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Quantum-classical transition in dissipative systems through scaled trajectories

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Abstract
A nonlinear quantum–classical transition wave equation is proposed for dissipative systems within the Caldirola-Kanai model. Equivalence of this transition equation to a scaled Schrödinger equation is proved. The dissipative dynamics is then studied in terms of what we call scaled trajectories following the standard procedure used in Bohmian mechanics. These trajectories depend on a continuous parameter allowing us a smooth transition from Bohmian to classical trajectories. Arrival times and actual momentum distribution functions are also analyzed. The propagation of a Gaussian wave packet in a viscid medium under the presence of constant, linear and harmonic potentials is studied. The gradual decoherence process and localization are easily visualized and understood within this theoretical framework.

1. Introduction
Decoherence and emergence of the classical behavior in open quantum systems is a very broad and active field of research [1–7]. Three main different approaches to deal with quantum dissipative/stochastic dynamics are considered in literature [7]. First, the system-plus-environment model where the physical system of interest and its surrounding is considered as a isolated system, both parts being in continuous interaction. Open system dynamics are described following one of the five standard pictures of quantum mechanics: Schrödinger, Heisenberg, Dirac (interaction), Feynman and de Broglie-Bohm. In the Schrödinger and Dirac pictures, master equations for the reduced system where the bath degrees of freedom have been traced out are obtained. A Markovian behavior of the corresponding equations is reached when the bath is assumed to have no memory, as displayed by the Lindblad equation. Non-Markovian equations lead to a loss of analytical expressions as well as larger computational times. Following the path integral formulation, the time evolution of the density matrix is also used with great success. In the Heisenberg picture, the quantum (generalized or standard) Langevin equation framework has been widely developed. Nowadays, the application of these stochastic quantum formalisms are being addressed to more and more complex systems. If the wave function is seen as a stochastic process in Hilbert space, the dynamics is no longer described by a master equation but by the Itô stochastic differential equation and the realizations of the underlying stochastic processes are called quantum trajectories. Unfortunately, this is the same denomination used in Bohmian mechanics and can lead to confusion [5, 6, 8]. The Bohmian picture allows us to also describe the corresponding dynamics in terms of trajectories, enabling a quite straightforward correspondence with classical trajectories. Second, nonlinear wave mechanics which constitutes one of the most investigated topics in quantum mechanics [7]. And third, effective time dependent Hamiltonians. Dissipation is described by explicitly time dependent Lagrangians and/or Hamiltonians, thus avoiding to deal with the environment degrees of freedom. This approach allows one to preserve the canonical formalism which can be a good starting point to reach the quantum analog of the corresponding dissipative dynamics. The so-called Caldirola-Kanai (CK) model [9, 10] is considered a Hamiltonian formulation of the Langevin equation with zero fluctuations. In this framework, the system undergoes a gradual decay until its total
energy is completely and irreversibly lost by dissipation. The dynamics is then stopped and localized. This model can provide us a nice illustration of the dynamics of a particle on quantum viscous media [11].

In this work, the CK model is considered to analyze the quantum–classical transition in dissipative systems. For this goal, a continuous and gradual transition is proposed through trajectories within the Bohmian framework. Our starting point is the so-called classical Schrödinger equation [12]. Rozen argued that, in the large mass limit, the Schrödinger equation should be replaced by another one, known as classical wave equation, which is equivalent to the continuity and classical Hamilton–Jacobi equations. This classical wave equation contains a non-linear term in the wave function which in general prevents superposition of different states. Following Rozen, these authors have studied quantum–classical transition in terms of arrival times in the scattering of a non-Gaussian wave packet by a rectangular barrier. Recently, a generalization of Rozen’s work has been carried out. In this regard, a wave equation known as transition equation which can be tuned to describe both classical and quantum behavior in a continuous way has been proposed [14], and quantum-like behaviour of the corresponding equation has been shown to be equivalent to a linear Schrödinger equation with a rescaled Planck’s constant [14]. Chou [15] has studied this quantum–classical transition for wave packet interference within the Bohmian framework. All of this work has been carried out for conservative systems. The same procedure can be used for dissipative systems following the standard procedure of Bohmian mechanics. The resulting scaled trajectories are obtained in terms of a continuous parameter within the interval [0, 1] allowing us a smooth transition from Bohmian to classical trajectories by reducing the parameter from 1 to 0. Arrival times and actual momentum distribution functions are also analyzed from these scaled trajectories. The propagation of a Gaussian wave packet in a viscous medium under the presence of constant, linear and harmonic potentials is studied in order to simply illustrate the consequences of this approach. The gradual decoherence process and localization are easily visualized and understood within this theoretical framework. Finally, we would like to emphasize that this approach used for conservative and dissipative systems could be considered as an alternative and efficient way to the WKB approximation.

2. Classical transition equation and scaled Schrödinger equation in the Caldirola-Kanai model

In studying the motion of a body in a dissipative medium, it is assumed that the friction force is a function of its velocity. These forces are not conservative and kinetic energy is usually lost as heat. To simplify theoretical considerations, a linear friction force in velocity is assumed. For slow, laminar, non-turbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments’ [16]. Thus, the classical equation of motion for a particle with mass \( m \) in a viscous medium with friction \( \gamma \) under the influence of a one dimensional potential \( V(x) \) reads as

\[
mx'' + m\gamma x' + \frac{\partial}{\partial x} V(x) = 0.
\]

This equation of motion can be derived from the following Lagrangian [4]

\[
\mathcal{L} = \left[ \frac{m}{2} x'^2 - V(x) \right] e^{-\gamma t}
\]

where the canonical momentum \( p_c \) is given by

\[
p_c = \frac{\partial \mathcal{L}}{\partial x'} = mx e^{-\gamma t}
\]

which is explicitly time-dependent. Notice the difference between canonical momentum \( p_c \) and kinematic one \( mx = p e^{-\gamma t} \). From this Lagrangian, the so-called CK Hamiltonian is given by [4]

\[
H = \dot{x} p_c - \mathcal{L} = \frac{p_c^2}{2m} e^{-\gamma t} + V(x) e^{\gamma t}.
\]

The Hamiltonian operator \( \hat{H} \) corresponding to the classical function (3) can be obtained from the standard quantization rule by substituting the canonical momentum \( p_c \) by \( \frac{i}{\hbar} \frac{\partial}{\partial x'} \)

\[
\hat{H} = -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2}{\partial x'^2} + e^{\gamma t} V(x).
\]

The commutation relation is now given by \([x, p_c] = i \hbar\) and the uncertainty principle \( \Delta x \Delta p_c \sim \hbar \) is formally satisfied. However, the commutation relation of the position operator and kinematic momentum is \( i\hbar e^{-\gamma t} \), leading to the violation of the uncertainty principle. Notice that as long as quantities related to the momentum are not computed, the use of the wave equation in the physical coordinate space is formally correct [11]. In other words, this violation of uncertainty principle can be justified for friction, simulating the action of an almost
macroscopic medium coupled to the particle, since it makes the motion as time proceeds more and more predictable [17]. The time-dependent Schrödinger equation within the CK framework then reads as

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)e^{-\gamma t} \right] \psi(x, t). \]  

(5)

By using the polar form of the wave function as in Bohmian mechanics, \( \psi(x, t) = R(x, t)e^{iS(x,t)/\hbar} \), one obtains

\[ \frac{\partial R^2}{\partial t} + \frac{\partial}{\partial x} \left( R^2 \frac{1}{m} \frac{\partial S}{\partial x} e^{-\gamma t} \right) = 0, \]  

(6)

\[ \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 e^{-\gamma t} + V(x)e^{iS} + Qe^{-\gamma t} = 0, \]  

(7)

where the quantum potential \( Q(x, t) \) is defined as

\[ Q(x, t) = -\frac{\hbar^2}{2m} \frac{\partial R}{\partial x}. \]  

(8)

Equation (6) is the continuity equation which can be written as

\[ \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0, \]

where the probability density and probability current density are

\[ \rho(x, t) = R^2(x, t) = |\psi(x, t)|^2, \]

(9)

\[ j(x, t) = R^2 \frac{1}{m} \frac{\partial S}{\partial x} e^{-\gamma t} = \frac{\hbar}{m} \text{Im} \left( \psi^* \frac{\partial \psi}{\partial x} \right) e^{-\gamma t}. \]  

(10)

Furthermore, the Bohmian velocity field is given by the guiding condition

\[ v_B(x, t) = \frac{j(x, t)}{\rho(x, t)} = \frac{1}{m} \frac{\partial S}{\partial x} e^{-\gamma t}, \]

(11)

from which Bohmian trajectories \( x(x^{(0)}, t) \) are obtained by integrating this guidance equation,

\[ \frac{dx}{dt} = v_B(x, t) \bigg|_{x=x(x^{(0)}, t)}. \]  

(12)

Throughout this work, \( x^{(0)} \) indicates the initial position of the particle. It should be noted that, in the context of Bohmian mechanics, the only initial condition is the position of the particle since its initial momentum is specified by the phase of the wave function according to equation (11). From equation (12), we have that

\[ m\ddot{x} = \frac{d}{dt} \left( e^{-\gamma t} \frac{\partial S}{\partial x} \right) = e^{-\gamma t} \left[ -\frac{\partial S}{\partial x} + \left( \frac{\partial}{\partial t} + \dot{\gamma} \frac{\partial}{\partial \dot{x}} \right) \frac{\partial S}{\partial \dot{x}} \right]. \]  

(13)

Now, using equations (12) and (7), equation (13) can be rewritten as

\[ m\ddot{x} + m\gamma \dot{x} + \frac{\partial}{\partial x} \left( V + e^{-\gamma t} Q \right) = 0, \]  

(14)

which resembles Newton’s law of motion (1) except for the extra term \( e^{-\gamma t} \frac{\partial Q}{\partial x} \), where \(-\frac{\partial Q}{\partial x}\) represents the quantum force derived from the quantum potential. Without the time exponential factor, this equation can be seen as a quantum Langevin equation where the stochastic term given by the noise is absent.

Following Rozen's procedure, by subtracting the non-linear term \( e^{-\gamma t} Q \) to the classical potential term \( e^{-\gamma t} V \) in the Schrödinger equation (5), the classical wave equation in the CK framework is expressed as

\[ i\hbar \frac{\partial}{\partial t} \psi_\text{cl}(x, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)e^{iS} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left( \frac{\partial}{\partial x} \right)^2 \left( \frac{\partial}{\partial x} \right) e^{-\gamma t} \right] \psi_\text{cl}(x, t). \]  

(15)

We also regard classical mechanics as the limiting case of quantum mechanics. In this sense, the classical non-linear differential equation (15) is the limiting case of the Schrödinger linear equation (5). The last term in (15) which is proportional to the quantum potential is responsible for the non-linearity. As we have shown in appendix A.1, it must be there in order to recover the classical Hamilton-Jacobi equation where \( \hbar \) does not appear. In this way, the classical statistical mechanics of a single particle is reformulated in a form which is similar to the formal structure of quantum mechanics. Wave-particle duality is present in classical mechanics, but the wave has a purely passive role: the state of a classical system is still determined by its position and momentum. In this scheme, the meaning of \( \psi_\text{cl} \) is purely descriptive or mathematical. On the contrary, in Bohmian mechanics, the state of a system is given by its wave function and position and the later is guided by the former. After Holland [8], "the quantum and the classical world are both aspects of a single, undivided universe. One continuously passes from one regime to the other by varying the effectiveness of the quantum potential." A
similar approach is also used by Allori et al [18]. Throughout this work, the subindex ‘cl’ stands for classical motion.

A quantum–classical transition wave equation can also be introduced according to [14]

$$\frac{i\hbar}{\partial t} \tilde{\psi}(x, t) = \left[-\frac{\hbar^2}{2m} e^{-\alpha} \frac{\partial^2}{\partial x^2} + V(x) e^{i\alpha} + (1 - \varepsilon) \frac{\hbar^2}{2m} \frac{\partial^2}{|\tilde{\psi}(x, t)|} e^{-i\alpha}\right] \tilde{\psi}(x, t), \quad (16)$$

where a degree of quantumness represented by $\varepsilon$ (with $0 \leq \varepsilon \leq 1$) has been included. This equation provides a continuous description for the transition process of physical systems from purely quantum to classical regime.

Thus, for $\varepsilon = 0$, this transition equation reduces to the classical wave equation (15) while, for $\varepsilon = 1$, it reduces to the Schrödinger equation (5) in the CK framework.

By substituting the polar form $R_x(x, t)e^{iS(x, t)/\hbar}$ of the wave function in equation (16) and after some straightforward manipulations, the following equations are reached

$$-\frac{\partial S}{\partial t} \tilde{\psi} = \frac{1}{2m} e^{-\alpha} \left(\frac{\partial S}{\partial x}\right)^2 \tilde{\psi} + V(x) e^{i\alpha} \tilde{\psi} - \frac{\hbar^2}{2m} e^{-i\alpha} \frac{1}{R_x} \frac{\partial^2}{\partial x^2} \tilde{\psi}, \quad (17)$$

$$i\hbar \frac{\partial R_x}{\partial t} e^{iS/\hbar} = -\frac{\hbar^2}{2m} e^{-\alpha} \left[2i \frac{\partial R_x}{\hbar} \frac{\partial S}{\partial x} e^{iS/\hbar} + i \frac{\partial^2 S}{\partial x^2} \tilde{\psi}\right], \quad (18)$$

where the scaled Planck’s constant

$$\hbar = \hbar \sqrt{\alpha}, \quad (19)$$

as well as the scaled wave function in polar form

$$\tilde{\psi}(x, t) = R_x(x, t) e^{iS(x, t)/\hbar} \quad (20)$$

have been defined. By adding equations (17) and (18), the scaled linear Schrödinger equation is obtained

$$i\hbar \frac{\partial \tilde{\psi}(x, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} e^{-\alpha} \frac{\partial^2}{\partial x^2} + V(x) e^{i\alpha}\right] \tilde{\psi}(x, t). \quad (21)$$

Thus, the nonlinear transition equation (16) is equivalent to the scaled linear Schrödinger equation (21) and has the same structure than equation (5). This will be our working equation for the scaled wave function, which can also be expressed in terms of the transition wave function as

$$\tilde{\psi}(x, t) = \psi(x, t) \exp\left[i \hbar \left(\frac{1}{\sqrt{\alpha}} - 1\right) S_x(x, t)\right]. \quad (22)$$



### 3. Probability distribution in Bohmian mechanics: arrival time and actual momentum

As is well known, the complete description of a system in Bohmian mechanics is given by its particle position and wave function. Quantum trajectories $x(x^{(0)}, t)$ are obtained from the guidance equation. Assuming that the initial distribution function for particle positions is given by the Born rule, it is concluded by the continuity equation that such a rule holds at any time

$$\rho(x, t) = \int dx^{(0)} \rho(x^{(0)}, 0) \delta(x - x(x^{(0)}, t)).$$

Moreover, the probability distribution function for a particle property $f$ is given by [19]

$$\Pi(f) = \int dx^{(0)} \rho(x^{(0)}, 0) \delta(f - f(x^{(0)})), \quad (23)$$

$f(x^{(0)})$ being the value of $f$ along the quantum trajectory $x(x^{(0)}, t)$. Thus, for the arrival time distribution at a given location $X$, one has

$$\Pi_t(t) = \int dx^{(0)} \rho(x^{(0)}, 0) \delta(t - T_X(x^{(0)})), \quad (24)$$

with $T_X(x^{(0)})$ giving the time at which particles arrive at the detector location $X$ and the mean arrival time is given by

$$\tau(X) = \int_0^\infty dt \ t \ \Pi_t(t) \quad (25)$$
By using the properties of the Dirac delta function and non-crossing property of Bohmian trajectories, Leavens [19] obtained

\[ \Pi_\chi(t) = \frac{|j(X, t)|}{\int_0^\infty dt' |j(X, t')|} \]  

(27)

for the intrinsic (apparatus independent) arrival time distribution for those particles that actually arrive at the detector location \( X \). It should be notice that this relation is naturally obtained within the Bohmian mechanics. There are several theoretical treatments for the arrival time distribution within the standard theory of quantum mechanics due to the absence of a clear treatment to include time observables into its formalism (for a review see, for example, [20]). As pointed out in this review, different theoretical methods approach to equation (27) at asymptotic distances and times from the particle source and existing scatterers.

On the other hand, the Bohmian momentum filed is given by

\[ \dot{p}_x = \frac{\hbar}{m} \frac{\partial}{\partial x} \langle \psi | \hat{\psi} \rangle \]  

and the coefficient of \( \hat{p}(x, t) \) in equation (33) is the scaled velocity field

\[ v_s(x, t) = \frac{\hbar}{2m\sigma_0} \frac{\partial}{\partial x} \frac{e^{-\gamma t}}{\sigma_t^2} (x - x_t) + \frac{p_t}{m}. \]  

(35)
Scaled trajectories are then expressed from the guidance equation (12) as
\[
\frac{d}{dt}(x - x_t) = \frac{\hbar}{2 m \sigma_0} e^{-\gamma t} \operatorname{Im}(\mathcal{g}_t) (x - x_t),
\]
which has the solution,
\[
\ln \left( \frac{x(x^{(0)}, t) - x_t}{x^{(0)} - x_0} \right) = \frac{\hbar}{2 m \sigma_0} \int_0^t dt' e^{-\gamma t'} \operatorname{Im}(\mathcal{g}_{t'}) .
\]
or written differently
\[
x(x^{(0)}, t) = x_t + g(t) \frac{\sigma_t}{\sigma_0} (x^{(0)} - x_0), \tag{36}
\]
with
\[
g(t) = \sigma_0 \exp \left[ -\frac{\hbar}{2 m \sigma_0} \int_0^t dt' e^{-\gamma t'} \operatorname{Im}(\mathcal{g}_{t'}) \right]. \tag{37}
\]
Equation (36) shows that if the particle is initially located at the center of the wave packet, it remains there forever, following its classical path \(x_c\). The structure of this equation presents a general dressing scheme \([7]\) where the scaled trajectory consists of a classical trajectory (a particle property) plus a term involving the width of the wave packet (a wave property). Bohmian and classical trajectories are obtained for the two extreme values of the continuous parameter \(\epsilon = 1, 0\), respectively.

On the other hand, from equation (36) one also obtains
\[
m \dot{x}(x^{(0)}, t) = p_t + m \int f(t) \frac{\sigma_t}{\sigma_0} (x^{(0)} - x_0) \tag{38}
\]
for the momentum along the quantum trajectory, where we have introduced the auxiliary function
\[
f(t) = g(t) + \frac{\sigma_t}{\sigma_0} \dot{g}(t) \tag{39}
\]
for simplicity. An alternative way to obtain the momentum along a trajectory is substituting equation (36) into equation (35). Thus, from equation (28), the actual momentum distribution can be expressed as
\[
\tilde{\Pi}(p, t) = \frac{1}{\sqrt{2\pi} \sigma_0} \int dx^{(0)} \exp \left[ -\frac{(x^{(0)} - x_0)^2}{2\sigma_0^2} \right] \delta \left( p - p_t - m f(t) \frac{\sigma_t}{\sigma_0} (x^{(0)} - x_0) \right)
\]
\[
= \frac{1}{\sqrt{2\pi} \sigma_0} \int dy \exp \left[ -\frac{y^2}{2\sigma_0^2} \right] \delta \left( mf(t) \frac{\sigma_t}{\sigma_0} + p_t - p \right)
\]
\[
= \frac{1}{\sqrt{2\pi} \Sigma_t} \exp \left[ -\frac{(p - p_t)^2}{2\Sigma_t^2} \right], \tag{40}
\]
with
\[
\Sigma_t = mf(t) \sigma_t \tag{41}
\]
being the width of the momentum distribution. In A.3, the Fourier transform of the free damped Gaussian wave packet whose module-squared is the quantum distribution function for the momentum is derived. For the product of widths, we have
\[
\sigma_t \Sigma_t = \frac{m}{2} f(t) \frac{d}{dt} \sigma_t^2 . \tag{42}
\]

Now, let us consider some special cases.

4.1. Free scaled Gaussian wave packet in non-viscid medium

For a free particle, \(V(x) = 0\), in a non-viscid medium, \(\gamma = 0\), the results are similar to those found in [8] for a nonscaled wave function. One has
the auxiliary functions being given by

\[
\begin{aligned}
g(t) &= 1, \\
f(t) &= 1.
\end{aligned}
\]

For the widths in configuration and actual momentum spaces, we have

\[
\begin{aligned}
\tilde{\sigma}_t &= \sigma_0 \sqrt{1 + \frac{\hbar^2 t^2}{4m\sigma_0^2}}, \\
\tilde{\Sigma}_t &= \frac{\hbar t}{4m\sigma_0^2}.
\end{aligned}
\]

and at asymptotic times, \( t \to \infty \),

\[
\begin{aligned}
\tilde{\sigma}_\infty &\approx \frac{\sqrt{\hbar}}{2m\sigma_0} t, \\
\tilde{\Sigma}_\infty &\approx \frac{\sqrt{\hbar}}{2\sigma_0} \equiv \sigma_p.
\end{aligned}
\]

Thus, in this limit, one easily recognizes that the actual momentum distribution coincides with the usual quantal one

\[
\lim_{t \to \infty} \tilde{\Pi}(p, t) = \frac{1}{\sqrt{2\pi\sigma^2_p}} \exp \left[ -\frac{(p - p_0)^2}{2\sigma^2_p} \right] \equiv |\tilde{\chi}(p, 0)|^2,
\]

where \( \tilde{\chi}(p, 0) \) is the Fourier transform of the initial wave function (29) which is evaluated in A.3.

Scaled trajectories follow the dressing scheme

\[
x(x^{(0)}, t) = x_0 + \frac{p_0}{m} t + (x^{(0)} - x_0) \sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma_0^2}},
\]

using equations (43)–(45) in equation (36). Equation (48) has also been obtained in [15]. The \( \epsilon \) parameter is accompanying the wave feature as a factor. Bohmian trajectories are obtained from these scaled trajectories for \( \epsilon = 1 \).

In the classical regime, when \( \epsilon = 0 \), from equations (32), (33) and (40), one has

\[
\rho_c(x, t) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp \left[ -\frac{(x - x_0 - \frac{p_0}{m} t)^2}{2\sigma_0^2} \right],
\]

\[
\tilde{j}_d(x, t) = \frac{p_0}{m} \rho_c(x, t),
\]

\[
\Pi_d(p, t) = \delta(p - p_0),
\]

which shows that we have a Gaussian wave packet in configuration space which moves with constant velocity \( \frac{p_0}{m} \) and maintains its width. However, the actual momentum has a Dirac delta distribution around \( p = p_0 \), i.e., all particles in the ensemble have the same momentum \( p_0 \), which is apparent from equation (38) in the classical limit where \( \tilde{\sigma}_t = 0 \); and noting that \( p_1 = p_0 \). The arrival time distribution at the detector position \( X \) has a simple form in this classical limit,

\[
\Pi_{d, X}(t) = \frac{2p_0^2}{\pi m^2\sigma_0^2} \exp \left[ -\frac{p_0^2}{2m^2\sigma_0^2} \left( t - \frac{m}{p_0} (X - x_0) \right)^2 \right],
\]

showing a Gaussian shape with a center at \( m(X - x_0)/p_0 \). From equation (36), the classical arrival time at the detector position \( X \) is given by

\[
T_{X, cl}(x^{(0)}) = \frac{m}{p_0}(X - x^{(0)}),
\]
and the classical mean arrival time is finally expressed as

\[ \tau_{cl}(X) = \frac{m}{P_0} (X - \langle x \rangle_0) = \frac{m}{P_0} (X - \sigma_0) \]  

from equations (49) and (53).

4.2. Free scaled Gaussian wave packet in a viscous medium

In this case, the free evolution of the Gaussian wave packet (29) in a viscous medium is considered. From the free propagator of the Schrödinger equation which we have derived in A.2 (equations (A6), (A8) and (29)), we have

\[ \tilde{\psi}(x, t) = \left( \frac{m\gamma}{2\pi i\hbar(1 - e^{-\gamma t})} \right)^{1/4} \frac{1}{(2\pi\sigma_0^2)^{1/4}} \int dx' \exp \left[ \frac{im}{\hbar} \frac{\gamma}{1 - e^{-\gamma t}} (x - x')^2 - \frac{(x' - x_0)^2}{4\sigma_0^2} + \frac{i}{\hbar} P_0 (x' - x_0) \right] \]

\[ = \left( \frac{m\gamma}{2\pi i\hbar(1 - e^{-\gamma t})} \right)^{1/4} \frac{1}{(2\pi\sigma_0^2)^{1/4}} \left[ \frac{\pi}{d_1} \right] \exp \left[ \frac{d_1^2}{4d_2 + d_3} \right] \]  

where

\[ d_1 = \frac{1}{4\sigma_0^2} \frac{im}{\hbar} \frac{\gamma}{1 - e^{-\gamma t}}, \]

\[ d_2 = \frac{im}{\hbar} \frac{\gamma}{1 - e^{-\gamma t}} x + \frac{1}{2\sigma_0^2} x_0 + \frac{i}{\hbar} P_0, \]

\[ d_3 = \frac{im}{\hbar} \frac{\gamma}{1 - e^{-\gamma t}} x^2 - \frac{1}{4\sigma_0^2} x_0^2 - \frac{i}{\hbar} P_0 x_0. \]

After straightforward but lengthy calculations, one arrives at the shape (30) for the wave function but now with

\[ \begin{cases} \tilde{x}_t = \sigma_0 \left( 1 + \frac{i \hbar \gamma}{2 m \sigma_0^2} \right), \\
\quad \tilde{x}_t = x_0 + \frac{P_0}{m} \frac{1 - e^{-\gamma t}}{\gamma}, \\
\quad \tilde{p}_t = m \tilde{x}_t = P_0 e^{\gamma t}, \\
\quad A_{cl,t} = \frac{p_0^2}{2m} \frac{1 - e^{-\gamma t}}{\gamma}, \end{cases} \]

instead of equations (43). From the expression $e^{\gamma t}p_t = e^{\gamma t}m \tilde{x}_t$ for the canonical momentum, it has been argued that the CK Hamiltonian describes a particle of exponentially growing mass, $m(t) = m e^{\gamma t}$ [21]. For the auxiliary function, we have that

\[ g(t) = \frac{\sigma_0}{\tilde{\sigma}_t} \exp \left( \frac{1}{2} \log \frac{\tilde{\sigma}_t^2}{\sigma_0^2} \right) = 1, \]

by noting that

\[ \frac{d}{dt} \tilde{\sigma}_t^2 = \frac{\hbar}{m\sigma_0} e^{-\gamma t} \text{Im}(\tilde{x}_t) \]

and $f(t) = 1$ again, meaning that equations (44) are still valid. Analogously, for the widths of the distributions in configuration and actual momentum, one has

\[ \begin{cases} \tilde{\sigma}_t = \sigma_0 \sqrt{1 + \frac{\epsilon \hbar^2}{4m^2 \sigma_0^2} \left( \frac{1 - e^{-\gamma t}}{\gamma} \right)^2}, \\
\quad \tilde{\sigma}_t = \sigma_0 \sqrt{1 + \frac{\epsilon \hbar^2}{4m^2 \sigma_0^2} \left( \frac{1 - e^{-\gamma t}}{\gamma} \right)^2}. \end{cases} \]

Scaled trajectories within the dressing scheme are now given by

\[ x(\chi^{(0)}, t) = x_0 + \frac{P_0}{m} \frac{1 - e^{-\gamma t}}{\gamma} + (\chi^{(0)} - x_0) \sqrt{1 + \frac{\epsilon \hbar^2}{4m^2 \sigma_0^2} \left( \frac{1 - e^{-\gamma t}}{\gamma} \right)^2}, \]

using equations (56), (44) and (57) in equation (36). For $\epsilon > 0$, these trajectories correspond to an intermediate regime, leading to the Bohmian trajectories when $\epsilon = 1$. Again, this parameter appears as a factor in the non-classical part. This intermediate regime is critical to better understand the gradual decoherence process in this global dynamics. For a given value of friction $\gamma$ and at a given instant of time, widths behave like
where $c$ is a constant. Thus, in the transition from the quantum to classical regime, $\epsilon \to 0$, both widths decrease from its maximum value in the quantum regime to the minimum one in the classical regime.

In viscous media, $\gamma > 0$, at $t = 0$, $\Sigma_0 = 0$ which shows that all particles of the ensemble initially have the same velocity $p_0/m$. As $t \to \infty$, from equations (57), (32) and (40) one has

$$\tilde{\sigma}_c \approx \sigma_0 \left[ 1 + \frac{\epsilon \hbar^2}{4m^2 \sigma_0^2 \gamma^2} \right],$$

$$\tilde{\Sigma}_c \approx 0,$$

$$\tilde{\rho}(x, \infty) \approx \frac{1}{\sqrt{2\pi \tilde{\sigma}_c^2}} \exp \left[ -\frac{(x - x_0 - \frac{p_0}{m\gamma})^2}{2\tilde{\sigma}_c^2} \right],$$

$$\tilde{\Pi}(p, t) = \delta(p),$$

showing that at the end, the distribution function in the position representation stops at the point $x_0 + \frac{p_0}{m\gamma}$, and the distribution function for the actual momentum takes the form of the Dirac delta around zero momentum, revealing all particles of the ensemble stop at the end. In other words, the wave packet becomes localized, that is, motionless and with the spreading being frozen. For comparison, we provide the momentum space representation of the scaled wave function and the corresponding distribution function in A.3. Moreover, from equation (58), one sees that particles stop (localization) finally at

$$x(x^{(0)}, \infty) = x^{(0)} + \frac{p_0}{m\gamma} + \left( x^{(0)} - x_0 \right) \left[ 1 + \frac{\epsilon \hbar^2}{4m^2 \sigma_0^2 \gamma^2} \right],$$

and to arrive at the detector location $X$, the detector must be placed at $X \leq x(x^{(0)}, \infty)$.

In the classical limit, $\epsilon \to 0$, one has

$$\tilde{\sigma}_c \to \sigma_0,$$

$$\tilde{\Sigma}_c \to 0,$$

and from equations (32) and (40) the classical distribution functions read as

$$\rho_c(x, t) = \frac{1}{\sqrt{2\pi \sigma_0^2}} \exp \left[ -\frac{(x - x_0 - \frac{p_0 e^{-\gamma t}}{m})^2}{2\sigma_0^2} \right],$$

$$\Pi_c(p, t) = \delta(p - p_0 e^{-\gamma t}).$$

These relations show that the classical distribution function in configuration space retains its Gaussian shape during the motion with velocity $p_0 e^{-\gamma t}/m$, while the momentum distribution is again a delta function, with all particles of the classical ensemble having the same momentum $p_0 e^{-\gamma t}$. Furthermore, from equation (58) one obtains

$$x_{cl}(x^{(0)}, t) = x^{(0)} + \frac{p_0}{m\gamma} (1 - e^{-\gamma t})$$

for the classical trajectory and

$$T_{X,cl}(x^{(0)}) = \frac{1}{\gamma} \ln \left[ 1 - \frac{m\gamma}{p_0} (X - x^{(0)}) \right]$$

for the arrival time at the detector location $X$ which by the same reasoning as above, one should not worry about the argument of the ln-function and positivity of $T_{X,cl}(x^{(0)})$.

All of results of this subsection reduce to the corresponding ones in the previous subsection in the non-viscid limit. The corresponding scaled trajectories are identical to those of [11] obtained from a given anzat for the wave function when $\epsilon = 0, 1$.

### 4.3. Propagation of a scaled Gaussian wave packet in a damped linear potential

Let us consider now the propagation of the wave packet (29) in viscous media but in the presence of an external linear potential.
a being a constant acceleration. The corresponding propagator of the scaled Schrödinger equation (21) is given by equation (A9) with \( \hbar \) instead of \( \h \). So, by using equation (A6), the time evolution of the wave function is obtained by straightforward but lengthy calculations as equation (36). The resulting equations are

\[
\frac{\dot{z}_i}{\tau} = \sigma_0 \left( 1 + \frac{\gamma^2 \hbar}{2m \sigma_0^2} 1 - e^{-\gamma t} \right),
\]

\[
x_i = x_0 + \frac{p_0}{m} \frac{1 - e^{-\gamma t}}{\gamma} - \frac{a \gamma^2}{\gamma^2} t^2 - 1 + e^{-\gamma t},
\]

\[
p_i = m x_i = p_0 e^{-\gamma t} - ma \frac{1 - e^{-\gamma t}}{\gamma},
\]

\[
A_{cl,t} = \frac{p_i^2}{2m} \frac{1 - e^{-\gamma t}}{\gamma^2} - a \left[ \frac{p_0^2 e^{-\gamma t} + c^2}{\gamma^2} + m x_0 \frac{1 - e^{-\gamma t}}{\gamma^2} + a^2 m^2 (2\gamma t - 1) e^{\gamma t} - e^{-\gamma t} \right].
\]

A direct comparison with equations (56) shows that \( \tilde{\beta} \) has the same expression for both propagation in zero and in constant force field. Thus, equations (44) are still valid and widths are given by equation (57).

On the other hand, scaled trajectories within the dressing scheme are given by

\[
x(x(0), t) = x_0 + \frac{p_0}{m} \frac{1 - e^{-\gamma t}}{\gamma} - \frac{a \gamma^2}{\gamma^2} t^2
\]

\[
+ (x(0) - x_0) \sqrt{1 + \frac{e^\hbar^2}{4m^2 \sigma_0^2} \left( 1 - e^{-\gamma t} \right)^2},
\]

when using equations (70), (44) and (57) in equation (36). This equation shows that particles, just as the classical ones, move with constant velocity \(-a/\gamma\) at large times, \(t \to \infty\). Furthermore, the probability current density at the detector location \(X\), non-normalized arrival time distribution, is given by

\[
\tilde{j}(X, t) = \left[ \frac{e^\hbar^2}{4m^2 \sigma_0^4 \tilde{\sigma}_t} \frac{1 - e^{-\gamma t}}{\gamma} (X - x_i) + \frac{p_i}{m} \right] \times \frac{1}{\sqrt{2\pi \tilde{\sigma}_t^2}} \exp \left[ -\frac{(X - x_i)^2}{2\tilde{\sigma}_t^2} \right],
\]

from equations (32), (33) and (57). In the classical limit established by equation (64) we obtain that

\[
\tilde{j}_{cl}(X, t) = \frac{p_i}{m} \frac{1}{\sqrt{2\pi \sigma_0^2}} \exp \left[ -\frac{(X - x_i)^2}{2\sigma_0^2} \right],
\]

which does not have a Gaussian shape in time.

It is very instructive to analyze in more detail this dissipative dynamics within the scaled wave function framework or transition quantum–classical regime. In this regard, we work in a system of units where \( \hbar = 1 \) and \( m = 1 \). We place the detector at \( X = 0 \). Parameters of the initial Gaussian wave packet are chosen to be \( \sigma_0 = 1 \); and \( x_0 = -10 \) and \( p_0 = 5 \). Seven scaled trajectories are calculated for each dynamical regime considered: \( \epsilon = 1 \), Bohmian; \( \epsilon = 0.5 \), intermediate; \( \epsilon = 0 \), classical. As pointed above, arrival times of those particles that actually reach the detector are given by the modulus of the probability current density. For computations, one has to make sure that the denominator of equation (27) is non-zero implying that the number of particles in the ensemble reaching the detector is not zero. Due to the equivariance principle, particles distribute according to the Born rule. The non-crossing property of Bohmian paths implies that if a particle in the far left tail of the Gaussian packet (for example, \( x(0) = x_0 - 5\sigma_0 \)) reaches the detector, all other particles have certainly reached before. If we want that all particles arrive at the detector, one must choose the friction coefficient such that \( \gamma \leq 0.22 \) in the quantum regime, \( \epsilon = 1 \), and \( \gamma \leq 0.33 \) for the classical one, \( \epsilon = 0 \). In figure 1, arrival time distributions for the free particle and a uniform accelerating (\( a < 0 \) in equation (69)) force field are showed. Two different values of \( \gamma \) and \( a \) are used for each regime: \( \epsilon = 1 \) (Bohmian, black curve), \( \epsilon = 0.5 \) (intermediate, red curve), \( \epsilon = 0 \) (classical, green curve). From this figure, one observes that the maximum of the distribution is shifted towards longer times when passing from the quantum to classical regime, while its width (full width at half maximum) becomes narrower. The maximum locates at shorter times and the distribution becomes narrower for an accelerating force compared to the free case, revealing that particles arrive sooner at the detector location. The difference between arrival times of different particles of the ensemble becomes smaller. Figure 2 shows that mean arrival time increases with friction \( \gamma \) for three different accelerations (the same color code is used for the three dynamical regimes than in figure 1). This time is always shorter for the classical regime. This behavior is a clear manifestation of the non-Gaussian shape before alluded.

In the following two figures, figures 3 and 4, the corresponding scaled trajectories derived from equations (58) and (71) are plotted. In each row, the friction coefficient \( \gamma \) is constant: \( \gamma = 0 \) (first row), \( \gamma = 0.1 \)
In each column, the dynamical regime $\epsilon$ is constant: $\epsilon = 1$ (first column), $\epsilon = 0.5$ (second column), $\epsilon = 0$ (third column). Scaled trajectories do not cross and bend with friction as a function of time. Furthermore, the distance between two specific Bohmian trajectories, from equations (58) and (71) can be expressed as

$$x(x_1^{(0)}, t) - x(x_2^{(0)}, t) = (x_1^{(0)} - x_2^{(0)}) \frac{\delta t}{\sigma_0}$$

for the free and linear potential cases. Since $\delta t$ increases with time then trajectories diverge in the quantum case, but width of the wave packet in the classical case is constant meaning that in this case the distance between

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**Figure 1.** Arrival time distributions for two different values of $\gamma$ and $a$ and three regimes: $\epsilon = 1$ (Bohmian, black curve), $\epsilon = 0.5$ (intermediate, red curve), $\epsilon = 0$ (classical, green curve).

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**Figure 2.** Mean arrival times versus $\gamma$ for different values of $\epsilon$ (regimes) and $a$: $\epsilon = 1$ (Bohmian, black curve), $\epsilon = 0.5$ (intermediate, red curve), $\epsilon = 0$ (classical, green curve).
trajectories remains constant; trajectories never cross each other even in the classical case. This feature is apparent in figures 3 and 4. It should also be noticed that for a given time, $\dot{\beta}$ decreases with $\epsilon$ and the rate of divergence of trajectories decreases continuously along the quantum–classical transition.

The corresponding Bohmian and classical trajectories are identical to those of [11] obtained from a given anzat for the wave function.
4.4. Propagation of a scaled Gaussian wave packet in a damped harmonic potential

Let us analyze now the time evolution of the Gaussian wave packet (29) in the damped harmonic potential 
\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2 e^{-\gamma t}. \] 
By using equations (A6) and (A10), we obtain

\[
\tilde{\psi}(x, t) = \frac{1}{(2\pi \sigma_0^2)^{1/4}} \exp \left[ \alpha_t (x - x_t)^2 + \frac{i}{\hbar} p_t (x - x_t) + \frac{i}{\hbar} \eta_t \right],
\]

where

\[
\alpha_t = \frac{m}{2\hbar} e^{-\gamma t/2} \left[ -\frac{\gamma}{2} + \frac{\omega}{\sin \omega} \left( \cos \omega t + \frac{\gamma}{2} \sin \omega t \right) \right],
\]

\[
\eta_t = \frac{m}{4} \sin^2 \omega t \left[ 2 \omega \cot(\omega t) \left( \frac{p_0^2}{m^2} - \omega_0^2 x_0^2 \right) - \gamma \left( \frac{p_0}{m} + \frac{\gamma x_0}{2} \right)^2 - 4\omega^2 x_0 \left( \frac{p_0}{m} + \frac{\gamma x_0}{4} \right) \right],
\]

with \( \omega, x, \) and \( p_t \) given respectively by

\[ \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}, \]

\[ x_t = e^{-\gamma t/2} \left[ x_0 \left( \cos \omega t + \frac{\gamma}{2} \sin \omega t \right) + \frac{p_0}{m} \sin \omega t \right], \]

\[ p_t = m \dot{x}_t = e^{-\gamma t/2} \left[ -m \omega_0^2 x_0 \sin \omega t + \frac{p_0}{m} \left( \cos \omega t - \frac{\gamma}{2} \sin \omega t \right) \right], \]

which corresponds to the classical trajectory \( x_t \) for the classical damped harmonic oscillator with initial position \( x_0 \) and initial momentum \( p_0 \). Only the underdamped case, \( \omega_0 > \gamma/2 \), is considered here. The center of the Gaussian wave packet follows \( x_t \) with velocity \( p_t/m \). One can easily check that \( \langle \hat{x} \rangle(t) = x_t \) and \( \langle \hat{p} \rangle(t) e^{-\gamma t} = p_t \) which are acceptable results according to the Ehrenfest theorem. One can also see that

\[ \alpha_0 = \frac{-1}{4\sigma_0^2}, \]

which is a reasonable result due to the relation \( \tilde{\psi}(x, 0) = \tilde{\psi}_0(x) \).

The probability density is given by equation (32) with \( x \) given by equation (78) and width

\[ \sigma_t = \sigma_0 e^{-\gamma t/2} \left[ \left( \cos \omega t + \frac{\gamma}{2} \sin \omega t \right)^2 + \frac{\hbar^2}{4m^2 \sigma_0^4} \sin^2 \omega t \right]. \]

Finally, the scaled velocity field is given by

\[ v_t(x, t) = \theta(t) (x - x_t) + \frac{p_t}{m}, \]

from which the corresponding trajectories are given by

\[ x(x^{(0)}, t) = x_t + (x^{(0)} - x_0) \Theta(t), \]

where

\[ \Theta(t) = \exp \left[ \int_0^t dt' \theta(t') \right]. \]

Here, the dressing scheme is again reproduced but in a more involved way through \( \Theta(t) \). The physical meaning of this function is better understood from the frictionless and classical limits. The density function for the actual momentum has also the Gaussian shape (40) with \( p_t \) given by equation (79) and the width by

\[ \Sigma_t = m \sigma_0 \Theta(t) = m \sigma_0 |\theta(t) \Theta(t)| \].

Note that even for \( \omega_0 < \frac{\gamma}{2} \) where \( \omega \) is zero or imaginary, still \( \sigma_t \), \( \theta(t) \), \( \Theta(t) \) and \( \Sigma_t \) are all real functions of time. From equation (80), it follows that \( \Sigma_0 = 0 \), i.e., the actual momentum distribution function is initially a Dirac delta function \( \Pi(p, 0) = \delta(p - p_0) \), meaning that all particles of the ensemble have the same momentum \( p_0 \).
For propagation in a non-viscous medium, $\gamma = 0$, one obtains

$$
\Theta(t) = \frac{\bar{\sigma}_t}{\sigma_0},
$$

(88)

and

$$
x(x^{(0)}, t) = x_t + (x^{(0)} - x_0) \frac{\bar{\sigma}_t}{\sigma_0},
$$

(89)

where $\Theta(t)$ gives the ratio of the Gaussian wave packet width at $t$ and $t = 0$. This function is precisely the ratio appearing in equation (36) as a factor of the second sum of the dressing scheme, which can be seen as a normalized time dependent width. In the classical limit $\epsilon \rightarrow 0$ we have

$$
\bar{\sigma}_t = \sigma_0 \left| \frac{\cos \omega t}{\epsilon} + \frac{\gamma}{2} \frac{\sin \omega t}{\epsilon} \right|,
$$

(91)

and

$$
\Theta(t) = \cos \omega t + \frac{\gamma \sin \omega t}{\omega},
$$

(92)

where again $\Theta(t)$ is given by the normalized time dependent width. In this case, this width displays a damped oscillatory motion with the frequency given by $\omega$. Thus, we can claim that $\Theta(t)$, for the scaled trajectories, represents the time evolution of the normalized width of the Gaussian wave packet, which is a purely wave property. Moreover, by considering equation (93) in (83), the dissipative classical trajectory is expressed as

$$
x_{cl}(x^{(0)}, t) = x_t + (x^{(0)} - x_0)e^{-\gamma t/2} \left( \cos \omega t + \frac{\gamma}{2} \frac{\sin \omega t}{\omega} \right),
$$

(95)

indicating that all particles of the classical ensemble have initially the same momentum. According equation (83), $\Theta(t)$ could also be seen as the deviation (or difference) of the scaled trajectory with respect to the $x_t$-trajectory. It is noticeable to observe that widths do not reduce to constant values in the classical limit. In figure 5, $\Theta(t)$ is plotted as a function of time for different values of $\epsilon$ and $\gamma$ for $\omega_0 = 0.6$. The different dynamical regimes are: $\epsilon = 0.5$ (intermediate, red curve), and $\epsilon = 0$ (classical, green curve).
However, the scaled trajectories do not cross because $\Theta(t)$ approaches asymptotically to zero but never reaches it. As a result of this behaviour, trajectories are becoming more dense at larger friction coefficients. These features are clearly observed in figure 6. In this case, we have taken a motionless wave packet, $p_0 = 0$, with center at $x_0 = 1$ for numerical calculations. The corresponding scaled trajectories are plotted for the same values of $\gamma$ and $\omega$ as before. The resulting trajectories are also identical to those of [11] obtained from a given anzat for the wave function when $\omega = 0,1$.

5. Concluding remarks

The field of open quantum systems is being very active and fruitful to better understand the decoherence process. In this work, we have used one of the three approaches mentioned in the Introduction, the effective Hamiltonian approach, where all the degrees of freedom of the bath coordinates have been suppressed. The CK model represents the paradigm for dissipative dynamics since it is simple and very intuitive. When it is applied to simple interaction potentials, the dynamics is also solved by analytical methods. For this reason, we have used this model to analyze the quantum–classical transition through trajectories in a continuous way. Starting from the nonlinear classical wave equation for conservative systems due to Rozen and introducing a scaled Planck’s constant, the dissipative dynamics is proposed to be described by a linear Schrödinger equation. This differential equation is solved following the Bohmian procedure in order to obtain what we call scaled trajectories. The continuous parameter $\omega$ responsible for the smooth quantum–classical transition covers the interval $[0, 1]$, where $\omega = 1$ provides the standard Bohmian (quantum) trajectories and $\omega = 0$, the corresponding classical ones. These scaled trajectories follow always the same dressing scheme consisting of a classical trajectory (particle property) plus a term involving the width of the wave packet (wave property). Thus, the decoherence as well as localization can be seen as gradual processes enabling us the understanding and visualization of this dissipative dynamics. This procedure could be easily extended to deal with other interesting problems such as, for example, tunneling, gas collisions at high pressure and surface diffusion.

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Appendix

In this appendix some useful relations are given within the CK model.

A.1. Classical wave equation

By using the polar form of classical wave function, \( \psi_{cl}(x, t) = R_d(x, t) \exp[iS_d(x, t) / \hbar] \) in the classical wave equation \((\ref{15})\), we obtain two coupled differential equations

\[
\frac{\partial R_d}{\partial t} = -\frac{1}{2m} \left( 2 \frac{\partial R_d}{\partial x} + \frac{\partial^2 S_d}{\partial x^2} \right) e^{-\gamma t}, \tag{A1}
\]

\[
-\frac{\partial S_d}{\partial t} = \frac{1}{2m} \left( \frac{\partial S_d}{\partial x} \right)^2 e^{-\gamma t} + V(x) e^{\gamma t}; \tag{A2}
\]

By multiplying both sides of the first equation by \(2R_d\) and re-arranging terms, one obtains the continuity equation

\[
\frac{\partial R_d^2}{\partial t} + \frac{\partial}{\partial x} \left( \frac{R_d^2}{m} \frac{\partial S_d}{\partial x} e^{-\gamma t} \right) = 0, \tag{A3}
\]

whereas equation \((A2)\) is the classical Hamilton-Jacobi equation. Noting the continuity equation \((A3)\), the classical velocity is defined as

\[
\dot{x} = \frac{1}{m} \frac{\partial S_d}{\partial x} e^{-\gamma t}, \tag{A4}
\]

from which one obtains the classical equation of motion

\[
m \ddot{x} = \frac{d}{dt} \left( \frac{\partial S_d}{\partial x} e^{-\gamma t} \right) = e^{-\gamma t} \left[ -\gamma \frac{\partial S_d}{\partial x} + \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \dot{x} \right) \frac{\partial S_d}{\partial x} \right]
\]

\[
= e^{-\gamma t} \left[ -m \gamma \dot{x} e^{\gamma t} - \frac{1}{2m} \left( \frac{\partial S_d}{\partial x} \right)^2 e^{-\gamma t} + V(x) e^{\gamma t} \right] - \frac{1}{m} \frac{\partial S_d}{\partial x} e^{-\gamma t} \frac{\partial^2 S_d}{\partial x^2} \tag{A5}
\]

where we have used equations \((A4)\) and the partial derivative of \((A2)\) with respect to the space coordinate.

A.2. Propagators

Since the Hamiltonian of the CK model in free space commutes with itself in different times, the formal solution of the Schrödinger equation is given by \([22]\)

\[
\psi(x, t) = \int dx' G(x, x'; t) \psi(x', 0), \tag{A6}
\]

where

\[
G(x, x'; t) = \langle x | e^{-i \int_0^t dt' \hat{H}(t') / \hbar} | x' \rangle \tag{A7}
\]

is called the propagator of the system. For the free Hamiltonian of CK model, one has

\[
G_{free}(x, x'; t) = \langle x | \exp \left[ -i \int_0^t dt' \frac{\hat{p}^2}{2m} e^{-\gamma t'} \right] | x' \rangle
\]

\[
= \langle x | \exp \left[ -i \int_0^t dt' \frac{\hat{p}^2}{2m} e^{-\gamma t'} \right] \int_{-\infty}^{\infty} dp' \langle p' | \psi(x') \rangle
\]

\[
= \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dp' e^{ip(x-x')/\hbar} \exp \left[ -i \frac{p'^2}{\hbar 2m} \int_0^t dt' e^{-\gamma t'} \right],
\]

where we have used the fact that

\[
\langle x | p \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{ipx/\hbar},
\]
Finally, by calculating the integrals, one obtains
\[ G_{\text{free}}(x, x'; t) = \sqrt{\frac{m\gamma}{2\pi\hbar}} \exp\left[ \frac{im\gamma}{2\hbar} \left( x - x' \right)^2 \right] \] (A8)
for freely propagating particle in viscid media within the CK framework. In the non-viscid limit, \( \gamma \rightarrow 0 \), (A8) gives the known result of the free particle propagator. It should be mentioned that in the case of scaled Schrödinger equation one must replace \( \hbar \) by \( \hbar' \) in the above equations.

It is remarkable that in the linear potential, Hamiltonian \( \hat{H}(t) = \frac{\hat{p}^2}{2m}e^{-\gamma t} + ma \hat{x}e^{\gamma t} \) does not commute at different times, \( [\hat{H}(t), \hat{H}(t_2)] = 2i\hbar \sinh[\gamma(t_2 - t_2)] \). Thus, the evolution operator \( U(t) \) does not have the simple form \( e^{-i\hat{H}t} \). Therefore, by the above method we cannot compute the propagator of the linear potential in CK model.

Moeira [23] has already computed the propagator of linear potential \( V(x) = ma \hat{x} \) within the framework of CK using the Lagrangian formulation:
\[ G(x, x'; t) = \exp\left[ -\frac{im}{2\hbar} \left( x e^{2\gamma t} - \gamma t - 1 \right) + x' e^{-2\gamma t} + \frac{im}{2\hbar} \left( e^{2\gamma t} - e^{-2\gamma t} - \gamma^2 t^2 - 2 \right) \right] \]
\[ \times \ G_{\text{free}}(x, x'; t) \]
(A9)
which has the correct form in the non-viscid limit.

Moeira [23] has also given propagator for the damped harmonic oscillator with Hamiltonian
\[ \hat{H} = \frac{\hat{p}^2}{2m}e^{-\gamma t} + \frac{1}{2} \omega_0^2 \hat{x}^2e^{\gamma t} \] as follows,
\[ G(x, x'; t) = \sqrt{\frac{m\omega_0\gamma^{3/2}}{2\pi\hbar \sinh(\omega t)}} \exp\left[ -\frac{im\gamma}{2\hbar} (x^2 - x'^2 e^{\gamma t}) + \frac{im\omega}{2\hbar \sinh(\omega t)} (x^2e^{2\gamma t} + x'^2 \cos(\omega t) - 2xx' e^{3\gamma t/2}) \right] \]
(A10)
with \( \omega = \sqrt{\omega_0^2 - \gamma^2/4} \).

### A.3. Fourier transform of the scaled free Gaussian wave packet
As usual, we define the Fourier transform of the scaled wave function \( \tilde{\psi}(x, t) \) as follows,
\[ \tilde{\phi}(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \ e^{-ipx/\hbar} \tilde{\psi}(x, t). \] (A11)

At first, we derive the scaled Schrödinger equation in momentum space. To this end, we compute the partial derivative of \( \tilde{\phi}(p, t) \) with respect to time
\[ i\hbar \frac{\partial}{\partial t} \tilde{\phi}(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \ e^{-ipx/\hbar} \left( i\hbar \frac{\partial}{\partial t} \tilde{\psi}(x, t) \right) \]
\[ = \frac{p^2}{2m} e^{-\gamma t} \tilde{\phi}(p, t) + \frac{1}{\sqrt{2\pi\hbar}} \int dx \ e^{-ipx/\hbar} \exp\left[ \frac{ip}{\hbar} \left( x - x_0 \right) - \frac{i}{\hbar} \gamma t \right] \]
where we have used the scaled Schrödinger equation (21) and the integration by parts. Thus, for the free particle in viscid media, the wave function in momentum space satisfies
\[ i\hbar \frac{\partial}{\partial t} \tilde{\phi}(p, t) = \frac{p^2}{2m} e^{-\gamma t} \tilde{\phi}(p, t). \] (A12)

Now, we compute the Fourier transform of the free damped scaled Gaussian wave packet (30) using (A11)
\[ \tilde{\phi}(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \ e^{-ipx/\hbar} \frac{1}{(2\pi\tilde{\sigma}_p)^{1/4}} \exp\left[ -\frac{(x - x_0)^2}{4\tilde{\sigma}_p}, \frac{i}{\hbar} p_0 (x - x_0) + \frac{i}{\hbar} \frac{p^2}{2m} 1 - e^{-\gamma t} \right] \]
\[ = \frac{(p - p_0)^2}{(2\pi\tilde{\sigma}_p)^{1/4}} \exp\left[ -\frac{i}{\hbar} p_0 x_0 - \frac{i}{\hbar} \frac{p^2}{2m} 1 - e^{-\gamma t} \right], \] (A13)
with
\[ \tilde{\sigma}_p = \frac{\hbar}{2\sigma_0^2}. \] (A14)
Thus, the canonical momentum distribution function reads
\[
|\tilde{\psi}(p, t)|^2 = \frac{1}{\sqrt{2\pi} \sigma_p} \exp\left[-\frac{(p - p_0)^2}{2\sigma_p^2}\right].
\] (A15)

This distribution function has a time-independent width $\sigma_p$ and is centered at $p_0$ and is independent on the friction coefficient $\gamma$.

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