Time-reversal and the adjoint method with an application in telecommunication

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Abstract. We establish a direct link between the time-reversal technique and the so-called adjoint method for imaging. Using this relationship, we derive new solution strategies for an inverse problem which arises in telecommunication. These strategies are all based on iterative time-reversal experiments, which try to solve the inverse problem experimentally instead of computationally. We will focus in particular on examples from underwater acoustic communication and wireless communication in a Multiple-Input Multiple-Output (MIMO) setup.
1. Introduction

Time-reversal techniques have attracted great attention recently due to the large variety of interesting potential applications. The basic idea of time-reversal (often also referred to as ‘phase-conjugation’ in the frequency domain) can be roughly described as follows: A localized source emits a short pulse of acoustic, electromagnetic or elastic energy which propagates through a richly scattering environment. A receiver array records the arriving waves, typically as a long and complicated signal due to the complexity of the environment, and stores the time-series of measured signals in memory. After a short while, the receiver array sends the recorded signals time-reversed (i.e. first in - last out) back into the same medium. Due to the time-reversibility of the wave fields, the emitted energy backpropagates through the same environment, practically retracing the paths of the original signal, and refocuses, again as a short pulse, on the location where the source emitted the original pulse. This, certainly, is a slightly oversimplified description of the rather complex physics which is involved in the real time-reversal experiment. In practice, the quality of the refocused pulse depends on many parameters, as for example the randomness of the background medium, the size and location of the receiver array, temporal fluctuations of the environment, etc. Surprisingly, the quality of the refocused signal increases with increasing complexity of the background environment. This was observed experimentally by M. Fink and his group at the Laboratoire Ondes et Acoustique at Université Paris VII in Paris by a series of laboratory experiments (see for example [17, 20, 21]), and by W. A. Kuperman and co-workers at the Scripps Institution of Oceanography at University of California, San Diego, by a series of experiments performed between 1996 and 2000 in a shallow ocean environment (see for example [9, 18, 26]).

The list of possible applications of this time-reversal technique is long. In an iterated fashion, resembling the power method for finding maximal eigenvalues of a square matrix, the time-reversal method can be applied for focusing energy created by an ultrasound transducer array on strongly scattering objects in the region of interest. This can be used for example in lithotripsy for localizing and destroying gall-stones in an automatic way, or more generally in the application of medical imaging problems. Detection of cracks in the aeronautical industry, or of submarines in the ocean are other examples. See for example [39, 40], and for related work [7, 12]. The general idea of time-reversibility, and its use in imaging and detection, is certainly not that new. Looking into the literature for example of seismic imaging, the application of this basic idea can be found in a classical and very successful imaging strategy for detecting scattering interfaces in the earth, the so-called ‘migration’ technique [4, 8]. However, the systematic use and investigation of the time-reversal phenomenon and its experimental realizations started more recently, and has been carried out during the last 10–15 years or so by different research groups. See for example [17, 18, 20, 21, 26, 31, 46, 49] for experimental demonstrations, and [1, 2, 3, 5, 6, 9, 15, 16, 23, 28, 30, 38, 39, 44, 45, 47, 50] for theoretical and numerical approaches.

One very young and promising application of time-reversal is communication. In this paper we will mainly concentrate on that application, although the general results should carry over also to other applications as mentioned above.

The paper is organized as follows. In section 2 we give a very short introduction into time-reversal in the ocean, with a special emphasis on underwater sound
communication. Wireless communication in a MIMO setup, our second main application in this paper, is briefly presented in section 3. In section 4 we discuss symmetric hyperbolic systems in the form needed here, and examples of such systems are given in section 5. In section 6 the basic spaces and operators necessary for our mathematical treatment are introduced. The inverse problem in communication, which we are focusing on in this paper, is then defined in section 7. In section 8 we derive the basic iterative scheme for solving this inverse problem. Section 9 gives practical expressions for calculating the adjoint communication operator, which plays a key role in the iterative time-reversal schemes presented in this paper. In section 10 the acoustic time-reversal mirror is defined, which will provide the link between the 'acoustic time-reversal experiment' and the adjoint communication operator. The analogous results for the electromagnetic time-reversal mirror are discussed in section 11. Section 12 combines the results of these two sections, and explicitly provides the link between time-reversal and the adjoint imaging method. Sections 13, 14 and 15 propose then several different iterative time-reversal schemes for solving the inverse problem of communication, using this key relationship between time-reversal and the adjoint communication operator. The practically important issue of partial measurements (and generalized measurements) is treated in section 16. Finally, section 17 summarizes the results of this paper, and points out some interesting future research directions.

2. Time-reversal and communication in the ocean

The ocean is a complex wave-guide for sound. In addition to scattering effects at the top and the bottom of the ocean, also its temperature profile and the corresponding refractive effects contribute to this wave-guiding property and allow acoustic energy to travel large distances. Typically, this propagation has a very complicated behaviour. For example, multipathing occurs if source and receiver of sound waves are far away from each other, since due to scattering and refraction there are many possible paths on which acoustic energy can travel between them. Surface waves and air bubbles at the top of the ocean, sound propagation through the rocks and sedimentary layers at the bottom of the ocean, and other effects further contribute to the complexity of sound propagation in the ocean. When a source (e.g. the base station of a communication system in the ocean) emits a short signal at a given location, the receiver (e.g. a user of this communication system) some distance away from the source typically receives a long and complicated signal due to the various influences of the ocean to this signal along the different connecting paths. If the base station wants to communicate with the user by sending a series of short signals, this complex response of the ocean to the signal needs to be resolved and taken into account.

In a classical communication system, the base station which wants to communicate with a user broadcasts a series of short signals (e.g. a series of 'zeros' and 'ones') into the environment. The hope is that the user will receive this message as a similar series of temporally well-resolved short signals which can be easily identified and decoded. However, this almost never occurs in a complex environment, due to the multipath and the resulting delay-spread of the emitted signals. Typically, when the base station broadcasts a series of short signals into such an environment, intersymbol interference occurs at the user position due to the temporal overlap of the multipath contributions of these signals. In order to recover the individual signals, a significant amount of signal processing is necessary at the user side, and, most
importantly, the user needs to have some knowledge of the propagation behaviour of the signals in this environment (i.e. he needs to know the 'channel'). Intersymbol interference can in principle be avoided by adding a sufficient delay between individual signals emitted at the base station which takes into account the delay-spread in the medium. That, however, slows down the communication, and reduces the capacity of the environment as a communication system. An additional drawback of simply broadcasting communication signals from the base station is the obvious lack of interception security. A different user who also knows the propagation behaviour of signals in the environment, can equally well resolve the series of signals arriving at his location and decode them.

Several approaches have been suggested to circumvent the above mentioned drawbacks of communication in multiple-scattering environments. Some very promising techniques are based on the time-reversibility property of propagating wave-fields. The basic idea is as follows. The user who wants to communicate with the base station, starts the communication process by sending a short pilot signal through the environment. The base station receives this signal as a long and complex signal due to multipathing. It time-reverses the received signal and sends it back into the environment. The backpropagating waves will produce a complicated wave-field everywhere due to the many interfering parts of the emitted signal. However, due to the time-reversibility of the wave-fields, we expect that the interference will be constructive at the position of the user who sent the original pilot signal, and mainly destructive at all other positions. Therefore, the user will receive at his location a short signal very similar to (ideally, a time-reversed replica of) the pilot signal which he sent for starting the communication. All other users who might be in the environment at the same time will only receive noise speckle due to incoherently interfering contributions of the backpropagating field. If the base station sends the individual elements ('ones' and 'zeros') of the intended message in a phase-encoded form as a long overlapping string of signals, the superposition principle will ensure that, at the user position, this string of signals will appear as a series of short well-separated signals, each resembling some phase-shifted (and time-reversed) form of the pilot signal.

In order to find out whether this theoretically predicted scenario actually takes place in a real multiple-scattering environment like the ocean, Kuperman et al have performed a series of four experiments in a shallow ocean environment between 1996 and 2000, essentially following the above described scenario. The experiments have been performed at a Mediterranean location close to the Italian coast. (A similar setup was also used in an experiment performed in 2001 off the coast of New England which has been reported in Yang.) A schematic view of these experiments is shown in figure. The single 'user' is replaced here by a 'probe source', and the 'source-receive array' (SRA) plays the role of the 'base station'. An additional vertical receive array (VRA) was deployed at the position of the probe source in order to measure the temporal and spatial spread of the backpropagating fields in the neighbourhood of the probe source location. In this shallow environment (depths of about 100–200 m, and distances between 10 and 30 km) the multipathing of the waves is mostly caused by multiple reflections at the surface and the bottom of the ocean. The results of the experiments have been reported in. They show that in fact a strong spatial and temporal focusing of the backpropagating waves occurs at the source position. A theoretical explanation of the temporal and spatial refocusing of time-reversed waves in random environments has been given in Blomgren et al.
3. The MIMO setup in wireless communication

The underwater acoustics scenario described above directly carries over to situations which might be more familiar to most of us, namely to the more and more popular wireless communication networks using mainly electromagnetic waves in the microwave regime. Starting from the everyday use of cell-phones, ranging to small wireless-operating local area networks (LAN) for computer systems or for private enterprise communication systems, exactly the same problems arise as in underwater communication. The typically employed microwaves of a wavelength at about 10–30 cm are heavily scattered by environmental objects like cars, fences, trees, doors, furniture etc. This causes a very complicated multipath structure of the signals received by users of such a communication system. Since bandwidths are limited and increasingly expensive, a need for more and more efficient communication systems is imminent. Recently, the idea of a so-called multiple-input multiple-output (MIMO) communication system has been introduced with the potential to increase the capacity and efficiency of wireless communication systems [22]. The idea is to replace a single antenna at the base station which is responsible for multiple users, or even a system where one user communicates only with his own dedicated base antenna, by a more general system where an array of multiple antennas at the base station is interacting simultaneously and in a complex way with multiple users. A schematic description of such a MIMO system (with seven base antennas and seven users) is given in figure 2. See for example [24, 42] for recent overviews on MIMO technology.

Time-reversal techniques are likely to play also here a key role in improving communication procedures and for optimizing the use of the limited resources (especially bandwidth) which are available for this technology [19, 29, 32, 37]. One big advantage of time-reversal techniques is that they are automatically adapted to the complex environment and that they can be very fast since they do not require heavy signal processing at the receiver or user side. In [34, 35], an iterated time-reversal scheme for the optimal refocusing of signals in such a MIMO communication system was proposed, which we will describe in more details in section 13.
In the present paper we will establish a direct link between the time-reversal technique and solution strategies for inverse problems. As an application of this relationship, we will derive iterative time-reversal schemes for the optimization of wireless or underwater acoustic MIMO communication systems. The derivation is performed completely in time-domain, for very general first order symmetric hyperbolic systems describing wave propagation phenomena in a complex environment. One of the schemes which we derive, in a certain sense the ‘basic one’, will turn out to be practically equivalent to the scheme introduced in [34, 35], although the derivation uses different tools. Therefore, it provides a new interpretation of that scheme. The other schemes which we introduce are new in this application, and can be considered as either generalizations of the basic scheme, or as independent alternatives to that scheme. Each of them addresses slightly different objectives and has its own very specific characteristics.

4. Symmetric hyperbolic systems

We treat wave propagation in communication systems in the general framework of symmetric hyperbolic systems of the form

\[ \Gamma(x) \frac{\partial u}{\partial t} + \sum_{i=1}^{3} D^i \frac{\partial u}{\partial x_i} + \Phi(x)u = q \]  

(1)

\[ u(x, 0) = 0. \]  

(2)

Here, \( u(x, t) \) and \( q(x, t) \) are real-valued time-dependent \( N \)-vectors, \( x \in \mathbb{R}^3 \), and \( t \in [0, T] \). \( \Gamma(x) \) is a real, symmetric, uniformly positive definite \( N \times N \)-matrix, i.e., \( \Gamma(x) \geq \epsilon \) for some \( \epsilon > 0 \). Moreover, \( \Phi(x) \) is a symmetric positive semi-definite \( N \times N \) matrix, i.e., \( \Phi(x) \geq 0 \). It models possible energy loss through dissipation in the medium. The \( D^i \) are real, symmetric and independent of \( (x, t) \). We will also use the
short notation
\[ \Lambda := \sum_{i=1}^{3} D_i \frac{\partial}{\partial x^i}. \] (3)

In addition to the above mentioned assumptions on the coefficients \( \Gamma(x) \) and \( \Phi(x) \), we will assume throughout this paper that all quantities \( \Gamma(x), \Phi(x), u(x,t) \) and \( q(x,t) \) are 'sufficiently regular' in order to safely apply for example integration by parts and Green’s formulas. For details see for example [11, 33]. Throughout this paper we will assume that no energy reaches the boundaries \( \partial \Omega \) during the time \([0,T]\), such that we will always have
\[ u(x,t) = 0 \text{ on } \partial \Omega \times [0,T]. \] (4)

The energy density \( \mathcal{E}(x,t) \) is defined by
\[ \mathcal{E}(x,t) = \frac{1}{2} \langle \Gamma(x)u(x,t), u(x,t) \rangle_N \]
\[ = \frac{1}{2} \sum_{m,n=1}^{N} \Gamma_{mn}(x)u_m(x,t)u_n(x,t). \]

The total energy \( \hat{\mathcal{E}}(t) \) in \( \Omega \) at a given time \( t \) is therefore
\[ \hat{\mathcal{E}}(t) = \frac{1}{2} \int_{\Omega} \langle \Gamma(x)u(x,t), u(x,t) \rangle_N \, dx. \]

The flux \( \mathcal{F}(x,t) \) is given by
\[ \mathcal{F}_i(x,t) = \frac{1}{2} \langle D_i u(x,t), u(x,t) \rangle_N, \quad i = 1, 2, 3. \]

5. Examples for symmetric hyperbolic systems

In the following, we want to give some examples for symmetric hyperbolic systems as defined above. Of special interest for communication are the system of the linearized acoustic equations and the system of Maxwell’s equations. We will discuss these two examples in detail in this paper. Another important example for symmetric hyperbolic systems is the system of elastic waves equations, which we will however leave out in our discussion for the sake of shortness. We only mention here that all the results derived here apply without restrictions also to linear elastic waves. Elastic wave propagation becomes important for example in ocean acoustic communication models which incorporate wave propagation through the sedimentary and rock layers at the bottom of the ocean.

5.1. Linearized acoustic equations

As a model for underwater sound propagation, we consider the following linearized form of the acoustic equations in an isotropic medium
\[ \rho(x) \frac{\partial v}{\partial t} + \text{grad} \, p = q_v, \] (5)
\[ \kappa(x) \frac{\partial p}{\partial t} + \text{div} \, v = q_p, \] (6)
\[ p(x,0) = 0, \quad v(x,0) = 0. \] (7)
Here, $v$ is the velocity, $p$ the pressure, $\rho$ the density, and $\kappa$ the compressibility. We have $N = 4$, $u = (v, p)^T$ and $q = (q_v, q_p)^T$ (where '$T$' as a superscript always means 'transpose'). Moreover, we have
\[
\Gamma(x) = \text{diag}(\rho(x), \rho(x), \rho(x), \kappa(x)) \quad \text{and} \quad \Phi(x) = 0.
\]
With the notation $\varphi = (\partial_1, \partial_2, \partial_3)^T$, we can write $\Lambda$ as
\[
\Lambda = \begin{pmatrix} 0 & \varphi^T \\ \varphi & 0 \end{pmatrix}.
\]
The operators $D^i$, $i = 1, 2, 3$, can be recovered here from $\Lambda$ by putting
\[
D^i_{m,n} = \begin{cases} 1 & \text{where } \Lambda_{m,n} = \partial_i, \\ 0 & \text{elsewhere} \end{cases}
\]
The energy density $E(x, t)$ is given by
\[
E(x, t) = \frac{1}{2} (\rho(x)|v(x, t)|^2 + \kappa(x)p^2(x, t)),
\]
and the energy flux $F(x, t)$ is
\[
F(x, t) = p(x, t)v(x, t).
\]
We mention that the dissipative case $\Phi(x) \neq 0$ can be treated as well in our framework, and yields analogous results to those presented here.

5.2. Maxwell’s equations

As a second example, we will consider Maxwell’s equations for an anisotropic medium with some energy loss due to the inherent conductivity. This can for example model wireless communication in a complex environment.

\[
\begin{align*}
\epsilon(x) \frac{\partial E}{\partial t} - \nabla \times H + \sigma(x)E &= q_E \\
\mu(x) \frac{\partial H}{\partial t} + \nabla \times E &= q_H
\end{align*}
\]
We have $N = 6$, and $u = (E, H)^T$, $q = (q_E, q_H)^T$. Moreover,
\[
\begin{align*}
\Gamma(x) &= \text{diag}(\epsilon(x), \mu(x)) \\
\Phi(x) &= \text{diag}(\sigma(x), 0).
\end{align*}
\]
Here, $\epsilon$ and $\mu$ are symmetric positive definite $3 \times 3$ matrices modelling the anisotropic permittivity and permeability distribution in the medium, and $\sigma$ is a symmetric positive semi-definite $3 \times 3$ matrix which models the anisotropic conductivity distribution. In wireless communication, this form can model for example dissipation by conductive trees, conductive wires, rainfall, pipes, etc. The operator $\Lambda$ can be written in block form as
\[
\Lambda = \begin{pmatrix} 0 & -\Xi \\ \Xi & 0 \end{pmatrix},
\]
with
\[
\Xi = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ -\partial_3 & 0 & -\partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}.
\]
The operators $D^i, i = 1, 2, 3,$ can be recovered here from $\Lambda$ by putting

$$D^i_{m,n} = \begin{cases} 
1 & \text{where } \Lambda_{m,n} = \partial_i, \\
-1 & \text{where } \Lambda_{m,n} = -\partial_i, \\
0 & \text{elsewhere} 
\end{cases}$$

The energy density $E(x,t)$ is given by

$$E(x,t) = \frac{1}{2} \left( \epsilon(x)|E(x,t)|^2 + \mu(x)|H(x,t)|^2 \right).$$

The energy flux $F(x,t)$ is described by the Poynting vector

$$F(x,t) = E(x,t) \times H(x,t).$$

5.3. Elastic waves equations

As already mentioned, also elastic waves can be treated in the general framework of symmetric hyperbolic systems. For more details we refer to [33, 41].

6. The basic spaces and operators

For our mathematical treatment of time-reversal we introduce the following spaces and inner products. We assume that we have $J$ users $U_j, j = 1, \ldots, J$ in our system, and in addition a base station which consists of $K$ antennas $A_k, k = 1, \ldots, K$. Each user and each antenna at the base station can receive, process and emit signals which we denote by $s_j(t)$ for a given user $U_j, j = 1, \ldots, J$, and by $r_k(t)$ for a given base antenna $A_k, k = 1, \ldots, K$. Each of these signals consists of a time-dependent $N$-vector indicating measured or processed signals of time-length $T$. In our analysis we will often have to consider functions defined on the time interval $[T, 2T]$ instead of $[0, T]$. For simplicity, we will use the same notation for the function spaces defined on $[T, 2T]$ as we use for those defined on $[0, T]$. It will always be obvious which space we refer to in a given situation.

Lumping together all signals at the users on the one hand, and all signals at the base station on the other hand, yields the two fundamental quantities

$$s = (s_1, \ldots, s_J) \in \hat{Z},$$

$$r = (r_1, \ldots, r_K) \in Z,$$

with

$$\hat{Z} = (L_2([0, T])^N)^J, \quad Z = (L_2([0,T])^N)^K.$$ 

The two signal spaces $Z$ and $\hat{Z}$ introduced above are equipped with the inner products

$$\langle s^{(1)}, s^{(2)} \rangle_Z = \sum_{j=1}^{J} \int_{[0,T]} \langle s^{(1)}_j(t), s^{(2)}_j(t) \rangle_N dt,$$

$$\langle r^{(1)}, r^{(2)} \rangle_Z = \sum_{k=1}^{K} \int_{[0,T]} \langle r^{(1)}_k(t), r^{(2)}_k(t) \rangle_N dt.$$

The corresponding norms are

$$\|s\|^2_Z = \langle s, s \rangle_Z, \quad \|r\|^2_Z = \langle r, r \rangle_Z.$$

Each user $U_j$ and each antenna $A_k$ at the base station can send a given signal $s_j(t)$ or $r_k(t)$, respectively. This gives rise to a source distribution $\hat{q}_j(x,t), j = 1, \ldots, J,$
or \( q_k(x,t), k = 1, \ldots, K \), respectively. Here and in the following we will use in our notation the following convention. If one symbol appears in both forms, with and without a 'hat' (³) on top of this symbol, then all quantities with the 'hat' symbol are related to the users, and those without the 'hat' symbol to the antennas at the base station.

Each of the sources created by a user or by a base antenna will appear on the right hand side of (1) as a mathematical source function and gives rise to a corresponding wave field which satisfies (1), (2). When solving the system (1), (2), typically certain Sobolev spaces need to be employed for the appropriate description of the underlying function spaces (see for example [11]). For our purposes, however, it will be sufficient to assume that both, source functions and wave fields, are members of the following canonical function space

\[ U = \{ u \in L^2(\Omega \times [0,T])^N, u = 0 \text{ on } \partial \Omega \times [0,T], \|u\|_U < \infty \}, \]

and which we have equipped with the usual energy inner product

\[ \langle u(t), v(t) \rangle_U = \int_{[0,T]} \int_{\Omega} \langle \Gamma(x)u(x,t), v(x,t) \rangle_N \, dx \, dt, \]

and the corresponding energy norm

\[ \|u\|_U^2 = \langle u, u \rangle_U. \]

Also here, in order to simplify the notation, we will use the same space when considering functions in the shifted time interval \([T, 2T]\) instead of \([0, T]\).

Typically, when a user or an antenna at the base station transforms a signal into a source distribution, it is done according to a very specific antenna characteristic which takes into account the spatial extension of the user or the antenna. We will model this characteristic at the user by the functions \( \hat{\gamma}_j(x), j = 1, \ldots, J \), and for base antennas by the functions \( \gamma_k(x), k = 1, \ldots, K \). With these functions, we can introduce the linear 'source operators' \( \hat{Q} \) and \( Q \) mapping signals \( s \) at the set of users and \( r \) at the set of base antennas into the corresponding source distributions \( \hat{q}(x,t) \) and \( q(x,t) \), respectively. They are given as

\[ \hat{Q} : \hat{Z} \rightarrow U, \quad \hat{q}(x,t) = \hat{Q}s = \sum_{j=1}^{J} \hat{\gamma}_j(x)s_j(t), \]

\[ Q : Z \rightarrow U, \quad q(x,t) = Qr = \sum_{k=1}^{K} \gamma_k(x)r_k(t). \]

We will assume that the functions \( \hat{\gamma}_j(x) \) are supported on a small neighbourhood \( \hat{V}_j \) of the user location \( \hat{d}_j \), and that the functions \( \gamma_k(x) \) are supported on a small neighbourhood \( V_k \) of the antenna location \( d_k \). Moreover, all these neighbourhoods are strictly disjoint to each other. For example, the functions \( \hat{\gamma}_j(x) \) could be assumed to be \( L_2 \)-approximations of the Dirac delta measure \( \delta(x - \hat{d}_j) \) concentrated at the user locations \( \hat{d}_j \), and the functions \( \gamma_k(x) \) could be assumed to be \( L_2 \)-approximations of the Dirac delta measure \( \delta(x - d_k) \) concentrated at the antenna locations \( d_k \).

Both, users and base antennas can also record incoming fields \( u \in U \) and transform the recorded information into signals. Also here, this is usually done according to very specific antenna characteristics of each user and each base antenna. For simplicity (and without loss of generality), we will assume that the antenna characteristic of a user or base antenna for receiving signals is the same as for
transmitting signals, namely $\hat{\gamma}_j(x)$ for the user and $\gamma_k(x)$ for a base antenna. (The case of more general source and measurement operators is discussed in section 16.) With this, we can define the linear 'measurement operators' $\hat{M} : U \to \hat{Z}$ and $M : U \to Z$, respectively, which transform incoming fields into measured signals, by

$$s_j(t) = (\hat{M}u)_j = \int_{\Omega} \hat{\gamma}_j(x)u(x,t)dx, \quad (j = 1, \ldots, J)$$

$$r_k(t) = (Mu)_k = \int_{\Omega} \gamma_k(x)u(x,t)dx, \quad (k = 1, \ldots, K)$$

Finally, we define the linear operator $F$ mapping sources $q$ to states $u$ by

$$F : U \to U, \quad Fq = u,$$

where $u$ solves the problem (1), (2). As already mentioned, we assume that the domain $\Omega$ is chosen sufficiently large and that the boundary $\partial \Omega$ is sufficiently far away from the users and base antennas, such that there is no energy reaching the boundary in the time interval $[0, T]$ (or $[T, 2T]$) due to the finite speed of signal propagation. Therefore, the operator $F$ is well-defined.

Formally, we can now introduce the two linear communication operators $A$ and $B$ which are at the main focus of this paper. They are defined as

$$A : Z \to \hat{Z}, \quad Ar = \hat{MFQ}r,$$

$$B : \hat{Z} \to Z, \quad Bs = MFQs.$$ 

The operator $A$ models the following situation. The base station emits the signal $r(t)$ which propagates through the complex environment. The users measure the arriving wave fields and transform them into measurement signals. The measured signals at the set of all users is $s(t) = Ar(t)$. The operator $B$ describes exactly the reversed situation. All users emit together the set of signals $s(t)$, which propagate through the given complex environment and are received by the base station. The corresponding set of measured signals at all antennas of the base station is just $r(t) = Bs(t)$. No time-reversal is involved so far.

7. An inverse problem arising in communication

In the following, we outline a typical problem arising in communication, which gives rise to a mathematically well-defined inverse problem.

A specified user of the system, say $U_1$, defines a (typically but not necessarily short) pilot signal $\alpha(t)$ which he wants to use as a template for receiving the information from the base station. The base station wants to emit a signal $\tilde{r}(t)$ which, after having travelled through the complex environment and arriving at the user $U_1$, matches this pilot signal as closely as possible. Neither the base station nor any other user except of $U_1$ are required (or expected) to know the correct form of the pilot signal $\alpha(t)$ for this problem. As an additional constraint, the base station wants that at the other users $U_j, j > 1$, as little energy as possible arrives when communicating with the specified user $U_1$. This is also in the interest of the other users, who want to use a different 'channel' for communicating at the same time with the base antenna, and want to minimize interference with the communication initiated by user $U_1$. The complex environment itself in which the communication takes place (i.e. the 'channel') is assumed to be unknown to all users and to the base station.
In order to arrive at a mathematical description of this problem, we define the 'ideal signal' $\tilde{s}(t)$ received by all users as

$$\tilde{s}(t) = (\alpha(t), 0, \ldots, 0)^T. \quad (11)$$

Each user only knows his own component of this signal, and the base antenna does not need to know any component of this ideal signal at all.

**Definition 7.1 The inverse problem of communication:** In the terminology of inverse problems, the above described scenario defines an inverse source problem, which we call for the purpose of this paper the 'inverse problem of communication'. The goal is to find a 'source distribution' $\tilde{r}(t)$ at the base station which satisfies the 'data' $\tilde{s}(t)$ at the users:

$$A\tilde{r} = \tilde{s}. \quad (12)$$

The 'state equation' relating sources to data is given by the symmetric hyperbolic system $\mathbf{1}$, $\mathbf{2}$.

**Remark 7.1** Notice that the basic operator $A$ in (12) is unknown to us since we do not know the complicated medium in which the waves propagate. If the operator $A$ (together with $\tilde{s}$) would be known at the base station by some means, the inverse source problem formulated above could be solved using classical inverse problems techniques, which would be computationally expensive but in principle doable. In our situation, we are able to do physical experiments, which amounts to 'applying' the communication operator $A$ to a given signal. Determining the operator $A$ explicitly by applying it to a set of basis functions of $Z$ would be possible, but again too expensive. We will show in the following that, nevertheless, many of the classical solution schemes known from inverse problems theory can be applied in our situation even without knowing this operator. The basic tool which we will use is a series of time-reversal experiments, applied to carefully designed signals at the users and the base station.

**Remark 7.2** A practical template for an iterative scheme for finding an optimal signal at the base station can be envisioned as follows. User $U_1$ starts the communication process by emitting an initial signal $s_1^{(0)}(t)$ into the complex environment. This signal, after having propagated through the complex environment, finally arrives at the base station and is received there usually as a relatively long and complicated signal due to the multiple scattering events it experienced on its way. When the base station receives such a signal, it processes it and sends a new signal $r^{(1)}(t)$ back through the environment which is received by all users. After receiving this signal, all users become active. The user $U_1$ compares the received signal with the pilot signal. If the match is not good enough, it processes the received signal in order to optimize the match with the pilot signal when receiving the next iterate from the base station. All other users identify the received signal as unwanted noise, and process it with the goal to receive in the next iterate from the base station a signal with lower amplitude, such that it does not interfere with their own communications. All users send now their processed signals, which together define $s^{(1)}(t)$, back to the base station. The base station receives them all at the same time, again usually as a long and complicated signal, processes this signal and sends a new signal $r^{(2)}(t)$ back into the environment which ideally will match the desired signals at all users better than the previously emitted signal $r^{(1)}(t)$. This iteration stops when all users are satisfied, i.e., when user $U_1$ receives a signal which is sufficiently close to the pilot signal $\alpha(t)$, and the energy or amplitude of the signals arriving at the other users has decreased.
enough in order not to disturb their own communications. After this learning process of the channel has been completed, the user $U_1$ can now start communicating safely with the base antenna using the chosen pilot signal $\alpha(t)$ for decoding the received signals.

Similar schemes have been suggested in [18, 19, 29, 32, 37, 44] in a single-step fashion, and in [34, 35] performing multiple steps of the iteration. The main questions to be answered are certainly which signals each user and each base antenna needs to emit in each step, how these signals need to be processed, at which stage this iteration should be terminated, and which optimal solution this scheme is expected to converge to. One goal of this paper is to provide a theoretical framework for answering these questions by combining basic concepts of inverse problems theory with experimental time-reversal techniques.

8. The basic approach for solving the inverse problem of communication

A standard approach for solving problem (12) in the situation of noisy data is to look for the least-squares solution

$$ r_{LS} = \min_r \|Ar - \hat{s}\|_Z^2 $$

(13)

In order to practically find a solution of (13), we introduce the cost functional

$$ J(r) = \frac{1}{2} \langle Ar - \hat{s}, Ar - \hat{s} \rangle_Z $$

(14)

In the basic approach we propose to use the gradient method for finding the minimum of (14). In this method, in each iteration a correction $\delta r$ is sought for a guess $r$ which points into the negative gradient direction $-A^*(Ar - \hat{s})$ of the cost functional (14). In other words, starting with the initial guess $r^{(0)} = 0$, the iteration of the gradient method goes as follows:

$$ r^{(0)} = 0 $$

$$ r^{(n+1)} = r^{(n)} - \beta^{(n)} A^*(Ar^{(n)} - \hat{s}), $$

(15)

where $\beta^{(n)}$ is the step-size at iteration number $n$. Notice that the signals $r^{(n)}$ are measured at the base station, whereas the difference $Ar^{(n)} - \hat{s}$ is determined at the users. In particular, $\hat{s}$ is the pilot signal only known by the user who defined it, combined with zero signals at the remaining users. $Ar^{(n)}$ is the signal received by the users at the $n$-th iteration step.

9. The adjoint operator $A^*$

We see from (15) that we will have to apply the adjoint operator $A^*$ repeatedly when implementing the gradient method. In this section we provide practically useful expressions for applying this operator to a given element of $\hat{Z}$.

**Lemma 9.1** We have

$$ A^* = Q^* F^* \hat{M}^*. $$

(16)

**Proof:** This follows from $A = \hat{M} F Q$.

**Theorem 9.1** We have

$$ M^* = \Gamma^{-1} Q, \quad \hat{M}^* = \Gamma^{-1} \hat{Q}, $$

$$ Q^* = M \Gamma, \quad \hat{Q}^* = \hat{M} \Gamma. $$

(17)
Proof: The proof is given in appendix A.

Next, we want to find an expression for \( F^* v \), \( v \in U \), where \( F^* \) is the adjoint of the operator \( F \).

**Theorem 9.2** Let \( z \) be the solution of the adjoint symmetric hyperbolic system

\[
- \Gamma(x) \frac{\partial z}{\partial t} - \sum_{i=1}^{3} D_i \frac{\partial z}{\partial x_i} + \Phi(x) z = \Gamma(x) v(x,t),
\]

(18)

\[
z(x,T) = 0,
\]

(19)

\[
z(x,t) = 0 \quad \text{on} \quad \partial \Omega \times [0,T].
\]

(20)

Then

\[
F^* v = \Gamma^{-1}(x) z(x,t).
\]

(21)

**Proof:** The proof is given in appendix B.

\[\square\]

**Remark 9.1** This procedural characterization of the adjoint operator is often used in solution strategies of large scale inverse problems, where it naturally leads to so-called 'backpropagation strategies'. See for example [13, 14, 25, 27, 36, 48] and the references given there.

**Remark 9.2** Notice that in the adjoint system [13, 20] 'final value conditions' are given at \( t = T \) in contrast to [11]–[13] where 'initial value conditions' are prescribed at \( t = 0 \). This corresponds to the fact that time is running backward in [13]–[20] and forward in [11]–[13].

10. The acoustic time-reversal mirror

In the following we want to define an operator \( S_a \) such that \( F^* = \Gamma^{-1} S_a F S_a \Gamma \) holds. We will call this operator \( S_a \) the 'acoustic time-reversal operator'. We will also define the 'acoustic time-reversal mirrors' \( T_a \) and \( \hat{T}_a \), which act on the signals instead of the sources or fields.

We consider the acoustic system

\[
\rho \frac{\partial v_f}{\partial t} + \text{grad} p_f(x,t) = q_v,
\]

(22)

\[
\kappa \frac{\partial p_f}{\partial t} + \text{div} v_f(x,t) = q_p,
\]

(23)

\[
v_f(x,0) = 0, \quad p_f(x,0) = 0 \quad \text{in} \quad \Omega,
\]

(24)

with \( t \in [0,T] \) and zero boundary conditions at \( \partial \Omega \times [0,T] \). We want to calculate the action of the adjoint operator \( F^* \) on a vector \( (\phi,\psi)^T \in U \).

**Theorem 10.1** Let \( (\phi,\psi)^T \in U \) and let \( (v_a,p_a)^T \) be the solution of the adjoint system

\[
- \rho \frac{\partial v_a}{\partial t} - \text{grad} p_a(x,t) = \rho(x) \phi(x,t),
\]

(25)

\[
-\kappa \frac{\partial p_a}{\partial t} - \text{div} v_a(x,t) = \kappa(x) \psi(x,t)
\]

(26)

\[
v_a(x,T) = 0, \quad p_a(x,T) = 0 \quad \text{in} \quad \Omega,
\]

(27)

with \( t \in [0,T] \) and zero boundary conditions at \( \partial \Omega \times [0,T] \). Then we have

\[
F^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \rho^{-1} v_a(x,t) \\ \kappa^{-1} p_a(x,t) \end{pmatrix}.
\]

(28)
which just corresponds to the application of the operator $S \in \tau$

In order to show that the so defined time-reversal experiment is correctly modelled by call this experiment the 'acoustic time-reversal experiment'. Notice that the time is running forward in this experiment.

For given $q = (q_v, q_p)^T \in U$ (and similarly for all $u = (v, p)^T \in U$)

\[
(S_a q)(x, t) = \begin{pmatrix}
-q_v(x, 2T - t) \\
q_p(x, 2T - t)
\end{pmatrix}
\]  

Definition 10.2 We define the acoustic time-reversal mirrors $T_a$ and $\hat{T}_a$ by putting for all $r = (r_v, r_p)^T \in Z$ and all $s = (s_v, s_p)^T \in \hat{Z}$

\[
(T_a r)(t) = \begin{pmatrix}
-r_v(2T - t) \\
r_p(2T - t)
\end{pmatrix}, 
(\hat{T}_a s)(t) = \begin{pmatrix}
-s_v(2T - t) \\
s_p(2T - t)
\end{pmatrix}
\]

The following lemma is easy to verify.

Lemma 10.1 We have the following commutations

\[
MS_a = T_a M, \quad S_a Q = Q T_a, \\
MS_a = \hat{T}_a M, \quad S_a Q = \hat{Q} T_a.
\]  

Theorem 10.2 For $(\phi, \psi)^T \in U$ we have

\[
\Gamma^{-1} S_a F S_a \Gamma \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = F^* \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right)
\]

Proof: For the proof it is convenient to make the following definition.

Definition 10.3 Acoustic time-reversal experiment: For given $(\phi, \psi)^T \in U$ define $q_v(x, t) = \rho \phi$ and $q_p(x, t) = \kappa \psi$ and perform the following physical experiment

\[
\rho \frac{\partial}{\partial s} v_tr(x, s) + \text{grad} p_tr(x, s) = -q_v(x, 2T - s)
\]  

\[
\kappa \frac{\partial}{\partial s} p_tr(x, s) + \text{div} v_tr(x, s) = q_p(x, 2T - s)
\]  

\[
v_tr(x, T) = 0, \quad p_tr(x, T) = 0 \quad \text{in} \quad \Omega,
\]

with $s \in [T, 2T]$ and zero boundary conditions. Doing this experiment means to process the data in the following way: Time-reverse all data $(\phi, \psi)^T$ according to $t \to 2T - s$, $t \in [0, T]$, and, in addition, reverse the directions of the velocities $\phi \to -\phi$. We call this experiment the 'acoustic time-reversal experiment'. Notice that the time is running forward in this experiment.

The solution $(v_tr, p_tr)^T$ of this experiment can obviously be represented by

\[
\begin{pmatrix}
v_tr \\
p_tr
\end{pmatrix} = F S_a \Gamma \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right).
\]

In order to show that the so defined time-reversal experiment is correctly modelled by the adjoint system derived above, we make the following change in variables:

\[
\tau = 2T - s, \quad \hat{v}_tr = -v_tr, \quad \hat{p}_tr = p_tr,
\]

which just corresponds to the application of the operator $S_a$ to $(v_tr, p_tr)^T$. We have $\tau \in [0, T]$. In these variables the time-reversal system \cite{35, 36} gets the form

\[
-\rho \frac{\partial}{\partial \tau} \hat{v}_tr(x, \tau) - \text{grad} \hat{p}_tr(x, \tau) = q_v(x, \tau)
\]  

\[
-\kappa \frac{\partial}{\partial \tau} \hat{p}_tr(x, \tau) - \text{div} \hat{v}_tr(x, \tau) = q_p(x, \tau)
\]
\[
\hat{v}_{tr}(x, T) = 0, \quad \hat{p}_{tr}(x, T) = 0 \quad \text{in} \ \Omega. \quad (40)
\]

Taking into account the definition of \(q_v\) and \(q_p\), we see that

\[
\begin{pmatrix}
\hat{v}_{tr} \\
\hat{p}_{tr}
\end{pmatrix}
= \begin{pmatrix}
v_a \\
p_a
\end{pmatrix} \quad \text{where} \ v_a \text{ and } p_a \text{ solve the adjoint system (25)–(27). Therefore, according to theorem 10.1:}
\]

\[
\begin{pmatrix}
\hat{v}_{tr} \\
\hat{p}_{tr}
\end{pmatrix} = \Gamma F^* \begin{pmatrix}
\phi \\
\psi
\end{pmatrix}. 
\]

Since we have with (36), (37) also

\[
\begin{pmatrix}
\hat{v}_{tr} \\
\hat{p}_{tr}
\end{pmatrix} = S_a F S_a \Gamma \begin{pmatrix}
\phi \\
\psi
\end{pmatrix},
\]

the theorem is proven. \(\square\)

11. The electromagnetic time-reversal mirror

In the following we want to define an operator \(S_e\) such that \(F^* = \Gamma^{-1} S_e F S_e \Gamma\) holds. We will call this operator \(S_e\) the 'electromagnetic time-reversal operator'. We will also define the 'electromagnetic time-reversal mirrors' \(T_e\) and \(\hat{T}_e\), which act on the signals instead of the sources or fields.

We consider Maxwell's equations

\[
\begin{align*}
\epsilon \frac{\partial E_f}{\partial t} - \text{curl} H_f + \sigma E_f &= q_E \quad (41) \\
\mu \frac{\partial H_f}{\partial t} + \text{curl} E_f &= q_H \quad (42) \\
E_f(x, 0) = 0, \quad H_f(x, 0) &= 0 \quad (43)
\end{align*}
\]

with \(t \in [0, T]\) and zero boundary conditions. We want to calculate the action of the adjoint operator \(F^*\) on a vector \((\phi, \psi)^T \in U\).

**Theorem 11.1** Let \((\phi, \psi)^T \in U\) and let \((E_a, H_a)^T\) be the solution of the adjoint system

\[
\begin{align*}
-\epsilon \frac{\partial E_a}{\partial t} + \text{curl} H_a(x, t) + \sigma E_a &= \epsilon(x)\phi(x, t) \quad (44) \\
-\mu \frac{\partial H_a}{\partial t} - \text{curl} E_a(x, t) &= \mu(x)\psi(x, t) \quad (45) \\
E_a(x, T) &= 0, \quad H_a(x, T) = 0 \quad \text{in} \ \Omega \quad (46)
\end{align*}
\]

with \(t \in [0, T]\) and zero boundary conditions. Then we have

\[
F^* \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = \begin{pmatrix}
\epsilon^{-1} E_a(x, t) \\
\mu^{-1} H_a(x, t)
\end{pmatrix},
\]

**Proof:** This theorem is just an application of theorem 10.1 to the electromagnetic symmetric hyperbolic system. Again, we will give a direct proof in appendix D as well.
Definition 11.1 We define the electromagnetic time-reversal operator \( S_e \) by putting for all \( q = (q_E, q_H)^T \in U \) (and similarly for all \( u = (E, H)^T \in U \))

\[
(S_e q)(x, t) = \begin{pmatrix} -q_E(x, 2T - t) \\ q_H(x, 2T - t) \end{pmatrix}
\]  

(48)

Definition 11.2 We define the electromagnetic time-reversal mirrors \( T_e \) and \( \hat{T}_e \) by putting for all \( r = (r_E, r_H)^T \in Z \) and for all \( s = (s_E, s_H)^T \in \hat{Z} \)

\[
(T_e r)(t) = \begin{pmatrix} -r_E(2T - t) \\ r_H(2T - t) \end{pmatrix}, \quad \hat{T}_e s)(t) = \begin{pmatrix} -s_E(2T - t) \\ s_H(2T - t) \end{pmatrix}
\]  

(49)

The following lemma is easy to verify.

Lemma 11.1 We have the following commutations

\[
MS_e = T_e M, \quad S_e Q = Q T_e, \\
\hat{M} S_e = \hat{T}_e M, \quad S_e \hat{Q} = \hat{Q} \hat{T}_e.
\]  

(50)

Theorem 11.2 For \((\phi, \psi)^T \in U\) we have

\[
\Gamma^{-1} S_e F S_e \Gamma \begin{pmatrix} \phi \\ \psi \end{pmatrix} = F^* \begin{pmatrix} \phi \\ \psi \end{pmatrix}
\]  

(51)

Proof: For the proof it is convenient to make the following definition.

Definition 11.3 Electromagnetic time-reversal experiment: For a given vector \((\phi, \psi)^T \in U\) define \( q_E(x, t) = c\phi \) and \( q_H(x, t) = \mu \psi \) and perform the physical experiment

\[
\epsilon \frac{\partial}{\partial s} E_{tr}(x, s) - \text{curl} E_{tr}(x, s) + \sigma H_{tr}(x, s) = -q_E(x, 2T - s) \\
\mu \frac{\partial}{\partial s} H_{tr}(x, s) + \text{curl} E_{tr}(x, s) = q_H(x, 2T - s)
\]  

(52)

(53)

\[
E_{tr}(x, T) = 0, \quad H_{tr}(x, T) = 0 \quad \text{in} \quad \Omega,
\]  

(54)

with \( s \in [T, 2T] \) and zero boundary conditions. Doing this experiment means to process the data in the following way: Time-reverse all data according to \( t \rightarrow 2T - s, \ t \in [0, T] \), and, in addition, reverse the directions of the electric field component by \( \phi \rightarrow -\phi \). We call this experiment the 'electromagnetic time-reversal experiment'. Notice that the time is running forward in this experiment.

The solution \((E_{tr}, H_{tr})^T\) of this experiment can obviously be represented by

\[
\begin{pmatrix} E_{tr} \\ H_{tr} \end{pmatrix} = F S_e \Gamma \begin{pmatrix} \phi \\ \psi \end{pmatrix}.
\]  

(55)

In order to show that the so defined time-reversal experiment is correctly modelled by the adjoint system derived above, we make the following change in variables:

\[
\tau = 2T - s, \quad \hat{E}_{tr} = -E_{tr}, \quad \hat{H}_{tr} = H_{tr},
\]  

(56)

which just corresponds to the application of the operator \( S_e \) to \((E_{tr}, H_{tr})^T\). We have \( \tau \in [0, T] \). In these variables the time-reversal system \((\hat{E}_{tr}, \hat{H}_{tr})^T\) gets the form

\[
-\epsilon \frac{\partial}{\partial \tau} \hat{E}_{tr}(x, \tau) + \text{curl} \hat{H}_{tr}(x, \tau) + \sigma \hat{E}_{tr}(x, s) = q_E(x, \tau),
\]  

(57)

\[
-\mu \frac{\partial}{\partial \tau} \hat{H}_{tr}(x, \tau) - \text{curl} \hat{E}_{tr}(x, \tau) = q_H(x, \tau)
\]  

(58)
Taking into account the definition of \( q_E \) and \( q_H \), we see that

\[
\begin{pmatrix}
\hat{E}_{tr} \\
\hat{H}_{tr}
\end{pmatrix} = \begin{pmatrix}
E_a \\
H_a
\end{pmatrix}
\]

where \( E_a \) and \( H_a \) solve the adjoint system (44)–(46). Therefore, according to theorem 11.1:

\[
\begin{pmatrix}
\hat{E}_{tr} \\
\hat{H}_{tr}
\end{pmatrix} = \Gamma F^* \begin{pmatrix}
\phi \\
\psi
\end{pmatrix}.
\]

Since we have with (55), (56) also

\[
\begin{pmatrix}
\hat{E}_{tr} \\
\hat{H}_{tr}
\end{pmatrix} = S e F S e \Gamma \begin{pmatrix}
\phi \\
\psi
\end{pmatrix},
\]

the theorem is proven. \( \square \)

Remark 11.1 For electromagnetic waves, there is formally an alternative way to define the electromagnetic time-reversal operator, namely putting for all \( q = (q_E, q_H)^T \in U \)

\[
(S e q)(x, t) = \begin{pmatrix}
q_E(x, 2T - t) \\
-q_H(x, 2T - t)
\end{pmatrix},
\]

accompanied by the analogous definitions for the electromagnetic time-reversal mirrors. With these alternative definitions, theorem 11.2 holds true as well, with only very few changes in the proof. Which form to use depends mainly on the preferred form for modelling applied antenna signals in the given antenna system. The first formulation directly works with applied electric currents, whereas the second form is useful for example for magnetic dipole sources.

12. Time-reversal and the adjoint operator \( A^* \)

Define \( S = S_a, T = T_a, \hat{T} = \hat{T}_a \) for the acoustic case, and \( S = S_e, T = T_e, \hat{T} = \hat{T}_e \) for the electromagnetic case. We call \( S \) the time-reversal operator and \( T, \hat{T} \) the time-reversal mirrors. We combine the results of lemma 10.1 and lemma 11.1 into the following lemma.

Lemma 12.1 We have the following commutations

\[
MS = TM, \quad SQ = QT,
\]

\[
MS = \hat{T} M, \quad SQ = \hat{Q} T.
\]

Moreover, combining theorem 10.2 and theorem 11.2 we get

Theorem 12.1 For \( (\phi, \psi)^T \in U \) we have

\[
\Gamma^{-1} S F S \Gamma \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = F^* \begin{pmatrix}
\phi \\
\psi
\end{pmatrix}.
\]
Theorem 12.2 We have
\[ A^* = TB\hat{T}. \] (62)

Proof: Recall that the adjoint operator \( A^* \) can be decomposed as \( A^* = Q^*F^*M^* \). With theorem 9.1, theorem 12.1, and lemma 12.1, it follows therefore that
\[ A^* = M\Gamma\Gamma^{-1}SF\Gamma\Gamma^{-1}\hat{Q} \]
\[ = MSFS\hat{Q} \]
\[ = TMF\hat{Q}\hat{T} \]
\[ = TB\hat{T}, \]
which proves the theorem. \( \square \)

Remark 12.1 The above theorem provides a direct link between the adjoint operator \( A^* \), which plays a central role in the theory of inverse problems, and a physical experiment modelled by \( B \). The expression \( TB\hat{T} \) defines a 'time-reversal experiment'. We will demonstrate in the following sections how we can make use of this relationship in order to solve the inverse problem of communication by a series of physical time-reversal experiments.

Remark 12.2 We only mention here that the above results hold as well for elastic waves with a suitable definition of the elastic time-reversal mirrors. We leave out the details for brevity.

13. Iterative time-reversal for the gradient method

13.1. The basic version

The results achieved above give rise to the following experimental procedure for applying the gradient method (15) to the inverse problem of communication as formulated in sections 7 and 8. First, the pilot signal ̂s(t) is defined by user \( U_1 \) as described in (11). Moreover, we assume that the first guess \( r(0) \) at the base station is chosen to be zero. Then, using theorem 12.2, we can write the gradient method (15) in the equivalent form
\[ r(0) = 0 \]
\[ r(n+1) = r(n) + \beta(n)TB\hat{T}(\hat{s} - Ar(n)), \] (63)
or, expanding it,
\[ r(0) = 0 \]
\[ s(n) = \hat{s} - Ar(n) \]
\[ r(n+1) = r(n) + \beta(n)TB\hat{T}s(n) \] (64)
In a more detailed form, we arrive at the following experimental procedure for implementing the gradient method, where we fix in this description \( \beta(n) = 1 \) for all iteration numbers \( n \) for simplicity.

(i) The user \( U_1 \) chooses a pilot signal \( \alpha(t) \) which he wants to use for communicating with the base station. The objective signal at all users is then \( \hat{s}(t) = (\alpha(t), 0, \ldots, 0)^T \). The initial guess \( r(0) \) at the base station is defined to be zero, such that \( s(0) = \hat{s} \).

(ii) The user \( U_1 \) initiates the communication by sending the time-reversed pilot signal into the environment. This signal is \( \mathcal{T}\hat{s}(0)(t) \). All other users are quiet.
(iii) The base station receives the pilot signal as $B \hat{T} s^{(0)}(t)$. It time-reverses this signal and sends this time-reversed form, namely $r^{(1)}(t) = T B \hat{T} s^{(0)}(t)$, back into the medium.

(iv) The new signal arrives at all users as $A r^{(1)}(t)$. All users compare the received signals with their components of the objective signal $\hat{s}(t)$. They take the difference $s^{(1)}(t) = \hat{s}(t) - A r^{(1)}(t)$, and time-reverse it. They send this new signal $\hat{T} s^{(1)}(t)$ back into the medium.

(v) The base station receives this new signal, time-reverses it, adds it to the previous signal $r^{(n)}(t)$, and sends the sum back into the medium as $r^{(n+1)}(t)$.

(vi) This iteration is continued until all users are satisfied with the match between the received signal $A r^{(n)}(t)$ and the objective signal $\hat{s}(t)$ at some iteration number $n$. Alternatively, a fixed iteration number $n$ can be specified a-priori for stopping the iteration.

Needless to say that, in practical implementations, the laboratory time needs to be reset to zero after each time-reversal step.

**Remark 13.1** The experimental procedure which is described above is practically equivalent to the experimental procedure which was suggested and experimentally verified in [34, 35]. Therefore, our basic scheme provides an alternative derivation and interpretation of this experimental procedure.

**Remark 13.2** We mention that several refinements of this scheme are possible and straightforward. For example, a weighted inner product can be introduced for the user signal space $\hat{Z}$ which puts different preferences on the satisfaction of the user objectives during the iterative optimization process. For example, if the 'importance' of suppressing interferences with other users is valued higher than to get an optimal signal quality at the specified user $U_i$, a higher weight can be put into the inner product at those users which did not start the communication process. A user who does not care about these interferences, simply puts a very small weight into his component of the inner product of $\hat{Z}$.

**Remark 13.3** Notice that there is no mechanism directly built into this procedure which prevents the energy emitted by the base antenna to increase more than the communication system can support. For example, if the subspace of signals

$$Z_0 := \{r(t) : A r = 0\}$$

is not empty, then it might happen that during the iteration described above (e.g. due to noise) an increasing amount of energy is put into signals emitted by the base station which are in this subspace and which all produce zero contributions to the measurements at all users. More generally, elements of the subspace of signals

$$Z_\varepsilon := \{r(t) : \|A r\|_{\hat{Z}} < \varepsilon \|r\|_{Z}\},$$

for a very small threshold $0 < \varepsilon << 1$, might cause problems during the iteration if the pilot signal $\hat{s}(t)$ chosen by the user has contributions in the subspace $AZ_\varepsilon$ (i.e. in the space of all $s = A r$ with $r \in Z_\varepsilon$). This is so because in the effort of decreasing the mismatch between $A r$ and $\hat{s}(t)$, the base antenna might need to put signals with high energy into the system in order to get only small improvements in the signal match at the user side. Since the environment (and therefore the operator $A$) is unknown a-priori, it is difficult to avoid the existence of such contributions in the pilot signal.
Time-reversal and the adjoint method with an application in telecommunication

One possible way to prevent the energy emitted by the base station to increase artificially would be to project the signals \( r^{(n)}(t) \) onto the orthogonal complements of the subspaces \( Z_0 \) or \( Z_\varepsilon \) (if they are known or can be constructed by some means) prior to their emission. Alternatively, the iteration can be stopped at an early stage before these unwanted contributions start to build up. (This in fact has been suggested in [34, 35]).

In the following subsection we introduce an alternative way of ensuring that the energy emitted by the base station stays reasonably bounded in the effort of fitting the pilot signal at the users.

13.2. The regularized version

Consider the regularized problem

\[
\mathbf{r}_{LSr} = \text{Min}_r (\|A\mathbf{r} - \tilde{\mathbf{s}}\|^2_Z + \lambda \|\mathbf{r}\|^2_Z)
\]  \hspace{1cm} (65)

with some suitably chosen regularization parameter \( \lambda > 0 \). In this problem formulation a trade-off is sought between a signal fit at the user side and a minimized energy emission at the base station. The trade-off parameter is the regularization parameter \( \lambda \). Instead of (14) we need to consider now

\[
\tilde{J}(\mathbf{r}) = \frac{1}{2} \langle A\mathbf{r} - \tilde{\mathbf{s}}, A\mathbf{r} - \tilde{\mathbf{s}} \rangle_Z + \frac{\lambda}{2} \langle \mathbf{r}, \mathbf{r} \rangle_Z
\]  \hspace{1cm} (66)

The negative gradient direction is now given by \(-A^*(A\mathbf{r} - \tilde{\mathbf{s}}) - \lambda \mathbf{r}\), such that the regularized iteration reads:

\[
\mathbf{r}^{(0)} = 0
\]

\[
\mathbf{r}^{(n+1)} = \mathbf{r}^{(n)} + \beta^{(n)} \left( T B \tilde{T} (\tilde{\mathbf{s}} - A\mathbf{r}^{(n)}) - \lambda \mathbf{r}^{(n)} \right),
\]  \hspace{1cm} (67)

where we have replaced \( A^* \) by \( T B \tilde{T} \). The time-reversal iteration can be expanded into the following practical scheme

\[
\mathbf{r}^{(0)} = 0
\]

\[
\tilde{\mathbf{s}}^{(n)} = \tilde{\mathbf{s}} - A\mathbf{r}^{(n)}
\]

\[
\mathbf{r}^{(n+1)} = \mathbf{r}^{(n)} + \beta^{(n)} T B \tilde{T} \tilde{\mathbf{s}}^{(n)} - \beta^{(n)} \lambda \mathbf{r}^{(n)}.
\]  \hspace{1cm} (68)

Comparing with (64), we see that the adaptations which need to be applied in the practical implementation for stabilizing the basic algorithm can easily be done.

14. The minimum norm solution approach

14.1. The basic version

In this section we want to propose an alternative scheme for solving the inverse problem of communication. As mentioned above, a major drawback of the basic approach is that the energy emitted by the base station is not limited explicitly when solving the optimization problem. The regularized version presented above alleviates this problem. However, we want to mention here that, under certain assumptions, there is an alternative scheme which can be employed instead and which has an energy constraint directly built in. Under the formal assumption that there exists at least one (and presumably more than one) solution of the inverse problem at hand (i.e. the 'formally underdetermined case'), we can look for the minimum norm solution

\[
\text{Min}_r \|\mathbf{r}\|^2_Z \text{ subject to } A\mathbf{r} = \tilde{\mathbf{s}}.
\]  \hspace{1cm} (69)
In Hilbert spaces this solution has an explicit form. It is

\[ r_{MN} = A^*(AA^*)^{-1} \hat{s}. \]  

(70)

Here, the operator \((AA^*)^{-1}\) acts as a filter on the pilot signal \(\hat{s}\). Instead of sending the pilot signal to the base station, the users send the filtered version of it. Certainly, a method must be found in order to apply the filter \((AA^*)^{-1}\) to the pilot signal. One possibility of doing so would be to try to determine the operator \(AA^*\) explicitly by a series of time-reversal experiments on some set of basis functions of \(\hat{Z}\), and then invert this operator numerically. However, this might not be practical in many situations. (It certainly would be slow and it would involve a significant amount of signal-processing, which we want to avoid here.) Therefore, we propose an alternative procedure. First, we notice that there is no need to determine the whole operator \((AA^*)^{-1}\), but that we only have to apply it to one specific signal, namely \(\hat{s}\). Let us introduce the short notation

\[ C = AA^*. \]

In this notation, we are looking for a signal \(\hat{s} \in \hat{Z}\) such that \(Cs = \hat{s}\). We propose to solve this equation in the least squares sense:

\[ \hat{s} = \min_{s} \|Cs - \hat{s}\|_2^2. \]  

(71)

Moreover, as a suitable method for practically finding this solution, we want to use the gradient method. Starting with the initial guess \(s_0 = 0\), the gradient method reads

\[ s^{(0)} = 0 \]
\[ s^{(n+1)} = s^{(n)} - \beta^{(n)} C^*(Cs^{(n)} - \hat{s}) , \]  

(72)

where \(C^*\) is the adjoint operator to \(C\) and \(\beta^{(n)}\) is again some step-size. Expanding this expression, and taking into account \(C^* = C\) and \(A^* = TB\hat{T}\), we arrive at

\[ s^{(0)} = 0 \]
\[ s^{(n+1)} = s^{(n)} + \beta^{(n)} ATB\hat{T}(\hat{s} - ATB\hat{T}s^{(n)}) . \]  

(73)

In the practical implementation, we arrive at the following iterative scheme:

**Initialize gradient iteration:** \(s^{(0)} = 0\)

for \(n = 0, 1, 2, \ldots\) do:

\[ r^{(n+\frac{1}{2})} = TB\hat{T}s^{(n)} \]
\[ s^{(n+\frac{1}{2})} = \hat{s} - Ar^{(n+\frac{1}{2})} \]
\[ r^{(n+1)} = TB\hat{T}s^{(n+\frac{1}{2})} \]
\[ s^{(n+1)} = s^{(n)} + \beta^{(n)} Ar^{(n+1)} \]

**Terminate iteration at step \(\hat{n}\):** \(\hat{s} = s^{(\hat{n})}\)

**Final result:** \(r_{MN} = TB\hat{T}\hat{s}\).

This iteration can be implemented by a series of time-reversal experiments, without the need of heavy signal-processing. The final step of the above algorithm amounts to applying \(A^*\) to the result of the gradient iteration for calculating \(\hat{s} = (AA^*)^{-1}\hat{s}\), which yields then \(r_{MN}\). This will then be the signal to be applied by the base station during the communication process with the user \(U_1\).
14.2. The regularized version

In some situations it might be expected that the operator \( C \) is ill-conditioned, such that its inversion might cause instabilities, in particular when noisy signals are involved. For those situations, a regularized form of the minimum norm solution is available, namely

\[
\mathbf{r}_{MNr} = A^*(AA^* + \lambda I_Z)^{-1} \hat{\mathbf{s}} \tag{75}
\]

where \( I_Z \) denotes the identity operator in \( Z \) and \( \lambda > 0 \) is some suitably chosen regularization parameter. The necessary adjustments in the gradient iteration for applying \((AA^* + \lambda I_Z)^{-1}\) to \( \hat{\mathbf{s}} \) are easily done. We only mention here the resulting procedure for the implementation of this gradient method by a series of time-reversal experiments:

Initialize gradient iteration: \( \mathbf{s}^{(0)} = 0 \)

for \( n = 0, 1, 2, \ldots \) do:

\[
\begin{align*}
\mathbf{r}^{(n+\frac{1}{2})} &= T \hat{B} \hat{T} \mathbf{s}^{(n)} \\
\mathbf{s}^{(n+\frac{1}{2})} &= \hat{\mathbf{s}} - A \mathbf{r}^{(n+\frac{1}{2})} - \lambda \mathbf{s}^{(n)} \\
\mathbf{r}^{(n+1)} &= T \hat{B} \hat{T} \mathbf{s}^{(n+\frac{1}{2})} \\
\mathbf{s}^{(n+1)} &= \mathbf{s}^{(n)} + \beta^{(n)} A \mathbf{r}^{(n+1)} + \beta^{(n)} \lambda \mathbf{s}^{(n+\frac{1}{2})}.
\end{align*}
\]

Final result: \( \mathbf{r}_{MNr} = T \hat{B} \hat{T} \hat{\mathbf{s}}. \)

Terminate iteration at step \( \hat{n} \): \( \hat{\mathbf{s}} = \mathbf{s}^{(\hat{n})} \)

Again, the last step shown above is a final application of \( A^* \) to the result of the gradient iteration for calculating \( \hat{\mathbf{s}} = (AA^* + \lambda I_Z)^{-1} \hat{\mathbf{s}} \), which yields then \( \mathbf{r}_{MNr} \). This will then be the signal to be applied by the base station during the communication process with the user \( U_1 \).

15. The regularized least squares solution revisited

We have introduced above the regularized least squares solution of the inverse problem of communication, namely

\[
\mathbf{r}_{LSr} = \text{Min}_{\mathbf{r}} \left( \| A \mathbf{r} - \hat{\mathbf{s}} \|_Z^2 + \lambda \| \mathbf{r} \|_Z^2 \right) \tag{77}
\]

with \( \lambda > 0 \) being the regularization parameter. In Hilbert spaces, the solution of (77) has an explicit form. It is

\[
\mathbf{r}_{LSr} = (A^* A + \lambda I_Z)^{-1} A^* \hat{\mathbf{s}}, \tag{78}
\]

where \( I_Z \) is the identity operator in \( Z \). It is therefore tempting to try to implement also this direct form as a series of time-reversal experiments and compare its performance with the gradient method as it was described above. As our last strategy which we present in this paper, we want to show here that such an alternative direct implementation of (77) is in fact possible.

Notice that in (78) the filtering operator \((A^* A + \lambda I_Z)^{-1}\) is applied at the base station, in contrast to the previous case where the user signal was filtered by the operator \((AA^* + \lambda I_Z)^{-1}\). Analogously to the previous case, we need to find a practical way to apply this filter to a signal at the base station. We propose again to solve the equation

\[
(A^* A + \lambda I_Z) \tilde{\mathbf{r}} = \tilde{\mathbf{r}} \tag{79}
\]
in the least squares sense, where \( \tilde{r} = A^*\hat{s} \). Defining

\[
C = A^*A + \lambda I_Z,
\]

and using \( C^* = C \) and \( A^* = TB\hat{T} \), we arrive at the following gradient iteration for solving problem \( \mathbf{r}(0) = 0 \):

\[
\begin{aligned}
\mathbf{r}^{(n)} &= 0 \\
\mathbf{r}^{(n+1)} &= \mathbf{r}^{(n)} + \beta^{(n)}(TB\hat{T}A + \lambda I_Z) \left( \tilde{r} - (TB\hat{T}A + \lambda I_Z)\mathbf{r}^{(n)} \right).
\end{aligned}
\]  

(80)

This gives rise to the following practical implementation by a series of time-reversal experiments:

User sends pilot signal to base station: \( \tilde{r} = TB\hat{T}\hat{s} \)

Initialize gradient iteration: \( \mathbf{r}^{(0)} = 0 \)

for \( n = 0, 1, 2, \ldots \) do:

\[
\begin{aligned}
\mathbf{s}^{(n+\frac{1}{2})} &= A\mathbf{r}^{(n)} \\
\mathbf{r}^{(n+\frac{1}{2})} &= \tilde{r} - TB\hat{T}\mathbf{s}^{(n+\frac{1}{2})} - \lambda \mathbf{r}^{(n)} \\
\mathbf{s}^{(n+1)} &= A\mathbf{r}^{(n+\frac{1}{2})} \\
\mathbf{r}^{(n+1)} &= \mathbf{r}^{(n)} + \beta^{(n)}TB\hat{T}\mathbf{s}^{(n+1)} + \beta^{(n)}\lambda \mathbf{r}^{(n+\frac{1}{2})}.
\end{aligned}
\]  

(81)

Terminate iteration at step \( \hat{n} \):

\( \tilde{r} = \mathbf{r}^{(\hat{n})} \)

Final result:

\( r_{LSr} = \tilde{r} \).

\( r_{LSr} \) will then be the signal to be applied by the base station during the communication process with the user \( U_1 \).

16. Partial and generalized measurements

In many practical applications, only partial measurements of the whole wave-field are available. For example, in ocean acoustics often only pressure is measured, whereas the velocity field is not part of the measurement process. Similarly, in wireless communication only one or two components of the electric field might be measured simultaneously, but the remaining electric components and all magnetic components are missing. We want to demonstrate in this section that all results presented above are valid also in this situation of partial measurements, with the suitable adaptations.

Mathematically, the measurement operator needs to be adapted for the situation of partial measurements. Let us concentrate here on the special situation that only one component \( u_\nu (\nu \in \{1, 2, 3, \ldots \}) \) of the incoming wave field \( \mathbf{u} \) is measured by the users and the base station. All other possible situations will then just be combinations of this particular case. It might also occur the situation that users can measure a different partial set of components than the base station. That case also follows directly from this canonical situation.

We introduce the new signal space at the base station \( Y = (L_2[0, T])^K \) and the corresponding ‘signal projection operator’ \( P_\nu \) by putting

\[
P_\nu : Z \rightarrow Y, \quad P_\nu (r_1(t), \ldots, r_K(t))^T = r_\nu(t).
\]

We see immediately that its adjoint \( P_\nu^* \) is given by

\[
P_\nu^* : Y \rightarrow Z, \quad P_\nu^* r_\nu(t) = (0, \ldots, 0, r_\nu(t), 0, \ldots, 0)^T
\]

where \( r_\nu(t) \) appears on the right hand side at the \( \nu \)-th position. Our new measurement operator \( M_\nu \), and the new source operator \( Q_\nu \), are then defined by

\[
\begin{array}{ll}
M_\nu : U \rightarrow Y, & M_\nu \mathbf{u} = P_\nu \mathbf{M} \mathbf{u} \\
Q_\nu : Y \rightarrow U, & Q_\nu r_\nu = Q P_\nu^* r_\nu.
\end{array}
\]  

(82)
Analogous definitions are done for $\hat{Y}$, $\hat{P}_\nu$, $\hat{M}_\nu$ and $\hat{Q}_\nu$ at the users.

Obviously, we will have to replace now in the above derivation of the iterative time-reversal procedure all measurement operators $M$ by $M_\nu$ (and $\hat{M}$ by $\hat{M}_\nu$) and all source operators $Q$ by $Q_\nu$ (and $\hat{Q}$ by $\hat{Q}_\nu$). In particular, the new ‘communication operators’ are now given by

$$
A_\nu : Y \rightarrow \hat{Y}, \quad A_\nu r_\nu = \hat{M}_\nu FQ_\nu r_\nu,
B_\nu : \hat{Y} \rightarrow Y, \quad B_\nu s_\nu = M_\nu F\hat{Q}_\nu s_\nu.
$$

(83)

In the following two theorems we show that the main results of this paper carry over to these newly defined operators.

**Theorem 16.1** We have

$$
M^*_\nu = \Gamma^{-1}Q_\nu, \quad \hat{M}^*_\nu = \Gamma^{-1}\hat{Q}_\nu,
Q^*_\nu = M_\nu \Gamma, \quad \hat{Q}^*_\nu = \hat{M}_\nu \Gamma.
$$

(84)

**Proof:** The proof is an easy exercise using (82) and theorem 9.1. □

**Theorem 16.2** It is

$$
A^*_\nu = TB_\nu \hat{T}.
$$

(85)

**Proof:** The proof is now identical to the proof of theorem 12.2, using theorem 16.1 instead of theorem 9.1. □

**Remark 16.1** In fact, it is easy to verify that all results of this paper remain valid for arbitrarily defined linear measurement operators

$$
M_U : U \rightarrow Z_U, \quad M_A : U \rightarrow Z_A,
$$

where $Z_U$ and $Z_A$ are any meaningful signal spaces at the users and the base antennas, respectively. The only requirement is that it is experimentally possible to apply signals according to the source operators defined by

$$
Q_U : Z_U \rightarrow U, \quad Q_U = \Gamma M_U^*,
Q_A : Z_A \rightarrow U, \quad Q_A = \Gamma M_A^*,
$$

where $M_U^*$ and $M_A^*$ are the formal adjoint operators to $M_U$ and $M_A$ with respect to the chosen signal spaces $Z_U$ and $Z_A$. In addition, the measurement and source operators as defined above are required to satisfy the commutation relations as stated in lemma 12.1. Under these assumptions, we define the generalized communication operators $\hat{A}$ and $\hat{B}$ by

$$
\hat{A} : Z_A \rightarrow Z_U, \quad \hat{A}r = M_U FQ_A r,
\hat{B} : Z_U \rightarrow Z_A, \quad \hat{B}s = M_AFQ_U s.
$$

Now the proof to theorem 12.2 directly carries over to this generalized situation, such that we have also here

$$
\hat{A}^* = TB\hat{T}.
$$

This yields iterative time-reversal schemes completely analogous to those presented above.
17. Summary and future directions

We have derived in this paper a direct link between the time-reversal technique and the adjoint method for imaging. Using this relationship, we have constructed several iterative time-reversal schemes for solving an inverse problem which arises in ocean acoustic and wireless communication. Each of these schemes can be realized physically as a series of time-reversal experiments, without the use of heavy signal processing or computations. One of the schemes which we have derived (and which we call the 'basic scheme'), is practically equivalent to a technique introduced earlier in [34, 35] using different tools. Therefore, we have given an alternative theoretical derivation of that technique, with a different mathematical interpretation. The other schemes which we have introduced are new in this application. They represent either generalizations of the basic scheme, or alternatives which follow different objectives.

Many questions related to these and similar iterative time-reversal approaches for telecommunication are still open. The experimental implementation has been investigated so far only for one of these techniques in [34, 35], for the situation of underwater sound propagation. A thorough experimental (or, alternatively, numerical) verification of the other schemes is necessary for their practical evaluation. An interesting and practically important problem is the derivation of quantitative estimates for the expected focusing quality of each of these schemes, for example following the ideas of the work performed for a single step in [5]. Certainly, it is expected that these estimates will again strongly depend on the randomness of the medium, on the geometry and distribution of users and base antennas, and on technical constraints as for example partial measurements. Also, different types of noise in the communication system need to be taken into account. The performance in a time-varying environment is another interesting issue of practical importance. All schemes presented here can be adapted in principle to a dynamic environment by re-adjusting the constructed optimal signals periodically. Practical ways of doing so need to be explored.

Finally, we want to mention that speed is an important factor in the application of communication. In the time-reversal experiment, in a certain sense a physical process is used for applying a mathematical operator (namely $A^*$) in order to solve a given inverse problem. In a complex environment, this can be much faster than using a powerful computer, even if the complex environment would be known to the users and the base station.

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Appendix A: Proof of theorem 9.1

For arbitrary $r \in \mathbb{Z}$ and $u \in U$ we have

$$\langle Mu, r \rangle_Z = \sum_{k=1}^{K} \int_{[0,T]} \int_{\Omega} \gamma_k(x)u(x,t)dx \cdot \langle r(t) \rangle_N dt$$

$$= \int_{[0,T]} \int_{\Omega} \langle \Gamma(x)u(x,t), \Gamma^{-1}(x) \sum_{k=1}^{K} \gamma_k(x)r(t) \rangle_N dx dt.$$

Therefore,

$$\langle Mu, r \rangle_Z = \langle u, M^*r \rangle_U$$

with

$$M^*r = \Gamma^{-1}(x) \sum_{k=1}^{K} \gamma_k(x)r(t) = \Gamma^{-1}(x)Qr.$$

Doing the analogous calculation for $\hat{M}^*$ we get

$$\hat{M}^* = \Gamma^{-1}Q, \quad \hat{\hat{M}}^* = \Gamma^{-1}\hat{Q}.$$ (A.1)

Taking the adjoint of (A.1) we see that

$$Q^* = MT, \quad \hat{Q}^* = \hat{M}T$$

holds as well. This proves the theorem. \hfill \Box

Appendix B: Proof of theorem 9.2

We have the following version of Green’s formula

$$\int_{[0,T]} \int_{\Omega} \left( \langle \Gamma(x) \frac{\partial u}{\partial t} + \sum_{i=1}^{3} D^i \frac{\partial u}{\partial x_i} + \Phi(x)u, z \rangle_N - \Gamma(x)u(x,T), z(x,T) \rangle_N dx dt \right. \right.$$

$$+ \int_{[0,T]} \int_{\Omega} \langle \Gamma(x)u(x,t), v(x,t) \rangle_N dx dt + \int_{[0,T]} \int_{\Omega} \langle q(x,t), z(x,t) \rangle_N dx dt$$

$$= \int_{[0,T]} \int_{\Omega} \left( u, -\Gamma(x) \frac{\partial z}{\partial t} - \sum_{i=1}^{3} D^i \frac{\partial z}{\partial x_i} + \Phi(x)z \right) \rangle_N dx dt$$

$$+ \int_{[0,T]} \int_{\Omega} \langle \Gamma(x)u(x,t), v(x,t) \rangle_N dx dt + \int_{[0,T]} \int_{\Omega} \langle q(x,t), z(x,t) \rangle_N dx dt$$

$$+ \int_{\Omega} \langle \Gamma(x)u(x,T), z(x,T) \rangle_N dx - \int_{\Omega} \langle \Gamma(x)u(x,0), z(x,0) \rangle_N dx$$

$$+ \int_{[0,T]} \int_{\partial \Omega} \sum_{i=1}^{3} \langle D^i u, z \rangle_N \nu_i(x) d\sigma dt,$$

where $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward normal at $\partial \Omega$ in the point $x$. Notice that we have augmented Green’s formula in (A.2) by some terms which appear in identical form on the left hand side and on the right hand side.
We will assume here that the boundary is far away from the sources and receivers and that no energy enters $\Omega$ from the outside, such that during the time interval of interest $[0, T]$ all fields along this boundary are identically zero. This is expressed by the boundary conditions given in (4) and (20). Let $u(x, t)$ be a solution of (1), (2), (4), and $z(x, t)$ a solution of (18), (20). Then the first term on the left hand side of (A.2) and the third term on the right hand side cancel each other because of (4) and (20). The second term on the left hand side and the first term on the right hand side cancel each other because of (1). The second term on the left hand side and the first term on the right hand side cancel each other because of (18). The $(t = T)$-term and the $(t = 0)$ term vanish due to (10) and (2), respectively, and the boundary integral vanishes because of (1) and (20). The remaining terms (i.e. the third term on the left hand side and the second term on the right hand side) can be written as

$$\langle Fq, v \rangle_U = \langle q, F^*v \rangle_U,$$

with $F^*v = \Gamma^{-1}(x)z(x, t)$ as defined in (21).

**Appendix C: Direct proof of theorem 10.1**

We prove the lemma by using Greens formula:

$$\int_0^T \int_{\Omega} \left[ \rho \frac{\partial v_f}{\partial t} - v_f \rho \frac{\partial \phi}{\partial t} - \nabla p + \nabla f + \nabla v_f \right] dx dt$$

(A.3)

$$+ \int_0^T \int_{\Omega} \left[ \rho \nabla \phi + \kappa \nabla \psi \right] dx dt + \int_0^T \int_{\partial \Omega} [ q_v v_a + q_p p_a ] dx dt$$

$$= \int_0^T \int_{\Omega} \left[ -v_f \rho \frac{\partial v_a}{\partial t} - p_f \nabla v_a - p_f \kappa \frac{\partial p_a}{\partial t} - v_f \nabla p_a \right] dx dt$$

$$+ \int_0^T \int_{\Omega} [ q_v v_a + q_p p_a ] dx dt$$

$$+ \int_0^T \int_{\partial \Omega} (v_a \cdot n) p_f d\sigma dt + \int_0^T \int_{\partial \Omega} (v_f \cdot n) p_a d\sigma dt$$

$$+ \int_{\Omega} \rho \left[ (v_f v_a)(x, T) - (v_f v_a)(x, 0) \right] dx + \int_{\Omega} \kappa \left[ (p_f p_a)(x, T) - (p_f p_a)(x, 0) \right] dx.$$

This equation has the form (A.2). Notice that we have augmented Green’s formula in (A.2), as already shown in (A.2), by some terms which appear in identical form on the left hand side and on the right hand side.

The first term on the left hand side of equation (A.3) and the third term on the right hand side cancel each other due to (22), (23). The second term on the left hand side and the first term on the right hand side cancel each other because of (24), (20). The $(t = T)$-terms and the $(t = 0)$-terms vanish due to (27), (24), respectively, and the boundary integral vanishes because of (1) and (20). The remaining terms (i.e. the third term on the left hand side and the second term on the right hand side) can be written as

$$\langle Fq, v \rangle_U = \langle q, F^*v \rangle_U,$$

with $F^*v = \Gamma^{-1}(x)z(x, t)$ as defined in (21).

Therefore, $F^*$ is in fact the adjoint of $F$, and the lemma is proven.
Appendix D: Direct proof of theorem\ref{thm:direct}

We prove the lemma by using Green’s formula:

\[
\int_0^T \int_\Omega \left[ \frac{\partial E_f}{\partial t} E_a - \text{curl} H_f E_a + \sigma E_f E_a + \mu \frac{\partial H_f}{\partial t} H_a + \text{curl} E_f H_a \right] \, dx \, dt
+ \int_0^T \int_\Omega [\epsilon E_f \phi + \mu H_f \psi] \, dx \, dt
+ \int_0^T \int_\Omega [q_E E_a + q_H H_a] \, dx \, dt
= \int_0^T \int_\Omega \left[ - E_f \frac{\partial E_a}{\partial t} - H_f \text{curl} E_a + E_f \sigma E_a - H_f \mu \frac{\partial H_a}{\partial t} + E_f \text{curl} H_a \right] \, dx \, dt
+ \int_0^T \int_\Omega [\epsilon E_f \phi + \mu H_f \psi] \, dx \, dt
+ \int_0^T \int_{\partial \Omega} E_a \times H_f \cdot n \, d\sigma \, dt
+ \int_0^T \int_{\partial \Omega} E_f \times H_a \cdot n \, d\sigma \, dt
+ \int_\Omega \epsilon [E_f E_a(T) - E_f E_a(0)] \, dx
+ \int_\Omega \mu [H_f H_a(T) - H_f H_a(0)] \, dx
\]

This equation has the form (A.2). Notice that we have augmented Green’s formula in (A.4), as already shown in (A.2), by some terms which appear in identical form on the left hand side and on the right hand side.

The first term on the left hand side of equation (A.4) and the third term on the right hand side cancel each other because of (41) and (42). The second term on the left hand side and the first term on the right hand side cancel each other because of (44), (45). The \((t = 0)\)-terms and the \((t = T)\)-terms vanish due to (43) and (46). The boundary terms vanish because of zero boundary conditions. We are left over with the equation

\[
\int_0^T \int_\Omega [\epsilon E_f \phi + \mu H_f \psi] \, dx \, dt
= \int_0^T \int_\Omega [q_E E_a + q_H H_a] \, dx \, dt.
\]

Defining \(F^*\) by (47), this can be written as

\[
\left\langle F \begin{pmatrix} q_E \\ q_H \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_U = \left\langle \begin{pmatrix} q_E \\ q_H \end{pmatrix}, F^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_U.
\]

Therefore, \(F^*\) is in fact the adjoint of \(F\), and the lemma is proven. \(\square\)

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