Nonlinear porous medium flow with fractional potential pressure

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Abstract

We study a porous medium equation with nonlocal diffusion effects given by an inverse fractional Laplacian operator:

$$\partial_t u - \nabla \cdot (u \nabla p) = 0, \quad p = (-\Delta)^{-s} u, \quad 0 < s < 1.$$  

We pose the problem for $x \in \mathbb{R}^n$ and $t > 0$ with bounded and compactly supported initial data, and prove existence of weak and bounded solutions that propagate with finite speed, a property that is not shared by other fractional diffusion models.

1 Introduction

We study a nonlinear diffusion model with nonlocal effects described by the system

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = K(u).$$

Here, $u$ is a function of the variables $(x, t)$ to be thought of as a density or concentration, and therefore nonnegative, while $p$ is the pressure, which is related to $u$ via a linear positive operator $K$, which we assume to be the inverse of a fractional Laplacian. To be specific, the problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n,$$

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where \( u_0 \) is a nonnegative and bounded function in \( \mathbb{R}^n \) with compact support or fast decay at infinity.

The model arises from the consideration of a continuum, say, a fluid or a population, represented by a density distribution \( u(x,t) \) that evolves with time following a velocity field \( v(x,t) \), according to the continuity equation

\[
 u_t + \nabla \cdot (u \, v) = 0.
\]

We now assume that \( v \) derives from a potential, \( v = -\nabla p \), as happens for instance in fluids in porous media according to Darcy’s law, and in that case \( p \) denotes the pressure. But potential velocity fields are also found in many other instances, like Hele-Shaw cells.

We still need a closure relation to relate \( u \) and \( p \). In the case of gases in porous media, as modeled in the 1930’s by Leibenzon and Muskat [25, 27], the closure relation takes the form of a state law: \( p = f(u) \), where \( f \) is a nondecreasing scalar function, which is linear when the flow is isothermal, and a higher power of \( u \) if it is adiabatic. The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. See [34] for a description of these and other applications. Summing up, we get the standard porous medium equation, \( u_t = c \Delta (u^2) \), or more generally, \( u_t = \Delta (u^m) \) with \( m > 1 \).

In this paper we propose to consider the case where \( p \) es related to \( u \) through a linear fractional potential operator, \( K = (-\Delta)^{-s} \) with kernel \( K(x,y) = c|x-y|^{-(n-2s)} \) (i. e., a Riesz operator, cf. [23, 33], see also Appendix for precise definitions and some comments). The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory. This literature is mostly elliptic, cf. [6, 11, 31], but cf. works like [4, 28] for related parabolic problems.

More generally, it could be assumed that \( K \) is an operator of integral type defined by convolution on all of \( \mathbb{R}^n \), with the assumptions that is positive and symmetric. The fact the \( K \) is a homogeneous operator of degree \( 2s \), \( 0 < s < 1 \), will be important in the proofs given below. An interesting variant would be \( K = (-\Delta + cI)^{-s} \). We are not exploring here such extensions.

**Extreme cases.** (1) If we take in our model \( s = 0 \), so that \( K = \) the identity operator, we get the standard porous medium equation with \( m = 2 \), whose behavior is well-known, see [2, 34] for the mathematical theory and the applications.
(2) In the other end of the $s$ interval, when $s = 1$ and we take $\mathcal{K} = (-\Delta)^{-1}$, we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u.$$  

In one dimension this leads to $u_t = u_x p_x - u^2, p_{xx} = -u$. In terms of $v = -p_x = \int u \, dx$ we have

$$v_t = up_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + vv_x = 0$ which generates shocks in finite time if we allow for $u$ to have two signs.

As a related precedent we may mention the model studied by Lions and Mas-Gallic [26], who are interested in the regularization of the velocity field in the standard porous medium equation by means of a convolution kernel. They get a system which is formally like ours, but there is a big difference in the study since they assume the kernel to be smooth and integrable, and in fact an approximation of the Dirac delta, in other words, a short-range interaction. Since the kernel of the fractional operator $(-\Delta)^{-s}$ is $k(x, y) \sim |x - y|^{-(n-2s)}$, i.e., a long-range interaction, we are far away from that situation, but it may serve as a previous regularization step.

A model from superconductivity arises in recent work by Ambrosio and Serfaty [1] describing the evolution of the vortex-density in superconductor modeling. The system is similar to our system with $\mathcal{K} = (-\Delta)^{-s}, s = 1$ and their mathematical tools are quite different. On the other hand, the equation with $s = 1/2$ has been proposed by Head [21] as the equation of motion of a dislocation continuum, and then $u$ is the dislocation density and the space dimension is $n = 1$. The mathematical investigation of this case is performed by Biler et al. in [8], though in terms of the integrated equation $v_t + |v_x| \Lambda(v) = 0$, where $\Lambda$ is the Lévy operator of order 1, which is equivalent to $(-\partial_{xx}^{1/2})^{1/2}$.

Models of this kind arise in other contexts. Some variants will be indicated at the end of the paper.

**Organization of the paper.** Section 2 derives the basic estimates in a formal way. The proof of existence of a weak solution proceeds by approximation, whereby the degeneracy of the equation is eliminated, diffusion is added and the kernels are regularized. This is technically delicate, so we first prove existence of weak solutions for the approximate problems posed in bounded domains in Section 3, and at the time basic estimates are rigorously derived. In Section 4 weak solutions of the original problem are constructed in the whole space by passage to the limit after a tail control step based on a novel argument with so-called suitable “true upper barriers”, cf.
Theorem 4.1. Such barrier method in new and turns out to be well adapted to obtain comparison results in the presence of nonlocal operators.

We then establish the main properties of the solutions: Section 5 establishes the property of finite propagation, which is a main feature of porous media equations and gives rise to the appearance of a free boundary. We discuss the persistence of positivity in Section 6.

Two sections close the paper: the Appendix, Section 7 gathers some useful definitions. A final Section 8 contains comments on variants, extensions or ongoing work on the topic of this paper: this refers in particular to the pending questions of uniqueness, smoothness or asymptotic behaviour.

Notation. We will use the notation $L_s = (-\Delta)^s$ with $0 < s < 1$ for the fractional powers of the Laplace operator defined on smooth functions in $\mathbb{R}^n$ by Fourier transform and extended in a natural way to functions in the Sobolev space $H^{2s}(\mathbb{R}^n)$. Technical reasons imply that in one space dimension the restriction $s < 1/2$ will be observed. The inverse operator is denoted by $K_s = (-\Delta)^{-s}$ and can be realized by convolution

$$K_s = K_s \star u, \quad K_s(x) = c(n, s)|x|^{-(n-2s)}.$$

as described in the appendix. $K_s$ is a positive self-adjoint operator. We will write $\mathcal{H}_s = K_s^{1/2}$ which has kernel $K_{s/2}$. The subscript $s$ will be omitted when $s$ is fixed and known. For functions that depend on $x$ and $t$, convolution is applied for every fixed $t$ with respect to the space variables. We then use the abbreviated notation $u(t) = u(\cdot, t)$.

2 Basic estimates

The existence theory of weak solutions needs a lengthy process based on several approximations and passage to the limit that may obscure to the reader the main properties of the solutions. These are however very clear from usual considerations in mathematical physics, and the authors think that it useful for the reader to have them in mind as a goal. Therefore, we do at this stage formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions and decreases fast enough as $|x| \to \infty$. The calculations are to be justified later by the approximation process. We fix $s \in (0, 1)$ and put $\mathcal{K} = (-\Delta)^{-s}$ and $\mathcal{H} = \mathcal{K}^{1/2}$. 

4
• Conservation of mass

\[
\frac{d}{dt} \int_{\mathbb{R}^n} u(x,t) \, dx = \int_{\mathbb{R}^n} \nabla \cdot (uKu) \, dx = 0.
\]

• First energy estimate:

\[
\frac{d}{dt} \int_{\mathbb{R}^n} u(x,t) \log u(x,t) \, dx = -\int_{\mathbb{R}^n} (\nabla u \cdot \nabla K u) \, dx = -\int_{\mathbb{R}^n} |\nabla H u|^2 \, dx,
\]
where we use the fact that \( K = H^2 \), and \( H \) is a positive self-adjoint operator that commutes with the gradient.

• Second energy estimate:

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |Hu(x,t)|^2 \, dx = \int_{\mathbb{R}^n} H u (Hu)_t \, dx = \int_{\mathbb{R}^n} Ku u_t \, dx = \int_{\mathbb{R}^n} (Ku) \nabla \cdot (u \nabla K u) \, dx = -\int_{\mathbb{R}^n} u |\nabla K u|^2 \, dx.
\]

• \( L^\infty \) estimate. We prove that the \( L^\infty \) norm does not increase in time.

Sketch of the proof. At a point of maximum of \( u \) at time \( t = t_0 \), say \( x = 0 \), we have

\[ u_t = \nabla u \cdot \nabla P + u \Delta K(u). \]

The first term is zero, and for the second we have \( -\Delta K = \mathcal{L} \) where \( \mathcal{L} = (-\Delta)^q \) with \( q = 1 - s \) so that

\[ \Delta K u(0) = -Lu(0) = -c \int_{\mathbb{R}^n} \frac{u(0) - u(y)}{|y|^{n+2(1-s)}} \, dy \leq 0, \]
where \( c(s,n) > 0 \).

• Conservation of positivity: \( u_0 \geq 0 \) implies that \( u(t) \geq 0 \) for all times. The argument is similar.

• We derive next the \( L^p \) estimates, \( 1 < p < \infty \):

\[
\frac{d}{dt} \int_{\mathbb{R}^n} u^p(x,t) \, dx = p \int_{\mathbb{R}^n} u^{p-1} u_t \, dx = -(p-1) \int_{\mathbb{R}^n} u^{p-1} \nabla u \cdot \nabla K u \, dx =
\]

\[
(p-1) \iint (u(x)^p - u(y)^p) \Delta K_s(x-y) u(y) \, dxdy =
\]

\[
-(p-1) \iint (u(x)^p - u(y)^p) \Delta K_s(x-y) (u(x) - u(y)) \, dxdy
\]

\[
+(p-1) \iint (u(x)^p - u(y)^p) \Delta K_s(x-y) u(x) \, dxdy = -I_1 + I_2.
\]
Since $u \geq 0$ and $\Delta K_s \geq 0$ for $0 < s < 1$, the first integral $I_1$ is positive. But symmetry means that $I_2 = I_1/2$. We conclude that $\int u^p \, dx$ is decreasing in time. See a related calculation in [15] and [18].

- A standard comparison result for parabolic equations does not seem to work. This is one of the main technical difficulties in the study of this equation. In fact, we will find special situations where some comparison holds. Some partial comparison allows us to prove two main results of the paper.

## 3 Existence I. Smooth approximate solutions

We want to solve the initial-value problem for the equation

\[ \partial_t u = \nabla (u \nabla K u), \quad K = (-\Delta)^{-s}, \]

posed in $Q = \mathbb{R}^n \times (0, \infty)$ or at least $Q_T = \mathbb{R}^n \times (0, T)$, with parameter $0 < s < 1$. We will take initial data $u_0(x) \geq 0$, $u_0 \in L^1(\mathbb{R}^n)$. We assume mostly for technical convenience that $u_0$ is bounded, and in the next section we will also impose decay conditions as $|x| \to \infty$.

We want to obtain a suitable weak solution $u(x,t)$ defined in $Q$. We approach this problem by a process that consists of regularization, elimination of the degeneracy, and reduction of the space domain. Once the approximate problems are solved, estimates are obtained that allow to pass to the limit step by step in all the approximations to obtain in the end a weak solution of the original problem.

**Definition.** We say that $u$ is a weak solution of equation (3.1) in $Q_T = \mathbb{R}^n \times (0, T)$ with initial data $u_0 \in L^1(\mathbb{R}^n)$ if $u \in L^1(Q_T)$, $K(u) \in L^1(0, T : W^{1,1}_{loc}(\mathbb{R}^n))$, and $u \nabla K(u) \in L^1(Q_T)$, and the identity

\[ \int \int u (\phi_t - \nabla K(u) \cdot \nabla \phi) \, dx \, dt + \int u_0(x) \phi(x, 0) \, dx = 0 \]

holds for all continuous test functions $\phi$ in $Q_T$ such that $\nabla x \phi$ is continuous, and $\phi$ has compact support in the space variable and vanishes near $t = T$.

**3.1.** The modifications that we use as a starting point are as follows: regularization is done by adding Laplacian term plus a kernel smoothing; the degeneracy is eliminated by raising the $u = 0$ level in the diffusion coefficient. Specifically, we take small numbers $\varepsilon, \delta, \mu \in (0, 1)$ and consider the equation $(E(\varepsilon, \delta, \mu))$:

\[ u_t = \delta \Delta u + \nabla \cdot (d(u) \nabla K_{\varepsilon}(u)), \]
posed in $Q_{T,R} = \{ x \in B_R(0), 0 < t < T \}$. A simple option for $d(u)$ is $d(u) = u + \mu$ with a small $\mu > 0$. Another option would be $d(u) = \mu$ for $0 \leq u \leq \mu$ and $d(u) = u$ for $u \geq \mu$ (we prefer the former one). Besides, the equation is posed in the spatial domain $B_R = B_R(0)$ for $0 < t < T$. We also take initial conditions

\begin{equation}
(3.4) \quad u(x, 0) = \hat{u}_0(x) \quad x \in B_R(0),
\end{equation}

where $\hat{u}_0 = u_0,\varepsilon, R$ is a nonnegative, smooth and bounded approximation of the initial data $u_0 \geq 0$. Finally, we take boundary data

\begin{equation}
(3.5) \quad u(x, t) = 0 \quad \text{for } |x| = R, \ t \geq 0.
\end{equation}

Let us explain now a convenient approximation of the kernel: $K_\varepsilon u(t) = \zeta_\varepsilon \ast u(t)$, and $\zeta_\varepsilon$ is a smooth approximation of the Riesz kernel $k_s(x) = c|x|^{-(n-2s)}$ corresponding to the inversion of the $s$-Laplacian on $\mathbb{R}^n$. In our implementation it acts on the extension of $u(x, t)$ to the whole domain $x \in \mathbb{R}^n$, which is done in the natural way, i.e., putting $u = 0$ for $|x| \geq R$ and $t > 0$. This approximation process is a bit similar to the approximation of the porous media performed by Lions and Mas-Gallic in [26], but their kernel was just a mollifying kernel representing short-range effects and the consequences of a long-range kernel with slow decay at infinity are quite different.

The existence and uniqueness of a solution $u(x, t) = u_{\varepsilon, \delta, \mu, R}(x, t)$ for the model is then more or less standard, and the solution is smooth. In the weak formulation we have

\begin{equation}
(3.6) \quad \int \int u(\phi_t - \delta \Delta \phi) \, dx \, dt - \int \int d(u)(\nabla K_\varepsilon(u) \cdot \nabla \phi) \, dx \, dt + \int u_0(x) \phi(x, 0) \, dx = 0
\end{equation}

with double integrals in $Q_{T,R} = B_R(0) \times (0, T)$, and valid for every smooth $\phi$ that vanishes at the lateral boundary and for all large $t$. It is also clear that a priori estimates like the ones in the previous section apply to this model. In particular we have

(E1.a) An easy computation gives the decay of the "total mass"

\begin{equation}
(3.7) \quad \int_{B_R} u(x, t) \, dx \leq \int_{B_R} \hat{u}_0(x) \, dx.
\end{equation}

Of course, we lose the expected mass conservation of the original problems because of the zero Dirichlet conditions of our present problem, but the estimate is still useful.

(E1.b) We also need conservation of nonnegativity, which is easy.
The $L^\infty_x$ bound is conserved, $0 \leq u(x,t) \leq \|\hat{u}_0\|_\infty$, and the argument is as in the previous section. As a consequence, the solutions $u(\cdot,t)$ at time $t$ belong to all $L^p(B_R)$ spaces with norm that is independent of the parameters $\delta,\varepsilon,\mu$ and $R$.

We now introduce a version of the first energy inequality of previous section. We select the function $F$ defined by the conditions $F''(u) = 1/d(u)$ and $F(0) = F'(0) = 0$. After some integrations by parts we will get

\begin{equation}
\frac{d}{dt} \int_{B_R} F(u) \, dx = -\delta \int_{B_R} \frac{\|
abla u\|^2}{d(u)} \, dx - \int_{B_R} |
abla \mathcal{H}_\varepsilon u|^2 \, dx,
\end{equation}

where $\mathcal{H}_\varepsilon = \mathcal{K}_\varepsilon^{1/2}$ and we have used the fact that $F''(u) = 0$ on $\Sigma = \{|x| = R\} \times [0,T]$ to annihilate the boundary term in the integration by parts. This formula implies that for all $0 < t < T$:

\begin{equation}
\int_{B_R} F(u(t)) \, dx + \delta \int_0^t \int_{B_R} \frac{\|
abla u\|^2}{d(u)} \, dx \, dt + \int_0^t \int_{B_R} |
abla \mathcal{H}_\varepsilon u|^2 \, dx \, dt = \int F(\hat{u}_0) \, dx.
\end{equation}

This implies estimates for $|\nabla \mathcal{H}_\varepsilon u|^2$ and $\delta|\nabla u|^2/d(u)$ in $L^1_{x,t}(Q_{T,R})$ and the bounds for such norms are independent of $\varepsilon, \delta, R$, and $T$. They do depend on $\mu > 0$ through the value of $F = F_\mu$. Indeed, the explicit formula for $F(u)$ is defined for $u > 0$ as:

\begin{equation}
F_\mu(u) = (u + \mu) \log(1 + (u/\mu)) - u, \quad F'_\mu(u) = \log(1 + (u/\mu)).
\end{equation}

This offers difficulties when $\mu \to 0$. (In case we take the second option, we would have

\begin{equation}
F_\mu(u) = (1/2\mu)u^2 \quad \text{for } 0 \leq u \leq \mu, \quad F_\mu(u) = u \log(u/\mu) + (\mu/2) \quad \text{otherwise},
\end{equation}

which is no better)

**Note.** We take as $\mathcal{K}_\varepsilon$ the operator obtained by convolution with a standard mollification of $k_s$ of the form $\zeta_\varepsilon = k_s \ast \rho_\varepsilon$ where $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ and $\rho$ is a $C^\infty_c(\mathbb{R}^n)$, $\rho \geq 0$, $\rho$ radially symmetric and decreasing; moreover, if $\rho = \sigma \ast \sigma$ where $\sigma$ has the same properties. Then, we can write $\mathcal{H}_\varepsilon$ as the operator with kernel $k_{s/2} \ast \sigma_\varepsilon$.  

3.2. We have to pass to the limit in four parameters: $\delta, \varepsilon, \mu$ and $R$. The last two limits are the most delicate. In order to examine the convergence arguments, we can try to pass to the limit $\varepsilon \to 0$ as a next step to obtain a solution of the equation

\begin{equation}
u_t = \delta \Delta u + \nabla \cdot (d(u)\nabla \mathcal{K}(u)),
\end{equation}
with same initial and boundary data. Using in a precise way the above estimates we get convergence of $u_\varepsilon \to u$ in $L^\infty_t(L^1_x)$ weak. This is enough for the limit of the first integral of the formula of weak solution. The second integral contains the product $u \nabla K u$ and we need to study better the consequences of the estimates:

(i) Since $K(u) = H(H(u))$ and $\nabla K(u) = \nabla H(H(u)) = H(\nabla H(u))$, we derive from the bound for $\nabla H u$ in $L^2_{x,t}$ the needed estimates for $K(u)$. Recall that $H(u)$ and $K(u)$ are defined on all of $\mathbb{R}^n$. We recall that $\nabla H(u)$ is a "derivative of order 1 - $s$ of $u$", and since $u$ is bounded, $u \in L^\infty_{x,t}$, we conclude that $u \in L^2_t H^{1-s}_{x,loc}$. By potential theory, it is then clear that $K(u) \in L^2_t H^{1+s}_x$.

(ii) We also need some continuity in time. Using the equation, which expresses $u_t$ as the divergence of $u \nabla K u$, together with the boundedness of $u$ and the bound for $K(u)$ in $H^{1+s}_x$, we conclude that $u_t \in L^2_t H^{-1+s}_{x,loc}$. Hence, the family of approximations to $u$ is relatively compact in the sense of the parabolic compactness theorems by Aubin and Simon, cf. [7], [32]. This means that a limit $u$ exists (along suitable subsequences) and $u \in C([0,T] : L^2(B_R))$.

All of this is used in passing to the limit of the term $\iint u(\nabla K u) \nabla \phi \, dx \, dt$ as follows: we have the convergence of $u_\varepsilon$ in $C([0,T] : L^2(B_R))$ together with the weak convergence of $p_\varepsilon = K_\varepsilon u_\varepsilon$ and $\nabla p_\varepsilon$. So we can pass to the limit in this term. The conclusion is that we have obtained a weak solution of the initial value problem for the equation we have mentioned, posed in $Q_R$ with zero Dirichlet boundary conditions. The regularity of $u$, $H u$ and $K u$ is as stated before. We also have the energy formula

$$ (3.12) \quad \int_{B_R} F_\mu(u(t)) \, dx + \delta \int_0^t \int_{B_R} \frac{\nabla u^2}{d(u)} \, dx \, dt + \int_0^t \int_{B_R} |\nabla H u|^2 \, dx \, dt = \int_{B_R} F_\mu(u_0) \, dx. $$

**Remark.** If we pass also to the limit $\delta \to 0$, which is feasible with no extra effort, then we lose the $H^1$ estimates for $u$, and besides we have a problem with the boundary data that is maybe important. Therefore, we will keep the term $\delta \Delta u$ for the moment to avoid the problem.

### 3.3
We will now try to pass to the limit, either as $\mu \to 0$ or as $R \to \infty$, the order depending on convenience. In that sense we recall the detailed form of the energy identity

$$ \delta \int_0^t \int_{B_R} \frac{\nabla u^2}{d(u)} \, dx \, dt + \int_0^t \int_{B_R} |\nabla H u|^2 \, dx \, dt + \int_{B_R} (u_0 - u(t)) \, dx $$
$$ \int_{B_R} (u(t) + \mu) \log(1 + \frac{u(t)}{\mu}) \, dx = \int_{B_R} (u_0 + \mu) \log(1 + \frac{u_0}{\mu}) \, dx. $$
(Note: we write \( u \) but we should write \( u_R \) since the solutions changes here with the radius of the ball). Terms 1, 2 and 3 are positive and behave well in both limits. It remains to examine the behavior of Terms 4 and 5. Both are nonnegative which is good for us in the case of Term 4, but Term 5 diverges as \( \mu \to 0 \). Therefore, our choice is letting \( R \to \infty \). Now Term 5 is bounded by hypothesis (though the bound depends on \( \mu \)). In the limit we easily get a solution of the problem in the whole space, for the equation

\[
(3.13) \quad u_t = \delta \Delta u + \nabla \cdot ((u + \mu) \nabla K(u)), \quad x \in \mathbb{R}^n, \ t > 0.
\]

The limit \( \delta \to 0 \) offers then no difficulty if one wants to take it. However, the limit \( \mu \to 0 \) needs some extra properties.

3.4. One of such useful properties is the Conservation of Mass, that we establish next for the solutions of \((3.13)\).

**Lemma 3.1** Under the assumption that \( u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) the constructed nonnegative solutions of the previous problem satisfy

\[
(3.14) \quad \int u(x, t) \, dx = \int u_0(x) \, dx \quad \text{for all } t > 0.
\]

**Proof.** Recall that we assume \( s < 1/2 \) if \( n = 1 \). We integrate against a cutoff function to get \( \varphi \in C^\infty(\mathbb{R}^n) \) supported in \( R \leq |x| \leq 2R \) with \( \varphi = 1 \) for \( |x| \leq R \). We get

\[
\int u_t \varphi \, dx = \delta \int u \Delta \varphi \, dx - \int (u + \mu)(\nabla K u \cdot \nabla \varphi) \, dx = I_1 + I_2.
\]

For the typical cutoff choice we estimate the first integral as \( I_1 = O(R^{-2}) \) using the fact that \( u(t) \in L^1(\mathbb{R}^n) \) and then \( I_1 \to 0 \) as \( R \to \infty \). As for the last integral, we do

\[
I_2 = \int K u \nabla u \nabla \varphi \, dx + \int u K u \Delta \varphi \, dx + \mu \int K u \Delta \varphi \, dx.
\]

The latter integral can be estimated as

\[
I_{23} = \mu \int u (K \Delta \varphi) \, dx = \mu \|u\|_1 O(R^{-2(1-s)})
\]

(where we use the fact that \( K\Delta \) has homogeneity \( 2(1-s) \) as a differential operator) and this goes to zero as \( R \to \infty \).
Before we estimate the other two integrals we split the kernel $K$ into a bounded part $K_1 = \min\{1, K\}$ and the rest $K_2 = K - K_1$ which is supported in a small ball. Both parts are nonnegative and moreover $K_1 \in L^\infty$ and $K_2 \in L^1$. It means that $K_1 * u \in L^q \ast L^p$ for $q > q_0 = n/(n - 2s)$ (recall that we always have $2s < n$) and for every $p \geq 1$, hence $K_1 u \in L^p$ for all $p > q_0$, while $K_2 * u \in L^q \ast L^p$ with $q < q_0$ and $p \geq 1$ which means that $K_2 * u \in L^p$ for all $p \geq 1$. We conclude that $K u \in L^p$ for $p > q_0$. We then have

$$I_{22} = \int uKu \Delta \varphi \, dx \leq \frac{C}{R^2} \|u\|_q \|Ku\|_p$$

which has in particular a bound of the form $|I_{22}| \leq C\|u\|_1 R^{-2}$ that goes to zero as $R \to \infty$. Finally,

$$I_{21} = \int Ku (\nabla u \cdot \nabla \varphi) \, dx.$$

Since $\nabla u \in L^2$ and $\nabla \varphi = O(R^{-1})$ and $\nabla \varphi \in L^p$ with $p > n$ we only need $Ku \in L^q$ with a small $q < 2n/(n - 2) + 4s < n + 2$.

In the limit $R \to \infty$ when $\varphi = 1$, we get (3.14).

**Theorem 3.2** Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $u_0 \geq 0$. Then there exists a weak solution $u = u_{\delta, \varepsilon, \mu}$ of the approximate equation (3.11) posed in $Q_T$ with initial data $u_0$, and $u \in L^\infty(0, \infty : L^1(\mathbb{R}^n))$, $u \in L^\infty(Q)$, $\nabla H(u) \in L^2(Q)$. Moreover, for all $t > 0$ we have

$$(3.15) \quad \int_{\mathbb{R}^n} u(x, t) \, dx = \int_{\mathbb{R}^n} u_0(x) \, dx,$$

and $\|u(t)\|_\infty \leq \|u_0\|_\infty$. The first energy inequality holds, in the form

$$\delta \int_0^t \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{u + \mu} \, dxdt + \int_0^t \int_{\mathbb{R}^n} |\nabla H u|^2 \, dxdt + \int_{\mathbb{R}^n} u(t) \log(u(t) + \mu) \, dx + \mu \int_{\mathbb{R}^n} \log(1 + (u/\mu)) \, dx \leq \int_{\mathbb{R}^n} u_0 \log(u_0 + \mu) \, dx + \mu \int_{\mathbb{R}^n} \log(1 + (u_0/\mu)) \, dx.$$

Note that the last integral is less than $\int u_0 \, dx$. The parameters $\delta, \varepsilon$ and $\mu$ are larger than 0. Passing to the limits $\delta \to 0, \varepsilon \to 0$ offers now no difficulties. But we recall that before taking those limits the solution is smooth in $x$ and $t$, and this may be convenient in justifying calculations to be done below.

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4 Tail control. Existence of weak solutions

We start the section by stating the main existence theorem that will be proved here by passage to the limit $\mu \to 0$ in the construction of the last section. The limit that we want to take is not trivial because of the possible behaviour of the constructed solutions for large $|x|$ which affects the convergence of some of the resulting integrals. We will pass to the limit in the last formula for the energy using as an extra tool a nice decay at infinity, so that the term $\int u \log^-(u + \mu) \, dx$ will be bounded uniformly as $\mu \to 0$.

**Theorem 4.1** Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that

\[ 0 \leq u_0(x) \leq Ae^{-a|x|} \quad \text{for some } A,a > 0. \]

Then there exists a weak solution $u$ of Equation (3.1) with initial data $u_0$. Besides, $u \in C([0, \infty) : L^1(\mathbb{R}^n))$, $u \in L^\infty(Q)$, $\nabla H(u) \in L^2(Q)$. For all $t > 0$ we have

\[ \int_{\mathbb{R}^n} u(x,t) \, dx = \int_{\mathbb{R}^n} u_0(x) \, dx, \]

and $\|u(t)\|_\infty \leq \|u_0\|_\infty$. The solution decays exponentially as $|x| \to \infty$ as explained in the next tail control subsection. The first energy inequality holds in the form

\[ \int_0^t \int_{\mathbb{R}^n} |\nabla H u|^2 \, dx \, dt + \int_{\mathbb{R}^n} u(t) \log(u(t)) \, dx \leq \int_{\mathbb{R}^n} u_0 \log(u_0) \, dx, \]

while the second says that for all $0 < t_1 < t_2 < \infty$

\[ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u |\nabla Ku|^2 \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^n} |H(u(t_2))|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^n} |H(u(t_1))|^2 \, dx. \]

We recall that $Q = \mathbb{R}^n \times (0, \infty)$, and we take $0 < s < 1$ for all $n \geq 2$, $0 < s < 1/2$ for $n = 1$. The result is first proved for the solutions of the approximate equation with $\delta, \varepsilon, \mu > 0$ constructed in Theorem 3.2 and the information is then transferred to the original equation once we show that we can pass to the limit and the estimates are uniform.
4.1 Tail control

In order to prove the above existence theorem we need to estimate a certain rate of decay of our solutions as \(|x| \to \infty\). This will of course depend on a similar assumption that we are imposing on the initial data. In the case of the PME a possible proof proceeds by constructing explicit weak solutions (or supersolutions) with that property and then using the comparison principle, that holds for that equation. Since we do not have such a general comparison principle here, we have to devise a comparison method with a suitable family of barrier functions that work because they are some kind of “exaggerated supersolutions”. What we need is to make sure that they do not admit a first contact from below. We will give to functions with such a property the more serious name of true supersolutions. This original technique has to be adapted to the peculiar form of the integral kernels involved in operator \(K\). We prove two kinds of results, the stronger one for small \(s\).

**Theorem 4.2** Let \(0 < s < 1/2\) and assume that our solution \(u\) is bounded \(0 \leq u(x, t) \leq L\), and \(u_0\) lies below a function of the form

\[
U_0(x) = Ae^{-a|x|}, \quad A, a > 0.
\]

If \(A\) is large then there is a constant \(C > 0\) that depends only on \((n, s, a, L, A)\) such that for any \(T > 0\) we will have the comparison

\[
u(x, t) \leq Ae^{Ct-a|x|} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } 0 < t \leq T.
\]

**Proof.** In order to have enough regularity in the comparison argument below we make the proof for the solutions constructed in Theorem 3.2 in the whole space with parameters \(\delta, \varepsilon\) and \(\mu > 0\) and we will show that the constants in the upper estimate are uniform with respect to such parameters if the mentioned parameters are small.

- **Reduction.** By scaling we may put \(a = L = 1\). This is done by considering instead of \(u\) the function \(\tilde{u}\) defined as

\[
u(x, t) = L \tilde{u}(ax, bt), \quad b = La^{2-2s},
\]

which satisfies the equation \(\tilde{u}_t = \delta_1 \Delta \tilde{u} + \nabla \cdot (\tilde{d}(\tilde{u})\nabla K(\tilde{u}))\) with \(\delta_1 = a^{2s}\delta/L\). Note that then \(\tilde{u}(x, 0) \leq A_1 e^{-|x|}\) with \(A_1 = A/L\). A simple calculation then shows that \(C(a, L, A) = La^{2-2s}C(1, 1, A/L)\).
• **Contact analysis.** Therefore, we assume that $0 \leq u(x, 0) \leq 1$ and also that

$$u(x, 0) \leq Ae^{-r} \quad r = |x| > 0,$$

where $A > 0$ is a constant that will be chosen below, say larger than $2$. Given constants $C, \varepsilon$ and $\eta > 0$ we consider a radially symmetric candidate to upper barrier function of the form

$$(4.8) \quad \hat{U}(x, t) = Ae^{Ct-r} + \varepsilon Ae^{\eta t},$$

and we take $\varepsilon$ small. Then $C$ will have to be determined in terms of $A$ to satisfy a “true supersolution condition” which is obtained by contradiction at the first point $(x_c, t_c) \in Q_T$ of possible contact of $u$ and $\hat{U}$. Note that if there is contact it cannot happen only at $|x| = \infty$ since $u$ is integrable and $\hat{U}$ converges to a positive constant. This is the reason to add the correction term in $\varepsilon$ (use of the correction term can be avoided if we start the comparison argument with the solutions of the approximate problem posed in a ball $B_R(0)$ with zero boundary data and then pass to the limit $R \to \infty$ as done in the previous section). The contact does not happen at $x_c = 0$ if $A$ is large since, putting $r_c = |x_c|$, we have at the contact point equality $\hat{U} = Ae^{Ct_c-r_c} + \varepsilon Ae^{\eta t_c} = u$ and also $u \leq 1$, so that $A \leq Ae^{Ct_c} \leq e^{r_c}$ so $r_c$ is big if $A$ is.

We need at least $A > 1$. Note that we are assuming $0 < t_c \leq T$ and $T$ fixed (in this argument).

At the point and time of contact we have $u = Ae^{Ct_c-r_c} + \varepsilon Ae^{\eta t_c}$. Assuming also that $u$ is $C^2$ smooth, a standard argument also gives

$$\partial_r u = -Ae^{Ct_c-r_c}, \quad \Delta u \leq Ae^{Ct_c-r_c}, \quad u_t \geq AC e^{Ct_c-r_c} + \varepsilon \eta Ae^{\eta t_c},$$

(all of them computed at the point $(x_c, t_c)$), and the spatial derivatives of $u$ at $x_c$ in directions perpendicular to the radius are zero. Next, we put $p = Ku$ and use the equation in full approximate form:

$$(4.9) \quad u_t = \delta \Delta u + \nabla u \cdot \nabla p + (u + \mu) \Delta p,$$

to get the basic inequality

$$CAe^{Ct_c-r_c} + \varepsilon \eta e^{\eta t_c} \leq \delta A e^{Ct_c-r_c} - A e^{Ct_c-r_c} \overline{\partial_r p} + (A e^{Ct_c-r_c} + \varepsilon A e^{\eta t_c} + \mu) \overline{\Delta p}.$$  

Cleaning up this expression, we get at the contact point the following condition:

$$(4.10) \quad C + \varepsilon \eta e^\xi \leq \delta - \overline{\partial_r p} + (1 + \varepsilon e^\xi + \mu_1) \overline{\Delta p},$$
where \( \xi = r_c + (\eta - C)t_c \), \( \mu_1 = (\mu/A)e^{r_c - Ct_c} \) and the overline indicates that the values of \( \partial_r p \) and \( \Delta p \) are calculated at the point of contact.

**Summing up the main ideas.** In (4.9) we have written the nonlinear term of the equation involving the fractional operator in “non-divergence form”, precisely as the sum of a transport term involving first derivatives on \( p \) and another term containing the Laplacian of \( p \). When we evaluate those terms at the contact point with the barrier and arrive at (4.10), the latter term is not difficult in terms of global integrals of \( u \) and the other two first terms in (4.9) have also simple contributions. However, the transport term does not have a sign and its control becomes more difficult. We proceed with the simple cases and leave the difficult case for the end.

- In order to get a contradiction with inequality (4.10) we will estimate the values of \( \partial_r p \) and \( \Delta p \). For \( n \geq 1, 0 < s < 1 \), we use the formula

\[
(4.11) \quad p(x, t) = c \int \frac{u(x - y, t)}{|y|^{n-2s}} dy = c \int \frac{u(x + y, t)}{|y|^{n-2s}} dy,
\]

with some \( c = c(s) > 0 \), which produces the singular integrals

\[
(4.12) \quad p_{x_i}(x, t) = c_1 \int \frac{(u(x + y, t) - u(x - y, t)) y_i}{2|y|^{n+2-2s}} dy,
\]

\((i = 1, \cdots, n)\) that for \( s < 1/2 \) can also be written as

\[
(4.13) \quad p_{x_i}(x, t) = c_1 \int \frac{(u(x + y, t) - u(x, t)) y_i}{|y|^{n+2-2s}} dy,
\]

and finally,

\[
(4.14) \quad \Delta p(x, t) = c_2 \int \frac{u(x + y, t) + u(x - y, t) - 2u(x, t)}{|y|^{n+2-2s}} dy.
\]

Here and in the sequel of the proof we denote by \( c, c_i, K, K_i \) different absolute positive constants, i.e., constants that depend only on \( n \) and \( s \).

- We now estimate \( \Delta p \) at \( x_c \). In view of inequality (4.10) we need a control from above. Using formula (4.14) we see that the integral on the set \( |y| \geq 1 \) is absolutely convergent and absolutely bounded (since \( u \) is bounded by 1). Besides, to bound from above the part of integral on the set \( |y| \leq 1 \) we will use the fact that \( u(t_c) \) lies below \( Ae^{Ct_c - |x|} \) with tangency at \( |x| = r_c \); since the numerator of the integrand is the
discretization of the second derivative for \( u \) at \( x_c \) we use the fact that it is bounded above by the same expression for \( \bar{U} \) to get
\[
\int_{|y| \leq 1} \frac{u(x_c + y, t_c) + u(x - y, t_c) - 2u(x, t_c)}{|y|^{n+2-2s}} \, dy \leq \int_{|y| \leq 1} \frac{2Ae^{Ct_c-r_c}|y|^2}{|y|^{n+2-2s}} \, dy,
\]
which is bounded since \( s > 0 \). Noting that \( Ae^{Ct_c-r_c} \leq 1 \) we conclude that the term in (4.10) containing \( \bar{u} \) contributes with at most \( K_1 \) to the inequality.

- Next, we estimate the transport term involving \( \partial_t p \). We have
\[
\bar{\partial_t p} = c_1 \int \frac{(u(x + y, t) - u(x, t)) \cdot (\hat{x}_c \cdot \hat{y})}{|y|^{n+1-2s}} \, dy,
\]
where \( \hat{x} = x/|x| \) and \( \hat{y} = y/|y| \). Since we want to get a contradiction in formula (4.10) and \( \bar{\partial_t p} \) appears with a minus sign, we need to estimate this term from below. This integral is delicate so we split the calculation into several pieces. At this moment we make the further assumption \( s < 1/2 \). Then the integral for \( |y| \geq 1/2 \) is bounded as:
\[
|I_{\{|y| > 1/2\}}(\bar{\partial_t p})| \leq c \int_{|y| > 1/2} \left| \frac{u(x_c + y, t_c) - u(x_c - y, t_c)}{|y|^{n+2-2s}} \right| \, dy \leq c \int_{|y| \geq 1} \frac{1}{|y|^{n+1-2s}} \, dy \leq K_2.
\]
Hence, this part has the desired control.

- The integral on the ball \( \{|y| \leq 1/2\} \) is split again into two parts. By rotation we may assume that \( x_c \) is directed along the first axis, \( x_c = (r,0,\cdots,0) \), where \( r = |x| \). Then the integral is calculated on \( \Omega_1 = \{ y : |y| \leq 1/2, y_1 > 0 \} \) and on \( \Omega_2 = \{ y : |y| \leq 1/2, y_1 < 0 \} \). The last is easily bounded since \( u \) touches \( \bar{U} \) at \( x_c \) and lies below everywhere at time \( t_c \). Hence,
\[
-I_{\Omega_2}(\bar{\partial_t p}) = \int_{\Omega_2} \frac{(u(x_c, t_c) - u(x_c + y, t_c))}{|y|^{n+2-2s}} \, y_1 \, dy = \int_{\Omega_1} \frac{(u(x_c - y, t_c) - u(x_c, t_c))}{|y|^{n+2-2s}} \, y_1 \, dy \leq \int_{\Omega_1} \frac{2Ae^{Ct_c-r_c}}{|y|^{n+1-2s}} \, dy \leq K_3.
\]
where in the last inequality we have used the linear approximation for \( \bar{U} \). Again, this is a good control.

- Now, the difficult part. The integral on \( \Omega_1 \) (i.e., the “half integral looking outside near \( x_c \)) is more delicate since the difference \( u(x, t_c) - u(x + y, t_c) \) could in principle drop quite abruptly even at a relatively short distance from \( x_c \) and this would make the integral very big. However, we have a miraculous control by combining the estimate on the Laplacian with the good part of the estimate of \( \bar{\partial_t p} \).
Lemma 4.3 With the previous assumptions and notations we have

\begin{equation}
- I_{\Omega_i}(\partial_r p) + \frac{1}{2} I_{\Omega_i}(\Delta p) \leq - \frac{1}{2} I_{\Omega_i}(\partial_r p).
\end{equation}

Proof. We combine the integral of $-\partial_r p$ with a part of the integral for $\Delta p$ as follows:

\begin{equation}
Y = - \int_{\Omega_i} \frac{(u(x_c + y, t_c) - u(x_c, t_c))y_1 + \frac{1}{2} (u(x_c + y, t_c) + u(x_c - y, t_c) - 2u(x_c, t_c))}{|y|^{n+2-2s}} dy
+ \frac{1}{2} \int_{\Omega_i} \frac{u(x_c + y, t_c) + u(x_c - y, t_c) - 2u(x_c, t_c)}{|y|^{n+2-2s}} dy
\end{equation}

and we study carefully the integrand of $Y$. We have a numerator of the form

\begin{equation}
(u(x_c, t_c) - u(x_c + y, t_c))y_1 + \frac{1}{2} (u(x_c + y, t_c) + u(x_c - y, t_c) - 2u(x_c, t_c)) =
-(\frac{1}{2} - y_1) (u(x_c, t_c) - u(x_c + y, t_c)) + \frac{1}{2} (u(x_c - y, t_c) - u(x_c, t_c))
\end{equation}

Luckily, the first term is negative, hence we conclude that estimate (4.15) holds.

• We can now finish the contradiction argument. All these estimates allow to conclude that $-I(\partial_r p) \leq K\delta$. We go now back to (4.10) and conclude that the inequality implies that

\begin{equation}
C + \varepsilon (\eta - K) e^{r_c + (\eta - C)t_c} \leq \delta + \frac{K \mu}{A} e^{r_c},
\end{equation}

This cannot happen if we choose $\varepsilon \geq \frac{\delta}{4}$ and $C = \eta \geq 2K$, where we keep the extra condition that $\mu$ and $\delta$ must be small (at least less than 1). Then $\varepsilon$ may go to zero as $\mu \to 0$.

• Finally, we worry about the regularity of the solutions. To make the above proof fully rigorous, we apply the argument to the solutions of the regularized problem where we smooth the velocity field $\nabla p$ by regularizing the kernel. We have to make the estimates for $p_x$ and $\Delta p$ when

\begin{equation}
p(x, t) = \int K_\varepsilon(y) u(x + y) dy.
\end{equation}

where for instance $K_\varepsilon(y) = K(y)$ for $\varepsilon \leq |y| \leq 1/\varepsilon$, $K_\varepsilon(y)$ is a parabolic cap with $C^1$ fit in $|y| \leq \varepsilon$, and finally $K_\varepsilon(y) = 0$ for $|y| \leq 2/\varepsilon$. The regularization mentioned in Section 3 will also do. The solutions $u_\varepsilon$ to this problem have bounded speeds, they are smooth and bounded with smooth and bounded $p_x$ and the previous estimates for $p_x$ and $\Delta p$ hold uniformly in $\varepsilon$. Passing to the limits $\varepsilon \to 0$, the previous conclusions hold for any weak limit solution as constructed above, cf. equation (3.13). The extra limit $\delta \to 0$ offers then no difficulty.
**Theorem 4.4** Let now $1/2 < s < 1$. Under the assumptions of the previous theorem the stated tail estimate works locally in time. The global statement must be replaced by the following: there exists an increasing function $C(t)$ such that

$$
(4.16) \quad u(x, t) \leq Ae^{C(t)[t-|x|]} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } 0 < t \leq T.
$$

**Proof.** (1) In the previous proof we had to put $s < 1/2$ only because of the problem in estimating the integral for $\partial_x p$ on an exterior domain, away from the contact point. When $s \geq 1/2$ we can estimate such integral as a convolution integral between $u$ and the kernel $K_1(y) = y_1|y|^{-n-2+2s} \chi(|y| \geq 1)$. Now, this kernel belongs to $L^p(\mathbb{R}^n)$ for all $p > n/(n + 1 - 2s)$, hence we only need to bound $u(t)$ in an $L^q(\mathbb{R}^n)$ norm with $1 \leq q < n/(2s-1)$ to get

$$
(4.17) \quad I_* := |I_{\{|y|>1/2\}}(\partial_x p)| \leq \|u(t)\|_q \|K_1\|_p
$$

Moreover, since $u \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ we have

$$
\|u(t)\|_q \leq \|u(t)\|^{1/q}_1 \|u(t)\|^{(q-1)/q}_\infty
$$

We know that $\|u(t)\|_\infty \leq 1$ and $u(x, t) \leq Ae^{Ct}e^{-|x|}$ hence $\|u(t)\|_1 \leq cAe^{Ct}$. Therefore the term contributes

$$
I_* \leq KA^{1/q}e^{Ct}/q
$$

which allows to go back to (4.10) and get

$$
(4.18) \quad C + \varepsilon(\eta - K)e^{\varepsilon e^{(\eta - C)t_c}} \leq \delta + K + KA^{1/q}e^{Ct_c}/q + \frac{K\mu}{A} e^{c\varepsilon},
$$

The contradiction argument works as before with only one big difference. Once we put $C = \eta = KA^{1/q}$ the contradiction is gotten if we restrict the time so that $e^{Ct_c}/q \leq 2$ which happens if

$$
t_c \leq T_1 = c_2/C = c_3K^{-1}A^{-1/q} = c_4A^{1/q}.
$$

We do not play with $A$ here, only $A$ bigger than 2 or the like.

(2) Once we prove estimate (4.16) for $0 < t < T_1$ we can repeat the argument for another time interval, but now with constant $A_1 = Ae^{CT_1} = Ae^{c_2}$, and then $C_1 = KA_1^{1/q}$. We get a valid time interval

$$
T_2 = c_4A_1^{1/q} = c_4A^{1/q}e^{-c_2/q}
$$

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where a new factor will appear every time we repeat the iterations. In this way we can extend the bound up to a certain time that depends on the initial data through the value of $A$.

(3) In order to prove that the time for which the estimate is valid goes forever we need to improve the $L^1$ norm of $u(t)$ by using the fact that $u \leq 1$ together with $u(x,t) \leq Ae^{Ct}e^{-|x|}$. Summing the contributions of both upper bounds, we now get

$$\|u(t)\|_1 \leq c_0(\log(Ae^{Ct}))^n (1 + A^{-1}e^{-Ct})$$

so that

$$I_* \leq K(\log A + Ct)^{n/q} (1 + A^{-1}e^{-Ct})^{1/q}$$

and then Inequality (4.18) may be replaced by

$$C + ... \leq K + K(\log A + Ct)^{n/q} (1 + A^{-1}e^{-Ct})^{1/q} + \frac{K\mu}{A} e^{c_e},$$

where we have dropped the terms in $\delta, \varepsilon$ and $\mu$ that add no novelties or problems. Now we may put $C = K[(\log A)^{n/q} + 2]$ and $CT_1 = c_2$ so that

$$T_1 = c_4(\log A)^{-n/q}$$

At the new starting time, we have $A_1 = Ae^{CT_1} = Ae^{c_2}$ and then

$$T_2 = c_4(\log A_1)^{-n/q} = c_4(\log A + c_2)^{-n/q}$$

and so on. Since for large $k$ we get $T_k \sim ck^{-n/q}$ and we may always take $q > n$, the series $\sum T_k$ diverges, so that Estimate (4.16) is global in time. 

4.2 Proof of the existence result

We may now pass to the limit in the weak formulation of the equation satisfied by $u = u_{\delta, \varepsilon, \mu}$ constructed at the end of previous section. The exponential decay bound on the solutions, which is uniform in $\mu$ allows to improve the consequences of the energy inequality that is now written as

$$\delta \int_0^t \int \frac{|\nabla u_\mu|^2}{u_\mu + \mu} \, dx \, dt + \int_0^t \int |\nabla H u_\mu|^2 \, dx \, dt + \int u_\mu(t) \log^+(u_\mu(t) + \mu) \, dx$$

$$\leq \int u_0 \log(u_0 + \mu) \, dx + \mu \int \log(1 + (u_0/\mu)) \, dx + \int u_\mu(t) \log^-(u_\mu(t) + \mu) \, dx.$$
where we use the notation $u_\mu$ for the solution for clarity. Since the right-hand side is bounded uniformly in $\mu$ we get a uniform estimate for the family $|\nabla H u_\mu|$ in $L^2(Q)$. We also have a uniform bound on $\nabla u_\mu$ in $L^2(Q)$ if $\delta > 0$. We can therefore pass to the weak limit in $u_\mu \to \tilde{u}$, and then $H u_\mu \to H \tilde{u}$ and $\nabla H u_\mu \to \nabla H \tilde{u}$. The same happens to $K u_\mu$ and $\nabla K u_\mu$. The weak solution is obtained and we have

$$
\delta \int_0^t \int \frac{|\nabla \tilde{u}|^2}{\tilde{u}} \, dx \, dt + \int_0^t \int |\nabla H \tilde{u}|^2 \, dx \, dt + \int \tilde{u}(t) \log^+(\tilde{u}(t)) \, dx 
\leq \int u_0 \log(u_0) \, dx + \int \tilde{u}(t) \log^-(\tilde{u}(t)) \, dx.
$$

Note that the term $\int \mu \log(1 + (u_0/\mu)) \, dx$ disappears in the limit by the Dominated Convergence Theorem since the integrand is uniformly bounded by $u_0$. Due to the decay at infinity the integral $\int \tilde{u}(t) \log \tilde{u}(t) \, dx$ is absolutely convergent.

The constructed solution has the same $L^\infty$ bound as $u_0$ and conservation of mass holds by the proof of Lemma 3.1.

Passing to the limit $\delta \to 0$ and $\varepsilon \to 0$ offers no difficulty, so Theorem 4.1 is proved but for the second energy estimate.

\[ \Box \]

### 4.3 Second energy estimate

Let us establish the second energy estimate for weak solutions with tail decay. We compute formally of the approximations where everything is justified

$$
\frac{1}{2} \frac{d}{dt} \int \varphi |H u(x,t)|^2 \, dx = \int \varphi H u H u_t \, dx = \int (\varphi H u) u_t \, dx = \int H(\varphi H u) u_t \, dx = \int H(\varphi H u) \nabla(\varphi H u) \cdot \nabla K u \, dx = \int \varphi H(\varphi H u) \cdot \nabla K u \, dx - \int u H(\varphi H u) \cdot \nabla K u \, dx,
$$

so that, putting $\psi = 1 - \varphi$ we have

$$
\frac{1}{2} \frac{d}{dt} \int \varphi |H u(x,t)|^2 \, dx + \int \varphi |\nabla K u|^2 \, dx = \int u (\psi \nabla K u - H(\psi \nabla H u)) \cdot \nabla K u \, dx - 2 \int H u \nabla \varphi \cdot H(u \nabla K u) \, dx,
$$

With the properties we have for $u$ the right-hand side must go to zero as $\varphi \to 1$ along the typical cutoff sequence, at least after integration in time.
5 Finite propagation. Solutions with compact support

One of the most important features of the porous medium equation and other related degenerate parabolic equations is the property of finite propagation, whereby compactly supported initial data $u_0(x)$ give rise to solutions $u(x,t)$ that have the same property for all positive times, i.e., the support of $u(\cdot, t)$ is contained in a ball $B_{R(t)}(0)$ for all $t > 0$ and $R(t)$ is bounded on bounded intervals $0 < t < T$.

A possible proof in the case of the PME proceeds by constructing explicit weak solutions with that property (and possibly larger initial data) and then using the comparison principle, which holds for that equation. Since we do not have such a general principle here, we have to devise a comparison method with a suitable family of upper barriers that behave as “true (exaggerate) supersolutions”. The technique has been presented in whole detail in the tail analysis of the previous section and is here adapted to the peculiar needs of bounded support. Here is the end result.

**Theorem 5.1** Assume that $u$ is a bounded solution, $0 \leq u \leq L$, of equation (3.1) with $K = (-\Delta)^{-s}$ with $0 < s < 1$ ($0 < s < 1/2$ if $n = 1$), as constructed in Theorems 4.2 and 4.4. Assume that $u_0$ has compact support. Then $u(\cdot, t)$ is compactly supported for all $t > 0$. More precisely, if $0 < s < 1/2$ and $u_0$ is below the ”parabola-like” function

$$U_0(x) = a(|x| - b)^2,$$

for some $a, b > 0$, with support in the ball $B_b(0)$, then there is $C$ large enough, such that

$$u(x,t) \leq a(Ct - (|x| - b))^2$$

Actually, we can take $C = C_0(n,s)L^{(1/2)+s}a^{(1/2)-s}$. For $1/2 \leq s < 2$ a similar conclusion is true, but now $C$ is an increasing function of $t$ and we do not obtain a scaling formula for its dependence of $L$ and $a$.

**Proof.** The application of the method is very similar to the case worked out the tail control section. Therefore, we will dispense with some of the technicalities of regularization to gain space and clarity. We assume that our solution $u(x,t) \geq 0$ has bounded initial data $u_0(x) = u(x,t_0) \leq L$ and also that $u_0$ is below the parabola $U_0(x) = a(|x| - b)^2$, $a, b > 0$; moreover, the support of $u_0$ in the ball of radius $b$ and
the graphs of \( u_0 \) and \( U_0 \) are strictly separated in that ball. We take as comparison function \( U(x, t) = a(Ct - (|x| - b))^2 \) and argue at the first point and time where \( u(x, t) \) touches \( U \) from below. The fact that such a first contact point happens for \( t > 0 \) and does not happen at \( x = \infty \) is justified by regularization as before. We put \( r = |x| \).

By scaling we may put \( a = L = 1 \). See detail of the reduction step below. We examine in detail the situation in which the touching point \((x_c, t_c)\) is not the minimum, say, \( x_c \) lies at a distance from the front \( |x_f(t)| := b + Ct \), so that \( b + Ct_c - |x_c| = h > 0 \). Note that since \( u \leq 1 \) we must have \( |h| \leq 1 \). Assuming also that \( u \) is \( C^2 \) smooth, a standard argument gives

\[
 u = h^2, \quad u_r = -2h, \quad \Delta u \leq 2n, \quad u_t \geq 2Ch, 
\]

all of them computed at the point \((x, t) = (x_c, t_c)\). Putting \( p = Ku \) and using the equation \( u_t = \nabla u \cdot \nabla p + u \Delta p \), we get the inequality

\[
 (5.3) \quad 2Ch \leq -2hp_r + h^2 \Delta p, 
\]

where the overline indicates that the values of \( p_r \) and \( \Delta p \) are calculated at the point of contact. Moreover, \( u(x, t) \leq (x_f - x)^2 \) for all \( x \in \mathbb{R}^n \). In order to get a contradiction we will estimate the values of \( \bar{p}_r \) and \( \Delta \bar{p} \). The formulas for \( p, p_r \) and \( \Delta p \) in terms of \( u \) are given in (4.11), (4.12), (4.13), (4.14).

- Estimating \( \Delta p \) at the contact point offers no novelties. As before and in view of inequality (5.3) we need a control from above. Using formula (4.14) we see that the integral on \( |y| \geq 1/2 \) is absolutely convergent and bounded (since \( u \) is bounded). Besides, to bound the integral for \( |y| \leq 1 \) from above we will use the fact that \( u \) lies below a parabola, hence

\[
 \int_{|y| \leq 1} \frac{u(x + y, t_c) + u(x - y, t_c) - 2u(x, t_c)}{|y|^{n+2-2s}} \, dy \leq \int_{|y| \leq 1} \frac{2|y|^2}{|y|^{n+2-2s}} \, dy, 
\]

which is bounded since \( s > 0 \). We conclude that the term in (5.3) containing \( \bar{p}_{xx} \) contributes with at most \( Kh^2 \) to the inequality, where \( K > 0 \) is an absolute constant.

- Next, we estimate \( \bar{p}_r \). Much of the argument is similar to the tail analysis but the end is more delicate. Since we want to get a contradiction in formula (5.3) and the coefficient \( u_r = -2h \) of \( \bar{p}_r \) is negative, we need to estimate this term from below. Using formula (4.12) we see again that the integral for \( |y| \geq 1/2 \) is absolutely and uniformly bounded by a constant \( K_2 \).
The integral on the ball \( \{|y| \leq 1/2\} \) is split into several parts. We will drop for convenience of writing the dependence on \( t_c \) in the formulas the follow. By rotation we may assume that \( x_c \) is directed along the first axis, \( x_c = (r,0,\ldots,0) \), where \( r = |x| \). Then the integral is calculated on \( \Omega_1 = \{ y : |y| \leq 1/2, y_1 > 0 \} \) and on \( \Omega_2 = \{ y : |y| \leq 1/2, y_1 < 0 \} \). The last is easily bounded since \( u \) touches \( U \) at \( x_c \) and lies below everywhere at time \( t_c \). As in the tail analysis we have,

\[
-I_{\Omega_2}(\partial_r p) = \int_{\Omega_2} \frac{(u(x_c) - u(x_c + y))y_1}{|y|^{n+2-2s}} \, dy \leq K_3.
\]

The integral on \( \Omega_1 \) (i.e., the “half integral looking outside near \( x_c \)) is more delicate as we have said, since the difference \( u(x,t_c) - u(x + y,t_c) \) could in principle drop quite abruptly even at a relatively short distance from \( x_c \) and this would make the integral very big. There is a part with \( |y| \) between \( \theta h \) and 1 for any given \( \theta \in (0,1) \) that is easy (note that \( 0 \leq u(x,0) - u(x + y,t) \leq u(x,0) = h^2 \)):

\[
\left| \int_{\theta h}^{1} \frac{u(x_c + y) - u(x_c)}{|y|^{n+1-2s}} \, dy \right| \leq h^2 \int_{\theta h}^{1} \frac{1}{|y|^{n+1-2s}} \, dy = c_3(\theta) h^{1+2s}
\]

and this is good even for small \( h \).

The last part of the integral for \( \nabla_* p \), over the half-ball \( H_1 = \{ y : y_1 > 0 \} \) with \( h = 0 \), is in principle bad since \( u(x_c + y) \) could drop abruptly, thus making the integral very negative or even divergent near \( y = 0 \). We are going to combine the integral of \( -\partial_r p \) with a part of the integral for \( \Delta p \) as follows:

\[
Y = -\int_{H_1} \frac{(u(x_c + y) - u(x_c))y_1}{|y|^{n+2-2s}} + \frac{h}{2} \int_{H_1} \frac{u(x_c + y) + u(x_c - y) - 2u(x_c)}{|y|^{n+2-2s}} \, dy
\]

and we study carefully the integrand of \( Y \). We have a numerator of the form

\[
(u(x_c) - u(x_c + y))y_1 + \frac{h}{2}(u(x_c + y) + u(x_c - y) - 2u(x_c)) =
\]

\[
-\left( \frac{h}{2} - y_1 \right)(u(x_c) - u(x_c + y)) + \frac{h}{2}(u(x_c - y) - u(x_c))
\]

Luckily, the first term is negative (we take \( 0 < \theta < 1/2 \) (a security factor), hence we conclude that

\[
-I_{H_1}(\partial_r p) + \frac{h}{2} I_{H_1}(\Delta p) \leq -\frac{h}{2} I_{\Omega_3}(\partial_r p).
\]

We may now sum up all the terms in the right-hand side of (5.3) and show that they are bounded above by \( Kh \) where \( K \) is a uniform constant. Therefore, for large
C inequality (5.3) is impossible, hence there cannot be a contact point with $h \neq 0$. In this way we get a minimal constant $C = C_0(n, s)$ for which such contact does not take place.

- **Reduction Step.** We use it to get the dependence on $L$ and $a$. Here, it goes as follows: since the equation scaling is

$$\tilde{u}(x, t) = Au(Bx, Tt)$$

with parameters $A, B, T > 0$ such that $T = AB^{2-2s}$, if we have done the proof for $u$ that has height 1 and is below $(|x| - b_0)^2$ initially, and get a comparison with a $U$ as above with speed $C_0 > 0$, then the assumptions $\tilde{u}(x, t) \leq L$ and $\tilde{u}(x, t_0) \leq a(|x| - b)^2$ initially, are satisfied if we put

$$A = L, \quad AB^2 = a, \quad b = b_0/B,$$

i.e., $A = L, B = (a/L)^{1/2}$, and then $T = a^{1-s}L^s$. The new speed is then

$$\tilde{C} = \frac{C_0 T}{B} = C_0 a^{1/2-s}L^{1/2+s}.$$

- We still have to consider the modification of the proof when $1/2 \leq s < 1$. The only problem is the estimate of $\partial_r p$ on the exterior of a ball. This is done as in the tail control case.

We next lemmas complete the details of the comparison proof that have been left out in the previous lemma.

**Lemma 5.2** Under the assumptions of Prop. 5.1 there is no contact either at $u = 0$, in the sense that strict separation of $u$ and $U$ holds for all $t > 0$ if $C$ is large enough as in the previous lemma.

**Proof.** Precisely, what we want is to eliminate the possible contact of the supports at the lower part of the parabola. Instead of doing this by analyzing the possible contact point, we proceed by a change in the test function that we replace by

$$U_\varepsilon = (Ct - (|x| - b))^2 + \varepsilon(1 + Dt) \quad \text{for} \quad |x| \leq b + Ct,$$

and $U_\varepsilon = \varepsilon(1 + Dt)$ for $1x| \geq b + Ct$. Here $\varepsilon > 0$ is a small constant and $D > 0$ will be suitably chosen. Assume that the solution starts at $t = 0$ and touches this test function $U_\varepsilon$ for the first time at the time $t = t_\varepsilon$. 

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The argument for a contact at the points $|x| < b + Ct_c$ where $U_\varepsilon$ is a parabola works like previously.

The argument at contact points $|x| \geq b + Ct_c$ leads to

\begin{equation}
D\varepsilon \leq \varepsilon(1 + Dt_c)\overline{\Delta p},
\end{equation}

where the overline indicates as before that the value $\Delta p$ are calculated at the point of contact. Moreover, $u(x, t_c) \leq U_1(x, t_c)$ for all $x \in \mathbb{R}$. If we are able to prove again that $\overline{\Delta p} \leq K$ we will get

\begin{equation}
D\varepsilon \leq \varepsilon(1 + Dt_c)K
\end{equation}

which is contradictory if $D > K$, say $D = 2K$, and $t_c < 1/D = 1/(2K)$. This estimate is uniform in $\varepsilon$ and gives in the limit and upper bound of the form $u(x, t) \leq (Ct - (|x| - b))^2$ for all $0 < t < 1/2K$, and as a corollary, the support of $u$ bounded on the right by the line $|x| = Ct + b$ in that time interval. After this time the process can be repeated and the conclusion is true for all times. □

- We now reflect for a moment on the regularity requirements. Arguing as in the tail control case by using the smooth solutions of the approximate equations, the previous conclusions hold for any weak limit solution.

### 5.1 Consequences. Growth estimates of the support

The following analysis is done for $s < 1/2$ and concerns bounded solutions with compactly supported initial data. By free boundary we mean, as usual, the topological boundary of the support of the solution $S(u) := \{(x, t) : u > 0\}$.

**Corollary 5.3** Let $u_0$ be bounded above by $L$ with $u_0(x) = 0$ for $|x| \leq R$. Then, we get an estimate for the free boundary points of the form $|x(t)| \leq R + C_2 t^{1/(2-2s)}$ if $s < 1/2$.

**Proof:** We know that the support of $u(\cdot, t)$ is bounded for all times, say, it is contained in a ball of radius $r(t)$. We take a time $t_1$ and find a parabolic barrier as before, with coefficient $a > 0$ that is initially above and separated from $u(\cdot, t_1)$. This can be done by choosing first $r_1$ such that $ar_1^2 = L$ and then putting in the formula of the parabolic barrier $b = r(t_1) + r_1 + \varepsilon$, and then we can go forever in time in the comparison. Using the speed estimate in Theorem 5.1 we get in the limit $\varepsilon \rightarrow 0$:

\[ r(t) - r(t_1) - r_1 \leq C(t - t_1) = C_0 L(t_1)^{1/2+s} a^{1/2-s}(t - t_1) = C_0 L(t_1)(t - t_1)/r_1^{1-2s} \]
Here $r_1 > 0$ is free by moving $a$. We can use the $L^\infty$ bound $L(t_1) \leq L(0)$ We get

$$r(t) \leq r(t_1) + r_1 + C_0 L(t - t_1)/r_1^{1-2s}$$

Now take $t_1 = 0$ and optimize the right-hand expression in $r_1 > 0$ for $s < 1/2$. Notice that in the limit $s = 1/2$ we would get linear growth, while for $s = 0$ we get the standard $t^{1/2}$ growth of the Porous Medium Equations under these assumptions.

### 6 Persistence of positivity

We establish another property that plays an interesting role in the theory of porous medium equations to avoid degeneracy points for the solutions. It is called persistence of positivity. For continuous solutions it implies non-shrinking of the support.

**Theorem 6.1** Let $u$ be a weak solution as constructed in Theorem 4.1 and assume that $u_0(x)$ is positive in a neighborhood of a point $x_0$. Then $u(x_0, t)$ is positive for all times $t > 0$.

**Proof.** This issue allows for another use of the technique presented in the tail analysis, but this time with a true subsolutions. We assume that $u_0(x) \geq c > 0$ in a ball $B_R(x_0)$. By translation and scaling we may assume that $x_0 = 0$ and $c = R = 1$. The idea is to study the contact point with a parabola that shrinks quickly in time, like

$$U(x, t) = e^{-at}F(|x|),$$

with $F$ suitably chosen and $a > 0$ large enough. Firstly, we choose $F$ to be radially symmetric and decreasing, with $F(0) = 1/2$ and $F(|x|) = 0$ for $|x| > 1/2$. The contact point $(x_c, t_c)$ is sought in $B_{1/2}(0) \times (0, \infty)$. By approximation we may assume that $u$ is positive everywhere so no contact at the parabolic border is assumed. At a positive contact point we have $u_t \leq U_t = -au$, $\nabla u = e^{-at}F'(|x|)e_x$ and the standard arguments on the equation imply

$$-aF(|x|) \geq F'(|x|)\overline{p} + F(|x|)\Delta_x p$$

Now we have

$$p(x, t) = Ku(x, t) = e^{-at}(KF)(r)$$

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As in the tall control analysis we prove that $\Delta_x \bar{p}$ is bounded uniformly. The novelty is that $F' \leq 0$ implies that $p_r \leq 0$ so that the term $F''(|x|)p_r^2 \geq 0$. In this way the inequality implies

$$a \leq Ke^{-at}$$

which is false if $a > K$. Hence, under this size assumption there can be no contact point, and $U$ is a true subsolution, and positive for all times in $B_{1/2}(0)$. □

7 Appendix. Fractional Laplacians

We collect some data on fractional Laplacians for the reader’s convenience

**Fractional Laplacians and potentials.** According to Stein [33], Chapter V, the definition of $(-\Delta)^{\beta/2}$ is done by means of Fourier series

$$((-\Delta)^{\beta/2} f)(x) = (2\pi|x|^\beta \hat{f}(x))$$

and can be used for positive and negative values of $\beta$. For $\beta = -\alpha$ negative, with $0 < \alpha < n$, we have the equivalence with the Riesz potentials [30]

$$(-\Delta)^{-\alpha/2} f = I_\alpha(f) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}dy$$

(acting on functions of the class $\mathcal{S}$ for instance) with precise constant

$$\gamma(\alpha) = \pi^{n/2}2^n \Gamma(\alpha/2)/\Gamma((n - \alpha)/2).$$

Note that $\gamma \to \infty$ as $\alpha \to n$, but $\gamma/(n - \alpha)$ converges to a nonzero constant, $\pi^{n/2}2^{n-1}\Gamma(n/2)$. Stein also mentions the Bessel potentials which are associated to the modified inverse Laplace operators

$$((I - \Delta)^{-\alpha/2} f) = \mathcal{I}_\alpha(f)$$

The Bessel potential has a kernel $G_\alpha(x)$ that is better behaved at infinity, though not given by a simple kernel (see [33], page 132).

**Fractional Laplacians via extensions**

In the case $s = 1/2$ it is the well-known technique of harmonic extension to the upper plane of the space with one more dimension and then taking the boundary normal derivative, and has been used in the study or variational problems with thin obstacles as in [3], see also [9, 10]. For $s \neq 1/2$ the method has been recently developed in [12], it involves elliptic equations with weights and weighted normal derivatives.
8 Comments and extensions

**Extensions.** Very different versions to the evolution process are obtained when the pressure is related to \( u \) in other ways. Thus, we can use a pressure-density relation of the form \( p = f(u) \) with \( f \) an increasing function, and then the model equation would be

\[
\partial_t u = \nabla \cdot (u \nabla K(f(u))).
\]  

(8.1)

Many of the results proved here should apply to this model. Another possibility consists of equations of the form \( \partial_t u = \nabla \cdot (f(u) \nabla K(u)) \).

**Relaxation. Chemotaxis models.** We could also relax the relation of \( p \) to \( u \) into the form

\[
\partial_t p + (-\Delta)^s p = u.
\]  

(8.2)

This reflection is motivated by a very important system, the Keller-Segel chemotaxis model, [23, 22], in which the phenomenon which is modeled by \( u \) is not diffusion but the concentration of a certain population and \( p \) is replaced by variable \( c \) proportional to the concentration of the chemical substance responsible for the aggregation of the population. A suitable general system is proposed in the form

\[
u_t = \varepsilon \Delta u - \nabla \cdot (u \nabla c), \quad \delta c_t + K^{-1} c = f(u, c).
\]  

(8.3)

The standard chemotaxis model uses \( K = -\Delta \) and \( f(u, c) = u - bc \). In the limit case where \( \delta \) and \( b \) are zero, and if we use as \( K \) an integral operator as described above, we get a model with a term like ours but note the different sign, \( u_t = \varepsilon \Delta u - \nabla \cdot (u \nabla K(u)) \), which is a consequence of the fact that we are dealing with aggregation and not diffusion. The study of these equations is also of interest.

**Finite propagation.** The finite propagation property is not true for other alternative models of porous medium equation with fractional diffusion like the ones studied in [4], [28]. The last reference deals with the model

\[
\frac{\partial u}{\partial t} + (-\Delta)^{1/2}(|u|^{m-1}u) = 0,
\]  

(8.4)

For any \( m > m_* = (n - 1)/n \), it is proved that a unique nonnegative strong solution of this problem exists for data in \( L^1_t(L^m) \) and is strictly positive. The maximum principle applies to this problem.
Uniqueness and comparison. These are widely open issues. We have used in the present paper comparison with what we call “true supersolutions” and “true subsolutions”. In one space dimension, a uniqueness proof has been obtained in [8] by integrating the equation (with respect to $x$) and using then solutions in the sense of viscosity. Such trick is not available in several dimensions.

Evolution and regularity of free boundaries. This is quite important topic motivated by the property of finite propagation.

Smoothing $L^1$ into $L^\infty$ and $C^\alpha$ regularity. These topics will be treated in [13].

Asymptotic behaviour. This topic is under study. Let us outline the main details. There exists a family of self-similar solutions for this problem, in the spirit of the fundamental solution of the linear problems or the Barenblatt solutions of the standard Porous Medium Equation. The spatial profile of such solutions is obtained by solving for the pressure $p$ an obstacle problem with a truncated paraboloid as obstacle; the corresponding density $u$ is then the mass of the negative fractional Laplacian of $p$, supported on the contact set; all this fits perfectly into the elliptic theory described [6]). The details of the construction, as well as the convergence of a typical solution to such asymptotic profiles after suitable scaling, will be established in an upcoming publication, [14].

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