EXPONENTIAL ENERGY DECAY FOR KERR–DE SITTER
BLACK HOLES BEYOND EVENT HORIZONS

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ABSTRACT. We establish an exponential decay estimate for linear waves on the Kerr–de Sitter slowly rotating black hole. Combining the cutoff resolvent estimate of [10] with the red-shift effect and a parametrix near the event horizons, we obtain exponential decay on the whole domain of outer communications.

We study the decay of linear waves on the Kerr–de Sitter metric, corresponding to a rotating black hole in a spacetime with positive cosmological constant. (See, for example, [6] for the motivation for the problem and a survey of recent results.) Although in the original coordinates \((t, r, \theta, \phi)\) the metric is only defined on \(M = \{r_- < r < r_+\}\) and becomes singular on the event horizons \(\{r = r_\pm\}\), we will use a different coordinate system \((t^+, r, \theta, \phi^+)\), in which the metric can be extended beyond the event horizons to \(M_\delta = \{r^- - \delta < r < r^+ + \delta\}\). In particular, \(t^+ \sim t + C_\pm \ln |r - r_\pm|\) near \(r = r_\pm\), with \(C_\pm\) positive constants; see (1.4) for precise formulas and Section 1.2 in general for the description of the metric. The speed of rotation of the black hole is described by the parameter \(a\); for \(a = 0\), we get the spherically symmetric Schwarzschild–de Sitter metric. We establish the following

**Theorem 0.1.** Let \(\Box_g\) be the d’Alembert–Beltrami operator of the Kerr–de Sitter metric on \(M_\delta\). Fix \(\varkappa > 0\). For \(a, \delta > 0, \nu > 0\) small enough and any \(s \geq 1\), there exists a constant \(C\) such that if \(u \in H^{s+\varkappa+1}_\text{loc}(M_\delta)\) is a solution to the equation \(\Box_g u = f \in H^{s+\varkappa}_\text{loc}(M_\delta)\), with \(\text{supp } u \subset \{t^+ > -T\}\) for some \(T\), then

\[
\|e^{\nu t}(u - \Pi_0 f)\|_{H^{s}(M_\delta)} \leq C\|e^{\nu t} f\|_{H^{s+\varkappa}(M_\delta)}.
\]

Here

\[
\Pi_0 f = \frac{1 + \alpha}{4\pi(r^4_+ + r^4_- + 2a^2)} \int_M f \, d\text{Vol}
\]

is a constant (note that we integrate over \(M\), not the whole \(M_\delta\); \(H^s\) norms are taken with respect to the \((t^+, r, \theta, \phi^+)\) coordinates (here \((\theta, \phi^+)\) are treated as spherical coordinates on \(S^2\)).

The main ingredient of the proof, which gives us exponential decay, is the scattering resolvent constructed in [10]. We modify the argument of [10, Theorem 6] to get exponential decay for \(u\) on a certain compact subset \(K_\delta \subset M\), under the condition that \(f\) is supported in \(K_\delta\) as well (Proposition 2.1). In the present paper, we specify \(\Box_g u\), rather than the Cauchy data of \(u\), to facilitate the proofs. However, it is not hard to convert Theorem 0.1 to an exponential decay estimate for the Cauchy problem, such as the one given in [10].

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The second ingredient, described in Section 1.3, is the energy estimate produced by
the red-shift effect at the event horizons, first introduced by Dafermos and Rodnianski
in [4, 7, 8]. Paper [4] in particular introduced the idea that the red-shift implies that
boundedness and decay properties propagate from the event horizon to a neighbor-
hood of it in the black hole interior. The vector field approach to the red-shift effect
was introduced in [7]; in [8], the method was extended to higher-order estimates using
the remark that commutation generated further terms of favorable sign. Paper [7]
established red-shift estimates for the Schwarzschild black hole, while [8] considered
the case of slowly rotating Kerr; the (subextremal) Kerr–de Sitter horizons are in par-
ticular covered by [6, Theorems 7.1 and 7.2]. It should also be noted that in certain
cases, such as extremal Reissner–Nordström spacetimes considered by Aretakis [1, 2],
the failure of the red-shift is directly related to instabilities of linear waves at the
event horizon.

In our presentation, we follow both [6, Section 3.3] and the paper [17] by Tataru
and Tohaneanu on integrated decay for the Kerr black hole. Combining the red-shift
effect with Proposition 2.1, we obtain an estimate on \( u \) on the whole \( M_\delta \), provided
that \( f \) is still supported in \( K_\delta \). Finally, we use a Morawetz-type argument together
with red-shift (Proposition 2.3) to construct an exponentially decaying parametrix
for the wave equation near the event horizons and reduce the general problem to the
case \( \text{supp} f \subset K_\delta \).

Compared to the energy estimate for the Minkowski spacetime (Proposition 1.4),
we lose \( 1 + \kappa \) derivatives in Theorem 1, where \( \kappa > 0 \) can be arbitrarily small. This
is related to the exponent in the polynomial resolvent estimate of [10], which in turn
is determined by the separation of variables procedure employed there [10, Proposi-
tion 3.4]. It is possible that a more careful analysis will yield a smaller loss in deriva-
tives; however, the presence of trapping indicates that loss of regularity is inevitable
(see [15] for a precise statement in the now classical case of obstacle scattering).

Exponential decay of linear waves on the Schwarzschild–de Sitter metric has been
studied in [3, 5, 14]. Dafermos and Rodnianski [5], using vector field multipliers,
proved that linear waves decay faster than every negative power of \( t_+ \). Bony and
Häfner [3], building on earlier work on the scattering resolvent by Sá Barreto and
Zworski [16], showed exponential decay away from the event horizons. Finally, Melrose
et al. [14] proved exponential decay up to the event horizons. The latter result,
combined with the recent work on normally hyperbolic trapping [19] and gluing semi-
classical resolvent estimates [9], can be applied to certain short-range stationary per-
turbations of the Schwarzschild–de Sitter spacetime; see [9, Corollary 6.1]. It should
be noted, however, that Kerr–de Sitter is not an acceptable perturbation, in particu-
lar because the theorem of Mazzeo and Melrose [13] does not apply to the low-energy
situation anymore. Therefore, at the moment, the results of [10] seem necessary for
obtaining exponential decay of waves on Kerr–de Sitter.

1. Kerr–de Sitter metric and the red-shift effect

1.1. Energy estimates. We recall some well-known facts from Lorentzian geo-
metry; see, for example, [6, Appendices] or [18, Section 2.8] for a more detailed account.

Let \( M \) be an \( n \)-dimensional smooth manifold and \( g \) be a Lorentzian metric; that is,
a symmetric \((0, 2)\)-tensor \( g \) of signature \((1, n - 1)\). (Sometimes a different convention
is used, in which the metric has signature \((n - 1, 1)\). The basic example is the space \(\mathbb{R}_{t,x}^n\) with the Minkowski metric

\[ dt^2 - \sum_{j=1}^{n-1} dx_j^2. \]

A tangent vector \(X\) is called timelike if \(g(X, X) > 0\), null if \(g(X, X) = 0\), and spacelike if \(g(X, X) > 0\). If \(X\) and \(Y\) are two timelike vectors, then we say that they point in the same direction if \(g(X, Y) > 0\) and they point in opposite directions if \(g(X, Y) < 0\). This definition can be extended to cases when \(X\) and/or \(Y\) is a nonzero null vector.

We now describe a way of obtaining energy estimates for the wave equation on Lorentzian manifolds. Let \(\Omega \subset M\) be a bounded domain and \(u \in C^\infty(\bar{\Omega})\). Define the symmetric \((0, 2)\)-tensor \(T_{\nabla u}\) by the formula

\[
T_{\nabla u}(X, Y) = (Xu)(Yu) - \frac{1}{2} g(\nabla u, \nabla u) g(X, Y),
\]

valid for all vector fields \(X, Y\) on \(\Omega\). Note that for fixed \(X\) and \(Y\), \(T_{\nabla u}(X, Y)\) is a quadratic form in \(\nabla u\). If \(X\) and \(Y\) are both timelike, then this form is positive definite in \(\nabla u\) for \(X\) and \(Y\) pointing in the same direction and negative definite otherwise. Same is true if \(X\) and/or \(Y\) is null, with the form being nonnegative or nonpositive, respectively.

Fix a vector field \(X\) on \(\Omega\) and consider the vector field \(J_X(u)\), given by the formula

\[ g(J_X(u), Y) = T_{\nabla u}(X, Y), \]

valid for all vector fields \(Y\). The divergence theorem then gives

\[
\int_{\partial \Omega} T_{\nabla u}(X, \vec{n}) \, dS = \int_{\Omega} \operatorname{div} J_X(u) \, d\operatorname{Vol}. \tag{1.2}
\]

Here \(\vec{n}\) is the unit normal vector pointing outward (in the sense that \(g(\vec{n}, Z) > 0\) for every vector \(Z\) pointing outside of \(\bar{\Omega}\)); \(dS\) is the area measure induced by the restriction of \(g\) to \(\partial \Omega\), and \(d\operatorname{Vol}\) is the volume measure induced by \(g\). One has to take care when defining the left-hand side of (1.2) at the points where \(\partial \Omega\) is null, as \(\vec{n}\) blows up, being both unit and null, and \(dS\) is equal to zero; see [6, Appendix C] for details. The discussion following (1.1) implies

**Proposition 1.1.** Let \(C\) be an open subset of \(\partial \Omega\) whose tangent space is either spacelike or null at every point. Moreover, assume that \(X\) is timelike and points outside of \(\Omega\) on \(C\). Then for every \(u\),

\[
\int_C T_{\nabla u}(X, \vec{n}) \, dS \geq 0. \tag{1.3}
\]

The sign of the flux of \(J_X\) over a timelike piece of \(\partial \Omega\) cannot be determined in general; however, we can find it if \(u\) satisfies a boundary condition:

**Proposition 1.2.** Let \(C\) be an open timelike subset of \(\partial \Omega\) and assume that \(u|_C = 0\). If \(X\) points inside of \(\Omega\) on \(C\), then (1.3) holds.
Proof. We have \( \nabla u = v \vec{n} \) on \( C \), for some function \( v \). Then
\[
T_{\nabla u}(X, \vec{n}) = \frac{v^2}{2} g(X, \vec{n})g(\vec{n}, \vec{n}) = -\frac{v^2}{2} g(X, \vec{n}) \geq 0.
\]
\[ \square \]

Finally, we relate the divergence of \( J_X \) to the d’Alembert–Beltrami operator \( \square_g \):

**Proposition 1.3.** Let \( \mathcal{L}_X g \) be the Lie derivative of \( g \) with respect to \( X \), and consider the symmetric \((0,2)\)-tensor \( K^X \) given by
\[
K^X = \frac{1}{2} \mathcal{L}_X g - \frac{1}{4} \text{Tr}(g^{-1} \mathcal{L}_X g) g.
\]
Then
\[
\text{div} J_X(u) = (Xu) \square_g u + K^X(\nabla u, \nabla u).
\]

As a basic application, we prove the energy estimate for the constant-coefficient wave equation:

**Proposition 1.4.** Fix \( 0 < T < R \) and consider the domain
\[
\Omega = \{ 0 < t < T, \ |x| < R - t \}
\]
in the Minkowski spacetime. Let \( u \in C^\infty(\bar{\Omega}) \) and define the energy
\[
E(s) = \frac{1}{2} \int_{\{ t = s \}} \frac{1}{|x| < R - s} |u_t|^2 + |\partial_x u|^2 \, dx, \quad 0 \leq s \leq T.
\]
Then
\[
E(T) \leq E(0) + \int_{\Omega} u_t \square_g u \, dt \, dx.
\]

Proof. We take \( X = \partial_t \) and apply (1.2) on \( \Omega \). Since \( X \) is Killing, \( K^X = 0 \) and thus
\[
\int_{\partial \Omega} T_{\nabla u}(X, \vec{n}) \, dS = \int_{\Omega} u_t \square_g u \, dt \, dx.
\]
Now, the boundary of \( \Omega \) consists of the following pieces:
\[
\mathcal{P}_0 = \{ t = 0, \ |x| < R \}, \quad \mathcal{P}_T = \{ t = T, \ |x| < R - T \}, \quad \mathcal{C} = \{ 0 \leq t \leq T, \ |x| = R - t \}.
\]
The integral over \( \mathcal{P}_0 \) is equal to \(-E(0)\) and the integral over \( \mathcal{P}_T \) is equal to \( E(T) \). Finally, the integral over \( \mathcal{C} \) is nonnegative by Proposition 1.1, as \( \mathcal{C} \) is null and \( \partial_t \) points outside of \( \Omega \) on \( \mathcal{C} \). \[ \square \]

1.2. Kerr–de Sitter metric. The Kerr–de Sitter metric is given by
\[
g = -\rho^2 \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) - \frac{\Delta_\theta \sin^2 \theta}{(1 + \alpha)^2 \rho^2} \left( a \, dt - \left( r^2 + a^2 \right) d\varphi \right)^2 + \frac{\Delta_r}{(1 + \alpha)^2 \rho^2} \left( dt - a \sin^2 \theta \, d\varphi \right)^2.
\]
Here $M_0$ is the mass of the black hole, $\Lambda$ is the cosmological constant (both of which we assume to be fixed), and $aM_0$ is the angular momentum (which we assume to be small);

$$\Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda r^2}{3}\right) - 2M_0r,$$

$$\Delta_\theta = 1 + \alpha \cos^2 \theta,$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \alpha = \frac{\Lambda a^2}{3}.$$

The metric in the $(t, r, \theta, \varphi)$ coordinates is defined for $\Delta_r > 0$; we assume that $r_\pm$ are two positive roots of the equation $\Delta_r = 0$, such that $\Delta_r > 0$ on the open interval $0 < r_- < r < r_+ < \infty$. The variables $\theta \in [0, \pi]$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ are the spherical coordinates on the sphere $\mathbb{S}^2$. The spacetime is then

$$M = \mathbb{R}_t \times (r_-, r_+) \times \mathbb{S}^2_{\theta, \varphi}.$$

(Note the difference in notation with [10].) The volume form is

$$d\text{Vol} = \frac{\rho^2 \sin \theta}{(1 + \alpha)^2} dt dr d\theta d\varphi.$$

For $a = 0$, we get the Schwarzschild–de Sitter metric:

$$g_0 = -\frac{r^2}{\Delta_r} dr^2 + \frac{\Delta_r}{r^2} dt^2 - r^2 g_S,$$

where

$$g_S = d\theta^2 + \sin^2 \theta d\varphi^2$$

is the round metric on the sphere of radius 1.

Next, we introduce a modification of the Kerr-star coordinates (see [6, Section 5.1]). We remove the singularities at $r = r_\pm$ by making the change of variables $(t, r, \theta, \varphi) \rightarrow (t_+, r, \theta, \varphi_+)$, where

$$(1.4) \quad t_+ = t - F_t(r), \quad \varphi_+ = \varphi - F_\varphi(r).$$

Note that $\partial_{t_+} = \partial_t$ and $\partial_{\varphi_+} = \partial_\varphi$. The functions $F_t$ and $F_\varphi$ are required to be smooth on $(r_-, r_+)$ and satisfy the following condition:

$$F'_t(r) = \pm \frac{(1 + \alpha)(r^2 + a^2)}{\Delta_r}, \quad F'_\varphi(r) = \pm \frac{(1 + \alpha) a}{\Delta_r} \text{ for } |r - r_\pm| < \varepsilon.$$  

Here $\varepsilon > 0$ is some fixed small constant. The metric $g$ in the $(t_+, r, \theta, \varphi_+)$ coordinates takes the following form for $|r - r_\pm| < \delta$:

$$-\rho^2 \frac{d\theta^2}{\Delta_\theta} - \frac{\Delta_\theta \sin^2 \theta}{(1 + \alpha)^2 \rho^2} (a dt_+ - (r^2 + a^2) d\varphi_+)^2$$

$$+ \frac{\Delta_r}{(1 + \alpha)^2 \rho^2} (dt_+ - a \sin^2 \theta d\varphi_+)^2 \pm \frac{2}{(1 + \alpha)} (dt_+ - a \sin^2 \theta d\varphi_+) dr.$$
We see that the metric is smooth up to the event horizons \( \{ r = r_\pm \} \); moreover, for \( \delta \) small enough, we can extend it to

\[
M_\delta = \{ r_- - \delta \leq r \leq r_+ + \delta \}.
\]

The event horizons are null, while the surfaces \( \{ r = r_0 \} \) are spacelike for \( r_0 \not\in [r_-, r_+] \).

The time surfaces \( \{ t = \text{const} \} \) are null near the event horizons; however, one can shift the time variable a little bit (see [10, Section 1]) to make the problem

\[
\Box_g u = f \in C^\infty_0(M_\delta), \quad \text{supp } u \subset \{ t > -T \}
\]

well posed. We call \( u \) the forward solution of the equation \( \Box_g u = f \).

Finally, note that the field \( \partial_t \) (which is the same in the \( (t, r, \theta, \varphi) \) and \( (t_+, r, \theta, \varphi_+) \) coordinates) is not timelike on \( M \) inside the two surfaces located \( O(a) \)-close (in the \( r \) variable) to the event horizons; these surfaces are called the ergospheres.

### 1.3. Red-shift effect.

In this section, we prove the following energy estimate:

**Proposition 1.5.** For \( \delta > 0 \), define

\[
K_\delta = \{ r_- + \delta < r < r_+ - \delta \} \subset M.
\]

Then for \( \delta, a, \) and \( \nu > 0 \) small enough, \( s \) a nonnegative integer, and every forward solution \( u \) to the equation \( \Box_g u = f \in C^\infty_0(M_\delta) \), we have\(^1\)

\[
\| e^{\nu t} u \|_{H^{s+1}(M_\delta)} \lesssim \| e^{\nu t} f \|_{H^s(M_\delta)} + \| e^{\nu t} u \|_{H^{s+1}(K_\delta)}.
\]

We start the proof with the construction of a special vector field; see also [6, Proposition 3.3.1].

**Proposition 1.6.** For \( \delta > 0 \) and a small enough, there exists a vector field \( X \) defined on \( M_\delta \setminus K_{2\delta} \), with the following properties:

- \( X \) is stationary; that is, \( [X, \partial_t] = 0 \).
- \( X \) is timelike and \( X t_+ > 0 \), \( \pm Xr > 0 \) on \( M_\delta \setminus K_{2\delta} \).
- The tensor \( K^X \), defined in Proposition 1.3, is negative definite on \( M_\delta \setminus K_{2\delta} \).

**Proof.** We will construct \( X \) for \( a = 0 \); same field will work for small \( a \) since the components of the Kerr–de Sitter metric near the event horizons are continuous functions of \( a \). Moreover, since \( \delta > 0 \) can be chosen arbitrarily small, we only need to verify properties of \( X \) at the event horizons. We use the \( (t_+, r, \theta, \varphi_+) \) coordinates. The metric for \( a = 0 \) has the form

\[
\Delta_r \left( \frac{r^2}{r^2} dt_+^2 + 2dt_+ dr - r^2 g_S \right)
\]

for \( |r - r_+| < 2\delta \); if we take

\[
X = X_r(r) \partial_r + X_t(r) \partial_t,
\]

\(^1\)We write \( (A) \lesssim (B) \), if there exists some constant \( C \), independent of the choice of \( f \), such that \( A \leq C(B) \). Here \( C \) might depend on the parameters of the problem such as \( \nu, s, \) and \( \kappa \).
where $X_r, X_t$ are some functions, then at $r = r_\pm$,

$$L_X g = X_r \left( \frac{\Delta r}{r^2} dt^2 - 2r g_s \right) \pm 2[\partial_r X_t \, dr^2 + \partial_r X_r \, dr \, dt_+],$$

$$K^X = \frac{X_r \Delta r}{2r^2} dt^2_+ \pm \partial_r X_t \, dr^2 \mp \frac{2X_r}{r} \, dr \, dt_+ + \frac{r^2}{2} \partial_r X_r g_s.$$  

We put $X_t = 1$ and $X_r = \pm 1$ at $r = r_\pm$; then the field $X$ is timelike and $dt_+(X) > 0$. To make $K^X$ negative definite, it then suffices to take $\mp \partial_r X_t$ positive and large enough and $\partial_r X_r$ negative at the event horizons. □

**Remark.** Note that the only component of $K^X$ whose sign is definite independently of the choice of $\partial_r X$ is

$$K^X(\partial_t, \partial_t) = \frac{1}{2}(L_X g)_{t_+ t_+} = -g(X, \nabla_{\partial_t} \partial_t).$$

One can compute

(1.6) $\nabla_{\partial_t} \partial_t = \kappa \partial_t$

for some constant $\kappa > 0$; then,

$$K^X(\partial_t, \partial_t) = \mp \kappa X_r, \, r = r_\pm.$$  

Equation (1.6) can be interpreted as follows: the momentum is exponentially decaying as a function of the geodesic parameter on the family of trapped geodesics $\{r = r_\pm, (\theta, \varphi) = \text{const}\}$. This is related to the classical red-shift effect; see [6, Sections 3.3.2 and 7.1] for more details.

We are now ready to prove Proposition 1.5. To facilitate the inductive argument for estimating higher derivatives, we show the following more general fact:

**Proposition 1.7.** Assume that $\psi(r)$ is a function on $M_\delta$ such that $\psi \geq 0$ outside of $K_\delta$, and $u$ is a forward solution to the equation

$$(\Box g + \psi X) u = f \in C^0(M_\delta).$$

Here $X$ is the field constructed in Proposition 1.6. Then for $a, \delta > 0, \nu > 0$ small enough and each nonnegative integer $s$,

(1.7) $\|e^{\nu t} u\|_{H^{s+1}(M_\delta)} \lesssim \|e^{\nu t} f\|_{H^s(M_\delta)} + \|e^{\nu t} u\|_{H^{s+1}(K_\delta)}.$

**Proof.** We use induction on $s$. First, assume that $s = 0$. Take a nonnegative function $\chi(r)$ on $M_\delta$ such that $\chi = 0$ near $K_{2\delta}$, but $\chi = 1$ away from $K_\delta$. Let $T > 0$ and apply the divergence theorem in the region

$$\Omega_T = M_\delta \cap \{t_+ < T\}$$

to the vector field

$$V = e^{2\nu t_+} \chi J_X(u).$$

Here $J_X$ is defined in Section 1.1. (The divergence theorem holds, despite $\Omega_T$ being noncompact, since $u$ is a forward solution.) We compute by Proposition 1.3

$$\text{div } V = e^{2\nu t_+} [2\nu \chi dt_+(J_X(u)) + e^{2\nu t_+} \chi' dr(J_X(u)) + \chi(Xu)f - \chi \psi(Xu)^2 + \chi K^X(\nabla u, \nabla u)].$$
The flux of $V$ is nonnegative by Proposition 1.1; therefore, by letting $T \to +\infty$ we get
\[
\int_{M^3 \setminus K^3} -e^{2\nu t + 1} K^X (\nabla u, \nabla u) \, d\text{Vol} \lesssim \nu \|e^{\nu t + 1} u\|_{H^1(M^3)}^2 + \|e^{\nu t + 1} u\|_{H^1(K^3)}^2 + \|e^{\nu t + 1} u\|_{H^1(M^3)} \cdot \|f\|_{L^2(M^3)} \|
\]
Since $K^X$ is negative definite on $M^3 \setminus K^3$ and by Poincaré inequality, we have for $\nu$ small enough,
\[
\|e^{\nu t + 1} u\|_{H^1(M^3)}^2 \lesssim \|e^{\nu t + 1} u\|_{H^1(K^3)}^2 + \|e^{\nu t + 1} u\|_{H^1(M^3)} \cdot \|f\|_{L^2(M^3)} \|
\]
This finishes the proof of (1.7) for $s = 0$.

Now, assume that $s \geq 1$ and (1.7) is true for $s = 1$; we will prove it for $s$ following [8, Sections 1.7.5 and 10] (see also [17, Theorem 4.4]). First, let $Y$ be equal to either $\partial_t$ or a Killing field on $S^2$; then $[\psi X, Y] = 0$ and, since the metric is spherically symmetric for $a = 0$, $[\Box_g, Y]$ is a second-order differential operator with $O(a)$ coefficients. We have
\[(\Box_g + \psi X) Y u = Yf + [\Box_g, Y] u;\]
therefore, by (1.7),
\[
\|e^{\nu t + 1} Y u\|_{H^s(M^3)} \lesssim \|e^{\nu t + 1} Y f\|_{H^{s-1}(M^3)} + O(a) \|e^{\nu t + 1} u\|_{H^{s+1}(M^3)} + \|e^{\nu t + 1} Y u\|_{H^s(K^3)}.
\]
Therefore, if $\hat{\partial}_u$ is composed of derivatives of $u$ with respect to $t_+, \theta, \varphi_+$, then
\[(1.8) \quad \|e^{\nu t + 1} \hat{\partial}_u\|_{H^s(M^3)} \lesssim \|e^{\nu t + 1} f\|_{H^s(M^3)} + O(a) \|e^{\nu t + 1} u\|_{H^{s+1}(M^3)} + \|e^{\nu t + 1} u\|_{H^{s+1}(K^3)}.
\]
Now, we estimate $\partial_r u$. We can write
\[[\Box_g + \psi X, \partial_r] = -\eta X \partial_r + L,
\]
where $L$ is a second-order differential operator not containing any $\partial^2_r$ terms and $\eta$ is positive near the event horizons. Then
\[
\|e^{\nu t + 1} L u\|_{H^{s-1}(M^3)} \lesssim \|e^{\nu t + 1} \hat{\partial}_u\|_{H^s(M^3)} + \|e^{\nu t + 1} u\|_{H^s(M^3)}.
\]
We have
\[(\Box_g + (\psi + \eta) X) \partial_r u = \partial_r f + Lu;\]
since $\psi + \eta \geq 0$ near the event horizons, by (1.7) applied to $\partial_r u$ and (1.8) we get
\[
\|e^{\nu t + 1} u\|_{H^{s+1}(M^3)} \lesssim \|e^{\nu t + 1} \partial_r u\|_{H^s(M^3)} + \|e^{\nu t + 1} \hat{\partial}_u\|_{H^s(M^3)} + \|e^{\nu t + 1} f\|_{H^s(M^3)} + \|e^{\nu t + 1} u\|_{H^{s+1}(K^3)} + \|e^{\nu t + 1} \hat{\partial}_u\|_{H^s(M^3)} + \|e^{\nu t + 1} u\|_{H^{s+1}(M^3)};
\]
it remains to take $a$ small enough.
2. Proof of exponential decay

Throughout this section, $u$ is a forward solution to the equation $\Box_g u = f$, with $f \in C_0^\infty(M_\delta)$. (The estimates for general $f$ can then be obtained by a density argument.)

2.1. Case of $f$ supported in $K_\delta$. First of all, we use the resolvent estimates of [10] to obtain exponential decay away from the event horizons:

**Proposition 2.1.** Fix $\delta > 0$, $\kappa > 0$ and assume that $\chi(r) \in C_0^\infty(r_- + \delta, r_+ - \delta)$. Then for a small enough and every $s \geq 0$, we have

$$\|e^{\nu t}\chi(r)(u - \Pi_0 f)\|_{H^s} \lesssim \|e^{\nu t} f\|_{H^{s+\kappa}}$$

for every $f \in C_0^\infty(K_\delta)$. (We can use $e^{\nu t}$ in place of $e^{\nu t + s}$, as $|t - t_+|$ is bounded and the two weights are equivalent in $K_\delta$.)

**Proof.** We use the argument of [10, Theorem 6]. By [10, Proposition 1.1], $e^{-Ct}u$ is tempered in the time variable for some constant $C$; therefore, the Fourier–Laplace transform

$$\hat{u}(\omega, \cdot) = \int_{-\infty}^{\infty} e^{it\omega} u(t, \cdot) \, dt$$

is well defined and holomorphic in $\{\text{Im } \omega > C\}$. Let $K_S = (r_- + \delta, r_+ - \delta) \times S^2$ be the space slice of $K_\delta$. We choose a small enough so that [10, Theorem 2] provides us with the scattering resolvent $R_g(\omega) : L^2(K_S) \to L^2(K_S)$; it is a family of operators meromorphic in the entire complex plane. By [10, Proposition 1.2],

$$\chi(r)\hat{u}(\omega) = \chi(r)R_g(\omega)(\rho^2 \hat{f}(\omega)),$$

where $\rho(r, \theta)$ is the smooth function defined in Section 1.2 and $\hat{f}(\omega)$ is an entire function that is rapidly decaying in $\omega$ for $\text{Im } \omega$ bounded, with values in $C_0^\infty(K_S)$. Now, there exists $\nu > 0$ such that $R_g(\omega)$ is holomorphic and polynomially bounded in $\{\text{Im } \omega \geq -\nu\}$, except for a pole at zero [10, Theorems 4 and 5]. Therefore, we can use Fourier inversion formula and contour deformation to get

$$\chi(r)(u(t, \cdot) - \Pi_0 f) = \frac{1}{2\pi} \int_{\text{Im } \omega = -\nu} e^{-it\omega} \chi(r)R_g(\omega)(\rho^2 \hat{f}(\omega)) \, d\omega,$$

(2.1)

the residue at zero being exactly $\Pi_0 f$. Now, let $s \in \mathbb{R}$, put $h = \langle \omega \rangle^{-1}$, and introduce the semiclassical Sobolev space $H^s_{h,\text{comp}}(K_S) \subset \mathcal{E}'(K_S)$; the norm of $v \in \mathcal{E}'(K_S)$ in this space is given by $\|\langle h D\rangle^s v\|_{L^2}$, where $\langle h D \rangle^s$ is a Fourier multiplier and $v$ is extended by zero to $\mathbb{R}^3 \supset K_S$. Then the norm of $e^{\nu t} f$ in $H^s(K_\delta)$ is equivalent to the norm of $\langle \omega \rangle^s \hat{f}$ in $L^2_h H^s_h$, where

$$\|v\|^2_{L^2_h H^s_h} = \int_{\text{Im } \omega = -\nu} \|v(\omega)\|^2_{H^s_h} \, d\omega.$$

This, together with the resolvent estimate of [10, Theorem 5] gives for $\kappa$ fixed and $\nu$ small enough,

$$\|\langle \omega \rangle^s \chi(r)R_g(\omega)(\rho^2 \hat{f}(\omega))\|_{L^2_h H^s_h} \lesssim \|e^{\nu t} f\|_{H^{s+\kappa}}.$$
Now, we use that $R_g(\omega)$ is a right inverse to the second-order differential operator [10, Section 1]

$$P_g(\omega) = D_r(D_r D_r) - \frac{(1 + \alpha)^2}{\Delta r}((r^2 + a^2)\omega - a D_\phi)^2 + \frac{1}{\sin \theta} D_\theta(\Delta \theta \sin \theta D_\theta) + \frac{(1 + \alpha)^2}{\Delta \theta \sin^2 \theta}(a \omega \sin^2 \theta - D_\phi)^2;$$

then $h^2P_g(\omega)$ is a semiclassical pseudodifferential operator and for a small enough, it is elliptic on $K_S$ outside of some $\omega$-independent compact set. (This is equivalent to saying that $K_\delta$ does not intersect the ergosphere.) We can construct a semiclassical parametrix (see, for example, [11, Section 4.5] or [10, Proposition 5.1]); i.e., a properly supported semiclassical pseudodifferential operator on classical parametrix (see, for example, [11, Section 4.5] or [10, Proposition 5.1]); i.e., a properly supported semiclassical pseudodifferential operator on $Q$ on $K_S$ that maps $H^s_{h, \text{loc}}(K_S) \to H^{s+2}_{h, \text{loc}}(K_S)$ for all $s$ and such that $I - Q h^2 P_g(\omega)$ maps $H^{-N}_{h, \text{loc}}(K_S) \to H^N_{h, \text{loc}}(K_S)$ with norm $O(1)$ for all $N$. Then for any $\chi_1(r) \in C_0^\infty((r-, r_+))$ that is nonzero near supp $\chi$ and any $v \in C_0^\infty(K_S)$, we can apply $I - Q h^2 P_g(\omega)$ to $R_g(\omega)v$ to get

$$\|\chi(r)R_g(\omega)v\|_{H^s_h} \lesssim \langle \omega \rangle^{-2} \|v\|_{H^{s-2}_h} + \|\chi_1(r)R_g(\omega)v\|_{H^0_h}. \tag{2.2}$$

Therefore,

$$\|\langle \omega \rangle^s \chi(r)R_g(\omega)(\rho^2 \hat{f}(\omega))\|_{L^2_t H^s_h} \lesssim \|e^{\nu t}f\|_{H^{s+\kappa}};$$

it remains to combine this with (2.1).

Combining the above fact with the red-shift estimate, we get

**Proposition 2.2.** Fix $\delta > 0$ such that Proposition 1.5 holds and choose a small enough so that Proposition 2.1 holds for $\delta/2$ in place of $\delta$. Take $\nu > 0$ small enough so that both propositions above hold. Then for $s \geq 1$ and every $\kappa > 0$, we have

$$\|e^{\nu t}(u - \Pi_0 f)\|_{H^s(M_\delta)} \lesssim \|e^{\nu t}f\|_{H^{s+\kappa}(K_\delta)};$$

for every $f \in C_0^\infty(K_\delta)$.

**Proof.** We consider the case of integer $s$; the general case follows by interpolation in Sobolev spaces (see, for example, [18, Section 4.2]). Let $\psi(t_\pm)$ be a smooth function that is equal to 1 for $t_\pm$ large positive and to 0 for $t_\pm$ large negative; take large $T \in \mathbb{R}$ and apply Proposition 1.5 to $u - \psi(t_\pm + T)\Pi_0 f$:

$$\|e^{\nu t}(u - \Pi_0 f)\|_{H^s(M_\delta)} \lesssim \|e^{\nu t}(u - \psi(t_\pm + T)\Pi_0 f)\|_{H^s(M_\delta)} + \|e^{\nu t}(1 - \psi(t_\pm + T))\Pi_0 f\|_{H^s(M_\delta)}$$

$$\lesssim \|e^{\nu t}f\|_{H^{s-1}(M_\delta)} + \|e^{\nu t}(u - \Pi_0 f)\|_{H^s(K_\delta)} + e^{-\nu T}|\Pi_0 f|;$$

the second term above is estimated by Proposition 2.1 and the third one tends to zero at $T \to +\infty$. \qed
2.2. General case. The idea is to construct an exponentially decaying function $u_1$ solving the equation $\Box_g u_1 = f$ near the event horizons and then estimate the difference $u - u_1$ by Proposition 2.2. We let $u_1 \in C^\infty(M_\delta \setminus K_2\delta)$ solve the following initial/boundary value problem:

$$\Box_g u_1 = f \text{ in } M_\delta \setminus K_2\delta,$$

$$\text{supp } u_1 \subset \{ t_+ > -T \} \text{ for some } T,$$

$$u_1|_{\partial K_{2\delta}} = 0.$$

Note that the surfaces $\partial K_{2\delta} = \{ r = r_{\pm\delta} \}$ are timelike; therefore, this problem has a unique solution (see, for example, [12, Theorem 24.1.1]). This solution is exponentially decaying in time:

**Proposition 2.3.** For $\delta > 0, \nu > 0$ small enough, $\nu$ small enough depending on $\delta$, and every $s \geq 0$,

$$\| e^{\nu t} u_1 \|_{H^{s+1}(M_\delta \setminus K_{2\delta})} \lesssim \| e^{\nu t} f \|_{H^s(M_\delta \setminus K_{2\delta})}. \quad (2.3)$$

**Proof.** First, consider the case $s = 0$. We argue as in the proof of Proposition 1.7, using the vector field $X$ constructed in Proposition 1.6. Namely, we apply the divergence theorem to the vector field $V = e^{2\nu t} J_X(u_1)$ in the region

$$\Omega_T = (M_\delta \setminus K_{2\delta}) \cap \{ t_+ \leq T \}.$$

The flux of $V$ over $\{ t_+ = T \}$ and $\partial M_\delta$ is nonnegative by Proposition 1.1; the flux over $\partial K_{2\delta}$ is nonnegative by Proposition 1.2. Computing the divergence of $V$ by Proposition 1.3, we get (2.3).

Now, we assume that (2.3) is true for $s - 2$ and prove it for $s$; the rest follows by induction and interpolation in Sobolev spaces. For a small enough, $\partial_t$ is timelike in $K_\delta \setminus K_{2\delta}$; therefore, for a large-enough constant $C_0$, the operator

$$L = C_0 \partial_t^2 - \Box_g$$

is elliptic on $K_\delta \setminus K_{2\delta}$. Since $u_1$ satisfies the Dirichlet boundary condition on $\partial K_{2\delta}$, we have

$$\| e^{\nu t} u_1 \|_{H^{s+1}(K_\delta \setminus K_{2\delta})} \lesssim \| e^{\nu t} L u_1 \|_{H^{s-1}(M_\delta \setminus K_{2\delta})} + \| e^{\nu t} u_1 \|_{H^{s-1}(M_\delta \setminus K_{2\delta})}$$

$$\lesssim \| e^{\nu t} \partial_t^2 u_1 \|_{H^{s-1}(M_\delta \setminus K_{2\delta})} + \| e^{\nu t} f \|_{H^s(M_\delta \setminus K_{2\delta})}$$

$$\lesssim \| e^{\nu t} \partial_t^2 f \|_{H^{s-2}(M_\delta \setminus K_{2\delta})} + \| e^{\nu t} f \|_{H^{s-1}(M_\delta \setminus K_{2\delta})}$$

$$\lesssim \| e^{\nu t} f \|_{H^s(M_\delta \setminus K_{2\delta})}.$$

Here we applied (2.3) to $u_1$ and $\partial_t^2 u_1$ and used that $\Box_g$ commutes with $\partial_t^2$.

Now, take a nonnegative function $\chi_\delta(r) \in C^\infty$ such that $\chi_\delta = 0$ near $K_{2\delta}$, but $\chi_\delta = 1$ away from $K_\delta$. We can use the above estimate and apply Proposition 1.5 to $\chi_\delta u_1$ to get

$$\| e^{\nu t} u_1 \|_{H^{s+1}(M_\delta \setminus K_{2\delta})} \lesssim \| e^{\nu t} \chi_\delta u_1 \|_{H^{s+1}(K_\delta \setminus K_{2\delta})} + \| e^{\nu t} \chi_\delta u_1 \|_{H^{s+1}(M_\delta \setminus K_{2\delta})}$$

$$\lesssim \| e^{\nu t} f \|_{H^s(M_\delta \setminus K_{2\delta})} + \| e^{\nu t} \chi_\delta u_1 \|_{H^{s+1}(K_\delta \setminus K_{2\delta})}$$

$$+ \| e^{\nu t} [\Box_g, \chi_\delta] u_1 \|_{H^s(K_\delta \setminus K_{2\delta})}$$

$$\lesssim \| e^{\nu t} f \|_{H^s(M_\delta \setminus K_{2\delta})},$$

as required. $\square$
We are now ready to prove Theorem 0.1. Take $\chi_\delta$ from the proof of Proposition 2.3 and consider $u_2 = u - \chi_\delta(r)u_1$. Then

$$\Box_g u_2 = (1 - \chi_\delta) f - [\Box_g, \chi_\delta] u_1$$

is supported in $K_\delta$; moreover, by Proposition 2.3,

$$\|e^{\nu t} \Box_g u_2\|_{H^{s+\kappa}(M_\delta)} \lesssim \|e^{\nu t} f\|_{H^{s+\kappa}(M_\delta)}.$$

Therefore, we may apply Proposition 2.2 to $u_2$ to get

$$\|e^{\nu t} (u_2 - \Pi_0(\Box_g u_2))\|_{H^s(M_\delta)} \lesssim \|e^{\nu t} f\|_{H^{s+\kappa}(M_\delta)}.$$

Note that $\Pi_0(\Box_g u_2) = \Pi_0 f$, as

$$\Pi_0 \Box_g \chi_\delta(r) u_1 = \lim_{T \to +\infty} \int_{\delta(M \cap \{t_+ = T\})} g(\nabla(\chi_\delta(r) u_1), \vec{n}) dS.$$

The integral over the cap $M \cap \{t_+ = T\}$ converges to zero, as $u_1$ is exponentially decaying in time. As for the timelike piece $\partial M \cap \{t_+ \leq T\}$, the normal vector $\vec{n}$ is tangent to $\partial M$ and we can use this to replace the integral of $g(\nabla u_1, \vec{n})$ over the timelike piece by a certain integral over the spheres $\partial M \cap \{t_+ = T\}$; the latter will decay exponentially as $T \to +\infty$. We now get

$$\|e^{\nu t} (u - \Pi_0 f)\|_{H^s(M_\delta)} \lesssim \|e^{\nu t} \chi_\delta u_1\|_{H^s(M_\delta)} + \|e^{\nu t} (u_2 - \Pi_0(\Box_g u_2))\|_{H^s(M_\delta)} \lesssim \|e^{\nu t} f\|_{H^{s+\kappa}(M_\delta)},$$

which finishes the proof.

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