GOOD ACTION ON A FINITE GROUP

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Abstract. Let $G$ and $A$ be finite groups with $A$ acting on $G$ by automorphisms. In this paper we introduce the concept of “good action”; namely we say the action of $A$ on $G$ is good, if $H = [H, B]C_H(B)$ for every subgroup $B$ of $A$ and every $B$-invariant subgroup $H$ of $G$. This definition allows us to prove a new noncoprime Hall-Higman type theorem.

If $A$ is a nilpotent group acting on the finite solvable group $G$ with $C_G(A) = 1$, a long standing conjecture states that $h(G) \leq \ell(A)$ where $h(G)$ is the Fitting height of $G$ and $\ell(A)$ is the number of primes dividing the order of $A$ counted with multiplicities. As an application of our result we prove the main theorem of this paper which states that the above conjecture is true if $A$ and $G$ have odd order, the action of $A$ on $G$ is good and some other fairly general conditions are satisfied.

1. Introduction

Let $A$ be a finite group that acts on the finite group $G$ by automorphisms. We write $h(G)$ for the Fitting height of $G$, and $\ell(A)$ for the length of the longest chain of subgroups of $A$ which coincides with the number of primes dividing the order of $A$ counted with multiplicities if $A$ is solvable. Thompson [14] proved that in case where $A$ and $G$ are both solvable and $(|G|, |A|) = 1$, $h(G)$ is bounded in terms of $h(C_G(A))$ and $\ell(A)$. Thompson’s result has inspired the work of many authors and has been refined in particular in [13], [10], and [17]. Namely, in [17] Turull obtained that $h(G) \leq h(C_G(A)) + 2\ell(A)$.

Due to the lack of some nice consequences of coprime action the situation is very difficult to handle without the coprimeness condition $(|G|, |A|) = 1$. An example obtained by Bell and Hartley [11] shows that for any nonnilpotent finite group $A$, there exists a finite group $G$ of arbitrarily large Fitting height on which $A$ acts fixed point freely and noncoprimely. However if $A$ is nilpotent and $C_G(A) = 1$, a special case of Dade’s theorem [4] provides an exponential bound for $h(G)$ in terms of $\ell(A)$. Some improvements of this bound are obtained in particular cases, e.g. see [12] for cyclic $A$. But apparently, improving to a linear bound is a difficult problem.

A celebrated work of Thompson [15] asserts that every finite group admitting a fixed point free automorphism of prime order is nilpotent. This result gave birth to the long standing conjecture that, under the coprimeness condition $(|G|, |A|) = 1$, if $C_G(A) = 1$ then $h(G)$ is at most $\ell(A)$. There have been a great amount of work on this problem for various cases of $A$ and finally Turull settled the conjecture for almost all $A$ in a sequence of papers (see in particular [16], [17] and [18]) by proving the following result.

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Theorem Let $A$ be a finite group acting by automorphisms on the finite solvable group $G$ such that $(|G|, |A|) = 1$ and $C_G(A) = 1$. If every proper subgroup of $A$ acts with regular orbits on $G$ then $h(G) \leq \ell(A)$.

Here a group $B$ is said to act with regular orbits on another group $G$ if for any $B$-invariant section $S$ of $G$ on which $B$ acts irreducibly there exists $x \in S$ such that $C_B(x) = C_B(S)$ so that the $B$-orbit (which is actually an $B/C_B(S)$-orbit) is a regular orbit, that is an orbit of length $|B/C_B(S)|$. It should be noted that there are large classes of finite groups $A$ always acting with regular orbits on any finite group $G$ of coprime order on which it acts. But there also exist many finite groups which do not need to act with regular orbits.

The example due to Bell and Hartley [1] mentioned above forced to state the noncoprime version of the conjecture as follows:

Conjecture If $A$ is a finite nilpotent group acting fixed point freely on a finite (solvable) group $G$ by automorphisms then $h(G) \leq \ell(A)$.

Although the noncoprime version has been proven in some special cases ([3], [6], [7], [8], [5]), it is still unproven even in the case where $A$ is cyclic.

In the present paper we introduce the concept of a good action of $A$ on $G$; namely we say the action is “good” if $H = [H, B]C_H(B)$ and for every subgroup $B$ of $A$ and for every $B$-invariant subgroup $H$ of $G$. It can be regarded as a generalization of the coprime action due to the fact that every coprime action is good. Some other features of the coprime action, e.g. the existence of an $A$-invariant Hall subgroups, are actually consequences of the fact that coprime action is a good action as we show in Proposition 2.2. On the other hand there are noncoprime actions which are good (see Remark 2.4). So it is natural to ask whether one can get results in the noncoprime case which are similar to the above theorem due to Turull if some nice consequences of a coprime action are kept. In other words one can ask whether the relative easiness of the proofs in the coprime case is due to “goodness” of the action or not. Our main result is a partial answer which provides the best possible upper bound for the Fitting height of a solvable group of odd order admitting a good and fixed point free action. Namely we prove the following.

Theorem (Theorem 4.5) Let $A$ be a finite nilpotent group of odd order which is $C_q \wr C_q$-free for any prime $q$. Suppose that $A$ acts with regular orbits on the finite group $G$ such that this action is good and fixed point free. If $\bigcap_{a \in A} [G, B]^a = 1$ for some subgroup $B$ of $A$, then $G$ is a solvable group of Fitting height at most $\ell(A : B)$ where $\ell(A : B)$ is the number of prime divisors of $|A : B|$ counted with multiplicities.

Notice that we always have $\bigcap_{a \in A} [G, B]^a = 1$ for $B = 1$ which yields the following.

Corollary Let $A$ be a finite nilpotent group of odd order which is $C_q \wr C_q$-free for any prime $q$. Suppose that $A$ acts with regular orbits on the finite group $G$ such that this action is good and fixed point free. Then $G$ is a solvable group of Fitting height at most $\ell(A)$. 

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The proof of this theorem follows the model of the proof of Theorem 2.2 in [16]. Although at some points we could have avoided the details by adding “by an argument similar to…” and referring to [16]; for the reader’s convenience, we have formulated them so that they can be easily followed without any further references.

It should also be noted that in order to overcome some of the main difficulties arising from noncoprimeness we needed a new Hall-Higman type theorem. In this direction we obtained Theorem 3.1 and Theorem 3.3 which are of independent interest in the study of noncoprime action problems.

Throughout the paper all groups are finite, notation and terminology are standard.

2. Good Action

In this section we introduce the concept of a good action. Some of its immediate consequences are obtained below.

**Definition 2.1.** Let $G$ and $A$ be groups where $A$ acts on $G$ by automorphisms. We say “the action of $A$ on $G$ is good” if the equality $H = [H, B]C_H(B)$ holds for any subgroup $B$ of $A$ and for any $B$-invariant subgroup $H$ of $G$.

**Proposition 2.2.** Let $G$ and $A$ be groups where $A$ acts on $G$ by automorphisms, and suppose that the action of $A$ on $G$ is good. If $B \trianglelefteq A$ and $N$ is a normal $B$-invariant subgroup of $G$ then

1. the action of $B$ on every $B$-invariant subgroup of $G$ is good;
2. the equality $[H, B, B] = [H, B]$ holds for any $B$-invariant subgroup $H$ of $G$;
3. the equality $C_{H/N}(B) = C_H(B)N/N$ holds for any $B$-invariant subgroup $H$ of $G$ containing $N$;
4. the induced action of $B$ on $G/N$ is good.

**Proof.** (1) and (2) are straightforward. To prove (3) set $X/N = C_{H/N}(B)$. Since $X$ is $B$-invariant, we have $X = [X, B]C_X(B)$ with $[X, B] \trianglelefteq N$. It follows that $X = NC_X(B)$ whence $C_H(B)N/N = X/N$. Notice next that $H = [H, B]C_H(B)$ implies

$$H/N = [H, B]C_H(B)N/N = [H, B]N/NC_{H/N}(B) = [H/N, B]C_{H/N}(B)$$

as $C_H(B)N/N = C_{H/N}(B)$ proving (4). \hfill $\Box$

**Proposition 2.3.** Let $G$ and $A$ be solvable groups where $A$ acts on $G$ by automorphisms. The action of $A$ on $G$ is good whenever $(|[G, A]|, |GA/[G, A]|) = 1$.

**Proof.** Let $B \trianglelefteq A$. Then for any $B$-invariant subgroup $H$ of $G$, $H \cap [G, A] = H_0 \trianglelefteq H$ and $H/H_0 \cong [G, A]H/[G, A]$ has order coprime to $H_0$. Therefore $HB$ has a Hall subgroup $T$ containing $B$ which is a complement of $H_0$ in $HB$. Now $H = H_0(T \cap H)$ and clearly we have $H_0 = [H_0, B]C_{H_0}(B)$ as $(|H_0|, |B|) = 1$. We also have $[T \cap H, B] \trianglelefteq T \cap H_0 = 1$, yielding that $H = [H, B]C_H(B)$. \hfill $\Box$

**Remark 2.4.** Motivated by Proposition 2.3 we can construct some concrete examples of good action as follows:

**Example 1.** Let $H = (C_5 \times C_5) \rtimes SL(2, 3)$ be the Frobenius group with kernel isomorphic to $C_5 \times C_5$. Since $H$ has a unique minimal normal subgroup and $O_2(H) = 1$ we there exists a faithful irreducible $H$-module $V$ over $GF(2)$. Let $G = V \rtimes H$ and
\langle x \rangle$ be a Sylow 3-subgroup of $G$. If $\alpha$ is the inner automorphism induced by $x$ on $G$ then the action of $A = \langle \alpha \rangle$ on $G$ is good by the above proposition.

A slight modification of Example 1 shows that $A$ need not be a subgroup of $\text{Inn}G$:

**Example 2.** Let $G = R\langle x \rangle$ be the group built in Example 1 where $R = O_p(G)$, and let $T = \langle t \rangle \cong C_2$. Set $\Gamma = G\langle T \rangle$ and let $\Gamma_0 \cong (R_1 \times R_2) \rtimes S_3$ where $R_1 \cong R \cong R_2$ with $R_1^t = R_2$ and $S_3$ is the symmetric group of degree 3 generated by $\{x^{-1}x^t, t\}$. If $\beta$ is the automorphism induced by conjugation by $xx^t$ on $\Gamma_0$, then one can verify that the action of $A = \langle \beta \rangle$ on $\Gamma_0$ is good. Moreover $\beta$ is not an inner automorphism of $\Gamma_0$.

**Proposition 2.5.** Let $G$ and $A$ be groups where $A$ acts on $G$ by automorphisms, and suppose that the action of $A$ on $G$ is good. Let $p \in \pi(A)$ and let $B$ be a $p$-subgroup of $A$. If $G$ is $p$-solvable then $[G, B]$ is a $p'$-group.

**Proof.** Set $H = [G, B]$ and $\overline{H} = H/O_p(H)$. By Proposition 2.2 the action of $B$ on $\overline{H}$ and hence on $X = O_p(\overline{H})$ is good. Then $[X, B, B] = [X, B]$ by Proposition 2.2 and so $[X, B] = 1$. As $[B, X, \overline{H}] = 1 = [X, \overline{H}, B]$, by the three subgroups lemma we have $[\overline{H}, B, X] = 1$. Since $G$ is $p$-solvable we get $[\overline{H}, B] \leq C_{\overline{H}}(X) \leq X$. Then $[\overline{H}, B] = [\overline{H}, B, B] \leq [X, B] = 1$ by Proposition 2.2. This establishes the claim. □

**Proposition 2.6.** Let $G$ be a solvable group acted on by a nilpotent group $A$. If the action is good, then $G$ contains an $A$-invariant Sylow $p$-subgroup for any prime $p \in \pi(G)$.

**Proof.** We proceed by induction on $|G| + |\pi(A)|$. Let $p \in \pi(G)$. If $O_p(G) \neq 1$ then we see by induction that $\overline{G} = G/O_p(G)$ contains an $A$-invariant Sylow $p$-subgroup $\overline{P}$, and the inverse image $P$ of $\overline{P}$ is an $A$-invariant Sylow $p$-subgroup of $G$.

Suppose that $O_p(G) = 1$ and let $N$ be a minimal normal $A$-invariant subgroup of $G$. We observe that $N \leq O_q(G)$ for some $q \in \pi(G) \setminus \{p\}$. By induction $G/N$ contains an $A$-invariant Sylow $p$-subgroup and hence we may assume that $G = NP$ with $P \in \text{Syl}_p(G)$.

Suppose first that $A$ is an $r$-group for some prime $r$. If $r = q$, then $[N, A] = 1$ by Proposition 2.5, and so $[G, A] = [P, A] \leq C_G(N) = N$ by the three subgroups lemma. Then $[G, A] = [G, A, A] \leq [N, A] = 1$ by Proposition 2.2 and the claim follows. If $r = p$, let $S$ be a Sylow $p$-subgroup of $GA$ containing $A$. Then $S \cap G$ is an $A$-invariant Sylow $p$-subgroup of $G$ as desired. If $r \notin \{p, q\}$, then $(|G|, |A|) = 1$ and the conclusion holds.

Suppose now that $|\pi(A)| > 1$. If $A_q \neq 1$ then $[G, A_q] = 1$ and by induction applied to the action of $A_q'$ on $G$, one can assume that $A_q = 1$. If $A_p \neq 1$ then $[G, A_p] \leq N$ by Proposition 2.5 and hence $G = NC_G(A_p)$ due to good action. This yields that $\text{Syl}_p(C_G(A_p)) \subseteq \text{Syl}_p(G)$. Since $A$ is nilpotent $C_G(A_p)$ is an $A$-invariant subgroup of $G$ and, by the inductive hypothesis, it contains an $A$-invariant Sylow $p$-subgroup. Finally if $p, q \notin \pi(A)$, then $(|G|, |A|) = 1$ and the theorem follows. □

In the rest of this section we study the relation between the Fitting heights of $G$ and $C_G(A)$ in case of a good action.
Definition 2.7. (Definition 1.1 and 1.2 of [17]) Let $G$ and $A$ be finite groups where $A$ acts on $G$. We say that a sequence $(S_i)_{i=1}^h$ of $A$-invariant subgroups of $G$ is an $A$-tower of $G$ of height $h$ if the following are satisfied:

1. $S_i$ is a $p_i$-group, $p_i$ is a prime, for $i = 1, \ldots, h$;
2. $S_i$ normalizes $S_j$ for $i \leq j$;
3. Set $P_h = S_h$, $P_i = S_i/C_{S_i}(P_{i+1})$, $i = 1, \ldots, h-1$ and we assume that $P_i$ is not trivial for $i = 1, \ldots, h$;
4. $p_i \neq p_{i+1}$, $i = 1, \ldots, h-1$.

An $A$-tower $(S_i)_{i=1}^h$ of $G$ is said to be irreducible if the following are satisfied:

5. $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$ and, if $p_i \neq 2$, then $P_i$ has exponent $p_i$ for $i = 1, \ldots, h$. Moreover $P_{i-1}$ centralizes $\Phi(P_i)$;
6. $P_i$ is elementary abelian;
7. There exists $H_i$, an elementary abelian $A$-invariant subgroup of $P_{i-1}$ such that $[H_i, P_i] = 1$;
8. $(\prod_{i=1}^{h-1} S_i)A$ acts irreducibly on $P_i/\Phi(P_i)$.

Remark 2.8. Moreover we can easily verify that Lemma 1.9 of [17] requires that $G$ contains an $A$-invariant Sylow $p$-subgroup for every prime $p$ dividing $|G|$. Hence our Proposition 2.6 and Lemma 1.9 of [17] yields

$$h(G) = \max \{ h : \text{there is an } A\text{-tower of height } h \text{ in } G \}.$$ 

Theorem 2.9. Let $A$ be a group of prime order $p$ acting on the solvable group $G$ such that $C_G(A)$ is of odd order. If the action of $A$ on $G$ is good, then $h(G) \leq h(C_G(A)) + 4$.

Proof. Let $h = h(G)$. By the above remark there exists an irreducible $A$-tower $(S_i)_{i=1}^h$ of height $h$ in $G$, with $P_i = S_i/T_i$, $i = 1, \ldots, h$ and $P_0 = 1$. Let $k$ be the largest integer with the property that $[P_k, A] = 1$. As the action is good, by Proposition 2.2 (3) we have $C_{P_i}(A) = C_{S_i}(A)T_i/T_i$ for each $i$, and hence we may assume that $S_i$ is centralized by $A$ for each $i = 1, \ldots, k$ and that $[S_{k+1}, A] = S_{k+1}$. It follows by Proposition 2.5 that $p_i \neq p$ for each $i = k+1, \ldots, h$, that is $\prod_{i=k+1}^h S_i$ is a $p'$-group. Notice also that $h - k > 4$ because $k \leq h(C_G(A))$.

Set $C_i = C_{S_i}(A)$ for each $i$. Then $C_i = S_i$ for $i = 1, \ldots, k$. Also note that $p_k$ is odd as $|C_G(A)|$ is odd.

Suppose first that $p_k \neq p$, by Theorem 3.1 in [17] applied to $S_h, \ldots, S_k$, we obtain $j \in \{ k + 1, \ldots, h \}$ such that the sequence $C_h, \ldots, C_{j+1}, C_{j-1}, \ldots, C_k$ satisfies the conditions (1), (2) and (3) of Definition 2.7, possibly with $p_{j+1} = p_{j-1}$. It follows that either the sequence

$$C_h, \ldots, C_{j+1}, C_{j-1}, \ldots, C_k, \ldots, C_1$$

forms an $A$-tower of height $h - 1$ or the sequence

$$C_h, \ldots, C_{j+2}, C_{j-1}, \ldots, C_k, \ldots, C_1$$

forms an $A$-tower of height $h - 2$, and hence $h(G) \leq h(C_G(A)) + 2$.

Suppose next that $p_k = p$. Note that if $k = 1$ then $S_h, \ldots, S_2$ forms an $A$-tower whose terms are $p'$-subgroups, and by the main result of [17], we get $h - 1 =
\[ h(\prod_{i=2}^{h} S_i) \leq h(C_G(A)) + 2. \] Therefore we may assume that \( k > 1 \). Notice that we have either \( p_{k-1} \) and \( p_{k+1} \) are equal, or not. If the former holds then the sequence \( S_h, \ldots, S_{k+2}, S_{k-1} \) forms an \( A \)-tower. As \( p_{k-1} \) is odd, we apply Theorem 3.1 in [17], and obtain \( j \in \{ k + 2, \ldots, h \} \) such that \( C_{h}, \ldots, C_{j+1}, C_{j-1}, \ldots, C_{k-1} \) is a sequence satisfying the conditions (1), (2) and (3) of Definition 2.7, possibly with \( p_{j+1} = p_{j-1} \). It follows that either the sequence

\[ C_h, \ldots, C_{j+1}, C_{j-1}, \ldots, C_{k+2}, C_{k-1}, \ldots, C_1 \]

forms an \( A \)-tower of height \( h - 3 \) or the sequence

\[ C_h, \ldots, C_{j+2}, C_{j-1}, \ldots, C_{k+2}, C_{k-1}, \ldots, C_1 \]

forms an \( A \)-tower of height \( h - 4 \). Then we have \( h \leq h(C_G(A)) + 4 \) and the theorem follows. Finally suppose that \( p_{k-1} \neq p_{k+1} \). Now \( S_h, \ldots, S_{k+1}, S_{k-1} \) forms an \( A \)-tower. Notice again that as \( p_{k-1} \) is odd Theorem 3.1 in [17] gives \( j \in \{ k + 1, \ldots, h \} \) such that the sequence \( C_{h}, \ldots, C_{j+1}, C_{j-1}, \ldots, C_{k+1}, C_{k-1} \) satisfies the conditions (1), (2) and (3) of Definition 2.7. It follows that either the sequence

\[ C_h, \ldots, C_{j+1}, C_{j-1}, \ldots, C_{k+1}, C_{k-1}, \ldots, C_1 \]

forms an \( A \)-tower of height \( h - 2 \) or the sequence

\[ C_h, \ldots, C_{j+2}, C_{j-1}, \ldots, C_{k+1}, C_{k-1}, \ldots, C_1 \]

forms an \( A \)-tower of height \( h - 3 \). Then we have \( h(G) \leq h(C_G(A)) + 3 \) and this completes the proof. \( \square \)

**Theorem 2.10.** Let \( A \) be a solvable group acting on the solvable group \( G \) such that \( C_G(A) \) is of odd order. If the action is good then \( h(G) \leq h(C_G(A)) + 4\ell(A) \).

**Proof.** Let \( G \) be a minimal counterexample to the theorem and let \( B \triangleleft A \) such that \( A/B \) is of prime order. By induction applied to the action of \( B \) on \( G \) we see that \( h(G) \leq h(C_G(B)) + 4\ell(B) \). Theorem 2.9 applied to the action of \( A/B \) on \( C_G(B) \) yields that \( h(C_G(B)) \leq h(C_G(A)) + 4 \), whence \( h(G) \leq h(C_G(A)) + 4\ell(A) \). \( \square \)

3. A noncoprime Hall-Higman Type Theorem

This section is devoted to the study of some technical problems pertaining to the proof of our main theorem. The following results are also of independent interest because they seem to be effectively applicable in other situations of noncoprime action.

**Theorem 3.1.** Let \( A \) be a nilpotent group acting with regular orbits on the group \( G \). Let \( V \) be a complex \( GA \)-module so that \( V_G \) is homogeneous on which \( A \) acts fixed point freely. Suppose that \( G/N \) is a \( GA \)-chief factor of \( G \) which is an elementary abelian \( r \)-group for some prime \( r \). If \( A \) normalizes a Hall \( r \)-subgroup of \( GA \), then there exists a homogeneous component \( U \) of \( V_N \) and a subgroup \( B \) of \( A \) such that \( B \leq N_A(U), C_V(B) = 0 \) and \( [G, B] \leq N_G(U) \). In particular we have \( [G, B] \leq N \) in case where \( V_N \) is not homogeneous.

**Proof.** It is useful to proceed in a series of steps.
(1) We may assume that \( V_n = W_1 \oplus \ldots \oplus W_s \) where \( W_i = mY_i, \ i = 1, \ldots, s, \) for some positive integer \( m \) and for pairwise nonisomorphic irreducible \( N \)-submodules \( Y_1, \ldots, Y_s \). Furthermore, \( G \) acts transitively and \( G/N \) acts regularly on \( \Omega = \{ W_1, \ldots, W_s \} \).

Proof. Let \( X \) be an irreducible submodule of \( V_G \). Then \( X_N \) is irreducible or homogeneous or a sum of nonisomorphic irreducible submodules by [13], 6.18. In the former and the second cases, \( V_n \) is homogeneous and we take \( V = U \) and \( B = A \).

We may therefore assume that the latter holds, that is, \( X_N = Y_1 \oplus \cdots \oplus Y_s \), where \( Y_i \) are pairwise nonisomorphic irreducible \( N \)-submodules. Then \( V_N = mY_1 \oplus \cdots \oplus mY_s \) for some positive integer \( m \). Set \( W_i = mY_i, \ i = 1, \ldots, s. \) As \( V_G \) is homogeneous, we may assume that \( G \) acts transitively on \( \Omega = \{ W_1, \ldots, W_s \} \). Hence \( GA = N_G(W_i)G \) for each \( i = 1, \ldots, s \). We may also observe that \( N = N_G(W_i) \) for each \( i = 1, \ldots, s \), as \( G/N \) is a \( GA \)-chief factor, that is, \( G/N \) acts regularly on \( \Omega \). \( \square \)

(2) \( G/N \) is an elementary abelian \( r \)-group for some prime \( r \). Set \( A = A_r \times A_r \). Then \( A_r \) centralizes \( G/N \).

Proof. This follows from the irreducibility of \( G/N \) as a \( GA \)-module. \( \square \)

(3) Set \( M = N_G(W_1) \). Then \( G/N \) is centralized by \( K = Core_G(M) \) and hence \( K = C_M(G/N) \). Note that \( N = K \cap G \).

Proof. Now \( N \leq M \). Notice that \( MG = GA \) as \( G \) acts transitively on \( \Omega \) by (1). Then \( G \not\leq M \). We have \( N \leq K \cap G \leq GA \) and hence \( K \cap G \cap M = N \) by the irreducibility of \( G/N \) as a \( GA \)-module. It follows that \( [G, K] \leq N \), that is, \( K \) centralizes \( G/N \). Then \( K \leq L = C_M(G/N) \). Now, \( L \) is normalized by \( G \) as \( [G, L] \leq N \leq L. \) \( L \) is also left invariant by \( M \). Therefore \( L \) is \( GA \)-invariant as \( GA = MG \). This shows that \( L \leq \bigcap_{x \in GA} x^F = K \) and hence we have the equality \( L = K \), as desired. \( \square \)

(4) Set \( \overline{GA} = GA/K \). Then \( G/N \cong \overline{G} = O_r(GA) \). Furthermore \( O_{r,r'}(GA) = \overline{GA} \).

Proof. By (3), \( N = K \cap G \) and hence \( G/N \) is an \( r \)-group. Then \( \overline{G} \leq O_r(GA) \). On the other hand, \( \frac{GA}{KG} = \frac{KGA}{KG} \cong A/A \cap KG \) and \( A_r(A \cap KG)/(A \cap KG) \) is trivial, because \( A_r \leq C_G(A/N) = C_{MG}(G/N) = GC_M(G/N) = KG \) by (3). This yields that \( |GA/KG| \) is not divisible by \( r \) and hence \( \overline{G} = O_r(GA) \). In fact, \( O_{r,r'}(\overline{GA}) = \overline{GA} \). \( \square \)

(5) \( M = QK = N_G(Q) \) for some Hall \( r' \)-subgroup \( Q \) of \( GA \) normalized by \( A \) and hence \( A \leq M^{x_0} = M_0 \) for some \( x_0 \in GA \).

Proof. Let now \( Q \) be a Hall \( r' \)-subgroup of \( GA \) normalized by \( A \). Then \( \overline{Q} \) is an \( A \)-invariant Hall \( r' \)-subgroup of \( GA \) and hence \( QO_r(GA) = QG = GA \) by (4). On the other hand, we also have \( \overline{GA} = \overline{MG} \).

Thus \( \overline{Q} \) and \( \overline{M} \) are conjugate in \( \overline{GA} \), that is, \( \overline{Q} = \overline{M^{x_0}} \) for some \( x_0 \in x_0K \) in \( \overline{GA} \). It follows that \( M^{x_0} = QK \). Set \( M_0 = M^{x_0} \) and \( W_0 = W_1^{x_0} \).

We observe next that \( N_G(Q) = QK \): Assume that \( \overline{Q} \) is properly contained in \( N_G(\overline{Q}) \). Then there exists \( \overline{y} = yK \in O_r(\overline{GA}) \) such that \( [\overline{Q}, \overline{y}] = 1 \). Recall
that $GA$ acts transitively on $\Omega$. As a consequence $GA$ acts transitively on $\Omega' = \{N_{GA}(W_i) : i = 1, \ldots, s\}$. Since $\overline{\sigma}$ fixes $M_0$ as $QK = M_0$, we see that $\overline{\sigma}$ fixes $N_{GA}(W_0)$. Notice that $G/N \cong O_r(GA)$ acts regularly on $\Omega$ and hence on $\Omega'$. Thus we have $Q = N_{GA}(Q)$ implying that $QK = N_{GA}(Q) = M_0$. Now $A \subseteq M_0$ holds, as claimed.

(6) Theorem follows.

Proof. We have $V = W_0^{GA}$ where $W_0$ is an irreducible $M_0$-module. Choose a subset $R$ of $G$ which consists of double coset representatives of $(M_0, A)$ in $G$. We observe

$C_A(xN) = A \cap M_0^x$ for each $x \in R$. To see this, let $b \in C_A(xN)$. Then $b \in M_0^x$ as $A \subseteq M_0$. Conversely, pick an element $b$ from $A \cap M_0^x$. Then $[b, x^{-1}] \in N$, that is, $[b, xN] = 1$ which yields the desired equality.

Recall that $A$ acts with regular orbits on $G/N$ by hypothesis. More precisely, there exists $x_1 \in R$ such that $C_A(x_1N) = C_A(G/N)$. Note that

$V_A = \bigoplus_{x \in R} W_0^x|_{A \cap M_0^x}$

by Mackey’s theorem. Taking now $U = W_0^{x_1}$ and $B = C_A(G/N) = A \cap M_0^{x_1}$, we see that $U_B \subseteq V_A$ and hence $C_V(A) = 0$ implies $C_U(B) = 0$. Consequently $B \neq 1$ with $[G, B] \leq N \leq N_G(U)$ proving the theorem.

Under the assumption that the action of $A$ on $G$ is good, the above theorem would imply that $G = N_G(W)C_G(B)$. The following example shows that we can not expect this equality without the assumption of “goodness” of the action.

Example 3.2. Let $\langle \sigma \rangle \cong \mathbb{Z}_9$ and $\alpha \in Aut \langle \sigma \rangle$ given by $\alpha(\sigma) = \sigma^4$. Then the group $S = \langle \sigma, \alpha \rangle$ is extraspecial of order 27 and of exponent 3 where $Z(S) = \langle \sigma^3 \rangle$. Let $R$ be a group isomorphic to $\mathbb{Z}_7$. Then $S$ acts on $R$ so that $\langle \sigma^3, \alpha \rangle$ forms the kernel of this action. Put $G = R \langle \sigma \rangle$ and $N = RZ(S)$. Now $F = G/Z(S)$ is a Frobenius group of order 21, acted on by $A = \langle \alpha \rangle$. Here, $[G, A] \leq Z(S)$. $F$ has an irreducible faithful character $\chi$ with $\chi(1) = 3$. Considering $\chi$ as an irreducible character of $G$ with kernel $Z(S)$, we see that $\chi_N = \chi_1 + \chi_2 + \chi_3$, a sum of distinct irreducible characters $\chi_1, \chi_2, \chi_3$, each of which is fixed by $\alpha$. Let $V$ be a complex $GA$-module and $W_1$ be a complex $N$-module affording $\chi$ and $\chi_1$, respectively. It is obvious that $G \neq N_G(W_1)C_G(A) = N$ as $N_G(W_1) = N = C_G(A)$. Note also that $C_V(A)$ is $GA$-invariant as $[G, A] \leq \ker(G$ on $V$). Therefore $C_V(A) = 0$ as $V$ is an irreducible $GA$-module.

Theorem 3.3. Let $G$ be a group on which a nilpotent group $A$ acts with regular orbits. Suppose that the action of $A$ on $G$ is good. Let $V$ be a complex $GA$-module such that $V_G$ is homogeneous, $C_V(A) = 0$ but $C_V(A_0) \neq 0$ for every proper subgroup $A_0$ of $A$. Let $N \triangleleft GA$ such that $[Z(N), A]$ acts nontrivially on $V$, and let $W$ be a homogeneous component of $V_N$. Then $W$ is $A$-invariant and $[G, A] \leq N_G(W)$, that is, $G = N_G(W)C_G(A)$.

Proof. Let $M \supseteq N$ be such that $G/M$ is a $GA$-chief factor of $G$. By Theorem 3.1 there exists a homogeneous component $U$ of $V_M$ and a subgroup $B$ of $A$ such that $B \leq N_A(U)$, $C_V(B) = 0$ and $[G, B] \leq N_G(U)$. By hypothesis we have $B = A$ and hence $[G, A] \leq N_G(U)$. 

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As $M \leq N_G(U) \leq G$ we get either $M = N_G(U)$ and $N_G(U) = G$. Suppose first that $N_G(U) = G$. Then $U = V$, that is $V_M$ is homogeneous. Let $W$ be a homogeneous component of $V_N$. By induction, applied to the action of $MA$ on $V$, we get $[M, A] \leq N_G(W)$. Since the action is good, this yields $M = N_G(W)C_M(A)$. Let now $T$ be a transversal for $N_G(W)$ in $M$. Clearly we may assume that $T \subseteq C_G(A)$. Now $V_N = \bigoplus_{t \in T} W^t$. Therefore for each $t \in T$, $W^t$ is $A$-invariant whence

$$[Z(N), A] \leq \bigcap_{t \in T} Ker(W^t) = Ker(V).$$

This contradiction shows that $V_M$ is not homogeneous. Thus we have $M = N_G(U)$ and hence $[G, A] \leq M$, that is $G = MC_G(A)$ as the action is good. Let now $S$ be a transversal for $M$ in $G$. We may assume that $S \subseteq C_G(A)$ and hence $V_M = \bigoplus_{s \in S} U^s$. It also follows that $U^s$ is $A$-invariant for all $s \in S$ as $U$ is $A$-invariant. If $[Z(N), A]$ is trivial on $U$ then it is trivial on $U^s$ for all $s \in S$ as $S \subseteq C_G(A)$ and hence on $V$. Therefore $[Z(N), A]$ is nontrivial on $U$. Note that if $C_U(A_0) = 0$ for some proper subgroup $A_0$ of $A$ then $C_{U^s}(A_0) = 0$ for all $s \in S$ and so $C_V(A_0) = 0$, which is not the case. By applying induction to the action of $MA$ on $U$ we see that there exists a homogeneous component $W_1 \leq W$ of $U_N$ and a subgroup $B$ of $A$ such that $B \leq N_A(W_1)$, $C_U(B) = 0$ and $[M, B] \leq N_M(W_1)$. Clearly $B = A$ and hence $[G, A] = [M, A] \leq N_G(W_1)$. Then $U \cap W$ is $A$-invariant whence $W$ is $A$-invariant and

$$[G, A] = [G, A, A] \leq [M, A] \leq N_G(W_1) \leq N_G(W).$$

This completes the proof. \qed

4. Fixed Point Free Good Action

Remark 4.1. As an immediate consequence of Theorem 2.10 we observe that if $A$ is a nilpotent group acting fixed point freely on the solvable group $G$ and the action is good, then $h(G) \leq 4\ell(A)$. The next theorem improves this bound.

Theorem 4.2. Let $G$ be a solvable group and $A$ be a nilpotent group acting fixed point freely on $G$. If the action is good, then $h(G) \leq 2\ell(A)$.

Proof. Suppose that $|A| = \prod_{i=1}^{k} p_i^{r_i}$.

We use induction on $k$. If $k = 1$, then $(|G|, |A|) = 1$ and the result is well-known by Corollary 3.2 in [17]. Suppose that $p_j \notin \pi(G)$ for some $j \in \{1, 2, \ldots, k\}$. Then $A_j$ is a group of automorphisms of coprime order of $G$. By inductive hypothesis and Corollary 3.2 in [17] we have

$$h(C_G(A_j)) \leq 2\ell(A/A_j).$$

Then the same corollary in [17] implies that

$$h(G) \leq h(C_G(A_j)) + 2\ell(A_j) \leq 2\ell(A/A_j) + 2\ell(A_j) = 2\ell(A).$$

Hence one can suppose that $\{p_1, p_2, \ldots, p_k\} \subseteq \pi(G)$. By Proposition 2.5 the group $A_i$ centralizes $G/O_{p_i}(G)$ and we have $h(O_{p_i}(G)) \leq 2\ell(A)$ by the previous argument. Let $N = \prod_{i=1}^{k} O_{p_i}(G)$. Then $h(N) = \max \{h(O_{p_i}(G)) : i = 1, \ldots, k\} \leq 2\ell(A)$. Moreover $G/N$ is centralized by every $A_i$, and so it is centralized by $A$, that is $G = N$. \qed
Remark 4.3. It can be seen that Theorem A in [5] can be extended as follows: Let $A$ be an abelian group of squarefree exponent coprime to 6 acting fixed point freely on a group $G$ whose Sylow 2-subgroups are abelian. Then $h(G) \leq \ell(A)$. The following theorem is similar to this result in the sense that the assumption $([A], 6) = 1$ can be replaced by the goodness of the action.

**Theorem 4.4.** Let $G$ and $A$ be solvable groups such that the action of $A$ on $G$ is good. Suppose that $G$ has abelian Sylow 2-subgroups and that $A$ is abelian of squarefree exponent. If $C_G(A) = 1$ then $h(G) \leq \ell(A)$.

**Proof.** We proceed by induction on $\ell = \ell(A)$. The claim is well known in case where $\ell = 1$. Thus we may assume that $\ell > 1$. Let $p \in \pi(A)$ and let $\alpha$ be a $p$-element of $A$. Then, by the fundamental result on the structure of abelian groups, there is $B \leq A$ such that $A = B \oplus \langle \alpha \rangle$. Let $C = C_G(\alpha)$. By induction applied to the action of $B$ on $C$ we deduce that $h(C) \leq \ell - 1$. On the other hand appealing to Satz 3 in [13] we get $h([G, \alpha]) \leq \ell$. Set $N = \prod_{\alpha \notin A}[G, \alpha]$. Clearly, $N$ is a normal subgroup of $G$ with $h(N) \leq \ell$. Since $A$ acts trivially on $G/N$ we have $G = N$. \qed

The rest of this section is devoted to the proof of the main result of this paper.

**Theorem 4.5.** Let $A$ be a finite nilpotent group of odd order which is $C_q \setminus C_q$-free for any prime $q$. Suppose that $A$ acts with regular orbits on the finite group $G$ such that this action is good and fixed point free. If $\bigcap_{\alpha \in A}[G, B]^a = 1$ for some subgroup $B$ of $A$, then $G$ is a solvable group of Fitting height at most $\ell(A : B)$ where $\ell(A : B)$ is the number of prime divisors of $|A : B|$ counted with multiplicities.

**Proof.** We proceed by induction on $|G| + |A| + \ell(A : B)$. Set $h = h(G)$. The group $G$ is solvable by [2]. As in the proof of Theorem 2.9 we see the existence of a sequence of sections $P_1, \ldots, P_h$ of $G$ with $P_i = S_i/T_i$ where $S_i$ and $T_i$ are $A$-invariant subgroups of $G$ satisfying conditions (1)-(8) of Definition 2.7. It should be noted that we may assume that $T_i = 1$ and $S_i \leq F(G)$.

To simplify the notation we set $V = P_h$ and $P = S_{h-1}$. By induction we have $G = VP S_{h-2} \ldots S_1$. Then we may assume that $\Phi(V) = 1 = T_{h-1} = 1$ by corresponding induction arguments. Set now $X = PS_{h-2} \ldots S_1$. By (8) of Definition 2.7 $V$ is an irreducible $XA$-module. We shall proceed in a series of steps:

(1) $A$ acts faithfully on $G$, $A_1 = C_A(P) \leq B$ and $(|P|, |A : B|) = 1$.

**Proof.** By induction applied to the action of $A/Ker(A)$ on $G$ with respect to the subgroup $BKer(A)$ on $G/Ker(A)$ we get $h \leq \ell(A : BKer(A) on G)$ which yields that $Ker(A)$ on $G) \leq B$. Therefore we may assume that $Ker(A)$ on $G) = 1$.

We can observe that $A_1 = C_A(P)$ centralizes all the subgroups $P, S_{h-2}, \ldots, S_1$ due to good action: Firstly we have $[S_{h-2}/T_{h-2}, A_1] = 1$ by the three subgroups lemma, whence $[S_{h-2}, A_1] = 1$ by Proposition 2.2 (3). Repeating the same argument we get the claim.

Clearly $A_1 \triangleleft A$. If $A_1 \not\leq B$, by induction applied to the action of $A/A_1$ on the group $PS_{h-2} \ldots S_1$ with respect to the subgroup $B/A_1$ we have $h - 1 \leq \ell(A/A_1 : B/A_1)$, which is a contradiction. Thus $A_1 \leq B$ and hence $(|P|, |A : B|) = 1$ because $A_1$ centralizes $P$ by Proposition 2.5. \qed
(2) For any subgroup \(C\) of \(A\) containing \(B\) properly we have \(P = \langle \langle P, C \rangle \rangle^X\).

Proof. Set \(P_0 = \langle \langle P, C \rangle \rangle^X\) and \(X_0 = S_{h-2} \ldots S_1\). Suppose that \(P_0 \neq P\). Note that \(P_0 \triangleleft XC\) and set \(K = C_{X_0}(P/P_0)\). Then \(P_0K \triangleleft PX_0C \triangleleft XC\). Since \([P, C] \leq P_0\) we have \([X_0, C] \leq K\) by the three subgroups lemma. Then \([X, C] \leq PX_0K\). Notice that \(P_0S_{h-2}\) is normalized by \(P_0K\). If \(P \leq P_0K\) then \(P\) normalizes \(P_0S_{h-2}\) and so

\[P = \langle \langle P, S_{h-2} \rangle \rangle \leq \langle \langle P, P_0S_{h-2} \rangle \rangle \leq P_0S_{h-2} \cap P = P_0,\]

which is impossible. Thus we have \(P \not\subseteq P_0K\) and so \(P \not\subseteq \bigcap_{a \in A}[X, C]^a\). This forces that \(P \cap \bigcap_{a \in A}[X, C]^a \leq \Phi(P)\) by condition (8) of Definition 2.7. Set \(Y = \bigcap_{a \in A}[X, C]^a\) and \(X = X/Y\). An induction argument applied to the action of \(A\) on \(X\) with respect to \(C\) yields that

\[h(\bar{X}) = h - 1 \leq \ell(A : C) = \ell(A : B) - \ell(C : B)\]

whence \(h \leq \ell(A : B)\). This completes the proof of step (2).

(3) There exists \(B_1 \leq A\) such that \(B \triangleleft B_1\); and an irreducible complex \(XB_1\)-submodule \(M\) such that \(M_X\) is homogeneous, \(P \not\subseteq \text{Ker}(X\text{ on } M)\), \([X, B] \leq \text{Ker}(X\text{ on } M)\), \(C_M(B) = M\) and \(C_M(B_1) = 0\).

Proof. Clearly \(V \not\subseteq \langle G, B \rangle = \bigcap_{a \in A}[G, B]^a = 1\). Note that \(V_{XB}\) is completely reducible as \(XB \triangleleft \triangleleft XA\). Let \(W\) be an irreducible \(XB\)-submodule of \(V\) on which \(P\) acts nontrivially. If \(W \not\subseteq \langle G, B \rangle\) then \([W, B] \leq W \cap [G, B] = 1\) and so \([X, B] \leq \text{Ker}(X\text{ on } W)\) by the three subgroups lemma. Therefore \(W_X\) is irreducible. Let \(U\) be the \(X\)-homogeneous component of \(V\) containing \(W_X\). Then \(C_U(B) \neq 0\), \(P \not\subseteq \text{Ker}(X\text{ on } U)\), and \([X, B] \leq \text{Ker}(X\text{ on } U)\). Set now \(B_0 = N_A(U)\). Then \(U\) is an irreducible \(XB_0\)-module, and \(B\) is properly contained in \(B_0\) as \(C_U(B_0) = 0\).

Let \(\bar{k}\) be the algebraic closure of \(k = \mathbb{F}_{p_h}\). Let \(I\) be an irreducible submodule of \(U \otimes_k \bar{k}\). By the Fong-Swan theorem we may take an irreducible complex \(XB_0\)-module \(M_0\) such that \(\text{Ker}(X\text{ on } M_0) = KER(X\text{ on } U)\) and \(M_0\) gives \(I\) when reduced modulo \(p_h\). Thus \(C_{M_0}(B) \neq 0\), \(P \not\subseteq \text{Ker}(X\text{ on } M_0)\), \([X, B] \leq \text{Ker}(X\text{ on } M_0)\) and \(C_{M_0}(B_0) = 0\). Observe that \(B\) normalizes each \(X\)-homogeneous component of \(M_0\) as \([X, B] \leq \text{Ker}(X\text{ on } M_0)\). Let now \(M\) be an \(X\)-homogeneous component of \(M_0\) such that \(C_M(B) \neq 0\). Set \(B_1 = N_{B_0}(M)\). Then we have \(C_M(B_1) = 0\) as \(C_{M_0}(B_0) = 0\).

Suppose that \(B\) is not normal in \(B_1\), and let \(C = \langle B_{B_1} \rangle\). Then \([X, C] \leq \text{Ker}(X\text{ on } M)\). On the other hand, by (2) we have \(P = \langle \langle P, C \rangle \rangle^X\). This forces that \(P \leq [X, C] \triangleleft \text{Ker}(X\text{ on } M)\), which is not the case. Thus \(B \triangleleft B_1\) whence \(C_M(B)\) is a nonzero \(XB_1\)-submodule of \(M\), and so \([M, B] = 0\). Then \(M\) is an irreducible \(\mathbb{C}XB_1\)-module such that \(M_X\) is homogeneous, \(P \not\subseteq \text{Ker}(P\text{ on } M)\), \([X, B]B \leq \text{Ker}(XB\text{ on } M)\), and \(C_M(B_1) = 0\).

(4) Theorem follows.

Proof. We consider the set of all pairs \((M_a, C_a)\) such that \(B \leq C_a \subseteq B_1\), \(M_a\) is an irreducible \(XC_a\)-submodule of \(M_{XC_a}\) and \(C_{M_a}(C_a) = 0\). Choosing \((M_1, C)\) with \(|C|\) minimum. Then \(C_{M_1}(C_0) \neq 0\) for every \(B \leq C_0 < C\), \((M_1)_X\) is homogeneous and \(\text{Ker}(X\text{ on } M_1) = \text{Ker}(X\text{ on } M)\).

Set now \(\bar{X} = X/\text{Ker}(P\text{ on } M)\). We can observe that \([Z(\bar{P}), C] = 1\); Otherwise, it follows by Theorem 3.3 that for any \(P\)-homogeneous component \(U\) of \((M_1)_P\), the
module $U$ is $C$-invariant and $X = N_X(U)C_X(C)$. Then $C_X(C)$ acts transitively on the set of all $P$-homogeneous components of $M_1$. Clearly we have $[Z(P), C] \leq Ker(P|U)$ and hence $[Z(P), C] = 1$, as claimed. Thus if $P$ is abelian, then $[P, C] \leq Ker(P|M)$ and hence $P = \langle [P, C] \rangle^X \leq Ker(P|M)$ by (2), which is not the case. Therefore $P$ is nonabelian.

Let now $U$ be a homogeneous component of $(M_1)_{\Phi(P)}$. Notice that $\Phi(P) \leq Z(P)$ by (5) of Definition 2.7 and so $[\Phi(P), C] = 1$. Then $U$ is $C$-invariant. Set $\hat{P} = \hat{P}/Ker(P|U)$. Now $\Phi(\hat{P}) = \hat{\Phi(P)}$ is cyclic of prime order $p$. Since $[Z(P), C] = 1$ we get $[X, C] \leq C_X(Z(P))$ by the three subgroups lemma. Now clearly we have $[X, C] \leq N_X(U)$. That is $X = N_X(U)C_X(C)$ as the action is good and so $C_X(C)$ acts transitively on the set of all homogeneous components of $(M_1)_{\Phi(P)}$. Hence $M_1 = \bigoplus_{t \in T} U^t$ where $T$ is a transversal for $N_X(U)$ in $X$ contained in $C_X(C)$. Notice that $N_{XC}(U) = N_X(U)C$. Set $X_1 = C_X(\Phi(P))$. Now $C_{XC}(\Phi(P)) = X_1C < XC$ and we have $[X, C] \leq X_1$ by the three subgroups lemma. Then $X = X_1C_X(C)$. Clearly we have $PS_{h-2} \leq X_1 \leq N_X(U)$ and $X_1C < XC$. Recall that $P/\Phi(P)$ is an irreducible $XC$-module and hence $P/\Phi(P)$ is completely reducible as an $X_1C$-module. Note that $\hat{P}/\Phi(\hat{P}) \cong P/\Phi(P)C_P(U)$. As $P/\Phi(P)$ is completely reducible we see that so is $P/\Phi(P)C_P(U)$. Hence $\hat{P}/\Phi(\hat{P})$ is also completely reducible.

Suppose that $\Phi(P) \neq Z(P))$. Then there is an $X_1C$-invariant subgroup $E$ containing $\Phi(P)$ so that

$$\hat{P}/\Phi(\hat{P}) = Z(\hat{P})/\Phi(\hat{P})) \oplus E/\Phi(\hat{P}).$$

Then $\hat{P} = Z(\hat{P})E$ and hence $Z(\hat{P}) \cap E = Z(E)$. Clearly we have $(\hat{P})' = \Phi(\hat{P})) \leq Z(E)$. Also,

$$E/\Phi(\hat{P}) \cap Z(\hat{P})/\Phi(\hat{P}) = 1$$

and hence $Z(E) \leq \Phi(\hat{P})$. Thus we have $Z(E) = \Phi(\hat{P}) = (\hat{P})'$. As $E \leq \hat{P}$ we get $\Phi(E) \leq \Phi(\hat{P}) = Z(E)$. It follows that $Z(E) = E' = \Phi(E) = \Phi(\hat{P})$ is cyclic of prime order and hence $E$ is extraspecial. Now $[Z(\hat{P}), C] = 1$ gives $[Z(\hat{P}), C] = 1$. Thus $[Z(E), C] = 1$.

Next we observe that $C_C(E) \leq B$: Otherwise there is a nonidentity element $b$ in $C \setminus B$ such that $[\hat{P}, b] = 1$ and hence $[\hat{P}, b] \leq Ker(\hat{P}|U)$. Since $X = X_1C_X(C) \leq N_X(U)C_X(C)$ we get $[\hat{P}, b] \leq Ker(\hat{P}|M)$. Set $C_1 = B(b)$. Recall that $[P, B] \leq Ker(P|M)$ by (3). Then, by (2), we have $P = \langle [P, C_1] \rangle^X \leq Ker(P|M)$ which is not the case. Therefore $C_C(E) \leq B$ as claimed.

Notice that $p$ divides $|B/C_C(E)|$ if and only if $B_p \not\leq C_C(E)$ which is impossible by Proposition 2.5 due to good action. This means by (1) that $p$ is coprime to $|C/C_C(E)|$. Note also that $B$ and hence $C_C(E)$ acts trivially on $U$ by (3). We apply now Lemma 2.1 in [3] to the action of the semidirect product $E(C/C_C(E))$ on the module $U$ and see that $C_U(C/C_C(E)) \neq 0$. This final contradiction completes the proof. \[\square\]
REFERENCES

[1] S.D. Bell, B. Hartley, A note on fixed-point-free actions of finite groups, Quart. J. Math. Oxford Ser. (2) 41 no. 162, 127–130 (1990).

[2] V. V. Belyaev, B. Hartley, Centralizers of finite nilpotent subgroups in locally finite groups. Algebra Logika 35 (1996) 389–410; English transl. Algebra Logic 35 (1996) 217–228.

[3] K. N. Cheng, Finite groups admitting fixed point free automorphisms of order $pq$, Groups St. Andrews 1985 Proc. Edinburgh Math. Soc. (2) 30 no. 1 (1987) 51–56.

[4] E. C. Dade, Carter subgroups and Fitting heights of finite solvable groups, Illinois J. Math. 13 (1969) 449–514.

[5] G. Ercan, İ. Ş. Güloğlu, Fixed point free action on groups of odd order, J. Algebra 320 (2008), no. 1, 426–436.

[6] G. Ercan, İ. Ş. Güloğlu, On finite groups admitting a fixed point free automorphism of order $pqr$, J. Group Theory 7 4(2004) 437–446.

[7] G. Ercan, İ. Ş. Güloğlu, and Ö. Sağdıçoğlu, Fixed point free action of an abelian group of odd nonsquarefree exponent, Proc. Edinburgh Math. Soc. 54 no.1 (2011) 77–89.

[8] G. Ercan, A Fitting length conjecture without the coprimeness condition, Monatsh. Math. 167 no. 2 (2012) 175–187.

[9] A. Espuelas, Regular orbits on symplectic modules, J.Algebra 138 (1991) 1–12.

[10] B. Hartley, I. M. Isaacs, On characters and fixed points of coprime operator groups, J.Algebra 131 (1990) 342–358.

[11] I. M. Isaacs, Character theory of finite groups, AMS Chelsea Publishing, Providence, RI, 2006. xii+310 pp. ISBN: 978-0-8218-4229-4; 0-8218-4229-3.

[12] E. Jabara, The Fitting length of finite soluble groups II: Fixed-point-free automorphisms, J. Algebra 487 161–172.

[13] H. Kurzweil, p-Automorphismen von auflösbaren $p'$-Gruppen, Math. Z. 120 (1971), 326–354.

[14] J. G. Thompson, Automorphisms of soluble groups, J.Algebra 1 (1964) 259–267.

[15] J. G. Thompson, Finite Groups with Fixed-Point-Free Automorphisms of Prime Order, Proceedings of the National Academy of Sciences of the United States of America, 45 no. 4 (1959) 578–581.

[16] A. Turull, Fixed point free action with regular orbits, J. Reine Angew. Math. 371 (1986) 67–91.

[17] A. Turull, Fitting height of groups and of fixed points, J. Algebra 86 (1984), 555–556.

[18] A. Turull, Character theory and length problems. Finite and locally finite groups (Istanbul, 1994), 377-400, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 471, Kluwer Acad. Publ., Dordrecht (1995)

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