Remarks on pseudo-vertex-transitive graphs with small diameter

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Abstract

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with vertex set $X$ and diameter $D$. Let $A$ denote the adjacency matrix of $\Gamma$. For a vertex $x \in X$ and for $0 \leq i \leq D$, let $E_i(x)$ denote the projection matrix to the $i$th subconstituent space of $\Gamma$ with respect to $x$. The Terwilliger algebra $T(x)$ of $\Gamma$ with respect to $x$ is the semisimple subalgebra of Mat$_X(\mathbb{C})$ generated by $A, E_0^*(x), E_1^*(x), \ldots, E_D^*(x)$. Let $V$ denote a $\mathbb{C}$-vector space consisting of complex column vectors with rows indexed by $X$. We say $\Gamma$ is pseudo-vertex-transitive whenever for any vertices $x, y \in X$, there exists a $\mathbb{C}$-vector space isomorphism $\rho : V \rightarrow V$ such that $(\rho A - A \rho)V = 0$ and $(\rho E_i^*(x) - E_i^*(y) \rho)V = 0$ for all $0 \leq i \leq D$. In this paper, we discuss pseudo-vertex transitivity for distance-regular graphs with diameter $D \in \{2, 3, 4\}$. For $D = 2$, we show that a strongly regular graph is pseudo-vertex-transitive if and only if all its local graphs have the same spectrum. For $D = 3$, we consider the Taylor graphs and show that they are pseudo-vertex transitive. For $D = 4$, we consider the antipodal tight graphs and show that they are pseudo-vertex transitive.

Keywords: Distance-regular graph; Terwilliger algebra; Pseudo-vertex-transitive; Strongly regular graph; Taylor graph; Antipodal tight graph

2020 Mathematics Subject Classification: 05E30

1 Introduction

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with vertex set $X$ and diameter $D$. We note that $\Gamma$ can be regarded as a combinatorial analogue of a compact rank one symmetric space [3, p. 311]. We recall the Terwilliger algebra of $\Gamma$. Let $A$ denote the adjacency matrix of $\Gamma$. Fix a base vertex $x \in X$. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x)$ denote the projection matrix onto the $i$th

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subconstituent space of $\Gamma$ with respect to $x$. The Terwilliger algebra (or subconstituent algebra) $T(x)$ of $\Gamma$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A, E^*_0, E^*_1, \ldots, E^*_D$ [25]. Note that $T(x)$ is finite-dimensional and semisimple. Let $T(y)$ denote the Terwilliger algebra of $\Gamma$ with respect to another base vertex $y \in X$. We say the Terwilliger algebras $T(x)$ and $T(y)$ are isomorphic whenever there exists a $\mathbb{C}$-algebra isomorphism from $T(x)$ to $T(y)$ that sends $A$ to $A$ and $E^*_i$ to $E^*_i(y)$ for $0 \leq i \leq D$. We remark that the isomorphism class of $T(x)$ may depend on the choice of the base vertex $x$. For instance, the twisted Grassmann graph, introduced by van Dam and Koolen [7], has two orbits of the automorphism group on its vertex set, say $X_1$ and $X_2$. For vertices $x \in X_1$ and $y \in X_2$, the Terwilliger algebras $T(x)$ and $T(y)$ of the twisted Grassmann graph are not isomorphic; cf. [2].

Let $V$ denote the $\mathbb{C}$-vector space consisting of complex column vectors with rows indexed by $X$. Observe that $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. View $V$ as a left module for $T(x)$ and call this the standard module of $T(x)$ (or standard $T(x)$-module). Since $T(x)$ is semisimple, $V$ decomposes into a direct sum of irreducible $T(x)$-modules. We say $\Gamma$ is pseudo-vertex-transitive whenever for every pair of vertices $x, y \in X$ there exists a $\mathbb{C}$-vector space isomorphism $\rho : V \to V$ such that $(\rho A - A \rho)V = 0$ and $(\rho E^*_i - E^*_i(y) \rho)V = 0$ for all $0 \leq i \leq d$. In other words, when $\Gamma$ is pseudo-vertex-transitive, it means that the isomorphism class of the standard $T(x)$-module of $\Gamma$ does not depend on the base vertex $x$; consequently, the multiplicities of irreducible $T(x)$-modules do not depend on the base vertex $x$.

We recall the notion of the thinness of $\Gamma$. Let $W$ denote an irreducible $T(x)$-module. Then $W$ is a direct sum of nonzero spaces among $E^*_0 W, E^*_1 W, \ldots, E^*_D W$, and also a direct sum of nonzero spaces among $E_0 W, E_1 W, \ldots, E_D W$, where $E_i$ is the $i$th primitive idempotent of $\Gamma$. We note that the dimension of $E^*_i W$ is at most 1 for $0 \leq i \leq D$ if and only if the dimension of $E_i W$ is at most 1 for $0 \leq i \leq D$ [25, Lemma 3.9]; in this case, $W$ is called thin. The graph $\Gamma$ is called thin whenever every irreducible $T(x)$-module is thin for every vertex $x \in X$.

In the present paper, we discuss the thinness and pseudo-vertex transitivity of $Q$-polynomial distance-regular graphs with small diameter $D \in \{2, 3, 4\}$. For $D = 2$, a distance-regular graph is strongly regular. We show that pseudo-vertex transitivity of a strongly regular graph is determined by the spectrum of its local graph; cf. Theorem 5.11. For $D \in \{3, 4\}$, we discuss only the distance-regular antipodal double covers. Let $\Gamma$ denote a distance-regular antipodal double cover with diameter $D$. If $D = 3$, then $\Gamma$ is a Taylor graph. We discuss the thinness of $\Gamma$ and show that the isomorphism class of the standard module of the Terwilliger algebra of $\Gamma$ is determined by its intersection numbers; thus, it is pseudo-vertex-transitive; cf. Theorem 7.6. If $D = 4$, $\Gamma$ is $Q$-polynomial if and only if $\Gamma$ is tight [16]. We discuss the thinness of $\Gamma$ and show that $\Gamma$ is pseudo-vertex-transitive, provided that $\Gamma$ is tight; cf. Theorem 8.7. We note that bipartite and/or antipodal $Q$-polynomial distance-regular graphs are pseudo-vertex-transitive; cf. Proposition 6.3. In a future paper, we will discuss pseudo-vertex transitivity of general $Q$-polynomial distance-regular graphs with diameter $D \geq 3$.

We remark that the thin property of an antipodal $Q$-polynomial double cover with $D \in \{3, 4\}$ plays an important role in determining pseudo-vertex transitivity. From this point of view, we could ask the following question: for $D \geq 3$ does the thinness of a $Q$-polynomial distance-regular graph guarantee its pseudo-vertex transitivity? Recently, Ito and Koolen suggested the following.
with rows indexed by $X$ let $\text{Mat}_X$ indexed by $X$.

Section 9.

and $c_0 \leq b$ called the intersection array $D$ tight graphs with $D$. Section 8 discusses the antipodal 6 reviews some preliminaries concerning the antipodal distance-regular graphs. Section 7 deals with Taylor graphs and their thinness and pseudo-vertex transivity. Section 8 discusses the antipodal tight graphs with $D = 4$. The paper ends with a brief summary and directions for future work in Section 9.

Throughout the paper, we use the following notation. Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all complex matrices whose rows and columns are indexed by $X$. Let $V = \mathbb{C}^X$ denote the $\mathbb{C}$-vector space consisting of all complex column vectors with rows indexed by $X$. We endow $V$ with the standard Hermitian inner product $\langle u, v \rangle = u^\dagger \overline{v}$ for $u, v \in V$. We view $V$ as a left module for $\text{Mat}_X(\mathbb{C})$, called the standard module.

2 Preliminaries: distance-regular graphs

In this section, we recall some definitions and notation. Let $\Gamma$ denote a finite, simple, undirected, connected graph with vertex set $X$ and diameter $D = \max\{\partial(x, y) \mid x, y \in X\}$, where $\partial$ denotes the path-length distance function for $\Gamma$. For a vertex $x \in X$, define $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ for $0 \leq i \leq D$. The subgraph $\Delta_i(x), 0 \leq i \leq D$, of $\Gamma$ induced by $\Gamma_i(x)$ is called the $i$th subconstituent of $\Gamma$ with respect to $x$. The number $|\Gamma_1(x)|$ is called the valency of $x$ in $\Gamma$. A graph $\Gamma$ is said to be regular with valency $k$ (or $k$-regular) if each vertex of $\Gamma$ has valency $k$. We abbreviate $\Delta(x) = \Delta_1(x)$, the first subconstituent of $\Gamma$ with respect to $x$, and call this the local graph of $\Gamma$ at $x$. Let $\lambda$ denote an eigenvalue of $\Delta(x)$. We call $\lambda$ a local eigenvalue of $\Gamma$ with respect to $x$. We say a graph $\Gamma$ is locally $\Delta$ whenever all local graphs of $\Gamma$ are isomorphic to $\Delta$.

We say $\Gamma$ is distance-regular whenever for all integers $0 \leq h, i, j \leq D$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number $p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of $x$ and $y$. The constants $p^h_{ij}$ are called the intersection numbers of $\Gamma$. We abbreviate $a_i = p^1_{1,i}(0 \leq i \leq D)$, $b_i = p^i_{1,i+1}(0 \leq i \leq D-1)$ and $c_i = p^i_{1,i-1}(1 \leq i \leq D)$. Observe $\Gamma$ is regular with valency $k = b_0$, and $c_i + a_i + b_i = k$ for $0 \leq i \leq D$, where we define $c_0 = 0$ and $b_D = 0$. The sequence $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_{D}\}$ is called the intersection array of $\Gamma$. We say $\Gamma$ is bipartite whenever $a_i = 0$ for $0 \leq i \leq D$. Problem 1.1 (The Ito-Koolen Problem [22]). Classify all thin pseudo-vertex-transitive $Q$-polynomial distance-regular graphs with large enough diameter.

The ultimate goal of our study is to classify thin pseudo-vertex-transitive $Q$-polynomial distance-regular graphs. The present paper shows that such graphs with small diameter are difficult to classify.
We assume that $\Gamma$ is distance-regular with diameter $D$. For $0 \leq i \leq D$, let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x,y) = i, \\
0 & \text{if } \partial(x,y) \neq i, 
\end{cases} \quad (x,y \in X).$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. We abbreviate $A := A_1$, called the adjacency matrix of $\Gamma$. Observe (i) each $A_i$ is real symmetric; (ii) $A_0 = I$; (iii) $\sum_{i=0}^{D} A_i = J$, the all-ones matrix; (iv) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \ (0 \leq i, j \leq D)$. By these facts, we find that $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$, which we call the Bose-Mesner algebra of $\Gamma$. It is known that $A$ generates $M$. The algebra $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1} J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $E_i E_j = \delta_{ij} E_i$; cf. [1, p.45]. We call $E_i$ the $i$th primitive idempotent of $\Gamma$. Since $\{E_i\}_{i=0}^{D}$ is a basis for $M$, there exist complex scalar $\{\theta_i\}_{i=0}^{D}$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Observe $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. The scalars $\{\theta_i\}_{i=0}^{D}$ are real [3, p.197], and mutually distinct as $A$ generates $M$. We call $\theta_i$ the eigenvalue of $\Gamma$ associated with $E_i$ for $0 \leq i \leq D$. Observe $V = E_0 V + E_1 V + \cdots + E_D V$, an orthogonal direct sum. For $0 \leq i \leq D$, $E_i V$ is the eigenspace of $A$ associated with $\theta_i$. For $0 \leq i \leq D$, we denote by $m_i$ the rank of $E_i$ and observe $m_i = \dim(E_i V)$. We call $m_i$ the multiplicity of $\theta_i$. By the spectrum of $\Gamma$, we mean the multiset containing its eigenvalues, each with its multiplicity, denoted by $\text{Spec}(\Gamma) = \{\theta_0^m, \theta_1^m, \ldots, \theta_D^m\}$.

We recall the notion of the $Q$-polynomial property of $\Gamma$. Let $\circ$ denote the entrywise product in $\text{Mat}_X(\mathbb{C})$. Since $A_i \circ A_j = \delta_{ij} A_i \ (0 \leq i, j \leq D)$, the Bose-Mesner algebra $M$ is closed under $\circ$. Since $\{E_i\}_{i=0}^{D}$ is a basis for $M$, there exist complex scalars $q_{ij}^h$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h, \quad (0 \leq i, j \leq D).$$

By [1, p.48, 49] the scalars $q_{ij}^h$ are real and nonnegative. The $q_{ij}^h$ are called the Krein parameters of $\Gamma$. We say $\Gamma$ is $Q$-polynomial (with respect to the ordering $E_0, E_1, \ldots, E_D$) whenever for all integers $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) if one of $h, i, j$ is greater than (resp. equal to) the sum of the other two [1, p.235]. From now on, unless otherwise stated, we assume that a $Q$-polynomial distance-regular graph discussed in this paper has the ordering $E_0, E_1, \ldots, E_D$. For more background information about distance-regular graphs, we refer the reader to [1,3,8].

### 3 Preliminaries: the Terwilliger algebra

In this section, we recall the Terwilliger algebra of a $Q$-polynomial distance-regular graph. Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with vertex set $X$ and diameter $D$. Fix a vertex $x \in X$. We refer to $x$ as a “base” vertex. For $0 \leq i \leq D$, we define the diagonal matrix $E_i^* = E_i^*(x) \in \text{Mat}_X(\mathbb{C})$ with diagonal entry

$$(E_i^*)_{yy} = \begin{cases} 
1 & \text{if } \partial(x,y) = i, \\
0 & \text{if } \partial(x,y) \neq i, 
\end{cases} \quad (y \in X).$$
We call $E_i^*$ the $i$th dual primitive idempotent of $\Gamma$ with respect to $x$. Observe $\sum_{i=0}^{D} E_i^* = I$ and $E_i^* E_j^* = \delta_{ij} E_i^*$. By these facts, $E_0^*, E_1^*, \ldots, E_D^*$ is a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$ with respect to $x$, which we call the dual Bose-Mesner algebra of $\Gamma$. Recall the primitive idempotents $\{E_i^*_j\}_{i=0}^{D}$ of $\Gamma$. For $0 \leq i \leq D$, define the diagonal matrix $A_i^* = A_i^*(x) \in \text{Mat}_X(\mathbb{C})$ with diagonal entry $(A_i^*)_yy = |X|(E_i)_{xy}$ for $y \in X$. By [25, p.379], $A_0^*, A_1^*, \ldots, A_D^*$ is a basis for $M^*$. Moreover $A_0^* = I$ and $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij} h_i A_h^*$. We call $A_i^*$ the $i$th dual distance matrix of $\Gamma$ with respect to $x$. We abbreviate $A^* = A_1^*$ and call this the dual adjacency matrix of $\Gamma$ with respect to $x$. The matrix $A^*$ generates $M^*$ [25, Lemma 3.11]. Since $\{E_i^*_i\}_{i=0}^{D}$ is a basis for $M^*$, there exist complex scalars $\{\theta_i^*\}_{i=0}^{D}$ such that $A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Observe $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq D$. The scalars $\{\theta_i^*\}_{i=0}^{D}$ are real [25, Lemma 3.11] and mutually distinct. We call $\theta_i^*$ the dual eigenvalue of $\Gamma$ associated with $E_i^*$. Observe $V = E_0^* V + E_1^* V + \cdots + E_D^* V$, an orthogonal direct sum. For $0 \leq i \leq D$, $E_i^* V$ is the eigenspace of $A^*$ associated with $\theta_i^*$, which we call the $i$th subconstituent space of $\Gamma$ with respect to $x$.

Recall the Bose-Mesner algebra $M$ of $\Gamma$ and the dual Bose-Mesner algebra $M^*(x)$ of $\Gamma$ with respect to $x$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the Terwilliger algebra (or subconstituent algebra) of $\Gamma$ with respect to $x$ [25]. Note that $A$ and $A^*$ generate $T$. The algebra $T$ is finite-dimensional and noncommutative. The algebra $T$ is semisimple since it is closed under the conjugate transpose map. By [25, Lemma 3.2], we have the following relations in $T$. For $0 \leq i, j \leq D$,

$$E_i^* A E_j^* = 0 \quad \text{if} \quad |i - j| > 1,$$
$$E_i A^* E_j = 0 \quad \text{if} \quad |i - j| > 1.$$

By a $T$-module, we mean a subspace $W$ of $V$ such that $BW \subseteq W$ for all $B \in T$. Observe that $V$ is a $T$-module, called the standard module of $T$ (or standard $T$-module). A $T$-module $W$ is called irreducible if $W \neq 0$ and $W$ contains no $T$-modules other than 0 and $W$. Two $T$-modules $W, W'$ are isomorphic if there exists a $\mathbb{C}$-vector space isomorphism $\sigma : W \rightarrow W'$ such that

$$(\sigma B - B\sigma)W = 0,$$

for all $B \in T$. Let $W$ be a $T$-module and let $U$ be a $T$-submodule of $W$. The orthogonal complement of $U$ in $W$ is a $T$-module since $T$ is closed under the conjugate transpose map. It follows that $W$ decomposes into an orthogonal direct sum of irreducible $T$-modules. We observe that $V$ decomposes an orthogonal direct sum of irreducible $T$-modules; choose one of the irreducible $T$-modules in this decomposition of $V$, denoted by $W$. By the multiplicity of $W$, we mean the number of irreducible $T$-modules in this decomposition which are isomorphic to $W$ as $T$-modules.

Let $W$ be an irreducible $T$-module. Then $W$ decomposes into a direct sum of nonzero spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$, and also a direct sum of nonzero spaces among $E_0 W, E_1 W, \ldots, E_D W$. By the endpoint of $W$, we mean $\min\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}$. By the dual endpoint of $W$, we mean $\min\{i \mid 0 \leq i \leq D, E_i W \neq 0\}$. By the diameter of $W$, we mean $|\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}| - 1$. Let $r$ denote the endpoint of $W$ and $d$ the diameter of $W$. By [25, Lemma 3.9], $E_r^* W \neq 0$ if and only if $r \leq i \leq r + d$, and $W = \sum_{h=0}^{d} E_{r+h}^* W$, an orthogonal direct sum; in addition, $\dim(E_i^* W) \leq 1$ for $0 \leq i \leq D$ if and only if $\dim(E_i W) \leq 1$ for $0 \leq i \leq D$. An irreducible $T$-module $W$ is said to
be thin whenever $\dim(E_i^* W) \leq 1$ for $0 \leq i \leq D$. There exists a unique thin irreducible $T$-module with endpoint 0 and diameter $D$, which we call the primary $T$-module. Let $W$ denote the primary $T$-module. Let $v_0, v_1, \ldots, v_D$ be a sequence of vectors of $W$, not all zero. This sequence is said to be a standard basis for $W$ whenever both (i) $v_i \in E_i^* W$ for $0 \leq i \leq D$; and (ii) $\sum_{i=0}^{D} v_i \in E_0 W$. For instance, the sequence $E_0^* \mathbb{1}, E_1^* \mathbb{1}, \ldots, E_D^* \mathbb{1}$ is a standard basis for $W$ [25, Lemma 3.6], where $\mathbb{1}$ is the all-ones vector. We note that the action of $A$ on a standard basis for $W$ is given by

$$Av_i = b_{i-1} v_{i-1} + a_i v_i + c_i v_{i+1} \quad (0 \leq i \leq D),$$

where $a_i, b_i, c_i$ are the intersection numbers of $\Gamma$ and $b_{-1} v_{-1} := 0$ and $c_{D+1} v_{D+1} := 0$.

Let $W$ be a thin irreducible $T$-module with endpoint $r$ and diameter $d$. We recall the actions of $A$ and $A^*$ on $W$. Take a nonzero vector $v_0 \in E_r^* W$. For $1 \leq i \leq d$, define $v_i = E_{r+i}^* A v_{i-1}$ in $E_{r+i}^* W$. Observe that the vector $v_i$ is a basis for $E_r^* W$ for each $0 \leq i \leq d$, and thus $v_0, v_1, \ldots, v_d$ is an orthogonal basis for $W$. Define the scalars $a_i(W), 0 \leq i \leq d$, and $x_i(W), 1 \leq i \leq d$, by

$$a_i(W) = \text{trace}(E_{r+i}^* A | W), \quad x_i(W) = \text{trace}(E_{r+i}^* A E_{r+i-1}^* A | W),$$

where $B|_W$ denotes the restriction of $B$ to $W$. Let $\theta_i$ be the eigenvalue of $\Gamma$ associated with $E_i$ for $0 \leq i \leq D$. Note that (cf. [5, Lemma 5.10])

$$\sum_{i=0}^{d} a_i(W) = \sum_{i=0}^{d} \theta_{t+i},$$

where $t$ is the dual endpoint of $W$. For $0 \leq i \leq d$, the action of $A$ on $v_i$ is given as

$$Av_i = v_{i+1} + a_i(W) v_i + x_i(W) v_{i-1},$$

where $v_{d+1} := 0$ and $x_0(W) v_{-1} := 0$; cf. [5, Theorem 5.7]. The action of $A^*$ on $v_i$ is given as $A^* v_i = \theta_{r+i}^* v_i$, where $\theta_j^*$ is the dual eigenvalue of $\Gamma$ associated with $E_j^*$.

The graph $\Gamma$ is said to be thin with respect to $x$ whenever every irreducible $T(x)$-module is thin. The graph $\Gamma$ is said to be thin whenever $\Gamma$ is thin with respect to every vertex $x$ of $\Gamma$. See [26, Section 6] for examples of thin $Q$-polynomial distance-regular graphs.

Recall the local graph $\Delta(x)$ of $\Gamma$ at $x \in X$. Observe that $\Delta(x)$ has $b_0 = k$ vertices and is $a_1$-regular. Let $a_1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ denote the local eigenvalues of $\Gamma$ with respect to $x$. The spectrum of $\Delta(x)$ is called the local spectrum of $\Gamma$ at $x$. Let $W$ denote a thin irreducible $T$-module with endpoint 1. We observe that $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$ with corresponding eigenvalue, say $\lambda$. Note that $\lambda$ is one of $\lambda_2, \ldots, \lambda_k$. We call $\lambda$ the local eigenvalue of $W$.

We finish this section with a comment, which will be useful later when calculating the dimension of the Terwilliger algebra.

**Proposition 3.1.** Let $T = T(x)$ be the Terwilliger algebra of $\Gamma$ with respect to $x$ and $V$ the standard $T$-module. Suppose that $V$ has exactly $r$ non-isomorphic irreducible $T$-modules $W_{11}, W_{21}, \ldots, W_{r1}$...
with \( \dim(W_{i1}) = n_i \). For \( 1 \leq i \leq r \), let \( m_i \) be the multiplicity of \( W_{i1} \). Then \( V \) decomposes into a direct sum of irreducible \( T \)-modules:

\[
V = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{m_i} W_{ij},
\]

where \( W_{ij} \) and \( W_{i'j'} \) are isomorphic as \( T \)-modules if and only if \( i = i' \). The algebra \( T \) is isomorphic to the semisimple algebra

\[
\operatorname{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \operatorname{Mat}_{n_r}(\mathbb{C})
\]

and

\[
\dim(T) = \sum_{i=1}^{r} n_i^2.
\]

(3.5)

**Proof.** Follows from Wedderburn’s theory [6].

4 **Pseudo-vertex transitivity**

In this section, we recall the notion of a pseudo-vertex-transitive graph. Throughout this section, we denote by \( \Gamma \) a \( Q \)-polynomial distance-regular graph with vertex set \( X \) and diameter \( D \). Let \( A \) denote the adjacency matrix of \( \Gamma \). For \( x \in X \), let \( A^*(x) \) denote the dual adjacency matrix of \( \Gamma \) with respect to \( x \) and let \( T(x) \) denote the Terwilliger algebra of \( \Gamma \) with respect to \( x \). Recall the standard module \( V = \mathbb{C}^X \). We now give the following definition.

**Definition 4.1.** The graph \( \Gamma \) is said to be **pseudo-vertex-transitive** whenever for every pair of vertices \( x, y \in X \), there exists a \( \mathbb{C} \)-vector space isomorphism \( \rho : V \rightarrow V \) such that

\[
(\rho A - A \rho)V = 0, \quad (\rho A^*(x) - A^*(y) \rho)V = 0.
\]

(4.1)

Suppose that for any two vertices \( x, y \in X \) there exists a \( \mathbb{C} \)-vector space isomorphism \( \rho : V \rightarrow V \) satisfying (4.1). Then we say that the standard \( T(x) \)-module and the standard \( T(y) \)-module are isomorphic.

**Lemma 4.2.** \( \Gamma \) is pseudo-vertex-transitive if and only if for every pair of vertices \( x, y \in X \) there exist ordered bases \( B_x \) and \( B_y \) for \( V \) such that the matrix representing \( A \) (resp. \( A^*(x) \)) with respect to \( B_x \) is equal to the matrix representing \( A \) (resp. \( A^*(y) \)) with respect to \( B_y \).

**Proof.** Immediate from Definition 4.1.

For any two vertices \( x, y \in X \), consider the Terwilliger algebras \( T(x) \) and \( T(y) \) of \( \Gamma \). We say \( T(x) \) and \( T(y) \) are isomorphic whenever there exists a \( \mathbb{C} \)-algebra isomorphism from \( T(x) \) to \( T(y) \) that sends \( A \mapsto A \) and \( A^*(x) \mapsto A^*(y) \). By Lemma 4.2, \( T(x) \) and \( T(y) \) are isomorphic, provided that \( \Gamma \) is pseudo-vertex-transitive. That is, the isomorphism class of the Terwilliger algebra of a pseudo-vertex-transitive \( \Gamma \) does not depend on the base vertex.
Remark 4.3. (i) Let Aut(\Gamma) denote the automorphism group of \Gamma. If Aut(\Gamma) has a single group orbit on \textit{X}, then \Gamma is pseudo-vertex-transitive. From this, it follows that every vertex-transitive graph is pseudo-vertex-transitive. The converse of this statement is not true; see Example 5.15 and Remark 7.7.

(ii) The twisted Grassmann graph $\tilde{J}_q(2D + 1, D)$ is not vertex-transitive; it has two orbits of the automorphism group on its vertex set, say $X_1$ and $X_2$. In [2], Bang et al. showed that all irreducible modules for the Terwilliger algebra $T(x)$ of $\tilde{J}_q(2D + 1, D)$ with respect to $x \in X_2$ are thin, and there are non-thin irreducible modules for the Terwilliger algebra $T(y)$ of $\tilde{J}_q(2D + 1, D)$ with respect to $y \in X_1$. It implies that $\tilde{J}_q(2D + 1, D)$ is not pseudo-vertex-transitive. We remark that Tanaka and Wang determined all irreducible $T(x)$-modules of $\tilde{J}_q(2D + 1, D)$ for the thin case; cf [23].

(iii) With reference to (ii), let $x \in X_2$ be a base vertex of $\tilde{J}_q(2D + 1, D)$ and let $T(x)$ be the Terwilliger algebra of $\tilde{J}_q(2D + 1, D)$. We recall the Grassmann graph $J_q(2D + 1, D)$. We note that $J_q(2D + 1, D)$ is thin [26]. For a base vertex $y$ of $J_q(2D + 1, D)$, let $T(y)$ be the Terwilliger algebra of $J_q(2D + 1, D)$. The local eigenvalues of $J_q(2D + 1, D)$ at $x$ equal to the local eigenvalues of $J_q(2D + 1, D)$ at $y$. From this, it follows that the isomorphism classes of the irreducible $T(x)$-modules with endpoint 1 are the same as the isomorphism classes of the irreducible $T(y)$-modules with endpoint 1. However, their multiplicities are different; cf. [2]. In addition, it turns out that the Terwilliger algebras $T(x)$ and $T(y)$ are not isomorphic to each other; cf. [23].

Suppose that \Gamma is pseudo-vertex-transitive. For $x \in X$ and for $0 \leq i \leq D$, let $\Delta_i(x)$ denote the $i$th subconstituent of \Gamma with respect to $x$. Then the spectrum of $\Delta_i(x)$ is determined by the characteristic polynomial for $E_i^*(x)AE_i^*(x)$. Let $y$ be another vertex in \textit{X}. Since \Gamma is pseudo-vertex-transitive, by Lemma 4.2 there are ordered bases $B_x$ and $B_y$ for $V$ such that the matrix representing $E_i^*(x)AE_i^*(x)$ with respect to $B_x$ is equal to the matrix representing $E_i^*(y)AE_i^*(y)$ with respect to $B_y$. It follows that the characteristic polynomial for $E_i^*(x)AE_i^*(x)$ is equal to the characteristic polynomial for $E_i^*(y)AE_i^*(y)$. Therefore we obtain the following lemma.

Lemma 4.4. Suppose that \Gamma is pseudo-vertex-transitive. Then for $0 \leq i \leq D$ and for all vertices $x, y \in X$, the spectrum of $\Delta_i(x)$ and the spectrum of $\Delta_i(y)$ are equal to each other.

Next, we generalize the concept of pseudo-vertex transitivity. Let $\Gamma'$ denote a $Q$-polynomial distance-regular graph with vertex set $X'$ and diameter $D'$. Fix a vertex $x' \in X'$. Denote by $A'$ the adjacency matrix of $\Gamma'$ and $A'^*(x')$ the dual adjacency matrix of $\Gamma'$ with respect to $x'$. Let $T'(x')$ be the Terwilliger algebra of $\Gamma'$ with respect to $x'$. Then $T(x)$ and $T'(x')$ are said to be $T$-algebra isomorphic whenever $D = D'$ and $|X| = |X'|$ and there exists a $\mathbb{C}$-algebra isomorphism from $T(x)$ to $T'(x')$ that sends $A \mapsto A'$ and $A^*(x) \mapsto A'^*(x')$. In this case, such an algebra isomorphism is called a $T$-algebra isomorphism. Let $W$ denote a $T(x)$-module of $\Gamma$ and let $W'$ denote a $T'(x')$-module of $\Gamma'$. Then $W$ and $W'$ are said to be $T$-isomorphic whenever (i) there exists a $T$-algebra isomorphism $f : T(x) \rightarrow T'(x')$; and (ii) there exists a $\mathbb{C}$-vector space isomorphism $\rho : W \rightarrow W'$ such that $(\rho A - A\rho)W = 0$ and $(\rho A^*(x) - A'^*(x'))\rho W = 0$. Recall $V = \mathbb{C}^X$ the standard $T(x)$-module of $\Gamma$. Let $V' = \mathbb{C}^{X'}$ denote the standard $T'(x')$-module of $\Gamma'$. With the above notation, we give the following definition.
**Definition 4.5.** Two $Q$-polynomial distance-regular graphs $\Gamma$ and $\Gamma'$ are said to be $T$-isomorphic whenever for every pair of vertices $x \in X$ and $x' \in X'$, (i) there exists a $T$-algebra isomorphism $f : T(x) \to T'(x')$; and (ii) there exists a $\mathbb{C}$-vector space isomorphism $\rho : V \to V'$ such that $(\rho B - f(B)\rho)V = 0$ for all $B \in T(x)$; that is, the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\rho} & V' \\
\downarrow B & & \downarrow f(B) \\
V & \xrightarrow{\rho} & V'
\end{array}
$$

commutes for all $B \in T(x)$.

**Lemma 4.6.** With the above notation, the following (i)-(iii) are equivalent.

(i) $\Gamma$ and $\Gamma'$ are $T$-isomorphic.

(ii) For any vertices $x \in X$ and $x' \in X'$, there exists a $\mathbb{C}$-vector space isomorphism $\rho : V \to V'$ such that

$$(\rho A - A'\rho)V = 0, \quad (\rho A^*(x) - A'^*(x')\rho)V = 0.$$  

That is, the standard $T(x)$-module $V$ and the standard $T'(x')$-module $V'$ are $T$-isomorphic.

(iii) There exist ordered bases $\mathcal{B}$ for $V$ and $\mathcal{B}'$ for $V'$ such that the matrix representing $A$ (resp. $A^*(x)$) with respect to $\mathcal{B}$ is equal to the matrix representing $A'$ (resp. $A'^*(x')$) with respect to $\mathcal{B}'$.

**Proof.** Routine. \hfill $\blacksquare$

**Lemma 4.7.** If $\Gamma$ and $\Gamma'$ are $T$-isomorphic, then the intersection array of $\Gamma$ is equal to the intersection array of $\Gamma'$.

**Proof.** Fix vertices $x \in X$ and $x' \in X'$. Let $V$ denote the standard $T(x)$-module of $\Gamma$ and $V'$ the standard $T'(x')$-module of $\Gamma'$. Write $T = T(x)$, $T' = T'(x')$, $E^*_i = E^*_i(x)$, $E'^*_i = E'^*_i(x')$ for $0 \leq i \leq D$. Since $\Gamma$ and $\Gamma'$ are $T$-isomorphic, there exists a $\mathbb{C}$-vector space isomorphism $\rho : V \to V'$ such that $(\rho B - f(B)\rho)V = 0$ for all $B \in T$, where $f$ is a $T$-algebra isomorphism from $T$ to $T'$. Let $W$ denote the primary $T$-module. Set $W' = \rho(W)$. Then $W'$ is a $T'$-module since for all $B' \in T'$ we have $B'W' = B'\rho(W) = \rho(BW) \subseteq \rho(W) = W'$. Clearly, the $T'$-module $W'$ is irreducible. Moreover, $W'$ has endpoint 0 since $E^*_0 W \neq 0$ and $\rho(E^*_0 W) = E'^*_0 \rho(W) = E'^*_0 W' \neq 0$. By these comments, $W'$ is the primary $T'$-module; cf. [10, Proposition 8.4].

Let $\bar{\rho}$ denote the restriction of $\rho$ to $W$. Then $W$ and $W'$ are $T$-isomorphic because for all $w \in W$ and for all $B \in T$ we have $\bar{\rho}(Bw) = \rho(Bw) = f(B)\rho(w) = f(B)\bar{\rho}(w)$. Recall the standard basis $\{E^*_i \mathbf{1}\}_{i=0}^{D}$ for $W$. We observe that $\bar{\rho}(E^*_i \mathbf{1}) = E'^*_i \bar{\rho}(\mathbf{1})$ for $0 \leq i \leq D$ and the sequence $\{E'^*_i \bar{\rho}(\mathbf{1})\}_{i=0}^{D}$ is a standard basis for $W'$. Since $\bar{\rho}(AE^*_i \mathbf{1}) = A'E'^*_i \bar{\rho}(\mathbf{1})$ for $0 \leq i \leq D$, the matrix representing $A$ with respect to $\{E^*_i \mathbf{1}\}_{i=0}^{D}$ is equal to the matrix representing $A'$ with respect to $\{E'^*_i \bar{\rho}(\mathbf{1})\}_{i=0}^{D}$. By (3.1), these matrices are tridiagonal whose entries are the intersection numbers of $\Gamma$ and $\Gamma'$, respectively. The result follows. \hfill $\blacksquare$
Lemma 4.8. Suppose that $\Gamma$ and $\Gamma'$ are $T$-isomorphic. Then for $0 \leq i \leq D$ and for all vertices $x \in X$, $x' \in X'$, the spectrum of $\Delta_i(x)$ and the spectrum of $\Delta_i(x')$ are equal to each other.

Proof. Similar to Lemma 4.4. Use Lemma 4.6. ■

5 Strongly regular graphs

In this section, we summarize the results of [27] and give a characterization of pseudo-vertex-transitive strongly regular graphs. In addition, we give several examples of cospectral strongly regular graphs and discuss their pseudo-vertex transitivity.

5.1 Preliminaries

We begin by recalling the notion of a strongly regular graph. Let $\Gamma$ be a $k$-regular graph with $n$ vertices. We say $\Gamma$ is strongly regular with parameters $(n, k, a, c)$ whenever each pair of adjacent vertices has the same number $a$ of common neighbors, and each pair of distinct non-adjacent vertices has the same number $c$ of common neighbors. Note that a connected strongly regular graph with parameters $(n, k, a, c)$ is distance-regular with diameter two and intersection array \{k, k-a-1; 1, c\}.

For the rest of this section, we denote by $\Gamma$ a connected strongly regular graph with parameters $(n, k, a, c)$. The graph $\Gamma$ has exactly three eigenvalues $k, \sigma, \tau$ with

$$\sigma = \frac{a-c + \sqrt{D}}{2}, \quad \tau = \frac{a-c - \sqrt{D}}{2},$$

where $D = (a-c)^2 + 4(k-c)$. The multiplicities $m_\sigma$ and $m_\tau$ of $\sigma$ and $\tau$, respectively, are given by

$$m_\sigma = \frac{(n-1)\tau + k}{\tau - \sigma}, \quad m_\tau = \frac{(n-1)\sigma + k}{\sigma - \tau}.$$

Let $X$ denote the vertex set of $\Gamma$. Fix a vertex $x \in X$. Consider the distance partition $\{\Gamma_i(x)\}_{i=0}^2$ of $X$. With respect to this partition, we write the adjacency matrix $A$ of $\Gamma$ in the partitioned matrix form:

$$A = \begin{pmatrix} 0 & 1^t & 0 \\ 1 & B_1 & N \\ 0 & N^t & B_2 \end{pmatrix},$$

where $B_i (i = 1, 2)$ is the adjacency matrix of the $i$th subconstituent $\Delta_i(x)$ of $\Gamma$. For notational convenience, we denote by

$$\bar{A}_1 := E_i^* A E_i^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_2 := E_2^* A E_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_2 \end{pmatrix},$$

where $E_i^* = E_i^*(x)$ ($i = 1, 2$) is the $i$th dual primitive idempotent of $\Gamma$. Note that $\Delta(x)$ is $a$-regular with $k$ vertices and that $\Delta_2(x)$ is $(k-c)$-regular with $n-k-1$ vertices. It follows that $\Delta(x)$ and $\Delta_2(x)$ have trivial eigenvalues $a$ and $k-c$, respectively. We say that an eigenvalue of $\Delta_i(x)(i = 1, 2)$ is local if it is not equal to an eigenvalue of $\Gamma$ and has an eigenvector orthogonal to $1$. The following is a characterization of a local eigenvalue of $\Delta_i(x)$ ($i = 1, 2$).
Lemma 5.1 (cf. [12, Theorem 10.6.3]). Let \((i,j) = (1,2)\) or \((i,j) = (2,1)\). Then \(\lambda\) is a local eigenvalue of \(\Delta_i(x)\) if and only if \(a - c - \lambda\) is a local eigenvalue of \(\Delta_j(x)\), with equal multiplicities.

Lemma 5.2 (cf. [27, Proposition 3.1, Lemma 3.2]). Let \(\sigma, \tau\) be eigenvalues of \(\Gamma\) as in (5.1). Fix a vertex \(x\) of \(\Gamma\) and write \(E_i^x = E_i^x(x)\) for \(0 \leq i \leq 2\) and \(T = T(x)\). Let \(A\) be the adjacency matrix of \(\Gamma\) as in (5.3). Let \(V\) be the standard \(T\)-module. Then the following (I) and (II) hold.

(I) Let \(v \in E_i^x V\) be an eigenvector of \(\tilde{A}_1\) with an eigenvalue \(\lambda\) such that \(\langle 1, v \rangle = 0\). Then \(E_2^x AE_i^x v = 0\) if and only if \(\lambda \in \{\sigma, \tau\}\). Let \(W\) be the subspace of \(V\) spanned by \(v, E_2^x AE_i^x v\). Then \(W\) is a thin irreducible \(T\)-module. Moreover, the following (i) and (ii) hold.

(i) Suppose \(\lambda \notin \{\sigma, \tau\}\). Then \(E_2^x AE_i^x v\) is an eigenvector of \(\tilde{A}_2\) with an eigenvalue \(\sigma + \tau - \lambda\). The \(T\)-module \(W\) has endpoint 1 and diameter 1. The vector \(v\) is a basis for \(E_1^x W\) and \(E_2^x AE_i^x v\) is a basis for \(E_2^x W\). The matrix representing \(A\) with respect to a basis \(\{v, E_2^x AE_i^x v\}\) for \(W\) is

\[
\begin{pmatrix}
\lambda & -(\lambda - \sigma)(\lambda - \tau) \\
1 & \sigma + \tau - \lambda
\end{pmatrix}
\]  

(ii) If \(\lambda \in \{\sigma, \tau\}\), then \(W = E_1^x W\). The \(T\)-module \(W\) has endpoint 1 and diameter 0. The action of \(A\) on \(W\) is given by \(Av = \lambda v\).

(II) Let \(u \in E_2^x V\) be an eigenvector of \(\tilde{A}_2\) with an eigenvalue \(\lambda'\) such that \(\langle 1, u \rangle = 0\). Then \(E_1^x AE_2^x u = 0\) if and only if \(\lambda' \in \{\sigma, \tau\}\). Let \(W\) be the subspace of \(V\) spanned by \(u, E_1^x AE_2^x u\). Then \(W\) is a thin irreducible \(T\)-module. Moreover, the following (i) and (ii) hold.

(i) Suppose \(\lambda' \notin \{\sigma, \tau\}\). Then \(E_1^x AE_2^x u\) is an eigenvector of \(\tilde{A}_1\) with an eigenvalue \(\sigma + \tau - \lambda'\). The \(T\)-module \(W\) has endpoint 1 and diameter 1. The vector \(u\) is a basis for \(E_2^x W\) and \(E_1^x AE_2^x u\) is a basis for \(E_1^x W\). The matrix representing \(A\) with respect to a basis \(\{E_1^x AE_2^x u, u\}\) for \(W\) is

\[
\begin{pmatrix}
\lambda' & -(\lambda' - \sigma)(\lambda' - \tau) \\
1 & \sigma + \tau - \lambda'
\end{pmatrix}
\]

(ii) If \(\lambda' \in \{\sigma, \tau\}\), then \(W = E_2^x W\). The \(T\)-module \(W\) has endpoint 2 and diameter 0. The action of \(A\) on \(W\) is given by \(Au = \lambda' u\).

Recall the first subconstituent space \(E_1^x V\) of \(\Gamma\). Note that \(\dim(E_1^x V) = k\). Let

\[
v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_k
\]

denote a sequence of eigenvectors of \(\tilde{A}_1\) in \(E_1^x V\) such that \(v_1 := E_1^x 1\) and \(\langle v_i, v_j \rangle = 0\) for \(i \neq j\). Observe that \(\{v_i\}_{i=1}^k\) is an orthogonal basis for \(E_1^x V\). Let

\[
a = \lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_k
\]

denote the eigenvalues of \(\tilde{A}_1\) corresponding to the eigenvectors in (5.5), respectively. Assume that \(\lambda_i \notin \{\sigma, \tau\}\) for \(1 \leq i \leq r\) and \(\lambda_j \in \{\sigma, \tau\}\) for \(r+1 \leq j \leq k\). We consider the second subconstituent
depends on the base vertex $x$ and (5.7), the algebra $T$ decomposes into an orthogonal direct sum of thin irreducible $T$-modules:

$$V = W_1 \oplus (\bigoplus_{i=2}^{r} W_i) \oplus (\bigoplus_{j=r+1}^{k} W_j) \oplus (\bigoplus_{h=r+1}^{s} W'_h). \tag{5.7}$$

The following proposition classifies thin irreducible $T$-modules of $\Gamma$ up to isomorphism.

**Proposition 5.3** (cf. [27, Lemma 3.4]). Let $W_i$ (resp. $W'_h$) denote an irreducible $T$-module associated with an eigenvalue $\lambda_i$ (resp. $\lambda'_h$) as in (5.6). Then the following (i) and (ii) hold.

(i) For $2 \leq i, j \leq k$, $W_i$ and $W_j$ are isomorphic as $T$-modules if and only if $\lambda_i = \lambda_j$.

(ii) For $r + 1 \leq h, l \leq s$, $W'_h$ and $W'_l$ are isomorphic as $T$-modules if and only if $\lambda'_h = \lambda'_l$.

Let $\ell_1$ (resp. $\ell_2$) denote the number of distinct eigenvalues of $\Delta(x)$ (resp. $\Delta_2(x)$) contained in $\{\sigma, \tau\}$ with respect to the eigenvectors orthogonal to $E^{\ast}_1$ (resp. $E^{\ast}_2$). Let $\ell'_1$ (resp. $\ell'_2$) denote the number of distinct eigenvalues of $\Delta(x)$ (resp. $\Delta_2(x)$) not contained in $\{\sigma, \tau\}$ with respect to the eigenvectors orthogonal to $E^{\ast}_1$ (resp. $E^{\ast}_2$). By Proposition 5.3, the isomorphism classes of irreducible $T$-modules of $\Gamma$ are determined by $\text{Spec}(\Gamma)$, $\text{Spec}(\Delta(x))$, and $\text{Spec}(\Delta_2(x))$. It follows that there are $\ell_1 + \ell_2$ irreducible $T(x)$-modules of dimension one up to isomorphism, and there are $\ell'_1 (= \ell'_2)$ thin irreducible $T(x)$-modules of dimension two up to isomorphism. There is only one thin irreducible $T(x)$-module of dimension three, namely the primary $T(x)$-module. By Proposition 3.1 and (5.7), the algebra $T(x)$ is isomorphic to the semisimple algebra

$$(\text{Mat}_1(\mathbb{C})^{\oplus \ell_1}) \oplus (\text{Mat}_1(\mathbb{C})^{\oplus \ell_2}) \oplus (\text{Mat}_2(\mathbb{C})^{\oplus \ell'_1}) \oplus \text{Mat}_3(\mathbb{C}). \tag{5.8}$$

We call $(\ell_1, \ell'_1, \ell_2, \ell'_2)$ the dimension sequence of $T(x)$. Note that the dimension sequence of $T(x)$ depends on the base vertex $x$ of $\Gamma$. 

| Thin irreducible $T$-module | Basis | Dimension | Endpoint | Diameter |
|-----------------------------|-------|-----------|----------|----------|
| $W_1$           | $\{E_n^a A E_1^a v_1\}_{n=0}^2$ | 3         | 0        | 2        |
| $W_i$ ($2 \leq i \leq r$) | $v_i, E_n^a A E_1^a v_i$ | 2         | 1        | 1        |
| $W_j$ ($r + 1 \leq j \leq k$) | $v_j$ | 1         | 1        | 0        |
| $W'_h$ ($r + 1 \leq h \leq s$) | $u_h$ | 1         | 2        | 0        |

Since $T$ is semisimple, the standard $T$-module $V$ decomposes into an orthogonal direct sum of thin irreducible $T$-submodules:

$$V = W_1 \oplus (\bigoplus_{i=2}^{r} W_i) \oplus (\bigoplus_{j=r+1}^{k} W_j) \oplus (\bigoplus_{h=r+1}^{s} W'_h). \tag{5.7}$$
Lemma 5.4 (cf. [27, Theorem 1.1]). Let $x$ be a base vertex of $\Gamma$ and let $T(x)$ be the Terwilliger algebra of $\Gamma$ with respect to $x$. Then

$$\dim T(x) = \ell_1 + \ell_2 + 4\ell'_1 + 9$$

where $(\ell_1, \ell'_1, \ell_2, \ell'_2)$ is the dimension sequence of $T(x)$.

Proof. By (5.8) and Proposition 3.1. □

5.2 A characterization of pseudo-vertex-transitive strongly regular graphs

In this subsection, we characterize pseudo-vertex-transitive strongly regular graphs. First, we discuss a relationship between $\text{Spec}(\Delta(x))$ and $\text{Spec}(\Delta_2(x))$ of $\Gamma$.

Lemma 5.5. Let $k, \sigma, \tau$ be eigenvalues of $\Gamma$ as in (5.1) with multiplicities $m_\sigma, m_\tau$, respectively. Fix a vertex $x$ of $\Gamma$. The spectrum of the second subconstituent $\Delta_2(x)$ of $\Gamma$ is determined by the spectrum of the local graph $\Delta(x)$ of $\Gamma$ and the parameters $(n, k, a, c)$.

Proof. Let us denote the spectrum of $\Delta(x)$ by

$$\{a^1, \sigma^{f_\sigma}, \tau^{f_\tau}, \lambda_1^{m_1}, \ldots, \lambda_s^{m_s}\},$$

for some $s$. Here, it is possible that $f_\sigma = 0$ and/or $f_\tau = 0$. By Lemma 5.1, we can write the spectrum of $\Delta_2(x)$ as

$$\{(k-c)^1, \sigma^{g_\sigma}, \tau^{g_\tau}, (a-c-\lambda_1)^{m_1}, \ldots, (a-c-\lambda_s)^{m_s}\},$$

where it is possible that $g_\sigma = 0$ and/or $g_\tau = 0$. Since $|\Gamma_1(x)| = k$ and $|\Gamma_2(x)| = n - k - 1$, we have

$$1 + f_\sigma + f_\tau + m_1 + \cdots + m_s = k, \quad 1 + g_\sigma + g_\tau + m_1 + \cdots + m_s = n - k - 1. \quad (5.9)$$

Using the equations in (5.9), we obtain

$$g_\sigma + g_\tau = n - 2k - 1 + f_\sigma + f_\tau. \quad (5.10)$$

Since the sum of all the eigenvalues of $\Delta_2(x)$ is the trace of $\mathcal{A}_2$, which is zero, it follows

$$\sigma g_\sigma + \tau g_\tau = -(k-c) - m_1(a-c-\lambda_1) - \cdots - m_s(a-c-\lambda_s). \quad (5.11)$$

Solve the system of equations (5.10) and (5.11) for $g_\sigma$ and $g_\tau$ to obtain

$$g_\sigma = -k + m_\sigma + f_\tau, \quad g_\tau = -k + m_\tau + f_\sigma. \quad (5.12)$$

The result follows. □

Remark 5.6. By (5.12) we have

$$f_\sigma = k - m_\tau + g_\tau, \quad f_\tau = k - m_\sigma + g_\sigma.$$ 

Thus, the spectrum of $\Delta(x)$ is also determined by the spectrum of $\Delta_2(x)$. 

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Example 5.7. Consider the Johnson graph $J(8, 2)$ with vertex set $X = \binom{[8]}{2}$, the collection of all 2-subsets of $\Omega = \{1, 2, \ldots, 8\}$. The graph $J(8, 2)$ is strongly regular with parameters $(28, 12, 6, 4)$. Fix a vertex $x \in X$. The spectrum of $J(8, 2)$ is $\{12^1, 4^7, -2^{20}\}$ and the spectrum of the local graph $\Delta(x)$ is given by $\{6^1, 4^1, -2^3, 0^5\}$. Using Lemma 5.5, we obtain the spectrum of $\Delta_2(x)$ of $J(8, 2)$:

$$\{8^1, 4^0, -2^9, 2^5\}.$$

Proposition 5.8. A connected strongly regular graph $\Gamma$ is thin.

Proof. By (5.6), $\Gamma$ is thin with respect to $x$. By Proposition 5.3 and Lemma 5.5, the result follows. □

Recall $E_i^* = E_i^*(x)$ the $i$th dual primitive idempotent of $\Gamma$ ($i = 0, 1, 2$). Consider another connected strongly regular graph $\Gamma'$ with parameters $(n', k', a', c')$ and vertex set $X'$. Fix $x' \in X'$. Let $A'$ be the adjacency matrix of $\Gamma'$ and $A'' = A''(x')$ the dual adjacency matrix of $\Gamma'$ with respect to $x'$. Write $E_i' = E_i'(x')$ for $0 \leq i \leq 2$ and $T' = T'(x')$. With these notations, we state the following two lemmas.

Lemma 5.9. Suppose that the parameters of $\Gamma$ and $\Gamma'$ are same, that is, $(n, k, a, c) = (n', k', a', c')$. Let $v = E_1^* \mathbb{1}$ and $v' = E_1^* \mathbb{1}$. Then $Tv$ and $T'v'$ are $T$-isomorphic.

Proof. Observe that $Tv$ is the primary $T$-module with a basis $\{E_i^* A E_1^* v\}_{i=0}^2$; see (5.6). Define the vectors $\{v_i\}_{i=0}^2$ by

$$v_0 = E_0^* A E_1^* v, \quad v_1 = a^{-1} E_1^* A E_1^* v, \quad v_2 = c^{-1} E_2^* A E_1^* v.$$ 

Observe that $v_i = E_i^* \mathbb{1}$ $(0 \leq i \leq 2)$, and thus $\{v_i\}_{i=0}^2$ is a standard basis for $Tv$. Similarly, we define the vectors $\{v_i'\}_{i=0}^2$ by

$$v_0' = E_0^* A' E_1^* v', \quad v_1' = a^{-1} E_1^* A' E_1^* v', \quad v_2' = c^{-1} E_2^* A' E_1^* v'.$$

Then $\{v_i'\}_{i=0}^2$ is a standard basis for the primary $T'$-module $T'v'$. Define a vector space isomorphism $\rho : Tv \rightarrow T'v'$ that sends $v_i$ to $v_i'$. Since $\Gamma$ and $\Gamma'$ have same parameters $(n, k, a, c)$, from (3.1) the matrix representing $A$ with respect to $\{v_i\}_{i=0}^2$ and the matrix representing $A'$ with respect to $\{v_i'\}_{i=0}^2$ are the same as the tridiagonal matrix with entries of the intersection numbers of $\Gamma$:

$$
\begin{pmatrix}
0 & k & 0 \\
1 & a & k - a - 1 \\
0 & c & k - c
\end{pmatrix}.
$$

Therefore, we have $(\rho A - A' \rho)v_i = 0$ for $0 \leq i \leq 2$. Moreover, we readily check that $(\rho E_j^* - E_j^* \rho)v_i = 0$ for $0 \leq i, j \leq 2$. The result follows. □

Lemma 5.10. Suppose that the parameters of $\Gamma$ and $\Gamma'$ are same, that is, $(n, k, a, c) = (n', k', a', c')$. Let $i \in \{1, 2\}$. Let $v \in E_i^* V$ denote an eigenvector of $E_i^* A E_1^*$ with an eigenvalue $\lambda$ such that $\langle \mathbb{1}, v \rangle = 0$. Let $v' \in E_i^* V$ denote an eigenvector of $E_i^* A' E_1^*$ with an eigenvalue $\lambda'$ such that $\langle \mathbb{1}, v' \rangle = 0$. Then $\lambda = \lambda'$ if and only if $Tv$ and $T'v'$ are $T$-isomorphic.
Proof. Set $i = 1$. Suppose that $\lambda = \lambda'$. If $\lambda \notin \{\sigma, \tau\}$, then by Lemma 5.2(I)(i) $Tv$ has a basis $v, E_1 v$ and $T'v'$ has a basis $v', E_1 v'$. Abbreviate $w = E_1 v$ and $w' = E_1 v'$. Define a vector space isomorphism $\rho : Tv \rightarrow T'v'$ by $\rho(v) = v'$ and $\rho(w) = w'$. By Lemma 5.2(I)(i) and since $\lambda = \lambda'$, the matrix representing $A$ with respect to a basis $v, w$ and the matrix representing $A'$ with respect to a basis $v', w'$ are both equal to the matrix (5.4). From this, it follows that $(\rho A - A' \rho)Tv = 0$. It is clear that $(\rho E_1 - E_1 \rho)Tv = 0$ for $0 \leq i \leq 2$. Thus, $Tv$ and $T'v'$ are $T$-isomorphic. If $\lambda \in \{\sigma, \tau\}$, use Lemma 5.2(I)(ii) in a similar manner as demonstrated above to get the result. Conversely, if $Tv$ and $T'v'$ are $T$-isomorphic, then the matrix representing $A$ with respect to a basis $v, w$ must be the same as the matrix representing $A'$ with respect to a basis $v', w'$. By this and using Lemma 5.2(I), we have $\lambda = \lambda'$. For $i = 2$, use Lemma 5.2(II). The result follows.

We now characterize pseudo-vertex-transitive strongly regular graphs.

**Theorem 5.11.** A connected strongly regular graph $\Gamma$ is pseudo-vertex-transitive if and only if for every pair of vertices $x, y \in X$, the spectrum of the local graph $\Delta(x)$ of $\Gamma$ at $x$ is equal to the spectrum of the local graph $\Delta(y)$ of $\Gamma$ at $y$.

Proof. If $\Gamma$ is pseudo-vertex-transitive, by Lemma 4.4 it immediately follows that $\text{Spec}(\Delta(x)) = \text{Spec}(\Delta(y))$. Conversely, for $x, y \in X$ suppose that $\text{Spec}(\Delta(x)) = \text{Spec}(\Delta(y))$. By Lemma 5.5, it follows that $\text{Spec}(\Delta_2(x)) = \text{Spec}(\Delta_2(y))$. Using these, we will show that the standard $T(x)$-module and the standard $T(y)$-module are isomorphic. First, choose an eigenvalue $\lambda \in \Delta(x) = \Delta(y)$. Let $v \in E_1(x)V$ denote an eigenvector of $E_1(x)AE_1(x)$ corresponding to $\lambda$, and let $v' \in E_1(y)V$ denote an eigenvector of $E_1(y)AE_1(y)$ corresponding to $\lambda$. If $\lambda = a$, then $Tv$ and $T'v'$ are the primary modules of $T(x)$ and $T(y)$, respectively. By Lemma 5.9, $T(x)v$ and $T(y)v'$ are isomorphic. If $\lambda \neq a$, we observe that $T$-modules $T(x)v$ and $T(y)v'$ have both endpoint 1. By Lemma 5.10, $T(x)v$ and $T(y)v'$ are isomorphic. Next, choose an eigenvalue $\mu \in \Delta_2(x) = \Delta_2(y)$ such that $\mu \in \{\sigma, \tau\}$. Let $u \in E_2(x)V$ denote an eigenvector of $E_2(x)AE_2(x)$ corresponding to $\mu$, and let $u' \in E_2(y)V$ denote an eigenvector of $E_2(y)AE_2(y)$ corresponding to $\mu$. Observe that both $T$-modules $T(x)v$ and $T(y)v'$ have endpoint 2 and diameter 0. By Lemma 5.10, $T(x)u$ and $T(y)u'$ are isomorphic. By these comments and (5.7), the standard modules for $T(x)$ and $T(y)$ are isomorphic. Since $x, y$ is an arbitrary pair of vertices in $X$, $\Gamma$ is pseudo-vertex-transitive.

In the following, we characterize connected strongly regular graphs which are $T$-isomorphic.

**Theorem 5.12.** Let $\Gamma$ and $\Gamma'$ be connected strongly regular graphs with parameters $(n, k, a, c)$ and $(n', k', a', c')$, respectively. Then $\Gamma$ and $\Gamma'$ are $T$-isomorphic if and only if both (i) $(n, k, a, c) = (n', k', a', c')$ and (ii) for every pair of vertices $x \in X$ and $x' \in X'$ the spectrum of the local graph $\Delta(x)$ of $\Gamma$ is equal to the spectrum of the local graph $\Delta'(x')$ of $\Gamma'$.

Proof. Suppose that $\Gamma$ and $\Gamma'$ are $T$-isomorphic. By Lemma 4.7, the intersection arrays of $\Gamma$ and $\Gamma'$ are the same. From this, it follows $(n, k, a, c) = (n', k', a', c')$. Also, by Lemma 4.8, it follows $\text{Spec}(\Delta(x)) = \text{Spec}(\Delta'(x'))$. Conversely, suppose that both (i) and (ii) hold. We claim that
the dimension of the Terwilliger algebra $T_{\Gamma}$ vertex-transitive. The following table shows the spectrum of the local graph $\Delta_k$ eigenvalues $K$

Example 5.13. In this subsection, we give some concrete examples of cospectral strongly regular graphs and discuss their pseudo-vertex transitivity.

Example 5.13. Let $\Gamma$ denote the Shrikhande graph, a Cayley graph of $\mathbb{Z}_4 \times \mathbb{Z}_4$ relative to the generating set $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$. Let $\Gamma'$ denote the $4 \times 4$-grid graph, the line graph of $K_{4,4}$. The graphs $\Gamma$ and $\Gamma'$ are cospectral strongly regular with parameters $(16, 6, 2, 2)$ and the eigenvalues $k = 6, \sigma = 2, \tau = -2$. Moreover, $\Gamma$ and $\Gamma'$ are both vertex-transitive and thus pseudo-vertex-transitive. The following table shows the spectrum of the local graph $\Delta_G$ of $G \in \{\Gamma, \Gamma'\}$ and the dimension of the Terwilliger algebra $T_G$ of $G$.

| graph $G$ | Spec($\Delta_G$) | dim$T_G$ |
|-----------|------------------|---------|
| $\Gamma$  | $\{2^1, 1^2, (-1)^2, (-2)^1\}$ | 20      |
| $\Gamma'$ | $\{2^2, (-1)^4\}$       | 15      |

By Theorem 5.12, it follows that $\Gamma$ and $\Gamma'$ are not $T$-isomorphic.

Example 5.14 (cf. [27, Example 4.2]). Recall the Johnson graph $\Gamma = J(8, 2)$ with vertex set $X = \binom{\Omega}{2}$, where $\Omega = \{1, 2, \ldots, 8\}$. The automorphism group $\text{Aut}(\Gamma)^1$ acts on $X$ transitively, and hence $\Gamma$ is pseudo-vertex-transitive. We recall the Chang graphs. The three Chang graphs $\Gamma', \Gamma''$, $\Gamma'''$ can be obtained from $\Gamma$ by Seidel switching with respect to one of the sets (cf. [1, pp.105])

\footnote{Note that $\text{Aut}(\Gamma) = S_8$.}
(i) \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\} for \(\Gamma^\prime\),
(ii) \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 1\}\} for \(\Gamma^\prime\prime\),
(iii) \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 4\}\} for \(\Gamma^\prime\prime\prime\).

Note that none of three Chang graphs (i)–(iii) is vertex-transitive. For case (i), the action of \(\text{Aut}(\Gamma^\prime)\) on \(X\) has two orbits:

\[
U_1^\prime = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}, \quad U_1'' = X \setminus U_1^\prime.
\]

For case (ii), the action of \(\text{Aut}(\Gamma^\prime\prime)\) on \(X\) has two orbits:

\[
U_1'' = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}, \quad U_1'' = X \setminus U_1''.
\]

For case (iii), the action of \(\text{Aut}(\Gamma^\prime\prime\prime)\) on \(X\) has three orbits

\[
U_1''' = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}, \quad U_2''' = \{(i, j) \mid 4 \leq i < j \leq 8\}, \quad U_3''' = X \setminus \{U_1''' \cup U_2''\}.
\]

We remark that the Johnson graph \(J(8, 2)\) and three Chang graphs (i)–(iii) are cospectral strongly regular with parameters \((28, 12, 6, 4)\) and the eigenvalues \(k = 12, \sigma = 4, \tau = -2\). In the following table we display the spectrum of the local graph \(\Delta_G\) of \(G \in \{\Gamma, \Gamma^\prime, \Gamma^\prime\prime, \Gamma^\prime\prime\prime\}\) and the dimension of the Terwilliger algebra \(T_G\) of \(G\).

|\(G\)  | base vertex \(x\) | \(\text{Spec(}\Delta_G(x)\)) | \(\dim T_G(x)\) |
|-------|--------------------|----------------------------------|------------------|
|\(\Gamma\) | \(x \in X\) | \(\{6^1, 0^5, 4^1, (-2)^5\}\) | 16 |
|\(\Gamma^\prime\) | \(x \in U_1^\prime\) | \(\{6^1, 2^2, 0^2, (-2)^6\}\) | 20 |
| | \(x \in U_2^\prime\) | \(\{6^1, (1 + \sqrt{5})^1, 2^1, 0^3, (1 - \sqrt{5})^1, (-2)^5\}\) | 27 |
|\(\Gamma^\prime\prime\) | \(x \in U_1''\) | \(\{6^1, (1 + \sqrt{3})^2, 2^2, 0^2, (-1)^1, (-2)^5\}\) | 23 |
| | \(x \in U_2''\) | \(\{6^1, (1 + \sqrt{3})^1, 2^1, \sqrt{3}^2, 0^1, (1 - \sqrt{3})^1, (-\sqrt{3})^1, (-2)^5\}\) | 35 |
| \(x \in U_1'''\) | \(\{6^1, 3^1, (1 + \sqrt{3})^2, (1 - \sqrt{3})^2, (-1)^1, (-2)^5\}\) | 27 |
|\(\Gamma^\prime\prime\prime\) | \(x \in U_2'''\) | \(\{6^1, (1 + \sqrt{3})^2, 2^2, (-1)^2, (1 - \sqrt{3})^2, (-2)^5\}\) | 23 |
| | \(x \in U_3'''\) | \(\{6^1, (1 + \sqrt{3})^1, 2^1, \sqrt{3}^2, 0^1, (1 - \sqrt{3})^1, (-\sqrt{3})^1, (-2)^5\}\) | 35 |

From this table and by Proposition 5.11, we find that the three Chang graphs are not pseudo-vertex-transitive. We finish this example with comments.

(i) For \(x \in U_2^\prime\) and \(y \in U_1'''\), the Terwilliger algebras \(T_{\Gamma^\prime}(x)\) and \(T_{\Gamma'''}(y)\) have the same dimension 27, but they are not \(T\)-algebra isomorphic since their local spectra are not equal to each other. Note that the dimension sequences of \(T_{\Gamma^\prime}(x)\) and \(T_{\Gamma'''}(y)\) are equal to \((1, 4, 1, 4, \ldots)\), and thus they are semisimple algebra isomorphic to each other.

\(^2\)In [27, Example 4.2], the orbit \(U_1''\) was missed.
(ii) In a similar way to (i), for \(x \in U''_1\) and \(y \in U''_2\), the Terwilliger algebras \(T_{\Gamma'}(x)\) and \(T_{\Gamma''}(y)\) are not \(T\)-algebra isomorphic, but their dimension sequences are equal to \((1,3,1,3)\). Thus they are semisimple algebra isomorphic to each other.

(iii) For \(x \in U''_2\) and \(y \in U''_2\), the local graphs \(\Delta_{\Gamma'}(x)\) and \(\Delta_{\Gamma''}(y)\) have the same spectrum. Therefore, the Terwilliger algebras \(T_{\Gamma'}(x)\) and \(T_{\Gamma''}(y)\) are \(T\)-algebra isomorphic.

**Example 5.15.** A generalized quadrangle is an incidence structure such that: (i) any two points are on at most one line, and hence any two lines meet in at most one point, (ii) If \(p\) is a point not on a line \(L\), then there is a unique point \(p'\) on \(L\) such that \(p\) and \(p'\) are collinear. If every line contains \(s+1\) points, and every point lies on \(t+1\) lines, we say that the generalized quadrangle has order \((s, t)\), denoted by \(GQ(s, t)\).

The point graph of a generalized quadrangle is the graph with the points of the quadrangle as its vertices, with two points adjacent if and only if they are collinear. The point graph \(\Gamma\) of a \(GQ(s, t)\) is strongly regular with parameters \(((s+1)(st+1), s(t+1), s-1, t+1)\) and eigenvalues \(k = s(t+1), \sigma = s-1, \tau = -t-1\), with respective multiplicities

\[
1, \frac{s(t+1)(t+1)}{s+t}, \frac{s^2(st+1)}{s+t}.
\]

By construction, \(\Gamma\) is locally a disjoint union of the \((t+1)\) cliques of size \(s\). Thus the spectrum of a local graph \(\Delta\) of \(\Gamma\) is determined by \(s, t\):

\[
\text{Spec}(\Delta) = \{(s-1)^{t+1}, -1^{s-1(t+1)}\}.
\]

From these comments and by Theorem 5.11, it follows that \(\Gamma\) is pseudo-vertex-transitive. Moreover, the point graph \(\Gamma'\) of a \(GQ(s', t')\) is \(T\)-isomorphic to \(\Gamma\) if and only if \((s, t) = (s', t')\). The dimension of the Terwilliger algebra \(T(x)\) of \(\Gamma\) with respect to a vertex \(x\) is given as follows.

\[
\dim T(x) = \begin{cases} 
10, & \text{if } t = s = 1; \\
15, & \text{if } t = 1, s \neq 1, \text{ or } t \neq 1, s \neq 1, s^2 = t; \\
11, & \text{if } t \neq 1, s = 1; \\
16, & \text{if } t \neq 1, s \neq 1, s^2 \neq t.
\end{cases}
\]

We finish this example with a comment. The isomorphism class of the Terwilliger algebra for the point graph of a \(GQ(s, t)\) only depends on \(s\) and \(t\), and thus it is pseudo-vertex-transitive. However, there exist generalized quadrangles whose point graphs are not vertex-transitive. For example, there are at least three pairwise non-isomorphic \(GQ(q, q^2)\) of which at least one is not vertex-transitive, provided that \(q = 2^{2h+1}, h \geq 1\); see [20, Page 45].

6 Preliminaries: antipodal distance-regular graphs

In this section, we recall some preliminaries concerning antipodal distance-regular graphs, which will be used in the next sections. Let \(\Gamma\) denote a distance-regular graph with vertex set \(X\) and
diameter $D \geq 3$. The graph $\Gamma$ is called antipodal whenever the vertices at distance $D$ from a given vertex are all at distance $D$ from each other. Note that $\Gamma$ is antipodal if and only if $\Gamma$ has the intersection array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$, where $b_i = c_{D-i}$ for $0 \leq i \leq D - 1$, except possibly $i = \lfloor D/2 \rfloor$; cf. [1, Proposition 4.2.2]. Note also that being at distance $D$ or zero induces an equivalence relation on $X$. With respect to this equivalence relation, the equivalence classes are called antipodal classes. We say $\Gamma$ is an antipodal $r$-cover if the equivalence class has size $r$. Let $\Gamma$ be an antipodal 2-cover (or double cover). Then for each vertex $x \in X$ there is a unique vertex $y \in X$ with $\partial(x, y) = D$. Such a unique vertex is called the antipode of $x$, denoted by $\hat{x}$.

**Lemma 6.1.** Let $\Gamma$ be an antipodal double cover with diameter $D \geq 3$. Let $x$ be a vertex of $\Gamma$ and $\hat{x}$ the antipode of $x$. Then $W$ is an irreducible $T(x)$-module if and only if $W$ is an irreducible $T(\hat{x})$-module.

**Proof.** Let $W$ denote an irreducible $T(x)$-module. Then $W$ is invariant under the actions of $A$ and $E_j^*(x)$ for $0 \leq j \leq D$. Since $\Gamma$ is an antipodal double cover, it follows that $E_j^*(x) = E_{D-j}(\hat{x})$ for $0 \leq j \leq D$. Thus $W$ is invariant under the actions of $A$ and $E_h^*(\hat{x})$ for $0 \leq h \leq D$. The result follows. ■

Suppose that $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$. Then $\Gamma$ is antipodal if and only if $\Gamma$ is dual bipartite, that is, the Krein parameters of $\Gamma$ satisfy $q_{li}^i = 0$ for $0 \leq i \leq D$; cf. [24, Theorem 3] (or [19, Theorem 4.2]). The antipodal $Q$-polynomial distance-regular graphs are completely classified for diameter $\geq 6$ and valency $\geq 3$ cases; cf. [9].

**Lemma 6.2** (cf. [4, Corollary 11.5]). Let $\Gamma$ be an antipodal $Q$-polynomial distance-regular graph with $D \geq 3$. Fix a vertex $x$ of $\Gamma$ and write $T = T(x)$. Let $W$ be an irreducible $T$-module with diameter $d$. Recall the scalars $\{a_i(W)\}_{i=0}^d$ from (3.2). Then

$$a_i(W) = a_{d-i}(W) \quad (0 \leq i \leq d).$$

**Proof.** Apply the result of [4, Corollary 11.5] to the dual bipartite $Q$-polynomial distance-regular graphs. ■

We give a remark. Caughman [4] showed that a bipartite $Q$-polynomial distance-regular graph $\Gamma$ with $D \geq 3$ is thin and pseudo-vertex-transitive. More specifically, he proved that any irreducible $T(x)$-module is both thin and dual thin [4, Lemma 9.2] and that the isomorphism class of an irreducible $T(x)$-module is completely determined by its endpoint and diameter [4, Theorem 13.1]. He also found a recurrence that gives the multiplicities of irreducible $T(x)$-modules; this recurrence is expressed as the intersection numbers, eigenvalues, and dual eigenvalues of $\Gamma$ [4, Section 14]. Thus the isomorphism class of the standard $T$-module of $\Gamma$ does not depend on the base vertex $x$. By these comments, $\Gamma$ is thin and pseudo-vertex-transitive. Dualizing the results of [4], we can find that the dual bipartite (or antipodal) $Q$-polynomial distance-regular graphs are thin and pseudo-vertex-transitive. These comments are summarized as follows.

**Proposition 6.3** ([4]). A bipartite and/or dual bipartite $Q$-polynomial distance-regular graph with $D \geq 3$ is thin and pseudo-vertex-transitive.
In the next two sections, we will discuss pseudo-vertex transitivity of the antipodal $Q$-polynomial distance-regular graphs with diameter three and four. We can see that these graphs are pseudo-vertex-transitive by Proposition 6.3, but we will prove it in detail to explain how the thin property plays in determining their pseudo-vertex transitivity.

We now recall the tight distance-regular graphs. Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. In [14], Jurisić et al. proved that the intersection numbers $a_i, b_i$ satisfy
\[
\left( \theta_1 + \frac{k}{a_1 + 1} \right) \left( \theta_D + \frac{k}{a_1 + 1} \right) \geq -\frac{ka_1 b_1}{(a_1 + 1)^2}.
\]
We say $\Gamma$ is tight whenever $\Gamma$ is not bipartite and equality holds in (6.2). Define
\[
b^+ := -1 - \frac{b_1}{1 + \theta_D}, \quad b^- := -1 - \frac{b_1}{1 + \theta_1}.
\]
We say $\Gamma$ is tight with respect to $x$ whenever every irreducible $T(x)$-module with endpoint 1 is thin with local eigenvalue $b^+$ or $b^-$. The tight distance-regular graphs are characterized by using the Terwilliger algebra [11]. We state the following lemma.

**Lemma 6.4** ([11, Theorem 13.6], [14, Theorem 12.6]). Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. The following (i)–(iii) are equivalent.

(i) $\Gamma$ is tight.

(ii) For $x \in X$ the local graph of $\Gamma$ at $x$ is connected strongly regular with eigenvalues $a_1, b^+, b^-$. 

(iii) $\Gamma$ is non-bipartite and tight with respect to each vertex.

Pascasio [18] showed that the non-bipartite antipodal $Q$-polynomial distance-regular graphs are tight. For instance, the Johnson graph $J(2D, D)$, the halved $2D$-cube, the non-bipartite Taylor graphs, and the Meixner1 graph are non-bipartite antipodal $Q$-polynomial distance-regular, and therefore they are tight; see [9, 14].

We finish this section with the following lemmas, which will be used later.

**Lemma 6.5** (cf. [11, Theorem 10.1]). Let $\Gamma$ be a $Q$-polynomial distance-regular graph with $D \geq 3$. Fix $x \in X$ and write $T = T(x)$ the Terwilliger algebra of $\Gamma$. Let $W$ be a thin irreducible $T$-module with endpoint 1. If $W$ has the local eigenvalue $b^+$ (resp. $b^-$), then $W$ has dual endpoint 1 (resp. 2).

**Lemma 6.6** (cf. [11, Theorem 10.7]). Let $\Gamma$ be a $Q$-polynomial distance-regular graph with $D \geq 3$. Fix $x \in X$ and write $T = T(x)$ and $E_i = E_i(x)$ for $0 \leq i \leq D$. Let $W$ be a thin irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{b^+, b^-\}$. Then $W$ has dimension $D - 1$. For $0 \leq i \leq D$, $E_i W$ is zero if $i \in \{0, D\}$ and has dimension 1 if $i \notin \{0, D\}$. Moreover, $E_0 W = 0$ if $i \in \{0, n\}$ and has dimension 1 if $i \notin \{0, n\}$, where $n = 1$ if $\lambda = b^-$; and $n = D$ if $\lambda = b^+$.

**Lemma 6.7** (cf. [11, Theorem 11.1]). Let $\Gamma$ be a $Q$-polynomial distance-regular graph with $D \geq 3$. Fix a vertex $x$ of $\Gamma$ and write $T = T(x)$. Let $W$ be a thin irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{b^+, b^-\}$. Let $W'$ be an irreducible $T$-module. Then $W$ and $W'$ are isomorphic as $T$-modules if and only if $W'$ is thin with endpoint 1 and local eigenvalue $\lambda$. 20
7 Taylor graphs

In this section, we discuss the pseudo-vertex transitivity of the Taylor graphs. A Taylor graph is a distance-regular antipodal double cover of a complete graph with diameter three, that is, a distance-regular graph with intersection array \{k, b, 1; 1, b, k\}, where \(b < k - 1\). We note that the non-bipartite Taylor graphs are exactly tight distance-regular graphs with diameter three [16].

For the rest of this section, we denote by \(\Gamma\) a Taylor graph with vertex set \(X\) and the eigenvalues \(k = \theta_0 > \theta_1 > \theta_2 > \theta_3\). The spectrum of \(\Gamma\) is

\[\{k^1, \theta_1^f, \theta_2^g, \theta_3^g\},\]

where

\[\theta_1 = \frac{k - 2b - 1 + \sqrt{D}}{2}, \quad \theta_2 = -1, \quad \theta_3 = \frac{k - 2b - 1 - \sqrt{D}}{2},\]  

(7.1)

and

\[f = \left(\frac{1}{2} - \frac{k - 2b - 1}{2\sqrt{D}}\right)(k + 1), \quad g = \left(\frac{1}{2} + \frac{k - 2b - 1}{2\sqrt{D}}\right)(k + 1),\]

where \(D = (k - 2b - 1)^2 + 4k\). By Lemma 6.4, for each \(x \in X\) the local graph \(\Delta = \Delta(x)\) of \(\Gamma\) is connected strongly regular with parameters \((n_\Delta, k_\Delta, a_\Delta, c_\Delta)\), where

\[n_\Delta = k, \quad k_\Delta = k - b - 1, \quad a_\Delta = \frac{2k - 3b - 4}{2}, \quad c_\Delta = \frac{k - b - 1}{2}.\]

From (5.1), the three eigenvalues of \(\Delta\) are given by

\[k_\Delta, \quad \sigma = \frac{k - 2b - 3 + \sqrt{D}}{4}, \quad \tau = \frac{k - 2b - 3 - \sqrt{D}}{4},\]  

(7.2)

where \(D = (k - 2b - 1)^2 + 4k\). One can verify that

\[\sigma = -1 - \frac{b}{1 + \theta_3}, \quad \tau = -1 - \frac{b}{1 + \theta_1}.\]  

(7.3)

Thus, \(\sigma = b^+\) and \(\tau = b^-\), where \(b^+\) and \(b^-\) are from (6.3). By (5.2) the multiplicities \(m_\sigma\) and \(m_\tau\) of \(\sigma\) and \(\tau\), respectively, are given by

\[m_\sigma = \frac{k - 1}{2} - \frac{(k + 1)(k - 2b - 1)}{2\sqrt{D}}, \quad m_\tau = \frac{k - 1}{2} + \frac{(k + 1)(k - 2b - 1)}{2\sqrt{D}}.\]  

(7.4)

By these comments we have the following lemma.

**Lemma 7.1.** For each \(i \in \{1, 2\}\) and for all \(x \in X\), the spectrum of \(\Delta_i(x)\) of \(\Gamma\) is determined by the intersection numbers \(k\) and \(b\).

**Proof.** For the case \(i = 1\), it follows from (7.2) and (7.4). For the case \(i = 2\), since \(\Gamma\) is an antipodal double cover it follows \(\Delta_1(x) \cong \Delta_1(\hat{x}) = \Delta_2(x)\). The result follows.

**Lemma 7.2.** Fix a vertex \(x \in X\) and let \(T = T(x)\) be the Terwilliger algebra of \(\Gamma\). Let \(W\) be a \(T\)-module. Then the endpoint of \(W\) is either 0 or 1.
Lemma 7.3. Recall the eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3$ of $\Gamma$. Fix $x \in X$ and write $T = T(x)$. Let $W$ be an irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{\sigma, \tau\}$, where $\sigma, \tau$ are from (7.3). Then the following (i)-(iii) hold.

(i) $W$ has diameter 1 and is thin.

(ii) Let $w_0 \in E^*_1 W$ be a nonzero vector. Set $w_1 = E^*_2 Aw_0$. Then $\{w_0, w_1\}$ is a basis for $W$. With respect to this basis, the matrix representing $A$ is given by

$$
\begin{pmatrix}
\lambda & (\lambda - \theta_t)^2 \\
1 & \lambda
\end{pmatrix}
$$

(7.5)

where $t = 1$ if $\lambda = \sigma$ and $t = 2$ if $\lambda = \tau$.

(iii) The local eigenvalues $\sigma, \tau$ of $W$ are given by $\sigma = (\theta_1 - \theta_2)/2$ and $\tau = (\theta_2 - \theta_3)/2$.

Proof. (i): For the first assertion, it is similar to the proof of Lemma 7.2. For the second assertion, one knows that $\Gamma$ is tight. By Lemma 6.4(iii), $W$ is thin.

(ii): Clearly, $\{w_0, w_1\}$ is an orthogonal basis for $W$. By (3.4), the action of $A$ on $w_0$ is given by $Aw_0 = a_0(W)w_0 + w_1$. Observe that $a_0(W) = \lambda$. Next, the action of $A$ on $w_1$ is given by $Aw_1 = x_1(W)w_0 + a_1(W)w_1$, where the scalars $a_1(W)$ and $x_1(W)$ are from (3.2). As $\Gamma$ is an antipodal double cover, by (6.1) we find $a_1(W) = a_0(W) = \lambda$. Recall the intersection numbers $b_0(W)$ and $c_1(W)$ of $W$. Referring to [5, Section 9], we find that $a_0(W) + b_0(W) = \theta_t$ and $c_1(W) + a_1(W) = \theta_t$, where $t$ is dual endpoint of $W$. Using these two equations and the fact that $x_1(W) = b_0(W)c_1(W)$, we find $x_1(W) = (\lambda - \theta_t)^2$. By Lemma 6.5, the dual endpoint of $W$ is $t = 1$ (resp. $t = 2$) if the local eigenvalue $\lambda$ is $\sigma$ (resp. $\tau$). By these comments, we obtain the matrix (7.5).

(iii): By (3.3), we have $a_0(W) + a_1(W) = 2\lambda = \theta_t + \theta_{t+1}$, where $t$ is the dual endpoint of $W$. By Lemma 6.5, the result follows.

Proposition 7.4. A Taylor graph is thin.

Proof. By Lemma 7.2 and Lemma 7.3.

We recall the following lemma, which classifies all irreducible $T$-modules of $\Gamma$ with endpoint one.
Lemma 7.5. Fix $x \in X$ and write $T = T(x)$. Let $W$ be a thin irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{\sigma, \tau\}$, where $\sigma, \tau$ are from (7.2). Let $W'$ denote an irreducible $T$-module. Then $W$ and $W'$ are isomorphic as $T$-modules if and only if $W'$ is thin with endpoint 1 and local eigenvalue $\lambda$.

Proof. Apply Lemma 6.7 to $\Gamma$. The result immediately follows. ■

Recall the standard $T$-module $V$. Since $T$ is semisimple and using Lemma 7.2, Lemma 7.3, and Lemma 7.5, $V$ decomposes into an orthogonal direct sum of thin irreducible $T$-modules

$$V = W_0 \oplus \bigoplus_{i} W_i \oplus \bigoplus_{j} W'_j,$$

where we denote by

| $T$-module | dimension | endpoint | diameter | comment |
|------------|-----------|----------|----------|---------|
| $W_0$      | 4         | 0        | 3        | the primary module |
| $W_i$      | 2         | 1        | 1        | local eigenvalue $\sigma$ |
| $W'_j$     | 2         | 1        | 1        | local eigenvalue $\tau$ |

where $\sigma$ and $\tau$ are from (7.3).

Theorem 7.6. A Taylor graph is pseudo-vertex-transitive.

Proof. For a given vertex $x \in X$, let $T(x)$ be the Terwilliger algebra of $\Gamma$. By Lemma 7.1 and Lemma 7.5, the isomorphism class of an irreducible $T(x)$-module is determined by the parameters $k$ and $b$. Moreover, by Lemma 7.3 and (7.4), the multiplicities of irreducible $T(x)$-modules are determined by $k$ and $b$. Therefore, the isomorphism class of the standard $T(x)$-module (7.6) of $\Gamma$ is completely determined by $k$ and $b$. The result follows. ■

Remark 7.7. Taylor graphs are not vertex-transitive in general. For example, there are four different Taylor graphs with intersection array $\{25,12,1;1,12,25\}$ of which only one is vertex-transitive; cf. [21].

We finish this section with a comment.

Proposition 7.8. For each $x \in X$, the dimension of $T(x)$ of $\Gamma$ is given by

$$\dim(T(x)) = 24.$$ 

Proof. Applying Proposition 3.1 to (7.6), the algebra $T(x)$ is isomorphic to

$$\text{Mat}_4(\mathbb{C}) \oplus \text{Mat}_2(\mathbb{C}) \oplus \text{Mat}_2(\mathbb{C}).$$

From (3.5), it follows that $\dim(T(x)) = 4^2 + 2^2 + 2^2 = 24$. ■

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### 8. Antipodal tight graphs $\text{AT4}(p, q, 2)$

In this section, we consider a distance-regular antipodal double cover with diameter four. We recall some properties of this graph that are needed in this section. For more information, we refer the reader to [13, 15, 16]. Let $\Gamma$ denote a non-bipartite distance-regular antipodal double cover with diameter four. Then $\Gamma$ is $Q$-polynomial if and only if $\Gamma$ is tight [16]. Suppose that $\Gamma$ is $Q$-polynomial. The intersection array of $\Gamma$ is parameterized by $p, q$ (cf. [16, Section 5]):

$$\{q(pq + p + q), (q^2 - 1)(p + 1), q(p + q)/2, 1; 1, q(p + q)/2, (q^2 - 1)(p + 1); q(pq + p + q)\}, \quad (8.1)$$

where $p \geq 1$, $q \geq 2$ are integers. The spectrum of $\Gamma$ is given by $\{\theta_0 = q(pq + p + q), \theta_1 = pq + p + q, \theta_2 = p, \theta_3 = -q, \theta_4 = -q^2\}, \quad (8.2)$

and $m_0 = 1$,

$$\begin{align*}
    m_1 &= \frac{q(pq^2 + q^2 + pq - p)}{p + q}, \\
    m_2 &= \frac{q(pq + p + q)(q^2 - 1)(2q + pq + p)}{(p + q)(p + q^2)}, \\
    m_3 &= \frac{(pq^2 + q^2 + pq - p)(pq + p + q)}{p + q}, \\
    m_4 &= \frac{(p + 1)(pq + p + q)(pq^2 + q^2 + pq - p)}{(p + q)(p + q^2)}.
\end{align*}$$

For each $x \in X$ the local graph $\Delta = \Delta(x)$ of $\Gamma$ is connected strongly regular with parameters $(n_\Delta, k_\Delta, a_\Delta, c_\Delta)$, where (cf. [16, Section 5])

$$n_\Delta = q(pq + p + q), \quad k_\Delta = p(q + 1), \quad a_\Delta = 2p - q, \quad c_\Delta = p. \quad (8.3)$$

The nontrivial eigenvalues of $\Delta$ are $b^+ = p$ and $b^- = -q$ and their multiplicities are

$$m_{b^+} = \frac{(q^2 - 1)(pq + p + q)}{p + q}, \quad m_{b^-} = \frac{pq(q + 1)(p + 1)}{p + q}. \quad (8.4)$$

Let $\text{AT4}(p, q, r)$ denote an antipodal tight $r$-cover of diameter four with parameters $p$ and $q$. Clearly, $\Gamma$ is the same thing as $\text{AT4}(p, q, 2)$. There are three known examples of an $\text{AT4}(p, q, 2)$, namely the following graphs with array $\{c_1, c_2, c_3, c_4\}$:

(i) $\text{AT4}(2, 2, 2)$, the Johnson graph $J(8, 4)$ with $\{1, 4, 9, 16\};$

(ii) $\text{AT4}(4, 2, 2)$, the half-cube $\frac{1}{2}H(8, 2)$ with $\{1, 6, 15, 28\};$

(iii) $\text{AT4}(8, 4, 2)$, the Meixner1 graph with $\{1, 24, 135, 176\}$.

**Remark 8.1.** In [9, Theorem 1.1], Dickie and Terwilliger gave a family of graphs with array $\{1, \beta \eta, (\beta^2 - 1)(2\eta - \beta + 1), \beta(2\eta + 2n\beta - \beta^2)\}$, where $\beta \geq 2, \eta \geq 3\beta/4$ are integers and $\eta$ divides $\beta^2(\beta^2 - 1)/2$. We remark that $\beta \eta$ is an even integer by [15, Corollary 3.2]. With respect to the graph $\text{AT4}(p, q, 2)$, we note that $p = 2\eta - \beta$ and $q = \beta$. Using this and by (8.2), the eigenvalues of $\text{AT4}(p, q, 2)$ are given in terms of $\beta$ and $\eta$ as follows.

$$\begin{align*}
    \theta_0 &= \beta(2\eta + 2n\beta - \beta^2), \quad \theta_1 = 2\eta + 2n\beta - \beta^2, \quad \theta_2 = 2\eta, \quad \theta_3 = -\beta, \quad \theta_4 = -\beta^2.
\end{align*}$$

If $(\beta, \eta) = (2, 2)$ we have $\text{AT4}(2, 2, 2)$, and if $(\beta, \eta) = (2, 3)$ we have $\text{AT4}(4, 2, 2)$; these graphs are all examples known for $\beta = 2$. In addition, if $(\beta, \eta) = (4, 6)$ we have the Meixner1 graph $\text{AT4}(8, 4, 2)$, which is the only known example for $\beta > 2$ so far.
For the rest of this section, we denote by $\Gamma$ an antipodal tight graph $\text{AT}4(p,q,2)$ with vertex set $X$. Fix $x \in X$ and write $T = T(x)$ the Terwilliger algebra of $\Gamma$ and $E_i^* = E_i^*(x)$ the $i$th dual primitive idempotent of $\Gamma$ for $0 \leq i \leq 4$. We recall a classification of all thin irreducible $T$-modules of $\Gamma$ with endpoint 1.

**Lemma 8.2.** Let $W$ denote a thin irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{p, -q\}$. Then $W$ has diameter 2. If $W'$ is an irreducible $T$-module, then $W$ and $W'$ are isomorphic as $T$-modules if and only if $W'$ is thin with endpoint 1 and local eigenvalue $\lambda$.

*Proof.* Apply Lemma 6.6 and Lemma 6.7.

Next, we discuss irreducible $T$-modules of $\Gamma$ with endpoint 2.

**Lemma 8.3.** If $W$ is an irreducible $T$-module with endpoint 2, then $W$ has diameter 0 and is thin.

*Proof.* Let $d$ denote the diameter of $W$. Observe that the possible value for $d$ is 0, 1, or 2. If $d = 2$, then $\dim(W) = 3$ and $E_2^* W = E_0^*(\hat{x}) W \neq 0$. Take a nonzero vector $w \in E_0^*(\hat{x}) W$. Since $W$ is a $T$-module, we have $Mw \subseteq W$. Observe that $Mw$ is the primary $T(\hat{x})$-module. It follows $\dim(Mw) = 5$, which is greater than $\dim(W) = 3$, a contradiction. If $d = 1$, then $E_3^* W = E_1^*(\hat{x}) W$ is nonzero. It follows that $W$ is an irreducible $T(\hat{x})$-module with endpoint 1. As $\Gamma$ is tight, by Lemma 6.4(iii) $W$ is thin and has local eigenvalue $\lambda \in \{p, -q\}$. By Lemma 8.2, we have $d = 2$, a contradiction. By these comments, we have $d = 0$. Since $W$ is an irreducible $T$-module, the dimension of $W$ is one. The result follows.

**Lemma 8.4.** Let $W$ be an irreducible $T$-module with endpoint 2. Let $\eta$ be an eigenvalue of $E_2^* AE_2^*$ with an eigenspace $W$. If $W'$ is another irreducible $T$-module with endpoint 2, then $W$ and $W'$ are isomorphic as $T$-modules if and only if $W'$ is an eigenspace associated with the eigenvalue $\eta$.

*Proof.* Let $\eta'$ denote an eigenvalue of $E_2^* AE_2^*$ with the eigenspace $W'$. By Lemma 8.3, $W$ and $W'$ are both one-dimensional. Let $\rho$ denote a vector space isomorphism from $W$ and $W'$. Apparently, $(E_2^* \rho - \rho E_2^*) W = 0$ for all $0 \leq i \leq 4$. Also, we find that $(A \rho - \rho A) W = 0$ if and only if $\eta = \eta'$. The result follows.

**Lemma 8.5.** Recall $\Gamma$ an antipodal tight graph $\text{AT}4(p,q,2)$. For each vertex $x \in X$ and for each $1 \leq i \leq 4$, the spectrum of $i$th subconstituent $\Delta_i(x)$ of $\Gamma$ is determined by the parameters $p, q$.

*Proof.* Fix a vertex $x$ in $\Gamma$. Since $\Gamma$ is tight, $\Delta_1(x) = \Delta_3(x)$ is strongly regular with the parameters $(q(pq + p + q), p(q + 1), 2p - q, p)$, and has the nontrivial eigenvalues $p, -q$. It suffices to show that the spectrum of the second subconstituent $\Delta_2 = \Delta_2(x)$ of $\Gamma$ is determined by $p$ and $q$. We first assume the Krein parameter $q_{44}^1 = 0$. By [13, Theorem 5.5], $\Delta_2$ is an antipodal distance-regular graph with diameter four, and intersection array and the spectrum of $\Delta_2$ are determined by $p$ and $q$. Now we assume $q_{44}^1 \neq 0$. As the graph $\Gamma$ is 1-homogeneous in the sense of Nomura (cf. [14, Theorem 11.7]), $\Delta_2$ is edge-regular. We claim that (i) $\Delta_2$ has at most 7 distinct eigenvalues; and (ii) each eigenvalue and its multiplicity are determined by $p$ and $q$.

First, we find that $a_2$ is the eigenvalue of $\Delta_2$ with multiplicity 1. Next, let $W$ denote a thin irreducible $T$-module with endpoint 1 and local eigenvalue $\lambda \in \{p, -q\}$. By Lemma 8.2, $W$ has
diameter 2. Take a nonzero vector $w_0 \in E_1^*W$. Set $w_i = E_i^*Aw_{i-1}$ for $i = 1, 2$. Then $\{w_i\}^2_{i=0}$ is a basis for $W$, and from (3.4) the action of $A$ on the basis $\{w_i\}^2_{i=0}$ is given by

$$Aw_i = w_{i+1} + a_i(W)w_i + x_i(W)w_{i-1}.$$  

Observe that $a_0(W) = \lambda$ and $a_0(W) = a_2(W) = \lambda$ by (6.1). Observe also that $a_1(W)$ is an eigenvalue of $E_2^*AE_2^*$, i.e., an eigenvalue of $\Delta_2$. By this and using (3.3) and Lemma 6.5, we have

$$a_1(W) = \theta_t + \theta_{t+1} + \theta_{t+2} - 2\lambda,$$

where $t = 1$ if $\lambda = p$ and $t = 2$ if $\lambda = -q$. From this and by (8.2), we find that $a_1(W)$ is determined by $p,q$. Since each $\lambda$ yields the scalar $a_1(W)$, multiplicities of $\lambda$ and $a_1(W)$ are equal to each other.

Finally, let $W$ denote a thin irreducible $T$-module with endpoint 1. By Lemma 8.3, $W$ is a one-dimensional eigenspace for $E_2^*AE_2^*$. It follows that the corresponding eigenvalue of $\Delta_2$ becomes an eigenvalue of $\Gamma$. From this, we find that $\Delta_2$ has four possible eigenvalues $\theta_1, \theta_2, \theta_3, \theta_4$. For each $1 \leq i \leq 4$, let $m_i$ denote the multiplicity of $\theta_i$. Abbreviate $B = E_2^*AE_2^*$. Since $\Delta_2$ is edge-regular and $\text{trace}(B^\ell) = \sum_{i=1}^4 \theta_i^\ell m_i$, $\ell = 0, 1, 2, 3$, we can solve this system of equations for $m_i$, $1 \leq i \leq 4$, where $m_i$ are determined by $p,q$. The result follows. 

We remark that the proof of Lemma 8.5 is motivated to give a new feasibility condition for an $AT_4(p,q)$; see [28]

**Proposition 8.6.** An antipodal tight graph $AT_4(p,q,2)$ is thin.

**Proof.** For any vertex $x$, an irreducible $T(x)$-module of $AT_4(p,q,2)$ with endpoint 1 is thin since $AT_4(p,q,2)$ is tight. In addition, every irreducible $T(x)$-module of $AT_4(p,q,2)$ with endpoint 2 is thin by Lemma 8.3. The result follows. 

Recall the standard $T$-module $V$. Since $T$ is semisimple and using Lemma 8.2 and Lemma 8.4, $V$ decomposes into an orthogonal direct sum of thin irreducible $T$-modules

$$V = W_0 \oplus \bigoplus_i W_i \oplus \bigoplus_j W'_j \oplus \bigoplus_h W''_h,$$

where we denote by

| $T$-module | dimension | endpoint | diameter | comment |
|------------|-----------|----------|----------|---------|
| $W_0$      | 5         | 0        | 4        | the primary module |
| $W_i$      | 3         | 1        | 2        | local eigenvalue $p$ |
| $W'_j$     | 3         | 1        | 2        | local eigenvalue $-q$ |
| $W''_h$    | 1         | 2        | 0        | eigenspace of $E_2^*AE_2^*$ |

**Theorem 8.7.** An antipodal tight graph $AT_4(p,q,2)$ is pseudo-vertex-transitive.
Proof. For a given vertex \( x \in X \), let \( T(x) \) be the Terwilliger algebra of \( AT4(p, q, 2) \). By Lemma 8.2 and Lemma 8.4, the isomorphism class of an irreducible \( T(x) \)-module is determined by the parameters \( p \) and \( q \). Moreover, by Lemma 8.5 together with (8.4), the multiplicities of irreducible \( T(x) \)-modules are determined by \( p \) and \( q \). Therefore, the isomorphism class of the standard \( T(x) \)-module (8.5) is completely determined by \( p \) and \( q \). The result follows. ■

We finish this section with a comment.

**Proposition 8.8.** For each \( x \in X \) the dimension of \( T(x) \) of \( AT4(p, q, 2) \) is given by

\[
\dim(T(x)) = \ell + 43,
\]

where \( \ell(\leq 7) \) is the number of distinct eigenvalues of \( \Delta_2(x) \) except for the eigenvalue \( a_2 \).

**Proof.** Applying Proposition 3.1 to (8.5), the algebra \( T(x) \) is isomorphic to

\[
\text{Mat}_5(\mathbb{C}) \oplus \text{Mat}_3(\mathbb{C}) \oplus \text{Mat}_3(\mathbb{C}) \oplus (\text{Mat}_1(\mathbb{C}))^{\ell},
\]

where \( \ell \) is the number of distinct eigenvalues of \( \Delta_2(x) \) except for the eigenvalue \( a_2 \). From (3.5), it follows that

\[
\dim(T(x)) = \ell(1)^2 + 3^2 + 3^2 + 5^2 = \ell + 43.
\]

\[\blacksquare\]

9 Concluding remarks

In this paper, we discussed pseudo-vertex transitivity of \( Q \)-polynomial distance-regular graphs with small diameter \( D \in \{2, 3, 4\} \). We briefly summarize the results. For the case \( D = 2 \), we showed that the local spectrum of a strongly regular graph characterizes its pseudo-vertex transitivity. In addition, we showed that two strongly regular graphs are \( T \)-isomorphic if and only if they have the same parameters and their local spectra are equal to each other. For the case \( D = 3 \), we were concerned with a Taylor graph \( \Gamma \) with intersection array \( \{k, b, 1; 1, b, k\} \). We showed that \( \Gamma \) is thin and that the isomorphism classes of the standard module of the Terwilliger algebra \( T(x) \) of \( \Gamma \) only depend on the parameters \( k, b \), and hence \( \Gamma \) is pseudo-vertex-transitive. For the case \( D = 4 \), we were concerned with an antipodal tight graph \( AT4(p, q, 2) \). We proved that \( AT4(p, q, 2) \) is thin and that the isomorphism classes of the standard module of Terwilliger algebra \( T(x) \) of \( AT4(p, q, 2) \) only depend on the parameters \( p, q \), and hence \( AT4(p, q, 2) \) is pseudo-vertex-transitive. In both cases, \( D = 3 \) and \( D = 4 \), the thin property of distance-regular antipodal double covers played an important role in determining their pseudo-vertex transitivity. We also noted that bipartite and/or dual bipartite \( Q \)-polynomial distance-regular graphs are thin and pseudo-vertex-transitive. We anticipate that the thinness of a distance-regular graph still plays a significant role in determining pseudo-vertex transitivity for \( D \geq 3 \). We give the following conjecture.

**Conjecture 9.1.** A thin \( Q \)-polynomial distance-regular graph with diameter \( D \geq 3 \) is pseudo-vertex-transitive.
As we saw in Problem 1.1, the Ito-Koolen problem asks to classify all thin pseudo-vertex-transitive $Q$-polynomial distance-regular graphs with large enough diameter. As a first step towards solving the Ito-Koolen problem, Tan et al. [22] found an upper bound of the intersection number $c_2$ and parameter $\alpha$, respectively, for a thin $Q$-polynomial distance-regular graph with classical parameters $(D, b, \alpha, \beta)$. Each upper bound is given by a function in $b$, where $b = b_1/(\theta_1 + 1)$ and where $\theta_1$ is the second largest eigenvalue of $\Gamma$. We give some suggestions for further research.

**Problem 9.2.** With reference to the result of [22], improve the upper bound on $c_2$.

**Problem 9.3.** Assume $\Gamma$ has the same parameters as a Grassmann graph $J_q(2D + 1, D)$ and that $\Gamma$ is thin. Prove or disprove that $\Gamma$ and $J_q(2D + 1, D)$ are $T$-isomorphic. We remark that Liang et al. [17] have proved this problem for large enough diameter $D$.

**Acknowledgements**

The authors would like to express many thanks to the anonymous referees for their corrections and valuable comments. J.-H. Lee would like to thank Paul Terwilliger for several valuable conversations and comments. J.H. Koolen is partially supported by the National Key R and D Program of China (No. 2020YFA0713100), the National Natural Science Foundation of China (No. 12071454), and the Anhui Initiative in Quantum Information Technologies (No. AHY150000). Y.-Y Tan is supported by the National Natural Science Foundation of China (No. 11801007, 12171002) and Natural Science Foundation of Anhui Province (No. 1808085MA17) and the foundation of Anhui Jianzhu University (No. 2018QD22).

**References**

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[2] S. Bang, T. Fujisaki and J. H. Koolen, The spectra of the local graphs of the twisted Grassmann graphs, *European J. Combin.* **30**(3) (2009) 638-654.

[3] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin-Cummings, Menlo Park, 1984.

[4] J. S. Caughman IV, The Terwilliger algebra of bipartite $P$- and $Q$- polynomial schemes, *Discrete Math.*, **196** (1999), 65–95.

[5] D. R. Cerzo, Structure of thin irreducible modules of a $Q$-polynomial distance-regular graph, *Linear Algebra Appl.* **433** (2010), 1573–1613.

[6] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.
[7] E. R. van Dam and J. H. Koolen, A new family of distance-regular graphs with unbounded
diameter, *Invent. Math.* **162** (2005) 189–193.

[8] E. R. van Dam, J. H. Koolen and H. Tanaka, Distance-regular graphs, *Electron. J. Combin.* (2016) #DS22.

[9] G. Dickie and P. Terwilliger, Dual bipartite $Q$-polynomial distance-regular graphs, *European J. Combin.* **17** (1996) 613–623.

[10] E. Egge, A generalization of the Terwilliger algebra, *J. Algebra* **233** (2000), 213–252.

[11] J. T. Go and P. Terwilliger, Tight distance-regular graphs and the subconstituent algebra, *European J. Combin.* **23** (2002), no. 7, 793–816.

[12] C. D. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, Berlin, 2001.

[13] A. Jurisić, AT4 family and 2-homogeneous graphs, *Discrete Math.* **264** (2003), 127–148.

[14] A. Jurisić, J. H. Koolen and P. Terwilliger, Tight distance-regular graphs, *J. Algebra Combin.* **12**(2) (2000) 163-197.

[15] A. Jurisić and J. H. Koolen, Nonexistence of some Antipodal Distance-regular Graphs of Diameter Four, *European J. Combin.* **21** (2000), no. 8, 1039–1046.

[16] A. Jurisić and J. H. Koolen, Krein parameters and antipodal tight graphs with diameter 3 and 4, *Discrete Math.*, **244**(1–3) (2002) 181–202.

[17] X. Liang, Y.-Y Tan and J. H. Koolen, Thin distance-regular graphs with classical parameters $(D, q, q, \frac{d^4-1}{q^2-1})$ with $t > D$ are the Grassmann graphs, *Electron. J. Combin.* (2021), #P4.45.

[18] A. A. Pascasio, Tight distance-regular graphs and the $Q$-polynomial property, *Graphs Combin.* **17** (2001), 149–169.

[19] A. A. Pascasio, An inequality on the cosines of a tight distance-regular graph, *Linear Algebra Appl.* **325** (2001), 147–159.

[20] S. E. Payne and J. A. Thas, *Finite generalized quadrangles: Second Edition*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2009.

[21] E. Spence, Regular Two-graphs, [http://www.maths.gla.ac.uk/~es/twograph/reg2Graph.php](http://www.maths.gla.ac.uk/~es/twograph/reg2Graph.php)

[22] Y.-Y Tan, J. H. Koolen, M.-Y Cao and J. Park, Thin $Q$-polynomial distance-regular graphs have bounded $c_2$, *preprint*.

[23] H. Tanaka and T. Wang, The Terwilliger algebra of the twisted Grassmann graph: the thin case, *Electron. J. Combin.* **28** (2020) #P4.15 (22pp.)

[24] P. Terwilliger, Balanced sets and $Q$-polynomial association schemes, *Graphs Combin.* **4** (1988) 87–94.
[25] P. Terwilliger, The subconstituent algebra of an association scheme I, *J. Algebra Combin.* 1(4)(1992) 363–388.

[26] P. Terwilliger, The subconstituent algebra of an association scheme III, *J. Algebra Combin.* 2(2)(1993) 177–210.

[27] M. Tomiyama and N. Yamazaki, The subconstituent algebra of a strongly regular graph, *Kyushu J. Math.* 48(2)(1994) 323–334.

[28] Z.-J. Xia, J.-H. Lee and J. H. Koolen, A new feasible condition for the AT4 family, *submitted.*