Deletion to Induced Matching

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Abstract

In the Deletion to Induced Matching problem, we are given a graph \( G \) on \( n \) vertices, \( m \) edges and a non-negative integer \( k \) and asks whether there exists a set of vertices \( S \subseteq V(G) \) such that \( |S| \leq k \) and the size of any connected component in \( G - S \) is exactly 2. In this paper, we provide a fixed-parameter tractable (FPT) algorithm of runtime \( O^*(1.748^k) \) for the Deletion to Induced Matching problem using branch-and-reduce strategy and path decomposition. We also extend our work to the exact-exponential version of the problem.

2012 ACM Subject Classification Theory of computation \( \rightarrow \) Fixed parameter tractability

Keywords and phrases Fixed Parameter Tractable, Parameterized Algorithms, Complexity Theory

1 Introduction

In the classic Vertex Cover problem, the input is a graph \( G \) and integer \( k \), and the task is to determine whether there exists a vertex set \( S \) of size at most \( k \) such that every edge in \( G \) has at least one endpoint in \( S \). Such a set is called a vertex cover of the input graph \( G \). An equivalent definition of a vertex cover is that every connected component of \( G - S \) has at most 1 vertex. This view of the Vertex Cover problem gives rise to a natural generalization: can we delete at most \( k \) vertices from \( G \) such that every connected component in the resulting graph has at most \( \ell \) vertices? Kumar et al [13] studied this generalization as \( \ell \)-COC (\( \ell \)-Component Order Connectivity). In this work, we would study a special case of this generalization where \( \ell \) is exactly 2. Formally, we consider the following problem, called Deletion to Induced Matching (IND).

Deletion to Induced Matching (IND)

Input: A graph \( G \) on \( n \) vertices and \( m \) edges, and a positive integer \( k \).
Task: determine whether there exists a set \( S \subseteq V(G) \) such that \( |S| \leq k \) and the maximum size of a component in \( G - S \) is exactly 2.

From the work of Stockmeyer and Vazirani [17], it is evident that IND is NP-complete. This motivates the study of IND within paradigms for coping with NP-hardness, such as approximation algorithms [20], exact exponential time algorithms [10], parameterized algorithms [4] [7] and kernelization [12] [14]. In this work we focus on IND from the perspective of parameterized complexity and exact exponential algorithms. As our main result, we provide an algorithm that given an instance \( (G, k) \) of IND such that degree of any vertex is at most 3, runs in polynomial time to output a path decomposition such that the path width is bounded by \( O(k) \). We provide an application of branching technique to convert an arbitrary instance \( (G, k) \) of IND to \( (G', k') \) such that degree of any vertex in \( G' \) is at most 3.

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Related Work. If the component size is 1, then the problem converts to finding a VERTEX COVER, which is extremely well studied from the perspective of approximation algorithms [20, 21], exact exponential time algorithms [1, 16, 22], parameterized algorithms [1, 6] and kernelization [21, 15]. The relaxed version of IND where component sizes are bounded by 2 (also 2-COC [13]) is also well studied, and has been considered under several different names. The problem, or rather the dual problem of finding a largest possible set \( S \) that induces a subgraph in which every connected component has order at most 2, was first defined by Yannakakis [23] under the name Dissociation Set. The problem has attracted attention in exact exponential time algorithms [11, 21], the fastest currently known algorithm [21] has running time \( O(1.3659^n) \). 2-COC has also been studied from the perspective of parameterized algorithms [11, 18] (under the name VERTEX COVER \( P_3 \)) as well as approximation algorithms [19]. The fastest known parameterized algorithm, due to Chang et al. [1] has running time \( 1.7485^k n^{O(1)} \), while the best approximation algorithm, due to Tu and Zhou [19] has factor 2.

To the best of our knowledge, IND has not been studied in the parameterized setting. We achieve the runtime of \( 1.7485^k n^{O(1)} \) in the polynomial space. We also provide a 1.5098\( ^8 \) runtime algorithm for the exact exponential version of the problem, called EXTEND.

Our Method. The key to our solution is reducing an IND instance \((G, k)\) to \((G', k')\) using branch-and-reduce strategy such that the maximum degree of any vertex \( v \in G' \) is 3. We achieve this by analysing certain properties of a solution set \( S \subseteq V(G) \). Thus, we only apply the branch-and-reduce strategy on vertices with degree more than 4. Consider a search tree \( T^* \) and the subtree rooted at some node with IND instance \((G', k')\). Now, for any \( u \in V(G) \), either \( u \in S \) or \( u \notin S \). In the former case, the algorithm creates a child with instance \((G'[V(G') \setminus u], k' - 1)\). In the later case, we study the \( e \in E(G') \) such that \( e := uv \) for some \( v \in N(u) \) (neighbours of \( u \) in \( G' \)). Our branching rules cover all the possible cases of \( \{u, v\} \not\subseteq S \) where \( v \in N(u) \).

Now, the motivation behind reducing the IND instances to maximum degree 3 instances, call \((G', k')\), is because we can efficiently solve those instances using ideas from path decomposition. We are able to show that there exists a path decomposition of \( G' \) with pathwidth having constant dependency on \( k' \). Not only that we provide an algorithm that runs in polynomial time to construct one.

We note that Fomin and Hågström [8] proved a tight bound on a path decomposition of a graph \( G \) with maximum degree at most three. They show that for any \( \epsilon > 0 \), \( \text{pw}(G) \leq (\frac{1}{6} + \epsilon)|V(G)| \) (pathwidth of the graph, cf. [2]). We first show that in the case of the special degree 3 graphs, we can bound the number of vertices of degree exactly 3 if \((G, k)\) is a YES-instance, and vice versa. Now the key is to show a path decomposition of the instance of width bounded by \( O(k) \). The main insight is in using the decomposition given by Fomin and Hågström [8], call it \( \mathcal{P}' \), and then applying color coding of edges and vertices to appropriately group the bags of the decomposition \( \mathcal{P}' \) (cf. [2]) to construct the desired path decomposition \( \mathcal{P} \).

Now, the only question left to be answered is how to construct an efficient solution for the instance \((G, k)\)? We provide a dynamic programming algorithm to solve an IND instance \((G, k)\) if a path decomposition of \( G \) if given. We define a coloring map of any bag \( X_t \in \mathcal{P} \) i.e. the path decomposition \( \mathcal{P} \) of \( G \) in the following manner: \( f : X_t \to \{0, 1, 2\} \) assigning three different colors to vertices of the bag. For the path decomposition \( \mathcal{P} \) represented as \((\mathcal{P}, \{X_{e}\}_{e \in E(\mathcal{P})})\). The idea is to dynamically find partial solution \( S_t \subseteq G[V_t] \) where \( V_t := \bigcup_{i=1}^{t} X_i \) for any \( t \in V(\mathcal{P}) \). Thus, the coloring set \( \{0, 1, 2\} \) induces a canonical meaning with the color 0 is when the vertex is in the partial solution, the color 1 is assigned when the vertex is not in the partial solution but to be paired later in the DP, and the color 2 is
We show the function $d_{-1}$ to ensure that the path width of a path with at least one edge is 1, not 2.

Let $G$ of a graph of bags where $u$ is adjacent to $v$. $X_{i,j}$ is said to be incident on an edge $e$. For every vertex in $G_i - S_i$ is at most 1.

We show the function $c[\cdot,\cdot]$ could be dynamically constructed. Since, the maximum possible colorings $f$ of a bag $X_i$ is bounded by $3^{|X_i|}$ thus the main claim could be shown.

The exact-exponential algorithm of the problem which involves finding the minimum set of bags containing a single element, we simply write $c$. $t$ such that the following two properties hold:

1. $S_t \cap X_i = f^{-1}(0)$, i.e. the set of vertices of $X_i$ that belong to the partial solution.
2. Degree of every vertex in $G_i - S_i$ is at most 1.

We further define a cost function $c[t, f]$ to denote the union $u \cup N(u)$, where $d(u)$ denotes the degree of the vertex $u$. For any subgraph $X \subseteq G$, by $N(X)$ we denote the set of neighbors of vertices in $X$ outside $X$, i.e. $N(X) := (\bigcup_{u \in X} N(u)) \setminus X$. An induced subgraph on $X \subseteq V(G)$ is denoted by $G[X]$.

A path $P$ is a graph, denoted by a sequence of vertices $v_1v_2\ldots v_t$ such that for any $i,j \in [t]$, $v_i, v_j \in E(P)$ if and only if $|i-j| = 1$. A cycle $C$ is a graph, denoted either by a sequence of vertices $v_1v_2\ldots v_t$ or by a sequence of edges $e_1e_2\ldots e_t$, such that for any $i,j \in [t]$ $u_i, u_j \in E(C)$ if and only if $|i-j| = 1 \mod t$ or in terms of edges, for any $i,j \in [t]$, $e_i$ is adjacent to $e_j$ if and only if $|i-j| = 1 \mod t$. The length of a path(cycle) is the number of edges in the path(cycle).

**Path Decomposition.** A path decomposition of a graph $G$ is a sequence $P = (X_1, X_2, \ldots, X_r)$ of bags where $X_i \subseteq V(G)$ for each $i \in \{1, 2, \ldots, r\}$, such that the following conditions hold:

- $\bigcup_{i=1}^r X_i = V(G)$. In other words, every vertex of $V(G)$ is in at the least one of the bag.
- For every $uv \in E(G)$, there is an $\ell \in \{1, 2, \ldots, r\}$ such that the bag $X_\ell$ contains both $u$ and $v$.
- For every $u \in V(G)$, if $u \in X_i \cap X_k$ for some $i \leq k$, then $u \in X_j$ also for each $j$ such that $i \leq j \leq k$. In other words, the indices of the bags containing $u$ form an interval in $\{1, 2, \ldots, r\}$.

The width of a path decomposition $(X_1, X_2, \ldots, X_r)$ is $\max_{1 \leq i \leq r} |X_i| - 1$. The pathwidth of a graph $G$, denoted by $pw(G)$, is the minimum possible width of a path decomposition of $G$. The reason for subtracting 1 in the definition of the width of the path decomposition is to ensure that the path width of a path with at least one edge is 1, not 2.
Fixed Parameter Tractability. A parameterized problem $\Pi$ is a subset of $\Sigma^* \times \mathbb{N}$. A parameterized problem $\Pi$ is said to be fixed parameter tractable (FPT) if there exists an algorithm that takes as input an instance $(I, k)$ and decides whether $(I, k) \in \Pi$ in time $f(k) \cdot n^c$, where $n$ is the length of the string $I$, $f(k)$ is a computable function depending only on $k$ and $c$ is a constant independent of $n$ and $k$.

A data reduction rule, or simply, reduction rule, for a parameterized problem $Q$ is a function $\phi: \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ that maps an instance $(I, k)$ of $Q$ to an equivalent instance $(I', k')$ of $Q$ such that $\phi$ is computable in time polynomial in $|I|$ and $k$. We say that two instances of $Q$ are equivalent if $(I, k) \in Q$ if and only if $(I', k') \in Q$; this property of the reduction rule $\phi$, that it translates an instance to an equivalent one, is referred as the safeness of the reduction rule.

A fixed-parameter algorithm based on branch-and-reduce strategy consists of a collection of reduction rules and branching rules. The branching rules are used to recursively solve the smaller instances of the problem with smaller parameter. We analyze each branching rule and use the worst-case time complexity over all branching rules as an upper bound of the running time. We represent the execution of a branching algorithm via search tree. The root of a search tree represents the input of the problem, every child of the root represents a smaller instance reached by applying a branching rule associated with the instance of the root. One can recursively assign a child to a node in the search tree when applying a branching rule. Notice that we do not assign a child to a node when applying a reduction rule. The running time of a branching algorithm is usually measured by the maximum number of leaves in its corresponding search tree. Let $b$ be any branching rule. When rule $b$ is applied, the current instance $(G, k)$ is branched into $s \geq 2$ instances $(G_i, k_i)$ where $|G_i| \leq |G|$ and $k_i = k - t_i$. Notice that fixed-parameter algorithms return “No” when the parameter $k \leq 0$. We call $b = (t_1, t_2, \cdots, t_s)$ the branching vector of branching rule $b$. This can be formulated in a linear recurrence: $T(k) \leq T(k - t_1) + T(k - t_2) + \cdots + T(k - t_s)$, where $T(k)$ is the number of leaves in the search tree depending on the parameter $k$. The running time of the branching algorithm using only branching rule $b$ is $O(poly(n) \cdot T(k)) = O^*(c^k)$ where $c$ is the unique positive real root of $x^k - x^{k-t_1} - x^{k-t_2} - \cdots - x^{k-t_s} = 0$ [10]. The number $c$ is called the branching number of the branching vector $(t_1, t_2, \cdots, t_s)$.

3 Faster FPT algorithm for Deletion to Induced Matching

In this section, we would provide the fixed parameter tractable algorithm for DELETION TO INDUCED MATCHING. First, we construct a set of branching rules for problem instances with at the least one vertex with degree $\geq 4$. The branching rules are based on a novel observation. Using these rules, we get a problem instance $(G', k')$ from $(G, k)$ such that any vertex $u \in V(G')$ has degree bounded by $3$. We provide an efficient solution for the instance $(G', k')$ using ideas from path decomposition.

We denote by $S$ a potential solution of size at most $k$. Now we proceed to write the reduction rules and branching rules. Note that while stating a reduction rule or a branch rule we assume that previous rules are not applicable. Furthermore, each rule changes the instance from $(G, k)$ to $(G', k')$ where $|V(G')| < |V(G)|$ and $k' \leq k$, but we use the same symbols $(G, k)$ to represent the modified instance. We note the following key observation:

\textbf{Observation 1.} If $\exists u, v \in V(G)$ such that $N[v] \setminus u \subseteq N(u)$, then there exists a solution $S$ such that either $u \in S$ or $\{u, v\} \notin S$. 


Indeed if \( S \) is a solution such that \( u \notin S \) and \( v \in S \) such that \( \exists w \in N(u) \cap N(v) \) and \( w \notin S \), then \( S' := (S \setminus v) \cup u \) is also a solution where \( v \notin S \). If \( \exists x \notin N(v) \) such that \( x \notin S \), then note that all the neighbors of \( x \) outside that of \( u \) has to be deleted. Thus, \( S' := (S \setminus v) \cup x \) where \( x \in S \) and \( v \notin S \). Hence, even for \( u \notin S \), we get a solution such that either \( u \in S \) or \( \{u,v\} \subset S \).

Observation \([1]\) suggests the following branching rule where we assume that the degree of \( u \) is at the least 4 i.e. \( d(u) \geq 4 \), where the branching subtree is rooted at \( u \):

**Branching Rule 1.** If \( \exists v \in V(G) \) such that \( N[v] \subseteq N[u] \). Then, make nodes in the branch tree for the following cases: \( u \in S \) or \( u \notin S \). In the second case, we pair the vertex \( u \) with \( v \) and delete at least \( d(u) - 1 \) many vertices. The recurrence relation is \( T(k) \leq T(k-1) + T(k-d+1) \) whose solution is bounded by \( T(k) \leq 1.4656^k \).

From now on we assume that for every vertex \( u \) of degree at least 4, we have that for every vertex \( v \in N(u) \), \( |N(v) \cap N[u]| \geq 1 \). If \( u \notin S \), then one neighbor \( v \in N(u) \) does not belong to \( S \). In that case, \( N(v) \cap N[u] \subseteq S \). This is the basis for the following branching rule:

**Branching Rule 2.** Let \( u \) be a vertex of degree \( d \geq 4 \) such that \( \forall v \in N(u) \ |N(v) \cap N[u]| \geq 1 \). Create \( d+1 \) branch nodes: one for the case when \( u \in S \) and one for each vertex \( v \in N(u) \) such that \( (N(u) \cup N(v)) \setminus \{u,v\} \subseteq S \) where at least \( d \) vertices are deleted. Then, the recurrence relation is \( T(k) \leq T(k-1) + d \cdot T(k-d) \) whose solution is bounded by \( T(k) \leq 1.748^k \).

Using the above branching rules \([1, 2]\) we can reduce any IND instance \((G, k)\) to \((G', k')\) such that \( G' \) has maximum degree 3 for any vertex \( u \in V(G') \). Now, we would provide the construction of a path decomposition for \((G', k')\). First, we state the following result by Fomin and Høie \([3]\):

**Theorem 1** (Fomin and Høie). For any \( \epsilon > 0 \), there exists an integer \( n_\epsilon \) such that for every graph \( G \) with maximum vertex degree at most three and with \( |V(G)| > n_\epsilon \), \( pw(G) \leq (\frac16 + \epsilon)|V(G)| \). Furthermore, such a decomposition can be obtained in polynomial time.

Now, we would prove a lemma which ascertains a key property on the number of vertices with degree 3 for an IND problem instance \((G, k)\) to be a YES-instance:

**Lemma 2.** Let \( G \) be a graph of maximum degree 3. Then \((G, k)\) is a yes-instance if and only if there are at most \( 2.5k \) vertices of degree 3 in \( G \).

**Proof.** Let \((G, k)\) be a yes-instance of IND and \( S \) be a solution of size at most \( k \). Since the degree of any vertex is at most 3, the number of edges between \( S \) and \( V \setminus S \) is at most \( 3k \).

Now, the degree of any vertex in \( G - S \) is 1. Hence, for every vertex of degree 3 in \( V \setminus S \), the number of edges to \( S \) is 2. Hence, the maximum number of degree 3 vertices is bounded by \( k + \frac{2k}{6} = 2.5k \).

If number of degree 3 vertices is more than \( 2.5k \), then for any set \( S \) of size \( k \), \( G - S \) contains a vertex of degree greater than 1 and hence \((G, k)\) is a no-instance. \( \square \)

Using Theorem \([1]\) and Lemma \([2]\) we can bound the path width of the path decomposition of an IND instance \((G, k)\) as follows:

**Lemma 3.** There exists a polynomial time algorithm that given an IND instance \((G, k)\) obtains a path decomposition of \( G \) of width at most \( \frac{2.5k}{6} + 2 \).
Deletion to Induced Matching

Proof. By Lemma 2, the maximum number of vertices of degree 3 in $G$ is at most $2.5k$. We obtain a graph $G'$ by first recursively contracting all edges incident on two degree-2 vertices and then recursively contracting edges incident on one vertex of degree at most 2. If the degree of a contracted vertex is 3, we color the edge representing the contracted path red. If a vertex of degree 1 gets merged to a vertex of degree 3, we color the vertex red. Clearly, the maximum degree of any vertex in $G'$ is at most 3 and $|V(G')| \leq 2.5k$. By Theorem 1, we can obtain a path decomposition $P'$ of width at most $p' = \frac{2.5k}{6}$ in polynomial time.

Now we show how to obtain a path decomposition $P$ of width at most $p' + 1$ of $G$ using the decomposition $P'$ in polynomial time. Suppose there is a red edge in bag $X_t$. Let $x, y$ be the endpoints of this edge. Without loss of generality, assume that $y$ is introduced after $x$ has been introduced and $X_{t'}$ be the bag just before $y$ is introduced. Clearly, $x \in X_{t'}$. Let $P = xu_1u_2 \ldots u_ay$ be the path in $G$ corresponding to the contracted edge is $G'$. We introduce the following bags in order in $P'$ after $X_{t'}$: $X_{u_1}, X_{u_2}, X_{u_3}, X_{u_4} \ldots$ where $X_u$ denotes an introduce bag for $u$ and $X_u$ denotes a forget node for $u$. Here $X_{u_1} = X_{t'} \cup \{u_1\}$. Note that the size of any bag is at most $p' + 2$. This split of a bag containing a red edge is applied exhaustively. Note that there can be at most one red edge between two vertices that also share an edge in the original graph. If there is a red edge between two vertices which are not adjacent in the original graph, then in the above sequence when we introduce extra bags, the size of the bags does not exceed the original size. Now we consider the case when there are two $xy$-paths in $G$ that get contracted. If $xy \in E(G)$, then pathwidth of $G$ is at most 3 and hence the decomposition satisfies the requirements of the lemma. Otherwise, we follow above procedure of introducing extra bags for the second path before $y$ is introduced.

For red vertices, the procedure is similar. Let $v$ be a red vertex in some bag $X_t$ and let $au_1u_2 \ldots v$ be the path that got contracted to $v$. Let $X'_t$ be the bag just before $v$ is introduced. We insert the following bags in the decomposition after $X'_t$: $X_u, X_{u_1}, X_{u_2}, X_{u_3} \ldots$. We apply this operation exhaustively for each red vertex. The width of the decomposition is at most $p' + 2$. It is easily seen that $P$ is a valid path decomposition of $G$.

Before, we show our algorithm for solving an IND instance with every vertex $u \in G$ having maximum degree 3, we would give an algorithm to solve an IND instance in $O^*(3^p)$ time where $p$ is the pathwidth of a path decomposition for $G$:

**Theorem 4.** There exists an algorithm that given a path decomposition of $G$ solves IND in $O^*(3^p)$ time where $p$ is the width of the decomposition.

Proof. We would assume that we are given a nice path decomposition $(P, \{X_t\}_{t \in V(P)})$ of width $p$ of $G$. We define $V_t := \bigcup_{i=1}^{\frac{p}{t}} X_t$ and $G_t := G[V_t]$ for any $t \in V(P)$. We define a coloring of a bag $X_t$ as a mapping $f : X_t \rightarrow \{0, 1, 2\}$ assigning three different colors to vertices of the bag, which is described extensively as follows:

- If $f(v) = 0$, then $v$ must be contained in the partial solution in $G_t$.
- If $f(v) = 1$, then $v$ does not belong to the partial solution, but it is isolated in $G_t - S_t$ where $S_t$ is the partial solution in $G_t$. These are the vertices which are to be paired later in the DP (dynamic programming of the algorithm).
- If $f(v) = 2$, then $v$ does not belong to the partial solution, but it has degree one in $G_t - S_t$. Note that $d(v) \leq 1$ in $G[X_t - f^{-1}(0)]$.

For a subset $X \subseteq V(G)$, consider a coloring $f : X \rightarrow \{0, 1, 2\}$. For a vertex $v \in V(G)$ and a color $\alpha \in \{0, 1, 2\}$ we define a new coloring $f_{v \rightarrow \alpha} : X \cup \{v\} \rightarrow \{0, 1, 2\}$ as follows:

$$f_{v \rightarrow \alpha}(x) = \begin{cases} f(x) & \text{when } x \neq v, \\ \alpha & \text{when } x = v. \end{cases}$$
Similarly, we define another coloring \( f_{v\leftarrow \alpha} : X \rightarrow \{0, 1, 2\} \) as follows:

\[
f_{v\leftarrow \alpha}(x) = \begin{cases} 
  f(x) & \text{when } x \neq v, \\
  \alpha & \text{when } x = v.
\end{cases}
\]

For a coloring \( f \) of \( X \) and \( Y \subseteq X \), we use \( f|_Y \) to denote the restriction of \( f \) to \( Y \). For a coloring \( f \) of \( X_t \), we denote by \( c[t, f] \) the minimum size of a set \( S_t \subseteq V_t \) such that the following two properties hold:

1. \( S_t \cap X_t = f^{-1}(0) \), i.e. the set of vertices of \( X_t \) that belong to the partial solution.
2. Degree of every vertex in \( G_t - S_t \) is at most 1.

Since the decomposition is nice, we specify the values of the recursive function \( c[\cdot, \cdot] \) to various node types as shown below:

**Leaf node:** For a leaf node \( t \) we have that \( X_t = \emptyset \). Hence there is only one, empty coloring, and we have \( c[t, 0] = 0 \).

**Introduce node:** Let \( t \) be an introduce node with a child \( t' \) such that \( X_t = X_{t'} \cup \{v\} \) for some \( v \notin X_{t'} \). We write the recursive formulae for various cases. Note that we put infinity as a value for \( c[t, f] \) whenever a feasible solution is not possible.

\[
c[t, f] = \begin{cases} 
  1 + c[t', f|_{X_{t'}}] & \text{if } f(v) = 0 \\
  c[t', f|_{X_{t'}}] & \text{if } f(v) = 1 \text{ and } \exists u \in X_t \text{ s.t. } uv \in E(G) \text{ and } f(u) \neq 0 \\
  c[t', f_{v\leftarrow 1}|_{X_{t'} \setminus w}] & \text{if } f(v) = 2, \exists w \text{ s.t. } f(w) = 1 \text{ and } uv \in E(G) \text{ and } \exists u \text{ s.t. } f(u) = 2 \text{ and } uv \in E(G) \\
  \infty & \text{otherwise.}
\end{cases}
\]

Note that when \( vw \in E(G) \) and \( f(w) = 1 \) in \( X_{t'} \), then at the **introduce node** \( X_t \), we have \( f(w) = 2 \).

**Forget node:** Let \( t \) be a forget node with a child \( t' \) such that \( X_t = X_{t'} \setminus \{v\} \) for some \( v \in X_{t'} \). Since multiple colorings of \( X_{t'} \) can lead to the same coloring of \( X_t \), it suffices to keep the one that leads to minimum size solution.

\[
c[t, f] = \min \{ c[t', f_{v\rightarrow 0}], c[t', f_{v\rightarrow 1}], c[t', f_{v\rightarrow 2}] \}
\]

If an introduced vertex is colored 2 then, the coloring is valid unless there is a vertex \( w \in X_{t'} \) such that the restriction \( f|_{X_{t'}} \) satisfies \( f|_{X_{t'}}(w) = 1 \).

Since there are at most \( 3^{|X_t|} \) number of colorings \( f \) for any bag \( X_t \), the time to process any node is at most \( 3^{|X_t|+1} \). Hence, IND can be solved in \( O^*(3^p) \) time.

**Lemma 5.** IND on graphs of maximum degree 3 can be solved in \( O^*(1.581^k) \) time.

**Proof.** Run the algorithm of Lemma 3 to get a nice path decomposition of width at most \( \frac{2n}{k} + 2 \) and then use the algorithm of Theorem 4. Hence, the runtime of the algorithm is bounded by \( O^*(3^{\frac{2n}{k} + 2}) = O^*(1.581^k) \).

We have provided a faster FPT solution for problem instances with maximum degree 3. Using the branching rules and Lemma 5 we would state and proof the main claim of the section in which we show a fixed-parameter tractable algorithm of runtime \( O^*(1.748^k) \) for any IND instance \( (G, k) \). In the following, we state the claim below and then we complete the proof.
Theorem 6 (Main Theorem). \textsc{ind} can be solved in \(O^*(1.748^k)\) time.

Proof. Algorithm solves \textsc{ind} in two phases. In phase one, it applies some reduction rules to eliminate degree four vertices. In phase two, it solves \textsc{ind} on graphs of maximum degree 3.

After exhaustively applying branching rules 1 and 2 on an \textsc{ind} instance \((G, k)\), we can assume that for the reduced instance \((G', k')\) the maximum degree of a vertex in the reduced graph \(G'\) is 3. At this point we run the algorithm of Lemma 5 to solve \textsc{ind} optimally. Hence, if \((G, k)\) is a \textsc{yes}-instance of \textsc{ind}, then in at least one branch, we get a solution of size at most \(k\).

Now we proceed to the runtime analysis of the algorithm. First, we write the recurrence relations for each of the branch rules we have used so far.

Branch rule 1 \[
T(k) \leq T(k-1) + T(k-d+1) \quad T(k) \leq 1.465^k
\]

Branch rule 2 \[
T(k) \leq T(k-1) + d \cdot T(k-d) \quad T(k) \leq 1.748^k
\]

Consider the branch tree at the end of the branch phase. Let \(s\) be the parameter with which the algorithm of Lemma 5 is called. Clearly, \(0 < s \leq k\). It is easily shown by induction on \(k\) that the number of leaves with parameter \(s\) is bounded by the worst case branching in the branch tree which is \(1.748^{k-s}\): indeed the base case \(k = 1\) is trivial. Consider that the statement is true for any value less than \(k\). Let \(T\) represent the branch tree. Then consider all the nodes with parameter \(s\) in \(T\). If there is a path on which there is no such node then, delete the subtree rooted at the node that has an edge to the path that leads to a node with parameter \(s\). Now delete all subtrees rooted at nodes with parameter \(s\) except the root. In this truncated tree defined as \(T^*\), leaves have parameter \(s\). Now, change \(k\) to \(k' = k - s\). Due to this parameter change, all leaves are with parameter 1. Since, \(k' < k\), by induction, the number of nodes with parameter value 1 is bounded by \(1.748^{k'} = 1.748^{k-s}\). Hence, the total runtime of the algorithm is bounded as follows

\[
\sum_{s=0}^{s=k} 1.748^{k-s} \times 1.581^s \times n^{O(1)} \leq \sum_{s=0}^{s=k} 1.748^s \times 1.748^{k-s} \times n^{O(1)} \leq O^*(1.748^k)
\]

4 Faster Exact-Exponential Algorithm

In this section, we would discuss an exact-exponential version of the Deletion to Induced Matching problem. We provide an exact-exponential algorithm of runtime \(O^*(1.5098^n)\) which uses the key ideas from the parameterized version of \textsc{ind}. We state the the exact-exponential version of the problem, also called \textsc{extend}, as follows:

Deletion to Induced Matching (\textsc{extend})

\textbf{Input:} A graph \(G\) on \(n\) vertices and \(m\) edges.

\textbf{Task:} determine the minimum cardinality subset \(S \subseteq V(G)\) such that the maximum size of any component in \(G - S\) is exactly 2.

Notice that since we get a fixed parameter tractable algorithm of running time of \(O^*(1.748^k)\) for \textsc{ind} thus the running time of the exact-exponential problem is bounded by \(O(1.748^n)\) where we solve for \(n\) instead of \(k\). We would use the stated Observation 1 (cf §3) to find the minimum possible set \(S\) such that the task is fulfilled. Note that the observation has a particular property that it doesn’t involve increase in the cardinality of a solution set.
S' if a solution S exists.

We approach the problem in the similar manner where the main idea is to exhaust all the vertices v such that \( d(v) \geq 4 \). We state the following observation

\[ \text{Observation 2. If } \exists u, v \notin S \text{ such that } uv \in E(G) \text{ then we can delete } \{u, v\} \text{ where } G \rightarrow G' \text{ such that } n \rightarrow n - 2. \]

Observations 1 and 2 suggest the following branching rule 1, 2 and 3 such that \( d(u) \geq 4 \):

\[ \text{Branching Rule 3. If } \exists v \in V(G) \text{ such that } N[v] \subseteq N[u]. \text{ Then, make nodes in the branch tree for the following cases: } u \in S \text{ or } u, v \notin S \text{ using Observation 4. In the second case we delete at least 3 vertices as } d(u) \geq 4 \text{ along with uv edge using Observation 4. The recurrence relation is } T(n) \leq T(n - 1) + T(n - 5) \text{ which solves to } 1.3247^n. \]

From now on we assume that for every vertex u of degree at least 4, we have that for every vertex \( v \in N(u), |N(v) \cap N[u]| \geq 1 \). If \( u \notin S \), then one neighbor \( v \in N(u) \) does not belong to S. In that case, \( N(v) \cap N[u] \subseteq S \). This is the basis for the following branching rule:

\[ \text{Branching Rule 4. Let } u \text{ be a vertex of degree } d \geq 4 \text{ such that } \forall v \in N(u), |N(v) \cap N[u]| \geq 1. \text{ Create } d + 1 \text{ branch node: one for the case when } u \in S \text{ and one for each vertex } v \in N(u) \text{ such that } N(u) \cup N(v) \setminus \{u, v\} \subseteq S \text{ where at least } d \text{ vertices are deleted along with } \{u, v\} \text{ using Observation 4. Then, the recurrence relation is } T(n) \leq T(n - 1) + d \cdot T(n - d - 2) \text{ whose solution is bounded by } T(n) \leq 1.5098^n. \]

Using the above branching rules 3, 4 we can reduce any EXTEND instance \((G, k)\) to \((G', k')\) such that \( G' \) has maximum degree 3 for any vertex \( u \in V(G') \). Now, we would prove a theorem which ascertains an algorithm on an EXTEND instance where every vertex of \( G \) has bounded degree 3.

\[ \text{Theorem 7. There exists an algorithm that given an instance } G \text{ of max degree 3 solves EXTEND in } O^*(3^\frac{n}{5}) \text{ time.} \]

\[ \text{Proof. Note that in Lemma 3 we find a path decomposition of the IND instance } (G, k) \text{ with width } p \text{ such that } p \text{ is bounded by } 2.5k + 2. \text{ But note that using Lemma 2, } G \text{ can have at max } 2.5k \text{ many vertices of degree 3 which is indeed bounded by } n. \text{ Now, if we run the algorithm in Theorem 4 for all possible IND instance } (G, k) \forall k \in [n] \text{ where the path decomposition of } G \text{ has pathwidth } p \text{ bounded by } (\frac{n}{5} + 2). \text{ Note, using this procedure the runtime is bounded by } n \ast 3^{(\frac{n}{5} + 2)} \ast n^c \text{ where } c \text{ is a constant from Theorem 4. Thus, overall the running time for EXTEND is bounded by } O(3^\frac{n}{5}) \text{ (i.e. } O(1.2009^n)) \text{ .} \]

\[ \text{Theorem 8 (Main Theorem). EXTEND can be solved in } O^*(1.5098^n) \text{ runtime.} \]

\[ \text{Proof. Algorithm solves EXTEND in two phases. In phase one, it applies some reduction rules to eliminate degree four vertices. In phase two, it solves EXTEND on graphs of maximum degree 3.} \]

\[ \text{After exhaustively applying Branching rules 3 and 4 we can assume that the maximum degree of a vertex in the remaining graph is 3. Hence, this instance can be seen as an instance of EXTEND } (G', n - n') \text{ with maximum degree three. At this point we run the algorithm of Theorem 7 to solve EXTEND optimally. Now we proceed to the runtime analysis of the} \]

We believe that there could be further improvement to the runtime for involving an induced bipartite subgraph. We leave the improvement for future work.

The bottleneck could be analysing a matching case ideas one could provide similar results for various exact version of decomposition, which gives an efficient solution to this problem. We note that using similar matching problem. We provide a novel way to combine ideas from branching and path decomposition.

In this work, we provide a fixed parameter tractable algorithm to the deletion to induced matching problem. First, we write the recurrence relations for each of the branch rules we have used so far.

Branch rule \( T(n) \leq T(n-1) + T(n-5) \) \( T(n) \leq 1.3247^n \)

Branch rule \( T(n) \leq T(n-1) + 5098 \cdot T(n-d-2) \) \( T(n) \leq 1.5098^n \)

Consider the branch tree at the end of the branch phase. Let \( G' \) be the instance with which the algorithm of Theorem 7 is called. Let \( |V(G')| = n' \). It is easily shown by induction on \( n \) that the number of leaves with \( |V(G')| = n' \) is bounded by the worst case branching in the branch tree which is \( 1.5098^{n-n'} \): indeed in the base case \( n = 1 \) is trivial. Consider that the statement is true for any value less than \( n \). Let \( T \) represent the branch tree. Then consider all the nodes with \( |V(G')| = n' \) in \( T \). If there is a path on which there is no such node then, delete the subtree rooted at the node that has an edge to the path that leads to a node with \( |V(G')| = n' \). Now delete all subtrees rooted at nodes with \( |V(G')| = n' \) except the root. In this truncated tree \( T^* \), leaves have \( G' \) where \( |V(G')| = n' \). Now, change \( n \) to \( n_0 = n - n' \).

Due to this change, all leaves are with \( |V(G')| = n' \). Since, \( n_0 < n \), by induction, the number of nodes with \( |V(G')| = 1 \) is bounded by \( 1.5098^{n_0} = 1.5098^{n-n'} \). Hence, the total runtime of the algorithm is bounded by

\[
\sum_{s=0}^{n} 1.5098^{n-s} \times 1.2009^{n-s} \leq \sum_{s=0}^{n} 1.5098^{s} \times 1.5098^{n-s} \leq O(1.5098^n)
\]

5 Conclusion

In this work, we provide a fixed parameter tractable algorithm to the deletion to induced matching problem. We provide a novel way to combine ideas from branching and path decomposition, which gives an efficient solution to this problem. We note that using similar ideas one could provide similar results for various exact version of \( \ell \)-COC (2-COC in our case). We believe that there could be further improvement to the runtime for IND if the branching rule [2] is improved to a better bound. The bottleneck could be analysing a matching case involving an induced bipartite subgraph. We leave the improvement for future work.

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