Local rings with quasi-decomposable maximal ideal

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Abstract

Let $(R, m)$ be a commutative noetherian local ring. In this paper, we prove that if $m$ is decomposable, then for any finitely generated $R$-module $M$ of infinite projective dimension $m$ is a direct summand of (a direct sum of) syzygies of $M$. Applying this result to the case where $m$ is quasi-decomposable, we obtain several classifications of subcategories, including a complete classification of the thick subcategories of the singularity category of $R$.

1. Introduction

Let $R$ be a commutative noetherian local ring with maximal ideal $m$. First, we investigate the structure of syzygies of finitely generated $R$-modules in the case where $m$ is decomposable as an $R$-module. Our main result in this direction is Theorem 3·6, which includes the following remarkable statement.

**Theorem A.** Let $R$ be a commutative noetherian local ring with decomposable maximal ideal $m$. Let $M$ be a finitely generated $R$-module with infinite projective dimension. Then $m$ is a direct summand of $\Omega^3 M \oplus \Omega^4 M \oplus \Omega^5 M$.

Here, $\Omega^n(−)$ stands for the $n$th syzygy in the minimal free resolution. This result (Theorem 3·6 strictly) recovers and refines main theorems of the first author and Sather–Wagstaff [16] on the vanishing of Tor and Ext modules over fiber products.

Next, we apply the above theorem to classification problems of subcategories over a Cohen–Macaulay local ring with quasi-decomposable maximal ideal. Here, we say that an ideal $I$ of $R$ is quasi-decomposable if there exists an $R$-regular sequence $x = x_1, \ldots, x_n$ in $I$ such that $I/(x)$ is decomposable. Examples of rings with quasi-decomposable maximal ideal include the 2-dimensional non-Gorenstein normal local domains with a rational singularity (see Example 4·8); more examples of such rings are given in Section 4. Our main
result in this direction is Theorem 4.5, which especially yields a complete classification of the thick subcategories of the singularity category $\mathcal{D}_{sg}(R)$ of such $R$:

**Theorem B.** Let $R$ be a Cohen–Macaulay singular local ring whose maximal ideal is quasi-decomposable. Suppose that on the punctured spectrum $R$ is either locally a hypersurface or locally has minimal multiplicity. Then taking the singular supports induces a one-to-one correspondence between the thick subcategories of $\mathcal{D}_{sg}(R)$ and the specialization-closed subsets of $\text{Sing } R$.

Here, $\text{Sing } R$ stands for the singular locus of $R$. The singularity category $\mathcal{D}_{sg}(R)$, which is also called the stable derived category of $R$, is a triangulated category that has been introduced by Buchweitz [4] and Orlov [18], and studied deeply and widely so far; see [12] and references therein. The singular support of an object $C$ of $\mathcal{D}_{sg}(R)$ is defined as the set of prime ideals $p$ such that $C_p$ is non-zero in $\mathcal{D}_{sg}(R_p)$. The inverse map of the bijection in the above theorem can also be given explicitly.

There are indeed a lot of examples satisfying the assumptions of Theorems A and B (more precisely, Theorems 3-6 and 4-5), and we shall present some of them.

The organisation of this paper is as follows. Section 2 is devoted to notation, definitions and some basic properties which are used in later sections. In Section 3, we prove a structure result of syzygies of modules over a local ring with decomposable maximal ideal, which gives rise to Theorem A as an immediate corollary. In Section 4, for a Cohen–Macaulay local ring with quasi-decomposable maximal ideal, we classify resolving/thick subcategories of module/derived/singularity categories, including Theorem B. Sections 5 and 6 state several other applications of our main results, including the main theorems of [16].

2. **Background and conventions**

This section contains the terminology and some of the definitions and their basic properties that will tacitly be used in this paper. For more details see [20, 21, 22].

2.1. Throughout this paper, $R$ is a commutative noetherian local ring with maximal ideal $m$ and residue field $k$, and all modules are finitely generated.

2.2. Let $M$ be an $R$-module. All syzygies of $M$ are calculated by using the minimal free resolution of $M$, and the $i$-th syzygy of $M$ is denoted by $\Omega^i_R M$. The minimal number of generators for $M$ is denoted by $\nu_R(M)$. For an integer $i \geq 0$ the $i$th Betti number of $M$ is denoted by $\beta^R_i(M)$; note by definition that $\beta^R_i(M) = \nu_R(\Omega^i_R M)$.

2.3. We denote by $\text{Sing}(R)$ the singular locus of $R$, namely, the set of prime ideals $p$ of $R$ for which the local ring $R_p$ is singular (i.e. non-regular). Also, $\text{NonGor}(R)$ stands for the non-Gorenstein locus of $R$, that is, the set of prime ideals $p$ of $R$ such that $R_p$ is not Gorenstein.

2.4. We denote the punctured spectrum $\text{Spec } R \setminus \{m\}$ of $R$ by $\text{Spec}^0 R$. Recall that $R$ is said to have an isolated singularity if $R_p$ is a regular local ring for all $p \in \text{Spec}^0 R$, in other words, if $\text{Sing } R \subseteq \{m\}$.

2.5. A subset $S$ of $\text{Sing } R$ is called specialization-closed provided that for prime ideals $p \subseteq q$ of $R$ if $p$ is in $S$, then so is $q$. 
2-6. Throughout this paper, all subcategories are full and closed under isomorphism. We denote by mod \( R \) the category of (finitely generated) \( R \)-modules and by PD(\( R \)) the subcategory of mod \( R \) consisting of modules of finite projective dimension. Also CM(\( R \)) is the subcategory of mod \( R \) consisting of maximal Cohen–Macaulay \( R \)-modules. By add \( R \) we denote the subcategory of mod \( R \) consisting of direct summands of finite direct sums of copies of \( R \).

2-7. We denote by \( D^b(\mathcal{X}) \) the bounded derived category of mod \( R \), and by D\( _{sg}(\mathcal{X}) \) the singularity category of \( R \), that is, the Verdier quotient of \( D^b(\mathcal{X}) \) by perfect complexes. We denote by \( \pi : D^b(\mathcal{X}) \to D_{sg}(\mathcal{X}) \) the canonical functor.

For a subcategory \( \mathcal{X} \) of \( D^b(\mathcal{X}) \) we denote by \( \pi \mathcal{X} \) the subcategory of \( D_{sg}(\mathcal{X}) \) consisting of objects \( M \) such that \( M \cong \pi X \) for some \( X \in \mathcal{X} \). For a subcategory \( \mathcal{Y} \) of \( D_{sg}(\mathcal{X}) \) we denote by \( \pi^{-1} \mathcal{Y} \) the subcategory of \( D^b(\mathcal{X}) \) consisting of objects \( N \) such that \( \pi N \in \mathcal{Y} \).

2-8. For an object \( M \in D_{sg}(\mathcal{X}) \), the singular support ssupp \( M \) of \( M \) is defined as the set of prime ideals \( \mathfrak{p} \) of \( R \) such that \( M_{\mathfrak{p}} \not\cong 0 \) in \( D_{sg}(\mathcal{X})_{\mathfrak{p}} \).

For a subcategory \( \mathcal{X} \) of \( D_{sg}(\mathcal{X}) \), the singular support ssupp \( \mathcal{X} \) of \( \mathcal{X} \) is defined by ssupp \( \mathcal{X} := \bigcup_{X \in \mathcal{X}} \text{ssupp} \ X \).

For a subset \( S \) of Spec \( R \), we denote by ssupp\(^{-1} \) \( S \) the subcategory of \( D_{sg}(\mathcal{X}) \) consisting of objects whose singular supports are contained in \( S \).

2-9. For an \( R \)-module \( M \), the non-free locus NF(\( M \)) (resp. infinite projective dimension locus IPD(\( M \))) of \( M \) is defined as the set of prime ideals \( \mathfrak{p} \) of \( R \) such that \( M_{\mathfrak{p}} \) is non-free (resp. of infinite projective dimension) over \( R_{\mathfrak{p}} \).

For a subcategory \( \mathcal{X} \) of mod \( R \), the non-free locus NF(\( \mathcal{X} \)) (resp. infinite projective dimension locus IPD(\( \mathcal{X} \))) of \( \mathcal{X} \) is defined by NF(\( \mathcal{X} \)) := \bigcup_{X \in \mathcal{X}} \text{NF} \( X \) (resp. IPD(\( \mathcal{X} \)) := \bigcup_{X \in \mathcal{X}} \text{IPD} \( X \)).

For a subset \( S \) of Spec \( R \), we denote by NF\(_{CM}(\mathcal{X})\(^{-1} \) \( S \)) (resp. IPD\(_{CM}(\mathcal{X})\(^{-1} \) \( S \)) the subcategory of CM(\( R \)) (resp. mod \( R \)) consisting of modules whose non-free (resp. infinite projective dimension) loci are contained in \( S \).

2-10. There are three types of restriction maps rest that we will consider in this paper. We clarify each one for the convenience of the reader.

(i) For a subcategory \( \mathcal{X} \) of mod \( R \), we set \( \text{rest}_{CM(\mathcal{X})}(\mathcal{X}) := \mathcal{X} \cap \text{CM}(\mathcal{X}) \).

(ii) For a subcategory \( \mathcal{X} \) of D\( ^b(\mathcal{X}) \), we set \( \text{rest}_{\text{mod}(\mathcal{X})}(\mathcal{X}) := \mathcal{X} \cap \text{mod}(\mathcal{X}) \).

(iii) For a subcategory \( \mathcal{X} \) of D\( _{sg}(\mathcal{X}) \), we set \( \text{rest}_{\text{CM}(\mathcal{X})}(\mathcal{X}) := \pi^{-1} \mathcal{X} \cap \text{CM}(\mathcal{X}) \).

2-11. Let \( \mathcal{A} \) be an abelian category with enough projective objects. We say that a subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is resolving if it contains the projective objects and is closed under direct summands, extensions, and kernels of epimorphisms. Note that a resolving subcategory of mod \( R \) is nothing but a subcategory containing \( R \) and closed under direct summands, extensions and syzygies.

For an object \( M \in \mathcal{A} \), we denote by res \( M \) the smallest resolving subcategory of \( \mathcal{A} \) that contains \( M \). This subcategory is called the resolving closure of \( M \).

2-12. Let \( \mathcal{A} \) (resp. \( \mathcal{T} \)) be an abelian category (resp. a triangulated category). A subcategory \( \mathcal{X} \) of \( \mathcal{A} \) (resp. \( \mathcal{T} \)) is called thick provided that \( \mathcal{X} \) is closed under direct summands, and for any exact sequence \( 0 \to L \to M \to N \to 0 \) in \( \mathcal{A} \) (resp. exact triangle \( L \to M \to N \hookrightarrow \) in \( \mathcal{T} \), if two of \( L, M, N \) are in \( \mathcal{X} \), then so is the third. The thick closure of a subcategory \( \mathcal{X} \)
of \( \mathcal{A} \) (resp. \( \mathcal{T} \)), denoted by \( \text{thick}_{\mathcal{A}} \mathcal{X} \) (resp. \( \text{thick}_{\mathcal{T}} \mathcal{X} \)), is by definition the smallest thick subcategory of \( \mathcal{A} \) (resp. \( \mathcal{T} \)) containing \( \mathcal{X} \). Note that \( \text{thick}_{\text{mod-}R} R = \text{PD}(R) \), and that \( \text{thick}_{\text{D}^b(R)} R \) consists of the perfect complexes, that is, bounded complexes of free \( R \)-modules (of finite rank). For a subcategory \( \mathcal{X} \) of \( \text{D}^b(R) \), we simply write \( \text{thick}_{\text{D}^b(R)} \mathcal{X} \) for \( \text{thick}_{\text{D}^b(R)}(\pi \mathcal{X}) \).

2.13. We say that \( R \) is a hypersurface if its \( m \)-adic completion \( \hat{R} \) is isomorphic to the quotient of a regular local ring by an element. (Hence, by definition, any regular local ring is a hypersurface.)

2.14. Let \( R \) be a Cohen–Macaulay local ring. Then \( R \) satisfies the inequality

\[
\text{e}(R) \geq \text{edim } R - \dim R + 1,
\]

where \( \text{e}(R) \) and \( \text{edim } R \) stand for the multiplicity of \( R \) and the embedding dimension of \( R \), respectively. We say that \( R \) has minimal multiplicity (or maximal embedding dimension) if the equality of (2.14.1) holds. If the residue field \( k \) of \( R \) is infinite, then this is equivalent to the existence of a system of parameters \( x = x_1, \ldots, x_d \) of \( R \) such that \( m^2 = x_m \); see [3, exercise 4.6.14].

2.15. Let \( R \) be a Cohen–Macaulay local ring. We say that \( \text{Spec}^0 R \) is a hypersurface (resp. has minimal multiplicity, has finite Cohen–Macaulay representation type) if the Cohen–Macaulay local ring \( R_p \) is a hypersurface (resp. has minimal multiplicity, has finite Cohen–Macaulay representation type) for every \( p \in \text{Spec}^0 R \). A Cohen–Macaulay local ring with an isolated singularity is a typical example satisfying all of these three conditions, so they are mild conditions.

3. Local rings with decomposable maximal ideal

This section is devoted to the proof of Theorem A. This theorem is a consequence of Theorem 3.6, which is the main result of this section.

The following fact was established by Ogoma [17, lemma 3.1] and gives a characterization of local rings with decomposable maximal ideal in terms of fiber products.

**Fact 3.1.** The maximal ideal \( m \) has a direct sum decomposition \( m = I \oplus J \) in which \( I, J \) are non-zero ideals of \( R \) if and only if \( R \) is isomorphic to the non-trivial fiber product \( (R/I) \times_k (R/J) \), that is, \( R \) is isomorphic to the pull-back of the natural surjections \( R/I \twoheadrightarrow k \leftarrow R/J \). This isomorphism is naturally defined by \( r \mapsto (r + I, r + J) \) for \( r \in R \). In this case, by [5, remark 3.1] we have the equality

\[
\text{depth } R = \min\{\text{depth } R/I, \text{ depth } R/J, 1\}.
\]

In particular, if \( m \) is decomposable and \( M \) is an \( R \)-module, then it follows from the Auslander–Buchsbaum formula that \( \text{pd}_R M \geq 2 \) if and only if \( \text{pd}_R M = \infty \).

**Lemma 3.2.** Suppose that \( m \) has a direct sum decomposition \( m = I \oplus J \). Let \( N \) be an \( R/I \)-module, and let \( n := \nu_{R/I} N = \nu_R (N) \). Then there is an isomorphism \( \Omega_R N \cong I^\oplus n \oplus \Omega_{R/I} N \) of \( R \)-modules.
Proof. There is a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
I^\oplus n & \rightarrow & I^\oplus n \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Omega_R N \\
\downarrow & & \downarrow \\
0 & \rightarrow & R^\oplus n \\
\downarrow & & \downarrow \\
0 & \rightarrow & N \rightarrow 0 \\
\end{array}
\]

with exact rows and columns. Now we have

\[
\Omega_{R/I} N = \text{Ker } \sigma \subseteq m(R/I)^\oplus n = (m/I)^\oplus n \cong J^\oplus n \subseteq R^\oplus n.
\]

Call this composite injection \( \eta: \Omega_{R/I} N \rightarrow R^\oplus n \). Then we have \( \theta = \varepsilon \eta, \rho \eta = \sigma \varepsilon \eta = \sigma \theta = 0 \). Hence, there is a homomorphism \( \zeta: \Omega_{R/I} N \rightarrow \Omega_R N \) such that \( \eta = \lambda \zeta \). Since \( \theta \zeta \bar{\varepsilon} = \varepsilon \lambda \zeta \bar{\varepsilon} = \varepsilon \eta = \theta \) and \( \theta \) is injective, we get \( \zeta \bar{\varepsilon} = 1 \). Therefore, the left column in the above diagram splits and we get \( \Omega_R N \cong I^\oplus n \oplus \Omega_{R/I} N \).

**Lemma 3-3.** Suppose that \( m \) has a direct sum decomposition \( m = I \oplus J \) such that \( J \neq 0 \). Then the following hold.

1. If \( R/I \) is a discrete valuation ring, then there is an isomorphism \( R/I \cong J \) of \( R \)-modules.
2. Conversely, if \( J \) is a free \( R/I \)-module, then \( R/I \) is a discrete valuation ring.

Proof. Note that \( m/I \cong J \) as \( R \)-modules.

1. By assumption we have \( m/I \cong R/I \). Hence, the assertion follows.
2. By assumption, \( m/I \) is a free \( R/I \)-module. Hence, the global dimension of \( R/I \) is at most one. If it is zero, then \( R/I \) is a field and we have \( I = m \), which contradicts \( J \) being non-zero. Thus, \( R/I \) is a discrete valuation ring.

The following lemma is shown by a similar argument as in [16, lemma 2-5].

**Lemma 3-4.** Let \( I \) be a proper ideal of \( R \). Let \( M \) be an \( R \)-module with \( \text{pd}_R M \geq 2 \). If \( \Omega_R^2 M \) is \( R/I \)-free, then \( R/I \) has positive depth.

Proof. By assumption there are injections \( R/I \rightarrow \Omega_R^2 M \rightarrow m^{\oplus e} = m^{\oplus e} / e \) for some \( e > 0 \), and call this composition \( \phi: R/I \rightarrow m^{\oplus e} \). Set \( \bar{s} = \phi(1) \in m^{\oplus e} \). If \( R/I \) has depth zero, then it contains a non-zero socle \( \bar{s} \), and we have \( \phi(\bar{s}) = s \bar{\varepsilon} = 0 \), which contradicts the injectivity of the map \( \phi \). Hence, \( R/I \) has positive depth.

**Proposition 3-5.** Suppose that \( m \) has a direct sum decomposition \( m = I \oplus J \) in which \( I \) and \( J \) are non-zero ideals of \( R \). Let \( M \) be an \( R \)-module with \( \text{pd}_R M \geq 2 \). If either \( R/I \) or \( R/J \) has depth zero, then \( \Omega_R^2 M \) is neither \( R/I \)-free nor \( R/J \)-free.

Proof. Assume that depth \( R/I = 0 \). (The case where depth \( R/J = 0 \) is handled similarly.) It follows from Lemma 3-4 that \( \Omega_R^2 M \) is not \( R/I \)-free. Suppose that \( \Omega_R^2 M \) is \( R/J \)-free.
Then again by Lemma 3.4 the local ring $R/J$ has positive depth. There is an exact sequence $0 \to (R/J)^{\oplus a} \to R^{\oplus b} \to R^{\oplus d} \to M \to 0$ where each $c_{ij}$ is an element of $m$ and $a, b, d$ are positive integers. Applying the functor $\text{Hom}_R(k, -)$ to this, we have an exact sequence

$$0 \to \text{Hom}_R(k, (R/J)^{\oplus a}) \to \text{Hom}_R(k, R^{\oplus b}) \to 0 \to \text{Hom}_R(k, R^{\oplus d}).$$

As $R/J$ has positive depth, $\text{Hom}_R(k, (R/J)^{\oplus a}) = 0$. We see from this exact sequence that $\text{Hom}_R(k, R^{\oplus b}) = 0$, and hence depth $R > 0$. This is a contradiction because by (3.1.1) we must have depth $R = 0$. Thus, $\Omega^4_R M$ is not $R/J$-free.

Theorem A is a consequence of the following result, which is the main result of this section.

**Theorem 3.6.** Let $(R, m)$ be a local ring, and suppose that there is a non-trivial direct sum decomposition $m = I \oplus J$. Let $M$ be an $R$-module with $\text{pd}_R M \geq 2$.

1. One of the following holds true:
   
   (i) $m$ is a direct summand of $\Omega^3_R M$ or $\Omega^4_R M$;
   
   (ii) $m$ is a direct summand of $\Omega^3_R M$, and $\Omega^2_R M$ is $R/I$-free;
   
   (iii) $m$ is a direct summand of $\Omega^3_R M$, and $\Omega^2_R M$ is $R/J$-free;
   
   (iv) $m$ is a direct summand of $\Omega^3_R M \oplus \Omega^4_R M$, $R/I$ is a discrete valuation ring, and $\Omega^2_R M$ is $R/J$-free;
   
   (v) $m$ is a direct summand of $\Omega^3_R M \oplus \Omega^4_R M$, $R/J$ is a discrete valuation ring, and $\Omega^2_R M$ is $R/I$-free.

2. If either $R/I$ or $R/J$ has depth 0, then none of (ii)-(v) occurs.

**Proof.** (1) It follows from [7] (see also [15, proposition 4.2]) that

$$\Omega^2_R M \cong A \oplus B, \quad \Omega^3_R M \cong V \oplus W, \quad \Omega^4_R M \cong X \oplus Y$$

for some $R/I$-modules $A, V, X$ and $R/J$-modules $B, W, Y$. By Lemma 3.2, we have

$$\Omega^2_R A \cong I^{\oplus v(A)} \oplus \Omega^2_R A, \quad \Omega^3_R V \cong I^{\oplus v(V)} \oplus \Omega^2_R I V, \quad \Omega^2_R X \cong I^{\oplus v(X)} \oplus \Omega^2_R I X, \quad \Omega^3_R B \cong J^{\oplus v(B)} \oplus \Omega^2_R B, \quad \Omega^3_R W \cong J^{\oplus v(W)} \oplus \Omega^2_R J W, \quad \Omega^3_R Y \cong J^{\oplus v(Y)} \oplus \Omega^2_R J Y.$$

We consider the following five cases. Note that since $\text{pd}_R M \geq 2$, the second Betti number $\beta := \beta^2_2(M)$ is non-zero.

Case 1. $A \neq 0 \neq B$. In this case, $I$ and $J$ are direct summands of $\Omega^2_R A$ and $\Omega^2_R B$, respectively. Hence, $m = I \oplus J$ is a direct summand of $\Omega^2_R A \oplus \Omega^2_R B = \Omega^3_R M$.

Case 2. $A = 0 = V$. In this case, $\Omega^2_R M = B$ and $\Omega^3_R M = W$ are both $R/J$-modules, and annihilated by the ideal $J$ of $R$. Hence, by the short exact sequence $0 \to \Omega^3_R M \to R^{\oplus \beta} \to \Omega^2_R M \to 0$ we observe that $J^2$ annihilates $R^{\oplus \beta}$. Since $\beta \neq 0$ we get $J^2 = 0$. Therefore, $mJ = (I + J)J = 0$. This implies that $J$ is a $k$-vector space. Since $B = \Omega^2_R M \neq 0$, the ideal $J$ is a direct summand of $\Omega^3_R B = \Omega^3_R M$. Thus $k$ is a direct summand of $\Omega^2_R M$, and $m$ is a direct summand of $\Omega^3_R M$.

Case 3. $B = 0 = W$. This case is treated similarly to Case 2, and we see that $m$ is a direct summand of $\Omega^3_R M$.

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1. We use $v$ instead of $v_R$ to keep the notation simpler.
Case 4. $A = 0 \neq V$. In this case, we must have $B \neq 0$, and hence, $n := \nu(B) > 0$. If $W \neq 0$, then $V \neq 0 \neq W$, and Case 1 shows that $m$ is a direct summand of $\Omega^4_R M$. Thus, we may assume that $W = 0$. Then we have $\Omega^2_R M \cong B$ and

$$V \cong \Omega^3_R M \cong \Omega^2_R B \cong J^{\oplus n} \oplus \Omega_R/B. \tag{3.6-1}$$

Since $V$ is annihilated by $I$, so is the direct summand $\Omega_R/B$. As $\Omega_R/B$ is also annihilated by $J$, and by $m = I \oplus J$, we see that $\Omega_R/B$ is a $k$-vector space. If $\Omega_R/B$ is non-zero, then $k$ is a direct summand of $\Omega_R/B$, and hence, of $\Omega^4_R M$. In this case $m$ is a direct summand of $\Omega^4_R M$, and we are done. Thus, we may assume that $\Omega_R/B = 0$. In this case, $\Omega^2_R M \cong B \cong (R/J)^{\oplus n}$ and we have

$$V \cong \Omega^3_R M \cong \Omega^2_R B \cong J^{\oplus n}. \tag{3.6-1}$$

If $Y = 0$, since we already assumed that $W = 0$, Case 3 shows that $m$ is a direct summand of $\Omega^5_R M$. Hence, we may assume $Y \neq 0$. If $X \neq 0 \neq Y$, then Case 1 shows that $m$ is a direct summand of $\Omega^5_R M$. Therefore, we may assume $X = 0$. Now we have

$$Y \cong \Omega^4_R M \cong \Omega_R V \cong I^{\oplus m} \oplus \Omega_R/I V,$$

where $m := \nu(V)$. We make a similar argument as above. Since $Y$ is annihilated by $J$, so is $\Omega_R/I V$, which is also annihilated by $I$. Hence, $\Omega_R/I V$ is a $k$-vector space. If $\Omega_R/I V$ is non-zero, then $k$ is a direct summand of $\Omega_R/I V$, and thus, of $\Omega^5_R M$. In this case $m$ is a direct summand of $\Omega^5_R M$. So, we may assume $\Omega_R/I V = 0$. Under this assumption, $V$ is a free $R/I$-module and we get $V \cong (R/I)^{\oplus m}$. Therefore, by (3.6-1) we have $J^{\oplus n} \cong (R/I)^{\oplus m}$. Since $n > 0$ we see that $J$ is a free $R/I$-module. Now Lemma 3.3(2) implies that $R/I$ is a discrete valuation ring. Then $J$ is isomorphic to $R/I$ by Lemma 3.3(1). We obtain isomorphisms $\Omega^3_R M \cong J^{\oplus n} \cong (R/I)^{\oplus n}$ and $\Omega^4_R M \cong I^{\oplus n}$. Therefore, $m = I \oplus J$ is a direct summand of $\Omega^5_R M \oplus \Omega^5_R M$.

Case 5. $B = 0 \neq W$. This case is treated similarly to Case 4, and we see that one of the following holds: $m$ is a direct summand of $\Omega^4 M$, or $\Omega^2_R M$ is $R/I$-free and $m$ is a direct summand of $\Omega^2_R M$, or $\Omega^2_R M$ is $R/I$-free and $R/J$ is a discrete valuation ring and $m$ is a direct summand of $\Omega^3_R M \oplus \Omega^4_R M$.

(2) This part follows from (1) and Proposition 3.5.

In the remainder of this section, we present several examples showing that each of the cases (i)-(v) in Theorem 3.6 occurs and these conditions are independent.

Convention 3.7. In the following examples, $k$ will be a field as before, and $X, Y, Z$ will be indeterminates over $k$. We will use the lower-case letters $x, y, z$ to represent the residues of the variables $X, Y, Z$ in the ring $R$.

We start by providing examples satisfying (iv) and (v), but not (i)-(iii).

Example 3.8. Consider the fiber product

$$R = k[[X]] \times_k k[[Y]] \cong \frac{k[[X, Y]]}{(XY)}$$

and set $I = (x)$ and $J = (y)$. We then have $m = I \oplus J$. We also have $R/J \cong k[[x]]$ and $R/I \cong k[[y]]$, which are discrete valuation rings. Take $M = R/I$. Then

$$\Omega^i_R M \cong \begin{cases} R/J & \text{if } i \geq 0 \text{ is odd,} \\ R/I & \text{if } i \geq 0 \text{ is even.} \end{cases}$$
In particular, $\Omega^2_R M$ is $R/I$-free, and $\Omega^3_R M \oplus \Omega^4_R M \cong m$. Thus, (v) holds. Moreover, $m$ is not a direct summand of $\Omega^i_R M$ for all $i \geq 0$. Hence, none of (i)-(iii) holds.

Similarly, if we set $M = R/J$, then (iv) holds, but none of (i)-(iii) holds.

The following is also an example that satisfies (iv), but not (i)-(iii). Comparing with Example 3.8, this example has the advantage that $R/J$ is not a discrete valuation ring, hence, it does not satisfy (v) as well. (By symmetry, one can construct an example that satisfies (v), but not (i)-(iv).)

**Example 3.9.** Consider the fiber product

$$R = k[[X]] \times_k k[[Y, Z]] \cong \frac{k[[X, Y, Z]]}{(XY, XZ)}.$$ 

Setting $I = (y, z)$ and $J = (x)$, we have $m = I \oplus J$. The ring $R/I = k[[x]]$ is a discrete valuation ring, while $R/J = k[[y, z]]$ is not. One can easily check (by hand) that the minimal free resolution of the $R$-module $R/(y)$ has the form:

$$\cdots \longrightarrow R^5 \xrightarrow{\begin{pmatrix} y & z & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 \\ 0 & 0 & 0 & 0 & x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & 0 & z \\ 0 & 0 & y-x \end{pmatrix}} R^2 \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{y} R/(y) \longrightarrow 0$$

Note that it contains a minimal free presentation of $I$ and $\Omega_R I \cong (R/I)^{\oplus 2} \oplus R/(x)$. Let $M$ be the (Auslander) transpose of $I$, that is, the $R$-module $M$ defined by the following exact sequence

$$0 \longrightarrow I^* \longrightarrow R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}} R^3 \xrightarrow{x} M \longrightarrow 0$$

where $I^* := \text{Hom}_R(I, R)$. Then $\Omega^2_R M = I^*$. From the natural short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, we have an exact sequence

$$0 \longrightarrow (R/I)^* \xrightarrow{f} R^* \longrightarrow I^* \longrightarrow \text{Ext}^1_R(R/I, R) \longrightarrow 0.$$ 

Note that the map $f$ can be described by the inclusion map $J = (0 : I) \rightarrow R$, and hence, there is an exact sequence

$$0 \longrightarrow R/J \longrightarrow I^* \longrightarrow \text{Ext}^1_R(R/I, R) \longrightarrow 0. \quad (3.9.1)$$

We claim that $\text{Ext}^1_R(R/I, R) = 0$. Indeed, from the natural short exact sequence $0 \longrightarrow R \longrightarrow (R/I) \oplus (R/J) \rightarrow k \longrightarrow 0$ we get an exact sequence

$$\text{(R/I)}^* \oplus \text{(R/J)}^* \xrightarrow{g} R^* \longrightarrow \text{Ext}^1_R(k, R) \longrightarrow \text{Ext}^1_R((R/I) \oplus (R/J), R) \longrightarrow 0.$$ 

Since the map $g$ can be described by the inclusion map

$$m = J \oplus I = (0 : I) \oplus (0 : J) \longrightarrow R$$

its cokernel is isomorphic to $k$. The element $x - y \in R$ is $R$-regular and annihilates $k$. Using [3, lemma 3.1-16] and the isomorphism $R/(x - y) \cong k[[y, z]]/(y^2, yz)$, we easily see that $\text{Ext}^1_R(k, R) \cong \text{Hom}_{R/(x-y)}(k, R/(x-y)) \cong k$. Thus, we obtain $\text{Ext}^1_R((R/I) \oplus (R/J), R) = 0$, which implies that $\text{Ext}^1_R(R/I, R) = 0$, as we claimed.

This claim and (3.9.1) imply that $\Omega^2_R M = I^* \cong R/J$. Combining this with the isomorphism $\Omega^3_R M \oplus \Omega^4_R M \cong m$ shows that (iv) holds.
On the other hand, none of the syzygies $\Omega_3^2 R M \neq J, \Omega_4^4 R M \neq I,$ and $\Omega_5^5 R M = \Omega_R I = (R/I)^2 \oplus (R/J)$ contain $m$ as a direct summand. In fact, taking into account the minimal numbers of generators, we see that $m$ is not a direct summand of $J$ or $I$. Also, $m \cong (R/I)^{\oplus 2} \oplus (R/J)$ if we assume that $m$ is a direct summand of $\Omega_R I$. Then, however, localization at the prime ideal $I$ yields $R_I = mR_I \cong (R/I \cdot R_I)^{\oplus 2}$ which contradicts the fact that the local ring $R_I$ is indecomposable as a module over itself. Consequently, none of (i)-(iii) and (v) holds.

The following is an example that satisfies (ii), but not (i) and (iii)-(v). (By symmetry, one can construct an example that satisfies (iii), but not (i)-(ii) and (iv)-(v).)

**Example 3.10.** Let $R, I, J$ be as in Example 3.9, and let $M = R/(y)$. Then

$$\Omega_3^3 R M = I, \quad \Omega_4^4 R M = (R/I)^{\oplus 2} \oplus (R/J), \quad \Omega_5^5 R M = I^{\oplus 2} \oplus J.$$  

By the argument in Example 3.9, $m$ is not a direct summand of $\Omega_3^3 R M$ or $\Omega_5^5 R M$, while it is a direct summand of $\Omega_4^4 R M$. We also have $\Omega_2^2 R M = R/I$, which is not $R/J$-free, and $R/J = k[[y, z]]$ is not a discrete valuation ring. Thus, (i) and (iii)-(v) do not hold.

The next examples satisfy (i), but they do not satisfy (ii)-(v).

**Example 3.11.** (a) Consider the fiber product

$$R = \frac{k[[X]]}{(X^2)} \times_k \frac{k[[Y]]}{(X^2, XY)}$$

and set $I = (x)$ and $J = (y)$. Then we have $m = I \oplus J$. Since $R/J$ has depth zero, it follows from Theorem 3.6 that none of (ii)-(v) holds. Let $M = R/J$. The minimal free resolution of $M$ is

$$\cdots \longrightarrow R^3 \begin{pmatrix} x & y & 0 \\ 0 & 0 & x \end{pmatrix} \longrightarrow R^2 \begin{pmatrix} x \\ 0 \end{pmatrix} \longrightarrow R \longrightarrow R \longrightarrow M \longrightarrow 0.$$  

It follows from this that $\Omega_3^3 R M = m \cong k \oplus (R/I)$, and we observe that $m$ is a direct summand of $\Omega_3^3 R M$ for all $i \geq 3$. In particular, statement (i) holds true.

We can also construct an example satisfying (i) but not (ii)-(v) using the ring of Example 3.9.

(b) Let $R, I, J$ be as in Example 3.9, and let $M = (y) \cong R/(x)$. Then Example 3.9 shows that $m$ is not a direct summand of $\Omega_3^3 R M = (R/I)^{\oplus 2} \oplus (R/J),$ but it is a direct summand of $\Omega_4^4 R M = I^{\oplus 2} \oplus J$. As $I^2 = (y^2, yz, z^2)$ is non-zero, $\Omega_5^5 R M = I$ is not $R/I$-free. Since $R/J \cong k[[y, z]]$ is not a discrete valuation ring, $\Omega_5^5 R M = I$ is not $R/J$-free by Lemma 3.3(2). Thus, (ii)-(v) do not hold.

4. Local rings with quasi-decomposable maximal ideal

This section contains the proof of Theorem B. Recall that the definitions and terminology have been introduced in Section 2.

**Definition 4.1.** If there exists an $R$-sequence $x$ of length $n \geq 0$ such that $m/(x)$ is decomposable, then we say that $m$ is quasi-decomposable (with $x$).

If $m$ is decomposable, then of course $m$ is quasi-decomposable. However, the converse is far from being true. We shall give several examples of a local ring with indecomposable quasi-decomposable maximal ideal at the end of this section.

We prove the following lemmas for the future use.
Lemma 4.2. Let $M$ be an $R$-module, and let $x$ be an $R$-sequence of length $n \geq 0$ annihilating $M$. Then for all $t \geq 0$ and $u \geq n$ there exists $v \geq 0$ such that

$$\Omega^u_R \Omega^{t}_{R/(x)} M \cong \Omega^{u+t}_R M \oplus R^{\oplus v}.$$ 

**Proof.** There is an exact sequence

$$0 \rightarrow \Omega^t_R M \rightarrow F_{t-1} \rightarrow F_{t-2} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

of $R/(x)$-modules with each $F_i$ free over $R/(x)$. Taking the $u$-th syzygies over $R$, we get an exact sequence

$$0 \rightarrow \Omega^u_R \Omega^t_{R/(x)} M \rightarrow \Omega^u_R F_{t-1} \oplus R^{\oplus a_{t-1}} \rightarrow \cdots$$

$$\rightarrow \Omega^u_R F_1 \oplus R^{\oplus a_1} \rightarrow \Omega^u_R F_0 \oplus R^{\oplus a_0} \rightarrow \Omega^u_R M \rightarrow 0$$

for some $a_i \geq 0$. As $u \geq n$, each $\Omega^u_R F_i$ is $R$-free. This exact sequence then says that $\Omega^u_R \Omega^t_{R/(x)} M$ is isomorphic to the direct sum of $\Omega^u_R \Omega^t_{R/(x)} M = \Omega^{u+t}_R M$ and some free $R$-module, as desired.

Lemma 4.3. Let $M$ be an $R$-module, and let $x$ be an $M$-sequence of length $n \geq 0$. Then $\Omega^t_R (M/xM)$ belongs to the resolving closure $\text{res} M$ of $M$ for all $i \geq n$.

**Proof.** Since $x$ is an $M$-sequence, the Koszul complex $K(x, M)$ is acyclic with $H_0(K(x, M)) = M/xM$. Each $K(x, M)$ is a direct sum of copies of $M$, which belongs to $\text{res} M$. Hence, $\Omega^u_R (M/xM)$ is in $\text{res} M$ by [20, lemma 4.3]. Therefore, $\Omega^t_R (M/xM)$ is in $\text{res} M$ for all $i \geq n$.

Lemma 4.4. Let $R$ be a $d$-dimensional Cohen–Macaulay local ring with quasi-decomposable maximal ideal. Let $\mathcal{X}$ be a resolving subcategory of $\text{mod} R$ contained in $\text{CM}(R)$. If $\mathcal{X}$ contains a non-free $R$-module, then $\mathcal{X}$ also contains $\Omega^k_R k$.

**Proof.** Let $x$ be an $R$-sequence of length $n \geq 0$ such that $m/(x)$ is decomposable, and let $X \in \mathcal{X}$ be a non-free $R$-module. Then $X$ is maximal Cohen–Macaulay, and $x$ is an $X$-sequence. Hence, $\text{pd}_{R/\langle x \rangle} X/xX = \text{pd}_R X = \infty$ by [3, lemma 1-3-5]. Theorem A implies that $\Omega^k_{R/\langle x \rangle} k = m/(x)$ is a direct summand of

$$\Omega^3_{R/\langle x \rangle} (X/xX) \oplus \Omega^4_{R/\langle x \rangle} (X/xX) \oplus \Omega^5_{R/\langle x \rangle} (X/xX) = Y/xY,$$

where $Y := \Omega^3_{R} X \oplus \Omega^4_{R} X \oplus \Omega^5_{R} X \in \mathcal{X}$.

If $n \leq d - 1$, then by Lemma 4.2 we have $\Omega^{d-1}_{R} \Omega^t_{R/(x)} k \cong \Omega^{d+1}_R k \oplus F$ for some free $R$-module $F$. The module $\Omega^{d-1}_{R} \Omega^t_{R/(x)} k$ is a direct summand of $\Omega^{d-1}_{R} (Y/xY)$. As $Y$ is maximal Cohen–Macaulay, $x$ is a $Y$-sequence. It follows from Lemma 4.3 that $\Omega^{d-1}_{R} (Y/xY)$ is in $\mathcal{X}$. Hence, so is $\Omega^{d-1}_{R} \Omega^t_{R/(x)} k$, and therefore, $\Omega^t_R k$ is in $\mathcal{X}$.

Now let us consider the case $n = d$. Similarly as above, we see that $\Omega^{d}_{R} \Omega^t_{R/(x)} k \cong \Omega^{d+1}_R k \oplus F'$ for some free $R$-module $F'$, that $\Omega^d_{R} \Omega^t_{R/(x)} k$ is a direct summand of $\Omega^{d}_R (Y/xY)$, and that $\Omega^d_{R} (Y/xY)$ is in $\mathcal{X}$. We also see that $\Omega^{d+1}_R k$ belongs to $\mathcal{X}$. Since $\text{depth}_R \Omega^{d+1}_R k = d$, it follows from [6, proposition 4.2] that $\Omega^d_R k$ is in $\text{res}(\Omega^{d+1}_R k \oplus \Omega^{d+1}_R k) = \text{res} \Omega^{d+1}_R k$. Hence, $\Omega^d_R k$ is in $\mathcal{X}$, as desired.

Now we can state and prove the main result of this section.

**Theorem 4.5.** Let $(R, m)$ be a $d$-dimensional Cohen–Macaulay singular local ring and assume that $m$ is quasi-decomposable.
Suppose that $\text{Spec}^0 R$ is a hypersurface or has minimal multiplicity. Then one has the following commutative diagram of mutually inverse bijections:

\[
\begin{array}{ccc}
\{ \text{Resolving subcategories of} \} & \xrightarrow{\text{NF}} & \{ \text{Specialization-closed subsets of} \} \\
\text{mod } R \text{ contained in } \text{CM}(R) & & \text{Sing } R \\
\end{array}
\]

\[
\begin{array}{ccc}
\{ \text{Thick subcategories of} \} & \xrightarrow{\text{thick}_{\text{mod } R}} & \{ \text{Thick subcategories of} \} \\
\text{CM}(R) \text{ containing } R & & \text{mod } R \text{ containing } R \\
\end{array}
\]

\[
\begin{array}{ccc}
\{ \text{Thick subcategories of} \} & \xrightarrow{\pi^{-1}} & \{ \text{Thick subcategories of} \} \\
\text{D}_{\text{sg}}(R) \text{ containing } \omega & & \text{D}^b(R) \text{ containing } \omega \\
\end{array}
\]

Assume that $R$ is excellent and has a canonical module $\omega$. Suppose that $\text{Spec}^0 R$ has finite Cohen–Macaulay representation type. Then one has the following commutative diagram of mutually inverse bijections:

\[
\begin{array}{ccc}
\{ \text{Resolving subcategories of} \} & \xrightarrow{\text{NF}} & \{ \text{Specialisation-closed subsets of} \} \\
\text{mod } R \text{ contained in } \text{CM}(R) \text{ and containing } \omega & & \text{Sing } R \text{ containing NonGor } R \\
\end{array}
\]

\[
\begin{array}{ccc}
\{ \text{Thick subcategories of} \} & \xrightarrow{\text{thick}_{\text{mod } R}} & \{ \text{Thick subcategories of} \} \\
\text{CM}(R) \text{ containing } R \text{ and } \omega & & \text{mod } R \text{ containing } R \text{ and } \omega \\
\end{array}
\]

\[
\begin{array}{ccc}
\{ \text{Thick subcategories of} \} & \xrightarrow{\pi^{-1}} & \{ \text{Thick subcategories of} \} \\
\text{D}_{\text{sg}}(R) \text{ containing } \omega & & \text{D}^b(R) \text{ containing } R \text{ and } \omega \\
\end{array}
\]

**Proof.** Note that $\text{NonGor}(R) = \text{NF}(\omega) = \text{IPD}(\omega)$ for part (2). Hence, by [21, lemma 2.6] we observe that all the twelve maps in Theorem 4.5 are well-defined, and that

\[
\text{NF}(\text{NF}_{\text{CM}}^{-1}(W)) = W, \quad \text{IPD}(\text{IPD}^{-1}(W)) = W \tag{4.5.1}
\]

for each specialisation-closed subset $W$ of $\text{Spec } R$ contained in $\text{Sing } R$.

We first show that

\[
\mathcal{X} = \text{thick}_{\text{mod } R}(\text{rest}_{\text{CM}(R)} \mathcal{X}) = \text{thick}_{\text{mod } R}(\mathcal{X} \cap \text{CM}(R)) \tag{4.5.2}
\]

for each thick subcategory $\mathcal{X}$ of $\text{mod } R$ containing $R$. Since $\mathcal{X}$ is thick and contains $\mathcal{X} \cap \text{CM}(R)$, we see that $\mathcal{X}$ contains $\text{thick}_{\text{mod } R}(\mathcal{X} \cap \text{CM}(R))$. Take any module $X \in \mathcal{X}$. Then there is an exact sequence $0 \to \Omega^d X \to F_{d-1} \to \cdots \to F_0 \to X \to 0$ of $R$-modules with each $F_i$ free. Note that $\Omega^d X$ and $F_i$ for $0 \leq i \leq d - 1$ are in $\mathcal{X} \cap \text{CM}(R)$. Thus, these modules belong to $\text{thick}_{\text{mod } R}(\mathcal{X} \cap \text{CM}(R))$. Therefore, $X$ is also in $\text{thick}_{\text{mod } R}(\mathcal{X} \cap \text{CM}(R))$, and we have the equalities in (4.5.2).
Next we show that
\[ \text{IPD}(\text{thick}_{\text{mod} R} \mathcal{X}) = \text{NF}(\mathcal{X}) \]  
for any subcategory \( \mathcal{X} \) of \( \text{CM}(R) \). It is straightforward to check that
\[ \text{IPD}(\text{thick}_{\text{mod} R} \mathcal{X}) = \text{IPD}(\mathcal{X}) \subseteq \text{NF}(\mathcal{X}). \]

Let \( p \) be a prime ideal such that \( X_p \) is a non-free \( R_p \)-module for some \( X \in \mathcal{X} \). Since the \( R_p \)-module \( X_p \) is maximal Cohen–Macaulay, it has infinite projective dimension. Hence, \( \text{NF}(\mathcal{X}) \) is contained in \( \text{IPD}(\mathcal{X}) \). Therefore, we have the equality in (4.5-3).

In view of (4.5-1), (4.5-2) and (4.5-3), we show the following three statements:

(a) \( \text{NF}_{\text{CM}}^{-1} \cdot \text{NF} = 1; \)
(b) \( \text{IPD}^{-1} \cdot \text{IPD} = 1; \)
(c) The equalities in the diagrams hold.

For (a) fix a resolving subcategory \( \mathcal{X} \) of \( \text{mod} R \) contained in \( \text{CM}(R) \). If \( \mathcal{X} = \text{add} R \), then \( \text{NF}(\mathcal{X}) \) is empty, and we have \( \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X})) = \text{add} R = \mathcal{X} \). Thus, we may assume that \( \mathcal{X} \neq \text{add} R \). Then \( \mathcal{X} \) contains a non-free \( R \)-module, and it follows from Lemma 4-4 that \( \mathcal{X} \) contains \( \Omega^d k \). If \( \text{Spec}^0 R \) is a hypersurface (resp. has minimal multiplicity), then by virtue of \([20, \text{theorem 5-13}] \) (resp. \([21, \text{theorem 5-6}] \)) we obtain the equality \( \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X})) = \mathcal{X} \). If \( R \) is excellent, \( \text{Spec}^0 R \) has finite Cohen–Macaulay representation type, and \( \mathcal{X} \) contains a canonical module, then the equality \( \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X})) = \mathcal{X} \) follows from \([21, \text{theorem 6-11 and corollary 6-12}] \).

For (b) let \( \mathcal{X} \) be a thick subcategory of \( \text{mod} R \) containing \( R \). Then \( \mathcal{X} \) is contained in \( \text{IPD}^{-1}(\text{IPD}(\mathcal{X})) \). Now for an \( R \)-module \( M \) in \( \text{IPD}^{-1}(\text{IPD}(\mathcal{X})) \) we claim that
\[ \text{NF}(\Omega^d_R M) \subseteq \text{NF}(\mathcal{X} \cap \text{CM}(R)). \]

Indeed, let \( p \) be a prime ideal such that \( (\Omega^d_R M)_p \) is non-free. Then the \( R_p \)-module \( \Omega^d_R (M_p) \) is non-free, and \( M_p \) has infinite projective dimension. Hence, \( p \) is in \( \text{IPD}(M) \), which is contained in \( \text{IPD}(\mathcal{X}) \). This implies that \( X_p \) has infinite projective dimension for some \( X \in \mathcal{X} \). Therefore, \( (\Omega^d_R X)_p \) is non-free, which says that \( p \) belongs to \( \text{NF}(\Omega^d_R \mathcal{X}) \). Since \( \Omega^d_R \mathcal{X} \) is in \( \mathcal{X} \cap \text{CM}(R) \), the claim follows.

This claim implies that \( \Omega^d M \) belongs to \( \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X} \cap \text{CM}(R))) \). Note that \( \mathcal{X} \cap \text{CM}(R) \) is a resolving subcategory contained in \( \text{CM}(R) \). In each of the cases where \( \text{Spec}^0 R \) is a hypersurface, where \( \text{Spec}^0 R \) has minimal multiplicity, and where \( R \) is excellent, \( \text{Spec}^0 R \) has finite Cohen–Macaulay representation type and \( \mathcal{X} \) contains a canonical module, it follows from (a) that \( \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X} \cap \text{CM}(R))) = \mathcal{X} \cap \text{CM}(R) \). Hence, \( \Omega^d M \) is in \( \mathcal{X} \), and so is \( M \) as \( \mathcal{X} \) is thick and contains \( R \).

Finally, for (c) let \( \mathcal{X} \) be a resolving subcategory of \( \text{mod} R \) contained in \( \text{CM}(R) \). By (a) we have \( \mathcal{X} = \text{NF}_{\text{CM}}^{-1}(\text{NF}(\mathcal{X})) \); setting \( W = \text{NF}(\mathcal{X}) \), we have \( \mathcal{X} = \text{NF}_{\text{CM}}^{-1}(W) \). Let \( 0 \to A \to B \to C \to 0 \) be an exact sequence of maximal Cohen–Macaulay \( R \)-modules. If \( A \) and \( B \) are in \( \mathcal{X} \), then \( \text{NF}(A) \) and \( \text{NF}(B) \) are contained in \( W \), and we see that \( \text{NF}(C) \) is also contained in \( W \) by the Auslander–Buchsbaum formula. Hence \( C \) is in \( \mathcal{X} \), which shows that \( \mathcal{X} \) is a thick subcategory of \( \text{CM}(R) \). The proof of (c) is now complete.

The one-to-one correspondence \( \text{rest}_{\text{mod} R}(\text{thick}_{\text{mod} R}) \) in (1) and (2) follows from \([11, \text{theorem 1}] \), while the one-to-one correspondence \( (\pi^{-1}, \pi) \) is shown by noting that for two objects \( M, N \in \text{D}^b(R) \) with \( \pi M \cong \pi N \) one has \( M \in \text{thick}_{\text{mod} R}(R, N) \).

It is straightforward that \( \text{NF}_{\text{CM}(R)}(\mathcal{X}) = \text{rest}_{\text{CM}(R)}(\text{rest}_{\text{mod} R}(\pi^{-1}(\mathcal{X}))) \) for a (thick) subcategory \( \mathcal{X} \) of \( \text{D}_{\text{sg}}(R) \). Let \( \mathcal{Y} \) be a thick subcategory of \( \text{CM}(R) \) containing \( R \). Then
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\[ \pi \left( \text{thick}^{\text{pd}}_{R}(\text{thick}_{\text{mod}} R Y) \right) = \pi \left( \text{thick}^{\text{pd}}_{R}(Y) \right) \text{ is a thick subcategory of } D_{\text{sg}}(R) \text{ containing } \pi Y. \]

If \( Z \) is a thick subcategory of \( D_{\text{sg}}(R) \) containing \( \pi Y \), then \( Y \) is contained in \( \pi^{-1} Z \), and so is \( \text{thick}^{\text{pd}}_{R}(Y) \). It follows that \( \pi \left( \text{thick}^{\text{pd}}_{R}(Y) \right) = \text{thick}_{D_{\text{sg}}(R)}(Y) \). Thus, the lower squares in the diagrams in (1) and (2) are commutative. This completes the proof of Theorem 4-5.

Our Theorem B can now be deduced from Theorem 4-5.

4-6 (Proof of Theorem B). Let \( X \) be a thick subcategory of \( D_{\text{sg}}(R) \), and let \( W \) be a specialization-closed subset of \( \text{Sing} R \). According to Theorem 4-5, it suffices to show the equalities:

(a) \( \text{IPD}(\text{rest}_{\text{mod}} R(\pi^{-1} X)) = \text{ssupp} X \); and

(b) \( \pi(\text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1} W)) = \text{ssupp}^{-1} W. \)

For (a), pick a prime ideal \( p \in \text{IPD}(\text{rest}_{\text{mod}} R(\pi^{-1} X)) = \text{IPD}(\pi^{-1} X \cap \text{mod} R). \) Then there is an \( R \)-module \( M \) such that \( X := \pi M \in X \) and \( \text{pd}_{R_{p}} M_{p} = \infty \). Hence, \( X_{p} \not\subseteq 0 \) in \( D_{\text{sg}}(R_{p}) \), which implies that \( p \) is in \( \text{ssupp} X \). Conversely, pick a prime ideal \( p \in \text{ssupp} X \). Then there is an object \( X \in X \) such that \( X_{p} \not\subseteq 0 \) in \( D_{\text{sg}}(R_{p}) \). Choose an object \( Y \in D^{b}(R) \) with \( \pi Y = X \). There exists an exact triangle \( P \to Y \to M[n] \to \text{in} D^{b}(R) \) such that \( P \in \text{thick}^{\text{pd}}_{R}(R), M \in \text{mod} R \), and \( n \in \mathbb{Z} \). Hence \( X = \pi Y = (\pi M)[n] \), which implies that \( \pi M \) is in \( X \); and hence, \( M \) is in \( \pi^{-1} X \cap \text{mod} R = \text{rest}_{\text{mod}} R(\pi^{-1} X) \). Since \( p \) belongs to \( \text{IPD}(M) \), it belongs to \( \text{IPD}(\text{rest}_{\text{mod}} R(\pi^{-1} X)). \)

For (b), let \( A \) be an object in \( \pi(\text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1} W)) \). Then \( A \cong \pi B \) for some \( B \in \text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1} W) \). Hence, \( \text{ssupp} A = \text{ssupp}(\pi B) \) = \( \text{IPD}(B). \) If \( p \) is a prime ideal outside \( W \), then \( (\text{IPD}^{-1}(W))_{p} \) is contained in \( \text{PD}(R_{p}) \), and so is \( \text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1}(W))_{p} \), which contains \( B_{p} \). Hence, \( B_{p} \) has finite projective dimension as an \( R_{p} \)-module, which shows that \( \text{IPD}(B) \) is contained in \( W \). Therefore, \( A \) belongs to \( \text{ssupp}^{-1} W \). Conversely, let \( A \) be an object in \( \text{ssupp}^{-1} W \), and write \( A = \pi B \) with \( B \in D^{b}(R) \). Then it is easy to see that \( \text{IPD}(B) \) is contained in \( W \), and \( B \) belongs to \( \text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1} W) \). Hence, \( A \) is in \( \pi(\text{thick}^{\text{pd}}_{R}(\text{IPD}^{-1} W)) \).

We close this section by presenting several examples of a Cohen–Macaulay non-Gorenstein local ring with quasi-decomposable maximal ideal; actually, it turns out that there are plenty of such examples.

Example 4-7. Let \( R \) be a \( d \)-dimensional Cohen–Macaulay local ring which is not a hypersurface. Suppose that \( k \) is infinite and that \( R \) has minimal multiplicity. Then \( m \) is quasi-decomposable. In fact, we find a system of parameters \( x \) of \( R \) with \( m^{2} = x m \). Then we have \( m(m/(x)) = 0 \), which means that \( m/(x) \) is a \( k \)-vector space. Since \( R \) is not a hypersurface, the dimension of \( m/(x) \) as a \( k \)-vector space is at least two. Hence \( m/(x) \) is decomposable.

Example 4-8. Let \( R \) be a 2-dimensional non-Gorenstein normal local domain with a rational singularity. Then by [10, theorem 3-1], the ring \( R \) has minimal multiplicity. Therefore, by Example 4-7 it has a quasi-decomposable maximal ideal.

Example 4-9. Let \( H = \langle pq + p + 1, 2q + 1, p + 2 \rangle \) be the numerical semigroup with \( p, q > 0 \) and \( \gcd(p + 2, 2q + 1) = 1 \), and let \( R = k[[H]] \) be (the completion of) the numerical semigroup ring of \( H \) over a field \( k \). Then \( R \) is a non-Gorenstein almost-Gorenstein (see [9]) Cohen–Macaulay local domain of dimension 1, and \( m \) is quasi-decomposable. In fact, by virtue of [14, proposition 2(1)], we have

\[
R \cong \frac{k[[x, y, z]]}{I_{2}(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})},
\]

(4-9-1)
where $I_2(−)$ stands for the ideal generated by $2 \times 2$-minors. This implies $R/(z) \cong k[[x, y]]/(x^2, xy, y^{p+1})$, and hence we obtain a non-trivial direct sum decomposition $m/(z) \cong x(R/(z)) \oplus y(R/(z))$.

**Example 4.10.** Let $R = k[[H]]$ be a non-Gorenstein almost-Gorenstein numerical semigroup ring with embedding dimension 3 and multiplicity $\leq 6$. Then $m$ is quasi-decomposable. Indeed, by [14, proposition 2(2)] we again have the isomorphism (4.9.1) with $p, q > 0$, and a similar argument as in Example 4.9 applies.

**Example 4.11.** For each of the Cohen–Macaulay local rings $R$ given in [21, examples 7.1, 7.2, 7.4 and 7.5] the maximal ideal is quasi-decomposable, since the quotient of a suitable system of parameters is isomorphic to the ring

$$k[s, t]/(s^2, st, t^2) = k[s]/(s^2) \times_k k[t]/(t^2),$$

where $s, t$ are indeterminates over the field $k$. We should notice that these four examples also explain the independence of the assumptions of Theorem 4.5 on the punctured spectrum. (None of them has an isolated singularity.)

### 5. More on thick subcategories of mod $R$

As another application of Theorem 3.6, in this section we study the thick subcategories of mod $R$ that contain a module of infinite projective dimension where $m$ is quasi-decomposable. We begin with making a lemma.

**Lemma 5.1.** Let $M$ be an $R$-module and $x = x_1, \ldots, x_n$ be an $R$-sequence. Then $x$ is an $\Omega^i_R M$-sequence.

**Proof.** We use induction on $n$. The assertion is trivial when $n = 0$. Let $n > 0$. The induction hypothesis implies that the sequence $x' := x_1, \ldots, x_{n-1}$ is regular on both $\Omega^{n-1} M$ and $\Omega^n M = \Omega^{n-1}(\Omega M)$. The exact sequence

$$0 \longrightarrow \Omega^n M \longrightarrow F \longrightarrow \Omega^{n-1} M \longrightarrow 0$$

in which $F$ is free induces an exact sequence

$$0 \longrightarrow \Omega^n M/x'\Omega^n M \longrightarrow F/x' F \longrightarrow \Omega^{n-1} M/x'\Omega^{n-1} M \longrightarrow 0.$$

This shows that the multiplication map of $\Omega^n M/x'\Omega^n M$ by $x_n$ is injective. Thus, $x = x_1, \ldots, x_n$ is regular on $\Omega^n M$.

The next proposition plays a central role in this section.

**Proposition 5.2.** Suppose that $m$ is quasi-decomposable with an $R$-sequence $x$ of length $n$. If $M$ is an $R$-module with $\text{pd}_R M \geq n + 2$, then $k$ is in $\mathcal{X} := \text{thick}_{\text{mod} R}(R, M)$.

**Proof.** We consider the following cases.

Case 1. $n = 0$. In this case $m$ is decomposable, and we can apply Theorem 3.6. Since $\mathcal{X}$ contains all the syzygies of $M$ and their direct summands, we are done.

Case 2. $n > 0$. It follows from Lemma 5.1 that $x$ is an $\Omega^n_R M$-sequence. Using [3, lemma 1.3.5], we have

$$\text{pd}_{R/(x)}\Omega^n_R M/x\Omega^n_R M = \text{pd}_R \Omega^n_R M \geq (n + 2) - n = 2.$$

By Case 1 we have $k$ is in $\text{thick}_{\text{mod} R/(x)}(R/(x), \Omega^n_R M/x\Omega^n_R M)$. Therefore, $k$ is
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in \text{thick}_{\text{mod}} R \{ R/(x), \Omega^n_R M/x\Omega^n_R M \}. The modules \( R/(x) \) and \( \Omega^n_R M/x\Omega^n_R M \) are in \text{thick}_{\text{mod}} R \{ R, \Omega^n_R M \} = \mathcal{X}. Therefore, \( k \) belongs to \( \mathcal{X} \), as desired.

**Corollary 5.3.** Suppose that \( R \) has an isolated singularity (that is, \( \text{Sing} R \) is a point), and that \( m \) is quasi-decomposable with an \( R \)-sequence \( x \) of length \( n \).

1. If \( \mathcal{X} \) is a thick subcategory of \( \text{mod} R \) containing \( R \) and at least one module of projective dimension \( \geq n + 2 \), then \( \mathcal{X} = \text{mod} R \).

2. There exists no thick subcategory \( \mathcal{X} \) with \( \text{PD}(R) \not\subseteq \mathcal{X} \not\subseteq \text{mod} R \).

**Proof.** (1) Since we assume that \( R \) has an isolated singularity, by \([19, \text{theorem VI.8}]\) (see also \([11, \text{proposition 9}]\)) we have \text{thick}_{\text{mod}} R \{ R, k \} = \text{mod} R. The assertion now follows from Proposition 5.2.

(2) If \( \mathcal{X} \) is a thick subcategory strictly containing \( \text{PD}(R) \), then \( \mathcal{X} \) contains a module of infinite projective dimension, and \( \mathcal{X} = \text{mod} R \) by (1).

**Remark 5.4.** Corollary 5.3(2) holds whenever \( R \) is a hypersurface with an isolated singularity. In fact, since \( R \) is a hypersurface, by \([22, \text{theorem 5-1(1)}]\) there is a one-to-one correspondence between the thick subcategories of \( \text{mod} R \) containing \( R \) and the specialization-closed subsets of \( \text{Sing} R \). As \( R \) has an isolated singularity, \( \text{Sing} R \) is trivial, and thus there is no thick subcategory \( \mathcal{X} \) with \( \text{PD}(R) \not\subseteq \mathcal{X} \not\subseteq \text{mod} R \).

The following is also a corollary of Proposition 5.2 (hence one of Theorem 3-6), which is a variant of \([21, \text{proposition 5.2}]\).

**Corollary 5.5.** Let \( R \) be a Cohen–Macaulay non-Gorenstein local ring with minimal multiplicity and infinite residue field. Let \( M \) be an \( R \)-module with infinite projective dimension. Then \( k \) belongs to \text{thick}_{\text{mod}} R \{ R, M \} \).

**Proof.** In view of Example 4.7, the maximal ideal of \( R \) is quasi-decomposable. Hence the assertion follows from Proposition 5.2.

6. Vanishing of Ext and Tor

In this section we give some Ext and Tor vanishing results over local rings with quasi-decomposable maximal ideal as applications of Theorem 3-6. Some of these results refine the existing results of \([16]\) as we explain below. We start by a lemma.

**Lemma 6.1.** Let \( M \) and \( N \) be \( R \)-modules. Let \( t \geq 0 \) be an integer.

1. Suppose that \( m \) is a direct summand of \( \Omega^t M \).
   (a) If \( \text{Tor}_\ell^R(M, N) = 0 \) for some \( \ell \geq t + 1 \), then \( \text{pd}_R N < \infty \).
   (b) If \( \text{Ext}_\ell^R(M, N) = 0 \) for some \( \ell \geq t + \max \{ 1, \text{depth}_R N \} \), then \( \text{id}_R N < \infty \).

2. Suppose that \( m \) is a direct summand of \( \Omega^t M \oplus \Omega^{t+1} M \).
   (a) If \( \text{Tor}_\ell^R(M, N) = \text{Tor}_{\ell+1}^R(M, N) = 0 \) for some \( \ell \geq t + 1 \), then \( \text{pd}_R N < \infty \).
   (b) If \( \text{Ext}_\ell^R(M, N) = \text{Ext}_{\ell+1}^R(M, N) = 0 \) for some \( \ell \geq t + \max \{ 1, \text{depth}_R N \} \), then \( \text{id}_R N < \infty \).

**Proof.** (1a) Since \( \ell - t > 0 \), we have \( \text{Tor}_\ell^R(M, N) \cong \text{Tor}_{\ell-t}^R(\Omega^t M, N) \). By assumption we get \( \text{Tor}_{\ell-t}^R(k, N) \cong \text{Tor}_{\ell-t}^R(m, N) = 0 \). Hence, \( \text{pd}_R N < \infty \).

(1b) As \( \ell - t > 0 \), we have \( \text{Ext}_\ell^R(M, N) \cong \text{Ext}_{\ell-t}^R(\Omega^t M, N) \). The assumption implies \( \text{Ext}_{\ell-t}^R(k, N) \cong \text{Ext}_{\ell-t}^R(m, N) = 0 \). Since \( \ell - t + 1 > \text{depth} N \), by \([8, \text{theorem (1.1)}]\) we conclude that \( \text{id}_R N < \infty \).
(2a) Since $\ell - t > 0$, we have
\[
0 = \text{Tor}^R_{\ell}(M, N) \oplus \text{Tor}^R_{\ell+1}(M, N) \cong \text{Tor}^R_{\ell-t}(\Omega^1 M \oplus \Omega^{t+1} M, N),
\]
which contains $\text{Tor}^R_{\ell-t}(m, N) \cong \text{Tor}^R_{\ell-t+1}(k, N)$ as a direct summand. Hence $N$ has finite projective dimension.

(2b) This follows from a similar argument to the proofs of (2a) and (1b).

Our Theorem 3.6 recovers [16, theorem 1.1] as we state next.

**Corollary 6.2** (Nasseh and Sather–Wagstaff). Assume that $R = S \times_k T$ is a fiber product, where $S$ and $T$ are local rings with common residue field $k$ and $S \neq k \neq T$. Let $M$ and $N$ be $R$-modules.

1. If $S$ or $T$ has depth 0 and $\text{Tor}^R_{\ell}(M, N) = 0$ for some $\ell \geq 5$, then $M$ or $N$ is $R$-free.
2. If $\text{Tor}^R_{\ell}(M, N) = \text{Tor}^R_{\ell+1}(M, N) = 0$ for some $\ell \geq 5$, then $\text{pd}_R M \leq 1$ or $\text{pd}_R N \leq 1$.

**Proof.** (1) By Fact 3.1 the ring $R$ has depth 0. If $M$ is not $R$-free, then $\text{pd}_R M = \infty$. Theorem 3.6 implies that $m$ is a direct summand of $\Omega^3_R M$ or $\Omega^4_R M$. Lemma 6.1(1) now implies that $\text{pd}_R N < \infty$, and $N$ is $R$-free.

(2) By Fact 3.1 the ring $R$ has depth $\leq 1$. Theorem 3.6 implies that $m$ is a direct summand of $\Omega^3_R M \oplus \Omega^4_R M$ or $\Omega^5_R M$. Assume that $\text{pd}_R M \geq 2$. Note that $\ell \geq 3 + 1$ and $\ell + 1 \geq 5 + 1$. It is observed from (1a) and (2a) in Lemma 6.1 that $\text{pd}_R N < \infty$. Hence, $\text{pd}_R N \leq 1$.

An analogous argument to the proof of Corollary 6.2 yields its Ext version. Note that this highly refines the Ext vanishing results in [16].

**Corollary 6.3.** Assume that $R = S \times_k T$ is a fiber product, where $S$ and $T$ are local rings with common residue field $k$ and $S \neq k \neq T$. Let $M$ and $N$ be $R$-modules.

1. If $S$ or $T$ has depth 0 and $\text{Ext}^R_{\ell}(M, N) = 0$ for some $\ell \geq 4 + \max\{1, \text{depth}_R N\}$, then $M$ is $R$-free or $N$ is $R$-injective.
2. If $\text{Ext}^R_{\ell}(M, N) = \text{Ext}^R_{\ell+1}(M, N) = 0$ for some $\ell \geq 4 + \max\{1, \text{depth}_R N\}$, then $\text{pd}_R M \leq 1$ or $\text{id}_R N \leq 1$.

To get our next applications, we prepare a lemma.

**Lemma 6.4.** Let $x = x_1, \ldots, x_n$ be an $R$-sequence such that $m/(x)$ is decomposable. Then $n$ is equal to either depth $R - 1$ or depth $R$.

**Proof.** By Fact 3.1 we have depth $R/(x) \leq 1$. This shows the assertion.

The following results are generalizations of Corollaries 6.2 and 6.3 to a local ring with quasi-decomposable maximal ideal.

**Corollary 6.5.** Let $R$ be a local ring with depth $R = d$ such that $m$ is quasi-decomposable with an $R$-sequence $x = x_1, \ldots, x_n$. Let $M$ and $N$ be $R$-modules for which there exists an integer $t \geq \max\{5, n + 1\}$ such that $\text{Tor}^R_{i}(M, N) = 0$ for all $t + n \leq i \leq t + n + d$. Then $\text{pd}_R M < \infty$ or $\text{pd}_R N < \infty$.

**Proof.** Note that $t - n > 0$. By Lemma 5.1 we see that $x$ is a regular sequence on both $X := \Omega^n_M$ and $Y := \Omega^n_N$. We also have $\text{Tor}^R_{i}(X, Y) = 0$ for all $t - n \leq i \leq t - n + d$. Using the long exact sequences arising from the short exact sequences
\[
0 \longrightarrow Y/(x_1, \ldots, x_{j-1})Y \xrightarrow{x_j} Y/(x_1, \ldots, x_{j-1})Y \longrightarrow Y/(x_1, \ldots, x_j)Y \longrightarrow 0
\]
for $1 \leq j \leq n$, we observe that $\text{Tor}_i^R(X, Y / x Y) = 0$ for all $t \leq i \leq t - n + d$. Applying [13, section 18, lemma 2(i)], we obtain $\text{Tor}_{i}^{R/(x)}(X/xX, Y/xY) = 0$ for all $t \leq i \leq t - n + d$.

According to Lemma 6-4, the integer $n$ is either $d$ or $d - 1$. If $n = d$, then $\text{Tor}_{i}^{R/(x)}(X/xX, Y/xY) = 0$ and $R/(x)$ has depth zero. Fact 3-1 and Corollary 6-2(1) imply that either $X/xX$ or $Y/xY$ is $R/(x)$-free. If $n = d - 1$, then we have $\text{Tor}_{i}^{R/(x)}(X/xX, Y/xY) = \text{Tor}_{i+1}^{R/(x)}(X/xX, Y/xY) = 0$. Fact 3-1 and Corollary 6-2(2) imply that $\text{pd}_{R/(x)}(X/xX) \leq 1$ or $\text{id}_{R/(x)}(Y/xY) \leq 1$. Hence, $\text{pd}_{R} X < \infty$ or $\text{id}_{R} Y < \infty$ by [3, lemma 1-3-5]. Therefore, $\text{pd}_{R} M < \infty$ or $\text{id}_{R} N < \infty$.

**Corollary 6-6.** Let $R$ be a $d$-dimensional Cohen–Macaulay local ring such that $m$ is quasi-decomposable with an $R$-sequence $x = x_1, \ldots, x_n$. Let $M$ and $N$ be $R$-modules, and set $s := d - \text{depth } M (\geq 0)$. Suppose that there exists an integer $t \geq 5$ such that $\text{Ext}_{i}^{R}(M, N) = 0$ for all $t + s \leq i \leq t + s + d$. Then $\text{pd}_{R} M < \infty$ or $\text{id}_{R} N < \infty$.

**Proof.** Without loss of generality we may assume that $R$ is complete, so that $R$ admits a canonical module. Let $L := \Omega_{R}^{s} M$ and note then that $L$ is maximal Cohen–Macaulay. We now have $\text{Ext}_{i}^{R}(L, N) = 0$ for all $t \leq i \leq t + d$. By virtue of [1, theorem 1-1], we get an exact sequence $0 \to Y \to X \to N \to 0$ in which $X$ is maximal Cohen–Macaulay and $Y$ has finite injective dimension. Since $\text{Ext}_{i}^{R}(L, Y) = 0$ for all $m > 0$, we get $\text{Ext}_{i}^{R}(L, X) = 0$ for all $t \leq i \leq t + d$. Using the long exact sequences arising from the short exact sequences

$$0 \to L/(x_1, \ldots, x_{j-1})L \xrightarrow{j} L/(x_1, \ldots, x_{j-1})L \to L/(x_1, \ldots, x_{j})L \to 0$$

for $1 \leq j \leq n$, we see that $\text{Ext}_{i}^{R}(L/xL, X) = 0$ for all $t + n \leq i \leq t + d$. It follows from [13, section 18, lemma 2(i)] that $\text{Ext}_{i}^{R}(L/xL, X/xX) = 0$ for all $t \leq i \leq t + d - n$.

Lemma 6-4 says that $n = d$ or $n = d - 1$, and note that

$$5 = 4 + 1 = 4 + \max\{1, \text{depth}_{R/(x)} X/xX\}.$$ 

When $n = d$, the ring $R/(x)$ has depth zero and we have $\text{Ext}_{i}^{R/(x)}(L/xL, X/xX) = 0$. Combining Fact 3-1 and Corollary 6-3(1), we conclude that $L/xL$ is $R/(x)$-free or $X/xX$ is $R/(x)$-injective. When $n = d - 1$, we have $\text{Ext}_{i}^{R/(x)}(L/xL, X/xX) = \text{Ext}_{i+1}^{R/(x)}(L/xL, X/xX) = 0$. In this case, Fact 3-1 and Corollary 6-3(2) imply that $\text{pd}_{R/(x)}(L/xL) \leq 1$ or $\text{id}_{R/(x)}(X/xX) \leq 1$. Therefore, $\text{pd}_{R} L < \infty$ or $\text{id}_{R} X < \infty$; see [3, lemma 1-3-5 and corollary 3-1-15]. Thus, $\text{pd}_{R} M < \infty$ or $\text{id}_{R} N < \infty$.

**Remark 6-7.** We do not know how to get a complete generalisation of Corollary 6-3 to the higher-dimensional case (with no Cohen–Macaulay assumption) like what we did in Corollary 6-5 as a generalisation of Corollary 6-2. However, if we assume infinitely many Ext vanishings, then we can prove the following result using Corollary 6-5 and [2, proposition 6-5] (see also the proof of [16, proposition 4-1]):

**Corollary 6-8.** Let $R$ be a local ring with quasi-decomposable maximal ideal $m$. Let $M$ and $N$ be $R$-modules such that $\text{Ext}_{i}^{R}(M, N) = 0$ for all $i \geq 0$. Then $\text{pd}_{R} M < \infty$ or $\text{id}_{R} N < \infty$.

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