Ground Motion of two triangular hills and a semi-cylindrical canyon near a cavity

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Abstract. Wave function expansion method is developed to study ground motion of two triangular hills and a semi-cylindrical canyon near embedded cavity by incident SH-waves. Firstly the analytical model is divided into 3 parts and the displacement fields in the three regions are constructed. Based on the “conjunction” condition at shared boundary, three districts are then conjoined. Combined with the free-stress condition of semi-cylindrical canyon, a series of infinite algebraic equations is obtained. Lastly, some numerical examples are provided to discuss mechanics law of ground motion impacted by different parameters.

1. Introduction
Local terrain has been demonstrated to be the key one to cause surface earthquake damage or ground motion. Study on estimating the dependence of seismology on local terrain has always been a growing hotspot. The change pattern of terrain mainly involves hills and canyons. In the last 40 years, the methods of wave function, moving coordinate system, complex variables and the region-matching technique have been developed in a certain amount of literatures[1-9] on ground motion of surface geometry such as canyons and hills incident by SH-waves, plane P waves or Rayleigh waves, which have provided theoretical guidance for anti-seismic and antidetonation engineering.

Research on estimating the effects of local topography near shallow buried cavities on ground motion is an important research topic in earthquake engineering. On the basis of the theoretical results obtained by Yang and Xu’s work[9], ground motion of two scalene triangular hills and a semi-cylindrical canyon near an embedded cavity under incident SH-waves is further studied.

2. Analytical model
As Figure 1 given is an elastic half-space model of two scalene triangular hills and a semi-cylindrical canyon near an embedded cavity. To solve SH-waves scattering problem by this complex terrain, the key is to derive the governing equations of SH-waves which should satisfy stress boundary condition at bevel edge $C_1$ and $C_2$ of the hills, the edge of the cavity and the semi-cylindrical canyon, and the horizontal surface $S$. Based on “division”, the analytical model is divided into three parts as shown as Figure 2. At the shared boundary, the stresses and displacements satisfy continuous condition.
Suppose that the distance between \( O_1 \) and \( O_2 \) (\( O_3 \) and \( O_4 \)) are \( d_1 \), the distance between \( O_1 \) and \( O_3 \) are \( d_2 \), \( R_1 = R_3 \), and the projection of \( |O_1O_2| \) and \( |O_3O_4| \) on the bottom are \( \Delta_1 \) and \( \Delta_2 \), and \( R_1 = 1.0, R_2 = 0.5 \), then

\[
d_1 = R_1 + R_3,
\]

\[
d_2 = 2R_1 + 2R_3, \quad \Delta_1 = (n_2 - n_1)/(n_2 + n_1), \quad \Delta_2 = (n_4 - n_3)/(n_4 + n_3).
\]

**Figure 1.** Analytical model of two triangle hills and a semi-cylindrical canyon near an embedded cavity

**Figure 2.** The division of the solution domain

3. Fundamental theory

3.1. Governing equation

In an isotropic, uniformity and continuous medium, the displacement field \( W \) excited by incident SH-waves should satisfy the governing equation at complex plane \((Z, \bar{Z})\)

\[
\frac{\partial^2 W}{\partial Z \partial \bar{Z}} + \frac{1}{4} k^2 W = 0,
\]

wherein, the dependency relationship between \( W \) and the time \( t \) is \( \exp(-i\omega t) \) (omitted hereinafter), \( k = \omega/c_s \), in which \( \omega \) denotes the circular frequency of wave function, \( c_s = \sqrt{\mu/\rho} \) the propagation
velocity of the shear wave, and $\rho$ and $\mu$ the mass density and shear modulus of the medium, respectively.

The expressions of corresponding stresses in polar coordinates system take the form of

$$\tau_{r\theta} = \mu \left( \frac{\partial W}{\partial r} e^{i\theta} + \frac{\partial W}{\partial \theta} e^{-i\theta} \right), \tau_{\theta \theta} = i \mu \left( \frac{\partial W}{\partial r} e^{i\theta} - \frac{\partial W}{\partial \theta} e^{-i\theta} \right).$$

(2)

3.2. Standing waves in domain I and III.

In domain I, under incident SH-waves, the standing wave $W^0_{\text{I}}$, namely the total wave field, should obey the stress free condition at bevel edge

$$\tau_{r\theta}^{0}\left(\theta = \left(\theta_i + \theta_j\right)/2\right) = 0, \quad \tau_{\theta \theta}^{0}\left(\theta = -\left(\theta_i + \theta_j\right)/2\right) = 0.$$  

(3)

In complex coordinates $(Z, \overline{Z})$ and $(\overline{Z}, Z_i)$, the expression of $W^0_{\text{I}}$ and the associated stress expression $\tau_{r\theta}^{0}\left(Z_i, \overline{Z}_i\right)$ satisfying Eq. (1) and boundary condition equation (3) can be defined in Reference [9].

Similarly, in the complex coordinate system $(Z, \overline{Z})$, the standing wave $W^0_{\text{III}}\left(Z_i, \overline{Z}_i\right)$ in domain III and the corresponding stresses $\tau_{r\theta}^{0}\left(Z_i, \overline{Z}_i\right)$ have also been derived in Reference[].

3.3. Scattering waves in domain II

In domain II, the scattering waves generated by three semi-cylindrical canyons (shown as Figure 2), which should satisfy stress free boundary condition on the horizontal surface $S_i$, can be respectively constructed in the complex system $(Z, \overline{Z}_i)$ ($i=1, 2, 3$). The expressions of the scattering waves $W^{s1}(Z_i, \overline{Z}_i), W^{s2}(Z_i, \overline{Z}_i)$ and $W^{s3}(Z_i, \overline{Z}_i)$ along with the corresponding stresses of the scattering waves can be defined in Reference[9].

Besides, the other scattering waves will be excited by the embedded cavity, in complex plane $(Z, \overline{Z}_i)$, which takes the form of

$$W_{i}^{(s)}(\xi_i, \overline{\xi}_i) = W_0 \sum_{m=-\infty}^{\infty} \left[ F_m^{(1)} \left( H^{(1)}_{2m} \left( k \xi_i - h_i \right) \right) \left( \xi_i - h_i \right)^{2m} + H^{(1)}_{2m} \left( k \xi_i + h_i \right) \left( \xi_i + h_i \right)^{2m} \right] + F_m^{(2)} \left( H^{(1)}_{2m+1} \left( k \xi_i - h_i \right) \right) \left( \xi_i - h_i \right)^{2m+1} - H^{(1)}_{2m+1} \left( k \xi_i + h_i \right) \left( \xi_i + h_i \right)^{2m+1} \right],

(4)

Substituting above expression into Eq. (2), we can obtain the corresponding stress $\tau_{r\theta}^{(s)}\left(\xi_i, \overline{\xi}_i\right)$.

3.4. Incident wave and reflected wave.

Given as Figure 1, in $XOY$ coordinate system, SH-waves are incident with an angle $\alpha$. In complex plane $(Z, \overline{Z}_i)$ ($i=1, 2, 3, 4$), the expressions of the incident wave and reflected wave take the form of

$$W^{ir}\left(Z_i, \overline{Z}_i\right) = W_0 \left\{ 2 J_0 \left( k \xi_i \right) + 2 \sum_{m=1}^{\infty} (-1)^n J_{2m} \left( k \xi_i \right) \cos 2n\alpha \left( \frac{\xi_i}{\left| \xi_i \right|} \right)^{2m} + \left( \frac{\xi_i}{\left| \xi_i \right|} \right)^{2m} \right\}.$$
\[
+2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(k, |\xi|) \sin(2m+1)\alpha \left[ \left( \frac{\xi}{|\xi|} \right)^{2n+1} - \left( \frac{\xi}{|\xi|} \right)^{-(2n+1)} \right]
\]

(\(\xi_i = Z_i, Z_2 + id_1, Z_3 + id_2, Z_4 + h_i; \ i = 1, 2, 3, 4\)).

Substitution of Eq. (5) into Eq. (2), the relevant stresses \(\tau_{xx}'(Z_i, \bar{Z}_j)\) can be obtained.

4. Solutions

Based upon the “conjunction” idea, the displacement and stress must be guaranteed to be continuous at the shared boundary of domain II (domain III and II), and combined with zero-stress condition on boundary edge of the semi-cylindrical canyon and the embedded cavity, a series of definite equations for solving unknown coefficients \(D_n^{(j)}, E_n^{(j)}, A_n^{(j)}, B_n^{(j)}, C_n^{(j)}\) and \(F_m^{(j)}\) \((j = 1, 2)\) can be derived

\[
\begin{align*}
W^{O1}(Z_i, \bar{Z}_i) &= W^{S1}(Z_i, \bar{Z}_i) + W^{S2}(Z_i, \bar{Z}_i) + W^{S3}(Z_i, \bar{Z}_i) + W^{S4}(Z_i, \bar{Z}_i) + W^{S5}(Z_i, \bar{Z}_i) + W^{S6}(Z_i, \bar{Z}_i) + W^{S7}(Z_i, \bar{Z}_i) + W^{S8}(Z_i, \bar{Z}_i) \quad Z_i \in D_1, \\
\tau_{xx}^{O1}(Z_i, \bar{Z}_i) &= \tau_{xx}^{S1}(Z_i, \bar{Z}_i) + \tau_{xx}^{S2}(Z_i, \bar{Z}_i) + \tau_{xx}^{S3}(Z_i, \bar{Z}_i) + \tau_{xx}^{S4}(Z_i, \bar{Z}_i) + \tau_{xx}^{S5}(Z_i, \bar{Z}_i) + \tau_{xx}^{S6}(Z_i, \bar{Z}_i) + \tau_{xx}^{S7}(Z_i, \bar{Z}_i) + \tau_{xx}^{S8}(Z_i, \bar{Z}_i) \quad Z_i \in D_1, \\
W^{O2}(Z_i, \bar{Z}_i) &= W^{S1}(Z_i, \bar{Z}_i) + W^{S2}(Z_i, \bar{Z}_i) + W^{S3}(Z_i, \bar{Z}_i) + W^{S4}(Z_i, \bar{Z}_i) + W^{S5}(Z_i, \bar{Z}_i) + W^{S6}(Z_i, \bar{Z}_i) + W^{S7}(Z_i, \bar{Z}_i) + W^{S8}(Z_i, \bar{Z}_i) \quad Z_i \in D_2, \\
\tau_{xx}^{O2}(Z_i, \bar{Z}_i) &= \tau_{xx}^{S1}(Z_i, \bar{Z}_i) + \tau_{xx}^{S2}(Z_i, \bar{Z}_i) + \tau_{xx}^{S3}(Z_i, \bar{Z}_i) + \tau_{xx}^{S4}(Z_i, \bar{Z}_i) + \tau_{xx}^{S5}(Z_i, \bar{Z}_i) + \tau_{xx}^{S6}(Z_i, \bar{Z}_i) + \tau_{xx}^{S7}(Z_i, \bar{Z}_i) + \tau_{xx}^{S8}(Z_i, \bar{Z}_i) \quad Z_i \in D_2, \\
\tau_{xx}^{O1}(Z_2, \bar{Z}_2) &= \tau_{xx}^{S1}(Z_2, \bar{Z}_2) + \tau_{xx}^{S2}(Z_2, \bar{Z}_2) + \tau_{xx}^{S3}(Z_2, \bar{Z}_2) + \tau_{xx}^{S4}(Z_2, \bar{Z}_2) + \tau_{xx}^{S5}(Z_2, \bar{Z}_2) + \tau_{xx}^{S6}(Z_2, \bar{Z}_2) + \tau_{xx}^{S7}(Z_2, \bar{Z}_2) + \tau_{xx}^{S8}(Z_2, \bar{Z}_2) = 0 \quad Z_2 \in D_3, \\
\tau_{xx}^{O1}(Z_2, \bar{Z}_3) &= \tau_{xx}^{S1}(Z_2, \bar{Z}_3) + \tau_{xx}^{S2}(Z_2, \bar{Z}_3) + \tau_{xx}^{S3}(Z_2, \bar{Z}_3) + \tau_{xx}^{S4}(Z_2, \bar{Z}_3) + \tau_{xx}^{S5}(Z_2, \bar{Z}_3) + \tau_{xx}^{S6}(Z_2, \bar{Z}_3) + \tau_{xx}^{S7}(Z_2, \bar{Z}_3) + \tau_{xx}^{S8}(Z_2, \bar{Z}_3) = 0 \quad Z_2 \in D_3.
\end{align*}
\]

Insert the corresponding displacement and stress expressions into above equations, multiply both of sides with \(e^{-i\omega t}\), and above equations are turned into Fourier series which can be solved by utilizing the limitation truncation to satisfy the computation precision.

5. Displacement amplitudes

Under incident SH-waves, the total displacement fields in domain land III include two standing waves,

\[
W_i = W^{O1} + W^{O2}.
\]

In domain II, the total wave field involves five displacement fields

\[
W_\| = W^{S1} + W^{S2} + W^{S3} + W^{S4} + W^{S5}.
\]

6. Numerical results and discussions

For numerical example, as shown as Figure 1, suppose that the amplitude of incident wave \(W_0 = 1.0\), \(R_1 = R_2 = R_3 = 1.0\), \(\rho_1 = \rho_2 = \rho_3\), \(\mu_1 = \mu_2\), \(k_1 = k_2\), \(n_1 = n_2\), \(n_1 = n_2\), \(h_1 = h_2\), \(\Delta_1 = \Delta_2\). In the data graphs, \(y_i / R_i \leq -1.0\) represents the horizontal surface at the left hand side and \(4 \leq y_i / R_i \leq 5\) the right hand side, \(|y_i / R_i| < 1.0\) the first hill, \(1 \leq y_i / R_i < 2\) the canyon, \(2 \leq y_i / R_i < 4\) the second hill and \(y_i / R_i = \Delta_i\) the vertex of the first triangular hill. The vertex angles of triangular hill are defined as \(\theta_0\). Based on above theoretical derivation, the scattering problem of two scalene triangular hills and a semi-cylindrical canyon near an embedded cavity under incident SH-waves is reported. In this part, the influence of different parameters on surface displacements is discussed, and a comparison is made with the obtained results in Yang and Xu’s work\([9]\). The expressions of surface displacements \(|W|\) are defined respectively by Eqs. (7) and (8).

Variation of the surface displacement amplitudes \(|W|\) impacted by different frequency \(\eta\) and incident angles \(\alpha\) when \(\theta_1 = \theta_2 = \theta_3 = 0^\circ, 10^\circ\) and \(20^\circ\) is discussed. From Figure 3, it can be seen that the surface displacement amplitudes \(|W|\) shows manifest static characteristic analysis, namely the
quasi-static characteristic case. When \( \eta = 0.25 \) and 1.0, the distribution of the surface displacement amplitudes reveal more and more distinct dynamic characteristics, as depicted by Figure 4 to 5. From Figure 4(a) and Figure 5(a), it can be observed that the maximum displacement amplitude \( |W|_{\text{max}} \) always occurs at \( y_i/R_i = \Delta_i \) (the vertex of the first triangular hill) when \( \alpha = 0^\circ \) namely SH-waves incident vertically. Additionally, it can be observed from Figure 5 that the surface displacement amplitudes \( |W| \) decrease when \( \eta > 0.5 \). Taking \( \eta = 1.0 \) as an example, \( |W|_{\text{max}} = 6.04, 5.97 \) and 5.12 that correspond to \( \theta_{21} = 0^\circ, 10^\circ \) and 20\(^\circ\), as described in Figure 5(a).

As given by Figure 1, the embedded cavity is supposed be located below the first triangular hill. Observed from the data graphs, the variation of \( |W| \) can be demonstrated to appear more and more evidently, and by comparing with the graphs described in Reference [9], the surface displacement amplitudes of horizontal surface at the left hand side and the first hill seem to be much greater. A possible reason for this is the complex scattering waves excited by the embedded cavity. The producing strong dynamic response can impact manifestly on ground vibration.

![Figure 3](image1.png)

**Figure 3.** Variation of surface displacements \( |W| \) with \( y_i/R_i \) when \( \eta = 0.1 \)

![Figure 4](image2.png)

**Figure 4.** Variation of surface displacements \( |W| \) with \( y_i/R_i \) when \( \eta = 0.25 \)
7. Conclusions
By comparison with the numerical results for the analytical model of two scalene triangular hills and a semi-cylindrical canyon reported in Reference [9], the strong dependence of ground motion of the complex terrain on the existence of the embedded cavity can be revealed from the numerical results obtained in current paper. Therefore, more attention should be paid to the existence of the embedded cavity in anti-seismic study of complex topography.

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