Partitions into Piatetski-Shapiro sequences

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Abstract

Let $\kappa$ be a positive real number and $m \in \mathbb{N} \cup \{\infty\}$ be given. Let $p_{\kappa, m}(n)$ denote the number of partitions of $n$ into the parts from the Piatetski-Shapiro sequence $(\lfloor \ell^\kappa \rfloor)_{\ell \in \mathbb{N}}$ with at most $m$ times (repetition allowed). In this paper we establish asymptotic formulas of Hardy-Ramanujan type for $p_{\kappa, m}(n)$, by employing a framework of asymptotics of partitions established by Roth-Szekeres in 1953, as well as some results on equidistribution.

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1 Introduction and statement of results

1.1 Background

Let $\lfloor u \rfloor$ denote the integral part of the real number $u$. The Piatetski-Shapiro sequence of parameter $\kappa (\in \mathbb{R}_+)$ (in short $PS(\kappa)$) is the sequence $(\lfloor \ell^\kappa \rfloor)_{\ell \in \mathbb{N}}$, which is studied in various directions. Let $m \in \mathbb{N} \cup \{\infty\}$ be given and $p_{\kappa, m}(n)$ denote the number of partitions of $n$ into the sequence $PS(\kappa)$ with at most $m$ times (repetition allowed). We simply write $p_{\kappa}(n) = p_{\kappa, \infty}(n)$ and $q_{\kappa}(n) = p_{\kappa, 1}(n)$. When $\kappa = 1$, $p_1(n)$ is the well-known unrestricted partition function $p(n)$ and $q_1(n)$ is partition function $q(n)$ with partitions into distinct parts.

Since the time of Euler, we have known that $p_{\kappa, m}(n)$ has the following generating function

$$G_{\kappa, m}(z) := \sum_{n \geq 0} p_{\kappa, m}(n)e^{-nz} = \prod_{\ell \geq 1} \sum_{0 \leq r \leq m} e^{-rz\lfloor \ell^\kappa \rfloor},$$

where $\Re(z) > 0$.

Hardy and Ramanujan [9] established the following celebrated asymptotic formulas for $p(n)$ and $q(n)$:

$$p(n) \sim \frac{1}{4 \sqrt{3n}} e^{\sqrt{n}/3} \quad \text{and} \quad q(n) \sim \frac{1}{4 \sqrt{3n}^{3/4}} e^{\sqrt{n}/3},$$

where $n$ is an integer.

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as $n \to \infty$. We call these types asymptotic formulas of Hardy-Ramanujan type. After Hardy and Ramanujan, such type problems have been widely investigated in many works in the literature. For example, Ingham [10], Meinardus [18], Schwarz [25, 26] and Richmond [20, 21, 22], Roth and Szekeres [24], Wright [30] has investigated and established powerful asymptotic results for various types of integer partition functions.

Of particular interest are functions related to partitions into certain special sequences. As an application of a powerful asymptotic results for general functions, Roth and Szekeres [24] established asymptotic formulas for partition functions with partitions into rather general polynomial, whenever the argument of this polynomial taking from positive integers or primes. For more, see for examples [1, 4, 6, 8, 27, 32] and [28, 31] for partitions into polynomials and primes, respectively. Erdős and Richmond [7] considered the Hardy-Ramanujan type for partitions into the Beatty sequences $\lfloor \alpha \ell \rfloor_{\ell \in \mathbb{N}}$, where $\alpha > 1$ is an irrational number with finite irrationality measure. The first author [33] improved their result to any irrational number $\alpha > 1$.

We now go back to this paper’s topic, partitions into the sequence $\text{SP}(\kappa)$. Recently, Chen and the second author in [2, 13, 14] investigated the asymptotics of $r$-th root partition function with any $r > 1$, which corresponds to the partition function $p_{1/r}(n)$. They established upper and low bound and some asymptotics for $p_{1/r}(n)$. For $r = 2$, Luca and Ralaivaosaona [17] established an asymptotic formula of Hardy-Ramanujan type for $p_{1/2}(n)$, that is,

$$p_{1/2}(n) \sim 2^{5/18}3^{-1/2}\pi^{-1/2}\zeta(3)^{7/18}e^{2\zeta(-1)+\zeta'(0)}n^{-8/9} \times \exp\left(\frac{3\zeta(3)^{1/3}}{2^{1/3}}n^{2/3} + \frac{\zeta(2)}{2^{2/3}\zeta(3)^{1/3}}n^{1/3} - \frac{\zeta(2)^2}{24\zeta(3)}\right), \quad (1.3)$$

as $n \to \infty$, where $\zeta(s)$ is the Riemann zeta function. In [3], Chern gave the asymptotic formula of Hardy-Ramanujan type for $q_{1/2}(n)$, the square-root partitions into distinct parts, by adjusting the well-known approach of Meinardus [18], that is,

$$q_{1/2}(n) \sim 2^{-7/6}3^{-1/3}\pi^{-1/2}\zeta(3)^{1/6}n^{-2/3} \times \exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3} + \frac{\zeta(2)}{2 \cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3} - \frac{\zeta(2)^2}{72\zeta(3)}\right). \quad (1.4)$$

The asymptotic formula of Hardy-Ramanujan type for $k$-th root partition $p_{1/k}(n)$ with any $k \in \mathbb{N}$ was established by Wu and the second author of this paper in a very recent work [15].

### 1.2 Main results

In this paper, we shall investigate and establish the asymptotic formula of Hardy–Ramanujan type for $p_{\kappa,m}(n)$, for any $\kappa \in \mathbb{R}^+$ and any $m \in \mathbb{N} \cup \{\infty\}$. Throughout this
paper, we write \( \alpha = 1/k \). To state our main results, we need the following so-called Piatetski-Shapiro zeta function:

\[
\zeta_k(s) = \sum_{n \geq 1} \frac{1}{[n^k]^s}, \quad \Re(s) > \alpha.
\]  

(1.5)

Furthermore, we need the following notations and definitions. Let \( 1_{\text{event}} \) be the indicator function. For \( u \in \mathbb{R} \), let \([u]\) and \(\lfloor u \rfloor\) denote the smallest integer \(\geq u\) and the fractional part of \(u\), respectively. For \( z \in \mathbb{C} \), let

\[
E_0(z) := (1 - z), \quad E_h(z) = (1 - z) \exp \left( \sum_{1 \leq j \leq h} z^j/j \right) (h \geq 1)
\]

be the canonical factor. For any function \( f \) and \( x \in \mathbb{R} \), let \( f(\infty) := \lim_{m \to +\infty} f(m) \) and \( \widetilde{B}_1(x) := \lfloor x \rfloor - 1/2 \). We first prove the following proposition.

**Proposition 1.1.** For some real number \( \sigma_k \in (0,1) \), \( \zeta_k(s) \) can be monomorphic continued analytically to \( \Re(s) \geq -\sigma_k \) whose singularities are simple poles at \( s = \alpha, \alpha - 1, \ldots, \alpha + 1 - \lfloor \alpha \rfloor \), and their residues are

\[
\text{Res}_{s=\alpha-h} (\zeta_k(s)) = \frac{\Gamma(\alpha+1)}{(h+1)! \Gamma(\alpha-h)}, \quad h = 0, 1, \ldots, \lfloor \alpha \rfloor - 1.
\]

Furthermore, for all \( s \) with \( \Re(s) \geq \epsilon - \sigma_k \) and \( |\Im(s)| \geq 1 \) with any \( \epsilon > 0 \), we have

\[
\zeta_k(s) \ll_\epsilon |s|(|s| + 1)^{|\alpha|+1}.
\]

Moreover,

\[
\zeta_k(0) = -\frac{1_{\beta \notin \mathbb{N}}}{2} - \frac{1_{\beta \in \mathbb{N}}}{1 + \kappa'}
\]

and

\[
\zeta_k' (0) = \begin{cases} 
\sum_{0 \leq h < \alpha} \frac{\binom{\alpha}{h} \zeta'(-h)}{h} & \alpha \in \mathbb{N}, \\
\sum_{1 \leq h \leq |\alpha|} \frac{\zeta(\alpha-h-1)}{h} + \sum_{n \geq 2} \log \left( \frac{E_0(1/n)^{\beta_1(n^{1/k})+1} \beta_1(n^{1/k})}{E_0(1/n)^{\beta_1(n^{1/k})}} \right), & \alpha \notin \mathbb{N}.
\end{cases}
\]

**Remark 1.1.** Since for \( k \in \mathbb{N} \), we have \( \zeta_k(s) = \zeta(ks) \). Note that \( 2\zeta'(0) = -\log(2\pi) \), by the above expression of \( \zeta_k'(0) \) we obtain that

\[
-\frac{k}{2} \log(2\pi) + 1 = \zeta(1 - 1/k) + \sum_{n \geq 2} \log \left( \frac{E_0(1/n^k)E_0(1/n)^{\beta_1(n^{1/k})}}{E_1(1/n)^{\beta_1(n^{1/k})}} \right),
\]

for all integers \( k \geq 2 \). For \( k = 2, 3 \), the above identities has passed the numerical examination of Mathematica.

Under the above proposition, our main results are stated as follows.
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Theorem 1.2. For any $m \in \mathbb{N} \cup \{\infty\}$, as $n \to \infty$

$$p_{\kappa,m}(n) \sim \lambda_{\kappa,m}(\beta_{\kappa,m})^{-\delta_{\kappa,m}} \frac{\beta_{\kappa,m}}{n} \exp \left( \sum_{0 \leq h \leq \alpha} \lambda_{\kappa,m}(h) \left( \frac{n}{\beta_{\kappa,m}} \right)^{\frac{\delta_{\kappa,m}}{\beta_{\kappa,m}}} \right),$$

where $\delta_{\kappa,m} = \frac{1}{2} + \frac{\alpha}{2} - \frac{1}{2} \arg \left( \frac{\zeta(\alpha + 1)}{\zeta(\alpha)} \right)$.

Let $\lambda_{\kappa,m}(h) = \left( \frac{(m + 1)^\alpha \zeta(\alpha + 1)}{\alpha^2 \Gamma(\alpha + 1)} \right)^{\frac{\alpha}{\beta_{\kappa,m}}}$.

\begin{align*}
\lambda_{\kappa,m}(2) &= \frac{(m + 1)^\alpha \zeta(\alpha + 1) \lambda_{\kappa,m}(2)}{2 \alpha^2 \lambda_{\kappa,m}(0)^2}, \\
\lambda_{\kappa,m}(3) &= \frac{(m + 1)^\alpha \zeta(\alpha + 1) \lambda_{\kappa,m}(3)}{24 \alpha^2 \lambda_{\kappa,m}(0)^3}.
\end{align*}

Remark 1.2. For the details of the calculation of $\lambda_{\kappa,m}(h)$, see Lemma 2.3 of Section 2. In particular, with the help of Mathematica, we can get

$$\left( \lambda_{\kappa,m}(0), \lambda_{\kappa,m}(1) \right) = \left( (1 + \kappa) \beta_{\kappa,m}, \frac{(m + 1)^\alpha \zeta(\alpha + 1)}{2 \alpha^2 \lambda_{\kappa,m}(0)^2} \right),$$

and

$$\left( \lambda_{\kappa,m}(2), \lambda_{\kappa,m}(3) \right) = \left( (1 + \kappa) \beta_{\kappa,m}, \frac{(m + 1)^\alpha \zeta(\alpha + 1) \lambda_{\kappa,m}(2)}{2 \alpha^2 \lambda_{\kappa,m}(0)^3} \right).$$

Corollary 1.3. Let $\kappa > 1$. As $n \to \infty$

$$p_{\kappa}(n) \sim c_{\kappa} n^{-\frac{\alpha + 1}{\alpha + m}} \exp \left( (1 + \kappa) \beta_{\kappa}^{-\frac{1}{\alpha + m}} n^{\frac{1}{\alpha + m}} \right),$$

and

$$q_{\kappa}(n) \sim c_{\kappa} n^{-\frac{\alpha + 1}{\alpha + m}} \exp \left( (1 + \kappa) (1 - 2^{-\alpha}) \beta_{\kappa}^{-\frac{1}{\alpha + m}} n^{\frac{1}{\alpha + m}} \right),$$

where $\beta_{\kappa} = \alpha \Gamma(\alpha + 1) \zeta(\alpha + 1)$ and

$$\left( c_{\kappa}, c_{\kappa} \right) = \left( \frac{\sqrt{2} e^{\zeta(0)} \beta_{\kappa}^{-\frac{1}{\alpha + m}}, \zeta(4) = \pi^4 / 90 \text{ and } \zeta'(-2) = \zeta(3) / 4 \pi^2} {2 \sqrt{\pi (1 + \alpha)}} \right).$$

Using Theorem 1.2, we can find the same asymptotics as Luca-Ralaivaosaona [17] and Chern [3], that is, asymptotics of $p_{1/2}(n)$ and $q_{1/2}(n)$ are (1.3) and (1.4), respectively. Moreover, by Theorem 1.2 we can also find the same leading asymptotic formulas as Wu and the second author of this paper in [15], for the $k$-th root partition functions $p_{1/k}(n)$ \footnote{In [15], some coefficients of asymptotic formula for $p_{1/2}(n)$ are not explicitly given.}. In particular, by noting $\zeta(2) = \pi^2 / 6$, $\zeta(4) = \pi^4 / 90$ and $\zeta'(-2) = \zeta(3) / 4 \pi^2$, we have the following exactly asymptotic formulas.
Corollary 1.4. As $n \to \infty$

$$p_{1/3}(n) \sim \frac{(25\pi)^{1/4} e^{\frac{25\pi (3)^3}{\pi^8} \frac{2\pi (3)^3}{n^2} + 3\zeta'(-1)}}{4(5n)^{13/16}} \times \exp \left( \frac{4\pi (5n)^{3/4}}{15} + \frac{3\zeta(3)(5n)^{1/2}}{\pi^2} + \frac{\left(\pi^6 - 15\zeta(3)^2\right)(5n)^{1/4}}{6\pi^5} \right),$$

and

$$q_{1/3}(n) \sim \frac{(5/14)^{1/2} e^{\frac{60\pi (3)^3}{49\pi^8} \frac{15\pi (3)^3}{28\pi^2}}}{2^{11/4}(5n/14)^{5/8}} \times \exp \left( \frac{28\pi \left(\frac{5n}{14}\right)^{3/4}}{15} + \frac{9\zeta(3)\left(\frac{5n}{14}\right)^{1/2}}{\pi^2} + \frac{\left(7\pi^6 - 1215\zeta(3)^2\right)\left(\frac{5n}{14}\right)^{1/4}}{42\pi^5} \right).$$

Our paper is organized as follows: In Section 2 we will prove Theorem 1.2, thus establish an asymptotic formula $p_{\kappa,m}(n)$. In Section 3 we use some results on equidistribution to check the sequence $PS(\kappa)$ meets the conditions of the framework of Roth and Szekeres [24]. In Section 4 we use a theorem of van der Corput [5], the classical Taylor’s theorem to prove Proportion 1.1, that is, the analytic continuation for the Piatestki-Shapiro zeta function $\zeta_\kappa(s)$.

## 2 The proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. Our proof is based on the framework of Roth and Szekeres [24] and its generalization, that is, the work of Liardet and Thomas [16]. In order to be able to use this framework, we need to prove that the sequence $PS(\kappa)$ meets the following conditions.

(I) There exist $\mu > 0$ such that

$$\mu = \lim_{H \to \infty} \frac{\log[H^\kappa]}{\log H},$$

(II) We have\(^2\)

$$\lim_{H \to \infty} \inf_{\frac{1}{2} \leq y \leq \frac{1}{2}} \left( \frac{1}{\log H} \sum_{1 \leq \ell \leq H} \|\ell^\kappa \|_y \right)^2 = \infty.$$ \(^2\)Note that the condition corresponding to $p_{\kappa,\infty}(n)$ in Roth and Szekeres [24, p.259, (II*)] is

$$\lim_{H \to \infty} \inf_{\frac{1}{2} \leq y \leq \frac{1}{2}} \left( \frac{1}{\log H} \sum_{1 \leq \ell \leq H} \frac{\left[H^\kappa / \ell^\kappa \right]}{\log \ell^\kappa} \|\ell^\kappa \|_y \right)^2 = \infty.$$ 

Obviously, this condition is weaker than that required by $p_{\kappa,n}(n)$ with $m \in \mathbb{N}$. 

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Clearly, the condition (I) is easily to verified. For the condition (II), we shall check it in Section 3, by employ the equidistribution properties of the sequence \((\ell^\kappa)_{\ell \geq 1}\), as well as some works of van der Corput [5].

Define that for any \(\kappa > 0\)

\[
L_\kappa(z) = - \sum_{\ell \geq 1} \log \left( 1 - e^{-z\ell^\kappa} \right), \quad \Re(z) > 0.
\]

Then, since the sequence \(PS(\kappa)\) satisfies the above two conditions, following the works of Roth and Szekeres [24, Theorem 2] and Liardet and Thomas [16, Theorem 14.2], we can get the following proposition immediately.

**Proposition 2.1.** Let \(m \in \mathbb{N}\). For a small positive number \(\epsilon\), as \(n \to +\infty\)

\[
p_\kappa(n) = \frac{\exp (L_\kappa(x) + nx)}{\sqrt{2\pi L''_\kappa(x)}} \left( 1 + O\left( n^{\frac{1}{m+1} + \epsilon} \right) \right),
\]

and

\[
p_{\kappa,m}(n) = \frac{\exp (L_\kappa(y) - L_\kappa((m+1)y) + ny)}{\sqrt{2\pi (L''_\kappa(y) - (m+1)^2 L''_\kappa((m+1)y))}} \left( 1 + O\left( n^{\frac{1}{m+1} + \epsilon} \right) \right),
\]

where \(x\) and \(y\) are positive numbers which solves the equations

\[
L_\kappa(x) + n = 0 \quad \text{and} \quad L'_\kappa(y) - (m+1)L'_\kappa((m+1)y) + n = 0,
\]

respectively.

According to the Mellin transform representation of \(L_\kappa(x)\) and the analytic properties of \(\zeta_\kappa(s)\), that is, Proposition 1.1, as well as some well fact on analytic number theory, we will prove the following proposition.

**Proposition 2.2.** Let \(\epsilon\) be a sufficiently small positive number. For some real number \(\sigma_\kappa \in (0, 1]\) which is in Proposition 1.1 and each integer \(j \geq 0\), as \(x \to 0^+\)

\[
\frac{d^j}{dx^j} L_\kappa(x) = \frac{d^j}{dx^j} \left( \sum_{0 \leq h < \alpha} \frac{\Gamma(\alpha + 1)\zeta(\alpha - h + 1)}{(h + 1)!x^{\alpha-h}} - \zeta_\kappa(0) \log x + \zeta'_\kappa(0) \right) + O(x^{\sigma_\kappa - \epsilon - j}).
\]

**Proof.** By the Mellin’s inversion formula, for any \(c > \alpha\) we have

\[
\frac{d^j}{dx^j} L_\kappa(x) = \sum_{0 \leq h < \alpha} \frac{\Gamma(\alpha + 1)\zeta(\alpha - h + 1)}{(h + 1)!x^{\alpha-h}} - \zeta_\kappa(0) \log x + \zeta'_\kappa(0) + O(x^{\sigma_\kappa - \epsilon - j}).
\]

Therefore, for each integer \(j \geq 0\),

\[
(-1)^j L^{(j)}_\kappa(z) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\Gamma(s)\zeta(s+1)\zeta_\kappa(s)z^{-s}}{s-j} ds.
\]

(2.1)
It is well known (see [19, p.38, p.92]) that fixed any \(a, b \in \mathbb{R}\), for all \(\sigma \in [a, b]\) and any \(|t| \geq 1\),
\[
\Gamma(\sigma + it) \ll |t|^{|\sigma|/2} \exp\left(-\frac{\pi}{2} |t|\right)
\]
and
\[
\zeta(\sigma + it) \ll |t|^{|\sigma|+1/2}.
\]
From the estimation of \(\zeta(s)\) in Proposition 1.1 we have
\[
\Gamma(s)\zeta(s + 1)\zeta_\kappa(s) \ll \varepsilon |t|^{O(1)} \exp\left(-\frac{\pi}{2} |t|\right),
\]
for \(t \in \mathbb{R} \setminus (-1, 1)\) and \(\sigma \geq \varepsilon - \sigma_\kappa\). Thus, by the residue theorem, we can move the line of integration (2.1) to the \(\Re(s) = \varepsilon - \sigma_\kappa\) and taking into account the possible poles on \(\Re(s) > -\sigma_\kappa\) of \(\Gamma(s)\zeta(s + 1)\zeta_\kappa(s)z^{-s-j}\), we obtain
\[
(-1)^j/L_k^j(z) = \lim_{\Re(s) > -\sigma_\kappa} \Gamma(s + j)\zeta(s + 1)\zeta_\kappa(s)z^{-s-j} + O_{\varepsilon}\left(|z|^{-j+\sigma_\kappa-\varepsilon}\right).
\]
Note that the only poles of gamma function \(\Gamma(s)\) are at \(s = -k\) \((k \in \mathbb{Z}_{\geq 0})\), and all are simple; \(s = 1\) is the only pole of \(\zeta(s)\) and is simple; from Proposition 1.1, all poles of \(\zeta_\kappa(s)\) lies on \(\Re(s) > -\sigma_\kappa\) are simple listed as follows
\[
\alpha, \alpha - 1, \ldots, \alpha + 1 - [\alpha],
\]
and for all integer \(h\) with \(0 \leq h < \alpha\),
\[
\lim_{s=\alpha-h} \Gamma(s)\zeta(s + 1)\zeta_\kappa(s) = \frac{\Gamma(\alpha + 1)}{(h + 1)!\Gamma(\alpha - h)}.
\]
Thus for \(j = 0\), we have
\[
\lim_{\Re(s) > -\sigma_\kappa} \frac{\Gamma(s)\zeta(s + 1)\zeta_\kappa(s)}{z^s} = \lim_{s=0} \frac{\Gamma(s)\zeta(s + 1)\zeta_\kappa(s)z^{-s}}{} + \sum_{0 \leq h < \alpha} \lim_{s=\alpha-h} \frac{\Gamma(s)\zeta(s + 1)\zeta_\kappa(s)z^{-s}}{}
\]
\[
= \zeta_\kappa(0) - \zeta_\kappa(0) \log z + \sum_{0 \leq h < \alpha} \frac{\Gamma(\alpha + 1)\zeta(\alpha - h + 1)}{(h + 1)!z^{\alpha-h}},
\]
and for integer \(j \geq 1\), we have
\[
\lim_{\Re(s) > -\sigma_\kappa} \frac{\Gamma(s + j)\zeta(s + 1)\zeta_\kappa(s)}{z^{s+j}} = \frac{\Gamma(j)\zeta_\kappa(0)}{z^j} + \sum_{0 \leq h < \alpha} \frac{\Gamma(\alpha + 1)\Gamma(\alpha - h + j)\zeta(\alpha - h + 1)}{(h + 1)!\Gamma(\alpha - h)z^{\alpha-h+j}}.
\]
This completes the proof. \(\square\)
Under Proposition 2.1, the proof of Theorem 1.2 immediately follows from the sharp asymptotics of $L_\kappa(z)$ stated in Proposition 2.2 and the following Lemma 2.3.

**Lemma 2.3.** Let $a, b \in \mathbb{R}$ and let $(c_h)_{h \geq 0}$ be a sequence of real numbers which has finite support. Let $\alpha, c_0 \in \mathbb{R}_+$ and $\sigma \in (0, 1]$ and real function $L(t)$ satisfy

$$
\frac{d^j}{dt^j} L(t) = \frac{d^j}{dt^j} \left( a - b \log t + \sum_{h \geq 0} c_h t^{h-\alpha} \right) + O(t^{\sigma-j}), \ j = 0, 1, 2
$$

as $t \to 0^+$. If $u > 0$ solves the equation $L'(u) + n = 0$, then as $n \to +\infty$ there exist computable real constants $\lambda_h$ such that

$$
e^{L(u)+nu} \sqrt{2\pi L''(u)} = \sqrt{\frac{\lambda_0}{\pi}} \frac{1}{n^{\lambda_0/2}} e^{\lambda_0} \exp \left( \sum_{0 \leq h \leq 3} \lambda_h \left( \frac{n}{c_0 \alpha} \right)^{\frac{\lambda_h}{\alpha+1}} + O \left( \frac{n^{-\min(1-\sigma, \sigma/\alpha)}}{1+\alpha} \right) \right).
$$

In particular,

$$
(\lambda_0, \lambda_2) = \left( c_0 (\alpha + 1), c_2 - \frac{(\alpha - 1)^2 \lambda_1^2}{2 \alpha \lambda_0} \right)
$$

and

$$
(\lambda_1, \lambda_3) = \left( c_1, c_3 - \frac{(\alpha - 1)(\alpha - 2) \lambda_1 \lambda_2}{\alpha \lambda_0} - \frac{(\alpha - 4)(\alpha - 1)^3 \lambda_3}{6 \alpha^2 \lambda_0^2} \right).
$$

**Proof.** Let $n \to +\infty$. From $n + L'(u) = 0$ and $c_0 > 0$, it is clear that $u \to 0^+$ and

$$
n = \sum_{h \geq 0} \frac{(\alpha - h) c_h}{u^{\alpha+1-h}} + \frac{b}{u} + O \left( u^{-1+\sigma} \right).
$$

This means $u \sim (\alpha c_0 / n)^{1/\alpha}$. By (2.2) it implies $L''(u) = c_0 \alpha (\alpha + 1) u^{-\alpha-2} (1 + O(u))$. By further reduction formula we obtain

$$
e^{L(u)+nu} \sqrt{2\pi L''(u)} = u^{1+\alpha/2-b} (1 + O(u)) \frac{\exp \left( b + \sum_{h \geq 0} c_h (\alpha - h + 1) u^{h-\alpha} \right)}{e^{\alpha} \sqrt{2\pi c_0 \alpha (\alpha + 1)}}.
$$

Let $P(u) = \sum_{h \geq 1} \frac{(\alpha - h) c_h}{\alpha c_0} u^{h-1}$ and $t_n = (\alpha c_0 / n)^{1/\alpha}$. Using generalized binomial theorem we obtain

$$
t_n = u \left( 1 + u P(u) + bu^\alpha / \alpha c_0 + O \left( u^{\alpha+\sigma} \right) \right)^{-1/\alpha}
$$

$$
= u \left( 1 + u P(u) \right)^{-1/\alpha} - \frac{u^{\alpha+1}}{c_0 \alpha (\alpha + 1)} \left( b + O \left( u^{\min(1, \sigma)} \right) \right).
$$

This means that

$$
u = t_n \left( 1 + O \left( t_n^{\min(1, \alpha)} \right) \right).
$$
Therefore, as 2.5

\[ \hat{t}_n = t_n + u^{\alpha + 1} \left( b + O \left( u^{\min(\alpha, \sigma)} \right) \right) / c_0 \alpha (\alpha + 1) \]

\[ = t_n + bt_n^{\alpha + 1} / c_0 \alpha (\alpha + 1) + O \left( t_n^{\alpha + 1 + \min(\alpha, \sigma)} \right). \]

(2.5)

Since \((c_h)_h\) have finite support, \(P(u)\) is a polynomial and hence \((1 + zP(z))^{1/(1+\alpha)}\) is analytic at \(z = 0\). Recall that the Bürmann Theorem (see [29, p. 129]) states that: Suppose that \(f(z)\) is analytic at \(z = 0\) and \(f(0) = 0\), if \(t = uf(u)^{-1}\) then for \(t \to 0\)

\[ u = \sum_{j \geq 0} \frac{t^{j+1}}{(j+1)!} \left| \frac{d^j}{dz^j} \right|_{z=0} f(z)^{j+1} \].

Therefore, as \(\hat{t}_n \to 0\),

\[ u = \sum_{j \geq 0} \frac{t^{j+1}}{(j+1)!} \left| \frac{d^j}{dz^j} \right|_{z=0} \left( 1 + zP(z) \right)^{j+1}. \]

Then, for each integer \(h \geq 0\) it is clear that \((\hat{t}_n u)^{h-\alpha}\) is analytic at \(\hat{t}_n = 0\). Hence the finite sum function \(\sum_{k \geq 0} c_k (\alpha - h + 1) u^{h-\alpha}\) is analytic at \(t_n = 0\). So, as \(\hat{t}_n \to 0\),

\[ \sum_{h \geq 0} c_h (\alpha - h + 1) u^{h-\alpha} = \hat{t}_n^{-\alpha} \sum_{j \geq 0} \lambda_j \hat{t}_n^j \]

(2.6)

holds for a computable sequence \((\lambda_j)_j\) of real numbers which can be obtained with the help of Mathematica. Inserting (2.5) and using the generalized binomial theorem, we have

\[ \hat{t}_n^{\alpha-\alpha} = t_n^{\alpha-\alpha} \left( 1 + (j - \alpha)bt_n^\alpha / c_0 \alpha (\alpha + 1) + O \left( t_n^{\alpha + \min(\alpha, \sigma)} \right) \right). \]

Inserting (2.6) we have

\[ \sum_{h \geq 0} c_h (\alpha - h + 1) u^{h-\alpha} = \hat{t}_n^{-\alpha} \sum_{0 \leq j \leq \alpha} \lambda_j t_n^j - b + O \left( t_n^{\alpha + 1 - \alpha} + t_n^{\min(\alpha, \sigma)} \right) \]

(2.7)

Substituting (2.7) and \(t_n = (c_0 \alpha / n)^{1/\alpha}\) to (2.3), we complete the proof. \(\square\)

3 The check of the condition (II)

For the case of \(\kappa \in \mathbb{N}\), Roth and Szekeres [24, P.241] have mentioned that for any integer \(\kappa > 0\), \((P^\kappa)_{\kappa \in \mathbb{N}}\) satisfies the condition (II). Hence we only verify the case of \(\kappa \in \mathbb{R}_+ \setminus \mathbb{N}\). We begin with the following lemmas.

Lemma 3.1. For \(\kappa \in \mathbb{R}_+ \setminus \mathbb{N}\) and \(|y| \leq 1/2\), we have

\[ \left| \sum_{H/2 \leq \ell < H} e^{2\pi i y(\ell^\kappa)} \right| \leq \left| \sum_{H/2 \leq \ell < H} e^{2\pi i y \ell^\kappa} \right| + \frac{\pi H}{8} + o(H), \]

as \(H \to +\infty\).
Partitions into Piatetski-Shapiro sequences

Proof. By the triangle inequality, we have
\[ \left| \sum_{H/2 \leq \ell < H} e^{2\pi i y(\ell^x)} \right| \leq \sum_{H/2 \leq \ell < H} e^{2\pi i y(\ell^x - \frac{1}{2})} + \sum_{H/2 \leq \ell < H} e^{2\pi i y(\ell^x - \frac{1}{2})} \left( e^{2\pi i y(\frac{1}{2} - \ell^x)} - 1 \right) \]
\[ \leq \sum_{H/2 \leq \ell < H} e^{2\pi i y\ell^x} + \sum_{H/2 \leq \ell < H} \left| e^{2\pi i y(1/2 - \ell^x)} - 1 \right|. \]

Notice that
\[ \left| e^{2\pi i y(1/2 - \ell^x)} - 1 \right| = \left| \int_0^{2\pi i y(1/2 - \ell^x)} e^{iu} \, du \right| \]
\[ \leq 2\pi y(1/2 - \ell^x) \leq \pi |1/2 - \ell^x|, \]
and \((\ell^x)_{\ell \in \mathbb{N}}\) is uniformly distributed in \((0, 1)\) for \(\kappa \in \mathbb{R}_+ \setminus \mathbb{N}\), see for example [12, p.31, Exercises 3.9]. It follows that
\[ \sum_{H/2 \leq \ell < H} \left| e^{2\pi i y(1/2 - \ell^x)} - 1 \right| \leq \pi \sum_{H/2 \leq \ell < H} |1/2 - \ell^x| \]
\[ \sim \frac{\pi H}{2} \int_0^1 |1/2 - x| \, dx = \frac{\pi H}{8}, \]
as \(H \to \infty\). Thus, we complete the proof of Lemma 3.1.

Lemma 3.2. For \(\kappa \in \mathbb{R}_+ \setminus \mathbb{N}\) and \((\log H)^{-2\kappa} \ll y \leq 1/2\), we have
\[ \sum_{H/2 \leq \ell < H} e^{2\pi i y\ell^x} = o(H), \]
as \(H \to \infty\).

Proof. For the cases of \(\kappa \in (0, 1]\), it follows from [11, p.206, Lemma 8.8] and \((\log H)^{-2\kappa} \ll y \leq 1/2\) that
\[ \sum_{H/2 \leq \ell < H} e^{2\pi i y\ell^x} = \int_{H/2}^H e^{2\pi i y\ell^x} \, dt + O\left( \frac{1}{1 - y H^{\kappa - 1}} \right) \]
\[ = \frac{1}{\kappa} \int_{(H/2)^x}^{H^x} e^{2\pi i yu} u^{1/\kappa - 1} \, du + O(1) \]
\[ = \frac{1}{2\pi i \kappa} \int_{(H/2)^x}^{H^x} u^{1/\kappa - 1} \, du^2 \pi i y u + O(1) \]
\[ = O\left( y^{-1} H^{1-\kappa} \right) + O(1) = o(H). \]
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For the cases of \( \kappa \in (1, \infty) \), we have \( \lceil \kappa \rceil \geq 2 \). It follows from [11, p.213, Theorem 8.20] and \((\log H)^{-2\kappa} \ll y \leq 1/2\) that

\[
\sum_{H/2 \leq \ell \leq H} e^{2\pi i \ell y} \ll H^{1-2^{\kappa}} (yH^{1-k})^{\frac{1}{3\sqrt{2}}} + H(yH^{1-k})^{\frac{1}{3\sqrt{2}}}
\]

\[
= H(yH^{1-k+4-8^{\kappa}})^{\frac{1}{3\sqrt{2}}} + o(H)
\]

\[
\ll H(yH^{1+\{\kappa\}})^{\frac{1}{3\sqrt{2}}} + o(H) = o(H).
\]

Thus, we complete the proof of Lemma 3.2. \( \square \)

We now verify that the sequence \( \text{PS}(\kappa) \) with \( \kappa \in \mathbb{R}_+ \setminus \mathbb{N} \) satisfies the condition (II). For \((\log H)^{-2\kappa} \ll y \leq 1/2\), using Lemma 3.1 and Lemma 3.2, we have

\[
\sum_{1 \leq \ell \leq H} ||(\ell\kappa) y||^2 \geq \frac{1}{\pi^2} \sum_{1 \leq \ell \leq H} \sin^2 \left( \frac{\pi^2 [(\ell\kappa) y]}{\ell\kappa} \right)
\]

\[
\geq \frac{1}{2\pi^2} \sum_{H/2 \leq \ell \leq H} \left( 1 - e^{2\pi i [(\ell\kappa) y]} \right)
\]

\[
\geq \frac{H}{4\pi^2} - \frac{1}{2\pi^2} \sum_{H/2 \leq \ell \leq H} \left| e^{2\pi i [(\ell\kappa) y]} \right| + O(1)
\]

\[
= \frac{H}{4\pi^2} - \frac{1}{2\pi^2} \cdot \frac{\pi H}{8} + o(H) = \frac{H}{4\pi^2} \left( 1 - \frac{\pi}{4} \right) + o(H),
\]

as \( H \to \infty \). Note that \( 1 - \pi/4 > 0 \), which implies that for \((\log H)^{-2\kappa} \ll y \leq 1/2\), one has

\[
\frac{1}{\log H} \sum_{1 \leq \ell \leq H} ||(\ell\kappa) y||^2 \gg \frac{H}{\log H} \to \infty, \quad H \to \infty.
\] (3.1)

On the other hand, for \((2[H\kappa])^{-1} \leq y \leq (\log H)^{-2\kappa}\), using the fact that \( ||x|| = x \) for \( x \in (0, 1/2) \), we have the estimate that

\[
\frac{1}{\log H} \sum_{1 \leq \ell \leq H} ||(\ell\kappa) y||^2 \gg \frac{1}{\log H} \sum_{1 \leq \ell \leq (2y)^{-1/\kappa}} \ell^{2\kappa} y^{2} \gg \frac{y^{-1/\kappa}}{\log H} \gg (\log H) \to \infty,
\]

as \( H \to \infty \). Therefore, combining with (3.1), we obtain that the sequence \( \text{PS}(\kappa) \) with \( \kappa \in \mathbb{R}_+ \setminus \mathbb{N} \) satisfies the condition (II).

4 The proof of Proposition 1.1

In this section, we use a theorem of van der Corput [5], the classical Taylor’s theorem to study the analytic properties of \( \zeta_{\kappa}(s) \).
4.1 On a theorem of van der Corput

Define the Dirichlet series

$$\hat{\zeta}_\alpha(s) = \sum_{n \geq 1} \frac{\widetilde{B}_1(n^\alpha)}{n^s},$$

for all $\Re(s) > 1$. Recall that $\widetilde{B}_1(x) = \{x\} - 1/2$, we have $\hat{\zeta}_\alpha(s) = -\zeta(s)/2$ when $\alpha$ is a positive integer. For $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, we prove the following lemma.

**Lemma 4.1.** Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. We have

$$\sum_{1 \leq n \leq N} \widetilde{B}_1(n^\alpha) \ll N^{1-\sigma_\alpha} \log N, \text{ for } N \geq 2,$$

where

$$\sigma_\alpha = \max_{m \in \mathbb{N}_2} \min \left( \frac{2^{1-m} - \alpha}{2^{m-1}}, \frac{2^{1-m} - 1}{2^{m-1}}, \frac{2^{1-m} - \alpha}{2^{m-1}} \right) \in (0, 1).$$

**Proof.** Fixed $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, let $f(x) = x^{\alpha}$, $x > 0$. Then for any given integer $m \geq 2$ and positive number $X \geq 2$ we have $f^{(m)}(x) \asymp X^{\alpha-m}$, for all $x \in (X, 2X]$. Using a theorem of van der Corput, see [23, Theorem (van der Corput)] or [5, Satz 3], we have

$$\sum_{X < n \leq 2X} \widetilde{B}_1(n^\alpha) \ll X \left( X^{-2^{1-m}} \log X + X^{2^{m-1} - \alpha} \right).$$

Choosing integer $m \geq 2$ such that the exponent of $X$ above takes the smallest value, we obtain

$$\sum_{X < n \leq 2X} \widetilde{B}_1(n^\alpha) \ll X^{1-\sigma_\alpha} \log X.$$

Finally we replace the dyadic segment $X < n \leq 2X$ by the whole interval $1 \leq n \leq N$ by subdividing latter, we have

$$\sum_{1 \leq n \leq N} \widetilde{B}_1(n^\alpha) \ll 1 + \sum_{0 \leq j \leq \log_2 N-1} \sum_{2^{j-1}N < n \leq 2^jN} \widetilde{B}_1(n^\alpha) \ll N^{1-\sigma_\alpha} \log N.$$

This completes the proof. $\square$

Using integration by parts for a Riemann-Stieltjes integral, we obtain the following proposition.

**Proposition 4.2.** Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and let $\sigma_\alpha(\in (0, 1))$ be given by Lemma 4.1. Then $\hat{\zeta}_\alpha(s)$ can be holomorphic continued analytically to $\Re(s) > 1 - \sigma_\alpha$. Moreover,

$$\hat{\zeta}_\alpha(s) \ll_{s, \varepsilon} 1 + |s|, \text{ for all } \Re(s) \geq \varepsilon + 1 - \sigma_\alpha \text{ with any sufficiently small } \varepsilon > 0.$$
4.2 Analytic continuation of $\zeta_\kappa(s)$

**Proposition 4.3.** For $\alpha \in \mathbb{N}$ we have

$$\zeta_\kappa(s) = \sum_{0 \leq h < \alpha} (\alpha_h) \zeta(s - h).$$

*(4.1)*

**Proof.** Since $\alpha \in \mathbb{N}$, the series for $\zeta_\kappa(s)$ can be rewritten as

$$\zeta_\kappa(s) = \sum_{n \geq 1} \frac{\#\{\ell \in \mathbb{N} : \lfloor \ell^\kappa \rfloor = n\}}{n^s}$$

$$= \sum_{n \geq 1} \frac{(n + 1)^\alpha - n^\alpha}{n^s}$$

$$= \sum_{n \geq 1} \frac{1}{n^s} \sum_{0 \leq h < \alpha} \left(\frac{\alpha}{h}\right) n^h,$$

which completes the proof by interchanging the order of the above summation. \(\square\)

**Proposition 4.4.** For $\alpha \notin \mathbb{N}$, we have

$$\zeta_\kappa(s) = \sum_{1 \leq \ell \leq [\alpha]} \frac{\Gamma(s + \ell)}{\ell! \Gamma(s)} \zeta(s + \ell - \alpha) - \frac{s}{2} \zeta(s + 1) - s\zeta_\alpha(s + 1) + sR_\alpha(s),$$

*(4.2)*

where $R_\alpha(s)$ is given as below *(4.4)* and holomorphic function for all $\Re(s) > -\min(1/2, 1 - \{\alpha\})$.

Moreover, for all $\Re(s) \geq \epsilon - \min(1/2, 1 - \{\alpha\})$ with any small $\epsilon > 0$,

$$R_\alpha(s) \ll \epsilon (|s| + 1)^{[\alpha]}.$$

**Proof.** The series for $\zeta_\kappa(s)$ can be rewritten as

$$\zeta_\kappa(s) = \sum_{n \geq 1} \frac{\#\{\ell \in \mathbb{N} : \lfloor \ell^\kappa \rfloor = n\}}{n^s}$$

$$= \sum_{n \geq 1} \frac{[\left(\frac{n + 1}{n}\right)^\alpha] - \left[\frac{n^\alpha}{n}\right]}{n^s}$$

$$= \sum_{n \geq 2} \left(\frac{1}{(n - 1)^s} - \frac{1}{n^s}\right) \left[\frac{n^\alpha}{n}\right] - 1).$$

Let us define that

$$d_\alpha = \begin{cases} \infty, & \alpha \notin \mathbb{Q}, \\ \text{the denominator of reduced fraction of } \alpha, & \alpha \in \mathbb{Q}. \end{cases}$$

*(4.3)*

Since $\alpha \notin \mathbb{N}$, it is clear that $d_\alpha \geq 2$. Using the fact that for all $x \in \mathbb{R}$,

$$\lfloor x \rfloor - 1 = x - \mathbf{1}_{x \in \mathbb{Z}} - \lfloor x \rfloor,$$
we have
\[
\zeta_n(s) = \sum_{n \geq 2} \frac{\left(1 - \frac{1}{n}\right)^s - 1}{n^s} - \sum_{n \geq 2} \frac{(1 - \frac{1}{n})^{-s} - 1}{n^{s-a}} - \sum_{n \geq 2} \frac{(1 - \frac{1}{n})^{-s} - 1}{n^s}.
\]

Using the classical Taylor’s theorem, we have
\[
(1 - t)^{-s} = \sum_{0 \leq j \leq h} \frac{(s)_j t^j}{j!} + \frac{(s)_{h+1} t^{h+1}}{h!} \int_0^1 \frac{(1 - \theta)^h d\theta}{(1 - t\theta)^{s+h}}, \quad |t| < 1,
\]
for each integer \( h \geq 0 \), where \((s)_\ell = s(s+1) \cdots (s+\ell-1) = \Gamma(s+\ell)/\Gamma(s)\) is the Pochhammer symbol. Thus, we have
\[
\zeta_n(s) = \sum_{n \geq 1} \frac{1}{n^{s-a}} \sum_{1 \leq \ell \leq [\alpha]} \frac{(s)_\ell}{\ell!} \frac{1}{n^\ell} + s \sum_{n \geq 1} \frac{1/2 - \{n^{\alpha}\}}{n^{s+1}} - \frac{s(s+1)}{2} + sR_\alpha(s),
\]
where
\[
R_\alpha(s) = -\sum_{n \geq 2} \frac{1}{n^{d_\alpha(s)+1}} \int_0^1 \frac{d\theta}{(1 - \theta/n^{d_\alpha})^{s+1}} - \sum_{n \geq 2} \frac{(s+1)\{n^{\alpha}\}}{n^{s+2}} \int_0^1 \frac{(1 - \theta) d\theta}{(1 - \theta/n)^{s+2}}
+ \sum_{n \geq 2} \frac{(s+1)_{[\alpha]}}{n^{s-\alpha+[\alpha]+1}[\alpha]!} \int_0^1 \frac{(1 - \theta)^{[\alpha]} d\theta}{(1 - \theta/n)^{s+[\alpha]+1}} - \sum_{1 \leq \ell \leq [\alpha]} \frac{(s+1)_{\ell-1}}{\ell!}.
\]

It is clear that \( R_\alpha(s) \) holomorphic for all \( \Re(s) > -\min(1 - [\alpha], 1/2) \). Moreover, for \( \sigma = \Re(s) \geq \varepsilon - \min(1 - [\alpha], 1/2) \) with any \( \varepsilon > 0 \), we have
\[
|R_\alpha(s)| \ll \sum_{n \geq 2} \frac{1}{n^{d_\alpha(s)+1}} + \sum_{n \geq 2} \frac{(|s| + 1)}{n^{s+2}} + \sum_{n \geq 2} \frac{(|s| + 1)_{[\alpha]}}{n^{\sigma-\alpha+[\alpha]+1}} + \sum_{0 \leq \ell \leq [\alpha]} \frac{(|s| + 1)_{\ell}}{(\ell + 1)!}
\ll \zeta(2\sigma + 2) + (|s| + 1)\zeta(\sigma + 2) + (|s| + 1)_{[\alpha]}\zeta(\sigma + 2 - [\alpha])
\ll \varepsilon \cdot (|s| + 1)_{[\alpha]} \ll (|s| + 1)^{[\alpha]}.
\]
This completes the proof of this proposition. \( \square \)

It is well-known that \( \zeta(s) \) can be monomorphic continued analytically to whole complex plane \( \mathbb{C} \) with only pole \( s = 1 \). From Proposition 4.2, \( \tilde{\zeta}_n(s) \) can be holomorphic continued analytically to \( \Re(s) > 1 - \sigma_\alpha \). Therefore, from Proposition 4.3 and Proposition 4.4, we find that there exists
\[
\sigma_\kappa = \min(\sigma_\alpha, 1/2, 1 - [\alpha]) \in (0, 1)
\]
such that \( \zeta_\kappa(s) \) can be monomorphic continued analytically to \( \Re(s) > -\sigma_\kappa \) and all poles lies on \( \Re(s) > -\sigma_\kappa \) which are simple listed as follows

\[
\alpha, \alpha - 1, \ldots, \alpha + 1 - \lceil \alpha \rceil.
\]

Combing with (4.1) and (4.2), it is clear that

\[
\text{Res}_{s=\alpha-h} (\zeta_\kappa(s)) = \frac{\Gamma(\alpha + 1)}{(h + 1)!\Gamma(\alpha - h)},
\]

and for all \( \Re(s) \geq \varepsilon - \sigma_\kappa \) with any \( \varepsilon > 0 \) and \( |\Im(s)| \geq 1 \),

\[
\zeta_\kappa(s) \ll \varepsilon |s|(|s| + 1)^{|\alpha|+1}.
\]

Then, under the above discussion, the proof of of Proposition 1.1 will follows from below subsection about the evaluating of \( \zeta_\kappa(0) \) and \( \zeta'_\kappa(0) \).

### 4.3 The evaluating of \( \zeta_\kappa(0) \) and \( \zeta'_\kappa(0) \)

**Proposition 4.5.** For \( \alpha \in \mathbb{N} \) we have

\[
\zeta_\kappa(0) = -\frac{\alpha}{\alpha + 1} \quad \text{and} \quad \zeta'_\kappa(0) = \sum_{0 \leq h < \alpha} \binom{\alpha}{h} \zeta'(-h).
\]

**Proof.** For \( \alpha \in \mathbb{N} \), from (4.1)

\[
\zeta_\kappa(s) = \sum_{0 \leq h < \alpha} \binom{\alpha}{h} \zeta(s - h),
\]

and the well-known facts that \( \zeta(0) = -1/2 \) and \( \zeta(-h) = -B_{h+1}/(h + 1), h \in \mathbb{N} \), where \( B_\ell \) is the usual \( \ell \)-th Bernoulli number, we have

\[
\sum_{0 \leq h < \alpha} \binom{\alpha}{h} \zeta(-h) = -\frac{1}{2} - \sum_{1 \leq h < \alpha} \binom{\alpha}{h} \frac{B_{h+1}}{h + 1}
\]

\[
= -\frac{1}{2} - \frac{1}{1 + \alpha} \left( \sum_{0 \leq h \leq \alpha} \binom{\alpha + 1}{h} B_h - 1 - \binom{\alpha + 1}{1} B_1 \right)
\]

\[
= -\frac{1}{2} - \frac{1}{1 + \alpha} \cdot \frac{\alpha - 1}{2} = -\frac{\alpha}{\alpha + 1}.
\]

Here we use the fact that \( B_1 = -\frac{1}{2} \) and for \( \alpha \in \mathbb{N} \),

\[
\sum_{0 \leq h \leq \alpha} \binom{\alpha + 1}{h} B_h = 0.
\]

This completes the proof. \( \square \)
Lemma 4.6. For \( \alpha \notin \mathbb{N} \) we have
\[
\zeta_{\kappa}(0) = -1/2
\]
and
\[
\zeta'_{\kappa}(0) = \sum_{1 \leq h \leq [\alpha]} \frac{\zeta(h - \alpha) - 1}{h} + \sum_{n \geq 2} \log \left( \frac{E_{0}(1/n) B_{1}(n^{\alpha}) + 1_{n^{\alpha} \in \mathbb{N} \cap (1/n)^{\alpha}}}{E_{[\alpha]}(1/n)^{\alpha}} \right). \tag{4.5}
\]

Proof. It directly follows from (4.2) and the fact that \( \zeta(s + 1) = 1/s + \gamma + O(s) \) that \( \zeta_{\kappa}(0) = -1/2 \) and
\[
\zeta'_{\kappa}(0) = \sum_{1 \leq \ell \leq [\alpha]} \frac{\zeta(\ell - \alpha)}{\ell} - \frac{\gamma}{2} - \hat{\zeta}(1) + R_{\alpha}(0). \tag{4.6}
\]

Here \( \gamma \) is the Euler-Mascheroni constant. From (4.4), we have
\[
R_{\alpha}(0) = \sum_{n \geq 2} \log \left( 1 - \frac{1}{n^{\alpha}} \right) - \sum_{n \geq 2} \frac{[n^{\alpha}]}{n^2} \int_{0}^{1} (1 - \theta)(1 - \theta/n)^{-\alpha} \, d\theta
+ \sum_{n \geq 2} \frac{1}{n^{2-\alpha}} \int_{0}^{1} (1 - \theta)^{\alpha}(1 - \theta/n)^{-[\alpha]-1} \, d\theta - \sum_{1 \leq \ell \leq [\alpha]} \frac{1}{\ell}.
\]
Note that for each integer \( h \geq 0 \),
\[
\int_{0}^{1} (1 - \theta)^{h} \, d\theta = \frac{1}{(1 - \theta/n)^{h+1}} = -n^{h+1} \left( \sum_{1 \leq j \leq h} \frac{1}{j} + \log \left( \frac{1}{1 - \frac{1}{n}} \right) \right),
\]
we further have
\[
R_{\alpha}(0) = \sum_{n \geq 2} \log \left( 1 - \frac{1}{n^{\alpha}} \right) + \sum_{n \geq 2} \frac{[n^{\alpha}]}{n} \left( \frac{1}{n} + \log \left( 1 - \frac{1}{n} \right) \right)
- \sum_{n \geq 2} n^{[\alpha]-1+[\alpha]} \left( \sum_{1 \leq j \leq [\alpha]} \frac{1}{jn} + \log \left( 1 - \frac{1}{n} \right) \right) - \sum_{1 \leq \ell \leq [\alpha]} \frac{1}{\ell}.
\]
Using the fact that
\[
\sum_{n \geq 2} \left( \frac{1}{n} + \log \left( 1 - \frac{1}{n} \right) \right) = \gamma - 1,
\]
it implies that
\[
R_{\alpha}(0) = \sum_{n \geq 2} \log \left( 1 - \frac{1}{n^{\alpha}} \right) + \sum_{n \geq 2} ([n^{\alpha}] - 1/2) \log \left( 1 - \frac{1}{n} \right) + \frac{\gamma}{2} + \hat{\zeta}(1)
- \sum_{n \geq 2} n^{[\alpha]} \left( \sum_{1 \leq j \leq [\alpha]} \frac{1}{jn} + \log \left( 1 - \frac{1}{n} \right) \right) - \sum_{1 \leq \ell \leq [\alpha]} \frac{1}{\ell}.
\]
Recall that \( E_{k}(z) = (1 - z)e^{\sum_{1 \leq j \leq k} z^{j}/j} \) and \( B_{1}(x) = \lfloor x \rfloor - 1/2 \) and we can complete the proof. \( \Box \)
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