On the Clustering Properties of Mini-Jet and Mini-Dijet in High-Energy pp Collisions

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Mini-jets and mini-dijets provide useful information on multiple parton interactions in the low transverse-momentum (low-$p_T$) region. As a first step to identify mini-jets and mini-dijets, we study the clustering properties of produced particles in the pseudorapidity and azimuthal angle space, in high-energy pp collisions. We develop an algorithm to find mini-jet-like clusters by using the k-means clustering method, in conjunction with a k-number (cluster number) selection principle. We test the clustering algorithm using minimum-bias events generated by PYTHIA8.1, for pp collision at $\sqrt{s} = 200$ GeV. We find that multiple mini-jet-like and mini-dijet-like clusters of low-$p_T$ hadrons occur in high multiplicity events. However similar clustering properties are also present for particles produced randomly in a finite pseudorapidity and azimuthal angle space. The ability to identify an azimuthally back-to-back correlated mini-jet-like clusters as physical mini-jets and mini-dijets will therefore depend on the additional independent assessment of the dominance of the parton-parton hard-scattering process in the low-$p_T$ region.

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I. INTRODUCTION

The mechanism of relativistic parton-parton hard-scattering is an important basic perturbative QCD process in particle production in high-energy nucleon-nucleon collisions \cite{1,2}. Because of the composite nature of a nucleon, multiple hard-scattering between projectile nucleon partons and target nucleon partons will lead to the production of jets and dijets whose subsequent fragmentation gives rise to the production of particle clusters. It is distinctly different from the nonperturbative flux-tube fragmentation process \cite{6,10,23,30,40} in which a quark of one nucleon and the diquark of the other nucleon (or a gluon of one nucleon and the gluon of the other nucleon \cite{43,52}) form one flux tube and the subsequent fragmentation of the flux tube leads to the production of hadrons. It is also different from the direct-fragmentation process \cite{53} in which the partons from the composite nucleon fragment directly into the detected particles.

The hard-scattering process was originally proposed as the dominant process for the production of high-$p_T$ jet clusters, of order many tens of GeV/$c$ \cite{1,7}. However, the UA1 Collaboration found that it is also the dominant process for the production of particle clusters with a total $p_T$ of a few GeV/$c$, for $\bar{p}p$ collisions at $\sqrt{s}=0.2$ to 0.9 TeV \cite{15}. The term “mini-jet” was introduced to describe low-$p_T$ jet clusters \cite{14}. The dominance of jet production can be extended to lower and lower $p_T$ domains at high collision energies because (i) the fraction of particles produced by such a process increases rapidly with collision energies $\sqrt{s}$, and (ii) the jet-production invariant cross section at mid-rapidity varies as an inverse power of $p_T$ \cite{8,16,17,27,28,54}.

Recently, the region of dominance of the hard-scattering process has been found to extend to the production of hadrons in the even lower $p_T$ region of a few tenths of a GeV/$c$ \cite{26,29}. An indirect piece of evidence comes from the observation on the transverse momentum spectra of produced hadrons: For the production of particles with $p_T$ within the range from a few tenths of a GeV to a few hundred GeV in high-energy pp and $p\bar{p}$ collisions at $\sqrt{s}=0.9$ to 7 TeV, the hadron transverse spectra, whose magnitude spans over 14 decades of magnitude, can be described by a simple Tsallis inverse-power-law type distribution with only three degrees of freedom \cite{26,28}. The simplicity of the power-law type transverse spectra suggests that only a single mechanism, the hard-scattering process, dominates over the extended $p_T$ domain. An additional piece of direct evidence comes from the jet-like structure in the two-hadron angular ($\Delta\eta, \Delta\phi$) correlation data in a minimum-$p_T$-bias measurement of the STAR Collaboration in pp collisions at $\sqrt{s} = 200$ GeV \cite{55,58}. The two-particle angular correlations of these low-$p_T$ particles exhibit the signature of parton-parton scattering. Furthermore, the momentum distributions of hadrons associated with a hadron trigger of a few GeV/$c$ in pp collisions at $\sqrt{s}=200$ GeV exhibit a jet-like cluster structure within a cone, as observed by the STAR Collaboration \cite{54,55} and the PHENIX Collaboration \cite{60,67}.

The extension of the dominance of the hard-scattering model to the low-$p_T$ domain of a few tenths of GeV/$c$ raises serious questions on the large and divergent pQCD corrections at low $p_T$ and the competition from the nonperturbative flux tube fragmentation process associated with low-$p_T$ phenomena. We need additional theoretical and experimental comparisons of the hard-scattering model to construct the proper phenomenological descrip-
and azimuthal angle space and search for an algorithm clustering properties of mini-jets in the pseudorapidity at high energies. As a first step, we examine here the jets and mini-dijets in the low-scattering processes, for the production of multiple mini-remains lacking.

Experimental evidence for multiple parton interactions

Additional higher-order diagrams with the radiation and 1c, or odd, as in Figs. 1d, 1e, and 1f. There can also be of produced mini-jets can be even, as in Figs. 1a, 1b, and 1c. Furthermore, a parton of the other proton, as depicted in Figs. 1d, 1e, and 1f. The numbers of one proton can make multiple collisions (known also as multiple collision MPI, known also as multiple collision processes \[1, 7, 8, 11–17\]). Among many other diagrams, the hard-scattering process can lead to the production of one, two, three pairs of mini-dijets as depicted in Figs. 1a, 1b, and 1c. Furthermore, a parton of one proton can make multiple collisions (known also as re-scattering \[13\] ) with different partons of the other proton, as depicted in Figs. 1d, 1e, and 1f. The numbers of produced mini-jets can be even, as in Figs. 1a, 1b, and 1c, or odd, as in Figs. 1d, 1e, and 1f. There can also be additional higher-order diagrams with the radiation and the absorption of gluon partons, which lead to additional mini-jets.

For the production of high-\(p_T\) jets, the multiple parton scattering processes have been observed in high energy \(pp\) or \(p\bar{p}\) collisions \[68, 71\]. Theoretical discussions on the production of mini-jets beyond the leading order has also been investigated, and hard inclusive dijet production with multiparton interactions has also been considered \[18, 24, 27, 72\]. However, in the low-\(p_T\) region, the experimental evidence for multiple parton interactions with the production of multiple mini-jets and mini-dijets remains lacking.

We would like to develop tools to study multiple hard-scattering processes, for the production of multiple mini-jets and mini-dijets in the low-\(p_T\) domain in \(pp\) collisions at high energies. As a first step, we examine here the clustering properties of mini-jets in the pseudorapidity and azimuthal angle space and search for an algorithm for finding mini-jet clusters.

The production processes for low-\(p_T\) particles are not only of intrinsic importance with regard to the underlying mechanism for low-\(p_T\) particle production, they are also of extrinsic application values because the nucleon-nucleon collision lies at the heart of a nucleus-nucleus collision, and the low-\(p_T\) particle production dominates the particle production process. An understanding of the mechanism of low-\(p_T\) particle production in nucleon-nucleon collisions provide vital information on the initial conditions that may exist at the early stage of nucleus-nucleus collisions, for which much interest has been focused recently.

With the observation of the near-side ridge in high-multiplicity events in high-energy \(pp\) collisions \[59, 66, 73, 80\], the initial dynamics of the system after the production of a jet or a mini-jet \[80\] depends on the initial configuration of the system. The examination of such a system also calls for an event-by-event study of the multiple mini-jet and mini-dijet productions in \(pp\) collisions.

Our event-by-event study has been stimulated by a similar investigation for particle production at lower \(pp\) collision energies where the particle production process may be dominated by flux-tube fragmentation \[81\]. There, the basic conservation laws and the semi-classical picture of the fragmentation process provide powerful tools to reconstruct the space-time dynamics of the pair production processes that may occur, if exclusive data for the production process can become available. In the present investigation, the space-time dynamics of parton-parton hard-scatterings may provide useful experimental information on the multiple collision processes and on the constituent nature of the colliding nucleons.

This paper is organized as follows. In Section II, we summarize the properties of a mini-jet from previous studies. In Section III, we introduce the algorithm for finding mini-jet-like clusters in the pseudorapidity and azimuthal angle space. The algorithm consists of the k-means clustering method and the k-number (cluster number) selection principle, based on the physical properties of mini-jet clusters. We illustrate the usage of such an algorithm in Section IV, using sample events with high multiplicities generated by PYTHIA8.1. We examine the change of the clustering behavior as a function of increasing multiplicities in PYTHIA8.1 events in Section V. We investigate whether similar properties of clustering can be found in a random distribution within the same finite \((\eta, \phi)\) phase space in Section VI. We present our conclusions and discussions in Section VII. We discuss another method of finding the cluster number, the elbow method, and note its ambiguities in the Appendix.

II. PROPERTIES OF A MINI-JET

Information on the signatures and the structure of a mini-jet in the \((\eta, \phi)\) scatter plot can be inferred from the distribution of the two-hadron angular correlation as

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![Diagram of multiple collision processes](image_url)

**FIG. 1.** Various multiple collision diagrams in a \(pp\) hard-scattering leading to the production of jets, which are called mini-jets when the transverse momentum of the jet is small. Shown here are diagrams for the production of (a) a dijet pair, (b) two dijet pairs, and (c) three dijet pairs. Furthermore, a scattered parton can make an additional collision with a different parton of the other proton, as shown in diagrams (d), (e), and (f).
a function of the pseudorapidity difference $\Delta \eta = \eta_2 - \eta_1$ and the azimuthal angular differences $\Delta \phi = \phi_2 - \phi_1$ of the two particles detected with angular coordinates $(\eta_1, \phi_1)$ and $(\eta_2, \phi_2)$ in coincidence \[62, 63\]. The mini-jet structure appears as a cluster of particles in the $(\eta, \phi)$ space (and a cone in three-dimensional configuration space) as indicated by a two-hadron Gaussian distribution in $\Delta \eta$ and $\Delta \phi$ in the form

$$\frac{dN}{d\Delta \eta d\Delta \phi}(\Delta \eta, \Delta \phi) \propto \exp \left\{ \frac{(\Delta \eta)^2 + (\Delta \phi)^2}{2\sigma^2} \right\},$$  \tag{1}

where the quantity $\sigma$ was found to be \[66\]

$$\sigma = \frac{\sigma_{\phi0} m_a}{\sqrt{m_a^2 + p_{T,trigger}^2}}, \quad \sigma_{\phi0} = 0.5, \quad m_a = 1.1 \text{ GeV},$$ \tag{2}

when triggered by a hadron with transverse momentum $p_{T,trigger}$. In the minimum-bias data at RHIC energies we shall consider, the quantity $p_{T,trigger}$ takes on the average value of $(p_T)$, which is of order 0.4 GeV/c, and Eq. \[2\] therefore yields $\sigma_{\phi0}=0.5$. The two-particle distribution of Eq. \[1\] has a half-width at half maximum at $R=\sqrt{(\Delta \eta)^2 + (\Delta \phi)^2}=1.2 \sigma_{\phi0}=0.6$.

We can consider a circle of radius $R$ in the $(\eta, \phi)$ plane. The minimum separation between any two points inside the circle is zero and the maximum separation is $2R$. Setting $2R=2.4\sigma_{\phi0}$ (or $R=0.6$) will allow the circle to contain a large fraction (about 95%) of the Gaussian distribution \[1\] within the circular domain. One of the signatures of a mini-jet cluster of particles can be indicated by a cluster of particles within a radius of $R=0.6$ in the plane of $(\eta, \phi)$.

In the hard-scattering process in the collision of two partons, $a + b \rightarrow a' + b'$, the partons $a'$ and $b'$ materialize subsequently as mini-jets. The initial $a$ and $b$ partons may be endowed with an intrinsic transverse momentum $k_T$ of the order of 0.6 to 1.0 GeV/c \[3, 32, 33\]. To detect final partons $a'$ and $b'$ in the central rapidity region, the conservation of 4-momentum requires that the initial longitudinal momenta of the colliding partons will be converted essentially into the magnitudes of the transverse momenta and the scattered partons $a'$ and $b'$ will come out azimuthally in nearly back-to-back directions. For the scattering of low-$p_T$ partons whose longitudinal momenta are comparable to their intrinsic transverse momenta as we envisage here, we expect approximate back-to-back correlation with considerable fluctuations. The signature of a mini-dijet can be taken to be a pair of mini-jets whose azimuthal angles are approximately correlated within the range of $\pi - R$ to $\pi + R$.

III. ALGORITHM FOR FINDING MINI-JET-LIKE CLUSTERS

As discussed in the last section, a mini-jet shows up as a cluster of particles with a cone radius of $R=0.6$ in the $(\eta, \phi)$ space. Such a cluster can be searched by the k-means clustering method \[33, 34\], in conjunction with an additional k-number (cluster-number) specification principle. In such a search, we ascribe the characteristic mini-jet radius $R = 0.6$ to a cluster and we name it a “mini-jet-like” cluster. In practical terms, two mini-jet-like clusters that are azimuthally correlated in a back-to-back manner may be identified as a physical mini-dijet of two correlated mini-jets at high collision energies, if the mini-jet producing hard-scattering process has been confirmed to be dominant in the low-$p_T$ region as suggested in earlier studies \[20, 24, 53, 58\].

For a given set of $M$ produced particles specified by their angular positions, \{x$_i$=(\eta$_i$, \phi$_i$), i=1,2,3,...,M\}, and a given $K$ number of clusters, the k-means clustering method consists of (i) partitioning the set of $M$ particles into $K$ cluster subsets, $S_k=\{x_k^i\}, k=1,2,...,K$, and (ii) finding for each cluster subset the corresponding cluster center \{C$_k$, k=1,2,...,K\} so as to minimize the potential function

$$\Phi(K) = \sum_{k=1}^K \left\{ \sum_{x_i^k \in S_k} (x_i^k - C_k)^2 \right\},$$ \tag{3}

which is defined as the total subset sum of the squares of the distances between the cluster subset points and their corresponding cluster center $C_k$.

For a fixed value of $K$, the variation of the above potential function $\Phi(K)$ with respect to the cluster center $C_k$ is given by

$$\delta \Phi(K) = -\sum_{k=1}^K \left\{ \sum_{x_i^k \in S_k} 2(x_i^k - C_k) \cdot \delta C_k \right\},$$ \tag{4}

Because all $\delta C_k$ are independent, the minimization of $\Phi(K)$ with respect to the variation of the positions of the cluster centers $C_k$ leads to $\delta \Phi(K)/\delta C_k = 0$ and

$$\sum_{x_i^k \in S_k} 2(x_i^k - C_k) = 0.$$ \tag{5}

This yields $C_k$ as the centers of gravity of the subset of points of $S_k = \{x_k^i\}, k = 1, 2, ..., K$,

$$C_k = \frac{1}{M_k} \sum_{x_i^k \in S_k} x_i^k,$$ \tag{6}

where $M_k = (\sum_{x_i^k \in S_k} 1)$ is the number (multiplicity) of particles in the subset $S_k$.

In numerical implementation of the k-means clustering method, one choose randomly the first cluster center as one of the data points and choose randomly the other $K-1$ cluster centers in the other data points with probability proportional to the square of the distance from the first cluster center \[33\]. For each data point, the knowledge of the positions of the initial cluster centers then...
allows one to calculate the squares of the distance between the data point and all $K$ cluster centers. To each data point, one then assigns the data point to the subset $S_k$ with the smallest square of distance to its cluster center $C_k$. After all subset assignments to $S_k$ have been completed for all data points, the center of gravity of the data points in each new subset $S_k$ is then re-calculated to give the new cluster centers $C_k$, with which the iterative procedure will proceed until it is convergent. One then calculates the potential function $\Phi(K)$ of Eq. (5) as the sum of squared distances.

The above standard procedure is then repeated with other random initializations of the initial cluster centers. After many cluster center random initializations and the corresponding convergent solutions and the potential functions $\Phi(K)$ have been obtained, the proper solution for the case of a given value of $K$ can be found and selected as the solution with the minimum value of the potential function $\Phi(K)$. For a given value of $K$, the k-means clustering method then yields uniquely the cluster subsets of particles $S_k = \{x_i^k\}, k = 1, 2, \ldots, K$ associated with each cluster and the corresponding cluster center location $C_k$.

The k-means clustering method needs an amendment to make it applicable for mini-jet-like cluster searches because the method will lead to poorly displaced and inaccurate cluster centers, if particle points that are obviously not part of a cluster and quite far away from a cluster have been included into the particle data set in the clustering algorithm. The presence of these ‘non-cluster’ particles are possible because there may be other sources of particle production in addition to those from clusters within the narrow window of acceptance. We need to use our knowledge on the structure of the mini-jet in Eq. (11) to sieve out these non-cluster data points in the set of $M$ particles. We calculate the distances between any data point and all other data points in the $(\eta, \phi)$ plane. The knowledge of these distances allows us to exclude any data point whose minimum separation to all other data points exceeds a distance $2R$, presumably the maximum separation for two data points in a mini-jet. After these points are excluded to yield a reduced set of particles belonging to clusters in this modification, the k-means clustering method becomes very efficient, set of particles belonging to clusters in this modification, jet. After these points are excluded to yield a reduced set of data points outside of the cluster circles.

IV. ILLUSTRATION OF THE ALGORITHM FOR FINDING MINI-JET-LIKE CLUSTERS

We shall apply the above algorithm for finding mini-jet-like clusters from charged hadrons generated by the PYTHIA8.1 for high-energy $pp$ collisions at $\sqrt{s} = 200$ GeV. The event generators PYTHIA8.1 [9] and PYTHIA6.4 [8] include the multiple parton interaction processes as described in Ref. [8], with additional considerations on color correlations, flavour correlations, junction topology, beam remnant configurations [11], and interleave initial state radiations [12]. The fully interleave evolution [13] and re-scattering [14] are further included in PYTHIA8.2 [10].

In the series of PYTHIA programs, the basic picture of the multiple collision process arises from the composite nature of the proton which possesses a parton spatial density distribution in addition to the standard parton momentum distribution function (parton PDF). The parton-parton collisions between the constituents of the projectile proton and the target proton are assumed to
be independent of each other, and the number of collisions in an event is therefore given by a Poisson distribution. The probability of parton-parton collisions is then a function of the parton-parton cross section and the impact parameter. To extend the parton-parton scattering cross section to the low-$p_T$ region for minimum-bias studies, the divergent parton-parton scattering cross section at low transverse momenta has to be regularized with a cut-off parameter that can be chosen to yield the appropriate charged-hadron multiplicity distribution. We expect finite multiple parton-parton multiple collision probabilities for the independent collisions of projectile partons with target partons, as depicted in the diagrams in Fig. 1. They lead to the production of multiple mini-jets and mini-dijets in the angular scatter plot of produced charged particles.

The probability for the occurrence of multiple mini-jets and mini-dijets depends on the charge multiplicity of the event, which is part of the total hadron multiplicity. For brevity of notation and its frequent usage, we shall abbreviate “charge multiplicity” or “charged-particle multiplicity” simply by “multiplicity”, when ambiguities do not arise or are not pertinent. We shall restore back the term “charge multiplicity” when it is properly needed.

In order to predict what may be expected experimentally for multiple mini-jet and mini-dijet productions, we generate minimum-bias events using the PYTHIA8.1 and accept primary charged particles with $|\eta|\leq 1$. For each event multiplicity, we select 5 random events for analysis. Each event will be labeled by $p Me I$, where $p M$ stands for PYTHIA minimum-bias event with charge multiplicity $M$, and $e I$ denotes the $I$th event with the charge multiplicity $M$. We would like to search for the presence of the expected mini-jet-like and mini-dijet-like clusters from the angular scatter plots of charged particles in these events.

The detected and identified charged particles include not only charged hadrons but also a small percentage (of about 12%) of $e^+$ or $e^-$. By convention, we include these leptons in our charged multiplicity counts. However, because the $e^+$ and $e^-$ particles arise from many different hadronic and non-hadronic sources, and the relations between these particles and their hadron parents, if they arise from hadronic decays, are non-trivial, we shall exclude them in our mini-jet finding algorithm. Their presence in the scattered $(\eta, \phi)$ plot provides a sense of possible hadronic activities in the vicinity of their angular locations.

In Figs. 2, 4, 5, and 6, we shall show sample scatter plots of charged particles in the $(\eta, \phi)$ plane from minimum-bias events generated by the PYTHIA8.1 event generator. We display the particle labels of kaons, protons, electrons, and muons while the other particles are all charged pions. The solid and open points denote positive and negative particles respectively, and circular and square points denote $p_T \geq 0.5$ GeV/c and $p_T < 0.5$ GeV/c, respectively.

We shall illustrate the algorithm for finding mini-jet-like clusters with concrete examples. We consider three randomly selected minimum-bias PYTHIA8.1 events with $M=20$ in Fig. 2. For each of these events, we assume different cluster numbers $K$ and, obtain $K$ clusters and their corresponding cluster centers $C_k$ using the k-means clustering method. We then construct cluster circles with a radius $R = 0.6$ circumscribing the cluster centers.

In Fig. 2 for event p20e2, for the cases of $K = 4$, 5, 6, and 7, the number of points outside of the cluster circles are 10, 6, 4, and 2 respectively. For the case of $K=8$, there is no k-means clustering solution without one of
the clusters possessing only a single particle. $K=8$ is therefore excluded from our consideration for this p20e2 event because we do not consider a single particle to be a cluster. If the clusters are mini-jet clusters, then almost all particle points should be inside the cluster circles. The case of $K=7$ leads to the fewest number of particles outside of the cluster circles. According to the principle of fewest outside points, $K=7$ is the proper number of clusters for event p20e2. Similarly, for the event p20e4 in Fig. 2 and $K=4, 5, 6,$ and 7, the number of points outside of the cluster circles are $8, 5, 3,$ and 0 respectively. We infer that $K=7$ leads to mini-jet-like clusters. For the event p20e5 and $K=3, 4, 5,$ and 6, the number of outside points are $11, 6, 1,$ and 0. We infer that $K=6$ is the proper cluster number with zero points outside of the cluster circles.

It should be mentioned that there is another method, the “elbow method”, to select the cluster number $K$ by studying the $K$-dependence of the potential function $\Phi(K)$ [55, 60]. The method consists of determining the cluster number by the location of the “kink” where there is a sudden change of the slope of the potential function. The method suffers from the ambiguities in finding where the “kink” lies, and will not be used in the present context. We shall discuss the ambiguities in such a method in Appendix A.

V. SCATTER PLOTS OF PRODUCED CHARGED PARTICLES FROM PYTHIA8.1

We shall study the clustering properties of charged particles produced in $pp$ collisions in minimum-bias events with $-1 \leq \eta \leq 1$, generated by PYTHIA8.1 at $\sqrt{s} = 200$ GeV. In reviewing the scatter plots of these particles in the $(\eta, \phi)$ space as a function of the charged particle multiplicity, it should be kept in mind that those events with larger charge multiplicity numbers $M$ are events with lower occurrence frequencies. The multiplicity distribution for this set of particles generated by PYTHIA8.1 is given in Fig. 8. The average number of charged particles within the window of $|\eta| \leq 1$ is $(M)=6.94$.

We plot in Figs. 4 to 6 mini-jet-like clusters of particles within a radius of $R=0.6$ obtained from the clustering algorithm of finding mini-jet-like clusters. As the multiplicity increases beyond $M=6$, there appears to be a gradual onset of the production of multiple mini-jet-like clusters, using minimum-bias events generated by PYTHIA8.1, for $pp$ collision at $\sqrt{s} = 200$ GeV. Therefore, the search for the non-mini-jet production mechanism will need to focus on events with multiplicity $M$ less than about 6. To examine whether the non-mini-jet mechanism is the flux tube fragmentation, we need more information of identified particles along a greater region of the rapidity space as suggested in [81].

An interesting question arises whether the angular clustering of data points at $(\Delta \eta, \Delta \phi) \sim 0$ may arise from the decay of resonances. As the invariant mass of two relativistic hadrons with $(\Delta \eta, \Delta \phi) \sim 0$ is nearly zero, the clustering of hadrons around $(\Delta \eta, \Delta \phi) \sim 0$ may not likely arise from resonance decays.

The partitioning of the set of charged particles into mini-jet-like clusters can be carried out on an event-by-event basis in Figs. 4 to 6 by identifying a mini-jet-like cluster as an assembly of particles, represented by a circle in the $(\eta, \phi)$ plane with a radius of $R=0.6$. We can furthermore identify a mini-dijet-like pair of clusters as two correlated mini-jet-like clusters whose centers are separated azimuthally within the range from $\pi - R$ to $\pi + R$. In Figs. 1-6 we indicate a mini-jet-like cluster and its corresponding associated partner by circles of the same line type and color. At the end edges of $\phi = \pm \pi$, the scatter plot are sometimes wrapped around so as to facilitate the partitioning particles into mini-jet-like clusters, as in events p11e2,p11e4,p11e5,....

The data in Figs. 4 to 6 reveal that as the multiplicity increases, mini-jet-like clusters of more than 2 particles within a radius of $R=0.6$ occur with a greater probability. In most of the events with $M = 7$ to 9 and higher multiplicities, a single mini-jet-like cluster appears often to correlate roughly with an associated mini-jet-like partner in azimuthally nearly back-to-back directions. There may be a fluctuation of the back-to-back correlation due to the intrinsic transverse momentum of the partons. We conclude from these figures that mini-dijet-like clusters commence at $M \sim 7$, with the probability increasing gradually as $M$ increases, and appear nearly consistently for $M \gtrsim 11$, as indicated in Figs. 4 and 5.

We show in Figs. 5 the scatter plots of charged particles in events with high multiplicities, $11 \leq M \leq 15$. As the multiplicity number $M$ increases beyond $M \gtrsim 13$ there is a transition from the production of one pair of mini-dijet-like clusters to the production of two pairs of mini-dijet-like clusters, with each pair of mini-dijet-like cluster approximately azimuthally back-to-back with respect to each other. The transition region is not sharp, as

![Figure 3](image-url)
FIG. 4. (Color Online) Scatter plots in the ($\eta, \phi$) plane for produced charged particles in events, with multiplicities $M=5$, 7, and 9, within $-1 \leq \eta \leq 1$, generated by the PYTHIA8.1 for pp collisions at $\sqrt{s}=200$ GeV. Circular curves indicate the locations of the mini-jet-like clusters.

FIG. 5. (Color Online) Scatter plots in the ($\eta, \phi$) plane for $M=11$, 13, and 15, within the window of $-1 \leq \eta \leq 1$, generated by the PYTHIA8.1 for pp collisions at $\sqrt{s}=200$ GeV. Circular curves indicate the locations of the mini-jet-like clusters.

many events contain only a single pair of mini-dijet-like cluster, while many other events in Fig. 5 contain double correlated mini-dijet-like clusters. We conclude from these figures that two mini-dijet-like cluster pairs begin to set in with $M \gtrsim 14$, with the probability increasing gradually as $M$ increases.

We show in Figs. 5 the scatter plots of charged particles in events with ultra-high multiplicities, $17 \leq M \leq 21$. As the multiplicity number $M$ increases beyond $M \gtrsim 17$, the production of two sets of mini-jet-like clusters appears nearly consistently, with occasional production of 5 mini-jet-like clusters. In Fig. 5 events with $M \gtrsim 20$ appear to contain events with three pairs of mini-dijet-like clusters.

The results from the present analysis indicates that multiple mini-jet-like clusters and mini-dijet-like clusters are common occurrences for events with high multiplicities and their numbers increase with the increasing multiplicity $M$.

Fig. 6a) shows that for events generated by PYTHIA8.1 within $p_T \leq 0.15$ GeV/c and $|\eta| \leq 1$, the number of mini-jet-like clusters $K$ appears to be a linear function of charge multiplicity $M$ given by

$$K = (0.268 \pm 0.024)[M + (0.816 \pm 1.272)]. \quad (7)$$
FIG. 6. (Color Online) Scatter plots in the \((\eta, \phi)\) plane for \(M = 17, 19,\) and \(21\), within the window of \(-1 < \eta < 1\), generated by the PYTHIA8.1 for \(pp\) collisions at \(\sqrt{s} = 200\) GeV. Circular curves indicate the locations of the mini-jet-like clusters.

VI. CLUSTERING OF PARTICLES IN A RANDOM DISTRIBUTION WITH UNIFORM PROBABILITIES

The results in the last section indicate the copious production of clusters in the theoretical model of PYTHIA8.1, which contains mainly the mini-jet production mechanism for high-energy \(pp\) collisions at \(\sqrt{s} = 200\) GeV. Many of these clusters also exhibit back-to-back azimuthal correlations to make them good candidates for physical mini-dijets, and multiple mini-dijets. These theoretical clusters as well as their corresponding experimental counterparts will likely represent physical mini-jets and mini-dijets, if the dominance of the parton-parton hard scattering process for mini-jet production is extended to the low-\(p_T\) region, as suggested by [26, 29, 55–58].

It is worth noting that the clustering property by itself is not sufficient to definitively identify a cluster as mini-jet cluster because similar clustering properties are also present in other particle production models. It is necessary to have other independent collaborative supports for the mini-jet occurrence in order to identify the observed mini-jet-like clusters as likely physical mini-jet clusters.

In order to bring the need for independent collaborative supports into a sharp focus, it is illustrative to examine the clustering properties of particles produced in a simple schematic model in which a total of \(M'\) particles are randomly and independently produced with a uniform probability in the \((\eta, \phi)\) phase space within the window of \(|\eta| \leq \Delta \eta_{\text{window}}/2\) and \(|\phi| \leq \pi\),

\[
\frac{dP_{\text{random}}}{d\eta \, d\phi} = \frac{\Theta(\Delta \eta_{\text{window}}/2 - |\eta|) \, \Theta(\pi - |\phi|)}{2\pi \Delta \eta_{\text{window}}}. \tag{8}
\]

This can be the approximate mode of production when particles are produced independently with a uniform probability in rapidity, as from the fragmentation of a flux tube at very high energies [6, 10, 25, 30–46]. It can also be the probability distribution used to describe noise particles randomly produced within the experimental \((\eta, \phi)\) phase space. We use different symbols \(\{M, K\}\) from PYTHIA 8.1 Events

\[
K = 0.268(M + 0.816)\quad \text{From PYTHIA 8.1 Events}
\]

\[
K' = 0.309(M' - 0.938)\quad \text{From the Random Distribution}
\]

FIG. 7. (Color Online) Relations between charge multiplicity and the number of mini-jet-like clusters: (a) for \(pp\) collisions at \(\sqrt{s} = 200\) GeV as extracted from events generated by PYTHIA 8.1 in Figs. [Figs. 4-6], (b) for randomly distributed particles within \(|\eta| \leq 1\) and \(|\phi| \leq \pi\).
...and \( \{M', K'\} \) to emphasize that they are different sources of particle production that are likely to be collision-energy and detector-noise dependent.

We find in this case of random distribution that particle clustering also occurs when a large \( M' \) number of particles are produced randomly over a small phase space. To understand such a clustering, we can pick any two produced particles. The probability that a pair of particles falling randomly within the circle of radius \( R \) with respect to each other is

\[
P_{\text{random}} = \left( \frac{\pi R^2}{2\pi \Delta \eta_{\text{window}}} \right). \tag{9}
\]

In an event with multiplicity \( M' \), the number of distinct pairs is

\[
(\text{number of distinct pairs}) = \frac{M'(M' - 1)}{2}. \tag{10}
\]

Therefore, in an event with multiplicity \( M' \), the (average) number of clusters for the random distribution, \( K'(2, M') \), is the product of Eqs. (9) and (10),

\[
K'(2, M') = \frac{M'(M' - 1)}{2} \left( \frac{\pi R^2}{2\pi \Delta \eta_{\text{window}}} \right), \tag{11}
\]

upon identifying a cluster as two particles falling within a radius of \( R = 0.6 \). However, because clusters can be formed with more than two particles, the above quantity \( K'(2, M') \) represents therefore only the upper limit of the number of clusters when particles fall into and join other clusters.

More generally, the number \( K'(n, M') \) of clusters of random coincidence for a cluster of \( n \) particles within a radius of \( R \) in an event with multiplicity \( M \) is

\[
K'(n, M') = C_n^{M'} \left( \frac{\pi R^2}{2\pi \Delta \eta_{\text{window}}} \right)^{n-1}. \tag{12}
\]

For the a detector with \( \Delta \eta_{\text{window}} = 2 \) such as the STAR detector, we have

\[
K'(2, M') = \frac{M'(M' - 1)}{2} \times 0.09. \tag{13}
\]

Thus, the upper limit of the number of clusters from the random distribution Eq. (13) increases quadratically as a function of the multiplicity \( M' \). This upper limit can be quite large for large \( M' \). For example, one expects the upper limit of \( K'(2, M') = 0.9 \) and \( 4.95 \) clusters for \( M' = 5 \) and \( M' = 11 \) respectively. Thus, we would not be surprised to find clusters even for randomly and independently distributed particles as the multiplicity \( M' \) increases.

In our numerical example, we generate particles randomly with the uniform probability distribution of Eq. (8) within \(-\pi \leq \phi \leq \pi \) and \(-1 \leq \eta \leq 1 \). We label the events as \( \pm M' e I \) and show sample events with multiplicity \( M' = 5 \) to \( M' = 21 \) in Figs. 8 to 11, where we shall not distinguish the charges and the types of particles. We then use the mini-jet finding algorithm of Sections III and IV to locate mini-jet-like cluster centers and circumscribe the mini-jet-like cluster in circles, with approximately back-to-back clusters in circles of the same type.

The average number of clusters is \(<K'> = 0.8 \) for \( M' = 5 \) in Fig. 8 and \(<K'> = 3.2 \) for \( M' = 11 \) in Fig. 9, which approximately agree with the general trend on the increase in the upper limit of the number of clusters for randomly distributed events as \( K'_{\text{random}} \leq 0.9 \) for \( M' = 5 \), and \( K'_{\text{random}} \leq 4.95 \) for \( M' = 11 \), estimated from Eq. (13).

Figs. 8 to 11 show that as the multiplicity increases, the number of clusters \( K' \) also increases. In Fig. (7b), we shows that for events generated by the random distribution in the phase space of \( |\eta| \leq 1 \) and \( |\phi| \leq \pi \), the number of mini-jet-like clusters \( K' \) appears to be a linear...
FIG. 9. (Color Online) Scatter plots in the \((\eta, \phi)\) plane for produced particles in events with multiplicities \(M' = 11, 13,\) and 15 generated by an event generator with a uniform and independent production within \(|\eta| \leq 1\) and \(|\phi| \leq \pi\). Circular curves indicate the locations of the cluster circles with \(R = 0.6\).

FIG. 10. (Color Online) Scatter plots in the \((\eta, \phi)\) plane for produced particles in events with multiplicities \(M' = 17, 19,\) and 21 generated an event generator with a uniform and independent production within \(|\eta| \leq 1\) and \(|\phi| \leq \pi\). Circular curves indicate the locations of the cluster circles with \(R = 0.6\).

function of charge multiplicity \(M'\) given by

\[
K' = (0.309 \pm 0.030)(M' - (0.938 \pm 1.304)),
\]

which is similar to the dependence of the cluster number \(K\) and the multiplicity number \(M\) for events generated by PYTHIA8.1 in Eq. (7). We note that the number of clusters \(K'\) estimated by Eq. (11) represents only an upper limit, because a cluster with more than two particles can be formed in high multiplicity events. The number of clusters increases only linearly with multiplicity \(M'\), instead of the quadratic dependence of Eq. (11), as shown in Fig. 7(b).

One way to study the clusters that are formed is by way of the \((\Delta \eta = \eta_1 - \eta_2, \Delta \phi = \phi_1 - \phi_2)\) correlations between clusters located at \((\eta_1, \phi_1)\) and \((\eta_2, \phi_2)\). Figs. 8 to 10 for the random and uniformly distributed particles also exhibit azimuthal corrections for some of the pairs, as cluster circles of similar types in these figures indicate. Thus, the clusters in the random distribution also exhibit approximate azimuthal back-to-back correlations, as can be observed in Figs. 9 and 10.

We can estimate the number of azimuthally back-to-back correlated mini-jet-like clusters \(D'\) as a function of the number of clusters \(K'\). We consider a pair of mini-jet-like clusters. The probability that the pair of mini-jet-like clusters can be considered back-to-back correlated in
azimuthal angles is

\[ P_{\text{random}} = \frac{2R}{2\pi}. \]  

(15)

In an event with \( K' \) number of mini-jet-like clusters, the number of distinct mini-jet-like pairs is

\[ \text{(number of distinct pairs)} = \frac{K'(K' - 1)}{2}. \]  

(16)

Therefore, in such an event with \( K' \) number of mini-jet-like clusters, the (average) number of mini-dijet-like pairs \( D'(K') \) for the random distribution, is the product of Eqs. (15) and (16).

\[ D'(K') = \frac{K'(K' - 1)}{2} \left( \frac{R}{\pi} \right). \]  

(17)

Thus, the number of mini-dijet-like pair is \( D'(K') = 1.15 \) for \( K' = 4 \). This means that when \( K' \) exceeds about 4 the number of mini-dijet-like pair of clusters \( D' \sim 1 \) and back-to-back correlated mini-dijet-like pair will begin to set in, as one can observe from the number of mini-dijet–like clusters in events x11e3, x13e2, and x15e1 with \( K' \gtrsim 4 \) in Fig. 9.

Results in Figs. 8 to 10 indicate that by distributing particles densely within a small angular phase space, clustering and azimuthal correlations occur also for randomly distributed source of particle. Thus, clustering and azimuthal correlation by themselves cannot be the only means of identifying mini-jets and mini-dijets. The identification of these clusters as such arises from other independent supports for the dominance of the hard-scattering model for mini-jet production of low-\( p_T \) particles.

VII. CONCLUSIONS AND DISCUSSIONS

The parton-parton hard scattering is an important process in high-energy nucleon-nucleon collisions. Although originally conceived to involve only the production of high-\( p_T \) particles, it has been suggested that the dominance of the hard-scattering process may extend to the low-\( p_T \) region, with the production of mini-jets and mini-dijets, as the collision energy increases.

As a first attempt to identify mini-jet and mini-dijets, we develop an algorithm to search for mini-jet-like clusters using the \( k \)-means clustering method, supplemented with a \( k \)-number (cluster-number) selection principle. The method adopts a scheme of random initialization of the initial centers, minimizing the potential function \( \Phi(K') \) for a fixed \( K \), and looking for the \( K \) number of clusters that leaves the fewest particles outside the cluster circles. The method is stable, fast, accurate, and yields mini-jet-like clusters and their associated particles.

Using such a method, we have located mini-jet-like clusters in the \((\eta, \phi)\) plane on an event-by-event basis, using events generated by PYTHIA8.1, which contains the dynamics of multiple parton interactions. To a mini-jet-like cluster identified by the procedures, one often find an associated mini-jet-like cluster located at approximately \( |\Delta\phi_{\text{jet-jet}}| \sim \pi \pm R \). Their azimuthal angular correlation suggests that they may be identified as the two partners of a mini-dijet-like pair. We find that mini-jet-like clusters, mini-dijet-like pairs, and multiple mini-dijet-like pairs of low-\( p_T \) hadrons are common occurrences for PYTHIA8.1 events with high multiplicities. The number of multiple mini-jet-like clusters and mini-dijet-like pairs increases with increasing multiplicity \( M \).

It must be pointed out however that clustering and azimuthal correlations alone cannot be the only means to identify mini-jet and mini-dijets. A randomly distributed set of particles in large multiplicities also exhibit clustering properties similar to those from the PYTHIA8.1 program with mini-jets. The ability to separate out the mini-jet of multiplicity \( M \) from other sources of particles of multiplicity \( M' \) will depend on the ratio \( M/M' \), which is likely to be collision-energy and detector-noise dependent. In this regard, the quantitative assessment of the dominance of the relativistic hard-scattering process in the low-\( p_T \) region needs to be independently established in order to identify the mini-jet-like clusters as physical mini-jets. The success of such an identification will provide a tool to investigate mini-jet and mini-dijet properties, for which not too much detail information has been collected. Furthermore, quantitative predictions based on first principles of perturbative QCD for the low-\( p_T \) region is difficult because the multiple collision probability involves higher-order corrections beyond the leading order [22]. Other possible non-perturbative QCD effects may also be present. The present investigation seeking ways to identify the multiple hard-scattering process may pave the way to a future semi-empirical phenomenological description of the multiple scattering process, with information obtained by direct mini-jet and mini-dijet analysis of the angular scatter plot of experimental event-by-event data.

From our investigations, one may also wish to develop parallel strategies to study mini-jets and mini-dijets. One way is to apply the proposed algorithm to examine experimental data at various energies and consider tentatively the mini-jet-like clusters to be physical mini-jets and study quantitative information on the production cross sections and the phase-space distribution of these objects, for comparison with the theory of multiple mini-jet production as a function of the collision energies. In this regard, we should note that the higher the \( pp \) collision energy, the greater is the probability of the dominance of the hard-scattering process for the production of low-\( p_T \) particles, and the greater will be the probability of the mini-jet-like clusters to be indeed physical mini-jets. It will be of interest is to see whether even higher multiple collisions of partons with a greater number of mini-dijets may lead to a black-disk type behavior as the path-length of the parton traversing through the target proton increases.
On a parallel track, one may like to lower the noise multiplicity $M'$ by increasing the value of the lower bound of $p_T$, as mini-jets and harder jets becomes more and more dominant as the value of $p_T$ increase. This will reduce the multiplicity of the noise or non-mini-jet particles. Alternatively, one can also consider only mini-jet clusters with a large number of particles. The increase in the number of particles in each cluster cuts down on the number of noise clusters and lower the noise in the identification of the mini-jets.

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**Appendix A: The “Elbow” Method of Cluster Number Selection**

There is another method to select the cluster number $K$ by studying the $K$-dependence of the potential function $\Phi(K)$. For a given $K$ value, after the minimization of the potential function $\Phi(K)$ with respect to the random initialization of the cluster centers and the variations of the cluster center positions, the quantity $\Phi(K)$ of Eq. (3) is then evaluated. The potential function $\Phi(K)$ is on the whole a decreasing function of increasing $K$ (Fig. 11), as it reaches the limiting value of zero when the number of clusters $K$ is the same as the number of data points $M$. An inefficient and slow decrease of $\Phi(K)$ occurs, if a cluster is subdivided into smaller sub-clusters with a subsequently smaller change of the $\Phi(K)$ slope. On the other hand, a large and abrupt change of $\Phi(K)$ as a function of $K$ signifies a significant change of the structure of the clustering configuration and may be the location of the appropriate cluster number. Hence, it has been suggested that the proper cluster number $K$ occurs at the kink (or ‘elbow’) of the curve of $\Phi(K)$ as a function of $K$ or at the location of an abrupt change of the slope of $\Phi(K)$.

We calculate the potential function $\Phi(K)$ as a function of the cluster number $K$ for events with $M = 20$ as shown in Fig. 2. For event $p20e2$ shown in Fig. 4, a kink of $P(K)$ occurs at $K = 3$ and a very weak kink also appears to occur at $K = 6$. The determination of the location of the “kink” is not without ambiguity. The elbow method would suggest the cluster number of $K = 3$ or 6 but as we observed in Fig. 2, the proper cluster number as determined from the principle of fewest outside points is $K = 7$. For event $p20e4$, kinks of $\Phi(K)$ occur at $K = 3$ and 5, but the appropriate cluster number as determined from the principle of fewest outside points is 7. For $p20e5$, the potential function shows a sharp kink at $K = 3$, and weaker kinks at 5, and 6 whereas the method of the principle of fewest outside points gives $K = 6$. The method of the sharpest kink has the difficulty of recognizing the location of the kink, as many changes of slopes occur at different locations. If one takes the method to be given by the location with the greatest change of the magnitude of the slope, it would give $K$ numbers which differ from the $k$-number selection principle of the fewest number of outside points.

We conclude that in the elbow method the determination on the location of the “kink” is ambiguous and there is no obvious method to resolve the ambiguities. The principle of the fewest outside points should be the proper criterion for the selection of the proper cluster number $K$ as it is based on the physical property of the clustering of a mini-jet.

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