We analyze maximal supersymmetry in eleven-dimensional supergravity from the point of view of the oriented matroid theory. The mathematical key tools in our discussion are the Englert solution and the chirotope concept. We argue that chirotopes may provide other solutions not only for eleven-dimensional supergravity but for any higher dimensional supergravity theory.

Keywords: eleven-dimensional supergravity, oriented matroid theory, chirotopes

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1.- Introduction

The main purpose of this brief note is to discuss the importance of applying the oriented matroid theory [1] to supergravity theories. In particular, we focus on the possibility of relating the chirotope concept [1] of oriented matroid theory to the Freund-Rubin-Engler solution of eleven-dimensional supergravity [2]-[4].

As it is known, the Freund-Rubin and Englert solutions compactify $d = 11$ spacetime $M^{11}$ into a product of two spaces: 4-dimensional anti-de Sitter manifold $AdS_4$ and the seven sphere $S^7$. It turns out that while the Freund-Rubin solution corresponds to maximally supersymmetric solution preserving the full supersymmetry of eleven-dimensional supergravity action, and in that sense can be considered as a ”trivial” solution [5]-[6], the Englert solution leads to spontaneous breakdown of maximal supersymmetry and therefore can be interpreted as a ”non-trivial solution” associated with $S^7$-geometry.

A key object in the above solutions is a four-form field strength $F = dA$ or $F^{ABCD}$, with $A, B, C, D = 0, ..., 10$. In fact, if one assumes that the only non-vanishing components of $F^{ABCD}$ are proportional to the completely antisymmetric symbol $\varepsilon^{\mu\nu\alpha\beta}$, with $\mu, \nu, \alpha, \beta = 0, ..., 3$, then the trivial solution arises from the bosonic sector of eleven-dimensional supergravity field equations. While if in addition one assumes non-vanishing values for $F^{ijkl}$, with $ijkl = 4, ..., 10$, one obtains the non-trivial solution. From this perspective it becomes evident that it is important to study, deeply, the algebraic properties of $F^{ABCD}$ and their relation with the trivial and non-trivial solutions.

One can observe, for instance, that since in the case of maximally supersymmetric solutions $F^{ABCD}$ is decomposable, it must be possible to relate it to the chirotope concept via the Grassmann-Plücker relations (see Ref. [7]). Here, we are interested in investigating the connection between the object $F^{ABCD}$ associated with the Englert solution and the chirotope concept. We argue that, in principle, our analysis can be used to find new solutions for eleven-dimensional supergravity.

It is worth mentioning that the oriented matroid theory has been connected with a number of topics, including $p$-branes [8], Chern-Simons theory [9], superstrings [10], gravity [11] and two time physics [12]. In particular, using the phirotope concept [13]-[15], which is a generalization of chirotopes concept, in Ref. [7] a relation with super$p$-branes has been established. These progresses have motivated a proposal of considering the oriented matroid the-
ory as a mathematical framework for $M$-theory [16].

In this paper, after a brief review of maximally supersymmetric solution, we review the Englert solution putting special emphasis in the algebraic identities of the structure constants for octonions which allow a connection with the oriented matroid theory via the chirotope concept. Specifically, we prove that not only in the case of the Freund-Rubin solution of eleven-dimensional supergravity the four-form field $F$ admits an interpretation of chirotope but also in the case of the Englert solution. The key idea in this proof is that both sectors of $D = 11$ supergravity, 4-dimensional anti-de Sitter manifold $AdS_4$ and the seven sphere $S^7$, admit a realizable chirotope interpretation, although the full four-form field $F$ may correspond to nonrealizable chirotopes. Since to each chirotope one can associate its dual we find that our investigation may open the possibility to find dual solutions of $D = 11$ supergravity.

2.- Maximally Supersymmetric Solutions

Let $(M^{11}, g, F)$ be a maximally supersymmetric solution of eleven-dimensional supergravity. In the non-degenerate case, Figueroa-O’Farrill and Papadopoulos proved the theorem [5] that such a solution must be isometric to either $AdS_4 \times S^7$ or $AdS_7 \times S^4$. Their starting point in this result was the vanishing of the curvature $\mathcal{R}$ of the supercovariant connection $\mathcal{D}$. In fact, demanding the vanishing of the curvature $\mathcal{R}$ they found that $(M^{11}, g, F)$ is a maximally supersymmetric solution if and only if $(M^{11}, g)$ is locally symmetric space and $F$ is parallel and decomposable, and from this results such a theorem follows (see Ref. [5] for details).

Here, we are interested in revisiting the fact that $F$ is decomposable. From the formula $\mathcal{R} = 0$ one can essentially derive two algebraic formulae for $F$, namely

$$F \wedge F = 0$$  \hspace{1cm} (1)

and

$$\iota_X F \wedge \iota_Y F = 0,$$ \hspace{1cm} (2)

where $\iota_X$ and $\iota_Y$ denote an inner product for the two vectors $X$ and $Y$, respectively. From these two formulae one then shows that $F$ satisfies the relation
\[ \iota_Z \iota_Y \iota_X F \wedge F = 0. \quad (3) \]

Conversely, if (3) is satisfied then (1) and (2) follow. The formula (3) means that \( F \) is decomposable, that is, (3) implies that \( F \) can be written as the wedge product of four one-forms.

The way that Figueroa-O’Farrill and Papadopoulos prove that (1) and (2) imply (3) is by first observing that contracting (1) with respect to the three vectors \( X, Y \) and \( Z \) one obtains

\[ \iota_Z \iota_Y \iota_X F \wedge F = -\iota_Y \iota_X F \wedge \iota_Z F. \quad (4) \]

While, contracting equation (2) with a third vector field one gets

\[ \iota_Y \iota_X F \wedge \iota_Z F = \iota_Y \iota_Z F \wedge \iota_X F. \quad (5) \]

Thus, comparing (4) and (5) one sees that whereas (5) implies that the expression \( \iota_Y \iota_X F \wedge \iota_Z F \) is symmetric in \( X \) and \( Z \), (4) means that it is skew-symmetric. This implies that the term \( \iota_Y \iota_X F \wedge \iota_Z F \) must vanish.

### 3.- Figueroa-O’Farrill-Papadopoulos formalism revisited

Let us first write the algebraic expressions (1), (2) and (3) in the alternative way

\[ F_{[A_1 A_2 A_3 A_4 F_{B_1 B_2 B_3 B_4}] = 0,} \quad (6) \]

\[ F_{A_1[A_2 A_3 A_4 F_{B_1 B_2 B_3}B_4] = 0,} \quad (7) \]

and

\[ F_{A_1 A_2 A_3 [A_4 F_{B_1 B_2 B_3}B_4] = 0,} \quad (8) \]

respectively. Here, the bracket \([,]\) means completely antisymmetric.

If we are not interested in using differential forms notation as in section 2, the contraction of (6) and (7) with respect to different vectors forces to us to define the bracket \([,]\) in the following form

\[ G_{[A_1 \ldots A_{d+1}] \equiv G_{C_1 \ldots C_{d+1}} \delta_{A_1 \ldots A_{d+1}}^{C_1 \ldots C_{d+1}}.} \quad (9) \]
Here, $G_{C_1...C_{d+1}}$ is any $d + 1$-rank tensor and $\delta_{A_1...A_{d+1}}^{C_1...C_{d+1}}$ is the generalized delta symbol. The advantage of this notation is that several properties of the generalized delta can be used. For instance, considering the fact that

$$\delta_{A_1...A_{d+1}}^{C_1...C_{d+1}} = \delta_A^{C_1} \delta_{A_2...A_{d+1}}^{C_2...C_{d+1}} + \sum_{k=2}^{d+1} (-1)^k \delta_A^{A_k} \delta_{A_2...A_{k-1}}^{C_2...C_{d+1}}$$

(10)

where $\hat{A}_k$ means omitting this index, one can easily prove that (6) is satisfied if and only if

$$F_{A_1C_3C_4F_{D_1D_2D_3D_4}} = 0,$$

which is equivalent to

$$F_{A_1[A_2A_3A_4F_{B_1B_2B_3B_4}] = 0.}$$

(12)

In the notation of section 2 this result can be written as

$$\iota_X F \wedge F = 0.$$  

(13)

Properly, applying (10) to the expression (11) once again we get

$$F_{A_1C_3C_4F_{D_1D_2D_3D_4} \delta_{A_2A_3A_4B_1B_2B_3B_4}} = 3F_{A_1A_2[A_3A_4F_{B_1B_2B_3B_4}]} + 4F_{A_1[A_3A_4B_1F_{B_2B_3B_4}]A_2}.$$

(14)

Thus, considering the fact that the second term of this expression corresponds to formula (2) we can set

$$F_{A_1[A_3A_4B_1F_{B_2B_3B_4}]A_2} = 0,$$

and therefore using (12) we obtain the result

$$F_{A_1A_2[A_3A_4F_{B_1B_2B_3B_4}] = 0.}$$

(16)

Similar technique leads to the identity

$$F_{A_1A_2[A_3A_4F_{B_1B_2B_3B_4}] = 2F_{A_1A_2A_3[A_4F_{B_1B_2B_3B_4}]} + 4F_{A_1A_2[A_4B_1F_{B_2B_3B_4}]A_3}.}$$

(17)

Thus, using (16) we find

$$F_{A_1A_2A_3[A_4F_{B_1B_2B_3B_4}] = -2F_{A_1A_2[A_4B_1F_{B_2B_3B_4}]A_3}.}$$

(18)
This expression implies that the right hand side of (18) is antisymmetric in the indices $A_1$ and $A_3$.

On the other hand we have

$$F_{A_1[A_2A_4B_1}F_{B_2B_3B_4]A_3} = 3F_{A_1A_2[A_4B_1}F_{B_2B_3B_4]A_3} - 3F_{A_1[A_4B_1B_2}F_{B_3B_4]A_2A_3}$$

$$= 3F_{A_1A_2[A_4B_1}F_{B_2B_3B_4]A_3} - 3F_{A_3A_2[A_4B_1}F_{B_2B_3B_4]A_1}.$$  

(19)

From (15) we see that the left hand side of (19) vanishes and therefore we obtain

$$F_{A_1A_2[A_4B_1}F_{B_2B_3B_4]A_3} = F_{A_3A_2[A_4B_1}F_{B_2B_3B_4]A_1}. \number{20}$$

This means that $F_{A_1A_2[A_4B_1}F_{B_2B_3B_4]A_3}$ is symmetric in the indices $A_1$ and $A_3$ which contradicts the conclusion below (18). Thus we have found that the only consistent possibility is to set

$$F_{A_1A_2A_3[A_4}F_{B_1B_2B_3B_4]} = 0. \number{21}$$

Summarizing, we have shown that (12) and (15) imply (21). Conversely, using once again the properties of the delta generalized $\delta_{A_1...A_{d+1}}^{C_1...C_d...}$, one can show that both $F_{A_1[A_2A_3A_4}F_{B_1B_2B_3B_4]}$ and $F_{A_1[A_3A_4B_1}F_{B_2B_3B_4]A_2}$ can be written in terms of $F_{A_1A_2A_3[A_4}F_{B_1B_2B_3B_4]}$ and therefore (21) implies (12) and (15). This means that the expression (21) is equivalent to the two formulae (12) and (15). Thus, we have complete an alternative proof of such an equivalence.

The formula (21) implies that $F_{A_1A_2A_3A_4}$ is decomposable. This means that $F_{A_1A_2A_3A_4}$ can be written in the form

$$F^{A_1A_2A_3A_4} = \varepsilon^{a_1a_2a_3a_4}v^A_{a_1}v^A_{a_2}v^A_{a_3}v^A_{a_4}, \number{22}$$

where $v^A_a$ is an arbitrary $4 \times d + 1$-matrix. Thus, one may conclude, as Figueroa-O’Farrill and Papadopoulos did, that the maximal supersymmetry of eleven-dimensional supergravity implies that $F^{A_1A_2A_3A_4}$ can be written as (22). In the non-degenerate case, spontaneous compactification allows to assume that the only nonvanishing components of $v^A_a$ are $v^\mu_a \sim \delta^\mu_a$, with $\mu = 0, 1, 2, 3$ or $v^{\tilde{\mu}}_a \sim \delta^{\tilde{\mu}}_a$, with $\tilde{\mu} = 8, 9, 10, 11$ leading to the two possible solutions $AdS_4 \times S^7$ or $AdS_7 \times S^4$, respectively (see Ref. [5] for details). In fact, in the first case one gets from (22) that the only nonvanishing components
of $F^{A_1 A_2 A_3 A_4}$ are $F^{\mu \alpha \beta} \sim \varepsilon^{\mu \alpha \beta}$ and therefore one obtains from the eleven-dimensional field equations

$$
\varepsilon^{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4 NPQ} F^{NPQM} ; M = \frac{1}{24} F_{[A_1 A_2 A_3 A_4} F_{B_1 B_2 B_3 B_4]},
$$

$$
R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{6} F_{MPQR} F^P_{QR} - \frac{1}{16} g_{MN} F_{SPQR} F^{SPQR},
$$

the Freund-Rubin solution $AdS_4 \times S^7$. While in the second case $F^{\hat{\mu} \hat{\nu} \hat{\alpha} \hat{\beta}} \sim \varepsilon^{\hat{\mu} \hat{\nu} \hat{\alpha} \hat{\beta}}$, the field equations (23) lead to the solution $AdS_7 \times S^4$.

4.- Englert solution revisited

In the case of Englert solution we have also $AdS_4 \times S^7$, but (1) or (6) are no longer satisfied and therefore the right hand side of the first field equation in (23) is different from zero. This means that according to the discussion of the previous section maximal supersymmetry is broken. In turn, this implies that $F^{A_1 A_2 A_3 A_4}$ cannot be written in the form (22) or in other words $F^{A_1 A_2 A_3 A_4}$ does not satisfy (21). However we would like to emphasize that in spite of $F^{A_1 A_2 A_3 A_4}$ does not satisfy (21) the background solution is still the same as the Freund-Rubin solution, namely $AdS_4 \times S^7$. How is this possible? The answer comes from one of the division algebras: the octonionic structure.

Consider the octonion identity

$$
f_{abcd} f_{efgd} = \delta^e_b \delta^d_g + \frac{1}{4} f_{[ab}^{[ef} \delta^{g]}_c],
$$

(24)

with the indices $a, b, ...etc$ running from 4 to 11. Here, $f_{abcd}$ is a self dual object. Furthermore, $f_{abcd}$ is defined in terms of the octonion structure constants $\psi_{ijk}$ and its dual $\varphi_{ijkl}$ through the relations

$$
f_{ijk11} = \psi_{ijk}
$$

(25)

and

$$
f_{ijkl} = \varphi_{ijkl}.
$$

(26)

From (24) it is not difficult to see that

$$
f_{[ijk} f_{lmn]} r = 0.
$$

(27)
This expression can be understood as a solution for

\[ f_{s[ijk}f_{lmn]}r = 0, \]  

(28)

which remains us the formula (15) reduced to seven dimensions. In fact, following a Gürsey and Tze [6], introducing a sieben-bein \( h^r_k \) one can make this identification

\[ F_{ijkl} = h^r_i h^s_j h^t_k h^m_l f_{rstm} \]  

(29)

and therefore (28) leads to

\[ F_{s[ijk}F_{lmn]}r = 0. \]  

(30)

Starting from (24) and following similar arguments we may establish that

\[ F_{s[ijk}F_{lmnr]} = 0 \]  

(31)

and

\[ F_{[sijk}F_{lmnr]} = 0. \]  

(32)

Thus, according to the discussion of previous sections (30) and (31) imply that \( F_{ijkl} \) satisfies the relation

\[ F_{sj[k}F_{lmnr]} = 0 \]  

(33)

which means that \( F_{ijkl} \) is decomposable.

On the other hand, in four dimensions as we already mentioned, we can take

\[ F_{\mu\nu\alpha\beta} = \lambda \varepsilon_{\mu\nu\alpha\beta}, \]  

(34)

where \( \lambda \) is an arbitrary function. Since \( \varepsilon_{\mu\nu\alpha\beta} \) is a maximal completely antisymmetric object in four dimensions we get the formula

\[ F_{\mu\nu\alpha[\beta F_{\sigma\rho\gamma]} = 0, \]  

(35)

which implies

\[ F_{[\mu\nu\alpha\beta F_{\sigma\rho\gamma]} = 0. \]  

(36)

Thus, \( F_{\mu\nu\alpha\beta} \) is also decomposable.
Our main observation is that despite both $F_{ijkl}$ and $F_{\mu\nu\alpha\beta}$ are decomposable the eleven-dimensional four-form $F_{ABCD}$ is not. The reason comes from the fact that if $F_{ijkl}$ and $F_{\mu\nu\alpha\beta}$ are decomposables and the only nonvanishing components of $F_{ABCD}$, the relation $F_{A_1A_2A_3A_4}F_{B_1B_2B_3B_4}$ is different from zero and therefore the full $F_{ABCD}$ is not decomposable. The result follows from the expression

$$F_{\mu\nu\alpha[\beta}F_{ijkm]} \neq 0,$$

or

$$F_{[\mu\nu\alpha\beta}F_{ijkm]} \neq 0.$$  \hspace{1cm} (37)

In fact, since $\varepsilon^{\mu\nu\alpha\beta}$ and $f^{ijkm}$ take values in the set $\{-1, 0, 1\}$ in general we have that

$$\varepsilon_{\mu\nu\alpha[\beta}f_{ijkm]} \neq 0,$$

or

$$\varepsilon_{[\mu\nu\alpha\beta}f_{ijkm]} \neq 0.$$  \hspace{1cm} (38)

In turn this means that $F_{[A_1A_2A_3A_4}F_{B_1B_2B_3B_4]} \neq 0$ or $F \wedge F \neq 0$ and, according to the discussion of section 2, consequently we no longer have maximal supersymmetric solution. Nevertheless, as Englert showed, although the right hand side of the first field equation in (23) is not vanishing the field equations still admit the solution $AdS_4 \times S^7$.

5.- Connection with chirotopes

The aim of this section is to discuss the formalism described in section 2, 3 and 4 from the point of view of the oriented matroid theory. Indeed, our discussion will focus on the chirotope concept which provides one possible definition of an oriented matroid.

Chirotopes had been a major subject of investigation in mathematics during the last 25 years [1]. Roughly speaking a chirotope is a combinatorial abstraction of subdeterminants of a given matrix. More formally, a realizable $p$-rank chirotope is an alternating function $\chi : \{1, \ldots, n\}^p \rightarrow \{-1, 0, 1\}$ satisfying the Grassmann-Plücker relation.
\[ \chi_{\hat{A}_1 \ldots \hat{A}_{n-1}[\hat{A}_p, \chi_{\hat{B}_1 \ldots \hat{B}_p}] = 0, \quad (41) \]

while nonrealizable \( p \)-rank chirotope corresponds to the case

\[ \chi_{\hat{A}_1 \ldots \hat{A}_{n-1}[\hat{A}_p, \chi_{\hat{B}_1 \ldots \hat{B}_p}] \neq 0. \quad (42) \]

It is worth mentioning that there is a close connection between chirotopes and Grassmann variety. In fact, the Grassmann-Plücker relations describe a projective embedding of the Grassmannian of planes via decomposable \( p \)-forms (see Ref. [1] for details).

Thanks to our revisited review of Freund-Rubin and Englert solutions given in the previous sections we find that the link between this these solutions and the chirotope is straightforward. In fact, our first observation is that any \( \varepsilon \)-symbol is in fact a realizable chirotope (see Refs. [12] and [7]), since it is always true that

\[ \varepsilon_{\hat{A}_1 \ldots \hat{A}_{n-1}[\hat{A}_n, \varepsilon_{\hat{B}_1 \ldots \hat{B}_n}] = 0. \quad (43) \]

From this perspective we recognize that the formula (21) indicates that in the case of maximal supersymmetry the four-form \( F_{ABCD} \) is a realizable 4-rank chirotope. While in the case of Englert solution, from (39) and (42) we discover that one may identify \( F_{ABCD} \) with a nonrealizable 4-rank chirotope. We think that this identification open the possibility to introduce other chirotopes no necessarily related to octonions as a solution for eleven-dimensional gravity.

6.- Final Remarks

We have identified the Freund-Rubin-Englert solution for eleven-dimensional supergravity with the chirotope concept. In the case of maximally supersymmetric solution the four-form \( F_{ABCD} \) can be identified with a realizable 4-rank chirotope, while in the case of Englert solution, \( F \) may correspond to a nonrealizable 4-rank chirotope. However, there are many possible 4-rank chirotopes in eleven dimensions and therefore there must be many new and unexpected solutions for eleven dimensional gravity.

One of our key tools in our formalism is the octonionic structure. This division algebra was already related to the Fano matroid and therefore, a
possible connection with supergravity was established in Ref [17]. Here, we have been more specific and through the chirotope concept we established the relation between the Freund-Rubin-Englert solution and oriented matroid theory. However, it may be interesting to understand the possible role of the Fano matroid in this scenario.

Here, we focused on eleven-dimensional supergravity but according to the results given in Ref. [6] in principle, one may expect to apply similar procedure in the case of ten-dimensional supergravity and other higher dimensional supergravities such as Type I supergravity and massive IIA supergravity.

An important property in the oriented matroid theory is that one can associate any chirotopes with its dual. Thus, working on the framework of oriented matroids we can assure that any possible solution for eleven-dimensional gravity in terms of chirotopes shall have a dual solution. This means that this kind of solution contains automatically a dual symmetry.

Using the idea of matroid bundle [18]-[22], Guha [23] has observed that chirotopes can be related to Nambu-Poisson structure. It may be interesting for further research to see whether the present formalism can be useful to bring the Nambu-Poisson structure to eleven-dimensional supergravity.

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