ζ-function regularization and one-loop renormalization of field fluctuations in curved space-times

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Abstract:
A method to regularize and renormalize the fluctuations of a quantum field in a curved background in the ζ-function approach is presented. The method produces finite quantities directly and finite scale-parametrized counterterms at most. These finite counterterms are related to the presence of a particular pole of the effective-action ζ function as well as to the heat kernel coefficients. The method is checked in several examples obtaining known or reasonable results. Finally, comments are given for as it concerns the recent proposal by Frolov et al. to get the finite Bekenstein-Hawking entropy from Sakharov’s induced gravity theory.

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Introduction

This letter is devoted to develop a possible ζ-function approach to regularize and renormalize the averaged square field \( \langle \phi^2(x) \rangle \) in a general curved background. That quantity has been studied by several authors \[1\] because its knowledge is an important step to proceed with the point-splitting renormalization of the stress tensor and also due to its importance in cosmological theories. The knowledge of the value of \( \langle \phi^2(x) \rangle \) is also necessary to get the renormalized Hamiltonian from the renormalized stress tensor in non minimally coupled theories and this could be important dealing with thermodynamical considerations \[2, 3, 4, 5\]. Recently, it has also been studied in relation to the black hole entropy physics \[2, 3, 4\].

Let us consider a generic Euclidean field theory in curved background \( \mathcal{M} \) corresponding to the Euclidean action

\[
S_A[\phi] = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{g} \phi A \phi,
\]

where Euclidean motion operator \( A \) is supposed self-adjoint and positive-definite. The local ζ function related to the operator \( A \), \( \zeta(s, x|A) \), can be defined as the analytic continuation of the series \[6, 7\]

\[
\zeta(s, x|A) = \sum_n \lambda_n^{-s} \phi_n^*(x) \phi_n(x).
\]

where \( \{\phi_n(x)\} \) is a complete set of normalized eigenvectors of \( A \) with eigenvalues \( \lambda_n \), and in the sum the null eigenvectors are omitted. In Eq. (2), the right hand side is supposed to be analytically continued in the whole \( s \)-complex plane. Indeed, it is well-known that the ζ function is a meromorphic function with, at most, simple poles situated at \( s = 1 \) and \( s = 2 \) in case of a four dimensional spacetime. We are working formally with a discrete spectrum, but all considerations can be trivially extended to operators with continuous spectrum. The most importance of the ζ function is that the derivative of such a function evaluated at \( s = 0 \) defines a regularization of the one-loop effective action:

\[
S_{\text{eff}}[\phi, g] = \frac{1}{2} \frac{d}{ds} \zeta(s, x|A/\mu^2)
\]

where \( \mu \) is an arbitrary parameter with the dimensions of a mass necessary from dimensional considerations. Recently, the method has been extended in order to define a similar ζ-function regularization directly for the renormalized stress tensor \[4\].

The usual approach to compute the field fluctuation by the ζ-function techniques, leads to the naive definition for the one-loop averaged square field

\[
\langle \phi^2(x) \rangle := \zeta(s, x|A/\mu^2)|_{s=1},
\]

This definition follows directly from (2) taking account of the spectral decomposition of the two-point function. Anyhow, barring exceptional situations (e.g. see \[4\]), this definition is not available because the presence of a pole at \( s = 1 \) in the analytically continued ζ function
and further infinite subtraction procedures seem to be necessary. Conversely, in the cases of
the effective action and stress tensor regularization, the $\zeta$-function approach leads naturally to
the complete cancellation of divergences maintaining the finite $\mu$-parameterized counterterms
physically necessary [5]. We shall see shortly that, also in the case of $\langle \phi^2(x) \rangle$, it is possible
to improve the $\zeta$-function approach to get the same features: cancellation of all divergences
maintenance of the finite $\mu$-parameterized counterterms.

Finally we shall check the approach in some cases of physical interest, producing some
comments on the recent proposal to explain the Bekenstein-Hawking entropy in the framework
of Sakharov’s induced gravity [2, 3].

1 The general approach

Let us define the $\zeta$ function of $\langle \phi^2(x) \rangle$. The way we will follow is very similar to that followed
in [5] to compute the $\zeta$ function of the stress tensor. The one-loop effective action is given as

$$S_{\text{eff}}[g_{ab}] = \ln \int D\phi \ e^{S_A} = \ln \int D\phi \ e^{S_A}|_{m=0} - \frac{1}{2} \int d^4 x \sqrt{g} m^2 \phi^2. $$

(4)

Let us temporarily suppose to go “off-shell” as far as the field mass is concerned, namely let us
put $m^2 \rightarrow m^2(x)$, where $m^2(x)$ is a general smooth function. Then we have

$$\langle \phi^2(x) \rangle = \frac{2}{\sqrt{g(x)}} \frac{\delta S_{\text{eff}}}{\delta m^2(x)}|_{m^2(x)=m^2},$$

(5)

where $m^2$ is the actual value of the field mass. From the $\zeta$-function regularization, we can
formally write

$$\langle \phi^2(x) \rangle = \left. \frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta m^2(x)} \right|_{m^2(x)=m^2} \frac{d}{ds}|_{s=0} \zeta(s|A/\mu^2)$$

or, supposing to be possible to interchange the order of the derivatives

$$\langle \phi^2(x) \rangle = \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\sqrt{g(x)}} \frac{\delta \zeta(s|A/\mu^2)}{\delta m^2(x)}|_{m^2(x)=m^2}.$$

A definition of the functional derivative of the $\zeta$ function is now necessary. We get such
a definition following the same way as in [5] for the case of stress tensor. Let us first notice
that one can quite simply prove (see Appendix in [5] where proofs are carried out for similar
identities)

$$\frac{\delta \lambda_n}{\delta m^2(x)} = - \phi_n^*(x) \phi_n(x) \sqrt{g(x)}.$$ 

(6)

Then, we can define

$$\frac{\delta \zeta(s|A/\mu^2)}{\delta m^2(x)} := \sum_n \frac{\delta \lambda_n^-}{\delta m^2(x)}.$$ 

(7)
where we suppose to analytically continue the right hand side as far as is possible in the complex $s$ plane. As usual, the summation does not include null eigenvalues. Then, employing (6) we have (notice that the eigenvalues of $A/\mu^2$ are $\lambda_n/\mu^2$)

$$\frac{\delta \zeta(s|A/\mu^2)}{\delta m^2(x)} = \sqrt{g(x)} \frac{s}{\mu^2} \zeta(s + 1, x|A/\mu^2).$$

Substituting this result in the right hand side of (5) we have our definition of the $\zeta$-renormalized $\langle \phi^2(x) \rangle$

$$\langle \phi^2(x) \rangle := \frac{d}{ds}|_{s=0} s \zeta(s + 1, x|A/\mu^2),$$

(8)

where the $\zeta$ function is that evaluated at the actual value of the mass $m$. The identity above can be written down also as

$$\langle \phi^2(x) \rangle := \frac{d}{ds}|_{s=0} Z(s|A/\mu^2),$$

where

$$Z(s|A/\mu^2) := \frac{s}{\mu^2} \zeta(s + 1, x|A/\mu^2)$$

is the $\zeta$ function of the field fluctuation.

Notice that the simple pole at $s = 0$ in $\zeta(s + 1, x|A/\mu^2)$, whenever it exists, is canceled out by the factor $s$; moreover, when $\zeta(s + 1, x|A/\mu^2)$ is regular at $s = 0$, the definition in (8) coincides with the naive one, Eq. (3).

It is worth while stressing that, in general, the scale $\mu$ can remain into the final result. It represents the usual ambiguity due to the remaining finite renormalization already found as far as the effective action and the stress tensor are concerned [5]. In general, the scale could remain into the final result whenever another fixed scale is already present into the theory, e.g. the mass of the field or the curvature of the spacetime. Actually, it can disappear also in those cases provided particular conditions hold true (see below). The disappearance of the scale $\mu$ from the final result is equivalent to the possibility of using the definition (3). Indeed, (8) can be also written as

$$\langle \phi^2(x) \rangle := \frac{d}{ds}|_{s=0} s \zeta(s + 1, x|A) + s \zeta(s + 1, x|A)|_{s=0} \ln \mu^2.$$  

(9)

We see that $\ln \mu^2$ disappears if and only if $\zeta(s + 1, x|A)$ is analytic at $s = 0$, namely, if and only if we can use the definition (3).

Finally, it is interesting to investigate the relation between the presence of a pole in the $\zeta$ function and the heat-kernel coefficients. From the well-known asymptotic expansion of the local heat kernel on manifolds without boundary

$$K_t(x|A/\mu^2) \simeq \frac{1}{(4\pi)^{D/2}} \sum_{j=0}^{\infty} k_j(x|A/\mu^2) t^{j-D/2},$$

4
it is easy to show that the \( \zeta \) function has a pole in \( s = 1 \) if and only if the dimension \( D \) is even and the coefficient \( k_{D-1}(x|A) \) is non-vanishing. Therefore, in odd dimensions we can use the definition (3) of the fluctuations, while in even dimensions we can rewrite the definition (8) as

\[
\langle \phi^2(x) \rangle := \lim_{s \to 1} \left[ \zeta(s, x|A/\mu^2) - \frac{k_{D-1}(x|A/\mu^2)}{(4\pi)^{D/2}\Gamma(s)(s - 1)} \right] + \frac{\gamma}{(4\pi)^{D/2}} k_{D-1}(x|A/\mu^2),
\]

where \( \gamma \) is Euler’s constant. In four dimensions we have that \( k_1 = \frac{1}{6}(1 - 6\xi)R(x) - m^2 \), and so it is clear that the naive definition, Eq. (3), is available only in the massless case with vanishing scalar curvature or in the massless conformal coupling case, \( \xi = 1/6 \).

2 Simple applications and comments

As a first application we consider a scalar field in Minkowski space time. It is well known \cite{9} that for a massless field the corresponding \( \zeta \) function can be considered as vanishing, and so we pass directly to the massive case. The local \( \zeta \) function in four dimensions reads

\[
\zeta(s, x|A/\mu^2) = \frac{m^4(\mu/m)^{2s}}{16\pi^2(s - 1)(s - 2)},
\]

which has a simple pole at \( s = 1 \). Using Eq. (8) we obtain

\[
\langle \phi^2(x) \rangle = \frac{m^2}{16\pi^2} \left[ 2 \ln \frac{m}{\mu} - 1 \right].
\]

Then we consider a scalar massless field in Minkowski spacetime contained in a large box at the temperature \( \beta \). The local \( \zeta \) function is simply obtained (see \cite{6}) and reads

\[
\zeta(s, x|A/\mu^2) = \frac{\sqrt{\pi} \mu^4}{(2\pi)^3} \left( \frac{2\pi}{\beta \mu} \right)^{4-2s} \frac{\Gamma(s - 3/2)}{\Gamma(s)} \zeta_R(2s - 3),
\]

where \( \zeta_R(s) \) is the usual Riemann zeta function. Notice that no pole appears at \( s = 1 \), hence we could also use the naive definition (3) instead of (8). In both cases the result is

\[
\langle \phi^2(x) \rangle_\beta = \frac{1}{12\beta^2}.
\]

This result is the same which follows from other approaches (e.g., subtracting the Minkowski massless zero temperature two-point function from the thermal one and performing the limit of coincidence of the arguments).

Now we consider the Casimir effect due to two infinite parallel planes on which the field is constrained to vanish, namely we consider a massless scalar field in the Euclidean manifold \([0, L] \times R^3\). In this case the local \( \zeta \) function can be computed taking the Mellin transform of
the corresponding heat kernel, which is given, e.g., in [9]. A straightforward computation yields
\[ 0 < x < L/2 \]
\[ \zeta(s, x|A/\mu^2) = \frac{L^{2s-4} \Gamma(2-s)}{16\pi^2 \Gamma(s)} \left[ 2\zeta_R(4-2s) + \left( \frac{x}{L} \right)^{2s-4} - 2\zeta_R(4-2s, x/L) \right]. \] (11)

We see that the \( \zeta \) function is regular at \( s = 1 \) and so the fluctuations can be computed in the
naive way:
\[ \langle \phi^2(x) \rangle_\beta = \frac{1}{48L^2} - \frac{1}{8\pi^2 L^2} \left[ \zeta_R(2, x/L) - \frac{L^2}{2x^2} \right] \]
\[ = \frac{1}{48L^2} \left[ 1 - 3\csc^2 \frac{\pi x}{L} \right] \]
\[ \sim - \frac{1}{16\pi^2 x^2} + O(x^0), \]
in agreement with the known result, see, e.g., [10].

As the simplest application on a curved background we may consider the field fluctuation
of a massless conformally coupled scalar field on the open static Einstein universe with spatial
section metric
\[ a^2 \left[ dX^2 + \sinh^2 X (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \]
The modes can be built up following [11]. A
few calculations lead to the thermal local \( \zeta \) function
\[ \zeta(s, x|A) = \sqrt{\frac{\pi \mu}{2(2\pi)^2}} \left( \frac{2\pi}{\beta \mu} \right)^{4-2s} \frac{\Gamma(s-3/2)}{\Gamma(s)} \zeta_R(2s-3). \]

In practice, this is the \( \zeta \) function in the Minkowski large box. Hence we have similarly
\[ \langle \phi^2(x) \rangle_\beta = \pi \frac{\mu}{12 \beta^2}. \]

and thus at zero temperature \( \beta \to +\infty \) and \( \langle \phi^2(x) \rangle_\beta \to \langle \phi^2(x) \rangle_{\text{vacuum}} = 0 \) as well-known [11].

Another interesting application is the case of a massive scalar field near an idealized GUT
cosmic string with deficit angle \( 2\pi(1 - 1/\nu) \), whose local \( \zeta \) function has been recently computed
[14] in the limit \( mr \ll 1 \), \( r \) being is the distance from the core of the string. Keeping terms up
to \( O((mr)^2) \), we get the following expression for the fluctuations
\[ \langle \phi^2(x) \rangle = \frac{1}{48\pi r^2} \left\{ \nu^2 - 1 + 6(mr)^2 \left[ (\nu - 1) \left( \gamma - \ln \frac{r \mu}{2} \right) + \ln \nu + \frac{\nu}{2} \left( 2 \ln \frac{m}{\mu} - 1 \right) \right] \right\}. \] (12)

The above expression is different from the result obtained subtracting the Minkowski value
[13, 14], and in fact reduces to the Minkowski value, Eq. (10), rather than vanishing when the
conical singularity is removed, namely \( \nu \to 1 \). We note that in the massless case both procedures
give the same result, since the massless \( \zeta \) function is regular at \( s = 1 \).

The expression of the fluctuations given in Eq. (12) may also be interpreted as giving the
fluctuations of a massive scalar field in the Rindler space at temperature \( T = 1/\beta = \nu/2\pi \). This
is a consequence of the well-known correspondence between the cosmic string background and
the Rindler space, identifying the cone angle \( \beta = 2\pi/\nu \) with the inverse temperature. In this case the expression (12) is valid near the event horizon, \( r = 0 \).

It is interesting to use this expression to evaluate the Bekenstein-Hawking entropy of a black hole in the framework of Sakharov’s induced gravity [15], as done by Frolov et al. [2, 3]. In this appealing approach, Einstein action arises as the low-energy limit of the effective action of some quantum fields of large mass, \( N_s \) non minimally coupled scalar fields and \( N_d \) fermion fields. The Bekenstein-Hawking entropy is identified with the entropy of these fields propagating in the black hole background, which can be computed using the conical manifold method corrected with a surface term:

\[
S_{BH}(\beta_H) = \beta^2 \partial_\beta F_\beta|_{\beta=\beta_H} - \sum_s 2\pi \xi_s \int_\Sigma d\sigma \langle \phi^2_s \rangle,
\]

where \( \beta_H \) is the Hawking temperature for which the conical singularity is absent, \( F_\beta \) is the free energy of the fields propagating in the conical manifold and \( \Sigma \) is the horizon.

We want to discuss that problem very briefly within our approach (related massless cases without to consider Sakharov’s theory have been recently discussed in [8, 12]). Approximating the black-hole metric near the horizon with the Rindler metric, we can compute the free energy of a single scalar field from the local \( \zeta \) function as

\[
Z_\beta = \int \mathcal{L}(x) \sqrt{g} d^4x = \int \frac{1}{2} \frac{d}{ds} \zeta(s, x|A/\mu^2)|_{s=0} \sqrt{g} d^4x,
\]

where the \( \zeta \) function is the one given in [14]. Employing the result (12) at the Hawking temperature, namely \( \nu = 1 \), we get

\[
S_{BH}(\beta_H) = \frac{A_H}{120\pi \epsilon^2} \left[ 2 - 5m_s^2 \epsilon^2 \ln(R/\epsilon) + 15\pi m_s^2 \xi_s \epsilon^2 (2 \ln(m_s/\mu) - 1) \right],
\]

where \( A_H \) is the area of the horizon, \( \epsilon \) is a minimal distance from the horizon (‘horizon cut off’) and \( R \) is a large radius needed to control the volume divergence. As before, we have considered only terms up to order \( O((m_s r)^2) \), which is justified near the horizon.

It is clear from Eq. (14) that the surface terms \( \int_\Sigma d\sigma \langle \phi^2_s \rangle \) in Eq. (13) cannot cancel out the horizon divergences (as \( \epsilon \to 0 \)) coming from \( \beta^2 \partial_\beta F_\beta \), no matter how we choose fixed values of the masses \( m_s, m_d \) and couplings \( \xi_s \) and thus it seems that, within our regularization procedure, Frolov Fursaev Zelnikov’s idea does not works.

With regard of this point, it is worth noticing that in the local \( \zeta \) function computation of the black-hole entropy there is a very clear distinction between ultraviolet and horizon divergences: the former are cured already in the local quantities by the \( \zeta \) function procedure, while the latter arise only in global quantities obtained integrating over the manifold local quantities with non-integrable divergences at the horizon. In order to control this divergences the integrations are stopped at a distance \( r = \epsilon \) form the horizon.

Instead, in the global heat-kernel plus proper-time regularization employed in [2, 3] no such distinction is available (see also [17], where the ultraviolet cut-off \( L^{-1} \) and the minimal distance form the horizon \( \epsilon \) are identified, \( \epsilon = L^{-1} \)). Indeed, if the global \( \zeta \) function is computed taking
the Mellin transform of the global heat kernel, then the theory seems not to have any horizon divergence, in contrast with the results of others local approaches \[18\]. Obviously the global $\zeta$ function obtained through this way disagrees with the global $\zeta$ function obtained by integrating the local one. This disagreement between global approach and local approach is peculiar of Euclidean conical manifolds.

We stress further that the direct global approach produces an apparently unphysical temperature dependence for $\beta \neq \beta_H$ \[18\] and it is not clear how to relate the global quantities to local quantities as the renormalized stress tensor in such an approach.

We finally notice that the expression of $\langle \phi_s^2 \rangle_{\beta_H}$ employed in \[2, 3\] is simply the flat space result, as can be easily checked, and so it shows the usual ultraviolet divergences, but it has no horizon divergence. As a consequence, it seems to us that the method employed in \[2, 3\] to compute $S_{BH}$ allows the cancellation of the ultraviolet divergences, as possible with less assumptions, within the Rindler approximation, using the local $\zeta$ function, but not the cancellation of the horizon divergences, which do not appear in \[2, 3\].

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