrow{\alpha_{\Sigma_2,\Sigma_1}^G} \mathcal{W}_{\Sigma_2}
\]

\[
W(f_0^{\Sigma_2}, f_1^{\Sigma_2}) \xrightarrow{\alpha_{\Sigma_2,\Sigma_1}^G} W(f_0^{\Sigma_2}, f_1^{\Sigma_2}):= W(f_0^{\Sigma_1}, f_1^{\Sigma_1})
\]

which can be pulled back to the their state spaces $(\alpha_{\Sigma_2,\Sigma_1}^G)^*: \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_2}$. From and (2.49) and (2.50), we infer that

\[
((\alpha_{\Sigma_2,\Sigma_1}^G)^* \omega_{\Sigma_2})_{2,0}(f_1^{\Sigma_1}, f_1^{\Sigma_1}) = \omega_{\Sigma_2}^{2,0,0}(\nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1}), \nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1})))
\]

\[
- \omega_{\Sigma_2}^{2,01}(\nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1}), \nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1})))
\]

\[
- \omega_{\Sigma_2}^{2,10}(\nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1}), \nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1})))
\]

\[
+ \omega_{\Sigma_2}^{2,00}(\nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1}), \nu_{\Sigma_2}^* (G' \circ (\nu_{\Sigma_1}^*)' (f_1^{\Sigma_1})))
\]

etc.

Thus, $*$-automorphisms of this form associated with two Cauchy surface are another way to state the correspondence between initial data and solutions. Now, we may say that Hadamard condition gives additional constraints on the initial data for two-point function(s), such that these data fit together via the dynamical law (3.10) to yield the positive/negative frequencies for the singularities.

In [39], we find explicit prescriptions in terms of pseudo-differential calculus how to construct and characterize Hadamard states on globally hyperbolic spacetimes, but these methods rely on the metric structure of the given spacetime, as we would expect, and therefore do not directly transfer to settings without such a structure, e.g. loop quantum gravity. More precisely, these construction use factorizations of the differential operator $P$ in terms of pseudo-differential operators to construct an explicit parametrization of Hadamard states. Interestingly, the construction in [39] works with a characterization of Hadamard states in terms of optimal data as above, which is obtained from generic data by pullback qith a pseudo-differential operator (see theorem 7.1 of [39]). We observe, that the bound (3.1) is compatible with the conditions for the construction of a Hadamard state given in the said theorem.

In loop quantum gravity, such methods could only be applied in a certain semi-classical regime, where one reconstructs a spacetime metric $g$, or at least a spatial metric $g$ on $\Sigma$, from the geometric operators of the quantum theory.

To further elaborate on this point, let us consider a quantum algebra $\mathfrak{A}_0$ of initial data on a Cauchy surface $\Sigma$ for a matter field on $M$, which is classically defined by a normally hyperbolic operator $P$ (or a quasi-linear version to include interactions), e.g. a Klein-Gordon field, a (gauge fixed) Maxwell-Yang-Mills field or a Higgs field. A state $\omega_\Phi: \mathfrak{A}_0 \to C$ of the quantum field may or may not depend on the spacetime metric $g$ or its restriction $q$ to $\Sigma$, e.g. a Hadamard state $\omega_H$ in the first case, or a background independent state based on a cylindrical measure for functions of point-holonomies $\exp(i\Phi)$ in the second case (cf. [51, 62]). The second possibility is what we expect to happen in loop quantum gravity, or any theory, where a (classical) spacetime metric is not directly available in the quantum
theory. In such cases, an application of the microlocal spectrum condition to quantum states of matter will require the restriction of the theory to some sort of semi-classical sector, which provides us with an effective spacetime metric $g_{\text{eff}}$. Assuming this, we can ask whether the quantum state $\omega_\Phi$ is, at least approximately, Hadamard w.r.t. $g_{\text{eff}}$. Clearly, we can only expect the microlocal spectrum condition to be satisfied in an approximate sense, if we use irregular states for the quantum matter field as proposed in $[51]$, because the microlocal spectrum condition requires the existence of the two-point function(al) of the state $\omega_\Phi$. A detailed discussion of the latter issue for the Maxwell field, quantized by loop quantum gravity methods, will be presented elsewhere.

A proposal by the authors for the construction of a semi-classical sector within (canonical) loop quantum gravity, which roughly follows ideas presented in $[53-55]$, will be put forward soon $[56]$. Methods that achieve this in symmetry reduced models, i.e. loop quantum cosmology, and make contact with the theory of adiabatic vacua, have already been established and been applied to cosmological perturbation theory $[14,15]$. Alternatively, one could try to adapt the factorization techniques to deparametrizing models and their quantum Hamiltonians (see $[11]$) to find analogs of the positive/negative frequency condition.

Finally, let us point out that the microlocal spectrum condition is tailored to (linear) quantum fields defined by normally hyperbolic operators, because the characteristic set of such operators is the co-light cone bundle of $(M,g)$. This means, there is a countable exhaustion $\{K_n\}_{n=1}^\infty$ of $M$ by compact sets $K_n \subset M$. A detailed discussion of the latter issue for the Maxwell field, quantized by loop quantum gravity methods, will be presented elsewhere.

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5 Appendix

This appendix is intended to provide some mathematical background material and key results from microlocal analysis (cf. $[24]$, see also $[8,19,58]$). The definitions for distributions on manifolds follow those in $[17]$ (see also $[21,59,60]$).

5.1 Distributions on manifolds

Let $M$ be a finite dimensional, Hausdorff, second-countable, $\sigma$-compact $C^\infty$-manifold. Given a vector bundle $E \xrightarrow{\pi} M$, we denote by $\mathcal{D}(M,E) := \Gamma_c^\infty(M,E)$ the space of smooth, compactly supported sections. This space can be made into a nuclear, strict LF-space (see the explanation following (5.2)) by the following semi-norms:

Fix an arbitrary Riemannian metric $g$ on $M$ and an arbitrary fibre metric $g_E$ (hermitian in the complex case) on $E$. Additionally, choose arbitrary connections in $T^*M$ and $E$, such that we have induced connections $\nabla : \Gamma^\infty(M, T^*M \otimes E) \to \Gamma^\infty(T^*M \otimes^{k+1} E)$, $\forall k \in \mathbb{N}_0$. The metrics $g, g_E$ induce norms $\| \cdot \|_{g,g_E} : T^*M \otimes E \to \mathbb{R}_\geq 0$, $\forall k \in \mathbb{N}_0$, and we define for $f \in \mathcal{D}(M,E)$:

$$\|f\|_{C^n(K,E)} := \max_{j=1,\ldots,n} \sup_{x \in K} \|\nabla^j f(x)\|_{g,g_E}, \ n \in \mathbb{N}_0, K \subset M \text{ compact.}$$

(5.1)

Clearly, different choices of metrics and connections lead to equivalent semi-norms, because $K$ is compact. Now, we introduce the spaces:

$$\mathcal{D}_K(M,E) := \{f \in \mathcal{D}(M,E) \mid \text{supp}(f) \subset K\}, \ K \subset M \text{ compact},$$

(5.2)

which we turn into Frechét spaces with the families of semi-norms $\{\| \cdot \|_{C^n(K,E)}\}_{n \in \mathbb{N}}$. The nuclear, strict LF-topology on $\mathcal{D}(M,E)$ is defined as the topology generated by all semi-norms $p : \mathcal{D}(M,E) \to \mathbb{R}_\geq 0$, s.t. all the restrictions $p|\mathcal{D}_K(M,E), \ K \subset M \text{ compact,}$ are continuous (in $\mathcal{D}_K(M,E)$). This topology has the important property

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9This means, there is a countable exhaustion $\{K_n\}_{n=1}^\infty$ of $M$ by compact sets $K_n \subset M$.
10All connections are denoted by the same symbol $\nabla$. 
that it turns $\mathcal{D}(M, E)$ into a barreled space (in which the Banach-Steinhaus theorem or principle of uniform boundedness holds, cf. [61]), and entails the following notions of convergence in $\mathcal{D}(M, E)$:

**Proposition 5.1:**
A sequence $\{f_j\}_{j=1}^{\infty} \subset \mathcal{D}(M, E)$ converges to $f \in \mathcal{D}(M, E)$, if and only if

1. $\exists K \subseteq M$ compact : $\forall j : \text{supp}(f_j), \text{supp}(f) \subseteq K$,
2. $\forall n \in \mathbb{N}_0 : \lim_{j \to \infty} \|f_j - f\|_{C^n(K, E)} = 0$.

$\mathcal{D}(M, E)$ with its nuclear, strict LF-topology is called the space of test section in $E$ on $M$. Distributions in $E^\star[11]$ on $M$ with values in a real or complex, finite dimensional vector space $V$ can be defined as sequentially continuous maps $\mathcal{D}(M, E) \to V$, where we fix some arbitrary norm $\| \cdot \|_V$ on $V$.

**Definition 5.2:**
We denote the space of sequentially continuous maps $\mathcal{D}(M, E) \to V$ endowed with the weak*-topology is by $\mathcal{D}'(M, E^\star, V)$, and call it the space of distributions in $E^\star$ on $M$ with values in $V$. If $V = \mathbb{R}, \mathbb{C}$, we abbreviate the notation by $\mathcal{D}'(M, E^\star)$.

Equivalently, we can characterize distributions in the following way.

**Proposition 5.3:**
For a map $u : \mathcal{D}(M, E) \to V$ the following conditions are equivalent:

1. $u \in \mathcal{D}'(M, E^\star, V)$,
2. $\forall K \subseteq M$ compact $\exists k \in \mathbb{N}_0, \infty > C > 0 : \forall f \in \mathcal{D}(M, E) : \|u(f)\|_V \leq ||f||_{C^\infty(K, E)}$.

If $M$ is orientable, we may choose a (smooth) volume form $dV$ on $M$ [12]. This gives rise to a continuous embedding:

$$\mathcal{D}(M, E^\star) \subset \mathcal{D}'(M, E^\star)$$  \hspace{1cm} (5.3)

$$f \mapsto (f' \mapsto \int_M (f, f')dV =: u_{f, dV}(f'))$$

where $f' \in \mathcal{D}(M, E)$. Since two volume forms $dV, dV'$ on $M$ differ by a nowhere vanishing function $f_{dV, dV'} \in C^\infty(M)$, any two embeddings of this kind are equivalent. This motivates the definition of derivatives of distributions.

**Definition 5.4:**
A linear differential operator $P : \Gamma^\infty(M, E) \to \Gamma^\infty(M, E)$ uniquely extends to a continuous, linear operator $P' : \mathcal{D}'(M, E^\star) \to \mathcal{D}'(M, E^\star)$ by

$$\forall u \in \mathcal{D}'(M, E^\star) : (P'u)(f) := u(Pf), f \in \mathcal{D}(M, E).$$  \hspace{1cm} (5.4)

Equation (5.3) is compatible with the definition of the formal adjoint $P^\star : \Gamma^\infty(M, E^\star) \to \Gamma^\infty(M, E^\star)$ of $P$ relative to $dV$, because of the identity:

$$\int_M (P^*f, f')dV = \int_M (f, Pf')dV, f \in \mathcal{D}(M, E^\star), f' \in \mathcal{D}(M, E).$$  \hspace{1cm} (5.5)

Next, we define the support of a distribution, as the generalization of the support of a function resp. section.

**Definition 5.5:**
The support $\text{supp}(u)$ of a distribution $u \in \mathcal{D}'(M, E^\star, V)$ is the complement of the set

$$\{x \in M \mid \exists U \subseteq M \text{ open}, x \in U : u(f) = 0 \forall f \in \mathcal{D}(M, E), \text{supp}(f) \subseteq U\}.$$  \hspace{1cm} (5.6)

Clearly, $\text{supp}(u)$ is closed in $M$.

---

[11] $E^\star$ is the fibrewise dual of $E$.

[12] More generally, we can use a nowhere vanishing density on $M$. 
The distributions with compact support $D_0'(M, E^*)$ can be considered as the (distributional) dual of the smooth section in $E$, $\Gamma^\infty(M, E)$, because of the identity:

$$u(f) = u(\varphi f),$$

(5.7)

where $\varphi \in \mathcal{D}(M)$ is a test function with $\varphi \equiv 1$ on a neighborhood of $\text{supp}(u)$. We can turn the smooth sections $\Gamma^\infty(M, E)$ into a nuclear Frechét space $\mathcal{E}(M, E)$ by the semi-norms $\|f\|_{C^m(K, E)}$, s.t. its weak*-topological, $V$-valued dual is the space of distributions with compact support $\mathcal{E}'(M, E^*, V) = D_0'(M, E^*, V)$. The spaces $\mathcal{D}_K(M, E)$ are closed subspaces of $\mathcal{E}(M, E)$. The notion of convergence in $\mathcal{E}(M, E)$ is given by:

**Proposition 5.6:**

A sequence $\{f_j\}_{j=1}^\infty \subset \mathcal{E}(M, E)$ converges to $f \in \mathcal{E}(M, E)$, if and only if

$$\forall K \subset M \text{ compact, } n \in \mathbb{N}_0 : \lim_{j \to \infty} \|f_j - f\|_{C^n(K, E)} = 0.$$  

(5.8)

There is a characterization of the elements in $\mathcal{E}'(M, E^*, V)$ similar to proposition 5.3 as well.

**Proposition 5.7:**

For a map $u : \mathcal{E}(M, E) \to V$ the following conditions are equivalent:

1. $u \in \mathcal{E}'(M, E^*, V)$,
2. $\exists K \subset M$ compact, $k \in \mathbb{N}_0, \infty > C > 0 : \forall f \in \mathcal{D}(M, E) : \|u(f)\|_V \leq \|f\|_{C^0(K, E)}.$

### 5.2 The wavefront set - Tools from microlocal analysis

A main advantage in the theory of distributions on $\mathbb{R}^n$ is the applicability of the Fourier transform to investigate smoothness properties. This can be, at least partly, cast into a local notion generalizable to ($C^\infty$-)manifolds, namely the so-called wave front set

$$\text{WF}(u) \subset T^*M \setminus \{0\} , u \in \mathcal{D}'(M)$$

(5.9)

which will be as indicated a subset of the cotangent bundle of $M$. This set captures information on the (co-)directions along which the singularities of $u$ “propagate”, and e.g. allows for a refined analysis of the operations possible with distributions.

To define the wave front set explicitly we need the following “localization” of the decay properties of distributions on $\mathbb{R}^n$:

**Definition 5.8:**

For $u \in \mathcal{D}'(\mathbb{R}^n)$ we call $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ a regular direction of $u$ at $x$, if there exists $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi(x) \neq 0$ and an open conic neighborhood $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ of $k$, s.t.

$$\forall N \in \mathbb{N} : \sup_{k' \in \Gamma} (1 + |k'|^N |\hat{\phi}(k')| \leq C_N < \infty$$

(5.10)

Recall that a set $\Gamma$ is called conic if $k \in \Gamma \Leftrightarrow rk \in \Gamma$, $r \in \mathbb{R}_{>0}$.

Let $\Sigma_x(u)$ denote the complement of the regular directions at $x$.

We observe that this definition is local in the sense that $\text{supp} \phi$ can arbitrarily concentrated around $x$, i.e. $\forall \phi \in \mathcal{D}(\mathbb{R}^n) : \phi(x) \neq 0 : \Sigma_x(\phi u) = \Sigma_x(u)$.

**Definition 5.9:**

The wave front set of $u \in \mathcal{D}'(\mathbb{R}^n)$ is given by the set

$$\text{WF}(u) = \{(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) | k \in \Sigma_x\}$$

(5.11)

Clearly, $\text{WF}(u)$ is conic in the sense that it is invariant under multiplication of the second component by positive scalars, i.e. $(x, k) \in \text{WF}(u) \Leftrightarrow (x, rk) \in \text{WF}(u), r > 0$.

An immediate consequence of the definition is that the wave front set naturally generalizes the notion of singular support of a distribution.
Corollary 5.10:
The projection of $\text{WF}(u)$ onto the first component is $\text{sing supp } u$.

Another observation following from the interplay of the Fourier transform and the complex conjugation is:

Corollary 5.11:
For $u \in \mathcal{D}'(\mathbb{R}^n)$ one has

$$\text{WF}(\bar{u}) = \{(x, k) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \mid (x, -k) \in \text{WF}(u)\} = -\text{WF}(u),$$

(5.12)

where $\bar{u}$ denotes the complex conjugate.

Moreover, in analogy with the support of a distribution, we have the following local behavior of the wave front set.

Corollary 5.12:
For any linear ($C^\infty$-)differential operator $P$ the wave front set has the property

$$\text{WF}(Pu) \subset \text{WF}(u)$$

(5.13)

To realize the wave front set as part of the cotangent bundle of a manifold one needs its transformation behavior under ($C^\infty$-)maps $\Phi : U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^m$ between open sets.

Definition 5.13:
The co-normal of $\Phi$ is the set

$$N_\Phi = \{(\Phi(x), \eta) \in V \times \mathbb{R}^m \mid (d\Phi_x)^* \eta = 0\}.$$ 

(5.14)

Obviously we have $N_\Phi = \{0\}$ if $\Phi$ is a submersion, i.e. $d\Phi$ is everywhere onto.

As the main obstacle in defining the composition of distributions with ($C^\infty$-)maps is due to the presence of singularities one is led to consider spaces of distributions with certain restrictions on their wave front set. This paves the way to extending operations (e.g. multiplication) from ($C^\infty$-)functions to distributions.

Definition 5.14 (Hörmander’s pseudo-topology):
For an open subset $U \subset \mathbb{R}^n$ and a closed cone $\Gamma \subset U \times (\mathbb{R}^n \setminus \{0\})$ consider the set

$$\mathcal{D}'_\Gamma(U) = \{u \in \mathcal{D}'(U) \mid \text{WF}(u) \subset \Gamma\}.$$ 

(5.15)

A sequence $\{u_i\} \subset \mathcal{D}'_\Gamma(U)$ is said to converge to $u \in \mathcal{D}'_\Gamma(U)$ ($u_i \rightarrow u$ within $\mathcal{D}'_\Gamma(U)$) if

(i) $u_i \rightarrow u$ in $\mathcal{D}'(U)$

(ii) $\forall N \in \mathbb{N} : \forall \phi \in \mathcal{D}(U) : \forall V \subset \mathbb{R}^n$ closed cone : $\Gamma \cap (\text{supp } \phi \times V) = \emptyset$

$$\sup_{\Gamma} |k|^N |\hat{\phi}u_i(k) - \hat{\phi}u(k)| \rightarrow 0, \ i \rightarrow \infty.$$ 

(5.16)

There are several topologies on $\mathcal{D}'_\Gamma(U)$ compatible with this notion of convergence (cf. [19]). Now we are in the position to state the main theorem:

Theorem 5.15 (Theorem 8.2.4, [24]):
Let $\Phi : U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^m$ be as above. There is one and only one way to define the pullback $\Phi^* u$ for $u \in \mathcal{D}'(\mathbb{R}^n)$ with

$$N_\Phi \cap \text{WF}(u) = \emptyset$$

(5.17)

such that $\Phi^* u = u \circ \Phi$ for $u \in C^\infty$, and for any closed conic subset $\Gamma \subset V \times (\mathbb{R}^m \setminus \{0\})$ with $\Gamma \cap N_\Phi = \emptyset$

$$\Phi^* : \mathcal{D}'_\Gamma(V) \rightarrow \mathcal{D}'_{\Phi^* \Gamma}(U)$$

(5.18)

$\Phi^* \Gamma = \{(x, (d\Phi_x)^* \eta) \mid (\Phi(x), \eta) \in \Gamma\}$ is continuous.

Moreover the wave front set satisfies

$$\text{WF}(\Phi^* u) \subset \Phi^* \text{WF}(u)$$

(5.19)
Interestingly this makes precise the intuition that the singularities of a distribution (as a geometrical object on a manifold) should “propagate” along tangential direction and not along the co-normal.

Consider now a \((C^\infty)\)-manifold \(M\) and a distribution \(u \in \mathcal{D}'(M)\). Utilizing theorem \ref{thm:5.15}, we define the wave front set \(WF(u) \subset T^* M \setminus \{0\}\) by

\[
(x, k) \in WF(u) \Leftrightarrow (\kappa(x), (d\kappa^{-1})^* k) \in WF((\kappa^{-1})^* u)
\]

for any chart \(\kappa : U \subset M \rightarrow V \subset \mathbb{R}^n\). In case of a \((C^\infty)\)-vector bundle \(E\) over \(M\) and \(u \in \mathcal{D}'(M, E)\) one defines \(WF(u) := \bigcup_{i=1,...,e} WF(u_i)\) w.r.t. a local trivializations s.t. \((u = (u_1, ..., u_e))\). This is independent of the trivialization since the passage between two trivialization is given by the multiplication of \((u_1, ..., u_e)\) by an invertible \((C^\infty)\)-matrix.

Another important implication of theorem \ref{thm:5.15} is the possibility to define restrictions of distributions to submanifolds in certain cases:

**Corollary 5.16:**

Let \(\iota : S \rightarrow M\) be an (embedded) submanifold. For every \(u \in \mathcal{D}'(M)\) with \(WF(u) \cap N = \emptyset\) the restriction

\[
u_i | S = \iota^* u\]

is a well defined distribution in \(S (u_i | S \in \mathcal{D}'(S))\).

Next we take a closer look at the wave front set of the (exterior) tensor product of distributions, which will be important due to the fact, that the product of \((C^\infty)\)-functions can be given as

\[
fg(x) = \Delta^*(f \otimes g)(x),
\]

where \(\Delta : M \rightarrow M \times M\) is the diagonal map.

**Proposition 5.17:**

For \(u \in \mathcal{D}'(M), v \in \mathcal{D}'(M')\) the wave front set of the (exterior) tensor product \(u \otimes v \in \mathcal{D}'(M \times M')\) obeys the restriction

\[
WF(u \otimes v) \subset (WF(u) \times WF(v)) \cup ((\text{supp} u \times \{0\}) \times WF(v)) \cup (WF(u) \times (\text{supp} v \times \{0\})).
\]

**Proof:**

The Fourier transform of \((\phi u) \otimes (\psi v)\) is given by \(\hat{\phi u} \hat{\psi v} (\phi(x) \neq 0, \psi(y) \neq 0)\). According to relation \ref{eq:5.10} we have for the regular directions at \((x, y)\) (w.r.t. to a local coordinate system):

\[
\forall N \in \mathbb{N} : \sup_{(k, k') \in \Omega \subset (T^*_x M \times T^*_y M') \setminus \{0\}} (1 + \sqrt{\frac{|k| + |k'|}})^N |\hat{\phi u(k)}\hat{\psi v(k')}| \leq C_N < \infty.
\]

So we infer that

\[
\Sigma_x (u \otimes v) \subset (\Sigma_x(u) \times \Sigma_y(v)) \cup \{(0 \times \Sigma_y(v)) \cup (\Sigma_x(u) \times \{0\})
\]

which implies the result.

Obviously, the co-normal of the diagonal map \(\Delta\) is given by

\[
N_{\Delta} = \left\{(\Delta(x), (k, l)) \in T^* M^2 | k = -l \right\},
\]

leading together with proposition \ref{prop:5.17} to the extension theorem for multiplication:

**Theorem 5.18 (Theorem 8.2.10. \cite{24}):**

For \(u, v \in \mathcal{D}'(M)\) the product \(uv\) is well-defined if \(WF(u \otimes v) \cap N_{\Delta} = \emptyset\), i.e. there is no \((x, k) \in T^* M\) s.t. \((x, k) \in WF(u)\) and \((x, -k) \in WF(v)\), and given by

\[
uv = \Delta^*(u \otimes v).
\]
Furthermore the wave front $WF(uv)$ satisfies
\[
WF(uv) \subset WF(u) \oplus WF(v) \cup WF(u)|_{supp\,v} \cup WF(v)|_{supp\,u}.
\]

Another class of important theorems concerns the composition of distributions as linear maps. The first is essentially a refined version of theorem 8.2.12. in [24].

**Theorem 5.19:**
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $K \in \mathcal{D}'(U \times V)$. Denote by $\mathcal{K}$ the corresponding map $\mathcal{D}(V) \rightarrow \mathcal{D}'(U)$. Then the wave front set $WF(\mathcal{K})$ for $\psi \in \mathcal{D}(V)$ satisfies
\[
\{(x, k) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, y; k, 0) \in WF(K), \ y \in supp^o \psi\} \subset WF(\mathcal{K}) \quad \subset \tag{5.29}
\]
\[
\{(x, k) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, y; k, 0) \in WF(K), \ y \in supp \psi\}, \tag{5.30}
\]
where $^o$ denotes the interior of a set. Defining
\[
WF(K)_{U|V} = \{(x, k) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, y; k, 0) \in WF(K), \ y \in V\}
\]

one has
\[
WF(K)_{U|supp\,\psi} \subset WF(\mathcal{K}) \subset WF(K)_{U|supp\,\psi}. \tag{5.33}
\]

**Proof:**
For $x_0 \in U$ take $\phi \in \mathcal{D}(U)$ with $\phi(x_0) = 1$ and define
\[
K_1 = (\phi \otimes \psi)K \in \mathcal{D}'(U \times V), \ \psi \in \mathcal{D}(V). \tag{5.34}
\]
To analyze the wave front set we look at $\widehat{\phi(\mathcal{K})}$:
\[
\widehat{\phi(\mathcal{K})}(k) = \phi(\mathcal{K}) \left( e^{-ik \cdot } \right) = K \left( \phi e^{-ik \cdot } \cdot , \psi \right) = \widehat{K_1}(k, 0). \tag{5.35}
\]
If $u \in \mathcal{D}'(\mathbb{R}^n)$ is a compactly supported distribution we denote by $\Sigma(u)$ the complement of the regular directions of its Fourier transform, s.t. $\pi_2(WF(u)) = \Sigma(u)$\footnote{$\pi_i$, $i = 1, 2$ denotes the projection on the respective component.}. Moreover one can show\footnote{see [24] p. 253 et seq.} for $\chi \in \mathcal{D}(\mathbb{R}^n)$
\[
\Sigma(\chi u) \subset \Sigma(u) \text{ and } \Sigma(\chi u) \to \Sigma_x(u) \text{ for } supp \chi \to \{x\}, \ \chi(x) \neq 0. \tag{5.36}
\]
This directly leads to $\bigcup_{x,y} \Sigma \chi_x \chi_y \subset \Sigma(K_1) \text{ and } \Sigma_{x,y} \chi_x \chi_y \subset \Sigma_x \chi_y \subset \Sigma_{x,y} K_1$. So we find:
\[
\pi_2 (WF(K)_{supp\,\phi \otimes \psi}) \subset \pi_2 (WF(K)) \subset \pi_2 (WF(K)_{supp\,\phi \otimes \psi}) \tag{5.37}
\]
\[
\Rightarrow \pi_2 (WF(K)_{supp\,\phi \otimes \psi}) |_{l=0} \subset \Sigma(\phi(\mathcal{K})) \subset \pi_2 (WF(K)_{supp\,\phi \otimes \psi}) |_{l=0}. \tag{5.38}
\]
Letting $supp \phi \to \{x_0\}$ proves the theorem. \qed

Along similar lines one obtains an extension theorem to the latter

**Theorem 5.20 (cf. Theorem 8.2.13. [24]):**
The exists a unique extension of $\mathcal{K}$ to those $u \in \mathcal{E}'(V)$ with $WF(u) \cap (\neg WF(K)V \cup U) = \emptyset$, s.t.
\[
\mathcal{E}'(M) \cap \mathcal{D}'(V) \ni u \rightarrow \mathcal{K} u \in \mathcal{D}'(U) \tag{5.39}
\]
is continuous for all compact sets $M \subset V$ and closed cones $\Gamma$ with $\Gamma \cap (\neg WF(K)V \cup U) = \emptyset$. Define
\[
WF'(K) = \{ (x, y; k, l) \mid (x, y; k, -l) \in WF(K) \}, \quad WF(K) = \{ (x, y; k, l) \mid (x, y; -k, l) \in WF(K) \}. \tag{5.40}
\]
Finally we need a theorem shedding light on the interplay between wave front sets on \( \mathcal{WF}(\mathcal{X}u) \subset \mathcal{WF}'(K) \circ \mathcal{WF}(u) \cup \mathcal{WF}(K)_{\| \text{supp} u \|} \),

\[ \text{(5.41)} \]

where \( \circ \) denotes the composition, i.e. \( \mathcal{WF}'(K) \circ \mathcal{WF}(u) = \{(x, k) \mid (x, y; k, -l) \in \mathcal{WF}(K), \ (y, l) \in \mathcal{WF}(u)\} \).

and a composition theorem for this type of maps.

**Theorem 5.21** (cf. Theorem 8.2.14, [24]):

Let \( U \subset \mathbb{R}^n, V \subset \mathbb{R}^m \) and \( W \subset \mathbb{R}^p \) be open sets and \( K_1 \in \mathcal{D}'(U \times V), K_2 \in \mathcal{D}'(V \times W) \). Furthermore assume the projection

\[ \pi_2 : \text{supp} K_2 \to W \]

(5.42)

to be proper, i.e. preimages of compact sets are compact, and

\[ \mathcal{WF}'(K_1)_{\| U \| \cap \mathcal{WF}(K_2)_{\| V \|} = \emptyset. \]

(5.43)

Then the composition \( \mathcal{X}_1 \circ \mathcal{X}_2 \) is defined and its kernel \( K \) satisfies the following condition on its wave front set

\[ \mathcal{WF}(K) \subset \mathcal{WF}'(K_1) \circ \mathcal{WF}(K_2) \]

(5.44)

\[ \cup \{(x, z; k, 0) \mid (x, y; k, 0) \in \mathcal{WF}(K_1), \ (y, z) \in \text{supp} K_2 \} \]

(5.45)

\[ \cup \{(x, z; 0, m) \mid (x, y) \in \text{supp} K_1, \ (y, z; 0, m) \in \mathcal{WF}(K_2)\} \]

(5.46)

\[ \subset \mathcal{WF}'(K_1) \circ \mathcal{WF}(K_2) \cup \left( \mathcal{WF}(K_1)_{\| \pi_1(\text{supp} K_1) \|} \times \{ \pi_2(\text{supp} K_2) \times \{0\} \} \right) \]

(5.47)

\[ \cup \left( (\pi_1(\text{supp} K_1) \times \{0\}) \times \mathcal{WF}(K_2)_{\| \pi_2(\text{supp} K_1) \|} \right) \]

(5.48)

Finally we need a theorem shedding light on the interplay between wave front sets on \( (C^\infty,\cdot) \)-differential operators (cf. Theorem 8.3.1, [24] and Theorem 26.1.1, [62]).

**Theorem 5.22**:

Let \( P = \sum_{n \leq m} P_n(x, \partial_x) \) be a linear \( (C^\infty,\cdot) \)-differential operator of order \( m \) on a \( (C^\infty,\cdot) \)-manifold \( M \), then

\[ \mathcal{WF}(u) \setminus \mathcal{WF}(Pu) \subset \text{Char} P, \ u \in \mathcal{D}'(M), \]

(5.49)

where \( \text{Char} P = \{(x, k) \in T^*M \setminus \{0\} \mid P_m(x, k) = 0\} \) denotes the characteristic set of \( P \).

If additionally the principal symbol \( P_m \) is real and homogeneous of degree \( m \), \( \mathcal{WF}(u) \setminus \mathcal{WF}(Pu) \) will be invariant under the flow of the Hamiltonian vector field associated with \( P_m \) w.r.t. to the natural symplectic structure on \( T^*M \).

6 References

[1] Klaus Fredenhagen and Katarzyna Rejzner. Perturbative Algebraic Quantum Field Theory. *arXiv preprint arXiv:1208.1428*, 2012.

[2] Romeo Brunetti, Klaus Fredenhagen, and Katarzyna Rejzner. Quantum gravity from the point of view of locally covariant quantum field theory. *arXiv preprint arXiv:1306.1058*, 2013.

[3] Robert M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. Chicago Lectures in Physics. The University of Chicago Press, 1994.

[4] Romeo Brunetti and Klaus Fredenhagen. Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds. *Commun. Math. Phys.*, 208:623 – 661, 2000.

[5] Stefan Hollands and Robert M. Wald. Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime. *Commun. Math. Phys.*, 231:309–345, 2002.

[6] Romeo Brunetti, Klaus Fredenhagen, and Rainer Verch. The Generally Covariant Locality Principle: A New Paradigm for Local Quantum Physics. *Commun. Math. Phys.*, 237:31–68, 2003.

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16 From theorem [5.19] one additionally has \( \mathcal{WF}((1 \otimes u) K)_{U \| \text{supp} u \|} \subset \mathcal{WF}(\mathcal{X}u) \)
[7] Stephen A. Fulling, Francis J. Narcowich, and Robert M. Wald. Singularity structure of the two-point function in quantum field theory in curved spacetime, II. *Annals of Physics*, 136(2):243–272, 1981.

[8] Wolfgang Junker. Hadamard states, Adiabatic Vacua and the Construction of Physical States for Scalar Quantum Fields on Curved Spacetime. *Reviews in Mathematical Physics*, 8(08):1091–1159, 1996.

[9] Christian Gérard and Wrochna Michał. Construction of Hadamard states by pseudo-differential calculus. *arXiv preprint arXiv:1209.2604*, 2012.

[10] Marek J. Radzikowski. Micro-local Approach to the Hadamard Condition in Quantum Field Theory on Curved Space-Time. *Commun. Math. Phys.*, 179:529–553, 1996.

[11] Kristina Giesel and Thomas Thiemann. Scalar material reference systems and loop quantum gravity. *arXiv preprint arXiv:1206.3807*, 2012.

[12] Christian Lüders and John E Roberts. Local Quasiequivalence and Adiabatic Vacuum States. *Communications in mathematical physics*, 134(1):29–63, 1990.

[13] Wolfgang Junker and Elmar Schrohe. Adiabatic Vacuum states on General Spacetime Manifolds: Definition, Construction, and Physical Properties. In *Annales Henri Poincaré*, volume 3, pages 1113–1181. Springer, 2002.

[14] Ivan Agullo, Abhay Ashtekar, and William Nelson. Extension of the quantum theory of cosmological perturbations to the Planck era. *Physical Review D*, 87(4):043507, 2013.

[15] Ivan Agullo, Abhay Ashtekar, and William Nelson. The pre-inflationary dynamics of loop quantum cosmology: Confronting quantum gravity with observations. *Classical and Quantum Gravity*, 30(8):085014, 2013.

[16] John Dimock. Algebras of local observables on a manifold. *Commun. Math. Phys.*, 77:219–228, 1980.

[17] Christian Bär, Nicolas Ginoux, and Frank Pfaffle. *Wave Equations on Lorentzian Manifolds and Quantization*. ESI Lectures in Mathematics and Physics. European Mathematical Society, 2007.

[18] Lars Hörmander. Fourier Integral Operators. I. *Acta mathematica*, 127(1):79–183, 1971.

[19] Yoann Dabrowski and Christian Brouder. Functional properties of Hörmander’s space of distributions having a specified wavefront set. *arXiv preprint arXiv:1308.1061*, 2013.

[20] Andreas Kriegl and Peter W. Michor. *The Convenient Setting for Global Analysis*. American Mathematical Society, 1997.

[21] F.G. Friedlander. *The Wave Equation on a Curved Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1975.

[22] Antonio N. Bernal and Miguel Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Commun. Math. Phys.*, 257:43–50, 2005.

[23] Helga Baum and Ines Kath. Normally hyperbolic operators, the Huygens property and conformal geometry. *Annals of Global Analysis and Geometry*, 14:315–371, 1996. SFB-288-212.

[24] Lars Hörmander. *The Analysis of Linear Partial Differential Operators: Vol. 1.: Distribution Theory and Fourier Analysis*. Springer-Verlag, 1983.

[25] Hans Lindblad. Counterexamples to local existence for semi-linear wave equations. *American Journal of Mathematics*, 118(1):1–16, 1996.

[26] Johannes Jisse Duistermaat and Lars Hörmander. Fourier Integral Operators. II. *Acta mathematica*, 128(1):183–269, 1972.

[27] Detlev Buchholz and Hendrik Grundling. The Resolvent Algebra: A New Approach to Canonical Quantum Systems. *Journal of Functional Analysis*, 254(11):2725–2779, 2008.

[28] Detlev Buchholz. The Resolvent Algebra: Ideals and Dimension. *Journal of Functional Analysis*, 2013.
[29] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras, Symmetry Groups, Decomposition of States*. Texts and Monographs in Physics. Springer Verlag, 2nd edition, 1987.

[30] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States, Models in Quantum Statistical Mechanics*. Texts and Monographs in Physics. Springer Verlag, 2nd edition, 1997.

[31] R. Brunetti, K. Fredenhagen, and M. Köhler. The Microlocal Spectrum Condition and Wick Polynomials of Free Fields on Curved Spacetimes. *Commun. Math. Phys.*, 180:633–652, 1996.

[32] Kai Kratzert. *Singuläritätsstruktur der Zweiunglückfunktion des freien Diracfeldes in einer global hyperbolischen Raumzeit*. PhD thesis, DESY, 1999.

[33] Stefan Hollands. The Hadamard Condition for Dirac Fields and Adiabatic States on Robertson–Walker Spacetimes. *Communications in Mathematical Physics*, 216(3):635–661, 2001.

[34] Claudio D’Antoni and Stefan Hollands. Nuclearity, Local Quasiequivalence and Split Property for Dirac Quantum Fields in Curved Spacetime. *Communications in mathematical physics*, 261(1):133–159, 2006.

[35] Claudio Dappiaggi, Thomas-Paul Hack, and Nicola Pinamonti. The extended algebra of observables for Dirac fields and the trace anomaly of their stress-energy tensor. *Reviews in Mathematical Physics*, 21(10):1241–1312, 2009.

[36] Ko Sanders. The locally covariant Dirac field. *Reviews in Mathematical Physics*, 22(04):381–430, 2010.

[37] Katarzyna Rejzner. Fermionic fields in the functional approach to classical field theory. *Reviews in Mathematical Physics*, 23(09):1009–1033, 2011.

[38] Stefan Hollands. Renormalized quantum Yang–Mills fields in curved spacetime. *Reviews in Mathematical Physics*, 20(09):1033–1172, 2008.

[39] Klaus Fredenhagen and Katarzyna Rejzner. Batalin-vilkovisky formalism in the functional approach to classical field theory. *Communications in Mathematical Physics*, 314(1):93–127, 2012.

[40] Klaus Fredenhagen and Katarzyna Rejzner. Batalin-vilkovisky formalism in perturbative algebraic quantum field theory. *Communications in Mathematical Physics*, 317(3):697–725, 2013.

[41] Klaus Fredenhagen and Katarzyna Rejzner. Local covariance and background independence. In *Quantum Field Theory and Gravity*, pages 15–23. Springer, 2012.

[42] Barrett O’Neill. *Semi-Riemannian Geometry - With Applications to Relativity*. Pure and Applied Mathematics. Academic Press, 1983.

[43] Stephen A. Fulling. *Aspects of Quantum Field Theory in Curved Space-Time*. Number 17 in London Mathematical Society Student Texts. Cambridge University Press, 1989.

[44] Paul R Chernoff. Essential self-adjointness of powers of generators of hyperbolic equations. *Journal of Functional Analysis*, 12(4):401–414, Apr 1973.

[45] Lars Hörmander. *The Analysis of Linear Partial Differential Operators: Vol. 3.: Pseudo-Differential Operators*. Springer-Verlag, 1985.

[46] Israel M. Gel’fand and N. Ya. Vilenkin. *Generalized Function Vol. 4: Applications of Harmonic Analysis*. Academic Press, 1964.

[47] Nelson Dunford and Jacob T. Schwartz. *Linear Operators, Part II: Spectral Theory, Self-Adjoint Operators in Hilbert Space*. Wiley, 1963.

[48] Niayesh Afshordi, Siavash Aslanbeigi, and Rafael D. Sorkin. A Distinguished Vacuum State for a Quantum Field in a Curved Spacetime: Formalism, Features, and Cosmology. *JHEP*, 1208:137, 2012.

[49] Christopher J. Fewster and Rainer Verch. On a Recent Construction of ‘Vacuum-like’ Quantum Field States in Curved Spacetime. *Class.Quant.Grav.*, 29:205017, 2012.
[50] Marcos Brum and Klaus Fredenhagen. ”Vacuum-like” Hadamard states for quantum fields on curved spacetimes. 2013.

[51] Thomas Thiemann. Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories. Classical and Quantum Gravity, 15(6):1487, 1998.

[52] Thomas Thiemann. Quantum Spin Dynamics (QSD): V. Quantum gravity as the natural regulator of the hamiltonian constraint of matter quantum field theories. Classical and Quantum Gravity, 15(5):1281, 1998.

[53] Hanno Sahlmann and Thomas Thiemann. Towards the QFT on curved spacetime limit of QGR. I: A general scheme. Class. Quant. Grav., 23:867–908, 2006.

[54] Hanno Sahlmann and Thomas Thiemann. Towards the QFT on curved spacetime limit of QGR. II: A concrete implementation. Class. Quant. Grav., 23:909–954, 2006.

[55] Kristina Giesel, Johannes Tambornino, and Thomas Thiemann. Born-Oppenheimer decomposition for quantum fields on quantum spacetimes. 2009.

[56] Alexander Stottmeister and Thomas Thiemann. ”to appear”. 2014.

[57] Stefan Hollands and Robert M. Wald. Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime. Commun. Math. Phys., 223:289–326, 2001.

[58] Hanno Sahlmann and Rainer Verch. Microlocal Spectrum Condition and Hadamard Form for Vector-Valued Quantum Fields in Curved Spacetime. Rev.Math.Phys., 13:1203–1246, 2001.

[59] Walter Rudin. Functional Analysis. McGraw-Hill Book Company, 1973.

[60] Dirk Werner. Funktionalanalysis. Springer Verlag, 6ed edition, 1995.

[61] A. P. Robertson and Wendy Robertson. Topological Vector Spaces. Cambridge University Press, 1964.

[62] Lars Hörmander. The Analysis of Linear Partial Differential Operators: Vol.: 4.: Fourier Integral Operators. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1985.