Gradient Descent for One-Hidden-Layer Neural Networks: Polynomial Convergence and SQ Lower Bounds

Santosh Vempala
Georgia Institute of Technology
vempala@gatech.edu

John Wilmes
Brandeis University
wilmes@brandeis.edu

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Abstract

We study the complexity of training neural network models with one hidden nonlinear activation layer and an output weighted sum layer. We analyze Gradient Descent applied to learning a bounded target function on \( n \) real-valued inputs. We give an agnostic learning guarantee for GD: starting from a randomly initialized network, it converges in mean squared loss to the minimum error (in 2-norm) of the best approximation of the target function using a polynomial of degree at most \( k \). Moreover, for any \( k \), the size of the network and number of iterations needed are both bounded by \( n^{O(k)} \log(1/\epsilon) \). The core of our analysis is the following existence theorem, which is of independent interest: for any \( \epsilon > 0 \), any bounded function that has a degree \( k \) polynomial approximation with error \( \epsilon_0 \) (in 2-norm), can be approximated to within error \( \epsilon_0 + \epsilon \) as a linear combination of \( n^{O(k)} \cdot \text{poly}(1/\epsilon) \) randomly chosen gates from any class of gates whose corresponding activation function has nonzero coefficients in its harmonic expansion for degrees up to \( k \). In particular, this applies to training networks of unbiased sigmoids and ReLUs. We also rigorously explain the empirical finding that gradient descent discovers lower frequency Fourier components before higher frequency components.

We complement this result with nearly matching lower bounds in the Statistical Query model. GD fits well in the SQ framework since each training step is determined by an expectation over the input distribution. We show that any SQ algorithm that achieves significant improvement over a constant function with queries of tolerance some inverse polynomial in the input dimensionality \( n \) must use \( n^{\Omega(k)} \) queries even when the target functions are restricted to a set of \( n^{O(k)} \) degree-\( k \) polynomials, and the input distribution is uniform over the unit sphere; for this class the information-theoretic lower bound is only \( \Theta(k \log n) \).

Our approach for both parts is based on spherical harmonics. We view gradient descent as an operator on the space of functions, and study its dynamics. An essential tool is the Funk-Hecke theorem, which explains the eigenfunctions of this operator in the case of the mean squared loss.

1 Introduction

It is well known that artificial neural networks (NNs) can approximate any real-valued function. Fundamental results [20, 7, 4] show that a NN with a single hidden layer provides a universal representation up to arbitrary approximation, with the number of hidden units needed depending on the function being approximated and the desired accuracy.

Besides their generality, an important feature of NNs is the ease of training them — gradient descent (GD) is used to minimize the error of the network, measured by a loss function of the current weights. This seems to work across a range of labeled data sets. Yet despite its tremendous success, there is no satisfactory explanation for the efficiency or effectiveness of this generic training algorithm[1].

The difficulty is that even for highly restricted classes of NNs, natural loss functions such as the mean squared loss have a highly non-convex landscape with many nonoptimal local minima. However, when data is generated from a

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1Indeed, one might consider this a miraculous feat of engineering and even ask, is there anything to explain rigorously? We are not entirely comfortable with this view and optimistic of some life beyond convexity.
model with random weights, GD (the stochastic version with a small batch size) seems to consistently learn a network with error close to zero. This raises the prospect of a provable guarantee, but there are two complicating experimental observations. First, the randomness of the initialization appears essential (standard in practice) as in experiments it is possible to remain stuck at higher error. Second, we observe smaller error (and it decreases more quickly) when the model size used for training is made larger; in particular, for the realizable case (when the data is itself labeled by a NN), we train using many more units than the original. This aspect is also commonly encountered in the training of large NNs on real data — even with huge amounts of data, the size of the model used can be larger.

In this paper we give nearly matching upper and lower bounds that help explain the phenomena seen in practice when training NNs. The upper bounds are for GD and the lower bounds are for all statistical query algorithms. We summarize them here, and present them formally in the next section.

Our algorithmic result is an agnostic upper bound on the approximation error and time and sample complexity of GD with the standard mean squared loss function. Despite training only the output layer weights, our novel proof techniques avoid using any convexity in the problem. Since our analysis does not rely on reaching a global minimum, there is reason to hope the techniques will extend to nonconvex settings where we can in general expect only to find a local minimum. Prior results along this line were either for more complicated algorithms or more restricted settings; the closest is the work of Andoni et al.\cite{Andoni2017} where they assume the target function is a bounded degree polynomial. A detailed comparison of results is given in Section\ref{sec:experiments} As a corollary of our convergence analysis, we obtain a rigorous proof of the “spectral bias” of gradient descent observed experimentally in \cite{Hernández-Lobato2015}.

The upper bound shows that to get close to the best possible degree \(k\) polynomial approximation of the data, it suffices to run GD on a NN with \(n^{O(k)}\) units, using the same number of samples. It suffices to train the output layer weights alone. This is an agnostic guarantee. We prove a matching lower bound for solving this polynomial learning problem over the uniform distribution on the unit sphere, for any statistical query algorithm that uses tolerance inversely proportional to \(n^{\Omega(k)}\). Thus, for this general agnostic learning problem, GD is as good as it gets.

\subsection{Results}

We consider NNs with on inputs from the sphere \(S^{n-1} \subseteq \mathbb{R}^n\), a single hidden layer with \(m\) units having some nonlinear activation \(\phi: \mathbb{R} \to \mathbb{R}\), and a single linear output unit. All units are without additive bias terms. We will consider inputs drawn from the uniform distribution \(D\) on \(S^{n-1}\).

We denote by \(W\) the set of units in the hidden layer, and abuse notation to write \(u \in W\) for both the hidden layer unit and its corresponding weight vector in \(\mathbb{R}^n\). The output-layer weight corresponding to \(u \in W\) will be denoted \(b_u \in \mathbb{R}\). Hence, the NN computes a function of the form

\[
\phi(x) = \sum_{u \in W} b_u \phi(u \cdot x). \tag{1}
\]

We initialize our NNs by choosing the vectors \(u \in W\) independently from \(D\), and setting each \(b_u\) to 0.

For two functions \(f, g: S^{n-1} \to \mathbb{R}\), the mean squared loss with respect to \(D\) is \(\mathbb{E}_{x \sim S^{n-1}}((f(x) - g(x))^2)\). Given data \((x, y)\) with \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}\), we analyze GD to minimize the mean squared loss of the current model with respect to the given data. The specific GD procedure we consider is as follows: in each iteration, the gradient of the loss function is computed using a finite sample of examples, with the entire sample reused for each iteration. The output-layer weights \(b_u\) are then modified by adding a fixed multiple of the estimated gradient, and the hidden-layer weights \(u \in W\) are kept fixed.

\textbf{Convergence guarantees.} Our first theorem is for training networks of sigmoid gates. The same statement holds for ReLU activation units and even functions \(g\).

\textbf{Theorem 1.1.} Let \(\varepsilon_0 > 0\), \(k \in \mathbb{N}\), and \(g: S^{n-1} \to \mathbb{R}\) an odd bounded function such that \(\|g - g^{(\leq k)}\|_2 \leq \varepsilon_0\), where \(g^{(\leq k)}\) denotes the best polynomial of degree at most \(k\) approximation of \(g\) in \(L^2\) norm on \(S^{n-1}\). Then for any \(\varepsilon > 0\), for some \(m = n^{O(k)}\poly\left(\|g\|_2 / \varepsilon\right)\) the following holds: A randomly initialized single-hidden-layer NN with \(m\) sigmoid gates in the hidden layer and a linear output layer, with high probability, will have mean squared loss of at most \(\varepsilon_0 + \varepsilon\) after at most \(n^{O(k)}\log(\|g\|_2 / \varepsilon)\) iterations of GD applied to the output layer weights, re-using a set of \(m\) samples in each iteration.
Next we state a more general theorem. This will apply to a large class of activation functions. The main property we need of the activation function is that it should not be a low-degree polynomial. We first introduce additional notation. (See Section 2 for related definitions and background.) We denote by \( \mathcal{H}_{n,k} \) the set of spherical harmonics of degree \( k \) on the sphere \( S^{n-1} \).

**Definition 1.2.** Given an \( L^2 \) function \( f \) on \( S^{n-1} \), we denote by \( f^{(k)} \) the projection of \( f \) to \( \mathcal{H}_{n,k} \), so \( f = \sum_{k=0}^{\infty} f^{(k)} \). We also write \( f^{(\leq k)} = \sum_{i=0}^{k} f^{(i)} \), and for \( S \subseteq \mathbb{N} \), we write \( f^{(S)} = \sum_{i \in S} f^{(i)} \). For \( S \subseteq \mathbb{N} \) and \( \alpha > 0 \), an \( (n,S,\alpha) \)-activation is a function \( \phi : \mathbb{R} \to \mathbb{R} \) with the property that for any \( u \in S^{n-1} \), the map \( f(x) = \phi(u \cdot x) \) has a harmonic polynomial expansion with \( \|f^{(k)}\| \geq \alpha \) for all \( k \in S \).

Since the dimension \( n \) is uniform throughout this paper, we will abbreviate our notation and refer to \( (S,\alpha) \)-activations. The set \( S \) will not generally depend on \( n \), but the quantity \( \alpha = \alpha(n) \) generally will (see, e.g., Lemma 2.6).

For example, the commonly used sigmoid gate \( \sigma_{\text{sig}}(x) = 1/(1 + e^{-x}) \) is an \( (S,\alpha) \)-activation function for \( S \) the odd integers less than \( k \) and \( \alpha = n^{-O(k)} \). Similarly, ReLU gates are \( (S,\alpha) \)-activation functions for subsets \( S \) of the even integers.

**Theorem 1.3.** Let \( \varepsilon_0 > 0 \) and \( g : S^{n-1} \to \mathbb{R} \) a bounded function such that \( \|g - g^{(S)}\|_2 \leq \varepsilon_0 \). Then for any \( \varepsilon > 0 \), and any \( (S,\alpha) \)-activation function \( \phi \) with \( \|\phi\|_{\infty} \leq 1 \), for some \( m = \text{poly}(1/\alpha, \|g\|_2/\varepsilon) \) the following holds: A randomly initialized single-hidden-layer NN with \( m \) \( \phi \)-gates in the hidden layer and a linear output layer, with high probability, will have mean squared loss of at most \( \varepsilon_0 + \varepsilon \) after at most \( \text{poly}(1/\alpha) \log(\|g\|_2/\varepsilon) \) iterations of GD applied to the output layer weights, re-using a set of \( m \) samples in each iteration.

This general theorem has the following corollary in the realizable case, when data is generated by a one-hidden-layer NN. In this case, the function can be approximated by a low-degree polynomial. In order to allow for this approximation guarantee, and to side-step previous statistical query lower bounds [30], we guarantee some degree on nondegeneracy by focusing on unbiased NNs, i.e., networks without additive bias terms (as in Eq. (1)).

**Corollary 1.4.** Let \( g \) be computed by an unbiased one-hidden-layer NN with sigmoid units in the hidden layer and a linear output. Suppose the \( \ell_1 \) norm of the output layer weights is \( \alpha \), and each hidden layer weight vector has \( \ell_2 \) norm at most \( b \). Then for every \( \varepsilon > 0 \), for some \( m = \text{poly}(1/\alpha, \|g\|_2/\varepsilon) \) the following holds: A randomly initialized single-hidden-layer NN with \( m \) sigmoid units in the hidden layer and a linear output layer, with high probability, will have mean squared loss of at most \( \varepsilon_0 + \varepsilon \) after at most \( \text{poly}(1/\alpha) \log(\|g\|_2/\varepsilon) \) iterations of GD applied to the output layer weights, re-using a set of \( m \) samples in each iteration.

The use of sigmoid units in Corollary 1.4 is not essential, but the bounds on network size and training time will depend on the specific activation function chosen.

**Spectral bias.** As a consequence of our techniques, we give a proof of the “spectral bias” phenomenon observed experimentally in [26]. The experiments of [26] showed that neural networks trained via gradient descent learned low Fourier frequencies more quickly than higher frequencies, which the authors propose as a mechanism to explain generalization performance of deep learning. We prove that low frequencies are indeed learned more quickly than high frequencies, where “low frequencies” and “high frequencies” are understood as low and high degree harmonic components of a function.

To quantify the relative speed of learning, we introduce additional notation.

**Definition 1.5.** Let \( H_i \) denote the residual after training a NN via GD for \( i \) iterations, and let \( \Delta_{i}^{(j)} = H_{i}^{(j)} - H_{i-1}^{(j)} \) denote the change in the \( i \)th residual in degree \( j \). Suppose \( H_i^{(k)}, H_i^{(\ell)} \neq 0 \). We say \( r = r_i^{(k,\ell)} > 0 \) is the rate of progress in degree \( k \) relative to degree \( \ell \) if
\[
\frac{\|\Delta_{i}^{(k)}\|}{\|\Delta_{i}^{(\ell)}\|} = r \frac{\|H_i^{(k)}\|}{\|H_i^{(\ell)}\|}.
\]

Thus, if the rate of progress in degree \( k \) relative to degree \( \ell \) is \( r_i^{(k,\ell)} > 1 \), the network learns degree-\( k \) information more quickly—and degree-\( \ell \) information less quickly—compared to what would be expected based on the relative sizes of the residual in degrees \( k \) and \( \ell \). Conversely, if the rate \( r_i^{(k,\ell)} < 1 \), the network learns the degree-\( k \) information more slowly.
Lipschitzness assumption. We say an SQ algorithm makes inner product queries \( h \) than any fixed constant with probability at least 1 of variance \( \varepsilon \) which the oracle replies with the value of oracle, even with Gaussian noise, responds honestly and is not allowed to make any adversarial changes or 1-STAT.

Our lower bounds concern three query models. We extend these oracles to more general problems over distributions.

Theorem 1.7. Fix \( \varepsilon > 0 \) and \( \ell > k \). For some \( m = n^{O(\ell)} \text{poly}(1/\varepsilon) \), the following holds for any odd bounded function \( g : S^{n-1} \to \mathbb{R} \). A randomly initialized single-hidden-layer NN with \( m \) sigmoid gates in the hidden layer and a linear output layer, trained via GD applied to the output layer weights, re-using a set of \( m \) samples in each iteration, with high probability will have rate of progress in degree \( k \) relative to degree \( \ell \) at least \( r_i^{(k,\ell)} \geq n^{\Omega(\ell-k)} \), assuming the \( i \)-th residual \( H_i \) satisfies \( \|H_i^{(\ell)}\|_2, \|H_i^{(k)}\|_2 \geq \varepsilon \).

Lower bounds. Our lower bounds hold in the very general Statistical Query (SQ) model, first defined by Kearns [22]. An SQ algorithm solves a computational problem over an input distribution and interacts with the input only by querying the expected value of \( f \) to an oracle that replies with an approximation of \( f \) with \( \varepsilon \)-error.

The bound is the standard deviation of \( t \) independent Bernoulli coins with desired expectation, i.e., the error that even a random sample of size \( t \) would yield. The SQ complexity of an algorithm is given by the number of queries and the batch size \( t \). The remaining computation is unrestricted and can use randomization. We will also give lower bounds against the 1-STAT oracle, which responds to queries with a single honest bit. Given a distribution \( D \) over \( X \) and a query function \( h : X \to [0, 1] \), the 1-STAT oracle responds with a single value \( h(x) \), where \( x \sim D \).

The SQ framework was introduced by Kearns for supervised learning problems [22] using the STAT(\( \tau \)) oracle, which, for \( \tau \in \mathbb{R}_+ \), responds to a query function \( h : X \to [0, 1] \) with a value \( v \) such that \( |\mathbb{E}_D(h) - v| \leq \tau \). The STAT(\( \sqrt{\tau} \)) oracle can be simulated by the VSTAT(\( O(1/\tau) \)) oracle. The VSTAT oracle was introduced by [12] who extended these oracles to more general problems over distributions.

Choosing a useful SQ model for regression problems is nontrivial. We discuss some of the pitfalls in Section 4. Our lower bounds concern three query models.

The first allows quite general query functions. We say an SQ algorithm (for regression) makes \( L^\infty \)-normalized \( \lambda \)-Lipschitz queries concerning an unknown concept \( g : X \to \mathbb{R} \) if it makes queries of the form \( h : X \times [-1, 1] \to [0, 1] \), where \( h \) is \( \lambda \)-Lipschitz at any fixed \( x \in X \), to which the SQ oracle should respond with a value \( v \) approximating \( \mathbb{E}_{x \sim D}(h(x, g(x))/\|g\|_\infty) \). We get similar lower bounds for a natural family of inner product queries with no Lipschitzness assumption. We say an SQ algorithm makes inner product queries concerning an unknown concept \( g : X \to \mathbb{R} \) if it makes queries of the form \( h : X \to [0, 1] \) to an oracle that replies with an approximation of \( \mathbb{E}_{x \sim D}(g(x)h(x)) \). Finally, we say an SQ algorithm makes \( L^\infty \)-normalized queries to 1-STAT with Gaussian noise of variance \( \varepsilon \) concerning an unknown concept \( g : X \to \mathbb{R} \) if it makes queries of the form \( h : X \times \mathbb{R} \to [0, 1] \), to which the oracle replies with the value of \( h(x, g(x))/\|g\|_\infty + \zeta \) where \( x \sim D \) and \( \zeta \sim N(0, \varepsilon) \). Unlike VSTAT, the 1-STAT oracle, even with Gaussian noise, responds honestly and is not allowed to make any adversarial changes or coordinate its responses to multiple queries.

Theorem 1.7. Let \( \varepsilon > 0 \). For all \( k, \lambda > 0 \) and all sufficiently large \( n \) and \( d < \exp(n^{1/2-\varepsilon}) \), there exists a family \( \mathcal{C} \) of degree-\( k \) polynomials on \( S^n \) with \( |\mathcal{C}| = d \) such that if a randomized SQ algorithm learns \( \mathcal{C} \) to regression error less than any fixed constant with probability at least 1/2:

1. it requires at least \( \Omega(d) \) queries, if the queries are inner product queries to VSTAT(\( n^{\Omega(k)} \));
2. it requires at least \( \Omega(d) \) queries, if the queries are \( L^\infty \)-normalized \( \lambda \)-Lipschitz queries to VSTAT(\( n^{\Omega(k)}/\lambda \));
3. for \( d = n^{\Omega(k)} \), it requires at least \( n^{\Omega(k)}/\lambda \) queries, if the queries are \( L^\infty \)-normalized queries to 1-STAT with Gaussian noise of variance \( 1/\lambda^2 \)

where all the hidden constants depend on \( \varepsilon \) only.

In the case of training a NN via GD, the relevant queries should yield gradients of the loss function with respect to the current weights. In the case of mean squared loss, the gradients are of the form \( \mathbb{E}((g - f)^2 \nabla_w f) \), where \( g \) is the unknown concept, \( f \) is the current network, and \( w \) represents parameters of \( f \). These gradients can be estimated via queries in any of the models we consider. The lower bound on the 1-STAT oracle in particular implies that as long as the function values of queried inputs are perturbed by a random Gaussian, any training algorithm needs \( n^{\Omega(k)} \) queries.
1.2 Approach and techniques

The gradient of the loss function with respect to any outer layer weight can be viewed as a spherical transform of the current residual error. More precisely, if the current function $f$ is computed by an unbiased single hidden-layer NN with output-layer weights $b_u$, as in Eq. (1), and the residual error with respect to the target function $g$ is $H = g - f$, then for any $u$,

$$
\nabla_{b_u} \|H\|^2 = 2 \mathbb{E}_x (\phi(u \cdot x) H(x)).
$$

(2)

The latter expectation is quite special when the domain of integration is the unit sphere. Different choices of the function $\phi$ correspond to different spherical transformations. For example, $\phi(u \cdot x)$ being the indicator of $u \cdot x \geq 0$ is the hemispherical transform, while $\phi(u \cdot x) = 1$ iff $u \cdot x = 0$ is the Radon transform, etc. This type of transformation

$$
\mathcal{J}_\phi(H)(u) = \mathbb{E}_{x \in S^{n-1}} (\phi(x \cdot u) H(x))
$$

has a closed form expression whenever the function $H$ is a harmonic polynomial (see definitions in Section 2). By the classical Funk-Hecke theorem, for any bounded function $H$ has a closed form expression whenever the function $\phi$ is a harmonic polynomial (see Section 2). By the classical Funk-Hecke theorem, for any bounded function $\phi$ and any harmonic polynomial $P$ of degree $k$ on the sphere $S^{n-1} \subseteq \mathbb{R}^n$, there is an explicit constant $\alpha_{n,k}(\phi)$ such that

$$
\mathcal{J}_\phi(P)(u) = \alpha_{n,k}(\phi) P(u).
$$

In particular, the harmonic polynomials are eigenfunctions of the operator $\mathcal{J}_\phi$. Moreover, since there exists an orthonormal basis of harmonic polynomials for $L^2$ functions over the unit sphere, any function (in our case the residual $H$) has zero norm iff the corresponding transform has zero norm (assuming the function $\phi$ has nonzero coefficients $\alpha_{n,k}(\phi)$).

With the above observations in hand, we can now outline our analysis. We focus on the dynamics of GD as an operator on a space of functions. In particular, for a set $Z \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \to \mathbb{R}$, we define an operator

$$
T_Z(f)(u) = \frac{1}{|Z|} \sum_{z \in Z} f(z) \phi(u \cdot z).
$$

(3)

Thus, if the current residual error is given by some function $H$, then the empirical gradient of the mean-squared loss with respect to a set $X$ of labeled examples is $T_X(H)$ (see Section 3).

Our analysis proceeds in three stages:

1. Show that, with a large enough set $X$ of samples, the empirical gradient operator $T_X$ approximates the Funk transform $\mathcal{J}_\phi$ as an operator on the space of residual error functions (Lemmas 3.5 and 3.6).

2. Bound the rate at which error from the approximation of $T_X$ by $\mathcal{J}_\phi$ accumulates over multiple rounds of GD (Lemmas 3.7 and 3.8).

3. Estimate the final loss in terms of the distance of the target function from the space of low-degree harmonic polynomials — i.e., the distance from the most significant eigenspaces of $\mathcal{J}_\phi$ (see proof of Lemma 3.3).

A crucial observation that simplifies our analysis is that when $f$ is given by a NN as in Eq. (1), then $f$ itself is obtained by applying the operator $T_W$, where $W$ is the set of hidden weights in $f$, to a function $a : S^{n-1} \to \mathbb{R}$ that computes the output-layer coefficients for each gate (see Equation (8)).

Our analysis does not use the fact that the optimization produces an approximate global minimum; hence, there is a greater hope of generalizing to nonconvex regimes where we expect to instead only reach a local minimum in general. Another pleasant feature of our analysis is that we need not directly prove a “representation theorem” showing that the hypothesis minimizing the population loss is a good approximation to the target function; instead, we can derive such a result for free, as a corollary to our analysis. That is, since we prove directly that GD on the output layer weights of a single-layer NN with randomly-initialized gates results in small loss, it follows that any low-degree harmonic polynomial is in fact approximated by such a network. Our hope is that this new approach offers an interesting possibility for understanding GD in more difficult settings.
A practical consequence of our method of analysis is that we can easily prove a “spectral bias” result, showing the lower degrees are learned more quickly than higher degrees, as was suggested experimentally in [26] (see Theorem 1.5).

The upper bound guarantees hold for the agnostic learning problem of minimizing the least squares error, and the bound is with respect to the best degree $k$ polynomial approximation. The size of the network needed grows as $n^k \Omega(k)$, as does the time and sample complexity. We show that this unavoidable for any SQ algorithm, including GD and its variants on arbitrary network architectures. The “hard” functions used for the lower bound will be generated by spherical harmonic polynomials. Specifically, we use the univariate Legendre polynomial of degree $k$ in dimension $n$, denoted as $P_{n,k}$, and also called the Gegenbauer polynomial (see Section 2 for more background). We pick a set of unit vectors $u$ and for each one we get a polynomial $f_u(x) = P_{n,k}(u \cdot x)$. We choose the vectors randomly so that most have a small pairwise inner product. Then querying one of these polynomials gives little information about the others (on the same input $x$), and forces an algorithm to make many queries. As in the work on SQ regression algorithms of [30], it is essential not only to bound the pairwise correlations of the “hard” functions themselves, but also of arbitrary “smoothed” indicator functions composed with the hard family. This is accomplished by using a concentration of measure inequality on the sphere to avoid regions where these indicators are in fact correlated. In contrast to those earlier SQ regression lower bounds, we obtain bounds on the sensitivity parameter $t$ for the VSTAT($t$) oracle that scales with the number of queries $d$ and the degree $k$.

1.3 Related work

Explaining the success of deep NNs and GD for training NNs has been a challenge for several years. The trade-off between depth and size for the purpose of representation has been rigorously demonstrated [31, 11]. Moreover, there are strong complexity-theoretic and cryptographic-assumption based lower bounds to contend with [5, 9, 23]. These lower bounds are typically based on Boolean functions and “hard” input distributions. More recent lower bounds hold even for specific distributions and smooth functions, for basic GD [29], and even realizable smooth functions for any SQ algorithm and any product logconcave input distribution [30]. These earlier lower bound constructs are degenerate in the sense that they rely on data generated by networks whose bias and weight vectors have unbounded Euclidean norm as the dimension increases. In contrast, the constructions used in this paper match a corresponding upper bound almost exactly by making use of generic harmonic polynomials in the construction, apply to a significantly broader family of functions, and achieve a much stronger bound on the sensitivity parameter $t$.

Upper bounds have been hard to come by. Standard loss functions, even for one-hidden-layer networks with an output sum gate, are not convex and have multiple disconnected local minima. One body of work shows how to learn more restricted functions, e.g., polynomials [11] and restricted convolutional networks [6]. Another line of work investigates classes of such networks that can be learned in polynomial time, notably using tensor methods [21, 28] and polynomial kernels [15, 17], more direct methods with assumptions on the structure of the network [16, 13] and a combination of tensor initialization followed by GD [33]. A recent paper shows that the tensor method can be emulated by GD by adding a sufficiently sophisticated penalty to the objective function [14]. Earlier work gave combinatorial methods to learn random networks [2], guarantees for learning linear dynamical systems by GD [19] and ReLU networks with more restrictive assumptions [24]. Representation theorems analogous to our own were also proved in [3], and a very general analysis of GD is given in [8].

Our analysis is reminiscent of the well-known random kitchen sinks paper [27], which showed that GD using a hard upper bound on the magnitude of coefficients (in practice, an $L_1$ penalty term) with many random features from some distribution achieves error that converges to the best possible error among functions whose coefficients are not much higher than those of the corresponding densities of the sampling distribution. While this approach has been quite insightful (and effective in practice), it (a) does not give a bound for standard GD (with no penalty) and (b) does not address functions that have very different support than the sampling distribution. Our bounds compare with the best possible polynomial approximations and are essentially the best possible in that generality for randomly chosen features.

The work of Andoni et al. [11] shows that GD applied to learn a bounded degree polynomial, using a 1-hidden-layer network of exponential gates, converges with roughly the same number of gates (and a higher iteration count, poly($1/\varepsilon$) instead of log($1/\varepsilon$) to achieve error $\varepsilon$). A crucial difference is that our analysis is agnostic and we show that
GD converges to the error of the best degree $k$ approximation of the target function given sufficient many gates. We also state our results for general and commonly-used activation functions, rather than the $e^x$ gate analyzed in [1], and obtain explicit sample complexity bounds. Of course, the proof technique is also novel; we obtain our representation theorem as a side effect of our direct analysis of GD, rather than the other way around.

## 2 Spherical Harmonics

We now recall the basic theorems of spherical harmonics we will require. A more detailed treatment can be found in [18].

A homogeneous polynomial $p$ of degree $k$ in $\mathbb{R}^n$ is said to be harmonic if it satisfies the differential equation $\Delta p = 0$, where $\Delta$ is the Laplacian operator. We denote by $\mathcal{H}_{n,k}$ the set of spherical harmonics of degree $k$ on the sphere $S^{n-1}$, i.e., the projections of all harmonic polynomials of degree $k$ to the sphere $S^{n-1}$. The only properties of harmonic polynomials used in this paper are that they are polynomials, form an orthogonal basis for $L^2(S^{n-1})$, and are eigenfunctions of Funk transforms, as we now explain. We denote by $P_{n,k} : \mathbb{R} \rightarrow \mathbb{R}$ the (single-variable) Legendre polynomial of degree $k$ in dimension $n$, which is also called the Gegenbauer polynomial. We note that $|P_{n,k}(t)| \leq 1$ for all $|t| \leq 1$.

We define $N(n, k) = |\mathcal{H}_{n,k}|$.

**Definition 2.1.** Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and integrable. We define the Funk transformation for functions $H : S^{n-1} \rightarrow \mathbb{R}$ as $\mathcal{J}_\phi(H)(u) = \mathbb{E}_{x \in S^{n-1}}(\phi(x \cdot u)H(x))$. For $n, d \in \mathbb{N}$ we define the constant

$$\alpha_{n,k}(\phi) = \int_{-1}^{1} \phi(t)P_{n,k}(t)(1 - t^2)^{(n-3)/2} dt .$$

**Theorem 2.2 (Funk–Hecke).** Let $\phi : [-1, 1] \rightarrow \mathbb{R}$ be bounded and integrable, and let $P \in \mathcal{H}_{n,k}$. Then, for $\mathcal{J}_\phi$ and $\alpha_{n,k}(\phi)$ as in Definition 2.1, $\mathcal{J}_\phi(P)(u) = \alpha_{n,k}(\phi)P(u)$.

The following proposition is immediate from Cauchy-Schwarz.

**Proposition 2.3.** We have $\|\mathcal{J}_\phi(G)\|_\infty \leq \|\phi\|_2\|G\|_2$.

**Lemma 2.4.** Let $\phi : [-1, 1] \rightarrow \mathbb{R}$ be bounded and integrable, and let $H : \mathbb{R}^n \rightarrow \mathbb{R}$. Then for any $k \in \mathbb{N}$, $(\mathcal{J}_\phi H)(k) = \alpha_{n,k}(\phi)H(k)$.

**Proof.** By Proposition 2.3, $\mathcal{J}_\phi$ has bounded norm as an operator on $L^2(S^{n-1})$ and so by Theorem 2.2

$$\mathcal{J}_\phi(H) = \mathcal{J}_\phi \left( \sum_{k=0}^{\infty} H^{(k)} \right) = \sum_{k=0}^{\infty} \mathcal{J}_\phi(H^{(k)}) = \sum_{k=0}^{\infty} \alpha_{n,k}(\phi)H^{(k)} .$$

**Lemma 2.5.** Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an $(S, \alpha)$-activation. Then for any $f : S^{n-1} \rightarrow \mathbb{R}$, we have

$$\|f - \mathcal{J}_\phi^2 f\|_2^2 \leq \|f\|_2^2 - \alpha^2 \|f(S)\|_2^2 .$$

**Proof.** By Lemma 2.4

$$\|f - \mathcal{J}_\phi^2 f\|_2^2 = \sum_{k=0}^{\infty} (1 - \alpha_{n,k}(\phi)^4)\|f^{(k)}\|_2^2$$

$$= \|f\|_2^2 - \sum_{k \in S} \alpha_{n,k}(\phi)^4\|f^{(k)}\|_2^2$$

$$\leq \|f\|_2^2 - \alpha^4 \|f(S)\|_2^2 .$$
2.1 Spectra for specific activation functions

We first prove a general lemma describing the harmonic spectrum of a wide class of functions, and then derive estimates of the spectra for commonly used activation functions.

Lemma 2.6. Suppose \( \phi : [-1, 1] \to \mathbb{R} \) has an absolutely convergent Taylor series \( \phi(t) = \sum_{i=0}^{\infty} a_i t^i \) on \([-1, 1]\). Suppose that for all \( i > j \), we have \( a_i < a_j \) whenever \( a_i \) and \( a_j \) are nonzero. Then for any positive integer \( d \), \( \phi \) is an \((S, a_d n^{-d-O(1)})\)-activation, where \( S = \{1 \leq d : a_i \neq 0\} \).

**Proof.** Define

\[
r_{n,k} = \frac{(-1)^k \Gamma ((n-1)/2)}{2^k \Gamma (k + (n-1)/2)}
\]

By Rodrigues’ formula (see [18 Proposition 3.3.7]),

\[
\alpha_{n,k}(\phi) = \int_{-1}^{1} \phi(t) P_{n,k}(t)(1 - t^2)^{(n-3)/2} dt = r_{n,k} \int_{-1}^{1} \phi(t) \frac{d^k}{dt^k} (1 - t^2)^{(n-3)/2}.
\]

Hence, by the bounded convergence theorem,

\[
\alpha_{n,k}(\phi) = r_{n,k} \sum_{i=0}^{\infty} a_i \int_{-1}^{1} t^i \frac{d^k}{dt^k} (1 - t^2)^{(n-3)/2}.
\]

We claim that

\[
\int_{-1}^{1} t^i \frac{d^k}{dt^k} (1 - t^2)^{(n-3)/2} = \begin{cases} 0 & \text{if } i < k \text{ or } i \equiv k \mod 2 \\ (-1)^k k! B((i - k + 1)/2, (n-3)/2 + k+1) & \text{otherwise} \end{cases}
\]

where \( B(a,b) \) is the Euler beta function. Indeed, integrating by parts, we see that if \( i < k \) the expression is 0, and otherwise

\[
\int_{-1}^{1} t^i \frac{d^k}{dt^k} (1 - t^2)^{(n-3)/2} = (-1)^k k! \int_{-1}^{1} t^{i-k} (1 - t^2)^{(n-3)/2}.
\]

After a change of variables \( u = t^2 \), this latter integral is by definition \( B((i - k + 1)/2, (n-3)/2 + k+1) \).

Therefore, we compute for all \( k \geq 0 \),

\[
\alpha_{n,k}(\phi) = r_{n,k} (-1)^k k! \sum_{i=0}^{\infty} a_i B((i - k + 1)/2, (n-3)/2 + k+1)
\]

Now for any \( i > j \geq k \) of the same parity \( \mod 2 \), if \( a_j \neq 0 \) we estimate

\[
|a_i B((i - k + 1)/2, (n-3)/2 + k+1)| < 1/2.
\]

In particular, whenever \( a_k \neq 0 \) we have

\[
|\alpha_{n,k}(\phi)| = \Theta(r_{n,k} k! a_k B(1/2, (n-3)/2 + k+1)) = \Theta(a_k n^{-k-O(1)})
\]

□

Lemma 2.7. 1. Let \( \phi(t) = 1/(1 + e^{-t}) \) be the standard sigmoid function. Then for any positive integer \( d \), \( \phi \) is an \((S, n^{-d-O(1)})\)-activation function, where \( S \) contains 0 and all odd integers less than \( d \).

2. Let \( \phi(t) = \log(e^t + 1) \) be the “softplus” function. Then for any positive integer \( d \), \( \phi \) is an \((S, n^{-d-O(1)})\)-activation function, where \( S \) contains 1 and all even integers less than \( d \).

**Proof.** The statement follows from Lemma 2.6 by computing the relevant Taylor series. □

We can also perform a similar computation for ReLU activations. (A more general estimate is given in [3 Appendix D.2].)

Lemma 2.8. Let \( \phi(t) = \max\{t, 0\} \) be the ReLU function. Then for any positive integer \( d \), \( \phi \) is an \((S, n^{-d-O(1)})\)-activation function, where \( S \) contains 1 and all even integers less than \( d \).
3 Analysis of Gradient Descent

In this section, we fix a function \( g : \mathbb{R}^n \to \mathbb{R} \) we wish to learn. We also fix an \((S, \alpha)\)-activation function \( \phi \) with \( \|\phi\|_\infty \leq 1 \), for some finite \( S \subseteq \mathbb{N} \).

We let \( W \subseteq \mathbb{R}^n \) be a finite set of independent points drawn from the uniform distribution \( D \) on the sphere \( S^{n-1} \).

Similar to Eq. 1, we define \( f : \mathbb{R}^W \times \mathbb{R}^n \to \mathbb{R} \) as \( f(b, x) = f_0(x) = \sum_{u \in W} b_u \phi(u \cdot x) \), so \( f \) is computed by an unbiased single-hidden-layer neural network with hidden layer weight matrix given by \( W \) and linear output layer weights given by \( b \).

We will study how \( f \) changes as we update \( b \) according to gradient descent on the mean-squared loss function \( \mathbb{E}_{x \sim D}(f(b, x) - g(x))^2 \).

We let \( \epsilon > 0 \), the approximation error we will achieve over the projection of \( g \) to harmonics of degrees in \( S \). We define quantities \( t, \delta, \) and \( m \) as follows, using absolute constants \( c_t, c_\delta, \) and \( c_m \) to be defined later in the proof. The maximum number of iterations of gradient descent will be

\[
t = c_t \alpha^{-4} \log \left( \frac{\|g\|_2}{\epsilon} \right) .
\]

We define \( \delta \) to be an error tolerance used in certain estimates in the proof,

\[
\delta = c_\delta \alpha^4 \left( \frac{\epsilon}{\|g\|_2 t} \right)^2 .
\]

Finally, we define \( m \) to be the number of hidden units (so \( |W| = m \)), as well as the number of samples,

\[
m = c_m \frac{\|g\|_\infty}{\delta^2} \log \left( \frac{\|g\|_\infty}{\delta} \right) .
\]

Let \( X \) be a collection of \( m \) random independent samples \( x \in \mathbb{R}^n \), the set \( X \), along with the labels \( g(x) \) for \( x \in X \), will be the training data used by the algorithm.

We recall the definition in Eq. 3 of the operator

\[
T_Z(f)(u) = \frac{1}{|Z|} \sum_{z \in Z} f(z) \phi(u \cdot z) .
\]

defined for sets \( Z \subseteq \mathbb{R}^n \) and functions \( f : \mathbb{R}^n \to \mathbb{R} \). As described in Section 1.2, the empirical gradient is given by the operator \( T_X \) applied to the residual error, i.e., the gradient of \( (g - f)^2 \) with respect to the output layer weight for the gate \( u \) is estimated as \( T_X(g - f)(u) \), where \( X \) is the set of sample inputs. On the other hand, we will observe below that the neural network \( f \) itself can also be understood as the result of applying the operator \( mT_W \) to a function representing the output-layer weights.

For integers \( i \geq 0 \) we shall define functions \( f_i, a_i : \mathbb{R}^n \to \mathbb{R} \) recursively, corresponding to the model function \( f \) and its coefficients after \( i \) rounds of gradient descent. In particular, we let \( f_i(x) = f(a_i, x) \), i.e.,

\[
f_i(x) = \sum_{u \in W} a_i(u) \phi(u \cdot x) = mT_W(a_i)(x)
\]

We denote by \( H_i = g - f_i \) the \( i \)th residual. We define \( a_0(u) = 0 \) and, for \( i \geq 1 \), set \( a_i(u) = a_{i-1}(u) + (1/m)T_X H_i(u) \).

We therefore have the following two propositions which describe how the neural network evolves over multiple iterations of gradient descent.

**Proposition 3.1.** Suppose the output-layer weights \( b \in \mathbb{R}^W \) are initially 0. Then after \( i \) rounds of gradient descent with learning rate \( 1/(2m) \), we have \( b_u = a_i(u) \).

We remark that there is no need to have \( |X| = |W| \); our analysis simply achieves the same bound for both. It is, however, useful to have separate sets \( X \) and \( W \), so that these samples are independent.
Proof. Indeed, as we have observed in Eq. (2), for each \( u \in W \), the true gradient of the loss \((g - f)^2\) with respect to the output-level weight \( b_u \) is \( 2 \mathbb{E}_x (\phi(u \cdot x)(g - f)(x)) \). So the empirical gradient using the samples in \( X \) is indeed

\[
\frac{2}{|X|} \sum_{x \in X} \phi(u \cdot x)(g - f)(x) = 2T_X(g - f)(u).
\]

Thus, a single iteration of gradient descent with learning rate \( 1/(2m) \) will update the weight \( b_u \) by adding \((1/m)T_X(g - f)(u)\). The proposition now follows by induction on \( i \).

**Proposition 3.2.** For all \( i \geq 0 \), \( f_{i+1} = f_i + T_W T_X H_i \).

Proof. By the definitions of \( f_i \) and \( a_i \), we have

\[
f_{i+1}(x) = \sum_{u \in W} a_{i+1}(u)\phi(u \cdot x)
= \sum_{u \in W} (a_i(u) + (1/m)T_X H_i(u))\phi(u \cdot x)
= f_i(x) + T_W T_X H_i(x)
\]

as desired.

Having introduced and explained the necessary notation, we now state our main technical estimate, the following Lemma 3.3 which will be proved at the end of this section. For the rest of Section 3, we write \( \Delta H_i = H_{i+1} - H_i = f_{i+1} - f_i \) for the change in the residual at step \( i \), and we abbreviate \( \mathcal{J} = \mathcal{J}_0 \).

**Lemma 3.3.** Suppose \( i \leq t \) and \( \|H_i^{(S)}\|_2 \geq \varepsilon \) for all \( j \leq i \). Then with high probability

\[
\|\Delta H_i - \mathcal{J}^2 H_i\|_2 = O(\delta \|g\|_2^t).
\]

### 3.1 Proof of Main Results

Given Lemma 3.3 proved in the following Section 3.2, the main results stated in Section 1.1 are straightforward.

**Proof of Theorem 1.3.** By Lemma 3.3 as long as \( \|H_i\|_2^2 \) remains larger than \( \varepsilon \) and \( i \leq t \), we have \( \|\Delta H_i - \mathcal{J}^2 H_i\|_2 \leq O(\delta \|g\|_2^t) \). Now \((\Delta H_i - \mathcal{J}^2 H_i)^{(S)}\) and \((\Delta H_i - \mathcal{J}^2 H_i^{(S)})\) are orthogonal, so also \( \|\Delta H_i - \mathcal{J}^2 H_i^{(S)}\|_2 \leq O(\delta \|g\|_2^t) \). Therefore, rewriting \( \Delta H_i = H_{i+1} - H_i \), we have

\[
\|H_{i+1}^{(S)}\|_2^2 \leq \|H_i^{(S)}\|_2^2 + O(\delta \|g\|_2^t).
\]

Combining with Lemma 2.5

\[
\|H_{i+1}^{(S)}\|_2^2 \leq (1 - \alpha^4)\|H_i^{(S)}\|_2^2 + O(\delta \|g\|_2^t).
\]

For a sufficiently small choice of the constant \( c_4 \) defining \( \delta \) (Eq. (6)), under the assumption that \( \|H_i^{(S)}\|_2^2 \geq \varepsilon \), we can take the \( O(\delta \|g\|_2^t) \) term to be at most \( (\alpha^4/2)\|H_i^{(S)}\|_2^2 \). Therefore,

\[
\|H_{i+1}^{(S)}\|_2^2 \leq \left(1 - \frac{\alpha^4}{2}\right)\|H_i^{(S)}\|_2^2.
\]

Since \( \|H_0^{(S)}\|_2 \leq \|H_0\|_2 = \|g\|_2 \), for some \( s = O(\alpha^{-4} \log(\|g\|/\varepsilon)) < t \) we have \( \|H_s^{(S)}\|_2^2 < \varepsilon \) (assuming a sufficiently large choice of the constant \( c_4 \) defining \( t \) in Eq. (5)). Then

\[
\|H_s\|_2^2 = \|H_s^{(S)}\|_2^2 + \|H_s^{(S)}\|_2^2 < \|H_s^{(S)}\|_2^2 + \varepsilon
\]

as desired.
Theorem 1.1 now follows from Theorem 1.3 in view of Lemma 2.7.
To prove Corollary 1.4 we first recall an approximation lemma of Livni et al. [25, Lemma 2]:

**Lemma 3.4.** Let \( \phi(t) = 1/(1 + e^{-t}) \) denote the sigmoid function. For every \( \varepsilon > 0 \), there is a polynomial \( p \) of degree \( d = O(L \log(L/\varepsilon)) \) such that \( |p(t) - \phi(t)| < \varepsilon \) for all \( t \in [-L, L] \).

**Proof of Corollary 1.4.** Set \( \delta' = \varepsilon/a \). By Lemma 3.4 there is a polynomial \( p \) of degree \( O(b \log(b/\varepsilon')) \) such that \( |p(t) - \phi(t)| < \varepsilon \) for all \( t \in [-b, b] \). Therefore, for every \( u \in \mathbb{R}^n \) with \( |u|_2 \leq b \) and every \( x \in S^{n-1} \), we have \( |p(u \cdot x) - \phi(u \cdot x)| \leq \delta' \). Hence, \( \sum_i a_i (p(u_i x) - \phi(u_i x)) < \varepsilon \) whenever \( \sum_i a_i < a \) and each \( u_i \) satisfies \( |u_i|_2 \leq b \).

In particular, the functions computed by the networks described in the statement of the corollary can be approximated to within \( \varepsilon \) error by polynomials of degree \( O(b \log(ab/\varepsilon)) \). The Corollary now follows from Theorem 1.1. \( \square \)

Finally, we prove Theorem 1.6.

**Proof of Theorem 1.6.** In Eqs. (6) and (7) defining \( \delta \) and \( m \), we defined \( m \) in terms of \( \delta \), but what matters for the proof of Lemma 3.3 is that \( m = \Omega(\delta^{-2}) \), and if the number of units \( m \) is increased beyond the bound necessary for Theorem 1.3 we may decrease \( \delta \) to preserve the relationship between \( m \) and \( \delta \) (and similarly \( t \)). So without loss of generality, we may take \( \delta \) to be \( O(\log(b/\varepsilon')) \). Therefore, taking \( m = n^{O(t)} \) for \( j \in \{k, \ell \} \)

\[
\alpha_{n,j}(\phi)^2 ||H^{(j)}_i||_2 < n^{-\Omega(t)} \varepsilon < \delta ||g||_2 t^2.
\]

Using Lemma 3.3 we compute

\[
\begin{align*}
F_k^{(k, \ell)} &= \frac{||\Delta^{(k)}|| ||H^{(k)}_i||}{||\Delta^{(\ell)}|| ||H^{(\ell)}_i||} > \frac{\left( ||J^2 H^{(k)}_i - O(\delta ||g||_2 t^2)||H^{(\ell)}_i|| \right)}{\left( ||J^2 H^{(\ell)}_i + O(\delta ||g||_2 t^2)||H^{(k)}_i|| \right)} \\
&= \frac{\left( \alpha_{n,k}(\phi)^2 ||H^{(k)}_i|| - O(\delta ||g||_2 t^2) \right) ||H^{(\ell)}_i||}{\left( \alpha_{n,\ell}(\phi)^2 ||H^{(\ell)}_i|| + O(\delta ||g||_2 t^2) \right) ||H^{(k)}_i||} \\
&= \frac{\Omega(\alpha_{n,k}(\phi)^2)}{\alpha_{n,\ell}(\phi)^2} \\
&= n^{\Omega(t-k)}.
\end{align*}
\]

**3.2 Proof of Lemma 3.3**

We now prove Lemma 3.3. Essentially, the lemma states that the operator \( T_Z \) approximates \( \mathcal{J} \) for sufficiently large sets \( Z \). We will prove Lemma 3.3 via a sequence of gradually improving estimates of the approximation of \( \mathcal{J} \) by \( T_Z \). Lemma 3.5 gives a very general approximation, which we use to prove the finer approximation described in Lemma 3.6.

**Lemma 3.5.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( u \in \mathbb{R}^n \), and let \( \eta, p > 0 \). There is some

\[
\ell = O \left( \frac{||f||^2_2 + \eta ||f||_\infty}{\eta^2} \log \left( \frac{1 + ||f||_\infty}{\eta p} \right) \right)
\]

such that if \( Z \subseteq \mathbb{R}^n \) is a set of \( \ell \) independent random points drawn from \( D \), then with probability at least \( 1 - p \), we have both \( ||T_Z(f) - \mathcal{J}(f)||_2 \leq \eta \) and

\[
P_{u \sim D} (||T_Z(f)(u) - \mathcal{J}(f)(u)|| > \eta/2) < p.
\]

**Proof.** Without loss of generality assume \( \eta < 1 \) and let \( p_0 = \eta^2 p^2 / (4(1 + ||f||_\infty)^2) \). Fix \( u \in \mathbb{R}^n \). Let \( z_1, \ldots, z_\ell \) be drawn independently from \( D \), and let \( \zeta_i \) denote the random variable \( f(z_i) \phi(u \cdot z_i) - \mathcal{J}(f)(u) \) for \( 1 \leq i \leq \ell \). We have \( \mathbb{E}(\zeta_i) = 0 \) by definition of \( \mathcal{J} \). Since \( ||\phi||_\infty \leq 1 \), we have by Hölder’s inequality

\[
\text{Var}(\zeta_i) \leq ||\phi||_\infty^2 ||f||_2^2 \leq ||f||_2^2.
\]

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Furthermore, by Proposition 2.3

\[ |\zeta_i| \leq \|\phi_\infty\|_\infty + \|\phi\|_2 \|f\|_2 \leq 2\|f\|_\infty. \]

By a Bernstein bound, we therefore have

\[ P_k \left( \frac{1}{\ell} \sum_{i=1}^{\ell} \zeta_i > \eta/2 \right) < 2 \exp \left( -\Omega \left( \frac{\ell \eta^2}{\|f\|_2^2 + \eta \|f\|_\infty} \right) \right) < p_0, \]

with the second inequality holding for an appropriate choice of the constant hidden in the definition of \( \ell \).

Let \( Z = \{z_1, \ldots, z_\ell\} \), so

\[ \frac{1}{\ell} \sum_{i=1}^{\ell} \zeta_i = T_Z(f)(u) - J(f)(u). \]

Let \( B(u, Z) = 1 \) if \( |T_Z(f)(u) - J(f)(u)| > \eta/2 \) and 0 otherwise. By the preceding inequality, we have \( \mathbb{E}_{u, Z}(B(u, Z)) < p_0 \). Therefore, by Markov’s inequality, the probability over the choice of \( Z \) that \( \mathbb{E}_{u, Z}(B(u, Z)) > p_0/p \) is at most \( p \).

Hence, with probability \( 1 - p \) over the choice of \( Z \), we have

\[ P_u(|T_Z(f)(u) - J(f)(u)| > \eta/2) = \mathbb{E}_u(B(u, Z)) < \frac{\eta^2 p}{4(1 + 2\|f\|_\infty)^2}. \tag{9} \]

In particular, the second inequality of the present lemma holds.

We now complete the proof of the first inequality. For all choices of \( Z \) and \( u \), using Proposition 2.3, we have

\[ |T_Z(f)(u) - J(f)(u)| < |T_Z(f)(u)| + |J(f)(u)| \leq 2\|f\|_\infty. \]

Combined with Eq. (9), we have

\[ \|T_Z(f) - J(f)\|_2^2 \leq (2\|f\|_\infty)^2 P(|T_Z(f)(u) - J(f)(u)| > \eta/2) + \eta^2 / 4 \leq \eta^2. \]

We denote by \( \phi_x : \mathbb{R}^n \to \mathbb{R} \) the function \( \phi_x(u) = \phi(u \cdot x) \).

In the following Lemma 3.6 we prove a finer-tuned approximation of the operator \( J \) by both \( T_X \) and \( T_W \). Since Lemma 3.5 doesn’t give a sufficiently tight approximation between the operators simultaneously for every \( L^2 \)-function on \( S^{n-1} \), we restrict our attention to the subspace we care about, namely, the functions spanned by the \( \phi_x \) for \( x \in W \cup X \).

**Lemma 3.6.** With probability \( 1 - 1/m \) over the choice of \( W \) and \( X \), the following statements are all true:

1. \( \|T_X g - Jg\|_2 \leq \delta; \)
2. For all \( u \in W \), we have \( \|T_X \phi_u - J\phi_u\|_2 \leq \delta; \)
3. For all \( x \in X \) we have \( \|T_W \phi_x - J\phi_x\|_2 \leq \delta; \)
4. For all \( x \neq y \in X \), we have \( |T_W(\phi_x)(y) - J(\phi_x)(y)| \leq \delta/2; \)

**Proof.** We will use Markov’s inequality to bound the probability that \( T_W(\phi_x) \) is far from \( J(\phi_x) \) at a random input, followed by a union bound over the choice of \( W \) and the choice of \( X \). We require the constant \( c_m \) to be sufficiently large.

In detail, we have \( m \geq c_m\log(\|g\|_\infty/\delta/\|g\|_\infty^2\delta^2) \), so for any fixed \( k, c_0 > 0 \), we can set \( c_m \) sufficiently large that there is some \( p < 1/m^k \) also satisfying

\[ m \geq c_p \frac{\|g\|_\infty}{\delta^2} \log \left( \frac{\|g\|_\infty}{\delta p} \right). \]
The same statement also holds (for appropriate choice of \( c_n \)) with \( \phi \) in place of \( g \), since \( \| \phi \|_{\infty} \leq 1 \). Then since \( |W| \geq m \), by Lemma 3.5 for any fixed \( x \in \mathbb{R}^n \), we have with probability \( 1 - 1/(16m^3) \) over the choice of \( W \) that

\[
\| T_W \phi_x - \mathcal{J} \phi_x \|_2 < \delta.
\] (10)

and

\[
P_{z \sim D} (\| T_W (\phi_x)(z) - \mathcal{J}(\phi_x)(z) \| > \delta/2) < 1/(16m^3).
\] (11)

Therefore, by Markov’s inequality, with probability \( 1 - 1/(2m) \) over the choice of \( W \), Eqs. (10) and (11) both hold for a random \( x \sim D \) with probability \( 1 - 1/(8m^2) \).

Similar to Eq. (10), with \( X \) in place of \( W \) and \( g \) in place of \( \phi_x \), statement (1) of the present lemma holds with probability \( 1 - 1/(16m^3) > 1 - 1/(8m) \) over the choice of \( X \). Furthermore, for any fixed \( X \), taking a union bound over \( W \), we have with probability \( 1 - m/(16m^3) > 1 - 1/(8m) \) that statement (2) holds.

Now suppose \( W \) is such that Eq. (10) holds for a random \( x \sim D \) with probability at least \( 1 - 1/(8m^2) \); as we have already observed, this is the case with probability at least \( 1 - 1/(2m) \) over the choice of \( W \). Then by a union bound over \( X \), it then follows that with probability \( 1 - 1/(8m) \) over the choice of \( X \), statement (3) holds. Finally, suppose similarly that \( W \) is such that Eq. (11) holds for a random \( x \sim D \) with probability at least \( 1 - 1/(8m^2) \). By a union bound, we have with probability at least \( 1 - 1/(8m) \) that for all \( x \in X \),

\[
P_{z \sim D} (\| T_W (\phi_x)(z) - \mathcal{J}(\phi_x)(z) \| > \delta/2) < 1/(16m^3).
\]

Now, fixing such an \( x \in X \), a union bound over all \( y \in X \) with \( y \neq x \) gives that

\[
\| T_W (\phi_x)(y) - \mathcal{J}(\phi_x)(y) \| \leq \delta/2
\]

with probability \( 1 - 1/(16m^2) \). Taking another union bound over all \( x \in X \), we get statement (4) with probability \( 1 - 1/(16m) \) as well. Overall, statements (1)–(4) hold with probability at least \( 1 - 1/m \).

For the remainder of this section, we use the notation \( \alpha_i = \max_{u \in W} |a_i(u)| \) and \( \beta_i = \max_{x \in X} |H_i(x)| \).

We focus on the second step of our analysis, as outlined in Section 1.2, bounding the rate at which error from the approximations of \( \mathcal{J} \) described above accumulates over multiple iterations of GD. More precisely, we control the norm of \( f \), measured via \( \alpha_i \) and \( \beta_i \). The statements are given in the following two lemmas.

**Lemma 3.7.** Suppose statements (1)–(3) of Lemma 3.6 all hold. Then for all \( i \in \mathbb{N} \), we have both \( \| T_X H_i - \mathcal{J} H_i \|_2 \leq \delta (m\alpha_i + 1) \) and \( \| \Delta H_i - \mathcal{J}^2 H_i \|_2 \leq \delta (\beta_i + m\alpha_i + 1) \).

**Proof.** By Lemma 3.6(3), since \( T_W \) and \( \mathcal{J} \) are linear operators,

\[
\|(T_W - \mathcal{J})(T_X H_i)\|_2 = \left\| (T_W - \mathcal{J}) \left( \frac{1}{m} \sum_{x \in X} H_i(x) \phi_x \right) \right\|_2
\]

\[
\leq \frac{1}{m} \sum_{x \in X} \beta_i \|(T_W - \mathcal{J}) \phi_x\|_2 \leq \delta \beta_i.
\]

Similarly, by Lemma 3.6(2), we have

\[
\|(T_X - \mathcal{J})f_i\|_2 = \left\| (T_X - \mathcal{J}) \left( \sum_{u \in W} a_i(u) \phi_u \right) \right\|_2
\]

\[
\leq \sum_{u \in W} \alpha_i \|(T_X - \mathcal{J}) \phi_u\|_2 \leq m\delta \alpha_i.
\]

Finally, by Lemma 3.6(1), we have

\[
\|(T_X - \mathcal{J})H_i\|_2 = \|(T_X - \mathcal{J})g - (T_X - \mathcal{J})f_i\|_2
\]

\[
\leq \|T_X g - \mathcal{J} g\|_2 + \|(T_X - \mathcal{J})f_i\|_2
\]

\[
\leq \delta + m\delta \alpha_i.
\]
By Proposition \[3.2\] we have \(\Delta H_i = T_W T_X H_i\). Therefore, since \(\|J(h)\|_2 \leq \|h\|_2\) for all functions \(h\), we have altogether that

\[
\|\Delta H_i - J^2 H_i\|_2 = \|T_W T_X H_i - J^2 H_i\|_2 \\
\leq \|J(T_X H_i - J^2 H_i)\|_2 + \delta \beta_i \\
\leq \|J(T_X - J) H_i\|_2 + \delta \beta_i \\
\leq \|T_X - J\| H_i\|_2 + \delta \beta_i \\
\leq \delta (\beta_i + m \alpha_i + 1) .
\]

**Lemma 3.8.** For all \(i \geq 0\), we have \(\alpha_{i+1} \leq \alpha_i + \beta_i / m\). Furthermore, if statement (4) of Lemma \[3.6\] holds, then for all \(i \geq 0\), we have

\[
\beta_{i+1} \leq \beta_i + \|H_i\|_2 + \delta (\beta_i/2 + m \alpha_i + 1) + 2 \beta_i/m.
\]

**Proof of Lemma 3.8** For the first inequality, we have by definition that for all \(u \in W\)

\[
a_{i+1}(u) = a_i(u) + \frac{1}{m^2} \sum_{x \in X} H_i(x) \phi(u \cdot x) \leq \alpha_i + \beta_i / m.
\]

For the second inequality, fix \(y \in X\). Using statement (4) of Lemma \[3.6\] we compute

\[
| (T_W - J) H_i(y) | = \left| (T_W - J) \left( \frac{1}{m} \sum_{x \in X} H_i(x) \phi_x \right)(y) \right| \\
\leq \frac{1}{m} \sum_{x \in X} | H_i(x) (T_W - J)(\phi_x)(y) | \\
\leq \frac{\beta_i}{m} \left( | (T_W - J)(\phi_y)(y) | + \sum_{y \neq x} | (T_W - J)(\phi_x)(y) | \right) \\
\leq \frac{\beta_i}{m} \left( 2 + (m - 1) \frac{\delta}{2} \right).
\]

By Proposition \[3.3\] and the first statement of Lemma \[3.7\]

\[
\|J T_X H_i\|_\infty \leq \|T_X H_i\|_2 \leq \|J H_i\|_2 + \delta (m \alpha_i + 1) \leq \|H_i\|_2 + \delta (m \alpha_i + 1).
\]

Therefore, by Proposition \[3.2\]

\[
| H_{i+1}(y) | \leq | H_i(y) | + | T_W T_X H_i(y) | \\
\leq \beta_i + \|J T_X H_i(y)\| + |(T_W - J) H_i(y) | \\
\leq \beta_i + \|H_i\|_2 + \|H_i\|_2 + \delta (m \alpha_i + 1) + \frac{\beta_i}{m} \left( 2 + (m - 1) \frac{\delta}{2} \right) \\
\leq \beta_i + \|H_i\|_2 + \delta (\beta_i / 2 + m \alpha_i + 1) + 2 \beta_i/m.
\]

Finally, we complete the proof.

**Proof of Lemma 3.3** Let \(S = \mathbb{N} \setminus S\). We argue by induction that for all \(i \leq t\), as long as \(|H_i^{(S)}| \geq \varepsilon\), the following are all true:

1. \(\beta_i \leq O((i + 1)|g|_2)\)
2. \(\alpha_i \leq O((i + 1)^2|g|_2/m)\)
3. \(\|\Delta H_i - J^2 H_i\|_2 \leq O(|g|_2(i + 1)^2)\)

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In particular, with a constant error tolerance $\tau$:

Proposition 4.1. But such a model is in fact far too general, as the following proposition shows.

For a sufficiently small choice of the constant $\alpha$, and let $\beta_i = \frac{\alpha_i}{m}$, and let $\beta_i \leq \beta_{i-1} + \|H_{i-1}\| + \frac{\beta_{i-1}}{m} + \frac{\alpha_{i-1}}{m} + 2\beta_{i-1}/m$

This latter expression is at most $\beta_{i-1} + \|g\| + O(\|g\|/t)$, using the fact that $i < t$ and the definitions of $t$, $\delta$, and $m$ in Eqs. (5), (6), and (7). Estimate (1) now follows by induction.

Similarly, from the first statement of Lemma 3.8 and from estimate (1), we have

$\quad \|H_i\| \leq \|H_{i-1}\| + \frac{\beta_{i-1}}{m} + \frac{\alpha_{i-1}}{m} + O(\|g\|/t)$,

which gives estimate (2) by induction.

By the second statement of Lemma 3.7 and using estimates (1) and (2), we have

$\quad \|\Delta H_i - \mathcal{J}^2 H_i\| \leq \delta (\beta_{i-1} + \|g\| + 2\beta_{i-1}/m)$,

Giving estimate (3) by induction. Rewriting $\Delta H_i = H_{i+1} - H_i$, we have $\|H_{i+1}\| \leq \|H_i\| + O(\|g\|/t(i+1)^2)$. Now by estimate (4), $\|H_{i+1}\| \leq \|H_i\| + O(\|g\|/t(i+1)^2)$. Combining with Lemma 2.5

$\quad \|H_{i+1}\| \leq \|H_i\| - \frac{\alpha}{m} \|H_i\| + O(\|g\|/t(i+1)^2)$.

For a sufficiently small choice of the constant $c_4$ defining $\delta$ (Eq. 5), under the assumption that $\|H_i(\cdot)\| \geq \varepsilon$, we can take the $O(\|g\|/t(i+1)^2)$ term to be at most $(\alpha^2/2)\|H_i(\cdot)\|^2$. Therefore,

$\quad \|H_{i+1}\| \leq \|H_i\| - \frac{\alpha^2}{2} \|H_i(\cdot)\|^2$.

The norm $\|H_i\|$ of the residual is therefore monotonically decreasing in $i$, giving estimate (4).

4 Statistical query models

We now prove our statistical query lower bounds stated in Theorem 1.7. First, we remark on some of the difficulty of choosing an appropriate statistical query model for regression problems. Let $D$ be a probability distribution over a domain $X$, and let $f : X \to \mathbb{R}$ be an unknown concept. A natural and very general statistical query model for the regression problem of learning $f$ might allow as queries arbitrary measurable functions $h : X \times \mathbb{R} \to [0, 1]$. The SQ oracle should respond to such a query with a value $v$ approximating $\mathbb{E}_{x \sim D} h(x, f(x))$ to within the error tolerance. But such a model is in fact far too general, as the following proposition shows.

Proposition 4.1. Let $D$ be a probability distribution over some domain $X$, and let $C$ be a finite family of functions $f : X \to \mathbb{R}$ such that for every pair $f, g \in C$, the probability over $x \sim D$ that $f(x) = g(x)$ is 0. There is a measurable function $h$ such that for every $v \in [0, 1]$,

$$\left| \{ f \in C \mid \mathbb{E}_{x \sim D} h(x, f(x)) - v \leq \tau \} \right| = O(\tau |C|).$$

In particular, with a constant error tolerance $\tau$, such a family $C$ can be learned using $\log |C|$ statistical queries.
**Proof.** We arbitrarily choose evenly spaced values \( v_f \in [0, 1] \) corresponding to each \( f \in \mathcal{C} \). It suffices to find a measurable function \( h(x, y) \) such that \( h(x, f(x)) = v_f \) for all \( x \in X \), excluding perhaps a subset of \( X \) of measure 0 where two different functions have equal values. Since this condition specifies the value of \( h \) only on a set of measure 0 in \( X \times \mathbb{R} \), it is straightforward to find such a function \( h \).

In particular, a statistical query model allowing arbitrary measurable and bounded queries would allow efficiently learning any finite class of real-valued functions, perhaps perturbed slightly to ensure the functions disagree pairwise almost everywhere.

Furthermore, arbitrary measurable query functions don’t have concise descriptions anyway. So it is reasonable to require “well-behaved” query functions. We now describe three “well-behaved” statistical query settings, and prove strong lower bounds against algorithms learning degree-\( k \) polynomials on \( S^{n-1} \) in each setting.

The approach taken in [30], are revisited here, is to require the query function \( h : X \times \mathbb{R} \rightarrow [0, 1] \) to be Lipschitz for every fixed \( x \in X \). However, the Lipschitz-ness of the query function is sensitive to the scale of the concepts \( f : X \rightarrow \mathbb{R} \); a Lipschitz constant of 1 for the query function is far more meaningful when the concepts have bounded range than when their outputs are spread across the entirety of \( \mathbb{R} \). Hence, we consider \( L^\infty \)-normalized Lipschitz queries in Section 4.2.

Another approach is to insist on noisy concepts. In the model considered above, the query function \( h \) will receive inputs of the form \( (x, f(x)) \) where \( x \) is drawn from a distribution over the input space \( X \) and \( f : X \rightarrow \mathbb{R} \) is the unknown concept. In a noisy setting, it would be reasonable to replace the query input with a noised version \( (x, f(x) + \zeta) \) where \( \zeta \) is drawn from a noise distribution, such as a standard Gaussian. Clearly, the statistical oracle becomes weaker as the variance of \( \zeta \) increases, but as in the Lipschitz queries considered above, the relationship between the strength of the oracle and the amount of noise is sensitive to the scale of the concept \( f : X \rightarrow \mathbb{R} \). Again, in this setting, we will consider \( L^\infty \)-normalized queries in the presence of Gaussian noise of variance \( 1/\lambda^2 \). In fact, since the noise on the input has the effect of smoothing the expectation of the query function, this setting amounts to a special case of the more general \( L^\infty \)-normalized Lipschitz queries described previously. However, the noise approach allows for binary query functions \( h : X \rightarrow \{0, 1\} \), so this is the model we consider for lower bounds on 1-STAT, given in Section 4.3.

A final plausible query model would restrict the form of the queries. As noted in the proof of Proposition 4.1, the only important values of a query function \( h : X \times \mathbb{R} \rightarrow [0, 1] \) are on the zero measure subset \( \{(x, f(x)) : f \in \mathcal{C}, x \in X\} \subseteq X \times \mathbb{R} \). Hence, instead of working in the product space \( X \times \mathbb{R} \) where only a measure-zero set is relevant, we might instead allow query functions \( h : X \rightarrow [0, 1] \) to an oracle that responds with an approximation of the inner product with the concept, i.e., with a value \( v \) approximating \( \mathbb{E}_{x \in D}(h(x)f(x)) \) where \( f \) is the unknown concept. Such a query is called an inner product query, and these queries suffice, for example, to train a neural network using gradient descent against mean squared error. Lower bounds against such queries are given in Section 4.1.

### 4.1 Inner product queries

We recall the definition of statistical dimension, denoting by \( \rho_D(\mathcal{C}) \) the average correlation among the functions of \( \mathcal{C} \), i.e.,

\[
\rho_D(\mathcal{C}) = \frac{1}{|\mathcal{C}|^2} \sum_{f,g \in \mathcal{C}} \rho_D(f, g)
\]

where \( \rho_D(f, g) = \text{Cov}_D(f, g)/\sqrt{\text{Var}(f) \text{Var}(g)} \).

**Definition 4.2.** Let \( \gamma > 0 \), let \( D \) be a probability distribution over some domain \( X \), and let \( \mathcal{C} \) be a family of functions \( f : X \rightarrow \mathbb{R} \). The **statistical dimension** of \( \mathcal{C} \) relative to \( D \) with average correlation \( \gamma \), denoted by \( \text{SDA}(\mathcal{C}, D, \gamma) \), is defined to be the largest integer \( d \) such that for every subset \( \mathcal{C}' \subseteq \mathcal{C} \) of size at least \( |\mathcal{C}'| \geq |\mathcal{C}|/d \), we have \( \rho_D(\mathcal{C}') \leq \gamma \).

The following theorem can be proved in a manner almost identical to the proof of [12] Theorem 2.7.

**Theorem 4.3.** Let \( D \) be a distribution on a domain \( X \) and let \( \mathcal{C} \) be a family of functions \( f : X \rightarrow \mathbb{R} \). Suppose there for some \( d, \gamma > 0 \) we have \( \text{SDA}(\mathcal{C}, D, \gamma) \geq d \). Let \( A \) be a randomized algorithm learning \( \mathcal{C} \) over \( D \) with probability greater than \( 1/2 \) to within regression error less than \( \Omega(1) \). If \( A \) only uses inner product queries to \( \text{VSTAT}(1/(3\gamma)) \), then \( A \) uses \( \Omega(d) \) queries.
In what follows, \( f^{(k)}_u : S^n \rightarrow \mathbb{R} \) is defined by \( f^{(k)}_u(x) = \sqrt{N(n,k)}P_{n,k}(u \cdot x) \).

**Lemma 4.4.** Let \( u, v \in S^n \) be such that \(|u \cdot v| = t\). Then

\[
\rho(f^{(k)}_u, f^{(k)}_v) \leq \left( 1 + \frac{k - 1}{k + n - 3} \right) |t| + \sqrt{\frac{k - 1}{k + n - 3}} k
\]

**Proof.** By the Funk–Hecke theorem, \( \| f^{(k)}_u \|_2 = 1 \) We therefore have, again by the Funk–Hecke theorem,

\[
\rho_{x \in S^n}(f^{(k)}_u(x), f^{(k)}_v(x)) = N(n, k) \left( E_{x \in S^n} (P_{n,k}(u \cdot x)P_{n,k}(v \cdot x)) - E_{x \in S^n} (P_{n,k}(u \cdot x)) E_{x \in S^n} (P_{n,k}(v \cdot x)) \right)
\]

\[
= P_{n,k}(u \cdot v) = P_{n,k}(t). \]

Furthermore, for all \( \ell \), the Legendre polynomials satisfy the recurrence relation (see \cite{18} Proposition 3.3.11)

\[
(\ell + n - 2)P_{n,\ell+1}(t) - (2\ell + n - 2)\ell P_{n,\ell}(t) + \ell P_{n,\ell-1}(t) = 0.
\]

Since \( P_{n,0}(t) = 1 \) and \( P_{n,1}(t) = t \) (by \cite{18} Proposition 3.3.7), the result follows.

We can now prove the SQ lower bound for this class of queries.

**Proof of Theorem 4.7 (1).** Taking a random (uniform) set \( B \) of \( d \) vectors \( u \in S^n \), let \( C = \{ f^{(k)}_u : u \in B \} \). For any pair of distinct vectors \( u, v \in B \), we have with probability \( 1/d^2 \) that \( t_{uv} = |u \cdot v| = O(\sqrt{(\log d)/n}) \). Thus, with positive probability, we have \( t_{uv} = O(\sqrt{(\log d)/n}) \) for all distinct \( u, v \in B \). Then by Lemma 4.4 we have

\[
\rho(f^{(k)}_u, f^{(k)}_v) = O \left( \sqrt{\frac{\log d}{n}} \right)^k \leq n^{-\Omega(k)}
\]

for all \( u, v \in B \). The theorem now follows from Theorem 4.3.

**4.2 Lipschitz queries**

We now recall the Lipschitz query model introduced in \cite{30}. The functions learned in that paper were already bounded, so no \( L^\infty \) normalization is performed. We state an \( L^\infty \)-normalized version of the relationship between statistical dimension and statistical query complexity, which are an immediate consequence of those proved in \cite{30}.

For \( y \in \mathbb{R} \) and \( \varepsilon > 0 \), we define the \( \varepsilon \)-soft indicator function \( \chi^{(\varepsilon)}_y : \mathbb{R} \rightarrow \mathbb{R} \) as

\[
\chi^{(\varepsilon)}_y(x) = \chi_y(x) = \max\{0, 1/\varepsilon - (1/\varepsilon)^2|x - y|\}.
\]

So \( \chi_y \) is \((1/\varepsilon)^2\)-Lipschitz, is supported on \((y - \varepsilon, y + \varepsilon)\), and has norm \( \| \chi_y \|_1 = 1 \).

**Definition 4.5.** Let \( \bar{\gamma} > 0 \), let \( D \) be a probability distribution over some domain \( X \), and let \( \mathcal{C} \) be a family of functions \( f : X \rightarrow \mathbb{R} \) that are identically distributed as random variables over \( D \). The **statistical dimension** of \( \mathcal{C} \) relative to \( D \) with average covariance \( \bar{\gamma} \) and precision \( \varepsilon \), denoted by \( \varepsilon\text{-SDA}(\mathcal{C}, D, \bar{\gamma}) \), is defined to be the largest integer \( d \) such that the following holds: for every \( y \in \mathbb{R} \) and every subset \( C' \subseteq C \) of size \(|C'| > |C|/d \), we have \( \rho_D(C') \leq \bar{\gamma} \). Moreover, \( \text{Cov}_D(C'_y) \leq (\max\{\varepsilon, \mu(y)\})^2 \bar{\gamma} \) where \( C'_y = \{\chi^{(\varepsilon)}_y \circ f : f \in C\} \) and \( \mu(y) = \|f\|_\infty \text{E}_D(\chi^{(\varepsilon)}_y \circ f) \) for some \( f \in C \).

**Theorem 4.6.** Let \( D \) be a distribution on a domain \( X \) and let \( \mathcal{C} \) be a family of functions \( f : X \rightarrow \mathbb{R} \) identically distributed as unit-variance random variables over \( D \). Suppose there is \( d \in \mathbb{R} \) and \( \lambda, \bar{\gamma} > 0 \) such that \( \lambda \geq 1 \geq \bar{\gamma} \) and \( \varepsilon\text{-SDA}(\mathcal{C}, D, \bar{\gamma}) \geq d \), where \( \varepsilon \leq \bar{\gamma}/(2\lambda) \). Let \( A \) be a randomized algorithm learning \( \mathcal{C} \) over \( D \) with probability greater than \( 1/2 \) to regression error less than \( \Omega(1 - 2/\sqrt{\gamma}) \). If \( A \) only uses \( L^\infty \)-normalized queries to \( \text{VSTAT}(t) \) for some \( t = O(1/\bar{\gamma}) \), which are \( \lambda \)-Lipschitz at any fixed \( x \in X \), then \( A \) uses \( \Omega(d) \) queries.
We can now prove the lower bound for this query model. We use the same family of functions as for the inner product query model, but we must now also estimate the covariances of the soft indicators of these functions, as in the following lemma. We recall our notation $f_{u}^{(k)}(x) = \sqrt{N(n,k)}P_{n,k}(u \cdot x)$.

**Lemma 4.7.** Let $u, v \in S^n$, let $y \in \mathbb{R}$, and let $\ell, \varepsilon > 0$. Let $$\mu_0 = \mathbb{E}_{x \sim S^n}(\lambda_0^{(e)}(f_{u}^{(k)}(x))).$$ Then $$\left| \text{Cov}(\lambda_0^{(e)}(f_{u}^{(k)}(x)), \lambda_0^{(e)}(f_{v}^{(k)}(x))) \right| = O(\ell(\ell \cdot v)^2 \log n \mu_0^2 + n^{-\ell}/\varepsilon^2)$$

**Proof.** For $w, x \in S^n$, we write $z_{u}(x) = \lambda_0^{(e)}(f_{w}^{(k)}(x))$.

We will pass from $x \sim S^n$ to $x$ sampled from the subset $R$ of the sphere such that $x \cdot u$ and $x \cdot v$ are each in the range $[-a, a]$ where $a = \sqrt{2n / \log n}$. Let $E = \mathbb{E}_{x \sim R}(z_{u}(x)z_{v}(x))$. We wish to estimate the quantity $$\mathbb{E}_{x \sim S^n}(z_{u}(x)z_{v}(x)) = E + \mathbb{E}_{x \sim S^n \backslash R}(z_{u}(x)z_{w}(x))$$ $$\leq E + (1/\varepsilon)^2 \text{vol}(S^n \backslash R) \leq E + O(n^{-\ell}/\varepsilon^2).$$

Next, let $\alpha = u \cdot v$ and write $v = \sqrt{1-\alpha^2}v' + \alpha u$ where $v' \perp u$ and $u, v'$ can be completed to some basis $u_1 = u, u_2 = v', w_3, \ldots, w_n$ of $\mathbb{R}^n$. For a point $x$, let the coordinates in this basis be $x_1, x_2, \ldots, x_n$. In the set $R$, we have $x_1 \in [-a, a]$ and similarly $\alpha x_1 + \sqrt{1-\alpha^2}x_2 \in [-a, a]$. Let $\zeta(t) = \chi_0^{(e)}(\sqrt{N(n,k)} \mu_{a,k}(t))$, so $z_{u}(x) = \zeta(w \cdot x)$. Then $$E = \mathbb{E}_{x \sim R}(z_{u}(x)z_{v}(x)) = \frac{\text{vol}(S^{n-2})}{\text{vol}(S^n)} \int_{x_1=-a}^{a} \int_{x_2=(-a-x_1)/\sqrt{1-\alpha^2}}^{a} \zeta(w)(1-\alpha^2x_1 + \alpha x_2)(1-x_1^2-x_2^2)^{(n-3)/2} \, dx_1 \, dx_2.$$ Now substituting $w = \sqrt{1-\alpha^2}x_2 + \alpha x_1$, we have $$\int_{x_1=-a}^{a} \int_{x_2=(-a-x_1)/\sqrt{1-\alpha^2}}^{a} \zeta(x_1)(\sqrt{1-\alpha^2}x_1 + \alpha x_2)(1-x_1^2-x_2^2)^{(n-3)/2} \, dx_1 \, dx_2$$ $$= \frac{1}{\sqrt{1-\alpha^2}} \int_{x_1=-a}^{a} \int_{x_2=-a}^{a} \zeta(x_1)(w)(1-\alpha^2x_1^2 - \alpha^2x_2^2 - w^2 + 2\alpha x_1 w) \, dx_1 \, dw$$ $$\leq \frac{1}{\sqrt{1-\alpha^2}} \int_{x_1=-a}^{a} \int_{x_2=-a}^{a} \zeta(x_1)(w)(1-x_1^2 - x_2^2 - w^2(1-2\alpha^2)) \, dx_1 \, dw$$ Next we note that in our range of $x_1, w$, $$\max \left( \frac{(1-x_1^2)(1-\alpha^2) - w^2(2-2\alpha^2)}{(1-x_1^2-w^2)^{(n-3)/2}} \right) \leq \left( 1 + \frac{\alpha^2(x_1^2+2w^2)}{1-x_1^2-w^2} \right)^{(n-3)/2} \leq 1 + O(\ell \alpha^2 \log n)$$ using the fact that $x_1, w \in [-\sqrt{2n \log n/n, \sqrt{2n \log n/n}}$. Therefore, $$\frac{1}{\sqrt{1-\alpha^2}} \int_{x_1=-a}^{a} \int_{x_2=-a}^{a} \zeta(x_1)(w)(1-x_1^2(1-\alpha) - w^2(2-2\alpha)) \, dx_1 \, dw$$ $$\leq (1 + O(\ell \alpha^2 \log n)) \int_{x_1=-a}^{a} \int_{x_2=-a}^{a} \zeta(x_1)(w)(1-x_1^2-w^2)^{(n-3)/2} \, dx_1 \, dw$$ $$\leq (1 + O(\ell \alpha^2 \log n)) \int_{x_1=-a}^{a} \int_{x_2=-a}^{a} \zeta(x_1)(w)(1-x_1^2)^{(n-3)/2}(1-w^2)^{(n-3)/2} \, dx_1 \, dw.$$
We therefore have
\[ E_{x \sim R} \left( z_u(x) z_v(x) \right) \leq (1 + O(\ell(u \cdot v)^2 \log n)) \mu^2. \]

We conclude that
\[ \left| \text{Cov}_{x \sim S^n} (z_u(x), z_v(x)) \right| = O(\ell(u \cdot v)^2 \log n) \mu_0^2 + O(n^{-\ell/\varepsilon^2}) \]
as desired. \( \square \)

**Proof of Theorem 4.7 (2).** Taking a random (uniform) set \( B \) of \( d \) vectors \( u \in S^n \), let \( C = \{ f_u^{(k)} : u \in B \} \). As seen in the proof of Theorem 4.7 (1), we can take every pair \( u, v \in B \) to satisfy \( u \cdot v = O(\sqrt{(\log d)/n}) \). By Lemma 4.4 we have
\[ \rho(f_u^{(k)}, f_v^{(k)}) = O \left( \sqrt{\frac{\log d}{n}} \right) \leq n^{-\Omega(k)} \]
Fix \( y \in \mathbb{R} \) and let
\[ \mu_0 = \mathbb{E}_{x \sim S^n} \left( \chi_y((z_u^{(k)}(x))) \right). \]
and
\[ \mu(y) = \|f_u^{(k)}\|_\infty \mu_0 = n^{-\Theta(k)} \mu_0 \]
Take \( \tilde{\gamma} = n^{-\Theta(k)} \) and \( \varepsilon = \tilde{\gamma}/(2\lambda) \). By Lemma 4.7 using \( \ell = \Theta(k \log \lambda) \) in the statement of the lemma, we furthermore have
\[ \left| \text{Cov}_{x \sim S^n} (z_u(x), z_v(x)) \right| = O(\ell(u \cdot v)^2 \log n) \mu_0^2 + O(n^{-\ell/\varepsilon^2}) = n^{-\Omega(k)} (\mu_0^2 + \varepsilon^2). \]
We therefore have \( \text{SDA}(C, S^n, n^{-\Omega(k)}) = d \). The result now follows by Theorem 4.6. \( \square \)

### 4.3 Queries with Gaussian noise

We conclude this section with our lower bounds against 1-STAT. These lower bounds rely on the simulation of 1-STAT using VSTAT proved in [12, Theorem 3.13].

In the regression context we have so far considered, we have a family \( C \) of concepts \( g : X \rightarrow \mathbb{R} \) for some domain \( X \) and distribution \( D \) on \( X \), and the SQ query functions \( h : X \times \mathbb{R} \rightarrow \mathbb{R} \) take pairs \( (x, g(x)) \) (or \( (x, g(x)/\|g\|_\infty) \) in the \( L^\infty \)-normalized case) as input. We now consider SQ algorithms that make queries \( h : X \times \mathbb{R} \rightarrow \mathbb{R} \) that will take pairs \( (x, g(x) + \zeta) \) as input, where \( \zeta \) is some random noise. Equivalently, we could consider replacing \( C \) with noisy concepts \( g \), for which \( g(x) \) is not a precise number but rather a distribution. Formally, we say an SQ algorithm makes \( L^\infty \)-normalized queries to 1-STAT in the presence of Gaussian noise of variance \( \varepsilon \) if it makes queries of the form \( h : X \times \mathbb{R} \rightarrow \{0, 1\} \), to which the oracle replies with the value of \( h(x, g(x)/\|g\|_\infty + \zeta) \) where \( x \sim D \) and \( \zeta \sim \mathcal{N}(0, \varepsilon) \). We similarly define queries to VSTAT in the presence of Gaussian noise, replacing the codomain of \( h \) with \([0, 1]\); in this case, as previously, the oracle must reply with a value \( v \) such that
\[ |p - v| \leq \max \left\{ \frac{1}{t}, \sqrt{\frac{p(1 - p)}{t}} \right\}. \]
where
\[ p = \mathbb{E}_{x \sim D, \zeta \sim \mathcal{N}(0, \varepsilon)} h(x, g(x)/\|g\|_\infty + \zeta). \]

In order to give our lower bounds against 1-STAT oracles, we first give the following lower bounds for queries to VSTAT in the presence of Gaussian noise, which by Lemma 4.9 are in effect a special case of those proved in Section 4.2 for Lipschitz queries.

**Lemma 4.8.** Let \( \varepsilon > 0 \). For all \( k, \lambda > 0 \) and all sufficiently large \( n \) and \( d < \exp(n^{1/2 - \varepsilon}) \), there exists a family \( C \) of degree-\( k \) polynomials on \( S^n \) with \( |C| = d \) such that if a randomized SQ algorithm learns \( C \) to regression error less than any fixed constant with probability at least \( 1/2 \), it requires at least \( \Omega(d) \) queries, if the queries are \( L^\infty \)-normalized queries to VSTAT \( (n^{\Omega(k)}/\lambda) \) in the presence of Gaussian noise of variance \( 1/\lambda^2 \). (All the hidden constants depend on \( \varepsilon \) only.)
Lemma 4.9. Let $D$ be a probability distribution over some domain $X$, let $f : X \to \mathbb{R}$, and let $\zeta \sim \mathcal{N}(0, \sigma^2)$ be a Gaussian random variable. Let $h : X \times \mathbb{R} \to [0, 1]$ and define

$$\tilde{h}(x, y) = \mathbb{E}_{\zeta} h(x, y + \zeta).$$

Then $\tilde{h}(x, y)$ is $1/(2\sigma)$-Lipschitz in $y$.

Proof. Fix $x \in X$ and let $y_1, y_2 \in \mathbb{R}$. We estimate

$$|\tilde{h}(x, y_1) - \tilde{h}(x, y_2)| = \|\mathbb{E}_{z_1 \sim \mathcal{N}(y_1, \sigma^2)} h(x, z_1) - \mathbb{E}_{z_2 \sim \mathcal{N}(y_2, \sigma^2)} h(x, z_2)\| \leq D_{TV}(\mathcal{N}(y_1, \sigma^2), \mathcal{N}(y_2, \sigma^2))$$

where $D_{TV}$ denotes the total variation distance. This distance is bounded above by $|y_1 - y_2|/2\sigma$ (see, e.g., [10]).

Proof of Lemma 4.8. Given an $L^\infty$-normalized query $h : x \times \mathbb{R} \to [0, 1]$ to VSTAT in the presence of Gaussian noise of variance $1/\lambda^2$, let $\tilde{h}$ be as in Lemma 4.9. We have

$$\mathbb{E}_{x \sim D, \zeta \sim \mathcal{N}(0, \epsilon)} h(x, g(x)/\|g\|_{\infty} + \zeta) = \mathbb{E}_{x \sim D} \tilde{h}(x, g(x)).$$

Therefore, since $\tilde{h}$ is $1/(2\lambda)$-Lipschitz by Lemma 4.9, the query $h$ to VSTAT in the presence of Gaussian noise can be simulated by a $1/(2\lambda)$-Lipschitz query to VSTAT. The result is now immediate from Theorem 1.7 (2).

Proof of Theorem 1.7 (3). By [12, Theorem 3.13], if there is an algorithm solving the problem using $m$ queries to 1-STAT, there is an algorithm solving the problem using $m$ queries to VSTAT($O(m)$). By Lemma 4.8, at least $\Omega(d)$ queries to VSTAT($n^{\Omega(k)}/\lambda$) are required. Hence, at least $n^{\Omega(k)}/\lambda$ queries to 1-STAT are also required in the presence of Gaussian noise of variance $1/\lambda^2$, as long as the number $d$ of polynomials in $C$ is at least $n^{\Omega(k)}$.

5 Discussion

We have given a polynomial-time analysis of gradient descent for training a neural network in an agnostic setting. In particular, we show that functions that are approximated by polynomials can be learned by gradient descent, as well as functions computed by single-hidden-layer neural networks. These results build on a long line of work by many authors studying the power of random initialization combined with output-layer training.

We show that our analysis is essentially tight, in the sense that no statistical query algorithm can have significantly better time complexity.

Extending the training to hidden-layer weights cannot offer an asymptotic improvement in the number of gates needed to achieve small error in the general setting we consider. However, experiments suggest that training hidden-layer weights might allow for tighter bounds in the realizable case. In particular, it would be interesting to give a fully polynomial analysis of gradient descent for learning data labeled by a single-hidden layer neural network with $m$ neurons. An extension for networks with bounded bias parameters, rather than unbiased networks, would also be interesting.

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