A SIMPLE SOLUTION OF THE LOTKA-VOLterra EQUATIONS

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Abstract

In this work we consider a simple, approximate, tending toward exact, solution of the system of two usual Lotka-Volterra differential equations. Given solution is obtained by an iterative method. In any finite approximation order of this solution, exponents of the corresponding Lotka-Volterra variables have simple, time polynomial form. When approximation order tends to infinity obtained approximate solution converges toward exact solution in some finite time interval.

1 Introduction

As it is well-known [1]-[5] the following system of two nonlinear differential equations of the first order

\[
\frac{dx}{dt} = ax - bxy \\
\frac{dy}{dt} = -cy + dxy
\]

represents the remarkable, usual Lotka-Volterra differential equations system with significant applications in the ecology, biology, medicine etc.. Here \(x, y\), represent the real, positive variables that depends of the time, \(t\), while \(a, b, c, d\), represent real, positive time independent constants, i.e. parameters. It is supposed that the initial values

\[
x(0) = x_0
\]
of $x, y$ are known. Existing, exact solution of the system (1), (2) presentable in the simple (closed) form (combination of the simple functions of the time, eg. a polynomial of the time etc.) is unknown to this day. For this reason different approximate, especially numerical (Runge-Kutta etc.), methods for solution of the system (1), (2) are used. But, even in this case approximate solution of the system (1), (2) presentable in a simple form is unknown.

In this work we shall suggest a simple, approximate, tending toward exact, solution of the Lotka-Volterra equations system (1), (2). Given solution will be obtained by an iterative method. In any finite approximation order of this solution, exponents of the corresponding Lotka-Volterra variables have simple, time polynomial form. When approximation order tends to infinity obtained approximate solution converges toward exact, simple solution in some finite time interval. All this can be very useful for the applications in many domains of the ecology, biology, medicine, etc., but explicit consideration of such applications goes over basic intentions of this work.

2 A simple approximate solution of the usual Lotka-Volterra equations system

As it is well-known [1]-[5] usual Lotka-Volterra equations system (1), (2) can be simply transformed in the following system

$$\frac{d \ln x}{dt} = a - by$$

(5)

$$\frac{d \ln y}{dt} = -c + dx$$

(6)

or, after well-known changes of the variables,

$$u = \ln x \Leftrightarrow x = \exp u$$

(7)

$$v = \ln y \Leftrightarrow y = \exp v$$

(8)

(so that $u, v$ represent the real variables, exponents of $x, y$), in the system

$$\frac{du}{dt} = a - b \exp v$$

(9)

$$\frac{dv}{dt} = -c + d \exp u$$

(10)

It is supposed that the initial values

$$u(0) = u_0$$

(11)

$$v(0) = v_0$$

(12)

of $u, v$ corresponding to initial values $x_0, y_0$ (3), (4) are known.

Solution of the system (9), (10) for conditions (11), (12) will be supposed in the following approximate form characteristic for $n$-th approximation step

$$u_{(n)} = A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n \quad \text{for} \quad n = 0, 1, 2,$$

(13)

$$v_{(n)} = B_0 + B_1 t + B_2 t^2 + \ldots + B_n t^n \quad \text{for} \quad n = 0, 1, 2,$$

(14)
where $A_n$ and $B_n$ for $n = 0, 1, 2, \ldots$ are unknown real coefficients. These coefficients will be determined by supposition

\begin{align*}
A_0 &= u_0 \\
B_0 &= v_0
\end{align*}

and by approximate iterative method

\begin{align*}
\frac{du(n)}{dt} &= a - b \exp[v_{n-1}] \quad \text{for} \quad n = 1, 2, \ldots \quad (17) \\
\frac{dv(n)}{dt} &= -c + d \exp[u_{n-1}] \quad \text{for} \quad n = 1, 2, \ldots \quad (18)
\end{align*}

In the first approximation order, i.e. for $n = 1$, introduction of (15), (16) in (17), (18) yields immediately

\begin{align*}
A_1 &= a - b \exp[v_0] \\
B_1 &= -c + d \exp[u_0]
\end{align*}

In this way coefficients $A_1$ and $B_1$ are completely determined.

In the second approximation order, i.e. for $n = 2$, introduction of (15), (16), (19), (20) in (17), (16) yields

\begin{align*}
A_1 + 2A_2t &= a - b \exp[v_0 + B_1t] = a - b \exp[v_0] \exp[B_1t] \\
B_1 + 2B_2t &= -c + d \exp[u_0 + A_1t] = -c + d \exp[u_0] \exp[A_1t]
\end{align*}

For additional conditions

\begin{align*}
|B_1t| &\ll 1 \\
|A_1t| &\ll 1
\end{align*}

i.e. for

\begin{align*}
t &\ll \left| \frac{1}{B_1} \right| \\
t &\ll \left| \frac{1}{A_1} \right|
\end{align*}

right hands of (21), (22) can be approximated by theirs linear Taylor expansions which yields

\begin{align*}
A_1 + 2A_2t &= a - b \exp[v_0](1 + B_1t) \\
B_1 + 2B_2t &= -c + d \exp[u_0](1 + A_1t)
\end{align*}

It, according to (19), (20), yields

\begin{align*}
A_2 &= -\frac{bB_1 \exp[v_0]}{2} \\
B_2 &= \frac{dA_1 \exp[u_0]}{2}
\end{align*}

In this way coefficients $A_2$ and $B_2$ are completely determined.

In $n + 1$-th approximation order introduction of (13), (14) in (17), (18) yields

\begin{align*}
A_1 + 2A_2t + \ldots + (n + 1)A_{n+1}t^n &= a - b \exp[v_0 + B_1t + B_2t^2 + \ldots + B_nt^n] = \\
&= a - b \exp[v_0] \exp[B_1t + B_2t^2 + \ldots + B_nt^n] \quad \text{for} \quad n = 2, 3, \ldots 
\end{align*}
\[ B_1 + 2B_2 t + ... + (n+1)B_{n+1}t^n = -c + d \exp[u_0 + A_1 t + A_2 t^2 + ... + A_n t^n] = -c + \sum_{i=1}^n \exp[u_i] \exp[A_{i+1} + A_{i+2} t^2 + ... + A_n t^n] \quad \text{for} \quad n = 2, 3, ... \quad (32) \]

Also, suppose that the following additional approximation conditions are satisfied

\[ 1 \gg |B_1 t + B_2 t^2 + ... + B_n t^n| \quad \text{for} \quad n = 2, 3, ... \quad (33) \]

\[ 1 \gg |B_1 t| \gg |B_2 t^2| \gg ... \gg |B_n t^n| \quad \text{for} \quad n = 2, 3, ... \quad (34) \]

\[ 1 \gg |A_1 t + A_2 t^2 + ... + A_n t^n| \quad \text{for} \quad n = 2, 3, ... \quad (35) \]

\[ 1 \gg |A_1 t| \gg |A_2 t^2| \gg ... \gg |A_n t^n| \quad \text{for} \quad n = 2, 3, ... \quad (36) \]

Obviously, condition (34) ensures the convergence of (14), while condition (36) ensures the convergence of (13). As it is not hard to see, all conditions (33)-(36) can be changed by one condition

\[ t \ll \min\left(\frac{1}{B_1}, \frac{B_2}{B_1}, ..., \frac{B_n}{B_{n-1}}, \frac{1}{A_1}, \frac{A_2}{A_1}, ..., \frac{A_n}{A_{n-1}}\right) \quad \text{for} \quad n = 2, 3, ... \quad (37) \]

where right hand of the inequality (37) represents the minimum function equivalent to the minimal arguments.

According to (33)-(36), right hands of (31),(32) can be approximated by theirs linear Taylor expansions which yields

\[ A_1 + 2A_2 t + ... + nA_n t^n = a - b \exp[u_0] [1 + B_1 t + B_2 t^2 + ... + B_n t^n] \quad \text{for} \quad n = 2, 3, ... \quad (38) \]

\[ B_1 + 2B_2 t + ... + nB_n t^n = -c + d \exp[u_0] [1 + A_1 t + A_2 t^2 + ... + A_n t^n] \quad \text{for} \quad n = 2, 3, ... \quad (39) \]

Then, it follows

\[ A_n = -\frac{bB_{n-1} \exp[u_0]}{n} \quad \text{for} \quad n = 2, 3, ... \quad (40) \]

or

\[ A_n = -\frac{b \exp[u_0 + v_0]}{n(n-1)} A_{n-2} \quad \text{for} \quad n = 2, 3, ... \quad (42) \]

\[ B_n = -\frac{b \exp[u_0 + v_0]}{n(n-1)} B_{n-2} \quad \text{for} \quad n = 2, 3, ... \quad (43) \]

Or,

\[ A_{2k} = \frac{(-bd \exp[u_0 + v_0])^k}{(2k)!} A_0 \quad \text{for} \quad k = 1, 2, 3, ... \quad (44) \]

\[ A_{2k+1} = \frac{(-bd \exp[u_0 + v_0])^k}{(2k+1)!} A_1 \quad \text{for} \quad k = 1, 2, 3, ... \quad (45) \]

\[ B_{2k} = \frac{(-bd \exp[u_0 + v_0])^k}{(2k)!} B_0 \quad \text{for} \quad k = 1, 2, 3, ... \quad (46) \]
\[ B_{2k+1} = \frac{(-bd \exp[u_0 + v_0])^k}{(2k+1)!} B_1 \quad \text{for} \quad k = 1, 2, 3, ... \] (47)

In this way coefficients \( A_n \) and \( B_n \) are completely determined by (44)-(47) for \( n = 2, 3, \ldots \).

So, according to previous analysis, coefficients \( A_n \) and \( B_n \) are completely determined for any \( n \), i.e. for \( n = 0, 1, 2, \ldots \). Obviously, coefficients \( A_n \) and \( B_n \) appear explicitly in \( n \)-th approximation order but their form stand conserved in any next approximation order, for \( n = 0, 1, 2, \ldots \).

Further, from (40)-(47) it follows

\[ \left| \frac{A_n}{A_{n+1}} \right| (n + 1) \quad \text{for} \quad n = 2, 3, ... \] (48)

\[ \left| \frac{B_n}{B_{n+1}} \right| (n + 1) \quad \text{for} \quad n = 2, 3, ... \] (49)

It admits that, in a rough approximation, (37) be reduced in

\[ t \ll \min\left(\left| \frac{1}{B_1} \right|, \left| \frac{1}{A_1} \right| \right) \] (50)

It demonstrates roughly that suggested approximate solution (13), (14) of the system (9), (10) for conditions (11), (12) is consistent and convergent in a time interval \([0, \tau]\) (whose explicit form will not be considered here) that is not infinite small, i.e. infinitesimal.

Obviously, when \( n \) tends toward infinity consistent and convergent approximate solution (13), (14) of the system (9), (10) for conditions (11), (12) in a finite time interval \([0, \tau]\) tends toward exact solution.

Finally, without detailed analysis, we can observe the following. We do not consider here the explicit form of \([0, \tau]\), i.e. \( \tau \) in general case. Also, explicit form of the period, \( T \), of the Lotka-Volterra periodical variables \( x, y \) is unknown in general case. For this reason we cannot compare directly \( \tau \) and \( T \) in general case. Nevertheless, we can generally suppose either \( \tau > T \) or \( \tau \leq T \).

In the first case suggested solution (7), (8), (13)-(16), (19), (20), (44)-(47) represents a complete solution of the Lotka-Volterra equations system (1), (2) with initial conditions (3),(4). In the second case, according to (7), (8), (13)-(16), (19), (20), (44)-(47), we can determine exactly \( x(\tau) \) and \( y(\tau) \). Then, \( x(t), y(t) \) solutions of (1), (2) in a time moment \( t = \tau + t' \) for initial conditions (3), (4), can be presented by \( x(t'), y(t') \) solution of (1), (2) where \( t' \) represents new (translated) time variable for new initial conditions \( x(t' = 0) = x(\tau) \) and \( y(t' = 0) = y(\tau) \). Given solution can be obtained by expressions analogous to (7), (8), (13)-(16), (19),(20), (44)-(47) and this solution is consistent and convergent in some new, finite time interval \([0, \tau']\). If \( \tau + \tau' \) is larger than \( T \) or equivalent to \( T \) obtained solution represents final solution. In opposite case described procedure of the solution by "translation" of the time variable must be further repeated. It can be supposed that final solution would be obtained after finite number of the repetitions of the suggested solution procedure. Simply speaking, exact, simple solution of the usual Lotka-Volterra equations would be obtained by a finite number of the suggested analytical continuations.

3 Conclusion

In conclusion we can shortly repeat and point out the following. In this work we suggest a simple, approximate, tending toward exact, solution of the usual Lotka-Volterra differential equations
system. Given solution is obtained by an iterative method. In any finite approximation order of this solution, exponents of the corresponding Lotka-Volterra variables have simple, time polynomial form. When approximation order tends to infinity obtained approximate solution converges toward exact solution in some finite time interval. All this can be very useful for the applications in many domains of the ecology, biology, medicine, etc., but such applications are not considered explicitly in this work.

4 References

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