Coupling the Hexagonal B1-grid and B2-grid to Avoid a Computational Mode Problem of the Hexagonal ZM-grid

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Abstract

A shallow water model is developed on the regular hexagonal mesh by combining the hexagonal B1-grid and B2-grid schemes. The new scheme called the hexagonal synchronized B-grid (SB-grid) scheme in this work allows avoiding a computational mode problem of the ZM-grid scheme. It is known that the problem is caused by the mismatch of degrees of freedoms of the prognostic variables. The SB-grid uses the same variable arrangement as the ZM-grid, placing fluid depths and fluid velocities at the centers and corners of hexagonal cells, respectively, but the nonlinear terms of the momentum equation are discretized using wider spatial stencils than those of the ZM-grid. This change results in the inhibition of extra interactions in the velocity fields that enhances a computational mode in the ZM-grid. Geostrophic adjustment tests on a regular hexagonal mesh confirm that the SB-grid shallow water model behaves almost equivalently to the Z-grid model, and the computational mode problem is certainly settled down.

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1. Introduction

The icosahedral mesh, which can be generated by iterative subdivisions of an icosahedron (Heikes and Randall 1995), provides quasi-homogeneous and quasi-isotropic coverage over the sphere (Miura and Kimoto 2005). This property could be useful for global atmospheric models, especially ones that target high-resolution simulations. Several atmospheric models that use the icosahedral mesh as the basis for horizontal discretization are being used for scientific research and weather predictions. A few examples are the Nonhydrostatic Icosahedral Atmospheric Model (NICAM) (Satoh et al. 2008, 2014), the Model for Prediction Across Scales-Atmosphere (MPAS-A) (Skamarock et al. 2012), and the Icosahedral Non-hydrostatic (ICON) modeling framework (Zängl et al. 2015). In terms of the variable arrangement for a shallow water model, NICAM uses the hexagonal A-grid, which collocates the fluid depth \( h \) at the centers of the hexagonal cells. MPAS-A uses the hexagonal C-grid, which places \( h \) at the centers of the hexagonal cells and the normal component of the velocity field \( u \), at the midpoints of the edges of the hexagonal cells. Alternatively, ICON uses the triangular C-grid, in which \( h \) is located at the centers of the triangular cells and \( u \) is located at the edges of the triangular cells.

More than a decade ago, the hexagonal ZM-grid was proposed as another variable arrangement method on the hexagonal mesh (Ringler and Randall 2002a). The ZM-grid places \( h \) at the cell centers and \( v \) at the cell corners (Fig. 1a). The ZM-grid seemed prospective because it has the same dispersion relation as the Z-grid (Masuda and Ohnishi 1986; Heikes and Randall 1995) that places vorticity and divergence at the hexagonal cell centers, and no Poisson equation is required, similarly to the A-grid (Tomita et al. 2001). Note that two Poisson equations must be solved to reconstruct the velocity field in the Z-grid. In the ZM-grid, however, there is a drawback that the number of \( v \) points is twice the number of \( h \) points. The ratio of degrees of freedom of the velocity components to those of the fluid depth is four, which should be two ideally, and this extra degree of freedom in \( v \) allows computational modes to be enhanced in the velocity field. This may be the reason why the ZM-grid is not adopted for the dynamical core of any atmospheric models currently.

It has been demonstrated that the undesirable influence of the computational mode of the ZM-grid can be mitigated by adding hyperviscosity terms to the momentum equation (Ringler and Randall 2002b) or by using an upwind-biased transport scheme for momentum components (Danilov 2012; Danilov and Androsov 2015). Even with these mitigations, however, there remain concerns that the computational mode is enhanced and causes unphysical behavior of the models. In a test case initialized by a balanced, solid-body rotational flow (Williamson et al. 1992), for example, a ZM-grid shallow water model on an icosahedral mesh suffers from a gradual but continuous acceleration or deceleration of velocity around the poles, accompanied by a corresponding decrease or increase of the fluid depth at the pole points (result not shown). A hyperviscosity and/or an upwind-biased transport can slow down the growth of the mode, but the ascent or descent of the fluid eventually becomes noticeable after a longer-term integration.

In this work, we seek for a more fundamental solution on the computational mode problem of the ZM-grid. We start from a shallow water system on the regular hexagonal mesh. On the regular hexagonal mesh, we are free from difficulties due to the mesh distortions and special treatments of the twelve pentagons of the spherical icosahedral mesh, and can concentrate exclusively on the computational mode problem.

2. The synchronized B-grid shallow water model

We introduce the B1-grid and B2-grid models first because we derive the SB-grid scheme by combining the B1-grid and B2-grid models (Figs. 1b and 1c). In this work, we call the HB/ E 1 and HB/E 2 schemes (Mesinger 2000) as the B1-grid and B2-grid, respectively, for simplicity. The B1-grid places \( h \) at the centers of the hexagonal cells and \( v \) at every other vertex of the hexagonal cells. The B2-grid also places \( h \) at the cell centers, but \( v \) is assigned in the B1-grid and B2-grid is free of extra modes caused by a mismatch of the numbers of \( v \) and \( h \) points.

Given zero surface elevation, the vector-invariant formulation of the shallow water equations is as follows:

\[
\frac{\partial h}{\partial t} = -\nabla \cdot (hv) \quad \text{and} \quad \frac{\partial v}{\partial t} = -\eta k \times v - \nabla (K + gh),
\]

where \( \nabla \) is the horizontal gradient operator, \( k \) is the vertical unit vector, \( K = \frac{1}{2} |v|^2 \) is the kinetic energy, and \( g \) is the gravitational constant. The definition of \( \eta \) is \( \eta = f - \zeta \), where \( f \) is the Coriolis parameter and \( \zeta \) is the vertical component of the relative vorticity.

Here, variables are denoted with the subscripts 1 and 2 for a clear distinction of the B1-grid and B2-grid models. The prognos-
tic equations of the B1-grid are

\[
\frac{\partial h}{\partial t} = -[\nabla \cdot (h \mathbf{v})]_{h_0}, \quad \text{and} \\
\frac{\partial v_i}{\partial t} = -(f + k \cdot \nabla \times \mathbf{v}_i)_{h_0} - [\nabla' K_i]_{h_0} - [\nabla (gh_i)]_{h_0}.
\]  

(3)

Similarly, the prognostic equations of the B2-grid model are

\[
\frac{\partial h_j}{\partial t} = -[\nabla'' (h_j \mathbf{v}_j)]_{h_0}, \quad \text{and} \\
\frac{\partial v_{i,j}}{\partial t} = -(f + k \cdot \nabla'' \times \mathbf{v}_{i,j})_{h_0} - [\nabla'' K_{i,j}]_{h_0} - [\nabla (gh_{i,j})]_{h_0}.
\]  

(4)

The terms with subscripts \( P_i, Q_j, \) or \( Q_k \) are evaluated at a cell center or at a cell corner (Fig. 2a). Note that the B1-grid model does not depend on the variables of the B2-grid model and vice versa.

The discrete form of the gradient operator is defined at a cell corner \( Q_{0} \) (Fig. 2b) as

\[
\nabla' \cdot \mathbf{v} \approx \frac{1}{2A} \sum_{\alpha=0}^{6} \left( \alpha \mathbf{v}_{i+1/2(i-1/2)} \right) \mathbf{e}_{\alpha,i},
\]  

(7)

where \( j + 1 \) is cyclic, \( \alpha_i \) are the values of the function \( \alpha \) at the cell corners \( P_i \) \((j = 0,1,2), S \) is the area of the triangular cells, \( L \) is the distance between the neighboring cell centers, and \( \mathbf{e}_{\alpha,i} \) is the unit vector normal to the \( i \)-th cell edge of the triangular cell.

The flux-divergence operator with the single asterisk in (3) is discretized at the cell center \( P_j \) (Fig. 2a) as

\[
\nabla' \cdot (h \mathbf{v}) \approx \frac{1}{2A} \sum_{\alpha=0}^{6} \left( \alpha \mathbf{v}_{i+1/2(i-1/2)} \right) \mathbf{e}_{\alpha,i},
\]  

(8)

where \( i - 1 \) is cyclic, \([i - 1/2] \) means the integer part of \((i - 1)/2\), \( \mathbf{v}_{i,j} \) is the velocity at the \( i \)-th cell corner, \( \bar{\alpha}_i \) is the value of \( \alpha \) at the \( i \)-th cell corner, \( A \) is the area of the hexagonal cells, \( l \) is the distance between the neighboring cell corners, and \( \mathbf{e}_{\alpha,i} \) is the unit vector normal to the \( i \)-th cell edge of the \( P_j \) hexagonal cell. The interpolation from the cell centers \( P_j \) \((j = 0,1,2)\) to the cell corner

\[
Q_{0} \quad (Q_{0} \text{Fig. 2b}) \text{ is defined as}
\]

\[
\bar{\alpha}_i = \frac{1}{2} \sum_{j=1}^{2} \alpha_j,
\]

(9)

The gradient and curl operators with a single asterisk in (4) are defined at the cell corner \( Q_{0} \) (Fig. 2c) as

\[
\nabla' \alpha_{ij} \approx \frac{L}{2A} \sum_{k=0}^{6} \left( \alpha_i + \alpha_{j+1} \right) \mathbf{e}_{\alpha_{i,j}},
\]

(10)

\[
k \cdot \nabla' \times \mathbf{v}_{i,j} \approx \frac{L}{2A} \sum_{k=0}^{6} \left( \mathbf{v}_{i,k} + \mathbf{v}_{i,j+1,k} \right) \mathbf{e}'_{\alpha_{i,j}},
\]

(11)

where \( k - 1 \) is cyclic, \( \alpha_i \) is the values of \( \alpha \) at the cell corner \( Q_k \) \((k = 1,2,\ldots,6)\) where \( \mathbf{v} \) is placed, \( \Lambda = 3A \) is the area of the large hexagon cell around \( Q_0 \) configured by connecting \( Q_k \) \((k = 1,2,\ldots,6)\), and \( \mathbf{e}'_{\alpha_{i,j}} \) are the unit vectors normal and tangential, respectively, to the \( k \)-th edge of the large hexagonal cell.

Due to the rotational symmetry of the B1-grid and B2-grid systems, the double asterisk operators in (5) and (6) can be obtained from the single asterisk operators (8), (10) and (11) with slight modifications of the indices.

To obtain the SB-grid model, we couple the B1-grid and B2-grid models. Nudging terms are added to the right-hand-side of (3) and (5) to synchronize the B1-grid and B2-grid models as

\[
\frac{\partial h_i}{\partial t} = -[\nabla' \cdot (h_i \mathbf{v}_i)]_{h_i} - \frac{h_i - h_2}{2\tau} - \frac{h_1 - h_i}{2\tau},
\]

(12)

\[
\frac{\partial v_{i,j}}{\partial t} = -(f + k \cdot \nabla'' \times \mathbf{v}_{i,j})_{h_i} - \frac{h_i - h_2}{2\tau} - \frac{h_1 - h_i}{2\tau},
\]

(13)

where \( \tau > 0 \) is the timescale of the synchronization. Note that the domain integral of \( h_i \) plus \( h_2 \) is conserved under the time integration because of the antisymmetric form of the nudging terms. The coupled B1–B2 system with a finite value of \( \tau \) is constituted by the four prognostic equations: (12), (13), (4), and (6). The ratio of degrees of freedom of \( \mathbf{v} \) plus \( \mathbf{v}_i \) to those of \( h_i \) plus \( h_2 \) is two, and thus there is no mismatch of the degrees of freedoms.

Here, the pair of \( h_1 \) and \( h_2 \) are transformed to another pair of prognostic variables \( h_i = (h_1 + h_2)/2 \) and \( h_j = (h_1 - h_2)/2 \), which represent the common variation of and the difference between \( h_1 \) and \( h_2 \), respectively. The equations of \( h_i \) and \( h_j \) are
\[
\frac{\partial h}{\partial t} \bigg|_{h_1} = -\frac{1}{2}[\nabla^\top(h v_1) + \nabla^\top(h v_2)]_{h_1}, \quad \text{and} \quad (14)
\]
\[
\frac{\partial h}{\partial t} \bigg|_{h_0} = -\frac{1}{2}[\nabla^\top(h v_1) - \nabla^\top(h v_2)]_{h_0} - \frac{h_0}{\tau} h_0. \quad (15)
\]

If the B1-grid and B2-grid systems are stable in the sense that the values of all prognostic boundaries are bounded, the magnitudes of the terms \(\nabla^\top(h v_1)\) and \(\nabla^\top(h v_2)\) are upper bounded and there exists a finite real number \(\varepsilon\) that satisfies
\[
\max \left\{ |\nabla^\top(h v_1)|, |\nabla^\top(h v_2)| \right\} < \varepsilon.
\]

Let \(\hat{h}\) be the characteristic scale of the variation of \(h\). If \(\hat{h} = 0\) is satisfied, this is the condition that we want to obtain. For \(\hat{h} > 0\), we can choose \(0 < \tau < \hat{h}/\varepsilon\) so that the time evolution of \(h\) is dominated by the damping term as
\[
\frac{\partial h}{\partial t} \sim \frac{h_0}{\tau}. \quad (17)
\]

This means that the time evolution of \(h\) can be approximated as \(h_t \sim h_0 (t = 0) e^{-\tau t}\). As a result, it is found that (15) converges to the condition
\[
h_0 = 0 \quad (18)
\]
in the infinitesimal limit of \(\tau\). From the definition, (18) is equivalent to \(h_t = h_0 = h_2\). Thus, for \(\tau \to 0\), the equations (14), (4), and (6) becomes
\[
\frac{\partial h}{\partial t} \bigg|_{h_1} = -\left[\nabla^\top\left(h v_1 + \frac{v_1}{2}\right)\right]_{h_1}, \quad (19)
\]
\[
\frac{\partial v_1}{\partial t} = -(f + k \cdot \nabla \times v_1)_{h_0} (k \times v_1)_{h_0} - [\nabla^\top K_{h_0}]_{h_0} - [\nabla(gh)]_{h_0}, \quad (20)
\]
and
\[
\frac{\partial v_2}{\partial t} = -(f + k \cdot \nabla \times v_2)_{h_0} (k \times v_2)_{h_0} - [\nabla^\top K_{h_0}]_{h_0} - [\nabla(gh)]_{h_0}. \quad (21)
\]

The discrete form of the flux-divergence is
\[
\nabla^\top\left(\frac{v_1 + v_2}{2}\right) \bigg|_{h_0} = \frac{1}{2}[\nabla^\top(\alpha v_1)]_{h_0} + \frac{1}{2}[\nabla^\top(\alpha v_2)]_{h_0} \approx \frac{l}{\Delta h} \sum_{i=1}^{l} \left(\hat{a}_i + \hat{a}_{i+1}\right) \left(\frac{v_i + v_{i+1}}{2}\right) \epsilon_{x,i}. \quad (22)
\]

In summary, the SB-grid model is composed of the three prognostic equations and one constraint: (19), (20), (21) and (18). Although the prognostic equation of \(h\) is degenerated, the prognostic variables, \(h, v_1, v_2\), must satisfy the constraint \(h_0 = 0\). Therefore, the ratio of degrees of freedom of \(v_1\) plus \(v_2\) to those of \(h + v_0\) is still two, and the mismatch of the degrees of freedoms does not exist.

It has already been found that the B1-grid and B2-grid are unsuitable for the atmospheric models because the dispersion relation of the linearized shallow water system is unrealistic (Ničković et al. 2002). However, the complete synchronization of the B1-grid and B2-grid systems recovers the same form of the dispersion relation as the ZM-grid and Z-grid models (Ringler and Randall 2002b). This fact is obvious since the discrete forms of the linearized shallow water equations are the same between the ZM-grid and the SB-grid models.

The difference between the ZM-grid and the SB-grid is in the discrete form of the nonlinear terms of the momentum equation. As it is known (Ringler and Randall 2002b), a computational mode in the velocity field is enhanced due to the extra coupling of the velocity fields through nonlinear interactions. If we denote the extra mode in the \(v_1\) and \(v_2\) fields as \(\hat{v}_1\) and \(\hat{v}_2\), their evolution follows
\[
\frac{\partial \hat{v}_1}{\partial t} = g(v_1, v_2) + f k \times \hat{v}_1, \quad (23)
\]
\[
\frac{\partial \hat{v}_2}{\partial t} = -g(v_1, v_2) + f k \times \hat{v}_2, \quad (24)
\]
where \(g(v_1, v_2)\) represents a contribution from the nonlinear interaction between \(v_1\) and \(v_2\). When \(g(v_1, v_2) = 0\), a system composed of (23) and (24) allows modes of \(v_1\) and \(v_2\) to be enhanced without coupling with \(h\), as it is in the ZM-grid model. In the SB-grid model, the nonlinear terms are discretized in a manner to exclude direct interactions between the \(v_1\) and \(v_2\) fields. Thus, \(g(v_1, v_2) = 0\) holds and \(\hat{v}_1\) and \(\hat{v}_2\) are not enhanced if a stable time discretization scheme is chosen.

3. Numerical results

A numerical test is performed to compare geostrophic adjustments of the new SB-grid model, the Z-grid model (Masuda and Ohnishi 1986; Heikes and Randall 1995), the ZM-grid model (Ringler and Randall 2002a) and the A-grid model (Tomita et al. 2001). The equations are nondimensionalized, and the parameters are set to \(g = 1.0\) and \(f = 1.0\). The characteristic fluid depth is \(H = 1.0\). The cell interval is \(l = 2.0\), and the time interval is \(\Delta t = 0.2\). Thus, the ratio of the deformation radius \(\lambda = \sqrt{gH}/f\) to the cell interval is \(\lambda L = 0.5\), and the Courant number is \(\Delta t \sqrt{gH}/L = 0.1\) regarding linear gravity waves. The number of hexagonal cells is 20 for both the \(x\) and \(y\) directions. The lateral boundary conditions are doubly periodic. Three-stage Runge-Kutta (RK3) time stepping (Wicker and Skamarock 2002) is used for time integrations. No hyperviscosity term is added explicitly, except for an optional experiment that adds the fourth-order hyperviscosity to the ZM-grid model with viscosity coefficients \(K_2 = 10^{-4}\) and \(10^{-5}\).

The initial fluid depths are \(h = H\) uniformly, except for one cell where \(h = H + 0.01\); hereafter we refer to this cell as the central hexagon. To include an inertial motion of the background field so that the computational mode of the ZM-grid is readily enhanced, the initial fluid velocities are \(u = 0.1\) and \(v = 0.0\) uniformly. Note that this background flow is subtracted in the figures.

The time evolutions of \(h\) are very similar among the Z-grid, ZM-grid, and SB-grid models (Fig. 3). This is a natural consequence of the fact that the three models share the same frequency equation about the linearized system. Gravity waves propagate outward from the central hexagon nearly isotropically. The solutions of the A-grid model are largely different from the other models, the indication being that emissions of gravity waves are obviously inefficient. The velocity vectors around the central hexagon are nearly rotationally symmetric for the Z-grid, SB-grid and A-grid, pointing outward with clockwise rotation, while this rotational symmetry is largely impaired for the ZM-grid.

To facilitate understanding of the behavior of the computational mode, the time evolution of \(h\) at the central hexagon and those of the \(x\) and \(y\) components of \(v\) at the upper right corner of the central hexagon are compared (Fig. 4). The temporal variances of \(h\) are nearly equivalent among the Z-grid, ZM-grid, and SB-grid, while the amplitude of the variation is obviously smaller for the A-grid. The potential energy anomaly given at the central hexagon is not effectively radiated away in the A-grid, supporting the concern that the A-grid model cannot respond appropriately to a grid-scale forcing. Although evolutions of \(v\) are similar among the Z-grid, ZM-grid, and SB-grid before about 100 steps, a lower frequency oscillation with larger amplitude dominates after about 200 steps for the ZM-grid. This computational mode in \(v\) develops without influencing the physical mode of \(h\) in this short-term simulation. The fractional change of the domain-integrated mass-weighted potential enstrophy shows that the Z-grid is the best...
among the four models, while the SB-grid is comparable to or slightly better than the ZM-grid.

The fourth-order hyperviscosity does not solve the computational mode problem of the ZM-grid (Supplement Fig. S1). The stronger hyperviscosity works to reduce the amplitude of the computational mode, but a convergence to the solutions of the Z-grid is not observed. Even with the hyperviscosity, a slow oscillation gradually emerges as time progresses. When the hyperviscosity is stronger, the departure from the Z-grid in $h$ is non-negligible, and the conservation of the potential enstrophy is damaged.

To confirm that the $v_1$ and $v_2$ fields are not decoupled in the SB-grid system, an additional test is performed with a spike-like forcing given at the upper right corner of the central hexagon initially as $v_x = (0.1/\sqrt{2}, 0.1/\sqrt{2})$, while $v_x = v_y = 0$ elsewhere (Supplemental Figs. S2 and S3). For a comparison with the Z-grid model, the initial flow fields are reconstructed by computing the
stream function $\psi$ and the velocity potential $\chi$. According to the Helmholtz-Hodge decomposition, $v = k \times \nabla \psi + \nabla \chi + \mathbf{v}$. The discrete gradient (7) is used and the harmonic component is $\nabla = 0$.

The time evolution of $h$ is again very similar among the Z-grid, ZM-grid, and SB-grid models. Gravity waves propagate outward with a spiral structure. No visible difference exists in the variation of $h$ at the central hexagon (Supplemental Fig. S3a). However, a difference emerges in $v$ between the ZM-grid and the other models as time proceeds. The computational mode of the ZM-grid is enhanced strongly at the upper right corner of the central hexagon, where the spike-like forcing is given. Although the amplitude is relatively smaller, the computational mode is also enhanced at the lower right corner of the central hexagon. It is notable for the SB-grid model that the $v$ field responds appropriately to the forcing given to the $v$ field. The temporal changes of the flow fields of the Z-grid and SB-grid are nearly the same. The conservation of potential enstrophy of the SB-grid model is no worse than the Z-grid and ZM-grid models.

4. Summary and discussion

A shallow water model that uses hexagonal SB-grid staggering has been developed. The coupling between the B1-grid and B2-grid systems by using (12) and (13) can be categorized as a bidirectional coupling between two chaotic systems (Fujisaka and Yamada 1983; Boccaletti et al. 2002). If the infinitesimal limit of the nudging timescale is taken, the synchronization strength becomes infinite and a complete synchronization of the two systems is attained. The difference between the SB-grid and the ZM-grid is in the nonlinear terms: the SB-grid uses wider spatial stencils than the ZM-grid. This change removes the extra interactions in the velocity fields, and allows bypassing the computational mode problem of the ZM-grid.

The numerical results demonstrate that the coarse resolution nonlinear terms can be a more fundamental remedy on the computational mode problem of the ZM-grid, comparing to the addition of a hyperviscosity term. In the numerical tests performed, the time evolutions of the fluid depths and the velocities were very similar between the SB-grid and Z-grid models. Conservation of the domain-integrated mass-weighted potential enstrophy was comparable to or slightly better than the ZM-grid.

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Supplement

Figure S1. The same as Fig. 4, except for the ZM-grid model including the fourth-order hyperviscosity term.

Figure S2. The same as Fig. 3, except for the second test. The solutions after 125 steps.

Figure S3. The same as Fig. 4, except for the second test. The $x$ and (f) $y$ components of the velocity vector at the lower right corner of the central hexagon are added.

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