MULTI-PERIOD PORTFOLIO OPTIMIZATION IN A DEFINED CONTRIBUTION PENSION PLAN DURING THE DECUMULATION PHASE

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ABSTRACT. This paper studies a multi-period portfolio selection problem for retirees during the decumulation phase. We set a series of investment targets over time and aim to minimize the expected losses from the time of retirement to the time of compulsory annuitization by using a quadratic loss function. A target greater than the expected wealth is given and the corresponding explicit expressions for the optimal investment strategy are obtained. In addition, the withdrawal amount for daily life is assumed to be a linear function of the wealth level. Then according to the parameter value settings in the linear function, the withdrawal mechanism is classified as deterministic withdrawal, proportional withdrawal or combined withdrawal. The properties of the investment strategies, targets, bankruptcy probabilities and accumulated withdrawal amounts are compared under the three withdrawal mechanisms. Finally, numerical illustrations are presented to analyze the effects of the final target and the interest rate on some obtained results.

1. Introduction. According to the contract of contributions and benefits, pension schemes can be divided into the defined benefit scheme and the defined contribution (DC for short hereafter) scheme. Nowadays DC plans have received substantial attention because they are more effective in easing the pressure on public financial system by transferring the financial risk from the sponsor to the retiree. In a DC plan, the financial risk borne by the pension member can be divided into two parts: the investment risk during the accumulation phase and the annuity risk at retirement. The investment risk during the accumulation phase is well known. As

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for the annuity risk, when DC members retire at a time with high annuity prices, they may have to accept a lower than expected pension income. To reduce the annuity risk at retirement, in many countries, including Canada, Denmark, Japan, the UK and the USA, a retiree in the DC scheme is permitted to buy a life annuity immediately after retirement or to defer annuitization. When the retiree decides to use all of his accumulated pension fund to buy a life annuity immediately after retirement, he can receive a lifelong consumption stream that cannot be outlived. Nonetheless, according to the findings in Brown [4, 5], Finkelstein and Poterba [13], Inkmann, Lopes and Michaelides [19], Lockwood [20] and Mitchell et al. [27], the levels of the voluntary annuitization exhibited among retirees are extremely low due to the strong bequest motive, the poor health situation of the retiree or the low implied rate of the annuity products. Therefore, in the countries where immediate annuitization is not the only option available, the retiree can defer annuitization after retirement, periodically withdraw some money from the fund for daily life and invest the remaining wealth in assets with higher returns until the time of compulsory annuitization. This process is usually called the income drawdown option.

In the income drawdown option, there are two main strands of research objectives. One is to minimize the probability of shortfall, and the other is to examine utility-maximizing annuitization strategies. We start by recalling research studying how to minimize the shortfall probability. On the one hand, the income drawdown option has some advantages over the life annuity in terms of the greater liquidity and the possibility to bequeath some of the funds to heirs in the case of an early death. On the other hand, this option cannot hedge the longevity risk; that is, the pensioner might outlive his assets before the uncertain date of death or the time of compulsory annuitization. Therefore, some research focuses on minimizing the shortfall probability. Among others, Milevsky [21] finds the optimal time to annuitize based on a deterministic model and a stochastic model with a stochastic interest rate and random mortality with the aim of minimizing the shortfall probability. Milevsky and Robinson [24] obtain the lifetime probability and the eventual probability of ruin for pensioners, where the lifetime probability of ruin is the probability that the net wealth hits zero prior to a stochastic time of death, and the eventual probability of ruin is the probability that net wealth will ever hit zero for an infinitely lived individual. Albrecht and Maurer [1] study the personal probability of consumption shortfall with respect to German insurance and capital market conditions. Milevsky, Moore and Young [23] assume that the withdrawal amount is a constant and obtain the optimal investment strategy and the optimal time to annuitize with the aim of minimizing the probability that the wealth reaches zero while the retiree is still alive. In addition to the shortfall probability minimization, some papers focus on the optimal decision with the CARA or CRRA utility. In some well-cited papers on this topic, Milevsky and Young [25] seek to maximize the expected power utility of discounted consumption over the time of annuitization, the consumption amount and the investment amount. Stabile [28] and Milevsky and Young [26] aim to maximize the expected sum of the power utility of consumption and the power utility of terminal wealth over the triples of annuitization time, consumption amount and investment amount. Emms and Haberman [12] obtain the optimal investment policies to minimize the expected loss of the pensioner as measured by the performance of the fund against a benchmark when the income drawdown rate is defined as a function of time and wealth level. They consider cases
with exponential or power loss functions. In addition to the exponential and power utilities, another prevailing utility adopted is the quadratic loss function. Compared with the CARA or CRRA utility functions, whose utility-based risk preference cannot be known clearly by the individuals, the quadratic loss function is an intuitive measure of risk because most individuals can assess their desired targets of consumption or wealth in the future. In the decision-making models with quadratic loss functions, the consumption amount and wealth level that a retiree wishes to achieve are given in advance. Then, the retiree attempts to minimize the deviations of the obtained consumption and wealth from the desired levels. In this strand of research, Gerrard, Haberman and Vigna [15] investigate a continuous-time optimal investment choice for a DC pension scheme, assuming that the withdrawal amount is constant over time and the pensioner has no bequest motive. In a follow-up paper, Gerrard et al. [14] generalize the basic model of Gerrard, Haberman and Vigna [15], where the interim consumption and annuitization time are fixed, to the case with a bequest motive. Using the same quadratic loss function and following a target-based approach, Gerrard, Haberman and Vigna [16] find the optimal investment and consumption choices when the annuitization time is fixed and the time of death of the retiree is random. Emms [10] incorporates the idea of relative choice into a target-based model of income drawdown and finds the optimal investment strategy and optimal drawdown rate. Gerrard, Højgaard and Vigna [17] aim to find the optimal annuitization time and optimal investment-consumption strategy to minimize the weighted sum of expected deviances of the running consumption rate and the final wealth from the desired values. Di Giacinto et al. [9] extend Gerrard, Haberman and Vigna [15] by adding a no short-selling constraint on the investment strategy and a final capital requirement constraint on the wealth level. For more information on the income drawdown option, interested readers are referred to Blake, Cairns and Dowd [2], Di Giacinto et al. [8] and Milevsky [22].

The current research on investment risk during the accumulation phase in DC pension plans is rich; see, for example, Emms [11], Vigna [29], Wu and Zeng [30], Zhao, Wang and Wei [32]. However, the research on optimal control during the decumulation phase is relatively limited and deserves more attention. Moreover, to the best of our knowledge, the existing research on the optimal control for the income drawdown option is performed under the continuous-time framework. We believe that the discrete-time counterpart is also worth studying. One reason is that the discrete-time model is more practical in reality, because we cannot adjust our strategy continuously in the real world. Another reason is that, there exist some differences between the results in the discrete-time model and those in the continuous-time model. For example, the optimal investment amount in our discrete-time model does not completely coincide with that in the continuous-time version. Moreover, the target without penalty of surplus does not exist in the discrete-time model because the returns of risky assets over time do not have a continuous trajectory. Therefore, to fill the gap in the research, this paper focuses on a discrete-time optimal investment problem for a DC pension plan during the decumulation phase by using a quadratic loss function and following a target-based approach. In addition, when the withdrawal amount is not a control variable, it is set to a constant or deterministic function of time in the vast majority of cases. The exceptions are Emms and Haberman [12] and Young [31], whose withdrawal amount for consumption is assumed to be a function of time and wealth level. Emms and Haberman [12] consider a continuous-time model with exponential or power loss functions. Young
aims to minimize the probability that retirees outlive their wealth during the decumulation phase. In this paper, we assume that the withdrawal amount is a linear function of the fund. More specifically, at time $n$, the withdrawal amount is $\rho_n X_n + c_n$ where $X_n$ is the wealth, $\rho_n$ is the withdrawal proportion and $c_n$ represents the amount withdrawn from the fund regardless of the performance of the investment. When $\rho_n$ and $c_n$ take different values, we obtain three different withdrawal mechanisms, called deterministic withdrawal, proportional withdrawal and combined withdrawal. Specifically, when $\rho_n = 0$, the commonly used withdrawal mechanism is derived. These withdrawal mechanisms have different withdrawal amounts in each period. For example, deterministic withdrawal gives retirees a predictable consumption amount regardless of the investment performance of the pension. Under the proportional withdrawal mechanism, the consumption amount in each period fluctuates with the investment return, so retirees sometimes face a low consumption amount due to low investment returns. In contrast to proportional withdrawal, combined withdrawal ensures the retirees a minimum standard of living regardless of the performance of the investment. Because most retirees care about the potential differences among withdrawal methods, we compare, in detail, the shortfall probabilities, the accumulated withdrawal amounts, the distance between the target and the obtained wealth, etc under three different withdrawal mechanisms. To the best of our knowledge, no existing literature has studied this topic, and our work provides some insights into choosing a final target and withdrawal mechanism if possible.

The remainder of this paper is organized as follows. Assumptions and the problem formulation are described in Section 2. The optimal investment strategy and the expression of target $F_n$ that expected wealth cannot exceed are obtained in Section 3, where some basic properties of the investment policies and the target $F_n$ are also presented. Some numerical illustrations are presented in Section 4 to compare the investment policies, the bankruptcy probabilities and the withdrawal amounts under three withdrawal mechanisms. We also analyze the effects of the terminal target and the interest rate on the obtained results in this section. Conclusions are given in Section 5. Proofs of the theorems are provided in Appendixes A-D.

2. Problem formulation. In this paper, we assume that a pensioner retires at time 0 with initial wealth $x_0$ and withdraws a certain income from the pension fund until the compulsory time $T$ for annuitization. Further, assume that the pension fund is invested in one risk-free asset and one risky asset and denote their gross returns over period $[n, n + 1)$ by $r^f_n$ and $R_n$, respectively, where $r^f_n$ is a positive constant and $R_n$ is a non-negative random variable. Moreover, $R_n$ is assumed to be independent of $R_m$ for all $n \neq m$. Moreover, it is reasonable to assume that $\mathbb{E}[R_n] > r^f_n > 1$ for all $n = 0, 1, \ldots, T - 1$. Over period $[n, n + 1)(n = 0, 1, \ldots, T - 1)$, the pensioner withdraws an amount for daily-life consumption that is a linear function of wealth $X_n$ at time $n$, i.e., the withdrawal amount is $\rho_n X_n + c_n$, where $\rho_n$ is the withdrawal proportion of the fund and $c_n$ represents the amount that the pensioner can obtain regardless of the performance of the investment. In this paper, when $c_n = 0$ and $\rho_n > 0$ for $n = 0, 1, \ldots, T - 1$, the withdrawal mechanism is called proportional withdrawal. When $c_n > 0$ and $\rho_n = 0$ for $n = 0, 1, \ldots, T - 1$, the withdrawal mechanism is called deterministic withdrawal in the sense that the amount $c_n$ is independent of the investment return and is therefore relatively deterministic. The optimal investment choice with deterministic withdrawal has...
received substantial attention in the continuous-time model. When \( \rho_n > 0 \) and \( c_n > 0 \) for \( n = 0, 1, \ldots, T-1 \), the withdrawal mechanism is called combined withdrawal. Here, we note that different withdrawal mechanisms have different values of \( \rho_n \) and \( c_n \). For example, \( c_n \) in the deterministic withdrawal mechanism is usually larger than that in the combined withdrawal mechanism. Immediately after withdrawal, a proportion \( \alpha_n \) of the remaining wealth is invested in the risky asset, and the wealth under the strategy \( \alpha \) evolves over time according to:

\[
X_{n+1}^\alpha = ((1 - \rho_n)X_n^\alpha - c_n) \left(r_n^\alpha + \alpha_n R_n^\alpha\right), \quad n = 0, 1, \ldots, T-1, 
\]

where \( R_n^\alpha = R_n - r_n^\epsilon \) is the excess return of the risky asset. Moreover, let \( r_n^\epsilon = E(R_n^\epsilon) \).

Referring to Gerrard et al. [14] and Gerrard, Haberman and Vigna [15, 16], we use a quadratic loss function to measure the deviation of the running wealth from the desired value. At any time \( n(n = 0, 1, \ldots, T-1) \), the objective of the pensioner is to minimize the expected sum of deviations from the target:

\[
\min_{\alpha_n, \ldots, \alpha_T} E_{n,x_n} \left( \sum_{k=n}^{T-1} (F_k - X_k^\alpha)^2 + \varepsilon (F_T - X_T^\alpha)^2 \right), 
\]

where \( E_{n,x_n}(\cdot) = E(\cdot | X_n = x_n) \), \( F_k \) is the investment target at time \( k \), \( k = 0, 1, \ldots, T \). In addition, the weighting factor \( \varepsilon \) can be used to reflect the different evaluations of the deviation of \( X_T^\alpha \) from the target \( F_T \) when the final deviation is considered to be more important than the previous deviations at time \( n = 0, 1, \ldots, T-1 \). The use of this loss function is not novel in the management of pension funds. For example, Boulier, Michel and Wisnia [3] and Cairns [6] also adopt this function. Compared with the CARA and CRRA utility functions, whose utility-based risk preference cannot be known clearly by the individual, the quadratic loss function is relatively intuitive because most individuals more or less know their desired future consumption or wealth targets.

To solve the optimization problem, we define the performance function \( J_n \) and the value function \( V_n \) as follows:

\[
J_n(x_n; (\alpha_n, \ldots, \alpha_{T-1})) = E_{n,x_n} \left( \sum_{k=n}^{T-1} (F_k - X_k^\alpha)^2 + \varepsilon (F_T - X_T^\alpha)^2 \right), 
\]

\[
V_n(x_n) = \min_{\alpha_n, \ldots, \alpha_{T-1}} J_n(x_n; (\alpha_n, \ldots, \alpha_{T-1})), \quad n = 0, 1, \ldots, T-1. 
\]

According to dynamic programming and Bellman’s principle of optimality, we have the recursive formula of \( V_n(x_n) \) as follow:

\[
V_n(x_n) = \min_{\alpha_n} \left\{ (F_n - x_n)^2 + E_{n,x_n} \left[ V_{n+1}(X_{n+1}^{\alpha_n}) \right] \right\}, \quad n = 0, 1, \ldots, T-1 
\]

with the boundary condition

\[
V_T(x_T) = \varepsilon (F_T - x_T)^2 = \varepsilon (x_T)^2 - 2\varepsilon F_T x_T + \varepsilon F_T^2. 
\]

In the next section, we first find the explicit expressions of the optimal investment strategy and a series of targets that the expected wealth cannot exceed. Then, we perform experiments to study the properties of the investment strategies, the targets and the bankruptcy probabilities. Additionally, the accumulated withdrawal amounts under three withdrawal mechanisms are compared. Finally, we analyze the effects of the final target and the interest rate on some obtained results.
3. Optimal investment strategy and expression of $F_n$. This section aims to derive explicit expressions for the optimal investment strategy and the targets greater than the expected wealth and then studies their properties. First, the optimal investment proportion $\alpha_n^*$ is obtained, as demonstrated in Theorem 3.1.

**Theorem 3.1.** For problem (2), the optimal investment proportion $\alpha_n^*$ is

$$\alpha_n^* = \left( \frac{B_{n+1}}{A_{n+1}((1-\rho_n)x_n-c_n)} - r_n^f \right) \frac{\sigma_n^C}{\mathbb{E}(R_n^C)^2}, \quad n = 0, 1, \ldots, T-1. \quad (5)$$

The value function $V_n(x_n)$ is given by

$$V_n(x_n) = A_n(x_n)^2 - 2B_n x_n + C_n, \quad (6)$$

where

$$A_n = A_{n+1} \frac{\text{Var}(R_n^C)}{\mathbb{E}(R_n^C)^2} (r_n^f)^2 (1-\rho_n)^2 + 1, \quad n = 0, 1, \ldots, T-1, \quad (7)$$

$$A_T = \varepsilon, \quad (8)$$

$$B_n = \frac{\text{Var}(R_n^C)}{\mathbb{E}(R_n^C)^2} r_n^f (1-\rho_n) \left( A_{n+1} r_n^f c_n + B_{n+1} \right) + F_n, \quad n = 0, 1, \ldots, T-1, \quad (9)$$

$$B_T = \varepsilon F_T, \quad (10)$$

$$C_n = C_{n+1} - \frac{(B_{n+1})^2}{A_{n+1}} + A_{n+1} \left( \frac{B_{n+1}}{A_{n+1}} + r_n^f c_n \right)^2 \frac{\text{Var}(R_n^C)}{\mathbb{E}(R_n^C)^2} + (F_n)^2, \quad (11)$$

for $n = 0, 1, \ldots, T-1,$

$$C_T = \varepsilon (F_T)^2.$$  

*Proof.* See Appendix A. □

According to (2), the quadratic loss function also penalizes the deviation above the target. Therefore, some continuous-time models, such as Di Giacinto et al. [9], Gerrard et al. [14] and Gerrard, Haberman and Vigna [16], attempt to find a target greater than the corresponding wealth level at each time to avoid the surplus penalty. However, in contrast to the continuous-time models, the discrete-time counterpart cannot find a series of targets that satisfy $F_n \geq X_n^{\alpha^*}$ for any $n = 0, 1, \ldots, T$ when the risky return $R_n$ can take an infinite value with a positive probability. In what follows, we first explain why this phenomenon occurs in the discrete-time model and then give our solution to address the problem. First, we obtain the wealth process under the optimal investment proportion (5), which is denoted by $X_n, n = 0, 1, \ldots, T$ for simplicity:

$$X_{n+1} = ((1-\rho_n)X_n - c_n) \left( r_n^f + \frac{B_{n+1}}{A_{n+1}((1-\rho_n)X_n-c_n)} - r_n^f \right) \frac{\sigma_n^C}{\mathbb{E}(R_n^C)^2}$$

$$= r_n^f ((1-\rho_n)X_n - c_n) + \left( \frac{B_{n+1}}{A_{n+1}} - r_n^f ((1-\rho_n)X_n-c_n) \right) \frac{\sigma_n^C}{\mathbb{E}(R_n^C)^2}, \quad (12)$$

$n = 0, 1, \ldots, T-1.$

When $n = T-1$, we have

$$X_T = r_{T-1}^f ((1-\rho_{T-1})X_{T-1} - c_{T-1})$$

$$+ \left( \frac{B_T}{A_T} - r_{T-1}^f ((1-\rho_{T-1})X_{T-1} - c_{T-1}) \right) \frac{\sigma_{T-1}^C}{\mathbb{E}(R_{T-1}^C)^2}. \quad (13)$$
If the risky return $R_n(n = 0, 1, \ldots, T - 1)$ can take an infinite value with a positive probability, then for any series of targets $F_0, F_1, \ldots, F_T$, regardless of the relationship between $F_k$ and $X_k$ $(0 \leq k \leq T - 1)$, we have the following conclusions. At time $T - 1$, when given wealth $x_{T-1}$, if

$$\frac{B_T}{A_T} > r^f_{T-1} \left( (1 - \rho_{T-1})x_{T-1} - c_{T-1} \right),$$

then $X_T$ can be greater than any given final target $F_T$ with a positive probability because $R^n_{T-1} = R_{T-1} - r^f_{T-1}$ can take an infinite value. Otherwise, if

$$\frac{B_T}{A_T} \leq r^f_{T-1} \left( (1 - \rho_{T-1})x_{T-1} - c_{T-1} \right),$$

then according to (13), we obtain

$$E_{T-1, x_{T-1}}(X_T) - F_T = r^f_{T-1} \left( (1 - \rho_{T-1})x_{T-1} - c_{T-1} \right) \frac{\text{Var}(R^n_{T-1})}{E(R^n_{T-1})^2} + \frac{B_T}{A_T} \frac{(r^f_{T-1})^2}{E(R^n_{T-1})^2} - F_T \geq \frac{B_T}{A_T} - F_T = 0.$$

When $E_{T-1, x_{T-1}}(X_T) \geq F_T$ and $X_T$ is not deterministic, the target $F_T$ cannot be greater than $X_T$ with probability one. Consequently, under the assumption that the risky return $R_n(n = 0, 1, \ldots, T - 1)$ can take an infinite value with a positive probability, we cannot find the targets without the surplus penalty. Here, it should be noted that the assumption about the risky return is not exaggerated. In most of the literature, for example, Cheung and Yang [7] and Hardy [18], the risky return is assumed to be log-normally distributed. To address this problem, we tend to find a target greater than the expected wealth at each time. Although this technique cannot avoid the penalty of surplus from the perspective of the mathematical analysis, we justify this approach as follows. Avoidance of the penalty of surplus is not the only concern in a target-based model with a quadratic loss function. It is natural for retirees to reduce deviation below the target, but sometimes deviation above the target is also considered risky and is not worthy of excessive pursuit. Usually, to derive a wealth level greater than the target, the pensioner will invest a large amount of wealth in the risky asset. However, this behavior may make the pensioner suffer a great loss. Therefore, the targets can be taken as the rational choice for the desired wealth, and any deviation from the target deserves equal attention. That is why Gerrard, Haberman and Vigna [15] adopt the price of a level annuity as the target, even though this type of target might have a penalty of surplus. In conclusion, some reasonable targets, even those with a penalty of surplus, are justified. In our model, we adopt a series of targets that the expected wealth cannot exceed. In fact, these targets have some similar properties to those without a penalty of surplus, which is shown in Theorem 3.4. It follows from Theorem 3.4 that once the target is reached at time $n$, the pensioner will invest all the available wealth in the risk-free asset for the remaining periods, and the fund is equal to the target since then. In addition, although the running wealth might exceed the target from the perspective of the mathematical analysis under our setting, a numerical experiment in Section 4 indicates that this probability is almost zero. In other words, the event that the running wealth deviates from its normal trajectory is abnormal. This experiment, in a way or to a certain extent, supports the assumption that the penalty of surplus
is sometimes necessary. Consequently, the choice of the quadratic loss function is justified, even if a target strictly greater than the fund cannot be derived in the discrete-time model.

Next, we find an explicit expression for \( F_n \), which the expected wealth cannot exceed. As mentioned above, this type of target can be regarded as a rational choice for the desired wealth. However, a more important reason to adopt \( F_n \) is that \( F_n \) is the natural target or the safety level for the pensioner. Specifically, when a retiree takes \( F_n \) as a target, he can achieve the target from time \( m > n \) once the target is reached at time \( n \). This conclusion will be seen shortly in Theorem 3.4. Interested readers are referred to Gerrard, Haberman and Vigna [16] for more information about the “natural target” and the “safety level”.

According to (12), we obtain

\[
E_{0,x_0}(X_{n+1}) - \frac{B_{n+1}}{A_{n+1}} = \left[ r_n^f ((1 - \rho_n)E_{0,x_0}(X_n) - c_n) - \frac{B_{n+1}}{A_{n+1}} \right] \frac{\text{Var}(R_n^e)}{E(R_n^e)^2}. \tag{14}
\]

If \( F_n = B_n/A_n \), then according to (7) and (9), we have

\[
F_{n+1} = \frac{B_{n+1}}{A_{n+1}} = \frac{1}{A_{n+1}} \left( \frac{B_n - F_n}{r_n^f(1 - \rho_n)} \frac{E(R_n^e)^2}{\text{Var}(R_n^e)} - A_{n+1}r_n^f c_n \right)
= \frac{r_n^f(1 - \rho_n) (B_n - F_n)}{A_n - 1} - r_n^f c_n
= \frac{r_n^f ((1 - \rho_n)F_n - c_n)}. \tag{15}
\]

Substituting (15) into (14) leads to

\[
E_{0,x_0}(X_{n+1}) - F_{n+1} = r_n^f (1 - \rho_n) \frac{\text{Var}(R_n^e)}{E(R_n^e)^2} (E_{0,x_0}(X_n) - F_n)
= (x_0 - F_n) \prod_{k=0}^{n} r_k^f (1 - \rho_k) \frac{\text{Var}(R_k^e)}{E(R_k^e)^2}. \tag{16}
\]

It follows from (17) that if \( x_0 < F_0 \), then \( E_{0,x_0}(X_n) < F_n \), \( n = 0, 1, \ldots, T \), which indicates that the expected wealth at each time \( n \) never exceeds the target \( F_n \) in (15). In reality, it is natural for a pensioner to have a terminal target at the age of compulsory annuitization. In the following, we assume that the terminal target \( F_T \) is predetermined; then, the explicit expressions for \( F_n \), \( n = 0, 1, \ldots, T - 1 \) are obtained by the recursive formula (15):

\[
F_n = F_T \prod_{k=n}^{T-1} \frac{1}{r_k^f (1 - \rho_k)} + \sum_{k=n}^{T-1} \frac{c_k}{1 - \rho_k} \prod_{m=n}^{k-1} \frac{1}{r_m^f (1 - \rho_m)}. \tag{18}
\]

Referring to (18), Eq. (17) can be written as

\[
F_{n+1} - E_{0,x_0}(X_{n+1}) = \left[ F_T \prod_{k=0}^{T-1} \frac{1}{r_k^f (1 - \rho_k)} + \sum_{k=0}^{T-1} \frac{c_k}{1 - \rho_k} \prod_{m=0}^{k-1} \frac{1}{r_m^f (1 - \rho_m)} - x_0 \right] \prod_{k=0}^{n} r_k^f (1 - \rho_k) \frac{\text{Var}(R_k^e)}{E(R_k^e)^2}. \tag{19}
\]

In view of (18) and (19), the expected wealth at each time can be written as

\[
E_{0,x_0}(X_{n+1}) = F_T \left( 1 - \prod_{k=0}^{n} \frac{\text{Var}(R_k^e)}{E(R_k^e)^2} \right) \prod_{k=n+1}^{T-1} \frac{1}{r_k^f (1 - \rho_k)}.
\]
First, we investigate how the terminal target $F_T$ affects the expected wealth at each time and the deviation of the expected wealth from the target. (19) shows that a higher terminal target $F_T$ leads to a larger deviation of the expected wealth from the target. (20) indicates a higher terminal target $F_T$ results in a higher expected wealth level over time regardless of the withdrawal mechanism, which is an expected result. A higher terminal target $F_T$ has a larger target $F_n$, $n = 0, 1, \ldots, T - 1$ and therefore a higher expected wealth level because our model is target-oriented. In addition, $E_{0,x_0}(X_{n+1})$ under the proportional withdrawal mechanism is the most sensitive to $F_T$ because this mechanism has a largest withdrawal proportional $\rho_n$ and the derivative with respect to $F_T$ increases along with $\rho_n$. Second, some properties of $F_n$ are studies. It follows from the expression of $F_n$ that the target is independent of the risky return. It depends on the final target $F_T$, the withdrawal proportion $\rho_n$, the deterministic withdrawal amount $c_n$ and the risk-free return $r_n^f$. This phenomenon is reasonable. A retiree usually determines his current desired wealth level based on future consumption and the final target. He would also consider the impact of the interest rate but usually would not take into account the future performance of the stock market. Moreover, according to (18), a larger $F_T$ and larger $c_k, \rho_k$, $k = n, n + 1, \ldots, T - 1$ in the future can lead to a larger current target $F_n$. Meanwhile, higher risk-free returns $r_k^f$, $k = n, n + 1, \ldots, T - 1$ result in smaller $F_n$. We explain this as follows. On the one hand, if a greater amount is expected to be withdrawn in the future, then one will set a higher target $F_n$ at the current time to prevent the worst-case scenario where the fund cannot pay the pension. On the other hand, the risk-free return is usually regarded as the discount factor, with the consequence that higher risk-free returns have lower discount values of $F_T$ and $c_k$. We continue by analyzing some properties of $F_n$. According to (16), if $r_n^f(1 - \rho_n)Var(R_n^e)/E(R_n^e)^2 < 1$, then

$$F_{n+1} - E_{0,x_0}(X_{n+1}) < F_n - E_{0,x_0}(X_n).$$

(21)

According to $E_{0,x_0}(X_n) < F_n$, $n = 0, 1, \ldots, T$ and (21), we have the following conclusions.

(i) If the initial wealth $x_0$ is less than the initial target $F_0$, then the expected wealth at any time $n$ is no more than the corresponding target $F_n$. Specifically, if $E_{0,x_0}(X_n) = F_n$, then the expected wealth at any future time $k = n + 1, \ldots, T$ is also equal to its corresponding target.

(ii) When

$$r_n^f(1 - \rho_n)Var(R_n^e)/E(R_n^e)^2 < 1, \ n = 0, 1, \ldots, T - 1,$$

(22)

the distance between the expected wealth and the target declines over time. It is not difficult to achieve condition (22) in the real world. Specifically, when $\rho_n = 0$, the left-hand side of (22) achieves its maximum value, and (22) can be rewritten as

$$r_n^f - 1 < \left(\frac{r_n^e}{\sqrt{Var(R_n^e)}}\right)^2,$$
which means that the one-period interest rate of the risk-free asset is less than the squared value of the expected excess return per unit of risk.

(iii) As mentioned above, at time $n$, the withdrawal amount is a function of the available wealth, i.e., $\rho_n X_n + c_n$. Based on the values of $\rho_n$ and $c_n$, there are three types of withdrawal mechanism. For convenience, deterministic withdrawal, proportional withdrawal and combined withdrawal are, respectively, denoted by $(\rho_{n,1} = 0, c_{n,1})$, $(\rho_{n,2}, c_{n,2} = 0)$ and $(\rho_{n,3}, c_{n,3})$. Generally, the proportion $\rho_{n,2}$ is greater than $\rho_{n,3}$ because in combined withdrawal, regardless of the investment performance, there is still an amount $c_{n,3}$ that can be withdrawn from the fund for daily life. As a result, according to condition (22), the commonly used deterministic withdrawal mechanism has the largest values.

**Theorem 3.2.** $F_n > F_n \rho_n + c_n$, $n = 0,1,\ldots,T - 1$.

*Proof.* See Appendix B. □

The term $F_n \rho_n + c_n$ is the target amount withdrawn from the fund at time $n$ for the daily survival of the retiree. In this sense, Theorem 3.2 indicates that if the wealth level $X_n$ is near to the target $F_n$, the pensioner will not spend in excess of his income. Next, we consider the trend of $F_n$ over time, which is summarized in the following theorem.

**Theorem 3.3.** If $r^f(1 - \rho_n) < 1$, then $F_{n+1} < F_n$. When $r^f$, $\rho_n$ and $c_n$ are constant, $F_{n+1} < F_n$, $n = 0,1,\ldots,T - 1$ holds if and only if $F_T (r^f(1 - \rho) - 1) < cr^f$.

*Proof.* See Appendix C. □

The term $r^f(1 - \rho_n)$ in Theorem 3.3 can be explained as the actual risk-free return after withdrawal according to the proportion $\rho_n$. Theorem 3.3 indicates that, when the actual risk-free return is less than 1, the target wealth is declining over time. When $r^f$, $\rho_n$ and $c_n$ are constant, by (18), we derive

$$F_n (r^f(1 - \rho) - 1) - cr^f = \frac{F_T (r^f(1 - \rho) - 1) - cr^f}{(r^f(1 - \rho))^{T-n}}.$$  \hspace{1cm} (23)

Consequently, $F_T (r^f(1 - \rho) - 1) < cr^f$ is equivalent to $F_n (r^f(1 - \rho) - 1) < cr^f$, $n = 0,1,\ldots,T$, which means that the one-period interest of the current target $F_n$ is less than the one-period return of the constant withdrawal $c$. Under these circumstances, the target is decreasing over time.

In what follows, we analyze the investment strategy. Denote by $\pi_n^*$ the investment amount; then, we have

$$\pi_n^* = \alpha_n^* ((1 - \rho_n)x_n - c_n) = r^f_n (1 - \rho_n) \frac{r^c_n}{E(R^c_n)^{r^c_n}} (F_n - x_n).$$ \hspace{1cm} (24)

First, it follows from (24) that if $F_n > x_n$, then a larger deviation between $F_n$ and $x_n$ leads to a larger amount of wealth invested in the risky asset, with the aim of returning the running wealth close to the target as soon as possible. Otherwise, if $F_n < x_n$, then the deviation above the target becomes a warning to the retirees, and the risky asset will be shorted to reduce the running wealth to a rational level. When
Proof. See Appendix D.

Taking the expectation of both sides of (24) gives
\[
E_{0,x_0}(\pi^*_n) = \frac{r_f^n(1 - \rho_n)\pi^*_n}{E(R^n_0)^2}(F_n - E_{0,x_0}(X_n)).
\]

According to (17), when \( x_0 < F_0 \), \( E_{0,x_0}(X_n) < F_n, \ n = 0, 1, \ldots, T \), which leads to \( E_{0,x_0}(\pi^*_n) > 0 \). Therefore, when the initial wealth is less than the initial target, the pensioner will not short sell the risky asset on average. Up to this point, through a mathematical analysis, we have obtained some basic properties about the target \( F_n \) and the optimal strategy. In the following, we aim to compare three types of targets and strategies under different withdrawal mechanisms via numerical analysis.

4. Numerical analysis. To conduct the simulations, we assume that the pensioner retires at the age of 60 and that the compulsory age for annuitization is 75. Furthermore, assume that \( x_0 = 150000 \), the pensioner withdraws an amount from the fund monthly, the gross monthly return of the risk-free asset is a constant \( r_f = 1.0025 \) and the monthly return of the risky asset \( R_n \) has a log-normal distribution with mean 0.005 and standard deviation 0.2. The proportions and the constant amounts in deterministic withdrawal, proportional withdrawal and combined withdrawal are denoted by \( (\rho_1(=0), c_1) \), \( (\rho_2, c_2(=0)) \) and \( (\rho_3, c_3) \), respectively. To enable comparison of the results in this paper under different withdrawal mechanisms, we assume that these three mechanisms have the same initial information and final target. Let \( \bar{a}_{60} \) be the price of the annuity calculated by the life table of the USA\(^1\) in 2013. Then, \( x_0/\bar{a}_{60} \) is the yearly withdrawal amount and \( c_1 = x_0/\bar{a}_{60}/12 \) is the monthly withdrawal amount. To have the same withdrawal amount at time 0, let \( \rho_2 = c_1/x_0, c_3 = c_1/2 \) and \( \rho_3 = (c_1 - c_3)/x_0 \). In addition, the terminal targets take the same value \( F_T = 1.5(x_0/\bar{a}_{60})\bar{a}_{75} \). The reason for a pensioner to defer annuitization is the hope of being able to buy a better annuity at terminal time \( T \), so we assume that the final target helps the pensioner to receive a higher retirement income after the time of compulsory annuitization, which is approximately 1.5 times of the initial yearly withdrawal amount \( x_0/\bar{a}_{60} \). The following subsections focus on the differences in the targets, the running wealth and the strategies among the three withdrawal mechanisms.

4.1. Properties of the targets \( F_n \) and the wealth \( X_n \). In this subsection, the first experiment illustrates the targets over time under different withdrawal mechanisms. Here, we adopt the life table of the USA in 2013 to calculate the constant amount \( c_1 \) that an annuity at retirement would provide each month. We obtain the value \( c_1 = 754.6 \) after calculation, and the other parameters are \( \rho_2 = 0.5\%, \ c_3 = 377.3, \ \rho_3 = 0.25\% \) and \( F_T = 141411 \). In the following, we conduct three experiments to illustrate the properties of the targets and the running wealth.

Experiment 1. The trends of the targets \( F_n, \ n = 0, 1, \ldots, T \)

\(^1\)data from: http://www.mortality.org/
In this experiment, we aim to find the differences in the targets over time among the three withdrawal mechanisms. The resulting trends of $F_n$ from $n = 0$ to $T = 15 \times 12 = 180$ are shown in Figure 1, which indicates that:

(i) All the targets are declining over time. Additionally, $F_T (r^f (1 - \rho) - 1) - cr^f < 0$, $i = 1, 2, 3$, which means the results obtained by numerical analysis coincide with Theorem 3.3, and the explanation behind this phenomenon has been given below Theorem 3.3.

(ii) The commonly used deterministic withdrawal mechanism with $\rho_1 = 0$ has the smallest value of $F_n$ in the first approximately 11 years but the largest value of $F_n$ in the last approximately four years. The value of $F_n$ in combined withdrawal is always between those in deterministic withdrawal and proportional withdrawal, which can be explained as follows. According to formula (35) in Appendix C, the decreasing speed $(r^f (1 - \rho))^{n-T} (cr^f - F_T (r^f (1 - \rho) - 1))$ in one period is increasing with respect to $c$ and $\rho$. However, deterministic withdrawal mechanism has the smallest withdrawal proportion and the largest withdrawal amount, so the decreasing speed is a trade-off between the withdrawal amount $c$ and the withdrawal proportional $\rho$. With our parameter settings, the speed $(r^f (1 - \rho))^{n-T} (cr^f - F_T (r^f (1 - \rho) - 1))$ in deterministic(proportional) withdrawal has the smallest(largest) value initially but the largest(smallest) value in the later periods. Consequently, we obtain the trends of the targets shown in Figure 1.

(iii) The distances among these three targets decrease as the terminal time approaches because the three withdrawal mechanisms have the same final target $F_T$.

In addition to the trends of the targets, many individuals also care about the distance between the target and the running wealth, which is presented in Experiment 2.

**Experiment 2. What is the distance between the target and the wealth?**

In this experiment, we analyze the distance between the target $F_n$ and the expected wealth $E_{0,x_0}(X_n)$. We first find from Figure 2 that regardless of the withdrawal mechanism, the distance between the target and the expected wealth decreases over time, which means the target-oriented model can help retirees to reach their final investment target with high possibility. Second, the expected terminal wealth in the commonly used withdrawal mechanism, i.e., deterministic withdrawal...
mechanism, has the maximum deviation from the terminal target \( F_T \). More specifically, \( F_T - E_{0,x_0}(X_T) \) are, respectively, equal to 8900, 5313 and 6840 for deterministic withdrawal, proportional withdrawal and combined withdrawal. In this sense, the proportional withdrawal mechanism has an advantage over the other withdrawal mechanisms in guaranteeing that the final target is achieved.

![Graphs showing deterministic, proportional, and combined withdrawal mechanisms](image)

(a) Deterministic withdrawal mechanism  (b) Proportional withdrawal mechanism  (c) Combined withdrawal mechanism

**Figure 2.** The targets and the expected wealths over time

Now, we have analyzed the distance between the target and the wealth level from the viewpoint of the expectation. However, the probability that the terminal wealth \( X_T \) is within a neighborhood of \( F_T \) is also a principle concern of the pensioner. Most retirees may ask the question: How high is the probability that the terminal wealth is within a neighborhood of the preset targets? Here we raise one numerical example to answer this question. To this end, we produce 10000 sample values of \( X_T \) to obtain the frequencies of the various events in Table 1. For simplicity, in the forthcoming Tables 1-5, deterministic withdrawal, combined withdrawal and proportional withdrawal are, respectively, abbreviated to “deter. withdrawal”, “comb. withdrawal” and “prop. withdrawal”.

As shown in Figure 2, the expected wealth \( E_{0,x_0}(X_T) \) at terminal time \( T \) has the minimum deviation from the corresponding target compared with other wealth levels at time \( n = 0, 1, \ldots, T - 1 \). In this sense, the terminal wealth \( X_T \) is closest to \( F_T \) and has the highest probability of being larger than \( F_T \), i.e., being subject to the penalty of surplus. However, Table 1 shows that in the 10000 samples of \( X_T \), none are larger than the target \( F_T \). This suggests that it is more difficult for the other
Table 1. Frequencies of some events that $X_T$ is in the neighborhood of $F_T$

| $X_T \in [a, b)$ | Deter. withdrawal Frequencies | Comb. withdrawal Frequencies | Prop. withdrawal Frequencies |
|------------------|--------------------------------|-----------------------------|-----------------------------|
| $(-\infty, F_T - 6000)$ | 0.3024 | 0.2472 | 0.1985 |
| $[F_T - 6000, F_T - 4000)$ | 0.0997 | 0.0885 | 0.0791 |
| $[F_T - 4000, F_T - 2000)$ | 0.1811 | 0.1801 | 0.1724 |
| $[F_T - 2000, F_T)$ | 0.4168 | 0.4842 | 0.5500 |
| $[F_T, +\infty)$ | 0 | 0 | 0 |

wealth levels $X_n, n = 1, 2, \ldots, T - 1$ to be higher than their corresponding targets. In summary, the penalty of surplus under our target settings is a small-probability event. Moreover, Table 1 shows that the proportional withdrawal mechanism has the highest frequency that $X_T$ is within the interval $[F_T - d, F_T)$. For example, under the mechanism of proportional withdrawal, $X_T$ is within $[F_T - 2000, F_T)$ and $[F_T - 4000, F_T)$ at the highest frequencies of 55% and 72.24%, respectively. Therefore, proportional withdrawal mechanism gives the highest probability of being within the neighborhood of $F_T$.

Experiment 3. Can the running wealth guarantee the withdrawal?

In addition to the distance of $X_T$ from $F_T$, the pensioner also wants to know whether the investment can guarantee the withdrawal, i.e., whether the wealth is greater than the withdrawal amount. To this end, we want to determine the first time and the total number of times the wealth level is lower than the withdrawal amount during the entire investment period. Let $\tau$ be the first time the wealth is below the withdrawal amount, i.e., the first time bankruptcy occurs, and denote by $S$ the total number of bankruptcies in the time series $n = 0, 1, 2, \ldots, 180$. Then, we produce 10000 sample trajectories of the wealth processes $\{X_n, n = 0, 1, 2, \ldots, 180\}$ and analyze the properties of $\tau$ and $S$. The results are presented in Tables 2-3.

Table 2. The total number of bankruptcies–10000 simulations

| Events | Deter. withdrawal Number of simulations | Comb. withdrawal Number of simulations | Prop. withdrawal Number of simulations |
|--------|----------------------------------------|---------------------------------------|---------------------------------------|
| $S = 0$ | 9692 | 9726 | 9729 |
| $S \in (0, 50]$ | 265 | 247 | 246 |
| $S \in (50, 100]$ | 39 | 23 | 22 |
| $S \in (100, 150]$ | 4 | 4 | 3 |
| $S \in (150, +\infty)$ | 0 | 0 | 0 |
| Mean of $S$ | 0.7341 | 0.5870 | 0.5060 |

Table 2 shows that the bankruptcy is a rare event in the three withdrawal mechanisms. In the 10000 simulations, bankruptcy does not occur in more than 9692 simulations, and the expected number of bankruptcies is less than 1 of the total 180 decision points. This phenomenon may be attributed to the target orientation of our model. The pensioners try their best to reduce the deviation of wealth from the target at any time, leading to wealth being close to the target with high probability. Consequently, the wealth is seldom below the withdrawal amount which is much smaller than the target. In addition, the proportional withdrawal mechanism has
the smallest expected number of bankruptcies while the deterministic withdrawal mechanism has the largest one. We believe this result is reasonable. In proportional withdrawal, the withdrawal amount is proportional to the wealth level, so bankruptcy occurs only when the wealth level is less than zero. However, in deterministic withdrawal mechanism, the pensioner goes bankrupt when the wealth level is below the constant withdrawal amount.

Table 3. The first occurrence of bankruptcy

| Events          | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|-----------------|-------------------|------------------|-----------------|
| $\tau \in [0, 60)$ | 100               | 117              | 153             |
| $\tau \in (60, 120)$ | 142              | 118              | 92              |
| $\tau \in [120, 180]$ | 66               | 39               | 26              |
| Mean of $\tau$  | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|                 | 86                | 76               | 65              |

For the three withdrawal mechanisms, Table 2 shows that there are respectively 308, 274 and 271 simulations in which bankruptcy occurs. When the bankruptcy does occur, we calculate the first time the bankruptcy happens in the whole investment period. Table 3 shows that there are 21.43% (66/308), 14.23% (39/274) and 9.59% (26/271) chances of bankruptcy in the last five years of deferring annuitization for deterministic withdrawal, combined withdrawal and proportional withdrawal. Furthermore, the expected first time of bankruptcy under the deterministic withdrawal mechanism is roughly the 7th year of the investment period and the 6th year and the 5th year for the other two mechanisms. Consequently, a pensioner under deterministic withdrawal is last to go bankruptcy. These results, together with the conclusions obtained in Figure 2 and Table 1, show that on the one hand, deterministic withdrawal has the largest distance from the final target $F_T$ and the largest number of bankruptcies; on the other hand, it is last mechanism to reach bankruptcy. Therefore, no withdrawal mechanism has the absolute advantage.

4.2. Properties of the investment strategies. In this subsection, we study the properties of the investment strategies and compare the strategies under three withdrawal mechanisms. It first follows from (25) that, on average, a pensioner will not short sale the risky asset regardless of the withdrawal mechanism. To obtain more properties of $\pi_n^*$ in (24), we conduct the following experiments with the same parameter settings as above.

First, we study the relationship of the expected remaining wealth after withdrawal $E_{0,x_0}((1 - \rho_n)X_n - c_n)$ and the expected investment amount $E_{0,x_0}(\pi_n^*)$ under three withdrawal mechanisms; the results are demonstrated in Figure 3. In Figure 3, $E_{0,x_0}(\pi_n^*)$ is larger than 0 and less than the expected wealth throughout the investment process. Therefore, on average, the pensioner will not short sale the risk-free and risky assets when a target-oriented investment model is adopted. This experiment is conducted to study the relationship of $E_{0,x_0}((1 - \rho_n)X_n - c_n)$ and $E_{0,x_0}(\pi_n^*)$ over time. Next, we perform 10000 simulations to determine the relationship between $(1 - \rho_n)X_n - c_n$ and $\pi_n^*$. With our parameter settings, we first obtain that $\pi_n^* > 0$ for all $n = 0, 1, \ldots, T - 1$ and all simulations, which indicates that if the pensioner wants to guarantee the withdrawal amount every month, he will always invest his wealth in the risky asset. Second, the average number of times and the average first time that the optimal investment amount is greater than the
remaining wealth are calculated. Denote by $\hat{\tau}$ and $\hat{S}$ the first time and the total number of times that $\pi^*_n > X_n$, $n = 0, 1, \ldots, T - 1$; the results are shown in Table 4.

**Table 4.** The mean of $\hat{\tau}$ and $\hat{S}$—10000 simulations

|              | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|--------------|-------------------|------------------|------------------|
| Mean of $\hat{\tau}$ | 64.4634           | 51.1655          | 41.9057          |
| Mean of $\hat{S}$     | 24.1194           | 21.6262          | 18.9368          |

In view of Table 4, for deterministic withdrawal, combined withdrawal and proportional withdrawal, the investment amount is greater than the remaining wealth, respectively, in 24.1194, 21.6262 and 18.9368 of the $15 \times 12 = 180$ investment decisions, on average. These three numbers indicate that none of the investment behaviors are aggressive because no more than $24.1194/180 \approx 13.4\%$ decisions result in a short sale of the risk-free asset. In addition, the investment decision with proportional withdrawal is the most conservative. In the 180 decisions over time, only 18.9368 decisions borrow money from the bank for the risky investment. We explain this as follows. Under the mechanism of proportional withdrawal, the withdrawal amount fluctuates with the wealth level. The less the wealth is, the less the withdrawal amount. As a result, the decision maker under the proportional withdrawal mechanism has a greater sense of security and invests wealth in the risky
asset more conservatively. In terms of the first time a short sale of the risk-free asset occurs, Table 4 shows that, on average, proportional withdrawal results in the earliest short sale at approximately \( n = 42 \). According to (24), we know that the larger the deviation of \( X_n \) from \( F_n \) is, the larger the amount invested in the risky asset. From Figure 2, it is clear that the average distance between \( F_n \) and \( X_n \) under the mechanism of proportional withdrawal is initially the largest, so the decision maker will short sale the risk-free asset as soon as possible to reduce the deviation. This behavior, to some degree, also explains why the pensioner with proportional withdrawal is the first one to go bankrupt in Table 3.

4.3. Properties of the withdrawal amount. We care about not only the deviation of the wealth from the target and the bankruptcy frequency, but also the total amount that can be obtained from time 0 to time \( T \). This subsection compares three types of expected withdrawal amounts throughout the whole investment period. To this end, let \( AWA_n \) be the random accumulated withdrawal amounts up to time \( n \) and \( WA_n \) be the random withdrawal amount at time \( n \).

More specifically, \( AWA_n = \sum_{k=0}^{n} (X_k \rho_k + c_k) \) and \( WA_n = X_n \rho_n + c_n \). Furthermore, denote by \( AWA_{n,1}, AWA_{n,2} \) and \( AWA_{n,3} \) the accumulated amount corresponding to deterministic withdrawal, combined withdrawal and proportional withdrawal, respectively, and let \( WA_{n,1}, WA_{n,2} \) and \( WA_{n,3} \) be the amounts corresponding to deterministic withdrawal, combined withdrawal and proportional withdrawal, respectively. Referring to (17), i.e.,

\[
E_{0,x_0}(X_n) = F_n + (x_0 - F_0) \prod_{k=0}^{n-1} r_k (1 - \rho_k) \frac{\text{Var}(R_k)}{\text{E}(R_k)^2}, \quad n = 0, 1, \ldots, T \quad (26)
\]

where \( F_n \) is in (18), we can derive the explicit expressions for \( E_{0,x_0}(AWA_{n,i}) \) and \( E_{0,x_0}(WA_{n,i}), i = 1, 2, 3 \).

![Figure 4. Accumulated withdrawal amounts over time](image)

Table 5. Accumulated withdrawal amounts and their relative changes

| Time | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|------|-------------------|-----------------|-----------------|
| \( n = 36 \) | 927,922           | 928,338         | 929,112         |
| \( n = 72 \) | 555,088 (27166)   | 556,223 (27885) | 558,294 (29182) |
| \( n = 108 \) | 822,255 (27167)   | 838,865 (27642) | 866,875 (28851) |
| \( n = 144 \) | 109,422 (27167)   | 110,911 (27046) | 114,109 (27234) |
| \( n = 180 \) | 136,589 (27167)   | 137,134 (26223) | 139,640 (25531) |
Referring to (26) and maintaining the assumption about the parameter settings in this section, we present the trends of $E_{0,x_0}(AW_{A_{n,1}})$, $i = 1, 2, 3$ in Figure 4 and the expected accumulated withdrawal amounts at the end of the 3rd year, the 6th year, the 9th year, the 12th year and the 15th year in Table 5. Figure 4 indicates that, on average, the proportional withdrawal mechanism has an advantage over the other mechanisms in guaranteeing the largest accumulated withdrawal amount at each time point. We are also interested in the relative change$^2$ of the accumulated withdrawal amount over time in each withdrawal mechanism, which is shown in Table 5 where the value in a parenthesis is the relative change of the expected accumulated withdrawal amount. Table 5 indicates that the relative change in deterministic withdrawal mechanism is very stable and declines over time in the other two mechanisms. In addition, most of the time, proportional withdrawal mechanism has the largest variation of the withdrawal amount. We explain these phenomena as follows. The withdrawal amount in the deterministic withdrawal mechanism is independent of the wealth, so the relative change is not changeable in this withdrawal. However, the other accumulated withdrawal amounts depend on the expected wealth level. It follows from Figure 2 that the expected wealth in proportional withdrawal mechanism is the most volatile, leading to the most volatile relative change of the accumulated withdrawal amount.

In addition to the average value of the accumulated withdrawal amount up to time $n$, we also care about the expected withdrawal amount at time $n$. Although the accumulated withdrawal amount up to time $n$ in the proportional withdrawal is the largest, Figure 5 indicates that the withdrawal amount obtained at time $n$ with the proportional withdrawal mechanism is the smallest in the last approximately 4 years. By contrast, the withdrawal amount at time $n$ for deterministic withdrawal becomes the largest at approximately the end of the 11th year of the deferring annuitization period. Figure 5 also shows that the withdrawal amount at each time for the proportional withdrawal mechanism is the most volatile. It increases for approximately 4 years and then decreases until the terminal time, and the difference between the highest and the lowest points is approximately 120. Most people may not be willing to accept a decreasing withdrawal amount over time, but our study shows that the proportional withdrawal actually has the largest accumulated withdrawal amount up to any time $0 < n \leq T$.

$^2$The relative change is defined as the current value minus the previous value.
The previous experiments are conducted under the assumption that the interest rate and the targets are fixed. In the following, we aim to study the distinct effects of the interest rate and the final target on some obtained results.

4.4. Effects of the interest rate and the final target. This subsection analyzes the effects of the interest rate and the final target on the results obtained in the previous subsections. We choose to investigate the interest rate and the final target because the interest rate is a key financial index in a country and has significant impacts on the decision making and the targets at each time. Meanwhile, the target is affected not only by the financial environment but also by the individual valuation, so we also want to determine how the results change along with the final target. Due to space limitations, we study the effects of the interest rate and the target on only the most important issues: the bankruptcy probability, the accumulated withdrawal amount, and the distance between $F_n$ and $E_{0,x_0}(X_n)$. We start with the effects of the interest rate. To this end, we let the monthly interest rate change from 1.002 to 1.003 with a step size of 0.0005 while the other parameters are kept the same as above.

First, the distances between $F_n$ and $E_{0,x_0}(X_n)$ are shown in Figure 6 from which we see that a higher interest rate results in a lower distance between the target and the expected wealth at the beginning of the investment period and then leads to a higher distance until the end of the investment. The inflection points are roughly $n = 60, 40$ and 20 under the mechanisms of deterministic withdrawal, combined
withdrawal and proportional withdrawal, respectively. Moreover, the deterministic withdrawal mechanism is the most sensitive to the interest rate, followed by combined withdrawal and proportional withdrawal. Therefore, if the proportional withdrawal mechanism is adopted, when the interest rate is changed, we can avoid large fluctuation in $F_n - E_{0,x_0}(X_n)$.

Second, the effect on the expected number of bankruptcy is considered. Table 6 indicates that the bankruptcy remains a small-probability event even when the yearly return of the risk-free asset is increased from $1.002^{12} = 1.0243$ to $1.003^{12} = 1.0366$. Table 6 shows that for all the withdrawal mechanisms, a higher risk-free return results in a lower mean number of the bankruptcies in the total $15 \times 12 = 180$ decision times. This phenomenon is reasonable because a higher risk-free return results in higher investment returns when all the other parameters are fixed, leading to a lower probability of bankruptcy.

| $r^f$ | Events | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|-------|--------|-------------------|------------------|-----------------|
| 1.0020 | $S = 0$ | 9672 | 9697 | 9701 |
| Mean of S | 0.8268 | 0.6933 | 0.6123 |
| 1.0025 | $S = 0$ | 9692 | 9726 | 9729 |
| Mean of S | 0.7341 | 0.5870 | 0.5060 |
| 1.0030 | $S = 0$ | 9690 | 9724 | 9718 |
| Mean of S | 0.6565 | 0.5304 | 0.5001 |

Finally, the effect of the risk-free return on the accumulated withdrawal amount up to time $n$, $n = 0, 1, \ldots, T$ is studied. The results are illustrated in Figure 7, from which we can see that the risk-free return has a positive effect on the expected accumulated withdrawal amount. This result is expected because a higher risk-free asset results in a lower price of the annuity and then a larger amount of the deterministic withdrawal amount $c_1$ each month. Moreover, according to $\rho_2 = c_1/x_0$, $c_3 = c_1/2$ and $\rho_3 = (c_1 - c_3)/x_0$, a larger $c_1$ yields larger $\rho_2$, $\rho_3$ and $c_3$. Consequently, the higher obtained wealth level resulting from the higher risk-free return produces the results in Figure 7. In addition, we also want to know which withdrawal mechanism is more sensitive to the fluctuation of the interest rate. To this end, we compare the growth ranges of the expected accumulated withdrawal amounts when $r^f$ is increased. The growth ranges of the accumulated withdrawal amounts when $r^f$ is increased from $1.002$ to $1.0025$ and from $1.0025$ to $1.003$ are presented in Figure 8(a) and Figure 8(b), respectively. In Figure 8, proportional withdrawal is the most sensitive to the interest rate while deterministic withdrawal is the least sensitive.

In the following, we investigate the effects of the terminal target $F_T$. By (20), it is clear that a higher terminal target $F_T$ results in a higher expected wealth level over time regardless of the withdrawal mechanism. In terms of the effect of $F_T$ on the accumulated withdrawal amount, its definition $E_{0,x_0}(AWA_n) = \sum_{k=0}^{n} [\rho_k E_{0,x_0}(X_k) + c_k]$ and (20) show that: (a) the accumulated withdrawal amount is not affected by $F_T$ under the deterministic withdrawal mechanism; (b) For the proportional and combined withdrawal mechanisms, a larger $F_T$ leads to higher accumulated withdrawal amounts over time. In addition, the proportional withdrawal mechanism is more sensitive to $F_T$. In order to study the effect of $F_T$ on the bankruptcy, we have to fall back on the numerical analysis. To this end, the final target
Figure 7. Expectation of the accumulated withdrawal amount over time

(a) Deterministic withdrawal mechanism  
(b) Proportional withdrawal mechanism  
(c) Combined withdrawal mechanism

Figure 8. Growth ranges of the accumulated withdrawal amounts

\( F_T \) is allowed to increase from \( 1.4(x_0/\bar{a}_{60})\bar{a}_{75} \), \( 1.5(x_0/\bar{a}_{60})\bar{a}_{75} \) to \( 1.6(x_0/\bar{a}_{60})\bar{a}_{75} \) and we perform 10000 simulations. Then, the obtained results are illustrated in Table 7.

Table 7 shows that the larger the terminal target \( F_T \) is, the higher expectation of the total number of bankruptcies, which is reasonable because a higher \( F_T \) can lead the retiree to invest more remaining wealth in the risky asset, resulting in greater fluctuations in the wealth level and increasing the total number of bankruptcies.
Table 7. The total number of bankruptcies–10000 simulations

| $F_T = 1.4(x_0/\bar{a}_{60})\bar{a}_{75}$ | Events | Deter. withdrawal | Comb. withdrawal | Prop. withdrawal |
|------------------------------------------|--------|-------------------|------------------|-----------------|
| $S = 0$ Mean of $S$ | 9771 simulations | 0.4555 | 9820 simulations | 0.3011 | 9850 simulations | 0.2060 |
| $F_T = 1.5(x_0/\bar{a}_{60})\bar{a}_{75}$ | $S = 0$ Mean of $S$ | 9692 simulations | 0.7341 | 9726 simulations | 0.5870 | 9729 simulations | 0.5060 |
| $F_T = 1.6(x_0/\bar{a}_{60})\bar{a}_{75}$ | $S = 0$ Mean of $S$ | 9608 simulations | 0.9601 | 9611 simulations | 0.8639 | 9540 simulations | 0.8636 |

Furthermore, (19) shows that a higher terminal target leads to a larger deviation of the expected wealth from the target. In summary, on the one hand, the final target $F_T$ has positive effects on the expected running wealth and the accumulated withdrawal amount. On the other hand, a larger $F_T$ might lead to a higher probability of bankruptcy and greater deviation of the expected running wealth from the target. This analysis shows that a higher $F_T$ does not always yield a better result. Therefore, a retiree should determine his final target based on multiple individual requirements, such as his tolerances for bankruptcy and deviation from the target.

5. Conclusions. In this paper, we have studied a multi-period investment problem for a DC pension scheme member during the decumulation phase. The pension member is allowed to defer annuitization after retirement, periodically withdraw some money from the fund for daily life and invest the remaining wealth in the assets with higher returns until the time of compulsory annuitization. The withdrawal amount is assumed to be a linear function of the wealth level of the fund, that is, the withdrawal amount is $\rho_n X_n + c_n$. When $\rho_n$ and $c_n$ take different values, we obtain three different withdrawal mechanisms: deterministic withdrawal, proportional withdrawal and combined withdrawal. We set a series of investment targets over time and aim to minimize the expected losses from retirement to the time of compulsory annuitization by using the quadratic loss function. To highlight the effects of the targets and the different withdrawal mechanisms, the time of compulsory annuitization is predetermined. By dynamic programming, the explicit expressions for the optimal investment strategy and the targets greater than the expected wealth are obtained. By mathematical and numerical analysis, the main results and conclusions in this paper are listed as follows: (i) The distance between the target and the expected wealth decreases over time. Compared with the deterministic and combined withdrawal mechanisms, the proportional withdrawal mechanism has an advantage in achieving the final target $F_T$ and has the highest probability to be within the neighborhood of $F_T$. (ii) Bankruptcy is a rare event for all three withdrawal mechanisms. Moreover, the proportional withdrawal mechanism has the smallest expected number of bankruptcies while the deterministic withdrawal mechanism has the largest one. However, the deterministic withdrawal mechanism is the last to go bankrupt. (iii) On average, the pensioner will not short sale the risk-free and risky assets when the target-oriented investment model is adopted. In addition, the investment decision with proportional withdrawal is the most conservative in the sense that only 18.9368 of the total 180 decisions result in borrowing money from the bank for the risky investment. (iv) On average, the proportional withdrawal mechanism has the largest accumulated withdrawal amount up to time $n$. However, the withdrawal amount obtained at time $n$ for proportional withdrawal mechanism is the smallest in the last approximately 4 years. (v) A higher interest
rate results in a lower deviation from the targets at the beginning of the investment period and then leads to a higher deviation until the end of the investment period. Furthermore, the risk-free return has a negative effect on the expected number of the bankruptcies and a positive effect on the expected accumulated withdrawal amount. Compared with the deterministic and combined withdrawal mechanisms, the proportional withdrawal is the most sensitive to the interest rate. (vi) The terminal target \( F_T \) has a positive effect on the expected terminal wealth and the expected accumulated withdrawal amounts under the proportional and combined withdrawal mechanisms but a negative effect on the frequency of bankruptcy and the deviation from the target at each time.

While most existing papers focus on the continuous-time control model for DC pension plans during the decumulation phase, the present paper studies a discrete-time counterpart and assumes that the withdrawal amount is a linear function of the running wealth. By mathematical and numerical analysis, we have obtained many results listed above. However, our paper also has some limitations: (1) The inflation risk is not considered; however, the inflation risk will have an essential effect on the investment behavior of pensioners. (2) The mortality risk of the pensioners and the motivation of the bequest are neglected. In future works, we will relax these assumptions and consider more general models.

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**Appendix A. The proof of Theorem 3.1.**

**Proof.** When \( n = T - 1 \), we have

\[
V_{T-1}(x_{T-1}) = (F_{T-1} - x_{T-1})^2 + \varepsilon \min_{\alpha_{T-1}} E_{T-1,x_{T-1}} \left( F_T - X_T^{\alpha_{T-1}} \right)^2.
\]

Substituting (1) into \( E_{T-1,x_{T-1}} \left( F_T - X_T^{\alpha_{T-1}} \right)^2 \) yields

\[
E_{T-1,x_{T-1}} \left( F_T - X_T^{\alpha_{T-1}} \right)^2
= E_{T-1,x_{T-1}} \left[ F_T - (1 - \rho_{T-1})x_{T-1} - c_{T-1} \left( r_{T-1}^f + \alpha_{T-1} R_{T-1}^e \right) \right]^2
\]

\[
= (F_T)^2 - 2F_T (1 - \rho_{T-1})x_{T-1} - c_{T-1} \left( r_{T-1}^f + \alpha_{T-1} R_{T-1}^e \right) + (1 - \rho_{T-1})x_{T-1} - c_{T-1}^2 \left( (r_{T-1}^f)^2 + 2r_{T-1}^f \alpha_{T-1} R_{T-1}^e + \alpha_{T-1}^2 E(R_{T-1}^e)^2 \right)
\]

The optimal solution of \( \min_{\alpha_{T-1}} E_{T-1,x_{T-1}} \left( F_T - X_T^{\alpha_{T-1}} \right)^2 \) is

\[
\alpha_{T-1}^* = \left( \frac{F_T}{(1 - \rho_{T-1})x_{T-1} - c_{T-1}} - r_{T-1}^f \right) \frac{r_{T-1}^f}{E(R_{T-1}^e)^2}.
\] (27)

Substituting (27) into \( V_{T-1}(x_{T-1}) \) results in

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Therefore, the optimal solution of
\[
\min_{\alpha_n} V_n(x_n) = (F_n - x_n)^2 + C_{n+1} + A_{n+1}(r^f_n e^\alpha_n)^2 + 2B_{n+1}x_n^\alpha_n + C_{n+1}
\]
\[
= (F_n - x_n)^2 + C_{n+1} + A_{n+1}(r^f_n e^\alpha_n)^2 + 2B_{n+1}x_n^\alpha_n + C_{n+1}
\]
\[
\Gamma_n(\alpha_n) = A_{n+1}((1 - \rho_n)x_n - c_n)^2 (2r^f_n \alpha_n r^e_n + (\alpha_n)^2 E(R_n)^2)
\]
\[
- 2B_{n+1}((1 - \rho_n)x_n - c_n)\alpha_n r^e_n.
\]
The optimal solution of \(\min_{\alpha_n} \Gamma_n(\alpha_n)\) clearly exists and is
\[
\alpha_n^* = \left( \frac{B_{n+1}}{A_{n+1}((1 - \rho_n)x_n - c_n)} - r^f_n \right) \frac{r^e_n}{E(R_n)^2}. \tag{29}
\]
Therefore,
\[
V_n(x_n) = (F_n - x_n)^2 + C_{n+1} + A_{n+1}(r^f_n e^\alpha_n)^2 + 2B_{n+1}x_n^\alpha_n + C_{n+1}
\]
\[
\Gamma_n(\alpha_n) = A_{n+1}((1 - \rho_n)x_n - c_n)^2 (2r^f_n \alpha_n r^e_n + (\alpha_n)^2 E(R_n)^2)
\]
\[
- 2B_{n+1}((1 - \rho_n)x_n - c_n)\alpha_n r^e_n.
\]
The optimal solution of \(\min_{\alpha_n} \Gamma_n(\alpha_n)\) clearly exists and is
\[
\alpha_n^* = \left( \frac{B_{n+1}}{A_{n+1}((1 - \rho_n)x_n - c_n)} - r^f_n \right) \frac{r^e_n}{E(R_n)^2}. \tag{29}
\]
Therefore,
According to (18), we have

\[ -2 \left( \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} \right) r_n^f (1 - \rho_n) \left( A_{n+1} r_n^f c_n + B_{n+1} \right) + F_n \right) x_n + C_{n+1} - \frac{(r_n^c)^2}{E(R_n^c)^2} A_{n+1} (B_{n+1})^2 + 2 \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} A_{n+1} (r_n^f)^2 (c_n)^2 \]

+ \frac{2 \text{Var}(R_n^c)}{E(R_n^c)^2} B_{n+1} r_n^f c_n + (F_n)^2 := A_n (x_n)^2 - 2 B_n x_n + C_n. \tag{30} \]

Consequently, it is clear that

\[ A_n = \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} A_{n+1} (r_n^f)^2 (1 - \rho_n)^2 + 1, \tag{31} \]

\[ B_n = \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} r_n f (1 - \rho_n) (A_{n+1} r_n^f c_n + B_{n+1}) + F_n, \tag{32} \]

\[ C_n = C_{n+1} - \frac{(B_{n+1})^2}{A_{n+1}} \]

\[ + \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} A_{n+1} \left( \frac{(B_{n+1})^2}{A_{n+1}} + 2 \frac{B_{n+1} r_n^f c_n + (r_n^f)^2 (c_n)^2}{A_{n+1}} \right) + (F_n)^2 \]

\[ = C_{n+1} - \frac{(B_{n+1})^2}{A_{n+1}} + A_{n+1} \left( \frac{B_{n+1}}{A_{n+1}} + r_n^f c_n \right)^2 \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} + (F_n)^2. \tag{33} \]

\[(29)-(33)\) indicate that \((5)-(11)\) also hold for \(n\). By mathematical induction, we complete the proof of Theorem 3.1. \qed

**Appendix B. The proof of Theorem 3.2.**

**Proof.** According to (18), we have

\[ F_n - F_n \rho_n - c_n = (1 - \rho_n) \left( \prod_{k=n}^{T-1} \frac{1}{r_k^f} \frac{1}{1 - \rho_k} + \sum_{k=n}^{T-1} c_k \prod_{m=n}^{k-1} \frac{1}{r_m^f} \frac{1}{1 - \rho_m} \right) - c_n \]

\[ = \frac{1}{r_n^f} \left( \prod_{k=n+1}^{T-1} \frac{1}{r_k^f} \frac{1}{1 - \rho_k} + \sum_{k=n+1}^{T-1} c_k \prod_{m=n+1}^{k-1} \frac{1}{r_m^f} \frac{1}{1 - \rho_m} \right) \tag{34} \]

> 0. \qed

**Appendix C. The proof of Theorem 3.3.**

**Proof.** In view of (18), we obtain

\[ F_{n+1} - F_n \]

\[ = F_T \prod_{k=n+1}^{T-1} \frac{1}{r_k^f} \frac{1}{1 - \rho_k} + \sum_{k=n+1}^{T-1} c_k \prod_{m=n+1}^{k-1} \frac{1}{r_m^f} \frac{1}{1 - \rho_m} \tag{34} \]

\[ - F_T \prod_{k=n}^{T-1} \frac{1}{r_k^f} \frac{1}{1 - \rho_k} - \sum_{k=n}^{T-1} c_k \prod_{m=n}^{k-1} \frac{1}{r_m^f} \frac{1}{1 - \rho_m} \]
Therefore, when $F = F_n$ according to (24), we have

$$\pi_n = \pi_{n+1}.$$ 

Similarly, in view of (15), we have $X_n = X_{n+1}$, with the consequence that $\pi_{n+1}^* = 0$.

Then, it is obvious that $F_{n+1} < F_n$ when $r_n^f (1 - \rho_n) < 1$. When $r_n^f$, $\rho_n$ and $c_n$ are constant over time, we derive

$$F_{n+1} - F_n = \left(1 - \frac{1}{r^f (1 - \rho_n)} \right) \left(\frac{F \prod_{k=n+1}^{T-1} \frac{1}{r^f_k (1 - \rho_k)}}{\sum_{k=n+1}^{T-1} c_k \prod_{m=n+1}^{k-1} \frac{1}{r^f_m (1 - \rho_m)}}\right) - \frac{c_n}{1 - \rho_n}.$$ 

Therefore, when $F_T (r^f (1 - \rho_n) - 1) < cr^f$, $F_{n+1} < F_n$ for $n = 0, 1, \ldots, T - 1$.

**Appendix D. The proof of Theorem 3.4.**

**Proof.** For any $n = 0, 1, \ldots, T - 1$ and given wealth level $x_n$, if $F_n = x_n$, then according to (24), we have $\pi_n^* = 0$. Substituting $\pi_n^* = 0$ into (1) yields

$$X_{n+1} = r_n^f ((1 - \rho_n)x_n - c_n) = r_n^f ((1 - \rho_n)F_n - c_n).$$

In view of (15), we have $X_{n+1} = F_{n+1}$, with the consequence that $\pi_{n+1}^* = 0$. Similarly, $\pi_k^* = 0$ can lead to $F_{k+1} = X_{k+1}$ for $k = n + 1, n + 2, \ldots, T$ and then $\pi_{k+1}^* = 0$ for $k = n + 1, \ldots, T - 1$.

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