A simple general proof of gauge invariance in QED is given in the framework of causal perturbation theory. It illustrates a method which can also be used in non-abelian gauge theories.
The existing rigorous proofs of gauge invariance in perturbative QED [1, 2] are rather involved due to ultraviolet and infrared problems. The latter are severe enough, so that these proofs do not apply to the case of massless fermions. We present a simple general proof to fill this gap. It covers both the massless and massive case.

The general idea of proving gauge invariance is well-known: Gauge invariance can only be spoiled by certain local anomalous terms, one then tries to absorb those terms by (finite!) renormalizations. It is non-trivial that this is possible, because there are more anomalies than there is freedom in normalization. Some further property of QED must be used. Here different possibilities exist. We will show that charge conjugation (C-invariance) is sufficient for our purpose. We do not discuss the problem of fixing all normalization constants of the theory by further physical conditions. This would require a careful study of the infrared problems in the adiabatic limit, which lies beyond the scope of this note. Such problems were studied by Hurd [6] using the methods of [1]. In our approach to gauge invariance, the infrared problem plays no role.

We work with causal perturbation theory [3-5] where there is no ultraviolet problem and only well defined, finite quantities appear. In this framework, gauge invariance is expressed as follows [3]: Let $g(x) \in S(\mathbb{R}^4)$ be a test function (switching function) and $S(g)$ the perturbatively defined S-matrix

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \cdots g(x_n),$$

and let the $n$-point function $T_n$ be normally ordered with respect to the (free) photon operators

$$T_n(x_1, \ldots, x_n) = \sum_{l=0}^{n} \sum_{1 \leq k_1 < \ldots < k_l \leq n} \left( \sum_{\mu_1, \ldots, \mu_l} \right) t_{k_1 \ldots k_l}^{\mu_1 \cdots \mu_l}(x_1, \ldots, x_n)$$

$$\times : A_{\mu_1}(x_{k_1}) \cdots A_{\mu_l}(x_{k_l}) :.$$  

Then we must have

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} t_{k_1 \ldots k_l}^{\mu_1 \cdots \mu_l}(x_1, \ldots, x_n) = 0,$$

for all $1 \leq l \leq n$, all $1 \leq j \leq l$, all $1 \leq k_1 < \ldots < k_l \leq n$ and all $(x_1, \ldots, x_n) \in \mathbb{R}^{in}$. The $t$'s in (2) contain the Fermi operators: $t_{k_1 \ldots k_l}^{\mu_1 \cdots \mu_l}$ is the sum of all graphs of order $n$ with external photon lines at the vertices $x_{k_1}, \ldots, x_{k_l}$ and no other external photon lines, the external fermions being arbitrary. It follows from (2) that $t_{k_1 \ldots k_l}^{\mu_1 \cdots \mu_l}$ is symmetrical in $(x_{k_1}, \mu_1) \ldots (x_{k_l}, \mu_l)$.

It is our aim to prove gauge invariance (3) by induction on $n$. For the beginning of the induction we refer to [3], Chap.3.11, and we consider $n > 3$ here. Let us assume that (3) holds for all $m$-point distributions with $m \leq n - 1$. Going from $n - 1$ to $n$ according to the inductive construction [3], we have first to form

$$R'_n(x_1, \ldots, x_n) = \sum_X T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X),$$

where $\tilde{T}_{n_1}$ comes from the perturbation expansion (1) of $S(g)^{-1}$. Each term in (4) is a product of $T_m$'s with $m \leq n - 1$ and disjoint arguments. In virtue of the induction assumption,
each term is gauge invariant, because the normal ordering in the photon operators does not affect it. The same is true for

\[ A'_n(x_1, \ldots, x_n) = \sum_X \hat{T}_{n_1}(X)T_{n-n_1}(Y, x_n), \]

and for

\[ D_n = R'_n - A'_n. \]

This distribution has causal support with respect to \( x_n \):

\[ \text{supp} D_n \subseteq \Gamma^+_n(x_n) \cup \Gamma^-_n(x_n), \]

where \( V^\pm(x) \) is the closed forward or backward cone of \( x \), respectively. The essential step in the inductive construction is the splitting of \( D_n = R_n - A_n \) into a retarded and advanced part. There remains to prove that gauge invariance is preserved under this operation. The final steps \( T_n = R_n - R'_n \) and symmetrization with respect to \( x_n \) do not affect gauge invariance.

Our starting point is the relation

\[ \frac{\partial}{\partial x_n^\nu} \text{def} = \frac{\partial}{\partial x_n^\nu} d_{k_1 \ldots k_l}^{\mu_1 \ldots \mu_{l-1} \nu}(x_1, \ldots, x_{n-1}, x_n) = 0, \]

where we have introduced a shorthand notation and have taken \( x_{k_j} = x_n, \mu_j = \nu \) for simplicity. Since the retarded part is equal to

\[ r_\nu(x_1, \ldots, x_n) = \begin{cases} d_\nu(x_1, \ldots, x_n) & \text{on } \Gamma^+_n(x_n) \setminus (x_n, \ldots, x_n) \\ 0 & \text{on } \Gamma^-_n(x_n) \cup C. \end{cases} \]

we conclude from (9) that \( \partial_\nu r_\nu \) can only have a point support:

\[ \text{supp} \partial_\nu r_\nu(x_1, \ldots, x_n) \subseteq (x_n, \ldots, x_n). \]

We decompose

\[ d_\nu = \sum_{f=0}^n d_{f'}^\nu, \quad r_\nu = \sum_{f=0}^n r_{f'}^\nu \]

into contributions of all graphs of order \( n \) with \( 2 \cdot f \) external fermion lines and \( l \) external photon lines. According to (9), we must separately have

\[ \partial_\nu d_{f'}^\nu(x_1, \ldots, x_n) = 0, \quad \forall 0 \leq f' \leq n. \]

It is important to note that there are no terms with derivative \( \partial_\nu \) acting on a Fermi field operator for the following reason: This derivative comes from an external photon operator \( A_\nu(x_n) \). If there is also a Fermi operator \( \psi(x_n) \), the vertex \( x_n \) is connected with the rest of the graph by a single fermion line, represented by a Fermi propagation function \( S_{\text{ret}}(x_n - x_j) \), \( S_{\text{av}} \) or \( S_F \). Such a contribution can be reduced by means of the Dirac equation

\[ \frac{\partial}{\partial x_n^\nu} \left( \bar{\psi}(x_n) \gamma^\nu S_{\text{ret}}(x_n - x_j) \right) = i\bar{\psi}(x_n) \delta^4(x_n - x_j), \]
\[
\frac{\partial}{\partial x_n} \left( S_{\text{ret}}(x_j - x_n) \gamma^n \psi(x_n) \right) = -i\delta^4(x_j - x_n)\psi(x_n). \tag{15}
\]

Next we carry out the normal product decomposition and split all numerical distributions. Now (11) must also hold for every \( r^f_j \), separately. Using a well-known theorem on distributions with point-like support, we arrive at

\[
\partial_{\nu} r^f_j(x_1, \ldots, x_n) = \sum_g : \overline{\psi}(x_{i_1}) \ldots \overline{\psi}(x_{i_{f'}}) \left( \sum_{|a| \leq \omega(g) + 1} K^g_a D^a \delta(x_1 - x_n) \right) \ldots \delta(x_{n-1} - x_n) \psi(x_{i_1}) \ldots \psi(x_{i_{f'}}) :, \tag{16}
\]

where the sum runs over all graphs \( g \) with \( 2f' \) external fermions and \( l \) photons. \( \omega(g) \) is the singular order of the graph \( g \) which is given by the simple expression \[3\]
\[
\omega(g) = 4 - 3f' - l. \tag{17}
\]

In (16) we have used the fact that the derivative increases \( \omega \) by 1.

For \( \omega < -1 \), the inner sum in (16) contains no term, hence the expression vanishes, which proves the desired divergence relation in this case. There remains to investigate the possible cases of \( \omega \geq -1 \). In virtue of (17), there are only the following four cases of this kind:

\[
(f', l) = (0, 2) \ (0, 4) \ (1, 1) \ (1, 2)
\]

\[
\omega = 2 \ 0 \ 0 \ -1 \ \text{case : I II III IV.} \tag{18}
\]

These cases must now be examined. According to the lemma in Sect.3.2 of ref.[2], we have only to consider connected diagrams. However, combinations of field operators which differ from each other in \( d_{\nu} d_{\nu} \) may agree in \( \partial_{\nu} d_{\nu} \), in virtue of the identities (14) and (15).

**Case I:** \( (f', l) = (0, 2), \omega = 2 \)

This is vacuum polarization. Since \( f' = 0 \), Eq.(16) does not contain any field operator, so that we have to deal with numerical distributions only. We write down the anomaly relation (16) for the \( t \)-distribution, assuming the external photon operators at \( x_1 \) and \( x_2 \) for convenience:

\[
\partial_{1\nu} \Pi^{\mu
u}(x_1, x_2; x_3, \ldots, x_n) = \left( \sum_{i,j,k} K_{ijk} \partial^\mu_i \partial_{j\alpha} \partial^\nu_k + \sum_k L_k \partial^\mu_k \right) \delta^{n-1}, \tag{19}
\]

where \( \delta^{n-1} = \delta(x_1 - x_n) \ldots \delta(x_{n-1} - x_n) \). Different from (16), \( x_n \) is an inner vertex here. The r.h.s. is the most general covariant local distribution with \( \omega = 3 \). We now claim:

**Proposition 1.** The anomaly (19) can be restricted to the following form

\[
\partial_{1\nu} \Pi^{\mu\nu} = \left[ K_1 \sum_{i=3}^n \partial^\mu_i \partial_{i\alpha} \partial^\nu_1 + K_2 \sum_{i=3}^n \partial^\mu_i \partial_{1\alpha} \partial^\nu_i + K_3 \partial^\mu_2 \partial_{2\alpha} \partial^\nu_1 + K_4 \partial^\mu_2 \partial_{2\alpha} \partial^\nu_1 + (K_3 + K_4) \partial^\mu_3 \partial_{1\alpha} \partial^\nu_1 + (K_5 + K_6) \partial^\mu_4 \partial_{1\alpha} \partial^\nu_1 + K_5 \partial^\mu_4 \partial_{2\alpha} \partial^\nu_1 + K_6 \partial^\mu_4 \partial_{2\alpha} \partial^\nu_1 + K_7 \partial^\mu_1 \right] \delta^{n-1}. \tag{20}
\]
Proof. Calculating the divergence of (19) with respect to \(x_2\), the result must be symmetric in \(x_1, x_2\):

\[
(\partial_{\mu} \partial_{\nu} \Pi^\mu\nu)(x_1, x_2; \ldots) = (\partial_{\mu} \partial_{\nu} \Pi^\mu\nu)(x_2, x_1; \ldots).
\]

(21)

This implies \(L_k = 0\) for \(k \neq 1\) and \(K_{ijk} = 0\) for \(i, j, k > 2\). Furthermore, in virtue of (21), only the following \(K_{ijk}\) can be \(\neq 0\): (i) \(K_{i1j}, K_{i1k}, K_{1jk}\), (ii) \(K_{mpk}, K_{mjp}, K_{imp}\), (iii) \(K_{imp}\), where \(l, m, p \leq 2, i, j, k > 2\).

We now use the property that (19) is symmetric in all inner vertices \(x_3, \ldots x_n\). This allows to express case (ii), depending on one internal index \(> 2\) only, by derivatives with respect to \(x_1, x_2\), for example:

\[
\sum_{k=3}^{n} K_{mnk} \partial_{\mu}^k \delta^{n-1} = K_{mn} \sum_{3}^{n} \partial_{\mu}^k \delta^{n-1} = K_{mn}(\partial_{\mu}^1 - \partial_{\mu}^2) \delta^{n-1},
\]

(22)

because

\[
\sum_{k=1}^{n} \partial_{\mu}^k \delta(x_1 - x_n) \cdot \delta(x_{n-1} - x_n) = 0.
\]

(23)

This reduces this case to case (iii). In case (i) the symmetry in the inner vertices implies

\[
K_{ij1} = K_{\pi i \pi j 1}, \quad K_{i1k} = K_{\pi i 1 \pi k}
\]

(24)

for all permutations \(\pi\). This means that the diagonal elements \(i = j, i = k\) are equal and all off-diagonal elements also. By redefining these constants, the latter sum over \(i \neq j\) can be transformed into an independent summation over \(i, j = 3, \ldots n\), which, by (23), is also reduced to case (iii). There remains the summation over diagonal terms, leading to the first two terms in (20). Finally, in case (iii) we have \(K_{111} = K_{122} + K_{221}, K_{222} = 0\) due to (21). Then we arrive at the remaining terms in (20).

In the anomaly (20) the derivative \(\partial_{\mu} \Pi^\mu\nu\) can now be taken out

\[
\partial_{\mu} \Pi^\mu\nu = \partial_{\mu} \left[ g^{\nu\alpha} K_1 \sum_{i=3}^{n} \partial_{\alpha}^i \partial_{\alpha}^i + K_2 \sum_{i=3}^{n} \partial_{\nu}^i \partial_{\mu}^i \\
+ K_3 (\partial_{\mu}^1 \partial_{\nu}^2 + \partial_{\mu}^2 \partial_{\nu}^1) + K_4 g^{\mu\nu}(\mathbf{1} + \mathbf{2}) + K_5 \partial_{\nu}^1 \partial_{\mu}^2 + K_6 \partial_{\nu}^2 \partial_{\mu}^1 + K_7 g^{\mu\nu} \right] \delta^{n-1}.
\]

(25)

The square bracket is a polynomial of degree \(\omega(\Pi^{\mu\nu}) = 2\) and has the symmetry properties of \(\Pi^{\mu\nu}\). Therefore it can be transformed away by renormalization of \(\Pi^{\mu\nu}\) which completes the proof of case I.

Case II: \((f', l) = (0, 4), \omega = 0\)

This is photon-photon scattering where we have again to deal with one numerical distribution only. As in the proof of Prop.1 (21), the anomaly can be expressed by derivatives with respect to the external coordinates \(x_1, \ldots x_4\), using the symmetry in the inner vertices:

\[
\partial_{\mu} t^{\nu\alpha\beta}(x_1, x_2, x_3, x_4; x_5, \ldots x_n) = \]
\begin{equation}
\sum_{k=1}^{4} \left( K_{k1} \partial_{\mu} g^{\alpha \beta} + K_{k2} \partial_{\nu} g^{\mu \beta} + K_{k3} \partial_{\rho} g^{\mu \alpha} \right) \delta^{n-1}.
\end{equation}

Again, \( x_n \) is an inner vertex here. Since \( \partial_{1\nu} \partial_{2\mu} \partial_{3\alpha} \partial_{4\beta} t^{\nu \mu \alpha \beta} \) must be symmetric in \( x_1, \ldots, x_4 \), it follows that \( K_{11} = K_{12} = K_{13} = K \) and all other \( K_{kj} = 0 \). Then

\begin{equation}
\partial_{1\nu} t^{\nu \mu \alpha \beta} = K \partial_{1\nu} \left( g^{\nu \mu} g^{\alpha \beta} + g^{\nu \alpha} g^{\mu \beta} + g^{\nu \beta} g^{\mu \alpha} \right) \delta^{n-1},
\end{equation}

and this anomaly can again be transformed away by renormalization of \( t^{\nu \mu \alpha \beta} \).

\textit{Case III}: \( (f', l) = (1, 1), \omega = 0 \)

We have one external photon operator in this case which is now attached to \( x_n \), which is the differentiation variable in agreement with (16). The decomposition of (16) leads to the following classes of diagrams: (i) the vertex function with external Fermi operators \( \overline{\psi}(x_i) \psi(x_j) \), \( 1 \leq i \neq j \leq n - 1 \), (ii) taking (14) and (15) into account, we have to include reducible diagrams containing the self-energy \( \Sigma \), (iii) there is an additional class of reducible diagrams containing the vacuum polarization tensor \( \Pi^{\mu \nu} \). The diagrams (ii) and (iii) have external Fermi operators \( \overline{\psi}(x_i) \psi(x_i) \). For fixed \( x_i, x_j \), the anomaly relation for the \( t \)-distribution reads

\begin{equation}
\partial_{n\nu} \left[ \overline{\psi}(x_i) \Delta'(\ldots) \psi(x_j) : + \overline{\psi}(x_n) \gamma^{\nu} S_F(x_n - x_i) \Sigma(\ldots) \psi(x_j) : + \overline{\psi}(x_i) \Sigma(\ldots) S_F(x_j - x_n) \gamma^{\nu} \psi(x_n) : + \overline{\psi}(x_i) \Pi^{\nu \mu}(\ldots) D_F(x_j - x_i) \gamma^{\mu} \psi(x_i) : \right] = \\
= \overline{\psi}(x_i) \left( K_0 1 + K_1 \partial_1 + K_2 \partial_2 + K_3 \partial_3 \right) \partial_4 \delta^{n-1} \psi(x_j) : .
\end{equation}

Here we have used gauge invariance \( \partial_{4\nu} \Pi^{\nu \mu} = 0 \) (case I) in order \( n - 1 \).

\textbf{Proposition 2.} The anomaly (28) can be restricted to

\begin{equation}
K \cdot \overline{\psi}(x_i) \partial_4 \partial_1 \delta^{n-1} \psi(x_j) : .
\end{equation}

\textit{Proof.} The total anomaly, i.e. the sum over all \( 1 \leq i \neq j \leq n - 1 \) of (28), is symmetric in \( x_1, \ldots, x_{n-1} \). Instead of explicit symmetrization, it is simpler to consider (28) on totally symmetric test functions \( \varphi(x_1, \ldots, x_n) \). Then the term with \( \partial_1 \delta \) can be transformed by means of the Dirac equation as follows

\begin{equation}
\int \overline{\psi}(x_i) \partial_1 \delta^{n-1} \psi(x_j) \cdot \varphi(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = \\
= - \int \overline{\psi}(x_n) \gamma^{\nu} \psi(x_n) : \left( \partial_{4\mu} \varphi \right)(x_n, \ldots, x_n) \, dx_n + \\
+ \text{im} \int \overline{\psi}(x_n) \psi(x_n) : \varphi(x_n, \ldots, x_n) \, dx_n = \\
= \int \overline{\psi}(x_i) \partial_4 \delta^{n-1} \psi(x_j) : \varphi(x_1, \ldots, x_n) \, dx_1 \ldots dx_n + \\
+ \text{im} \int \overline{\psi}(x_n) \psi(x_n) : \varphi(x_n, \ldots, x_n) \, dx_n,
\end{equation}
for \( k \neq i,j,n \), and similarly for \( \bar{\theta} \delta \). That means \( \bar{\theta} \delta \) can be substituted by \((\bar{\theta}_1 + \ldots + \bar{\theta}_{n-1})\delta/(n-1)\) plus the mass terms. The latter may be included in \( K_0 \) in (28). Using (22),
the anomaly assumes the form

\[
: \bar{\psi}(x_i)(K_0' \mathbf{1} + K \bar{\theta}_n)\delta^{n-1}\psi(x_j) : .
\] (31)

Now we use charge conjugation invariance

\[
U_c T_n(x_1, \ldots, x_n) U_c^{-1} = T_n(x_1, \ldots, x_n).
\] (32)

Since (28) is multiplied by the photon operator \( A(x_n) \), which changes sign under charge conjugation, the anomaly (28), (31) must also change sign if the \( C \)-conjugated term of (28) is enclosed. This implies \( K_0' = 0 \), which completes the proof of the proposition.

The remaining anomaly (29) can be transformed away by renormalization of the vertex function \( \Lambda' \).

**Case IV:** \( (f', l) = (1, 2), \omega = -1 \)

In this case we have two external Fermi and two photon operators. If the former have coordinates \( x_i, x_j \) and the latter \( x_n \) and \( x_{n-1} \), the anomaly relation for the \( t \)-distribution is of the following form:

\[
\partial_{\mu\nu} \left[ : \bar{\psi}(x_i)\Gamma_4^{\mu\nu}(\ldots)\psi(x_j) : + \text{reducible terms} \right] = K_{ij} : \bar{\psi}(x_i)\gamma^\mu\psi(x_j) : \delta^{n-1}.
\] (33)

Here \( \Gamma_4^{\mu\nu} \) is the contribution of the irreducible diagrams, the reducible terms may have other arguments in the spinor operators, similar to (28). Adding the \( C \)-conjugated equations, the l.h.s. is even under charge conjugation, because it is multiplied by two photon operators in the total \( n \)-point distribution \( T_n \). The anomalous terms on the r.h.s. add up to one local term with support \( x_i = \ldots = x_j = x_n \). The latter is odd under charge conjugation, hence, the factor \( K \) in front must vanish. This completes the proof of gauge invariance.

It should be emphasized that gauge invariance in the sense of (3) is weaker than the \( C \)-number Ward identities [2]. The latter are proven here in case III between spinor fields only. To prove them in general needs some input about the infrared behaviour, as the existence of the central splitting solution in the massive case [2]. We will return to this question elsewhere. But condition (3) is all what is needed for the proof of unitarity [3]. The method described here can also be used in Yang-Mills theories [7].

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