The continuum limit of $sl(N/K)$ integrable super spin chains.

H. Saleur

Department of Physics
University of Southern California
Los Angeles, CA 90089-0484

I discuss in this paper the continuum limit of integrable spin chains based on the superalgebras $sl(N/K)$. The general conclusion is that, with the full "supersymmetry", none of these models is relativistic. When the supersymmetry is broken by the generator of the sub $u(1)$, Gross Neveu models of various types are obtained. For instance, in the case of $sl(N/K)$ with a typical fermionic representation on every site, the continuum limit is the GN model with $N$ colors and $K$ flavors. In the case of $sl(N/1)$ and atypical representations of spin $j$, a close cousin of the GN model with $N$ colors and $j$ flavors with flavor anisotropy is obtained. The Dynkin parameter associated with the fermionic root, while providing solutions of the Yang Baxter equation with a continuous parameter, thus does not give rise to any new physics in the field theory limit.

These features generalize to the case where an impurity is embedded in the system.
1. Introduction

The continuum (field theory) limit of quantum spin chains with ordinary symmetries, whether integrable or not, is generally well understood, and described by field theories whose symmetries closely match the ones of the underlying lattice model. The most striking example of this phenomenon is furnished by the case of $sl(N)$ integrable spin chains, whose continuum limits are $SU(N)$ Wess Zumino models, with a level that depends on the representation used to build up the chain [1]. Quantum group deformations of the lattice symmetries are also known to give rise smoothly to similar deformations in the field theory and the associated scattering matrices.

In contrast, the continuum limit of spin chains based on superalgebras is rather poorly understood. This is unfortunate, since the question is related to physical problems of the highest interest, in particular in the context of disordered systems [2], [3], [4], and maybe of $N = 2$ supersymmetry [5]. One thing that seems clear, is that, in the integrable case, none of these continuum limits have to do with the corresponding Wess Zumino models on supergroups: this is only expected, since these WZW models can present very pathological non unitary properties [6], [7]: if not, what, then, are these continuum limits? The same question arises after quantum group deformation. Here, some preliminary results have indicated a very rich structure: it was indeed shown in [8] that the continuum limit of a particular $osp(2/2)_q$ model [9] coincided with the continuum limit of the well known Bukhvostov Lipatov model [10], providing the first example of an integrable “double sine-Gordon” model [11], [12]. The case of $sl(N/K)_q$ models based on fundamental representations has also been studied in some details [13], enough to show that they have a relativistic limit, but far from providing a complete identification of the latter. It is thus pretty clear that a bunch of solvable field theories of the highest interest are lurking behind super spin chains, and this paper is a first step at clarifying the situation.

Putting the question of field theory aside, integrable lattice models based on superalgebras have a rather long history, starting with the t-J model, which corresponds to $sl(2/1)$ and the fundamental representation. Generalized t-J models, based on $sl(N/1)$ and still the fundamental representation have also been studied in the context of strongly interacting electrons, and - although maybe the algebraic origin of the models was not so clear - in the study of quantum impurity problems with the degenerate Anderson model. Following developments in superconductivity, models based on more complex algebras or representations have been considered: for instance, the model based on the algebra $sl(2/1)$
and typical four dimensional representations was introduced in [14], while the model based on \( sl(2/2) \) and the fundamental (which is also typical) was introduced in [15].

The paper is organized as follows. In sections 2, 3 and 4, I introduce the general Bethe ansatz for chains based on typical fermionic representations. The ground state and physical excitations are studied in section 6, while in section 7, I determine the S matrix and the mapping onto the Gross-Neveu model. Sections 7 and 8 are devoted to special cases, in particular atypical fermionic representations and “bosonic” representations. The whole study is extended in section 9 to the case of chains with impurities. A few conclusions are gathered in section 10.

Before starting, I would like to stress that, although the continuum limit of these models has not been systematically studied before, the following has some overlap with known partial results from [16] and [17].

2. The Bethe equations

Recall that \( sl(N/K) \) has \( N + K - 1 \) Cartan generators, the first \( N - 1 \) belonging to \( sl(N) \), the last \( K - 1 \) to \( sl(K) \), the special generator \( H_N \) being associated with the odd root. The Dynkin diagram decomposes into \( sl(N) \) and \( sl(K) \) parts, connected by the odd root:

I wish to consider first integrable hamiltonians with a fermionic representation of \( sl(N/K) \) on every site. This is a priori the most interesting case, since these representations exhibit, in the typical case, a continuous parameter - a feature that is absent in models based on ordinary algebras. One might hope that this parameter describes some interesting new physics - maybe giving rise to a multiparameter integrable quantum field theory. As we will see shortly, the detailed study of the exact solution does not support that expectation unfortunately.

The Dynkin parameters of what I call (a bit incorrectly) fermionic representations are \((0, \ldots, 0, t, 0, \ldots, 0)\), \( t \) being a real number, which I assume positive in what follows: the case \( t < 0 \) would follow simply by exchanging \( N \) and \( K \). For \( t \) generic, this representation
is typical, with dimension $2^{NK}$, and vanishing super dimension. Atypical cases correspond to $t$ integer, $-(K-1) \leq t \leq N-1$.

I call the roots of the Bethe equations $\mu_{N-1}, \ldots, \mu_1, \mu_0, \lambda_1, \ldots, \lambda_{K-1}$, and introduce the function

$$ e_t(\nu) = \frac{\nu + it/2}{\nu - it/2}, \quad (2.1) $$

The Bethe equations read then

$$
1 = \prod e_2(\mu_{N-1} - \mu'_{N-1})e_{-1}(\mu_{N-1} - \mu_{N-2}) \\
1 = \prod e_{-1}(\mu_p - \mu_{p-1})e_2(\mu_p - \mu'_p)e_{-1}(\mu_p - \mu_{p+1}), \ p = N-2, \ldots, 1 \\
e_t^L(\mu_0) = \prod e_1(\mu_0 - \mu_1)e_{-1}(\mu_0 - \lambda_1) \\
1 = \prod e_{-1}(\lambda_1 - \lambda_2)e_2(\lambda_1 - \lambda'_1)e_{-1}(\lambda_1 - \mu_0) \\
1 = \prod e_{-1}(\lambda_p - \lambda_{p-1})e_2(\lambda_p - \lambda'_p)e_{-1}(\lambda_p - \lambda_{p+1}), \ p = 2, \ldots, K-2 \\
1 = \prod e_{-1}(\lambda_{K-1} - \lambda_{K-2})e_2(\lambda_{K-1} - \lambda'_{K-1}), \quad (2.2)
$$

where as usual \((13)\), the pattern of $e$ labels reproduces the Cartan matrix: the two salient features are the absence of $\mu_0, \mu_0$ coupling, and the opposite couplings of $\mu_0$ to $\mu_1$ and $\lambda_1$ respectively. The notation is obvious but implicit. For instance in the first equation the product is taken over all Bethe roots $\mu_{N-2}$ and all Bethe roots $\mu'_{N-1}$ different from $\mu_{N-1}$.

To get some intuition about these equations, one can think of the dimension of the typical representation $2^{NK}$ as the number of possible ways of putting fermions with $N$ colors and $K$ flavors on a given site of the chain. For instance in the case of $sl(2/1)$, the four states can be interpreted as empty, one fermion with spin up or down, and finally a pair of fermions. The parameter $N_{\mu_0}$ can be interpreted as the number of fermions; the numbers of fermions with a given color are then given by $N_{\mu_0} - N_{\mu_1}, N_{\mu_1} - N_{\mu_2}, \ldots, N_{\mu_{N-2}} - N_{\mu_{N-1}}, N_{\mu_{N-1}}$, and the numbers of fermions with a given flavor by $N_{\mu_0} - N_{\lambda_1}, N_{\lambda_1} - N_{\lambda_2}, \ldots, N_{\lambda_{K-2}} - N_{\lambda_{K-1}}, N_{\lambda_{K-1}}$. Dynkin parameters can easily be deduced from this and the knowledge of the Cartan generators in the typical representation.

The energy takes the form

$$ E = \epsilon \sum_{\mu_0} \frac{t}{\mu_0^2 + \frac{\epsilon}{4}} + A \sum_{\mu_0} 1, \quad (2.3) $$

where I have put a chemical potential for the number of fermionic Bethe roots, and $\epsilon = \pm 1$. Explicit expressions for the hamiltonians themselves can be found in the references below.
for some special cases; they are, of course, quite intricate, except for the simplest values of $N$ and $K$.

I am not aware of a general derivation of equations (2.2), even though it is presumably possible using the general techniques developed in [19], and the form is very natural from algebraic considerations [18]. A number of particular cases have already been studied however; besides $sl(2/1)$ [20] and $sl(2/2)$, recall that the fermionic representations we are considering can become atypical for special values of the parameter $t$. In the case of $sl(N/1)$, the value $t = 1$ corresponds in fact to the fundamental representation, and the model we are interested in coincides then with the $su(N)$ t-J model which was extensively studied by Schlottmann [21]. There also exists by now a huge literature of quantum deformation of super groups and various considerations about graded inverse scattering method, with motivations ranging from properties of electronic materials to knot theory.

The solutions of the Bethe equations are as follows. Consider first the fermionic Bethe roots $\mu_0$. Because there is no $(\mu_0, \mu_0)$ coupling on the right hand side of the Bethe equations (the corresponding element of the Cartan matrix vanishes), the usual string solutions, well known for ordinary algebras, are not possible. However because the coupling between $\mu_0$ and $\mu_1$ has a sign opposite to the one for ordinary algebras, it is possible to compensate for the growth or decay of the left hand side of the Bethe equations when $\mu_0$ has an imaginary part by having complexes of “strings over strings”. Such complexes were probably first introduced by Takahashi [22] in his study of one dimensional fermions interacting with an attractive delta function potential; they have been widely used since, in particular by Schlottmann in his study of models based on the fundamental of $sl(N/1)$. They are of the type

$$
\begin{align*}
\mu_0 &= \mu_0^{p-1} + (- (p-1)i/2, \ldots, (p-1)i/2) \\
\mu_q &= \mu_0^{p-1} + (- (p-1 - q)i/2, \ldots, (p-1 - q)i/2), \quad q = 1, \ldots, p-2 \\
\mu_{p-1} &= \mu_{p-1}^{p-1},
\end{align*}
$$

for $p = 1, \ldots, N$. Hence, for $\mu_0$ there are string solutions of length smaller or equal to $N$. Of course the patterns (2.4) are obeyed only in the large $L$ limit. For finite $L$ the solutions of the Bethe equations differ from (2.4) by exponentially small amounts. One has to use and eliminate these small deviations to rewrite Bethe equations involving the complexes. As for the $\mu_p$, $p > 0$ roots that are not involved in such complexes and the $\lambda_p$ roots, they are determined by the same arguments as for $sl(N)$ and $sl(K)$ respectively ie they can be strings of any possible length. Observe that the role of $sl(N)$ and $sl(K)$ are exchanged if $t$ is negative; once again, in the following, I shall assume that $t > 0$. 

4
By taking the logarithm of the Bethe equations and differentiating we get a system of integral equations. Let us introduce notations for densities. I call $\rho_p$ ($p = 1, N$) the density per unit length of real centers of $\mu_0$ strings of length $p$ (ie the density of $\mu_0^{p-1}$ in (2.4)). The density of real centers of $l$ strings of $\mu_p$ roots that are not in one of the complexes (2.4) we call $\sigma_p^{(l)}$. The density of real centers of $l$ strings of $\lambda_p$ solutions we call $\tau_p^{(l)}$. I will usually reserve the labels $p, q$ for the colors of roots and $l, m$ for the types of strings solutions. We also use the labels $p, q$ for the complexes of strings over strings because they behave in many ways like new roots colors.

I define the Fourier transform as

$$\hat{f}(x) = \int dv e^{ivx} f(v), \quad f(v) = \frac{1}{2\pi} \int dx e^{-ivx} \hat{f}(x), \quad (2.5)$$

and introduce the following notation

$$a_t(v) = \frac{i}{2\pi} d\nu \ln [e_t(v)] = \frac{1}{2\pi} \frac{t}{\nu^2 + \frac{t^2}{4}}, \quad (2.6)$$

with

$$\hat{a}_t(x) = e^{-t|x|/2}. \quad (2.7)$$

I also define for $r, s$ integers (all these notations are rather standard)

$$G_{rs} = a_{r+s-2} + a_{r+s-4} + \ldots a_{r-s}, \quad r \geq s; \quad G_{rs} = G_{sr}, \quad r \leq s, \quad (2.8)$$

(with $a_0 = 0$) and

$$A_{rs} = (G_{rs} + \delta_{rs}) * (1 + a_2). \quad (2.9)$$

where $*$ denotes convolution. Their Fourier transforms are

$$\hat{G}_{rs} = \frac{\sinh(xs/2)}{\sinh(x/2)} e^{-(r-1)|x|/2} - \delta_{rs}, \quad r \geq s, \quad (2.10)$$

and

$$\hat{A}_{rs} = \frac{2 \cosh(x/2)}{\sinh(x/2)} \sinh(xs/2) e^{-r|x|/2}, \quad r \geq s, \quad (2.11)$$

I also introduce the kernel

$$s(\nu) = \frac{\pi/2}{\cosh \pi \nu}, \quad \hat{s}(x) = \frac{1}{2 \cosh x/2}, \quad (2.12)$$

and define

$$a_{rs} = s * A_{rs}, \quad (2.13)$$
with
\[ \hat{a}_{rs}(x) = \frac{\sinh(xs/2)}{\sinh(x/2)} e^{-r|x|/2}, \quad r \geq s. \] (2.14)

I can now write the continuum version of the Bethe equations. Introducing the symbol \( G_{t+1,p} \) which is defined by a formula similar to (2.8) even when \( t \) is not integer (which is usually the case)
\[ G_{t+1,p} = a_{t+p-1} + a_{t+p-3} + \ldots a_{t-p+1}, \] (2.15)
(so, for instance, \( G_{2,p} = a_p \)) we have, for the fermionic root, a set of \( N \) equations, one for each string
\[ G_{t+1,p} = \rho_p + \tilde{\rho}_p + \sum_{q=1}^{N} G_{pq} \ast \rho_q + \sum_{l \geq 1} a_l \ast \sigma_p^{(l)} - \sum_{l \geq 1} a_{pl} \ast \tau_1^{(l)}, \quad p = 1, \ldots, N. \] (2.16)

For \( p = N \) recall that there is no density \( \sigma_N^{(l)} \) so the corresponding term has to be suppressed from the equation. For the \( N - 1 \) roots of \( sl(N) \) we have an infinity of equations, one for each type of string
\[ a_l \ast \rho_p = \tilde{\rho}_p + \beta_p + \sum_{m \geq 1} A_{lm} \ast \sum_{q=1}^{N-1} C_{pq} \ast \sigma_q^{(m)}, \quad p = 1, \ldots, N - 1, \quad l \geq 1 \] (2.17)
where
\[ C_{pq}(\nu) = \delta(\nu)\delta_{pq} - s(\nu)\delta_{p,q-1} - s(\nu)\delta_{p,q+1}. \] (2.18)
Finally for the \( K - 1 \) roots of \( sl(K) \) roots we have
\[ \delta_{p1} \sum_{q=1}^{N} a_{1q} \ast \rho_q = \tilde{\tau}_p^{(l)} + \sum_{m \geq 1} A_{lm} \ast \sum_{q=1}^{K-1} C_{pq} \ast \tau_q^{(m)}, \quad p = 1, \ldots, K - 1. \] (2.19)

3. The thermodynamic equations

We now write the thermodynamic equations. The energy term is
\[ E = \sum_{p=1}^{N} \int (\epsilon G_{t+1,p} + pA) \rho_p, \] (3.1)
where for notational simplicity, we have not written the variables that are integrated over (the real centers of the fermionic strings). The chemical potential breaks the "supersymmetry", leaving as a symmetry the bosonic part \( sl(N) \otimes sl(K) \otimes u(1) \).
We introduce pseudo energies \( \epsilon_p, \kappa_p^{(l)}, \zeta_p^{(l)} \) defined by
\[
\frac{\rho_p}{\tilde{\rho}_p} = \exp(-\epsilon_p / T), \quad p = 1, \ldots, N
\]
\[
\frac{\sigma_p^{(l)}}{\tilde{\sigma}_p^{(l)}} = \exp(-\kappa_p^{(l)} / T), \quad p = 1, \ldots, N - 1, \quad l = 1, \ldots, \infty \quad (3.2)
\]
\[
\frac{\tau_p^{(l)}}{\tilde{\tau}_p^{(l)}} = \exp(-\zeta_p^{(l)} / T), \quad p = 1, \ldots, K - 1, \quad l = 1, \ldots, \infty.
\]
The minimization of the free energy leads to the system of thermodynamic Bethe ansatz (TBA) equations:
\[
0 = p A - G_{t+1,p} - \epsilon_p + \sum_{q=1}^{N} G_{pq} \star T \ln \left( 1 + e^{-\epsilon_q / T} \right) - \sum_{l \geq 1} a_l \star T \ln \left( 1 + e^{-\kappa_p^{(l)} / T} \right) - \sum_{l \geq 1} a_{lp} \star T \ln \left( 1 + e^{-\zeta_p^{(l)} / T} \right), \quad (3.3)
\]
and
\[
0 = -T \ln \left( 1 + e^{\kappa_p^{(l)} / T} \right) + \sum_{m \geq 1} A_{ml} \star \sum_{q=1}^{N-1} C_{qp} \star T \ln \left( 1 + e^{-\kappa_q^{(m)} / T} \right) + a_l \star T \ln \left( 1 + e^{-\epsilon_p / T} \right), \quad (3.4)
\]
and
\[
0 = -T \ln \left( 1 + e^{\zeta_p^{(l)} / T} \right) + \sum_{m \geq 1} A_{ml} \star \sum_{q=1}^{K-1} C_{qp} \star T \ln \left( 1 + e^{-\zeta_q^{(m)} / T} \right) \nonumber \\
- \delta_{p1} \sum_{q=1}^{N} a_{ql} \star T \ln \left( 1 + e^{-\epsilon_q / T} \right), \quad (3.5)
\]
The free energy reads then
\[
F = -\frac{T}{2\pi} \int \sum_{p=1}^{N} G_{p+1,t} \ln \left( 1 + e^{-\epsilon_p / T} \right). \quad (3.6)
\]

4. Large temperature entropy

Before going any further, it is useful to check the completeness of the solutions by studying the large temperature entropy. The general case is a bit heavy, so I will simply discuss some particular examples here. Let us start with \( sl(2/1) \), and introduce the quantities
\[
x_1 = e^{\kappa_1^{(l)} / T}, \quad l = 1, \ldots, \infty
\]
\[
y_1 = e^{\epsilon_1 / T}
\]
\[
y_2 = e^{\epsilon_2 / T}.
\]

In the large temperature limit, these go to constants, which are solution of the system

\[ x_l = [(1 + x_{l-1}) (1 + x_{l+1})]^{1/2} \left( 1 + \frac{1}{y_1} \right)^{\delta_{l1}/2} \]

\[ y_1 = \left( 1 + \frac{1}{y_2} \right) \prod_l \left( 1 + \frac{1}{x_l} \right)^{-1} \]  \hspace{1cm} (4.2)

\[ y_2 = \left( 1 + \frac{1}{y_2} \right) \left( 1 + \frac{1}{y_1} \right). \]

The solution of this system is

\[ x_l = \left( l + \frac{3}{2} \right)^2 - 1, \quad y_1 = \frac{4}{5}, \quad y_2 = 3. \]  \hspace{1cm} (4.3)

Meanwhile, at large temperature, one has, for \( t \neq 1 \)

\[ F \approx -T \left[ 2 \ln \left( 1 + \frac{1}{y_2} \right) + \ln \left( 1 + \frac{1}{y_1} \right) \right] = -T \ln 4, \]  \hspace{1cm} (4.4)

in agreement with the dimension of the typical representations of \( sl(2/1), \ d = 4 \). If, however, \( t = 1 \), one finds

\[ F \approx -T \left[ \ln \left( 1 + \frac{1}{y_2} \right) + \ln \left( 1 + \frac{1}{y_1} \right) \right] = -T \ln 3, \]  \hspace{1cm} (4.5)

in agreement with the dimension of the fermionic atypical representations of \( sl(2/1), \ d = 3 \).

We have performed the same exercise for \( sl(3/1) \). It is a bit more laborious, so we will only give the final result here. Introducing \( y_i = y_1 = e^{\epsilon_i/T} \), we found \( y_1 = \frac{7}{9}, \ y_2 = \frac{32}{17}, \ y_3 = 7 \). The large temperature free energy for typical representations is then

\[ F \approx -T \left[ 3 \ln \left( 1 + \frac{1}{y_3} \right) + 2 \ln \left( 1 + \frac{1}{y_2} \right) + \ln \left( 1 + \frac{1}{y_1} \right) \right] = -T \ln 8. \]  \hspace{1cm} (4.6)

There are now two types of atypical representations. If \( t = 2 \), one has

\[ F \approx -T \left[ 2 \ln \left( 1 + \frac{1}{y_3} \right) + 2 \ln \left( 1 + \frac{1}{y_2} \right) + \ln \left( 1 + \frac{1}{y_1} \right) \right] = -T \ln 7, \]  \hspace{1cm} (4.7)

while if \( t = 1 \),

\[ F \approx -T \left[ \ln \left( 1 + \frac{1}{y_3} \right) + \ln \left( 1 + \frac{1}{y_2} \right) + \ln \left( 1 + \frac{1}{y_1} \right) \right] = -T \ln 4. \]  \hspace{1cm} (4.8)

All of these coincide with known results of \( sl(3/1) \) representation theory. The general relation between the \( sl(N/K) \) TBA and representation theory seems quite interesting, but I won’t comment any more on it here [23].
5. The ground state and physical excitations

In this paragraph, I will restrict to the “generic case” \( t \geq N \). Some special cases are studied further below. I will also concentrate on the case \( \epsilon = -1 \), and comment briefly on the case \( \epsilon = 1 \) - which happens to be quite similar - at the end.

5.1. Equations as \( T \to 0 \)

As \( T \to 0 \) we find the system

\[
0 = pA - G_{t+1,p} - \epsilon_p - \sum_{q=1}^{N} G_{pq} \star \epsilon_q^- + \sum_{l \geq 1} a_l \star \kappa_p^{(l)-} + \sum_{l \geq 1} a_{lp} \star \zeta_1^{(l)-},
\]

and

\[
0 = -\kappa_p^{(l)+} - \sum_{m \geq 1} A_{ml} \star \sum_{q=1}^{N-1} C_{qp} \star \kappa_q^{(m)-} - a_l \star \epsilon_p^-,
\]

and

\[
0 = -\zeta_p^{(l)+} - \sum_{m \geq 1} A_{lm} \star \sum_{q=1}^{K-1} C_{qp} \star \zeta_q^{(m)-} + \delta_{p1} \sum_{q=1}^{N} a_{ql} \star \epsilon_q^-.
\]

where I have introduced as usual the positive and negative parts of the pseudoenergies. Whatever the value of \( A \) it is easy to see that one has

\[
\epsilon_1^- = \ldots = \epsilon_{N-1}^- = 0,
\]

together with

\[
\kappa_p^{(l)\pm} = 0, \ p = 1, \ldots, N-1, \ l = 1, \ldots, \infty.
\]

and

\[
\zeta_p^{(l)+} = 0, \ l = 1, \ldots, \infty, ; \quad \zeta_p^{(l)-} = 0, \ l \neq N.
\]

The equation (5.2) is then satisfied identically. The equation (5.3) now reads

\[
0 = -A_{lN} \star \sum_{q=1}^{K-1} C_{qp} \star \zeta_q^{(N)-} + \delta_{p1} a_{N1} \star \epsilon_N^-.
\]

Using \( a_{Nl} = s \star A_{Nl} \) (2.13), eq. (5.7) can be rewritten

\[
0 = -\sum_{q=1}^{K-1} C_{qp} \star \zeta_q^{(N)-} + \delta_{p1} s \star \epsilon_N^-.
\]
that is, using the form of $C_{pq}$,

\[
\begin{align*}
\zeta_1^{(N)} - s \ast \zeta_2^{(N)} &= s \ast \epsilon_N^- \\
\zeta_2^{(N)} - s \ast [s_1^{(N)} + s_3^{(N)} - ] \\
\ldots \\
\zeta_{k-1}^{(N)} &= s \ast \xi_{K-2}^-,
\end{align*}
\]

whose solution is

\[
\hat{\zeta}_p^{(N)} = \frac{\sinh(K - p)x/2}{\sinh K x/2} \hat{\epsilon}_N^-.
\]

or

\[
\hat{\zeta}_p^{(N)} = s_{K-p,K} \ast \epsilon_N^-,
\]

where I defined

\[
\hat{s}_{rs} = \frac{\sinh r x/2}{\sinh s x/2}.
\]

Similar results hold for the densities, that is

\[
\rho_1 = \ldots = \rho_{N-1} = 0,
\]

together with

\[
\sigma_p^{(l)} = \tilde{\sigma}_p^{(l)} = 0, \quad p = 1, \ldots, N - 1, \quad l = 1, \ldots, \infty,
\]

and

\[
\tilde{\tau}_p^{(l)} = 0, \quad p = 1, \ldots, K - 1, \quad l = 1, \ldots, \infty,
\]

and

\[
\tau_p^{(l)} = 0, \quad p = 1, \ldots, K - 1, \quad l \neq N.
\]

From (5.11) we get also

\[
\tau_p^{(N)} = s_{K-p,K} \ast \rho_N.
\]

We thus see that, whatever the value of $A$, the only non-vanishing particle densities are $\rho_N$ and $\tau_p^{(N)}$. To proceed further we have to distinguish the cases $A = 0$ and $A > 0$. 

5.2. The case $A = 0$

I consider first the case $A = 0$. In the ground state one has then, in addition to (5.4)

$$\epsilon^+_1 = \ldots \epsilon^+_N = 0,$$

so all the hole densities vanish, leaving the system

$$0 = -G_{t+1,p} - (\delta_{pN} + G_{pN}) \ast \epsilon^-_N + a_{NP} \ast \zeta_1^{(N)^-},$$

From (5.11) we replace $\zeta(N) - p$ by its expression in terms of $\epsilon_1^-$ to get, in terms of Fourier transforms,

$$0 = -\sinh \frac{px}{2} e^{-t|x|/2} \frac{\sinh pN}{\sinh x/2} e^{-(N-1)|x|/2} \epsilon^-_N + \sinh \frac{px}{2} e^{-N|x|/2} \sinh(K-1)x/2 \frac{\sinh x/2}{\sinh Kx/2} \epsilon^-_N.$$  (5.20)

As expected, $p$ disappears and we get

$$\hat{\epsilon}^-_N = -\frac{\sinh Kx/2}{\sinh x/2} e^{(N-K-t)|x|/2}, \quad \hat{\zeta}_p^{(N)^-} = -\frac{\sinh(K-p)x/2}{\sinh x/2} e^{(N-K-t)|x|/2},$$

and therefore

$$\epsilon^-_N = -a_{t-N+K,K}, \quad \zeta_p^{(N)^-} = -a_{t-N+K,K-p},$$

all other pseudoenergies being zero. In the ground state we therefore have as well, by comparing the Bethe equations and the limit $T \to 0$ of the thermodynamic equations,

$$\hat{\rho}_N = \frac{\sinh Kx/2}{\sinh x/2} e^{(N-K-t)|x|/2}, \quad \rho_N = a_{t+K-N,K},$$

$$\hat{\tau}_p^{(N)} = \frac{\sinh(K-p)x/2}{\sinh x/2} e^{(N-K-t)|x|/2}, \quad \tau_p^{(N)} = a_{t+K-N,K-p},$$

all other densities being zero. The ground state is thus filled with complexes of $N$ strings over strings (2.4) and $N$ strings for all the $sl(K)$ roots $\lambda_p$.

Excitations are made of holes in the distributions (5.23), with excitation energies exactly equal to $-\epsilon^-_N$ and $-\zeta_p^{(N)^-}$. By a standard argument the momentum is given by $p = 2\pi \int (\text{density})$. Taking inverse fourier transform we see that these excitation energies are expressed as sums of terms of the form $\alpha_t(\nu)$ while momenta are sum of terms of the type $i \ln \epsilon_t(\nu)$. At large values of the bare rapidity $\nu$ where the gap vanishes we therefore have $\epsilon \propto \frac{1}{\nu^2}$ and $p \propto \frac{1}{\nu}$ i.e $e \propto p^2$. The excitations are therefore non relativistic. Such a
The dispersion relation is characteristic of quantum ferromagnets. However I have not chosen the "wrong sign" of the Hamiltonian: identical features are observed for $\epsilon = 1$ (see below). There does not seem to be a very physical reason why the excitations are not relativistic. Technically, what happens is that the fermionic Bethe roots having no self coupling, the dispersion relation of the associated dressed excitations is almost the same as the one of the bare excitations. This can be seen especially clearly in the case of $sl(1/1)$ (more generally, $N = K$), where the Bethe equation reduces to $e_t(\mu_0)^L = 1$, and the energy of excitations is $\epsilon_N^+ = -\omega_t$. The Hamiltonian is the one of a XX chain with a magnetic field, $H = \sum_j \sigma_j^+ \sigma_j^- + \sigma_j^- \sigma_j^+ - 2 \sum_j \sigma_j^z$. After fermionization and Fourier transform, it becomes $H = \sum_k 2 \cos k a_k^\dagger a_k - 2F$, $F$ the number of fermions. In that language, the ground state is obtained by filling up all modes $-\pi \leq k \leq \pi$, and the gapless excitations occur near $k = 0$, where the energy goes like $\epsilon \propto k^2$.

I thus conclude that, if the supersymmetry $sl(N/K)$ is not broken, integrable lattice models based on fermionic representations do not have a relativistic limit (we will see later that this is true for other representations as well). This result is of course disappointing, and in sharp contrast with the situation for ordinary Lie algebras. To get some non trivial results, we do in fact need to break the supersymmetry, as I now demonstrate.

5.3. The case $A > 0$

Suppose now $A > 0$. The first difference with the case $A = 0$ is that $\epsilon$ has also a non vanishing positive part obeying

$$G_{t+1,N} - NA = -\epsilon_N^+ - (\delta + G) \star \epsilon_N^-,$$  \hspace{1cm} (5.24)

with the kernel

$$\hat{G} = \frac{\sinh(N - K)x/2}{\sinh Kx/2} e^{-N|x|/2}.$$  \hspace{1cm} (5.25)

In particular when $N = K$ we have simply (much like in the $N = K = 1$ case)

$$G_{t+1,N} - NA = -\epsilon_N^+ - \epsilon_N^-.$$  \hspace{1cm} (5.26)

The function $\epsilon_N$ is now negative on a finite interval $[-Q, Q]$. We have

$$\epsilon_N^-(\lambda) + \int_{-Q}^{Q} G(\lambda - \mu) \epsilon_N^-(\mu) = -G_{t+1,N} + NA, \ \lambda \in [-Q, Q],$$  \hspace{1cm} (5.27)
and
\[ \epsilon_N^+ + \int_{|\mu| \geq Q} H(\lambda - \mu) \epsilon_N^+(\mu) d\mu = -a_{t-N+K,K} + KA, \quad |\lambda| \geq Q \quad (5.28) \]

where \( 1 + \hat{H} = \frac{1}{1 + \hat{G}} \).

These equations can be solved perturbatively in the limit \( Q \gg 1 \) using standard Wiener-Hopf techniques. At dominant order in this limit, the cut-off \( Q \) is related to the magnetic field by
\[ A \approx \frac{1}{K} a_{t-N+K,K}(Q) \approx \frac{\text{cst}}{Q^2}, \quad Q \gg 1, \quad (5.29) \]
(this result is exact in the case \( N = K \)). The system is still gapless but the gap now vanishes at finite rapidity. For \( \mu \) close to \( \pm Q \) one has
\[ \epsilon_N(\mu) \approx |a'_{t-N+K,K}(Q)|(|\mu| - Q), \quad |\mu| \approx Q. \quad (5.30) \]

In the presence of a magnetic field there is no simple relation between the ground state pseudoenergies and densities. The latter obey, instead of (5.27),
\[ \rho_N(\lambda) + \int_{-Q}^Q G(\lambda - \mu) \rho_N(\mu) d\mu = G_{t+1,N}, \quad \lambda \in [-Q, Q], \quad (5.31) \]
and
\[ \tilde{\rho}_N + \int_{|\mu| \geq Q} H(\lambda - \mu) \tilde{\rho}_N(\mu) d\mu = a_{t-N+K,K}, \quad |\lambda| \geq Q. \quad (5.32) \]

In particular these densities are discontinuous at the cutoff \( Q \). For large \( Q \) we have approximately \( \rho_N(Q) \approx \tilde{\rho}_N(Q) \approx a_{t-N+K,K}(Q) \). On the other hand the relation between momenta and densities still holds so we get for the momentum of excitations (5.30)
\[ p_N(\mu) \approx 2\pi a_{t-N+K,K}(Q)(|\mu| - Q), \quad |\mu| \approx Q \quad (5.33) \]

The massless excitations in the \( \epsilon_N^+ \) branch therefore now are relativistic, with the sound velocity
\[ v_s = \frac{1}{2\pi} \frac{|a'_{t-N+K,K}(Q)|}{a_{t-N+K,K}(Q)} \approx \frac{1}{\pi Q}, \quad Q \gg 1. \quad (5.34) \]

The \( \epsilon_N^+ \) part induces also non vanishing \( \epsilon_p^+ \). We have
\[ 0 = pA - G_{t+1,p} - \epsilon_p^+ - s_{pN} \ast (\delta + G) \ast \epsilon_N^-, \quad (5.35) \]
where we used
\[ \hat{G}_{pN} - \hat{a}_{pN} \frac{\sinh(K-1)x/2}{\sinh Kx/2} = \frac{\sinh px/2}{\sinh Kx/2} e^{(K-N)|x|/2} \]
Observing that
\[ G_{t+1,p} = \frac{\sinh px/2}{\sinh Nx/2} G_{t+1,N} \]
and using (5.24) we get
\[ \epsilon_+^p = s_{pN} \star \epsilon_N^+, \quad (5.36) \]

Therefore, we have now new particle-like excitations in the system, with energy \( \epsilon_+^p \). Their physical nature is easy to understand. At a given rapidity, when \( Q >> 1 \) only the tails at \( \pm \infty \) of \( s_{pN} \) gives significant contributions because \( \epsilon_N^+ \) vanishes in \([-Q, Q]\). At large argument the behaviour of \( s_{pN} \) is determined by the pole of its Fourier transform nearest the real axis, that is \( x = 2i\pi/N \), and we can approximate
\[ s_{pN}(\nu) \approx \frac{2}{N} \sin \left( \frac{p\pi}{N} \right) e^{-2\pi|\nu|/N}, \quad \nu \to \pm \infty. \quad (5.37) \]

Expanding \( \epsilon_N^+ \) close to \( Q \), we get therefore
\[ \epsilon_+^p(\mu) \approx \frac{N}{\pi^2} \sin \left( \frac{p\pi}{N} \right) |a'_{-N+K,K}(Q)| e^{-2\pi Q/N} \cosh(2\pi \mu/N), \quad |\mu| << Q, \quad p = 1, \ldots, N - 1 \quad (5.38) \]

The momentum of these excitations is given by \( p = 2\pi \int \tilde{\rho}_p \). From (5.35) we deduce as well
\[ \tilde{\rho}_p = s_{pN} \star \rho_N, \quad (5.39) \]
and thus
\[ p = \frac{2N}{\pi} \sin \left( \frac{p\pi}{N} \right) a_{-N+K,K}(Q) e^{-2\pi Q/N} \sinh(2\pi \mu/N), \quad |\mu| << Q. \quad (5.40) \]

Hence the excitations are relativistic once again, with a mass
\[ m_p = \frac{N}{\pi^2} \sin \left( \frac{p\pi}{N} \right) |a'_{-N+K,K}(Q)| e^{-2\pi Q/N}, \quad p = 1, \ldots, N - 1, \quad (5.41) \]
and the same sound velocity as before (which was quite obvious from (5.36) and (5.39)).

The \( \zeta \) excitations are also modified due to the existence of the cut-off \( Q \). This is easily studied using (5.11) which still holds. The gap still vanishes at rapidities much bigger (in absolute value) than \( Q \), but with a different dependence on rapidities. In this limit the behaviour of \( \left( \epsilon_{p(N)}^\zeta \right)^- \) is determined by the tail of the kernel (5.12) \( s_{K-p,K} \). For \( \lambda > 0 \) \( (\lambda < 0) \), only the region close to \( Q(-Q) \) contributes, so we find
\[ \left( \epsilon_{p(N)}^\zeta \right)^- \approx \frac{K}{\pi^2} \sin \left( \frac{(K-p)\pi}{K} \right) |a'_{-N+K,K}(Q)| e^{-2\pi Q/K} e^{\pm 2\pi \mu/K}, \quad |\mu| >> Q. \quad (5.42) \]
The momentum of these excitations is given by \( p = 2\pi \int \tau_p^{(N)} \). From (5.17) we check that these excitations are now left and right moving relativistic massless excitations with a sound velocity that is still given by (5.34) and a mass parameter

\[
m_p = \frac{K}{\pi^2} \sin \left( \frac{(K - p)\pi}{K} \right) |a_{t-N+K,K}(Q)| e^{-2\pi Q/K}, \quad p = 1, \ldots, K - 1
\]  

(5.43)

To conclude this section, I would like to notice that the techniques I used are entirely similar to the ones developed by Tsvelik in his study of \( sl(2) \) chains with a magnetic field [24].

5.4. The case \( \epsilon = 1 \)

When \( \epsilon = 1 \) and \( A = 0 \), it is easy to see that the solution of the Bethe equations is \( \epsilon_p^+ = G_{t+1,p} \) for \( p = 1, \ldots, N - 1 \), while all other pseudo energies vanish: the ground state is empty, but there are hole densities for all the fermionic strings. When \( A = -B \), \( B > 0 \) is turned on, only \( \epsilon_N \) acquires a negative part, while the relations \( \epsilon_p^+ = s_{p,N} \ast \epsilon_N^+ \) and \( \zeta_p^{(N)-} = s_{K-p,K} \ast \epsilon_N^- \) still hold. Things are thus very much similar to the case \( \epsilon = -1 \); the difference is that \( \epsilon_N \) is now positive (instead of negative) in a finite interval, and thus it is the \( \epsilon_p^+ \) excitations that are massless, while the \( \zeta_p^{(N)} \) excitations are massive. In effect, the roles of \( N \) and \( K \) are thus exchanged.

6. S matrices in the scaling limit

We now discuss the way the various excitations interact in the case \( \epsilon = -1 \) and \( A \) a small positive number (similar results would hold for \( \epsilon = 1 \) and \( A = -B \), \( B \) a small positive number, up to the exchange of \( N \) and \( K \)). In the following we shall be interested in the scaling limit of the lattice model. In this limit the three types of excitations we have identified \( (\epsilon_p^+, \epsilon_N^\pm, \zeta_p^{(N)-}) \) become decoupled since they occur respectively for rapidities \( \mu \) such that \( |\mu| << Q, |\mu| \approx Q \) and \( |\mu| >> Q \). We shall generally write equations where this coupling has been neglected by the symbol \( \approx \). Densities evaluated in the ground state are denoted by \( |0\).

We now get back to the equations that involve densities (and are magnetic field independent), and consider first the physics in the vicinity of \( |\mu| << Q \). In that region, it
turns out that physical densities are hole densities $\tilde{\tau}$, so our first task is to invert (2.19) and express instead the densities $\tau$ in terms of hole densities $\tilde{\tau}$. We find

$$B_{p1} * s * \rho_{n} = \tau^{(n)}_{p} - \sum_{q=1}^{K-1} B_{pq} * \sum_{m \geq 1} C_{nm} * \tilde{\tau}_{q}^{(m)}, \quad p, q = 1, \ldots, K - 1,$$

where

$$B_{pq} = \frac{2 \cosh x/2}{\sinh x/2 \sinh Kx/2} \sinh [(K - p)x/2 - \delta_{pq}], \quad p \geq q,$$

and $C_{lm}$ is defined exactly like in (2.18), but acting on upper indices. Also, when $n \geq N$, there is no density $\rho_{n}$, and the source term disappears from the equation (6.1).

The next step is then to replace $\tau_{l}^{(l)}$ in (2.16) by the expression (6.1) for $p = 1$. We then use the last equation (2.16) for $p = N$ to eliminate $\rho_{N}$. Replacing in the equations (2.16) for $p = 1, \ldots, N - 1$ leads to

$$G_{t+1,p} - (G_{t+1,N} * H_{pN} * (1 + H_{NN})^{-1}) + H_{pN} * (1 + H_{NN})^{-1} * \tilde{\rho}_{N} =$$

$$\rho_{p} + \tilde{\rho}_{p} + \sum_{q=1}^{N-1} (H_{pq} - H_{pN} * H_{Nq} * (1 + H_{NN})^{-1}) * \rho_{q} + \sum_{l \geq 1} a_{l} * \sigma_{p}^{(l)}$$

$$+ \sum_{l \geq 1} (H_{pN} * a_{Nl} * (1 + H_{NN})^{-1} - a_{pl}) * \sum_{q=1}^{K-1} B_{1q} * \sum_{m \geq 1} C_{lm} * \tilde{\tau}_{q}^{(m)},$$

where we introduced the kernels

$$H_{pq} = \frac{\sinh qx/2}{\sinh Kx/2} e^{(K-p)|x|/2 - \delta_{pq}}, \quad p \geq q.$$

The last term vanishes for $l \geq N$; moreover, in the approximation we are considering, it is not possible to make holes in the distributions $\tau_{q}^{(N)}$, which would cost a very large energy. The term $\tilde{\tau}_{q}^{(N)}$ thus disappears from the equation (6.3), leaving a finite set of $\tilde{\tau}$ densities. The constant term on the left hand side of (6.3) vanishes identically too. After simplifying the expressions slightly, we thus end up with the system

$$\rho_{p} + \tilde{\rho}_{p} \approx s_{pN} * \tilde{\rho}_{N} |_{0} + \sum_{q=1}^{N-1} Z_{pq} * \rho_{q} - \sum_{q=1}^{K-1} s_{K-q-K} * \tilde{\tau}_{q}^{(p)} - \sum_{l \geq 1} a_{l} * \sigma_{p}^{(l)}, \quad p = 1, \ldots, N - 1$$

where

$$Z_{rs} = \delta_{rs} - \frac{\sinh rx/2 \sinh (N - s)x/2}{\sinh Kx/2 \sinh Nx/2} e^{K|x|/2}, \quad r \leq s, \quad Z_{rs} = Z_{sr}.$$
This has to be supplemented by the equations

\[ \tilde{\sigma}_p^{(l)} + \sum_{m \geq 1} A_{lm} \sum_{q=1}^{N-1} C_{pq} \star \sigma_q^{(m)} = a_l \star \rho_p, \quad p = 1, \ldots, N - 1, \quad l \geq 1, \quad (6.7) \]

and

\[ \tau_q^{(p)} - \sum_{r=1}^{K-1} B_{qr} \sum_{s=1}^{N-1} C_{ps} \star \tilde{\tau}_r^{(s)} \approx s_{K-q,K} \star \rho_p, \quad p = 1, N - 1, \quad q = 1, K - 1. \quad (6.8) \]

We can identify the source terms \( s_{pn} \star \tilde{\rho}_N|_0 \) with \( \dot{\rho}/2\pi \) thanks to (5.40). These equations are then easily shown to coincide with the system obtained in the study of an \( su(N) \) scattering theory at level \( K \). Recall in particular that there are particles of mass \( m_p = m \sin(p\pi/N) \) associated with every fully antisymmetric representation and carrying a charge which is a weight of these representations. The S matrix is discussed in [25]: it is the tensor product of an \( sl(N) \) level \( K \) RSOS S matrix and a \( sl(N) \) “vertex” (soliton) S matrix. This structure is transparent on (6.5): the \( \tilde{\tau}_q^{(p)} \) are the densities involved in the diagonalization of the RSOS part of the scattering matrix, while the \( \sigma_p^{(l)} \) are those from the diagonalization of the vertex part.

For the excitations at rapidities \( |\mu| >> Q \), things are a bit simpler, since the densities \( \rho \) and \( \sigma \) are totally frozen in that limit. The physics is thus fully described by the equations

\[ \delta_{p1} a_{1p} \star \rho_N|_0 \approx \tilde{\tau}_p^{(l)} + \sum_{m \geq 1} A_{lm} \sum_{q=1}^{K-1} C_{pq} \star \tau_q^{(m)}. \quad (6.9) \]

These equations are similar to the ones one would write for a \( sl(K) \) lattice model with the fully symmetric representation \( N\omega_1 \) on every site, in the limit appropriate to study the massless right moving excitations [26]. The scattering theory can then be easily extracted. This time one has massless excitations of mass parameter \( m_q = m \sin(q\pi/K) \), and the S matrix is the tensor product of an \( sl(K) \) level \( N \) RSOS S matrix and a \( sl(K) \) soliton S matrix. This massless theory is well known to describe the \( SU(K) \) level \( N \) WZW theory.

The idea of describing a conformal field theory by a massless scattering theory has a long history going back to [27]. It has been a subject of intense interest recently in the context of quantum impurity problems [28] and the quantum KdV equation [29].

Finally the \( \epsilon_N \) excitations are completely free in this limit, describing a massless \( U(1) \) degree of freedom.
The different pieces of scattering theory found in this section coincide with the known results \[30\] for the \(N\) colors, \(K\) flavors chiral Gross Neveu model. We thus conclude that, in the limit of large \(Q\) (that is, infinitesimally small symmetry breaking field), the continuum limit of the \(sl(N/K)\) quantum spin chain with generic fermionic representations obeying \(t > N - 1\) coincides with the chiral Gross Neveu model

\[
L = i\bar{\psi}^{jf}\partial_t \psi^{jf} + g\psi^{jf}_{L}\psi^{kg}_{R}\psi^{kj}_{L},
\]

where \(j\) the color index runs from 1 to \(N\) and \(f\) the flavor index from 1 to \(K\). In terms of currents, the interaction reads \(J_L J_R + J^a_L J^a_R\), where \(J\) is the \(U(1)\), chirality carrying current, \(J^a\) are the \(sl(N)\) currents, with \(J^a_R = \psi^{jf}_{R} (T^a)^j_k \psi^{kf}_{R}\). The parameter \(g\) in (6.10) varies as \(g \propto 1/Q\), and the true scaling limit is obtained when \(q \to \infty\), that is \(g \to 0\). In that limit, using the equation determining \(\epsilon_\pm\), and well known considerations on dressed charges \[31\], I found that the \(U(1)\) degree of freedom has a radius \(R = \frac{K}{4\pi^2}\). Except when \(N = K\), this is not the radius that is expected for the action (6.10), and the latter requires an additional \(J_L J_R\) coupling to be correct. As is well known, such a coupling does not change any of the physical or integrability properties in a significant way.

7. Some particular cases

We concentrated so far on the case of fermionic representations with \(t > N - 1\) (which, in particular, are typical). When \(t \leq N - 1\), things can get quite complicated, due to the different possible structures of the source term \(G_{t+1,p}\) in the Bethe equations.

The simplest situation occurs when \(\epsilon = 1\). In that case, it is easy to see that when \(A = 0\), the ground state is always given by \(\epsilon^+_p = G_{t+1,p}\) (all others being zero), irrespective of the value of \(t\) (and thus excitations are not relativistic). When the field \(A = -B\) is turned on, \(\epsilon_N\) is the only fermionic pseudo energy to acquire a negative part, which also gives rise to negative parts \(\zeta^{(N)-}_p = s_{K-p,K} \star \epsilon^-_N\). The situation is thus very similar to the case \(t \geq N - 1\): the \(\epsilon^+_p\) excitations are massless and described by an \(SU(N)\) level \(k\) Wess Zumino model, while the \(\zeta^{(N)-}_p\) are massive, and described by an \(SU(K)\) level \(N\) massive theory; the \(\epsilon_N\) excitations still correspond to a simple \(U(1)\) theory. The only difference with the case \(t > N - 1\) is that the Fermi velocities of these excitations are not in general equal anymore. To see this, let us restrict to the case \(K = 1\) for simplicity. The equations for the fermionic roots are then

\[
0 = -pB + G_{t+1,p} - \epsilon^+_p - G_{pN} \epsilon^-_N, \quad p \leq N - 1
\]

\[
0 = -NB + G_{t+1,N} - \epsilon^+_N - (1 + G_{NN}) \epsilon^-_N
\]
The pseudo energy $\epsilon_N$ acquires a negative part at large rapidities, while one has

$$\epsilon_p^+ = \frac{G_{pN}}{1 + G_{NN}} \epsilon_N^+ + G_{t+1,p} - \frac{G_{pN} G_{t+1,N}}{1 + G_{NN}}. \quad (7.2)$$

In contrast with the case $t > N - 1$ where the second term vanishes, the massless relativistic region now corresponds to rapidities much larger than $Q$ (the Fermi rapidity for $\epsilon_N$), where the first term of (7.2) is negligible: the behaviour of $\epsilon_p$ is thus fully determined by the second term: it is independent of $\epsilon_N$ and $Q$, and thus is bound to have a different Fermi velocity than the massless $U(1)$ excitations. More careful study of this second term shows that it reproduces a set of massless excitations with mass parameters proportional to $\sin \frac{p\pi}{N}$, for any value of $t$. The continuum limit of this model is thus the tensor product of a level 1 $SU(N)$ WZW model and a $U(1)$ boson, each with its own sound velocity. Notice that it is not necessary to take the $Q \to \infty$ limit here (since there are no excitations at small rapidity to decouple) and as $Q$ varies, the radius of compactification of the boson changes. When $Q \to 0$ (low density limit in the t-J language to be discussed below), it goes to the point $R = \sqrt{\frac{N}{4\pi}}$, of the $c = 1$. The whole theory thus reproduces the well known system of free fermions with $U(1) \times SU(N)$ symmetry (see eg [32]). When instead $Q \to \infty$ (corresponding to the limit of half filling in the t-J model), the radius goes to $R = \sqrt{\frac{1}{4\pi}}$, the same value obtained when bosonizing a single free fermion (which corresponds to the case $N = 1$). This value arises also when considering the $U \to \infty$ limit of the $SU(N)$ Hubbard model [33].

Indeed, the case $t = 1$, $K = 1$ is nothing but the so called $SU(N)$ t-J model [21]. It is interesting to notice here that the Bethe equations we started with coincide with those of Schlottmann, and therefore to a point of view where there is an atypical fermionic representation on every site. As is well known, other Bethe ansätze can be written for this model, as was done first by Sutherland [34]: they correspond to a point of view where there is a fundamental representation on every site [35]. The continuum limit of the t-J model has been worked out in [36]: the results are identical to what we found here, although they are not formulated in terms of massless scattering, but rather using conformal dimensions and dressed charge matrices.

The case $\epsilon = -1$ is considerably more involved, and it is not clear where the ground state lies as $t$ varies [19]. An exception is $t = N - 1$, where, when $A = 0$, the ground state is obtained by filling up the sea of length $N - 1$ fermionic roots, $\tilde{\epsilon}_{N-1} = -\frac{\sinh Kx/2}{\sinh x/2} e^{-Kx/2}$, and excitations are, as usual, not relativistic. When a field is added, $\epsilon_{N-1}$ acquires a
positive part, together with the other $\epsilon$’s which obey $\epsilon^+_p = \frac{\sinh px/2}{\sinh(N-1)x/2} \epsilon^+_N$. The relation $\zeta_p^{(N-1)-} = s_{K-p,K} \star \epsilon_N^-$ holds, too. It follows that both $\zeta_p^{(N-1)-}$ and $\epsilon_p^+$ give rise to relativistic massless (resp. massive) excitations. In addition however, one has $\kappa_p^{(l)+} = -a_l \star \epsilon_N^-$, giving rise to non relativistic excitations: from the field theory point of view, this is thus a not very interesting situation. I suspect similar conclusions hold for other values of $t$.

8. Other representations

In the $K = 1$ case, the choice $t = 1$ corresponds, in fact, to putting a fundamental representation on every site. It is interesting to study more generally the case where the chain is built up using representations with Dynkin parameters

$$
\begin{array}{ccccccc}
  j & 0 & 0 & 0 & 0 \\
  \bigcirc & - & - & \bigotimes & - & - & \bigcirc
\end{array}
$$

The equations then look as in (2.2), except for the source terms: the left hand side for the $\mu_0$ root is now equal to one, while the left hand side for the $\mu_{N-1}$ root is $e_j^L$, where $j$ is the (integer) Dynkin parameter, $j = a_1$. The $su(N)$ t-J model corresponds to $K = 1$, $j = 1$ already discussed above. In the presence of a chemical potential, the energy reads

$$
E = - \sum_{\mu_{N-1}} \frac{j}{\mu_{N-1}^2 + \frac{j}{4}} + A \sum_{\mu_{N-1}} 1. \quad (8.1)
$$

The solutions to the Bethe equations are simpler than in the fermionic case: there are the usual strings for every bosonic root, while the $\mu_0$ for the fermionic root are all real. With the same notations as before (setting $\rho_1 \equiv \rho$) we now have, going to the continuum version

$$
G_{j+1,l} \delta_{p,N-1} = \tilde{\sigma}_p^{(l)} + \sum_{m \geq 1} A_{lm} \star \sum_{q=1}^{N-1} C_{pq} \star \sigma_q^{(m)} - \delta_{p1} a_l \star \rho, \quad \text{(8.2)}
$$

together with

$$
0 = \rho + \tilde{\rho} - \sum_{l \geq 1} a_l \star \sigma_1^{(l)} + \sum_{l \geq 1} a_l \star \tau_1^{(l)}, \quad p = 1, \ldots, N-1, \quad \text{(8.3)}
$$
and
\[ \delta p_1 a_l \ast \rho = \tilde{\tau}^{(l)}_p + \sum_{m \geq 1} a_{lm} \ast \sum_{q=1}^{K-1} C_{pq} \ast \tau^{(m)}_q, \quad p = 1, \ldots, K - 1. \] (8.4)

To discuss what is going on, let us consider as an example the case \( N = K = 2 \). The TBA equations at \( T = 0 \) are, setting \( \kappa_1^{(l)} \equiv \kappa^{(l)}, \zeta_1^{(l)} = \zeta^{(l)}, \)

\[
G_{j+1,l} = -\kappa^{(l)}+ - \sum A_{ml} \ast \kappa^{(m)-} + a_l \ast \epsilon^{-} \]
\[
0 = A - \epsilon + \sum a_l \ast \kappa^{(l)-} - \sum a_l \ast \zeta^{(l)-} \]
\[
0 = -\zeta^{(l)+} - \sum A_{lm} \ast \zeta^{(m)+} + a_l \ast \epsilon^{-}.
\] (8.5)

This system can be transformed into

\[
\kappa^{(l)} = s \ast \left( \kappa^{(l-1)+} + \kappa^{(l+1)+} \right) + \delta_{l1} s \ast \epsilon^{-} - \delta_{lj} s
\]
\[
\epsilon = A + \sum a_l \ast \kappa^{(l)-} - \sum a_l \ast \zeta^{(l)-} \]
\[
\zeta^{(l)} = s \ast \left( \zeta^{(l-1)+} + \zeta^{(l+1)+} \right) + \delta_{l1} s \ast \epsilon^{-}.
\] (8.6)

In the ground state, all positive parts of the pseudo energie s vanish. It follows that \( \kappa^{(1)-} = s \ast \epsilon^{-} \) and \( \kappa^{(j)-} = -s, \zeta^{(1)-} = s \ast \epsilon^{-} \). These can be put back in the equation for \( \epsilon \), which reads in that case (it is particularly simple because \( N = K \) here)

\[
\epsilon = A - s \ast a_j.
\] (8.7)

When \( A \) vanishes, the \( \epsilon \) excitations are non relativistic. When \( A \) is positive, and provided it is not too big, \( \epsilon \) has a positive and a negative part; the corresponding \( u(1) \) excitations are then massless; calling \( Q \) the Fermi rapidity for the \( \epsilon \) excitations, their sound velocity is \( u = \frac{1}{2\pi} \frac{|\epsilon'(Q)|}{\rho(Q)} \). From (8.6), it also follows that \( \kappa^{(j)-} = -s, \kappa^{(1)-} = s \ast \epsilon^{-} \) and \( \zeta^{(1)-} = s \ast \epsilon^{-} \). The \( \kappa \) excitations are thus also massless and relativistic at rapidities much larger than \( Q \); notice however that they have different Fermi velocities: the one for \( \kappa^{(j)-} \) is independent of \( Q \) (it turns out to be \( v_j = \pi \) in our conventions), while the one for \( \kappa^{(1)-} \) does depend on it, and reads \( v_1 = \frac{1}{2\pi} \frac{|\kappa^{(1)-}'(\mu)|}{\rho^{(1)}(\mu)} \), \( \mu \to \infty \). As for the \( \zeta \) excitations, they are also massless, with a similar sound velocity.

Much like in the previous section, and in contrast with the case of generic fermionic representations (with \( t \geq N - 1 \)), relativistic invariance does not require \( Q \) to be large, and thus is obtained for an entire range of values of \( A \). Here, this means that the associated scattering theory has a continuous parameter \( A \), whose meaning we now partly elucidate.
To do so, we observe that the equations for the densities are identical to a decoupled system for a pair of $sl(2)$ spin chains, the first $sl(2)$ chain having a source term on the first and $j^{th}$ node, the second $sl(2)$ chain on the first node. The second system has thus the $su(2)$ level 1 WZW model as continuum limit. As for the first, it is similar to a general class of lattice models with mixtures of several spins. These models were discussed in details in [37] and [38]. In [38], the sound velocities for all excitations were the same, and the continuum limit was described as the tensor product of an $SU(2)$ level $j$ WZW model (with $c = \frac{3j}{j+2}$) and of an $SU(2)$ minimal coset model with $c = 1 - \frac{6}{(j+1)(j+2)}$. In the limit of small $Q$, this result essentially still holds, but now the two types of excitations each have different Fermi velocities, $v_j$ and $v_1$ respectively. The scattering theory is as in [38], and the field theory can be related with a 2 colors $j$ flavors Gross Neveu model with flavor anisotropy. Away from that limit, things are more complicated: the contribution to the free energy from the $\kappa$ degrees of freedom can be written as $f_\kappa \approx -\frac{\pi T^2}{6} \left( \frac{c_j}{v_j} + \frac{c_1}{v_1} \right)$, and although the sum $c_j + c_1$ stays the same (it is controlled by the overall shape of the TBA diagram), the individual values of these two parameters evolve with $A$. In the limit where $Q \to \infty$, one finds that $c_1 = 1$ while $c_j = \frac{3(j-1)}{j+1}$. One should not however think that this model always decomposes into the sum of two independent CFTs with different sound velocities, and central charges $c_j$ and $c_1$, except for very small and very large $Q$. To get an idea of what happens, I will restrict to the case $j = 2$, which was also partly treated in [17]. Consider therefore a theory made initially of an Ising model and a level 2 WZW model, which is the right description of the system at small $A$. This theory can be written in terms of 4 species of Majorana fermions $\chi_i$, $i = 0, 1, 2, 3$. Suppose now one adds, as suggested in [17], a four fermion coupling in each chiral sector. The hamiltonian say for the right movers reads then

$$H = -ia \sum_{i=1}^{3} \chi_i \partial_x \chi_i - ib \chi_0 \partial_x \chi_0 + c \chi_0 \chi_1 \chi_2 \chi_3.$$ 

To handle this model, the best is to bosonize, representing the fermions as $\chi_0 \propto \cos \phi_1$, $\chi_1 \propto \sin \phi_1$ [39] (and similarly for $\chi_2, \chi_3$ in terms of another boson $\phi_2$). The hamiltonian reads then, schematically,

$$H = A \left[ (\partial \phi_1)^2 + 2 (\partial \phi_2)^2 + i\alpha_0 \partial^2 \phi_1 \right] + B \left[ (\partial \phi_1)^2 - i\alpha_0 \partial^2 \phi_1 \right] + C \partial \phi_1 \partial \phi_2$$

where $\partial \equiv \partial_x$. For $C = 0$, and a choice of $\alpha_0$ corresponding to the bosonized Ising stress energy tensor, the $A$ term is a $c = 3/2$ theory, the $B$ term a $c = 1/2$ theory. Moreover, these two theories are independent (the short distance expansion of their two stress energy
tensors is regular). For $\alpha_0 = 0$, and $A = B$, the theory decomposes in two independent
bosons $\phi_1 \pm \phi_2$ with different sound velocities, obtained by diagonalizing the quadratic
form. In general however, the theory defined by $\mathcal{H}$ is not the sum of two independent
conformal field theories. The massless excitations identified previously have to be thought
of as a way to define the excitations of the whole theory, which is therefore not conformal
invariant, since it does not have a well defined sound velocity. It would definitely be of
interest to study the finite size spectrum of the lattice model to push this investigation
further.

The foregoing results generalize easily to arbitrary $N, K$. The key point is that the $\epsilon$
excitations give rise to a massless $U(1)$ (charge) degree of freedom, and that this excitation
in turn feeds source terms on the $l = 1$ strings for both the (color) $sl(N)$ and (flavor) $sl(K)$
excitations. As a result, one always gets the $SU(K)$ level one WZW model in the flavor
sector, and a more complex theory in the color sector, that reduces to a mixture of the
$SU(N)$ level $j$ WZW model and an $SU(N)$ coset model in limit of small $A$.

9. Impurities

By a very general construction, it is possible to build an integrable impurity model by
inserting a different representation, or a representation with a different spectral parameter,
in the general quantum inverse scattering framework. This is the same trick that has often
been used to study models with mixtures of representations, as well as models with spectral
parameter heterogeneities [40], [26]. In the context of impurities, the method was probably
first used in [41]. It was applied for instance to the t-J model with a four dimensional
impurity (the $sl(2/1)$ case) in [42].

Note that I am only discussing impurities in a periodic chain here; this is not the same
(although results in the continuum limit are quite related) than having a chain with eg
open boundaries, nor boundary impurities. For some recent results in that direction, see
eg [43].

9.1. Fermionic impurities in a generic fermionic chain

Let us first suppose that we insert in a chain based on fermionic representations with
$t > N - 1$, a fermionic representation with a different value of the Dynkin parameter,
and a shifted spectral parameter. The Bethe equations now look like (2.2), except that
for the $\mu_0$ root, the left hand side contains an additional term $e_{\nu'}(\mu_0 - \Upsilon)$. The energy
takes the same form as before, and all the impurity term changes is the equations for the densities: equations (2.17) and (2.19) are unchanged, while (2.16) contains an additional \( \frac{1}{L} G_{t'p}(\mu_0 - \Upsilon) \), \( L \) the size of the system.

Since the equations for the densities appear in the thermodynamic Bethe equations only through their variations, the equations determining the ground state are unchanged: (5.1), (5.2) and (5.3) still hold, together with the analysis of the previous sections. Proceeding further to analyze the scattering of the excitations, equations (6.5), (6.7) and (6.9) still hold, too. The only role of the impurity is that \( \tilde{\rho}_N \) and \( \rho_N \) in the left hand sides of (6.5) and (6.9) have now a \( 1/L \) part, which follows from the solution of the equation generalizing (5.31):

\[
\rho_N(\lambda) + \int_{-Q}^{Q} K(\lambda - \mu) \rho_N(\mu) d\mu = G_{t+1,N}(\lambda) + \frac{1}{L} G_{t'+1,N}(\lambda - \Upsilon).
\] (9.1)

Note that the Fermi cut-off \( Q \) is not changed, since it follows from the condition \( \epsilon^\pm_N(Q) = 0 \), and the latter is a TBA equation, independent of any impurity terms. The Fermi velocity has a \( 1/L \) correction that we neglect in the limit \( L \) large. The only effect of the impurity is thus to modify \( \tilde{\rho}_N \) and \( \rho_N \) in the equations (6.5) and (6.9) by a term of order \( 1/L \).

Consider for instance equation (6.5). Since \( \tilde{\rho}_N \) still vanishes for rapidities smaller or equal to \( Q \), the impurity term does not introduce any non trivial phase shift for the densities \( \rho_p \); its only effect at leading order in the scaling limit \( Q \rightarrow \infty \) is to renormalize the mass of the excitations (keeping their ratios constant) by a \( 1/L \) term which becomes negligible in the limit \( L \) large. The impurity does not introduce any flow in the renormalization group sense, and roughly corresponds to changing the length of the system by a finite amount (I’ll refer to this, not quite correctly, as being “irrelevant”). The same conclusion holds for the \( u(1) \) sector.

In contrast, suppose now that \( t' < N - 1 \). In that case, as has been noticed before, the combination

\[
G_{t'+1,p} - \frac{H_{pN} G_{t'+1,N}}{1 + H_{NN}} = G_{t'+1,p} - \frac{G_{pN} G_{t'+1,N}}{1 + G_{NN}}
\]

does not vanish. Calling this combination \( I_{t',p} \), it follows that for large \( L \) the right hand side of equation (6.3) contains now two source terms, \( s_{pN} \bar{\rho}_N \) and \( \frac{1}{L} I_{t',p}(\mu - \Upsilon) \).

In the case \( t' = 1 \), the impurity term is \( \frac{1}{L} s_{pN}(\mu - \Upsilon) \): the equations exactly coincide with those of the exactly screened \( su(N) \) Kondo model with \( K \) channels [44],[45], although the bulk now appears massive. If however one concentrates on the massless limit of the
bulk degrees of freedom by letting the rapidity \( \mu \to \infty \), while at the same time also sending the impurity rapidity \( \Upsilon \to \infty \), the Kondo equations are then exactly recovered, \( 2\pi \Upsilon/N \) being related in a simple way to the Kondo rapidity.

In fact, it is easy to see that our Bethe equations coincide with those of the degenerate Anderson model when \( K = 1 \) - this has already been observed by Schlottmann in some cases [46]. Indeed, the equations for the degenerate Anderson model as written say in [44] are the same as the ones we are considering, with \( t' = 1, \Upsilon = \frac{\epsilon}{2\Gamma}, \mu_0 = \frac{k}{2\Gamma} \), except that the source term for the Anderson model is \( \exp(ik_jL) \equiv \exp(2i\Gamma\mu_0L) \) instead of our \( [e_{t}(\mu_0)]^L \). To match these too, it is enough to send \( t \to \infty \) with \( t \propto 1/\Gamma \), and rescale the length of the system appropriately. Since the physical properties are independent of \( t \) for \( t > N - 1 \), we should indeed obtain the same results as for the degenerate Anderson model when \( t' = 1 \).

Introducing the physical rapidity \( \theta = \frac{2\pi \mu}{N} \), such that the dispersion relation of massless excitations is \( p = e \propto e^\theta \), we find that

\[
s_{N-p,N}(\theta) = \frac{1}{i} \frac{d}{d\theta} \sin \left( \frac{\theta}{2N} - \frac{\pi p}{2N} \right) \equiv \frac{1}{i} \frac{d}{d\theta} (p).
\]

In the case of higher values of \( t' \), we find more complex reflexion matrices. For \( t' = 2 \) for instance, one has \( (p - 1)(p + 1) \), etc. The meaning of these scattering matrices will be discussed further in [47].

To conclude, we see that, while the continuum limit without impurity was a Gross Neveu model with \( N \) colors and \( K \) flavors, either the fermionic impurity is irrelevant if \( t' > N - 1 \), or it affects the \( su(N) \) sector in the same way as in a pure \( su(N) \) theory (as for the \( su(K) \) and \( u(1) \) sector, they are still unaffected).

Non fermionic impurities in the fermionic chain could also be considered, but their effect is quite straightforward: they affect the \( SU(N) \) or \( SU(K) \) sectors of the Gross Neveu model as in pure \( SU(N) \) or \( SU(K) \) theories, giving rise to various \( N \) or \( K \) channel Kondo models (a solid state physics model corresponding to this situation was proposed in [48]).

9.2. Fermionic impurities in a non fermionic chain

Another example of interest is a fermionic impurity in a non fermionic chain. Consider for instance again the case \( N = K = 2 \) with a representation having Dynkin parameters \( a_1 = j \) in the bulk. The only difference with the analysis of section 8 is that a \( 1/L \) term appears in the right hand side of equation (8.3): the situation is thus very similar to the case of a fermionic representation in a fermionic chain: the impurity renormalizes the source terms for the two \( SU(2) \) systems by a term of order \( 1/L \) is rapidity independent: no flow is generated. This conclusion generalizes to any \( N, K \), and seems to agree with the results of [42].
10. Conclusion

In conclusion, it does not seem possible to observe any interesting “supersymmetric” properties in the continuum limit of integrable lattice models based on $sl(N/K)$ superalgebras\footnote{Models based on $osp(N/2M)$ seem more promising, since, according to [49], the ones based on the fundamental representation are conformal invariant.}. In fact, with the whole superalgebra symmetry, the continuum limit of these chains is not even relativistic, in sharp contrast with what happens in the case of ordinary algebras, where this continuum limit coincides with Wess Zumino models on the corresponding group. Interesting continuum limits can be obtained only when the superalgebra symmetry is broken. The case we have considered in details here leaves the $sl(N) \otimes sl(K) \otimes u(1)$ symmetry, and, in the continuum limit, gives various instances of color and flavor Gross Neveu models. The most “symmetric” case is obtained with fermionic representations with Dynkin parameter $t \geq N - 1$: in that case, one truly gets the $N$ colors, $K$ flavors, Gross Neveu model, with the remarkable property that all the excitations have the same sound velocity indeed. Other cases lead to continuum field theories with less symmetry, typically involving mixtures of massive and massless excitations with different sound velocities. We have also considered impurity models, concluding that in that case too, nothing really new is observed, impurities either leading to irrelevant perturbations, or reproducing known Kondo models in the $SU(N)$ or $SU(K)$ sectors of the GN model. In particular, the continuous parameter that is our disposal when using typical representations does not give rise to interesting tunable parameters in the field theory limit; in general, it simply affects the sound velocity, or the overall mass scale.

It must be stressed that models based on the quantum deformations $sl_q(N/K)$ would be relativistic, even without the introduction of a chemical potential (this is especially easy to see in the case of $sl(1/1)$, where turning on the quantum group deformation is equivalent to adding up the chemical potential $A$). It is not known what the continuum limit of these models is in general, nor what happens to them as $q \to 1$; this is actually a very intriguing question, on which I hope to report soon.

An issue that is somewhat related is what would happen for models based on, for instance, alternating fundamental representation and its conjugate. Unlike in the $sl(N)$ case, the conjugate of the fundamental does not behave like another fundamental representation. While, for instance, the $sl(3)$ model with an alternance of 3 and $\bar{3}$ is integrable, I do not know whether the $sl(2/1)$ model is, nor what the Bethe equations would look like.
This problem deserves more study, as it seems related with important issues in disordered systems [50].

**Acknowledgments:** I am grateful to A. Tsvelik, who kindly encouraged me to publish these results despite their somewhat disappointing nature. I also would like to thank N. Yu Reshetikhin for interesting discussions on super algebras and the Bethe ansatz. This work was supported by the DOE and the NSF (under the NYI program).
References

[1] I. Affleck, Nucl. Phys. B265 (1986) 409; H. M. Babujian, Nucl. Phys. B215 (1983) 317.
[2] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. 60 (1988) 956.
[3] M. Milovanovic and N. Read, Phys. Rev. B53 (1996) 13559.
[4] M. R. Zirnbauer, J. Math. Phys. 38 (1997) 2007.
[5] Z. Maassarani, J. Phys. A28 (1995) 1305.
[6] L. Rozansky and H. Saleur, Nucl. Phys. B376 (1992) 461.
[7] J.S. Caux, I. I. Kogan and A. M. Tsvelik, Nucl. Phys. B489 (1996) 444.
[8] H. Saleur, hep-th/9811023, to appear in J. Phys. A Lett.
[9] M. J. Martins, P. B. Ramos, Phys. Rev. B56 (1997) 6376.
[10] A. P. Bukhvostov, L. N. Lipatov, Nucl. Phys. B180 (1981) 116.
[11] V. A. Fateev, Nucl. Phys. B473 (1996) 509.
[12] F. Lesage, H. Saleur, P. Simonetti, Phys. Rev. B57 (1998) 4694.
[13] H. de Vega and E. Lopes, Phys. Rev. Lett. 67 (1991) 489.
[14] A. J. Bracken, M. D. Gould, J. R. Links and Y.Z. Zhang, Phys. Rev. Lett. 74 (1995) 2768; Z. Maassarani, J. Phys. A. 28 (1995) 1305.
[15] F. Essler, V. Korepin and K. Schoutens, Phys. Rev. Lett. 68 (1992) 2960.
[16] A. M. Tsvelik, Sov. Phys. JETP 66 (1987) 754.
[17] H. Frahm, M. Pfannmüller, A. M. Tsvelik, Phys. Rev. Lett. 81 (1998) 2116; H. Frahm, cond-mat/9904157
[18] N. Yu Reshetikhin, Lett. Math. Phys. 14 (1987) 235.
[19] P.B. Ramos and M. J. Martins, Nucl. Phys. B500 (1997) 579.
[20] M. P. Pfannmüller, H. Frahm, Nucl. Phys. B479 (1996) 575.
[21] K. Lee and P. Schlottmann, J. Physique Coll. 49 (1988) C8 709; P. Schlottmann, J. Phys. C4 (1992) 7565.
[22] M. Takahashi, Prog. Th. Phys. 44 (1970) 899.
[23] I. Bars, B. Morel and H. Ruegg, J. Math. Phys. 24 (1983) 2253.
[24] A. M. Tsvelik, Nucl. Phys. B305 (1988) 675.
[25] C. Ahn, D. Bernard and A. Leclair, Nucl. Phys. B346 (1990) 409; V. V. Bazhanov and N.Yu Reshetikhin, Prog. Theor. Phys. 102 (1990) 301.
[26] N. Yu Reshetikhin, H. Saleur, Nucl. Phys. B419 (1994) 507
[27] L. D. Fadeev and L. A. Takhtajan, Phys. Let. A85 (1981) 375.
[28] P. Fendley, Phys. Rev. Lett. 71 (1993) 2485; P. Fendley, H. Saleur, N. P. Warner, Nucl. Phys. B430 (1994) 597
[29] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, Commun.Math.Phys. 200 (1999) 297-324, and references therein.
[30] A. Polyakov and P. B. Wiegmann, Phys. Lett. 131B (1983) 121; P. B. Wiegmann, Phys. Lett. 141B (1984) 217. I. Affleck, Nucl. Phys. B265 (1986) 448, and references therein.

[31] V. Korepin, N. M. Bogoliubov, and A. G. Izergin, “Quantum inverse scattering method and correlation functions”, Cambridge (1993).

[32] A. Tsvelik, “Quantum field theory in condensed physics”, Cambridge (1995).

[33] N. Kawakami, Phys. Rev. B47 (1993) 2928.

[34] B. Sutherland, Phys. Rev. B12 (1975) 3795.

[35] F. Essler and V. Korepin, Phys. Rev. B46 (1992) 4197.

[36] P. A. Bares and G. Blatter, Phys. Rev. Lett. 64 (1990) 2567; N. Kawakami and S. K. Yang, Phys. Rev. Lett. 65 (1990) 2309; N. Kawamaki, Phys. Rev. B47 (1993) 2928.

[37] N. Andrei, M. Douglas and A. Jerez, cond-mat/9502082.

[38] H. Saleur, P. Simonetti, Nucl. Phys. B 535 (1998) 596.

[39] E. Kiritsis, Phys. Lett. B198 (1987) 379.

[40] L. D. Faddeev and N. Yu Reshetikhin, Ann. Phys. 167 (1986) 227; A. G. Izergin and V. E. Korepin, Lett. Math. Phys. 5 (1981) 199; C. Destri and H. de Vega, J. Phys. A22 (1989) 1329.

[41] N. Andrei and H. Johannesson, Phys. Lett. A100 (1984) 108.

[42] G. Bedürftig, F. H. Essler, and H. Frahm, Nucl. Phys. B489 (1997) 697.

[43] H. Q. Zhou and M. D. Gould, cond-mat/9809055; H. Fan, B. Y. Hou and K. J. Shi, cond-mat/9803213.

[44] A. M. Tsvelik and P. Wiegmann, Adv. Phys. 32 (1983) 453.

[45] N. Andrei, K. Furuya and J. Lowenstein, Rev. Mod. Phys. 55 (1983) 331.

[46] P. Schlottmann, Z. Phys. B49 (1982) 109.

[47] P. Fendley, H. Saleur, “Integrable boundary theories with $SU(n)$ symmetry”, in preparation.

[48] I.N. Karnaukhov, “Integrable model of orbital degenerate electrons interacting with impurity”, preprint March 15, 1995.

[49] M. J. Martins, B. Nienhuis, R. Rietman, Phys. Rev. Lett. 81 (1998) 504.

[50] I. Gruzberg, A. Ludwig and N. Read, cond-mat/9902063.