APPELL-CARLITZ NUMBERS

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Abstract. In this paper, we introduce the concept of the (higher order) Appell-Carlitz numbers which unifies the definitions of several special numbers in positive characteristic, such as the Bernoulli-Carlitz numbers and the Cauchy-Carlitz numbers. Their generating function is named Hurwitz series in the function field arithmetic ([11, p. 352, Definition 9.1.4]). By using Hasse-Teichmüller derivatives, we also obtain several properties of the (higher order) Appell-Carlitz numbers, including a recurrence formula, two closed forms expressions, and a determinant expression.

The recurrence formula implies Carlitz’s recurrence formula for Bernoulli-Carlitz numbers. Two closed from expressions implies the corresponding results for Bernoulli-Carlitz and Cauchy-Carlitz numbers. The determinant expression implies the corresponding results for Bernoulli-Carlitz and Cauchy-Carlitz numbers, which are analogues of the classical determinant expressions of Bernoulli and Cauchy numbers stated in an article by Glaisher in 1875.

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1. Introduction. The Bernoulli numbers $B_n \in \mathbb{Q}$ ($n = 0, 1, 2, \ldots$) are defined by the generating function

$$
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
$$

The Bernoulli numbers may also be defined by the recursive formula

$$
B_0 = 1, \quad B_n = -\sum_{j=0}^{n-1} \frac{n!}{j!(n+1-j)!} B_j \quad \text{for } n \geq 1,
$$

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which can be obtained by comparing the coefficients in the expansion of \( t = (e^t - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \). The Bernoulli numbers have many applications in modern number theory, such as the Eisenstein series in modular forms (see [1]), and the arithmetic of algebraic number fields, especially Kummer’s notion of regularity and the class number of \( p \)th cyclotomic fields (see [25, p. 62, Theorem 5.16]).

It is well-known that there exist close analogues between the rational number field \( \mathbb{Q} \) and the rational function fields \( \mathbb{F}_r(T) \) over a finite field \( \mathbb{F}_r \) (see [10]). In 1935, Carlitz [4] gave an analogue of Bernoulli numbers for rational function field \( \mathbb{F}_r(T) \), denoted by \( BC_n \), which is now known as the Bernoulli-Carlitz numbers. In subsequent works, he also found many interesting properties of them, including the analogue of the well-known von Staudt–Clausen theorem (see [5, 6] and [20]).

The definition of Bernoulli-Carlitz numbers is as follows. Let \( [i] = T^{r^i} - T, D_i = [i][i-1]^r \cdots [1]^{r^{i-1}} \) with \( D_0 = 1 \). The Carlitz exponential is defined by

\[
e_C(z) = \sum_{j=0}^{\infty} \frac{z^{r^j}}{D_j}.
\]

For a nonnegative integer \( i \) with \( r \)-ary expansion \( i = \sum_{j=0}^{m} c_j r^j \) (\( 0 \leq c_j < r \)), the Carlitz factorial \( \Pi(i) \) is defined by

\[
\Pi(i) = \prod_{j=0}^{m} D_j^{c_j}.
\]

In analogue with (1.1), the Bernoulli-Carlitz numbers \( BC_n \in \mathbb{F}_r(T) \) (\( n = 0, 1, 2, \ldots \)) are defined by the generating function

\[
\frac{z}{e_C(z)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} z^n.
\]

By comparing the coefficients in the expansion of \( z = e_C(z) \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} z^n \), Carlitz found the following recursive formula of the Bernoulli-Carlitz numbers \( BC_n \) which are analogues of (1.2)

\[
BC_0 = 1, \quad BC_n = - \sum_{j=1}^{[\log_r(n+1)]} \frac{\Pi(n)}{\Pi(r^j)\Pi(n+1-r^j)} BC_{n+1-r^j} \quad \text{for } n \geq 1,
\]

where \([ \cdot ]\) is the greatest integer function. As in the classical case, the Bernoulli-Carlitz numbers have many deep connections with the arithmetic of function fields, especially the class groups of cyclotomic function fields (see [11, Section 9.2], [24, Section 5.2] or [13, 14]).

For \( \ell \in \mathbb{N} \), in 1924, Nörlund [21] defined the higher order Bernoulli numbers \( B_n^{(\ell)} \in \mathbb{Q} \) (\( n = 0, 1, 2, \ldots \)) by the generating function

\[
\left( \frac{t}{e^t - 1} \right)^\ell = \sum_{n=0}^{\infty} \frac{B_n^{(\ell)} t^n}{n!}.
\]
and in 2005, Jeong, Kim and Son [18] defined the higher order Bernoulli-Carlitz numbers $BC_n^{(\ell)} \in \mathbb{F}_r(T)$ ($n = 0, 1, 2, \ldots$) by the generating function

$$
(1.6) \quad \left( \frac{z}{e^{C}(z)} \right)^\ell = \sum_{n=0}^{\infty} \frac{BC_n^{(\ell)}}{\Pi(n)} z^n.
$$

Letting $l = 1$ in (1.5) and (1.6), we recover the Bernoulli numbers $B_n$ and the Bernoulli-Carlitz numbers $BC_n$, respectively.

The classical Cauchy numbers $c_n \in \mathbb{Q}$ ($n = 0, 1, 2, \ldots$) are defined by the generating function

$$
(1.7) \quad \frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!}
$$

(see [9]).

Let $L_i = [i][i - 1] \cdots [1]$ ($i \geq 1$) with $L_0 = 1$, let

$$
(1.8) \quad \log_C(z) = \sum_{i=0}^{\infty} (-1)^i \frac{z^{r_i}}{L_i}
$$

be the Carlitz logarithm. In 2016, Kaneko and Komatsu [19] defined the Cauchy-Carlitz numbers $CC_n$ ($n = 0, 1, 2, \ldots$) by the generating function

$$
(1.9) \quad \frac{z}{\log_C(z)} = \sum_{n=0}^{\infty} \frac{CC_n}{\Pi(n)} z^n
$$

(see [19, p. 240, (12)]), and for $\ell \in \mathbb{N}$, they also defined the higher order Cauchy-Carlitz numbers $CC_n^{(\ell)}$ ($n = 0, 1, 2, \ldots$) by

$$
(1.10) \quad \left( \frac{z}{\log_C(z)} \right)^\ell = \sum_{n=0}^{\infty} \frac{CC_n^{(\ell)}}{\Pi(n)} z^n
$$

(see [19, p. 249, (28)]).

Recently, in order to unify the definitions of several special numbers in the classical setting such as the (higher order) Bernoulli numbers and the (higher order) Cauchy numbers, Hu and Komatsu [17] introduced the concept of the related numbers of higher order Appell polynomials. Their definition is as follows.

Let $C$ be the field of complex numbers, let $S(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ be any formal power series in $C[[t]]$ and $a_0 \neq 0$, the Appell polynomials $A_n(z)$ are defined by the generating function

$$
(1.11) \quad S(t) e^{zt} = \sum_{n=0}^{\infty} A_n(z) \frac{t^n}{n!}
$$

(see [2]). Since $a_0 \neq 0$, there exists the formal power series (for some $d_n \in \mathbb{C}$)

$$
(1.12) \quad f(t) = \frac{1}{S(t)} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}
$$
in $\mathbb{C}[[t]]$, and (1.11) becomes

\begin{equation}
\frac{e^{zt}}{f(t)} = \sum_{n=0}^{\infty} A_n(z) \frac{t^n}{n!}.
\end{equation}

For $\ell \in \mathbb{N}$, we can also define the higher order Appell polynomials by the generating function

\begin{equation}
\frac{e^{zt}}{(f(t))^{\ell}} = \sum_{n=0}^{\infty} A_n^{(\ell)}(z) \frac{t^n}{n!}
\end{equation}

(see [3, Théorème 1.1]). As in the classical case, $a_n^{(\ell)} = A_n^{(\ell)}(0)$ is defined to be the related numbers of higher order Appell polynomials, that is,

\begin{equation}
\frac{1}{(f(t))^{\ell}} = \sum_{n=0}^{\infty} a_n^{(\ell)} \frac{t^n}{n!}
\end{equation}

and $a_n = a_n^{(1)}$ the related numbers of Appell polynomials (see [17, p. 3, (6)]).

In (1.15), let $f(t) = \frac{e^t - 1}{t}$ and $\frac{\log(1+t)}{t}$, we obtain the (higher order) Bernoulli numbers and the (higher order) Cauchy numbers, respectively.

To unify the definitions of several special numbers in positive characteristic, such as the Bernoulli-Carlitz numbers and the Cauchy-Carlitz numbers, we here define the Appell-Carlitz numbers to be a sequence $\{AC_n\}_{n=0}^{\infty}$ in $k = \mathbb{F}_r(T)$ with a normalization $AC_0 = 1$. Let $S(z) \in k((z))$ be the generating function of $\{AC_n\}_{n=0}^{\infty}$, that is,

\begin{equation}
S(z) = \sum_{n=0}^{\infty} \frac{AC_n}{\Pi(n)} z^n.
\end{equation}

If $S(z) = \left(\frac{e^z}{e^{C(z)}}\right)^{\ell}$ and $S(z) = \left(\frac{e^z}{\log C(z)}\right)^{\ell}$, then we obtain the higher order Bernoulli-Carlitz numbers and the higher order Cauchy-Carlitz numbers, respectively. It needs to mention that in Goss’s book [11, p. 352, Definition 9.1.4], the above generating function $S(z)$ is named as the Hurwitz series.

For $\ell \in \mathbb{N}$, we may also define the higher order Appell-Carlitz numbers $AC_n^{(\ell)}(z) (n = 0, 1, 2, \ldots)$ as the generating function

\begin{equation}
(S(z))^{\ell} = \sum_{n=0}^{\infty} \frac{AC_n^{(\ell)}}{\Pi(n)} z^n.
\end{equation}

Denote by $f(z) = \frac{1}{S(z)}$, we have

\begin{equation}
\frac{1}{(f(z))^{\ell}} = \sum_{n=0}^{\infty} \frac{AC_n^{(\ell)}}{\Pi(n)} z^n,
\end{equation}

which is an analogue of (1.15) in $\mathbb{F}_r(T)$. It should be noted that the definitions of the Appell-Carlitz numbers and their higher order counterparts depend on the series $S(z)$. 
2. Main results and their corollaries. “In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.” (See [22, p. 91]). During the recent years, there are many results concerning closed form expressions for special numbers and polynomials in characteristic 0 case, such as Bernoulli, Euler, Cauchy, Apostol–Bernoulli, hypergeometric Bernoulli numbers and polynomials, see [7, 8, 16, 17, 22] and the references therein.

In this paper, we shall address our attention to the characteristic $p$ case and obtain several properties of the (higher order) Appell-Carlitz numbers, including a recurrence formula, two closed form expressions, a determinant expression. The recurrence formula (Theorem 2.1) implies Carlitz’s recurrence formula for Bernoulli-Carlitz numbers (see (1.4) above). Two closed from expressions (Theorems 2.4 and 2.9) implies the corresponding results for Bernoulli-Carlitz and Cauchy-Carlitz numbers (see Corollaries 2.5, 2.6, 2.7 and 2.8 below). The determinant expression (Theorem 2.10) implies the corresponding results for Bernoulli-Carlitz and Cauchy-Carlitz numbers (see Corollaries 2.12 and 2.14 below).

Suppose that $f(z) = \frac{1}{S(z)}$ has the following power series expansion

\begin{equation}
    f(z) = \sum_{n=0}^{\infty} \lambda_n z^n.
\end{equation}

Then we have the following recurrence formula for the higher order Appell-Carlitz numbers.

**Theorem 2.1.** (Recurrence formula for higher order Appell-Carlitz numbers)

\[
    AC_{m}^{(\ell)} = -\Pi(m) \sum_{i=0}^{m-1} \frac{AC_{i}^{(\ell)}}{\Pi(i)} D_{\ell}(m - i)
\]

with $AC_{0}^{(\ell)} = 1$, where

\[
    D_{\ell}(e) = \sum_{i_{1}+\ldots+i_{\ell}=e} \lambda_{i_{1}} \cdot \ldots \cdot \lambda_{i_{\ell}}.
\]

Letting $\ell = 1$ in the above result, we get a recurrence formula for Appell-Carlitz numbers.

**Corollary 2.2.** (Recurrence formula for Appell-Carlitz numbers)

\[
    AC_{m} = -\Pi(m) \sum_{i=0}^{m-1} \frac{AC_{i}}{\Pi(i)} \lambda_{m-i}.
\]

**Remark 2.3.** In the case of Bernoulli-Carlitz numbers, we have

\[
    f(z) = \frac{e_{C}(z)}{z/2} = \sum_{j=0}^{\infty} \frac{z^{r_{j}} - 1}{D_{j}}.
\]
Define

\[ \delta^*_e = \begin{cases} \frac{1}{D_n} & \text{if } e = r^n - 1 \text{ for some } n \\ 0 & \text{if } e \neq r^n - 1 \text{ for any } n \end{cases} \]  

then comparing with (2.1), we have

\[ \lambda_j = \delta^*_j \]  

for \( j = 0, 1, 2, \ldots \). By Corollary 2.2 and (2.4), we obtain Carlitz’s recurrence formula for Bernoulli-Carlitz numbers (see (1.4) above)

\[ BC_m = -\Pi(m) \sum_{i=0}^{m-1} \frac{BC_i}{\Pi(i)} \delta^*_m \]  

\[ = -\Pi(m) \sum_{i=0}^{m-1} \frac{BC_{m-i}}{\Pi(m-i)} \delta^*_i \]  

\[ = -\Pi(m) \sum_{j=0}^{[\log_e(m+1)]} \frac{BC_{m+1-r^j}}{\Pi(m+1-r^j)} r^{j-1} \]  

\[ = -\sum_{j=1}^{[\log_e(m+1)]} \frac{\Pi(m)}{\Pi(r^j)\Pi(m+1-r^j)} BC_{m+1-r^j}, \]  

since \( D_j = \Pi(r^j) \), for \( m \geq 1 \).

We also have a closed form expression for the higher order Appell-Carlitz numbers.

**Theorem 2.4.** (Closed form expression for higher order Appell-Carlitz numbers) For \( m \geq 1 \), we have

\[ AC_m^{(\ell)} = \Pi(m) \sum_{k=1}^{m} (-1)^k \sum_{\substack{e_1 + \cdots + e_k = m \\
\ell(e_1) \cdots \ell(e_k) \geq 1}} D_\ell(e_1) \cdots D_\ell(e_k), \]  

where \( D_\ell(e) \) are given in (2.2).

Letting \( \ell = 1 \) in the above result, we have a closed form expression for Appell-Carlitz numbers.

**Corollary 2.5.** (Closed form expression for Appell-Carlitz numbers) For \( m \geq 1 \), we have

\[ AC_m = \Pi(m) \sum_{k=1}^{m} (-1)^k \sum_{\substack{e_1 + \cdots + e_k = m \\
\ell(e_1) \cdots \ell(e_k) \geq 1}} \lambda_{e_1} \cdots \lambda_{e_k}. \]
Then by (2.4), we have

\[ BC_m = \Pi(m) \sum_{k=1}^{m} (-1)^k \sum_{e_1 + \cdots + e_k = m, e_1, \ldots, e_k \geq 1} \delta_{e_1}^* \cdots \delta_{e_k}^* \]

\[ = \Pi(m) \sum_{k=1}^{m} (-1)^k \sum_{r^{i_1} + \cdots + r^{i_k} = m+k, r^{i_1}, \ldots, r^{i_k} > 1} \frac{1}{D_{i_1}} \cdots \frac{1}{D_{i_k}}. \]

Since \( D_i = \Pi(r^i) \), we have the following closed form expression for Bernoulli-Carlitz numbers by Jeong, Kim and Son (see [18, p. 63, Theorem 4.2]).

**Corollary 2.6.** (Closed form expression for Bernoulli-Carlitz numbers) For \( m \geq 1 \), we have

\[ BC_m = \Pi(m) \sum_{k=1}^{m} (-1)^k \sum_{r^{i_1} + \cdots + r^{i_k} = m+k, r^{i_1}, \ldots, r^{i_k} > 1} \frac{1}{\Pi(r^{i_1})} \cdots \frac{1}{\Pi(r^{i_k})}. \]

More generally, from Theorem 2.4 and (2.4), we may also recover the following closed form expression for higher order Bernoulli-Carlitz numbers (see [18, p. 65, Proposition 4.5]). We would like to refer Thakur’s book [24, p. 145, the second last paragraph] on a discussion of this formula.

**Corollary 2.7.** (Closed form expression for higher order Bernoulli-Carlitz numbers) For \( m \geq 1 \), we have

\[ BC_m^{(\ell)} = \Pi(m) \sum_{j=1}^{m} (-1)^j \sum_{i_1, \ldots, i_j \geq 1, i_1 + \cdots + i_j = m} M^{(\ell)}(i_1) \cdots M^{(\ell)}(i_j), \]

where for each \( i \),

\[ M^{(\ell)}(i) := \sum_{e_1, \ldots, e_\ell \geq 0, r^{e_1} + \cdots + r^{e_\ell} = i} \frac{1}{\Pi(r^{e_1}) \Pi(r^{e_2}) \cdots \Pi(r^{e_\ell})}. \]

In the case of Kaneko and Komatsu’s Bernoulli-Carlitz numbers, we have

\[ f(z) = \frac{\log C(z)}{z} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{r^j-1}}{L_j}. \]

Define

\[ \delta_{e^*}^* = \begin{cases} (-1)^n \frac{1}{L_n} & \text{if } e = r^n - 1 \text{ for some } n \\ 0 & \text{if } e \neq r^n - 1 \text{ for any } n, \end{cases} \]
then comparing with (2.1), we have

\[(2.7) \quad \lambda_j = \delta_j^{**} \]

for \(j = 0, 1, 2, \ldots \).

From Theorem 2.4 and (2.4), we also recover the following closed form expression for Kaneko and Komatsu’s higher order Cauchy-Carlitz numbers (see [19, p. 249, Proposition 6]).

**Corollary 2.8.** (Closed form expression for higher order Cauchy-Carlitz numbers) For \(m \geq 1\), we have

\[CC_m^{(\ell)} = \Pi(m) \sum_{j=1}^{m} (-1)^j \sum_{\substack{i_1, \ldots, i_j \geq 1 \atop i_1 + \cdots + i_j = m}} M^{(\ell)}(i_1) \cdots M^{(\ell)}(i_j), \]

where for each \(i\),

\[M^{(\ell)}(i) := \sum_{\substack{j_1, \ldots, j_\ell \geq 0 \atop r^1 + \cdots + r^\ell = i}} (-1)^{j_1 + \cdots + j_\ell} \frac{(1)^{j_1 + \cdots + j_\ell}}{L_{j_1} \cdots L_{j_\ell}}.\]

Generalizing Jeong, Kim and Son’s result for Bernoulli-Carlitz numbers [18, p. 63, Theorem 4.1], we get another closed form expression for Appell-Carlitz numbers.

**Theorem 2.9.** (Another closed form expression for Appell-Carlitz numbers) For \(m \geq 1\), we have

\[AC_m = \Pi(m) \sum_{j=1}^{m} (-1)^j \sum_{\substack{i_1, \ldots, i_m \geq 0 \atop i_1 + \cdots + i_m = j \atop i_1 + 2i_2 + \cdots + m_{i_m} = m}} \left(\begin{array}{c} j \\ i_1, \ldots, i_m \end{array}\right) \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_m^{i_m}.\]

We have a determinant expression of the higher order Appell-Carlitz numbers.

**Theorem 2.10.** (Determinant expression of higher order Appell-Carlitz numbers) For \(m \geq 1\), we have

\[AC_m^{(\ell)} = (-1)^m \Pi(m) \begin{vmatrix} D_{\ell}(1) & 1 \\ D_{\ell}(2) & D_{\ell}(1) \\ \vdots & \vdots & \ddots & 1 \\ D_{\ell}(n-1) & D_{\ell}(n-2) & \cdots & D_{\ell}(1) \\ D_{\ell}(n) & D_{\ell}(n-1) & \cdots & D_{\ell}(2) & D_{\ell}(1) \end{vmatrix},\]

where \(D_{\ell}(e)\) are given in (2.2).

Letting \(\ell = 1\) in the above result, we have the following determinant expression for the related numbers of Appell-Carlitz numbers.
Corollary 2.11. (Determinant expression of Appell-Carlitz numbers) For \( m \geq 1 \), we have

\[
AC_m = (-1)^m \Pi(m) = \begin{vmatrix}
\lambda_1 & 1 & & & \\
\lambda_2 & \lambda_1 & & & \\
& \vdots & \ddots & \ddots & \\
\lambda_{m-1} & \lambda_{m-2} & \cdots & \lambda_1 & 1 \\
\lambda_m & \lambda_{m-1} & \cdots & \lambda_2 & \lambda_1 \\
\end{vmatrix}
\]

Then by (2.4), we obtain a determinant expression of Bernoulli-Carlitz numbers.

Corollary 2.12. (Determinant expression of Bernoulli-Carlitz numbers) For \( m \geq 1 \), we have

\[
BC_m = (-1)^m \Pi(m) = \begin{vmatrix}
\delta_1^* & 1 & & & \\
\delta_2^* & \delta_1^* & & & \\
& \vdots & \ddots & \ddots & \\
\delta_{m-1}^* & \delta_{m-2}^* & \cdots & \delta_1^* & 1 \\
\delta_m^* & \delta_{m-1}^* & \cdots & \delta_2^* & \delta_1^* \\
\end{vmatrix},
\]

where

\[
\delta_e^* = \begin{cases} 
\frac{1}{\Pi_n} & \text{if } e = r^n - 1 \text{ for some } n \\
0 & \text{if } e \neq r^n - 1 \text{ for any } n.
\end{cases}
\]

Remark 2.13. Since \( D_n \) equals to \( \Pi(r^n) \), the Carlitz factorial, the above result is an analogue of the following classical determinant expression of Bernoulli numbers \( B_m \) by Glaisher in 1875 (see [9, p. 53]):

\[
B_m = (-1)^m m! = \begin{vmatrix}
\frac{1}{2!} & \frac{1}{2!} & & & \\
\frac{1}{3!} & \frac{1}{2!} & & & \\
& \vdots & \ddots & \ddots & \\
\frac{1}{(m-1)!} & \frac{1}{2!} & \cdots & \frac{1}{2!} & 1 \\
\frac{1}{m!} & \frac{1}{m!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \\
\frac{1}{(m+1)!} & \frac{1}{m!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \\
\end{vmatrix}
\]

Similarly, by (2.7), we obtain the following determinant expression of Cauchy-Carlitz numbers.

Corollary 2.14. (Determinant expression of Cauchy-Carlitz numbers) For \( m \geq 1 \), we have

\[
CC_m = (-1)^m \Pi(m) = \begin{vmatrix}
\delta_1^{**} & 1 & & & \\
\delta_2^{**} & \delta_1^{**} & & & \\
& \vdots & \ddots & \ddots & \\
\delta_{m-1}^{**} & \delta_{m-2}^{**} & \cdots & \delta_1^{**} & 1 \\
\delta_m^{**} & \delta_{m-1}^{**} & \cdots & \delta_2^{**} & \delta_1^{**} \\
\end{vmatrix},
\]
where
\[
\delta^{*\ast}_{e} = \begin{cases} 
(-1)^n \frac{1}{L_n} & \text{if } e = r^n - 1 \text{ for some } n \\
0 & \text{if } e \neq r^n - 1 \text{ for any } n,
\end{cases}
\]

**Remark 2.15.** Since the classical Cauchy numbers are defined by the generating function

\begin{equation}
\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}
\end{equation}

(see [9]), by applying the power series expansion of

\begin{equation}
f(t) = \frac{\log(1 + t)}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n + 1}
\end{equation}

to [17, Theorem 3], we get the following determinant expression of Cauchy numbers

\begin{equation}
c_m = (-1)^m m!
\end{equation}

This is equivalent to Glaisher’s following determinant expression in 1875 (see [9, p. 50]):

\begin{equation}
c_m = m!
\end{equation}

if considering the generating function

\begin{equation}
\frac{-t}{\log(1 - t)} = \sum_{n=0}^{\infty} (-1)^n c_n \frac{t^n}{n!}
\end{equation}

and applying the power series expansion

\[ f(t) = \frac{\log(1 - t)}{-t} = \sum_{n=0}^{\infty} \frac{t^n}{n + 1} \]

to [17, Theorem 3]. By comparing the power series expansion of the Carlitz logarithm

\[ \log_C(z) = \sum_{i=0}^{\infty} (-1)^i \frac{z^{r_i}}{L_i} \]
and the power series expansion of the classical logarithm
\[
\log(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n},
\]
we have seen the analogue between (2.10) and (2.13).

3. Hasse-Teichmüller derivatives ([17, Section 2] and [18, Section 2]).
Since \( n! = 0 \) in a field of characteristic \( p \) if \( n \geq p \), and \( \frac{d}{dt}(t^n) = 0 \) if \( p \) divides \( n \), the classical differential calculus faces essential difficulties in positive characteristics.

In 1936, Hasse [15] introduced the concept of hyperdifferentials to overcome these, now known as the Hasse-Teichmüller derivatives. In this section, we shall recall the definition and basic properties of these derivatives which serves as the main tool for our proof.

Let \( F \) be a field of any characteristic, \( F[[z]] \) the ring of formal power series in one variable \( z \), and \( F((z)) \) the field of Laurent series in \( z \). Let \( m \) be a nonnegative integer. The Hasse-Teichmüller derivative \( H^{(m)} \) of order \( m \) is defined by

\[
H^{(m)} \left( \sum_{n=R}^{\infty} c_n z^n \right) = \sum_{n=R}^{\infty} c_n \binom{n}{m} z^{n-m}
\]

for \( \sum_{n=R}^{\infty} c_n z^n \in F((z)) \), where \( R \) is an integer and \( c_n \in F \) for any \( n \geq R \). Note that \( \binom{n}{m} = 0 \) if \( n < m \).

The Hasse-Teichmüller derivatives satisfy the product rule [23], the quotient rule [12] and the chain rule [15]. One of the product rules can be described as follows.

**Lemma 3.1.** ([23, 18]) For \( f_i \in F[[z]] \) (\( i = 1, \ldots, k \)) with \( k \geq 2 \) and for \( m \geq 1 \), we have
\[
H^{(m)}(f_1 \cdots f_k) = \sum_{i_1 + \cdots + i_k = m \atop i_1, \ldots, i_k \geq 0} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k).
\]

The quotient rules can be described as follows.

**Lemma 3.2.** ([12, 18]) For \( f \in F[[z]] \setminus \{0\} \) and \( m \geq 1 \), we have
\[
H^{(m)} \left( \frac{1}{f} \right) = \sum_{k=1}^{m} \frac{(-1)^k}{f^{k+1}} \sum_{i_1 + \cdots + i_k = m \atop i_1, \ldots, i_k \geq 1} H^{(i_1)}(f) \cdots H^{(i_k)}(f)
\]

\[
= \sum_{k=1}^{m} \frac{(m+1)}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{i_1 + \cdots + i_k = m \atop i_1, \ldots, i_k \geq 0} H^{(i_1)}(f) \cdots H^{(i_k)}(f).
\]
Lemma 3.3. ([23, 18]) For $f \in F[[z]]$ and for $m \geq 1$, $j \geq 2$, we have

$$H^{(m)}(f^j) = \sum_{k=1}^{j} f^{j-k} \sum_{i_1, \ldots, i_m \geq 0 \atop i_1+\cdots+i_m = k} \frac{j(j-1)\cdots(j-k+1)}{i_1! \cdots i_m!} \times (H^{(1)}(f))^{i_1} \cdots (H^{(m)}(f))^{i_m}.$$  

4. Proofs of the main results. In this section, we shall proof our main results which have been introduced in Section 2.

Lemma 4.1. For $m \geq 1$, we have

$$\sum_{i_{\ell+1}=0}^{m} \frac{AC_{i_{\ell+1}}^{(\ell)}}{\Pi(i_{\ell+1})} \sum_{i_1+\cdots+i_{\ell} = m-i_{\ell+1}} \lambda_{i_1} \cdots \lambda_{i_{\ell}} = 0.$$  

Proof. Put $f(z) = \frac{1}{S(z)}$. From (1.17) and (2.1), we have

$$(4.1) \quad 1 = (f(z))^\ell (S(z))^\ell = \left(\sum_{n=0}^{\infty} \lambda_n z^n\right)^\ell \left(\sum_{n=0}^{\infty} \frac{AC_n^{(\ell)}}{\Pi(n)} z^n\right).$$  

Applying the Hasse-Teichmüller derivative $H^{(m)}$ of order $m$ to (4.1), we have

$$(4.2) \quad H^{(m)} \left[ (f(z))^\ell (S(z))^\ell \right]_{z=0} = H^{(m)}(1)_{z=0} = 0.$$  

Note that for $j = 1, 2, \ldots, \ell$, by the definition of the Hasse-Teichmüller derivative (3.1), we have

$$H^{(i_j)}(f(z))_{z=0} = H^{(i_j)} \left( \sum_{n=0}^{\infty} \lambda_n z^n \right)_{z=0}$$

$$= \sum_{n=i_j}^{\infty} \lambda_n \binom{n}{i_j} z^{n-i_j}$$

$$= \lambda_{i_j}$$

and

$$H^{(i_{\ell+1})} \left[ (S(z))^\ell \right]_{z=0} = H^{(i_{\ell+1})} \left( \sum_{n=0}^{\infty} \frac{AC_n^{(\ell)}}{\Pi(n)} z^n \right)_{z=0}$$

$$= \sum_{n=i_{\ell+1}}^{\infty} \frac{AC_n^{(\ell)}}{\Pi(n)} \binom{n}{i_{\ell+1}} z^{n-i_{\ell+1}}$$

$$= \frac{AC_{i_{\ell+1}}^{(\ell)}}{\Pi(i_{\ell+1})}.$$
Then by Lemma 3.1, we have

\[ H^{(m)} \left[ (f(z))^\ell (S(z))^\ell \right] \bigg|_{z=0} = \sum_{i_1 + \cdots + i_{\ell+1} = m} H^{(i_1)} (f(z)) \bigg|_{z=0} \cdots H^{(i_\ell)} (f(z)) \bigg|_{z=0} H^{(i_{\ell+1})} \left[ (S(z))^\ell \right] \bigg|_{z=0} \]

(4.3)

\[ = \sum_{i_1 + \cdots + i_{\ell+1} = m} \lambda_{i_1} \cdots \lambda_{i_\ell} \frac{AC^{(\ell)}_{i_{\ell+1}}}{\Pi(i_{\ell+1})}. \]

Comparing with (4.2), we get

(4.4) \[ \sum_{i_1 + \cdots + i_{\ell+1} = m} \lambda_{i_1} \cdots \lambda_{i_\ell} \frac{AC^{(\ell)}_{i_{\ell+1}}}{\Pi(i_{\ell+1})} = 0, \]

which is the desired formula. \( \Box \)

**Proof of Theorem 2.1.** From Lemma 4.1, we have

\[ AC^{(\ell)}_m = -\Pi(m) \sum_{i=0}^{m-1} \frac{AC^{(\ell)}_i}{\Pi(i)} D_\ell(m-i) \]

with \( AC^{(\ell)}_0 = 1 \), where

(4.5) \[ D_\ell(e) = \sum_{i_1 + \cdots + i_{\ell+1} = e} \lambda_{i_1} \cdots \lambda_{i_\ell}, \]

which is Theorem 2.1. \( \Box \)

**Proof of Theorem 2.4.** Denote by

(4.6) \[ h(z) = (f(z))^\ell, \]

where

\[ f(z) = \sum_{n=0}^{\infty} \lambda_n z^n. \]

Since by (3.1), the definition of the Hasse–Teichmüller derivative, we have

\[ H^{(i)}(f(z)) \bigg|_{z=0} = \sum_{n=1}^{\infty} \lambda_n \binom{n}{i} z^{n-i} \bigg|_{z=0} \]

\[ = \lambda_i. \]
Then applying the product rule of the Hasse-Teichmuller derivative in Lemma 3.1, we get
\[
H(e)(h(z))|_{z=0} = \sum_{\substack{i_1+\ldots+i_\ell=e \\text{ such that } i_1, \ldots, i_\ell \geq 0}} H^{(i_1)}(f(z))|_{z=0} \cdots H^{(i_\ell)}(f(z))|_{z=0}
\]
(4.7)
\[
= \sum_{\substack{i_1+\ldots+i_\ell=e \\text{ such that } i_1, \ldots, i_\ell \geq 0}} \lambda_{i_1} \cdots \lambda_{i_\ell}
\]
:= \mathcal{D}_\ell(e).

Since by (4.6) and (1.18)
\[
\frac{1}{h(z)} = \frac{1}{(f(z))^\ell} = \sum_{n=0}^{\infty} \frac{AC_n(\ell)}{\Pi(n)} z^n,
\]
we have
(4.8)
\[
H^{(m)} \left( \frac{1}{h(z)} \right) \bigg|_{z=0} = H^{(m)} \left( \sum_{n=0}^{\infty} \frac{AC_n(\ell)}{\Pi(n)} z^n \right) \bigg|_{z=0} = \frac{AC_m(\ell)}{\Pi(m)}.
\]
And from (3.2), the quotient rule of the Hasse-Teichmuller derivative, and (4.7), we get
(4.9)
\[
H^{(m)} \left( \frac{1}{h(z)} \right) \bigg|_{z=0} = \sum_{k=1}^{m} \frac{(-1)^k}{h^{k+1}} \sum_{\substack{e_1+\ldots+e_k=m \\text{ such that } e_1, \ldots, e_k \geq 1}} H^{(e_1)}(h(z))|_{z=0} \cdots H^{(e_k)}(h(z))|_{z=0}
\]
\[
= \sum_{k=1}^{m} (-1)^k \sum_{\substack{e_1+\ldots+e_k=m \\text{ such that } e_1, \ldots, e_k \geq 1}} \mathcal{D}_\ell(e_1) \cdots \mathcal{D}_\ell(e_k).
\]
Comparing (4.8) and (4.9) we get the desired formula. \(\square\)

**Proof of Theorem 2.9.** From the geometric series expansion, we have
(4.10)
\[
S(z) = \frac{1}{1 + (f(z) - 1)} = \sum_{j=0}^{\infty} (-1)^j (f(z) - 1)^j.
\]
And by (1.16) and the definition of the Hasse-Teichmuller derivative (3.1), we get
\[
H^{(m)}(S(z))|_{z=0} = H^{(m)} \left( \sum_{n=0}^{\infty} \frac{AC_n}{\Pi(n)} z^n \right) \bigg|_{z=0} = \frac{AC_m}{\Pi(m)}.
\]
Then applying the Hasse-Teichmuller derivative \(H^{(m)}\) of order \(m \geq 1\) to both sides of (4.10), we get
(4.11)
\[
\frac{AC_m}{\Pi(m)} = \sum_{j=1}^{\infty} (-1)^j H^{(m)}(g^j)|_{z=0},
\]
where

$$g := f(z) - 1 = \sum_{i=1}^{\infty} \lambda_i z^i. $$

Lemma 3.3 yields

$$H^{(m)}(g^j)|_{z=0} = \sum_{k=1}^{j} g^{j-k} \sum_{i_1, \ldots, i_m \geq 0 \atop i_1 + \cdots + i_m = k \atop i_1 + 2i_2 + \cdots + mi_m = m} \frac{j(j-1) \cdots (j-k+1)}{i_1! \cdots i_m!}$$

$$\times (H^{(1)}(g))^{i_1} \cdots (H^{(m)}(g))^{i_m} \bigg|_{z=0}. $$

By (4.12), we have $g(0) = 0$ and $g^{j-k}|_{z=0} = 0$ if $j \neq k$, thus the right hand side of the above equality equals to

$$\sum_{i_1, \ldots, i_m \geq 0 \atop i_1 + \cdots + i_m = j \atop i_1 + 2i_2 + \cdots + mi_m = m} \frac{j!}{i_1! \cdots i_m!} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_m^{i_m}. $$

Substituting to (4.11) and also noticing that for $j > m$, the summation index of the above sum becomes empty thus the sum equals to 0, we get the desired formula. □

**Proof of Theorem 2.10.** At this stage, we show that the proof of [17, Theorem 2] which based on the inductive method can also be applied to our situation.

Denote by $A_m^{(\ell)} = \frac{(-1)^m AC_m^{(\ell)}}{\Pi(m)}$. Then, we shall prove that for any $m \geq 1$

$$A_m^{(\ell)} = \begin{vmatrix} D_\ell(1) & 1 \\ D_\ell(2) & D_\ell(1) \\ \vdots & \vdots & \ddots & 1 \\ D_\ell(n-1) & D_\ell(n-2) & \cdots & D_\ell(1) & 1 \\ D_\ell(n) & D_\ell(n-1) & \cdots & D_\ell(2) & D_\ell(1) \end{vmatrix}. $$

When $m = 1$, (4.13) is valid, because by Corollary 2.5 we have

$$AC_1 = (-1)\Pi(1)D_\ell(1). $$

Assume that (4.13) is valid up to $m - 1$. Notice that by Corollary 2.2, we have

$$A_m^{(\ell)} = \sum_{i=1}^{m} (-1)^{i-1} A_{m-i}^{(\ell)} D_\ell(i). $$
Thus, by expanding the first row of the right-hand side (4.13), it is equal to

\[
D_\ell(1) D_{m-1}^{(\ell)} - D_\ell(2) D_{m-2}^{(\ell)} \\
= D_\ell(1) A_{m-1}^{(\ell)} - D_\ell(2) A_{m-2}^{(\ell)} \\
+ (-1)^{m-2} \sum_{i=1}^{m} (-1)^{i-1} D_\ell(i) A_{m-i}^{(\ell)} = A_m^{(\ell)}.
\]

Note that \( A_1^{(\ell)} = D_\ell(1) \) and \( A_0^{(\ell)} = 1 \). \( \square \)

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