Abstract: We prove homogenization properties of random Hamilton-Jacobi-Bellman (HJB) equations on continuum percolation clusters, almost surely w.r.t. the law of the environment when the origin belongs to the unbounded component in the continuum. Here, the viscosity term carries a degenerate matrix, the Hamiltonian is convex and coercive w.r.t. the degenerate matrix and the underlying environment is non-elliptic and its law is non-stationary w.r.t. the translation group. We do not assume uniform ellipticity inside the percolation cluster, nor any finite-range dependence (i.i.d.) assumption on the percolation models and the effective Hamiltonian admits a variational formula which reflects some key properties of percolation. The proof is inspired by a method of Kosygina-Rezakhanlou-Varadhan [KRV06] developed for the case of HJB equations with constant viscosity and uniformly coercive Hamiltonian in a stationary, ergodic and elliptic random environment. In the non-stationary and non-elliptic set up, we leverage the coercivity property of the underlying Hamiltonian as well as a relative entropy structure (both being intrinsic properties of HJB, in any framework) and make use of the random geometry of continuum percolation.

1. Introduction

Consider a continuum percolation model resulting from the realizations $\omega \in \Omega$ of a point process in $\mathbb{R}^d$ with $d \geq 2$. The translation group $\{\tau_x\}_{x \in \mathbb{R}^d}$ acts on $\Omega$ and it is natural to assume that the underlying law $\mathbb{P}$ of the point process is invariant and ergodic under this action. If there is a unique infinite unbounded component $C_\infty(\omega)$ containing the origin 0, then the event $\Omega_0 = \{0 \in C_\infty\}$ has strictly positive probability, allowing us to define the conditional probability

$$
\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot | \Omega_0). \tag{1.1}
$$

Note that, because of conditioning on 0 being in the infinite cluster, the probability measure $\mathbb{P}_0$, in contrast to its unconditional counterpart $\mathbb{P}$, is not invariant under the action of $\{\tau_x\}_{x \in \mathbb{R}^d}$.

For $\omega \in \Omega_0$, we now consider the Hamilton-Jacobi-Bellman (HJB) equation on the infinite cluster $C_\infty(\omega)$

$$
\partial_t u_\varepsilon = \frac{\varepsilon}{2} \text{div}\left( a\left(\frac{x}{\varepsilon}, \omega\right) \nabla u_\varepsilon\right) + H\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \omega\right), \quad \text{in } (0, T) \times \varepsilon C_\infty(\omega) \tag{1.2}
$$

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with an initial condition \( f(\cdot) \) – we refer to \( \text{(2.10)} \) for a precise formulation. Here the diffusion coefficient \( a(x, \omega) = a(\tau_x \omega) \) is degenerate elliptic in the sense that the support of \( x \mapsto a(\tau_x \omega) \) is contained in the closure of the infinite cluster \( C_\infty(\omega) \), the Hamiltonian \( H(x, p, \omega) = H(p, \tau_x \omega) \) is convex and coercive in \( p \in \mathbb{R}^d \) w.r.t. the semi-norm induced by the degenerate matrix \( a \) and the initial condition \( f \) is uniformly continuous. These conditions are natural and they guarantee that, for any fixed \( \varepsilon > 0 \) and \( T > 0 \), there is actually a unique viscosity solution \( u_\varepsilon \) of \( \text{(1.2)} \).

Given this background, the goal of the present article is to develop a method for studying homogenization of \( u_\varepsilon \) almost surely w.r.t. the conditional probability \( P_0 \) – we will show that, \( P_0 \)-almost surely and as \( \varepsilon \to 0 \), \( u_\varepsilon \to u_{\text{hom}} \) with \( u_{\text{hom}} \) solving the homogenized equation

\[
\begin{aligned}
\partial_t u_{\text{hom}} &= \mathcal{H}(\nabla u_{\text{hom}}), & & \text{in } (0, \infty) \times \mathbb{R}^d, \\
u_{\text{hom}}(0, x) &= f(x), & & \text{on } \mathbb{R}^d.
\end{aligned}
\]

The effective Hamiltonian \( \mathcal{H} \) admits a variational representation

\[
\mathcal{H}(\theta) = \inf_G \left( \sup_{P_0} \left[ \frac{1}{2} \text{div}(a(G+\theta)) + H(G+\theta) \right] \right) \quad \forall \theta \in \mathbb{R}^d, \tag{1.3}
\]

and determines the effective equation \( u_{\text{hom}} \) as a viscosity solution

\[
u_{\text{hom}}(t, x) = \sup_{y \in \mathbb{R}^d} \left[ f(y) - t \mathcal{I}(\frac{y-x}{t}) \right], \quad \text{with } \mathcal{I}(y) := \sup_{\theta \in \mathbb{R}^d} \langle \theta, y - \mathcal{H}(\theta) \rangle,
\]

we refer to Theorem \( \text{2.21} \) for a precise statement, and to Corollary \( \text{2.22} \) for an application to an \( P_0 \)-a.s. large deviation principle for a degenerate diffusion with a random drift and on a percolation cluster.

The study of homogenization of HJB equations was initiated in a fundamental work of Lions, Panico

loulou and Varadhan [LPV87] which treated first order Hamilton-Jacobi equations in the periodic setting – that is, when \( a \equiv 0 \) and \( H(\cdot + z, \cdot) \equiv H(\cdot, \cdot) \) for all \( z \in \mathbb{Z}^d \). Since then, there have been very important works in the field by Souganidis [So99], Ishii [I99], Evans [E92], Rezakhanlou-Tarver [RT00], Lions-Sougandis [LS05, LS10], Kosygina-Rezakhanlou-Varadhan [KRV06] (see also Kosygina-Varadhan [KV08] for time-dependent case), Armstrong-Sougandis [AS12] and Armstrong-Tran [AT14, AT15]. While particular conditions vary from paper to paper, the main assumptions in these works on the Hamiltonian \( H \) involve convexity and super-linearity in \( p \), some regularity in \( p \) and \( x \) as well as uniform continuity on the initial condition \( f \). In all these works, homogenization holds in an almost sure (i.e., quenched) sense w.r.t. the law of the random environment, which is assumed to be stationary and ergodic under the translation group. For aforementioned reasons, the latter framework does not cover the conditional measure \( P_0 \), which is relevant for studying almost sure behavior of \( \text{(1.2)} \) on percolation clusters, where homogenization of elliptic equations of the form

\[-\text{div}(a(\nabla u + \varepsilon)) = 0 \]

in a reversible and discrete framework have been studied quite extensively in the recent years [SS04, MP07, BB07, PRS15, LNO15, AD18].

In this context and to the best of our knowledge, the present paper is the first instance where homogenization of HJB equations have been studied on percolation clusters – a fundamental class of models for studying statistical mechanics of random media. As we will see, their inherent properties like non-translation-invariance and non-ellipticity pervade through the sequel (including in the variational formula \( \text{(1.3)} \) of the homogenized limit) and manifest into fundamental difficulties – we refer to Section \( \text{2.3} \) for the main ideas of the proof. Also, the current method does not require any \( \text{finite-range dependence} \) (i.e. i.i.d.) assumption on the percolation models which are allowed to have long-range correlations, we refer to Appendix \( \text{A} \) for concrete examples of such models. Before turning to formal statements of the main results (cf. Section \( \text{2} \)), it is instructive to first provide the precise mathematical layout of the point processes.
1.1 Point processes and Palm measures.

1.1.1. Point processes. Fix an integer $d \geq 2$, and let $\Omega$ be the space of all locally finite subsets of $\mathbb{R}^d$. We denote by $\mathcal{B}(\mathbb{R}^d)$ to its Borel $\sigma$-algebra. The Lebesgue measure will be denoted by $\lambda$ (or by $\lambda_d$ when we need to emphasize on the dimension). We endow $\Omega$ with the smallest $\sigma$-algebra $\mathcal{G}$ that makes the maps $\omega \mapsto \#(\omega \cap A)$ measurable for all $A \in \mathcal{B}(\mathbb{R}^d)$, where $\#(\omega \cap A)$ denotes the cardinality of $\omega \cap A$. A Point process is stationary with respect to ($\tau$) if

$$P(\tau x \omega := \omega - x = \{y - x : y \in \omega\}).$$

We say a point process is stationary if

$$P \circ \tau_x = P \quad \forall x \in \mathbb{R}^d. \quad (1.4)$$

A stationary point process is ergodic with respect to ($\tau_x$) if

$$\forall A \in \mathcal{G} \forall x \in \mathbb{R}^d : \quad \tau_x A = A \implies P(A) \in \{0, 1\}. \quad (1.5)$$

We also define the intensity measure of $P$ as the measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given by

$$\Theta(A) := \int \#(\omega \cap A)P(d\omega) = \mathbb{E}[\#(\omega \cap A)]. \quad (1.6)$$

Here and throughout the sequel, $\mathbb{E}$ will denote expectation w.r.t. $P$. Notice that when $P$ is stationary and $\Theta$ is locally finite, then there exists some $\zeta \in (0, \infty)$ such that $\Theta = \zeta \lambda$. We call $\zeta$ the intensity of the point process.

1.1.2. Palm measures. We now turn to the definition of Palm measures, which, on an intuitive level, formalizes the idea of the distribution of a Point process conditioned on containing some fixed point $x \in \mathbb{R}^d$. First, we define the measure $\mathcal{C}$ on $\mathbb{R}^d \otimes \Omega$ as

$$\mathcal{C}(A) := \mathbb{E}\left[\sum_{x \in \omega} \mathbb{1}_A(x, \tau_x \omega)\right] \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}. \quad (1.7)$$

The measure $\mathcal{C}$ can be decomposed when $P$ is stationary: Indeed, by [SW08, Theorem 3.3.1], if $P$ is a stationary point process (i.e., if (1.4) holds) with intensity $\zeta \in (0, \infty)$, then there exists a unique measure $P^{(0)}$ on $(\Omega, \mathcal{G})$ such that

$$\mathcal{C} = \zeta \lambda \otimes P^{(0)}. \quad (1.8)$$

We call $P^{(0)}$ the Palm measure corresponding to $P$. It can be seen as the distribution of a point process conditioned on containing the origin (see [LP18, Proposition 9.5]). In particular, $P^0(0 \notin \omega) = 0 \in (\mathbb{R}^d)$. We can define more generally

$$P^{(x)} := P^{(0)} \circ \tau_x \quad \text{for} \quad x \in \mathbb{R}^d. \quad (1.9)$$

The aforementioned decomposition allows us to disintegrate $P$ in terms of ($P^{(x)}_{x \in \mathbb{R}^d}$): Indeed, by [SW08, Theorem 3.3.3], if $P$ is a stationary point process with intensity $\zeta \in (0, \infty)$, then for all $f \in L^1(\mathbb{R}^d \times \Omega)$, $\omega \mapsto \sum_{x \in \omega} f(x, \omega)$ is measurable, and

$$\mathbb{E}\left[\sum_{x \in \omega} f(x, \omega)\right] = \zeta \int_{\mathbb{R}^d} \mathbb{E}^{(0)}[f(x, \tau_x \omega)]dx = \zeta \int_{\mathbb{R}^d} \mathbb{E}^{(x)}[f(x, \omega)]dx. \quad (1.9)$$
Similarly, one can define the \( n \)-fold Palm distribution \( \mathbb{P}^{(x_1, \ldots, x_n)} \) for \( x_1, \ldots, x_n \in \mathbb{R}^d \). In this case, we have the equality

\[
E \left[ \sum_{x_1, \ldots, x_n \in \omega} f(x_1, \ldots, x_n, \omega) \right] = \zeta^n \int_{(\mathbb{R}^d)^n} \mathbb{E}^{(x_1, \ldots, x_n)} [f(x_1, \ldots, x_n, \omega)] dx_1 \cdots dx_n \tag{1.10}
\]

for \( f \in L^1((\mathbb{R}^d)^n \times \Omega) \), where the sign \( \neq \) above the sum indicates that the sum is taken over pairwise distinct elements. Also, \( \mathbb{E}^{(x_1, \ldots, x_n)} \) stands for expectation w.r.t. the \( n \)-fold Palm distribution \( \mathbb{P}^{(x_1, \ldots, x_n)} \).

1.1.3. *Assumptions on the point process* \( \mathbb{P} \). For any \( \omega \in \Omega \), which denotes a locally finite point set in \( \mathbb{R}^d \), we define a random domain \( \mathcal{C}(\omega) \), which is an open set

\[
\mathcal{C}(\omega) := \bigcup_{x \in \omega} B_{\frac{1}{2}}(x) \subset \mathbb{R}^d, \tag{1.11}
\]

where \( B_r(x) = \{ y \in \mathbb{R}^d : |y-x| < r \} \) denotes an open ball centered at \( x \) of radius \( r > 0 \). The set \( \mathcal{C}(\omega) \) can be decomposed into a disjoint union of connected components. If there is a unique unbounded connected component, then this component is denoted by \( \mathcal{C}_\infty(\omega) \subset \mathcal{C}(\omega) \subset \mathbb{R}^d \). Its boundary will be denoted by \( \partial \mathcal{C}_\infty \), with \( \text{int}(\mathcal{C}_\infty) \) denoting the interior. Moreover, we define

\[
\Omega_0 := \{ \omega \in \Omega : \mathcal{C}_\infty(\omega) \text{ exists}, 0 \in \mathcal{C}_\infty(\omega) \} \subset \Omega. \tag{1.12}
\]

If \( \mathbb{P}(\Omega_0) > 0 \) (which we will assume in condition \( \text{(P3)} \) stated below), then we can define the conditional probability measure \( \mathbb{P}_0 \) on \( \Omega_0 \) via

\[
\mathbb{P}_0(\cdot) := \mathbb{P}(\cdot | \Omega_0), \quad \text{viz.} \quad \mathbb{P}_0(A) = \frac{\mathbb{P}(A)}{\mathbb{P}(\Omega_0)} \quad \text{for all} \ A \in \mathcal{G} \cap \Omega_0.
\]

By the openness and the connectedness of \( \mathcal{C}_\infty(\omega) \), every two points \( x, y \in \mathcal{C}_\infty \) can be connected by a curve in \( C^1([0,1];\mathbb{R}^d) \) such that the interior distance \( d_\omega \) is defined on \( \mathcal{C}_\infty(\omega) \) via

\[
d_\omega(x,y) = \inf \left\{ \int_0^1 \| \dot{r}(s) \| ds : r \in C^1([0,1];\mathbb{R}^d), \ r(0) = x, \ r(1) = y, \right. \\
\left. \quad \text{and} \ r(s) \in \mathcal{C}_\infty(\omega) \text{ for all } s \in [0,1] \right\}.
\]

To state the condition \( \text{(P6)} \) below we define \( n(\omega, e) \in \mathbb{N} \) for all \( e \in \mathbb{Z}^d \) with \( |e| = 1 \) and \( \omega \in \Omega_0 \) to be the “successive arrivals” of \( \mathcal{C}_\infty \) along a certain direction \( e \), i.e., we set

\[
n(\omega, e) = \min\{ k \in \mathbb{N} : ke \in \mathcal{C}_\infty(\omega) \}. \tag{1.13}
\]

We are now ready to state the following assumptions on the point process \( \mathbb{P} \):

(P1) \( \mathbb{P} \) is stationary ergodic with respect to \( (\tau_e)_{e \in \mathbb{R}^d} \). Moreover, \( \mathbb{P} \) is also stationary ergodic with respect to \( \tau_e \) for all \( e \in \mathbb{Z}^d \) with \( |e|_1 = 1 \) (namely, any \( A \in \mathcal{G} \) such that \( \tau_e A = A \) satisfies \( \mathbb{P}(A) \in \{0,1\} \)).

(P2) Recall the definition of \( \Theta \) from (1.6). Then for any compact set \( A \subset \mathcal{B}(\mathbb{R}^d) \) we have \( \Theta(A) < \infty \). In particular, \( \Theta = \zeta A \) for some \( \zeta \in (0,\infty) \).

(P3) With the above definition of \( \mathcal{C}(\omega) \) and \( \Omega_0 \) we assume \( \mathbb{P}(\Omega_0) > 0 \), i.e. with positive \( \mathbb{P} \)-probability, the set \( \mathcal{C}(\omega) \) has a unique open, unbounded and connected component \( \mathcal{C}_\infty(\omega) \) containing the origin \( 0 \in \mathbb{R}^d \).
There are constants $c_0, c_1, c_2 > 0$ such that for each $x, y \in \mathbb{R}^d$,  
\begin{equation}
\mathbb{P}^{(x,y)}(d_\omega(x,y) \geq c_0|x-y|_\infty; 0, x, y \in \mathcal{C}_\infty) \leq c_1 e^{-c_2|x-y|_\infty}.
\end{equation}

where $\mathbb{P}^{(x,y)}$ refers to the two-fold Palm distribution defined above (1.10).

The FKG-inequality is satisfied, i.e., if $A_1, A_2 \subset \Omega$ are increasing events (meaning that if $\omega_1 \in A_1$ and $\omega_1 \subset \omega_2$, then $\omega_2 \in A_1$ for $i = 1, 2$), then $\mathbb{P}(A_1 \cap A_2) \geq \mathbb{P}(A_1)\mathbb{P}(A_2)$.

We let $v_\varepsilon(\omega) := n(\omega, e)e$ and assume that there exist constants $c_3, c_4 > 0$ such that
\begin{equation}
\forall \theta > 0 \forall e \in \mathbb{Z}^d \text{ with } |e|_1 = 1 : \quad \mathbb{P}_0\{|v_\varepsilon(\omega)| > \theta\} \leq c_3 e^{-c_4 \theta}.
\end{equation}

Condition (P1) guarantees that $d_\omega$ is comparable to the Euclidean distance with high probability, while (P5) is a standard monotonicity inequality satisfied by many models. Since $\mathbb{P}$ is ergodic with respect to $\tau_\varepsilon$ (by (P1)) and since $\mathbb{P}(0 \in \mathcal{C}_\infty) > 0$, by the Poincaré recurrence theorem (cf. [P89, Sec. 2.3]) we have $n(\omega, e) < \infty$. Then (P6) implies that moving along the coordinate axes has good recurrence properties and that $v_\varepsilon$ possesses all moments under $\mathbb{P}_0$. These assumptions are natural and are satisfied by many well-studied models – we refer to Appendix A.

2. Main results

In Section 2.1 we will introduce the equation (1.2) in a precise form and record the necessary assumptions. In Section 2.2 we will announce our main results, while in Section 2.3 we will outline the principal ingredients of the proof.

2.1 The HJB equation. We now state the assumptions on the diffusion coefficient $a$, the Hamiltonian $H$ and the initial value $f$ appearing in (1.2), which guarantee existence and uniqueness of a viscosity solution (cf. Proposition 3.3). Stating these assumptions require some further notation.

Denote by $\mathcal{S}_d$ the space of $d \times d$ symmetric matrices. There is a natural partial order in $\mathcal{S}_d$: we say that for $A, B \in \mathcal{S}_d, A \leq B$ if $B - A$ is positive semidefinite, i.e., all its eigenvalues are nonnegative. For any symmetric, positive semidefinite matrix $a$ (which will be defined below in (F1)) denote by $\sigma \in \mathcal{S}_d$ the unique (symmetric and positive semidefinite) matrix such that
\begin{equation}
a = \frac{1}{2} \sigma \sigma^T \quad \text{on } \Omega_0.
\end{equation}

We also define the inner product $\langle \cdot, \cdot \rangle_a = \langle \cdot, \cdot \rangle_{a(\omega)}$ as
\begin{equation}
\langle x, y \rangle_a := \langle a(\omega)x, y \rangle = \langle x, a(\omega)y \rangle \quad \forall x, y \in \mathbb{R}^d,
\end{equation}

which defines a semi-norm
\begin{equation}
\|x\|_a := \sqrt{\langle x, x \rangle_a}.
\end{equation}

We impose the following assumptions on $a, H$ and $f$:

(F1) The matrix $a : \Omega \to \mathcal{S}_d$ is positive semidefinite, and
\begin{equation}
a(x, \omega) := a(\tau_x \omega)
\end{equation}
defines a stationary process with respect to $\{\tau_x\}_{x \in \mathbb{R}^d}$. The restriction of $a$ to $\Omega_0$ (defined in (1.12)) satisfies the following: there exists $c_5 \in (0, \infty)$ and a measurable function $\xi : \Omega_0 \to (0, \infty)$ such that $\mathbb{P}_0$-a.s.
\begin{equation}
\xi(\omega)|x|^2 \leq \langle a(\omega)x, x \rangle \leq c_5 |x|^2 \quad \forall x \in \mathbb{R}^d.
\end{equation}
Furthermore, there exists $\delta > 0$, $\alpha > 1 + \delta$, $\gamma > d$ such that
\[
\mathbb{E}_0[\xi(\omega)^{-\chi}] < \infty,
\] (2.3)
where
\[
\chi = \chi(\alpha, \gamma, \delta) := \frac{\alpha}{2} \max \left\{ \frac{1 + \delta}{\alpha - (1 + \delta)}, \frac{\gamma}{\alpha - 1} \right\}.
\] (2.4)

For $\mathbb{P}_0$-a.s. $\omega$, the maps $x \in \mathbb{R}^d \mapsto \sigma(x, \omega) := \sigma(\tau_x \cdot) \in \mathcal{S}_d$ and $x \in \mathbb{R}^d \mapsto \text{div} a(x, \omega) \in \mathbb{R}^d$ are Lipschitz continuous, with Lipschitz constant independent on $\omega$. Moreover,
\[
\text{supp}(a) \subset \overline{C}_\infty
\]
and $|\text{div} a|$ is uniformly bounded. Since $a$ is Lipschitz, we can assume that $x \mapsto \xi(\tau_x \omega)$ is also Lipschitz by taking the minimum eigenvalue of $a$.

*(F2)* The Hamiltonian $H : \mathbb{R}^d \times \Omega \to \mathbb{R}$ satisfies for each $\omega \in \Omega$ that $p \mapsto H(\pm p, \omega)$ is convex. Moreover, there are constants $c_6, \ldots, c_9 > 0$ such that for all $(p, \omega) \in \mathbb{R}^d \times \Omega_0$,
\[
c_6 \|p\|_a^2 - c_7 \leq H(p, \omega) \leq c_8 \|p\|_a^2 + c_9,
\] (2.5)
and $H(\cdot, \omega) \equiv 0$ outside $\Omega_0$. Here, $\alpha > 1 + \delta$ and $\delta > 0$ are arbitrary (but specified in (2.3)). Equivalently, for $\alpha' := \frac{\alpha}{\alpha - 1}$ and constants $c_{10}, \ldots, c_{13}$,
\[
c_{10} \|q\|_{\alpha'} - c_{11} \leq L(q, \omega) \leq c_{12} \|q\|_{\alpha'} + c_{13},
\] (2.6)
where $L(q, \omega) := \sup_{p \in \mathbb{R}^d} \langle p, q \rangle - H(p, \omega)$. In particular, $L(\cdot, \omega) \equiv 0$ outside $\Omega_0$.

*(F3)* The map $x \mapsto H(x, p, \omega) := H(p, \tau_x \omega)$ defines a stationary process with respect to translations. Moreover, there are constants $c_{14}, c_{15}, c_{16} > 0$ such that for any $\omega \in \Omega_0$, $x, y \in C_\infty(\omega)$ and $p \in \mathbb{R}^d$,
\[
|H(x, p, \omega) - H(y, p, \omega)| \leq (c_{14} \|p\|_{\alpha} + c_{15}) |x - y|,
\] (2.7)
\[
|H(x, p, \omega) - H(x, q, \omega)| \leq c_{16} (\|p\| + \|q\| + 1)^{\alpha - 1} |p - q|.
\] (2.8)

*(F4)* The initial condition $f : \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous. In particular, for any $\delta > 0$, there exists some $K_\delta > 0$ such that for any $x, y \in \mathbb{R}^d$,
\[
|f(x) - f(y)| \leq K_\delta |x - y| + \delta.
\] (2.9)

### 2.2 Main results.

For any $\varepsilon > 0$, $T > 0$ and $\omega \in \Omega_0$, consider the Hamilton-Jacobi-Bellman equation
\[
\begin{aligned}
\partial_t u_\varepsilon &= \frac{\varepsilon}{2} \text{div} \left( a(\frac{x}{\varepsilon}, \omega) \nabla u_\varepsilon \right) + H(\frac{\varepsilon}{x}, \nabla u_\varepsilon, \omega), \\
&M \in (0, T) \times \varepsilon C_\infty(\omega), \\
u_\varepsilon(t, x) = f(x), \quad &\text{in } \{0\} \times \varepsilon C_\infty(\omega) \cup ((0, T) \times \varepsilon \partial C_\infty) \Omega_0.
\end{aligned}
\] (2.10)

We are now ready to state our first main result.

**Theorem 2.1.** Assume *(P1),(P6)* on the point process $\mathbb{P}$, *(F1),(F4)* on $a$, $H$ and $f$, and let $u_\varepsilon$ be the unique viscosity solution of (2.10). Then $\mathbb{P}_0$-almost surely and as $\varepsilon \to 0$, we have that $u_\varepsilon \to u_{\text{hom}}$ uniformly on compact sets, where $u_{\text{hom}}$ is the unique viscosity solution of
\[
\begin{aligned}
\partial_t u_{\text{hom}} &= \overline{P} \nabla u_{\text{hom}}, \\
u_{\text{hom}}(t, x) = f(x), \quad &\text{in } (0, \infty) \times \mathbb{R}^d, \\
u_{\text{hom}}(0, x) = f(x), \quad &\text{on } \mathbb{R}^d.
\end{aligned}
\] (2.11)
Here, the effective Hamiltonian \( \overline{H} \) is given by the variational formula
\[
\overline{H}(\theta) = \inf_{G \in \mathcal{G}_\delta} \left( \operatorname{esssup}_{y \in \mathbb{R}^d} \left[ \frac{1}{2} \text{div}(a(G+\theta)) + H(G+\theta) \right] \right),
\]
where the class \( \mathcal{G}_\delta \) contains \( L^{1+\delta}(\mathbb{P}_0) \) functions (with \( \delta > 0 \) being arbitrary, but specified in \( (\ref{subadd}) \) defined in Section 4. Moreover, the homogenized limit \( u_{\text{hom}} \) is given by the Hopf-Lax formula
\[
u_{\text{hom}}(t, x) = \sup_{y \in \mathbb{R}^d} \left( f(y) - t \mathcal{I} \left( \frac{y-x}{t} \right) \right), \quad \text{with} \quad \mathcal{I}(y) := \sup_{\theta \in \mathbb{R}^d} \left[ (\theta, y) - \overline{H}(\theta) \right].
\]

**Remark 1** The equation \( (1.2) \) can also be rewritten in a non-divergence form as
\[
\partial_t u_\varepsilon = \frac{1}{2} \text{Tr} \left( a(\frac{X}{\varepsilon}, \omega) \text{Hess}_x u_\varepsilon \right) + \mathcal{H} \left( \frac{X}{\varepsilon}, \nabla u_\varepsilon, \omega \right), \quad \text{with} \quad \mathcal{H}(x, p, \omega) = H(x, p, \omega) + \frac{1}{2} \text{div}(a(x, \omega)) \cdot p.
\]
By \( (1.1) \) \( |\text{div} a| \leq C \) is bounded and therefore our assumptions on \( H \) translate to that of \( \mathcal{H} \). Consequently, homogenization of the above equation is covered also by Theorem 2.1.

A particular case of \( H \), which is appealing from a probabilistic viewpoint, is the quadratic Hamiltonian
\[
H_b(p, \omega) := \frac{1}{2} \| p \|^2_a + \langle b(\omega), p \rangle_a.
\]
With this choice, Theorem 2.1 leads to the following result. For any configuration \( \omega \in \Omega_0 \), let \( P_0^\omega \) denote the law of the diffusion
\[
dX_t = \sigma(X_t, \omega)dW_t + \text{div} a(X_t, \omega)dt + a(X_t, \omega)b(X_t, \omega)dt
\]
starting at 0 in the environment \( \omega \), where \( (W_t)_{t \geq 0} \) is a standard Brownian motion in \( \mathbb{R}^d \) (whose law is independent of \( \mathbb{P} \)). Our next main result is a quenched large deviation principle for the degenerate diffusion with a random drift on a continuum percolation cluster:

**Corollary 2.2.** Assume \( (P1)(P6) \) on the point process \( \mathbb{P} \), and \( (F1) \) on \( a(\cdot) \) such that \( (2.3) \) holds with \( \chi = \frac{\alpha(1+\delta)}{\alpha(1+\delta))} \). Let \( b : \Omega \rightarrow \mathbb{R}^d \) be a map so that \( x \mapsto b(\tau_x \omega) \) defines a stationary process w.r.t. translations and \( H_0 \) defined in \( (2.11) \) satisfies \( (2.7)-(2.8) \) (for these to hold, it suffices to assume that \( x \mapsto b(x, \omega) \) is bounded and Lipschitz). Then for \( \mathbb{P}_0 \)-almost every realization \( \omega \in \Omega_0 \), the distribution \( P_0^\omega \left[ X_t \in \cdot \right] \) satisfies a large deviation principle with rate function
\[
I(x) = \sup_{\theta \in \mathbb{R}^d} \left\{ \langle \theta, x \rangle - \overline{H}(\theta) \right\}, \quad \text{with} \quad \mathcal{H}(x, p, \omega) = H(x, p, \omega) + \frac{1}{2} \text{div}(a(x, \omega)) \cdot p.
\]

**Remark 2** We can also consider Hamiltonians of the type
\[
H_{b,v}(p, \omega) := \frac{1}{2} \| p \|^2_a + \langle b(\omega), p \rangle_a - V(\omega),
\]
and show an \( \mathbb{P}_0 \)-almost sure large deviation principle for the distribution of \( X_t/t \) under the measure \( dQ_0^\omega \propto e^{-\int_0^t V(X_s, \omega)ds}dP_0^\omega \) if we assume some moment condition on the potential \( V \) w.r.t. \( \mathbb{P}_0 \), which provides an absorbing random environment. We refer to a Armstrong-Tran [ATT14, Corollary 2] where such a result has been obtained (in a stationary ergodic setting) using the sub-additive ergodic theorem,
which was developed in a detailed study for Brownian motion \( (a \equiv \text{Id}, b \equiv 0) \) in a Poissonian potential by Sznitman [S94] (see also Kosygina [K08] Section 7).

**Remark** Note that the moment condition (2.3) would hold, for instance, for \( \delta \sim 0, \alpha = 2 \) and any \( \gamma > d \) (so that \( \chi = \max\left(\frac{1+\delta}{2-\delta}, \gamma\right) = \gamma > d \)). We also remark that (2.3) with the exponent \( \chi = (\alpha \gamma)/2(\alpha - 1) \) for \( \gamma > d \) is needed for Theorem 2.1 at one step in its lower bound (cf. the discussion on p.10 in Section 2.3), while the other exponent with \( \chi = \frac{(1+\delta)}{2(\alpha-(1+\delta))} \) is used to obtain a weak limit \( G \in G_\delta \). Note that the former assumption (carrying the term \( \gamma > d \)) is not needed for Corollary 2.2 (see Section 5.4.2 for its proof). Also, if we required \( \xi(\cdot) \geq c_0 > 0 \) on \( \Omega_0 \) (i.e., if \( a \) were uniformly elliptic just inside the cluster, which we do not assume), then the condition (2.5) would hold for any \( \alpha > 1 \). Finally, we remark that when the framework is discrete and \( b \equiv 0 \) (i.e., reversible), a more specific case of Corollary 2.2 corresponds to studying large deviations for simple random walk on percolation clusters in \( \mathbb{Z}^d \) [K12, BMO16], see also [SS04, MP07, BB07] for CLT results. However, by definition, this setup is automatically uniformly elliptic inside the cluster (the transition probability \( \pi_{\omega}(0 \leftrightarrow e) \geq 1/2d \) if the edge \( 0 \leftrightarrow e \) in the discrete lattice is present in the environment \( \omega \) and also reversible where [KV86] plays a crucial role (this is different from treating HJB equations). Also, for large deviations one uses a change of measure argument that is not applicable to a general Hamiltonian as in Theorem 2.1.

### 2.3 Ingredients of the proof.

The goal of this section is to underline the main ingredients of the proof, for which, as a guiding philosophy we will follow a novel method developed by Kosygina-Resakhanlou-Varadhan [KRV06] for treating viscous HJB equation (when \( a(\omega) \equiv \text{Id} \)) in a stationary ergodic setting, see also Kosygina [K08, Sec. 6] for a review on this approach and Kosygina-Varadhan [KV08] for an extension of this method to a time-dependent set up. The root of this approach goes back to the seminal work of [LPV87] and the framework of environment seen from the particle developed in [PV81, K85, KV86]. In the present scenario, fundamental difficulties stem from a combination of non-translational invariance and degeneracy of HJB equations on percolation clusters. For the convenience of the reader, we will briefly outline the [KRV06] approach and subsequently underline the new input of the current method.

**The previous approach of [KRV06]:** Let us denote by \( \mathbb{P} \) the law of a stationary and ergodic random environment, with \( \tilde{u}_\varepsilon \) solving \( \partial_t \tilde{u}_\varepsilon = \frac{\varepsilon}{2} \Delta \tilde{u}_\varepsilon + \tilde{H}(\varepsilon^{-1} x, \nabla \tilde{u}_\varepsilon, \omega) \) with \( \tilde{u}_\varepsilon(0, x) = \tilde{f}(x) \) being uniformly continuous and \( p \mapsto \tilde{H}(p, \omega) \) being convex and satisfying uniformly in \( \omega \), \( \tilde{H}(p, \omega) \sim |p|\alpha \) for \( \alpha > 1 \) suitably large. To avoid technicalities we drop recalling further conditions which were assumed earlier. The method consists of three main steps.

**Lower bound.** The optimal control representation

\[
\tilde{u}_\varepsilon(t, x, \omega) = \sup_{c \in C_B} E^{Q^c}_{x, \varepsilon} \left[ \tilde{f}(\varepsilon \bar{X}(t/\varepsilon)) - \int_0^{t/\varepsilon} \tilde{L}(\bar{X}(s), c(s, \bar{X}(s)), \omega) ds \right]. \tag{2.18}
\]

Here \( C_B \) denotes the space of all bounded controls \( c : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) and \( Q^c_x \) denotes the law of the \( \mathbb{R}^d \)-valued diffusion \( \bar{X}(t) = x + \int_0^t c(s, \bar{X}(s)) ds + B(t) \) starting at \( x \in \mathbb{R}^d \). Then for fixed \( x = 0 \), a lower bound on \( \tilde{u}_\varepsilon(t, 0, \omega) \) is obtained by restricting to controls of the form \( c(s, x, \bar{X}(s)) = b(s, \omega) \) for some \( b \in L^\infty(\mathbb{P}) \) and invoking the ergodic theorem:

\[
\liminf_{\varepsilon \to 0} \tilde{u}_\varepsilon(t, 0, \omega) \geq \sup_{(b, \phi)} \left[ f \left( t \int \mathbb{P}(d\omega) \phi(\omega) b(\omega) \right) - t \int \mathbb{P}(d\omega) \phi(\omega) \tilde{L}(b, 0) \right]. \tag{2.19}
\]
Here the supremum is taken over those pairs \((b, \phi)\) such that \(\phi \, d\mathbb{P}\) is an invariant measure for \(A_b := \frac{1}{2} \Delta + b \cdot \nabla\) with \(b, \phi, \nabla \phi, \nabla^2 \phi \in L^\infty(\mathbb{P}).\) The above lower bound (at \(x = 0\)) is extended to a locally uniform bound using the uniform ellipticity of the matrix \(a(\omega) \equiv \text{Id}\) in this set up and also translation-invariance of \(\mathbb{P}\).

**Convex variational analysis:** Note that, for linear initial data \(f(x) = \langle p, x \rangle\), the lower bound (2.19) is of the form \(\overline{\mathcal{P}}(p)\), where \(\overline{\mathcal{P}}(p) := \sup_{(b, \phi)} \mathbb{E}^p[\phi(\omega)(\langle p, b(\omega) \rangle - L(b(\omega), \omega))]\). Because \(\inf_u \mathbb{E}^p[\phi(\omega)A_b u(\omega)] = -\infty\) unless \(\phi \, d\mathbb{P}\) is an invariant measure for \(A_b\), the supremum over \((b, \phi)\) can be decoupled by adding a Lagrange multiplier, leading to

\[
\overline{\mathcal{P}}(p) = \sup_{\phi} \sup_{b} \inf_{u} \mathbb{E}^p[\phi(\omega)(\langle p, b(\omega) \rangle - L(b(\omega), \omega) + A_b u(\omega))].
\]

Starting from the above observation, \([KRV06]\) developed convex variational analysis by successively applying min-max theorems. The success of this min-max method relies on, among other requirements, “compactness” of the underlying variational problems. In the stationary ergodic setting of \([KRV06]\), this compactness becomes readily available if one restricts the relevant variational problem(s) (e.g. the supremum over \(u\)) to bounded regions. Then one can successively pass to further lower bounds \(\overline{\mathcal{P}}_k(p)\) at finite truncation level \(k\), which leads to approximate gradients \(v_k := \nabla u_k\). The uniform super-linearity assumption \(p \mapsto H(p, \omega) \gtrsim |p|^\alpha\) leads to a moment condition which implies existence of a \(\lim_{k \to \infty} v_k \in L^\alpha(\mathbb{P})\), which (in the stationary ergodic set up) is a stationary gradient and satisfies a mean-zero property \(\mathbb{E}^p[v] = 0\). It is worth noting that, in this set up, both properties are direct consequences of the invariant action of \(\tau_x\) w.r.t. the environment law \(\mathbb{P}\). Construction of such a \(v\) and successive application of min-max theorems then lead to a suitable variational lower bound on \(\overline{\mathcal{P}}(p)\).

**Upper bound:** Using the stationary gradients \(v\) constructed above, one then considers the path integral \(V(\omega, x) = \int_{0 \to x} \langle v, dz(s) \rangle\) with the normalization \(V(\omega, 0) = 0\) a.s. An important step for obtaining a matching upper bound then entails showing the sub-linear growth property \(V(x, \omega) = o(|x|)\) as \(|x| \to \infty\) almost surely w.r.t. \(\mathbb{P}\). In the stationary ergodic set up, this result was shown using the aforementioned mean-zero property of \(v\) (w.r.t. \(\mathbb{P}\)) and the ergodic theorem, combined with uniform coercivity and further assumptions imposed on \(\mathcal{H}\) (e.g. by using uniform gradient estimates for sufficiently regular \(H\), or by the Sobolev embedding theorem when \(\alpha > d\), or by a perturbation method when \(\alpha > 2\) and \(H\) satisfies \(D^2 H(p, \omega) \geq c I\) on \(\{p \in \mathbb{R}^d : |p| \geq k\}\) for \(c, k > 0\)). Then by comparison with a super-solution \(\tilde{u}_c(t, x, \omega) := \langle p, x \rangle + t\overline{\mathcal{P}}(p) + c V(x/c, \omega)\) and using that the perturbation caused by \(V\) is negligible, thanks to its sub-linear growth, provide a “matching” upper bound.

**The current method:** In the current set up, we also follow the earlier philosophy for the lower bound and consider a variational representation (3.8) of the solution \(u_x\) of (1.2). For reasons to be explained below, instead of deterministic and bounded \(c \in C_b\), we choose progressively measurable

---

1. Since the supremum in (2.19) is taken for every fixed \(\omega\), one can allow the control \(c \in C_b\) to be \(\omega\)-dependent, and for a lower bound, restrict to those \(c\) which are independent of the time variable \(s\) and stationary in the space variable \(x\), i.e., \(c(s, x, \omega) = b(\tau_s \omega)\) for some \(b \in L^\infty(\mathbb{P})\). Working with such controls allows one to study the *environment seen from the particle*, which is a diffusion process \(\mathcal{X}(\omega) = \tau_X(\omega) \in \Omega\) taking values in the environment space \(\Omega\), starting at \(\omega \in \Omega\) with generator \(A_b\). By restricting further to those \(b \in L^\infty(\mathbb{P})\) with an invariant density \(\phi\) (for the generator \(A_b\) with \(\nabla \phi, \nabla^2 \phi \in L^\infty(\mathbb{P})\), one uses ergodic properties of the environment process \(\mathcal{X}\). These ergodic properties then translate also to those of the original diffusion \(X(t)\), leading to (2.19). Let us also note that, here gradient \(\nabla = (\nabla_{i})_{i=1}^d\) (and likewise \(\Delta\)) is defined in a weak sense: \(\nabla_i\) is the infinitesimal generator of the translation group \(\{\tau_x\}_{x \in \mathbb{R}^d}\) acting on \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) via

\[
(\nabla_i f)(\omega) = \lim_{h \to 0} \frac{f(\tau_{x, h}\omega) - f(\omega)}{h}.
\]
controls sampled from an auxiliary probability space \((\mathcal{X}, \mathcal{F}, P)\) \(^2\) and work with Lipschitz maps \(b \in L_0^1(\phi dP_0)\) (instead of bounded \(b\)), see Section (3.2.2). Due to the non-invariance and non-ellipticity of \(P\), the ergodic theorem (shown in Theorem [3.1]) needs extra care which leads to a lower bound on \(\liminf_{\varepsilon \to 0} u_\varepsilon(t, 0, \omega)\) at \(x = 0\), see Lemma [3.9] and Lemma [3.10]. The usual step then is to obtain (uniform in \(\varepsilon\)) Lipschitz estimates for the solutions of (1.2) in order to upgrade this inequality to locally uniform convergence. However, since the Hamiltonian is not uniformly coercive in our case (recall (2.5)), the Lipschitz estimates are only local. Nevertheless, one can control the oscillations of \(u_\varepsilon\) around \(x = 0\) uniformly in \(t\) and \(\varepsilon\) for \(|x|\) small by applying Morrey’s inequality and a comparison principle from [AT15] and [D19], cf. Lemma 3.11. To apply this inequality, we need the moment assumption (2.3) with the exponent \(\chi = (\alpha \gamma)/2(\alpha - 1)\) for \(\gamma > d\). The locally uniform lower bound, which is first obtained for smooth initial condition, is extended to any uniformly continuous \(f\) in Lemma [3.12]-Lemma [3.15].

Now for the variational analysis, we are not allowed to apply the min-max route using restriction to bounded regions. Indeed, first note that, one does not expect a mean-zero property of a prospective weak limit w.r.t. \(P_0\) which is not shift-invariant. However, one can refine the mean-zero condition by studying shifts defined by the successive arrivals of the continuum cluster along coordinate directions (see (1.13)). But then, restriction to a bounded region is incompatible with non-translation invariance of \(P_0\) – for any prospective limit point, the latter would deny the gradient condition (recall from (2.20)) that gradients are defined on \(\Omega_0\) with respect to the usual shifts \(\tau_x\) that leaves \(P_0\) non-invariant), while the former would be discordant with the refined mean-zero condition which requires keeping track of arbitrarily long cluster excursions. Therefore, we use a different route. For the first step of the min-max theorem, we exploit the intrinsic coercivity properties of the Hamiltonian (w.r.t. \(a\)), which propagates to the accompanying variational formula. At this step, our choice of the class \((b, \phi) \in \mathcal{E}\) in (3.17) is important where we work with Lipschitz maps \(b \in L_0^1(\phi dP_0)\). If we were to use uniformly bounded maps \(b \in L^\infty(P_0)\) as considered previously, this class would not be closed in \(L^p(P_0)\) for \(p \geq 1\). This closeness is crucial for showing weak compactness in the first min-max step in Lemma 5.3. For the second step, we introduce a subtractive relative entropy term which is structurally well-suited to the optimal control variational formula accompanying from the preceding steps. This entropy term provides the requisite coercivity in order to apply the second min-max theorem, see Lemma 5.4. Combined with the moment assumption (2.3) for \(\chi = \frac{\alpha(1+\delta)}{2(\alpha-1+\delta)}\), subsequently we are able to deduce existence of a weak limit \(G \in L^{1+\delta}(P_0)\).

An advantage of this approach is that, the weak limit \(G\) is now both curl-free and refined mean-zero, being conformant to the properties of \(P_0\) and that of the cluster, see Lemma 5.6-Lemma 5.8 for details. We note that, while this technique seems to be a natural approach for treating degenerate HJB for non-stationary set up, it is also unifying with the earlier [KRV06] approach in the sense that the coercivity of \(H\), used in the first min-max step, is an intrinsic assumption for HJB (regardless of the set up). Similarly, the relative entropy structure invoked in the second min-max step is well-suited to the preceding variational formulas from here that are applicable to both frameworks, and the desired limiting properties are established in a natural way, the properties being determined by the respective set up.

Turning to the upper bound, an important step here also involves showing sub-linearity of the path integral \(V_0(\omega, x) = \int_{0 \to x} G, dz(s) = o(|x|), P_0\text{-a.s.}\) Note that our assumptions on \(a, H\), the geometry of the continuum percolation as well as properties of the limit points \(G\) are different from [KRV06]. Therefore, the proof of this step is also quite different here for which we build on the assumptions

\(^2\)For conceptual reasons it might be useful to note that the progressively measurable control \(c\) is sampled from a fixed auxiliary probability space, and therefore, the SDE (3.7) admits a strong solution which is not Markovian. In contrast, the SDE \(\tilde{X}\) underneath (2.18) admits a weak solution for \(c \in C_0\).
Using these, quite some technical effort is needed to show also this step in the current scenario, which constitutes Section 4. Note that in the continuum framework we are not allowed to invoke arguments based on combinatorial counting, neither do not assume uniform ellipticity inside the infinite cluster, which are key properties used in the aforementioned works on limit theorems for simple random walks on discrete percolation clusters (in a reversible set up, which is different from studying HJB equations). Then using a mollification and continuity argument, and combined with the arguments from the lower bound part, the requisite upper bound is shown in Proposition 5.9 and Section 5.4.

An orthogonal approach to the [KRV06] method for treating HJB in a stationary ergodic set up involves sub-additivity [So99, RT00, LS05, LS10, AT14]. We believe that such a method could also be extended to the current percolation set up (with extra work). While sub-additivity does not immediately yield a variational formula for the effective Hamiltonian (in contrast to the present method), we refer to the very interesting work [AT14, Remark 3] in the stationary ergodic where such a variational formula has been obtained also using the sub-additive ergodic theorem.

Organization of the rest of the article: In Section 3 we will provide the lower bound for the solution of HJB equations. Section 4 is devoted to studying properties of the “correctors” and in Section 5 we will carry out the variational analysis and complete the proof of the upper bound and that of Theorem 2.1 and Corollary 2.2. In Appendix A we will provide examples of percolation models covered by our set up and in Appendix B-C we will collect some auxiliary arguments that are used in the sequel.

3. Viscosity solutions, ergodic theorem and the lower bound

3.1 Viscosity solutions of H-J-B equations on percolation clusters.

The goal of the section is to provide existence and uniqueness of the PDE (2.10), by explicitly representing its solution as the value function of an optimal control problem. Notice that by (F1) the HJB equation (2.10) with $\varepsilon = 1$ can be written as, for any $\omega \in \Omega_0$,

$$\begin{cases}
\frac{\partial u}{\partial t} = \mathcal{H}(x, \nabla u, \text{Hess}_x u) & \text{in } V, \\
u(t, x) = f(x) & \text{on } \partial V
\end{cases}$$

for the open set

$$V := (0, T) \times C_\infty(\omega),$$

and the Hamiltonian

$$\mathcal{H}(x, p, q) := \frac{1}{2} \text{div}(a(x, \omega)) \cdot p + \frac{1}{2} \text{Trace}(a(x, \omega) q) + H(x, p, \omega).$$

Here and it what follows, we denote by $\partial V$ the parabolic boundary. In the case when $V = (0, T) \times U$ for an open set $U \subset \mathbb{R}^d$, $\partial V = ((0, T) \times \partial U) \cup \{0\} \times U$.

To define the notion of viscosity solutions, we need some further notation. First, we recall the definition of the upper and lower semicontinuous envelopes. For a set $V \subset \mathbb{R}_+ \times \mathbb{R}^d$, denote by $\text{USC}(\overline{V})$ the set of upper semicontinuous functions $w : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$. Similarly, $\text{LSC}(\overline{V})$ denotes the space of lower semicontinuous functions $w : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$.

**Definition 3.1.** Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $u : \overline{V} \mapsto \mathbb{R}$ be a locally bounded function. We define the upper semicontinuous envelope $u_* : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$ as

$$u^*(x) := \inf \{w(x) : w \in \text{USC}(\overline{V}) \text{ and } w \geq u\}.$$

The lower semicontinuous envelope is defined as $u_* := -(u^*)^*$. 

(P1) (P6)
It follows directly from the definition that $u_\ast \leq u \leq u^\ast$. Moreover, $u_\ast \in \text{LSC}(\overline{V})$ while $u^\ast \in \text{USC}(\overline{V})$.

**Definition 3.2 (Viscosity sub/super-solutions).** Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$.

- We say that a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity subsolution of
  
  \[
  \frac{\partial u}{\partial t} = \mathcal{H}(x, \nabla u, \text{Hess}_x u) \quad \text{in } V
  \]

  if for all $(s, y) \in V$ and smooth function $\phi$ in a neighborhood of $(s, y)$ such that the map $(t, x) \mapsto (u^\ast - \phi)$ has a local maximum at $(s, y)$, one has

  \[
  \frac{\partial \phi(t_0, x_0)}{\partial t} - \mathcal{H}(x_0, \nabla \phi(t_0, x_0), \text{Hess}_x \phi(t_0, x_0)) \leq 0.
  \]

- Similarly, we say a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity supersolution of

  \[
  \frac{\partial u}{\partial t} = \mathcal{H}(x, \nabla u, \text{Hess}_x u) \quad \text{in } V
  \]

  if for all $(s, y) \in V$ and smooth function $\phi$ in a neighborhood of $(s, y)$ such that the map $(t, x) \mapsto (u_\ast - \phi)$ has a local minimum at $(s, y)$, one has

  \[
  \frac{\partial \phi(t_0, x_0)}{\partial t} - \mathcal{H}(x_0, \nabla \phi(t_0, x_0), \text{Hess}_x \phi(t_0, x_0)) \geq 0.
  \]

- We say a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity solution of \((3.4)\) if it is both a subsolution and supersolution.

The existence of viscosity solution itself follows from Perron’s method (see for example [IS7]). However, as mentioned above, our goal is to characterize the solution as the value function of an optimal control problem of diffusions on percolation clusters. This is carried out as follows: we fix an auxiliary probability space $(\mathcal{X}, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \geq 0}$ on $\mathcal{F}$ and an auxiliary $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ (whose law $P$ is independent of the law $\mathbb{P}$ of the point process). Let

\[
C_T = \left\{ c : [0, T] \times \mathcal{X} \to \mathbb{R}^d : c \text{ is progressively measurable and } (3.6) \text{ holds.} \right\},
\]

where

\[
E^P \left[ \int_0^T c(s)^2 ds \right] < \infty,
\]

By \((F1)\), for each $c \in C_T$ and $x_0 \in C_\infty$, there is a unique strong solution to the stochastic differential equation (SDE)

\[
X_t = x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t (\text{div } a)(X_s) ds + \int_0^t a(X_s) c(s) ds \quad \text{a.s. } \forall t \geq 0,
\]

see [T13 Theorem 3.1]. Let us also mention that the above display is understood to hold in a pointwise sense for every fixed $\omega$ (and for each realization of the auxiliary probability space $\mathcal{X}$) which is suppressed from the notation. Now, note that by our assumptions, $X_t \in \mathcal{X}_\infty$ for all $t \geq 0$ (cf. [BGHJ21 Lemma 3.4]). Denote by $P^x_\omega$ the law of the solution of \((3.7)\) and set

\[
u(t, x, \omega) := \sup_{c \in C_T} J(t, x, c),
\]

where

\[
J(t, x, c) := E^{P^x_\omega} \left[ f(X_t) - \int_0^t L(X_s, c(s)) ds \right].
\]

The characterization of the solution to \((2.10)\) as an optimal control problem is given by the next proposition.
Proposition 3.3. Assume \(F1\)(F4) Then for \(\mathbb{P}_0\)-almost every realization \(\omega \in \Omega_0\), the function \(u\) in (3.3) is the unique viscosity solution (cf. 3.2) of
\[
\begin{cases}
\partial_t u = \frac{1}{2} \text{div} \left( a(x, \omega) \nabla u \right) + H(x, \nabla u, \omega), & \text{in } (0, T) \times C_\infty(\omega), \\
 u(t, x, \omega) = f(x), & \text{on } \{(0) \times C_\infty(\omega)\} \cup ((0, T) \times \partial C_\infty(\omega))
\end{cases}
\]  
(3.10)
of at most linear growth.

Proof. The existence and uniqueness follow from dynamic programming and a comparison principle, respectively, see Appendix C for details. \(\square\)

3.2 The ergodic theorem for the environment process on percolation clusters. The goal of this section is to prove an ergodic theorem (cf. Proposition 3.4 below) for the so-called environment process which, for homogenization of stationary ergodic random media (at least in the elliptic setting), goes back to the works of Kozlov [K85] and Papanicolau-Varadhan [PV82]. In our context, this environment process is a diffusion taking values in the space of conditioned environments \(\Omega_0\).

3.2.1. The environment process. Recall that the group \(\{\tau_x\}_{x \in \mathbb{R}^d}\) acts on \((\Omega, \mathcal{G}, \mathbb{P})\) via translations. This action allows us to define, for any \(u : \Omega \to \mathbb{R}\), its weak gradient via
\[(\nabla u)(\omega) := \lim_{\varepsilon \to 0} \frac{u(\tau_{\varepsilon e_i}(\omega)) - u(\omega)}{\varepsilon}, \quad i = 1, \ldots, d.
\]
Likewise, we also define the corresponding divergence. Now for \(a : \Omega \to \mathbb{R}^{d \times d}\) satisfying (F1) we set
\[(L^{(b)}u)(\omega) := \frac{1}{2} \text{div} \left( a(\omega) \nabla u(\omega) \right) + \langle b(\omega), \nabla u(\omega) \rangle_a \quad \forall \omega \in \Omega_0.
\]  
(3.11)
For a reasonable class of maps \(b : \Omega \to \mathbb{R}^d\) (which do not depend on the probability space \((\mathcal{X}, \mathcal{F}, \mathbb{P})\)) and class of test functions \(u\), \(L^{(b)}\) is the generator of a Markov process taking values on \(\Omega_0\) which can be defined as follows. Set \(b(x, \omega) := b(\tau_x \omega)\) and let \(X_t\) denote the \(\mathbb{R}^d\)-valued diffusion solving the SDE
\[X_t = \int_0^t \sigma(X_s) dB_s + \int_0^t (\text{div} a)(X_s) ds + \int_0^t a(X_s) b(X_s) ds \quad \text{a.s. } \forall t \geq 0,
\]
with quenched law \(\mathbb{P}^{b,\omega}_0\) and generator
\[(L^{(b,\omega)}u)(x) = \frac{1}{2} \text{div} \left( a(x, \omega) \nabla u(x) \right) + \langle b(x, \omega), \nabla u(x) \rangle_a.
\]  
(3.12)
Then
\[\bar{\omega}_t := \tau_{X_t} \omega \]  
(3.13)
is the \(\Omega_0\)-valued diffusion process with generator \(L^{(b)}\) defined in (3.11). We call \((\bar{\omega}_t)_{t \geq 0}\) the environment process with generator \(L^{(b)}\), and its law with initial condition \(\delta_\omega\) is denoted by \(Q^{b,\omega}\).

3.2.2. Invariant density for the environment process. Recall that \(\mathbb{P}_0 = \mathbb{P}(\cdot|\Omega_0)\). We write \(L^1_+ (\mathbb{P}_0)\) for the space of all non-negative and \(\mathbb{P}_0\)-integrable functions on \(\Omega\). Any probability density \(\phi \in L^1_+ (\mathbb{P}_0)\) with \(\int \phi d\mathbb{P}_0 = 1\) is an invariant density with respect to \(Q^{b,\omega}\) if
\[
\frac{1}{2} \text{div}(a \nabla \phi) = \text{div}(\phi(ab)), \quad \text{i.e., } (L^{(b)}\phi)^* = 0, \quad \text{in } \Omega_0,
\]  
(3.14)
with the generator \(L^{(b)}\) defined in (3.11). For any probability density \(\phi\), we also set
\[L^a_0(\phi d\mathbb{P}_0) := \left\{ b : \Omega_0 \to \mathbb{R}^d \text{ measurable: } \int d\mathbb{P}_0 \phi \|b\|_a < \infty \right\}.
\]  
(3.15)
In particular, for $P$ invariant under the induced shift, so
and $\omega \in C_x$ to the environmental process, we have

$$\text{We first show that Proposition 3.4:}$$

Proof of Proposition 3.4:

See Appendix B.

Proof.

and ergodic with respect to $P$. Proposition 3.5.

For every $\sigma$, $\phi$ shift

Then

Thus, for $P_0(A)$-a.s $\omega$, $E^{b,\omega}(\mathbb{1}_{A^c}(\tau_{X_i} \omega)) = 0$. Equivalently, for $P_0(A)$-a.s $\omega$, $E^{b,\omega}(\mathbb{1}_{A}(\tau_{X_i} \omega)) = 1$. In particular, for $P_0(A)$-a.s $\omega$, $\mathbb{1}_{A}(\tau_{X_i} \omega) = 1$ $Q^{b,\omega}$-a.s. We claim that this implies that $A$ is $P_0$-a.s. invariant under the induced shift, so $P_0(A) \in \{0, 1\}$. Since $P_0(A) > 0$, the equivalence between $Q$ and $P_0$ would be complete. To show the claim, notice that for $\omega$ as above, $\tau_{x;\omega} A$ for almost all $x \in C_\infty(\omega)$. Indeed, if there is a subset $V$ of $C_\infty(\omega)$ of positive Lebesgue measure satisfying $\tau_{x;\omega} \notin A$
for \( x \in V \), then since the diffusion visits every set of positive Lebesgue measure inside \( C_\infty(\omega) \), we would have \( P_{b,\omega}(X_1 \in V) > 0 \), so that \( Q_{b,\omega}(X_{1,2} \notin A) > 0 \), which would be a contradiction. Thus, for \( P_0 \)-a.s. \( \omega \in A \) and almost all \( x \in C_\infty(\omega) \), we have \( \tau_{x,\omega} \in A \). In other words,

\[
\int_A \int_{\mathbb{R}^d} I_{\{x: \tau_{x,\omega} \notin A\}} \, dxdP_0 = 0.
\]

By Fubini’s theorem,

\[
\int_{\mathbb{R}^d} \int_A I_{A^c}(\tau_{x,\omega}) \, dx \, dP_0 = 0.
\]

Hence, for almost all \( x \in \mathbb{R}^d \),

\[
\int_A I_{A^c}(\tau_{x,\omega}) \, dP_0 = \frac{1}{P(0 \in C_\infty)} \int_{\Omega} I_{A}(\omega) I_{A^c}(\tau_{x,\omega}) \, dP = 0.
\]

By the continuity of the map \( \mathbb{R}^d \ni y \mapsto I_{A}(\omega) I_{A^c}(\tau_{x,\omega}) \in L^1(P) \), we deduce that for all \( x \in \mathbb{R}^d \), \( I_{A}(\omega) I_{A^c}(\tau_{x,\omega}) = 0 \) \( P_0 \)-a.s. In particular, \( P_0 \)-a.s., for all \( x \in Q^d \) we have \( I_{A}(\omega) I_{A^c}(\tau_{x,\omega}) = 0 \). By definition of the induced shift (see \( \ref{3.18} \)), \( \phi(\omega, e) \in Q^d \) and we conclude that \( P_0 \)-a.s., \( I_{A}(\omega) I_{A^c}(\sigma(\omega)) = 0 \). In other words, \( A \) is invariant under the induced shift \( P_0 \)-a.s., which proves that \( Q \sim P_0 \). The other two assertions follow from standard arguments. \( \square \)

The following consequence of the last theorem is a law of large numbers for the trajectory of the diffusion.

**Corollary 3.6.** Fix \( (b, \phi) \in \mathcal{E} \). Then \( P_0 \times P_{0,\omega}^b \)-a.s.,

\[
\lim_{t \to \infty} \frac{X_t}{t} = \mathbb{E}_0 \left[ \phi(\omega) \left( \frac{1}{2} \text{div} a(\omega) + a(\omega)b(\omega) \right) \right]. \tag{3.19}
\]

**Proof.** By definition, \( X_t \) satisfies

\[
X_t = \int_0^t \sigma(X_s) \, dB_s + \int_0^t \left( \frac{1}{2} \text{div} a + ab \right)(X_s) \, ds. \tag{3.20}
\]

Since \( \sigma \) is bounded, the stochastic integral divided by \( t \) goes to 0 \( P_0 \times P_{0,\omega}^b \)-a.s. Moreover, Proposition \( \ref{3.3} \) yields

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( \frac{1}{2} \text{div} a + ab \right)(X_s) \, ds = \mathbb{E}_0 \left[ \frac{1}{2} \text{div} a + ab \right] \phi \quad P_0 \times P_{0,\omega}^b \)-a.s. \tag{3.21}
\]

This finishes the proof. \( \square \)

The following immediate consequence of Proposition \( \ref{3.4} \) and Corollary \( \ref{3.6} \) will be used several times in the sequel:

**Corollary 3.7.** Fix \( (b, \phi) \in \mathcal{E} \). Then for \( P_0 \) almost every \( \omega \in \Omega_0 \) and \( P_{0,\omega}^b \)-a.s. and in \( L^1(P_{0,\omega}^b) \), we have

\[
\lim_{\varepsilon \to 0} \int_0^{t/\varepsilon} b(X_s, \omega) \, ds = t \int P_0(\, d\omega) \phi(\omega) b(\omega),
\]

\[
\lim_{\varepsilon \to 0} \int_0^{t/\varepsilon} L(X_s, b(X_s, \omega), \omega) \, ds =: t h(b, \phi) = t \int P_0(\, d\omega) \phi(\omega) L(b(\omega), \omega) \phi(\omega), \tag{3.22}
\]

\[
\lim_{\varepsilon \to 0} \varepsilon X_{t/\varepsilon} =: t m(b, \phi) = t \int P_0(\, d\omega) \phi(\omega) \left( \frac{1}{2} \text{div}(a(\omega)) + b(\omega) \right),
\]

uniformly on \( [0, T] \).
3.3 The lower bound. The main result of this section is the following the lower bound:

**Theorem 3.8.** Under \((F1),(F4)\) let \(u_\varepsilon(t,x)\) be the solution of \((1.2)\) and \(u_{\text{hom}}\) as in \((2.13)\). Then \(\mathbb{P}_0\)-a.s., for any \(T, \ell > 0\),

\[
\liminf_{\varepsilon \to 0} \inf_{0 \leq t \leq T} \inf_{x \in \mathbb{C}_\infty \mid x \leq \ell} (u_\varepsilon(t,x,\omega) - u_{\text{hom}}(t,x)) \geq 0, \tag{3.23}
\]

where

\[
u_{\text{hom}}(t,x) = \sup_{y \in \mathbb{R}^d} \left[ f(y) - tI\left(\frac{y - x}{t}\right) \right], \quad I(x) = \sup_{\theta \in \mathbb{R}^d} \|\langle \theta, x \rangle - \mathcal{P}(\theta)\|,
\]

and

\[
\mathcal{P}(\theta) := \sup_{(b,\phi) \in \mathcal{E}} \left( \int \phi d\mathbb{P}_0 \left[ \frac{1}{2} \text{div}(a\theta) + \langle \theta, b \rangle - L(b, \omega) \right] \right). \tag{3.25}
\]

The rest of this section is devoted to the proof of the above theorem, for which we will need some preliminary results contained in Lemmas 3.9, 3.11. First, we recall the definition of the space \(C_T\) of progressively measurable functions \(c : [0,T] \times \mathcal{F}^\varepsilon \rightarrow \mathbb{C}_\infty\) such that \((3.6)\) holds. Then \((cf. (3.7))\)

\[
v(t,x,\omega) := \sup_{c \in C_T} E^{P_\varepsilon}_{\omega} \left[ f(X_t) - \int_0^t L(X_s,c(s))ds \right]
\]
solves \((3.10)\). Let \(v_\varepsilon\) be the solution of \((3.10)\) with initial data \(\varepsilon^{-1}f(\varepsilon x)\) and domain \((0, T/\varepsilon) \times \mathbb{C}_\infty\).

Then

\[
u_\varepsilon(t,x,\omega) := \varepsilon v_\varepsilon(t \frac{x}{\varepsilon}, \omega) \tag{3.26}
\]
solves \((2.10)\). By the uniqueness of the viscosity solution (recall Proposition 3.3) \(u_\varepsilon\) can then be written as

\[
u_\varepsilon(t,x,\omega) = \sup_{c \in C_T} E^{P_\varepsilon}_{\omega} \left[ f(\varepsilon X_{t/\varepsilon}) - \varepsilon \int_0^{t/\varepsilon} L(X_s,c(s))ds \right]. \tag{3.27}
\]

An alternative representation of the above expression is given by

\[
u_\varepsilon(t,x,\omega) = \sup_{c \in C_T} E^{P_\varepsilon}_{\omega} \left[ f(X_t) - \int_0^t L\left(\frac{X_s}{\varepsilon}, c\left(\frac{s}{\varepsilon}\right)\right)ds \right], \tag{3.28}
\]

where \(P_\varepsilon^{x,c,\omega}\) is the law of the diffusion satisfying the SDE

\[
X_t = x + \sqrt{\varepsilon} \int_0^t \sigma \left( \frac{X_s}{\varepsilon} \right) dB_s + \int_0^t (\text{div} a) \left( \frac{X_s}{\varepsilon} \right) ds + \int_0^t a \left( \frac{X_s}{\varepsilon} \right) c\left(\frac{s}{\varepsilon}\right) ds. \tag{3.29}
\]

In this section, we use constants \(C, C''\) independent on \(\omega, t, \varepsilon\) that may change from line to line.

**Lemma 3.9.** Assume \((F1),(F2)\) and \((F4)\). Then we can replace the supremum of \(c \in C_T\) in \((3.28)\) by a supremum over \(c \in C_T^* \subset C_T\) of functions satisfying the following: for each \(\delta > 0\), there exists a constant \(C_\delta\) depending only on \(\delta\) and the constants \(\alpha, \alpha'\) appearing in \((F2)\) and \((F4)\) such that for all \(\omega \in \Omega_0\),

\[
\sup_{x \in \mathbb{C}_\infty} \varepsilon E^{P_\varepsilon}_{x/\varepsilon} \left[ \int_0^{t/\varepsilon} |L(X_s,c(s))|ds \right] \leq C_\delta (t + \sqrt{\varepsilon}t) + 2\alpha \delta. \tag{3.30}
\]

In particular, for all \(c \in C_T^*\),

\[
\sup_{x \in \mathbb{C}_\infty} \varepsilon E^{P_\varepsilon}_{x/\varepsilon} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha'} ds \right] \leq C_\delta (t + \sqrt{\varepsilon}t) + 2\alpha \delta. \tag{3.31}
\]
Proof. First, recall that under $P_{x/\varepsilon}^{c,\omega}$, the diffusion satisfies

$$
\varepsilon X_{t/\varepsilon} = x + \varepsilon \int_0^{t/\varepsilon} \sigma(X_s)dB_s + \varepsilon \int_0^{t/\varepsilon} (\text{div} a)(X_s)ds + \varepsilon \int_0^{t/\varepsilon} a(X_s)c(s)ds.
$$

(3.32)

We will now use the upper bound (2.2) from (F1) to deduce that

$$
|a(X_s)c(s)| \leq C|\sigma(X_s)c(s)| = C\|c(s)\|_{\alpha}.
$$

Note that the norm above implicitly depends on $X_s$. Also using (F1) we have uniformly $|\text{div} a| \leq C'$ for some $C' < \infty$. Using these two bounds,

$$
E_{x/\varepsilon}^{P_{x}^{c,\omega}}[\|eX_{t/\varepsilon} - x\|] \leq \varepsilon E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \left( \int_0^{t/\varepsilon} \sigma(X_s)dB_s \right)^2 \right]^{1/2} + C't + \varepsilon CE_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha}ds \right].
$$

Using Itô isometry, followed by employing the upper bound from (2.2), we have

$$
E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \left( \int_0^{t/\varepsilon} \sigma(X_s)dB_s \right)^2 \right] \leq C' t/\varepsilon.
$$

Hence,

$$
E_{x/\varepsilon}^{P_{x}^{c,\omega}}[\|eX_{t/\varepsilon} - x\|] \leq C(t + \varepsilon \sqrt{t}) + \varepsilon CE_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha}ds \right].
$$

(3.33)

By Hölder’s inequality and (2.6), we obtain the inequalities

$$
\varepsilon E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha}ds \right] \leq t^{1/\alpha} \varepsilon E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha'}ds \right]^{1/\alpha'},
$$

(3.34)

$$
\varepsilon E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|c(s)\|_{\alpha'}ds \right] \leq c_{10} \varepsilon E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^{t/\varepsilon} \|L(X_s, c(s))\|ds \right] + c_{11}t.
$$

(3.35)

Notice that by (3.30) and (3.35) we obtain (3.31). Thus, we only need to prove (3.30).

Using the formula (3.28) with $c \equiv 0$, by (2.6), (2.9) and (3.33), for any $\delta > 0$ we obtain the lower bound

$$
u_{\varepsilon}(t, x, \omega) - f(x) \geq E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ f(X_t) - f(x) - \int_0^t L \left( \frac{X_s}{\varepsilon}, 0 \right) ds \right]
$$

$$
\geq -K_\delta E_{x/\varepsilon}^{P_{x}^{c,\omega}}[|X_t - x|] - c_{13}t - \delta
$$

$$
\geq -K_\delta (\sqrt{\varepsilon t} + \varepsilon t) - c_{13}t - \delta.
$$

(3.36)

So, we only need to consider $c \in C_T$ such that

$$
E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ f(X_t) - f(x) - \int_0^t L \left( \frac{X_s}{\varepsilon}, c \left( \frac{s}{\varepsilon} \right) \right) ds \right] \geq -K_\delta (\sqrt{\varepsilon t} + \varepsilon t) - c_{13}t - \delta,
$$

and because of (2.9), such $c$ has to fulfill

$$
E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ K_\delta |X_t - x| - \int_0^t L \left( \frac{X_s}{\varepsilon}, c \left( \frac{s}{\varepsilon} \right) \right) ds \right] \geq -K_\delta (\sqrt{\varepsilon t} + \varepsilon t) - c_{13}t - 2\delta.
$$

Set $\Theta(t) := E_{x/\varepsilon}^{P_{x}^{c,\omega}} \left[ \int_0^t L \left( \frac{X_s}{\varepsilon}, c \left( \frac{s}{\varepsilon} \right) \right) ds \right] + c_{11}t$. Applying the inequalities (3.33), (3.34) and (3.35) to the last display, we only need to consider $c \in C_T$ satisfying

$$
K_\delta (t + \sqrt{\varepsilon t} + 2c_9^{-1/\alpha} t^{1/\alpha} \Theta(t)^{1/\alpha'} - \Theta(t) + K_\delta (\sqrt{\varepsilon t} + \varepsilon t) + (c_{11} + c_{13})t + 2\delta \geq 0.
$$
If $A := 2K_4c_0^{-1/α'}t^{1/α}$ and $B := 2K_4(\sqrt{ε}t + t) + (c_{11} + c_{13})t + 2δ$, we can write the last inequality as

$$A\Theta(t)^{1/α'} - Θ(t) + B \geq 0.$$  

By Young’s inequality and using that $\frac{1}{α} + \frac{1}{α'} = 1$, we deduce that

$$\frac{A^α}{α} + \frac{Θ(t)}{α'} - Θ(t) + B = \frac{A^α}{α} + B - Θ(t)/α \geq 0,$$

so that $Θ(t) ≤ αB + A^α$. Recalling the definitions of $A$ and $B$, we deduce (3.30), finishing the proof of the lemma.

**Lemma 3.10.** Assume (F1), (F2) and (F4). For any $η > 0$, there exists a set $N_η$ with $P_0(N_η) > 1 - η$ such that for any $(b, φ) ∈ E$ (recall 3.17),

$$\liminf_{ε → 0} \inf_{ω ∈ N_η} \inf_{0 ≤ t ≤ T} [u_ε(t, 0, ω) - f(m(b, φ)t) + th(b, φ)] \geq 0,$$

where $h(b, φ)$ and $m(b, φ)$ are defined in (3.22).

**Proof.** For each $ε > 0$, $(b, φ) ∈ E$ and $(t, \frac{ξ}{ε}) ∈ (0, T) × C_∞$,

$$u_ε(t, 0, ω) ≥ E_p^{b, ω} \left[ f(εX_t/ε) - ε \int_0^{t/ε} L(X_s, b(X_s))ds \right] P_0 - a.s.$$

Since $f$ is assumed to be uniformly continuous, recalling (6.22) we obtain

$$\liminf_{ε → 0} u_ε(t, 0, ω) ≥ \liminf_{ε → 0} E_p^{b, ω} \left[ f(εX_t/ε) - ε \int_0^{t/ε} L(X_s, c(s))ds \right] = f(tm(b, φ)) - th(b, φ)$$

for any $(b, φ) ∈ E$. The lemma now follows from Egorov’s theorem. □

The next lemma requires Lipschitz estimates to control the oscillation of $u_ε$ around zero, uniformly in $ε$, on balls of radius $r$, as $r → 0$. We will need a stronger condition on $f$. For proving Theorem 3.8 for any $f$ satisfying (F4) this condition will be relaxed in Lemmas 3.12-3.14.

**Lemma 3.11.** Assume (F1), (F3) and that the initial condition $f ∈ C^∞(ℝ^d) \cap W^{2,∞}(ℝ^d)$. Then for any $(b, φ) ∈ E$,

$$\liminf_{r → 0} \liminf_{ε → 0} \inf_{0 ≤ t ≤ T} \inf_{y ∈ C_∞ \cap |y| ≤ r} \left[ u_ε(t, y, ω) - f(tm(b, φ)) + th(b, φ) \right] \geq 0 \quad P_0-a.s.$$

**Proof.** By Lemma 3.10 it is enough to prove that $P_0$-a.s.,

$$\limsup_{r → 0} \limsup_{ε → 0} \sup_{0 ≤ t ≤ T} \sup_{y ∈ C_∞ \cap |y| ≤ r} \left[ u_ε(t, y, ω) - u_ε(t, 0, ω) \right] = 0. \quad (3.37)$$

Recall that $u_ε(t, x, ω) = εv_ε(t, x, ω)$, where $v_ε(t, x, ω)$ solves (3.10) with initial condition $ε^{-1}f(εx)$. Then

$$\sup_{y ∈ C_∞ \cap |y| ≤ r} \left[ v_ε(t, y, ω) - v_ε(t, 0, ω) \right] = ε \sup_{y ∈ C_∞ \cap |y| ≤ r} \left| v_ε \left( \frac{t}{ε}, y, ω \right) - v_ε \left( \frac{t}{ε}, 0, ω \right) \right|.$$

By Morrey’s inequality (see [EI0] Section 5.6.2), for any $γ > d$, there is a constant $C = C(γ, d)$ such that

$$\sup_{y ∈ C_∞ \cap |y| ≤ r} \left| v_ε \left( \frac{t}{ε}, y, ω \right) - v_ε \left( \frac{t}{ε}, 0, ω \right) \right| ≤ \frac{Cr}{r} \left( \frac{∫_{B_{r/ε}(0)} |∇v_ε(t/ε, x, ω)|dx}{\lambda_d(B_{r/ε}(0))} \right)^{1/γ}.$$

Since $|∇v_ε(t/ε, x, ω)| = |∇f(εx)| ≤ Cr$ if $x ∈ B_{r/ε}(0) \cap C_∞$, the main contributing part in the integral comes from $B_{r/ε}(0) \cap C_∞$. In view of the assumption that $f ∈ C^∞(ℝ^d) \cap W^{2,∞}(ℝ^d)$, we can apply
Theorem 2.5], and the comparison principle Theorem C.1 to conclude that uniformly on t,
\[ |\nabla v_\varepsilon(t/\varepsilon, x, \omega)| \leq C \xi(x, \omega)^{-\alpha/2(\alpha - 1)}, \]
where C is a constant depending on \( d, \alpha \), the (uniform) Lipschitz constant of \( \sigma \), and the constants \( c_0, \ldots, c_{16} \) defined in Eqs. (2.5)–(2.8).

Therefore, by the ergodic theorem and using (2.3), we obtain
\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \sup_{y \in \mathbb{C}_\infty: |y| \leq \ell} \left| v_\varepsilon \left( \frac{t}{\varepsilon}, x, \omega \right) - v_\varepsilon \left( \frac{t}{\varepsilon}, 0, \omega \right) \right| \leq C t^{1/\gamma} \xi(\omega)^{-\alpha/2(\alpha - 1)} [\xi(\omega) - \alpha/2(\alpha - 1)]^{1/\gamma}.
\]

We let \( r \to 0 \) to conclude.

\[ \square \]

We are now ready to provide the proof.

**Proof of Theorem 3.8** Let us first prove the result assuming that the initial condition \( f \in C^\infty(\mathbb{R}^d) \cap W^{2, \infty}(\mathbb{R}^d) \).

For any \( \ell \geq 1 \), we can consider the family of functions \( \{ f^{(y)}, |y| \leq \ell \} \), where \( f^{(y)}(x) := f(x + y) \).
Notice that \( f^{(y)} \) is uniformly continuous with a constant \( K_f \) as in (F4). Then exactly as in the proof of Lemma 3.10 we find a family of functions \( u_\ell^y \) such that for any \( \eta > 0 \), there exists some \( N_\eta \) with \( \mathbb{P}_0(N_\eta) \geq 1 - \eta \) and
\[
\lim_{\varepsilon \to 0} \inf_{\omega \in N_\eta} \inf_{0 \leq t \leq T} \inf_{y \in \mathbb{C}_\infty: |y| \leq \ell} \inf_{\tau \in \mathbb{R}} \left[ u_\varepsilon^y(t, 0, \omega) - f(y + m(b, \phi) t) + th(b, \phi) \right] \geq 0.
\]

By the ergodic theorem, in a set \( N \) of \( \mathbb{P}_0 \)-probability 1 (we can assume it is contained in \( \bigcup_{\eta > 0} N_\eta \)), it holds that
\[
\lim_{\varepsilon \to 0} \frac{\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1}, \tau_x \omega \in N_\eta \})}{\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1} \})} = \frac{\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1}, \tau_x \omega \in N_\eta \})/\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1} \})}{\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1} \})} = \mathbb{P}[N_\eta \cap \Omega_0] / \mathbb{P}[\Omega_0] = \mathbb{P}_0(N_\eta) \geq 1 - \eta.
\]

For each \( \omega \in N \), \( \varepsilon \leq \varepsilon_0(\eta) \),
\[
\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1}, \tau_x \omega \in N_\eta \}) \geq (1 - 2\eta)\lambda_d(\{ x \in \mathbb{C}_\infty : |x| \leq \ell \varepsilon^{-1} \}).
\]

In particular, every \( x \in \mathbb{C}_\infty \) satisfying \( |x| \leq \varepsilon_0 \varepsilon^{-1} \ell \) is within distance \( \ell \varepsilon^{-1}(3\delta)^{1/d} \) from some \( x' \in \mathbb{C}_\infty \) satisfying \( \tau_{x'} \omega \in N_\eta \). Thus, by Lemma 3.11 and noting that \( u_\varepsilon^y(t, 0, \omega) = u_\varepsilon(t, x, \tau_{-x/\varepsilon} \omega) \), we deduce that for each \( \omega \in N \),
\[
\lim_{\varepsilon \to 0} \inf_{0 \leq t \leq T} \inf_{x \in \mathbb{C}_\infty: |x| \leq \ell} \inf_{(b, \phi) \in \mathcal{E}} \left[ u_\varepsilon(t, x, \omega) - f(x + m(b, \phi) t) + th(b, \phi) \right] \geq 0.
\]

Let
\[
u(t, x) := \sup_{(b, \phi) \in \mathcal{E}} [f(x + m(b, \phi) t) - th(b, \phi)], \quad (3.38)
\]

We claim that
\[
u(t, x) = u^\text{hom}(t, x), \quad \text{with } u^\text{hom} \text{ defined in (3.24)}, \quad (3.39)
\]
Indeed, note first that by definition of $\mathcal{H}$ and (3.22),
\[
\overline{H}(\theta) = \sup_{(b, \phi) \in \mathcal{E}} [(\theta, m(b, \phi)) - h(b, \phi)] \\
= \sup_{y \in \mathbb{R}^d} \sup_{(b, \phi) \in \mathcal{E}: m(b, \phi) = y} [(\theta, y) - h(b, \phi)] \\
= \sup_{y \in \mathbb{R}^d} \left( (\theta, y) - \inf_{(b, \phi) \in \mathcal{E}: m(b, \phi) = y} h(b, \phi) \right).
\]

On the other hand, since $\mathcal{I}$ is the convex conjugate of $\mathcal{H}$, we conclude that
\[
\mathcal{I}(y) = \inf_{(b, \phi) \in \mathcal{E}: m(b, \phi) = y} h(b, \phi).
\]

(3.40)

As a result, and using (3.40),
\[
\begin{align*}
u(t, x) &= \sup_{(b, \phi) \in \mathcal{E}} \left[ t(x + m(b, \phi)t) - th(b, \phi) \right] \\
&= \sup_{y \in \mathbb{R}^d} \sup_{(b, \phi) \in \mathcal{E}: m(b, \phi) = y} \left[ t(x + yt) - th(b, \phi) \right] \\
&= \sup_{y \in \mathbb{R}^d} [t(x + yt) - \mathcal{I}(y)] \\
&= u_{\text{hom}}(t, x),
\end{align*}
\]

which proves the claim. As a consequence,
\[
\liminf_{\varepsilon \to 0} \inf_{0 \leq t \leq T} \inf_{x \in C_{\mathcal{E}}: |x| \leq \varepsilon} \left[ u_{\varepsilon}(t, x, \omega) - u_{\text{hom}}(t, x) \right] \geq 0 \quad \mathbb{P}_0 \ - \ a.s.
\]

This finishes the proof as long as the initial condition $f \in C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. The next three lemmas will extend the result to a uniformly continuous initial condition $f$ (i.e., $f$ satisfying (F4)), concluding the proof of Theorem 3.8.

\[\text{Lemma 3.12. Given any uniformly continuous function } f : \mathbb{R}^d \to \mathbb{R}, \text{ there exists a sequence } (f_k)_k \subset C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d) \text{ such that } f_k \to f \text{ uniformly.}\]

\[\text{Proof. First, we check that } f \text{ can be approximated by Lipschitz functions. An explicit construction is given by}
\]
\[ f_k(x) := \inf_{y \in \mathbb{R}^d} \{ f(y) + k|x-y| \}. \]

Since $f_k$ is the infimum of $k$-Lipschitz functions over a convex set, then $f_k$ is also $k$-Lipschitz. Clearly $f_k \leq f$. To verify that $f_k \to f$ uniformly, we note that
\[
f(x) - f_k(x) = \sup_{y \in \mathbb{R}^d} \{ f(x) - f(y) - k|x-y| \}.
\]

For each $\delta > 0$, let $K_\delta > 0$ such that $|f(x) - f(x)| \leq K_\delta |x-y| + \delta$ for any $x, y \in \mathbb{R}^d$. Then
\[
f(x) - f_k(x) \leq \sup_{y \in \mathbb{R}^d} [(K_\delta - k)|x-y| + \delta].
\]

Therefore, for every $\delta > 0$, if $k > K_\delta$, we have $f(x) - f_k(x) \leq \delta$ for all $x \in \mathbb{R}^d$. Since $\delta$ is arbitrary, this shows that $f$ can be approximated by Lipschitz functions. On the other hand, any Lipschitz function can be approximated by functions in $C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ by mollification. \qed
Lemma 3.13. Let \( V \subset \mathbb{R}_+ \times \mathbb{R}^d \) be an open set. Let \( f_1, f_2 \) be two uniformly continuous functions in \( V \). Let \( u_1, u_2 \) be the corresponding viscosity solutions to (3.1) with initial conditions \( f_1 \) and \( f_2 \) respectively. Then
\[
\sup_{(t,x) \in V} |u_1(t,x) - u_2(t,x)| \leq \sup_{(t,x) \in \partial V} |f_1(x) - f_2(x)|. 
\] (3.41)

Proof. If the right-hand side is infinite, the claim is trivial. Otherwise, note that it is enough to show the inequality
\[
\sup_{(t,x) \in V} |u_1(t,x) - u_2(t,x)| \leq \sup_{(t,x) \in \partial V} |f_1(x) - f_2(x)|.
\]
By subtracting a constant, we can assume that \( \sup_{(t,x) \in \partial V} |f_1(x) - f_2(x)| = 0 \), that is, \( f_1(x) \leq f_2(x) \) for all \( (t,x) \in \partial V \). Then the result follows from the comparison principle, Theorem C.1. \( \square \)

Lemma 3.14. Let \( V \subset \mathbb{R}_+ \times \mathbb{R}^d \) be an open set, \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), and \( (f_n)_{n \in \mathbb{N}}, f_n : \mathbb{R}^d \rightarrow \mathbb{R} \) such that \( f \) is uniformly continuous and \( (f_n)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}^d) \cap W^{2, \infty}(\mathbb{R}^d) \) such that \( f_n \rightarrow f \) uniformly. For each \( n \in \mathbb{N} \), let \( u_n \) be the viscosity solution to (3.1) with initial condition \( f_n \) and \( u \) the viscosity solution to (2.1) with initial condition \( f \). Then \( u_n \rightarrow u \) uniformly in \( V \).

Proof. By Lemma 3.13, we have for each \( n \)
\[
\sup_{(t,x) \in V} |u_n(t,x) - u(t,x)| \leq \sup_{(t,x) \in \partial V} |f_n(x) - f(x)| \leq \sup_{x \in \mathbb{R}^d} |f_n(x) - f(x)|.
\]
Letting \( n \rightarrow \infty \) and using that \( f_n \rightarrow f \) uniformly finishes the proof. \( \square \)

The following lemma will conclude the proof of Theorem 3.8.

Lemma 3.15. The conclusion of Theorem 3.8 holds if the initial condition \( f \) is uniformly continuous.

Proof. Using Lemma 3.12, let \( (f_n)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}^d) \cap W^{2, \infty}(\mathbb{R}^d) \) satisfy such that \( f_n \rightarrow f \) uniformly. Given \( \varepsilon > 0 \), let \( u^n_{\varepsilon}, \ v_{\varepsilon} \) be the solutions to (2.11) with initial conditions \( f_n \) and \( f \) respectively. Similarly, let \( u^n_{\text{hom}}, u^\varepsilon_{\text{hom}} \) be the solutions to (2.11) with initial conditions \( f_n \) and \( f \) respectively. By Lemma 3.14, \( \mathbb{P}_0 \)-a.s., \( u^n \rightarrow u^\varepsilon \) uniformly (and uniformly on \( \varepsilon \)). Similarly, \( u^n_{\text{hom}} \rightarrow u^\varepsilon_{\text{hom}} \) uniformly. Moreover, by Theorem 2.1, \( \mathbb{P}_0 \)-a.s. we know that \( u^n_{\varepsilon} \) converges as \( \varepsilon \rightarrow 0 \) to \( u^n_{\text{hom}} \) uniformly on compact sets. By the triangle inequality, we can deduce that \( \mathbb{P}_0 \)-a.s., \( u^\varepsilon \) converges \( u^\varepsilon_{\text{hom}} \) uniformly on compact sets. \( \square \)

4. Correctors

Given any \( \delta > 0 \), we start this section by defining the class of gradients \( G \in \mathcal{G}_\delta \) and the corresponding “correctors” \( V_G : \mathbb{R}^d \times \Omega_0 \rightarrow \mathbb{R}^d \). Let \( \mathcal{G}_\delta \) be the class of functions \( G : \Omega_0 \rightarrow \mathbb{R}^d \) satisfying the following properties:

- **\( L^{1+\delta}(\mathbb{P}_0) \)-boundedness:** The following inequalities hold:
  \[
  \|G\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty, 
  \] (4.1)
  and
  \[
  \text{esssup}_{\mathbb{P}_0} \left[ \frac{1}{2} \text{div}(a(G + \theta)) + H(G + \theta) \right] < \infty. 
  \] (4.2)

- **Curl-free property on the cluster:** Given any \( G : \Omega_0 \rightarrow \mathbb{R}^d \), with a slight abuse of notation we will continue to write
  \[
  G : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \text{with} \quad G(\omega, x) = G(\tau x, \omega). 
  \]
Now, for $\mathbb{P}_0$-almost every $\omega \in \Omega_0$, we require that $G$ is curl-free, meaning $\nabla \times G(\omega, \cdot) = 0$ on $\mathcal{C}_\infty$, or simply, for $\mathbb{P}_0$-almost every $\omega \in \Omega_0$, we have

$$\int_{\mathcal{C}} G(\omega, \cdot) \cdot d\mathbf{r} = 0 \quad (4.3)$$

for every rectifiable simple closed path $\mathcal{C}$ on $\mathcal{C}_\infty$. For any $G$ satisfying (4.3) we define $V_G : \Omega_0 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$V_G(\omega, x) := \int_{0 \sim x} G(\omega, \cdot) \cdot d\mathbf{r}, \quad (4.4)$$

where $0 \sim x$ is any piecewise smooth curve contained in $\mathcal{C}_\infty$ (and 0 when $x \notin \mathcal{C}_\infty$). Note that the choice of the smooth curve is irrelevant, thanks to (4.3).

- **Zero induced mean:** Recall the definition of $n(\omega, e)$ from (1.13) and set $\mathbf{v}_e = \mathbf{v}_e(\omega) = n(\omega, e)e$. Then we require that

$$\mathbb{E}_0[V_G(\cdot, \mathbf{v}_e)] = 0. \quad (4.5)$$

**Definition 4.1.** For any $\delta > 0$, we say that $G \in \mathcal{G}_\delta$ if (4.1)-(4.3) and (4.5) hold. Similarly, we declare that $G \in \mathcal{G}_\infty$ if the above conditions hold, but replace (4.1), by

$$\text{ess sup}_{\mathbb{P}_0} |G(\omega)| < \infty. \quad (4.6)$$

In this section we will prove the following result.

**Theorem 4.2.** Fix $d \geq 2$ and $G \in \mathcal{G}_\infty$. Then for $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$ and every $\ell > 0$, we have

$$\lim_{\varepsilon \to 0} \sup_{|x| \leq \ell} \varepsilon |V_G(\frac{x}{\varepsilon}, \omega)| = 0 \quad \mathbb{P}_0 - \text{a.s.}$$

The rest of this section is devoted to the proof of Theorem 4.2, which will be carried out in few steps.

### 4.1 Controlling directional growth.

The main result of this section, Theorem 4.3 stated below provides a control on the growth of $V_G$ along coordinate directions. For this purpose, we fix a unit coordinate vector $e$ and for $\omega \in \Omega_0$, define the successive arrivals $(n_k(\omega))_{k \in \mathbb{N}}$ of the cluster recursively as follows: Recall (1.13) and define

$$n_1(\omega) = n(\omega, e), \quad \text{and for } k \geq 1 \text{ we set } n_{k+1}(\omega) := \min\{l \in \mathbb{N} : l > n_k(\omega), le \in \mathcal{C}_\infty(\omega)\}.$$

**Theorem 4.3.** Let $e$ be any unit coordinate vector. If $G \in \mathcal{G}_\delta$, then $\mathbb{P}_0$-a.s.,

$$\lim_{k \to \infty} \frac{|V_G(n_k(\omega)e, \omega)|}{k} = 0.$$

The proof of Theorem 4.3 will need the following result.

**Proposition 4.4.** For any unit coordinate vector $e$, recall that we denote the first successive arrival in direction $e$ by $\mathbf{v}_e = \mathbf{v}_e(\omega) = n(\omega, e)e$. Then for any $G \in \mathcal{G}_\delta$, we have $\mathbb{E}_0|V_G(\mathbf{v}_e, \cdot)| < \infty$. More precisely, there is a constant $C = C(d, \delta, \mathbb{P}_0)$ such that for any $G \in \mathcal{G}_\delta$, $\mathbb{E}_0|V_G(\mathbf{v}_e, \cdot)| \leq C\|G\|_{L^{1+\delta}(\mathbb{P}_0)}$.

**Proof of Theorem 4.3 (assuming Proposition 4.4).** For each $k \in \mathbb{N}$, set $x_0 = 0$ and $x_j = nje$ for $1 \leq j \leq k$. We choose a path $0 \sim x_k$ from 0 to $x_k$ contained in $\mathcal{C}_\infty(\omega)$ such that, for some
0 = t_0 < t_1 < ... < t_k = 1 and r : [0, 1] → (x_0 ∼ x_k), it holds r(t_j) = x_j. Then by the definition of V_G in (4.3),

\[ V_G(n_k e, \omega) = \int_{x_0 \sim x_k} G(r, \omega) dr = \sum_{j=0}^{k-1} \int_{x_j \sim x_{j+1}} G(r, \omega) dr = \sum_{j=0}^{k-1} V_G(x_{j+1} - x_j, \tau_{x_j} \omega) = \sum_{j=1}^{k-1} V_G((n_j(\omega) - n_{j-1}(\omega))e, \tau_{n_{j-1}} e \omega) = \sum_{j=0}^{k-1} V_G(n_1(\sigma^j_x(\omega))e, \sigma^j_x(\omega)). \]

Recall (3.18) for the definition of the induced shift and Definition (4.3) for that of the corrector V_G. We define the function \( F(\omega) = V_G(n(\omega), e, \omega) \), so that

\[ V_G(n_k(\omega)e, \omega) = \sum_{j=0}^{k-1} F \circ \sigma^j_x(\omega). \quad (4.7) \]

From Proposition 3.3 the induced shift \( \sigma_x \) is \( \mathbb{P}_0 \)-preserving and ergodic. Furthermore, from Proposition 4.1 the function \( F \in L^1(\mathbb{P}_0) \). Then by Birkhoff’s Ergodic Theorem,

\[ \lim_{k \to \infty} \frac{\sum_{j=0}^{k-1} F \circ \sigma^j_x(\omega)}{k} = \mathbb{E}_0[V_G(n(\omega), e, \omega)] = 0, \quad (4.8) \]

where the last equality comes from the induced mean-zero property (4.5) of \( G \in \mathcal{G}_\delta \).

To show Proposition 4.4 we require first a lemma.

**Lemma 4.5.** Let \( \ell = \ell(\omega) = d_\omega(0, v_e(\omega)) \) be the graph distance between 0 and \( v_e = n(\omega)e \). Then there exist constants \( a, C > 0 \) such that for any \( t > 0 \),

\[ \mathbb{P}_0 \left( \sup_{0 \leq s \leq n_1(\omega)} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\}d_\omega(0, se) > t \right) \leq Ce^{-at}. \quad (4.9) \]

In particular,

\[ \mathbb{P}_0(\ell > t) < Ce^{-at}. \]

**Proof.** Let \( \varepsilon > 0 \). For \( t > 0 \) we write \( t_\varepsilon := \lfloor \varepsilon t \rfloor \). Then

\[ \mathbb{P}_0 \left( \sup_{0 \leq s \leq n_1} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\}d_\omega(0, se) > t \right) \leq \mathbb{P}_0(n_1(\omega) \geq t\varepsilon) + \mathbb{P}_0 \left( \sup_{0 \leq s \leq t\varepsilon} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\}d_\omega(0, se) > t \right). \]

By (P6) the claim follows once we prove that the second term goes to zero at an exponential rate. This probability is bounded above by \( \sum_{i=1}^{\lfloor t\varepsilon \rfloor} \mathbb{P}_0 \left( \sup_{t_{i-1} \leq s \leq t_i} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\}d_\omega(0, se) > t \right) \). Since the number of summands is growing only polynomially in \( t \), it suffices to show that each summand there decays exponentially in \( t \). We will proceed as follows:

We define

\[ m := \min\{l \in \mathbb{N} : l > t_\varepsilon, -le \in \mathcal{C}_\infty\}, \quad A_{x,y} = \{d_\omega(x, y) \geq t/2, x, y \in \mathcal{C}_\infty\}. \]

Now we observe that on the event \( \{\sup_{t_{i-1} \leq s \leq t_i} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\}d_\omega(0, se) > t \} \), one of the following cases must hold:

- \( m > 2t_\varepsilon \), or
• at least one of the points \( le \) with \( l \in \mathbb{Z} \) and \( |l| \leq 2t_\varepsilon \) is in \( \mathcal{C}_\infty \) and for some \( i - 1 \leq s \leq i, \max\{d_\omega(0, -le), d_\omega(-le, se)\} \geq t/2\).

In the first of the two cases above we have \( |v_e \circ \sigma^m_e| > t_\varepsilon \) for at least one \( m = 1, \ldots, t_\varepsilon \). Hence,

\[
\mathbb{P}_0\left( \sup_{i-1 \leq s \leq i} \mathbb{I}\{se \in \mathcal{C}_\infty(\omega)\} d_\omega(0, se) > t \right) \\
\leq \sum_{m=1}^{t_\varepsilon} \mathbb{P}_0(\sigma^m_e(\{|v_e| \geq t_\varepsilon\})) + \sum_{\ell=t_\varepsilon}^{2t_\varepsilon} \mathbb{P}_0(\exists i - 1 \leq s \leq i : A_{0, le} \cup A_{le, se}).
\]

By (P6) the probabilities of the events in the first sum are equal and exponentially small. The second sum is bounded by

\[
t_\varepsilon \mathbb{P}_0(A_{0, le}) + \sum_{\ell=t_\varepsilon}^{2t_\varepsilon} \mathbb{P}_0(\exists i - 1 \leq s \leq i : A_{le, se}).
\]

To bound the first term, we use (1.10) and (P4) to obtain the bound

\[
\mathbb{P}_0(A_{0, le}) \leq \frac{1}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \mathbb{P}\left( \exists x \neq y \in \mathcal{C}_\infty(\omega) : |x| \leq \frac{1}{2}, |y - le| \leq \frac{1}{2}, d_\omega(x, y) \geq \frac{t}{2} + 1; 0, x, y \in \mathcal{C}_\infty \right) \\
\leq \frac{1}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \mathbb{E}\left[ \sum_{x, y \in \omega} \mathbb{I}\{|x| \leq 1/2, |y - le| \leq 1/2, d_\omega(x, y) \geq t/2 + 1; 0, x, y \in \mathcal{C}_\infty \} \right] \\
= \frac{\zeta^2}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \int_{[-1/2,1/2]^d} \int_{[le-1/2,le+1/2]^d} \mathbb{P}^{x,y}\left( d_\omega(x, y) \geq \frac{t}{2} + 1; 0, x, y \in \mathcal{C}_\infty \right) dx dy \\
\leq Ce^{-C' t_\varepsilon}
\]

for some constants \( C, C' > 0 \) which are independent of \( l \), and for \( \varepsilon > 0 \) small enough, with \( \zeta \) defined in (P2). Following the same calculations as in the last display, we can also show that

\[
\mathbb{P}_0(\exists i - 1 \leq s \leq i : A_{le, se}) \leq Ce^{-C' t_\varepsilon}
\]

for constants \( C, C' > 0 \) independent of \( i \) and \( l \), for \( \varepsilon > 0 \) small enough. After estimating the probabilities of all events by an exponential upper bound, from the unions we get another factor that is linear in \( l \), which can be absorbed by the exponential bound for \( t \) large enough. Thus the proof of Lemma 4.3 is complete.

\[ \square \]

Now we are ready to show Proposition 4.4.

**Proof of Proposition 4.4.** Let \( \mathcal{B} = \mathcal{B}(\omega) \) be an enumeration of the balls that appear in the construction of \( \mathcal{C}(\omega) \) (recall (P3)). Then define the random variable

\[
\tilde{d}_\omega(x, y) := \min\left\{ n \in \mathbb{N} : \exists (B_i)_{i=1}^n \subset \mathcal{B} \text{ such that } x \in B_1, y \in B_n, \text{ and } B_{i-1} \cap B_i \neq \emptyset \forall 1 \leq i \leq n \right\}
\]

and set

\[
\tilde{\ell} := \tilde{d}_\omega(0, v_\varepsilon).
\]
Note that there is some constant \( c > 0 \) such that for all \( n > 0 \),
\[
\mathbb{P}_0\left( \ell > n \right) \leq e^{-cn}.
\]
(4.12)

For \( j \in \mathbb{N} \), let \( N_j := \mathbb{Z}/2 \cap [-j,j]^d \). We consider this set as a graph, where for \( x, y \in N_j \), \( x \sim y \) iff \( |x - y|_1 = 1/2 \). Note that if \( \ell = j \), then there is a nearest-neighbor path on \( N_j \) of length \( k \leq 3^d j \) such that for all \( 1 \leq i \leq k - 1 \), the line segment between \( x_i \) and \( x_{i+1} \) is contained in the cluster. Thus, we write
\[
\{ \ell = j \} = \bigcup_{k=1}^{3^d j} \bigcup_{x_1, \ldots, x_k} \left\{ \ell = j \cap A(x_1, \ldots, x_k) \right\},
\]
where
\[
A(x_1, \ldots, x_k) := \left\{ x_1, \ldots, x_k \text{ is a nearest-neighbor path on } N_j, \forall 1 \leq i \leq k - 1 \text{ the line segment between } x_{i-1}, x_i \text{ is inside } C_\infty \right\}.
\]
(4.13)

If \( \ell = j \), then one can write for some nearest neighbor path \( 0 = x_0, x_1, \ldots, x_k \) on \( N_j \) \((1 \leq k \leq 3^d j)\) such that the line segment between \( x_i \) and \( x_{i+1} \) is inside \( C_\infty \) for all \( 0 \leq i \leq k - 1 \),
\[
\begin{align*}
\int_{0 \to v_e} |G(\omega, \cdot)| \cdot d\mathbf{r} &\leq \sum_{i=0}^{k-1} \int_0^1 |G(\tau_{x_i} \omega, t(x_{i+1} - x_i))| dt \\
&\leq 2 \sum_{i=0}^{k-1} \sum_{|e|=1} \int_0^{1/2} |G(\tau_{x_i} \omega, te)| dt \\
&\leq 2(3^d j) \sum_{x \in C_\infty \cap N_j : |x| \leq 3^d j} \sum_{|e|=1} \int_0^{1/2} |G(\tau_{x} \omega, te)| dt.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\mathbb{E}_0|V_G(v_e, \cdot)| &= \sum_{j=1}^{\infty} \mathbb{E}_0\left[ \left| \int_{0 \to v_e} G(\omega, \cdot) \cdot d\mathbf{r}, \ell = j \right| \right] \\
&\leq 2 \sum_{j=1}^{\infty} \sum_{|e|=1} \sum_{x \in N_j : |x| \leq 3^d j} (3^d j) \int_0^{1/2} \mathbb{E}_0\left[ |G(\tau_{x} \omega, te)|, x \sim x + e \subset C_\infty, \ell = j \right] dt \\
&\leq 2 \sum_{j=1}^{\infty} \sum_{|e|=1} \sum_{x \in N_j : |x| \leq 3^d j} (3^d j) \int_0^{1/2} \mathbb{E}_0\left[ |G(\tau_{x} \omega, te)|^{1+\delta}, x \sim x + e \subset C_\infty \right]^{1/(1+\delta)} \mathbb{P}_0(\ell = j)^{\frac{1}{1+\delta}} dt.
\end{align*}
\]
(4.14)

Since \( G \in G_\delta \), then for any \( x \in \mathbb{R}^d \),
\[
\mathbb{E}_0\left[ |G(\omega, x)|^{1+\delta}, x \subset C_\infty \right] \leq \|G\|_{L^{1+\delta}(\mathbb{P}_0)}.
\]

As a consequence, (4.14) can be bounded by
\[
C(d)\|G\|_{L^{1+\delta}(\mathbb{P}_0)} \sum_{j=1}^{\infty} j^2 \mathbb{P}_0(\ell = j)^{\delta/(1+\delta)} \leq C(d, \delta, \mathbb{P}_0)\|G\|_{L^{1+\delta}(\mathbb{P}_0)}
\]
due to (4.12). This finishes the proof of the proposition.
Corollary 4.6. Let $G \in \mathcal{G}_\infty$. Then for any unit coordinate vector $e$ and $\mathbb{P}_0$-a.s.,
\[
\lim_{s \to \infty} \frac{\mathbb{P}\{se \in C_\infty(\omega)\}}{s} = 0.
\]

Proof. If $se \in C_\infty(\omega)$, then there exists $k \geq 0$ such that $n_k(\omega) \leq s < n_{k+1}(\omega)$. Note that $k \not\to \infty$ as $s \not\to \infty$. Then we have
\[
\frac{|V_G(se, \omega)|}{s} \leq \frac{|V_G(n_k(\omega)e, \omega)|}{n_k(\omega)} + \frac{|V_G((s - n_k(\omega))e, \tau_{n_k(\omega)}e\omega)|}{n_k(\omega)}.
\]

By the ergodic theorem (as in the proof of Theorem 4.3) and $K_6$, it suffices to verify that for any $\varepsilon > 0$,
\[
\lim_{n \to \infty} \sup_{n_k(\omega) \leq s \leq n_{k+1}(\omega)} \mathbb{P}\{se \in C_\infty(\omega)\} \frac{|V_G((s - n_k(\omega))e, \tau_{n_k(\omega)}e\omega)|}{k} = 0 \text{ } \mathbb{P}_0\text{-a.s.}
\]

Since $G \in \mathcal{G}_\infty$, it is enough to prove that
\[
\lim_{k \to \infty} \sup_{n_k(\omega) \leq s \leq n_{k+1}(\omega)} \mathbb{P}\{se \in C_\infty(\omega)\} \frac{d_\omega(n_k(\omega)e, se)}{k} = 0 \text{ } \mathbb{P}_0\text{-a.s.} \tag{4.15}
\]

By the Borel-Cantelli lemma, it suffices to verify that for any $\varepsilon > 0$,
\[
\sum_{k=1}^{\infty} \mathbb{P}_0 \left( \sup_{n_k(\omega) \leq s \leq n_{k+1}(\omega)} \mathbb{P}\{se \in C_\infty(\omega)\} \frac{d_\omega(n_k(\omega)e, se)}{k} > \varepsilon k \right) < \infty. \tag{4.16}
\]

Since $\mathbb{P}_0$ is invariant under $\tau_{n_ke}$, the sum above is equal to
\[
\sum_{k=1}^{\infty} \mathbb{P}_0 \left( \sup_{0 \leq s \leq n_1(\omega)} \mathbb{P}\{se \in C_\infty(\omega)\} \frac{d_\omega(0, se)}{k} > \varepsilon k \right).
\]

By Lemma 4.5, this sum is finite for each $\varepsilon > 0$, concluding the proof. \hfill \Box

4.2 Controlling density of growth.

The main result of this section is the following result:

Proposition 4.7. Let $d \geq 2$ and $G \in \mathcal{G}_\infty$. Then for all $\varepsilon > 0$ and $\mathbb{P}_0$-almost all $\omega$,
\[
\lim_{r \to \infty} \frac{1}{(2\pi)^d} \int_{x \in C_\infty(\omega), |x| \leq r} \mathbb{P}\{|V_G(x, \omega)| \geq \varepsilon r\} dx = 0. \tag{4.17}
\]

The proof of Proposition 4.7 consists of three main steps.

Step 1: We start this section with a definition: Given $K > 0$ and $\varepsilon > 0$, we say that a point $x \in \mathbb{R}^d$ belongs to $G_{K,\varepsilon}(\omega)$ for $\omega \in \Omega$ if $x \in C_\infty(\omega)$ and
\[
|V_G(x + te, \omega) - V_G(x, \omega)| \leq K + \varepsilon |t| \tag{4.18}
\]
for each $t \in \mathbb{R}$, and $e$ is a unit coordinate vector such that $x + te \in C_\infty(\omega)$. We will use the following consequence of Corollary 4.6 in the sequel: for every $\varepsilon > 0$, $\mathbb{P}(0 \in C_\infty) = \lim_{K \to \infty} \mathbb{P}(0 \in G_{K,\varepsilon})$. For $k \in \{1, \ldots, d\}$, let us also define
\[
\Lambda^k_r = \{x \in \mathbb{R}^k : |x|_\infty \leq r\}, \tag{4.19}
\]
which is the $k$-dimensional section of the $d$-dimensional box $\{ x \in \mathbb{R}^d : |x|_\infty \leq r \}$, and set
\[
\psi_{k,\varepsilon}(\omega) := \limsup_{r \to \infty} \inf_{y \in C_\infty(\omega) \cap \Lambda^1_r} \frac{1}{|\Lambda^1_r|} \int_{x \in C_\infty(\omega) \cap \Lambda^1_r} \mathbb{I}\{|V_G(x, \omega) - V_G(y, \omega)| \geq \varepsilon r\} \, dx,
\]
\[
\psi_k(\omega) := \lim_{\varepsilon \to 0} \psi_{k,\varepsilon}(\omega).
\]

Lemma 4.8. Let $1 \leq k < d$. If $\psi_k = 0$ $\mathbb{P}$-almost surely, then also $\psi_{k+1} = 0$ $\mathbb{P}$-almost surely.

**Step 2: Proof of Lemma 4.8.** For $k \leq d$, we consider the $k$-dimensional Lebesgue measure on $\mathbb{R}^k$ and we call it $\lambda_k$. We assume that $\mathbb{P}$-a.s. $\psi_1 = 0$. In particular, for each $\varepsilon > 0$ and large enough $r$, there is some set $\Delta \subset C_\infty \cap \Lambda^1_r$ satisfying
\[
\lambda_1(\Lambda^1_r \cap C_\infty \setminus \Delta) \leq \varepsilon \lambda_1(\Lambda^1_r),
\]
\[
|V_G(x, \omega) - V_G(y, \omega)| \leq \varepsilon r \quad x, y \in \Delta.
\]
Moreover, for $K > 0$ large enough (but deterministic), replacing $\Delta$ by $\Delta \cap \mathcal{G}_{K,\varepsilon}$ grant us the following properties for large $r$:

1. $\lambda_1(\Lambda^1_r \cap C_\infty \setminus \Delta) \leq \varepsilon \lambda_1(\Lambda^1_r)$,
2. $|V_G(x, \omega) - V_G(y, \omega)| \leq \varepsilon r \quad x, y \in \Delta$,
3. $\Delta \subset \mathcal{G}_{K,\varepsilon}$, and
4. $\Delta \cap \Lambda^1_r \neq \emptyset$.

This is a consequence of the fact that $\lim_{K \to \infty} \mathbb{P}(0 \in C_\infty \setminus \mathcal{G}_{K,\varepsilon}) = 0$, and the ergodic theorem. We stress that even though these conditions are easily satisfied in dimension one, the construction will allow us to obtain the same properties in larger dimensions. In particular, we want that the “base” $\Delta$ is contained in each successive step, so that (iv) will be always valid.

Next, for $L \in \mathbb{N}$ and $r > 0$, define
\[
\Xi_{L,r}(\omega) := \{ x \in \Lambda^1_r : \# \{ 0 \leq i \leq L - 1 : x + ie_2 \in C_\infty(\omega) \} > 0 \}.
\]
we claim that for each $\delta > 0$, there exists some $L = L(\delta)$ (deterministic) that satisfies $\mathbb{P}$-a.s. $\lambda(\Xi_{L,r}) \geq (1 - \delta)\lambda(\Lambda^1_r)$ for large $r$ (which may depend on $\omega$). Indeed, by the ergodic theorem, the following equality holds $\mathbb{P}$-a.s. for all $L \in \mathbb{N}$:
\[
\lim_{r \to \infty} \frac{\lambda(\Xi_{L,r})(\omega)}{\lambda(\Lambda^1_r)} = \mathbb{P}( \# \{ 0 \leq i \leq L : ie_2 \in C_\infty(\omega) \} > 0).
\]
(4.22)

Since
\[
\lim_{L \to \infty} \frac{1}{L} \# \{ i \in \{0, \ldots , L - 1 \} : ie_2 \in C_\infty(\omega) \} = \mathbb{P}(0 \in C_\infty) > 0 \quad \mathbb{P}$-a.s.
\]
as $L \to \infty$, the probability on the right in (4.22) converges to 1, so the claim holds. For fixed $L$, choose $K > 0$ large enough so that $\mathbb{P}$-a.s., for all $i = 0, \ldots , L - 1$ the conditions (i)-(iv) above will hold some $\Delta_i \subset \tau_{ie_2}(\Lambda^1_r)$ (replacing $\Lambda^1_r$ with $\tau_{ie_2}(\Lambda^1_r)$ in (i) and (iv) for $r$ large enough. Next, we define for $r > 0$ (and setting $\Delta_0 := \Delta$)
\[
\Lambda := \Lambda_r := \{ x \in \Lambda^2_r \cap C_\infty : \exists 0 \leq i \leq L - 1 \in [-r, r]^2 \text{ such that } x = ye_1 + te_2 \text{ and } ye_1 + ie_2 \in \Delta_i \}.
\]
(4.23)

In words, $\Lambda$ represents the points in $x \in \Lambda^2_r$ which have some $\tilde{x} \in \Delta_i$ that shares the same projection over $\mathbb{R}e_1$. Note that $\Delta \subset \Lambda$, so in particular, $\Lambda \cap \Lambda^1_r \neq \emptyset$ for large $r$. We show that the density $\Lambda$ is close to 1. More precisely, if $x \in (\Lambda^2_r \cap C_\infty) \setminus \Lambda$, then $x = ye_1 + te_2$ for some $(y, t) \in [-r, r]^2$, and either
Since Step 3: Proof of Proposition 4.7. and by Corollary 4.6, it holds for large enough $r$

$$
\frac{\lambda_2((\Lambda_r^2 \cap C_\infty) \setminus \Lambda)}{\lambda_2(\Lambda_r^2)} \leq \frac{1}{2r} \int_{-r}^{r} \mathbb{P}(y_2 \in \Lambda_r^1 \setminus \Xi_{L,r})dy + \sum_{i=0}^{L-1} \frac{1}{2r} \int_{-r}^{r} \mathbb{P}(y_1 + s \in \Lambda_r^1 \setminus \Delta_i)dy \\
= \frac{1}{2r} \left[ \lambda_1(\Lambda_r^1 \setminus \Xi_{L,r}) + \sum_{i=0}^{L-1} \lambda_1((\Lambda_r^1 \cap C_\infty) \setminus \Delta_i) \right] \leq L \varepsilon + \delta. \tag{4.24}
$$

At this point, we choose $\varepsilon$ and $\delta$. Let $\varepsilon, \delta > 0$ small enough so that $L \varepsilon + \delta < \frac{1}{2} \mathbb{P}(0 \in C_\infty)^2$. By the FKG-inequality in $(\text{P5})$ (note that $\{x \in C_\infty\}$ is an increasing event), for every $x, y \in \mathbb{R}^d$ we have

$$
\mathbb{P}(x \in C_\infty(y), y \in C_\infty(x)) \geq \mathbb{P}(x \in C_\infty(y)) \mathbb{P}(y \in C_\infty(x)) = \mathbb{P}(0 \in C_\infty)^2.
$$

Moreover, for $K$ large enough, by the ergodic theorem we have for any $s, t \in \{0, \ldots, L - 1\}$

$$
\lim_{r \to \infty} \frac{1}{\lambda_1(\Lambda_r^1)} \lambda_1(x \in \Lambda_r^1 : x + s \in \mathcal{G}_{K,\varepsilon}, x + te \in \mathcal{G}_{K,\varepsilon}) = \mathbb{P}(s \in \mathcal{G}_{K,\varepsilon}, te \in \mathcal{G}_{K,\varepsilon}) > L \varepsilon + \delta. \tag{4.25}
$$

Thus, for large enough $r$, for every $s, t \in \{0, \ldots, L - 1\}$, the density of points $x \in \Lambda_r^1$ such that $x + s \in \Delta_s$ and $x + te \in \Delta_t$ is positive. To finish the proof, we verify that for each $u, v \in \Lambda$,

$$
|V_G(u, \omega) - V_G(v, \omega)| \leq 7r \varepsilon \text{ for large } r \text{ such that all the above holds (in particular, (4.24), (4.25))}.
$$

Indeed, if $u = x_1 e_1 + y_1 e_2$ and $v = x_2 e_1 + y_2 e_2 \in \Lambda$, then there are $s, t \in \{0, \ldots, L - 1\}$ such that if $u' := x_1 e_1 + s e_2$ and $v' := x_2 e_1 + t e_2$, then $u', v' \in \mathcal{G}_{K,\varepsilon}(\omega)$ (for $K = K(\omega)$ independent on $r$ that satisfies the conditions listed above). Moreover, by (4.25), there exists some $x_3 \in \Lambda_r^1$ satisfying $u'' = x_3 e_1 + s e_2$ and $v'' = x_3 e_1 + t e_2$. Putting all together, we have

$$
|V_G(u, \omega) - V_G(v, \omega)| \leq |V_G(u, \omega) - V_G(u', \omega)| + |V_G(u', \omega) - V_G(u'', \omega)| + |V_G(u'', \omega) - V_G(v'', \omega)| + |V_G(v'', \omega) - V_G(v, \omega)|
\leq K + \varepsilon|x_2 - s| + K + \varepsilon|x_1 - x_3| + K + \varepsilon|s - t| + K + \varepsilon|x_2 - x_3| + K + \varepsilon|y_2 - t|
\leq 5K + 3\varepsilon L + 6\varepsilon r \leq 7\varepsilon r
$$

for large enough $r$. In conclusion, by the last computation, the fact that $\Lambda \cap \Lambda_r^1 \neq \emptyset$ and (4.24), $q_{2,7\varepsilon} \leq L \varepsilon + \delta$. By letting first $\varepsilon \searrow 0$ and then $\delta \searrow 0$, we deduce that $q_{2} = 0$ $\mathbb{P}_0$-a.s.

We can use the same construction to go to higher dimensions. More precisely, the element $\Lambda$ for dimension $\rho$ becomes the element $\Delta$ in dimension $\rho + 1$. The base case guarantees that properties (i) and (iv) that appear at the beginning of the proof remain true for $\rho > 1$. This finishes the proof of Lemma 4.8.

\textbf{Step 3: Proof of Proposition 4.7} Theorem 4.7 will follow from Corollary 4.6 and Lemma 4.8. Since

$$
\inf_{y} \lambda_1(\{x \in C_\infty \cap \Lambda_r^1 : |V_G(x, \omega) - V_G(y, \omega)| \geq \varepsilon r\})
\leq \lambda_1(\{x \in C_\infty \cap \Lambda_r^1 : |V_G(x, \omega)| \geq \varepsilon r - |V_G(0, \omega)|\}),
$$

and by Corollary 4.6, it holds $q_{1} = 0$ for $\mathbb{P}_0$-almost every $\omega$. By changing over to appropriate shifts, we also have $q_{1} = 0$ for $\mathbb{P}$-almost every $\omega$. We use Lemma 4.8 repeatedly, which shows that $q_{d} = 0$ $\mathbb{P}$-a.s. and thus, $\mathbb{P}_0$-a.s. Again by Corollary 4.6, there exists $r_0 = r_0(\omega)$ with $\mathbb{P}_0(r_0 < \infty) = 1$ such
that \( |V_G(y, \omega)| \leq \varepsilon r/2 \) for any \( r \geq r_0 \) and any \( y \in \Lambda^d \cap C^\infty(\omega) \). Therefore,
\[
\lambda_d(\{x \in C^\infty \cap \Lambda^d: |V_G(x, \omega)| \geq \varepsilon r\}) \\
\leq \inf_y \lambda_d(\{x \in C^\infty \cap \Lambda^d: |V_G(x, \omega) - V_G(y, \omega)| \geq \varepsilon r - |V_G(y, \omega)|\}) \\
\leq \inf_y \lambda_d(\{x \in C^\infty \cap \Lambda^d: |V_G(x, \omega) - V_G(y, \omega)| \geq \varepsilon r/2\}),
\]
and (4.17) holds for any \( \varepsilon > 0 \). This finishes the proof of Theorem 4.7. \( \square \)

4.3 Proof of Theorem 4.2

We will prove an equivalent version of Theorem 4.2, namely:

**Theorem 4.9.** Fix \( d \geq 2 \) and \( G \in \mathcal{G}_\infty \). Then for \( P_0 \)-a.e. \( \omega \in \Omega_0 \),
\[
\lim_{r \to \infty} \sup_{x \in C^\infty \cap [-r,r]^d} \frac{|V_G(x, \omega)|}{r} = 0.
\]

Some preliminary lemmas will be required for the proof of the above result. Before that, let us set some notation that will be useful in the sequel. We will be interested in considering sets on \( \mathbb{R}^d \times \mathbb{R}^d \), so we endow this space with the standard product Lebesgue measure, which we denote by \( \lambda_d^\otimes 2 \). The section on the “first” coordinate of a measurable set \( A \subset \mathbb{R}^d \times \mathbb{R}^d \) is
\[
A^{(x)} := \{y \in \mathbb{R}^d: (x, y) \in A\} \quad \forall x \in \mathbb{R}^d.
\]

Given \( a \in (0,1) \) and \( r, \delta, \rho > 0 \), we also define
\[
C_r(a) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d: ar < |x - y|_\infty < r\}, \\
D(\rho) = D(\rho, \omega) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d: d_\omega(x, y) \geq \rho|x - y|_\infty; x, y \in C^\infty\}, \\
E(r) := (C^\infty)^2 \cap ([-r, r]^d)^2.
\]

**Lemma 4.10.** For any \( a \in (0,1) \), there exists a constant \( \rho = \rho(a, d) \) such that for all \( \delta > 0 \), \( P_0 \)-a.s. for large enough \( n \in \mathbb{N} \), for every \( x, y \in C^\infty \cap [-n, n]^d \) satisfying \( a\delta n < |x - y|_\infty < \delta n \), we have \( d_\omega(x, y) \leq \rho|x - y|_\infty \).

**Proof.** Fix any \( \delta' > \delta \) and \( \rho < c_0 \) (as in (1.14)), and choose \( 0 < a' < a \) such that \( a'a' < a\delta \), so that for \( n \in \mathbb{N} \) large enough, \( \delta n + 1 \leq \delta' n \) and \( \alpha \delta n = 1 > a'\delta' n \). By (P4) and (1.10), we have
\[
\mathbb{P}_0(\exists x, y \in C^\infty \cap [-n, n]^d, a\delta n < |x - y|_\infty < n\delta, d_\omega(x, y) \geq \rho|x - y|_\infty) \\
\leq \frac{1}{\mathbb{P}(0 \in C^\infty)} \mathbb{E} \left[ \sum_{x, y \in \omega} \mathbb{1}_\{(x, y) \in C_\delta(\omega') \cap [-n, n+1]^d, a'\delta n < |x - y|_\infty < n\delta, \right.
\]
\[
\left. \quad d_\omega(x, y) \geq \rho|x - y|_\infty; 0, x, y \in C^\infty\} \right] \\
= \frac{1}{\mathbb{P}(0 \in C^\infty)} \mathbb{E} \left[ \mathbb{1}_{C_\delta(\omega')} \mathbb{1}_{[-n, n+1]^d} \mathbb{1}_{\mathbb{1}_{C^\infty}(x, y, \omega)} \mathbb{1}_{\lambda_d^\otimes 2}(dx, dy) \mathbb{P}(0, x, y \in C^\infty, d_\omega(x, y) \geq \rho|x - y|_\infty) \right] \\
\leq \frac{\zeta^2}{\mathbb{P}(0 \in C^\infty)} \int_{C_\delta(\omega') \cap [-n, n+1]^2} \lambda_d^\otimes 2(dx, dy) \mathbb{P}(0, x, y \in C^\infty, d_\omega(x, y) \geq \rho|x - y|_\infty) \\
\leq C e^{-C'n}
for some $C = c(a, d, ρ), C' = C'(a, d, ρ) > 0$, with $ζ$ defined in (P2). The claim follows by the Borel-Cantelli lemma. □

Lemma 4.11. Let $C ⊂ R^d$ be any box of the type $[a_1, b_1] × [a_2, b_2] × \cdots × [a_d, b_d]$. Then $P$-a.s.,

$$\lim_{r \to \infty} \frac{\lambda_d(C_∞ \cap rC)}{\lambda_d(rC)} = p_∞.$$

Proof. This is an application of [K02, Theorem 10.14]. □

We are now ready to prove Theorem 4.9 which will also prove Theorem 4.2.

Proof of Theorem 4.9. We consider some $ℓ = ℓ(d, P) ∈ N$ satisfying

$$p_∞ > \frac{1}{2d(ℓ-1)}. \quad (4.28)$$

We claim the proof is complete once we show the following: in a measurable set $A$ such that $P_0(A) = 1$, for all $ε > 0$ and $ω ∈ A$, there exists some $r_0 = r_0(ω)$ such that if $r ≥ r_0$, for all $x ∈ [-r, r]^d \cap C_∞$ with $|V_G(x_ω)|_∞ > εr$,

$$\lambda_d \left( \left[ E(r) \cap C_δρ(2^{-λ}) \right] \cap \{ |V_G(\cdot, ω)|_∞ ≤ εr \} \right) > 0 \quad (4.29)$$

for some $δ = δ(d, P, ε)$ that vanishes as $ε → 0$ (recall the notation (4.20) and (4.27)). Indeed, for any $x, y ∈ C_∞$,

$$|V_G(x_ω) - V_G(y_ω)|_∞ ≤ d_ω(x, y) \text{ess sup}_{P_0} |G(ω, x)|_∞. \quad (4.30)$$

For a fixed $x ∈ C_∞ \cap [-r, r]^d$, if $|V_G(x_ω)|_∞ ≤ εr$ for all $r ≥ r_0$, there is nothing else to do. Otherwise, choose $r_1(ω)$ large enough so that Lemma 4.10 is true for $r ≥ r_1$ and $a = 2^{-λ}$ (of course the lemma is still true if we replace $n ∈ N$ by $r ∈ R$). Now, if $r ≥ r_0 \lor r_1$, by (4.29), for every $x ∈ [-r, r]^d \cap C_∞$ satisfying $|V(x_ω)|_∞ > εr$, we find some $y ∈ [-r, r]^d \cap C_∞$ such that $2^{-λ}δr < |x - y|_∞ ≤ δr$ and $|V(y_ω)|_∞ ≤ εr$. In particular, $d_ω(x, y) ≤ ρ|x - y|_∞ ≤ ρδr$. Hence, by (4.30), we deduce that

$$|V_G(x_ω)|_∞ ≤ |V_G(y_ω)|_∞ + |V_G(x_ω) - V_G(y_ω)|_∞ ≤ εr + d_ω(x, y) \text{ ess sup}_{P_0} |G(ω, x)|_∞ \quad (4.31)$$

$$≤ εr + δrρ \text{ ess sup}_{P_0} |G(ω, x)|_∞.$$

Since $δ → 0$ as $ε → 0$, this finishes the proof, once we prove the claim (4.29).

Now we turn to the proof of (4.29). By Theorem 4.7 there is a measurable set $A_1$ with $P_0(A_1) = 1$, so that for all $ω ∈ A_1$,

$$\limsup_{r \to \infty} \frac{1}{r^d} \int_{C_∞ \cap [-r, r]^d} 1\{ |V_G(x_ω)|_∞ > εr \} dx = 0. \quad (4.32)$$

On the other hand, by Lemma 4.11 we know that for a fixed box $C = [a_1, b_1] × [a_2, b_2] × \cdots × [a_d, b_d]$, there exists a measurable set $A_C$ satisfying $P(A_C) = 1$ and

$$\lim_{r \to \infty} \frac{\lambda_d(C_∞ \cap rC)}{\lambda_d(rC)} = p_∞ \quad \text{for all } ω ∈ A_C. \quad (4.33)$$

Choose any $κ = κ(d, P) ∈ (0, 1)$ (which exists due to (4.28)) and $c = (d, P) > 0$ satisfying

$$1 - κ > \frac{1}{p_∞2d(ℓ-1)} \quad \text{and}, \quad c \left( 1 - κ - \frac{1}{p_∞2d(ℓ-1)} \right) > 1. \quad (4.34)$$
Next, for each $\varepsilon > 0$, let
\[
\delta := \left( \frac{\varepsilon c}{p_\infty} \right)^{1/d}.
\] (4.36)
We can cover $[-1,1]^d$ with finitely many cubes $C_1, \ldots, C_m \subset [-1,1]^d$ of side $\delta$. In particular, for every $x \neq y$ in the same box we will have $|x - y|_\infty < \delta$. By Lemma 4.11 applied to these boxes, we deduce that there exists a measurable set $A_2$ with $\mathbb{P}_0(A_2) = 1$ such that for all $\omega \in A_2$ and $1 \leq i \leq m$ we have
\[
\lim_{r \to \infty} \frac{\lambda_d(C_\infty \cap rC_i)}{\lambda_d(rC_i)} = p_\infty.
\] (4.37)
Let $A := A_1 \cap A_2$. Then for every $\omega \in A$ there exists some $r_0 = r_0(\omega)$ such that for all $r \geq r_0$ and $1 \leq i \leq m$
\[
\lambda_d(C_\infty \cap [-r, r]^d \cap \{|V_G(\cdot, \omega)|_\infty > \varepsilon r\}) < \varepsilon r^d,
\] and
\[
\lambda_d(C_\infty \cap rC_i) \geq p_\infty(1 - \kappa)\lambda_d(rC_i)
\] (4.38)
\[
= r^d\delta^dp_\infty(1 - \kappa)
\]
\[
= \varepsilon c(1 - \kappa)r^d.
\]
For every fixed $x \in [-r, r]^d \cap C_\infty$ that satisfies $|V_G(x, \omega)|_\infty > \varepsilon r$, we have $x \in rC_i$ for some $1 \leq i \leq m$, so that $x \in C_\infty \cap rC_i$. We decompose $\lambda_d(C_\infty \cap rC_i)$ as
\[
\lambda_d(C_\infty \cap rC_i) = \lambda_d(C_\infty \cap rC_i \cap \{|V_G(\cdot, \omega)|_\infty > \varepsilon r\})
\]
\[
+ \lambda_d(C_\infty \cap rC_i \cap \{|V_G(\cdot, \omega)|_\infty \leq \varepsilon r\}).
\]
By (4.38), and noting that $rC_i \subset [-r, r]^d$, we know that $\lambda_d(C_\infty \cap rC_i \cap \{|V_G(\cdot, \omega)|_\infty > \varepsilon r\}) < \varepsilon r^d$. This inequality, combined with the equality above allow us to deduce that
\[
\lambda_d(C_\infty \cap rC_i \cap \{|V_G(\cdot, \omega)|_\infty \leq \varepsilon r\}) \geq \varepsilon r^d(c(1 - \kappa) - 1) > 0,
\] (4.39)
and the last inequality holds since $c > \frac{1}{1 - \kappa}$ by (4.35). Next, we decompose the Lebesgue measure of $D' := C_\infty \cap rC_i \cap \{|V_G(\cdot, \omega)|_\infty \leq \varepsilon r\}$ as
\[
\lambda_d(D') = \lambda_d(D' \cap B_{r\delta/2^\ell}^\infty(x)) + \lambda_d(D' \cap B_{r\delta/2^\ell}^\infty(x)^C),
\] (4.40)
where $B_{r\delta/2^\ell}^\infty(x)$ is the ball centered at $x$ of radius $r\delta/2^\ell$ with respect to the $|\cdot|_\infty$ norm, which is a cube of side $r\delta/2^{\ell-1}$.

We conclude that
\[
\lambda_d(D' \cap B_{r\delta/2^\ell}^\infty(x)) \leq (r\delta/2^{\ell-1})^d = \frac{\varepsilon c r^d}{p_\infty 2^{d(\ell-1)}}.
\] (4.41)
Therefore, by (4.39), (4.40) and (4.41),
\[
\lambda_d(D' \cap B_{r\delta/2^\ell}^\infty(x)^C) \geq \varepsilon r^d\left(c(1 - \kappa) - 1 - \frac{c}{2^{d(\ell-1)p_\infty}}\right).
\] (4.42)
By the choice of $c$ in (4.35), we deduce that
\[
c(1 - \kappa) - 1 - \frac{c}{2^{d(\ell-1)p_\infty}} > 0.
\]
Thus,
\[
\lambda_d(D' \cap B_{r\delta/2^\ell}^\infty(x)^C) > 0.
\] (4.43)
Finally, recall the notation of $A^{(\omega)}$ from (4.26) and that of $E(r)$ from (4.27). Then by construction,
\[
\{D' \cap B_{r\delta/2^\ell}^\infty(x)^C\} \subset (E(r) \cap C_{br}(2^{-\ell})^{(\omega)}) \cap \{|V_G(\cdot, \omega)|_\infty \leq \varepsilon r\},
\]
so by (4.43), the claim (4.29) follows, completing the proof of Theorem 4.9, therefore that of Theorem 4.2. □

5. Entropic variational analysis

Recall the variational formula defined in Theorem 3.8:

\[ \mathcal{H}(\theta) = \sup_{(b, \phi) \in \mathcal{E}} \left( \int \phi dP_0 \left[ \frac{1}{2} \text{div}(a\theta) + \langle \theta, b \rangle_a - L(b, \omega) \right] \right). \] (5.1)

We also set

\[ \mathcal{X}(\theta) := \inf_{G \in \mathcal{G}_\delta} \left( \text{ess sup}_{P_0} \left[ \frac{1}{2} \text{div}(a(G + \theta)) + H(G + \theta) \right] \right), \] (5.2)

with the class \( \mathcal{G}_\delta \) from Definition 4.1. The goal of this section is to show the equivalence of lower bound \( \mathcal{H}(\cdot) \) proved in Section 3 and the upper bound which will be provided by \( \mathcal{X}(\cdot) \) (see Section 5.3 below). The equivalence is presented in the following theorem.

Theorem 5.1. Assume (F1), (F2) and (F4). Then for any \( \theta \in \mathbb{R}^d \),

\[ \mathcal{H}(\theta) = \mathcal{X}(\theta). \]

The proof is divided into several steps. In Section 5.1 is proved the estimate \( \mathcal{H}(\theta) \geq \mathcal{X}(\theta) \) in Theorem 5.2 that crucially relies on Proposition 5.5, which is then proved in Section 5.2. In Section 5.3 complete the proof of Theorem 5.1 by establishing the bound \( \mathcal{H}(\theta) \leq \mathcal{X}(\theta) \). Finally, Section 5.4 is devoted to the completion of the proof of the main result in Theorem 2.1.

From now on we assume the same hypotheses from Theorem 5.1.

5.1 Proving the lower bound \( \mathcal{H}(\cdot) \geq \mathcal{X}(\cdot) \).

We will first prove

Theorem 5.2. Under the assumptions of Theorem 5.1, for any \( \theta \in \mathbb{R}^d \),

\[ \mathcal{H}(\theta) \geq \mathcal{X}(\theta). \]

The rest of Section 5.1 and Section 5.2 are devoted to the proof of the above theorem. We set

\[ \mathcal{D} := \{ g : C^2_c(\Omega_0) : g : \Omega_0 \to \mathbb{R} \} \] (5.3)

to be the linear space of functions on \( \Omega_0 \) with compact support, such that their first and second weak derivatives (defined in Section 3.2.1) exists and are continuous. For any \( g \in \mathcal{D} \), define

\[ R_\theta g(\omega) = \frac{1}{2} \text{div}(a(\omega)(\nabla g(\omega) + \theta)), \quad \text{and write} \quad R = R_0. \] (5.4)

As already observed in [KRV06], we have that for any \( g \in \mathcal{D} \),

\[ \int dP_0 \phi(Rg + \langle b, \nabla g \rangle_a) \begin{cases} = 0 & \forall g \in \mathcal{D} \text{ if } (b, \phi) \in \mathcal{E}, \\ \neq 0 & \text{for some } g \in \mathcal{D} \text{ if } (b, \phi) \notin \mathcal{E}, \end{cases} \] (5.5)

and hence, by taking constant multiples if \( (b, \phi) \notin \mathcal{E} \), we conclude that the infimum over \( g \in \mathcal{D} \) in (5.5) is 0 if \( (b, \phi) \in \mathcal{E} \), and \( -\infty \) otherwise. Therefore,

\[ \mathcal{H}(\theta) = \sup_{\phi \in \Phi} \sup_{b \in B_a} \inf_{g \in \mathcal{D}} \left[ \int dP_0 \phi \left( \frac{1}{2} \text{div}(a\theta) + \langle \theta, b \rangle_a - L(b, \omega) \right) + \langle Rg + \langle b, \nabla g \rangle_a \rangle \right], \] (5.6)

where

\[ \Phi := \left\{ \phi \in L^1_+(P_0) : \int \phi dP_0 = 1 \right\}. \] (5.7)
Furthermore, for \( \phi \in \Phi \),
\[
B_\phi := \left\{ b \in L^1_a(\phi d\mathbb{P}_0) : \forall \omega \in \Omega_0 : x \mapsto b(\tau_x \omega) \in \text{Lip} \right\},
\]
with \( L^1_a(\phi d\mathbb{P}_0) \) being defined in (5.15). We remark that, for any \( \phi \in \Phi \), the set \( B_\phi \) contains constant functions \( b \). First, we will prove

**Lemma 5.3.** Let \( \overline{H}(\theta) \) be the variational formula defined in (5.1) (or equivalently, in (5.6)). Then
\[
\overline{H}(\theta) = \sup_{\phi \in \Phi} \inf_{b \in B_\phi} \left\{ \int d\mathbb{P}_0 \phi(R_\theta g + H(\theta + \nabla g(\omega), \omega)) \right\}.
\]

**Proof.** By (5.6) and (5.4),
\[
\overline{H}(\theta) = \sup_{\phi \in \Phi} \sup_{b \in B_\phi} \inf_{g \in \mathcal{D}} \left\{ \int d\mathbb{P}_0 \phi(\langle \theta + \nabla g, b \rangle + R_\theta g - L(b, \omega)) \right\}.
\]

First, we need to exchange the supremum over \( b \) with the infimum over \( g \), for which we would like to apply the min-max theorem from [AE84, Theorem 8, p. 319]. The requirements for which are verified as follows. Note that the map
\[
b \mapsto \int d\mathbb{P}_0 \phi(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega))
\]
is concave and upper semicontinuous, while the map
\[
g \mapsto \int d\mathbb{P}_0 \phi(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega))
\]
is convex and lower semicontinuous.

We need to verify the remaining compactness (resp. coercivity): we will show that for a fixed \( \phi \in \Phi \) and \( g \in \mathcal{D} \), the level sets
\[
E_c := \left\{ b \in B_\phi : \int d\mathbb{P}_0 \phi(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega)) \geq c \right\}
\]
are weakly compact in \( L^1_a(\phi d\mathbb{P}_0) \). Indeed, by the Eberlein-Šmulian theorem (see [DS58, p.430]), checking the latter condition is equivalent to verifying that the set \( E_c \) above is weakly closed and sequentially weakly compact in \( L^1_a(\phi d\mathbb{P}_0) \). For the second condition, it is enough to show that \( E_c \) is bounded and uniformly integrable, but both these conditions follow from the coercivity of \( L \). Indeed, recall (2.16) from (F2)
\[
c_{10} ||g||_{a}^{\alpha'} - c_{11} \leq L(g, \omega) \leq c_{12} ||g||_{a}^{\alpha'} + c_{13}, \quad \alpha' = \frac{\alpha}{\alpha - 1}, \quad 1 < \alpha < \infty.
\]

On the other hand, using that \( g \in \mathcal{D} \) has compact support and \( \nabla g \) is continuous, \( |\theta + \nabla g| \leq (|\theta| + ||\nabla g||_{L^\infty(\mathbb{P}_0)} =: C_1(\theta, g) < \infty \). Moreover, by (F1) \( |\text{div}(a)| \leq C \), so we can find a constant \( C_2(\theta, g) \) such that by (5.4), \( |R_\theta g| \leq C_2(\theta, g) \). Hence,
\[
\int d\mathbb{P}_0 \phi(\langle \theta + \nabla g, b \rangle_a + R_\theta g) \leq C_2(\theta, g) + C_1(\theta, g) \int d\mathbb{P}_0 \phi \|b\|_a < \infty
\]
since \( b \in L^1_a(\phi d\mathbb{P}_0) \), recall (3.15).
Thus, it remains to show that $E_\varepsilon$ is weakly closed. Since $E_\varepsilon$ is convex, it suffices to show that $E_\varepsilon$ is strongly closed. Indeed, suppose that $(b_n)_n \subset E_\varepsilon$ such that $b_n \to b$ in $L^1_\alpha(\phi d\mathbb{P}_0)$. Passing to a subsequence, since $L$ is lower semicontinuous and by Fatou’s lemma, one can easily verify that $\int \mathbb{P}_0 \phi((\theta + \nabla g, b)_a + R_\theta g - L(b, \omega)) \geq c$. We will construct a function $\tilde{b}$ such that $\tilde{b} = b$ $\mathbb{P}_0$-a.s. and for all $\omega \in \Omega_0$, $\mathbb{R}^d \ni x \mapsto \tilde{b}(x, \omega) \in \text{Lip}$. Let $\Omega_0'$ with $\mathbb{P}_0(\Omega_0') = 1$ such that $b_n(\omega) \to b(\omega)$ for all $\omega \in \Omega_0'$. For a fixed $\omega \in \Omega_0'$, we know that the family $(b_n(\cdot, \omega))_n$ is uniformly equicontinuous and on any compact set $K \subset \mathbb{R}^d$ and $x \in K$, 

$$|b_n(x, \omega)| \leq |x| + |b_n(0, \omega)| \leq \text{diam}(K) + \sup_n |b_n(0, \omega)|.$$ 

As $b_n(0, \omega) \to b(0, \omega)$, the supremum above is finite. Hence, for fixed $\omega$, the family of continuous functions $(b_n(\cdot, \omega))_n$ is globally uniformly equicontinuous and uniformly bounded on compact sets. By the Arzelà-Ascoli theorem, the sequence $(b_n(\cdot, \omega))$ converges uniformly on compact sets and therefore converges pointwise to some function $\tilde{f}(\cdot, \omega) \in \text{Lip}$. By definition, $\tilde{f}(0, \omega) = b(\omega)$ $\mathbb{P}_0$-a.s. Now let us consider the set

$$\Omega_0' := \{\omega \in \Omega_0 : \exists x \in \mathbb{R}^d, \omega' \in \Omega_0' : \omega = \tau_x \omega'\}.$$

Then we define

$$\tilde{b}(\omega) := \begin{cases} \tilde{f}(x, \omega') & \text{if } \omega = \tau_x \omega' \text{ for some } x \in \mathbb{R}^d, \omega' \in \Omega_0', \\ 0 & \text{otherwise.} \end{cases} \quad (5.12)$$

Let us first check that $\tilde{b}$ is well-defined. Indeed, suppose that $\omega = \tau_x \omega' = \tau_y \omega''$ for some $x, y \in \mathbb{R}^d$ and $\omega', \omega'' \in \Omega_0'$. Then

$$\tilde{f}(x, \omega') = \lim_{n \to \infty} b_n(x, \omega') = \lim_{n \to \infty} b_n(0, \tau_x \omega') = \lim_{n \to \infty} b_n(0, \tau_y \omega'') = \lim_{n \to \infty} b_n(y, \omega'') = \tilde{f}(y, \omega'').$$

Hence, the function $\tilde{b}$ is well defined. Notice that on $\Omega_0'$, $\tilde{b}(\omega) = f(0, \omega) = b(\omega)$, so $\tilde{b} = b$ $\mathbb{P}_0$-a.s. Finally, let us check that for all $\omega \in \Omega_0$, $C_\infty(\omega) \ni x \mapsto \tilde{b}(x, \omega) \in \text{Lip}$. Indeed,

(i) If $\omega \in \Omega_0'$, then $\omega = \tau_z \omega'$ for some $z \in \mathbb{R}^d$ and $\omega' \in \Omega_0$ and for $x, y \in \mathbb{R}^d$,

$$|\tilde{b}(x, \omega) - \tilde{b}(y, \omega)| = |\tilde{b}(x, \tau_z \omega') - \tilde{b}(y, \tau_z \omega')| = |\tilde{f}(x + z, \omega') - \tilde{f}(y + z, \omega')| \leq |x - y|.$$

(ii) On the other hand, if $\omega \in \Omega_0 \setminus \Omega_0'$, then the same holds for $\tau_x \omega$ for all $x \in C_\infty(\omega)$, and the Lipschitz condition is trivially satisfied.

This finishes the proof that $E_\varepsilon$ is weakly compact in $L^1_\alpha(\phi d\mathbb{P}_0)$, and therefore, by the aforementioned min-max theorem, we can exchange the $\sup_{b \in B_\phi}$ and $\inf_{g \in D}$ in $(5.10)$ to obtain

$$\overline{H}(\theta) = \sup_{\phi \in \Phi} \sup_{g \in D} \sup_{b \in B_\phi} \left[ \int \mathbb{P}_0 \phi((\theta + \nabla g, b)_a + R_\theta g - L(b, \omega)) \right].$$
Since the integrand depends locally in $b$, we can bring the supremum over $b$ inside the integral, and use the duality between $H$ and $L$ to conclude that
\[ \overline{\Pi}(\theta) = \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[ \int dP_0\phi \left( R_{\theta}g + \sup_{b \in B_\phi} \left[ (\theta + \nabla g, b)_a - L(b, \omega) \right] \right) \right] \]
\[ = \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[ \int dP_0\phi \left( R_{\theta}g + H(\theta + \nabla g(\omega), \omega) \right) \right]. \quad (5.13) \]

In the last equality we used that, for any $\phi \in \Phi$, the set $B_\phi$ defined in (5.8) contains constants, so that
\[ \sup_{b \in B_\phi} \left[ (\theta + \nabla g, b)_a - L(b, \omega) \right] = \sup_{y \in \mathbb{R}^d} \left[ (\theta + \nabla g, y)_a - L(y, \omega) \right] = H(\theta + \nabla g(\omega), \omega). \]

\[ \square \]

We would like to now swap the order of $\sup_{\phi}$ and $\inf_g$ in (5.13).

**Lemma 5.4.** With $R_{\theta}$ defined in (5.4), let
\[ S_\theta(g)(\omega) := R_{\theta}g(\omega) + H(\theta + \nabla g(\omega), \omega), \quad g \in \mathcal{D}. \quad (5.14) \]

Then for any $\theta \in \mathbb{R}^d$,
\[ \overline{\Pi}(\theta) = \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[ \int dP_0\phi S_\theta(g) \right] \geq \lim_{\epsilon \to 0} \inf_{g \in \mathcal{D}} \left[ \epsilon \log \int dP_0 \exp \left[ \epsilon^{-1} S_\theta(g) \right] \right]. \quad (5.15) \]

**Proof.** For any probability density $\varphi \geq 0$ on $\Omega_0$ (i.e., $\int_{\Omega_0} \varphi dP_0 = 1$), let
\[ \text{Ent}_{P_0}(\varphi) = \int \varphi \log \varphi dP_0 \geq 0 \]
be the entropy of $\varphi$. Its non-negativity is a consequence of the Jensen's inequality. Moreover, the map $\varphi \mapsto \text{Ent}_{P_0}(\varphi)$ is convex, weakly lower semicontinuous and has weakly compact sub-level sets, meaning, for any $\ell > 0$, $\{ \varphi : \text{Ent}_{P_0}(\varphi) \leq \ell \}$ is compact in the weak topology. Thus, for any $\epsilon > 0$ we have the lower bound
\[ \overline{H}(\theta) \geq \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[ \int dP_0\phi S_\theta(g) - \epsilon \text{Ent}_{P_0}(\phi) \right] = \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[ \int dP_0\phi (S_\theta(g) - \epsilon \log \phi) \right]. \]

Similarly as in (5.11), we use the fact that $g \in \mathcal{D}$ together with the assumptions $[\text{F1}]$ to conclude that there is a constant $C_2(\theta, g)$ such that $|R_{\theta}(g)| \leq C_2(\theta, g)$, so that
\[ \int dP_0\phi S_\theta(g) \leq C_2(\theta, g) + \int dP_0\phi H(\theta + \nabla g) \leq C_2(\theta, g) + c_8 \int dP_0\phi \| \theta + \nabla g \|_a + c_9 \]
\[ \leq C_2(\theta, g) + c_8 C(\alpha, \theta, g) + c_9 < \infty. \]

where for the second inequality we used the upper bound from (2.5) in (F2) and for the third inequality we used $|a(\omega)x, x| \leq c_5 |x|^2$ from (F1) and again that $\sup_{\omega} |\theta + \nabla g(\omega)| \leq |\theta| + \|\nabla g\|_{L^\infty(P_0)}$ for $g \in \mathcal{D}$. Now, for any fixed $\phi \in \Phi$, the map
\[ g \mapsto \int dP_0\phi (S_\theta(g) - \epsilon \log \phi) \]
is convex and continuous, while for any fixed $g \in \mathcal{D}$, the map
\[ \varphi \mapsto \int dP_0\varphi (S_\theta(g) - \epsilon \log \varphi) \]
is concave, upper-semicontinuous and has compact superlevel sets in the weak $L^1_+(\mathbb{P}_0)$ topology, we can again use Von-Neumann’s minimax theorem to justify changing the order of sup$_\phi$ and inf$_g \in D$. This means,

$$\mathcal{P}(\theta) \geq \inf_{g \in D} \sup_{\phi} \left[ \int d\mathbb{P}_0 \phi (S_\theta(g) - \epsilon \log \phi) \right].$$

The above variational problem over $\phi$ subject to the condition $\int \phi d\mathbb{P}_0 = 1$ can be solved explicitly and the maximizing density is

$$\phi = \frac{\exp[\epsilon^{-1} S_\theta(g)]}{\mathbb{P}_0[\exp[\epsilon^{-1} S_\theta(g)]]}.$$

We replace this value of $\phi$ in the last lower bound for $\mathcal{P}(\theta)$ to obtain

$$\mathcal{P}(\theta) \geq \inf_{g \in D} \left[ \epsilon \log \int d\mathbb{P}_0 \exp[\epsilon^{-1} S_\theta(g)] \right].$$

We let $\epsilon \to 0$, to deduce the lower bound claimed in (5.15).

Given the above results, the lower bound in Theorem 5.2 will now be a consequence of the following technical result that will be established in Section 5.2.

**Proposition 5.5.** For any given $\epsilon > 0$, there exists a sequence $\epsilon_n \to 0$ and a sequence of functions $(g_n)_n \subset D$ so that

$$\mathcal{P}(\theta) \geq \epsilon_n \log \mathbb{E}_0 \left[ e^{\epsilon_n^{-1} S_\theta(g_n, \cdot)} \right] - \epsilon,$$

and $G_n(\omega) := \nabla g_n$ converges weakly in $L^{1+\delta}(\mathbb{P}_0)$ (with $\delta > 0$ as in (2.3)) and in distribution (along a subsequence) to some $G$. Furthermore, $G \in \mathcal{G}_\delta$, which is defined in Section 4.

**Proof of Theorem 5.2** (assuming Proposition 5.5): By Proposition 5.5, for $r > 0$, we pick some sequence $\epsilon_n \to 0$ and $g_n \in D$ satisfying

$$\mathcal{P}(\theta) \geq \epsilon_n \log \mathbb{E}_0 \left[ e^{\epsilon_n^{-1} S_\theta(g_n, \cdot)} \right] - r.$$

For fixed $n$, the map $\lambda \in [0, \infty) \to \frac{1}{\lambda} \log \mathbb{E}_0 [e^{\lambda S_\theta(g_n, \cdot)}]$ is increasing, so for each $\eta, \lambda > 0$, if $n$ is large enough,

$$\mathcal{P}(\theta) \geq \frac{1}{\lambda} \log \mathbb{E}_0 [e^{\lambda S_\theta(g_n, \cdot)}] - r = \frac{1}{\lambda} \log \mathbb{E}_0 \left[ e^{\lambda \left(\frac{1}{2} \text{div}(a(\omega)(G_n(\omega) + \theta)) + H(\theta + \nabla G_n(\omega), \omega))} \right] - r.$$

For any $M, \lambda > 0$, the map

$$x \mapsto e^{\lambda \left( M \left( \frac{1}{2} \text{div}(a(\omega)(x + \theta)) + H(\theta + x, \omega) \right) \right)}$$

is continuous and bounded. Thus, letting $n \to \infty$ and using the fact (from Proposition 5.5) that $G_n$ converges to $G$ in distribution, we conclude from the above bound that

$$\mathcal{P}(\theta) \geq \frac{1}{\lambda} \log \mathbb{E}_0 \left[ e^{\lambda \left( M \left( \frac{1}{2} \text{div}(a(\omega)(G(\omega) + \theta)) + H(\theta + G(\omega), \omega) \right) \right)} \right] - r.$$

Now by letting $M \nearrow \infty$ and using monotone convergence theorem, we obtain

$$\mathcal{P}(\theta) \geq \log \left\| e^{\frac{1}{2} \text{div}(a(\omega)(G(\omega) + \theta)) + H(\theta + G(\omega), \omega)} \right\|_{L^1(\mathbb{P}_0)} - r. \quad (5.17)$$
Finally, letting $\lambda \to \infty$, we obtain
\[
\overline{H}(\theta) \geq \text{ess sup}_{P_0} \left[ \frac{1}{2} \text{div}(a(G + \theta)) + H(G + \theta) \right] - r 
\geq \inf_{G \in \mathcal{G}} \overline{\Lambda}(\theta, G) - r.
\] (5.18)

Since $r > 0$ is arbitrary, we are done with the proof of Theorem 5.2.

5.2 Proof of Proposition 5.5

We divide the proof of Proposition 5.5 into three subsequent lemmas.

**Lemma 5.6.** For any given $\varepsilon > 0$, there exists a sequence $\varepsilon_n \to 0$ and a sequence of functions $(g_n)_n \in \mathcal{D}$ so that (5.16) holds, and $G_n(\omega) := \nabla g_n(\omega)$ converges weakly in $L^{1+\delta}(P_0)$ (with $\delta > 0$ as in (2.3)) and in distribution along a subsequence to some random variable $G \in L^{1+\delta}(P_0)$.

**Proof.** We start with the bound (5.15) in Lemma 5.4 which implies that there exist sequences $\varepsilon_n \to 0$ and $(g_n)_n \subset \mathcal{D}$ satisfying
\[
\varepsilon_n \log E_0 [e^{\varepsilon_n \mathcal{S}_\theta(g_n, \cdot)}] \leq \overline{H}(\theta).
\] (5.19)

Using this we will first show that
\[
\sup_n \|G_n\|_{L^{1+\delta}(\Omega_0)} < \infty.
\] (5.20)

In particular, the above bound will imply that $G_n$ converges weakly in $L^{1+\delta}(P_0)$ along a subsequence to some $G$.

We now prove (5.20). Note that the map $\lambda \in [0, \infty) \mapsto \frac{1}{\lambda} \log E_0 [e^{\lambda \mathcal{S}_\theta(g_n, \cdot)}]$ is increasing. Thus, recalling the definition of $\mathcal{S}_\theta$ from (5.14) and using (5.19), we obtain that for $n$ large enough,
\[
\log E_0 \left[ e^{R_\theta g_n(\omega) + H(\theta + \nabla g_n(\omega), \omega)} \right] \leq \overline{H}(\theta).
\]

The lower bound on $H(\cdot, \omega)$ from (F2) implies
\[
\log E_0 \left[ e^{R_\theta g_n(\omega) + c_6 \|\theta + \nabla g_n\|^2_{L^2} - c_7} \right] \leq \overline{H}(\theta).
\]

Set $G_n := \nabla g_n$. Then Jensen’s inequality applied to the bound bound and the definition of $R_\theta g_n = \frac{1}{2} \text{div}(a(\nabla g_n + \theta))$ from (5.14) leads to
\[
E_0 \left[ \frac{1}{2} \text{div}(a(G_n + \theta)) + c_6 \|\theta + G_n\|^2_a \right] \leq \overline{H}(\theta) + c_7.
\]

Since $G_n = \nabla g_n$, we have $E_0 [\text{div}(aG_n)] = 0$. Thus by (F1) we conclude that for some constant $C = C(\theta, \eta)$,
\[
\sup_n E_0 [\|G_n\|^\alpha_a] \leq C, \quad \alpha > 1.
\]

But by (2.22),
\[
\|G_n\|^\alpha_a = \langle a(\omega), G_n, G_n \rangle^{\alpha/2} \geq \xi(\omega)^{\alpha/2} |G_n|^\alpha.
\]

Combining the last two displays, we have
\[
\sup_n E_0 [\xi(\omega)^{\alpha/2} |G_n|^\alpha] \leq C.
\] (5.21)

Hence,
\[
E_0 [G_n^{1+\delta}] = E_0 \left[ |G_n|^{1+\delta} \xi(\omega)^{\frac{\alpha(1+\delta)}{2}} \xi(\omega)^{-\frac{1+\delta}{2}} \right] \leq E_0 \left[ |G_n|^\alpha \xi(\omega)^{\alpha/2} \right]^{\frac{\alpha(1+\delta)}{2}} E_0 \left[ \xi(\omega)^{-\frac{\alpha(1+\delta)}{2(\alpha-1)}} \right]^{\frac{\alpha-1-\delta}{\alpha}} < \infty.
\] (5.22)
In the first upper bound we used Hölder’s inequality with exponents \( \alpha \frac{\alpha}{\alpha - 1 - \delta} > 1 \) (recall that \( \alpha > 1 + \delta \)) and \( \frac{\alpha}{\alpha - 1 - \delta} \), and for the second bound we invoked (5.21) and (2.3) with \( \chi = \frac{2 \cdot \alpha - 1 + \delta}{\alpha - (1 + \delta)} \). Hence,

\[
\sup_n \mathbb{E}_0[\|G_n\|^{1+\delta}] < \infty,
\]

with \( \delta > 0 \). Consequently, \( G_n \) converges weakly in \( L^{1+\delta}(\mathbb{P}_0) \) and in distribution along a subsequence to some random variable \( G \in L^{1+\delta}(\mathbb{P}_0) \), as claimed.

**Lemma 5.7.** The limit \( G \) of \( G_n \) from Lemma 5.6 satisfies the closed loop condition defined in (4.3), i.e., for any simple closed path \( C \) contained in the infinite cluster \( C_\infty \), we have \( \int_C G(\omega, \cdot) \cdot dr = 0 \), almost surely w.r.t. \( \mathbb{P}_0 \).

**Proof.** We first assert that it suffices to prove that for any measurable set \( A \subset \Omega_0 \),

\[
\mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} \int_C G(\omega, \cdot) \cdot dr \right] = 0 \quad \text{for each simple closed path } C \subset C_\infty. \tag{5.23}
\]

Indeed, (5.23) says that for any fixed simple closed path, \( \mathbb{P}_0 \)-a.s., \( \mathbb{1}_{C \subset C_\infty} \int_C G(\omega, \cdot) \cdot dr = 0 \). We want to show that this holds \( \mathbb{P}_0 \)-a.s. uniformly on each closed loop. Since line integrals are independent of the parametrization of the path, for each \( C \), we choose any (but fixed from now) smooth function

\[
f_C : [0, 1] \to \mathbb{R}^d \quad \text{satisfying } f(0) = f(1).
\]

The space

\[
X := \{ f \in C^\infty[0, 1] : f(0) = f(1) \}
\]

is separable under the \( \| \cdot \|_\infty \) norm, so there exists some countable dense subset \( Y \subset X \). If (5.23) holds, we can show that \( \mathbb{P}_0 \)-a.s., the closed loop condition holds for each curve \( C \) such that \( f_C \in Y \). To extend this to all simple closed curves in \( \mathbb{R}^d \), we can approximate each curve by a sequence \( C_n \) such that \( f_{C_n} \in Y \). Since the convergence is uniform, it is easy to deduce that \( \mathbb{P}_0 \)-a.s., \( \mathbb{1}_{C \subset C_\infty} \int_C G(\omega, \cdot) \cdot r = 0 \) for any simple closed curve \( C \). Thus, we only need to show that (5.23) holds for fixed \( A \subset \Omega_0 \) and simple closed curve \( C \).

Let \( f : [0, 1] \to \mathbb{R}^d \) be any smooth function that parametrizes \( C \). For each fixed \( n \in \mathbb{N} \), we know that \( G_n = \nabla g_n \) satisfies the closed loop condition (because it is a gradient). By Fubini’s theorem we have

\[
0 = \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} \int_C G_n(\omega, \cdot) \cdot dr \right] = \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} \int_0^1 G_n(\omega, f(x)) \cdot f'(x) dx \right]
\]

\[
= \int_0^1 f'(x) \cdot \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} G_n(\omega, f(x)) \right] dx.
\]

Since \( G_n \) converges weakly to \( G \) in \( L^{1+\delta}(\mathbb{P}_0) \) (as shown in Lemma 5.6), for fixed \( x \in [0, 1] \),

\[
\lim_{n \to \infty} \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} G_n(\omega, f(x)) \right] = \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} G(\omega, f(x)) \right].
\]

Using that \( \sup_n \mathbb{E}_0[\|G_n\|^{1+\delta}] < \infty \), and that \( f' \) is bounded on \( [0, 1] \), we can apply dominated convergence theorem to conclude that

\[
0 = \lim_{n \to \infty} \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} G_n(\omega, f(x)) \right] = \int_0^1 f'(x) \cdot \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} G(\omega, f(x)) \right] dx.
\]

As \( G \in L^{1+\delta}(\mathbb{P}_0) \), we can again exchange the order of integration using Fubini’s theorem so the right-hand side in the last display is \( \mathbb{E}_0 \left[ \mathbb{1}_{A \cap (C \subset C_\infty)} \int_C G(\omega, \cdot) \cdot dr \right] \). This shows (5.23) and concludes the proof of the lemma. \( \square \)
The following result will complete the proof of Proposition 5.5.

**Lemma 5.8.** The limit $G$ of $G_n$ from Lemma 5.6 belongs to the class $G_0$ from Definition 4.1.

**Proof.** We have already proved (4.1). On the other hand, the proof of (4.2) follows from the first inequality in (5.13) (note that for this part we are only using the weak convergence of $G_n$ towards $G$ in $L^{1+\delta}(\mathbb{P}_0)$ and in distribution, which have been established in Lemma 5.6). Also, the closed loop property (4.3) was shown in Lemma 5.7. Thus it remain to check that $G$ satisfies the zero induced mean property $\mathbb{E}_0[V_{G}(\cdot, v_e)] = 0$, recall (4.5).

Let us fix a coordinate unit vector $e$, and recall the definitions of $v_e = n(\omega, e)e$ from (1.13), that of $\tilde{\ell}(\omega) = \tilde{d}_n(0, v_e)$ from (4.11) and of the sets $A(x_1, \cdots, x_k)$ from (1.13). Choose $\tilde{A}(x_1, \cdots, x_k) \subset A(x_1, \cdots, x_k)$ so that

$$\{\tilde{\ell} = j\} = \bigcup_{k=1}^{3^d j} \{\tilde{\ell} = j \cap \tilde{A}(x_1, \cdots, x_k}\},$$

where $\bigcup$ represents disjoint union. Next, for any $R > 0$ define

$$\eta_R := \mathbb{E}_0[V_{G}(\omega, v_e), \tilde{\ell} \leq R].$$

By dominated convergence theorem, the required identity $\mathbb{E}_0[V_{G}(\cdot, v_e)] = 0$ follows once we show that $\eta_R \to 0$ as $R \to \infty$. For this purpose, we further claim that

$$\eta_R = \lim_{n \to \infty} \mathbb{E}_0 \left[ \int_{0 \sim v_e} G_n(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} \leq R \right]$$

$$= - \lim_{n \to \infty} \mathbb{E}_0 \left[ \int_{0 \sim v_e} G_n(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} > R \right] .$$

We observe that the second equality in (5.25) follows from the fact that

$$\mathbb{E}_0 \left[ \int_{0 \sim v_e} G_n(\omega, \cdot) \cdot d\tilde{r} \right] = \mathbb{E}_0 \left[ g_n(\sigma_{v_e} \omega) - g_n(\omega) \right] = 0,$$

because $\sigma_{v_e}$ is measure-preserving under $\mathbb{P}_0$ (recall Proposition 3.5) and for each fixed $n$, $g_n$ is bounded and continuous. Thus, the only nontrivial claim is the first equality in (5.25). We decompose $\eta_R$ as (below, $x_0 := 0$)

$$\eta_R = \sum_{j=1}^{R} \sum_{k=1}^{3^d j} \sum_{x \in N_j} \mathbb{E}_0 \left[ \int_{0 \sim v_e} G(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_k) \right]$$

$$= \sum_{j=1}^{R} \sum_{k=1}^{3^d j} \sum_{x \in N_j} \sum_{i=1}^{k} \mathbb{E}_0 \left[ \int_{\tilde{A}(x_1, \cdots, x_i)} G(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_k) \right] .$$

On $\tilde{A}(x_1, \cdots, x_j)$, we can always choose the straight line between these two points as a curve. Using that $G_n$ converges to $G$ weakly in $L^{1+\delta}(\mathbb{P}_0)$ (cf. Lemma 5.6) we deduce that

$$\lim_{n \to \infty} \mathbb{E}_0 \left[ \int_{\tilde{A}(x_1, \cdots, x_j)} G_n(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_j) \right] = \mathbb{E}_0 \left[ \int_{\tilde{A}(x_1, \cdots, x_j)} G(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_j) \right] .$$

Therefore,

$$\eta_R = \sum_{j=1}^{R} \sum_{k=1}^{3^d j} \sum_{x \in N_j} \sum_{i=1}^{k} \lim_{n \to \infty} \mathbb{E}_0 \left[ \int_{\tilde{A}(x_1, \cdots, x_j)} G_n(\omega, \cdot) \cdot d\tilde{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_j) \right] .$$
Finally, we can exchange the limit with the sum over $x_1, \ldots, x_k$ by noting that

$$E_0 \left[ \int_{x_{i-1} \tilde{\ell} x_i} G_n(\omega, \cdot) \cdot \text{d}r, \tilde{\ell} = j, \tilde{A}(x_1, \ldots, x_j) \right]$$

is uniformly bounded because $\sup_n \|G_n\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty$. This shows (5.25). To conclude proving that $\eta_R \to 0$ as $R \to \infty$, we use (5.25) to estimate $|\eta_R|$ as

$$|\eta_R| = \lim_{n \to \infty} \left| E_0 \left[ \int_{0 \tilde{\ell} \mathbb{P}_0} G_n(\omega, \cdot) \cdot \text{d}r, \tilde{\ell} > R \right] \right| \leq \lim_{n \to \infty} \sup_{j=R} \sum_{j=R}^{\infty} \mathbb{E}_0 \left[ \int_{0 \tilde{\ell} \mathbb{P}_0} |G_n(\omega, \cdot)| \cdot \text{d}r, \tilde{\ell} = j \right].$$

Now, following the arguments exactly as in the proof of Proposition 4.4 and using that $\sup_n \|G_n\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty$, we can show that the last display is bounded above by $C_1 e^{-C_2 R}$ for some constants $C_1, C_2 > 0$, implying that $|\eta_R| \to 0$, which in turn completes the proof that $G$ satisfies the induced mean zero property. Thus Lemma 5.8 and therefore Proposition 5.5 are proved.

5.3 Proof of Theorem 5.1

Note that the bound $\Lambda(\cdot) \leq H(\cdot)$ has already been proved in Theorem 5.2. The proof of Theorem 5.1 will be complete once we show the reversed bound

$$\Lambda(\theta) \geq H(\theta). \tag{5.26}$$

The above inequality will follow once we prove an upper bound for Theorem 2.1 w.r.t. a linear initial condition $f(x) = \langle \theta, x \rangle$ for any $\theta \in \mathbb{R}^d$ (The upper bound for a general initial condition $f$ satisfying (F4) will be provided in Section 5.4 in Theorem 5.10 there).

Proposition 5.9. Assume (F1) and (F2). Let $u_{\varepsilon, \theta}$ be the unique viscosity solution to (2.10) with initial condition $f(x) = \langle \theta, x \rangle$. Then

$$\limsup_{\varepsilon \to 0} u_{\varepsilon, \theta}(t, 0, \omega) \leq t \overline{\Lambda}(\theta) \, \mathbb{P}_0\text{-a.s.}, \tag{5.27}$$

where $\overline{\Lambda}$ is defined in (5.22).

Assuming the above fact, we can conclude

Proof of Theorem 5.1 (assuming Proposition 5.9): Combining the lower bound from Theorem 3.8 for the particular case $f(x) = \langle \theta, x \rangle$ with Proposition 5.9 we conclude that $\overline{H}(\theta) \leq \overline{\Lambda}(\theta)$. The reverse bound has been shown in Theorem 5.2 which proves Theorem 5.1.

Proof of Proposition 5.9: Let us first sketch the main idea of the proof. To simplify notation, for a fixed $\theta \in \mathbb{R}^d$, we will simply write

$$u_{\varepsilon}(t, x, \omega) = u_{\varepsilon, \theta}(t, x, \omega).$$

Recall that for a fixed $t > 0$, by Lemma 3.9

$$u_{\varepsilon}(t, 0, \omega) = \varepsilon \sup_{c \in \mathcal{C}_{T_0}^{r*}} \mathbb{E}_0^{0, \omega} \left[ \langle \theta, X_{t/\varepsilon} \rangle - \int_0^{t/\varepsilon} L(X_s, c(s)) \text{d}s \right]. \tag{5.28}$$

Next, let us fix any $G \in \mathcal{G}_\delta$ as defined in (4.11), with $V_G(\omega, x) := \int_{0 \tilde{\ell} x} G(\omega, \cdot) \cdot \text{d}r$ as defined in (4.14), and set

$$h_G(x) := \langle \theta, x \rangle + V_G(x, \omega)$$
for a fixed $\omega \in \Omega_0$. If $V_G$ were smooth enough, $\nabla h_G = \theta + G$ and by Itô’s formula,

$$\langle \theta, X_{t/\varepsilon} \rangle + V_G(X_{t/\varepsilon}, \omega) = \int_0^{t/\varepsilon} (\theta + G(X_s)) \sigma(X_s) dB_s + \frac{1}{2} \int_0^{t/\varepsilon} \text{div}(a(\theta + G))(X_s) ds + \int_0^{t/\varepsilon} \langle c(s), \theta + G(X_s) \rangle_a ds.$$ 

Therefore, we would obtain

$$E^{\varepsilon, \omega}_0 E^{\varepsilon, \omega}_0 [\langle \theta, X_{t/\varepsilon} \rangle - \int_0^{t/\varepsilon} L(X_s, c(s)) ds] = -E^{\varepsilon, \omega}_0 [V_G(X_{t/\varepsilon}, \omega)]$$

$$+ E^{\varepsilon, \omega}_0 \left[ \int_0^{t/\varepsilon} \left( \frac{1}{2} \text{div}(a(X_s))(\theta + G(X_s)) + \langle c(s), \theta + G(X_s) \rangle_a - L(X_s, c(s)) \right) ds \right].$$

(5.29)

Together with (5.28), we would have the bound

$$E^{\varepsilon, \omega}_0 \left[ \int_0^{t/\varepsilon} \left( \frac{1}{2} \text{div}(a(G + \theta)) + H(G + \theta) \right) ds \right].$$

(5.30)

If $V_G$ were bounded, we could apply Theorem 4.9 and deduce that $\mathbb{P}_0$-a.s., for all $r > 0$ there exists some $c_r = c_r(\omega)$ such that for all $x \in C_\infty$, $|V_G(x, \omega)| \leq r|x| + c_r$, leading to

$$u_\varepsilon(t, 0, \omega) \leq t\lambda(\theta) + \varepsilon c_r + r \sup_{c \in C^*_T} E^{\varepsilon, \omega}_0 [\langle \varepsilon X_{t/\varepsilon} \rangle].$$

(5.31)

By Lemma 3.9 and the inequalities (3.34), one can deduce (see (5.34) below for details) that $E^{\varepsilon, \omega}_0 [\langle \varepsilon X_{t/\varepsilon} \rangle]$ is uniformly bounded over $0 < \varepsilon \leq 1$ and $c \in C^*_T$. Thus, one simply let first $\varepsilon \to 0$ and then $r \to 0$ to conclude the proof.

However, a priori $G \in G_{1+\delta}$ is neither smooth enough nor bounded. Nevertheless, we can mollify $G$ to get a smooth and bounded version, so that we can apply the same reasoning as above.

If $\rho$ is any spherically symmetric mollifier with support the unit ball and such that $\int_{\mathbb{R}^d} \rho(y) dy = 1$ and $r > 0$, we set

$$G_r(\omega) := \int_{\mathbb{R}^d} G(\tau_r y) \rho(y) dy = G * \rho_r,$$ 

(5.32)

where $\rho_r(y) := \delta^{-d} \rho(y/\delta)$. Similarly, define $V^r_G(x, \omega) := \int_{\mathbb{R}^d} V_G(x + \delta y, \omega) \rho(y) dy = V_G * \rho_r$. In particular, $\nabla V^r_G = G_r$ and by Young’s inequality, $|G_r|_{\infty} \leq C_r$ for some constant depending on $r$. As a consequence, for any $x \in C_\infty$ and any path $0 \sim x$ inside $C_\infty$,

$$V^r_G(x, \omega) - V^r_G(0, \omega) = \int_{0 \sim x} G_r(\omega, \cdot) \cdot d\mathbf{r},$$

(5.33)

so that the line integral is independent of the path $0 \sim x$. By Proposition 4.4, it also holds that $\mathbb{E}_0 \left[ \int_{0 \sim \delta x} |G_r(\omega, \cdot) - G(\omega, \cdot)| \cdot d\mathbf{r} \right] \to 0$ as $r \to 0$. Therefore, we can replace $G_r$ by $G_r - c_r$ with some constant vector $c_r$ such that $|c_r| \to 0$ as $r \to 0$ to obtain a smooth, bounded element in $G_\infty$ on which
Theorem 4.2 applies. Repeating the arguments in (5.29) with $G_r - c_r$ (equivalently, replacing $\theta$ by $\theta - c_r$), we obtain

$$\begin{align*}
E^{\epsilon, \omega}_0 \left[ \langle \theta, \epsilon X_{t/\epsilon} \rangle - \epsilon \int_0^{t/\epsilon} L(X_s, c(s)) ds \right] \\
\leq -\epsilon E^{\epsilon, \omega}_0 \left[ V^*_G(X_{t/\epsilon}, \omega) - V^*_G(0, \omega) - \langle c_r, X_{t/\epsilon} \rangle \right] \\
+ \epsilon E^{\epsilon, \omega}_0 \left[ \int_0^{t/\epsilon} \frac{1}{2} \left( \rho(a(X_s)) \langle \theta + G_r(X_s) - c_r \rangle + H(X_s, \theta + G_r(X_s) - c_r) \right) \right] ds.
\end{align*}$$

The first term can be bounded exactly as before. To handle the second and third terms, we use the convexity of $H$ and Jensen’s inequality to bound the sum of the second and third expectations above by

$$\begin{align*}
\epsilon E^{\epsilon, \omega}_0 \left[ \int_0^{t/\epsilon} \int_{\mathbb{R}^d} \left( \frac{1}{2} \right) \text{div}(a(X_s)) \langle \theta + G(X_s + ry) - c_r \rangle \right] \rho(y) dy ds \\
+ \epsilon E^{\epsilon, \omega}_0 \left[ \int_0^{t/\epsilon} \int_{\mathbb{R}^d} \left[ H(X_s, \theta + G(X_s + ry) - c_r) \right] \rho(y) dy ds \right] \\
\leq t \text{ess sup}_{F_0} \left[ \frac{1}{2} \text{div}(a(G + \theta - c_r)) + H(G + \theta - c_r) \right].
\end{align*}$$

Letting $\epsilon \to 0$, then $r \to 0$ and using the continuity of the map $\theta \mapsto \text{ess sup}_{F_0} \left[ \frac{1}{2} \text{div}(a(G + \theta)) + H(G + \theta) \right]$, we conclude the proof of the proposition.

5.4 Proof of Theorem 2.1 and Corollary 2.2

In this section we will complete the proof of Theorem 2.1 and Corollary 2.2.

5.4.1. Proof of Theorem 2.1: Given Theorem 3.8 it remains to show the following result:

**Theorem 5.10.** Let $u_\epsilon(t, x)$ be the solution of (1.2) and $u_{\text{hom}}$ as in (3.24). If (F1)-(F4) hold, then $P_0$-a.s., for any $T, \ell > 0$,

$$\limsup_{\epsilon \to 0} \sup_{0 \leq t \leq T} \sup_{x \in \mathcal{C}_\infty; |x| \leq \ell} \quad (u_\epsilon(t, x, \omega) - u_{\text{hom}}(t, x)) \leq 0. \quad (5.33)$$

**Proof:** For any fixed $t, \epsilon, x$ and $\omega$,

$$\begin{align*}
u_\epsilon(t, x, \omega) - u_{\text{hom}}(t, x) \\
= \sup_{\theta \in \mathcal{C}_T} \left( E^{\epsilon, \omega}_0 \left[ \frac{1}{t} \int_0^{t/\epsilon} f(\epsilon X_{t/\epsilon}) ds \right] - \sup_{y \in \mathbb{R}^d} \left( f(y) - t \mathcal{I} \left( \frac{y - x}{t} \right) \right) \right) \\
\leq \sup_{\theta \in \mathcal{C}_T} E^{\epsilon, \omega}_0 \left[ t \mathcal{I} \left( \frac{\epsilon X_{t/\epsilon} - x}{t} \right) - \epsilon \int_0^{t/\epsilon} L(X_s, c(s)) ds \right] \\
= \sup_{\theta \in \mathcal{C}_T} E^{\epsilon, \omega}_0 \left[ \sup_{\theta \in \mathcal{C}_T} \left( \frac{\epsilon X_{t/\epsilon} - x}{t} \right) - t \mathcal{I} \left( \frac{\theta - \epsilon X_{t/\epsilon} - x}{t} \right) - \epsilon \int_0^{t/\epsilon} L(X_s, c(s)) ds \right].
\end{align*}$$
Since by Lemma 3.9 and Eqs. (3.34) - (3.35)
\[ S := \sup_{\varepsilon \in \varepsilon} \sup_{0 \leq x \leq 1} \sup_{0 \leq t \leq T} E^{P_{\varepsilon}} \left[ |\varepsilon X_t(\varepsilon) + |\varepsilon \int_0^{t/\varepsilon} L(X_s, c(s))ds | \right] < \infty \] (5.34)

and $\overline{H}$ also satisfies the estimates of (2.5) with the Euclidean norm, we deduce that it is enough to show that for any fixed $\theta \in \mathbb{R}^d$,
\[ \limsup_{\varepsilon \to 0} \sup_{\varepsilon \in \varepsilon} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{C}_{\infty} : |x| \leq \ell} E^{P_{\varepsilon}} \left[ (\theta, \varepsilon x - x) - t\overline{H}(\theta) - \varepsilon \int_0^{t/\varepsilon} L(X_s, c(s))ds \right] \leq 0. \] (5.35)

Following as in the proof of Proposition 5.9 for any $G \in \mathcal{G}_\delta$ and $r > 0$, we apply Itô’s formula to $\theta + V_r - c_r$ with $|c_r| \to 0$ as $r \to 0$, obtaining
\[ E^{P_{\varepsilon}} \left[ (\theta, \varepsilon x_t(\varepsilon) - x) - \varepsilon \int_0^{t/\varepsilon} L(X_s, c(s))ds \right] - t\overline{H}(\theta) \]
\[ \leq -\varepsilon E^{P_{\varepsilon}} \left[ V_r^G(\varepsilon x_t(\varepsilon), \omega) - V_r^G(x_t(\varepsilon), \omega) - \langle c_r, x_t(\varepsilon) \rangle \right] \]
\[ + \varepsilon E^{P_{\varepsilon}} \left[ \int_0^{t/\varepsilon} \frac{1}{2} \text{div}(a(X_s))(\theta + G_r(X_s) - c_r) + H(X_s, \theta + G_r(X_s) - c_r)ds \right] - t\overline{H}(\theta). \] (5.36)

To bound the first term (5.36), we recall that, thanks to Theorem 4.2, $P_0$-a.s., for all $\tau > 0$ there is some $C_\tau = C_\tau(\omega)$ such that for all $x \in \mathbb{C}_{\infty}$, $|V_r^G(x, \omega)| \leq \tau|x| + C_\tau$. Then the first expectation (5.36) is bounded above by
\[ 2\varepsilon C_\tau + (\tau + |c_r|)E^{P_{\varepsilon}} \left[ |\varepsilon X_t(\varepsilon) + |\varepsilon x| \right]. \]

By (5.34), we deduce that
\[ \limsup_{\tau \to 0} \limsup_{\varepsilon \to 0} \sup_{\varepsilon \in \varepsilon} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{C}_{\infty} : |x| \leq \ell} \left( -\varepsilon E^{P_{\varepsilon}} \left[ V_r^G(\varepsilon x_t(\varepsilon), \omega) - V_r^G(x_t(\varepsilon), \omega) - \langle c_r, x_t(\varepsilon) \rangle \right] \right) \]
\[ \leq S|c_r|. \]

We bound the second term as in (5.32), implying that (5.37) is bounded above by
\[ t \sup_{\tau \geq 0} \left[ \frac{1}{2} \text{div}(a(G + \theta - c_r)) + H(G + \theta - c_r) \right] - t\overline{H}(\theta). \]

By Theorem 5.1 for any $\varepsilon' > 0$, there is some $G \in \mathcal{G}_\delta$ so that the last display is bounded by
\[ \varepsilon' + t\overline{H}(\theta - c_r) - t\overline{H}(\theta), \]
so that the final bound is $S|c_r| + \varepsilon' + t\overline{H}(\theta - c_r) - t\overline{H}(\theta)$. As $\overline{H}$ is continuous, letting $\varepsilon' \to 0$, then $\varepsilon' \to 0$ and finally $r \to 0$, we deduce (5.33), concluding thus the proof of Theorem 5.10 and Theorem 2.1.

5.4.2. Proof of Corollary 2.12. Let $u_\varepsilon$ solve (2.10) for the particular choice (2.14) and initial condition $f(x) = (\theta, x)$. We set
\[ v(t, x) := \exp \left\{ \frac{u_\varepsilon(\varepsilon t, \varepsilon x)}{\varepsilon} \right\}, \] (5.38)
then $v(t, x)$ solves
\[ \frac{\partial}{\partial t} v(t, x) = (\mathcal{L}^{(b, \omega)} v)(t, x), \quad v(0, x) = e^{(\theta, x)}, \quad \text{where} \]
\[ \mathcal{L}^{(b, \omega)} = \text{div}(a(\cdot, \omega) \nabla \cdot) + \langle b(\cdot, \omega), \nabla \cdot \rangle_a \] (5.39)
is the generator of the $\mathbb{R}^d$-valued diffusion $X_t$. By Feynman-Kac formula, we have $v(t, x) = E_x[\exp\{(\theta, X(t))\}]$ with $E_x$ denoting expectation with respect to the diffusion with generator $\mathcal{L}^b_\omega$ starting at $x \in \mathbb{R}^d$. Since $u_\varepsilon(t, 0) = \frac{1}{t} \log v(t/\varepsilon, 0)$, we have
\[
\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon(t, 0) = \lim_{t \to \infty} \frac{1}{t} \log v(t, 0),
\]
and the result follows from Theorem 2.1. Note that to prove Corollary 2.2, we only need to prove (5.40) for $x = 0$, the lower bound for which follows from Lemma 3.10. Hence, we require (2.3) only for $\chi = \frac{\gamma + \delta}{\alpha - (1 + \delta)}$ for some $\alpha > 1 + \delta$ and $\delta > 0$ which is needed for (5.22) to show weak compactness of the gradients and that $G \in G_8$. As explained earlier, the other moment assumption in (2.4) with $\gamma > d$ is necessary to deduce Lipschitz estimates, which guarantee locally uniform convergence in the lower bound for Theorem 2.1.

\section*{Appendix A. Percolation models satisfying assumptions $\text{(P1)}$-$\text{(P6)}$}

Let us illustrate some important percolation models that satisfy the assumptions imposed earlier.

The **Boolean model**: The simplest continuum percolation model, known as the Boolean model, is defined as $\mathcal{C}(\omega) := \bigcup_{x \in \omega} B_{1/2}(x)$, where $\omega$ is sampled with respect to a probability measure $\mathbb{P} = \mathbb{P}^\zeta$ that satisfies the following (recall the notation from Section 1.1):

- For any bounded $A \subset \mathbb{R}^d$ and any $n \in \mathbb{N}_0$,
  \[
  \mathbb{P}^\zeta(\#(\omega \cap A) = n) = \frac{(\zeta | A|)^n}{n!} e^{-\zeta | A|}. \tag{A.1}
  \]

- For any collection of disjoint, bounded Borel sets $A_1, \ldots, A_k \subset \mathbb{R}^d$,
  \[
  \mathbb{P}^\zeta(\#(\omega \cap A_1) = n_1, \ldots, \#(\omega \cap A_k) = n_k) = \prod_{i=1}^k \mathbb{P}(\#(\omega \cap A_i) = n_i). \tag{A.2}
  \]

The Boolean model satisfies the above assumptions $\text{(P1)}$-$\text{(P6)}$ for $\zeta$ large enough.

**Proposition A.1.** Fix $d \geq 2$. Then there exists $\zeta_c \in (0, \infty)$ such that for all $\zeta > \zeta_c$, $(\Omega, \mathcal{G}, \mathbb{P}^\zeta)$ satisfies $\text{(P1)}$-$\text{(P5)}$ and also $\text{(P6)}$ if $d \geq 3$.

**Proof.** The proof of these results are well-known: $\text{(P1)}$ is a consequence of [MR96] Propositions 2.6-2.7. $\text{(P2)}$ is consequence of (A.1), $\text{(P3)}$ follows from [MR96] Theorems 3.5-3.6. Property $\text{(P4)}$ can be found in [CGY11] Lemma 3.4, and $\text{(P5)}$ appears in [MR96] Theorem 2.2. To verify $\text{(P6)}$ in $d \geq 3$, given $\zeta > \zeta_c$, let $L > 0$ be large enough so that in $\mathbb{R}^{d-1} \times [0, L]$ there almost surely exists an infinite cluster $\mathcal{C}_\infty$. That existence is guaranteed by the fact that the critical intensity for $\mathbb{R}^d$ coincides with the limit of the critical intensities for $\mathbb{R}^2 \times [0, L]^{d-2}$ as $L \to \infty$ (see [T93] Theorem 1). By the uniqueness of the percolation cluster in $\mathbb{R}^d$, the cluster $\mathcal{C}_\infty$ is almost surely a subset of $\mathcal{C}_\infty$. Now let $A_L$ be the event that at least one element of the set $\{j \in \mathbb{N}_0 : j = 1, \ldots, L\}$ is an element of $\mathcal{C}_\infty$. Then
\[
\{|v_\varepsilon| \geq Lt\} \cap \{0 \in \mathcal{C}_\infty\} \subset \bigcap_{k \leq |t|} \tau_{kLe}(A_L^C).
\]
Since all the events in the intersection above are independent, with $p_L = \mathbb{P}(A_L)$, we have
\[
\mathbb{P}(|v_\varepsilon| \geq Lt, 0 \in \mathcal{C}_\infty) \leq (1 - p_L)^{|t|}
\]
for all $t > 0$, and $\text{(P6)}$ follows.
Continuum random cluster model: Besides the Boolean model, we mention briefly a model which presents long-range correlations, namely the continuum random cluster model (CRCM) [DH15, H18]. We proceed to define the model and give a sketch of the validity of properties (P1)-(P6). For \( \omega \in \Omega \) and \( \Lambda \subset \mathbb{R}^d \), set \( \omega_\Lambda := \omega \cap \Lambda \). A probability measure \( P \) on \( (\Omega, \mathcal{G}) \) is a continuum random cluster model with parameters \( q \) and \( \Lambda \subset \omega \) and each bounded set \( \Lambda \subset \mathbb{R}^d \), the conditional law of \( P \) given \( \omega_\Lambda \) is absolutely continuous with respect to a Poisson point process restricted to \( \Lambda \), \( \mathbb{P}_\Lambda^\zeta \), with density

\[
\frac{q^{N_\Lambda(\omega_\Lambda)}}{Z_\Lambda(\omega_\Lambda)}.
\]

Here, \( N_\Lambda^\Lambda \) is the \( \Lambda \)-local number of connected components of a configuration [H18, Definition 2.1] and \( Z_\Lambda \) is the partition function

\[
Z_\Lambda(\omega_\Lambda) := \int_\Omega q^{N_\Lambda(\omega_\Lambda)} \mathbb{P}_\Lambda^\zeta(d\omega_\Lambda).
\]

Equivalently, the DLR equations are satisfied: for every bounded and measurable function \( f \) and bounded set \( \Lambda \subset \mathbb{R}^d \),

\[
\int_\Omega f(\omega)dP = \int_\Omega \int_\Omega f(\omega' \cup \omega_\Lambda) \frac{q^{N_\Lambda(\omega' \cup \omega_\Lambda)}}{Z_\Lambda(\omega_\Lambda)} \mathbb{P}_\Lambda^\zeta(d\omega_\Lambda)P(d\omega).
\]

By [DH15, Theorem 1], there exists at least one stationary CRCM\((q, \zeta)\), which can be chosen ergodic, so that (P1) holds, while (P2) is satisfied by construction. Assumption (P3) is a consequence of [H18, Theorems 1 and 2].

Next, we sketch the ideas behind the proof of (P4) for which we need some definitions.

- Given a finite subset \( \Lambda \subset \mathbb{Z}^d \), the outer boundary of \( \Lambda \) is given by \( \partial^{\text{out}}\Lambda := \{ x \in \Lambda^c : \exists y \in \Lambda, |x - y|_1 = 1 \} \).
- If \( B := \prod_{i=1}^d [a_i, b_i] \) is a box in \( \mathbb{R}^d \), we say a connected component \( C \) contained in \( B \) is crossing for \( B \) if for all \( i \in \{1, \cdots, d\} \), there exist vertices \( x(i) = (x_1(i), \cdots, x_d(i)) \in C \) and \( y(i) = (y_1(i), \cdots, y_d(i)) \in C \) such that \( |x(i) - a| \leq \frac{1}{2} \) and \( |y(i) - b| \leq \frac{1}{2} \).
- For each \( M > 0 \), we define a random field \( \{X_z : z \in \mathbb{Z}^d\} \) as follows. Let \( B_z \) and \( B_z^+ \) be concentric cubes centered at \( z \) of radius \( M \) and \( \frac{5M}{4} \) respectively. Define the event

\[
A_z := \{ \exists ! \text{ crossing component } C_z \text{ in } B_z^+, \text{ each subbox of radius } M/4 \text{ and center } z + h, h \in \mathbb{Z}^d \cap [-M/2, M/2]^d \text{ contains a unique crossing component, and all of these crossing components are connected to } C_z \},
\]

and set \( X_z := 1_{A_z} \). If one can prove that for some integer \( k \geq 1 \),

\[
\lim_{M \to \infty} \sup_{z \in \mathbb{Z}^d} \sup_{p \in (0,1)} \mathbb{P}(X_z = 1 | \sigma(X_y : |y - z|_\infty > k)) = 1,
\]

then for all \( p \in (0,1) \), if \( M = M(p) \) is large enough, the product measure of i.i.d random variables \( \{Y_z : z \in \mathbb{Z}^d\} \) such that \( Y_z = 1 \) with probability \( p \) and 0 otherwise, is stochastically dominated by the law of \( \{X_z : z \in \mathbb{Z}^d\} \) [LLS97, Theorem 1.3]. Then following as in [CGY11] pp. 160-161 one can conclude the proof of (P4). To check (A.3), see for example [P96, Theorem 3.1] when \( d \geq 3 \) and [CM04, Theorem 9] when \( d = 2 \).

Finally, we mention that, at least for \( \zeta \) sufficiently large (large enough so that it surpasses a slab critical parameter), it can shown that (P6) is satisfied. Regarding the FKG inequality,
we could not find in the literature the version for the CRCM, but following ideas from [G06, Theorem 4.17] and [MR96, Theorem 2.2] we believe it should yield the desired result.

Beyond the models discussed above, there are many other models which exhibit long-range correlations and which satisfy assumptions (P1)(P6) — examples of such models include random interlacements, the vacant sets of random interlacements and the level sets of the Gaussian free field in \( d \geq 3 \), see [Sz10, T09, T09a, CP12, DRS14, PRS15].

**Appendix B. Ergodic properties.**

Here we will provide the proof of Proposition 3.5, which is a consequence of the following known result from ergodic theory (see [P89, BB07]). We include it here for the sake of completeness.

**Lemma B.1.** Let \((\mathcal{X}, \mathcal{F}, \mu)\) be a probability space and let \(T: \mathcal{X} \to \mathcal{X}\) be invertible, measure preserving and ergodic with respect to \(\mu\). Let \(A \in \mathcal{F}\) with \(\mu(A) > 0\). If \(n: A \to \mathbb{N} \cup \{\infty\}\) is defined by

\[
n(x) = \min\{k > 0 : T^k(x) \in A\}
\]

and \(S: A \to A\) by \(S(x) = T^{n(x)}(x)\) for \(x \in A\), then \(S\) is measure preserving and ergodic with respect to \(\mu|_A\) and almost surely invertible with respect to the same measure.

**Proof.** We first prove that \(S\) is measure preserving. By the Poincaré Theorem, \(n(x) < \infty\) almost surely. For any \(j \geq 1\) we define \(A_j = \{x \in A : n(x) = j\}\). By definition, the \(A_j\) are disjoint and as \(n(x) < \infty\) almost surely, \(\mu(A \setminus \bigcup_{j \geq 1} A_j) = 0\). As the restriction of \(S\) to \(A_j\) is \(T^j\) and since \(T^j\) is measure preserving, \(S\) is measure preserving on \(A_j\). We claim that \(S(A_i) \cap S(A_j) = \emptyset\). This, together with the fact that \(S\) is measure preserving on \(A_j\), proves that \(S\) is measure preserving on the disjoint union \(\bigcup_{j \geq 1} A_j\) and therefore on \(A\).

Thus, we only owe the claim \(S(A_i) \cap S(A_j) = \emptyset\). We assume that there exists \(x \in S(A_i) \cap S(A_j)\) for \(1 \leq i < j\). This requires the existence of \(y, z \in A\) with \(n(y) = i\), \(n(z) = j\) and \(x = T^i(y) = T^j(z)\). As \(T\) is invertible, \(y = T^{j-i}(z)\). Thus, \(n(z) \leq j - i < j\), which is a contradiction to \(n(z) = j\) and the desired claim follows.

Next, we note that \(T\) is invertible. Thus, \(S\) is almost surely invertible, as the intersection \(S^{-1}(\{x\}) \cap \{S\text{-well defined}\}\) is a one-point set.

We finally want to show that \(S\) is ergodic. Let \(B \in \mathcal{F}\) such that \(B \subseteq A\) is \(S\)-invariant. Then if \(x \in B\) and \(n \geq 1\), it follows that \(S^n(x) \notin A \setminus B\). This implies that for any \(x \in B\) and \(k \geq 1\), if \(T^k(x) \in B\), then \(T^k(x) \notin A \setminus B\). We conclude that \(C = \bigcup_{k \geq 1} T^k(B)\) is \(T\)-invariant and \(B \subseteq C \subseteq (X \setminus A) \cup B\). In particular, \(\mu(B) \leq \mu(C) \leq 1 + \mu(B) - \mu(A)\). Therefore, ergodicity of \(T\) implies \(\mu(C) \in \{0, 1\}\), which forces \(\mu(B) \in \{0, \mu(A)\}\) and thus, the ergodicity of \(S\) with respect to \(\mu|_A\).

**Proof of Proposition 3.5.** The shift \(\tau_\varepsilon\) is invertible, measure preserving and ergodic with respect to \(\mathbb{P}\). It follows from Lemma B.1 that the induced shift \(\sigma_\varepsilon\) is \(\mathbb{P}_0\)-preserving, almost surely invertible and ergodic with respect to \(\mathbb{P}_0\).

**Appendix C. Proof of Proposition 3.3.**

We now sketch the ideas behind the proof of existence and uniqueness of solutions of (3.10). While these arguments are well-known, we include them here for the sake of completeness. Since \(\omega \in \Omega_0\) is fixed, we omit it from the notation.

Let us first address the existence of solution. Recall from Section 3.1 that we fix a probability space \((\mathcal{X}, \mathcal{F}, P)\), a filtration \((\mathcal{F}_t)_{t \geq 0}\) and a Brownian motion \((B_t)_{t \geq 0}\) adapted to the filtration. We
set \( S := (0, T) \times C_{\infty} \), where \( T > 0 \) is fixed. The controls take values in \( U = \mathbb{R}^d \). The diffusion will be governed by controlled functions \( b : S \times U \rightarrow \mathbb{R}^d \) and \( \sigma : S \times U \rightarrow S_d \), where \( \sigma \) is defined as in Section 2.1 and \( b(t, x, u) := a(x)u + \text{div} a(x) \). Note that both \( b \) and \( \sigma \) are independent on \( t \), so we omit such dependence in what follows. Then for each \( c \in C_T \), there exists a unique solution of the SDE
\[
dX_t = b(X_t, c_t)dt + \sigma(X_t, c_t)dB_t
\]for each initial condition \( x \in \mathbb{R}^d \). We follow the set up from [13], where the cost function \( J : [0, T] \times \mathbb{R}^d \times C_T \rightarrow \mathbb{R} \) is defined as
\[
J(t, x, c) := \mathbb{E}^P \left[ f(X_T^{t,x,c}) - \int_t^T L(X_s^{t,x,c}, c_s)ds \right],
\]with \( X_s^{t,x,c} \) being the solution to (C.1) with initial condition \( X_t^{t,x,c} = x \). Then the value function \( V : S \rightarrow \mathbb{R} \) is defined by
\[
V(t, x) := \sup_{c \in C_T} J(t, x, c).
\]
The corresponding Hamiltonian \( \tilde{H} : \mathbb{R}^d \times \mathbb{R}^d \times S_d \rightarrow \mathbb{R} \) is given by
\[
\tilde{H}(x, p, A) := \sup_{u \in \mathbb{R}^d} \left[ b(x, u) \cdot p - L(x, u) + \frac{1}{2} \text{Trace}(a(x)A) \right] = \frac{1}{2} \text{Trace}(a(x)A) + \frac{1}{2} \text{div} a(x) \cdot p + H(x, p),
\]with \( H \) defined in Section 2.1. By [13, Theorem 7.4], \( V \) is a viscosity solution of the (backward) equation
\[
\begin{align*}
\partial_t V &= -\frac{1}{2} \text{Trace}(a(\cdot) \text{Hess}_x V) - \frac{1}{2} \text{div} a(x) \cdot \nabla V - H(\cdot, \nabla V), & \text{in } [0, T) \times C_{\infty}, \\
V(T, x) &= f(x), & \text{on } C_{\infty}.
\end{align*}
\]But note that defining \( u(t, x) := V(T - t, x) \) transforms (C.2) into (3.10) and \( \tilde{J} \) into \( J \) from (3.9). This completes the existence proof.

To verify uniqueness, we will appeal to the following comparison principle:

**Theorem C.1.** [AT15, Theorem 2.3] Let \( U \subset \mathbb{R}^d \) open and \( T > 0 \). Assume that \( u \in \text{USC}([0, T) \times \overline{U}) \), \( v \in \text{LSC}([0, T) \times \overline{U}) \) are of at most linear growth. Suppose that \( u \) is a viscosity subsolution and \( v \) is a viscosity supersolution of
\[
\partial_t u - \frac{1}{2} \text{Trace}(a(x) \text{Hess}_x u) - H(x, \nabla u),
\]in \( (0, T) \times U \) such that \( u(\cdot, 0) \leq v(\cdot, 0) \) on \( \overline{U} \) and \( u \leq v \) on \([0, T) \times \partial U\). Then \( u \leq v \) in \([0, T) \times U\).

Note that our equation (3.10) is of the form
\[
\partial_t u = \frac{1}{2} \text{Trace}(a(x) \text{Hess}_x u) + H(x, \nabla u).
\]But (C.3) and (C.4) are equivalent by mapping \( u \) to \(-u\) if \( p \rightarrow H(\cdot, -p) \) is also convex, which we are assuming. This change leaves the hypothesis from (F2) and (F3) invariant, so that it is enough to obtain a comparison principle for (C.3). The assumptions on \( a, \sigma, H \) in the mentioned articles are similar to ours, except that the coercivity assumption (2.5) in our case is defined with respect to the \( a \)-norm instead of the usual Euclidean norm. But this coercivity with respect to the \( a \)-norm is enough to use the above comparison principle. Indeed, (2.2)-(2.5) imply that \( \mathbb{P}_0 \)-a.s. \( \xi(\omega) > 0 \) and for all \( x \in C_{\infty} \), there exists a neighborhood of \( x \) such that for all \( y \) in such a neighborhood, \( \xi(\tau_y \omega) > \xi(\tau_x \omega)/2 \).

In
particular, for each \( x \in C_\infty \), there is a neighborhood of \( x \) such that the lower bound in (2.5) can be replaced by \( c(x)|p|^\alpha \). The rest of the argument follows from the above result.

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References

[AD18] S. Armstrong and P. Dario. Elliptic regularity and quantitative homogenization on percolation clusters. Comm. Pure Appl. Math. 71 (2018), 1717-1849

[AS12] S. Armstrong and P. Souganidis. Stochastic homogenization of Hamilton–Jacobi and degenerate Bellman equations in unbounded environments. J. Math. Pures Appl. 97, no. 5, 460-504 (2012)

[AS13] S. Armstrong and P. Souganidis. Stochastic Homogenization of Level-Set Convex Hamilton-Jacobi Equations. International Mathematics Research Notices. 2013, no.15, 3420?3449 (2013)

[AT14] S. Armstrong and H. Tran. Stochastic homogenization of viscous Hamilton-Jacobi equations and applications. Anal. PDE. 7, no. 8, 1969-2007 (2014)

[AT15] S. Armstrong and H. Tran. Viscosity solutions of general viscous Hamilton-Jacobi equations. Math. Ann. 361 no. 3-4, 647-687 (2015)

[AE84] J.-P Aubin and I. Ekeland. Applied nonlinear analysis. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York (1984)

[BGHJ21] C. Beck, L. Gonon, M. Hutzenthaler and A. Jentzen. On existence and uniqueness properties for solutions of stochastic fixed point equations. Discrete Contin. Dyn. Syst. Ser. B 26, no. 9, 4927-4962 (2021) (2012)

[BB07] N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. Probab. Theory Relat. Fields 137, no. 3-4, 83-120 (2007)

[BMO16] N. Berger, C. Mukherjee and K. Okamura. Quenched large deviations for simple random walks on percolation clusters including long range correlations. Comm. Math. Phys. 358, no. 2, 633-673 (2018)

[CP12] J. Cerny and S. Popov. On the internal distance in the interlacement set Electron. J. Probab. 17, no. 29, 1-25.

[CGY11] G. Chen, T-D. Guo and C-L. Yao. Large Deviations for the Graph Distance in Super-critical Continuum Percolation. J. Appl. Prob. 48, no. 1, 154-172 (2011)

[CM04] A. Couronné and R. J. Messikh. Surface order large deviations for 2D FK- percolation and Potts models. Stoc. Proc. Appl. 113, no. 1, 81-99 (2004)

[D19] A. Davini. Existence and uniqueness of solutions to parabolic equations with superlinear Hamiltonians. Contemp. Math. 21, no. 1, 1-25 (2019)

[DH15] D. Dereudre and P. Houdebert. Infinite volume continuum random cluster model. Electron. J. Probab. 20, no. 125, 1-24 (2015)

[DRS14] A. Drewitz, B. Ráth, and A. Sapozhnikov. On chemical distances and shape theorems in percolation models with long-range correlations. J. Math. Phys. 55 (2014), no. 8, 083307, 30 pp.

[DS58] N. Dunford and J.T. Schwartz Linear operators, part 1: general theory. John Wiley & Sons (1958)
L. Evans Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A.* **120** (3-4) 245-265 (1992)

L. Evans Partial differential equations. Graduate Studies in Mathematics, 19. American Mathematical Society (2010)

G. R. Grimmett. The random-cluster model. Springer-Verlag (2006).

P. Houdebert. Percolation results for the continuum random cluster model. *Adv. Appl. Prob.* **50**, no. 1, 231-244 (2018)

H. Ishii. Perron’s method for Hamilton-Jacobi equations. *Duke Math. J.* **55**, no. 2, 369-384 (1987)

H. Ishii. Homogenization of the Cauchy problem for Hamilton-Jacobi equations. In *Stochastic analysis, control, optimization and applications, Systems Control Found. Appl.* 305? 324. Birkhäuser Boston, Boston, MA (1999).

V. V. Jikov, S. M. Kozlov and O. A. Oleinik. Homogenization of differential operators and integral functionals. Springer Verlag (1994)

O. Kallenberg Foundations of Modern Probability. Springer Science & Business Media. (2002)

C. Kipnis and S.R.S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104**, no. 1, 1-19 (1986)

E. Kosygina. Homogenization of stochastic Hamilton-Jacobi equations: brief review of methods and applications. *Stochastic Analysis and Partial Differential Equations, Series of Contemporary Mathematics* 429 (2007), American Mathematical Society, 189-204

E. Kosygina, F. Rezakhanlou and S. R. S. Varadhan. Stochastic homogenization of Hamilton-Jacobi-Bellman equations, *Comm. Pure Appl. Math.*, **59**, no. 10, 1489-1521 (2006)

E. Kosygina, and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium. *Comm. Pure Appl. Math.* **61**, no. 6, 816-847 (2008)

S. M. Kozlov. The method of averaging and walks in inhomogeneous environments. *Uspekhi Mat Nauk, (Russian Math. Surveys),* **40**, no. 12, 73-145, (1985)

N. Kubota. Large deviations for simple random walks on supercritical percolation clusters. *Kodai Math. J.* **35** (3), 560?575 (2012)

A. Lamacz, S. Neukamm and F. Otto. Moment bounds for the corrector in stochastic homogenization of a percolation model. *Electron. J. Probab.* **20** (2015), no.106, 1-30

G. Last and M. Penrose. *Lectures on the Poisson process.* Cambridge University Press. (2018)

T. Liggett, R. Schonmann and A. Stacey. Domination by product measures. *Ann. Probab.* **25**, no. 1, 71-95 (1997)

P.L. Lions, G. Papanicolaou and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi equation. Unpublished preprint, circa (1987)

P. L. Lions and P. Souganidis. Homogenization for viscous Hamilton-Jacobi equations in stationary, ergodic media. *Comm. Partial Differential Equations*, **30**, no. 1-3, 335-376 (2005)
[LS10] P. L. Lions and P. Souganidis. Stochastic homogenization for Hamilton-Jacobi and viscous Hamilton-Jacobi equations with convex nonlinearities-revisited. Comm. Mathematical Sciences, 8, no. 2, 627-637 (2010)

[MP07] P. Matheiu and A. Piatnitski. Quenched invariance principle for random walks on percolation clusters, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 463, no. 2085, 2287-2307 (2007)

[MR96] R. Meester and R. Roy. Continuum percolation. Cambridge University Press. (1996)

[PV81] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients, Random fields, Vol. I, II (Esztergom, 1979) 27, 835-873 (1981)

[PV82] G. Papanicolau and S. R. S. Varadhan. Diffusion with random coefficients. Statistics and probability: essays in honor of C. R. Rao 253-262 (1982)

[P89] K. Petersen. Ergodic Theory. Corrected reprint of the 1983 original. Cambridge Studies in Advanced Mathematics, vol 2. Cambridge University Press, Cambridge, 1989.

[P96] A. Pisztora. Surface order large deviations for Ising, Potts and percolation models, Probab. Theory Relat. Fields 104, no. 4, 427-466, (1996)

[PRS15] E. B. Procaccia, R. Rosenthal and A. Sapozhnikov. Quenched invariance principle for simple random walk on clusters in correlated percolation models, Probab. Theory Relat. Fields 166, no. 3-4, 619-657 (2016)

[RT00] F. Rezakhanlou and J. E. Tarver. Homogenization for stochastic Hamilton-Jacobi equations. Arch. Ration. Mech. Anal. 151, 277-309 (2000)

[SW08] R. Schneider and W. Weil. Stochastic and integral geometry. Springer-Verlag, Berlin. (2008)

[SS04] V. Sidoravicius and A. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. Probab. Theory Relat. Fields 129, no. 2, 219-244, (2004)

[So99] P. E. Souganidis. Stochastic homogenization of Hamilton-Jacobi equations and some applications. Asymptot. Anal., 20 (1):1-11 (1999)

[S94] A. S. Sznitman. Shape theorem, Lyapounov exponents, and large deviations for Brownian motion in a Poissonian potential. Comm. Pure. Appl. Math. 47, no. 12, 1655-1688, (1994)

[Sz10] A. S. Sznitman. Vacant set of random interlacements and percolation. Ann. Math. (2), 171, no. 6, 2039-2087 (2010)

[T93] H. Tanemura. Behavior of a supercritical phase of a continuum percolation model on $\mathbb{R}^d$. J. Appl. Probab. 30, no.2, 382-396 (1993)

[T09] A. Teixeira. Interlacement percolation on transient weighted graphs. Elec. J. Probab. 14, no. 54, 1604-1628 (2009)

[T09a] A. Teixeira. On the uniqueness of the infinite cluster of the vacant set of random interlacements, Ann. Appl. Probab. 19, no. 1, 454-466 (2009)

[T13] N. Touzi. Optimal stochastic control, stochastic target problems, and backward SDE. Springer, New York. (2013)