An Extended Family of Slant Curves in $S$–manifolds

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Abstract
In this paper, we define an extended family of slant curves (i.e. $\theta_\alpha$–slant curves) in $S$–manifolds. We give two examples of such curves in $\mathbb{R}^{2n+3s}(-3s)$, where we choose $n = 1$, $s = 2$. Finally, we study biharmonicity of these curves in $S$–space forms.

Keywords: $\theta_\alpha$–slant curve; $S$–manifold; biharmonic curve.

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1. Introduction

In [6], J. Eells and L. Maire studied selected topics in harmonic maps. In this paper, they suggested $k$-harmonic maps. G. Y. Jiang dealt with the case $k = 2$ in [11]. He derived the first and second variational formulas for 2-harmonic maps. On the other hand, in [4], B. Y. Chen published a survey article, which is divided into 25 sections. In one of these sections, he considered a biharmonic submanifold of Euclidean space as $\Delta H = 0$, where $\Delta$ denotes the Laplace operator and $H$ denotes the mean curvature vector field. If the ambient space is considered as Euclidean, then Jiang’s and Chen’s results match.

In [5], J. T. Cho, J. Inoguchi and J. E. Lee defined and studied slant curves in Sasakian manifolds. They proved a theorem, which is similar to the classical theorem of Lancret for curves in Euclidean 3-space. They showed that a non-geodesic curve in a Sasakian 3-manifold is a slant curve if and only if the ratio of $(\tau \pm 1)$ and $\kappa$ is constant, where $\kappa$ and $\tau$ denotes the geodesic curvature and torsion of the curve, respectively. They gave some interesting examples. Notably, in the Heisenberg group with an appropriate metric, they exhibited slant helix and a slant curve which is not a helix.

In [8], D. Fetcu and C. Oniciuc obtained a method of producing biharmonic submanifolds in a Sasakian space form using the flow of characteristic vector field $\xi$. They showed that under the flow action of $\xi$ a biharmonic integral submanifold is carried to a biharmonic anti-invariant submanifold. Following their idea, the present author and C. Özgür considered biharmonic slant curves in $S$–space forms [9].

It is a natural motivation to generalize the results of slant curves to $\theta_\alpha$–slant curves in $S$–manifolds. In Section 2, we give the fundamental definitions and theorems of $S$–space forms, biharmonic maps and Frenet curves. In Section 3, we define an extended family of slant curves, namely $\theta_\alpha$–slant curves, in $S$–manifolds and give two examples. In Section 4, we obtain the necessary and sufficient conditions for $\theta_\alpha$–slant curves in $S$–space forms to be proper biharmonic.
2. Preliminaries

Let \((M, g)\) be a \((2n + s)\)-dimensional Riemann manifold. \(M\) is called a framed metric manifold with a framed metric structure \((f, \xi_{\alpha}, \eta^{\alpha}, g)\), \(\alpha \in \{1, ..., s\}\), if it satisfies:

\[
f^2 X = -X + \sum_{\alpha=1}^{s} \eta^{\alpha}(X)\xi_{\alpha}, \quad \eta^{\alpha}(f(X)) = 0, \quad \eta^{\alpha}(\xi_{\beta}) = \delta_{\alpha\beta}, \quad f(\xi_{\alpha}) = 0,
\]

\[
g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta^{\alpha}(X)\eta^{\alpha}(Y),
\]

\[
\eta^{\alpha}(X) = g(X, \xi_{\alpha}), \quad d\eta^{\alpha}(X, Y) = -d\eta^{\alpha}(Y, X) = g(X, fY),
\]

where \(f\) is a \((1, 1)\)-type tensor field of rank \(2n\); \(\xi_{1}, ..., \xi_{s}\) are vector fields; \(\eta^{1}, ..., \eta^{s}\) are 1-forms and \(g\) is a Riemannian metric on \(M; X, Y \in TM\) and \(\alpha, \beta \in \{1, ..., s\}\) (see [13], [15]). \((f, \xi_{\alpha}, \eta^{\alpha}, g)\) is said to be an \(S\)-structure, if the Nijenhuis tensor of \(f\) is equal to \(-2d\eta^{\alpha} \otimes \xi_{\alpha}\), for all \(\alpha \in \{1, ..., s\}\) [1].

If \(s = 1\), a framed metric structure is the same as an almost contact metric structure and an \(S\)-structure is the same as a Sasakian structure. For an \(S\)-structure, we have the following equations [1]:

\[
(\nabla_X f)Y = \sum_{\alpha=1}^{s} \{g(fX, fY)\xi_{\alpha} + \eta^{\alpha}(Y)f^2 X\},
\]

and

\[
\nabla \xi_{\alpha} = -f,
\]

for all \(\alpha = 1, ..., s\). In case of \(s = 1\), (2.3) can be calculated from (2.2).

Let \(X \in T_pM\) be orthogonal to \(\xi_{1}, ..., \xi_{s}\). The plane section spanned by \(\{X, fX\}\) is called an \(f\)-section in \(T_pM\) and its sectional curvature is called an \(f\)-sectional curvature. Let \((M, f, \xi_{\alpha}, \eta^{\alpha}, g)\) be an \(S\)-manifold. If \(M\) has constant \(f\)-sectional curvature, its curvature tensor \(R\) is given by

\[
R(X, Y)Z = \sum_{\alpha, \beta} \{\eta^{\alpha}(X)\eta^{\beta}(Z)f^2 Y - \eta^{\beta}(Y)\eta^{\alpha}(Z)f^2 X
\]

\[
-\frac{1}{2}\{g(fX, fZ)f^2 Y + g(fY, fZ)f^2 X + g(fX, fZ)f^2 X + g(fY, fZ)f^2 Y\},
\]

for \(X, Y, Z \in TM\) [3]. In this case, \(M\) is called an \(S\)-space form and it is denoted by \(M(c)\). In case of \(s = 1\), an \(S\)-space form is the same as a Sasakian space form [2].

Let \((M, g)\) and \((N, h)\) be Riemannian manifolds and \(\varphi : M \rightarrow N\) a differentiable map. A harmonic map is a critical point of the energy functional of \(\varphi\), which is defined as

\[
E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g,
\]

(see [7]). Furthermore, a biharmonic map is a critical point of the bienergy functional

\[
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,
\]

where \(\tau(\varphi) = \text{trace} \nabla d\varphi\) and it is called the first tension field of \(\varphi\). Jiang derived the biharmonic map equation [11]

\[
\tau_2(\varphi) = -J^c(\tau(\varphi)) = -\Delta \tau(\varphi) - \text{trace} R^N(d\varphi, \tau(\varphi))d\varphi = 0,
\]

where \(J^c\) denotes the Jacobi operator of \(\varphi\). It is obvious that harmonic maps are biharmonic. So, non-harmonic biharmonic maps are called proper biharmonic.

Let \(\gamma : I \rightarrow M\) be a unit-speed curve in an \(n\)-dimensional Riemannian manifold \((M, g)\). The curve \(\gamma\) is called a Frenet curve of osculating order \(r\) \((1 \leq r \leq n)\), if there exists orthonormal vector fields \(T, E_2, ..., E_r\) along the curve
validating the Frenet equations
\[
\begin{align*}
T &= \gamma', \\
\nabla_T T &= \kappa_1 E_2, \\
\nabla_T E_2 &= -\kappa_1 T + \kappa_2 E_3, \\
\nabla_T E_r &= -\kappa_r E_{r-1},
\end{align*}
\]

where \(\kappa_1, \ldots, \kappa_{r-1}\) are positive functions called the curvatures of \(\gamma\). If \(\kappa_1 = 0\), then \(\gamma\) is called a geodesic. If \(\kappa_1\) is a non-zero positive constant and \(r = 2\), \(\gamma\) is called a circle. If \(\kappa_1, \ldots, \kappa_{r-1}\) are non-zero positive constants, then \(\gamma\) is called a helix of order \(r\) \((r \geq 3)\). If \(r = 3\), it is shortly called a helix.

A submanifold of an \(S\)-manifold is said to be an integral submanifold if \(\eta^\alpha(X) = 0\), \(\alpha \in \{1, \ldots, s\}\), where \(X\) is tangent to the submanifold \(12\). A Legendre curve is a 1-dimensional integral submanifold of an \(S\)-manifold \((M^{2n+s}, f, \xi, \eta, g)\). More precisely, a unit-speed curve \(\gamma : I \to M\) is a Legendre curve if \(T\) is \(g\)-orthogonal to all \(\xi\) \((\alpha = 1, \ldots, s)\), where \(T = \gamma'\) \([14]\).

### 3. \(\theta_\alpha\)–Slant Curves in \(S\)-manifolds

In this section, we define an extension of slant curves in \(S\)-manifolds. Firstly, we give the following definition:

**Definition 3.1.** Let \(M = (M^{2n+s}, f, \xi, \eta, g)\) be an \(S\)-manifold and \(\gamma : I \to M\) a unit-speed curve. \(\gamma\) is called a \(\theta_\alpha\)–slant curve, if there exist constant angles \(\theta_\alpha\) \((\alpha = 1, \ldots, s)\) such that \(\eta^\alpha(T) = \cos \theta_\alpha\). Here, we call \(\theta_\alpha\) the contact angles of \(\gamma\).

One can easily see that Definition 3.1 extends the family of slant curves to \(\theta_\alpha\)–slant curves. In particular, a \(\theta_\alpha\)–slant curve is called slant if its all contact angles are equal (see \([9]\)).

For a \(\theta_\alpha\)–slant curve, if we differentiate \(\eta^\alpha(T) = \cos \theta_\alpha\) along the curve \(\gamma\), we obtain
\[
\eta^\alpha(E_2) = 0,
\]
for all \(\alpha = 1, \ldots, s\). From now on, we use the following notations:

\[
A = \sum_{\alpha=1}^{s} \cos^2 \theta_\alpha, \quad B = \sum_{\alpha=1}^{s} \cos \theta_\alpha, \quad V = \sum_{\alpha=1}^{s} \cos \theta_\alpha \xi_\alpha.
\]

The following corollary is directly obtained:

**Corollary 3.1.** If \(\gamma\) is slant, then
\[
A = s \cos^2 \theta, B = s \cos \theta, V = \cos \theta \sum_{\alpha=1}^{s} \xi_\alpha,
\]
where \(\theta\) denotes the equal contact angles of \(\gamma\).

Let \(\gamma\) be a non-geodesic unit-speed \(\theta_\alpha\)–slant curve. Using equation 2.1, we find
\[
g(fT, fT) = 1 - A \geq 0.
\]
If \(A = 1\), then we have \(fT = 0\), that is, \(T = V\). Hence, we get
\[
\nabla_T T = \nabla_T V = 0,
\]
which means \(\gamma\) is a geodesic. As a result, we can give the following proposition:

**Proposition 3.1.** For a non-geodesic unit-speed \(\theta_\alpha\)–slant curve in an \(S\)-manifold,
\[
A = \sum_{\alpha=1}^{s} \cos^2 \theta_\alpha < 1.
\]
Note that, if $\gamma$ is slant, we obtain Proposition 3.1 in [9].

From equations 2.1 and 2.5, we obtain the following statement:

**Proposition 3.2.** For a non-geodesic unit-speed $\theta_\alpha-$slant curve in an $S$-manifold $(M, f, \xi, \eta^\alpha, g)$, we have

$$\nabla_T fT = (1 - A) \sum_{\alpha=1}^s \xi_\alpha + B (-T + V) + \kappa_1 fE_2.$$  \hspace{1cm} (3.2)

Now we give the following examples of non-trivial $\theta_\alpha-$slant curves in $\mathbb{R}^{2n+s}(-3s)$, choosing $n = 1$, $s = 2$. For detailed information on $\mathbb{R}^{2n+s}(-3s)$, see [10].

**Example 3.1.** $\gamma : I \rightarrow \mathbb{R}^4(-6)$, $\gamma(t) = (t, 0, t, \sqrt{2}t)$ is a $\theta_\alpha-$slant curve with the contact angles $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$. In fact, $\gamma$ is a $\theta_\alpha-$slant circle with $\kappa_1 = \frac{\sqrt{2}+1}{2}$.

**Example 3.2.** Let $c_i$ be arbitrary constants $(i = 1, \ldots, 4)$, $t_0 \in I$, $\theta_1$ and $\theta_2$ constants such that $A = \cos^2 \theta_1 + \cos^2 \theta_2 < 1$. Let us consider a smooth function $u : I \rightarrow \mathbb{R}$ and define $\gamma_i : I \rightarrow \mathbb{R}$ $(i = 1, \ldots, 4)$ as

$$\gamma_1(t) = c_1 + 2\sqrt{1 - A} \int_{t_0}^t \cos(u(p)) \, dp,$$

$$\gamma_2(t) = c_2 + 2\sqrt{1 - A} \int_{t_0}^t \sin(u(p)) \, dp,$$

$$\gamma_3(t) = c_3 + 2t \cos \theta_1$$

$$+ 2\sqrt{1 - A} \int_{t_0}^t \cos(u(q)) \left( c_2 + 2\sqrt{1 - A} \int_{t_0}^q \sin(u(p)) \, dp \right) dq,$$

$$\gamma_4(t) = c_4 + 2t \cos \theta_2$$

$$+ 2\sqrt{1 - A} \int_{t_0}^t \cos(u(q)) \left( c_2 + 2\sqrt{1 - A} \int_{t_0}^q \sin(u(p)) \, dp \right) dq.$$

Then $\gamma : I \rightarrow \mathbb{R}^4(-6)$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is a $\theta_\alpha-$slant curve with the contact angles $\theta_1$ and $\theta_2$.

**4. Biharmonic $\theta_\alpha-$Slant Curves in $S-$Space Forms**

In this section, we consider proper biharmonic $\theta_\alpha-$slant curves in $S$-space forms. Let $\gamma$ be a unit-speed $\theta_\alpha-$slant curve in an $S$-space form $(M, f, \xi, \eta^\alpha, g)$. Then, we have

$$R(T, \nabla_T T) T = -\kappa_1 \left[ B^2 + \frac{c + 3s}{4} (1 - A) \right] E_2 - 3\kappa_1 \frac{c - s}{4} g(fT, E_2) fT,$$

$$\tau_2(\gamma) = \nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T) T$$

$$= -3\kappa_1 \kappa_1' T$$

$$+ \left( \kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + \kappa_1 \left[ B^2 + \frac{c + 3s}{4} (1 - A) \right] \right) E_2$$

$$+ (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4$$

$$+ 3\kappa_1 \frac{c - s}{4} g(fT, E_2) fT.$$  \hspace{1cm} (4.1)

As a result, we can state the following theorem:
Theorem 4.1. $\gamma$ is a proper-biharmonic $\theta_\alpha$–slant curve in an $S$–space form $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if $\kappa_1 = \text{constant} > 0$ and
\[
3 \frac{c - s}{4} g(fT, E_2) fT = \left[ \kappa_1^2 + \kappa_2^2 - B^2 - \frac{c + 3s}{4} (1 - A) \right] E_2 - \kappa_2^\prime E_3 - \kappa_2 \kappa_3 E_4.
\] (4.2)

Proof. Let $\gamma$ be a proper-biharmonic $\theta_\alpha$–slant curve. Then $\kappa_1 > 0$ and $\tau_2(\gamma) = 0$. If we take the inner-product of both sides with $T$, we find $\kappa_1 = \text{constant} > 0$. Hence, from equation (4.1), we obtain equation (4.2). Conversely, if $\kappa_1 = \text{constant} > 0$ and equation (4.2) is satisfied, we find $\tau_2(\gamma) = 0$, which completes the proof.

We will consider equation (4.2) from all points of view. Our discussions are based on the question: ”When do the coefficients of $fT$ vanish?” First discussion is for the absence of the term with $fT$ in equation (4.2). Second discussion is for the non-vanishing coefficients.

First Discussion: The term with $fT$ vanishes.

i) $c = s$.

In this case, equation (4.2) becomes
\[
0 = \left[ \kappa_1^2 + \kappa_2^2 - B^2 - s (1 - A) \right] E_2 - \kappa_2^\prime E_3 - \kappa_2 \kappa_3 E_4.
\] (4.3)

As a result, we give the following Theorem:

Theorem 4.2. Under the assumption $c = s$; $\gamma$ is a proper-biharmonic $\theta_\alpha$–slant curve in $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if either $\gamma$ is a circle with $\kappa_1 = \sqrt{B^2 + s(1 - A)}$ or a helix with $\kappa_1^2 + \kappa_2^2 = B^2 + s(1 - A)$.

Proof. From equation (4.3), since $\{E_2, E_3, E_4\}$ is $g$–orthonormal, the proof is clear.

ii) $c \neq s$ and $fT \perp E_2$.

Under these assumptions, equation (4.2) gives us
\[
0 = \left[ \kappa_1^2 + \kappa_2^2 - B^2 - \frac{c + 3s}{4} (1 - A) \right] E_2 - \kappa_2^\prime E_3 - \kappa_2 \kappa_3 E_4.
\] (4.4)

Firstly, we need to prove the following Lemma:

Lemma 4.1. Let $\gamma$ be a $\theta_\alpha$–slant curve of order $r = 3$ in an $S$–space form $(M, f, \xi_\alpha, \eta^\alpha, g)$ and $fT \perp E_2$. Then, $\{T, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\}$ is linearly independent.

Proof. Let $r = 3$ and $fT \perp E_2$. Let us denote $S_1 = \{T, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\}$. In view of equations (2.5), (3.1) and (3.2), we have
\[
g(E_2, T) = g(E_2, E_3) = g(E_2, fT) = g(E_2, \nabla_T fT) = g(E_2, \xi_\alpha) = 0,
\]
for all $\alpha = 1, \ldots, s$. Thus, $S_1$ is linearly independent if and only if $S_2 = \{T, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\}$ is linearly independent. From the assumption, we have $fT \perp E_2$. If we differentiate $g(fT, E_2) = 0$, we find $g(fT, E_3) = 0$. Since $g(fT, fT) = 1 - A > 0$ is a constant, we obtain $g(fT, \nabla_T fT) = 0$. $f$ is skew-symmetric, so $g(fT, T) = 0$. From equation (2.1), we also have $g(fT, \xi_\alpha) = 0$, for all $\alpha = 1, \ldots, s$. Then, omitting $fT$, we get that $S_2$ is linearly independent if and only if $S_3 = \{T, E_3, \nabla_T fT, \xi_1, \ldots, \xi_s\}$ is linearly independent. Now, let us investigate whether $T$ is linearly dependent with other vector fields in $S_3$. From Frenet equations, $g(T, E_3) = 0$. Equation (3.2) gives us $g(T, \nabla_T fT) = 0$. Assume that $T \in \text{sp} \{\xi_1, \ldots, \xi_s\}$. If we differentiate
\[
T = \sum_{\alpha=1}^s \cos \theta_\alpha \xi_\alpha,
\]
along the curve $\gamma$, we get $\kappa_1 = 0$, which is a contradiction. As a result, $T \notin \text{sp} \{\xi_1, \ldots, \xi_s\}$. Hence, $S_3$ is linearly independent if and only if $S_4 = \{E_3, \nabla_T fT, \xi_1, \ldots, \xi_s\}$ is linearly independent. If we differentiate $g(fT, E_3) = 0$, we find $g(\nabla_T fT, E_3) = 0$. Now, let us assume $E_3 \in \text{sp} \{\xi_1, \ldots, \xi_s\}$. If we differentiate
\[
E_3 = \sum_{\alpha=1}^s \eta^\alpha (E_3) \xi_\alpha,
\]
we obtain
\[-\kappa_2 E_2 = \sum_{\alpha=1}^s \{\nabla_T [\eta^\alpha(E_3)] \xi_{\alpha} - \eta^\alpha(E_3)f_T\}.\]

If we take the inner-product of both sides with $E_2$, we find $\kappa_2 = 0$, which contradicts $r = 3$. Then, $E_3 \not\in \text{sp}\{\xi_1, ..., \xi_s\}$. So, $S_4$ is linearly independent if and only if $S_5 = \{\nabla_T f T, \xi_1, ..., \xi_s\}$ is linearly independent. Equation (3.2) can be rewritten as
\[\nabla_T f T = \sum_{\alpha=1}^s [(1 - A) + B \cos \theta_{\alpha}] \xi_{\alpha} - BT + \kappa_1 f E_2.\]

Since $f T \perp E_2$ and $f$ is skew-symmetric, we have $f E_2 \perp T$. As a result, the term $(-BT + \kappa_1 f E_2)$ does not vanish, that is, $\nabla_T f T \not\in \text{sp}\{\xi_1, ..., \xi_s\}$. Consequently, $S_5$ is linearly independent and the proof is complete. \hfill \Box

In view of Lemma 4.1, we can state the following theorem:

**Theorem 4.3.** Under the assumptions $c \neq s$ and $f T \perp E_2$; $\gamma$ is a proper-biharmonic $\theta_{\alpha}$-slant curve in $(M, f, \xi_{\alpha}, \eta^\alpha, g)$ if and only if either

a) $\dim(M) \geq 4 + s$ and $\gamma$ is a circle with $\kappa_1 = \dfrac{1}{2} \sqrt{4B^2 + (c + 3s)(1 - A)}$, where $\{T, E_2, f T, \nabla_T f T, \xi_1, ..., \xi_s\}$ is linearly independent; or

b) $\dim(M) \geq 5 + s$ and $\gamma$ is a helix with $\kappa_1^2 + \kappa_2^2 = B^2 + \dfrac{c + 3s}{4}(1 - A)$, where $\{T, E_2, E_3, f T, \nabla_T f T, \xi_1, ..., \xi_s\}$ is linearly independent.

**Proof.** If we consider Lemma 4.1 and equation (4.4) together, the proof is directly obtained. \hfill \Box

**Second Discussion: The term with $f T$ does not vanish.**

i) $c \neq s$ and $f T \parallel E_2$.

In this case, since $g(f T, f T) = 1 - A$ and $f T \parallel E_2$, we can write
\[f T = \varepsilon \sqrt{1 - AE_2},\] (4.5)
where $\varepsilon = \text{sgn}(g(f T, E_2))$. Then, equation (4.2) becomes
\[3 \frac{c - s}{4} (1 - A) E_2 = \left[\kappa_1^2 + \kappa_2^2 - B^2 - \frac{c + 3s}{4}(1 - A)\right] E_2 - \kappa_2 E_3 - \kappa_1 \kappa_2 E_4.\] (4.6)

Firstly, we can state the following Lemma:

**Lemma 4.2.** Let $\gamma$ be a non-geodesic $\theta_{\alpha}$-slant curve in an $S$-space form $(M, f, \xi_{\alpha}, \eta^\alpha, g)$ and $f T \parallel E_2$. If $\kappa_1$ is a constant, then $\gamma$ is either a circle or a helix.

**Proof.** Let $\kappa_1 = \text{constant} > 0$. From equations (2.5), (3.2) and (4.5), after some calculations, we get
\[\kappa_2 \varepsilon \sqrt{1 - AE_3} = (1 - A) \sum_{\alpha=1}^s \xi_{\alpha} - (B + \varepsilon AD) T + (B + \varepsilon D) V,\] (4.7)
where we denote $D = \kappa_1 / \sqrt{1 - A}$. Note that
\[g(T, T) = 1, \ g(T, \sum_{\alpha=1}^s \xi_{\alpha}) = B, \ g(T, V) = A,\]
\[g(\sum_{\alpha=1}^s \xi_{\alpha}, \sum_{\alpha=1}^s \xi_{\alpha}) = s,\]
\[g(\sum_{\alpha=1}^s \xi_{\alpha}, V) = B, \ g(V, V) = A.\]

As a result, if we denote the norm of the right-hand side of equation (4.7) by $C$, we have
\[C = \sqrt{1 - A} \sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s},\]
which gives us
\[ \kappa_2 = \sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s}. \]
So, \( \kappa_2 \) is a constant. If \( \kappa_2 = 0 \), then \( \gamma \) is a circle. If \( \kappa_2 \neq 0 \), equation (4.7) gives us
\[ E_3 = a_0 T + a_1 \xi_1 + \ldots + a_s \xi_s, \]
for some constants \( a_0, \ldots, a_s \). If we differentiate this last equation, we obtain
\[ -\kappa_2 E_2 + \kappa_3 E_4 = a_0 \kappa_1 E_2 - a_1 fT - \ldots - a_s fT. \tag{4.8} \]
If we take the inner-product of equation (4.8) with \( E_4 \), considering the fact that \( fT \parallel E_2 \), we find \( \kappa_3 = 0 \). In this case, \( \gamma \) is a helix.

In view of Lemma 4.2, we have the following result:

**Theorem 4.4.** Under the assumptions \( c \neq s \) and \( fT \parallel E_2 \); \( \gamma \) is a proper-biharmonic \( \theta_\alpha \)-slant curve in \( (M, f, \xi_\alpha, \eta^\alpha, g) \) if and only if either

- (a) it is a circle with \( \kappa_1 = \sqrt{B^2 + c(1 - A)} \) with the Frenet frame field
  \[ \{ T, \frac{\varepsilon fT}{\sqrt{1 - A}} \} \]
  where \( B^2 + c(1 - A) > 0 \); or

- (b) it is a helix with \( \kappa_1 = \sqrt{1 - AD} \), \( \kappa_2 = \sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s} \) with the Frenet frame field
  \[ \{ T, \frac{\varepsilon fT}{\sqrt{1 - A}} \frac{\varepsilon}{1 - A\sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s}} W \} \]
  where \( AD^2 - As + B^2 + 2\varepsilon BD + s > 0 \), \( D > 0 \) is a constant satisfying
  \[ D(2\varepsilon B + D) = (1 - A)(c - s) \]
  and \( W \) denotes
  \[ W = (1 - A) \sum_{\alpha=1}^{s} \xi_\alpha - (B + \varepsilon AD) T + (B + \varepsilon D) V. \]

**Proof.** Let \( \gamma \) be proper-biharmonic. Then, \( \kappa_1 = \text{constant} > 0 \) and equation (4.6) must be satisfied. If we take the inner-product of equation (4.6) with \( E_2, E_3 \) and \( E_4 \), we get
\[ \kappa_1^2 + \kappa_2^2 = B^2 + c(1 - A) \]
\[ \kappa_2 = \text{constant}, \kappa_3 = 0, \]
respectively. From the proof of Lemma 4.2, using equation (4.10), we obtain the curvatures and the Frenet frame field of \( \gamma \). Furthermore, if \( \gamma \) is a helix, if we replace \( \kappa_1 \) and \( \kappa_2 \) in equation (4.10), we find equation (4.9).

Conversely, let \( \gamma \) be a one of the curves given in (a) or (b). Then, one can easily show that equation (4.4) is verified. So, \( \gamma \) is proper-biharmonic.

**ii)** \( c \neq s \) and \( g(fT, E_2) \neq 0, 1, -1 \).

Since the equality cases are previously investigated, we complete our discussions under the assumptions \( c \neq s \) and \( g(fT, E_2) \neq 0, 1, -1 \). Let us consider a smooth function \( m(t) \) such that
\[ g(fT, E_2) = \sqrt{1 - A} \cos m(t). \tag{4.11} \]
Differentiating this equation, we have
\[ \kappa_2 g(fT, E_3) = -\sqrt{1 - A} m'(t) \sin m(t). \tag{4.12} \]
If we take the inner-product of equation (4.2) with \( E_2, E_3 \) and \( E_4 \), we find
\[ \kappa_1^2 + \kappa_2^2 = B^2 + \frac{c + 3s}{4} (1 - A) + \frac{3(c - s)}{4} g(fT, E_2)^2, \tag{4.13} \]
\[
\kappa'_2 + \frac{3}{4} \left( c - s \right) g(fT, E_2) g(fT, E_3) = 0, \\
\kappa_2 \kappa_3 + \frac{3}{4} \left( c - s \right) g(fT, E_2) g(fT, E_4) = 0,
\]
respectively. If we multiply equation (4.14) with \(2\kappa_2\), equations (4.11) and (4.12), we have
\[
2\kappa_2 \kappa'_2 + (1 - A) \frac{3}{4} \left( c - s \right) [-2m'(t) \sin m(t) \cos m(t)] = 0.
\]
If we integrate the last equation, we get
\[
\kappa_2^2 = - (1 - A) \frac{3}{4} \left( c - s \right) \cos^2 m(t) + h_0,
\]
where \(h_0\) is an arbitrary constant. If we write equation (4.16) in (4.13), we obtain \(m(t)\) is constant. As a result, we can write
\[
fT = \sqrt{1 - A} \left( \cos m E_2 + \sin m E_4 \right),
\]
where \(m \in (0, 2\pi) - \{ \frac{\pi}{2}, 0, \frac{3\pi}{2} \}\). Now, we can give the following theorem:

**Theorem 4.5.** Under the assumptions \(c \neq s\) and \(g(fT, E_2) \neq 0, 1, -1\); \(\gamma\) is a proper-biharmonic \(\theta_\alpha\)-slant curve in \((M, J, \xi, \eta^\alpha, g)\) if and only if \(\kappa_1, \kappa_2\) and \(\kappa_3\) are constants such that
\[
\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = B^2 + \frac{c + 3s}{4} (1 - A) + \frac{3}{4} \left( c - s \right) \cos^2 m, \\
\kappa_2 \kappa_3 + \frac{3}{8} \left( 1 - A \right) \sin 2m = 0,
\]
where \(fT = \sqrt{1 - A} \left( \cos m E_2 + \sin m E_4 \right)\) and \(m \in (0, 2\pi) - \{ \frac{\pi}{2}, 0, \frac{3\pi}{2} \}\).

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