Possible existence of time machines in a five-dimensional spacetime

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Abstract

The idea of constructing a time machine is not new and even received a boost thanks to the realization that a traversable wormhole could be converted to a time machine. This also implied that one could only travel back to the time of the conversion. This paper addresses these issues, as well as the concomitant causality violations, by starting with a physically acceptable model, a spacetime that is anti-de Sitter due to an extra time-like dimension, thereby allowing the existence of closed time-like curves. By assuming that the extra dimension is independent of the radial coordinate, the wormhole retains its basic geometric properties regardless of its location and can therefore serve as a shortcut for any closed time-like curve, which, in turn, can extend indefinitely into the past. The same independence of the radial coordinate suggests that the wormhole could connect a region in the present to a local region that (1) lies in the past and (2) does not contain any closed time-like curves. Previous studies have suggested that if such a region is sufficiently localized, it may be possible to avoid a causality violation.

Keywords: closed time-like curves, time travel, causality violation

1 Introduction

The idea of constructing a time machine is not new. One of the first proposals, the van Stockum time machine [1, 2], starts with an infinitely long rigidly rotating cylinder of dust surrounded by a vacuum. A more recent example is the Gott time machine [3], which employs two spinning cosmic strings to induce closed time-like curves. Both are considered unphysical [4]. Another approach is the use of wormholes [5], elaborated on in this paper. For this approach to work, traversable wormholes would have to exist and be available when and where needed.

Wormholes are handles or tunnels connecting widely separated regions of our Universe or different universes altogether. These are often referred to as intra-universe or inter-universe wormholes, respectively. Since we are primarily interested in the former, we will
assume that an intra-universe wormhole is strictly a shortcut for an otherwise ordinary trip through spacetime.

A wormhole may be described by the static and spherically symmetric line element

\[ ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]  

(1)

using units in which \( c = G = 1 \). Here \( \Phi = \Phi(r) \) is called the redshift function, which must be everywhere finite to avoid the appearance of an event horizon. The function \( b = b(r) \) is called the shape function, since it determines the spatial shape of a wormhole when viewed, for example, in an embedding diagram \[6\]. The spherical surface \( r = r_0 \) is the throat of the wormhole. The shape function must satisfy the following conditions:

\[ b(r_0) = r_0, \quad b(r) < r \quad \text{for} \quad r > r_0, \quad \text{and} \quad b'(r_0) \leq 1, \] called the flare-out condition in Ref. \[6\]. For a Morris-Thorne wormhole, this condition can only be met by violating the null energy condition (NEC), which states that for the energy-momentum tensor \( T_{\alpha\beta} \),

\[ T_{\alpha\beta}k^\alpha k^\beta \geq 0 \quad \text{for all null vectors} \quad k^\alpha. \]  

(2)

Matter that violates the NEC is called “exotic” in Ref. \[6\]. In particular, for the outgoing null vector \((1,1,0,0)\), the violation takes on the form

\[ T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \]  

(3)

Here \( T_t^t = -\rho \) is the energy density, \( T_r^r = p_r \) is the radial pressure, and \( T_\theta^\theta = T_\phi^\phi = p_t \) is the lateral pressure. A final requirement is asymptotic flatness: \( \lim_{r \to \infty} \Phi(r) = 0 \) and \( \lim_{r \to \infty} b(r)/r = 0 \).

A different approach to wormholes can be found in Kuhfittig \[7\], which assumes the existence of an extra spatial dimension leading to the following line element:

\[ ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{2\mu(r,l)} dl^2, \]  

(4)

Symmetry considerations suggested that the extra term should have the same exponential form as the first two terms; \( l \) is the extra coordinate. We also assume that \( e^{2\lambda(r)} = 1 - b(r)/r \), as before. In Ref. \[7\] the throat of the wormhole is threaded with ordinary (nonexotic) matter. According to Ref. \[8\], however, referring back to the Raychaudhuri equation, a violation of the NEC cannot be avoided. It is interesting to note that in Ref. \[7\] this violation can be attributed to the existence of the extra dimension.

We now turn to a completely different issue regarding the extra dimension. If this dimension is indeed spacelike, then the signature is \(-+++-\), as in Eq. \(1\). It is proposed in Ref. \[9\], however, that the signature \(-+++-\) is in principle allowed, thereby yielding two time-like components. In other words, the line element

\[ ds^2 = -e^{2\Phi(r)} dt^2 - e^{2\mu(r,l)} dl^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \]  

(5)

would be consistent with Einstein’s theory. The resulting five-dimensional spacetime is an example of an anti-de Sitter space and is characterized by a negative cosmological constant. (In fact, in the absence of matter and energy, the curvature of a spacelike section is negative, corresponding to hyperbolic non-Euclidean geometry.)

For our purposes, the most important property of an anti-de Sitter space is another well-known property, the existence of closed time-like curves. We will pursue this topic in Sec. \[3\].
2 Preliminary calculations

We start this section with the Einstein field equations in the orthonormal frame:

\[ G_{\hat{\alpha}\hat{\beta}} = R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2} R g_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}; \]  

thus

\[ g_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}. \]  

(7)

Using indices 0 – 4, we can write

\[ T_{00} = -T^t_t = \rho \quad \text{and} \quad T_{11} = T^r_r = p_r. \]

Since we wish to follow Ref. [7], we need to rewrite line element (5) in the following convenient form:

\[ ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) - e^{2\mu(r,l)} dl^2. \]  

(8)

To study the effect of the extra time-like dimension, we choose an orthonormal basis \( \{ e_{\hat{\alpha}} \} \) which is dual to the following 1-form basis:

\[ \theta^0 = e^{\Phi(r)} dt, \quad \theta^1 = \left[ 1 - b(r)/r \right]^{-1/2} dr, \quad \theta^2 = r d\theta, \quad \theta^3 = r \sin\theta d\phi, \quad \theta^4 = e^{\mu(r,l)} dl; \]

this basis does not depend on the signature.

The connection 1-forms, the curvature 2-forms, and the components of the Riemann curvature tensor are derived in Ref. [7]. In this paper, we need only the components of the Ricci tensor [7], listed next.

\[ R_{00} = -\frac{1}{2} \frac{d\Phi(r)}{dr} \frac{rb' - b}{r^2} + \frac{d^2\Phi(r)}{dr^2} \left( 1 - \frac{b}{r} \right) \]

\[ + \left( \frac{d\Phi(r)}{dr} \right)^2 \left( 1 - \frac{b}{r} \right) + 2 \frac{d\Phi(r)}{r} \frac{dr}{dr} \left( 1 - \frac{b}{r} \right)
\]

\[ + \frac{d\Phi(r)}{dr} \frac{\partial \mu(r,l)}{\partial r} \left( 1 - \frac{b}{r} \right), \]  

(10)

\[ R_{11} = \frac{1}{2} \frac{d\Phi(r)}{dr} \frac{rb' - b}{r^2} - \frac{d^2\Phi(r)}{dr^2} \left( 1 - \frac{b}{r} \right) \]

\[ - \left( \frac{d\Phi(r)}{dr} \right)^2 \left( 1 - \frac{b}{r} \right) + \frac{rb' - b}{r^3} - \frac{\partial^2 \mu(r,l)}{\partial r^2} \left( 1 - \frac{b}{r} \right)
\]

\[ + \frac{1}{2} \frac{\partial \mu(r,l)}{\partial r} \frac{rb' - b}{r^2} - \left( \frac{\partial \mu(r,l)}{\partial r} \right)^2 \left( 1 - \frac{b}{r} \right), \]  

(11)
\[ R_{22} = R_{33} = -\frac{1}{r} \frac{d\Phi(r)}{dr} \left( 1 - \frac{b}{r} \right) + \frac{1}{2} \frac{rb'}{r^3} - \frac{b}{r^3} - \frac{1}{r} \frac{\partial \mu(r, l)}{\partial r} \left( 1 - \frac{b}{r} \right), \]  

(12)

\[ R_{44} = -\frac{1}{r} \frac{d\Phi(r)}{dr} \frac{\partial \mu(r, l)}{\partial r} \left( 1 - \frac{b}{r} \right) - \frac{\partial^2 \mu(r, l)}{\partial r^2} \left( 1 - \frac{b}{r} \right) + \frac{1}{2} \frac{\partial \mu(r, l)}{\partial r} \frac{rb'}{r^2} - \frac{b}{r^2} \left[ \frac{\partial \mu(r, l)}{\partial r} \right]^2 \left( 1 - \frac{b}{r} \right) - \frac{2}{r} \frac{\partial \mu(r, l)}{\partial r} \left( 1 - \frac{b}{r} \right). \]  

(13)

With Eq. (3) in mind, we now obtain

\[ 8\pi (\rho + p_r) = [R_{00} - \frac{1}{2} R(-1)] + [R_{11} - \frac{1}{2} R(1)] = R_{00} + R_{11}. \]  

(14)

Since we are primarily interested in the vicinity of the throat, we assume that \( 1 - b(r_0)/r_0 = 0 \). So

\[ \rho + p_r |_{r = r_0} = \frac{1}{8\pi} \frac{b'(r_0)}{r_0^2} - 1 \left[ 1 + \frac{r_0}{2} \frac{\partial \mu(r_0, l)}{\partial r} \right]. \]  

(15)

At this point we need to introduce a key requirement: we will assume that the extra dimension is independent of \( r \); thus \( \partial \mu(r, l)/\partial r = 0 \) and Eq. (15) becomes

\[ \rho + p_r |_{r = r_0} = \frac{1}{8\pi} \frac{b'(r_0)}{r_0^2} - 1 < 0, \]  

(16)

which is the usual flare-out condition for a Morris-Thorne wormhole, indicating that the NEC has been violated. So the location of the wormhole is not restricted, even in the presence of closed time-like curves.

### 3 Closed time-like curves

As noted in the Introduction, we are primarily interested in wormholes that provide a shortcut for an already existing path. We have also seen that the extra time-like dimension, resulting in an anti-de Sitter space, assures the existence of closed time-like curves somewhere. Since \( u(r, l) \) is assumed to be independent of \( r \), it is possible in principle to place a wormhole anywhere, even along a closed time-like curve. The result is a wormhole that functions as a time machine.

One normally associates time travel with causality violations, which is easy to illustrate by pretending that the pockets of a pool table are wormholes that act as time machines. Suppose a ball is tossed toward a pocket with the initial direction and velocity so chosen that, if the ball is undisturbed, it exits the second mouth of the wormhole in the past and knocks its younger self off course, thereby preventing it from even entering the wormhole. So the above initial conditions would be forbidden. According to Refs. [10] and [11], however, this conclusion is at odds with classical physics: what local mechanism could prevent an experimenter from tossing a ball in a certain direction at a particular velocity?

Possibly against initial expectations, self-consistent solutions have been found by assuming that the ball does not necessarily enter the wormhole undisturbed [12], also discussed in Ref. [10].
As a result of these considerations, more general mathematical solutions have been proposed in Refs. [10] and [11]: certain global features such as the existence of closed time-like curves should not prevent possible actions in line with classical physics in a sufficiently localized region, provided that this region does not itself contain closed time-like curves. So it may be possible to avoid a causality violation.

This conclusion shows, from a practical standpoint, the advantage of using traversable wormholes. We saw at the end of Sec. 2 that the independence of $u(r, l)$ of $r$ allows a wormhole to be placed anywhere. So the wormhole could serve as a shortcut for any closed time-like curve. Furthermore, the wormhole would not need to terminate directly on this curve. By avoiding this curve, it could be made to terminate in the type of localized region discussed above.

4 Conclusion

This paper starts by assuming the existence of an extra dimension. According to Ref. [13], the field equations in terms of the Ricci tensor are

$$R_{AB} = 0, \quad A, B = 0, 1, 2, 3, 4.$$ 

This vacuum solution includes the Einstein field equations containing matter, sometimes referred to as the induced-matter theory: what we perceive as matter can be viewed as the impingement of the higher-dimensional space onto ours. According to Ref. [13], the introduction of the extra dimension has produced many other key insights and should therefore be seen as an extremely useful mathematical model, even though the extra dimension cannot be directly observed.

As noted in Sec. 1, Einstein’s theory allows the extra dimension to be space-like or time-like. If time-like, the result is a five-dimensional anti-de Sitter space. Accordingly, the line element for the wormhole spacetime becomes

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{2u(r, l)} dl^2.$$ 

Because of the extra temporal dimension, the spacetime contains closed time-like curves. The assumption that $u(r, l)$ is independent of $r$ leads to

$$\rho + p_r|_{r=r_0} = \frac{1}{8\pi} \frac{b'(r_0) - 1}{r_0^2} < 0,$$

the usual flare-out condition at the throat. So the wormhole could be placed anywhere. This approach necessitates the existence of traversable wormholes.

It is well known that a traversable wormhole can in principle be converted to a time machine [5]. It would not be possible, however, to travel back in time beyond the age of the time machine, that is, before the time machine was created. In this paper, this restriction no longer applies since the wormhole makes use of closed time-like curves in an anti-de Sitter space.
Finally, since the wormhole can be placed anywhere in the spacetime, it can serve as a shortcut for any closed time-like curve. Moreover, the wormhole would not need to terminate directly on this curve; instead, it could be made to connect the present to a sufficiently localized region that does not contain a closed time-like curve. According to Refs. [10] and [11], in this region, classical physics operations would be allowed. So it may be possible to avoid the time-travel paradoxes.

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