Two-loop two-point functions with masses: asymptotic expansions and Taylor series, in any dimension

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Abstract In all mass cases needed for quark and gluon self-energies, the two-loop master diagram is expanded at large and small $q^2$, in $d$ dimensions, using identities derived from integration by parts. Expansions are given, in terms of hypergeometric series, for all gluon diagrams and for all but one of the quark diagrams; expansions of the latter are obtained from differential equations. Padé approximants to truncations of the expansions are shown to be of great utility. As an application, we obtain the two-loop photon self-energy, for all $d$, and achieve highly accelerated convergence of its expansions in powers of $q^2/m^2$ or $m^2/q^2$, for $d = 4$.
1 Introduction

The agreement between LEP data [1] and standard-model predictions is overwhelming [2]. Deviations from the standard model would signal new physics and are therefore intensively sought. For this reason, high-precision standard-model radiative corrections must be computed, entailing not only two-loop diagrams but also the effects of particle masses. In many cases, radiative corrections to leading mass-dependent effects are significant; in some cases [3] they can reach 50%. Thus, in recent years, the attention devoted to two-loop diagrams with masses has increased considerably [4]–[12].

Whilst massless multi-loop diagrams can be dealt with by essentially algebraic methods [13], the situation is more complex in the case of massive diagrams. The first successful algebraic approach to massive two-loop two-point functions achieved a systematic evaluation of diagrams with external massive particles on the mass-shell [6, 7], using recurrence relations obtained from integration by parts in \( d \equiv 4 - 2\varepsilon \) spacetime dimensions. These recurrence relations reduced a large variety of integrals, depending on only one mass parameter, to combinations of \( \Gamma \)-functions and a single so-called master integral that contains all the problematic analytic properties of the Feynman diagrams.

As a next step, we investigate in this paper all the master integrals, with one massive particle and an arbitrary external momentum, that occur in the two-loop self-energy diagrams of QED and QCD, in an arbitrary dimension \( d \). Applications of the corresponding \( d = 4 \) diagrams are made, for example, in QCD sum rules [14, 15]. There are several reasons for evaluating them with arbitrary \( d \).

First, it turns out [8] that \( d \)-dimensional calculations are often much easier than their 4-dimensional counterparts. Recurrence relations are best derived within the framework of dimensional regularization, where contributions from surface terms in a partial integration are disregarded. Moreover, these recurrence relations are systematically implementable as computer-algebra algorithms. Hence the algebraic methods of this paper are far less taxing than previous 4-dimensional analytical methods (as the author of [1] is only too painfully aware).

Secondly, our methods yield not only two-loop results in QED [16, 17, 18], QCD [19, 20] and QCD\(_2\) [21], but are also applicable to the dimensional regularization of higher-loop calculations, where two-loop terms of order \( \varepsilon^{L-2} \) are needed to obtain an \( L \)-loop result.

Thirdly, many of our results are analytically simpler than their specializations to \( d = 4 \). We find that the \( d \)-dimensional master integrals are generally expressible in terms of simple hypergeometric functions; only when these are expanded in \( \varepsilon \) do we encounter complicated polylogarithms. From this point of view, the fearsome analytic complexity of trilogarithms [22], first encountered in the two-loop QED calculation of [10] and later surfacing in QCD [19], is merely a consequence of taking a singular limit of a few otherwise very tractable hypergeometric functions of the type \( _2F_1 \) and \( _3F_2 \).

Finally, the analytic continuation of hypergeometric functions giving a Taylor series in \( q^2/m^2 \), to obtain those giving an asymptotic expansion in \( m^2/q^2 \), is simple for arbitrary \( d \), yet almost impossibly difficult if one knows only the \( d = 4 \) Taylor coefficients. Thus the lesser labour of determining the \( d \)-dimensional small-\( q^2 \) expansion also yields the greater benefit of a pair of complementary expansions, to use for \( d = 4 \).
The objectives of this paper are as follows:

(a) to obtain, whenever possible, hypergeometric representations of the \(d\)-dimensional two-loop master integrals of QED and QCD;

(b) when this appears impossible, to obtain the differential equation satisfied by the \(d\)-dimensional master integral;

(c) to generate, thereby, any desired number of terms in both the small-\(q^2\) Taylor series and also the large-\(q^2\) asymptotic expansion, for any \(d\);

(d) to demonstrate the utility of applying methods of accelerated convergence to each of the two series thereby obtained;

(e) to exemplify these algebraic and numerical methods in the concrete case of the self-energy of the photon.

These aims are complementary to those of [12], which extends the calculation of leading terms in the asymptotic expansion, with \(d = 4\), to mass cases more complicated than those considered in [4]. Here we restrict attention to the mass cases of [4], whilst both significantly generalizing and also greatly simplifying the analysis, by working with arbitrary \(d\). Moreover, the knowledge we derive from the complete Taylor series and from the complete asymptotic expansion enables us to investigate methods of accelerated convergence [23], which we find to be of great utility. We therefore proceed as follows.

In Section 2 we consider all the mass cases of [4], for arbitrary \(d\). These are illustrated in Fig. 1. For all cases except that of Fig. 1(c), we achieve objectives (a) and (c), above. The irreducibility to simple hypergeometric series of integral \(I_3\) of Fig. 1(c) is to be expected; for \(d = 4\) it is not reducible to polylogarithms, being rather a function whose discontinuity has a derivative involving an elliptic integral [4]. For \(I_3\), we achieve objectives (b) and (c), using differential equations of the type developed in [4, 24].

In Section 3 we study the \(d \rightarrow 4\) and \(d \rightarrow 3\) limits of the most difficult integral, \(I_3\), whose Taylor series diverges for \(|q^2| > m^2\), whilst its asymptotic expansion diverges for \(|q^2| < 9m^2\). A method for separating the sources of discontinuity is given in Section 3.1, allowing series expansions throughout the entire complex plane. This is shown to entail working to high numerical precision at intermediate stages of the computations.

In Section 3.2 we find a more efficient solution to the problem of intermediate space-like values of \(q^2\), namely the application of Padé approximants to the Taylor series, and their extension, via the so-called \(\varepsilon\)-algorithm [23], to achieve accelerated convergence of the asymptotic expansion. The improvement of the convergence of the expansions, within their respective domains of validity, is impressive. Even more significant is the high
numerical accuracy achieved in the space-like region $q^2 \in [m^2, 9m^2]$, where neither of the original series was valid. This extension of the domain of convergence is known to occur when one applies Padé approximants to the Taylor series of a function with a positive spectral density, namely to a function of the so-called Stieltjes type \[23\]. We have no right to expect it to occur for the asymptotic expansion, since that involves both powers and logarithms of $m^2/q^2$, when $d = 4$. We regard the numerical results of Section 3.2 as highly significant; such dual application of methods of accelerated convergence to small-$q^2$ \[11\] and large-$q^2$ \[12\] series may well be the next step to take in extending the scope and accuracy of practical standard-model calculations.

In Section 4 we apply the methods of previous sections to the photon self-energy. First we show how analytically simple are the new $d$-dimensional results for the self-energy and its spectral density. Spin is truly an inessential complication in $d$ dimensions; the methods that yield the corresponding master integral also yield the full self-energy in terms of precisely three simple hypergeometric functions. Secondly, we show how impressively accurate are the results of accelerating the convergence of the Taylor series and the asymptotic expansion.

In Section 5 we summarize our findings and draw conclusions from the efficiency of our algebraic methods and the accuracy of our numerical approximations.

## 2 Master integrals

We consider the generalization to $d = 4 - 2\varepsilon$ space-time dimensions of integrals analyzed in \[11\] in the case $d = 4$, namely the (euclidean) integrals

\[
I_0 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, 0) P(k_2, 0) P(k_1 - q, 0) P(k_2 - q, 0) P(k_1 - k_2, 0), \quad (1)
\]

\[
I_1 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, m) P(k_2, 0) P(k_1 - q, 0) P(k_2 - q, 0) P(k_1 - k_2, 0), \quad (2)
\]

\[
I_2 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, m) P(k_2, m) P(k_1 - q, 0) P(k_2 - q, 0) P(k_1 - k_2, 0), \quad (3)
\]

\[
I_3 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, 0) P(k_2, m) P(k_1 - q, m) P(k_2 - q, 0) P(k_1 - k_2, m), \quad (4)
\]

\[
I_4 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, m) P(k_2, m) P(k_1 - q, m) P(k_2 - q, m) P(k_1 - k_2, 0), \quad (5)
\]

\[
\tilde{I}_3 = \int \int \frac{d^d k_1 d^d k_2}{\pi^d T^2 (1 + \varepsilon)} P(k_1, m) P(k_2, 0) P(k_1 - q, m) P(k_2 - q, 0) P(k_1 - k_2, m), \quad (6)
\]

where $P(k, m) \equiv 1/(k^2 + m^2)$ and the subscript of $I_N$ denotes the number of massive lines. The denominator structure of integrals \(\{2, 3\}\) is illustrated in Fig. 1. (Note that the $d \to 4$ limits of these integrals, given in \[11\], differ by a factor of $q^2$.) Integrals \(\{I_0, \tilde{I}_3, I_4\}\) are needed for the two-loop gluon self-energy; integrals \(\{I_2, I_3\}\) for the quark self-energy. As in \[11\], we include the integral $I_1$ for illustrative purposes.
2.1 Integral $I_1$: an example of Mellin–Barnes techniques

With only one massive line, the integral (2) of Fig. 1(a) is most easily evaluated by using the Mellin–Barnes representation [25]:

$$P(k_1, m) \equiv \frac{1}{k_1^2 + m^2} = \frac{1}{2\pi i k_1^2} \int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} ds \Gamma(-s) \Gamma(1 + s) \left[ \frac{m^2}{k_1^2} \right]^s,$$

where the contour separates the poles of the integrand on the right, with $\text{Re} s \geq 0$, from those on the left, with $\text{Re} s \leq -1$. Inserting (7) in (2), one obtains an integral over $s$ whose integrand is given in terms of $\Gamma$-functions by the triangle rule of [13]. Closing the contour to the left, we find the small-$q^2$ expansion

$$q^2 m^4 \varepsilon^3 (1 - 2\varepsilon) I_1 = \frac{1}{2} \delta_2 F_1 \left[ \varepsilon, 2\varepsilon; \frac{-q^2}{m^2} \right] + (\beta - \frac{1}{2} \delta) \frac{1}{2} F_1 \left[ 1, \varepsilon; \frac{-q^2}{m^2} \right]$$

$$+ \frac{1}{2} \gamma \frac{1}{2} F_1 \left[ 1, -\varepsilon, \frac{-q^2}{m^2} \right] - \beta - \frac{1}{2} \gamma,$$

where

$$\beta \equiv \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left( \frac{m^2}{q^2} \right)^\varepsilon = 1 - \varepsilon \ln(q^2/m^2) + O(\varepsilon^2),$$

$$\gamma \equiv \frac{\Gamma(1 + 2\varepsilon) \Gamma^3(1 - \varepsilon) \left( \frac{m^2}{q^2} \right)^{2\varepsilon}}{\Gamma(1 + \varepsilon) \Gamma(1 - 3\varepsilon)} = \beta^2 - 6\zeta(3)\varepsilon^3 + O(\varepsilon^4),$$

$$\delta \equiv \frac{\Gamma(1 - \varepsilon) \Gamma(1 + 2\varepsilon)}{\Gamma(1 + \varepsilon)} = 1 + 2\zeta(2)\varepsilon^2 + O(\varepsilon^3)$$

are structures that will recur in our analysis. Closing the contour to the right, we find the large-$q^2$ expansion

$$q^2 m^4 \varepsilon^3 (1 - 2\varepsilon) I_1 = -\frac{1}{2} \delta + \beta^2 \left( 1 + \frac{m^2}{q^2} \right)^{-2\varepsilon} + (\frac{1}{2} \beta - \delta) \frac{1}{2} F_1 \left[ 1, \varepsilon; \frac{m^2}{1 - \varepsilon; -q^2} \right]$$

$$+ \beta_2 F_1 \left[ \varepsilon, 2\varepsilon; \frac{m^2}{1 - \varepsilon; -q^2} \right] - \frac{1}{2} \gamma_2 F_1 \left[ 2\varepsilon, 3\varepsilon; \frac{m^2}{1 + \varepsilon; -q^2} \right] - \frac{1}{2} \gamma_2 F_1 \left[ 1, 3\varepsilon; \frac{m^2}{1 + \varepsilon; -q^2} \right],$$

which may also be obtained by inverting the hypergeometric functions of (8).

As $d \to 4$, we find that the same function appears in the small-$q^2$ expansion (8) and the large-$q^2$ expansion (12):

$$q^2 I_1 \big|_{d=4} = -F_1(q^2/m^2) = 6\zeta(3) + F_1(m^2/q^2),$$

where

$$F_1(x) = \sum_{n=1}^{\infty} \left( \frac{\ln^2 x + 2\zeta(2)}{2n} - \frac{2\ln x}{n^2} + \frac{3}{n^3} + 2 \sum_{r=1}^{n-1} \frac{n - r}{r^2 n^2} \right) (-x)^n,$$

for $|x| < 1$, in agreement with Eq. (14) of [4].
2.2 Integral $I_4$: for abelian boson self-energies

The small-$q^2$ expansion of integral (5) of Fig. 1(d) is obtained by extending the method of integration by parts, used in [13] for the massless integral (11). The triangle rule eliminates either the massless line or two of the massive lines. In the former case, one has the product of two one-loop integrals; in the latter case, the external momentum may be routed through the massless line, making the small-$q^2$ expansion straightforward. In terms of hypergeometric functions that are regular as $q^2 \rightarrow 0$, the $d$-dimensional result is

$$m^{2+4\epsilon}(1-2\epsilon)I_4 = \left(1 + \frac{q^2}{4m^2}\right)G_4 - \frac{H_4}{(1+2\epsilon)(1-\epsilon)},$$

$$G_4 \equiv 2F_1\left[1, 1 + \epsilon; -\frac{q^2}{4m^2}\right], \quad H_4 \equiv 3F_2\left[1, 1 + \epsilon, 1 + 2\epsilon; -\frac{q^2}{4m^2}\right],$$

which gives the small-$q^2$ expansion in Eq. (44) of [4], as $d \rightarrow 4$.

This result may be transformed into one involving hypergeometric series that are regular as $q^2 \rightarrow \infty$, by inverting the series (14), to obtain:

$$q^2m^{4\epsilon}e^3(1-2\epsilon)I_4 = \left(1 + \frac{4m^2}{q^2}\right)^{-2\epsilon} \left(\beta - 2F_1\left[\frac{1}{2} - \epsilon, -\epsilon; \frac{4m^2}{-q^2}\right]\right)^2$$

$$- \gamma 2F_1\left[\frac{1}{2} + \epsilon, 3\epsilon; \frac{4m^2}{-q^2}\right] + 2\beta 2F_1\left[\frac{1}{2} + 2\epsilon; \frac{4m^2}{-q^2}\right] - 3F_2\left[1, \frac{1}{2} - \epsilon, -\epsilon; \frac{4m^2}{-q^2}\right],$$

from which the large-$q^2$ expansion of Eq. (44) of [4] is obtained in the limit $d \rightarrow 4$.

2.3 Integral $I_2$: for non-abelian fermion self-energies

The key to obtaining results (13, 14) for $I_4$ was the generalization of the triangle rule of [13], which applies when a massless particle is exchanged between particles whose masses do not change [23]. This method is therefore applicable to integral (3) of Fig. 1(b), for which one obtains the small-$q^2$ expansion

$$m^{2+4\epsilon}e^2(1-\epsilon)(1-2\epsilon)I_2 = \delta H_2 - G_2 + \frac{\epsilon}{1-\epsilon} \frac{q^2}{m^2} G_2^2,$$

with $\delta$ given by (14) and

$$G_2 \equiv 2F_1\left[1, 1 + \epsilon; -\frac{q^2}{m^2}\right], \quad H_2 \equiv 2F_1\left[1 + \epsilon, 1 + 2\epsilon; -\frac{q^2}{m^2}\right].$$

Inverting the hypergeometric series, we obtain the large-$q^2$ expansion

$$q^2m^{4\epsilon}e^3(1-2\epsilon)I_2 = \beta 2F_1\left[1 + \epsilon, 2\epsilon; \frac{m^2}{-q^2}\right] - \gamma 2F_1\left[1 + 2\epsilon, 3\epsilon; \frac{m^2}{-q^2}\right] - F_2 + F_2^2,$$

with

$$F_2 \equiv 2F_1\left[1, \epsilon; \frac{m^2}{-q^2}\right] - \beta \left(1 + \frac{m^2}{q^2}\right)^{-2\epsilon} \frac{\epsilon}{1-\epsilon} \frac{q^2}{m^2} G_2^2.$$

The expansions (18, 20) give the coefficients of Eq. (22) of [4], as $d \rightarrow 4$. 
2.4 Integral $\tilde{I}_3$: for non-abelian boson self-energies

More difficult than either of the integrals (3) of Fig. 1(e). In this case, the triangle rule fails to simplify the integral, since the exchanged particle is massive. Nevertheless, the small-$q^2$ expansion of a combination of the integral and its derivative w.r.t. $q^2$ can be reduced to $\mathbf{3} F_2$ and $\mathbf{2} F_1$ functions, using integration by parts. From this result, an expansion in terms of $\mathbf{4} F_3$ and $\mathbf{3} F_2$ functions is obtained by integration, which yields

$$m^{2+4\varepsilon}(1+\varepsilon)(1-2\varepsilon)\tilde{I}_3 = \frac{1}{(1-\varepsilon)(1+2\varepsilon)}4F_3\left[\begin{array}{c}1,1+\varepsilon,1+\varepsilon,1+2\varepsilon;\frac{-q^2}{4m^2} \\
\frac{3}{2}+\varepsilon,2+\varepsilon,2-\varepsilon;\frac{4m^2}{4m^2}\end{array}\right]$$

$$+ \frac{1+\varepsilon}{2\varepsilon}2F_2\left[\begin{array}{c}1,1,1+\varepsilon;\frac{-q^2}{4m^2} \\
\frac{3}{2},2;\frac{4m^2}{4m^2}\end{array}\right] - \frac{1}{2\varepsilon}3F_2\left[\begin{array}{c}1,1+\varepsilon,1+\varepsilon;\frac{-q^2}{4m^2} \\
\frac{3}{2},2+\varepsilon;\frac{4m^2}{4m^2}\end{array}\right]. \quad (22)$$

The presence of $(m^2/q^2)^\varepsilon$, via $\beta$ in the second term of (22), reveals the branchpoint at $q^2 = 0$, which prohibits the development of a simple Taylor series. Instead, one obtains

$$\tilde{I}_3 = \frac{1}{2m^2} \sum_{n=0}^\infty \frac{n! \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} \left(-\frac{q^2}{4m^2}\right)^n \left(\frac{3}{(n+1)^2} - \frac{\ln(q^2/m^2)}{n+1}\right) + O(\varepsilon), \quad (23)$$

in agreement with Eq. (45) of [4].

The inversion of (22), to produce a large-$q^2$ expansion, is likewise difficult, since each hypergeometric function has a repeated argument. To outflank this difficulty, one returns to the differential equation relating the integral and its derivative w.r.t. $m^2$ to simpler hypergeometric functions, whose inversion is not frustrated by repeated arguments. Inverting the latter and integrating the resultant series, one obtains

$$q^2m^{4\varepsilon}\varepsilon^3(1-2\varepsilon)\tilde{I}_3 = 4F_3\left[\begin{array}{c}1,\frac{1}{2}-\varepsilon,\varepsilon,-\varepsilon;\frac{4m^2}{1-\varepsilon,1-\varepsilon,1-2\varepsilon;\frac{-q^2}{4m^2}} \\
\frac{3}{2}+\varepsilon,1+\varepsilon;\frac{4m^2}{4m^2}\end{array}\right] - \gamma 3F_2\left[\begin{array}{c}1+\varepsilon,1+\varepsilon;\frac{4m^2}{1+\varepsilon,1+\varepsilon;\frac{-q^2}{4m^2}} \\
\frac{3}{2},2+\varepsilon;\frac{4m^2}{4m^2}\end{array}\right]$$

$$+ \beta^2 2F_1\left[\begin{array}{c}1+\varepsilon,\varepsilon;\frac{4m^2}{1+\varepsilon;\frac{-q^2}{4m^2}} \\
\frac{3}{2},\frac{3}{2};\frac{4m^2}{4m^2}\end{array}\right] - 3F_2\left[\begin{array}{c}1,\frac{1}{2},\varepsilon,-\varepsilon;\frac{4m^2}{1-\varepsilon,1-\varepsilon;\frac{-q^2}{4m^2}} \\
\frac{3}{2},-\varepsilon,2-\varepsilon;\frac{4m^2}{4m^2}\end{array}\right] + \left(\frac{2\varepsilon}{1-\varepsilon}\right) 3F_2\left[\begin{array}{c}1,\frac{3}{2},\frac{3}{2};\frac{4m^2}{2,2-\varepsilon;\frac{-q^2}{4m^2}} \\
\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{4m^2}{4m^2}\end{array}\right] \beta m^2 \frac{\ln(q^2/m^2)}{q^2}. \quad (24)$$

The $d \to 4$ limit agrees with Eq. (45) of [4]; the $m \to 0$ limit agrees with the massless result of [3] for integral (3):

$$I_0 = \frac{q^{2(d-5)}(1+2\varepsilon)}{1-2\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^3(-\varepsilon)}{\Gamma(1+3\varepsilon)} \left(1 - \frac{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)}{\Gamma(1-2\varepsilon)} \cos \pi \varepsilon\right) = \frac{6\zeta(3)}{q^2} + O(\varepsilon), \quad (25)$$

which is likewise obtained from (12, 24, 21).

2.5 Integral $I_3$: for abelian fermion self-energies

The remaining integral (4) of Fig. 1(e) is by far the most difficult. Symptomatic of its intractability are the features (4) found for $d = 4$, where the small-$q^2$ expansion involves the maximum value of Clausen’s integral, which is not reducible to $\zeta$-functions, and the
derivative of the discontinuity in the time-like region $-q^2 > 9m^2$ is an elliptic integral. Accordingly, we generalize the method of [4] to arbitrary derivative of the discontinuity in the time-like region.

First we treat the simplest two-point function with three massive propagators:

\[ J_3 \equiv \int \int \frac{d^d k_1 d^d k_2}{\pi^d \Gamma^2(1 + \varepsilon)} P(k_2, m) P(k_1 - q, m) P(k_1 - k_2, m), \]  

for which a Mellin–Barnes transform of one propagator yields

\[ -m^{4\varepsilon - 2\varepsilon^2}(1 - \varepsilon)(1 - 2\varepsilon) J_3 = \sum_{n=0}^{\infty} \frac{A(n, q^2) + (1 - 2\varepsilon) B(n, q^2)}{4^n}, \]  

which \( (a)_n \equiv \Gamma(a + n)/\Gamma(a). \) At \( q^2 = -m^2, \) we obtain from (27, 28) the on-shell result

\[ -m^{4\varepsilon - 2\varepsilon^2}(1 - \varepsilon)(1 - 2\varepsilon) J_3 \big|_{q^2 = -m^2} = 3 F_2 \left[ \begin{array}{c} n + 1, n + \varepsilon, n - 1 + 2\varepsilon; \frac{1}{2} + \varepsilon; \frac{1}{4} \\ n - 1 + 2\varepsilon, \frac{1}{2} + \varepsilon; \frac{1}{4} \end{array} \right] + (1 - 2\varepsilon) 3 F_2 \left[ \begin{array}{c} n + 1, n + \varepsilon, n + 2 - \varepsilon; \frac{3}{2} - 2\varepsilon; \frac{3}{2} \\ n - 1 + 2\varepsilon, \frac{3}{2} - 2\varepsilon; \frac{3}{2} \end{array} \right] = \frac{3}{2} - \frac{1}{\varepsilon} - \frac{19}{8}\varepsilon^2 + O(\varepsilon^3), \]  

whose \( \varepsilon \)-expansion may extended, through \( O(\varepsilon^6), \) in terms of \( \{\text{Li}_n(1), \text{Li}_n(\frac{1}{2})| n \leq 5 \}. \)

After some transformations, (27) can be rewritten as

\[ -m^{4\varepsilon - 2\varepsilon^2}(1 - \varepsilon)(1 - 2\varepsilon) J_3 = \sum_{n=0}^{\infty} C(n) \left( \frac{-q^2/m^2} {2 - \varepsilon} \right)^n n!, \]  

where \( C(0) = 3 F_2 \left[ \begin{array}{c} 1, 1 - 2\varepsilon; \frac{1}{4} \\ \frac{1}{2} + \varepsilon; \frac{1}{2} \end{array} \right] + (1 - 2\varepsilon) 3 F_2 \left[ \begin{array}{c} 1, \frac{3}{2}; \frac{3}{2} \\ \frac{1}{2} + \varepsilon; \frac{1}{2} \end{array} \right] = \frac{3}{2} (1 - 9\varepsilon^2 S_2) + O(\varepsilon^3), \]  

where \( S_2 \) is a multiple of \( \text{Cl}_2(\pi/3), \) the maximum value of Clausen’s integral:

\[ S_2 = \sum_{n=1}^{\infty} \frac{2n - 1}{(3n - 1)^2(3n - 2)^2} = -\frac{4}{9\sqrt{3}} \pi/3 \int_0^{\pi/3} \sin^2 \theta \, d\theta \]  

and is needed to high precision, for the numerical analysis of Section 3.1. It is tedious to obtain higher coefficients from (32). It is easier to use the second-order differential equation that follows from (28, 29), namely

\[ 2(q^2 + m^2)(q^2 + 9m^2)m^2 J_3'' - \left( (d - 4)q^4 + 10(3d - 10)q^2 m^2 + 9(5d - 16)m^4 \right) J_3' + 3(d - 8)(d - 3)(q^2 + 3m^2) J_3 = \frac{48q^2 m^{2d-6}}{(d - 4)^2}, \]  

(36)
where primes denote differentiation w.r.t. $m^2$. This differential equation may also be obtained directly, using integration by parts. It yields the recurrence relation

$$C(n+1) = \frac{1}{3}(d-2)(d-3)\delta_{n,0} - \frac{1}{18} \{ (d+4)(d-3) + 10n(d-2n-4) \} C(n) - \frac{1}{36} n(d+2n-2)(d-2n-2)(d-n-2)C(n-1),$$

(37)

which enables efficient computation of any desired number of small-$q^2$ coefficients.

The expansion at large $q^2$ is of the form

$$-q^2 m^4 \varepsilon^2 (1-2\varepsilon) J_3 = \sum_{n=0}^{\infty} \left( \gamma q^n C_1(n) + \beta q^n m^2 C_2(n) + m^4 C_3(n) \right) \frac{(-m^2/q^2)^n}{n!},$$

(38)

where $C_1(0) = \varepsilon / [2(1-3\varepsilon)(2-3\varepsilon)]$, $C_2(0) = 3/(1-\varepsilon)$, and the remaining coefficients are rational functions of $\varepsilon$, determined by (39). These coefficients may be obtained in closed form, by transforming the hypergeometric series (28, 29) into large-$q^2$ expansions.

This inversion generates the structures (38, 39). Collecting powers of $-m^2/q^2$, we obtain the terminating series

$$C_1(n) = \frac{(-1 + 2\varepsilon)n(3\varepsilon - n - 2)}{2(1+\varepsilon)_{n-1}} {}_3F_2 \left[ \begin{array}{c} -\frac{1}{2} + \varepsilon, 1 - \varepsilon - n, -n; \\ -1 + 2\varepsilon, \varepsilon; \end{array} \right] 4,$$

(39)

$$C_2(n) = \frac{(-1 + 2\varepsilon)n(\varepsilon)}{(1-\varepsilon)(2-\varepsilon)_n} {}_3F_2 \left[ \begin{array}{c} -\frac{1}{2} + \varepsilon, 1 - \varepsilon - n, -n; \\ -1 + 2\varepsilon, \varepsilon; \end{array} \right] 4 + \frac{2(-1 + 2\varepsilon)n}{1-\varepsilon} {}_3F_2 \left[ \begin{array}{c} 1 + \varepsilon - n, -n; \\ 2 - \varepsilon, \varepsilon; \end{array} \right] 4,$$

(40)

$$C_3(n) = -\frac{(1-2\varepsilon)n!}{(1-\varepsilon)^2} {}_3F_2 \left[ \begin{array}{c} \frac{3}{2} - \varepsilon, 1 - \varepsilon - n, -n; \\ 2 - \varepsilon, 3 - 2\varepsilon; \end{array} \right] 4 + \frac{2(\varepsilon)n}{(2-\varepsilon)_n} {}_3F_2 \left[ \begin{array}{c} \frac{1}{2}, -1 + \varepsilon - n, -n; \\ 2 - \varepsilon, \varepsilon; \end{array} \right] 4,$$

(41)

for the coefficients of (38), which we have checked by substitution in the differential equation (36).

From the expansions of $J_3$, we generate those of

$$K_3 \equiv \int \int \frac{d^d k_1 d^d k_2}{\pi^d \Gamma^2(1+\varepsilon)} P(k_1,0) P(k_2,m) P(k_2-q,m) P(k_1-k_2,m),$$

(42)

using integration by parts, which yields

$$(q^2 + m^2)K_3 - \frac{1}{3}(q^2 + 9m^2) J_3' + (3d-8)J_3$$

$$= \frac{4m^{2d-10}}{(d-3)(d-4)} \left( \frac{m^2(q^2 - m^2)}{d-4} + \frac{(q^2 + m^2)^2 G_2}{d-2} \right),$$

(43)

with $G_2$ given by (19) at small $q^2$ and by (21) at large $q^2$. Finally we obtain $I_3$ from integration by parts, which yields

$$(d-4)((q^2 + 3m^2)I_3 - \frac{1}{3}(q^2 - 3m^2) I_3) - m^2(q^2 + m^2)(2I_3' + I_2') + K_3 - J_3'$$

$$= \frac{4m^{2d-12}}{d-3} \left( \frac{2m^4}{(d-4)^2} + \frac{m^2(q^2 + m^2) G_2}{(d-2)(d-4)} - \frac{q^2(q^2 + m^2) G_2}{(d-2)^2} \right),$$

(44)
with $I_2$ given by (1820). From this first-order equation the expansion coefficients may be obtained, at both large and small $q^2$, for any $d$.

The small-$q^2$ expansion is of the form

$$-m^{4\varepsilon^2}\varepsilon^2(1-\varepsilon)(1-2\varepsilon)I_3 = \frac{\delta}{2} \sum_{n=0}^{\infty} \frac{(-q^2/2m^2)^n}{(d/2)n!} P_1(n) + C(0) \sum_{n=0}^{\infty} \frac{(q^2/18m^2)^n}{(d/2)n!} P_2(n) - 6 \sum_{n=0}^{\infty} \frac{(q^2/18m^2)^n}{(d/2)n!} P_3(n) \prod_{n/2 < k \geq 0} \frac{1}{d+2k-2},$$  \hspace{1cm} (45)$$

where $\delta$ is given by (11), $C(0)$ is the constant of (18), the final product is set to unity for $n=0$, and $P_i(n)$ are polynomials in $d$. Table 1 gives the first 5 terms in the Taylor series. (We have computed the first 20.)

**Table 1: Expansion coefficients of $I_3$ at small $q^2$.**

| $P_1(0)$ | $P_1(1)$ | $P_1(2)$ | $P_1(3)$ | $P_1(4)$ |
|----------|----------|----------|----------|----------|
| $d^2 - 9d + 22$ | $d^4 - 21d^3 + 168d^2 - 604d + 832$ | $d^6 - 36d^5 + 541d^4 - 4338d^3 + 19588d^2 - 47304d + 48000$ | $d^8 - 54d^7 + 1271d^6 - 17022d^5 + 141884d^4 - 754056d^3 + 2497966d^2 - 4722432d + 3916800$ | $d^{10} - 104d^9 + 10944d^8 - 62540d^7 + 1058280d^6 - 1053280d^5 + 28901060d^4 - 4722432d^3 + 3916800d^2 - 10944d + 1$ |

The large-$q^2$ expansion is of the form

$$q^2m^{4\varepsilon^2}\varepsilon^2(1-2\varepsilon)I_3 = \sum_{n=0}^{\infty} \left( \beta^2 M_0(n) - \gamma M_1(n) + \beta M_2(n) + M_3(n) \right) \frac{(-m^2/q^2)^n}{n!},$$  \hspace{1cm} (46)$$

where the coefficients are given by the recurrence relations

$$
(n + \varepsilon)M_0(n) &= n(n-1+3\varepsilon)M_0(n-1) + (-1+4\varepsilon)_n, \\
(n + \varepsilon)M_1(n) &= 2n(n-1+2\varepsilon)M_1(n-1) - n(n-1)(n-2+3\varepsilon)M_1(n-2) + \frac{1}{3} C_1(n+1) + (n-2+3\varepsilon)C_1(n) + \frac{1}{3} (2n-1+3\varepsilon)(-1+2\varepsilon)_n(3\varepsilon)_{n-1}/(1+\varepsilon)_n, \\
(n + \varepsilon)M_2(n) &= 2n(n-1+\varepsilon)M_2(n-1) - n(n-1)(n-2+2\varepsilon)M_2(n-2) + \frac{1}{3} (n+1-\varepsilon)C_2(n) + n(n-2+2\varepsilon)C_2(n-1) + (2n-1+\varepsilon)(-1+2\varepsilon)_n(\varepsilon)_{n-1}/(1-\varepsilon)_n - 2(-2+2\varepsilon)_n F_2 \left[ \frac{1, \varepsilon, -n;}{1-\varepsilon, 3-2\varepsilon - n; 1} \right], \\
(n - \varepsilon)M_3(n) &= 2n(n-1)M_3(n-1) - n(n-1)(n-2+\varepsilon)M_3(n-2) + 2M_4(n) - \frac{1}{3} n(n+1-2\varepsilon)C_3(n-1) - n(n-1)(n-2+\varepsilon)C_3(n-2), \\
(n - 2\varepsilon)M_4(n) &= n(n-3+2\varepsilon)(M_4(n-1) - \delta_{n,2}) - 2\varepsilon(1-2\varepsilon)(n! - \delta_{n,0} - \frac{1}{2} \delta_{n,1})(\varepsilon)_{n-2}/(2-\varepsilon)_{n-1},
$$
with $M_{0,1}(0) = 1/\varepsilon$, $M_{2,3,4}(0) = 0$, and $C_{1,2,3}(n)$ given by (39)-(41). The extra set of coefficients in (34) serves to expand the square of series (21), needed for the $I_2$ term on the left-hand side of (14) and for the $G^2_2$ term on the right. Relations (47-51) allow very rapid computation of the asymptotic expansion (10); the hard work has been done in obtaining the closed forms (39-41), for the expansion (38) of the irreducibly difficult integral (26), together with the differential equations (43,44). Table 2 gives the first 5 terms in the asymptotic expansion. (We have computed the first 20.)

Table 2: Expansion coefficients of $I_3$ at large $q^2$.

\[
M_0(0) = \frac{2}{(d-4)}, \quad M_0(1) = \frac{4(d-5)}{(d-6)}, \quad M_0(2) = \frac{-4(2d^3-33d^2+176d-308)}{(d-6)(d-8)}, \\
M_0(3) = \frac{8(2d^3-37d^2+210d-384)(d-5)(d-7)}{(d-6)(d-8)(d-10)}, \\
M_0(4) = \frac{-8(4d^5-144d^4+1097d^3-13158d^2+42064d-52416)(d-5)}{(d-6)(d-8)(d-10)(d-12)}, \\
M_1(0) = \frac{2}{(d-4)}, \quad M_1(1) = \frac{12(d-4)(d-5)}{(d-6)^2}, \quad M_1(2) = \frac{-6(5d^3-86d^2+472d-800)(3d-14)(d-5)}{(d-6)^4(d-8)^2(d-10)^2}, \\
M_1(3) = \frac{12(7d^4-195d^3+1952d^2-8172d+11712)(3d-14)(3d-16)(d-5)(d-7)}{(d-6)^4(d-8)^2(d-10)^2(d-12)^2}, \\
M_1(4) = \frac{-18(41d^6-2046d^5+41532d^4-436860d^3+2489536d^2-7211904d+8128512)(3d-14)(3d-16)(d-5)(d-7)}{(d-6)^4(d-8)^2(d-10)^2(d-12)^2}, \\
M_2(0) = 0, \quad M_2(1) = \frac{2(2d-3)(d-8)}{(d-2)(d-6)}, \quad M_2(2) = \frac{-2(3d^3-15d^2+22d+152)(d-3)}{(2d)(d-6)}, \\
M_2(3) = \frac{4(2d^5-45d^4+197d^3+350d^2-2304d)(d-3)}{3(d+1)(d+2)(d-2)(d-6)}, \\
M_2(4) = \frac{-4(2d^7-51d^6+96d^5+3063d^4-6180d^3+42192d^2-1300496d+466992)(d-3)}{3(d+1)(d+2)(d-2)(d-6)}, \\
M_3(0) = 0, \quad M_3(1) = \frac{4}{(d-2)^2}, \quad M_3(2) = \frac{-8(d^2-19d+44)}{(d-2)^2(d-6)}, \\
M_3(3) = \frac{-24(39d^2-482d+1072)}{(d+2)(d-2)^2(d-6)}, \quad M_3(4) = \frac{192(d^3+21d^2-824d^2+7104d^2-11672d-8352)}{(d+4)(d+2)(d-2)^2(d-6)^2}.
\]

Taking the limit $d \to 4$, we obtain, from Tables 1 and 2, the expansions given in (4) for

\[
q^2 I_3 \bigg|_{d=4} = \int_0^\infty \frac{4w \, dw}{w^2 - m^2} \left( \ln \frac{w}{m} - \frac{w^2 - m^2}{w^2} \ln \frac{w^2 - m^2}{m^2} \right) \ln \frac{w^2 + q^2}{w^2} \\
+ \int_0^\infty \frac{4 \, dw}{(w^2 - 4m^2)^{1/2}} \left( 3 \ln \frac{w}{m} - \frac{3w^2 - 3m^2 + q^2}{W_+ W_-} \ln \frac{W_+ + W_-}{W_+ - W_-} \right) \ln \frac{w}{2m},
\]

with $W_\pm \equiv ((w \pm m)^2 + q^2)^{1/2}$. (Note that the sign of $q^2$ in (4) is opposite to that chosen here.) In the case $d = 3$, we find that $I_3$ is given by the imaginary part of a complex dilogarithm, throughout the space-like region $q^2 > 0$, and is hence reducible to three instances of Clausen’s integral [22]:

\[
m^4 I_3 \bigg|_{d=3} = \frac{2 \text{ Im Li}_2 \left( r e^{i\theta} \right)}{r \tan(\theta/2)} = \frac{2\omega \ln r + \text{ Cl}_2(2\omega) + \text{ Cl}_2(2\theta) - \text{ Cl}_2(2\omega + 2\theta)}{r \tan(\theta/2)},
\]

with

\[
r = \frac{m^2 + q^2}{4m^2}, \quad \cos \theta = \frac{m^2 - q^2}{m^2 + q^2}, \quad \cos \omega = \frac{3m^2 + q^2}{(9m^2 + q^2)^{1/2}(m^2 + q^2)^{1/2}}.
\]
3 The problem of intermediate $q^2$

In the cases of integrals $I_1$, $I_4$, $I_2$ and $\tilde{I}_3$, the small-$q^2$ expansions [8,13,18,22] and the large-$q^2$ expansions [12,17,20,24] together cover the entire complex $q^2$-plane, cut along the negative real axis, with the possible exception of the circle $|q^2| = m^2$, in the cases of $I_1$ and $I_2$, or the circle $|q^2| = 4m^2$, in the cases of $I_4$ and $\tilde{I}_3$. On these circles, the series may converge only conditionally, or may diverge, depending on the value of $d$. Otherwise, one has a method for computing the integrals by truncation of the appropriate series, for any $d$ and $q^2$.

This happy circumstance is unfortunately far from generic. In the general mass case, the integral

$$I_5 = \int \int \frac{d^4k_1 d^4k_2}{\pi^4 \Gamma^2(1+\varepsilon)} P(k_1,m_1)P(k_2,m_2)P(k_1 - q,m_3)P(k_2 - q,m_4)P(k_1 - k_2,m_5), \quad (55)$$

has four contributions, with distinct branchpoints, corresponding to the four ways of cutting the master diagram [9]. The small-$q^2$ expansion converges for

$$|q^2| < \min \left( (m_1 + m_3)^2, (m_2 + m_4)^2, (m_1 + m_4 + m_5)^2, (m_2 + m_3 + m_5)^2 \right), \quad (56)$$

whilst the large-$q^2$ expansion converges for

$$|q^2| > \max \left( (m_1 + m_3)^2, (m_2 + m_4)^2, (m_1 + m_4 + m_5)^2, (m_2 + m_3 + m_5)^2 \right). \quad (57)$$

In the case of a branchpoint at the origin, as occurs for $I_1$ and $\tilde{I}_3$, a small-$q^2$ expansion may be found, valid up to the next branchpoint, as in (8,22). In the general mass case, however, each method fails at intermediate values of $|q^2|$. In particular, $I_3$ cannot be computed directly from the results of Tables 1 and 2 when $q^2$ is in the problematic annulus $9m^2 > |q^2| > m^2$.

Tables 3 and 4 show the accuracies to which truncations of the small and large $q^2$ series approximate the exact results [5,23], and their analytic continuations to the time-like region $-q^2 \in [0, m^2]$. They show the relative error, $\Delta_d(n) \equiv |I_3^{\text{trunc}}/I_3 - 1|$, when $I_3^{\text{trunc}}$ is obtained from the first $n$ terms in the series, with $d = 4$ or $d = 3$.

The domain of convergence of the small-$q^2$ Taylor series of Table 3 is $|q^2| \leq m^2$, for $d = 4$, and $|q^2| < m^2$, for $d = 3$. There is a branchpoint at $q^2 = -m^2$, where a cut begins, due to intermediate states with only one massive particle. The limit of (52) exists as $q^2 \to -m^2$, giving $\tilde{I}_3 \to 6\zeta(2) \ln 2 - \frac{5}{2} \zeta(3)$, for $d = 4$. The limit of (53) does not exist; instead one finds that $m^2(m^2 + q^2)\tilde{I}_3 \to 4\zeta(2)$, for $d = 3$. Note that the convergence of the $d = 4$ Taylor series is slow for $|q^2| = m^2$.

One expects the asymptotic truncation errors to become substantial for $q^2 \approx 9m^2$, since the intermediate states with three massive particles generate a branchpoint at $q^2 = -9m^2$. The last row of Table 4 shows that one may trespass only a small distance inside the annulus horribilis before the divergence of the asymptotic expansion becomes a problem. The task of computing the integral reliably, by series expansions at intermediate $q^2$, is clearly a delicate one, which we accomplish in Section 3.1, by separating the sources of discontinuity. In Section 3.2, we show how to avoid this separation, for space-like $q^2$, by using methods of accelerated convergence [23].
Table 3: Small-$q^2$ truncation errors.

| $q^2/m^2$ | $\Delta_4(5)$     | $\Delta_4(10)$    | $\Delta_4(15)$    | $\Delta_3(5)$     | $\Delta_3(10)$    | $\Delta_3(15)$    |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1.0     | $6.7 \times 10^{-2}$ | $2.3 \times 10^{-2}$ | $1.2 \times 10^{-2}$ | $5.1 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | $5.7 \times 10^{-2}$ |
| 0.8     | $2.3 \times 10^{-2}$ | $2.6 \times 10^{-3}$ | $4.4 \times 10^{-4}$ | $1.2 \times 10^{-1}$ | $9.3 \times 10^{-3}$ | $7.3 \times 10^{-4}$ |
| 0.6     | $5.6 \times 10^{-3}$ | $1.5 \times 10^{-4}$ | $6.1 \times 10^{-6}$ | $1.5 \times 10^{-2}$ | $1.6 \times 10^{-4}$ | $1.6 \times 10^{-6}$ |
| 0.4     | $7.7 \times 10^{-4}$ | $2.8 \times 10^{-6}$ | $1.5 \times 10^{-8}$ | $4.5 \times 10^{-4}$ | $1.5 \times 10^{-7}$ | $4.8 \times 10^{-11}$ |
| 0.2     | $2.5 \times 10^{-5}$ | $2.9 \times 10^{-9}$ | $4.8 \times 10^{-13}$ | $4.2 \times 10^{-4}$ | $1.4 \times 10^{-7}$ | $4.4 \times 10^{-11}$ |
| -0.2    | $2.9 \times 10^{-5}$ | $3.4 \times 10^{-9}$ | $5.8 \times 10^{-13}$ | $1.3 \times 10^{-2}$ | $1.3 \times 10^{-4}$ | $1.4 \times 10^{-6}$ |
| -0.4    | $1.0 \times 10^{-3}$ | $4.0 \times 10^{-6}$ | $2.2 \times 10^{-8}$ | $9.1 \times 10^{-2}$ | $7.2 \times 10^{-3}$ | $5.7 \times 10^{-4}$ |
| -0.6    | $8.8 \times 10^{-3}$ | $2.7 \times 10^{-4}$ | $1.2 \times 10^{-5}$ | $9.1 \times 10^{-2}$ | $7.2 \times 10^{-3}$ | $5.7 \times 10^{-4}$ |
| -0.8    | $4.6 \times 10^{-2}$ | $6.4 \times 10^{-3}$ | $1.2 \times 10^{-3}$ | $3.6 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $3.9 \times 10^{-2}$ |
| -1.0    | $2.7 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | $1.2 \times 10^{-1}$ |

Table 4: Large-$q^2$ truncation errors.

| $q^2/m^2$ | $\Delta_4(5)$     | $\Delta_4(10)$    | $\Delta_4(15)$    | $\Delta_3(5)$     | $\Delta_3(10)$    | $\Delta_3(15)$    |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 15      | $1.8 \times 10^{-5}$ | $2.5 \times 10^{-7}$ | $6.4 \times 10^{-9}$ | $1.8 \times 10^{-3}$ | $3.3 \times 10^{-5}$ | $1.1 \times 10^{-6}$ |
| 13      | $4.8 \times 10^{-5}$ | $1.3 \times 10^{-6}$ | $6.7 \times 10^{-8}$ | $3.8 \times 10^{-3}$ | $1.4 \times 10^{-4}$ | $1.0 \times 10^{-5}$ |
| 11      | $1.5 \times 10^{-4}$ | $8.6 \times 10^{-6}$ | $1.0 \times 10^{-6}$ | $9.1 \times 10^{-3}$ | $8.0 \times 10^{-4}$ | $1.3 \times 10^{-4}$ |
| 9       | $5.3 \times 10^{-4}$ | $8.1 \times 10^{-5}$ | $2.5 \times 10^{-5}$ | $2.6 \times 10^{-2}$ | $6.2 \times 10^{-3}$ | $2.7 \times 10^{-3}$ |
| 7       | $2.6 \times 10^{-3}$ | $1.3 \times 10^{-3}$ | $1.4 \times 10^{-3}$ | $9.9 \times 10^{-2}$ | $8.1 \times 10^{-2}$ | $1.2 \times 10^{-1}$ |

3.1 Separating the discontinuities

Denoting the two contributions to \( I_a \) and \( I_b \), we find separate asymptotic expansions of the form

\[
I_{a,b} = 3\zeta(3) \mp \left( \frac{1}{6} L^3 - 2\zeta(2)L + 7\zeta(3) \right) + \sum_{n=1}^{\infty} \left( \frac{a_{a,b}(n)}{n} L^2 + \frac{b_{a,b}(n)}{n^2} L + \frac{c_{a,b}(n)}{n^3} \right) \left( -\frac{m^2}{q^2} \right)^n,
\]

with \( L \equiv \ln(q^2/m^2) \). The coefficients of the simpler term, \( I_a \), with no contribution from intermediate states with three massive particles, may be obtained in closed form:

\[
a_a(n) = -\frac{1}{2}, \quad b_a(n) = -3, \quad c_a(n) = -4 - 2n^2\zeta(2) + \sum_{r=1}^{n-1} \frac{n}{r} \left( 2 - \frac{n}{r} \right),
\]

from which we obtain the coefficients of \( I_b \) by subtraction, given the \( d \to 4 \) limit of the asymptotic expansion \( \left[ \right] \). The small-$q^2$ expansions are of the form

\[
I_{a,b} = -\sum_{n=1}^{\infty} \frac{d_{a,b}(n)}{n} \left( -\frac{q^2}{m^2} \right)^n, \quad d_a(n) = \zeta(2) + \sum_{r=1}^{n-1} \frac{1}{rn} \left( 2 - \frac{n}{r} \right),
\]

from which we obtain the coefficients of \( I_a \) from the \( d \to 4 \) limit of the Taylor series \( \left\{ \right\} \).

The strategy now is clear: in the annulus one uses the large-$q^2$ expansion of \( I_a \) and the small-$q^2$ expansion of \( I_b \). (The small-$q^2$ expansion of \( I_a \) and the large-$q^2$ expansion of
$I_b$ are never needed; outside the annulus one deals with the full integral.) A computational difficulty becomes apparent, however, when one evaluates the small-$q^2$ coefficients of the troublesome term, $I_b$: the coefficient $d_b(n)$ is numerically undetermined unless the constant (34) is known to an accuracy substantially better than 1 part in $9^n$. For example,

$$d_b(20) = -2\zeta(2) - \frac{57479895451350079074519109851}{144156399118288650182006800} + \frac{33449771497435781}{3334363393393138} \approx 6.074 \times 10^{-22} \quad (61)$$

is 22 orders of magnitude smaller than its $S_2$-dependent term. Yet, at $q^2 = 9m^2$, its relative contribution to the Taylor series is $2 \times 10^{-4}$.

It may now be seen that the method of (11), which gives the Taylor series of Feynman integrals in terms of $S_2$, requires one to work with intermediate numerical precision much greater than the desired final accuracy: 18 orders of magnitude greater in the above example. Fortunately one may exploit this novel feature and use the near vanishing of higher coefficients in the series to obtain a suitably accurate approximation to $S_2$, which is how we computed the value (35), using the analytical expression for $d_b(30)$. By high precision computation of the appropriate series, we then obtained the acceptably small truncation errors of Table 5.

| $q^2/m^2$ | $\Delta_4(5)$ | $\Delta_4(10)$ | $\Delta_4(15)$ | $\Delta_3(5)$ | $\Delta_3(10)$ | $\Delta_3(15)$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 8        | $1.0 \times 10^{-3}$ | $8.7 \times 10^{-5}$ | $1.5 \times 10^{-5}$ | $7.6 \times 10^{-3}$ | $1.3 \times 10^{-3}$ | $3.4 \times 10^{-4}$ |
| 7        | $4.9 \times 10^{-4}$ | $2.2 \times 10^{-5}$ | $2.0 \times 10^{-6}$ | $3.6 \times 10^{-3}$ | $3.0 \times 10^{-4}$ | $4.1 \times 10^{-5}$ |
| 6        | $2.2 \times 10^{-4}$ | $4.5 \times 10^{-6}$ | $1.9 \times 10^{-7}$ | $2.0 \times 10^{-3}$ | $5.7 \times 10^{-5}$ | $3.6 \times 10^{-6}$ |
| 5        | $9.9 \times 10^{-5}$ | $7.0 \times 10^{-7}$ | $1.2 \times 10^{-8}$ | $2.1 \times 10^{-3}$ | $8.4 \times 10^{-6}$ | $2.0 \times 10^{-7}$ |
| 4        | $8.8 \times 10^{-5}$ | $1.0 \times 10^{-7}$ | $4.2 \times 10^{-10}$ | $5.9 \times 10^{-3}$ | $6.9 \times 10^{-6}$ | $1.2 \times 10^{-8}$ |
| 3        | $3.6 \times 10^{-4}$ | $7.8 \times 10^{-7}$ | $2.2 \times 10^{-9}$ | $3.0 \times 10^{-2}$ | $1.4 \times 10^{-4}$ | $5.9 \times 10^{-7}$ |
| 2        | $4.0 \times 10^{-3}$ | $6.9 \times 10^{-5}$ | $1.5 \times 10^{-6}$ | $3.2 \times 10^{-1}$ | $1.1 \times 10^{-2}$ | $3.6 \times 10^{-4}$ |

For $d = 3$, one must separate (33) as follows:

$$m^4I_3 \bigg|_{d=3} = G_a + G_b, \quad G_a = \frac{\theta \ln r + 4\text{Cl}_2(\pi - \theta) - \zeta(2) \tan(\theta/2)}{r \tan(\theta/2)} \quad (62)$$

and use the asymptotic expansion of $G_a$ and the Taylor series of $G_b$ in the annulus. For the latter, high precision is again required, to deal with cancellations of the type

$$\zeta(2) + \frac{929138162069420913518612265142}{3909282043945987467032995529425} - \frac{1650990499861696}{166966668933225} \ln \frac{3}{2} \approx 1.076 \times 10^{-22}, \quad (63)$$

which occurs in the coefficient analogous to (61).

### 3.2 Accelerated convergence

Another strategy for dealing with the problematic region $q^2 \in [m^2, 9m^2]$ is the use of techniques that accelerate the convergence of the large and small $q^2$ expansions. A convenient technique is the $\varepsilon$-algorithm of (23). In general, given a sequence of approximations, \{S_n| n = 0, 1, 2, \ldots\}, one constructs a table of approximants using:

$$T(m, n) = T(m - 2, n + 1) + 1/\{T(m - 1, n + 1) - T(m - 1, n)\}, \quad (64)$$
with \( T(0, n) \equiv S_n \) and \( T(-1, n) \equiv 0 \). If the sequence \( \{S_n\} \) is obtained by successive truncation of a Taylor series, the approximant \( T(2k, j) \) is identical to the \([k + j/k]\) Padé approximant \( [23] \), derived from the first \( 2k + j + 1 \) terms in the Taylor series. Moreover, in the case of Stieltjes series, with a constant-sign spectral density, the \( n \to \infty \) limit of diagonal Padé approximants converges throughout the space-like region, giving the analytic continuation of the function whose derivatives at the origin are the Taylor coefficients. We have verified that (52, 53) are of Stieltjes type.

Table 6: Accelerated convergence of the Taylor series of \( I_3 \), with \( d = 4 \).

| \( q^2/m^2 \) | \( E_4(2) \) | \( E_4(4) \) | \( E_4(6) \) | \( E_4(8) \) |
|-------------|----------|----------|----------|----------|
| 3.0 | \( 6.0 \times 10^{-3} \) | \( 5.5 \times 10^{-5} \) | \( 5.8 \times 10^{-7} \) | \( 5.9 \times 10^{-9} \) |
| 2.0 | \( 1.9 \times 10^{-3} \) | \( 7.5 \times 10^{-6} \) | \( 3.4 \times 10^{-8} \) | \( 1.5 \times 10^{-10} \) |
| 1.0 | \( 1.9 \times 10^{-4} \) | \( 1.3 \times 10^{-7} \) | \( 1.0 \times 10^{-10} \) | \( 8.2 \times 10^{-14} \) |
| 0.8 | \( 8.3 \times 10^{-5} \) | \( 3.0 \times 10^{-8} \) | \( 1.3 \times 10^{-11} \) | \( 5.2 \times 10^{-15} \) |
| 0.6 | \( 2.7 \times 10^{-5} \) | \( 4.1 \times 10^{-9} \) | \( 7.0 \times 10^{-13} \) | \( 1.2 \times 10^{-16} \) |
| 0.4 | \( 5.0 \times 10^{-6} \) | \( 2.0 \times 10^{-10} \) | \( 9.3 \times 10^{-15} \) | \( 4.3 \times 10^{-19} \) |
| 0.2 | \( 2.3 \times 10^{-7} \) | \( 8.2 \times 10^{-13} \) | \( 3.2 \times 10^{-18} \) | \( 1.3 \times 10^{-23} \) |
| \(-0.2\) | \( 5.7 \times 10^{-7} \) | \( 4.7 \times 10^{-12} \) | \( 4.2 \times 10^{-17} \) | \( 3.9 \times 10^{-22} \) |
| \(-0.4\) | \( 3.2 \times 10^{-5} \) | \( 7.3 \times 10^{-9} \) | \( 1.8 \times 10^{-12} \) | \( 4.4 \times 10^{-16} \) |
| \(-0.6\) | \( 5.1 \times 10^{-4} \) | \( 1.1 \times 10^{-6} \) | \( 2.8 \times 10^{-9} \) | \( 6.5 \times 10^{-12} \) |
| \(-0.8\) | \( 5.7 \times 10^{-3} \) | \( 1.1 \times 10^{-4} \) | \( 2.2 \times 10^{-6} \) | \( 3.8 \times 10^{-8} \) |
| \(-1.0\) | \( 1.2 \times 10^{-1} \) | \( 4.6 \times 10^{-2} \) | \( 1.8 \times 10^{-2} \) | \( 4.3 \times 10^{-3} \) |

Table 6 shows the errors \( E_4(n) \equiv |I_3^n/I_3 - 1| \), with \( I_3^n \) obtained, via the \( \varepsilon \)-algorithm, from \( T(2n, 0) \), i.e. from the \( [n/n] \) Padé approximant to the first \( 2n + 1 \) terms of the Taylor series of \( I_3 \), with \( d = 4 \). Comparison of Tables 3 and 6 reveals the accelerated convergence for \( |q^2| \leq m^2 \). Table 6 also shows the extension of the domain of convergence to \( q^2 > m^2 \).

For larger values of \( q^2 \), we apply the \( \varepsilon \)-algorithm to the asymptotic expansion of \( I_2 \). Now the \( n \)th term in the series is \( -(m^2/q^2)^n \) times a quadratic polynomial in \( \ln(q^2/m^2) \), and hence there is no Padé approximant corresponding to the first \( 2n + 1 \) terms, nor any theorem, known to us, to guarantee convergence of the \( \varepsilon \)-algorithm for \( |q^2| < 9m^2 \), outside the domain of convergence of the asymptotic expansion. Nevertheless, remarkably good results are obtained in the space-like region \( q^2 \geq 3m^2 \), as can be seen from Table 7.

Comparison of Tables 4 and 7 reveals accelerated convergence, in an enlarged domain, as was observed at small \( q^2 \). Together, Tables 6 and 7 show that results accurate to a few parts in \( 10^9 \) can be obtained \textit{throughout} the space-like region, \( q^2 > 0 \). Thus, if one needs only space-like values, the delicate separation method of Section 3.1 may be avoided; it suffices to use the \( \varepsilon \)-algorithm, to achieve accelerated convergence of the large-\( q^2 \) expansion, for \( q^2 \geq 3m^2 \), or the small-\( q^2 \) expansion, for \( q^2 \leq 3m^2 \). Only if one requires the real and imaginary parts on the problematic region of the cut, with \( -q^2 = s + i0 \) and \( s \in [m^2, 9m^2] \), does one need to disentangle the separate sources of discontinuity.
Table 7: Accelerated convergence of the asymptotic expansion of $I_3$, with $d = 4$.

| $q^2/m^2$ | $E_4(2)$ | $E_4(4)$ | $E_4(6)$ | $E_4(8)$ |
|------------|-----------|-----------|-----------|-----------|
| 12.0       | $8.9 \times 10^{-7}$ | $2.2 \times 10^{-9}$ | $5.7 \times 10^{-14}$ | $9.0 \times 10^{-17}$ |
| 11.0       | $2.0 \times 10^{-6}$ | $7.6 \times 10^{-9}$ | $8.6 \times 10^{-14}$ | $5.7 \times 10^{-16}$ |
| 10.0       | $4.2 \times 10^{-6}$ | $4.5 \times 10^{-8}$ | $8.0 \times 10^{-13}$ | $6.0 \times 10^{-15}$ |
| 9.0        | $9.0 \times 10^{-6}$ | $1.4 \times 10^{-7}$ | $4.1 \times 10^{-12}$ | $3.6 \times 10^{-14}$ |
| 8.0        | $2.0 \times 10^{-5}$ | $9.7 \times 10^{-8}$ | $2.0 \times 10^{-11}$ | $4.6 \times 10^{-14}$ |
| 7.0        | $4.6 \times 10^{-5}$ | $1.4 \times 10^{-7}$ | $1.1 \times 10^{-10}$ | $1.7 \times 10^{-13}$ |
| 6.0        | $1.2 \times 10^{-4}$ | $2.7 \times 10^{-7}$ | $6.7 \times 10^{-10}$ | $4.2 \times 10^{-15}$ |
| 5.0        | $3.3 \times 10^{-4}$ | $6.1 \times 10^{-7}$ | $5.3 \times 10^{-9}$ | $1.4 \times 10^{-12}$ |
| 4.0        | $1.1 \times 10^{-3}$ | $1.4 \times 10^{-6}$ | $6.1 \times 10^{-8}$ | $4.6 \times 10^{-11}$ |
| 3.0        | $5.3 \times 10^{-3}$ | $1.8 \times 10^{-6}$ | $1.7 \times 10^{-6}$ | $1.7 \times 10^{-9}$ |

4 Photon self-energy

Evaluating two-loop diagrams for the photon self-energy, one encounters the integrals

$$
\int \int \frac{d^d k_1 d^d k_2}{\pi^d \Gamma^2(1 + \varepsilon)} P^{j_1}(k_1, m) P^{j_2}(k_2, m) P^{j_3}(k_1 - q, m) P^{j_4}(k_2 - q, m) P^{j_5}(k_1 - k_2, 0)
$$

$$
\equiv V_Q(j_1, j_2, j_3, j_4, j_5)(m^2)^{d-j_1-j_2-j_3-j_4-j_5},
$$

with the propagators of Fig. 1(d) raised to integer powers $j_1$ to $j_5$. Integration by parts yields the recurrence relations

$$
T(j_5, j_2, j_4)V_Q = \left\{ j_2 2^+(5^- - 1^-) + j_4 4^+(5^- - 3^-) \right\} V_Q,
$$

$$
T(j_5, j_1, j_3)V_Q = \left\{ j_1 1^+(5^- - 2^-) + j_3 3^+(5^- - 4^-) \right\} V_Q,
$$

where $T(j_k, j_i, j_m) \equiv d - 2j_k - j_i - j_m$ and $1^\pm V_Q(j_1, ...)$ $\equiv V_Q(j_1 \pm 1, ...)$, etc. Using (66) to remove one propagator, carrying the momentum $k_1$, and then (67) to remove another, carrying the momentum $k_2$, every $V_Q$ integral can be reduced to combinations of a rational function of $q^2/m^2$ and $d$ (deriving from tadpole diagrams, through which $q$ does not flow) and rational multiples of integrals of the type

$$
V_Q(2, 2, 1, 0, 0) = \frac{G_4}{4 - d},
$$

$$
V_Q(2, 2, 1, 1, 0) = \frac{1}{4} G_4^2,
$$

$$
V_Q(0, \alpha, \beta, 0, \gamma) = \frac{\Gamma(\alpha + \beta + \gamma - d) \Gamma(\frac{d}{2} - \gamma) \Gamma(\beta + \gamma - \frac{d}{2}) \Gamma(\alpha + \gamma - \frac{d}{2})}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\frac{d}{2}) \Gamma(\alpha + \beta + 2\gamma - d) \Gamma^2(3 - \frac{d}{2})}
\cdot 4 F_3\left[ \begin{array}{c}
\gamma, \alpha + \beta + \gamma - d, \beta + \gamma - \frac{d}{2}, \alpha + \gamma - \frac{d}{2} ; \frac{-q^2}{4m^2} \end{array} \right].
$$

Moreover, the $4 F_3$ hypergeometric functions of (70), with integer values of $\alpha$, $\beta$ and $\gamma$, are always reducible to two contiguous $3 F_2$ functions.

It follows that every two-loop calculation of the perturbative [19, 20] contribution, or of a gluon-condensate [13] contribution, to the correlator of a current connecting quarks of
equal mass, amounts to no more than the evaluation of 5 rational functions, corresponding
to the coefficients of unity, \( G_1 \), \( G_2 \), and two suitably chosen \( _3F_2 \) functions. In the case
of the master integral \( I_4 \), with unit exponents, the functions \( G_4 \) and \( H_4 \) were chosen
in (13) in such a way that only \( G_2 \) and \( H_4 \) appeared in the result (13). In general, a basis
\( \{ 1, G_4, G_2^2, I_4, I_4' \} \) gives coefficients that are regular as \( d \to 4 \), enabling one to obtain as
many terms as desired in the small-\( q^2 \) expansion, or the large-\( q^2 \) expansion, merely by
taking the \( d \to 4 \) limits of the 5 rational coefficients, in this basis, and using the \( d = 4 \) expansions
of \( G_4 \) and \( I_4 \). In this sense, \( I_4 \) is truly the master integral, for this mass case.

### 4.1 Vacuum polarization function

In the on-shell renormalization scheme of conventional QED [16, 17, 18], the renormalized
photon propagator has a denominator \((1 + \Pi)\), where the vacuum polarization function,
\( \Pi \), vanishes at \( q^2 = 0 \). Generalizing to arbitrary \( d \), we write

\[
\Pi(z) = \frac{T(d)}{4} \left\{ \left( \frac{\alpha}{4\pi} \right) \Pi_1(z) + \left( \frac{\alpha}{4\pi} \right)^2 \Pi_2(z) + O(\alpha^3) \right\},
\]

where \( z \equiv -q^2/4m^2 \), traces are normalized by \( \text{Tr}(\gamma_\mu\gamma_\nu) = T(d)g_{\mu\nu} \), and we form the
dimensionless coupling \( \alpha \equiv \Gamma(3-d/2)m^{d-4}e^2/(4\pi)^{d/2-1} \) from the on-shell charge
and mass. The renormalization-scheme-invariant coupling is given by \( \alpha/(1 + \Pi) \). For \( d = 4 \),
\( \alpha = e^2/4\pi \) is the fine structure constant of QED, measured at \( q^2 = 0 \).

To give a compact form for \( d \)-dimensional two-loop vacuum polarization, we choose
the basis \( \{ 1, \phi_1, \phi_1^2, \phi_2, \phi_3 \} \), where

\[
\begin{align*}
\phi_1(z) &= z \frac{(d-6)}{5} _2F_1\left[ 1, 4 - \frac{d}{2}; \frac{5}{2}, z \right], \\
\phi_2(z) &= (d-3)(d-4)(d-6) \left( \frac{(d+2)\frac{d}{2}(d+1)}{(d+2)d(d-5)(d-7)} \right) _3F_2\left[ 1, 4 - \frac{d}{2}, 5 - d; \frac{2}{2}, \frac{9-d}{2}, z \right], \\
\phi_3(z) &= (d-6) \left( \frac{(d+2)d(d-1)}{(d+2)d(d-5)(d-7)} \right) _3F_2\left[ 1, 4 - \frac{d}{2}, 6 - d; \frac{2}{2}, \frac{9-d}{2}, z \right].
\end{align*}
\]

We implemented (X) [70] in both [28] REDUCE, running on a VAX4100, and FORM,
running on a AT486. In each case, a few seconds of CPU time yielded

\[
\begin{align*}
\Pi_1(z) &= \frac{4}{3(d-1)} \left\{ \left( z(d-2) - [1 + z(d-2)] \phi_1(z) \right) \right\}, \\
\Pi_2(z) &= \frac{-32z}{9(d-4)} \left\{ \frac{d^3 - 12d^3 + 48d^2 - 53d - 24}{d(d-1)(d-5)} - \frac{z(d-2)(d^2 - 7d + 16) [1 - \phi_1(z)]}{4(d-1)} \\
&\quad + \frac{(d - 6 - (d - 3)^2)\phi_1(z)^2}{2(d-3)(1-z)} - \frac{(d - 2)(d - 5)\phi_1(z)}{2(d-1)(d-3)} \phi_1(z) - \frac{(d - 2)^2}{2(d-1)} \phi_1^2(z) \\
&\quad - \frac{9}{d - 3} \phi_2(z) + 18 \frac{(d - 2)(d - 5) - z(d^2 - 5d + 8)}{d - 3} \phi_3(z) \right\}. \\
\end{align*}
\]

The \( d \to 4 \) limit of \( \Pi_2(z) \) is best taken after transforming to the basis \( \{ 1, G_4, G_2^2, I_4, I_4' \} \),
in which the coefficients are regular, though rather lengthy (on account of factors of \( (3d-8) \)
and \((3d - 10)\) in their denominators. At \(d = 4\), with \(T(4) = 4\), we obtain

\[
\Pi_1(z)_{|d=4} = \frac{20}{9} + \frac{4}{3z} - \frac{4(1-z)(1+2z)}{3z} G(z),
\]

\[
\Pi_2(z)_{|d=4} = \frac{10}{3} + \frac{26}{3z} - \frac{4(1-z)(3+2z)}{z} G(z) + \frac{2(1-z)(1-16z)}{3z} G^2(z)
- \frac{2(1+2z)}{3z} \left[ 1 + 2z(1-z) \frac{d}{dz} \right] \frac{I(z)}{z},
\]

\[
G(z) \equiv G_{4,d=4} = _2F_1\left[ \frac{1}{3}, 1; \frac{2}{3}; z \right] = \frac{2y \ln y}{y^2 - 1},
\]

\[
I(z) \equiv q^2 I_{4,d=4} = 6 [\zeta(3) + 4 \text{Li}_3(-y) + 2 \text{Li}_3(y)] - 8 [2 \text{Li}_2(-y) + \text{Li}_2(y)] \ln y
- 2 [2 \ln(1+y) + \ln(1-y)] \ln^2 y,
\]

\[
y \equiv \sqrt{\frac{1-1/z-1}{1-1/z+1}},
\]

with polylogarithms \([22]\) in \((80)\) that were obtained in \([13]\) and later derived dispersively in \([4]\). In the next section we derive the discontinuities of \((78,80)\) from our hypergeometric functions \((72–74)\).

For \(d = 3\), \(\Pi_2(z)\) has an infrared divergence, attributable to on-shell singularities in the fermion propagator, which prohibit on-shell mass renormalization. A finite result may, however, be obtained in terms of the bare mass.

For \(d = 2\), with \(T(2) = 2\), we obtain

\[
\Pi_1(z)_{|d=2} = -\frac{4}{3} - \frac{2}{z} + \frac{2}{z} G(z),
\]

\[
\Pi_2(z)_{|d=2} = \frac{16}{3} + \frac{2(1-2z)}{z(1-z)} G(z) - \frac{2}{z(1-z)} G^2(z).
\]

### 4.2 Spectral density

For the vacuum polarization function \(\Pi(z)\), we write the spectral representation

\[
\Pi(z) = z \int_1^\infty \frac{\rho(t) dt}{t - z},
\]

with spectral density

\[
\rho(t) = \frac{\text{Im} \Pi(t + i0)}{\pi t} = \frac{T(d)}{4} \left\{ \left( \frac{\alpha}{4\pi} \right) \rho_1(t) + \left( \frac{\alpha}{4\pi} \right)^2 \rho_2(t) + O(\alpha^3) \right\}.
\]

From the general integral representations

\[
_2F_1\left[ \begin{array}{c} a, b \\ c \end{array}; z \right] = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_1^\infty \frac{(t-1)^{c-b-1}t^{a-c} dt}{(t-z)^a},
\]

\[
_3F_2\left[ \begin{array}{c} a, b, c \\ d, e \end{array}; z \right] = \frac{\Gamma(d) \Gamma(e)}{\Gamma(b) \Gamma(c) \Gamma(d+e-b-c)} \int_1^\infty \frac{(t-1)^{d+e-b-c-1}t^{a-d} dt}{(t-z)^a} \int_1^\infty \frac{(t-1)^{d+e-b-c-1}t^{a-d} dt}{(t-z)^a}._2F_1\left[ \begin{array}{c} e-c, e-b \\ d+e-b-c \end{array}; 1-t \right],
\]

\(17\)
we obtain the spectral representations of the hypergeometric functions in (72, 73, 74), which all have a = 1. To obtain the contribution to the spectral density of the φ_t^2(z) terms in (75), we use the transformation

\[ 2F1 \left[ a, b; c; z \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} 2F1 \left[ a, b; 1+a+b-c; 1-z \right] + z^{1-c} (1-z)^{c-a-b} \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} 2F1 \left[ 1-a, 1-b; 1-a-b+c; 1-z \right], \]  

which enables one to split \( \phi_1(t+i0) \) into its real and imaginary parts, for \( t \in [1, \infty] \). In this particular case, with \( a = 1 \), the second hypergeometric function in (88) is trivial.

Our results for the \( d \)-dimensional spectral density are as follows:

\[
\begin{align*}
\rho_1(t) &= \frac{1 + t(d-2)}{2t^2} r_1(t), \\
\rho_2(t) &= \frac{4}{(d-3)(d-4)t^2} \left\{ 2t\rho_1(t) - \frac{2(d-2)(d-5) - 2t(d^2 - 5d + 8)}{d-1} r_3(t) - \frac{2 + t(d-2)(d-4)(d-5) - t^2(d-2)(d^2 - 7d + 16)}{2(d-5)} r_1(t) r_4(t) \right\}, \\
\rho_3(t) &= \frac{\Gamma \left( \frac{1}{2} \right) t^{-\frac{d}{2}} (t-1)^{-\frac{d-2}{2}}}{\Gamma \left( \frac{d+1}{2} \right)} \frac{\Gamma \left( \frac{3-d}{2} \right) t^{1-\frac{d}{2}} (t-1)^{-\frac{d-7}{2}}}{\Gamma \left( 3-d \right) \Gamma \left( \frac{3d-5}{2} \right) \Gamma \left( \frac{3-d}{2} \right)} 2F1 \left[ \frac{d-1}{2d-5}, \frac{1}{2}, 1-t \right], \\
\rho_4(t) &= \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{2-d}{2} \right) \sin \left( \frac{\pi d}{2} \right)}{1-t} r_1(t) + 2F1 \left[ 1, \frac{3-d}{2d-7}, \frac{1}{2}, 1-t \right],
\end{align*}
\]

where \( \rho_2, \rho_3(t) \) derive from the imaginary parts of (73, 74), whilst \( \rho_1, \rho_4(t) \) derive from the real and imaginary parts of (72), with the real part entering via \( \text{Im}(\phi_1^2) = 2\text{Re}(\phi_1) \text{Im}(\phi_1) \).

We now investigate the limits as \( d \to 2 \) and \( d \to 4 \). The former case was considered in the QCD2 sum rules of [21], the latter was the subject of the classic QED calculation of Källén and Sabry [16].

As \( d \to 2 \), we obtain

\[
\begin{align*}
r_1(t) &\to \frac{2}{tv}, \quad r_2(t) \to \frac{1}{2tv}, \quad r_3(t) \to \frac{-1}{8t^2v^3}, \quad r_4(t) \to \frac{3}{2tv^2} + \frac{3}{4t^2v^3} \ln x, \quad (95)
\end{align*}
\]

with

\[
x \equiv \frac{1-v}{1+v}, \quad v \equiv \sqrt{1 - 1/t}. \quad (96)
\]

The spectral density is obtained, from (89, 90), as

\[
t^3 \rho(t)_{d=2} = \frac{1}{2v} \left( \frac{\alpha}{4\pi} \right) + \left\{ \frac{1+v^2}{2v^3} + \frac{(1-v^2)^2}{2v^4} \ln x \right\} \left( \frac{\alpha}{4\pi} \right)^2 + O(\alpha^3). \quad (97)
\]
Making the transformations \( z \to t + i0, \ y \to -x + i0, \ \ln y \to \ln x + i\pi \), in (93), we verify that (97) corresponds to the discontinuity of the \( d = 2 \) result of (32, 33). Transforming to the bare charge and mass, we verify the QCD result of (11). Since \( \rho(t) \) has a non-integrable singularity at \( t = 1 \), the dispersion relation should be written for \( (z-1)\Pi(z) \). The corresponding inclusion of an additional factor of \( (t-1) \) in the spectral integral will not destroy its convergence at large \( t \).

To obtain the \( d = 4 \) spectral density of (16), we need to expand the hypergeometric functions of (22, 24) to first order in \( \varepsilon \equiv 2 - d/2 \). The required expansions are

\[
2F1\left[ \frac{d-1}{2d-5}, \frac{1}{2}; 1 - t \right] = \frac{15\sqrt{x}}{(1-x)^3} \left\{ (1-x^2)\left(\frac{1}{4} + x + \frac{1}{4}x^2\right) \left[ 1 - \frac{27}{10}\varepsilon - 4\varepsilon \ln \frac{1+x}{2} \right] + (x + x^2 + x^3)[\ln x + \varepsilon f(x)] - \varepsilon (\frac{31}{5}x + \frac{8\varepsilon}{10}x^2 + \frac{11}{5}x^3 + x^4) \ln x + O(\varepsilon^2) \right\} \tag{98}
\]

\[
2F1\left[ \frac{d-3}{2d-7}, \frac{1}{2}; 1 - t \right] = \frac{-3\sqrt{x}}{2(1-x)^3} \left\{ (1-x^2) \left[ 1 - 3\varepsilon - 4\varepsilon \ln \frac{1+x}{2} \right] + (1 + x^2)[\ln x + \varepsilon f(x)] - 2\varepsilon (3 + x + 5x^2) \ln x + O(\varepsilon^2) \right\}, \tag{99}
\]

\[
2F1\left[ \frac{1, 3-d}{2, d-1}; 1 - t \right] = \frac{-2x}{(1-x^2)} \left\{ (1 + 2\varepsilon) \ln x - \varepsilon g(x) + O(\varepsilon^2) \right\}, \tag{100}
\]

\[
f(x) = 6 \ln x \ln(1-x) - \frac{7}{2} \ln^2 x + 6 \text{Li}_2(x) - 4\zeta(2) + 4 \text{Li}_2(-x) + 4 \ln 2 \ln x, \tag{101}
\]

\[
g(x) = 2 \ln x \ln(1-x) - \frac{7}{2} \ln^2 x + 2 \text{Li}_2(x) - 2\zeta(2), \tag{102}
\]

from which we obtain

\[
t\rho(t)|_{d=4} = \frac{2v(3-v^2)}{3} \left( \frac{\alpha}{4\pi} \right) + \left\{ 2v(5-3v^2) + \frac{(1-v^2)(15+v^2)}{3} \right\} \ln x + \frac{16v(3-v^2)}{3} \left[ \frac{1+v^2}{2v} + x \frac{d}{dx} \right] L_2(x) \left( \frac{\alpha}{4\pi} \right)^2 + O(\alpha^3), \tag{103}
\]

in terms of the dilogarithmic discontinuity

\[
L_2(x) \equiv m^2 t \left. I_4(-q^2/4m^2 = t + i0) - I_4(-q^2/4m^2 = t - i0) \right|_{d=4} 2\pi i \frac{d}{dx} \ln x, \tag{104}
\]

which was found by explicit integration in (18) and has here been obtained from the \( O(\varepsilon) \) terms of the hypergeometric functions (98, 100). Differentiating (104) in our compact result (103), we easily reproduce the lengthier formulae of (16, 17, 18).

Here we have obtained the \( d = 4 \) result of (103, 104) by developing the \( \varepsilon \)-expansion of the hypergeometric functions in our algebraically computed \( d \)-dimensional result (99). This is the generic method we propose for unsolved problems, such as the two-loop gluon-condensate contributions to equal-mass current correlators (13).

Note that our use of the regular basis \{1, G_4, G_4^2, I_4, I_4^2\}, to obtain (78) in the previous section, reduces the \( d = 4 \) spectral problem of this section to the calculation of the discontinuity of the master integral (15), for \( d = 4 \). Once this is obtained from hypergeometric
functions, as in (104), all other spectral calculations of this type are reduced to algebra. Once one has used Cauchy’s theorem to construct the function with this discontinuity, as was done in [4] in the case of (80), all calculations of similar two-point functions are reduced to $d$-dimensional algebra, amounting to no more than using recurrence relations to find 5 rational coefficients that are regular as $d \to 4$.

4.3 Accelerated convergence of the Taylor series

For small $q^2/m^2 = -4z$, we find the coefficients of the Taylor series

$$\Pi_2(z) = \sum_{n=1}^{\infty} c(n) [-4z]^n,$$

from the hypergeometric series of (72,73,74) that enter our $d$-dimensional result (76). Running REDUCE 3.4.1 [28] on a VAX 4100, it took 20 seconds of CPUtime to obtain the first 10 Taylor coefficients, the numerator of $c(10)$ being a polynomial in $d$ of degree 27. We list only

$$c(1) = \frac{4(3d^5 - 48d^4 + 227d^3 - 58d^2 - 1728d + 1872)}{9(d + 2)d(d - 3)(d - 5)(d - 7)},$$

$$c(2) = \frac{11d^7 - 227d^6 + 1289d^5 + 1619d^4 - 31840d^3 + 30036d^2 + 263952d - 311040}{90(d + 4)(d + 2)d(d - 3)(d - 7)(d - 9)},$$

higher-order coefficients being too cumbersome to write here. The $d \to 4$ limit of the leading coefficient (106) gives the classic result of [29]: $\frac{1}{4}c(1) = -\frac{82}{81} + O(\varepsilon)$.

Taking the $d \to 4$ limit of the first 20 coefficients of (105), we find agreement with [21], where a closed form is given for $d = 4$. We use these 4-dimensional coefficients to compute Padé approximants to the Taylor series. Computation of the two-loop spectral density reveals it to be non-negative. Hence $\Pi_2$ is of Stieltjes type and convergence of the Padé approximants is guaranteed [23] throughout the region $z \in [-\infty, 1]$ and hence for arbitrarily large space-like $q^2$. Table 8 demonstrates the quality of the convergence. The notation is the same as in Section 3.2: truncating the sum after $2n + 1$ terms, with the first term vanishing, we tabulate $E_4(n)$, the modulus of the relative error in the 4-dimensional $[n/n]$ Padé approximant, for values of $q^2/m^2$ up to 12, well outside the disk $|q^2/m^2| < 4$, to which the convergence of the original Taylor series was restricted.

With just four non-vanishing terms in the Taylor series, the error $E_4(2)$ is less than 1%, for $q^2/m^2 \leq 10$. The inclusion of each new set of four terms, reduces the error by at least two orders magnitude. Tables 6 and 8 clearly demonstrate the utility of applying Padé approximants to Taylor series, for obtaining accurate values of two-point functions, in the space-like region. Since the calculation of the Taylor coefficients, from vacuum diagrams, can be performed in the general mass case [11], this is a most welcome result. We remark that the technology exists to obtain three-loop QCD results by this method, since the calculation of the relevant vacuum diagrams has been automated in the course of work on three-loop terms in QED [20, 30]. Methods of accelerated convergence may also prove useful in the case of multi-loop diagrams with more than two external particles, by improving expansions in powers of all the scalar products of the external momenta.
The error has been obtained without accelerating the convergence. As in Section 3.2, the application of only the first three asymptotic coefficients, for arbitrary $d$, with $z = \rho + i0$, and $\rho \in (-\infty, \infty)$, gives a finite coefficient that is quadratic in $\ln(q^2/m^2)$. For large $q^2/4m^2 = -z$, we invert the series in (72,73,74) and obtain the coefficients of $\Pi_2(z) = \sum_{n=0}^{\infty} p(n)/z^n$, (108)

for arbitrary $d$. Each coefficient involves the structures $\beta$ and $\gamma$, given in (9,10). In general, the terms $\{1, \gamma, \beta, \beta^2\}$ occur, with rational coefficients that are singular as $d \to 4$. The $d \to 4$ limit then gives a finite coefficient that is quadratic in $\ln(q^2/m^2)$. On the cut, with $z = -q^2/4m^2 = t + i0$ and $t \in [1, \infty]$, one may extract the asymptotic expansions of the real and imaginary parts of $\Pi_2(t + i0)$, using $\ln(q^2/m^2) = \ln(4t) - i\pi$. Here we list only the first three asymptotic coefficients, for arbitrary $d$:

$$p(0) = \frac{8(d^3 - 12d^2 + 41d - 34)}{d(d-3)(d-4)(d-5)} - \frac{32(d-2)(d^2 - 4d + 8)}{(d-1)(d-4)(3d-8)(3d-10)} \gamma$$

$$+ \frac{8(d-2)(d^2 - 7d + 16)}{(d-1)(d-3)^2(d-4)^3} \beta^2,$$

$$p(1) = \frac{16(d-2)(d^3 - 8d^2 + 21d - 26)}{(d-1)(d-3)(d-4)(d-6)(3d-10)} \gamma + \frac{8(d-1)}{(d-3)^2(d-4)} \beta$$

$$- \frac{8(d-2)(d^2 - 6d + 11)}{(d-1)(d-3)^2(d-4)^2} \beta^2,$$

$$p(2) = \frac{32(d-1)}{d(d-2)(d-3)(d-4)^2} - \frac{4(d-2)(d-5)(d^4 - 14d^3 + 65d^2 - 128d + 128)}{(d-1)(d-3)(d-4)^3(d-6)(d-8)} \gamma$$

$$- \frac{2(d^3 - 3d^2 - 26d + 40)}{d(d-3)(d-4)^2} \beta + \frac{4(d^3 - 10d^2 + 32d - 36)}{(d-1)(d-4)^3} \beta^2.$$ (111)

Table 9 shows accelerated convergence of the large-$q^2$ expansion of $\Pi_2$, for $q^2/m^2 \geq 6$. The error $E_4(4)$ is at least three orders of magnitude smaller than that which would have been obtained without accelerating the convergence. As in Section 3.2, the application of

| $q^2/m^2$ | $E_4(2)$ | $E_4(4)$ | $E_4(6)$ | $E_4(8)$ |
|-----------|--------|--------|--------|--------|
| 12.0      | $1.1 \times 10^{-2}$ | $1.5 \times 10^{-4}$ | $1.9 \times 10^{-6}$ | $2.4 \times 10^{-8}$ |
| 11.0      | $9.4 \times 10^{-3}$ | $1.1 \times 10^{-4}$ | $1.1 \times 10^{-6}$ | $1.2 \times 10^{-8}$ |
| 10.0      | $7.8 \times 10^{-3}$ | $7.2 \times 10^{-5}$ | $6.3 \times 10^{-7}$ | $5.4 \times 10^{-9}$ |
| 9.0       | $6.3 \times 10^{-3}$ | $4.6 \times 10^{-5}$ | $3.2 \times 10^{-7}$ | $2.2 \times 10^{-9}$ |
| 8.0       | $4.9 \times 10^{-3}$ | $2.7 \times 10^{-5}$ | $1.4 \times 10^{-7}$ | $7.6 \times 10^{-10}$ |
| 7.0       | $3.6 \times 10^{-3}$ | $1.5 \times 10^{-5}$ | $5.7 \times 10^{-8}$ | $2.2 \times 10^{-10}$ |
| 6.0       | $2.5 \times 10^{-3}$ | $7.0 \times 10^{-6}$ | $1.8 \times 10^{-8}$ | $4.8 \times 10^{-11}$ |
| 5.0       | $1.6 \times 10^{-3}$ | $2.8 \times 10^{-6}$ | $4.5 \times 10^{-9}$ | $7.3 \times 10^{-12}$ |
| 4.0       | $8.9 \times 10^{-4}$ | $8.3 \times 10^{-7}$ | $7.3 \times 10^{-10}$ | $6.4 \times 10^{-13}$ |
| 3.0       | $3.9 \times 10^{-4}$ | $1.6 \times 10^{-7}$ | $6.0 \times 10^{-11}$ | $2.1 \times 10^{-14}$ |
| 2.0       | $1.1 \times 10^{-4}$ | $1.2 \times 10^{-8}$ | $1.3 \times 10^{-12}$ | $6.2 \times 10^{-15}$ |
the \(\varepsilon\)-algorithm, to both series, allows one to cover the entire space-like region, to good accuracy, using only a few terms from each series. Moreover, we find the \(\varepsilon\)-algorithm to be similarly effective in accelerating the convergence of the asymptotic expansions of the real and imaginary parts on the cut.

Table 9: Accelerated convergence of the asymptotic expansion of \(\Pi_2\), with \(d = 4\).

| \(q^2/m^2\) | \(E_4(1)\) | \(E_4(2)\) | \(E_4(3)\) | \(E_4(4)\) |
|-------------|-------------|-------------|-------------|-------------|
| 40.0        | 1.1 \times 10^{-3} | 9.7 \times 10^{-4} | 2.7 \times 10^{-10} | 2.3 \times 10^{-13} |
| 36.0        | 1.4 \times 10^{-3} | 1.6 \times 10^{-6} | 5.1 \times 10^{-10} | 5.4 \times 10^{-13} |
| 32.0        | 1.8 \times 10^{-3} | 2.7 \times 10^{-6} | 1.1 \times 10^{-9}  | 1.4 \times 10^{-12} |
| 28.0        | 2.5 \times 10^{-3} | 5.0 \times 10^{-6} | 2.4 \times 10^{-9}  | 4.1 \times 10^{-12} |
| 24.0        | 3.5 \times 10^{-3} | 9.9 \times 10^{-6} | 5.9 \times 10^{-9}  | 1.4 \times 10^{-11} |
| 20.0        | 5.3 \times 10^{-3} | 2.2 \times 10^{-5} | 1.7 \times 10^{-8}  | 6.0 \times 10^{-11} |
| 16.0        | 8.6 \times 10^{-3} | 5.8 \times 10^{-5} | 5.4 \times 10^{-8}  | 3.3 \times 10^{-10} |
| 12.0        | 1.6 \times 10^{-2} | 1.9 \times 10^{-4} | 1.7 \times 10^{-7}  | 2.7 \times 10^{-9}  |
| 10.0        | 2.4 \times 10^{-2} | 3.6 \times 10^{-4} | 9.7 \times 10^{-8}  | 6.3 \times 10^{-9}  |
| 8.0         | 3.9 \times 10^{-2} | 6.5 \times 10^{-4} | 2.8 \times 10^{-6}  | 4.9 \times 10^{-8}  |
| 6.0         | 7.2 \times 10^{-2} | 1.1 \times 10^{-4} | 1.7 \times 10^{-4}  | 2.0 \times 10^{-7}  |

5 Summary and conclusions

There is a pressing need for techniques to calculate multi-loop radiative corrections in processes with a wide variety of masses and momenta, in order to probe the standard model. Here we have made a second step in harnessing the economy of \(d\)-dimensional algebraic methods to that end. The first step was made in [6, 7], which dealt with on-shell massive two-loop two-point functions. Here we have tackled all the off-shell two-loop master integrals needed for QED and QCD, using integration by parts to reduce all but one of them to hypergeometric functions, and dealing with the troublesome fermion-propagator integral of Fig. 1(c) equally effectively, by using differential equations, likewise obtained from integration by parts.

As might have been expected from previous successes [2, 4, 5] of dimensional regularization of massive diagrams, our new \(d\)-dimensional results are both easier to obtain and simpler in structure than their 4-dimensional specializations. Just as the complexity of polylogarithms [22] results from a singular case of hypergeometric series, so 4-dimensional calculations are harder than \(d\)-dimensional ones. The moral to be drawn from this is clear: take the limit \(d \to 4\) as late as possible. The cost of carrying an extra variable in a computer-algebra program is a small one to pay, for the avoidance of analytic complexity, and can, in any case, be made even smaller, by retaining only an appropriate number of powers of \((d - 4)\).

We have applied algebraic methods to the two-loop photon self-energy. As promised, a thorough knowledge of the analytic properties of the master integral turns out to be sufficient to solve problems with arbitrary spinorial or tensorial complexities, without need
of further analysis. All problems of the same mass structure (such as the problem of the two-loop gluon-condensate contribution to vacuum polarization, which still lacks a full solution) amount to no more than the determination of 5 rational functions of $d$ and $q^2/m^2$, multiplying structures that have been thoroughly analyzed in Section 4.

We have shown examples in which the whole of the space-like region may be accurately covered by applying Padé approximants to the small-$q^2$ Taylor series, and their generalization, via the $\varepsilon$-algorithm, to the large-$q^2$ asymptotic expansion. Tables 8 and 9 exhibit such accelerated convergence, in the case of two-loop vacuum polarization.

In the time-like region, our methods are similarly effective. In the case of vacuum polarization, accelerated convergence of the Taylor series, for $-q^2 \in [0, 4m^2]$, is achieved by applying the $\varepsilon$-algorithm to truncations of (105). On the cut, with $-q^2 = s + i0$ and $s \in [4m^2, \infty]$, accelerated convergence of the asymptotic expansions of the real and imaginary parts is achieved by applying the $\varepsilon$-algorithm to truncations of (108). Thus we can approximate the two-loop vacuum polarization function (78) to good accuracy, throughout the entire complex plane, using only a few of its algebraically computed Taylor and asymptotic coefficients. Even near the branchpoint, with $-q^2 \approx 4m^2$, we find the $\varepsilon$-algorithm to be effective in accelerating the convergence of the original series.

These efficient methods of algebraic computation and numerical approximation are now available for all the self-energy diagrams of QED and QCD, in all kinematical regions, thanks to our explicit large and small $q^2$ expansions, in (17,20,24) and (15,18,22), and our analysis of the remaining fermion-propagator integral of Section 2.5, with contributions that were conveniently separated in Section 3.1. Given this separation, one may accurately approximate the real and imaginary parts in the demanding time-like region $-q^2 = s + i0$, with $s \in [m^2, 9m^2]$, by applying the $\varepsilon$-algorithm separately to the asymptotic expansion of the contribution with the lower threshold and to the Taylor series of the contribution with the higher threshold, paying attention to numerical cancellations in the latter’s coefficients, as in (11,23). Elsewhere, one need not make this delicate separation. Tables 6 and 7 of Section 3.2 show how good is the coverage of the entire region $q^2 \in [-m^2, \infty]$, thanks to the enlarged domain of convergence of each series. The application of such methods to the more general mass cases of (11, 12) should help to extend the scope, speed and accuracy of standard-model calculations.

It now requires only careful algebraic programming to reduce the bosonic and fermionic two-loop propagators of QED and QCD, in an arbitrary covariant gauge, to the minimal set of master integrals of Fig 1, together with a few of their derivatives. The application of methods of accelerated convergence to the expansions of these propagators in powers of $q^2/m^2$ and $m^2/q^2$ will then yield economical and accurate parametrizations, throughout the entire complex plane. Moreover, terms of order $(d - 4)^{L-2}$ may also be obtained, for use in the dimensional regularization of higher-order calculations, with $L$ loops.

It seems most likely that some degree of the analytical economy and numerical accuracy achieved in this paper is also achievable in wider areas, involving more loops, or more masses, or more external particles. We plan to study three-loop diagrams.

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