Research Article

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Towards a Ring Analogue of the Leftover Hash Lemma

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Abstract: The leftover hash lemma (LHL) is used in the analysis of various lattice-based cryptosystems, such as the Regev and Dual-Regev encryption schemes as well as their leakage-resilient counterparts. The LHL does not hold in the ring setting, when the ring is far from a field, which is typical for efficient cryptosystems. Lyubashevsky et al. (Eurocrypt ’13) proved a “regularity lemma,” which can be used instead of the LHL, but applies only for Gaussian inputs. This is in contrast to the LHL, which applies when the input is drawn from any high min-entropy distribution. Our work presents an approach for generalizing the “regularity lemma” of Lyubashevsky et al. to certain conditional distributions. We assume the input was sampled from a discrete Gaussian distribution and consider the induced distribution, given side-channel leakage on the input. We present three instantiations of our approach, proving that the regularity lemma holds for three natural conditional distributions.

Keywords: Lattice-based cryptography, leakage resilience, Ring-LWE, regularity lemma

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1 Introduction

The leftover hash lemma (LHL) is used in the analysis of various lattice-based cryptosystems. Specifically, it is often useful to argue that for high-min entropy input $x \in \mathbb{Z}_q^m$ and random matrix $A \leftarrow \mathbb{Z}_n^m \times \mathbb{Z}_q^m$, $Ax$ is uniform random, given $A$. The above fact is used in the proof of security for both the Regev and Dual-Regev encryption schemes. More sophisticated proof approaches that utilize the LHL along with the structure of the matrix $A$ have been used to argue leakage resilience of these cryptosystems, such as in [1, 13].¹

Analogues of the statement above do not necessarily hold in the ring setting. Specifically, assuming a high min-entropy input $x = x_1, \ldots, x_l$, setting $a_1 = 1$, and $a_2, \ldots, a_l$ chosen uniformly at random from the ring, the uniformity of $a_{l+1} = \sum_{i \in [l]} a_i x_i$ does not follow from the LHL lemma, in cases where the ring is far from a field, which is the typical case for efficient cryptosystems.

Fortunately, Lyubashevsky et al. [25, 26] proved a “regularity lemma” showing that the distribution over $a_{l+1}$ as above is (close to) uniform random, even given $a_2, \ldots, a_l$, but only for the case where the input $x$ is drawn from a discrete Gaussian distribution of sufficiently high standard deviation. While sufficient for proving the security of certain cryptosystems, unlike the more general leftover hash lemma, the statement of

¹ For example, techniques include decomposition of the matrix $A$ into two random matrices of varying dimensions [1].
the regularity lemma of [25] implies nothing about uniformity of \( a_{i+1} \) in the case that \( \mathbf{x} \) is a high min-entropy input from another distribution.

The ring setting.
Consider the number field \( K = \mathbb{Q}[x]/\Phi_m(x) \), where \( \Phi_m(x) \) is the \( m \)-th cyclotomic polynomial of degree \( \varphi(m) \). The ring of integers, \( \mathbb{Z} \subset K \), is defined as \( \mathbb{Z} = \mathbb{Z}[x]/\Phi_m(x) \). \( \mathbb{Z}_q := \mathbb{Z}[x]/\Phi_m(x) \) denotes the set of polynomials obtained by taking an element of \( \mathbb{Z}[x]/\Phi_m(x) \) and reducing each coefficient modulo \( q \). In this paper, we further assume that \( m \) is a power of two, so \( \Phi_m(x) = x^m + 1 \) has degree \( n = m/2 \), and set \( q \) to be a prime such that \( q \equiv 1 \mod m \). In this case \( \Phi_m(x) \) completely splits into \( n \) factors in \( \mathbb{Z}_q[x] \). This is the setting favored in practice since it allows for optimizations in the implementation, such as fast arithmetic over the ring \( \mathbb{Z}_q \).

A Ring Analogue of the LHL.
For rings \( \mathbb{Z}_q \) such as the above, a result analogous to the leftover hash lemma—proving that \( a_{i+1} = \sum_{l=0}^{n} a_l x_l \) is indistinguishable from random, given \( a_2, \ldots, a_i \), as long as \( x_1, \ldots, x_i \) has sufficiently high min-entropy—is impossible. For example, if the \( j \)-th NTT coordinate of each ring element in \( \mathbf{x} = x_1, \ldots, x_i \) is leaked, then the \( j \)-th NTT coordinate of \( a_{i+1} = \sum_{l=0}^{n} a_l x_l \) is known\(^2\), and so \( a_{i+1} \) is very far from uniform. Yet this is only a \( 1/n \) leakage rate!\(^1\)

Nevertheless, Lyubashevsky et al. [25, 26] proved a “regularity lemma” showing that for matrix \( A = [I_k | \hat{A}] \in (\mathbb{Z}_q)^{k \times l} \), where \( I_k \in (\mathbb{Z}_q)^{k \times k} \) is the identity matrix and \( \hat{A} \in (\mathbb{Z}_q)^{k \times (l-k)} \) is uniformly random, and \( \mathbf{x} \) chosen from a discrete Gaussian distribution (centered at 0) over \( \mathbb{Z}_q^l \), the distribution over \( A \mathbf{x} \) is (close to) uniform random. A similar result was proven by Micciancio [28], but requires super-constant dimension \( l \), thus yielding non-compact cryptosystems. In contrast, the regularity lemma of [25] holds even for constant dimension \( l \) as small as 2. The fundamental technical question we consider in this work is:

For which distributions \( \mathcal{D} \) over \( \mathbf{x} \in \mathbb{Z}_q^l \), is the distribution over \( A \mathbf{x} \) (close to) uniform random, for \( R, q, A \) as above and constant \( l \)?

### 1.1 Our Results

We prove a “regularity lemma” for three conditional distributions, which we describe next. Only the parameter \( s \)—the standard deviation of the discrete Gaussian for sampling each coordinate of \( \mathbf{x} \)—differs in each setting.

#### Conditional Distribution I.

We assume a secret key \( \mathbf{x} = (x_1, \ldots, x_l) \), where each \( x_i \in \mathbb{Z}_q \). Moreover, each \( x_i \) itself is represented as an \( n \)-dimensional vector. So in total, \( \mathbf{x} \) is an \( l \cdot n \)-dimensional vector. We consider the conditional distribution on \( \mathbf{x} \) when the sum of \( \mathbf{x} \) and \( \mathbf{e} \) is revealed, where each coordinate of \( \mathbf{e} \) is a Gaussian random variable with standard deviation at least \( s \). This setting captures leakage on \( \mathbf{x} \) by an adversary who uses a fast, but inaccurate device to obtain noisy measurements of each sampled coordinate of the secret key (e.g. through a power or timing channel). We prove that it is sufficient to set \( s \geq \sqrt{2} \cdot 2n \cdot q^{k(l-2)/ln} \), See Theorem 2.1 and Corollary 2.2.

\(^2\) Applying NTT to \( a_i, x_i \in \mathbb{Z}_q \)—resulting in \( n \)-dimensional vectors, \( \tilde{a}_i, \tilde{x}_i \in \mathbb{Z}_q^n \)—allows for component-wise multiplication/addition, so the \( j \)-th NTT coordinate of \( a_ix_i, i \in [l] \) will be known and so the \( j \)-th NTT coordinate of \( a_{i+1} \) is known.

\(^3\) We thank an anonymous reviewer for pointing out this counterexample to us.
Conditional Distribution II.

We consider the conditional distribution over \( x = (x_1, \ldots, x_l) \) when we leak \( \ell \) coordinates from each \( x_i, i \in [l] \), and we set parameters such that the fraction of leaked coordinates \( \frac{\ell l}{n} \) is constant. The \( \ell \) leaked coordinates are arbitrary, but the same \( \ell \) coordinates must be leaked from each \( x_i, i \in [l] \).4 Low noise is added to each leaked coordinate (only \( 2n \) standard deviation, as opposed to \( \sqrt{2} \cdot 2n \cdot q^{k l 1+2/(n l)} \) standard deviation as in Conditional Distribution I). No information at all is leaked about the remaining coordinates. This setting corresponds to a side-channel attack launched during the sampling of \( x \), where the attacker has a slower, but more accurate device which allows it to obtain more accurate measurements for a constant fraction of the coordinates of the secret key, but no information for the remaining coordinates.5 We prove that it is sufficient to set \( s \geq 2n \cdot q^{k l 1+2/(n l)} \), where \( \ell \cdot l \) is the number of leaked coordinates. See Theorem 2.3 and Corollary 2.6.

Conditional Distribution III.

Here, we consider the conditional distribution on \( x \), when the magnitude of \( x \) with Gaussian channel error \( e \) is revealed (note that \( e \) is a scalar). We assume \( e \) is sampled from a univariate Gaussian with standard deviation \( s \). A motivation for this type of leakage is that (discrete) Gaussian sampling of \( x \) is often implemented via rejection sampling in practice [7, 12]. E.g. a vector could be sampled from a “close” multi-dimensional binomial distribution and rejection sampling then used to obtain a sample from the correct distribution. The rejection condition depends on the weight of \( x \) under the target distribution, which in turn depends on the magnitude of \( x \), and so this information is vulnerable to leakage during computation.6 We prove that it is sufficient to set \( s \geq \sqrt{14/5 \cdot (n l’/n) \cdot \ln n} \cdot 2n \cdot q^{k l 1+2/(n l)} \), where \( n’ = n \cdot l + 1 \). See Theorem 2.9 and Corollary 2.10.

Applications to leakage resilience.

Since applications of the LHL/Regularity Lemma in lattice-based cryptography are widespread, a number of Ring-LWE (RLWE) cryptosystems achieve certain leakage resilience properties using our results. Such cryptosystems include the ring analogues of Regev encryption [24], Dual-Regev encryption [25], and identity-based encryption (IBE) based on Dual-Regev encryption [19] (see ring version in [3]). Specifically, by substituting our “regularity lemma” for the original “regularity lemma” in the security proofs, those schemes still enjoy security guarantees even given certain leakage on the randomness for encryption (for Regev) the secret key (for Dual-Regev), and the secret key corresponding to the challenge identity (for IBE).

### 1.2 Our High-Level Approach

For a matrix \( A = [I_k | \bar{A}] \in (R_q)^{kl} \), where \( I_k \in (R_q)^{k \times k} \) is the identity matrix and \( \bar{A} \in (R_q)^{k \times (l-k)} \) is uniformly random, we define \( \Lambda^{-1}(A) = \{ z \in R^l : Az = 0 \text{ mod } qR \} \). If \( [x \mod \Lambda^{-1}(A)] \) is uniform random (over cosets of \( \Lambda^{-1}(A) \)), then the distribution of \( Ax \) is also uniform random over cosets of \( qR^k \). The input/output distributions can then be discretized over the ring \( R \). Therefore, the goal is to show that when \( x \) is sampled from continuous distribution \( D \), we have that \( [x \mod \Lambda^{-1}(A)] \) is uniform random. Consider the case where the distribution \( D \) is exactly a Gaussian distribution with mean 0 and standard deviation \( s \). In this case, if \( s \) is greater than or equal to the smoothing parameter of \( \Lambda^{-1}(A) \), this by definition ensures that the distribution \( [x \mod \Lambda^{-1}(A)] \) is uniform random. Thus, [25] prove their regularity lemma by showing that with high probability over choice of \( A \), the smoothing parameter, \( \eta_e(\Lambda^{-1}(A)) \), is upperbounded by \( s \).

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4 Alternatively, we can view the leakage as \( \ell \) completely arbitrary coordinates, with leakage rate of \( \ell l/(n l) \), which remains constant for constant \( l \).

5 Here we assume that the secret key is stored as a vector in the canonical embedding (in the other leakage scenarios, the result holds when the secret key is stored in using the polynomial representation or is stored as a vector in the canonical embedding).

6 For example, a power analysis attack on the BLISS signature scheme [18] exploited the rejection sampling procedure to recover the magnitude (norm) of certain secret values, which then led to a full break of the scheme.
Before presenting our approach to extending the above result, it is instructive to give a high-level recap of how to derive upper bounds on the smoothing parameter.

Let $\rho_s := e^{-\frac{(x_0)^2}{2s^2}}$ and let $\psi_s$ (the normalization of $\rho_s$) correspond to the probability density function (PDF) of the normalized $n$-dimensional Gaussian distribution with mean 0 and standard deviation $s$. In the following, for a function $f$ we concisely represent $\sum_{v \in A} f(v)$ by $f(A)$. To show that the distribution over $[x \mod A]$ is (close to) uniform when $x$ is sampled from a distribution with PDF $\psi_s$, one needs to show that for every coset $(A + c)$ of the lattice, $\psi_s(A + c) = \frac{1}{\det(A)}$. Focusing on the zero coset, where $c = 0$, we can prove this using the Poisson summation formula, which says that for any lattice $A$ and integrable function $\rho_s$: $\psi_s(A) = \frac{1}{\det(A)} \cdot \hat{\psi_s}(A^\vee)$, where for a function $f$, $\hat{f}$ denotes the $n$-dimensional Fourier transform of $f$ and $A^\vee$ is the dual lattice of $A$ (see Appendix A.2). It remains to show that $\hat{\psi_s}(A^\vee)$ is close to 1 (i.e. is upperbounded by 1 + $\varepsilon$).

The proof approach outlined above can be applied to (integrable) normalized PDF $\Psi$ that are not Gaussians centered at 0: To show that the distribution over $[x \mod A]$ is (close to) uniform when $x$ is sampled from a distribution with PDF $\Psi$, it is sufficient to show that $\hat{\Psi}(A^\vee)$ is upperbounded by 1 + $\varepsilon$.

In this work, we consider PDF’s, $\Psi$, that correspond to the PDF of $x$, from the point of view of the adversary, given the leakage. The technical contribution of this work is to show that, for each conditional distribution, (with overwhelming probability over choice of $A$) $\hat{\Psi}(A^\perp(A^\vee))$ is close to 1. Specifically, for each distribution, our approach requires: (1) Determining the PDF $\Psi$, (2) Computing (an upper bound for) the multi-dimensional Fourier transform of $\Psi$ (denoted $\hat{\Psi}$), (3) Proving that $\hat{\Psi}((A^\perp(A^\vee))$ is upperbounded by 1 + $\varepsilon$ (or, equivalently that $\hat{\Psi}((A^\perp(A^\vee)) \setminus \{0\})$ is upperbounded by $\varepsilon$).

1.3 Related Work

**Leakage-resilient cryptography.**

There is a significant body of work on leakage-resilient cryptographic primitives, beginning with the work of Dziembowski and Pietrzak [16] on leakage-resilient stream-ciphers. Other constructions include [1, 5, 6, 14, 22, 22, 23, 23, 27, 30, 31]. With the exception of [1], most of these results construct new cryptosystems from the bottom up. In our work, we consider whether we can prove that an existing cryptosystem enjoys leakage resilience, without modification of the scheme.

**Lattice-based & leakage-resilient cryptography.**

Goldwasser et al. [20] initiated the study of leakage resilience of lattice based cryptosystems. This was followed by series of works [1, 13, 15], all these papers however study leakage resilience of schemes based on standard LWE problem in both symmetric as well as public key setting.

**Robustness of Ring-LWE**

To the best of our knowledge the ePrint version [10] of this work is the first effort to study the robustness of RLWE based cryptosystems under leakage. Subsequent to the publishing of ePrint [10], interest has sparked in analyzing the RLWE-based schemes and their leakage resilience. Albrecht et al. [2] implemented cold boot attack on RLWE based KEM schemes and compared the number of operations required to mount the attack when secret is stored with different encodings. Recently, Bolboceanu et al. [4] studied the hardness of RLWE problem in cases where the secret is sampled from distributions other than uniform random distribution over the ring. In [11], it is shown that under specific structured leakage on the NTT encoding of secret key, it is possible to recover the entire secret key given multiple RLWE samples and they implement the attack to recover the secret in real world parameter settings.
Other variants of LHL
Stehlé and Steinfeld [34] studied the leftover hash lemma in the ring setting for power of 2 cyclotomics and Rosca et al. [33] generalized their result to non-cyclotomic rings. However, both these results study the case where input is sampled from discrete Gaussian distribution.

2 Extending the Regularity Lemma

For a positive integer \( n \), we denote by \([n]\) the set \( \{1, \ldots, n\} \). We denote vectors in boldface \( x \) and matrices using capital letters \( A \). For vector \( x \) over \( \mathbb{R}^n \) or \( \mathbb{C}^n \), define the \( \ell_2 \) norm as \( \|x\|_2 = (\sum |x_i|^2)^{1/2} \). We write as \( \|x\| \) for simplicity. Background and standard definitions related to lattices and algebraic number theory are in Appendix A. Our results are applicable when \( R \) is the ring of integers in the \( m \)-th cyclotomic number field \( K \) of degree \( n \), \( m = 2n \) is a power of 2 and prime \( q \) is s.t. \( q \equiv 1 \mod m \). We denote by \( I_k \in (R_q)^{k \times k} \) the identity matrix.

2.1 Conditional Distribution I

Recall that \( x = (x_1, \ldots, x_l) \), where each coordinate of each \( x_i \in R_q \) is sampled from a discrete Gaussian with standard deviation \( s \) and each \( x_i \) is represented as a vector in either the polynomial or canonical basis.\(^7\) We assume leakage of all coordinates, with Gaussian noise of standard deviation \( \sigma \). \( x \) is a vector in either the polynomial or canonical basis.

Theorem 2.1. For positive integers \( k \leq l \leq \text{poly}(n) \), let \( A = [I_k|\bar{A}] \in (R_q)^{k \times l} \), where \( \bar{A} \in (R_q)^{k \times (l-k)} \) is uniformly random. Then for all \( \sigma \geq 2n \cdot q^{k(l+2)/(2n)} \) and \( c \in \mathbb{R}^{nl} \) then

\[
\rho_{\sigma,c} \left( A^{\perp} (A)^{\vee} \right) \leq 1 + 2^{-O(n)},
\]

except with probability at most \( 2^{-O(n)} \) over choice of \( \bar{A} \).

Proof. The theorem follows from Lemma B.7 and the regularity lemma from [26]. \( \square \)

The following corollary follows from Lemmas B.12 and B.13 and Theorem 2.1.

Corollary 2.2. Let \( R, n, q, k, l, c, \sigma \) be as in Theorem 2.1. Assume that \( A = [I_k|\bar{A}] \in (R_q)^{k \times l} \) is chosen as in Theorem 2.1. Then, with probability \( 1 - 2^{-O(n)} \) over the choice of \( \bar{A} \), the distribution of \( AX \in R_q^n \), where \( x \in R^n \) is chosen from \( D_{l,\sigma,c} \), the discrete Gaussian probability distribution over \( R^n \) with parameter \( \sigma \) and center \( c \), satisfies that the probability of each of the \( q^n \) possible outcomes is in the interval \( (1 \pm 2^{-O(n)})q^{-nk} \) (and in particular is within statistical distance \( 2^{-O(n)} \) of the uniform distribution over \( R_q^n \)).

\(^7\) Either representation works since for power-of-two cyclotomics, spherical Gaussians in the polynomial basis correspond to spherical Gaussians in the canonical basis.
In particular, this means that the standard deviation used to sample \( x \) should be increased from \( 2n \cdot q^{k/\ell+2/(\ell n)} \) (as in [26]) to \( \sqrt{\frac{1+\eta^n}{\ell/n}} \cdot 2n \cdot q^{k/\ell+2/(\ell n)} \). Setting \( \tau = 1 \), we obtain the parameters described in the introduction.

### 2.2 Conditional Distribution II

Recall that \( x = (x_1, \ldots, x_l) \), where each \( x_i \in R_q \) and each \( x_i \) is represented as a vector in the canonical embedding. We assume leakage of \( \ell \) coordinates—with low noise added—of each \( x_i \) for \( i \in [l] \) and restrict the coordinates leaked across each \( x_i \) to be the same. Let \( S \subseteq [n] \), where \( |S| = \ell \) denote the set of positions (from each \( x_i \)) that are leaked. Lemma D.1 shows that, conditioned on leakage, each component \( x^i_j \), \( i \in [l], j \in S \), (resp. \( \not\in S \)) is sampled from Gaussian distribution with mean \( c^j_i :\leq \frac{n^2}{\ell^2/n^2} \) (resp. 0), and variance \( \sigma^2_x \geq 4n^2 \) (resp. \( \sigma^2_x = s^2 \)).

**Theorem 2.3.** For positive integers \( k \leq l \leq \text{poly}(n) \), let \( A = [I_k][\bar{A}] \in (R_q)^{k \times l} \), where \( \bar{A} \in (R_q)^{k \times (l-k)} \) is uniformly random. Let \( \sigma := (\sigma_1, \ldots, \sigma_n) \in R^n \) and \( c := (c_1, \ldots, c_n) \in R^n \) be vectors, where \( \ell \) positions in \( \sigma \) are set to \( 1 \), while the other positions are set to \( 0 \). Let \( k, l, \ell \) be such that \( l - k \cdot l \cdot \ell/n > 0 \) and \( l - k - 1 \geq 1 \), and let \( s \geq 2n \cdot q^{3+\eta} \)

\[ \rho_{\sigma,c} \left( \bar{A}^{\top}(A) \right) \leq 1 + 2^{-D(n)} \] except with probability at most \( 2^{-D(n)} \) over choice of \( \bar{A} \).

For proving Theorem 2.3, we begin with exposition on the forms of the Ideals \( qR^\vee \subseteq \mathfrak{J} \subseteq R^\vee \) in power-of-two cyclotomics as well as some lemmas.

To generate the set \( T \) of ideals \( \mathfrak{J} \) such that \( qR^\vee \subseteq \mathfrak{J} \subseteq R^\vee \) we take each ideal \( \mathfrak{J} \) s.t. \( qR \subseteq \mathfrak{J} \subseteq R \) and set \( \mathfrak{J} := q\mathfrak{J}^\vee \). Recall from Fact A.3 that \( (q) \) splits completely into \( n \) distinct ideals of norm \( q \), i.e. \( qR = \Pi_{i \in [n] \mathfrak{p}_i} \). Therefore, the set of all ideals \( \mathfrak{J} \) such that \( qR \subseteq \mathfrak{J} \subseteq R \), is exactly the set \( \mathfrak{J} := \{ \Pi_{i \in [n] \mathfrak{p}_i} | S \subseteq [n] \} \). Thus, the number of ideals \( \mathfrak{J} \) such that \( qR \subseteq \mathfrak{J} \subseteq R \) (and hence also the number of ideals \( \mathfrak{J} \subseteq T \)) is exactly \( 2^n \). Now, note that for each ideal \( \mathfrak{J} \subseteq T \),

\[ |qR^\vee| = |R/q\mathfrak{J}^\vee| = N(q\mathfrak{J}^\vee). \]

Thus, we see that for each \( \mathfrak{J} \subseteq T \), \( 1 \leq |qR^\vee| \leq q^n \).

Let \( T_1 \) denote the set of ideals \( \mathfrak{J} \subseteq T \) such that \( |qR^\vee| < 2^n \). Let \( T_2 \) denote the set of ideals \( \mathfrak{J} \) such that \( |qR^\vee| \geq 2^n \). Furthermore, let \( T_1^j \) be the set of \( \mathfrak{J} \subseteq T_1 \) such that \( s \geq \eta_{2^{-2n}}((\frac{1}{2} |\mathfrak{J}|)^\vee) \) (where \( \eta_{2^{-2n}} \) denotes the smoothing parameter and \( s \) is fixed as above). Let \( T_2^j := T_2 \setminus T_1^j \). Let \( \sigma := (\sigma_1, \ldots, \sigma_n) \in R^n \) be a vector with \( \ell \) positions are set to \( 2n \), while the other positions are set to value \( s \).

**Lemma 2.4.** For ideals \( \mathfrak{J} \subseteq T_1 \),

\[ \eta_{2^{-2n}}((\frac{1}{q} |\mathfrak{J}|)^\vee) \leq 2n. \]

The proof of Lemma 2.4 can be found in Appendix E.1.

**Lemma 2.5.** For ideals \( \mathfrak{J} \subseteq T_2^j \)

\[ |\mathfrak{J}/qR^\vee|^{-(l-k)} \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} |\mathfrak{J}| \right)^i \right) \leq 2^{-n(l-k)}, \]

where \( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \) is an \( n \)-dimensional Gaussian function with coordinate-wise standard deviation \( 1/\sigma_i \), \( i \in [n] \) and center \( 0 \) (see beginning of Appendix B).

The proof of Lemma 2.5 can be found in Appendix E.1. We now conclude the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Since by Lemma B.7 we have that for any \( (n \cdot l) \)-dimensional vectors, \( c, x \) and any \( n \)-dimensional vector \( \sigma = (\sigma_1, \ldots, \sigma_n) \):

\[ \rho_{\sigma,c}(x) \leq \rho_{\sigma,c}(x) = \rho_{(1/\sigma_1, \ldots, 1/\sigma_n)}(x), \]
then following the proof of [26] step-by-step, it is sufficient to show that
\[
\sum_{\beta \in T} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{(1/\sigma_1, \ldots, 1/\sigma_n)} \left( \frac{1}{q} \beta \right)^l - 1 \right) \leq 2^{-\Omega(n)}.
\]

We will show that
\[
\sum_{\beta \in T_2} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{(1/\sigma_1, \ldots, 1/\sigma_n)} \left( \frac{1}{q} \beta \right)^l - 1 \right) \leq 2^{-\Omega(n)},
\]
and that
\[
\sum_{\beta \in (T_1 \cup T_2)} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l - 1 \right) \leq 2^{-\Omega(n)}.
\]

To show (2), note that by Lemma 2.4, for ideals \( \beta \in T_1 \) (we have that \( \eta_{2^{-n}}((\frac{2}{q})^\vee) \leq 2n \)). This means that for each \( i \in [n] \), \( \sigma_i \geq \eta_{2^{-n}} \), which implies that \( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l \leq (1 + 2^{-2n})^l \).

On the other hand, by definition of \( T_2 \), for ideals \( \beta \in T_2 \), we have that \( \sigma_i < \eta_{2^{-n}} \), for each \( i \in [n] \). Thus, by Lemma B.6 we have that \( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l \leq \left( \frac{\eta_{2^{-n}}((\frac{2}{q})^\vee)}{\sigma_i} \right) \cdot (1 + 2^{-2n})^l \).

Similarly to [26], using the lower bound of \( s \) from Theorem 2.3, we bound
\[
\sum_{\beta \in (T_1 \cup T_2)} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l - 1 \right)
\leq \sum_{\beta \in T} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l - 1 \right)
\leq \sum_{\beta \in T} |\beta| q^{R^\vee} |^{-l-k} \cdot \max\left(1, \left( |\beta| q^{R^\vee} |A_K| s^{-n} \cdot (2n)^{-\ell} \right) \right)^l \cdot (1 + \varepsilon)^l
\leq 2^{-\Omega(n)} + 2(s/n)^{-n} q^{kn+2} \left( \frac{s}{2n} \right)^{-\ell} \in 2^{-\Omega(n)}.
\]

Moreover, by Lemma 2.5 and the fact that \( |T_2| \leq |T| = 2^n \), we can bound
\[
\sum_{\beta \in T_2} |\beta| q^{R^\vee} |^{-l-k} \cdot \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \beta \right)^l - 1 \right) \leq 2^n \cdot 2^{-n(l-k)} \in 2^{-\Omega(n)},
\]
where the last line follows from the setting of parameters in Theorem 2.3.

This completes the proof. \( \square \)

The following corollary follows from Lemmas B.12 and B.13 and Theorem 2.3.

**Corollary 2.6.** Let \( k, l, \ell, \sigma \) and \( c \) be as in Theorem 2.3. Assume that \( A = [I_K| \bar{A}] \in (R_q)^{k+l} \) is chosen as in Theorem 2.3. Then, with probability \( 1 - 2^{-\Omega(n)} \) over the choice of \( \bar{A} \), the distribution of \( \bar{A} \) is \( R_q^k \), where \( x \in R^l \) is chosen from \( D_{R^l, \ell, \sigma, c} \), the discrete Gaussian probability distribution over \( R^l \) with parameter \( \sigma \) and center \( c \), satisfies that the probability of each of the \( q^{nk} \) possible outcomes is in the interval \( (1 \pm 2^{-\Omega(n)})q^{nk} \) (and in particular is within statistical distance \( 2^{-\Omega(n)} \) of the uniform distribution over \( R_q^k \)).

In particular, this means that the standard deviation used to sample \( x \) should be increased from \( 2n \cdot q^{k(l+2)/(nl)} \) (as in [26]) to \( 2n \cdot q^{kn/n} \).
2.3 Conditional Distribution III

We slightly change the dimensions so that \( x \) is represented by a vector of dimension \( n' := l \cdot n + 1 \). When \( n \) is a power of two, a spherical Gaussian in the coefficient representation is also a spherical Gaussian in the canonical embedding representation [24]. So we can assume that \( x \) is generated using the coefficient representation, where each coordinate is sampled independently from a discrete Gaussian, \( D_{Z,x'} \). During sampling of \( x \), an additional coordinate is sampled and stored together with the remainder of the secret. We compute the PDF corresponding to the conditional distribution on \( x \), given \( z = |r + e| \), where \( r = \|x\| \) as:

\[
F_{X|x||X|+E=z}(\|X\|=r) = \frac{e^{-\frac{(\|x\|-r)^2}{2\sigma^2}} + e^{-\frac{(\|x\|-r)^2}{2\sigma^2}}}{N},
\]

where \( N \) is the normalization factor. For details on how the PDF is computed, see Appendix E.2. \( F_{X|x||X|+E=z}(\|X\|=r) \) is the sum of two Gaussian functions centered at \( \frac{x^T}{\sqrt{1+x^T}} \) and \( \frac{-x^T}{\sqrt{1+x^T}} \) respectively with the same standard deviation \( \sigma \). Suppose \( \nu = s \), we have \( \sigma = \frac{s}{\sqrt{2}} \).

**Lemma 2.7.** Suppose \( \nu = s \), we bound the center \( \frac{x^T}{\sqrt{1+x^T}} \) from Equation 3 by \( \Pr \left( \frac{x^T}{\sqrt{1+x^T}} \geq s \sqrt{\pi} \right) \in 2^{-O(n)} \), where the probability is taken over choice of \( x \) and \( e \).

The proof is found in Appendix E.2.

Let \( \Psi_{\sigma,c}(x) := F_{X|x||X|+E=z}(\|X\|=|\|x\||) \) be the normalization of the function \( f(x) := e^{-\frac{1}{2}(x^T|c|^2)} + e^{-\frac{1}{2}(x^T|c|^2)} \). By Lemma 2.7, we have that with all but negligible probability, \( c := \frac{x^T}{\sqrt{1+x^T}} \leq \sqrt{2} \cdot \sigma \sqrt{n} \).

For the proof, we will require certain properties of the Fourier transform of \( \Psi_{\sigma,c} \), when \( c \) is bounded as above. We state those properties in the following theorem, which is proved in Appendix C.

**Theorem 2.8.** Let \( n' := l \cdot 2^a + 1 \), where \( l, a \) are positive integers and \( a > 2 \), and \( c \leq \sqrt{2} \cdot \sigma \sqrt{n} \). Let \( \Psi_{\sigma,c} \) denote the normalized PDF corresponding to the non-normalized function \( f(x) := e^{-\frac{1}{2}(x^T|c|^2)} + e^{-\frac{1}{2}(x^T|c|^2)} \), where \( x \) is a vector over \( n' \) dimensions. and let \( \overline{\Psi}_{\sigma,c}(y) \) denote the \( n' \)-dimensional Fourier transform of \( \Psi_{\sigma,c} \). Then \( |\overline{\Psi}_{\sigma,c}(y)| \leq n'^{\sigma} \cdot e^{-n\|y\|^2/\sigma^2} \) for \( \|y\| > 1/\sigma \).

We next present the main theorem of this section.

**Theorem 2.9.** For positive integers \( k \leq l \leq \text{poly}(n) \), let \( A = [I_l|\tilde{A}] \in \mathbb{R}_q^{k \times l} \), where \( \tilde{A} \in \mathbb{R}_q^{k \times (l-k)} \) is uniformly random. Let \( c \leq \sqrt{2} \cdot \sqrt{n} \cdot \sigma \) and let \( \sigma \leq \sqrt{\frac{1}{l \cdot 2^{a}}} \ln n' \cdot 2^n \cdot q^{k(l-k)/2} \). Define \( A^+ := A^+ \times Z \), where \( Z \), written as \( \Lambda^+(A) := \Lambda^+(A) \times Z \). Then \( \overline{\Psi}_{\sigma,c}(A^+) \leq 1/\text{det}(\Lambda^+(A))^\frac{1}{2} = 1 + 2^{-O(n)} \) except with probability at most \( 2^{-O(n)} \).

**Proof.** Note that \( \Lambda^+(A) \) is a lattice of even dimension \( l \cdot n \) (where \( n \) is a power of two), but Theorem 2.8 holds only for \( n' \) equal to \( l \cdot 2^a + 1 \). Therefore, we define \( n' := l \cdot n + 1 \), and we have the \( n' \)-dimensional lattice \( \Lambda^+(A) := \Lambda^+(A) \times Z \). We have the following properties of \( \Lambda^+(A) \), which can be verified by inspection:

- (a) \( (\Lambda^+(A)^+)^\vee := \Lambda^+(A)^\vee \times Z \);
- (b) the shortest non-zero vector in \( (\Lambda^+(A)^+)^\vee \) is at least \( \min(\lambda_1(\Lambda^+(A)^\vee), 1) \), where \( \lambda_1(\Lambda^+(A)^\vee) \) denotes the shortest non-zero vector in \( \Lambda^+(A)^\vee \);

By Poisson summation formula, it is sufficient to show that with probability \( 1 - 2^{-O(n)} \) over choice of \( A \), \( |\overline{\Psi}_{\sigma,c}(\Lambda^+(A)^\vee)| \leq 1 + 2^{-O(n)} \), where \( \overline{\Psi}_{\sigma,c} \) denotes the Fourier transform of \( \Psi_{\sigma,c} \) over \( n' \) dimensions and the notation \( |\overline{\Psi}_{\sigma,c}| \) means the summation of the absolute value of the function over the lattice \( \Lambda^+(A)^\vee \).

We first note that, over \( n' \) dimensions, \( \overline{\Psi}_{\sigma,c}(0) = 1 \). This follows due to the fact that by definition of Fourier transform, \( \overline{\Psi}_{\sigma,c}(0) := \int_{\mathbb{R}^{n'}} \Psi_{\sigma,c}(x) \, dx \). Since \( \Psi_{\sigma,c} \) is a normalized PDF, it must be the case that \( \int_{\mathbb{R}^{n'}} \Psi_{\sigma,c}(x) \, dx = 1 \).
Thus, it remains to show that $|\Psi_{\sigma,c}|((A^\perp(A))^\perp \setminus \{0\}) \leq 2^{-\Omega(n)}$.

Towards showing this, we first let $\beta = 2n \cdot q^{k/2}/(nl)$ for simplicity, and then use Theorem 2.8 to show that, when $\kappa = |y| \geq \sqrt{n/\pi}$,

$$|\Psi_{\sigma,c}(y)| \leq n^{n'} \cdot e^{-(\sigma^2 \cdot n \cdot x^2)/7} \leq e^{-(\sigma^2 \cdot n \cdot x^2)/7},$$

where the last line follows since $\sigma := \sqrt{2n^6 \ln n^6 \cdot 2n \cdot q^{k/2}/(nl)} = \sqrt{(2n^6/\pi) \ln n^6 \cdot \beta}$ is chosen so that when $\kappa = \sqrt{n/\pi} \cdot e^{5(\sigma^2 \cdot n \cdot x^2)/7} \geq n^{n'} = e^{\kappa \ln n^6}$.

Let $Q := \sum_{y \in (A^\perp(A))^\perp \setminus \{0\}} e^{-(\sigma^2 \cdot n \cdot x^2)/7}$. Combining the above inequalities which hold when $\kappa = \sqrt{n/\pi}$, together with (b) and Corollary B.17, which states that with probability $1 - 2^{-\Omega(n)}$ over choice of $A$, the shortest non-zero vector in $A^\perp(A)^\perp$ has length $\kappa = \sqrt{n/\pi}$, we conclude that an upper bound on $Q$ yields an upper bound on the desired quantity, $|\Psi_{\sigma,c}|((A^\perp(A))^\perp \setminus \{0\})$.

Additionally note that when $\kappa = \sqrt{n/\pi}$, then

$$e^{-(\sigma^2 \cdot n \cdot x^2)/7} = e^{-(\sigma^2 \cdot n \cdot x^2)/7} \cdot e^{-(\sigma^2 \cdot n \cdot x^2)/7} \leq e^{-1/5 \ln n^6 \cdot e^{-(\sigma^2 \cdot n \cdot x^2)/7}},$$

where the inequality follows since (by above) $e^{5(\sigma^2 \cdot n \cdot x^2)/7} \geq n^{n'} = e^{\kappa \ln n^6}$. So $e^{-(\sigma^2 \cdot n \cdot x^2)/7} \leq n^{-1/5 \ln n^6} = e^{-1/5 \ln n^6}$. Moreover, recall that two applications of Poisson summation give:

$$\sum_{y \in (A^\perp(A))^\perp} e^{-(\sigma^2 \cdot n \cdot x^2)/7} \leq n^6 \cdot \sum_{y \in (A^\perp(A))^\perp} e^{-(\sigma^2 \cdot n \cdot x^2)/7} \leq 2^{n^6} \cdot \sum_{y \in (A^\perp(A))^\perp} e^{2(\sigma^2 \cdot n \cdot x^2)/7}$$

Combining the above, we have that

$$Q \leq \sum_{y \in (A^\perp(A))^\perp} e^{-1/5 \ln n^6 \cdot e^{-(\sigma^2 \cdot n \cdot x^2)/7}} \leq e^{-1/5 \ln n^6 \cdot 2^{n^6}} \cdot \sum_{y \in (A^\perp(A))^\perp} e^{2(\sigma^2 \cdot n \cdot x^2)/7} = e^{-1/5 \ln n^6 \cdot 2^{n^6}} \cdot (1 + Q),$$

where the first inequality follows from (4) and the definition of $Q$, the second inequality from (5), and the final equality from the definition of $Q$.

Thus we have that $(1 - e^{-1/5 \ln n^6 \cdot 2^{n^6}})Q \leq e^{-1/5 \ln n^6 \cdot 2^{n^6}}$ which implies that $Q \leq 2 \cdot e^{-1/5 \ln n^6 \cdot 2^{n^6}} \leq 2^{-n^6 + 1} \leq 2^{-\Omega(n)}$, assuming $n^6$ is at least $2^{10}$.

\begin{corollary}
Let $l, k, \sigma$ and $c$ be as in Theorem 2.9. Assume that $A = \{I_k|A\} \in \mathbb{R}^k$ is chosen as in Theorem 2.9. Then, with probability $1 - 2^{-\Omega(n)}$ over the choice of $A$, the distribution of $AX \in \mathbb{R}^k$ where $(x, x_n') \in R^l \times \mathbb{Z}$ is chosen from $D_{R^l \times \mathbb{Z}, \rho_{\sigma}}$, satisfies that the probability of each of the $q^{nk}$ possible outcomes is in the interval $(1 \pm 2^{-\Omega(n)})q^{-nk}$ (and in particular is within statistical distance $2^{-\Omega(n)}$ of the uniform distribution over $R^k$).
\end{corollary}

The proof appears in Appendix E.2.

Given the corollary, the analysis of Conditional Distribution III is complete. In particular, this means that the standard deviation used to sample $x$ should be increased from $2n \cdot q^{k/2}/(nl)$ (as in [26]) to $\sqrt{1 + \frac{\pi^2}{12}} \cdot 2n \cdot q^{k/2}/(nl)$.

\section{Conclusions and Future Directions}

In this work, we present a general approach for analyzing the leakage resilience of RLWE-based cryptosystems, by determining and analyzing the explicit PDF resulting from the conditional distribution of the RLWE secret given the leakage. Our approach can be used to provide a security analysis for existing cryptosystems in...
the presence of leakage, with appropriate choice of parameters (and without any modifications to the scheme). We instantiate our approach by considering three leakage settings and corresponding conditional distributions I, II and III.

A key technical tool in the analysis of conditional distribution II is extending the regularity lemma of [25]; to cases where $x$ is drawn from a non-spherical Gaussian with standard deviation significantly smaller than the smoothing parameter in a constant fraction of the dimensions and larger than the smoothing parameter in the remaining dimensions. In the analysis of conditional distribution III we find applications of the Radial Fourier Transform to lattice-based cryptography.

Future Directions.

We believe that our approach of generalizing the regularity lemma to conditional distributions can be used as an important tool in the security analysis of RLWE-based cryptosystems. In future work, we plan to extend our analysis to other conditional distributions, with implications for other leakage settings. A first candidate is generalizing conditional distribution II to (certain types of) multivariate Gaussians with covariance matrices that are not diagonal. Such a generalization would allow us to capture leakage of coordinates in the polynomial instead of canonical representation.

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A Preliminaries and Definitions

A.1 Notation

For a positive integer \( n \), we denote by \([n]\) the set \( \{1, \ldots, n\} \). We denote vectors in boldface \( \mathbf{x} \) and matrices using capital letters \( A \). For vector \( \mathbf{x} \) over \( \mathbb{R}^n \) or \( \mathbb{C}^n \), define the \( \ell_2 \) norm as \( \|\mathbf{x}\|_2 = (\sum |x_i|^2)^{1/2} \). We write as \( \|\mathbf{x}\| \) for simplicity.

A.2 Lattices and background

Let \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) denote the cycle, i.e. the additive group of reals modulo 1. We also denote by \( \mathbb{T}_q \) its cyclic subgroup of order \( q \), i.e., the subgroup given by \( \{0, 1/q, \ldots, (q-1)/q\} \).

Let \( H \) be a subspace, defined as \( H \subseteq \mathbb{C}^{m*} \), (for some integer \( m \geq 2 \)),

\[
H = \{ \mathbf{x} \in \mathbb{C}^{m*} : x_i = x_{m-i}, \forall i \in \mathbb{Z}_m^* \}.
\]

A lattice is a discrete additive subgroup of \( H \). We exclusively consider the full-rank lattices, which are generated as the set of all linear integer combinations of some set of \( n \) linearly independent basis vectors \( B = \{\mathbf{b}_j\} \subset H \):

\[
\Lambda = \mathcal{L}(B) = \left\{ \sum_j z_j \mathbf{b}_j : z_j \in \mathbb{Z} \right\}.
\]

The determinant of a lattice \( \mathcal{L}(B) \) is defined as \( |\det(B)| \), which is independent of the choice of basis \( B \). The minimum distance \( \lambda_1(\Lambda) \) of a lattice \( \Lambda \) (in the Euclidean norm) is the length of a shortest nonzero lattice vector.

The dual lattice of \( \Lambda \subset H \) is defined as following, where \( \langle \cdot, \cdot \rangle \) denotes the inner product.

\[
\Lambda^\vee = \{ \mathbf{y} \in H : \forall \mathbf{x} \in \Lambda, \langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i \in \mathbb{Z} \}.
\]

Note that, \( (\Lambda^\vee)^\vee = \Lambda \), and \( \det(\Lambda^\vee) = 1/\det(\Lambda) \).

Discretization

Discretization is an important procedure used in applications based on lattices, such as converting continuous Gaussian distribution (defined in Appendix B) into a discrete Gaussian distribution (Definition B.9). Given a lattice \( \Lambda = \mathcal{L}(B) \) represented by some “good” basis \( B = \{\mathbf{b}_j\} \), a point \( \mathbf{x} \in H \), and a point \( \mathbf{c} \in H \) representing a lattice coset \( \Lambda + \mathbf{c} \), the discretization process outputs a point \( \mathbf{y} \in \Lambda + \mathbf{c} \) such that the length of \( \mathbf{y} - \mathbf{x} \) is not too large. This is denoted as \( \mathbf{y} \leftarrow |\mathbf{x}|_{\Lambda+c} \). A discretization procedure is called valid if it is efficient; and depends only on the lattice coset \( \Lambda + (\mathbf{c} - \mathbf{x}) \), not on particular representative used to specify it. Note that for a valid discretization, \( |\mathbf{z} + \mathbf{x}|_{\Lambda+c} \) and \( |\mathbf{z}|_{\Lambda+c} \) are identically distributed for any \( \mathbf{z} \in \Lambda \). For more details and actual description of algorithms used for discretization we refer the interested reader to [26].

A.3 Algebraic Number Theory

For a positive integer \( m \), the \( m^{th} \) cyclotomic number field is a field extension \( K = \mathbb{Q}(\zeta_m) \) obtained by adjoining an element \( \zeta_m \) of order \( m \) (i.e. a primitive \( m^{th} \) root of unity) to the rationals. The minimal polynomial of \( \zeta_m \) is the \( m^{th} \) cyclotomic polynomial

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

The

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
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the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
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\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
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the

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\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

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\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
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the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

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\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

the

\[
\text{the} \text{of} \text{an} \text{element} \text{of} \text{order} \text{m} \text{i.e. a primitive m^{th} root of unity} \text{to the rationals. The minimal polynomial of \zeta_m is the m^{th} cyclotomic polynomial}
\]

\[
\Phi_m(X) = \prod_{i \in \mathbb{Z}_m^*} (X - \omega_m^i) \in \mathbb{Z}[X],
\]

the
where \( \omega_m \in \mathbb{C} \) is any primitive \( m \)th root of unity in \( \mathbb{C} \).

For every \( i \in \mathbb{Z}_m \), there is an embedding \( \sigma_i : K \to \mathbb{C} \), defined as \( \sigma_i(\zeta_m) = \omega_m^i \). Let \( n = \varphi(m) \), the totient of \( m \). The trace \( \text{Tr} : K \to \mathbb{Q} \) and norm \( N : K \to \mathbb{Q} \) can be defined as the sum and product, respectively, of the embeddings:

\[
\text{Tr}(x) = \sum_{i \in \mathbb{Z}} \sigma_i(x) \quad \text{and} \quad N(x) = \prod_{i \in \mathbb{Z}} \sigma_i(x).
\]

For any \( x \in K \), the \( \ell_p \) norm of \( x \) is defined as \( ||x||_p = ||\sigma(x)||_p = (\sum_{i \in \mathbb{Z}} |\sigma_i(x)|^p)^{1/p} \). We omit \( p \) when \( p = 2 \). Note that the appropriate notion of norm \( ||\cdot|| \) is used throughout this paper depending on whether the argument is a vector over \( \mathbb{C}^n \), or whether the argument is an element from \( K \); whenever the context is clear.

### A.4 Ring of Integers and Its Ideals

Let \( R \subseteq K \) denote the set of all algebraic integers in a number field \( K \). This set forms a ring (under the usual addition and multiplication operations in \( K \)), called the ring of integers of \( K \). Ring of integers in \( K \) is written as \( K = \mathbb{Z}[\zeta_m] \).

The (absolute) discriminant \( \Delta_K \) of \( K \) measures the geometric sparsity of its ring of integers. The discriminant of the \( m \)th cyclotomic number field \( K \)

\[
\Delta_K = \left( \prod_{\text{prime } p | m} p^{1/(p-1)} \right)^n \leq n^n,
\]

in which the product in denominator runs over all the primes dividing \( m \).

An (integral) ideal \( \mathcal{J} \subseteq R \) is a non-trivial (i.e., \( \mathcal{J} \neq \varnothing \) and \( \mathcal{J} \neq \{0\} \)) additive subgroup that is closed under multiplication by \( R \), i.e., \( r \cdot a \in \mathcal{J} \) for any \( r \in R \) and \( a \in \mathcal{J} \). The norm of an ideal \( \mathcal{J} \subseteq R \) is the number of cosets of \( \mathcal{J} \) as an additive subgroup in \( R \), defined as index of \( \mathcal{J} \), i.e., \( N(\mathcal{J}) = \frac{|R|}{|\mathcal{J}|} \). Note that \( N(\mathcal{J}) = N(\mathcal{J}) \).

A fractional ideal \( \mathcal{J} \) in \( K \) is defined as a subset such that \( \mathcal{J} \subseteq R \) is an integral ideal for some nonzero \( d \in R \). Its norm is defined as \( N(\mathcal{J}) = N(\mathcal{J})/N(d) \). An ideal lattice is a lattice \( \sigma(\mathcal{J}) \) embedded from a fractional ideal \( \mathcal{J} \) by \( \sigma \) in \( H \). The determinant of an ideal lattice \( \sigma(\mathcal{J}) \) is \( \det(\sigma(\mathcal{J})) = N(\mathcal{J}) \cdot \sqrt{\Delta_K} \). For simplicity, however, most often when discussing about ideal lattice, we omit mention of \( \sigma \) since no confusion is likely to arise.

**Lemma A.1** ([26]). For any fractional ideal \( \mathcal{J} \) in a number field \( K \) of degree \( n \),

\[
\sqrt{n} \cdot N^{1/n}(\mathcal{J}) \leq \lambda_1(\mathcal{J}) \leq \sqrt{n} \cdot N^{1/n}(\mathcal{J}) \cdot \sqrt{\Delta_K/n}.
\]

For any fractional ideal \( \mathcal{J} \) in \( K \), its dual ideal is defined as

\[
\mathcal{J}^\vee = \{ a \in K : \text{Tr}(a\mathcal{J}) \subseteq \mathbb{Z} \}.
\]

**Definition A.2.** For \( R = \mathbb{Z}[\zeta_m] \), define \( g = \prod_p (1 - \zeta_p) \in R \), where \( p \) runs over all odd primes dividing \( m \).

Also, define \( t = \frac{\mathfrak{m}}{\mathfrak{q}} \in R \), where \( \mathfrak{m} = \frac{\mathfrak{q}}{2} \) if \( m \) is even, otherwise \( \mathfrak{m} = m \).

The dual ideal \( \mathcal{J}^\vee \) of \( R \) is defined as \( \mathcal{J}^\vee = (t^{-1}) \), satisfying \( R \subseteq \mathcal{J}^\vee \subseteq \mathfrak{m}^{-1} R \). For any fractional ideal \( \mathcal{J} \), its dual is \( \mathcal{J}^\vee = \mathcal{J}^{-1} \cdot \mathcal{J}^\vee \). The quotient \( R_q(\mathcal{J}) \) is defined as \( R_q(\mathcal{J}) = R^\vee / q R^\vee \).

**Fact A.3** ([26]). Assume that \( q \) is a prime satisfying \( q = 1 \mod m \), so that \( (q) \) splits completely into \( n \) distinct ideals of norm \( q \). The prime ideal factors of \( (q) \) are \( q_i = (q) + (\zeta_m - \omega_m^i) \), for \( i \in \mathbb{Z}_m \). By Chinese Reminder Theorem, the natural ring homomorphism \( R/(q) \to \prod_{i \in \mathbb{Z}_m} (R/q_i) \cong (\mathbb{Z}_q^m) \) is an isomorphism.

**Lemma A.4.** [26, Lemma 2.23] Let \( p \) and \( q \) be positive coprime integers, and \( [\cdot] \) be a valid discretization to (cosets of) \( p R^\vee \). There exists an efficient transformation that on input \( w \in R_q^\vee \) and a pair in \((a', b') \in \)}
$R_q \times (K_B / qR^\vee)$, outputs a pair $(a = pa^e \mod qR, b) \in R_q \times R_q^\vee$ with the following guarantees: if the input pair is uniformly distributed then so is the output pair; and if the input pair is distributed according to the RLWE distribution $A_{s,\psi}$ for some (unknown) $s \in R^\vee$ and distribution $\psi$ over $K_B$, then the output pair is distributed according to $A_{s,\hat{\psi}}$, where $\hat{\chi} = |p \cdot \hat{\psi}|_{w+pR^\vee}$.

**Lemma A.5.** [26, Lemma 2.24] Let $p$ and $q$ be positive coprime integers, $|\cdot|$ be a valid discretization to (cosets of) $pR^\vee$, and $w$ be an arbitrary element in $K_B$. If $R$DLWE$_{q,\psi}$ is hard given $l$ samples, then so is the variant of $R$DLWE$_{q,\psi}$ in which the secret is sampled from $\chi := |p \cdot \hat{\psi}|_{w+pR^\vee}$, given $l - 1$ samples.

## B Regularity and Fourier Transforms

Let $\rho_{s,c}$ denote an $n$-dimensional Gaussian function with standard deviation $s$ and mean $c$.

### One and Multi-Dimensional Gaussians.

For $s > 0$, $c \in \mathbb{R}$, $x \in \mathbb{R}$, define the Gaussian function $\rho^1_{s,c} : \mathbb{R} \to (0, 1]$ as

$$\rho^1_{s,c}(x) := e^{-\frac{s^2}{2}(x-c)^2}.$$

When $c = 0$, we write for simplicity,

$$\rho^1_s(x) := e^{-\frac{s^2}{2}x^2}.$$

By normalizing this function we obtain the continuous Gaussian probability distribution $\psi^1_{s,c}$ (resp. $\psi^1_s$) of parameter $s$, whose density is given by $s^{-1} \cdot \rho^1_{s,c}(x)$ (resp. $s^{-1} \cdot \rho^1_s(x)$).

We denote by $\rho_{(s_1, \ldots, s_n), (c_1, \ldots, c_n)}$ the distribution over $\mathbb{R}^n$ with the following pdf:

$$\rho_{(s_1, \ldots, s_n), (c_1, \ldots, c_n)}(x_1, \ldots, x_n) := \rho^1_{s_1, c_1}(x_1) \cdots \rho^1_{s_n, c_n}(x_n).$$

When $c = 0$, we again write for simplicity, $\rho_{(s_1, \ldots, s_n)}$. Moreover, when $s_1 = \cdots = s_n$ and the dimension is clear from context we write for simplicity $\rho_{s, (c_1, \ldots, c_n)}$ (resp. $\rho_s$). Normalizing as above, we obtain the corresponding continuous Gaussian probability distribution $\psi_{(s_1, \ldots, s_n), (c_1, \ldots, c_n)}$ (resp. $\psi_s$).

### Definition B.1 (Fourier Transform).

Given an integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we denote by $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ the Fourier transform of $f$, defined as

$$\widehat{f}(y) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} \, dx.$$

### Theorem B.2 (Poisson Summation Formula).

Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary lattice of dimension $n$, and let $f : \mathbb{R}^n \to \mathbb{C}$ be an appropriate function. Then

$$f(\Lambda) = \frac{1}{\det(\Lambda)} \widehat{f}(\Lambda^\vee),$$

where $\Lambda^\vee$ is the dual lattice of $\Lambda$ and $\widehat{f}$ is a Fourier transform of $f$.

### Definition B.3.

For an $n$-dimensional lattice $\Lambda$, and positive real $\varepsilon > 0$, we define its smoothing parameter $\eta_\varepsilon(\Lambda)$ to be the smallest $s$ such that $\rho_{1/s}(\Lambda^\vee \setminus \{0\}) \leq \varepsilon$.

---

8 Assume that (1). $\int_{\mathbb{R}^n} |f(x)| \, dx < \infty$. (2). Function $f(\Lambda + u)$ is continuous on $\mathbb{R}^n$. (3). The series $\widehat{f}(\Lambda^\vee)$ is absolutely convergent. (See [17] for details)
Lemma B.4. [9, 29] For any $n$-dimensional lattice $\Lambda$, we have $\sqrt{n} \eta_e(\Lambda) \leq \eta_e(\Lambda') \leq \sqrt{n} \eta_e(\Lambda)$, for $e \in [2^{-n}, 1]$.

Claim B.5 ([26]). For any $n$-dimensional lattice $\Lambda$ and $\epsilon, s > 0$,

$$\rho_{1/s}(\Lambda) \leq \max \left(1, \left(\frac{\eta_e(\Lambda')}{s}\right)^n\right) (1 + \epsilon).$$

Lemma B.6. For any $n$-dimensional lattice $\Lambda$ and $\epsilon > 0$, $s := (s_1, \ldots, s_n) \in \mathbb{R}_{>0}^n$, and $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$, if all of $s_1, \ldots, s_n < \eta_e(\Lambda')$ then

$$\rho_{(1/s_1, \ldots, 1/s_n), (c_1, \ldots, c_n)}(\Lambda) \leq \left(\frac{\eta_e(\Lambda')}{s_1} \ldots \frac{\eta_e(\Lambda')}{s_n}\right) (1 + \epsilon).$$

Proof. Applying Poisson summation formula twice, using the fact that for all vectors $x \in \mathbb{R}^n$, $\tilde{\rho}_{(1/s_1, \ldots, 1/s_n), (c_1, \ldots, c_n)}(x) \leq (s_1)^{-1} \cdots (s_n)^{-1} \cdot \rho_{(s_1, \ldots, s_n)}(x)$, and the fact that $\tilde{\rho}_{\eta_e(\Lambda')}(\Lambda') = \eta_e(\Lambda')^n \cdot \rho_{1/\eta_e(\Lambda')}$, we have:

$$\rho_{(1/s_1, \ldots, 1/s_n), (c_1, \ldots, c_n)}(\Lambda) \leq \det(\Lambda)^{-1}(s_1)^{-1} \cdots (s_n)^{-1} \cdot \rho_{(s_1, \ldots, s_n)}(\Lambda') \leq \det(\Lambda)^{-1}(s_1)^{-1} \cdots (s_n)^{-1} \cdot \eta_e(\Lambda')^n \cdot \rho_{1/\eta_e(\Lambda')}(\Lambda')$$

$$= (s_1)^{-1} \cdots (s_n)^{-1} \cdot \eta_e(\Lambda')^n \cdot \rho_{1/\eta_e(\Lambda')}(\Lambda) \leq \left(\frac{\eta_e(\Lambda')}{s_1} \ldots \frac{\eta_e(\Lambda')}{s_n}\right) (1 + \epsilon).$$

where the last inequality follows from the definition of $\eta_e(\Lambda')$.

Lemma B.7. [29, Lemma 3.6] For any lattice $\Lambda$, positive real $s > 0$ and a vector $c$, $\rho_{s, c}(\Lambda) \leq \rho_{s}(\Lambda)$.

Definition B.8. Let $\Lambda$ be an $n$-dimensional lattice and $\Psi$ a probability distribution over $\mathbb{R}^n$. Define the discrete probability distribution of $\Psi$ over $\Lambda$ to be:

$$D_{\Lambda, \Psi}(x) = \frac{\Psi(x)}{\Psi(\Lambda)}, \forall x \in \Lambda.$$ 

Definition B.9. Let $\Lambda$ be an $n$-dimensional lattice, define the discrete Gaussian probability distribution over $\Lambda$ with parameter $(s_1, \ldots, s_n)$ and center $(c_1, \ldots, c_n)$ as

$$D_{\Lambda,(s_1, \ldots, s_n), (c_1, \ldots, c_n)}(x) = \frac{\rho_{(s_1, \ldots, s_n), (c_1, \ldots, c_n)}(x)}{\rho_{(s_1, \ldots, s_n), (c_1, \ldots, c_n)}(\Lambda)}, \forall x \in \Lambda.$$ 

Remark B.10. Whenever $\Psi$ is Gaussian with parameter $(s_1, \ldots, s_n)$ and center $(c_1, \ldots, c_n)$ we denote it’s discrete Gaussian probability by $D_{\Lambda,(s_1, \ldots, s_n), (c_1, \ldots, c_n)}$. If $s = s_1 = \cdots = s_n$ (resp. $c = c_1 = \cdots = c_n$) we write $D_{\Lambda,(s_1, \ldots, s_n), (c_1, \ldots, c_n)}$ (resp. $D_{\Lambda,(s_1, \ldots, s_n), (c_1, \ldots, c_n)}$).

Lemma B.11. [29, Lemma 4.4] For any $n'$-dimensional lattice $\Lambda$, and reals $0 < \epsilon < 1$, $s \geq \eta_e(\Lambda)$, we have

$$\Pr_{x \sim D_{\Lambda, \psi}} \left(\|x\| > s \sqrt{n'}\right) \leq \frac{1 + \epsilon}{1 - \epsilon} \cdot 2^{-n'}.$$

The following is a modified version of Lemma 3.8 from [32].

Lemma B.12. Let $\Lambda$ be an $n$-dimensional lattice and $\Psi$ a probability distribution over $\mathbb{R}^n$. If $\hat{\Psi}(\Lambda' \setminus \{0\}) \leq \epsilon$, then for any $c \in \mathbb{R}^n$, $\Psi(\Lambda + c) \in \det(\Lambda')(1 \pm \epsilon)$, where $|\hat{\Psi}(\Lambda' \setminus \{0\})|$ denotes the summation of the absolute value of the function at each point in $\Lambda' \setminus \{0\}$.

Proof. First, since $\Psi$ is a pdf, we have that $\hat{\Psi}(0) = 1$. We have:

$$\Psi(\Lambda + c) = \det(\Lambda') \sum_{y \in \Lambda'} \hat{\Psi}(y)e^{2\pi i \langle c, y \rangle} \in \det(\Lambda') \left(1 \pm \sum_{y \in \Lambda' \setminus \{0\}} |\hat{\Psi}(y)e^{2\pi i \langle c, y \rangle}|\right)$$
We next state an additional corollary of the regularity theorem from [26].

Theorem B.15. Then, with probability

\[ \eta_{2^k n} \left( \Lambda^\perp(A) \right) \leq 1 + 2^2 - O(n), \]

In particular, if \( s > 2n \cdot q^{k(l+2)/(n)} \) then \( \mathbb{E}_A \left[ \rho_{1/s} \left( A^\perp(A)^\perp \right) \right] \leq 1 + 2^{-O(n)} \), and so by Markov’s inequality, \( \eta_{2^k n} \left( A^\perp(A) \right) \leq s \) except with probability at most \( 2^{-O(n)} \).

The following corollary was presented in [26].

Corollary B.16. Let \( R \), \( n \), \( q \), \( k \), and \( l \) be as in Theorem B.15. Assume that \( A = [I_k | \tilde{A}] \in (R_q)^{k+l} \) is chosen as in Theorem B.15. Then, with probability \( 1 - 2^{-O(n)} \) over the choice of \( \tilde{A} \), the distribution of \( Ax \in R_q^k \), where each coordinate of \( x \in R_q^k \) is chosen from a discrete Gaussian distribution of parameter \( s > 2n \cdot q^{k(l+2)/(n)} \) over \( R \), satisfies that the probability of each of the \( q^nk \) possible outcomes is in the interval \( (1 \pm 2^{-O(n)})q^{-nk} \) (and in particular is within statistical distance \( 2^{-O(n)} \) of the uniform distribution over \( R_q^k \)).

We next state an additional corollary of the regularity theorem from [26].

Corollary B.17. Let \( R \), \( n \), \( q \), \( k \), and \( l \) be as in Theorem B.15. Assume that \( A = [I_k | \tilde{A}] \in (R_q)^{k+l} \) is chosen as in Theorem B.15. Then, with probability \( 1 - 2^{-O(n)} \) over the choice of \( \tilde{A} \), the shortest non-zero vector in \( \Lambda^\perp(A)^\perp \) has length at least \( \frac{\sqrt{n}}{2nq^{\frac{k(l+2)}{n}}} \).

C Proof of Theorem 2.8

In this section, we prove the following theorem, which provides an upper bound on the Fourier transform of a pdf for the analysis of Conditional Distribution III in Section 2.3.

Theorem 2.8. Let \( n' := l \cdot 2^a + 1 \), where \( l \), \( a \) are positive integers and \( a > 2 \), and \( c \leq \sigma \cdot \sqrt{2} \cdot \sqrt{2l} \). Let \( \Psi_{\sigma,c} \) denote the normalized pdf corresponding to the non-normalized function \( f(x) := e^{-\frac{||x||^2}{2\sigma^2}} + e^{-\frac{||x||^2}{2\sigma^2}} \), where \( x \) is a vector over \( n' \) dimensions. and let \( \bar{\Theta}_{\sigma,c} \) denote the \( n' \)-dimensional Fourier transform of \( \Psi_{\sigma,c} \). Then \( |\bar{\Theta}_{\sigma,c}(y)| \leq n'^n \cdot e^{-n||y||^2\sigma^2} \) for \( ||y|| > 1/\sigma \).
The following lemma computes a lower bound of the normalization factor of the pdf in Theorem 2.8. Once we prove the lemma, we proceed to the proof of Theorem 2.8.

**Lemma C.1.** Let \( n' \in \mathbb{N} \) be odd, \( x \in \mathbb{R}^{n'} \), \( c \in \mathbb{R} \). Then
\[
\int_{\mathbb{R}^{n'}} e^{-\frac{n'(x-c)^2}{\sigma^2}} + e^{-\frac{\|x\|^2}{\sigma^2}} \, dx \geq \sigma^{n'}.
\]

**Proof.** Let \( f(x) := e^{-\frac{n'(x-c)^2}{\sigma^2}} + e^{-\frac{\|x\|^2}{\sigma^2}} \). Let \( r = \|x\| \). Since \( f \) is a radial function, we slightly abuse notation and denote by \( f(r) := e^{-\frac{n'(r-c)^2}{\sigma^2}} + e^{-\frac{r^2}{\sigma^2}} \). Now, we have that
\[
\int_{\mathbb{R}^{n'}} f(x) \, dx = n' V_{n'} \int_0^\infty r^{n'-1} f(r) \, dr,
\]
where \( V_{n'} \) denotes the volume of \( n' \)-dimensional ball \( V_{n'} = \frac{\pi^{(n'\!\!)/2}}{\Gamma((n'+1)/2)} \). Since \( f \) is an even function and \( n' \) is odd, so \( r^{n'-1} \) is an even function, we have that \( r^{n'-1} f(r) \) is even and so
\[
\int_0^\infty r^{n'-1} f(r) \, dr = 1/2 \int_{-\infty}^\infty r^{n'-1} f(r) \, dr.
\]

Let \( a = \pi/\sigma^2 \). Since \( n' \) is odd, we now have that
\[
\int_{-\infty}^\infty e^{-a(r-c)^2} r^{n'-1} \, dr = \int_{-\infty}^\infty e^{-a r^2} (t + c) r^{n'-1} \, dt = \int_{-\infty}^\infty e^{-a r^2} \sum_{j=0}^{n'-1} \binom{n'-1}{j} c^j t^{n'-1-j} \, dt
\]
\[
= \sum_{j=0}^{n'-1} \binom{n'-1}{j} c^j \int_{-\infty}^\infty e^{-a r^2} r^{n'-1-j} \, dr = \sum_{j=0}^{n'-1} \binom{n'-1}{j} c^j \left( \frac{1}{2} \right) \frac{1}{\sqrt{\pi}} \frac{\Gamma((n'-j)/2)}{\Gamma(n'/2)}
\]
\[
= \sum_{j=0}^{n'-1} \binom{n'-1}{j} \frac{c^j}{\sqrt{\pi}} a^{-j} \frac{\Gamma(n'/2)}{\Gamma((n'-j)/2)} \geq a^{-n'} \frac{\Gamma(n'/2)}{\Gamma((n'-1)/2)}.
\]

Combining the above with (C1) and (C2) and substituting for \( a \), we get that \( \int_{\mathbb{R}^{n'}} f(x) \, dx \geq \sigma^{n'} \), which completes the proof of the lemma.

**Proof of Theorem 2.8.** Let \( N \) be the normalization of \( f(x) \) over \( n' \) dimensions. We have from Lemma C.1 that \( N \geq \sigma^{n'} \) Thus, it remains to show that for \( n' := 1 \cdot 2^a + 1 \) and \( c \leq \sigma \cdot \sqrt{2} \cdot \sqrt{n'} \), \( \hat{f}(y) \leq \sigma^{n'} \cdot n'^{5/4} \cdot e^{-\pi\|y\|^2/\sigma^2} \).

Let \( r := \|x\| \), we slightly abuse notation and view \( f \) as a function of \( r, f(r) := e^{-\frac{n'(r-c)^2}{\sigma^2}} + e^{-\frac{r^2}{\sigma^2}} \). Since \( \Psi_{a,c} \) is a radial function, so is its Fourier transform, thus, we again slightly abuse notation and view \( F := \hat{f} \) as a function of \( x := \|y\| \). We may now use the formula for the radial Fourier transform of an \( n' \)-dimensional, radial function \( f \) to find \( F \) [21]:
\[
F(x) = x^{\frac{n'+2}{2}} (2\pi)^{\frac{n'-1}{2}} \int_0^\infty t^{n'-1} f(r) J_{n'-\frac{1}{2}}(2\pi r) r \, dr,
\]
where \( J_{n'-\frac{1}{2}} \) denotes the Bessel function of the first kind of order \( n'-\frac{1}{2} \). The Bessel function of first kind of order \( \nu \) is defined as [35, Page 40]:
\[
J_{\nu}(z) := \sum_{j=0}^\infty (-1)^j (\frac{1}{2}z)^{\nu+2j} / \Gamma(\nu+j+1)!
\]

For half-integer order \( \nu := n + \frac{1}{2} \), there is a closed-form representation of \( J_\nu \). Specifically, it can be expressed as [35, Page 298]:
\[
J_{n+\frac{1}{2}}(z) := J_{n+\frac{1}{2}}(z) \left( \frac{2}{\pi z} \right)^{1/2} \sin z - J_{n-\frac{1}{2}}(z) \left( \frac{2}{\pi z} \right)^{1/2} \cos z.
\]
where \( R_{\nu,\beta}(z) \) and \( R_{\nu-1,\beta}(z) \) are Lommel polynomials defined as [35, Page 296]:

\[
R_{\nu,\beta}(z) = \sum_{j=0}^{[\nu/2]} \frac{(-1)^j (n-j)! \Gamma(v+n-j)}{j!(n-2j)! \Gamma(v+j)} \left( \frac{z}{2} \right)^{2j-n},
\]

where the \([x]\) means the largest integer not exceeding \(x\).

We now have:

\[
|F(x)| = \left| \kappa^{-\nu-(\beta-2)}(2\pi) \int_0^\infty r^{\nu-2} f(r) I_{\nu+\frac{1}{2}} (2\pi kr) r \, dr \right|
\]

\[
= \kappa^{-\nu-(\beta-2)}(2\pi) \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \sin(2\pi kr) r \, dr
\]

\[
\leq \kappa^{-\nu-(\beta-2)}(2\pi) \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \sin(2\pi kr) r \, dr
\]

\[
+ \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c'_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \cos(2\pi kr) r \, dr,
\]

where the first equality follows from (C3), the second equality follows from (C5), (C6) and the settings of

\( c_j := \frac{(-1)^j (\nu-2j)! \Gamma(\nu+1) \Gamma(j+1)}{j!(\nu-2j)! \Gamma(1+j)} \) and \( c'_j := \frac{(-1)^j (\nu'-2j)! \Gamma(\nu'+1) \Gamma(j+1)}{j!(\nu'-2j)! \Gamma(1+j)} \).

In order to bound (C7), we will individually upper bound

\[
I_1 := \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \sin(2\pi kr) r \, dr
\]

and

\[
I_2 := \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c'_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \cos(2\pi kr) r \, dr.
\]

Recalling that \( f(r) = e^{-\frac{\rho_0^2}{\sigma^2} r^2} + e^{-\frac{\rho_0^2 c_0^2}{\sigma^2} r^2} \), we have that

\[
I_2 = \left| \int_0^\infty r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c'_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \left( e^{i2\pi kr} + e^{-i2\pi kr} \right) r \, dr \right|
\]

\[
= \frac{1}{2} \left( \frac{1}{4\pi^2 K} \right)^{1/2} \int_{-\infty}^{\infty} r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c'_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu-1} \right) \left( \frac{2}{2\pi^2 kr} \right)^{1/2} \left( e^{i2\pi kr} + e^{-i2\pi kr} \right) r \, dr
\]

\[
= \frac{1}{2} \left( \frac{1}{4\pi^2 K} \right)^{1/2} \int_{-\infty}^{\infty} r^{\nu-2} f(r) \left( \sum_{j=0}^{[\nu-1/2]} c'_j \left( \frac{2\pi kr}{2} \right)^{2j-\nu+1/2} \right) r \left( e^{i2\pi kr} + e^{-i2\pi kr} \right) \, dr
\]

\[
\leq \frac{1}{2} \left( \frac{1}{4\pi^2 K} \right)^{1/2} \left[ \sum_{j=0}^{[\nu-1/2]} c'_j (\pi k)^{2j-\nu+1/2} \right] \int_{-\infty}^{\infty} r^{2j+2} \left( e^{-\frac{\rho_0^2 c_0^2}{\sigma^2} r^2} + e^{-\frac{\rho_0^2}{\sigma^2} r^2} \right) \left( e^{i2\pi kr} + e^{-i2\pi kr} \right) \, dr.
\]
where the second equality follows since $f(r)$ is an even function, $\cos(2\pi kr)$ is an even function and for $n' = l \cdot 2^a + 1$, all powers of $r$ in the integrand are even, which means that the entire integrand is an even function.

To compute an upper bound on

$$\left| \int_{-\infty}^{\infty} e^{-\frac{\sigma \rho^2}{\sigma^2}} + e^{-\frac{\sigma \rho^2}{\sigma^2}} (e^{i2\pi kr} + e^{-i2\pi kr}) \, dr \right|$$

(C9)

as above, we integrate each term separately. Since the analysis is essentially the same for each term, we focus on upper bounding the term $A := \left| \int_{-\infty}^{\infty} e^{-\frac{\sigma \rho^2}{\sigma^2} + \frac{1}{2}} e^{-\frac{\rho^2}{\sigma^2}} e^{-i2\pi kr} \, dr \right|$

$$= e^{-\frac{\sigma^2}{\sigma^2}} \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\frac{\rho^2}{\sigma^2}} e^{-i2\pi kr} \, dr \right|$$

$$\leq e^{-\frac{\sigma^2}{\sigma^2}} \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\frac{\rho^2}{\sigma^2}} e^{-i2\pi kr} \, dr \right|$$

$$\leq e^{-\frac{\sigma^2}{\sigma^2}} \left| \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\frac{\rho^2}{\sigma^2}} e^{-i2\pi kr} \, dr \right|$$

Thus, we have that

(C9) $\leq \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+3} e^{-\frac{\sigma^2}{\sigma^2} + \frac{1}{2}} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+3} e^{-\frac{\rho^2}{\sigma^2}} \Gamma \left( \frac{3}{2} + j \right) \left( \frac{2j+2}{j+1} \right) \left( \frac{2j+2}{j+1} \right)^{2j+2} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \left( \frac{\rho^2}{\sigma^2} \right)^{2j+2}$

Plugging the above back into (C8), and recalling that $|c_j| = \left( \frac{2j+2}{j+1} \right) \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \left( \frac{\rho^2}{\sigma^2} \right)^{2j+2}$, we have that

$|\Pi| \leq 1/2 \left( \frac{1}{2\pi} \right)^{2j+2} \sum_{j=0}^{n' - 1} |c_j| \left( \frac{2j+2}{j+1} \right) \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} e^{-\frac{\rho^2}{\sigma^2}} \Gamma \left( \frac{3}{2} + j \right) \left( \frac{2j+2}{j+1} \right) \left( \frac{2j+2}{j+1} \right)^{2j+2} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \left( \frac{\rho^2}{\sigma^2} \right)^{2j+2}$

$\leq 1/2 \left( \frac{1}{2\pi} \right)^{2j+2} \sum_{j=0}^{n' - 1} \left( \frac{2j+2}{j+1} \right) \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} e^{-\frac{\rho^2}{\sigma^2}} \Gamma \left( \frac{3}{2} + j \right) \left( \frac{2j+2}{j+1} \right) \left( \frac{2j+2}{j+1} \right)^{2j+2} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \left( \frac{\rho^2}{\sigma^2} \right)^{2j+2}$

$\leq 1/2 \left( \frac{1}{2\pi} \right)^{2j+2} \left( \frac{2j+2}{j+1} \right) \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} e^{-\frac{\rho^2}{\sigma^2}} \Gamma \left( \frac{3}{2} + j \right) \left( \frac{2j+2}{j+1} \right) \left( \frac{2j+2}{j+1} \right)^{2j+2} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \left( \frac{\rho^2}{\sigma^2} \right)^{2j+2}$

Where the last inequality follows since $\binom{n'}{j} \leq 2^n$ and $n! \leq n^n$. We now turn to upper-bounding $I$. Recalling that $f(r) = e^{-\frac{\sigma \rho^2}{\sigma^2} r^2} + e^{-\frac{\sigma \rho^2}{\sigma^2} r^2}$, we have that

$$I = \left| \int_{0}^{\infty} r^\frac{1}{2j+2} f(r) \left( \sum_{j=0}^{n' - 1} c_j \left( \frac{2\pi kr}{2} \right)^{2j+2} \right) \left( \frac{2}{2\pi^2 kr} \right)^{j+1} \sin(2\pi kr) \, dr \right|$$

(C10)
where the second equality follows since \( f(r) \) is an even function, \( \sin(2\pi kr) \) is an odd function and for \( n' = l \cdot 2^a + 1 \), all powers of \( r \) in the integrand are odd, which means that the entire integrand is an even function.

To compute an upper bound on

\[
\int_{-\infty}^{\infty} r^{2j+1} \left( e^{-\frac{\pi j \sigma r^2}{\sigma^2}} + e^{-\frac{\pi j \sigma r^2}{\sigma^2}} \right) \left( e^{i2\pi kr} - e^{-i2\pi kr} \right) \, dr
\]

as above, we integrate each term separately. Since the analysis is essentially the same for each term, we focus on the term \( B := \int_{-\infty}^{\infty} e^{-\frac{\pi j \sigma r^2}{\sigma^2}} e^{i2\pi kr} \, dr = e^{-\pi \sigma^2 r^2 + i2\pi kc} \int_{-\infty}^{\infty} e^{-\pi \sigma^2 r^2 (r - (c + i\sigma \alpha))]^2} \, dr \):

\[
B = e^{-\pi \sigma^2 \alpha^2} \int_{-\infty}^{\infty} r^{2j+1} e^{-\pi \sigma^2 (r-(c+i\sigma \alpha))]^2} \, dr \leq e^{-\pi \sigma^2 \alpha^2} \int_{-\infty}^{\infty} r^{2j+1} e^{-\pi \sigma^2 (r-(c+i\sigma \alpha))]^2} \, dr
\]

Thus, we have that

\[
(C11) \leq \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \Gamma \left( \frac{1}{2} + j \right) \left( \frac{2j+1}{j+1} \right) \left[ 4 \frac{\xi}{\alpha} \right]^{2j+1}
\]

Plugging the above back into (C10), and recalling that \( |c_j| = \frac{(c_j - i)(c_j + 1)\Gamma \left( \frac{1}{2} + j \right)}{j!(2j+1)\Gamma \left( \frac{1}{2} + j \right)} \), we have that

\[
I \leq 1/2 \left( \frac{1}{4 \pi^2 K} \right)^{\frac{1}{2}} \sum_{j=0}^{[n']/2-1} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{2j+2} \frac{2j+1 \xi}{j+1} \frac{2j+1}{j+1} \frac{1}{(2j+1)!} \frac{\Gamma \left( \frac{1}{2} + j \right)}{\Gamma \left( \frac{1}{2} + j \right)} \frac{1}{(2j+1)!} \left( \frac{\xi}{\alpha} \right)^{2j+1}
\]

\[
\leq 1/2 \left( \frac{1}{2\pi} \right) e^{-\pi \sigma^2 \alpha^2} \sum_{j=0}^{[n']/2-1} a_{j}^{2j+1} c_{j}^{2j+1} \left( \frac{2j+1}{j+1} \right)^{\frac{1}{2}} \left( \frac{2j+1}{j+1} \right)^{\frac{1}{2}} \Gamma \left( \frac{n'}{2} - 1 - j \right) \sigma^{2j+2} e^{i2j+1} \left( \frac{\xi}{\alpha} \right)^{4j+3}
\]

\[
\leq 1/2 \left( \frac{1}{2\pi} \right) e^{-\pi \sigma^2 \alpha^2} \left( n' \cdot 2^{n'} \cdot n'^{2} \right) \sum_{j=0}^{[n']/2} a_{j}^{2j+1} c_{j}^{2j+1} \left( \frac{2j+1}{j+1} \right)^{\frac{1}{2}} \left( \frac{2j+1}{j+1} \right)^{\frac{1}{2}} \Gamma \left( \frac{n'}{2} - 1 - j \right) \sigma^{2j+2} e^{i2j+1} \left( \frac{\xi}{\alpha} \right)^{4j+3}
\]
Where the last inequality follows since \( \binom{n}{2} \leq 2^n \) and \( n! \leq n^m \). Finally, plugging into (C7), and recalling that \( c \leq \sigma \sqrt{n} \cdot \sqrt{n'} \) and \( \kappa > \frac{1}{3} \), we obtain:

\[
|F(\kappa)| \leq \frac{1}{2} e^{-nx^2 \sigma^2} \left( n' \cdot 2^{\frac{n}{2}} \cdot n'^{\frac{n}{2}} \right) \left( \sum_{j=0}^{\frac{n^2-1}{2}} \sigma^{2j+3} \cdot 2^{j+2} \cdot 2^{n'-j+2} + \sum_{j=0}^{\frac{n'-1}{2}} \sigma^{2j+2} \cdot 2^{j+1} \cdot 2^{n'-j+4} \right)
\]

\[
\leq \sigma^n \cdot n'^{n'} \cdot e^{-nx^2 \sigma^2}
\]

\[ \square \]

## D Manipulating Gaussians

We begin by defining some notation, which will be useful in all of the Conditional Distributions when manipulating Gaussian-distributed random variables. We write probability density function of random variable \( X \) at value \( x \), sampled from \( n \)-dimensional Gaussian distribution with each component of variable pairwise independent, as

\[
\psi_{s,u}(X = x) = \prod_{i \in [n]} \frac{1}{s_i} \exp \left( -\frac{\pi(x_i - u_i)^2}{s_i} \right),
\]

with mean \( u = (u_1, \ldots, u_n) \) and standard deviation \( s = (s_1, \ldots, s_n) \). The probability density function of \( Y \) at value \( y \), sampled from \( n \)-dimensional Gaussian distribution with each component of variable pairwise independent, can be written as

\[
\psi_{v,\mu}(Y = y) = \prod_{i \in [n]} \frac{1}{v_i} \exp \left( -\frac{\pi(y_i - \mu_i)^2}{v_i} \right),
\]

with mean \( \mu = (\mu_1, \ldots, \mu_n) \) and standard deviation \( v = (v_1, \ldots, v_n) \).

We now consider the distribution of \( X \), conditioned on knowledge of \( X + Y \). We proceed with the following straightforward lemma:

**Lemma D.1.** Given two independent random variables \( X \) and \( Y \). Suppose that the distribution of \( X \) is a \( n \)-dimensional Gaussian random variable with mean \( u \) and standard deviation \( s \), each component of \( X \) pairwise independent, and the distribution of \( Y \) is a \( n \)-dimensional Gaussian distribution with mean \( \mu \) and standard deviation \( v \), each component of \( Y \) pairwise independent. Then the distribution of \( X \) conditioned on \( X + Y \) is also a \( n \)-dimensional Gaussian distribution, where each component of \( X \) is pairwise-independent with mean \( c := (c_1, \ldots, c_n) \) where \( c_i := \frac{\frac{1}{s_i^2} - \frac{1}{v_i^2} + \frac{1}{\sigma^2}}{\frac{1}{s_i^2} + \frac{1}{v_i^2}} \) and standard deviation \( \sigma := (\sigma_1, \ldots, \sigma_n) \), where \( \sigma_i := \sqrt{\frac{1}{s_i^2} + \frac{1}{v_i^2}} \).

**Proof.** We have \( F_{Z|A}(Z = b) \) generically represent the probability density function of random variable \( Z \) at value \( b \), conditioned on event \( A \).

We can then derive the density function of \( X \) given the value \( z = (z_1, \ldots, z_n) \) of \( X + Y \) by computing

\[
F_{X|X+Y=z}(X = x) = \frac{\psi_{s,u}(X = x)\psi_{v,\mu}(Y = y)}{f_{\mu}(X = x)f_{\nu}(Y = y)} = \prod_{i \in [n]} \frac{1}{s_i} e^{-\frac{\pi(x_i - u_i)^2}{s_i}} e^{-\frac{\pi(y_i - \mu_i)^2}{v_i}} = \prod_{i \in [n]} \frac{1}{s_i} e^{-\frac{\pi(x_i - u_i)^2}{s_i}} e^{-\frac{\pi(y_i - \mu_i)^2}{v_i}} \frac{1}{v_i} e^{-\frac{\pi(y_i - \mu_i)^2}{v_i}} \]

\[
\prod_{i \in [n]} \left( \frac{1}{s_i^2} + \frac{1}{v_i^2} \right) \exp \left( -\pi \left( \frac{1}{s_i^2} + \frac{1}{v_i^2} \right) \left( x_i - \frac{\frac{1}{s_i^2} - \frac{1}{v_i^2} + \frac{1}{\sigma^2}}{\frac{1}{s_i^2} + \frac{1}{v_i^2}} \right)^2 \right)
\]

Hence \( F_{X|X+Y=z}(X = x) \) is also in the form of probability density function of \( X \) on value \( x \) sampled \( n \)-dimensional Gaussian distribution, where each component \( x_i \) is generated independently with mean \( \frac{\frac{1}{s_i^2} - \frac{1}{v_i^2} + \frac{1}{\sigma^2}}{\frac{1}{s_i^2} + \frac{1}{v_i^2}} \), and variance parameter \( \frac{1}{\sigma^2} \).
E Additional Proofs for Section 2

E.1 Additional Proofs in Conditional Distribution II

Lemma 2.4. For ideals $J \in T_1$,
\[ \eta_2^{-n} \left( \frac{\mathfrak{d}}{q} \right)^\vee \leq 2n. \]

Proof.\[ \eta_2^{-n} \left( \frac{\mathfrak{d}}{q} \right)^\vee \leq \frac{\sqrt{n}}{\lambda_1 \left( \frac{\mathfrak{d}}{q} \right)^\vee} \leq \left( N \left( \frac{\mathfrak{d}}{q} \right) \right)^{-1/n} \leq (|J/qR| \cdot n^n)^{1/n} \leq (2^n \cdot n^n)^{1/n} = 2n, \]

where (E1) follows from Lemma B.4, (E2) follows from Lemma A.1, and (E3) follows from the fact that $\left( N \left( \frac{\mathfrak{d}}{q} \right) \right)^{-1} = |J/qR| = |R^\vee / R| \cdot |\mathfrak{d}/qR^\vee| = \Delta_K |\mathfrak{d}/qR|$ (for example, see [8, page 63]), and (E4) follows from the definition of $T_1$. \[ \square \]

Lemma 2.5. For ideals $J \in T_1^2$,
\[ |J/qR^\vee|^{-(l-k)} \left( p_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \right)^\vee \right) \leq 2^{-n(l-k)} \]

Proof. Recall that $\sigma := (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}_{>0}^n$ is defined as a vector such that $\ell$ positions are set to $2n$, while the other positions are set to $s$. Define $z_1, \ldots, z_n$ in the following way: For $i \in [n]$, if $\sigma_i = s$ then $z_i = \sigma_i$. Otherwise, $z_i = \eta_2^{-n} \left( \frac{1}{q} \right)^\vee$. Applying Poisson summation twice we arrive at:
\[
\begin{align*}
\rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \right)^\vee &= 1/\det(\frac{1}{q} \mathbf{1}) \cdot (1/\sigma_1 \cdots 1/\sigma_n) p_{\sigma_1, \ldots, \sigma_n} \left( \frac{1}{q} \mathbf{1} \right)^\vee \leq 1/\det(\frac{1}{q} \mathbf{1}) \cdot (1/\sigma_1 \cdots 1/\sigma_n) p_{z_1, \ldots, z_n} \left( \frac{1}{q} \mathbf{1} \right)^\vee \leq (\eta_2^{-n}((\frac{1}{q} \mathbf{1})^\vee)^{\ell} \cdot \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q} \mathbf{1} \right)^\vee \leq (1 + 2^{-2n}) \cdot \left( \eta_2^{-n}((\frac{1}{q} \mathbf{1})^\vee)^{\ell}, \right) \end{align*}
\]

where (E6) follows from definitions of $\rho$ and $z_i$. To derive (E7), let us first introduce the following claim.

Claim E.1. For any lattice $L^\vee$,
\[ \rho_{s_1, \ldots, s_n}(L) = s_1 \cdot s_2 \cdots s_n \cdot \frac{1}{\det(L)} \cdot \rho_{1/s_1, \ldots, 1/s_n}(L^\vee) \]
Proof. It can be easily verified by combining Poisson Summation formula and the fact that \( \hat{\rho}_{s_1, \ldots, s_n} = s_1 \cdots s_n \rho_{1/s_1, \ldots, 1/s_n}. \)

By replacing \( s_i \) with \( 1/z_i \) for all \( i \) and replacing \( L \) with \( \frac{q}{g} \), we have

\[
1/\det(\frac{q}{g}) \cdot \rho_{z_1, \ldots, z_n} \left( \frac{1}{q/g} \right) = z_1 \cdots z_n \cdot \rho_{1/z_1, \ldots, 1/z_n} \left( \frac{1}{q/g} \right).
\]

By plugging into (E6), we have

\[
\left( \frac{z_1}{\sigma_1} \cdots \frac{z_n}{\sigma_n} \right) \cdot \rho_{1/z_1, \ldots, 1/z_n} \left( \frac{1}{q/g} \right)
\]

By definition of \( z_i, \frac{z_i}{\sigma_i} = 1 \) when \( \sigma_i = s \) and \( \frac{z_i}{\sigma_i} = \eta_{2n}(\frac{1}{q/g})^{1/2n} \), when \( \sigma_i = 2n \). Since there are \( \ell \) positions in \( \sigma \) when \( \sigma_i = 2n \), we obtain (E7). Finally (E8) follows by definition of smoothing parameter \( \eta_{2n}(\frac{1}{q/g})^{1/2n} \).

Now, using the fact that \( \eta_{2n} \leq (\Delta_K |\beta||qR^V|)^{1/n} \), the fact that \( \Delta_K = n^q \) and the fact that \( |\beta||qR^V| \geq 2^n \), and the set of parameters, we have that

\[
|\beta||qR^V|^{-(l-k)} \left( \rho_{1/\sigma_1, \ldots, 1/\sigma_n} \left( \frac{1}{q/g} \right) \right) \leq |\beta||qR^V|^{-(l-k-1/l) (1 + 2^{-2n^q})} \cdot 2^{-\ell - 1} \leq 2^{-n(l-k)}
\]

which completes the proof of the lemma. \( \square \)

E.2 Additional Proofs in Conditional Distribution III

Recall that a generic PDF of one dimensional Gaussian distribution is defined as:

\[
\psi_{s,u}(x) = \frac{1}{s} \exp \left( -\frac{(x-u)^2}{s^2} \right),
\]

where \( u \) is mean, and \( s \) is standard deviation of the distribution. We write probability density function of secret key \( X \) at value \( x = (x_1, \ldots, x_{n'}) \), of which each coordinate is independently sampled from a Gaussian distribution with center at 0 and standard deviation \( s \), as

\[
\psi_s(X = x) = \prod_{i \in [n']} \frac{1}{s} \exp \left( -\frac{\pi x_i^2}{s^2} \right) = \frac{1}{s^n} \exp \left( -\frac{\pi r^2}{s^2} \right) = \psi_s(\|X\| = r),
\]

where \( r \) is the magnitude of \( x \). It also can be viewed as probability density function of secret key for its magnitude \( \|X\| = r \), denoted as \( \psi_s(\|X\| = r) \). The error is sampled from a 1-dimensional Gaussian distribution with center at 0. We write probability density function of error \( E \) at value \( y \) is

\[
\psi_s(E = y) = \frac{1}{v} \exp \left( -\frac{\pi y^2}{v^2} \right).
\]

Let \( F_{Z|A}(f(Z) = b) \) generically represent the probability density function of random variable \( Z \) at value \( b \) of \( f(Z) \), conditioned on event \( A \).

We now derive the density function of secret key \( X \) given the value \( z \) of \( \|X\| + E \). The weight placed on a value \( x = (x_1, \ldots, x_{n'}) \) by the conditional distribution depends only on the magnitude of \( x \) (i.e. \( r = \|x\| \)) and can be computed as:

\[
\frac{F_{X, \|X\| + E = z}}{\psi_s(\|X\| = r) F_{X, \|X\| = r} + \psi_s(\|X\| = r)} \frac{F_{X, \|X\| + E = z}}{\psi_s(\|X\| = r) F_{X, \|X\| = r} + \psi_s(\|X\| = r)}
\]

(E9)
We now restate and prove Lemma 2.7.

Proof. Using union bound, we have
\[
\Pr(\|X\|^2 > n) \leq \sum_{i} \Pr(\|X_i\|^2 > n/2) \leq \sum_{i} \frac{e^{-\frac{n}{2} \lambda_i^2}}{\lambda_i^2}. 
\]

where $N$ is the normalization factor.

Lemma E.2. Given a random variable $Y$ chosen from a Gaussian distribution $G_\mathcal{E}(y, v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2v^2}\right)$, $Y$ is upper bounded by $\sqrt{\mathcal{E}}$ except for negligible probability, written as $\Pr(Y \geq \sqrt{\mathcal{E}}) \in 2^{-O(n)}$.

Proof. $\Pr(Y \geq y) = \Pr(X \geq x)$, where $X = \sqrt{\mathcal{E}}$ is a standard normal, $x = \sqrt{\mathcal{E}}v$. By using Chernoff bound and calculating exponential moment of standard normal distribution, we have, for any $\lambda > 0$.

\[
\Pr(X \geq x) \leq \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda x}} = e^{\lambda^2/2},
\]

Set $\lambda = x$ and $y = \sqrt{\mathcal{E}}$, then $\Pr(Y \geq \sqrt{\mathcal{E}}) \leq e^{-x^2/2} = e^{-\pi v/2}$. The lemma follows.

We now restate and prove Lemma 2.7.

Lemma 2.7. Suppose $\mathcal{E} = s$, we can bound a center $\frac{s^2}{\mathcal{E} + s^2}$ from Equation E9 by $\Pr\left(\frac{s^2}{\mathcal{E} + s^2} \geq s\sqrt{\mathcal{E}}\right) \in 2^{-O(n)}$.

Proof. Using union bound, we have
\[
\Pr\left(\frac{s^2}{\mathcal{E} + s^2} \geq s\sqrt{\mathcal{E}}\right) = \Pr\left(\frac{s}{2} \leq s\sqrt{\mathcal{E}}\right) \leq \Pr\left(R + E \geq 2s\sqrt{\mathcal{E}}\right) + \Pr(\lambda \leq R) + \Pr\left(E \geq \sqrt{\mathcal{E}}\right) + \Pr\left(E \geq v\sqrt{\mathcal{E}}\right).
\]

Note that since $s > n$, and using the fact that $\lambda_1 ((R^t \times Z)^t) \geq \lambda_1 (R^\perp) \geq \sqrt{n}N_\mathcal{E}(\mathcal{E}^\perp) = \sqrt{n} \cdot (\mathcal{E}^\perp)^\perp = \sqrt{n}(\frac{1}{\sqrt{n}})^\perp = 1$ (See Lemma A.1), by Lemma B.4, we ensure $s > \eta_{2-}(R^t \times Z)$. Then by Lemma B.11 and Lemma E.2, we deduce that $\Pr\left(\frac{s^2}{\mathcal{E} + s^2} \geq s\sqrt{\mathcal{E}}\right) \in 2^{-O(n)}$.

Corollary 2.10. Let $k, l, \sigma$ and $c$ be as in Theorem 2.9. Assume that $A = [I_k | \hat{A}] \in (R_q)^{k \times l}$ is chosen as in Theorem 2.9. Then, with probability $1 - 2^{-O(n)}$, over the choice of $\hat{A}$, the distribution of $Ax \in R_q^l$ is chosen from $D_{\mathcal{E} + Z, \Psi}$, which satisfies that the probability of each of the $q^{-\eta/\sqrt{n}}$ possible outcomes is

in the interval $(1 \pm 2^{-\Omega(n)})q^{-\eta/\sqrt{n}}$ (and in particular is within statistical distance $2^{-\Omega(n)}$ of the uniform distribution over $R_q^l$).

Proof. $\Psi, c \left( A^\perp (A^\perp)^\perp + (b, b^\perp) \right) \in \det((A^\perp (A^\perp)^\perp)(1 \pm 2^{-\Omega(n)}))$, which means that if we choose a $n'$-dimensional vector from distribution $D_{\mathcal{E} + Z, \Psi}$, written as $x' = (x, x')$, and let $(b, b') = x' \mod (A^\perp (A^\perp)^\perp)$, then the resulting distribution is within statistical distance $2^{-\Omega(n)}$ to uniform distribution over $(R^l \times Z)$ modulo $(A^\perp (A^\perp)^\perp)$. Due to the structure of $A^\perp (A^\perp)^\perp$, this also implies that the marginal distribution over $b$ is uniform over $(R^l)$ modulo $(A^\perp (A^\perp)^\perp)$. Moreover, we can easily see that if $x' = (x, x')$, if $x' \mod (A^\perp (A^\perp)^\perp) = (b, b')$, then $Ax = Ab$. Finally, since when $b$ is uniform random over $R^l$ modulo $A^\perp (A^\perp)^\perp$, we have that $Ab$ is uniform random over $R_q^l$, the corollary follows.