Some bounds for quantum copying with multiple copies

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We study the relative error of the state-dependent $N \to L$ cloning. A copying transformation and dimension of state space are not specified. Only the unitarity of quantum mechanical transformations is used. The proposed approach is based on the notion of the angle between two states. Firstly, the notion of the angle between two states is discussed. The lower bound on the relative error of copying with multiple copies is examined. In addition, the lower bound on the absolute error is then studied. We compare the obtained bounds with the case of maximizing the global fidelity.

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I. INTRODUCTION

A copying of the quantum information has some severe constraints. First of all, an arbitrary quantum state cannot be perfectly copied [1]. For copying of set of two non-orthogonal pure states some bounds already appear. The two-state problem was first considered by Hillery and Bužek [2]. They examined approximate cloning machines destined for copying of prescribed two non-orthogonal states. In paper [3] such devices were called 'state-dependent cloners'. Writers of [3] introduced the notion of 'global fidelity' and constructed the optimal symmetric state-dependent cloner which optimizes the global fidelity. The problem is stated in the following way. Let us assume that our our auxiliary device CM (the copying machine) must produce two copies (one actual copy plus the original) of particle secretly prepared in some state from a set $\mathcal{A} = \{|\phi\rangle, |\psi\rangle\}$. How well CM can do? If states $|\phi\rangle$ and $|\psi\rangle$ are not orthogonal, then errors will be inevitably introduced. The general lower bound on the absolute error of copying of two-state set was obtained in paper [3]. Hillery and Bužek also considered the lower bound on the absolute error in the case, when CM produces $n + 1$ copies (i.e. $n$ actual copies plus the original) of single input particle. These results were extended in papers [4,5] those examined $N \to L$ state-dependent cloning. (Note that pointed works also concern another questions.) The universal copying machines, producing multiple copies, were considered in papers [6].

Of course, the evaluation of copying quality is dependent on the used measure of 'closeness' to ideality. Measure used by Hillery and Bužek can be named 'absolute error' of copying of two-state set. In work [4–6] the global fidelity was maximized and the optimal symmetric state-dependent cloner was constructed.

Thus, the state-dependent cloning was mostly examined from the 'global fidelity' viewpoint. However, the state-dependent cloning is a complex subject with many facets. Important as the notion of the global fidelity is, it does not cover the problem on the whole. In paper [7] author proposed and motivated the notion of the relative error for $1 \to 2$ state-dependent cloning. The lower bound on the relative error was deduced. Our approach based on the notion of the angle between two states. In present paper the developed us approach is applied to the copying machine, giving a fixed number $L$ of copies from a fixed number $N$ of identically prepared particles (it is clear, $L > N$). In submitted analysis a copying transformation and dimension of state space are not specified. Only the unitarity of quantum mechanical transformation is used. The lower bound on the relative error is derived. The lower bound on the relative error allows to elucidate a tradeoff between a quality of copies and the part of actual copies. We also improve results obtained by Hillery and Bužek [8]. We describe the optimal asymmetric state-dependent cloner which minimizes both the relative and absolute errors. In our examination all the state vectors are normalized to unity. In calculations non-unit vectors will sometimes occur, and these cases will be expressly stated. The norm of the vector $|\Phi\rangle$ is defined as $|||\Phi||| = (\langle \Phi | \Phi \rangle)^{1/2}$.

II. PRELIMINARY

We shall now discuss the notion of angle between two states. Angle $\delta(\Phi, \Psi) \in [0; \pi/2]$ between two states $|\Phi\rangle$ and $|\Psi\rangle$ is defined by

$$
\delta(\Phi, \Psi) \overset{\text{def}}{=} \arccos \left( \left| \langle \Phi | \Psi \rangle \right| \right).
$$

(2.1)
For brevity we shall also often write \( \delta_{\Phi}\Psi \). In paper \([7]\) the following useful inequality was proven:

\[
|\langle \Phi | \Pi | \Phi \rangle - |\langle \Psi | \Pi | \Psi \rangle| \leq \sqrt{1 - |\langle \Phi | \Psi \rangle|^2} = \sin \delta_{\Phi}\Psi, \tag{2.2}
\]

where \( \Pi \) is any projector. So that if angle \( \delta_{\Phi}\Psi \) is small, then the probability distributions generated by states \( |\Phi\rangle \) and \( |\Psi\rangle \) for an arbitrary measurement are close to each other:

\[
|P(R | \Phi) - P(R | \Psi)| \leq \sin \delta_{\Phi}\Psi. \tag{2.3}
\]

It must be stressed that for mixed states there is a similar relation. Suppose that \( \chi \) and \( \omega \) are density operators describing states of a quantum system \( I \). We can imagine that these mixed states arise by a partial trace operation from pure states of an extended system \( "I+II" \). That is, there are states describing states of a quantum system \( I \). We can imagine that these mixed states arise by a partial trace operation from pure states of an extended system \( "I+II" \). That is, there are states \( |u\rangle \) and \( |v\rangle \), for which

\[
\chi = \text{Tr}_u[|u\rangle\langle u|] \quad \text{and} \quad \omega = \text{Tr}_u[|v\rangle\langle v|].
\]

These pure states \( |u\rangle \) and \( |v\rangle \) are called “purifications” of \( \chi \) and \( \omega \) respectively. Fidelity

\[
F(\chi, \omega) \overset{\text{def}}{=} \sup \{ |\langle u|v\rangle|^2 | \text{\( |u\rangle \) and \( |v\rangle \) are purifications of \( \chi \) and \( \omega \)} \}
\]

was introduced by Jozsa \([10]\). Fidelity \( F(\chi, \omega) \) ranges between 0 and 1, \( F(\chi, \omega) = 1 \) if and only if \( \chi = \omega \). The measurement over system \( I \) in mixed state \( \rho \) produces result \( R \) with probability

\[
P(R | \rho) = \text{Tr}_I[\Pi_R \rho],
\]

where \( \Pi_R \) is the corresponding projector. Let the supremum in Eq. \((2.4)\) is reached by purifications \( |u'\rangle \) and \( |v'\rangle \).

Because

\[
P(R | \chi) = \langle u' | \Pi_R \otimes 1 | u' \rangle \quad \text{and} \quad P(R | \omega) = \langle v' | \Pi_R \otimes 1 | v' \rangle,
\]

where \( 1 \) is the identity operator, Eq. \((2.2)\) then gives

\[
|P(R | \chi) - P(R | \omega)| \leq \sqrt{1 - |\langle u'|v' \rangle|^2} = \sqrt{1 - F(\chi, \omega)}. \tag{2.5}
\]

The last relation extends Eq. \((2.3)\) to the case of mixed states.

The measurement of transition probability is an other main form of experiment with a quantum system. In this case we for a time allow the studied system \( I \) to interact unitarily with the auxiliary system \( II \). The system \( II \) will cause the system \( I \) to perform some transitions. The probability that after the expiry of time \( t \) the system \( I \) will have some property described by projector \( II \) is equal to

\[
\text{Tr}_I[\Pi R(t)] = \text{Tr}[\langle 1 II | \sigma(t) | II \rangle], \tag{2.6}
\]

where density operator \( \rho(t) = \text{Tr}_I[\sigma(t)] \) of system \( I \) is the partial trace of density operator \( \sigma(t) \) of composite system \"I+II" over system \( II \). If in initial moment the system \( I \) resides in pure state \( |s\rangle \) and the system \( II \) resides in pure state \( |m\rangle \), then after the expiry of time \( t \) the state of composite system \"I+II" is described by density operator

\[
\sigma^{(s)}_m(t) = \langle V^{(s)}(t)| V^{(s)}_m(t) \rangle,
\]

where vector \( |V^{(s)}_m(t)\rangle = U(t) |s\rangle \otimes |m\rangle \), \( U(t) \) is the evolution operator of composite system \"I+II". We then get that probability is

\[
\text{Tr}[\langle 1 II | \sigma^{(s)}_m(t) | II \rangle] = \langle V^{(s)}(t)| V^{(s)}_m(t) \rangle. \tag{2.7}
\]

The unitarity of transformation \( U(t) \) implies that \( \langle V^{(s)}(t)| V^{(s)}_m(t) \rangle = \langle \phi | \psi \rangle \), and by the use of Eq. \((2.3)\) we then have

\[
|\text{Tr}[\langle 1 II | \sigma^{(s)}_m(t) | II \rangle] - \text{Tr}_{1-II}[\langle 1 II | \sigma^{(s)}_m(t) | II \rangle]| \leq \sin \delta_{\phi\psi}. \tag{2.7}
\]
The idempotency of the above projector implies

\[ \phi(0) = \sum_m \mu_m |m\rangle \langle m|, \quad \sum_m \mu_m = 1. \]

Then after the expiry of time \( t \) composite system \( "I+\Pi" \) will reside in the state described by density operator

\[ \sigma^{(s)}(t) = U(t) \left( \sum_m \mu_m |s\rangle \langle s| \otimes |m\rangle \langle m| \right) U^\dagger(t) = \sum_m \mu_m \sigma_m^{(s)}(t). \]

Applying property \( \text{Tr} (A + B) = \text{Tr} A + \text{Tr} B \), the triangle inequality and Eq. (2.7), we then get

\[ \left| \text{Tr} \left[ (\Pi \otimes 1) \sigma^{(\phi)}(t) \right] - \text{Tr} \left[ (\Pi \otimes 1) \sigma^{(\psi)}(t) \right] \right| \leq \sin \delta_{\phi\psi}. \quad (2.8) \]

Taking \( \Pi = |\varphi\rangle \langle \varphi| \), for probability \( w(\varphi|s;t) \) of transition of system I from state \( |s\rangle \) to state \( |\varphi\rangle \) in a time \( t \) we get such an inequality:

\[ |w(\varphi|s;t) - w(\varphi|\psi;t)| \leq \sin \delta_{\phi\psi}. \quad (2.9) \]

Eqs. (2.8) and (2.9) show that if the angle between two states is small, then experimental manifestations of these states are close to each other. Thus, the angle between two pure states is the reasonable measure of their closeness.

### III. BASIC DEFINITIONS

Let us assume that the copying machine has as input \( N \) particles, each from which is prepared in state \( |s\rangle \), and the copying machine must output a fixed number \( L = M + N \) of similar prepared particles (ideally, number \( L \) of particles, prepared each in state \( |s\rangle \)). Input state \( |s\rangle \) is in a set \( \mathfrak{A} = \{|\phi\rangle, |\psi\rangle\} \) of two pure states, which we would like to copy. If initial state of the copying machine is described by vector \( |m\rangle \), then a CM action is

\[ \forall |s\rangle \in \mathfrak{A} : \quad |s^{\otimes N} \rangle \otimes |m\rangle \longmapsto |V_m^{(s)}\rangle. \quad (3.1) \]

The unitarity of a copying transformation implies that

\[ ((|\phi\rangle |\psi\rangle)^N = (V_m^{(s)} |V_m^{(s)}). \quad (\cos \delta_{\phi\psi})^N = \cos \delta (V_m^{(s)}; V_m^{(s)}). \quad (3.2) \]

Following to paper [3], we act on the output \( |V_m^{(s)}\rangle \) by projector \( |s^{\otimes L}\rangle \langle s^{\otimes L}| \otimes 1 \), where 1 is the identity operator. Let vector \( |s^{\otimes L}\rangle \otimes |q_m^{(s)}\rangle \) be a result of this action, then the output can be expressed as

\[ |V_m^{(s)}\rangle = |s^{\otimes L}\rangle \otimes |q_m^{(s)}\rangle + |\perp_m^{(s)}\rangle. \quad (3.3) \]

The idempotency of the above projector implies

\[ \{ |s^{\otimes L}\rangle \langle s^{\otimes L}| \otimes 1 \} |\perp_m^{(s)}\rangle = 0. \quad (3.4) \]

In general, \( |q_m^{(s)}\rangle \) and \( |\perp_m^{(s)}\rangle \) are not unit, but there is

\[ ||q_m^{(s)}||^2 + ||\perp_m^{(s)}||^2 = 1 \quad (3.5) \]

according to the unitarity. Quantity \( X_m^{(s)} = |||\perp_m^{(s)}||| \) was introduced by Hillery and Bužek [3] as the size of error of copying of state \( |s\rangle \). We shall now consider a relationship between \( X_m^{(s)} \) and the deviation of the resulting probability distribution from the desired probability distribution. Let us introduce magnitude

\[ \delta_m^{(s)} = \inf \{ \delta(V_m^{(s)}; s^{\otimes L} \otimes k) \mid \langle k|k\rangle = 1 \}. \quad (3.6) \]

Using relation (3.4), we see that the inner product of unit vectors \( \langle V_m^{(s)}| \) and \( |s^{\otimes L}\rangle \otimes |k\rangle \) is equal to \( \langle q_m^{(s)}|k\rangle \). Because \( |||k||| = 1 \), the Schwarz inequality gives

\[ ||\langle q_m^{(s)}|k\rangle|| \leq ||q_m^{(s)}||. \]
where the equality takes place if and only if $|q_m^{(s)}⟩ = c|k⟩$ for some complex number $c$. The maximal value of the modulus of the inner product of two unit vectors corresponds to the minimal value of angle between these vectors, so that if stated in Eq. (3.6) infimum is reached by vector $|k⟩$, then unit vector $|k⟩$ and vector $|q_m^{(s)}⟩$ are collinear. For $||q_m^{(s)}⟩|| \neq 0$ let us define vectors

$$|k_m^{(s)}⟩ = \frac{|q_m^{(s)}⟩}{||q_m^{(s)}⟩||} \quad \text{and} \quad |Id_m^{(s)}⟩ = |s⟩^{\otimes L} \otimes |k_m^{(s)}⟩. \quad (3.7)$$

Stated in Eq. (3.6) infimum is reached for each vector $|k⟩ = u|k_m^{(s)}⟩$ with complex unit $u$, and $δ_m^{(s)}$ is angle between unit vectors $|V_m^{(s)}⟩$ and $|Id_m^{(s)}⟩$. Let Hermitian operator $A$ describes some observable for particle 1. Its measurement over particle in state $|s⟩$ produces result $a$ with probability $p(a|s) = ⟨s|Π_a|s⟩$, where $Π_a$ is the corresponding projector. Consider now this observable for composite system "1+⋯+L+CM". In accordance with Eq. (2.4), the measurement of such an observable over system "1+⋯+L+CM" in pure state $|V⟩$ gives result $a$ with probability

$$P(a \text{ for } 1 \mid V) = ⟨V|Π_a \otimes 1^{\otimes L}|V⟩,$$

where $Π_a \otimes 1^{\otimes L}$ is the projector on the corresponding subspace of the composite system state space. In a similar manner, the expression for measurement of observable for particle $j$ is obtained. For state $|Id_m^{(s)}⟩$, the probability of outcome $a$ is

$$P(a \text{ for } j \mid Id_m^{(s)}) = ⟨s|Π_a|s⟩ = p(a|s), \quad (3.8)$$

where $j = 1, \ldots, L$ and $s = φ, ψ$. Thus, $|Id_m^{(s)}⟩$ corresponds to the ideal output. In line with Eqs. (3.6) and (3.7) we have

$$\cos δ_m^{(s)} = |⟨V_m^{(s)}|Id_m^{(s)}⟩| = |⟨q_m^{(s)}|k_m^{(s)}⟩| = ||q_m^{(s)}⟩||. \quad (3.9)$$

Then Eq. (3.7) gives $||⊥_m^{(s)}⟩|| = sin δ_m^{(s)}$. As Eqs. (2.3) and (3.8) and definition of $X_m^{(s)}$ show,

$$P(a \text{ for } j \mid V_m^{(s)}) - p(a|s)| ≤ X_m^{(s)}, \quad (3.10)$$

i.e. magnitude $X_m^{(s)}$ characterizes as a whole the deviation of the resulting probability distribution from the desired probability distribution. Sum $X_m^{(φ)} + X_m^{(ψ)}$ evaluates the total error of copying of set $A$, when the initial state of the copying machine is described by vector $|m⟩$.

**Definition 1** The size $AE(A) = X_m^{(φ)} + X_m^{(ψ)}$ is the absolute error of copying for set $A = \{|φ⟩, |ψ⟩\}$.

However, this criterion loses sight of closeness of states $|φ⟩$ and $|ψ⟩$. To understand this better we shall argue in the form of a game. Fixed parameters are the set $A = \{|φ⟩, |ψ⟩\}$ of two non-orthogonal states, the number $N$ of input qubits and the number $L$ of output qubits. Two persons called Alice and Clare play the game. (Following to paper [8], we shall call the paradigmatical cloner Clare.) Both players know the game parameters. At first, Alice chooses one state from the set $A$ without Clare’s knowledge. The next Alice’s move is to prepare each of $N$ qubits in the chosen state. Then Alice sends the prepared qubits to Clare. Clare’s step is to make $L > N$ qubits of the given $N$ qubits. The next Clare’s step is to guess the Alice’s choice by measurement made on the output of the copying machine. The game is repeated. How many chances has Clare?

Let us take that states $|φ⟩$ and $|ψ⟩$ are sufficiently close to each other. Then the lower bound on the absolute error is close to 0. Next, both the upper bounds on the global fidelity and the local fidelity are close to 1. All these criteria assert that copying process can be made near to the ideality. Clare knows both the ideal outputs corresponding to choice of $|φ⟩$ and choice of $|ψ⟩$ respectively. She will compare given output to this ideal output and to that one. At first sight it seems that Clare can simply recognize the chosen state. It would be a rashness to think so. Indeed, the closeness of states $|φ⟩$ and $|ψ⟩$ implies certain closeness of the ideal outputs. But if so, is Clare able to decide that given output should be related to this ideal output and not to that one? How are Clare’s chances dependent on the game parameters?

To express this in quantitative form we should use some measure of closeness for states $|Id_m^{(φ)}⟩$ and $|Id_m^{(ψ)}⟩$. Since according to Eq. (2.3)

$$|P(R \mid Id_m^{(φ)}) - P(R \mid Id_m^{(ψ)})| ≤ \sin δ(Id_m^{(φ)}, Id_m^{(ψ)}),$$
the quantity \( \sin(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \) provides such a measure. It stands to reason, this quantity is depending on similarity of states \(|\phi\rangle\) and \(|\psi\rangle\). Let us take that \( \sin(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \) is small. Is Clare willing to decide that given output \(|V_m^{(\psi)}\rangle\) should be related to ideal output \(|I_m^{(\phi)}\rangle\) and not to \(|I_m^{(\psi)}\rangle\)? The closeness of \(|V_m^{(\psi)}\rangle\) to \(|I_m^{(\phi)}\rangle\) is measured by \(X_m^{(\phi)} = \sin(\delta_m)\), the closeness of \(|V_m^{(\psi)}\rangle\) to \(|I_m^{(\psi)}\rangle\) is measured by \(X_m^{(\psi)} = \sin(\delta_m)\). It is not without significance that \(\sin(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \) is size of the same kind. Therefore, it is advisable to compare the absolute error with pointed quantity.

**Definition 2** The relative error of \(N \to L\) copying for set \(\mathfrak{A} = \{|\phi\rangle, |\psi\rangle\}\) is

\[
RE(\mathfrak{A}) \overset{\text{def}}{=} AE(\mathfrak{A}) / \sin(\delta(I_m^{(\phi)}, I_m^{(\psi)})) .
\]  

We shall now derive the angle relations, from which bounds on the errors are simply obtained. In order to be rid of bulky expressions we shall below use the notation

\[
\delta_N = \delta(\phi^{\otimes N}, \psi^{\otimes N}) .
\]  

Recall that the spherical triangle inequality holds \([7]\):

\[
\delta(X, Y) \leq \delta(X, Z) + \delta(Y, Z) .
\]  

Using Eq. \((3.13)\) twice, we have

\[
\delta(I_m^{(\phi)}, I_m^{(\psi)}) \leq \delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta(V_m^{(\phi)}, V_m^{(\psi)}) .
\]  

In accordance with the Schwarz inequality, there is

\[
|\langle I_m^{(\phi)}| I_m^{(\psi)} \rangle| = |\langle \phi| \psi \rangle|^L |\langle k^{(\phi)}| k^{(\psi)} \rangle| \leq |\langle \phi| \psi \rangle|^L ,
\]  

whence we obtain \(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \geq \delta_L\). Therefore, \(\delta_L \leq \delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta(V_m^{(\phi)}, V_m^{(\psi)})\), or simply

\[
\delta_m^{(\phi)} + \delta_m^{(\psi)} \geq \delta_L - \delta_N
\]  

in line with Eq. \((3.2)\). Since \(0 \leq |\langle \phi| \psi \rangle| \leq 1\) and \(N < L\), there is \(\delta_N \leq \delta_L\). Eqs. \((3.14)\) and \((3.15)\) contain the restrictions imposed by the laws of the quantum theory. In particular, the ones allow to derive the lower bounds on both the relative error and absolute error.

**IV. LOWER BOUNDS ON THE ABSOLUTE AND RELATIVE ERRORS**

In this section the lower bounds on the absolute and relative errors will be obtained. In order to minimize \(RE(\mathfrak{A})\) the quantity \(\sin(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \) must be as increased as possible. We shall individually consider two cases:

(i) \(\delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta_N \leq \pi/2\),

(ii) \(\delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta_N > \pi/2\).

Using Eqs. \((3.2)\) and \((3.14)\), for the case (i) we have

\[
\sin(\delta(I_m^{(\phi)}, I_m^{(\psi)}) \leq \sin(\delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta_N) .
\]  

In addition, there is (see trigonometric formula for sine of sum \([11]\))

\[
\sin(\delta_m^{(\phi)}) + \sin(\delta_m^{(\psi)}) \geq \sin(\delta_m^{(\phi)}) + \sin(\delta_m^{(\psi)}) .
\]  

By the two last inequalities,

\[
RE(\mathfrak{A}) \geq \cos \delta_N - \sin \delta_N \cot(\delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta_N) .
\]  

It must be stressed that the equality in Eq. \((1.2)\) is necessary for the equality in Eq. \((1.3)\). We want minimizing the right-hand side of Eq. \((1.3)\) in the interval \(\delta_L \leq \delta_m^{(\phi)} + \delta_m^{(\psi)} + \delta_N \leq \pi/2\) established by Eq. \((3.15)\) and the case (i) condition. The minimum is reached at the left boundary point of the above interval. In fact, the right-hand
The expression for particle \( A \) can be rewritten as
\[
\sin \delta \leq \sin \left( \frac{\delta}{m} \right) + \sin \left( \frac{\delta}{m} \right).
\]
(4.4)

Further, in the case (i) inequality \( \Delta E(\mathcal{A}) \geq \sin(\delta_L - \delta_N) \) holds. In the case (ii) \( \Delta E(\mathcal{A}) \geq \sin(\delta_m^\phi + \sin(\delta_m^\phi) \right) \) because \( \sin(\delta_\text{Id}_{m}^\phi (\phi, \psi)) \leq 1 \). Next, the case (ii) condition can be separated into two alternatives, \( \pi/2 - \delta_N < \delta_m^\phi + \delta_m^\psi \leq \pi/2 \) and \( \pi/2 < \delta_m^\phi + \delta_m^\psi \leq \pi \). The first alternative contains
\[
\Delta E(\mathcal{A}) \geq \sin(\delta_m^\phi + \delta_m^\psi) \geq \cos \delta_N.
\]
(4.5)

In the second alternative the conditions \( \delta_m^\phi \leq \pi/2 \) and \( \pi/2 < \delta_m^\phi + \delta_m^\psi \leq \pi \) ensure \( \sin(\delta_m^\phi + \delta_m^\psi) \geq 1 \). So, in the case (ii) \( \Delta E(\mathcal{A}) \geq \cos \delta_N \) and \( \Delta E(\mathcal{A}) \geq \cos \delta_N \). To sum up, we see that the lower bound on the relative error is given by the right-hand side of Eq. (4.4). Designating \( z = \cos \delta_{\phi \psi} \), hence \( \cos \delta_L = z^L \) and \( \cos \delta_N = z^N \), Eq. (4.4) can be rewritten as
\[
\Delta E(\mathcal{A}) \geq F(z|N, L) \equiv \frac{z^N - z^L \sqrt{(1 - z^{2N})/(1 - z^{2L})}}{N}.
\]
(4.6)

In addition, we have established that \( A(\mathcal{A}) \geq \sin(\delta_L - \delta_N) \). This inequality can be reformulated as the following inequality which improves the results of Ref. [3]:
\[
\Delta E(\mathcal{A}) \geq z^N \sqrt{1 - z^{2L}} - z^L \sqrt{1 - z^{2N}}.
\]
(4.7)

Note that the derived lower bounds remain valid in the case, when the initial state of the copying machine is mixed. This extension has meaning, because the copying machine, most likely, be the macroscopic system. If the initial state of the copying machine is described by density operator
\[
\rho = \sum_m \mu_m |m\rangle \langle m|, \quad \sum_m \mu_m = 1,
\]
(4.8)

then after the copying procedure the composite system ”1+⋯+L+CM” will reside in the state described by density operator
\[
\sigma^{(s)} = \sum_m \mu_m |V_s^m\rangle \langle V_s^m|.
\]
(4.9)

Then the size of the error of copying of state \( |s\rangle \) is defined in a reasonable way as
\[
X^{(s)} = \sum_m \mu_m X_m^{(s)}.
\]
(4.10)

Then the measurement of particle observable over particle 1 produces result \( a \) with probability
\[
\mathbb{P}(a \text{ for } 1 | \sigma^{(s)}) = \text{Tr} \left[ (\Pi_a \otimes 1^L) \sigma^{(s)} \right],
\]
(4.11)

The expression for particle \( j \) is simply obtained by obvious changes. Using Eqs. (3.10) and (4.9), we then get by a way, which is similar to reason for (2.3), such an inequality
\[
\left| \mathbb{P}(a \text{ for } j | \sigma^{(s)}) - \mathbb{P}(a |s) \right| \leq X^{(s)}.
\]
(4.12)

where \( j = 1, \ldots, L \) and \( s = \phi, \psi \). In this case the absolute error \( \Delta E(\mathcal{A}) = X^{(\phi)} + X^{(\psi)} \), the relative error
\[
\Delta E(\mathcal{A}) = \sum_m \mu_m \frac{X_m^{(\phi)} + X_m^{(\psi)}}{\sin \delta(\text{Id}_m^{(\phi)}, \text{Id}_m^{(\psi)})}\]
(4.13)

Note that the lower bounds given by theorems 1 and 2 are tightest. Indeed, we shall below describe the cloner that reaches the ones. For example, \( F(z|N, L) \) is plotted as function of \( z \) for \( N = 1 \) and five values of \( L \) in Fig. 1. In the greater part of interval \( z \in [0; 1] \) the function increases and only in the vicinity of the right boundary point
the one becomes decreasing. The maximum of function and the limiting value \(1 - \sqrt{N/L}\) as \(z \to 1\) are values of the same order. So, quantity \(1 - \sqrt{N/L}\) can estimate our possibilities for the state-dependent \(N \to L\) copying. If the relative number of actual copies is small, i.e. \(N/L \approx 1\), then for all \(z \in [0; 1]\) the lower bound on the relative error is also small. Theoretically, in this case we can attain the good quality of state-dependent cloning. Conversely, if the relative number of actual copies is close to 1, i.e. \(N/L \ll 1\), then the relative error will be perceptible (except almost orthogonal states). In the limit \(L \to \infty\) we have \(F(z|N, L) \to z^N\). Thus, there is, in general, a tradeoff between a quality of copies and the relative number of actual copies. As to the game played by Alice and Clare, we note the following. If for given parameters of the game the quantity \(F(z|N, L)\) is value of order 1 then Clare hardly has chances. Too strong a closeness of ideal outputs will prevent her from guessing Alice’s choise.

V. OPTIMAL ASYMMETRIC CLONER

We shall now describe the asymmetric state-dependent cloner reaching the presented lower bounds. In principle, both lower bounds given by Eqs. (4.4) and (4.6) can be reached without ancilla. Then a unitary operator \(U\) acts on the Hilbert space of \(L\) qubits:

\[
|\psi^{(s)}\rangle = U \{ |s\rangle \otimes |0\rangle \}
\]

for \(s = \phi, \psi\). The ideal output \(|\text{Id}^{(s)}\rangle = |s\rangle \otimes |0\rangle\), and sin \(\delta(Id^{(\phi)}, Id^{(\psi)}) = \sqrt{1 - z^{2L}}\). The equality in Eq. (3.15) is necessary to minimize the relative error. Recall that the equality in Eq. (3.13) holds only if the triplet is coplanar [7]. Therefore the equality in Eq. (3.15) holds only if both final states \(|V^{(\phi)}\rangle\) and \(|V^{(\psi)}\rangle\) lie in plane \(\text{span}\{ |\phi\rangle \otimes |L\rangle, |\psi\rangle \otimes |L\rangle\}\). This is also necessary to maximize the global fidelity [3]. Because unitary operations preserve angles, we have

\[
\delta(V^{(\phi)}, V^{(\psi)}) = \delta(|\psi\rangle \otimes 0^\otimes M, |\psi\rangle \otimes 0^\otimes M).
\]

If states \(|\phi\rangle\) and \(|\psi\rangle\) are not orthogonal or identical then angle \(\delta(|\phi\rangle \otimes |L\rangle, |\psi\rangle \otimes |L\rangle)\) is larger than the right-hand side of Eq. (3.16), and the ideal copying is impossible. In fact, it is impracticable that angle between \(|\phi\rangle \otimes |0^\otimes M\rangle\) and \(|\psi\rangle \otimes |0^\otimes M\rangle\) should be properly increased. To superpose the plane \(\text{span}\{ |\phi\rangle \otimes |0^\otimes M\rangle, |\psi\rangle \otimes |0^\otimes M\rangle\}\) onto the plane \(\text{span}\{ |\phi\rangle \otimes |L\rangle, |\psi\rangle \otimes |L\rangle\}\) by rigid rotation \(U\) is at most that we can achieve. The transformation with characteristics

\[
\text{span}\{ |V^{(\phi)}\rangle, |V^{(\psi)}\rangle\} = \text{span}\{ |\phi\rangle \otimes |L\rangle, |\psi\rangle \otimes |L\rangle\},
\]

\[
\delta^{(\phi)} = \delta^{(\psi)} = (\delta_L - \delta_N)/2
\]

is the optimal 'global' cloner constructed in Refs. [4] (for the 1 \(\to\) 2 case) and [3]. This cloner produces equal errors for both states \(|\phi\rangle\) and \(|\psi\rangle\). The absolute error \(AE_S(A) = 2 \sin (|\delta_L - \delta_N|/2)\). Using the standard trigonometric formulae we find that the relative error

\[
RE_S(A) = \sqrt{2 \left[ \frac{1 - z^{N+L}}{1 - z^{2L}} - \frac{1 - z^{2N}}{1 - z^{2L}} \right]^{1/2}}.
\]

This cloner does not reach the equality in Eq. (4.2) (except when states \(|\phi\rangle\) and \(|\psi\rangle\) are orthogonal or identical). Therefore, the optimal 'global' cloner minimizes neither the relative error nor the absolute error.

We shall now propose an asymmetric cloner optimizing the relative error. Such a optimal asymmetric state-dependent cloner is defined by

\[
\text{span}\{ |V^{(\phi)}\rangle, |V^{(\psi)}\rangle\} = \text{span}\{ |\phi\rangle \otimes |L\rangle, |\psi\rangle \otimes |L\rangle\},
\]

\[
\delta^{(\phi)} = 0 \land \delta^{(\psi)} = \delta_L - \delta_N.
\]

This cloner makes the ideal copying of one from pair \(A\) of prescribed states. Both the equality in Eq. (3.15) and the equality in Eq. (4.2) are reached. Therefore, for the cloner defined by Eqs. (5.2) and (5.3), the relative error \(RE_A(A) = F(z|N, L)\) and the absolute error \(AE_A(A)\) is equal to the right-hand side of Eq. (4.5). In other words, the optimal asymmetric state-dependent cloner minimizes both the relative and absolute errors.

Thus, if states \(|\phi\rangle\) and \(|\psi\rangle\) are not orthogonal or identical then \(RE_A(A) < RE_S(A)\). Is this distinction significant? In order to study the question we consider the relative value of difference between \(RE_S(A)\) and \(RE_A(A)\), that is

\[
f(z|N, L) = \{ RE_S(A) - RE_A(A) \}/RE_A(A).\]
For example, $f(z|N, L)$ is plotted as function of $z$ for $N = 1$ and five values of $L$ in Fig. 2. One sees that the distinction between $RE_S(\mathfrak{A})$ and $RE_A(\mathfrak{A})$ reaches several interest and becomes perceptible as the part of actual copies increases. Quantity $f(z|N, L)$ also illustrates the distinction between $AE_S(\mathfrak{A})$ and $AE_A(\mathfrak{A})$. Thus, the cloner defined by Eqs. (5.4) and (5.5) is not insignificant.

VI. CONCLUSION

We have studied new optimality criterion for the state-dependent $N \rightarrow L$ cloning. We have beforehand presented a few useful inequalities. Using physical reasons, the notion of the relative error has been then introduced. We have found that optimizing the relative error principally differs from optimizing other criteria. The tightest lower bounds on both the relative error and the absolute error have been obtained. The ones depend on the number $N$ of input qubits, the number $L$ of input qubits and argument $z$ that is the modulus of the inner product of states to be copied. We have described the optimal asymmetric state-dependent cloner that minimizes both the relative error and the absolute error. For studying with respect to 'fidelity' viewpoint, optimizing the relative error is complementary rather than competitive. Thus, the study of the relative error has allowed to complement a portrait of the state-dependent $N \rightarrow L$ cloning.

[1] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature 299, 802–803 (1982)
[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa and B. Schumacher, Noncommuting mixed states cannot be broadcast, Phys. Rev. Lett. 76, 2818–2821 (1996)
[3] M. Hillery and V. Bužek, Quantum copying: fundamental inequalities, Phys. Rev. A 56, 1212–1216 (1997)
[4] D. Bruß, D.P. DiVincenzo, A. Ekert, C.A. Fusch, C. Macchiavello, J.A. Smolin. Optimal universal and state-dependent quantum cloning, Phys. Rev. A 57, 2368–2378 (1998), see also quant-ph/9705038
[5] A. Chefles and S. M. Barnett, Strategies and Networks for State-Dependent Quantum Cloning, LANL report quant-ph/9812035
[6] C. Macchiavello, Bounds on the efficiency of cloning for two-state quantum systems, J. Optics B 2, 144–150 (2000)
[7] A. E. Rastegin, Some bounds for quantum copying, LANL report quant-ph/0108014
[8] R. F. Werner, Optimal cloning of pure state, Phys. Rev. A 58, 1827–1832 (1998)
[9] P. Zanardi, A Note on Quantum Cloning in d dimensions, LANL report quant-ph/9804011
[10] R. Jozsa, Fidelity for mixed quantum states, J. Mod. Optics 41, 2315–2323 (1994)
[11] Handbook of Mathematical Functions, edited by M. Abramovitz and I. A. Stegun (National Bureau of Standards, Washington, 1964)
FIG. 1. The function defined by Eq. (4.5) for $N = 1$ and five values of $L$, namely $L = 3, 5, 8, 13, 39$. 
FIG. 2. The function defined by Eq. (5.6) for $N = 1$ and five values of $L$, namely $L = 3, 5, 7, 11, 17$. 