Abstract

I introduce a simple continuous probability theory based on the Ginzburg-Landau equation that provides for the first time a common analytical basis to relate and describe the main features of two seemingly different phenomena of condensed-matter physics, namely self-organized criticality and multifractality. Numerical support is given by a comparison with reported simulation data. Within the theory the origin of self-organized critical phenomena is analysed in terms of a nonlinear singularity spectrum different from the typical convex shape due to multifractal measures.

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Singularity spectrum of self-organized criticality

E. Canessa

ICTP-International Centre for Theoretical Physics, Trieste, Italy
The concept of self-organized criticality (SOC) \[1\] has attracted great interest recently, both analytically \[2, 3\] and experimentally \[4, 5\]. The idea behind SOC is that a certain class of dynamical many-body systems drive themselves into a statistically stationary critical state, with no intrinsic length or time scale, where they exhibit fractal behaviour and generate $1/f$ noise. Besides SOC, the generalization of fractal growth to self-similar multifractals has also attracted considerable attention from the physics community over the past years (see, e.g., \[6, 7\]). Theoretical models to describe MF have been concerned with mean-field arguments \[8\] and standard renormalization group methods \[9\]. These approaches, however, do not allow for the direct analysis of the (parabolic-shaped) singularity spectrum typical for a multifractal structure \[6, 7, 10\].

Motivated by the recent suggestion that SOC supports the appearance of fractal structures \[1\], it is natural to ask then if there is a common principle underlying the seemingly unrelated phenomena of SOC and MF. So far as I know a fixed scale transformation method \[11\], developed for fractal growth, has been used to investigate analytically the nature of 2D clusters in SOC. Henceforth, it is also tempting to search for a unifying scenario that underpins a plausible link between MF and SOC. In fact this is the motivation for this work in which I only take a step in that direction.

In this rapid communication I propose a simple continuous probability theory based on the Ginzburg-Landau (GL) equation \[12\] that combines together the concepts of SOC and MF for the first time. In this goal I explore a new analytical basis which, on the one hand, allows to understand the genesis of SOC from the point of view of a nonlinear singularity spectrum and, on the other hand, reveals further insight into the physics governing this crossover. In short, I make here an attempt to unravel the basic phenomena at the origin of power laws and multifractal dimensions.
A crucial feature of the present formalism is to consider that all random variables in a one-dimensional (1D) space, $\mathcal{R}^1$, are functions of the coordinate variable $\chi$ which I map into an equivalent independent variable $\zeta$—say, energy/unit force—characterizing a random system. Then, all probabilities may become expressible in terms of the uniform probability distribution function

$$G(\zeta_2) - G(\zeta_1) = P\{\zeta_1 < \zeta \leq \zeta_2\} \approx \int_{\zeta_1}^{\zeta_2} \phi(\zeta) \, d\zeta, \quad (1)$$

where $\{\}$ indicates the function interval and $\phi$ is a uniform probability density on the line (or $\mathcal{R}^1$) which needs to be specified.

Assume, within this continuous probability model, that the order parameter $\phi$ satisfies

$$\phi(\zeta) \equiv \frac{\phi_0}{2} \{1 + \mu H(\zeta)\}, \quad (2)$$

such that $\phi(\zeta \to +\infty)/\phi_0 \to 0$ and $\phi(\zeta \to -\infty)/\phi_0 \to 1$, (i.e., $\mu H(+\infty) = -1$ and $\mu H(+\infty) = 1$), and $\mu$ is a coefficient as discussed below (c.f., Eq.(19)).

I postulate $H(\zeta)$ in Eq.(2) to be given by the real solutions of the following static, dimensionless GL-like equation [12]:

$$\frac{\partial^2 H(\zeta^*)}{\partial (\zeta^*)^2} + pH(\zeta^*) - qH^3(\zeta^*) = 0, \quad \zeta^* \in D,$$

where $\zeta^* \equiv \zeta/\zeta_o$. $D$ is the $\zeta$-domain, $[p, q] > 0$ and $\zeta_o$ is a positive coefficient of dim[length]. It is not unreasonable to consider the (1D) domain $D$ to be infinite. Then, the GL possesses the well-known stable, kink solution

$$H(\zeta^*) = \pm \sqrt{\frac{p}{q}} \tanh(\zeta^* \sqrt{\frac{p}{2}}). \quad (3)$$

Using this result together with Eqs.(1) and (2), I will establish a simple relation for the probability distribution $P$. 

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The integral of Eq.(1) over the limits: \( \zeta_2 \equiv \lambda_1 \zeta_o \geq \zeta + \lambda_2 \zeta_o \equiv \zeta_1 \) is

\[
G(\lambda_1 \zeta_o) - G(\zeta + \lambda_2 \zeta_o) = \frac{\phi_0}{2} \int_{\zeta + \lambda_2 \zeta_o}^{\lambda_1 \zeta_o} \{1 \pm \mu \tanh(\frac{\zeta'}{\zeta_o})\} \, d\zeta' = -\tau(\zeta),
\]

where I introduced the new function \( \tau(\zeta) \) and set \( p = q = 2 \) to reduce the number of free parameters. These integration limits lead to the condition

\[
\lambda_2 - \lambda_1 + \zeta^* \leq 0.
\]

Note that the sign in Eq.(4) implies that the single functions \( G \) are here assumed to satisfy the condition \( G(\zeta + \lambda_2 \zeta_o) > G(\lambda_1 \zeta_o) \) for \( \zeta \neq 0 \), which throughout the theory are undefined, whereas the free parameters \( \lambda_i \) (i=1,2) will restrict the range of \( \zeta^* \) via Eq.(5).

Suppose \( \lambda_2 \neq \lambda_1 \), then it is straightforward to solve the above integral and get

\[
\tau(\zeta^*) \approx (1 + \frac{\zeta^*}{\lambda_2 - \lambda_1})\{\tau(0) \mp \mu \phi_0^* \ln \cosh \lambda_2\}
\]

\[
\pm \mu \phi_0^* \{\frac{\zeta^*}{\lambda_2 - \lambda_1} \ln \cosh \lambda_1 + \ln \cosh(\lambda_2 + \zeta^*)\},
\]

in which \( \phi_0^* \equiv \zeta_o \phi_0 / 2 \) and

\[
\tau(0) \equiv G(\lambda_2 \zeta_o) - G(\lambda_1 \zeta_o) = (\lambda_2 - \lambda_1) \phi_0^* \{1 \mp \mu \Gamma_\lambda\},
\]

such that

\[
\Gamma_\lambda \equiv \frac{\ln \cosh \lambda_1 - \ln \cosh \lambda_2}{\lambda_2 - \lambda_1}.
\]

When \( \zeta^* = 0 \), MF will be described via Eq.(7) assuming \( G(\lambda_2 \zeta_o) < G(\lambda_1 \zeta_o) \), whereas for SOC: \( G(\lambda_2 \zeta_o) > G(\lambda_1 \zeta_o) \).

To analyse the possible multifractal features of \( P \) let me define the function \( D_{\zeta^*} \) as

\[
\tau(\zeta^*) \equiv \{\lambda_2 - \lambda_1 + \zeta^*\} D_{\zeta^*}.
\]
From Eqs.(6) and (9) it follows that

\[
D \zeta^* \equiv \frac{1}{\lambda_2 - \lambda_1} \left\{ \tau(0) \mp \mu \phi_0^* \ln \cosh \lambda_2 \right\} \\
\pm \frac{\mu \phi_0^*}{\lambda_2 - \lambda_1 + \zeta^*} \left\{ \frac{\zeta^*}{\lambda_2 - \lambda_1} \ln \cosh \lambda_1 + \ln \cosh (\lambda_2 + \zeta^*) \right\} ,
\] (10)

such that \( \lambda_2 - \lambda_1 + \zeta^* \neq 0 \). From this relation I obtain

\[
D_{\zeta^* \to 0} = \frac{\tau(0)}{\lambda_2 - \lambda_1} ,
\] (11)

and

\[
D_{\zeta^* \to +\infty} = D_{\zeta^* \to 0} \pm \mu \phi_0^* \{1 + \Gamma_\lambda\} ,
\]

\[
D_{\zeta^* \to -\infty} = D_{\zeta^* \to 0} \pm \mu \phi_0^* \{1 - \Gamma_\lambda\} .
\] (12)

However if \( \lambda_2 - \lambda_1 + \zeta^* = 0 \) then

\[
D_{\zeta^* \to (\lambda_2 - \lambda_1)} = D_{\zeta^* \to 0} \pm \mu \phi_0^* \{\Gamma_\lambda + \tanh \lambda_1\} .
\] (13)

Complementary to Eq.(4) I also define the following dimensionless function

\[
\alpha(\zeta^*) \equiv \frac{\partial}{\partial \zeta^*} \tau(\zeta^*) = D_{\zeta^* \to 0} \pm \mu \phi_0^* \{\Gamma_\lambda + \tanh (\lambda_2 + \zeta^*)\} .
\] (14)

Therefore, using Eq.(14) in conjunction with Eq.(12), it can be easily shown that

\[
\alpha_{\text{max}} \equiv \alpha(\zeta^* \to -\infty) = D_{\zeta^* \to -\infty} ,
\]

\[
\alpha_{\text{min}} \equiv \alpha(\zeta^* \to +\infty) = D_{\zeta^* \to +\infty} .
\] (15)

In order to relate these equations to the principle of MF [7, 14], I consider the following condition for \( \alpha \) and \( D \). If \( \zeta^* \to +\infty \), then \( D_{\zeta^* \to +\infty} \) and \( \alpha(\zeta^*) \) are allow to take one of the two values, namely 0 or \( 2D_{\zeta^* \to 0} \), depending on the phenomena to be considered. The
particular case for which \( \alpha(\zeta^* \to +\infty) \approx 0 \) and \( D_{\zeta^* \to +\infty} \approx 0 \) will be in correspondence with the features of MF. To achieve this I approximate
\[
\mp \mu \phi_0^* \approx \frac{\varepsilon D_{\zeta^* \to 0}}{(1 + \Gamma \lambda)},
\]
where the integer factor \( \varepsilon \equiv \pm 1 \) is included to distinguish these two limiting values.

According to the original definitions used in MF [15, 16] I also define the dimensionless functions
\[
f(\alpha) \equiv \zeta^* \alpha(\zeta^*) - \tau(\zeta^*),
\]
and
\[
C_{\zeta^*} \equiv -\frac{\partial^2}{\partial (\zeta^*)^2} \tau(\zeta^*) = \mp \mu \phi_0^* \text{sech}^2(\lambda_2 + \zeta^*).
\]
When \( f(\alpha) \) and \( D_{\zeta^*} \) are smooth functions of \( \alpha \) and \( \zeta^* \), then \( f(\alpha) \) can be related to \( \tau(\zeta^*) \) by a Legendre transformation [10]. As is well known this reflects a deep connection with the thermodynamic formalism of equilibrium statistical mechanics where \( \tau \) and \( C_{\zeta^*} \) are the analogous ‘free energy’ and ‘specific heat’, respectively. Within this characterization \( \alpha \) becomes an analogous ‘internal energy’ and \( f \) an analogous ‘entropy’ [15].

Having established a physical meaning for these functions and assuming \( \phi_0^* > 0 \) as discussed below, I realize then that \( \mu \) can take the values
\[
\mu \to \mp 1,
\]
since it is physically satisfactory to set \( C_{\zeta^*} \geq 0 \). This, in turn, implies that the reduced probability density \( \phi(\zeta)/\phi_0 \) of Eq.(4) can take the desired values 0 and 1 when \( \zeta^* \to +\infty \) and \( \zeta^* \to -\infty \), respectively. Within these limits, in the case of MF, the quantities \( \alpha(\zeta^*) \) of Eq.(14), \( D_{\zeta^*} \) of Eq.(10) and \( C_{\zeta^*} \) of Eq.(18) converge to zero whereas \( \tau(\zeta^*) \) of Eq.(6) increases or decreases monotonically depending on \( \zeta^* \).
Besides this, the choice of Eq.(19) implies that I may also obtain Eq.(16) from Eq.(11) in conjunction with Eq.(7) provided that $\varepsilon \to +1$. However, in the case that $\varepsilon \to -1$ is chosen, Eq.(16) can only be recovered using again the aforementioned equations but changing the sign of the previous value for $\tau(0)$. This will became more clear latter in the calculations. The present continuous probability theory is thus dependent on $\lambda_i$ ($i=1,2$) and (the sign and magnitude of) $\tau(0)$ in Eq.(7), where the reduced variable $\zeta$ satisfies the condition given by Eq.(5) and $\phi_0^*$ is positive satifying Eq.(19).

The case $\lambda_1 > \lambda_2 > 0$ (MF): In Fig.1(a) I display the dependence of the analogous ‘free energy’ $\tau$ on the reduced coordinate variable $\zeta^*$ for different values of $\lambda_1$ and $\lambda_2$, such that $\tau(0) = -1$ and $\varepsilon = 1$. Noting that the difference between $\lambda_2$ and $\lambda_1$ is $-1$ in all three curves illustrated, so as to have $D_{\zeta^*\to 0} = 1$, then from Eq.(5) $\zeta \leq 1$. Hitherto, the present GL-based approach to MF allows $\zeta^*$ to take on negative as well as positive values $\leq 1$. However, in the following plots concerning MF, the range of $\zeta^*$ is extended to 3 for illustrative purposes.

From Fig.1(a) it can be seen that, on increasing the value of $\lambda_1$, there is a more rapid convergence of $\tau(\zeta^*)$ for positive $\zeta^*$ than within the region $\zeta^* < 0$; displaying thus typical features of MF [15]. Such a nontrivial behaviour of $\tau$ illustrates the data collapse or breakdown of MF at $\zeta^* > 0$ where $\tau(\zeta^*) > 0$. This is in accordance with the MF structure of the function $D_{\zeta^*}$ shown in Fig.2. In fact, this corresponds to a spectrum of fractal dimensions. In this figure I point out that $D_{\zeta^*}$ -defined via Eq.(5)- is not constant for positive and negative values of $\zeta^*$ so that it may correspond to a multifractal dimension. Hence, $\tau$ becomes also a nonlinear function of $\zeta^*$. In fact this approach yields, e.g. for the case $\lambda_1 = 1, \lambda_2 = 0$ (full line in Fig.1(a)) the values $D_{\zeta^*\to 1} = 0.421$ and $D_{\zeta^*\to -\infty} = 3.53$ using Eqs.(13) and (12), respectively.
On the other hand, in Fig.3(a) I display the analogous 'internal energy' $\alpha$ as a function of $\zeta^*$ for the same set of values of $\lambda_1$ and $\lambda_2$ as in Fig.1(a). When $\lambda_1 = 1$ and $\lambda_2 = 0$ this function exhibits sharp variations around $\zeta^* = 0$ with a maximum value that shows a stronger dependence on $\lambda_1$ for positive $\zeta^*$ than for negative $\zeta^*$. This rules out the possibility that the actual position for a critical value of $\zeta$ (or analogous critical inverse 'temperature' [15]), at which the multifractal formalism actually breaks down, is obtained at $\zeta^*_c = 0$. For other values of $\lambda_1$ and $\lambda_2$, $\zeta^*_c$ changes towards negative values. One should also bear in mind that for $\zeta^* < \zeta^*_c$, the analogous 'free energy' $\tau(\zeta^*)$ is dominated by $\alpha$ which, in turn, varies with the magnitude of $\lambda_2$.

Characteristic features of a phase transition at $\zeta^*_c$ can be figured out by examining the shape of the analogous 'specific heat' $C_{\zeta^*}$ of Eq.(18) which is illustrated in Fig.4. It can be easily visualized that there is a sharp peak around the value $\zeta^*_c = 0$ for the case corresponding to the full line in Fig.1(a). The heights and positions of these curves are strongly dependent on $\lambda_1 > \lambda_2$. Nicely, this finding is also similar to reported MF results [15]. Finally, I investigate the behaviour of the analogous 'entropy' $f$ against $\alpha$ for several values of $\lambda_1$ and $\lambda_2$. As can be seen from Fig.5(a) I find that, on increasing the magnitude of $\lambda_1$, the left-hand side of these plots converge more rapidly than the right-hand sides which converge poorly. This is in complete agreement with the MF signal observed in the context of self-similar random resistor networks (open and full circles in Fig.5(a) from [17]) or, e.g., in DLA when $\lambda_1$ is taken to be related to the system size of the simulation box [7, 16]. In fact, this singularity $f(\alpha)$ shows the characteristic normalized convex shape found in MF. Given its interpretation of a multifractal character for fractal subsets, each with a different fractal dimension having singularity strength $\alpha$, it is expected that $f(\alpha) \geq 0$.  

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Let me remark again that all of these predictions resemble qualitatively many of the intriguing results observed in MF, such as the breakdown of multifractal behaviour and the existence of a phase transition at $\zeta^*$, which may not necessarily be at $\zeta^* = 0$. In the examples considered above the maximum and minimum values of the analogous 'internal energy' $\alpha$ can be estimated from Eqs.(15) in conjunction with Eq.(12), respectively. This allows for the existence of a critical point $\zeta^*$ above which the infinite hierarchy (or broad distribution) of phases can be found, but below which a single phase appears characterized by the maximum 'energy' $\alpha_{max}$.

The case $\lambda_1 > \lambda_2$ such that $\lambda_2 < 0$ (SOC): In Fig.1(b) the dependence of the normalized probability distribution function $\tau$ of Eq.(6) on the energy/unit force variable $\zeta^*$ is plotted for different values of $\lambda_1$ and $\lambda_2$ such that, as a main difference with respect to MF, $\tau(0) = 1$ and $\varepsilon = -1$. This new choice of $\tau(0)$ allows to normalize $\tau(\zeta^*)$ and the choice of $\varepsilon$ enables to mimic the main features of SOC, namely a power-law behaviour, provided the reduced variable $\zeta^*$ is associated with the log-function of some measured event. As an illustrative example, in Fig.1(c) I show a reasonable description of the probability $\pi$ that the water flow intensity of a randomly chosen site is larger than $s$, as obtained from the SOC signal calculated within a model of erosion [18]. This demonstrates that the theory do apply to self-organizing systems. For a class of continuous, cellular automaton models of earthquakes [19, 20], the function $\tau$ is simply reinterpreted as being the number of events with reduced released energy $E \sim e^{\varepsilon^*}$.

The several theoretical curves in Fig.1(b) refer to different values of ( ) $\lambda_1 = 3$ and $\lambda_2 = -6$; (−−−) $\lambda_1 = 3$, $\lambda_2 = -7$; (⋯⋯) $\lambda_1 = 3$, $\lambda_2 = -8$. Using Eq.(5), then the GL-based theory is valid for $\zeta \leq 9, 10, 11$, respectively, where I deal with the physically interesting interval $0 < \tau(\zeta^*) \leq 1$. A given slope of the linear behaviour of
the curves in this figure is determined by fixing $\lambda_1 > 0$. In particular, this parameter may be associated with the elastic parameter of the spring-block model for earthquakes which links the rate of occurrence of earthquakes of magnitude $M$ greater than $m$ to the energy (seismic moment) $E$ released during the earthquake via the famous empirical Gutenberg-Richter law \cite{19}. And the cutoff in the axes $\zeta^*$ may be related to the system size of cellular automaton modeling for arrays of threshold elements.

To see more clearly power-law features in the behaviour of $\tau$ over a wide range of positive values of $\zeta^*$ (as those in Figs.1(b) and (c)), it is necessary to investigate the behavior of the derivative of $\tau(\zeta^*)$, defined through $\alpha$ of Eq.\((14)\), which is plotted in Fig.3(b). A glance into the behaviour of $\alpha(\zeta^*)$ indicates that for the smallest displayed values of $\zeta^* > 0$ the $\alpha$-function converges to a constant negative value, revealing in this way the constant nature of the negative slopes in the ($\tau$-$\zeta^*$) curves of Fig.1(b). On increasing $\zeta^*$ each curve smoothly approach a smaller value. Clearly, due to the probabilistic nature of $\tau$ for SOC (i) such convergences of $\alpha$ towards smaller negative values need not to be considered and (ii) the relations between $\alpha_{\max,min}$ and $D_{\zeta^*\rightarrow\mp\infty}$ (c.f., Eq.(15)), become meaningless.

After establishing this resemblance of a power-law description, it is tempting to continue applying anew the above formalism of MF to analyse SOC. As already mentioned, this may help to understand the origin of SOC from the novel viewpoint of a nonlinear singularity spectrum different from what is common to multifractal objects (c.f., Fig.5(a)).

In view of the features in $\alpha(\zeta^*)$ of Fig.3(b), the second derivative of $\tau$ (not shown), namely $C_{\zeta^*}$ of Eq.(18), present a sharp peak around an inflection point of the function $\alpha(\zeta^*)$ of Fig.3(b), say $\zeta_{inf}$. As a difference to the above results of MF (c.f., Fig.4),
these peak heights reduce their magnitude on decreasing $\lambda_2 < \lambda_1$ and shift their position towards positive values of $\zeta^*$. In this case no phase transition as in the case of MF is expected because $\tau > 0$ restricts the range of valid $\zeta^* < \zeta_{inf}$. Moreover, as mentioned above, $D_{\zeta^*}$ remains undefined for SOC due to the re-interpretation of $\tau(\zeta^*)$. Similarly to what is shown in Fig.2 this function for SOC may simply become not constant on increasing $\zeta^*$ as in MF.

On the other hand, the continuous singularity spectrum $f(\alpha)$ plays an alternative role when dealing with SOC which can be assessed from Fig.5(b). In this plot it can be seen that this function exhibits a rather nonlinear behaviour different from the parabolic behaviour typical of multifractal entities (c.f., Fig.5(a)). In MF $f$ takes its maximum at the value $\alpha(\zeta^* = 0)$ whereas in SOC this spectrum becomes a monotonically increasing (negative) function of (negative) $\alpha$. On decreasing the magnitude of $\lambda_2$ these curves converge to $-1$ and separate out as a function of decreasing $\alpha$ within the range of validity of $\alpha$ in Fig.3(b). Since $\tau(\zeta^*)$ is positive, then $-2 < f(\alpha) < -1$. I suggest this new aspect of the behaviour of $f(\alpha)$ to be a fundamental property for the additional characterization of SOC. It is therefore most likely that the linear behaviour displayed by $\tau(\zeta^*)$ (c.f., Fig.1(b)), that is quantified via $\alpha(\zeta^*)$ (c.f., Fig.3(b)), finds its root through the behaviour of Eq.(17) for $f(\alpha)$ (c.f., Fig.5(b)).

To some extent, I have been able to shed light on a basic mechanism leading to both concepts of MF and SOC from the continuous density probability $\phi(\zeta)$ as given in Eq.(2). This function has been assumed to be related to $H(\zeta)$ via Eq.(2), which I postulated to be given by the real kink solutions of a dimensionless GL-like equation. Such solutions are known to minimize the GL free energy functional when $|\phi|^2$ is related to a particle concentration [12]. Of course to use a continuous probability theory may
be seen as being heuristic, specially so if reported simulation results have been done using discretized cell configurations. But, as I have discussed, a great deal of relevant information can be extracted from a continuos approach which, essentially, does relay on the sign of $\lambda_2$, $\tau(0)$ and $\varepsilon$ only.

The thermodynamic treatment of MF provides a convenient way to quantify the complexity of multifractals by characterizing scale-invariant singularities and, in most cases, allowing for an effective comparison between theory and experiments \cite{1, 4, 7, 15}. Nevertheless, I have proposed here that this formalism may be extended as a tool to rationalize SOC on the basis of a singularity spectrum that turns out to be nonlinear as well (when comparing Figs.5(a) and (b)). While the present (static) GL-based theory is extremely simple, it is important to emphasize that it gives information about the complex origin of self-organized critical phenomena whose physics is assumed -to a good approximation- to be analog to that required to describe MF. This theory also reflects the minimal ingredients than can give rise to an intrinsically critical state. In passing, I add that multifractal structures can also emerge from GL equations with random initial conditions for the temporal evolution \cite{21}.

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Figure captions

• **Figure 1**: (a) Dependence of the analogous ‘free energy’ $\tau$ on the coordinate variable $\zeta^*$ displaying typical features of MF phenomena [7]. Different values of $(\lambda_1 = 1, \lambda_2 = 0)$; $(- - - -) \lambda_1 = 1.5, \lambda_2 = 0.5$; $(\cdots \cdots) \lambda_1 = 2, \lambda_2 = 1$, illustrates the data collapse at $\zeta^* > 0$ where $\tau > 0$. In these examples $\tau(0) = -1$ and $\varepsilon = 1$.

(b) Dependence of the normalized probability distribution function $\tau$ on the variable $\zeta^*$ displaying typical features of SOC. Different curves refer to different values of $(\lambda_1 = 3, \lambda_2 = -6)$; $(- - - -) \lambda_1 = 3, \lambda_2 = -7$; and $(\cdots \cdots) \lambda_1 = 3, \lambda_2 = -8$. In these examples $\tau(0) = 1$ and $\varepsilon = -1$.

(c) Present description of the probability $\pi$ that the water flow intensity of a randomly chosen site is larger than the event $s$. Open circles are the SOC signal in a model of erosion [18].

• **Figure 2**: Plot of the multifractal dimension $D_{\zeta^*}$ against $\zeta^*$ showing a non-constant behaviour for the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(a).

• **Figure 3**: (a) Dependence of the analogous 'internal energy' $\alpha$ on $\zeta^*$ displaying characteristic features of a phase transition for the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(a).

(b) The function $\alpha$ of Eq.(14) as a function of $\zeta^*$ for the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(b). These curves reflect the power-law features in the behaviour
of $\tau(\zeta^*)$ for SOC.

- **Figure 4**: Analogous 'specific heat' $C_{\zeta^*}$ as a function of $\zeta^*$ displaying characteristic features of a phase transition for the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(a).

- **Figure 5**: (a) Dependence of the analogous 'entropy' $f$ on the analogous 'internal energy' $\alpha$ for the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(a). Open and full circles are the MF signal calculated in the context of self-similar random resistor networks for different cell sizes (see [17]).

(b) The nonlinear $f(\alpha)$ singularity spectrum obtained from Eq.(17) proposed to characterize SOC using the same set of values for $\lambda_1$ and $\lambda_2$ as in Fig.1(b).