Impact of Vertex Addition on Italian Domination Number

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Abstract

An Italian dominating function (IDF), of a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that for every $v \in V(G)$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an IDF on $G$ is the sum $f(V) = \sum_{v \in V(G)} f(v)$ and Italian domination number, $\gamma_I(G)$ is the minimum weight of an IDF. In this paper, we study the impact of corona operator and addition of twins on Italian domination number.

Keywords: Italian domination number, corona operator, twin vertex.

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1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G then we write $V(G)$ and $E(G)$ as $V$ and $E$ respectively. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. A subset $S \subseteq V$ of vertices in a graph is called a dominating set if every $v \in V$ is either an element of $S$ or is adjacent to an element of $S$ [8]. The domination number, $\gamma(G)$ is the minimum cardinality of a dominating set of $G$.

An Italian dominating function, in short IDF, of a graph $G$ is a function $f : V \rightarrow \{0, 1, 2\}$ which satisfies the condition that for every $v \in V$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$, i.e.; either $v$ is adjacent to a vertex $u$ with $f(u) = 2$ or to at least two vertices $x$ and $y$ with $f(x) = f(y) = 1$. The weight of an Italian dominating function is $f(V) = \sum_{u \in V} f(u)$. The Italian domination number, $\gamma_I(G)$ is the minimum weight of an Italian dominating function. An IDF with weight $\gamma_I(G)$ is called $\gamma_I$-function. Let $V_i^f$ or simply $V_i$, denote the set of vertices assigned $i$ by the function $f$. The Italian domination number was first introduced in [3] with the name Roman-\{2\}-domination. For any graph $G$, the Italian domination number is bounded by $\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G) \leq 2\gamma(G)$ which was given in [3, 12]. M. A. Henning and W. F. Klostermeyer studied the Italian domination number of trees [7]. The Italian domination number of generalized Petersen graph, $P(n, 3)$ is found in [4]. In [11], it is proved that $\gamma_I(G) + 1 \leq \gamma_I(M(G)) \leq \gamma_I(G) + 2$, where $M(G)$ is the Mycielskian graph of $G$. It is also proved that $\gamma_I(S(K_n, 2)) = 2n - 1$ and $n^{t-2}\alpha(G)\gamma_I(G) \leq \gamma_I(S(G, t)) \leq n^{t-2}(n - \gamma_I(G) - |V_2| - E_2)$ where $S(G, t)$ is the Sierpinski graph of $G$, $\alpha(G)$ is the independence number of $G$ and $E_2$ is the set of non-isolated vertices in $< V_2 >$. To know more about Italian domination number, the interested readers can refer [1, 5, 6, 7, 11, 13].

The corona of two graphs $G_1 = (V_1; E_1)$ and $G_2 = (V_2; E_2)$, denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of $G_1$ and $|V_1|$ copies of $G_2$, and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$. We denote the complete graph on $n$ vertices by $K_n$. If $G_2$ is $K_1$, then $G_1 \odot G_2$ is $G_1$ together with one pendant vertex each attached to all the vertices of $G_1$. A false twin of a vertex $u$ is a vertex $u'$ which is adjacent to all vertices in $N(u)$. A true twin of a vertex $u$ is a vertex $u'$ which is adjacent to all vertices in $N(u)$. Two vertices $u$ and $u'$ are said to be twins if either they are true twins or false twins. For any graph theoretic terminology and notations not mentioned here, the readers may refer to [2].
The following result is useful in this paper.

**Theorem 1.1.** For the class of $P_n$, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$.

## 2 Corona Operator on Italian Domination

In this section, we find the value of Italian domination number of corona operator of any two graphs $G$ and $H$, where $H \nRightarrow K_1$. The upper and lower bounds for $\gamma_I(G \odot K_1)$ is obtained and the corresponding realization problem is also settled. Also, the exact value of $\gamma_I(G \odot K_1)$, when $G$ belongs to some special classes of graphs is obtained.

**Lemma 2.1.** Let $G$ be a graph and $u$ be a pendent vertex of $G$. Then there exists a $\gamma_I$-function $f$ of $G$ in which $f(u) \neq 2$.

*Proof.* If possible assume that there exists a $\gamma_I$-function with $f(u) = 2$. Note that the weight of neighbor of $u$, say $v$ is zero, due to the minimality of $f$. Then we can reassign $f(u) = f(v) = 1$ or $f(u) = 0$, $f(v) = 2$, which is again a $\gamma_I$-function on $G$ with $f(u) \neq 2$. Hence the lemma.

In this paper, here onwards, we consider $\gamma_I$-functions for which the weight of a pendent vertex is not equal to 2.

**Theorem 2.2.** For every graph $G$ and $H \nRightarrow K_1$, $\gamma_I(G \odot H) = 2n$, where $n$ is the order of $G$.

*Proof.* Define an Italian domination function $f$ of $G \odot H$ as follows.

\[
f(v) = \begin{cases} 
2, & \text{for } v \in V(G), \\
0, & \text{otherwise}.
\end{cases}
\]

Therefore, $\gamma_I(G \odot H) \leq 2n$. There are $n$ mutually exclusive copies of $H$ each of which requires at least weight 2 in IDF. So $\gamma_I(G \odot H) \geq 2n$. Hence the theorem.

**Theorem 2.3.** For any graph $G$, $n + 1 \leq \gamma_I(G \odot K_1) \leq 2n$. 

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Proof. Let \( V(G) = \{v_1, v_2, ... v_n\} \) and let \( u_i \) be the leaf neighbor of \( v_i \) in \( G \odot K_1 \). Define an IDF of \( G \odot K_1 \) as follows.

\[
f(u) = \begin{cases} 
2, & \text{for } u = u_i, \\
0, & \text{otherwise.}
\end{cases}
\]

Therefore, \( \gamma_I(G) \leq 2n \). To prove the left inequality, let \( f \) be any IDF of \( G \odot K_1 \). By Lemma 2.4, each \( u_i \) must be either in \( V_1 \) or adjacent to a vertex in \( V_2 \). If \( u_i \in V_1 \), for all \( i = 1, 2, 3, ... n \), none of the vertices in \( G \) can be Italian dominated by \( u_i \) alone. Therefore, \( f(V) \geq n + 1 \). If \( u_i \notin V_1 \) for some \( i \), then \( u_i \) is adjacent to a vertex in \( V_2 \) which further increases the value of \( f(V) \). Hence the theorem.

\[\blacksquare\]

Theorem 2.4. Any positive integer \( a \) is realizable as the Italian domination number of \( G \odot K_1 \), for some \( G \) if and only if \( n + 1 \leq a \leq 2n \), where \( n \) is the number of vertices in \( G \).

Proof. Let \( G \) be a graph with \( |V(G)| = n \). If \( \gamma_I(G \odot K_1) = a \) then by theorem 2.3, \( n + 1 \leq a \leq 2n \). To prove the converse, let \( G \) be the graph \( K_{1,m} \cup (n - m - 1)K_1 \) where \( 0 \leq m \leq n - 1 \). Let \( v_1, v_2, ... v_{m+1} \) be the vertices of \( K_{1,m} \) in which \( v_1 \) is the universal vertex and \( v_{m+2}, v_{m+3}, ..., v_n \) be the isolated vertices in \( G \). Let \( v' \) be the leaf neighbor of \( v_i \) in \( G \odot K_1 \). Define an IDF \( f \) on \( V(G \odot K_1) \) as follows.

\[
f(u) = \begin{cases} 
2, & \text{for } u = v_i, \ i = 1, m + 2, m + 3, ... , n, \\
1, & \text{for } u = v', \ i = 2, 3, ... , m + 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Therefore, \( f \) is a \( \gamma_I \)-function with weight \( 2(n - m) + m = 2n - m \), \( 0 \leq m \leq n - 1 \). So \( \gamma_I(G \odot K_1) \) varies from \( n + 1 \) to \( 2n \). Hence the theorem.

\[\blacksquare\]

Theorem 2.5. \( \gamma_I(G \odot K_1) = n + 1 \) if and only if \( G \) has a universal vertex.

Proof. Let \( G \) be a graph with vertices \( v_1, v_2, v_3, ... v_n \) and let \( u_i \) be the leaf neighbor of \( v_i \). Let \( v_1 \) be the universal vertex in \( G \). Define an IDF of \( G \odot K_1 \) as follows.

\[
g(v) = \begin{cases} 
2, & v = v_1, \\
1, & v = u_i, \ i = 2, 3, ... , n, \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( g(V) = n + 1 \) which is the minimum possible and hence \( \gamma_I(G \odot K_1) = n + 1 \).
To prove the converse part, assume that $\gamma_f(G \oplus K_1) = n + 1$. Let $f$ be a $\gamma_f$-function of $G \oplus K_1$. If possible assume that $G$ does not have a universal vertex. Out of $n$ pendent vertices in $G \oplus K_1$, let $k$ vertices be in $V_1^f$ so that the remaining $n - k$ pendent vertices are adjacent to vertices in $V_2^f$. Then $f(V) = n + 1 \geq k + 2(n - k) = 2n - k$. Therefore, $k \geq n - 1$. If $k = n - 1$ then there exists a pendent vertex $u_1$ which is adjacent to a vertex in $V_2^f$. If $u_1$ is adjacent to a vertex in $V_2^f$ then, since $u_1$ is not a universal vertex we need more vertices with non zero weight to Italian dominate vertices in $G$, which is a contradiction to the fact that $\gamma_f(G \oplus K_1) = n + 1$. Therefore our assumption is wrong. Hence $G$ has a universal vertex.

**Theorem 2.6.** $\gamma_f(G \oplus K_1) = 2n$ if and only if $G = K_n^c$.

**Proof.** Let $v_1, v_2, ... v_n$ be the vertices of $G$ and let $u_i$ be the pendent vertex adjacent to $v_i$ in $G \oplus K_1$ for $i = 1, 2, ..., n$. If possible assume that there exists an edge $v_i v_j$ in $G$. Then $u_i v_i v_j u_j$ is a $P_4$ in $G \oplus K_1$ which can be Italian dominated by assigning 2 to $v_i$ and 1 to $u_j$. Now, assigning 2 to every $v_k$ for $k = 1, 2, ..., n$ and $k \neq i, j$ gives an IDF of $G \oplus K_1$ with weight $3 + 2(n - 2) = 2n - 1$, which contradicts the fact that $\gamma_f(G \oplus K_1) = 2n$. Hence $G$ does not have an edge. i.e., $G = nK_1 = K_n^c$. It is trivial that if $G = K_n^c$ then $\gamma_f(G \oplus K_1) = 2n$.

**Theorem 2.7.**

\[
\gamma_f(K_{p,q} \oplus K_1) = \begin{cases} 
  p + q + 1, & p = 1 \text{ or } q = 1, \\
  p + q + 2, & \text{otherwise.} 
\end{cases}
\]

**Proof.** Let $V(K_{p,q}) = u_1, u_2, ... u_p, v_1, v_2, ..., v_q$ and $u_i'$ be the leaf neighbor of $u_i$, $i = 1, 2, ..., p$ and $v_j'$ be that of $v_j$ for $j = 1, 2, ... q$ in $K_{p,q} \oplus K_1$. By the left inequality of 2.3 $p + q + 1 \leq \gamma_f(K_{p,q} \oplus K_1)$.

**Case 1:** $p = 1$ or $q = 1$.

Without loss of generality, let $p = 1$. Define an IDF of $K_{p,q} \oplus K_1$ as follows.

\[
f(u) = \begin{cases} 
  2, & \text{for } u = u_1, \\
  1, & \text{for } u = v_j', j = 1, 2, 3, ..., q, \\
  0, & \text{otherwise.} 
\end{cases}
\]

The weight $f(V) = 2 + q = p + q + 1$. Therefore, $\gamma_f(K_{p,q} \oplus K_1) \leq p + q + 1$. Hence $\gamma_f(K_{p,q} \oplus K_1) = p + q + 1$. 

Define an IDF of $K_k$ vertices be in $K_{p,q} \odot K_1$ as follows.

$$f(u) = \begin{cases} 
2, & \text{for } u = u_1 \text{ and } u = v_1, \\
1, & \text{for } u = u'_i \text{ if } i = 2, 3, ..., p, \text{ and } u = v'_j \text{ if } j = 2, 3, ..., q, \\
0, & \text{otherwise.}
\end{cases}$$

The weight $f(V) = 4 + p - 1 + q - 1 = p + q + 2$. Therefore, $\gamma_I(K_{p,q} \odot K_1) \leq p + q + 2$.

**Case 2:** $p, q \geq 2$.

To prove the reverse inequality, if possible assume that there exists an IDF $g$ of $K_{p,q} \odot K_1$ with weight $p + q + 1$. Out of $p + q$ pendent vertices in $K_{p,q} \odot K_1$, let $k$ vertices be in $V'_1$. Note that, by Lemma 2.1 we can always find a $\gamma_I$-function in which pendent vertices are assigned values either 0 or 1. So that the remaining $p + q - k$ pendent vertices are adjacent to vertices in $V_2^g$. Hence the weight of $g$, $g(V) = p + q + 1 \geq k + 2(p + q - k)$. Hence, $k \geq p + q - 1$. If $k > p + q - 1$ then $k = p + q$ so that all the pendent vertices are in $V'_1^g$ and none of them can be Italian dominated by any of the non-pendent vertices. Therefore, we need more vertices having non-zero values under $g$, which contradicts $g = p + q + 1$. If $k = p + q - 1$, then one pendent vertex, say $x$, is adjacent to a vertex in $V^g_2$, say $y$. Then $y$ can not Italian dominate any of the vertices in its partite set of $K_{p,q}$ containing $y$. Therefore, we need more vertices having non-zero values under $g$ which is a contradiction. Hence the theorem.

**Theorem 2.8.** For any graph $G$, $\gamma_I((G \odot K_1) \odot K_1) = 3n$ where $n = |V(G)|$.

**Proof.** Let $G$ be a graph with vertex set $V(G) = v_1, v_2, ..., v_n$ and let $u_i$ be the leaf neighbor of $v_i$ in $G \odot K_1$. Let $v'_i$ and $u'_i$ be the leaf neighbors of $v_i$ and $u_i$, respectively, in $(G \odot K_1) \odot K_1$. There are $n$ vertex disjoint $P_4$’s, $v'_i u'_i u_i v_i$ for $i = 1, 2, ..., n$ in $(G \odot K_1) \odot K_1$. Let $f$ be an IDF on $(G \odot K_1) \odot K_1$. Then the 2 pendent vertices $v'_i$ and $u'_i$ in each $P_4$ should be either in $V'_1^f$ or adjacent to a vertex in $V'_2^f$. If all the pendent vertices are in $V'_1^f$, then to Italian dominate non-pendent vertices $v_i$ and $u_i$ we need more vertices with non-zero weight in $P_4$. The pendent vertices have no common neighbors. Hence, under $f$, the sum of the values of vertices in each $P_4$ must be at least 3. Therfore, $f(V) \geq 3n$. To prove the reverse inequality, define $g$ as follows.

$$g(u) = \begin{cases} 
1, & \text{for } u = v'_i, u'_i, u_i, \text{ if } i = 1, 2, ..., n, \\
0, & \text{otherwise.}
\end{cases}$$

Then $g$ is an IDF on $(G \odot K_1) \odot K_1$ with $g(V) = 3n$. Hence the theorem.
Theorem 2.9. \( \gamma_I(P_n \odot K_1) = \lceil \frac{4n}{3} \rceil \).

Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( P_n \) and let \( u_i \) be the pendent vertex corresponding to \( v_i \), for \( i = 1, 2, \ldots, n \). If possible assume that there exists a \( \gamma_I \)-function \( g \) of \( P_n \odot K_1 \) such that \( g(V) < \frac{4n}{3} \). Note that we can always find a \( \gamma_I \)-function in which pendent vertices are assigned values either 0 or 1 by Lemma 2.1. Out of \( n \) pendent vertices in \( P_n \odot K_1 \) let \( p \) vertices be in \( V_1^0 \), so that the remaining \( n - p \) pendent vertices are assigned value 0 and hence adjacent to vertices in \( V_1^0 \). i.e., \( n - p \) vertices in \( P_n \) are assigned the value 2. These \( n - p \) vertices can Italian dominate atmost \( 3(n - p) \) vertices of \( P_n \), i.e., at least \( n - (3(n - p)) = 3p - 2n \) vertices are not yet Italian dominated. To Italian dominate these \( 3p - 2n \) vertices we need atleast \( \frac{3p - 2n}{3} \) more vertices of weight 1 in \( P_n \). Therefore, \( g(V) > p + 2(n - p) + \frac{3p - 2n}{3} \). i.e., \( g(V) > \frac{4n}{3} \), which is a contradiction. So \( g(V) \geq \frac{4n}{3} \). Define an IDF of \( P_n \odot K_1 \) as follows.

When \( n = 3k \).

\[
 f(u) = \begin{cases} 
 2, & \text{if } u = v_{3j-1}, \text{ for } j = 1, 2, \ldots, k, \\
 1, & \text{if } u = u_j, \text{ for every } j \text{ such that } f(v_j) \neq 2, \\
 0, & \text{otherwise}.
\end{cases}
\]

Then \( f \) is an IDF with \( f(V) = \frac{4n}{3} \).

So that, \( \gamma_I(P_n \odot K_1) \leq \frac{4n}{3} \). Therefore, \( \gamma_I(P_n \odot K_1) = \frac{4n}{3} \).

When \( n = 3k + 1 \).

\[
 f(u) = \begin{cases} 
 2, & \text{if } u = v_n, \text{ or } v_{3j-1} \text{ for } j = 1, 2, \ldots, k, \\
 1, & \text{if } u = u_j, \text{ for every } j \text{ such that } f(v_j) \neq 2, \\
 0, & \text{otherwise}.
\end{cases}
\]

Then \( f \) is an IDF with \( f(V) = \frac{4n+2}{3} \). So that, \( \gamma_I(P_n \odot K_1) \leq \frac{4n+2}{3} \). Therefore, \( \gamma_I(P_n \odot K_1) = \frac{4n+2}{3} \).

When \( n = 3k + 2 \).

\[
 f(u) = \begin{cases} 
 2, & \text{if } u = v_{3k+2}, \text{ for } k = 0, 1, 2, \ldots \\
 1, & \text{if } u_j \text{ with } f(v_j) \neq 2, \\
 0, & \text{otherwise}.
\end{cases}
\]

Then \( f \) is an IDF with \( f(V) = \frac{4n+1}{3} \). So that, \( \gamma_I(P_n \odot K_1) \leq \frac{4n+1}{3} \). Therefore, \( \gamma_I(P_n \odot K_1) = \frac{4n+1}{3} \).

Hence, \( \gamma_I(P_n \odot K_1) \leq \lceil \frac{4n}{3} \rceil \).

Theorem 2.10. \( \gamma_I(C_n \odot K_1) = \lceil \frac{4n}{3} \rceil \).

Proof. The proof is similar to that of \( P_n \).
3 Addition of twin vertex

In this section, we discuss the impact of addition of twin vertices to a graph \( G \) on the Italian domination number of a graph.

**Lemma 3.1.** Let \( u \) and \( u' \) be true twins in a graph \( G \). Then there exists a \( \gamma_I \)-function of \( G \) in which \( f(u') = 0 \).

**Proof.** Let \( f \) be a \( \gamma_I \)-function of \( G \). If \( f(u') = 2 \), then \( f(u) = 0 \), due to the minimality of \( f \). Now, reassign \( f(u) = 2 \) and \( f(u') = 0 \), so that \( f \) is a \( \gamma_I \)-function of \( G \) with the required property.

If \( f(u') = 1 \) then \( f(u) \) can be either 1 or 0. Note that due to the minimality it can not be 2. If \( f(u) = 1 \) then we can reassign \( f(u) = 2 \) and \( f(u') = 0 \) so that \( f \) is still a \( \gamma_I \)-function of \( G \) with the required property. If \( f(u) = 0 \) then to Italian dominate \( u \) there esists a \( v \in N(u) \) such that \( f(v) = 1 \). Since \( N(u) = N(u') \), in this case also we can interchange the weights of \( u \) and \( u' \) to get a \( \gamma_I \)-function in which \( f(u') = 0 \). Hence the lemma.

**Theorem 3.2.** Let \( G \) be a graph and \( u \in V(G) \). Let \( H \) be the graph obtained from \( G \) by attaching a true twin \( u' \) to \( u \). Then \( \gamma_I(H) = \gamma_I(G) \) or \( \gamma_I(G) + 1 \).

**Proof.** Let \( f \) be a \( \gamma_I \)-function of \( G \). If \( u \in V_0^f \cup V_2^f \) then \( f \) can be extended to an IDF of \( H \) by assigning 0 to \( u' \) so that

\[
\gamma_I(H) \leq \gamma_I(G). \tag{1}
\]

If \( u \in V_1^f \) and there exists \( v \in N(u) \) such that weight of \( v \) not equal to 0 then \( f \) can be extended to an IDF of \( H \) by assigning 0 to \( u' \) so that

\[
\gamma_I(H) \leq \gamma_I(G). \tag{2}
\]

Now assume that there does not exist any \( \gamma_I \)-function of \( G \) for which \( u \in V_0^f \cup V_2^f \) or \( |N(u) \cap (V_1^f \cup V_2^f)| \geq 1 \), then we can extend \( f \) to an IDF of \( H \) by assigning 1 to \( u' \) so that

\[
\gamma_I(H) \leq \gamma_I(G) + 1. \tag{3}
\]

Let \( g \) be a \( \gamma_I \)-function of \( H \). Then by Lemma 3.1 there exists an IDF \( g \) in which \( g(u') = 0 \). Then the restriction of \( g \) to \( V(G) \) is an IDF of \( G \) so that

\[
\gamma_I(G) \leq \gamma_I(H). \tag{4}
\]

The weight of \( u \) in \( H \) can be \( g(u) = 0, 1 \) or 2. If \( g(u) = 2 \), all the vertices in the neighborhood of \( u \) other than \( u' \) are Italian dominated by some other vertices
in $H$. Then the restriction of $g$ to $G$ by assigning weight 1 to $u$ is an IDF of $G$. Therefore, $\gamma_I(G) \leq \gamma_I(H) - 1$. i.e.,

$$\gamma_I(G) + 1 \leq \gamma_I(H).$$

(5)

From equations (1), (2), (3), (4) and (5), we get, $\gamma_I(H) = \gamma_I(G)$ or $\gamma_I(G) + 1$. $\Box$

**Lemma 3.3.** Let $u$ and $u'$ be false twins in a graph $G$. Then there exists a $\gamma_I$-function $f$ of $G$ in which $f(u') \neq 2$.

**Proof.** Let $f$ be a $\gamma_I$-function with $f(u') = 2$. Then $f(u) = 0$, by the minimality of $f$. If $f(u) = 0$ then there exists a $v \in N(u)$ such that $f(v) = 2$ or two vertices $x, y \in N(u)$ such that $f(x) = f(y) = 1$. Since, $u$ and $u'$ have the same neighborhood, exchange weights of $u$ and $u'$. Then we get a $\gamma_I$-function with same weight and $f(u') = 0$. $\Box$

**Theorem 3.4.** Let $G$ be a graph and $u \in V(G)$. Let $H$ be a graph obtained from $G$ by attaching a false twin $u'$ to $u$. Then $\gamma_I(H) = \gamma_I(G)$ or $\gamma_I(G) + 1$.

**Proof.** Let $f$ be a $\gamma_I$-function of $G$. If $u \in V_0^I$ or $|N(u) \cap V_1^I| \geq 1$ or $|N(u) \cap V_2^I| \geq 2$ then $f$ can be extended to an IDF of $H$ by assigning 0 to $u'$ so that

$$\gamma_I(H) \leq \gamma_I(G).$$

(6)

Now, assume that there does not exist any $\gamma_I$-function of $G$ for which any of the above conditions are satisfied. Then we can extend $f$ to an IDF of $H$ by assigning 1 to $u'$ so that

$$\gamma_I(H) \leq \gamma_I(G) + 1.$$

(7)

Let $g$ be a $\gamma_I$-function of $H$. Then by Lemma 3.3 there exists a $\gamma_I$-function with $g(u') \neq 2$. Therefore, $g(u') = 1$ or 0. If $g(u') = 0$ then the restriction of $g$ to $G$ is an IDF of $G$. Therefore,

$$\gamma_I(G) \leq \gamma_I(H)$$

(8)

If $g(u') = 1$, but all the neighbors of $u'$ are Italian dominated by some other vertices (i.e., $u'$ is assigned value 1 to Italian dominate itself), then the restriction of $g$ to $G$ will be an IDF with $\gamma_I(G) \leq \gamma_I(H) - 1$. i.e.,

$$\gamma_I(G) + 1 \leq \gamma_I(H).$$

(9)

From equations (6), (7), (8) and (9), we get, $\gamma_I(H) = \gamma_I(G)$ or $\gamma_I(G) + 1$. $\Box$
4 Conclusion

In this paper, the impact of corona operator and addition of twins on the Italian domination number is studied. The following problems may also be worth investigating.

Problem 1: The effect of other graph operations on Italian domination number.

Problem 2: The effect of corona operator on Italian domination number of some other graph classes.

Problem 3: The effect of corona operator on other domination parameters.

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