OPPOSITE FILTRATIONS, VARIATIONS
OF HODGE STRUCTURE,
AND FROBENIUS MODULES

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Dedicated to Professor Yuri I. Manin, on the occasion of his sixty fifth birthday

Abstract. In this article we describe three constructions of complex variations of
Hodge structure, proving the existence of interesting opposite filtrations that general-
ize a construction of Deligne. We also analyze the relation between deformations of
Frobenius modules and certain maximally degenerate variations of Hodge structures.
Finally, under a certain generation hypothesis, we show how to construct a Frobenius
manifold starting from a deformation of a Frobenius module.

1. Introduction

Let $X$ be a complex manifold. Then, an unpolarized complex variation of Hodge
structure $(E, \nabla, F, \Phi)$ over $X$ consists of a flat, $C^\infty$ complex vector bundle $(E, \nabla)$
over $X$ equipped with a decreasing Hodge filtration $F$ and an increasing filtration
$\Phi$ such that

1. $F$ is holomorphic with respect to $\nabla$, and $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_X$;
2. $\Phi$ is anti-holomorphic with respect to $\nabla$, and $\nabla(\Phi_q) \subseteq \Phi_{q+1} \otimes \Omega^1_X$;
3. $E = F^p \oplus \Phi_{p-1}$ for each index $p$ (i.e. $F$ is opposite to $\Phi$).

Alternatively, conditions (1.1)–(1.3) are equivalent to the assertion that the $C^\infty$
decomposition

$$E = \bigoplus_p U^p$$

defined by the Hodge bundles $U^p = F^p \cap \Phi_p$ satisfy Griffiths’ transversality:

$$\nabla : \mathcal{E}^0(U^p) \rightarrow \mathcal{E}^{0,1}(U^{p+1}) \oplus \mathcal{E}^{1,0}(U^p) \oplus \mathcal{E}^{1,1}(U^p) \oplus \mathcal{E}^{1,0}(U^{p-1})$$

In the present article, we describe three constructions of complex variations of Hodge
structure, and their relationship with quantum cohomology and Frobenius manifolds. More precisely, in §2 we recall that:

1. Given a variation of graded-polarized mixed Hodge structure $\mathcal{V}$, the Hodge
filtration $F$ of $\mathcal{V}$ pairs with the convolution

$$\Phi = (F^\vee \ast \mathcal{W})$$

of the weight filtration $\mathcal{W}$ of $\mathcal{V}$ and $F^\vee_q = F^{-q}$ to define an unpolarized
$\mathbb{C}$VHS on the underlying $C^\infty$ flat vector bundle of $\mathcal{V}$. 

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(b) Near a maximally unipotent boundary point, the $B$-model variation of Hodge structure attached to a degenerating family of Calabi–Yau threefolds gives rise to a variation of mixed Hodge structure of Hodge–Tate type, whose extension data encodes the quantum cohomology of the mirror.

(c) The Higgs field $\theta$ obtained by application of construction (a) to (b) also determines the quantum product of the mirror, i.e. in this case mirror symmetry can be understood as a duality of CVHS.

Following [D3], in §3 we generalize the construction (b) by showing that given any admissible variation of graded-polarized mixed Hodge structure $V \to \Delta^n$, there exists a canonical filtration $\Psi$ which is opposite to the Hodge filtration $F$ near zero, and hence defines an unpolarized CVHS. Moreover, when the limiting mixed Hodge structure is Hodge–Tate, this opposite filtration coincides with the relative weight filtration of $V$.

The even cohomology of a compact smooth manifold $X$ of dimension $k$ is naturally a Frobenius algebra with respect to the cup product and intersection pairing. For Kähler manifolds, the quantum product provides a deformation of this algebra. These constructions restrict to $V = \oplus_p H^{p,p}(X)$ if $X$ is Calabi-Yau. A Frobenius module of weight $k$ is an abstract version of the module structure obtained when the product operation on $V$ is restricted to a module over $\text{Sym} H^{1,1}(X)$. In §4 we review these notions as well as the equivalence between families of Frobenius modules and variations of Hodge structure with special degenerating behavior (Theorem (4.14)).

In general, the construction of a Frobenius module arising from an algebra as described above implies a loss of information. Yet, in some cases, it is possible to recover the full product structure. In §5 we discuss how this is the case when the weight of the module is at most 5 and, also, when the algebra is “generated by $H^{1,1}$”, a condition that has already been used in the context of quantum cohomology [KM,Kr] as well as in the recent work of C. Hertling and Yu. Manin [HM].

If a family of Frobenius modules generates a family of Frobenius algebras, a natural question to ask is if it is possible to unfold this last family into a Frobenius manifold. We study some cases where families of Frobenius modules can be unfolded into (germs) of Frobenius manifolds. We do this in two ways: an explicit construction is given for low weight families (5.6), and a general argument (Theorem (5.8)) is presented using the techniques of Sections 2 and 3, as well as [HM].

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## 2. Variations of Mixed Hodge Structure

Let $V$ be a finite dimensional vector space. Then, a decreasing filtration of $V$ is an exhaustive sequence of subspaces

$$0 \subseteq \cdots \subseteq F^p \subseteq F^{p-1} \subseteq \cdots \subseteq V$$

of $V$ such that $F^p \subseteq F^{p-1}$ for all $p$. Likewise, an increasing filtration $\Psi$ of $V$ is an exhaustive sequence of subspaces of $V$ such that $\Psi_q \subseteq \Psi_{q+1}$ for each index $q$. As alluded to in the introduction, a decreasing filtration $F$ is then said to be opposite to an increasing filtration $\Psi$ if and only if

$$V = F^p \oplus \Psi_{p-1}$$  \hspace{1cm} (2.1)
for each index $p$. Similarly, given an integer $k$, a pair of decreasing filtrations $F$ and $G$ are said to be $k$-opposed if and only if $F$ is opposite to the increasing filtration $\Psi_q := G^{k-q}$.

**Definition 2.2.** Let $A$ be a subfield of $\mathbb{R}$. Then, a pure $A$–Hodge structure of weight $k$ is pair $(V_A,F)$ consisting of a finite dimensional $A$-vector space $V_A$ and a decreasing Hodge filtration $F$ of $V_\mathbb{C} = V_A \otimes \mathbb{C}$ by complex subspaces such that the conjugate filtration $\bar{F}$ defined by the real structure $V_\mathbb{R} = V_A \otimes \mathbb{R}$ is $k$-opposed to $F$.

By the Hodge decomposition theorem, the primitive $1$st cohomology of a smooth complex projective variety $X$ carries a pure Hodge structure of weight $k$. Accordingly, one defines a polarization of a pure Hodge structure $(V_A, F)$ of weight $k$ to be a $(-1)^k$–symmetric bilinear form $Q : V_A \otimes V_A \to A$ such that

1. $Q(F^p, F^{k-p+1}) = 0$;
2. $i^{p-q}Q(v, \bar{v})$ is positive definite on $F^p \cap \bar{F}^{k-p}$;

for each index $p$.

**Definition 2.3.** Let $S$ be a complex manifold and $A$ be a subfield of $\mathbb{R}$. Then, a variation of pure, polarized $A$–Hodge structure of weight $k$ over $S$ consists of a local system of $V_A$ of finite dimensional $A$-vector spaces over $S$ equipped with a decreasing Hodge filtration $F$ of $V = V_A \otimes \mathcal{O}_S$ by holomorphic subbundles, and a flat $(-1)^k$–symmetric bilinear form $Q : V_A \otimes V_A \to A$ such that

1. $F$ and $\bar{F}$ are $k$-opposed;
2. $F$ is horizontal, i.e. $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega_S^1$;
3. $Q$ polarizes each fiber of $V$.

In particular, by the work of Griffiths [G], each smooth projective morphism $f : X \to S$ of complex algebraic varieties gives rise to a variation of pure $\mathbb{Q}$–Hodge structure of weight $k$ on $V_\mathbb{Q} = R^k_f(\mathbb{Q})$. On the primitive part of $R^k_{f*}(\mathbb{Q})$ the variation is, also, polarized. Dropping the requirement that $f$ be smooth and/or projective, one then obtains the notion of a variation of graded-polarized mixed Hodge structure:

**Definition 2.4** [D2]. Let $A$ be a subfield of $\mathbb{R}$. Then, an $A$–mixed Hodge structure consists of a finite dimensional $A$-vector space equipped with a decreasing Hodge filtration $F$ of $V_\mathbb{C}$ together with an increasing weight filtration $W(V_A)$ of $V_A$ such that $F$ induces a pure $A$–Hodge structure of weight $k$ on each non-trivial quotient $Gr^W_k = W_k/W_{k-1}$ of the complexification $W$ of $W(V_A)$ via the rule

$$F^p Gr^W_k = \frac{F^p \cap W_k + W_{k-1}}{W_{k-1}}$$

Likewise, a graded-polarized mixed Hodge structure is just a mixed Hodge structure endowed with a choice of polarization $Q_k$ for each non-trivial quotient $Gr^W_k$.

**Definition 2.5** [SZ]. Let $S$ be a complex manifold and $A$ be a subfield of $\mathbb{R}$. Then, a variation of graded-polarized $A$–mixed Hodge structure over $S$ consists of a local system $V_A$ of finite dimensional $A$-vector spaces over $S$ equipped with a decreasing Hodge filtration $F$ of $V = V_A \otimes \mathcal{O}_S$ by holomorphic subbundles, and an increasing

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\[1\] In fact, the same is true without restricting to the primitive cohomology (see Chapter 5, Section 6 of [We]).
weight filtration $\mathcal{W}(\mathcal{V}_A)$ of $\mathcal{V}_A$ by flat subbundles, together with a collection of flat, non-degenerate bilinear forms $Q_k$ such that:

1. $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_S$;
2. $(\text{Gr}_W^k(\mathcal{V}_A), F \text{Gr}_k^W, Q_k)$ is a variation of pure polarized $A$-Hodge structure of weight $k$.

Remark. In definitions (2.2)--(2.5) one may replace $A$ by a finitely generated noetherian subring of $\mathbb{R}$ such that $A \otimes \mathbb{Q}$ is a field. However, in this context, one only requires $W$ to be a filtration of $V_A \otimes \mathbb{Q}$.

Let $\mathcal{V}$ be a variation of pure, polarized Hodge structure of weight $k$ and $\bar{\Phi}$ be the increasing filtration of $\mathcal{V}$ defined by the rule

$$\bar{\Phi}_p = \bar{F}^{k-p}$$

Then, the fact that $\bar{\nabla} = \nabla$ and $F$ is holomorphic, horizontal and $k$-opposed to $\bar{F}$ implies that $\bar{\Phi}$ is an anti-holomorphic filtration of $\mathcal{V}$ which is opposite to $F$ such that

$$\nabla(\bar{\Phi}_q) \subseteq \bar{\Phi}_{q+1} \otimes \bar{\Omega}^1$$

i.e. $(F, \bar{\Phi})$ defines an unpolarized $C^\infty$VHS on the underlying flat $C^\infty$ bundle of $\mathcal{V}$. Accordingly [Si], one obtains an associated Higgs bundle structure $\bar{\partial} + \bar{\theta}$ on the underlying $C^\infty$ vector bundle of $\mathcal{V}$ as follows: Let

$$\mathcal{V} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

denote the Hodge decomposition (1.4) defined by the Hodge bundles

$$U^p = \mathcal{H}^{p,q} = F^p \cap \bar{\Phi}_p$$

Then, by virtue of equation (1.5), we can write

$$\nabla = \tau + \bar{\partial} + \partial + \theta$$

where

$$\tau(U^p) \subseteq \mathcal{E}^{0,1}(U^{p+1}), \quad \bar{\partial}(\mathcal{E}^0(U^p)) \subseteq \mathcal{E}^{0,1}(U^p),$$

$$\partial(\mathcal{E}^0(U^p)) \subseteq \mathcal{E}^{1,0}(U^p), \quad \theta(U^p) \subseteq \mathcal{E}^{1,0}(U^{p-1})$$

Moreover, upon expanding out the flatness condition $\nabla^2 = 0$ relative to (2.7) and (2.8), one obtains the Higgs field condition

$$(\bar{\partial} + \theta)^2 = \partial^2 + \bar{\partial}\theta + \theta \wedge \theta = 0$$

More generally, mutatis mutandis, given an unpolarized $C^\infty$VHS, one obtains a corresponding Higgs bundle structure $\bar{\partial} + \theta$ on the underlying $C^\infty$ vector bundle of the variation via the above constructions.

Let $\mathcal{V}$ be a variation of graded-polarized mixed Hodge structure. Then, as consequence of the following result, one obtains a system of Hodge bundles (1.4) on the underlying $C^\infty$ bundle of $\mathcal{V}$ which satisfy the transversality condition (1.5), and hence a corresponding Higgs bundle structure $\bar{\partial} + \theta$. 

Theorem 2.10 [D2,D4]. Let \((F,W)\) be a mixed Hodge structure. Then, there exists a unique, functorial bigrading

\[ V_C = \bigoplus_{p,q} I^{p,q} \]  

(2.11)

of the underlying complex vector space \(V_C\) such that

(a) \(F^p = \bigoplus_{a \geq p} I^{a,b}\) for each index \(p\);
(b) \(W_k = \bigoplus_{a+b \leq k} I^{a,b}\) for each index \(k\);
(c) For each bi-index \((p,q)\),

\[ T_{p,q} = I^{q,p} \mod \bigoplus_{a<q,b<p} I^{a,b} \]  

(2.12)

Moreover,

\[ \bigoplus_{r \leq p} I^{r,s} = \sum_k \tilde{F}^{k-p} \cap W_k \]  

(2.13)

Proof. The existence of a direct sum decomposition (2.11) satisfying conditions (a)-(c) is shown in [D2]. That this decomposition also satisfies (2.13) is shown in [D4] (cf. [P]). Namely, in order to establish (2.13), observe that

\[ \bigoplus_{r \leq p} I^{r,s} = \bigoplus_{r \leq p} \tilde{T}^{r,s} \]  

(2.14)

since, for any bi-index \((r,s)\) with \(r \leq p\), the left hand side of (2.14) contains every bi-index \((r',s') < (r,s)\). Thus, it is sufficient to show that

\[ \bigoplus_{r \leq p} \tilde{T}^{r,s} = \sum_k \tilde{F}^{k-p} \cap W_k \]  

(2.15)

By conditions (a) and (b), for each pair of indices \(p\) and \(k\),

\[ \tilde{F}^{k-p} \cap W_k = \tilde{F}^{k-p} \cap W_k = \bigoplus_{r+s \leq k, s \geq k-p} \tilde{T}^{s,r} \]  

(2.16)

and hence

\[ \tilde{F}^{k-p} \cap W_k \subseteq \bigoplus_{r \leq p} \tilde{T}^{r,s} \]  

(2.17)

since \(r+s \leq k\) and \(s \geq k-p \implies r+s \leq k \leq s+p \implies r \leq p\). Thus, fixing \(p\) and taking the sum of (2.17) over all \(k\) yields:

\[ \sum_k \tilde{F}^{k-p} \cap W_k \subseteq \bigoplus_{r \leq p} \tilde{T}^{r,s} \]

Conversely, by equation (2.16),

\[ \bigoplus_{r \leq p} \tilde{T}^{r,s} \subseteq \sum_k \tilde{F}^{k-p} \cap W_k \]  

(2.18)

Indeed, given any bi-index \((s,r)\) with \(r \leq p\), there exists an index \(k\) such that \(r+s \leq k \leq s+p\) (i.e. \(r+s \leq k\) and \(s \geq k-p\)), and hence \(\tilde{T}^{s,r} \subseteq \tilde{F}^{k-p} \cap W_k\).
Corollary 2.19. Let $\mathcal{V}$ be a variation of mixed Hodge structure, and

$$\mathcal{V} = \bigoplus_{p,q} T^{p,q}$$

(2.20)

denotes the $C^\infty$ decomposition of $\mathcal{V}$ into a sum of $C^\infty$ subbundles defined by the pointwise application of Theorem (2.10). Then, the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ pairs with the increasing filtration

$$\Phi_q = \sum_k \mathcal{F}^{k-q} \cap W_k$$

(2.21)

to define an unpolarized $\mathbb{C}$VHS for which the resulting $C^\infty$ decomposition (1.4) is given by the formula

$$\mathcal{V} = \bigoplus_p \mathcal{U}^p, \quad \mathcal{U}^p = \bigoplus_q T^{p,q}$$

(2.22)

Proof. As in the pure case, the fact that $\mathcal{F}$ is holomorphic and horizontal, implies that $\Phi$ is an anti-holomorphic filtration of $\mathcal{V}$ such that $\nabla(\Phi_p) \subseteq \Phi_{p+1} \otimes \Omega^1$. Moreover, by Theorem (2.10),

$$\mathcal{F}^p = \bigoplus_{a \geq p} T^{a,b}, \quad \Phi_p = \bigoplus_{a \leq p} T^{a,b}$$

(2.23)

Consequently, $\mathcal{V} = \mathcal{F}^p \oplus \Phi_{p-1} = \bigoplus_{p,q} T^{p,q}$ and hence $(\mathcal{F}, \Phi, \nabla)$ satisfy the axioms of an unpolarized CVHS. Likewise, by equation (2.23), $\mathcal{U}^p := \mathcal{F}^p \cap \Phi_p = \bigoplus_q T^{p,q}$

Remark. Given a pair of increasing filtrations $A$ and $B$ of a vector space $V$, one defines the convolution $A * B$ to be the increasing filtration of $V$ given by the rule

$$(A * B)_q = \sum_{r+s=q} A_r \cap B_s = \sum_k A_{q-k} \cap B_k$$

(2.24)

In particular, if for any decreasing filtration $F$ of $V$ one defines $F^r = F^{-r}$ then the increasing filtration (2.21) is given by the formula $\Phi = (\mathcal{F}^* \ast \mathcal{W})$

Following [D8], we now relate the above constructions with mirror symmetry. To this end, let $\mathcal{V} \to \Delta^*$ be a variation of pure, polarized Hodge structure of weight $k$ over a product of punctured disks which has unipotent monodromy, and $T_j = e^{-N_j}$ denote the monodromy of $\mathcal{V}$ about the $j$th punctured disk. Then, as a consequence of Schmid’s orbit theorems [Sc], one obtains an associated limiting mixed Hodge structure $(\mathcal{F}_\infty, \mathcal{W}[-k])$ on the central fiber [D1] of the canonical extension of $\mathcal{V}$, where $\mathcal{W}[-k] = W_{j-k}$ denotes the shifted monodromy weight filtration of $\mathcal{V}$:

Theorem 2.25 [CK]. Let $\mathcal{V} \to \Delta^*$ be a variation of pure, polarized $A$–Hodge structure with unipotent monodromy, and

$$\mathcal{C} = \{ \sum_j a_j N_j \mid a_j > 0 \}$$

denote the corresponding monodromy cone. Then, there exists a unique increasing filtration $\mathcal{W}(\mathcal{V}_A)$ of $\mathcal{V}_A$ by sub-local systems, called the monodromy weight filtration of $\mathcal{V}$, such that for every element $N \in \mathcal{C}$:

1. $N(\mathcal{W}_j) \subseteq \mathcal{W}_{j-2}$;
2. $N^j : Gr_j\mathcal{W} \to Gr_{j-2}\mathcal{W}_j$.
Suppose now that $\mathcal{V} \to \Delta^{*r}$ is a variation of pure Hodge structure for which the corresponding limiting mixed Hodge structure is Hodge–Tate, i.e.

$$I_{p,q}^{p,q}(F_\infty, W[-k]) = 0$$

unless $p = q$. Then, following [D3], $\mathcal{V}$ should be regarded as maximally degenerate.

Moreover, as consequence of the following result, to each such maximally degenerate variation $\mathcal{V}$, one can associate a corresponding Hodge–Tate variation $\mathcal{V}^\circ$:

**Theorem 2.26** [D3]. Let $\mathcal{V} \to \Delta^{*r}$ be a variation of pure polarized Hodge structure of weight $k$ for which the associated limiting mixed Hodge structure is Hodge–Tate. Then, the Hodge filtration $F$ pairs with the shifted monodromy weight filtration $W[-k]$ of $\mathcal{V}$ to define a Hodge–Tate variation $\mathcal{V}^\circ$ over a neighborhood of zero in $\Delta^{*r}$.

**Remark.** By shrinking $\Delta^{*r}$ as necessary, we shall henceforth assume that $(F, W)$ is Hodge–Tate over all of $\Delta^{*r}$.

To apply Theorem (2.26) to the study of mirror symmetry, we now recall the following definition:

**Definition 2.27.** Let $X$ be a polarized Calabi–Yau manifold of dimension $d$, and $\overline{M}_X$ be a smooth partial compactification of the moduli space $M_X$ [B,Ti,To] of complex structures on $X$, such that

$$D = \overline{M}_X - M_X$$

is a normal crossing divisor. Let $p \in D$ be such that, in a neighborhood of $p$, $D = \sum_{j=1}^r D_j$ with $r = \dim M_X = \dim H^{d-1,1}(X) + \{p\} = \bigcap_{j=1}^r D_j$. Then, $p$ is said to be a maximally unipotent boundary point of $M_X$ if and only if the variation of Hodge structure $\mathcal{V}_s = H^d(X_s)$ (2.29) over $M_X$ has maximal unipotent monodromy on a neighborhood of $p$ in $M_X$, i.e. the associated monodromy weight filtration $W$ satisfies the following two conditions:

1. $\dim Gr^{W}_{d} = 1$, $\dim Gr^{W}_{d-1} = 0$, $\dim Gr^{W}_{d-2} = r$;
2. $Gr^{W}_{d-2} = \bigoplus_{j=1}^r N_j(Gr^{W}_{d-2})$.

Let $X$ be a smooth Calabi–Yau threefold, and $p$ be a maximally unipotent boundary point of $M_X$. Then, it can be shown that the limiting mixed Hodge structure of the variation (2.29) at $p$ is Hodge–Tate. Moreover, according to [D3], the resulting variation $\mathcal{V}^\circ$ produced by Theorem (2.26) should be viewed as the $A$-model variation of the mirror $X^\circ$ of $X$.

To extract the quantum cohomology of $X^\circ$ using this approach, we recall [D3] that given a pair of variations of $\mathbb{Z}$–Hodge structure $A$, and $B$ over $S$ of pure type $(p, p)$ and $(p-1, p-1)$ respectively,

$$\text{Ext}^1(A, B) = \text{Hom}(A, B) \otimes O^*(S).$$

(2.30)

More generally, given a $\mathbb{Z}$-variation $\mathcal{V}$ of Hodge–Tate type, let $E_k$ denote the extension class (2.30) attached to the extension

$$0 \to Gr^{W}_{2k-2} \to W_{2k}/W_{2k-4} \to Gr^{W}_{2k} \to 0.$$
Lemma 2.31 [D3]. Let $\mathcal{V} \to \Delta^r$ be a $\mathbb{Z}$-variation of Hodge–Tate type, and $\tilde{\mathcal{V}}$ denote the canonical extension of $\mathcal{V}$ over $\Delta^r$ defined in [D1]. Then, the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ extends holomorphically to $\tilde{\mathcal{V}}$, and remains opposite to the weight filtration $\mathcal{W}$ over $0 \in \Delta^r$ if and only if

$$E_k \in \text{Hom}(Gr^W_{2k}, Gr^W_{2k-2}) \otimes \mathcal{O}_{mer}(\Delta^*)$$

for each index $k$, where $\mathcal{O}_{mer}(\Delta^r)$ denotes the group of invertible holomorphic functions on $\Delta^r$ which extend meromorphically to $\Delta^r$.

Remark. The variation $\mathcal{V}^\circ$ produced by Theorem (2.26) always satisfies the hypothesis of Lemma (2.31).

Thus, given a smooth Calabi–Yau threefold $X$ and a maximal unipotent boundary point $p$ of $\mathcal{M}_X$, one obtains via application of Lemma (2.31) to the resulting Hodge–Tate variation $\mathcal{V}^\circ$, a corresponding set of extension classes $E_3$, $E_2$ and $E_1$ which encode the quantum product of $X^\circ$ as follows: Let $\mathcal{W}$ be the weight filtration of $\mathcal{V}^\circ$. Then, $\mathcal{W}$ is defined over $\mathbb{Z}$ with weight graded quotients:

$$Gr^\mathcal{W}_6(V_2^\circ) \cong \mathbb{Z}, \quad Gr^\mathcal{W}_4(V_2^\circ) \cong \mathbb{Z}^r, \quad Gr^\mathcal{W}_2(V_2^\circ) \cong \mathbb{Z}^r, \quad Gr^\mathcal{W}_0(V_2^\circ) \cong \mathbb{Z}^r$$

where $r = \dim H^2(X)$. Select a generator $1$ of $Gr^W_0(V_2^\circ)$. Then, on account of maximal unipotent monodromy, there exists a unique system of “canonical coordinates” $(q_1, \ldots, q_r)$ on the base of $\mathcal{V}^\circ$ relative to which the extension class $E_3$ assumes the form

$$E_3(1) = \sum_j q_j N_j(1)$$

(2.32)

Moreover, as discussed in [D3], upon expanding the logarithmic derivative of $E_2$ relative to the system of canonical coordinates (2.32), one obtains a generating function for the number of rational curves in the mirror $X^\circ$.

Alternatively, as described in [P], given a maximal unipotent boundary point $p$ as above, both the canonical coordinates (2.32) and the quantum product of the mirror $X^\circ$ can be described in terms of the Higgs field $\theta$ attached to $\mathcal{V}^\circ$ by (2.19) as follows: Let

$$H^{p,p} = Gr^\mathcal{W}_{6-2p}(V_2^\circ),$$

be a generator of $H^{0,0}$, and $T_j = N_j(1)$. Then, $\theta$ induces a map

$$H^{p,p} \to H^{p+1,p+1} \otimes \Omega^1(\Delta^r)$$

such that

(a) $\theta(\xi) 1 = \sum_j T_j \otimes \Omega_j(\xi)$;
(b) $T_a \ast T_b = \theta(\xi_a) \circ \theta(\xi_b) 1$;

where $(\Omega_1, \ldots, \Omega_r)$ denotes the coframe of $\Omega^1(\Delta^r)$ defined by the 1-forms

$$\Omega_j = \frac{1}{2\pi i q_j} dq_j.$$

$T_a \ast T_b$ represents [M,CoK] the quantum product on $H^{1,1}(X^\circ)$, and $\Omega_j(\xi_k) = \delta_{jk}$.

In §3, we shall extend the correspondence $\mathcal{V} \leftrightarrow \mathcal{V}^\circ$ given by Theorem (2.26) to arbitrary variations of graded-polarized mixed Hodge structures $\mathcal{V} \to \Delta^m$ which are admissible in the sense of [SZ] by constructing a suitable opposite filtration $\Psi$ from the limiting mixed Hodge structure of $\mathcal{V}$.
3. Asymptotic Behavior

Let \((E, \nabla)\) be a flat \(\mathbb{C}\)-vector bundle over \(\Delta^* n\) with unipotent monodromy. Then \([D1]\), up to isomorphism, there exists a unique extension \(\tilde{E} \to \Delta^n\) of \(E\) such that \(\nabla\) has at worst simple poles with nilpotent residues along the normal crossing divisor

\[
D = \Delta^n - \Delta^{*n} = \cup_j D_j
\]

Equivalently, given a system of local coordinates \((s_1, \ldots, s_n)\) on \(\Delta^n\) relative to which the divisor (3.1) assumes the form

\[
s_1 \cdots s_n = 0
\]

with \(s_j = 0\) on \(D_j\), and flat multivalued frame \((\sigma_1, \ldots, \sigma_m)\) of \(E\), the canonical extension \(\tilde{E}\) described above can be identified with the locally free sheaf generated by the sections

\[
\tilde{\sigma}_j = e^{\frac{1}{N_j} \sum_i (\log s_i) N_i} \sigma_j
\]

where \(T_j = e^{-N_j}\) denotes the monodromy action of \(E\) about \(s_j = 0\).

In particular, if \(V \to \Delta^{*n}\) is a variation of graded-polarized mixed Hodge structure for which the Hodge filtration \(F\) of \(V\) extends holomorphically to the canonical extension \(\tilde{V}\), one can define a corresponding nilpotent orbit \(F_{\text{nilp}}\) by simply extending \(F(0)\) to a filtration of \(V\) which is constant with respect to the frame (3.2).

Alternatively, \(F_{\text{nilp}}\) may be described in terms of the associated period map

\[
\varphi : \Delta^{*n} \to M/\Gamma
\]

defined by parallel translation of the Hodge filtration \(F\) of \(V\) to a fixed reference fiber \(V = V_{s_0}\) of \(V\) as follows:

Let \(V \to \Delta^{*n}\) be a variation of graded-polarized mixed Hodge structure, and \(V = V_{s_0}\) denote a fixed reference fiber of \(V\). Let \(W\) denote the specialization of the weight filtration of \(V\) to \(V\), and \(Q = \{Q_k\}\) denote the corresponding specialization of the graded-polarizations of \(V\) to \(Gr^W\). Define \(X\) to be the flag variety consisting of all decreasing filtrations \(F\) of \(V\) such that

\[
dim(F^p) = \text{rank}(F^p)
\]

and let \(M\) be the classifying space \([P]\) consisting of all filtrations \(F \in X\) such that \((F, W)\) is a mixed Hodge structure which is graded-polarized by \(Q\). Then, the period map (3.3) takes values in the quotient of \(M\) by the action of the monodromy group \(\Gamma\) of \(V\).

**Theorem 3.4** \([P]\). The classifying space \(M\) defined above is a complex manifold upon which the real Lie group

\[
G = \{ g \in GL(V)^W \mid Gr(g) \in Aut_{\mathbb{R}}(Q) \}
\]

acts transitively by automorphisms, where \(Gr(g) : Gr^W \to Gr^W\) denotes the map induced by an element of the stabilizer \(GL(V)^W\) of \(W\) on \(Gr^W\).
In particular, on account of Theorem (3.4), the orbit
\[ \tilde{M} = G_C.F_o \subseteq X \]  
(3.5)
of a point \( F_o \in \mathcal{M} \) under the complex Lie group
\[ G_C = \left\{ g \in GL(V)^W \mid Gr(g) \in Aut_C(Q) \right\} \]
(3.6)
is well defined, independent of \( F_o \in \mathcal{M} \).

**Warning.** In general, \( G_C \) is not the complexification of \( G \).

To continue, observe that the period map (3.3) is locally liftable. Accordingly, there exists a holomorphic map \( F : U^n \to \mathcal{M} \) which makes the following diagram commute
\[
\begin{array}{c}
\Delta^n \xrightarrow{\varphi} \mathcal{M}/\Gamma \\
p \downarrow \downarrow \\
U^n \xrightarrow{F} \mathcal{M}
\end{array}
\]
(3.7)
where \( U^n \subseteq \mathbb{C}^n \) denotes the product of upper half-planes on which the imaginary parts of the standard coordinates \( (z_1, \ldots, z_n) \) are positive, and \( p : U^n \to \Delta^n \) denotes the standard covering map defined by the coordinates \( s_j = e^{2\pi i z_j} \).

Thus, on account of the commutativity of (3.7), the map
\[
\psi(z) = e^{-\sum_j z_j N_j}.F(z) 
\]
(3.8)
satisfies the periodicity condition
\[
\psi(z_1, \ldots, z_j + 1, \ldots, z_n) = \psi(z_1, \ldots, z_n)
\]
and hence descends to a well defined holomorphic map
\[
\psi(s) : \Delta^n \to \tilde{M} 
\]
(3.9)

**Lemma 3.10.** Let \( \mathcal{V} \to \Delta^n \) be a variation of graded-polarized mixed Hodge structure. Then, the Hodge filtration \( F \) of \( \mathcal{V} \) extends holomorphically to the canonical extension \( \mathcal{V} \to \Delta^n \) if and only if
\[
F_\infty := \lim_{s \to 0} \psi(s) 
\]
(3.11)
extists.

**Proof.** Relative to the trivialization (3.2), \( F \) coincides with the filtration (3.9).

Likewise, after unraveling the above definitions, on finds that the pull back \( p^*(F_{nilp}) \) of \( F_{nilp} \) to \( U^n \) coincides with the nilpotent orbit
\[
F_{nilp} = e^{\sum_j z_j N_j}.F_\infty
\]
(3.12)

**Remark.** To be coordinate free, (3.11) and (3.12) should be viewed as follows: Let \( (s_1, \ldots, s_n) \) be a system of local coordinates on \( \Delta^n \) which are compatible with
the given divisor structure (3.1), and \((\lambda_1, \ldots, \lambda_n)\) be the corresponding system of coordinates on \(T_0(\Delta^n)\) defined by the basis vectors \(e_j = (\frac{\partial}{\partial s_j})_0\). Then, the period map
\[
\varphi_{nilp}(\lambda_1, \ldots, \lambda_n) = e^{2\pi i \sum_j \log(\lambda_j) N_j / F_\infty}
\]
determines a variation of mixed Hodge structure over the complement of the divisor \(\lambda_1 \cdots \lambda_n = 0\), with monodromy action \(T_j = e^{-N_j}\) about \(\lambda_j = 0\), which is well defined, independent of the choice of local coordinates \((s_1, \ldots, s_n)\) as above.

For variations of graded-polarized mixed Hodge structure, the analog of the monodromy weight filtration (2.25) is the relative weight filtration
\[
^rW = ^rW(N, W)
\]
discussed in [SZ]. Moreover, based upon the study of degenerating families of varieties, Steenbrink and Zucker proposed the following, now standard, definition of an admissible variation of graded-polarized mixed Hodge structure over \(\Delta^*\):

**Definition 3.13 [SZ].** Let \(V \to \Delta^*\) be a variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then, \(V\) is admissible if

(a) The limiting Hodge filtration (3.11) exists;

(b) The relative weight filtration \(^rW = ^rW(N, W)\) exists.

The admissibility conditions (3.11) always hold in the pure case as a consequence of Schmid’s nilpotent orbit theorem. For multivariable degenerations, one defines admissibility via curve test using (3.13) [K]. Moreover, one has the following result:

**Theorem 3.14 [K].** Let \(V \to \Delta^{*n}\) be an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy, and
\[
\mathcal{C} = \{ \sum_j a_j N_j | a_j > 0 \}
\]
denote the corresponding monodromy cone. Let \(W\) denote the specialization of the weight filtration of \(V\) to some fixed reference fiber \(V = V_{s_0}\). Then,

1. \(^rW(N, W)\) exists for every element \(N \in \mathcal{C}\);
2. \(^rW = ^rW(N, W)\) is well defined, independent of \(N \in \mathcal{C}\);
3. \((F_\infty, ^rW)\) is a mixed Hodge structure;
4. \(N_1, \ldots, N_n\) are \((-1, -1)\)-morphisms of \((F_\infty, ^rW)\).

Let \(V \to \Delta^{*n}\) be an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy, and
\[
V = \bigoplus_{p, q} I^{p, q}
\]
denote the corresponding decomposition (2.11) defined by the limiting Hodge filtration \(F_\infty\) and the relative weight filtration \(^rW\) of \(V\). Define,
\[
\Psi_p = \bigoplus_{a \leq p} I^{a, b}
\]
and \(g^\Psi = \{ \alpha \in g_C | \alpha(\Psi_p) \subseteq \Psi_{p-1} \}\) denote the subalgebra of \(g_C = Lie(G_C)\) consisting of those elements which preserve the increasing filtration \(\Psi\) defined by (3.16), and act trivially on each layer \(Gr^\Psi_p = \Psi_p / \Psi_{p-1}\) of \(Gr^\Psi\).
Lemma 3.17. $\Psi$ is opposite to $F_\infty$. Moreover, relative to the decomposition

$$g_C = \bigoplus_{r,s} g^{r,s}$$

induced by the bigrading (3.15),

$$g^\Psi = \bigoplus_{r<0} g^{r,s}$$

Proof. That $\Psi$ is opposite to $F_\infty$ is a simple consequence of definition (3.16), and the fact [cf. Theorem (2.10)] that $F^\infty_C = \oplus_{a \geq p} I^{a,b}$. Likewise, since $g^{r,s}(I^{a,b}) \subseteq I^{a+r,b+s}$ the sum $\oplus_{r<0} g^{r,s}$ maps $\Psi_p$ to $\Psi_{p-1}$, and hence is contained in $g^\Psi$. Conversely, by equation (3.16), if $\alpha \in g^\Psi$ then the components $\alpha^{r,s}$ with respect to the decomposition (3.18) can be non-zero only if $r < 0$.

Corollary 3.20. Let $\mathcal{V} \to \Delta^\ast n$ be an admissible variation with unipotent monodromy. Then, on a neighborhood of the origin, the associated function (3.9) admits a unique representation of the form

$$\psi(s) = e^{\Gamma(s)} F_\infty$$

with respect to $g^\Psi$-valued holomorphic function $\Gamma(s)$ which vanishes at the origin.

Proof. By (3.19), $g^\Psi$ is a vector space complement to $\text{Lie}(G^\infty_C)$ in $g_C$. Consequently, the map $u \mapsto e^u.F_\infty$ is a biholomorphism from a neighborhood of zero in $g^\Psi$ onto a neighborhood of $F_\infty$ in $\hat{\mathcal{M}}$. Accordingly, $\psi(s)$ admits a unique representation of the type described above on a neighborhood of zero in $\Delta^n$.

A priori, the filtration $\Psi$ defined above depends on the choice of coordinates used in the construction of the limiting Hodge filtration (3.11). However, as the following result shows, this is in fact not the case, since the monodromy logarithms $N_1, \ldots, N_n$ preserve $\Psi$.

Theorem 3.22. $\Psi$ is independent of the choice of coordinates used in the definition of $F_\infty$. Moreover,

$$\Psi = (F_{nilp}^\vee) \ast (\ast W) = (F^\infty_C) \ast (\ast W)$$

Proof. Without loss of generality, any two systems of local coordinates $(s_1, \ldots, s_n)$ and $(\tilde{s}_1, \ldots, \tilde{s}_n)$ on $\Delta^n$ which are compatible with the divisor structure (3.1) may be assumed to be of the form $\tilde{s}_j = f_j s_j$ for some collection of holomorphic functions $f_1, \ldots, f_n$ which do not vanish at the origin. Moreover, direct calculation shows that, under such changes of coordinates, the corresponding filtrations (3.11) are related by the equation:

$$\tilde{F}_\infty = e^{-\frac{1}{\pi i} \sum_j \log(f_j(0)) N_j} F_\infty$$

(3.24)
Let \( \mathcal{N} = \text{span}_\mathbb{C}(N_1, \ldots, N_n) \). Then, \( \mathcal{N} \subseteq \mathfrak{g}^{-1,-1} \) because each \( N_j \) is a \((-1, -1)\)-morphism of \((F_\infty, r^W)\). Accordingly,

\[
N \in \mathcal{N} \implies \tilde{I}^{a,b}_{(e^N, F_\infty, r^W)} = e^N \tilde{I}^{a,b}_{(F_\infty, r^W)}
\]

and hence

\[
\Psi(\tilde{F}_\infty, r^W) = e^{\tilde{N}} \Psi(F_\infty, r^W)
\]

where

\[
\tilde{N} = -\frac{1}{2\pi i} \sum_j \log(f_j(0))N_j
\]

On the other hand,

\[
e^{\tilde{N}} \Psi(F_\infty, r^W) = \Psi(F_\infty, r^W)
\]

since \( \mathcal{N} \subseteq \mathfrak{g}^{-1,-1} \) and \( \mathfrak{g}^{-1,-1} \subseteq \mathfrak{g}^\Psi \) by Lemma (3.17). Consequently,

\[
\Psi = \Psi(F_\infty, r^W) = e^{\tilde{N}} \Psi(F_\infty, r^W) = \Psi(\tilde{F}_\infty, r^W)
\]

is independent of the choice of coordinates used in the construction of the limiting Hodge filtration (3.11).

To verify (3.23), observe that \( \Psi = (\tilde{F}_\infty) * (r^W) \) by virtue of equations (3.16) and (2.13). To show that

\[
\Psi = (\tilde{F}_{\text{nilp}}) * (r^W)
\]

observe that \( \mathcal{N} \) is closed under complex conjugation since \( \tilde{N}_j = N_j \) for all \( j \). Consequently, when evaluated at any particular point in \( U^n, \tilde{F}_{\text{nilp}} = e^{\tilde{N}} \tilde{F}_\infty \) for some element \( N \in \mathcal{N} \). On the other hand, by definition [SZ], \( \mathcal{N} \) preserves the relative weight filtration \( r^W \). Thus,

\[
(F_{\text{nilp}}) * (r^W) = (e^{\tilde{N}} \tilde{F}_\infty^\vee) * (r^W) = e^{\tilde{N}}(\tilde{F}_\infty^\vee) * (r^W) = e^{\tilde{N}}e^N = \Psi
\]

Finally, as a consequence of the above results, we obtain the following generalization of Theorem (2.26):

**Theorem 3.28.** Let \( \mathcal{V} \to \Delta^{*n} \) be an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then, \( \Psi \) extends to a filtration \( \bar{\Psi} \) of \( \mathcal{V} \) by flat subbundles which pairs with the Hodge filtration \( F \) of \( \mathcal{V} \) to define an unpolarized \( \mathbb{C} \) VHS on a neighborhood of the origin.

**Proof.** Since each \( N_j \) preserves \( \Psi \), and \( \mathcal{V} \) has only local monodromy, \( \Psi \) extends to a filtration \( \bar{\Psi} \) of \( \mathcal{V} \) by flat subbundles. To see that \( \bar{\Psi} \) is opposite to \( \bar{F} \), observe that after pulling everything back to the upper half-plane, we can write

\[
F(z) = e^{\sum_j z_j N_j} \psi(s) = e^{\sum_j z_j N_j} e^{\Gamma(s)} F_\infty
\]

relative to a \( \mathfrak{g}^\Psi \)-valued holomorphic function \( \Gamma(s) \) by virtue of equations (3.8) and (3.21). Moreover, since \( \mathfrak{g}^\Psi \) is nilpotent, we can write

\[
e^{\sum_j z_j N_j e^{\Gamma(s)}} = e^{X(z)}
\]

for some function \( X(z) \) with values in \( \mathfrak{g}^\Psi \). Thus,

\[
V = F_\infty^p \oplus \Psi_{p-1} = e^{X(z)}(F_\infty^p \oplus \Psi_{p-1})
\]

\[
= (e^{X(z)} F_\infty^p) \oplus (e^{X(z)} \Psi_{p-1}) = F^p(z) \oplus \Psi_{p-1}
\]

since \( F(z) = e^{X(z)} F_\infty \) and \( \mathfrak{g}^\Psi \) preserves \( \Psi \).
4. Frobenius Modules

Let $X$ be a compact Kähler manifold. Then the even part of $H^*(X, \mathbb{C})$ is naturally a Frobenius algebra with product given by the cup product and the bilinear pairing arising from the intersection form\(^2\). The quantum cohomology defines a deformation of this ("classical") structure parameterized by the complexified Kähler cone of $X$. In this case, the deformed ("quantum") product is defined in terms of a function known as the Gromov-Witten potential.

In the case where $X$ is also Calabi-Yau, the cohomology decomposes $H^*(X, \mathbb{C}) = \oplus_{p,q} H^{p,q}$ and the quantum product preserves this bigrading. In particular, $\oplus_p H^{p,p}$ inherits a Frobenius algebra structure. We should mention that these spaces also satisfy nondegeneracy and positivity conditions encoded in the Hard Lefschetz Theorem and the Hodge-Riemann relations.

In his analysis of mirror symmetry, D. Morrison constructed a polarized variation of pure Hodge structure using the data described above. This variation is known as the A-model variation ([M], [CoK, Chapter 8]). A close inspection of his construction shows that not all of the algebra structure is used: it is only the $\text{Sym}^2$ pairings arising from the intersection form $\langle \cdot, \cdot \rangle$. Clearly, the fact that $\langle \cdot, \cdot \rangle$ is Frobenius — the right hand side is Kronecker’s $\delta$ — for all $b = 0, \ldots, m$. We also set $\tilde{a} := p$ if and only if $T_a \in V_p$ and assume that the map $\sim : \{0, \ldots, m\} \to \{0, \ldots, 2k\}$ is increasing.

**Definition 4.1.** $(V, \mathcal{B}, e, \ast)$ is a graded $V_2$-Frobenius module of weight $k$ if

1. $e \neq 0$ and $V_0 = \langle e \rangle$.
2. $V$ is a graded $\text{Sym} V_2$-module under $\ast$.
3. For all $v_1, v_2 \in V$ and $w \in V_2$

$$\mathcal{B}(w \ast v_1, v_2) = \mathcal{B}(v_1, w \ast v_2) \quad (4.2)$$

4. $w \ast e = w$ for all $w \in V_2$.

Since $T_0 \in V_0$, it must be a non-zero multiple of $e$ and we assume that an adapted basis satisfies $T_0 = e$. Clearly, the fact that $V$ is a Sym $V_2$-module is equivalent to

$$T_j \ast (T_l \ast T) = T_l \ast (T_j \ast T) \text{ for all } T_j, T_l \in V_2 \text{ and } T \in V. \quad (4.3)$$

We say that $V$ is real if $V$ has a real structure, $V_\mathbb{R}$, compatible with its grading, $\ast$ is real, $e \in V_\mathbb{R}$, and $\mathcal{B}$ is defined over $\mathbb{R}$.

To any real Frobenius module we can associate a Hodge–Tate mixed Hodge structure, split over $\mathbb{R}$, whose canonical bigrading is

$$I_0^{p,p} := V_2(k-\mathbb{p}). \quad (4.4)$$

\[^2\text{The full } H^*(X, \mathbb{C}) \text{ is a } \mathbb{Z}_2\text{-graded Frobenius algebra, as considered by Kontsevich and Manin in [KM].} \]
The multiplication operator $L_w \in \text{End}(V)$, $w \in V_2$, is an infinitesimal automorphism of the bilinear form

$$Q(v_a, v_b) := (-1)^{k + \bar{a}/2} B(v_a, v_b),$$

as well as a $(-1, -1)$-morphism of the associated mixed Hodge structure. We will say that $w \in V_2 \cap V_\R$ polarizes $V$ if the mixed Hodge structure $(I_{\ast \ast}, Q, L_w)$ is polarized [CK]. A real Frobenius module $V$ is said to be polarizable if it contains a polarizing element. Given a polarizing element $w$, the set of polarizing elements is an open cone in $V_2 \cap V_\R$. We can then choose a basis $T_1, \ldots, T_r$ of $V_2 \cap V_\R$ spanning a simplicial cone $C$ contained in the closure of the polarizing cone and with $w \in C$. Such a choice of a basis of $V_2$ will be called a framing of the polarized Frobenius module.

**Example 4.6.** If $X$ is a compact Kähler manifold of dimension $k$, let $V_{2p} := H^{p,p}(X)$, $B_{\text{int}}$ the intersection pairing on $V := \oplus_{p=0}^k V_{2p}$, and $\sim$ the restriction of the cup product to $V$. Then, $(V, B_{\text{int}}, 1, \sim)$ defines a polarizable Frobenius module. The real structure is induced by $H^*(X, \R)$.

Given an adapted basis $\{T_0, \ldots, T_m\}$ of $V$, let $z_0, \ldots, z_m$ be the corresponding linear coordinates on $V$ and set $q_j := \exp(2\pi i z_j)$ for $j = 1, \ldots, r := \dim V_2$. If $U$ is the upper-half plane, we may identify $U^r \cong (V_2 \cap V_\R) \oplus iC$ and view the correspondence

$$\sum_{j=1}^r z_j T_j \in (V_2 \cap V_\R) \oplus iC \mapsto (q_1, \ldots, q_r) \in (\Delta^*)^r.$$

Let $V$ be a framed Frobenius module of weight $k$. Then the action of $V_2$ on $V$ can be recovered from a homogeneous polynomial $\phi_0 \in \C[z_0, \ldots, z_m]$ of degree three, called the classical potential. Indeed, if we let

$$\phi_0(z_0, \ldots, z_m) := \sum_{j=2, 0 \leq \bar{a}, \bar{b} \leq 2k} z_j z_\bar{a} z_\bar{b} \frac{C(\bar{a})}{12} B(T_j * T_{\bar{a}}, T_{\bar{b}}),$$

with

$$C(\bar{a}) := \begin{cases} 2 & \text{if } k = 3 \text{ and } \bar{a} = 2, \\ 3 & \text{if } k \neq 3 \text{ and } \bar{a} = 2 \text{ or } \bar{a} = 2k - 4, \\ 6 & \text{otherwise}, \end{cases}$$

then we recover the $V_2$-action by:

$$T_j * T_a := \sum_{\bar{c} = \bar{a} + 2} \frac{\partial^3 \phi_0}{\partial z_j \partial z_a \partial z_\bar{c}} T_c; \quad j = 1, \ldots, r.$$

**Example 4.7.** In the case of weight $k = 3$, the classical potential is

$$\phi_0(z_0, \ldots, z_m) = \sum_{j=2} z_0 z_j z_\delta(j) + \frac{1}{6} \sum_{j=2, l=2, k=4} B(T_j * T_l, T_{\delta(k)}) z_j z_l z_\delta(k).$$

In the special case where $\dim V_{2p} = 1$ for $p = 0, \ldots, 3$, we obtain $\phi_0(z_0, \ldots, z_3) = z_0 z_1 z_2 + \kappa z_3^3$ for $\kappa = \frac{1}{6} B(T_1 * T_1, T_1)$. This is the case of, for instance, the central part of the cohomology of the quintic threefold in $\P^4$.

Next we consider deformations of a framed Frobenius module induced from a potential. Let $R := \C[q_1, \ldots, q_r]_0$ denote the ring of convergent power series vanishing for $q_1 = \cdots = q_r = 0$ and $R'$ be its image under the map induced by $q_j \mapsto e^{2\pi i z_j}$ for $1 \leq j \leq r$. 
Definition 4.8. Let \((V, \mathcal{B}, e, \ast)\) be a framed Frobenius module of weight \(k > 3\) with classical potential \(\phi_0\). A quantum potential on \(V\) is a function \(\phi : V \to \mathbb{C}\) of the form \(\phi = \phi_0 + \phi_h\), where

\[
\phi_h(z) := \sum_{a=2k-4} z_a \phi_h^a(z_1, \ldots, z_r) + \sum_{2 < a < 2k-4, \tilde{a} + \tilde{b} = 2k-2} z_a z_b \phi_h^{ab}(z_1, \ldots, z_r),
\]

(4.9)

with \(\phi_h^a, \phi_h^{ab} \in R'\) and such that the action

\[
T_j \cdot_q T_a := \sum_{\tilde{c} = \tilde{a} + 2} \frac{\partial^3 \phi}{\partial z_j \partial z_a \partial z_{\tilde{c}(c)}} T_c, \quad \text{with } q = (q_1, \ldots, q_r) \in \Delta^r.
\]

(4.10)

turns \((V, \mathcal{B}, e, \cdot_q)\) into a graded \(V_2\)-Frobenius module for all \(q\).

For weight \(k = 3\) modules, the form of the potential is just \(\phi = \phi_0 + \phi_h\) where \(\phi_h \in R'\), and requiring that (4.10) defines a graded \(V_2\)-module imposes no constraint on \(\phi_h\), contrary to the situation of higher weights.

Remark. For \(V\) of weight 1 or 2, the Frobenius module is determined by \(\mathcal{B}\) and \(e\); hence no deformations in the sense of Definition (4.8) are possible.

Remark 4.11. The form (4.9) of the quantum potential is motivated by the Gromov-Witten potential, except that in the setup of Frobenius modules only a graded part of the potential is relevant.

The condition for a quantum potential to induce a Frobenius module can be interpreted as a WDVV type of equation, only that, compared to the original setup in quantum cohomology, here we only see a graded part of the full system.

Example 4.12. In the weight \(k = 3\) case we have, for \(j = \tilde{l} = 2\),

\[
T_j \cdot_q T_l = T_j \ast T_l + (2\pi i)^3 \sum_{\tilde{a} = 4} \frac{\partial^3 \phi_h(q_1, \ldots, q_r)}{\partial q_j \partial q_l \partial q_{\tilde{a}(a)}} q_j q_l q_{\tilde{a}(a)} T_a,
\]

while the action of \(T_j\) on all the other graded parts of \(V\) is the one given by \(\ast\).

Notice that, since \(\phi_h\) is assumed to be convergent at \(q_1 = \cdots = q_r = 0\), we have \(u_0 = \ast\).

As we stated at the beginning of this section, there is a correspondence between deformations of a framed Frobenius module and a certain type of maximally degenerating polarized variation of Hodge structure. To make precise the notion of maximally degenerating variation we reformulate Definition (2.27) in the context of abstract variations.

Definition 4.13. Given a polarized variation of Hodge structure of weight \(k\) over \((\Delta^*)^r\) whose monodromy is unipotent, we say that \(0 \in \Delta^r\) is a maximally unipotent boundary point if

1. \(\dim I_{k,k}^1 = 1, \dim I_{k-1,k-1}^1 = r\) and \(\dim I_{k,k-1}^1 = \dim I_{k-2,k}^1 = 0\), as well as \(I_{p,q}^1 = 0\) for all \(p, q < 0\), where \(I^{p,q}_{\ast}\) is the bigrading associated to the limiting mixed Hodge structure and,

2. \(\text{span}_{\mathbb{C}} \{N_1(I_{k,k}^1), \ldots, N_r(I_{k,k}^1)\} = I_{k-1,k-1}^1\), where \(N_j\) are the monodromy logarithms of the variation.
**Theorem 4.14.** There is a 1-1 correspondence between

(a) Deformations of a framed Frobenius module \((V, B, e, \ast)\) of weight \(k\) arising from a quantum potential, and

(b) Germs of polarized variations of pure Hodge structure of weight \(k\) on \(V\) degenerating at a maximally unipotent boundary point to a limiting mixed Hodge structure of Hodge-Tate type, split over \(\mathbb{R}\), and together with a marked real point \(e \in F^k\).

The proof of this correspondence can be found in [CF2, Theorem 4.1]. The cases of weight 3 and 4 have been analyzed in [P] and [CF1]. Below we will describe the main constructions that set up the correspondence. The key technical step in proving the theorem is the asymptotic description of the Hodge filtration of an admissible variation presented in §3.

Let \(V \to (\Delta^*)^r\) be a variation of pure, polarized Hodge structure with unipotent monodromy. Then, by Corollary (3.20), the germ of \(V\) at zero is determined by the following data:

1. The nilpotent orbit (3.12);
2. The function \(\psi(s) = e^{\Gamma(s)} \cdot F_\infty\).

Moreover, to such a nilpotent orbit (3.12), we can associate the corresponding limiting mixed Hodge structure \((F_\infty, W^{-k})\), which turns out to be polarized by the monodromy cone (3.14) (cf. [CK]). Furthermore, this polarized mixed Hodge structure is equivalent to the nilpotent orbit [Sc], [CKS], [CF1, Theorem 2.3]. Also, [CF1, Theorem 2.7], on account of the horizontality of the period map of \(V\), the function \(\Gamma(s)\) can be recovered from its projection \(\Gamma_1\) onto the subspace \(\mathfrak{p} := \oplus_s \mathfrak{g}^{-1,s}\) of \([\text{cf. Lemma (3.17)}]\)

\[ \mathfrak{g} := \mathfrak{g}^\Psi = \bigoplus_{r < 0, s} \mathfrak{g}^{r,s} \]

and the monodromy logarithms \(N_1, \ldots, N_r\) of \(V\).

The period mapping of the variation \(V\) is written as \(e^{X(s)} \cdot F_\infty\) for a (multivalued) map \(X : (\Delta^*)^r \to \mathfrak{g}^\mathfrak{p}\), as in (3.29)-(3.30). The horizontality condition of the variation \(V\) can be written, in terms of \(X\), as

\[ e^{-X(s)} \cdot d(e^{X(s)}) = dX_{-1} \quad \text{for} \quad X_{-1} = \sum_{j = 1}^r \frac{\log(s_j)}{2\pi i} N_j + \Gamma_{-1}(s), \quad (4.15) \]

and, in turn, this is equivalent to \(dX_{-1}\) being a Higgs field.

**Remark.** Theorem (4.14) can now be refined to show that under the same correspondence, the nilpotent orbit of the variation of Hodge structure corresponds to the framed Frobenius module and that the function \(\Gamma_{-1}\) corresponds to the quantum potential. Moreover, the transversality condition of the variation is equivalent to the graded part of the WDVV equation alluded to in Remark (4.11).

In what follows we will describe the two constructions that establish the equivalence between degenerating variations of Hodge structure and Frobenius modules.

**Construction 4.16 (Variation of Hodge structure from a Frobenius module).** This is the transcription of the well known \(A\)-model variation of Hodge structure into the language of families of Frobenius modules. See [CF2, §5] for more details as well as [CoK, Chapter 8] for the standard \(A\)-model variation.
Consider a deformation of the framed Frobenius module \((V, B, e, *)\) of weight \(k\) generated by the potential \(\phi = \phi_0 + \phi_\hbar\), with the framing \(\{T_1, \ldots, T_r\}\); let \((q_1, \ldots, q_r)\) be the corresponding system of local coordinates on \((\Delta^*)^r\). Define

1. A free sheaf \(V := V \otimes \mathcal{O}_{(\Delta^*)^r}\).
2. A Hodge filtration \(F^p := (\oplus_{a \geq p} V_{2(k-a)}) \otimes \mathcal{O}_{(\Delta^*)^r} \subseteq V\).
3. The Dubrovin connection: for \(T \in V\) and \(j = 1, \ldots, r\),
   \[\nabla_{\frac{i}{\pi q_j}} T := \frac{1}{2\pi i q_j} T_j \cdot q T.\]

It follows from the symmetry condition (4.3) that \(\nabla\) is flat. Also, it has simple poles at \(q_j = 0\) where the residue \(\text{Res}_{q_j=0}(\nabla)\) is, up to a constant, the endomorphism of \((V, B, e, *)\) given by the action of \(T_j\) on \(V\). The logarithms of the monodromy of \(\nabla, N_j\), are also given by that same action.

4. A polarization \(Q\) given by (4.5).
5. A real form \(V_R\) defined as follows. Let \(\tilde{V} = V \otimes \mathcal{O}_{\Delta^r}\) and \(\tilde{\nabla} = \nabla - \frac{1}{2\pi i} \sum_{j=1}^r N_j \otimes dq_j / q_j\).

Then \(\tilde{\nabla}\) is a flat connection on \(\tilde{V}\); for \(v \in V\) we define \(\tilde{\sigma}_v\) to be the \(\tilde{\nabla}\)-flat section such that \(\tilde{\sigma}_v(0) = v\). Then \(V_R \subseteq V\) is the local system generated by the sections \(\exp(-\frac{1}{2\pi i} \sum_j \log(q_j) N_j)\tilde{\sigma}_v\) for all \(v \in V_R\).

**Theorem 4.17.** The tuple \((V, \nabla, F, V_R, Q)\) constructed above is a polarized variation of Hodge structure of weight \(k\) having a maximally unipotent boundary point at \(0 \in \Delta^r\) where the mixed Hodge structure is of Hodge–Tate type split over \(\mathbb{R}\).

A proof of Theorem (4.17) can be found in [CF2, Theorem 5.10]. See [CoK, Theorem 8.5.11] for a version of this result in the original setup of quantum cohomology of Calabi-Yau manifolds.

**Remark.** The construction described above does not explicitly use the fact that the deformation of the Frobenius module comes from a potential. Nevertheless, the quoted proof does depend on that fact.

**Construction 4.18 (Frobenius module from a variation of Hodge structure).** The starting point is a polarized variation of Hodge structure of weight \(k\) degenerating to a maximally unipotent boundary point where the limiting mixed Hodge structure is of Hodge–Tate type, split over \(\mathbb{R}\), together with a real element \(e \in F^k\).

The construction of a Frobenius module uses only the information of the variation near the degeneration, so that we can restrict all considerations to the case of a variation over \((\Delta^*)^r\) having the origin as a maximal degeneration.

Define:

1. A graded vector space \(V := \bigoplus_{p=0}^k V_{2p}\), with \(V_{2p}\) defined by (4.4).
2. A symmetric nondegenerate pairing \(B\) on \(V\) by (4.5).
3. A real structure \(V_R\) on \(V\) defined by the real structure carried, by hypothesis, by all \(I_p\).
4. A framing \(\{T_1, \ldots, T_r\}\) of \(V\), where \(T_j := N_j(e) \in V_2\) for \(j = 1, \ldots, r\).
5. An action \(*\) of \(V\) on \(V\) by \(T_j * T := N_j(T)\) for \(j = 1, \ldots, r\).
6. A “unit” \(e \in F^k = I^{k,k} = V_0\).
Then, \((V, B, *, e)\) is a \(V_2\) Frobenius module framed by \(\{T_1, \ldots, T_r\}\). This follows from the commutativity of the monodromy logarithms, the fact that these are infinitesimal isometries of \(Q\), and that they polarize the limiting mixed Hodge structure.

To define a deformation of \((V, B, *, e)\), we need special coordinates \((q_1, \ldots, q_r)\) on \(\Delta^r\), known as canonical coordinates. Such a coordinate system is characterized by the fact that the function \(\Gamma_{-1}\) is normalized with respect to these coordinates to satisfy \(\Gamma_{-1}(T^{11}) = 0\). Canonical coordinates exist because of the maximal unipotency condition on the variation (see [CF1, §3]). Finally, for \(j = 1, \ldots, r\) and \(T \in V\), define

\[
T_j \cdot q T := \left( \frac{\partial X_{-1}}{\partial z_j} \right)_q(T)
\]

(4.19)

where the right hand side is the evaluation of the endomorphism of \(V, \frac{\partial X_{-1}}{\partial z_j}\), on the vector \(T\). The fact that this action defines a family of \(\text{Sym} V_2\) modules is equivalent to (4.15). The quantum potential \(\phi_h\) that generates the deformation is:

\[
\phi_h^{ab}(q) := \frac{1}{2} B(\Gamma_{-1}(T_a), T_b) \quad \text{for} \quad 2 < \tilde{a} < 2k - 4 \quad \text{and} \quad \tilde{a} + \tilde{b} = 2k - 2
\]

\[
\phi_h^0(q) := B(-\Gamma_{-2}(T_a), T_0) \quad \text{for} \quad \tilde{a} = 2k - 4
\]

\[
\phi_h := \sum_{\tilde{a} = 2k-4} z_a \phi_h^a + \sum_{2 < \tilde{a} < 2k-4, \tilde{a} + \tilde{b} = 2k-2} z_a z_b \phi_h^{ab}
\]

\[
\phi := \phi_0 + \phi_h,
\]

(4.20)

where \(\Gamma_{-2}\) is determined by \(\Gamma_{-1}\) and the monodromy logarithms.

**Theorem 4.21.** \((V, B, e, *)\) as constructed above is a graded Frobenius module framed by \(\{T_1, \ldots, T_r\}\). Moreover, the \(V_2\) action defined by (4.19) is a deformation of \((V, B, e, *)\) generated by the potential (4.20), whose quantum part is uniquely determined by the function \(\Gamma_{-1}\) of the variation.

**Remark.** In the weight \(k = 1\) and \(k = 2\) cases, the use of canonical coordinates implies the normalization \(\Gamma = 0\). This choice parallels the fact that there are no deformations of Frobenius modules for the same weights.

## 5. Frobenius Algebras

Following Dubrovin, a Frobenius algebra over the finite dimensional \(\mathbb{C}\)-vector space \(V\) consists of a tuple \((V, \hat{\ast}, e)\), where \(\hat{\ast}\) defines a commutative and associative \(\mathbb{C}\)-algebra structure on \(V\) with unit \(e\), and \(B\) is a non-degenerate symmetric bilinear form on \(V\) such that

\[
B(v_1 \hat{\ast} v_2, v_3) = B(v_2, v_1 \hat{\ast} v_3) \quad \text{for all} \quad v_1, v_2, v_3 \in V.
\]

(5.1)

Additionally, \((V, B, \hat{\ast}, e)\) is a graded Frobenius algebra of weight \(k\) if \(V\) decomposes as \(V = \oplus_{p=0}^k V_{2p}\), \(V_0 = \langle e \rangle\), the product \(\hat{\ast}\) is graded and \(B\) pairs \(V_{2p}\) with \(V_{2(k-p)}\) for all \(p\).

If \((V, B, \hat{\ast}, e)\) is a graded Frobenius algebra of weight \(k\) we can, by restricting the action of \(\hat{\ast}\), define a graded \(V_2\)-module \((V, B, *, e)\). In general, this process can not be reversed since part of the product structure is lost. Nevertheless, in some cases a full Frobenius algebra can be constructed from a graded \(V_2\)-module. We will illustrate this phenomenon in the following examples.
Example 5.2. Let \((V, \mathcal{B}, *, e)\) be a graded \(V_2\) Frobenius module of degree \(k = 3\). Then, for homogeneous elements \(v_a, v_b, v_c\) in \(V_2\) and \(V_5\) we define a multiplication \(*\) by:

\[
v_a \ast e = e \ast v_a = v_a; \quad v_b \ast v_a = v_a \ast v_b = v_a \ast v_b = 0\text{ if } \hat{a} > 2\text{ and } \hat{b} > 2.
\]

We notice that \(\ast\) is graded and commutative. Associativity can be seen as follows: if any vector in a triple product is (a multiple of) the identity, associativity is immediate. If all vectors are in \(V_2\), associativity follows from (4.2). Finally, if none of these conditions hold, all triple products vanish by the grading properties. Finally, a short analysis shows that \(\mathcal{B}(v_a \ast v_b, v_c) = \mathcal{B}(v_b, v_a \ast v_c)\).

Similar arguments show that the same result applies in the weight \(k = 4\) and \(k = 5\) cases. The cases of weight \(k = 1\) and \(k = 2\) are trivial since the product structure is determined by \(\mathcal{B}\) and \(e\).

Example 5.3. Let \((V, \mathcal{B}, *, e)\) be a graded \(V_2\) Frobenius module of degree \(k\). We say that \(V\) is generated by \(V_2\) if the linear map \(\text{Sym} V_2 \to V\) defined by \(P \mapsto P \ast e\) is onto. In this case, the algebra structure of \(\text{Sym} V_2\) can be transferred to \(V\). Explicitly, for homogeneous vectors \(v_a = P_a \ast e\) and \(v_b = P_b \ast e\) we have \(v_a \ast v_b = (P_a \ast P_b) \ast e\). Clearly, \(P_a\) is homogeneous, so that \(\ast\) defines a graded multiplication that is commutative, associative and with unit \(e\). Condition (5.1) follows immediately from the iteration of (4.2).

If \((V, \mathcal{B}, \gamma, e)\) is a (continuous) family of Frobenius modules such that \(\gamma_0 = \ast\) generates \(V\) in \(V_2\), then for all \(q\) near \(0\) the corresponding Frobenius module also generates and we obtain a family of graded Frobenius algebras defined over an open neighborhood of \(q = 0\) of the same parameter space.

In quantum cohomology the standard constructions produce the Frobenius manifold structure on \(H^*(X, \mathbb{C})\) whose product is the \textit{big quantum product}. Then, considering \(i : H^2(X, \mathbb{C}) \subset H^*(X, \mathbb{C})\), the product obtained by pulling back to \(i^*(TH^*(X, \mathbb{C})) \cong H^2 \times H^*\) the big quantum product is known as the \textit{small quantum product}.

Starting from a family of graded \(V_2\)-Frobenius modules we have seen that in some cases it is possible to construct a family of Frobenius algebras over an open subset of the same base. In Section 4 we took the parameter space of such a family to be \(\Delta^r\), with \(r = \dim V_2\). We can also consider the covering of the polydisk by the product of upper half planes, \(U^r\), given by \(z \mapsto q = \exp(2\pi iz)\); in this case we pull back our constructions and obtain a family of modules (or algebras) over \(U^r\), that we view as contained in \(V_2\) via the framing. In this case, the family is defined near the point at infinity.

One can pose then the following question: If \(V\) is generated by \(V_2\) under \(\ast\), the construction described in the previous paragraph resembles the small quantum product with a family of Frobenius algebras defined over (an open set contained in) \(V_2\). Is it possible to “unfold” this structure to obtain a full Frobenius manifold analogous to the big quantum product? We will show that the answer to this question is positive in some cases.

Before going into the details, recall that a \textit{Frobenius manifold} \([Du]\) is a complex manifold \(M\) of dimension at least 1 with a commutative and associative multiplication \(\circ\) on the holomorphic tangent bundle \(TM\), a unit field \(e\) and a symmetric, nondegenerate bilinear pairing \(q\) on \(TM\) such that the Levi-Civita connection of \(g\), \(\nabla^q\), is flat, the unit field \(e\) is \(\nabla^q\)-flat and for all vector fields \(X, Y, Z\) on \(M\) the
following conditions hold:

\[ g(X:Y, Z) = g(Y, X:Z) \]  \hspace{1cm} (5.4)

and

\[ \nabla^g_X (Y : Z) - \nabla^g_Y (X : Z) + X \cdot \nabla^g_Y Z - Y \cdot \nabla^g_X Z - [X, Y] : Z = 0. \]  \hspace{1cm} (5.5)

Remark. Frobenius manifolds usually carry a vector field known as the “Euler field”. We will not describe this field in our construction below, but such object can readily be constructed in terms of the grading that is part of a Frobenius module.

Construction 5.6. Let \((V, \mathcal{B}, *, e)\) be a framed Frobenius module of weight \(k\) that is deformed via the potential \(\phi = \phi_0 + \phi_h\) and assume that the module structure of \(*\) can be extended to an algebra that we denote by \(\hat{*}\). As we mentioned above the family can be lifted to a family of modules on \(U^r \subseteq V_2\). An adapted basis \(T_0, \ldots, T_m\) of \(V\) provides coordinates \(z_0, \ldots, z_m\) and the algebra structure \(\hat{*}\) corresponds to \(\text{Im}(z_j) = \infty\) for \(j = 1, \ldots, r\).

The algebra structure \(\hat{*}\) can be encoded into a classical potential \(\hat{\phi}_0\), and we define a multiplication on the tangent bundle \(TV\) by

\[
T_a \hat{*} T_b := \sum_c \frac{\partial^3 (\hat{\phi}_0 + \hat{\phi}_h)}{\partial z_a \partial z_b \partial z_c} T_c = T_a \hat{\ast} T_b + \sum_c \frac{\partial^3 \hat{\phi}_h}{\partial z_a \partial z_b \partial z_c} T_c, \hspace{1cm} (5.7)
\]

for all \(\sum_a z_a T_a \in V\). We remark that this definition does not, in general, define a graded product.

We also consider the (constant) metric on \(TV\) induced by \(\mathcal{B}\) and its Levi-Civita connection \(\nabla\) characterized by \(\nabla T = 0\) for all \(T \in V\) regarded as a constant vector field. In particular, the unit \(e \in V_0\) extends to a flat vector field that we keep denoting by \(e\).

To see if the previous data makes \(TV\) into a Frobenius manifold we have to check the following

(1) Compatibility between \(\hat{*}\) and \(\mathcal{B}\) (5.4):

\[
\mathcal{B}(T_a \hat{*} T_b, T_c) = \mathcal{B}(T_a \hat{\ast} T_b + \sum_d \frac{\partial^3 \hat{\phi}_h}{\partial z_a \partial z_b \partial z_d} T_d, T_c)
\]

\[
= \mathcal{B}(T_a \hat{\ast} T_b, T_c) + \sum_d \frac{\partial^3 \hat{\phi}_h}{\partial z_a \partial z_b \partial z_d} \mathcal{B}(T_d, T_c)
\]

\[
= \mathcal{B}(T_a \hat{\ast} T_b, T_c) + \sum_d \frac{\partial^3 \phi_h}{\partial z_a \partial z_b \partial z_d},
\]

from where (5.4) follows by applying (5.1) to \(\hat{*}\) and by the symmetry of the second summand.

(2) The potentiality condition (5.5):

\[
\nabla \frac{\partial \hat{\phi}_h}{\partial z_a}(T_b \hat{*} T_c) = \nabla \frac{\partial \hat{\phi}_h}{\partial z_a}(T_b \hat{\ast} T_c + \sum_d \frac{\partial^3 \phi_h}{\partial z_a \partial z_b \partial z_d} T_d)
\]

\[
= \nabla \frac{\partial \hat{\phi}_h}{\partial z_a}(T_b \hat{\ast} T_c) + \nabla \frac{\partial \hat{\phi}_h}{\partial z_a}(\sum_d \frac{\partial^3 \phi_h}{\partial z_a \partial z_b \partial z_d} T_d)
\]

\[
= \sum_d \frac{\partial^4 \phi_h}{\partial z_a \partial z_b \partial z_c \partial z_d} T_d.
\]
By the linearity of (5.5) it suffices to check it for \( X = T_a, Y = T_b \) and \( Z = T_c \). The result then follows from the symmetry of the last expression.

(3) Commutativity, associativity and unit: commutativity is clear from formula (5.7). That \( e \) is a unit follows from \( e \) being a unit for \( \ast \) and that there is no dependence on \( z_0 \) in \( \phi_h \). The associativity of \( \cdot \), will be discussed below.

We will only check the associativity in the cases of weight 3, 4 and 5. The fact that \( \cdot \) is not graded makes the computations hard, and this is only partially eased by the weaker property \( V_{2p} \pi z V_{2q} \subseteq \oplus_{a \geq (p+q)} V_{2a} \).

Weight \( k = 3 \). In this case \( \cdot \) is graded and can be computed from

\[
T_a \cdot z T_b = \begin{cases} 
T_a \cdot z T_b & \text{if } \hat{a} = \hat{b} = 2, \\
T_a \ast T_b & \text{otherwise},
\end{cases}
\]

where \( z' = \pi_2(z) \) is the projection of \( \sum z_aT_a \) on \( V_2 \). Associativity follows immediately.

Weight \( k = 4 \). The more involved of the triple products is that of \( T_a \cdot z(T_b \cdot z T_c) \) with \( \hat{a} = \hat{b} = \hat{c} = 2 \). We proceed as follows. Remember that the quantum potential for weight 4 is \( \phi_h = \sum_{\bar{a}=4} z_{\bar{a}}\phi_{\bar{a}}'(z_1, \ldots, z_r) \). Then:

\[
T_a \cdot z(T_b \cdot z T_c) = T_a \cdot z(T_b \ast T_c) + \sum_{d=4} \frac{\partial^3 \phi_h}{\partial z_b \partial z_c \partial z_{d(4)}} T_d + \sum_{d=6} \frac{\partial^3 \phi_h}{\partial z_a \partial z_c \partial z_{d(4)}} T_d
\]

\[
= T_a \cdot z(T_b \cdot z' T_c) + \sum_{d=6} \frac{\partial^3 \phi_h}{\partial z_b \partial z_c \partial z_{d(6)}} T_d
\]

\[
= T_a \cdot z(T_b \cdot z' T_c) + \sum_{d=6} \frac{\partial^3 \phi_h}{\partial z_b \partial z_c \partial z_{d(6)}} T_d
\]

\[
= T_a \cdot z'(T_b \cdot z' T_c) + \frac{\partial^3 \phi_h}{\partial z_b \partial z_c \partial z_{d(6)}} T_{d(0)}
\]

where, as before, \( z' = \pi_2(z) \) denotes the projection of \( z \) to \( V_2 \). Then, using the commutativity of \( \cdot \) we have

\[
(T_a \cdot z T_b) \cdot z T_c = T_c \cdot z(T_a \cdot z T_b) = T_c \cdot z' (T_a \cdot z' T_b) + \frac{\partial^3 \phi_h}{\partial z_a \partial z_b \partial z_c} T_{d(0)},
\]

and, since \( \cdot' \) defines a family of Frobenius modules, relation (4.3) implies the associativity of \( \cdot' \).

The weight \( k = 5 \) case is similar to the previous one, with more painful computations.

Remark. All together, the construction described in (5.6) provides unfoldings of the original algebra structure defined on \( V_2 \times V \) to a full Frobenius manifold on \( TV \) for weights \( k = 3, 4, 5 \). We note that these results do not follow from the application of [HM, Theorem 4.5] because in our case we are working with germs at infinity or, in terms of families defined over \( \Delta^r \) our Higgs fields have logarithmic singularities. Perhaps, Construction (5.6) should be regarded as evidence for an extended version of [HM, Theorem 4.5].
It seems unlikely that associativity—hence this explicit construction—can be extended to weights \( k \geq 6 \). The reason for that is that the homogeneity properties of a potential defining the product on a Frobenius manifold imply that, for weight \( k \leq 5 \) all potentials have the form (4.9), whereas for \( k \geq 6 \) new homogeneous terms can be present.

Another connection between families of Frobenius modules generated in \( V_2 \) and Frobenius manifolds can be established as follows.

**Theorem 5.8.** Consider a deformation of the framed Frobenius module \((V, B, *, e)\) induced by a quantum potential defined over a neighborhood of \( 0 \in \Delta^* \) \((r = \dim V_2)\) and such that \( V \) is generated by \( V_2 \) under \(*\). Then, for each \( \hat{s} \in (\Delta^*)^r \) near 0 the family of Frobenius modules generates a germ of a family of algebras at \( \hat{s} \). Moreover, this germ can be unfolded to a germ of a Frobenius manifold.

**Proof.** By Theorem (4.14) the deformation of \((V, B, *, e)\) generates a polarized variation of Hodge structure \((\mathcal{V}, \nabla, \mathcal{F}, \mathcal{V}_R, \mathcal{Q})\) over a neighborhood of \( 0 \in \Delta^* \), with a marked element \( e \in F^k \). The limiting mixed Hodge structure of this variation is given by (4.4). As was observed in (3.13), this variation defines an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy.

As in (3.16) let \( \Psi_p = \oplus_{a,p,b} I^{a, b} = \oplus_{a,p} V_{2(k-a)} \). Then \( \Psi_p \) is \( \mathfrak{g}_-\)-invariant and opposite to \( F_\infty \). Moreover, by Theorem (3.28) \( \Psi_p \) extends to a \( \nabla \)-flat increasing filtration \( \Psi_p \) of \( \mathcal{V} \) that pairs with \( F^p \) to define an unpolarized CVHS in a neighborhood of the origin. In particular, by (1.3), \( F^p \) and \( \Psi_p \) are opposite.

By parallel transporting the bundles and additional structure to a fixed fiber of \( \mathcal{V} \) we write

\[
F^p(s) = e^{X(s)} \cdot F^p_\infty \quad \text{with} \quad F^p_\infty = \oplus_{a \geq p} V_{2(k-p)}.
\]

Recalling that \( \Psi_p \) is \( \mathfrak{g}_-\)-invariant we can find the \( C^\infty \) decomposition (1.4) of the complex variation as follows:

\[
U^p(s) = F^p(s) \cap \Psi_p(s) = (e^{X(s)} \cdot F^p_\infty) \cap \Psi_p = e^{X(s)} \cdot (F^p_\infty \cap \Psi_p)
\]

\[
= e^{X(s)} \cdot V_{2(k-p)}.
\]

Then, using (1.5), the Higgs field \( \theta \) of the \( \mathbb{C} \)VHS is

\[
\theta(s) = \exp(X(s)) dX_{-1} \exp(-X(s)). \quad (5.9)
\]

We notice that since \( X(s) \) is an infinitesimal automorphism of \( Q \),

\[
Q(F^p, F^{k-p+1}) = Q(F^p_\infty, F^{k-p+1}_\infty) = 0
\]

and, by the \( \mathfrak{g}_-\)-invariance of \( \Psi_p \),

\[
Q(\Psi_p(s), \Psi_{e-p-1}(s)) = Q(\Psi_p, \Psi_{k-p-1}) = 0.
\]

From (5.9) it follows that, for a fixed \( \hat{s} \), the \( \mathbb{C} \)-span of the iterated application of \( \theta_Y \) for \( Y \in T_{\hat{s}}(\Delta^*)^r \) is conjugate to that of \((dX_{-1})_Y\). But the span for these last operators as \( \hat{s} \to 0 \) is the span of the iterated action of the monodromy logarithms which act as \( V_2 \) acts on \( V \) via \(*\) as was remarked in (3) of Construction (4.16).
Then, under the hypothesis that $V$ is generated by $V_2$ under $\ast$, we conclude that the same condition holds for the Higgs field $\theta(\hat{s})$, for $\hat{s}$ near 0.

Next we use the machinery of [HM, §5] to construct a Frobenius manifold. We claim that $(((\Delta^\ast)^r, \hat{s}), V, \nabla, \mathcal{F})$ is a germ of an $H^2$-generated variation of filtrations of weight $k$, in the language of [HM, Definition 5.3]. The only thing that remains to be proved is that the Higgs field $C : \mathcal{F}^p / \mathcal{F}^{p+1} \to \mathcal{F}^{p-1} / \mathcal{F}^p$ induced by $\nabla$ satisfies the generation condition. But, under the isomorphism introduced by parallel transport to a fixed fiber of $V$ and the canonical isomorphism $\mathcal{F}^p(s) / \mathcal{F}^{p+1}(s) \simeq U^p(s)$ we see that $C$ is, in fact, isomorphic to $\theta$ and, since this last field satisfies the generation condition, so does $C$, and the claim is proved.

But then, the $H^2$-generated variation of filtrations of weight $k$ defined above, together with the pairing $\mathcal{Q}$, the opposite filtration $\Psi_p$, and the “fixed generator” $e$ satisfy the conditions of [HM, Theorem 5.6]. Hence, we conclude that there is a unique unfolding of this structure to a germ of a Frobenius manifold.

**Remark.** The filtration $\Psi$ constructed in Theorem (5.8) is related to the relative weight filtration $rW$ by the rule $\Psi_p = rW_p^2$ since the associated limiting mixed Hodge structure is Hodge–Tate.

In view of Theorem (4.14) there is the following immediate corollary to Theorem (5.8).

**Corollary 5.10.** Let $\mathcal{V} \to (\Delta^\ast)^r$ be a variation of Hodge structure of weight $k$ which satisfies the hypothesis of Theorem (4.14), and assume that the iterated action of the monodromy cone of $\mathcal{V}$ on $e \in F^k_\infty \subseteq F_\infty^0$ spans $F^0_\infty$. Then, for each $\hat{s} \in \Delta^\ast$ sufficiently close to the origin, there exists a corresponding germ of a Frobenius manifold $M_{\hat{s}}$, which is completely determined by the asymptotic behavior of $\mathcal{V}$.

**Remark.** Because of (5.9) we see that there is a simple connection between the Higgs field $\theta$ and the one that appears in the correspondence described in Section 4, namely, $dX_{-1}$. In fact,

$$\theta = \exp(X) dX_{-1} \exp(-X) \equiv dX_{-1} \mod F^{-2}g.$$

Alternatively, $\theta$ and $dX_{-1}$ define isomorphic Higgs bundles.

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