Abstract. We show that a certain type of quasi finite, conservative, ergodic, measure preserving transformation always has a maximal zero entropy factor, generated by predictable sets. We also construct a conservative, ergodic, measure preserving transformation which is not quasi finite; and consider distribution asymptotics of information showing that e.g. for Boole's transformation, information is asymptotically mod-normal with normalization \( \propto \sqrt{n} \). Lastly we see that certain ergodic, probability preserving transformations with zero entropy have analogous properties and consequently entropy dimension of at most \( \frac{1}{2} \).

\section{Introduction}

Let \((X, B, m, T)\) be a conservative, ergodic, measure preserving transformation and let \(F := \{ F \in B : m(F) < \infty \}\).

Call a set \(A \in F\) \(T\)-predictable if it is measurable with respect to its own past in the sense that \(A \in \sigma(\{T^{-n}A : n \geq 1\})\) (the \(\sigma\)-algebra generated by \(\{T^{-n}A : n \geq 1\}\)) and let \(P = P_T := \{T\text{-predictable sets}\}\).

If \(m(X) < \infty\), Pinsker’s theorem ([Pi]) says that

- \(P\) is the maximal, zero-entropy factor algebra
  i.e. \(P \subseteq B\) is a factor algebra \((T\text{-invariant, sub-}\sigma\text{-algebra}), h(T, P) = 0\) (see §1) and if \(C \subseteq B\) is a factor algebra, with \(h(T, C) = 0\), then \(C \subseteq P\). \(P\) is aka the Pinsker algebra of \((X, B, m, T)\).

When \((X, B, m, T)\) is a conservative, ergodic, measure preserving transformation with \(m(X) = \infty\), the above statement fails and indeed \(\sigma(P) = B\): Krengel has shown ([K2]) that:

- \(\forall A \in F, \epsilon > 0, \exists B \in F, m(A \Delta B) < \epsilon\), a strong generator in the sense that \(\sigma(\{T^{-n}B : n \geq 1\}) = B\), whence \(\sigma(P_T) = B\).

It is not known if there is always a maximal, zero-entropy factor algebra (in case there is some zero-entropy factor algebra).

We recall the basic properties of entropy in §1 and define the class of log lower bounded conservative, ergodic, measure preserving transformations in §2.

1991 Mathematics Subject Classification. 37A40, 60F05.

Key words and phrases. measure preserving transformation; conservative; ergodic; entropy; quasi finite; predictable set; entropy dimension.

©2007 Preliminary version.

Typeset by \texttt{AMSL-T\TeX}
These are quasi finite in the sense of [K1] and are discussed in §2 in this context where also examples are constructed including a conservative, ergodic, measure preserving transformation which is not quasi finite.

A log lower bounded conservative, ergodic, measure preserving transformation with some zero-entropy factor algebra has a maximal, zero-entropy factor algebra generated by a specified hereditary subring of predictable sets (see §5).

We obtain information convergence (in §4) for quasi finite transformations (cf [KS]).

For quasi finite, pointwise dual ergodic transformations with regularly varying return sequences, we obtain (in §6) distributional convergence of information. Lastly, we construct a probability preserving transformation with zero entropy with analogous distributional properties and estimate its entropy dimension in the sense of [FP]. This example is unusual in that it has a generator with information function asymptotic to a non degenerate random variable (the range of Brownian motion).

§1 Entropy

We recall the basic entropy theory of a probability preserving transformation \((\Omega, \mathcal{A}, P, S)\). Let \(\alpha \subset \mathcal{A}\) be a countable partition.

- The entropy of \(\alpha\) is \(H(\alpha) := \sum_{a \in \alpha} P(a) \log \frac{1}{P(a)}\);
- the \(S\)-join of \(\alpha\) from \(k\) to \(\ell\) (for \(k < \ell\)) is
  \[
  \alpha^\ell_k(S) := \{ \bigcap_{j=k}^{\ell} S^{-j} a_j : a_k, a_{k+1}, \ldots, a_\ell \in \alpha \}.
  \]
  By subadditivity, \(\exists \lim_{n \to \infty} \frac{1}{n} H(\alpha_0^{n-1}(S)) =: h(S, \alpha)\) (the entropy\(^1\) of \(S\) with respect to \(\alpha\)).
- The entropy of \(S\) with respect to the factor algebra (\(S\)-invariant, \(\sigma\)-algebra), \(\mathcal{C} \subset \mathcal{A}\) is \(h(S, \mathcal{C}) := \sup_{\alpha \subset \mathcal{C}} h(S, \alpha)\).
- By the generator theorem, if \(\alpha\) is a partition, then \(h(S, \alpha) = h(S, \sigma(\{S^n \alpha : n \in \mathbb{Z}\}))\).
- The information of the countable partition \(\alpha \subset \mathcal{A}\) is the function \(I(\alpha) : \Omega \to \mathbb{R}\) defined by
  \[
  I(\alpha)(x) := \log \frac{1}{P(\alpha(x))}
  \]
  where \(\alpha(x) \in \alpha\) is defined by \(x \in \alpha(x) \in \alpha\). Evidently
  \[
  H(\alpha) = \int_\Omega I(\alpha)dP.
  \]

- **Convergence of information** is given by the celebrated Shannon- McMillan-Breiman theorem (see [S], [M], [Br] respectively), the statement (3) here being due to Chung [C] (see also [IT]).

\(^1\)mean entropy rate
Let $(\Omega, A, P, S)$ be an ergodic probability preserving transformation and let $\alpha$ be a partition with $H(\alpha) < \infty$, then

$$\frac{1}{n} I(\alpha_i^N(S)) \longrightarrow h(S, \alpha) \quad \text{a.s. as } n \to \infty;$$

equivalently $P(\alpha_i^N(S)(x)) = e^{-nh(S,\alpha)(1+o(1))}$ for a.e. $x \in \Omega$ as $n \to \infty$ where $x \in \alpha_i^N(S)(x) \in \alpha_i^N(s)$.

- We’ll need Abramov’s formula for the entropy of an induced transformation of an ergodic probability preserving transformation $(\Omega, A, P, S)$:

$$h(S_A) = \frac{1}{m(A)} h(S) \forall A \in A$$

where $S_A : A \to A$ is the induced transformation on $A$ defined by

$$S_A x := S^{\varphi_A(x)} x, \ \varphi_A(x) := \min \{ n \geq 1 : S^n x \in A \} \quad (x \in A).$$

- Abramov’s formula can be proved using convergence of information (see [Ab] and §4 here).

KRENGEL ENTROPY. Suppose that $(X, B, m, T)$ is a conservative, ergodic, measure preserving transformation then using Abramov’s formula (as shown in [K1])

$$m(A) h(T_A) = m(B) h(T_B) \quad \forall A, B \in \mathcal{F} := \{ F \in B, \ 0 < m(F) < \infty \}.$$ 

Set $\underline{h}(T) := m(A) h(T_A)$, (any $A \in B, \ 0 < m(A) < \infty$) – the Krengel entropy of $T$.

More generally, the Krengel entropy of $T$ with respect to the factor (i.e. $\sigma$-finite, $T$-invariant sub-$\sigma$-algebra) $\mathcal{C} \subset B$ is

$$\underline{h}(T, \mathcal{C}) := m(A) h(T_A, \mathcal{C} \cap A) \quad (A \in \mathcal{C}, \ 0 < m(A) < \infty).$$

- Another definition of entropy is given in [Pa].

It is shown in [Pa] that for quasi finite (see §2 below) conservative, ergodic, measure preserving transformations, the two entropies coincide.

§2 Quasifiniteness and Log lower boundedness

QUASIFINITENESS.

Let $(X, B, m, T)$ be a conservative, ergodic, measure preserving transformation .

Recall from [K1] that a set $A \in \mathcal{F}$ is called quasi finite (qf) if $H_A(\rho_A) < \infty$ where $\rho_A := \{ A \cap T^{-n} A \setminus \bigcup_{j=1}^{n-1} T^{-j} A : \ n \geq 1 \}$ and that $T$ is so called if $\exists$ such a set. As shown in proposition 7.1 in [K1],

- for $A \in \mathcal{F}$ quasi finite, $A \in \mathcal{P}_T \iff h(T_A, \rho_A) = 0$.

There are conservative, ergodic, measure preserving transformation $s$ which are not quasi finite. An unpublished example of such by Ornstein is mentioned in [K2, p. 82].

Here we construct a conservative, ergodic, measure preserving transformation with no quasi finite extension. To do this we first establish a saturation property for the collection of quasi finite sets:
Proposition 2.0.
Suppose that $(X, B, m, T)$ is a conservative, ergodic, quasi finite, measure preserving transformation, then $\forall F \in \mathcal{F}$, $\exists A \in \mathcal{B} \cap F$ such that $m(A) > 0$ and such that each $B \in \mathcal{B} \cap A$ is quasi finite.

Proof. We show first that $
\exists 1$ if $F \in \mathcal{F}$ is quasi finite, then $\forall \epsilon > 0$, $\exists A \in \mathcal{B} \cap F$ such that $m(F \setminus A) < \epsilon$ and such that each $B \in \mathcal{B} \cap A$ is quasi finite.

Proof. By (3), $\frac{1}{n} I(\rho_F) n^{-1} (T_F) \rightarrow h(T_F, \rho_F)$ a.e. as $n \rightarrow \infty$. By Egorov’s theorem, $\exists A \in \mathcal{B} \cap F$ such that $m(F \setminus A) < \epsilon$ and such that the convergence is uniform on $A$.

For $B \in \mathcal{B} \cap A$, let $N_{n,B} := \# \{ a \in (\rho_F)^{n-1}(T_F) : m(a \cap B) > 0 \}$ (where $\#F$ means the number of elements in the set $F$), then $N_{n,B} = e^{nh(T_F, \rho_F)(1 + o(1))}$ as $n \rightarrow \infty$.

Define $\psi : B \rightarrow \mathbb{N}$ by $\psi(x) := \min \{ n \geq 1 : T_F^nx \in B \}$, then
- $\int_B \psi dm = \sum_{n=1}^{\infty} mn([\psi = n]) = m(F) < \infty$ (by Kac’s formula);
- $\varphi_B(x) = \sum_{j=0}^{\psi(x)-1} \varphi_F(T_F^j x)$ whence

$$\rho_B \leq \gamma_B := \bigcup_{n=1}^{\infty} \{ \psi = n \cap a : a \in (\rho_F)^{n-1}(T_F) \}.$$  

Thus

$$H_{m_B}(\rho_B) \leq H_{m_B}(\gamma_B)$$

$$= \sum_{n=1}^{\infty} m_B([\psi = n]) H_{m_{[\psi=n]}}((\rho_F)^{n-1}(T_F))$$

$$\leq \sum_{n=1}^{\infty} m_B([\psi = n]) \log N_{n,B} < \infty : \log N_{n,B} \sim nh(T_F, \rho_F) \sqrt{\n} \Box \n$$

To complete the proof, let $F \in \mathcal{F}$. Suppose that $Q \in \mathcal{F}$ is quasi finite, then evidently so is $T^{-n}Q \forall n \geq 1$. By ergodicity, $\exists n \geq 1$ such that $m(F \cap T^{-n}Q) > 0$.

By $\n$, $\exists G \in \mathcal{B} \cap T^{-n}Q$ such that $m(T^{-n}Q \setminus G) < \epsilon := \frac{m(F \cap T^{-n}Q)}{9}$ and such that each $B \in \mathcal{B} \cap G$ is quasi finite. The set $A = G \cap F$ is as required. $\Box$

Example 2.1. Let $(X_0, B_0, m_0, T_0)$ be the conservative, ergodic, measure preserving transformation defined as in [Fr] by the cutting and stacking construction

$$B_0 = 1, \ B_n = \bigoplus_{k=1}^{N_n} B_{n-1}^0 0^{L_n,k}$$

where $N_n, L_{n,k}, 1 \leq k \leq N_n$ satisfy

$$N_{n+1} \geq e^{nN_1 \ldots N_n}, \ L_{n,k+1} > \sum_{j=1}^{k} L_{n,j} + kh_{n-1},$$

where $h_n := |B_n|$. 
Proposition 2.1.
No extension $T$ of the conservative, ergodic, measure preserving transformation $T_0$ defined in example 2.1 is quasi finite.

Proof Suppose otherwise, that $(X, \mathcal{B}, m, T)$ is a (WLOG) conservative, ergodic extension of $T_0$ and that $F \in \mathcal{F}$ is quasi finite, then evidently so is $T^n F \forall n \geq 1$. By proposition 2.0 $\exists A \in \mathcal{B}$, $A \subset B_0$ quasi finite. We’ll contradict this (and therefore the assumption that $\exists F \in \mathcal{F}$ quasi finite).

\[ \begin{align*}
\xi 1 & \text{ Write } B_n = \bigcup_{j=0}^{b_n-1} T^j b_n \text{ where } b_n \subset B_0, \ m(b_n) = \frac{1}{N_1 N_2 \ldots N_n} \text{ and } B_n = \bigcup_{k=1}^{N_n+1} B^{(k)}_n = \bigcup_{k=1}^{N_n+1} T^{(n+1)} B^{(1)}_n \text{ where } \kappa(n+1, k) = (k-1)|B_n| + \sum_{j=1}^{k-1} L_{n+1, j} \text{ (i.e. the } B^{(k)}_n (1 \leq k \leq N_{n+1}) \text{ are the subcolumns of } B_n \text{ appearing in } B_{n+1}). \\
\xi 2 & \text{ For } n \geq 1, \text{ let } \xi_n := \{0 \leq j \leq b_n - 1 : T^j b_n \subset B_0\}, \text{ then } \\
B_0 = \bigcup_{j \in \xi_n} T^j b_n, \ |\xi_n| = N_1 N_2 \ldots N_n \text{ and } \\
\text{for } x \in b_n, \ \{T^k x\}_{k=0}^{N_1 N_2 \ldots N_n - 1} = \{T^j x : j \in \xi_n\}. \\
\xi 3 & \text{ Fix } 0 < \epsilon < \frac{1}{4} \text{ and let } \\
b_{n, \epsilon} := \{x \in b_{n+1} : \left| \frac{1}{|\xi_{n+1}|} \sum_{k \in \xi_{n+1}} 1_A(T^k x) - m(A) \right| < \epsilon m(A)\}. 
\end{align*} \]

By $\xi 2$ above, for $x \in b_{n+1},$

\[ \begin{align*}
\frac{1}{|\xi_{n+1}|} \sum_{k \in \xi_{n+1}} 1_A(T^k x) = \frac{1}{N_1 N_2 \ldots N_n} \sum_{k=0}^{N_1 N_2 \ldots N_n - 1} 1_A(T^k x) 
\end{align*} \]

and a standard argument using the ergodic theorem for $T_{B_0}$ shows that $\exists M$ so that $m(b_{n, \epsilon}) > (1 - \epsilon) m(b_{n+1}) \ \forall n \geq M.$

$\xi 4$ Fix $n \geq M$ and $x \in b_{n, \epsilon},$ let $\xi_{A, n, x} := \{k \in \xi_{n+1} : T^k x \in A\}$ and $A_{n, x} := \{T^j x : j \in \xi_{A, n, x}\},$ then for $x \in b_{n, \epsilon},$

\[ \#\{1 \leq k \leq N_{n+1} : A_{n, x} \cap B_{n}^{(k)} \neq \phi\} \geq (1 - \epsilon) m(A) \frac{|\xi_{n+1}|}{\xi_n} = (1 - \epsilon) m(A) N_{n+1}. \]

- For $n \geq M, x \in b_{n, \epsilon},$ write

\[ \{1 \leq k \leq N_{n+1} : A_{n, x} \cap B_{n}^{(k)} \neq \phi\} =: \{\kappa_i(x) : 1 \leq i \leq \nu\} \]

where $\nu - 1 > (1 - \epsilon) m(A) N_{n+1}$ and $\kappa_i(x) < \kappa_{i+1}(x) \ \forall i.$

- For $1 \leq i \leq \nu,$ let $\xi_{A, n, x}^{(i)} := \{k \in \xi_{n+1} : T^k b_{n+1} \subset A_{n, x} \cap B_{n}^{(k)}\}$ and let $m_i := \min \xi_{A, n, x}^{(i)}, \ \overline{m}_i := \max \xi_{A, n, x}^{(i)} ; y_i := \overline{m}_i - \underline{m}_i, \ (1 \leq i \leq \nu - 1).$ Note that $y_i \leq \sum_{j=1}^{\kappa_i} L_{n+1, j} + \kappa_i b_n < L(n + 1, \kappa_i + 1) \leq L(n + 1, \kappa_{i+1}) \leq y_{i+1}.$

$\xi 5$ For $K \subset \xi_{n+1},$ let $a_K := \{x \in b_{n+1} : \xi_{A, n, x} = K\}$ and let

\[ \beta_n := \{a_K : K \subset \xi_{n+1}\}, \ \alpha_n := \bigcup_{j \in \xi_n} T^j a : a \in \beta_n\].
• For $a \in \beta_n$, $a \subset b_{n,e}$, $1 \leq i \leq \nu - 1$,
$$m(a \cap [\varphi_A = y_i(a)]) = \frac{m(a)}{N_1...N_{n+1}}.$$  
• Thus

$$H(\rho A) \geq H(\rho A||\alpha_n)$$

$$\geq \sum_{a \in \beta_n, \ a \subset b_{n,e}} m(a) \sum_{i=1}^{\nu-1} m([\varphi_A = y_i(a)]|a) \log \frac{1}{m([\varphi_A = y_i(a)]|a)}$$

$$\geq m(\beta_{n,e}) \frac{(\nu-1) \log (N_{n+1})}{N_1...N_{n+1}}$$

$$\geq (1 - \epsilon)^2 m(A) \frac{\log N_{n+1}}{N_1N_2...N_n}$$

Thus

$$H(\rho A) \geq H(\rho A || \alpha_n) \geq m(\beta_{n,e}) \frac{(\nu-1) \log (N_{n+1})}{N_1...N_{n+1}}$$

$$\geq (1 - \epsilon)^2 m(A) \frac{\log N_{n+1}}{N_1N_2...N_n}$$

$$\geq (1 - \epsilon)^2 m(A) n \uparrow \infty. \quad \square$$

**Log lower boundedness.**

For $(X, B, m, T)$ a conservative, ergodic, measure preserving transformation; set

$$\mathcal{F}_{\log, T} := \{ A \in B : 0 < m(A) < \infty, \int_A \log \varphi_A dm < \infty \}.$$  

• Note that

$$\mathcal{F}_{\log, T} \subset \{\text{quasi finite sets}\}; \text{ because}$$

$$p_n \geq 0, \sum_{n=1}^{\infty} p_n \log n < \infty \implies \sum_{n=1}^{\infty} \frac{1}{p_n} < \infty.$$  

• Call $T$ log-lower bounded (LLB) if $\mathcal{F}_{\log, T} \neq \emptyset$.

**Proposition 2.2.**

(i) $T$ is LLB iff $\frac{1}{\log n} \sum_{k=0}^{n-1} f \circ T^n \to \infty$ a.e. as $n \to \infty$ for some and hence all $f \in L^1(m)_+: = \{ f \in L^1, f \geq 0, \int_X f dm > 0 \}$;

(ii) $T$ is not LLB iff $\liminf_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^{n-1} f \circ T^n = 0$ a.e. for some and hence all $f \in L^1_+$;

(iii) If $(X, B, m, T)$ is LLB and $C \subset B$ is a factor, then $C \cap \mathcal{F}_{\log, T} \neq \emptyset$.

(iv) $\mathcal{F}_{\log, T}$ is a hereditary ring.

**Proof** Statements (i) and (ii) follow from theorem 2.4.1 in [A] and (iii) follows from these. We prove (iv).

Suppose that $A \in \mathcal{F}_{\log, T}$, $B \in B$, $B \subset A$, then $\varphi_B(x) = \sum_{k=0}^{\psi(x)-1} \varphi_A(T_A^k x)$ ($x \in B$) where $\psi : B \to \mathbb{N}$, $\psi(x) := \min \{ n \geq 1 : T_A^n x \in B \}$.

By Kac formula,

$$\int_B \sum_{k=0}^{\psi-1} f \circ T_A^k dm = \int_A f dm \quad \forall f \in L^1(m).$$
To see that $B \in \mathcal{F}_{\log,T}$, we use this and $\log(k + \ell) \leq \log(k) + \log(\ell)$:

$$\int_B \log \varphi_B dm = \int_B \psi^{-1} \log(\sum_{k=0}^{\psi-1} \varphi_A \circ T_A^k) dm$$

$$\leq \int_B \sum_{k=0}^{\psi-1} \log(\varphi_A \circ T_A^k) dm$$

$$= \int_A \log \varphi_A dm < \infty.$$

Suppose that $A, B \in \mathcal{F}_{\log,T}$, then $\varphi_{A \cup B} \leq 1_A \varphi_A + 1_B \varphi_B$ whence

$$\int_{A \cup B} \log(\varphi_{A \cup B}) dm = \int_A \log(\varphi_{A \cup B}) dm + \int_B \log(\varphi_{A \cup B}) dm$$

$$\leq \int_A \log(\varphi_A) dm + \int_B \log(\varphi_B) dm$$

$$< \infty. \quad \square$$

§3 Examples of LLB transformations

Pointwise dual ergodic transformations.

A conservative, ergodic, measure preserving transformation $(X, \mathcal{B}, m, T)$ is called pointwise dual ergodic if there is a sequence of constants $(a_n(T))_{n \geq 1}$ (called the return sequence of $T$) so that

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \hat{T}^k f \rightarrow \int_X f dm \quad \text{a.e. for some (and hence all) } f \in L^1(m)_+$$

where $\hat{T} : L^1(m) \rightarrow L^1(m)$ is the transfer operator defined by

$$\int_A \hat{T} f dm = \int_{T^{-1}A} f dm \quad (f \in L^1(m), A \in \mathcal{B}).$$

See [A, 3.8].

**Proposition 3.1.**

Let $(X, \mathcal{B}, m, T)$ be a pointwise dual ergodic, conservative, ergodic, measure preserving transformation, then $T$ is LLB $\iff \sum_{n=1}^{\infty} \frac{1}{a_n(T)} < \infty$.

**Proof**

Let $A \in \mathcal{F}$ be a uniform set in the sense that for some $f \in L^1(m)_+$

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \hat{T}^k f \rightarrow \int_X f dm \quad \text{uniformly on } A.$$

By lemma 3.8.5 in [A],

$$\int_A (\varphi_A \wedge n) dm = m\left( \bigcup_{k=0}^{n} T^{-k}A \right) \geq \frac{n}{a_n(T)}$$

whence

$$A \in \mathcal{F}_{\log} \iff \sum_{n=1}^{\infty} \frac{m(\bigcup_{k=0}^{n} T^{-k}A)}{n} < \infty \iff \sum_{n=1}^{\infty} \frac{1}{a_n(T)} < \infty. \quad \square$$
Remarks.

1) For example, the simple random walk on \( \mathbb{Z} \) is LLB (\( \therefore a_n(T) \propto \sqrt{n} \)); whereas the simple random walk on \( \mathbb{Z}^2 \) is not LLB (\( \therefore a_n(T) \propto \log n \)).

2) It is not known whether the simple random walk on \( \mathbb{Z}^2 \) is quasi finite, or even has a factor with finite entropy.

Example 3.2.

There is a quasi finite, conservative, ergodic, Markov shift \( (X, \mathcal{B}, m, T) \) with \( a_n(T) \asymp \sqrt{\log n} \).

- Note that by proposition 3.1, this \( T \) is not LLB.

Proof of example 3.2: Let \( f_k := \frac{1}{2^k} \) \( n \geq 1 \) and \( f_k := 0 \) \( \forall k \in \mathbb{N} \setminus 4^k \), then \( f \in \mathcal{P}(\mathbb{N}) \).

- Let \( \Omega := \mathbb{N}^2 \) and let \( P = f^\mathcal{P} \in \mathcal{P}(\Omega, \mathcal{B}(\Omega)) \) be product measure, then \( (\Omega, \mathcal{B}(\Omega), P, S) \) is an ergodic, probability preserving transformation where \( S : \Omega \to \Omega \) is the shift.

- Define \( \varphi : \Omega \to \mathbb{N} \) by \( \varphi(\omega) := \omega_0 \) and let \( (X, \mathcal{B}, m, T) \) be the tower over \( (\Omega, \mathcal{B}(\Omega), P, S) \) with height function \( \varphi \).

- It follows that \( (X, \mathcal{B}, m, T) \) is a conservative, ergodic, Markov shift with \( a_n(T) \asymp \sum_{k=0}^n u_k \) where \( u \) is defined by the renewal equation: \( u_0 = 1 \), \( u_n = \sum_{k=1}^n f_k u_{n-k} \).

- To see that \( (X, \mathcal{B}, m, T) \) is quasi finite, we check that \( \Omega \) is quasi finite. Indeed

\[
H_{\Omega}(\rho_\Omega) = \sum_{k \geq 1} f_k \log \frac{1}{f_k} = \sum_{n=1}^{\infty} \frac{a \log 2}{2^n} < \infty.
\]

- To estimate \( a_n(T) \), recall that by lemma 3.8.5 in [A], \( a_n(T) \asymp \frac{n}{L(n)} \) where

\[
L(n) := \min_{\ell=0}^{n} T^{-\ell} \mathcal{O} = \sum_{k=0}^{n} \sum_{\ell=k+1}^{\infty} f_\ell.
\]

Now,

\[
\sum_{\ell=k+1}^{n} f_\ell = \sum_{n > \log_4 \log_4 k} \frac{1}{2^n} \asymp \frac{1}{2^{n+4} \log_4 k} = \frac{1}{\sqrt{n} \log_4 k}.
\]

Thus \( L(n) \asymp \frac{n}{\sqrt{n} \log n} \) and \( a_n(T) \asymp \sqrt{\log n} \). \( \square \)

The Hajian-Ito-Kakutani Transformations.

- Let \( \Omega = \{0,1\}^\mathbb{N} \), \( \ell(\omega) := \min \{ n \geq 1 : \omega_n = 0 \} \) and let \( \tau : \Omega \to \Omega \) be the adding machine defined by

\[
\tau(1, \ldots, 1, 0, \omega_{\ell(\omega)+1}, \ldots) := (0, \ldots, 0, 1, \omega_{\ell(\omega)+1}, \ldots).
\]

For \( p \in (0,1) \), define \( \mu_p \in \mathcal{P}(\Omega) \) by \( \mu_p([a_1, \ldots, a_n]) := p_{a_1} \cdots p_{a_n} \) where \( p_0 := 1-p \), \( p_1 := p \). It follows that \( (\Omega, \mathcal{A}, \mu_p, \tau) \) is an ergodic, nonsingular transformation with \( \frac{d \mu_p \circ \tau}{d \mu_p} = (1-p)^\phi \) where \( \phi := \ell - 2 \).

Now let \( X := \Omega \times \mathbb{Z} \) and define \( T : X \to X \) by \( T(x,n) = (\tau x, n + \phi(x)) \). For \( p \in (0,1) \), define \( m_p \in \mathcal{B}(X) \) by \( m_p(A \times \{n\}) := \mu_p(A) \left( \frac{1}{1-p} \right)^{-n} \).

As shown in [HIK] (see also [A]) \( T_p = (X, \mathcal{B}, m_p, T) \) is a conservative, ergodic, measure preserving transformation (aka the Hajian-Ito-Kakutani transformation). The entropy is given by \( \underline{h}(T_p) = h(T_p|_{\Omega \times \{0\}}) = 0 \) by [MP] since \( T_p|_{\Omega \times \{0\}} \) is the Pascal adic transformation.
Proposition 3.3.

\((X, B, m_p, T)\) is LLB \(\forall \ 0 < p < 1\).

Proof  As in the proof of proposition 5.1 in [A1],
\begin{equation*}
\sum_{k=0}^{2^n-1} 1_{\Omega \times \{0\}} \circ T^k(x, 0) = \#\{0 \leq k \leq 2^n - 1 : \sum_{j=0}^{k-1} \phi(\tau^j x) = 0\} \\
\geq \#\{0 \leq K \leq n - 1 : \sum_{j=0}^{2^K-1} \phi(\tau^j x) = 0\}
\end{equation*}

Now \(\sum_{j=0}^{2^K-1} \phi(\tau^j x) = \phi(S^K x)\) where \(S : \Omega \to \Omega\) is the shift, and so
\begin{equation*}
\sum_{k=0}^{2^n-1} 1_{\Omega \times \{0\}} \circ T^k(x, 0) \geq \#\{0 \leq K \leq n - 1 : \phi(S^K x) = 0\} \sim (1 - p)n
\end{equation*}

for \(\mu_p\)-a.e. \(x \in \Omega\) by Birkhoff’s theorem for the ergodic, probability preserving transformation \((\Omega, B(\Omega), \mu_p, S)\). The LLB property now follows from proposition 2.2.  \(\square\)

- Let \(\mathfrak{S}\) be the Polish group of measure preserving transformations of \((\mathbb{R}, B(\mathbb{R}), m_\mathbb{R})\) equipped with the weak topology.

Proposition 3.4.

The collection of LLB measure preserving transformations is meagre in \(\mathfrak{S}\).

Proof  Let
\begin{equation*}
\mathcal{L} := \{T \in \mathfrak{S} : \exists n_k \to \infty, \frac{S_{n_k}(f)}{\log n_k} \to 0 \ \text{a.e.} \ \forall f \in L^1\}
\end{equation*}

where \(S_n(f) = S_n^T(f) := \sum_{j=0}^{n-1} f \circ T^j\).

By proposition 2.2, it suffices to show that \(\mathcal{L}\) is a dense \(G_\delta\) set in \(\mathfrak{S}\).

By example 3.2, \(\exists\) a conservative, ergodic, measure preserving transformation \(T \in \mathcal{L}\). \(\mathcal{L}\) is conjugacy invariant, and so dense in \(\mathfrak{S}\) by the conjugacy lemma (e.g. 3.5.2 in [A]).

To see that \(\mathcal{L}\) is a \(G_\delta\) set, let

- \(P \sim m\) be a probability;
- fix \(\{A_n : n \in \mathbb{N}\} \subset \mathcal{F} := \{A \in B : m(A) < \infty\}\) so that \(\sigma(\{A_n : n \in \mathbb{N}\}) = B\) and let
\begin{equation*}
\mathcal{L}' := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcap_{\nu=1}^{k} \{T \in \mathfrak{S} : P([S_n(1_{A_\nu}) > \frac{1}{n} \log n]) < \frac{1}{2^n}\},
\end{equation*}

then \(\mathcal{L}'\) is a \(G_\delta\). We claim \(\mathcal{L}' = \mathcal{L}\).

Evidently,
\begin{equation*}
\mathcal{L}' = \{T \in \mathfrak{S} : \exists n_k \to \infty \text{ such that } \frac{S_{n_k}(1_{A_{\nu}})}{\log n_k} \to 0 \ \text{a.e.} \ \forall \nu \geq 1\}
\end{equation*}
whence \( L' \supset L \).

Now suppose that \( T \in L' \), that \( \frac{S_{nk}(1_A\cap)}{\log nk} \to 0 \) a.e. \( \forall \nu \geq 1 \) and let \( f \in L^1 \).

Evidently \( \frac{S_n(f)}{\log n} \to 0 \) a.e. on \( \mathcal{D} \), the dissipative part of \( T \). The conservative part of \( T \) is

\[
\mathcal{C} = \bigcup_{\nu=1}^{\infty} A_{\nu} \text{ where } A_{\nu} := \left\lfloor \sum_{n=1}^{\infty} 1_{A_{\nu}} \circ T^n = \infty \right\rfloor.
\]

By Hopf’s theorem, \( \frac{S_{nk}(f(x))}{\log nk} \to h_{\nu}(f) \) a.e. on \( A_{\nu} \forall \nu \geq 1 \) where \( h_{\nu}(f) \circ T = h_{\nu}(f) \)

\[
\int_{A_{\nu}} h_{\nu}(f) dm = \int_{X} f dm, \text{ whence, a.e. on } A_{\nu},
\]

\[
\frac{S_{nk}(f)}{\log nk} = \frac{S_{nk}(f(x))}{\log nk} \to 0. \quad \square
\]

\[\Box\]

§4 Information convergence

Let \((X, \mathcal{B}, m, T)\) be a conservative, ergodic, measure preserving transformation.

- A countable partition \( \xi \subset \mathcal{B} \) is called cofinite if \( \exists A = A_\xi \in \mathcal{F} \) with \( A^c \in \xi \). We call \( A^c \) the cofinite atom of \( \xi \) and \( A \) the (finite) core of \( \xi \).

- If \( \xi \subset \mathcal{B} \) is cofinite, then \( \xi_{\xi_1}(T) \) is also cofinite, with core \( A_{\xi_{\xi_1}} = \bigcup_{n=1}^{\infty} T^{-n}A \).

The \( T \)-process generated by a cofinite partition \( \xi \) restricted to its core \( A \) is given by

Krengel’s formula [K1]:

\[\bullet\] \( \xi_{\xi_1}(T(x)) = (\rho_A \lor ((\xi \cap A) \lor \rho_A)^n(T_A))(x) \) for a.e. \( x \in A \)

where for \( x \in X, \alpha \) a partition of \( X \), \( \alpha(x) \) is defined by \( x \in \alpha(x) \in \alpha \);

\( \varphi_n(x) := \sum_{k=0}^{n-1} \varphi_A(T^n x) \); and

\( \rho_A := \{ A \lor T^{-n}A \lor \bigcup_{k=1}^{n-1} T^{-k}A : n \in \mathbb{N} \} \).

- A cofinite partition \( \xi \subset \mathcal{B} \) is called quasi-finite (qf) if \( A = A_{\xi} \) is quasi finite and \( H_A(\xi) < \infty \).

- Note that \( \xi \) quasi finite \( \Rightarrow H_A(\xi \lor \rho_A \lor T_A \rho_A) < \infty \).

Convergence of information for quasi finite partitions.

**Proposition 4.1 (c.f. [KS]).**

Let \((X, \mathcal{B}, m, T)\) be a conservative, ergodic, measure preserving transformation, let \( \xi \subset \mathcal{B} \) be a quasi finite partition and let \( p \in L^1(m), \ p > 0, \ \int_X p dm = 1 \), then for a.e. \( x \in X, \)

\[
\frac{1}{S_n(p)(x)} \int \xi_{\xi_1}(T)(x) \to h(T, \xi)
\]

where \( S_n(p)(x) := \sum_{k=0}^{n-1} p(T^k x) \) and \( I(\xi_{\xi_1}(T))(x) := \log \frac{1}{m(\xi_{\xi_1}(T)(x))} \).

**Proof**  Let \( A \) be the core of \( \xi \) and set \( \zeta := (\xi \cap A) \lor \rho_A \), then by (\( \bullet \))

\[
\xi_{\eta}(T_A)(x) \subseteq \xi_{\eta_1}(T)(x) \subseteq \xi_{\eta}(T_A)(x) \text{ a.e. } x \in A
\]

where \( x \in \xi(x) \in \xi, \ s_n := S_n(1_A) \).
• By (3), for $T_A$, a.e. on $A$, $I(\xi_1^N(T_A)) \sim N h(T_A, \zeta)$, whence for a.e. $x \in A$,

$$
\log \frac{1}{m(\xi_1^N(T)(x))} \sim \log \frac{1}{m(\xi_1^{n(x)}(T_A)(x))} \\
\sim s_n(x) h(T_A, \zeta) \\
\sim S_n(p)(x) m(A) h(T_A, \zeta) \\
= S_n(p)(x) h(T, \xi).
$$

We obtain convergence a.e. on $\bigcup_{k=0}^N T^{-k} A$ by substituting $\xi_1^N(T)$ for $\xi$; whence convergence a.e. on $X$ as $\bigcup_{k=0}^N T^{-k} A \uparrow X$. □

• Abramov’s formula is proved analogously in case $(X, B, m, T)$ is an ergodic, probability preserving transformation. As in [Ab]:

$$
\frac{1}{n} \log \log \frac{1}{m(\xi_1^N(T)(x))} \approx \frac{1}{n} s_n(x) \log h(T_A, \zeta) \xrightarrow{\text{Birkhoff's PET}} m(A) h(T_A, \zeta).
$$

§5 Pinsker algebra

Let $(X, B, m, T)$ be a LLB, conservative, ergodic, measure preserving transformation.

Define

$$\mathcal{F}_\Pi := \{A \in \mathcal{F}_{\log, T} : A \in \sigma(\{T^{-k} A : k \geq 1\}) \} = \mathcal{P} \cap \mathcal{F}_{\log, T}.$$

In this section, we show that (in case $\mathcal{F}_\Pi \neq \emptyset$) $\mathcal{B}_\Pi := \sigma(\mathcal{F}_\Pi)$ is the maximal zero entropy factor of $T$.

To do this, we’ll need

Krengel’s predictability lemma. [K1]:

Let $(X, B, m, T)$ be a quasi finite, conservative, ergodic, measure preserving transformation, let $\xi \subset B$ be a quasi finite partition with core $A$, and let $\zeta = \xi \cap A$, then

$$\xi \subset \xi_1^\infty(T) \mod m \iff h(T_A, \zeta \vee \rho_A) = 0.$$

In particular

$$A \in \sigma(\{T^{-n} A : n \geq 1\}) \iff h(T_A, \rho_A) = 0.$$

• For $F \in \mathcal{F}$, set

$$\mathcal{P}_F = \mathcal{P}_{T_F} := \{A \in B \cap F : A \in \sigma(\{T_F^{-k} A : k \geq 1\})\}.$$

By Pinsker’s theorem ([Pi]),

• $\mathcal{P}_F$ is a $T_F$-factor algebra of subsets of $F$, $h(T_F, \mathcal{P}_F) = 0$ and

• if $A \subset B \cap F$ is another $T_F$-factor algebra of subsets of $F$ with $h(T_F, A) = 0$, then $A \subset \mathcal{P}_F$.

Theorem 5.1.

(i) $\mathcal{F}_\Pi$ is a ring and $\mathcal{F}_\Pi \cap F = \mathcal{P}_F \forall F \in \mathcal{F}_\Pi$.

(ii) If $\mathcal{F}_\Pi \neq \emptyset$, then $\sigma(\mathcal{F}_\Pi)$ is the maximal factor of zero entropy.
Proof.

1. Let $A \in \mathcal{F}_{\log}$. By Krengel’s predictability lemma, $F \in \mathcal{F}_{\Pi}$ iff $h(T_F, \rho_F) = 0$.
   Thus, $F \in \mathcal{F}_{\Pi}$ if and only if $F$ is a factor of $F_B$ such that $h(T, B) = 0$.

2. Next, fix $F \in \mathcal{F}_{\Pi}$. We claim that $\rho_F \subseteq \mathcal{P}_F$. This is because $F \in \mathcal{F}_{\Pi} \Rightarrow h(T_F, \rho_F) = 0$.

3. We now show that $\mathcal{P}_F \subseteq \mathcal{F}_{\Pi} \cap F \forall F \in \mathcal{F}_{\Pi}$.
   Proof: Fix $F \in \mathcal{F}_{\Pi}$ and let $B_0 := \sigma(T^n A \colon n \in \mathbb{Z}, A \in \mathcal{P}_F)$, then $B_0$ is a factor, $F \in B_0$ and $B_0 \cap F = \mathcal{P}_F$. Thus $h(T, B_0) = h(T_F, \rho_F) = 0$ and by 1, $\mathcal{P}_F \subseteq \mathcal{F}_{\Pi} \cap F$.\@ \notag

4. Now we claim that $A, B \in \mathcal{F}_{\Pi} \Rightarrow A \cup B \in \mathcal{P}_F$.
   Proof: Set $C := A \cup B$, then $C \in \mathcal{F}_{\Pi}$. Set $\zeta := \{A \cap B, A \setminus B, B \setminus A\}$ and $\xi := \zeta \cup \{C^c\}$. By (8),
   $$
   \xi^\infty(T) \cap C = \rho_C \vee (\zeta \vee \rho_C)^\infty(T_C).
   $$
   By assumption, $\zeta \subseteq \xi^\infty(T) \cap C$, whence also $\rho_C \subseteq \xi^\infty(T) \cap C$. Thus
   $$
   \zeta \vee \rho_C \subseteq \rho_C \vee (\zeta \vee \rho_C)^\infty(T_C); \quad \xi \subseteq \rho_C \vee T_C \rho_C \subseteq (\zeta \vee \rho_C \vee T \rho_C)^\infty(T_C),
   $$
   and (using $H_C(\zeta \vee \rho_C \vee T \rho_C) < \infty$) we have
   $$
   h(T_C, \rho_C) \leq h(T_C, \zeta \vee \rho_C \vee T \rho_C) = 0
   $$
   whence $C \in \sigma(\{T^k \colon k \geq 1\})$ and $C \in \mathcal{F}_{\Pi}$.\@ \notag

5. Now we show that $\mathcal{F}_{\Pi}$ is a ring by proving that $A, B \in \mathcal{F}_{\Pi} \Rightarrow \zeta := \{A \cap B, A \setminus B, B \setminus A\} \subseteq \mathcal{F}_{\Pi}$.
   Proof: By 3, it suffices to show that $\zeta \subseteq \mathcal{P}_C$ where $C := A \cup B$. To see this, fix $a \in \zeta$, then
   $$
   h(T_C, \{a, C \setminus a\}) \leq h(T_C, \zeta) \leq h(T_C, \zeta \vee \rho_C \vee T \rho_C) = 0
   $$
   (as above) and $a \in \mathcal{P}_C$.\@ \notag

6. To complete the proof of (i), we show that $\mathcal{F}_{\Pi} \cap F \subseteq \mathcal{P}_F \forall F \in \mathcal{F}_{\Pi}$.
   Proof: Fix $F \in \mathcal{F}_{\Pi}$, $A \in \mathcal{F}_{\Pi} \cap F$. Let $\zeta := \{A, F \setminus A\}$, $\xi := \zeta \cup \{F^c\}$.
   By the ring property, $A \in \mathcal{F}_{\Pi}$, whence $\xi \subseteq \xi^\infty(T) \mod m$. By proposition 4, $h(T_F, \zeta \vee \rho_F) = 0$, whence
   $$
   h(T_F, \zeta) \leq h(T_F, \zeta \vee \rho_F) = 0
   $$
   and $A \in \mathcal{P}_F$.\@ \notag

7. To see (ii), fix $F \in \mathcal{F}_{\Pi}$, then by (i), $\mathcal{F}_{\Pi} \cap F = \mathcal{P}_F = \mathcal{F}_{\Pi} \cap F \cap F$ whence $h(T, \sigma(\{F\})) = m(F)h(T, \rho_F) = 0$ and if $C \subseteq B$ is a factor with $h(T, C) = 0$, then by 1, $C \cap \mathcal{F}_{\log} \subseteq \mathcal{F}_{\Pi}$, whence $C \subseteq \sigma(\{F\})$.\@
§6 Asymptotic distribution of information
with infinite invariant measure

Pointwise dual ergodic transformations.

Let \((X, \mathcal{B}, m, T)\) be a pointwise dual ergodic measure preserving transformation and assume that the return sequence \(a_n = a_n(T)\) is regularly varying with index \(\alpha (\alpha \in [0, 1])\), then by the Darling-Kac theorem (theorem 3.6.4 in [A] – see also references therein),

\[
\frac{1}{a_n} \frac{S_n^T(f)}{S_n^T(X)} \xrightarrow{\text{d}} \int_X f dm \cdot X_\alpha \quad \text{as } n \to \infty \forall f \in L^1(m)_+ \]

where

- \(X_\alpha\) is a Mittag-Leffler random variable of order \(\alpha\) normalised so that \(E(X_\alpha) = 1\); and
- \(F_n \xrightarrow{\text{d}} Y\) means

\[
\int_X G(F_n) dP \to E(G(Y)) \quad \forall P \in \mathcal{P}(X, \mathcal{B}), \ P \ll m, \ G \in C([0, \infty]).
\]

Note that \(X_1 \equiv 1, X_0\) has exponential distribution and for \(\alpha \in (0, 1)\), \(X_\alpha = \frac{1}{\sqrt{\pi}}\) where \(E(e^{-tX_\alpha}) = e^{-ct^\alpha}\) (some \(c = c_\alpha > 0\)). In particular \(X_{\frac{1}{2}} = |N|\) where \(N\) is a centered Gaussian random variable on \(\mathbb{R}\).

**Proposition 6.1.** Suppose that \((X, \mathcal{B}, m, T)\) is a quasi finite, pointwise dual ergodic measure preserving transformation and assume that the return sequence \(a_n = a_n(T)\) is regularly varying with index \(\alpha \ (\alpha \in [0, 1])\).

If \(\xi \subset \mathcal{B}\) is quasi finite, then

\[
\frac{1}{a_n(T)} \log \frac{1}{m(\xi \cap (T)^n(x))} \xrightarrow{\text{d}} h(T, \xi)X_\alpha
\]

as \(n \to \infty\).

**Proof** This follows from proposition 4.1 and \((\xi)\). \(\square\)

**Example 6.2:** Boole’s transformation.

Let \((X, \mathcal{B}, m, T)\) be given by \(X = \mathbb{R}, \ m = \text{Lebesgue measure and } Tx = x - \frac{1}{2},\) then \(T\) (see [A]) is a pointwise dual ergodic, measure preserving transformation with \(a_n(T) \sim \frac{\sqrt{2\pi}}{n}\), so \(\mathcal{F}_T \neq \emptyset\) and \(T\) is LLB, whence quasi finite.

- By Proposition 6.1, if \(\xi \subset \mathcal{B}\) is quasi finite, then

\[
\frac{1}{a_n(T)} \log \frac{1}{m(\xi \cap (T)^n(x))} \xrightarrow{\text{d}} h(T, \xi)|N|
\]

as \(n \to \infty\).
§7 Analogous properties of probability preserving transformations

The last part of this paper is devoted to the construction of an ergodic, probability preserving transformation having a generating partition with properties analogous to (C2). The “measure theoretic invariant” related to this is entropy dimension as in [FP].

Let \((\mathbb{T}, T, m_\mathbb{T}, R)\) be an irrational rotation of the circle (equipped with Borel sets and Lebesgue measure).

Let \(f \in L^2(\mathbb{T})\) satisfy the weak invariance principle i.e. \(B_n(t) \to B(t)\) in distribution on \(C([0,1])\) where \(B\) is Brownian motion and

\[
B_n(t) := f_{[nt]-1} + (nt - [nt])f \circ T^{[nt]}
\]

(where \(f_k := \sum_{j=0}^{k-1} f \circ R^j\)). Existence of such \(f \in L^2(\mathbb{T})\) is shown in [V].

- In particular,

\[
\frac{\sum_{i=1}^n f}{\sqrt{n}} \to |\mathcal{N}|, \quad \frac{\sum_{i=1}^n f \circ T}{\sqrt{n}} \to \mathcal{R}
\]

where \(R_n := \max_{1 \leq k \leq n} f_k, L_n := \min_{1 \leq k \leq n} (-f_k)\) and \(\mathcal{R} := \max_{t \in [0,1]} B(t) - \min_{t \in [0,1]} B(t)\).

The random variable \(\mathcal{R}\) is known as the range of Brownian motion. Its (non-Gaussian) distribution of is calculated in [Fe].

Let \((Y, \mathcal{C}, \mu, S)\) be the 2-shift with generating partition \(Q = \{Q_0, Q_1\}\) and symmetric product measure.

Let \(\rho : Y \to \mathbb{R}\) be defined by \(\rho = \alpha_0 1_{Q_0} + \alpha_1 1_{Q_1}\) where \(\alpha_0 < \alpha_1, \int_Y \rho \mu = 1\) and \(\alpha_0, \alpha_1\) are rationally independent, then the special flow (under \(\rho\)) \((Y^\rho, \mathcal{C}^\rho, q, S^\rho)\) is Bernoulli where

\[
Y^\rho := \{(y, s) : y \in Y, s \in [0, \rho(y))\}, \quad \mathcal{C}^\rho := \mathcal{C} \times \text{Lebesgue}, \quad q := \mu \times \lambda,
\]

and

\[
S^\rho_t (y, s) := (S^\alpha y, s + t - \rho_n(y))
\]

where \(0 \leq s + t - \rho_n(y) < \rho(S^n y), \rho_n := \sum_{j=0}^{n-1} \rho \circ S^j\).

- Note that the “vertical” partition \(Q := \{Q_0, Q_1\}\) where \(Q_i := Q_i \times [0, \alpha_i)\) \((i = 0, 1)\) generates \(\mathcal{C}\) under \(S^\rho\).

Define the probability preserving transformation \((X, \mathcal{B}, m, T)\) by

\[
(\mathcal{B}) \quad X := \mathbb{T} \times Y^\rho, \quad m = m_\mathbb{T} \times q, \quad \mathcal{B} := \mathcal{T} \times \mathcal{C}^\rho, \quad T(x, (y, s)) := (R(x), S^\rho_{t(x)}(y, s)).
\]

For \(P\) a finite partition of \(\mathbb{T}\) into intervals (which generates \(\mathcal{T}\) under \(R\)), define the partition \(\xi = \xi_P\) of \(X\) by

\[
(\xi) \quad \xi(\omega, y, s) := P(\omega) \times \mathcal{V}(t \in (0, f(\omega)) S^\rho_{t(x)}(y, s)
\]

where for \(x, y \in \mathbb{R}, t(x, y) := \lfloor x + y \rfloor \) (the closed interval joining \(x\) and \(y\)).

Next, we show that that \(\xi\) is measurable and \(H(\xi) < \infty\).
Proposition 7.1.
The partition $\xi$ is measurable, generates $\mathcal{B}$ under $T$, $H(\xi) < \infty$ and

\[ \frac{1}{\sqrt{n}} I(\xi_0^{n-1}(T)) \xrightarrow{b} h(S^pR) \]

where $R$ is the range of Brownian motion.

Proof The proof is in stages. We claim first that

\[ \xi_0^{n-1}(T)(\omega, y, s) = P_0^{n-1}(R)(\omega) \times \left( \bigvee_{t \in [-L_n(\omega), R_n(\omega)]} S^p_{-tQ}(y, s) \right). \]

Proof of (\textbullet): Note that for $n \geq 1$,

\[ (T^{-n}\xi)(\omega, y, s) = \xi(R^n(\omega), S^p_{f_n(\omega)}(y, s)) \]

\[ = P(R^n(\omega)) \times \left( \bigvee_{t \in (f(R^n(\omega)), f_n(\omega) + f(R^n(\omega)))} S^p_{-tQ}(y, s) \right) \]

To continue, we need the following (elementary) proposition:

Let $a_n \in \mathbb{R}$ $(n \geq 1)$ then $\bigcup_{k=0}^{n-1} \{t_k, s_{k+1}\} = [m_n, M_n]$ where $a_0 := 0$, $s_n := \sum_{k=0}^{n} a_k$, $m_n := \min_{0 \leq k \leq n} s_k$, $M_n := \max_{0 \leq k \leq n} s_k$.

To finish the proof of (\textbullet):

\[ \xi_0^{n-1}(T)(\omega, y, s) = \bigvee_{k=0}^{n-1} T^{-k}\xi(\omega, y, s) \]

\[ = \bigcap_{k=0}^{n-1} P(R^k(\omega)) \times \left( \bigvee_{t \in (f_k(\omega), f_{k+1}(\omega))} S^p_{-tQ}(y, s) \right) \]

Now consider $\rho_n : Y \to \mathbb{R}$ defined by

\[ \rho_n(y) := \begin{cases} 
\sum_{k=0}^{n-1} \rho(S^k y) & n > 0, \\
0 & n > 0, \\
\sum_{k=1}^{\lvert n \rvert} \rho(S^{-k} y) & n < 0,
\end{cases} \]
then \( \rho_n(y) < \rho_{n+1}(y) \) and \( \forall \ y \in Y \), \( \rho_n(y) \to \pm \infty \) as \( n \to \pm \infty \).

For \( y \in Y \) and \( t \in \mathbb{R} \), define \( \lceil t \rceil_y \in \mathbb{Z} \) be so that \( \rho_{\lceil t \rceil_y}(y) \leq t < \rho_{\lceil t \rceil_y + 1}(y) \).

It follows that for \( t \in \mathbb{R} \):

- \( \frac{|t|}{\alpha_1} - 1 \leq \lceil t \rceil_y \leq \frac{|t|}{\alpha_1} \), and
- \( S^0_t(y, s) = ([S^{\lceil t \rceil_y} + t, y, s - \rho_{\lceil t \rceil_y + 1}(y)] \).

Our next claim is that

\[
\mathcal{S}_{0}^{-1}(T)(\omega, y, s) = P_{0}^{-1}(R)(\omega) \times Q_{\text{y} = [S^{\lceil t \rceil_y} + t, y, s - \rho_{\lceil t \rceil_y + 1}(y)]}(S)(y) \times \eta_n(\omega, y)(s)
\]

where for each \( (\omega, y) \in \Omega \times Y \), \( \eta_n(\omega, y) \) is a partition of \([0, \rho(y)]\) into at most \( \frac{R_{n}(\omega)+L_{n}(\omega) + 1}{\alpha_0} \) intervals.

**Proof of (\zeta).**

Fixing \((\omega, y, s) \in X \) and \( n \geq 1 \), we have

\[
\left( \bigvee_{t \in [-L_n(\omega), R_n(\omega)]} S^0_t \right)(y, s) = \bigcap_{t \in [-L_n(\omega), R_n(\omega)]} \mathcal{Q}(S^0_t(y, s))
\]

where for each \((\omega, y) \in \Omega \times Y \), \( \eta_n(\omega, y) \) is a partition of \([0, \rho(y)]\) into at most \( \frac{R_{n}(\omega)+L_{n}(\omega) + 1}{\alpha_0} \) intervals. \( \square \) (\zeta).

- **Observation of (\zeta) with \( n = 1 \) shows that**

\[
\xi(\omega, y, s) := \rho_n(y) \times Q_{\nu_+ - \nu_-, \nu_+ - \nu_-}(S)(y) \times \eta_1(\omega, y)(s)
\]

where

\[
\nu_+(w, y, s) = [s + \rho(w) \lor 0]_y, \quad \nu_-(w, y, s) = [s + \rho(w) \land 0]_y.
\]

Thus, \( \xi \) is measurable.

Moreover, writing \( Z := \{[\nu_- = k, \nu_+ = \ell] : k, \ell \in \mathbb{Z} \} \), we see that

\[
I(\xi | Z)(\omega, y, s) = I(P)(\omega) + I(Q_{\rho_\nu_+} \times \rho_\nu_-)(S)(y) + I(\eta_1)(\omega, y)(s)
\]

where

\[
1 + \frac{1 + |f(\omega)|}{\alpha_0} \leq \frac{1 + |f(\omega)|}{\alpha_0} \leq \frac{1 + |f(\omega)|}{\alpha_0} \cdot \log 2 + \log \frac{1 + |f(\omega)|}{\alpha_0}
\]

and

\[
H(\xi | Z) \leq H(P) + \frac{\log 2}{\alpha_0} \cdot \|f\| + \int_{\Omega} \frac{1 + |f(\omega)|}{\alpha_0} dm < \infty.
\]
Now \( |\nu_\pm(\omega, y, s)| \leq \frac{|I(\omega)|+1}{\alpha n} \) and

\[
(\nu_+(w, y, s), \nu_-(w, y, s)) = \begin{cases} 
([s + f(\omega) \lor 0], 0) & f(\omega) \geq 0, \\
(0, [s + f(\omega) \land 0]) & f(\omega) < 0;
\end{cases}
\]

whence using (\*\*) (see page 6) \( H(\mathcal{Z}) < \infty \) and

\[
H(\xi) = H(\xi | \mathcal{Z}) + H(\mathcal{Z}) < \infty.
\]

- Since \( \xi \) is measurable, (\*\*) now shows that it generates \( \mathcal{B} \) under \( T \).
- To establish (\#\#), we claim that for a.e. \((x, y, s)\), for any \( \epsilon > 0 \), for sufficiently large \( n = n(x, y, s) \),

\[
P_0^n \cap (R)(x) \times Q \cap \frac{R_n(x)(1+\epsilon)}{\alpha n}(S,y) \times \eta_n(x,y)(s) \subseteq \epsilon n^{-1}(T)(x, y, s)
\]

(\#\#)

where for each \((\omega, y) \in \Omega \times Y\), \( \eta_n(\omega, y) \) is a partition of \([0, \rho(y)]\) into at most \( \frac{R_n(\omega) + L_n(\omega) + 1}{\alpha n} \)

Proof of (\#\#): For a.e. \((x, y, s) \in X, R_n(x), L_n(x) \uparrow \infty \) and \( \rho_n(y) \sim n \), whence \( [s - L_n(x)]y \sim L_n(\omega) \) and \([s + R_n(\omega)]y \sim R_n(x)\). (\#\#) follows from (\*\*) using

- We claim next that \( \forall (x, y) \in T \times Y \),

(\*\*) \[
\frac{1}{\sqrt{n}} (I(P_0^{n-1}(R)) + I(\eta_n(x,y)) \xrightarrow{m} 0.
\]

Proof \( \#\eta_n(x,y) \leq E_n(x) := \frac{R_n(x)+L_n(x)+1}{\alpha n} \) and \( \#P_0^{n-1}(R) \leq Mn \) for some

\( M > 0 \), \( \forall n \geq 1 \), whence

\[
m(I(P_0^{n-1}(R)) \geq t\sqrt{n}) \leq \frac{1}{t\sqrt{n}} H(P_0^{n-1}(R)) \leq \frac{\log n}{t\sqrt{n}} \xrightarrow{m} 0 \text{ as } n \to \infty
\]

and \( \forall (x, y) \),

\[
m(I(\eta_n(x,y)(s))) \geq t\sqrt{n}) \leq \frac{1}{t\sqrt{n}} H(\eta_n(x,y)) \leq \frac{\log E_n(x)}{t\sqrt{n}} \xrightarrow{m} 0 \text{ as } n \to \infty
\]

proving (\*\*). \( \checkmark \)

Using (\#\#), (\*\*) and (\#\#) for \( S \) we have, as \( n \to \infty \),

\[
\frac{1}{\sqrt{n}} I(\xi_0^{n-1}(T))(x, y, s) = \frac{1}{\sqrt{n}} I(Q \cap \frac{R_n(x)(1+\epsilon)}{\alpha n}(S,y) + O(\frac{\log n}{\sqrt{n}}) = \frac{1}{\sqrt{n}}(L_n(x) + R_n(x)) \log 2(1 + o(1)) + O(\frac{\log n}{\sqrt{n}}) \xrightarrow{\delta} \mathcal{R} \log 2 = \mathcal{R} h(S^\rho). \quad \Box (\#\#)
\]
Estimation of entropy dimension.

Let \((Z, \mathcal{D}, \nu, R)\) be a probability preserving transformation and let \(P \subset \mathcal{D}\) be a countable partition of \(Z\).

As in [FP], let for \(n \geq 1\), \(\epsilon > 0\),
\[
\alpha(n, P, a, \epsilon) := \bigcup_{a' \in P_0^{n-1}(R), \overline{d}(a, a') < \epsilon} a
\]
where \(\overline{d}(a, a') := \frac{1}{n} \# \{0 \leq k \leq n - 1 : a_k \neq a'_k\}\) is Hamming distance, and let
\[
K(P, n, \epsilon) := \min \{\# F : F \subset P_0^{n-1}(R), \nu(\bigcup_{a \in F} B(n, P, a, \epsilon)) > 1 - \epsilon\}.
\]

- The ergodic, probability preserving transformation is said to have *upper entropy dimension* \(\Delta \in [0, 1]\) if for some countable, measurable generating partition \(P\) with finite entropy (and hence – as proved in [FP]– for all such),
\[
\lim_{n \to \infty} \frac{\log \log K(P, n, \epsilon)}{\log n} \to \Delta.
\]

**Proposition 7.2.** Let \((X, B, m, T)\) be as in (3), then the upper entropy dimension is at most \(\frac{1}{2}\).

**Proof** Let \(\xi = \xi_P\) be as in (5) and let \(h = h(S^p)\). For \(n \geq 1\), \(J \subset \mathbb{R}_+\) an interval bounded away from 0 and \(\infty\), define \(\xi_n(J) := \{a \in \xi_0^{n-1}(T) : \frac{1}{\sqrt{n}} \log \frac{1}{m(a)} \in hJ\}.

We claim that \(\# \xi_n(J) \sim E(1_J(R)) e^{hR \sqrt{n}} e^{c(n)}\) as \(n \to \infty\).

**Proof** Suppose that \(J = [r - \delta, r + \delta]\), then
\[
P(R \in J) \sim m([\frac{1}{\sqrt{n}} I(\xi_0^{n-1}(T) \in hJ)]) = \sum_{a \in \xi_n(J)} m(a) = \# \xi_n(J) e^{-h \sqrt{n}(r \pm \delta)}
\]
(because \(m(a) = e^{-h \sqrt{n}(r \pm \delta)} \forall a \in \xi_n(J)\)); whence
\[
E(e^{h \sqrt{n} R - 2\delta}) 1_J(R) \lesssim \# \xi_n(J) \lesssim E(e^{h \sqrt{n} (R + 2\delta}) 1_J(R)).
\]
Using this on a decomposition of \(J\) into a finite union of disjoint short enough intervals proves \(\# \xi_n(J) = E(e^{h \sqrt{n} R} 1_J(R)) e^{\pm \epsilon \sqrt{n}} \forall \epsilon > 0\), whence \(\Box\).

- Evidently \(K(\xi, n, \epsilon) \leq \# \xi_n([\frac{1}{\sqrt{n}} M])\) for some \(M = M_\epsilon > 0\) whence \(K(\xi, n, \epsilon) \leq e^{c_\epsilon \sqrt{n}(1+o(1))}\) and \(\lim_{n \to \infty} \frac{\log \log K(\xi, n, \epsilon)}{\log n} \leq \frac{1}{2} \forall \epsilon > 0\). \(\Box\)

**Remark on the lower bound.**

The upper estimate for the entropy dimension follows from the the weak invariance principle for the “random walk” \(f_n\). In a similar manner, a lower estimate would follow from an analogous result for the “local time” of the random walk. Such a result is not available for the present example. However, such considerations show that the “relative entropy dimension” of an aperiodic, centered random walk in random scenery over its Bernoulli factor is \(1/2\).
REFERENCES

[A] Aaronson, Jon., *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50., American Mathematical Society, Providence, RI, 1997.

[A1] ———, *The intrinsic normalising constants of transformations preserving infinite measures*, J. D’Analyse Math. 49 (1987), 239-270.

[AL] ———; Lemańczyk, M., *Exactness of Rokhlin endomorphisms and weak mixing of Poisson boundaries*, Contemp. Math. 385 (2005), 77-87.

[Ab] Abramov, L.M., *Entropy of a derived automorphism*, Doklady Akad. Nauk SSSR 128 (1959), 647-650 [Russian]; Amer. Math. Soc. Transl. Ser. II 49 (1960), 162-176.

[AR] ———; Rohlin, V. A., *Entropy of a skew product of mappings with invariant measure*, Vestnik Leningrad. Univ. 17 (1962), No. 7, 5-13.

[Br] Breiman, L., *The individual ergodic theorem of information theory*, Ann. Math. Statist. 28 (1957), 809-811.

[C] Chung, K. L., *A note on the ergodic theorem of information theory*, Ann. Math. Statist. 32 (1961), 612-614.

[Fe] Feller, W., *The asymptotic distribution of the range of sums of independent random variables*, Ann. Math. Statistics 22 (1951), 427-432.

[FP] Ferenczi, S; Park, K. K, *Entropy dimensions and a class of constructive examples*, Discrete Contin. Dyn. Syst. 17 (2007, no. 1), 133–141.

[Fr] Friedman, N. A., *Introduction to Ergodic Theory*, Van Nostrand Reinhold, New York, 1970.

[HJK] Hajian, Arshag; Ito, Yuji; Kakutani, Shizuo, *Invariant measures and orbits of dissipative transformations*, Advances in Math. 9 (1972), 52–65.

[IT] Ionescu Tulcea, A., *Contributions to information theory for abstract alphabets*, Ark. Mat. 4 (1961), 235–247.

[KS] Klimko, E. M.; Sucheston, Louis, *On convergence of information in spaces with infinite invariant measure*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 10 (1968), 226–235.

[K1] Krengel, Ulrich, *Entropy of conservative transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 7 (1967), 161–181.

[K2] ———, *On certain analogous difficulties in the investigation of flows in a probability space and of transformations in an infinite measure space*, Functional Analysis (Caroll O’Wilde, ed.), Proc. Sympos., Monterey, Calif., 1969, Academic Press, New York, 1970, pp. 75–91.

[LL] Lemańczyk, M.; Lesigne, E., *Ergodicity of Rokhlin cocycles*, J. Anal. Math. 85 (2001), 43–86.

[M] McMillan, B., *The basic theorems of information theory*, Ann. Math. Statistics 24 (1953), 196–219.

[MP] Mela, X; Petersen, K, *Dynamical properties of the Pascal adic transformation*, Ergod. Th. and dynam. sys. 25 (2005), 227–256.

[Pa] Parry, W., *Entropy and generators in ergodic theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Pl] Pinsker, M. S., *Dynamical systems with completely positive or zero entropy*, Dokl. Akad. Nauk SSSR 135, 1025-1026 (Russian); Soviet Math. Dokl. 1 (1960), 937–938.

[S] Shannon, C. E., *A mathematical theory of communication*, Bell System Tech. J. 27 (1948), 379–423, 623–656.

[V] Voiculescu, D., *Invariance Principles and Gaussian Approximation for Strictly Stationary Processes*, Trans. Amer. Math. Soc. 351 (1999), No. 8, 3351-3371.

[Aaronson] School of Math. Sciences, Tel Aviv University, 69978 Tel Aviv, Israel.

E-mail address: aaro@tau.ac.il

[Park] Dept. of Math., Ajou University, Suwon 442-729, South Korea.

E-mail address: kkpark@ajou.ac.kr