CLT for biorthogonal ensembles and related combinatorial identities

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Abstract
We study the fluctuations of certain biorthogonal ensembles for which the underlying family \( \{P, Q\} \) satisfies a finite-term recurrence relation of the form \( xP(x) = JP(x) \). For polynomial linear statistics of such ensembles, we reformulate the cumulants' method introduced in [Sos00a] in terms of counting lattice paths on the graph of the adjacency matrix \( J \). In the spirit of [BD], we show that the asymptotic fluctuations of polynomial linear statistics are described by the right-limits of the matrix \( J \). Moreover, whenever the right-limit is a Laurent matrix, we prove that the CLT obtained in [BD] is equivalent to Soshnikov’s main combinatorial lemma. We discuss several applications to unitary invariant Hermitian random matrices. In particular, we provide a general Central Limit Theorem (CLT) in the one-cut regime. We also prove a CLT for square singular values of product of independent complex rectangular Ginibre matrices. Finally, we discuss the connection with the Strong Szegő theorem where this combinatorial method originates.

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1 Introduction

Random matrix ensembles were introduced by Wigner who suggested that the distribution of scattering resonances of heavy nuclei is described by eigenvalues of certain random matrices, \[\text{Wig51}\]. Now-days, random matrix models and their generalizations have found a wide range of applications in physics and mathematics, \[\text{ABF11}\]. Mathematically, Hermitian ensembles are easier to analyze because of their determinantal structure. The connection between unitary invariant Hermitian ensembles and orthogonal polynomials was developed by Gaudin and Met ha in the early stages of the theory, \[\text{MG60}\]. Namely, if the space of $N \times N$ Hermitian matrices, denoted $\mathcal{H}_N$, is equipped with the probability measure
\[
d\mathbb{P}_{N,V}(M) = Z_{N,V}^{-1}e^{-N\text{Tr}V(M)}dM ,
\]
where $dM$ is the Lebesgue measure on the manifold $\mathcal{H}_N$ and $V : \mathbb{R} \to \mathbb{R}$ is a function which satisfies the condition
\[
\lim_{|x| \to \infty} \frac{V(x)}{\log(1 + |x|^2)} = +\infty .
\]

Then, the eigenvalues of a random matrix sampled from $\mathbb{P}_{N,V}$ have a joint density on $\mathbb{R}^N$ with respect to the product of $N$ copies of the measure $d\mu_N/dx = e^{-NV(x)}$ which is given by
\[
\varrho_N(x_1, \ldots, x_N) = \frac{1}{N!} \text{det}_{N \times N} \left[K_N(x_i, y_j)\right] ,
\]
where the function $K_N$ is expressed in terms of the orthogonal polynomials $(P_k^N)_{k=0}^\infty$ with respect to $d\mu_N/dx = e^{-NV(x)}$ as follows,
\[
K_N(x, y) = \sum_{k=0}^{N-1} P_k^N(x)P_k^N(y) .
\]

We will use the conventions,
\[
P_k^N(x) = \kappa_k^Nx^k + \cdots \quad \text{and} \quad \int P_k^N(x)P_j^N(x)d\mu_N(x) = \delta_{k,j} ,
\]
where $\kappa_k^N > 0$. The most well known example is the Gaussian Unitary Ensemble (GUE) which corresponds to the potential $V(x) = x^2/2$, in which case $(P_k^N)_{k=0}^\infty$ are the Hermite polynomials. Generally, if $\mathcal{X}$ is a complete separable metric space equipped with a Radon measure $\mu$, a determinantal process is a point process on $(\mathcal{X}, \mu)$ whose correlation functions with respect to $\mu$ are of the form
\[
\rho_k(x_1, \ldots, x_k) = \text{det}_{k \times k} \left[K(x_i, y_j)\right] ,
\]
where the function $K$ is called the correlation kernel. There is an extensive literature available on determinantal processes, e.g. \[\text{Sos00b, HK06, Joh00}\], and the determinantal form of the correlation functions \[\text{(1.6)}\] implies that many natural observables can be expressed explicitly in terms of the kernel $K$, e.g. formula \[\text{(1.6)}\]. Using orthogonality, it is simple to check by integrating $N-k$ variables that the correlation functions of the j.p.d.f. \[\text{(1.3)}\] satisfies \[\text{(1.6)}\] for all $k \leq N$. Hence, eigenvalues of the Hermitian ensemble $\mathbb{P}_{N,V}$ form a determinantal process with correlation kernel \[\text{(1.4)}\] with respect to $\mu_N$. In this article, we will consider a generalization of these ensembles introduced by Borodin, \[\text{Bor99}\], called Biorthogonal Ensembles (BOE).

**Definition 1.1.** A biorthogonal family $\{P_k, Q_k\}_{k=0}^\infty$ is a set of functions defined on $\mathcal{X}$ such that $P_kQ_n \in L^1(\mu)$ for all $n, k \in \mathbb{N}_0$ and
\[
\int P_k(x)Q_n(x)d\mu_N(x) = \delta_{k,n} .
\]
A biorthogonal ensemble $\Xi_N = (\xi_k)_{k=1}^N$ is a determinantal process on $(\mathcal{X}, \mu_N)$ with correlation kernel

$$K_N(x, y) = \sum_{k=0}^{N-1} P_k^N(x) Q_k^N(y) ,$$

where $(P_k^N, Q_k^N)_{k=0}^\infty$ is a biorthogonal family.

In the following we will only consider the case $\mathcal{X} = \mathbb{R}$, although the results could be formulated in a broader context. An alternative definition of a biorthogonal ensemble is that it is a point process with the j.p.d.f. $(1.11)$. Interestingly, biorthogonal ensembles arise in various other contexts beyond Hermitian ensembles, e.g. non-intersecting paths, domino tilings, multiple orthogonal polynomials, etc. We do not intend to review the theory here and we refer to [Ko05, BD, Kui10] or [ABF11, chap. 11] for more in-depth introductions. In many examples, there is more structure than just the biorthogonality relation $(1.7)$. For instance, for the Hermitian models $(1.1)$, the functions $Q_k^N = P_k^N$ are polynomials of degree $k$ and the kernel $(1.4)$ is symmetric. In this case, the determinantal process $\Xi_N$ is entirely characterized by the reference measure $d\mu_N/ dx = e^{-NV(x)}$ and $K_N$ called the Christoffel-Darboux kernel. In the following, we will consider the general setting where $\mu_N$ is an arbitrary Borel measure on $\mathbb{R}$ which satisfies, for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} |x|^k d\mu_N(x) < \infty .$$

This condition guarantees that the polynomials $(1.5)$ exist and $(P_k^N)_{k=0}^\infty$ is an orthogonal basis of the space of polynomials $\mathbb{R}(x)$ equipped with the inner product inherited from $L^2(\mu_N)$. In the literature, such a point process is usually called the Orthogonal Polynomial Ensemble (OPE) with reference measure $\mu_N$ and it is of interest to know which properties of the measure $\mu_N$ lead to $\Xi_N$ having certain universal features as $N \to \infty$. For instance, one may consider the well-known problem of finding for which potential $V(x)$, the Hermitian model $(1.1)$ falls in the GUE universality class. In particular, we will revisit the question of universality of fluctuations for a linear statistic

$$\Xi_N(f) = \sum_{k=1}^N f(\xi_k) ,$$

where $\xi_k$ refers to the point configuration of a biorthogonal ensemble. Our approach is inspired by [BD] where the authors prove that, if the Jacobi matrix $J(\mu_N)$ of the reference measure $\mu_N$ has a right-limit which is a Laurent matrix, see definitions $(2.2)$ and $(2.5)$ below, then for any polynomial $F \in \mathbb{R}(x)$,

$$\Xi_N(F) - \mathbb{E}[\Xi_N(F)] \Rightarrow \mathcal{N}\left(0, \sum_{k\geq 1} k \left| \hat{F}_k \right|^2 \right)$$

as $N \to \infty$, where

$$\hat{F}_k = \int_T F(2\cos \theta) e^{-ik\theta} d\theta .$$

The authors insight was that, for OPEs, it is possible to express the moment generating function of the random variable $\Xi_N(F)$ in terms of a Fredholm determinant involving the Jacobi matrix $J(\mu_N)$. If $F$ is a polynomial and $|\lambda|$ is sufficiently small,

$$\mathbb{E}[e^{\lambda \Xi_N(F)}] = \det [1 + P_N(\exp \lambda F(J) - 1)P_N] ,$$

where $P_N$ is the projection on $l^2(\mathbb{Z}_+)$ onto span$(e_0, \ldots, e_{N-1})$. Then, the concept of right-limit establishes the link with the theory of Toeplitz determinants. In fact, one can interpret $(1.11)$ as the counterpart of the Strong Szegő theorem for Hermitian matrix models. This connection is very
interesting and discussed in section 4.2. Like [BD], the starting point of our analysis is based on formula (1.12) and the cumulant method introduced in [Sos00a] and presented in section 4.1. Given a test function \( f \), the cumulants of a linear statistic \( \Xi_N(f) \) are defined by the formal power series

\[
\log \mathbb{E} \left[ e^{\lambda \Xi_N(f)} \right] \sim \sum_{n=1}^{\infty} C_n^N[f] \frac{\lambda^n}{n!} .
\]  

(1.13)

In [Sos00a], using Fourier analysis, Soshnikov reduced the problem of computing the limits of cumulants of linear statistics of the eigenvalues of a Haar distributed random matrix in the unitary group \( U_N \) (CUE) to certain combinatorial identities known as the Main Combinatorial Lemma (MCL - theorem 2.1). In this paper we apply this method to OPEs and certain BOEs. In particular, using an elementary combinatorial approach based on the tridiagonal structure of the Jacobi matrix \( J(\mu_N) \), we derive explicit formulae for the cumulants of the random variable \( \Xi_N(F) \) for any polynomial \( F \); see lemma 2.1 below. When \( J \) has a right-limit which is a Laurent matrix as \( N \to \infty \), these formulae have simple limits and the proof of the CLT (1.11) boils down to the main combinatorial lemma.

The method developed by Breuer and Duits in [BD] allows them to treat BOEs as well, when the functions \( (P_k^N)_{k=0}^{\infty} \) satisfy a recurrence relation:

\[
x \begin{pmatrix} P_0^N(x) \\ P_1^N(x) \\ \vdots \end{pmatrix} = J \begin{pmatrix} P_0^N(x) \\ P_1^N(x) \\ \vdots \end{pmatrix} \quad \forall x \in \mathbb{R} .
\]  

(1.14)

Suppose that there exists a sequence of (infinite) matrices \( J(\mu) \) such that (1.14) holds for all \( N \in \mathbb{N} \) and the matrices \( J(\mu) \) have a fixed number of non-zero diagonals. Then, if \( J(\mu) \) has a right-limit as \( N \to \infty \), which is a Laurent matrix, then the biorthogonal ensemble \( \Xi_N \) satisfies a CLT analogous to (1.11); see theorem 2.6 below. The combinatorial approach developed in this paper works in this more general framework as well and we are able to show that the results of [BD] are a consequence of the main combinatorial lemma.

This generalization is a significant step since it provides the first example of a CLT for linear statistics of biorthogonal ensembles which are not Hermitian. In particular, in [BD], the authors discussed an application to eigenvalues' statistics of both \( M \) and \( \tilde{M} \) for the two matrix model

\[
dP_N[V, V'] = Z^{-1} e^{-N \text{Tr} \left[ V(M) + \tilde{M}^2 - \tau M \tilde{M} \right]} dM d\tilde{M}
\]  

(1.15)

on \( \mathcal{H}_N \times \mathcal{H}_N \). In section 3, we present another important application to square singular values of products of complex Ginibre matrices following the work of [AIK13, KZ14]. In fact, if the recurrence matrix \( J(\mu) \) has a right-limit \( M \) as \( N \to \infty \) which is not a Laurent matrix, we still obtain a limit theorem but generically we can not expect the fluctuations to be Gaussian; see theorem 2.4. It would be worth to investigate further the distribution of the random variable (2.10) under various assumptions on \( M \). In particular, in view of the results [Pas06, Shc13, BG] for OPEs in the multi-cut regime, it would be interesting to understand the case where \( M \) is periodic or quasi-periodic along its main diagonals, see section 3.4. This case will be treated in another article.

The rest of this paper is organized as follows. The main results, limit theorems for biorthogonal ensembles which satisfies a finite-term recurrence relation, are formulated in section 2 and proofs are given in section 3. Two applications are discussed in section 3. First, we recall from [BD] that, as a by-product of theorem 2.6, we obtain universality for orthogonal polynomial ensembles when
the reference measure $\mu_N$ satisfies the so-called one-cut assumption. In this regime, we show that the CLT holds for $C^1$ test function as well, see theorem 3.1. Then, we present a new application to products of rectangular complex Ginibre matrices as the dimensions of the matrices go to infinity. In section 4.1, we give a short introduction to the theory of biorthogonal ensembles and we recall the results of \cite{Sos00a}. In section 4.2, we review the connection between the main combinatorial lemma of Soshnikov, the *Strong Szegő* theorem and theorem 2.6. In the appendix A for completeness, we give a proof of Soshnikov’s main combinatorial lemma which emphasizes on the connection with the Dyson-Hunt-Kac (DHK) formulae and the Bohnenblust-Spitzer combinatorial lemma.

## 2 Main result

The general framework we consider is that of a BOE denoted $\Xi = (\xi_k)_{k=1}^N$ (cf. definition 1.1) where the functions $(P_k^N)_{k=0}^\infty$ satisfy a recurrence relation of the form (1.14) for a given sequence of matrices $J^{(N)}$. We also suppose that there exists $w > 0$ such that for all $N \in \mathbb{N}$,

$$J_{ij}^{(N)} = 0 \quad \text{if} \quad |i - j| \geq w . \quad (2.1)$$

The main examples are OPEs which are discussed in more details in section 3.1. Below, $J$ is called *the recurrence matrix* and we omit the superscript $(N)$ when the dimension is fixed. In \cite{BD}, the recurrence matrix $J$ is interpreted as an operators acting formally on $l^2(\mathbb{Z}_+)$ and the authors established conditions under which the limit of the Fredholm determinant (1.12) exists and can be evaluated; see section 4.2. We have a different approach and interpret $J$ as the *weighted adjacency matrix* of a directed graph $G(J)$. In general, to any (infinite) matrix $M$ corresponds a weighted graph $G(M) = (V, \bar{E})$ where $V \subseteq \mathbb{Z}$ and $\bar{E} = \{ (j, i) \in V \times V : M_{ij} \neq 0 \}$. The vertex set $V$ inherits the total ordering of $\mathbb{Z}$. Edges are oriented, i.e. $(j, i) \neq (i, j)$, and they are weighted or labelled by entries of $M$, see e.g. figure 3.3. Using this formalism, we express the cumulants $C_n[F]$ of a linear statistic $\Xi_N[F]$ for an arbitrary polynomial $F \in \mathbb{R}(x)$ in terms of sums over lattice-paths on the adjacency graph of the matrix $F(J)$. In the following, we let

$$\Lambda_n = \bigcup_{\ell=1}^{n-1} \{ n \in \mathbb{N}^\ell : n_1 < \cdots < n_\ell < n \} .$$

For any $n \geq 2$, the set $\Lambda_n$ is isomorphic to the set of compositions of the integer $n$ under the change of variables $\Psi : \bigcup_{\ell=1}^{n-1} \{ k \in \mathbb{N}^{\ell+1} : k_1 + \cdots + k_{\ell+1} = n \} \mapsto \Lambda_n$ given by

$$n_1 = k_1, \quad n_2 = k_1 + k_2, \quad \ldots, \quad n_\ell = k_1 + \cdots + k_\ell. \quad (2.2)$$

Then, for any $n = (n_1, \ldots, n_\ell) \in \Lambda_n$, we define

$$\bar{\psi}(n) = \frac{(-1)^{\ell+1}}{\ell+1} \binom{n}{k} = \frac{(-1)^{\ell+1} n!}{\ell+1 \cdot k_1! \cdots k_{\ell+1}!} . \quad (2.3)$$

Given a band matrix $M$, let $r \in \mathbb{Z}$ be the root of $G(M) = (V, \bar{E})$. Motivated by lemma 2.1 we define for any $n \in \Lambda_n$, $\Gamma^r_n[M]$ to be the set of all paths $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ on the graph $G(M)$ such that

$$\begin{align*}
(i) & \quad \pi(0) = \pi(n) < r \\
(ii) & \quad \max \{ \pi(n_1), \ldots, \pi(n_\ell) \} \geq r .
\end{align*}$$

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Lemma 2.1. For any polynomial $F \in \mathbb{R}(x)$. If $M = F(J)$, we have for $n \geq 2$,

$$C_{N}^{n}[F] = \sum_{n \in \Lambda_{n}} \mathcal{O}(n) \sum_{\pi \in \Gamma_{n}^{t}(M)} \prod_{i=1}^{n} M_{\pi(i) \pi(i-1)} . \quad (2.4)$$

Proof. Section 5.1

Observe that formula (2.4) only depends on finitely many entries of $J$ and we do not need to require that $J$ is bounded. However, in order to prove convergence of the moment generating function of the linear statistics $\Xi_{N}[F]$, we will suppose that there exists constants $C, \alpha > 0$ such that for all $i, j \geq -N$,

$$|f_{N+i,N+j}(n)| \leq \exp(\alpha \max\{0, i, j\}) . \quad (H.1)$$

In section 5.1, see lemma 5.1, using this condition, we derive a uniform estimate for the cumulants of the random variable $\Xi_{N}(F)$. Namely, for any polynomial $F$, there exists a constant $C_{F} > 0$ such that for any $n \geq 2$,

$$|C_{N}^{n}(F)| \leq n! \exp(nC_{F}) . \quad (2.5)$$

Hence, if (H.1) holds, the power series (1.13) converges uniformly in the disk $|\lambda| < e^{-C_{F}}$ and the sequence of random variables $\Xi_{N}(F)$ is tight. Then, given a subsequence $(N_{k})_{k \in \mathbb{N}}$, we provide a sufficient condition for the convergence in distribution of $\Xi_{N_{k}}(F)$ as $k \to \infty$.

Remark 2.1. It is important to consider unbounded matrix $J$ to treat OPEs with respect to a measure $\mu_{N}$ supported on the whole real line. The hypothesis (H.1) is not restrictive since, in general, if the measures $\mu_{N}$ are properly rescaled, then the recurrence coefficients grow at most polynomially.

Definition 2.2. Let $(M^{(n)})_{n \in \mathbb{N}}$ be a sequence of (infinite) matrices. We say that $M$ is a right-limit of $M^{(n)}$ along a subsequence $N_{k}$ if for all $i, j \in \mathbb{Z}$,

$$M_{ij} = \lim_{k \to \infty} M_{N_{k}+i,N_{k}+j}^{(N_{k})} .$$

Then, we will denote $M_{N_{k}}^{(N_{k})} \overset{N}{\to} M$ as $k \to \infty$. Then, we simply say that $M$ is the right-limit of $M^{(n)}$ if $M^{(N)} \overset{N}{\to} M$ as $N \to \infty$.

This concept comes from the spectral theory of Jacobi operators and its relevance to fluctuations of unitary invariant random matrices has been pointed out in [BD]. In the context of this paper, it is interesting to reformulate definition 2.2 in terms of convergence of graphs. A sequence of weighted graph $G_{k}$ rooted at $r_{k}$ is said to converge locally to a rooted graph $G_{\infty}$ if, for every $n \in \mathbb{N}$, the $n$-neighborhood of the root $r_{k}$ stabilizes and the weights in this neighborhood converge as $k \to \infty$; see e.g. [Vir14] section 3 for another application to random matrix theory. Hence, a sequence $(M^{(n)})_{n \in \mathbb{N}}$ of band matrices has a right-limit of $M$ along a subsequence $(N_{k})_{k \in \mathbb{N}}$ if and only if the sequence of graphs $G(M^{(N_{k})})$ rooted at $N_{k}$ converges locally to $G(M)$ rooted at 0.

Motivated by lemma 2.1 define for all $n \in \mathbb{N}$,

$$\varpi_{r}^{n}(M) = \sum_{n \in \Lambda_{n}} \mathcal{O}(n) \sum_{\pi \in \Gamma_{n}^{t}(M)} \prod_{i=1}^{n} M_{\pi(i) \pi(i-1)} . \quad (2.6)$$

The definitions of right-limit and local convergence implies that, if $M^{(N_{k})} \overset{N}{\to} M$ as $k \to \infty$, then for any $n \in \mathbb{N}$,

$$\lim_{k \to \infty} \varpi_{N_{k}}^{n}(M^{(N_{k})}) = \varpi_{0}^{n}(M) , \quad (2.7)$$

and this yields a limit theorem.
\textbf{Theorem 2.3.} Let $\Xi_N$ be a BOE with recurrence matrix $J^{(N)}$ which satisfies the condition \((H.1)\). Suppose that for a given polynomial $F \in \mathbb{R}(x)$, the matrix $F(J^{(N)})$ has a right-limit $M$ along a subsequence $(N_k)_{k \in \mathbb{N}}$. Then
\[
\lim_{k \to \infty} \left( \log \mathbb{E} \left[ e^{\lambda \Xi_{N_k}(F)} \right] - \lambda \mathbb{E} \left[ \Xi_{N_k}(F) \right] \right) = \sum_{n=2}^{\infty} \omega_n^0(M) \frac{\lambda^n}{n!}, \tag{2.8}
\]
uniformly in compact sets of the disk $|\lambda| < e^{-C_F}$.

\textbf{Proof.} Let $M^{(N_k)} = F(J^{(N_k)})$. According to lemma 2.1, for any $N \in \mathbb{N}$, the cumulants of the linear statistic $\Xi_N(F)$ are given by $C_N^N[F] = \omega_N^N(M^{(N)})$ for any $n \geq 2$. Moreover, by (2.7), we have
\[
\lim_{k \to \infty} C_{N_k}^0[F] = \omega_0^0(M). \tag{2.9}
\]
Then, if we subtract the first term $C^1_N[F] = \mathbb{E} \left[ \Xi_{N_k}(F) \right]$, the estimate (2.5) which is uniform in $N$ implies that we can pass to the limit term by term in the series (1.13). This yields formula (2.8). \qed

In fact, it follows from (2.9) that if we merely suppose that $F(J^{(N)}) \xrightarrow{\ell} M$ as $k \to \infty$ and the entries of the right limit have exponential growth, then we still have a limit theorem,
\[
\Xi_{N_k}(F) - \mathbb{E} \left[ \Xi_{N_k}(F) \right] \Rightarrow X(F) \quad \text{as } k \to \infty, \tag{2.10}
\]
where the distribution of the random variable $X(F)$ is characterized by
\[
\mathbb{E} \left[ e^{\lambda X(F)} \right] = \exp \left( \sum_{n=2}^{\infty} \omega_n^0(M) \frac{\lambda^n}{n!} \right), \tag{2.11}
\]
for all $|\lambda|$ sufficiently small. When the recurrence matrix itself has a right-limit, we obtain the following corollary.

\textbf{Theorem 2.4.} Let $\Xi_N$ be a BOE with recurrence matrix $J^{(N)}$. If $J^{(N)}$ has a right-limit $L$ along a subsequence $(N_k)_{k \in \mathbb{N}}$ which satisfies $|L_{ij}| \leq C \exp \left( \alpha \max\{|i|,|j|\} \right)$ for all $k, j \in \mathbb{Z}$, then for any polynomial $F \in \mathbb{R}(x)$,
\[
\Xi_{N_k}(F) - \mathbb{E} \left[ \Xi_{N_k}(F) \right] \Rightarrow X(F) \quad \text{as } k \to \infty, \tag{2.12}
\]
and the Laplace transform of the random variable $X(F)$ is given by formula (2.11) where $M = F(L)$.

\textbf{Proof.} Section 5.2. \qed

A similar limit theorem was established in [BD] using asymptotics of regularized Fredholm determinants. However, for technical reasons, their proof works only for a non-varying measure $\mu_N = \mu$; see theorem 2.4 and remark 2.6 therein. In particular, using the conventions of [Sim05] chap. 9, we have
\[
\sum_{n=2}^{\infty} \omega_n^0(M) \frac{\lambda^n}{n!} = \lim_{m \to \infty} \log \left( \det_2 \left[ 1 + Q_m(e^{\lambda M} - 1)Q_m \right] \right), \tag{2.12}
\]
where $Q_m$ is the projection on $l^2(\mathbb{Z})$ onto span$(e_{-1}, \ldots, e_{-m})$. For any $m \geq 1$, the determinant on the r.h.s. of (2.12) is analytic for sufficiently small $|\lambda|$ and the coefficients of its Taylor series can be computed using formula (1.3), replacing $J$ by $M$ and the projection $P_N$ by $Q_m$, or using the Plemelj-Smithies formulae. Then, adapting the proof of lemma 2.1, it is not difficult to check that these coefficients converge to $\omega_n^0(M)$ for any $n \geq 2$ as $m \to \infty$. Moreover, it follows from the proof of theorem 2.4 that the series (2.12) converges uniformly in a small disk around 0.
Formula (2.8) shows that, for the class of BOEs we consider, the fluctuations are described by the right-limits of the recurrence matrix $J$. For the matrix model (1.1), it is known that for a generic potential $V$, the entries of the matrix $J$ oscillates and linear eigenvalues statistics do not converge as $N \to \infty$. However, $J$ has right-limits along certain subsequences and theorem 2.3 implies that, along such a subsequence, for any polynomial $F$, $\Xi_{N_k}(F)$ converges in distribution as $k \to \infty$. This mechanism was investigated heuristically by Pastur in [Pas06] using semiclassical formulae for orthonormal polynomials and he showed that, when the asymptotic eigenvalue distribution fills up several intervals, the fluctuations are generically not Gaussian. In fact, analogous but more explicit limit theorems have recently been proved for general $\beta$-ensembles as well, [BG13, BG]. According to formula (2.8), the limit law of the random variable $\Xi_{N_k}(F)$ is Gaussian only if $w_n^0(M)$ vanishes for all $n > 2$, cf. (2.6). Hence, the right-limit $M$ must have a very special structure to obtain such subtle cancellations. In fact, the natural condition is that $M$ is constant along its diagonals.

**Definition 2.5.** A Laurent polynomial is a function $s(z) = \sum_{k \in \mathbb{Z}} s_k z^k$ such that only finitely many coefficients $s_k \in \mathbb{R}$ are non-zero. We let $L(s) = (s_{i-j})_{i,j \in \mathbb{Z}}$ be the Laurent matrix with symbol $s(z)$.

Given a Laurent matrix $L$, the graph $G(L)$ is translation-invariant and it allows us to simplify formula (2.6). Namely, it is proven in section 4.2 that

$$w_n^0(L(s)) = \sum_{\omega_1+\cdots+\omega_n=0, \omega_i \in \mathbb{Z}} \hat{s}_{\omega_1} \cdots \hat{s}_{\omega_n} G_n(\omega_1, \ldots, \omega_n), \quad (2.13)$$

where for any $x \in \mathbb{R}^n$,

$$G_n(x_1, \ldots, x_n) = \sum_{\nu \in \Lambda_n} \hat{\nu}(n) \max \left\{ 0, \sum_{i=1}^{n_1} x_i, \sum_{i=1}^{n_2} x_i, \ldots, \sum_{i=1}^{n_l} x_i \right\}, \quad (2.14)$$

and $\hat{\nu}(n)$ is given by (2.3). Formula (2.13) is very similar to the limit that Soshnikov obtained for cumulants of linear eigenvalue statistics of Haar distributed random matrices; see formula (4.12). For OPEs, this analogy is motivated by the fact that these models fall in the sine process universality class at local and mesoscopic scale, see [Lam] and reference therein. However, it was rather unexpected that, at the global scale, we obtain the same combinatorial structure as for the CUE. As a consequence of the Main Combinatorial Lemma of [Sos00a], lemma 4.1 below, for any Laurent polynomial $s(z)$,

$$w_n^0(L(s)) = \delta_{n,2} \frac{1}{2} \sum_{\omega \in \mathbb{Z}} \hat{s}_\omega \hat{s}_{-\omega} |\omega|. \quad (2.15)$$

This means that, if the matrix $F(J^{(N)})$ has a right-limit along a subsequence $(N_k)_{k \in \mathbb{N}}$ which is a Laurent matrix, then the linear statistic $\Xi_{N_k}(F)$ converges in distribution to a Gaussian random variable as $k \to \infty$. In random matrix theory, the MCL and its variations have become an important device to prove that fluctuations of eigenvalue statistics are asymptotically Gaussian. Besides his work on linear statistics from the classical group, the techniques developed in [Sos00a] also applies to the sine process and some other generalization, see [Sos01]. The MCL was also used in [RV07] to keep track of eigenvalues fluctuations for the Gignibre ensemble. In fact, the approach of Rider and Virag is rather similar to the one used in this article and it can be generalized to complex orthogonal polynomial ensembles with respect to a rotationally invariant measure. A similar lemma appeared first in the work of Spohn on linear statistics of the sine process which comes up as the invariant measure for Dyson’s Brownian motion; see Lemma 2 in [Spo87]. However, the first correct proof of Spohn’s lemma was given in [Sos00a] as well. Another related lemma (lemma A.1) appeared in [RS96] to show that, assuming the Riemann Hypothesis, the correlation functions for the spacings between $N$ consecutive appropriately rescaled zeroes of the Riemann Zeta function converge to that of the GUE as $N \to \infty$. In the appendix A we show that all these lemmas are consequences of
the DHK formulae \( [A.5] \) which were used in \( [Kac64] \) to provide one of the first proofs of the Strong Szegő theorem, cf. formula (4.10).

Even though it appears in several different contexts, it was pointed out in \( [ML] \) that the combinatorial structure behind the MCL is very sensitive. Namely, if we modify the correlation kernel \( (1.3) \), e.g. by removing the mode \( N - m \) for a given \( m \geq 2 \), we generically obtain a non-Gaussian process in the limit. In this case, the limits (2.9) still hold but we need to ‘remove’ all the paths such that \( \pi(0) = m \) or \( \pi(n_j) = -m \) for some \( j \in \{1, \ldots, \ell \} \) from the definition of \( \varpi_0^s(M) \) and, even if \( M \) is a Laurent matrix, there is no cancellation like (2.14).

As a consequence of (2.15), we obtain a CLT for biorthogonal ensembles.

**Theorem 2.6.** Let \( \Xi_N \) be a BOE with recurrence matrix \( J^{(N)} \). Suppose that there exists a subsequence \( (N_k)_{k \in \mathbb{N}} \) and a Laurent polynomial \( s(z) \) such that \( J^{(N_k)} \xrightarrow{\mathcal{L}} L(s) \). Then, for any polynomial \( F \in \mathbb{R}(z) \),

\[
\Xi_{N_k}(F) - \mathbb{E}[\Xi_{N_k}(F)] \xrightarrow{\mathcal{L}} \mathcal{N}(0, \|F\|^2_s) \quad \text{as } k \to \infty.
\]

The variance is given by

\[
\|F\|^2_s = \sum_{k=1}^{\infty} k \widehat{F}(s)_k \widehat{F}(s)_{-k}
\]

(2.16)

where

\[
\widehat{F}(s)_k = \frac{1}{2\pi i} \oint F(s(z)) z^{-k} dz.
\]

(2.17)

In formula (2.17) and in the rest of this article, \( \oint \) denotes an integral over the closed contour \( \{|z| = 1\} \).

**Proof.** Section 5.2

Theorem 2.6 first appeared in \( [BD] \) as Corollary 2.2 where several applications were provided. In particular, the authors’ results for OPEs are reviewed in section 3.1. We also provide an extension of theorem 2.6 to rather general test functions for the varying weights \( (1.1) \) when the support of the equilibrium is simply connected; see theorem 3.1. Actually, this extension follows rather easily from the variance estimates derived in \( [Lam] \). For the two-matrix model \( (1.15) \), it was proved in \( [BD] \) that the eigenvalues of \( M \) form a BOE whose recurrence matrix is not symmetric and has \( \deg(V) + 1 \) non-zero main diagonals. Under suitable assumptions on the potential \( V \), this matrix has a right-limit which is Laurent matrix and there is a CLT. In section 3.2, we apply theorem 2.6 to another BOE which consists of square singular values of a product of rectangular Ginibre matrices. In this case, we obtain a family of CLTs which depend on the asymptotic ratios between the dimensions of the matrices, see theorem 3.2.

**Remark 2.2.** If the Laurent matrix \( L(s) \) is self-adjoint, then \( \widehat{F}(s)_{-k} = \overline{\widehat{F}(s)_k} \) and by Devinatz’s formula, Proposition 6.1.10 in \( [Sim05] \),

\[
\|F\|^2 = \frac{1}{8\pi^2} \iint_{[-\pi, \pi]^2} \left| \frac{F(s(e^{i\theta})) - F(s(e^{i\phi}))}{e^{i\theta} - e^{i\phi}} \right|^2 d\theta d\phi.
\]

(2.18)

It provides an expression for the variance \( \|F\|^2 \) directly in terms of the test function \( F \). Moreover, since \( s(e^{i\theta}) \) is real-valued when \( \theta \in [-\pi, \pi] \), this formula makes sense for any function \( F \in C(\mathbb{R}) \).

For BOEs, the right-limit \( L(s) \) need not be self-adjoint, so that in general \( \widehat{F}(s)_{-k} \neq \overline{\widehat{F}(s)_k} \) and it is not evident from formula (2.17) that \( \|F\|_s \geq 0 \). Moreover, since \( s(z) \) is not real-valued on the contour \( \{|z| = 1\} \), it is not clear either how to extend formula (2.17) to test functions which are not analytic.
3 Examples

3.1 Orthogonal Polynomial Ensembles

Let \( (\mu_N) \) be a sequence of Borel measures on \( \mathbb{R} \) satisfying the condition (1.9) for all \( N \in \mathbb{N} \). We consider the determinantal process \( \Xi_N \) with correlation kernel of the form (1.4) where \( P_k^N \) are the OPs with respect to \( \mu_N \). There are two main classes of models, either \( \mu_N \) is a sequence of probability measures with a given support (e.g. the Jacobi ensembles), or the exponential weights \( \mu_N/dx = e^{-NV(x)} \) which corresponds to the Hermitian ensembles (1.1). We will focus on the second class which is a natural generalization of the GUE with a surprisingly rich behavior. However, non-varying measures with compact support have a similar behavior and fit in the context of the discussion below. Orthogonal polynomials satisfy a three term recurrence relation,

\[
xP_n^N(x) = a_n^N P_{n+1}^N(x) + b_n^N P_n^N(x) + a_{n-1}^N P_{n-1}^N(x),
\]

which is of the type (1.14) where

\[
J = \begin{bmatrix}
  b_0^N & a_1^N & 0 & 0 & 0 \\
  a_0^N & b_1^N & a_2^N & 0 & 0 \\
  0 & a_1^N & b_2^N & a_3^N & 0 \\
  0 & 0 & a_2^N & b_3^N & a_4^N \\
  0 & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

is called the Jacobi matrix. Note that \( J \) may depend on the dimension \( N \in \mathbb{N} \), and we may denote \( J = J(\mu_N) \) to emphasize the dependency. We can interpret \( J \) as a symmetric operator acting formally on \( L^2(\mathbb{N}_0) \). In fact, the reference measure \( \mu_N \) is the spectral measure of \( J \) at \( e_0 \). Hence, \( J \) is bounded if and only if the measure \( \mu_N \) is compactly supported and when the support of the measure \( \mu_N \) is unbounded, \( J \) is essentially self-adjoint if and only if the moment problem for \( \mu_N \) is determinate; see [Dei99a, chap. 2]. In the framework of this paper, the definiteness of \( J \) is not relevant since we view of \( J \) as the adjacency matrix of a labelled graph

\[
\mathcal{G}(J) = \begin{array}{ccccccc}
\bullet & \gamma & \bullet & \gamma & \bullet & \gamma & \bullet \\
0 & 1 & 2 & 3 & 4 & 5
\end{array}
\]

The first fact is that a typical point configuration reaches an equilibrium as \( N \to \infty \). Namely, there exists a probability measure \( \mu_* \) on \( \mathbb{R} \) such that for any \( f \in C \cap L^\infty(\mathbb{R}) \),

\[
N^{-1} \Xi_N(f) \xrightarrow{N \to \infty} \int f(x) d\mu_*(x) dx.
\]

The convergence holds almost surely and \( \mu_* \) is called the equilibrium measure. It is defined as the minimizer of a certain logarithmic energy, it is compactly supported and absolutely continuous. In particular, for the matrix ensembles (1.1), the equilibrium density, denoted \( \rho_V \), exists as long as the potential \( V(x) \) is lower-semicontinuous and satisfies the condition (1.2), see e.g. [Joh98] or [Dei99a, chap. 6]. The law of large numbers (3.4) usually follows from concentration estimates, see e.g. [BD14, Har15] in the context of this paper or [AGZ10] for general results. In particular, the lattice path method developed in section 5.1 is inspired from [Har15] where it was used to estimate the variance of polynomial linear statistics and show that the zeroes of \( P_N^N \) are also distributed according to \( \mu_* \) as \( N \to \infty \). Actually, this result implies that if there exists two continuous functions \( b : \mathbb{R}_+ \to \mathbb{R} \) and \( a : \mathbb{R}_+ \to [0, \infty) \) such that

\[
a_n^N \to a(t) \quad \text{and} \quad b_n^N \to b(t)
\]

(3.5)
as \( n/N \to t \), then the equilibrium density is given by

\[
\rho_V(x) = \int_0^1 \frac{1}{\sqrt{\gamma_+(t) - x} \sqrt{\gamma_-(t) - x}} \, dt \quad \text{where} \quad \gamma_{\pm}(t) = b(t) \pm 2a(t) .
\]

The assumption (3.5) is not necessary for the existence of the equilibrium density, however it holds for many of the classical examples from the literature both in the continuous and discrete setting, see [KV99]. Moreover, it implies that the recurrence matrix \( J(\mu_N) \) has a right-limit in the sense of definition 2.2 which is a Laurent matrix with symbol \( s(z) = az + b + az^{-1} \) where \( a = a(1) \) and \( b = b(1) \). In this case, theorem 2.6 describes the fluctuations around the equilibrium configuration given by \( \rho_V \). For general OPEs, the fluctuation are expected to remain bounded for large \( N \). However, there is a CLT only if the support of the equilibrium measure is simply connected. Otherwise, generically, the variance is quasi-periodic and there are limits only along some particular subsequences. Moreover, the asymptotic distribution need not be Gaussian, see [Pas06] for a heuristic argument. This oscillatory behavior is explained by the fact that the fluctuations of the number of eigenvalues in the different components of the support do not stabilize as \( N \to \infty \). In [She13, BG], these results have been rigorously established based on the 1/\( N \) expansion of the integral

\[
\mathcal{Z}_N^\beta[V] = \int e^{-\beta \mathcal{H}_V(x_1, \ldots, x_N)/2} dx_1 \cdots dx_N ,
\]

where, if \( V \) is a real-analytic and confining potential, the Hamiltonian is given by

\[
\mathcal{H}_V(x_1, \ldots, x_N) = -\sum_{i \neq j} \log |x_i - x_j| + N \sum_i V(x_i) .
\]

These results go beyond the context of this paper, since there are valid for general \( \beta > 0 \) when there is no determinantal structure. Moreover, the work of Borot and Guionnet, [BG13, BG], goes beyond the CLT and established the existence of the all order asymptotic expansion for the partition function 3.6. Formally, we briefly summarize their results regarding the asymptotic distribution of linear statistics. Let \( n \geq 1 \) and suppose that \( \text{supp}(\rho_V) = \bigcup_{j=1}^n I_j \) where \( I_j \) are closed non-empty intervals. If we let \( \mathcal{E} = (\rho_V(I_1), \ldots, \rho_V(I_n)) \in (0, 1)^n \) be the so-called filling fractions and \( \bar{N} = N \mathcal{E} \), then

\[
\mathbb{E} \left[ e^{\Xi_N(f)} \right] \sim e^{-N \langle f, \rho_V \rangle} e^{-(\beta/2 - 1) \mathcal{P}[f, f] + \frac{\beta}{4} \mathcal{Q}[f, f]} \frac{\partial^2 \mathcal{Q}[\bar{N}] - \mathcal{Q}[\bar{N}] |\tau|}{\partial \mathcal{Q}[\bar{N}] |\tau|} ,
\]

where \( c_{\bar{N}} \in \mathbb{R}^n, u_{\bar{N}}, v_{\bar{N}} \) are linear functionals, \( \mathcal{Q} \) is a quadratic functional, \( \tau \) is a positive definite \( n \times n \) matrix, and

\[
\partial_{\bar{N}}(v | \tau) = \sum_{k \in \mathbb{N}^+ \mathbb{Z}^n} e^{-\frac{1}{2} (k,v) + (v,k)} .
\]

For now on, we suppose that \( \beta = 2 \) which corresponds to the Hermitian matrix models 1.1. This values is special, since on the r.h.s. of 3.7, the linear term vanishes and \( c_{2, V} \equiv 0 \). This asymptotics implies that the random variable \( \Xi_N(f) - N \langle f, \rho_V \rangle \) has a limit and this limit is Gaussian if and only if \( u_{\bar{N}}[f] = 0 \), i.e. the test function \( f \) lies in the kernel of an \( n \)-dimensional linear system. Otherwise, there are limits only along subsequences such that

\[
\{(Nk\epsilon_1), \ldots, (Nk\epsilon_n)\} \to \bar{q} \quad \text{as} \quad k \to \infty ,
\]

where \( \{x\} \) denotes the integer part of \( x > 0 \). Note that, by periodicity, we may replace \( \bar{N} \) by \( \left\{ (N\epsilon_1), \ldots, (N\epsilon_n) \right\} \) in the definition of the Theta function 4.3. Then, the condition 3.9 implies that \( \Xi_N(f) - N \langle f, \rho_V \rangle \) converges in distribution to the sum of two independent random variables. A real-valued Gaussian with mean-zero and variance \( \mathcal{Q}[f, f] \) and a random variable \( \Gamma \) which is the
projection on the vector $u_V[f]$ of a discrete Gaussian random variable with mean $\bar{q}$ and covariance matrix $\tau$. Namely if $Y \in \mathbb{Z}^n$ is a random variable with distribution

$$P[Y = k] = e^{-\frac{1}{2}(k, \tau k)},$$

then $\Gamma = (\bar{q} + Y, u_V[f])$ has the moment generating function

$$\mathbb{E}[e^{\lambda \Gamma}] = \frac{\partial q(u_V[f]|\tau)}{\partial q(0|\tau)} = \frac{\partial_0(\tau q + u_V[f]|\tau)}{\partial_0(\tau q|\tau)}.$$ 

In [Pas00], Pastur already noticed that (3.9) was the natural condition to obtain limits of the Laplace transform of $\Xi_N(f)$ and that the limiting laws depend on the parameter $\bar{q} \in [0, 1]^n$. Like theorem 2.4, his results are based on the hypothesis that the Jacobi matrix $J$ has right-limits. Namely, motivated by the asymptotics of Deift et al, [DK+99] he supposed that the recurrence coefficients satisfy for any $k \in \mathbb{Z}$,

$$a_{N+k}^N \sim R(N\bar{c} + k\bar{\alpha}) \quad b_{N+k}^N \sim S(N\bar{c} + k\bar{\alpha}),$$

where $R : \mathbb{T}^n \to \mathbb{R}$ and $S : \mathbb{T}^n \to \mathbb{R}$ are continuous functions, $\bar{\alpha} \in \mathbb{R}^n$, and $T = \mathbb{R}/\mathbb{Z}$. This hypothesis implies that the Jacobi matrix $J(\mu_N)$ has a right-limit $M_{\bar{q}}$ along the subsequence (3.9). Hence, by theorem 2.4 once centered, the sequence $\Xi_N(f)$ converges in distribution as $k \to \infty$. Generically, the matrix $M_{\bar{q}}$ is quasi-periodic and there is no reason why the cumulants $\omega_0^f(M_{\bar{q}})$ would vanish for all $n \geq 3$. In fact, it would be very interesting to investigate the moment generating function (2.11) in such cases to see if one can recover the description (3.7). In particular, in the case of the quartic potential $V(z) = z^4/4 - tz^2/2$ where the asymptotic of the recurrence coefficients is known explicitly and the right-limits of the Jacobi matrix are two-periodic along its diagonals, see [BI99] Thm 1.1. We leave these computations for future work.

In the one-cut case, i.e. when $\text{supp}(\mu_N) = [b - 2a, b + 2a]$ for some $b \in \mathbb{R}$ and $a > 0$, the situation is simpler, the recurrence coefficients have a limit:

$$a_n^N \to a \quad \text{and} \quad b_n^N \to b \quad \text{as} \quad \frac{n}{N} \to 1.$$ (3.11)

For the exponential weights $\mu_N/dx = e^{-NV(x)}$, it is a consequence of the work of Johansson [Joh98] and for non-varying measure $d\mu_N/dx = w I_I$ where $I$ is an interval and $w > 0$ on $I$, it follows from the celebrated Denisov-Rakhmanov theorem, [Sim11] Thm. 1.4.2. The asymptotics (3.11) implies that $J(\mu_N) \xrightarrow{\tau} L(s)$ as $N \to \infty$ where $L(s)$ is a Laurent matrix with symbol $s(z) = az + b + az^{-1}$. Hence, by theorem 2.6 we obtain a CLT as $N \to \infty$,

$$\Xi_N(F) - \mathbb{E}[\Xi_N(F)] \Rightarrow \mathcal{N}(0, \|F\|^2).$$ (3.12)

For general $\beta > 0$ and $C^2$ test functions, the counterpart of this CLT also appeared in [Joh98] if the potential $V(x)$ is an even degree polynomial. Then, it was generalized to any analytic potential in [KS10]. For certain tridiagonal $\beta$-ensembles, similar CLTs have also been derived using the method of moments and lattice-path counting, see [DE06] [DPT12]. This shows that fluctuations of Hermitian random matrices are universal. By an affine transformation of the potential, we can always suppose that $a = 1/2$ and $b = 0$. Then, according to formulae (2.10 - 2.17), the limiting variance of the linear statistics $\Xi_N(F)$ is given by

$$\Sigma(F)^2 = \frac{1}{4} \sum_{k=1}^{\infty} k|c_k(F)|^2$$ (3.13)

where, if $T_k$ denote the Chebychev polynomials of the first kind ($T_k(\cos \theta) = \cos k\theta$), then

$$c_k(F) = \frac{2}{\pi} \int_{-1}^{1} F(x)T_k(x) \frac{dx}{\sqrt{1-x^2}}.$$ (3.14)
are the Fourier-Chebychev coefficients of the function $F$. A more explicit formula for the variance of $\Sigma(F)^2$ can be deduced from Devintz's formula (2.18). We have

\[
\Sigma(F)^2 = \int \left| F(\cos \theta) - F(\cos \phi) \right|^2 \frac{1}{e^{\theta} - e^{\phi}} \, d\theta d\phi ,
\]  

(3.15)

and by symmetry,

\[
\Sigma(F)^2 = \int \left| F(\cos \theta) - F(\cos \phi) \right|^2 \left\{ \frac{1}{e^{\theta} - e^{\phi}} + \frac{1}{e^{\theta} - e^{-\phi}} \right\} d\theta d\phi .
\]

Making the change of variables $x = \cos \theta$ and $y = \cos \phi$, we obtain

\[
\frac{1}{\left| e^{\theta} - e^{\phi} \right|^2} + \frac{1}{\left| e^{\theta} - e^{-\phi} \right|^2} = \frac{1}{(x - y)^2 + (\sqrt{1 - x^2} - \sqrt{1 - y^2})^2} + \frac{1}{(x - y)^2 + (\sqrt{1 - x^2} + \sqrt{1 - y^2})^2} = \frac{4 - 4xy}{4(x - y)^2}
\]

after some elementary but non-obvious simplifications. This implies that

\[
\Sigma(F)^2 = \int \left| F(x) - F(y) \right|^2 \frac{1 - xy}{\sqrt{1 - x^2} \sqrt{1 - y^2}} \, dx dy
\]

(3.16)

which is a weighted $H^{1/2}$ Sobolev semi-norm. Optimally, the CLT (3.12) should hold for any function $f : \mathbb{R} \to \mathbb{R}$ such that $\Sigma(f) < \infty$ and has some reasonable growth at $\infty$. However, because one needs to control the fluctuations the edges of the spectrum, it is not known if this holds even for the GUE. In [BD], when $\mu_N$ have a fixed compact support, the authors discussed the extension of (3.12) to test function $f \in C^1(\mathbb{R})$ with polynomial growth. In [Lam], some general variance estimates for OPs have been derived in the one-cut regime. Namely, if the measures $\mu_N$ have densities $w_N$ and the OPs satisfy a Plancherel-Rotach asymptotics,

\[
P_{N-\delta}^N(x) = \sqrt{\frac{2}{\pi}} \frac{\cos \left( N\pi F(x) + \psi_\delta(x) \right)}{\sqrt{w_N(x)(1 - x^2)^{1/4}}} + o(1)
\]

(3.17)

uniformly for all $x$ in compact subset of $[-1, 1]$ for $\delta = 0, 1$. Suppose also that $\psi_0, \psi_1 \in C^1$ and $F' = \rho_\nu$ on $[-1, 1]$. Then, for any $f \in C^1(\mathbb{R})$ such that there exists $Q, k > 0$ and $|f'(x)| \leq Q|x|^k$ for all $|x| \geq 1$, we obtain

\[
\lim_{N \to \infty} \text{Var} \left[ \Xi_N(f) \right] \leq \frac{4}{\pi^2} \int \left| f(x) - f(y) \right|^2 \frac{dx dy}{\sqrt{1 - x^2} \sqrt{1 - y^2}} .
\]

(3.18)

In fact, the estimate (3.18) is valid under more general assumptions but we prefer to keep a simple formulation. For the weight $w_N(x) = e^{-NV(x)}$, if supp $\rho_\nu = [-1, 1]$, the asymptotics (3.17) has been derived in [DK+99] using the Riemann-Hilbert steepest descent method. For fixed measures, $d\mu_N / dx = w_{-1[-1, 1]}$ where $w > 0$ and satisfies mild regularity condition on $[-1, 1]$, the Plancherel-Rotach asymptotics is classical and $F(x) = \arccos(x)$, see [Sze13, Thm. 12.1.4]. In fact, in the later case, a similar estimate can be obtained without using the precise asymptotics (3.17). see [BD] formula (5.7). As discussed in [Lam] appendix A, this allows us to extend (3.12) to $C^1$ test functions.

**Theorem 3.1.** Let $V : \mathbb{R} \to \mathbb{R}$ be a real analytic function which satisfies the condition (1.2) and such that the equilibrium measure satisfies supp $\rho_V = [-1, 1]$. If $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of a random matrix distributed according to $P_{N, V}$, then for any $f \in C^1(\mathbb{R})$ such that there exists $Q, k > 0$ and $|f'(x)| \leq Q|x|^k$ for all $|x| \geq 1$, we have

\[
\sum_{k=1}^N f(\lambda_k) - \mathbb{E} \left[ \sum_{k=1}^N f(\lambda_k) \right] \xrightarrow{N \to \infty} \mathcal{N}(0, \Sigma(f)^2) .
\]

(3.19)
Theorem 3.1 have the following interpretation. Viewing \( \Xi_N = \sum_{k=1}^{\infty} \delta_{\lambda_k} \) as a random measure on \( \mathbb{R} \), once centered, it converges in distribution to a Gaussian process \( \Phi \) supported on the interval \([-1, 1]\) with covariance

\[
\mathbb{E}[\Phi(f)\Phi(g)] = \frac{1}{4} \sum_{k=1}^{\infty} k c_k(f)c_k(g) . \tag{3.20}
\]

The proof of theorem 3.1 and the estimate (3.18) are given in appendix A, we just sketch the main arguments. First, (3.18) implies that the sequence of random variables \( X_N = \Xi_N(f) - \mathbb{E}[\Xi_N(f)] \) is tight. Then, we approximate the test function \( f \) by a sequence of polynomial \( (F_k)_{k \in \mathbb{N}} \) so that

\[
\sup \{|f'(x) - F'_k(x)| : |x| \leq 1\} \leq 1/k .
\]

By theorem 2.6 since \( \Xi_N(F_k) - \mathbb{E}[\Xi_N(F_k)] \Rightarrow \Phi(F_k) \) as \( N \to \infty \) for any \( k \in \mathbb{N} \), we conclude that the limiting distributions of \( X_N \) as \( N \to \infty \) and \( \Phi(F_k) \) as \( k \to \infty \) coincide and are equal to \( \Phi(f) \).

**Remark 3.1.** The Chebychev polynomials \((T_j)_{j=1}^{\infty}\) are orthonormal with respect to the measure \( dv = \frac{2}{\pi} (1 - x^2)^{-1/2} \delta_{|x|<1} dx \). According to formula (3.13), \( c_k(T_j) = \delta_{k,j} \) and by (3.20), this implies that the random variables \( \left( \frac{\sqrt{\pi}}{2} T_j \right)_{j=1}^{\infty} \) are independent identically distributed \( \mathcal{N}(0, 1) \). In the appendix B we provide an alternative proof which emphasis on the fact that \( \nu \) is the spectral measure of the right limit of \( J(\mu_N) \) if the condition (3.11) holds with the normalization \( a = 1/2 \) and \( b = 0 \).

### 3.2 Product of independent complex Ginibre matrices

Recently, it was established that the square singular values of a product of rectangular complex Ginibre matrices forms a biorthogonal ensemble, [AIK13]. In [KZ14], it was proved that this point process can be interpreted as a multiple orthogonal polynomial ensemble of type II and the authors found explicit formulae for the corresponding polynomials and their recurrence coefficients. Then, they used a double contour integral representation for the correlation kernel to obtain its scaling limit at the hard edge. It describes a new universality class which generalizes the classical Bessel kernels. In this section, we derive the asymptotics of the recurrence coefficients and deduce from theorem 2.6 that the fluctuations of the square singular values process are described by a CLT.

Let \( X_j \) be an \( N_j \times N_j-1 \) random matrix whose entries are independent complex Gaussian with mean 0 and variance 1. We consider the square singular values of the product

\[
W_N = X_m \cdots X_1 . \tag{3.21}
\]

We denote \( N_j = N + \eta_j \) and we suppose that \( \eta_0 = 0 \) and \( \eta_1, \ldots, \eta_m \geq 0 \), so that \( W_N \) is an \( N \times N \) random matrix and, almost surely, all its eigenvalues are positive. In [AIK13], it was proved that the square singular values \( \mu_1, \ldots, \mu_N \) of the matrix (3.21) have a j.p.d.f. on \( (\mathbb{R}^+)^N \) which is given by

\[
\varrho_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det_{N \times N} [x_i^{-1}] \det_{N \times N} [w_{j-1}(x_i)] , \tag{3.22}
\]

where, for any \( n \in \mathbb{N}_0 \),

\[
w_n(x) = \frac{1}{2\pi i} \int_{c+iy} \Gamma(s + \eta_1 + n) \prod_{l=2}^{m} \Gamma(s + \eta_l) x^{-s} ds .
\]

The contour of integration is \( L = \{c+iy, y \in \mathbb{R}\} \) with \( c > 0 \) and \( w_0, w_1, \ldots \) are Meijer G-functions, although we will not need this fact in the following. Akemann et al. used the determinantal structure to investigate the one-point correlation function and its moments. They proved that, in a certain scaling limit, it converges to a compactly supported density as \( N \to \infty \). The largest square singular
value behaves asymptotically like \( \mu_N \sim \gamma \prod_{j=1}^{m} N_j \) where \( \gamma > 0 \) and we consider the rescaled point process

\[
\Xi_N = \sum_{k=1}^{N} \delta_{\mu_k/M(N)} \quad \text{where} \quad M(N) = \prod_{j=1}^{m} N_j .
\] (3.23)

The results of [AIK13] suggest that the process \( \Xi_N \) should satisfy a law of large number like (3.4).

The next theorem describes the fluctuations around this equilibrium configuration.

**Theorem 3.2.** If there exists \( \theta_1, \ldots, \theta_m \in [0, 1] \) such that \( N_j/N \to 1/\theta_j \) as \( N \to \infty \), then for any polynomial \( F \in \mathbb{R}[x] \), we have

\[
\Xi_N(F) - \mathbb{E}[\Xi_N(F)] \Rightarrow \mathcal{N}\left(0, \sum_{k=1}^{\infty} kC^{(\theta)}_k(F)C^{(\theta')}_{-k}(F)\right)
\] (3.24)

where the coefficients are given by

\[
C^{(\theta)}_k(F) = \frac{1}{2\pi i} \oint F(z^m \prod_{l=0}^{m-1} (z^{-1} + \theta_l)) z^{-k} \frac{dz}{z} .
\]

**Proof.** This CLT is a direct of proposition 3.4 below and theorem 2.6. This provides a new class of CLTs for global smooth statistics of random matrices which generalizes theorem 3.1. Indeed, if \( m = 1 \), \( W_N^2 W_N \) is a Wishart matrix and, if \( \theta_1 = 1 \), the eigenvalues process \( \Xi_N \) is described by the Laguerre ensemble which fits in the framework of section 3.1. Although, as we pointed out in remark 2.2, it is not obvious how one would generalize the CLT (3.2) to test functions which are not analytic and it is a problem that we do not address here. It would be interesting to see whether theorem 3.2 is generic for multiple orthogonal polynomials which satisfy an \( m \)-term recurrence relation. The rest of this section gives an account on the biorphogonal structure of the ensemble (3.22), the corresponding recurrence coefficients and their asymptotics. The result are summarized in proposition 3.4 below. We follow [KZ14], sections 3 and 4, although our definitions are slightly different and we derive the asymptotics of the recurrence coefficients in a more general regime.

Given polynomial \( q_0, q_1, \ldots \) such that \( \deg q_n = n \), define

\[
Q_n(x) = \frac{1}{2\pi i} \int_L q_n(s) \prod_{l=1}^{m} \Gamma(s + \eta_l) x^{-s} ds .
\] (3.25)

By performing linear combinations in the columns of the determinants of formula (3.22),

\[
q_N(x_1, \ldots, x_N) = \frac{1}{Z_N^{N \times N}} \det_{N \times N} [P_{j-1}(x_i)] \det_{N \times N} [Q_{j-1}(x_i)] ,
\] (3.26)

where \( P_0, P_1, \ldots \) are polynomials such that \( \deg P_n = n \) and \( \{P_n, Q_n\}_{n \geq 0} \) is a biorthogonal family (cf. definition 1.1). Then, it turns out that \( Z_N^{N \times N} = N! \) and the point process \( (\mu_1, \ldots, \mu_N) \) is determinantal with correlation kernel \( K_N(x, y) = \sum_{n=0}^{N-1} P_n(x)Q_n(y) \) on \( L^2(\mathbb{R}_+) \), cf. formula (4.1). A natural choice is given by \( q_0 = 1 \) and for any \( n \geq 1 \),

\[
q_n(s) = (s-1) \cdots (s-n) .
\] (3.27)

Then, the functions \( Q_0, Q_1, \ldots \) satisfy the Rodriguez’s type formula:

\[
Q_n(x) = \left( -\frac{d}{dx} \right)^n \{ x^n Q_0(x) \} .
\]
Moreover, this also provides an integral formula for the Polynomial $P_n$. For any $x > 0$,

$$P_n(x) = \frac{1}{2\pi i} \oint_{\Sigma} \frac{\Gamma(z - n)}{\prod_{l=0}^{m} \Gamma(z + \nu_l + 1)} x^z dz . \quad (3.28)$$

The contour $\Sigma$ goes around the poles of the function $\Gamma(z - n)$ at $0, 1, \cdots, n$, so that by computing the residues, we see that $P_n$ is indeed a polynomial of degree $n$. The fact that $\{P_n, Q_n\}_{n \geq 0}$ is a biorthogonal family is checked below, see proposition 3.6. Since $P_0, \ldots, P_{n+1}$ is a basis of $\mathbb{R}_{n+1}(x)$, we have

$$xP_n(x) = \alpha_{n,1}P_{n+1}(x) + \sum_{k=0}^{n} \alpha_{n,-k}P_{n-k}(x). \quad (3.29)$$

where

$$\alpha_{n,-k} = \int_0^{\infty} xP_n(x)Q_{n-k}(x) dx .$$

Another result of [KZ14] is the following explicit formula

**Proposition 3.3.** Let $n \geq m$. For any $k > m$, we have $\alpha_{n,-k} = 0$ and if $-1 \leq k \leq m$,

$$\alpha_{n,-k} = \frac{1}{(k+1)!} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \prod_{l=0}^{m} (n + \eta_l - i + 1) . \quad (3.30)$$

**Proof.** Using the Mellin-Barnes inversion formula, by formula (3.25), when $\Re z < -1$,

$$\int_0^{\infty} x^zQ_n(x)dx = q_n(z+1)\prod_{l=1}^{m} \Gamma(z + \nu_l + 1) . \quad (3.31)$$

Using formula (3.27) and the functional equation of the $\Gamma$-function, since $\eta_0 = 0$, it implies that

$$\int_0^{\infty} x^{z+1}Q_n(x)dx = q_{n-1}(z+1)\prod_{l=0}^{m} (z + \nu_l + 1) \prod_{l=1}^{m} \Gamma(z + \nu_l + 1) ,$$

and, by formula (3.28), we obtain for any $k \geq -1$,

$$\int_0^{\infty} xP_{n+k}(x)Q_n(x)dx = \frac{1}{2\pi i} \oint_{\Sigma} \prod_{l=0}^{m} (z + \nu_l + 1) \frac{\Gamma(z - n - k)q_{n-1}(z+1)}{\Gamma(z+1)} dz$$

$$\alpha_{n+k,-k} = \frac{1}{2\pi i} \oint_{\Sigma} \prod_{l=0}^{m} (z + \nu_l + 1) \frac{1}{(z+1)\cdots(z+n-k)} dz . \quad (3.32)$$

If $k > m$, the integrand is $O_{z \to \infty}(z^{-2})$ and all its poles lie inside $\Sigma$. Hence, we can move the contour to $\infty$ and the integral vanishes. Otherwise, by the residue theorem,

$$\frac{1}{2\pi i} \oint_{\Sigma} \prod_{l=0}^{m} (z + \nu_l + 1) \frac{1}{(z-n+1)\cdots(z-n-k)} dz = \sum_{j=0}^{k+1} \prod_{l=0}^{m} (n+j+\nu_l) \prod_{i=0}^{k+1} i^{-j-i}$$

$$= \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{j!(k+1-j)!} \prod_{l=0}^{m} (n+j+\nu_l) . \quad (3.33)$$

By formulae (3.32) and (3.33), if $n \geq m$, we conclude that for any $-1 \leq k \leq m$,

$$\alpha_{n,-k} = \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{j!(k+1-j)!} \prod_{l=0}^{m} (n-k+j+\nu_l)$$

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and we see that formula (3.30) holds by the change of variable $i = k + 1 - j$. 

We have seen that $(\mu_1, \ldots, \mu_N)$ is a determinantal process with kernel $K_N(x, y) = \sum_{n=0}^{N-1} P_n(x)Q_n(y)$. It follows that the rescaled process $\Xi_N$, (3.29), has correlation kernel

$$
\tilde{K}_N(x, y) = \sum_{n=0}^{N-1} \tilde{P}_n^N(x)\tilde{Q}_n^N(y),
$$

(3.34)

where $\tilde{P}_n = \kappa_n(N)P_n(x/M(N))/\sqrt{M(N)}$ and $\tilde{Q}_n = \kappa_n(N)^{-1}Q_n(x/M(N))/\sqrt{M(N)}$. According to proposition 3.6 below, $\{\tilde{P}_n^N, \tilde{Q}_n^N\}_{n \geq 0}$ is still a biorthogonal family on $L^2(\mathbb{R}_+)$ and, by formula (3.29), the polynomials $\tilde{P}_0^N, \tilde{P}_1^N, \ldots$ satisfy a recurrence relation of the form (1.14) where the matrix $\mathbf{J}$ is given by

$$
\mathbf{J}_{n, n-k} = \frac{\kappa_n(N)}{\kappa_{n-k}(N)} \alpha_{n-k} \alpha_{n-k}^{-1}(N). 
$$

(3.35)

By proposition 3.3, $\mathbf{J}$ has at most $m+1$ non-zero diagonals and, to obtain a limit theorem, it remains to check whether it has a right-limit. We need to compute the asymptotics of the coefficients $\mathbf{J}_{N, N-k}$ as $N \to \infty$. We will be interested in the regime $N_t/N = 1 + \eta_t/N \to 1/\theta_t$ where $\theta_1, \ldots, \theta_m \in [0, 1]$. First, we simplify formula (3.30) using an elementary combinatorial lemma. For any $-1 \leq k \leq m$, we have

$$
\alpha_{n, k} = \frac{1}{(k+1)!} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \sum_{S \subseteq \{0, \ldots, m\}} (1-i)^{|S|} \prod_{i \in S} N_i.
$$

Since $M(N) = \prod_{i=1}^{m} N_i$, by lemma 3.5 below applied to the polynomials $(1-x)^{|S|}$, we obtain

$$
\frac{\alpha_{n, k}}{M(N)} = \frac{N}{(k+1)!} \sum_{S \subseteq \{0, \ldots, m\}} \prod_{i \in S} \frac{1}{N_i} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (1-i)^{|S|} \prod_{i \in S} \frac{1}{N_i}.
$$

Hence, if we choose $\kappa_n(N) = N^n$, by formula (3.35), we conclude that, if $N_t/N \to 1/\theta_t$ as $N \to \infty$ for all $j = 0, \ldots, m$ (by convention, $\theta_0 = 1$), then for all $-1 \leq k \leq m$,

$$
\mathbf{J}_{N, N-k} = \sum_{S \subseteq \{0, \ldots, m\}} \prod_{i \in S} \theta_i + O_{N \to \infty}(N^{-1}). 
$$

(3.36)

Otherwise, if $k > m$, then $\mathbf{J}_{N, N-k} = 0$ for all $N > 0$. The next proposition summarizes these results.

**Proposition 3.4.** Let $\mu_1, \ldots, \mu_N$ be the square singular values of the matrix (3.24). The rescaled point process (3.29) is a biorthogonal ensemble with correlation kernel (3.34) where $\tilde{P}_0^N, \tilde{P}_1^N, \ldots$ are polynomials which satisfies a recurrence relation of the form (1.14) and the matrix $\mathbf{J}$ has a right-limit $\mathbf{L}$ which is a Laurent matrix with symbol $s(z) = z^m \prod_{i=0}^{m} (z^{-1} + \theta_i)$.

**Proof.** By formula (3.36), the recurrence matrix $\mathbf{J}$ has a right-limit which is given by

$$
\mathbf{L}_{i, j} = \lim_{N \to \infty} \mathbf{J}_{N+i, N+j} = \sum_{S \subseteq \{0, \ldots, m\}} \prod_{i \in S} \theta_i,
$$

where

$$
\mathbf{J}_{n, n-k} = \frac{\kappa_n(N)}{\kappa_{n-k}(N)} \alpha_{n-k} \alpha_{n-k}^{-1}(N).
$$
If \(-1 \leq i - j \leq m\) and 0 otherwise. Hence, according to definition 2.5, \(L\) is a Laurent matrix with symbol

\[
s(z) = \sum_{k=1}^{m} z^k \sum_{S \subseteq \{0,\ldots,m\}} \prod_{l \in S} \theta_l
= z^m \sum_{S \subseteq \{0,\ldots,m\}} z^{-m+|S|} \prod_{l \in S} \theta_l
= z^{-m} \prod_{l=0}^{m} (z^{-1} + \theta_l)
\]

**Lemma 3.5.** For any \(n \in \mathbb{N}_0\) and for any polynomial \(R(x)\) of degree \(\leq n\),

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} R(k) = R^{(n)}(0) .
\]

**Proof.** Let \(0 \leq r \leq n\), if we differentiate the binomial identity \(r\) times, we obtain

\[
\binom{n}{r} (1 + x)^{n-r} = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{r} x^{k-r} ,
\]

and, if we evaluate at \(x = -1\), this implies that

\[
\sum_{k=0}^{n} (-1)^{k-r} \binom{n}{k} \binom{k}{r} = \delta_{r,n} . \tag{3.37}
\]

Since \(\left\{ \binom{x}{r} \right\}_{r=0}^{n}\) is a basis of \(\mathbb{R}_n[x]\) and \(R(x) = R^{(n)}(0) \binom{x}{n} + \cdots\), the lemma follows from equation (3.37). \(\square\)

**Proposition 3.6.** Let \(Q_0, Q_1, \ldots\) be the functions defined by (3.25) and (3.27) and \(P_0, P_1, \ldots\) be the polynomials defined by (3.28). Then, \(\{P_n, Q_n\}_{n \geq 0}\) is a biorthogonal family on \(L^2(\mathbb{R}_+)\).

**Proof.** We need to check that for all \(k, n \in \mathbb{N}_0\),

\[
\int_{0}^{\infty} P_k(x) Q_n(x) dx = \delta_{k,n} . \tag{3.38}
\]

Combining formula (3.28) with the Mellin-Barnes inversion formula (3.31), we have

\[
\int_{0}^{\infty} P_k(x) Q_n(x) dx = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Gamma(z - k)}{\prod_{m=0}^{m} \Gamma(z + in + 1)} \int_{0}^{\infty} x^z Q_n(x) dz
= \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Gamma(z - k) q_n(z + 1)}{\Gamma(z + 1)} dz .
\]

By definition of the polynomials \(q_n\), (3.27), this implies that

\[
\int_{0}^{\infty} P_k(x) Q_n(x) dx = \frac{1}{2\pi i} \oint_{\Gamma} \frac{q_n(z + 1)}{q_{k+1}(z + 1)} dt . \tag{3.39}
\]

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First, if \( n > k \), the integrand is an entire function and, by Cauchy’s theorem, the integral \( (3.39) \) vanishes. Second, if \( k < n \), the integrand is \( O(z^{-2}) \) and all its poles lie inside \( \Sigma \). Hence, we can move the contour to infinity and the integral \( (3.39) \) vanishes. Finally, if \( k = n \), we obtain

\[
\int_0^\infty P_n(x)Q_n(x) \, dx = \int_{\Sigma} \frac{dt}{t-n} = 1 ,
\]

and it completes the proof of formula \( (3.38) \).

4 Discussion

In this section, we provide some background on biorthogonal ensembles. In particular, we derive formula \( (1.12) \) which is the starting point of our cumulants analysis. Then, we present Soshnikov’s results on eigenvalues of Haar-distributed unitary random matrices and we discuss the connection between his result, our results and the Strong Szegő theorem.

4.1 The cumulants method

Let \( \Xi_N \) be a BOE introduced in definition \( (1.1) \). By formula \( (1.6) \), the joint density of the process is given by

\[
\rho_N(x_1, \ldots, x_N) = \frac{1}{N!} \det_{N \times N} [K(x_i, x_j)] = \frac{1}{N!} \det_{N \times N} [P^N_j(x_i)] \det_{N \times N} [Q^N_j(x_i)] , \tag{4.1}
\]

where the index \( i = 1, \ldots, N \) and \( j = 0, \ldots, N - 1 \). Hence, for any \( g \in L^\infty(\mu_N) \), we have

\[
\mathbb{E} \left[ \prod_{k=1}^N g(\xi_k) \right] = \int \rho_N(x_1, \ldots, x_N) \prod_{k=1}^N g(x_k) \, d\mu_N(x_k) = \frac{1}{N!} \int \det_{N \times N} [g(x_i)P^N_j(x_i)] \det_{N \times N} [Q^N_j(x_i)] \prod_{k=1}^N d\mu_N(x_k) ,
\]

and by Andréief’s formula,

\[
\mathbb{E} \left[ \prod_{k=1}^N g(\xi_k) \right] = \det_{N \times N} \left[ \int g(x)P^N_j(x)Q^N_i(x) \, d\mu_N(x) \right] .
\]

Formally, if we take \( g(x) = e^{f(x)} \) for some test function \( f : \mathbb{R} \to \mathbb{C} \) we obtain

\[
\mathbb{E} \left[ e^{\Xi_N(f)} \right] = \det_{N \times N} \left[ \int e^{f(x)}P^N_j(x)Q^N_i(x) \, d\mu_N(x) \right] . \tag{4.2}
\]

On the other hand, the recurrence relation \( (1.14) \) implies that for any \( n \in \mathbb{N} \) and \( j \geq 0 \),

\[
x^n P^N_j(x) = \sum_{i \geq 0} J^n_{ij} P^N_i(x) .
\]

Note that, since we assume that the matrix \( J \) is a band matrix, the previous sum has only finitely many terms. Besides, when \( J \) is bounded, for any real-analytic function \( g \), we obtain

\[
g(x)P^N_j(x) = \sum_{i \geq 0} g(J)_{ij} P^N_i(x) .
\]
In particular, because of the biorthogonal structure \([1.7]\),
\[
g(J)_{ij} = \int g(x) P_j^N(x) Q_i^N(x) d\mu_N(x) . \tag{4.3}
\]
Combining formulae \([4.2]\) and \([4.3]\), we obtain
\[
E \left[ e^{\Xi_N(f)} \right] = \det_{N \times N} \left[ \exp f(J) \right] , \tag{4.4}
\]
where we take the determinant of the principal \( N \times N \) minor of the infinite matrix \( \exp f(J) \). In order to investigate the asymptotics of formula \([4.4]\) as \( N \to \infty \), it is convenient rewrite the determinant over \( l^2(\mathbb{N}) \). If \( P_N \) denotes the projection onto \( \text{span}(e_1, \ldots, e_N) \), we have
\[
E \left[ e^{\Xi_N(f)} \right] = \det \left[ 1 - P_N + P_N \exp (f(J)) P_N \right] . \tag{4.5}
\]
Note that it yields formula \([1.12]\). In fact, if the operator \( J \) is bounded and the test function \( f \) is analytic, then the r.h.s. of \([4.5]\) is the Fredholm determinant of a finite rank operator acting on \( l^2(\mathbb{N}) \). This determinant is unitary equivalent to
\[
E \left[ e^{\Xi_N(f)} \right] = \det \left[ 1 + K_N \left( \exp f - 1 \right) \right]_{L^2(\mu_N)} , \tag{4.6}
\]
where \( K_N \) is the integral operator acting on \( L^2(\mu_N) \) with kernel \( K_N, \([1.8]\) \), and \( e^f - 1 \) is interpreted as a multiplier. Formula \([4.6]\) is well-known, see e.g. \([Joh06]\) formula (2.33) for another derivation.

Using formula \((5.12)\) in \([Sim05]\), we can express the cumulants of the random variable \( \Xi_N(f) \) in terms of the Jacobi matrix \( J \). Namely, if \( \lambda \in \mathbb{C} \),
\[
\log \det \left[ 1 + P_N \left( \exp \lambda f(J) - 1 \right) P_N \right] = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \text{Tr} \left[ (P_N (\exp \lambda f(J) - 1) P_N)^\ell \right] ,
\]
and, if we expand the exponential, we obtain
\[
\log E \left[ e^{\lambda \Xi_N(f)} \right] \sim \sum_{n=1}^{\infty} C_N^n[f] \frac{\lambda^n}{n!} , \tag{4.7}
\]
where
\[
C_N^n[f] = \sum_{\ell=1}^{n} \frac{(-1)^{\ell+1}}{\ell} \sum_{k_1 + \cdots + k_\ell = n} \frac{n!}{k_1! \cdots k_\ell!} \text{Tr} \left[ P_N f(J)^{k_1} P_N \cdots P_N f(J)^{k_\ell} P_N \right] . \tag{4.8}
\]
If the Jacobi matrix \( J \) is unbounded, when the test function \( f \) is a polynomial, it is not a priori plain that the series \([4.7]\) is convergent. However, since \( J \) is a band matrix, the cumulants \( C_N^n[f] \) are finite for all \( n, N \in \mathbb{N} \). In section \( 5.1 \) we provide a combinatorial interpretation of the traces on the r.h.s. of \([4.8]\) and by keeping track of cancellations, we can pass to the limit as \( N \to \infty \). The use of cumulants to investigate fluctuations of linear statistics of large random matrices goes back to Costin and Lebowitz. \([CL95]\), and has been subsequently developed by many authors. Much in the spirit of this article is the work of Soshnikov on Haar distributed random matrices from the Unitary group \( U_N \) which is presented in the next section. The combinatorial approach developed in \([RV07]\) for polynomial linear statistics of the Ginibre ensemble is also related. Namely, for a rotationally invariant measure on the plane, it is also possible to exploit the recursive structure of the OPs \((\gamma_k z^k)_{k=0}^{\infty}\) to compute the cumulants of any linear statistic of the form \( \sum c_k z^{\alpha_k} z^{\beta_k} \) where \( \alpha_k, \beta_k \in \mathbb{N}_0 \).
4.2 The Strong Szegő theorem

The circular unitary ensemble is a complex biorthogonal ensemble where \( P_j(z) = \overline{Q_j(z)} = z^j \) and \( \mu_N = \nu \) is the uniform probability measure on the unit circle \( \mathbb{T} \). Hence, if \( \mathcal{F} \) denotes the Fourier transform on \( \mathbb{T} \), we obtain

\[
\int e^{\lambda f(z)} P_j(z)Q_i(z) d\nu(z) = \mathcal{F}(e^{\lambda f})_{i-j}.
\]

By formula (4.12), this implies that the Laplace transform of a linear statistic \( \Xi_{d\nu}(f) = \sum_{k=1}^{N} f(\theta_k) \) is given by a Toeplitz determinant with symbol \( e^{\lambda f} \):

\[
E\left[e^{\lambda \Xi_{d\nu}(f)}\right] = D_N[e^{\lambda f}] .
\] (4.9)

If \( \|f\|^2_{H^{1/2}(\mathbb{T})} = \sum_{n \in \mathbb{Z}} |n| |\hat{f}_n|^2 < \infty \) where \( \hat{f} = \mathcal{F}(f) \), then it follows from the the Strong Szegő that

\[
\lim_{N \to \infty} \log D_N[e^{\lambda f}] - \lambda N \int f(z) d\nu(z) = \frac{1}{2} \lambda^2 \|f\|^2_{H^{1/2}(\mathbb{T})} .
\] (4.10)

According to (4.9), this has the following probabilistic interpretation,

\[
\Xi_{d\nu}(f) - E[\Xi_{d\nu}(f)] \Rightarrow \mathcal{N}(0,\|f\|^2_{H^{1/2}(\mathbb{T})}) \quad \text{as } N \to \infty .
\] (4.11)

We refer to Simon’s book [Sim04] for five different proofs of the Strong Szegő, some historical background and interesting motivations. There is yet at least two other proofs. The first by Deift using a Riemann-Hilbert formulation to compute the resolvent of the CUE correlation kernel, [Dei99b]. The second by Soshnikov which is based on the cumulant expansion, [Sos00a]. In fact, this proof does not rely on the fact that the Laplace transform of \( \Xi_{d\nu}(f) \) is given explicitly by a Toeplitz determinant but use instead formula (4.7), translation invariance of the CUE kernel and Fourier analysis. In fact, Lemma 1 in [Sos00a] implies that

\[
\lim_{N \to \infty} C^n_{d\nu} [f] = \sum_{\omega_1 + \cdots + \omega_n = 0, \omega_i \in \mathbb{Z}} \hat{f}(\omega_1) \cdots \hat{f}(\omega_n) G_n(\omega_1,\ldots,\omega_n) ,
\] (4.12)

where the function \( G_n \) is given by (2.14) and the limit holds for any function \( f \in H^{1/2}(\mathbb{T}) \) such that

\[
\sum_{\omega \in \mathbb{Z}} |\hat{f}(\omega)| < \infty .
\] (4.13)

We check that according to definition (2.9), if \( f(z) \) is a Laurent polynomial, the r.h.s. of formula (4.12) turns out to be equal \( \wp_0^0(\mathcal{L}(f)) \) where \( \mathcal{L}(f) \) denotes the Laurent matrix with symbol \( f(z) \). The difficult part to obtain the CLT (4.11) is to prove the Main combinatorial Lemma.

Lemma 4.1 (MCL, [Sos00a]). For any \( x \in \mathbb{R}^n \) such that \( x_1 + \cdots + x_n = 0 \), we have

\[
\sum_{\sigma \in S_n} G_n(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \begin{cases} |x_1| & \text{if } n = 2 \\ 0 & \text{else} \end{cases} .
\]

For completeness, a proof of the MCL inspired from [Sos00a] is given in the appendix [A]. We also discuss the connection with the seminal work of Kac, [Kac54], on the Strong Szegő theorem.

There is an explicit formula in terms of a Fredholm determinant for the error term in the Strong Szegő theorem, (4.10), which is usually called the Borodin-Okounkov formula. Namely, if \( \hat{f}_0 = 0 \), we have

\[
D_N[e^{\lambda f}] = Z(f) \det [Q_N(1 - H(f))Q_N]_{\ell^2(\mathbb{N}_0)} ,
\] (4.14)

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where \( Z(f) = \exp\left( \sum_{k=1}^{\infty} k f_k \hat{f}_{-k} \right) \), \( H(f) \) is a trace class operator acting \( l^2(N_0) \) and \( Q_N \) denotes the projection onto \( \text{span}\{e_N, e_{N+1}, \ldots\} \). The identity \( (4.14) \) appeared in [BO00] with a remarkable proof based on the theory of Schur processes developed by the authors. Although it was proved many years earlier in a less explicit form in [GC79]. We refer to [Sim05] section 6.2 for different proofs and historical references. A different approach to the asymptotics of Toeplitz determinant using operator theoretic technics was introduced by Widom in [Wid73]. Immediately after the work of Borodin and Okounkov, it was later realized that Widom’s method also allows to prove formula \( (4.14) \) and certain generalizations when \( f \notin H^{1/2}(T) \); see [BW00] or [Bas05] for a rather elementary introduction and further references. In the context of this article, this method was used in [BD] to analyze the biorthogonal ensembles which satisfy a recurrence relation of the form \( (1.14) \) for some band matrix \( J \). The authors where the first to realize that the existence of a right-limit for \( J \) implies a limit theorem for polynomial statistics. In particular, they already obtained theorem \( 2.6 \) and a weaker version of theorem \( 2.4 \) One of their main tool is the following asymptotic formula for certain Fredholm determinant. Lemma 3.1 in [BD] states that for any trace-class matrix \( A \) acting on \( l^2(N_0) \) and for any Laurent polynomial \( s \), if \( T(s) \) is the Toeplitz matrix with entries \( T(s)_{ij} = \hat{s}_{i-j} \) for all \( i, j \in N_0 \), then

\[
\lim_{N \to \infty} \det\left[ 1 + P_N(e^{T(s)+A} - 1)P_N \right] e^{-\sum_{i,j} \langle P_N(T(s)+A) \rangle} = Z(s). \tag{4.15}
\]

The consequence of this asymptotics are well-illustrated by looking at the Chebyshev process of the second kind. Namely, we consider the OPE, \( \Xi_N \), with respect to the reference measure \( d\mu/dx = \frac{1}{\pi} \sqrt{1-x^2} 1_{|x|<1} \). The Jacobi matrix \( J \) of the measure \( d\mu \) is the Toeplitz matrix with symbol \( s(z) = \frac{z+1/z}{2} \):

\[
J = \begin{pmatrix}
0 & 1/2 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 \\
& & & \ddots
\end{pmatrix}.
\tag{4.16}
\]

It is easy to check that, for any polynomials \( F \in \mathbb{R}(x) \), \( F(J) = T(F(s)) + A \) where \( A \) is a finite rank matrix. Hence, since \( \mathbb{E}[\Xi_N(F)] = \text{Tr}[P_N F(J)] \), according to formula \( (4.15) \),

\[
\mathbb{E}\left[ e^{\Xi_N(F)} - \mathbb{E}[\Xi_N(F)] \right] = \det\left[ 1 + P_N(e^{T(s)+A} - 1)P_N \right] e^{-\sum_{i,j} \langle P_N(T(s)+A) \rangle}
\]

and, by \( (4.15) \), we obtain

\[
\lim_{N \to \infty} \mathbb{E}\left[ e^{\Xi_N(F)} - \mathbb{E}[\Xi_N(F)] \right] = Z(F(s)).
\]

Since, by definition, \( Z(\lambda F(s)) = \lambda^2 \| F \|^2 \) for any \( \lambda \in \mathbb{R} \), we get a special case of the CLT of theorem \( 2.6 \). The general case follows from a universality principle. Breuer and Duits showed that, if two recurrence matrices have the same right-limit, then for a given polynomial \( f \), the cumulants \( (4.8) \) also have the same limits. Hence, if \( J^{(N_N)} \xrightarrow{N} L(s) \) where \( L(s) \) is a Laurent matrix, we may replace \( J^{(N)} \) by a Toeplitz matrix \( T(s) \) to compute the limit of the moment generating function of a given polynomial statistic and the central limit theorem \( 2.6 \) follows from \( (4.15) \).

5 Proofs

5.1 Paths formulation and the proof of lemma 2.1

Let \( \Xi_N \) be a BOE with recurrence matrix \( J = J^{(N)} \). In section 4.1, we have proved that the cumulants of a linear statistic \( \Xi_N(F) \), where \( F \) is a polynomial, can be expressed in terms of the
we see that
\[ M \ell \pi = \nu \]
and for any collection \( \Gamma \) of paths on \( \mathcal{G}(M) \), we let
\[ M \{ \Gamma \} = \sum_{\pi \in \Gamma} M \{ \pi \}. \]  
(5.3)

In particular, if we denote by \( M^n_{ij} \) the entries of the matrix \( M^n \), we get
\[ M^n_{ij} = \sum_{\pi \in \Gamma^n_{j \rightarrow i}} M \{ \pi \} = M \{ \Gamma^n_{j \rightarrow i} \}. \]  
(5.4)

This implies that for any \( n \in \mathbb{N} \),
\[ \text{Tr} \left[ P_N M^n P_N \right] = \sum_{j=0}^{N-1} M^n_{jj} = \sum_{j=0}^{N-1} M \{ \Gamma^n_{j \rightarrow j} \}. \]  
(5.5)

Hence, we have an expression for the first cumulant \( C_N^1[F] = \mathbb{E} [ \Xi_N(F) ] \). Then we discuss how to obtain similar formulae for
\[ \text{Tr} \left[ P_N M^k P_N \cdots P_N M^k P_N \right] = \sum_{j=0}^{N-1} \cdots \sum_{j=0}^{N-1} \prod_{i=1}^{\ell} M^k_{j_i+1 j_i}, \]  
(5.6)

where by convention \( \ell + 1 = 1 \). If \( \pi \in \Gamma^{k_1}_{i \rightarrow s} \) and \( \kappa \in \Gamma^{k_2}_{s \rightarrow j} \) are two paths, then we define a new path \( \pi \oplus \kappa \in \Gamma^{k_1+k_2}_{i \rightarrow s} \) which is the concatenation of \( \pi \) and \( \kappa \). Moreover, by definition of the weight \( M \{ \cdot \} \), we see that \( M \{ \pi \oplus \kappa \} = M \{ \pi \} M \{ \kappa \} \). Conversely, any path \( \nu \in \Gamma^{k_1+k_2}_{i \rightarrow s} \) has a unique decomposition \( \nu = \pi \oplus \kappa \) such that \( \pi \in \Gamma^{k_1}_{i \rightarrow s} \) and \( \kappa \in \Gamma^{k_2}_{s \rightarrow j} \). Hence, we can write \( \Gamma^{k_1+k_2}_{i \rightarrow s} = \Gamma^{k_1}_{i \rightarrow s} \oplus \Gamma^{k_2}_{s \rightarrow j} \) and if we iterate, we obtain
\[ \Gamma^{k_1}_{j_1 \rightarrow j_2} \oplus \Gamma^{k_2}_{j_2 \rightarrow j_3} \oplus \cdots \oplus \Gamma^{k_{\ell+1}}_{j_{\ell+1} \rightarrow j_{\ell+1}} = \Gamma^{k_1+k_2+\cdots+k_{\ell}}_{j_1 \rightarrow j_{\ell+1}} \]  
and
\[ M \{ \Gamma^{k_1}_{j_1 \rightarrow j_2} \} M \{ \Gamma^{k_2}_{j_2 \rightarrow j_3} \} \cdots M \{ \Gamma^{k_{\ell}}_{j_{\ell} \rightarrow j_{\ell+1}} \} = M \{ \Gamma^{k_1+k_2+\cdots+k_{\ell}}_{j_1 \rightarrow j_{\ell+1}} \}. \]  

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By formula (5.10), this implies that for any \( k \in \mathbb{N}_0^n \),
\[
\prod_{i=1}^{\ell} M_{j_{i+1}j_i}^k = M\{\Gamma_{j_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_{\ell + 1}}^{k_1+k_2+\cdots+k_\ell}\}
\]
and, by formula (5.6),
\[
\text{Tr} [P_N M^{k_1} P_N \cdots P_N M^{k_\ell} P_N] = \sum_{j_1=0}^{N-1} \cdots \sum_{j_\ell=0}^{N-1} M\{\Gamma_{j_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_{\ell + 1}}^{k_1+k_2+\cdots+k_\ell}\}.
\]

Observe that for any \( j \in \mathbb{N}_0 \),
\[
\sum_{j_2=1}^{N-1} \sum_{j_{\ell+1}=1}^{N-1} \prod_{j=1}^{\ell} \Gamma_{j_{j_2} \rightarrow j_2 \cdots \rightarrow j_{j_{\ell+1}}} = \{\pi \in \Gamma^{n}_1 \rightarrow j_1 : \pi(k_1) < N, \pi(k_1+k_2) N, \cdots, \pi(k_1+\cdots+k_{\ell-1}) < N\}
\]
and, if \( n = k_1 + \cdots + k_\ell \), then
\[
\sum_{j_2=0}^{N-1} \sum_{j_{\ell+1}=0}^{N-1} M\{\Gamma_{j_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_{\ell + 1}}^{k_1+k_2+\cdots+k_\ell}\} = \sum_{\pi \in \Gamma^{n}_1 \rightarrow j_1} M\{\pi\} \mathbb{I}_{\max(\pi(k_1), \cdots, \pi(k_1+\cdots+k_{\ell-1})) < N}
\]
Hence, we can rewrite the r.h.s. of formula (5.7) as a single sum under constraints,
\[
\text{Tr} [P_N M^{k_1} P_N \cdots P_N M^{k_\ell} P_N] = \sum_{j_1=0}^{N-1} \sum_{\pi \in \Gamma^{n}_1 \rightarrow j_1} M\{\pi\} \mathbb{I}_{\max(\pi(k_1), \cdots, \pi(k_1+\cdots+k_{\ell-1})) < N}.
\]

According to formula (5.11), we obtain for any \( n \in \mathbb{N} \),
\[
C^n_N[F] = -\sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell!} \sum_{k_1+\cdots+k_\ell = n} \frac{n!}{k_1! \cdots k_\ell!} \sum_{j_2=0}^{N-1} \sum_{j_{\ell+1}=0}^{N-1} M\{\pi\} \mathbb{I}_{\max(\pi(k_1), \cdots, \pi((k_1+\cdots+k_{\ell-1})) < N}.
\]

Because of the identity
\[
\sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell!} \sum_{k_1+\cdots+k_\ell = n} \frac{n!}{k_1! \cdots k_\ell!} = \mathbb{I}_{n=1},
\]
the contributions of most of the paths in formula (5.9) cancel. Formula (5.10) is classical, see e.g. formula (1.14) in [Sos00a] for a proof or formula (A.13) in the appendix. Combining formulae (5.5) and (5.8), we have for any composition \( n = k_1 + \cdots + k_\ell \),
\[
\text{Tr} [P_N M^{k_1} P_N \cdots P_N M^{k_\ell} P_N] = \text{Tr} [P_N M^{n} P_N] - \sum_{k_0=0}^{N-1} \sum_{\pi \in \Gamma^{n}_1 \rightarrow j_1} M\{\pi\} \mathbb{I}_{\max(\pi(k_1), \cdots, \pi(k_1+\cdots+k_{\ell-1})) \geq N}.
\]

Hence, by (5.10), we obtain for any \( n \geq 2 \),
\[
C^n_N[F] = \sum_{\ell=2}^{n} \frac{(-1)^\ell}{\ell!} \sum_{k_1+\cdots+k_\ell = n} \frac{n!}{k_1! \cdots k_\ell!} \sum_{j_2=0}^{N-1} \sum_{j_{\ell+1}=0}^{N-1} M\{\pi\} \mathbb{I}_{\max(\pi(k_1), \cdots, \pi(k_1+\cdots+k_{\ell-1})) \geq N}.
\]

Making the change of variables \( n = \Psi(k) \) given by (2.22) in the previous formula, we conclude that for any \( n \geq 2 \),
\[
C^n_N[F] = \sum_{n \in \Lambda^n} \varnothing(n) \sum_{j_2=0}^{N-1} \sum_{j_{\ell+1}=0}^{N-1} M\{\pi\} \mathbb{I}_{\max(\pi(n_1), \cdots, \pi(n_\ell)) \geq N}.
\]
This completes the proof of lemma 2.1. We end up this section by proving the estimate (2.5). The asymptotics of formula (5.11) as \( N \to \infty \) is performed in the next section.

**Lemma 5.1.** Suppose that \( L \) is a doubly-infinite matrix which satisfies the condition (2.1) with bandwidth \( w \) and that there exists \( C > 0, \alpha \in \mathbb{R} \) such that \(|L_{ij}| \leq C \exp (\alpha \max \{|k|, |j|\})\) for all \( i, j \in \mathbb{Z} \). Then, for any polynomial \( F \), there exists a constants \( C_F > 0 \) such that for any \( n \geq 2 \),

\[
|\varpi_0^n(F(L))| \leq n! \exp(nC_F).
\]

Moreover, \( C_F \leq \log(2dCAw^2) + \alpha w^d \) if \( F(x) = \sum_{k=0}^d c_k x^k \) and \( A = \max \{|c_k| : k \leq d\} \).

**Proof.** According to formula (2.7), letting \( M = F(L) \), we may rewrite

\[
\varpi_0^n(M) = \sum_{n \in \Lambda_n} \mathcal{U}(n) \sum_{j < 0} \sum_{\pi \in \Gamma_{j+1}^j} M(\pi) I_{\max(\pi(n_1), \ldots, \pi(n_d))} \geq 0.
\]

First observe that, since the graph \( G(L) \) has degree at most \( w^d \), only the paths which start at a vertex \( j \geq -w^{nd}/2 \) contribute to the sum (5.1), otherwise their maximum is non-negative. This implies that

\[
\left| \varpi_0^n(M) \right| \leq \sum_{n \in \Lambda_n} \mathcal{U}(n) \sum_{-w^{nd}/2 < j < 0} \left| \sum_{\pi \in \Gamma_{j+1}^j} M(\pi) \right|.
\]

According to (2.2 - 2.3), for any \( n \in \mathbb{N} \),

\[
\sum_{n \in \Lambda_n} \mathcal{U}(n) = \sum_{n=1}^{n-1} \frac{1}{n+1} \sum_{k_1 + \cdots + k_{n+1} = n} \frac{n!}{k_1! \cdots k_{n+1}!} \leq n!2^n,
\]

since \( 2^n \) is the number of compositions of \( n \). Thus, we get

\[
\left| \varpi_0^n(M) \right| \leq n!2^n \sum_{-w^{nd}/2 < j < 0} \left| M(\Gamma_{j+1}^j) \right|.
\]

(5.12)

Since \( M = F(L) \) with \( F(x) = \sum_{k \leq d} c_k x^k \), formula (5.3) implies that for any \( j \geq 0 \)

\[
M(\Gamma_{j+1}^j) = M_n^{\pi_j} = \sum_{k_1, \ldots, k_n \leq d} c_{k_1} \cdots c_{k_n} \prod_{i=0}^{n-1} L_{ij}^{k_i}.
\]

If we denote by \( |L| \) the matrix with entries \( |L|_{ij} = |L_{ij}| \) and \( A = \max \{|a_k| : k \leq d\} \), by applying the triangle inequality,

\[
\left| \prod_{i=0}^{n-1} c_{k_i} \cdots c_{k_n} \right| \leq A^n |L|_{ij}^{k_1 + \cdots + k_n},
\]

and we obtain

\[
\left| M(\Gamma_{j+1}^j) \right| \leq A^n d^n |L|_{jj}^{nd}.
\]

(5.13)

Since the matrix \( L \) has at most \( w \) non-zero diagonals and \( |L_{ij}| \leq C \exp (\alpha \max \{|i|, |j|\}) \), for any \( q \in \mathbb{N} \) and \(|j| \leq w^{nd} \),

\[
|L|_{jj}^q \leq C^d w^{q-1} \exp (\alpha \max \{|q|, |j|\} w^{nd}/2),
\]

and we deduce from the upper-bound (5.13) that \( \left| M(\Gamma_{j+1}^j) \right| \leq (Ca)^n w^{nd} \exp (\alpha w^{nd}) \). If we combine this estimate and (5.12), we conclude

\[
\left| \varpi_0^n(M) \right| \leq n!(2CAw^{2nd} \exp (\alpha w^{nd})).
\]

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When \( F \mid \mathcal{O} \) observe that by lemma 5.1, if \( F \mid j \mathcal{O} \) holds for an arbitrary polynomial \( j \mathcal{O} \) are given by \[ \varpi^N_{ij}(F(J)) = \pi \begin{cases} J_{N+i,N+j} & \text{if } i,j \geq -N \\ 0 & \text{else} \end{cases} \], then \( \varpi^N_{ij}(F(J)) = \pi \begin{cases} J_{N+i,N+j} & \text{if } i,j \geq -N \\ 0 & \text{else} \end{cases} \). Moreover, if the recurrence matrix \( J \) satisfies (5.1) and the hypothesis (H.1), the matrix \( L \) satisfies the assumptions of lemma 5.1 and this yields the estimate (2.5).

5.2 Gaussian fluctuations: proof of theorem 2.4 and theorem 2.6

We first prove Theorem 2.4 much in the spirit of the proof of theorem 2.3. Then, we will deduce theorem 2.6 using the MCL. First of all, if the recurrence matrix \( J^{(N_k)} \) has a right-limit \( L \) along a subsequence \( N_k \), then for any polynomial \( F \),

\[
\lim_{k \to \infty} F(J^{(N_k)})_{N_k+i,N_k+j} = F(L)_{ij} \quad \forall i,j \in \mathbb{Z}.
\] (5.14)

When \( F(x) = x^q \) for \( q \in \mathbb{N} \), this follows from formula (5.3). Namely, for any \( i,j \geq -N_k \),

\[
J^{q}_{N_k+i,N_k+j} = J\{\Gamma^q_{N_k+i\to N_k+j}\}.
\]

If \( J^{(N_k)} \xrightarrow{k} L \), the graph \( \mathcal{G}(J) \) rooted at \( N_k \) converges locally to the graph of \( L \) rooted at 0 and this implies that for any \( i,j \in \mathbb{Z} \),

\[
\lim_{k \to \infty} J\{\Gamma^q_{N_k+i\to N_k+j}\} = L\{\Gamma^q_{i\to j}\} = L^q_{ij}.
\]

By taking linear combinations, we conclude that (5.14) holds for an arbitrary polynomial \( F \). This means that the right-limit of the matrix \( F(J) \) along the subsequence \( N_k \) is equal to \( F(L) \). By lemma 2.1, \( C^N_F[F] = \varpi^N_F(F(J^{(N_k)})) \) for any \( n \geq 2 \) and by (2.7), we obtain

\[
\lim_{k \to \infty} C^N_{N_k}[F] = \varpi^N_F(F(L)).
\] (5.15)

Observe that by lemma 5.1 if \( |L_{ij}| \leq C \exp \left( \alpha \max \{|i|,|j|\} \right) \), then \( |\varpi^N_F(F(L))| \leq n! \exp(nC_F) \) and by a standard argument, see e.g. lemma 4.8 in [1, 1], there exists a random variable whose cumulants are given by \( \varpi^N_F(F(L)) \) for all \( n \geq 2 \). It implies that, when centered, the linear statistic \( \Xi_{N_k}(F) \) converges in distribution to a random variable \( X(F) \) whose moment generating function is given by

\[
\mathbb{E} \left[e^{\lambda X(F)}\right] = \exp \left( \sum_{n=2}^{\infty} \varpi^N_F(M) \frac{\lambda^n}{n!} \right),
\] (5.16)

where \( M = F(L) \). This completes the proof of theorem 2.4.

By formula (2.6), we know that

\[
\varpi^N_F(M) = \sum_{n \in \Lambda_n} \tilde{O}(n) \sum_{k=1}^{\infty} \sum_{\pi \in \Gamma^{\Lambda_n}_{n-k \to -k}} M\{\pi\} \mathbb{1}_{\max\{\pi(n_1),\ldots,\pi(n_{\ell-1})\} \geq 0}.
\]

By lemma 5.2 below, if \( L = L(s) \) is a Laurent matrix, then \( M = F(L) \) is also a Laurent Matrix with symbol \( F(s) \). In particular, the weighted graph \( \mathcal{G}(M) = (\mathbb{Z}, \tilde{E}) \) is translation invariant and there is a bijection \( \Gamma^\Lambda_{0 \to 0} \mapsto \Gamma^{\Lambda_n}_{n-k \to -k} \) given by \( \pi = \tilde{\pi} - k \) so that \( M\{\pi\} = M\{\tilde{\pi}\} \) and

\[
\max\{\pi(n_1),\ldots,\pi(n_{\ell-1})\} = \max\{\tilde{\pi}(n_1),\ldots,\tilde{\pi}(n_{\ell-1})\} - k.
\]
This implies that
\[
\varpi^n_0(M) = \sum_{n \in \Lambda_n} \mathcal{O}(n) \sum_{\bar{\pi} \in \Gamma_{n \to 0}^0} M(\bar{\pi}) \sum_{k=1}^{\infty} I_{\max\{\bar{\pi}(n_1), \ldots, \bar{\pi}(n_{\ell-1})\} = k}.
\]

We denote \( F(s(z)) = \sum_{k \in \mathbb{Z}} \beta_k z^k \). Since \( s(z) \) is a Laurent polynomial, only finitely many of the coefficients \( \beta_k \neq 0 \) and, by definition, \((j, i) \in E\) if and only if \( \beta_{j-i} \neq 0 \). Hence, given \( \omega \in \mathbb{Z}^n \) such that \( \omega_1 + \cdots + \omega_n = 0 \), if we let
\[
\pi_\omega(k) = \omega_1 + \cdots + \omega_k \quad \forall k = 0, \ldots, n,
\]
then \( \pi_\omega \in \Gamma_{n \to 0}^n \) if and only if \( M(\pi_\omega) = \beta_{\omega_1} \cdots \beta_{\omega_n} \neq 0 \). Making the change of variables \( \bar{\pi} = \pi_\omega \) in formula (5.17), we obtain
\[
\varpi^n_0(M) = \sum_{n \in \Lambda_n} \mathcal{O}(n) \sum_{\omega_1 + \cdots + \omega_n = 0 \atop \omega_i \in \mathbb{Z}} \beta_{\omega_1} \cdots \beta_{\omega_n} \max\{0, \pi_\omega(n_1), \ldots, \pi_\omega(n_{\ell-1})\}.
\]

It remains to observe that by (5.18) and (2.14), for any \( \omega \in \mathbb{Z}^n \), we have
\[
\sum_{n \in \Lambda_n} \mathcal{O}(n) \max\{0, \pi_\omega(n_1), \ldots, \pi_\omega(n_{\ell-1})\} = G_n(\omega_1, \ldots, \omega_n),
\]
so that formula (5.19) implies that
\[
\varpi^n_0(M) = \sum_{\omega_1 + \cdots + \omega_n = 0 \atop \omega_i \in \mathbb{Z}} \beta_{\omega_1} \cdots \beta_{\omega_n} G_n(\omega_1, \ldots, \omega_n).
\]

Since \( M = L(F(s)) \) and \( \beta_\omega = \hat{F}(s)_\omega \), this is nothing but formula (2.13) given in the introduction. By symmetry, we can rewrite formula (5.20),
\[
\varpi^n_0(M) = \sum_{\omega_1 + \cdots + \omega_n = 0 \atop \omega_i \in \mathbb{Z}} \beta_{\omega_1} \cdots \beta_{\omega_n} \frac{1}{n!} \sum_{\sigma \in S(n)} G_n(\omega_\sigma(1), \ldots, \omega_\sigma(n)).
\]

By the Main Combinatorial Lemma, lemma 4.1, we conclude that \( \varpi^n_0(M) = 0 \) for all \( n \geq 3 \) and
\[
\varpi^n_0(M) = \frac{1}{2} \sum_{\omega \in \mathbb{Z}} |\omega| \beta_\omega \beta_{-\omega}.
\]

According to formula (5.16), this shows that, if \( L = L(s) \) is a Laurent matrix, then the limit \( X(F) \) of the random variable \( \Xi_{N_k}(F) \) is Gaussian and the variance is given by
\[
\text{Var} \left[ X(F) \right] = \sum_{\omega > 0} \omega \beta_\omega \beta_{-\omega},
\]
where
\[
\beta_\omega = \hat{F}(s)_\omega = \frac{1}{2\pi i} \oint F(s(z)) z^{-\omega} \frac{dz}{z}.
\]
This completes the proof of the central limit theorem 2.6.
Lemma 5.2. Let $L$ be a Laurent matrix with symbol $s(z)$. For any $n \in \mathbb{N}$, we have

$$s(z)^n = \sum_{j \in \mathbb{Z}} z^j L^{\{\Gamma_0 \rightarrow j\}}.$$  \hfill (5.22)

Moreover, if $F \in \mathbb{R}[x]$, then $F(L)$ is a Laurent matrix with symbol $F(s)$.

Proof. Recall that we denote $s(z) = \sum_{\omega \in \mathbb{Z}} \hat{s}_\omega z^\omega$ so that for any $n \in \mathbb{N}$, we have

$$s(z)^n = \sum_{\omega \in \mathbb{Z}^n} \prod_{k=1}^n s_{\omega_k} z^{\omega_k}.$$  

Then, using the notation (5.18), $\prod_{k=1}^n s_{\omega_k} = L\{\pi_{\omega}\}$, and we obtain

$$s(z)^n = \sum_{\omega \in \mathbb{Z}^n} L\{\pi_{\omega}\} z^{\pi_{\omega}(n)}.$$  \hfill (5.23)

Since the sum (5.23) is over all path of length $n$ starting at 0 on the graph $G(L)$, this proves formula (5.22). Then, according to formula (5.4), for all $i, j \in \mathbb{Z}$,

$$L_{i,j}^n = L\{\Gamma_{j \rightarrow i}^n\} = L\{\Gamma_0 \rightarrow i - j\}$$

since the weighted graph $G(L)$ is translation invariant. By formula (5.22), this implies that for all $i, j \in \mathbb{Z}$,

$$L_{i,j}^n = \hat{s}(z)^{n}_{i-j}.$$  

By definition 2.5 we conclude that $L^n$ is a Laurent matrix with symbol $s(z)^n$. The same property hold for any polynomial by taking linear combinations. \hfill \□

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A Appendix: Proof of the Main Combinatorial Lemma

In this section, we present the proof of the MCL, lemma 4.1. We mostly follow the argument of [Sos00a] but we use different notations and provide more details. Moreover, a factor $1/2$ seems to be missing in the formulation of the MCL in [Sos00a]. First, we use the seminal results of Spitzer, [Spi56], on the geometry of simple random walks to give an alternative proof of a formula due to Rudnick and Sarnak, cf. lemma A.1, which plays a key role in Soshnikov’s proofs of both the MCL and Spohn’s lemma, [Spo87]. Along the way, we also show that lemma A.1 follows from the classical DHK formula, (A.1). This fact was certainly known to experts. However its proof seems to be missing from the literature. Moreover, this emphasizes that the combinatorial machinery behind Kac and Soshnikov proofs of the Strong Szegő theorem is the same.

In section 4.2 we have seen that, for the CUE, the fluctuations of linear statistics are described by the the Strong Szegő theorem. In [Kac54], Kac gave a combinatorial proof of the Strong Szegő theorem and a continuum analog for Fredholm determinant, also know as the Akhietzer-Kac formula. In fact, this approach is the basis of further generalization to pseudo-differential operators on
manifolds; see [Gio01] or [Sim04] section 6.5 for further references. Kac showed that, for sufficiently smooth functions, the Strong Szegő theorem boils down to the following identity

$$\sum_{\sigma \in S(n)} M(0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \sum_{\sigma \in S(n)} x_{\sigma(1)} \sum_{k=1}^{n} \theta(x_{\sigma(1)} + \cdots + x_{\sigma(k)}),$$  \hspace{1cm} (A.1)$$

where $\theta$ denotes the Heaviside step function and for any $x \in \mathbb{R}^n$,

$$M(x_1, \ldots, x_n) = \max \left\{ x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_n \right\}. \hspace{1cm} (A.2)$$

In [Kac54], Kac gave an elementary proof of formula (A.1) which is due to Dyson and Hunt. Therefore it is common to call (A.1) the Dyson-Hunt-Kac (DHK) formula, see [Sim04] Thm. 6.5.3. He also provided an interpretation of (A.1) in terms of the expected value of the maximum of a $n$-step simple random walks, cf. formula (A.4) below. The general mechanism to keep track of the distribution of the maximum of a random walk with independent identically distributed increments was understood [Spi56]. Namely, let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables, let $(S_k)_{k \geq 0}$ be their partial sums, and for any $u \in \mathbb{R}$, let $u^+ = \max \{0, u\}$. Then, for any $t \in \mathbb{R}$ and $|\lambda| < 1$,

$$\sum_{n=0}^{\infty} E \left[ e^{tu \max\{S_0, \ldots, S_n\}} \right] \lambda^n = \exp \left( \sum_{n=1}^{\infty} E \left[ e^{tS_n^+} \right] \frac{\lambda^n}{n} \right). \hspace{1cm} (A.3)$$

The proof of formula (A.3) is based on a beautiful bijection due to Bohnenblust, valid for any $x \in \mathbb{R}^n$, between the set $\{M(0, x_{\sigma(1)}, \ldots, x_{\sigma(n)})\}_{\sigma \in S(n)}$ and $\{T(x; \sigma)\}_{\sigma \in S(n)}$ where

$$T(x; \sigma) = \sum_{\text{cycles } \tau \text{ of } \sigma} \left( \sum_{j \in \tau} x_j \right)^+. \hspace{1cm} (A.4)$$

Formula (A.3) has several applications in the theory of random walks. For instance, a formula for the joint distribution of $\max\{S_0, \ldots, S_n\}, S_n)$ and a nice proof of the strong law of large numbers are given in [Spi56]; see also [Ste02] for a modern reference. It also leads to formula (A.1). If we differentiate formula (A.3) with respect to the parameter $t$ and evaluate at $t = 0$, we obtain

$$\sum_{n=0}^{\infty} E [M(0, X_1, \ldots, X_n)] \lambda^n = \frac{1}{1-\lambda} \sum_{n=1}^{\infty} E [S_n^+] \frac{\lambda^n}{n},$$

since $\exp \left( \sum_{n=0}^{\infty} \frac{\lambda^n}{n} \right) = \frac{1}{1-\lambda}$ and $\max\{S_0, \ldots, S_n\} = M(0, X_1, \ldots, X_n)$. If we identify the coefficients of $\lambda^n$ on both sides, we conclude that

$$E [M(0, X_1, \ldots, X_n)] = \sum_{k=1}^{n} \frac{E [(X_1 + \cdots + X_k)^+]}{k}. \hspace{1cm} (A.4)$$

Now, let $x \in \mathbb{R}^n$, $p_1, \ldots, p_n > 0$ be distinct numbers such that $\sum_{j=1}^{n} p_j = 0$ and suppose that $P[X_1 = x_j] = p_j$ for all $j = 1, \ldots, n$. If we identify the coefficient of $p_1 \cdots p_n$ in formula (A.4), we obtain

$$\sum_{\sigma \in S(n)} M(0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \sum_{k=1}^{n} \frac{1}{k} \sum_{\sigma \in S(n)} (x_{\sigma(1)} + \cdots + x_{\sigma(k)})^+. \hspace{1cm} (A.5)$$

Note that, if $\theta$ denotes the Heaviside step function, then for any $k = 1, \ldots, n$,

$$\sum_{\sigma \in S(n)} (x_{\sigma(1)} + \cdots + x_{\sigma(k)})^+ = k \sum_{\sigma \in S(n)} x_{\sigma(1)} \theta(x_{\sigma(1)} + \cdots + x_{\sigma(k)}),$$

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and formulae (A.1) and (A.5) are equivalent.

The connection with random matrix theory was realized in [RS96] where the Bohnenblust-Spitzer method was used to derive the following identity.

**Lemma A.1.** If \( x_1 + \cdots + x_n = 0 \), then

\[
\sum_{\sigma \in S_n} M(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \frac{n}{4} \sum_{F \subset [n]} |F| (|F| - 1)! (n - |F| - 1)! \left| \sum_{k \in F} x_k \right|^2, \quad (A.6)
\]

where the sum is over all subsets of \([n] = \{1, \ldots, n\}\) and, by convention, \((-1)! = 0\).

**Proof.** First observe that for any \( x \in \mathbb{R}^n \) such that \( x_1 + \cdots + x_n = 0 \), we have

\[
\sum_{\sigma \in S_n} M(-x_{\sigma(1)}, \ldots, -x_{\sigma(n)}) = \sum_{\sigma \in S_n} M(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) . \quad (A.7)
\]

This follows from symmetry and the simple observation that, since \(-x_1 - \cdots - x_k = x_{k+1} + \cdots + x_n\), by formula (A.2),

\[
M(-x_1, \ldots, -x_n) = \max \{ x_2 + \cdots + x_n, x_3 + \cdots + x_n, \ldots, x_n, 0 \} .
\]

Formula (A.7) and (A.5) implies that, if \( x_1 + \cdots + x_n = 0 \), then

\[
\sum_{\sigma \in S(n)} M(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \sum_{k=1}^{n-1} \frac{1}{2k} \sum_{\sigma \in S(n)} \left| x_{\sigma(1)} + \cdots + x_{\sigma(k)} \right| . \quad (A.8)
\]

Moreover, for any \( x \in \mathbb{R}^n \), we have

\[
\sum_{k=1}^{n} \frac{1}{k} \sum_{\sigma \in S(n)} \left| x_{\sigma(1)} + \cdots + x_{\sigma(k)} \right| = \sum_{k=1}^{n} \frac{1}{k} \sum_{F \subset [n]: |F| = k} \# \left\{ \sigma \in S(n) : \{\sigma(1), \ldots, \sigma(k)\} = F \right\} \left| \sum_{k \in F} x_k \right| = \sum_{F \subset [n]} \left( |F| - 1 \right)! |F^c|! \left| \sum_{k \in F} x_k \right| ,
\]

where \( F^c \) denotes the complement of \( F \) and \( |F| \) is the cardinal of \( F \). By convention, the last sum equals 0 if \( F = \emptyset \). Finally, if \( x_1 + \cdots + x_n = 0 \), \( |\sum_{k \in F} x_k| = |\sum_{k \in F^c} x_k| \) and we obtain

\[
\sum_{k=1}^{n} \frac{1}{k} \sum_{\sigma \in S(n)} \left| x_{\sigma(1)} + \cdots + x_{\sigma(k)} \right| = \frac{1}{2} \sum_{F \subset [n]} \left( |F| - 1 \right)! |F^c|! + (|F^c| - 1)! |F|! \left| \sum_{k \in F} x_k \right| = \frac{n}{2} \sum_{F \subset [n]} \left( |F| - 1 \right)! (|F^c| - 1)! \left| \sum_{k \in F} x_k \right| . \quad (A.9)
\]

Hence, combining formula (A.8) and (A.9), we have proved formula (A.6). \( \square \)

In [RS96], Rudnick and Sarnak investigated the statistical properties of the non-trivial zeroes of primitive \( L \)-function viewed as a pseudo-random point process. They used lemma A.1 to prove that in the correct scaling, the correlation functions of the spacings between the zeroes agree with the ones of the sine process. Hence, it is not surprising that the proof of the MCL in [Sos00a] is based on lemma A.1 as well.
Proof of lemma 4.7  We fix \( x \in \mathbb{R}^n \). We want to prove that, if \( x_1 + \cdots + x_n = 0 \), then

\[
\sum_{\sigma \in S_n} G_n(x_{\sigma(1)}, \ldots , x_{\sigma(n)}) = \begin{cases} |x_1| & \text{if } n = 2 \\ 0 & \text{else} \end{cases}, \tag{A.10}
\]

where, according to the definitions (2.14) and (2.3), we have

\[
G_n(x_1, \ldots , x_n) = \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell} \sum_{n_1 + \cdots + n_\ell = n} \binom{n}{\ell} \max \left\{ 0, \sum_{i=1}^{n_1} x_i, \sum_{i=1}^{n_1+n_2} x_i, \ldots , \sum_{i=1}^{n_1+\cdots+n_\ell} x_i \right\}. \tag{A.11}
\]

For any finite set \( A \), we let \( \Pi(A) \) be the set of all partitions of \( A \). It means that \( \pi = \{\pi_1, \ldots , \pi_\ell\} \in \Pi(A) \) if and only if \( \emptyset \neq \pi_j \subseteq A \) for all \( j \leq \ell \) and \( \pi_1 \cup \cdots \cup \pi_\ell = A \). In the following, we will also denote \( \ell = |\pi| \). We refer to [Sta12] Example 3.10.4 for an account on the partition lattice \( \Pi(A) \). In particular, its Möbius function is given by

\[
\mu(\pi) = \mu(\hat{0}, \pi) = (-1)^{|\pi|-1}(|\pi| - 1)! \tag{A.12}
\]

where \( \hat{0} = \{A\} \) is the trivial partition and

\[
\sum_{\pi \in \Pi(A)} \mu(\pi) = 1_{|A|=1}. \tag{A.13}
\]

Indeed, if the set \( A \) has a single element, then there is only 1 partition and the sum \( (A.13) \) is equal to \( \mu(\hat{0}) = 1 \). Otherwise, the sum vanishes by definition of the Möbius function. In fact, it is simple exercise to check that formulae \( (A.13) \) and \( (5.10) \) are the same. For any \( n \in \mathbb{N} \), we let \( \Pi[n] \) be the partition lattice of \( [n] = \{1, \ldots , n\} \) and for any \( \pi = \{\pi_1, \ldots , \pi_\ell\} \in \Pi[n] \), we define

\[
\Upsilon_\pi(x_1, \ldots , x_n) = \sum_{\tau \in \mathbb{B}(n)} \max \left\{ \sum_{i \in \pi_{\tau(1)}} x_i + \cdots + \sum_{i \in \pi_{\tau(j)}} x_i \mid j = 1, \ldots , \ell \right\}.
\]

Lemma [A.1] implies that if \( x_1 + \cdots + x_n = 0 \),

\[
\Upsilon_{\pi}(x_1, \ldots , x_n) = \frac{|\pi|}{4} \sum_{S \subset \{1, \ldots , |\pi|\}} (|S| - 1)! (|S^c| - 1)! \sum_{i \in \bigcup_{j \in \pi} S} x_i. \tag{A.14}
\]

On the other hand, for any \( x \in \mathbb{R}^n \), given a composition \( n = (n_1, \ldots , n_\ell) \) of \( n \), we have

\[
\sum_{\sigma \in \mathbb{B}(n)} \max \left\{ \sum_{i=1}^{n_1} x_{\sigma(i)}, \ldots , \sum_{i=1}^{n_1+n_2} x_{\sigma(i)} \right\} = n_1! \cdots n_\ell! \sum_{F_1 \cup \cdots \cup F_\ell = \{1, \ldots , n\}} \max \left\{ \sum_{i \in F_1} x_i , \ldots , \sum_{i \in F_\ell} x_i \right\}.
\]

This formula comes from the fact that for any disjoint subsets \( F_1, \ldots , F_\ell \subset \{1, \ldots , n\} \) such that \( |F_j| = n_j \),

\[
\# \left\{ \sigma \in \mathbb{S}(n) \mid \sigma(1), \ldots , \sigma(n_1) \in F_1 \right\}
\]

\[
\sigma(n_1 + 1), \ldots , \sigma(n_1+n_2) \in F_2 \\
\vdots \\
\sigma(n_1 + \cdots + n_\ell - 1) , \ldots , \sigma(n_1 + \cdots + n_\ell) \in F_\ell \right\} = n_1! \cdots n_\ell!.
\]

This implies that for any \( \ell = 1, \ldots , n \),

\[
\sum_{n_1 + \cdots + n_\ell = n} \frac{n!}{n} \sum_{\sigma \in \mathbb{S}(n)} \max \left\{ \sum_{i=1}^{n_1} x_{\sigma(i)}, \ldots , \sum_{i=1}^{n_1+n_\ell} x_{\sigma(i)} \right\} = n! \sum_{F_1 \cup \cdots \cup F_\ell = \{1, \ldots , n\}} \max \left\{ \sum_{i \in F_1} x_i , \ldots , \sum_{i \in F_\ell} x_i \right\} = n! \sum_{F \in \Pi[n]} \Upsilon_F(x_1, \ldots , x_n). 
\]
According to formula (A.11), we have proved that
\[
\sum_{\sigma \in \mathfrak{S}(n)} G_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = n! \sum_{F \in \Pi[n]} \frac{(-1)^{|F|}}{|F|} \gamma_F(x_1, \ldots, x_n).
\]

If \(x_1 + \cdots + x_n = 0\), by formula (A.14), this implies that
\[
\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}(n)} G_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \frac{1}{4} \sum_{F \in \Pi[n]} (-1)^{|F|} \sum_{S \subseteq \{1, \ldots, |F|\}} (|S| - 1)!(|S^c| - 1)! \left| \sum_{i \in j \in S} x_i \right|.
\]

(A.15)

In the previous sum, we make the change of variables \((F, S) \in \Pi[n] \times 2^{[F]}\) to \((A, \pi, \tilde{\pi}) \in 2^{[n]} \times \Pi(A) \times \Pi(A^c)\) given by
\[
A = \bigcup_{j \in S} F_j, \quad \pi = \{F_j \mid j \in S\}, \quad \tilde{\pi} = \{F_j \mid j \notin S\}.
\]

By (A.12), we have \((-1)^{|F|} (|S| - 1)!(|S^c| - 1)! = \mu(\pi)\mu(\tilde{\pi})\) and we obtain
\[
\sum_{F \in \Pi[n]} (-1)^{|F|} \sum_{S \subseteq \{1, \ldots, |F|\}} (|S| - 1)!(|S^c| - 1)! \left| \sum_{i \in j \in S} x_i \right| = \sum_A \left| \sum_{\pi \in \Pi(A)} \mu(\pi) \sum_{\tilde{\pi} \in \Pi(A^c)} \mu(\tilde{\pi}) \right|.
\]

By formulae (A.15) and (A.13), we conclude that
\[
\sum_{\sigma \in \mathfrak{S}(n)} G_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \frac{n!}{4} \sum_{A \subseteq \{1, \ldots, n\}} \left| \sum_{i \in A} x_i \right| \mathbf{1}_{|A| = 1} \mathbf{1}_{|A^c| = 1}
\]
\[
= \frac{1}{2} \beta_n^2 (|x_1| + |x_2|).
\]

Since \(|x_1| = |x_2|\) when \(x_1 + x_2 = 0\), this completes the proof of formula (A.10). \(\square\)

**B  The covariance structure of OPEs**

In this section, we deal with OPE in the case when the Jacobi matrix \(J(\mu_N)\) has a right-limit \(L\) which is the adjacency matrix of the graph

\[
\hat{Z} = \begin{array}{cccccc}
-2 & \bullet & 1/2 & \bullet & 1/2 & \bullet & 1/2 & \bullet & 1/2 & \bullet & 2 \\
\end{array}
\]

As discussed in section 5.1, it corresponds to the case where the equilibrium measure for the point process \(\Xi_N\) satisfies \(\operatorname{supp}(\mu_+) = [-1, 1]\). The aim is to prove by elementary means that the OPs with respect to the spectral measure of the graph \(\hat{Z}\) are the basis which diagonalizes the covariance matrix of the process \(\mathfrak{S}(X) = (\mathfrak{S}(x), \mathfrak{S}(x^2), \ldots)\) where \(X = (x^1, x^2, x^3, \ldots)^e\). The results of section 5.2 implies that for any polynomial \(F \in \mathbb{R}(x)\), the linear statistic \(\Xi_N(F)\), once centered, converges in distribution to a mean-zero Gaussian random variable \(\mathfrak{S}(F)\) with variance
\[
\text{Var} [\mathfrak{S}(F)] = \sum_{k > 0} k \beta_k \beta_{-k},
\]

32
where $\beta_k$ are the coefficients of the Laurent polynomial $F(\frac{z+1/z}{2})$. Let $n, m \in \mathbb{N}$ and $F(x) = x^n + tx^m$, keeping track of the coefficient of $t$, we obtain

$$
\mathbb{E}[\Theta(x^n)\Theta(x^m)] = \sum_{k>0} k\alpha_k^n\alpha_{-k}^m,
$$

where $\alpha_k^n$ denotes the coefficients of the Laurent polynomial $\left(\frac{z+1/z}{2}\right)^n$. Using the Binomial formula, we obtain

$$
\alpha_k^n = 2^{-n} \binom{n}{\frac{n}{2}},
$$

and by convention $\binom{n}{j} = 0$ if $j \notin \{0, 1, \ldots, n\}$. These coefficients may also be obtained from formula (5.22) by counting the number of paths of length $n$ on the graph $\hat{Z}$ which starts at 0 and end up at the vertex $k$. Observe that for any $k > 0$, we have $\alpha_k^n = \alpha^n_{-k}$, so that if we let $A = (\alpha_k^n)_{k,n \in \mathbb{N}}$ and $\Delta = (\sqrt{n}\mathbb{I}_{k=n})_{k,n \in \mathbb{N}}$, by formula (B.1), we obtain

$$
\mathbb{E}[\Theta(X)\Theta(X)^*] = A\Delta A^*.
$$

Observe that the matrix $A$ is lower triangular with entries 1 on its diagonal so that all its principle minor are invertible and, for any $c \in \mathbb{R}^N$, we can define

$$
Y = A^{-1}(X-c).
$$

In fact, we will choose $c_n = \alpha_0^n = \binom{n}{n/2}$ but it is not relevant for now. For any $n \in \mathbb{N}$, $Y_n$ is polynomial of degree $n$, so that $Y$ is a basis of $\mathbb{R}(x)$. Moreover, by linearity, $\Theta(Y) = A^{-1}\Theta(X)$ since $\Theta(c) = 0$ because $c$ is constant. According to formula (B.2), we get

$$
\mathbb{E}[\Theta(Y)\Theta(Y)^*] = \Delta.
$$

Hence, in the basis $Y$, the covariance matrix of the process $\Theta$ is diagonal.

On the other hand, we have

$$
\sum_{k \in \mathbb{Z}} \alpha_k^n\alpha_{-k}^m = 2^{-n-m}\sum_{k>0} \binom{n}{\frac{n}{2}} \binom{m}{\frac{m}{2}} = 2^{-n-m} \binom{n+m}{\frac{n+m}{2}}
$$

and, if $\nu$ denotes the spectral measure of the graph $\hat{Z}$, for any $n \geq 0$,

$$
\int x^nd\nu = 2^{-n}\#\{\text{paths of length } n \text{ on } \hat{Z} \text{ starting at 0 and at 0}\} = \alpha_0^n.
$$

Hence, since $\alpha_k^n = \alpha^n_{-k}$, we have

$$
\int(x^n - \alpha_0^n)(x^m - \alpha_0^m)d\nu(x) = \sum_{k \in \mathbb{Z}} \alpha_k^n\alpha_{-k}^m - \alpha_0^n - \alpha_0^m
$$

$$
= 2(AA^*)_{nm}
$$

By (B.3), this implies that

$$
\int YY^*d\nu = A \left(\int (X-c)(X-c)^*d\nu\right) A^{-*}
$$

$$
= 2I
$$

where $I$ is the identity matrix. The last identity shows that $Y$ is an eigenbasis for $\mathbb{R}(x)$ equipped with the inner product inherited from $L^2(\nu)$. It is a well-know fact that the spectral measure of
the graph $\tilde{Z}$ is the arcsine distribution $d\nu = \frac{2}{\pi} (1 - x^2)^{-1/2} \mathbb{1}_{|x|<1} dx$, so that, up to a factor 2, $Y$ are the Chebychev polynomials of the first kind. In fact, if $Y_0 \equiv 1$, formula (B.3) boils down to the well-known identity,

$$x^n = \sum_{k \geq 0} a_k^n Y_k(x) = 2^{1-n} \sum_{k=1}^{n} \binom{n}{n-k} T_k(x) + 2^{1-n} \binom{n}{n/2}.$$ 

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