Boundedness of Monge–Ampère singular integral operators on Besov spaces

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1. Introduction

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex smooth function and consider the Monge–Ampère measure $\mu$ generated by $\phi$

$$\mu := \det D^2 \phi,$$

where $D^2 \phi$ denotes the Hessian matrix of $\phi$. For a given function $u$,

$$\det D^2 (\phi + tu) = \det D^2 \phi + t \text{trace} (\Phi D^2 u) + \cdots + t^n \det D^2 u,$$

where $\Phi = (\Phi)_{ij}$ is the matrix of cofactors of $D^2 \phi$. The linearized Monge–Ampère operator is denoted by

$$L_\phi u = \text{trace} (\Phi D^2 u).$$

To study the properties of the solutions for the linearized Monge–Ampère equation $L_\phi u = 0$, Caffarelli and Gutiérrez \cite{Caffarelli1993} introduced a family of sections as follows. Let $\rho(x, y) = \phi(y) - \phi(x) - \nabla \phi(x) \cdot (y - x)$. Given $x \in \mathbb{R}^n$ and $t > 0$, the section is defined by

$$S_\phi(x, t) = \{y \in \mathbb{R}^n : \rho(x, y) < t\}.$$ 

These sets are convex and play crucial role in the study of linearized Monge–Ampère equation (see \cite{Caffarelli1993, Caffarelli1994}). Indeed, if the Monge–Ampère measure $\mu$ satisfies the geometric conditions, namely doubling and a uniform continuity conditions, Caffarelli and Gutiérrez \cite{Caffarelli1993, Caffarelli1994} proved a variant of the
There exist positive constants $K_1, K_2, K_3$ and $\epsilon_1, \epsilon_2$ such that, given two sections $S(x_0, t_0)$, $S(x, t)$ with $t \leq t_0$ satisfying $S(x_0, t_0) \cap S(x, t) \neq \emptyset$, and given an affine transformation $T$ that normalizes $S(x_0, t_0)$, there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x, t)$, which satisfies $B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x, t)) \subset B(z, K_1(t/t_0)^{\epsilon_1})$ and $T(x) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2})$. Here and below $B(x, t)$ denotes the Euclidean ball centered at $x$ with radius $t$.

**A** There exist positive constants $K_1, K_2, K_3$ and $\epsilon_1, \epsilon_2$ such that, given two sections $S(x_0, t_0)$, $S(x, t)$ with $t \leq t_0$ satisfying $S(x_0, t_0) \cap S(x, t) \neq \emptyset$, and given an affine transformation $T$ that normalizes $S(x_0, t_0)$, there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x, t)$, which satisfies $B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x, t)) \subset B(z, K_1(t/t_0)^{\epsilon_1})$ and $T(x) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2})$. Here and below $B(x, t)$ denotes the Euclidean ball centered at $x$ with radius $t$.

**B** There exists a constant $v > 0$ such that given a section $S(x, t)$ and $y \notin S(x, t)$, if $T$ is an affine transformation that normalizes $S(x, t)$, then for any $0 < \epsilon < 1$,

$$B(T(y), \epsilon^\nu) \cap T(S(x, (1 - \epsilon)t)) = \emptyset.$$ 

**C** $\bigcap_{t > 0} S(x, t) = \{x\}$ and $\bigcup_{t > 0} S(x, t) = \mathbb{R}^n$.

Based on the above properties on sections, Caffarelli and Gutiérrez [5] introduced the Monge–Ampère singular integral operators as follows. Suppose that $0 < \gamma \leq 1$ and $c_1, c_2 > 0$. Let $\{k_i(x, y)\}_{i \in \mathbb{Z}}$ be a sequence of kernels satisfying the following conditions:

**D1** $\supp k_i(\cdot, y) \subset S(y, 2^i)$ for all $y \in \mathbb{R}^n$;

**D2** $\sup \{ k_i(x, \cdot) \subset S(x, 2^i) \}$ for all $x \in \mathbb{R}^n$;

**D3** $\int_{\mathbb{R}^n} k_i(x, y) d\mu(y) = \int_{\mathbb{R}^n} k_i(x, y) d\mu(x) = 0$ for all $x, y \in \mathbb{R}^n$;

**D4** $\sup_i \int_{\mathbb{R}^n} |k_i(x, y)| d\mu(y) \leq c_1$ for all $x \in \mathbb{R}^n$;

**D5** $\sup_i \int_{\mathbb{R}^n} |k_i(x, y)| d\mu(x) \leq c_1$ for all $y \in \mathbb{R}^n$;

**D6** If $T$ is an affine transformation that normalizes the section $S(y, 2^i)$, then

$$|k_i(u, y) - k_i(v, y)| \leq \frac{c_2}{\mu(S(y, 2^i))} |T(u) - T(v)|^\gamma;$$

**D7** If $T$ is an affine transformation that normalizes the section $S(x, 2^i)$, then

$$|k_i(x, u) - k_i(x, v)| \leq \frac{c_2}{\mu(S(x, 2^i))} |T(u) - T(v)|^\gamma.$$

Denote $K(x, y) = \sum_{i \in \mathbb{Z}} k_i(x, y)$. The Monge–Ampère singular integral operator $H$ is defined by

$$H(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y) d\mu(y).$$

Caffarelli and Gutiérrez [5] proved that $H$ is bounded on $L^2(\mathbb{R}^n, d\mu)$. Subsequently, Incognito [6] established the $L^p(\mathbb{R}^n, d\mu)$, $1 < p < \infty$, and weak type $(1,1)$ estimates of $H$. It was also showed that $H$ is bounded from $H^1_{\mathcal{F}}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n, d\mu)$ and is bounded on $H^1_{\mathcal{F}}(\mathbb{R}^n)$ in [7] and [8], respectively. Recently, Lin [9] proved the boundedness of $H$ acting on $H^p_{\mathcal{F}}(\mathbb{R}^n)$, $1/2 < p < 1$, and their dual spaces which can be realized as Carleson measure spaces, Campanato spaces, and Lipschitz spaces.

Let $\phi$, $\rho$ and $\mu$ be as defined at the beginning of the article. Then $\tilde{\rho}(x, y) := \frac{1}{2}(\rho(x, y) + \rho(\rho(x, y)))$ is a quasi-metric on $\mathbb{R}^n$ and $(\mathbb{R}^n, \tilde{\rho}, \mu)$ is a space of homogeneous type. From now on, for simplicity we still use the same notation $S(x, t) := \{y \in \mathbb{R}^n : \tilde{\rho}(x, y) < t\}$ to denote the sections induced by $\tilde{\rho}$ and let $\mathcal{S} = \{S(x, t) : x \in \mathbb{R}^n \text{ and } t > 0\}$ be the family of sections. We assume the quasi-metric $\tilde{\rho}$ satisfies...
the regularity condition
\[
|\tilde{\rho}(x, y) - \tilde{\rho}(x', y)| \leq C \tilde{\rho}(x, x')^\varepsilon \left[\tilde{\rho}(x, y) + \tilde{\rho}(x', y)\right]^{1-\varepsilon}
\]  
(1)

for some constant \(C\), some regularity exponent \(\varepsilon \in (0, 1)\), and for all \(x, x', y \in \mathbb{R}^n\). We also assume the measure \(\mu\) satisfies the doubling property.

Under these assumptions, we can construct \(\{S_k\}_{k \in \mathbb{Z}}\), the approximation to the identity associated with \(\mathcal{F}\) (see Lemma 3.1 below for the existence). Let \(D_k = S_k - S_{k-1}\) and suppose that \(\mu(\mathbb{R}^n) = \infty\). Applying Coifman’s decomposition to the identity yields
\[
I = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D_k D_j = \sum_{k=-\infty}^{\infty} \sum_{j \in \mathbb{Z}: |k-j| \leq N} D_k D_j + \sum_{k=-\infty}^{\infty} \sum_{j \in \mathbb{Z}: |k-j| > N} D_k D_j := T_N + R_N.
\]

By Cotlar–Stein almost orthogonal estimates, one obtains a similar Calderón-type reproducing formula
\[
f = \sum_{k=-\infty}^{\infty} T_N^{-1} D_k^N D_k(f) = \sum_{k=-\infty}^{\infty} D_k^N D_k T_N^{-1}(f),
\]  
(2)

where \(N\) is a fixed large integer, \(D_k^N = \sum_{|j| \leq N} D_{k+j}\) and \(T_N^{-1}\) is the inverse of \(T_N\), and the series converges in \(L^2(\mathbb{R}^n, d\mu)\) (see the argument right after Lemma 3.3). In this paper we do not consider the \(L^p\) convergence for \(1 < p < \infty\) with \(p \neq 2\), but we still show that the above Calderón-type reproducing formula (2) holds for certain subspace of \(L^2(\mathbb{R}^n, d\mu)\), namely the following

**Theorem 1.1:** Let \(\{S_k\}_{k \in \mathbb{Z}}\) be an approximation to the identity associated with \(\mathcal{F}\) on \((\mathbb{R}^n, \tilde{\rho}, \mu)\), \(\mu(\mathbb{R}^n) = \infty\), and \(D_k = S_k - S_{k-1}\) for \(k \in \mathbb{Z}\). For \(|\alpha| < \varepsilon / 4\) and \(1 \leq p, q \leq \infty\), if \(f \in L^2(\mathbb{R}^n, d\mu)\) and satisfies
\[
\left\{ \sum_{k \in \mathbb{Z}} \left(2^{k\alpha} \|D_k(f)\|_{L^p_\mu}\right)^q \right\}^{1/q} < \infty,
\]

then (2) holds with respect to the norm defined by \(\left\{ \sum_{k \in \mathbb{Z}} (2^{k\alpha} \|D_k(f)\|_{L^p_\mu})^q \right\}^{1/q}\), where we make an appropriate modification for \(q = \infty\).

This result leads to introduce a new test function space as follows.

**Definition:** Let \(\{S_k\}_{k \in \mathbb{Z}}\) be an approximation to the identity associated with \(\mathcal{F}\) and \(D_k = S_k - S_{k-1}\) for \(k \in \mathbb{Z}\). For \(|\alpha| < \varepsilon / 4\) and \(1 \leq p, q \leq \infty\), define
\[
\mathcal{B}^{\alpha,q}_{p,\mathcal{F}} = \{f \in L^2(\mathbb{R}^n, d\mu) : \|f\|_{\mathcal{B}^{\alpha,q}_{p,\mathcal{F}}} < \infty\},
\]

where
\[
\|f\|_{\mathcal{B}^{\alpha,q}_{p,\mathcal{F}}} = \begin{cases} 
\left( \sum_{k \in \mathbb{Z}} \left(2^{k\alpha} \|D_k(f)\|_{L^p_\mu}\right)^q \right)^{1/q} & \text{if } 1 \leq q < \infty, \\
\sup_{k \in \mathbb{Z}} 2^{k\alpha} \|D_k(f)\|_{L^p_\mu} & \text{if } q = \infty.
\end{cases}
\]

It is clear that the test function space \(\mathcal{B}^{\alpha,q}_{p,\mathcal{F}}\) is a subspace of \(L^2(\mathbb{R}^n, d\mu)\). Applying the above Calderón-type reproducing formula (2), one can show that the test function space \(\mathcal{B}^{\alpha,q}_{p,\mathcal{F}}\) is independent of the choice of the approximation to the identity (see Proposition 5.1). Let \((\mathcal{B}^{\alpha,q}_{p,\mathcal{F}})'\) denote
the distribution space (dual of \( \dot{B}^{\alpha,q}_{p,p} \)). Note that for each fixed \( k \) and \( x \), the function \( D_k(x, \cdot) \) belongs to \( \dot{B}^{\alpha,q}_{p,p} \) for all \( |\alpha| < \varepsilon/4 \), \( 1 \leq p, q \leq \infty \), and thus \( D_k(f) \) is well defined for all \( f \in (\dot{B}^{\alpha,q}_{p,p})' \) (see the proof in Section 5). Moreover, applying the second difference property (v) in Lemma 3.1, we will show that the Calderón-type reproducing formula (2) still holds on the distribution (dual) space as follows.

**Theorem 1.2:** Under the same assumptions as Theorem 1.1, for each \( f \in (\dot{B}^{\alpha,q}_{p,p})' \),

\[
\langle f, g \rangle = \sum_{k \in \mathbb{Z}} (T_N^{-1} D_k D_k^N (f), g) = \sum_{k \in \mathbb{Z}} (D_k D_k^N T_N^{-1} (f), g), \quad \forall \ g \in \dot{B}^{\alpha,q}_{p,p}.
\]

Once this reproducing formula is established, we can define the Besov space \( \dot{B}^{\alpha,q}_{p,p} \) as follows.

**Definition:** For \( |\alpha| < \varepsilon/4 \) and \( 1 \leq p, q \leq \infty \), let \( p' \) and \( q' \) denote the conjugate index of \( p \) and \( q \), respectively. Suppose that \( \{S_k\}_{k \in \mathbb{Z}} \) is an approximation to the identity associated with \( \mathcal{F} \) on \( (\mathbb{R}^n, \tilde{\rho}, \mu) \) and set \( D_k = S_k - S_{k-1} \). The Besov spaces associated with \( \mathcal{F} \) are defined to be

\[
\dot{B}^{\alpha,q}_{p,p} = \left\{ f \in (\dot{B}^{-\alpha,q}_{p,p})' : \|f\|_{\dot{B}^{\alpha,q}_{p,p}} := \left\{ \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \|D_k f\|_{L^p_{\mu}} \right)^q \right\}^{1/q} < \infty \right\}
\]

with an appropriate modification for \( q = \infty \).

Again, applying the reproducing formula for distribution spaces, we can develop a theory of the Besov spaces on \( (\mathbb{R}^n, \tilde{\rho}, \mu) \). The main result of this theory is the following.

**Theorem 1.3:** Let \( \varepsilon_1 \) be the constant given in condition (A), \( \gamma \) be the constant given in conditions (D6) and (D7), and \( \varepsilon \) be the regularity exponent given in (1). For \( |\alpha| < \min\{\varepsilon, \gamma \varepsilon_1\}/4 \) and \( 1 \leq p, q \leq \infty \), the Monge–Ampère singular integral operator \( H \) is bounded on \( \dot{B}^{\alpha,q}_{p,p} \).

We will give preliminary and historical remarks in the next section. Then we construct the approximation to the identity associated to sections and obtain the almost orthogonality estimate in Section 3. In Section 4 the proofs of Calderón–type reproducing formulae on test function spaces \( \dot{B}^{\alpha,q}_{p,p} \) and their duals are given. We discuss the dense subspaces of Besov spaces \( \dot{B}^{\alpha,q}_{p,p} \) and their dual spaces as well in Section 5. Finally, Theorem 1.3 is proved in Section 6.

We use \( a \wedge b \) and \( a \vee b \) to denote \( \min\{a, b\} \) and \( \max\{a, b\} \) respectively. We also write \( a \lesssim b \) to indicate that \( a \) is majorized by \( b \) times a constant independent of \( a \) and \( b \), while the notation \( a \approx b \) indicates that both \( a \lesssim b \) and \( b \lesssim a \) hold.

### 2. Preliminary and historical remarks

Let \( \phi, \rho \) and \( \mu \) be defined as before. It is known that conditions (A) and (B) imply the following engulfing property: there exists a constant \( \theta \geq 1 \), depending only on \( \nu, K_1, \) and \( \varepsilon_1 \), such that if \( x \in S(y, \varepsilon) \) then \( S(y, \varepsilon) \subset S(x, \theta \varepsilon) \). From this property it is easy to show that

\[
\rho(y, x) \leq \theta \rho(x, y)
\]

and

\[
\rho(x, y) \leq \theta^2 (\rho(x, z) + \rho(z, y)).
\]

Then \( \tilde{\rho}(x, y) := \frac{1}{2}(\rho(x, y) + \rho(y, x)) \) is a quasi-metric on \( \mathbb{R}^n \); that is,
(i) $\bar{\rho}(x, y) = \bar{\rho}(y, x) \geq 0$ for all $x, y \in \mathbb{R}^n$;
(ii) $\bar{\rho}(x, y) = 0$ if and only if $x = y$;
(iii) the quasi-triangle inequality holds: there is a constant $A_0 \in [1, \infty)$ such that
\[
\bar{\rho}(x, y) \leq A_0[\bar{\rho}(x, z) + \bar{\rho}(z, y)] \quad \text{for all } x, y, z \in \mathbb{R}^n.
\] (4)

Moreover, it is easy to see that $\bar{\rho}(x, y)$ and $\rho(x, y)$ are geometrically equivalent. Therefore, all results obtained by Caffarelli and Gutiérrez as mentioned in Section 1 still hold with replacing $\rho(x, y)$ by $\bar{\rho}(x, y)$. Since $\bar{\rho}$ is a space of homogeneous type, one might expect that the Besov spaces and the boundedness of singular integrals associated with sections deduced by $\bar{\rho}$ would follow from known results on spaces of homogeneous type. However, this is not the case. To see this, let us recall the theory of classical Besov spaces on $\mathbb{R}^n$. It was well known that the Littlewood–Paley theory plays a crucial role for developing function spaces on $\mathbb{R}^n$. Let $\psi$ be a Schwartz function satisfying

(i) $\text{supp } \hat{\psi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \}$;
(ii) $|\hat{\psi}(\xi)| \geq C > 0$ for $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

The classical Besov space $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, the space of tempered distributions modulo polynomials, satisfying
\[
\|f\|_{\dot{B}_p^{\alpha,q}} := \left( \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \|\psi_k \ast f\|_p \right)^q \right)^{1/q} < \infty,
\]
where $\psi_k(x) = 2^{kn} \psi(2^k x)$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

A crucial tool for the study of Besov spaces is the Calderón reproducing formula which was first provided by Calderón [10]. This formula says that, for any given function $\psi$ satisfying the above conditions (i) and (ii), there exists a function $\phi$ with the properties similar to $\psi$ such that
\[
f = \sum_{k=-\infty}^{\infty} \phi_k \ast \psi_k \ast f,
\] (5)
where the series converges not only in $L^2(\mathbb{R}^n)$, but also in $\mathcal{S}'(\mathbb{R}^n) = \{ f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^\alpha dx = 0 \text{ for all } |\alpha| \geq 0 \}$ and in $\mathcal{S}'(\mathbb{R}^n)$, the dual of $\mathcal{S}(\mathbb{R}^n)$. See [11] for more details.

Applying this reproducing formula, one can show that the definition of $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is independent of the choice of functions $\psi$ which satisfy the above conditions (i) and (ii). Moreover, using this formula, one also can study the theory of Besov spaces which includes the embedding, interpolation, duality, atomic decomposition, and the boundedness of singular integrals on $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$. See [11–14] for more details.

The classical theory of Calderón–Zygmund singular integral operators as well as the theory of function spaces on $\mathbb{R}^n$ were based on extensive use of convolution operators and on the Fourier transform. In order to carry over these results of Fourier analysis on $\mathbb{R}^n$ to more general setting, in the early 1970s, Coifman and Weiss [15] introduced spaces of homogeneous type. We say that $(X, d, \nu)$ is a space of homogeneous type in the sense of Coifman and Weiss if $d$ is a quasi-metric on $X$ and $\nu$ is a non-zero measure satisfying the doubling condition. We say that a non-zero measure $\nu$ satisfies the doubling condition if there is a constant $C_\nu$ such that, for all $x \in X$ and $r > 0$,
\[
\nu(B_d(x, 2r)) \leq C_\nu \nu(B_d(x, r)) < \infty,
\] (6)
where $B_d(x, r) = \{ y \in X : d(x, y) < r \}$.

Spaces of homogeneous type include many special spaces in analysis and have many applications in the theory of singular integrals and function spaces. See [15–18] for more details.
By the end of the 1970s, it was well recognized that much contemporary real analysis requires little structure on the underlying space. For instance, to obtain the maximal function characterizations for the Hardy spaces on spaces of homogeneous type, Macías and Segovia [19] proved that one can replace the quasi-metric $d$ by another quasi-metric $d'$ on $X$ such that the topologies induced on $X$ by $d$ and $d'$ coincide, and $d'$ has the following regularity property:

$$|d'(x, y) - d'(x', y)| \leq C_0 d'(x, x')^\varepsilon [d'(x, y) + d'(x', y)]^{1-\varepsilon}$$

for some constant $C_0$, some regularity exponent $\varepsilon \in (0, 1)$, and for all $x, x', y \in X$. Moreover, the measure $\nu$ satisfies

$$C_1^{-1} r \leq \nu(B_d(x, r)) \leq C_1 r \quad \text{for some constant } C_1.$$  \hfill (7)

Note that property (7) is much stronger than the doubling condition (6). Macías and Segovia [20] established the maximal function characterization for Hardy spaces $H^p(X)$, $(1+\varepsilon)^{-1} < p \leq 1$, on spaces of homogeneous type $(X, d', \nu)$ whenever the quasi-metric $d'$ and the measure $\nu$ satisfy the regularity condition (1) and property (7), respectively.

The seminal result on spaces of homogeneous type $(X, d', \nu)$, where $d'$ and $\nu$ satisfy condition (1) and $\nu$ satisfies property (7), is the $Tb$ theorem given by David et al. [21]. The key step to establish such a $Tb$ theorem is Coifman’s construction of the approximation to the identity and the decomposition of the identity. Coifman’s construction of the approximation to the identity is as follows. Take a smooth function $h$ defined on $[0, \infty)$, equals to 1 on $[1, 2]$, and 0 on $[0, 1/2] \cup [4, \infty)$. Let $T_k$ be the operator with kernel $2^k h(2^k d'(x, y))$. Property (7) of the measure $\nu$ implies that $C^{-1} \leq T_k(1) \leq C$ for some $C > 0$. Let $M_k$ and $W_k$ be the operators of multiplications by $1/T_k(1)$ and $[T_k(1/T_k(1))]^{-1}$, respectively, and let $S_k := M_k T_k W_k T_k M_k$. Then the regularity property (1) on the metric $d'$ and property (7) on the measure $\nu$ imply that the kernels $S_k(x, y)$ of $S_k$ satisfy the following conditions: for some constant $C > 0$,

(i) $S_k(x, y) = 0$ for $d'(x, y) \geq C 2^{-k}$, and $\|S_k\|_\infty \leq C 2^k$;
(ii) $|S_k(x, y) - S_k(x', y)| \leq C 2^{k(1+\varepsilon)} d'(x, x')^\varepsilon$;
(iii) $|S_k(x, y) - S_k(x, y')| \leq C 2^{k(1+\varepsilon)} d'(y, y')^\varepsilon$;
(iv) $\int_X S_k(x, y) \, d\nu(y) = \int_X S_k(x, y) \, d\nu(x) = 1$.

Let $D_k := S_k - S_{k-1}$. Coifman’s decomposition of the identity is given as follows. If $\nu(X) = \infty$, the identity operator $I$ can be written as

$$I = \sum_{k=-\infty}^{\infty} D_k = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D_k D_j = T_N + R_N,$$

where $T_N = \sum_{k=-\infty}^{\infty} \sum_{|j| \leq N} D_k D_j$ and $R_N = \sum_{k=-\infty}^{\infty} \sum_{|j| > N} D_k D_j$.

David, Journé, and Semmes showed that if $N$ is a fixed large integer, then $R_N$ is bounded on $L^p(X)$, $1 < p < \infty$, with the operator norm less than 1. Therefore, if $N$ is a fixed large integer and $D_k^N = \sum_{|j| \leq N} D_{k+j}$, they obtained the following Calderón-type reproducing formula

$$f = \sum_{k=-\infty}^{\infty} T_N^{-1} D_k^N D_k f = \sum_{k=-\infty}^{\infty} D_k D_k T_N^{-1} f,$$

where $T_N^{-1}$ is the inverse of $T_N$ and the series converges in $L^p(X)$, $1 < p < \infty$. Using this Calderón-type reproducing formula, they provided the Littlewood–Paley theory for $L^p(X)$, $1 < p < \infty$.  


Namely, for each $1 < p < \infty$, there exists a positive constant $C_p$ such that

$$C_p^{-1} \|f\|_p \leq \left\| \left\{ \sum_k |D_k(f)|^2 \right\}^{1/2} \right\|_p \leq C_p \|f\|_p.$$  

The above estimates were the key tool in [21] for proving the $T(b)$ theorem on $(X, d, \nu)$.

In [22], the Besov space was developed via the Littlewood–Paley theory on spaces of homogeneous type $(X, d, \nu)$ with the regularity property (1) on the metric $d$ and property (7) on the measure $\nu$. They first introduced a space of test function $\mathcal{M}(X)$, and then proved that $R_N$ defined as above in Coifman’s decomposition of the identity is bounded on $\mathcal{M}(X)$ with the operator norm less than 1 for a fixed large integer $N$. They showed that, for a fixed large integer $N$ and for each $k$, $T_N^{-1}D_k^N$ is a test function; that is, it satisfies similar conditions as $D_k$ does. Therefore, they obtained the following Calderón-type reproducing formula. Let $\{S_k\}_{k=-\infty}^{\infty}$ be any approximation to the identity as in [22] and $D_k = S_k - S_{k-1}$. There exist families of operators $\{\tilde{D}_k\}_{k=-\infty}^{\infty}$ and $\{\tilde{\tilde{D}}_k\}_{k=-\infty}^{\infty}$ such that

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k(f) = \sum_{k=-\infty}^{\infty} D_k \tilde{\tilde{D}}_k(f),$$

where the both series converge in the space $L^p(X), 1 < p < \infty$, the space $\mathcal{M}(X)$, and the dual $(\mathcal{M}(X))^\prime$ of $\mathcal{M}(X)$.

Note that the formula (8) is similar to (5). Thus, the theory of Besov spaces on spaces of homogeneous type $(X, d, \nu)$ with properties (1) and (7) can be developed as in the case of $\mathbb{R}^n$. More precisely, the Besov space on such a space of homogeneous type $(X, d, \nu)$, $B^{\alpha,q}_p(X)$ for $1 \leq p, q \leq \infty$ and $|\alpha| < \varepsilon$, where $\varepsilon$ depends on the regularity of the approximation to the identity $S_k$, is defined to be the collection of all $f \in (\mathcal{M}(X))^\prime$ such that

$$\|f\|_{B^{\alpha,q}_p(X)} = \left( \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \|D_k(f)\|_p \right)^q \right)^{1/q} < \infty.$$  

Again, applying formula (8), one can show that Besov spaces $B^{\alpha,q}_p(X)$ are independent of the choice of approximations to the identity $\{S_k\}$ and, moreover, all properties such as embedding, interpolation, duality, atomic decomposition and the $T1$ theorem were obtained (see [22–25]). Assume that the quasi-metric $d$ satisfies (1). If the measure $\nu$ satisfies the doubling and the additional reverse doubling condition; that is, there are constants $\kappa > 0$ and $c \in (0, 1)$ such that

$$c \lambda^\kappa \nu(B_d(x, r)) \leq \nu(B_d(x, \lambda r))$$  

for all $x \in X, 0 < r < \sup_{x \neq y \in X} d(x, y)/2$ and $1 \leq \lambda < \sup_{x \neq y \in X} d(x, y)/2r$, then the theory of the Besov space can be established as well. The key point is that, when $\nu$ satisfies the doubling and the reverse doubling conditions, one can still introduce test function spaces and distributions; moreover, the formula (8) still holds on $L^p, 1 < p < \infty$, test function spaces and distributions. See [26,27] for more details.

We now return to the current situation in this paper. As mentioned, $(\mathbb{R}^n, \bar{\rho}, \mu)$ is space of homogeneous type in the sense of Coifman and Weiss. Note that the quasi-metric $\bar{\rho}(x, y)$ may have no regularity and the measure $\mu$ satisfies the doubling property only. Therefore, the method mentioned above can not be carried over to our situation. To achieve our goal, a new approach is required.

The departure of our new approach is the following result proved by Macias and Segovia in [19].

**Theorem 2.1:** Let $d(x, y)$ be a quasi-metric on a set $X$. There exists a quasi-metric $d'(x, y)$ on $X$ such that
(i) \( d'(x, y) \) is geometrically equivalent to \( d(x, y) \); that is, \( C^{-1}d(x, y) \leq d'(x, y) \leq Cd(x, y) \) for some constant \( C > 0 \) and for all \( x, y \in X \);

(ii) \( d' \) satisfies the regularity property (1).

Based on the above theorem, in the space of homogeneous type \( (\mathbb{R}^n, \bar{\rho}, \mu) \), we may assume that the quasi-metric \( \bar{\rho} \) satisfies the regularity condition (1) and the measure \( \mu \) satisfies the doubling property.

3. Existence of the approximation to the identity

In this section, we construct an approximation to the identity associated to sections deduced by \( \bar{\rho} \) and \( \mu \). Here and throughout this paper, \( V_k(x) \) always denotes the measure \( \mu(S(x, 2^{-k})) \) for \( k \in \mathbb{Z} \) and \( x \in \mathbb{R}^n \). Let \( \psi : \mathbb{R} \to [0, 1] \) be a smooth function which is 1 on \((-1, 1)\) and vanishes on \((-\infty, -2) \cup (2, \infty)\). We define

\[
T_k(f)(x) = \int_{\mathbb{R}^n} \psi(2^k \bar{\rho}(x, y)) f(y) \, d\mu(y), \quad k \in \mathbb{Z}.
\]

Then

\[
T_k(1)(x) \leq \int_{\bar{\rho}(x,y) \leq 2^{1-k}} d\mu(y) \leq \mu(S(x, 2^{1-k})) \leq C \mu(S(x, 2^{-k})).
\]

Conversely,

\[
T_k(1)(x) \geq \int_{\bar{\rho}(x,y) < 2^{-k}} d\mu(y) = \mu(S(x, 2^{-k})).
\]

Hence, \( T_k(1)(x) \approx \mu(S(x, 2^{-k})) \). It is easy to check \( V_k(x) \approx V_k(y) \) whenever \( \bar{\rho}(x, y) \leq (A_0)^3 2^{5-k} \). Thus,

\[
T_k \left( \frac{1}{T_k(1)} \right)(x) = \int_{\mathbb{R}^n} \psi(2^k \bar{\rho}(x, y)) \frac{1}{T_k(1)(y)} \, d\mu(y)
\approx \int_{\mathbb{R}^n} \psi(2^k \bar{\rho}(x, y)) \frac{1}{V_k(y)} \, d\mu(y)
\approx \frac{1}{V_k(x)} \int_{\mathbb{R}^n} \psi(2^k \bar{\rho}(x, y)) \, d\mu(y)
= \frac{1}{V_k(x)} T_k(1)(x) \approx 1.
\]

Let \( M_k \) be the operator of multiplication by \( M_k(x) := 1/(T_k(1)(x)) \) and let \( W_k \) be operator of multiplication by \( W_k(x) := [T_k(1/T_k(1))(x)]^{-1} \). We set \( S_k = M_k T_k W_k T_k M_k \). Then the kernel of \( S_k \) is

\[
S_k(x, y) = \int_{\mathbb{R}^n} M_k(x) \psi(2^k \bar{\rho}(x, z)) W_k(z) \psi(2^k \bar{\rho}(z, y)) M_k(y) \, d\mu(z).
\]

Lemma 3.1: There exists a sequence of operators \( \{S_k\}_{k \in \mathbb{Z}} \) with kernels \( S_k(x, y) \) defined on \( \mathbb{R}^n \times \mathbb{R}^n \) satisfying the following properties.

(i) \( S_k(x, y) = S_k(y, x) \);

(ii) \( S_k(x, y) = 0 \) if \( \bar{\rho}(x, y) > A_0 2^{2-k} \) and \( |S_k(x, y)| \leq (C/(V_k(x) + V_k(y))) \), where \( A_0 \) is the constant in (4);

(iii) \( |S_k(x, y) - S_k(x', y)| \leq C(((2^k \bar{\rho}(x, x'))^5)/(V_k(x) + V_k(y))) \) for \( \bar{\rho}(x, x') \leq (A_0)^3 2^{5-k} \);

(iv) \( |S_k(x, y) - S_k(x, y')| \leq C(((2^k \bar{\rho}(y, y'))^5)/(V_k(x) + V_k(y))) \) for \( \bar{\rho}(y, y') \leq (A_0)^3 2^{5-k} \).
\[(v) \quad |[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')]| \leq C((2^k \tilde{\rho}(x,x'))^\varepsilon (2^k \tilde{\rho}(y,y'))^\varepsilon) / (V_k(x) + V_k(y)) \quad \text{for} \quad \tilde{\rho}(x,x') \leq (A_0)^{2^{3-k}} \text{and} \quad \tilde{\rho}(y,y') \leq (A_0)^{3 2^{5-k}};\]
\[(vi) \quad \int_{\mathbb{R}^n} S_k(x,y) d\mu(x) = 1 \quad \text{for all} \quad y \in \mathbb{R}^n;\]
\[(vii) \quad \int_{\mathbb{R}^n} S_k(x,y) d\mu(y) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.\]

Such a sequence of operators \(\{S_k\}_{k \in \mathbb{Z}}\) is called an approximation to the identity associated with \(\mathcal{F}\) on \((\mathbb{R}^n, \tilde{\rho}, \mu)\).

**Proof:** Property (i) is obvious since \(\tilde{\rho}(x,y) = \rho(y,x)\). (ii) If \(S_k(x,y) \neq 0\), then \(\tilde{\rho}(x,z) \leq 2^{1-k}\) and \(\tilde{\rho}(z,y) \leq 2^{1-k}\). Hence \(\tilde{\rho}(x,y) \leq A_0 2^{2-k}\). That is, \(S_k(x,y) = 0\) when \(\tilde{\rho}(x,y) > A_0 2^{2-k}\). The definition of \(M_k\) gives
\[
|S_k(x,y)| \leq \frac{1}{T_k(1)(x)} \frac{1}{T_k(1)(y)} \int_{\tilde{\rho}(x,z) \leq 2^{1-k}} \psi(2^k \tilde{\rho}(x,z)) W_k(z) \psi(2^k \tilde{\rho}(z,y)) d\mu(z)
\leq C \frac{1}{V_k(x)} \frac{1}{V_k(y)} \mu(S(x,2^{1-k}))
\leq \frac{C}{V_k(y)},
\]
which implies \(|S_k(x,y)| \leq (C/(V_k(x) + V_k(y)))\) whenever \(\tilde{\rho}(x,y) \leq A_0 2^{2-k}\).

To estimate (iii), we write
\[
S_k(x,y) - S_k(x',y)
= \int_{\mathbb{R}^n} [M_k(x) \psi(2^k \tilde{\rho}(x,z)) - M_k(x') \psi(2^k \tilde{\rho}(x',z))] W_k(z) \psi(2^k \tilde{\rho}(z,y)) M_k(y) d\mu(z)
= \int_{\mathbb{R}^n} [M_k(x) - M_k(x')] \psi(2^k \tilde{\rho}(x,z)) W_k(z) \psi(2^k \tilde{\rho}(z,y)) M_k(y) d\mu(z)
+ \int_{\mathbb{R}^n} M_k(x') [\psi(2^k \tilde{\rho}(x,z)) - \psi(2^k \tilde{\rho}(x',z))] W_k(z) \psi(2^k \tilde{\rho}(z,y)) M_k(y) d\mu(z)
:= I_1 + I_2.
\]

For \(I_1\), we have
\[
|M_k(x) - M_k(x')| = \frac{|T_k(1)(x') - T_k(1)(x)|}{T_k(1)(x') T_k(1)(x)} \approx \frac{|T_k(1)(x') - T_k(1)(x)|}{V_k(x') V_k(x)}.
\]
Since \(|\tilde{\rho}(x,z) - \tilde{\rho}(y,z)| \leq C(\tilde{\rho}(x,y))^\varepsilon [\tilde{\rho}(x,z) + \tilde{\rho}(y,z)]^{1-\varepsilon}\), we have
\[
|\psi(2^k \tilde{\rho}(x,y)) - \psi(2^k \tilde{\rho}(x',y))| \leq C 2^k(\tilde{\rho}(x,x'))^\varepsilon [\tilde{\rho}(x,y) + \tilde{\rho}(x',y)]^{1-\varepsilon}
\leq C 2^{k(1-\varepsilon)} (\tilde{\rho}(x,x'))^\varepsilon
= C(2^k \tilde{\rho}(x,x'))^\varepsilon \quad \text{for} \quad \tilde{\rho}(x,x') \leq (A_0)^{3 2^{5-k}}. \quad (9)
\]

Then for \(\tilde{\rho}(x,x') \leq (A_0)^{3 2^{5-k}}\),
\[
|T_k(1)(x') - T_k(1)(x)| \leq \int_{\mathbb{R}^n} |\psi(2^k \tilde{\rho}(x,y)) - \psi(2^k \tilde{\rho}(x',y))| d\mu(y)
\leq CV_k(x')(2^k \tilde{\rho}(x,x'))^\varepsilon,
\]
which yields

\[ |M_k(x) - M_k(x')| \leq C(2^k \tilde{\rho}(x, x'))^\epsilon \frac{1}{V_k(x)}. \quad (10) \]

Therefore,

\[ |I_1| \leq C(2^k \tilde{\rho}(x, x'))^\epsilon \frac{1}{V_k(x)} \leq C(2^k \tilde{\rho}(x, x'))^\epsilon \frac{1}{V_k(x) + V_k(y)} \quad \text{for } \tilde{\rho}(x, x') \leq (A_0)^3 2^{5-k}. \]

For \( \tilde{\rho}(x, x') \leq (A_0)^3 2^{5-k} \), it follows from (9) that

\[ |I_2| \leq \int_{\mathbb{R}^n} [M_k(x') [\psi(2^k \tilde{\rho}(x, z)) - \psi(2^k \tilde{\rho}(x', z))] W_k(z) \psi(2^k \tilde{\rho}(z, y)) M_k(y)] \mu(z) \]

\[ \leq C(2^k \tilde{\rho}(x, x'))^\epsilon \frac{1}{V_k(y)} \leq C(2^k \tilde{\rho}(x, x'))^\epsilon \frac{1}{V_k(x) + V_k(y)}. \]

The proof of (iv) is similar to (iii).

To verify (v), we write

\[ [S_k(x, y) - S_k(x', y)] - [S_k(x, y') - S_k(x', y')] \]

\[ = \int_{\mathbb{R}^n} [M_k(x) [\psi(2^k \tilde{\rho}(x, z)) - \psi(2^k \tilde{\rho}(x', z))] W_k(z) \]

\[ \times [\psi(2^k \tilde{\rho}(z, y)) M_k(y) - \psi(2^k \tilde{\rho}(z, y')) M_k(y')] \mu(z) \]

\[ = \int_{\mathbb{R}^n} [M_k(x) - M_k(x')] [\psi(2^k \tilde{\rho}(x, z)) W_k(z) [\psi(2^k \tilde{\rho}(z, y)) - \psi(2^k \tilde{\rho}(z, y'))] M_k(y)] \mu(z) \]

\[ + \int_{\mathbb{R}^n} [M_k(x) - M_k(x')] [\psi(2^k \tilde{\rho}(x, z)) W_k(z) \psi(2^k \tilde{\rho}(z, y')) M_k(y) - M_k(y')] \mu(z) \]

\[ + \int_{\mathbb{R}^n} M_k(x') [\psi(2^k \tilde{\rho}(x, z)) - \psi(2^k \tilde{\rho}(x', z)) M_k(z) \]

\[ \times [\psi(2^k \tilde{\rho}(z, y)) - \psi(2^k \tilde{\rho}(z, y'))] M_k(y) \mu(z) \]

\[ + \int_{\mathbb{R}^n} M_k(x') [\psi(2^k \tilde{\rho}(x, z)) - \psi(2^k \tilde{\rho}(x', z)) W_k(z) \psi(2^k \tilde{\rho}(z, y')) [M_k(y) - M_k(y')] \mu(z) \]

\[ := J_1 + J_2 + J_3 + J_4. \]

To estimate \( J_1 \), we use (9) and (10) for \( \tilde{\rho}(x, x') \leq (A_0)^3 2^{5-k} \) and \( \tilde{\rho}(y, y') \leq (A_0)^3 2^{5-k} \) combined with the support condition of \( \psi \) to get

\[ |J_1| \leq C(2^k \tilde{\rho}(x, x'))^\epsilon (2^k \tilde{\rho}(y, y'))^\epsilon \frac{1}{V_k(x) + V_k(y)}. \]

Similarly, for \( \tilde{\rho}(x, x') \leq (A_0)^3 2^{5-k} \) and \( \tilde{\rho}(y, y') \leq (A_0)^3 2^{5-k} \),

\[ |J_2| + |J_3| + |J_4| \leq C(2^k \tilde{\rho}(x, x'))^\epsilon (2^k \tilde{\rho}(y, y'))^\epsilon \frac{1}{V_k(x) + V_k(y)}. \]
For (vi), we have
\[
\int S_k(x, y) d\mu(x) = \int \int M_k(x) \psi(2^k \hat{\rho}(x, z)) W_k(z) \psi(2^k \hat{\rho}(z, y)) M_k(y) d\mu(z) d\mu(x)
\]
\[
= \int \left( \int \psi(2^k \hat{\rho}(z, x)) M_k(x) d\mu(x) \right) W_k(z) \psi(2^k \hat{\rho}(z, y)) M_k(y) d\mu(z)
\]
\[
= \int \left[ T_k \left( \frac{1}{T_k(1)} \right) \right] W_k(z) \psi(2^k \hat{\rho}(z, y)) M_k(y) d\mu(z)
\]
\[
= M_k(y) \int \psi(2^k \hat{\rho}(z, y)) d\mu(z)
\]
\[
= M_k(y) T_k(1)(y) = 1,
\]
and (vii) is obtained by the same argument.

Lemma 3.2: Let \(\{S_k\}_{k \in \mathbb{Z}}\) be an approximation to the identity associated with \(\mathcal{F}\) and set \(D_k = S_k - S_{k-1}\) for \(k \in \mathbb{Z}\). There exists a constant \(C\) such that
\[
|D_j D_k(x, y)| \leq C 2^{-|j-k|} \frac{1}{V_{\min(j,k)}(x) + V_{\min(j,k)}(y)}.
\]

Proof: For \(k \geq j\), we use vanishing condition of \(D_k\) and Lemma 3.1 (ii), (iv) to get
\[
|D_j D_k(x, y)| \leq \int_{\hat{\rho}(y,z) \leq A_0 2^{3-k}} |D_j(x, z) - D_j(x, y)||D_k(z, y)| d\mu(z)
\]
\[
\leq C \int_{\hat{\rho}(y,z) \leq A_0 2^{3-k}} (2^j \rho(z, y))^e \frac{1}{V_j(y)} \frac{1}{V_k(y)} d\mu(z)
\]
\[
\leq C 2^{-(k-j)e} \frac{1}{V_j(y)}.
\]
Similarly, for \(k < j\), the vanishing condition of \(D_j\) and Lemma 3.1 (ii), (iii) show
\[
|D_j D_k(x, y)| \leq \int_{\hat{\rho}(x,z) \leq A_0 2^{3-j}} |D_j(x, z)||D_k(z, y) - D_k(x, y)| d\mu(z)
\]
\[
\leq C \int_{\hat{\rho}(x,z) \leq A_0 2^{3-j}} \frac{1}{V_j(x)} \left( 2^k \rho(z, x) \right)^e \frac{1}{V_k(x)} d\mu(z)
\]
\[
\leq C 2^{-(j-k)e} \frac{1}{V_k(x)}.
\]
Since \(V_k(x) \approx V_k(y)\) when \(\hat{\rho}(x, y) \leq (A_0)^2 2^{4-k}\), the proof is finished.

By Lemmas 3.1 (ii) and 3.2, we immediately have the following result.

Lemma 3.3: Let \(\{S_k\}_{k \in \mathbb{Z}}\) be an approximation to the identity associated with \(\mathcal{F}\) and set \(D_k = S_k - S_{k-1}\) for \(k \in \mathbb{Z}\). For \(1 \leq p \leq \infty\), there exists a constant \(C\) such that
\[
\|D_j D_k\|_{L^p_{\mu} \to L^p_{\mu}} \leq C 2^{-|j-k|}.
\]
In this section, we show Theorems 1.1 and 1.2, which are the Calderón-type reproducing formula for \( \mathcal{F} \). Calderón-type reproducing formulae for \( \mathcal{F} \) and its dual.

Proof of Theorem 1.1: We prove the first equality in (2) only because the proof for the second one is similar. We first show that if \( N \) is chosen to be large enough then there exists a constant \( C \) such that

\[
\| R_N \( f \) \|_{L^2_\mu} \leq C N^{2/3} \| f \|_{L^2_\mu}.
\]

(11)

To do this, since \( R_N \) is bounded on \( L^2(\mathbb{R}^n, d\mu) \) and \( f = \sum_{k \in \mathbb{Z}} T_N^{-1} D_k^N D_k(f) \) in \( L^2(\mathbb{R}^n, d\mu) \), we write

\[
\| R_N(\hat{f}) \|_{\mathcal{F}^{a,q}_{p,\mathfrak{F}}} = \left\{ \sum_{k \in \mathbb{Z}} \left( 2^k \| D_k R_N(\hat{f}) \|_{L^2_\mu} \right)^q \right\}^{1/q}
\]

\[
= \left\{ \sum_{k \in \mathbb{Z}} \left( 2^k \| D_k R_N(\sum_{k' \in \mathbb{Z}} T_N^{-1} D_{k'}^N D_{k'}(\hat{f})) \|_{L^2_\mu} \right)^q \right\}^{1/q}
.
\]

Observing

\[
D_k R_N \left( \sum_{k'} T_N^{-1} D_{k'}^N D_{k'}(\hat{f}) \right)(x) = D_k R_N \sum_{k'} \sum_{m=0}^\infty (R_N)^m D_{k'}^N D_{k'}(\hat{f})(x)
\]

\[
= \sum_{k'} \sum_{m=0}^\infty D_k (R_N)^{m+1} D_{k'}^N D_{k'}(\hat{f})(x)
\]

and plugging \( R_N = \sum_{k=-\infty}^\infty \sum_{|k-j|>N} D_k D_{k'} \) yield

\[
D_k (R_N)^{m+1} D_{k'}^N = D_k \sum_{|k_0-\ell_0|>N} D_{k_0} D_{\ell_0} \sum_{|k_1-\ell_1|>N} D_{k_1} D_{\ell_1} \cdots \sum_{|k_m-\ell_m|>N} D_{k_m} D_{\ell_m} D_{k'}^N
\]

\[
= \sum_{|k_0-\ell_0|>N} \cdots \sum_{|k_m-\ell_m|>N} D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D_{k'}^N.
\]

Thus

\[
\| R_N(\hat{f}) \|_{\mathcal{F}^{a,q}_{p,\mathfrak{F}}} \leq \left\{ \sum_k \left( \sum_{k'} \sum_{m=0}^\infty \sum_{|k_0-\ell_0|>N} \cdots \sum_{|k_m-\ell_m|>N} 2^k \| D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D_{k'}^N \|_{L^2_\mu \rightarrow L^2_\mu} \| D_{k'}(\hat{f}) \|_{L^2_\mu} \right) \right\}^{1/q}.
\]
Note that $D^N_k = \sum_{|j| \leq N} D_{k+j}$. Applying Lemma 3.3 gives that

\[
\|D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D^N_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq \sum_{|j| \leq N} \|D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D^j_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq C \sum_{|j| \leq N} 2^{-|k-k_0|} 2^{-|\ell_0-k_1|} \cdots 2^{-|\ell_{m-1}-k_m|} 2^{-|\ell_{m-j}|} 2^{-|k-j|} \varepsilon
\]

and

\[
\|D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D^N_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq CN \|D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m}\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq CN 2^{-|k_0-\ell_0|} 2^{-|\ell_0-\ell_1|} \cdots 2^{-|\ell_{m-1}-k_m|} 2^{-|\ell_{m-j}|} 2^{-|k-j|} \varepsilon/2.
\]

Taking an average of these two estimates yields

\[
2^{k \alpha} \|D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D^N_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq CN^{1/2} 2^{-|k-k_0|} 2^{-|\ell_0-\ell_1|} \cdots 2^{-|\ell_{m-1}-k_m|} 2^{-|\ell_{m-j}|} 2^{-|k-j|} \varepsilon/2.
\]

Inserting

\[
2^{k \alpha} = 2^{(k-k_0)\alpha} 2^{(k_0-\ell_0)\alpha} 2^{(\ell_0-\ell_1)\alpha} \cdots 2^{(\ell_{m-1}-k_m)\alpha} 2^{(k_m-\ell_m)\alpha} 2^{(\ell_{m-j})\alpha} 2^{(k+j)\alpha}
\]

into the above last estimate implies

\[
2^{k \alpha} \|D_k D_{k_0} D_{\ell_0} D_{k_1} D_{\ell_1} \cdots D_{k_m} D_{\ell_m} D^N_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq CN^{1/2} 2^{-|k-k_0|} 2^{-|\ell_0-\ell_1|} \cdots 2^{-|\ell_{m-1}-k_m|} 2^{-|\ell_{m-j}|} 2^{-|k-j|} \varepsilon/2.
\]

Hence,

\[
2^{k \alpha} \|D_k (R_N)^{m+1} D^N_k\|_{L^p_{\mu} \rightarrow L^p_{\mu}} \\
\leq CN^{1/2} 2^{-|k-k_0|} 2^{-|\ell_0-\ell_1|} \cdots 2^{-|k-k_0|} 2^{-|\ell_0-\ell_1|} \cdots 2^{-|\ell_{m-j}|} 2^{-|k-j|} \varepsilon/2.
\]
Applying Hölder’s inequality gives

\[ \| R_N(f) \|_{L^3_p,q}^{\alpha} \leq C N^{1/2} \left\{ \sum_k \left( \sum_{k'} \sum_{m=0}^{\infty} \sum_{|k_0-k_0|>N} \sum_{|k_m-k_m|>N} 2^{-|k-k_0|s/2-|a|} \| 2^{-|k-k_0|s/2-|a|} \| \right) \right\} \]

\[ \times 2^{-|k_0-k_1|s/2-|a|} \cdots 2^{-|k_m-k_m|s/2-|a|} \]

\[ \times \left( \sum_{k'} \sum_{m=0}^{\infty} \sum_{|k_0-k_0|>N} \sum_{|k_m-k_m|>N} 2^{-|k-k_0|s/2-|a|} \right) \]

\[ \times 2^{-|k_0-k_1|s/2-|a|} \cdots 2^{-|k_m-k_m|s/2-|a|} \]

\[ \times \left( 2^{k'+j|s|} \| D_k(f) \|_{l^p_m}^{q} \right) \] \]

Observe that if we choose \( N \) large enough so that \( 2^{-N(s/2-|a|)} < 1 \) then

\[ \sum_{k'} \sum_{m=0}^{\infty} \sum_{|k_0-k_0|>N} \sum_{|k_m-k_m|>N} 2^{-|k-k_0|s/2-|a|} \]

\[ \times 2^{-|k_0-k_1|s/2-|a|} \cdots 2^{-|k_m-k_m|s/2-|a|} \]

\[ \leq C 2^{-N(s/2-|a|)} \]

and

\[ \sum_{k} \sum_{m=0}^{\infty} \sum_{|k_0-k_0|>N} \sum_{|k_m-k_m|>N} 2^{-|k-k_0|s/2-|a|} \]

\[ \times 2^{-|k_0-k_1|s/2-|a|} \cdots 2^{-|k_m-k_m|s/2-|a|} \]

\[ \leq C 2^{-N(s/2-|a|)} \]

Note that \( \sum_{|j|\leq N} 2^{|j|} \leq C N^{2|a|} \). Finally, we have

\[ \| R_N(f) \|_{L^3_p,q}^{\alpha} \leq C N^{3/2} 2^{-N(s/2-2|a|)} \left\{ \sum_{k'\in\mathbb{Z}} \left( 2^{k'|s|} \| D_k(f) \|_{l^p_m}^{q} \right) \right\}^{1/q} \]

\[ \leq C N^{3/2} 2^{-N(s/2-2|a|)} \| f \|_{L^3_p,q}^{\alpha} \]

which gives the estimate in (11).

Note that \( T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} (R_N)^m \) and choose \( N \) large enough such that \( C N^{3/2} 2^{-N(s/2-2|a|)} < 1 \), so (11) implies

\[ \| T_N^{-1}(f) \|_{L^3_p,q}^{\alpha} \leq C_N \| f \|_{L^3_p,q}^{\alpha} \] (12)
which shows that $T_{N}^{-1}$ is bounded on $\tilde{\mathcal{B}}_{p,F}^{\alpha,q}$. In order to prove that $\sum_{k \in \mathbb{Z}} T_{N}^{-1} D_{k}^{N} D_{k}(f)$ converges to $f$ in $\tilde{\mathcal{B}}_{p,F}^{\alpha,q}$, we observe that

$$f(x) - \sum_{|k| \leq M} T_{N}^{-1} D_{k}^{N} D_{k}(f)(x) = \sum_{|k| > M} T_{N}^{-1} D_{k}^{N} D_{k}(f)(x) \quad \text{for } f \in L_{\mu}^{2}.$$ 

Therefore, we only need to show

$$\lim_{M \to \infty} \left\| \sum_{|k| > M} T_{N}^{-1} D_{k}^{N} D_{k}(f) \right\|_{\tilde{\mathcal{B}}_{p,F}^{\alpha,q}} = 0.$$ 

Indeed, by (12),

$$\left\| \sum_{|k| > M} T_{N}^{-1} D_{k}^{N} D_{k}(f) \right\|_{\tilde{\mathcal{B}}_{p,F}^{\alpha,q}} \leq C_{N} \left\| \sum_{|k| > M} D_{k}^{N} D_{k}(f) \right\|_{\tilde{\mathcal{B}}_{p,F}^{\alpha,q}}.$$ 

The same argument as the proof of (11) yields

$$\left\| \sum_{|k| > M} T_{N}^{-1} D_{k}^{N} D_{k}(f) \right\|_{\tilde{\mathcal{B}}_{p,F}^{\alpha,q}} \leq C_{N} \left\{ \sum_{|k| > M} \left( 2^{k\alpha} \| D_{k}(f) \|_{L_{\mu}^{p}} \right)^{q} \right\}^{1/q}.$$ 

The assumption of $f$ shows that the right-hand side of the above inequality goes to 0 as $M \to \infty$, and hence the first equality in (2) holds.

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** For $g \in \tilde{\mathcal{B}}_{p,F}^{\alpha,q}$ and $f \in (\tilde{\mathcal{B}}_{p,F}^{\alpha,q})'$, Theorem 1.1 says

$$\langle f, g \rangle = \left\{ \sum_{k \in \mathbb{Z}} T_{N}^{-1} D_{k}^{N} D_{k}(g) \right\} = \sum_{k \in \mathbb{Z}} \langle f, T_{N}^{-1} D_{k}^{N} D_{k}(g) \rangle. \quad (13)$$

Since $S_{k}$ is self-adjoint (Lemma 3.1 (i)), operators $D_{k}$, $D_{k}^{N}$, and $T_{N}^{-1}$ are all self-adjoint. Therefore, it suffices to show

$$\langle f, T_{N}^{-1} D_{k}^{N} D_{k}(g) \rangle = \langle (D_{k})^{*}(D_{k}^{N})^{*} (T_{N}^{-1})^{*}(f), g \rangle = \langle D_{k} D_{k}^{N} T_{N}^{-1}(f), g \rangle. \quad (14)$$

Taking the summation for $k \in \mathbb{Z}$ on both sides of (14) yields the second equality of (3). The argument for the first equality in (3) is similar, and we omit the details.

To give a rigorous proof of (14), we claim

$$\langle f, D_{k}(g) \rangle = \langle D_{k}(f), g \rangle \quad \text{for } g \in \tilde{\mathcal{B}}_{p,F}^{\alpha,q}, \; f \in (\tilde{\mathcal{B}}_{p,F}^{\alpha,q})'.$$ 

(15)
Assuming the claim for the moment, we have

\[
\{ f, D_{k}^m (R_N)^{m-1} D_k D_k^N (g) \} = \{ D_{k}^m (R_N)^{m-1} D_k^N D_k^m (g) \} = \{ D_{k} D_k^m (f), (R_N)^{m-1} D_k^N D_k^m (g) \}.
\]

Since \( R_N \) can be expressed to be \( R_N = \sum_{k' \in \mathbb{Z}} \sum_{|\ell| > N} D_{k'}^m D_k^m \), we take the summation \( \sum_{k' \in \mathbb{Z}} \sum_{|\ell| > N} D_{k'}^m D_k^m \), we take the summation \( \sum_{k' \in \mathbb{Z}} \sum_{|\ell| > N} D_{k'}^m D_k^m \), we take

\[
\{ f, R_N (R_N)^{m-1} D_k^N D_k^m (g) \} = \{ R_N (f), (R_N)^{m-1} D_k^N D_k^m (g) \}.
\]

Repeating the same process \( m \) times, we obtain

\[
\{ f, T_N^{-1} D_k^N D_k^m (g) \} = \{ T_N^{-1} (f), D_k^N D_k^m (g) \}
\]

and then

\[
\{ f, T_N^{-1} D_k^N D_k^m (g) \} = \{ D_k D_k^N T_N^{-1} (f), g \},
\]

which and (13) give us

\[
\{ f, g \} = \sum_{k \in \mathbb{Z}} \{ D_k D_k^N T_N^{-1} (f), g \}.
\]

The first equality of (3) can be obtained similarly.

We now return to the proof of claim (15), which contains three steps:

Step 1. Show that each \( D_k \) is bounded on \( L_{p,q}^{\alpha} \) for all \( |\alpha| < (\varepsilon/4) \) and \( 1 \leq p, q \leq \infty \).

Step 2. Show that \( \{ f, D_k (g) \} = \{ D_k (f), g \} \) for all \( f \in (L_{p,q}^{\alpha})' \) and \( g \in L_{p,q}^{\alpha} \cap L_{p} \).

Step 3. Show that \( L_{p,q}^{\alpha} \subset L_{p,q}^{\alpha} \cap L_{p}^{\alpha} \), where \( L_{p,q}^{\alpha} \cap L_{p}^{\alpha} \) denotes the closure of \( L_{p,q}^{\alpha} \cap L_{p}^{\alpha} \) with respect to \( \| \cdot \|_{L_{p,q}^{\alpha}} \).

To prove step 1, we use Theorem 1.1 to write

\[
\| D_k (f) \|_{L_{p,q}^{\alpha}} = \left\{ \sum_{\ell \in \mathbb{Z}} 2^{\ell \alpha} \left\| \sum_{k' \in \mathbb{Z}} D_k D_{k'}^N T_N^{-1} (f) \right\|_{L_{p,q}^{\alpha}}^q \right\}^{1/q} \leq \left\{ \sum_{\ell \in \mathbb{Z}} 2^{\ell \alpha} \left( \sum_{k' \in \mathbb{Z}} \| D_k D_{k'}^N \|_{L_{p,q}^{\alpha}} \right) \right\}^{1/q} \leq \left\{ \sum_{|j| \leq N} 2^{2j} \left\| D_{k'}^N T_N^{-1} (f) \right\|_{L_{p,q}^{\alpha}}^q \right\}^{1/q} \leq N \left\{ \sum_{|j| \leq N} 2^{2j} \left\| D_{k'}^N T_N^{-1} (f) \right\|_{L_{p,q}^{\alpha}}^q \right\}^{1/q} \leq N^{3/2} 2^{N|\alpha|} \| T_N^{-1} (f) \|_{L_{p,q}^{\alpha}} \leq N^{3/2} 2^{N|\alpha|} C_N \| f \|_{L_{p,q}^{\alpha}}.
\]
To show step 2, for \( g \in B_{p,F}^{\sigma,q} \cap L_1^p \), we define
\[
gkM(x) = \int_{S(0, M)} D_k(x, y)g(y)d\mu(y), \quad M > 0,
\]
where \( S(0, M) \) denotes the section \( \{ y \in \mathbb{R}^n : \hat{\rho}(0, y) < M \} \). By step 1,
\[
\| D_k(g) - gkM \|_{B_{p,F}^{\sigma,q}} = \| D_k(g|_{\mathbb{R}^n \setminus S(0, M)})\|_{B_{p,F}^{\sigma,q}} \lesssim N^{3/2}2^{|\alpha|}C_N\|g|_{\mathbb{R}^n \setminus S(0, M)}\|_{B_{p,F}^{\sigma,q}} \to 0
\]
as \( M \to \infty \). Thus,
\[
\langle f, D_k(g) \rangle = \lim_{M \to \infty} \langle f, gkM \rangle. \tag{16}
\]
Since \( \{ \text{int } S(z, 2^{-(k+j)}) \}_{z \in S(0, M)} \) is an open covering of \( S(0, M) \), there exist finite number of sections \( \{ S(zj, 2^{-(k+j)}) \}_{j=1}^{N_j}, z_j \in S(0, M) \), such that \( S(0, M) \subset \bigcup_{j=1}^{N_j} S(z_j, 2^{-(k+j)}) \). Let
\[
\begin{align*}
Q_1 &= S(0, M) \cap S(z_1, 2^{-(k+j)}); \\
Q_2 &= S(0, M) \cap S(z_2, 2^{-(k+j)}) \setminus Q_1; \\
Q_3 &= S(0, M) \cap S(z_3, 2^{-(k+j)}) \setminus (Q_1 \cup Q_2); \\
& \vdots \\
Q_{N_j} &= S(0, M) \cap S(z_{N_j}, 2^{-(k+j)}) \setminus \bigcup_{j=1}^{N_j-1} Q_j.
\end{align*}
\]
Then \( \{ Q_j \}_{j=1}^{N_j} \) are disjoint and \( \bigcup_{j=1}^{N_j} Q_j = S(0, M) \). Now we write
\[
gkM(x) = \sum_{j=1}^{N_j} \int_{Q_j} D_k(x, y)g(y)d\mu(y)
\]
\[
= \sum_{j=1}^{N_j} \int_{Q_j} [D_k(x, y) - D_k(x, y_j)]g(y)d\mu(y) + \sum_{j=1}^{N_j} D_k(x, y_j) \int_{Q_j} g(y)d\mu(y)
\]
\[
:= g_{1,k,M,j}(x) + g_{2,k,M,j}(x),
\]
where \( y_j \) is any point in \( Q_j \). To consider \( \| g_{1,k,M,j} \|_{B_{p,F}^{\sigma,q}} \), the second difference smoothness condition (v) in Lemma 3.1 will be used. For simplicity of notations, we denote by
\[
F_{k,j}(x, y) = [D_k(x, y) - D_k(x, y_j)]\chi_{Q_j}(y).
\]
Lemma 3.1 tells us that
\[
\begin{align*}
(a) & \quad \text{supp } F_{k,j}(\cdot, y) \subset S(y, 16(A_0)^2 2^{-k}) \text{ and supp } F_{k,j}(x, \cdot) \subset S(x, 8A_0 2^{-k}); \\
(b) & \quad \int_{\mathbb{R}^n} F_{k,j}(x, y)d\mu(x) = 0; \\
(c) & \quad |F_{k,j}(x, y)| \leq C 2^{-J\varepsilon} \left(1/(V_k(x) + V_k(y))\right); \\
(d) & \quad |F_{k,j}(x, y) - F_{k,j}(x', y)| \leq C 2^{-J\varepsilon} \left(2^{k\varepsilon} (\hat{\rho}(x, x'))^\varepsilon (1/(V_k(x) + V_k(y)))\right).
\end{align*}
\]
where $x'$ satisfies $\bar{\rho}(x, x') \leq 32(A_0)^3 2^{-k}$. Under the above conditions (a)–(d), using a similar argument to the proofs of Lemmas 3.1 and 3.2, we obtain that for all $k, \ell \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$,

$$\text{supp}(D_\ell F_{k,j})(\cdot, y) \subset S(y, 32(A_0)^3(2^{-\ell} \lor 2^{-k}));$$  \hspace{1cm} (17)

$$\text{supp}(D_\ell F_{k,j})(x, \cdot) \subset S(x, 16(A_0)^2(2^{-\ell} \lor 2^{-k}));$$  \hspace{1cm} (18)

$$|D_\ell F_{k,j}(x, y)| \leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p} \left( \frac{1}{1} V_{\ell \wedge k}(x) + V_{\ell \wedge k}(y) \right).$$  \hspace{1cm} (19)

Set

$$F(x, y) = \sum_{j=1}^{N_j} (D_\ell F_{k,j})(x, y).$$

By (18) and (19),

$$\int_{\mathbb{R}^n} |F(x, y)| d\mu(y) \leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p} \sum_{j=1}^{N_j} \int_{Q_j \cap S(x, 16(A_0)^2(2^{-\ell} \lor 2^{-k})))} \frac{d\mu(y)}{V_{\ell \wedge k}(x) + V_{\ell \wedge k}(y)}$$

$$\leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p} \mu(S(x, 16(A_0)^2(2^{-\ell} \lor 2^{-k})))$$

$$\leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p} \cdot \cdot$$

Similarly, (17) and (19) yield

$$\int_{\mathbb{R}^n} |F(x, y)| d\mu(x) \leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p}.$$

The above two inequalities imply

$$\|D_\ell (g_{k,M,j}^1)\|_{L^p_\mu} \leq C 2^{-J_\varepsilon} 2^{-|\ell - k|_p} \|g\|_{L^p_\mu},$$

and then

$$\|g_{k,M,j}^1\|_{L^{\infty}_{p,q}} \leq C 2^{-J_\varepsilon} \left\{ \sum_{\ell \in \mathbb{Z}} 2^{\ell q - |\ell - k|_q} \right\}^{1/q} \|g\|_{L^p_\mu}$$

$$\leq C 2^{-J_\varepsilon} 2^{k\varepsilon} \|g\|_{L^p_\mu}$$

$$\rightarrow 0 \hspace{1cm} \text{as} \hspace{1cm} J \rightarrow \infty.$$  \hspace{1cm} (20)

By (16) and (20), we have

$$\langle f, D_k(g) \rangle = \lim_{M \to \infty} \lim_{J \to \infty} \langle f, g_{k,M,j}^2 \rangle$$

$$= \lim_{M \to \infty} \lim_{J \to \infty} \sum_{j=1}^{N_j} D_k(f)(y_j) \int_{Q_j} g(y) d\mu(y).$$  \hspace{1cm} (21)
where we use Lemma 3.1 (i) to know that $D_k$ is self-adjoint. We now write

$$
\sum_{j=1}^{N_j} D_k(f)(y) \int_{Q_j} g(y) d\mu(y)
$$

$$
= \sum_{j=1}^{N_j} \int_{Q_j} D_k(f)(y) g(y) d\mu(y)
$$

$$
+ \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^{N_j} [D_k(f)(y) - D_k(f)(y)] \chi_{Q_j} \right\} g(y) d\mu(y).
$$

Using the second difference property (v) in Lemma 3.1 again and a similar proof of (20), we can show that

$$
\| [D_k(y_j, \cdot) - D_k(y, \cdot)] \chi_{Q_j} \|_{L^{p,q}_{\mu}} \leq C2^{-jL}2^{k\alpha} V_k(y)^{(1/p) - 1}
$$

and hence

$$
\left| [D_k(f)(y_j) - D_k(f)(y)] \chi_{Q_j} \right| \leq \| [D_k(y_j, \cdot) - D_k(y, \cdot)] \chi_{Q_j} \|_{L^{p,q}_{\mu}} \|f\|_{L^{p,q}_{\mu}}
$$

$$
\leq C2^{-jL}2^{k\alpha} V_k(y)^{(1/p) - 1} \|f\|_{L^{p,q}_{\mu}}.
$$

The Lebesgue dominated convergence theorem shows that

$$
\lim_{J \to \infty} \int_{\mathbb{R}^n} \left\{ \sum_{j=1}^{N_j} [D_k(f)(y_j) - D_k(f)(y)] \chi_{Q_j} \right\} g(y) d\mu(y) = 0,
$$

which together with (21) shows

$$
\langle f, D_k(g) \rangle = \lim_{M \to \infty} \lim_{J \to \infty} \sum_{j=1}^{N_j} \int_{Q_j} D_k(f)(y) g(y) d\mu(y)
$$

$$
= \int_{\mathbb{R}^n} D_k(f)(y) g(y) d\mu(y)
$$

$$
= \langle D_k(f), g \rangle.
$$

For the proof of step 3, given $g \in B^{\alpha,q}_{p,F}$, let

$$
\tilde{g}_{k,M}(x) = \int_{S(0,M)} D_N^N(x, y) D_k T_N^{-1}(g)(y) d\mu(y), \quad M > 0.
$$

Then $\tilde{g}_{k,M} \in L^p_{\mu} \cap B^{\alpha,q}_{p,F}$. It follows from Theorem 1.1 that

$$
\left\| g - \sum_{|k| \leq M} \tilde{g}_{k,M} \right\|_{L^{p,q}_{\mu}} = \left\| g - \sum_{|k| \leq M} D_N^N D_k T_N^{-1}(g) \chi_{S(0,M)} \right\|_{L^{p,q}_{\mu}} \to 0 \quad \text{as } M \to \infty.
$$

Hence, claim (15) is proved, and the proof of Theorem 1.2 is completed.
5. Besov spaces associated with sections

In this section, we study the basic properties of Besov spaces. We first apply the Calderón-type reproducing formula for $L^2_{\mu}$ to prove that the definition of $\dot{B}^{\alpha,q}_{p,F}$ is independent of the choice of approximations to the identity.

**Proposition 5.1:** Let $|\alpha| < \varepsilon / 4$ and $1 \leq p, q \leq \infty$. Suppose that $\{S_k\}_{k \in \mathbb{Z}}$ and $\{P_k\}_{k \in \mathbb{Z}}$ are approximations to the identity associated with $\mathcal{F}$. Set $D_k = S_k - S_{k-1}$ and $E_k = P_k - P_{k-1}$. Then for $f \in L^2_{\mu}$,

$$\left\{ \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \| D_k(f) \|_{L^p_{\mu}} \right)^q \right\}^{1/q} \approx \left\{ \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \| E_k(f) \|_{L^p_{\mu}} \right)^q \right\}^{1/q}.$$

**Proof:** For $f \in L^2_{\mu}$, we have $f = \sum_{k' \in \mathbb{Z}} T^{-1}_N E^N_{k'} E_k'(f)$ in $L^2_{\mu}$. Hence

$$D_k(f) = \sum_{k' \in \mathbb{Z}} D_k T^{-1}_N E^N_{k'} E_k'(f) = \sum_{k^* \in \mathbb{Z}} \sum_{m=0}^\infty D_k(R_N)^m E^N_{k'} E_k'(f).$$

Applying the same argument as the proof of (11), we obtain

$$\left\{ \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \| D_k(f) \|_{L^p_{\mu}} \right)^q \right\}^{1/q} \leq C \left\{ \sum_{k^* \in \mathbb{Z}} \left( 2^{k^\prime \alpha} \| E_k'(f) \|_{L^p_{\mu}} \right)^q \right\}^{1/q}$$

and hence the proof follows. ■

It is well known that the space of Schwartz functions is dense in the classical Besov space on $\mathbb{R}^n$. The following result is one of the main results in this section, which shows that the test function space $\dot{B}^{\alpha,q}_{p,F}$ is dense in $\dot{B}^{\alpha,q}_{p,F}$ as well.

**Theorem 5.2:** Let $|\alpha| < \varepsilon / 4$ and $1 \leq p, q \leq \infty$. Then

$$\overline{\dot{B}^{\alpha,q}_{p,F}} = \dot{B}^{\alpha,q}_{p,F},$$

where $\overline{\dot{B}^{\alpha,q}_{p,F}}$ denotes the closure of $\dot{B}^{\alpha,q}_{p,F}$ with respect to $\| \cdot \|_{\dot{B}^{\alpha,q}_{p,F}}$.

To show the above theorem, we need the following lemma.

**Lemma 5.3:** Let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{F}$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. For $|\alpha| < \varepsilon / 4$ and $1 \leq p, q \leq \infty$, both $D_k(x, \cdot)$ and $D_k(\cdot, y)$ are in $\dot{B}^{\alpha,q}_{p,F}$ for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

**Proof:** Since $D_k(x, \cdot) = D_k(\cdot, x)$ for any fixed $x \in \mathbb{R}^n$, it suffices to verify the lemma for $D_k(x, \cdot)$. By Lemma 3.2,

$$\| D_j(D_k(x, \cdot)) \|_{L^1_{\mu}} \leq C 2^{-|j-k|\varepsilon}.$$
and
\[ \|D_j(D_k(x, \cdot))\|_{L^p_{\mu}} \leq C 2^{-|j-k|\varepsilon} \frac{1}{V_k(x)}. \]
If \(1 < p < \infty\), then
\[ \|D_j(D_k(x, \cdot))\|_{L^p_{\mu}} \leq \|D_j(D_k(x, \cdot))\|_{L^1_{\mu}}^{1/p} \|D_j(D_k(x, \cdot))\|_{L^{1-(1/p)}_{\mu}}^{1-(1/p)} \leq C 2^{-|j-k|\varepsilon} V_k(x)^{(1/p)-1}. \]
Combining above three estimates yields
\[ \|D_k(x, \cdot)\|_{L^{p,q}_{\mu,\mathcal{F}}} = \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|D_j(D_k(x, \cdot))\|_{L^p_{\mu}} \right)^q \right\}^{1/q} \]
\[ \leq C \frac{1}{V_k(x)^{1-(1/p)}} \left\{ \sum_{j \in \mathbb{Z}} 2^{|j\alpha - |j-k|\varepsilon|q} \right\}^{1/q} \]
\[ \leq C 2^{\alpha\varepsilon} \frac{1}{V_k(x)^{1-(1/p)}}, \]
and the proof of Lemma 5.3 is completed. \( \square \)

**Remark 5.4:** The same argument as the proof of Lemma 5.3 shows that if \(f \in C^1(\mathbb{R}^n)\) with compact support and
\[ \int_{\mathbb{R}^n} f(x) d\mu(x) = 0, \]
then \(f \in B^{\alpha,q}_{p,\mathcal{F}}\) for \(|\alpha| < \varepsilon/4\) and \(1 \leq p, q \leq \infty\).

We now show Theorem 5.2.

**Proof of Theorem 5.2:** To show \(\overline{B^{\alpha,q}_{p,\mathcal{F}}} \subset B^{\alpha,q}_{p,\mathcal{F}}\), let \(\{f_m\}_{m \in \mathbb{N}}\) be a Cauchy sequence in \(B^{\alpha,q}_{p,\mathcal{F}}\) with respect to the norm \(\|\cdot\|_{B^{\alpha,q}_{p,\mathcal{F}}}\). We will prove that there is an \(f \in B^{\alpha,q}_{p,\mathcal{F}}\) such that \(f_m\) converges to \(f\) in \(B^{\alpha,q}_{p,\mathcal{F}}\) as \(m \to \infty\).

We first claim that, if \(f \in B^{\alpha,q}_{p,\mathcal{F}}\), then \(f \in (B^{\alpha,q}_{p,\mathcal{F}})’\) and \(\|f\|_{(B^{\alpha,q}_{p,\mathcal{F}})’} \leq C\|f\|_{B^{\alpha,q}_{p,\mathcal{F}}}\). Given \(f \in B^{\alpha,q}_{p,\mathcal{F}}\) and \(g \in B^{\alpha,q}_{p,\mathcal{F}}\), let \(\{S_k\}_{k \in \mathbb{Z}}\) be an approximation to the identity associated to sections and set \(D_k = S_k - S_{k-1}\). By Calderón-type reproducing formula \(f = \sum_{k \in \mathbb{Z}} D_N^k D_k T_N^{-1}(f)\) in \(L^2_{\mu}\) and Hölder’s inequality,
\[ |\langle f, g \rangle| = \left| \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} D_k T_N^{-1}(f) D_N^k(g) d\mu \right| \]
\[ \leq \sum_{k \in \mathbb{Z}} \|D_k T_N^{-1}(f)\|_{L^p_{\mu}} \|D_N^k(g)\|_{L^q_{\mu}’} \]
\[ \leq \left\{ \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|D_k T_N^{-1}(f)\|_{L^p_{\mu}}^q \right\}^{1/q} \left\{ \sum_{k \in \mathbb{Z}} 2^{-k\alpha q'} \|D_N^k(g)\|_{L^{q’}_{\mu}’}^q \right\}^{1/q’}. \]
Since \(D_N^k = \sum_{|j| \leq N} D_{k+j}\), we have
\[ \left\{ \sum_{k \in \mathbb{Z}} 2^{-k\alpha q'} \|D_N^k(g)\|_{L^{q’}_{\mu}’}^q \right\}^{1/q’} \leq C N^{\alpha|\alpha|} \|g\|_{B^{\alpha,q}_{p,\mathcal{F}}}’, \]
and hence

$$|\langle f, g \rangle| \leq CN2^{N|\alpha|}C_N \|f\|_{\mathcal{B}^{\alpha,q}_{p,F}} \|g\|_{\mathcal{B}^{-\alpha,q'}_{p',F}}$$  \hspace{1cm} (22)

due to (12). Thus, the claim follows.

Let \( \{f_m\}_{m \in \mathbb{N}} \subset \mathcal{B}^{\alpha,q}_{p,F} \) be a Cauchy sequence with respect to the norm \( \| \cdot \|_{\mathcal{B}^{\alpha,q}_{p,F}} \). The above claim implies that \( \{f_m\}_{m \in \mathbb{N}} \) is also Cauchy with respect to the norm \( \| \cdot \|_{\mathcal{B}^{-\alpha,q'}_{p',F}} \) and \( \|f_m\|_{\mathcal{B}^{\alpha,q}_{p,F}} \) is bounded by \( C \) independent of \( m \). Since \( \mathcal{B}^{\alpha,q'}_{p',F} \) is a Banach space, there is an \( f \in (\mathcal{B}^{\alpha,q'}_{p',F})' \) such that \( f_m \to f \) in \( (\mathcal{B}^{\alpha,q}_{p,F})' \) as \( m \to \infty \). It follows from Lemma 5.3 that

$$|D_k(f_m - f)(x)| \leq \|D_k(x, \cdot)\|_{\mathcal{B}^{-\alpha,q'}_{p',F}} \|f_m - f\|_{(\mathcal{B}^{\alpha,q}_{p,F})'},$$

which shows

$$\lim_{m \to \infty} D_k(f_m)(x) = D_k(f)(x).$$  \hspace{1cm} (23)

By Fatou's lemma and (23),

$$\|f\|_{\mathcal{B}^{\alpha,q}_{p,F}} \leq \liminf_{m \to \infty} \|f_m\|_{\mathcal{B}^{\alpha,q}_{p,F}} \leq C,$$

which shows \( f \in \mathcal{B}^{\alpha,q}_{p,F} \). By the Lebesgue dominated convergence theorem, we obtain that \( \{f_m\} \) converges to \( f \) in \( \mathcal{B}^{\alpha,q}_{p,F} \).

To prove \( \mathcal{B}^{\alpha,q}_{p,F} \subset \mathcal{B}^{\alpha,q}_{p,F} \), given \( f \in \mathcal{B}^{\alpha,q}_{p,F} \), the same argument as proof of Theorem 1.2 shows

$$f = \sum_{k \in \mathbb{Z}} D_k D_k^N T_N^{-1}(f),$$  \hspace{1cm} (24)

where the series converges in \( \mathcal{B}^{\alpha,q}_{p,F} \). Define \( f_{k,M} \) by

$$f_{k,M}(x) = \int_{S(0,M)} D_k(x,y)(D_k^N T_N^{-1}(f)(y))d\mu(y).$$

Then

$$\left\| f - \sum_{|k| \leq M} f_{k,M} \right\|_{\mathcal{B}^{\alpha,q}_{p,F}} \leq \sum_{|k| \leq M} D_k D_k^N T_N^{-1}(f) - \sum_{|k| \leq M} f_{k,M} \right\|_{\mathcal{B}^{\alpha,q}_{p,F}} + \sum_{|k| > M} D_k D_k^N T_N^{-1}(f) \right\|_{\mathcal{B}^{\alpha,q}_{p,F}}.$$

Minkowski's inequality and Lemma 3.3 yield

$$\left\| \sum_{|k| \leq M} D_k D_k^N T_N^{-1}(f) - \sum_{|k| \leq M} f_{k,M} \right\|_{\mathcal{B}^{\alpha,q}_{p,F}} \leq \sum_{\ell \in \mathbb{Z}} \left( 2^{\ell \alpha} \sum_{|k| \leq M} \left\| D_k D_k^N T_N^{-1}(f(x)) \chi_{\mathbb{R}^n \setminus S(0,M)} \right\|_{L^p_\mu} \right)^{q} \right\}^{1/q} \leq C \sum_{\ell \in \mathbb{Z}} \left( \sum_{|k| \leq M} 2^{(\ell-k)\alpha} 2^{-|\ell-k|} 2^{|\ell-k|} \left\| D_k D_k^N T_N^{-1}(f(x)) \chi_{\mathbb{R}^n \setminus S(0,M)} \right\|_{L^p_\mu} \right)^{q} \right\}^{1/q}. $$
By Hölder’s inequality,

\[
\left\| \sum_{|k| \leq M} D_k D_k^N T_N^{-1}(f) - \sum_{|k| \leq M} f_{k,M} \right\|_{B^\alpha,q_{\mathcal{F}}} \leq C \left\{ \sum_{\ell \in \mathbb{Z}} \sum_{|k| \leq M} 2^{(\ell-k)\alpha} 2^{-(\ell-k)\gamma} 2^{k\alpha q} \| D_k^N T_N^{-1}(f) \chi_{\mathbb{R}^n,(S_{0,M}^\mathcal{F})} \|_{L_p^q} \right\}^{1/q}
\]

Using (12) and (24), we obtain

\[
\left\| f - \sum_{|k| \leq M} f_{k,M} \right\|_{B^\alpha,q_{\mathcal{F}}} \to 0 \quad \text{as } M \to \infty.
\]

It follows from Remark 5.4 that \( f_{k,M} \) belongs to \( \dot{B}^\alpha,q_{\mathcal{F}} \), so we have \( \dot{B}^\alpha,q_{\mathcal{F}} \subset \overline{B}^\alpha,q_{\mathcal{F}} \) and the proof is completed.

The following duality argument of \( \dot{B}^\alpha,q_{\mathcal{F}} \) is another main result in this section.

**Theorem 5.5:** Let \(|\alpha| < \varepsilon/4\).

(a) For \(1 \leq p, q \leq \infty\) and each \( g \in \dot{B}^\alpha,q_{\mathcal{F}} \), the mapping \( \mathcal{L}_g : f \mapsto \int_{\mathbb{R}^n} f(x)g(x) d\mu(x) \), defined initially on \( \dot{B}^\alpha,q_{\mathcal{F}} \), extends to a bounded linear functional on \( \dot{B}^\alpha,q_{\mathcal{F}} \) and satisfies \( \| \mathcal{L}_g \| \lesssim \| g \|_{1-p,q} \).

(b) Conversely, for \(1 \leq p, q < \infty\), every bounded linear functional \( \mathcal{L} \) on \( \dot{B}^\alpha,q_{\mathcal{F}} \) can be realized as \( \mathcal{L} = \mathcal{L}_g \) with \( g \in \dot{B}^\alpha,q_{\mathcal{F}} \) and \( \| g \|_{1-p,q} \lesssim \| \mathcal{L} \| \).

To show the above theorem, we need the following

**Lemma 5.6:** Let \( \{S_k\}_{k \in \mathbb{Z}} \) be an approximation to the identity associated with \( \mathcal{F} \) and \( D_k = S_k - S_{k-1} \) for \( k \in \mathbb{Z} \). For \(|\alpha| < \varepsilon/4\) and \(1 \leq p, q \leq \infty\), if a sequence of functions \( \{g_k\}_{k \in \mathbb{Z}} \) satisfies \( \| \{2^{k\alpha} \| g_k \|_{L_p^q} \}_{k \in \mathbb{Z}} \|_{\ell^q} < \infty \), then \( \sum_{k \in \mathbb{Z}} D_k(g_k) \in \dot{B}^\alpha,q_{\mathcal{F}} \) and \( \| \sum_{k \in \mathbb{Z}} D_k(g_k) \|_{p,q} \lesssim \| \{2^{k\alpha} \| g_k \|_{L_p^q} \}_{k \in \mathbb{Z}} \|_{\ell^q} \).

**Proof:** For \( m_1, m_2 \in \mathbb{Z} \) with \( m_1 < m_2 \), define \( g_{m_1}^{m_2} = \sum_{k=m_1}^{m_2} D_k(g_k) \). Given \( f \in \dot{B}^\alpha,q_{\mathcal{F}} \), Hölder’s inequality yields

\[
\| g_{m_1}^{m_2} \|_{\dot{B}^\alpha,q_{\mathcal{F}}} \leq \sum_{k=m_1}^{m_2} \| g_k, D_k(f) \|_{L_p^q} \leq \left\{ \sum_{k=m_1}^{m_2} \left( 2^{k\alpha} \| g_k \|_{L_p^q} \right)^q \right\}^{1/q} \leq \left\{ \sum_{k=m_1}^{m_2} \left( 2^{k\alpha} \| D_k(f) \|_{L_p^q} \right)^q \right\}^{1/q},
\]

which shows \( g_{m_1}^{m_2} \in (\dot{B}^\alpha,q_{\mathcal{F}})' \) and

\[
\| g_{m_1}^{m_2} \|_{(\dot{B}^\alpha,q_{\mathcal{F}})'} \leq \left\{ \sum_{k=m_1}^{m_2} \left( 2^{k\alpha} \| g_k \|_{L_p^q} \right)^q \right\}^{1/q}.
\]
If we set $g = \sum_{k \in \mathbb{Z}} D_k(g_k)$, then $g \in (\mathcal{B}_{p',q}^{\alpha,q})'$ as well. Using Lemma 3.3 and Hölder’s inequality, we get
\[
\sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \|D_j(g)\|_{L^p_{\mu}} \right)^q \leq \sum_{j \in \mathbb{Z}} \left( 2^{j\alpha} \sum_{k \in \mathbb{Z}} \|D_j D_k(g_k)\|_{L^p_{\mu}} \right)^q \lesssim \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|g_k\|_{L^p_{\mu}}^q,
\]
which completes the proof. ■

Now we return to proving the duality for $\mathcal{B}_{p,\mathcal{F}}^{\alpha,q}$.

**Proof of Theorem 5.5.** (a) follows from (22) and Theorem 5.2. For (b), given a bounded linear functional $\mathcal{L}$ on $\mathcal{B}_{p,\mathcal{F}}^{\alpha,q}$, by Theorem 5.2 again, $\mathcal{L}$ is also a bounded linear functional on $\mathcal{B}_{p,\mathcal{F}}^{\alpha,q}$ and
\[
|\mathcal{L}(f)| \leq \|\mathcal{L}\| \|f\|_{\mathcal{B}_{p,\mathcal{F}}^{\alpha,q}} \quad \text{for } f \in \mathcal{B}_{p,\mathcal{F}}^{\alpha,q}.
\]
Let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity associated with $\mathcal{F}$ and set $D_k = S_k - S_{k-1}$. Then, for each $f \in \mathcal{B}_{p,\mathcal{F}}^{\alpha,q}$, $\{D_k(f)\}_{k \in \mathbb{Z}}$ is in the sequence space
\[
\ell^q_{\mathcal{F}}(L^p_{\mu}) = \left\{ \{f_k\}_{k \in \mathbb{Z}} : \|\{f_k\}_{k \in \mathbb{Z}}\|_{\ell^q_{\mathcal{F}}(L^p_{\mu})} := \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|f_k\|_{L^p_{\mu}}^q \right)^{1/q} < \infty \right\}.
\]
Define $\mathcal{L}_i$ on a subset of $\ell^q_{\mathcal{F}}(L^p_{\mu})$ by
\[
\mathcal{L}_i(\{D_k(f)\}_{k \in \mathbb{Z}}) = \mathcal{L}(f) \quad \text{for } f \in \mathcal{B}_{p,\mathcal{F}}^{\alpha,q}.
\]
Hence,
\[
|\mathcal{L}_i(\{D_k(f)\}_{k \in \mathbb{Z}})| \leq \|\mathcal{L}\| \|f\|_{\mathcal{B}_{p,\mathcal{F}}^{\alpha,q}} = \|\mathcal{L}\| \|\{D_k(f)\}_{k \in \mathbb{Z}}\|_{\ell^q_{\mathcal{F}}(L^p_{\mu})}.
\]
The Hahn–Banach theorem shows that $\mathcal{L}_i$ can be extended to a functional $\overline{\mathcal{L}_i}$ on $\ell^q_{\mathcal{F}}(L^p_{\mu})$. Since $(\ell^q_{\mathcal{F}}(L^p_{\mu}))' = \ell^{-\alpha}_{q'}(L^p_{\mu})$ for $1 \leq p, q < \infty$ (see [13, p.178]), there exists a unique sequence $\{g_k\}_{k \in \mathbb{Z}} \in \ell^{-\alpha}_{q'}(L^p_{\mu})$ such that
\[
\overline{\mathcal{L}_i}(\{f_k\}_{k \in \mathbb{Z}}) = \sum_{k \in \mathbb{Z}} \langle f_k, g_k \rangle \quad \text{for all } \{f_k\}_{k \in \mathbb{Z}} \in \ell^q_{\mathcal{F}}(L^p_{\mu})
\]
and
\[
\|\{g_k\}_{k \in \mathbb{Z}}\|_{\ell^{-\alpha}_{q'}(L^p_{\mu})} \lesssim \|\overline{\mathcal{L}_i}\| \leq \|\mathcal{L}\|.
\]
For $f \in \mathcal{B}_{p,\mathcal{F}}^{\alpha,q}$, we have
\[
\mathcal{L}(f) = \mathcal{L}_i(\{D_k(f)\}_{k \in \mathbb{Z}}) = \sum_{k \in \mathbb{Z}} \langle D_k(f), g_k \rangle = \sum_{k \in \mathbb{Z}} \langle f, D_k(g_k) \rangle = \left\{ f, \sum_{k \in \mathbb{Z}} D_k(g_k) \right\}.
\]
Let $g = \sum_{k \in \mathbb{Z}} D_k(g_k)$. Lemma 5.6 says that $g \in \mathcal{B}_{p,\mathcal{F}}^{-\alpha,q'}$ and
\[
\|g\|_{\mathcal{B}_{p,\mathcal{F}}^{-\alpha,q'}} \lesssim \|\{g_k\}_{k \in \mathbb{Z}}\|_{\ell^{-\alpha}_{q'}(L^p_{\mu})} \lesssim \|\mathcal{L}\|.
\]
This completes the proof. ■
6. The boundedness on $\dot{B}^{\alpha,q}_{p,\infty}$

To prove the boundedness of Monge–Ampère singular integral operator $H$ acting on $\dot{B}^{\alpha,q}_{p,\infty}$, the key tool is the almost orthogonality estimate. A weak version of an almost orthogonality estimate was obtained in [9, Lemma 9.1]. We now show a pointwise almost orthogonality estimate as follows. Let $\{E_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity associated to sections with regularity exponent $\varepsilon$ and

$$D_k^\varepsilon := D_{-k} = E_{-k} - E_{-k-1}.$$ 

Denote by $\gamma$ the number satisfying conditions (D6) and (D7), and by $\epsilon_1$ the constant given in condition (A). The kernel $K(x, y) = \sum_i k_i(x, y)$ of Monge–Ampère singular integral operator $H$ satisfies conditions (D1)–(D7), and write

$$H_i(f)(x) := \int_{\mathbb{R}^n} k_i(x, y)f(y)d\mu(y).$$

**Lemma 6.1:** For $0 < \varepsilon' < \min\{\varepsilon, \gamma \epsilon_1\}$,

$$|D_k^\varepsilon HD_k^{\varepsilon'}(x, y)| \lesssim \frac{2^{-|k-k'|\varepsilon'}}{\mu(S(x, 2^{k\gamma k'})) + \mu(S(y, 2^{k\gamma k'})) + \mu(S(x, \tilde{\rho}(x, y)))} \left(\frac{2^{k\gamma k'}}{2^{k\gamma k'} + \tilde{\rho}(x, y)}\right)^{(\min\{\varepsilon, \gamma \epsilon_1\} - \varepsilon')/2}.$$

**Proof:** Obviously

$$|D_k^\varepsilon HD_k^{\varepsilon'}(x, y)| \leq \sum_j |D_k^\varepsilon HD_k^{\varepsilon'}(x, y)|.$$

To estimate $|D_k^\varepsilon HD_k^{\varepsilon'}(x, y)|$, we consider six cases. As before, we write $V_k(x) = \mu(S(x, 2^{-k}))$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

**Case 1:** $j \leq k < k'$. In this case, we use Lemma 3.1 and conditions (D3), (D4) to deduce

$$\left|\int_{\mathbb{R}^n} k_j(u, v)D_k^\varepsilon(v, y)d\mu(v)\right| = \left|\int_{\mathbb{R}^n} k_j(u, v)[D_k^\varepsilon(v, y) - D_k^{\varepsilon'}(u, y)]d\mu(v)\right| \lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k'}(y)} \int_{\mathbb{R}^n} |k_j(u, v)|d\mu(v) \lesssim \frac{2^{-|j-k'|\varepsilon'} |k_j(u, v)|}{V_{-k'}(y)}.$$

Note also that the integrand $k_j(u, v)D_k^\varepsilon(v, y)$ is zero when $\tilde{\rho}(u, y) > 9(A_0)^2 2^{k'}$, where $A_0$ is the constant satisfying (4). For $\tilde{\rho}(u, y) \leq 9(A_0)^2 2^{k'}$, the engulfing property of sections implies $V_{-k'}(u) \approx V_{-k'}(y)$. Therefore

$$\left|\int_{\mathbb{R}^n} k_j(u, v)D_k^\varepsilon(v, y)d\mu(v)\right| \lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k'}(y)} X_{S(u, 9(A_0)^2 2^{k'})}(y) \lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k'}(u) + V_{-k'}(y) + \mu(S(y, \tilde{\rho}(u, y)))} \left(\frac{2^{k'}}{2^{k'} + \tilde{\rho}(u, y)}\right)^{\varepsilon'}.$$
Since $|D_k^x(x, u)| \lesssim (1/(V_{-k}(x) + V_{-k}(u) + \mu(S(x, \rho(x, u)))))(2^k/(2^k + \rho(x, u)))^\varepsilon$, we have

$$
|D_k^x H_j D_k^x(x, y)| \lesssim \int_{\mathbb{R}^n} \frac{1}{V_{-k}(x) + V_{-k}(u) + \mu(S(x, \rho(x, u)))} \left(\frac{2^k}{2^k + \rho(x, u)}\right)^\varepsilon \times \frac{2^{-|j-k|\varepsilon}}{V_{-k}(y) + V_{-k}(y) + \mu(S(y, \rho(y, y)))} \left(\frac{2^k}{2^k + \rho(y, y)}\right)^\varepsilon \, d\mu(u),
$$

and hence, for $|j - k'| = |j - k| + |k - k'|$,

$$
|D_k^x H_j D_k^x(x, y)| \lesssim \frac{2^{-|j-k|\varepsilon} 2^{-|k-k'|\varepsilon}}{V_{-k}(x) + V_{-k}(y) + \mu(S(x, \rho(x, y)))} \left(\frac{2^k}{2^k + \rho(x, y)}\right)^\varepsilon.
$$

Summation over $j \in \mathbb{Z}$ yields the desired estimate.

**Case 2:** $j < k' \leq k$. Note that the kernel $H_j$ is symmetric for $j < k$. The same argument as in Case 1 gets

$$
\left| \int_{\mathbb{R}^n} D_k^x(x, u)k_j(u, v) d\mu(u) \right| \lesssim \frac{2^{-|j-k|\varepsilon}}{V_{-k}(x) + V_{-k}(v) + \mu(S(x, \rho(x, v)))} \left(\frac{2^k}{2^k + \rho(x, v)}\right)^\varepsilon.
$$

Therefore,

$$
|D_k^x H_j D_k^x(x, y)| \lesssim \frac{2^{-|j-k|\varepsilon} 2^{-|k-k'|\varepsilon}}{V_{-k}(x) + V_{-k}(y) + \mu(S(x, \rho(x, y)))} \left(\frac{2^k}{2^k + \rho(x, y)}\right)^\varepsilon
$$

and the desired estimate is obtained by taking summation over $j \in \mathbb{Z}$.

**Case 3:** $k' \leq k < j$. In this case, we use the smoothness condition of $H_j$ and both the cancelation and size conditions of $D_k^x$ to deduce

$$
\left| \int_{\mathbb{R}^n} k_j(u, v) D_k^x(v, y) d\mu(v) \right| = \left| \int_{S(y, A_0 2^j + k')} [k_j(u, v) - k_j(u, y)] D_k^x(v, y) d\mu(v) \right| \lesssim \int_{S(y, A_0 2^j + k')} \frac{1}{V_{-j}(u)} |T_j(v) - T_j(y)| |D_k^x(v, y)| d\mu(v).
$$

(25)

We may assume that $S(u, 2^j) \cap S(y, 2^k) \neq \emptyset$; otherwise, the integrand is zero. Hence, by property (A) of the sections and $j > k'$,

$$
T_j(S(y, 2^k)) \subset B \left(z, K_1 \left(\frac{2^{k'}}{2} \right)^\varepsilon_1 \right),
$$

where $|z| \leq K_2$ and $T_j$ is an affine transformation that normalizes $S(u, 2^j)$. Therefore,

$$
|T_j(v) - T_j(y)| \lesssim 2^{-|j-k'|\varepsilon_1},
$$

which yields

$$
\left| \int_{\mathbb{R}^n} k_j(u, v) D_k^x(v, y) d\mu(v) \right| \lesssim \frac{2^{-|j-k'|\varepsilon_1}}{V_{-j}(u)} \int_{\mathbb{R}^n} |D_k^x(v, y)| d\mu(v) \lesssim \frac{2^{-|j-k'|\varepsilon_1}}{V_{-j}(u)} \chi_{S(u, A_0^2) \cap (y)}(y),
$$

respectively.
where $\varepsilon'' = \frac{1}{2}(\min\{\varepsilon, \gamma \varepsilon_1\} + \varepsilon')$. Let $\delta = \frac{1}{2}(\min\{\varepsilon, \gamma \varepsilon_1\} - \varepsilon')$. We have

\[
\left| \int_{\mathbb{R}^n} k_j(u, v) D_k^p(v, y) d\mu(v) \right| \lesssim \frac{2^{-|j-k|\varepsilon''}}{V_j(u)} \left( \frac{2^j}{2^j + \rho(u, y)} \right)^\delta \\
\lesssim \frac{2^{-|j-k|\varepsilon'}}{V_{-k}'(u) + V_{-k}'(y) + \mu(S(y, \rho(u, y)))} \left( \frac{2^k}{2^k + \rho(u, y)} \right)^\delta.
\]

Arguing as in Case 1, we obtain

\[
|D_k^p HD_k^p(x, y)| \lesssim \frac{2^{-|k-k'|\varepsilon'}}{V_{-k}(x) + V_{-k}(y) + \mu(S(x, \rho(x, y)))} \left( \frac{2^k}{2^k + \rho(x, y)} \right)^\delta.
\]

Case 4: $k < k' \leq j$. Similar to Case 3.

Case 5: $k \leq j \leq k'$. Using the cancelation conditions for $D_k^p$ and $H_j$ in the second variables, we write

\[
|D_k^p H_j D_k^p(x, y)| = \left| \int_{S(x, \kappa(A_0)/2)} D_k^p(x, u)[k_j(u, v) - k_j(x, v)][D_k^p(v, y) - D_k^p(u, y)] d\mu(u) d\mu(v) \right|
\]

\[
\lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k}'(y)} \int_{S(x, \kappa(A_0)/2)} \left( \int |D_k^p(x, u)||k_j(u, v) - k_j(x, v)| d\mu(u) \right) d\mu(v)
\]

A similar argument to the estimate for (25) gives

\[
|D_k^p H_j D_k^p(x, y)| \lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k}'(y)} \int_{S(x, \kappa(A_0)/2)} \frac{2^{-|j-k|\varepsilon'}}{V_j(x)} d\mu(v)
\]

\[
\lesssim \frac{2^{-|j-k'|\varepsilon'}}{V_{-k}'(y)} 2^{-|j-k'|\varepsilon'}
\]

\[
= \frac{2^{-|k-k'|\varepsilon'}}{V_{-k}'(y)} 2^{-|j-k'|\varepsilon'}.
\]

Note that the support of $D_k^p H_j D_k^p$ forces $\rho(x, y) \lesssim (A_0)^3 2^k$, which implies $V_{-k}(y) \approx V_{-k}(x)$. Thus,

\[
|D_k^p H_j D_k^p(x, y)| \lesssim \frac{2^{-|k-k'|\varepsilon'}}{V_{-k}'(x)} 2^{-|j-k'|\varepsilon'} \chi_{S(x, (A_0)^3 2^k)}(y).
\]

Summation over $j \in \mathbb{Z}$ gives the desire estimate.

Case 6: $k' < j < k$. Similar to Case 5.

We now are ready to demonstrate Theorem 1.3.

**Proof of Theorem 1.3.** Since $B_{p, F}^{\alpha, q}$ is dense in $B_{p, F}^{\alpha, q}$, it suffices to show Theorem 1.3 for $f \in B_{p, F}^{\alpha, q}$.

Given $f \in B_{p, F}^{\alpha, q}$, we note that $f \in L^2(\mathbb{R}^n, d\mu)$ and $H$ is bounded on $L^2(\mathbb{R}^n, d\mu)$. Applying the Calderón-type reproducing formula (2) yields

\[
D_k(Hf)(x) = D_k H \left( \sum_{k' \in \mathbb{Z}} D^{\gamma}_{k'} D_k T^{\gamma}_{N}^{-1}(f) \right)(x) = \sum_{k' \in \mathbb{Z}} D_k H D^{\gamma}_{k'} D_k T^{\gamma}_{N}^{-1}(f)(x).
\]

By Lemma 6.1 and Minkowski’s inequality, we have

\[
\|D_k(Hf)\|_{L^p_{\mu}} \lesssim \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|\varepsilon'} \|D_k T^{\gamma}_{N}^{-1}(f)\|_{L^p_{\mu}}.
\]
Hence, Hölder’s inequality gives

\[
\|H(f)\|_{\dot{B}^{\alpha,q}_{p,q},F} \lesssim \left\{ \sum_{k \in \mathbb{Z}} \left( 2^{k\alpha} \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|} \|D_{k'} T_N^{-1}(f)\|_{L^p_\mu} \right)^q \right\}^{1/q}.
\]

By (12), \(\|H(f)\|_{\dot{B}^{\alpha,q}_{p,q}} \lesssim C_N \|f\|_{\dot{B}^{\alpha,q}_{p,q}}\).

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