Some Analogues of a Result of Vasconcelos

DAVID EARL DOBBS*  
Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, USA  
e-mail: dobbs@math.utk.edu

JAY ALLEN SHAPIRO  
Department of Mathematics, George Mason University, Fairfax, Virginia 22030-4444, USA  
e-mail: jshapiro@gmu.edu

Abstract. Let $R$ be a commutative ring with total quotient ring $K$. Each monomorphic $R$-module endomorphism of a cyclic $R$-module is an isomorphism if and only if $R$ has Krull dimension 0. Each monomorphic $R$-module endomorphism of $R$ is an isomorphism if and only if $R = K$. We say that $R$ has property (⋆) if for each nonzero element $a \in R$, each monomorphic $R$-module endomorphism of $R/Ra$ is an isomorphism. If $R$ has property (⋆), then each nonzero principal prime ideal of $R$ is a maximal ideal, but the converse is false, even for integral domains of Krull dimension 2. An integral domain $R$ has property (⋆) if and only if $R$ has no $R$-sequence of length 2; the “if” assertion fails in general for non-domain rings $R$. Each treed domain has property (⋆), but the converse is false.

1. Introduction

All rings considered in this note are commutative; all rings and modules are unital. Our starting point is the following result of Vasconcelos [14, Theorem]: a ring $R$ has (Krull) dimension 0 if and only if each monomorphic $R$-module endomorphism of a finitely generated $R$-module is surjective (that is, is an isomorphism). While

* Corresponding Author.
Received June 21, 2013; accepted September 12, 2013.
2010 Mathematics Subject Classification: Primary 13A15, 13C99; Secondary 13G05, 13F05, 20M14, 13B30.
Key words and phrases: Commutative ring, cyclic module, monomorphism, Krull dimension, monoid ring, integral domain, pullback, treed domain, pseudo-valuation domain, total quotient ring, localization.
The second-named author thanks the University of Tennessee for its warm hospitality during his visit in 2013.
the statement of this result concerns module-theoretic behavior, its proof involves a
central ring-theoretic result of Cohen-Seidenberg (cf. [4]), namely, that the partners
of any integral ring extension must have the same dimension. In considering whether
a result like that of Vasconcelos could be proved without recourse to such ring-
theoretic tools, we found an elementary module-theoretic proof of the following
analogue (see Proposition 2.1): a ring $R$ has (Krull) dimension 0 if and only if each
monomorphic $R$-module endomorphism of a cyclic $R$-module is an isomorphism.
The rest of this note examines what can be said about a ring $R$ if one imposes such
a condition on only some of its cyclic modules.
For instance, Proposition 2.2 shows each monomorphic $R$-module endomor-
phism of (the cyclic $R$-module) $R$ is an isomorphism if and only if $R$ is a total
quotient ring. This is perhaps of greater interest because total quotient rings can
be of arbitrary dimension (cf. [12]), but one would seek greater generality if $R$
is a (commutative integral) domain, since domains which are total quotient rings
must be fields (and hence of no ideal-theoretic interest). For this reason, we focus
on rings $R$ that have the following property, which we call property ($\star$): for each
nonzero element $a \in R$, each monomorphic $R$-module endomorphism of (the cyclic
$R$-module) $R/Ra$ is an isomorphism.

It is shown in Proposition 2.4 that rings having property ($\star$) retain some of the
low-dimensional flavor of the rings studied by Vasconcelos, in the sense that when-
ever $R$ has property ($\star$), each nonzero principal prime ideal of $R$ is a maximal ideal.
However, Example 2.6 shows that the converse is false, even for two-dimensional
domains. A very useful result for domains is given in Proposition 2.5, where it is
shown that a domain $R$ has property ($\star$) if and only if there is no $R$-sequence on $R$
of length 2. Examples of domains of arbitrary dimension that have property ($\star$) are
provided by Corollary 2.9: if $R$ is a pseudo-valuation domain (in the sense of [11]),
then $R$ has property ($\star$). In addition, Example 2.6 is shown to be best-possible by
Corollary 2.8, where it is shown that each one-dimensional domain has property ($\star$).
More generally, Corollary 2.7 establishes that each treed domain has property ($\star$).
Additional examples of domains having property ($\star$) are given in Theorem 2.10,
which provides some pullbacks, not all of which are treed domains, that have prop-
erty ($\star$). Remark 2.12 (a) shows that the converse of Corollary 2.7 is false, while
Remark 2.12 (d) shows that the criterion in Corollary 2.5 involving $R$-sequences
does not characterize property ($\star$) for non-domains. Finally, Remark 2.12 (e) shows
that property ($\star$) is somewhat more fragile than zero-dimensionality in regard to
its possible stability under globalization.
As usual, if $A$ is a ring, it will be convenient to denote the (Krull) dimension
of $A$ by $\dim(A)$; the set of units of $A$ by $U(A)$; the set of prime ideals of $A$ by
$\text{Spec}(A)$; the radical of an ideal $I$ of $A$ by $\sqrt{I}$; the set of zero-divisors of $A$ by $Z(A)$;
and the total quotient ring of $A$ by $\text{tq}(A) := A_{A \setminus Z(A)}$. Also, if $E$ is an $A$-module
and $P \in \text{Spec}(A)$, we let $E_P$ denote the localization of $E$ at $A \setminus P$. Parts (d) and
(e) of Remark 2.12 use the idealization construction of Nagata; a useful reference
for basic facts about idealization is [12]. Any unexplained material is standard, as
in the textbooks listed in the bibliography.
2. Results

We begin in Proposition 2.1 by showing how a focus on cyclic modules can lead to a simple analogue of the motivating result of Vasconcelos.

**Proposition 2.1.** Let $R$ be a ring. Then the following conditions are equivalent:

1. If $E$ is a cyclic $R$-module and $f \in \text{Hom}_R(E, E)$ is an injection, then $f$ is an isomorphism;
2. $\dim(R) = 0$.

*Proof.* (1) $\Rightarrow$ (2): The contrapositive of this implication can be proved as in the proof of [14, Theorem]. For the sake of completeness, we next provide the details.

If (2) fails, pick distinct prime ideals $P \subset Q$ of $R$, pick an element $b \in Q \setminus P$, put $E := R/P$, let $f$ be the element of $\text{Hom}_R(E, E)$ defined by multiplication by $b$, and note that $f$ is an injection but not a surjection, so that (1) fails.

(2) $\Rightarrow$ (1): Suppose that $\dim(R) = 0$. Let $E$ be a cyclic $R$-module and suppose that $f \in \text{Hom}_R(E, E)$ is an injection. We will show that $f$ is surjective. By hypothesis, there exists an ideal $I$ of $R$ such that $E \cong R/I$. Of course, $f$ is surjective (respectively, injective) if and only if the induced map $f_M \in \text{Hom}_{R_M}(E_M, E_M)$ is surjective (respectively, injective) for each $M \in \text{Spec}(R)$; and for any such $M$, we have $E_M \cong R_M/IR_M$, which is a cyclic $R_M$-module. Thus, without loss of generality, we may take $(R, M)$ quasi-local and $E = R/I$. Also without loss of generality, $I \neq R$ (since the only endomorphism of the module 0 is surjective), and so $I \subseteq M$. In fact, $\sqrt{I} = M$ by [9, Corollary 2.10], since the zero-dimensionality of $R$ implies that $M$ is the unique prime ideal of $R$. Next, note that $f(1 + I) = b + I$ for some $b \in R$. (If $I \neq 0$, then $b$ is not uniquely determined, but it suffices to pick any such $b$.) Then $f$ is simply multiplication by $b$; that is, if $r \in R$, then $f(r + I) = br + I$ ($= b \cdot (r + I)$. If $b \in U(R)$, then $f(b^{-1} + I) = bb^{-1} + I = 1 + I$, and so $f$ is surjective since $f$ is $R$-linear. Hence, without loss of generality, $b \in R \setminus U(R) = M$. It suffices to obtain a contradiction in this case. As above, we have that $\sqrt{Rb} = M$, and so $b \in \sqrt{I}$. Hence, there is a smallest positive integer $n$ such that $b^n \in I$. Consider $\rho := b^{n-1} \in R$. Then $\rho + I \neq 0 \in E$, although $f(\rho + I) = b \rho + I = b^n + I = 0 \in E$, which contradicts the assumption that $f$ is a monomorphism. The proof is complete. \qed

We next consider rings that satisfy various weaker versions of condition (1) of Proposition 2.1. By focusing on just the cyclic $R$-module $R$, Proposition 2.2 gives a new characterization of total quotient rings. As is well known (cf. [12, Theorems 26.2 and 26.4]), such rings can be of arbitrary Krull dimension.

**Proposition 2.2.** Let $R$ be a ring. Then the following conditions are equivalent:

1. If $f \in \text{Hom}_R(R, R)$ is an injection, then $f$ is an isomorphism;
2. $R = \text{tq}(R)$.

*Proof.* The result holds for the zero ring (as its total quotient ring must sensibly be defined to also be the zero ring). Thus, without loss of generality, $R \neq 0$. If $b \in R,$
let $\tilde{b} \in \text{Hom}_R(R,R)$ denote multiplication by $b$, namely, the map given by $r \mapsto br$. As $b$ varies over the elements of $R$, $\tilde{b}$ varies over the elements of $\text{Hom}_R(R,R)$. Moreover (since $R \neq 0$), $f = \tilde{b}$ is a monomorphism if and only if $b \notin Z(R)$. On the other hand, $f = \tilde{b}$ is surjective if and only if $b \in U(R)$ (in which case, $f = \tilde{b}$ is clearly an isomorphism). Thus, (1) holds if and only if $R \setminus Z(R) = U(R)$. Since it is clear that (2) holds if and only if $R \setminus Z(R) = U(R)$, the proof is complete.

Seeking a condition that is, in a sense, midway between condition (1) of Proposition 2.1 and condition (1) of Proposition 2.2, we next introduce the main object of study in this note. We will say that a ring $R$ has property ($\star$) if, for each nonzero element $a \in R$, each monomorphic $R$-module endomorphism of (the cyclic $R$-module) $R/Ra$ is an isomorphism. Our first examples of such rings are given in Corollary 2.3, which is an immediate consequence of Proposition 2.1 (or of Vasconcelos’ motivating result [14, Theorem]).

**Corollary 2.3.** Each ring of Krull dimension 0 has property ($\star$).

We next show that any ring $R$ that has property ($\star$) must exhibit certain behavior that was established in [6, Corollary 2.4] for any going-down domain, namely, that each nonzero principal prime ideal of $R$ is a maximal ideal.

**Proposition 2.4.** Let $R$ be a ring that has property ($\star$). Then each nonzero principal prime ideal of $R$ is a maximal ideal.

**Proof.** It suffices to rework the proof of the implication (1) $\Rightarrow$ (2) in Proposition 2.1 (taking $P$ in that proof to be the given nonzero principal prime ideal and $Q$ to be, if possible, a strictly larger prime ideal of $R$).

Example 2.6 will show that the converse of Proposition 2.4 is false. First, it is convenient to note that for domains, property ($\star$) admits the following reformulation involving $R$-sequences (also known as regular sequences).

**Proposition 2.5.** Let $R$ be a domain. Then $R$ has property ($\star$) if and only if there is no $R$-sequence (on $R$) of the form $a,b$.

**Proof.** For all $a,b \in R$, let $\widehat{(a,b)}$ denote the $R$-module homomorphism $\widehat{(a,b)} : R/Ra \rightarrow R/Ra$ given by $r + Ra \mapsto br + Ra$, for each $r \in R$. Note that $R$ fails to have property ($\star$) if and only if there exist elements $a,b$ in $R$ such that $a \neq 0$ and $\widehat{(a,b)}$ is injective but not surjective. On the other hand (cf. [13, page 84]), if $a$ and $b$ are elements of $R$, then $a,b$ is an $R$-sequence (on $R$) if and only if $Ra + Rb \neq R$, $a \neq 0$ and $\widehat{(a,b)}$ is injective.

Suppose first that the “if” assertion fails. Hence, $R$ has no $R$-sequences consisting of two elements, but there exist elements $a,b$ in $R$ such that $a \neq 0$ and $\widehat{(a,b)}$ is injective but not surjective. Consequently $Ra + Rb = R$, and so $R/Ra = R(b + Ra) = \widehat{(a,b)}(R/Ra)$, whence $\widehat{(a,b)}$ is surjective, the desired contradiction.

Next, suppose that the “only if” assertion fails. Then $R$ has an $R$-sequence $a,b$, but $R$ has property ($\star$). The $R$-sequence condition gives us that $Ra + Rb \neq R$,
Some Analogues of a Result of Vasconcelos

821

\(a \neq 0\) and \((a, b)\) is injective. But then property \((\ast)\) ensures that \((a, b)\) is surjective. Thus \(R/Ra = (a, b)(R/Ra) = (Ra + Rb)/Ra\), and so a standard homomorphism theorem yields that \(R = Ra + Rb\), the desired contradiction, which completes the proof.

Recall from [3] that a domain is said to be an antimatter domain if it has no irreducible elements. Note that any antimatter domain \(D\) vacuously satisfies the conclusion of Proposition 2.4; Example 2.6 will show that such \(D\) need not satisfy the hypothesis of Proposition 2.4. Also, it will be convenient to let \(\mathbb{Q}^+\) denote the monoid consisting of the non-negative rational numbers under addition.

**Example 2.6.** Let \(K\) be a perfect field of positive characteristic, let the monoid \(S\) be the cardinal sum of two copies of \(\mathbb{Q}^+\), and let \(R\) denote the monoid ring \(K[X; S]\). Then \(R\) is a domain of Krull dimension 2 and each nonzero principal prime ideal of \(R\) is a maximal ideal, but the ring \(R\) does not have property \((\ast)\).

**Proof.** By [1, Theorem 5 (1)], \(R\) is an antimatter domain (and a GCD-domain). By combining [10, Theorems 21.4 and 17.1] with [9, Theorem 30.5], we see that the dimension of this domain is the torsion-free rank of \(S\), namely, 2. Since \(R\) is an antimatter domain, it has no nonzero principal prime ideals (and so, vacuously, any such ideal is maximal). It remains only to show that \(R\) does not have property \((\ast)\). By Proposition 2.5, it therefore suffices to produce an \(R\)-sequence \(a, b\) (on \(R\)). The reader can easily check that \(a := X(0, 1), b := X(1, 0)\) form the desired \(R\)-sequence.

Recall that a domain \(R\) is said to be treed if \(\text{Spec}(R)\), as a poset under inclusion, forms a tree; that is, if no prime ideal of \(R\) can contain incomparable prime ideals of \(R\). Corollary 2.7 will lead to many examples of rings that have property \((\ast)\) by showing that any treed domain has property \((\ast)\). In this vein, and with an eye to the comment that preceded Proposition 2.4, we recall that any going-down domain is a treed domain [5, Theorem 2.2] (and the converse is false).

**Corollary 2.7.** If \(R\) is a treed domain, then \(R\) has property \((\ast)\).

**Proof.** By [7, Theorem 2.2], there is no \(R\)-sequence (on \(R\)) of the form \(a, b\). Therefore, an application of Proposition 2.5 completes the proof.

By Corollary 2.7 and the comment preceding it, each going-down domain has property \((\ast)\). The applications in the next two corollaries could be inferred from this fact about going-down domains, but we prefer to present those corollaries as consequences of Corollary 2.7, as treed domains are perhaps more widely known than going-down domains.

**Corollary 2.8.** If \(R\) is a domain of Krull dimension at most 1, then \(R\) has property \((\ast)\).

**Proof.** Since \(R\) is a treed domain, the assertion follows from Corollary 2.7. For an alternate proof, combine Proposition 2.5 with the well-known fact (cf. [13, Exercise
that for any non-negative integer \( n \), an \( n \)-dimensional ring cannot contain an \( R \)-sequence of length \( n + 1 \).

In view of Corollary 2.8 and Example 2.6, the following result is of some interest, as pseudo-valuation domains can have arbitrary Krull dimension. (To see this, either use [6, Lemma 4.5 (v), Proposition 4.9 (i)] or consider a valuation domain whose value group has rank equal to the desired dimension.)

**Corollary 2.9.** Each pseudo-valuation domain has property \( (\star) \).

**Proof.** Each pseudo-valuation domain is a treed domain [11, Corollary 1.3]. Therefore, an application of Corollary 2.7 completes the proof.

The conclusion in Corollary 2.9 applies as well, and more generally, to the class of locally pseudo-valuation domains (the so-called LPVDs). Since the property of being treed is a local property of domains, this raises the question whether property \( (\star) \) itself is a local property. We will address some related issues in Remark 2.12.

**Theorem 2.10.** Let \( (T, M) \) be a quasi-local domain with property \( (\star) \), let \( D \) be a subring of \( T/M \), let \( \pi : T \rightarrow T/M \) denote the canonical projection map, and let \( R := D \times_{T/M} T \) \((= \pi^{-1}(D))\). Then \( R \) has property \( (\star) \) if and only if \( D \) has property \( (\star) \).

**Proof.** We have that \( R/M \cong D \), and so \( M \in \text{Spec}(R) \). Also, by the order-theoretic impact of the topological description of \( \text{Spec}(R) \) (with the Zariski topology) in [8, Theorem 1.4], each prime ideal of \( R \) is comparable with \( M \) under inclusion, and the set of prime ideals of \( R \) that contain (resp., are contained in) \( M \) is isomorphic as a poset to \( \text{Spec}(D) \) (resp., \( \text{Spec}(T) \)).

We will prove the “if” assertion first. So, suppose that \( D \) has property \( (\star) \). By Proposition 2.5, it suffices to derive a contradiction from the assumption that \( a, b \) is an \( R \)-sequence on \( R \). Our analysis will break naturally into two cases that are determined by where \( a \) and \( b \) reside relative to \( M \). In any event, note that \( 0 \neq a, b \in R \setminus U(R) \); and that \( b, a \) is also an \( R \)-sequence on \( R \), by [7, Lemma 2.1 (b)].

Case 1: either \( a \notin M \) or \( b \notin M \). By the preceding observation, we may assume, without loss of generality, that \( a \notin M \). Now, by a standard homomorphism theorem, \( D/Da \cong R/(Ra + M) \) as \( R \)-modules. It will simplify the notation to point out that \( Ra + M = Ra \); that is, \( M \subseteq Ra \). (In detail, note that \( a \in T \setminus M = U(T) \), so that, working in \( T \), we have \( M = (a^{-1}M)a \subseteq (TM)a = Ma \subseteq Ra \).) Let \( f \) denote multiplication by \( b \) on \( D/Da \). We claim that \( f \) is a monomorphism. Suppose, on the contrary, that there exists \( t \in R \setminus Ra \) such that \( bt \in Ra \). Since \( a, b \) is an \( R \)-sequence on \( R \), we conclude that \( t \in Ra \), the desired contradiction. This proves the above claim; that is, \( f \) is a monomorphism.

Since \( D \) has property \( (\star) \) and \( f \) is a monomorphism, \( f \) must be surjective.
Therefore, there exists \( u \in R \) such that \( bu - 1 \in Ra + M = Ra \). Thus, \( 1 \in Ra + Rb \), contradicting that \( a, b \) is an \( R \)-sequence on \( R \).

Case 2: both \( a, b \) are elements of \( M \). Since \( a, b \) is an \( R \)-sequence on \( R \), we have that \( b \not\in Ra \). We claim that \( b \not\in Ta \). To see this, suppose, on the contrary, that \( b = qa \) for some \( q \in T \setminus R \). Then \( q \in U(T) \) (since \( T \setminus U(T) = M \subseteq R \)). Observe that \( b \in M \setminus Ra \), and so \( b + Ra \neq 0 \in R/Ra \). However, \( b \cdot (b + Ra) = b^2 + Ra = (q^2a)a + Ra = 0 \in R/Ra \), since \( q^2a \in Ta \subseteq M \subseteq R \). This contradicts that \( a, b \) is an \( R \)-sequence on \( R \), thus proving the above claim; that is, \( b \not\in Ta \).

Note that multiplication by \( b \) on \( T/Ta \) is not surjective, since the image of this endomorphism is contained in \( M/Ta \). Consequently, since \( T \) has property (\( \ast \)), this endomorphism is not injective. Hence, there exists \( c \in T \setminus Ta \) such that \( bc \in Ta \). Write \( bc = ta \), with \( t \in T \). If \( c \in U(T) \), then \( b^2 = (tc^{-1}b)a \in Ra \) (since \( tc^{-1}b \in Tb \subseteq M \subseteq R \)), so that multiplication by \( b \) annihilates the nonzero element \( b + Ra \in R/Ra \), which contradicts that \( a, b \) is an \( R \)-sequence on \( R \). Therefore, \( c \in T \setminus U(T) = M \subseteq R \). As \( bc = ta \) and \( c \not\in Ra \), the fact that \( a, b \) is an \( R \)-sequence on \( R \) ensures that \( t \not\in R \). In particular, \( t \not\in M \), and so \( t^{-1} \in T \). Then \( d := ct^{-1} \) satisfies \( bd = a \) and \( d \in MT = M \subseteq R \). As \( a, b \) is an \( R \)-sequence on \( R \), it follows that \( d \in Ra \). Thus, \( c = td \in T(Ra) = Ta \), the desired contradiction. This completes the proof of the “if” assertion.

Suppose the “only if” assertion fails. Then \( R \) has property (\( \ast \)) but \( D \) does not have property (\( \ast \)). Use \( \pi|R \) to identify \( D = R/M \). By Proposition 2.5, there exist \( a, b \in R \), such that \( \overline{a} := a + M, \overline{b} := b + M \) is a regular sequence on \( D \) (that is, an \( \mathcal{R} \)-sequence of \( D \)-modules). Necessarily, neither \( a \) nor \( b \) is an element of \( M \). As above, a standard homomorphism theorem gives that \( D/D\overline{a} \cong R/(Ra + M) \). By Proposition 2.5, it cannot be the case that \( a, b \) is an \( R \)-sequence on \( R \). It also cannot be the case that \( Ra + Rb = R \) (since \( D\overline{a} + D\overline{b} \neq D \)). Hence, there exists \( s \in R \setminus Ra \) such that \( bs \in Ra \). Then, \( a \text{ a fortiori, } bs \in Ra + M \), whence \( \overline{s} := s + M \) satisfies \( \overline{b} \overline{s} \in D\overline{a} \). Since \( \overline{a}, \overline{b} \) is a regular sequence on \( D \), it follows that \( \overline{s} \in D\overline{a} \); that is, \( s \in Ra + M \). However, \( M = (Ma^{-1})a \subseteq (MT)a = Ma \subseteq Ra \), and so \( s \in Ra \), the desired contradiction, which completes the proof.

\[ \square \]

**Corollary 2.11.** Let \( R \subseteq T \) be domains such that \( \text{Spec}(R) = \text{Spec}(T) \) as sets. If \( T \) has property (\( \ast \)), then \( R \) has property (\( \ast \)).

**Proof.** As any field has property (\( \ast \)), we may assume that \( R = T \) and \( R \) are not fields. Also without loss of generality, \( R \neq T \). Then, by [2, Proposition 3.3 and Theorem 3.25], \( R \) is a quasi-local ring, say with maximal ideal \( M \), such that if \( K \) denotes the quotient field of \( R \) and \( (M :_K M) := \{ u \in K \mid uM \subseteq M \} \), with \( \pi : (M :_K M) \rightarrow (M :_K M)/M \) the canonical projection map, then there exists a field \( k \subseteq (M :_K M)/M \) such that \( T \) equals the pullback \( \pi^{-1}(k) \). As \( T \) is quasi-local and \( K \) has property (\( \ast \)), the assertion now follows from Theorem 2.10. For an alternate proof, one could combine [7, Proposition 2.5] with the criterion for property (\( \ast \)) in Proposition 2.5.

\[ \square \]

Since any pseudo-valuation domain can be obtained as the pullback of its canonically associated valuation domain and a subfield of the residue field of that valuation
domain (cf. [2, Proposition 2.6]), Corollary 2.11 reduces the assertion in Corollary 2.9 to the case of a valuation domain. An additional application of Theorem 2.10 will be given in Remark 2.12 (c).

Remark 2.12. (a) The converse of Corollary 2.7 is false, even for two-dimensional Noetherian domains. Indeed, let $R$ be any two-dimensional Noetherian local domain which is not a Cohen-Macaulay ring. An example of such $R$ was given in [7, Remark 2.4], where it was also shown (for any such $R$) that $R$ has no $R$-sequence of the form $a, b$. Hence, by Proposition 2.5, $R$ has property ($\star$). Of course, $R$ is not treeed (cf. [13, Theorem 144]).

(b) Many examples of valuation domains that are also antimatter domains were given in [3]. By Corollary 2.7 (or Corollary 2.9), each of those domains has property ($\star$). Their behavior is to be contrasted with that of the antimatter domain given in Example 2.6. Noetherian domains form another prominent class of domains that contains some domains having property ($\star$) and some domains that do not have property ($\star$). Indeed, recall from (a) that any two-dimensional (Noetherian) local non-Cohen-Macaulay domain has property ($\star$), while one can show, by combining Proposition 2.4 with [13, Theorems 5 and 142], that any Noetherian unique factorization domain of Krull dimension at least 2 must fail to have property ($\star$). These facts help to show the diversity of the class of domains that have property ($\star$). Our point here is that if $D$ is a Noetherian domain (more generally, a domain satisfying the ascending chain condition on principal ideals) but not a field, then $D$ is far from being an antimatter domain, since each nonzero nonunit of $D$ is a product of irreducible elements of $D$.

(c) Consider the data in Theorem 2.10, with $D$ also assumed to be quasi-local and treeed. Once again, we use the order-theoretic impact of the topological description of $\text{Spec}(R)$ (with the Zariski topology) in [8, Theorem 1.4]. This shows that $R$ is quasi-local and also that $R$ is treeed if and only if $T$ is treeed; in addition, it follows from [8, Proposition 2.1 (5)] that $\dim(R) = \dim(D) + \dim(T)$. Thus, if $T$ is one of the two-dimensional Noetherian local non-Cohen-Macaulay domains considered in (a) and $D$ is a $d$-dimensional valuation subring of the field $T/M$, then Theorem 2.10 implies that the pullback $R := D \times_{T/M} T$ is a $(d + 2)$-dimensional non-treeed domain having property ($\star$), where, as usual, we take $\infty + 2 := \infty$.

(d) Proposition 2.5 does not extend to arbitrary rings. In fact, the “if” assertion of Proposition 2.5 fails for the following data. Consider the ring $A := \mathbb{Z}_{2\mathbb{Z}}$ and the idealization $R := A(+).A$. Then $R$ is a non-domain (since $Z(R) = 0(+)A \neq 0$); $R$ is a Noetherian local one-dimensional ring with unique maximal ideal $2A(+).A$; and Proposition 2.4 shows that $R$ does not have property ($\star$), since $0(+)A = A(0, 1)$ is a nonzero principal prime ideal of $R$ which is not a maximal ideal of $R$. Finally, since $\dim(R) < 2$, it follows from [13, Theorem 132] that $R$ has no $R$-sequence of the form $a, b$.

(e) We consider here the behavior of property ($\star$) under globalization. In this paragraph, we get a positive result for domains. To wit: if $R$ is a domain such that $R_M$ has property ($\star$) for each maximal ideal $M$ of $R$, then $R$ has property ($\star$). For
a proof, we may assume, without loss of generality, that $R$ is not a field. Then the assertion follows by combining [7, Lemma 2.1 (a)] with the criterion for property (\*) in Proposition 2.5.

Next, if the ambient ring $R$ is not necessarily a domain, we have the following positive result. If $R$ is a ring whose Jacobson radical $J(R)$ contains $Z(R)$ and if $R_M$ has property (\*) for each maximal ideal $M$ of $R$, then $R$ has property (\*). For a proof, let $0 \neq a \in R$ and consider an arbitrary monomorphism $f \in \text{Hom}_R(R/Ra, R/Ra)$. Our task is to show that $f$ is an isomorphism or, equivalently, that the induced map $f_M : (R/Ra)_M \rightarrow (R/Ra)_M$ is an isomorphism for each maximal ideal $M$ of $R$. Of course, $f_M$ inherits the property of being a monomorphism from $f$, for each $M$. Note also that for each $M$, we have a canonical isomorphism $(R/Ra)_M \cong R_M/(Ra)_M$ and $(Ra)_M = R_M(a/1)$. Therefore, since each $R_M$ is assumed to have property (\*), we will be done if $(a/1) \neq (0/1) \in R_M$ for each $M$. This, in turn, follows from the hypothesis that $Z(R) \subseteq J(R)$ (for if $(a/1) \neq (0/1) \in R_M$, then $(0 :_R a) \nsubseteq M$, although $(0 :_R a) \subseteq Z(R) \subseteq J(R) \subseteq M$, the desired contradiction).

We pause to give an example of a quasi-local non-domain $R$ satisfying the hypothesis that $Z(R) \subseteq J(R)$ from the preceding paragraph. Let $A$ and $R = A(+)A$ be as constructed in (d). Then $Z(R) = 0(+)A \subseteq 2A(+)A = J(R)$. Notice that we showed in (d) that $R$ does not have property (\*).

Finally, despite the positive results in the first two paragraph of this part (e), we show next that property (\*) does not globalize in general. In other words, there exists a ring $R$ such that $R_M$ has property (\*) for each maximal ideal $M$ of $R$ but $R$ does not have property (\*). Necessarily, by the above remarks, any such $R$ must fail to be a domain and must fail to satisfy that $Z(R) \subseteq J(R)$. To construct a satisfactory $R$, we let $(A, N)$ be a quasi-local tree domain which is not a field, and put $R := A \times A$. The maximal ideals of $R$ are $N \times A$ and $A \times N$; the localizations of $R$ at these maximal ideals are each canonically isomorphic as rings to $A_N \cong A$. In particular, both of these localizations are tree domains and so, by Corollary 2.7, $R_M$ has property (\*) for each maximal ideal $M$ of $R$. It remains to show that $R$ does not have property (\*). To that end, first use the hypothesis that $A$ is not a field to pick a nonzero nonunit $\rho \in N$, and then let $a := (1, 0)$, $b := (0, \rho) \in R$. It suffices to show that if $f$ denotes multiplication by $b$ on $E := R/Ra$, then $f$ is a monomorphism but not an isomorphism. By means of the canonical isomorphism $E \cong A$, $f$ can be identified with the $A$-module homomorphism $A \rightarrow A$ given by multiplication by $\rho$. This map is clearly a monomorphism but not surjective.

In closing, we raise the following question about localization which is, in a sense, dual to the question that was studied in Remark 2.12(d). Can one find useful conditions on a ring $R$ that has property (\*) and a multiplicatively closed subset $S$ of $R$ so as to ensure that $R_S$ also has property (\*)?
References

[1] D. D. Anderson, J. Coykendall, L. Hill and M. Zafrullah, *Monoid domain constructions of antimatter domains*, Comm. Algebra, 35(2007), 3236–3241.

[2] D. F. Anderson and D. E. Dobbs, *Pairs of rings with the same prime ideals*, Canad. Math. J., 2(1980), 363–384.

[3] J. Coykendall, D. E. Dobbs and B. Mullins, *On integral domains with no atoms*, Comm. Algebra, 27(1999), 5813–5831.

[4] I. S. Cohen and A. Seidenberg, *Prime ideals and integral dependence*, Bull. Amer. Math. Soc., 52(1946), 252–261.

[5] D. E. Dobbs, *On going-down for simple overrings, II*, Comm. Algebra, 1(1974), 439–458.

[6] D. E. Dobbs, *Coherence, ascent of going-down and pseudo-valuation domains*, Houston J. Math., 4(1978), 551–567.

[7] D. E. Dobbs, *Treed domains have grade 1*, Internat. J. Commut. Rings, 2(2003), 43–46.

[8] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl., 123(1980), 331–355.

[9] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[10] R. Gilmer, *Commutative Semigroup Rings*, Univ. Chicago Press, Chicago/London, 1984.

[11] J. R. Hedstrom and E. G. Houston, *Pseudo-valuation domains*, Pacific J. Math., 75(1978), 137–147.

[12] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Dekker, New York, 1988.

[13] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago/London, 1974.

[14] W. V. Vasconcelos, *Injective endomorphisms of finitely generated modules*, Proc. Amer. Math. Soc., 25(1970), 900–901.