COHEN-MACAUŁAY-NESS IN CODIMENSION FOR BIPARTITE GRAPHS

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Abstract. Let $G$ be an unmixed bipartite graph of dimension $d - 1$. Assume that $K_{n,n}$, with $n \geq 2$, is a maximal complete bipartite subgraph of $G$ of minimum dimension. Then $G$ is Cohen-Macaulay in codimension $d - n + 1$. This generalizes a characterization of Cohen-Macaulay bipartite graphs by Herzog and Hibi and a result of Cook and Nagel on unmixed Buchsbaum graphs. Furthermore, we show that any unmixed bipartite graph $G$ which is Cohen-Macaulay in codimension $t$, is obtained from a Cohen-Macaulay graph by replacing certain edges of $G$ with complete bipartite graphs. We provide some examples.

1. Introduction

Cohen-Macaulay simplicial complexes are among central research topics in combinatorial commutative algebra. While characterization of such complexes is a far reaching problem, one appeals to study specific families of Cohen-Macaulay simplicial complexes. Flag complexes are among important families of complexes recommended to study [10, page 100]. However, it is known that a simplicial complex is Cohen-Macaulay if and only if its barycentric subdivision is a Cohen-Macaulay flag complex. Therefore, a characterization of Cohen-Macaulay flag complexes is equivalent to a characterization of Cohen-Macaulay simplicial complexes. Nevertheless, after all, the ideal of a flag complex is generated by quadratic square-free monomials, which are simpler compared with arbitrary square-free monomial ideals. Furthermore, it seems that, expressing many combinatorial properties in terms of graphs are more convenient. As some evidences, the characterization of unmixed bipartite graphs by Villarreal [11] and Cohen-Macaulay bipartite graphs by Herzog and Hibi [5] are well expressed in terms of graphs.

On the other hand, in the hierarchy of families of graphs with respect to Cohen-Macaulay property, Buchsbaum complexes appear right after Cohen-Macaulay ones. Unmixed bipartite Buchsbaum graphs were characterized by Cook and Nagel [1] (also by the authors [3]). Natural families of graphs in this hierarchy are bipartite CM$_t$ graphs, i.e., graphs that their independence complexes are pure and Cohen-Macaulay in codimension $t$. The concept of CM$_t$ simplicial complexes were introduced in [4] which is the pure version of simplicial complexes Cohen-Macaulay in codimension $t$ studied by Miller, Novik and Swartz [6]. In this note, we give characterizations of unmixed bipartite CM$_t$ graphs in terms of its dimension and the minimum dimension of its maximal nontrivial complete bipartite subgraphs. Cook and Nagel showed that the only non-Cohen-Macaulay unmixed bipartite graphs are

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complete bipartite graphs \cite{1} Theorem 4.10 and \cite{3} Theorem 1.3. Our results are generalizations of this fact to unmixed bipartite graphs which are Cohen-Macaulay in arbitrary codimension. In the next section we gather necessary definitions and known results to be used in the rest of the paper. In Section 3 we improve some results on joins of simplicial complexes and disjoint unions of graphs with respect to the CM\(_t\) property. Section 4 is devoted to two characterizations of bipartite CM\(_t\) graphs and some examples.

2. Preliminaries

For basic definitions and general facts on simplicial complexes we refer to the book of Stanley \cite{10}. By a complex we will always mean a simplicial complex. Let \(G = (V, E)\) be a simple graph with vertex set \(V\) and edge set \(E\). The \emph{inclusive neighborhood} of \(v \in V\) is the set \(N[v]\) consisting of \(v\) and vertices adjacent to \(v\) in \(G\). The \emph{independence complex} of \(G = (V, E)\) is the complex \(\text{Ind}(G)\) with vertex set \(V\) and with faces consisting of independent sets of vertices of \(G\), i.e., sets of vertices of \(G\) where no two elements of them are adjacent. These complexes are called \emph{flag complexes}, and their Stanley-Reisner ideal is generated by quadratic square-free monomials. By \emph{dimension} of a graph \(G\) we mean the dimension of the complex \(\text{Ind}(G)\). A graph \(G\) is said to be \emph{ unmixed} if \(\text{Ind}(G)\) is pure.

For an integer \(t \geq 0\), a complex \(\Delta\) is called CM\(_t\) if it is pure and for every face \(F \in \Delta\) with \(\#(F) \geq t\), \(\text{link}_\Delta(F)\) is Cohen-Macaulay. This is the same as pure complexes which are Cohen-Macaulay in codimension \(t\). Accordingly, CM\(_0\) and CM\(_1\) complexes are precisely Cohen-Macaulay and Buchsbaum complexes, respectively. Clearly, a CM\(_t\) complex is CM\(_r\) for all \(r \geq t\) and a complex of dimension \(d - 1\) is always CM\(_{d-1}\). One uses the convention that for \(t < 0\), CM\(_t\) would mean CM\(_0\).

A graph \(G\) is called CM\(_t\) if \(\text{Ind}(G)\) is CM\(_t\). A basic tool for checking CM\(_t\) property of complexes is the following lemma.

**Lemma 2.1.** (\cite{4} Lemma 2.3) Let \(t \geq 1\) and let \(\Delta\) be a nonempty complex. Then the following are equivalent:

(i) \(\Delta\) is a CM\(_t\) complex.

(ii) \(\Delta\) is pure and \(\text{link}_\Delta(v)\) is CM\(_{t-1}\) for every vertex \(v \in \Delta\).

By the straightforward identity \(\text{link}_{\text{Ind}(G)}(v) = \text{Ind}(G \setminus N[v])\), the counterpart of this lemma for graphs would be the following:

**Lemma 2.2.** Let \(t \geq 1\) and let \(G\) be a graph. Then the following are equivalent:

(i) \(G\) is a CM\(_t\) graph.

(ii) \(G\) is unmixed and \(G \setminus N[v]\) is a CM\(_{t-1}\) graph for every vertex \(v \in G\).

We recall some basic relevant facts on bipartite graphs. A graph \(G = (V, E)\) is called \emph{bipartite} if \(V\) is a disjoint union of a partition \(V_1\) and \(V_2\) and \(E \subset V_1 \times V_2\). If \(\#(V_1) = m\) and \(\#(V_2) = n\) and \(E = V_1 \times V_2\), then \(G\) is the \emph{complete} bipartite graph \(K_{m,n}\). We will be interested in unmixed complete bipartite graphs \(K_{m,n}\).

Unmixed bipartite graphs are characterized by Villarreal in the following result.

**Theorem 2.3.** \cite{11} Theorem 1.1) Let \(G\) be a bipartite graph without isolated vertex. Then \(G\) is unmixed if and only if there is a partition \(V_1 = \{x_1, \ldots, x_n\}\) and \(V_2 = \{y_1, \ldots, y_n\}\) of vertices of \(G\) such that
(1) $x_iy_i$ is an edge in $G$ for $1 \leq i \leq n$ and
(2) If $x_iy_j$ and $x_jy_k$ are edges in $G$, for some distinct $i$, $j$ and $k$, then $x_iy_k$ is an edge in $G$.

In this case, such a partition and ordering is called a pure order of $G$. The edges $x_iy_i$, $i = 1, \cdots, n$ are called a perfect matching edges of $G$. A pure order is said to have a cross if, for some $i \neq j$, $x_iy_j$ and $x_jy_i$ are both edges in $G$. Otherwise, the order is called cross-free (see [1, §4]). For unmixed bipartite graphs, being cross-free is independent of an ordering of vertices of $G$. More precisely, if $G$ has a cross in some pure ordering, it has a cross in every pure ordering [1, Lemma 4.5].

An immediate consequence of Theorem 2.3 is the following useful lemma.

**Lemma 2.4.** Let $G$ be an unmixed bipartite graph with pure order of vertices $(\{x_1, \cdots, x_d\}, \{y_1, \cdots, y_d\})$ and let $K_{n,n}$ be a complete bipartite subgraph of $G$ on $(\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\})$.

(i) If $x_jy_k$ is an edge in $G$ for some $j$ and $k$, then $x_jy_k$ is an edge in $G$ for all $l = 1, \cdots, n$.

(ii) If $x_ky_j$ is an edge in $G$ for some $k$ and $j$, then $x_ky_j$ is an edge in $G$ for all $l = 1, \cdots, n$.

**Proof.** The assertion (i) is immediate by Theorem 2.3 because $x_iy_i$ is an edge in $K_{n,n} \subset G$ for all $l = 1, \cdots, n$. Also (ii) follows because $x_iy_i$ is an edge in $K_{n,n} \subset G$ for all $l = 1, \cdots, n$. □

There are also at least two nice characterization of Cohen-Macaulay bipartite graphs.

**Theorem 2.5.** [5, Theorem 3.4] Let $G$ be a bipartite graph without isolated vertices. Then $G$ is Cohen-Macaulay if and only if there is a pure ordering $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_n\}$ of vertices of $G$ such that $x_iy_j$ being in $G$ implies $i \leq j$.

The ordering in Theorem 2.5 is called a Macaulay order of vertices of $G$.

**Proposition 2.6.** [8, Proposition 4.8] Let $G$ be a bipartite graph. Then $G$ is Cohen-Macaulay if and only if $G$ has a cross-free pure order.

Bipartite Buchsbaum graphs are also classified. First recall that a complex is Buchsbaum if and only if it is pure and the link of each vertex is Cohen-Macaulay [9]. Thus, a graph is Buchsbaum if and only if $G$ is unmixed and for each vertex $v \in G$, $G \setminus N[v]$ is Cohen-Macaulay. For bipartite graphs there is a sharper result. Complete bipartite graphs are well-known to be Buchsbaum (e.g., see [12, Proposition 2.3]). But indeed, the converse is also true.

**Theorem 2.7.** (see [1, Theorem 4.10] or [8, Theorem 1.3]) Let $G$ be a bipartite graph. Then $G$ is Buchsbaum if and only if $G$ is a complete bipartite graph $K_{n,n}$ for some $n \geq 2$, or $G$ is Cohen-Macaulay.

3. Joins of CM$_t$ complexes and disjoint unions of CM$_t$ graphs

It is known that the join of two complexes is Cohen-Macaulay if and only if they are both Cohen-Macaulay (see [8] and [2]). If $\Delta$ is a CM$_r$ complex of dimension $d-1$ and $\Delta'$ is a CM$_r$ complex of dimension $d'-1$, then their join $\Delta \ast \Delta'$ is a CM$_t$ complex where $t = \max\{d+r',d'+r\}$ [4, Proposition 2.10]. However, if one
of the complexes is Cohen-Macaulay, this result could be strengthened. Below we combine this with relevant known results.

**Theorem 3.1.** Let $\Delta$ and $\Delta'$ be two complexes of dimensions $d - 1$ and $d' - 1$, respectively. Then

(i) The join complex $\Delta \ast \Delta'$ is Cohen-Macaulay if and only if both $\Delta$ and $\Delta'$ are so.

(ii) If $\Delta$ is Cohen-Macaulay and $\Delta'$ is $CM_{r'}$ for some $r' \geq 1$, then $\Delta \ast \Delta'$ is $CM_{d+r'}$ (independent of $d'$). This is sharp, i.e., if $\Delta'$ is not $CM_{r'-1}$, then $\Delta \ast \Delta'$ is not $CM_{d+r'-1}$. In particular, a cone on $\Delta'$ is $CM_{r'+1}$.

(iii) If $\Delta$ is $CM_r$ and $\Delta'$ is $CM_{r'}$, for some $r, r' \geq 1$, then $\Delta \ast \Delta'$ is $CM_t$ where $t = \max\{d + r', d' + r\}$. Conversely, if $\Delta \ast \Delta'$ is $CM_t$, then $\Delta$ is $CM_{t-d'}$ and $\Delta'$ is $CM_{t-d}$.

**Proof.** The statement in (i) is proved by Sava [S] and Fröberg [F]. The assertion (iii) is proved in [C] Theorem 2.10. We prove (ii) using induction on $d + r' \geq 2$.

Let $d + r' = 2$, i.e., $d = 1$ and $r' = 1$. Then $\Delta = \{v\}$ is a singleton. Thus $\text{link}_{\Delta \ast \Delta'}(v) = \Delta'$, which is $CM_1$. For $v \in \Delta$, $\text{link}_{\Delta \ast \Delta'}(v) = \Delta \ast \text{link}_{\Delta'}(v)$, which is Cohen-Macaulay by (i). Thus by Lemma 2.1, $\Delta \ast \Delta'$ is $CM_2$. Now let $d + r' \geq 2$. Let $v \in \Delta$. Then, $\text{link}_{\Delta \ast \Delta'}(v) = \text{link}_{\Delta}(v) \ast \Delta'$. But $\text{link}_{\Delta}(v)$ is Cohen-Macaulay of dimension less than $d - 1$, and $\Delta'$ is $CM_{r'}$. Thus by induction hypothesis $\text{link}_{\Delta \ast \Delta'}(v) = CM_{d-1+r'}$. If $v \in \Delta'$, then $\text{link}_{\Delta \ast \Delta'}(v) = \Delta' \ast \text{link}_{\Delta'}(v)$. But $\text{link}_{\Delta'}(v)$ is $CM_{r'-1}$ and hence $\text{link}_{\Delta \ast \Delta'}(v)$ is again $CM_{d+r'-1}$. Therefore, $\Delta \ast \Delta'$ is $CM_{d+r'}$. To prove that this result is sharp, proceed by induction on $d \geq 1$. Indeed, in this case, for any $v \in \Delta$, $\text{link}_{\Delta}(v)$ has dimension less than $d - 1$ and hence by induction hypothesis, $\text{link}_{\Delta \ast \Delta'}(v) = \text{link}_{\Delta}(v) \ast \Delta$ is not $CM_{d+r'-2}$. Therefore, $\Delta \ast \Delta'$ is not $CM_{d+r'-1}$.

Let $G \sqcup G'$ denote the disjoint union of graphs $G$ and $G'$. By the fact that $\text{Ind}(G \sqcup G') = \text{Ind}(G) \ast \text{Ind}(G')$, the counter-part of Theorem 3.1 for graphs will be the following.

**Theorem 3.2.** Let $G$ and $G'$ be two graphs on disjoint sets of vertices and of dimensions $d - 1$ and $d' - 1$, respectively. Then

(i) The graph $G \sqcup G'$ is Cohen-Macaulay if and only if both $G$ and $G'$ are so.

(ii) If $G$ is Cohen-Macaulay and $G'$ is $CM_{r'}$ for some $r' \geq 1$, then $G \sqcup G'$ is $CM_{d+r'}$. If $G'$ is not $CM_{r'-1}$, then $G \sqcup G'$ is not $CM_{d+r'-1}$.

(iii) If $G$ is $CM_r$ and $G'$ is $CM_{r'}$ for some $r, r' \geq 1$, then $G \sqcup G'$ is $CM_t$ where $t = \max\{d + r', d' + r\}$. Conversely, if $G \sqcup G'$ is $CM_t$, then $G$ is $CM_{t-d'}$ and $G'$ is $CM_{t-d}$.

4. Two characterizations of bipartite $CM_t$ graphs

We now restrict to the case of bipartite graphs. Since Cohen-Macaulay bipartite graphs are characterized by Herzog and Hibi [H] Theorem 3.4], and also in a different version by Cook and Nagel [C] Proposition 4.8], we consider the non-Cohen-Macaulay case.

**Theorem 4.1.** Let $G$ be an unmixed bipartite graph of dimensions $d - 1$. Let $K_{n,n}$, with $n \geq 2$, be a maximal complete bipartite subgraph of $G$ of minimum dimension. Then $G$ is $CM_{d-n+1}$ but it is not $CM_{d-n}$.
Proof. We prove both assertions by induction on \( d \geq 2 \). If \( d = 2 \) then \( G = K_{2,2} \) which is CM, but it is not Cohen-Macaulay. Assume that \( d > 2 \). We show that for every \( v \in G \), \( G \setminus N[v] \) is CM and for some \( v \in G \) it is not CM.

Let \( (\{x_1, \ldots, x_d\}, \{y_1, \ldots, y_d\}) \) be a pure order of \( G \). Let \( x_i \) be a vertex of some maximal bipartite subgraph \( K_{m,m} \) with \( m \geq n \). Then \( G \setminus N[x_i] \) is a disjoint union of \( c \geq m - 1 \) isolated vertices and an unmixed bipartite graph \( H \) of dimension \( d - c - 2 \). The graph \( H \) is unmixed because \( \text{Ind}(G \setminus N[x_i]) = \text{link}_x(\text{Ind}(G)) \), and any link of a pure complex is pure. But \( G \setminus N[x_i] = (\{x_1, \ldots, x_i\} \cup H \) is unmixed if and only if \( H \) is so. Observe that if \( y_{j_0} \) is a vertex of a maximal bipartite subgraph of \( G \) and \( y_{j_0} \in N[x_i] \), then by Lemma 2.3 all \( y_j \) vertices of this subgraph belong to \( N[x_i] \).

Thus if \( H \) has no crosses, by Proposition 2.6 it is Cohen-Macaulay. Otherwise, the minimum dimension of maximal complete bipartite subgraphs of \( H \) will not be less than the minimum dimension of such subgraphs in \( G \). Hence by the induction hypothesis \( H \) is CM, and by Theorem 3.2(ii), \( G \setminus N[x_i] \) is CM for all \( x_i \in G \). If \( x_i \) does not belong to any maximal bipartite subgraph of \( G \) of positive dimension, then \( G \setminus N[x_i] \) is a disjoint union of \( c \geq 0 \) isolated vertices and an unmixed bipartite graph \( H \) of dimension \( d - c - 2 \). Hence \( H \) is CM, and by Theorem 3.2(ii), \( G \setminus N[x_i] \) is CM. A similar argument reveals that for any \( y_j \in G \), the graph \( G \setminus N[y_j] \) is CM. Therefore, by Lemma 2.2 \( G \) is CM. We now proceed the induction step to show that this result is sharp. Let \( d > 2 \) and let \( K_{n,n} \), \( n \geq 2 \), be a maximal bipartite subgraph of \( G \) of minimum dimension. Take \( x_i \in G \setminus K_{n,n} \). First assume that \( x_i \) is not adjacent to any vertex in \( K_{n,n} \) and consider \( G \setminus N[x_i] \). Let \( G \setminus N[x_i] \) be the disjoint union of \( c \geq 0 \) isolated vertices and an unmixed bipartite graph \( H \) of dimension \( d - c - 2 \). Then \( H \) contains \( K_{n,n} \) and hence by induction hypothesis \( H \) is sharp. CM and \( G \setminus N[x_i] \) is sharp CM. Therefore, \( G \) cannot be CM. Now assume that \( x_i y_j \in G \) for some \( j \) with \( y_j \in K_{n,n} \). Then by purity of the order, all \( y_j \in K_{n,n} \) is adjacent to \( x_i \). But then \( y_j \) is not adjacent to any vertex of \( K_{n,n} \), because otherwise, \( K_{n,n} \) will not be maximal. In this case, consider \( G \setminus N[y_j] \) and proceed similar to the previous case.

As a second characterization of bipartite CM graphs, we show that any CM graph is obtained from a Cohen-Macaulay graph \( H \) by replacing the perfect matching edges of \( H \) by complete bipartite graphs. This statement will be more precise in the next theorem. But first we provide a definition and a lemma.

Definition 4.2. Let \( H \) be an unmixed bipartite graph with pure order
\[
(\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_r\}).
\]
For a fixed \( i \), by replacing the edge \( x_i y_i \in H \) with a complete bipartite graph
\[
K_{n_i,n_i} = \{x_{1i}, \ldots, x_{ni}\} \times \{y_{1i}, \ldots, y_{ni}\}
\]
we mean a bipartite graph \( H' \) with vertex set
\[
(\{x_1, \ldots, x_i-1, x_{1i}, \ldots, x_{ni}, x_{i+1}, \ldots, x_r\}, \{y_1, \ldots, y_i, y_{i+1}, \ldots, y_{ri}\}),
\]
preserving all adjacencies, i.e.,
(i) \( x_s y_t \in H' \) for all \( s, t \neq i \) if and only if \( x_s y_t \in H \),
(ii) \( x_i y_j \in H' \) for all \( k \) if and only if \( x_i y_j \in H \),
(iii) \( x_j y_k \in H' \) for all \( k \) if and only if \( x_j y_k \in H \).
Lemma 4.3. Let $G$ be an unmixed bipartite graph with pure order on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \ldots, x_d\}$ and $W = \{y_1, \ldots, y_d\}$. Let $n_1, \ldots, n_d$ be any positive integers. Let $G' = G(n_1, \ldots, n_d)$ be the graph obtained by replacing each edge $x_iy_i$ with the complete bipartite graph $K_{n_i,n_i} = \{x_{i_1}, \ldots, x_{i_{n_i}}\} \times \{y_{i_1}, \ldots, y_{i_{n_i}}\}$ for all $i = 1, \ldots, d$. Then $G'$ is also unmixed.

Proof. Let $K_{n_i,n_i} = \{x_{i_1}, \ldots, x_{i_{n_i}}\} \times \{y_{i_1}, \ldots, y_{i_{n_i}}\}$. Then $V(G') = \{(x_{i_1}, \ldots, x_{i_{n_i}}), (y_{i_1}, \ldots, y_{i_{n_i}}), (x_{i_1}, \ldots, x_{i_{n_i}}), (y_{i_1}, \ldots, y_{i_{n_i}})\}$ is a pure order of $G'$. In fact, for all $i, r$, $x_iryir \in G'$. Also if $x_iryjr \in G'$ and $x_jykr \in G'$, then $x_jyj \in G$ and $x_jyk \in G$, and hence, $x_jyk \in G$. Thus by the construction of $G'$, $x_irykr \in G'$.

Theorem 4.4. Let $G$ be a Cohen-Macaulay bipartite graph with a Macaulay order on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \ldots, x_d\}$ and $W = \{y_1, \ldots, y_d\}$. Let $n_1, \ldots, n_d$ be any positive integers with $n_i \geq 2$ for at least one $i$. Let $G' = G(n_1, \ldots, n_d)$ be the graph obtained by replacing each edge $x_iy_i$ with the complete bipartite graph $K_{n_i,n_i}$ for all $i = 1, \ldots, d$. Let $n_{i_0} = \min\{n_i > 1 : i = 1, \ldots, d\}$, $n = \sum_{i=1}^{d} n_i$. Then $G'$ is exclusively a $CM_{n-n_{i_0}+1}$ graph. Furthermore, any bipartite $CM_t$ graph is obtained by such a replacement of complete bipartite graphs in a unique bipartite Cohen-Macaulay graph.

Proof. The first claim follows by Lemma 4.3 and Theorem 4.1. We settle the second claim. Let $G$ be a bipartite $CM_t$ graph with a pure order of vertices. Let $K_{n_i,n_i}, \ldots, K_{n_d,n_d}$ be the maximal bipartite subgraphs of $G$, where $n_i \geq 1$ for all $i$. Observe that, by maximality, these complete subgraphs of $G$ are disjoint. Choose one edge $x_{i_1}y_{i_1}$ from each subgraph $K_{n_i,n_i}$ for all $i = 1, \ldots, d$. Let $H$ be the induced subgraph of $G$ on the vertex set $\{(x_{i_1}, \ldots, x_{i_{n_i}}), (y_{i_1}, \ldots, y_{i_{n_i}})\}$. By Lemma 2.4 $H$ is independent of the choice of particular edge $x_{i_1}y_{i_1}$ from $K_{n_i,n_i}$ and hence $H$ is unique. Since the ordering of vertices of $G$ is a pure order, its restriction to $H$ is also pure. Thus, $H$ is an unmixed bipartite graph. But by the maximality of the complete bipartite subgraphs $K_{n_i,n_i}$, and the construction of $H$, it is cross-free. Therefore, by Proposition 2.4 $H$ is Cohen-Macaulay. Now any edge $x_{i_1}y_{i_1}$ replace in $H$ with $K_{n_i,n_i}$ for all $i = 1, \ldots, d$, preserving all other adjacencies. Let $H'$ be the resulting graph. Then by the construction, $G = H'$, as required. □

Remark 4.5. Let $H$ be a bipartite Cohen-Macaulay graph and let $G = H'$ be a bipartite $CM_t$ graph obtained from $H$ by the replacing process described above. Assume that $G$ is not $CM_{t-1}$ and $t \geq 2$. Using the results of this section, the following observations are immediate.

First of all, $1 \leq \dim H \leq t - 1$. Because if $\dim H \geq t$ and we replace just one $K_{n,n}$ with $n \geq 2$, then $G$ is strictly $CM_r$ with $r \geq t + 1$. On the other hand, if $\dim H = 0$, then $G$ is $CM_1$.

If $\dim H = t - 1$, then only one $K_{n,n}$ with $n \geq 2$ can be replaced. Because replacing at least two $K_{n,n}$ with $n \geq 2$, $G$ is strictly $CM_r$ with $r \geq t + 1$.

If $\dim H = t - 1$, for replacing just one $K_{n,n}$, $n$ is arbitrary and hence $G$ is of dimension $n + t - 2$.

If $\dim H \leq t - 2$, the number of replacements should be at least 2. Again because if with one replacement of $K_{n,n}$, $n \geq 2$, $G$ would be $CM_r$ with $r \leq t - 1$.

When $\dim H \leq t - 2$, the maximum number of replacements of $K_{n,n}$, $n \geq 2$, is at most $t - \dim H$ which may occur replacing $K_{2,2}$'s.
For $\dim H \leq t - 2$, the maximum size of $K_{n,n}$ to be replaced is also $n = t - \dim H$ which may occur when we have two replacements.

Using these remarks we may easily distinguish all bipartite CM$_t$ graphs for $t = 2, 3, 4$.

**Example 4.6.** Bipartite CM$_2$ graphs which are not Buchsbaum. Using the notation of Remark 4.5 we have $\dim H = 1$. There are just two non-isomorphic bipartite Cohen-Macaulay graphs of dimension one. By replacing process, they produce two types of bipartite CM$_2$ graphs which are not Buchsbaum. They are of arbitrary dimensions. More precisely, one such graph is the disjoint union of an edge $x_1y_1$ with $K_{n_2,n_2} = \{x_{21}, \ldots, x_{2n_2}\} \times \{y_{21}, \ldots, y_{2n_2}\}$, $n_2 \geq 2$, and the other one consists of the first graph together with the edges $x_1y_{2i}$ for all $i = 1, \ldots, n_2$. The second graph with $n_2 = 3$ could be depicted in Figure 1.

![Figure 1](image.png)

**Example 4.7.** Bipartite CM$_3$ graphs which are not CM$_2$. For these graphs $\dim H = 1, 2$.

If $\dim H = 1$, by Example 4.6, there are just two bipartite CM$_3$ graphs by replacing two edges of a perfect matching by $K_{2,2}$’s. In this case, $\dim G = 3$. (see Figure 2, and Figure 3).

If $\dim H = 2$, then there are 4 non-isomorphic bipartite Cohen-Macaulay graphs of dimension 2. By replacing one perfect matching edge with $K_{n,n}$ of arbitrary size in each Cohen-Macaulay graph, they produce 7 types of bipartite CM$_3$ graphs which are not CM$_2$. Note that depending on the choice of the edge to be replaced in each case, we may get non-isomorphic bipartite graphs. In this case $\dim G = n + 1$. 
Example 4.8. Bipartite $CM_4$ graphs which are not $CM_3$. For these graphs $\text{dim}H = 1, 2, 3$.

If $\text{dim}H = 1$, there are two bipartite $CM_4$ graphs obtained by replacing two edges of a perfect matching by $K_{3,3}$’s. In this case, $\text{dim}G = 5$. And, similarly, there are two others obtained by replacing one edge with $K_{2,2}$ and another edge with $K_{3,3}$. In this case, $\text{dim}G = 5$.

If $\text{dim}H = 2$, then while there are 4 non-isomorphic bipartite Cohen-Macaulay graphs of dimension 2, by replacing two perfect matching edges with $K_{2,2}$’s in each Cohen-Macaulay graph, they produce 7 bipartite $CM_4$ graphs which are not $CM_3$. They all have dimension 4.

If $\text{dim}H = 3$, then there are 10 non-isomorphic bipartite Cohen-Macaulay graphs of dimension 3. Replacing one perfect matching edge with $K_{n,n}$, $n \geq 2$, in each Cohen-Macaulay graph, they produce 25 bipartite $CM_4$ graphs which are not $CM_3$. They all have dimension $n + 2$. Out of all 36 bipartite $CM_4$ graphs, 21 graphs are connected.

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