Stability of 4-dimensional Space-time from IIB Matrix Model via Improved Mean Field Approximation

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Abstract

The origin of our four-dimensional space-time has been pursued through the dynamical aspects of the IIB matrix model via the improved mean field approximation. Former works have been focused on the specific choice of configurations as ansatz which preserve SO(d) rotational symmetry. In this report, an extended ansatz is proposed and examined up to 3rd order of approximation which includes both SO(4) ansatz and SO(7) ansatz in their respective limits. From the solutions of self-consistency condition represented by the extrema of free energy of the system, it is found that a part of solutions found in SO(4) or SO(7) ansatz disappear in the extended ansatz. It implies that the extension of ansatz works as a device to distinguish the stable solutions from the unstable ones. It is also found that there is a non-trivial accumulation of extrema including the SO(4)-preserving solution, which may lead to the formation of plateau.

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I. INTRODUCTION AND SUMMARY

The IIB matrix model has been proposed as a constructive formulation of superstring theory [1, 2]. One of the significant features of the model is that the space-time itself is expressed by the eigenvalue distributions of 10 bosonic matrices, and thus treated as a dynamical variable. Therefore, the origin of our four-dimensional universe can be argued on the basis of this framework as a spontaneous symmetry breakdown of Lorentz symmetry [3, 4, 5, 6, 7, 8, 9]. The subject has been pursued up to now with a technique called the improved mean field approximation (IMFA) [10, 11, 12]. By the systematic application of the method, it was revealed that among SO($d$)-preserving configurations $d = 4$ case seems to be preferred and that the extent of space-time turns out to be large for the four-dimensional directions against the rest of six-dimensional part. This result suggests the spontaneous breakdown of Lorentz symmetry and the emergence of our four-dimensional space-time [5, 6, 7, 9].

The IMFA method is a systematic improvement of variational method. It introduces quadratic terms to the original action, which may be considered as artificial mean fields, and formulate a perturbative series expansion, which involves a number of parameters as coefficients of those quadratic terms. We then reorganize the series by resummation to obtain the forms of improved series. The determination of parameters is guided by the principle of minimal sensitivity [13]; the result should be least sensitive to those nominal parameters. It is found in several examples that the improved series is stable in some regions of parameters, in which the dependences on those parameters would be considered to vanish effectively, and the exact value should be reproduced. Thus, the solution to the consistency condition is formulated as identifying such a flat region. We denote it as plateau.

The analyses of the IIB matrix model based on the IMFA method have been done only for a restricted set of configurations. It becomes an enormous task to solve the above consistency conditions because there is quite a large number of parameters introduced along the prescription of the IMFA method. In order to reduce the number of parameters and to clarify the physical implications, restrictions were imposed on the set of parameters so that the SO($d$) subgroup of SO(10) symmetry stays intact. In particular, $d = 4$ and 7 cases have been examined intensively. At this stage we have to reflect whether or not those choices of configurations are reasonable and proper. In the present report we examine a wider set of parameters which covers both SO(4) and SO(7) cases as its subsets. It is referred as “4-3-3” ansatz below. With this particular ansatz we
can treat the solutions of both SO(4) and SO(7) ansatz on equal footings and discuss the problem which vacuum would be more preferred.

A wider class of configurations also sheds lights on the problem whether or not the plateau is actually realized in the analysis thus far carried out up to high orders. This issue may be translated to the following statement. If the IMFA were to realize the minimal sensitivity, the value on plateau should be independent of any nominal parameters. Then, plateau must be stable even if we adopt a wider class of ansatz.

We evaluated up to third order contributions of improved series and found the solutions of consistency conditions defined by the extrema of the improved free energy with respect to the artificial parameters. It turned out that some of the solutions of SO(4) ansatz and SO(7) ansatz also appear in the extended parameter space, while the others vanish. The latter set of extrema are considered to be “unstable” in the sense that they correspond to the saddle points of the improved series.

There is found a non-trivial accumulation of extrema including one of SO(4)-preserving solutions, which has been seen in the higher order calculations to belong to the (would-be) plateau in SO(4) ansatz \[7, 9\]. Therefore, this accumulation will give an affirmative support on the plausibility that the solution found in SO(4) ansatz forms a plateau. It implies that the four-dimensional universe is more preferred to the seven-dimensional universe.

It is noted along with the present analysis that the extension of artificial parameter space is as efficient as the calculation of higher order perturbation in the IMFA method. In the former analyses of SO(4) and SO(7) ansatz, higher-order calculations were required to obtain fair signal for development of plateau.

II. IMPROVED MEAN FIELD APPROXIMATION

The model we examine in this report is the IIB matrix model defined by the partition function with the action \( S \) as

\[
Z = \int dA \, d\psi \, e^{-S},
\]

\[
S = N \text{Tr} \left[ -\frac{\lambda}{4} [A_{\mu}, A_{\nu}]^2 - \frac{\sqrt{3}}{2} \bar{\psi} \Gamma^a [A_{\mu}, \psi] \right],
\]

where \( A_{\mu} \) and \( \psi \) are both \( N \times N \) Hermite matrices, and they are a SO(10) vector and a left-handed spinor, respectively. We choose the scale of \( A_{\mu} \) and \( \psi \) so that the action takes the above form. \( \lambda \) is
a coupling constant\textsuperscript{1}. Though $\lambda$ can be absorbed by the rescaling of the fields, it is instead kept at first and it will later be set to 1.

The IMFA prescription is applied to the IIB matrix model by the following steps \([5, 6]\). We first introduce the quadratic term as:

\[
S_0 = N \text{Tr} \left[ \frac{1}{2} M_{\mu\nu} A_\mu A_\nu + \frac{1}{2} m_{\mu\nu\rho} \bar{\psi} \Gamma^{\mu\nu\rho} \psi \right],
\]

which is chosen to be of most generic SU($N$) invariant form. $M_{\mu\nu}$ and $m_{\mu\nu\rho}$ are arbitrary parameters. The former is symmetric with the exchange of $\mu$ and $\nu$, while the latter is totally anti-symmetric with $\mu$, $\nu$, and $\rho$.

The original action $S$ is transformed nominally by adding and subtracting $S_0$ as

\[
S \longrightarrow S_0 + (S - S_0).
\]

Then the term $(S - S_0)$ would be viewed as an interaction term and the perturbative expansion is constructed by considering the term $S_0$ as a free part. Instead, we consider the deformed action $S' = S_0 + S$ and formulate the perturbation theory with reference to the coupling constant $\lambda$. Next we shift the parameters as follows by introducing the formal expansion parameter $g$:

\[
\lambda \longrightarrow g \lambda,
\]

\[
M_{\mu\nu} \longrightarrow M_{\mu\nu} - g M_{\mu\nu},
\]

\[
m_{\mu\nu\rho} \longrightarrow m_{\mu\nu\rho} - g m_{\mu\nu\rho},
\]

If $g$ is naively taken to be 1, the fictitious parameters vanish and the action $S'$ returns to the original action $S$. We reorganize the series in terms of $g$, disregard $O(g^{n+1})$ terms, and then set $g$ to 1. Thus we obtain the improved series of order $n$.

The artificial parameters, $M_{\mu\nu}$ and $m_{\mu\nu\rho}$, are determined according to the principle that the result should be least sensitive to those parameters, since the original model does not rely on them. The dependence is brought in due to the truncation at finite orders. Thus, we assume if there exists a region of parameters in which the physical quantity such as free energy becomes stable, the dependence on the artificial parameters should vanish effectively, and the true value would be

\textsuperscript{1} The coupling constant $\lambda$ is related to the Yang-Mills coupling $g_0$ by $\sqrt{\lambda} = \sqrt{g_0^2 N}$, when the IIB matrix model is seen as the dimensional reduction of 10-dimensional supersymmetric SU($N$) Yang-Mills theory to zero volume limit. We are considering the large-$N$ limit with $\lambda$ fixed to $O(1)$. 
reproduced. This consistency condition is called plateau condition. The emergence of plateau is a key feature to recognize whether or not the IMFA prescription works well.

Typically, the improved series of a finite order forms a flat region in which it fluctuates gently and accompanies a number of extrema. So, we adopt a criterion for identifying the plateau by the accumulation of extrema of the improved free energy. Furthermore, we estimate the values of the physical quantities at those extrema as the representatives of the estimates on the plateau. If the improved series were convergent, they would provide a good approximate value at high enough orders of the IMFA analysis. The values obtained along the IMFA prescription are considered to be non-perturbative, although the original series are based on perturbative expansions about a perturbative vacuum \[14\]. We can obtain a solution that corresponds to the non-perturbative vacuum of spontaneously broken symmetry.

III. ANSATZ

In the case of the IIB matrix model, the total number of artificial parameters are quite large, namely, 10 real numbers for $M_{\mu\nu}$ (assumed to be diagonalized by SO(10) rotation), and 120 for $m_{\mu\nu\rho}$. It will demand an enormous effort to search for the plateau in this vast space of parameters. Therefore we impose restrictions on the configuration by considering symmetry that remains unbroken to diminish the number of parameters.

In the former works \[5, 6, 7, 9\], the configurations called SO($d$) ansatz have been intensively examined which preserve SO($d$) rotational symmetry. In a practical sense, we impose the condition for the two-point functions of bosonic and fermionic fields:

$$
\langle (A_\mu)_{ij}(A_\nu)_{kl} \rangle = \frac{1}{N} C_{\mu\nu} \delta_{il} \delta_{jk},
$$
(6)

$$
\langle (\psi_\alpha)_{ij}(\psi_\beta)_{kl} \rangle = \frac{1}{\sqrt{120N}} u_{\mu\nu\rho} (C\tilde{\Gamma}^{\mu\nu\rho})_{ijl} \delta_{il} \delta_{jk}.
$$
(7)

The guideline of choice is described as follows. First, SO($d$) subgroup of SO(10) is fixed to which directions the expectation values of fermionic two-point function $u$ are zero. $d$ is chosen from 1 to 9. Toward the rest of the directions, $u$ may have non-zero value. Since $u$ is a rank three anti-symmetric tensor, a single non-zero component of $u_{\mu\nu\rho}$ is accompanied by three-dimensional subspace if the rotational symmetry is taken into account. Thus $(10 - d)$ dimensional part would naturally be decomposed into multiples of three-dimensional blocks. Furthermore, those blocks are subjected to the permutation symmetry of the interchange of each other.
Among those choice shown above, \( d = 4 \) and \( d = 7 \) cases are relevant; it is reported \([6]\) that \( d = 5,6 \) cases reduce to SO(7) ansatz, while \( d = 2,3 \) cases to SO(4) ansatz and \( d = 1 \) case has no solution.

The preserved symmetry and the explicit forms of the exact propagators for \( d = 4 \) and \( d = 7 \) cases are given as follows.

**SO(7) ansatz:** \( \text{SO}(7) \times \text{SO}(3) \)

\[
C_{\mu\nu} = \text{diag}(7c_1\text{'s, }3c_2\text{'s}), \quad \psi = u \Gamma^{8,9,10},
\]

**SO(4) ansatz:** \( \text{SO}(4) \times \text{SO}(3) \times \text{SO}(3) \times \mathbb{Z}_2 \)

\[
C_{\mu\nu} = \text{diag}(4c_1\text{'s, }6c_2\text{'s}), \quad \psi = \frac{u}{\sqrt{2}}(\Gamma^{5,6,7} + \Gamma^{8,9,10}),
\]

The \( \mathbb{Z}_2 \) factor stands for the permutation symmetry between two SO(3) factors.

Now we extend the ansatz to incorporate larger class of parameter space by relaxing some restrictions above. In this report we consider the configuration called “4-3-3” ansatz, in which the symmetry, \( \text{SO}(4) \times \text{SO}(3) \times \text{SO}(3) \), should be preserved. It is obtained by disregarding the permutation symmetry of SO(4) ansatz. The forms of the exact propagators are taken as follows:

\[
C_{\mu\nu} = \begin{pmatrix}
\ldots & 4 & \ldots \\
\ldots & c_1 & \ldots \\
\ldots & c_2 & \ldots \\
\ldots & c_3 & \ldots \\
\end{pmatrix}, \quad \psi = u_1 \Gamma^{5,6,7} + u_2 \Gamma^{8,9,10},
\]

The three-dimensional blocked form is still respected in the present choice so that the number of non-zero fermionic flux should be kept small.

This configuration reduces to SO(4) ansatz or SO(7) ansatz in their respective limits.

\[
c_2 = c_3, \quad u_1 = u_2 \left( = \frac{u}{\sqrt{2}} \right) \quad \Leftrightarrow \quad \text{SO}(4) \text{ ansatz},
\]
\[
c_1 = c_2, \quad u_1 = 0 \quad \Leftrightarrow \quad \text{SO}(7) \text{ ansatz},
\]
It has been argued by comparing the free energy, which of SO($d$) ansatz dominates in the configuration space, where the values of free energy are estimated individually for each ansatz. Since the 4-3-3 ansatz covers both SO(4) and SO(7) ansatz as subsets, the comparison will now be made on the same basis.

IV. RESULTS

We applied the IMFA method to the IIB matrix model and obtained the improved series of free energy for 4-3-3 ansatz up to third order.

In order to evaluate the free energy we use the fact that the free energy $F$ is related by Legendre transformation to 2PI free energy $G$ which is given by the sum of two-particle irreducible diagrams, in terms of exact propagators, $C_{\mu \nu}$ and $u$ \[15\]. This is mainly technical reason, for the number of diagrams reduces drastically by working with 2PI diagrams.

Since the large-$N$ limit of the model is considered, the dominant contribution derives from the planar diagrams \[16\], which we have only to evaluate. The number of 2PI planar vacuum diagrams are, 2 at zeroth, first and second order, and 4 at third order. The explicit expression of free energy up to third order is presented in the appendix.

From the 2PI free energy $G(c, u)$, the free energy $F(M, m)$ is obtained by:

$$F(M, m) = \left\{ G(c, u) + \frac{4}{2} M_1 c_1 + \frac{3}{2} M_2 c_2 + \frac{3}{2} M_3 c_3 - \frac{8}{2} \sum_{i=1,2} m_i u_i \right\}_{c=c(M,m), u=u(M,m)},$$

(12)

where $c_i (i = 1, 2, 3)$ and $u_j (j = 1, 2)$ are determined by the solution of the following relations:

$$M_i = \frac{\partial G(c, u)}{\partial c_i}, \quad m_j = \frac{\partial G(c, u)}{\partial u_j}.$$  

(13)

Next we performed the IMFA prescriptions to the free energy and obtained the improved series as a function of \{M\} and \{m\} by the transformation \[5\] and setting $g$ equal to 1. In order to determine those artificial parameters we search for the solution of plateau condition by identifying the accumulation of the extrema of improved series. The extrema are found by numerical means.

Fig.\[11\] shows the distribution of extrema of the improved free energy for 4-3-3 ansatz plotted on $m_1$-$m_2$ plane. The extrema are shown by bullets (filled •, shaded •, and unfilled ◦). Vertical and horizontal lines ($m_1 = 0$ or $m_2 = 0$) correspond to SO(7) ansatz, and a diagonal line ($m_1 = m_2$) does to SO(4) ansatz. The shaded bullets (•) on the diagonal line represent the SO(4)-preserving solutions ($M_2 = M_3$). The shaded bullets (•) on the vertical and horizontal lines represent the
FIG. 1: Distribution of extrema of the improved free energy for 4-3-3 ansatz plotted on $m_1$-$m_2$ plane (bullets). Vertical and horizontal lines correspond to SO(7) ansatz, and a diagonal line does to SO(4) ansatz.

SO(7)-preserving solutions which satisfy $M_1 = M_2$ (or $M_1 = M_3$), while the circles (○) correspond to the solutions in which SO(7) symmetry is not preserved, i.e. $M_1 \neq M_2$ (or $M_1 \neq M_3$).

There are also plotted the extrema of SO(4) ansatz (squares □) and those of SO(7) ansatz (diamonds ◊) for reference. It is seen that some of the SO(7) solutions (◊) coincide with those of 4-3-3 ansatz (●) on vertical or horizontal lines. The other solutions disappear from the solutions of 4-3-3 ansatz, which are considered to be unstable in the sense that they correspond to saddle points in extended space of parameters. It is consistent that SO(7)-nonpreserving solutions (○) do not coincide with the solutions of SO(7) ansatz. Similar consideration applies to the extrema on the diagonal line. The SO(4) solutions (□) that appear as extrema of 4-3-3 ansatz (●) are considered to be stable, while the others are to be unstable. This speculation may lead to the prospect that the extended parameter space works to distinguish the plausible plateau from the others.

It is yet unclear what configuration would become dominant at such low order of calculations. However, there seems to exist a non-trivial accumulation of extrema on and near the SO(4) symmetric subspace (enclosed by dashed circle in the figure). The SO(4) symmetric extrema in this region corresponds to that of SO(4) ansatz case which is known to belong to the would-be plateau in higher order analysis. It is found that each of those two extrema located near the diagonal line has the property $M_2 = M_3$ for the bosonic variables as well. It implies that SO(4) symmetry is al-
most restored in this region. This supports that the region would be hopeful for plateau. To clarify
the situation, higher order contributions for the 4-3-3 ansatz will be required as a future outlook.

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APPENDIX A: 2PI FREE ENERGY OF 4-3-3 ANSATZ

Here we present the two-particle irreducible free energy for 4-3-3 ansatz up to third order. The
additive constant is adjusted to the definition in \cite{5}. 
\[ G/N^2 = 3(1 + \log 2) - \frac{1}{2} \log c_1^4 c_2^3 c_3^3 + 8 \frac{1}{2} \log(u_1^2 + u_2^2) \]
\[ + \lambda \left\{ 6c_1^2 + 3c_2^2 + 3c_3^2 + 24c_2 u_1^2 + 12c_1 c_2 + 12c_1 c_3 + 9c_2 c_3 
- 24c_3 u_1^2 - 32c_1 u_1^2 - 24c_2 u_2^2 - 32c_1 u_2^2 + 24c_3 u_2^2 \right\} \]
\[ + \lambda^2 \left\{ -32c_1^2 u_1^4 - \frac{27}{2} c_2^2 c_3^2 - 12c_2^2 u_1^4 - 12c_2^2 u_1^4 - 18c_1^2 c_2^2 - 18c_1^2 c_2^2 
- 96c_1 c_2 u_1^4 - 96c_1 c_3 u_1^4 - 72c_2 c_3 u_1^4 - 9c_1^4 - \frac{9}{2} c_2^4 - \frac{9}{2} c_3^4 - 32c_1^2 u_2^4 
- 64c_1^2 u_2^4 - 96c_1 c_2 u_2^4 - 192c_1 c_2 u_1^2 u_2^2 - 96c_1 c_3 u_2^4 - 192c_1 c_3 u_1^2 u_2^2 
- 12c_2^2 u_2^4 + 72c_2^2 u_1^2 u_2^2 + 432c_2 c_3 u_1^2 u_2^2 - 72c_2 c_3 u_2^4 - 12c_2^2 u_2^4 + 72c_2^2 u_1^2 u_2^2 \right\} \]
\[ + \lambda^3 \left\{ -96c_2^4 u_1^4 - 576c_1 c_2 c_3 u_1^2 u_2^2 - 1728c_1 c_2 c_3 u_1^2 u_2^2 - 1728c_1 c_2 c_3 u_1^2 u_2^2 - 96c_1 c_2^2 u_2^6 
- 672c_1 c_2^2 u_2^6 + 1056c_1 c_2 c_3 u_1^4 u_2^2 - 96c_2^4 u_2^4 + 576c_2^4 u_2^4 - 24c_2^3 u_1^6 
- 1944c_2^3 c_1^2 u_2^4 - 96c_2^4 u_1^4 - 72c_2^2 c_3 u_2^6 - 1080c_2^2 c_3 u_1^2 u_2^2 + 1080c_2 c_3 u_1^2 u_2^2 
- 72c_2 c_3^2 u_2^6 + 1080c_2 c_3^2 u_1^2 u_2^2 - 1080c_2 c_3^2 u_1^2 u_2^2 - 96c_3^4 u_2^4 + 576c_3^4 u_2^4 
- 232c_3^4 u_2^6 + 1176c_1^3 u_1^4 u_2^2 - 1944c_3^3 u_1^4 u_2^2 + 14c_2^6 - 128c_1^4 u_1^6 + 39c_1^3 c_3^3 
- 232c_2^3 u_1^6 - 24c_3^4 u_1^4 + 36c_2^3 c_3^2 + 52c_2 c_3^2 + 52c_1^3 c_3^3 - 384c_1^2 c_3^2 c_1^4 
- 768c_1^2 c_3^2 u_1^6 - 576c_1 c_2 c_3 u_1^6 - 96c_1^2 c_3^2 u_1^6 - 9c_1 c_2 c_3 u_1^6 - 9c_1 c_2^2 c_3^2 u_1^6 
+ 72c_2 c_3^2 u_1^6 + 30c_1^6 + 14c_3^6 + 18c_4^4 c_3^2 + 18c_4^4 c_2^2 - 128c_3^4 u_2^6 - 384c_1^3 u_1^4 u_2^4 
- 384c_1^3 u_2^4 + 12c_2^4 c_2^2 - 1536c_2^2 c_2 u_1^2 u_2^2 - 768c_1^2 c_2 u_1^2 u_2^2 - 384c_1^2 c_2^2 u_2^4 
- 384c_1^2 c_2 c_2 u_1^2 u_2^2 - 768c_1^2 c_2^2 u_2^6 - 192c_1^2 u_1^4 + 12c_1^2 c_2^2 + 9c_2^4 c_3^2 + 9c_2^4 c_3^2 
- 192c_1^4 u_2^4 - 384c_1^4 u_1^2 u_2^4 - 384c_1^4 c_3^2 u_1^2 u_2^2 - 768c_1^2 c_3 u_1^2 u_2^4 - 1536c_1^2 c_3 u_1^4 u_2^2 
+ 1176c_2^3 u_1^4 u_2^2 - 480c_1 c_2^2 u_1^2 u_2^4 - 1056c_1 c_2 u_1^2 u_2^4 - 672c_1 c_2 u_1^4 u_2^2 \right\} \]
\[ + O(\lambda^4) \quad (A1) \]

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