Pricing European Options in Realistic Markets

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Abstract

We investigate the relation between the fair price for European-style vanilla options and the distribution of short-term returns on the underlying asset ignoring transaction and other costs. We compute the risk-neutral probability density conditional on the total variance of the asset’s returns when the option expires. If the asset’s future price has finite expectation, the option’s fair value satisfies a parabolic partial differential equation of the Black-Scholes type in which the variance of the asset’s returns rather than a trading time is the evolution parameter. By immunizing the portfolio against large-scale price fluctuations of the asset, the valuation of options is extended to the realistic case of assets whose short-term returns have finite variance but very large, or even infinite, higher moments. A dynamic Delta-hedged portfolio that is statically insured against exceptionally large fluctuations includes at least two different options on the asset. The fair value of an option in this case is determined by a universal drift function that is common to all options on the asset. This drift is interpreted as the premium for an investment exposed to risk due to exceptionally large variations of the asset’s price. It affects the option valuation like an effective cost-of-carry for the underlying in the Black-Scholes world would. The derived pricing formula for options in realistic markets is arbitrage free by construction. A simple model with constant drift qualitatively reproduces the often observed volatility -skew and -term structure.

1 Introduction

An important result of modern finance is that the fair (no-arbitrage) price $V$ for a European-style option is the expected present value (PV) of its future payoff,

$$ V = E^Q[PV\text{(payoff)}] . $$

The expectation in Eq. (1) is with respect to a risk-neutral (martingale) measure $Q$ on the space of price-paths. The fundamental theorem of asset pricing ensures the existence of the risk-neutral measure $Q$ in the absence of arbitrage.
opportunities, but does not explicitly relate it to the process for the underlying. We shall see that in some cases of interest, the validity of Eq. (1) is restricted to options with a bounded payoff.

Instead of directly computing the expectation in Eq. (1) for European-style options, we will consider the conditional expectation

$$\mathbb{E}^Q[PV(\text{payoff}) | v_f],$$

with respect to volatility paths with the same total final variance $v_f$ of the asset’s returns. Eq. (1) is recovered by taking the expectation over $v_f$. For asset prices that follow a diffusion process, the conditional expectation of Eq. (2) turns out to be unique, but we will see that this generally is not true for realistic processes.

The analysis of the fair value of European-style call and put options by Black and Scholes\textsuperscript{3} was based on a stochastic model in which the returns of the asset follow a random walk. The fair value $C_{BS}$ of an European-style call in this model depends only on the asset’s spot price $S_0$, the volatility $\sigma$ and risk-free rate $r$ (both assumed constant), time to exercise $T$ and the option’s strike $K$. Dimensional analysis requires that

$$C_{BS}(S_0, \sigma, r, T; K) = S_0c_{BS}(rT, \sigma^2T; K/S_0).$$

Note that the valuation (3) of a European-style call depends only on the final variance $v_f = \sigma^2T$, and the integrated discount factor $rT$, rather than separately on the volatility $\sigma$, risk-free rate $r$ and time to expiration $T$. Assuming that the (mean) risk-free rate is known, Eq. (3) can be inverted to give the (implied) volatility $\sigma_{BS}^\text{implied}$ with which the Black-Scholes model would reproduce the observed spot price $C$ of a call with time to expiration $T$ and strike $K$,

$$\sigma_{BS}^\text{implied} = \tilde{\sigma}(C/S_0, rT, K/S_0)/\sqrt{T},$$

where $\tilde{\sigma}$ is the dimensionless overall standard deviation of the distribution of returns. Limitations of the Black-Scholes option pricing formula are expressed by the fact that the implied volatility of European-type options generally is found to depend on the strike $K$ and time to expiration $T$. The volatility implied by calls that are in-the-money very often is higher than that implied by out-of-the-money calls. A graph of the implied volatility against the call’s strike therefore tends to ”smile” (somewhat crookedly) rather than frown. The effect is referred to as the volatility -smile or -skew. The dependence of the implied volatility on $T$ is known as the volatility’s term structure.

The observed volatility skew has been traced to a number of causes. All of them are related to a higher probability for exceptionally large fluctuations
in the returns than the random walk model admits. Stochastic models that simulate this effect have been considered\(^4\), but a quantitative explanation of the empirically observed fluctuations has only recently been proposed\(^5\). Since the observed large-scale fluctuations in returns are not quantitatively reproduced by a simple stochastic model it perhaps is of some interest to (re)examine the problem of option pricing assuming as little as possible. The fair price of an option in fact does not depend on many details of the process for the underlying asset. Only the short-term transition probability is relevant for a fully dynamic hedging strategy, but this strategy is quite different for the following three kinds of assets:

I. The asset’s expected future price is well-defined by the short-term transition probability of its returns.

II. The variance of the short-term returns on the asset is finite, but the asset’s expected future price diverges.

III. The variance of the short-term returns on the asset diverges.

A normal distribution of the short-term returns is an example in the first class, but not the only one. Any distribution of returns that falls off sufficiently rapidly and in particular any distribution with compact support belongs to this class. No-arbitrage arguments uniquely price European-style options on assets in this class if the final variance of the asset's returns is known. It is possible to construct a dynamic portfolio with just one kind of option (in addition to the asset) that is without appreciable risk.

The other two classes sub-divide the category of assets with sub-exponential short-term return distributions. Equities\(^1\), indices\(^6\) and commodities\(^7\) historically fall in the second class of assets and this case will therefore concern us most. It turns out that one still can construct a dynamic portfolio that is without appreciable risk, but the portfolio in this case includes at least two different options on the underlying asset. A portfolio with just one option (and the asset) cannot be insured against exceptionally large price fluctuations of the asset and is therefore not without risk. Although the risk-neutral conditional expectation of Eq. (2) exists for options with bounded payoffs, it no longer is uniquely related to the process for the underlying.

Very little can be said about the third possibility, the Paretian case. The construction of a risk-free dynamic portfolio from options on the underlying is no longer possible. Indeed, the notion that the variance of the returns is a measure of risk has to be reexamined and Eq. (1) may not be very meaningful. Since the variance of returns for assets on which vanilla options can be drawn apparently is finite\(^6,6,7\), the Paretian case will not be further investigated here.
We proceed as follows. Using the variance of the asset’s returns as the evolution parameter, the Black-Scholes analysis is extended to European-style options on any class I asset in the next section. In section 3 we extend the analysis to include options on assets that belong to class II (the realistic case). Section 4 summarizes and discusses some aspects of the results.

2 A Variation on the Black-Scholes Analysis of Option Prices

It is useful to slightly generalize the Black-Scholes analysis to the case where the volatility of the underlying can be an arbitrary function of time.

We will use the variance \( v \) of the asset’s returns rather than a (continuous) trading time to parameterize the evolution of an option’s fair value. The variance is a monotonically increasing quantity; a trading- or calendar-time \( t \), can be viewed as defining an instantaneous volatility \( \sigma(t) \):

\[
\sigma^2(t) := \frac{\partial v}{\partial t} \geq 0. \tag{5}
\]

On any given volatility path \( \{\sigma(t); 0 \leq t \leq T\} \) there is a one-to-one correspondence between the variance \( v \) and the "time" \( t \).

\[
v(t) = \int_0^t d\xi \sigma^2(\xi). \tag{6}
\]

[The origin of the time-scale here is chosen to coincide with the moment of vanishing uncertainty in the asset’s price.] Eq. (6) enables one to formally consider the evolution in "time" as an evolution in the variance of the underlying’s returns (if the volatility is finite).

To compensate for the time value of money, all prices will be stated as multiples of the price of an actively traded risk-free bond that matures when the European-style option expires. The spot price of the bond is \( S_B(t) \) and its nominal value \( N_B = S_B(\text{maturity}) \). Since the transition probability is for the returns rather than for the price of the underlying, it is convenient to convert to the dimensionless variables,

\[
x(t) := \ln[S(t)/S_B(t)] \ , \ k := \ln[K/N_B] \ . \tag{7}
\]

Changes in the log-price \( x \) give the return on the underlying relative to the return on the bond and \( k \) is the strike value of \( x \). We assume that the fair price \( C \) of a European-style call option at any moment depends only on the time to expiration, the strike price and the spot prices for the underlying asset and the bond. The fair call price in multiples of \( S_B \) and expressed in the above dimensionless quantities is denoted by,

\[
c_k(x, v) := C(S(t), S_B(t), t; K)/S_B(t) \ . \tag{8}
\]
2.1 Generic Properties of the Short-Term Transition Probability

The dynamic hedging strategy of Black and Scholes that assigns a fair value to a European-style call depends on the existence of a very simple portfolio that is without appreciable risk for a sufficiently short period of time.

Let the current log-price of the stock be \( x \) and the probability that the stock will have an excess return between \( y - x \) and \( y + dy - x \) a short time from now be described by the transition probability density,

\[
p_h(y|x, \ldots) = p_h(y|x, v).
\]

A small variance \( h \) of the transition probability corresponds to a short time interval. The ellipses denote all additional quantities on which the transition probability may depend, such as market- and economic- indicators, the weather and political environment, etc. In effect, the transition probability for the returns depends on the current time \( t \), respectively on the variance \( v \). One fortunately does not require detailed knowledge of \( p_h(y|x, v) \) to value an option on the asset.

The transition probability Eq. (9) furthermore depends only on the excess return \( y - x \) rather than on \( x \) and \( y \) individually. This financially plausible proposition can be cast in the form of a denominational argument: the probability for a certain change of the asset’s price should not depend on whether a single bond with value \( S_B(t) \) or a package of two, three, or for that matter 6.378 bonds is used as price reference. With the definition (7), this freedom in the reference denomination implies that the transition probability is invariant under (global) translations \( z \) of all log-prices,

\[
p_h(y - z|x - z, v) = p_h(y|x, v) = p_h(y - x|0, v), \quad \forall z,
\]

where the latter expression is obtained by setting \( z = x \). It is important that neither the variance \( h \) of the short-term returns nor the variance \( v \) of the overall returns are affected by this translation. Because the short-term transition probability density \( p_h \) depends only on the difference \( y - x \), the variance of the overall returns is additive: the current variance of the returns \( v \) increases to \( v + h \) after the short time interval we are considering.

Since \( h \to 0_+ \) as the time interval is shortened, the transition probability density has to approach Dirac’s distribution in this limit,

\[
\lim_{h \to 0_+} p_h(y|x, v) = \delta(y - x).
\]

Due to Eq. (11) the expected future log-price of the stock,

\[
\tilde{y}(h; v) = x + \mu(h; v) := \int_{-\infty}^{\infty} dy y p_h(y|x, v),
\]

5
approaches the current log-price $x$ and $\mu(h; v)$ must become arbitrarily small for $h \to 0_+$. $\mu(h; v)$ is the expected excess return on the asset at a given point in time when the variance of the asset’s returns increases by $h$. It may appear financially reasonable to assume that $\mu(h; v)$ for $h \sim 0$ has the expansion $\mu(h; v) = a(v)h + b(v)h^2 + \ldots$. However, ignoring transaction costs, very short-term stock investments could have a higher expected rate of return than long-term ones. To avoid the financially unstable situation that the return on very short-term investments becomes absolutely certain, it is sufficient to require that,

$$\lim_{h \to 0_+} \frac{\mu(h; v)}{\sqrt{h}} = 0.$$  

(13)

The mean return in other words should not outstrip the width of the distribution of short-term returns.

The second moment of the transition probability, by definition, is given by its variance $h$ and $\bar{y}(h; v)$

$$\mathbb{E}^p[y^2] = h + \bar{y}(h; v)^2 := \int_{-\infty}^{\infty} dy \, y^2 \, p_h(y|x, v).$$  

(14)

Somewhat surprisingly perhaps, one does not require detailed knowledge of the higher moments of the distribution of short-term returns.

2.2 The Black-Scholes Valuation of a European-Style Call

Emulating the analysis of Black and Scholes, the fair value of a European-style call is found by constructing a portfolio that is without appreciable risk for sufficiently small $h$. Consider a portfolio $P$ of one European-style call with strike $K$ and $-\Delta$ of the underlying. When the portfolio is set up at a log-price $x$ for the asset, the value $V_P$ of this position is,

$$V_P(x, v) = c_k(x, v) - \Delta(x, v)e^x$$  

(15)

bonds at $S_B(v)$. If the hedge ratio $\Delta(x, v)$ is not changed, the value of this portfolio (in bonds) when the variance of the asset’s returns has increased by $h$ becomes,

$$V_P(y, v + h) = c_k(y, v + h) - \Delta(x, v)e^y,$$  

(16)

if the stock’s excess return over this period is $y - x$. To avoid arbitrage, the value of this position when the hedge is set up should be its expected future

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We assume that the asset can be sold short and ignore transaction fees, dividends and other costs-of-carry as well as bid-ask spreads.
value discounted by a factor that accounts for the risk of investing in portfolio $P$. One thus quite generally comes to the conclusion that,

\[ V_P(x, v) = e^{-R_P(h, v)} \int_{-\infty}^{\infty} dy \, p_h(y|x, v) V_P(y, v + h) . \]  

The discount factor $e^{-R_P(h, v)}$ is compensation for the excess risk associated with holding the portfolio $P$ rather than the (risk-free) bonds [by pricing relative to the bond, we already took the time value of money into account]. $R_P(h, v)$ depends not only on the perceived risk of the portfolio, but also reflects the valuation of this risk by investors. The value of a certain risk generally depends on the circumstances, that is on the time $t$, respectively on the overall variance $v$. $R_P(0, v) = 0$ in order for Eq. (17) to be consistent.

The absence of arbitrage opportunities requires that $R_P(h, v) \geq 0$ for all $h$. If the portfolio is without appreciable risk over the interval $h$, $R_P(h, v) = 0$ and Eq. (17) becomes the martingale hypothesis. Note that the general form of Eq. (17) is valid for finite $h$ and could be the starting point for valuing hedge slippage.

Since the transition probability $p_h(y|x, v)$ is strongly peaked near $y \sim x$ for $h \to 0_+$, one is led to expand the portfolio’s future value (16) about $y = x$ and $h = 0$. The first few terms of this expansion are,

\[ V_P(y, v + h) = V_P(x, v) + (y - x)[c'_k(x, v) - \Delta(x, v)e^x] + h\hat{c}_k(x, v) \]

\[ + \frac{1}{2}(y - x)^2[c''_k(x, v) - \Delta(x, v)e^x] + O(h(y - x), (y - x)^3, h^2) , \]

where the shorthand notation,

\[ \hat{\phi}(x, v) := \frac{\partial}{\partial v} \phi(x, v) \quad \text{and} \quad \phi'(x, v) := \frac{\partial}{\partial x} \phi(x, v) , \]

denotes partial derivatives of a function with respect to $v$ and $x$.

The term proportional to $y - x$ in Eq. (18) vanishes for the particular hedge ratio$^b$

\[ \Delta(x, v) \to \Delta(x, v) = e^{-x}c'_k(x, v) , \]

and the corresponding portfolio will be denoted by $P$. Its value $V_P$ for $y \sim x$ and $h \sim 0$ has the simplified expansion,

\[ V_P(y, v + h) = V_P(x, v) + h\hat{c}_k(x, v) + \frac{(y - x)^2}{2}[c''_k(x, v) - c'(x, v)] \]

\[ + O(h(y - x), (y - x)^3, h^2) . \]

$^b$To verify that Eq. (20) is precisely the hedge of Black and Scholes, note that with definition (7), $e^{-x} \frac{\partial}{\partial x} = S_B \frac{\partial}{\partial S_B}$
We show below that the portfolio $P$ is without appreciable risk for sufficiently small $h$ if the short-term return distribution of the asset belongs to class I. It is important that this hedge depends only on the (observed) log-price $x$ at the time it is entered into.

Let us for the moment assume that the contribution to the integral in Eq. (17) from higher order terms in the expansion Eq. (18) becomes negligible for $h \to 0_+$ (see Appendix A). In this case the hedge (20) allows us to evaluate the RHS of Eq. (17) for sufficiently small $h$ as,

$$V_P(x, v) = e^{-R_P(h, v)} \int_{-\infty}^{\infty} dy p_h(y|x, v) V_P(y, v + h)$$

$$\sim V_P(x, v) - R_P(h, v) V_P(x, v) + h \dot{c}_k(x, v) + \frac{h + \mu^2(h; v)}{2} [c''_k(x, v) - c'_k(x, v)] ,$$

(22)

where we have used that by Eqs. (12) and (14),

$$h + \mu^2(h; v) = \int_{-\infty}^{\infty} dy (y - x)^2 p_h(y|x, v) .$$

(23)

Taking the limit $h \to 0_+$ of Eq. (22) and using Eq. (13), the fair option price $c_k(x, v)$ is seen to satisfy the partial differential equation,

$$\dot{c}_k(x, v) + \frac{1}{2} [c''_k(x, v) - c'_k(x, v)] = R_P(v) [c_k(x, v) - c'_k(x, v)] ,$$

(24)

with a mean excess portfolio return per unit of variance of,

$$r_P(v) = \lim_{h \to 0_+} h^{-1} R_P(h; v) .$$

(25)

Note that $c_{-\infty}(x, v) = e^x$, is a particular solution to Eq. (24), because a call with strike $K = 0$ has the same intrinsic value as the underlying. It should be emphasized that the mean return $\mu(h; v)$ of the underlying asset does not enter Eq. (24) as long as it satisfies Eq. (13).

Using the definitions (7) and (8) and assuming that the volatility is a known function of the trading time $t$, Eq. (24) assumes a more familiar form when the evolution is parameterized by $t$,

$$\left[ \frac{\partial}{\partial t} + \tilde{r}_P(t) S \frac{\partial}{\partial S} + \frac{\sigma^2(t)}{2} S^2 \frac{\partial^2}{\partial S^2} \right] C(S; t; K) = \tilde{r}_P(t) C(S; t; K) .$$

(26)
The portfolio’s instantaneous overall return rate $\tilde{r}_p(t)$ in Eq. (26) consists of two parts: the (risk-free) return rate of the bond $r(t) = S_B^{-1}dS_B(t)/dt$ and the risk-premium of the portfolio,

$$\tilde{r}_p(t) = r(t) + \sigma^2(t)r_p(v(t)).$$  \hspace{1cm} (27)$$

Eq. (26) is the partial differential equation of Black and Scholes for the valuation of options with an in general time-dependent volatility and a option-dependent discount rate $\tilde{r}_p(t)$. However, Eq. (24) is not merely Eq. (26) rewritten in terms of other variables: Eq. (24) remains valid even if the volatility is stochastic or an unknown function of the trading time. Eq. (24) can still be integrated in this case and shows that the fair value of a European-style option depends only on the overall variance of the asset’s returns when it expires.

We have yet to show that the portfolio $P$ with the hedge ratio (20) is (at least formally) without appreciable risk and that $\tilde{r}_p(v)$ therefore vanishes in the absence of arbitrage opportunities. $\tilde{r}_p(t) = r(t)$ in Eq. (26) then does not depend on the option and becomes the risk-free rate of the bond.

2.3 The Risk of Holding the Dynamically Hedged Portfolio $P$

A portfolio is without appreciable short-term risk compared to an investment in the asset alone, if the variance of the portfolio’s return decreases faster than the variance of the asset’s return, which is $h$. One thus has to show that

$$\lim_{h \to 0^+} h^{-1}\text{Var}[V_P(y, v + h)] = \frac{d}{dv}\text{Var}[V_P(y, v)] = 0$$ \hspace{1cm} (28)$$

We continue to assume (see Appendix A for details) that the transition probability is sufficiently sharply peaked about $y \sim x$ and again expand,

$$V_P^2(y, v + h) = V_P^2(x, v) + 2V_P(x, v) \left\{ h c_k(x, v) + \frac{(y-x)^2}{2} [c_k''(x, v) - c_k'(x, v)] \right\} + O(h(y-x), (y-x)^3, h^2).$$  \hspace{1cm} (29)$$

The expectation of $V_P^2(y, v + h)$ to order $h$ then is,

$$\mathbf{E}^{p_h}[V_P^2] = \int_{-\infty}^{\infty} dy \, p_h(y|x, v)V_P^2(y, v + h)$$

$$= V_P^2(x, v) + 2V_P(x, v) \left\{ h c_k(x, v) + \frac{h + \mu^2(h)}{2} [c_k''(x, v) - c_k'(x, v)] \right\} + O(h^{3/2})$$

$$= \mathbf{E}^{p_h}[V_P]^2 + O(h^{3/2}).$$  \hspace{1cm} (30)$$
The variance of the returns on portfolio $P$ therefore is of order $h^{3/2}$ and the risk of entering into this investment compared to an investment in the underlying can theoretically be made as small as one wishes by rebalancing the portfolio often enough (we are ignoring transaction costs). To avoid arbitrage, the discount rate therefore must be the risk-free one. The portfolio $P$ in this sense is perfectly hedged over the short-term and we should set $r_P(v) = 0$ in Eqs. (24) and (27).

2.4 A Comment on Stochastic Volatility

The parabolic partial differential equation (24) was derived without specifying a stochastic process for the asset’s price. Specific properties of the short-term returns enter the solution to Eq. (24) only through the boundary conditions. Integration of Eq. (24) for a European-style option requires knowledge of the payoff of the option and of the overall variance $v_f$ of the asset’s returns at exercise. The option payoff is readily expressed in terms of the value of a risk-free bond that matures when the option expires. For European-style options the only uncertainty thus is in the final variance $v_f$ of the asset’s returns.

By construction, the price of a European-style call option does not depend on the volatility path. Paths with the same overall variance $v_f$ of the asset’s returns when the option expires give the same fair option price.

Eq. (24) implies that the fair value of an European-style option for a given final variance of Eq. (2) is the risk-neutral conditional expectation of the option payoff with the pdf$^c$,

$$p_{BS}(y|x, v_f) = (2\pi \sqrt{v_f})^{-1} \exp\left[-(y-x+v_f/2)^2/(2v_f)\right]. \quad (31)$$

Denoting the risk-neutral marginal probability distribution for the overall variance of the returns by$^d q(v_f|T, \ldots)$, the risk-neutral probability measure $Q$ for European-style options on the asset is given by the pdf

$$p_Q(y|x, T, \ldots) = \int_0^\infty dv_f q(v_f|T, \ldots) p_{BS}(y|x, v_f). \quad (32)$$

The pdf $q(v_f|T, \ldots)$ is the only ingredient that specifically depends on market expectations. It therefore is not uniquely specified by the process for the asset. Since we do not have options on a particular asset to every strike $K$, the market is not complete. The distribution $q(v_f|T, \ldots)$ thus cannot be

$^c$With respect to the variance at expiration $v_f$, the distribution (31) solves the "backward" evolution equation that corresponds to (24).

$^d$The ellipses again represent any other pre-visible quantities (such as the current spot price $x$).
uniquely inferred from the observed option prices. One may, however, hope to obtain a reasonable estimate by using a trial distribution for \( q(v_f | T, \ldots) \) whose mean and variance are calibrated to reproduce a few observed option prices. Note that Eq. (32) represents the risk-free measure \( Q \) as a positively weighted superposition of Gaussian distributions with mean, \( x - v_f / 2 \) and variance \( v_f \). The mean and variance of each Gaussian are strictly correlated. In the next section we will argue that this unfortunately does not appear to be very realistic and Eq. (32) probably is not sufficiently general to reproduce the observed smile.

3 The Valuation of European Calls in Realistic Markets

In deriving Eq. (24) we tacitly assumed that the expectation in Eq. (17) is meaningful. Since the fair value of a call that is deep in-the-money approaches \( S - K \), we see that for \( \Delta \neq 1 \), the fair value of the portfolios we have been considering essentially becomes proportional to the price of the underlying \( S = S_B e^{y} \) for large values of \( y \). The expectation in Eq. (17) for such portfolios is finite only if the price of the underlying has finite expectation,

\[
E^{p_h}[S/S_B] = \int_{-\infty}^{\infty} dy \, e^y p_h(y| x, v) < \infty .
\]  

(33)

Together with the result of Appendix A that the contribution from higher moments becomes negligible in the limit \( h \to 0_+ \), we thus find that Eq. (24) (with \( r_P(v) = 0 \)) holds for options on class I assets only.

The historical distributions for equities\(^1\), indices\(^6\) and commodities\(^7\) do not belong to this class. Empirically the probability densities for short-term returns have tails that fall off as a power in \( x \) only. For time intervals between 5 minutes and three weeks the observed pdfs of the returns on equities are all shape-similar and well reproduced\(^6\) by a t-distribution for 3 degrees of freedom with mean \( \bar{y}(h; v) = x + \mu(h; v) \) and variance \( h \),

\[
p_{h}^{\text{emp.}}(y|x, \mu) \sim \frac{2h^{3/2}}{\pi((y - \bar{y})^2 + h)^2} .
\]  

(34)

The integral of Eq. (33) diverges in this case and the valuation of the previous portfolios is all but meaningless: being long a call apparently becomes a very attractive position – unfortunately, the risk associated with this position is not calculable. If the probability for exceptionally large fluctuations is sufficiently great, the expected future value of some portfolios no longer is determined by small fluctuations about \( y \sim x \), even as \( h \to 0_+ \). The short-term expected value of the portfolios we have been considering in this case
mainly comes from the exceptionally large fluctuations, even though these are not the most frequent. Truncating the expansion (18) about \( y = x \) in this case gives an inaccurate representation of the portfolio’s variation in price and the derivation of Eq. (24) is no longer valid.

The damage can be contained by considering only portfolios that are immune to large variations in the price of the underlying. It is sufficient to restrict to portfolios whose value is uniformly bounded by a finite constant \( V_{\text{max}}(P) \),

\[ |V_P(x, v)| < V_{\text{max}}(P), \quad \forall x, v < v_f. \]  

(35)

Examples of simple portfolios that are bounded in this manner are a vanilla put or a covered vanilla call. For class II assets one can select those portfolios from the above set that are bounded and without appreciable risk for sufficiently short periods of time\(^e\). Since the set of portfolios that satisfy (35) is smaller than the set of admissible portfolios in the case of class I processes, it is not surprising that the valuation of options on class II assets is less constrained.

The simplest bounded dynamic portfolio that is without appreciable risk contains two European-style options on the asset that differ in strike or time to expiration. We here discuss the case of a portfolio of two covered calls, \( \tilde{c}_1 \) and \( \tilde{c}_2 \) with the same expiration date but strikes \( k_1 \) and \( k_2 \) respectively.

The portfolio’s fair value in bonds when the variance of the return distribution is \( v \) and the asset’s log-price is \( y \) can be written,

\[ V_P(y, v) = \Delta_1 \tilde{c}_1(y, v) + \Delta_2 \tilde{c}_2(y, v), \]  

(36)

where the fair price of a covered call is,

\[ \tilde{c}_i(y, h) = c_i(y, h) - e^{\gamma}, \quad i = 1, 2. \]  

(37)

The weights \( \Delta_1 \) and \( \Delta_2 \) of the two covered calls are chosen so that the portfolio’s price does not change appreciably for small variations of the asset’s price about its current log-price \( x \),

\[ \frac{\partial}{\partial y} V_P(y, v) \bigg|_{y=x} = 0. \]  

(38)

The weights,

\[ \Delta_1 = \tilde{c}_2'(x, v) ; \quad \Delta_2 = -\tilde{c}_1'(x, v), \]  

(39)

give one possible solution to Eq. (38). When the variance increases by \( h \), the portfolio \( P \) with weights (39) assumes the value,

\[ V_P(y, v + h) = \tilde{c}_1(y, v + h)\tilde{c}_2'(x, v) - \tilde{c}_2(y, v + h)\tilde{c}_1'(x, v) \]

\[ = \left[ \frac{\tilde{c}_1(y, v + h)\tilde{c}_1'(x, v)}{\tilde{c}_2(y, v + h)\tilde{c}_2'(x, v)} \right] , \]  

(40)

\(^e\)This procedure is not possible for Pareto return distributions with a divergent variance\(^a\).
for a return on the asset of \( y - x \). About \( y = x \) and \( h = 0 \), \( V_P(y, v + h) \) has the expansion,

\[
V_P(y, v + h) = V_P(x, v) + h \left[ \frac{\tilde{c}_1(x, v)}{\tilde{c}_2(x, v)} \frac{\tilde{c}_1'(x, v)}{\tilde{c}_2'(x, v)} \right] + \frac{(x - y)^2}{2} \left[ \frac{\tilde{c}_1''(x, v)}{\tilde{c}_2''(x, v)} \frac{\tilde{c}_1'(x, v)}{\tilde{c}_2'(x, v)} \right] + O(h^2, (x - y)h, (x - y)^3) .
\]

(41)

Using that the value of the portfolio \( P \) is bounded, its expected future price with the pdf (34) and sufficiently short time intervals is,

\[
\int_{\bar{y} - 1}^{\bar{y} + 1} dy p_{h,\mu}(y|x) V_P(y, v + h) = \int_{y-1}^{y+1} dy p_{h,\mu}(y|x) V_P(y, v + h) + O(h^{3/2})
\]

\[
= V_P(x, v) + h \left[ \frac{\tilde{c}_1(x, v)}{\tilde{c}_2(x, v)} \frac{\tilde{c}_1'(x, v)}{\tilde{c}_2'(x, v)} \right] + O(h^{3/2} \ln(h), \mu^2) ,
\]

(42)

where we have assumed that the expected short-term return on the asset satisfies Eq. (13). The determinant of the 3 \( \times \) 3 matrix is the result of combining the expectations of the two determinants in Eq. (41). Because the portfolio value is immunized against large price fluctuations, the truncation of the transition probability in Eq. (42) induces an error of order \( h^{3/2} \) only (see Appendix A for details). For class II short-term returns, the valuation of a bounded Delta-hedged portfolio thus effectively is reduced to the class I case. One similarly can show that the variance of \( V_P \) is of order \( h^{3/2} \) and that \( P \) therefore is without appreciable risk.

In the limit \( h \to 0_+ \), the fair values of any two covered European-style calls on a class II asset thus satisfy,

\[
\begin{vmatrix}
-1 & 0 & 1 \\
\tilde{c}_1(x, v) & \tilde{c}_1'(x, v) & \frac{1}{2} \tilde{c}_1''(x, v) \\
\tilde{c}_2(x, v) & \tilde{c}_2'(x, v) & \frac{1}{2} \tilde{c}_2''(x, v)
\end{vmatrix} = 0 .
\]

(43)

This is one partial differential equation for two unknown functions. However, since the determinant vanishes only when the corresponding system of linear equations is dependent, we can disentangle Eq. (43) into two linear partial differential equations for each covered call separately – at the cost of introducing a function \( \alpha(x, v) \). Excluding the possibility that the value of a covered call
does not depend on the asset’s price, Eq. (43) is equivalent to the set of linear equations,

\[
\begin{pmatrix}
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{\tilde{c}}_1(x, v) \\
\dot{\tilde{c}}_2(x, v)
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2}\tilde{c}''_1(x, v) \\
\frac{1}{2}\tilde{c}''_2(x, v)
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\begin{pmatrix}
\alpha(x, v) - \frac{1}{2} \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\] (44)

The first of these equations is true for any \(\alpha(x, v)\). The latter two imply that call options on a class II asset satisfy the partial differential equation,

\[
\dot{c}(x, v) + (\alpha(x, v) - \frac{1}{2})c'(x, v) + \frac{1}{2}c''(x, v) = \alpha(x, v)e^x.
\] (45)

At any given moment, \(\alpha(x, v)\) can be expressed in terms of the "Greeks" for any European-style call on the underlying,

\[
\alpha(x, v) = -\frac{\dot{\tilde{c}}(x, v) - \frac{1}{2}\tilde{c}'(x, v) + \frac{1}{2}\tilde{c}''(x, v)}{\tilde{c}'(x, v)},
\] (46)

and in particular does not depend on the strike of the option.

Our considerations of course also apply to transition probabilities that fall off more rapidly than (34). One indeed recovers the partial differential equation (24) of Black and Scholes as the special case

\[
\alpha(x, v) = 0.
\] (47)

As noted before, since a call with strike \(K = 0\) is worth the stock at exercise, \(c_{-\infty}(x, v) = e^x\) must be a special solution to Eq. (45) that does not depend on \(\alpha(x, v)\). The inhomogeneous term in Eq. (45) for the valuation of call options therefore is a matter of consistency. Put-call parity implies that a European-style put with the same strike and expiration date as a call satisfies the homogeneous partial differential equation with the same \(\alpha(x, v)\). By repeating the arguments for two covered calls with different expiration dates, one concludes that \(\alpha(x, v)\) also does not depend on the expiration date of an option. \(\alpha(x, v)\) in this sense is an universal function that does not depend on specific properties of European-style options.

The function \(\alpha(x, v) \neq 0\), can be viewed as a risk-premium on a covered call (respectively a put). The reason for such a premium is evident from the derivation: it represents the cost of insuring a simple Delta-hedged portfolio with just one option against large fluctuations in the price of the underlying. Note that \(\alpha(x, v)\) enters the evolution equation for options with bounded payoffs as an effective cost-of-carry for the underlying asset in the Black-Scholes world would.
This interpretation of $\alpha$ becomes evident if we consider the stochastic process whose generator $\hat{A}$ is the evolution operator in Eq. (45),

$$\hat{A}(v)\phi(w, v) = \left\{ \frac{1}{2} \frac{\partial^2}{\partial w^2} + (\alpha(w, v) - \frac{1}{2} \frac{\partial}{\partial w}) \right\} \phi(w, v).$$  \hspace{1cm} (48)

The corresponding stochastic process is\(^9\),

$$dw = (\alpha(w, v) - \frac{1}{2})dv + dB_v$$ \hspace{1cm} (49)

where $B_v$ denotes Brownian motion with zero mean and variance $dv$. The measure $Q$ of Eq. (1) that corresponds to Eq. (49) is unique\(^{10}\) as long as the drift $\alpha(w, v)$ is finite for all $w, v$ and does not increase faster\(^{1}\) than $|w|$ for $|w| \sim \infty$. Assuming this to be the case, the fair price of a European-style option on a class II asset is uniquely specified by $\alpha(w, v)$ and the marginal risk-free stopping distribution $q(v_f\mid T, \ldots)$.

[The $\frac{1}{2}$ in the drift-term of Eq. (49) does not appear in the corresponding stochastic process for $n(v) := e^{w(v)}$, which follows geometric Brownian motion\(^9\)

$$\frac{dn}{n} = \alpha(n, v)dv + dB_v,$$ \hspace{1cm} (50)

with mean instantaneous drift $\alpha(w = \ln(n), v)$.]\(^{12}\)

A constant effective risk premium on options was recently interpreted by Derman\(^{11}\) as due to a stock’s intrinsic time-scale generated by short-term speculators. Although our argument apparently is somewhat different, the rather similar effect described here may have a common origin: large exceptional fluctuations in the short-term returns of the underlying perhaps can be traced to speculation. The asymptotic power law fall-off of the return distribution Eq. (34) has indeed recently\(^{12}\) been linked to the speculative actions of large investors such as mutual funds.

It is difficult to compare a risk due to exceptionally large fluctuations to any risk arising from ”normal” fluctuations described by the variance of a distribution. How this exceptional risk is valued furthermore depends on the perception of investors. The function $\alpha(x, v)$ thus probably is specified only by the observed option prices themselves. In the absence of options to every strike

\(^{1}\)The interpretation of $\alpha$ as an effective cost of carry makes this mathematical statement rather obvious: nobody will hold an asset whose cost of carry grows faster than its return.

\(^{9}\)The mean drift $\alpha(n, v)$ in Eq. (50) should not be confused with the mean return of the asset. The two are not even related: the drift $\alpha(n, v)$ is due to large fluctuations in the price of the underlying, not due to its mean return. The stochastic process Eq. (49) is not the one followed by the log-price $x(v)$ of the asset.
and exercise date, the problem of calibrating $\alpha(x, v)$ to the observed market prices is not complete. Additional assumptions are required – for instance that the relative entropy to the Black-Scholes model is minimum.

To better visualize the effect a non-vanishing drift has on option prices, let us consider constant $\alpha > 0$. One can explicitly solve Eq. (45) in this case and obtains that the overall Black-Scholes variance $v_{BS}(\tilde{k}, \alpha; v_f)$ at expiration implied by a European call is implicitly given by the relation,

$$\ln(v_{BS}/v_f) + \frac{(\tilde{k} + v_{BS}/2)^2}{v_{BS}} = \frac{(\tilde{k} + v_f(1/2 - \alpha))^2}{v_f}.$$  

(51)

Here $v_f$ is the total variance of the asset’s returns at the time of exercise of the option and $\tilde{k} = k - x = \ln(K_{SB}/N_{BS}) = \ln \tilde{K}$ is its discounted strike in terms of the spot price of the asset.

![Fig.1: The implied overall variance $v_{BS}(\tilde{K}, \alpha; v_f)$ for $v_f = 1\%$. The surface is for European vanilla calls. The drift-function $\alpha(x, v)$ in Eq. (45) is taken to be constant and the discounted call strike $\tilde{K}$ is in percent of the spot price for the underlying asset.](image)

A typical implied total variance surface for constant $\alpha \geq 0$ is shown in Fig. 1. For $\alpha = 0$ Eq. (45) reduces to the Black-Scholes equation and $v_{BS}(\tilde{k}, \alpha = 0; v_f) = v_f$. In the Black-Scholes world with $\alpha = 0$ there is no volatility-skew if one conditions on the total variance $v_f$ of the returns on expiry of the European-style options. As $\alpha$ increases the variance implied by deep in-the-money calls becomes progressively greater compared to the one implied
by out-of-the-money calls. This behavior can be understood by interpreting \( \alpha \) as an effective cost-of-carry for the underlying asset in the Black-Scholes world – this cost evidently is avoided by holding a deep in-the-money call instead of the asset itself and is incurred when holding a deep in-the-money put. The same reasoning also implies that for \( \alpha > 0 \) in-the-money calls should become more undervalued by the Black-Scholes model as the time to expiration (and thus \( v_f \)) increases. The simple model with constant \( \alpha \) thus tends to qualitatively reproduce the volatility smile and term structure that is generically observed for equities. The fact that the total variance \( v_f \) of the returns at the expiry of the option is itself uncertain has to be taken into account in a realistic valuation. It does not qualitatively change the picture if the risk-neutral uncertainty of \( v_f \) is not too great. A more richly structured variance surface can be modelled by non-constant \( \alpha(x, v) \).

4 A Summary and Discussion of the Results

We separated the problem of valuing European-style options from that of constructing a risk-free portfolio by conditioning on the overall variance of the asset’s returns when the option expires. For hedging purposes, assets with sub-exponential short-term return distributions can be divided into those with finite and infinite variance (classes II and III respectively). Most financially interesting assets historically\(^1\)\(^,\)\(^6\)\(^,\)\(^7\) belong to class II. Fully dynamic portfolios that are risk-free can be constructed for any asset with short-term returns of finite variance and in particular for class II assets. However, in contrast to the Black-Scholes world, a risk-free portfolio in this case has to be statically immunized against exceptionally large fluctuations of the asset’s returns. In effect this implies that the portfolio’s variation in value must be bounded. It therefore contains at least two different European-style options on the asset.

We use the variance overall \( v \) of the returns on the underlying instead of a "trading time". The final overall variance \( v_f \) when the European-style option expires is interpreted as a stochastic "stopping time" for the risk-neutral diffusion. The diffusion for the fair price of an option on a class II asset was found to be characterized by a drift \( \alpha(x, v) \). Constant \( \alpha > 0 \) for given \( v_f \) qualitatively reproduces the volatility smile and term-structure often observed in equity markets. For short-term return distributions that fall off exponentially or faster, \( \alpha = 0 \) and the diffusion reduces to the one of Black-Scholes, albeit evolving in \( v \) instead of in a trading time.

One might object to considering sub-exponential return distributions for the underlying, since the price variations of an asset, although perhaps large, can be thought of as restricted to a finite range in the finite lifespan of the
option. There are at least two objections to this argument. Firstly, the value of equities does sometimes change considerably in a short time and fluctuations may exceed several standard deviations when a company is forced into bankruptcy or announces new patents or acquisitions. These scenarios are not so rare that they can be disregarded in the valuation of options (except perhaps by insiders). One realistically therefore may wish to immunize a portfolio against large price-changes of the underlying. It also is operationally and financially quite impossible to balance a portfolio arbitrarily often. If the tails of the return distribution are sufficiently fat, higher moments can become relevant in the evaluation of Eq. (17) when the change in variance, \( h \), between updates is finite. If higher moments of the distribution of returns are sufficiently large (not necessarily infinite) it again is advisable to immunize the portfolio against large price fluctuations of the underlying asset, that is restrict the portfolio’s variation in value under large variations of the asset’s price. The same strategy was found to be useful in pricing options on class II assets: the portfolio in this case should be statically immunized against exceptionally large fluctuations of the underlying and dynamically hedged to make it insensitive to normal ones as well.

Thus, although it may in reality not be possible to distinguish sharply between assets of class I and II, the strategy employed here for class II assets is the more realistic one. The processes for the two kinds of assets evidently can be continuously deformed into each other and it is gratifying that the evolution equations satisfied by the corresponding option prices also can be continuously deformed from \( \alpha(x, v) \neq 0 \) (class II) to \( \alpha(x, v) = 0 \) (class I).

The returns on financial assets in realistic markets fortunately have finite variance and the Paretian (class III) scenario therefore is quite academic. The risk-free measure for the valuation of European-style options conditional on the final variance of the asset’s returns is unique for class I assets. The volatility smile and term structure in this case are entirely due to the risk-free distribution of the final (stopping) variance. An a priori unspecified drift function \( \alpha(x, v) \) complicates the valuation of European-style options on class II assets. By Eq. (46) \( \alpha(x, v) \) is given by the “Greeks” of an option and effectively measures the amount by which option prices at any moment violate the Black-Scholes pde. We have interpreted \( \alpha(x, v) \) as the investor’s compensation for the residual risk of a single-option Delta-hedged portfolio due to exceptionally large fluctuations in the asset’s returns. As such, this drift is not explicitly related to the process for the underlying. However, quite interestingly, the effect of this drift on option valuation is equivalent to that of an effective cost-of-carry for the underlying asset in the Black-Scholes world.

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A Estimates of the Remainders

To justify the estimates in the text we here show that the remainders in the expansions (21) and (41) indeed are negligible as $h \to 0_+$. We first consider a pdf $p_h(x)$ of class I with variance $h$ and vanishing mean. The argument of $p_h$ can always be shifted to obtain a distribution with non-vanishing mean. Being in class I implies that,

$$
\int_{-\infty}^{\infty} dy p_h(y) e^{\lambda y} < \int_{0}^{\infty} dy p_h(y) e^{y} + \int_{-\infty}^{0} dy p_h(y) < \infty, \quad \forall \lambda < 1 , \tag{52}
$$

that is, the moment generating function is analytic about $\lambda = 0$ and all moments of $p_h(y)$, in particular, are finite.

We are interested in integrals of the form

$$
\int_{-\infty}^{\infty} dy p_h(y) f(y) , \tag{53}
$$

for functions $f(y)$ that are analytic (almost) everywhere. The remainder $R_N(y)$ in the McLaurin series

$$
f(y) = \sum_{n=0}^{N} \frac{y^n}{n!} f^{(n)}(0) + R_N(y) , \tag{54}
$$

thus vanishes as $N \to \infty$ for (almost) all $y$. An expression for the remainder is,

$$
R_N(y) = \frac{y^{N+1}}{(N+1)!} f^{(N+1)}(y \xi) \tag{55}
$$

with $0 < \xi < 1$. Changing the scale of the integration variable $y \to y \sqrt{h}$ the expectation of $R_N$ for small $h$ is,

$$
\mathbb{E}^{p_h}[R_N] := \int_{-\infty}^{\infty} dy p_h(y) R_N(y)
= \frac{h^{(N+1)/2}}{(N+1)!} \int_{-\infty}^{\infty} dy \sqrt{h}^{(N+1)} \{ \sqrt{h} p_h(y \sqrt{h}) \} f^{(N+1)}(y \sqrt{h} \xi) . \tag{56}
$$

The pdf $\sqrt{h} p_h(y \sqrt{h})$ has unit variance and the higher moments of the limiting pdf,

$$
p_1(y) := \lim_{h \to 0_+} \sqrt{h} p_h(y \sqrt{h}) , \tag{57}
$$
are finite. Using that $f^{(N+1)}(x)$ is analytic about $x = 0$, the integral in Eq. (56) has a finite limit for $h \to 0_+$ and one concludes that,

$$\mathbb{E}^{ph}[R_N] = O(h^{(N+1)/2}).$$  \hspace{1cm} (58)

This estimate continues to hold when the function $f(x)$ is expanded about a point $x = \mu(h)$ that satisfies Eq. (13). For analytic portfolio values, the estimates in Eqs. (22) and (30) thus are justified and the neglected terms are of higher order in $h$.

The valuation of bounded portfolios with pdf’s of class II such as Eq. (34) can be reduced to the previous case if the error from truncating the pdf becomes negligible for $h \to 0_+$. To see this, consider a pdf with zero mean and variance $h$ that for sufficiently small $h$ is bounded by,

$$p_h(y) \leq \frac{D}{\sqrt{h}} \left( \frac{h}{y^2} \right)^\nu \quad \text{for } |y| > 1, h < h_0,$$

where $D > 0$ and $\nu$ are constants that do not depend on $h$. Note that if the variance of $p_h(y)$ is finite, Eq. (59) holds for some $\nu > 3/2$. The contribution of the tails of the distribution to the expectation of a bounded function $|f(y)| \leq f_{\text{max}}$ in this case is,

$$\int_{|y|>1} dy \ p_h(y) f(y) \leq f_{\text{max}} \int_{|y|>1} dy p_h(y) \leq 2D f_{\text{max}} h^{\nu-1/2}.$$  \hspace{1cm} (60)

The tails of any distribution in class II (with $\nu > 3/2$) therefore give a subleading contribution to the expectation of a bounded function and can be cut off. For bounded functions, a pdf of class II effectively can be replaced by one that vanishes for $|y| > 1$. This pdf of class II with truncated tails is a pdf of class I up to a normalization factor. Using Eq. (60) with $f(y) = 1$, the normalization correction is of order $h^{\nu-1/2}$ and is itself sub-leading. The estimate of the expectation of bounded functions for pdf’s of class II thus is reduced to the previous case of class I distributions. [Note that $\nu = 2$ for the realistic pdf of Eq. (34) – the error induced by cutting off the tails in this case is of the same order as that due to neglecting the remainder in the expansion of the portfolio’s value.] For short-term returns with a finite variance, the estimate of the order of the corrections in Eq. (42) thus is justified for portfolios with bounded values and a distribution of the returns on the asset that falls off like Eq. (34).

1. V. Plerou, P. Gopikrishnan, L.N. Amaral, M. Meyer and H.E. Stanley, Phys. Rev. **E60**, 6519 (1999); R.N. Mantegna and H.E. Stanley, *An Introduction to Econophysics; Correlations and Complexity in Finance* (Cambridge University Press, Cambridge;2000).
2. K. Back, S. R. Pliska, J. Math. Econ. 20,1; R. C. Dalang, A. Morton, W. Willinger (1990), Stochastics and Stochastic Reports 29,185; W. Schachermayer, Insurance: Mathematics and Economics 11, 249 (1992); F. Delbaen and W. Schachermayer, Mathematische Annalen 312, 215 (1999); see also M. Avellaneda, P. Laurence, Quantitative Modeling of Derivative Securities (Chapman & Hall/CRC, New York, 2000) p.176ff.

3. F. Black and M. Scholes, J. Polit. Econ. 81,637 (1973); R. Merton, Bell Journal of Economics and Management Science 4,141 (1973).

4. T. Bollerslev, J. Econometrics 31, 307 (1986). For a solvable stochastic volatility model see, S. L. Heston, Rev. Fin. Studies 6, 327 (1993). For other approaches and a discussion see, Quant. Fin. 1, p.558-649 (2001).

5. M. Schaden, The Stock Price Distribution in Quantum Finance, New York University preprint May 2002, http://xxx.lanl.gov/abs/physics/0205053.

6. P. Gopikrishnan, V. Plerou, L.N. Amaral, M. Meyer and H.E. Stanley, Phys. Rev. E60, 5305 (1999); R.N. Mantegna and H.E. Stanley, Nature (London) 376, 46 (1995) had previously observed a power law exponent of $\alpha \sim 1.7$ in an intermediate region.

7. K. Matia, L.N. Amaral, S.P. Goodwin and H.E. Stanley, Non-Lévy Distribution of Commodity Price Fluctuations, Boston University preprint Feb. 2002, http://xxx.lanl.gov/abs/cond-mat/0202028.

8. J. H. McCulloch, The Value of European Options with Log-Stable Uncertainty, Working Paper 1985: reproduced in E. E. Peters, Fractal market analysis: applying chaos theory to investment and economics (Wiley & Sons, New York, 1994) p.224ff.

9. B. Øksendal, Stochastic Differential Equations: An Introduction with Applications 5th ed. (Springer, New York, 2000).

10. D. W. Strook and S. R. S. Varadhan, Multidimensional Diffusion Processes (Springer, New York, 1979).

11. E. Derman, Quant. Finance 2(4), 282 (2002).

12. X. Gabaix, P. Gopikrishnan, V. Plerou, H.E. Stanley, A simple theory of the "cubic" laws of stock market activity Boston University preprint 8/14/2002.

13. M. Avellaneda, C. Friedman, R. Holmes and D. Samperi, Applied Math. Finance 4(1), 37 (1997); M. Avellaneda, R. Buff, C. Friedman, N. Grandechamp, L. Kruk and J. Newman, Int. J. Theor. App. Finance 4(1), 91 (2000).