A note on the splitting theorem for the weighted measure

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Abstract In this paper, we study complete manifolds equipped with smooth measures whose spectrum of the weighted Laplacian has an optimal positive lower bound and the \( m \)-dimensional Bakry–Émery Ricci curvature is bounded from below by some negative constant. In particular, we prove a splitting type theorem for complete smooth measure manifolds that have a finite-weighted volume end. This result is regarded as a study of the equality case of an author’s theorem (Wu, J Math Anal Appl 361:10–18, 2010).

Keywords Bakry–Émery curvature · Rigidity · Eigenvalue · Metric measure space

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1 Introduction and main result

The splitting phenomenon for complete manifolds is an interesting topic in geometric analysis. Perhaps the most notable result is the work of Cheeger and Gromoll [5,6], where they proved that if an \( n \)-dimensional complete manifold \( M \) with nonnegative Ricci curvature has a geodesic line, then it is isometric to \( \mathbb{R} \times N \) with the product metric, where \( N \) is an \( (n - 1) \)-dimensional complete manifold with nonnegative Ricci curvature. In a recent work of Wang [21], he proved a splitting theorem for complete smooth measure manifolds whose \( m \)-dimensional Bakry–Émery Ricci curvature is bounded from below by a negative multiple of the lower bound of the weighted spectrum. In particular, from Wang’s result, we have

**Theorem 1.1** Let \( (M, g) \) be an \( n \)-dimensional \((n \geq 3)\) complete Riemannian manifold and \( \varphi \) be a smooth function on \( M \). Assume that the \( m \)-dimensional \((m \geq n)\) Bakry–Émery Ricci...
curvature satisfies
\[ \text{Ric}_{m,n} \geq -(m - 1). \]

Let \( \lambda_1(M) \) be the lower bound of the spectrum of the weighted Laplacian \( \Delta \varphi = \Delta - \nabla \varphi \cdot \nabla \) on \( M \), and assume that
\[ \lambda_1(M) \geq (m - 2). \]

Then either
1. \( M \) has only one end with infinite-weighted volume; or
2. \( M = \mathbb{R} \times N \) with the warped product metric
\[ ds^2_M = dt^2 + \cosh^2 t \, ds^2_N, \]
where \( N \) is an \((n - 1)\)-dimensional compact Riemannian manifold. In this case, \( \lambda_1(M) = m - 2 \).

Theorem 1.1 generalized the work of Li–Wang [12] on Riemannian manifolds to the weighted measure case. If \( \varphi \) is constant, then \( \text{Ric}_{m,n} = \text{Ric} \) for all \( m \geq n \) and Theorem 1.1 returns to the Li–Wang’s theorem [12] by taking \( m = n \).

The weighted measure concept, used in Theorem 1.1, can be briefly described as follows. Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( \varphi \) be a smooth function. We may define the weighted Laplacian
\[ \Delta \varphi := \Delta - \nabla \varphi \cdot \nabla, \]
which is the infinitesimal generator of the Dirichlet form
\[ E(\phi_1, \phi_2) = \int_M (\nabla \phi_1, \nabla \phi_2) \, d\mu, \quad \forall \phi_1, \phi_2 \in \mathcal{C}_0^\infty(M), \]
where \( \mu \) is an invariant measure of \( \Delta \varphi \) given by \( d\mu = e^{-\varphi} \, dv(g) \). The weighted Laplacian \( \Delta \varphi \) is self-adjoint with respect to the weighted measure \( d\mu \). For the smooth metric measure manifold \((M, g, e^{-\varphi} \, dv)\), we can define the \( m \)-dimensional Bakry–Émery Ricci curvature (see [1–3, 14]) by
\[ \text{Ric}_{m,n} := \text{Ric} + \text{Hess}(\varphi) - \frac{\nabla \varphi \otimes \nabla \varphi}{m - n}, \]
where \( \text{Ric} \) and \( \text{Hess} \) denote the Ricci curvature and the Hessian of the metric \( g \), respectively. Here \( m := \dim_{\text{BE}}(\Delta \varphi) \geq n \) is called the Bakry–Émery dimension of \( \Delta \varphi \), which is a constant, and \( m = n \) if and only if \( \varphi \) is a constant [14, 15]. A remarkable feather of \( \text{Ric}_{m,n} \) is that volume comparison theorems hold for \( \text{Ric}_{m,n} \) in \((M^n, g, e^{-\varphi} \, dv)\) that look like the case of Ricci tensor in \( m \)-dimensional complete manifolds [14, 22].

If we let \( m \) be infinite, then the \( m \)-dimensional Bakry–Émery Ricci curvature becomes the \( \infty \)-dimensional Bakry–Émery Ricci curvature
\[ \text{Ric}_\infty := \lim_{m \to \infty} \text{Ric}_{m,n} = \text{Ric} + \text{Hess}(\varphi). \]
This curvature is closely related to the gradient Ricci soliton
\[ \text{Ric}_\infty = \rho g \]
for some constant \( \rho \), which plays an important role in the theory of Ricci flow [4].
Recently, Fang et al. [9] obtained two generalizations of Cheeger–Gromoll splitting theorem via the Bakry–Émery Ricci curvature. Munteanu and Wang [16] studied function theoretic and spectral properties on complete noncompact smooth metric measure space with nonnegative ∞-dimensional Bakry–Émery Ricci curvature. In particular, they obtained an interesting splitting result on complete noncompact gradient steady Ricci solitons.

Using the classical trick of deriving gradient estimates, which is originated by Yau [26] (see also [8, 18]) the author proved the following result by choosing $K = \frac{m-1}{n-1}$ in Theorem 2.1 of [23].

**Theorem 1.2** Let $(M, g)$ be an $n$-dimensional $(n \geq 2)$ complete Riemannian manifold and $\varphi$ be a smooth function on $M$. Assume that the $m$-dimensional Bakry–Émery Ricci curvature satisfies

$$\text{Ric}_{m,n} \geq -(m-1).$$

Then

$$\lambda_1(M) \leq \frac{(m-1)^2}{4}.$$

Moreover, if $f$ be a positive function satisfying

$$\Delta \varphi f = -\lambda f$$

for some constant $\lambda \geq 0$, then $f$ must satisfy the gradient estimate

$$|\nabla \ln f|^2 \leq \frac{(m-1)^2}{2} - \lambda + \sqrt{\frac{(m-1)^4}{4} - (m-1)^2\lambda}.$$

Theorem 1.2 was also independently proved by Wang [20]. This result can be viewed as a weighted version of Cheng’s theorem [7]. For the case of $\text{Ric}_{\infty}$, if $|\nabla \varphi|$ is bounded, then we have another version of gradient estimates [24]. Moreover, the following example shows that the above gradient estimate is sharp.

**Example 1.3** Consider the $n$-dimensional complete manifold $M = \mathbb{R} \times N$ endowed with the warped product metric

$$ds_M^2 = dt^2 + \exp(-2t)ds_N^2.$$

If $\{\bar{e}_\alpha\}$ for $\alpha = 2, \ldots, n$ form an orthogonal basis of the tangent space of $N$, then $e_1 = \frac{\partial}{\partial t}$ together with $\{e_\alpha = \exp(-t)\bar{e}_\alpha\}$ form an orthogonal basis for the tangent space of $M$. By the routine computation, we have

$$\text{Ric}_{M,1j} = -(n-1)\delta_{1j}$$

and
\[ \text{Ric}_{M,\alpha\beta} = \exp(2t)\text{Ric}_N,\alpha\beta - (n-1)\delta_{\alpha\beta}. \]

If we choose the weighted function
\[ \varphi := (m-n)t, \]
then the \(m\)-dimensional Bakry–Émery Ricci curvature of \(M\) is
\[ \text{Ric}_{mn,1j} = \text{Ric}_{M,1j} + \varphi_{1j} - \frac{\varphi_1\varphi_j}{m-n} \]
\[ = -(m-1)\delta_{1j} \]
and
\[ \text{Ric}_{mn,\alpha\beta} = \exp(2t)\text{Ric}_N,\alpha\beta - (n-1)\delta_{\alpha\beta}. \]

Hence we observe that if the Ricci curvature of manifold \(N\) is nonnegative, then the \(m\)-dimensional Bakry–Émery Ricci curvature satisfies
\[ \text{Ric}_{mn} \geq -(m-1). \]

In this setting, we claim that
\[ \lambda_1(M) = \frac{(m-1)^2}{4}. \]

Indeed, we may choose the function
\[ f = \exp\left(\frac{m-1}{2} - t\right). \]

A direct computation yields that
\[ \Delta_{\varphi} f = \frac{d^2 f}{dt^2} - (n-1) \frac{df}{dt} - \frac{d\varphi}{dt} \cdot \frac{df}{dt} \]
\[ = -\frac{(m-1)^2}{4} f, \]
since \(\Delta = \frac{\partial^2}{\partial t^2} - (n-1) \frac{\partial}{\partial t} + \exp(2t)\Delta_N\). On the other hand, we have the following proposition, which is a mild generalization for the classical case.

**Proposition 1.4** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and \(\varphi\) be a smooth function on \(M\). If there exists a positive function \(f\) satisfying
\[ \Delta_{\varphi} f \leq -\lambda f, \]
then \(\lambda_1(M)\), the lower bound of the spectrum of the weighted Laplacian \(\Delta_{\varphi}\), satisfies
\[ \lambda_1(M) \geq \lambda. \]

Combining this proposition and Theorem 1.2, we immediately conclude that
\[ \lambda_1(M) = \frac{(m-1)^2}{4} \]
as claimed.

Since \(\frac{(m-1)^2}{4} \geq m-2\) with equality holds only when \(m = 3\) (in this case, we return to the classical Laplacian case, see Remark 1.6), Theorem 1.1 in fact asserts that the equality case in Theorem 1.2 implies that the measure manifold belongs to the case (1) of Theorem 1.1. Namely, the measure manifold must only have one infinite-weighted volume end, unless \(m = 3\). In this case, the warped product given in Theorem 1.1 is the only exception.
Naturally, we would like to ask if the finite-weighted volume ends can be ruled out when a measure manifold satisfies the hypotheses of Theorem 1.1. In this paper, we follow the arguments of Li–Wang’s work [13], and show that the above example is the only case (it may be different from the weighted function $\varphi$) when $M$ has a finite weighted volume end if $M$ achieves equality in weighted spectrum upper bound of Theorem 1.2.

**Theorem 1.5** Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) complete Riemannian manifold and $\varphi$ be a smooth function on $M$. Assume that the $m$-dimensional ($m > 3$) Bakry–Émery Ricci curvature satisfies

$$\text{Ric}_{m,n} \geq -(m - 1).$$

and $\lambda_1(M)$, the lower bound of the spectrum of the weighted Laplacian $\Delta \varphi$, satisfies

$$\lambda_1(M) \geq \frac{(m - 1)^2}{4}.$$

Then either

1. $M$ has only one end; or
2. $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \exp(-2t)ds_N^2,$$

where $N$ is an $(n - 1)$-dimensional compact manifold. Moreover,

$$\varphi(t, x) = \varphi(0, x) + (m - n)t$$

for all $(t, x) \in \mathbb{R} \times N$.

**Remark 1.6** In Theorem 1.5, we assume that $m > 3$. Because when $m = 3$, we observe that $\frac{(m - 1)^2}{4} = m - 2$, $n = m = 3$ and hence $\varphi$ is constant. Therefore, this case is exact the Li-Wang classical result (Theorem 0.6 in [13]).

**Remark 1.7** Using similar trick, we can obtain splitting theorems for complete manifolds with $\infty$-dimensional Bakry–Émery curvature by gradient estimates in [24]. This was treated by the author in a separated paper [25]. Finally, we point out that Theorem 1.5 has been independently proved by Munteanu and Wang in [17], and Su and Zhang in [19].

## 2 Preliminary

In this section, we will give some important lemmas to prepare the proof of Theorem 1.5. At first, we recall some basic definitions in smooth metric measure manifolds, which are also introduced in [21].

**Definition 2.1** Let $(M, g)$ be a complete Riemannian manifold and $\varphi$ be a smooth function on $M$. A weighted Green’s function $G_\varphi(x, y)$ is a function defined on $(M \times M) \setminus \{(x, x)\}$ satisfying

1. $G_\varphi(x, y) = G_\varphi(y, x)$, and
2. $\Delta_{\varphi, x} G(x, y) = -\delta_{\varphi, x}(y)$,

for all $x \neq y$, where $\delta_{\varphi, x}(y)$ is defined by
\[
\int_M \psi(y) \delta_{\varphi,x}(y) d\mu = \psi(x)
\]
for every compactly supported function \(\psi\).

In fact, every smooth measure manifold admits a weighted Green’s function. Following Li–Tam [11], we can give a constructive argument for the existence of \(G_\varphi(x, y)\). But some measure manifolds admit weighted Green’s functions which are positive and others may not. This special property distinguishes the weighted function theory of complete measure manifolds into two classes.

**Definition 2.2** A complete measure manifold \((M, g, e^{-\varphi} dv)\) is said to be weighted non-parabolic if it admits a positive weighted Green’s function. Otherwise, it is said to be weighted parabolic.

Following the arguments of Theorem 2.3 in [10], we can easily show that a complete measure manifold is weighted non-parabolic if and only if there exists a positive weighted super-harmonic function whose infimum is achieved at infinitely. In the following, we will give the definition of an end of a complete manifold.

**Definition 2.3** An end, \(E\), with respect to a compact subset \(\Omega \subset M\) is an unbounded connected component of \(M \setminus \Omega\). The number of ends with respect of \(\Omega\), denoted by \(N_\Omega(M)\), is the number of unbounded connected component of \(M \setminus \Omega\).

It is easy to see that if \(\Omega_1 \subset \Omega_2\), then \(N_{\Omega_1}(M) \leq N_{\Omega_2}(M)\). Hence if \(\Omega_i\) is a compact exhaustion of \(M\), then \(N_{\Omega_i}(M)\) is a monotonically nondecreasing sequence. If this sequence is bounded, then we say that \(M\) has finitely many ends. In this case, the number of ends of \(M\) is defined by

\[
N(M) = \lim_{i \to \infty} N_{\Omega_i}(M).
\]

Obviously, the number of ends is independent of the compact exhaustion \(\{\Omega_i\}\).

**Definition 2.4** An end \(E\) is said to be weighted non-parabolic if it admits a positive weighted Green’s function with Neumann boundary condition on \(\partial E\). Otherwise, it is said to be weighted parabolic.

From the construction of Li–Tam [11], we can easily verify that a complete measure manifold is weighted non-parabolic if and only if it has a weighted non-parabolic end.

We now state a decay property about weighted harmonic functions on the end of a smooth metric measure manifold, which is a slight generalization of Lemma 1.1 in [12].

**Lemma 2.5** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and \(\varphi\) be a smooth function on \(M\). Suppose \(E\) is an end of \(M\) and the weighted spectrum \(\lambda_1(M) > 0\). Then for any weighted harmonic function \(f\) on \(E\) such that \(f = \lim_{R_i \to \infty} f_i\) with \(\Delta_\varphi f_i = 0\) on \(E(R_i)\) and \(f_i = 0\) on \(E \cap \partial B_p(R_i)\), \(f\) satisfies the decay estimate

\[
\int_{E(R+1) \setminus E(R)} f^2 d\mu \leq C \exp(-2\sqrt{\lambda_1(M)}R)
\]

for some constant \(C > 0\) depending on \(f\), \(\lambda_1(M)\) and \(n\), where \(B_p(R)\) denotes a geodesic ball centered at some fixed point \(p \in M\) with radius \(R > 0\), and \(E(R) = B_p(R) \cap E\).
The following lemma is an characterization for an end by its weighted volume due to Wang [21].

**Lemma 2.6** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and \(\varphi\) be a smooth function on \(M\). We assume that
\[
\lambda_1(M) \geq \frac{(m-1)^2}{4}.
\]
Let \(E\) be an end of \(M\), and let \(V_\varphi(E)\) be the simply weighted volume of end \(E\). We denote the weighted volume of the set \(E(R)\) by \(V_\varphi(E(R))\). \(R > 0\) is large enough.

1. If \(E\) is a weighted parabolic end, then \(E\) must have exponential weighted volume decay given by
\[
V_\varphi(E) - V_\varphi(E(R)) \leq C \exp(-(m-1)R)
\]
for some constant \(C > 0\) depending on the end \(E\).

2. If \(E\) is a weighted non-parabolic end, then \(E\) must have exponential volume growth given by
\[
V_\varphi(E(R)) \geq C \exp((m-1)R)
\]
for some constant \(C > 0\) depending on the end \(E\).

**Remark 2.7** Lemma 2.6 can be viewed as a refined version of a front part of Theorem 1.2. In fact, if the \(m\)-dimensional Bakry–Émery Ricci curvature satisfies
\[
\text{Ric}_{m,n} \geq -(m-1),
\]
then the weighted Bishop volume comparison theorem (see [3, 14]) asserts that
\[
V_\varphi(B_p(R)) \leq V_{H^m}(B_p(R)) \leq C \exp((m-1)R).
\]
Combining this and Lemma 2.6, we conclude that
\[
\lambda_1(M) \leq \frac{(m-1)^2}{4},
\]
as asserted in Theorem 1.2.

On the other hand, if the \(m\)-dimensional Bakry–Émery Ricci curvature is bounded from below by \(-(m-1)\), then the weighted Bishop volume comparison theorem, says that for any \(x \in M\),
\[
\frac{V_\varphi(B_x(R))}{V_{H^m}(B(R))}
\]
is nonincreasing in \(R\), where \(V_\varphi(B_x(R)) = \int_{B_x(R)} e^{-\varphi} \, d\nu(g)\) denotes the weighted volume of the geodesic ball \(B_x(R)\), and \(V_{H^m}(B(R))\) denotes the volume of a geodesic ball of radius \(R\) in the \(m\)-dimensional hyperbolic space form \(H^m\) with constant curvature \(-1\). Therefore for any \(R_1 < R_2\), we have
\[
\frac{V_\varphi(B_x(R_2))}{V_\varphi(B_x(R_1))} \leq \frac{V_{H^m}(B(R_2))}{V_{H^m}(B(R_1))}.
\]
In particular, if we let \(x = p\), \(R_1 = 0\), and \(R_2 = R\), then
\[
V_\varphi(B_p(R)) \leq C \exp((m-1)R)
\]
(2.1)
for sufficiently large $R$. If we let $x \in \partial B_{\rho}(R)$, $R_1 = 1$ and $R_2 = R + 1$, then

\begin{align}
V_{\varphi}(B_x(1)) &\geq CV_{\varphi}(B_x(R + 1)) \exp(-(m - 1)R) \\
&\geq CV_{\varphi}(B_{\rho}(1)) \exp(-(m - 1)R).
\end{align}

(2.2)

Combining (2.1), (2.2) and Lemma 2.6, we have that

**Corollary 2.8** Let $(M, g)$ be a complete Riemannian manifold and $\varphi$ be a smooth function on $M$, with the $m$-dimensional Bakry–Émery Ricci curvature satisfying

$$\text{Ric}_{m,n} \geq -(m - 1).$$

We assume that

$$\lambda_1(M) \geq \frac{(m - 1)^2}{4}.$$ 

Let $E$ be an end of $M$, and let $V_{\varphi}(E)$ be the simply weighted volume of end $E$. We denote the weighted volume of the set $E(R)$ by $V_{\varphi}(E(R))$. $R > 0$ is large enough.

1. If $E$ is a weighted-parabolic end, then $E$ must have exponential weighted volume decay given by

$$C_4 \exp(-(m - 1)R) \leq V_{\varphi}(E) - V_{\varphi}(E(R)) \leq C_1 \exp(-(m - 1)R)$$

for some constant $C_1 \geq C_4 > 0$ depending on the end $E$.

2. If $E$ is a weighted-non-parabolic end, then $E$ must have exponential volume growth given by

$$C_3 \exp((m - 1)R) \geq V_{\varphi}(E) \geq C_2 \exp((m - 1)R)$$

for some constant $C_3 \geq C_2 > 0$ depending on the end $E$.

3 Proof of Theorem 1.5

We are now ready to prove Theorem 1.5 in introduction. The proof method belongs to Li–Wang [13].

**Proof (Proof of Theorem 1.5)** Suppose that the manifold $M$ satisfies the hypothesis of Theorem 1.5. Then Theorem 1.1 asserts that $M$ must have only one infinite-weighted volume end because the warped product with the metric given by

$$ds_M^2 = dt^2 + \cosh^2 t ds_N^2$$

has $\lambda_1(M) = m - 2$, which does not satisfy the second hypothesis of Theorem 1.5.

Now we assume that manifold $M$ has a finite-weighted volume end. Since $\lambda_1(M) > 0$, $M$ must also have an infinite weighted volume end. By choosing the compact set $D$ appropriately, we may assume that $M \setminus D$ has one infinite-weighted volume, weighted non-parabolic end $E_1$, and one finite-weighted volume, weighted parabolic end $E_2$.

In an analogous way as Li–Tam’s arguments [11], our consideration is the weighted measure case. We assert that there exists a positive-weighted harmonic function $f$ with the the following properties:

- $\inf_{\partial E_1(R)} f \to 0$ as $R \to \infty$;
- $\sup_{\partial E_2(R)} f \to \infty$ as $R \to \infty$; and

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• $f$ is bounded and has finite-weighted Dirichlet integral on $E_1$.

Then the gradient estimate of Theorem 1.2 implies that

$$|\nabla f|^2 \leq (m - 1)^2 f^2.$$  

Combining this with the fact that function $f$ is weighted harmonic, we have

$$\Delta \phi f^{1/2} = -\frac{1}{4} f^{-3/2} |\nabla f|^2 \leq -\frac{(m - 1)^2}{4} f^{1/2}. \tag{3.1}$$

If we let $h = f^{1/2}$, then for any nonnegative cut-off function $\psi$ we have

$$\int_M |\nabla (\psi h)|^2 d\mu = \int_M |\nabla \psi|^2 h^2 d\mu + \int_M \psi^2 |\nabla h|^2 d\mu + 2 \int_M \psi h \nabla \psi \nabla h d\mu. \tag{3.2}$$

Since

$$\int_M \psi h \nabla \psi \nabla h d\mu = -\int_M \psi \nabla \psi h \nabla h d\mu - \int_M \psi^2 |\nabla h|^2 d\mu - \int_M \psi^2 h \Delta \psi h d\mu,$$

the integral equality (3.2) reduces to

$$\int_M |\nabla (\psi h)|^2 d\mu = \int_M |\nabla \psi|^2 h^2 d\mu - \int_M \psi^2 h \Delta \psi h d\mu$$

$$= \int_M |\nabla \psi|^2 h^2 d\mu + \frac{(m - 1)^2}{4} \int_M \psi^2 h^2$$

$$- \int_M \psi^2 h \left[ \frac{(m - 1)^2}{4} h + \Delta \psi h \right] d\mu. \tag{3.3}$$

Since $\lambda_1(M) \geq \frac{(m-1)^2}{4}$, the definition of $\lambda_1(M)$ gives us

$$\frac{(m-1)^2}{4} \int_M \psi^2 h^2 d\mu \leq \int_M |\nabla (\psi h)|^2 d\mu.$$

Hence

$$\int_M \psi^2 h \left[ \frac{(m - 1)^2}{4} h + \Delta \psi h \right] d\mu \leq \int_M |\nabla \psi|^2 h^2 d\mu. \tag{3.4}$$

Integrating the gradient estimate of Theorem 1.2 along geodesics, we know that $f$ must satisfy the growth estimate

$$f(x) \leq C \exp((m - 1)r(x)),$$

where $r(x)$ is the geodesic distance from $x$ to a fixed point $p \in M$. In particular, when restricted on the parabolic end $E_2$, together with the volume estimate of Lemma 2.6, we conclude that

$$\int_{E_2(R)} f d\mu \leq CR. \tag{3.5}$$
On the other hand, Lemma 2.5 asserts that on $E_1$, the function $f$ must satisfy the decay estimate

$$
\int_{E_1(R+1) \setminus E_1(R)} f^2 \, d\mu \leq C \exp(-(m-1)R)
$$

for $R$ sufficiently large. By the Schwarz inequality, we have

$$
\int_{E_1(R+1) \setminus E_1(R)} f \, d\mu \leq C \exp\left(-\frac{m-1}{2}R\right) V_{\varphi E_1}(R+1),
$$

where $V_{\varphi E_1}(r)$ denotes the weighted volume of $E_1(r)$. Combining this with the volume estimate of Corollary 2.8, we have that

$$
\int_{E_1(R+1) \setminus E_1(R)} f \, d\mu \leq C
$$

for some constant $C$ independent of $R$. In particular, we have

$$
\int_{E_1(R)} f \, d\mu \leq CR.
$$

Combining this with (3.5), we conclude that

$$
\int_{B_p(R)} f \, d\mu \leq CR.
$$

(3.6)

Now we define the cut-off function $\psi$ on $M$ in (3.4) by

$$
\psi(x) = \begin{cases} 
1 & x \in B_p(R) \\
\frac{2R-r}{R} & x \in B_p(2R) \setminus B_p(R) \\
0 & x \not\in B_p(2R).
\end{cases}
$$

Hence, the right hand side of (3.4) is given by

$$
\int_M |\nabla \psi|^2 h^2 \, d\mu = R^{-2} \int_{B_p(2R) \setminus B_p(R)} h^2 \, d\mu
$$

and (3.6) implies

$$
\int_M |\nabla \psi|^2 h^2 \, d\mu \to 0
$$

as $R \to \infty$. Therefore, we obtain

$$
\Delta_{\varphi} h = -\frac{(m-1)^2}{4} h
$$

and inequality (3.1) used in the above argument is an equality. In particular, we have

$$
|\nabla f| = (m-1) f
$$
and

$$|\nabla \ln f|^2 = (m-1)^2.$$  \hspace{1cm} (3.7)

Hence, the inequalities used to prove the gradient estimate of Theorem 1.2 are all equalities. Namely we must have equality (2.11) in [23] since

$$\Delta \varphi |\nabla \ln f|^2 = \Delta |\nabla \ln f|^2 - \nabla \varphi \cdot \nabla |\nabla \ln f|^2 = |\nabla \ln f|^2 = 0.$$  \hspace{1cm}

Moreover the inequalities used to derive (2.11) in [23] must all be equalities. More specifically, equality (2.6) in [23] implies

$$(\ln f)_{1j} = 0$$

for all $1 \leq j \leq n$, whereas equality (2.7) in [23] gives

$$\langle \nabla \varphi, \nabla \ln f \rangle = (m-1)(m-n)$$  \hspace{1cm} (3.8)

and

$$(\ln f)_{\alpha\beta} = -\frac{|\nabla \ln f|^2}{m-1} \delta_{\alpha\beta}$$

$$= -(m-1) \delta_{\alpha\beta}$$

for all $2 \leq \alpha, \beta \leq n$. Since $e_1$ is the unit normal to the level set of $\ln f$, the second fundamental form $\Pi$ of the level set is given by

$$\Pi_{\alpha\beta} = \frac{(\ln f)_{\alpha\beta}}{(\ln f)_1}$$

$$= \frac{-(m-1) \delta_{\alpha\beta}}{m-1}$$

$$= -\delta_{\alpha\beta}.$$  

Moreover, (3.7) implies that if we set $t = \frac{\ln f}{m-1}$, then $t$ must be the distance function between the level sets of $f$, hence also for $\ln f$. Since $\Pi_{\alpha\beta} = (-\delta_{\alpha\beta})$, this implies that the metric on $M$ can be written as

$$ds_M^2 = dt^2 + \exp(-2t) ds_N^2.$$  

By (3.8), we also have

$$\varphi(t, x) = \varphi(0, x) + (m-n)t,$$

where $(t, x) \in \mathbb{R} \times N$. Since we assume that the manifold $M$ has two ends, $N$ must be compact.

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