ITERATED CIRCLE BUNDLES AND INFRANILMANIFOLDS

IGOR BELEGRADEK

ABSTRACT. We give short proofs of the following two facts: Iterated principal circle bundles are precisely the nilmanifolds. Every iterated circle bundle is almost flat, and hence diffeomorphic to an infranilmanifold.

A infranilmanifold is a closed manifold diffeomorphic to the quotient space \( N/\Gamma \) of a simply-connected nilpotent Lie group \( N \) by a discrete torsion-free subgroup \( \Gamma \) of the semidirect product \( N \rtimes C \) where \( C \) is a maximal compact subgroup of \( \text{Aut}(N) \). If \( \Gamma \) lies in the \( N \) factor, the infranilmanifold is called a nilmanifold.

An iterated circle bundle is defined inductively as the total space of a circle bundle whose base is an iterated circle bundle of one dimension lower, and the base at the first step is a point. If at each step the circle bundle is principal, the result is an iterated principal circle bundle.

This note was prompted by a question of Xiaochun Rong who asked me to justify the following fact mentioned in \([BW02]\):

**Theorem 1.** A manifold is an iterated principal circle bundle if and only if it is a nilmanifold.

The proof of Theorem 1 combines some bundle-theoretic considerations with classical results of Mal’cev \([Mal49]\). The “if” direction was surely known since \([Mal49]\) but \([FH86, \text{Proposition 3.1}]\) seems to be the earliest reference. The statement of Theorem 1 is mentioned without proof in \([Wei94, \text{p.98}]\) and \([FOT08, \text{p.122}]\).

**Summary of previous work:**

(1) Every iterated principal circle bundle has torsion-free nilpotent fundamental group because the homotopy exact sequence converts a principal circle bundle into a central extension with infinite cyclic kernel.

(2) Theorem 1.2 of \([Nak14]\) implies that every iterated principal circle bundle is diffeomorphic to an infranilmanifold; this was explained to me by Xiaochun Rong. Thus \([Nak14]\) gives another (less elementary) proof of the “only if” direction in Theorem 1 because every iterated principal circle bundle is homotopy equivalent to

---

*Key words and phrases.* nilmanifold, almost flat, circle bundle, nilpotent group.

*2010 Mathematics Subject classification.* Primary 20F18, Secondary 57R22.

This work was partially supported by the Simons Foundation grant 524838.
a nilmanifold, and the diffeomorphism type of an infranilmanifold is determined by its homotopy type [LR84].

(3) According to [PS61] a manifold is a principal torus bundle over a torus if and only if it is a nilmanifold modelled on a two-step nilpotent Lie group.

(4) Every 3-dimensional infranilmanifold has a unique Seifert fiber space structure, see [Sco83, Theorem 3.8], hence it is an iterated circle bundle if and only if the base orbifold (of the Seifert fibering) is non-singular, i.e., the 2-torus or the Klein bottle. Thus iterated circle bundles are rare among 3-dimensional infranilmanifolds.

(5) In [LM13] it is proven that every iterated circle bundle is homeomorphic to an infranilmanifold. Their argument splits in two parts: finding a homotopy equivalence and upgrading it to a homeomorphism. The latter uses topological surgery, which does not extend to the smooth setting.

(6) A natural way to establish the smooth version of the above-mentioned result in [LM13] is to show that every iterated circle bundle is almost flat, and then apply the celebrated work of Gromov-Ruh [Gro78, Ruh82] that infranilmanifolds are precisely the almost flat manifolds. Recall that a closed manifold is almost flat if it admits a sequence of Riemannian metrics of uniformly bounded diameters and sectional curvatures approaching zero. To this end we prove:

**Theorem 2.** Any iterated circle bundle is almost flat, and therefore diffeomorphic to an infranilmanifold.

**Proof of Theorem 1.** We use [Rag72, Chapter II] as a reference for Mal’cev’s work. If $N/\Gamma$ is a nilmanifold, then $\Gamma$ is finitely generated, torsion-free, and nilpotent, and conversely, any such group is the fundamental group of a nilmanifold, see [Rag72, Theorem 2.18]. Every automorphism of $\Gamma$ extends uniquely to an automorphism of $N$, see [Rag72, Theorem 2.11]. Applying this to conjugation by an element of the center of $\Gamma$ we get the inclusion of centers $Z(\Gamma) \subset Z(N)$. Nilpotency of $\Gamma$ ensures that $Z(\Gamma)$ is nontrivial, and therefore, there is a one-parameter subgroup $R \leq Z(N)$ such that $R \cap Z(\Gamma)$ is nontrivial, and hence infinite cyclic. Clearly $R \cap \Gamma = R \cap Z(\Gamma)$. The left $R$-action on $N$ descends to a free $R/(R \cap \Gamma)$-action on $N/\Gamma$, which makes $N/\Gamma$ into a principal circle bundle whose base $B_{\Gamma}$ is a nilmanifold, namely, the quotient of $N/R$ by $\Gamma/(R \cap \Gamma)$. This proves the “if” direction.

Conversely, let $p: E \to B$ be a principal circle bundle over a nilmanifold $B$. Its homotopy exact sequence is a central extension, so $\pi_1(E)$ is finitely generated torsion-free nilpotent. Consider a nilmanifold $N/\Gamma$ with $\Gamma \cong \pi_1(E)$, and let $z \in Z(\Gamma)$ be the element corresponding to the circle fiber of $p$ through the basepoint. Let $R \leq N$ be the one-parameter subgroup that contains $z$. As above $R \subset Z(N)$ and $N/\Gamma$ is the total space of a principal circle bundle $p_{\Gamma}: N/\Gamma \to B_{\Gamma}$ whose base $B_{\Gamma}$ is a nilmanifold and the fibers are the $R/(R \cap \Gamma)$-orbits. The cyclic group $R \cap \Gamma$ is generated by $z$ because its generator projects to a finite order element in the torsion-free group $\Gamma/(z) \cong \pi_1(B)$. Thus the isomorphism $\pi_1(E) \cong \pi_1(N/\Gamma)$ descends to
an isomorphism $\pi_1(B) \to \pi_1(B_\Gamma)$. Since all these manifolds are aspherical, the fundamental group isomorphisms are induced by homotopy equivalences, and we get a homotopy-commutative square

$$
\begin{array}{ccc}
E & \xrightarrow{\varepsilon} & N/\Gamma \\
p & \downarrow & \downarrow p_\Gamma \\
B & \xrightarrow{\beta} & B_\Gamma
\end{array}
$$

where $\varepsilon$ and $\beta$ are homotopy equivalences. We can assume that $\beta$ is a diffeomorphism because by [Rag72, Theorem 2.11] any homotopy equivalence of nilmanifolds is homotopic to a diffeomorphism. The Gysin sequence implies that the Euler class of a circle bundle generates the kernel of the homomorphism induced on the second cohomology by the bundle projection. The map of the Gysin sequences of $p$ and $p_\Gamma$ induced by the commutative square shows that $\beta$ preserves their Euler classes up to sign, and after changing the orientation if necessary we can assume that the Euler classes are preserved by $\beta$. The isomorphism type of a principal circle bundle is determined by its Euler class. Since $p$ and the pullback of $p_\Gamma$ via $\beta$ have the same Euler class, they are isomorphic, which gives a desired diffeomorphism of $E$ and $N/\Gamma$ and completes the proof of the “only if” direction. □

**Proof of Theorem 2.** In view of [Gro78, Ruh82] it is enough to prove inductively that the total space of any circle bundle over an almost flat manifold is almost flat. This comes via the following standard argument. Let $p: E \to B$ be a smooth circle bundle over a closed manifold $B$. For any Riemannian metric $\hat{g}$ on $B$ there is a metric $g$ on $E$ such that $p$ is a Riemannian submersion with totally geodesic fibers which are isometric to the unit circle, see [Bes08, 9.59]. As in [Bes08, 9.67] let $g^t$ be the metric on $E$ obtained by rescaling $g$ by a positive constant $t$ along the fibers of $p$, i.e., $g^t$ and $g$ have the same vertical and horizontal distributions $V$, $H$, and $g^t|_V = tg|_V$ and $g^t|_H = g|_H$. The fibers of $p$ are $g^t$-totally geodesic [Bes08, 9.68] so the $T$ tensor vanishes. The diameters of $g^t$, $\hat{g}$ satisfy $\text{diam}(g^t) \leq \text{diam}(\hat{g}) + O(\sqrt{t})$. The following lemma finishes the proof of almost flatness of $E$. □

**Lemma 3.** The sectional curvatures $K^t$, $\hat{K}$ of $g^t$, $\hat{g}$ satisfy $|K^t| \leq |\hat{K}| + O(\sqrt{t})$.

**Proof.** Fix any 2-plane $\sigma$ tangent to $E$. Since $H$ has codimension one, $\sigma$ contains a $g^t$-unit horizontal vector $X$. Let $C$ be a $g^t$-unit vector in $\sigma$ that is $g^t$-orthogonal to $X$. Write $C = U + Y$ where $U \in \mathcal{V}$, $Y \in \mathcal{H}$. The sectional curvature of $\sigma$ with respect to $g^t$ is given by

$$K^t_\sigma = \langle R^t(C, X)C, X \rangle^t = \langle R^t(Y, X)Y, X \rangle^t + 2\langle R^t(Y, X)U, X \rangle^t + \langle R^t(U, X)U, X \rangle^t$$

where $\langle C, D \rangle^t := g^t(C, D)$ and $R^t$ is the curvature tensor of $g^t$.

Lemma 9.69 of [Bes08] relates the $A$ tensors $A^t$, $A$ of $g^t$, $g$ as follows: $A^t_YX = A_YX$ and $A^t_XU = tA_XU$. Recall that $A_YX$ is vertical and $A_XU$ is horizontal. The formulas in [Bes08, 9.28, 9.69] give
\[ g(\tilde{R}(Y, X)\tilde{Y}, \tilde{X}) - \langle R^t(Y, X)Y, X \rangle^t = 3\langle A^t_Y X, A^t_Y X \rangle^t = 3t g(A_Y X, A_Y X) \]

\[ \langle R^t(Y, X)U, X \rangle^t = -\left[\langle (D_X A)_Y X, U \rangle \right]^t = -t g((D_X A)_Y X, U) \]

\[ \langle R^t(U, X)U, X \rangle^t = \langle A^t_X U, A^t_X U \rangle^t + \left[\langle (D_U A)_X X, U \rangle \right]^t = t^2 g(A_X U, A_X U) \]

where \([\langle (D_U A)_X X, U \rangle]^t = 0\) by the last formula in [Bes08, 9.32].

Since \(g(X, X) = 1 = g^t(C, C) = g(Y, Y) + tg(U, U)\), the vectors \(X, Y, \sqrt{t}U\) lie in the \(g\)-unit disk bundle of \(TE\), which is compact, so the functions \(g(A_Y X, A_Y X)\), \(\sqrt{t}g((D_X A)_Y X, U)\), \(tg(A_X U, A_X U)\) are bounded.

Therefore, if \(Y \neq 0\) and \(\sigma\) is the projection of \(\sigma\) in \(TB\), then

\[ K^t_\sigma = \tilde{g}(\tilde{R}(Y, X)\tilde{Y}, \tilde{X}) + O(\sqrt{t}) = \sqrt{\tilde{g}(Y, Y)} K_\sigma + O(\sqrt{t}) \]

and if \(Y = 0\), then \(K^t_\sigma = t^2 g(A_X U, A_X U) = O(t)\). Thus \(|K^t_\sigma| \leq |K_\sigma| + O(\sqrt{t})\). \(\Box\)

References

[Bes08] A. L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition.

[BW02] I. Belegradek and G. Wei, *Metrics of positive Ricci curvature on vector bundles over nilmanifolds*, Geom. Funct. Anal. 12 (2002), no. 1, 56–72.

[FH86] E. Fadell and S. Husseini, *On a theorem of Anosov on Nielsen numbers for nilmanifolds*, Nonlinear functional analysis and its applications (Maratea, 1985), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 173, Reidel, Dordrecht, 1986, pp. 47–53.

[FOT08] Y. Félix, J. Oprea, and D. Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008.

[Gro78] M. Gromov, *Almost flat manifolds*, J. Differential Geom. 13 (1978), no. 2, 231–241.

[LM13] J. B. Lee and M. Masuda, *Topology of iterated \(S^1\)-bundles*, Osaka J. Math. 50 (2013), no. 4, 847–869.

[LR84] K. B. Lee and F. Raymond, *Geometric realization of group extensions by the Seifert construction*, Contributions to group theory, Contemp. Math., vol. 33, Amer. Math. Soc., Providence, RI, 1984, pp. 353–411.

[Mal49] A. I. Mal’cev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat. 13 (1949), 9–32.

[Nak14] M. Nakayama, *On the \(S^1\)-fibred nilBott tower*, Osaka J. Math. 51 (2014), no. 1, 67–87.

[PS61] R. S. Palais and T. E. Stewart, *Torus bundles over a torus*, Proc. Amer. Math. Soc. 12 (1961), 26–29.

[Rag72] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Ruh82] E. A. Ruh, *Almost flat manifolds*, J. Differential Geom. 17 (1982), no. 1, 1–14.

[Sco83] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), no. 5, 401–487.

[Wei94] S. Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.

Igor Belegradek, School of Mathematics, Georgia Tech, Atlanta, GA, USA 30332

E-mail address: ib@math.gatech.edu