A COORDINATE-FREE DEFINITION OF HURWITZ SPACES

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Abstract. Given a couple of subspaces \( Y \subset X \) of the complex plane \( \mathbb{C} \) satisfying some mild conditions (nice couple), and given a PMQ-pair \((Q, G)\), consisting of a partially multiplicative quandle (PMQ) \( Q \) and a group \( G \), we introduce a Hurwitz space \( \text{Hur}(X, Y; Q, G) \), containing configurations of points in \( X \setminus Y \) and in \( Y \) with monodromies in \( Q \) and in \( G \), respectively. The construction generalises classical Hurwitz spaces of points in the plane \( \mathbb{C} \) with monodromies in a group \( G \). We introduce a notion of morphisms between nice couples of subspaces of the plane, and prove that generalised Hurwitz spaces are functorial both in the nice couple and in the PMQ-group pair. For a locally finite PMQ \( Q \) we prove a homeomorphism between \( \text{Hur}((0, 1)^2; Q_+) \) and the simplicial Hurwitz space \( \text{Hur}^\Delta(Q) \), introduced in previous work of the author: this provides in particular \( \text{Hur}((0, 1)^2; Q_+) \) with a cell stratification in the spirit of Fox-Neuwirth and Fuchs.

1. Introduction

In [Bia21a] we introduced the notion of partially multiplicative quandle (PMQ) and defined a simplicial Hurwitz space \( \text{Hur}^\Delta(Q) \), depending on a PMQ \( Q \). The space \( \text{Hur}^\Delta(Q) \) is defined as a difference of geometric realisations of two bisimplicial sets, and is thus equipped with a cell stratification. The construction requires \( Q \) to be an augmented PMQ, in the sense of [Bia21a, Definition 4.9].

As briefly discussed in [Bia21a, Section 6], a point in \( \text{Hur}^\Delta(Q) \) can be thought of as a finite subset \( P \subset (0, 1)^2 \), together with the information of a monodromy \( \psi \), defined on certain loops of \( \mathbb{C} \setminus P \) and taking values in \( Q \). Here \((0, 1)^2 \subset \mathbb{C}\) denotes the standard, open unit square.

The great disadvantage of the simplicial approach to Hurwitz spaces is the lack of functoriality with respect to self-maps of the ambient space \((0, 1)^2\). For instance, if \( \xi: \mathbb{C} \to \mathbb{C} \) is a homeomorphism whose support is compactly contained in \((0, 1)^2\), there should be an induced homeomorphism \( \xi_*: \text{Hur}^\Delta(Q) \to \text{Hur}^\Delta(Q) \) sending the couple \((P, \psi)\) to a suitable couple \((\xi(P), \xi_*\psi)\), where \( \xi(P) \) is the image of \( P \) under \( \xi \); the main difficulty is in defining a suitable “push-forward” along \( \xi \) of the monodromy \( \psi \). More generally, for a subspace \( X \subset \mathbb{C} \) different from the unit...
square \((0,1)^2\), one would like to define a Hurwitz space \(\text{Hur}(\mathcal{X}; Q)\), containing configurations \((P, \psi)\) as above, with \(P \subset \mathcal{X}\), in a “functorial” way.

In this article we introduce, for a semi-algebraic subspace \(\mathcal{X} \subset \mathbb{H}\) of the closed upper half-plane \(\mathbb{H} = \{ \mathbb{R} \geq 0 \} \subset \mathbb{C}\), and for a PMQ \(Q\), a Hurwitz space \(\text{Hur}(\mathcal{X}; Q)\). More generally we introduce, for a nice couple \((\mathcal{X}, \mathcal{Y})\) of subspaces \(\mathcal{Y} \subseteq \mathcal{X}\) of \(\mathbb{H}\) (see Definition 2.3) and for a PMQ-group pair \((Q, G)\) (see [Bia21a, Definition 2.15]), a coordinate-free Hurwitz space \(\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)\). Roughly speaking, a configuration \((P, \psi, \varphi)\) in \(\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)\) consists of a finite subset \(P \subset \mathcal{X}\) together with the information of a \(G\)-valued monodromy \(\varphi\) on all loops spinning around a single point in \(P\), and a refined, \(Q\)-valued monodromy \(\psi\) defined only on those loops spinning around a single point in \(P \setminus \mathcal{Y}\). The adjective “coordinate-free” refers to the fact that the definition does not rely on the cartesian coordinates of the plane, differently from the simplicial Hurwitz spaces \(\text{Hur}^\Delta(Q)\) from [Bia21a], where a Fox–Neuwirth–Fuchs-type stratification, depending on horizontal and vertical lines of the plane, is used crucially in the definition.

One can think of the construction leading to \(\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)\) as a relative version of the construction leading to \(\text{Hur}(\mathcal{X}; Q)\): in the classical theory of configuration spaces, points of \(P\) disappear when they move inside the relative subspace \(\mathcal{Y}\); in our setting they do not disappear, but are “downgraded” to points around which only a \(G\)-valued monodromy, instead of a \(Q\)-valued monodromy, is defined.

1.1. Statement of results. We prove that the construction of \(\text{Hur}(\mathcal{X}; Q)\) is functorial both in \(Q\) and in \(\mathcal{X}\), in particular the following properties hold:

- a morphism of PMQs \(\Psi : Q \to Q'\) induces a continuous map \(\Psi_* : \text{Hur}(\mathcal{X}; Q) \to \text{Hur}(\mathcal{X}; Q')\);
- a semi-algebraic homeomorphism \(\xi : \mathbb{C} \to \mathbb{C}\) with compact support in \(\mathbb{H}\), sending \(\mathcal{X}\) inside \(\mathcal{X}'\), induces a continuous map \(\xi_* : \text{Hur}(\mathcal{X}; Q) \to \text{Hur}(\mathcal{X}'; Q)\).

In Definitions 4.2 and 4.3 we introduce morphisms and lax morphisms of nice couples: both types of morphisms arise as certain continuous maps \(\mathbb{C} \to \mathbb{C}\). The first, main result of the article is the following theorem, which combines Theorems 4.1, 4.4, and 4.7.

**Theorem 1.1.** The coordinate-free Hurwitz spaces \(\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)\) are functorial both in the nice couple \((\mathcal{X}, \mathcal{Y})\), with respect to morphisms of nice couples, and in the PMQ-group pair. If we restrict to complete PMQs, then functoriality holds also with respect to lax morphisms of nice couples.

We recall that the simplicial Hurwitz space \(\text{Hur}^\Delta(Q)\) was defined in [Bia21a] only when \(Q\) is an augmented PMQ. For such a PMQ, we can set \(\mathcal{X} = (0,1)^2\), the unit square in \(\mathbb{C}\), and consider also the coordinate-free Hurwitz space \(\text{Hur}((0,1)^2; Q)\). Inside the latter, we can identify a subspace \(\text{Hur}((0,1)^2; Q_+)\), containing configurations \((P, \psi)\) whose monodromies around points of \(P\) lie in the augmentation ideal \(Q_+ = Q \setminus \{1\}\). The second main result of the article is the following theorem, which is Theorem 9.1.

**Theorem 1.2.** If \(Q\) is augmented and also locally finite ([Bia21a, Definition 4.12]), then there is a homeomorphism between the simplicial Hurwitz space \(\text{Hur}^\Delta(Q)\) and the coordinate-free Hurwitz space \(\text{Hur}((0,1)^2; Q_+)\).

Thus the construction of coordinate-free Hurwitz spaces of this article generalises that of simplicial Hurwitz spaces from [Bia21a].
In [Bia21a] Definition 6.17 we also introduced the notion of Poincare PMQ: an augmented and locally finite PMQ \(Q\) is Poincare if each connected component of \(\text{Hur}^\Delta(Q)\) is a topological manifold. More generally, for a commutative ring \(R\), we introduce the notion of \(R\)-Poincare PMQ: each component of \(\text{Hur}^\Delta(Q)\) is now required to be an \(R\)-homology manifold (see Definition 9.4). It is easy to see that connected components of \(\text{Hur}^\Delta(Q)\) are indexed by the set \(\hat{Q}\), i.e. the completion of the PMQ \(Q\): for \(a \in \hat{Q}\) we denote by \(\text{Hur}^\Delta(Q)(a)\) the corresponding component in \(\text{Hur}^\Delta(Q)\), as in [Bia21a Section 6]. An \(R\)-Poincare PMQ \(Q\) has the following advantage: for all \(a \in \hat{Q}\) the \(R\)-homology groups of \(\text{Hur}^\Delta(Q)(a)\) are isomorphic by Poincare-Lefschetz duality to the reduced \(R\)-cohomology groups of the one point compactification \(\text{Hur}^\Delta(Q)(a)\)\(^\infty\); the latter space is endowed with a finite cell decomposition [Bia21a Section 6], which can in principle be used for actual composition.

The third main result of the article is the following theorem, giving a simple criterion to recognise when a PMQ \(Q\) is Poincare, or \(R\)-Poincare; this combines Theorems 9.3 and 9.6.

**Theorem 1.3.** In order to prove that \(Q\) is Poincare (respectively, \(R\)-Poincare), it suffices to check that for all \(a \in \hat{Q}\) the connected component \(\text{Hur}^\Delta(Q)(a)\) of \(\text{Hur}^\Delta(Q)\) is a topological manifold (respectively, an \(R\)-homology manifold).

Note that, unless \(Q\) is already complete, the set \(\hat{Q}\) is larger than \(Q\).

### 1.2. Outline of the article.**

In Section 2 we introduce the notion of nice couple \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) of subspaces \(\mathcal{Y} \subset \mathcal{X}\) of the closed upper half-plane \(\mathbb{H}\). For each finite subset \(P \subset \mathcal{X}\), we introduce several PMQs contained in the fundamental group \(\pi_1(\mathcal{X} \setminus P)\): the most important is \(\Omega(P) = \Omega_\mathcal{C}(P)\), which will allow us to define configurations in \(\text{Hur}(\mathcal{C}; Q, G)\) supported on the set \(P\). In order to be able later to define a topology on the set \(\text{Hur}(\mathcal{C}; Q, G)\), we introduce in this section also the notion of (adapted) covering \(\mathcal{U}\) of a finite subset \(P \subset \mathcal{X}\), and associate several PMQs also the datum of a configuration of points \(P\) and a covering \(\mathcal{U}\) of it.

In Section 3 we recall the definition of Ran spaces. We then define \(\text{Hur}(\mathcal{C}; Q, G)\), for each nice couple \(\mathcal{C}\) and each PMQ-group pair \((Q, G)\), first as a set and then as a Hausdorff topological space (see Proposition 3.3). Finally, we discuss a variation of the definition, using a contractible subspace \(\mathbb{T}\) of \(\mathbb{C}\) as “ambient space” instead of the entire complex plane.

In Section 4 after describing functoriality of \(\text{Hur}(\mathcal{C}; Q, G)\) with respect to morphisms of PMQ-group pairs, we introduce a suitable notion of *morphism of nice couples*, and prove that for two nice couples \(\mathcal{C}, \mathcal{C}'\) and a morphism of nice couples \(\xi : \mathcal{C} \rightarrow \mathcal{C}'\) there is an induced continuous map \(\xi_* : \text{Hur}(\mathcal{C}; Q, G) \rightarrow \text{Hur}(\mathcal{C}'; Q, G)\); we also introduce a notion of *lax morphism of nice couples*, and prove that also a lax morphism \(\xi : \mathcal{C} \rightarrow \mathcal{C}'\) induces a continuous map between Hurwitz spaces, provided that \(Q\) is complete. Finally, we discuss enriched variations of the above results, for families of (lax) morphisms \(\mathcal{C} \rightarrow \mathcal{C}'\), parametrised by a topological space.

In Section 5 we give some simple applications of the functoriality of Hurwitz spaces, in particular we study some local properties of the topology of Hurwitz spaces. Moreover we give the definition of absolute Hurwitz space \(\text{Hur}(\mathcal{X}; Q)\) as a special case of the relative definition \(\text{Hur}(\mathcal{C}; Q, G)\).

In Section 6 we define the total monodromy, which is the simplest discrete, continuous invariant of configurations in Hurwitz spaces. The total monodromy...
takes the form of a map $\text{Hur}(\mathcal{C}; \mathbb{Q}, G) \to G$ in the relative case, and $\text{Hur}(\lambda^2; \mathbb{Q}) \to \hat{Q}$ in the absolute case, where $\hat{Q}$ is the completion of $\mathbb{Q}$. In this section we also define three types of actions of $G$ on (parts of) the space $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)$: one action is the action by global conjugation, and together with the total monodromy makes $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)$ into a $G$-crossed space; the other two actions are only defined on certain subspaces of $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)$, and account for the possibility of changing the monodromies around a single point $z \in \mathbb{C}$ of a configuration, provided that this point is the leftmost (respectively, the rightmost) point of the configuration.

In Section 7 we introduce, in the hypothesis that $\mathbb{Q}$ is augmented, a subspace $\text{Hur}(\mathcal{C}; \mathbb{Q}^+, G)$ of $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)$; using the notion of explosion we prove that the inclusion $\text{Hur}(\mathcal{C}; \mathbb{Q}^+, G) \subset \text{Hur}(\mathcal{C}; \mathbb{Q}, G)$ is in several cases a homotopy equivalence.

In Section 8 for an augmented PMQ $\mathbb{Q}$, we construct a continuous bijection $\nu: |\text{Arr}(\mathbb{Q})| \to \text{Hur}([0,1]^2; \mathbb{Q}^+)$, where $\hat{\mathbb{Q}}$ denotes the completion of $\mathbb{Q}$; we show that $\nu$ restricts to a bijection $\text{Hur}_\Delta^+(\mathbb{Q}) \to \text{Hur}((0,1)^2; \mathbb{Q}^+)$. In Section 9 we prove that $\Delta^+: \text{Hur}(\mathbb{Q}) \to \text{Hur}((0,1)^2; \mathbb{Q}^+)$ is a homeomorphism under the additional hypothesis that $\mathbb{Q}$ is a locally finite PMQ. We then switch to the analysis of Hurwitz spaces associated with Poincaré PMQs, and prove that a PMQ $\mathbb{Q}$ is Poincaré, i.e. each component of $\text{Hur}((0,1)^2; \mathbb{Q}^+)$ is a manifold, if and only if each component $\text{Hur}((0,1)^2; \mathbb{Q}^+)_a$, for $a \in \mathbb{Q} \subset \hat{\mathbb{Q}}$, is a manifold: here $\text{Hur}((0,1)^2; \mathbb{Q}^+)_a$ denotes the subspace of configurations with total monodromy equal to $a$. A similar result is proved for $R$-Poincaré PMQs, i.e. those PMQs for which each component of $\text{Hur}((0,1)^2; \mathbb{Q}^+)$ is an $R$-homology manifold, for a fixed commutative ring $R$.

In Appendix A we briefly discuss how to extend the definition of coordinate-free Hurwitz spaces to the case of generic ambient surfaces endowed with a basepoint on the boundary.

Finally, Appendix B contains the proofs of the most technical lemmas and propositions of the article; these proofs have been deferred to help the reader focus on the general framework.

Throughout the article we make heavy use of the results of [Bia21a], Sections 2-6: we cite every time which specific fact we are employing, so that the reader does not need to be familiar with the details of [Bia21a].

1.3. Motivation. This is the second article in a series about Hurwitz spaces. Our motivation to define generalised Hurwitz spaces is two-fold.

- The classical setting for Hurwitz spaces takes an integer $k \geq 0$ and a quandle $\mathbb{Q}$ (usually, a union of conjugacy classes in a group $G$) as input, and gives a space $\text{hur}_k(\mathbb{Q})$ of configurations of $k$ points in the unit square $\mathcal{R} = (0,1)^2$ with the information of a monodromy in $\mathbb{Q}$. Classical Hurwitz spaces are related to several problems in algebraic geometry and in number theory, but can be studied also in purely topological terms, as for example in [EVW16]. In particular, it is convenient to consider the disjoint union $\text{hur}(\mathbb{Q}) := \bigsqcup_{k \geq 0} \text{hur}_k(\mathbb{Q})$, which admits an $E_3$-algebra structure, and to study its group completion $\Omega B\text{hur}(\mathbb{Q})$, which contains stable information about classical Hurwitz spaces. As we will see in a subsequent article [Bin21b], coordinate-free Hurwitz spaces are well-suited for modeling the homotopy type of $B\text{hur}(\mathbb{Q})$, and provide, in a certain sense, even a double delooping "$B^2(\text{hur}(\mathbb{Q}))$" of $\text{hur}(\mathbb{Q})$, as if the latter were an $E_2$-algebra.
• The connection between Hurwitz spaces and moduli spaces of Riemann surfaces is well-known, and by using a suitable family of PMQs $\mathcal{G}^{\text{geo}}_d$ (see [Bia21a, Section 7]), one can even model the homotopy type of moduli spaces $\mathcal{M}_{g,n}$ as components of Hurwitz spaces; here $\mathcal{M}_{g,n}$ denotes the moduli space of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ ordered and parametrised boundary components. This connection is established in a subsequent article [Bia21c], and is used there to give an alternative proof of the Mumford conjecture on the stable rational cohomology of moduli spaces $\mathcal{M}_{g,n}$, originally proved by Madsen and Weiss [MW07].

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2. Groups and PMQs from configurations in the plane

With every pointed, connected topological space \((X, \ast)\) one can associate a group, namely the fundamental group \(\pi_1(X, \ast)\). In this section we associate several PMQs to a space arising as the complement in \(\mathbb{C}\) of a finite collection \(P\) of points.

2.1. Nice couples.

**Notation 2.1.** We endow the complex plane \(\mathbb{C}\) with the basepoint \(\ast\) corresponding to the complex number \(-\sqrt{-1}\); this point is contained in the lower half-plane. The closed, upper half-plane is denoted by \(\mathbb{H} = \{z \in \mathbb{C} | \Im(z) \geq 0\}\). For any subspace \(T \subset \mathbb{C}\) we denote by \(\mathring{T}\) the interior of \(T\), i.e. the set of all \(z \in T\) for which there is an open disc \(z \in U \subset T\).
Definition 2.2. A subset $\mathcal{J} \subseteq \mathbb{C}$ is semi-algebraic if it can be expressed as a finite union of polynomial equalities and (weak or strict) inequalities in the real, affine coordinates $\mathbb{R}(z)$ and $\mathbb{Z}(z)$ of $\mathbb{C}$.

For two semi-algebraic sets $\mathcal{J}, \mathcal{J}' \subseteq \mathbb{C}$, a continuous map $\xi : \mathcal{J} \rightarrow \mathcal{J}'$ is semi-algebraic if $\mathcal{J}$ can be expressed as a finite union of semi-algebraic subsets $\mathcal{J}_1, \ldots, \mathcal{J}_r \subseteq \mathcal{J}$, and the coordinates of $\xi|_{\mathcal{J}}$ are expressed, in the real affine coordinates of $\mathcal{J}_i$, by fractions of polynomials with real coefficients.

Note that a semi-algebraic subset $\mathcal{J} \subseteq \mathbb{C}$ can be written locally as a finite union of subsets of $\mathbb{C}$ that are diffeomorphic to points, open segments and open triangles. Note also that finite unions and intersections of semi-algebraic subsets are again semi-algebraic.

Definition 2.3. A nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ is a couple of subspaces $\emptyset \subseteq \mathcal{Y} \subseteq \mathcal{X} \subseteq \mathbb{H}$, such that the following properties hold:

• $\mathcal{X}$ and $\mathcal{Y}$ are semi-algebraic;
• $\mathcal{Y}$ is closed in $\mathcal{X}$.

By abuse of notation, for $\mathcal{X} \subset \mathbb{H}$ we will denote by $\mathcal{X}$ also the nice couple $(\mathcal{X}, \emptyset)$.

We fix a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ for the rest of the section.

2.2. Configurations and coverings. We consider configurations of points in $\mathbb{C}$ which are contained in $\mathcal{X}$, and associate several PMQs with them.

Notation 2.4. We usually denote by $P = \{z_1, \ldots, z_k\} \subset \mathcal{X}$ a finite collection of distinct points, for some $k \geq 0$. We will usually assume that there is $0 \leq l \leq k$ such that $z_1, \ldots, z_l$ are precisely the points of $P$ lying in $\mathcal{X} \setminus \mathcal{Y}$.

Definition 2.5. Let $P$ as in Notation 2.4. A covering of $P$ is a sequence $\underline{U} = (U_1, \ldots, U_\kappa)$ of convex, semi-algebraic, disjoint open subsets of $\mathbb{C} \setminus \{\ast\}$, satisfying the following conditions:

• $P$ is contained in the union $U_1 \cup \cdots \cup U_\kappa$;
• each $U_i$ intersects $P$ at least in one point;
• the closures $\overline{U}_i \subseteq \mathbb{C}$ of the sets $U_i$ are disjoint, compact and do not contain $\ast$.

A covering of $P$ is adapted if the following additional properties hold:

• $\kappa = k$, and each $U_i$ contains exactly one point of $P$;
• for all $1 \leq i \leq l$, i.e. for all $i$ such that $z_i \notin \mathcal{Y}$, if $z_i \in U_j$ then the closure $\overline{U}_j$ is disjoint from $\mathcal{Y}$; here it is useful to recall that $\mathcal{Y}$ is closed in $\mathcal{X}$.

Note that if $\underline{U}$ is an adapted covering of $P$, then the inclusion $\mathbb{C} \setminus \overline{\underline{U}} \hookrightarrow \mathbb{C} \setminus P$ is a homotopy equivalence. See Figure 1.

Notation 2.6. Let $\underline{U} = (U_1, \ldots, U_\kappa)$ be a covering of $P$ as in Definition 2.5. By abuse of notation we denote also by $\underline{U}$ the union $U_1 \cup \cdots \cup U_\kappa \subseteq \mathbb{C}$. We assume that there is $0 \leq \lambda \leq \kappa$ such that $U_1, \ldots, U_\lambda$ are precisely the open sets of $\underline{U}$ with $\overline{U}_i$ disjoint from $\mathcal{Y}$. If $\underline{U}$ is an adapted covering of $P$, using Notation 2.4, we also assume $z_i \in U_i$ for $1 \leq i \leq \kappa$.

The notion of covering is useful in defining a topology on the set of configurations of points as in Notation 2.3. Given a configuration $P$ and an adapted covering $\underline{U}$ of $P$, we can “perturb” $P$ to a new configuration $P' = \{z'_1, \ldots, z'_l\}$, such that $P' \subset \underline{U}$.
and each connected component of $U'$ contains at least one point of $P'$: then $U'$ is a covering of $P'$ as well. Intuitively, we have obtained $P'$ by slightly moving the points of $P$ and by splitting some points $z_i \in P$ in two or more points of $P'$; all these splittings occur inside $U$. See Subsection 3.1 for more details.

2.3. Fundamental group and admissible generating sets.

Definition 2.7. Let $P$ be as in Notation 2.4 We denote by $\mathfrak{S}(P)$ the group

$$\mathfrak{S}(P) = \pi_1(\mathbb{C} \setminus P, *)$$

and call it the fundamental group of $P$.

For $P$ as in Notation 2.4 the group $\mathfrak{S}(P)$ is a free group on $k$ generators: in the following we construct an explicit set of free generators for $\mathfrak{S}(P)$. See Figure 2.

We choose an adapted covering $U'$ of $P$, and use Notation 2.6. The boundary curves of $\bar{U}_1, \ldots, \bar{U}_k$ are denoted by $\partial U_1, \ldots, \partial U_k$ respectively, and are oriented clockwise. We also choose embedded arcs $\zeta_1, \ldots, \zeta_k$ joining the basepoint $*$ with $\partial U_1, \ldots, \partial U_k$. We assume the following:

• for all $1 \leq i \leq k$, the arc $\zeta_i$ intersects $*$ only at one endpoint, and intersects $\partial U_i$ only at the other endpoint;
• for distinct $1 \leq i, j \leq k$, the interior of the arc $\zeta_i$ is disjoint from the curve $\partial U_j$ and from the interior of the arc $\zeta_j$.

For $1 \leq i \leq k$ consider the element $f_i \in \mathfrak{S}(P)$ represented by a loop that begins at $*$, runs along $\zeta_i$ until it reaches the intersection with $\partial U_i$, spins clockwise around $\partial U_i$ and runs back to $*$ along $\zeta_i$.

The elements $f_1, \ldots, f_k$ exhibit $\mathfrak{S}(P)$ as a free group $\mathbb{F}^k$ on $k$ generators.

Definition 2.8. Let $P$ be as in Notation 2.4 A set of generators $f_1, \ldots, f_k$ of $\mathfrak{S}(P)$ obtained as described above is called an admissible generating set.

2.4. Fundamental PMQ. In the following we introduce several PMQs arising as subsets of $\mathfrak{S}(P)$, for $P$ as in Notation 2.4 Recall that a conjugacy class in $\mathfrak{S}(P)$ corresponds to a free (i.e. unbased) homotopy class of maps $S^1 \to \mathbb{C} \setminus P$. 

![Figure 1. On left, a nice couple $C = (X, Y)$, and a covering $U'$ of a configuration $P \subset X$; on right, an adapted covering $U$ of $P$.](image)
Figure 2. On left, a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$, a configuration $P \subset \mathcal{X}$, the boundary curves of an adapted covering $\mathcal{U}$ of $P$ and a choice of arcs $\zeta_i$; on right, the loops representing the corresponding admissible generating set of $\mathcal{G}(P)$.

**Definition 2.9.** Let $P$ be as in Notation 2.4. For all $1 \leq i \leq l$ we denote by $\Omega_i(P, z_i) \subset \mathcal{G}(P)$ the conjugacy class corresponding to a small simple closed curve that spins once, clockwise, around $z_i$; we define

$$\Omega(P) = \Omega_{\mathcal{C}}(P) := \{1\} \cup \bigcup_{1 \leq i \leq l} \Omega(P, z_i) \subset \mathcal{G}(P),$$

and call it the *fundamental PMQ* of $P$ relative to the nice couple $\mathcal{C}$. We consider on $\Omega(P)$ the PMQ structure inherited from $\mathcal{G}(P)$ (see [Bia21a, Definition 2.8]).

Let $P$ be as in Notation 2.4 and fix an admissible generating set $f_1, \ldots, f_k$ of $\mathcal{G}(P)$ (see Definition 2.8): then the elements of $\Omega(P)$ are precisely $1$ and all conjugates of the elements $f_1, \ldots, f_k$. In particular the isomorphism of groups $\mathcal{G}(P) \cong F_k$, given by the choice of an admissible generating set, restricts to a bijection $\Omega(P) \cong F_0^k$ (see [Bia21a, Definition 3.2]), and the hypotheses required by [Bia21a, Definition 2.8] are fulfilled. The partial product of $\Omega(P)$ is trivial, and $(\Omega(P), \mathcal{G}(P))$ is a PMQ-group pair. See Figure 3, left, for examples of elements in $\Omega(P)$.

2.5. *Extended fundamental PMQ.* We extend Definition 2.9 by considering more general simple closed curves.

**Definition 2.10.** Let $P$ be as in Notation 2.4. We denote by $\Omega_{\text{ext}}(P) = \Omega_{\text{ext}}^{\mathcal{C}}(P) \subset \mathcal{G}(P)$ the union of all conjugacy classes corresponding to oriented simple closed
curves $\beta \subset \mathbb{C} \setminus \mathcal{Y}$, such that $\beta$ spins clockwise and $\beta$ bounds a disc contained in $\mathbb{C} \setminus \mathcal{Y}$. We consider on $\mathcal{Q}^{\text{ext}}(P)$ the PMQ structure inherited from $\mathcal{Q}(P)$, and call it the \textit{extended fundamental PMQ} of $P$ relative to the nice couple $\mathcal{C}$.

Note that there is an inclusion of sets $\mathcal{Q}(P) \subseteq \mathcal{Q}^{\text{ext}}(P)$. See Figure 3, right, for a non-trivial example of an element in $\mathcal{Q}^{\text{ext}}(P)$. The fact that the hypotheses of \cite[Definition 2.8]{Bin21a} are fulfilled by $\mathcal{Q}^{\text{ext}}(P) \subseteq \mathcal{Q}(P)$ needs some explanation: this is contained in the following two propositions.

\textbf{Proposition 2.11.} Let $P$ be as in Notation 2.4. The set $\mathcal{Q}^{\text{ext}}(P)$ is generated under partial product by $\mathcal{Q}(P)$, i.e., every element $g$ of $\mathcal{Q}^{\text{ext}}(P)$ admits a decomposition $(g_1, \ldots, g_r)$ with respect to $\mathcal{Q}(P)$ (see \cite[Definition 3.5]{Bin21a}).

The proof of Proposition 2.11 is in Subsection B.1 of the appendix.

\textbf{Proposition 2.12.} Let $P$ be as in Notation 2.4. Then $\mathcal{Q}^{\text{ext}}(P) \subset \mathcal{Q}(P)$ satisfies the hypotheses of \cite[Definition 2.8]{Bin21a}, and hence inherits a structure of PMQ; as a consequence $(\mathcal{Q}^{\text{ext}}(P), \mathcal{Q}(P))$ is a PMQ-group pair.

The proof of Proposition 2.12 is in Subsection B.2 of the appendix. It follows that the inclusion $\mathcal{Q}(P) \subseteq \mathcal{Q}^{\text{ext}}(P)$ is a map of PMQs, and the inclusion $(\mathcal{Q}(P), \mathcal{Q}(P)) \subset (\mathcal{Q}^{\text{ext}}(P), \mathcal{Q}(P))$ is a map of PMQ-group pairs. In the following we study the problem of extending to $\mathcal{Q}^{\text{ext}}(P)$ maps of PMQs defined over $\mathcal{Q}(P)$.

\textbf{Definition 2.13.} Let $P$ be as in Notation 2.4 let $\mathcal{Q}$ be a PMQ and let $\psi : \mathcal{Q}(P) \to \mathcal{Q}$ be a map of PMQ. Let $g \in \mathcal{Q}^{\text{ext}}(P)$ and let $(g_1, \ldots, g_r)$ be a decomposition of $g$.
with respect to $\Omega(P)$ (see [Bia21a, Definition 3.5]). We say that $\psi$ can be extended over $g$ if the product $\psi(g_1) \ldots \psi(g_r)$ is defined in $Q$.

We denote by $\Omega^{\text{ext}}(P)_\psi = \Omega^{\text{ext}}(P)_{\psi} \subseteq \Omega^{\text{ext}}(P)$ the subset containing all elements $g$ over which $\psi$ can be extended.

Some comments on Definition 2.13 are needed. For $g \in \Omega^{\text{ext}}(P)$, the existence of a decomposition $(g_1, \ldots, g_r)$ of $g$ with respect to $\Omega(P)$ is granted by Proposition 2.11. This decomposition is in general not unique; nevertheless, by [Bia21a, Proposition 3.7], if $(g_1', \ldots, g_r')$ is another decomposition of $g$ with respect to $\Omega(P)$, then the two decompositions are connected by a sequence of standard moves (see [Bia21a, Definition 3.6]). Since $\psi$ is a map of PMQs, we obtain that the sequence $(\psi(g_1), \ldots, \psi(g_r))$ of elements of $Q$ can be transformed into the sequence $(\psi(g_1'), \ldots, \psi(g_r'))$ by a sequence of standard moves; it is then a direct consequence of the definition of PMQ, that the product $\psi(g_1) \ldots \psi(g_r)$ is defined if and only if the product $\psi(g_1') \ldots \psi(g_r')$ is defined, and if both products are defined then they are equal to each other.

This shows that, whether $\psi$ can be extended over $g$, only depends on the element $g$ but not on the decomposition $(g_1, \ldots, g_r)$ of $g$ with respect to $\Omega(P)$; moreover the assignment $g \mapsto \psi(g_1) \ldots \psi(g_r)$ gives a well-defined map of sets $\psi^{\text{ext}}: \Omega^{\text{ext}}(P)_\psi \to Q$, which extends the map $\psi: \Omega(P) \to Q$.

Proposition 2.14. The subset $\Omega^{\text{ext}}(P)_\psi \subseteq \mathcal{G}(P)$ satisfies the requirements of [Bia21a, Definition 2.8], and therefore $\Omega^{\text{ext}}(P)_\psi$ inherits a structure of PMQ. The map $\psi^{\text{ext}}: \Omega^{\text{ext}}(P)_\psi \to Q$ is the unique map of PMQs $\Omega^{\text{ext}}(P)_\psi \to Q$ restricting to $\psi: \Omega(P) \to Q$ on $\Omega(P)$.

The proof of Proposition 2.14 is in Subsection B.3 of the appendix. As a consequence of Proposition 2.11, $(\Omega^{\text{ext}}(P)_\psi, \mathcal{G}(P))$ is naturally a PMQ-group pair.

2.6. PMQs from coverings. We extend the definitions of the previous subsections by replacing a configuration of points $P$ with a configuration of open, convex sets $\mathcal{U}$ in $\mathbb{C}$.

Definition 2.15. Let $\mathcal{U}$ be a covering of $P$ (see Definition 2.6). We define $\mathcal{G}(\mathcal{U})$ as the group $\pi_1(\mathbb{C} \setminus \mathcal{U}, *)$, and call it the fundamental group of $\mathcal{U}$.

We use Notation 2.6 and define $\Omega(\mathcal{U}) \subseteq \mathcal{G}(\mathcal{U})$ as the union of $\{1\}$ and the conjugacy classes corresponding to the simple closed curves $\partial U_1, \ldots, \partial U_\lambda$, oriented clockwise. Similarly as in Definition 2.9, we consider on $\Omega(\mathcal{U})$ the PMQ structure inherited from $\mathcal{G}(\mathcal{U})$. The PMQ $\Omega(\mathcal{U})$ is called the fundamental PMQ of $\mathcal{U}$.

Finally, we define $\Omega(P, \mathcal{U}) = \Omega(\mathcal{U}) \subset \Omega^{\text{ext}}(P)$ as the union of all conjugacy classes in $\mathcal{G}(P)$ represented by simple closed curves $\beta$ which are oriented clockwise and, up to free homotopy, lie in one of the $\lambda$ regions of the form $U_i \setminus P \subset \mathbb{C} \setminus \mathcal{Y}$, for some $1 \leq i \leq \lambda$. The set $\Omega(P, \mathcal{U})$ is called the relative fundamental PMQ of $P$ with respect to $\mathcal{U}$.

Note that, for $\mathcal{U}$ as in Notation 2.6, $\mathcal{G}(\mathcal{U})$ is a free group on $\kappa$ generators, and an admissible generating set $f_1, \ldots, f_\kappa$ can be constructed in the same way as in Subsection 2.3 to give an isomorphism $\mathcal{G}(\mathcal{U}) \cong \mathbb{F}^\kappa$. By the same arguments used in Subsection 2.3, the previous identification restricts to an identification $\Omega(\mathcal{U}) \cong \mathbb{F}Q^n$, and therefore, analogously as in the case of $\Omega(P) \subseteq \mathcal{G}(P)$, the set $\Omega(\mathcal{U})$ inherits from $\mathcal{G}(\mathcal{U})$ a structure of PMQ, and $(\Omega(\mathcal{U}), \mathcal{G}(\mathcal{U}))$ is a PMQ-group pair.
Lemma 2.16. Using the notation above, the set $\Omega(P, U) \subseteq \mathcal{G}(P)$ inherits a structure of PMQ from $\mathcal{G}(P)$ in the sense of [24, Definition 2.8], and thus $(\Omega(P, U), \mathcal{G}(P))$ is a PMQ-group pair.

Proof. For each $1 \leq i \leq \lambda$ consider the nice couple $\mathcal{C}_i = ([\mathbb{H}, \mathbb{H} \setminus U_i])$. Then $\Omega_{\mathcal{C}_i}(P, U) \subseteq \mathcal{G}(P)$ can be identified with the union of PMQs $\bigcup_{i=1}^{\lambda} \Omega_{\mathcal{C}_i}^{\text{ext}}(P)$. Consider the abelianisation map $\text{ab}: \mathcal{G}(P) \rightarrow \mathcal{G}(P)^{\text{ab}} \cong \mathbb{Z}^k$, where the latter isomorphism is given by considering the basis of $\mathcal{G}(P)^{\text{ab}}$ as free abelian group given by the classes of the elements of an admissible generating set of $\mathcal{G}(P)$. Let $g \in \Omega_{\mathcal{C}_i}(P, U)$; then $\text{ab}(g)$ is a vector with all entries equal to 0 or 1; if $g = 1$, then all entries are zero, and if $g \in \Omega_{\mathcal{C}_i} \setminus \{1\}$ for some $1 \leq i \leq \lambda$ then at least one entry is equal to 1, and all entries equal to 1 correspond to standard generators represented by simple loops spinning around points of $P \cap U_i$. This implies that each PMQ $\Omega_{\mathcal{C}_i}(P)$ is augmented and that for distinct $1 \leq i, i' \leq \lambda$ we have $\Omega_{\mathcal{C}_i}(P) \cap \Omega_{\mathcal{C}_i'}(P) = \{1\}$. Moreover if $g = g_1 \ldots g_r$ is a decomposition of $g$ with all $f^j g_j \in \Omega(P, U)$, then the previous argument shows that if $g = 1$ then all $g_j = 1$, and if $g \in \Omega_{\mathcal{C}_i} \setminus \{1\}$ for some $1 \leq i \leq \lambda$ then all $g_j \in \Omega_{\mathcal{C}_i}$. We can now apply Proposition 2.12 to $\Omega_{\mathcal{C}_i}^{\text{ext}}(P)$ to show that, for all $1 \leq j \leq j' \leq r$ the product $g_j \ldots g_{j'}$ also lies in $\Omega_{\mathcal{C}_i}^{\text{ext}}(P)$, and hence in $\Omega_{\mathcal{C}_i}(P, U)$.

We conclude the subsection by analysing which inclusions hold between the groups and PMQs introduced so far. Note that the inclusion $\mathbb{C} \setminus U \subseteq \mathbb{C} \setminus P$ induces an injection of groups $\mathcal{G}(U) \subseteq \mathcal{G}(P)$. On the other hand there are inclusions of PMQs $\Omega(P) \subseteq \Omega(P, U)$ and $\Omega(U) \subseteq \Omega(P, U)$, giving rise in particular to an inclusion of PMQ-group pairs $(\Omega(U), \mathcal{G}(U)) \subseteq (\Omega(P, U), \mathcal{G}(P))$. If $U$ is adapted to $P$, then all the previous inclusions are isomorphisms.

Finally, note that there is an inclusion of PMQs $\Omega(P, U) \subseteq \Omega^{\text{ext}}(P)$, which in general is not an isomorphism. Proposition 2.11 specialises to the fact that $\Omega(P, U)$ is generated by $\Omega(P) \cap \Omega(P, U)$ under partial multiplication. Note that, in general, $\Omega(P)$ is not contained in $\Omega(P, U)$.

2.7. Maps induced by forgetting points.

Notation 2.17. Let $\mathcal{C} = (X, Y)$ be a nice couple and let $P \subseteq P' \subset X$. We denote by $i_P^{P'}: \mathcal{G}(P') \rightarrow \mathcal{G}(P)$ the map induced by the inclusion $\mathbb{C} \setminus P' \subseteq \mathbb{C} \setminus P$.

Note that $i_P^{P'}$ restricts to maps $\Omega(P') \rightarrow \Omega(P)$ and $\Omega^{\text{ext}}(P') \rightarrow \Omega^{\text{ext}}(P)$. To see this, let $[\gamma] \in \Omega(P')$ (respectively $[\gamma] \in \Omega^{\text{ext}}(P')$) be represented by a loop $\gamma$ which is freely homotopic in $\mathbb{C} \setminus P'$ to a simple curve $\beta \subset \mathbb{C} \setminus (P' \cup Y)$ spinning clockwise around at most one point (respectively, some points) of $P' \setminus Y$; then the same properties hold for $i_P^{P'}([\gamma])$ in $\Omega(P)$ (respectively, in $\Omega^{\text{ext}}(P)$).

3. Hurwitz spaces with coefficients in a PMQ-group pair.

The aim of this section is to define, for a PMQ-group pair $(Q, G)$ and a nice couple $\mathcal{C} = (X, Y)$ as in Definition 2.3, the Hurwitz space $\text{Hur}(\mathcal{C}; Q, G)$ of configurations of points in $\mathcal{C}$ with monodromies in $(Q, G)$. This includes the non-relative case of a nice couple of the form $(X, \emptyset)$: we will see in Section 5 that in this case the essential information about monodromies takes place in $Q$ and the construction is, in a certain sense, independent of $G$. 


We will work with configurations of points \( P \) as in Notation 2.3 in particular these configurations lie in the closed upper half-plane \( \mathbb{H} \).

Throughout the section we fix a PMQ-group pair \((Q, G) = (Q, G, \varepsilon, \tau)\) as in [Bin21a, Definition 2.15] and a nice couple \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) as in Definition 2.3.

### 3.1. Ran spaces.

We recall the definition and the main properties of the Ran space Ran\((\mathcal{X})\), focusing on the case of a connected subspace \( \mathcal{X} \subset \mathbb{H} \). We use [Lur17, Subsection 5.5.1] as main reference.

**Definition 3.1.** Let \( \mathcal{X} \subset \mathbb{H} \) be a subspace. We define Ran\((\mathcal{X})\) as the set of all finite subsets \( P \subset \mathcal{X} \), including \( \emptyset \); we denote by Ran\(_+\)(\(\mathcal{X}\)) the set Ran\((\mathcal{X})\) \(\setminus\) \(\{\emptyset\}\).

We define a topology on Ran\((\mathcal{X})\). For \( P \in \text{Ran}(\mathcal{X}) \) and \( U \) an adapted covering of \( P \) with respect to the nice couple \((\mathcal{X}, \emptyset)\) (see Definition 2.5), we let \( \mathcal{U}(P, U) = \mathcal{U}_\mathcal{X}(P, U) \subset \text{Ran}(\mathcal{X}) \) be the subset of all \( P' \in \text{Ran}(\mathcal{X}) \) with the following properties:

- \( P' \subset U \);
- \( P' \cap U_i \neq \emptyset \) for all \( 1 \leq i \leq \kappa \), using Notation 2.6.

A subset of the form \( \mathcal{U}(P, U) \) is called a normal neighbourhood of \( P \) in Ran\((\mathcal{X})\). Normal neighbourhoods form the basis of a Hausdorff topology on Ran\((\mathcal{X})\).

For \( \emptyset \neq P_0 \subset \mathcal{X} \) we denote by Ran\((\mathcal{X})\)\(P_0 \subset \text{Ran}_+(\mathcal{X}) \) the subspace containing all \( P \subset \mathcal{X} \) with \( P_0 \subset P \). Similarly, for \( z_0 \in \mathcal{X} \) we denote Ran\((\mathcal{X})\)\(z_0 = \text{Ran}_+(\mathcal{X})(z_0) \).

Our definition of Ran\((\mathcal{X})\) differs from the usual one in the literature (e.g. [Lur17, Definition 5.5.1.2]) because we allow also \( \emptyset \) as a point in Ran\((\mathcal{X})\). Note however that our Ran\((\mathcal{X})\) is the topological disjoint union of the singleton \( \{\emptyset\} \) and Ran\(_+\)(\(\mathcal{X}\)).

The following is a particular case of [Lur17, Lemma 5.5.1.8], which is originally due to Beilinson and Drinfeld [BD04].

**Lemma 3.2.** Let \( \mathcal{X} \subset \mathbb{H} \) be path connected and let \( P_0 \subset \mathcal{X} \) be a finite non-empty subset. Then Ran\((\mathcal{X})\)\(P_0 \) is weakly contractible.

The idea of the proof is the following: first note that Ran\((\mathcal{X})\)\(P_0 \) is a path connected abelian topological monoid with multiplication \( \mu: (P, P') \mapsto P \cup P' \) and neutral element \( P_0 \); moreover every element \( P \in \text{Ran}(\mathcal{X})\)\(P_0 \) is idempotent, i.e. \( \mu(P, P) = P \); it follows that every element of the group \( \pi_n(\text{Ran}(\mathcal{X})\)\(z_0), P_0 \) is idempotent for all \( n \geq 1 \), i.e. \( \pi_n(\text{Ran}(\mathcal{X})\)\(z_0), P_0 \) is the trivial group.

With a little more work one can prove the following version of the statement, which does not depend on \( P_0 \). See [Lur17, Theorem 5.5.1.6] for the proof.

**Lemma 3.3.** Let \( \mathcal{X} \subset \mathbb{H} \) be path connected. Then Ran\(_+\)(\(\mathcal{X}\)) is weakly contractible.

**Notation 3.4.** For a nice couple \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) we write Ran\((\mathcal{C}) = \text{Ran}(\mathcal{X}) \) and similarly for the subspaces introduced in Definition 3.1.

### 3.2. Hurwitz sets.

We first define Hur\((\mathcal{C}; Q, G)\) as a set.

**Definition 3.5.** Let \((Q, G)\) be a PMQ-group pair and let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) a nice couple (see Definition 2.3). An element of the Hurwitz set Hur\((\mathcal{C}; Q, G)\) is a configuration \( c = (P, \psi, \varphi) \), where

- \( P = \{z_1, \ldots, z_k\} \) is a finite subset of \( \mathcal{X} \), i.e. \( P \in \text{Ran}(\mathcal{X}) \);
- \( (\psi, \varphi): (\mathcal{Q}(P), \mathcal{G}(P)) \to (Q, G) \) is a map of PMQ-group pairs (see Definitions 2.7 and 2.8 for the PMQ-group pair \((\mathcal{Q}(P), \mathcal{G}(P))\)).
If \( c = (P, \psi, \varphi) \), we say that \( c \) is supported on \( P \); if \( S \) is any subspace of \( \mathcal{X} \) and \( P \subset S \), we say that \( c \) is supported in \( S \). The maps \( \psi \) and \( \varphi \) are called the monodromies of \( c \), with values in \( \mathcal{Q} \) and \( G \) respectively.

Roughly speaking, the monodromy \( \varphi \), with values in \( G \), is defined around all points of \( P \), whereas the monodromy \( \psi \), with values in \( \mathcal{Q} \), is defined only around points of \( P \) which lie in \( \mathcal{X} \setminus \mathcal{J} \). We can think of \( \psi \) as a refinement of \( \varphi \) away from \( \mathcal{J} \): indeed the composition \( c \circ \psi : \Omega(P) \to G \) is equal to \( \varphi|_{\Omega(P)} \), where the map of PMQs \( c : \mathcal{Q} \to G \) is part of the structure of the PMQ-group pair of \((\mathcal{Q}, G)\).

**Notation 3.6.** We usually expand a configuration \( c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) by \( c = (P, \psi, \varphi) \) and use Notation 3.6. Let \( P \) and \( \mathcal{U} \) be adapted coverings of \( P \). Similarly we expand another configuration \( c' \) as \((P', \psi', \varphi')\), and write \( P' = \{z'_1, \ldots, z'_k\} \).

### 3.3. The topology on Hurwitz spaces

We introduce a topology on the set \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \), in the spirit of the topology of the Ran space \( \text{Ran}(\mathcal{C}) \).

**Definition 3.7.** Recall Definition 2.15 and use Notation 3.6. Let \( c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) and let \( \mathcal{U} = (U_1, \ldots, U_k) \) be an adapted covering of \( P = \{z_1, \ldots, z_k\} \). We denote by \( \mathcal{U}(c; \mathcal{U}) = \mathcal{U}_\mathcal{Q}(c; \mathcal{U}) \) the subset of \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) containing all configurations \( c' = (P', \psi', \varphi') \) satisfying the following conditions:

- \( P' \subset \Omega(\mathcal{U}) \) (see Definition 2.15): as a consequence there is a natural inclusion of PMQ-group pairs \( (\Omega(\mathcal{U}), \mathcal{G}(\mathcal{U})) \subset (\Omega(P', \mathcal{U}), \mathcal{G}(P')) \) (see Definition 2.15);
- \( \Omega(P', \mathcal{U}) \) is contained in \( \Omega^{\text{ext}}(P')_\psi \) (see Definition 2.15), and the following composition of maps of PMQ-group pairs is equal to \((\psi, \varphi)\):

\[
\begin{align*}
(\Omega(P), \mathcal{G}(P)) & \xrightarrow{\cong} (\Omega(\mathcal{U}), \mathcal{G}(\mathcal{U})) \subset (\Omega(P', \mathcal{U}), \mathcal{G}(P')) \\
& \xrightarrow{\subseteq} (\Omega^{\text{ext}}(P')_\psi, \mathcal{G}(P')) \xrightarrow{(\psi', \varphi')} (\mathcal{Q}, G),
\end{align*}
\]

where we use the isomorphism discussed in the remark after Definition 2.15 and the map \((\psi', \varphi') : \Omega^{\text{ext}}(P')_\psi \to \mathcal{Q} \) from Proposition 2.14.

Each subset \( \mathcal{U}(c; \mathcal{U}) \) is called a normal neighbourhood of \( c \) in \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \).

Roughly speaking, if \( c' \in \mathcal{U}(c; \mathcal{U}) \), then \( P' \) is obtained from \( P \) by splitting each \( z_i \) into \( r_i \geq 1 \) points \( z'_1, \ldots, z'_{i,r_i} \) inside the neighbourhood \( U_i \) of \( z_i \). The value of the monodromy \( \varphi \) around \( z_i \) is decomposed, as an element of \( G \), in the values of the monodromy \( \varphi' \) around the points \( z'_{i,1}, \ldots, z'_{i,r_i} \), for \( 1 \leq i \leq k \). Similarly, the value of the monodromy \( \psi \) around \( z_i \) is decomposed, as an element of \( \mathcal{Q} \), in the values of the monodromy \( \psi' \) around the points \( z'_{i,1}, \ldots, z'_{i,r_i} \), for \( 1 \leq i \leq l \). See Figure 4 for an example of a configuration \( c \) and another configuration \( c' \) in a normal neighbourhood of \( c \).

**Proposition 3.8.** The subsets \( \mathcal{U}(c; \mathcal{U}) \) for varying \( c \) and \( \mathcal{U} \) form the basis of a Hausdorff topology on the set \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \).

**Proof.** Let \( c_1 = (P_1, \psi_1, \varphi_1) \) and \( c_2 = (P_2, \psi_2, \varphi_2) \) denote two configurations in \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \), let \( k_1 = |P_1| \) and \( k_2 = |P_2| \), and let \( \mathcal{U}_1 = (U_{1,1}, \ldots, U_{1,k_1}) \) and \( \mathcal{U}_2 = (U_{2,1}, \ldots, U_{2,k_2}) \) be adapted coverings of \( P_1 \) and \( P_2 \) respectively; finally, let \( \mathcal{U}(c_1; \mathcal{U}_1) \) and \( \mathcal{U}(c_2; \mathcal{U}_2) \) be the corresponding normal neighbourhoods.

Suppose that \( c' = (P', \psi', \varphi') \) lies in the intersection \( \mathcal{U}(c_1; \mathcal{U}_1) \cap \mathcal{U}(c_2; \mathcal{U}_2) \). Then we can define \( \mathcal{U}' = (U'_{1,1}, \ldots, U'_{l,k}) \) as the family of all convex open sets of the form
Figure 4. On left, a configuration $c = (P, \psi, \varphi)$ in the space $\text{Hur}(X, Y, Q, G)$ and an adapted covering $\underline{U}$ of $P$; on right, another configuration $c'$ in the normal neighbourhood $\mathcal{U}(c, U)$. The drawn loops are labelled with their $Q$-valued monodromy if they belong to $\mathcal{Q}(P)$ and $\mathcal{Q}(P')$ respectively, and are labelled with their $G$-valued monodromy otherwise.

$U_{1,i} \cap U_{2,j}$ that contain at least one point of $P'$. By construction $\underline{U'}$ is a covering of $P'$; we can then find a covering $\underline{U''} = (U''_1, \ldots, U''_{\kappa''})$ of $P'$ which is adapted to $P''$ and is finer than $\underline{U'}$, i.e. each $U''_i$ is contained in some $U'_j$, for all $1 \leq i \leq \kappa''$. It follows from Definition 3.7 that

$$\mathcal{U}(c'; U'') \subseteq \mathcal{U}(c_1; \underline{U}_1) \cap \mathcal{U}(c_2; \underline{U}_2).$$

Hence normal neighbourhoods are the basis of a topology on $\text{Hur}(C; Q, G)$.

To see that this topology is Hausdorff, we fix two distinct configurations $c = (P, \psi, \varphi)$ and $c' = (P', \psi', \varphi')$ in $\text{Hur}(C; Q, G)$. If $P = P'$, then for any adapted covering $\underline{U}$ of $P$ the two normal neighbourhoods $\mathcal{U}(c; \underline{U})$ and $\mathcal{U}(c'; \underline{U})$ are disjoint. If $P \neq P'$, without loss of generality we can assume that there is a point $z \in P \setminus P'$; let $\underline{U}$ and $\underline{U'}$ be adapted coverings of $P$ and $P'$ respectively, such that the connected component of $\underline{U}$ containing $z$ is disjoint from $\underline{U'}$; then $\mathcal{U}(c; \underline{U})$ and $\mathcal{U}(c'; \underline{U'})$ are again disjoint. □

**Notation 3.9.** The space $\text{Hur}(C; Q, G)$ from Proposition 3.8 is called the *Hurwitz space* associated with the nice couple $C$ and the PMQ-group pair $(Q, G)$.

**Definition 3.10.** We define $\varepsilon : \text{Hur}(C; Q, G) \to \text{Ran}(C)$ as the map given by the assignment $\varepsilon : (P, \psi, \varphi) \mapsto P$. 
Note that the preimage of \( \U(P, U) \subset \text{Ran}(\mathcal{C}) \) along \( \varepsilon \) is the disjoint union of all normal neighbourhoods \( \U(\varepsilon, U) \) for \( \varepsilon \) varying in the configurations of \( \text{Hur}(\mathcal{C}; Q, G) \) supported on \( P \); this shows continuity of \( \varepsilon \). Note also that \( \varepsilon : \text{Hur}(\mathcal{C}; Q, G) \to \text{Ran}(\mathcal{C}) \) is a homeomorphism if \( (Q, G) = (1, 1) \), where we use the following notation.

**Notation 3.11.** We denote by \((1, 1)\) the initial and terminal PMQ-group pair, consisting of the trivial PMQ \( \{1\} \) and of the trivial group \( \{1\} \).

**Notation 3.12.** For all nice couples \( \mathcal{C} \) we denote by \((\emptyset, 1, 1) \in \text{Hur}(\mathcal{C}; Q, G)\) the unique configuration \((P, \psi, \varphi)\) with \( P = \emptyset \); note that in this case the maps \( \psi \) and \( \varphi \) are defined on the trivial PMQ and on the trivial group respectively, so they have as images \( \{1\} \subset Q \) and \( \{1\} \subset G \) respectively.

Note that \((\emptyset, 1, 1)\) is an isolated point of the space \( \text{Hur}(\mathcal{C}; Q, G) \); we denote by \( \text{Hur}^+(\mathcal{C}; Q, G) \) the closed subspace \( \text{Hur}(\mathcal{C}; Q, G) \setminus \{(\emptyset, 1, 1)\} \subset \text{Hur}(\mathcal{C}; Q, G) \).

**Definition 3.13.** Let \( P_0 \subset X \) be a finite non-empty subset. We denote by \( \text{Hur}(\mathcal{C}; Q, G)_{P_0} \subset \text{Hur}^+(\mathcal{C}; Q, G) \) the subspace containing all configurations \( \varepsilon = (P, \psi, \varphi) \) with \( P_0 \subseteq P \).

For \( \varepsilon = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; Q, G)_{P_0} \) and an adapted covering \( U \) of \( P \) we denote
\[
\U(\varepsilon, U)_{P_0} = \U(\varepsilon, U) \cap \text{Hur}(\mathcal{C}; Q, G)_{P_0}.
\]

3.4. **Change of ambient space.** Let \( T \subset C \) be a contractible space containing \( * \), and let \( \mathcal{C} = (X, \mathcal{Y}) \) be a nice couple with \( \mathcal{Y} \subset X \subset T \); then for all finite subsets \( P \subset X \) we can give the equivalent definition \( \Theta(P) := \pi_1(T \setminus P, *) \), since the map \( \pi_1(T \setminus P, *) \to \pi_1(C \setminus P, *) \) induced by the inclusion \( T \setminus P \subset C \setminus P \) is an isomorphism. Similarly we can define \( \Omega_\mathcal{C}(P) \) as a subset of \( \pi_1(T \setminus P, *) \), containing \( 1 \) and the conjugacy classes of loops \( \gamma \) in \( T \setminus P \) which are freely homotopic, inside \( T \setminus P \), to a small simple closed curve spinning clockwise around one of the points of \( P \setminus \mathcal{Y} \). We thus have an alternative construction of the Hurwitz set \( \text{Hur}(\mathcal{C}; Q, G) \), in which we use as ambient space not the entire complex plane \( C \), but its subspace \( T \). The topology on the Hurwitz set from Proposition 3.8 can be recovered as well by restricting to coverings \( U \subset T \).

**Definition 3.14.** We denote by \( \text{Hur}^T(\mathcal{C}; Q, G) \) the Hurwitz space constructed using \( T \) as ambient space. For \( P \subset X \) we denote \( \Theta^T(P) = \pi_1(T \setminus P, *) \), and we denote by \( \Omega^T_\mathcal{C}(P) \subset \Theta^T(P) \) the subset corresponding to \( \Omega_\mathcal{C}(P) \) under the natural identification \( \Theta^T(P) \cong \Theta(P) \) induced by the inclusion \( T \setminus P \subset C \setminus P \).

The Hurwitz space \( \text{Hur}(\mathcal{C}; Q, G) \) is by definition the space \( \text{Hur}^T(\mathcal{C}; Q, G) \). Suppose now that we have a nice couple \( \mathcal{C} = (X, \mathcal{Y}) \) and two contractible subspaces \( T, T_1 \subset C \) satisfying the following properties:

- \( * \in T_1 \subset T \);
- \( X \subset T \);
- \( X \) splits as a disjoint union \( X_1 \sqcup X_2 \), with \( X_1 \subset \overline{T}_1 \) and \( X_2 \) contained in the interior of \( T \setminus T_1 \).

Denote by \( \mathcal{Y}_1 = \mathcal{Y} \cap X_1 \) and by \( \mathcal{C}_1 \) the nice couple \((X_1, \mathcal{Y}_1)\); then every finite subset \( P \subset X \) naturally decomposes as a union of \( P_1 = P \cap X_1 \) and \( P_2 = P \cap X_2 \); moreover the inclusion \( T_1 \setminus P_1 \subset T \setminus P \) induces an inclusion of PMQ-group pairs
\[
\U^T_1(P_1, P) : (\Omega^T_{\mathcal{C}_1}(P_1), \Theta^T_1(P_1)) \to (\Omega^T_{\mathcal{C}}(P), \Theta^T(P)),
\]
and if \( (\varphi, \psi) : (\Omega^T_{\mathcal{C}}(P), \Theta^T(P)) \to (Q, G) \) is a map of PMQ-group pairs, we can consider the restriction \( (\varphi, \psi) \circ \U^T_1(P_1, P) : (\Omega^T_{\mathcal{C}_1}(P_1), \Theta^T_1(P_1)) \to (Q, G) \).
**Definition 3.15.** The above construction gives a map of sets

$$i_{T_1}^T : \text{Hur}^T(\mathcal{C}; \mathcal{Q}, G) \to \text{Hur}^{T_1}(\mathcal{C}_1; \mathcal{Q}, G),$$

defined by sending $c = (P, \psi, \varphi)$ to $c' = (P', \psi', \varphi')$, where $P' = P \cap T_1$ and $(\psi', \varphi') = (\varphi, \psi) \circ i_{T_1}^T(P, P)$. See Figure 5.

**Figure 5.** On left: a contractible subspace $T_1 \subset \mathbb{C}$; a nice couple $\mathcal{C} = (X, Y)$ decomposing into two nice couples $\mathcal{C}_1$ and $\mathcal{C}_2$, one contained in $\hat{T}_1$ and the other in the interior of the complement of $T_1$; a configuration $c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G)$. On right, the image of $c$ in $\text{Hur}^{T_1}(\mathcal{C}_1; \mathcal{Q}, G)$ along $i_{T_1}^T$.

To prove that $i_{T_1}^T$ is continuous, let $c$ and $c'$ be as in Definition 3.15 and choose an adapted covering $U' \subset T$ of $P'$ with respect to the nice couple $\mathcal{C}_1$; since by hypothesis $P_2$ is contained in the interior of $T \setminus T_1$, we can extend $U'$ to an adapted covering of $P$ with respect to the nice couple $\mathcal{C}$, by adjoining open sets contained in $T \setminus T_1 \subset \mathbb{C}$ and covering $P_2$. We then have that $i_{T_1}^T$ sends $\mathcal{U}(c; U) \subset \text{Hur}^T(\mathcal{C}; \mathcal{Q}, G)$ inside $\mathcal{U}(c'; U') \subset \text{Hur}^{T_1}(\mathcal{C}_1; \mathcal{Q}, G)$. The canonical homeomorphism from Definition 3.14 can be rewritten as

$$i_T^C : \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \cong \text{Hur}^T(\mathcal{C}; \mathcal{Q}, G).$$

Conversely, let $\mathcal{C} = (X, Y)$ be a nice couple, and let $T_1, T_2$ be contractible subspaces of $\mathbb{C}$ containing $\ast$, such that $T_1 \cap T_2$ is contractible and disjoint from $X$. Let $T = T_1 \cup T_2$, and assume that $T$ is also contractible.

Suppose that $X = X_1 \cup X_2$, with $X_1 \subseteq \hat{T}_1$ and $X_2 \subseteq \hat{T}_2$; denote $Y_1 = Y \setminus X_1$ and $Y_2 = Y \setminus X_2$, and denote $\mathcal{C}_1 = (X_1, Y_1)$ and $\mathcal{C}_2 = (X_2, Y_2)$.

Let $c_1 = (P_1, \psi_1, \varphi_1) \in \text{Hur}^{T_1}(\mathcal{C}_1; \mathcal{Q}, G)$ and $c_2 = (P_2, \psi_2, \varphi_2) \in \text{Hur}^{T_1}(\mathcal{C}_2; \mathcal{Q}, G)$; we can define a configuration $(P, \varphi, \psi) \in \text{Hur}^T(\mathcal{C}; \mathcal{Q}, G)$ as follows:

- $P = P_1 \cup P_2$;
by the theorem of Seifert and van Kampen the group $\Theta^T(P)$ decomposes naturally as a free product $\Theta^T_1(P_1) \ast \Theta^T_2(P_2)$; we define $\varphi: \Theta^T(P) \to G$ as $\varphi_1 \ast \varphi_2$.

• the inclusions $\Theta^T_1(P_1) \subset \Theta^T(P)$ and $\Theta^T_2(P_2) \subset \Theta^T(P)$ restrict to inclusions $\Omega^T_{\epsilon_1}(P_1) \subset \Omega^T(P)$ and $\Omega^T_{\epsilon_2}(P_2) \subset \Omega^T(P)$; using [Bia21n, Theorem 3.3] we can define $\psi: \Omega^T(P) \to G$ by imposing that it restricts to $\psi_1$ on $\Omega^T_{\epsilon_1}(P_1)$ and to $\psi_2$ on $\Omega^T_{\epsilon_2}(P_2)$, and that $(\psi, \varphi): (\Theta^T(P), \Omega^T_1(P)) \to (Q, G)$ is a map of PMQ-group pairs.

**Definition 3.16.** The above construction gives a map of sets

$$ - \sqcup - : \text{Hur}^T_1(\mathcal{C}_1; Q, G) \times \text{Hur}^T_2(\mathcal{C}_2; Q, G) \to \text{Hur}^T(\mathcal{C}; Q, G). $$

To prove that $- \sqcup -$ is continuous, note that if $\mathcal{U}_1 \subset \mathcal{T}_1$ is an adapted covering of $P_1$ with respect to $\mathcal{C}_1$, and $\mathcal{U}_2 \subset \mathcal{T}_2$ is an adapted covering of $P_2$ with respect to $\mathcal{C}_2$, then $- \sqcup -$ restricts to a bijection between $\mathcal{U}(\epsilon_1, \mathcal{U}_1) \times \mathcal{U}(\epsilon_2, \mathcal{U}_2)$ and $\mathcal{U}(\epsilon, \mathcal{U})$, where $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \subset \mathcal{T}$ is also an adapted covering of $P$. This argument shows in fact that $- \sqcup -$ is a homeomorphism, with inverse given by the map

$$(i_{\mathcal{T}_1}, i_{\mathcal{T}_2}): \text{Hur}^T(\mathcal{C}; Q, G) \to \text{Hur}^T_1(\mathcal{C}_1; Q, G) \times \text{Hur}^T_2(\mathcal{C}_2; Q, G).$$

In Appendix A we will briefly generalize the above discussion to the case in which $T_1$ and $T_2$ are generic orientable surfaces with a basepoint on the boundary.

4. Functoriality

The construction of the space $\text{Hur}(\mathcal{C}; Q, G)$ depends on the nice couple $\mathcal{C}$ and on the PMQ-group pair $(Q, G)$. In this section we study how maps of PMQ-group pairs and maps of nice couples induce maps on the corresponding Hurwitz spaces.

4.1. **Functoriality in the PMQ-group pair.** We fix a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ throughout the subsection (see Definition 2.3); our aim is to prove the following theorem.

**Theorem 4.1.** The assignment $(Q, G) \mapsto \text{Hur}(\mathcal{C}; Q, G)$ extends to a functor from the category of PMQ-group pairs to the category of topological spaces.

Let $(Q, G)$ and $(Q', G')$ be two PMQ-group pairs, and let $(\Psi, \Phi): (Q, G) \to (Q', G')$ be a morphism of PMQ-group pairs. In the following we define an induced map $(\Psi, \Phi)_*: \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q', G')$.

Given a configuration $\epsilon = (P, \psi, \varphi)$ in the set $\text{Hur}(\mathcal{C}; Q, G)$, we associate with it a configuration $\epsilon' = (P', \psi', \varphi')$ in the set $\text{Hur}(\mathcal{C}; Q', G')$, where:

• $P' = P$;
• $(\psi', \varphi'): (\Omega(P), \Theta(P)) \to (Q', G')$ is the composition $(\Psi, \Phi) \circ (\psi, \varphi)$:

$$(\psi', \varphi'): (\Omega(P), \Theta(P)) \xrightarrow{(\psi, \varphi)} (Q, G) \xrightarrow{(\Psi, \Phi)} (Q', G').$$

This construction gives a map of sets

$$(\Psi, \Phi)_*: \text{Hur}(\mathcal{C}; \Psi, \Phi): \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q', G'),$$

and we obtain a functor $\text{Hur}(\mathcal{C}; -)$ from the category $\text{PMQGrp}$ of PMQ-group pairs to the category $\text{Set}$ of sets.

To show that $(\Psi, \Phi)_*$ is a continuous map, let $\epsilon' = (P', \psi', \varphi') \in \text{Hur}(\mathcal{C}; Q', G')$ and let $\mathcal{U}(\epsilon', \mathcal{U}') \subset \text{Hur}(\mathcal{C}; Q', G')$ be a normal neighbourhood associated with an adapted covering $\mathcal{U}'$ of $P'$ (see Definition 3.7). Then the preimage of $\mathcal{U}(\epsilon', \mathcal{U}')$ along
$(\Psi, \Phi)_*$ is the disjoint union $\bigsqcup \mathcal{U}(\epsilon, L')$, where $\epsilon$ ranges over all configurations in the fibre $(\Psi, \Phi)^{-1}(\epsilon')$. Thus we have a functor $\operatorname{Hur}(\mathcal{C}; \cdot)$ from $\text{PMQGrp}$ to the category $\text{Top}$ of topological spaces.

Note also that if $(\Psi, \Phi)$ is an injective map of PMQ-group pairs, then it induces an inclusion of spaces

$$(\Psi, \Phi)_* = \operatorname{Hur}(\mathcal{C}; \Psi, \Phi): \operatorname{Hur}(\mathcal{C}; Q, G) \to \operatorname{Hur}(\mathcal{C}; Q', G'),$$
i.e., the map $(\Psi, \Phi)_*$ is a homeomorphism onto its image. In particular we can take $Q' = Q$ to be the completion of $Q$, and consider the inclusion of PMQ-group pairs $(\hat{Q}, G) \subset (Q, G)$, yielding an inclusion of $\operatorname{Hur}(\mathcal{C}; Q, G)$ into the Hurwitz space $\operatorname{Hur}(\mathcal{C}; Q, G)$ associated with a PMQ-group pair consisting of a complete PMQ and a group. We further notice that the inclusion $\operatorname{Hur}(\mathcal{C}; Q, G) \subset \operatorname{Hur}(\mathcal{C}; \hat{Q}, G)$ is open: given $\epsilon = (P, \psi, \varphi) \in \operatorname{Hur}(\mathcal{C}; Q, G)$ and an adapted covering $\mathcal{U}$ of $P$, the normal neighbourhood $\mathcal{U}(\epsilon; \mathcal{U}) \subset \operatorname{Hur}(\mathcal{C}; Q, G)$ is mapped bijectively onto the corresponding normal neighbourhood $\mathcal{U}'(\xi; \mathcal{U}) \subset \operatorname{Hur}(\mathcal{C}; \hat{Q}, G)$. This is ultimately a consequence of the fact that $\mathcal{J}(\hat{Q}) = \hat{Q} \setminus Q$ is an ideal in $\hat{Q}$.

Another consequence of the functoriality in the PMQ-group pair is the following. For all PMQ-group pairs $(\mathcal{Q}, G)$ there is a unique inclusion of PMQ-group pairs $(1, 1) \to (\mathcal{Q}, G)$ (see Notation 5.11). This induces an inclusion $\operatorname{Hur}(\mathcal{C}; 1, 1) \to \operatorname{Hur}(\mathcal{C}; Q, G)$, and using the homeomorphism $\varepsilon: \operatorname{Hur}(\mathcal{C}; 1, 1) \cong \operatorname{Ran}(\mathcal{C})$ (see Definition 3.10), we obtain a natural inclusion $\operatorname{Ran}(\mathcal{C}) \subset \operatorname{Hur}(\mathcal{C}; Q, G)$, for all nice couples $\mathcal{C}$ and all PMQ-group pairs $(\mathcal{Q}, G)$.

Viceversa, we can consider the natural projection $\varepsilon: \operatorname{Hur}(\mathcal{C}; Q, G) \to \operatorname{Ran}(\mathcal{C})$ as the natural map $\operatorname{Hur}(\mathcal{C}; Q, G) \to \operatorname{Hur}(\mathcal{C}; 1, 1)$ induced by the unique map of PMQ-group pairs $(\mathcal{Q}, G) \to (1, 1)$.

4.2. **Two categories of nice couples.** We now fix a PMQ-group pair $(\mathcal{Q}, G)$ throughout the rest of the section. To discuss functoriality of Hurwitz spaces in the nice couple $\mathcal{C}$, we first need a good notion of map between nice couples.

**Definition 4.2.** Recall Definition 2.3 and let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ be two nice couples. A **morphism of nice couples** $\xi: \mathcal{C} \to \mathcal{C}'$ is a continuous, pointed map $\xi: (\mathcal{C}, *) \to (\mathcal{C}', *)$ such that the following properties hold:

1. $\xi$ is semi-algebraic;
2. $\xi$ is a proper map $\mathcal{C} \to \mathcal{C}'$, and is orientation-preserving in the sense that the induced map $\xi^*: H^2_c(\mathcal{C}) \to H^2_c(\mathcal{C})$ in cohomology with compact support is the identity;
3. $\xi$ restricts to maps $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$;
4. for all $z \in \mathcal{C}$ the fibre $\xi^{-1}(z) \subset \mathcal{C}$ is non-empty, compact and contractible;
5. for all $z \in \mathcal{X}' \setminus \mathcal{Y}'$ the fibre $\xi^{-1}(z)$ contains at most one point of $\mathcal{X} \setminus \mathcal{Y}$.

The composition of two morphisms $\xi: \mathcal{C} \to \mathcal{C}'$ and $\xi': \mathcal{C}' \to \mathcal{C}''$ is defined as the composition of maps $\xi' \circ \xi: (\mathcal{C}, *) \to (\mathcal{C}, *)$. We obtain a category $\text{NC}$ of nice couples.

Property (4) ensures that a morphism $\xi$ of nice couples is in particular a **local homotopy equivalence** in the following sense: if $\mathcal{J} \subset \mathcal{C}$ is a semi-algebraic set, then the restriction $\xi: \mathcal{X}^{-1}(\mathcal{J}) \to \mathcal{X}$ is a homotopy equivalence; more generally, if $\mathcal{J} \subset \mathcal{J}' \subset \mathcal{C}$ are two semi-algebraic sets, then $\xi: (\mathcal{X}^{-1}(\mathcal{J}), \mathcal{X}^{-1}(\mathcal{J}')) \to (\mathcal{J}, \mathcal{J}')$ is a homotopy equivalence of couples. This is an application of the main theorem of [Sma57].
The previous remark holds in particular when \( \mathcal{J} = (\xi')^{-1}(z) \) is a fibre of another morphism of nice couples \( \xi' \), over some point \( z \in \mathbb{C} \): thus the composition \( \xi \circ \xi' \) also satisfies property (4) of Definition 4.4, properties (1),(2),(3) and (5) are also automatically satisfied by the composition \( \xi \circ \xi' \).

It is useful to remark also the following property: if \( D' \subset \mathbb{C} \) is homeomorphic to a disc and \( \zeta \colon (\mathbb{C}, *) \rightarrow (\mathbb{C}, *) \) satisfies properties (1),(2) and (4), then also \( \xi^{-1}(D') \) is homeomorphic to a disc.

We will sometimes need to relax condition (5) in Definition 4.2, hence we give the following definition.

**Definition 4.3.** Let \( \mathcal{C} \) and \( \mathcal{C}' \) be nice couples as in Definition 4.2. A lax morphism of nice couples is a map \( \zeta \colon (\mathbb{C}, *) \rightarrow (\mathbb{C}, *) \) satisfying all conditions in Definition 4.2 except, possibly, condition (5). We obtain a category \( \mathcal{LNC} \) of nice couples with lax morphisms.

Note that \( \mathcal{NC} \) is a subcategory of \( \mathcal{LNC} \) containing all objects, but not all morphisms. Whenever we refer to a morphism of nice couples without specifying the word “lax”, we will assume that condition (5) in Definition 4.2 holds.

### 4.3. Functoriality in \( \mathcal{NC} \)

In this subsection we prove the following theorem.

**Theorem 4.4.** The assignment \( \mathcal{C} \mapsto \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) extends to a functor from the category \( \mathcal{NC} \) to the category of topological spaces.

Fix two nice couples \( \mathcal{C} \) and \( \mathcal{C}' \) and let \( \zeta \colon \mathcal{C} \rightarrow \mathcal{C}' \) be a morphism of nice couples. In the following we construct an induced map \( \xi_* \colon \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \rightarrow \text{Hur}(\mathcal{C}'; \mathcal{Q}, G) \).

Given a configuration \( \xi = (P, \psi, \varphi) \) in the set \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \), we associate with it a configuration \( \xi' = (P', \psi', \varphi') \) in the set \( \text{Hur}(\mathcal{C}'; \mathcal{Q}, G) \) as follows. First, we define \( P' = \xi(P) \), which is a finite subset of \( \mathcal{X} \), i.e. \( P' \in \text{Ran}(\mathcal{X}) \).

To define \( \varphi' \), note that \( \xi \) restricts to a homotopy equivalence \( \mathcal{C} \setminus \xi^{-1}(P') \rightarrow \mathcal{C} \setminus P \); in particular we obtain an isomorphism of groups

\[
\mathcal{G}(P') \cong \pi_1 \left( \mathcal{C} \setminus \xi^{-1}(P'), * \right). 
\]

Note also that the inclusion \( \mathcal{C} \setminus \xi^{-1}(P') \subseteq \mathcal{C} \setminus P \) induces a map of groups

\[
\pi_1 \left( \mathcal{C} \setminus \xi^{-1}(P'), * \right) \rightarrow \mathcal{G}(P).
\]

We denote by \( \xi^* \colon \mathcal{G}(P') \rightarrow \mathcal{G}(P) \) the composition \( \mathcal{G}(P') \cong \pi_1 \left( \mathcal{C} \setminus \xi^{-1}(P'), * \right) \rightarrow \mathcal{G}(P) \). We then define \( \varphi' \colon \mathcal{G}(P') \rightarrow G \) as the composition \( \varphi \circ \xi^* \).

**Lemma 4.5.** The map of groups \( \xi^* \colon \mathcal{G}(P') \rightarrow \mathcal{G}(P) \) restricts to a map of PMQs \( \mathcal{Q}_{\mathcal{C}'}(P') \rightarrow \mathcal{Q}_\mathcal{C}(P) \).

The proof of Lemma 4.5 is in Subsection 4.4 of the appendix. We can now define \( \psi' = \psi \circ \xi^* \colon \mathcal{Q}_{\mathcal{C}'}(P') \rightarrow \mathcal{Q} \). Note that \( \xi^* \colon (\mathcal{Q}_{\mathcal{C}'}(P'), \mathcal{G}(P')) \rightarrow (\mathcal{Q}_\mathcal{C}(P), \mathcal{G}(P)) \) is a map of PMQ-group pairs. Therefore \( \xi' \) is a well-defined configuration in the set \( \text{Hur}(\mathcal{C}'; \mathcal{Q}, G) \). This construction gives a map of sets

\[
\xi_* = \text{Hur}(\xi; \mathcal{Q}, G) \colon \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \rightarrow \text{Hur}(\mathcal{C}'; \mathcal{Q}, G),
\]

and we obtain a functor \( \text{Hur}(\cdot; \mathcal{Q}, G) \) from \( \mathcal{NC} \) to \( \text{Set} \). See Figure 6.

To show that \( \xi_* \) is continuous, let \( \epsilon \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) and \( \epsilon' = \xi_*(\epsilon) \), and use Notation 3.2 to let \( \mathcal{U}(\epsilon, \mathcal{U}') \subset \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) be a normal neighbourhood associated with an adapted covering \( \mathcal{U}' \) of \( P' \) (see Definition 3.7). We have \( \xi(P) \subset \mathcal{U}' \), hence we can find an adapted covering \( \mathcal{U} \) of \( P \), such that \( \xi(\mathcal{U}) \subset \mathcal{U}' \). Then the entire
Figure 6. On left, a configuration $\zeta \in \text{Hur}(\mathcal{X}, \mathcal{Y}, Q, G)$; on right, its image $\zeta' \in \text{Hur}(\mathcal{X}', \mathcal{Y}', Q, G)$ along the map $\xi^*$ induced by a morphism of nice couples $\xi$. The morphism $\xi$ has the effect of collapsing horizontally a rectangular region of $\mathcal{X}$ onto the vertical segment $\mathcal{Y}'$, and of expanding horizontally the complement of this rectangular region. The thick horizontal segment is the preimage along $\xi$ of $\xi(z_2) = \xi(z_3)$. The dashed loop on left is the image of the dashed loop on right along $\xi^*$.

normal neighbourhood $\U(c; \mathcal{U}) \subseteq \text{Hur}(\mathcal{C}; Q, G)$ is mapped along $\xi^*$ inside $\U(c'; \mathcal{U}')$. Thus we obtain a functor $\text{Hur}(\mathcal{C}); Q, G)$ from $\mathcal{C}$ to $\text{Top}$. Note that if $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ are two nice couples, and if $\mathcal{X} \subseteq \mathcal{X}'$, $\mathcal{Y} \subseteq \mathcal{Y}'$ and $\mathcal{Y} = \mathcal{X} \cap \mathcal{Y}'$, then $\text{Id}_\mathcal{C}$ is a morphism of nice couples $\mathcal{C} \to \mathcal{C}'$. The corresponding map $(\text{Id}_\mathcal{C})_* : \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}'; Q, G)$ is an inclusion of spaces: more precisely, $\text{Hur}(\mathcal{C}; Q, G)$ contains all configurations $\zeta \in \text{Hur}(\mathcal{C}'; Q, G)$ supported on $\mathcal{X}$ (see Definition 3.5).

In particular, given a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$, by Definition 2.3 the space $\mathcal{Y}$ is closed in $\mathcal{X} \subseteq \mathbb{H}$: this means that the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $\mathbb{H}$ is contained in $\mathcal{X}$, and $\mathcal{Y} = \mathcal{X} \cap \overline{\mathcal{Y}}$. Then $\text{Id}_\mathcal{C}$ is a morphism of nice couples $\mathcal{C} = (\mathcal{X}, \mathcal{Y}) \to \mathcal{C} := (\mathcal{X}, \mathcal{Y})$. Thus every Hurwitz space $\text{Hur}(\mathcal{C}; Q, G)$ can be regarded as a subspace of a Hurwitz space $\text{Hur}(\mathcal{C}; Q, G)$ associated with a nice couple of closed subspaces of $\mathbb{H}$.

4.4. A weak form of enriched functoriality. One can try to consider $\mathcal{C}$ as a category enriched in topological spaces: for all nice couples $\mathcal{C}$ and $\mathcal{C}'$ one can consider the compact-open topology on the set of morphisms $\xi : \mathcal{C} \to \mathcal{C}'$, considered as a subset of all continuous maps $\xi : \mathbb{C} \to \mathbb{C}$. The functor $\text{Hur}(\mathcal{C}); Q, G)$ is then likely to be a $\text{Top}$-enriched functor from $\mathcal{C}$ to $\text{Top}$. We shall not attempt to
prove this property in general, rather we shall restrict our attention to the following proposition.

**Proposition 4.6.** Let $\mathcal{C}=(\mathcal{X},\mathcal{Y})$ and $\mathcal{C}'=(\mathcal{X}',\mathcal{Y}')$ be nice couples and let $(\mathcal{Q},\mathcal{G})$ a PMQ-group pair. Let $\mathcal{S}$ be a topological space, and let $\mathcal{H}:\mathbb{C} \times \mathcal{S} \to \mathbb{C}$ be a continuous map, such that for all $s \in \mathcal{S}$ the map $\mathcal{H}(-,s):\mathbb{C} \to \mathbb{C}$ is a morphism of nice couples $\mathcal{C} \to \mathcal{C}'$ (see Definition 4.3). Let

$$\mathcal{H}_*: \text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G}) \times \mathcal{S} \to \text{Hur}(\mathcal{C}';\mathcal{Q},\mathcal{G})$$

be the map of sets defined by $\mathcal{H}_*(\varepsilon,s) = (\mathcal{H}(-,s))_*(\varepsilon)$. Then $\mathcal{H}_*$ is continuous.

**Proof.** Fix $(\varepsilon,s) \in \text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G}) \times \mathcal{S}$, let $\varepsilon' = \mathcal{H}_*(\varepsilon,s)$ and use Notation 3.6. Let $\mathcal{U}'$ be an adapted covering of $\mathcal{P}'$ and let $\mathcal{U}(\varepsilon',\mathcal{U}') \subset \text{Hur}(\mathcal{C}';\mathcal{Q},\mathcal{G})$ be the corresponding normal neighbourhood. By continuity of $\mathcal{H}$ we can find a neighbourhood $V \subset \mathcal{S}$ of $s$ and an adapted covering $\mathcal{U}$ of $\mathcal{P}$ such that $\mathcal{H}$ sends $\mathcal{U} \times V$ inside $\mathcal{U}'$: here we regard $\mathcal{U}$ and $\mathcal{U}'$ as subsets of $\mathcal{C}$. Then $\mathcal{H}_*$ sends the product neighbourhood $\mathcal{U}(\varepsilon,\mathcal{U}) \times V \subset \text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G}) \times \mathcal{S}$ inside $\mathcal{U}(\varepsilon',\mathcal{U}')$. $\square$

4.5. **Functoriality in LNC.** In this subsection we assume that $\mathcal{Q}$ is complete, and write $\mathcal{Q} = \mathcal{Q}$ to stress this choice; hence we work with the PMQ-group pair $(\mathcal{Q},\mathcal{G})$. We prove the following theorem.

**Theorem 4.7.** The assignment $\mathcal{C} \mapsto \text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G})$ extends to a functor from the category $\text{LNC}$ to the category of topological spaces.

Let $\mathcal{C}=(\mathcal{X},\mathcal{Y})$ be a nice couple, and let $\varepsilon = (P,\psi,\varphi) \in \text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G})$. By Definition 3.5 $\psi$ is a map of PMQs defined on $\Omega(P)$; using the completeness of $\mathcal{Q}$, Proposition 2.11 implies the equality $\Omega^{ext}(P) = \Omega^{ext}(P)_{\psi}$ (see also Definitions 2.10 and 2.13). Proposition 2.13 yields a map of PMQ-group pairs

$$(\psi^{ext},\varphi): (\Omega^{ext}(P),\mathcal{G}(P)) \to (\mathcal{Q},\mathcal{G})$$

extending $(\psi,\varphi): (\Omega(P),\mathcal{G}(P)) \to (\mathcal{Q},\mathcal{G})$.

**Definition 4.8.** We define a set $\text{Hur}^{ext}(\mathcal{C};\mathcal{Q},\mathcal{G})$: it contains triples $\varepsilon = (P,\psi,\varphi)$, where $P \in \text{Ran}(\mathcal{X})$ is a finite subset of $\mathcal{X}$, and $(\psi,\varphi): (\Omega^{ext}(P),\mathcal{G}(P)) \to (\mathcal{Q},\mathcal{G})$ is a map of PMQ-group pairs.

The previous discussion implies that the sets $\text{Hur}^{ext}(\mathcal{C};\mathcal{Q},\mathcal{G})$ and $\text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G})$ are in natural bijection. We can use this bijection to transfer the topology of $\text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G})$ to $\text{Hur}^{ext}(\mathcal{C};\mathcal{Q},\mathcal{G})$: in particular, for a configuration $\varepsilon = (P,\psi,\varphi) \in \text{Hur}^{ext}(\mathcal{C};\mathcal{Q},\mathcal{G})$ and for an adapted covering $\mathcal{U}$ of $\mathcal{P}$, the normal neighbourhood $\Omega(\varepsilon,\mathcal{U}) \subset \text{Hur}^{ext}(\mathcal{C};\mathcal{Q},\mathcal{G})$ contains all configurations $(P',\psi',\varphi')$ such that

- $P' \subset \mathcal{U}$; as a consequence there is a natural inclusion of PMQ-group pairs $(\Omega(\mathcal{U}),\mathcal{G}(\mathcal{U})) \subseteq (\Omega(P',\mathcal{U}),\mathcal{G}(P'))$ (see Definition 2.13);
- the following composition of maps of PMQ-group pairs is equal to the restriction of $(\psi,\varphi)$ on the PMQ-group pair $(\Omega(P),\mathcal{G}(P))$: $\Omega^{ext}(P',\mathcal{U}),\mathcal{G}(P')) \xrightarrow{((\psi')^{ext},\varphi')} (\mathcal{Q},\mathcal{G})$. 

\[
\begin{align*}
\text{Hur}(\mathcal{C};\mathcal{Q},\mathcal{G}) & \xrightarrow{\cong} (\Omega(\mathcal{U}),\mathcal{G}(\mathcal{U})) & \text{l.c.} & (\Omega(P',\mathcal{U}),\mathcal{G}(P')) \\
\xrightarrow{\subseteq} & (\Omega^{ext}(P'),\mathcal{G}(P')) & \xrightarrow{((\psi')^{ext},\varphi')} & (\mathcal{Q},\mathcal{G}).
\end{align*}
\]
Given a lax morphism of nice couples $\xi: \mathcal{C} \to \mathcal{C}'$, we can now follow the same procedure used in Subsection 4.3 and define a continuous map $\xi_\ast: \text{Hur}^{\text{ext}}(\mathcal{C}, \hat{Q}, G) \to \text{Hur}^{\text{ext}}(\mathcal{C}', \hat{Q}, G)$. The only difference is that Lemma 4.5 is replaced by the following lemma, whose proof is in Subsection 5.3 of the appendix.

**Lemma 4.9.** Let $\xi: \mathcal{C} \to \mathcal{C}'$ be a lax morphism of nice couples, let $P \subset X$ and let $P' = \xi(P) \subset X'$. Then the map of groups $\xi^\ast: \mathcal{G}(P') \to \mathcal{G}(P)$ restricts to a map $\Omega_{\xi^\ast}(P') \to \Omega_{\xi^\ast}(P)$ of PMQs.

Continuity of $\xi_\ast: \text{Hur}^{\text{ext}}(\mathcal{C}, \hat{Q}, G) \to \text{Hur}^{\text{ext}}(\mathcal{C}', \hat{Q}, G)$ is proved in the same way as in the case of a (non-lax) morphism of nice couples; similarly one can generalise Proposition 4.6 to the following.

**Proposition 4.10.** Let $\mathcal{C} = (X, \mathcal{Y})$ and $\mathcal{C}' = (X', \mathcal{Y}')$ be nice couples and let $(\hat{Q}, G)$ be a complete PMQ-group pair. Let $S$ be a topological space, and let $\mathcal{H}: \mathbb{C} \times S \to \mathbb{C}$ be a continuous map, such that for all $s \in S$ the map $\mathcal{H}(-, s): \mathbb{C} \to \mathbb{C}$ is a lax morphism of nice couples $\mathcal{C} \to \mathcal{C}'$ (see Definition 4.2). Let

$$\mathcal{H}_s: \text{Hur}(\mathcal{C}, \hat{Q}, G) \times S \to \text{Hur}(\mathcal{C}', \hat{Q}, G)$$

be the map of sets defined by $\mathcal{H}_s(\epsilon, s) = (\mathcal{H}(-, s))_s(\epsilon)$. Then $\mathcal{H}_s$ is continuous.

5. Applications of functoriality

In this section we apply the results from Section 4 to obtain basic information about Hurwitz spaces; moreover we introduce the operation of external product.

5.1. **Product structure for normal neighbourhoods.** The first application combines the discussion of Subsection 4.3 with the functoriality with respect to inclusions of nice couples.

Let $\mathcal{C} = (X, \mathcal{Y})$ be a nice couple, let $(Q, G)$ be a PMQ-group pair, let $\epsilon = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; Q, G)$, let $\mathcal{U}$ be an adapted covering of $P$, use Notations 2.4 and 2.6. We are interested in the topology of the normal neighbourhood $\mathcal{U}(\epsilon, \mathcal{U}) = \mathcal{U}_\epsilon(\mathcal{C}; \mathcal{U}) \subset \text{Hur}(\mathcal{C}; Q, G)$.

We can fix arcs $\zeta_1, \ldots, \zeta_k$ as in Definition 2.10, the arc $\zeta_i$ joins $*$ with a point on $\partial U_i$. For all $1 \leq i \leq k$ we define $T_i = \zeta_i \cup U_i$, $X_i = X \cap U_i$ and $Y_i = Y \cap U_i$. Let moreover $T = \bigcup_{i=1}^k T_i$, $X = \bigcup_{i=1}^k X_i$ and $Y = \bigcup_{i=1}^k Y_i$. Finally, let $\mathcal{C}_i$ be the nice couple $(X_i, Y_i)$, and let $\mathcal{C}$ be the nice couple $(X, Y)$.

We have an open inclusion $(\text{Id}_{\mathcal{C}})_\ast: \text{Hur}(\mathcal{C}; Q, G) \subset \text{Hur}(\mathcal{C}; Q, G)$ restricting to a homeomorphism of normal neighbourhoods $\mathcal{U}_\epsilon(\mathcal{C}; \mathcal{U}) \cong \mathcal{U}_\epsilon(\epsilon, \mathcal{U})$. In fact, the normal neighbourhood $\mathcal{U}_\epsilon(\epsilon, \mathcal{U})$ coincides with the entire space $\text{Hur}(\mathcal{C}; Q, G)$, because $X$ is contained in $\mathcal{U}$.

We can now consider $\mathcal{C}$ as a nice couple of spaces that are contained in the interior of $T$, and identify $\text{Hur}(\mathcal{C}; Q, G)$ with $\text{Hur}^T(\mathcal{C}; Q, G)$ along $\text{Id}_{\mathcal{C}}$.

We can then use the decomposition given by Definition 3.16 and write

$$\text{Hur}^T(\mathcal{C}; Q, G) \cong \prod_{i=1}^k \text{Hur}^{T_i}(\mathcal{C}_i; Q, G).$$
where the homeomorphism is given by the product of the maps $\bar{t}_i$. The above sequence of homeomorphisms reads
\[
\mathcal{U}(\zeta; U) \cong \text{Hur}(\mathcal{C}; Q, G) \cong \text{Hur}^\tau(\mathcal{C}; Q, G) \cong \prod_{i=1}^{k} \text{Hur}^{\tau_i}(\mathcal{C}_i; Q, G).
\]

Let $\zeta \in \mathcal{U}(\zeta; U)$ correspond to the sequence $(\zeta_1, \ldots, \zeta_k)$ along the above identification, where $\zeta_i \in \text{Hur}^{\tau_i}(\mathcal{C}_i; Q, G)$.

Note that $\zeta_i$ is supported on the singleton $\{z_i\}$. We consider the composition of the homeomorphism $(\mathcal{C}_1^\zeta)_1^{-1}: \text{Hur}^{\tau_1}(\mathcal{C}_1; Q, G) \to \text{Hur}(\mathcal{C}_1; Q, G)$ with the open inclusion $(\text{Id}_{\mathcal{C}_1})_*: \text{Hur}(\mathcal{C}_1; Q, G) \subset \text{Hur}(\mathcal{C}; Q, G)$, giving an embedding of $\text{Hur}^{\tau_1}(\mathcal{C}_1; Q, G)$ in $\text{Hur}(\mathcal{C}; Q, G)$. If we denote by $\zeta_i' = (\text{Id}_{\mathcal{C}_1})_* \circ (\mathcal{C}_1^\zeta)_1^{-1}(\zeta_i)$ the image of $\zeta_i$ along this embedding, we get a homeomorphism
\[
(\text{Id}_{\mathcal{C}_1})_* \circ (\mathcal{C}_1^\zeta)_1^{-1}: \text{Hur}^{\tau_1}(\mathcal{C}_1; Q, G) \cong \mathcal{U}(\zeta_i'; U_i) \subset \text{Hur}(\mathcal{C}; Q, G).
\]

Putting all the previous homeomorphisms together, we obtain the following theorem.

**Theorem 5.1.** Let $\zeta \in \text{Hur}(\mathcal{C}; Q, G)$, use Notations 3.6 and 2.6 and let $U$ be an adapted covering of $P$. Then there exist configurations $\zeta'_i \in \text{Hur}(\mathcal{C}; Q, G)$ supported on $\{z_i\}$ and a homeomorphism
\[
\mathcal{U}(\zeta, U) \cong \prod_{i=1}^{k} \mathcal{U}(\zeta'_i, U_i).
\]

The homeomorphism of Theorem 5.1 depends in general on the choice of arcs $\zeta_i$.

### 5.2. Three useful homeomorphisms

In this subsection we prove three homeomorphisms between Hurwitz spaces.

First, let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple and let $(Q, G, \epsilon, \tau)$ be a PMQ-group pair. The map of PMQs $\epsilon: Q \to G$ has an adjoint map of groups $G(\epsilon): G(Q) \to G$, so that the couple of maps $\text{Id}_Q: Q \to Q$ and $G(\epsilon): G(Q) \to G$ yields a map of PMQ-group pairs $(\text{Id}_Q, G(\epsilon))$: $(Q, G(Q)) \to (Q, G)$. By functoriality we obtain a continuous map
\[
(\text{Id}_Q, G(\epsilon))_*: \text{Hur}(\mathcal{C}; Q, G(Q)) \to \text{Hur}(\mathcal{C}; Q, G).
\]

**Lemma 5.2.** Let $\mathcal{C}$ be a nice couple of the form $(\mathcal{X}, \emptyset)$; then the above map $(\text{Id}_Q, G(\epsilon))_*$ is a homeomorphism.

The proof of Lemma 5.2 is in Subsection 3.6 of the appendix. Roughly speaking, Lemma 5.2 says that if we consider a nice couple of the form $(\mathcal{X}, \emptyset)$, then the space $\text{Hur}(\mathcal{C}; Q, G)$ only depends on $Q$: the monodromy $\psi$ uniquely determines the monodromy $\varphi$. This motivates the following notation, which can be thought of as an absolute definition of Hurwitz space, whereas the general one, given in Definition 3.5 and depending on a nice couple and a PMQ-group pair, can be considered as the general, relative definition.

**Notation 5.3.** For a subspace $\mathcal{X} \subset \mathbb{H}$ and a PMQ $Q$ we denote by $\text{Hur}(\mathcal{X}; Q)$ the space $\text{Hur}(\mathcal{X}; Q, G(Q))$. A configuration $\zeta \in \text{Hur}(\mathcal{X}; Q)$ is usually expanded as $(P, \psi)$ instead of $(P, \psi, \varphi)$ as in Notation 3.6 since $\varphi$ is uniquely determined by $\psi$. 
For the second homeomorphism, let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) be any nice couple, and consider the PMQ-group pair \((G, G)\), where \( G \) is considered as a PMQ with full product, the first \( G \) maps to the second \( G \) by \( \text{Id}_G \) and the second \( G \) acts on the first \( G \) by right conjugation.

Then \( \text{Id}_G \) is a morphism of nice couples \((\mathcal{X}', \mathcal{Y}') \rightarrow (\mathcal{X}, \mathcal{X})\). By functoriality we obtain a continuous map
\[
(\text{Id}_G)_* : \text{Hur}(\mathcal{C}; G, G) \rightarrow \text{Hur}(\mathcal{X}, \mathcal{X}; G, G).
\]

**Lemma 5.4.** The above map \((\text{Id}_G)_*\) is a homeomorphism.

The proof of Lemma 5.4 is in Subsection 3.7 of the appendix.

For the third homeomorphism, let \((\mathcal{Q}, G)\) be any PMQ-group pair and let \(\mathcal{C}\) be a nice couple of the form \((\mathcal{X}, \mathcal{X})\); then for all finite subset \(P \subset \mathcal{X}\) we have \(\mathcal{X}_P = \{1\}\); in particular for all \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) we have that \(\psi : \mathcal{Q}_P \rightarrow \mathcal{Q}\) is the trivial map of PMQs: roughly speaking, this means that we can replace \(\mathcal{Q}\) by another PMQ fitting with \(G\) into a PMQ-group pair, without changing the topology of \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\). For instance we can consider the map of PMQ-group pairs \((c, \text{Id}_G) : (\mathcal{Q}, G) \rightarrow (G, G)\), thus replacing \(\mathcal{Q}\) by \(G\). We obtain the following lemma.

**Lemma 5.5.** For all \(\mathcal{X} \subset \mathbb{H}\) and all PMQ-group pair \((\mathcal{Q}, G)\) the following map is a homeomorphism
\[
(\iota, \text{Id}_G)_* : \text{Hur}(\mathcal{X}, \mathcal{X}; \mathcal{Q}, G) \rightarrow \text{Hur}(\mathcal{X}, \mathcal{X}; G, G).
\]

Using Lemmas 5.2, 5.4 and 5.5 we can simplify our notation for Hurwitz spaces \(\text{Hur}(\mathcal{X}, \mathcal{Y}; \mathcal{Q}, G)\) whenever one of the following conditions is satisfied:

- \(\mathcal{Y} = \emptyset\), then we identify \(\text{Hur}(\mathcal{X}, \emptyset; \mathcal{Q}, G) \equiv \text{Hur}(\mathcal{X}; \mathcal{Q})\);
- \(\mathcal{Y} = \mathcal{X}\), then we identify \(\text{Hur}(\mathcal{X}, \mathcal{X}; \mathcal{Q}, G) \equiv \text{Hur}(\mathcal{X}, \mathcal{X}; G, G) \equiv \text{Hur}(\mathcal{X}; G)\);
- \(\mathcal{Q} = G\), then we identify \(\text{Hur}(\mathcal{X}, \mathcal{Y}; G, G) \equiv \text{Hur}(\mathcal{X}, \mathcal{X}; G, G) \equiv \text{Hur}(\mathcal{X}; G)\).

### 5.3. Functoriality and change of ambient space.

Recall Definition 4.14 let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) and \(\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')\) be two nice couples, and let \(\mathcal{T}\) and \(\mathcal{T}'\) two contractible semi-algebraic subspaces of \(\mathbb{C}\) containing *, such that \(\mathcal{X} \subset \mathcal{T}\) and \(\mathcal{X}' \subset \mathcal{T}'\).

Suppose that \(\xi : (\mathcal{T}, *) \rightarrow (\mathcal{T}', *)\) is a semi-algebraic homeomorphism restricting to an orientation-preserving homeomorphism \(\xi : \mathcal{T} \rightarrow \mathcal{T}'\), and to maps \(\xi : \mathcal{X} \rightarrow \mathcal{X}'\) and \(\xi : \mathcal{Y} \rightarrow \mathcal{Y}'\). Here we restrict to the case of a homeomorphism for simplicity, but any map \(\xi\) satisfying a suitable analogue of conditions (1)-(5) in Definition 4.2 may be used.

We define an induced map \(\xi_* : \text{Hur}^\mathcal{T}(\mathcal{C}; \mathcal{Q}, G) \rightarrow \text{Hur}^\mathcal{T}'(\mathcal{C}'\mathcal{Q}, G)\). Given a configuration \(\iota = (P, \psi, \varphi) \in \text{Hur}^\mathcal{T}(\mathcal{C}; \mathcal{Q}, G)\), we define \(\iota' = \xi_*(\iota) = (P', \psi', \varphi') \in \text{Hur}^\mathcal{T}'(\mathcal{C}'\mathcal{Q}, G)\) as follows:

- \(P' = \xi(P)\); note that \(\xi\) restricts to a homeomorphism \(\mathcal{T} \setminus P \rightarrow \mathcal{T}' \setminus P'\);
- \((\psi', \varphi') : (\mathcal{Q}_\mathcal{T}'(P')) \rightarrow (\mathcal{Q}, G)\) is the following composition of maps of PMQ-group pairs
\[
(\mathcal{Q}_\mathcal{T}'(P'), \mathcal{G}_{\mathcal{C}}(P')) \xrightarrow{(\xi^{-1})_*} (\mathcal{Q}_\mathcal{T}(P), \mathcal{G}_\mathcal{C}(P)) \xrightarrow{(\psi, \varphi)} (\mathcal{Q}, G).
\]

The same arguments used in Subsection 4.3 show that \(\xi_*\) is continuous. In the next articles of this series we will use this fact in the particular case in which \(\xi\) restricts also to homeomorphisms \(\mathcal{X} \rightarrow \mathcal{X}'\) and \(\mathcal{Y} \rightarrow \mathcal{Y}'\); then we can use the inverse homeomorphism \(\xi^{-1} : \mathcal{T}' \rightarrow \mathcal{T}\) to define a map \((\xi^{-1})_* : \text{Hur}^\mathcal{T}'(\mathcal{C}', \mathcal{Q}, G) \rightarrow \text{Hur}^\mathcal{T}(\mathcal{C}; \mathcal{Q}, G)\).
Hur$^\mathfrak{p}(\mathcal{C}; Q, G)$. The maps $\xi_*$ and $(\xi^{-1})_*$ are inverse homeomorphism, and we obtain in particular the following proposition.

**Proposition 5.6.** Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ be nice couples, and let $\xi: \mathcal{C} \to \mathcal{C}'$ be a morphism of nice couples (i.e., $\xi$ is a map $\mathcal{C} \to \mathcal{C}'$). Assume that $\xi$ restricts to a homeomorphism $\mathcal{T} \to \mathcal{T}'$, for two contractible semi-algebraic subspaces $\mathcal{T}, \mathcal{T}'$ containing $\ast$, and assume also that $\mathcal{X} \subset \mathcal{T}$, $\mathcal{X}' \subset \mathcal{T}'$ and $\xi$ restricts to homeomorphisms $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{Y} \to \mathcal{Y}'$. Then the map $\xi_*: \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}', Q, G)$ is a homeomorphism.

In Appendix A we will briefly generalize the above discussion to the case in which $\mathcal{T}$ and $\mathcal{T}'$ are homeomorphic orientable surfaces with non-empty boundary.

### 5.4. External products of Hurwitz spaces

In this subsection we fix a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and two PMQ-group pairs $(Q, G)$ and $(Q', G')$.

**Definition 5.7.** Recall [Bia21a, Definition 2.16]. We define an external product

$$- \times -: \text{Hur}(\mathcal{C}; Q, G) \times \text{Hur}(\mathcal{C}; Q', G') \to \text{Hur}(\mathcal{C}; (Q, G) \times (Q', G'))$$

Let $(\epsilon, \epsilon') \in \text{Hur}(\mathcal{C}; Q, G) \times \text{Hur}(\mathcal{C}; Q', G')$, and use Notation 5.4. We define $\epsilon \times \epsilon'$ as the configuration $(P'', \psi'', \varphi'') \in \text{Hur}(\mathcal{C}; (Q, G) \times (Q', G'))$, where:

- $P'' = P \cup P' \subset \mathcal{X}$;
- $(\psi'', \varphi''): (\Omega(P''), \text{st}(P'')) \rightarrow (Q, G) \times (Q', G)$ is the map of PMQ-group pairs given by $(\psi, \varphi) \circ i_{p''}^\ast, (\psi', \varphi') \circ i_{p''}^\ast)$ (see Notation 2.17).

**Proposition 5.8.** The external product $- \times -$ from Definition 5.7 is continuous and is a retraction of the map

$$(p_*, p'_*) : \text{Hur}(\mathcal{C}; (Q, G) \times (Q', G')) \to \text{Hur}(\mathcal{C}; (Q, G) \times \text{Hur}(\mathcal{C}; Q', G'))$$

induced by the projections $p : (Q, G) \times (Q', G') \to (Q, G)$ and $p' : (Q, G) \times (Q', G') \to (Q', G')$.

**Proof.** Let $\epsilon, \epsilon', \epsilon''$ be as in Definition 5.7, and let $U''$ be an adapted covering of $P''$. Then we can obtain an adapted covering $U'$ of $P$ (respectively, $U'$ of $P'$) by selecting the components of $U''$ containing one point of $P$ (respectively, of $P'$). We note that the product of normal neighbourhoods $\text{U}(\epsilon, U') \times \text{U}(\epsilon', U'')$ is mapped by the external product inside $\text{U}((\epsilon', U') \times (\epsilon', U''))$: this shows continuity of the external product.

For the second statement, let $\epsilon = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; (Q, G) \times (Q', G'))$, then both $p_*(\epsilon) = (P, p \circ (\psi, \varphi))$ and $p'_*(\epsilon) = (P, p' \circ (\psi, \varphi))$ are supported on the set $P \subset \mathcal{X}$, so that by Definition 5.7 also $p_*(\epsilon) \times p'_*(\epsilon)$ is supported on $P \cup P = P$. It now follows directly from Definition 5.7 that $p_*(\epsilon) \times p'_*(\epsilon)$ is equal to $\epsilon$. □

We will use the external product only in the special case in which $(Q', G') = (1, 1)$, and hence $\text{Hur}(\mathcal{C}; Q', G') \cong \text{Ran}(\mathcal{C})$.

**Notation 5.9.** By abuse of notation we will denote by $- \times -$ also the following composition

$$\text{Hur}(\mathcal{C}; Q, G) \times \text{Ran}(\mathcal{C}) \xrightarrow{\sim} \text{Hur}(\mathcal{C}; Q, G) \times \text{Hur}(\mathcal{C}; 1, 1)$$

which is obtained by composing $\text{Hur}(\mathcal{C}; Q, G) \times \text{Ran}(\mathcal{C})$ with $- \times -$.
5.5. Contractible normal neighbourhoods. The following lemma gives an effective way to prove contractibility of normal neighbourhoods in concrete situations.

**Lemma 5.10.** Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple and $(\mathcal{Q}, G)$ a PMQ-group pair. Let $c \in \text{Hur}(\mathcal{X}, \mathcal{Q})$, use Notations 3.6 and 2.6, and let $\mathcal{U}$ be an adapted covering of $\mathcal{P}$. Assume that there is a homotopy $\mathcal{H}^\mathcal{U}: \mathbb{C} \times [0, 1] \to \mathbb{C}$ satisfying the following:

1. $\mathcal{H}^\mathcal{U}(-, t)$ is a lax morphism of nice couples $\mathcal{C} \to \mathcal{C}$ for all $0 \leq t \leq 1$;
2. $\mathcal{H}^\mathcal{U}(-, 0) = \text{Id}_\mathcal{C}$;
3. $\mathcal{H}^\mathcal{U}(-, t)$ restricts to a map $U_i \to U_i$, for all $1 \leq i \leq k$ and all $0 \leq t \leq 1$;
4. $\mathcal{H}^\mathcal{U}(-, 1)$ maps $U_i$ constantly to $z_i$ for all $1 \leq i \leq k$.

Then the normal neighbourhood $\mathcal{U}(c, \mathcal{U})$ is contractible.

**Proof.** We include $\mathcal{Q}$ into its completion $\hat{\mathcal{Q}}$, and consequently include $(\mathcal{Q}, G)$ into $(\hat{\mathcal{Q}}, G)$. Recall that the inclusion $\text{Hur}(\mathcal{X}; \mathcal{Q}, G) \subset \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G)$ is open, and more precisely it maps normal neighbourhoods bijectively onto normal neighbourhoods. Thus suffices to prove that $\mathcal{U}(c, \mathcal{U})$ is contractible when considered as a normal neighbourhood in $\text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G)$.

Proposition 4.10 and property (1) of $\mathcal{H}^\mathcal{U}$ give a homotopy

$$\mathcal{H}^\mathcal{U}: \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \times [0, 1] \to \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G),$$

Consider now the union $\coprod_\mathcal{U} \mathcal{U}(\tilde{c}; \mathcal{U}) \subset \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G)$, where $\tilde{c}$ ranges among all configurations of $\text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G)$ supported on $\mathcal{P}$. By Property (3) the map $\mathcal{H}^\mathcal{U}$ restricts to a homotopy

$$\mathcal{H}^\mathcal{U}_c: \coprod_{\tilde{c}} \mathcal{U}(\tilde{c}; \mathcal{U}) \times [0, 1] \to \coprod_{\tilde{c}} \mathcal{U}(\tilde{c}; \mathcal{U}).$$

The argument in the proof of Proposition 3.8 shows that $\coprod_{\tilde{c}} \mathcal{U}(\tilde{c}; \mathcal{U})$ is the topological disjoint union of its open subspaces $\mathcal{U}(\tilde{c}; \mathcal{U})$.

The map $\mathcal{H}^\mathcal{U}(-; 0)$ is the identity of $\text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G)$ by property (2), in particular $\mathcal{H}^\mathcal{U}_c(-; 0)$ preserves each subspace $\mathcal{U}(\tilde{c}; \mathcal{U})$. It follows that $\mathcal{H}^\mathcal{U}_c$ restricts to a homotopy

$$\mathcal{H}^\mathcal{U}_c: \mathcal{U}(\tilde{c}, \mathcal{U}) \times [0, 1] \to \mathcal{U}(\tilde{c}, \mathcal{U})$$

for each $\tilde{c}$ supported on $\mathcal{P}$, in particular for $\tilde{c} = c$. By Property (4) the map $\mathcal{H}^\mathcal{U}_c(-; 1)$ takes values in configurations in $\text{Hur}(\mathcal{C}; \mathcal{Q}', G)$ supported on the set $\mathcal{P} = \{z_1, \ldots, z_k\}$, and the only such configuration inside $\mathcal{U}(c, \mathcal{U})$ is $c$. 

\[ \square \]

The hypothesis that the spaces $\mathcal{X}$ and $\mathcal{Y}$ occurring in a nice couple $\mathcal{C}$ are semialgebraic implies that, given $c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G)$ supported on a finite set $\mathcal{P}$, one can choose a small enough adapted covering of $\mathcal{P}$ for which a homotopy $\mathcal{H}^\mathcal{U}$ exists as in lemma 5.10 exists. It follows that the space $\text{Hur}(\mathcal{C}; \mathcal{Q}, G)$ is locally contractible.

6. Total monodromy and group actions

In this section we define the total monodromy of configurations in $\text{Hur}(\mathcal{C}; \mathcal{Q}, G)$ and describe several actions of $G$ on $\text{Hur}(\mathcal{C}; \mathcal{Q}, G)$ and on certain subspaces of it.
6.1. **Total monodromy.** The total monodromy is the simplest invariant of connected components of $\text{Hur}(\mathcal{C}; Q, G)$.

**Definition 6.1.** Let $\mathcal{C}$ be a nice couple, $(Q, G)$ a PMQ-group pair, and let $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}, Q, G)$. Let $\gamma: [0, 1] \to \mathbb{C}$ be a simple closed loop spinning clockwise around $P$, i.e., $\gamma$ bounds a disc in $\mathbb{C}$ that contains $P$. We define $\omega(c) = \varphi([\gamma]) \in G$, and call it the **total monodromy** of the configuration $c$: this gives a function $\omega: \text{Hur}(\mathcal{C}, Q, G) \to G$. See Figure 7, left.

Note that the loop $\gamma$ is well-defined up to homotopy, so that $\omega$ is well-defined as a map of sets. Since for any given covering $U$ of $P$ we can choose $\gamma$ spinning clockwise also around $U$, we note that $\omega$ is constant on the normal neighbourhood $\Omega(c; U)$; hence the total monodromy is locally constant and therefore an invariant of connected components of $\text{Hur}(\mathcal{C}; Q, G)$.

Note also that if $\xi: \mathcal{C} \to \mathcal{C}'$ is a map of nice couples, then $\xi(\gamma)$ is homotopic to a simple loop spinning clockwise around $\xi(P)$ (see Definition 4.2), in particular $\omega(c) = \omega(\xi_*(c))$, i.e. the total monodromy is preserved under maps of Hurwitz spaces induced by maps of nice couples. If $(\Psi, \Phi): (Q, G) \to (Q', G')$ is a morphism of PMQ-group pairs, then for all $c \in \text{Hur}(\mathcal{C}; Q, G)$ we have $\Phi(\omega(c)) = \omega((\Psi, \Phi)_*(c))$.

**Notation 6.2.** For a nice couple $\mathcal{C}$, a PMQ-group pair $(Q, G)$ and $g \in G$ we denote by $\text{Hur}(\mathcal{C}; Q, G)_{g} \subset \text{Hur}(\mathcal{C}; Q, G)$ the preimage of $g$ along $\omega$. If $P_{b}$ is as in Definition 3.13 we denote by $\text{Hur}(\mathcal{C}; Q, G)_{P_{b}; g}$ the corresponding subspace of $\text{Hur}(\mathcal{C}; Q, G)_{P_{b}}$. 

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**Figure 7.** On left, a configuration $c$ in $\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)$, whose total monodromy is the $G$-valued monodromy of the dashed loop; on right, a configuration $\hat{c}$ in $\text{Hur}(\mathcal{X}, Q)$, whose total monodromy is the $\hat{Q}$-valued monodromy of the dashed loop.
For $P_0 \subset \mathcal{X}$ (possibly $P_0 = \emptyset$) we obtain a natural decomposition

$$\text{Hur}(\mathcal{X}; Q, G)_{P_0} = \coprod_{g \in G} \text{Hur}(\mathcal{X}; Q, G)_{P_0^g}.$$ 

In the case $\mathcal{Y} = \emptyset$, we can refine Definition 6.3 by taking the values of $\omega$ in the completion $\hat{\mathcal{Q}}$ of $Q$, instead of $G$. Let $\mathcal{X} \subset \mathbb{H}$ be a semi-algebraic subset, and let $\epsilon = (P, \psi) \in \text{Hur}(\mathcal{X}; Q)$; if $\gamma$ is a simple loop in $\mathbb{C} \setminus P$ spinning clockwise around $P$, then $[\gamma] \in \Omega^{\text{ext}}(P)$, and since $\hat{\mathcal{Q}}$ is complete (and contains $Q$), we can extend $\psi: \Omega(P) \to \hat{\mathcal{Q}}$ to a map $\psi^{\text{ext}}: \Omega^{\text{ext}}(P) \to \hat{\mathcal{Q}}$, compare also with Subsection 4.3.

**Definition 6.3.** We define a locally constant map $\hat{\omega}: \text{Hur}(\mathcal{X}; Q) \to \hat{\mathcal{Q}}$ by setting $\hat{\omega}(\epsilon) := \psi^{\text{ext}}([\gamma])$, using the notation above. See Figure 7, right.

For $a \in \hat{\mathcal{Q}}$ and $\emptyset \subseteq P_0 \subset \mathcal{X}$ we define $\text{Hur}(\mathcal{X}; Q)_{P_0^a} \subset \text{Hur}(\mathcal{X}; Q)$ as the preimage of $a$ under the restriction of $\hat{\omega}$.

We obtain a natural decomposition

$$\text{Hur}(\mathcal{X}; Q)_{P_0} = \coprod_{a \in \hat{\mathcal{Q}}} \text{Hur}(\mathcal{X}; Q)_{P_0^a}.$$ 

The maps $\omega: \text{Hur}(\mathcal{X}; Q) \to \mathcal{G}(Q)$ and $\hat{\omega}: \text{Hur}(\mathcal{X}; Q) \to \hat{\mathcal{Q}}$ are related by the equality $\omega = \eta_{\hat{\mathcal{Q}}} \circ \hat{\omega}$, where we the groups $\mathcal{G}(Q)$ and $\mathcal{G}(\hat{\mathcal{Q}})$ are canonically identified, and $\eta_{\hat{\mathcal{Q}}}: \hat{\mathcal{Q}} \to \mathcal{G}(\hat{\mathcal{Q}})$ is the unit of the adjunction.

As a first application of the total monodromy, we prove the following proposition.

**Proposition 6.4.** Let $\mathcal{X}$ be a non-empty, semi-algebraic, convex and bounded subset of $\mathbb{H}$, and let $\hat{\mathcal{Q}}$ be a complete PMQ. Then the connected components of $\text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})$ are contractible and there is a bijection

$$\hat{\omega}: \pi_0(\text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})) \cong \hat{\mathcal{Q}}.$$ 

**Proof.** Since $\mathcal{X}$ is bounded, we can find a bounded, convex, semi-algebraic open set $\mathcal{X} \subset U \subset \mathbb{C} \setminus \{\ast\}$. Fix a point $z_0 \in \mathcal{X}$: for all $a \in \hat{\mathcal{Q}}$ we can define a configuration $\epsilon_a = (\{z_0\}, \psi_a) \in \text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})$ by setting $\psi_a([\gamma]) = a$ for a simple loop $\gamma$ spinning clockwise around $z_0$. By Lemma 5.10 each normal neighbourhood $\mathcal{U}(\epsilon_a; U)$ deformation retracts onto the configuration $\epsilon_a$; the statement follows from the observation that each $\epsilon \in \text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})$ is contained in one of these normal neighbourhoods, namely in $\mathcal{U}(\epsilon_{\omega(\epsilon)}; U)$. \hfill \Box

In fact the bijection $\hat{\omega}: \pi_0(\text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})) \cong \hat{\mathcal{Q}}$ holds whenever $\mathcal{X}$ is path connected; we will not use this more general fact and leave the proof to the reader. Proposition 6.4 has the following corollary.

**Corollary 6.5.** Let $\mathcal{X} \subset \mathbb{H}$ be semi-algebraic, convex and bounded; then for all $a \in \mathcal{Q}$ the space $\text{Hur}_+(\mathcal{X}; \mathcal{Q})_a$ is contractible.

**Proof.** The fact that $\mathcal{Q} \setminus \mathcal{Q}$ is an ideal of the complete PMQ $\hat{\mathcal{Q}}$ implies that, for all $a \in \mathcal{Q}$, the natural inclusion $\text{Hur}(\mathcal{X}; \mathcal{Q})_a \subset \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}})_a$ is in fact a homeomorphism. \hfill \Box
6.2. **Action by global conjugation.** Let \((Q, G)\) be a PMQ-group pair. Then the group \(G\) acts (on right) by conjugation on \(Q\) and on \(G\) itself. In particular the right action of \(G\) on \(Q\) takes the form of a map of groups \(\tau : G \to \text{Aut}_{\text{PMQ}}(Q)^{op}\), which is part of the structure of PMQ-group pair. The actions of \(G\) on \(Q\) by conjugation is compatible with respect to the map of PMQs \(\varepsilon : Q \to G\), which is also part of the structure of PMQ-group pair. In the following we define a corresponding right action of \(G\) on the space \(\text{Hur}(\mathcal{C}; Q, G)\), for any nice couple \(\mathcal{C}\).

**Definition 6.6.** For \(g \in G\) and \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C})\) we define \(c^g = (P, \psi^g, \varphi^g) \in \text{Hur}(\mathcal{C}; Q, G)\) as follows:

- \(\psi^g\) is the composition \(\Omega(P) \xrightarrow{\psi} Q \xrightarrow{t(g)} Q\) of maps of PMQs.
- \(\varphi^g\) is the composition \(\Theta(P) \xrightarrow{\varphi} G \xrightarrow{(-)^g} G\) of maps of groups, where \((-)^g : g' \mapsto g^{-1} g' g\).

The maps \((-)^g : \text{Hur}(\mathcal{C}, Q, G) \to \text{Hur}(\mathcal{C}; Q, G)\) are homeomorphisms (they map normal neighbourhoods bijectively to normal neighbourhoods) and assemble into a right action of \(G\) on the space \(\text{Hur}(\mathcal{C}; Q, G)\), called action by global conjugation.

Note that the total monodromy \(\omega\) (see Definition 6.1) satisfies the formula \(\omega(c^g) = \omega(c)^g \in G\). Note also that for \(P_0 \subset X\) as in Definition 3.13 the action by global conjugation restricts to the subspace \(\text{Hur}(\mathcal{C})_{P_0}\).

In the case \(\mathcal{Y} = \emptyset\), we can refine Definition 6.6 and let the completion \(\hat{Q}\) of \(Q\) act on \(\text{Hur}(\mathcal{X}; Q)\). By definition, a (right) action of \(\hat{Q}\) on \(\text{Hur}(\mathcal{X}; Q)\) is a map of PMQs \(\hat{Q} \to \text{Aut}_{\text{top}}(\text{Hur}(\mathcal{X}; Q))^{op}\). For \(a \in \hat{Q}\) and \(c = (P, \psi) \in \text{Hur}(\mathcal{X}; Q)\) we define \(c^a = (P, \psi^a) \in \text{Hur}(\mathcal{X}; Q)\) by setting \(\psi^a\) to be the composition \(\Omega(P) \xrightarrow{\psi} Q \xrightarrow{(-)^a} Q\). Here we use that \(Q \subset \hat{Q}\) is closed under conjugation by elements in \(\hat{Q}\).

6.3. **Left and right-based nice couples.** In this subsection we consider other natural actions of \(G\), defined on suitable subspaces of Hurwitz spaces \(\text{Hur}(\mathcal{C}; Q, G)\).

**Definition 6.7.** For \(t \in \mathbb{R}\) we define a homeomorphism \(\tau_t : (\mathbb{C}, *) \to (\mathbb{C}, *)\) by:

\[
\tau_t(z) = \begin{cases} 
z & \text{if } \Re(z) \leq -1 \\
z + t & \text{if } \Re(z) \geq 0 \\
z + (\Re(z) + 1)t & \text{if } -1 \leq \Re(z) \leq 0.
\end{cases}
\]

Note that \(\tau_t(*) = *\) for all \(t \in \mathbb{R}\). Note also that the assignment \(t \mapsto \tau_t\) defines a continuous, piecewise linear action of \(\mathbb{R}\) on \(\mathbb{C}\).

**Notation 6.8.** For \(t \in \mathbb{R}\) we denote by \(\mathbb{C}_{R \geq t} \subset \mathbb{C}\) the subspace containing all \(z \in \mathbb{C}\) with \(\Re(z) \geq t\). Similarly we define \(\mathbb{C}_{R > t}, \mathbb{C}_{R < t}, \mathbb{C}_{R = t}\) and \(\mathbb{C}_{R = t}\), the latter being a vertical line. For all \(-\infty \leq t < t' \leq +\infty\) we define a subspace \(S_{t, t'} \subset \mathbb{C}\) by:

\[
S_{t, t'} = \tau_t(\mathbb{C}_{R \geq 0}) \cap \tau_{t'}(\mathbb{C}_{R \leq 0}),
\]

where we use the conventions \(\tau_{-\infty}(\mathbb{C}_{R \geq 0}) = \tau_{+\infty}(\mathbb{C}_{R \leq 0}) = \mathbb{C}\) and \(\tau_{-\infty}(\mathbb{C}_{R \leq 0}) = \tau_{+\infty}(\mathbb{C}_{R \geq 0}) = \emptyset\).

**Definition 6.9.** A left-based nice couple is a nice couple \(\mathcal{C} = (X, \mathcal{Y})\) together with a choice of a point \(z^1 \in \mathcal{Y}\) satisfying the following property: there exists \(t \in \mathbb{R}\) such that \(\Re(z^1) = t\) and \(X\) is contained in the right half-plane \(\mathbb{C}_{R \geq t}\). We denote by \((z^1, \mathcal{C})\) a left-based nice couple.
Similarly, a right-based nice couple is a nice couple with a choice of a point $z^r \in \mathcal{Y}$ such that there exists $t \in \mathbb{R}$ with $\Re(z^r) = t$ and $\mathcal{X} \subseteq \mathbb{C}_{\Re \geq t}$. We denote it by $(\mathcal{C}, z^r)$.

A left-right-based nice couple (shortly, lr-based) is a nice couple which is both left- and right-based, such that $\Re(z^l) < \Re(z^r)$: we denote it by $(z^l, \mathcal{C}, z^r)$.

Note that if $(z^l, \mathcal{C})$ is a left-based (respectively, $(\mathcal{C}, z^r)$ is a right-based) nice couple, then the parameter $t$ in Definition is equal to $\min \mathcal{C} := \min \{ \Re(z) \mid z \in \mathcal{X} \}$ (resp. to $\max \mathcal{C} := \max \{ \Re(z) \mid z \in \mathcal{X} \}$), so it depends only on $\mathcal{C}$. On the other hand it is possible that for some nice couple $\mathcal{C}$ there are two or more possible choices of a point $z^l$ (respectively $z^r$) making it into a left-based (right-based, or lr-based) nice couple. It is also possible that a nice couple $\mathcal{C}$ cannot be improved to a left- or right-based nice couple. Note that if $(z^l, \mathcal{C}, z^r)$ is lr-based, then $z^l \neq z^r$.

6.4. Action by left and right multiplication. We define a left action of $G$ on the space $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^l}$, where $(z^l, \mathcal{C})$ is a left-based nice couple. Similarly, there is a right action of $G$ on $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^r}$ if $(\mathcal{C}, z^r)$ is a right-based nice couple. We will describe the construction focusing on the left-based case; the right-based case is analogous, and we will mention the differences in parentheses.

For the entire subsection fix a left-based (right-based) nice couple as in Definition 6.13. Recall Definitions 2.8 and Notation 6.8 and choose an arc $\zeta^l$ contained in $S_{-\infty, \min \mathcal{C}}$ and joining $\ast$ with $z^l$; assume also that the interior of $\zeta^l$ is contained in $S_{-\infty, \max \mathcal{C}}$. (In the right-based case, we would choose an arc $\zeta^r$ embedded in $S_{\min \mathcal{C}, +\infty}$, $\zeta^r$ with $z^r$ and whose interior is contained in $S_{\max \mathcal{C}, -\infty}$.)

**Definition 6.10.** Let $P$ be as in Notation 2.4 with $z^l \in P$ (resp. $z^r \in P$). An admissible generating set $f_1, \ldots, f_k$ for $\mathfrak{G}_C(P)$ is left-based (right-based) if it can be constructed as in Definition 2.8 using $\zeta^l$ (resp. $\zeta^r$) as the arc associated with $z^l$ (resp. $z^r$), and using only arcs $\zeta_i$ contained in $S_{\min \mathcal{C}, +\infty}$ (in $S_{-\infty, \max \mathcal{C}}$) for the other points of $P$.

**Notation 6.11.** We denote by $f^l$ (resp. $f^r$) the generator represented by a loop spinning around $z^l$ (resp. $z^r$).

**Definition 6.12.** Let $g \in G$ and let $(z^l, \mathcal{C})$ (resp. $(\mathcal{C}, z^r)$) be a left-based (right-based) nice couple. Let $\mathcal{C}$ be a configuration in $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^l}$ (in $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^r}$), use Notation 3.6 and let $f_1, \ldots, f_k$ be a left-based (right-based) admissible generating set for $\mathfrak{G}(P)$. We define $g \cdot \mathcal{C}$ (respectively $\mathcal{C} \cdot g$) as the configuration $(P, \psi', \varphi')$, where:

- $\varphi'$ is defined on the free group $\mathfrak{G}(P)$ by setting $\varphi'(f^l) = g \cdot \varphi(f^l)$ (by setting $\varphi'(f^r) = \varphi(f^r) \cdot g$) and by setting $\varphi'(f_i) = \varphi(f_i)$ for $1 \leq i \leq k$ such that $f_i \neq f^l$ (respectively $f_i \neq f^r$).
- $\psi'$ is defined on $\mathfrak{Q}(P)$ using [Bia21a] Theorem 3.3, by setting $\psi'(f_i) = \psi(f_i)$ for all $1 \leq i \leq l$ and imposing that $(\psi', \varphi'): (\mathfrak{Q}(P), \mathfrak{G}(P)) \to (\mathfrak{Q}, P)$ is a map of PMQ-group pairs. See Figure 8.

**Proposition 6.13.** For all $g \in \mathfrak{G}(\mathfrak{Q})$ the assignment $\mathcal{C} \mapsto g \cdot \mathcal{C}$ (respectively $\mathcal{C} \mapsto \mathcal{C} \cdot g$) does not depend on the choice of the left-based (right-based) admissible generating set, and gives rise to a continuous self-map $g \cdot -$ of $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^l}$ (respectively a self-map $- \cdot g$ of $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^r}$).

The collection of all maps $g \cdot -$ (all maps $- \cdot g$) gives a left (right) action of $G$ on $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^l}$ (on $\text{Hur}(\mathcal{C}; \mathbb{Q}, G)_{z^r}$).
Figure 8. On left, a configuration $c \in \text{Hur}(\mathcal{C}, Q, G)_z$; on right, its image under the right action of $g \in G$.

The proof of Proposition 6.13 is in Subsection B.8 of the appendix.

6.5. **Compatibilities of the left and right actions.**

**Lemma 6.14.** Let $\mathcal{C}$ be a left-based (right-based) nice couple. Then the total monodromy $\omega : \text{Hur}(\mathcal{C}, Q) \to G$ is a $G$-equivariant map, where $G$ acts on itself by left (right) multiplication.

**Proof.** We focus on the left-based case. Let $c \in \text{Hur}(\mathcal{C}, Q, G)_z$, let $c' = g \cdot c$ and use Notation 3.6. Let $f_1, \ldots, f_k$ be a left-based admissible generating set for $P = P'$, suppose $f_1 = f^l$ (see Notation 6.11), and suppose, up to permuting the indices from 2 to $k$, that the product $f_1 \ldots f_k$ represents an element $[\gamma] \in \mathcal{G}(P)$ as in Definition 6.1. Let $g = f_2 \ldots f_k$, so that $[\gamma] = f^l \cdot g$. Note that $\varphi'(g) = \varphi(g)$. Then $\omega(g \cdot c) = \varphi'(c) = \varphi'(f^l) \cdot \varphi'(g) = g \cdot \varphi(f^l) \cdot \varphi(g) = g \cdot \varphi([\gamma]) = g \cdot \omega(c)$.

Let now $(z^l, \mathcal{C}, z^r)$ be a lr-based nice couple: both spaces $\text{Hur}(\mathcal{C}, Q, G)_z$ and $\text{Hur}(\mathcal{C}, Q, G)_z$ contain $\text{Hur}(\mathcal{C}, Q, G)_z$ as subspace, and this subspace is preserved under both actions of $G$, on left on $\text{Hur}(\mathcal{C}, Q, G)_z$ and on right on $\text{Hur}(\mathcal{C}, Q, G)_z$.

**Lemma 6.15.** Let $(z^l, \mathcal{C}, z^r)$ be a lr-based nice couple. Then the left and the right actions of $G$ on $\text{Hur}(\mathcal{C}, Q, G)_z$ commute, i.e., for every $g, h \in \mathcal{G}(Q)$ the self-maps $g \cdot -$ and $- \cdot h$ of $\text{Hur}(\mathcal{C}, Q, G)_z$ commute.

**Proof.** Fix $c = (P, \varphi, \psi) \in \text{Hur}(\mathcal{C}, Q, G)_z$, and let $\zeta^l$ and $\zeta^r$ be arcs as in Definition 6.10. We can choose disjoint arcs $\zeta_i$ contained in $S_{\min \mathcal{C}, \max \mathcal{C}}$ completing $\zeta^l, \zeta^r$ to a
system of arcs as in Definition 6.13 yielding an admissible generating set for \( \mathfrak{G}(P) \) which is both left- and right-based. The equality \( g \cdot (c \cdot h) = (g \cdot c) \cdot h \) follows directly from Definition 6.12.

We thus obtain a (left) action of the group \( G \times G^{\text{op}} \) on \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \), and by Lemma 6.14 the map \( \omega : \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \to G \) is \( G \times G^{\text{op}} \)-equivariant.

**Lemma 6.16.** In the hypotheses of Lemma 6.15 the action of \( G \times G^{\text{op}} \) on the space \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) is free and properly discontinuous.

**Proof.** Let \( c = (P, \psi, \varphi) \in \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \), let \( U \) be an adapted covering of \( P \), and denote by \( U^1 \) and \( U^2 \) the components of \( U \) containing \( z^1 \) and \( z^2 \) respectively. Fix an admissible generating set for \( \mathfrak{G}(P) \) which is both left- and right-based, and let \( f^1 \) and \( f^2 \) be as in Notation 6.11.

Let \((g, h)\) be a non-trivial element in \( G \times G^{\text{op}} \), and denote by \( c' = (P, \psi', \varphi') \) the configuration \( g \cdot c \cdot h \). Then either \( \varphi'(f^1) = g \cdot \varphi(f^1) \neq \varphi(f^1) \) or \( \varphi'(f^2) = g \cdot \varphi(f^2) \neq \varphi(f^2) \), or both inequalities hold: in any case we conclude \( c' \neq c \), so the action of \( G \times G^{\text{op}} \) on \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) is free.

Recall Definition 6.13 and note that the normal neighbourhood \( U(c, U)_{z^1, z^2} \) is mapped by \( g \cdot - \cdot h \) to the normal neighbourhood \( U(g \cdot c \cdot h, U)_{z^1, z^2} \); since the configurations \( c \) and \( c' \) are supported on the same set \( P \), but \( \varphi' \neq \varphi \), the argument in the proof of Proposition 6.13 shows that \( U(c, U)_{z^1, z^2} \) and \( U(g \cdot c \cdot h, U)_{z^1, z^2} \) intersect trivially in \( \text{Hur}(\mathfrak{C}; Q, G) \), and a fortiori \( U(c, U)_{z^1, z^2} \) and \( U(g \cdot c \cdot h, U)_{z^1, z^2} \) intersect trivially in \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \). Hence the action of \( G \times G^{\text{op}} \) on \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) is properly discontinuous.

Recall Notation 6.2, we can decompose \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) as a disjoint union of subspaces \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) according to \( \omega \). If we act on \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) only on left, these subspaces will be permuted among each other, so that the quotient of \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) by the left action is homeomorphic to \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \). The same holds if we quotient \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) only by the right action of \( G \); we can define a more interesting space by quotienting \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) by both actions.

**Notation 6.17.** We denote by \( \text{Hur}(\mathfrak{C}; Q, G)_{G, G^{\text{op}}} \) the quotient of \( \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \) by the left and right actions of \( G \): the points \( z^1, z^2 \) will always be clear from the context and will thus be omitted from the notation. We denote by \( p_{G, G^{\text{op}}} : \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \to \text{Hur}(\mathfrak{C}; Q, G)_{G, G^{\text{op}}} \) the projection map.

By Lemma 6.16 we have in particular a normal covering map \( p_{G, G^{\text{op}}} : \text{Hur}(\mathfrak{C}; Q, G)_{z^1, z^2} \to \text{Hur}(\mathfrak{C}; Q, G)_{G, G^{\text{op}}} \), with \( G \) as group of deck transformations.

7. Hurwitz spaces with coefficients in augmented PMQs

In this section we introduce, for an augmented PMQ \( Q \) (see [Bin21a] Definition 4.9), a subspace \( \text{Hur}(\mathfrak{C}; Q^+, G) \) of the Hurwitz space \( \text{Hur}(\mathfrak{C}; Q, G) \); under suitable conditions on \( \mathfrak{C} \) the inclusion \( \text{Hur}(\mathfrak{C}; Q^+, G) \subset \text{Hur}(\mathfrak{C}; Q, G) \) is a weak homotopy equivalence.
Definition 7.1. Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple and let $(Q, G)$ be a PMQ-group pair, with $Q$ augmented. We define $\text{Hur}(\mathcal{C}; Q_+, G) \subset \text{Hur}(\mathcal{C}, Q, G)$ as the subspace containing all configurations $c = (P, \psi, \varphi)$ such that $\psi: Q(P) \to Q$ is an augmented map of PMQs, i.e. $\psi^{-1}(1_Q) = \{1_Q(P)\}$. If $\mathcal{Y}$ is empty we also write $\text{Hur}(\mathcal{X}; Q_+, G) \subset \text{Hur}(\mathcal{X}, Q, G)$ for the space $\text{Hur}(\mathcal{C}, Q_+, G(\mathcal{Q}))$.

Roughly speaking, a configuration $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}, Q, G)$ belongs to $\text{Hur}(\mathcal{C}; Q_+, G)$ if the monodromy $\psi$ attains “non-trivial” values around each point of $P \setminus \mathcal{Y}$: these are also all points of $P$ around which $\psi$ is defined. Here a “non-trivial” value is a value different from $1_Q$, i.e. a value in $Q_+$, whence the notation.

Note that if $\xi: \mathcal{C} \to \mathcal{C}'$ is a morphism of nice couples and $Q$ is augmented, then the induced map $\xi_*: \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}'; Q, G)$ restricts to a map of spaces $\xi_*: \text{Hur}(\mathcal{C}; Q_+, G) \to \text{Hur}(\mathcal{C}'; Q_+, G)$. This is true also for a lax morphism $\xi: \mathcal{C} \to \mathcal{C}'$ (see Definition 1.3), provided that $Q$ is complete and augmented.

Lemma 7.2. If $Q$ is augmented, then $\text{Hur}(\mathcal{C}; Q_+, G) \subset \text{Hur}(\mathcal{C}, Q, G)$.

Proof. Let $c \in \text{Hur}(\mathcal{C}, Q, G) \setminus \text{Hur}(\mathcal{C}; Q_+, G)$; then, using Notation 3.9, there is some $1 \leq i \leq l$ such that $\psi$ sends each element of $Q(P, z_i)$ to $1$ (see also Definition 2.8). Let $U_i$ be an adapted covering of $P$: then we claim that the entire normal neighbourhood $U(c; U_i)$ lies in the difference $\text{Hur}(\mathcal{C}; Q, G) \setminus \text{Hur}(\mathcal{C}; Q_+, G)$. To see this, let $c' = (P', \psi', \varphi') \in U(c; U_i)$, use Notation 2.8 and let $z'$ be a point in $P' \cap U_i$. Then each element $[\gamma'] \in Q(P', z')$ is sent by $\psi'$ to an element $\psi'([\gamma']) \in Q$ which occurs as a factor of a decomposition of $1_Q$ in the partial monoid $Q$: since $Q$ is augmented we have $\psi'([\gamma']) = 1$ and therefore $c'$ does not lie in $\text{Hur}(\mathcal{C}, Q_+, G)$. □

7.1. Homotopy equivalences from augmented PMQs. The rest of the section is devoted to the proof of the following technical propositions.

Proposition 7.3. Let $\mathcal{X} \subset \mathbb{H}$ be a semi-algebraic, non-empty and connected subspace, and let $Q$ be an augmented PMQ. Then the spaces $\text{Hur}(\mathcal{X}; Q_+, G)$ and $\text{Hur}(\mathcal{C}; Q_+, G)$ are homotopy equivalent.

Proposition 7.4. Let $(Q, G)$ be a PMQ-group pair with $Q$ augmented, and let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple with both $\mathcal{X}$ and $\mathcal{Y}$ non-empty and connected. Let $P_0 \subset \mathcal{Y}$ be a finite, non-empty subset. Then the inclusion $\text{Hur}(\mathcal{C}; Q_+, G)_{P_0} \subset \text{Hur}(\mathcal{C}, Q, G)_{P_0}$ is a homotopy equivalence.

In the rest of the section we fix a PMQ-group pair $(Q, G)$ with $Q$ augmented. Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple.

Definition 7.5. Let $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}, Q, G)$. A point $z \in P$ is inert for $c$ if $z \in \mathcal{X} \setminus \mathcal{Y}$ and $\psi$ maps each element of $Q(P, z)$ to $1_Q$ (see Definition 2.9).

We construct a retraction of sets $\rho: \text{Hur}(\mathcal{C}, Q, G) \to \text{Hur}(\mathcal{C}, Q_+, G)$ of the inclusion $\text{Hur}(\mathcal{C}; Q_+, G) \subset \text{Hur}(\mathcal{C}, Q, G)$: for each configuration $c \in \text{Hur}(\mathcal{C}, Q, G)$, we construct $\rho(c)$ by “forgetting” its inert points. More precisely, using Notation 3.9 if $P' \subset P$ is the subset of non-inert points for $c$, we note that $\varphi: \Theta(P') \to G$ and $\psi: Q(P') \to Q$ factor through maps $\varphi': \Theta(P') \to G$ and $\psi': Q(P') \to Q$ along the surjections $\iota_{P'}^P: \Theta(P) \to \Theta(P')$ and $\iota_{P'}^Q: Q(P) \to Q(P')$, (see Notation 2.17), and we define $\rho: (P, \psi, \varphi) \mapsto (P', \psi', \varphi')$.

Definition 7.6. For all nice couples $\mathcal{C}$ the previous assignment gives a map of sets $\rho: \text{Hur}(\mathcal{C}, Q, G) \to \text{Hur}(\mathcal{C}, Q_+, G)$. 

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Unfortunately, even assuming that \( \mathcal{Q} \) is augmented, \( \rho \) is in general not continuous: for instance, if \( P \) contains a point \( z_i \in \mathcal{Y} \) whose local monodromy with respect to \( \varphi \) is \( 1 \in G \), then \( \rho(P, \psi, \varphi) \) is a configuration supported also on the point \( z_i \); however if we perturb slightly \( z_i \) so that it “enters” in \( \mathcal{X} \setminus \mathcal{Y} \) (for this, suppose that \( z_i \) is an accumulation point for \( \mathcal{X} \setminus \mathcal{Y} \)), then in defining \( \rho(P, \psi, \varphi) \) we forget \( z_i \) and we do not replace it by any other point close to it.

**7.2. Explosions.** The previous issue can only occur when \( \mathcal{Y} \neq \emptyset \), and in fact if \( \mathcal{Y} = \emptyset \), then \( \rho: \text{Hur}(\mathcal{C}, \mathcal{Q}, G) \to \text{Hur}(\mathcal{C}; \mathcal{Q}_+, G) \) is continuous, as we will see in Corollary 7.9. In the general case we cannot just let an inert point \( z_i \in P \setminus \mathcal{Y} \) disappear; what we can do is to let every point \( z_i \in P \) explode (including non-inert points), by replacing \( z_i \) with one or more other points of \( \mathcal{X} \). This idea is elaborated in the following definition.

**Definition 7.7.** Let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) be a nice couple. An explosion \( \varepsilon \) of \( \mathcal{C} \) is a continuous map \( \varepsilon: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{C}) \) such that for all \( z \in \mathcal{Y} \) and all \( 0 \leq t \leq 1 \), \( z \in \varepsilon(z, t) \). An explosion \( \varepsilon \) is standard if \( \varepsilon(z, 0) = \{ z \} \in \text{Ran}(\mathcal{C}) \) for all \( z \in \mathcal{X} \).

Given an explosion \( \varepsilon \), a finite subset \( P \subset \mathcal{X} \) and a time \( 0 \leq t \leq 1 \), we can define a subset \( \varepsilon(P, t) \in \text{Ran}(\mathcal{C}) \) as the union of the subsets \( \varepsilon(z, t) \) for \( z \in P \). Thus an explosion \( \varepsilon \) induces a continuous map \( \text{Ran}(\mathcal{C}) \times [0, 1] \to \text{Ran}(\mathcal{C}) \), that by abuse of notation we still denote by \( \varepsilon \). Note that if \( \varepsilon \) is standard, then \( \varepsilon(-, 0) \) is the identity of \( \text{Ran}(\mathcal{C}) \).

**Proposition 7.8.** Let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) be a nice couple, let \( \varepsilon: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{C}) \) be an explosion, and let \((\mathcal{Q}, G)\) be a PMQ-group pair with \( \mathcal{Q} \) augmented. Denote by \( \varepsilon_\varepsilon: \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] \to \text{Hur}(\mathcal{C}; \mathcal{Q}_+, G) \) the following composition of maps of sets, where \( \varepsilon_\varepsilon \) was introduced in Definition 5.7.

\[
\text{Hur}(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] \xrightarrow{(\rho, \varepsilon) \times \text{Id}} \text{Hur}(\mathcal{C}; \mathcal{Q}_+, G) \times \text{Ran}(\mathcal{C}) \times [0, 1] \xrightarrow{\text{Id} \times \varepsilon} \text{Hur}(\mathcal{C}; \mathcal{Q}_+, G) \times \text{Ran}(\mathcal{C}) \xrightarrow{- \times -} \text{Hur}(\mathcal{C}; \mathcal{Q}, G).
\]

Then \( \varepsilon_\varepsilon \) is continuous. If moreover \( \varepsilon \) is standard, then \( \varepsilon_\varepsilon(-, 0) \) is the identity of \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \).

The proof of Proposition 7.8 is in Subsection 5.9 of the appendix. A particular application of Proposition 7.8 is the following:

**Corollary 7.9.** Let \( \mathcal{X} \subset \mathbb{H} \) be a semi-algebraic set and let \( \mathcal{Q} \) be an augmented PMQ; then the map \( \rho: \text{Hur}(\mathcal{X}; \mathcal{Q}) \to \text{Hur}(\mathcal{X}; \mathcal{Q}_+) \) is continuous.

**Proof.** Consider the explosion \( \varepsilon_\emptyset: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{X}) \) taking the constant value \( \emptyset \in \text{Ran}(\mathcal{X}) \); then \( \varepsilon_\emptyset(-, 0) = \rho \) is a continuous map. \( \square \)

**7.3. Proof of Propositions 7.3 and 7.4**

**Proof of Proposition 7.3.** Recall Definition 5.7 and Notation 5.9 and fix a point \( z_0 \in \mathcal{X} \). We claim that the map \( - \times z_0: \text{Hur}(\mathcal{X}; \mathcal{Q}_+) \to \text{Hur}_+(\mathcal{X}; \mathcal{Q}) \) is a homotopy equivalence, where we denote by \( z_0 \) also the singleton \( \{ z_0 \} \in \text{Ran}(\mathcal{X}) \).

Let \( \text{Hur}(\mathcal{X}; \mathcal{Q}_+) \) be the subspace of \( \text{Hur}_+(\mathcal{X}; \mathcal{Q}) \) containing all configurations \( c = (P, \psi) \) such that \( z_0 \in P \) and all points of \( P \setminus \{ z_0 \} \) are not inert; then \( - \times z_0 \)
gives a homeomorphism $\text{Hur}(\mathcal{X}; Q_{+}) \xrightarrow{\cong} \text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}}$, with inverse given by the restriction of $\rho$, which is continuous by Corollary 5.3. It suffices therefore to prove that the inclusion $\text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}} \subset \text{Hur}_{+}(\mathcal{X}; Q)$ is a homotopy equivalence. By Lemma 3.3 the space $\text{Ran}_{+}(\mathcal{X})$ is weakly contractible; since $\mathcal{X}$ is homeomorphic to a CW complex, there is a homotopy $\mathcal{E}^{\sim_{y}} : \mathcal{X} \times [0,1] \to \text{Ran}_{+}(\mathcal{X})$ with $\mathcal{E}^{\sim_{y}}(z,0) = \{z\}$ and $\mathcal{E}^{\sim_{y}}(z,1) = \{z_{0}\}$ for all $z \in \mathcal{X}$; in our language $\mathcal{E}^{\sim_{y}}$ is a standard explosion, giving rise to an extended explosion $\mathcal{E}^{z_{0}} : \text{Ran}(\mathcal{X}) \times [0,1] \to \text{Ran}(\mathcal{X})$ (see the remark after Definition 7.7).

Proposition 7.8 yields a homotopy $\mathcal{E}^{z_{0}} : \text{Hur}(\mathcal{X}; Q) \times [0,1] \to \text{Hur}(\mathcal{X}; Q)$, which restricts to a homotopy of $\text{Hur}_{+}(\mathcal{X}; Q)$. We note the following:

- $\mathcal{E}^{z_{0}}(-,0)$ is the identity of $\text{Hur}_{+}(\mathcal{X}; Q)$, again by Proposition 7.8.
- $\mathcal{E}^{z_{0}}(-,1)$ restricts to the identity on the subspace $\text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}}$: indeed if $c = (P, \psi) \in \text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}}$, then $\rho(c)$ is either equal to $c$, or is obtained by forgetting $z_{0} \in P$ in case $z_{0}$ is inert; since $\mathcal{E}^{z_{0}}(-,1)$ is constant on $\text{Ran}(\mathcal{X})$ with value $z_{0}$, we have anyway the equality $\rho(c) \times \mathcal{E}^{z_{0}}(P,1) = \rho(c) \times z_{0} = c$, i.e. the point $z_{0}$ is added again in the further composition defining $\mathcal{E}^{z_{0}}(-,1)$.
- $\mathcal{E}^{z_{0}}(-,1)$ takes values in $\text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}}$: this follows again from the equality $\mathcal{E}^{z_{0}}(c,1) = \rho(c) \times z_{0}$, holding for all $c \in \text{Hur}(\mathcal{X}; Q)$.

The homotopy $\mathcal{E}^{z_{0}}$ shows that the inclusion $\text{Hur}(\mathcal{X}; Q_{+})_{\sim_{y}} \subset \text{Hur}_{+}(\mathcal{X}; Q)$ is a homotopy equivalence.

Note that in the particular case $Q = \{1\}$, Proposition 7.3 implies that $\text{Ran}_{+}(\mathcal{X})$ is contractible; this is a mild improvement of the statement of Lemma 3.3.

**Proof of Proposition 7.4.** The proof is similar to the one of Proposition 7.3. By Lemma 5.3 the spaces $\text{Ran}_{+}(\mathcal{Y}) \subset \text{Ran}_{+}(\mathcal{X})$ are weakly contractible; since $(\mathcal{X}, \mathcal{Y})$ is homeomorphic to a couple of CW complexes, there is a homotopy $\mathcal{E}^{\mathcal{X}, \mathcal{Y}} : \mathcal{X} \times [0,1] \to \text{Ran}_{+}(\mathcal{X})$ with $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}(z,t) = \{z\}$ whenever $z \in \mathcal{Y}$ or $t = 0$, and such that $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}(z,1) \in \text{Ran}_{+}(\mathcal{Y})$ for all $z \in \mathcal{X}$. Note in particular that $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}$ is a standard explosion. We obtain an extended explosion $\mathcal{E}^{\mathcal{X}, \mathcal{Y}} : \text{Ran}(\mathcal{X}) \times [0,1] \to \text{Ran}(\mathcal{X})$, inducing by Proposition 7.3 a homotopy $\mathcal{E}^{\mathcal{X}, \mathcal{Y}} : \text{Hur}(\mathcal{E}; Q, G)_{P_{0}} \times [0,1] \to \text{Hur}(\mathcal{E}; Q, G)_{P_{0}}$ with the following properties:

- $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}(-,0)$ is the identity of $\text{Hur}(\mathcal{E}; Q, G)_{P_{0}}$;
- $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}(-,1)$ takes values in $\text{Hur}(\mathcal{E}; Q, Q_{+}, G)_{P_{0}}$.

It suffices now to prove that there is a homotopy of maps $\text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}} \to \text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}}$ from $\mathcal{E}^{\mathcal{X}, \mathcal{Y}}(-,1)_{\text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}}}$ to the identity of $\text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}}$.

Using weak contractibility of $\text{Ran}_{+}(\mathcal{Y})$ (see Lemma 5.3) together with the fact that $\mathcal{X}$ is homeomorphic to a CW complex, we can find a homotopy $\mathcal{E}^{\mathcal{Y}} : \mathcal{X} \times [0,1] \to \text{Ran}_{+}(\mathcal{Y})$ satisfying the following properties:

- $\mathcal{E}^{\mathcal{Y}}(-, 0) = \mathcal{E}^{\mathcal{X}, \mathcal{Y}}(-, 1)$;
- $\mathcal{E}^{\mathcal{Y}}(-, 1)$ is the constant map with value $P_{0} \in \text{Ran}_{+}(\mathcal{Y})$.

Denote by $\mathcal{E}^{\mathcal{Y}} : \text{Ran}(\mathcal{X})_{P_{0}} \times [0,1] \to \text{Ran}_{+}(\mathcal{Y})$ also the induced map on Ran spaces. Consider the homotopy $\mathcal{H}^{\mathcal{Y}} : \text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}} \times [0,1] \to \text{Hur}(\mathcal{E}; Q_{+}, G)_{P_{0}}$ given by
the composition
\[
\text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0} \times [0, 1] \xrightarrow{(\text{Id}, \varepsilon) \times \text{Id}} \text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0} \times \text{Ran}(\mathcal{X})_{P_0} \times [0, 1]
\]
\[
\text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0} \times \text{Ran}(\mathcal{Y}) \xrightarrow{- \times -} \text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0},
\]
where \(\varepsilon\) is from Definition 3.10 and in the last step we use Notation 5.9. Since \(\rho\) restricts to the identity on \(\text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0}\), the map \(\mathcal{H}^{\mathcal{Y}}(-, 0)\) coincides with the restriction of \(\mathcal{E}^{\mathcal{X}; \mathcal{Y}}(-, 1)\) to \(\text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0}\). On the other hand, \(\mathcal{H}^{\mathcal{Y}}(-, 1)\) is the identity of \(\text{Hur}(\mathcal{C}; \mathcal{Q}+, G)_{P_0}\).

7.4. An application of contractibility of \(\text{Ran}\) spaces. The following proposition deals with a generic PMQ-group pair \((\mathcal{Q}, G)\), with \(\mathcal{Q}\) possibly non-augmented, but is included in this section as it uses that \(\text{Ran}\) spaces are contractible, which is a consequence of Proposition 7.3.

**Proposition 7.10.** Let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) be a nice couple with \(\mathcal{X}\) non-empty and connected, let \(P_0 \subset \mathcal{X}\) be a non-empty finite subset, and let \((\mathcal{Q}, G)\) be a PMQ-group pair. Then the inclusion \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0} \subset \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\) is a homotopy equivalence.

**Proof.** By Proposition 7.3 there is a homotopy \(\mathcal{E}^{P_0} : \text{Ran}_+(\mathcal{C}) \times [0, 1] \to \text{Ran}_+(\mathcal{C})\) contracting \(\text{Ran}_+(\mathcal{C})\) onto the configuration \(P_0\).

We construct a homotopy \(\mathcal{H}^{P_0} : \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] \to \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\) as the composition
\[
\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] \xrightarrow{(\text{Id}, \varepsilon) \times \text{Id}} \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times \text{Ran}_+(\mathcal{C}) \times [0, 1]
\]
\[
\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times \text{Ran}_+(\mathcal{C}) \xrightarrow{- \times -} \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G),
\]
where \(\varepsilon\) is from Definition 3.10 and in the last step we use Notation 5.9. Note the following:

- \(\mathcal{H}^{P_0}(-; 0)\) is the identity of \(\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\);
- \(\mathcal{H}^{P_0}(-; 1)\) restricts to the identity on \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\);
- for all \(0 \leq s \leq 1\) and \(\varepsilon \in \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\), if we denote \(\varepsilon' = \mathcal{H}^{P_0}(\varepsilon; s)\) and use Notation 5.6 then \(P \subset P'\); if moreover we assume \(s = 1\), then \(P_0 \subset P'\).

In particular the map \(\mathcal{H}^{P_0}(-; 1)\) takes values in \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\) and restricts to the identity on \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\). This implies that the inclusion \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0} \subset \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\) is a homotopy equivalence.

8. Cell stratifications

We fix an augmented PMQ \(\mathcal{Q}\) throughout the section, and denote by \(\hat{\mathcal{Q}}\) its completion.

**Notation 8.1.** We denote by \(\hat{\mathcal{R}}\) the open unit square \((0, 1)^2 \subset \mathbb{H}\), and by \(\mathcal{R}\) the closed unit square \([0, 1]^2 \subset \mathbb{H}\).

In this section we introduce a cell stratification on \(\text{Hur}(\hat{\mathcal{R}}; \mathcal{Q}+)\). More precisely, we will do the following:
(1) we consider the completion $\hat{Q}$ of $Q$, and regard $\text{Hur}(\hat{R}; \hat{Q}_+)$ as an open subspace of $\text{Hur}(\hat{R}; \hat{Q}_+)$, by applying functoriality to the open inclusion $\hat{R} \subset R$ and to the inclusion of augmented PMQs $Q \subset \hat{Q};$

(2) we define a continuous bijection $v: |\text{Arr}(Q)| \to \text{Hur}(\hat{R}; \hat{Q}_+)$, where the bisimplicial complex $\text{Arr}(Q)$ was introduced in [Bia21a, Definition 6.6]; this will be given by defining, for every non-degenerate array $\underline{a} \in \text{Arr}_{p,q}(Q)$, a continuous map $\varphi: \Delta^p \times \Delta^q \to \text{Hur}(\hat{R}; \hat{Q}_+);

(3) the map $v$ restricts to a continuous bijection

$$v: \text{Hur}^\Delta = |\text{Arr}(Q)| \setminus |\text{NAdm}(Q)| \to \text{Hur}(R; Q),$$

and in the additional hypothesis that $Q$ is locally finite PMQ, this latter bijection is a homeomorphism.

8.1. A construction with simplices. We start by fixing some notation and by making some constructions with simplices and products of simplices. For $p \geq 0$, we regard the standard $p$-simplex $\Delta^p$ as the subspace of $[0,1]^p$ containing all $p$-tuples $\underline{s} = (s_1, \ldots, s_p)$ with $0 \leq s_1 \leq \cdots \leq s_p \leq 1$.

**Notation 8.2.** Whenever needed, we extend each $p$-tuple $\underline{s} = (s_1, \ldots, s_p)$ representing a point in $\Delta^p$ to a $p+2$-tuple $s_0, s_0, s_{p+1}$ by setting $s_0 = 0$ and $s_{p+1} = 1$.

**Notation 8.3.** For $p \geq 0$ we denote by $\text{bar}^p = (\text{bar}_1^p, \ldots, \text{bar}_{p+1}^p) = (\frac{1}{p+1}, \ldots, \frac{1}{p+1})$ the barycentre of $\Delta^p$.

**Definition 8.4.** We denote by $\tilde{\Delta}^{p,p} \subset \Delta^p \times \Delta^p$ the subspace containing all pairs $(\underline{s}, \underline{s}')$ such that the following holds: for all $0 \leq i \leq p$, either $s_i \neq s_{i+1}$ or both equalities $s_i = s_{i+1}$ and $s_i' = s_{i+1}'$ hold.

Roughly speaking, $(\underline{s}, \underline{s}') \in \tilde{\Delta}^{p,p}$ if and only if whenever $\underline{s}$ belongs to a facet of $\Delta^p$, then also $\underline{s}'$ belongs to the same facet of $\Delta^p$.

**Definition 8.5.** We define a continuous map $\mathcal{H}^p: \mathbb{R} \times \tilde{\Delta}^{p,p} \to \mathbb{R}$ by the formula

$$\mathcal{H}^p(x; \underline{s}, \underline{s}') = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus (0,1); \\
\frac{x-s_i}{s_{i+1}-s_i}s_{i+1}'+\frac{s_{i+1}-x}{s_{i+1}-s_i}s_i' & \text{if } s_i \neq s_{i+1} \text{ and } x \in [s_i, s_{i+1}]; \\
s_i' &= s_{i+1}' & \text{if } x = s_i = s_{i+1}. \end{cases}$$

Roughly speaking, $\mathcal{H}^p(\cdot; \underline{s}, \underline{s}') : \mathbb{R} \to \mathbb{R}$ is constructed by fixing $(-\infty,0] \cup [1,\infty)$ pointwise, by mapping each $s_i \mapsto s_i'$ and by extending by linear interpolation on the segments $[0,s_1], \ldots, [s_p,1]$; some of these segments might be degenerate, in this case no extension is needed. The subspace $\tilde{\Delta}^{p,p} \subset \Delta^p \times \Delta^p$ is essentially defined as the subspace of couples $(\underline{s}, \underline{s}')$ for which $\mathcal{H}^p(\cdot; \underline{s}, \underline{s}')$ is well-defined and continuous. The following property of the map $\mathcal{H}^p$ follows immediately, and we state it as lemma for future reference.

**Lemma 8.6.** Let $\underline{s}, \underline{s}', \underline{s}'' \in \Delta^p$ such that both pairs $(\underline{s}, \underline{s}')$ and $(\underline{s}', \underline{s}'')$ lie in $\tilde{\Delta}^{p,p}$; then also $(\underline{s}, \underline{s}'') \in \tilde{\Delta}^{p,p}$, and the map $\mathcal{H}^p(\cdot; \underline{s}, \underline{s}') : \mathbb{R} \to \mathbb{R}$ coincides with the composition $\mathcal{H}^p(\cdot; \underline{s}', \underline{s}'') \circ \mathcal{H}^p(\cdot; \underline{s}, \underline{s}')$.

**Definition 8.7.** For all $p, q \geq 0$ we define a map $\mathcal{H}^{p,q} : C \times \tilde{\Delta}^{p,p} \times \tilde{\Delta}^{q,q} \to \mathbb{C}$ by the formula

$$\mathcal{H}^{p,q}(x + y\sqrt{-1}; \underline{s}, \underline{s}', \underline{t}, \underline{t}') = \mathcal{H}^p(x; \underline{s}, \underline{s}') + \mathcal{H}^q(y; \underline{t}, \underline{t}').$$
Note that for all $(s, g' : l, l') \in \bar{\Delta}^{p \times \bar{q}} \times \bar{\Delta}^{q \times q}$ the map $H^{p \times q}(-; s, g' : l, l') : C \to C$ is a lax self morphism of the nice couple $(\mathcal{R}, \emptyset)$ (see Definition 4.3). By Proposition 4.10 we obtain a map

$$H^{p \times q} : \text{Hur}(\mathcal{R}; \hat{Q}_+) \times \bar{\Delta}^{p \times \bar{q}} \times \bar{\Delta}^{q \times q} \to \text{Hur}(\mathcal{R}; \hat{Q}_+)$$

Note also that $\Delta^p$ embeds diagonally into $\bar{\Delta}^{p \times \bar{q}}$; the restricted map

$$H^{p \times q} : \text{Hur}(\mathcal{R}; \hat{Q}_+) \times \Delta^p \times \Delta^q \to \text{Hur}(\mathcal{R}; \hat{Q}_+)$$

is just the projection on the first factor, since for all $(s, l) \in \Delta^p \times \Delta^q$ the map $H^{p \times q}(-; s, g' : l, l') : C \to C$ is the identity of $C$.

Lemma 8.8. Let $s, s', s'' \in \Delta^p$ such that $(s, s'), (s', s'') \in \bar{\Delta}^{p \times p}$, and let $l, l', l'' \in \Delta^q$ such that $(l, l'), (l', l'') \in \bar{\Delta}^{q \times q}$. Then $(s, s', s'', l, l', l'') \in \Delta^{p \times q}$, and the following equality of maps $\text{Hur}(\mathcal{R}; \hat{Q}_+) \to \text{Hur}(\mathcal{R}; \hat{Q}_+)$ holds:

$$H^{p \times q}(-; s, s', s'', l, l', l'') \circ H^{p \times q}(-; s, s', s', l, l') = H^{p \times q}(-; s, s', s'', l, l').$$

8.2. The array filtration. The next step is to define a filtration on $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ by closed subspaces $F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$, for $\nu \geq -1$.

Definition 8.9. Let $P \in \text{Ran}(\mathcal{R})$ be a finite, possibly empty subset of $\mathcal{R}$, and use Notation 2.3. We define the horizontal array degree of $P$, denoted $\text{arr}_{\text{hor}}(P) \geq 0$, as the cardinality of the finite set $\Re(P) \setminus \{0, 1\} = \{\Re(z_1), \ldots, \Re(z_k)\} \setminus \{0, 1\}$; similarly we define the vertical array degree of $P$, denoted $\text{arr}_{\text{ver}}(P) \geq 0$, as the cardinality of the finite set $\Im(P) \setminus \{0, 1\} = \{\Im(z_1), \ldots, \Im(z_k)\} \setminus \{0, 1\}$. The array bidegree $\text{arr}(P)$ is defined as the couple $(\text{arr}_{\text{hor}}(P), \text{arr}_{\text{ver}}(P))$, and the total array degree is defined as $|\text{arr}|(P) = \text{arr}_{\text{hor}}(P) + \text{arr}_{\text{ver}}(P)$.

For $c = (P, \psi) \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$ we define $\text{arr}(c) = (\text{arr}_{\text{hor}}(c), \text{arr}_{\text{ver}}(c)) := \text{arr}(P)$, and $|\text{arr}|(c) = |\text{arr}|(P)$. For $\nu \geq -1$ we define $F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$ as the subspace of $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ containing all configurations $c$ with $|\text{arr}|(c) \leq \nu$. For $\nu \geq 0$ we define $\mathfrak{F}_\nu^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+) := \text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F_{\nu-1}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$. Roughly speaking, $F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$ contains configurations $c = (P, \psi)$ such that the total number of horizontal and vertical lines passing through some point of $P$, excluding the sides of $\mathcal{R}$, does not exceed $\nu$. Similarly, $\mathfrak{F}_\nu^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+) := \text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F_{\nu-1}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$ contains those configurations $c$ for which this total number of lines is equal to $\nu$.

Lemma 8.10. For $\nu \geq -1$ the subspace $F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+) \subset \text{Hur}(\mathcal{R}; \hat{Q}_+)$ is closed.

Proof. We prove that $\text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$ is open. Let $c \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$ be a configuration with $|\text{arr}|(c) \geq \nu + 1$, use Notations 3.3 and 2.6 and let $U(c)$ be an adapted covering of $P$ with the following property: for all $1 \leq i \leq k$ the projection $\Re(U_i) \subset \mathbb{R}$ intersects the finite set $\Re(P) \cup \{0, 1\}$ only in the point $\Re(z_i)$, and the projection $\Im(U_i) \subset \mathbb{R}$ intersects the finite set $\Im(P) \cup \{0, 1\}$ only in the point $\Im(z_i)$. We claim that $M(c; U(c))$ is contained in $\text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F_{\nu}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{Q}_+)$. Let $c' \in M(c; U(c))$; then $P'$ intersects each $U_i$ in at least one point; by our choice of $U(c)$ we have that $\text{arr}_{\text{hor}}(c') \geq \text{arr}_{\text{hor}}(c)$ and $\text{arr}_{\text{ver}}(c') \geq \text{arr}_{\text{ver}}(c)$, hence $|\text{arr}|(c') \geq |\text{arr}|(c) \geq \nu + 1$ as desired.}

The filtration $F_{\nu}^{\text{arr}}$ on $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ plays in the following discussion a similar role as the skeletal filtration of a cell complex.
8.3. **Standard generating set.** In this subsection we introduce, for a finite set $P \subset \mathcal{R}$, a particular admissible generating set of $\mathfrak{G}(P)$, the *standard generating set*. 

Fix $P \subset \mathcal{R}$, and let $\mathcal{R}(P) \cup \{0, 1\}$ consist of the points $0 = x_0 < x_1 < \cdots < x_p < x_{p+1} = 1$, where $p = \text{arr}_{\text{hor}}(P)$; similarly let $0 = y_0 < y_1 < \cdots < y_q < y_{q+1} = 1$ be the elements of $\mathcal{R}(P) \cup \{0, 1\}$, where $q = \text{arr}_{\text{ver}}(P)$.

For all $(i,j) \in \{0, \ldots, p+1\} \times \{0, \ldots, q+1\}$ denote by $z_{i,j}$ the complex number $x_i + y_j \sqrt{-1} \in \mathbb{C}$, and let $I(P) \subset \{0, \ldots, p+1\} \times \{0, \ldots, q+1\}$ be the subset of pairs $(i,j)$ such that $z_{i,j}$ is a point of $P$.

Recall Notation 6.8. For all $(i,j) \in I(P)$ with $0 \leq i \leq p$ let $\zeta_{i,j}^{p,\text{std}}$ be an arc contained in $S_{x_i,x_{i+1}}$ and joining $*$ with $z_{i,j}$. Similarly, for all $(p+1, j) \in I(P)$ let $\zeta_{p+1,j}^{p,\text{std}}$ be an arc contained in $S_{1,\infty}$ joining $*$ with $z_{p+1,j}$. Up to changing the arcs by an isotopy, we may assume that the arcs $\zeta_{i,j}^{p,\text{std}}$ are disjoint away from $*$, note also that these arcs are uniquely determined up to an ambient isotopy of $\mathbb{C}$ that fixes $P$ pointwise and preserves each subspace $S_{x_i,x_{i+1}}$.

**Definition 8.11.** Recall Definition 2.8. We denote by $(f_{i,j}^{P,\text{std}})_{(i,j) \in I(P)}$ the admissible generating set of $\mathfrak{G}(P)$ associated with the arcs $\zeta_{i,j}^{p,\text{std}}$, and call it the *standard generating set* for $\mathfrak{G}(P)$. See Figure 9.

![Figure 9](image-url)
Notation 8.12. Use the notation from Definition 8.11. For $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 2$ we denote by $cf^p_{i,j}$ the product

$$cf^p_{i,j} = f^p_{i,0} \cdots f^p_{i,j-1} \in \mathcal{G}(P),$$

where we set $f^p_{i,j} = 1 \in \mathcal{G}(P)$ whenever $(i, j)$ does not belong to $I(P)$.

Note that $cf^p_{i,j}$ is represented by a simple loop $\gamma \subset \mathbb{C} \setminus P$ with the following properties:

- $\gamma \subset \mathbb{S}_{x_{i-1}, x_{i+1}} \cap \{z \in \mathbb{C} | \Im(z) < y_j\}$, where we use the conventions $x_{-1} = -\infty$ and $x_{q+2} = y_{q+2} = \infty$;
- $\gamma$ bounds a disc in $\mathbb{C}$ containing the points $z_{i,0}, \ldots, z_{i,j-1}$.

In particular $cf^p_{i,j} \in \Omega^{\text{ext}}(P)$, see Definition 2.10. Note also that for $j = 0$ we have $cf^p_{i,j} = 1$.

8.4. Characteristic maps of cells. In this subsection we introduce maps

$$\epsilon_{\Delta^p \times \Delta^q} : \text{Hur}(\mathcal{R}; \hat{\mathbb{Q}}_+) \to \text{Hur}(\mathcal{R}; \hat{\mathbb{Q}}_+)$$

depending on a non-degenerate array $\mathbf{a} \in \text{Arr}_{p,q}(\mathcal{Q})$. As we will see, each map $\epsilon_{\Delta}$ sends the interior of $\Delta^p \times \Delta^q$ injectively inside $\mathfrak{F}_{p,q}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{\mathbb{Q}}_+)$, and sends the boundary of $\Delta^p \times \Delta^q$ inside $F_{p,q-1}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{\mathbb{Q}}_+)$.

Recall from [Bin21a] Definitions 5.8 and 6.6 that the bisimplicial set $\text{Arr}(\mathcal{Q})$ consists of the sets $\text{Arr}_{p,q}(\mathcal{Q}) \cong \mathcal{Q}^{(p+2) \times (q+2)}$ for $p, q \geq 0$. An element $\mathbf{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ is an array of size $(p + 2) \times (q + 2)$ with entries in $\mathcal{Q}$. For $p \geq 1$ and $0 \leq i \leq p$, the $i$-th horizontal face map is denoted $d^i_{\text{hor}} : \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p-1,q}(\mathcal{Q})$, and for $q \geq 1$ and $0 \leq j \leq q$, the $j$-th vertical face map is denoted $d^j_{\text{ver}} : \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p,q-1}(\mathcal{Q})$. Similarly $s^i_{\text{hor}}$ and $s^j_{\text{ver}}$ denote the horizontal and vertical degeneracy maps. Formulas for face and degeneracy maps are given in [Bin21a] Lemma 6.8. An array $\mathbf{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ is non-degenerate if and only if it is not in the image of any horizontal or vertical degeneracy map of the bisimplicial set $\text{Arr}(\mathcal{Q})$.

Notation 8.13. For $\mathbf{a} \in \text{Arr}(p,q)$ we let $I(\mathbf{a}) \subset \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\}$ denote the set of pairs $(i,j)$ with $a_{ij} \neq 1$.

An array $\mathbf{a}$ is non-degenerate if and only if the following conditions hold, compare with [Bin21a] Subsection 6.3:

- for all $1 \leq i \leq p$ there is $0 \leq j \leq q + 1$ with $(i, j) \in I(\mathbf{a})$;
- for all $1 \leq j \leq q$ there is $0 \leq i \leq p + 1$ with $(i, j) \in I(\mathbf{a})$.

We fix a non-degenerate array $\mathbf{a} \in \text{Arr}(p,q)$ for the rest of the subsection. In the following we define a configuration $c_\mathbf{a} \in \text{Hur}(\mathcal{R}; \hat{\mathbb{Q}}_+)$ with $\text{arr}_{\text{hor}}(c_\mathbf{a}) = p$ and $\text{arr}_{\text{ver}}(c_\mathbf{a}) = q$.

Notation 8.14. We denote by $P^{p,q} \subset \mathcal{R}$ the set of complex numbers $z^p_{i,j} := \frac{i}{p+1} + \frac{j}{q+1} \sqrt{-1}$, with $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 1$.

Let $P_\mathbf{a} \subset P^{p,q}$ be the set containing all elements $z^p_{i,j}$ for $(i, j) \in I(\mathbf{a})$. Since $\mathbf{a}$ is admissible we have $\text{arr}_{\text{hor}}(P_\mathbf{a}) = p$ and $\text{arr}_{\text{ver}}(P_\mathbf{a}) = q$.

Notation 8.15. We denote by $(f^p_{i,j})_{(i,j) \in I(\mathbf{a})}$ the standard generating set $(f^p_{i,j})$ of $\mathcal{G}(P_\mathbf{a})$ (see Definition 8.11). For $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 2$ we denote by $cf^p_{i,j}$ the product $cf^p_{i,j}$ (see Notation 8.12).
Definition 8.16. We define $e_{\Delta}$ as the configuration $(P_a, \psi_a) \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$, where $\psi_a$ is defined by setting $\psi_a: \frac{\mathcal{P}}{I_{i,j}} \mapsto a_i, j$ for all $(i, j) \in I(q)$.

Note that since $\hat{Q}$ is complete we have an equality $\Omega_{\Delta}(P_a) = \Omega_{\Delta}(P_a)_{\psi_a}$, so that we can extend $\psi_a$ to a map of PMQs $\Omega_{\Delta}: \Omega_{\Delta}(P_a) \rightarrow \hat{Q}$; the element $\rho_{\Delta}^a \in \Omega_{\Delta}(P_a)$ is mapped to the product $a_{i,0} \ldots a_{i,j-1} \in \hat{Q}$ along $\psi_{\Delta}$.

Recall Notation 3.3 and Definition 8.4 and note that for all $\hat{z} \in \Delta^p$ the pair $(\bar{z}_p, s)$ belongs to $\Delta^p$.

Definition 8.17. For a non-degenerate array $a \in \text{Arr}(p, q)$ we define a continuous map $e_{\Delta}: \Delta^p \times \Delta^q \rightarrow \text{Hur}(\mathcal{R}; \hat{Q}_+)$ by the formula

$$e_{\Delta}(s; t) = \mathcal{H}^{\rho_{\Delta}}_{s, t} \left( e_{\Delta}; \bar{a}_p, s; \bar{a}_q, t \right).$$

Lemma 8.18. Let $a \in \text{Arr}_{p, q}$ be non-degenerate; then the map $e_{\Delta}$ has the following properties:

- it sends the interior of $\Delta^p \times \Delta^q$ injectively inside $\mathfrak{F}_{p+q} \text{Hur}(\mathcal{R}; \hat{Q}_+)$;
- it sends $\partial(\Delta^p \times \Delta^q)$ into $\mathfrak{F}_{p+q} \text{Hur}(\mathcal{R}; \hat{Q}_+)$.

Proof. Let $(s, t) \in \Delta^p \times \Delta^q$, and let $c = (P, \psi) := e_{\Delta}(s, t)$. Then the set $P$ is the image of the set $P_s$ under the map

$$\mathcal{H}^{\rho_{\Delta}}(-; \bar{a}_p, s; \bar{a}_q, t): \mathbb{C} \rightarrow \mathbb{C}$$

Note that the latter map sends $s_{i,j} \mapsto s_i + t_j \sqrt{-1}$ for all $0 \leq i \leq p+1$ and $0 \leq j \leq q+1$, in particular for $(i, j) \in I(a)$. It follows that $\mathcal{R}(P) \setminus \{0\} = \{s_1, \ldots, s_p\} \setminus \{0\}$ consists of at most $p$ points, and $\mathfrak{F}(P) \setminus \{0\} = \{t_1, \ldots, t_q\} \setminus \{0\}$ consists of at most $q$ points. More precisely, using also that $a$ is non-degenerate, we have that $|\mathcal{R}(P) \setminus \{0\}| = p$ if $0 < s_1 < \cdots < s_p < 1$, i.e. $s$ is in the interior of $\Delta^p$, and $|\mathcal{R}(P) \setminus \{0\}| < p$ if instead $s \in \partial \Delta^p$. Similarly $|\mathfrak{F}(P) \setminus \{0\}| = q$ if $t$ is in the interior of $\Delta^q$, and $|\mathfrak{F}(P) \setminus \{0\}| < q$ if instead $t \in \partial \Delta^q$.

It follows that $|\mathfrak{F}(P)| = p + q$ in all cases, and equality holds if and only if $(s, t)$ is in the interior of $\Delta^p \times \Delta^q$. □

Lemma 8.19. Let $\nu \geq 0$ and let $c \in \mathfrak{F}_{p+q} \text{Hur}(\mathcal{R}; \hat{Q}_+)$; then there is precisely one couple of indices $p, q \geq 0$ with $p + q = \nu$, and precisely one admissible array $a \in \text{Arr}(p, q)$, such that $c$ is in the image of $e_{\Delta}$.

Proof. We start by showing the existence, for a given configuration $c$, of $p, q$ and $a$ with the required properties. Use Notation 3.3. Then $\mathcal{R}(P) \cup \{0\}$ is a finite set $\{0 < s_1 < \cdots < s_p < 1\}$ of $p + 2$ elements, for some $p \geq 0$; similarly $\mathfrak{F}(P) \cup \{0\}$ is a finite set $\{0 < t_1 < \cdots < t_q < 1\}$ of $q + 2$ elements, for some $q \geq 0$. By the hypothesis that $c \in \mathfrak{F}_{p+q} \text{Hur}(\mathcal{R}; \hat{Q}_+)$ we have the equality $p + q = |\mathfrak{F}(P)| = \nu$.

Define an array $a$ of size $(p + 2) \times (q + 2)$ by letting $a_{i,j} = \psi(f_{i,j}^{P,\text{std}}) \in \hat{Q}$ for all $(i, j) \in I(P)$, and $a_{i,j} = 1$ for all $(i, j) \in \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\} \setminus I(P)$ (see Definition 8.11). The hypothesis that $c$ lies in $\text{Hur}(\mathcal{R}; \hat{Q}_+) \subset \text{Hur}(\mathcal{R}; \hat{Q}_+)$ ensures that $a$ is a non-degenerate array in $\text{Arr}_{p, q}(Q)$. Moreover we have $\mathcal{I}(a) = I(P)$.

We claim that $e_{\Delta}$ sends $(s, t) \in \Delta^p \times \Delta^q$ to $c$. This follows from the observation that the map $\mathcal{H}^{\rho_{\Delta}}(-; \bar{a}_p, s; \bar{a}_q, t)$ is a homeomorphism of $\mathbb{C}$, sending $P_a$ bijectively to $P$, and sending the standard generating set $(f_{i,j}^{P,\text{std}})_{(i,j) \in I(a)}$ of $\mathcal{G}(P_a)$ to the standard generating set $(f_{i,j}^{P,\text{std}})_{(i,j) \in I(a)}$ of $\mathcal{G}(P)$. This proves the existence of $p, q$ and $a$ as desired.
For uniqueness, suppose that we are given two integers \( p', q' \geq 0 \) and a non-degenerate array \( \bar{a}' \in \text{Arr}(p,q) \), such that \( p' + q' = \nu \) and such that there is a point \( (\bar{a}', \bar{t}') \in \Delta^{p'} \times \Delta^{q'} \) with \( \epsilon(a') : (\bar{s}', \bar{t}') \mapsto \epsilon \). Then by Lemma 8.18 we have that \( \langle \bar{s}', \bar{t}' \rangle \) lies in the interior of \( \Delta^{p'} \times \Delta^{q'} \), since \( \epsilon \) lies in \( \mathfrak{H}^\text{arr} \text{Hur}(\mathcal{R} ; \mathcal{Q}_+) \). Again

\[
\mathcal{H}^{p',q'} \left( \begin{array}{c}
\bar{s}', \bar{t}' ; \bar{a}' \end{array} \right) : \mathbb{C} \to \mathbb{C}
\]

is a homeomorphism of \( \mathbb{C} \) mapping the set \( P_{\bar{a}} \) bijectively to the set \( P \) by the formula

\[
z_{i,j} \mapsto s_i + \sqrt{-1} t_j .
\]

It follows that \( \{0 < s'_1 < \cdots < s'_{p'} < 1\} \) is equal to \( \Re(P) \cup \{0, 1\} \), and similarly \( \{0 < t'_1 < \cdots < t'_{q'} < 1\} \) is equal to \( \Im(P) \cup \{0, 1\} \); in particular, comparing with the construction above, we have \( p = p' \), \( q = q' \), \( \bar{s} = \bar{s}' \), \( \bar{t} = \bar{t}' \), and \( I(\epsilon(a')) = I(P) \).

Since \( \mathfrak{H}^{p',q'} \left( \begin{array}{c}
\bar{a}', \bar{s}' ; \bar{a}' \end{array} \right) \) gives a bijection between the standard generating sets \( (f^{p}_{i,j})_{i,j \in I(P)} \) and \( (f^{p',q'}_{i,j})_{i,j \in I(P)} \), it also follows that

\[
a_{i,j}' = \psi(a') (f^{p}_{i,j}) = \psi(f_{i,j}) = a_{i,j}
\]

for all \( (i,j) \in I(P) \), where \( \psi \) is the monodromy of the configuration \( c_\bar{a}' \); see Definition 8.16 hence \( a' = a \). ⊓⊔

8.5. Face restrictions and the bijection \( v \). In the following two propositions we analyse the restriction of \( c_\bar{a} \) to a face of \( \Delta^p \times \Delta^q \), and thus establish a link between the bisimplicial set \( \text{Arr}(\mathcal{Q}) \) and the cell stratification on \( \text{Hur}(\mathcal{R} ; \mathcal{Q}_+) \).

**Notation 8.20.** For \( 0 \leq i \leq p \) we denote by \( d^{\text{hor}}_{i} : \Delta^p \times \Delta^q \to \Delta^p \times \Delta^q \) the face \( (d_i \Delta^p) \times \Delta^q \subseteq \Delta^p \times \Delta^q \); for \( 0 \leq j \leq q \) we denote by \( d^{\text{ver}}_{j} : \Delta^p \times \Delta^q \to \Delta^p \times \Delta^q \) the face \( \Delta^p \times (d_j \Delta^q) \subseteq \Delta^p \times \Delta^q \).

Each face \( d_i \Delta^p \subseteq \Delta^p \) can be identified with the simplex \( \Delta^{p-1} \) by using either the coordinates \( (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p) \), for \( i \neq 0 \), or the coordinates \( (s_1, \ldots, s_i, s_{i+2}, \ldots, s_p) \), for \( i \neq p \); for \( 1 \leq i \leq p-1 \) the two choices give rise to the same identification. Similarly, there are canonical identifications of \( d^{\text{hor}}_{i} : \Delta^p \times \Delta^q \) with \( \Delta^{p-1} \times \Delta^q \), and of \( d^{\text{ver}}_{j} : \Delta^p \times \Delta^q \) with \( \Delta^p \times \Delta^{q-1} \).

**Proposition 8.21.** Let \( \bar{a} \) be a non-degenerate array in \( \text{Arr}_{p,q}(\mathcal{Q}) \), for some \( p \geq 1 \) and \( q \geq 0 \), and let \( 0 \leq i \leq p \). Then the restriction of \( c_\bar{a} : \Delta^p \times \Delta^q \to \text{Hur}(\mathcal{R} ; \mathcal{Q}_+) \) to the face \( d^{\text{hor}}_{i} \Delta^p \times \Delta^q \cong \Delta^{p-1} \times \Delta^q \) is equal to the map \( c_{\bar{a}'} \), where \( \bar{a}' = d^{\text{hor}}_{i} \bar{a} \).

**Proposition 8.22.** Let \( \bar{a} \) be a non-degenerate array in \( \text{Arr}_{p,q}(\mathcal{Q}) \), for some \( p \geq 0 \) and \( q \geq 1 \), and let \( 0 \leq j \leq q \). Then the restriction of \( c_\bar{a} : \Delta^p \times \Delta^q \to \text{Hur}(\mathcal{R} ; \mathcal{Q}_+) \) to the face \( d^{\text{ver}}_{j} \Delta^p \times \Delta^q \cong \Delta^p \times \Delta^{q-1} \) is equal to the map \( c_{\bar{a}'} \), where \( \bar{a}' = d^{\text{ver}}_{j} \bar{a} \).

The proof of Propositions 8.21 and 8.22 is in Subsections 3.10 and 3.11 of the appendix.

Recall from [Bia21a, Lemma 6.10] that there is a semi-bisimplicial set \( \text{Arr}_{n}(\mathcal{Q}) \) containing all non-degenerate arrays of \( \text{Arr}(\mathcal{Q}) \), and with vertical and horizontal face maps given by the restriction of those of \( \text{Arr}(\mathcal{Q}) \). Consider the geometric realisation \( |\text{Arr}_{n}(\mathcal{Q})| \) of the semi-bisimplicial complex \( \text{Arr}_{n}(\mathcal{Q}) \), and note that there is a homeomorphism \( |\text{Arr}_{n}(\mathcal{Q})| \cong |\text{Arr}(\mathcal{Q})| \).

By Propositions 8.21 and 8.22 we obtain a continuous map

\[
v : |\text{Arr}(\mathcal{Q})| \to \text{Hur}(\mathcal{R} ; \mathcal{Q}_+).
\]
Lemma \textbf{S.18} implies that $v$ is injective. If we consider on $|\text{Arr}(Q)|$ the skeletal filtration and on $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ the filtration $\mathcal{F}^{\text{arr}}_\bullet$, the same lemma implies that $v$ is a map of filtered spaces. Lemma \textbf{S.19} implies that $v$ is surjective. Hence $v$ is a continuous, filtered bijection.

We say that an entry $a_{i,j}$ of an array $a \in \text{Arr}_{p,q}(Q)$ is in boundary position if $i \in \{0, p+1\}$ or $j \in \{0, q+1\}$ (or both conditions hold). Recall from \textbf{Bia21a} Definition 6.11 and Lemma 6.12 that we have a sub-bisimplicial set $\text{NAdm}(Q)$. Lemma 8.18 implies that $\text{Arr}(\mathcal{R}; \hat{Q}_+) \subset \text{NAdm}(Q)$ of non-admissible arrays: an array $a \in \text{Arr}(p,q)$ is non-admissible if either of the following requirements is satisfied:

- there exists an entry $a_{i,j}$ lying in $\hat{Q} \setminus Q$
- there exists an entry $a_{i,j} \neq \emptyset$ in boundary position.

\textbf{Lemma 8.23.} The map $v$ restricts to continuous bijections

$$v: |\text{NAdm}(Q)| \to \text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus \text{Hur}(\mathcal{R}; Q_+);$$

$$v: \text{Hur}^\Delta(Q) = |\text{Arr}(Q)| \setminus |\text{NAdm}(Q)| \to \text{Hur}(\mathcal{R}; \hat{Q}_+).$$

\textbf{Proof.} Let $c \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$, and use Notation \textbf{3.6}. In the proof of Lemma \textbf{S.19} we have given a construction, depending on $c$, of a couple of numbers $p, q \geq 0$, a point $(\mathfrak{s}, \mathfrak{q})$ in the interior of $\Delta^p \times \Delta^q$ and a non-degenerate array $a \in \text{Arr}(p,q)$ such that $c = c(a, \mathfrak{s}, \mathfrak{q})$. The data $(a, \mathfrak{s}, \mathfrak{q})$ represent a point in $|\text{Arr}(Q)|$, which is precisely the preimage $v^{-1}(c)$; we have $v^{-1}(c) \in |\text{NAdm}(Q)|$ if and only if $a$ is non-admissible.

The array $a$ was constructed by considering the standard generating set of $\mathfrak{G}(P)$, and by setting $a_{i,j} = v(f^\text{std}_{i,j})$ for $(i,j) \in I(P)$, and $a_{i,j} = \emptyset$ otherwise. It follows that $a$ has all entries in $Q$ if and only if $v: \mathfrak{G}(P) \to \hat{Q}$ has image in $Q$, that is, $c \in \text{Hur}(\mathcal{R}; Q_+)$; and $a$ has all entries in boundary position equal to $\emptyset$ if and only if $P \subset \mathcal{R}$, that is, $c \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$. We have therefore

$$c \in \text{Hur}(\mathcal{R}; \hat{Q}_+) = \text{Hur}(\mathcal{R}; Q_+) \cap \text{Hur}(\mathcal{R}; \hat{Q}_+) \subset \text{Hur}(\mathcal{R}; \hat{Q}_+)$$

if and only if $a$ is admissible. \hfill \Box

9. Locally finite and Poincare PMQs

In this section we consider the Hurwitz spaces $\text{Hur}(\mathcal{R}; Q)$ in the special cases of a locally finite PMQ and of a Poincare PMQ $Q$. Recall that a PMQ $Q$ is Poincare if each connected component of $\text{Hur}^\Delta(Q)$ is a topological manifold; this condition implies that $Q$ is endowed with an intrinsic norm $h: \hat{Q} \to \mathbb{N}$ such that $\text{Hur}^\Delta(Q)(a)$ is an orientable manifold of dimension $2h(a)$ for all $a \in \hat{Q}$, see \textbf{Bia21a} Proposition 6.21. A Poincare PMQ is always locally finite, and a locally finite PMQ is always augmented.

9.1. Locally finite PMQs. The aim of this subsection is to prove the following theorem.

\textbf{Theorem 9.1.} Let $Q$ be a locally finite PMQ. Then the bijection $v: \text{Hur}^\Delta(Q) \to \text{Hur}(\mathcal{R}; \hat{Q}_+)$ is a homeomorphism.

We first note that both $|\text{Arr}(Q)|$ and $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ decompose as topological disjoint unions of subspaces

$$|\text{Arr}| = \coprod_{a \in \hat{Q}} |\text{Arr}(Q)(a)|, \quad \text{Hur}(\mathcal{R}; \hat{Q}_+) = \coprod_{a \in \hat{Q}} \text{Hur}(\mathcal{R}; \hat{Q}_+)(a).$$
The space $|\text{Arr}(Q)|$ is the geometric realisation of the bisimplicial set $\text{Arr}(Q)(a)$, which is the value at $a \in \hat{Q}/\hat{Q}$ of the $\hat{Q}$-crossed bisimplicial set $\text{Arr}(Q)$: concretely, $\text{Arr}_{p,q}(Q)(a)$ contains all arrays $\underline{a}$ satisfying $\prod_{i=0}^{p+1} \left( \prod_{j=0}^{q+1} a_{i,j} \right) = a \in \hat{Q}$. For the space $\text{Hur}(R; \hat{Q}_+)_a$, see Notation 6.2.

The map $\nu$ restricts for all $a \in \hat{Q}$ to a bijection $\nu_a: |\text{Arr}(Q)(a)| \to \text{Hur}(R; \hat{Q}_+)_a$. Consider first an element $a \in Q \subset \hat{Q}$. The hypothesis that $Q$ is locally finite implies that $\text{Arr}(Q)(a)$ is a bisimplicial complex with finitely many non-degenerate arrays; hence the geometric realisation $|\text{Arr}(Q)(a)|$ is compact. The bijection $\nu_a$ has thus a compact space as source and a Hausdorff space as target, and is therefore a homeomorphism. Restricting to $\text{Hur}^\Lambda(Q)(a)$ and $\text{Hur}(R; Q_+)_a$, we have a homeomorphism $\nu_a: \text{Hur}^\Lambda(Q)(a) \to \text{Hur}(R; Q_+)_a$.

Consider now a generic element $a \in \hat{Q}$, let $c \in \text{Hur}(R; Q_+)_a$, use Notation 3.6 and the notation of Subsection 8.3. Let $\mathcal{U}$ be an adapted covering of $P$, and assume that for all $(i, j) \in I(P)$ the component $U_{i,j} \subset \mathcal{U}$ containing $z_{i,j}$ satisfies the following properties:

- $\mathcal{R}(U_{i,j})$ intersects $\mathcal{R}(P) \cup \{0, 1\}$ only in $\mathcal{R}(z_{i,j})$;
- $\mathcal{S}(U_{i,j})$ intersects $\mathcal{S}(P) \cup \{0, 1\}$ only in $\mathcal{S}(z_{i,j})$.

Let $\hat{U}$ denote the union $\bigcup_{(i,j) \in I(P)} U_{i,j}$, i.e. the closure of $\mathcal{U}$, and note that $\hat{U}$ is compact. The normal neighbourhood $\mathcal{U}(c; \mathcal{U})$ can be regarded as an open subspace of $\text{Hur}(\hat{U}; Q_+)_a$, which by the argument of Theorem 6.1 is homeomorphic to a product of spaces

$$
\text{Hur}(\hat{U}; Q_+)_a \cong \prod_{(i,j) \in I(P)} \text{Hur}(\hat{U}_{i,j}; Q_+)_a_{i,j},
$$

for suitable elements $a_{i,j} \in Q$. Note that if Theorem 6.1 is applied using the arcs $(G_{i,j})_{(i,j) \in I(P)}$ yielding the standard generating set of $G(P)$, then the elements $a_{i,j}$ are precisely the entries different from 1 of the array $\underline{a}$ describing the cell of $\text{Hur}^\Lambda(Q)(a)$ containing $v^{-1}(c)$.

The previous analysis shows that each factor $\text{Hur}(\hat{U}_{i,j}; Q_+)_a_{i,j}$ is compact; it follows that $\text{Hur}(\hat{U}; Q_+)_a$ is compact i.e. $\text{Hur}(R; Q_+)_a$ is locally compact.

Consider $\text{Hur}(\hat{U}; Q_+)_a$ as a subspace of $\text{Hur}(R; Q_+)_a$: the hypothesis that $Q$ is locally finite implies that the preimage $v_a^{-1}(\text{Hur}(\hat{U}; Q_+)_a)$ intersects only finitely many cells in the cell decomposition of $|\text{Arr}(Q)(a)|$. Hence $v_a^{-1}(\text{Hur}(\hat{U}; Q_+)_a)$ is compact, being a closed subset of a finite cell sub-complex of $|\text{Arr}(Q)(a)|$. Since $v_a^{-1}(\text{Hur}(\hat{U}; Q_+)_a)$ contains the open neighbourhood $v_a^{-1}(\mathcal{U}(c; \hat{U}))$ of $v_a^{-1}(c)$, we obtain that $\text{Hur}^\Lambda(Q)(a)$ is also locally compact.

We conclude that $\nu_a: \text{Hur}^\Lambda(Q)(a) \to \text{Hur}(R; Q_+)_a$ is a proper continuous bijection between locally compact spaces, hence it is a homeomorphism.

9.2. A counterexample to Theorem 9.1 for non-locally finite PMQs. For an augmented but not locally finite PMQ $Q$, the bijection $\nu$ may not restrict to a homeomorphism $\text{Hur}^\Lambda(Q) \to \text{Hur}(R; Q_+)$: see the following example.

Example 9.2. Let $\hat{Q}$ be the complete PMQ from [Bin21a, Example 4.13], and let $c = (P, \psi) \in \text{Hur}(R; \hat{Q}_+)$ be a configuration supported on the set $P = \{z_c\} = \{\frac{1+i\sqrt{3}}{2}\}$ with $\psi$ defined by sending the unique element $|\gamma| \in \Omega(P) \setminus \{1\}$ to $w = f_1f_2 \in \hat{Q}$.
Note that for all $0 < \varepsilon \leq 1/2$ we have an adapted covering of $P$ of the form $U_\varepsilon = \{ z \in \mathbb{C} : |z - z_c| < \varepsilon \}$; the associated normal neighbourhoods $\mathcal{U}(c; U_\varepsilon)$ form a fundamental system of neighbourhoods of $c \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$.

For $\varepsilon > 0$, denote by $P_\varepsilon$ the set of two points $\{z_c \pm \varepsilon/2\}$, and note that $P_\varepsilon \subset U_\varepsilon$.

For every decomposition $w = a \cdot b$ with respect to $\mathcal{Q}_1$ we can define a configuration $\varsigma_{a,b} = (P_\varepsilon, \psi_{a,b}) \in \mathcal{U}(c; U_\varepsilon)$, where $\psi_{a,b}$ sends the standard generators $f_{1,1}^{\text{std}}$ and $f_{2,1}^{\text{std}}$ to $a$ and $b$ respectively. Using that $w$ has infinitely many non-trivial decompositions with respect to $\mathcal{Q}_1$, we obtain for all $0 < \varepsilon \leq 1/2$ an infinite family of configurations $\varsigma_{a,b}$ supported on the same set $P_\varepsilon$ and contained in an arbitrary small normal neighbourhood $\mathcal{U}(c; U_\varepsilon)$.

Note that the configurations $\psi^{-1}(\varsigma_{a,b})$, for fixed $\varepsilon$ and varying $a, b$ with $w = ab$, belong to different open cells of the cell stratification of $\text{Hur}^\Delta(\mathcal{Q})$. By a diagonal argument one can find a neighbourhood of $\psi^{-1}(c)$ in $\text{Hur}^\Delta(\mathcal{Q})$ containing, for all $\varepsilon > 0$, only finitely many points $(a; g, t)$ such that $\psi(a; g, t)$ has support precisely $P_\varepsilon$. Thus $\psi: \text{Hur}^\Delta(\mathcal{Q}) \to \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$ is not a homeomorphism.

In light of Example 9.2 one could argue that the topology on $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)$, described in Section 9.1, is not the correct topology to consider on Hurwitz spaces, and that one should rather consider the CW topology induced by $|\text{Arr}(\mathcal{Q})|$ along the bijection $\psi$. This would indeed simplify the discussion in this section, by making Theorem 9.1 tautological. Nevertheless it would become much more elaborate to replace the topology on $\text{Hur}(\mathcal{C}, \mathcal{Q}, G)$, for a generic nice couple $\mathcal{C}$ and a generic PMQ-group pair $(\mathcal{Q}, G)$, with the topology of a difference of CW complexes. Moreover, the functoriality of Hurwitz spaces with respect to morphisms of nice couplex, discussed Section 4, would also become much more complicated to prove.

9.3. Poincaré PMQs. In this subsection we prove the following theorem.

**Theorem 9.3.** Let $\mathcal{Q}$ be a locally finite PMQ, and suppose that for all $a \in \mathcal{Q} \subset \mathcal{Q}$ the space $\text{Hur}^\Delta(\mathcal{Q})(a)$ is a topological manifold of some dimension. Then $\mathcal{Q}$ is Poincaré.

**Proof.** By Theorem 9.1 the simplicial Hurwitz space $\text{Hur}^\Delta(\mathcal{Q})$ is homeomorphic to $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)$, so it suffices to prove that for all $b \in \mathcal{Q}$ the space $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b$ is a topological manifold. In the following we fix $b \in \mathcal{Q}$.

Let $c \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b$, use Notations 9.0 and 2.6 and let $\mathcal{U}$ be an adapted covering of $P$. By Theorem 5.1 the normal neighbourhood $\mathcal{U}(c; \mathcal{U}) \subset \text{Hur}(\mathcal{R}; \mathcal{Q})$ is homeomorphic to a product of normal neighbourhoods $\prod_{i=1}^k \mathcal{U}(c'_i; U_i)$ for suitable configurations $c'_i \in \text{Hur}(\mathcal{R}; \mathcal{Q})$; the argument of the proof of Theorem 5.1 shows in fact that $c'_i$ is a configuration in $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)$ supported on the single point $z_i$, and that there is a restricted homeomorphism

$$\mathcal{U}(c; \mathcal{U}) \cap \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b \cong \prod_{i=1}^k \left( \mathcal{U}(c'_i; \mathcal{U}) \cap \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b \right),$$

where $b_i = \omega(c'_i)$. A priori $b_i \in \mathcal{Q}$, but since $c'_i$ is supported on a single point we have $b_i \in \mathcal{Q}$.

The hypothesis on $\mathcal{Q}$ ensures that each space $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b$ is a topological manifold; thus also each open subset $\mathcal{U}(c'_i; \mathcal{U}) \cap \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b$ is a topological manifold,
and therefore \( \text{Hur}(\mathcal{L}; \mathcal{Q}_{+})_{b} \) is a topological manifold. This shows that each configuration in \( \text{Hur}(\mathcal{R}; \mathcal{Q}_{+})_{b} \) has a neighbourhood which is a topological manifold, and thus the space \( \text{Hur}(\mathcal{R}; \mathcal{Q}_{+})_{b} \), which is Hausdorff, is a topological manifold. \( \square \)

The proof of Theorem 9.3 can be generalised to homology manifolds as follows.

**Definition 9.4.** Let \( R \) be a commutative ring. A locally finite PMQ \( Q \) is \( R \)-Poincare if for all \( a \in Q \) the space \( \text{Hur}(\mathcal{R}; Q) \) is a \( R \)-homology manifold of some dimension, i.e. for all \( c \in \text{Hur}(\mathcal{R}; Q_{+}) \) the local homology

\[
\hat{H}_{*} \left( \text{Hur}(\mathcal{R}; Q_{+})_{a} \cap \text{Hur}(\mathcal{R}; Q_{+}) \setminus \{c\} ; R \right)
\]

is isomorphic to \( R \) in a single degree, and vanishes in all other degrees.

**Lemma 9.5.** Let \( Q \) be a locally finite PMQ, let \( a \in Q_{+} \) and let \( z_{0} \in \mathcal{R} \); then the space \( \text{Hur}(\mathcal{R}; Q_{+})_{a} \) is homeomorphic to the cone over the space

\[
\partial \text{Hur}(\mathcal{R}; Q_{+})_{a} := \text{Hur}(\mathcal{R}; Q_{+})_{a} \setminus \text{Hur}(\mathcal{R}; Q_{+})_{a},
\]

with vertex the unique configuration \( c_{z_{0}, a} \in \text{Hur}(\mathcal{R}; Q_{+})_{a} \) supported on \( z_{0} \).

**Proof.** Let \( \hat{Q} \) be the completion of \( Q \), and note that \( \text{Hur}(\mathcal{R}; Q_{+})_{a} \) is homeomorphic to \( \text{Hur}(\mathcal{R}; Q_{+})_{a} \). Without loss of generality, we may assume that \( Q \) is already complete. Note that the space \( \partial \text{Hur}(\mathcal{R}; Q_{+})_{a} \) is a closed subspace of \( \text{Hur}(\mathcal{R}; Q_{+})_{a} \), containing all configurations \( c \in \text{Hur}(\mathcal{R}; Q_{+})_{a} \) whose support intersects \( \partial \mathcal{R} \).

Fix a map \( \mathcal{H}^{\infty}_{z_{0}} : \mathbb{C} \times [0, 1] \to \mathbb{C} \) satisfying the following properties:

- \( \mathcal{H}^{\infty}_{z_{0}}(z, s) = sz_{0} + (1 - s)z \) for all \( z \in \mathcal{R} \) and \( 0 \leq s \leq 1 \);
- \( \mathcal{H}^{\infty}_{z_{0}}(-, s) \) is a lax morphism of nice couples \((\mathcal{R}, \emptyset) \to (\mathcal{R}, \emptyset) \) for all \( 0 \leq s \leq 1 \).

By Proposition 4.10 we obtain a continuous map \( \mathcal{H}^{\infty}_{z_{0}} : \text{Hur}(\mathcal{R}; Q_{+})_{a} \times [0, 1] \to \text{Hur}(\mathcal{R}; Q_{+})_{a} \), that we can restrict to a map

\[
\partial \mathcal{H}^{\infty}_{z_{0}} : \partial \text{Hur}(\mathcal{R}; Q_{+})_{a} \times [0, 1] \to \text{Hur}(\mathcal{R}; Q_{+})_{a}.
\]

The map \( \partial \mathcal{H}^{\infty}_{z_{0}} \) sends the subspace \( \partial \text{Hur}(\mathcal{R}; Q_{+})_{a} \times \{1\} \) constantly to the configuration \( c_{z_{0}, a} \). The quotient map

\[
\partial \mathcal{H}^{\infty}_{z_{0}} : \partial \text{Hur}(\mathcal{R}; Q_{+})_{a} \times [0, 1] \to \text{Hur}(\mathcal{R}; Q_{+})_{a}
\]

is a continuous bijection between compact Hausdorff spaces, hence it is a homeomorphism. \( \square \)

In particular for all \( z_{0} \in \mathcal{R} \) we have an isomorphism of homology groups, where \( R \)-coefficients are understood:

\[
\hat{H}_{*} \left( \text{Hur}(\mathcal{R}; Q_{+})_{a} \cap \text{Hur}(\mathcal{R}; Q_{+}) \setminus \{c_{z_{0}, a}\} \right) \cong \hat{H}_{*} \left( |\text{Arr}(Q)|(a) \cap |\text{NAdm}(Q)(a)| \right).
\]

The argument used in the proof of Theorem 9.3 together with the Künneth isomorphism, implies directly the following theorem.

**Theorem 9.6.** Let \( R \) be a commutative ring and let \( Q \) be a locally finite PMQ. Suppose that for all \( a \in Q \) the relative homology groups

\[
\hat{H}_{*} \left( |\text{Arr}(Q)(a)| \cap |\text{NAdm}(Q)(a)| ; R \right)
\]

are supported in a single degree, with corresponding homology group equal to \( R \). Then \( Q \) is \( R \)-Poincare.
Theorem 9.6 is the non-trivial arrow of an “if and only if” statement: if \( \mathcal{Q} \) is \( R \)-Poincaré, then in particular for all \( a \in \mathcal{Q} \) the space \( \text{Hur}^\Delta(\mathcal{Q})(a) \cong \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) is a \( R \)-homology manifold duality: since this space is contractible (see Proposition 6.24), by Poincaré-Lefschetz duality the relative homology groups
\[
\tilde{H}_*(\text{Arr}(\mathcal{Q})(a), |\text{Adm}(\mathcal{Q})(a)|; R)
\]
are supported in one degree, namely the \( R \)-homology dimension of \( \text{Hur}^\Delta(\mathcal{Q})(a) \), with corresponding group isomorphic to \( H^0(\text{Hur}^\Delta(\mathcal{Q})(a); R) \equiv R \).

The proof of [Bia21a Proposition 6.21] and [Bia21a Proposition 6.22] generalise to give the following Proposition.

**Proposition 9.7.** Let \( R \) be a commutative ring and let \( \mathcal{Q} \) be a \( R \)-Poincaré PMQ. Then \( \mathcal{Q} \) is coconnected and admits an intrinsic norm \( h: \mathcal{Q} \to \mathbb{N} \).

If we denote by \( h: \mathcal{Q} \to \mathbb{N} \) also the extension of the intrinsic norm to the completion \( \hat{\mathcal{Q}} \) of \( \mathcal{Q} \), then for all \( a \in \mathcal{Q} \) the space \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) is a \( R \)-homology manifold of dimension \( 2h(a) \).

**Proof.** Recall that for a locally finite PMQ \( \mathcal{Q} \) the candidate for the intrinsic norm \( h: \mathcal{Q} \to \mathbb{N} \) is the function of sets associating with \( a \in \mathcal{Q} \) the maximum \( r \geq 0 \) for which there exist a decomposition \( a = a_1 \ldots a_r \) with \( a_i \in \mathcal{Q}_+ \).

If \( a \in \mathcal{Q}_+ \) is irreducible, then \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) is homeomorphic to \( \mathcal{R} \), which is a \( R \)-homology manifold of dimension \( 2 = 2h(a) \); more generally, if \( a = a_1 \ldots a_r \) is a decomposition witnessing the equality \( h(a) = r \), then we can fix a configuration \( \mathcal{C} = (P, \psi) \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) supported on a subset \( P \subset \mathcal{R} \) of precisely \( r \) points. By Theorem 7.1 a normal neighbourhood of \( \mathcal{C} \) is homeomorphic to an open subset of \( (\mathcal{R})^r \); it follows that the \( R \)-homology dimension of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \), computed around \( \mathcal{C} \), is equal to \( 2h(a) \).

The same argument, applied to any decomposition \( a = bc \) in \( \mathcal{Q} \), shows that the \( R \)-homology dimension of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) is equal to the sum of the \( R \)-homology dimensions of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b \) and \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_c \); in fact we can find an open set of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) homeomorphic to the product of two open sets of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_b \) and \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_c \), respectively. It follows that \( h(a) = h(b) + h(c) \), i.e. \( h \) is an intrinsic norm. The \( R \)-homology dimension of \( \text{Hur}(\mathcal{R}; \mathcal{Q}_+)_a \) can be computed to be \( h(a) \) for a generic \( a \in \mathcal{Q} \) by the same argument, after fixing a decomposition of \( a \) as product of elements of \( \mathcal{Q} \).

This shows that \( \mathcal{Q} \) admits an intrinsic norm, and in particular it is maximally decomposable. To prove that \( \mathcal{Q} \) is coconnected, let \( \mathcal{Q}_{\leq 1} \subset \mathcal{Q} \) be the sub-PMQ containing elements of norm \( \leq 1 \); the inclusion of augmented PMQs \( \mathcal{Q}_{\leq 1} \subset \mathcal{Q} \) induces, for all \( a \in \mathcal{Q} \) a surjective, bisimplicial map \( |\text{Arr}(\mathcal{Q}_{\leq 1})(a)| \cong |\text{Arr}(\mathcal{Q})(a)| \), which is a bijection when restricted to bisimplices of dimension \( 2h(a) \) and \( 2h(a) - 1 \) (see the proof of [Bia21a Proposition 6.22]). Here we write \( |\text{Arr}(\mathcal{Q}_{\leq 1})(a)| \) for the disjoint union \( \bigcup \text{Arr}(\mathcal{Q}_{\leq 1})(a') \), where \( a' \) ranges among all elements of \( \mathcal{Q}_{\leq 1} \) which are sent to \( a \in \mathcal{Q} \) along the (surjective, but a priori not bijective) map \( \mathcal{Q}_{\leq 1} \to \mathcal{Q} \).

It follows that the induced map
\[
H_{2h(a)}(|\text{Arr}(\mathcal{Q}_{\leq 1})(a)|, |\text{Adm}(\mathcal{Q}_{\leq 1})(a)|) \to H_{2h(a)}(|\text{Arr}(\mathcal{Q})(a)|, |\text{Adm}(\mathcal{Q})(a)|),
\]
with \( R \)-coefficients for homology understood, is an isomorphism of \( R \)-modules. The rank of the second \( R \)-module is 1, because \( H_{2h(a)}(|\text{Arr}(\mathcal{Q})(a)|, |\text{Adm}(\mathcal{Q})(a)|; R) \cong H_0(\text{Hur}^\Delta(\mathcal{Q})(a)) \), and the space \( \text{Hur}^\Delta(\mathcal{Q})(a) \) is contractible. Similarly the rank of
the first $R$-module is the number of connected components of $\text{Hur}^\Delta (\mathcal{Q}_{\leq 1}) (a)$. It follows that there is exactly one element $a' \in \overline{Q}_{\leq 1}$ which is mapped to $a$ along the map $\overline{Q}_{\leq 1} \to \hat{Q}$, and that $\text{Hur}^\Delta (\mathcal{Q}_{\leq 1}) (a)$ is connected. This shows that $\mathcal{Q}$ is coconnected. \hfill \Box

APPENDIX A. HURWITZ SPACES ASSOCIATED WITH GENERIC SURFACES

The construction of coordinate-free Hurwitz spaces can be extended as follows to the setting in which the ambient space is not contained in $\mathbb{H}$.

Let $T$ be a connected, compact oriented surface with non-empty boundary $\partial T$, and fix a basepoint $* \in T$. We assume that $T$ is endowed with a semi-algebraic structure (an atlas with semi-algebraic transition maps) and we let $Y \subseteq X \subseteq T$ be semi-algebraic subsets with $Y$ closed in $X$. We denote by $\mathcal{C}$ the couple $(X, Y)$, which will play the role of a nice couple in this setting.

For a finite subset $P \subset X$, the fundamental group $\mathfrak{G} (P) := \pi_1 (T \setminus P, *)$ is a free group on $k' := |P| - \chi (T) + 1$ generators, and contains the trivial PMQ $\mathcal{Q}(P) = \mathcal{Q}_\mathcal{C} (P)$, which is the union of $\mathcal{Q}$ and all conjugacy classes of small simple closed curves spinning clockwise around a point in $P \setminus Y$. The word clockwise uses that $T$ is an oriented surface. Note that the PMQ-group pair $(\mathcal{Q}(P), \mathfrak{G}(P))$ is isomorphic, for some $1 \leq l \leq k$, to the PMQ-group pair $(\mathbb{F}_Q l^k, \mathbb{F}_K)$, where $k \leq k'$ is the cardinality of $P$.

For a PMQ-group pair $(Q, G)$, an element of the Hurwitz set $\text{Hur}^\mathcal{C} (\mathcal{C}; Q, G)$ takes the form $(P, \psi, \varphi)$, where $P \subset X$ and $(\psi, \varphi) : (\mathcal{Q}(P), \mathfrak{G}(P)) \to (Q, G)$ is a morphism of PMQ-group pairs. In the case $Y = \emptyset$ the morphism $\varphi$ is in general not uniquely determined by $\psi$ and one cannot write $(P, \psi)$ for an element of the Hurwitz set $\text{Hur}^\mathcal{C} (\mathcal{C}; Q)$: the difference is that, in general, $\mathcal{Q}(P)$ no longer generates $\mathfrak{G}(P)$ as a group.

The topology on $\text{Hur}^\mathcal{C} (\mathcal{C}; Q, G)$ can be defined by means of adapted coverings and normal neighbourhoods also in this setting. For two surfaces $T$ and $T'$ containing nice couples $\mathcal{C} = (X, Y)$ and $\mathcal{C}' = (X', Y')$ respectively, the class of maps $\xi : T \to T'$ inducing a map $\xi_* : \text{Hur}^\mathcal{C} (\mathcal{C}; Q, G) \to \text{Hur}^\mathcal{C}' (\mathcal{C}', Q, G)$ includes at least all orientation-preserving semi-algebraic embeddings $\xi : T \to T'$ restricting to inclusions $X \to X'$ and $Y \to Y'$, and sending $* \mapsto *$.

The total monodromy of $(P, \psi, \varphi)$ can be defined by evaluating $\varphi$ at the loop described by the boundary component of $T$ containing $*$; left and right based nice couples and the corresponding actions can be defined by using a suitable continuous function $\mathfrak{R} : T \to \mathbb{R} P^1$, sending $* \mapsto \infty$, restricting to an oriented homeomorphism between the boundary component of $T$ containing $*$ and $\mathbb{R} P^1$, and with fibre over $\infty$ given by $\{ * \}$. In the case of an augmented PMQ, one can define the subspace $\text{Hur}^\mathcal{C} (\mathcal{C}; Q_+, G)$ by requiring that $\psi$ restricts to a map of sets $\mathfrak{Q}(P)_+ \to Q_+$.

Most of the results of the current article, which focus on the setting of a contractible ambient space contained in $\mathfrak{C}$, should have an analogue version for Hurwitz spaces with other orientable surfaces as ambient space.

APPENDIX B. DEFERRED PROOFS

B.1. Proof of Proposition 3.11 Let $g \in \mathfrak{Q}^{\text{ext}} (P)$, and assume first that $g = [\gamma]$ is represented by a simple loop $\gamma$ in $\mathfrak{C} \setminus P$. The loop $\gamma$ is freely isotopic to a simple
closed curve in $\mathbb{C} \setminus \gamma$. In particular $\gamma$ bounds a disc $D$ in $\mathbb{C}$ which intersects $P$ only in points of $P \setminus \gamma$; without loss of generality, assume that $D \cap P$ consists of the points $z_1, \ldots, z_r$ for some $1 \leq r \leq l$.

We can then find an admissible generating set $f_1, \ldots, f_k$ of $\mathfrak{G}(P)$ such that $g = f_1 \cdots f_r \in \mathfrak{G}^\text{ext}(P)$ (see Definition 2.8): for this it suffices to choose the arcs $\zeta_1, \ldots, \zeta_r$ inside $D$ in a convenient way. This gives a decomposition $(f_1, \ldots, f_r)$ of $g$ with respect to $\mathfrak{G}(P)$ as required.

If $g \in \mathfrak{G}^\text{ext}(P)$ is not represented by a simple loop, we can still find a conjugate $g'$ of $g$ in $\mathfrak{G}^\text{ext}(P) \subset \mathfrak{G}(P)$, with $g' = [\gamma']$ represented by a simple loop $\gamma'$. By the previous argument we can decompose $g' = g'_1 \cdots g'_r$, with all $g'_i \in \mathfrak{G}(P)$; we can then conjugate the previous decomposition in $\mathfrak{G}(P)$ to obtain a decomposition $g = g_1 \cdots g_r$, with all $g_i$ still lying in $\mathfrak{G}(P)$.

**B.2. Proof of Proposition 2.12** Let $g \in \mathfrak{G}^\text{ext}(P)$ and assume first that $g = [\gamma]$ is represented by a simple loop $\gamma$ in $\mathbb{C} \setminus P$. Let $g = g_1 \cdots g_\rho$ be a decomposition of $g$ in elements $g_i \in \mathfrak{G}^\text{ext}(P)$. Each $g_i$ can be further decomposed, by Proposition 2.11 as $g_i = g_{i,1} \cdots g_{i,r_i}$, with $g_{i,j} \in \mathfrak{G}(P)$; therefore we obtain a decomposition

$$g = g_1,1 \cdots g_{1,r_1} \ g_2,1 \cdots g_{2,r_2} \cdots \ g_\rho,1 \cdots g_{\rho,r_\rho}$$

of $g$ with respect to $\mathfrak{G}(P)$. Our aim to show that, for all $1 \leq i < j \leq \rho$, the product $g_i \cdots g_j$ belongs to $\mathfrak{G}^\text{ext}(P)$. It suffices to prove the same statement for the second, finer decomposition involving the elements $g_{i,j}$. Hence, from now on, we assume that the elements $g_1, \ldots, g_\rho$ already belong to $\mathfrak{G}(P)$. According to Bia21a Definition 3.5], $(g_1, \ldots, g_\rho)$ is then a decomposition of $g$ with respect to $\mathfrak{G}(P)$.

By the same argument used in the proof of Proposition 2.11 we can find an admissible generating set $f_1, \ldots, f_k$ of $\mathfrak{G}(P)$ such that, for some $1 \leq r \leq l$, we have $g = f_1 \cdots f_r$, and such that $f_1, \ldots, f_r$ are contained in the subgroup $\pi_1(D \setminus P, *) \cong \mathbb{F}^r$ of $\mathfrak{G}(P)$, where $D$ is the disc bounded by $\gamma$.

We note that $(f_1, \ldots, f_r)$ is also a decomposition of $g$ with respect to $\mathfrak{G}(P)$, and a simple argument involving the projection onto the abelianisation of $\mathfrak{G}(P)$ shows that $\rho = r$ (see the remark after Bia21a Definition 3.5]). The decompositions $(f_1, \ldots, f_r)$ and $(g_1, \ldots, g_\rho)$ are connected by a sequence of standard moves (see Bia21a Definition 3.6, Proposition 3.7]).

A consequence of the previous argument is that $g_1, \ldots, g_\rho \in \mathfrak{G}(P)$ can be generated using the elements $f_1, \ldots, f_r$, and therefore $g_1, \ldots, g_\rho$ also lie in the subgroup $\pi_1(D \setminus P, *) \subseteq \mathfrak{G}(P)$.

In analogy with Definition 2.8 we say that $f_1, \ldots, f_r$ is an admissible generating set of $\pi_1(D \setminus P, *)$, meaning that each $f_i$ is represented by a simple loop that spins around one of the $r$ points of $D \cap P$, and these loops only intersect at $*$. It is now a classical fact that standard moves on admissible generating sets of $\pi_1(D \setminus P, *)$ can be implemented by homeomorphisms of $D$. More precisely, if $(\tilde{f}_1, \ldots, \tilde{f}_r)$ is an admissible generating set of $\pi_1(D \setminus P, *)$ and the sequence $(\tilde{g}_1, \ldots, \tilde{g}_r)$ of elements of $\pi_1(D \setminus P, *)$ is obtained from the sequence $(\tilde{f}_1, \ldots, \tilde{f}_r)$ by a standard move, then there is a homeomorphism $\xi: D \to D$ such that

- $\xi$ fixes $\gamma = \partial D$ pointwise: in particular $\xi(*) = *$;
- $\xi$ fixes $D \cap P$ as a set: in particular, $\xi$ restricts to a homeomorphism of $D \setminus P$;
- the map $\xi_*: \pi_1(D \setminus P, *) \to \pi_1(D \setminus P, *)$ sends $\tilde{f}_i \mapsto \tilde{g}_i$, for all $1 \leq i \leq r$. 


By applying this argument several times, we obtain that \( g_1, \ldots, g_r \) is also an admissible generating set of \( \pi_1(D \setminus P, *) \), and the fact that the product \( g = g_1 \cdots g_r \) is represented by a simple loop implies that the elements \( g_1, \ldots, g_r \) are ordered in a standard way, so that for all \( 1 \leq i < j \leq r \) also the product \( g_i \cdots g_j \) is represented by a simple loop in \( D \setminus P \subset \mathbb{C} \setminus P \); thus \( g_1, \ldots, g_j \in \mathcal{Q}^{\text{ext}}(P) \).

The case in which \( g \) is not represented by a simple loop \( \gamma \) is treated in the same way as in the proof of Proposition 2.11: we can find a conjugate by a simple loop in \( P \) represented by a simple loop \( \gamma' \), in particular \( g' \in \mathcal{Q}^{\text{ext}}(P) \); we conjugate the factorisation \( g = g_1 \cdots g_r \) to obtain a factorisation \( g' = g_1' \cdots g_r' \); by the previous argument each product \( g_i' \cdots g_j' \) lies in \( \mathcal{Q}^{\text{ext}}(P) \), and therefore also its conjugate \( g_i \cdots g_j \) lies in \( \mathcal{Q}^{\text{ext}}(P) \).

B.3. Proof of Proposition 2.14. Let \( g = g_1, \ldots, g_r \) be a decomposition of \( g \in \mathcal{Q}^{\text{ext}}(P) \) with \( g_i \in \mathcal{Q}^{\text{ext}}(P) \) for all \( 1 \leq i \leq r \). As in the proof of Proposition 2.12 we replace each \( g_i \) by a decomposition \( g_{i,1} \cdots g_{i,r_i} \), with \( g_{i,j} \in \mathcal{Q}(P) \); thus we obtain a decomposition of \( g \) with respect to \( \mathcal{Q}(P) \)

\[
\psi = g_{1,1} \cdots g_{1,r_1} g_{2,1} \cdots g_{2,r_2} \cdots g_{\rho,1} \cdots g_{\rho,r_\rho}.
\]

Since \( \psi \in \mathcal{Q}^{\text{ext}}(P) \), the following product is defined in \( \mathcal{Q} \):

\[
\psi^{\text{ext}}(g) = \psi(g_{1,1}) \cdots \psi(g_{1,r_1}) \psi(g_{2,1}) \cdots \psi(g_{2,r_2}) \cdots \psi(g_{\rho,1}) \cdots \psi(g_{\rho,r_\rho}).
\]

In particular for all \( 1 \leq i < j \leq \rho \), the sub-product \( \psi(g_{i,1}) \cdots \psi(g_{j,r_j}) \) is defined in \( \mathcal{Q} \). Together with Proposition 2.12 this shows that \( g_1 \cdots g_j \) lies in \( \mathcal{Q}^{\text{ext}}(P) \), hence \( \mathcal{Q}^{\text{ext}}(P) \) satisfies the hypotheses of [Bia21a, Definition 2.8].

The same argument shows also that \( \psi^{\text{ext}}(g) = \psi^{\text{ext}}(g_{1}) \cdots \psi^{\text{ext}}(g_{r}) \) in \( \mathcal{Q} \), hence \( \psi^{\text{ext}} \) is a map of partial monoids. It is also evident that \( \psi^{\text{ext}} \) restricts to \( \psi \) on \( \mathcal{Q}(P) \). To see that \( \psi^{\text{ext}} \) also preserves conjugation, let \( g, g' \in \mathcal{Q}^{\text{ext}}(P) \) and choose decompositions \( (g_1, \ldots, g_r) \) and \( (g'_1, \ldots, g'_r) \) of \( g \) and \( g' \) respectively with respect to \( \mathcal{Q}(P) \). We have a chain of equalities

\[
\psi^{\text{ext}}(g) = \psi^{\text{ext}}(g_1) \cdots \psi^{\text{ext}}(g_r) = \psi^{\text{ext}}(g_1) \cdots \psi^{\text{ext}}(g_r) = \psi^{\text{ext}}(g_1) \cdots \psi^{\text{ext}}(g_r).
\]

Thus \( \psi^{\text{ext}} : \mathcal{Q}^{\text{ext}}(P) \to \mathcal{Q} \) is a map of PMQs, restricting to the map \( \psi : \mathcal{Q}(P) \to \mathcal{Q} \). The fact that \( \psi^{\text{ext}} \) is the unique map of PMQs with these properties is a direct consequence of Proposition 2.11.

B.4. Proof of Lemma 4.5. Let \( z' \in P' \setminus \mathcal{Y}' \) and let \( \gamma' \) be a based loop in \( \mathbb{C} \setminus P' \) which is freely homotopic to a simple closed curve \( \beta' \subset \mathbb{C} \setminus P' \) spinning clockwise around \( z' \); in particular \( \beta' \) bounds a closed disc \( D' \subset \mathbb{C} \setminus P' \) with \( D' \cap P' = \{ z' \} \).

We have that \( D = \xi^{-1}(D') \) is also a disc in \( \mathbb{C} \setminus P \), and by property (5) in Definition 4.2 and by definition of \( P' := \xi(P) \), there is a unique \( z \in P \) with \( \xi(z) = z' \). We
consider \( \beta = \partial D \) as a simple closed curve in \( \mathbb{C} \setminus P \) spinning clockwise around \( z \): then \( \xi \) restricts to a homotopy equivalence \( \beta \to \beta' \), since:

- both spaces are homotopy equivalent to \( S^1 \), hence it suffices to prove that \( \xi \) induces a cohomology equivalence;
- the inclusions \( \beta \subset D \setminus z \) and \( \beta' \subset D' \setminus z' \) are homotopy equivalences, in particular cohomology equivalences;
- the map \( \xi : D \setminus z \to D' \setminus z' \) is a cohomology equivalence: this can be seen by comparing the cohomology long exact sequences of the couples \((D, D \setminus z)\) and \((D', D' \setminus z')\), using in particular that the map \( \xi^* : H^2(D', D' \setminus z') \to H^2(D, D \setminus z) \) can be rewritten as \( \xi^* : H^2_0(\mathbb{C}) \to H^2_0(\mathbb{C}) \), and is thus an isomorphism.

Moreover property (2) in Definition 4.2 implies that \( \xi : \beta \to \beta' \) is orientation-preserving, if both curves are oriented clockwise.

This implies that the conjugacy class represented by \( \beta' \) is mapped along \( \xi^* \) inside the conjugacy class represented by \( \beta \), which is contained in \( \Omega_e(P) \).

### B.5. Proof of Lemma 4.9

Let \( \gamma' \subset \mathbb{C} \setminus P' \) be a based loop homotopic to a simple closed curve \( \beta' \), with \( \beta' \) contained in \( \mathbb{C} \setminus Y' \) and \( \beta' \) oriented clockwise, such that \( \beta' \) bounds a disc \( D' \subset \mathbb{C} \setminus Y' \).

Let \( D = \xi^{-1}(D') \), which is a topological disc contained in \( \mathbb{C} \setminus Y \), and let \( \beta = \partial D \).

Let \( K' \subset D' \) be a smaller, closed disc containing \( P' \cap D' \), and denote \( K = \xi^{-1}(K') \).

Then \( \xi : \beta \to \beta' \) is a homotopy equivalence, since:

- both spaces are homotopy equivalent to \( S^1 \), hence it suffices to prove that \( \xi \) induces a cohomology equivalence;
- the inclusions \( \beta \subset D \setminus K \) and \( \beta' \subset D' \setminus K' \) are homotopy equivalences, in particular cohomology equivalences;
- the map \( \xi : D \setminus K \to D' \setminus K' \) is a cohomology equivalence: this can be seen by comparing the cohomology long exact sequences of the couples \((D, D \setminus K)\) and \((D', D' \setminus K')\), using in particular that the map \( \xi^* : H^2(D', D' \setminus K') \to H^2(D, D \setminus K) \) can be rewritten as \( \xi^* : H^2_0(\mathbb{C}) \to H^2_0(\mathbb{C}) \), and is thus an isomorphism.

Moreover property (2) in Definition 4.2 implies that \( \xi : \beta \to \beta' \) is orientation-preserving, if both curves are oriented clockwise.

It follows that \( \xi^* \) maps the conjugacy class of \( \beta' \) inside the conjugacy class of \( \beta \), which is contained in \( \Omega^e(P) \).

### B.6. Proof of Lemma 5.2

Let \( \epsilon = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}, \mathcal{Q}, \mathcal{G}(\mathcal{Q})) \), use Notation 2.3 and let \( f_1, \ldots, f_k \) be an admissible generating set for \( \mathcal{G}(P) \). Since we are dealing with the nice couple \((\mathcal{X}, \emptyset)\), whose second space is empty, we have that \( f_1, \ldots, f_k \in \Omega(P) \). By Definition 5.3 we have \( \varphi(f_i) = \eta_\mathcal{Q} (\psi(f_i)) \in \mathcal{G}(\mathcal{Q}) \); since \( f_1, \ldots, f_k \) exhibit \( \mathcal{G}(P) \) as a free group, we have that \( \varphi : \mathcal{G}(P) \to \mathcal{G}(\mathcal{Q}) \) is uniquely determined by \( \psi \).

On the other hand, by [Bin21a, Theorem 3.3], given any finite subset \( P \subset \mathcal{X} \) and a map of PMQs \( \psi : \Omega(P) \to \mathcal{Q} \), one can use the assignment \( f_i \mapsto \eta_\mathcal{Q}(\psi(f_i)) \) to define a group homomorphism \( \varphi : \mathcal{G}(P) \to \mathcal{G}(\mathcal{Q}) \) making \( (\psi, \varphi) : (\Omega(P), \mathcal{G}(P)) \to (\mathcal{Q}, \mathcal{G}(\mathcal{Q})) \) into a map of PMQ-group pairs.

Let \( \epsilon' = (P', \psi', \varphi') \) be the image of \( \epsilon \) along \( (\text{Id}_\mathcal{Q}, \mathcal{G}(\epsilon))_* \); then we have \( P' = P \) and \( \psi' = \psi \); from the previous discussion it follows that \( \epsilon \) can be reconstructed from \( \epsilon' \), and this proves injectivity of \( (\text{Id}_\mathcal{Q}, \mathcal{G}(\epsilon))_* \).

Viceversa, let \( \epsilon' = (P, \psi, \varphi') \) be any configuration in \( \text{Hur}(\mathcal{C}, \mathcal{Q}, \mathcal{G}) \); then the previous discussion shows that one can construct a configuration \( \epsilon \in \text{Hur}(\mathcal{C}, \mathcal{Q}, \mathcal{G}(\mathcal{Q})) \).
which is sent to \(c'\) along \((\text{Id}_\mathcal{Q}, \mathcal{G}(e))\); it suffices to take \(c = (P, \psi, \varphi)\), with \(\varphi\) defined as above by setting \(f_i \mapsto \eta_Q(\psi(f_i))\); this proves surjectivity of \((\text{Id}_\mathcal{Q}, \mathcal{G}(e))\).

To conclude, note that for all adapted coverings \(U\) of \(P\), the map \((\text{Id}_\mathcal{Q}, \mathcal{G}(e))\) restricts to a bijection from \(\mathcal{U}(\mathcal{e}; U) \subset \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G}(\mathcal{Q}))\) to \(\mathcal{U}(\mathcal{e'}; U) \subset \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G})\), where again we let \(\mathcal{e'}\) be the image of \(\mathcal{e}\) along \((\text{Id}_\mathcal{Q}, \mathcal{G}(e))\). This shows that \((\text{Id}_\mathcal{Q}, \mathcal{G}(e))\) is a homeomorphism.

**B.7. Proof of Lemma 5.4.** The proof is analogous to the one of Lemma 5.2. Let \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; \mathcal{G}, \mathcal{G})\), use Notation 2.4 and let \(f_1, \ldots, f_k\) be an admissible generating set for \(\mathcal{G}(P)\). Since we are dealing with the PMQ-group pair \((\mathcal{G}, \mathcal{G})\), the composition \(\mathcal{Q}_\mathcal{C}(P) \subset \mathcal{G}(P) \xrightarrow{\psi} \mathcal{G}\) equals \(\mathcal{Q}_\mathcal{C}(P) \rightarrow \mathcal{G}\). In particular \(\psi\) can be recovered from \(\varphi\).

On the other hand, by [Bia21a Theorem 3.3], given any finite subset \(P \subset \mathcal{X}\) and a map of groups \(\varphi: \mathcal{G}(P) \rightarrow \mathcal{G}\), one can use the assignment \(\psi: f_i \mapsto \varphi(f_i)\) for \(1 \leq i \leq l\) (using Notation 2.4) to define a map of PMQs \(\psi: \mathcal{Q}_\mathcal{C}(P) \rightarrow \mathcal{G}\) making \((\psi, \varphi): (\mathcal{Q}_\mathcal{C}(P), \mathcal{G}(P)) \rightarrow (\mathcal{G}, \mathcal{G})\) into a map of PMQ-group pairs.

Let \(c' = (P', \psi', \varphi')\) be the image of \(c\) along \((\text{Id}_\mathcal{C})\); then we have \(P' = P\) and \(\varphi' = \varphi\); from the previous discussion it follows that \(c\) can be reconstructed from \(c'\), and this proves injectivity of \((\text{Id}_\mathcal{C})\) on \(\mathcal{Q}_\mathcal{C}(P)\).

Viceversa, let \(c' = (P, \psi, \varphi')\) be any configuration in \(\text{Hur}(\mathcal{X}; \mathcal{X}; \mathcal{G}; \mathcal{G})\); then the previous discussion shows that one can construct a configuration \(c \in \text{Hur}(\mathcal{C}; \mathcal{G}, \mathcal{G})\) mapping to \(c'\) along \((\text{Id}_\mathcal{C})\); it suffices to take \(c = (P, \psi, \varphi)\), with \(\psi\) defined as above by setting \(f_i \mapsto \varphi(f_i)\); this proves surjectivity of \((\text{Id}_\mathcal{C})\).

To conclude, note that if \(U\) is an adapted covering of \(P\) with respect to \(\mathcal{C}\), then \(U\) is also adapted with respect to \((\mathcal{X}, \mathcal{X})\), and the map \((\text{Id}_\mathcal{C})\) restricts to a bijection from \(\mathcal{U}(\mathcal{C}; U) \subset \text{Hur}(\mathcal{C}; \mathcal{G}, \mathcal{G})\) to \(\mathcal{U}(\mathcal{e}; U) \subset \text{Hur}(\mathcal{C}; \mathcal{X}; \mathcal{G}, \mathcal{G})\), where again we let \(\mathcal{e}'\) be the image of \(\mathcal{e}\) along \((\text{Id}_\mathcal{C})\). This proves that \((\text{Id}_\mathcal{C})\) is a homeomorphism.

**B.8. Proof of Proposition 6.13.** We focus on the left-based case. Let \(c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G})\), use Notations 3.13 and 2.6 and let \(U\) be an adapted covering of \(P\). Denote by \(U^l\) the component of \(U\) containing \(z^l\); possibly up to shrinking \(U^l\), we can assume that the simple closed curve \(\partial U^l\) is cut by \(S_{\text{min}}\mathcal{C}, \mathcal{C}\) in two arcs. We decompose \(\mathcal{C}\) as the union of two subspaces: the first subspace is \(\mathcal{T}_1\), which is defined as the closure in \(\mathcal{C}\) of \(S_{\mathcal{C}, \mathcal{Q}, \mathcal{G}} \cup U^l\); the other subspace is \(\mathcal{T}_2 = S_{\text{min}}\mathcal{C}, \mathcal{G}, \mathcal{Q}, \mathcal{G} \setminus U^l\).

The first subspace contains \(P_1: = \{z^l\}\) in its interior, the second subspace contains \(P_2: = P \setminus \{z^l\}\) in its interior. The two subspaces \(\mathcal{T}_1\) and \(\mathcal{T}_2\) intersect in the contractible space \(S_{\text{min}}\mathcal{C}, \mathcal{C}\), which contains \(*\).

Using the theorem of Seifert and van Kampen we can write \(\mathcal{G}(P)\) as the free product \(\pi_1(\mathcal{T}_1 \setminus P_1, \ast) \ast (\pi_1(\mathcal{T}_2 \setminus P_2, \ast))\); the first factor is freely generated by \(f\), the second factor is freely generated by the other generators \(f_i\) in a left-based admissible generating set. The map \(\varphi'\) in Definition 6.12 can then be equivalently defined by setting \(\varphi'(f) = g \cdot \varphi(f^l)\), and by imposing that \(\varphi'\) and \(\varphi\) agree on the second factor.

Moreover, consider the nice couple \(\mathcal{C}_2 := (\mathcal{X} \cap \mathcal{T}_2, \mathcal{Y} \cap \mathcal{T}_2)\): then \(P_2\) is contained in \(\mathcal{X} \cap \mathcal{T}_2\). The composition

\[
\mathcal{Q}_{\mathcal{C}_2}(P_2) \xrightarrow{\subset} \mathcal{G}(P_2) \xleftarrow{\pi_1(\mathcal{T}_2 \setminus P_2, \ast)} \mathcal{Q}_{\mathcal{C}_2}(P_2) \xrightarrow{\subset} \mathcal{G}(P)
\]

has image in \(\mathcal{Q}_{\mathcal{C}}(P)\) and identifies \(\mathcal{Q}_{\mathcal{C}_2}(P)\) with the sub-PMQ of \(\mathcal{Q}_{\mathcal{C}}(P)\) containing homotopy classes which can be represented by a simple loop in \(\mathcal{T}_2 \setminus P_2\) spinning
covering is an adapted covering of $U$. The map $\psi': \Omega_c(P') \to Q$ from Definition 6.12 can be characterised by the following two properties:

- $\psi$ and $\psi'$ have the same restriction on $\Omega_c(P)$, regarded as a subset of $\Omega_c(P')$ as explained above.
- $(\psi', \varphi')$ is a map of PMQ-group pairs $(\Omega_c(P), \mathfrak{G}(P)) \to (Q, G)$.

The fact that this is a characterisation (i.e. existence and uniqueness of $\psi'$ with these properties) is shown using a choice of a left-based admissible generating set for $\mathfrak{G}(P)$ and using [Bin21a] Theorem 3.3; but the characterising properties of $\varphi'$ and $\psi'$ are now stated without reference to a left-based admissible generating set.

The fact that the collection of all maps $g: - \to Q$ gives an action of $G$ on the set $\text{Hur}(\mathfrak{C}, Q, G)_z$ follows directly from the formulas in Definition 6.12. To prove continuity of $g: -$, note that for all adapted coverings $U$ of $P$ the map $g: -$ establishes a bijection between the open subspaces $\mathfrak{U}(c, U)_z$ and $\mathfrak{U}(g \cdot c, U')_z$ of $\text{Hur}(\mathfrak{C}, Q, G)_z$. In particular $g: -$ is a homeomorphism of $\text{Hur}(\mathfrak{C}, Q, G)$ with inverse $g^{-1}$. –

The right-based case is analogous; the main difference is that, in the first part, one considers the component $U^T$ of $\mathcal{U}$ covering $z^T$, and decomposes $\mathcal{C}$ as the union of $T_1 = S_{-\infty, \max} e \setminus U^T$ and $T_2$ being the closure in $\mathcal{C}$ of $S_{\max} e, \infty \cup U^T$.

B.9. Proof of Proposition 7.8. Fix $(c, t) \in \text{Hur}(\mathfrak{C}, Q, G) \times [0, 1]$, denote $\varepsilon' = \varepsilon_s(c, t)$ and $\varepsilon'' = \rho(\varepsilon)$, and use Notation 3.6. Without loss of generality assume that $z_1, \ldots, z_r \in P \setminus \mathcal{V}$ are precisely the inert points of $c$, for some $0 \leq r \leq l$. Then $P'' = P \setminus \{z_1, \ldots, z_r\}$ and $P' = P'' \cup \varepsilon(P, t)$.

Let $\mathcal{U}'$ be an adapted covering of $P'$.

Let $\mathcal{U}_t$ be an adapted covering of $P'$.

Our aim is to find a neighbourhood of $(c, t) \in \text{Hur}(\mathfrak{C}, Q, G) \times [0, 1]$ which is mapped by $\varepsilon'$ inside $\mathfrak{U}(\varepsilon', \mathcal{U}')$. Let $\mathcal{U}'_{\varepsilon'(P, t)} \subseteq \mathcal{U}'$ denote the restriction of $\mathcal{U}'$ to $\varepsilon'(P, t) \subseteq P'$, i.e. the sequence of components of $\mathcal{U}'$ containing a point in $\varepsilon'(P, t)$. Then by continuity of $\varepsilon'$ we can find an adapted covering $\mathcal{U}'_t$ of $P$ and a neighbourhood $V$ of $t \in [0, 1]$ such that $\varepsilon'$ maps the entire product neighbourhood $\mathfrak{U}(P, \mathcal{U}_t) \times V$ into $\mathfrak{U}(\varepsilon'(P, t), \mathcal{U}'_{\varepsilon'(P, t)}) \subseteq \text{Ran}(\mathcal{C})$.

We claim that $\mathfrak{U}(c, \mathcal{U}_t) \times V$ is mapped by $\varepsilon''_s$ inside $\mathfrak{U}(\varepsilon', \mathcal{U}')$; the rest of the proof is devoted to this claim.

First, we prove that $P'' \subseteq \mathcal{U}'_t$. We can partition $P$ into subsets $P_1, \ldots, P_k$, with $P_i \subseteq U_i$. Note that for all $1 \leq i \leq r$ and for all $z \in P_i$, the point $z$ is inert for $z$: indeed $z \in U_i \subseteq \mathcal{C} \setminus \mathcal{V}$ because $\mathcal{U}$ is an adapted covering of $P$, hence $z \in \mathcal{X} \setminus \mathcal{V}$; moreover $\psi$ sends each element of $\mathfrak{Q}(\hat{P}, \hat{z})$ to a factor of $\mathfrak{A}$ in the augmented PMQ $\mathfrak{Q}$, i.e. to $\mathfrak{A}$. It follows that $P''$ is a subset of $\hat{P}_{r+1} \cup \cdots \cup \hat{P}_k \cup \varepsilon'(P, t)$, and the latter set is contained in $\mathcal{U}'_t$ by our choice of $\mathcal{U}$.

Second, we prove that every component of $\mathcal{U}'_t$ intersects $P''$ in at least one point. Let $U'_i$ be the component of $\mathcal{U}'_t$ containing the point $z'_i \in P'$, for some $1 \leq i \leq k'$. There are several cases to consider.

- If $z'_i \in \varepsilon'(P, t) \subseteq P'$, then there is $z_j \in P$ with $z'_i \in \varepsilon'(z_j, t)$, and there is $\hat{z} \in \hat{P} \cap U_j$. We can restrict $\mathcal{U}'$ to an adapted covering $\mathcal{U}'_{\varepsilon'(z_j, t)}$ of $\varepsilon'(z_j, t) \subseteq P'$, by selecting the relevant connected components; then our hypothesis on $\mathcal{U}$ implies that $\varepsilon'$ sends $(U_j \cap \mathcal{X}) \times V$ inside $\mathfrak{U}(\varepsilon'(z_j, t), \mathcal{U}'_{\varepsilon'(z_j, t)})$, where we use Notation 3.6.
The previous discussion shows that 

\[ (\cdot) \]

This shows that \( \bar{a} \). Let \( (\cdot) \) be a simple loop spinning clockwise around a component \( U' \).

For simplicity, in the rest of the proof we abbreviate by \( \rho(\cdot) \), thus deleting all inert points of \( \cdot \), and then by adding these inert points again through an external product of \( \rho(\cdot) \) with \( P = \bar{a} \).

B.10. Proof of Proposition 8.21. We introduce some notation for barycentres of faces of simplices. Recall Notation 8.3 for \( p \geq 1 \) and \( 0 \leq i \leq p \) we denote by

\[ \text{bar}^{p-1,i} = \left( \frac{1}{p}, \frac{2}{p}, \ldots, \frac{i-1}{p}, \frac{i}{p}, \frac{i+1}{p}, \ldots, \frac{p-1}{p} \right) \]

\( \in \Delta^p \) the barycentre of the face \( d_i \Delta^p \).

Lemma B.1. Recall Definition 8.16. The map \( e' \) sends \( \text{bar}^{p-1,i}, \text{bar}^q \) to \( \text{bar}' \), where \( \text{bar}' = \text{bar}^{p-1,i} \).

Before proving Lemma B.1 we will argue how Proposition 8.21 follows from it. Let \( (s, t) \in d_{\text{bar}}(\Delta^p \times \Delta^q) \subset \Delta^p \times \Delta^q \) (see Notation 8.20). Then the pair \( \text{bar}^{p-1,i}, \text{bar}^q \) belongs to \( \Delta^p \times \Delta^q \), and we can factor \( \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^q; s, t) \) as a composition \( \mathcal{H}^{p,q}(\cdot; \text{bar}^{p-1,i}, \text{bar}^q; s, t) \circ \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^q; \text{bar}^{p-1,i}, \text{bar}^q) \) by Lemma 8.8. Assuming Lemma B.1, we have that \( \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^q; \text{bar}^{p-1,i}, \text{bar}^q) \) sends \( \xi \to \bar{\xi} \); then by definition the second map \( \mathcal{H}^{p,q}(\cdot; \text{bar}^{p-1,i}, \text{bar}^q; s, t) \) sends \( \xi \to e'(s, t) \) (regarding \( s \) as a point in \( \Delta^{p-1} \times \Delta^q \)), and the composition \( \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^q; \text{bar}^{p-1,i}, \text{bar}^q) \) sends \( \xi \to e'(s, t) \).

The rest of the subsection is thus devoted to the proof of Lemma B.1. By definition, \( e'(\text{bar}^{p-1,i}, \text{bar}^q) \) is the image of \( \xi \) under \( \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^{p-1,i}, \text{bar}^q, \text{bar}^q) \). For simplicity, in the rest of the proof we abbreviate by \( \xi \to \xi \) the map \( \mathcal{H}^{p,q}(\cdot; \text{bar}^p, \text{bar}^{p-1,i}, \text{bar}^q, \text{bar}^q) \).

Recall Notation 8.14 the map \( \xi \to z_{i,j}^{p-1,q} \) for \( 0 \leq i \leq i' \leq \xi \), and \( z_{i,j}^{p-1,q} \) for \( i + 1 \leq i' \leq p + 1 \); it follows that the image of \( P_{\xi} \) along \( \xi \) is the set \( P_{\xi} \). This shows that \( e'(\text{bar}^{p-1,i}, \text{bar}^q) \) is a configuration supported on \( P_{\xi} \).

Consider now the two standard generating sets \( f_{(i,j)}^{p-1,q}(i,j) \in (\xi) \) of \( \mathcal{G}(P_{\xi}) \), and \( f_{(i,j)}^{p-1,q}(i,j) \in (\xi) \) of \( \mathcal{G}(P_{\xi}) \) (see also Notation 8.15), and consider the homomorphism \( \xi^* : \mathcal{G}(P_{\xi}) \to \mathcal{G}(P_{\xi}) \) from Subsection 4.3.
The key observation is that, for all $0 \leq i' \leq p$ and $0 \leq j \leq q + 2$, the homomorphism $\xi^*$ maps the product of standard generators $cf_{i,j}^p$ to

$$\xi^*(cf_{i,j}^p) \mapsto \begin{cases} 
 cf_{i,j}^p & \text{if } i' \leq i - 1; \\
 cf_{i,j}^p \cdot cf_{i+1,j}^p & \text{if } i' = i; \\
 cf_{i,j+1}^p & \text{if } i' \geq i + 1.
\end{cases}$$

This follows from the description of $cf_{i,j}^p$ as the class of a simple loop supported on $S_{x_{i-1}, x_{i+1}} \cap \{ z \leq y_j \}$ and spanning clockwise around the points $z_{i,j}^0, \ldots, z_{i,j}^q$.

For $i' = i$ we have in particular that $\xi^*(cf_{i,j}^p)$ is represented by a loop spanning around the horizontal segments joining $z_{i,j}^{q'}$ with $z_{i+1,j}^{q'}$, for $0 \leq j' \leq j$: these horizontal segments are the preimages along $\xi$ of the points $z_{i,j}^{q'}$ for $0 \leq j' \leq j$.

We can now use that $\xi^*$ is a group homomorphism and compute $\xi^*(f_{i,j}^p)$ for all $(i', j) \in I(u')$. In particular, for $i' = i$ we obtain

$$\xi^*(f_{i,j}^p) = \xi^* \left( cf_{i,j}^p \cdot cf_{i+1,j}^p \right) = \left( cf_{i,j}^p \cdot cf_{i+1,j}^p \right)^{-1} \cdot cf_{i,j+1}^p \cdot cf_{i+1,j+1}^p.$$

Similarly, for $i' < i$ we have $\xi^*(f_{i,j}^p) = f_{i',j}^p$, and for $i' > i$ we have $\xi^*(f_{i,j}^p) = f_{i',j}^p$.

Thus, applying $\psi_{u'}$ to $\xi^*(f_{i,j}^p)$, we obtain the equality $a_{i,j}^{u'} = \psi_{u'}(f_{i,j}^p)$, which together with [Bia21a, Lemma 6.8] yields the equality $u' = d_{i,\text{hor}}u$. This concludes the proof of Lemma B.1.

B.11. Proof of Proposition 8.22

The proof of this proposition is in many aspects analogue to the one of Proposition 8.21. Again it suffices to prove the following lemma.

Lemma B.2. The map $e_u$ sends $(bar_p, bar_{q-1}^e) \in \Delta_p \times \Delta^q$ to $e_u'$, where $u' = d_{i,\text{ver}}(u)$.

The deduction of Proposition 8.22 from Lemma B.2 is completely analogue as in the horizontal case, so we omit it.

The rest of the subsection is thus devoted to the proof of Lemma B.2. By definition, $e_u(bar_p, bar_{q-1}^e)$ is the image of $e_u$ under the map $\xi^*$, where for the rest of the proof we abbreviate by $\xi : C \to C$ the map $H^{p,q}(\bar{a}_u, \bar{a}_u'^e)$. The map $\xi$ sends $z_{i,j}^{q'} \mapsto z_{i,j}^{q-1}$ for $0 \leq j' \leq j$, and $z_{i,j}^{q'} \mapsto z_{i,j+1}^{q-1}$ for $j+1 \leq j' \leq q+1$; it follows that the image of $P_{u'}$ along $\xi$ is the set $P_{u'}$, and as in the horizontal case we obtain that $e_u(bar_p, bar_{q-1}^e)$ is a configuration supported on $P_{u'}$.

Consider now the two standard generating sets $(f_{i,j}^p)_{(i,j) \in I(u')}$ of $\mathfrak{G}(P_{u'})$, and $(f_{i,j}^p)_{(i,j) \in I(u')}$ of $\mathfrak{G}(P_{u'})$, and consider the homomorphism $\xi^* : \mathfrak{G}(P_{u'}) \to \mathfrak{G}(P_{u'})$.

The key observation is that, for all $0 \leq i \leq p + 1$ and $0 \leq j' \leq q + 1$, the homomorphism $\xi^*$ maps the product of standard generators $cf_{i,j}^p$ to

$$\xi^*(cf_{i,j}^p) \mapsto \begin{cases} 
 cf_{i,j}^p & \text{if } j' \leq j; \\
 cf_{i,j+1}^p & \text{if } j' \geq j + 1.
\end{cases}$$
This follows from the description of $c_{f_{i,j}^a}$ as the class of a simple loop supported on $S_{x_i-1,x_{i+1}} \cap \{ 3 \leq y_j' \}$ and spinning clockwise around the points $\frac{a_i}{x_i-1}, \frac{a_i}{x_i+1}$ and spinning clockwise around the points $z_{a_i}^0, \ldots, z_{a_i}^{j'}, z_{a_i}^{j'+1}, \ldots, z_{a_i}^{j-1}$.

For $j' \geq j+1$ we have in particular that $\xi^*(c_{f_{i,j}^a})$ is represented by a loop spinning around the points $z_{a_i}^0, \ldots, z_{a_i}^{j'}, z_{a_i}^{j'+1}, \ldots, z_{a_i}^{j-1}$ and around the vertical segment joining $z_{a_i}^{j'}$ with $z_{a_i}^{j+1}$; note that this vertical segment is in the preimage along $\xi$ of the point $z_{a_i}^{j'}$.

We can now use that $\xi^*$ is a group homomorphism and compute $\xi^*(f_{i,j}^a)$ for all $(i,j') \in I(a')$. In particular, for $j' = j$ we obtain

$$\xi^*(f_{i,j}^a) = \xi^* \left( \left( c_{f_{i,j}^a} \right)^{-1} \cdot c_{f_{i,j+1}^a} \right) = \left( c_{f_{i,j}^a} \right)^{-1} \cdot c_{f_{i,j+1}^a} = f_{i,j}^a \cdot f_{i,j+1}^a.$$  

Similarly, for $j' < j$ we have $\xi^*(f_{i,j}^a) = f_{i,j}^a$, and for $j' > j$ we have $\xi^*(f_{i,j}^a) = f_{i,j'}^{a}$.  

Thus, applying $\psi_{a'}$ to $\xi^*(f_{i,j}^a)$, we obtain the equality $a_{i,j'} = \psi_{a'}(f_{i,j}^a)$, which together with [Bia21a, Lemma 6.8] yields the equality $a' = a_1^{\text{hor}} a$. This concludes the proof of Lemma B.2.

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