The quantization of persistent current qubit. The role of inductance

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Abstract

The Hamiltonian of persistent current qubit is found within well known quantum mechanical procedure. It allows a selfconsistent derivation of the current operator in a two state basis. It is shown that the current operator is not diagonal in a flux basis. A non diagonal element comes from the finite inductance of the qubit. The results obtained in the paper are important for the circuits where two or more flux qubits are coupled inductively.

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I. INTRODUCTION

Josephson-junction qubits are known to be candidates for scalable solid-state quantum computing circuits [1]. Here we consider a superconducting flux qubit which has been first proposed in [2] and analyzed in [3] and [4]. The qubit consists of three Josephson junctions in a loop with very small inductance $L$, typically in the pH range. This insures effective decoupling from the environment. However, in the practical implementation of flux qubit circuitry it is important to have the loop inductance as much as possible consistent with a proper operation of a qubit. A relative large loop inductance facilitates a qubit control biasing schemes and the formation, control and readout of two-qubit quantum gates. These considerations stimulated some investigations of the role the loop inductance plays in the operation of a flux qubit [5], [6], [7]. The main goal of these works was the calculation of the corrections to the energy levels due to finite inductance of the loop. In the early work [5] these corrections have been obtained by perturbation expansion of the energy over small parameter $\beta = L/L_J$, where $L_J$ is the Josephson junction inductance. The extension to large $\beta$’s (up to $\beta \approx 10$) had been considered in [7]. However, it is important to realize that for finite loop inductance the interaction between two state qubit with its own LC circuit cannot in general be neglected. If $\beta$ is not small, as in [7], this interaction can have substantial influence on the energy levels. Unfortunately, this interaction in [7] has been completely neglected.

In principle, the account for a finite loop inductance (even if it is small) requires the correct construction of quantum mechanical Hamiltonian of a qubit, which contains all relevant interactions. This has been done in [6], where the effective Hamiltonian has been obtained by a rigorous expansion procedure in powers of $\beta$. As was shown in [6], one of the effect of the interaction of a flux qubit with its own LC oscillator is the renormalization of the Josephson critical current. The inclusion of circuit inductances in a systematic derivation of the Hamiltonian of superconducting circuits has been done in [8]. It allows the correct calculations of the effects of the finite inductance both for flux [9] and charge [10] qubits.

In this paper we investigate another physical effect which comes from finite loop inductance. Namely, we show that the finite loop inductance results in the additional term of the current operator in the flux basis:

$$\hat{I} = A\tau_Z + B\tau_X,$$  (1)
where $\tau_Z$ and $\tau_X$ are Pauli matrices in the flux basis. The quantities $A$ and $B$ in (1) are calculated in the paper the $B$ being conditioned by the finite loop inductance: for $L = 0$ the second term in (1) is absent. Though for the usual qubit design with small loop inductance this second term is relatively small, nevertheless, it might give noticeable effects for large $\beta$’s for the arrangements when two flux qubit are coupled either via a common inductance or inductively coupled via a term $M\hat{I}_1\hat{I}_2$ in the Hamiltonian, where $M$ is a mutual inductance between qubit’s loops, $\hat{I}_1$, $\hat{I}_2$ are the current operators of the respective qubits.

The paper is organized as follows. In Section II we start with the exact Lagrangian of a flux qubit with finite loop inductance. In section III with the aid of well known procedure we derive rigorously the quantum qubit Hamiltonian. The current operator is studied in Section IV, where we show that in general it is not diagonal in the flux basis. The matrix elements for the current operator are calculated in Section VI, where in order to obtain analytical results we consider a flux qubit with a small loop inductance.

II. LAGRANGIAN FOR THE FLUX QUBIT

We consider here a well known design of the flux qubit with three Josephson junctions [2], [3], [4], which is shown on Fig. 1.

Two junctions have equal critical current $I_c$ and (effective) capacitance $C$, while those of the third junction are slightly smaller: $\alpha I_c$ and $\alpha C$, with $0.5 < \alpha < 1$. If the Josephson energy $E_1 = I_c\Phi_0/2\pi$ is much larger than the Coulomb energy $E_C = e^2/2C$, the Josephson phase is well defined. Near $\Phi_x = \Phi_0/2$, this system has two low-lying quantum states [3, 4].

The Lagrangian of this qubit is the difference between the charge energy in the junction capacitors and the sum of Josephson and magnetic energy:

$$L = \sum_{i=1}^{3} \frac{C_i V_i^2}{2} + \sum_{i=1}^{3} E_i^{(i)} \cos \phi_i - \frac{\Phi^2}{2L}$$  \hspace{1cm} (2)

where $V_i$ is the voltage across the junction capacitance $C_i$, which is related to the phase $\phi_i$ by the Josephson relation $V_i = (\Phi_0/2\pi)\dot{\phi}_i$; $\Phi$ is the flux trapped in the loop:

$$\Phi = \frac{\Phi_0}{2\pi} \sum_{i=1}^{3} \phi_i - \Phi_x$$  \hspace{1cm} (3)
FIG. 1: A flux qubit, where an external magnetic flux $\Phi_X$ pierces the superconducting loop that contains three Josephson junctions and inductance $L$. Two Josephson junctions are considered to be identical, $E_{J1} = E_{J2} = E_J$, $C_1 = C_2 = C$, and $E_{J3} = \alpha E_J$, $C_3 = \alpha C$.

Next we make the following definitions: $\phi = \phi_1 + \phi_2 + \phi_3$, $\phi_1 + \phi_2 = 2\theta$, $\phi_1 - \phi_2 = 2\chi$. In terms of these new phases Lagrangian (2) takes the form:

$$L = \frac{\hbar^2}{16E_C} (\dot{\phi}_1^2 + \dot{\phi}_2^2) + \alpha \frac{\hbar^2}{16E_C} \dot{\phi}_3^2 + 2E_J \cos \theta \cos \chi + \alpha E_J \cos (\varphi - 2\theta) - \frac{E_J}{2\beta} (\varphi - \varphi_X)^2$$

where $\beta = 2\pi LI_C/\Phi_0$.

III. CONSTRUCTION OF HAMILTONIAN

Conjugate variables are defined in a standard way:

$$n_\varphi = \frac{1}{\hbar} \frac{\partial L}{\partial \dot{\varphi}} = \alpha \frac{\hbar}{8E_C} (\dot{\varphi} - 2\dot{\theta})$$

$$n_\theta = \frac{1}{\hbar} \frac{\partial L}{\partial \dot{\theta}} = \frac{\hbar}{4E_C} \dot{\theta} - \alpha \frac{\hbar}{4E_C} (\dot{\varphi} - 2\dot{\theta})$$

$$n_\chi = \frac{1}{\hbar} \frac{\partial L}{\partial \dot{\chi}} = \frac{\hbar}{4E_C} \dot{\chi}$$

From these equations we express phases in terms of conjugate variables:

$$\dot{\varphi} = \frac{8E_C}{h} 2\alpha + \frac{1}{\alpha} n_\varphi + \frac{8E_C}{h} n_\theta$$
\[ \dot{\theta} = \frac{4E_C}{\hbar} n_\theta + \frac{8E_C}{\hbar} n_\varphi \]  
\[ \dot{\chi} = \frac{4E_C}{\hbar} n_\chi \]  

In terms of conjugate variables Lagrangian (4) takes the form:

\[ L = 2E_C n_\theta^2 + 2E_C n_\chi^2 + 4E_C \frac{1 + 2\alpha}{\alpha} n_\varphi^2 + 8E_C n_\theta n_\varphi + 2E_J \cos \theta \cos \chi \]

\[ + 2E_J \cos \theta \cos \chi + \alpha E_J \cos (\varphi - 2\theta) - \frac{E_J}{2\beta} (\varphi - \varphi_X)^2 \]  

The Hamiltonian is constructed according to the well known rule:

\[ H = \hbar n_\varphi \dot{\varphi} + \hbar n_\theta \dot{\theta} + \hbar n_\chi \dot{\chi} - L \]  

Finally we obtain:

\[ H = 4E_C \frac{2\alpha + 1}{\alpha} n_\varphi^2 + 2E_C n_\chi^2 + 2E_C n_\theta^2 + 8E_C n_\theta n_\varphi + U(\chi, \theta, \varphi) \]  

where

\[ U(\chi, \theta, \varphi) = -2E_J \cos \theta \cos \chi - \alpha E_J \cos (\varphi - 2\theta) + \frac{E_J}{2\beta} (\varphi - \varphi_X)^2 \]  

Hence, equations of motion for the phases (8), (9), (10) are simply

\[ \dot{\varphi} = \frac{1}{\hbar} \frac{\partial H}{\partial n_\varphi}; \quad \dot{\theta} = \frac{1}{\hbar} \frac{\partial H}{\partial n_\theta}; \quad \dot{\chi} = \frac{1}{\hbar} \frac{\partial H}{\partial n_\chi}. \]  

The equations of motion for conjugate variables are:

\[ \dot{n}_\varphi = -\frac{1}{\hbar} \frac{\partial H}{\partial \varphi} = -\frac{\alpha E_J}{\hbar} \sin(\varphi - 2\theta) - \frac{E_J}{\hbar \beta} (\varphi - \varphi_X) \]  

\[ \dot{n}_\theta = -\frac{1}{\hbar} \frac{\partial H}{\partial \theta} = -2\alpha \frac{E_J}{\hbar} \sin(\varphi - 2\theta) - \frac{2E_J}{\hbar} \sin \theta \cos \chi \]  

\[ \dot{n}_\chi = -\frac{1}{\hbar} \frac{\partial H}{\partial \chi} = -\frac{2E_J}{\hbar} \cos \theta \sin \chi \]  

Below we consider Hamiltonian (13) as quantum mechanical with commutator relations imposed on its variables

\[ [\varphi, n_\varphi] = i; \quad [\theta, n_\theta] = i; \quad [\chi, n_\chi] = i \]
IV. CURRENT OPERATOR

From the first principles a current in the loop is equal to the first derivative of the state energy relative to external flux:

\[ I = \frac{\partial E_n}{\partial \Phi_X} \]  
(19)

This expression can be rewritten in terms of exact Hamiltonian of a system:

\[ I = \langle n | \frac{\partial \hat{H}}{\partial \Phi_X} | n \rangle \]  
(20)

From (20) we would make ansatz that the current operator is as follows:

\[ \hat{I} = \frac{\partial \hat{H}}{\partial \Phi_X} \]  
(21)

However (21) is not a consequence of (20). Therefore, the ansatz (21) must be proved in every case, since the current operator in the form of Eq. (21) has to be consistent with its definition in terms of variables of Hamiltonian \( H \). The prove for our case is given below.

The current operator across every junction is a sum of a supercurrent and a current through the capacitor:

\[ \hat{I}_i = I_0 \sin \phi_i + \frac{\hbar}{2e} C \ddot{\varphi}_i \quad (i = 1, 2) \]  
(22)

\[ \hat{I}_3 = \alpha I_0 \sin \varphi_3 + \alpha \frac{\hbar}{2e} C \ddot{\varphi}_3 \]  
(23)

Since the current in a loop is unique the equations (22) and (23) must give identical result. This is indeed the case if we express phases \( \phi_i (i = 1, 2, 3) \) in terms of \( \phi, \theta, \chi \) and use the equations (8), (9), (10), (15), (16), (17). For every \( I_i \) in (22), (23) we obtain the same expression

\[ \hat{I} = -I_0 \frac{\varphi - \varphi_X}{\beta} \]  
(24)

which is independent of parameters of a particular junction in the loop. From the other hand the expression (24) can be obtained from our Hamiltonian (13) with the aid of (21). Therefore, the equation (21) gives us the true expression for the current operator. It is important to note that the proper expression for the current operator (24) cannot be obtained without magnetic energy term in the original Lagrangian (2).

It follows from (13) and (24) that \( \left[ \hat{I}, \hat{H} \right] \neq 0 \). Therefore, an eigenstate of \( H \) cannot possess a definite current value.
A. Current operator in a two-state basis

Suppose a system is well described by two low lying states \( |\Psi_\pm\rangle \) with corresponding eigenenergies \( E_\pm \):

\[
\hat{H} |\Psi_\pm\rangle = E_\pm |\Psi_\pm\rangle
\]  
(25)

Within this subspace Hamiltonian can be expressed in terms of Pauli matrices \( \sigma_X, \sigma_Y, \sigma_Z \):

\[
\hat{H} = \frac{E_+ + E_-}{2} - \frac{E_+ - E_-}{2} \sigma_Z
\]  
(26)

with \( \sigma_Z |\Psi_\pm\rangle = \mp |\Psi_\pm\rangle \).

Now we calculate the matrix elements of the current operator \( \hat{I} \) within this subspace. According to (19), (20) and (21) diagonal matrix elements are:

\[
\langle \Psi_\pm | \hat{I} | \Psi_\pm \rangle = \frac{\partial E_\pm}{\partial \Phi_X} 
\]  
(27)

In order to find nondiagonal matrix elements of the current operator we use the expression

\[
\langle n | \frac{\partial \hat{H}}{\partial \lambda} | n' \rangle = (E_{n'} - E_n) \left\langle n \left| \frac{\partial n'}{\partial \lambda} \right. \rightangle 
\]  
(28)

which is obtained by differentiating of the identity \( \langle n | \hat{H} | n' \rangle = 0 \) with respect to parameter \( \lambda \). Hence, the nondiagonal elements of the current operator are:

\[
\langle \Psi_- | \hat{I} | \Psi_+ \rangle = \langle \Psi_+ | \hat{I} | \Psi_- \rangle = (E_+ - E_-) \left\langle \Psi_- \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right. \rightangle 
\]  
(29)

Therefore, we can express the current operator in terms of Pauli matrices:

\[
\hat{I} = \frac{\partial}{\partial \Phi_X} \left( \frac{E_+ + E_-}{2} \right) \mathbf{I} - \frac{\partial}{\partial \Phi_X} \left( \frac{E_+ - E_-}{2} \right) \sigma_Z + (E_+ - E_-) \left\langle \Psi_- \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right. \right\rangle \sigma_X 
\]  
(30)

where \( \mathbf{I} \) is the unity matrix.

Below we consider two low lying states of a flux qubit

\[
E_\pm = E_0 \pm \sqrt{\varepsilon^2 + \Delta^2} 
\]  
(31)

where \( E_0 \) and the tunneling rate \( \Delta \) are independent of the external flux \( \Phi_X \), and the quantity \( \varepsilon \) is linear function of the flux, \( \varepsilon = E_J \lambda f_X \), where \( \lambda \) is a numerical factor which depends on qubit parameters \( \alpha \) and \( g = E_J / E_C \), \( f_X = \Phi_X / \Phi_0 - 1/2 \).

Therefore, for the flux qubit we get in eigenstate basis:

\[
H = E_0 - \Delta \varepsilon \sigma_Z 
\]  
(32)
\[ \hat{I} = -\frac{\partial \Delta_\varepsilon}{\partial \Phi_X} \sigma_Z + 2\Delta_\varepsilon \left\langle \Psi_+ \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \sigma_X \right\rangle \] (33)

where \( \Delta_\varepsilon = \sqrt{\varepsilon^2 + \Delta^2} \).

Transformation to the flux basis is obtained via the rotation around \( y \) axes in a two level subspace with the aid of the matrix 
\[ R = \exp \left( i \xi \sigma_Y / 2 \right), \]
where \( \cos \xi = \varepsilon / \Delta_\varepsilon, \sin \xi = \Delta / \Delta_\varepsilon \):
\[ R^{-1} \sigma_Z R = \tau_Z \cos \xi + \tau_X \sin \xi, \]
\[ R^{-1} \sigma_X R = -\tau_Z \sin \xi + \tau_X \cos \xi, \]
where \( \tau_X, \tau_Z \) are Pauli matrices in a flux basis. Hence, we get for Hamiltonian (32) and current operator (33) in the flux basis:
\[ H = -\varepsilon \tau_Z - \Delta \tau_X \] (34)
\[ \hat{I} = - \left( \frac{\partial \Delta_\varepsilon}{\partial \Phi_X} \frac{\varepsilon}{\Delta_\varepsilon} + 2\Delta_\varepsilon \left\langle \Psi_+ \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \right\rangle \right) \tau_Z - \left( \frac{\partial \Delta_\varepsilon}{\partial \Phi_X} \frac{\Delta}{\Delta_\varepsilon} - 2\varepsilon \left\langle \Psi_+ \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \right\rangle \right) \tau_X \] (35)

Therefore, the current operator is not diagonal neither in the flux basis nor in the eigenstate basis.

The stationary state wave functions \( \Psi_\pm \) can be written as the superpositions of the wave functions in the flux basis, \( \Psi_L, \Psi_R \) where \( L, R \) stand for the left, right well, respectively:
\[ \Psi_\pm = a_\pm \Psi_L + b_\pm \Psi_R, \]
where
\[ a_\pm = \frac{\Delta}{\sqrt{2\Delta_\varepsilon (\Delta_\varepsilon \mp \varepsilon)}}, \quad b_\pm = \frac{\varepsilon \mp \Delta_\varepsilon}{\sqrt{2\Delta_\varepsilon (\Delta_\varepsilon \mp \varepsilon)}}. \] (36)
The coefficients \( a_\pm, b_\pm \) are defined in such a way, that \( \tau_Z |\Psi_L\rangle = -|\Psi_L\rangle, \tau_Z |\Psi_R\rangle = +|\Psi_R\rangle, \tau_X |\Psi_L\rangle = +|\Psi_R\rangle, \tau_X |\Psi_R\rangle = +|\Psi_L\rangle \). In terms of the functions \( \Psi_L, \Psi_R \) the cross term
\[ \left\langle \Psi_- \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \right\rangle \]
will read
\[ \left\langle \Psi_- \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \right\rangle = a_- \frac{\partial a_+}{\partial \Phi_X} + b_+ \frac{\partial b_+}{\partial \Phi_X} + a_+ a_- \left\langle \Psi_L \left| \frac{\partial \Psi_L}{\partial \Phi_X} \right| \right\rangle + b_- b_+ \left\langle \Psi_R \left| \frac{\partial \Psi_R}{\partial \Phi_X} \right| \right\rangle \] (37)

The results obtained up till now are exact in that we did not make any approximation to the Hamiltonian (13). However, in order to calculate \( \varepsilon \) and cross term \( \left\langle \Psi_- \left| \frac{\partial \Psi_+}{\partial \Phi_X} \right| \right\rangle \) in (33), (35) we need some approximate procedure.

V. APPROXIMATION TO QUANTUM MECHANICAL HAMILTONIAN

In order to calculate the matrix elements of the current operator we have to find the wave functions of two lowest levels of Hamiltonian (13). First we single out of the potential (12)
the fast variable $\varphi$, which describe the interaction of the qubit with its own LC circuit. The point of minimum $\varphi_C$ of $U(\chi, \theta, \varphi)$ with respect to $\varphi$ is defined from $\partial U/\partial \varphi = 0$:

$$\varphi_C = \varphi - \alpha \beta \sin (\varphi - 2\theta)$$

(38)

In the vicinity of $\varphi_C$ the potential $U(\chi, \theta, \varphi)$ can be written as:

$$U(\chi, \theta, \varphi) \approx U(\chi, \theta, \varphi_C) + \frac{E_J}{2\beta} \widehat{\varphi}^2 + \frac{\alpha E_J}{2} \cos (\varphi_C - 2\theta)$$

(39)

where $\widehat{\varphi}$ is a small operator correction to $\varphi_C$: $\varphi = \varphi_C + \widehat{\varphi}$.

As is known the potential $U(\chi, \theta, \varphi_C)$ has a degenerate point at $\Phi_x = \Phi_0/2$. Assuming $f_X \ll 1$, $\beta \ll 1$ we obtain near this point:

$$\varphi_C = \pi + 2\pi f_X - \alpha \beta \sin 2\theta$$

(40)

From (24) and (40) we find a current operator in "coordinate" representation:

$$\widehat{I} = I_0 \alpha \sin 2\theta$$

(41)

For $U(\chi, \theta, \varphi_C)$ we obtain near degeneracy point

$$U(\chi, \theta, \varphi_C) = -2E_J \cos \theta \cos \chi + \alpha E_J \cos 2\theta + \alpha 2\pi f_X E_J \sin 2\theta - \frac{\alpha^2 \beta E_J}{2} \sin^2 2\theta$$

(42)

Below we follow the procedure described in [11]. At $f_x = 0$, the potential (12) has two minima at $\chi = 0$, $\theta = \pm \theta_*$, with $\cos \theta_* = 1/2\alpha$ ($\theta_* > 0$). Tunnelling lifts their degeneracy, leading to energy levels $E_\pm = E_0 \pm \Delta$. However, at degenerate bias the current vanishes, forcing one to move slightly away from this point. In order to find the levels for $|f_x| \ll 1$ we expand Eq. (42) near its minima, retaining linear terms in $f_x$, $\beta$ and quadratic terms in $\chi, \theta$. Define $\theta_{s/r}^{s/l}$ as the minima, shifted due to $f_x$ and $\beta$:

$$\theta_{s/r}^{s/l} = \pm \theta_* + 2\pi f_x \frac{1-2\alpha^2}{4\alpha^2-1} \pm \beta \frac{1-2\alpha^2}{2\alpha(4\alpha^2-1)};$$

(43)

that is, the upper (lower) sign refers to the right (left) well. The potential energy (42) then reads:

$$\frac{U^{s/l}(\chi, \theta, \varphi_C)}{E_J} = U_0^{s/l} + A^{s/l} \chi^2 + B^{s/l} \widehat{\varphi}^2$$

(44)

where the operator correction $\widehat{\varphi} = \theta - \theta_{s/r}^{s/l}$,

$$U_0^{s/l} = -\alpha - \frac{1}{2\alpha} \pm 2\pi f_X \frac{\sqrt{4\alpha^2-1}}{2\alpha} - \beta \frac{4\alpha^2-1}{8\alpha^2}$$

(45)
\[
A^{r/l} = \frac{1}{2\alpha} \pm 2\pi f_X \frac{2\alpha^2 - 1}{2\alpha\sqrt{4\alpha^2 - 1}} + \beta \frac{2\alpha^2 - 1}{4\alpha^2}
\]

\[
B^{r/l} = 2\alpha - \frac{1}{2\alpha} \pm 2\pi f_X \frac{2\alpha^2 + 1}{2\alpha\sqrt{4\alpha^2 - 1}} + \beta \left( -\frac{1}{4} + \frac{5}{2}\alpha^2 - 2\alpha^4 \right)
\]

Combining (44) and (42) in (13) we obtain quadratic quantum mechanical Hamiltonian for the flux qubit in the left and right well near the degeneracy point:

\[
H^{r/l} = E_J U_0^{r/l} + \left[ \frac{4 (2\alpha + 1)}{\alpha} E_C \hat{n}_\phi^2 + \frac{E_J}{2\beta} \hat{\phi}^2 \right] + \left[ 2E_C \hat{n}_\chi^2 + E_J A^{r/l} \hat{\chi}^2 \right] + \left[ 2E_C \hat{n}_\theta^2 + E_J B^{r/l} \hat{\theta}^2 \right] + 8E_C \hat{n}_\phi \hat{n}_\phi + \frac{E_J}{2} C^{r/l} \hat{\phi}^2
\]

where

\[
C^{r/l} = \frac{1}{2\alpha} \left[ -1 \pm 2\pi f_X \left( \frac{2\alpha^2 - 1}{2\alpha\sqrt{4\alpha^2 - 1}} + \frac{\beta}{\alpha} \left( 1 - 2\alpha - \frac{4\alpha^2 - 1}{\alpha} \right) \right) \right] \quad (49)
\]

The first term in square brackets in (48) is the Hamiltonian of LC oscillator of the flux qubit, which is slightly modified by the last term in (48). The next two terms in square brackets are oscillator Hamiltonians for the flux qubit variables, \(\chi\) and \(\theta\), respectively. The interaction of the \(\theta\) degree of freedom with the qubit LC circuit is given by next-to-last term in (48).

Assuming the frequency \((LC)^{-1/2}\) of the qubit LC circuit is much higher than the junctions frequencies \(E_J/h, E_C/h\) we neglect the interaction of the qubit variables, \(\theta\) and \(\chi\), with the qubit LC oscillator. This is equivalent to the averaging of Hamiltonian (48) over the ground state of the LC Hamiltonian. Therefore, for the qubit Hamiltonian we obtain:

\[
H_{qb} = \langle H^{r/l} \rangle = E_J U_0^{r/l} + \frac{1}{2} \varepsilon_0 + \frac{E_J}{2} C^{r/l} \langle \phi^2 \rangle + \left[ 2E_C \hat{n}_\chi^2 + E_J A^{r/l} \hat{\chi}^2 \right] + \left[ 2E_C \hat{n}_\theta^2 + E_J B^{r/l} \hat{\theta}^2 \right]
\]

where

\[
\varepsilon_0 = \left( \frac{8 E_C E_J (2\alpha + 1)}{\beta \alpha} \right)^{1/2} \quad ; \quad \langle \phi^2 \rangle = \frac{1}{2} \left( \frac{8 \beta E_C (2\alpha + 1)}{E_J \alpha} \right)^{1/2}
\]

Next we confine ourself only to the ground state of (50) in either of the wells.

\[
\varepsilon^{r/l} = \frac{1}{2} \varepsilon_0 + E_J U_0^{r/l} + E_J \frac{C^{r/l}}{2} \langle \phi^2 \rangle + \frac{\hbar \omega_{\phi}^{r/l}}{2} + \frac{\hbar \omega_{\chi}^{r/l}}{2}
\]

where

\[
\hbar \omega_{\chi}^{r/l} = E_J \sqrt{\frac{4}{\alpha g} \left( 1 + 2\pi f_X \frac{2\alpha^2 - 1}{2\sqrt{4\alpha^2 - 1}} + \beta \frac{2\alpha^2 - 1}{4\alpha} \right)} \quad (52)
\]
The ground state wave functions in left (right) well are as follows:

\[
\Psi_{R/L} = \frac{1}{\sqrt{4\pi}} \left( E \frac{\hbar^2}{E_C^2} \right)^{1/4} \exp \left( -\frac{\hbar \omega_{r/l}}{8E_C} \chi^2 - \frac{\hbar \omega_{r/l}}{8E_C} (\theta - \theta_{r/l}^0)^2 \right),
\]

The tunneling between two wells lifts degeneracy yielding the well known result for eigenenergies \( E_{\pm} = (\varepsilon - \varepsilon^r)/2 \pm \sqrt{(\varepsilon - \varepsilon^r)^2/4 + \Delta^2} \), which was given above in Eq. (31). The Eqs. (51), (52), (53) allows us to calculate the numerical factor \( \lambda \) in (31):

\[
\lambda(\alpha, g) \frac{\alpha}{\pi} = \sqrt{4\alpha^2 - 1} - \sqrt{\alpha g} \left( \frac{2\alpha^2 - 1}{\sqrt{4\alpha^2 - 1} + \frac{2\alpha^2 + 1}{4\alpha^2 - 1}} \right) + \sqrt{\beta} \sqrt{\alpha g} \frac{\sqrt{8(2\alpha + 1)}}{8\alpha} \left( \frac{2\alpha^2 - 1}{\sqrt{4\alpha^2 - 1} + \frac{4\alpha^2 - 1}{\alpha}} \right)
\]

The average current in eigenstates \( E_{\pm} \) is calculated from (19):

\[
I_q = \frac{\partial E_{\pm}}{\partial \Phi_X} = \pm I_c f_X \frac{\lambda^2(\alpha, g)}{2\pi} \frac{E_J}{\sqrt{\varepsilon^2 + \Delta^2}}.
\]

If the deviation from the degeneracy point \( f_X = 0 \) is significant (\( \varepsilon \gg \Delta \)) then the current (56) reduces to its local value in a particular well \( I_q \to \pm I_c \lambda(\alpha, g)/2\pi = \partial \varepsilon_{r/l}/\partial \Phi_X \).

A. The matrix elements of the current operator

As is seen from Eqs. (33), (35), it is necessary to calculate the quantity \( \langle \Psi_+ | \partial \Psi_+ / \partial \Phi_X \rangle \).

The calculation yields the following result:

\[
\langle \Psi | \partial \Psi_+ / \partial \Phi_X \rangle = \frac{\partial \varepsilon}{\partial \Phi_X} \frac{\Delta}{2\Delta^2} + \beta \frac{\pi}{4\Phi_0} \frac{\Delta}{\Delta \varepsilon} F(\alpha),
\]

where

\[
F(\alpha) = \frac{(2\alpha^2 - 1)^2}{4\alpha \sqrt{4\alpha^2 - 1}} + \frac{\alpha(2\alpha^2 + 1)}{(4\alpha^2 - 1)^2} \left( -\frac{1}{4} + \frac{5}{2} \alpha^2 - 2\alpha^4 \right)
\]

The correction due to inductance (second term in r. h. s. of (58)) is usually small, however, it is responsible for nonzero value of non diagonal matrix elements of the current operator in the flux basis. In the eigenstate basis Eq. (33) transforms to:

\[
\dot{I} = \frac{\partial \varepsilon}{\partial \Phi_X} \frac{1}{\Delta \varepsilon} (-\varepsilon \sigma_Z + \Delta \sigma_X)
\]
where we neglect the correction due to inductance.

In the flux basis Eq. (35) transforms to:
\[
\hat{I} = -\frac{\partial \epsilon}{\partial \Phi_X} \left( \frac{\epsilon^2}{\Delta^2} + \frac{\Delta}{\Delta \epsilon} \right) \tau_Z + \beta \frac{\pi}{2 \Phi_0} \frac{\epsilon \Delta}{\Delta \epsilon} F(\alpha) \tau_X
\] (60)

The Eq. (60) is the main result of our calculations. It shows that the current operator in the flux qubit is not diagonal in the flux basis as it is implicitly assumed in most of papers on the subject. The non diagonal term comes from the finite inductance of the qubit loop. Though for the usual qubit design this term is relatively small, nevertheless, it might give noticeable effects for larger values of \( \beta \) in the arrangements when two flux qubit are inductively coupled via a term \( M \hat{I}_1 \hat{I}_2 \) in the Hamiltonian, where \( M \) is a mutual inductance between qubit’s loops, \( \hat{I}_1, \hat{I}_2 \) are the current operators of the respective qubits.

In conclusion, we show that the finite loop inductance of a flux qubit results in additional non diagonal term in the current operator in the flux basis. The result is important in the arrangements with magnetic coupling of two or more flux qubits.

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