External Gauge Invariance and Anomaly in BS Vertices and Boundstates

MASAKO BANDO, MASAYASU HARADA⋆ AND TAICHIRO KUGO

Department of Physics, Kyoto University
Kyoto 606, Japan

ABSTRACT

A systematic method is given for obtaining consistent approximations to the Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations which maintain the external gauge invariance. We show that for any order of approximation to the SD equation there is a corresponding approximation to the BS equations such that the solutions to those equations satisfy the Ward-Takahashi identities of the external gauge symmetry. This formulation also clarifies the way how we can calculate the Green functions of current operators in a consistent manner with the gauge invariance and the axial anomaly. We show which type of diagrams for the $\pi^0 \rightarrow \gamma\gamma$ amplitude using the pion BS amplitude give result consistent with the low-energy theorem. An interesting phenomenon is observed in the ladder approximation that the low energy theorem is saturated by the zeroth order terms in the external momenta of the pseudoscalar BS amplitude and the vector vertex functions.

⋆ Fellow of the Japan Society for the Promotion of Science for Japanese Junior Scientists.
1. Introduction

If we consider a strongly interacting fermion system, we have to deal with various boundstates and their interactions among themselves or to some external gauge fields. For instance, in QCD, the boundstates are hadrons, which are in fact the only observable states, and the external gauge fields are the photon and the $W$ and $Z$ bosons. In order to treat such boundstate problems, we are obliged to adopt some non-perturbative approximation scheme, and then it becomes a non-trivial issue whether and how to keep the external gauge invariance.

In this paper we discuss this problem for the case using the Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations as a non-perturbative approximation method [1]. We use, in a certain approximate manner, the SD equation for calculating fermion propagator and the inhomogeneous BS equations for the vertices of fermion with the external gauge bosons. Then the adopted approximation has to keep the external gauge invariance in order to give results consistent with, for instance, various low-energy theorems. This problem is actually not so trivial and, in fact, it will turn out that the approximations for the SD and BS equations cannot be independent of one another but have to satisfy a mutual consistency. This is because the fermion propagator and the vertex functions are connected to one another by the Ward-Takahashi (WT) identities resulting from the external gauge invariance. We shall present a general answer to this problem and find that for any order of approximation to the SD equation there is a corresponding approximation to the BS equations which satisfies the external gauge invariance.

A partial answer to this problem has actually been known to Maskawa and Nakajima [2, 3] for a long time: the axial-vector vertex function satisfies the axial-vector WT identity if it is determined by the ladder BS equation in which the fermion propagator determined by the ladder SD equation is used. They proved this result by a direct diagrammatic evaluation of the divergence of the axial-vector vertex in that approximation. Our result in this paper is to give a generalization to arbitrary order of approximations. The proof of the gauge invariance is given
in a very systematic manner.

Gauge invariance gives constraints not only on those fermion propagator and vertex functions but also the Green functions of the vector or axial-vector current operators coupled to the external gauge fields. We also discuss such Green functions of current operators in the same framework. We consider, for instance, a 3-point Green function of one axial-vector current and two vector currents, which gives a well known anomaly and is related the $\pi^0 \to \gamma\gamma$ amplitude by the low-energy theorem [4]. In the ladder approximation, as we show, the Green function simply obtained by a triangle diagram in which the vertices and fermion propagators are those in that approximation, satisfies the vector gauge invariance and reproduces the correct axial anomaly. This simple situation is, however, no longer true in any approximation beyond the ladder. We shall show that we have to include corrections intrinsic to the Green function which can be attributed neither to the fermion propagator nor to the vertex functions. This is also the case, even in ladder approximation, for general $n$-point Green function of current operators with $n \geq 4$. For instance, consider the 4-point Green function of vector currents corresponding to photon scattering $\gamma\gamma \to \gamma\gamma$. The simple box diagram consisting of the vertices and fermion propagators calculated in the ladder approximation does not satisfy gauge invariance, despite that the vertex function and the propagator themselves are mutually consistent with WT identity in the ladder approximation. We shall clarify in a general manner which types of diagrams should be included in order to obtain gauge-invariant Green functions (with correct anomaly in case it is present).

We discuss, in particular, the calculation of $\pi^0 \to \gamma\gamma$ decay amplitude by using pseudoscalar BS amplitude in detail, since it gives a typical example in which all these problems of gauge invariance and anomaly become relevant. We show that the resultant amplitude satisfies the low-energy theorem if the boundstate BS amplitude is the one determined by the BS equation in the ‘same’ order of approximation as those for the fermion propagator and the vector vertices, and if we include a particular set of diagrams which depend on the approximation. We also point out an interesting phenomenon in the ladder approximation that the low
energy theorem is saturated by the zeroth order terms in the external momenta of
the vector vertices and the pseudoscalar BS amplitude.

This paper is organized as follows. In section 2, we introduce a gauge invariant
effective action for the fermion propagator on arbitrary background of external
gauge fields. This provides us with a very systematic way to obtain mutually
consistent SD and BS equations. It is proved that the solutions to these equations
satisfy the WT identities in section 3. In section 4, we construct 3-point Green
function of one axial-vector and two vector current operators, and show that it
not only satisfies the vector gauge invariance but also reproduces correct axial
anomaly. Section 5 is devoted to consideration of the pseudoscalar boundstate
and the low-energy theorem for $\pi^0 \rightarrow \gamma\gamma$. We show which types of diagrams give
amplitude consistent with the low-energy theorem. In section 6, we discuss the
above mentioned phenomenon of zeroth order saturation of the low-energy theorem
in the ladder approximation. An appendix is added to show how the discussion of
the anomaly in section 4 goes when we adopt dimensional regularization instead
of the Pauli-Villars-Gupta’s one adopted in the text.

2. Gauge Invariant Effective Action

We consider an interacting fermion system like QCD in which the chiral sym-
metry is spontaneously broken dynamically. As usual we call the ‘gauge’ interaction
responsible for the formation of boundstates and for the spontaneous chiral sym-
metry breaking “color gauge interaction” and the gauge boson “gluon”, although
the present formulation also applies to more general systems than QCD.* In such
a system we want to calculate, in an approximate but non-perturbative manner, the
fermion propagator as well as the (color-singlet) vector and axial-vector vertices
to which external gauge fields couple. A method is to use the Schwinger-Dyson

* The ‘gauge’ boson need not be a true gauge boson; so, for example, it may have non-
zero masses and may be axial-vector. Then the Nambu-Jona-Lasinio [5] like models can be
discussed by considering a limit the vector and axial-vector ‘gauge’ boson masses go to very
large.
(SD) equation for the fermion propagator and the inhomogeneous Bethe-Salpeter
(BS) equation for the vertices. In this case, as announced in the Introduction, the
approximations adopted for the SD and BS equations cannot be independent one
another but have to satisfy a mutual consistency in order to meet the the external
gauge invariance requirement.

A systematic way to obtain those SD and BS equations satisfying the mutual
consistency, is provided if we consider the original system put in a general external
gauge field background. Let us denote the external background gauge fields as
$A_\mu \equiv A^a_\mu \lambda^a$ with flavor matrix $\lambda^a$ normalized as $\text{tr}(\lambda^a \lambda^b) = (1/2)\delta_{ab}$, and assume
a vector-like coupling to the fermion: $L_{\text{int}} = \bar{\psi}\gamma_\mu A^\mu \psi$. We assume this just for
notational simplicity and axial-vector case can be obtained simply by replacement
$\gamma_\mu \rightarrow \gamma_\mu \gamma_5$. An important assumption is that the flavor degrees of freedom to which
the external gauge fields couple are orthogonal to the color degrees of freedom to
which the ‘gluons’ (internal ‘gauge’ bosons) couple.

Effective action for fermion propagator $S_F$ in the presence of external gauge
field background is given by [6, 7]

$$
\Gamma[S_F, A] = i\text{Tr} \ln S_F - \text{Tr} (i\not{D} S_F) + i^{-1} \mathcal{K}_{2\text{PI}}[S_F],
$$

(2.1)

where we note that the external gauge field $A_\mu$ is present only at the covariant
derivative $D_\mu = \partial_\mu - iA_\mu$. Here $\mathcal{K}_{2\text{PI}}$ stands for the two particle irreducible (w.r.t.
fermion-line) diagram contributions: for example in QCD, we can expand the $\mathcal{K}_{2\text{PI}}$
into power series of the gauge coupling $\alpha_s = g_s^2/4\pi$,

$$
\mathcal{K}_{2\text{PI}} = \mathcal{K}_{2\text{PI}}^{(1)} + \mathcal{K}_{2\text{PI}}^{(2)} + \cdots
$$

(2.2)

and $\mathcal{K}_{2\text{PI}}^{(1)}$ and $\mathcal{K}_{2\text{PI}}^{(2)}$ are diagrammatically given by Fig. 1.
Fig. 1. Two particle irreducible (w.r.t. fermion-line) diagrams contributing to $\mathcal{K}_{2\text{PI}}^{(1)}$ and $\mathcal{K}_{2\text{PI}}^{(2)}$ in QCD. The double wavy line represents the gluon propagator $D_{\mu\nu}$ and the solid line represents the fermion propagator $S_F$.

More explicitly the first term $\mathcal{K}_{2\text{PI}}^{(1)}$ is given by\(^1\)

$$\mathcal{K}_{2\text{PI}}^{(1)} = -\frac{g_s^2}{2} \int d^4x d^4y \, \text{tr} \left( S_F(x,y)i\gamma_\mu T^a S_F(y,x)i\gamma_\nu T^a \right) D^{\mu\nu}(x-y), \quad (2.3)$$

where $T^a$ ($a = 1, \ldots, N_c^2 - 1$) are color matrices in the quark representation and $D_{\mu\nu}$ is tree level gluon propagator given by

$$D^{\mu\nu}(x) = \int \frac{d^4p}{i(2\pi)^4} e^{-ipx} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left( \frac{1}{p^2} - \frac{1}{p^2 - \Lambda^2} \right) \quad (\Lambda \to \infty), \quad (2.4)$$

where we have included an ultraviolet cutoff $\Lambda$ for definiteness. For later convenience, we refer to the first diagram contribution to the $\mathcal{K}_{2\text{PI}}^{(2)}$ in Fig. 1(b) as

\(^1\) If we use the running coupling constant, the coupling constant $g_s$ depends on the gluon momentum and hence we should understand that the running coupling factors $g_s^2$ in such a case are placed at the loop integrand. All the discussions in this paper apply equally both to the fixed and running coupling cases, provided that in the latter case the gluon momentum is used as the argument of the running coupling function [3].
\[ K_{2PI}^{(2a)} : \]

\[ K_{2PI}^{(2a)} = -\frac{g_s^4}{4} \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \ D^{\mu\nu}(y_1 - y_2) D^{\rho\sigma}(x_1 - x_2). \]

\[ \times \text{tr} \left( S_F(x_1, y_1) i\gamma_\mu T^a S_F(y_1, x_2)i\gamma_\rho T^b S_F(x_2, y_2)i\gamma_\nu T^a S_F(y_2, x_1)i\gamma_\sigma T^b \right). \]

(2.5)

By our assumption that the flavor is a freedom orthogonal to the color, we note that the flavor matrices \( \lambda^a \) commute with the color matrices \( T^a \). Then we have the following lemma:

**Lemma:** For any approximation for \( K_{2PI} \) by an arbitrary subset of diagrams contributing to \( K_{2PI} \), the effective action Eq.(2.1) is (external) gauge invariant:

\[ \Gamma[S_F, A] = \Gamma[S_F^U, A^U], \]

(2.6)

where the gauge transformation with \( U(x) = \exp \left( i\theta^a(x)\lambda^a \right) \) is given explicitly by

\[ A_\mu \rightarrow A_\mu^U = U A_\mu U^{-1} + \frac{1}{i} \partial_\mu U \cdot U^{-1}, \]

\[ S_F(x, y) \rightarrow S_F^U(x, y) = U(x)S_F(x, y)U^{-1}(y). \]

(2.7)

Proof) It is convenient to introduce the following functional notation:

\[ (U)_{xy} \equiv U(x)\delta^4(x - y), \]

\[ (\bar{\varphi})_{xy} \equiv (\bar{\varphi} - iA(x)) \delta^4(x - y), \]

\[ (S_F)_{xy} \equiv S_F(x, y). \]

(2.8)

In this notation we can rewrite the gauge transformation (2.7) simply in the form

\[ \bar{\varphi} \rightarrow U \bar{\varphi} U^{-1}, \]

\[ S_F \rightarrow U S_F U^{-1}. \]

(2.9)

Then we can see the gauge invariance of the first two terms in \( \Gamma \) as follows:

\[ \text{TrLn} S_F \rightarrow \text{TrLn} (U S_F U^{-1}) = \text{Tr} \left[ U \left( \text{Ln} S_F \right) U^{-1} \right] = \text{TrLn} S_F, \]

(2.10)
\[
\text{Tr} (\not D S_F) \rightarrow \text{Tr} (U \not D U^{-1} S_F U^{-1}) = \text{Tr} (\not D S_F) .
\]

(2.11)

The gauge invariance of \( K_{2\text{PI}}[S_F] \) is also shown similarly: for instance, for the lowest order diagram we have

\[
K_{2\text{PI}}^{(1)} \rightarrow - \frac{g_s^2}{2} \int d^4x d^4y \text{ tr} (U(x)S_F(x, y)U^{-1}(y)i\gamma_\mu T^aU(y)S_F(y, x)U^{-1}(x)i\gamma_\nu T^a) \times D^{\mu\nu}(x - y) = K_{2\text{PI}}^{(1)} .
\]

(2.12)

The point here is that the fermion propagators appear successively in the trace and the vertices there contain only color and \( \gamma \) matrices which commute with the flavor gauge transformation matrices \( U(x) \). This property clearly holds for any diagrams contributing to \( K_{2\text{PI}} \) and so the gauge invariance follows. This finishes the proof.

Because of the lemma, the Schwinger-Dyson (SD) equation derived from the action \( \Gamma \) is automatically (external) gauge covariant (or gauge invariant as a set of equations).

The SD equation is given by \( \delta \Gamma / \delta S_F = 0 \), which reads

\[
iS_F^{-1} = i\not D - i^{-1} \frac{\delta K_{2\text{PI}}}{\delta S_F} .
\]

(2.13)

If we take only the lowest order term in \( K_{2\text{PI}} \), \( K_{2\text{PI}}^{(1)} \), then this SD equation reduces to (see Fig. 2)

\[
iS_F^{-1} = i\not \phi + A + i^{-1} K * S_F ,
\]

(2.14)

with \( K * S_F \) defined by

\[
K * S_F \equiv g_s^2 (i\gamma_\mu T^a)S_F(y, x)(i\gamma_\nu T^a)D^{\mu\nu}(x - y) .
\]

(2.15)

Eq.(2.13) (or (2.14)) is the SD equation determining a solution \( S_F = S_F[A] \) for the fermion propagator, on an arbitrary external background gauge field \( A_\mu \).
\[ iS_F^{-1} = i \partial + A - i \begin{array}{c} \left( D_{\alpha\nu} \gamma^\mu T^a \right) S_F \end{array} \]

Fig. 2. Schwinger Dyson equation derived from the effective action \( \Gamma \) using \( K_{2\pi} = K_{2\pi}^{(1)} \).

The solution \( S_F[A] \) is expanded into a power series in the external gauge field \( A_\mu \):

\[ S_F[A] = S_F + i A_\mu^a G_3^{a\mu} + \frac{1}{2} A_\mu^a A_\nu^b G_4^{a\mu,b\nu} + \frac{i}{3} A_\mu^a A_\nu^b A_\rho^c G_5^{a\mu,b\nu, c\rho} + \cdots , \]  

(2.16)

where \( a, b \) and \( c \) denote the flavor indices. Here and henceforth the space-time coordinates and the integrations are suppressed, i.e., \( A_\mu^a G_3^{a\mu} \equiv \int d^4 z A_\mu^a(z) G_3^{a\mu}(x, y; z) \), etc. The function \( G_{n+2}^{a_1 \mu_1, \ldots, a_n \mu_n}(x, y; z_1, \ldots, z_n) \) defines a fermion 2-point function with \( n \) vector vertices inserted (see Fig. 3):

\[ G_3^{a\mu}(x, y; z) \equiv \left. \frac{1}{i} \frac{\delta S_F(x, y; A)}{\delta A_\mu^a(z)} \right|_{A=0} = \langle 0 | T j^{a\mu}(z) \bar{\psi}(x) \psi(y) | 0 \rangle , \]

\[ G_4^{a\mu,b\nu}(x, y; z, w) \equiv \left. \frac{1}{i^2} \frac{\delta S_F(x, y; A)}{\delta A_\mu^a(z) \delta A_\nu^b(w)} \right|_{A=0} = \langle 0 | T j^{a\mu}(z) j^{b\nu}(w) \bar{\psi}(x) \psi(y) | 0 \rangle , \ldots . \]

(2.17)

This is because \( \delta / \delta A_\mu^a \) yields an insertion of the vector current operator \( j^{a\mu} = \bar{\psi} \gamma^\mu \lambda^a \psi \) to which the external gauge boson \( A_\mu^a \) couples. Hereafter in this section, we suppress the flavor indices to denote \( G_{n+2}^{a_1 \mu_1, \ldots, a_n \mu_n} \) simply as \( G_{n+2}^{\mu_1 \ldots \mu_n} \), and write only \( \gamma^\mu \) in place of \( \gamma^\mu \lambda^a \) as vertex factors in the figures, accordingly.

Therefore the functional differentiation w.r.t. \( A_\mu \) (and then setting \( A = 0 \)) of the SD eq.(2.14) successively generates the Bethe-Salpeter (BS) equations for the \( G_{n+2}^{\mu_1 \ldots \mu_n} \) functions and they are automatically (external) gauge covariant. It is
convenient to define the following vertex function by amputating the fermion legs:

\[ \Gamma_{\mu_1 \cdots \mu_n} = S_{\mu_1}^{-1} G_{\mu_2 \cdots \mu_n} S_{\mu_n}^{-1}. \]  

(2.18)

To show what is going on as explicitly as possible, from here on in this section, we confine ourselves to the simplest case using the lowest order kernel \((K_{2PI} = K_{2PI}^{(1)})\).

The extension to more complicated case will be trivial. First differentiation \(\delta/\delta A_{\mu}\) of (2.14) gives (see Fig. 4)

\[ \Gamma_{3}^{\mu} = \gamma^{\mu} + \tilde{K} \ast \Gamma_{3}^{\mu}, \]  

(2.19)

where \(\tilde{K} \ast \Gamma_{3}^{\mu} \equiv K \ast (S_{F} \Gamma_{3}^{\mu} S_{F}) = K \ast G_{3}^{\mu}\) is defined in the same way as in Eq.(2.15).
Second differentiation of (2.14) gives (see Fig. 5)

\[
\Gamma_4^{\mu\nu} - \Gamma_3^\nu \sigma_3 - \Gamma_3^\mu \sigma_3 = \bar{K} \ast \Gamma_4^{\mu\nu} .
\] (2.20)

Third differentiation of (2.14) gives

\[
\begin{align*}
\Gamma_5^{\mu\nu\rho} &- (\Gamma_4^{\mu\nu} \sigma_3 \Gamma_3^\rho + \Gamma_3^\rho \sigma_3 \Gamma_4^{\mu\nu} + (\mu, \nu, \rho \text{ permutations})) \\
&+ (\Gamma_3^\mu \sigma_3 \Gamma_5^{\nu\rho} \sigma_3 + (\mu, \nu, \rho \text{ permutations})) \\
&= \bar{K} \ast \Gamma_5^{\mu\nu\rho} .
\end{align*}
\] (2.21)

We can see by the help of Eq.(2.20) that the last terms of the LHS in Eq.(2.21) play the role of cancelling the double counting of some diagrams contained in the second terms. Thus this equation reduces to

\[
\begin{align*}
\Gamma_5^{\mu\nu\rho} &- \left( (\bar{K} \ast \Gamma_4^{\mu\nu}) \sigma_3 \Gamma_3^\rho + \Gamma_3^\rho \sigma_3 (\bar{K} \ast \Gamma_4^{\mu\nu}) + (\mu, \nu, \rho \text{ permutations}) \right) \\
&- \left( \Gamma_3^\mu \sigma_3 \Gamma_5^{\nu\rho} \sigma_3 + (\mu, \nu, \rho \text{ permutations}) \right) \\
&= \bar{K} \ast \Gamma_5^{\mu\nu\rho} .
\end{align*}
\] (2.22)

Eqs.(2.19), (2.20) and (2.22) are the BS equations determining the vertices \( \Gamma_3^\mu, \Gamma_4^{\mu\nu} \) and \( \Gamma_5^{\mu\nu\rho} \), respectively, in the ladder approximation. It is in fact easy
to find formal solutions to these equations. Define the four-point fermion Green function $L$ in the ladder approximation as given in Fig. 6.

Then, inspection of the Eqs. (2.19), (2.20) and (2.22) shows that the formal solutions for $\Gamma^{\mu}_{3}$, $\Gamma^{\mu\nu}_{4}$ and $\Gamma^{\mu\nu\rho}_{5}$ are given by Figs. 7, 8 and 9, respectively.

Fig. 7. Formal solution of $\Gamma^{\mu}_{3}$ using $\mathcal{K}_{2\text{PI}} = \mathcal{K}_{2\text{PI}}^{(1)}$. 

3. Ward-Takahashi Identity

We now show explicitly that the (external) gauge invariance of the effective action (or the covariance of our SD and BS equations) implies that the vertex functions determined by those BS equations satisfy Ward-Takahashi identities.

Gauge invariance of the effective action $\Gamma[S_F, A]$ implies

$$\Gamma[S_F, A] = \Gamma[S'_F, A^U]$$  \hspace{1cm} (3.1)
with $S_F' \equiv US_FU^{-1}$, which leads to the gauge covariance of the SD equation:

$$\frac{\delta \Gamma[S_F, A]}{\delta S_F} = U^{-1} \frac{\delta \Gamma[S_F', A^U]}{\delta S_F'} U. \quad (3.2)$$

Therefore, if $S_F = S_F[A]$ is a solution to the SD equation on the background $A$, i.e.,

$$\frac{\delta \Gamma[S_F, A]}{\delta S_F} \bigg|_{S_F = S_F[A]} = 0, \quad (3.3)$$

then, its gauge-transformed one $S_F' = US_F[A]U^{-1}$ gives a solution to the SD equation on the gauge-transformed background $A^U$:

$$\frac{\delta \Gamma[S_F', A^U]}{\delta S_F'} \bigg|_{S_F' = US_F[A]U^{-1}} = 0. \quad (3.4)$$

Namely, we have shown

$$S_F[A^U] = US_F[A]U^{-1}. \quad (3.5)$$

If there are several solutions $S_F^i[A]$ ($i = 1, 2, \cdots$) for a single background $A$, we should have relation $S_F^i[A^U] = US_F^i[A]U^{-1}$ for each $i$, because of the continuity for $U \to 1$.

Substituting the expansion (2.16) into both sides of (3.5), we have

$$\text{LHS} = S_F + iA^U_\mu G_3^\mu + \frac{i^2}{2} A^U_\mu A^U_\nu G_4^{\mu\nu} + \cdots,$$

$$\text{RHS} = US_FU^{-1} + iA_\mu U G_3^\mu U^{-1} + \frac{i^2}{2} A_\mu A_\nu U G_4^{\mu\nu} U^{-1} + \cdots. \quad (3.6)$$

In particular, for an infinitesimal gauge transformation $U = 1 + i\theta \ (\theta = \theta^a \lambda^a)$, for
which $A^U_\mu$ is given by

$$A^{aU}_\mu = A^a_\mu + D_\mu \theta^a = A^a_\mu + \partial_\mu \theta^a + f_{abc} A^b_\mu \theta^c$$

(3.7)

with structure constant $f_{abc}$ of the flavor group, we find

$$\text{LHS} = \left( S_F + i \partial_\mu \theta^a G_3^{a\mu} \right) + i A^a_\mu \left[ (\delta_{ab} + f_{abc} \theta^c) G_3^{b\mu} + i \partial_\nu \theta^b G_4^{a\mu, b\nu} \right] + \cdots,$$

$$\text{RHS} = S_F + i (\theta S_F - S_F \theta) + i A^a_\mu \left[ G_3^{a\mu} + i \left( \theta G_3^{a\mu} - G_3^{a\mu} \theta \right) \right] + \cdots,$$

are equal with each other. Equating each power term in $A_\mu$ on both sides, we find

$$-i \partial_\mu G_3^{a\mu}(x, y; z) = i \delta^4(z - x) \lambda^a S_F(x - y) - i \delta^4(z - y) S_F(x - y) \lambda^a,$$

$$-i \partial_\nu G_4^{a\mu, b\nu}(x, y; z, w) = - f_{abc} \delta^4(w - z) G_3^{c\mu}(x, y; z) + i \delta^4(w - x) \lambda^b G_3^{a\mu}(x, y; z) - i \delta^4(w - y) G_3^{a\mu}(x, y; z),$$

(3.9)

$$\cdots,$$

and so on. These are just the Ward-Takahashi identities required by the external gauge invariance. For instance, Eq. (3.9) is nothing but the WT identity:

$$\partial_\mu \langle 0 | T j^{a\mu}(z) \psi(x) \bar{\psi}(y) | 0 \rangle = - \delta^4(z - x) \lambda^a \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle + \delta^4(z - y) \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \lambda^a.$$

Thus this proves that the fermion propagator $S_F$ and the vertices $\Gamma^{a_{\mu_1 \cdots \mu_n}}_{n+2}$ determined by our SD and BS equations satisfy the Ward-Takahashi identities giving relations among them; namely, our approximations for the SD and BS equations are mutually consistent and gauge invariant.

Note that we have proven that this external gauge invariance holds not only for the simplest ladder case (i.e., with $K_{2\Pi}^{(1)}$), but also for any order of approximations. For example, if we take $K_{2\Pi} = K_{2\Pi}^{(1)} + K_{2\Pi}^{(2a)}$, then the SD equation for $S_F$ and the BS equation for $\Gamma^{a\mu}_{3}$ are changed into the forms as shown in Figs. 10 and 11. These equations are much more complicated than the simple ladder ones, nevertheless they satisfy the gauge invariance. Important is the mutual consistency of the approximations between the SD equation and BS equations.

* The gauge invariance for the 3-point vertex $\Gamma^{a\mu}_{3}$ in the simplest ladder case has been known for a long time to Maskawa and Nakajima [2]. A refinement of the proof and the generalization to the running coupling case was given by Kugo and Mitchard [3].
\[ i S_F^{-1} [A] = i \mathcal{D} + \mathcal{A} - i \Gamma_3 - i \Gamma_3 \]

Fig. 10. SD equation using \( \mathcal{K}_{2\Pi} = \mathcal{K}_{2\Pi}^{(1)} + \mathcal{K}_{2\Pi}^{(2a)} \).

\[ \Gamma_3^\mu = \gamma^\mu + \Gamma_3^\mu \]

\[ \Gamma_3^\mu + \Gamma_3^\mu + \Gamma_3^\mu + \Gamma_3^\mu \]

Fig. 11. BS equation for \( \Gamma_3^\mu \) using \( \mathcal{K}_{2\Pi} = \mathcal{K}_{2\Pi}^{(1)} + \mathcal{K}_{2\Pi}^{(2a)} \).

4. Anomaly

We now turn to the problem of triangle anomaly. For definiteness, we work in \( SU(3)_c \) QCD with two flavors; we consider \( SU(2)_L \times SU(2)_R \) chiral symmetry limit in which the \( u \) and \( d \) quarks have zero masses.

Consider fermion 2-point function with a vector and an axial-vector current inserted, \( G_4^{Q\mu,\pi\alpha} \), where \( Q\mu \) denote the vector current to which photon couples,

\[ j^\mu = \bar{q} Q \gamma^\mu q , \]

\[ Q = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} + \frac{\tau_3}{2} \\ \frac{1}{6} + \frac{\tau_3}{2} \end{pmatrix} , \quad q = \begin{pmatrix} u \\ d \end{pmatrix} , \]

and \( \pi\alpha \) denotes the axial-vector current to which pion \( \pi^0 \) couples:

\[ j_5^\alpha = \bar{q} \frac{\tau_3}{2} \gamma^\alpha \gamma_5 q . \]
Then the WT identity Eq.(3.10), which was proven for the vector case in the previous section and can straightforwardly be extended to the axial-vector case, implies the following vector and axial-vector WT identity:

\[-i\partial_y \mu_{G^Q_\mu,\pi\alpha}(w, z; y, x) = i\delta^4(y - w) Q G^\pi_3(w, z; x) - i\delta^4(y - z) G^\pi_3(z, x) Q, \]

\[-i\partial_x \alpha_{G^Q_\mu,\pi\alpha}(w, z; y, x) = i\delta^4(x - w) \gamma_5 G^Q_3(w, z; y) + i\delta^4(x - z) G^Q_3(z, y) \gamma_5, \] (4.1a)

where the \( f_{abc} \) term does not appear because of \([Q, \gamma_3/2] = 0\). Therefore, if we close the open fermion legs of \( G^Q_4 \) by inserting another vector vertex factor \( Q^\gamma \), we obtain formally both vector and axial-vector gauge invariant 3-point function:

\[ T^{\alpha\mu\nu}(x, y, z) \equiv i^3 e^2 \text{tr} \left( G^Q_4(z, z; y) Q^\gamma \right) \]

\[ = i^3 e^2 \text{tr} \left( \langle 0 | T \mu (y) j_\alpha (x) \bar{\psi}(z) \psi(z) | 0 \rangle Q^\nu \right) \]

\[ = -i^3 e^2 \langle 0 | T \mu (x) j_\alpha (y) j_\nu (z) | 0 \rangle. \] (4.3)

Indeed the vector and axial-vector gauge invariance of \( T^{\alpha\mu\nu}(x, y, z) \) is immediately seen from Eqs.(4.1) and (4.2):

\[-i\partial_y T^{\alpha\mu\nu}(x, y, z) = e^2 \delta^4(y - z) \left[ \text{tr} \left\{ Q G^{5\alpha}_3(z, z; x) - G^{5\alpha}_3(z, z; x) Q \right\} Q^\nu \right] \]

\[ = 0, \] (4.4)

\[-i\partial_x T^{\alpha\mu\nu}(x, y, z) = e^2 \delta^4(x - z) \left[ \text{tr} \left\{ \frac{\tau_3}{2} \gamma_5 G^{\mu}_3(z, z; y) + G^{\mu}_3(z, z; y) \frac{\tau_3}{2} \gamma_5 \right\} Q^\nu \right] \]

\[ = 0 \ (\text{formally}). \] (4.5)

We should note two things here:
Despite its asymmetric looking in the above definition of $T^{\alpha\mu\nu}$, the two vector vertices $Q\mu$ and $Q\nu$ are in fact on the same footing. Indeed, if the lowest order $K_{2P1}^{(1)}$ is used, for instance, recall that the formal solution for $\Gamma_4$ was given in the form of Fig. 8. Taking it into account also that $\Gamma_3$ was given by Fig. 7, we see that the amplitude $T^{\alpha\mu\nu}$ is written in the following manifestly symmetric form with respect to the two vector vertices and stands for “triangle” diagram as shown in Fig. 12:

$$T^{\alpha\mu\nu} = i^3 e^2 \text{Tr} (S_F \Gamma^\mu S_F \Gamma_5^\alpha S_F \Gamma^\nu) + (\mu \leftrightarrow \nu \text{ cross term}) . \quad (4.6)$$

Here and henceforth, the vertices $\Gamma_3^\alpha$ and $\Gamma_3^{Q\mu}$ are simply denoted as $\Gamma_5^\alpha$ and $\Gamma^\mu$, respectively, for notational simplicity.

Fig. 12. Three point function in ladder approximation.

Inspection shows that the same is also true for more general $K_{2P1}$; for example, when we take $K_{2P1} = K_{2P1}^{(1)} + K_{2P1}^{(2a)}$, the formal solution for $T^{\alpha\mu\nu}$ take the form as given in Fig. 13, again showing manifest symmetry between the two vector vertices.

The above ‘proof’ of the vector and axial-vector gauge invariance is just formal. The three point function $T^{\alpha\mu\nu}$ given in Eq.(4.3) is not well-defined as it stands: the Green function $G_4(w, z; y, x)$ itself is well-defined but $G_4(z, z; y, x)$
at coincident point \( w = z \), appearing after closing the fermion legs, is a divergent quantity and no longer well-defined.

The latter point is of course the well-known problem of anomaly. In order to define the three point function \( T^{\alpha\mu\nu} \) properly, we need a regularization. Let us here adopt the Pauli-Villars-Gupta (PVG) regularization. (A brief discussion for the dimensional regularization is given in Appendix.) Since this regularization maintains the vector gauge-invariance, we have only to examine the axial-vector WT identity: \( q_\alpha T^{\alpha\mu\nu} = ? \). Hereafter we work with taking only the lowest order \( \mathcal{K}_{2\Pi} = \mathcal{K}^{(1)}_{2\Pi} \), for simplicity. Working in momentum space, we denote incoming momentum into the axial-vector vertex \( \Gamma_5^\alpha \) as \( q_\alpha \), and outgoing momenta from the vector vertices \( \Gamma^\mu \) and \( \Gamma^\nu \) as \( p_\mu \) and \( k_\nu \), respectively. In PVG regularization, the regularized 3-point function \( T^{\alpha\mu\nu}_{(\text{reg.})} \) is defined by
\[
T_{\alpha \mu \nu}^{\text{reg.}}(p, k) 
\equiv i^3 e^2 \int \frac{d^4 \ell}{(2\pi)^4} \text{tr} \left[ S_F(\ell) \Gamma^\mu S_F(\ell + p) \Gamma^5 S_F(\ell - k) \Gamma^\nu 
- S_F(\ell) Q \gamma^\mu S_F(\ell + p) \frac{\tau_3}{2} \gamma^\alpha \gamma_5 S_F(\ell - k) Q \gamma^\nu \right] + \text{(cross term)}
\]

(4.7)

where we have omitted the obvious momentum arguments of the vertices and \( S_F \) is the propagator of the PVG regulator field with mass \( M \):

\[
S_F(\ell) \equiv \frac{i}{\ell - M}.
\]

(4.8)

To see that this is really regularized, we actually need the high momentum behavior of the solutions \( S_F, \Gamma^\mu, \Gamma^\nu \) and \( \Gamma^5_5 \). Since the gluon propagator is well regularized as stated in Eq.(2.4), the loop integrals in the SD and BS equations are sufficiently convergent. From those equations one can convince oneself by a simple power counting argument that the solutions approach the free ones as \( \ell \to \infty \):

\[
S_F(\ell) \rightarrow \ell + O(\ell^{-1}), \\
\Gamma^\mu \rightarrow Q \gamma^\mu + O(\ell^{-2}), \\
\Gamma^5_5 \rightarrow \frac{\tau_3}{2} \gamma^\alpha \gamma_5 + O(\ell^{-2}).
\]

(4.9)

Therefore Eq.(4.7) gives a well regularized quantity: the degree of superficial divergence of the loop integral in this expression (4.7) is now only logarithmic (actually, the integral is convergent) and therefore the integral is independent of the shift of the loop momentum \( \ell \) [8], contrary to the unregularized case. For definiteness, however, we choose the momentum assignment for the \( \mu \leftrightarrow \nu \) cross term as shown in Fig. 12. Then, we can rewrite Eq.(4.7) into the form

\[
T_{\alpha \mu \nu}^{\text{reg.}}(p, k) = i^3 e^2 \text{Tr} \left[ (G_4^{Q\mu, \pi \alpha}(\ell, \ell - k; p, q) 
- S_F(\ell) Q \gamma^\mu S_F(\ell + p) \frac{\tau_3}{2} \gamma^\alpha \gamma_5 S_F(\ell - k) 
- S_F(\ell) \frac{\tau_3}{2} \gamma^\alpha \gamma_5 S_F(\ell - q) Q \gamma^\mu S_F(\ell - k) \right) Q \gamma^\nu \right]
\]

(4.10)
with functional trace $\text{Tr}$ implying also the momentum integration $\int d^4\ell/(2\pi)^4$.

Using the momentum space version of Eq. (4.2)

$$q_\alpha G_4^{Q\mu,\pi\alpha}(\ell, \ell - k; p, q) = i\frac{T_3}{2}\gamma_5 G_3^{Q\mu}(\ell - q, \ell - k) + G_3^{Q\mu}(\ell, \ell + p) i\frac{T_3}{2}\gamma_5$$

$$= i\frac{T_3}{2}\gamma_5 S_F(\ell - q)\Gamma^\mu S_F(\ell - k) + S_F(\ell)\Gamma^\mu S_F(\ell + p) i\frac{T_3}{2}\gamma_5$$

(4.11)

and an algebraic identity

$$q/\gamma_5 = S_F^{-1}(\ell + p) i\gamma_5 + i\gamma_5 S_F^{-1}(\ell - k) + 2M\gamma_5$$

(4.12)

we find (see Fig. 14)

$$q_\alpha T^{\alpha\mu\nu}_{\text{reg.}}(p, k)$$

$$= i^3 e^2 \text{Tr} \left[ \left( i\frac{T_3}{2}\gamma_5 S_F(\ell - q)\Gamma^\mu S_F(\ell - k) - i\frac{T_3}{2}\gamma_5 S_F(\ell - q)Q\gamma^\mu S_F(\ell - k) \right) Q\gamma^\nu \right.$$

$$+ \left( S_F(\ell)\Gamma^\mu S_F(\ell + p) i\frac{T_3}{2}\gamma_5 - S_F(\ell)Q\gamma^\mu S_F(\ell + p) i\frac{T_3}{2}\gamma_5 \right) Q\gamma^\nu \right]$$

$$- i^3 e^2 \lim_{M \to \infty} 2M \text{Tr} \left( i\frac{T_3}{2}\gamma_5 S_F(\ell - k)\gamma^\nu S_F(\ell)\gamma^\mu S_F(\ell + p) + \text{(cross term)} \right) .$$

(4.13)

The first trace term in the RHS vanishes if we can shift the integration variable $\ell$ into $\ell + q$ for the two terms in the first line since, then, they take the same form as the two terms in the second line and cancel them exactly by $\{i\frac{T_3}{2}\gamma_5, Q\gamma^\nu\} = 0$. This shift of the integration variable separately for the first and the second term yields nonvanishing surface terms [8], which are, however, the same for both terms and cancel each other thanks to the asymptotic behavior (4.9) of our solutions. The remaining last term reproduces the well-known anomaly:

$$q_\alpha T^{\alpha\mu\nu}_{\text{reg.}}(p, k) = e^2 N_c \frac{4\pi^2}{4\pi^2} \text{tr} (\tau_3 QQ) \epsilon^{\mu\nu\rho\sigma} p^\rho k^\sigma \equiv [\text{Anomaly}] .$$

(4.14)

We thus have shown that our 3-point function calculated by using the nonperturbative fermion propagator $S_F$ and vertices $\Gamma^\alpha_5$ and $\Gamma^\mu$, not only satisfy the
conservation for the vector channels but does in fact reproduce correct anomaly for the axial-vector channel, the latter being in accord with Adler-Bardeen theorem [9]. Note again that this nontrivial property is achieved only when the fermion propagator and the vertices are determined by such SD and BS equations with approximations mutually consistent with each other.

A comment may be in order: although we have discussed only the 3-point function in this section, it is clear that the same method as above can be applied to calculate arbitrary $n$-point Green functions of current operators. Namely, we can
simply close the open fermion legs of the unamputated vertex function $G^{\mu_1 \cdots \mu_{n-1}}_{(n-1)+2}$ by inserting an $n$-th vertex factor $\gamma^{\mu_n}$. The reader can apply this procedure, for instance, to $\Gamma^\mu_{\nu\rho}$ given in Fig. 9 in the ladder approximation and can see which types of the diagrams give gauge invariant 4-point Green function for photons corresponding to $\gamma\gamma \rightarrow \gamma\gamma$ scattering. This example shows that even in the ladder approximation there generally appear corrections intrinsic to the Green functions which can be attributed neither to the propagator nor to the gauge boson vertices.

5. Low Energy Theorem and Pseudoscalar Bound State

Thanks to these properties of our propagator and vertices, the decay amplitude for $\pi^0 \rightarrow 2\gamma$, for instance, satisfies the low-energy theorem. This holds, of course, provided that we use BS amplitude for the boundstate $\pi^0$ calculated by the ‘same’ approximation as that used for propagator and vertices. To see this is the purpose of this section.

We consider the 3-point function $T^{\alpha\mu\nu}(p, k)$ for on-shell photon case, $p^2 = k^2 = 0$, and recall the following fact which holds in the full theory and in our approximations at any order also. The vector gauge invariance ($p_\mu T^{\alpha\mu\nu}(p, k) = k_\nu T^{\alpha\mu\nu}(p, k) = 0$) and the bose symmetry $T^{\alpha\mu\nu}(p, k) = T^{\alpha\nu\mu}(k, p)$ together with odd parity property, determine the most general form of $T^{\alpha\mu\nu}$ as

$$T^{\alpha\mu\nu}(p, k) = \epsilon^{\mu\nu\rho\sigma} p_\rho k_\sigma q^\alpha F_1(q^2) + (\epsilon^{\alpha\mu\rho\sigma} k^\nu - \epsilon^{\alpha\nu\rho\sigma} p^\mu) p_\rho k_\sigma F_2(q^2) . \quad (5.1)$$

* Another possible form

$$A^{\alpha\mu\nu}(p, k) \equiv \epsilon^{\alpha\mu\rho\sigma}(p_\rho - k_\rho) q^2 + (\epsilon^{\alpha\mu\rho\sigma} p^\nu - \epsilon^{\alpha\nu\rho\sigma} k^\mu) p_\rho k_\sigma$$

can actually be expressed in terms of the above two amplitudes: Indeed, from the identity $\epsilon^{\mu\nu\rho\sigma} q^\alpha = 0$, we can prove

$$A^{\alpha\mu\nu}(p, k) = \epsilon^{\mu\nu\rho\sigma} p_\rho k_\sigma q^\alpha - (\epsilon^{\alpha\nu\rho\sigma} k^\mu - \epsilon^{\alpha\nu\rho\sigma} p^\mu) p_\rho k_\sigma .$$
Therefore generally we have

\[
q_{\alpha} T^{\alpha\mu\nu}(p, k) = \epsilon_{\mu\nu\rho\sigma} p_\rho k_\sigma q^2 F_1(q^2). \tag{5.2}
\]

The explicit \(q^2\) factor in the RHS implies that *only the \(\frac{1}{q^2}\) pole term in \(F_1(q^2)\) can contribute* to this amplitude \(q_{\alpha} T^{\alpha\mu\nu}\) in the limit of on-shell pion, \(q^2 \to 0\). (This is the reason why there exits a low-energy theorem to the \(\pi^0 \to 2\gamma\) decay amplitude.) It is important to note that the vector gauge invariance is essential here to this conclusion; indeed, otherwise, the other invariant amplitude term of the form \(\epsilon^{\alpha\mu\nu\rho}(p_\rho - k_\rho) F_3(q^2)\), for instance, could appear in Eq.(5.1) and contribute to this amplitude at \(q^2 = 0\) without having massless pole.

We now also recall the definition of the BS amplitude: the BS amplitude \(\chi\) as well as its conjugate \(\bar{\chi}\) for the pion boundstate \(|\pi(q)\rangle\) normalized invariantly as

\[
\langle \pi(q)|\pi(q')\rangle = (2\pi)^3 2|q|\delta^3(q - q'),
\]

are defined by

\[
\begin{align*}
\langle 0| T\bar{\psi}(x)\psi(y)|\pi(q)\rangle &= e^{-iqX} \int \frac{d^4p}{(2\pi)^4} e^{-ipr} \chi(p; q), \\
\langle \pi(q)| T\bar{\psi}(y)\psi(x)|0\rangle &= e^{iqX} \int \frac{d^4p}{(2\pi)^4} e^{ipr} \bar{\chi}(p; q),
\end{align*}
\tag{5.3}
\]

where \(X \equiv (x + y)/2\) and \(r \equiv x - y\). It is easy to see that CPT invariance leads to the relation \(\bar{\chi}(p; q) = \chi(p; -q)\), because of which we often denote \(\bar{\chi}\) as \(\chi\) below for notational simplicity. This boundstate generally appears as a massless pole term in the 4-point amplitude \(G_4 \equiv \langle 0| T\bar{\psi}\psi\bar{\psi}|0\rangle\) in the form:

\[
G_4 = \chi \frac{i}{q^2} \bar{\chi} + \text{regular term at } q^2 = 0. \tag{5.4}
\]

If we close the first \(\psi\) and \(\bar{\psi}\) legs of \(G_4\) by inserting \(\frac{\not{p}}{2} \gamma^\alpha\), it becomes the axial-vector vertex \(\Gamma^\alpha_5\) with fermion propagators attached. Therefore the pion pole appears in
the axial-vector vertex in the form (see Fig. 15):

$$\Gamma_5^\alpha = i f_\pi q^\alpha \frac{i}{q^2} \hat{\chi} + \text{(regular term)} ,$$  \hspace{1cm} (5.5)

where $\hat{\chi}$ is the amputated BS amplitude, $\hat{\chi} = S^{-1}_F \chi S^{-1}_F$, and the decay constant $f_\pi$ is given by

$$if_\pi q^\alpha = \langle 0 | \bar{\psi}(0) \frac{\tau_3}{2} \gamma^\alpha \gamma_5 \psi(0) | \pi(q) \rangle$$
$$= - \int \frac{d^4 p}{(2\pi)^4} \text{tr}[\frac{\tau_3}{2} \gamma^\alpha \gamma_5 \chi(p; q)].$$  \hspace{1cm} (5.6)

These are general arguments. If we use an approximation to determine $G_4$ and $\Gamma_5^\alpha$, the BS amplitude $\chi$ is also determined accordingly, via Eq.(5.4) or (5.5): for instance, if we use the ladder approximation ($K_{2\pi_1} = K_{2\pi_1}^{(1)}$), then the 4-point function $G_4$ is the ladder $L$ given in Fig. 6 and therefore the BS amplitude $\chi$ should be determined by the homogeneous ladder BS equation as shown in Fig. 16. The decay constant $f_\pi$ is also the one calculated by Eq.(5.6) using this BS amplitude.

Fig. 15. BS amplitude of the pion, $\hat{\chi}$, in the axial-vector vertex, $\Gamma_5^\alpha$.

Fig. 16. Graphical representation of homogeneous ladder BS equation for $\hat{\chi}$.
The massless pole in $T^{\alpha\mu\nu}$ required in Eq.(5.2) is provided by the pion pole in the axial-vector vertex $\Gamma_5^\alpha$ as given in (5.5). In the ladder approximation, the 3-point function $T^{\alpha\mu\nu}$ is given in the form (4.6) so that we find

$$
\lim_{q^2 \to 0} q_\alpha T^{\alpha\mu\nu}(p,k) = ie^2 f_\pi \text{Tr} (\hat{\chi} S_F \Gamma^\nu S_F \Gamma^\mu S_F) + \text{(cross term)},
$$

(5.7)

which is shown in Fig. 17.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig17.pdf}
\caption{Graphical representation of the function $T^{\mu\nu}(p,k)$.}
\end{figure}

Combining with Eq.(4.14), this proves that

$$
T^{\mu\nu}(p,k) \equiv i e^2 \text{Tr} (\hat{\chi} S_F \Gamma^\nu S_F \Gamma^\mu S_F) + \text{(cross term)} = \frac{1}{f_\pi} [\text{Anomaly}].
$$

(5.8)

Namely, when the BS amplitude $\chi$ is calculated in the ladder approximation, then the $\pi^0 \to 2\gamma$ amplitude consistent with the low energy theorem can be obtained by the LHS of Eq.(5.8) with vector vertices and fermion propagator which are also calculated by the same order of approximation. It will also be easy to convince oneself that in the next order of approximation taking $K_{2P1} = K_{2P1}^{(1)} + K_{2P1}^{(2a)}$, the correct amplitude satisfying the low-energy theorem can be obtained by calculating the diagrams given in the RHS of Fig. 13 with $\Gamma_5^\alpha$ vertex replaced by the BS amplitude $\hat{\chi}$. 
6. Saturation by zeroth order term

In the ladder approximation, there is a gauge in which the solution to the SD equation take the form

\[ iS_F^{-1}(p) = p - \Sigma(p^2) \]  

(6.1)

without wave function renormalization. [This is the case in Landau gauge for fixed coupling case \(^\star\), and there is a non-local gauge realizing Eq.(6.1) even for running coupling case \([3, 10]\).] In such a case, there appears an interesting phenomenon that the saturation of the low-energy theorem by the LHS of Eq.(5.8) is already realized by the zeroth order terms in the external momenta of the BS amplitude \(\hat{\chi}\) and the vector vertices \(\Gamma^\mu\) and \(\Gamma^\nu\). This is actually surprising because the linear terms in the external momenta individually give non-vanishing contributions to the amplitude. The net effect of those linear terms vanishes. Let us explain this fact.

First let us see what is the zeroth order term in the BS amplitude \(\chi\). The WT identity Eq.(3.9) for the axial-vector vertex reads

\[ q_\mu \Gamma^\mu_5 \left( p + \frac{q}{2}, p - \frac{q}{2} \right) = iS_F^{-1} \left( p + \frac{q}{2} \right) \frac{\tau_3}{2} \gamma_5 + i\gamma_5 \frac{\tau_3}{2} S_F^{-1} \left( p - \frac{q}{2} \right) . \]  

(6.2)

Consider \(q^\mu \to 0\) limit on both sides: For the LHS, only the \(1/q^2\) pseudoscalar pole term given in Eq.(5.5) survives in this limit and yields

\[ \lim_{q \to 0} \text{LHS} = -f_\pi \hat{\chi}(p; q = 0) . \]  

(6.3)

When \(S_F\) takes the form Eq.(6.1), the RHS, on the other hand, gives

\[ \lim_{q \to 0} \text{RHS} = -2\Sigma(p^2) \frac{\tau_3}{2} \gamma_5 . \]  

(6.4)

\(^\star\) If we use the regularized gluon propagator, \((2.4)\), there is a slight deviation from Landau gauge in the region \(p^2 \gtrsim \Lambda^2 [3]\).
So we find
\[ \hat{\chi}(p; q = 0) = \frac{2}{f_\pi} \Sigma(p^2) \frac{\tau_3}{2} \gamma_5 \equiv \hat{\chi}_0(p) . \] (6.5)

This is an exact result giving the zeroth order term in \( q \) as far as \( S_F \) takes the form Eq.(6.1). [Pagels-Stokar’s proposal [11] is essentially to regard this as an approximate BS amplitude up to order \( q^2 \):

\[ \hat{\chi}_{PS}(p; q) = \frac{2}{f_\pi} \Sigma(p^2) \frac{\tau_3}{2} \gamma_5 + O(q^2) . \] (6.6)

As a matter of fact, there appear three \( O(q) \) terms which generally take the form

\[ \hat{\chi}(p, q) = \frac{2}{f_\pi} \Sigma(p^2) \frac{\tau_3}{2} \gamma_5 + \hat{\chi}_{\text{linear}} + O(q^2) , \]

\[ \hat{\chi}_{\text{linear}} \equiv \left( \hat{P}(p^2)(p \cdot q)\gamma' + \hat{Q}(p^2)\gamma + \hat{T}(p^2)\frac{1}{2}(\gamma'q - q\gamma) \right) \gamma_5 . \] (6.7)

So the Pagels-Stokar approximation corresponds to discarding all those \( O(q) \) amplitudes.]

Next let us see the zeroth order form in the vector vertex \( \Gamma^\mu \). Again from the vector WT identity (3.9) and the form (6.1) of \( S_F \), we find

\[ p_\mu \Gamma^\mu \left( \ell + \frac{p}{2}, \ell - \frac{p}{2} \right) = i S_F^{-1} \left( \ell + \frac{p}{2} \right) Q - Q i S_F^{-1} \left( \ell - \frac{p}{2} \right) = p_\mu \left[ \gamma^\mu - 2\ell^\mu f(\ell^2, p \cdot \ell) \right] Q , \] (6.8)

with

\[ f(\ell^2, p \cdot \ell) \equiv \sum_{k=0}^{\infty} \frac{(p \cdot \ell)^2}{(2k+1)!} \left( \frac{d}{d\ell} \right)^{2k+1} \Sigma(\ell^2) \] (6.9)

for \( p^2 = 0 \) (on-shell photon). This tells us that

\[ \Gamma^\mu \left( \ell + \frac{p}{2}, \ell - \frac{p}{2} \right) = \left[ \gamma^\mu - 2\ell^\mu f(\ell^2, p \cdot \ell) + \Gamma^\mu_{\text{tr}} \right] Q , \] (6.10)

where \( \Gamma^\mu_{\text{tr}} \) stands for transversal part satisfying \( p_\mu \Gamma^\mu_{\text{tr}} = 0 \) by itself. Since \( \Gamma^\mu_{\text{tr}} \).
generally takes the form
\[
\Gamma_{\text{tr.}}^\mu = \frac{1}{2} \{ \gamma^\mu \not{p} - \not{p} \gamma^\mu \} A(\ell^2, p \cdot \ell) + \epsilon^{\mu \nu \rho \sigma} \not{\ell} \gamma_\rho \gamma_5 B(\ell^2, p \cdot \ell) \\
+ (p^2 \ell^\mu - \mu \not{p} \cdot \ell) [C_1(\ell^2, p \cdot \ell) + p C_2(\ell^2, p \cdot \ell) + f C_3(\ell^2, p \cdot \ell)] \tag{6.11}
\]
and is at least linear in the external momentum \( p \), the zeroth order term in \( p \) of \( \Gamma^\mu \) is given by
\[
\Gamma^\mu \bigg|_{p=0} = \gamma^\mu - 2 \ell^\mu \Sigma(\ell^2) \equiv \Gamma_0^\mu(\ell). \tag{6.12}
\]
Note that \( \ell^\mu \) here stands for the average momentum of those of incoming and outgoing fermions.

Let us now show that the zeroth order terms \( \hat{\chi}_0 \) and \( \Gamma_0^\mu \) of the BS amplitude and the vector vertices already saturate the low-energy theorem; namely, the equality
\[
\mathcal{T}^{\mu \nu}(p, k) = \mathcal{T}_0^{\mu \nu}(p, k) = \frac{1}{f_\pi} [\text{Anomaly}], \tag{6.13}
\]
holds for the on-shell photons and pion \( (p^2 = k^2 = q^2 = 0) \). A key observation for this equality is that
\[
q_\alpha \frac{\tau_3}{2} \gamma^\alpha \gamma_5 = (\not{\ell} + \not{p} - \Sigma(\ell + p)) \frac{\tau_3}{2} \gamma_5 + \gamma_5 \frac{\tau_3}{2} (\not{\ell} - \not{k} - \Sigma(\ell - k)) \\
+ (\Sigma(\ell + p) + \Sigma(\ell - k)) \frac{\tau_3}{2} \gamma_5
\]
\[
= i S_F^{-1}(\ell + p) \frac{\tau_3}{2} \gamma_5 + \frac{\tau_3}{2} \gamma_5 i S_F^{-1}(\ell - k) + \hat{\chi}_0 \left( \ell + \frac{p - k}{2} \right) + O(q^2). \tag{6.15}
\]
Eliminating \( \hat{\chi}_0 \) using this identity, we find
\[
f_\pi \mathcal{T}_0^{\mu \nu}(p, k) = -i e^2 \left[ \text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\mu S_F \Gamma_0^\nu \right) + \text{Tr} \left( S_F \Gamma_0^\mu S_F \frac{\tau_3}{2} \gamma_5 \Gamma_0^\nu \right) \right] \\
+ \text{Tr} \left( S_F i \frac{\tau_3}{2} \gamma_5 \Gamma_0^\mu S_F \Gamma_0^\nu \right) + \text{Tr} \left( S_F \Gamma_0^\mu i \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\nu \right) \right] \tag{6.16}
\]
\[
+ i e^2 \left[ q_\alpha \text{Tr} \left( \frac{\tau_3}{2} \gamma^\alpha \gamma_5 S_F \Gamma_0^\mu S_F \Gamma_0^\nu S_F \right) + \text{(cross term)} \right]
\]
since the last $O(q^2)$ term in Eq.(6.15) cannot contribute to the on-shell amplitude after the loop integration. Although we have suppressed the momentum arguments here, it is important to note that the factor $\frac{q^2}{2} \gamma_5$ indicates also the place at which the momentum $q$ flows in, so that it is not anticommutative with the vector vertex $\Gamma^\mu_0$ (or $\Gamma'^\nu_0$) which has nontrivial dependence of the leg momenta. The LHS is well-defined since the loop integration is convergent because of the presence of the BS amplitude $\hat{\chi}_0$ which damps sufficiently rapidly. But the terms in the RHS are not well-defined separately. In order to make each term well-defined, we add to the RHS the identity for the PVG propagator $S_F$,

$$0 = -ie^2 \left( S_F q_\alpha \frac{\tau_3}{2} \gamma_5 S_F - i \frac{\tau_3}{2} \gamma_5 S_F - S_F i \frac{\tau_3}{2} \gamma_5 - 2M S_F \frac{\tau_3}{2} \gamma_5 S_F \right) Q \gamma^\nu S_F Q \gamma^\mu$$

(following from Eq.(4.12) ) and its $\mu \leftrightarrow \nu$ exchanged one. Then Eq.(6.16) reads:

$$f_\pi T^\mu_0 (p, k) = -ie^2 \left[ \text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\mu S_F \Gamma_0^\nu \right)_{\text{PVG}} + \text{Tr} \left( S_F \Gamma_0^\mu S_F i \frac{\tau_3}{2} \gamma_5 \Gamma_0^\nu \right)_{\text{PVG}} \right]$$

$$+ \text{Tr} \left( S_F i \frac{\tau_3}{2} \gamma_5 \Gamma_0^\mu S_F \Gamma_0^\nu \right)_{\text{PVG}} + \text{Tr} \left( S_F \Gamma_0^\mu i \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\nu \right)_{\text{PVG}}$$

$$+ ie^2 \left[ q_\alpha \text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\nu S_F \Gamma_0^\gamma \right)_{\text{PVG}} + (\text{cross term}) \right]$$

$$+ ie^2 \lim_{M \to \infty} \left[ 2M \text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F Q \gamma^\nu S_F Q \gamma^\mu S_F \right) + (\text{cross term}) \right],$$

(6.17)

where each trace term with suffix PVG denotes the regularized one obtained by subtracting the same form term with replacements $S_F \rightarrow S_F$, $\Gamma_0^\mu(\nu) \rightarrow Q \gamma^\mu(\nu)$ made. For instance, the first term reads

$$\text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\mu S_F \Gamma_0^\nu \right)_{\text{PVG}} = \text{Tr} \left( \frac{\tau_3}{2} \gamma_5 S_F \Gamma_0^\mu S_F \Gamma_0^\nu - i \frac{\tau_3}{2} \gamma_5 S_F Q \gamma^\mu S_F Q \gamma^\nu \right).$$

(6.18)

In this regularized form we can make the shift of loop integration variables freely (\textit{i.e.}, producing no surface term) as explained before. Then the first four trace terms in Eq.(6.17) cancel among themselves and vanish: Indeed, in order for such
‘biangle’ diagrams with $\gamma_5$ inserted to give non-vanishing contribution, at least four $\gamma$ matrices have to be picked up, so that only the (momentum-independent) elementary vertex parts $Q\gamma^\mu$ and $Q\gamma^\nu$ in $\Gamma^\mu_0$ and $\Gamma^\nu_0$ can contribute. But then, the factors $i\frac{2}{3}\gamma_5$ anti-commute with $\Gamma^\mu_0$ or $\Gamma^\nu_0$ and those diagrams cancel with each other. This is the case since the shift of integration variables is allowed by the above ‘regularization’. But, this ‘regularization’ introduced an extra contribution given by the last term in Eq.(6.17), which is precisely the anomaly required by the low-energy theorem. So what now remains to be proved is that the second term in Eq.(6.17) vanishes:

$$q_\alpha T^{\alpha\mu\nu}_0 (p, k) = 0$$

$$T^{\alpha\mu\nu}_0 (p, k) \equiv i^3 e^2 \text{Tr} \left( \frac{2}{3} \gamma^\alpha \gamma_5 S_F \Gamma^\nu_0 \Gamma^\mu_0 S_F \right)_{\text{PVG}} + \text{(cross term)}.$$  \hspace{1cm} (6.19)

This can be proven if we can show that the amplitude $T^{\alpha\mu\nu}_0$ satisfies (vector) gauge invariance, $p_\mu T^{\alpha\mu\nu}_0 = k_\nu T^{\alpha\mu\nu}_0 = 0$. Indeed, then, together with the Bose symmetry, $T^{\alpha\mu\nu}_0$ has to have the general form Eq.(5.1) and hence $q_\alpha T^{\alpha\mu\nu}_0$ can be nonzero at $q^2 = 0$ only when a massless pole $1/q^2$ term exists in the axial-vector channel. But the diagram of $T^{\alpha\mu\nu}_0$ is clearly regular at $q^2 = 0$ since the fermion here carries non-zero mass function $\Sigma$, and hence $q_\alpha T^{\alpha\mu\nu}_0 = 0$ on the mass shell $q^2 = 0$.

To show the gauge-invariance of $T^{\alpha\mu\nu}_0$, we note that we can replace the zeroth order vertex function $\Gamma^\mu_0$ and $\Gamma^\nu_0$ in Eq.(6.19) by the full non-transverse parts $\gamma^\mu - 2\ell^\mu f(\ell^2, p \cdot \ell)$ and $\gamma^\nu - 2\ell^\nu f(\ell^2, k \cdot \ell)$, respectively, defined in Eq.(6.8), since the differences are of $O(p^2)$ and $O(k^2)$ and vanish on mass shell ($q^2 = p^2 = k^2 = 0$) after loop integration. Then, with these replacements performed, we can use the vector WT identity Eq.(6.8) to obtain

$$p_\mu T^{\alpha\mu\nu}_0 (p, k) = i^3 e^2 \left[ \text{Tr} \left( \frac{2}{3} \gamma^\alpha \gamma_5 S_F \Gamma^\nu_0 \Gamma^\mu_0 S_F iQ \right) - \text{Tr} \left( \frac{2}{3} iQ \gamma^\alpha \gamma_5 S_F \Gamma^\nu_0 \Gamma^\mu_0 S_F \right) \right]$$

$$- i^3 e^2 \left[ \text{Tr} \left( \frac{2}{3} \gamma^\alpha \gamma_5 S_F \Gamma^\nu_0 iQ S_F \right) - \text{Tr} \left( \frac{2}{3} \gamma^\alpha \gamma_5 S_F iQ \Gamma^\nu_0 S_F \right) \right].$$  \hspace{1cm} (6.20)

Here $Q$ indicates also the position from which the momentum $p$ flows out. The first two terms clearly cancel with each other. The second two terms also cancel.

31
since, again, only the (momentum-independent) elementary vertex part \( Q \gamma^\nu \) in \( \Gamma_0^\nu \) can contribute because of the presence of \( \gamma_5 \) in the trace and hence the factor \( Q \), indicating also the momentum-flow position, becomes commutative with \( \Gamma_0^\nu \). This finish the proof.

It is interesting to see explicitly how the diagram (6.14) with lowest order terms \( \hat{\chi}_0, \Gamma_0^\mu \) and \( \Gamma_0^\nu \) inserted actually reproduce the value of the low-energy theorem:

\[
\mathcal{T}_0^{\mu\nu}(p, k) = -ie^2 \int \frac{d^4 \ell}{i(2\pi)^4} \left\{ \text{tr} \left[ \hat{\chi}_0 \left( \ell + \frac{p - k}{2} \right) \frac{1}{\Sigma(\ell - k) - \ell + k} \Gamma_0^\mu \left( \ell - \frac{k}{2} \right) \right] \right. \\
\left. \times \frac{1}{\Sigma(\ell) - \ell} \Gamma_0^\nu \left( \ell + \frac{p}{2} \right) \frac{1}{\Sigma(\ell + p) - \ell - p} \right\} + (p \leftrightarrow k, \mu \leftrightarrow \nu) \\
= e^2 \frac{N_c}{4\pi^2} \text{tr} (\tau_3 QQ) \epsilon^{\mu\nu\rho\sigma} p^\rho k^\sigma \frac{2}{f_N} \int dx \frac{x \Sigma(x) (\Sigma(x) - 2x \Sigma'(x))}{(\Sigma^2(x) + x)^3},
\]  

(6.21)

where \( x = -\ell^2 \) denotes the Euclidean momentum. Interestingly enough, the value of the integral in the last line is \( \frac{1}{2} \) irrespectively of the detailed functional form of (nonzero) \( \Sigma(x) \). This can easily be seen if one can make a change of the integration variable \( x \) into dimensionless one \( y = x/\Sigma^2(x) \). Thus the lowest order terms correctly reproduce the value of the low-energy theorem.

**ACKNOWLEDGEMENTS**

T.K. is supported in part by the Grant-in-Aid for Scientific Research (#04640292) from the Ministry of Education, Science and Culture.
APPENDIX A. Dimensional Regularization

If we use the dimensional regularization instead of PVG regularization, the identity used in the algebraic manipulation deriving the WT identity Eq.(4.5) is changed to

\[ q \gamma_5 = \left[ \left( \ell + \frac{q}{2} \right) - \left( \ell - \frac{q}{2} \right) \right] \gamma_5 \]

\[ = \left[ \ell + \frac{q}{2} - \sum \left( \left( \ell + \frac{q}{2} \right)^2 \right) \right] \gamma_5 - \gamma_5 \left[ \ell - \frac{q}{2} - \sum \left( \left( \ell - \frac{q}{2} \right)^2 \right) \right] \]

\[ + \sum \left( \left( \ell + \frac{q}{2} \right)^2 \right) - \sum \left( \left( \ell - \frac{q}{2} \right)^2 \right) \gamma_5 - 2\ell \gamma_5 , \tag{A.1} \]

where \( \ell \) is the (now \( n \)-dimensional) loop momentum and \( l \) is its extra dimensional part \( (\ell = \ell^{(4)} + l) \). The first 3 terms in the RHS is the usual one and the usual proof of the WT identity applies to those terms: then, they vanish since now the integral is regularized and the shift of the integration variable is justified. So we have to evaluate only the contribution from the last term coming from the extra dimension part \( l \):

\[ q_\alpha T^{\alpha \mu \nu}(p,k) = 2ie^2 l_\alpha \text{Tr} (\Gamma_5 S_F \Gamma^\nu S_F \Gamma^\mu S_F) + \text{(cross term)} . \tag{A.2} \]

Since \( l \propto n - 4 \), only the divergent parts can contribute in the RHS. Since the propagator \( S_F \) and the vertices \( \Gamma^\mu \), \( \Gamma^\nu \) and \( \Gamma_5^0 \) asymptotically behave as free one as explained in (4.9), we can evaluate as follows:

\[ (\text{RHS}) = \lim_{n \to 4} 2ie^2 \text{Tr} (\ell \gamma_5 S_{FS=0} Q \gamma_\nu S_{FS=0} Q \gamma_\mu S_{FS=0}) \cdot \tag{A.3} \]

But this is again the same diagram which leads to the well-known anomaly.
REFERENCES

N1 For a recent review of these methods, see, for example, T. Kugo, in *Proc. of 1991 Nagoya Spring School of Dynamical Symmetry Breaking*, Apr. 23–27, 1991, ed. K. Yamawaki (World Scientific Pub. Co., Singapore, 1991). N2 T. Maskawa and H. Nakajima, Prog. Theor. Phys. 52 (1974) 1326; 54 (1975) 860. N3 T. Kugo and M.G. Mitchard, Phys. Lett. B282 (1992) 162; Phys. Lett. B286 (1992) 355. N4 S.L. Adler, Phys. Rev. 177 (1969) 2426; J.S. Bell and R. Jackiw, Nuovo Cim. 60A (1969) 46. N5 Y. Nambu and G. Jona-Lasinio, Phys. Rev. 10 (1961) 345. N6 C. De Dominicis and P.C. Martin, J. Math. Phys. 5 (1964) 14; 5 (1964) 31. N7 J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10 (1974) 2428. N8 R. Jackiw, in *Lectures on Current Algebra and its Applications*, by S.B. Treiman, R. Jackiw and D.J. Gross, (Princeton Univ. Press, 1972). N9 S.L. Adler and W.A. Bardeen, Phys. Rev. 182 (1969) 1517; W.A. Bardeen, Phys. Rev. 184 (1969) 1848. N10 H. Georgi, E.H. Simmons and A.G. Cohen, Phys. Lett. B236 (1990) 183. N11 H. Pagels and S. Stokar, Phys. Rev. D20 (1979) 2947.