Lorentz boosts of bispinor Bell-like states

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Abstract. We describe in this paper the effects of Lorentz boost on the quantum entanglement encoded in two-particle Dirac bispinor Bell-like states. Each particle composing the system described in this formalism has three degrees of freedom: spin, chirality, and momentum, and the joint state can be interpreted as a 6 qubit state. Given the transformation law of bispinor under boosts, we compute the change of the Meyer-Wallach global measure of quantum entanglement due to the frame transformation and study its equivalence to the results obtained for the relativistic spin 1/2 Bell-like states, constructed in the framework of the irreducible representations of the Lorentz group. We verify that the monotonic increase of the global entanglement under boosts for ultra-relativistic states is solely due to an increasing of the entanglement associated with the spins subsystems. For such ultra-relativistic states, the entanglement related to the chirality degrees of freedom is invariant, and the variation of the global entanglement of bispinor states is the same as the one calculated for relativistic spin 1/2 states. We also show that the particle-particle entanglement is invariant under boosts for any Bell-like state.

1. Introduction

Relativistic quantum mechanics and quantum information theory have been the framework to devise information protocols in relativistic setups, such as clock synchronization [1–3] and position verification [4]. In this context, transformation properties of quantum entanglement under frame transformations, in particular under Lorentz boosts, have been addressed and discussed in the last two decades [5–15]. The focus of many studies were on the transformation properties of spin $\frac{1}{2}$ entangled states [5–9], since spin states can be used to encode qubits, the basic quantum information unit.

In the relativistic quantum information framework, spin $\frac{1}{2}$ states are described by the irreducible representations (irreps) of the Poincarè group and a Lorentz boost acts on the spin degree of freedom as a momentum dependent rotation, the Wigner rotation [5–9]. Such transformation property is the basis of many interesting results, such as the generation of spin-momentum entanglement by frame transformations and non-invariance of the entanglement encoded in a pair of spin $\frac{1}{2}$ particles [5–9]. Despite the richness of this description, in the covariant Hamiltonian formulation of relativistic quantum mechanics of the Dirac equation, the inclusion of mass requires the consideration of the parity symmetry [16,17]. Spatial parity couples positive and negative parity states with positive and negative helicities and to have invariance under this symmetry one needs to consider the irreps of the extended Poincarè group [17]. In this case, spin $\frac{1}{2}$ is carried by Dirac four-component spinors (Dirac bispinors) satisfying the Dirac equation.
Such bispinorial structure can be used not only to describe charged massive fermions (quarks, electrons, muons, etc.) but also for investigating properties of Dirac-like systems, such as bilayer graphene [18–21].

Dirac bispinors are described in terms of a representation supported by a subgroup of the $SL(2, C) \otimes SL(2, C)$, the $SU(2) \otimes SU(2)$, accordingly carrying two discrete degrees of freedom (DoFs): spin and intrinsic parity [17], that are in general entangled [22–24]. In the scope of relativistic quantum information theory, the effects of Lorentz boosts on quantum entanglement encoded in a pair of bispinorial particles have been described in connection with Wigner rotations [25] and in the context of Fouldy-Wouthuysen (FW) spin operator [10]. Moreover, the role of the intrinsic parity-spin entanglement in the change of the total amount of entanglement shared by two bispinorial particles was addressed for general boosts scenarios and for antisymmetric states [13,14].

In this paper we consider the effects of Lorentz boosts on entangled two-particle states constructed as Bell-like superpositions of helicity eigenstates of the Dirac equation, including momentum superposition. We consider a scenario where two particles, described as helicity eigenstates of the free Dirac Hamiltonian, are in an entangled Bell-like state with respect to an inertial frame $S$ and we study the effects of a Lorentz boost on the correlational content of the state. The momentum degree of freedom is included as a discrete DoF and treated as an additional qubit carried by each particle. We verify that the particle-particle entanglement, encoded in the partition \{all DoFs of A; all DoFs of B\}, is invariant under boosts. Otherwise, for ultra-relativistic states the variation of the global entanglement, quantified via the Meyer-Wallach measure [26] is the same as obtained with the formalism using the irreps of the Lorentz group as presented in [8]. In this case, the amount of entanglement encoded in the partition (chirality vs all other DoFs) is invariant therefore not contributing for the change of the global entanglement.

### 2. Composite structure of Dirac equation and bispinor Bell-like states

The free Dirac equation that describes the dynamics of free fermionic particles reads (in momentum space) [27]

$$i\frac{\partial \psi}{\partial t} = \hat{H}_D \psi = (\mathbf{p} \cdot \mathbf{\alpha} + m\mathbf{\beta})\psi,$$

where $\mathbf{p}$ is the particle momentum and $m$ the particle mass. The $4 \times 4$ Dirac matrices $\mathbf{\alpha} = \{\hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z\}$ and $\mathbf{\beta}$ satisfy the anti-commutation relations $\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}\hat{I}_4$, $\hat{\alpha}_i\mathbf{\beta} + \mathbf{\beta}\hat{\alpha}_i = 0$ (for $i = x, y, z$) and $\mathbf{\beta}^2 = \hat{I}_4$. We have adopted natural units $\hbar = c = 1$, and from hereafter bold letters represents vector quantities with modulus $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and circumflexes denotes operators. $\hat{I}_N$ denotes the $N$ dimensional identity operator. Among the different representations of the Dirac matrices satisfying the required anti-commutation relations, we adopt in this paper the Chiral (or Weyl) representation, where the matrix $\hat{\gamma}_5 = i\mathbf{\beta}\hat{\alpha}_x\hat{\alpha}_y\hat{\alpha}_z$ is diagonal and given by $\hat{\gamma}_5 = \text{diag}\{\hat{I}_2, -\hat{I}_2\}$ \(^1\). In this representation the Dirac matrices are given by

$$\hat{\mathbf{\beta}} = \begin{bmatrix} 0 & \hat{I}_2 \\ \hat{I}_2 & 0 \end{bmatrix}, \quad \hat{\mathbf{\alpha}} = \begin{bmatrix} \hat{\sigma} & 0 \\ 0 & -\hat{\sigma} \end{bmatrix}. \quad (2)$$

The Dirac Hamiltonian exhibit symmetries that are supported by a group structure given in terms of the direct product of two algebras composing a subset of the $SL(2, C) \otimes SL(2, C)$, the

\(^1\) The chiral and the so called usual (or Dirac) representation of the Dirac matrices, where $\mathbf{\beta} = \text{diag}\{\hat{I}_2, -\hat{I}_2\}$, are related by the unitary transformation

$$\hat{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{I}_2 & \hat{I}_2 \\ \hat{I}_2 & -\hat{I}_2 \end{bmatrix}.$$
SU(2) ⊗ SU(2). The origin of such structure is related to the mass term of Dirac equation that requires the consideration of spatial parity symmetry, which couples two different irreducible representations of the Lorentz group, each one given in terms of the SU(2) group structure [16,17] (for more details see [14]). Correspondingly, the Hamiltonian (1) supported by such structure describes the dynamics of two discrete degrees of freedom: the chirality, associated to the Hilbert space \( H_C \), and the spin, associated with the Hilbert space \( H_S \). The joint dynamics is therefore described in terms of a composite Hilbert space \( \mathcal{H} = H_C \otimes H_S \), with \( \dim[H_C] = \dim[H_S] = 2 \). This structure becomes evident through the decomposition of the Dirac matrices in terms of tensor products of operators acting in each Hilbert space that for the chiral representation reads

\[
\hat{\alpha} = \hat{\sigma}^{(C)}_z \otimes \hat{\sigma}^{(S)}_z, \quad \hat{\beta} = \hat{\sigma}^{(C)}_x \otimes \hat{I}^{(S)}_2,
\]

where the subscripts \( C \) (\( S \)) indicates the chirality (spin) Hilbert space. In this framework, the Dirac Hamiltonian can be interpreted as a two-qubit operator and the particles described by such dynamics carry two discrete degrees of freedom with dynamics governed by a continuous variable, position or momentum.

Within the above prescription, the Dirac Hamiltonian eigenstates can be written as two-qubit states describing a bipartite system. For instance, the two degenerate positive energy eigenstate of the Dirac Hamiltonian (1) \( |u_s(p)\rangle \in H_C \otimes H_S \) (with \( s = 0, 1 \)) are given by

\[
|u_s(p)\rangle = \frac{1}{2} \sqrt{\frac{E_p + m}{E_p}} \left[ \left( 1 + \frac{p \cdot \hat{\sigma}^{(S)}_z}{E_p + m} \right) |0\rangle_C \otimes |\eta_s(p)\rangle_S + \left( 1 - \frac{p \cdot \hat{\sigma}^{(S)}_z}{E_p + m} \right) |1\rangle_C \otimes |\eta_s(p)\rangle_S \right],
\]

where \( E_p = \sqrt{p^2 + m^2} \) and \( |\eta_s(p)\rangle_S \) is a two-component spinor that depends on the spin polarization of the particle, and we have indicated the positive (negative) eigenstates of \( \hat{\sigma}^{(C)}_z \) by \( |0\rangle \) (\( |1\rangle \)). The solutions of the Dirac equation (4) cannot be decomposed as \( |\phi_C\rangle \otimes |\phi_S\rangle \), thus they are entangled states. The entanglement content of the Dirac equation solutions was characterized including external potentials [24] and in connection with low-energy systems [20, 21]. A convenient description of Dirac equations solutions that was adopted in previous studies [13, 14] are helicity eigenstates:

\[
\hat{j}^{(C)} \otimes \hat{\sigma}^{(S)}_z \cdot p \frac{p}{E_p} |u_s(p)\rangle = (-1)^s |u_s(p)\rangle,
\]

thus from now on the label \( s = 0 \) (1) denotes positive (negative) helicity. The helicity eigenstates are given explicitly as

\[
|u_s(p)\rangle = \frac{1}{2} \sqrt{\frac{E_p + m}{E_p}} \left[ \left( 1 + (-1)^s \frac{p}{E_p + m} \right) |0\rangle_C \otimes |\eta_s(p)\rangle_S + \left( 1 - (-1)^s \frac{p}{E_p + m} \right) |1\rangle_C \otimes |\eta_s(p)\rangle_S \right],
\]

which, different from the general solutions (4), are separable states.

With such particular solutions of the free Dirac equation, two-particle states are constructed as arbitrary superpositions of bispinors. Designating the two particles as \( A \) and \( B \), a general two-particle state reads

\[
|\psi_{A,B}\rangle = \frac{1}{N} \sum_i^n c_i |u_s_i(p_i)\rangle_A \otimes |u_{r_i}(q_i)\rangle_B.
\]

This is equivalent to the intrinsic parity-spin description of the Dirac equation reported previously [13, 14, 20, 21, 24], but here we choose to label one of the states with the chirality DoF, since we are working with the chiral representation of the Dirac matrices.
Such type of states can be created for example in quantum electrodynamical processes [28]. Since each of the particles are associated with a two-qubit state, the two-particle state is a 4 qubit state with the following DoFs: spins of particle A and B and chiralities of particle A and B. By further generalizing the bispinor superpositions (7) we can construct “Bell-like” states. A two-spins ½ Bell-like state has the form

$$\psi_{\text{Bell}} = \left( \cos \alpha |p\rangle_A \otimes |q\rangle_B + \sin \alpha (|q\rangle_A \otimes |p\rangle_B) \right) \otimes \left( \cos \beta |+\rangle_A \otimes |+\rangle_B + \sin \beta |-\rangle_A \otimes |+\rangle_B \right),$$

(8)

whose entanglement properties under relativistic transformations where studied in terms of Wigner rotations [8]. A bispinor Bell-like state equivalent to (8) is constructed by including in the superposition (7) the momentum DoF. Adopting the shorthand notation \(|u_s(p)\rangle \otimes |p\rangle \equiv |u_s(p), p\rangle\), a bispinor Bell-like state is written as

$$|\psi_{\text{Bell}}\rangle = \cos \alpha \cos \beta |u_0(p), p\rangle_A \otimes |u_0(q), q\rangle_B + \cos \alpha \sin \beta |u_1(p), p\rangle_A \otimes |u_1(q), q\rangle_B$$

$$+ \sin \alpha \cos \beta |u_1(q), q\rangle_A \otimes |u_1(p), p\rangle_B + \sin \alpha \sin \beta |u_0(q), q\rangle_A \otimes |u_0(p), p\rangle_B.$$  

(9)

The angles \(\alpha\) and \(\beta\) parametrize, as in (8), the degree of superposition between the momenta and between the helicities, respectively. The momentum eigenstates \(|p\rangle\) are normalized as

$$\int \frac{d^3p}{2E_p} |p\rangle = 1.$$  

(10)

Since each bispinor carries two DoFs (chirality and spin), the system described by (9) has 6 DoFs: the chiralities of A and of B (indicated by \(C_{A(B)}\)), the spins of A and B (\(S_{A(B)}\)) and the momenta of A B (\(P_{A(B)}\)). Moreover, from now on the letter \(\psi\) is used to refer to a state at the bispinor level, while the letter \(\phi\) will always refer to states in the spin \(\frac{1}{2}\) level of the irreps of the Lorentz group such as (8). We indicate the density matrix associated to a state \(|\xi\rangle\) by \(\rho_\xi = |\xi\rangle\langle\xi|\).

From now on we consider that the momenta distributions are sharply concentrated around their mean value, such that the overlap between \(|p\rangle\) and \(|q\rangle\) can be ignored. This supports an effective description of the momentum as a dichotomic variable, associated to an additional qubit of the systems, that is \(|p\rangle \equiv |0\rangle\) and \(|q\rangle \equiv |1\rangle\). This simplifying hypothesis was used in several investigations of transformation properties of quantum entanglement [8] and will be used through this paper. Moreover we will assume that the particles propagate in the \(e_z\) direction such that \(p\) and \(q\) are opposite, that is, \(p = -q = pe_z\). For this momenta configuration the helicity eigenstates are given by

$$|\eta_0(p)\rangle = |\eta_1(q)\rangle = |+\rangle, \quad |\eta_1(p)\rangle = |\eta_0(q)\rangle = |-\rangle,$$

(11)

where \(|\pm\rangle\) are the positive (negative) eigenstates of \(\sigma_z^{(S)}\).

Following the above considerations we verify that the state (9) is closely related to its counterpart (8) in the non-relativistic limit \(p \ll m\). In this case the bispinor helicity eigenstates (6) read

$$|\eta_s(p)\rangle \rightarrow \frac{|0\rangle_C + |1\rangle_C}{\sqrt{2}} \otimes |\eta_s\rangle_S,$$

(12)

and since for the considered momentum configuration \(|\eta_s\rangle_S\) is given by (11), one has

$$|\psi_{\text{Bell}}\rangle \rightarrow |\psi_{\text{NR}}\rangle = \frac{(|0\rangle + |1\rangle)_{CA} \otimes (|0\rangle + |1\rangle)_{CB}}{2} \otimes |\phi_{\text{Bell}}\rangle.$$  

(13)

Therefore, the subsystem associated with the DoFs \(\{S_A, P_A; S_B, P_B\}\) is described by the reduced density matrix \(\text{Tr}_{CA, CB}[|\psi_{\text{NR}}\rangle\langle\psi_{\text{NR}}|] = |\phi_{\text{Bell}}\rangle\langle\phi_{\text{Bell}}|\). One also notices that for \(m \gg p\) the
bispinors $|\alpha_{s}(p)|$ are eigenstates of the operator $\hat{s}_x^{(C)} \otimes \hat{I}^{(S)}$, which is the intrinsic parity operator in the chiral representation.

Under the approximation of the momentum as a dichotomic variable, the bispinor Bell-like state (9) can be interpreted as a 6 qubit state. Different from the entanglement encoded in states containing only two DoFs (bipartite states) whose characterization is supported by the Schmidt decomposition theorem [29], the description of quantum entanglement for multipartite states, such as (9) and (8), is an involved task [29]. The subsystems of a multipartite state can be correlated in nonequivalent ways such that multipartite entanglement can be studied following different approaches. Since the joint state (9) is a pure state, the multipartite entanglement shared in the partitions of the form $\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{N}\}$, $E_G$ is given by

$$E_G[\rho] = E[\rho^{\alpha_{i}}] = \frac{1}{N} \sum_{i=1}^{N} E_L[\rho^{\alpha_{i}}],$$

with the linear entropies $E_L$ given by

$$E_L[\rho] = \frac{d}{d-1} (1 - \text{Tr}[\rho^{2}]),$$

where $d$ is the dimension of the Hilbert space in which $\rho$ acts, and $\rho^{\alpha_{i}}$ is the reduced density matrix of the subsystem $\alpha_{i}$, obtained by tracing out all other subsystems $\rho^{\alpha_{i}} = \text{Tr}_{\{\alpha_{k}\} \neq \alpha_{i}}[\rho]$. The linear entropy $E_L[\rho^{\alpha_{i}}] \text{quantifies the amount of entanglement between the subsystem } \alpha_{i} \text{ and the subsystem composed by all } \alpha_{k} \text{ with } k \neq i, \text{ therefore } E_G \text{ is the mean value of the entanglement shared in the partitions of the form } \{\alpha_{i}; \{\alpha_{k}, \alpha_{i}\}\}. \text{ The global entanglement of Bell-like state (9) is given by}

$$E_G[\rho^{\psi_{\text{Bell}}}] = \frac{1}{6} \left[ 2 - \cos 4\alpha - \cos 4\beta + \frac{p^2}{2E_p^2} (3 - \cos 4\alpha - \cos 4\beta - \cos 4\alpha \cos 4\beta) \right].$$

The last term in the above expression is the contribution due to the chirality degree of freedom, which vanishes for $p \ll m$. In this limit, as highlighted by Eq. (12), the chirality factorizes out of the bispinorial structure, and therefore it is not entangled with all other subsystems. Moreover, for such non-relativistic limit $E_G[\rho^{\psi_{\text{Bell}}}] \rightarrow \frac{2}{3} E_G[\rho^{\phi_{\text{Bell}}}],$ and therefore the global entanglement properties of (9) are the same of (8).

3. Global entanglement of bispinor Bell-like states under Lorentz boosts

Once the general framework is set, we now proceed to describe the effects of Lorentz boosts on the entanglement content of bispinor Bell-like states (9) and its connections to the results obtained in the scope of Wigner rotations and spin $\frac{1}{2}$ states in the form (8).

Under a given Lorentz transformations $\Lambda$ connecting two inertial frames $S$ and $S'$, a state $|\psi(x)\rangle$ transforms as [16,17]

$$|\psi(x)\rangle \rightarrow \hat{S}[\Lambda]|\psi(\Lambda^{-1}x)\rangle,$$

where $\hat{S}[\Lambda]$ is an operator describing how $\Lambda$ acts on the state, which depends on the representation of the Lorentz group to which the state belongs and on the specific transformation. A Lorentz boost describes the transformation from an inertial frame $S$ to a frame $S'$ moving with constant speed $v$ with respect to $S$. The operator representing the action of Lorentz Boosts on bispinor states $\hat{S}_{\text{Boost}}$ reads [16,17,27]

$$\hat{S}_{\text{Boost}}[\omega] = \cosh \left( \frac{\omega}{2} \right) \hat{I}_4 + \sinh \left( \frac{\omega}{2} \right) n \cdot \hat{\alpha},$$

where $n$ is the null vector $n = (1, 1, 1, 1)$, and $\hat{\alpha}$ is the chiral representation.
where \( n = v/v \) and \( \omega = \text{arctanh} \left( \frac{v}{\sqrt{1-v^2}} \right) \) is the boost rapidity. The state constructed with the positive energy solutions of the free Dirac equation in momentum space \( |u_s(p), p\rangle \) transforms under boosts as

\[
|u_s(p)\rangle \otimes |p\rangle \rightarrow \hat{A}[\omega] \left( |u_s(p)\rangle \otimes |p\rangle \right) = \left( \frac{1}{N_p} \hat{S}_{\text{Boost}}[\omega] |u_s(\Lambda^{-1}p')\rangle \right) \otimes |p'\rangle,
\]

(19)

where \( N_p \) is a normalization factor and \( p' \) is the momentum with respect to \( S' \). To be in correspondence with the scenario described in the previous section, we consider now a boost in the direction \( n = (\sin \theta, 0, \cos \theta) \) in the \( x-z \) plane as depicted in Fig. 1. In this case, for the helicity eigenstates (Eq. 6) the normalization factors are given by

\[
N_p = \sqrt{\cosh \omega + \frac{p}{E_p} \sinh \omega \cos \theta} \quad \text{for} \quad p = pe_z,
\]

\[
N_q = \sqrt{\cosh \omega - \frac{p}{E_p} \sinh \omega \cos \theta} \quad \text{for} \quad q = -pe_z.
\]

We assume that the orthonormalization of the momentum qubits is preserved under boosts:

\[
\langle p|q\rangle \rightarrow \langle p'|q'\rangle = \delta_{p',q'},
\]

such that for the joint bispinor-momentum helicity eigenstates

\[
\langle u_s(p), p|u_r(q), q\rangle \rightarrow \frac{1}{N_p N_q} \langle u_s(\Lambda^{-1}p'), p'|\hat{S}_{\text{Boost}}^2[\omega]|u_r(\Lambda^{-1}q'), q'\rangle = \delta_{sr} \delta_{p',q'}.
\]

(20)

Figure 1. Schematic representation of the boost scenario. The Lorentz boost is performed to describe a frame transformation from \( S \) to \( S' \), which moves uniformly in the \( n = \sin \theta e_x + \cos \theta e_z \) direction with respect to \( S \). The transformation of the two-particle state (9) is described by the operator \( \hat{A}^{(A)} \otimes \hat{A}^{(B)} \) whose action in the bispinors are defined in (18), which induces non trivial changes in the quantum entanglement shared among the different DoFs of the system.

The action of boosts on two-particle states is given by \( \hat{A}^{(A)}[\omega] \otimes \hat{A}^{(B)}[\omega] \) and the bispinor Bell-like state transforms as

\[
|\psi_{\text{Bell}}\rangle \rightarrow |\psi'_{\text{Bell}}\rangle = \hat{A}^{(A)}[\omega] \otimes \hat{A}^{(B)}[\omega]|\psi_{\text{Bell}}\rangle,
\]

(21)

modifying in a non trivial way the superposition and consequently inducing changes in the entanglement encoded in the different partitions of the system. In particular, the entanglement
between all DoFs of particle $A$ and all DoFs of particle $B$, which we call particle-particle entanglement, is invariant. This can be verified by considering the reduced density matrix $\rho^A$ of subsystem $A$ obtained by tracing our all DoFs of $B$:

$$\rho^A_{\psiBell} = \text{Tr}_{B}[\rho_{\psiBell}] = \cos^2 \alpha (\cos^2 \beta |u_0(p), p\rangle\langle u_0(p), p| + \sin^2 \beta |u_1(p), p\rangle\langle u_1(p), p|) + \sin^2 \alpha (\cos^2 \beta |u_1(q), q\rangle\langle u_1(q), q| + \sin^2 \beta |u_0(q), q\rangle\langle u_0(q), q|)$$

Under a Lorentz boosts the orthonormalization of the bispinors is preserved (see Eq. 20), and thus

$$\text{Tr}_A[(\rho^A_{\psiBell})^2] = \text{Tr}_A[(\rho^A_{\psiBell})^2].$$

Therefore the linear entropy $E_L[\rho^A_{\psiBell}] = \frac{6}{5}(1 - \text{Tr}_A[(\rho^A_{\psiBell})^2])$ quantifying the entanglement in the partition \{all DoFs of $A$; all DoFs of $B$\} is invariant under boosts, in agreement with the invariance of particle-particle entanglement of spin $\frac{1}{2}$ Bell-like states (8) [8]. The linear entropies of the momenta subsystems $E_L[\rho^A_{\psiBell}]$ are also invariant under boosts, which is again in correspondence with the behavior reported for spin $\frac{1}{2}$ states $|\phi_{\psiBell}\rangle$. On the other hand, as verified in previous studies, bispinor superpositions like (7) exhibit invariance of particle-particle entanglement only if $A$ and $B$ have definite momenta [13].

Although the bispinor Bell-like state exhibits some invariant quantum correlations, the global measure of entanglement (14) is not invariant under boosts, with variation depending not only on the superposition and boost parameters but also on the energy regime of the state, encoded in the parameter $p/m$. Figure 2 shows the plot of the global entanglement of the transformed state $E_G[\rho_{\psiBell}]$ for a boost in the $e_x$ in function of the boost rapidity for several superposition angles for a non-relativistic state (left plot) and for a state with $p/m = 10$ (right plot). Non-relativistic states show an monotonic increase in the global entanglement. For states with $p \gg m$ the increase on $E_G$ is exclusively due to the variation in the linear entropy associated to the spins subsystems, since in this limit

$$E_G[\rho_{\psiBell}] = \frac{1}{4} \left( 3 - \cos 4\alpha - \cos 4\beta - \cos 4\alpha \cos 4\beta \right) = E_G[\rho_{\psiBell}].$$

For such ultra-relativistic states, $|u_+(p)| \rightarrow |s_C \otimes |\eta_\beta\rangle_S$ and because chirality is invariant under boosts, the contribution of the chirality subsystems entropy to the global entanglement, $E_L[\rho_{\psiBell}^C]$, is invariant. Additionally, for specific superposition angles the state has invariant $E_G$ and we also notice that the overall behavior of $E_G$ under boosts is similar to the one reported for anti-symmetric superpositions of the form (9), which do not include momentum entanglement.

The discussion of the previous paragraph sets the framework to relate the changes of the global entanglement of (9) under boosts with those of spin $\frac{1}{2}$ states (8). Although the states have the same amount of global entanglement for $p \ll m$, a Lorentz boost will increase the chirality subsystems entropy of bispinor Bell-like states, and the changes on the quantum entanglement are affected in a different way from the ones expected for spin $\frac{1}{2}$ states. Otherwise, for $p \gg m$ (24) holds and, although $|\psi_{\psiBell}\rangle$ and $|\phi_{\psiBell}\rangle$ have different amounts of global entanglement, the variation of $E_G$ induced by boosts is the same in both states. In fact, for $p \gg m$ one has

$$\rho_{\psiBell}^S = \rho_{\psiBell}^S,$$

as can be verified by considering the transformation properties of the spin $\frac{1}{2}$ states given in terms of Wigner rotations, with general formula given elsewhere (see for instance [9]). Therefore

$$E_G[\rho_{\psiBell}^S] = E_G[\rho_{\psiBell}^S]$$

and thus, since for the bispinor state in the limit $p \gg m$ (24) holds, the variations of $E_G$ due to Lorentz boosts are the same for $|\psi_{\psiBell}\rangle$ and for $|\phi_{\psiBell}\rangle$. 

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Global entanglement of the bispinor Bell-like state (9) under a Lorentz boost in the $e_x$ direction in function of the boost rapidity $\omega$, for non-relativistic states (left plot) and ultra-relativistic states (right plot) in the unboosted frame and for several superposition angles as indicated in the legend. The change on $E_G$ of ultra-relativistic state is due to an increase of the reduced entropy of only the spin subsystems, since for such states the entropies associated with the chiralities and with the momenta are invariant.

Taking as an example the Bell-like state with $\alpha = \pi/4$ and $\beta = 0$ (the red curve in Fig. 2), Figure 3 shows the difference between the variations of global entanglement

$$\delta E_G = 6\Delta E_G[\rho_{\psi\text{bell}}] - 4\Delta E_G[\rho_{\phi\text{bell}}]$$

where $\Delta E_G[\rho_\xi] = E_G[\rho_\xi] - E_G[\rho_\xi']$ is the variation of the global entanglement induced by a Boost in a given pure state $\rho_\xi$. The first row shows $|\delta E_G|$ as a function of both $p/m$ and the boost rapidity $\omega$ while the second row depicts it as a function of $p/m$ for several values of $\omega$ and for boost in two directions. For $p \gg m$, $\delta E_G \to 0$, and the variation of $E_G$ of the bispinor Bell state is the same of spin $\frac{1}{2}$ Bell state (8). We also notice that for small $\omega$ and small $p/m$, $\delta E_G \to 0$ since, as discussed in Sec. II, in this limit both states are equivalent and the chiralities are separable from the other subsystems.

4. Conclusions
In summary, we have described the effects of Lorentz boosts on the quantum entanglement encoded in two-particle states constructed as Dirac bispinors Bell-like superpositions. In this scenario, given the composite structure of the Dirac bispinors, each particle has 3 DoFs: spin, chirality and momentum, and we have considered the latter as a dichotomic variable, such that the joint two-particle state is interpreted as a 6 qubit state. Considering the non-relativistic regime, the constructed superposition has the same correlation properties of the relativistic spin $\frac{1}{2}$ state described in [8]. We verified that Lorentz boosts affect the Meyer-Wallach global measure of entanglement [26] shared among the different DoFs of the system. In general the frame transformation increases the amount of the global entanglement encoded in the state in a similar fashion to what was reported for bispinor superpositions not including the momentum DoF [14]. In correspondence to the boost effects on spin $\frac{1}{2}$ states as shown in [8], the entanglement in the partition \{ all DoFs of A; all DoFs of B\} is invariant under boosts. Moreover, for ultra-relativistic states the variation of the global entanglement is the same as the one calculated for spin $\frac{1}{2}$ states in the framework of Wigner rotations. In this limit, the entanglement of the chiralities with all other DoFs of the two-particle state is invariant and all changes in the global entanglement are due to an increase in the spins entanglement.
Figure 3. Absolute difference $|\delta E_G|$ between the variation of the global entanglement in bispinor Bell-like states (9) and in spin $1/2$ Bell-like states (8) under a Lorentz boost in the direction $\mathbf{n} = \sin \theta \mathbf{e}_x + \cos \theta \mathbf{e}_x$, as schematically depicted in Fig. 1. The plots are in function of $p/m$ and the boost rapidity $\omega$ (top row), and in function of $p/m$ for different values of $\omega$ (bottom row). The plots are for two different boost directions: $\pi/6$ (left column) and $\pi/2$ (right column). For ultra-relativistic states $p \gg m$ and the variation in both states are the same as a consequence of the invariance of the chirality subsystem entropy.

Our results generalize the preliminar discussion of [13] connecting the description of the transformation properties of quantum entanglement encoded in spin $1/2$ states, constructed in the framework of the irreps of the (proper) Lorentz group [5–9], with those of bispinor states, described via irreps of the complete Lorentz group [13,14], which includes the parity symmetry related to the mass term of the Dirac equation. The role played by the mass with respect to such connection between the frameworks is two-folded: although the states considered here have equal entanglement properties in the non-relativistic regime, the variation of the global entanglement under boosts is the same only in the ultra-relativistic regime, for which the bispinor states have definite chirality.

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