Investigation of wall-bounded turbulent flow using Dynamic mode decomposition

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Abstract. Dynamics mode decomposition (DMD) which is a method to construct a linear mapping describing the dynamics of a given time-series of any quantities is applied to the analysis of a turbulent channel flow. The flow fields are generated by direct numerical simulations for the friction Reynolds number $Re_\tau = 190$. The time-series of the flow fields in a short time-interval in the order of the wall-unit time-scale and in a small spatial domain that encloses a single near-wall structure are used as the inputs to DMD. In some datasets, linearly growing modes that seem to contribute to the well-known self-sustained cycle of the flow structures near the wall are detected.

1. Introduction

Understanding dynamics embedded in turbulent flows has been one of central issues in turbulence research so far. Modeling by a low-order dynamical system and marginally low Reynolds number simulations have contributed to this attempt, but it is still difficult to deal with the very large database generated by direct numerical simulations or experiments with high Reynolds numbers. One of the roadblocks in this way is the lack of methodology to extract useful information describing the dynamics from such enormous data.

Dynamic mode decomposition (DMD) provides a new mathematical tool for spectral-like analysis for nonlinear dynamics. In this method, a linear mapping is constructed from a time series of any field data. From a time-series $\mathbf{u}_n$, $n = 1, \cdots, N_t$, of a set of any physical quantities, we approximate its dynamics by a linear mapping $A$ satisfying

$$\mathbf{u}_{n+1} = A \mathbf{u}_n. \quad (1)$$

The eigenvalues and eigenvectors are calculated from the sequence $\mathbf{u}_n$ by a Krylov method. Through this method, the dynamics of the system are described by a linear mapping no matter whether it is linear or nonlinear, and are analysed as a usual linear system. Since only a time-series of data is required in this method, one of its major advantages is flexibility. We can focus on a subdomain instead of the entire flow field. Furthermore, any of the spatial axes may be treated as per the time-axis to examine the spatial evolution in that direction (Schmid, 2010). This method has been applied to a jet in crossflow (Rowley et al., 2009), cavity, and wake flows (Schmid, 2010).

Here DMD is applied to data from direct numerical simulations of a turbulent channel flow to assess its applicability.
2. Applicability of DMD to nonlinear systems

DMD decomposes a time-evolving field $u(x, y, z, t)$ into the form of

$$u(x, y, z, t) = \sum_{l} e^{\sigma_{l}t} f_{l}(x, y, z),$$

(2)

where $\sigma_{l} = \sigma_{R,l} + i\sigma_{I,l}$ and $f_{l}(x, y, z)$ are the eigenvalues and eigenfunctions of the mapping $A$. For simplicity, a one-dimensional field $u(x, t)$ is considered. If spatial periodicity is assumed as in some canonical turbulent flows, this is simplified to,

$$u(x, t) = \sum_{l} e^{\sigma_{l}t} \left\{ \sum_{k} a_{l,k} e^{ikx} \right\} = \sum_{k} e^{ikx} \sum_{l} a_{l,k} e^{\sigma_{l}t},$$

(3)

where $k$ is the wavenumber, and $a_{l,k}$ is the coefficient of the Fourier expansion. Under appropriate conditions DMD detects eigenvalues $\sigma$ accurately if $u$ has only a single mode $e^{ikx}e^{\sigma_{l}t}$ (Duke et al., 2011). DMD works as well if each $k$-mode has only a single eigenmode. However, if $k$ has different modes, DMD may not be able to decompose them, but gives an eigenmode which is a mixture of these modes. This may not be a serious issue for instability or weakly nonlinear problems, but it will be in general cases unless there exists a dominant mode that allows one to ignore the others. It is demonstrated through a simple case. A function $u(x, t)$ on $x = (0, L_{x}]$ and $t = (0, T]$ is used for the test. It is descritized on a uniform grid on $(x, t)$ plane as,

$$u_{mn} = u(m\Delta_{x}, n\Delta_{t}), \quad m = 1, \cdots, N_{x}, \quad n = 1, \cdots, N_{t},$$

where $\Delta_{x} = L_{x}/N_{x}$ and $\Delta_{t} = T/N_{t}$. The size of the domain and number of grid points are chosen as $L_{x} = 2\pi$, $T = 1$, $N_{x} = 100$ and $N_{t} = 100$. Results shown below are converged well at this resolution. Figure 1(a) shows an original signal that consists of three modes with the same intensity and spatial wavelength, and the DMD approximation. It illustrates that DMD cannot separate them, but gives a single mode with mixed growth rate and phase velocity. Though
this ideal case in which wavenumbers of some modes exactly coincide does not happen in real problems, they may be very close to each other, especially in turbulence whose energy spectra of the velocity fluctuations are typically dense. Practically the rank of the original system (1) is reduced during the computation with a given criterion to achieve the efficiency and robustness of the implementation (Schmid, 2010). At the same time, this reduction may lose a small difference in the wavenumbers, which leads to a failure to decompose them as in figure 1(a). The degree of the rank-reduction should be appropriately chosen for each case, depending on the quality of data and also the required accuracy of the results.

A more essential issue is whether the dynamics to be analyzed can be expressed in the form of (2) or (3). This is not the case in nonlinear systems in general. For example, though energy transfer between different scales is a key dynamical effect in turbulence, obviously (3) cannot express it. If the turbulent flow is statistically steady, that is, the trajectory of the solution in the phase space of the governing dynamical system is within an attractor, the real parts of the eigenvalues approach zero as the time-interval $T$ is made longer. The resulting set of eigenmodes with pure imaginary eigenvalues is equivalent to the Fourier expansion of the time-series of data (Mezić & Banaszuk, 2004; Mezić, 2005). However, this is out of our interest here. Instead, DMD may capture the short-time behaviour by locally approximating it by (2). First of all, the linear process must be dominant in that period to be approximated by linear mapping. Furthermore, the time interval must be short enough so that the functional form of $t$ can be locally approximated by $\exp(\sigma t)$. Figure 1(b) shows some examples in which actual waves in the data do not obey exponential decay or growth. DMD approximates well low-order polynomials and slower growing functions if the time-interval $T$ is short enough. The phase velocity is also detected quite well.

3. Application to turbulent channel flow

In the following, DMD is applied to a turbulent channel flow for $h^+ \equiv u_r h / \nu = 190$, where $u_r$ is the friction velocity, $h$ is the half-width and $\nu$ is the kinematic viscosity. The superscript + stands for quantities normalized by the wall-unit determined by $u_r$ and $\nu$. The velocity field is generated by direct numerical simulations. The wall-parallel size of the numerical domain is $6\pi h$ and $3\pi h$ in the streamwise and spanwise direction, and periodicity is assumed. In a well-developed statistically steady state flow, none of the fluctuations keeps growing for a long time. It has been shown that quasi-periodic behaviour of low-speed streaks accompanied by streamwise vortices, consisting of generation by bursting, growth, instability and decaying, maintains turbulent fluctuations in the near-wall region (Jiménez & Moin, 1991; Hamilton et al., 1995; Jiménez et al., 2005). For channel flows, del Álamo & Jiménez (2006) carried out a linear analysis with a turbulent mean velocity profile and an eddy viscosity model, and showed that, though all the modes eventually decay, some of them can initially grow due to the non-orthogonality of the system. The most amplified modes are found to organize flow structures in real channel flows. According to their results for the optimal mode in the near-wall region, the spanwise length-scale is $\lambda_z^+ = 100$, and its time-scale is $\lambda_z^+ / 10$. To detect this mode, the flow field is divided into 16 pieces in spanwise so that each piece fit to the size of this optimal mode.
Figure 2. An example of obtained eigenvalues.

Figure 3. Examples of the streamwise velocity component of (a) magnitude of eigenvector associated with the largest growth rate in figure 2, and (b) the original field at the beginning time, on the streamwise-spanwise plane at $y^+ = 25$. The velocity is normalized by (a) the maximum of the magnitude and (b) the friction velocity.

The total time interval is chosen to be $T^+ = 5$. Details of each flow field to be applied to DMD is summarized in Table 1.

Most of the obtained eigenvalues seem non-physical. Their growth rate depends on $\Delta t$ even when it is smaller than the time-resolution in the simulation of this flow. They may be due to the strong nonlinearity of the flow, and their interpretation is unclear. Such modes are ignored here, and only modes robust against the changes in the spatial and temporal resolution are picked up. Figure 2 shows examples of eigenvalues obtained from a data set which has a complex conjugate pair of growing modes. The eigenmodes with a larger imaginary part tend to decay faster. The eigenvalue of the growing mode is $\sigma_R^+ = 0.11$ and $\sigma_I^+ = \pm 4.5$. The approximated growth rate during the period is $\exp(\sigma_R^+ T^+) \approx 1.76$. Since such a linear amplification of fluctuations occurs randomly in time and space, growing modes are detected only by chance. It is necessary to search many realization to collect samples to obtain the statistics of these time-scales.

The eigenfunction of the most growing mode is compared with the original field in figure 3. Apart from the phase-shift, the eigenfunction seems to capture almost all the feature of the original field. Not only the growing mode, but the other eigenfunctions are quite similar to the original one as well. This is a consequence of the small time-interval $T$ to prohibit complex time-evolution involving different spatial scales, which cannot be represented by the form of (2).
4. Conclusions

Dynamic mode decomposition was applied to a turbulent channel flow. Though the flow itself is fully nonlinear, it succeeded in extracting an amplification of velocity fluctuations in the near-wall region by giving flow data local in time and space. Although it is difficult to depict the whole picture of its nonlinear dynamics, DMD provides a tool for data-based linear analysis of turbulent flows.

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