Cubicity, Degeneracy, and Crossing Number

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Abstract. A k-box $B = (R_1, \ldots, R_k)$, where each $R_i$ is a closed interval on the real line, is defined to be the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$. If each $R_i$ is a unit length interval, we call $B$ a $k$-cube. Boxicity of a graph $G$, denoted as $\text{box}(G)$, is the minimum integer $k$ such that $G$ is an intersection graph of $k$-boxes. Similarly, the cubicity of $G$, denoted as $\text{cub}(G)$, is the minimum integer $k$ such that $G$ is an intersection graph of $k$-cubes.

It was shown in [L. Sunil Chandran, Mathew C. Francis, and Naveen Sivadasan: Representing graphs as the intersection of axis-parallel cubes. MCDES-2008, IISc Centenary Conference, available at CoRR, abs/cs/0607092, 2006.] that, for a graph $G$ with maximum degree $\Delta$, $\text{cub}(G) \leq \lceil 4(\Delta + 1) \log n \rceil$. In this paper, we show that, for a $k$-degenerate graph $G$, $\text{cub}(G) \leq (k + 2)\lceil 2e \log n \rceil$. Since $k$ is at most $\Delta$ and can be much lower, this clearly is a stronger result. This bound is tight. We also give an efficient deterministic algorithm that runs in $O(n^2 k)$ time to output a $8k(\lceil 2.42 \log n \rceil + 1)$ dimensional cube representation for $G$.

The crossing number of a graph $G$, denoted as $\text{CR}(G)$, is the minimum number of crossing pairs of edges, over all drawings of $G$ in the plane. An important consequence of the above result is that if the crossing number of a graph $G$ is $t$, then $\text{box}(G)$ is $O(t^{4/3} \log t)$.

Let $(P, \preceq)$ be a partially ordered set and let $G_P$ denote its underlying comparability graph. Let $\dim(P)$ denote the poset dimension of $P$. Another interesting consequence of our result is to show that $\dim(P) \leq 2(k + 2)\lceil 2e \log n \rceil$, where $k$ denotes the degeneracy of $G_P$. Also, we get a deterministic algorithm that runs in $O(n^2 k)$ time to construct a $16k(\lceil 2.42 \log n \rceil + 1)$ sized realizer for $P$. As far as we know, though very good upper bounds exist for poset dimension in terms of maximum degree of its underlying comparability graph, no upper bounds in terms of the degeneracy of the underlying comparability graph is seen in the literature.

It was shown in [L. Sunil Chandran, Mathew C. Francis, and Naveen Sivadasan: Geometric Representation of Graphs in Low Dimension Using Axis Parallel Boxes. Algorithmica 56(2): 129-140, 2010.] that boxicity of almost all graphs in $\mathcal{G}(n, m)$ model is $O(d_{av} \log n)$, where $d_{av} = \frac{2m}{n}$ denotes the average degree of the graph under consideration. In this paper, we prove a stronger result. Using our bound for the cubicity of $k$-degenerate graphs, we show that cubicity of almost all graphs in $\mathcal{G}(n, m)$...
A graph \( G \) is an intersection graph of sets from a family of sets \( \mathcal{F} \), if there exists \( f : V(G) \to \mathcal{F} \) such that \( (u, v) \in E(G) \iff f(u) \cap f(v) \neq \emptyset \). Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs are the interval graphs. Interval graphs are the intersection graphs of closed intervals on the real line. A restricted form of interval graphs, that allow only intervals of unit length, are indifference graphs or unit interval graphs.

An interval on the real line can be generalized to a “\( k \)-box” in \( \mathbb{R}^k \). A \( k \)-box \( B = (R_1, \ldots, R_k) \), where each \( R_i \) is a closed interval on the real line, is defined to be the Cartesian product \( R_1 \times R_2 \times \cdots \times R_k \). If each \( R_i \) is a unit length interval, we call \( B \) a \( k \)-cube. Thus, 1-boxes are just closed intervals on the real line whereas 2-boxes are axis-parallel rectangles in the plane. The parameter boxicity of a graph \( G \), denoted as \( \text{box}(G) \), is the minimum integer \( k \) such that \( G \) is an intersection graph of \( k \)-boxes. Similarly, the cubicity of \( G \), denoted as \( \text{cub}(G) \), is the minimum integer \( k \) such that \( G \) is an intersection graph of \( k \)-cubes. Thus, interval graphs are the graphs with boxicity equal to 1 and unit interval graphs are the graphs with cubicity equal to 1. A \( k \)-box representation or a \( k \)-dimensional box representation of a graph \( G \) is a mapping of the vertices of \( G \) to \( k \)-boxes such that two vertices in \( G \) are adjacent if and only if their corresponding \( k \)-boxes have a non-empty intersection. In a similar way, we define \( k \)-cube representation (or \( k \)-dimensional cube representation) of a graph \( G \). Since \( k \)-cubes by definition are also \( k \)-boxes, boxicity of a graph is at most its cubicity.

The concepts of boxicity and cubicity were introduced by F.S. Roberts in 1969 [19]. Roberts showed that for any graph \( G \) on \( n \) vertices \( \text{box}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( \text{cub}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor \). Both these bounds are tight since \( \text{box}(K_{2,2,\ldots,2}) = \left\lfloor \frac{n}{2} \right\rfloor \) and \( \text{cub}(K_{3,3,\ldots,3}) = \left\lfloor \frac{2n}{3} \right\rfloor \) where \( K_{2,2,\ldots,2} \) denotes the complete \( n/2 \)-partite graph with 2 vertices in each part and \( K_{3,3,\ldots,3} \) denotes the complete \( n/3 \)-partite graph with 3 vertices in each part. It is easy to see that the boxicity of any graph is at least the boxicity of any induced subgraph of it.

Box representation of graphs finds application in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research (see [10]). Given a low dimensional box representation, some well known NP-hard problems become polynomial time solvable. For instance, the max-clique problem is polynomial time solvable for graphs with boxicity \( k \) because the number of maximal cliques in such graphs is only \( O((2n)^k) \).
1.1 Previous Results on Boxicity and Cubicity

It was shown by Cozzens [9] that computing the boxicity of a graph is \( \text{NP} \)-hard. Kratochvíl [14] showed that deciding whether the boxicity of a graph is at most 2 itself is \( \text{NP} \)-complete. It has been shown by Yannakakis [23] that deciding whether the cubicity of a given graph is at least 3 is \( \text{NP} \)-hard.

Researchers have tried to bound the boxicity and cubicity of graph classes with special structure. Scheinerman [20] showed that the boxicity of outerplanar graphs is at most 2. Thomassen [21] proved that the boxicity of planar graphs is bounded from above by 3. Upper bounds for the boxicity of many other graph classes such as chordal graphs, AT-free graphs, permutation graphs etc. were shown in [8] by relating the boxicity of a graph with its treewidth. The cube representation of special classes of graphs like hypercubes and complete multipartite graphs were investigated in [19, 15, 16].

Various other upper bounds on boxicity and cubicity in terms of graph parameters such as maximum degree, treewidth etc. can be seen in [5, 3, 4, 12, 8]. The ratio of cubicity to boxicity of any graph on \( n \) vertices was shown to be at most \( \lceil \log_2 n \rceil \) in [6].

1.2 Equivalent Definitions for Boxicity and Cubicity

Let \( G \) and \( G_1, \ldots, G_b \) be graphs such that \( V(G_i) = V(G) \) for \( 1 \leq i \leq b \). We say \( G = \bigcap_{i=1}^{b} G_i \) when \( E(G) = \bigcap_{i=1}^{b} E(G_i) \). Below, we state two very useful lemmas due to Roberts [19].

**Lemma 1.** For any graph \( G \), \( \text{box}(G) \leq k \) if and only if there exist \( k \) interval graphs \( I_1, \ldots, I_k \) such that \( G = I_1 \cap \cdots \cap I_k \).

**Lemma 2.** For any graph \( G \), \( \text{cub}(G) \leq k \) if and only if there exist \( k \) indifference graphs (unit interval graphs) \( I_1, \ldots, I_k \) such that \( G = I_1 \cap \cdots \cap I_k \).

1.3 Our Results

A graph \( G \) is \( k \)-degenerate if the vertices of \( G \) can be enumerated in such a way that every vertex is succeeded by at most \( k \) of its neighbors. The least number \( k \) such that \( G \) is \( k \)-degenerate is called the degeneracy of \( G \) and any such enumeration is referred to as a degeneracy order of \( V(G) \). For example, trees and forests are 1-degenerate and planar graphs are 5-degenerate. Series-parallel graphs, outerplanar graphs, non-regular cubic graphs, circle graphs of girth at least 5 etc. are subclasses of 2-degenerate graphs.

**Main Result:** It was shown in [3] that, for a graph \( G \) with maximum degree \( \Delta \), \( \text{cub}(G) \leq \lceil 4(\Delta + 1) \log n \rceil \). In this paper, we show that, for a \( k \)-degenerate graph \( G \), \( \text{cub}(G) \leq (k+2)\lceil 2e \log n \rceil \). Since \( k \) is at most \( \Delta \) and can be much lower, this clearly is a stronger result. We prove that this bound is tight. Moreover, we give an efficient deterministic algorithm that outputs a \( 8k(\lceil 2.42 \log n \rceil + 1) \) dimensional cube representation for \( G \) in \( O(n^2k) \) time.
Consequence 1: The crossing number of a graph $G$, denoted as $CR(G)$, is the minimum number of crossing pairs of edges, over all drawings of $G$ in the plane. We prove that, if $CR(G) = t$, then $box(G) \leq 66t^{4/3} \log 4t + 6$. This bound is tight up to a factor of $O((\log t)^{4/3})$. We also show that, if $G$ has $n$ vertices, then $cub(G)$ is $O(\log n + t^{1/4} \log t)$. See Section 5 for details.

Consequence 2: It was shown in [5] that boxicity of almost all graphs in $\mathcal{G}(n, m)$ model is $O(d_{av} \log n)$, where $d_{av} = \frac{2m}{n}$ denotes the average degree of the graph under consideration. What can we infer about the cubicity of almost all graphs from the result of [5]? It was shown in [6] that for every graph $G$, $cub(G) \leq \log_2 n \times box(G)$. Combining this result with that of [5], we can infer that cubicity of almost all graphs is $O(d_{av} \log^2 n)$. In this paper, we prove a stronger result. Using our bound for the cubicity of $k$-degenerate graphs, we show that cubicity of almost all graphs in $\mathcal{G}(n, m)$ model is $O(d_{av} \log n)$. See Section 6 for details.

Consequence 3: Let $(\mathcal{P}, \leq)$ be a poset (partially ordered set) and let $G_{\mathcal{P}}$ be the underlying comparability graph of $\mathcal{P}$. A linear order $L$ of $\mathcal{P}$ is a total order which satisfies $(x \leq y \in \mathcal{P}) \implies (x \leq y \in L)$. A realizer of $\mathcal{P}$ is a set of linear extensions of $\mathcal{P}$, say $\mathcal{R}$, which satisfy the following condition: for any two distinct elements $x$ and $y$, $x \leq y$ in $\mathcal{P}$ if and only if $x \leq y$ in $L$, $\forall L \in \mathcal{R}$. The poset dimension of $\mathcal{P}$, denoted by $dim(\mathcal{P})$, is the minimum integer $k$ such that there exists a realizer of $\mathcal{P}$ of cardinality $k$. Yannakakis [23] showed that it is NP-complete to decide whether the dimension of a poset is at most 3. The poset dimension is an extensively studied parameter in the theory of partial order (See [22] for a comprehensive treatment).

There are several research papers in the partial order literature which study the dimension of posets whose underlying comparability graph has some special structure – interval order, semi order and crown posets are some examples. While very good upper bounds (for example $c\Delta(\log \Delta)^2$, where $c$ is a constant) are known for poset dimension in terms of maximum degree $\Delta$ of its underlying comparability graph, as far as we know there are no upper bounds in terms of the degeneracy of the underlying comparability graph. Connecting our main result with a result in [1], we can get an upper bound for poset dimension in terms of the degeneracy of the underlying comparability graph as follows. It was shown in [1] that $dim(\mathcal{P}) < 2box(G_{\mathcal{P}})$. Therefore, if the degeneracy of the underlying comparability graph $G_{\mathcal{P}}$ is $k$, then our result says that $dim(\mathcal{P}) \leq 2(k + 2)/[2\epsilon \log n]$. Also, we get a deterministic algorithm that runs in $O(n^2 k)$ time to construct a $16k([2.42 \log n] + 1)$ sized realizer for $\mathcal{P}$.

2 Preliminaries

For any finite positive integer $n$, let $[n]$ denote the set $\{1, \ldots, n\}$. Unless mentioned explicitly, all logarithms are to the base $e$ in this paper. All the graphs that we consider are simple, finite and undirected. For a graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. For any vertex $u \in V(G)$,
$N_G(u) = \{ v \in V(G) \mid (u, v) \in E(G) \}$. We define $d_G(u) := |N_G(u)|$. The average degree of $G$ is denoted by $d_{av}(G)$.

Consider a graph $G$ whose vertices are partitioned into two parts, namely $V_A$ and $V_B$. That is, $V(G) = V_A \cup V_B$. We shall use $S_B(G)$ to denote the graph with $V(S_B(G)) = V(G)$ and $E(S_B(G)) = E(G) \setminus \{(u, v) \mid u, v \in V_B\}$. In other words, $S_B(G)$ is obtained from $G$ by making $V_B$ a stable set (or an independent set). Let $C_B(G)$ denote the graph with $V(C_B(G)) = V(G)$ and $E(C_B(G)) = E(G) \cup \{(u, v) \mid u, v \in V_B\}$. That is, $C_B(G)$ is obtained from $G$ by making $V_B$ a clique. Let $G_B$ denote the subgraph of $G$ induced on $V_B$.

Analogously, we define $S_A(G), C_A(G)$, and $G_A$.

Since an interval graph is the intersection graph of closed intervals on the real line, for every interval graph $I_a$, there exists a function $f_a : V(I_a) \rightarrow \{X \subseteq \mathbb{R} \mid X \text{ is a closed interval}\}$, such that $\forall u, v \in V(I_a), (u, v) \in E(I_a) \Leftrightarrow f_a(u) \cap f_a(v) \neq \emptyset$. The function $f_a$ is called an interval representation of the interval graph $I_a$. Note that the interval representation of an interval graph need not be unique. In a similar way, we call a function $f_b$ a unit interval representation of a unit interval graph $I_b$ if $f_b : V(I_b) \rightarrow \{X' \subseteq \mathbb{R} \mid X' \text{ is a unit length closed interval}\}$, such that $\forall u, v \in V(I_b), (u, v) \in E(I_b) \Leftrightarrow f_b(u) \cap f_b(v) \neq \emptyset$. Given a closed interval $X = [y, z]$, we define $l(X) := y$ and $r(X) := z$. We say that the interval $X$ has left end-point $l(X)$ and right end-point $r(X)$.

Given a graph $G$, a coloring $\mathcal{C}$ of $V(G)$ using colors $\chi_1, \ldots, \chi_a$ is a map $\mathcal{C} : V(G) \to \{\chi_1, \ldots, \chi_a\}$. For each $u \in V(G)$, we shall use $\mathcal{C}(u)$ to denote the color of $u$ in $\mathcal{C}$.

**Definitions, Notations and Assumptions used in Sections 3 and 4:** Recall that the degeneracy of a graph is the least number $k$ such that it has a vertex enumeration in which each vertex is succeeded by at most $k$ of its neighbors. Such an enumeration is called the degeneracy order. The graph $G$ that we consider in these sections is a $k$-degenerate graph having $V(G) = \{v_1, \ldots, v_n\}$, $|E(G)| = m$ and $\overline{m} = \binom{n}{2} - m$ denotes the number of non-edges in $G$. The enumeration $v_1, \ldots, v_n$ is a degeneracy order of $V(G)$ and is denoted by $\mathcal{D}$. For every $v_i, v_j \in V(G)$, we say $v_i \prec_D v_j$ if $v_i$ comes before $v_j$ in $\mathcal{D}$ i.e., $v_i \prec_D v_j$ if and only if $i < j$. Suppose $v_i \prec_D v_j$. If $(v_i, v_j) \in E(G)$, then we call $v_j$ a forward neighbor of $v_i$ and $v_i$ is referred to as a backward neighbor of $v_j$. Observe that since $G$ is $k$-degenerate, a vertex can have at most $k$ forward neighbors. If $(v_i, v_j) \notin E(G)$, then $v_j$ a forward non-neighbor of $v_i$ and $v_i$ is a backward non-neighbor of $v_j$.

For any $u \in V(G)$, $N_G^f(u) = \{w \in V(G) \mid w \text{ is a forward neighbor of } u\}$ and $N_G^b(u) = \{w \in V(G) \mid w \text{ is a backward neighbor of } u\}$.

**Support sets of a non-edge:** For each $(v_x, v_y) \notin E(G)$, where $v_x <_D v_y$, let $S_{xy} = \{ v_z \in N_G^f(v_x) \mid v_y <_D v_z \} \cup \{v_y\}$. We call $S_{xy}$ the weak support set of the non-edge $(v_x, v_y)$. Define $T_{xy} = S_{xy} \cup \{v_x\}$. We call $T_{xy}$ the strong support set of the non-edge $(v_x, v_y)$. Let $\mathcal{C}$ be a coloring (need not be proper) of $V(G)$. We say $S_{xy}$ is favorably colored in $\mathcal{C}$, if $\mathcal{C}(v_y) \neq \mathcal{C}(v_w), \forall v_w \in S_{xy} \setminus \{v_y\}$. We say $T_{xy}$ is favorably colored in $\mathcal{C}$, if $\mathcal{C}(v_y) \neq \mathcal{C}(v_w), \forall v_w \in T_{xy} \setminus \{v_y\}$.
3 Cube Representation and Coloring

Lemma 3. Let $G$ be a $k$-degenerate graph. Let $\chi = \{\chi_1, \ldots, \chi_a\}$ be a set of colors and let $C = \{C_1, \ldots, C_a\}$ be a family of colorings (need not be proper) of $V(G)$, where each $C_i$ uses colors from the set $\chi$. If the strong support set $T_{xy}$ of every non-edge $(v_x, v_y) \notin E(G)$, $v_x < v_y$, is favorably colored in some $C_i$, where $i \in [b]$, then $\text{cub}(G) \leq ab$.

Proof. We prove this by constructing $ab$ unit interval graphs $I_{i,j}$ on the vertex set $V(G)$, where $i \in [a]$ and $j \in [b]$, such that $G = \bigcap_{i=1}^{a} \bigcap_{j=1}^{b} I_{i,j}$. Then the statement will follow from Lemma 2. Let $f_{i,j}$ denote a unit interval representation of $I_{i,j}$. Let us partition the vertices of $I_{i,j}$ into two parts, namely $A^{ij}$ and $B^{ij}$, where $A^{ij} = \{v \in V(G) \mid C_i(v) = \chi_j\}$ and $B^{ij} = V(G) \setminus A^{ij}$. For every $i \in [a]$ and $j \in [b]$, a unit interval representation $f_{i,j}$ of $I_{i,j}$ is constructed from the coloring $C_i$ in the following way. For every $v_y \in V(G)$,

$$
\begin{align*}
\text{If } v_y & \in A^{ij}, \\
\text{then } f_{i,j}(v_y) & = [y + n, y + 2n] \\
\text{else } f_{i,j}(v_y) & = [g^{ij}_{\max}(v_y), g^{ij}_{\max}(v_y) + n],
\end{align*}
$$

where $g^{ij}_{\max}(v_y) = \max\{g \mid (v_y, v_g) \in E(G), v_g \in A^{ij}\} \cup \{0\}$.

Since the length of $f_{i,j}(v_y)$ is $n$, for every $v_y \in V(G)$, $I_{i,j}$ is a unit interval graph. It is easy to see that, $\forall v_x, v_y \in A^{ij}$, $2n \in f_{i,j}(v_x) \cap f_{i,j}(v_y)$ and therefore $A^{ij}$ forms a clique in $I_{i,j}$. Since $n \in f_{i,j}(v_x) \cap f_{i,j}(v_y)$, $\forall v_x, v_y \in B^{ij}$, $B^{ij}$ too forms a clique in $I_{i,j}$. For every $(v_x, v_y) \in E(G)$, with $v_x \in A^{ij}$ and $v_y \in B^{ij}$, we have $l(f_{i,j}(v_y)) = g^{ij}_{\max}(v_y) \leq n \leq l(f_{i,j}(v_x)) = n + x \leq n + g^{ij}_{\max}(v_y)$, where the last inequality is inferred from the fact that $(v_x, v_y) \in E(G)$ and $v_x \in A^{ij}$. But $n + g^{ij}_{\max}(v_y) = r(f_{i,j}(v_y))$. Therefore, we get $l(f_{i,j}(v_y)) \leq l(f_{i,j}(v_x)) \leq r(f_{i,j}(v_x))$ and hence $(v_x, v_y) \in E(I_{i,j})$. Hence $I_{i,j}$ is a supergraph of $G$.

Let $v_x <_D v_y$ and $(v_x, v_y) \notin E(G)$. We now have to show that there exists some unit interval graph $I_{i,j}$ such that $(v_x, v_y) \notin E(I_{i,j})$. We know that, by assumption, there exists a coloring, say $C_i$ (where $i \in [a]$), such that the strong support set $T_{xy}$ is favorably colored in $C_i$. Let $\chi_j = C_i(v_y)$. Let $g = g^{ij}_{\max}(v_x)$. We claim that $g < y$. Assume, for contradiction, that $g > y$. Then $g \neq 0$ and $v_y \in A^{ij}$. Since $y > x$, we get $g > x$. Therefore, $v_y \in N^{ij}_{C}(v_x)$ and $g > y$. This implies that $v_y \in T_{xy}$. Since $T_{xy}$ is favorably colored in $C_i$, $C_i(v_y) \neq \chi_j$. This contradicts the fact that $v_y \in A^{ij}$. Thus we prove the claim. Therefore, $r(f_{i,j}(v_x)) = n + g < n + y = l(f_{i,j}(v_y))$ and hence $(v_x, v_y) \notin E(I_{i,j})$. We infer that $G = \bigcap_{i=1}^{a} \bigcap_{j=1}^{b} I_{i,j}$.

Remark 1. Note that $\forall v \in V(G), i \in [a], j \in [b]$, either $f_{i,j}(v) \cap [n, n] \neq \emptyset$ or $f_{i,j}(v) \cap [2n, 2n] \neq \emptyset$ or both.

\[ \square \]
4 Cubicity and Degeneracy

4.1 An Upper Bound - Probabilistic Approach

**Theorem 1.** For every $k$-degenerate graph $G$, $\text{cub}(G) \leq (k + 2) \cdot \lceil 2e \log n \rceil$

**Proof.** Let $\chi = \{\chi_1, \ldots, \chi_{k+2}\}$ be a set of $k + 2$ colors. Generate a random coloring $C_1$ (need not be a proper coloring) of vertices of $G$ in the following way: For each vertex $v_x \in V(G)$, pick a color $\chi_j$, where $j \in [k + 2]$, uniformly at random from $\chi$ and set $C_1(v_x) = \chi_j$. In a similar way, independently generate random colorings $C_2, \ldots, C_b$, where $b = \lfloor 2e \log n \rfloor$.

For every $(v_x, v_y) \notin E(G)$ and $v_x <_D v_y$, since $G$ is $k$-degenerate we have $|T_{xy}| = t \leq k + 2$. $Pr[T_{xy} \text{ is favorably colored in } C_i] = \frac{(k+2)^{k+1}}{(k+2)!} = \binom{k+2}{k+1}^{-1} k+1 \leq \frac{1}{e} \cdot e^{-(k+2)^{k+1}}$. Therefore, $Pr[T_{xy} \text{ is not favorably colored in } C_i] \leq 1 - \frac{k+1}{k+2} k+1 \leq e^{-(k+2)^{k+1}}$. Now taking $b = \lfloor 2e \log n \rfloor$,

$$Pr\left[\bigcup_{x,y: (v_x, v_y) \notin E(G)} \bigcap_{i=1}^b (T_{xy} \text{ is not favorably colored in } C_i)\right] \leq n^2 e^{-b(k+2)^{k+1}} < 1.$$

Hence, $Pr[C_1, \ldots, C_b]$ satisfy the condition of Lemma 3 $> 0$. Therefore, there exists a coloring $C_1, \ldots, C_b$, with $b = \lfloor 2e \log n \rfloor$, of $V(G)$ using colors from the set $\{\chi_1, \ldots, \chi_{k+2}\}$ such that the condition of Lemma 3 is satisfied. Hence by Lemma 3, $\text{cub}(G) \leq (k + 2) \cdot \lfloor 2e \log n \rfloor$.

**Corollary 1.** Let $G$ be a $k$-degenerate graph with $n$ vertices and $G'$ a graph constructed from $G$ with $V(G') = V(G) \cup V'$, $E(G') = E(G) \cup \{(u, v) \mid u \in V', v \in V(G')\}$, and $V(G) \cap V' = \emptyset$. Then, $\text{cub}(G') \leq (k + 2) \cdot \lfloor 2e \log n \rfloor$.

**Proof.** From Theorem 1, we know that there exist $(k + 2) \cdot \lfloor 2e \log n \rfloor$ unit interval graphs $I_{i,j}$, where $i \in [k + 2]$, $j \in \lfloor 2e \log n \rfloor$, such that $G = \bigcap_i \bigcap_j I_{i,j}$. Let $f_{i,j}$ be the unit interval representation of each $I_{i,j}$ as per the construction in Lemma 3. We now construct $(k + 2) \cdot \lfloor 2e \log n \rfloor$ unit interval graphs $I'_{i,j}$, where $i \in [k + 2]$, $j \in \lfloor 2e \log n \rfloor$, such that $G' = \bigcap_i \bigcap_j I'_{i,j}$. Let $f'_{i,j}$ be a unit interval representation of $I'_{i,j}$. Then for each $v \in V(G')$,

$$f'_{i,j}(v) = f_{i,j}(v), \text{ if } v \notin V'$$

$$f'_{i,j}(v) = [n, 2n], \text{ if } v \in V'$$

From Remark 1 in Lemma 3, every $v \in V'$ is adjacent with every other vertex in each $I'_{i,j}$ since $f'_{i,j}(v) = [n, 2n], \forall v \in V'$. 

Tightness of Theorem 1 Recall that, a realizer of a partially ordered set (poset) $\mathcal{P}$ is a set of linear extensions of $\mathcal{P}$, say $\mathcal{R}$, which satisfy the following condition: for any two distinct elements $x$ and $y$, $x \leq y$ in $\mathcal{P}$ if and only if $x \leq y$ in $L$, $\forall L \in \mathcal{R}$. The poset dimension of $\mathcal{P}$, denoted by $\text{dim}(\mathcal{P})$, is the minimum integer $k$ such that there exists a realizer of $\mathcal{P}$ of cardinality $k$. Let $G_\mathcal{P}$ denote the underlying comparability graph of $\mathcal{P}$. Then by Theorem 1 in [1], $\text{box}(G_\mathcal{P}) \geq \frac{\text{dim}(\mathcal{P})}{2}$.

Let $\mathbb{P}(n,p)$ be the probability space of height-2 posets with $n$ minimal elements forming set $A$ and $n$ maximal elements forming set $B$, where for any $a \in A$ and $b \in B$, $Pr[a < b] = p$. Erdős, Kierstead, and Trotter in [11] proved that when $p = \frac{1}{\log n}$, for almost all posets $\mathcal{P} \in \mathbb{P}(n,p)$, $\Delta(G_\mathcal{P}) < \frac{\text{dim}(\mathcal{P})}{\log n}$ and $\text{dim}(\mathcal{P}) > 2\delta_2 n$, where $\delta_1$ and $\delta_2$ are some positive constants. Then by Theorem 1 in [1], $\text{cub}(G_\mathcal{P}) \geq \text{box}(G_\mathcal{P}) \geq \frac{\text{dim}(\mathcal{P})}{2} \geq \frac{2n}{2}$.

We know that $G_\mathcal{P}$ is $\Delta(G_\mathcal{P})$-degenerate. By Theorem 1, $\text{cub}(G_\mathcal{P}) \leq (\Delta(G_\mathcal{P}) + 2) \cdot [2e \ln n] \leq (\frac{2\Delta n}{\log n} + 2) \cdot [2e \ln n] \leq cn$, where $c$ is some constant. Hence the upper bound for cubicity given in Theorem 1 is tight.

4.2 Deterministic Algorithm

CONSTRUCT\_CUB\_REP($G$) is a deterministic algorithm which takes a simple, finite $k$-degenerate graph $G$ as input and outputs a cube representation in $8k\alpha$ dimensional space i.e., $8k\alpha$ unit interval graphs $I_{1,1}, \ldots, I_{1,8k}, \ldots, I_{\alpha,1}, \ldots, I_{\alpha,8k}$ such that $G = \bigcap_{i=1}^{n} \bigcap_{j=1}^{8k} I_{i,j}$. In order to achieve this, CONSTRUCT\_CUB\_REP ($G$) invokes the procedure CONSTRUCT\_COLORING i.e., Algorithm 4.2 (for a detailed version of this procedure, see Algorithm 4.4), $\alpha$ times and thereby generates $\alpha$ colorings $\mathcal{C}_1, \ldots, \mathcal{C}_\alpha$, where each coloring uses colors from the set $\{\chi_1, \ldots, \chi_{8k}\}$. Then from each coloring $\mathcal{C}_i$, it constructs $8k$ unit interval graphs $I_{i,1}, \ldots, I_{i,8k}$ using the construction described in Lemma 3, which is implemented in procedure CONSTRUCT\_UNIT\_INTERVAL\_GRAPHS.

Note that in order for $G$ to be equal to $\bigcap_{i=1}^{n} \bigcap_{j=1}^{8k} I_{i,j}$, Lemma 3 requires that the colorings $\mathcal{C}_1, \ldots, \mathcal{C}_\alpha$ satisfy the following property: for every $(v_x, v_y) \notin E(G)$, where $v_x <_\mathcal{P} v_y$, there exists an $i \in [\alpha]$ such that the strong support set $T_{xy}$ of this non-edge is favorably colored in $\mathcal{C}_i$. The colorings $\mathcal{C}_1, \ldots, \mathcal{C}_\alpha$ are generated one by one keeping this objective in mind. At the stage when we have just generated the $(i-1)$-th coloring $\mathcal{C}_{i-1}$, if a non-edge $(v_x, v_y)$ is such that its strong support set $T_{xy}$ is already favorably colored in some $\mathcal{C}_j$, where $j < i$, then we say that the non-edge $(v_x, v_y)$ is already DONE. Naturally at each stage we have to keep track of the non-edges that are not yet DONE. In order to do this, we introduce two data structures $BNN_i$ and $FNN_i$, for all $i \in [\alpha]$\footnote{$BNN$ - Backward Non-Neighbor, $FNN$ - Forward Non-Neighbor}. For each
Algorithm 4.1 CONSTRUCT\_CUB\_REP(G)

for \( y = n \) to 1 do
1. Initialize \( BNN_{v_y} \) \( \leftarrow \{ v_x \in V(G) \mid v_x \text{ is a backward non-neighbor of } v_y, \text{ and } (v_x, v_y) \} \).
2. Initialize \( FNN_{v_y} \) \( \leftarrow \{ v_z \in V(G) \mid v_y \text{ is a forward non-neighbor of } v_z, \text{ and } (v_y, v_z) \} \).
end for
3. SET \( \text{FLAG} \leftarrow \text{TRUE} \).
4. SET \( i \leftarrow 0 \).
while \( \text{FLAG} = \text{TRUE} \) do
5. \( i++ \).
6. \( C_i = \text{CONSTRUCT\_COLORING}(i) \).
   \( \text{for } y = 1 \) to \( n \) do
7. \( BNN_{v_y} \) \( \leftarrow BNN_{v_y} \setminus W(v_y, C_i) \).
8. \( FNN_{v_y} \) \( \leftarrow FNN_{v_y} \setminus Y(v_y, C_i) \).
   end for
9. If \( FNN_{v_y} = \emptyset, \forall v_y \in V(G) \), then \( \text{FLAG} = \text{FALSE} \).
end while
10. SET \( \alpha \leftarrow i \).
11. CONSTRUCT\_UNIT\_INTERVAL\_GRAPHS()

\( v_y \in V(G) \),
\( BNN_{v_y} = \{ v_x \in V(G) \mid v_x \text{ is a backward non-neighbor of } v_y, \text{ and } (v_x, v_y) \} \)
\( FNN_{v_y} = \{ v_z \in V(G) \mid v_z \text{ is a forward non-neighbor of } v_y, \text{ and } (v_y, v_z) \} \)

It is easy to see that, \( \bigcup_{v_y \in V(G)} BNN_{v_y} = \bigcup_{v_y \in V(G)} FNN_{v_y} \) and therefore,
\( \left( \bigcup_{v_y \in V(G)} BNN_{v_y} = \emptyset \right) \iff \left( \bigcup_{v_y \in V(G)} FNN_{v_y} = \emptyset \right) \). In Theorem 2, we show that if we select \( \alpha \) to be at least \( \lceil 2.42 \log n \rceil + 1 \), then \( FNN_{\alpha+1}[v_y] = \emptyset, \forall v_y \in V(G) \). This clearly would mean that all non-edges are DONE with respect to \( C_1, \ldots, C_\alpha \). In other words, the condition of Lemma 3 will be satisfied for \( C_1, \ldots, C_\alpha \).

The only thing that remains to be discussed now is how our coloring strategy (i.e. the procedure CONSTRUCT\_COLORING) achieves the above objective, namely \( BNN_{\alpha+1}[v_y] = \emptyset \) and \( FNN_{\alpha+1}[v_y] = \emptyset, \forall v_y \in V(G) \), if \( \alpha \geq \lceil 2.42 \log n \rceil + 1 \). To start with \( BNN_{v_y} \) (respectively \( FNN_{v_y} \)) contains all the backward (respectively forward) non-neighbors of \( v_y \). The procedure CONSTRUCT\_COLORING \( (i) \) generates the \( i \)-th coloring \( C_i \) as follows. It colors vertices in the reverse degeneracy order starting from vertex \( v_n \). The partial coloring at the stage when we have colored the vertices \( v_n \) to \( v_z \) is denoted by \( C_i^{v_z} \). Note that \( C_i^{v_z} = C_i \). Consider the stage at which the algorithm has already colored the vertices from \( v_n \) up to \( v_{y+1} \) and is about to color \( v_y \). That is, we have the partial coloring \( C_i^{v_y+1} \) and are about to extend it to the partial coloring \( C_i^{v_y} \) by assigning one of
Algorithm 4.2 CONSTRUCT\_COLORING\(i\) /* abridged */

/*For a detailed version of this procedure, see Algorithm 4.4.
All data structures are assumed to be global.

Notational Note:
Let \(C_i^{v_y}\) denote the partial coloring at the stage when we have colored the vertices \(v_n\) to \(v_x\). Let \(C_i^{v_y=x_c}\) denote the partial coloring that results if we extend \(C_i^{v_y+1}\) by assigning color \(\chi_c\) to \(v_y\).*

for \(y = n\) to 1 do
for each \(\chi_c \in \{\chi_1, \ldots, \chi_{8k}\}\) do
  1. Compute \(|X(v_y, C_i^{v_y=x_c})|\), \(|Y(v_y, C_i^{v_y=x_c})|\), and \(|Z(v_y, C_i^{v_y=x_c})|\) as per equations (2),(3), and (4) respectively.
  if \(|X(v_y, C_i^{v_y=x_c})| \geq \frac{1}{4}|BNN[v_y]|\) and \(|Y(v_y, C_i^{v_y=x_c})| \geq \frac{1}{4}|Z(v_y, C_i^{v_y=x_c})|\)
    then
      2. SET \(C_i^{v_y} \leftarrow C_i^{v_y=x_c}\) (i.e. SET \(C_i(v_y) \leftarrow \chi_c\)).
      3. SET \(Y(v_y, C_i^{v_y}) \leftarrow Y(v_y, C_i^{v_y=x_c})\)
      4. BREAK.
  end if
end for
end for
for \(y = 1\) to \(n\) do
  5. Compute \(W(v_y, C_i)\) as per equation (1)
  6. SET \(Y(v_y, C_i) \leftarrow Y(v_y, C_i^{v_y})\)
end for
7. Return \(C_i\).

The 8k possible colors to vertex \(v_y\). Let \(C_i^{v_y=x_c}\) denote the partial coloring that results if we extend \(C_i^{v_y+1}\) by assigning color \(\chi_c\) to \(v_y\). The coloring \(C_i\) and the partial colorings \(C_i^{v_y=x_c}\), \(\forall v_z \in V(G)\) and \(C_i^{v_y=x_c}\), \(\forall v_z \in V(G), \chi_c \in \{\chi_1, \ldots, \chi_{8k}\}\), will be generically called the colorings associated with the \(i\)-th stage (i.e. the \(i\)-th invocation of CONSTRUCT\_COLORING).

With respect to colorings \(C_1, \ldots, C_{i-1}\) and some coloring \(C_i'\) associated with the \(i\)-th stage, we define the following sets:

- \(W(v_w, C_i') = \{v_x \in BNN[v_w] \mid \text{the strong support set } T_{xw}\ \text{of non-edge (1)}\}
- \(X(v_w, C_i') = \{v_x \in BNN[v_w] \mid \text{the weak support set } S_{xw}\ \text{of non-edge (2)}\}
- \(Y(v_w, C_i') = \{v_x \in FNN[v_w] \mid \text{the strong support set } T_{wz}\ \text{of non-edge (3)}\}
- \(Z(v_w, C_i') = \{v_x \in FNN[v_w] \mid \text{the weak support set } S_{wz}\ \text{of non-edge (4)}\}

Naturally, we want to give a color \(\chi_c\) to \(v_y\) such that a large number of (not yet DONE) non-edges incident on \(v_y\) get DONE. With respect to the colorings \(C_1, \ldots, C_{i-1}\) and the partial coloring \(C_i^{v_y=x_c}\), we define the status of a non-edge
Algorithm 4.3 CONSTRUCT_UNIT_INTERVAL_GRAPHS()
/*All data structures are assumed to be global. */
1. INITIALIZE \( l(f_{i,j}(v_y)) \leftarrow 0, r(f_{i,j}(v_y)) \leftarrow n, \forall y \in [n], i \in \alpha, j \in [8k] \)
for \( i = 1 \) to \( \alpha \) do
   for \( y = n \) to \( 1 \) do
      2. SET \( j \leftarrow c \), such that \( C_i(v_y) = \chi_c \)
      3. SET \( l(f_{i,j}(v_y)) \leftarrow y + n \)
      4. SET \( r(f_{i,j}(v_y)) \leftarrow y + 2n \)
      for each \( v \in N_G(v_y) \) do
         if \( (C_i(v) \neq j) \cap (l(f_{i,j}(v)) = 0) \) then
            5. SET \( l(f_{i,j}(v)) \leftarrow y \)
            6. SET \( r(f_{i,j}(v)) \leftarrow y + n \)
      end if
   end for
end for
7. Output \( f_{i,j}(v_y), \forall y \in [n], i \in \alpha, j \in [8k] \)

incident on \( v_y \) as follows: A non-edge \((v_y, v_x) \in FNN_i[v_y]\) is DONE\(^2\) if \( T_{y,z} \) is favorably colored in \( C_i^{v_y = \chi_c} \) and is NOT-DONE if \( T_{y,z} \) is not favorably colored in \( C_i^{v_y = \chi_c} \). A non-edge \((v_x, v_y) \in BNN_i[v_y]\) is HOPELESS\(^3\) if \( S_{xy} \) (which happens to be a proper subset of \( T_{y,z} \)) is not favorably colored in \( C_i^{v_y = \chi_c} \) and is HOPEFUL if \( S_{xy} \) is favorably colored in \( C_i^{v_y = \chi_c} \). So when we decide a color for \( v_y \), our intention is to make a large fraction of the HOPEFUL non-edges of \( FNN_i[v_y] \) (i.e. the set \( Z(v_y, C_i^{v_y = \chi_c}) \)), DONE and to make a large fraction of \( BNN_i[v_y] \), HOPEFUL. More formally, we want the algorithm to assign a color \( \chi_c \) to \( v_y \) such that the following two conditions are satisfied:

(i) \( |X(v_y, C_i^{v_y = \chi_c})| \geq \frac{3}{4}|BNN_i[v_y]| \), and
(ii) \( |Y(v_y, C_i^{v_y = \chi_c})| \geq \frac{3}{4}|Z(v_y, C_i^{v_y = \chi_c})| \).

The obvious question then is whether such a color \( \chi_c \) always exists, for each \( v_y \in V(G) \). Lemma 4 answers this question in the affirmative. It follows that, the number of non-edges that are not yet DONE with respect to colorings \( C_1, \ldots, C_i \) is at most a constant fraction of the number of non-edges that were not DONE with respect to colorings \( C_1, \ldots, C_{i-1} \). This is formally proved in Lemma 5. That \( BNN_{\alpha+1}[v_y] = \emptyset \) and \( FNN_{\alpha+1}[v_y] = \emptyset, \forall v_y \in V(G) \), is a consequence of this and is formally proved in Theorem 2.

Lemma 4. For every \( i \in [\alpha], v_y \in V(G) \), (i) \( |X(v_y, C_i)| \geq \frac{3}{4}|BNN_i[v_y]| \), and
(ii) \( |Y(v_y, C_i)| \geq \frac{3}{4}|Z(v_y, C_i)| \).

\(^2\) Recall that we had defined earlier that a non-edge \((v_x, v_y)\) is DONE with respect to a list of colorings \( C_1, \ldots, C_{i-1} \) if \( T_{x,y} \) was favorably colored in some \( C_j \), where \( j < i \).
Here we extend this notion, by allowing the partial coloring \( C_i^{v_y = \chi_c} \) also in the list.

\(^3\) A HOPELESS non-edge \((v_x, v_y)\) will not be DONE with respect to \( C_1, \ldots, C_i \) if we set \( C_i(v_y) = \chi_c \), irrespective of the color given to \( v_y, \ldots, v_1 \).
Proof. The statement of the lemma is obvious if the BREAK statement in Step 4 of CONSTRUCT\_COLORING(i) (abridged version) is executed, for every \( i \in [\alpha] \) and \( v_y \in V(G) \). In order to prove that the BREAK statement will be executed, it is sufficient to show that there exists a color \( \chi_c \in \{ \chi_1, \ldots, \chi_{8k} \} \) such that \(|X(v_y, C_i^{\chi_c})| \geq \frac{3}{4}|BNN_i[v_y]|\) and \(|Y(v_y, C_i^{\chi_c})| \geq \frac{3}{4}|Z(v_y, C_i^{\chi_c})|\). Since the vertices in \( Z(v_y, C_i^{\chi_c}) \) or \( Z(v_y, C_i) \) do not depend on the colors given to \( v_1, \ldots, v_y \), we have \( Z(v_y, C_i^{\chi_c}) = Z(v_y, C_i) \). Hence, \( Z(v_y, C_i^{\chi_c}) \) and \( Z(v_y, C_i) \) can be used interchangeably.

Let \( A = BNN_i[v_y] \times Z(v_y, C_i) \). Let \( < v_x, v_z > \in A \) be an element of \( A \). We say a color \( \chi_c \) is good for \( < v_x, v_z > \), if \( v_x \in X(v_y, C_i^{\chi_c}) \) and \( v_z \in Y(v_y, C_i^{\chi_c}) \). In other words, \( \chi_c \) is good for \( < v_x, v_z > \), if both \( S_{xy} \) and \( T_{yz} \) are favorably colored in \( C_i^{\chi_c} \). \( S_{xy} \) is favorably colored in \( C_i^{\chi_c} \), if \( \chi_c \notin P \), where \( P = \{ C_i^{\chi_c}(v_w) \mid v_w \in N_G(v_x), v_y < v_w \} \). Since \(|N_G(v_x)| \leq k, |P| \leq k \). Therefore, there are at least \( 8k - k = 7k \) possible values that \( \chi_c \) can take such that \( S_{xy} \) is favorably colored in \( C_i^{\chi_c} \). For \( T_{yz} \) also to be favorably colored in \( C_i^{\chi_c} \), the only thing required is that \( \chi_c \neq C_i^{\chi_c}(v_z) \). Since \( v_z \in Z(v_y, C_i) \) and therefore \( S_{xy} \) is already favorably colored. This implies that there are at least \( 7k - 1 \) possible values that \( \chi_c \) can take such that both \( S_{xy} \) and \( T_{yz} \) are favorably colored in \( C_i^{\chi_c} \). In other words, there are at least \( 7k - 1 \) good colors for \( < v_x, v_z > \). Thus for each element in \( A \), there are at least \( 7k - 1 \) colors \( \chi_c \) good for it. For each color \( \chi_j \in \{ \chi_1, \ldots, \chi_{8k} \} \), let \( S_j = \{ < v_x, v_z > \in A \mid \chi_j \) is good for \( < v_x, v_z > \} = X(v_y, C_i^{\chi_c}) \times Y(v_y, C_i^{\chi_c}) \). Since there are at least \( (7k-1) \) colors \( \chi_j \) for each element in \( A \), \(|S_j| \geq (7k-1)|A|\). Then by pigeonhole principle, there exists a \( c \in [8k] \) such that \(|S_c| = |X(v_y, C_i^{\chi_c})| \times |Y(v_y, C_i^{\chi_c})| \geq \frac{(7k-1)|A|}{8k} = \frac{7k-1}{8k}|BNN_i[v_y]| \times |Z(v_y, C_i)| \geq \frac{3}{4}|BNN_i[v_y]| \times |Z(v_y, C_i)| \). In other words, \(|X(v_y, C_i^{\chi_c})| \geq \frac{3}{4}|BNN_i[v_y]| \) and \(|Y(v_y, C_i^{\chi_c})| \geq \frac{3}{4}|Z(v_y, C_i^{\chi_c})|\).

\( \square \)

Lemma 5. Let \( \overline{m}_i = \Sigma_{y \in [n]} FNN_i[v_y] \). Then \( \overline{m}_{i+1} \leq \frac{7}{16} \overline{m}_i \).

Proof. From Step 8 of CONSTRUCT\_CUB\_REP(G), we have \(|FNN_i[v_y]| = |FNN_i[v_y]| - |Y(v_y, C_i)| \leq |FNN_i[v_y]| - \frac{3}{4}|Z(v_y, C_i)| \) (using Lemma 4). Taking summation over all \( y \in [n] \), we get \( \overline{m}_{i+1} \leq \overline{m}_i - \frac{3}{4} \Sigma_{y \in [n]} |Z(v_y, C_i)| = \overline{m}_i - \frac{3}{4} \Sigma_{y \in [n]} |X(v_y, C_i)| \). The last equality comes from the fact that both \( \Sigma_{y \in [n]} |X(v_y, C_i)| \) and \( \Sigma_{y \in [n]} |Z(v_y, C_i)| \) represent the number of HOPEFUL non-edges in \( G \) with respect to colorings \( C_1, \ldots, C_\alpha \). From Lemma 4, we have \(|X(v_y, C_i)| \geq \frac{3}{4}|BNN_i[v_y]| \). Therefore, \( \overline{m}_{i+1} \leq \overline{m}_i - \frac{3}{4} \Sigma_{y \in [n]} |BNN_i[v_y]| \). Since \( \Sigma_{y \in [n]} |BNN_i[v_y]| = \Sigma_{y \in [n]} |FNN_i[v_y]| \), we get \( \overline{m}_{i+1} \leq \overline{m}_i - \frac{3}{4} \Sigma_{y \in [n]} |FNN_i[v_y]| = \overline{m}_i - \frac{3}{16} \overline{m}_i = \frac{7}{16} \overline{m}_i \). \( \square \)

Theorem 2. Let \( G \) be a \( k \)-degenerate graph. Algorithm CONSTRUCT\_CUB\_REP(G) constructs a valid \( 8k([2.42 \log n] + 1) \) dimensional cube representation for \( G \).

Proof. The algorithm constructs \( \alpha \) colorings \( C_1, C_2, \ldots, C_\alpha \) of \( V(G) \), where each coloring uses colors from the set \( \{ \chi_1, \ldots, \chi_{8k} \} \). From Lemma 5, we have \( \overline{m}_{i+1} \leq \frac{7}{16} \overline{m}_i \)
Running Time Analysis

Lemma 6. The procedure CONSTRUCT_COLORING(i) can be implemented to run in $O(km + kn)$ time, where $m = \sum_{y \in [n]} FNN_i[v_y]$.

Proof. A detailed description of the procedure is given in Algorithm 4.4. To implement the procedure efficiently, we make use of an $(n \times 8k)$ 0-1 matrix, hereafter called $\text{FNC}$ (Forward Neighbor Color), and two $(n \times n)$ 0-1 matrices named $\text{HOPE\_MATRIX}$ and $\text{DONE\_MATRIX}$ respectively. At the beginning of the procedure each of these matrices have all entries set to 0. As the procedure progresses, we change some of the entries to 1 in such a way that, $\forall w \in [n], j \in [8k], \text{FNC}[w][j] = 1 \iff \exists v_z \in N_G^i(w) \text{ such that } v_z \text{ is already colored by the procedure with color } \chi_j$. $\forall w, z \in [n], v_w, v_z \in BN_i[v_z], \text{HOPE\_MATRIX}[w][z] = 1 \iff S_{wz} \text{ is already favorably colored by the procedure}$. $\forall w, z \in [n], v_w, v_z \in BN_i[v_z], \text{DONE\_MATRIX}[w][z] = 1 \iff T_{wz} \text{ is already favorably colored by the procedure}$.

In order for the above matrices to satisfy their respective properties, the only thing that needs to be done is to update these matrices at each stage of the procedure. Consider the stage at which the procedure is extending partial coloring $C^i_{\chi_{v}}$ to $C^i_{\chi_{v+1}}$ by assigning color $\chi_{v+1}$ to $v$. At this stage, the matrices $\text{FNC}$, $\text{HOPE\_MATRIX}$ and $\text{DONE\_MATRIX}$ are updated as described in steps 11(a), 12(a) and 13(a) respectively. Note that this can be done in $O(|BN_i[v_y]| + |FNN_i[v_y]| + \Sigma^{v_y}_{v_z}N^{v_y}_{G}(v_y))$ time. Steps 4(a)-(b), 5(a)-(b) and 6(a)-(b) compute $X(v_y, C^i_{\chi_{v}}), Y(v_y, C^i_{\chi_{v}})$ and $Z(v_y, C^i_{\chi_{v}})$ respectively in $O(|BN_i[v_y]| + |FNN_i[v_y]|)$ time. Computing $W(v_y, C_i)$ is done in step 15(a)-(b) in $O(|BN_i[v_y]|)$ time.

Since steps 4 to 14, in the worst case, are run for each $v_y \in V(G), \chi_{v} \in \{\chi_1, \ldots, \chi_{8k}\}$, the procedure runs in $O(k(\Sigma_{y \in [n]}(|BN_i[v_y]| + |FNN_i[v_y]| + \Sigma^{v_y}_{v_z}N^{v_y}_{G}(v_y))))$ time. We know that $\Sigma_{y \in [n]}(|BN_i[v_y]| + |FNN_i[v_y]|) = 2m$ and $\Sigma_{y \in [n]}\Sigma^{v_y}_{v_z}N^{v_y}_{G}(v_y) = m \leq kn$. Hence the Lemma.

Theorem 3. CONSTRUCT\_CUB\_REP(G) runs in $O(n^2k)$ time.

Proof. The algorithm invokes the function CONSTRUCT\_COLORING(i) $\alpha$ times to construct colorings $C_1, \ldots, C_\alpha$ of $V(G)$. By Lemma 6, to construct these $\alpha$ colorings it requires $O(\Sigma^{\alpha}_{i=1}(m_i + \alpha kn))$ time. From Lemma 5, we get that $\Sigma^{\alpha}_{i=1}(m_i) = O(m)$. Since $\alpha = ([2.42 \log n] + 1)$, the running time of
Algorithm 4.4 CONSTRUCT\_COLORING(i) /* detailed */
/*All data structures are assumed to be global.
Notational Note:
Let $C_i^{v_{n}}$ denote the partial coloring at the stage when we have colored the vertices $v_n$ to $v_z$. Let $C_i^{v_{n-1}}$ denote the partial coloring that results if we extend $C_i^{v_{n-1}}$ by assigning color $\chi_c$ to $v_z$. */
1. Initialize $FNC[w][j] \leftarrow 0, \forall w \in [n], j \in [8k]$
2. Initialize $HOPE\_MATRIX[w][z] \leftarrow 0, \forall w, z \in [n]$
3. Initialize $DONE\_MATRIX[w][z] \leftarrow 0, \forall w, z \in [n]$
for $y = n$ to 1 do
for each $\chi_c \in \{\chi_1, \ldots, \chi_{8k}\}$ do
4. Compute $X(v_y, C_i^{v_{n-1}})$ /*as described in steps (a) and (b) below */
   (a) Initialize $X(v_y, C_i^{v_{n-1}}) \leftarrow \emptyset$
   (b) $\forall v_z \in BNN[v_y]$, if $FNC[x][c] = 0$, then
       SET $X(v_y, C_i^{v_{n-1}}) \leftarrow X(v_y, C_i^{v_{n-1}}) \cup \{v_z\}$
5. Compute $Y(v_y, C_i^{v_{n-1}})$ /*as described in steps (a) and (b) below */
   (a) Initialize $Y(v_y, C_i^{v_{n-1}}) \leftarrow \emptyset$
   (b) $\forall v_z \in FNN[v_y]$, if $HOPE\_MATRIX[y][z] = 1$ and
       $(C_i^{v_{n-1}}(v) \neq \chi_c)$, then SET $Y(v_y, C_i^{v_{n-1}}) \leftarrow Y(v_y, C_i^{v_{n-1}}) \cup \{v_z\}$
6. Compute $Z(v_y, C_i^{v_{n-1}})$ /*as described in steps (a) and (b) below */
   (a) Initialize $Z(v_y, C_i^{v_{n-1}}) \leftarrow \emptyset$
   (b) $\forall v_z \in FNN[v_y]$, if $HOPE\_MATRIX[y][z] = 1$, then
       SET $Z(v_y, C_i^{v_{n-1}}) \leftarrow Z(v_y, C_i^{v_{n-1}}) \cup \{v_z\}$
if $|X(v_y, C_i^{v_{n-1}})| \geq \frac{1}{4}|BNN[v_y]|$ and $|Y(v_y, C_i^{v_{n-1}})| \geq \frac{1}{4}|Z(v_y, C_i^{v_{n-1}})|$
then
7. SET $C_i^{v_{n-1}} \leftarrow C_i^{v_{n-1}}$ (i.e. SET $C_i(v_y) \leftarrow \chi_c$).
8. SET $X(v_y, C_i^{v_{n-1}}) \leftarrow X(v_y, C_i^{v_{n-1}})$
9. SET $Y(v_y, C_i^{v_{n-1}}) \leftarrow Y(v_y, C_i^{v_{n-1}})$
10. SET $Z(v_y, C_i^{v_{n-1}}) \leftarrow Z(v_y, C_i^{v_{n-1}})$
11. Update $FNC$ matrix. /*as described in step (a) below */
   (a) $\forall v_z \in N^2_2[v_y]$, SET $FNC[x][c] \leftarrow 1$
12. Update $HOPE\_MATRIX$ /*as described in step (a) below */
   (a) $\forall v_z \in X(v_y, C_i^{v_{n-1}})$, SET $HOPE\_MATRIX[x][y] \leftarrow 1$
13. Update $DONE\_MATRIX$ /*as described in step (a) below */
   (a) $\forall v_z \in Y(v_y, C_i^{v_{n-1}})$, SET $DONE\_MATRIX[y][z] \leftarrow 1$
14. BREAK.
end if
end for
for $y = 1$ to $n$ do
15. Compute $W(v_y, C_i)$ /*as described in steps (a) and (b) below */
   (a) Initialize $W(v_y, C_i) \leftarrow \emptyset$
   (b) $\forall v_z \in BNN[v_y]$, if $DONE\_MATRIX[y][z] = 1$, then
       SET $W(v_y, C_i) \leftarrow W(v_y, C_i) \cup \{v_z\}$
16. SET $Y(v_y, C_i) \leftarrow Y(v_y, C_i)$
end for
17. Return $C_i$.
the while loop in CONSTRUCT\_CUB\_REP\(G\) is \(\mathcal{O}(\overline{m}k + nk \log n)\). It is easy to see that the procedure CONSTRUCT\_UNIT\_INTERVAL\_GRAPHS() runs in \(\mathcal{O}(nk \log n)\) time. Since \(\overline{m} \leq n^2\), CONSTRUCT\_CUB\_REP\(G\) runs in \(\mathcal{O}(n^2k)\) time. 

\[\]

5 \textbf{Boxicity, Cubicity, and Crossing Number}

Crossing number of a graph \(G\), denoted as \(CR(G)\), is the minimum number of crossing pairs of edges, over all drawings of \(G\) in the plane. A graph \(G\) is planar if and only if \(CR(G) = 0\). Determination of the crossing number is an NP-complete problem.

The following theorem is due to Pach and Tóth [18]

\[\]

\textbf{Theorem 4.} For a graph \(G\) with \(n\) vertices and \(m \geq 7.5n\) edges, \(CR(G) \geq \frac{1}{33.75} \frac{m^3}{n^2}\), and this estimate is tight up to a constant factor.

The following claims follow from the above theorem.

\[\]

\textbf{Claim 1.} For a graph \(G\) on \(n\) vertices and \(m\) edges, if \(CR(G) \leq t\), then \(d_{av}(G) \leq 2\left(\frac{33.75m^4}{t}\right)^{1/3} + 15\).

\[\]

\textbf{Proof.} If \(m < 7.5n\), then \(d_{av}(G) < 15\). Otherwise, we have \(m \leq (33.75n^2t)^{1/3}\) implying that \(d_{av}(G) \leq 2\left(\frac{33.75m^4}{t}\right)^{1/3}\). \hfill \Box

\[\]

\textbf{Claim 2.} For a graph \(G\) on \(n\) vertices and \(m\) edges, if \(CR(G) = t\), then \(G\) is \((6.5t^{1/4} + 15)\)-degenerate.

\[\]

\textbf{Proof.} From the definition of crossing number we know that \(CR(G) \leq \binom{n}{2} \leq n^4\). Hence, \(n \geq t^{1/4}\). Then by Claim 1, \(d_{av}(G) \leq 6.5t^{1/4} + 15\). Thus \(G\) is \((6.5t^{1/4} + 15)\)-degenerate. \hfill \Box

\[\]

\textbf{Lemma 7.} Consider a graph \(G\) whose vertices are partitioned into two parts namely \(V_A\) and \(V_B\). That is, \(V(G) = V_A \cup V_B\). Then, \(\text{box}(C_B(G)) \leq 2\text{box}(S_B(G))\).

\[\]

\textbf{Proof.} Proof of this lemma is very similar to the proof of Lemma 3 in [7] and hence we only give a brief outline of it here. Assume \(\text{box}(S_B(G)) = r\). Then by Lemma 1, there exist \(r\) interval graphs \(I_1, \ldots, I_r\) such that \(S_B(G) = I_1 \cap I_2 \cap \cdots \cap I_r\). For each \(i \in [r]\), let \(f_i\) denote an interval representation of \(I_i\). From these \(r\) interval graphs we construct \(2r\) interval graphs \(I_1', \ldots, I_r', I_1'', \ldots, I_r''\) as outlined below. Let \(f_i', f_i''\) denote interval representations of \(I_i'\) and \(I_i''\) respectively, where \(i \in [r]\).

\[\]

\textbf{Construction of} \(f_i'\):

\[
\forall u \in V_A, \quad f_i'(u) = f_i(u).
\]

\[
\forall u \in V_B, \quad f_i'(u) = \left[\min_{v \in V_B} (l(f_i(v)), r(f_i(u)))\right].
\]

\[\]

\textbf{Construction of} \(f_i''\):

\[
\forall u \in A, \quad f_i''(u) = f_i(u).
\]

\[
\forall u \in B, \quad f_i''(u) = \left[l(f_i(u)), \max_{v \in V_B} (r(f_i(v)))\right].
\]

\[\]
We leave it to the reader to verify that \( C_B(G) = \bigcap_{i=1}^r (I_i' \cap I_i'') \).

\[ \square \]

**Lemma 8.** Consider a graph \( G \). Let vertices of \( G \) be partitioned into two parts namely \( V_A \) and \( V_B \). That is, \( V(G) = V_A \uplus V_B \). Then, \( box(G) \leq 2 box(S_B(G)) + box(G_B) \).

**Proof.** Let \( G' \) be the graph with \( V(G') = V(G) \) and \( E(G') = E(G) \cup \{(u, v) \mid u \in V_A, v \in V(G')\} \). That is, each \( u \in V_A \) is made a universal vertex in \( G' \). Observe that \( G = C_B(G) \cap G' \). Then by Lemma 1, we have \( box(G) \leq box(C_B(G)) + box(G') \). Applying Lemma 7, we get

\[
box(G) \leq 2 box(S_B(G)) + box(G')
\]

(5)

**Claim 3.** \( box(G') \leq box(G_B) \).

Clearly, \( G' \) is obtained from \( G_B \) by adding universal vertices one after the other. Since adding a universal vertex to a graph does not increase its boxicity, \( box(G') \leq box(G_B) \).

Combining Inequality 5 and Claim 3, we get \( box(G) \leq 2 box(S_B(G)) + box(G_B) \). \( \square \)

### 5.1 Boxicity and Crossing Number

**Theorem 5.** For a graph \( G \) with \( CR(G) = t \), \( box(G) \leq 66 \cdot t^4 |\log 4t|^{3/2} + 6 \).

**Proof.** Consider a drawing \( P \) of \( G \) with \( t \) crossings. We say a vertex \( v \) participates in a given crossing in \( P \), if at least one of the edges of the given crossing is incident on \( v \).

Partition the vertices of \( G \) into two parts, namely \( V_A \) and \( V_B \), such that \( V_B = \{ v \in V(G) \mid v \text{ participates in some crossing in } P \} \) and \( V_A = V(G) \setminus V_B \). Then by Lemma 8,

\[
box(G) \leq 2 box(S_B(G)) + box(G_B).
\]

Observe that \( S_B(G) \) is a planar graph and hence its boxicity is at most 3 (see [21]). Therefore, \( box(G) \leq 6 + box(G_B) \). For ease of notation, let \( H \equiv G_B \). Then,

\[
box(G) \leq 6 + box(H).
\]

(6)

We have \( CR(H) = CR(G) = t \). Let \( n = |V(H)| \) and \( m = |E(H)| \). At most 4 vertices participate in a given crossing. Since each vertex in \( H \) participates in some crossing in \( P \), we get

\[
n \leq 4t.
\]

Let \( V(H) = \{ v_1, \ldots, v_n \} \). Let \( v_1, \ldots, v_n \) be an ordering of the vertices of \( H \), such that for each \( i \in [n] \), \( d_{H_i}(v_i) \leq d_{H_i}(v) \), \( \forall v \in V(H_i) \), where \( H_i \) denotes the subgraph of \( H \) induced on vertex set \( \{ v_1, \ldots, v_n \} \). Let \( k = (\frac{17 \sqrt{2}}{3})^{\frac{3}{8}} (\frac{t}{\log 4t})^{\frac{1}{8}} \).
Let \( x = \min \{ i \in [n] \mid d_{H_i}(v_i) > k \} \). Partition \( V(H) \) into two parts, namely \( V_C = \{ v_1, \ldots, v_x-1 \} \) and \( V_D = \{ v_x, \ldots, v_n \} \). Then by Lemma 8,

\[
box(H) \leq 2 \text{box}(S_D(H)) + \text{box}(H_D).
\]

Note that \( S_{D}(H) \) is \( k \)-degenerate. If \( k = 1 \), then \( S_{D}(H) \) is a forest and hence its boxicity is at most 2. Suppose \( k > 1 \). Then by Theorem 1, 
\[
\text{box}(S_{D}(H)) \leq \text{cub}(S_{D}(H)) \leq (k + 2)|2e \log n| \leq 12k[\log(4t)] \leq 12 \left( \frac{33.75}{3} \right)^{1/2} t^{1/2} \left[ \log 4t \right]^{3/2}.
\]

Thus we have,

\[
box(H) \leq 24 \left( \frac{33.75}{3} \right)^{1/2} t^{1/2} \left[ \log 4t \right]^{3/2} + \text{box}(H_D). \tag{7}
\]

Since \( H_D \equiv H_x, v_x \) is a minimum degree vertex of \( H_D \). Therefore, \( d_{av}(H_D) > d_{H_D}(v_x) > k \). Then by Claim 1, we have

\[
k = \left( \frac{33.75}{3} \right)^{1/2} \left( \frac{t}{[\log 4t]} \right)^{1/2} \leq d_{av}(H_D) \leq 2 \left( \frac{33.75t}{|V(H_D)|} \right)^{1/3} + 15.
\]

From this, we get \( |V(H_D)| \leq 48^t \left( \frac{33.75t}{\log 4t} \right)^{1/2} \). Since boxicity of a graph is at most half the number of its vertices \( |V(G)| \), we get \( box(H_D) \leq 48^t \left( \frac{33.75t}{\log 4t} \right)^{1/2} \).

Substituting this in Inequality 7, we get

\[
box(H) \leq 66t^t \left[ \log 4t \right]^{3/2}.
\]

Therefore from Inequality 6, we get

\[
box(G) \leq 66t^t \left[ \log 4t \right]^{3/2} + 6.
\]

**Tightness of Theorem 5:** Let \( P(n, p) \) be the probability space of height-2 posets with \( n \) minimal elements forming set \( A \) and \( n \) maximal elements forming set \( B \), where for any \( a \in A \) and \( b \in B \), \( Pr[a < b] = p \). Erdős, Kierstead, and Trotter in [11] proved that when \( p = \frac{1}{\log n} \), for almost all posets \( P \in P(n, p) \),

\[
\Delta(G_P) < \frac{\delta_1 n^4}{(\log n)^2} \quad \text{and} \quad \text{dim}(P) > \delta_2 n,
\]

where \( \delta_1 \) and \( \delta_2 \) are some positive constants.

Then by Theorem 1 in [1],

\[
box(G_P) \geq \frac{\text{dim}(P)}{2} \geq \frac{\delta n}{2}.
\]

Let \( t = CR(G_P) \) and let \( m \) denote the number of edges of \( G_P \). It follows from definition of crossing number that

\[
t \leq \left( \frac{m}{2} \right) \leq m^2 \leq (n \Delta(G_P))^2 \leq \left( \frac{\delta_1 n^4}{(\log n)^2} \right)^2 \leq \frac{\delta_1^2 n^4}{(\log n)^4}.
\]

Since \( t \leq m^2 \leq n^4 \), we have \( n \geq t^{1/4} \) and thereby \( \log n \geq \frac{1}{4} \log t \). Thus,

\[
t \leq \left( \frac{\delta_1 n^4}{(\log n)^4} \right)^{1/2} = \frac{16 \delta_1 n^4}{(\log t)^2}.
\]

From Theorem 5, we have \( box(G_P) \leq c t^{1/4} (\log t)^{3/4} + d \leq \text{cn}(\log t)^{1/4} + d \), where \( c \) and \( d \) are some constants. Therefore, the bound given by Theorem 5 is tight up to a factor of \( O((\log t)^{3/4}) \).
5.2 Cubicity and Crossing Number

Theorem 6. For a graph $G$ with $CR(G) = t$, $cub(G) \leq 6 \log_2 n + (6.5 t^{1/4} + 17) \left[2e \log(4t)\right]$.

Proof. Consider a drawing $P$ of $G$ with $t$ crossings. As in Theorem 5, partition the vertices of $G$ into two parts, namely $V_A$ and $V_B$, such that $V_B = \{v \in V(G) \mid v \text{ participates in some crossing in } P\}$ and $V_A = V(G) \setminus V_B$. Let $G'$ be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{(u, v) \mid u \in V_A, v \in V(G')\}$. That is, each $u \in V_A$ is made a universal vertex in $G'$. Observe that $G = C_B(G) \cap G'$. Then by Lemma 2,

$$cub(G) \leq cub(C_B(G)) + cub(G')$$

It is shown in [6] that cubicity of a graph is at most $\log_2 n$ times its boxicity. Applying this result, we get

$$cub(G) \leq (\log_2 n)box(C_B(G)) + cub(G')$$

$$\leq (2 \log_2 n)box(S_B(G)) + cub(G') \text{ (by Lemma 7)}$$

Observe that $S_B(G)$ is a planar graph and hence its boxicity is at most 3 (see [21]). Therefore,

$$cub(G) \leq 6 \log_2 n + cub(G') \quad (8)$$

Observe that $G'$ is the graph with $V(G') = V(G_B) \cup V_A$ and $E(G') = E(G_B) \cup \{(u, v) \mid u \in V_A, v \in V(G')\}$. Since $CR(G_B) = CR(G) = t$, by Claim 2, $G_B$ is $(6.5 t^{1/4} + 15)$-degenerate. Then by Corollary 1, $cub(G') \leq (6.5 t^{1/4} + 17) \left[2e \log(|V_B|)\right]$. We know that at most 4 vertices participate in a given crossing. Since each vertex in $G_B$ participates in some crossing in $P$, we get $|V_B| \leq 4t$. Thus, $cub(G') \leq (6.5 t^{1/4} + 17) \left[2e \log(4t)\right]$. Substituting for $cub(G')$ in Inequality (8), we get

$$cub(G) \leq 6 \log_2 n + (6.5 t^{1/4} + 17) \left[2e \log(4t)\right].$$

6 Cubicity of Random Graphs

Given $n$ and $m$, in order to prove that almost all graphs in $G(n, m)$ model have cubicity $O\left(\frac{2m}{n} \log n\right)$, we first show that cubicity of almost all graphs in $G(n, p)$ model, where $p = \left(\frac{2m}{n}\right) \frac{1}{n-1} = \frac{m}{\binom{n}{2}}$, is $O\left(\frac{2m}{n} \log n\right)$. We then use a result in [2] to convert the result for graphs in $G(n, p)$ model to those in $G(n, m)$ model. To show that almost all graphs in $G(n, p)$ model have cubicity $O\left(\frac{2m}{n} \log n\right)$, we prove the following lemma. Then by Theorem 1, the desired result follows.

Lemma 9. For a random graph $G \in G(n, p)$, where $p = \frac{c}{n^{1-t}}$ and $1 \leq c \leq n - 1$, $Pr[G \text{ is } cec-degenerate] \geq 1 - \frac{1}{nt(n')}$.
Proof. In order to show that a given graph $G$ is $k$-degenerate it is enough to show that every induced subgraph of $G$ has average degree at most $k$. That is, for every $H$ which is an induced subgraph of $G$, $|E(H)| \leq \frac{|V(H)|k}{2}$. Below we prove that for almost all graphs $G \in \mathcal{G}(n, p)$, every induced subgraph $H$ of $G$ has $E(H) < |V(H)|2e$. 

We use the following version of the Chernoff bound (refer page 64 of [17]) in our proof

$$Pr[X \geq (1+\delta)\mu] < \left( \frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^\mu \leq \frac{1}{2(1+\delta)\mu \log_2 \frac{1}{1+\delta}}$$

, where $X$ is a summation of independent Bernoulli random variables, $\mu \geq E[X]$, and $\delta$ is any positive constant.

Let $G \in \mathcal{G}(n, p)$ be a random graph, where $p = \frac{\alpha}{n-1}$. Let $H$ be an induced subgraph of $G$ with $|V(H)| = n\alpha$, where $0 < \alpha \leq 1$. Let $Y_H$ be a random variable that represents the number of edges in $H$. For any $v \in V(H)$, let $d_H(v)$ denote the degree of $v$ in $H$. Then, $E[d_H(v)] = p(n\alpha - 1) = \frac{\alpha(n\alpha-1)}{n-1} \leq \frac{\alpha(n\alpha-1)}{n-1} \leq \alpha \log_{n\alpha}$ and $E[Y_H] = E[\frac{1}{2}\sum_{v \in V(H)} d_H(v)] = \frac{1}{2} \sum_{v \in V(H)} E[d_H(v)] \leq \frac{\alpha^2 c}{2}$. Let $\delta = \frac{4c}{\alpha} - 1$ and $\mu = \frac{\alpha n^2 c}{2} \geq E[Y_H]$. Applying Chernoff bound, we get $Pr[Y_H \geq 2\alpha n c] \leq \frac{1}{2^{2\alpha n c \log_2 (\frac{4c}{\alpha})}}$. Here we split the proof into two cases:

**case** $\frac{1}{4} \leq \alpha \leq 1$: Then, $Pr[Y_H \geq 2\alpha n c] \leq \frac{1}{2^{\log_2 \frac{1}{1+\delta}}} = \frac{1}{2^{\frac{\alpha}{\alpha}}}$. Since $c \geq 1$, we get $Pr[Y_H \geq 2\alpha n c] \leq \frac{1}{2^{\frac{\alpha}{\alpha}}}$. Applying union bound it follows that,

$$Pr\left[ \bigcup_{H:|V(H)| \geq \frac{n}{n}} Y_H \geq 2\alpha n c \right] \leq \frac{1}{2^{\frac{\alpha}{\alpha}}} \frac{1}{2^{\frac{\alpha}{\alpha}}} \leq \frac{1}{2^{\frac{\alpha}{\alpha}}} = \frac{1}{2^{\frac{\alpha}{\alpha}}} \leq \frac{1}{2^{\frac{\alpha}{\alpha}}} \leq \frac{1}{2^{\frac{\alpha}{\alpha}}}.$$

**case** $0 < \alpha < \frac{1}{4}$: Here we use the following expression given in page 17 of [13] while taking union bound: $n(\frac{n}{n}) \leq 2^{H(\alpha)}$, where $H(\alpha) = \alpha \log_{2} \left( \frac{1}{\alpha} \right) + (1 - \alpha) \log_{2} \left( \frac{1}{1-\alpha} \right)$ is the binary entropy function. This inequality can be proved using the Stirling’s formula for factorials. Since $\alpha < \frac{1}{4}$, we have $\alpha \log_{2} \left( \frac{1}{\alpha} \right) > (1 - \alpha) \log_{2} \left( \frac{1}{1-\alpha} \right)$. Therefore, $H(\alpha) \leq 2\alpha \log_{2} \left( \frac{1}{\alpha} \right)$. Hence,

$$Pr\left[ \bigcup_{1 \leq n \leq n} \bigcup_{H:|V(H)| = n\alpha} (Y_H \geq 2\alpha n c) \right] \leq \frac{n}{2^{\frac{\alpha}{\alpha}}} \frac{1}{2^{\frac{\alpha}{\alpha}}} \leq \frac{2^{\log_{2} \left( \frac{1}{\alpha} \right) 2^{\alpha} \log_{2} \left( \frac{1}{\alpha} \right)}}{2^{\alpha n c \log_{2} \left( \frac{1}{\alpha} \right)}} \leq \frac{2^{\alpha n c \log_{2} \left( \frac{1}{\alpha} \right) + \log_{2} n}}{2^{\alpha n c + \alpha n c \log_{2} \left( \frac{1}{\alpha} \right)}} = \frac{1}{2^{\alpha n c + (2\alpha - 2)\alpha n c \log_{2} \left( \frac{1}{\alpha} \right) - \log_{2} n}}$$
Since $c \geq 1$ and $\alpha \geq \frac{1}{n}$, we get
\[
Pr\left[ \bigcup_{1 \leq \alpha \leq \frac{1}{n}} \bigcup_{H : |V(H)| = \alpha} \left( Y_H \geq 2en\alpha \right) \right] \leq \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}} \tag{9}
\]

It is easy to see that the function $f(\alpha) = \alpha \log_2(\frac{1}{\alpha})$ is an increasing function, when $\alpha < \frac{1}{4}$. We have $\frac{1}{n} \leq \alpha < \frac{1}{4}$. Hence $f(\alpha) \geq f(1/n) = \frac{\log_2 n}{n}$, when $\alpha < \frac{1}{4}$.

Applying this to Inequality 9, we get
\[
Pr\left[ \bigcup_{1 \leq \alpha \leq \frac{1}{n}} \bigcup_{H : |V(H)| = \alpha} \left( Y_H \geq 2en\alpha \right) \right] \leq \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}
\]

Thus we say that the probability of any subgraph of $G$ to have its average degree greater than $(4ec+1)$ is at most $\frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$. In other words, $G$ is $4ec$-degenerate with probability at least $1 - \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$.

**Theorem 7.** For a random graph $G \in G(n, p)$, where $p = \frac{c}{n^2}$ and $1 \leq c \leq n-1$, $Pr[\text{cub}(G) \notin O(c \log n)] \leq \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$.

**Proof.** Proof follows directly from Theorem 1 and Lemma 9. \[\square\]

It is shown in page 35 of [2] that $$P_m(Q) \leq 3\sqrt{m}P_p(Q)$$
where $Q$ is a property of graphs of order $n$, and $P_m(Q)$ and $P_p(Q)$ are the probabilities of a graph chosen at random from the $G(n, m)$ or the $G(n, p)$ models respectively to have property $Q$ given that $p = \frac{m}{n^2} = \left(\frac{2m}{n}\right)\frac{1}{n-1}$. Note that for any connected graph $G$ with at least 2 vertices, $\frac{2m}{n} \geq \frac{2(n-1)}{n} \geq 1$. Since we are only interested in connected graphs, we assume $\frac{2m}{n} \geq 1$. Then by Theorem 7, for a random graph $G \in G(n, p)$, where $p = \left(\frac{2m}{n}\right)\frac{1}{n-1}$, $Pr[\text{cub}(G) \notin O(2m \log n)] \leq \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$. Combining this result with the result shown in [2], we say that for a random graph $G \in G(n, m)$, $Pr[\text{cub}(G) \notin O(2m \log n)] \leq \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$. Thus we have the following theorem.

**Theorem 8.** For a random graph $G \in G(n, m)$, $Pr[\text{cub}(G) \in O(2m \log n)] \geq 1 - \frac{1}{2^{4c+(2ec-2)n\log_2(1/n) - \log_2 n}}$.

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