1. Introduction

In this short paper we address the following question:

**Problem 1.1.** Given a compact constant scalar curvature Kähler manifold \((M, J, g, \omega)\), of complex dimension \(m := \dim \mathbb{C} M\), and having defined

\[
\Delta := \{(p_1, \ldots, p_n) \in M^n : \exists a \neq b \ p_a = p_b\},
\]

characterize the set \(\mathcal{P} W = \{(p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n)\} \subset (M^n \setminus \Delta) \times (0, +\infty)^n\) for which \(\tilde{M} = \text{Bl}_{p_1,\ldots,p_n} M\), the blow up of \(M\) at \(p_1, \ldots, p_n\) has a constant scalar curvature Kähler metric (cscK from now on) in the Kähler class

\[
\pi^*[\omega] - (\alpha_1 PD[E_1] + \cdots + \alpha_n PD[E_n]),
\]

where the \(PD[E_j]\) are the Poincaré duals of the \((2m - 2)\)-homology classes of the exceptional divisors of the blow up at \(p_j\).

This general problem is too complicated and its solution is likely to pass through the solution of well known conjectures relating the existence of cscK metrics with the \(K\)-stability of the polarized manifold.

Yet, more specific questions are treatable and could give light also on these ambitious programs. The first natural narrowing of Problem 1.1 is to require that not just one Kähler class has a cscK representative, but that this is the case for a whole segment in the Kähler cone of \(\tilde{M}\) touching the boundary at a point of the form \(\pi^*[\omega]\), where \(\omega\) is (necessarily) a cscK form on \(M\). Analytically this amounts to the following:

**Problem 1.2.** Given a compact Kähler constant scalar curvature manifold \((M, J, g, \omega)\) characterize the set \(\mathcal{A} \mathcal{P} W = \{(p_1, \ldots, p_n, a_1, \ldots, a_n)\} \subset (M^n \setminus \Delta) \times (0, +\infty)^n\) such that \(\tilde{M} = \text{Bl}_{p_1,\ldots,p_n} M\) has a constant scalar curvature Kähler metric in the class

\[
\pi^*[\omega] - \varepsilon^2 (a_1 PD[E_1] + \cdots + a_n PD[E_n]),
\]

for all \(\varepsilon\) sufficiently small. Here \(\mathcal{A} \mathcal{P} W\) refers to "asymptotic points and weights", namely points and weights in this singular perturbation setting.
Hence we can consider \((\alpha_1, \ldots, \alpha_n)\) as an asymptotic direction in the Kähler cone for which canonical representative can be found. It is immediate to extract from [1] the following:

**Theorem 1.1.** Assume that \((M, J, g, \omega)\) is a constant scalar curvature compact Kähler manifold without any nontrivial hamiltonian holomorphic vector field. Then \(\mathcal{APW} = (M^n \setminus \Delta) \times (0, +\infty)^n\).

The presence of hamiltonian holomorphic vector fields greatly enhances the difficulty and the interest of the problem. In [2] the authors have attacked this problem and found an interplay between its solution and the behavior of the hamiltonian holomorphic vector fields at the \(p_j\) that we briefly recall.

First recall that the Matsushima-Lichnerowicz Theorem asserts that the space of hamiltonian holomorphic vector fields on \((M, J, \omega)\) is also the complexification of the real vector space of holomorphic vector fields \(\Xi\) which can be written as

\[\Xi = X - i J X,\]

where \(X\) is a Killing vector field which vanish somewhere on \(M\). Let us denote by \(\mathfrak{h}\), the space of hamiltonian holomorphic vector field and by

\[\xi_\omega : M \rightarrow \mathfrak{h}^*\]

the moment map which is defined by requiring that if \(\Xi \in \mathfrak{h}\), the function \(\zeta_\omega := \langle \xi_\omega, \Xi \rangle\) is a (complex valued) Hamiltonian for the vector field \(\Xi\), namely the unique solution of

\[\bar{\partial} \zeta_\omega = \frac{1}{2} \omega(\Xi, \cdot),\]

which is normalized by

\[\int_M \zeta_\omega \, dvol_g = 0.\]

With these notations, the result we have obtained in [2] reads:

**Theorem 1.2.** Assume that \((M, J, g, \omega)\) is a constant scalar curvature compact Kähler manifold and that \(p_1, \ldots, p_n \in M\) and \(a_1, \ldots, a_n > 0\) are chosen so that:

(i) \(\xi_\omega(p_1), \ldots, \xi_\omega(p_n)\) span \(\mathfrak{h}^*\)

(ii) \(\sum_{j=1}^n a_j^{m-1} \xi_\omega(p_j) = 0 \in \mathfrak{h}^*\).

Then, there exist \(\varepsilon_0 > 0\) such that, for all \(\varepsilon \in (0, \varepsilon_0)\), there exists on \(\tilde{M} = Bl_{p_1, \ldots, p_n} M\), a constant scalar curvature Kähler metric \(g_\varepsilon\) associated to the Kähler form

\[\omega_\varepsilon \in \pi^* [\omega] - \varepsilon^2 (a_1, \varepsilon PD[E_1] + \ldots + a_n, \varepsilon PD[E_n])\],
Finaly, the sequence of metrics \((g_\varepsilon)_\varepsilon\) converges to \(g\) in \(C^\infty(M \setminus \{p_1, \ldots, p_n\})\).

Therefore, in the presence of nontrivial hamiltonian holomorphic vector fields, the number of points which can be blown up, their position, as well as the possible Kähler classes on the blown up manifold have to satisfy some constraints.

It is not hard to see from the proof in [2] that the Mapping
\[
(a_1, \ldots, a_n) \mapsto (a_1, \varepsilon, \ldots, a_n, \varepsilon)
\]
is continuous. Indeed, this follows from the construction itself which only uses fixed point theorems for contraction mappings and hence the metric we obtain depends smoothly on the parameters of the construction.

Theorem 1.2 has two major drawbacks : First, we lose control on the Kähler classes on \(\tilde{M}\) for which constant scalar curvature Kähler metrics can be constructed, second there are severe restrictions on the set of points and asymptotic directions.

The key idea to fill these gaps is to note that the construction of [2] is in fact a construction of the Riemannian metric \(g_\varepsilon\) and this is reflected by the fact that the sequence of metrics constructed converges to the initial metric \(g\) and also in the fact that condition (ii) really depends on the choice of the metric \(g\).

Now, on the one hand, the origin of (ii) stems from the existence of hamiltonian holomorphic vector fields on \((M, J)\) and in fact (ii) imposes on the choice of the asymptotic directions \((a_1, \ldots, a_n)\) as many constraints as the dimension of \(h\).

On the other hand, the existence of hamiltonian holomorphic vector fields is also related to the non-uniqueness of the constant scalar curvature Kähler metric on \(M\). More precisely, \(h\) is the Lie algebra of the group of automorphisms of \((M, J, g, \omega)\) and as such also parameterizes near \(g\) the space of constant scalar curvature Kähler metrics in a given Kähler class \([\omega]\) and for a given scalar curvature. Observe that this space has dimension equal to \(\dim h\). Therefore, we can Apply the result of Theorem 1.2 not only to the metric \(g\) itself but also to the pull back of \(g\) by any biholomorphic transformation.

Since condition (ii) depends on the choice of the metric, if we are only interested in the Kähler classes on the blown up manifold, we get more flexibility in the choice of the asymptotic parameters (observe that the dimension of the space of constant scalar curvature Kähler metrics near \(g\) (with fixed scalar curvature) is precisely equal to the number of constraints on the choice of the asymptotic parameters). This observation allows us to complement the result of Theorem 1.2 and get the: 

\[
|a_{j,\varepsilon} - a_j| \leq c \varepsilon^{\frac{2}{2m+1}}.
\]
Theorem 1.3. Assume that \((M, J, g, \omega)\) is a constant scalar curvature compact Kähler manifold and that \(p_1, \ldots, p_n \in M\) and \(a_1, \ldots, a_n > 0\) are chosen so that:

(i) \(\xi_\omega(p_1), \ldots, \xi_\omega(p_n)\) span \(\mathfrak{h}^*\) (genericity condition)

(ii) \(\sum_{j=1}^n a_j^{m-1} \xi_\omega(p_j) = 0 \in \mathfrak{h}^*\) (balancing condition)

(iii) no element of \(\mathfrak{h}\) vanishes at every point \(p_1, \ldots, p_n\). (general position condition)

Then \((p_1, \ldots, p_n, a_1, \ldots, a_n) \in \mathcal{APW}\).

Therefore, we can indeed prescribe the exact value of the asymptotic direction in which the Kähler classes in perturbed at the expense of imposing that no hamiltonian holomorphic vector field vanishes at every point we blow up.

The genericity condition is purely technical and it does not seem to hide any deep geometric nature. Indeed, as observed in [2]:

Lemma 1.1. With the above notations, assume that \(n \geq \dim \mathfrak{h}\). Then, the set of points \((p_1, \ldots, p_n) \in M^n \setminus \Delta\) satisfying the genericity condition is open and dense.

The balancing condition is certainly the heart of the problem, encoding the relevant stability property of \(\tilde{M}\). For example when all the \(a_j\) are rationals, the balancing condition is easily translated in the Chow polystability of the cycle \(\sum_j a_j^{m-1} p_j\) with respect to the action of the automorphism group of \(M\).

In a remarkable recent paper Stoppa [13] has proved, among other things, that if the cycle \(\sum_j a_j^{m-1} p_j\) is Chow unstable, then \((p_1, \ldots, p_n, a_1, \ldots, a_n)\) does not lie in \(\mathcal{APW}\). With a beautifully careful algebraic analysis he has in fact related a destabilizing configuration for the points to a destabilizing configuration of the blown up manifold giving a quantitative measure of the reciprocal unstabilities.

Going back to our problem, we first observe that the combination of the three above condition still leaves flexibility in the choices:

Theorem 1.4. With the above notations, assume that \(n \geq \dim \mathfrak{h} + 1\) then, the set of points \(((p_1, \ldots, p_n), (a_1, \ldots, a_n)) \in (M^n \setminus \Delta) \times (0, \infty)^n\) such that condition (i), (ii) and (iii) are fulfilled is open in \((M^n \setminus \Delta) \times (0, \infty)^n\).

Openness in the choice of the points was already contained in [2]. What we will prove in this short pPer is the openness in the choice of the asymptotic directions.
We can better understand the topology of $\mathcal{APW}$ by looking at

$$\pi_1 \nearrow \hspace{1cm} \searrow \pi_2$$

$M^n \setminus \Delta \hspace{1cm} (0, \infty)^n$

and define

$\mathcal{AP} = \pi_1(\mathcal{APW})$ and $\mathcal{AW} = \pi_2(\mathcal{APW})$.

With these notations, we obtain:

**Theorem 1.5.** Assume that $(p_1, \ldots, p_n) \in \mathcal{AP}$ and further assume that the general position condition holds, then

$\pi_2(\pi_1^{-1}(p_1, \ldots, p_n))$ is an open (nonempty) subset of $(0, \infty)^n$.

And we also have the:

**Theorem 1.6.** Assume that $(a_1, \ldots, a_n) \in \mathcal{AW}$ and further assume that there exists $(p_1, \ldots, p_n, a_1, \ldots, a_n) \in \pi_2^{-1}(a_1, \ldots, a_n)$ for which the general position condition holds, then $\pi_1(\pi_2^{-1}(a_1, \ldots, a_n))$ is an open dense subset of $M^n \setminus \Delta$.

Hence the general position condition shows that, by moving the cscK representative in $[\omega]$, the balancing condition is a very flexible one.

Theorems 1.5 and 1.6 are of completely different nature. Theorem 1.5 follows from an implicit function argument Applied to the set of solutions of the balancing condition, and it is of a local nature. On the other hand Theorem 1.6 follows from a suitable interpretation of the balancing condition in terms of the geometry of moment mPs. In this language, we can interpret $(p_1, \ldots, p_n) \in \pi_2^{-1}(a_1, \ldots, a_n)$ as a point in the zero set of a natural moment mP, and the general position condition is readily translated in the fact that this point is regular. The general theory then provides openness and density of the orbits of $(p_1, \ldots, p_n)$ through the action of the automorphisms group, which in turns implies the result.

We should note in this regard that it is a hopeless and confusing task to check conditions (i), (ii) and (iii) when the points move in these orbits but one should simply transport the solution associated to $(p_1, \ldots, p_n, a_1, \ldots, a_n)$ on $\tilde{M}$ to a solution at $(g(p_1), \ldots, g(p_n), a_1, \ldots, a_n)$ on $g(\tilde{M})$ where $g \in \text{Aut}(M)$.

It is important to emphasize that Theorem 1.5 cannot be improved to get a density result, as Stoppa [13] has found explicit bounds for the choice of weights to have cscK metrics in the blow ups of even deceptively simple examples as the projective plane (this result has then been strengthened by Della Vedova [6] to encompass the case of extremal metrics).
We conclude this paper by analyzing $\mathcal{APW}$ in the special case of manifolds for which $\dim \mathfrak{h} = 1$ and $n = 2$. We show that in this case $\pi_2(\mathcal{APW}) = (0, \infty)^2$. Recall that, among these type of manifolds, there are nontrivial explicit cscK metrics obtained by LeBrun [10] by the so-called moment construction. In this case we can also characterize $\pi_1(\mathcal{APW})$.

The analysis carried through in this note can be adapted to analyze the similar problem for extremal metrics in the sense of Calabi. In this case the algebraic analysis done by Stoppa [13] for $K$-stability has been completed by Della Vedova [6] for the relative $K$-stability in the sense of Szekelyhidi [14].

2. Proof of the results

We now proceed with the proof of the different results.

2.1. Proof of Theorems 1.3 and 1.5. Given a holomorphic vector field $\Xi \in \mathfrak{h}$ (close to 0) we consider $\phi_\Xi$ to be the bi-holomorphic transformation obtained by exponentiating the vector field $\Xi$, namely we consider the flow of the vector field $\Xi/||\Xi||$ at time $t = ||\Xi||$.

Here $|| \cdot ||$ is any norm on $\mathfrak{h}$, they are all equivalent since this space is finite dimension.

Observe that we have

$$[\phi_\Xi^* \omega] = [\omega],$$

and

$$s(\phi_\Xi^* \omega) = s(\omega).$$

Therefore, for all $\Xi \in \mathfrak{h}$, the Kähler form $\phi_\Xi^* \omega$ can be used in Theorem 1.2 to construct constant scalar curvature Kähler metrics on the blow up of $M$ at $p_1, \ldots, p_n$ in a Kähler class close to

$$\pi^* [\omega] - \varepsilon^2 (a_1 PD[E_1] + \ldots + a_n PD[E_n]).$$

The above discussion shows that we should be interested in the set Kähler forms $\bar{\omega} \in [\omega]$, with $s(\bar{\omega}) = s(\omega)$, points $(p_1, \ldots, p_n) \in M^n \setminus \Delta$ and asymptotic directions $(a_1, \ldots, a_n) \in (0, \infty)^n$ solution of the equation

$$a_1^{m-1} \xi_{\bar{\omega}}(p_1) + \ldots + a_n^{m-1} \xi_{\bar{\omega}}(p_n) = 0 \in \mathfrak{h}^*.$$ 

Let us assume that we have such a solution $(a_1, \ldots, a_n) \in (0, \infty)^n$ and $(p_1, \ldots, p_n) \in M^n \setminus \Delta$ for the Kähler form $\bar{\omega} = \omega$ itself. We would like to know if, close to this solution, we can move freely the coefficients $a_j$ and the points $p_j$. To this aim, we consider the Mapping

$$\mathcal{G} : (M^n \setminus \Delta) \times (0, +\infty)^n \times \mathfrak{h} \longrightarrow \mathfrak{h}^*$$

defined by

$$\mathcal{G}((b_1, \ldots, b_n), (q_1, \ldots, q_n), \Xi) = b_1^{m-1} \xi_{\bar{\omega}}(q_1) + \ldots + b_n^{m-1} \xi_{\bar{\omega}}(q_n),$$
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where
\[ \tilde{\omega} = \phi^* \omega. \]

Given \([\omega]\), the possible points which can be blown up and the possible asymptotic directions in which the Kähler class \([\omega]\) can be perturbed are just the projection over the first two components of the zeros of the Mapping \(\mathcal{S}\).

Let us assume that the differential of \(\mathcal{S}\) with respect to \(\Xi\), computed at the point \(((p_1, \ldots, p_n), (a_1, \ldots, a_n)) \in (M^n \setminus \Delta) \times (0, \infty)^n \times \mathfrak{h}\), is an isomorphism between \(\mathfrak{h}\) and \(\mathfrak{h}^*\). The implicit function theorem Applied to the Mapping \(\mathcal{S}\) guaranties that, close to \(((p_1, \ldots, p_n), (a_1, \ldots, a_n), 0)\), the set of solutions of
\[ \mathcal{S}((q_1, \ldots, q_n), (b_1, \ldots, b_n), \Xi) = 0, \]
is parameterized by \((q_1, \ldots, q_n)\) and \((b_1, \ldots, b_n)\). In other words, equation \((2)\) can be solved as
\[ \Xi = HV((b_1, \ldots, b_n), (q_1, \ldots, q_n)), \]
for some (smooth) Mapping \(HV\) defined from a neighborhood of \(((p_1, \ldots, p_n), (a_1, \ldots, a_n))\) in \((M^n \setminus \Delta) \times (0, \infty)^n\) into \(\mathfrak{h}\) and satisfying \(HV((p_1, \ldots, p_n), (a_1, \ldots, a_n)) = 0\).

To complete the proof of Theorems 1.3 and 1.5, we consider the Mapping
\[ (b_1, \ldots, b_n) \mapsto HV((p_1, \ldots, p_n), (b_1, \ldots, b_n)) \]
which is defined in a neighborhood \(U\) of \((a_1, \ldots, a_n)\), with values in \(\mathfrak{h}\). Observe that, by construction
\[ \sum_{j=1}^{n} b_j^{m-1} \xi_{\phi^* \omega}(p_j) = 0 \]
if \(\Xi = HV((p_1, \ldots, p_n), (b_1, \ldots, b_n))\). Moreover, reducing \(U\) if this is necessary, we can always assume that
\[ \xi_{\phi^* \omega}(p_1), \ldots, \xi_{\phi^* \omega}(p_n) \]
span \(\mathfrak{h}^*\), since this is true when \(\Xi = 0\) thanks to condition (i). Hence, we can use the result of Theorem 1.2 aApplied to the metric associated to \(\phi^* \omega\), the points \(p_1, \ldots, p_n\) and the coefficients \((b_1, \ldots, b_n) \in U\). This provides a Kähler metric in the Kähler class
\[ \pi^* [\omega] - \epsilon^2 (b_1^{m-1} PD[E_1] + \ldots + b_n^{m-1} PD[E_n]), \]
where the coefficients \(b_{j, \epsilon}\) depend (smoothly) on the points \(b_1, \ldots, b_n\). To summarize, we have defined a Mapping
\[ C_{\epsilon} : (b_1, \ldots, b_n) \in U \mapsto (b_1, \ldots, b_n, \epsilon) \in (0, \infty)^n. \]
As already mentioned, this Mapping is at least continuous and is close to the identity since \((1)\) implies that
\[ \|C_\epsilon((b_1, \ldots, b_n)) - (b_1, \ldots, b_n)\| \leq c \epsilon^{\frac{2}{m+2}}. \]
Clearly, deg($C_\varepsilon, (a_1, \ldots, a_n); U) = 1$ for $\varepsilon$ small enough and hence the image of $U$ by $C_\varepsilon$ contains an open neighborhood of $(a_1, \ldots, a_n)$, provided $\varepsilon$ is chosen small enough. This implies that, for all $\varepsilon$ small enough, it is possible to find $(b_1, \ldots, b_n) \in U$ so that

$$C_\varepsilon((b_1, \ldots, b_n)) = (a_1, \ldots, a_n),$$

and this completes the proof of Theorems 1.3 and 1.5.

Therefore, the only thing which remains to be understood is when the differential of $\mathcal{G}$ with respect of $\Xi$ is an isomorphism.

**The differential of $\mathcal{G}$ with respect of $\Xi$**

Let $\Xi \in \mathfrak{h}$ be given and $t \in \mathbb{R}$. We consider the one dimensional family of Kähler forms

$$\omega_t = \phi_t^* \omega.$$

First observe that, for $t$ small we can expand

$$\omega_t = \omega + i\partial \bar{\partial} (t f) + \mathcal{O}(t^2),$$

where $f$ is precisely the (real valued) potential associated to $\Xi$ given by

$$-\bar{\partial} f = \frac{1}{2} \omega(\Xi, -).$$

Recall that we can write

$$\Xi = X - iJX,$$

for some Killing vector field $X$ (for the metric $g$ associated to $\omega$) and we can also write

(3)

$$-d f = \omega(X, -).$$

Observe that if we consider the metric $\omega_t$, any fixed holomorphic vector field $\tilde{\Xi} \in \mathfrak{h}$ is associated to a (complex valued) potential (depending on $t$), which is defined by

(4)

$$-\bar{\partial} \tilde{f}_t = \frac{1}{2} \omega_t(\tilde{\Xi}, -)$$

and $\langle \tilde{\xi}_{\omega_t}, \tilde{\Xi} \rangle = \tilde{f}_t$, where $\xi_{\omega_t}$ is the moment $m\mathcal{P}$ associated to the Kähler form $\omega_t$.

Differentiating (4) with respect to $t$, at $t = 0$, we find

(5)

$$-\bar{\partial} (\tilde{\xi}_\omega, \tilde{\Xi}) = \frac{i}{2} \partial \bar{\partial} f(\tilde{\Xi}, -)$$

where $\tilde{\xi}_\omega$ is the first variation of $f \mapsto \xi_{\omega + i\partial \bar{\partial} f}$. Working in local coordinates and using the fact that $\Xi$ is holomorphic, one checks that the right hand side of (5) is equal to

$$\frac{i}{2} \partial \bar{\partial} f(\tilde{\Xi}, -) = -\frac{1}{2} \tilde{\Xi} f.$$

It is enough to consider the set of holomorphic vector fields $\Xi_2$ which can be written as

$$\tilde{\Xi} = X - iJX,$$
for some Killing vector field $\hat{X}$ (for the metric $g$). Given the definition of $f$, we get

\begin{align}
\langle \xi, \Xi \rangle &= -\frac{i}{2} \tilde{\Xi} f \\
&= -\frac{i}{2} df(\tilde{\Xi}) \\
&= -\frac{i}{2} df(\hat{X} - i J \hat{X}) \\
&= \frac{i}{2} \omega(X, \hat{X} - i J \hat{X}) \\
&= \frac{1}{2} \left( g(X, \hat{X}) + i g(J X, \hat{X}) \right)
\end{align}

(6)

The important point is that

$$
(\tilde{\Xi}, \Xi)_{Her} := g(X, \hat{X}) + i g(J X, \hat{X})
$$

is a positive definite Hermitian form. Alternatively, this corresponds to

$$
(\tilde{\Xi}, \Xi)_{Her} := -\frac{i}{2} \omega(\tilde{\Xi}, \Xi).
$$

We denote by $L$ the differential of $S$ with respect to $\Xi$, computed at $(a_1, \ldots, a_n)$, $(p_1, \ldots, p_n)$ and $\Xi = 0$. So that

$$
L : \mathfrak{h} \rightarrow \mathfrak{h}^*.
$$

and $L(\Xi) \in \mathfrak{h}^*$. Using the above computation, we conclude that

\begin{align}
L(\Xi) &= \frac{1}{2} \sum_{j=1}^{n} a_j^{m-1} (-, \Xi)_{Her}(p_j).
\end{align}

(7)

Now, $L$ generates a positive Hermitian form on $\mathfrak{h}$ by

$$
(\hat{X}, \Xi) = \frac{1}{2} \sum_{j=1}^{n} a_j^{m-1} (\Xi)_{Her}(p_j).
$$

Clearly, this form is non degenerate if and only if there is no holomorphic vector field $\Xi \in \mathfrak{h}$ which vanishes at every point $p_1, \ldots, p_n$ (therefore, we need $n \geq \dim \mathfrak{h}$). To summarize, we have proved the:

**Proposition 2.1.** Assume that there are no nontrivial element of $\mathfrak{h}$ vanishing at every $p_1, \ldots, p_n$, then the differential of $S$ with respect to $\Xi$, computed at $((a_1, \ldots, a_n), (p_1, \ldots, p_n), 0)$ is an isomorphism from $\mathfrak{h}$ into $\mathfrak{h}^*$.

All the above discussion seems to point out that the really important object is the zero set of the mapping $\mathfrak{G}$, or more precisely, its projection over the first two entries (the set of points which can be blown up and the set of asymptotic directions toward which the Kähler class can be deformed). This also explains the role of the zeros of the holomorphic
vector field, role which was completely occulted in [2] since the only important condition was associated to the potentials not the gradient of the potentials.

2.2. Proof of Theorem 1.6. We consider the action of the hamiltonian isometry group $H$ for some metric $g$. We can also consider $H$ acting on $M^n$ equipped with the weighted metric

$$a_1^{m-1} g + \ldots + a_n^{m-1} g.$$ 

The moment $\text{mP}$ for this action is then given by

$$\mu := a_1^{m-1} \xi_\omega + \ldots + a_n^{m-1} \xi_\omega.$$ 

In our case, Theorem 7.4 in [9] asserts that, if $\mu^{-1}(0) \neq \emptyset$ and if there exists $(p_1, \ldots, p_n) \in \mu^{-1}(0)$ satisfying the general position condition, then $(H \otimes \mathbb{C}) \cdot \mu^{-1}(0)$ is open and dense in $M^n \setminus \Delta$.

In other words, if we have a set of points $p_1, \ldots, p_n$ for which

$$\sum_{j=1}^n a_j^{m-1} \xi_\omega(p_j) = 0,$$

so that $(p_1, \ldots, p_n) \in \mu^{-1}(0)$ then the action of $H \otimes \mathbb{C}$, the complexification of $H$, provides an open dense set of points $U \subset M^n \setminus \Delta$ for which the condition

$$\sum_{j=1}^n a_j^{m-1} \xi_{\phi^*} \omega(q_j) = 0,$$

is fulfilled for some automorphism $\phi$ (depending on $(q_1, \ldots, q_n) \in U$). Now $\phi$ lifts to a biholomorphic $\text{mP} \hat{\phi} : Bl_{p_1,\ldots,p_n} M \rightarrow Bl_{q_1,\ldots,q_n} M$, and since we know that there exists a family of cscK forms $\omega_\epsilon \in \pi^* [\omega] - \epsilon^2 (a_1 PD[E_1] + \ldots + a_n PD[E_n])$ on $Bl_{p_1,\ldots,p_n} M$, the family $(\hat{\phi}^{-1})^* (\omega_\epsilon)$ is the seeked family of cscK forms on $Bl_{q_1,\ldots,q_n} M$. This completes the proof of the result.

3. An important example

Let us consider the simplest case where $\mathfrak{h} = \text{Span}\{\Xi\}$ and where we want to blow up 2 points. Then the condition on the points and the asymptotic directions which can be considered for the blow up procedure, becomes

$$a_1^{n-1} \xi_\omega(p_1) + a_2^{m-1} \xi_\omega(p_2) = 0$$

If

$$\zeta := \langle \xi_\omega, \Xi \rangle$$

this is just

$$(8) \quad a_1^{m-1} \zeta(p_1) + a_2^{m-1} \zeta(p_2) = 0.$$
Therefore, according to the result of [2] we can blow up any two points $p_1, p_2$ with directions $a_1, a_2$ satisfying $\xi$, provided $\zeta(p_1)$ and $\zeta(p_2)$ are not zero and have different signs. At this point, it looks like there is a constraint on the directions! However, as a consequence of our result, we see that we can locally move the coefficients $a_1, a_2$ freely provided the vector field $\Xi$ does not vanish at the two points $p_1, p_2$. Indeed, the formula that guarantees that $L$ is invertible just reduces to

$$L(\Xi, \Xi) \neq 0$$

which in this simple case reads

$$a_1^{m-1}(\Xi, \Xi)_{\text{Her}}(p_1) + a_2^{m-1}(\Xi, \Xi)_{\text{Her}}(p_2) \neq 0.$$ 

Therefore, if $\Xi$ does not vanish at both $p_1$ and $p_2$ (i.e. the general position condition is satisfied), then we can use the blow up procedure of [2] and we see that the set of directions of deformation of the Kähler classes is open. Observe that we obtain uniqueness of the corresponding constant scalar curvature Kähler metric since blowing up the points has "killed" the unique holomorphic vector field on $M$.

Observe that given $a_1, a_2 > 0$ it is always possible to find points $p_1, p_2 \in M$ satisfying $\xi$, i.e. $\pi_2(\mathcal{APW}) = (0, +\infty)^2$. Indeed, denote

$$a^- = \min \zeta < 0 < \max \zeta = a^+.$$ 

(Recall that the average of $\zeta$ over $M$ is zero). The intermediate value Theorem shows that one can find points $p_1, p_2$ such that

$$a^- < \zeta(p_1) = \frac{a^- a^+ a_2^{m-1}}{\sqrt{(a^- a_1^{m-1})^2 + (a^+ a_2^{m-1})^2}} < 0,$$

and

$$0 < \zeta(p_2) = \frac{-a^+ a^- a_1^{m-1}}{\sqrt{(a^- a_1^{m-1})^2 + (a^+ a_2^{m-1})^2}} < a^+,$$

and hence $\xi$ is satisfied for these two points.

The situation just described is far from an artificial speculation. Among these manifolds having only one holomorphic vector field vanishing somewhere, there is a well known class of examples of Kähler constant scalar curvature manifolds which are neither products nor Einstein, and have been discovered and investigated by Lebrun in [10] and also in [12], [8].

Let us recall that such surfaces are blow ups at finite set of points along the zero section of manifolds of the type $P(\mathcal{L} \oplus \mathcal{O})$, where $\mathcal{L}$ is a line bundle of positive degree over a Riemann surface of genus greater than 1. Such a procedure makes only the Euler vector field $\Xi$ survive on $M$. 


In this last setting, we define $\phi_t$ to be the flow associated to the Euler vector field $\Xi$. Given a point $p$ we write $p(t) = \phi_t(p)$ to be the image by flow of $\Xi$ passing through $p$ at time $t$. We have

$$\frac{d}{dt} \zeta(p(t)) = d\zeta(3\Xi) = -d\zeta(JX) = \omega(X, JX) = g(X, X),$$

by definition of $\zeta$. Observe that the flow $\phi$ preserves the fibers and thanks to this formula, the function $t \mapsto \zeta(p(t))$ is monotone increasing with $t$.

Now, if $p_1, p_2$ belong to the same fiber and do not belong to the zero section nor to the infinity section, given $a_1, a_2 > 0$ one can find $t \in \mathbb{R}$ such that

$$a_1 \zeta(p_1(t)) + a_2 \zeta(p_2(t)) = 0.$$

Hence, according to Theorem 1.3 we can blow up $(M, J, g_0, \omega)$ at the points $p_1(t)$ and $p_2(t)$ and find a cscK metric in the Kähler class corresponding to the weights $a_1$ and $a_2$. This is clearly equivalent to produce on the blow up of $(M, J, \phi_t^* g, \phi_t^* \omega)$ at the points $p_1, p_2$, a cscK metric in a Kähler class corresponding to the weights $a_1$ and $a_2$.

The metrics we consider have the remarkable property that $\omega(\Xi, \bar{\Xi})$, on each fiber of the line bundle, does not depend on the point chosen on the level set of the function $\zeta$. Since the image of a level set of $\zeta$ by the flow $\phi$ is another level of $\zeta$ and each fiber is preserved by the flow, it is enough to choose the points $p_1$ and $p_2$ at different ”moment heights” for the above discussion to hold. Since the genericity condition is obviously satisfied, this shows that $\pi_1(\mathcal{APW})$ contains $M^2 \setminus \mathcal{M}$, where

$$\mathcal{M} = \{(p_1, p_2) : \zeta_\omega(p_1) = \zeta_\omega(p_2), \text{ for some (hence any) cscK metric}\},$$

which is clearly open and dense in $M^2 \setminus \Delta$.

Conversely, if $(p_1, p_2) \in \mathcal{M}$, but not on the zero or infinity section of $L \oplus O$, then the balancing condition and the genericity condition cannot be simultaneously satisfied for any cscK metric on $M$, hence we cannot conclude that $(p_1, p_2)$ lies in $\mathcal{AP}$.

The last case to analyze is when both $p_1, p_2$ both lie on the zero section or on the infinity one. In this case, the idea is to work equivariantly as in [3] with respect to $K$, the group of isometries generated by $R\Xi$, and obtain extremal metric on $\tilde{M}$. This time we obtain extremal Kähler metrics without any constraint on the possible asymptotic directions toward which the Kähler class $\pi^*[\omega]$ can be deformed. But by Proposition 2.1 in [3] these metrics cannot be cscK (since the balancing condition is not satisfied) and hence these Kähler classes do not have cscK representatives by [3].

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