INTRODUCTION

Random Matrix Theory (RMT) has been a useful laboratory for simulating hamiltonians of statistical systems with random matrix elements, due possibly to impurities scattered around the material. An unphysical yet simplest choice is to assume each of matrix elements of the hamiltonian $M$ to be independently derived from the gaussian distribution,

$$Z = \int_{N \times N \text{ matrices}} dM \, e^{-N \text{ tr} \, M^2},$$

(1)
to which is associated Wigner’s single-banded, semi-circle spectrum. This macroscopic spectrum is, however, by no means realistic; physical spectra are far more complicated. Namely they may even have multi-band structures, which in the side of RMT correspond to non-gaussian distributions. Thus, it is only universal quantities insensitive to a chosen matrix measure that may be justifiably extracted from RMT for physical systems.

Various quantities concerning microscopic spectral correlation in the bulk of the spectral band have long been believed universal, as identical results are derived from non-gaussian ensembles corresponding to classical orthogonal polynomials other than Hermite. Here the term ‘microscopic’ refers to that the correlation is measured in the unit of the mean level spacing, which is of order $O(1/N)$ for ensembles of the type (1). This conjectured universality of the sine kernel

$$K_s(\lambda, \lambda') = \frac{\sin \pi(\lambda - \lambda')}{\pi(\lambda - \lambda')},$$

(2)
which comprises all spectral correlators
\[ \rho_s(\lambda_1, \ldots, \lambda_p) = \left\langle \prod_{a=1}^{p} \frac{1}{N} \operatorname{tr} \delta(\lambda_a - M) \right\rangle = \det_{1 \leq a, b \leq p} K_s(\lambda_a, \lambda_b), \]
is proved\(^3,4\) for unitary invariant ensembles with generic single-trace potentials \( e^{-N \operatorname{tr} V(M)} \).

RMT has recently extended its range of applicability toward QCD\(^5\). There the Dirac operator \( i\overline{\Psi}(i\partial_\mu + A_\mu) \), having gauge fields as random elements, is regarded as a random hamiltonian. Schematically, the Euclidean QCD partition function\(\int\)
\[ Z_{\text{QCD}} = \int [dA_\mu d\overline{\Psi}_f d\psi_f] e^{-\frac{1}{g^2} \int F^2 - \int \overline{\psi}_f (i\overline{\Psi}+m_f) \psi_f} \]
is transformed by the change of integration variables
\[ A_\mu \mapsto i\overline{\Psi} A \equiv M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} \]
into a RMT
\[ Z_{\text{RMT}} = \int_{N \times N \text{ matrices}} d\mu(W) \prod_f \det \left( \begin{array}{cc} m_f & -i\sigma_\mu (A_\mu + i\partial_\mu) \\ i\sigma_\mu (A_\mu + i\partial_\mu) & m_f \end{array} \right). \]
Here \( N \) stands for the size of the Dirac operator i.e. the volume of the spacetime, and \( d\mu(W) \) a measure invariant under
\[ W \to U W V^\dagger \quad (U, V \in U(N) \text{ for } W \text{ complex}). \]

Novel features of this RMT in addition to the conventional model \(^1\) are:
- Presence of the fermion determinant. For small \( m_f \), it expels Dirac eigenvalues of order \( O(m_f) \) from the origin.
- Chiral structure of the random matrix. Since \( \{M, \gamma_5\} = 0 \) each eigenvalue \( \lambda \) is accompanied by its mirror image \( -\lambda \). The Coulomb repulsion between these pairs prevents them from populating around the origin.

Among the quantities calculable from this type of RMTs, of particular interest are the distribution and correlation of soft eigenvalues of the Dirac operator as their accumulation is responsible for the spontaneous breaking of the chiral symmetry via\(^6\)
\[ \Sigma = |\langle \overline{\psi} \psi \rangle| = \frac{\pi \rho(0)}{N}. \]
Due to the above two features, correlations of soft eigenvalues are expected to deviate from the universal sine law \(^2\). Verbaarschot and his collaborators\(^5\) have calculated correlations of eigenvalues of order \( \lambda \sim O(1/N) \) from the gaussian ensemble
\[ d\mu(W) = dW e^{-N\Sigma \operatorname{tr} W^\dagger W}, \]
appealing to the conjecture of universality. These analytic predictions have been compared first to the numerical data from the simulation of the instanton liquid model\(^7\), and more recently to that of the lattice gauge theory\(^8,9\). The agreements are impressive.

\(^1\)For simplicity, we consider only the topologically trivial sector.
In this article I shall review the proof of universality of the microscopic spectral correlations of the model (6) for \( m_f = 0 \) and with single-trace potential measures, in order to justify partly the use of gaussian ensemble (9) and to corroborate the above mentioned agreements. Rather than to follow closely the original proof\(^{13,14}\), I shall disguise it with the conventional approach of \( Q, P \) operators\(^{15}\) and with the method employed in a recently proposed alternative proof\(^{16,17}\), in a hope that borrowing notions from quantum mechanics may add the proof some pedagogical flavor. At the end of the article I shall address a problem associated with multi-criticality.

**ORTHOGONAL POLYNOMIAL METHOD**

We start from recalling basic technology of random matrices: the orthogonal polynomial method. We consider the chiral unitary ensemble (\( \chi \)UE)

\[
Z_{\chi \text{UE}} = \int_{2N \times 2N \text{hermite}} dM \ e^{-N \text{tr} \ V(M^2)} |\det M|^\alpha, \quad M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix}
\]

(10)
capturing the global symmetries of \( N_c \geq 3 \) QCD with \( N_f = \alpha \) massless fundamental fermions, as well as the unitary ensemble (UE) without chiral structure

\[
Z_{\text{UE}} = \int_{N \times N \text{hermite}} dM \ e^{-N \text{tr} \ V(M^2)} |\det M|^\alpha
\]

(11)
modeling QCD. We allow non-integer \( \alpha > -1 \), for they can be treated on the same footing as integer cases. Both ensembles allow eigenvalue representations

\[
Z_{\chi \text{UE}} = \int_0^\infty \prod_{i=1}^N (d\lambda_i \lambda_i^\alpha e^{-NV(\lambda_i)}) \left| \det \lambda_j^{i-1} \right|^2, \quad (\lambda_i : \text{eigenvalues of } W^\dagger W)
\]

(12)
\[
Z_{\text{UE}} = \int_{-\infty}^\infty \prod_{i=1}^N (d\lambda_i |\lambda_i|^{\alpha} e^{-NV(\lambda_i^2)}) \left| \det \lambda_j^{i-1} \right|^2. \quad (\lambda_i : \text{eigenvalues of } M)
\]

(13)
Inside the integrals, \( \det \lambda_j^{i-1} \) may be replaced (up to an irrelevant constant) with polynomials \( \det P_{-1}(\lambda_j) \) orthonormal with respect to the measure \( d\lambda |\lambda|^{\alpha} e^{-NV(\lambda)} \):

\[
\delta_{nm} = \int d\lambda |\lambda|^{\alpha} e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = \int d\lambda \psi_n(\lambda) \psi_m(\lambda).
\]

(14)
Here we have introduced one-particle wave functions of a free fermion (at the \( n \)-th state)

\[
\psi_n(\lambda) = |\lambda|^{n/2} e^{-N V/2} P_n(\lambda).
\]

(15)
The fermionic nature comes from the Vandermonde determinant. The wave functions at first \( N \) levels comprise the Dirac sea, i.e. the \( N \)-particle ground-state wave function

\[
\Psi_N(\{\lambda\}) = \det_{1 \leq n,m \leq N} \psi_{n-1}(\lambda_m).
\]

(16)
Using this ground-state wave function, the vacuum expectation value of unitary invariant observables is written as

\[
\langle O \rangle = \int d^N \lambda \Psi_N(\{\lambda\}) O \Psi_N(\{\lambda\}).
\]

(17)

---

\(^{\dagger}\)Microscopic universality for the gaussian potential plus certain terms which breaks the invariance is proved in refs.10, 11. See also ref.12 for a perturbative treatment of a quartic potential.
Namely, spectral correlators are related to $\psi_n$ by

$$\rho_N(\lambda_1, \cdots, \lambda_p) \equiv \left\langle \prod_{a=1}^{p} \frac{1}{N} \text{tr} \delta(\lambda_a - M) \right\rangle_{\text{Wick’s theorem}} \pm \det_{1 \leq a, b \leq p} K_N(\lambda_a, \lambda_b) \quad (18)$$

where $K_N$ is the projector to the Dirac sea:

$$K_N(\lambda, \lambda') = \sum_{n=0}^{N-1} \psi_n(\lambda) \psi_n(\lambda') = q_N \frac{\psi_N(\lambda) \psi_{N-1}(\lambda') - \psi_{N-1}(\lambda) \psi_N(\lambda')}{\lambda - \lambda'} \quad (19)$$

Use is made of Christoffel-Dalboux formula in the last line, and $q_N$ is a constant defined immediately below.

We note that the orthogonality relation (14) for the $\chi$UE case ($\int_0^\infty d\lambda$) can be absorbed into the UE case ($\int_{-\infty}^{\infty} d\lambda$) by the change of variables $\lambda \to \sqrt{\lambda}$ accompanied by the redefinition $\alpha \to \frac{\alpha - 1}{2}$ (or $\alpha \to \frac{\alpha + 1}{2}$) and $P_{2n}(\sqrt{\lambda}) \to P_n(\lambda)$ (or $\sqrt{\lambda} P_{2n+1}(\sqrt{\lambda}) \to P_n(\lambda)$). Thus in the following we need only to consider the UE case, where orthogonal polynomials have definite parities.

Now let us for a while concentrate on $\alpha = 0$, where the logarithmic component of the potential is absent. In this case one may represent the Heisenberg algebra $[\frac{d}{d\lambda}, \lambda] = 1$ by matrices $Q$ and $P$ acting on the Hilbert space of orthonormal wave functions

$$\lambda \psi_n = q_{n+1} \psi_{n+1} + q_n \psi_{n-1} \equiv \sum_m Q_{nm} \psi_m, \quad (20)$$

$$\frac{d}{d\lambda} \psi_n = -\frac{N}{2} V' \psi_n + (\text{linear comb. of } \psi_{n-1}, \psi_{n-3}, \cdots) \equiv \sum_m P_{nm} \psi_m. \quad (21)$$

(Anti-)self adjointness of multiplication and differentiation operators inherits to the (anti-)symmetry of the representing matrices,

$$\int d\lambda \ (\lambda \psi_n) \psi_m = \int d\lambda \ \psi_n (\lambda \psi_m) \Rightarrow Q_{nm} = Q_{mn}, \quad (22)$$

$$\int d\lambda \ \left( \frac{d}{d\lambda} \psi_n \right) \psi_m = -\int d\lambda \ \psi_n \left( \frac{d}{d\lambda} \psi_m \right) \Rightarrow P_{nm} = -P_{mn}. \quad (23)$$

Although eq.(21) and the antisymmetry (23) are enough to determine the $P$ matrix as

$$P_{nm} = \mp \frac{N}{2} V'(Q)_{nm} \ (n < m), \quad (24)$$

we shall not use this fact except for fixing a integration constant in the sequel.

At this stage we consider the situation when the coefficient $q_n$ become single-valued in the limit $n, m \approx N \gg 1$,

$$q_n \sim q_{n+1} \sim q = O(1), \quad (25)$$

a crucial assumption which is violated in matrix ensembles with multi-band spectra. Then the matrix elements of $Q$, $P$ for $n, m = N + O(1)$ take the forms:

$$Q_{nm} = \begin{pmatrix} \vdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & q & 0 & q & \cdots & \cdots \\ q & 0 & q & \cdots & \cdots & \cdots \\ q & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad P_{nm} = N \begin{pmatrix} \vdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & -p_2 & 0 & \cdots & \cdots & \cdots \\ \cdots & 0 & -p_1 & 0 & -p_2 & \cdots \\ \cdots & \cdots & p_1 & 0 & -p_1 & 0 \\ \cdots & \ddots & \cdots & p_2 & 0 & p_1 & 0 & -p_1 \\ \cdots & \cdots & \cdots & \cdots & 0 & p_2 & 0 & p_1 & \cdots \end{pmatrix}. \quad (26)$$
A matrix of the above form of $P$ can always be written as
\[ P = \frac{N}{2} \sum_{k \geq 0} a_k Q^{2k} \Lambda \equiv \frac{N}{2} A(Q) \Lambda, \tag{27} \]
where the $\Lambda$ matrix stands for the antisymmetrizer
\[
\Lambda_{nm} = \begin{pmatrix}
... & 0 & -1 & ... \\
... & 1 & 0 & -1 & ... \\
1 & 0 & -1 & ... \\
1 & ... & ... \\
\end{pmatrix}.	ag{28}
\]
When $\alpha \neq 0$, $d/d\lambda$ cannot be represented on the space of orthogonal polynomials as it involves a term proportional to $1/\lambda$. However one may verify that the operator $d/d\lambda - \frac{(-)^n \alpha}{2\lambda}$ is indeed represented by the same $P$ matrix as in (27). Here $\hat{n}$ is the number operator $\hat{n} \psi_n = n \psi_n$, and the existence of the signature $(-)^n$ preserves the antisymmetry of $P$,
\[
\psi_{2m+1} (-\hat{n} \psi_{2n}) = - (-\hat{n} \psi_{2m+1}) \psi_{2n}.	ag{29}
\]
In the next section eqs. (26) and (27) are utilized to prove the microscopic universality.

**PROOF OF UNIVERSALITY**

Here we re-exhibit the asymptotic forms of recursion relations (26), (27) satisfied by the orthonormal wave functions ($\alpha = 0$):
\[
\lambda \psi_n = q (\psi_{n+1} + \psi_{n-1}), \tag{30}
\]
\[
d/d\lambda \psi_n = \frac{NA(\lambda)}{2} (-\psi_{n+1} + \psi_{n-1}). \tag{31}
\]
They are combined to yield unharmonic analogues of lowering and raising operators of a harmonic oscillator at highly excited levels $n \gg 1$:
\[
\left( \frac{1}{NA(\lambda)} \frac{d}{d\lambda} + \frac{\lambda}{2q} \right) \psi_n = \psi_{n-1}, \tag{32}
\]
\[
\left( -\frac{1}{NA(\lambda)} \frac{d}{d\lambda} + \frac{\lambda}{2q} \right) \psi_n = \psi_{n+1}. \tag{33}
\]
The requirement that two processes $\psi_n \xrightarrow{\psi_{n+1}} \psi_n$ must commute amounts to the suppression of the commutator of the raising and lowering operators in the large-$n$ and $N$ limit,
\[
\left[ \frac{\lambda}{2q} , \frac{1}{NA(\lambda)} \frac{d}{d\lambda} \right] = O\left( \frac{1}{N^*} \right) \ll 1. \tag{34}
\]
Thus we are lead to a second-order differential equation for $\psi_n$,
\[
\left[ \left( \frac{1}{NA(\lambda)} \frac{d}{d\lambda} \right)^2 + 1 - \left( \frac{\lambda}{2q} \right)^2 \right] \psi_n = 0. \tag{35}
\]
As mentioned in the end of the last section, the only modification for $\alpha \neq 0$ case is the replacement

$$\frac{d}{d\lambda} \psi_n \rightarrow \left( \frac{d}{d\lambda} - \frac{(-)^n \alpha}{2\lambda} \right) \psi_n. \quad (36)$$

The resulting equation takes the form

$$\left[ \frac{1}{N^2 A(\lambda)} \left( \frac{d}{d\lambda} + \frac{(-)^n \alpha}{2\lambda} \right) - \frac{1}{A(\lambda)} \left( \frac{d}{d\lambda} - \frac{(-)^n \alpha}{2\lambda} \right) + 1 - \left( \frac{\lambda}{2q} \right)^2 \right] \psi_n = 0. \quad (37)$$

Let us first take the macroscopic (large-$N$) limit:

$$N \rightarrow \infty, \quad \lambda: \text{fixed} \quad (38)$$

of (37). Since $1/N$ plays the role of $\hbar$ in this limit and thus eq.(37) is reduced to

$$\left[ \frac{1}{N^2 A(\lambda)^2} \frac{d^2}{d\lambda^2} + 1 - \left( \frac{\lambda}{2q} \right)^2 \right] \psi_n = 0, \quad (39)$$

we obtain the WKB solution:\footnote{This process was initiated in the context of stochastic quantization of matrix models\cite{Kristjansen}, although their Fokker-Planck equation differs from eq.(37). I thank C. F. Kristjansen for remarks on this point.}

$$\psi_n(\lambda) \propto \cos \left( N \int_0^\lambda d\lambda A(\lambda) \sqrt{1 - \left( \frac{\lambda}{2q} \right)^2 + \frac{n\pi}{2}} \right). \quad (40)$$

Substituting this solution to (39), we obtain the kernel in the large-$N$ limit:

$$K(\lambda, \lambda') \equiv \lim_{N \rightarrow \infty} K_N(\lambda, \lambda') \propto \sin \left( N \int_{\lambda'}^\lambda d\lambda A(\lambda) \sqrt{1 - \left( \frac{\lambda}{2q} \right)^2} \right) \frac{\sin \left( N \int_{-\lambda'}^{-\lambda} d\lambda A(\lambda) \sqrt{1 - \left( \frac{\lambda}{2q} \right)^2} \right)}{\lambda - \lambda'}, \quad (41)$$

$$\rho(\lambda) = K(\lambda, \lambda) = \text{const.} A(\lambda) \sqrt{1 - \left( \frac{\lambda}{2q} \right)^2}. \quad (42)$$

The constant is not fixed within this approach, though it can fixed to be $1/\pi$ if we use the explicit form of the $P$ operator (27). The meanings of $q$ and $A(\lambda)$ become clear at this stage: $\pm 2q$ stand for the edges of the spectrum and $A(\lambda)$ the deviation of the spectral envelope from Wigner’s semi-circle.

Now we proceed to take the microscopic limit:

$$N \rightarrow \infty, \quad \lambda \rightarrow 0, \quad z \equiv N\lambda: \text{fixed}. \quad (43)$$

We have already assumed the single-valuedness of the recursion coefficients $q_n$, which is known to be the sufficient condition for $A(0) = \pi \rho(0) > 0$. (In other words, we have assumed the chiral symmetry breaking as an input.) Then eq.(37) is reduced to

$$\left[ \frac{1}{A(0)^2} \left( \frac{d}{dz} + \frac{(-)^n \alpha}{2z} \right) - \frac{1}{A(0)} \left( \frac{d}{dz} - \frac{(-)^n \alpha}{2z} \right) + 1 \right] \psi_n = 0. \quad (44)$$

Its solution which is regular at $z = 0$ is a Bessel function:

$$\psi_n \propto \sqrt{z} J_{n \pm 1} (A(0)z), \quad \begin{cases} \text{n = even} \\ \text{n odd} \end{cases} \quad (45)$$
Within either even or odd sector of the wave functions, the $n$-dependence enters only through $A_n(0)$. Thus we have

$$\psi_n - \psi_{n-2} \sim \frac{d}{dA(0)}\psi_n \propto z^\frac{n}{2} \left( J_{\frac{n+2}{2}}(A(0)z) - J_{\frac{n+4}{2}}(A(0)z) \right). \quad (n = \text{even}) \quad (46)$$

Substituting the above solutions again into (49), we obtain universal forms of the microscopic kernels (called the Bessel kernels $22$)

$$K_s(z, z') \equiv \lim_{N \to \infty} \frac{1}{N} K_N\left( \frac{z}{N}, \frac{z'}{N} \right), \quad (47)$$

$$\chi \text{UE} : \quad K_s(z, z') = \pi \rho(0) \sqrt{z} \frac{z J_{\alpha+1}(2\pi \rho(0)z) J_{\alpha}(2\pi \rho(0)z') - (z \leftrightarrow z')}{z^2 - z'^2}, \quad (48)$$

$$\text{UE} : \quad K_s(z, z') = \frac{\pi \rho(0)}{2} \sqrt{z} \frac{J_{\alpha+1}(\pi \rho(0)z) J_{\alpha}(\pi \rho(0)z') - (z \leftrightarrow z')}{z - z'}. \quad (49)$$

and microscopic spectral densities $\rho_s(z) = K_s(z, z)$,

$$\chi \text{UE} : \quad \rho_s(z) = (\pi \rho(0))^2 |z| \left( J_{\alpha}^2 - J_{\alpha+1} J_{\alpha-1} \right) (2\pi \rho(0)z), \quad (50)$$

$$\text{UE} : \quad \rho_s(z) = \frac{(\pi \rho(0))^2}{2} z \left( J_{\alpha+1}^2 + J_{\alpha-1}^2 - J_{\alpha+1} J_{\alpha-1} - J_{\alpha+3} J_{\alpha-3} - J_{\alpha-1} J_{\alpha+3} \right) (\pi \rho(0)z). \quad (51)$$

In the above, the integration constants are fixed by requiring $\lim_{z \to \infty} \rho_s(z) = \rho(0)$. When measured in the unit of mean level spacing $1/\rho(0)$, $K_s(z, z') dz$ contains no free parameter. After this rescaling, the Bessel kernels (48) and (49) approach the sine kernel (8) in the limit $z, z' \to \infty$, $z - z' = O(1)$ as the repulsive effects due to the fermion determinant and the chiral structure become negligible.

**DISCUSSIONS**

In this article I have presented a proof of universality of microscopic correlations for RMT modeling QCD$_{3,4}$ coupled to massless quarks, within a single-trace potential class. The spectral correlation function is shown to be insensitive to the choice of matrix measures as long as the macroscopic spectrum is supported on a single interval.

Universality of the Bessel kernel is valid even beyond large-$N$ matrix models. Namely, the zero-dimensional reduction of the SU($N_f$) $\sigma$-model of pions also yields sum rules$^{23}$ which are identical to the moments of (50). Together with the numerical agreements mentioned in the introduction, it is very convincing that this wide range of universality encompasses QCD$_4$.

Recently it also became possible to incorporate quarks whose masses are within the same microscopic range as the Dirac eigenvalues$^{24, 25}$. We expect that the exactness of those universal correlations be verified by numerical simulations in a foreseeable future.

On the other hand, one may address a question: whether RMT can describe chiral symmetric phases of QCD such as Higgs phase or the large $N_f$ case. In these cases, accumulation of the soft eigenvalues is not strong enough to form a chiral condensate, although the theory is still interacting. There is no satisfactory answer to it so far, yet one may check the existence of universality within RMT. If the microscopic universality in the vicinity of the origin is broken for random matrix ensembles with $\rho(0) \propto |\langle \bar{\psi}\psi \rangle| = \ldots$
0, we may not expect to extract informations from RMT. In view of the above, we are lead to analyze a RMT with a double-well potential such as
\[ Z = \int_{N \times N} dM \, e^{-N \text{tr} (-M^2 + gM^4)}, \quad g = \frac{1}{4} \] (52)
whose coupling is tuned so as the macroscopic spectrum \( \rho(\lambda) \) to have a double zero at \( \lambda = 0 \). According to (22) it corresponds to \( A(\lambda) \) of the form
\[ A(\lambda) \sim \lambda^2 + O(\lambda^4). \] (53)
Then eq.(37) suggests a possible scaling limit:
\[ N \to \infty, \; \lambda \to 0, \; z \equiv N^{1/3} \lambda : \text{fixed}. \] (54)
However, the whole procedure must be reconsidered since the recursion coefficients \( q_n \) become double-valued in the large-\( n \) and \( N \) limit. A working hypothesis is to assume the following asymptotic form
\[ q_n = q_c + N^{-1/3} (-)^n f(t) + N^{-2/3} g(t) + O(N^{-1}), \quad t \equiv N^{2/3} \left(1 - \frac{n}{N}\right). \] (55)
Then the function \( A(\lambda) \) in the limit (54) is modified by a constant term involving \( f'(0) \) and \( g(0) \), which are the same order \( O(N^{-2/3}) \) as \( \lambda^2 \) under the limit (54),
\[ A(\lambda) = q_n (-2q_c^2 + q_n^2 + q_{n+1}^2 + \lambda^2) \sim N^{-2/3} (2g(0) - (-)^n f'(0) + z^2). \] (56)
The numerical values of \( f'(0) \) and \( g(0) \) are determined by solving Painlevé II equation (the microscopic limit of the Heisenberg algebra \([Q, P] = 1\)) under an appropriate boundary condition\(^{27}\). We may also have \( g \) run toward \( g_c = 1/4 \) with
\[ \gamma = N^{2/3} (g - g_c) : \text{fixed} \] (57)
while retaining the macroscopic spectrum intact. In this case the functions \( f'(t) \) and \( g(t) \) ought to be evaluated at \( t = \gamma \) instead of \( t = 0 \). These numerical values enter the resulting second order linear differential equation for the wave function \( \psi_n \) (the counterpart of (44)). Thus the macroscopic spectral density \( \rho(\lambda) \) is not sufficient to determine the microscopic correlation, but the latter depends upon the way how to approach the critical point. Microscopic universality holds in a weak sense for these cases, for Painlevé II equation is universally derived from a class of the critical potentials. Details of the analysis will appear elsewhere\(^{28}\).

ACKNOWLEDGEMENTS

I thank M. Praszalowicz and the organizers of the Workshop for their hospitality. Thanks are also due to P. H. Damgaard for collaboration and discussions summarized in this article. The work of SMN is supported in part by JSPS Postdoctoral Fellowships for Research Abroad, and by Nishina Memorial Foundation.
REFERENCES

1. M. L. Mehta, “Random Matrices”, 2nd Ed. (Academic Press, New York, 1991).
   For a recent review, see: T. Guhr, A. Müller-Groeling, H. A. Weidenmüller, cond-mat/9707301.
2. G. Szegö, “Orthogonal Polynomials” (Am. Math. Soc., Providence, 1939).
3. E. Brézin, A. Zee, Nucl. Phys. B402:613 (1993).
4. G. Moore, Prog. Theor. Phys. Suppl. 102:255 (1990).
5. E. V. Shuryak, J. J. M. Verbaarschot, Nucl. Phys. A560:306 (1993).
   For an updated review, see: J. J. M. Verbaarschot, this volume hep-th/9709032.
6. T. Banks, A. Casher, Nucl. Phys. B169:103 (1980).
7. J. J. M. Verbaarschot, Nucl. Phys. B427:534 (1994).
8. J. J. M. Verbaarschot, Phys. Lett. B368:137 (1996).
9. M. E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. J. M. Verbaarschot, T. Wettig, hep-lat/9704018.
10. A. D. Jackson, M. K. Şener, J. J. M. Verbaarschot, Nucl. Phys. B479:707 (1996).
11. J. Jurkiewicz, M. A. Nowak, I. Zahed, Nucl. Phys. B478:605 (1996).
12. E. Brézin, S. Hikami, A. Zee, Nucl. Phys. B464:411 (1996).
13. S. Nishigaki, Phys. Lett. B387:139 (1996).
14. G. Akemann, P. H. Damgaard, U. Magnea, S. Nishigaki, Nucl. Phys. B487:721 (1997).
15. P. Di Francesco, P. Ginsparg, J. Zinn-Justin, Phys. Rep. 254:1 (1995).
16. E. Kanzieper, V. Freilikher, Phys. Rev. Lett. 78:3806 (1997).
17. E. Kanzieper, V. Freilikher, cond-mat/9704149.
18. J. J. M. Verbaarschot, I. Zahed, Phys. Rev. Lett. 73:2288 (1994).
19. T. R. Morris, Nucl. Phys. B356:703 (1991).
20. G. M. Cicuta, L. Molinari, E. Montaldi, Mod. Phys. Lett. A1:125 (1986).
21. J. Ambjørn, C. F. Kristjansen, Int. J. Mod. Phys. A8:1259 (1993).
22. T. Nagao, K. Slevin, J. Math. Phys. 34:2075 (1993); P. J. Forrester, Nucl. Phys. B402:709 (1993).
23. H. Leutwyler, A. Smilga, Phys. Rev. D46:5607 (1992).
24. P. H. Damgaard, S. M. Nishigaki, hep-th/9711023.
25. P. H. Damgaard, S. M. Nishigaki, hep-th/9711096.
26. Y. Shimamune, Phys. Lett. B108:407 (1982).
27. C. Crnković, G. Moore, Phys. Lett. B257:322 (1991).
28. G. Akemann, P. H. Damgaard, U. Magnea, S. M. Nishigaki, hep-th/9712006.