THE CURVATURE IN Variant OF A HILBERT MODULE OVER $\mathbb{C}[z_1, \ldots, z_d]$

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Abstract. A notion of curvature is introduced in multivariable operator theory and an analogue of the Gauss-Bonnet-Chern theorem is established. Applications are given to the metric structure of graded ideals in $\mathbb{C}[z_1, \ldots, z_d]$, and the existence of “inner” sequences for closed submodules of the free Hilbert module $H^2(\mathbb{C}^d)$.

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Introduction.

Let $\hat{T} = (T_1, \ldots, T_d)$ be a $d$-tuple of mutually commuting operators acting on a common Hilbert space $H$. $\hat{T}$ is called a $d$-contraction if

$$\|T_1 \xi_1 + \cdots + T_d \xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2$$

for all $\xi_1, \ldots, \xi_d \in H$. The number $d$ will be fixed throughout this paper, and of course we are primarily interested in the cases $d \geq 2$. Let $A = \mathbb{C}[z_1, \ldots, z_d]$ be the complex unital algebra of all polynomials in $d$ commuting variables $z_1, \ldots, z_d$. A commuting $d$-tuple $T_1, \ldots, T_d$ of operators in the algebra $\mathcal{B}(H)$ of all bounded operators on $H$ gives rise to an $A$-module structure on $H$ in the natural way,

$$f \cdot \xi = f(T_1, \ldots, T_d) \xi, \quad f \in A, \quad \xi \in H;$$

and $(T_1, \ldots, T_d)$ is a $d$-contraction iff $H$ is a contractive $A$-module in the following sense,

$$\|z_1 \xi_1 + \cdots + z_d \xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2$$

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for all $\xi_1, \ldots, \xi_d \in H$. Thus it is equivalent to speak of $d$-contractions or of contractive Hilbert $A$-modules, and we will shift from one point of view to the other when it is convenient to do so.

For every $d$-contraction $\bar{T} = (T_1, \ldots, T_d)$ we have $0 \leq T_1 T_1^* + \cdots + T_d T_d^* \leq 1$, and hence

\begin{equation}
\Delta = (1 - T_1 T_1^* - \cdots - T_d T_d^*)^{1/2}
\end{equation}

is a positive operator on $H$ of norm at most one. The rank of $\bar{T}$ is defined as the dimension of the range of $\Delta$. Throughout this paper we will be primarily concerned with finite rank $d$-contractions (resp. finite rank contractive Hilbert $A$-modules).

We introduce several numerical invariants for finite rank contractive $A$-modules $H$, the principal ones being the curvature invariant $K(H)$, the Euler characteristic $\chi(H)$, and the degree $\deg(H)$. All of these quantities are real numbers (indeed, most are integers), and we develop their basic properties.

We now describe the main results of this paper, starting with a sketch of the definition of the curvature invariant $K(H)$. Let $H$ be a finite rank contractive Hilbert $A$-module with associated $d$-contraction $(T_1, \ldots, T_d)$. For every point $z = (z_1, \ldots, z_d)$ in complex $d$-space $\mathbb{C}^d$ we form the operator

\begin{equation}
T(z) = \bar{z}_1 T_1 + \cdots + \bar{z}_d T_d \in \mathcal{B}(H),
\end{equation}

$\bar{z}_k$ denoting the complex conjugate of the complex number $z_k$. Notice that the operator function $z \mapsto T(z)$ defines an antilinear mapping of $\mathbb{C}^d$ into $\mathcal{B}(H)$, and since $(T_1, \ldots, T_d)$ is a $d$-contraction we have

$$
\|T(z)\| \leq |z| = (|z_1|^2 + \cdots + |z_d|^2)^{1/2}
$$

for all $z \in \mathbb{C}^d$. In particular, if $z$ belongs to the open unit ball

$$
B_d = \{ z \in \mathbb{C}^d : |z| < 1 \}
$$

then $\|T(z)\| < 1$ and hence $1 - T(z)$ is invertible. Thus for every $z \in B_d$ we can define a positive operator $F(z)$ acting on the finite dimensional Hilbert space $\Delta H$ as follows,

$$
F(z)\xi = \Delta(1 - T(z)^*)^{-1}(1 - T(z))^{-1} \Delta \xi, \quad \xi \in \Delta H.
$$

We require the boundary values of the real-valued function $z \in B_d \mapsto \text{trace } F(z)$, which exist in the following sense. Let $\partial B_d = \{ z \in \mathbb{C}^d : |z| = 1 \}$ be the unit sphere in $\mathbb{C}^d$ and let $\sigma$ be normalized surface measure on $\partial B_d$.

**Theorem A.** For $\sigma$-almost every $\zeta \in \partial B_d$, the limit

$$
K_0(\zeta) = \lim_{r \uparrow 1}(1 - r^2)\text{trace } F(r \zeta)
$$

exists and satisfies $0 \leq K_0(\zeta) \leq \text{rank}(H)$.

Theorem A is proved in section 4. We define the curvature invariant by integrating $K_0$ over the sphere

\begin{equation}
K(H) = \int_{\partial B_d} K_0(\zeta) \, d\sigma(\zeta).
\end{equation}
Obviously, $K(H)$ is a real number satisfying $0 \leq K(H) \leq \text{rank}(H)$.

We now define the Euler characteristic $\chi(H)$ of a finite rank contractive $A$-module $H$. $\chi(H)$ depends only on the algebraic structure of the following $A$-submodule of $H$:

$$M_H = \text{span}\{ f \cdot \xi : f \in A, \xi \in \Delta H \}.$$ 

Notice that we have not taken the closure in forming $M_H$. Note too that if $r = \text{rank}(H)$ and $\zeta_1, \ldots, \zeta_r$ is a linear basis for $\Delta H$, then $M_H$ is the set of “linear combinations”

$$M_H = \{ f_1 \cdot \zeta_1 + \cdots + f_r \cdot \zeta_r : f_k \in A \}.$$ 

In particular, $M_H$ is a finitely generated $A$-module.

It is a consequence of Hilbert’s syzygy theorem for ungraded modules (cf. Theorem 182 of [18] or Corollary 19.8 of [14]) that $M_H$ has a finite free resolution; that is, there is an exact sequence of $A$-modules

\[ 0 \rightarrow F_n \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow M_H \rightarrow 0 \tag{0.4} \]

where $F_k$ is a free module of finite rank $\beta_k$,

$$F_k = A \oplus \cdots \oplus A_{\beta_k \text{times}}.$$ 

The alternating sum of the “Betti numbers” of this free resolution

\[ \sum_{k=1}^{n} (-1)^{k+1} \beta_k \]

does not depend on the particular finite free resolution of $M_H$, and hence we may define the Euler characteristic of $H$ by

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k, \tag{0.5}$$

where $\beta_k$ is the rank of $F_k$ in any finite free resolution of $M_H$ of the form (0.4).

One of the more notable results in the Riemannian geometry of surfaces is the Gauss-Bonnet theorem, which asserts that if $M$ is a compact oriented Riemannian 2-manifold and

$$K : M \rightarrow \mathbb{R}$$

is its Gaussian curvature function, then

\[ \frac{1}{2\pi} \int_M K \, dA = \beta_0 - \beta_1 + \beta_2 \tag{0.6} \]

where $\beta_k$ is the $k$th Betti number of $M$. In particular, the integral of $K$ depends only on the topological type of $M$. This remarkable theorem was generalized by Shiing-Shen Chern to compact oriented even-dimensional Riemannian manifolds in 1944 [6].

We will establish the following result in section 6, which we view as an analogue of the Gauss-Bonnet-Chern theorem for graded Hilbert $A$-modules. By a graded
Hilbert $A$-module we mean a pair $(H, \Gamma)$ where $H$ is a (finite rank, contractive) Hilbert $A$-module and $\Gamma : \mathbb{T} \to \mathcal{B}(H)$ is a strongly continuous unitary representation of the circle group such that

$$\Gamma(\lambda)T_k\Gamma(\lambda)^{-1} = \lambda T_k, \quad k = 1, 2, \ldots, d, \lambda \in \mathbb{T},$$

$T_1, \ldots, T_d$ being the $d$-contraction associated with the module structure of $H$. Thus, graded Hilbert $A$-modules are precisely those whose underlying operator $d$-tuple $(T_1, \ldots, T_d)$ possesses circular symmetry. $\Gamma$ is called the gauge group of $H$.

**Theorem B.** Let $H$ be a graded (contractive, finite rank) Hilbert $A$-module for which the spectrum of the gauge group is bounded below. Then $K(H) = \chi(H)$, and in particular $K(H)$ is an integer.

We remark that the hypothesis on the spectrum of the gauge group is equivalent to several other natural ones, see Proposition 6.4. Theorem B depends on the following asymptotic formulas for $K(H)$ and $\chi(H)$, which are valid for finite rank contractive Hilbert $A$-modules, graded or not. For such an $H$, let $(T_1, \ldots, T_d)$ be its associated $d$-contraction and define a completely positive normal map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\phi(A) = T_1 AT_1^* + \cdots + T_d AT_d^*.$$

Since $H$ is contractive and finite rank, $1 - \phi(1)$ is a positive finite rank operator, and a simple argument shows that $1 - \phi^n(1)$ is a positive finite rank operator for every $n = 1, 2, \ldots$.

**Theorem C.** For every contractive finite rank Hilbert $A$-module $H$,

$$\chi(H) = d! \lim_{n \to \infty} \frac{\text{rank} \left( 1 - \phi^{n+1}(1) \right)}{n^d}.$$

**Theorem D.** For every contractive finite rank Hilbert $A$-module $H$,

$$K(H) = d! \lim_{n \to \infty} \frac{\text{trace} \left( 1 - \phi^{n+1}(1) \right)}{n^d}.$$

Theorems C and D are proved in sections 3 and 5. The number $K(H)$ is actually the trace of a certain self-adjoint trace-class operator $d\Gamma$, which exists for any finite rank contractive Hilbert module. While the trace of this operator is therefore always nonnegative, it is noteworthy that $d\Gamma$ itself is never a positive operator. Indeed, we have found it useful to think of $d\Gamma$ as a higher dimensional operator-theoretic counterpart of the differential of the Gauss map $\gamma : M \to S^2$ of an oriented 2-manifold $M \subseteq \mathbb{R}^3$ (cf. [9], pp 136–146). We have glossed over some details in order to make the essential point; see section 5 for a more comprehensive discussion. In any case, the formula

$$K(H) = \text{trace} \ d\Gamma$$

is an essential component in the proofs of Theorems B and D (see Theorem 5.13 et seq).

These results have concrete implications about the invariant subspaces of $H^2$ and the algebraic structure of graded ideals in the polynomial algebra $\mathbb{C}[z_1, \ldots, z_d]$. The applications are discussed in section 8, and are briefly summarized as follows.
THE CURVATURE INVARIANT OF A HILBERT MODULE OVER $\mathbb{C}[z_1, \ldots, z_d]$

The dilation theory described in section 1 implies that for every closed submodule $M$ of the free Hilbert module $H^2$, there is a (finite or infinite) sequence $\Phi = \{\phi_1, \phi_2, \ldots\}$ of multipliers of $H^2$ whose associated multiplication operators $M_{\phi_n}$ satisfy

$$(0.9) \quad P_M = M_{\phi_1} M^*_{\phi_1} + M_{\phi_2} M^*_{\phi_2} + \ldots,$$

$P_M$ denoting the orthogonal projection of $H^2$ on $M$. Every multiplier $\phi$ can be regarded as a bounded holomorphic function defined on the open unit ball $B_d = \{z \in \mathbb{C}^d : |z| < 1\}$ and has a radial limit function $\hat{\phi} : \partial B_d \rightarrow \mathbb{C}$ defined almost everywhere $(d\sigma)$ by

$$\hat{\phi}(z) = \lim_{r \rightarrow 1} \phi(rz), \quad z \in \partial B_d.$$  

Formula (0.9) implies that the boundary values satisfy $\sum_n |\hat{\phi}_n(z)|^2 \leq 1$ almost everywhere on $\partial B_d$, and $\Phi$ is called an inner sequence if we have equality

$$\sum_n |\hat{\phi}_n(z)|^2 = 1$$

almost everywhere on $\partial B_d$.

In dimension $d = 1$, a familiar theorem of Beurling implies that there is a single multiplier $\phi$ which satisfies $P_M = M_{\phi} M^*_{\phi}$, and such a multiplier must be an inner function. By contrast, in dimension $d \geq 2$, there may be no single multiplier $\phi$ satisfying $P_M = M_{\phi} M^*_{\phi}$; indeed, in most cases the sequences $\{\phi_1, \phi_2, \ldots\}$ of (0.9) are necessarily infinite (see the corollary of Theorem F below). Moreover, we do not know if these infinite sequences associated with $M$ are inner in general. This problem is associated with the fact that in dimension $d \geq 2$ the canonical operators $S_1, \ldots, S_d$ associated with $H^2$ do not form a subnormal $d$-tuple, and are not usefully related to the $L^2$ space of any measure. Making use of the curvature invariant and a theorem of Auslander and Buchsbaum on the vanishing of the Euler characteristic of finitely generated modules over polynomial rings, we establish the following result which appears to cover many cases of interest.

**Theorem E.** Let $M$ be a closed submodule of $H^2$ which contains a nonzero polynomial. Then every sequence of multipliers $\Phi$ satisfying (0.9) is an inner sequence.

Let $I$ be an ideal in $\mathbb{C}[z_1, \ldots, z_d]$. Hilbert’s basis theorem implies that there is a finite set of elements $\phi_1, \ldots, \phi_r \in I$ which generates $I$ in the sense that

$$I = \{f_1 \cdot \phi_1 + \cdots + f_r \cdot \phi_r : f_k \in \mathbb{C}[z_1, \ldots, z_d]\}.$$  

If $I$ is graded in the sense that it is spanned by its homogeneous polynomials, then one can find a set $\phi_1, \ldots, \phi_r$ of generators such that

(0.7.1) each $\phi_k$ is a homogeneous polynomial of some degree $n_k$, and

(0.7.2) $\{\phi_1, \ldots, \phi_r\}$ is linearly independent.

Of course, systems of generators satisfying the conditions (0.7) are by no means unique.

We want to relate sets of generators of graded ideals to the natural norm on $\mathbb{C}[z_1, \ldots, z_d]$, obtained by restricting the Hilbert space norm on $H^2$ to polynomials. To do that effectively we must consider infinite generators. Let $\Phi$ be a (perhaps
infinite) linearly independent set of homogeneous polynomials in a graded ideal $I$ which is contractive in the sense that whenever $\phi_1, \ldots, \phi_r$ are distinct elements of $\Phi$ and $f_1, \ldots, f_r \in \mathbb{C}[z_1, \ldots, z_d]$ we have

$$\|f_1 \cdot \phi_1 + \cdots + f_r \cdot \phi_r\|^2 \leq \|f_1\|^2 + \cdots + \|f_r\|^2.$$ 

One can scale down any set $\Phi$ of polynomials so as to achieve this condition. A set $\Phi$ satisfying the above conditions is said to be a metric basis for $I$ if every polynomial $g$ of degree $n$ in $I$ can be represented as a sum $g = f_1 \cdot \phi_1 + \cdots + f_r \cdot \phi_r$ where $f_1, \ldots, f_r$ and $\phi_1, \ldots, \phi_r$ are as above and, in addition, satisfy

\begin{align*}
(0.8.1) \quad & \deg f_k + \deg \phi_k \leq n, \quad k = 1, \ldots, r, \text{ and} \\
(0.8.2) \quad & \|g\|^2 = \|f_1\|^2 + \cdots + \|f_r\|^2.
\end{align*}

Condition (0.8.1) controls the degrees of $\phi_1, \ldots, \phi_r$ and (0.8.2) asserts that the norms of $f_1, \ldots, f_r$ are as small as the contractive hypothesis allows.

In section 8 we show that every graded ideal in $\mathbb{C}[z_1, \ldots, z_d]$ has a metric basis, that the elements of a metric basis are mutually orthogonal, and that any two metric bases are equivalent in a natural sense. Thus, by giving up the requirement of finite generation of ideals, one obtains a uniqueness result for infinite generators satisfying (0.8.1) and (0.8.2).

Let $\Phi = \{\phi_1, \phi_2, \ldots\}$ be a metric basis for a graded ideal $I$ in the polynomial algebra $\mathbb{C}[z_1, \ldots, z_d]$, and let $\sigma$ be the natural measure on the unit sphere $\partial B_d \subseteq \mathbb{C}^d$. Theorem E implies that for $\sigma$-almost every point $\zeta \in \partial B_d$ we have

$$\sum_n |\phi_n(\zeta)|^2 = 1.$$ 

One cannot expect the preceding “almost everywhere” equation to hold everywhere on the unit sphere. Indeed, if

$$V = \{z \in \mathbb{C}^d : \phi_1(z) = \phi_2(z) = \cdots = 0\}$$

is the variety of common zeros of the polynomials $\phi_k$ (i.e., the zero set of the ideal $I$) then, since each $\phi_k$ is a homogeneous polynomial, $V$ is invariant under multiplication by positive scalars. So whenever $V$ contains something other than $(0, 0, \ldots, 0)$ it must intersect the unit sphere in $\mathbb{C}^d$, and in that case $V \cap \partial B_d$ is a nonvoid compact set of measure zero on which the $\phi_k$ all vanish.

The following result implies that a finite generating set for a graded ideal cannot be a metric basis except in a few insignificant cases.

**Theorem F.** Let $I$ be a graded ideal in $\mathbb{C}[z_1, \ldots, z_d]$ whose metric basis is a finite set $\{\phi_1, \ldots, \phi_n\}$. Then $I$ is of finite codimension in $\mathbb{C}[z_1, \ldots, z_d]$ and each of the canonical coordinates $z_1, \ldots, z_d$ is nilpotent modulo $I$.

Given any finite rank (contractive) Hilbert module $H$ over $\mathbb{C}[z_1, \ldots, z_d]$ and a closed submodule $K \subseteq H$, then both $K$ and its quotient $H/K$ are contractive Hilbert modules. It is quite easy to see that rank $H/K \leq$ rank $H$, and hence $H/K$ is also of finite rank. However one does not have control over the rank of the submodule $K$. Indeed, Theorem F has the following consequence, which stands in rather stark contrast with the assertion of Hilbert’s basis theorem.
Corollary. Let \( K \) be a (nonzero, closed) graded submodule of the free Hilbert module \( H^2 \) which is of infinite codimension in \( H^2 \). Then \( \text{rank } K = \infty \).

We end the paper with a discussion of some examples that serve to illustrate the properties of the invariants described above. In particular, we show that any variety in complex projective space \( \mathbb{P}^{d-1} \) gives rise to a pure graded rank-one Hilbert \( \mathbb{C}[z_1, \ldots, z_d] \)-module, and for some of these examples we compute all invariants in explicit terms (see section 9).

This work was initiated in order to obtain numerical invariants for normal completely positive maps of \( \mathcal{B}(H) \). Notice that all of the invariants introduced in this paper depend only on the properties of the map \( \phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \)

\[
(0.10) \quad \phi(A) = T_1 AT_1^* + \cdots + T_d AT_d^*
\]

associated with the canonical operators \( T_1, \ldots, T_d \) of the \( \mathbb{C}[z_1, \ldots, z_d] \)-module structure. Indeed, Theorems C and D express \( \chi(H) \) and \( K(H) \) explicitly in terms of \( \phi \), and Proposition 7.5 does the same for the secondary invariants \( \text{deg}(H) \), \( \mu(H) \).

Thus these numbers are actually invariants of certain completely positive maps \( \phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \). To be sure, not every weakly continuous completely positive map of \( \mathcal{B}(H) \) has the form (0.10) (with commuting operators \( T_k \)). Thus there remains an important question concerning the extent to which these results can be generalized to the case of noncommuting \( d \)-tuples of operators and their completely positive maps. Ultimately, there is an associated problem of finding new numerical invariants for noncommutative dynamics, that is, for semigroups of completely positive maps of \( \mathcal{B}(H) \) and their relatives, \( E_0 \)-semigroups. Until now, we have had only the index \([2]\) and the geometric structures constructed in \([3]\).

1. Free Hilbert modules and dilation theory.

The algebra of polynomials \( \mathbb{C}[z_1, \ldots, z_d] \) in \( d \) commuting variables (which we will abbreviate by \( A \) whenever it does not lead to confusion) has a natural inner product which can be defined in several ways. Here, we define this inner product in terms of the relation that exists between \( \mathbb{C}[z_1, \ldots, z_d] \) and the symmetric Fock space. Let \( T_+(E) \) be the symmetric tensor algebra of a \( d \)-dimensional complex vector space \( E \). Writing \( E^n \) for the symmetric tensor product of \( n \) copies of \( E \) for \( n \geq 1 \) (with \( E^0 \) defined as \( \mathbb{C} \)) one finds that the algebraic direct sum of vector spaces

\[
T_+(E) = E^0 + E^1 + E^2 + \ldots
\]

is a commutative algebra with unit with respect to the multiplication defined by symmetric tensoring. It has the following universal property: any linear mapping \( L : E \rightarrow B \) of \( E \) into a unital commutative complex algebra \( B \) extends uniquely to a unital homomorphism of complex algebras \( L : T_+(E) \rightarrow B \). In particular, if we choose a linear basis \( e_1, \ldots, e_d \) for \( E \) then there is a unique homomorphism of unital algebras \( \alpha : T_+(E) \rightarrow \mathbb{C}[z_1, \ldots, z_d] \) defined by \( \alpha(e_k) = z_k \) for \( k = 1, 2, \ldots, d \), and of course in this case \( \alpha \) is an isomorphism which we can use to identify \( T_+(E) \) with \( \mathbb{C}[z_1, \ldots, z_d] \) if we wish.

If we now fix an inner product on the one-particle space \( E \) then \( E \) becomes a finite dimensional Hilbert space, and so does the tensor product \( E^\otimes n \) of \( n \) copies of \( E \) for every \( n = 2, 3, \ldots \). Since the symmetric space \( E^n \) is a subspace of \( E^\otimes n \) for
every \( n \geq 1 \) it follows that \( E^n \) is naturally a Hilbert space; and for \( n = 0 \) we take the usual inner product on \( E^0 = \mathbb{C} \), \( \langle z, w \rangle = z\bar{w} \). Thus, the algebraic direct sum

\[
T_+ (E) = \mathbb{C} \oplus E^1 \oplus E^2 \oplus \ldots
\]

becomes an inner product space, which is dense in the symmetric Fock space over the Hilbert space \( E \).

Now we transport the inner product on \( T_+ (E) \) to an inner product on polynomials by picking an orthonormal basis \( e_1, \ldots, e_d \) for \( E \), and identifying \( T_+ (E) \) with \( \mathbb{C}[z_1, \ldots, z_d] \) by identifying \( e_k \) with \( z_k \) as above. The completion of the polynomials in this inner product is a Hilbert space we denote by \( H^2 (\mathbb{C}^d) \) or simply \( H^2 \) when, as will normally be the case, the dimension \( d \) is fixed.

The elements of \( H^2 \) can be realized as certain holomorphic functions in the open unit ball

\[
B_d = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z| = (|z_1|^2 + \cdots + |z_d|^2)^{1/2} < 1 \}
\]

which satisfy the following growth condition near the boundary

\[
|f(z)| \leq \frac{\|f\|}{\sqrt{1 - |z|^2}}, \quad z \in B_d.
\]

We refer the reader to part I of [1] for other characterizations of this Hilbert norm on polynomials and for a development of the function-theoretic properties of the space \( H^2 \). Here, we summarize a few of its basic features. For every \( \alpha \in B_d \) the function \( u_\alpha : B_d \to \mathbb{C} \) defined by

\[
u_\alpha(z) = \frac{1}{1 - \langle z, \alpha \rangle}, \quad |z| < 1
\]

belongs to \( H^2 \); \( H^2 \) is spanned by \( \{ u_\alpha : \alpha \in B_d \} \) and for every \( \alpha, \beta \in B_d \) we have

\[
\langle u_\alpha, u_\beta \rangle = \frac{1}{1 - \langle \beta, \alpha \rangle},
\]

(1.1.2)

\[
\langle f, u_\alpha \rangle = f(\alpha), \quad f \in H^2.
\]

We also have

\[
\|z_1f_1 + \cdots + z_df_d\|^2 \leq \|f_1\|^2 + \cdots + \|f_d\|^2
\]

for every \( f_1, \ldots, f_d \in H^2 \), so that in fact \( H^2 \) is a contractive Hilbert \( A \)-module. The \( d \)-tuple of operators \( S_1, \ldots, S_d \) obtained by multiplying by the \( d \) coordinate functions define a \( d \)-contraction on \( H^2 \). This \( d \)-contraction is called the \( d \)-shift in [1], and it has the property

\[
S_1S_1^* + \cdots + S_dS_d^* = 1 - E_0
\]

(1.2)

where \( E_0 \) denotes the projection of \( H^2 \) onto the one-dimensional space of constant functions. Using the terminology introduced in the introduction, \( H^2 \) is a contractive Hilbert \( A \)-module of rank one (the rank of Hilbert modules is defined later in this
section). The Hilbert $A$-module $H^2$ will occupy a central position throughout this paper.

We work with the category whose objects are contractive Hilbert modules over the algebra $A = \mathbb{C}[z_1, \ldots, z_d]$ of polynomials, and we will refer to such modules simply as Hilbert $A$-modules. Given two such modules $H_1$, $H_2$, $\text{hom}(H_1, H_2)$ will denote the convex set of all operators $A \in B(H_1, H_2)$ which are contractions ($\|A\| \leq 1$) and which intertwine the respective module actions, $A(f \cdot \xi) = f \cdot A\xi$, $\xi \in H_1$.

Notice that an isomorphism in $\text{hom}(H_1, H_2)$ is necessarily a unitary operator that intertwines the module actions, and when such an operator exists we say that $H_1$ and $H_2$ are isomorphic and write $H_1 \cong H_2$.

There are natural notions of (closed) submodule and quotient module in this category. A closed submodule $K$ of a contractive Hilbert $A$-module $H$ is a contractive Hilbert module. The quotient $H/K$ is of course a Banach space whose norm arises from an inner product on $H/K$; thus $H/K$ is also a contractive Hilbert $A$-module.

In more explicit operator-theoretic terms, let $T_1, \ldots, T_d$ be the canonical operators associated with the Hilbert $A$-module $H$ defined by $T_k \xi = z_k \xi$, $\xi \in H$. Given a closed submodule $K \subseteq H$, the operators $T_1, \ldots, T_d$ can be compressed to the coinvariant subspace $K^\perp \subset H$ to obtain a $d$-contraction $\hat{T}_1, \ldots, \hat{T}_d$ which acts on $K^\perp$ as follows

$$\hat{T}_k \eta = P_{K^\perp} T_d \eta, \quad \eta \in K^\perp,$$

$P_{K^\perp}$ denoting the orthogonal projection of $H$ on $K^\perp$. One finds that the Hilbert $A$-module structure on $K^\perp$ defined by $\hat{T}_1, \ldots, \hat{T}_d$ is isomorphic to the Hilbert $A$ module structure of the quotient $H/K$.

The Hilbert module point of view has been emphasized by Douglas and Paulsen in their work on representations of function algebras [12]. Significantly, the most important $d$-contractions cannot be dilated to normal $d$-contractions because of the failure of the von Neumann inequality for the unit ball in $\mathbb{C}^d$ (see [1, theorem 3.3]). It follows that contractive Hilbert $A$-modules cannot be profitably related to function algebras, and one must give up the idea of working with normal dilations. Instead, one seeks to relate Hilbert $A$ modules to $H^2$ and multiples of $H^2$. This dilation theory was worked out in [1], and is very effective for the category of (contractive) Hilbert $A$-modules. The purpose of this section is to reformulate those operator-theoretic results so that they are closer to the homological spirit of the central issues of this paper.

Suppose we are given a submodule $K \subseteq M$ of a Hilbert $A$-module $M$. The Hilbert modules $H$ which are isomorphic to the quotient $M/K$ are precisely those for which there is an exact sequence of Hilbert $A$-modules

$$0 \rightarrow K \rightarrow M \rightarrow U_H \rightarrow 0$$

where the connecting map $U$ is a coisometry, i.e., $UU^* = 1_H$. This leads us to an important notion.

**Definition 1.4.** Let $H$ be a Hilbert $A$-module. A dilation of $H$ is an exact sequence of Hilbert $A$-modules

$$M \rightarrow U_H \rightarrow 0$$

where $U$ is a coisometry.

Notice that the kernel of $U$ is left unspecified in this definition. Two dilations $M_k \rightarrow U_k \rightarrow 0$, $k = 1, 2$ are said to be equivalent if there is an isomorphism
W : M_1 \to M_2$ of Hilbert $A$-modules such that $U_2W = U_1$. There is no uniqueness of dilations in this generality. Indeed, if $M \xrightarrow[\to_U]{} H \xrightarrow[\to]{} 0$ is any dilation of $H$ and $N$ is an arbitrary Hilbert $A$-module, then we can construct an essentially different dilation

$$M \oplus N \xrightarrow[\to_V]{} H \xrightarrow[\to]{} 0$$

by taking $V$ to be the unique operator from $M \oplus N$ to $H$ which restricts to $U$ on $M$ and to $0$ on $N$.

Before discussing this phenomenon, we collect some terminology that will be used throughout the sequel. Let $H$ be Hilbert $A$-module and let $T_1, \ldots, T_d$ be the canonical operators associated with the module structure of $H$. Since

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2$$

for all $\xi \in H$, it follows that

$$0 \leq T_1T_1^* + \cdots + T_dT_d^* \leq 1_H,$$

hence we can define a positive operator $\Delta$ on $H$ by

$$\Delta = \left(1 - T_1T_1^* - \cdots - T_dT_d^*\right)^{1/2}.$$  

The rank of $H$ is defined as the rank of the operator $\Delta$,

$$\text{rank}(H) = \dim(\Delta).$$

rank$(H)$ can take on any of the values $0, 1, 2, \ldots, \infty$, and we have rank$(H) = 0$ iff $T_1T_1^* + \cdots + T_dT_d^* = 1$.

The operators $T_1, \ldots, T_d$ determine a completely positive map $\phi : B(H) \to B(H)$ by way of

$$\phi(A) = T_1AT_1^* + \cdots + T_dAT_d^*, \quad A \in B(H).$$

$\phi$ is continuous relative to the weak operator topology of $B(H)$, and by virtue of (1.6) we have $\|\phi\| = \|\phi(1)\| \leq 1$. It follows that $1 \geq \phi(1) \geq \phi^2(1) \geq \ldots$ is a decreasing sequence of positive operators and we write $\phi^\infty(1)$ for the limit

$$\phi^\infty(1) = \lim_{n \to \infty} \phi^n(1).$$

Of course, $\phi^\infty(A)$ is undefined for any operator $A$ other than the identity. If $\phi^\infty(1) = 0$ then $H$ is called a pure Hilbert $A$-module. The opposite extreme $\phi^\infty(1) = 1$ occurs only when rank$(H) = 0$.

Finally, the set of all polynomials $\{f(T_1, \ldots, T_d) : f \in A\}$ in the canonical operators $T_1, \ldots, T_d$ of $H$ is a commutative subalgebra of $B(H)$ which contains the identity operator, and we write this algebra of operators as $\text{alg}(H)$. $C^*(H)$ denotes the $C^*$-algebra generated by $\text{alg}(H)$. $C^*(H)$ is irreducible iff $H$ cannot be decomposed into an orthogonal direct sum $H = H_1 \oplus H_2$ of nonzero submodules $H_1, H_2$.

Now let

$$M \xrightarrow[\to_U]{} H \xrightarrow[\to]{} 0$$

be a dilation of $H$, and suppose that $M$ can be decomposed into a direct sum of submodules $M = M_1 \oplus M_2$ where the restriction of $U$ to $M_2$ vanishes. In this case we say that $M_2$ is a trivial summand of the dilation (1.8). For example, in the dilation (1.5), $N$ is a trivial summand.
Proposition 1.9. Let \( M \overset{U}{\longrightarrow} H \longrightarrow 0 \) be a dilation of a Hilbert \( A \)-module \( H \). The following are equivalent.

1. \( M \overset{U}{\longrightarrow} H \longrightarrow 0 \) has no nonzero trivial summands.
2. The set of vectors \( C^*(M)U^*H = \{AU^*\xi : A \in C^*(M), \xi \in H\} \) spans \( M \).
3. For every operator \( A \) in the commutant \( C^*(H)' \) we have \( AU^* = 0 \implies A = 0 \).

If \( M = H^2 \otimes E \) is a free Hilbert \( A \)-module (see the following paragraphs) then these conditions are equivalent to

4. The map of \( E \) to \( H \) defined by \( \zeta \mapsto U(1 \otimes \zeta) \) is one-to-one.

proof. This is fundamentally a restatement of the equivalence of properties (8.4.1), (8.4.1) and (8.4.3) of [1]. \( \qed \)

Definition 1.10. A dilation \( M \overset{U}{\longrightarrow} H \longrightarrow 0 \) of \( H \) is called minimal if it satisfies the conditions of Proposition 1.9.

Remark. It is clear from property (2) of Proposition 1.9 that any dilation \( M \overset{U}{\longrightarrow} H \longrightarrow 0 \) can be reduced to a minimal one \( M_0 \overset{U_0}{\longrightarrow} H \longrightarrow 0 \) by replacing \( M \) with the submodule \( M_0 = [C^*(M)U^*H] \) and \( U \) with its restriction to \( M_0 \).

We now summarize the main results on the existence and uniqueness of non-normal dilations for Hilbert \( A \)-modules. A free Hilbert \( A \)-module is a finite or countably infinite direct sum of copies of the rank-one module \( H^2 \). We write \( n \cdot H^2 \) for the direct sum of \( n \) copies of \( H^2 \), \( n = 1, 2, \ldots, \infty \). \( n \) is uniquely determined by the module \( n \cdot H^2 \); indeed, a simple computation (which we omit) shows that \( \text{rank}(n \cdot H^2) = n \) for every \( n \). Thus we will refer to \( n \cdot H^2 \) as the free Hilbert \( A \)-module of rank \( n = 1, 2, \ldots, \infty \). If \( E \) is a Hilbert space of dimension \( n \) and we make the Hilbert space \( H^2 \otimes E \) into a Hilbert \( A \)-module by setting \( f(g \otimes \zeta) = fg \otimes \zeta, \quad f \in A, \quad g \in H^2, \quad \zeta \in E \) then \( H^2 \otimes E \) is isomorphic to \( n \cdot H^2 \).

At the other extreme, a Hilbert \( A \)-module \( H \) is called spherical if its canonical operators \( \{T_1, \ldots, T_d\} \) are jointly normal in the sense that \( \{T_1, \ldots, T_d, T_1^*, \ldots, T_d^*\} \) is a commuting set of operators, and in addition satisfy \( T_1T_1^* + \cdots + T_dT_d^* = 1 \).
Spherical $d$-tuples $(T_1, \ldots, T_d)$ are the higher dimensional counterparts of unitary operators. One cannot avoid spherical modules in nonnormal dilation theory, and they represent a kind of degeneracy. By a standard Hilbert $A$-module we mean a direct sum of Hilbert $A$-modules of the form $H = F \oplus S$ where $F$ is free and $S$ is spherical. One or the other summand may be absent. A standard dilation of $H$ is a dilation

$$ M \xrightarrow{U} H \rightarrow 0 $$

in which $M$ is a standard module. Theorem 8.5 of [1] can now be reformulated as follows.

**Theorem 1.11.** Every Hilbert $A$-module $H$ has a minimal standard dilation

$$ F \oplus S \xrightarrow{U} H \rightarrow 0, $$

and any two such are equivalent. If

$$ F' \oplus S' \xrightarrow{U'} H \rightarrow 0 $$

is a second standard dilation then every isomorphism $W : F \oplus S \rightarrow F' \oplus S'$ satisfying $U'W = U$ decomposes into a direct sum $W = W_1 \oplus W_2$, where $W_1$ is an isomorphism of the free summands and $W_2$ is an isomorphism of the spherical summands.

There is somewhat more information available from [1, Theorem 8.5] concerning criteria for when one or the other of the summands $F$ or $S$ is missing.

**Theorem 1.12.** Let $H$ be a Hilbert $A$-module.

1. The minimal standard dilation of $H$ is free of rank $n = 1, 2, \ldots, \infty$

$$ n \cdot H^2 \xrightarrow{U} H \rightarrow 0 $$

iff $H$ is pure of rank $n$.

2. The minimal standard dilation of $H$ is spherical

$$ S \xrightarrow{U} H \rightarrow 0 $$

iff $\phi(1) = T_1 T_1^* + \cdots + T_d T_d^* = 1$ (i.e., iff $\text{rank}(H) = 0$).

Finally, we require more explicit information about the minimal standard dilation of a Hilbert $A$-module $H$

$$ F \oplus S \xrightarrow{U} H \rightarrow 0 $$

than is apparent from Theorems 1.11 and 1.12. Indeed, there is an explicit formula for $F$ and for the restriction of $U$ to $F$ which is described as follows. Consider the module $F = H^2 \otimes \Delta H$, where the $A$-module structure is defined by

$$ f \cdot (g \otimes \zeta) = fg \otimes \zeta, \quad f, g \in A, \zeta \in \Delta H. $$

We have already pointed out that $F = H^2 \otimes \Delta H$ is a free Hilbert $A$-module of rank $r = \text{rank}(H)$. Moreover, Theorem 4.5 of [1] implies that there is a unique bounded operator $U_0 : H^2 \otimes \Delta H \rightarrow H$ satisfying

$$ U_0(f \otimes \zeta) = f \cdot \Delta \zeta, \quad f \in A, \zeta \in \Delta H. $$
$U_0$ is obviously a homomorphism of Hilbert $A$-modules, and the proof of Theorem 4.5 of [1] shows that $U_0 U_0^* = 1_H - \phi^\infty (1_H)$. In particular, if $H$ is pure then $U_0$ is a coisometry and

\[(1.13) \quad H^2 \otimes \overline{\Delta H} \xrightarrow{U_0} H \rightarrow 0\]

provides a standard dilation of $H$ which has no spherical summand. Indeed, in this case (1.13) is actually the \textit{minimal} standard dilation (condition (4) of Proposition 1.9 is obviously satisfied).

If $H$ is not pure then (1.13) is not a standard dilation (nor is it even a dilation); but one can make it so by adding an appropriate spherical summand. The details are as follows.

\textbf{Theorem 1.14.} Let $H$ be a Hilbert $A$-module, let $F = H^2 \otimes \Delta H$, and let $U_0 : F \rightarrow H$ be the morphism defined in (1.13). Then there is a spherical module $S$ and a morphism $U_1 : S \rightarrow H$ such that $U_1 U_1^* = \phi^\infty (1_H)$, and such that if $U : F \oplus S \rightarrow H$ is defined by

\[U(\xi, \eta) = U_0 \xi + U_1 \eta\]

then

\[F \oplus S \xrightarrow{U} H \rightarrow 0\]

is the minimal standard dilation of $H$.

\textit{proof.} Let

\[F \oplus S \xrightarrow{V} H \rightarrow 0\]

be a minimal standard dilation of $H$. Given the formulas of the preceding paragraph, it suffices to show that the restriction of $V$ to $F$ can be identified with $U_0$ in the sense that there is an isomorphism of Hilbert $A$-modules $W : F \rightarrow H^2 \otimes \overline{\Delta H}$ such that $V \uparrow_F = U_0 W$. Now both $F$ and $H^2 \otimes \overline{\Delta H}$ are free modules of the same rank $r = \text{rank}(H)$, and thus we may identify $F$ with $H^2 \otimes \overline{\Delta H}$.

Having made that identification, let $V_0$ be the restriction of $V$ to the free summand $F = H^2 \otimes \overline{\Delta H}$ and consider the linear operator $A : \overline{\Delta H} \rightarrow H$ defined by

\[A \zeta = V_0 (1 \otimes \zeta),\]

1 denoting the constant function in $H^2$. We claim that $A$ has trivial kernel and $AA^* = \Delta^2$. Granting that for a moment, the polar decomposition provides a unitary operator $W_0 \in B(\overline{\Delta H})$ such that $A = \Delta W_0$, hence

\[V_0 (f \otimes \zeta) = f \cdot V_0 (1 \otimes \zeta) = f \cdot A \zeta = f \cdot \Delta W_0 \zeta = U_0 (f \otimes W_0 \zeta) = U_0 ((1_{H^2} \otimes W_0)(f \otimes \zeta)),\]

thus $W = 1_{H^2} \otimes W_0$ provides the required automorphism of $H^2 \otimes \overline{\Delta H}$ satisfying $V_0 = U_0 W$.

To see that $A$ is injective, pick $\zeta \in \overline{\Delta H}$ such that $A \zeta = 0$, and consider the operator $Z$ defined on the dilation module $M = H^2 \otimes \overline{\Delta H} \oplus S$ by

\[Z = 1_{H^2} \otimes (\zeta \otimes \overline{\zeta}) \oplus 0,\]
\( \zeta \otimes \bar{\zeta} \) denoting the rank-one operator on \( \Delta H \) associated with the vector \( \zeta \). \( Z \) is a self-adjoint operator in the commutant of \( C^*(M) \), and since \( V \) maps the vector \((1 \otimes \zeta, 0) \in M \) to \( A\zeta = 0 \) we have \( VZ = 0 \) and hence \( ZV^* = 0 \). By the third condition of Proposition 1.9, \( Z \) must be the zero operator and hence \( \zeta = 0 \).

Finally, we show that \( AA^* = \Delta^2 \). For that, let \( T_1, \ldots, T_d \) (resp. \( \tilde{T}_1, \ldots, \tilde{T}_d \)) be the canonical operators associated with the Hilbert module \( H \) (resp. \( M \)). Since \( M = H^2 \otimes \Delta H \oplus S \) and \( S \) is spherical, we see from (1.2) that the projection of \( M \) onto the subspace \( 1 \otimes \Delta H \oplus 0 \) is given by \( 1_M - \sum_{k=1}^d \tilde{T}_k \tilde{T}_k^* \). Since \( VV^* = 1_H \) and \( V\tilde{T}_k = T_k V \) for every \( k = 1, \ldots, d \) we have

\[
V(\sum_{k=1}^d \tilde{T}_k \tilde{T}_k^*)V^* = \sum_{k=1}^d T_k VV^* T_k^* = \sum_{k=1}^d T_k T_k^*,
\]

hence

\[
AA^* = V(1_M - \sum_{k=1}^d \tilde{T}_k \tilde{T}_k^*)V^* = VV^* - \sum_{k=1}^d T_k T_k^* = \Delta^2
\]
as required.

1.15. Dilations and free resolutions. We conclude this section with some remarks concerning free resolutions of pure Hilbert modules. Theorem 1.12 shows that \( H^2 \) and its multiples (free modules) occupy a key position in nonnormal dilation theory, in that a Hilbert module \( H \) is isomorphic to some quotient \( F/K \) of a free module \( F \) iff \( H \) is pure. This leads to the existence (and uniqueness) of minimal free resolutions in the category of pure Hilbert \( A \)-modules. More precisely, suppose we start with a pure Hilbert module \( H \). Applying Theorem 1.12 we find a minimal dilation of \( H \) of the form

\[
F \to H \to 0
\]

where \( F \) is free. Let \( K \subseteq F \) be the kernel of this dilation map. Free modules are pure, and submodules of pure modules are pure. Hence, we can reapply Theorem 1.12 to \( K \) itself and continue the sequence one step to the left. We then repeat the process on the kernel of the resulting map, continuing indefinitely or until some kernel \( K \) is zero.

Thus, every pure Hilbert \( A \)-module has a minimal free resolution, that is, there is an exact sequence of Hilbert \( A \)-modules

\[
\cdots \to F_3 \to F_2 \to F_1 \to H \to 0
\]

where each \( F_k \) is a free module and where each connecting map is a partial isometry which satisfies the conditions of Proposition 1.9. This minimal free resolution is unique in the sense that if

\[
\cdots \to F'_3 \to F'_2 \to F'_1 \to H \to 0
\]
is another one then after working from right to left with Theorem 1.11 we find that there is a sequence of isomorphisms \( W_k : F_k \to F'_k \) such that each subdiagram

\[
\begin{array}{ccc}
F_{k+1} & \longrightarrow & F_k \\
W_{k+1} \downarrow & & \downarrow W_k \\
F'_{k+1} & \longrightarrow & F'_k
\end{array}
\]
commutes.

Naturally, one would like to know if the resolution (1.16) is of finite length in the sense that \( F_k = 0 \) for sufficiently large \( k \). Moreover, if we start with a pure module \( H \) of finite rank, then by analogy with Hilbert’s syzygy theorem one might hope that the free resolution of \( H \) is a) of finite length and b) each of the free modules \( F_k \) is of finite rank. Unfortunately, nothing like that is true in this category. We will show in section 9 that every pure \( \text{graded} \) Hilbert \( A \)-module \( H \) of rank one which is not already isomorphic to \( H^2 \) has a minimal free resolution of the form

\[
\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow H \rightarrow 0
\]

where \( F_1 \) is isomorphic to \( H^2 \) but where \( F_2 \) is free of infinite rank. Thus, Hilbert’s syzygy theorem fails rather spectacularly for Hilbert \( A \)-modules.

Nevertheless, we will find in section 3 that algebraic free resolutions do play a central role in the theory of Hilbert \( A \)-modules.

2. Multipliers of free Hilbert modules.

Elements of free modules, and homomorphisms from one free module to another, can be “evaluated” at points in the open unit ball \( B_d \) in \( \mathbb{C}^d \). The purpose of this section is to discuss these evaluation maps and the relation between morphisms and multipliers. Throughout the section, \( \{ u_z : z \in B_d \} \) will denote the set of functions in \( H^2 \) defined in section 1 (see (1.1.1) and (1.1.2)).

Let \( E \) be a separable Hilbert space and consider the free Hilbert \( A \)-module \( F = H^2 \otimes E \) of rank \( r = \dim E \), where the module structure is defined by

\[
f \cdot (g \otimes \zeta) = fg \otimes \zeta, \quad f, g \in A, \zeta \in E.
\]

One can think of elements of \( H^2 \otimes E \) as \( E \)-valued holomorphic functions defined on \( B_d \) in the following way.

**Proposition 2.1.** For every element \( \xi \in H^2 \otimes E \) and every \( z \in B_d \) there is a unique vector \( \hat{\xi}(z) \in E \) satisfying

\[
\langle \hat{\xi}(z), \zeta \rangle = \langle \xi, u_z \otimes \zeta \rangle, \quad \zeta \in E.
\]

The function \( \hat{\xi} : B_d \to E \) is (weakly) holomorphic and satisfies

\[
\| \hat{\xi}(z) \| \leq \frac{\| \xi \|}{\sqrt{1 - |z|^2}}, \quad z \in B_d.
\]

**proof.** The argument is straightforward and we merely sketch the details. Since

\[
\| u_z \otimes \zeta \| = \| u_z \| \cdot \| \zeta \| = \frac{\| \zeta \|}{\sqrt{1 - |z|^2}}
\]

it follows that for fixed \( z \in B_d \), \( \zeta \mapsto \langle \xi, u_z \otimes \zeta \rangle \) is a bounded antilinear functional on \( E \). By the Riesz lemma there is a unique vector \( \hat{\xi}(z) \in E \) such that

\[
\langle \hat{\xi}(z), \zeta \rangle = \langle \xi, u_z \otimes \zeta \rangle, \quad \zeta \in E,
\]
and one has \( \| \hat{\xi}(z) \| \leq \| \xi \| (1 - |z|^2)^{-1/2} \). By the definition of \( u_z, \ z \mapsto \langle \xi, u_z \otimes \zeta \rangle \) is holomorphic in \( B_d \), hence \( \hat{\xi} : B_d \to E \) is weakly holomorphic. The estimate (2.2) is immediate from the definition of \( \hat{\xi} \).

**Remarks.** We lighten notation by writing \( \xi(z) \) for the value of \( \xi \) at \( z \), rather than the more pedantic \( \hat{\xi}(z) \). Notice that the \( A \)-module structure of \( H^2 \otimes E \) is expressed conveniently in terms of the values of \( \xi \) as follows

\[
(f \cdot \xi)(z) = f(z)\xi(z), \quad f \in A, \quad \xi \in H^2 \otimes E, \quad z \in B_d.
\]

Notice too that an element \( \xi \in H^2 \otimes E \) is uniquely determined by its functional representative \( z \in B_d \mapsto \xi(z) \) because \( \{u_z : z \in B_d\} \) spans \( H^2 \) [1], hence \( \{u_z \otimes \zeta : z \in B_d, \zeta \in E\} \) spans \( H^2 \otimes E \), and hence \( \xi(z) = 0 \) for all \( z \in B_d \) only when \( \xi \) is the zero element of \( H^2 \otimes E \).

Similarly, any bounded homomorphism of free modules can be evaluated at points of the unit ball so as to obtain a multiplier. In more detail, let \( E_1 \) and \( E_2 \) be two separable Hilbert spaces and let \( \Phi : B_d \to \mathcal{B}(E_1, E_2) \) be an operator-valued function defined on the open unit ball. We will say that \( \Phi \) is a **multiplier** if there is a bounded linear operator \( \hat{\Phi} : H^2 \otimes E_1 \to H^2 \otimes E_2 \) such that

\[
\Phi(z)\xi(z) = (\hat{\Phi}\xi)(z), \quad \xi \in H^2 \otimes E_1, \quad z \in B_d.
\]

The multiplier norm of \( \Phi \) is defined as the operator norm

\[
\| \Phi \|_{\mathcal{M}} = \| \hat{\Phi} \| = \sup_{\| \xi \|_{H^2 \otimes E_1} \leq 1} \| \hat{\Phi}\xi \|_{H^2 \otimes E_2}.
\]

Notice that the operator \( \hat{\Phi} \) is a homomorphism of the \( A \)-module structure of \( H^2 \otimes E_1 \) to that of \( H^2 \otimes E_2 \), since by the preceding remarks for \( f \in A \) and \( \xi \in H^2 \otimes E_1 \) we have

\[
\hat{\Phi}(f \cdot \xi)(z) = \Phi(z)((f \cdot \xi)(z)) = \Phi(z)(f(z)\xi(z)) = f(z)\Phi(z)\xi(z)
\]

\[
= f(z)\hat{\Phi}(\xi)(z) = (f \cdot \hat{\Phi}(\xi))(z)
\]

for all \( z \in B_d \), hence \( \Phi(f \cdot \xi) = f \cdot \hat{\Phi}(\xi) \). The space of all such multipliers is denoted \( \mathcal{M}(E_1, E_2) \), and we again simplify notation by dropping the circumflex over the operator \( \Phi \).

Now let \( E_1, E_2 \) be two separable Hilbert spaces. A bounded linear operator \( \Phi : H^2 \otimes E_1 \to H^2 \otimes E_2 \) satisfying \( \Phi(f \cdot \xi) = f \cdot \Phi(\xi) \) will be called a **homomorphism**, and in this section only the Banach space of all such will be written \( \mathcal{H}om(H^2 \otimes E_1, H^2 \otimes E_2) \) (recall that we have reserved the notation \( \text{hom}(H, K) \) for spaces of homomorphisms having norm at most 1).

Fix \( \Phi \in \mathcal{H}om(H^2 \otimes E_1, H^2 \otimes E_2) \). Then for every \( z \) in the open unit ball \( B_d \) we can define an operator \( \Phi(z) \in \mathcal{B}(E_1, E_2) \) by

\[
\langle \Phi(z)\zeta_1, \zeta_2 \rangle = \langle \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle, \quad \zeta_k \in E_k;
\]

indeed, this follows from an application of the Riesz lemma after taking note of the obvious estimate of the term on the right

\[
\left| \langle \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle \right| \leq \frac{\| \Phi \| \cdot \| \zeta_1 \| \cdot \| \zeta_2 \|}{\sqrt{1 - |z|^2}}.
\]
Proposition 2.4. For every homomorphism $\Phi : H^2 \otimes E_1 \rightarrow H^2 \otimes E_2$, the operator function $z \in B_d \mapsto \Phi(z) \in B(E_1, E_2)$ defined by (2.3) belongs to the multiplier space $M(E_1, E_2)$ and its associated operator is $\Phi$. Moreover,

1. $\sup_{z \in B} \|\Phi(z)\| \leq \|\Phi\| = \|\Phi\|_M$, and
2. the adjoint $\Phi^* \in B(H^2 \otimes E_2, H^2 \otimes E_1)$ of the operator $\Phi$ is related to the operator function $z \in B_d \mapsto \Phi(z)^* \in B(E_2, E_1)$ as follows,

$$\Phi^*(u_z \otimes \zeta) = u_z \otimes \Phi(z)^* \zeta, \quad z \in B_d, \quad \zeta \in E_2.$$

proof. We claim first that for every $\zeta_2 \in E_2$ and every $\xi \in H^2 \otimes E_1$ of the form $\xi = f \otimes \zeta$ we have

$$\langle \Phi(z)\xi(z), \zeta_2 \rangle = \langle \Phi(\xi), u_z \otimes \zeta_2 \rangle.$$ \hspace{1cm} (2.5)

Indeed, since $\xi(z) = f(z)\zeta_2$ the left side of (2.5) is $f(z) \langle \Phi(z)\zeta_1, \zeta_2 \rangle$ while since

$$\Phi(\xi) = \Phi(f \cdot (1 \otimes \zeta_1)) = f \cdot \Phi(1 \otimes \zeta_1),$$

the right side of (2.5) is

$$\langle f \cdot \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle = \langle f(z)\Phi(1 \otimes \zeta_1)(z), \zeta_2 \rangle = f(z) \langle \Phi(z)\zeta_1, \zeta_2 \rangle.$$

Hence (2.5) holds for elementary tensors $\xi$. Now for an arbitrary $\xi \in H^2 \otimes E_1$ the left side of (2.5) is bounded by

$$|\langle \Phi(z)\xi(z), \zeta_2 \rangle| \leq \|\Phi(z)\| \cdot \|\xi(z)\| \cdot \|\zeta_2\| \leq \|\Phi\| \cdot \|\xi\| \cdot \|\zeta_2\| \sqrt{1 - |z|^2},$$

and the right side is bounded by

$$|\langle \Phi(\xi), u_z \otimes \zeta_2 \rangle| \leq \|\Phi\| \cdot \|\xi\| \cdot \|u_z \otimes \zeta_2\| \leq \|\Phi\| \cdot \|\xi\| \cdot \|\zeta_2\| \sqrt{1 - |z|^2}.$$

Since $H^2 \otimes E_1$ is spanned by elements of the form $f \otimes \zeta_1$, (2.5) follows in general.

Now by definition of $\Phi(\xi)(z)$ the right side of (2.5) is $\langle \Phi(\xi)(z), \zeta_2 \rangle$, and since $\zeta_2$ is arbitrary in (2.5) we conclude that $\Phi(\xi)(z) = \Phi(z)\xi(z)$. Hence the function $\Phi(\cdot)$ is a multiplier with associated operator $\Phi \in B(h^2 \otimes E_1, H^2 \otimes E_2)$.

We now verify formula (2) of Proposition 2.4. Fix $f \in H^2$, $\zeta_k \in E_k$, $k = 1, 2$, and $z \in B_d$. Then we have

$$\langle f \otimes \zeta_1, \Phi^*(u_z \otimes \zeta_2) \rangle = \langle \Phi(f \otimes \zeta_1), u_z \otimes \zeta_2 \rangle = \langle f \cdot \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle$$
$$= \langle f(z)\Phi(1 \otimes \zeta_1)(z), \zeta_2 \rangle = f(z) \langle \Phi(z)\zeta_1, \zeta_2 \rangle = \langle f, u_z \rangle \langle \zeta_1, \Phi(z)^* \zeta_2 \rangle$$
$$= \langle f \otimes \zeta_1, u_z \otimes \Phi(z)^* \zeta_2 \rangle.$$

Since $H^2 \otimes E_1$ is spanned by vectors of the form $f \otimes \zeta_1$, the required formula (2) follows.

To prove (1) of Proposition 2.4 it suffices to show that for every $\zeta_k \in E_k$, $k = 1, 2$ with $\|\zeta_k\| \leq 1$ we have $|\langle \Phi(z)\zeta_1, \zeta_2 \rangle| \leq \|\Phi\|$. For that, write

$$(1 - |z|^2)^{-1} |\langle \Phi(z)\zeta_1, \zeta_2 \rangle| = \|u_z\|^2 |\langle \zeta_1, \Phi(z)^* \zeta_2 \rangle| = |\langle u_z \otimes \zeta_1, u_z \otimes \Phi(z)^* \zeta_2 \rangle|.$$
By the formula (2) just established, the right side is
\[ | \langle u_z \otimes \zeta_1, \Phi^*(u_z \otimes \zeta_2) \rangle | \leq \| u_z \| \| \Phi^* \| \left( 1 - |z|^2 \right)^{-1} \| \Phi \|, \]
from which the assertion of (1) follows.

Remarks. Experience with one-dimensional operator theory might lead one to expect that the space of all multipliers \( \mathcal{M}(E_1, E_2) \) should coincide with the space \( H^\infty(E_1, E_2) \) of all bounded holomorphic operator valued functions
\[ F : B_d \to \mathcal{B}(E_1, E_2). \]
However, the failure of von Neumann’s inequality for the ball \( B_d \) in dimension \( d \geq 2 \) (cf. [1], Theorem 3.3) implies that this is far from true. Indeed, if we consider the simplest case in which both spaces \( E_1 = E_2 = \mathbb{C} \) consist of scalars, then \( \mathcal{M}(\mathbb{C}, \mathbb{C}) = \mathcal{M} \) is the multiplier algebra introduced in [1], and it was shown there that \( \mathcal{M} \) is a proper subalgebra of the algebra \( H^\infty \) of all bounded holomorphic functions defined on the open unit ball in \( \mathbb{C}^d \) when \( d \geq 2 \). Indeed, examples are given in ([1], Theorem 3.3) of continuous functions defined on the closed unit ball \( f : B_d \to \mathbb{C} \) which are holomorphic in the interior \( B_d \), but which are not multipliers.

We conclude this section with a few remarks about boundary values. Let \( \sigma \) denote the natural normalized measure on the unit sphere \( \partial B_d \) in complex \( d \)-space, and let \( H^2(\partial B_d) \) denote the multivariate “Hardy space” defined as the closure in \( L^2(\partial B_d) \) of the holomorphic polynomials. Every element \( \tilde{f} \) of \( H^2(\partial B_d) \) has a natural holomorphic extension \( f \) to the interior the ball \( B_d \), and for \( \sigma \)-almost every \( z \in \partial B_d \) we have
\[ \lim_{r \uparrow 1} f(rz) = \tilde{f}(z). \]
Moreover,
\[ \lim_{r \uparrow 1} \int_{\partial B_d} |f(rz) - \tilde{f}(z)|^2 d\sigma(z) = 0 \]
(for example, see [26]).

These properties generalize to vector-valued functions as follows. Let \( E \) be a separable Hilbert space and let \( H^2(\partial B_d; E) \) denote the closure in \( L^2(\partial B_d; E) = L^2(\partial B_d) \otimes E \) of the linear span of all vector polynomials of the form \( f \otimes \zeta \), with \( f \in A \) and \( \zeta \in E \). Elements \( \tilde{\xi} \) of \( H^2(\partial B_d; E) \) extend in a similar way to holomorphic functions \( f : B_d \to E \) and there is a Borel set \( N \subseteq \partial B_d \) of measure zero such that for all \( z \in \partial B_d \setminus N \) we have
\[ (2.6) \quad \lim_{r \uparrow 1} \| \xi(rz) - \tilde{\xi}(z) \| = 0, \]
and moreover
\[ (2.7) \quad \lim_{r \uparrow 1} \int_{\partial B_d} \| \xi(rz) - \tilde{\xi}(z) \|^2 d\sigma(z) = 0; \]
for example, one can establish this by making use of the radial maximal function, see 5.4.11 of [26].

The preceding remarks lead immediately to the following conclusion about the boundary values of multipliers.
Proposition 2.8. Let $\Phi \in \mathcal{M}(E_1, E_2)$ be the multiplier of a homomorphism $\Phi$ in $\mathcal{H}om(H^2 \otimes E_1, H^2 \otimes E_2)$. Then for $\sigma$-almost every $z \in \partial B_d$, the net of operators $r \in (0, 1) \mapsto \Phi(rz) \in \mathcal{B}(E_1, E_2)$ is uniformly bounded and converges in the strong operator topology of $\mathcal{B}(E_1, E_2)$ to an operator $\tilde{\Phi}(z)$. The operator function $\tilde{\Phi} : \partial B_d \to \mathcal{B}(E_1, E_2)$ belongs to $L^\infty(\partial B_d; \mathcal{B}(E_1, E_2))$ and satisfies

$$\text{ess sup } \|\tilde{\Phi}(z)\| = \sup_{|z|<1} \|\Phi(z)\| \leq \|\Phi\|_{\mathcal{M}}.$$

proof. The argument is straightforward, and we merely sketch the details. For fixed $\zeta \in E_1$, consider the holomorphic vector-valued function $\xi : B_d \to E_2$ defined by $\xi(z) = \Phi(z)\zeta$. $\xi$ satisfies

$$\sup_{|z|<1} \|\xi(z)\| \leq \|\Phi(z)\|_{\mathcal{M}} \cdot \|\zeta\| < \infty$$

and therefore it is the restriction to $B_d$ of a unique element $\tilde{\xi} \in H^2(\partial B_d; E_2)$. Moreover, we have

$$\lim_{r \to 1} \|\xi(rz) - \tilde{\xi}(z)\| = 0$$

for $\sigma$-almost every $z \in \partial B_d$. Since $E_1$ is separable, a standard argument shows that the exceptional set can be made independent of $\zeta \in E_1$, and for all such nonexceptional points $z \in \partial B_d$ the net of operators $r \mapsto \Phi(rz)$ is strongly convergent to a limit operator $\tilde{\Phi}(z)$ satisfying 2.9.

3. Euler characteristic.

Throughout this section, $H$ will denote a finite rank Hilbert $A$-module. We will work not with $H$ itself but with the following linear submanifold of $H$

$$M_H = \text{span}\{ f \cdot \Delta \xi : f \in A, \xi \in H\}.$$ 

The definition and basic properties of the Euler characteristic are independent of any topology associated with the Hilbert space $H$, and depend solely on the linear algebra of $M_H$. As we have pointed out in the introduction, $M_H$ is a finitely generated $A$-module, and has finite free resolutions in the category of finitely generated $A$-modules

$$0 \to F_n \to \cdots \to F_2 \to F_1 \to M_H \to 0,$$

each $F_k$ being a sum of $\beta_k$ copies of the rank-one module $A$. The alternating sum of the ranks $\beta_1 - \beta_2 + \beta_3 - + \cdots$ does not depend on the particular free resolution of $M_H$, and we define the Euler characteristic of $H$ by

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k.$$

The main result of this section is an asymptotic formula (Theorem C) which expresses $\chi(H)$ in terms of the sequence of defect operators $1 - \phi^{n+1}(1)$, $n = 1, 2, \ldots$, where $\phi$ is the completely positive map on $\mathcal{B}(H)$ associated with the canonical operators $T_1, \ldots, T_d$ of $H$,

$$\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*.$$
The Hilbert polynomial is an invariant associated with finitely generated graded modules over polynomial rings \(k[x_1, \ldots, x_d]\), \(k\) being an arbitrary field. We require something related to the Hilbert polynomial, which exists in greater generality than the former, but whose existence can be deduced rather easily from Hilbert’s original work \([16], [17]\). While this polynomial is very fundamental (indeed, its existence might be described as \textit{the} fundamental result of multivariable linear algebra), it is less familiar to analysts than it is to algebraists.

We define this polynomial in a way suited to our needs below. It is convenient to work in terms of the following sequence of polynomials \(q_0, q_1, \ldots \in \mathbb{Q}[x]\), which are normalized so that \(q_k(0) = 1\), and which are defined recursively by

\begin{align*}
q_0(x) &= 1, \\
q_k(x) - q_k(x - 1) &= q_{k-1}(x), \quad k \geq 1.
\end{align*}

One finds that for \(k \geq 1\),

\[ q_k(x) = \frac{(x+1)(x+2)\ldots(x+k)}{k!}. \]

When \(x = n\) is a positive integer, \(q_k(n)\) is the binomial coefficient \(\binom{n+k}{n}\), and more generally \(q_k(\mathbb{Z}) \subseteq \mathbb{Z}, k = 0, 1, 2, \ldots\).

\textbf{Theorem 3.4.} Let \(V\) be a vector space over a field \(k\), let \(T_1, \ldots, T_d\) be a commuting set of linear operators on \(V\), and make \(V\) into a \(k[x_1, \ldots, x_d]\)-module by setting \(f \cdot \xi = f(T_1, \ldots, T_d)\xi, f \in k[x_1, \ldots, x_d], \xi \in V\).

Let \(G\) be a finite dimensional subspace of \(V\) and, for every \(n = 0, 1, 2, \ldots\) define a finite dimensional subspace \(M_n\) by

\[ M_n = \text{span}\{ f : \xi : f \in k[x_1, \ldots, x_d], \deg f \leq n, \xi \in G \}. \]

Then there are integers \(c_0, c_1, \ldots, c_d \in \mathbb{Z}\) and \(N \geq 1\) such that for all \(n \geq N\) we have

\[ \dim M_n = c_0 q_0(n) + c_1 q_1(n) + \cdots + c_d q_d(n). \]

In particular, the dimension function \(n \mapsto \dim M_n\) is a polynomial for sufficiently large \(n\).

\textbf{proof.} We may obviously assume that \(V = \cup_n M_n\), and hence \(V\) is a finitely generated \(k[x_1, \ldots, x_d]\)-module. The fact that the function \(n \mapsto \dim M_n\) is a polynomial of degree at most \(d\) for sufficiently large \(n\) follows from the result in section 8.4.5 of \([19]\); and the specific form of this polynomial follows from the discussion in \([19]\), section 8.4.4.

\textbf{Remark 3.4.} We emphasize that the dimension function \(n \mapsto \dim M_n\) is generally not a polynomial for all \(n \in \mathbb{N}\), but only for sufficiently large \(n \in \mathbb{N}\).

We also point out for the interested reader that one can give a relatively simple direct proof of Theorem 3.4 by an inductive argument on the number \(d\) of operators, along lines similar to the proof of Theorem 1.11 of \([14]\).

Suppose now that \(G\) is a finite dimensional subspace of \(V\) which \textit{generates} \(V\) as a \(k[x_1, \ldots, x_d]\)-module

\[ V = \text{span}\{ f : \xi : f \in k[x_1, \ldots, x_d], \xi \in G \}. \]
The polynomial
\[ p(x) = c_0 q_0(x) + c_1 q_1(x) + \cdots + c_d q_d(x) \]
defined by theorem 3.4 obviously depends on the generator \( G \); however, its top coefficient \( c_d \) does not. In order to discuss that, it is convenient to broaden the context somewhat. Let \( M \) be a module over the polynomial ring \( k[x_1, \ldots, x_d] \). A filtration of \( M \) is an increasing sequence \( M_1 \subseteq M_2 \subseteq \ldots \) of finite dimensional linear subspaces of \( M \) such that
\[
M = \bigcup_n M_n \quad \text{and} \quad x_k M_n \subseteq M_{n+1}, \quad k = 1, 2, \ldots, d, \quad n \geq 1.
\]
The filtration \( \{M_n\} \) is called proper if there is an \( n_0 \) such that
\[
M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_n, \quad n \geq n_0.
\]

**Proposition 3.7.** Let \( \{M_n\} \) be a proper filtration of \( M \). Then the limit
\[
c = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d}
\]
exists and defines a nonnegative integer \( c = c(M) \) which is the same for all proper filtrations.

**proof.** Let \( \{M_n\} \) be a proper filtration, choose \( n_0 \) satisfying (3.5), and let \( G \) be the generating subspace \( G = M_{n_0} \). One finds that for \( n = 0, 1, 2, \ldots \)
\[
M_{n_0+n} = \text{span}\{f \cdot \xi : \deg f \leq n, \quad \xi \in M_{n_0}\}
\]
and hence there is a polynomial \( p(x) \in \mathbb{Q}[x] \) of the form stipulated in Theorem 3.4 such that \( \dim M_{n_0+n} = p(n) \) for sufficiently large \( n \). Writing
\[
p(x) = c_0 q_0(x) + c_1 q_1(x) + \cdots + c_d q_d(x)
\]
and noting that \( q_k \) is a polynomial of degree \( k \) with leading coefficient \( 1/k! \), we find that
\[
c_d = d! \lim_{n \to \infty} \frac{p(n)}{n^d} = d! \lim_{n \to \infty} \frac{\dim M_{n_0+n}}{n^d} = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d},
\]
as asserted.

Now let \( \{M_n'\} \) be another proper filtration. Since \( M = \bigcup_n M_n' \) and \( M_{n_0} \) is finite dimensional, there is an \( n_1 \in \mathbb{N} \) such that \( M_{n_0} \subseteq M'_{n_1} \). Since \( \{M_n'\} \) is also proper we can increase \( n_1 \) if necessary to arrange the condition of (3.5) on \( M_n' \) for all \( n \geq n_1 \), and hence
\[
M'_{n_1+n} = \text{span}\{f \cdot \xi : \deg f \leq n, \quad \xi \in M'_{n_1}\}.
\]
Letting \( c'_d \) be the top coefficient of the polynomial \( p'(x) \) satisfying
\[
\dim M'_{n_1+n} = p'(n)
\]
for sufficiently large $n$, the preceding argument shows that

$$c'_d = d! \lim_{n \to \infty} \frac{\dim M'_n}{n^d}.$$ 

On the other hand, the inclusion $M_{n_0} \subseteq M'_{n_1}$, together with the condition (3.5) on both $\{M_n\}$ and $\{M'_n\}$, implies

$$M_{n_0+n} = \text{span}\{f \cdot \xi : \deg f \leq n, \quad \xi \in M_{n_0}\}$$

$$\subseteq \text{span}\{f \cdot \eta : \deg f \leq n, \quad \eta \in M'_{n_1}\} = M'_{n_1+n}.$$

Thus we have

$$\lim_{n \to \infty} \frac{\dim M_n}{n^d} = \lim_{n \to \infty} \frac{\dim M_{n_0+n}}{n^d} \leq \lim_{n \to \infty} \frac{\dim M'_{n_1+n}}{n^d} = \lim_{n \to \infty} \frac{\dim M'_n}{n^d},$$

from which we conclude that $c_d \leq c'_d$. By symmetry we also have $c'_d \leq c_d$. 

The following two results together constitute a variant of the Artin-Rees lemma of commutative algebra (cf. [29], page II-9). Since the result we require is formulated differently than the Artin-Rees lemma (normally a statement about the behavior of decreasing filtrations associated with ideals and their relation to submodules), and since we have been unable to locate an appropriate reference, we have included complete proofs.

Associated with any filtration $\{M_n\}$ of a $k[x_1, \ldots, x_d]$-module $M$ there is an associated $\mathbb{Z}$-graded module $\bar{M}$, which is defined as the (algebraic) direct sum of finite dimensional vector spaces

$$\bar{M} = \sum_{n \in \mathbb{Z}} \bar{M}_n,$$

where $\bar{M}_n = M_n/M_{n-1}$ for each $n \in \mathbb{Z}$, and where for nonpositive values of $n$, $M_n$ is taken as $\{0\}$. The $k[x_1, \ldots, x_d]$-module structure on $\bar{M}$ is defined by the commuting $d$-tuple of "shift" operators $T_1, \ldots, T_d$, where $T_k$ is defined on each summand $\bar{M}_n$ by

$$T_k : \xi + M_{n-1} \in M_n/M_{n-1} \mapsto x_k \xi + M_n \in M_{n+1}/M_n.$$ 

Remark 3.7. For our purposes, the essential feature of this construction is that for every $n \geq 1$, the following are equivalent

1. $M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_d$
2. $M_{n+1} = T_1 M_n + \cdots + T_d M_n$.

Lemma 3.8. Let $\{M_n\}$ be a filtration of a $k[x_1, \ldots, x_d]$-module $M$. The following are equivalent:

1. $\{M_n\}$ is proper.
2. The $k[x_1, \ldots, x_d]$-module $\bar{M}$ is finitely generated.

Proof of (1) $\implies$ (2). Find an $n_0 \in \mathbb{N}$ such that

$$M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_n.$$
for all $n \geq n_0$. From Remark 3.7 we have $\bar{M}_{n+1} = T_1 \bar{M}_n + \cdots + T_d \bar{M}_n$ for all $n \geq n_0$, hence $G = M_1 + \cdots + M_{n_0}$ is a finite dimensional generating space for $\bar{M}$.

**proof of (2) $\implies$ (1).** Assuming (2), we can find a finite set of homogeneous elements $\xi_k \in \bar{M}_{n_k}$, $k = 1, \ldots, r$ which generate $\bar{M}$ as a $k[x_1, \ldots, x_d]$-module. It follows that for $n \geq \max(n_1, \ldots, n_r)$ we have

$$\bar{M}_{n+1} = T_1 \bar{M}_n + \cdots + T_d \bar{M}_n.$$  

For such an $n$, Remark 3.7 implies that

$$\bar{M}_{n+1} = T_1 \bar{M}_n + \cdots + T_d \bar{M}_n.$$  

hence $\{M_n\}$ is proper.

**Lemma 3.9.** Let $\{M_n\}$ be a proper filtration of a $k[x_1, \ldots, x_d]$-module $M$, let $K \subseteq M$ be a submodule, and let $\{K_n\}$ be the filtration induced on $K$ by

$$K_n = K \cap M_n.$$  

Then $\{K_n\}$ is a proper filtration of $K$.

**proof.** Form the graded modules

$$\bar{M} = \sum_{n \in \mathbb{Z}} M_n/M_{n-1}$$  

and

$$\bar{K} = \sum_{n \in \mathbb{Z}} K_n/K_{n-1}.$$  

Because of the natural isomorphism

$$\bar{K}_n = K \cap M_n/K \cap M_{n-1} \cong (K \cap M_n + M_{n-1})/M_{n-1} \subseteq M_n/M_{n-1} = \bar{M}_n,$$

$\bar{K}$ is isomorphic to a submodule of $\bar{M}$. Lemma 3.8 implies that $\bar{M}$ is finitely generated. Thus by Hilbert’s basis theorem (asserting that graded submodules of finitely generated graded modules are finitely generated), it follows that $\bar{K}$ is finitely generated. Now apply Lemma 3.8 once again to conclude that $\{K_n\}$ is a proper filtration of $K$. 

We remark that the proof of Lemma 3.9 is inspired by Cartier’s proof of the Artin-Rees lemma [29], p II-9.

Let $M$ be a finitely generated $A$-module, choose a finite dimensional subspace $G \subseteq M$ which generates $M$ as an $A$-module, and set

$$M_n = \text{span}\{f \cdot \zeta : f \in A, \ \deg f \leq n, \ \zeta \in G\}.$$  

Since $\{M_n\}$ is a proper filtration, Proposition 3.7 implies that the number

$$c(M) = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d}$$  

exists as an invariant of $M$ independently of the choice of generator $G$. The following result shows that this invariant is additive on short exact sequences.
Proposition 3.10. For every exact sequence

\[ 0 \to K \to L \to M \to 0 \]

of finitely generated \( k[x_1, \ldots, x_d] \)-modules we have \( c(L) = c(K) + c(M) \).

**proof.** Since \( c(M) \) depends only on the isomorphism class of \( M \), we may assume that \( K \subseteq L \) is a submodule of \( L \) and \( M = L/K \) is its quotient. Pick any proper filtration \( \{L_n\} \) for \( L \) and let \( \{\hat{L}_n\} \) and \( \{K_n\} \) be the associated filtrations of \( L/K \) and \( K \)

\[
\hat{L}_n = (L_n + K)/K \subseteq L/K, \\
K_n = K \cap M_n \subseteq K.
\]

It is obvious that \( \{\hat{L}_n\} \) is proper, and Lemma 3.9 implies that \( \{K_n\} \) is proper as well.

Now for each \( n \geq 1 \) we have an exact sequence of finite dimensional vector spaces

\[ 0 \to K_n \to L_n \to \hat{L}_n \to 0, \]

and hence

\[ \dim L_n = \dim K_n + \dim \hat{L}_n. \]

Since each of the three filtrations is proper we can multiply the preceding equation through by \( d!/n^d \) and take the limit to obtain \( c(L) = c(K) + c(L/K) \). 

Remark 3.11. The addition property of Proposition 3.10 generalizes immediately to the following assertion. For every finite exact sequence

\[ 0 \to M_n \to \cdots \to M_1 \to M_0 \to 0 \]

of finitely generated \( k[x_1, \ldots, x_d] \)-modules, we have

\[ \sum_{k=0}^{n} (-1)^k c(M_k) = 0. \]

Corollary. Let \( M \) be a finitely generated \( k[x_1, \ldots, x_d] \)-module and let

\[ 0 \to F_n \to \cdots \to F_1 \to M \to 0 \]

be a finite free resolution of \( M \), where

\[ F_k = \beta_k \cdot k[x_1, \ldots, x_d] \]

is a direct sum of \( \beta_k \) copies of the rank-one free module \( k[x_1, \ldots, x_d] \). Then

\[ c(M) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k. \]
proof. Remark (3.11) implies that
\[ c(M) = \sum_{k=1}^{n} (-1)^{k+1} c(F_k), \]
and thus it suffices to show that if \( F = \beta \cdot k[x_1, \ldots, x_d] \) is a free module of rank \( \beta \in \mathbb{N} \), then \( c(F) = \beta. \)

By the additivity property of 3.10 we have
\[ c(\beta \cdot k[x_1, \ldots, x_d]) = \beta \cdot c(k[x_1, \ldots, x_d]) \]
and thus we have to show that \( c(k[x_1, \ldots, x_d]) \) is 1.

This follows from a computation of the dimensions of
\[ P_n = \{ f \in k[x_1, \ldots, x_d] : \deg f \leq n \} \]
and it is a classical result that
\[ \dim P_n = q_d(n) = \frac{(n+1)(n+d)}{d^d} \]
(see, for example, Appendix A of [1]). Thus
\[ c(k[x_1, \ldots, x_d]) = d! \lim_{n \to \infty} \frac{\dim P_n}{n^d} = \lim_{n \to \infty} \frac{(n+1)(n+d)}{n^d} = 1 \]
and the corollary is established.

We now deduce the main result of this section. Let \( H \) be a finite-rank Hilbert module over \( A = \mathbb{C}[z_1, \ldots, z_d] \), and let \( \phi : \mathcal{B}(H) \to \mathcal{B}(H) \) be its associated completely positive map \( \phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^* \).

**Theorem C.**
\[ \chi(H) = d! \lim_{n \to \infty} \frac{\text{rank}(1 - \phi^{n+1}(1))}{n^d}. \]

**proof.** Consider the module
\[ M_H = \text{span}\{ f \cdot \Delta \xi : f \in A, \ \xi \in H \} \]
and its natural (proper) filtration
\[ M_n = \text{span}\{ f \cdot \Delta \xi : \deg f \leq n, \ \xi \in H \}, \quad n = 1, 2, \ldots. \]
In view of the definition of \( \chi(H) \) in terms of free resolutions of \( M_H \), the preceding corollary implies that
\[ \chi(H) = c(M_H) = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d}. \]
Thus it suffices to show that
\[ \dim M_n = \text{rank}(1 - \phi^{n+1}(1)) \]
for every \( n = 1, 2, \ldots \). For that, we will prove
\[ (3.12) \quad M_n = (1 - \phi^{n+1}(1))H. \]

Indeed, writing
\[ (3.13) \quad 1 - \phi^{n+1}(1) = \sum_{k=0}^{n} \phi^k(1 - \phi(1)) = \sum_{k=0}^{n} \phi^k(\Delta^2), \]
we see in particular that \( 1 - \phi^{n+1}(1) \) is a positive finite rank operator for every \( n \) and hence
\[ (1 - \phi^{n+1}(1))H = \ker(1 - \phi^{n+1}(1))^\perp. \]
The kernel of \( 1 - \phi^{n+1}(1) \) is easily computed. We have
\[ \ker(1 - \phi^{n+1}(1)) = \{ \xi \in H : \langle (1 - \phi^{n+1}(1))\xi, \xi \rangle = 0 \}, \]
and by (3.13), \( \langle (1 - \phi^{n+1}(1))\xi, \xi \rangle = 0 \) iff
\[ \sum_{k=0}^{n} \langle \phi^k(\Delta^2)\xi, \xi \rangle = 0. \]
Since
\[ \phi^k(\Delta^2) = \sum_{i_1, \ldots, i_k=1}^{d} T_{i_1} \cdots T_{i_k} \Delta^2 T_{i_k}^* \cdots T_{i_1}^*, \]
the latter is equivalent to
\[ \sum_{k=0}^{n} \sum_{i_1, \ldots, i_k=1}^{d} \| \Delta T_{i_k}^* \cdots T_{i_1}^* \xi \|^2 = 0. \]
Thus the kernel of \( 1 - \phi^{n+1}(1) \) is the orthocomplement of the space spanned by
\[ \{ T_{i_1} \cdots T_{i_k} \Delta \eta : \eta \in H, \ 1 \leq i_1, \ldots, i_k \leq d, \ k = 0, 1, \ldots, n \}, \]

namely \( M_n = \text{span}\{ f \cdot \Delta \eta : \deg f \leq n, \ \eta \in H \} \). This shows that
\[ \ker(1 - \phi^{n+1}(1)) = M_n^\perp, \]
from which formula (3.12) is evident.

\[ \Box \]

Remark 3.14. Closed submodules of finite-rank Hilbert \( A \)-modules neet not have finite rank (see section 8, Corollary of Theorem F). However, if \( H_0 \) is a submodule of a finite rank Hilbert module \( H \) which is of finite codimension in \( H \), then \( \text{rank}(H_0) < \infty \). Indeed, if \( P_0 \) is the projection of \( H \) onto \( H_0 \), then
\[ \text{rank}(H_0) = \text{rank}((1_H - \phi_H)(1_{H_0})) = \text{rank}(P_0 - \phi_H(P_0)). \]
Since \( P_0 - \phi_H(P_0) = (1_H - \phi_H(1_H)) - P_0^\perp + \phi_H(P_0^\perp) \), we have
\[ \text{rank}(H_0) \leq \text{rank}(H) + \text{rank}(P_0^\perp) + \text{rank}(\phi_H(P_0^\perp)) < \infty. \]

On the other hand, given a submodule \( H_0 \subseteq H \) with \( \dim(\text{H} / H_0) < \infty \), the algebraic module \( M_{H_0} \) is not a submodule of \( M_H \), nor is it conveniently related to \( M_H \). Thus there is no direct way of relating \( \chi(H) \) to \( \chi(H_0) \) by way of their definitions. Nevertheless, Theorem C implies the following.
Corollary 1: stability of Euler characteristic. Let $H_0$ be a closed submodule of a finite rank Hilbert $A$-module $H$ such that $\dim(H/H_0) < \infty$. Then $\chi(H_0) = \chi(H)$.

**proof.** By estimating as in Remark 3.14 we have

$$\text{rank}(1_{H_0} - \phi_{[H_0]}^{n+1}(1_{H_0})) \leq \text{rank}(1_H - \phi_{[H]}^{n+1}(1_H)) + \text{rank}P_0^\perp + \text{rank}(\phi_{[H]}^{n+1}(P_0^\perp)),$$

$P_0$ denoting the projection of $H$ on $H_0$. Similarly,

$$\text{rank}(1_H - \phi_{[H]}^{n+1}(1_H)) \leq \text{rank}(1_{H_0} - \phi_{[H_0]}^{n+1}(1_{H_0})) + \text{rank}P_0^\perp + \text{rank}(\phi_{[H]}^{n+1}(P_0^\perp)).$$

Thus we have the inequality

$$|\text{rank}(1_H - \phi_{[H]}^{n+1}(1_H)) - \text{rank}(1_{H_0} - \phi_{[H_0]}^{n+1}(1_{H_0}))| \leq \text{rank}P_0^\perp + \text{rank}(\phi_{[H]}^{n+1}(P_0^\perp)).$$

One estimates the right side as follows. Note that

$$\langle \phi^{n+1}(P_0^\perp)\xi,\xi \rangle = \sum_{1 \leq i_1,\ldots,i_{n+1} \leq 1} \langle P_0^\perp T_{i_{n+1}}^*\cdots T_{i_1}^*\xi, T_{i_{n+1}}^*\cdots T_{i_1}^*\xi \rangle$$

vanishes iff $\xi$ belongs to the kernel of every operator of the form $P_0^\perp f(T_1,\ldots,T_d)^*$, where $f \in E_{n+1}H^2$ is a homogeneous polynomial of degree $n + 1$. Hence the range of the positive finite rank operator $\phi^{n+1}(P_0^\perp)$ is the orthocomplement of all such vectors $\xi$, and is therefore spanned linearly by the ranges of all operators $f(T_1,\ldots,T_d)P_0^\perp$, $f \in E_{n+1}H^2$, i.e.,

$$\text{span}\{ f \cdot \zeta : f \in E_{n+1}H^2, \ z \in P_0^\perp H \}.$$

It follows that

$$\text{rank}(\phi^{n+1}(P_0^\perp)) \leq \dim(E_{n+1}H^2) \cdot \text{rank}P_0^\perp = q_{d-1}(n+1)\text{rank}P_0^\perp.$$

Thus (3.15) implies that

$$\left|\frac{\text{rank}(1_H - \phi_{[H]}^{n+1}(1_H))}{n^d} - \frac{\text{rank}(1_{H_0} - \phi_{[H_0]}^{n+1}(1_{H_0}))}{n^d}\right|$$

is at most

$$\text{rank}(P_0^\perp) \frac{1 + q_{d-1}(n+1)}{n^d}.$$

Since $q_{d-1}(x)$ is a polynomial of degree $d - 1$, the latter tends to zero as $n \to \infty$, and the conclusion $|\chi(H) - \chi(H_0)| = 0$ follows from Theorem C after taking the limit on $n$.

For algebraic reasons, the Euler characteristic of a finitely generated $A$-module must be nonnegative ([18], Theorem 192) and hence $\chi(H) \geq 0$ for every finite rank Hilbert $A$-module $H$. One also has the following upper bound, which we collect here for later use.
Corollary 2. For every finite rank Hilbert $A$-module $H$, $0 \leq \chi(H) \leq \text{rank}(H)$.

proof. Let $M_H$ be the algebraic module associated with $H$ and let $M_1 \subseteq M_2 \subseteq \ldots$ be the proper filtration of it defined by

$$M_n = \text{span}\{ f : \xi : f \in A, \deg f \leq n, \xi \in \Delta H \}$$

$\Delta$ denoting the square root of $1_H - T_1 T_1^* - \cdots - T_d T_d^*$. Clearly

$$\dim M_n \leq \dim \{ f \in A : \deg f \leq n \} \cdot \dim \Delta H = q_d(n) \cdot \text{rank}(H).$$

From the corollary of Prop 3.10 which identifies $\chi(M_H)$ with

$$c(M_H) = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d} \leq d! \lim_{n \to \infty} \frac{q_d(n)}{n^d} \cdot \text{rank}(H) = \text{rank}(H),$$

the inequality follows. $\blacksquare$

4. Curvature invariant. Let $H$ be a Hilbert $A$-module with canonical operators $T_1, \ldots, T_d$. For every $z \in \mathbb{C}^d$ we define the operator $T(z) \in \mathcal{B}(H)$ as in (0.2),

$$T(z) = \bar{z}_1 T_1 + \cdots + \bar{z}_d T_d.$$ 

We have pointed out in the introduction that $\|T(z)\| \leq |z|$, and hence $1 - T(z)$ is invertible for all $z$ in the open unit ball $B_d$. Thus we can define an operator-valued function $F : B_d \to \mathcal{B}(\Delta H)$ as follows:

$$F(z)\xi = \Delta(1 - T(z)^*)^{-1}(1 - T(z))^{-1} \Delta \xi, \quad \xi \in \Delta H.$$  

Assuming that $\text{rank}(H) < \infty$, then $F(z)$ is a positive operator acting on a finite dimensional Hilbert space and we may take its trace to obtain a numerical function defined for all $z$ the open ball. We show in Theorem A below that the latter function has “renormalized” boundary values

$$K_0(z) = \lim_{r \to 1^+} (1 - r^2) \text{trace} F(rz)$$

for almost every point $z \in \partial B_d$ relative to the natural measure $d\sigma$ on $\partial B_d$. The curvature invariant $K(H)$ is defined by integrating the function $K_0(\cdot)$ over $\partial B_d$, and in general $K(H)$ is a real number satisfying $0 \leq K(H) \leq \text{rank}(H)$. Significantly, the curvature invariant is sufficiently sensitive that it detects precisely when the closed submodule of $H$ generated by $\Delta H$ is a free Hilbert module; the criterion for freeness is that the curvature should be the maximum possible value $K(H) = \text{rank}(H)$.

Fix a Hilbert $A$-module $H$, of arbitrary positive rank, and form the free Hilbert $A$-module $H^2 \otimes \Delta H$ of rank $r = \text{rank}H$. We have seen in section 1 that there is a unique operator $U_0 \in \text{hom}(H^2 \otimes \Delta H, H)$ satisfying

$$U_0(f \otimes \zeta) = f \cdot \Delta \zeta, \quad f \in A, \quad \zeta \in H.$$  

$U_0$ is a coisometry only when $H$ is pure; but it is a contraction in general and hence $1 - U_0^* U_0$ is a positive operator in $\mathcal{B}(H^2 \otimes \Delta H)$ of norm at most 1. The following result implies that $1 - U_0^* U_0$ can be associated with a multiplier, and that fact is essential to the proof of Theorems A and D.
**Proposition 4.3.** There is a free Hilbert module $F = H^2 \otimes E$ and a multiplier $\Phi \in \mathcal{M}(E, \Delta H)$ whose associated homomorphism satisfies

$$U_0^* U_0 + \Phi \Phi^* = 1_{H^2 \otimes \Delta H}.$$ 

**proof.** Let $F_0$ be the free module $H^2 \otimes \Delta H$. By Theorem 1.14 there is a spherical module $S_0$ and a map $U_1 \in \text{hom}(S_0, H)$ such that

$$U : (\xi, \eta) \in F_0 \oplus S_0 \mapsto U_0 \xi + U_1 \eta \in H$$

defines a minimal dilation

$$F_0 \oplus S_0 \xrightarrow{U} H \xrightarrow{0}.$$

We consider the kernel $K = \ker U$ of the dilation map $U$. $K$ is a Hilbert submodule of $F_0 \oplus S_0$ and therefore it too has a minimal dilation

$$F \oplus S \xrightarrow{V} K \xrightarrow{0},$$

where $V \in \text{hom}(F \oplus S, F_0 \oplus S_0)$ satisfies $VV^* = P_K$, $P_K$ denoting the projection of $F_0 \oplus S_0$ onto $K$. Define $\Phi \in \text{hom}(F, F_0)$ by $\Phi = P_{F_0} V \upharpoonright F$, $P_{F_0}$ denoting the projection of $F_0 \oplus S_0$ onto the first summand. We have to show that

$$U_0^* U_0 = 1_{F_0} - \Phi \Phi^*,$$

and for that we require

**Lemma 4.5.** Let $F_1, F_2$ be free Hilbert $A$-modules, let $S_1, S_2$ be spherical Hilbert $A$-modules, and let $V \in \text{hom}(F_1 \oplus S_1, F_2 \oplus S_2)$. Then $VS_1 \subseteq S_2$.

Assuming for the moment that Lemma 4.5 has been proved, we establish (4.4) as follows. Since $U_0$ is the restriction of $U$ to the free summand $F_0 \subseteq F_0 \oplus S_0$ we have $U_0^* U_0 = P_{F_0} U^* U \upharpoonright F_0$. Since $U$ is a coisometry, $U^* U$ is the projection $1 - P_K$ onto the orthocomplement of $K = \ker U \subseteq F_0 \oplus S_0$, hence

$$U_0^* U_0 = P_{F_0} (1 - P_K) \upharpoonright F_0 = 1_{F_0} - P_{F_0} P_K \upharpoonright F_0.$$ 

Applying Lemma 4.5 to the map $V \in \text{hom}(F \oplus S, F_0 \oplus S_0)$ we find that $VS \subseteq S_0$ or, equivalently, that $P_{F_0} V = P_{F_0} V P_F$. Since $VV^* = P_K$, the right side of (4.6) becomes

$$1_{F_0} - P_{F_0} V V^* \upharpoonright F_0 = 1_{F_0} - P_{F_0} V P_F V^* \upharpoonright F_0 = 1_{F_0} - \Phi \Phi^*,$$

as required.

**proof of Lemma 4.5.** Let $T_1, \ldots, T_d$ be the canonical operators associated with the Hilbert module $F_1 \oplus S_1$, and let $\phi_1 : \mathcal{B}(F_1 \oplus S_1) \to \mathcal{B}(F_1 \oplus S_1)$ be their associated completely positive map. Then $\phi_1(1) \geq \phi_1^2(1) \geq \phi_1^3(1) \geq \ldots$ is a decreasing sequence of projections with limit projection

$$\lim_{n \to \infty} \phi_1^n(1) = 0 \oplus 1_{S_1}.$$
Similarly, if \( \phi_2 \) is the corresponding completely positive map on \( B(F_2 \oplus S_2) \) then \( \phi_2^n(1) \downarrow 0 \oplus 1_{S_2} \). Since \( V \in \text{hom}(F_1 \oplus S_1, F_2 \oplus S_2) \) we have

\[
V \phi_1^n(1)V^* = \phi_2^n(VV^*) \leq \phi_2^n(1)
\]

for every \( n \), hence

\[
V(0 \oplus 1_{S_1})V^* = \lim_{n \to \infty} V \phi_1^n(1)V^* \leq \lim_{n \to \infty} \phi_2^n(1) = 0 \oplus 1_{S_2},
\]

from which \( VS_1 \subseteq S_2 \) follows.

From Proposition 4.3 we obtain

**Corollary.** Let \( F : B_d \to B(\Delta H) \) be the function (4.1) and let \( \Phi \in M(E, \Delta H) \) be the multiplier of Proposition 4.3. Then for all \( z \in B_d \) we have

\[
(1 - |z|^2)F(z) = 1 - \Phi(z)\Phi(z)^*.
\]

**Proof.** Fix \( \alpha \in B_d, \zeta_1, \zeta_2 \in \Delta H \). From (4.1) we can write

\[
\langle F(\alpha)\zeta_1, \zeta_2 \rangle = \langle (1 - T(\alpha))^{-1} \Delta \zeta_1, (1 - T(\alpha))^{-1} \Delta \zeta_2 \rangle.
\]

Consider the operator \( U_0 : H^2 \otimes \Delta H \to H \) given by \( U_0(f \otimes \zeta) = f \cdot \Delta \zeta \). Notice that for the element \( u_\alpha \in H^2 \) defined by

\[
u_\alpha(z) = (1 - (z, \alpha))^{-1}, \quad z \in B_d \]

we have

\[
u_0(u_\alpha \otimes \zeta) = (1 - T(\alpha))^{-1} \Delta \zeta.
\]

Indeed, the sequence of polynomials \( f_n \in H^2 \) defined by

\[
f_n(z) = \sum_{k=0}^n \langle z, \alpha \rangle^k
\]

converges in the \( H^2 \)-norm to \( u_\alpha \) since

\[
\|u_\alpha - f_n\|^2 = \sum_{k=n+1}^\infty |\alpha|^{2k} \to 0
\]

as \( n \to \infty \). Since

\[
U_0(f_n \otimes \zeta) = f_n \cdot \Delta \zeta = \sum_{k=0}^n T(\alpha)^k \Delta \zeta,
\]

formula (4.8) follows by taking the limit as \( n \to \infty \).

From (4.8) we find that

\[
\langle F(\alpha)\zeta_1, \zeta_2 \rangle = \langle U_0(u_\alpha \otimes \zeta_1), U_0(u_\alpha \otimes \zeta_2) \rangle = \langle U_0^* U_0 u_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2 \rangle.
\]

By Proposition 4.3 we have \( U_0^* U_0 = 1 - \Phi \Phi^* \), and using the formula \( \Phi^*(u_\alpha \otimes \zeta) = u_\alpha \otimes \Phi(\alpha)^* \zeta \) of Proposition 2.4 we can write

\[
\langle F(\alpha)\zeta_1, \zeta_2 \rangle = \langle (1 - \Phi \Phi^*)u_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2 \rangle
\]

\[
= \langle u_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2 \rangle - \langle u_\alpha \otimes \Phi(\alpha)^* \zeta_1, u_\alpha \otimes \Phi(\alpha)^* \zeta_2 \rangle
\]

\[
= \|u_\alpha\|^2 \langle \zeta_1, \zeta_2 \rangle - \langle \Phi(\alpha)^* \zeta_1, \Phi(\alpha)^* \zeta_2 \rangle
\]

\[
= (1 - |\alpha|^2)^{-1} \langle (1 - \Phi(\alpha)\Phi(\alpha)^*) \zeta_1, \zeta_2 \rangle,
\]

and the corollary follows after multiplying through by \( 1 - |\alpha|^2 \).
Lemma 4.9. Let $E_1, E_2$ be Hilbert spaces with $E_2$ finite dimensional, let $\Phi \in \mathcal{M}(E_1, E_2)$ be a multiplier, let $\tilde{\Phi} : \partial B_d \to \mathcal{B}(E_1, E_2)$ be its boundary function (see Proposition 2.8), and let $\sigma$ be normalized measure on $\partial B_d$. Then for $\sigma$-almost every point $z \in \partial B_d$ we have

$$\lim_{r \uparrow 1} \text{trace}(\Phi(rz)\Phi(rz)^*) = \text{trace}(\tilde{\Phi}(z)\tilde{\Phi}(z)^*).$$

**proof.** For any Hilbert-Schmidt operators $A, B \in \mathcal{B}(E_1, E_2)$ we have $\text{trace}AA^* = \text{trace}A^*A$ and

$$|\sqrt{\text{trace}(AA^*)} - \sqrt{\text{trace}(BB^*)}| \leq (\text{trace}(|A - B|^2))^{1/2}$$

(where we have written the usual $|X|^2$ for $X^*X$). Thus it suffices to show that

$$\lim_{r \uparrow 1} \text{trace}|\Phi(rz) - \tilde{\Phi}(z)|^2 = 0$$

almost everywhere $(d\sigma)$.

Now since $E_2$ is finite-dimensional, the operator norm on $\mathcal{B}(E_1, E_2)$ is equivalent to the Hilbert-Schmidt norm and since the space $\mathcal{L}^2(E_1, E_2)$ of Hilbert-Schmidt operators is a Hilbert space with its natural norm

$$\|A\|_2 = (\text{trace}A^*A)^{1/2},$$

we may consider that the multiplier $\Phi$ is a bounded holomorphic function from $B_d$ to the Hilbert space $\mathcal{L}^2(E_1, E_2)$. The required assertion now follows from (2.6).

Theorem A. Let $H$ be a Hilbert $A$-module of finite positive rank, let $F : B_d \to \mathcal{B}(\Delta H)$ be the operator function defined by (4.1), and let $\sigma$ denote normalized measure on the sphere $\partial B_d$. Then for $\sigma$-almost every $z \in B_d$, the limit

$$K_0(z) = \lim_{r \uparrow 1}(1 - r^2)\text{trace}F(r \cdot z)$$

exists and satisfies

$$0 \leq K_0(z) \leq \text{rank} H.$$

Moreover, the extreme case $K_0(z) = \text{rank} H$ (a.e.) occurs if, and only if, the closed submodule of $H$ generated by the range of $\Delta$ is free.

**proof.** The corollary of Proposition 4.3 implies that for all $z \in \partial B_d$ and every $r \in (0,1)$ we have

$$(1 - r^2)F(rz) = 1 - \Phi(rz)\Phi(rz)^*.$$  

From Lemma 4.9 we may conclude that for almost every $z \in \partial B_d$,

$$\lim_{r \uparrow 1}(1 - r^2)\text{trace}F(rz) = \text{rank} H - \text{trace}(\tilde{\Phi}(z)\tilde{\Phi}(z)^*),$$

and hence the limit function $K_0(\cdot)$ is expressed in terms of $\tilde{\Phi}$ by

$$K_0(z) = \text{trace}(1_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*).$$
Since $\|\tilde{\Phi}(z)\| \leq 1$ we have $0 \leq 1_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^* \leq 1_{\Delta H}$ and hence
\[
0 \leq K_0(z) \leq \text{trace} (1_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*) \leq \text{trace} 1_{\Delta H} = \text{rank}(H)
\]
for almost every $z \in \partial B_d$.

Now consider the extreme case in which (4.11)
\[
K_0(z) = \text{rank} H, \quad \text{a.e. } d\sigma(z).
\]
From formula (4.10) it follows that $\text{trace} (\tilde{\Phi}(z)\tilde{\Phi}(z)^*) = 0$ for almost every $z \in \partial B_d$. Since the trace is faithful and since $\Phi$ is uniquely determined by its boundary values, we conclude that $\Phi = 0$. Proposition 4.3 implies that $U_0$ must be an isometry. Thus $U_0$ is an isomorphism of $H^2 \otimes \Delta H$ onto the closed submodule of $H$ generated by $\Delta H$.

Conversely, if $M_H$ is isomorphic to a free Hilbert module $H^2 \otimes E$ then by a direct computation one finds that $\Delta$ is identified with the projection onto $1 \otimes E \subseteq H^2 \otimes E$, hence $\Delta H$ is identified with $E$ and $U_0$ is clearly an isometry. By Proposition 4.3, the multiplier $\Phi$ must be 0, and formula (4.10) shows that $K_0(z)$ is the constant function with value $\text{trace} 1_{\Delta H} = \text{rank}(H)$.

We now define the curvature invariant of a Hilbert $A$-module $H$ of finite rank,
\[
(4.12) \quad K(H) = \int_{\partial B_d} K_0(z) \, d\sigma(z).
\]
The basic property of $K(H)$ is that it is sensitive enough to detect exactly when a finite rank pure Hilbert module is free.

**Theorem 4.13.** For every finite rank Hilbert $A$-module $H$, we have
\[
0 \leq K(H) \leq \text{rank}(H).
\]
If $H$ is pure, then $K(H) = \text{rank}(H)$ if, and only if, $H \cong H^2 \otimes E$ is a free Hilbert $A$-module.

**proof.** This is immediate from the preceding discussion after noting that for a pure Hilbert $A$-module $H$, the map $U_0 : H^2 \otimes \Delta H \to H$ of (4.2) is a coisometry, and in particular
\[
H = U_0(H^2 \otimes \Delta H) = M_H = \text{span} \{ f \cdot \Delta \xi : f \in A, \ \xi \in H \}
\]
is generated as a Hilbert $A$-module by the range of $\Delta$.  

The curvature invariant also detects “inner sequences”. More precisely, let $M$ be a closed submodule of the rank-one free Hilbert $A$-module $H^2$. From the corollary of Theorem 5.9 below (Theorem 5.9 is proved independently of the discussion to follow) there is a Hilbert space $E$ and a multiplier $\Phi \in \mathcal{M}(E, \mathbb{C})$ whose morphism
satisfies $\Phi \Phi^* = P_M$. Choosing an orthonormal basis $e_1, e_2, \ldots$ for $E$ we obtain a sequence $\{\phi_n\}$ of multipliers of $H^2$ as follows
$$\phi_n(z) = (\Phi e_n)(z) = \Phi(z)e_n, \quad n = 1, 2, \ldots.$$ The associated multiplication operators $V_n f = \phi_n \cdot f$ satisfy
$$V_1 V_1^* + V_2 V_2^* + \cdots = \Phi \Phi^* = P_M. \tag{4.14}$$ For definiteness of notation, we can assume that the sequence $\phi_1, \phi_2, \ldots$ is infinite by adding harmless zero functions if it is not. Note that
$$\sup_{|z| < 1} \sum_{n=1}^{\infty} |\phi_n(z)|^2 \leq 1. \tag{4.15}$$ Indeed, if $\{u_\alpha : \alpha \in B_d\}$ denotes the family of functions in $H^2$ defined in (1.1), then $v_\alpha = (1 - |\alpha|^2)^{1/2}u_\alpha$ is a unit vector in $H^2$ which is an eigenvector for the adjoints of multiplication operators associated with any multiplier; thus for the operators $V_n$ we have $V_n^* v_\alpha = \phi_\alpha(z)v_\alpha$. Using (4.14) we find that
$$\sum_{n=1}^{\infty} |\phi_n(\alpha)|^2 = \sum_{n=1}^{\infty} \|V_n^* v_\alpha\|^2 = \sum_{n=1}^{\infty} \langle V_n V_n^* v_\alpha, v_\alpha \rangle = \langle P_M v_\alpha, v_\alpha \rangle \leq 1,$$ and (4.15) follows.

Therefore, the boundary functions $\tilde{\phi}_n : \partial B_d \to \mathbb{C}$ must satisfy
$$\sum_{n=1}^{\infty} |\tilde{\phi}_n(z)|^2 \leq 1$$ almost everywhere with respect to the natural normalized measure $\sigma$ on $\partial B_d$ and we will say that $\phi_n$ is an inner sequence if equality holds almost everywhere
$$\sum_{n=1}^{\infty} |\tilde{\phi}_n(z)|^2 = 1, \quad \text{a. e. } (d\sigma) \text{ on } \partial B_d.$$ Significantly, we do not know if every sequence $\phi_n$ associated with an invariant subspace of $H^2$ as in (4.14) must be an inner sequence (see section 8). The following result shows the relevance of the curvature invariant for this problem.

**Theorem 4.16.** Let $M \subseteq H^2$ be a closed submodule and let $\phi_1, \phi_2, \ldots$ be a sequence in the multiplier algebra of $H^2$ whose multiplication operators satisfy
$$P_M = M_{\phi_1} M_{\phi_1}^* + M_{\phi_2} M_{\phi_2}^* + \cdots.$$ Then $\phi_n$ is an inner sequence iff $K(H^2/M) = 0$.

**Proof.** Let $F$ be the direct sum of an infinite number of copies of $H^2$ and define $\Phi \in \text{hom}(F, H^2)$ by
$$\Phi(f_1, f_2, \ldots) = \sum_{n=1}^{\infty} \phi_n \cdot f_n.$$
Then $\Phi\Phi^* = P_M$. Letting $U : H^2 \to H^2/M$ be the natural projection, then $U$ defines a minimal dilation of $H^2/M$ and we have

$$U^* U + \Phi\Phi^* = 1_{H^2}.$$ 

Writing

$$K(H^2/M) = \int_{\partial B_d} K_0(z) d\sigma(z),$$

we see from formula (4.10) that in this case

$$K_0(z) = 1 - \Phi(z)\Phi(z)^* = 1 - \sum_{n=1}^{\infty} |\tilde{\phi}_n(z)|^2,$$

and therefore $K(H^2/M) = 0$ iff $\sum_n |\tilde{\phi}_n(z)|^2 = 1$ almost everywhere on $\partial B_d$. 

We will discuss applications of Theorem 4.16 in section 8.

5. Curvature operator: quantizing the Gauss map.

Let us recall a convenient description of the Gaussian curvature of a compact oriented Riemannian 2-manifold $M$. It is not necessary to do so, but for simplicity we will assume that $M \subseteq \mathbb{R}^3$ can be embedded in $\mathbb{R}^3$ in such a way that it inherits the usual metric structure of $\mathbb{R}^3$. After choosing one of the two orientations of $M$ (as a nondegenerate 2-form) we normalize it in the obvious way to obtain a continuous field of unit normal vectors at every point of $M$.

For every point $p$ of $M$ one can translate the normal vector at $p$ to the origin of $\mathbb{R}^3$ (without changing its direction), and the endpoint of that translated vector is a point $\gamma(p)$ on the unit sphere $S^2$. This defines the Gauss map

$$\gamma : M \to S^2$$

of $M$ to the sphere. Now fix $p \in M$. The tangent plane $T_pM$ is obviously parallel to the corresponding tangent plane $T_{\gamma(p)}S^2$ of the sphere (they have the same normal vector) and hence both are cosets of the same 2-dimensional subspace $V \subseteq \mathbb{R}^3$:

$$T_pM = p + V, \quad T_{\gamma(p)}S^2 = \gamma(p) + V.$$

Thus the differential $d\gamma(p)$ defines a linear operator on the two-dimensional vector space $V$, and the Gaussian curvature $K(p)$ of $M$ at $p$ is defined as the determinant of this operator $K(p) = \det d\gamma(p)$. $K(p)$ does not depend on the choice of orientation. The Gauss-Bonnet theorem asserts that the average value of $K(\cdot)$ is the alternating sum of the Betti numbers of $M$

$$\frac{1}{2\pi} \int_M K(p) = \beta_0 - \beta_1 + \beta_2.$$

In this section we define a curvature operator associated with any finite rank Hilbert $A$-module $H$. We discuss how it can be viewed as a quantized (higher-dimensional) analogue of the differential of the Gauss map $\gamma : M \to S^2$, we show
that the curvature operator is of trace class, that its trace agrees with the curvature invariant $K(H)$ of section 4, and we establish a key asymptotic formula for $K(H)$.

These considerations are best formulated in the following general setting. Let $C$ be a finite dimensional Hilbert space and let $F = H^2 \otimes C$ be the free Hilbert $A$-module of rank $r = \dim C$.

We may consider both $H^2 \otimes C$ and the “Hardy” space $H^2(\partial B_d; C)$ of $C$-valued functions as spaces of vector-valued holomorphic functions defined in the open unit ball $B_d$. For $H^2 \otimes C$ this is described in Proposition 2.1. $H^2(\partial B_d; C)$ is defined as the subspace of $L^2(\partial B_d, d\sigma; C)$ obtained by closing the space of $C$-valued holomorphic polynomials $f : \mathbb{C}^d \to C$ in the $L^2(\partial B_d, d\sigma; C)$-norm; and there is a natural way of extending functions in $H^2(\partial B_d; C)$ holomorphically to the open ball $B_d$ [26]. Theorem 4.3 of [1] implies that these two spaces of holomorphic functions are related as follows

$$H^2 \otimes C \subseteq H^2(\partial B_d; C).$$

Moreover, the inclusion map of (5.2) is a compact operator when $d \geq 2$. We will write $F$ instead of $H^2 \otimes C$, $\partial F$ instead of $H^2(\partial B_d; C)$, and $b : F \to \partial F$ for the inclusion map of (5.2). The nature of $b$ and $b^*b$ will be described more precisely in Proposition 5.7 below.

We define a linear map $\Gamma : \mathcal{B}(F) \to \mathcal{B}(\partial F)$ as follows

$$\Gamma(X) = bXb^*.$$  

**Remark 5.4.** We first record some simple observations about the operator mapping $\Gamma$. It is obvious that $\Gamma$ is a normal completely positive linear map. $\Gamma$ is also an order isomorphism because $b$ is injective. Indeed, if $\Gamma(X) \geq 0$ then $\langle X\xi, \xi \rangle \geq 0$ for every $\xi$ in the range $b^*(\partial F)$, and $b^*(\partial F)$ is dense in $F$ because $b$ has trivial kernel. A similar argument shows that $\Gamma$ is in fact a complete order isomorphism. However, in dimension $d \geq 2$ the range of $\Gamma$ is a linear space of compact operators which is norm-dense in $\mathcal{K}(\partial F)$ but proper: $\Gamma(\mathcal{B}(F)) \neq \mathcal{K}(\partial F)$. Indeed, if the range of $\Gamma$ were norm-closed then the closed graph theorem would imply that $\Gamma$ is a linear isomorphism of the Banach space $\mathcal{B}(F)$ onto $\mathcal{K}(\partial F)$, which is obviously absurd since $\mathcal{B}(F)$ is inseparable.

$\partial F = H^2(\partial B_d; C)$ is a Hilbert $A$-module whose canonical operators $Z_1, \ldots, Z_d$ are defined by $Z_k f(z) = \langle z, e_k \rangle f(z)$ for $z \in \partial B_d$, $e_1, \ldots, e_d$ being an orthonormal basis for $\mathbb{C}^d$. $(Z_1, \ldots, Z_d)$ is a pure subnormal $d$-contraction for which

$$Z_1^*Z_1 + \cdots + Z_d^*Z_d = 1,$$

while

$$Z_1Z_1^* + \cdots + Z_dZ_d^* = 1 - \mathcal{E}_0$$

$\mathcal{E}_0$ denoting the projection of $H^2(\partial B_d; C)$ onto the finite dimensional space of constant $C$-valued functions, and $\phi_{\partial F}(A) = Z_1AZ_1^* + \cdots + Z_dAZ_d^*$ is a normal completely positive map on $\mathcal{B}(H^2(\partial B_d; C))$.

**Definition 5.4.** $d\Gamma : \mathcal{B}(F) \to \mathcal{B}(\partial F)$ is defined as the following linear map

$$d\Gamma(X) = \Gamma(X) - \phi_{\partial F}(\Gamma(X)) = \Gamma(X) - \sum_{k=1}^d Z_k \Gamma(X) Z_k^*.$$  

We now define the curvature operator of a finite rank Hilbert $A$-module $H$. Let $U_0 : H^2 \otimes \Delta H \to H$ be the homomorphism defined in section 1,

$$U_0(f \otimes \zeta) = f \cdot \Delta \zeta, \quad f \in A, \quad \zeta \in \Delta H.$$
Definition 5.5. Let $H$ be a finite rank Hilbert $A$-module and take $F = H^2 \otimes \Delta H$, $\partial F = H^2(\partial B_d; \Delta H)$ above. The curvature operator of $H$ is defined as the self-adjoint operator
\[ d\Gamma(U_0^*U_0) \in \mathcal{B}(H^2(\partial B_d; \Delta H)). \]

Remarks. We have found it useful to think of the operator $\Gamma(U_0^*U_0)$ as a higher-dimensional “quantized” analogue of the Gauss map $\gamma : M \to S^2$ of (5.1), and of the curvature operator $d\Gamma(U_0^*U_0)$ as its “differential”. Of course, this is only an analogy. But we will also find that $d\Gamma(U_0^*U_0)$ belongs to the trace class, and
\[ \text{trace } d\Gamma(U_0^*U_0) = K(H), \]
the term $K(H)$ on the right being analogous to the average Gaussian curvature
\[ \frac{1}{2\pi} \int_M K = \frac{1}{2\pi} \int_M \det \gamma(p). \]

On the other hand, $K(H)$ is defined in section 4 as the integral of the trace (not the determinant) of an operator-valued function, and thus this analogy must not be carried to extremes.

We also remark that the curvature operator can be defined in somewhat more concrete terms as follows. Let $T(z)$ denote the operator function of $z \in \mathbb{C}^d$ defined in (0.2). $T(z)$ is invertible for $|z| < 1$, and hence every vector $\xi \in H$ gives rise to a function $\hat{\xi} : B_d \to \Delta H$ by way of
\[ \hat{\xi}(z) = \Delta(1 - T(z)^*)^{-1}\xi, \quad z \in B_d. \]
It is a fact that $\hat{\xi}$ belongs to $\partial F = H^2(\partial F; \Delta H)$, and thus we have defined a linear mapping $B : \xi \in H \mapsto \hat{\xi} \in \partial F$. Indeed, the reader can verify that $B$ is related to $b$ and $U_0$ by $B = bU_0^*$, and hence the curvature operator of Definition 5.5 is identical with
\[ BB^* - \phi_{\partial F}(BB^*) = BB^* - \sum_{k=1}^{d} Z_k BB^* Z_k^*. \]

We will not have to make use of the operator $B$ in the sequel.

Returning now to the general setting $F = H^2 \otimes C$, $\partial F = H^2(\partial B_d; C)$, where $C$ is a finite dimensional Hilbert space, we work out the basic properties of the operator mapping
\[ d\Gamma : \mathcal{B}(F) \to \mathcal{B}(\partial F). \]
The essential properties of the inclusion map $b : F \to \partial F$ are summarized as follows. We will write $E_n, n = 0, 1, 2, \ldots$ for the projection of $F = H^2 \otimes C$ onto its subspace of homogeneous (vector-valued) polynomials of degree $n$, and we have
\[ \text{trace } E_n = \dim \{ f \in H^2 : f(\lambda z) = \lambda^n f(z), \lambda \in \mathbb{C}, z \in \mathbb{C}^d \} \cdot \dim C \]
\[ = q_{d-1}(n) \cdot \dim C, \]
where $q_{d-1}(x)$ is the $d-1$st term in the sequence of polynomials $q_0(x), q_1(x), \ldots$ of (3.3). Since this polynomial will play an important part in the remainder of this section, we reiterate its definition here: for $d = 1$, $q_{d-1}(x) = 1$ and otherwise
\[ q_{d-1}(x) = \frac{(x+1) \ldots (x+d-1)}{(d-1)!}, \quad d \geq 2. \]
\[ (5.6) \]
Let $\tilde{E}_d$ be the corresponding sequence of projections acting on the Hilbert $A$-module $\partial F$. We will also write $N$ and $\tilde{N}$ for the respective number operators on $F$ and $\partial F$,

$$N = \sum_{n=0}^{\infty} nE_n, \quad \tilde{N} = \sum_{n=0}^{\infty} n\tilde{E}_n.$$ 

**Proposition 5.7.** Let $b : F \rightarrow \partial F$ be the natural inclusion. Then

1. $bE_n = \tilde{E}_n b, \quad n = 0, 1, 2, \ldots$
2. $b \in \text{hom}(F, \partial F)$.
3. $b^*b = q_{d-1}(N)^{-1} = \sum_{n=0}^{\infty} \frac{1}{q_{d-1}(n)} E_n$.

**proof.** Properties (1) and (2) are immediate from the definition of $b$. Property (3) follows from a direct comparison of the norms in $H_2$ and $H_2(\partial B_d)$. Indeed, if $f, g \in H_2$ are both homogeneous polynomials of degree $n$ of the specific form $f(z) = \langle z, \alpha \rangle^n, \quad g(z) = \langle z, \beta \rangle^n, \quad \alpha, \beta \in \mathbb{C}^d$,

then $\langle f, g \rangle_{H_2} = \langle \beta, \alpha \rangle^n$, whereas if we consider $f, g$ as elements of $H_2(\partial B_d)$ then we have

$$\langle bf, bg \rangle = \langle f, g \rangle_{H_2(B_d)} = q_{d-1}(n)^{-1} \langle \beta, \alpha \rangle^n,$$

see Proposition 1.4.9 of [26]. Since $E_n H_2$ is spanned by such $f, g$ we find that for all $f, g \in E_n H_2$,

$$\langle bf, bg \rangle = q_{d-1}(n)^{-1} \langle f, g \rangle_{H_2}.$$

Thus

$$E_n b^*b E_n = q_{d-1}(n)^{-1} E_n = q_{d-1}(N)^{-1} E_n,$$

and (3) follows for the case $C = \mathbb{C}$ from the fact that $b^*b$ commutes with $E_n$ and $\sum_n E_n = 1$.

If we now tensor both $H_2$ and $H_2(\partial B_d)$ with the finite dimensional space $C$ then we obtain (3) in general after noting that $\dim(K_1 \otimes K_2) = \dim K_1 \cdot \dim K_2$ for finite dimensional vector spaces $K_1, K_2$.

**Remark 5.8.** In the one-variable case $d = 1$, $q_{d-1}(x)$ is the constant polynomial 1, and hence 5.7 (3) asserts the obvious fact that $b$ is a unitary operator; i.e., there is no difference between $H_2$ and $H_2(S^1)$ in dimension one.

In dimension $d \geq 2$ however, $q_{d-1}(x)$ is a polynomial of degree $d - 1 \geq 1$ and hence

$$b^*b = q_{d-1}(N)^{-1}$$

is a positive compact operator. Significantly, the operator $b^*b$ is never trace class. Indeed, the computations of [1, Appendix A] imply that $b^*b \in \mathcal{L}^p$ iff $p > \frac{d}{d-1} > 1$.

Returning now to the discussion of $d\Gamma$, we begin by giving a description of the cone of all operators $X$ whose “differential” $d\Gamma(X)$ is positive.

**Theorem 5.9.** Let $F = H_2 \otimes C$ be a free Hilbert module. Then for any $X \in \mathcal{B}(F)$ the following are equivalent.

1. $d\Gamma(X) \geq 0$. 

There is a sequence $\Phi_1, \Phi_2, \ldots$ of bounded endomorphisms of $F$ such that

$$X = \sum_{n=1}^{\infty} \Phi_n \Phi_n^*.$$  

There is a free Hilbert module $\tilde{F}$ and a bounded homomorphism of Hilbert $A$-modules $\Phi : \tilde{F} \to F$ such that $X = \Phi \Phi^*$.  

**proof.** We prove the sequence of implications (3) \implies (2) \implies (1) \implies (3).

**proof of (3) \implies (2).** Let $\tilde{F}$ be a free Hilbert module and let $\Phi : \text{hom}(\tilde{F}, F)$ satisfy $\Phi \Phi^* = X$. We may assume that $\tilde{F}$ has infinite rank by adding a direct summand $H^2 \otimes \ell^2$ to it, and by extending $\Phi$ to the larger free Hilbert module by making it zero on the summand $H^2 \otimes \ell^2$.

Assuming that this has been arranged, and taking note of the isomorphism $\tilde{F} \cong F \oplus F \oplus \ldots$, we may assume that $\tilde{F} = F \oplus F \oplus \ldots$ is a direct sum of copies of $F$. Defining endomorphisms $\Phi_n \in B(F)$ by simply restricting $\Phi$ to the $n$th summand, we deduce the required representation

$$X = \Phi \Phi^* = \sum_{n=1}^{\infty} \Phi_n \Phi_n^*.$$

**proof of (2) \implies (1).** Since $d\Gamma$ is a normal operator mapping, it suffices to show that $d\Gamma(\Phi \Phi^*) \geq 0$ for every endomorphism $\Phi : F \to F$. For that, write

$$\Gamma(\Phi \Phi^*) = b\Phi \Phi^* b^* = (b\Phi)(b\Phi)^*,$$

where $b\Phi \in \text{hom}(F, \partial F)$ is the composite homomorphism of Hilbert $A$-modules. Thus if $T_1, \ldots, T_d$ are the canonical operators of $F$ and $Z_1, \ldots, Z_d$ are those of $\partial F$, then we have $Z_k b\Phi = b\Phi T_k$ for every $k$, hence

$$d\Gamma(\Phi \Phi^*) = b\Phi(1 - \sum_{k=1}^{d} T_k T_k^*)(b\Phi)^*$$

is a positive operator because $(T_1, \ldots, T_d)$ is a $d$-contraction.

**proof of (1) \implies (3).** Let $T_1, \ldots, T_d$ be the canonical operators of $F$, and suppose $d\Gamma(X) \geq 0$. We claim first that

$$(5.10) \quad 0 \leq \sum_{k=1}^{d} T_k X T_k^* \leq X.$$

To see that, note that for every $k = 1, \ldots, d$ we have $b T_k = Z_k b$, $Z_1, \ldots, Z_d$ being the canonical operators of $\partial F$. Hence

$$0 \leq d\Gamma(X) = b X b^* - \sum_{k=1}^{d} Z_k b X b^* Z_k^* = b X b^* - b(\sum_{k=1}^{d} T_k X T_k^*) b^* = \Gamma(X - \sum_{k=1}^{d} T_k X T_k^*).$$
Since $\Gamma$ is an order isomorphism the latter implies $X - \sum_k T_kX^* T_k \geq 0$, or
\begin{equation}
X - \phi(X) \geq 0,
\end{equation}

$\phi$ being the completely positive map of $\mathcal{B}(F)$ defined by
\[
\phi(A) = \sum_{k=1}^d T_kAT_k^*.
\]

Free Hilbert $A$-modules are pure, hence $\phi^n(1) \downarrow 0$ as $n \to \infty$. It follows that for every positive operator $A \in \mathcal{B}(F)$ we have $0 \leq \phi^n(A) \leq \|A\|\phi^n(1)$, and hence $\phi^n(A) \to 0$ in the strong operator topology of $\mathcal{B}(F)$, as $n \to \infty$. By taking linear combinations we find that $\lim_{n \to \infty} \phi^n(A) = 0$ in the strong operator topology for every $A \in \mathcal{B}(F)$.

Returning now to equation (5.11), we find that $X - \phi^{n+1}(X) = \sum_{k=1}^n \phi^k(X - \phi(X)) \geq 0$

for every $n = 1, 2, \ldots$ and since $\phi^{n+1}(X)$ must tend strongly to 0 by the preceding paragraph, we conclude that $X \geq 0$ by taking the limit on $n$ in the preceding inequality. Thus we may add the positive operator $\phi(X)$ to (5.11) to obtain the desired inequality (5.10).

Now consider the closed subspace $K \subseteq F$ obtained by closing the range of the positive operator $X^{1/2}$. We will make $K$ into a pure Hilbert $A$-module as follows.

We claim first that there is a unique $d$-contraction $\tilde{T}_1, \ldots, \tilde{T}_d$ acting on $K$ such that
\[
T_kX^{1/2} = X^{1/2}\tilde{T}_k, \quad k = 1, 2, \ldots, d.
\]

Indeed, the uniqueness of $\tilde{T}_1, \ldots, \tilde{T}_d$ is clear from the fact that if $\xi \in K$ and $X^{1/2}\xi = 0$, then $\xi = 0$; that is simply because $K$ is the closure of the range of $X^{1/2}$, hence the kernel of the restriction of $X^{1/2}$ to $K$ is trivial.

In order to construct the operators $\tilde{T}_k$ it is easier to work with adjoints, and we will define operators $A_k = T_k^*$ as follows. Fix $k = 1, \ldots, d$ and $\xi \in F$. Then by (5.10) we have
\[
\|X^{1/2}T_k^*\xi\|^2 \leq \sum_{k=1}^d \|X^{1/2}T_k^*\xi\|^2 = \sum_{k=1}^d \langle T_kXT_k^*\xi, \xi \rangle \leq \langle X\xi, \xi \rangle = \|X^{1/2}\xi\|^2,
\]

and hence there is a unique contraction $A_k \in \mathcal{B}(K)$ such that
\begin{equation}
A_kX^{1/2} = X^{1/2}T_k^*, \quad k = 1, \ldots, d.
\end{equation}

As in the previous estimate (5.12) implies
\[
\sum_{k=1}^d \|A_kX^{1/2}\xi\|^2 \leq \|X^{1/2}\xi\|^2, \quad \xi \in F,
\]
and hence $A_k^*A_1 + \cdots + A_d^*A_d \leq 1_K$. Since the $T_k^*$ mutually commute, (5.12) implies that the $A_k$ must mutually commute, and we set $\tilde{T}_k = A_k^*$, $k = 1, \ldots, d$.

Next, we claim that $(\tilde{T}_1, \ldots, \tilde{T}_d)$ is a pure $d$-contraction in the sense that if $\tilde{\phi} : \mathcal{B}(K) \to \mathcal{B}(K)$ is the map defined by

$$\tilde{\phi}(A) = \sum_{k=1}^d \tilde{T}_k A \tilde{T}_k^*,$$

then $\tilde{\phi}^n(1_K) \downarrow 0$ as $n \to \infty$. Since $\{\tilde{\phi}^n(1_K) : n \geq 0\}$ is a uniformly bounded sequence of positive operators, the claim will follow if we show that

$$\lim_{n \to \infty} \langle \tilde{\phi}^n(1_K) \eta, \eta \rangle = 0$$

for all $\eta$ in the dense linear manifold $X^{1/2}F$ of $K$. But for $\eta = X^{1/2}\xi$, $\xi \in F$, we have

$$\langle \tilde{\phi}^n(1_K)X^{1/2}\xi, X^{1/2}\xi \rangle = \langle X^{1/2}\tilde{\phi}^n(1_K)X^{1/2}\xi, \xi \rangle.$$

Since $X^{1/2}\tilde{T}_k = T_kX^{1/2}$ for all $k$ it follows that $X^{1/2}\tilde{\phi}^n(1_K)X^{1/2} = \phi^n(X)$ for every $n = 0, 1, 2, \ldots$, hence

$$\langle \tilde{\phi}^n(1_K)X^{1/2}\xi, X^{1/2}\xi \rangle = \langle \phi^n(X)\xi, \xi \rangle \leq \|X\| \langle \phi^n(1)\xi, \xi \rangle$$

and the right side decreases to zero as $n \to \infty$ because the free module $F$ is pure.

Using the operators $\tilde{T}_1, \ldots, \tilde{T}_d \in \mathcal{B}(K)$ we now consider $K$ to be a pure Hilbert $A$-module. By the basic dilation theory (see Theorem 1.14) there is a free Hilbert $A$-module $\tilde{F}$ and a coisometry $U \in \text{hom}(\tilde{F}, K)$. Thus if we denote the canonical operators of $\tilde{F}$ by $S_1, \ldots, S_d$, then we have $US_k = \tilde{T}_kU$ for $k = 1, \ldots, d$. Now define a linear operator $\Phi \in \mathcal{B}(\tilde{F}, F)$ by $\Phi = X^{1/2}U$. We have

$$\Phi S_k X^{1/2}US_k = X^{1/2}\tilde{T}_kU = T_kX^{1/2}U = T_k\Phi,$$

so that $\Phi \in \text{hom}(\tilde{F}, F)$. Finally, since $UU^* = 1_K$, it follows that

$$\Phi\Phi^* = X^{1/2}UU^*X^{1/2} = X,$$

and the proof of Theorem 5.9 is complete.

\textbf{Remark.} Notice that the proof of Theorem 5.9 made no use of the finite dimensionality of $C$, and in fact this result is valid \textit{verbatim} for free Hilbert $A$-modules of arbitrary rank $1, 2, \ldots, \infty$.

We digress momentarily to record the following (a fact associated with the dilation theory of [1, Theorem 8.5]) for later use.
Corollary. Let $M$ be a closed submodule of $H^2 \otimes C$ and let $P_M$ be the projection onto $M$. Then there is a Hilbert space $E$ and a partial isometry $\Phi \in \text{hom}(H^2 \otimes E, H^2 \otimes C)$ such that $P_M = \Phi \Phi^*$.

proof. Let $S_1, \ldots, S_d$ be the canonical operators of $H^2 \otimes C$. Since

$$S_1 P_M S_1^* + \cdots + S_d P_M S_d^* \leq P_M$$

and since $bS_k = Z_k b$, $k = 1, \ldots, d$ we have

$$d \Gamma(P_M) = b(P_M - \sum_{k=1}^d S_k P_M S_k^*) b^* \geq 0$$

and the conclusion follows from condition (3) of Theorem 5.9. 

The importance of the cone

$$P = \{ X \in \mathcal{B}(F) : d \Gamma(X) \geq 0 \}$$

for this work is that for every operator $X$ in the complex vector space spanned by $P$, $d \Gamma(X)$ is trace class, and trace $d \Gamma(X)$ can be expressed in terms of $X$ by way of an asymptotic formula.

Theorem 5.13. Let $F = H^2 \otimes C$, where $C$ is a finite dimensional Hilbert space. For every operator $X$ in the complex linear span of $\{ X \in \mathcal{B}(F) : d \Gamma(X) \geq 0 \}$, $d \Gamma(X)$ belongs to $L^1(\partial F)$ and

$$\text{trace}(d \Gamma(X)) = \dim C \cdot \lim_{n \to \infty} \frac{\text{trace}(X E_n)}{\text{trace} E_n},$$

$E_0, E_1, \ldots$ being the sequence of spectral projections of the number operator of $F$.

Theorem 5.13 depends on a general identity, which we establish first.

Lemma 5.14. Let $F = H^2 \otimes C$ be as in Theorem 5.13. Then for every $X \in \mathcal{B}(F)$ and $n = 0, 1, 2, \ldots$ we have

$$\text{trace}(d \Gamma(X) \bar{P}_n) = \dim C \cdot \frac{\text{trace}(X E_n)}{\text{trace} E_n},$$

where $\bar{P}_n = \bar{E}_0 + \bar{E}_1 + \cdots + \bar{E}_n$, $\{ \bar{E}_n \}$ being the spectral projections of the number operator of $\partial F$.

Remark. Notice that all of the operators $E_n, X E_n, d \Gamma(X) \bar{P}_n$ appearing in Lemma 5.14 are of finite rank. Note two that traces on the right refer to the Hilbert space $F$, while the trace on the left refers to the Hilbert space $\partial F$.

proof of Lemma 5.14. Let $\bar{E}_n$ be the projection of $H^2$ onto its space of homogeneous polynomials of degree $n$. Then $E_n = \bar{E}_n \otimes 1_C$, and hence

$$\text{trace} E_n = \text{trace} \bar{E}_n \cdot \dim C = q_{d-1}(n) \cdot \dim C,$$

where $q_{d-1}(n)$ is the degree of the polynomial $n$. Thus

$$\text{trace}(d \Gamma(X) \bar{P}_n) = \dim C \cdot \frac{\text{trace}(X E_n)}{\text{trace} E_n},$$

as desired.
for all \( n = 0, 1, \ldots \) where \( q_{d-1}(x) \) is the polynomial of (5.6). Thus we have to show that

\[
\text{trace} \left( d\Gamma(X) \hat{P}_n \right) = \frac{\text{trace} \left( X E_n \right)}{q_{d-1}(n)}, \quad n = 0, 1, \ldots
\]

Using Proposition 5.7 we have \( q_{d-1}(n)^{-1} E_n = b^* b E_n = b^* \hat{E}_n b \) and the right side of (5.15) can be rewritten

\[
\frac{\text{trace} \left( X E_n \right)}{q_{d-1}(n)} = \text{trace} \left( X b^* \hat{E}_n b \right) = \text{trace} \left( \Gamma(X) \hat{E}_n \right).
\]

Setting \( Y = \Gamma(X) \in B(\partial F) \), equation (5.15) becomes

\[
\text{trace} \left( (Y - \phi_{\partial F}(Y)) \hat{P}_n \right) = \text{trace} \left( Y \hat{E}_n \right),
\]

where \( \phi_{\partial F} \) is the completely positive map on \( B(\partial F) \) defined by the canonical operators of \( \partial F \), \( \phi_{\partial F}(A) = Z_1 AZ_1^* + \cdots + Z_d AZ_d^* \).

Notice that since \( Z_1^* Z_1 + \cdots + Z_d^* Z_d = 1_{\partial F} \), \( \phi_{\partial F} \) leaves the trace invariant in the sense that for \( A \in L^1(\partial F) \) we have

\[
\text{trace} \left( \phi_{\partial F}(A) \right) = \text{trace} \left( \sum_{k=1}^d Z_k^* Z_k A \right) = \text{trace}(A).
\]

Moreover, the relations \( Z_k^* \hat{P}_n = \hat{P}_{n-1} Z_k^* \) imply that \( \phi_{\partial F}(A) \hat{P}_n = \phi_{\partial F}(A \hat{P}_{n-1}) \) for \( n = 0, 1, \ldots \), where of course \( \hat{P}_{n-1} \) is taken as 0. Thus

\[
\text{trace} \left( \phi_{\partial F}(Y) \hat{P}_n \right) = \text{trace} \left( \phi_{\partial F}(Y \hat{P}_{n-1}) \right) = \text{trace} \left( Y \hat{P}_{n-1} \right)
\]

and the left side of (5.16) can be written

\[
\text{trace} \left( Y \hat{P}_n \right) - \text{trace} \left( \phi_{\partial F}(Y) \hat{P}_n \right) = \text{trace} \left( Y \hat{P}_n \right) - \text{trace} \left( Y \hat{P}_{n-1} \right)
\]

which agrees with the right side of (5.16) because \( \hat{P}_n - \hat{P}_{n-1} = \hat{E}_n \).

\[
\begin{proof}
\text{proof of Theorem 5.13.} \text{ It suffices to show that for any operator } X \text{ in } B(F) \text{ for which } d\Gamma(X) \geq 0, \text{ we must have } \text{trace} \left( d\Gamma(X) \right) < \infty \text{ as well as the limit formula of 5.13. From Lemma 5.14 we have }
\end{proof}

\(*\)

\[
\text{trace} \left( d\Gamma(X) \hat{P}_n \right) = \dim C \cdot \frac{\text{trace} \left( X E_n \right)}{\text{trace} E_n}, \quad n = 0, 1, 2, \ldots
\]

Theorem 5.9 implies that \( X \) must be a positive operator and hence

\[
0 \leq \frac{\text{trace} \left( X E_n \right)}{\text{trace} E_n} \leq \|X\|
\]

for every \( n \) because \( \rho_n(A) = \text{trace} \left( A E_n \right) / \text{trace} E_n \) is a state of \( B(F) \). Since the projections \( \hat{P}_n \) increase to \( 1_{\partial F} \) with increasing \( n \) we conclude from (5.17) that

\[
\text{trace} \left( d\Gamma(X) \right) = \sup_{n \geq 0} \text{trace} \left( d\Gamma(X) \hat{P}_n \right) \leq \dim C \cdot \|X\| < \infty.
\]

Moreover, since in this case

\[
\text{trace} \left( d\Gamma(X) \right) = \lim_{n \to \infty} \text{trace} \left( d\Gamma(X) \hat{P}_n \right),
\]

we may infer the required limit formula directly from (5.17) as well.

\[
\begin{proof}
In view of Theorem 5.9, the following lemma shows how to compute the trace of \( d\Gamma(X) \) in the most important cases.
\end{proof}\]

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\end{proof}\]
Lemma 5.18. Let $F_k = H^2 \otimes C_k$, $k = 1, 2$, where $C_2$ is finite dimensional, and let $\Phi \in \text{hom}(F_1, F_2)$. Considering $\Phi$ as a multiplier in $\mathcal{M}(C_1, C_2)$ with boundary value function $\tilde{\Phi} : \partial B_d \to \mathcal{B}(C_1, C_2)$ we have
\[
\text{trace } d\Gamma(\Phi^*) = \int_{\partial B_d} \text{trace } (\tilde{\Phi}(z)\tilde{\Phi}(z)^*) \, d\sigma(z).
\]

Remark. Note that for $\sigma$-almost every $z \in \partial B_d$, $\tilde{\Phi}(z)\tilde{\Phi}(z)^*$ is a positive operator in $\mathcal{B}(C_2)$, and since $C_2$ is finite dimensional the right side is well defined and dominated by $\|\Phi\|^2 \cdot \dim C_2$.

proof of Lemma 5.18. Consider the linear operator $A : C_1 \to H^2(\partial B_d; C_2)$ defined by $A\zeta = b(\Phi(1 \otimes \zeta))$, $\zeta \in C_1$. We claim first that
\[
d\Gamma(\Phi^*) = AA^*.
\]
Indeed, since $b\Phi \in \text{hom}(F_1, H^2(\partial B_d, C_2))$ we have
\[
\sum_{k=1}^d Z_k b\Phi^* b^* Z_k^* = \sum_{k=1}^d Z_k (b\Phi)(b\Phi)^* Z_k^* = b\Phi(\sum_{k=1}^d T_k T_k^*)(b\Phi)^*,
\]
where $T_1, \ldots, T_d$ are the canonical operators of $F_1 = H^2 \otimes C_1$, and hence
\[
d\Gamma(\Phi^*) = b\Phi(b\Phi)^* - \sum_{k=1}^d Z_k b\Phi^* b^* Z_k^* = b\Phi(1 - \sum_{k=1}^d T_k T_k^*)(b\Phi)^*.
\]
The operator $1 - \sum_k T_k T_k^*$ is the projection of $F_1 = H^2 \otimes C_1$ onto its space of $C_1$-valued constant functions and, denoting by $[1]$ the projection of $H^2$ onto the one dimensional space of constants $\mathbb{C}1$, the preceding formula becomes
\[
d\Gamma(\Phi^*) = b\Phi([1] \otimes 1_{C_1})(b\Phi)^* = AA^*,
\]
as asserted in (5.19).

Now fix an orthonormal basis $e_1, e_2, \ldots$ for $C_1$. By formula (5.19) we can evaluate the trace of $d\Gamma(\Phi^*)$ in terms of the vector functions $Ae_n \in H^2(\partial B_d, C_2)$ as follows,
\[
\text{trace } d\Gamma(\Phi^*) = \text{trace } H^2(\partial B_d; C_2)(AA^*) = \text{trace } C_1(A^*A)
\]
\[= \sum_n \|\Phi(1 \otimes e_n)\|_H^2(\partial B_d; C_2)^2.
\]

Turning now to the term on the right in Lemma 5.18, we first consider $Ae_n = b(\Phi(1 \otimes e_n))$ as a function from the open ball $B_d$ to $C_2$. In terms of the multiplier $\Phi(\cdot)$ of $\Phi$ we have
\[
Ae_n(z) = b(\Phi(1 \otimes e_n))(z) = \Phi(z)e_n
\]
and hence the boundary values $\tilde{A}e_n$ of $Ae_n$ are given by $\tilde{A}e_n(z) = \tilde{\Phi}(z)e_n$ for $\sigma$-almost every $z \in \partial B_d$. Thus for such $z \in \partial B_d$ we have
\[
\text{trace } C_2(\tilde{\Phi}(z)\tilde{\Phi}(z)^*) = \text{trace } C_1(\tilde{\Phi}(z)^*\tilde{\Phi}(z)) = \sum_n \|\tilde{\Phi}(z)e_n\|^2 = \sum_n \|\tilde{A}e_n(z)\|^2.
\]
Integrating the latter over the sphere we obtain
\[
\int_{\partial B_d} \text{trace } C_2(\tilde{\Phi}(z)\tilde{\Phi}(z)^*) \, d\sigma = \sum_n \|\tilde{A}e_n\|_H^2(\partial B_d; C_2)^2
\]
and from (5.20) we see that this coincides with trace $d\Gamma(\Phi^*)$.

We now establish the main asymptotic formula for $K(H)$. 

Theorem D. For every finite rank Hilbert $A$-module $H$, the curvature operator $d\Gamma(U_0^*U_0)$ belongs to the trace class $L^1(H^2(\partial B_d; \Delta H))$, and we have

$$K(H) = \text{trace } d\Gamma(U_0^*U_0) = d! \lim_{n \to \infty} \frac{\text{trace}(1 - \phi^{n+1}(1))}{n^d}$$

where $\phi : B(H) \to B(H)$ is the canonical completely positive map associated with the $A$-module structure of $H$.

Let $\Delta = (1 - \phi(1))^{1/2}$. We will actually prove a slightly stronger assertion, namely

$$(5.21) \quad K(H) = \text{trace } d\Gamma(U_0^*U_0) = (d - 1)! \lim_{n \to \infty} \frac{\text{trace}(\phi^n(\Delta^2))}{n^{d-1}}.$$

We first point out that it suffices to prove (5.21). For that, let $a_k = \text{trace} \phi^k(\Delta^2)$, $k = 0, 1, 2, \ldots$. Since

$$1 - \phi^{n+1}(1) = \sum_{k=0}^{n} \phi^k(1 - \phi(1)) = \sum_{k=0}^{n} \phi^k(\Delta^2)$$

and since for every $r = 1, 2, \ldots$

$$q_r(n) = \frac{(n + 1) \ldots (n + r)}{r!} \sim \frac{n^r}{r!},$$

we have

$$d! \frac{\text{trace}(1 - \phi^{n+1}(1))}{n^d} \sim \frac{\text{trace}(1 - \phi^{n+1}(1))}{q_d(n)} = \frac{a_0 + a_1 + \cdots + a_n}{q_d(n)}$$

while

$$(d - 1)! \frac{\text{trace}(\phi^n(\Delta^2))}{n^{d-1}} \sim \frac{\text{trace}(\phi^n(\Delta^2))}{q_{d-1}(n)} = \frac{a_n}{q_{d-1}(n)}$$

Thus the following elementary lemma allows one to deduce Theorem D from (5.21).

Lemma 5.22. Let $d = 1, 2, \ldots$ and let $a_0, a_1, \ldots$ be a sequence of real numbers such that

$$\lim_{n \to \infty} \frac{a_n}{q_{d-1}(n)} = L \in \mathbb{R}.$$  

Then

$$\lim_{n \to \infty} \frac{a_0 + a_1 + \cdots + a_n}{q_d(n)} = L.$$  

proof of Lemma 5.22. Choose $\epsilon > 0$. By hypothesis, there is an $n_0 \in \mathbb{N}$ such that

$$(5.23) \quad (L - \epsilon)q_{d-1}(k) \leq a_k \leq (L + \epsilon)q_{d-1}(k), \quad k \geq n_0.$$  

By the recursion formula (3.2.2) we have

$$\sum_{k=n_0}^{n} q_{d-1}(k) = \sum_{k=n_0}^{n} (q_d(k) - q_d(k - 1)) = q_d(n) - q_d(n_0 - 1).$$
Thus if we sum (5.26) from $n_0$ to $n$ and divide through by $q_d(n)$ we obtain

$$(L - \epsilon)(1 - \frac{q_d(n_0 - 1)}{q_d(n)}) \leq \frac{a_0 + \cdots + a_n}{q_d(n)} \leq (L + \epsilon)(1 - \frac{q_d(n_0 - 1)}{q_d(n)}).$$

Since $q_d(n) \to \infty$ as $n \to \infty$, the latter inequality implies

$$L - \epsilon \leq \liminf_{n \to \infty} \frac{a_0 + \cdots + a_n}{q_d(n)} \leq \limsup_{n \to \infty} \frac{a_0 + \cdots + a_n}{q_d(n)} \leq L + \epsilon,$$

and since $\epsilon$ is arbitrary, 5.24 follows.

\textit{proof of Theorem D.} Let $U_0 : H^2 \otimes \Delta H \to H$ be the homomorphism $U_0(f \otimes \zeta) = f \cdot \Delta \zeta$ defined in section 1 and discussed above. We claim that for every $n = 0, 1, \ldots$ (5.24) \[ \text{trace } \phi^n(\Delta^2) = \text{trace } (U_0^*U_0E_n), \]

$E_n \in \mathcal{B}(H^2 \otimes \Delta H)$ being the projection onto the space of homogeneous polynomials of degree $n$. Indeed, from the discussion preceding (1.13) we have

$$U_0U_0^* = 1 - \phi^\infty(1),$$

and since $\phi(\phi^\infty(1)) = \phi^\infty(1)$ we can write

$$\Delta^2 = 1 - \phi(1) = (1 - \phi^\infty(1)) - \phi(1 - \phi^\infty(1)) = U_0U_0^* - \phi(U_0U_0^*).$$

Thus

(5.25) \[ \phi^n(\Delta^2) = \phi^n(U_0U_0^*) - \phi^{n+1}(U_0U_0^*). \]

Write $F = H^2 \otimes \Delta H$, and let $\phi_F : \mathcal{B}(F) \to \mathcal{B}(F)$ denote its canonical completely positive map. Since $U_0 \in \text{hom}(F,H)$ we have

$$\phi^k(U_0U_0^*) = U_0\phi^k_F(1_F)U_0^*$$

for every $k = 0, 1, \ldots$. Moreover,

$$\phi^n_F(1_F) - \phi^{n+1}_F(1_F) = \phi^n_F(1_F - \phi_F(1_F)) = \phi^n_F(E_0) = E_n,$$

so that (5.25) implies

$$\phi^n(\Delta^2) = U_0E_nU_0^*.$$ 

The formula (5.24) follows immediately since

$$\text{trace }_H(U_0E_nU_0^*) = \text{trace }_F(U_0^*U_0E_n).$$

By Proposition 4.3 there is a free module $\tilde{F}$ and $\Phi \in \text{hom}(\tilde{F}, F)$ such that

$$U_0^*U_0 = 1_F - \Phi\Phi^*.$$
Since both $d\Gamma(1_F)$ and $d\Gamma(\Phi\Phi^*)$ are positive operators by Theorem 5.9 (indeed $d\Gamma(1_F)$ is the projection of $H^2(\partial B_d; \Delta H)$ onto its subspace of constant functions), it follows from Theorem 5.13 that the curvature operator $d\Gamma(U_0^*U_0)$ is trace class and, in view of (5.24), satisfies

$$
(5.26) \quad \text{trace } d\Gamma(U_0^*U_0) = \lim_{n \to \infty} \frac{\text{trace } (U_0^*U_0E_n)}{q_{d-1}(n)} = \lim_{n \to \infty} \frac{\text{trace } \phi^n(\Delta^2)}{q_{d-1}(n)}.
$$

Finally, we use (5.18) together with $U_0^*U_0 = 1_F - \Phi\Phi^*$ to evaluate the left side of (5.26) and we find that

$$
\text{trace } d\Gamma(U_0^*U_0) = \int_{\partial B_d} \text{trace } (1 - \tilde{\Phi}(z)\tilde{\Phi}(z)^*) d\sigma(z).
$$

Formula (4.10) shows that the term on the right is $K(H)$. □

As in the case of the Euler characteristic, the asymptotic formula of Theorem D leads to the following result on stability under finite dimensional perturbations.

\textbf{Corollary 1: stability of curvature.} Let $H_0$ be a closed submodule of a finite rank Hilbert $A$-module $H$ such that $\dim(H/H_0) < \infty$. Then $K(H) = K(H_0)$.

\textit{proof.} This is proved by estimating exactly as in the proof of the corollary of Theorem C. Indeed, the estimates here are simpler because the trace is a linear functional. One finds that

$$
|\text{trace } (1 - \phi_H^{n+1}(1_H)) - \text{trace } (1 - \phi_H^{n+1}(1_H))| \leq \dim(H/H_0)(1 + q_{d-1}(n + 1)).
$$

As in the proof of the corollary of Theorem C, one can multiply through by $d!/n^d$ and take the limit on $n$ to obtain the required relation $|K(H) - K(H_0)| \leq 0$. □

We also point out the following application to invariant subspaces of the $d$-shift $S_1, \ldots, S_d$ acting on $H^2$. In dimension $d = 1$ the invariant subspaces of the simple unilateral shift define submodules which are isomorphic to $H^2$ itself, and in particular they all have rank one. In higher dimensions, on the other hand, we can never have that behavior for the ranks of submodules of finite codimension.

\textbf{Corollary 2.} Suppose that $d \geq 2$, and let $M$ be a proper closed submodule of $H^2$ of finite codimension. Then $\text{rank } M \geq 2$.

\textit{proof.} Theorem 4.13 implies that $K(H^2) = 1$, and hence corollary 1 above implies that $K(M) = 1$ as well. By [1,Lemma 7.14], no proper submodule of $H^2$ can be a free Hilbert module in dimension $d > 1$, hence by the extremal property of $K(M)$ (Theorem 4.13) we must have $\text{rank } M > K(M) = 1$. □

\textit{Remark.} Of course, the ranks of finite codimensional submodules of $H^2$ must be finite by Remark 3.14, and they can be arbitrarily large.

Since the trace of a positive operator $A$ is dominated by $\|A\| \cdot \text{rank } A$, Theorems C and D together imply that $K(H) \leq \chi(H)$, and we conclude
Corollary 3. For every finite rank Hilbert $A$-module $H$,

$$0 \leq K(H) \leq \chi(H) \leq \text{rank}(H).$$

In the next section we will show that $K(H) = \chi(H)$ for graded Hilbert modules, but in section 9 we give examples of ungraded Hilbert modules for which $K(H) < \chi(H)$. The inequality of Corollary 3 is useful; a significant application is given in Theorem E of section 8.

6. Graded Hilbert modules.

In this section we prove an analogue of the Gauss-Bonnet-Chern theorem for Hilbert $A$-modules. The most general setting in which one might hope for such a result is the class of finite rank pure Hilbert $A$-modules. By the discussion of section 1, these are the Hilbert $A$-modules which are isomorphic to quotients $F/M$ of finite rank free modules $F$ by closed submodules $M$. However, in Proposition 9.2 we give examples of submodules $M \subseteq H^2$ for which $K(H^2/M) \neq \chi(H^2/M)$. In this section we establish the result (Theorem B) under the additional hypothesis that $H$ is graded. Examples are obtained by taking $H = F/M$ where $F$ is free of finite rank and $M$ is a closed submodule generated by a set of homogeneous polynomials (perhaps of different degrees). In particular, one can associate such a module $H$ with any algebraic variety in complex projective space $\mathbb{P}^{d-1}$ (see section 9).

By a graded Hilbert space we mean a pair $H, \Gamma$ where $H$ is a (separable) Hilbert space and $\Gamma : T \to B(H)$ is a strongly continuous unitary representation of the circle group $T = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$. $\Gamma$ is called the gauge group of $H$. Alternately, one may think of the structure $H, \Gamma$ as a $\mathbb{Z}$-graded Hilbert space by considering the spectral subspaces $\{ H_n : n \in \mathbb{Z} \}$ of $\Gamma$,

$$H_n = \{ \xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi, \ \lambda \in T \}.$$ 

The spectral subspaces give rise to an orthogonal decomposition

$$H = \cdots \oplus H_{-1} \oplus H_0 \oplus H_1 \oplus \cdots.$$ 

Conversely, given an orthogonal decomposition of a Hilbert space $H$ of the form (6.1), one can define an associated gauge group $\Gamma$ by

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n \quad \lambda \in T$$

$E_n$ being the orthogonal projection onto $H_n$.

A Hilbert $A$-module is said to be graded if there is given a distinguished gauge group $\Gamma$ on $H$ which is related to the canonical operators $T_1, \ldots, T_d$ of $H$ by

$$\Gamma(\lambda)T_k \Gamma(\lambda)^{-1} = \lambda T_k, \quad k = 1, \ldots, d, \ \lambda \in T.$$ 

Thus, graded Hilbert $A$-modules are those whose operators admit minimal (i.e., circular) symmetry. Letting $H_n$ be the $n$th spectral subspace of $\Gamma$, (6.2) implies that each operator is of degree one in the sense that

$$T_k H_n \subseteq H_{n+1}, \quad k = 1, \ldots, d, \ n \in \mathbb{Z}.$$
Conversely, given a $\mathbb{Z}$-graded Hilbert space which is also an $A$-module satisfying (6.3), then it follows that the corresponding gauge group

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n$$

satisfies (6.2), and moreover that the spectral projections $E_n$ of $\Gamma$ satisfy $T_k E_n = E_{n+1} T_k$ for $k = 1, \ldots, d$. Thus it is equivalent to think in terms of gauge groups satisfying (6.2), or of $\mathbb{Z}$-graded Hilbert $A$-modules with degree-one operators satisfying (6.3). Algebraists tend to prefer the latter description because it generalizes to fields other than the complex numbers. On the other hand, the former description is more convenient for operator theory on complex Hilbert spaces, and in this section we work mainly with gauge groups and (6.2).

Let $H$ be a graded Hilbert $A$-module. A linear subspace $S \subseteq H$ is said to be \textit{graded} if $\Gamma(\lambda) S \subseteq S$ for every $\lambda \in \mathbb{T}$. If $K \subseteq H$ is a graded (closed) submodule of $H$ then $K$ is a graded Hilbert $A$-module, and the gauge group of $K$ is of course the corresponding subrepresentation of $\Gamma$. Similarly, the quotient $H/K$ of $H$ by a graded submodule $K$ is graded in an obvious way. We require the following observation, asserting that several natural hypotheses on graded Hilbert modules are equivalent.

**Proposition 6.4.** For every graded finite rank Hilbert $A$-module $H$, the following are equivalent.

1. The spectrum of the gauge group $\Gamma$ is bounded below.
2. $H$ is pure in the sense that its associated completely positive map of $\mathcal{B}(H)$$\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*$ satisfies $\phi^n(1) \downarrow 0$ as $n \to \infty$.
3. The algebraic submodule

$$M_H = \operatorname{span}\{ f \cdot \Delta \xi : f \in A, \ \xi \in \Delta H \}$$

is dense in $H$.
4. There is a finite-dimensional graded linear subspace $G \subset H$ which generates $H$ as a Hilbert $A$-module.

Moreover, if (1) through (4) are satisfied then the spectral subspaces of $\Gamma$,

$$H_n = \{ \xi \in H : \Gamma(\lambda) \xi = \lambda^n \xi \ \ \ n \in \mathbb{Z} \},$$

are all finite dimensional.

**proof.** We prove that ($1 \implies 2 \implies 3 \implies 4 \implies 1$).

Let $E_n$ be the projection onto the $n$th spectral subspace $H_n$ of $\Gamma$ and let $T_1, \ldots, T_d$ be the canonical operators of $H$. From the commutation formula (6.2) it follows that $T_1 T_1^* + \cdots + T_d T_d^*$ commutes with $\Gamma(\lambda)$ and hence

$$(\lambda - 1) T_1 T_1^* - \cdots - T_d T_d^*)^{1/2}.$$

**proof of ($1 \implies 2$).** The hypothesis ($1$) implies that there is an integer $n_0$ such that $E_n = 0$ for $n < n_0$. By the preceding remarks we have $T_k E_p = E_{p+1} T_k$ for every $p \in \mathbb{Z}$. Thus $\phi(E_p) = E_{p+1} \phi(1) \leq E_{p+1}$, and hence $\phi^n(E_p) \leq E_{p+n}$. Writing

$$\phi^n(1) = \phi^n(\sum_{p=n_0}^{\infty} E_p) = \sum_{p=n_0}^{\infty} \phi^n(E_p) \leq \sum_{p=n_0+n}^{\infty} E_p,$$
the conclusion \( \lim_{n} \phi^n(1) = 0 \) is apparent.

**proof of (2) \( \Rightarrow \) (3).** Assuming \( H \) is pure, (1.13) implies that the natural map \( U_0 \in \text{hom}(H^2 \otimes \Delta H, H) \) defined by \( U_0(f \otimes \zeta) = f \cdot \Delta \zeta \) satisfies \( U_0U_0^* = 1 \), and therefore \( M_H = U_0(U_0(H^2 \otimes \Delta H) = H) \).

**proof of (3) \( \Rightarrow \) (4).** Assuming (3), notice that \( G = \Delta H \) satisfies condition (4). Indeed, \( G \) is finite dimensional because \( \text{rank}(H) < \infty \), it is graded because of (6.5), and it generates \( H \) as a closed \( A \)-module because the \( A \)-module \( M_H \) generated by \( G \) is dense in \( H \).

**proof of (4) \( \Rightarrow \) (1).** Let \( G \subseteq H \) satisfy (4). The restriction of \( \Gamma \) to \( G \) is a finite direct sum of irreducible subrepresentations, and hence there are integers \( n_0 \leq n_1 \) such that

\[
G = G_{n_0} \oplus G_{n_0+1} \oplus \cdots \oplus G_{n_1}
\]

where \( G_k = G \cap H_k \). In particular, \( G \subseteq H_{n_0} + H_{n_0+1} + \ldots \). Since the space \( H_{n_0} + H_{n_0+1} + \ldots \) is invariant under the operators \( T_1, \ldots, T_d \) by (6.3), we have

\[
H = \text{span}A \cdot G \subseteq H_{n_0} + H_{n_0+1} + \ldots
\]

Thus \( H = H_{n_0} + H_{n_0+1} + \ldots \), hence the spectrum of \( \Gamma \) is bounded below by \( n_0 \).

The finite dimensionality of all of the spectral subspaces of \( \Gamma \) follows from condition (4), together with the fact that for every \( n = 0, 1, 2, \ldots \), the space \( P_n \) of operators \( \{f(T_1, \ldots, T_d)\} \) where \( f \) is a homogeneous polynomial of degree \( n \) is finite dimensional and and \( P_n \) maps \( H_k \) into \( H_{k+n} \).

**Theorem B.** For every finite rank graded Hilbert \( A \)-module \( H \) satisfying the conditions of Proposition 6.4 we have \( K(H) = \chi(H) \).

**proof.** Because of the stability properties of \( \chi(\cdot) \) and \( K(\cdot) \) established in the corollaries of Theorems C and D, it suffices to exhibit a closed submodule \( H_0 \subseteq H \) of finite codimension for which \( K(H_0) = \chi(H_0) \). \( H_0 \) is constructed as follows.

Let \( \{E_n : n \in \mathbb{Z}\} \) be the spectral projections of the gauge group

\[
\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n.
\]

Since \( \Delta \) is a finite rank operator in the commutant of \( \{E_n : n \in \mathbb{Z}\} \), we must have \( E_n \Delta = \Delta E_n = 0 \) for all but a finite number of \( n \in \mathbb{Z} \), and hence there are integers \( n_0 \leq n_1 \) such that

\[
\Delta = \Delta_{n_0} + \Delta_{n_0+1} + \cdots + \Delta_{n_1}
\]

\( \Delta_k \) denoting the finite rank positive operator \( \Delta E_k \).

We claim that for all \( n \geq n_1 \) we have

\[
\phi(E_n) = E_{n+1}.
\]
Indeed, since $H$ is pure (Proposition 6.4 (2)) we can assert that

\[(6.8) \quad 1_H = \sum_{p=0}^{\infty} \phi^p(\Delta^2)\]

because

\[
\sum_{p=0}^{n} \phi^p(\Delta^2) = \sum_{p=0}^{n} \phi^p(1_H - \phi(1_H)) = 1_H - \phi^{n+1}(1_H)
\]

converges strongly to $1_H$ as $n \to \infty$. Multiplying (6.8) on the left with $E_n$ we find that

\[(6.9) \quad E_n = \sum_{p=0}^{\infty} E_n \phi^p(\Delta^2), \quad n \in \mathbb{Z}.
\]

Using (6.6) we have

\[
E_n \phi^p(\Delta^2) = \sum_{k=n_0}^{n_1} E_n \phi^p(\Delta_k^2).
\]

Now $\Delta_k^2 \leq E_k$ and hence $\phi^p(\Delta_k^2) \leq E_{k+p}$ for every $p = 0, 1, \ldots$. Thus for $n \geq n_1$,\[
\sum_{p=0}^{\infty} \sum_{k=n_0}^{n_1} E_n \phi^p(\Delta_k^2) = \sum_{k=n_0}^{n_1} \phi^{n-k}(\Delta_k^2) = \phi^{n-n_1} \left( \sum_{k=n_0}^{n_1} \phi^{-k}(\Delta_k^2) \right).
\]

This shows that when $n \geq n_1$, $E_n$ has the form

\[(6.10) \quad E_n = \phi^{n-n_1}(B),
\]

where $B$ is the operator \[
B = \sum_{k=n_0}^{n_1} \phi^{n_1-k}(\Delta_k^2),
\]

and (6.7) follows immediately from (6.10).

Now consider the submodule $H_0 \subseteq H$ defined by\[
H_0 = \sum_{n=n_1}^{\infty} E_n H.
\]

Notice that $H_0^\perp$ is finite dimensional. Indeed, that is apparent from the fact that \[
H_0^\perp = \sum_{n=-\infty}^{n_1-1} E_n H
\]

because by Proposition 6.4 (1) only a finite number of the projections \{$E_n : n < n_1$\} can be nonzero (indeed, here one can show that $E_n = 0$ for $n < n_0$), and Proposition 6.4 also implies that $E_n$ is finite dimensional for all $n$. 
Let $\phi_0 : B(H_0) \to B(H_0)$ be the completely positive map of $B(H_0)$ associated with the operators $T_1 |_{H_0}, \ldots, T_d |_{H_0}$. Then for every $k = 0, 1, \ldots$ we have

$$\phi_0^k(1_{H_0}) = \sum_{n=n_1}^{\infty} \phi^k(E_n).$$

From (6.7) we have $\phi^k(E_n) = E_{n+k}$ for $n \geq n_1$, and hence

$$\phi_0^k(1_{H_0}) = \sum_{p=n_1+k}^{\infty} E_p.$$ 

It follows that

$$1_{H_0} - \phi_0^{k+1}(1_{H_0}) = E_{n_1} + E_{n_1+1} + \cdots + E_{n_1+k}$$

is a projection for every $k = 0, 1, \ldots$. Thus for every $k \geq 0$,

$$\text{trace} (1_{H_0} - \phi_0^{k+1}(1_{H_0})) = \text{rank}(1_{H_0} - \phi_0^{k+1}(1_{H_0})), $$

and the desired formula $K(H_0) = \chi(H_0)$ follows immediately from Theorems C and D after multiplying through by $d!/k^d$ and taking the limit on $k$.

7. Degree.

Theorem C shows that the Euler characteristic (of a finite rank Hilbert $A$-module) vanishes whenever the rank function $\text{rank}(1 - \phi^{n+1}(1))$ grows relatively slowly. In such cases there are other numerical invariants which must be nontrivial and which can be calculated explicitly in certain cases. In this brief section we define these secondary invariants and summarize their basic properties.

Let $H$ be a finite rank Hilbert $A$-module. Consider the algebraic submodule

$$M_H = \text{span}\{ f \cdot \Delta \xi : f \in A, \ \xi \in H\}$$

and its natural filtration $\{M_n : n = 0, 1, 2, \ldots\}$

$$M_n = \text{span}\{ f \cdot \Delta \xi : \deg f \leq n, \ \xi \in H\}.$$ 

By Theorem 3.4 there are integers $c_0, c_1, \ldots, c_d$ such that

$$(7.1) \quad \dim M_n = c_0 q_0(n) + c_1 q_1(n) + \cdots + c_d q_d(n)$$

for sufficiently large $n$. Let $k$ be the degree of the polynomial on the right of (7.1). We observe first that the pair $(k, c_k)$ depends only on the algebraic structure of $M_H$.

**Proposition 7.2.** Let $M$ be a finitely generated $A$-module, let $\{M_n : n \geq 1\}$ be a proper filtration of $M$, and suppose $M \neq \{0\}$. Then there is a unique integer $k$, $0 \leq k \leq d$, such that the limit

$$\mu(M) = k! \lim_{n \to \infty} \frac{\dim M_n}{n^k}$$

is non-zero.
exists and is nonzero. \( \mu(M) \) is a positive integer and the pair \((k, \mu(M))\) does not depend on the particular filtration \( \{M_n\} \).

**proof.** By Theorem 3.4 there are integers \( c_0, c_1, \ldots, c_d \) such that

\[
\dim M_n = c_0 q_0(n) + c_1 q_1(n) + \cdots + c_d q_d(n)
\]

for sufficiently large \( n \). Let \( k \) be the degree of the polynomial on the right. Noting that \( q_r(x) \) is a polynomial of degree \( r \) with leading coefficient \( 1/r! \), it is clear that this \( k \) is the unique integer with the stated property and that

\[
\mu = k! \lim_{n \to \infty} \frac{\dim M_n}{n^k} = c_k
\]

is a (necessarily positive) integer.

To see that \((k, \mu)\) does not depend on the filtration, let \( \{M'_n\} \) be a second proper filtration. \( \{M'_n\} \) gives rise to a polynomial \( p'(x) \) of degree \( k' \) which satisfies \( \dim M'_n = p'(n) \) for sufficiently large \( n \). As in the proof of Proposition 3.7, there is an integer \( p \) such that \( \dim M_n \leq \dim M'_{n+p} \) for sufficiently large \( n \). Thus

\[
0 < k! \lim_{n \to \infty} \frac{\dim M_n}{n^k} \leq k! \limsup_{n \to \infty} \frac{\dim M'_{n+p}}{n^k}. 
\]

Now if \( k \) were greater than \( k' \) then the term on the right would be 0. Hence \( k \leq k' \) and, by symmetry, \( k = k' \).

We may now argue exactly as in the proof of Proposition 3.7 to conclude that the leading coefficients of the two polynomials must be the same, hence \( \mu = \mu' \).

**Definition 7.3.** Let \( H \) be a Hilbert \( A \)-module of finite positive rank. The degree of the polynomial (7.1) associated with any proper filtration of the algebraic module \( M_H \) is called the degree of \( H \), and is written \( \deg(H) \).

We will also write \( \mu(H) \) for the integer

\[
\mu(H) = \deg(H)! \lim_{n \to \infty} \frac{\dim M_n}{n^{\deg(H)}}
\]

associated with the degree of \( H \). If \( M_H \) is finite dimensional and not \( \{0\} \) then the sequence of dimensions \( \dim M_n \) associated with any proper filtration \( \{M_n\} \) is eventually a nonzero constant, hence \( \deg(H) = 0 \) and \( \mu(H) = \dim(H) \); conversely, if \( \deg(H) = 0 \) then \( M_H \) is finite dimensional. In particular, \( \deg H \) is a positive integer satisfying \( \deg(H) \leq d \) whenever the algebraic submodule \( M_H \) is infinite dimensional.

Note too that \( \deg(H) = d \) iff the Euler characteristic is positive, and in that case we have \( \mu(H) = \chi(H) \). In general, there is no obvious relation between \( \deg(H) \) and \( \rank(H) \), or between \( \mu(H) \) and \( \rank(H) \). In particular, \( \mu(H) \) can be arbitrarily large. The operator-theoretic significance of the invariant \( \mu(H) \) is not yet understood. An example for which \( 1 < \deg(H) < d \) is worked out in section 9.

Finally, let \( \phi \) be the completely positive map associated with the canonical operators \( T_1, \ldots, T_d \),

\[
\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*, \quad A \in B(H),
\]
and consider the generating function of the of the sequence of integers \(\operatorname{rank}(1 - \phi^{n+1}(1))\), \(n = 0, 1, 2, \ldots\), defined as the formal power series

\[
\hat{\phi}(t) = \sum_{n=0}^{\infty} \operatorname{rank}(1 - \phi^{n+1}(1)) t^n.
\]

We require the following description of \(\deg(H)\) and \(\mu(H)\) in terms of \(\hat{\phi}(t)\).

**Proposition 7.5.** The series \(\hat{\phi}(t)\) converges for every \(t\) in the open unit disk of the complex plane. There is a polynomial \(p(t) = a_0 + a_1 t + \cdots + a_s t^s\) and a sequence \(c_0, c_1, \ldots, c_d\) of real numbers, not all of which are 0, such that

\[
\hat{\phi}(t) = p(t) + \sum_{k=0}^{d} c_k (1 - t)^{k+1}, \quad |t| < 1.
\]

This decomposition is unique, and \(c_k\) belongs to \(\mathbb{Z}\) for every \(k = 0, 1, \ldots, d\). \(\deg(H)\) is the largest \(k\) for which \(c_k \neq 0\), and \(\mu(H) = c_k\).

**proof.** The proof of Theorem C shows that \(\operatorname{rank}(1 - \phi^{n+1}(1)) = \dim M_n\), where \(\{M_n : n = 1, 2, \ldots\}\) is the natural filtration of \(M_H\),

\[
M_n = \text{span}\{f \cdot \xi : \deg(f) \leq n, \quad \xi \in \Delta H\}.
\]

Since each \(q_r(x)\) is a polynomial of degree \(r\), formula (7.1) implies that there is a constant \(K > 0\) such that

\[
\dim M_n \leq Kn^d, \quad n = 1, 2, \ldots,
\]

and this estimate implies that the power series \(\sum_n \dim M_n t^n\) converges absolutely for every complex number \(t\) in the open unit disk.

Note too that for every \(k = 0, 1, \ldots, d\) the generating function for the sequence \(q_k(n), \ n = 0, 1, \ldots\) is given by

\[
\hat{q}_k(t) = \sum_{n=0}^{\infty} q_k(n) t^n = (1 - t)^{-k-1}, \quad |t| < 1.
\]

Indeed, the formula is obvious for \(k = 0\) since \(q_0(n) = 1\) for every \(n\); and for positive \(k\) the recurrence formula 3.2.2, together with \(q_k(0) = 1\), implies that

\[
(1 - t)\hat{q}_k(t) = q_{k-1}(t),
\]

from which (7.6) follows immediately.

Using (7.1) and (7.6) we find that there is a polynomial \(f(x)\) such that

\[
\hat{\phi}(t) = f(t) + \sum_{k=0}^{d} \frac{c_k}{(1 - t)^{k+1}},
\]

as asserted.

(7.7) implies that \(\hat{\phi}\) extends to a meromorphic function in the entire complex plane, having a single pole at \(t = 1\). The uniqueness of the representation of (7.7) follows from the uniqueness of the Laurent expansion of an analytic function around a pole. The remaining assertions of Proposition 7.5 are now obvious from the relation that exists between (7.1) and (7.6).
8. Applications: inner sequences and graded ideals.

Let $M$ be a closed submodule of the free Hilbert module $H^2 = H^2(\mathbb{C}^d)$ in dimension $d \geq 2$. Theorem 5.9 implies that there is a (finite or infinite) sequence of multipliers $\phi_n$ of $H^2$ whose associated multiplication operators $M_{\phi_n}$ satisfy

\[ \sum_n M_{\phi_n} M_{\phi_n}^* = P_M, \]

$P_M$ denoting the orthogonal projection of $H^2$ on $M$.

Remarks. If $\Phi = \{\phi_1, \phi_2, \ldots\}$ is a finite set then we require that it be linearly independent. If $\Phi$ is infinite then the appropriate condition is that for every sequence $\lambda = (\lambda_n) \in \ell^2$ we have

\[ \sum_{n=1}^{\infty} \lambda_n \phi_n = 0 \implies \lambda_1 = \lambda_2 = \cdots = 0. \]

It is best to think of conditions (8.1) and (8.2) in terms of dilation theory. (8.1) asserts that the operator $A \in \text{hom}(H^2 \otimes \ell^2, H^2)$ defined by $A(f \otimes \lambda) = f \cdot \sum_n \lambda_n \phi_n$ gives rise to a dilation of the pure Hilbert $A$-module $M$

\[ H^2 \otimes \ell^2 \overset{A}{\longrightarrow} M \longrightarrow 0 \]

and (8.2) asserts that it is the minimal dilation of $M$ in that condition (4) of Proposition 1.9 is satisfied.

The uniqueness of minimal dilations leads to the following description of all possible sequences $\Psi = \{\psi_1, \psi_2, \ldots\}$ which also satisfy (8.1) and (8.2). For definiteness, suppose that $\Phi$ is infinite. Then $\Psi$ must also be infinite and it is related to $\Phi$ through an infinite unitary matrix of scalars $(\lambda_{ij})$ as follows,

\[ \psi_k = \sum_{j=1}^{\infty} \lambda_{kj} \phi_j, \quad k = 1, 2, \ldots. \]

If $\Phi$ is a finite set with $N$ elements then so is $\Psi$, and the connecting matrix $(\lambda_{ij})$ in this case is an element of $U_N(\mathbb{C})$. Since we will obtain stronger results for graded submodules later in the section, we omit these details.

We have seen in (4.15) that any sequence of multipliers $\Phi = \{\phi_1, \phi_2, \ldots\}$ satisfying (8.1) obeys \( \sum_n |\phi(z)|^2 \leq 1 \) for every $z \in B_d$, and hence the associated sequence of boundary functions $\tilde{\phi}_n : \partial B_d \to \mathbb{C}$ satisfies \( \sum_n |\tilde{\phi}_n(z)|^2 \leq 1 \) almost everywhere $d\sigma$ on the boundary $\partial B_d$. $\Phi$ is called an inner sequence if equality holds

\[ \sum_n |\tilde{\phi}_n(z)|^2 = 1 \]

almost everywhere $(d\sigma)$ on $\partial B_d$.

We do not know if every nonzero closed submodule $M \subseteq H^2$ is associated with an inner sequence via (8.1); but the following result covers many cases of interest.
The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \ldots, z_d]$

**Theorem E.** Let $M$ be a closed submodule of $H^2$ which contains a nonzero polynomial. Then every sequence $\Phi = \{\phi_1, \phi_2, \ldots\}$ satisfying (8.1) is an inner sequence.

**proof.** Consider the rank-one Hilbert module $H = H^2/M$. The natural projection $U : H^2 \to H^2/M$ provides a minimal dilation

$$H^2 \xrightarrow{U} H^2/M \to 0$$

hence the algebraic submodule of $H$ is given by

$$M_H = U(A) = (A + M)/M \cong A/A \cap M.$$ 

Thus the annihilator of $M_H$ is $A \cap M \neq \{0\}$. A theorem of Auslander and Buchsbaum (Corollary 20.13 of [14], or Theorem 195 of [18]) implies that $\chi(M_H) = 0$. By Corollary 2 of Theorem C we have $K(H) = \chi(H) = \chi(M_H) = 0$, and the assertion follows from Theorem 4.16.

We turn now to the discussion of graded ideals in $A$ and graded submodules of $H^2$. Consider the free Hilbert module $H^2$ as a graded module with gauge group

$$(\Gamma_0(\lambda)f)(z_1, \ldots, z_d) = f(\lambda z_1, \ldots, \lambda z_d) \quad \lambda \in \mathbb{T}, \quad f \in H^2.$$ 

Every closed graded submodule $M \subseteq H^2$ decomposes into an orthogonal direct sum

$$M = M_0 \oplus M_1 \oplus M_2 \oplus \ldots$$

where $M_n = \{f \in M : \Gamma_0(\lambda)f = \lambda^n f\}$ consists of all elements of $M$ which are homogeneous polynomials of degree $n$. The (nonclosed) linear span

$$I = M_0 + M_1 + M_2 + \ldots$$

is a graded ideal in the polynomial algebra $A = \mathbb{C}[z_1, \ldots, z_d]$ which is dense in $M$ in the norm of $H^2$. This association $I \leftrightarrow M = \overline{I}$ between graded polynomial ideals and graded submodules of $H^2$ is bijective.

These remarks show that one can apply the Hilbert space methods of this paper to gain information about the structure of graded ideals in the algebra $A$. We base our approach to such questions on the notion of metric basis. We show that metric bases are inner sequences, and that they are necessarily infinite whenever the ideal is of infinite codimension in $A$. This has direct implications for infinite dimensional pure rank-one graded Hilbert modules $H$: such a Hilbert module $H$ is either free (i.e., $H \cong H^2$) or its minimal resolution into free Hilbert modules

$$\cdots \to F_2 \to F_1 \to H \to 0$$

becomes infinite at $F_2$ in that $F_2 \cong H^2 \otimes \ell^2$ is a free Hilbert module of infinite rank.

Let $I$ be a graded ideal in $A = \mathbb{C}[z_1, \ldots, z_d]$. By a *generator* for $I$ we mean a (finite or infinite) set $\Phi$ of homogeneous polynomials of $I$ which is linearly independent (i.e., every finite subset is linearly independent) and which generates $I$ in the sense that every element $g \in I$ can be written as a finite linear combination

$$(8.3) \quad g = f_1 \cdot \phi_1 + f_1 \cdot \phi_2 + \cdots + f_r \cdot \phi_r.$$
where \( f_1, f_2, \ldots, f_r \in A \) and \( \phi_1, \phi_2, \ldots, \phi_r \in \Phi \). Of course, Hilbert's basis theorem implies that every ideal of \( A \) has a finite generator; but since we want to relate generators closely to the natural norm on \( A \), we must work with infinite generators.

A generator \( \Phi \) is said to be contractive if for every finite set \( \phi_1, \ldots, \phi_r \) of distinct elements of \( \Phi \) and \( f_1, \ldots, f_r \in A \) we have

\[
\|\phi_1 \cdot f_1 + \cdots + \phi_r \cdot f_r\|^2 \leq \|f_1\|^2 + \cdots + \|f_r\|^2.
\]

Any generator can be made into a contractive one by scaling down its individual members appropriately.

**Definition 8.5.** Let \( I \) be a nonzero graded ideal in \( A \). A metric basis for \( I \) is a contractive generator with the following additional property: every polynomial \( g \in I \) of degree \( n \) admits a representation of the form (8.3) where \( \phi_1, \ldots, \phi_r \) are distinct elements of \( \Phi \) and where \( f_1, \ldots, f_r, \phi_1, \ldots, \phi_r \) together satisfy

1. \( \deg \phi_k + \deg f_k \leq n \), \( k = 1, \ldots, r \) and
2. \( \|g\|^2 = \|f_1\|^2 + \cdots + \|f_r\|^2 \).

**Remarks.** Property (1) implies that the degrees of \( \phi_1, \ldots, \phi_r \) are as small as they could possibly be, and (2) asserts that the norm of the \( r \)-tuple \( (f_1, \ldots, f_r) \) is as small as the contractivity hypothesis allows. The price one has to pay for these two favorable conditions is that metric bases are typically infinite (see Theorem F below), and in such cases the lengths \( r \) of the expressions appearing in (8.3) are unbounded as \( g \) varies. Nevertheless, if one uses a metric basis \( \Phi \) to represent polynomials \( g \) of degree at most \( n \) as in (8.3), then one has good control over the length of such expressions in terms of \( n \). More precisely, for every polynomial \( g \) of degree at most \( n \), there is a minimum length \( r = r(g) \) of expressions of the form (8.3). In remark 8.10 we exhibit explicit upper bounds for the sequence \( N_1, N_2, \ldots \) defined by

\[
N_n = \max\{ r(g) : g \in A, \ \deg g \leq n \}
\]

in terms of \( n \) and the defect operator \( \Delta \) of the Hilbert \( A \)-module \( I \subseteq H^2 \).

Another feature of metric bases is that they can be written down quite explicitly, as we now show by exhibiting a metric basis for an arbitrary graded ideal \( I \neq \{0\} \). Let \( M = I \) be the corresponding graded submodule of \( H^2 \), and let \( T_1, \ldots, T_d \) be the canonical operators of \( M \), \( T_k f = \tau_k f, f \in M \). Let \( \Delta = (1_M - T_1T_1^* - \cdots - T_dT_d^*)^{1/2} \) be the defect operator of the Hilbert \( A \)-module \( M \), and let \( G \) be the Hilbert space \( G = \Delta M \). The restriction of \( \Delta \) to \( G \) is a positive contraction with trivial kernel. Since the gauge group \( \Gamma_0 \) of \( H^2 \) leaves \( M \) invariant and commutes with \( \Delta \), it follows that

\[
\Gamma_G(\lambda) = \Gamma_0(\lambda) |_G, \quad \lambda \in T
\]

defines a grading on \( G \), and we obtain a decomposition

\[
\Delta |_G = \Delta_0 + \Delta_1 + \Delta_2 + \cdots
\]

where \( \Delta_n = \Delta P_{G_n} \) is a positive finite rank operator whose restriction to the space \( G_n \subseteq G \) of homogeneous polynomials of degree \( n \) in \( G \) is positive definite. For each \( n = 0, 1, 2, \ldots \) let \( \Psi_n \) be an orthonormal basis for \( G_n \) consisting of eigenvectors of \( \Delta \). Note that every element \( \psi \in \Psi \) is associated with a positive eigenvalue \( \lambda \) of \( \Delta \), \( \Delta \psi = \lambda \psi \). We define \( \Phi_n = \Delta \Psi_n \), and

\[
\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \cdots = \Delta \Psi.
\]

By construction, the elements of \( \Phi \) are nonzero mutually orthogonal homogeneous polynomials, and in particular \( \Phi \) is a linearly independent set.
Proposition 8.7. The set $\Phi$ defined by (8.6) is a metric basis for $I$.

proof. The proof is a direct application of the dilation theory summarized in section 1. Consider the free Hilbert module $F = H^2 \otimes G = H^2 \otimes \Delta M$, and define the operator $U \in \text{hom}(F, M)$ by

$$U(f \otimes g) = f \cdot \Delta g, \quad f \in A \quad g \in G.$$ 

Because $M$ is a pure Hilbert $A$-module, $U$ is a coisometry.

Since $M \subseteq H^2$ we can think of $U$ as an element of $\text{hom}(F, H^2)$, and then we have $UU^* = P_M$, $P_M$ denoting the projection of $H^2$ onto $M$. We can make $F = H^2 \otimes G$ into a graded Hilbert module by introducing the gauge group

$$\Gamma_F(\lambda) = \Gamma_0(\lambda) \otimes \Gamma_G(\lambda), \quad \lambda \in \mathbb{T}.$$ 

Notice that $U$ is a graded operator in the sense that $\Gamma_0(\lambda)U = U\Gamma_F(\lambda)$, $\lambda \in \mathbb{T}$. Indeed, for $f \in A$ and $g \in G$

$$U(\Gamma_F(\lambda)(f \otimes g)) = U(\Gamma_0(\lambda)f \otimes \Gamma_0(\lambda)g) = \Gamma_0(\lambda)f \cdot \Delta \Gamma_0(\lambda)g$$

$$= \Gamma_0(\lambda)f \cdot (\Gamma_0(\lambda)\Delta g) = \Gamma_0(\lambda)(f \cdot \Delta g) = \Gamma_0(\lambda)U(f \otimes g),$$

and the assertion follows because $H^2 \otimes G$ is spanned by such elements $f \otimes g$. Letting \{\(E_n : n = 0, 1, \ldots\) and \{\(\tilde{E}_n : n = 0, 1, \ldots\) be the spectral projections of $\Gamma_0$ and \(\Gamma_F\),

$$\Gamma_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n E_n, \quad \Gamma_F(\lambda) = \sum_{n=0}^{\infty} \lambda^n \tilde{E}_n$$

then we have

$$(8.8) \quad U\tilde{E}_n = E_n U, \quad n = 0, 1, 2, \ldots.$$ 

We claim now that $\Phi$ satisfies the contractivity condition (8.4). For that, let $\phi_1, \ldots, \phi_r$ be distinct elements of $\Phi$. Then there are positive numbers $\lambda_1, \ldots, \lambda_r$ and an orthonormal set $\psi_1, \ldots, \psi_r \in \Psi$ such that $\phi_k = \Delta \psi_k = \lambda_k \psi_k$, $k = 1, \ldots, r$. Thus for $f_1, \ldots, f_r \in A$ we have

$$\sum_{k=1}^{r} f_k \cdot \phi_k = \sum_{k=1}^{r} f_k \cdot \Delta \psi_k = U(\sum_{k=1}^{r} f_k \otimes \psi_k).$$

Since $\|U\| \leq 1$ and $\psi_1, \ldots, \psi_r$ is an orthonormal set in $G$ we have

$$\| \sum_{k=1}^{r} f_k \cdot \phi_k \|^2 \leq \| \sum_{k=1}^{r} f_k \otimes \psi_k \|^2 = \sum_{k=1}^{r} \| f_k \|^2,$$

and the assertion follows.

It remains to show that for every polynomial $p \in I$ with $\deg p = n$, there are elements $\phi_1, \ldots, \phi_r \in \Phi$ and $f_1, \ldots, f_r \in A$ satisfying $p = f_1 \cdot \phi_1 + \cdots + f_r \cdot \phi_r$ along with conditions (1) and (2) of Definition 8.5. Fixing $p$, define an element $\xi \in H^2 \otimes G$ by

$$\xi = (\tilde{E}_0 + \tilde{E}_1 + \cdots + \tilde{E}_n)U^*p.$$
Note first that $U\xi = p$. Indeed, since $UU^* = P_M$ we have $UU^*p = p$ and using (8.8) we have

$$U\xi = U((\sum_{k=0}^n \tilde{E}_k)U^*p) = (\sum_{k=0}^n E_k)UU^*p = \sum_{k=0}^n E_kp = p.$$  

It is also clear from its definition that $\xi$ belongs to $\tilde{E}_0F + \tilde{E}_1F + \ldots \tilde{E}_nF$. Since the gauge group of $F$ is a tensor product $\Gamma_F(\lambda) = \Gamma_0(\lambda) \otimes \Gamma_G(\lambda)$, we have

$$\tilde{E}_kF = \sum_{p=0}^k H^2_p \otimes G_{k-p}$$

and hence

$$\tilde{E}_0F + \tilde{E}_1F + \ldots \tilde{E}_nF \subseteq \sum\{H^2_p \otimes G_q : p, q \geq 0, \quad p + q \leq n\}.$$

Noting that $H^2_p \otimes G_q$ is spanned by elements of the form $f \otimes \psi$ with $f \in H^2_p$ and $\psi \in \Psi_q$, it follows that $\xi$ has a decomposition of the form

(8.9) \[ \xi = f_1 \otimes \psi_1 + f_2 \otimes \psi_2 + \cdots + f_r \otimes \psi_r \]

where $\psi_1, \ldots, \psi_r$ are distinct elements of $\Psi$ and where $\deg f_k + \deg \psi_k \leq n$. Setting $\phi_k = \Delta \psi_k \in \Phi$ we have $\deg \phi_k = \deg \psi_k$ and

$$p = U\xi = \sum_{k=1}^r f_k \cdot \Delta \psi_k = \sum_{k=1}^r f_k \cdot \phi_k.$$

Finally, we claim that $\|p\|^2 = \|f_1\|^2 + \cdots + \|f_r\|^2$. Indeed, since $\psi_1, \ldots, \psi_r$ are orthonormal in $G$, (8.9) implies that

$$\|f_1\|^2 + \cdots + \|f_r\|^2 = \|\xi\|^2 = \|U(\tilde{E}_0 + \cdots + \tilde{E}_n)U^*p\|^2 \leq \|U^*p\|^2 \leq \|p\|^2,$$

while since $\Phi$ satisfies (8.4) we must have the opposite inequality

$$\|f_1\|^2 + \cdots + \|f_r\|^2 \geq \|p\|^2,$$

and the proof is complete.

Remark 8.10. Let $I$ be a graded ideal in $A$ and let $g \in I$ be a polynomial of degree at most $n$. We can estimate the length $r$ of a representation $g = f_1 \cdot \phi_1 + \cdots + f_r \cdot \phi_r$ satisfying the conditions of Definition 8.5 as follows. Let $\Delta$ be the defect operator of $M = \bar{I}$ and let $\Delta_k$ denote its restriction to the subspace $M_k = I_k$ of homogeneous polynomials of degree $k$. Since by property (1) of Definition 8.5 the elements $\phi_1, \ldots, \phi_r$ must all belong to $\Phi_0 \cup \Phi_1 \cup \cdots \cup \Phi_n$ and since the cardinality of $\Phi_k$ is the rank of $\Delta_k$ we conclude that

$$r \leq \text{rank} \Delta_0 + \cdots + \text{rank} \Delta_n.$$
We now take up the issue of uniqueness of metric bases. Let $S = \{\xi_1, \ldots, \xi_m\}$ and $T = \{\eta_1, \ldots, \eta_n\}$ be two finite linearly independent sets of vectors in a Hilbert space $H$. We say that $S \sim T$ if $m = n$ and there is a unitary matrix $(u_{ij}) \in M_n(\mathbb{C})$ such that

$$
\eta_i = \sum_{j=1}^{n} u_{ij} \xi_j, \quad 1 = 1, \ldots, n.
$$

More specifically, if $\Phi$ and $\Psi$ are two linearly independent sets of homogeneous polynomials in $H^2$ then we say that $\Phi$ and $\Psi$ are equivalent (written $\Phi \sim \Psi$) if $\Phi_n \sim \Psi_n$ for every $n = 0, 1, 2, \ldots$ where $\Phi_n$ (resp. $\Psi_n$) denotes the set of all elements of $\Phi$ (resp. $\Psi$) which are homogeneous of degree $n$. We now show that any two metric bases for a graded ideal $I$ of $A$ are equivalent. That result is based on the following. We use the notation $\xi \otimes \bar{\eta}$ to denote the rank-one operator $\zeta \mapsto \langle \zeta, \eta \rangle \xi$ associated with a pair of vectors $\xi, \eta$ in a Hilbert space.

**Lemma 8.12.** Let $S = \{\xi_1, \ldots, \xi_m\}$ and $T = \{\eta_1, \ldots, \eta_n\}$ be linearly independent sets of vectors in a Hilbert space $H$ such that

$$
\sum_{k=1}^{m} \xi_k \otimes \bar{\xi}_k = \sum_{j=1}^{n} \eta_j \otimes \bar{\eta}_j.
$$

Then $S \sim T$ in the sense of (8.11).

**proof.** Consider the linear map $A : \mathbb{C}^m \to H$ defined by

$$
Az = z_1 \xi_1 + \cdots + z_m \xi_m.
$$

After endowing $\mathbb{C}^m$ with its usual inner product, one finds that the adjoint $A^* : H \to \mathbb{C}^m$ is given by

$$
A^* \zeta = (\langle \zeta, \xi_1 \rangle, \ldots, \langle \zeta, \xi_m \rangle),
$$

and therefore

$$
AA^* = \sum_{k=1}^{m} \xi_k \otimes \bar{\xi}_k.
$$

Because of linear independence we have $\ker A = \{0\}$, and hence $A^*H = \mathbb{C}^m$.

By the polar decomposition, there is a unique isometry $U : \mathbb{C}^m \to H$ having range $AC^m$ such that $A = PU$, where $P = (AA^*)^{1/2}$. Doing the same with the set $T$ we obtain a similar operator $B : \mathbb{C}^n \to H$ having trivial kernel such that $BB^* = AA^* = P^2$, and an isometry $V : \mathbb{C}^n \to H$ having range $BC^m$ such that $B = PV$.

Since $AC^m = BC^m = PH$, the operator $W = U^*V \in B(\mathbb{C}^n, \mathbb{C}^m)$ is unitary and satisfies $AW = B$. Thus $m = n$ and, letting $(w_{ij})$ be the matrix of $W$ relative to the usual orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ we find that

$$
\eta_i = Be_i = AW e_i = \sum_{j=1}^{n} w_{ij} A e_j = \sum_{j=1}^{n} w_{ij} \xi_j,
$$

for $1 = 1, \ldots, n$. Therefore

$$
\sum_{j=1}^{n} w_{ij} = \delta_{ij}, \quad 1 = 1, \ldots, n.
$$
as required.

Remarks. We point out that the converse of Lemma 8.12 is true in the sense that if \( \{\xi_1, \ldots, \xi_n\} \) is any set of vectors in a Hilbert space and \( \eta_1, \ldots, \eta_n \) is related to \( \{\xi_1, \ldots, \xi_n\} \) by way of a unitary \( n \times n \) matrix as in (8.11), then one verifies by direct computation that

\[
\sum_{k=1}^{n} \xi_k \otimes \bar{\xi}_k = \sum_{k=1}^{n} \eta_k \otimes \bar{\eta}_k.
\]

Let \( P \) be a finite rank positive operator on a Hilbert space \( H \). There is an orthonormal basis \( \{e_1, \ldots, e_r\} \) for the subspace \( PH \) consisting of eigenvectors for \( P \), \( Pe_j = \lambda_je_j \), \( j = 1, \ldots, r \), and one has \( \lambda_j > 0 \) for every \( j \). Setting \( \xi_k = \sqrt{\lambda_k}e_k \) we have \( P = \xi_1 \otimes \bar{\xi}_1 + \cdots + \xi_r \otimes \bar{\xi}_r \), and of course \( \{\xi_1, \ldots, \xi_r\} \) is linearly independent.

If \( \eta_1, \ldots, \eta_s \) is any other linearly independent set which represents \( P \) in a similar way \( P = \eta_1 \otimes \bar{\eta}_1 + \cdots + \eta_s \otimes \bar{\eta}_s \) then the \( \eta_k \) do not have to be eigenvectors of \( P \), but Lemma 8.12 implies that \( s = r \) and that \( \{\eta_1, \ldots, \eta_r\} \) is obtained from the eigenvector sequence \( \{\xi_1, \ldots, \xi_r\} \) by a simple rotation as in (8.11).

We also point out that the proof of Lemma 8.12 is closely related to the construction of the metric operator space associated with a normal completely positive map \( \phi : \mathcal{B}(H) \to \mathcal{B}(H) \) and in fact Lemma 8.12 extends in an appropriate way (with a proof similar to the one given) to infinite sets of vectors [2].

In view of Lemma 8.12, the uniqueness of metric bases will follow from

**Proposition 8.13.** Let \( \Psi \) be a metric basis for a nonzero graded ideal \( I \) in \( A \), let \( \Delta = (1 - T_1T_1^* - \cdots - T_dT_d^*)^{1/2} \) be the defect operator of \( M = \overline{I} \), and let \( E_n \) be the projection of \( H^2 \) onto the space of homogeneous polynomials of degree \( n \). Then \( \Delta E_n = 0 \) iff \( \Psi_n = \emptyset \), and when \( \Psi_n = \{\psi_1, \ldots, \psi_s\} \neq \emptyset \) we have

\[
\sum_{k=1}^{s} \psi_k \otimes \overline{\psi}_k = \Delta^2 E_n.
\]

**proof.** The idea of the proof is similar to that of Proposition 8.7, in that one constructs a graded minimal dilation of the Hilbert \( A \)-module \( \overline{I} \subseteq H^2 \) using the metric basis \( \Psi \), and then simply verifies that the required property holds. However, the details are different enough that we include them.

Let \( G \) be the graded Hilbert space \( \ell^2(\Psi) \) of all square summable functions \( f : \Psi \to \mathbb{C} \) with gauge group \( \Gamma_G \)

\[
(\Gamma_G(\lambda)(f))(\psi) = \lambda^{\deg \psi}f(\psi), \quad \psi \in \Psi.
\]

We identify the delta function concentrated at an element \( \psi \in \Psi \) with \( \psi \) itself, so that \( \Psi \) becomes an orthonormal basis for \( G \) consisting of eigenvectors of \( \Gamma_G \).

We will work with the graded free Hilbert module \( F = H^2 \otimes G, \Gamma_F = \Gamma_0 \otimes \Gamma_G \). Since \( \Psi \) is an orthonormal basis for \( G \), every element \( \xi \in F \) has unique expansion

\[
\xi = \sum_{\psi \in \Psi} f_\psi \otimes \psi
\]
where \( f_\psi, \psi \in \Psi \), is a sequence of elements of \( H^2 \) satisfying
\[
(8.14) \quad \| \xi \|^2 = \sum_{\psi \in \Psi} \| f_\psi \|^2.
\]

If \( f_1, \ldots, f_r \) is a set of polynomials in \( A \) and \( \psi_1, \ldots, \psi_r \) are distinct elements of \( \Psi \), then by (8.4) we have
\[
\| f_1 \cdot \psi_1 + \cdots + f_r \cdot \psi_r \|^2 \leq \| f_1 \|^2 + \cdots + \| f_r \|^2 = \| f_1 \otimes \psi_1 + \cdots + f_r \otimes \psi_r \|^2.
\]

It follows that there is a unique bounded operator \( U : F \to \bar{I} \) satisfying \( U(f \otimes \psi) = f \cdot \psi \) for every \( f \in A, \psi \in \Psi \), and of course \( \| U \| \leq 1 \) and \( U \in \text{hom}(F, \bar{I}) \).

Note that
\[
H^2 \otimes G \xrightarrow{U} \bar{I} \xrightarrow{0}
\]
is a graded minimal dilation. That is to say, \( UT_F(\lambda) = \Gamma_0(\lambda)U \) and \( U \) is a coisometry: \( UU^* = 1_{I} \). Indeed, one sees that \( U \) is graded exactly as in the proof of Proposition 8.7, and the conditions of Definition 8.5 imply that \( U \) is a partial isometry with dense range. In more detail, the range of \( U \) obviously contains \( I \), and for a given element \( g \in I \) we can find a representation
\[
g = f_1 \cdot \psi_1 + \cdots + f_r \cdot \psi_r
\]
which satisfies the conditions (1) and (2) of Definition 8.5. Setting
\[
\xi = f_1 \otimes \psi_1 + \cdots + f_r \otimes \psi_r \in F = H^2 \otimes G
\]
we find that \( U\xi = g \) and \( \| \xi \|^2 = \| f_1 \|^2 + \cdots + \| f_r \|^2 = \| g \|^2 \). Since \( \| U \| \leq 1 \), \( U \) must be a partial isometry. That this dilation is minimal is clear from condition (4) of Proposition 1.9.

The left side of the formula of Proposition 8.13 is calculated as follows. Since \( \Psi_n = \{ \psi_1, \ldots, \psi_s \} \) it follows that \( \{ \psi_1, \ldots, \psi_s \} \) is an orthonormal basis for \( G_n \). Letting \( D_n \) be the projection of \( G \) onto \( G_n \) and letting \( 1 \) be the constant function of \( H^2 \), then \( \{ 1 \otimes \psi_k : k = 1, \ldots, s \} \) is an orthonormal basis for the range of the projection \( E_0 \otimes D_n \in B(H^2 \otimes G) \). Since \( U(1 \otimes \psi_k) = \psi_k \in I \), we have
\[
\sum_{k=1}^{s} \psi_k \otimes \bar{\psi}_k = \sum_{k=1}^{s} U(1 \otimes \psi_k) \otimes \overline{U(1 \otimes \psi_k)} = U \sum_{k=1}^{s} (1 \otimes \psi_k) \otimes (1 \otimes \psi_k)^* U^* = U(E_0 \otimes D_n)U^*.
\]

Since the \( n \)th spectral projection \( \hat{E}_n \) of \( \Gamma_F \) is related to \( E_0 \otimes D_n \) by
\[
\hat{E}_n(E_0 \otimes 1_G) = E_0 \otimes D_n,
\]
the right side of the preceding formula is
\[
U\hat{E}_n(E_0 \otimes 1_G)U^* = E_nU(E_0 \otimes 1_G)U^*
\]
because $U$ is graded, $E_n$ being the projection of $H^2$ onto the space $H^2_n$. We conclude that

\[
\sum_{k=1}^s \psi_k \otimes \bar{\psi}_k = E_n U (E_0 \otimes 1_G) U^*.
\]

Recalling that $\Delta$ commutes with $E_n$, we can obtain the desired formula by showing that

\[
U (E_0 \otimes 1_G) U^* = \Delta^2.
\]

Indeed, (8.16) is a general formula satisfied by all minimal dilations such as $U$. Let $S_1, \ldots, S_d$ be the $d$-shift acting on $H^2$. We have seen that

\[
E_0 = 1_{H^2} - S_1 S_1^* - \cdots - S_d S_d^*.
\]

Since the canonical operators of $F = H^2 \otimes G$ are $S_1 \otimes 1_G, \ldots, S_d \otimes 1_G$, we have

\[
E_0 \otimes 1_G = 1_F - \sum_{k=1}^d (S_k \otimes 1_G)(S_k \otimes 1_G)^*.
\]

Since $U \in \text{hom}(F, \tilde{I})$ we find that for $T_1, \ldots, T_d$ the canonical operators of $\tilde{I}$ we have $U(S_k \otimes 1_G) = T_k U$, and thus

\[
U (E_0 \otimes 1_G) U^* = UU^* - T_1 UU^* T_1^* - \cdots - T_d UU^* T_d^* = 1_{\tilde{I}} - T_1 T_1^* - \cdots T_d T_d^* = \Delta^2,
\]

as required.

**Corollary 1.** Any two metric bases for a graded ideal $I \subseteq \mathbb{C}[z_1, \ldots, z_d]$ are equivalent.

**proof.** This is an immediate consequence of Proposition 8.13 and Lemma 8.12.

**Corollary 2.** Let $\Phi = \{\phi_1, \phi_2, \ldots\}$ be a metric basis for a graded ideal $I \subseteq A$, and let $M = \tilde{I}$ be its associated closed submodule of $H^2$. Then $\Phi$ is an inner sequence for $M$ satisfying (8.1) and (8.2).

**proof.** The minimal dilation

\[
H^2 \otimes G \xrightarrow{U} M \xrightarrow{0}
\]

exhibited in the proof of Proposition 8.13 shows that the sequence $\Phi$ satisfies (8.1) and (8.2), and Theorem E implies that $\Phi$ is an inner sequence.

The following result implies that metric bases are almost always infinite.
**Theorem F.** Let $I$ be a proper graded ideal in $A$ having a finite metric basis. Then $I$ is of finite codimension in $A$ and the canonical generators $z_1, \ldots, z_d$ of $A$ are all nilpotent modulo $I$.

**proof.** Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a metric basis for $I$ and let $s(z) = |\phi_1(z)|^2 + \cdots + |\phi_n(z)|^2$. By Theorem E, $s(z) = 1$ almost everywhere on the sphere $\partial B_d$ and since $s$ is continuous it must be identically 1 on $\partial B_d$. Thus the variety $V = \{z \in \mathbb{C}^d : \phi_1(z) = \cdots = \phi_n(z) = 0\}$ of common zeros does not intersect the unit sphere. $V$ cannot be empty since $I$ is proper. Since $V$ is a nonempty set invariant under multiplication by nonzero scalars which misses the unit sphere, it must consist of just the single point $(0, 0, \ldots, 0)$.

By Hilbert’s Nullstellensatz there is an integer $p \geq 1$ such that $z_1^p, \ldots, z_d^p \in I$ [14], Theorem 1.6. Since the $A$-module $A/I$ has a cyclic vector $1 + I$ and its canonical operators are nilpotent, it follows that $A/I$ is finite dimensional.

**Corollary.** Let $M \neq \{0\}$ be a closed graded submodule of $H^2$ such that $H^2/M$ is infinite dimensional. Then $M$ is an infinite rank Hilbert $A$-module.

**proof.** Contrapositively, assume that $\operatorname{rank}M < \infty$ and let

$$\Delta = (1_M - T_1 T_1^* - \cdots - T_d T_d^*)^{1/2}$$

where $T_k f = z_k \cdot f$, $f \in M$. $\Delta$ is a positive finite rank operator which commutes with the gauge group of $M$. Thus the metric basis $\Phi$ for the ideal $I = M \cap A$ exhibited in (8.6) must be a finite set, and Theorem F implies that $A/I$ is finite dimensional. Since $A$ is dense in $H^2$, the natural map of $A/I$ into $H^2/I = H^2/M$ must have dense range and thus $H^2/M$ is finite dimensional, contradicting the hypothesis on $M$.

9. Examples.

In this section we give examples of pure rank-one Hilbert modules illustrating (1) the failure of Theorem B for ungraded modules, and (2) the computation of the degree in cases where the Euler characteristic vanishes. We also give examples of pure rank-two graded Hilbert modules illustrating (3) the computation of nonzero values of $K(H) = \chi(H)$.

We begin with a discussion of the limits of Theorem B by presenting a class of examples for which $K(H) \neq \chi(H)$ (Proposition 9.2); a concrete example of such a Hilbert $A$-module is given in Example 9.3. We then describe a natural method for associating a graded Hilbert $A$-module with an algebraic variety in complex projective space, and we show that for some varieties one can calculate all numerical invariants of their associated Hilbert modules.

**Remark 9.1.** We will make use of the fact that if $K_1$ and $K_2$ are two closed submodules of the free Hilbert module $H^2$ for which $H^2/K_1$ is isomorphic to $H^2/K_2$, then $K_1 = K_2$. In particular, no nontrivial quotient of $H^2$ of the form $H^2/K$ with $K \neq \{0\}$ can be a free Hilbert $A$-module.

Indeed, this is part of Corollary 2 of [1, Theorem 8.5]. One can also deduce it from the material summarized in section 1 as follows. Let $U_j$ be the natural projection of $H^2$ onto $H^2/K_j$, $j = 1, 2$. Then both diagrams

$$H^2 \xrightarrow{U_j} H^2/K_j \rightarrow 0, \quad j = 1, 2$$
define minimal dilations. Applying Theorem 1.11, we see that any unitary isomorphism \( W : H^2/K_1 \to H^2/K_2 \) of Hilbert \( A \)-modules gives rise to a unique unitary operator \( \tilde{W} \) in the commutant of \( C^*(H^2) \) such that the diagram

\[
\begin{array}{ccc}
H^2 & \xrightarrow{U_1} & H^2/K_1 \\
\downarrow & & \downarrow W \\
H^2 & \xrightarrow{U_2} & H^2/K_2
\end{array}
\]

commutes. Since \( C^*(H^2) \) is the (irreducible) Toeplitz \( C^* \)-algebra [1, Theorem 5.7], \( \tilde{W} \) must be a scalar multiple of the identity operator, and the assertion follows from the fact that \( WK_1 = K_2 \).

**Proposition 9.2.** Let \( K \neq \{0\} \) be a closed submodule of \( H^2 \) which contains no nonzero polynomials, and consider the pure rank-one module \( H = H^2/K \). Then

\[
0 \leq K(H) < \chi(H) = 1.
\]

**proof.** We show first that \( \chi(H) = 1 \) by proving that the algebraic submodule \( M_H \) of \( H \) is free. Let \( U \in \text{hom}(H^2, H) \) be the natural projection onto \( H = H^2/K \). The kernel of \( U \) is \( K \), and \( U \) maps the dense linear subspace \( A \subseteq H^2 \) of polynomials onto \( M_H \), \( U(A) = M_H \). Since \( A \cap K = \{0\} \), the restriction of \( U \) to \( A \) gives an isomorphism of \( A \)-modules \( A \cong M_H \), and hence \( \chi(H) = \chi(A) = 1 \).

On the other hand, if \( K(H) \) were to equal 1 = \( \text{rank}(H) \) then by the extremal property (4.13) \( H \) would be isomorphic to the free Hilbert module \( H^2 \) of rank-one, which is impossible because of Remark 9.1.

**Problem.** We do not know if the curvature invariant \( K(H) \) of a pure finite rank Hilbert \( A \)-module is always an integer. Theorem B implies that this is the case for graded Hilbert modules, but Proposition 9.2 shows that Theorem B does not always apply. In particular, it is not known if \( K(H) = 0 \) for the ungraded Hilbert modules \( H \) of Prop. 9.2. In such cases, the equation \( K(H) = 0 \) is equivalent to the existence of an “inner sequence” for the invariant subspace \( K \) (see Theorem 4.16).

**Example 9.3.** It is easy to give concrete examples of submodules \( K \) of \( H^2 \) satisfying the hypothesis of Proposition 9.2. Consider, for example, the graph of the exponential function \( G = \{(z, e^z) : z \in \mathbb{C}\} \subseteq \mathbb{C}^2 \). Let \( d = 2 \), let \( H^2 = H^2(\mathbb{C}^2) \), and let \( K \) be the submodule of all functions in \( H^2 \) which vanish on the intersection of \( G \) with the unit ball

\[
K = \{ f \in H^2 : f \down|_{G \cap B_d} = 0 \}.
\]

Since \( f \in H^2 \mapsto f(z) = \langle f, u_z \rangle \) is a bounded linear functional for every \( z \in B_d \) it follows that \( K \) is closed, and it is clear that \( K \neq \{0\} \) (the function \( f(z_1, z_2) = e^{z_1} - z_2 \) belongs to \( H^2 \) and vanishes on \( G \cap B_d \)). After noting that the open unit disk about \( z = -1/2 \) maps into \( G \cap B_d \),

\[
\{(z, e^z) : |z + 1/2| < 1\} \subseteq G \cap B_d
\]
an elementary argument (which we omit) establishes the obvious fact that no nonzero polynomial can vanish on $G \cap B_d$.

An algebraic set in complex projective space $\mathbb{P}^{d-1}$ can be described as the set of common zeros of a finite set of homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{C}[z_1, \ldots, z_d]$.

\begin{align}
V = \{z \in \mathbb{C}^d : f_1(z) = \cdots = f_n(z) = 0\}
\end{align}

[14], pp 39–40. One can associate with $V$ a graded rank-one Hilbert $A$-module in the following way. Let $M_V$ be the submodule of $H^2 = H^2(\mathbb{C}^d)$ defined by

\begin{align}
M_V = \{f \in H^2 : f|_{V \cap B_d} = 0\}.
\end{align}

As in example 9.3, $M_V$ is a closed submodule of $H^2$, and because $\lambda V \subseteq V$ for complex scalars $\lambda$, $M_V$ is invariant under the action of the gauge group of $H^2$ and hence it is a graded submodule of $H^2$. Thus, $H = H^2/M_V$ is a graded, pure, rank-one Hilbert $A$-module.

We will show how to explicitly compute $H^2/M_V$ and its numerical invariants in certain cases, using operator-theoretic methods. The simplest member of this class of examples is the variety $V$ defined by the range of the quadratic polynomial $F : (x,y) \in \mathbb{C}^2 \mapsto (x^2, y^2, \sqrt{2}xy) \in \mathbb{C}^3$, that is,

\begin{align}
V = \{(x^2, y^2, \sqrt{2}xy) : x, y \in \mathbb{C}\} \subseteq \mathbb{C}^3.
\end{align}

However, one finds more interesting behavior in the higher dimensional example

\begin{align}
V = \{(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz) : x, y, z \in \mathbb{C}\} \subseteq \mathbb{C}^6,
\end{align}

and we will discuss the example (9.5) in some detail. We describe a more general context for these examples in Remark 9.12.

Notice first that $V$ can be described in the form (9.4) as the set

\begin{align}
V = \{z \in \mathbb{C}^6 : f_1(z) = f_2(z) = f_3(z) = f_4(z) = 0\}
\end{align}

of common zeros of the four homogeneous polynomials $f_k : \mathbb{C}^6 \to \mathbb{C}$,

\begin{align*}
f_1(z) &= z_1^2 - 2z_1z_2 = 0 \\
f_2(z) &= z_2^2 - 2z_1z_3 = 0 \\
f_3(z) &= z_3^2 - 2z_2z_3 = 0 \\
f_4(z) &= z_4z_5z_6 - 2^{3/2}z_1z_2z_3 = 0.
\end{align*}

The equivalence of (9.5) and (9.6) is an elementary computation which we omit. Note, however, that the fourth equation $f_4(z) = 0$ is necessary in order to exclude points such as $z = (1, 1, 1, -\sqrt{2}, \sqrt{2}, \sqrt{2})$, which satisfy the first three equations $f_1(z) = f_2(z) = f_3(z) = 0$ but which do not belong to $V$. Note too that $f_1, f_2, f_3$ are quadratic but that $f_4$ is cubic.

We will describe the Hilbert module $H = H^2(\mathbb{C}^6)/M_V$ by identifying its associated 6-contraction $(T_1, \ldots, T_6)$. These operators act on the even subspace
$H$ of $H^2(\mathbb{C}^3)$, defined as the closed linear span of all homogeneous polynomials $f(z_1, z_2, z_3)$ of even degree $2n$, $n = 0, 1, 2, \ldots$. Let $S_1, S_2, S_3 \in \mathcal{B}(H^2(\mathbb{C}^3))$ be the 3-shift. The even subspace $H$ is not invariant under the $S_k$, but it is invariant under any product of two of these operators $S_i S_j$, $1 \leq i, j \leq 3$. Thus we can define a 6-tuple of operators $T_1, \ldots, T_6 \in \mathcal{B}(H)$ by

$$(9.7) \quad (T_1, \ldots, T_6) = (S_1^2 |_H, S_2^2 |_H, S_3^2 |_H, \sqrt{2} S_1 S_2 |_H, \sqrt{2} S_1 S_3 |_H, \sqrt{2} S_2 S_3 |_H).$$

$(T_1, \ldots, T_6)$ is a 6-contraction because

$$\sum_{k=1}^{6} T_k T_k^* = \sum_{i,j=1}^{3} S_i S_j (P_H) S_j^* S_i^* \leq 1_H,$$

and in fact $H$ becomes a pure Hilbert $\mathbb{C}[z_1, \ldots, z_6]$-module.

If $f$ is a sum of homogeneous polynomials of even degrees then

$$\Gamma(e^{i\theta}) f(z_1, z_2, z_3) = f(e^{i\theta/2} z_1, e^{i\theta/2} z_2, e^{i\theta/2} z_3)$$

gives a well-defined unitary action of the circle group on the subspace $H \subseteq H^2(\mathbb{C}^3)$, and $H$ becomes a graded Hilbert module.

**Proposition 9.8.** $H$ is a rank-one graded Hilbert $\mathbb{C}[z_1, \ldots, z_6]$-module which is isomorphic to $H^2(\mathbb{C}^6)/M_V$. The invariants of $H$ are given by $K(H) = \chi(H) = 0$, $\deg(H) = 4$, $\mu(H) = 4$.

**proof.** Let $\phi(A) = T_1 A T_1^* + \cdots + T_6 A T_6^*$ be the canonical completely positive map of $\mathcal{B}(H)$ and, considering $H$ as a subspace of $H^2(\mathbb{C}^3)$, let $\sigma : \mathcal{B}(H^2) \to \mathcal{B}(H^2)$ be the map associated with the 3-shift

$$\sigma(B) = S_1 B S_1^* + S_2 B S_2^* + S_3 B S_3^*.$$  

$\phi$ and $\sigma$ are related in the following simple way: for every $A \in \mathcal{B}(H)$ we have

$$(9.9) \quad \phi(A) = \sum_{k=1}^{6} T_k A T_k^* = \sum_{i,j=1}^{3} S_i S_j (P_H) S_j^* S_i^* = \sigma^2(A P_H).$$

If $E_n \in \mathcal{B}(H^2)$ denotes the projection onto the subspace of homogeneous polynomials of degree $n$, then

$$\phi(1_H) = \sigma^2(\sum_{n=0}^{\infty} E_{2n}) = \sum_{n=0}^{\infty} E_{2n+2}.$$  

It follows that

$$\Delta^2 = 1_H - \phi(1_H) = E_0$$

is the one-dimensional projection onto the space of constants. Since

$$\phi^n(1_H) = \sigma^{2n}(\sum_{p=0}^{\infty} E_{2p}) = \sum_{p=n}^{\infty} E_{2p}$$
obviously decreases to 0 as \( n \to \infty \), we conclude that \( H \) is a pure Hilbert module of rank one.

Hence the minimal dilation of \( H \)

\[
H^2(\mathbb{C}^6) \xrightarrow{U_0} H \xrightarrow{\to} 0
\]

is given by

\[
U_0(f) = f \cdot \Delta 1 = f(T_1, \ldots, T_6) \Delta 1.
\]

If we evaluate this expression at a point \( z = (z_1, z_2, z_3) \in B_3 \) we find that

\[
U_0(f)(z_1, z_2, z_3) = f(z_1^2, z_2^2, z_3^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, \sqrt{2}z_2z_3).
\]

The argument on the right is a point in the ball \( B_6 \), and thus the preceding formula extends immediately to all \( f \in H^2(\mathbb{C}^6) \). Notice too that \( U_0 \) is a graded morphism in that \( U_0 \Gamma_0(\lambda) = \Gamma(\lambda) U_0 \), \( \lambda \in \mathbb{T} \), where \( \Gamma_0 \) is the gauge group of \( H^2(\mathbb{C}^6) \). The preceding formula shows that the kernel of \( U_0 \) is \( MV \), and thus we conclude that \( H \) is isomorphic to \( H^2(\mathbb{C}^6)/MV \), as asserted in Proposition 9.8.

It remains to calculate the power series \( \hat{\phi}(t) \) of Proposition 7.5 which determines the numerical invariants of \( H \). Since \( 1_H - \phi^{n+1}(1_H) \) is the projection

\[
1_H - \phi^{n+1}(1_H) = E_0 + E_2 + \cdots + E_{2n},
\]

it follows that

\[
\hat{\phi}(t) = \sum_{n=0}^{\infty} \dim(E_0 + E_2 + \cdots + E_{2n}) t^n,
\]

and therefore

(9.10) \( (1 - t) \hat{\phi}(t) = \sum_{n=0}^{\infty} \dim E_{2n} t^n. \)

Setting

\[
\hat{\sigma}(t) = \sum_{p=0}^{\infty} \dim E_p t^p,
\]

we find that for \( 0 < t < 1 \)

\[
\sum_{n=0}^{\infty} \dim E_{2n} t^n = 1/2 \sum_{p=0}^{\infty} \dim E_p (\sqrt{t})^p + \sum_{p=0}^{\infty} \dim E_p (-\sqrt{t})^p
\]

\[
= 1/2 (\hat{\sigma}(\sqrt{t}) + \hat{\sigma}(-\sqrt{t})),
\]

and hence from (9.10) we have

(9.11) \( \hat{\phi}(t) = \frac{\hat{\sigma}(\sqrt{t}) + \hat{\sigma}(-\sqrt{t})}{2(1 - t)}, \quad 0 < t < 1. \)

The dimensions \( \dim E_p \) were computed in [1, Appendix A], where it was shown that \( \dim E_p = q_2(p) \), \( q_2(x) \) being the polynomial defined by (3.3). Thus

\[
\hat{\sigma}(t) = \sum q_2(n) t^n = \frac{1}{(1 - t)^3}.
\]
and finally (9.11) becomes
\[ \hat{\phi}(t) = \frac{(1 - \sqrt{t})^{-3} + (1 + \sqrt{t})^{-3}}{2(1-t)} = \frac{(1 + \sqrt{t})^3 + (1 - \sqrt{t})^3}{2(1-t)^4} = \frac{1 + 3t}{(1-t)^4}. \]

The right side of the last equation can be rewritten
\[ \hat{\phi}(t) = \frac{-3}{(1-t)^3} + \frac{4}{(1-t)^4}, \]
hence the coefficients \((c_0, c_1, \ldots, c_6)\) of (7.1) are given by \((0, 0, 0, -3, 4, 0, 0)\). One now reads off the numerical invariants listed in Proposition 9.8.

\[ \text{Remark 9.12.} \]
One can easily write down a class of related examples (all having Euler characteristic zero) by considering powers \(\sigma^N : \mathcal{B}(H^2(\mathbb{C}^d)) \to \mathcal{B}(H^2(\mathbb{C}^d))\) of the completely positive map \(\sigma(B) = S_1 B S_1^* + \cdots + S_d B S_d^*\) of the \(d\)-shift for arbitrary powers \(N \geq 2\) and in arbitrary dimensions \(d\). The example (9.5) we have discussed is associated with the values \(N = 2\) and \(d = 3\). To explain this briefly, notice that if one expands the expression for \(\sigma^2\) in dimension \(d = 3\) into a sum of the form
\[ \sigma^2(B) = \sum_{k=1}^{6} T_k B T_k^*, \quad B \in \mathcal{B}(H^2(\mathbb{C}^3)), \]
then one finds that one set of candidates for the 6 operators \(T_1, \ldots, T_6\) is given by
\[ (9.13) \quad S_1^2, S_2^2, S_3^2, \sqrt{2} S_1 S_2, \sqrt{2} S_1 S_3, \sqrt{2} S_2 S_3. \]

These operators are the most natural ones, but of course one can replace them with certain linear combinations to obtain other 6-tuples which also serve to represent \(\sigma^2\); any two such 6-tuples must be related by a complex unitary \(6 \times 6\) matrix as in (8.9).

Once one settles on a 6-tuple such as (9.13), one finds that while it certainly defines a 6-contraction, it is not an irreducible 6-contraction because each of the operators \(T_1, \ldots, T_6\) leaves both the even subspace and its orthogonal complement (the odd subspace) of \(H^2(\mathbb{C}^3)\) invariant. The example leading to (9.5) was obtained by restricting the 6-tuple (9.13) to the irreducible even subspace.

If one chooses other values for \(N\) and the dimension \(d\), one finds that this method generates infinitely many higher dimensional examples for which one can, in principle, explicitly calculate \(\deg(H)\) and \(\mu(H)\).

Finally, we compute nontrivial values of the curvature invariant \(K(H)\) for certain examples of pure rank-two graded Hilbert modules \(H\). Let \(\phi\) be a homogeneous polynomial of degree \(N = 1, 2, \ldots\) in \(A = \mathbb{C}[z_1, \ldots, z_d]\) and let \(M\) be the graph of its associated multiplication operator
\[ M = \{(f, \phi \cdot f) : f \in H^2\} \subseteq H^2 \oplus H^2. \]

\(M\) is a closed submodule of the free Hilbert module \(F = H^2 \oplus H^2\), and \(H = F/M\) is a pure Hilbert module of rank 2 whose minimal dilation
\[ (9.14) \quad F \xrightarrow{U} H \xrightarrow{0} \]
is given by the natural projection $U$ of $F$ onto $H = F/M$.

We make $H$ into a graded Hilbert module as follows. Let $\Gamma$ be the gauge group defined on $F = H^2 \oplus H^2$ by

$$\Gamma(\lambda)(f,g) = (\Gamma_0(\lambda)f, \lambda^{-N}\Gamma_0(\lambda)g), \quad f, g \in H^2,$$

where $\Gamma_0$ is the natural gauge group of $H^2$ defined by

$$\Gamma_0(\lambda)f(z_1, \ldots, z_d) = f(\lambda z_1, \ldots, \lambda z_d).$$

One verifies that $\Gamma(\lambda)M \subseteq M$, $\lambda \in \mathbb{T}$. Thus the action of $\Gamma$ can be promoted naturally to the quotient $H = F/M$, and $H$ becomes a graded rank-two pure Hilbert module whose gauge group has spectrum $\{-N, -N+1, \ldots\}$. (9.14) becomes a graded dilation in that $U\Gamma(\lambda) = \Gamma(\lambda)U$ for all $\lambda \in \mathbb{T}$.

**Proposition 9.15.** For these examples we have $K(H) = \chi(H) = 1$.

**proof.** By Theorem B, $K(H) = \chi(H)$, and it suffices to show that $\chi(H) = 1$.

Let $H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi\}$, $n \in \mathbb{Z}$, be the spectral subspaces of $H$. It is clear that $H_n = \{0\}$ if $n < -N$, and (9.14) implies that the algebraic submodule $M_H$ is given by $M_H = U(A \oplus A)$. Hence $M_H$ is the (algebraic) sum

$$M_H = \sum_{n=\infty}^{-\infty} H_n.$$

Consider the proper filtration $M_1 \subseteq M_2 \subseteq \ldots$ of $M_H$ defined by

$$M_k = \sum_{n \leq k} H_n, \quad k = 1, 2, \ldots.$$

By the Corollary of Proposition 3.10 we have

$$(9.16) \quad \chi(M_H) = d! \lim_{k \to \infty} \frac{\dim M_k}{k^d},$$

and thus we have to calculate the dimensions

$$(9.17) \quad \dim M_k = \dim(\sum_{n \leq k} H_n) = \dim H_{-N} + \cdots + \dim H_{k-1} + \dim H_k$$

for $k = 1, 2, \ldots$.

In order to calculate the dimension of $H_n$ it is easier to realize $H$ as the orthogonal complement $M^\perp \subseteq F$, with canonical operators $T_1, \ldots, T_d$ given by compressing the natural operators of $F = H^2 \oplus H^2$ to $M^\perp$. Since $M$ is the graph of the multiplication operator $M_\phi f = \phi \cdot f$, $f \in H^2$, $M^\perp$ is given by

$$M^\perp = \{(-M^*_\phi g, g) : g \in H^2\}.$$

We compute

$$H_n = (M^\perp)_n = \{\xi \in M^\perp : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}\}.$$
Since $\Gamma_0(\lambda) M_\phi^* \Gamma_0(\lambda)^{-1} = (\Gamma_0(\lambda) M_\phi \Gamma_0(\lambda)^{-1})^* = (\lambda^N M_\phi)^* = \lambda^{-N} M_\phi^*$, we have

\[
\Gamma(\lambda)(-M_\phi^* g, g) = (-\Gamma_0(\lambda) M_\phi g, \lambda^{-N} \Gamma_0(\lambda) g) = (-\lambda^{-N} M_\phi^* \Gamma_0(\lambda) g, \lambda^{-N} \Gamma_0(\lambda) g),
\]

thus $\Gamma(\lambda)(-M_\phi^* g, g) = \lambda^n (-M_\phi^* g, g)$ iff $\Gamma_0(\lambda) g = \lambda^{n+N} g$, $\lambda \in \mathbb{T}$. For $n < -N$ there are no nonzero solutions of this equation, and for $n \geq -N$ the condition is satisfied iff $g$ is a homogeneous polynomial of degree $n+N$.

We conclude that $\dim H_n = 0$ if $n < -N$ and $\dim H_n = \dim A_{n+N} = q_{d-1}(n+N)$ if $n \geq -N$. Thus for $k \geq -N$ we see from (9.17) that

\[
\dim M_k = \sum_{n=-N}^{k} H_n = \sum_{n=-N}^{k} q_{d-1}(n+N).
\]

The identity $q_{d-1}(x) = q_d(x) - q_d(x - 1)$ of (3.2.2) implies that the right side of the preceding formula telescopes to $q_d(k+N) - q_d(-1) = q_d(k+N)$. Thus (9.16) implies that

\[
\chi(H) = \chi(M_H) = d! \lim_{k \to \infty} \frac{q_d(k+N)}{k^d} = \lim_{k \to \infty} \frac{(k+N+1)\ldots(k+N+d)}{k^d} = 1,
\]

as asserted. \hfill \blacksquare
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