A unified controllability/observability theory for some stochastic and deterministic partial differential equations

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Abstract. The purpose of this paper is to present a universal approach to the study of controllability/observability problems for infinite dimensional systems governed by some stochastic/deterministic partial differential equations. The crucial analytic tool is a class of fundamental weighted identities for stochastic/deterministic partial differential operators, via which one can derive the desired global Carleman estimates. This method can also give a unified treatment of the stabilization, global unique continuation, and inverse problems for some stochastic/deterministic partial differential equations.

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1. Introduction

We begin with the following controlled system governed by a linear Ordinary Differential Equation (ODE for short):

\[
\begin{aligned}
\frac{dy(t)}{dt} &= Ay(t) + Bu(t), \quad t > 0, \\
y(0) &= y_0.
\end{aligned}
\]  

In (1.1), \(A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m} \ (n, m \in \mathbb{N}), \ y(\cdot) \) is the state variable, \(u(\cdot)\) is the control variable, \(\mathbb{R}^n\) and \(\mathbb{R}^m\) are the state space and control space, respectively. System (1.1) is said to be exactly controllable at a time \(T > 0\) if for any initial state \(y_0 \in \mathbb{R}^n\) and any final state \(y_1 \in \mathbb{R}^n\), there is a control \(u(\cdot) \in L^2(0,T;\mathbb{R}^m)\) such that the solution \(y(\cdot)\) of (1.1) satisfies \(y(T) = y_1\).

The above definition of controllability can be easily extended to abstract evolution equations. In the general setting, it may happen that the requirement

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\( y(T) = y_1 \) has to be relaxed in one way or another. This leads to the approximate controllability, null controllability, and partial controllability, etc. Roughly speaking, the controllability problem for an evolution process is driving the state of the system to a prescribed final target state (exactly or in some approximate way) at a finite time. Also, the above \( B \) can be unbounded for general controlled systems.

The controllability/observability theory for finite dimensional linear systems was introduced by R.E. Kalman ([19]). It is by now the basis of the whole control theory. Note that a finite dimensional system is usually an approximation of some infinite dimensional system. Therefore, stimulated by Kalman’s work, many mathematicians devoted to extend it to more general systems including infinite dimensional systems, and its nonlinear and stochastic counterparts. However, compared with Kalman’s classical theory, the extended theories are not very mature.

Let us review rapidly the main results of Kalman’s theory. First of all, it is shown that: *System (1.1) is exactly controllable at a time \( T \) if and only if* \( \text{rank} [B, AB, \cdots, A^{n-1}B] = n \). However, this criterion is not applicable for general infinite dimensional systems. Instead, in the general setting, one uses another method which reduces the controllability problem for a controlled system to an observability problem for its dual system. The dual system of (1.1) reads:

\[
\begin{align*}
\frac{dw}{dt} &= -A^*w, \quad t \in (0, T), \\
w(T) &= z_0.
\end{align*}
\]

It is shown that: *System (1.1) is exactly controllable at some time \( T \) if and only if the following observability inequality (or estimate) holds*

\[
|z_0|^2 \leq C \int_0^T |B^*w(t)|^2 dt, \quad \forall \, z_0 \in \mathbb{R}^n.
\]

Here and henceforth, \( C \) denotes a generic positive constant, which may be different from one place to another. We remark that similar results remain true in the infinite dimensional setting, where the theme of the controllability/observability theory is to establish suitable observability estimates through various approaches.

Systems governed by Partial Differential Equations (PDEs for short) are typically infinite dimensional. There exists many works on controllability/observability of PDEs. Contributions by D.L. Russell ([40]) and by J.L. Lions ([29]) are classical in this field. In particular, since it stimulated many in-depth researches on related problems in PDEs, J.L. Lions’s paper [29] triggered extensive works addressing the controllability/observability of infinite dimensional controlled system. After [29], important works in this field can be found in [1, 4, 8, 11, 13, 17, 21, 25, 26, 43, 46, 55, 56]. For other related works, we refer to [18, 28] and so on.

The controllability/observability of PDEs depends strongly on the nature of the underlying system, such as time reversibility or not, and propagation speed of solutions, etc. The wave equation and the heat equation are typical examples. Now it is clear that essential differences exist between the controllability/observability theories for these two equations. Naturally, one expects to know whether some
relationship exist between the controllability/observability theories for these two equations of different nature. Especially, it would be quite interesting to establish, in some sense and to some extend, a unified controllability/observability theory for parabolic equations and hyperbolic equations. This problem was initially studied by D.L. Russell ([39]).

The main purpose of this paper is to present the author’s and his collaborators’ works with an effort towards a unified controllability/observability theory for stochastic/deterministic PDEs. The crucial analytic tool we employ is a class of elementary pointwise weighted identities for partial differential operators. Starting from these identities, we develop a unified approach, based on global Carleman estimate. This universal approach not only deduces the known controllability/observability results (that have been derived before via Carleman estimates) for the linear parabolic, hyperbolic, Schrödinger and plate equations, but also provides new/sharp results on controllability/observability, global unique continuation, stabilization and inverse problems for some stochastic/deterministic linear/nonlinear PDEs.

The rest of this paper is organized as follows. Section 2 analyzes the main differences between the existing controllability/observability theories for parabolic equations and hyperbolic equations. Sections 3 and 4 address, among others, the unified treatment of the controllability/observability problem for deterministic PDEs and stochastic PDEs, respectively.

2. Main differences between the known theories

In the sequel, unless otherwise indicated, \( G \) stands for a bounded domain (in \( \mathbb{R}^n \)) with a boundary \( \Gamma \in C^2 \), \( G_0 \) denotes an open non-empty subset of \( G \), and \( T \) is a given positive number. Put \( Q = (0, T) \times G \), \( Q_{G_0} = (0, T) \times G_0 \) and \( \Sigma = (0, T) \times \Gamma \).

We begin with a controlled heat equation:

\[
\begin{aligned}
& y_t - \Delta y = \chi_{G_0}(x)u(t, x) \quad \text{in } Q, \\
& y = 0 \quad \text{on } \Sigma, \\
& y(0) = y_0 \quad \text{in } G
\end{aligned}
\]  

and a controlled wave equation:

\[
\begin{aligned}
& y_{tt} - \Delta y = \chi_{G_0}(x)u(t, x) \quad \text{in } Q, \\
& y = 0 \quad \text{on } \Sigma, \\
& y(0) = y_0, \quad y_t(0) = y_1 \quad \text{in } G.
\end{aligned}
\]  

In (2.1), \( y \) and \( u \) are the state variable and control variable, the state space and control space are chosen to be \( L^2(G) \) and \( L^2(Q_{G_0}) \), respectively; while in (2.2), \((y, y_t)\) and \(u\) are the state variable and control variable, \( H^1_0(G) \times L^2(G) \) and \( L^2(Q_{G_0}) \) are respectively the state space and control space. System (2.1) is said to be null controllable (resp. approximately controllable) in \( L^2(G) \) if for any given \( y_0 \in L^2(G) \) (resp. for any given \( \varepsilon > 0, y_0, y_1 \in L^2(G) \)), one can find a control \( u \in L^2(Q_{G_0}) \)
such that the weak solution \( y(\cdot) \in C([0, T]; L^2(G)) \cap C((0, T]; H^1_0(G)) \) of (2.1) satisfies \( y(T) = 0 \) (resp. \( |y(T) - y_1|_{L^2(G)} \leq \varepsilon \)). In the case of null controllability, the corresponding control \( u \) is called a null-control (with initial state \( y_0 \)). Note that, due to the smoothing effect of solutions to the heat equation, the exact controllability for (2.1) is impossible, i.e., the above \( \varepsilon \) cannot be zero. On the other hand, since one can rewrite system (2.2) as an evolution equation in a form like (1.1), it is easy to define the exact controllability of this system. The dual systems of (2.1) and (2.2) read respectively

\[
\begin{align*}
\psi_t + \Delta \psi &= 0 \quad \text{in } Q, \\
\psi &= 0 \quad \text{on } \Sigma, \\
\psi(T) &= \psi_0 \quad \text{in } G
\end{align*}
\]

and

\[
\begin{align*}
\psi_{tt} - \Delta \psi &= 0 \quad \text{in } Q, \\
\psi &= 0 \quad \text{on } \Sigma, \\
\psi(T) &= \psi_0, \quad \psi_t(T) = \psi_1 \quad \text{in } G.
\end{align*}
\]

The controllability/observability theories for parabolic equations and hyperbolic equations turns out to be quite different. First of all, we recall the related result for the heat equation.

**Theorem 2.1.** ([25]) Let \( G \) be a bounded domain of class \( C^\infty \). Then: i) System (2.1) is null controllable and approximately controllable in \( L^2(G) \) at time \( T \); ii) Solutions of equation (2.3) satisfy

\[
|\psi(0)|_{L^2(G)} \leq C|\psi|_{L^2(Q_{G_0})}, \quad \forall \psi_0 \in L^2(G). \tag{2.5}
\]

Since solutions to the heat equation have an infinite propagation speed, the “waiting” time \( T \) can be chosen as small as one likes, and the control domain \( G_0 \) does not need to satisfy any geometric condition but being open and non-empty. On the other hand, due to the time irreversibility and the strong dissipativity of (2.3), one cannot replace \( |\psi(0)|_{L^2(G)} \) in inequality (2.5) by \( |\psi_0|_{L^2(G)} \).

Denote by \( \{\mu_i\}_{i=1}^\infty \) the eigenvalues of the homogenous Dirichlet Laplacian on \( G \), and \( \{\varphi_i\}_{i=1}^\infty \) the corresponding eigenvectors satisfying \( |\varphi_i|_{L^2(G)} = 1 \). The proof of Theorem 2.1 is based on the following observability estimate on sums of eigenfunctions for the Laplacian ([25]):

**Theorem 2.2.** Under the assumption of Theorem 2.1, for any \( r > 0 \), it holds

\[
\sum_{\mu_i \leq r} |a_i|^2 \leq C e^{C\sqrt{r}} \int_{G_0} \left| \sum_{\mu_i \leq r} a_i \varphi_i(x) \right|^2 dx, \quad \forall \{a_i\}_{\mu_i \leq r} \text{ with } a_i \in \mathbb{C}. \tag{2.6}
\]

Note that Theorem 2.2 has some other applications in control problems of PDEs ([32, 34, 44, 49, 55, 56]). Besides, to prove Theorem 2.1, one needs to utilize a time iteration method ([25]), which uses essentially the Fourier decomposition of solutions to (2.3) and especially, the strong dissipativity of this equation. Hence,
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this method cannot be applied to conservative systems (say, system (2.2)) or the system that the underlined equation is time-dependent.

As for the controllability/observability for the wave equation, we need to introduce the following notations. Fix any \( x_0 \in \mathbb{R}^n \), put

\[
\Gamma_0 \triangleq \{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0 \},
\]

(2.7)

where \( \nu(x) \) is the unit outward normal vector of \( G \) at \( x \in \Gamma \). For any set \( S \in \mathbb{R}^n \) and \( \varepsilon > 0 \), put \( \mathcal{O}_\varepsilon(S) = \{ y \in \mathbb{R}^n \mid |y - x| < \varepsilon \text{ for some } x \in S \} \).

The exact controllability of system (2.2) is equivalent to the following observability estimate for system (2.4):

\[
| (\psi_0, \psi_1) |_{L^2(G) \times H^{-1}(G)} \leq C |\psi|_{L^2(Q_{G_0})}, \quad \forall (\psi_0, \psi_1) \in L^2(G) \times H^{-1}(G).
\]

(2.8)

Note that the left hand side of (2.8) can be replaced by \( |(\psi(0), \psi_t(0))|_{L^2(G) \times H^{-1}(G)} \) (because (2.4) is conservative). The following classical result can be found in [29].

**Theorem 2.3.** Assume \( G_0 = O_\varepsilon(\Gamma_0) \cap G \) and \( T_0 = 2 \sup_{x \in G \setminus G_0} |x - x_0| \). Then, inequality (2.8) holds for any time \( T > T_0 \).

The proof of Theorem 2.3 is based on a classical Rellich-type multiplier method. Indeed, it is a consequence of the following identity (e.g. [47]):

**Proposition 2.4.** Let \( h \triangleq (h^1, \cdots, h^n) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a vector field of class \( C^1 \). Then for any \( z \in C^2(\mathbb{R} \times \mathbb{R}^n) \), it holds that

\[
\nabla \cdot \left\{ 2(h \cdot \nabla z)(\nabla z) + h \left[ z_t^2 - \sum_{i=1}^n z_{x_i}^2 \right] \right\} = -2(z_{tt} - \Delta z)h \cdot \nabla z + (2z_t h \cdot \nabla z)_t - 2z_t h_t \cdot \nabla z + (\nabla \cdot h) \left[ z_t^2 - \sum_{i=1}^n z_{x_i}^2 \right] + 2 \sum_{i,j=1}^n \left( \frac{\partial h_j}{\partial x_i} z_{x_i} z_{x_j} \right).
\]

The observability time \( T \) in Theorem 2.3 should be large enough. This is due to the finite propagation speed of solutions to the wave equation (except when the control is acting in the whole domain \( G \)). On the other hand, it is shown in [4] that exact controllability of (2.2) is impossible without geometric conditions on \( G_0 \). Note also that, the multiplier method rarely provides the optimal control/observation domain and minimal controll/observation time except for some very special geometries. These restrictions are weakened by the microlocal analysis ([4]). In [4, 5, 6], the authors proved that, roughly speaking, inequality (2.8) holds if and only if every ray of Geometric Optics that propagates in \( G \) and is reflected on its boundary \( \Gamma \) enters \( G_0 \) at time less than \( T \).

The above discussion indicates that the results and methods for the controllability/observability of the heat equation differ from those of the wave equation. As we mentioned before, this leads to the problem of establishing a unified theory for
the controllability/observability of parabolic equations and hyperbolic equations. The first result in this direction was given in [39], which showed that the exact controllability of the wave equation implies the null controllability of the heat equation with the same controller but in a short time. Further results were obtained in [32, 49], in which organic connections were established for the controllability theories between parabolic equations and hyperbolic equations. More precisely, it has been shown that: i) By taking the singular limit of some exactly controllable hyperbolic equations, one gives the null controllability of some parabolic equations ([32]); and ii) Controllability results of the heat equation can be derived from the exact controllability of some hyperbolic equations ([49]). Other interesting related works can be found in [34, 36, 43]. In the sequel, we shall focus mainly on a unified treatment of the controllability/observability for both deterministic PDEs and stochastic PDEs, from the methodology point of view.

3. The deterministic case

The key to solve controllability/observability problems for PDEs is the obtainment of suitable observability inequalities for the underlying homogeneous systems. Nevertheless, as we see in Section 2, the techniques that have been developed to obtain such estimates depend heavily on the nature of the equations, especially when one expects to obtain sharp results for time-invariant equations. As for the time-variant case, in principle one needs to employ Carleman estimates, see [17] for the parabolic equation and [47] for the hyperbolic equation. The Carleman estimate is simply a weighted energy method. However, at least formally, the Carleman estimate used to derive the observability inequality for parabolic equations is quite different from that for hyperbolic ones. The main purpose of this section is to present a universal approach for the controllability/observability of some deterministic PDEs. Our approach is based on global Carleman estimates via a fundamental pointwise weighted identity for partial differential operators of second order (It was established in [13, 15]. See [27] for an earlier result). This approach is stimulated by [24, 20], both of which are addressed for ill-posed problems.

3.1. A stimulating example. The basic idea of Carleman estimates is available in proving the stability of ODEs ([27]). Indeed, consider an ODE in $\mathbb{R}^n$: $\begin{cases} x_t(t) = a(t)x(t), & t \in [0, T], \\ x(0) = x_0, \end{cases}$ (3.1)

where $a \in L^\infty(0, T)$. A well-known simple result reads: Solutions of (3.1) satisfy $\max_{t \in [0, T]} |x(t)| \leq C|x_0|, \quad \forall x_0 \in \mathbb{R}^n$. (3.2)

A Carleman-type Proof of (3.2). For any $\lambda \in \mathbb{R}$, by (3.1), one obtains $\frac{d}{dt}(e^{-\lambda t}|x(t)|^2) = -\lambda e^{-\lambda t}|x(t)|^2 + 2e^{-\lambda t}x(t) \cdot x(t) = (2a(t) - \lambda)e^{-\lambda t}|x(t)|^2$. (3.3)
Choosing $\lambda$ large enough so that $2a(t) - \lambda \leq 0$ for a.e. $t \in (0, T)$, we find that

$$|x(t)| \leq e^{\lambda T/2} |x_0|, \quad t \in [0, T],$$

which proves (3.2).

\[\square\]

**Remark 3.1.** By (3.3), we see the following pointwise identity:

$$2e^{-\lambda t}x_t(t) \cdot x(t) = \frac{d}{dt}(e^{-\lambda t}|x(t)|^2) + \lambda e^{-\lambda t}|x(t)|^2. \quad (3.4)$$

Note that $x_t(t)$ is the principal operator of the first equation in (3.1). The main idea of (3.4) is to establish a pointwise identity (and/or estimate) on the principal operator $x_t(t)$ in terms of the sum of a “divergence” term $\frac{d}{dt}(e^{-\lambda t}|x(t)|^2)$ and an “energy” term $\lambda e^{-\lambda t}|x(t)|^2$. As we see in the above proof, one chooses $\lambda$ to be big enough to absorb the undesired terms. This is the key of all Carleman-type estimates. In the sequel, we use exactly the same method, i.e., the method of Carleman estimate via pointwise estimate, to derive observability inequalities for both parabolic equations and hyperbolic equations.

### 3.2. Pointwise weighted identity

We now show a fundamental pointwise weighted identity for general partial differential operator of second order. Fix real functions $\alpha, \beta \in C^1(\mathbb{R}^{1+m})$ and $b^j \in C^1(\mathbb{R}^{1+m})$ satisfying $b^j = b^k$ ($j, k = 1, 2, \ldots, m$). Define a formal differential operator of second order: $P_\alpha \triangleq (\alpha + i\beta)z_t + \sum_{j,k=1}^m (b^j z_{x_j})_x, i = \sqrt{-1}$. The following identity was established in [13, 15]:

**Theorem 3.2.** Let $z \in C^2(\mathbb{R}^{1+m}; \mathbb{C})$ and $\ell \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$. Put $\theta = e^\ell$ and $v = \theta z$. Let $a, b, \lambda \in \mathbb{R}$ be parameters. Then

$$\theta(P_\alpha T_1 + \bar{P}_\alpha T_1) + M_\ell + \sum_{k=1}^m \partial_{x_k} V^k$$

$$= 2|I_1|^2 + \sum_{j,k,j',k'=1}^m \left[2(b^{j,k} \ell_{x_j})_x v_x \cdot \partial_{x_k} V^k - (b^{j,k} + b^{j,k'}) \ell_{x_j} v_x \right] + \frac{1}{2} (\alpha b^j) \partial_{x_k} \ell_{x_j}$$

$$- ab^j \ell_{x_j} \ell_{x_k} v_x$$

$$+ \sum_{j,k=1}^m \left\{ (\beta b^j \ell_{x_j})_t + b^j (\beta \ell_{x_j})_x \right\} \left[ (\ell_{x_j} v) - (v_{x_j} \ell_{x_j}) \right]$$

$$+ [(\beta b^j \ell_{x_j})_x + a \beta b^j \ell_{x_j} v_x] \left[ (\ell_{x_j} v) - (v_{x_j} \ell_{x_j}) \right] + \sum_{j,k=1}^m b^{j,k} \partial_{x_k} (v_{x_j} v)$$

$$- a \sum_{j,k,j',k'=1}^m \left[ (b^{j,k} \ell_{x_j} v_x) \right] + B|v|^2,$$

where $I_1 = \int \left[ a \partial_{x_j} \ell_{x_j} v_x \right] (v_{x_j} v) \, dx dt$. This is a new Carleman-type estimate that holds for all second-order partial differential operators.
where

\[
I_1 \triangleq i\beta v_t - \alpha \ell_t v + \sum_{j,k=1}^{m} (b^{jk} v_{x_j})_{x_k} + Av,
\]

\[
A \triangleq \sum_{j,k=1}^{m} b^{jk} \ell_{x_j} \ell_{x_k} - (1 + a) \sum_{j,k=1}^{m} b^{jk} \ell_{x_j} v_{x_k} - b\lambda,
\]

\[
B \triangleq (\alpha^2 \ell_t + \beta^2 \ell_t - \alpha A) v
\]

\[
+ 2 \sum_{j,k=1}^{m} \left[ (b^{jk} \ell_{x_j} A)_{x_k} - (ab^{jk} \ell_{x_j} \ell_t)_{x_k} + a(A - \alpha \ell_t) b^{jk} \ell_{x_j} v_{x_k} \right],
\]

\[
M \triangleq \left[ (\alpha^2 + \beta^2) \ell_t - \alpha A \right] |v|^2 + \alpha \sum_{j,k=1}^{m} b^{jk} v_{x_j} \overline{v}_{x_k}
\]

\[
+ i\beta \sum_{j,k=1}^{m} b^{jk} \ell_{x_j} (\overline{v}_{x_k} v - v_{x_k} \overline{v}),
\]

\[
V^k \triangleq \sum_{j,j',k'=1}^{m} \left\{ - i\beta \left[ b^{jk} \ell_{x_j} (v \overline{v}_t - \overline{v} v_t) + b^{jk} \ell_t (v_{x_j} \overline{v} - \overline{v}_{x_j} v) \right]
\]

\[
- \alpha b^{jk} (v_{x_j} \overline{v}_t + \overline{v}_{x_j} v_t)
\]

\[
+ (2b^{jk} b^{j'} k' - b^{jk} b^{j' k'}) \ell_{x_j} (v_{x_j'} \overline{v}_{x_{k'}} + \overline{v}_{x_j'} v_{x_{k'}})
\]

\[
- \alpha b^{j' k'} \ell_{x_j} v_{x_{k'}} b^{ik} (v_{x_j} \overline{v} + \overline{v}_{x_j} v) + 2b^{jk} (A \ell_{x_j} - \alpha \ell_{x_j} \ell_t) |v|^2 \right\}.
\]

As we shall see later, Theorem 3.2 can be applied to study the controllability/observability as well as the stabilization of parabolic equations and hyperbolic equations. Also, as pointed out by [13], starting from Theorem 3.2, one can deduce the controllability/observability for the Schrödinger equation and plate equation appeared in [23] and [48], respectively. Note also that, Theorem 3.2 can be applied to study the controllability of the linear/nonlinear complex Ginzburg-Landau equation (see [13, 15, 38]).

### 3.3. Controllability/Observability of Linear PDEs

In this subsection, we show that, starting from Theorem 3.2, one can establish sharp observability/controllability results for both parabolic systems and hyperbolic systems.

We need to introduce the following assumptions.

**Condition 3.3.** Matrix-valued function \((p^{ij})_{1 \leq i,j \leq n} \in C^1(\mathbb{Q}; \mathbb{R}^{n \times n})\) is uniformly positive definite.

**Condition 3.4.** Matrix-valued function \((h^{ij})_{1 \leq i,j \leq n} \in C^1(\mathbb{Q}; \mathbb{R}^{n \times n})\) is uniformly positive definite.

Also, for any \(N \in \mathbb{N}\), we introduce the following
**Condition 3.5.** Matrix-valued functions $a \in L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))$ for some $p \in [n, \infty)$, and $a_1^1, \cdots, a_1^n, a_2 \in L^\infty(Q; \mathbb{R}^{N \times N})$.

Let us consider first the following parabolic system:

\[
\begin{align*}
\varphi_t - \sum_{i,j=1}^n (p^{ij} \varphi_{x_i})_{x_j} &= a \varphi + \sum_{k=1}^n a_k^1 \varphi_{x_k}, & \text{in } Q, \\
\varphi &= 0, & \text{on } \Sigma, \\
\varphi(0) &= \varphi^0, & \text{in } G,
\end{align*}
\]  

(3.6)

where $\varphi$ takes values in $\mathbb{R}^N$. By choosing $\alpha = 1$ and $\beta = 0$ in Theorem 3.2, one obtains a weighted identity for the parabolic operator. Along with [27], this identity leads to the existing controllability/observability result for parabolic equations ([9, 17]). One can go a little further to show the following result ([10]):

**Theorem 3.6.** Let Conditions 3.3 and 3.5 hold. Then, solutions of (3.6) satisfy

\[
|\varphi(T)||_{(L^2(G))^N} \leq \exp \left\{ C \left[ 1 + \frac{1}{T} + T|a|_{L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))} + \frac{1}{p-1} \right] + (1 + T) \left( \sum_{k=1}^N |a_k^1|_{L^\infty(Q; \mathbb{R}^{N \times N})} \right)^2 \right\} |\varphi|_{(L^2(Q; \varphi^0))}, \quad \forall \varphi^0 \in (L^2(G))^N. 
\]

(3.7)

Note that (3.7) provides the observability inequality for the parabolic system (3.6) with an explicit estimate on the observability constant, depending on the observation time $T$, the potential $a$ and $a_k^1$. Earlier result in this respect can be found in [9] and the references cited therein. Inequality (3.7) will play a key role in the study of the null controllability problem for semilinear parabolic equations, as we shall see later.

**Remark 3.7.** It is shown in [10] that when $n \geq 2$, $N \geq 2$ and $(p^{ij})_{1 \leq i, j \leq N} = I$, the exponent $\frac{2}{3}$ in $|a_k^1|_{L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))}$ (for the case that $p = \infty$ in the inequality (3.7)) is sharp. In [10], it is also proved that the quadratic dependence on $\sum_{k=1}^N |a_k^1|_{L^\infty(Q; \mathbb{R}^{N \times N})}$ is sharp under the same assumptions. However, it is not clear whether the exponent $\frac{2}{3} - \frac{n}{p}$ in $|a_k^1|_{L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))}$ is optimal when $p < \infty$.

Next, we consider the following hyperbolic system:

\[
\begin{align*}
v_t - \sum_{i,j=1}^n (h^{ij} v_{x_i})_{x_j} &= a v + \sum_{k=1}^n a_k^1 v_{x_k} + a_2 v_t, & \text{in } Q, \\
v &= 0, & \text{on } \Sigma, \\
v(0) &= v^0, & v_t(0) = v^1, & \text{in } G,
\end{align*}
\]  

(3.8)
where \( v \) takes values in \( \mathbb{R}^N \).

Compared with the parabolic case, one needs more assumptions on the coefficient matrix \((h^{ij})_{1 \leq i,j \leq n}\) as follows ([10, 16]):

**Condition 3.8.** There is a positive function \( d(\cdot) \in C^2(\overline{G}) \) satisfying

i) For some constant \( \mu_0 \geq 4 \), it holds

\[
\sum_{i,j=1}^{n} \left\{ \sum_{i',j'=1}^{n} \left[ 2h^{ij'}(h^{i'j}d_{x_{i'}})_x - h^{ij}(h^{i'j'}d_{x_{i'}})_x \right] \right\} \xi^i \xi^j \geq \mu_0 \sum_{i,j=1}^{n} h^{ij} \xi^i \xi^j,
\]

\( \forall (x, \xi^1, \cdots, \xi^n) \in \overline{G} \times \mathbb{R}^n \);

ii) There is no critical point of \( d(\cdot) \) in \( \overline{G} \), i.e., \( \min_{x \in \overline{G}} |\nabla d(x)| > 0 \);

iii) \( \frac{1}{4} \sum_{i,j=1}^{n} h^{ij}(x)d_{x_i}(x)d_{x_j}(x) \geq \max_{x \in \overline{G}} d(x), \quad \forall x \in \overline{G} \).

We put

\[
T^* = 2 \max_{x \in \overline{G}} \sqrt{d(x)}, \quad \Gamma^* \triangleq \left\{ x \in \Gamma \mid \sum_{i,j=1}^{n} h^{ij}(x)d_{x_i}(x)v_j(x) > 0 \right\}. \tag{3.9}
\]

By choosing \( b^{jk}(t, x) \equiv h^{jk}(x) \) and \( \alpha = \beta = 0 \) in Theorem 3.2 (and noting that only the symmetry condition is assumed for \( b^{jk} \) in this theorem), one obtains the fundamental identity derived in [16] to establish the controllability/observability of the general hyperbolic equations. One can go a little further to show the following result ([10]).

**Theorem 3.9.** Let Conditions 3.4, 3.5 and 3.8 hold, \( T > T^* \) and \( G_0 = G \cap O_\varepsilon(\Gamma^*) \) for some \( \varepsilon > 0 \). Then one has the following conclusions:

1) For any \((v^0, v^1) \in (H^1_0(G))^N \times (L^2(G))^N \), the corresponding weak solution \( v \in C([0, T]; (H^1_0(G))^N) \cap C^1([0, T]; (L^2(G))^N) \) of system (3.8) satisfies

\[
|v^0|_{H^1_0(G))^N} + |v^1|_{(L^2(G))^N} \leq \exp \left[ C \left( 1 + |a_1|_{L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))} \right. \right.
\]

\[
\left. + \left( \sum_{k=1}^{N} |a_k|_{L^\infty(Q; \mathbb{R}^{N \times N})} + |a_2|_{L^\infty(Q; \mathbb{R}^{N \times N})} \right)^2 \right] |\frac{\partial v}{\partial \nu}|_{L^2((0, T) \times \Gamma^*)}^N \tag{3.10}
\]

2) If \( a_k \equiv 0 \) (\( k = 1, \cdots, n \)) and \( a_2 \equiv 0 \), then for any \((v^0, v^1) \in (L^2(G))^N \times (H^{-1}(G))^N \), the weak solution \( v \in C([0, T]; (L^2(G))^N) \cap C^1([0, T]; (H^{-1}(G))^N) \) of system (3.8) satisfies

\[
|v^0|_{(L^2(G))^N} + |v^1|_{H^{-1}(G))^N} \leq \exp \left[ C \left( 1 + |a_1|_{L^\infty(0, T; L^p(G; \mathbb{R}^{N \times N}))} \right) \right] |v^1|_{L^2(Q; G_0))^N} \tag{3.11}
\]
A unified infinite-dimensional controllability/observability theory

As we shall see in the next subsection, inequality (3.11) plays a crucial role in the study of the exact controllability problem for semilinear hyperbolic equations.

Remark 3.10. As in the parabolic case, it is shown in [10] that the exponent \( \frac{2}{3} \) in the estimate \( |a|^2_{L^\infty(0,T;L^p(G;\mathbb{R}^N \times \mathbb{R}^N))} \) in (3.11) (for the special case \( p = \infty \)) is sharp for \( n \geq 2 \) and \( N \geq 2 \). Also, the exponent 2 in the term \( \left( \sum_{k=1}^N |a_k|^2_{L^\infty(Q;\mathbb{R}^N \times \mathbb{R}^N)} + |a_2|^2_{L^\infty(Q;\mathbb{R}^N \times \mathbb{R}^N)} \right)^2 \) in (3.10) is sharp. However, it is unknown whether the estimate is optimal for the case that \( p < \infty \).

By the standard duality argument, Theorems 3.6 and 3.9 can be applied to deduce the controllability results for parabolic systems and hyperbolic systems, respectively. We omit the details.

3.4. Controllability of Semi-linear PDEs. The study of exact/null controllability problems for semi-linear PDEs began in the 1960s. Early works in this respect were mainly devoted to the local controllability problem. By the local controllability of a system, we mean that the controllability property holds under some smallness assumptions on the initial data and/or the final target, or the Lipschitz constant of the nonlinearity.

In this subsection we shall present some global controllability results for both semilinear parabolic equations and hyperbolic equations. These results can be deduced from Theorems 3.6 and 3.9, respectively.

Consider first the following controlled semi-linear parabolic equation:

\[
\begin{align*}
\begin{cases}
y_t - \sum_{i,j=1}^n (p^{ij} y_{x_i})_{x_j} + f(y, \nabla y) = \chi_{G_0} u, & \text{in } Q, \\
y = 0, & \text{on } \Sigma, \\
y(0) = y_0, & \text{in } G.
\end{cases}
\end{align*}
\]

For system (3.12), the state variable and control variable, state space and control space, controllability, are chosen/defined in a similar way as for system (2.1).

Concerning the nonlinearity \( f(\cdot, \cdot) \), we introduce the following assumption ([9]).

Condition 3.11. Function \( f(\cdot, \cdot) \in C(\mathbb{R}^{1+n}) \) is locally Lipschitz-continuous. It satisfies \( f(0,0) = 0 \) and

\[
\begin{align*}
\begin{cases}
\lim_{|(s,p)| \to \infty} \frac{\int_0^1 f_s(\tau s, \tau p) d\tau}{\ln^\frac{1}{2} (1 + |s| + |p|)} = 0, \\
\lim_{|(s,p)| \to \infty} \frac{\left|\int_0^1 f_{p_1}(\tau s, \tau p) d\tau, \ldots, \int_0^1 f_{p_n}(\tau s, \tau p) d\tau\right|}{\ln^\frac{1}{2} (1 + |s| + |p|)} = 0,
\end{cases}
\end{align*}
\]

where \( p = (p_1, \ldots, p_n) \).
As shown in [9] (See [2] and the references therein for earlier results), linearizing the equation, estimating the cost of the control in terms of the size of the potential entering in the system (thanks to Theorem 3.6), and using the classical fixed point argument, one can show the following result.

**Theorem 3.12.** Assume that Conditions 3.3 and 3.11 hold. Then system (3.12) is null controllable.

In particular, Theorem 3.12 provides the possibility of controlling some blowing-up equations. More precisely, assume that \( f(s, p) \equiv f(s) \) in system (3.12) has the form

\[
f(s) = -s \ln'(1 + |s|), \quad r \geq 0.
\]

When \( r > 1 \), solutions of (3.12), in the absence of control, i.e. with \( u \equiv 0 \), blow-up in finite time. According to Theorem 3.12 the process can be controlled, and, in particular, the blow-up can be avoided when \( 1 < r \leq 3/2 \). By the contrary, it is proved in [2, 12] that for some nonlinearities \( f \) satisfying

\[
\lim_{|s| \rightarrow \infty} \frac{|f(s)|}{s \ln'(1 + |s|)} = 0,
\]

where \( r > 2 \), the corresponding system is not controllable. The reason is that the controls cannot help the system to avoid blow-up.

**Remark 3.13.** It is still an unsolved problem whether the controllability holds for system (3.12) in which the nonlinear function \( f(\cdot) \) satisfies (3.15) with \( 3/2 \leq r \leq 2 \). Note that, the growth condition in (3.13) comes from the observability inequality (3.7). Indeed, the logarithmic function in (3.13) is precisely the inverse of the exponential one in (3.7). According to Remark 3.7, the estimate (3.7) cannot be improved, and therefore, the usual linearization approach cannot lead to any improvement of the growth condition (3.13).

Next, we consider the following controlled semi-linear hyperbolic equation:

\[
\begin{aligned}
&y_{tt} - \sum_{i,j=1}^{n} (h_{ij} y_{x_i}) x_j = h(y) + \chi G_0 u \quad \text{in } Q, \\
y &= 0 \quad \text{on } \Sigma, \\
y(0) = y_0, \quad y_t(0) = y_1 \quad \text{in } G.
\end{aligned}
\]

(3.16)

For system (3.16), the state variable and control variable, state space and control space, controllability, are chosen/defined in a similar way as that for system (2.2). Concerning the nonlinearity \( h(\cdot) \), we need the following assumption ([10]).

**Condition 3.14.** Function \( h(\cdot) \in C(\mathbb{R}) \) is locally Lipschitz-continuous, and for some \( r \in [0, 1/2) \), it satisfies that

\[
\lim_{|s| \rightarrow \infty} \frac{\int_0^1 h_s(\tau s)d\tau}{\ln'(1 + |s|)} = 0.
\]

(3.17)
As mentioned in [10], proceeding as in the proof of [16, Theorem 2.2], i.e., by the linearization approach (thanks to the second conclusion in Theorem 3.9), noting that the embedding $H^1_0(G) \hookrightarrow L^2(G)$ is compact, and using the fixed point technique, one can show the following result.

**Theorem 3.15.** Assume that Conditions 3.4, 3.8 and 3.14 are satisfied, and $T$ and $G_0$ are given as in Theorem 3.9. Then system (3.12) is exactly controllable.

Due to the blow-up and the finite propagation speed of solutions to hyperbolic equations, one cannot expect exact controllability of system (3.12) for nonlinearities of the form (3.17) with $r > 2$. One could expect the system to be controllable for $r \leq 2$. However, in view of Remark 3.10, the usual fixed point method cannot be applied for $r \geq 3/2$. Therefore, when $n \geq 2$, the controllability problem for system (3.16) is open for $3/2 \leq r \leq 2$.

**Remark 3.16.** Note that the above “$3/2$ logarithmic growth” phenomenon (arising in the global exact controllability for nonlinear PDEs) does not occur in the pure PDE problem, and therefore the study of nonlinear controllability is of independent interest. More precisely, this means that for the controllability problem of nonlinear systems, there exist some extra difficulties.

### 3.5. Controllability of Quasilinear PDEs.

In this subsection, we consider the controllability of quasilinear parabolic/hyperbolic equations.

We begin with the following controlled quasilinear hyperbolic equation:

$$
\begin{aligned}
&y_{tt} - \sum_{i,j=1}^{n} (h_{ij} y_{x_i})_{x_j} = F(t, x, y, \nabla_{t,x} y, \nabla^2_{t,x} y) + qy + \phi_G u, \quad \text{in } Q, \\
y = 0, \quad \text{on } \Sigma, \\
y(0) = y_0, \quad y_t(0) = y_1, \quad \text{in } G.
\end{aligned}
$$

(3.18)

Here, $(h_{ij})_{1 \leq i,j \leq n} \in H^{s+1}(G; \mathbb{R}^{n \times n})$ and $q \in H^s(Q)$ with $s > \frac{n}{2} + 1$, and similar to [54], the nonlinear term $F(\cdot)$ has the form

$$
F(t, x, y, \nabla_{t,x} y, \nabla^2_{t,x} y) = \sum_{i=1}^{n} \sum_{\alpha=0}^{n} f_{i\alpha}(t, x, \nabla_{t,x} y)y_{x_i x_\alpha} + O(|y|^2 + |\nabla_{t,x} y|^2),
$$

where $f_{i\alpha}(t, x, 0) = 0$ and $x_0 = t, \phi_G$ is a nonnegative smooth function defined on $G$ and satisfying $\min_{x \in \partial G} \phi(x) > 0$. In system (3.18), as before, $(y, y_t)$ is the state variable and $u$ is the control variable. However, as we shall see later, the state space and control space have to be chosen in a different way from those used in the linear/semilinear setting.

The controllability of quasilinear hyperbolic equations is well understood in one space dimension ([26]). With regard to the multidimensional case, we introduce the following assumption.
Condition 3.17. The linear part in (3.18), i.e., hyperbolic equation
\[
\begin{cases}
y_{tt} - \sum_{i,j=1}^{n} (h^{ij} y_{x_i}) x_j = qy + \chi_{G_0} u, & \text{in } Q, \\
y = 0, & \text{in } \Sigma, \\
y(0) = y_0, \ y_t(0) = y_1, & \text{in } G.
\end{cases}
\] (3.19)

is exactly controllable in \(H^1_0(G) \times L^2(G)\) at some time \(T\).

Theorem 3.9 provides a sufficient condition to guarantee Condition 3.17 is satisfied. The following result is a slight generalization of that shown in [52].

Theorem 3.18. Assume Condition 3.17 holds. Then, there is a sufficiently small \(\varepsilon_0 > 0\) such that for any \((y_0, y_1), (z_0, z_1) \in (H^{s+1}(G) \cap H^1_0(G)) \times H^s(G)\) satisfying \(|(y_0, y_1)|_{H^{s+1}(G) \times H^s(G)} < \varepsilon_0\), \(|(z_0, z_1)|_{H^{s+1}(G) \times H^s(G)} < \varepsilon_0\) and the compatibility condition, one can find a control \(u \in \bigcap_{k=0}^{s-2} C^k([0,T]; H^{s-k}(G))\) such that the corresponding solution of system (3.18) verifies \(y(T) = z_0\) and \(y_t(T) = z_1\) in \(G\).

The key in the proof of Theorem 3.18 is to reduce the local exact controllability of quasilinear equations to the exact controllability of the linear equation by means of a new unbounded perturbation technique (developed in [52]), which is a universal approach to solve the local controllability problem for a large class of quasilinear time-reversible evolution equations.

Note however that the above approach does not apply to the controllability problem for quasilinear time-irreversible evolution equations, such as the following controlled quasilinear parabolic equation:
\[
\begin{cases}
y_{t} - \sum_{i,j=1}^{n} (a^{ij}(y) y_{x_i}) x_j = \chi_{G_0} u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } G.
\end{cases}
\] (3.20)

In (3.20), \(y\) is the state variable and \(u\) is the control variable, the nonlinear matrix-valued function \((a^{ij})_{1 \leq i,j \leq n} \in C^2(\mathbb{R}; \mathbb{R}^{n \times n})\) is locally positive definite. One can find very limited papers on the controllability of quasilinear parabolic-type equations ([31] and the references therein). One of the main difficulty to solve this problem is to show the “good enough” regularity for solutions of system (3.20) with a desired control.

We introduce the dual system of the linearized equation of (3.20).
\[
\begin{cases}
p_{t} - \sum_{i,j=1}^{n} (p^{ij} p_{x_i}) x_j = 0 & \text{in } Q, \\
p = 0 & \text{on } \Sigma, \\
p(0) = p_0 & \text{in } G.
\end{cases}
\] (3.21)
where \((p^{ij})_{1 \leq i, j \leq n}\) is assumed to satisfy Condition 3.3. Put \(B = 1 + \frac{1}{n} \sum_{i,j=1}^{n} |p^{ij}|^2_{C^1(\mathcal{Q})}\).

Starting from Theorem 3.2, one can show the following observability result ([31]).

**Theorem 3.19.** There exist suitable real functions \(\alpha\) and \(\varphi\), and a constant \(C_0 = C_0(\rho, n, G, T) > 0\), such that for any \(\lambda \geq C_0 e^{C_0 B}\), solutions of (3.21) satisfy

\[
|p(T)|_{L^2(G)} \leq C e^{\alpha \lambda} \varphi^{3/2} p_{L^2(Q_{G_0})}, \quad \forall p_0 \in L^2(G).
\]

(3.22)

In Theorem 3.19, the observability constant in (3.22) is obtained explicitly in the form of \(Ce^{CB}\) in terms of the \(C^1\)-norms of the coefficients in the principal operator appeared in the first equation of (3.21). This is the key in the argument of fixed point technique to show the following local controllability of system (3.20) ([31]).

**Theorem 3.20.** There is a constant \(\gamma > 0\) such that, for any initial value \(y_0 \in C^{2+1/2}(G)\) satisfying \(|y_0|_{C^{2+1/2}(G)} \leq \gamma\) and the first order compatibility condition, one can find a control \(u \in C^{2+1/2}(G)\) with \(\text{supp} u \subseteq [0, T] \times G_0\) such that the solution \(y\) of system (3.20) satisfies \(y(T) = 0\) in \(G\).

From Theorem 3.20, it is easy to see that the state space and control space for system (3.20) are chosen to be \(C^{2+1/2}(G)\) and \(C^{1+1/4}(Q)\), respectively. The key observation in [31] is that, thanks to an idea in [2], for smooth initial data, the regularity of the null-control function for the linearized system can be improved, and therefore, the fixed point method is applicable.

### 3.6. Stabilization of hyperbolic equations and further comments.

In this subsection, we give more applications of Theorem 3.2 to the stabilization of hyperbolic equations and comment other applications of this theorem and some related open problems.

One of the main motivation to introduce the controllability/observability theory is to design the feedback regulator ([19]). Stimulated by [29], there exist a lot of works addressing the stabilization problem of PDEs from the control point of view. To begin with, we fix a nonnegative function \(a \in L^\infty(\Gamma)\) such that \(\{x \in \Gamma \mid a(x) > 0\} \neq \emptyset\), and consider the following hyperbolic equation with a boundary damping:

\[
\begin{aligned}
&u_{tt} - \sum_{j,k=1}^{n} (h_{jk} u_{x_j})_{x_k} = 0 \quad \text{in } (0, \infty) \times G, \\
&\sum_{j,k=1}^{n} h_{jk} u_{x_j} v_k + a(x)u_t = 0 \quad \text{on } (0, \infty) \times \Gamma, \\
u(0) = u^0, \quad u_t(0) = u^1 \quad \text{in } G.
\end{aligned}
\]

(3.23)

Put \(H \triangleq \{(f, g) \in H^1(G) \times L^2(G) \mid \int_G f dx = 0\}\), which is a Hilbert space.
with the canonic norm. Define an unbounded operator \( A : H \to H \) by
\[
A \triangleq \left( \begin{array}{cc}
0 & I \\
\sum_{j,k=1}^n \partial_{x_k} (h^{jk} \partial_{x_j}) & 0
\end{array} \right),
\]
\[
D(A) \triangleq \left\{ u = (u^0, u^1) \in H \mid Au \in H, \left( \sum_{j,k=1}^n h^{jk} u^0_{x_j} \nu_k + au^1 \right) \big|_{\Gamma} = 0 \right\}.
\]

It is easy to show that \( A \) generates a \( C_0 \)-semigroup \( \{e^{tA}\}_{t \in \mathbb{R}} \) on \( H \). Hence, system (3.23) is well-posed in \( H \). Clearly, \( H \) is the finite energy space of system (3.23). One can show that the energy of any solution of (3.23) tends to zero as \( t \to \infty \) (There is no any geometric conditions on \( \Gamma \)).

Starting from Theorem 3.2, one can show the following result, which is a slight improvement of the main result in [14]:

**Theorem 3.21.** Assume Conditions 3.4 holds. Then solutions \( u \in C([0, \infty); D(A)) \cap C^1([0, \infty); H) \) of system (3.23) satisfy
\[
\|(u, u_t)\|_H \leq \frac{C}{\ln(2 + t)} \|(u^0, u^1)\|_{D(A)}, \quad \forall (u^0, u^1) \in D(A), \; \forall t > 0. \tag{3.24}
\]

Next, we consider a semilinear hyperbolic equation with a local damping:
\[
\begin{cases}
\begin{align*}
&u_t - \sum_{j,k=1}^n (h^{jk} u_{x_j})_{x_k} + f(u) + b(x)g(u, \nabla u) = 0 \quad \text{in} \ (0, \infty) \times G, \\
u(0) & = 0, \; u_t(0) = u^1 \quad \text{on} \ (0, \infty) \times \Gamma,
\end{align*}
\end{cases}
\]
\[
\begin{cases}
u \in C^1 \left([0, \infty) \times G, \mathbb{R}^n \right), \quad f : \mathbb{R} \to \mathbb{R} \text{ is a differentiable function satisfying} \ f(0) = 0, \ s f(s) \geq 0 \text{ and} \ \|f'(s)\| \leq C(1 + |s|^q) \text{ for any} \ s \in \mathbb{R}, \quad \text{where} \ q \geq 0 \text{ and} \ (n - 2)q \leq 2; \\
b : \mathbb{R}^n \to \mathbb{R} \text{ is a nonnegative function satisfying} \min b(x) \geq 0, \text{where} \ G_0 \text{ is given in Theorem 3.9; and} \ g : \mathbb{R}^{n+1} \to \mathbb{R} \text{ is a globally Lipschitz function satisfying} \ g(0, w) = 0, \ |g(r, w) - g(r_1, w_1)| \leq C(|r - r_1| + |w - w_1|) \ \text{and} \ g(r, w) \geq c_0 r^2 \text{ for some} \ c_0 > 0, \text{any} \ w, w_1 \in \mathbb{R}^n \text{and any} \ r, r_1 \in \mathbb{R}.
\end{cases}
\]

Define the energy of any solution \( u \) to (3.25) by setting
\[
E(t) = \frac{1}{2} \int_G \left[ |u_t|^2 + \sum_{j,k=1}^n h^{jk} u_{x_j} u_{x_k} \right] dx + \int_G \int_0^t f(s) ds dx.
\]

Starting from Theorem 3.2, one can show the following stabilization result for system (3.25) ([42]):

**Theorem 3.22.** Let \((u^0, u^1) \in H^1_0(G) \times L^2(G)\). Then there exist positive constants \( M \) and \( r \), possibly depending on \( E(0) \), such that the energy \( E(t) \) of the solution of (3.25) satisfies \( E(t) \leq Me^{-rt}E(0) \) for any \( t \geq 0 \).
Several comments are in order.

Remark 3.23. In [25], the authors need $C^\infty$-regularity for the data to establish Theorem 2.2. Recently, based on Theorem 3.2, this result was extended in [33] as follows: Denote by $\{\lambda_i\}_{i=1}^\infty$ the eigenvalues of any general elliptic operator of second order (with $C^1$-principal part coefficients) on $\Omega$ (of class $C^2$) with Dirichlet or Robin boundary condition, and $\{e_i\}_{i=1}^\infty$ the corresponding eigenvectors satisfying $|e_i|_{L^2(\Omega)} = 1$. Then, for any $r > 0$, it holds

$$\sum_{\lambda_i \leq r} |a_i|^2 \leq C e^{C \sqrt{r}} \int_{\Omega_0} \left| \sum_{\lambda_i \leq r} a_i e_i(x) \right|^2 dx, \quad \forall \{a_i\}_{\lambda_i \leq r} \text{ with } a_i \in \mathbb{C}.$$  

Remark 3.24. As indicated in [13, 22, 23], Theorem 3.2 can be employed to study the global unique continuation and inverse problems for some PDEs. Note also that this Carleman estimate based approach can be applied to solve some optimal control problems ([45]).

Remark 3.25. In practice, constrained controllability is more realizable. It is shown in [37] that the study of this problem is unexpectedly difficult even for the 1–d wave equation and heat equation. We refer to [30] for an interesting example showing that this problem is nontrivial even if the control is effective everywhere in the domain in which the system is evolved.

Remark 3.26. Note that the above mentioned approach applies mainly to the controllability, observability and stabilization of second order non-degenerate PDEs. It is quite interesting to extend it to the coupled and/or higher order systems, or degenerate systems but in general, this is nontrivial even for linear problems ([7, 53]).

Remark 3.27. Similar to other nonlinear problems, nonlinear controllability problems are usually quite difficult. It seems that there is no satisfactory controllability results published for nonlinear hyperbolic-parabolic coupled equations. Also, there exists no controllability results for fully nonlinear PDEs. In the general case, of course, one could expect only local results. Therefore, the following three problems deserve deep studies: 1) The characterization of the controllability subspace; 2) Controllability problem with (sharp) lower regularity for the data; 3) The problem that cannot be linearized. Of course, all of these problems are usually challenging.

4. The stochastic case

In this section, we extend some of the results/approaches in Section 3 to the stochastic case. As we shall see later, the stochastic counterpart is far from satisfactory, compared to the deterministic setting.

In what follows, we fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined. Let $H$ be a Fréchet space. Denote by $L^2_F(0,T;H)$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{L^2(0,T;H)}^2) < \infty$. 


with the canonical quasi-norms; by $L^2_x(0, T; H)$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded processes, with the canonical quasi-norms; and by $L^2(\Omega; C([0, T]; H))$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted continuous processes $X(\cdot)$ such that $E(\langle X(\cdot) \mathcal{C}(\mathcal{F}_t) \rangle_{[0, T]}^2) < \infty$, with the canonical quasi-norms (similarly, one can define $L^2(\Omega; C^k([0, T]; H))$ for $k \in \mathbb{N}$).

### 4.1. Stochastic Parabolic Equations

We begin with the following stochastic parabolic equation:

\[
\begin{cases}
  dz - \sum_{i,j=1}^{n} (p^{ij} z_{x_i})_{x_j} dt = [(a, \nabla z) + bz]dt + czdB(t) & \text{in } Q, \\
  z = 0 & \text{on } \Sigma, \\
  z(0) = z_0 & \text{in } G.
\end{cases}
\]  

with suitable coefficients $a, b$ and $c$, where $p^{ij} \in C^2(\Omega)$ is assumed to satisfy Condition 3.3 (Note that, technically we need here more regularity for $p^{ij}$ than the deterministic case). We are concerned with an observability estimate for system (4.1), i.e., to find a constant $C = C(a, b, c, T) > 0$ such that solutions of (4.1) satisfy

\[
|z(T)|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G))} \leq C|z|_{L^2(0, T; L^2(G_0))}, \quad \forall z_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\Omega)).
\]  

Similar to Theorem 3.2, we have the following weighted identity ([41]).

**Theorem 4.1.** Let $m \in \mathbb{N}$, $b^{ij} = b^{ji} \in L^2(\Omega; C^1([0, T]; W^{2, \infty}(\mathbb{R}^m)))$ $(i, j = 1, 2, \ldots, m)$, $\ell \in C^{1, 2}(\{(0, T) \times \mathbb{R}^m\})$ and $\Psi \in C^{1, 2}(\{(0, T) \times \mathbb{R}^m\})$. Assume $u$ is an $H^2(\mathbb{R}^m)$-valued continuous semi-martingale. Set $\theta = e^\ell$ and $v = \theta u$. Then for a.e. $x \in \mathbb{R}^m$ and $P$-a.s., $\omega \in \Omega$,

\[
\begin{align*}
2 \int_0^T \theta \left[ - \sum_{i,j=1}^{m} (b^{ij} v_{x_i})_{x_j} + Av \right] du - \sum_{i,j=1}^{m} (b^{ij} u_{x_i})_{x_j} dt + 2 \int_0^T \sum_{i,j=1}^{m} (b^{ij} u_{x_i})_{x_j} dv_x, \\
+ 2 \int_0^T \sum_{i,j=1}^{m} \left[ \sum_{i',j'=1}^{m} \left( 2 b^{ij} b^{i'j'} \ell_{x_{i'}} v_{x_{j'}} - b^{ij} b^{i'j'} \ell_{x_{i'}} v_{x_{j'}} \right) \right] dt \\
+ \Psi b^{ij} v_{x_i} v - b^{ij} \left( A \ell_{x_i} + \frac{\Psi_x}{2} \right) v^2 \bigg|_{x_j} dt \\
= 2 \int_0^T \sum_{i,j=1}^{m} \left[ \sum_{i',j'=1}^{m} \left[ 2 b^{ij} \left( b^{i'j'} \ell_{x_{i'}} v_{x_{j'}} \right) - b^{ij} b^{i'j'} \ell_{x_{i'}} \right]_{x_j} - \frac{b^{ij}}{2} + \Psi b^{ij} \right] v_{x_i} v_{x_j} dt \\
+ \int_0^T B v^2 dt + 2 \int_0^T \left[ - \sum_{i,j=1}^{m} (b^{ij} v_{x_i})_{x_j} + Av \right] \left[ - \sum_{i,j=1}^{m} (b^{ij} v_{x_i})_{x_j} + (A - \ell_t) v \right] dt \\
+ \left( \sum_{i,j=1}^{m} b^{ij} v_{x_i} v_{x_j} + Av^2 \right) \bigg|_0^T \\
- \int_0^T \theta^2 \sum_{i,j=1}^{m} b^{ij} \left[ (du_{x_i} + \ell_{x_i} du)(du_{x_j} + \ell_{x_j} du) \right] - \int_0^T \theta^2 A (du)^2, 
\end{align*}
\]
where
\[
A \triangleq - \sum_{i,j=1}^{m} (b^{ij} \ell_{x_i} \ell_{x_j} - b^{ij}_{x_i} \ell_{x_i} - b^{ij} \ell_{x_i} x_j) - \Psi,
\]
\[
B \triangleq 2 \left[ A \Psi - \sum_{i,j=1}^{m} (Ab^{ij}_{x_i}) x_j \right] - A_t - \sum_{i,j=1}^{m} (b^{ij} \Psi_{x_i}) x_i.
\]

Remark 4.2. Note that, in Theorem 4.1, we assume only the symmetry for matrix \((b^{ij})_{1 \leq i,j \leq n}\) (without assuming the positive definiteness). Hence, this theorem can be applied to study not only the observability/controllability of stochastic parabolic equations, but also similar problems for deterministic parabolic and hyperbolic equations, as indicated in Section 3. In this way, we give a unified treatment of controllability/observability problems for some stochastic and deterministic PDEs of second order.

Starting from Theorem 4.1, one can show the following observability result in [41] (See [3] and the references therein for some earlier results).

Theorem 4.3. Assume that
\[ a \in L^\infty_F(0,T;L^n(G;\mathbb{R}^n)), \quad b \in L^\infty_F(0,T;L^{n^*}(G)), \quad c \in L^\infty_F(0,T;W^{1,\infty}(G)), \]
where \( n^* \geq 2 \) if \( n = 1; \quad n^* > 2 \) if \( n = 2; \quad n^* \geq n \) if \( n \geq 3 \). Then there is a constant \( C = C(a,b,c,T) > 0 \) such that all solutions \( z \) of system (4.1) satisfy (4.2). Moreover, the observability constant \( C \) may be bounded as
\[ C(a,b,c,T) = Ce^{C[T^{-\frac{1}{2}} + T^{-\frac{3}{2}}]}, \]
with \( T \triangleq |a|_{L^\infty_F(0,T;L^n(G;\mathbb{R}^n))} + |b|_{L^\infty_F(0,T;L^{n^*}(G))} + |c|_{L^\infty_F(0,T;W^{1,\infty}(G))}. \]

As a consequence of Theorem 4.3, one can deduce a controllability result for backward stochastic parabolic equations. Unlike the deterministic case, the study of controllability problems for forward stochastic differential equations is much more difficult than that for the backward ones. We refer to [35] for some important observation in this respect. It deserves to mention that, as far as I know, there exists no satisfactory controllability result published for forward stochastic parabolic equations. Note however that, as a consequence of Theorem 2.2 and its generalization (see Remark 3.23), one can deduce a null controllability result for forward stochastic parabolic equations with time-invariant coefficients ([33]).

Theorem 4.1 has another application in global unique continuation of stochastic PDEs. To see this, we consider the following stochastic parabolic equation:
\[ Fz \triangleq dz - \sum_{i,j=1}^{n} (f^{ij} z_{x_i}) dt = [\langle a_1, \nabla z \rangle + b_1 z] dt + c_1 z dB(t) \quad \text{in } Q, \quad (4.3) \]
where \( f^{ij} \in C^{1,2}((0,T] \times G) \) satisfy \( f^{ij} = f^{ji} \) (\( i, j = 1, 2, \cdots, n \)) and for any open subset \( G_1 \) of \( G \), there is a constant \( s_0 = s_0(G_1) > 0 \) so that \( \sum_{i,j=1}^{n} f^{ij} \xi^i \xi^j \geq s_0 |\xi|^2 \)
for all \((t, x, \xi) \equiv (t, x, \xi^1, \cdots, \xi^n) \in (0, T) \times G_1 \times \mathbb{R}^n\); \(a_1 \in L^\infty_2(0, T; L^1_\text{loc}(G; \mathbb{R}^n))\), 
\(b_1 \in L^\infty_2(0, T; L^1_\text{loc}(G))\), and \(c_1 \in L^\infty_2(0, T; W^{1,\infty}_\text{loc}(G))\).

Starting from Theorem 4.1, one can show the following result ([50]).

**Theorem 4.4.** Any solution \(z \in L^2_\infty(\Omega; C([0, T]; L^2_\text{loc}(G))) \cap L^2_\infty(0, T; H^1_\text{loc}(G))\) of (4.3) vanishes identically in \(Q \times \Omega\), a.s. \(dP\) provided that \(z = 0\) in \(Q_{G_0} \times \Omega\), a.s. \(dP\).

Note that the solution of a stochastic equation is generally non-analytic in time even if all coefficients of the equation are constants. Therefore, one cannot expect a Holmgren-type uniqueness theorem for stochastic equations except for some very special cases. On the other hand, the usual approach to employ Carleman-type estimate for the unique continuation needs to localize the problem. The difficulty is that one cannot simply localize the problem as usual because the usual localization technique may change the adaptedness of solutions, which is a key feature in the stochastic setting. In equation (4.3), for the space variable \(x\), we may proceed as in the classical argument. However, for the time variable \(t\), due to the adaptedness requirement, we will have to treat it separately and globally. We need to introduce partial global Carleman estimate (indeed, global in time) even for local unique continuation for stochastic parabolic equation. Note that this idea comes from the study of controllability problem even though unique continuation itself is purely a PDE problem.

### 4.2. Stochastic Hyperbolic Equations

We consider now the following stochastic wave equation:

\[
\begin{cases}
 dz_t - \Delta z dt = (a_1 z_t + \langle a_2, \nabla z \rangle + a_3 z + f)dt + (a_4 z + g)dB(t) & \text{in } Q, \\
 z = 0 & \text{on } \Sigma, \\
 z(0) = z_0, \quad z_t(0) = z_1 & \text{in } G,
\end{cases}
\]

where \(a_1 \in L^\infty_2(0, T; L^\infty(G))\), \(a_2 \in L^\infty_2(0, T; L^\infty(G; \mathbb{R}^n))\), \(a_3 \in L^\infty_2(0, T; L^n(G))\), 
\(a_4 \in L^\infty_2(0, T; L^1(G))\), \(f \in L^2_2(0, T; L^2(G))\), \(g \in L^2_2(0, T; L^2(G))\) and \((z_0, z_1) \in L^2(\Omega, F_0, P; H^1_0(G) \times L^2(G))\). We shall derive an observability estimate for (4.3), i.e., find a constant \(C(a_1, a_2, a_3, a_4) > 0\) such that solutions of system (4.3) satisfy

\[
|\langle y(T), y_t(T) \rangle|_{L^2(\Omega, F_T, P; H^1_0(G) \times L^2(G))} \leq C(a_1, a_2, a_3, a_4) \left[ |\frac{\partial y}{\partial t}|_{L^2_2(0, T; L^2(G))} + |f|_{L^2_2(0, T; L^2(G))} + |g|_{L^2_2(0, T; L^2(G))} \right], \quad (4.5)
\]

\[
\forall (y_0, y_1) \in L^2(\Omega, F_0, P; H^1_0(G) \times L^2(G)).
\]

where \(\Gamma_0\) is given by (2.7) for some \(x_0 \in \mathbb{R}^d \setminus G\).

It is clear that, \(0 < R_0 \triangleq \min_{x \in \Gamma} |x - x_0| < R_1 \triangleq \max_{x \in \Gamma} |x - x_0|\). We choose a sufficiently small constant \(c \in (0, 1)\) so that \(\frac{(4+5c)R_1^2}{9c} > R_1^2\). In what follows, we take \(T\) sufficiently large such that \(\frac{(4+5c)R_1^2}{9c} > c^2T^2 > 4R_1^2\). Our observability estimate for system (4.3) is stated as follows ([51]).
Theorem 4.5. Solutions of system (4.4) satisfy (4.5) with
\[ C(a_1, a_2, a_3, a_4) = C \exp \left\{ \left[ \left( a_1, a_4 \right) \right] \Theta (0, T; L^\infty (G))^2 + \left| a_2 \right| \Theta (0, T; L^\infty (G))^2 + \left| a_3 \right| \Theta (0, T; L^\infty (G))^2 \right\} \]

Surprisingly, Theorem 4.5 was improved in [33] by replacing the left hand side of (4.5) by \( \left( \left| (y_0, y_1) \right| \right) \Theta (0, T; P; H^2 (G) \times L^2 (G)) \), exactly in a way of the deterministic setting. This is highly nontrivial by considering the very fact that the stochastic wave equation is time-irreversible.

The proof of Theorem 4.5 (and its improvement in [33]) is based on the following identity for a stochastic hyperbolic-like operator, which is in the spirit of Theorems 3.2 and 4.1.

Theorem 4.6. Let \( b^{ij} \in C^1 ((0, T) \times \mathbb{R}^n) \) satisfy \( b^{ij} = b^{ji} \) \((i, j = 1, 2, \ldots, n)\), \( \ell, \Psi \in C^2 ((0, T) \times \mathbb{R}^n) \). Assume \( u \) is an \( H^2 (\mathbb{R}^n) \)-valued \( \{ F_t \}_{t \geq 0} \)-adapted process such that \( u_t \) is an \( L^2 (\mathbb{R}^n) \)-valued semimartingale. Set \( \theta = e^\ell \) and \( v = \theta u \). Then, for a.e. \( x \in \mathbb{R}^n \) and \( P \)-a.s. \( \omega \in \Omega \),
\[
\theta \left( -2 \ell_t v_t + 2 \sum_{i,j=1}^n b^{ij} \ell_{x_i} v_{x_j} + \Psi v \right) \left[ du_t - \sum_{i,j=1}^n \left( b^{ij} u_{x_i} \right) x_j dt \right] \\
+ \sum_{i,j=1}^n \left[ \sum_{i', j'=1}^n \left( 2 b^{ij} b^{i' j'} \ell_{x_i} v_{x_j} - b^{ij} b^{i' j'} \ell_{x_i} v_{x_j} \right) - 2 b^{ij} \ell_{x_i} v_{x_j} v_t + b^{ij} \ell_{x_i} v_t^2 \right] \\
+ \Psi b^{ij} v_{x_i} v - \left( A \ell_{x_i} + \frac{\Psi x_j}{2} \right) b^{ij} v^2 \right] dt \\
+ \frac{d}{dt} \left[ \sum_{i,j=1}^n b^{ij} \ell_{x_i} v_{x_j} - \sum_{i,j=1}^n b^{ij} \ell_{x_i} v_{x_j} v_t + \ell_t v_t \right] - \Psi v_t v + \left( A \ell_t + \frac{\Psi t}{2} \right) v^2 \\
= \left\{ \left[ \ell_t + \sum_{i,j=1}^n \left( b^{ij} \ell_{x_i} \right) v_t^2 - 2 \sum_{i,j=1}^n \left( b^{ij} \ell_{x_i}, v_t \right) v_{x_i}, v_t \right] \\
+ \sum_{i,j=1}^n \left[ \left( b^{ij} \ell_t \right) t + \sum_{i', j'=1}^n \left( 2 b^{ij} \left( b^{i' j'} \ell_{x_i'} \right) v_{x_j'} - b^{ij} b^{i' j'} \ell_{x_i'} \right) \right] + \Psi b^{ij} \right] v_{x_i}, v_{x_j} \\
+ B v^2 + \left( -2 \ell_t v_t + 2 \sum_{i,j=1}^n b^{ij} \ell_{x_i} v_{x_j} + \Psi v \right) \left[ du_t + \theta^2 (du_t)^2 \right], \\
\right. \\
\text{where} \ (du_t)^2 \text{ denotes the quadratic variation process of } u_t,
\]
\[
\left\{ \begin{array}{l}
A \overset{\Delta}{=} \left( \ell_t^2 - \ell_t \right) - \sum_{i,j=1}^n \left( b^{ij} \ell_{x_i} \ell_{x_j} - b^{ij} \ell_{x_i} - b^{ij} \ell_{x_i x_j} \right) - \Psi, \\
B \overset{\Delta}{=} A \Psi + \left( A \ell_t \right) t - \sum_{i,j=1}^n \left( A b^{ij} \ell_{x_i} \right) v_t + \frac{1}{2} \left[ \Psi \ell_t - \sum_{i,j=1}^n \left( b^{ij} \Psi \ell_{x_i} \right) v_t \right].
\end{array} \right.
\]
4.3. Further comments. Compared to the deterministic case, the controllability/observability of stochastic PDEs is in its “enfant” stage. Therefore, the main concern of the controllability/observability theory in the near future should be that for stochastic PDEs. Some most relevant open problems are listed below.

- **Controllability of forward stochastic PDEs.** Very little is known although there are some significant progress in the recent work [33]. Also, it would be quite interesting to extend the result in [4] to the stochastic setting but this seems to be highly nontrivial.

- **Controllability of nonlinear stochastic PDEs.** Almost nothing is known in this direction although there are some papers addressing the problem in abstract setting by imposing some assumption which is usually very difficult to check for the nontrivial case.

- **Stabilization and inverse problems for stochastic PDEs.** Almost nothing is known in this respect.

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