1 INTRODUCTION

In writing this article, I have two main objectives. The first is to chart the evolution of the analytic approach to Quantum Chromodynamics (QCD) perturbation theory, focusing on the key turning points in its theoretical development. The second is to demonstrate how this approach works in terms of selected applications. Of course, I am forced to leave out many interesting applications and aspects of the approach, which in turn means that my presentation will not be complete. I would refer the reader to the original references for further reading and in order to study the subject in more technical detail. The sorts of ideas and issues to be discussed in this presentation include the following benchmarks:

— “Ultraviolet (UV) freedom”, “infrared (IR) slavery”, and Landau singularity.

— Renormalizability, analyticity (in $Q^2$ and $g^2$), causality, and summability.

— First remedies in the infrared: Color saturation, effective gluon mass.

— Screening of the Landau singularities via Sudakov form factors.

— Shirkov–Solovtsov analytic coupling—Euclidean and Minkowski space.

— From a recipe to a paradigm: Analytic Perturbation Theory (APT).

— More scales, more riddles: Logarithms of the factorization (evolution) scale—Naive and Maximal analytization.

— Generalization of the analyticity requirement: From the coupling to the whole hadronic amplitude and its spectral density.

— Creation of Fractional APT (FAPT) in the spacelike and timelike regions.

— Summation of perturbation series, Sudakov gluon resummation (exponentiation vs. analytization), power corrections.

— Crossing of heavy-flavor thresholds.

The article is organized as follows. In Section 2 I review the two core characteristics of QCD: ultraviolet freedom and infrared confinement. The first attempts how to deal with the Landau singularity of the strong
running coupling in the spacelike momentum region are described in Section 3. The analyticization procedure of the coupling by Shirkov and Solovtsov is presented in Section 4, while its generalization to hadronic observables in Section 5. Further extensions to accommodate more than one large scale are discussed in Section 6, followed in Section 7 by the analyticization treatment of noninteger powers of the coupling that give rise to fractional analytic perturbation theory in the Euclidean and Minkowski space. Applications of FAPT are presented in Section 8 (pion’s electromagnetic form factor) and Section 9 (decay into a bottom–antibottom quark pair of a scalar Higgs boson). Conclusions are drawn in Section 10.

2. ULTRAVIOLET FREEDOM AND INFRARED CONFINEMENT

Two key properties define QCD as the unbroken $SU(N_c = 3)$ (with $N_c$ denoting the number of colors) Yang–Mills theory describing strong interactions at the microscopic level: Asymptotic freedom and confinement. While the first property has been confirmed and was granted a Nobel prize in Physics in the year 2004, the other is still a matter of fact, with the underlying microscopic mechanism still at large. One claims that the reason for the non-observation of free quarks and gluons in nature is due to their color content. From this, one concludes that only color singlets are observable—hence, defacto color confinement.

The main ingredient in justifying the UV freedom of QCD is that the renormalized QCD coupling becomes small at large momenta: $\hat{g}^2 \rightarrow 0$ as $Q^2 \rightarrow \infty$ [1, 2]. This implies that though initially the coupling and the momentum scale are independent parameters they get linked to each other on account of the renormalizability of QCD. Hence, in the large-momentum region (or, equivalently, at short distances), QCD can be applied perturbatively, in analogy to QED in the sense of a weak-coupling expansion. A conventional way to define the running strong coupling in the $\overline{\text{MS}}$ scheme is given by ($\alpha_s = g^2/4\pi$)

$$\alpha_s(\mu) = \frac{12\pi}{(33 - 2N_f)\ln(\mu^2/\Lambda^2)} \left(1 - \frac{6(153 - 19N_f)}{(33 - 2N_f)^2} \ln\left(\frac{\mu^2}{\Lambda^2}\right)\right) + C\left[\frac{1}{\ln^3(\mu^2/\Lambda^2)}\right].$$  

(1)

which expresses it by means of the dimensional parameter $\Lambda = \Lambda_{\text{QCD}}$, whereas $N_f$ represents the number of active flavors. For our considerations to follow it is more convenient to use the abbreviation $L \equiv \ln(\mu^2/\Lambda^2)$ and recast the above expression in terms of the first two coefficients of the $\beta$ function

$$\mu^2 \frac{d}{d\mu^2}(\frac{\alpha_s(\mu)}{4\pi}) = \beta(\frac{\alpha_s(\mu)}{4\pi}),$$  

(2)

namely,

$$b_0 = \frac{11}{3} C_A - \frac{4}{3} T_R N_f = 11 - 2N_c/3$$  

(3)

and

$$b_1 = \frac{34}{3} C_A^2 - \left(4C_F + \frac{20}{3} C_A\right) T_R N_f = 102 - 38N_c/3,$$  

(4)

where $C_F = (N_c^2 - 1)/2N_c = 4/3$, $C_A = N_c = 3$, and $T_R = 1/2$. Then, Eq. (1) becomes

$$\alpha_s[L(\mu)] = \frac{4\pi}{b_0 L} \left(1 - \frac{b_1}{b_0} \ln]\right) + C\left[\frac{1}{L^3}\right].$$  

(5)

which is an approximate solution of the Renormalization-Group (RG) equation

$$\frac{d}{dL}(\frac{\alpha_s(L)}{4\pi}) = -b_0(\frac{\alpha_s(L)}{4\pi})^2 - b_1(\frac{\alpha_s(L)}{4\pi})^3,$$  

(6)

exhibiting the asymptotic-freedom property, i.e., $\alpha_s \sim 1/\ln(\mu^2/\Lambda^2) \rightarrow 0$ as $\mu^2/\Lambda^2 \rightarrow \infty$. The scale parameter $\Lambda$ is not a generic parameter of the QCD Lagrangian; its definition is arbitrary. However, once defined, it becomes the fundamental constant of QCD to be determined by experiment.

On the other hand, at $\mu^2 = \Lambda^2$ the running strong coupling has a Landau singularity that spoils analyticity. At the one-loop level this singularity is a simple pole, whereas at higher loops the corresponding Landau singularity has a more complicated structure (e.g., a square-root singularity at two loops, et cetera). To restore analyticity in the infrared region $0 \leq Q^2 \leq \Lambda^2$ and ensure causality in the whole $Q^2$ plane, one has to remove the Landau singularity at any loop order of the perturbative expansion. This may become a serious problem, both theoretically and phenomenologically, for several hadronic processes—in particular for exclusive processes like hadron form factors. For example, in the case of the spacelike pion’s electromagnetic form factor—a typical exclusive process—most experimental data are available in a momentum region, where repercussions of the Landau singularity can influence (actually contaminate) the perturbative calculation, as first shown and detailed in [4] and later.

\[\footnote{The first two coefficients $b_0$ and $b_1$ are scheme independent. The next coefficients $b_2$ and $b_3$ have also been analytically computed and confirmed by independent calculations at the level of 3 and 4 loops (see [3] for references).}\]
further analyzed in [5]. Another way to describe this behavior is to use the language of information theory and consider a perturbatively calculable quantity, like the scaled pion factor vs. the momentum transfer \( Q^2 \), as being a \textit{dynamic range} of quality of the perturbative expansion. In this context, the accuracy of the QCD perturbative result (\textit{the signal}) depends on the magnitude of the running coupling. The larger \( Q^2 \), the smaller the coupling and, hence, the higher the precision of the (perturbative) calculation is. As one approaches the Landau singularity, the coupling increases rendering the perturbative expansion less and less precise, i.e., creating a kind of \textit{noise}. When one reaches the Landau singularity, the “noise” (the uncertainty) becomes overwhelming and one is unable to extract any information (a clear “signal”) from perturbation theory. The next few sections will show in more detail how this problem can be averted.

Of course, one may interpret the breakdown of perturbation theory at low momenta \( \sim \Lambda \), as a “natural” consequence of the (unknown) confinement forces that prevail in this momentum region. The problem is that this breakdown is not universal but depends on the renormalization scheme adopted and the renormalization scale-fixing procedure used. For instance, adopting the Brodsky—Lepage—Mackenzie (BLM) commensurate-scale procedure, when including higher-order QCD perturbative corrections (see [4, 5] for further details), the BLM scale may become very low (compared to the original momentum scale) and deeply intruding into the “forbidden” nonperturbative region, where the running coupling blows up and perturbative QCD becomes invalid. An example: Using the BLM procedure in the calculation of the pion’s electromagnetic form factor at NLO [5], the QCD running coupling may still “feel” the Landau singularity at momenta as large as 6 GeV\(^2\), a region close to the largest momenta probed by experiment at present for this reaction. Hence, the (scheme-dependent) optimization of the QCD perturbative expansion of typical hadronic quantities may depend in a crucial way on the IR behavior of the running coupling. It becomes clear that any attempt to apply an optimized renormalization prescription (and scheme) in high fixed-order perturbative calculations should make sure that the result is not distorted by artificial logarithmic enhancements originating from the Landau singularity. Clearing away this obstruction is a real challenge.

3. \textsc{First Remedies in the Infrared}

The simplest way to avoid the Landau singularity in Euclidean space is to use a rigid IR cutoff on its value, say, \( \alpha_s^{\text{cutoff}} = 0.5–0.7 \), such as to render the perturbative expansion sound. A more sophisticated procedure is to assume that below some momentum scale \( \mu_{\text{IR}} \gg \Lambda \) color forces saturate, so that quarks and gluons are confined within color-singlet states. The IR cutoff scale \( \mu_{\text{IR}} \) can be conceived of as an effective gluon mass \( m_g \), generated by the dynamics of confinement, as proposed by Cornwall (1982) [6] (for further references and applications, see [7]). This mechanism tames the Landau singularity, but the intrinsic scale \( m_g \) is an extraneous parameter to the approach and has to be determined by other methods, e.g., on the lattice [8]. Indeed, the saturated coupling at one loop reads

\[
\alpha_s^{\text{sat}}(Q^2) = \frac{4\pi}{\beta_0 \ln [(Q^2 + \lambda^2)/\Lambda^2]}
\]  

(7)

with \( \lambda^2 = 4m_g^2 \) and is IR finite. Here \( m_g \) is a dynamical gluon mass with numerical values in the range (500 ± 200) MeV. This scale may be thought of as being the average transverse momentum of vacuum partons within the framework of nonlocal quark and gluon condensates. Technically, the role of the IR regulator \( m_g \) is to “freeze” \( \alpha_s(Q^2) \) at low momenta \( Q^2 \ll \lambda^2 \), so that the coupling ceases to increase and flattens out.

An alternative approach to protect perturbatively calculated observables was developed by Sterman and collaborators [9, 10], in which \( \alpha_s \) singularities were screened by Sudakov-type damping exponentials \( e^{-S} \) that encapsulate the effects of gluonic radiative corrections. These are resumed contributions stemming from two-particle reducible diagrams (giving rise to leading double logarithms), while two-particle irreducible diagrams (entailing subleading single logarithms) are absorbed into the hard-scattering part of the process. Because \( e^{-S} \) drops to zero faster than any power of \( \ln (\mu^2/\Lambda_{\text{QCD}}^2) \), it provides sufficient IR protection against Landau singularities both for mesonic amplitudes as well as for baryonic ones (see [7] for a review). Indeed, in the axial gauge, all Sudakov contributions due to unintegrated transverse momenta of the gluon propagators exponentiate into suppressing Sudakov factors that suffice to render the Landau singularity of the one-loop running coupling in the form factors of the pion and the nucleon harmless, though the latter case is theoretically much more involved—see [7, 11, 12]. The main advantage of this approach relative to the previous one is that it contains an intrinsic IR regulator in terms of the impact parameter characterizing the inter-quark separations in the hadrons. Hence, there is no need for external IR regulators (like \( m_g \)) because IR protection is provided in situ by the Sudakov effect, the underlying idea being that gluons with wavelengths larger than the impact parameter “see” the quark configuration as a whole, i.e., in a color-singlet state and hence decouple. It is worth noting that the Sudakov suppression can be simulated by a saturated running coupling of the form of Eq. (7), if we assume that the IR cutoff parameter (i.e., the effective gluon mass) has to be adjusted to each individual value of the varying external momentum—for a mathematical version of this idea see [7]. Let us close this dis-
cussion by remarking that the application of this type of approach to the timelike region using analytic continuation is not straightforward owing to the fact that not only the running coupling has to be analytically continued but also the whole Sudakov form factor.

A bridge here to what is to come in the next section with regard to analytization. In separate parallel developments, Krasnikov and Pivovarov (1982) [13], and independently Radyushkin (1982) [14], have obtained analytic expressions for the one-loop running coupling (and its powers) directly in Minkowski space using an integral transformation from the spacelike to the timelike region. In both approaches a resummed $\Lambda$-parametrization for $\alpha_s(Q^2)$ in the spacelike region was employed in order to construct in the timelike region an expansion for $R(q^2)$ in which all $(\pi^2/L^2)^N$-terms ($L = \ln(Q^2/\Lambda^2_{QCD})$) are summed explicitly. This way, an analytic coupling in the timelike region was derived. This sort of analytic coupling in Minkowski space was rediscovered in the context of the resummation of fermion bubbles by Ball, Beneke, and Braun (1995) [15], partly in connection with techniques and applications to the $\tau$ hadronic width [16] (more references are given in [17]). Still other approaches to avoid ghost singularities in the running coupling are referenced in [4, 17].

4. INTO ANALYTIZATION

Shirkov and Solovtsov (1996) [18] have invented an analytic coupling based solely on Renormalization-Group (RG) invariance and causality (spectrality) in terms of a dispersion relation, generalizing the earlier work by Bogoliubov, Logunov, and Shirkov for QED [19]. The key ingredient of their framework is the spectral density $p_\sigma(\sigma)$ with the aid of which an analytic running coupling can be defined in the Euclidean region ($-q^2 = Q^2$) using a Källén–Lehmann spectral representation:

$$[f(Q^2)]_{an} = \frac{1}{\sigma + Q^2 - i\epsilon} d\sigma. \quad (8)$$

The same spectral density $p_\sigma(\sigma) = \text{Im}[f(-\sigma)]/\pi$ is then used to define the running coupling also in the timelike region by means of a dispersion relation for the Adler function.

At the one-loop level, one obtains for the image of $f(Q^2) = a(Q^2) = (b_n/4\pi)\alpha_s(Q^2)$ in the Euclidean region [18]

$$\mathcal{A}_1(Q^2) = \frac{1}{\sigma + Q^2} d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}, \quad (9)$$

while its counterpart in Minkowski space reads

$$\mathcal{U}_1(s) = \int_0^\infty \frac{p_\sigma(\sigma)}{\sigma} d\sigma = \frac{1}{\pi} \arccos \frac{L_x}{\sqrt{\pi^2 + L_x^2}}, \quad (10)$$

where $L = \ln(Q^2/\Lambda^2_{QCD})$ and $L_x = \ln(s/\Lambda^2_{QCD})$. Both couplings are analytic functions of their arguments. The analyticity of $\mathcal{A}_1(Q^2)$ is ensured by the second power-behaved term in (9) which removes the Landau ghost in the coupling constant. The graphical representation of the removal of the Landau ghost in the coupling is shown in Fig. 1.

5. THE BIRTH OF APT. FROM A CONCEPT TO A PARADIGM

The simple analytization concept for the one-loop running coupling in the spacelike region has been extended by its inventors and their collaborators to higher loops and has been applied to several characteristic hadronic processes. Examples are the inclusive decay of a $\tau$-lepton into hadrons, the momentum-scale and scheme dependence of the Bjorken and the Gross–Llewellyn Smith sum rule, $\Upsilon$ decays into hadrons, et cetera (the corresponding references can be found in [17]).

In Fig. 2, we show the first power (left panel) and the second power (right panel) of the analytic couplings in the Euclidean (calligraphic notation) and in the Minkowski (Gothic notation) space. All couplings are evaluated at the one-loop level. One appreciates from this figure that the spacelike and timelike analytic couplings have a distorted symmetry for finite values of their arguments, though asymptotically, i.e., for $L, L_x \to \infty$, they tend to the conventional coupling: $\mathcal{A}_n(L \to \infty) \approx \mathcal{U}_n(L \to \infty) \Rightarrow a^n(L \to \infty)$ for $n \in \mathbb{N}$, bearing in mind that $a = (b_n/4\pi)\alpha$. As we will see later, this property is valid also for real indices $n \in \mathbb{R}$.

Meanwhile, a systematic approach—a new paradigm for perturbative calculations in QCD—has emerged, termed Analytic Perturbation Theory (APT). In a nutshell, it can be summarized as follows. The standard and the analytic couplings at one loop can be defined [17] recursively for any integer power (index) $n \in \mathbb{N}$ by means of

$$\begin{pmatrix} a^2(k) \\ \mathcal{A}_n(k) \\ \mathcal{U}_n(k) \end{pmatrix} = \frac{1}{(n-1)!} \left( \frac{d}{dk} \right)^{n-1} \begin{pmatrix} a^1(k) \\ \mathcal{A}_1(k) \\ \mathcal{U}_1(k) \end{pmatrix}, \quad (11)$$

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where

\[ a^n (\text{standard QCD with the scaled coupling } a = \alpha_s b_0/4\pi = 1/L, \ n \in \mathbb{N}: \text{power}) \]

\[ \mathcal{A}_n (\text{analytic in spacelike region, } n \in \mathbb{N}: \text{index);} \]
\[ k = L = \ln(Q^2/\Lambda^2) \]

\[ \mathcal{U}_n (\text{analytic in timelike region, } n \in \mathbb{N}: \text{index);} \]
\[ k = L_s = \ln(s/\Lambda^2). \]

On the other hand, in terms of the spectral density

\[ \rho_{\sigma}(\sigma) = \frac{1}{\pi} \text{Im}[a^n(-\sigma)] \]
\[ \rho_{\sigma}(\sigma) = \frac{\sin[n \arccos(L_s/\sqrt{L_s^2 + \pi^2})]}{n\pi[L_s^2 + \pi^2]^{n-1}}, \]

one has

\[ \mathcal{A}_n(Q^2) = \int_0^\infty \frac{\rho_{\sigma}(\sigma)}{\sigma + Q^2} d\sigma = \frac{1}{L^n} \frac{F(e^{-L}, 1 - n)}{\Gamma(n)}, \tag{13a} \]

\[ \mathcal{U}_n(s) = \int_s^\infty \frac{\rho_{\sigma}(\sigma)}{\sigma} d\sigma \]
\[ = \frac{\sin[(n-1)\arccos(L_s/\sqrt{L_s^2 + \pi^2})]}{(n-1)\pi[L_s^2 + \pi^2]^{n-1}}, \tag{13b} \]

where \( F(z, n) \) is the transcendental Lerch function serving in (13a) as a Landau-pole remover \cite{17}. In

**Fig. 1.** The Landau singularity, marked by the symbol \( \times \) in the left panel, is absent by construction in the analytic approach of Shirkov and Solovtsov \cite{18} (right panel).

**Fig. 2.** Analytic couplings in APT: spacelike region—\( \mathcal{A}_n(Q^2) \) and timelike region—\( \mathcal{U}_n(s) \). The left panel shows the first power (index), while the second power (index) is given in the right panel.
APT, hadronic quantities defined (in the Minkowski region) by
\[ R_{\text{PT}}(z) = \sum_n r_n \alpha_n^m(z) \]

have no Landau singularities in the Euclidean region, because the spacelike couplings \( A_n(z) \) are analytic functions. One appreciates that APT transforms a power-series expansion into a non-power-series, i.e., a functional expansion.

Let us close this section by making some important remarks (for references, see [17]):

—Analytization in Euclidean space means subtraction of the Landau pole (at one loop).

—Analytization in Minkowski space amounts to summation of \( \pi^2 \) terms.

—Observables become non-power-series expansions (i.e., \( n \) is an index not a power): \( \mathcal{D}(Q^2) = \sum_n d_n A_n(Q^2) \) (Euclidean space), \( \mathcal{R}(s) = \sum_n d_n |n|_{n}(Q^2) \) (Minkowski space).

—The elimination of ghost singularities in APT appears as a result of causality (spectrality) and RG invariance, i.e., the pole remover is not introduced by hand.

—Two-loop expressions for the analytic couplings are possible by means of the Lambert function (Magradze (2000)).

—Still higher orders can be obtained with an approximate spectral density and employing numerical integration (Shirkov (1999)).

—Incorporation of heavy-quark flavors is possible (“global APT” — Shirkov (2001)).

—Modifications of APT in the deep IR are possible via a spectral density which contains (nonperturbative) power corrections (Alekseev (2006); Nesterenko and Papavassiliou (2005); Cvetic and Valenzuela (2005)).

6. MORE SCALES, MORE RIDDLES

The above exposition suffices to demonstrate the usefulness of APT. But where are its limitations? Can the new paradigm be applied to more complicated cases that contain more than one large momentum scale? And can APT accommodate the crucial effect of evolution? As we cannot sidestep these problems, we will have to dig deeper into analytization. This is not a terminal difficulty of the analytic approach—and we can even benefit from it because doing so, we will ultimately expand APT to noninteger powers of the coupling. It is to these questions to which we focus now our attention.

To start with, we consider as a “theoretical laboratory” the factorized part of the spacelike pion’s electromagnetic form factor. In NLO perturbation theory, this quantity can be written as a convolution \( A \otimes B \equiv \int dz A(z) B(z) \) of a hard-scattering amplitude \( T \) and two pion distribution amplitudes, \( \varphi^\text{in}_\pi, \varphi^\text{out}_\pi \), representing the incoming and outgoing pion bound states of quarks with longitudinal momentum fractions \( x \) and \( y \), respectively. \( T \) is the amplitude for a collinear valence quark–antiquark pair, struck by a highly virtual photon with the large momentum \( Q^2 \), and contains hard-gluon exchanges characterized by the strong coupling \( \alpha_s \) to the considered order of the expansion. Then, we have

\[ F^\text{Fact}_\pi(Q^2) = \varphi^\text{in}_\pi(x, \mu_F^2) \otimes T^{\text{NLO}}_{\text{H}}(x, y, Q^2; \mu_F^2, \mu^2) \otimes \varphi^\text{out}_\pi(y, \mu_F^2), \]

where

\[ \varphi_\pi(x, \mu^2) = 6x(1-x) \]

is the leading twist-2 pion distribution amplitude containing the nonperturbative information on the pion quark structure in terms of the Gegenbauer coefficients \( a_n \) at the initial scale \( \mu^2 \approx 1 \text{ GeV}^2 \), and \( T \) is the sum of all quark-gluon subprocesses up to the order \( \alpha_s^2 \) (its explicit form can be found in the second paper in [4] and also in [5]). Note that \( T \) depends explicitly on \( \alpha_s \), whereas \( \varphi_\pi \) has a more complicated dependence controlled by the evolution equation. Besides, beyond the LO of perturbation theory, \( T \) depends not only on the large external momentum \( Q^2 \), but also on a second auxiliary momentum scale \( \mu_F \) which separates the hard (perturbative) from the soft (nonperturbative) dynamics on account of a factorization theorem. Invariably, the soft parts, i.e., the pion distribution amplitudes, have to be evolved with appropriate anomalous dimensions from the initial scale \( \mu^2 \) to the actual scale of observation \( Q^2 \). As we will discuss in the next section, this additional scale has influence on the analytization procedure that cannot be covered by APT.

The first application of APT to \( F^\text{Fact}_\pi(Q^2) \) was considered in [4], in which the strong coupling and its powers were replaced by their corresponding analytic images: \( a^\ast(L) \rightarrow (\mathcal{A}_L)^\ast \) (Naive analytization [5]). This treatment is strictly speaking incorrect because \( (\mathcal{A}_L)^\ast \neq [a^\ast(L)]_\text{an} \), but phenomenologically it works rather good [4]. An improvement of this approach was presented in [5], where a Maximal analytization of this quantity was performed. This proce-
duires become apparent by comparing the following two expressions:

\[ \frac{d\sigma}{d\lambda^2} \approx \sum_n \frac{\lambda^2}{n^2} \]

The differences between the two analytization procedures become apparent by comparing the following two expressions:

**Naive Analytization**

\[
[Q^2 T_{\text{fact}}(Q^2)]_{\text{Naive-An}} = \mathcal{A}_n^{(2)}(\lambda R, Q^2) I_{\text{H}}^{(0)}(x, y)
\]

**Maximal Analytization**

\[
[Q^2 T_{\text{fact}}(Q^2)]_{\text{Max-An}} = \mathcal{A}_n^{(2)}(\lambda R, Q^2) I_{\text{H}}^{(1)}(x, y; \mu_R, \mu_F^2, Q^2)
\]

where \(\lambda R Q^2 = \mu_R^2\) serves as a renormalization scale and \(\mu_F^2\) denotes the factorization scale of the process. The explicit form of the amplitudes \(I_{\text{H}}^{(0)}\) and \(I_{\text{H}}^{(1)}\) is irrelevant for our arguments here; it can be found in [4, 5, 21]. Furthermore, in [4], the (naive) analytization procedure was also applied after exponentiation to the Sudakov form factor in next-to-leading logarithmic approximation of the cusp anomalous dimension.

Some technical comments: (a) The pole remover in the one-loop running coupling does not change the leading double logarithmic behavior of the Sudakov factor, though the cusp anomalous dimension in the Sudakov exponent receives additional subleading contributions. (b) Due to the absence of ghost singularities, the Sudakov exponents are analytic functions of \(Q^2\). (c) The perturbatively calculable part of the pion’s electromagnetic form factor \(Q^2 T_{\text{fact}}(Q^2)\) receives the bulk of its contributions from small transverse distances \(b\) below \(\Lambda_{\text{QCD}} \approx 0.5\), i.e., it saturates in exactly that region, where the application of QCD perturbation theory is valid [4]. However, the full treatment of the Sudakov-type gluon resummation in impact space faces special subtleties which have not been fully resolved until now. First results within a toy model in momentum space (see Appendix C of [22]) seem to indicate a conflict between analytization and exponentiation of soft gluons.

Phenomenologically, already the “naive” analytization procedure and even more the “maximal” application of APT have important consequences: (i) the artificial increase of the spacelike electromagnetic pion form factor towards low \(Q^2\) values—caused by the Landau singularity—is absent, (ii) the sensitivity of the results on the choice of the renormalization scheme and the adopted scale setting is extremely reduced relative to the standard power-series expansion of perturbative QCD [Shirkov and Solovtsov (1998) and [5]]. This behavior is shown graphically in Fig. 3 for the scaled expression of \(Q^2 T_{\text{fact}}(Q^2)\) employing these two analytization procedures and including leading-order evolution of the pion distribution amplitude in APT. The predictions shown were calcu-
lated for different renormalization-scale settings, including the “default” choice, $\mu_R^2 = Q^2$ (dashed line), the Brodsky–Lepage–Mackenzie scale setting (dotted line), the so-called $\alpha_s$ scheme (dash-dotted line), et cetera, and employing the Bakulev–Mikhailov–Stefanis pion distribution amplitude which has $a_2(\mu_0^2 = 1 \text{ GeV}^2) = 0.20$, $a_4(\mu_0^2 = 1 \text{ GeV}^2) = -0.14$, $a_n(\mu_0^2 = 1 \text{ GeV}^2) \approx 0$ ($n > 2$). All these issues are discussed in detail in [5]. The main observation from this figure is that applying APT in its maximal form to a typical exclusive quantity, like $F_n^{\pi^0}(Q^2)$, not only provides IR stability, it also diminishes its renormalization scheme and scale-setting dependence already at the NLO level. Indeed, the predictions obtained in different schemes do not differ significantly from each other in a wide range of momenta from low to large $Q^2$ values.

To further appreciate the advantages of the analytic scheme relative to the standard power-series expansion, bear in mind our statements in Section 2 in the language of information theory. Within this context, one may consider the variation of the predictions—the uncertainty spread—obtained with different renormalization schemes, as “noise”, the prediction for $F_n^{\pi^0}(Q^2)$ itself being the “signal”. It becomes obvious from the right panel of Fig. 3 that maximal analytization within APT provides a much better dynamical range of quality because the signal-to-noise ratio is strongly enhanced in the whole $Q^2$ range, especially at low $Q^2$.

7. FRACTIONALIZING APT

In the previous section we have discussed the advantages of APT relative to the standard QCD perturbation theory and considered a typical exclusive quantity, notably, the factorizable part of the pion’s electromagnetic form factor that exhibits a dependence on potentially two large momentum scales: $Q^2$ and the factorization scale $\mu_F$ (that may also serve as the evolution parameter). Until now our discussion has not addressed the factorization-scale dependence under the proviso of analytization. But the considered example makes it obvious that, ultimately, one has to accommodate the analytization of terms like

$$Z[L] = e^{\int \frac{d^4 q}{(2\pi)^4} \frac{\gamma(q)}{\beta(q)}} \rightarrow [a_s(L)]^{\gamma_n/2|F_0|}$$

$$ \leftrightarrow \text{RG at one loop}$$

$$[a_s(L)]^{\gamma_n} \ln[a_s(L)]$$

$$ \leftrightarrow \text{RG at two loop}$$

$$[a_s(L)]^{\gamma_n} L^m$$

$$ \leftrightarrow \text{Factorization}$$

$$\exp[-a_s(L) F(x)]$$

$$ \leftrightarrow \text{Sudakov resummation}$$

Before studying the consequences of this generalized analyticity requirement for observables, let us first present the different analytization concepts, we have discussed above, in mutual comparison—Fig. 4. In this figure, the linear operations $A_E$ and $A_M$ define, respectively, the analytic running couplings in the Euclidean (spacelike)

$$A_E[a_n^{(l)}] = \mathcal{A}_E^{(l)}$$

with

$$\mathcal{A}_E^{(l)}(Q^2) \equiv \int_{\sigma = 0}^{\infty} \frac{\sigma^{\gamma_n^{(l)}(\sigma)}}{\sigma + Q^2} d\sigma$$

and the Minkowski (timelike) region

$$A_M[a_n^{(l)}] = \mathcal{Y}_n^{(l)}$$

with

$$\mathcal{Y}_n^{(l)}(s) \equiv \int_0^{\infty} \frac{\sigma^{\gamma_n^{(l)}(\sigma)}}{\sigma} d\sigma,$$
be represented by the following two integral transformations from the timelike region to the spacelike one
\[
\hat{D}[\mathcal{U}_n^{(i)}] = \mathcal{A}_n^{(i)} \quad \text{with} \quad \mathcal{A}_n^{(i)}(Q^2) = \frac{1}{2\pi i} \int_{-i \epsilon}^{i \epsilon} \frac{\mathcal{U}_n^{(i)}(\sigma)}{\sigma + Q^2} d\sigma.
\]
and for the inverse transformation
\[
\mathcal{U}_n^{(i)}(s) = \frac{1}{L^v} \frac{F(e^{-L}, 1 - v)}{\Gamma(v)},
\]
where the first term corresponds to the conventional term of perturbative QCD (with \(L = \ln(Q^2/\Lambda^2)\)) and the second one is entailed by the pole remover (cf. \(1/(e^L - 1)\) at one loop). The function \(\mathcal{A}_n(L)\) is an entire function in the index \(v\) and has the following properties:
\[
-\mathcal{A}_m(\pm \infty) = 0 \quad \text{for} \quad m \geq 2, m \in \mathbb{N}.
\]
\[
-\mathcal{A}_n(L) = -\int [1/\Gamma(v)] \times \sum_{r=0}^{\infty} \zeta(1 - v - r)[(-L')^r/r!] \quad \text{for} \quad |L| < 2\pi.
\]
Note that the reduced transcendental Lerch function \(F(z, v, m)\) is related to the Lerch function \(\Phi(z, v, m)\) via \(\Phi(z, v, 1) \equiv F(z, v, v)\). For a positive integer index, \(v = m \geq 2\), one has the relation
\[
F(z, 1 - m) = (-1)^m F(1, z, 1 - m),
\]
so that substituting Eq. (26) in (25), one arrives at
\[
\mathcal{A}_m(L) = (-1)^m \mathcal{A}_m(-L).
\]
In Minkowski space, the analytic images of the running coupling are completely determined by elementary functions [22] \((L_s = \ln(s/\Lambda^2)):\)
\[
\mathcal{U}_v(L_s) = \frac{\sin[(v - 1)\arccos(L_s/\sqrt{\pi^2 + L_s^2})]}{\pi(v - 1)\pi^2 + L_s^2(v - 1/2)^2},
\]
from which, for example, we get \(\mathcal{U}_0(L_s) = 1\), \(\mathcal{U}_1(L_s) = L_s\), \(\mathcal{U}_2(L) = L_s^2 - \pi^2/3\), \(\mathcal{U}_m(L) = (-1)^m \mathcal{U}_m(-L)\) for \(m \geq 2\), \(\mathcal{U}_{m}(-\infty) = \mathcal{A}_m(\infty) = 0\) for \(m \in \mathbb{N}\). For the value \(L = L_s = 0\), the analytic couplings in both regions are interrelated by the equation
\[
\mathcal{A}_m(0) = \left[\frac{(v - 1)\zeta(v)}{2^{v-1}}\right]\mathcal{U}_v(0),
\]
where
\[
\mathcal{U}_v(0) = \frac{\sin((v - 1)\pi/2)}{(v - 1)\pi^v}.
\]
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and where \( \zeta(\nu) \) is the Riemann \( \zeta \) function, with the coefficient in the bracket providing a quantitative measure for the magnitude of the distortion of the “mirror symmetry”, mentioned before, for any \( \nu \in \mathbb{R} \).

A graphical illustration of the analytic couplings \( \mathcal{A}_\nu \) and \( \mathfrak{A}_\nu \) is given in Fig. 5 which shows the rate of change of these functions with respect to the index \( \nu \) and the argument \( L \) (or \( L_s \)).

The major characteristics of FAPT in comparison with the standard perturbative expansion in QCD and the original APT are collected in Table 1, while further details can be found in [17, 21, 22].

Up to now we have actually concentrated our attention on the one-loop approximation of the running coupling and its analytic images. But the index derivative in FAPT gives us the possibility to obtain two-loop expressions from those at one loop. To achieve this goal, let us recall the two-loop RG equation for the standard QCD normalized coupling \( a^{(2)} = b_0 a^{(2)}/4\pi \)

\[
\frac{d a^{(2)}(L)}{dL} = -a^{(2)}(L)[1 + c_1 a^{(2)}(L)]
\]

with \( c_1 = \frac{b_1}{b_0} \)

which assumes the following functional form

\[
\frac{1}{a^{(2)}} + c_1 \ln \left[ \frac{a^{(2)}}{1 + c_1 a^{(2)}} \right] = L.
\]

This equation has the following exact solution (Magradze 2000)

\[
a^{(3)}(L) = -\frac{1}{c_1 1 + W_{-1}(z\eta(L))}.
\]

---

**Table 1.** FAPT versus standard QCD perturbation theory (SPT) and APT

| Theory          | SPT                       | APT                        | FAPT                          |
|-----------------|---------------------------|----------------------------|-------------------------------|
| Space           | \( \{ a^\nu \}_{\nu \in \mathbb{R}} \) | \( \{ \mathcal{A}_m, \mathfrak{A}_m \}_{m \in \mathbb{N}} \) | \( \{ \mathcal{A}_\nu, \mathfrak{A}_\nu \}_{\nu \in \mathbb{R}} \) |
| Series expansion| \( \sum_m f_m a^m(L) \)      | \( \sum_m f_m \mathcal{A}_m(L) \) | \( \sum_m f_m \mathcal{A}_m(L) \) |
| Inverse powers  | \( [a(L)]^{-m} \)                | –                          | \( \mathcal{A}_{-m}(L) = L^m \) |
| Multiplication  | \( a^\mu a^\nu = a^{\mu+\nu} \) | –                          | –                             |
| Index derivative| \( a^\nu \ln^k a \)          | –                          | \( \frac{d^k}{d\nu^k} \mathfrak{A}_\nu = [a^\nu \ln^k(a)]_{an} \) |
where we have expanded $\psi(z) = d\ln \Gamma(z)/dz$ denotes the Digamma function.

With the above expression in our hands, we can immediately obtain \(^{[22]}\) the spectral density at the two-loop level, $\rho_v^{(2)}(\sigma)$, by virtue of Eq. (12), viz.,

$$\rho_v^{(2)}(\sigma) = \frac{1}{\pi} \text{Im}[a_v^{(2)}(L - i\pi)].$$  \hspace{1cm} (34)

Using this expression in Eqs. (20) and (21), we have

$$\mathcal{A}_v^{(2)}(L) = \int_0^{\infty} \frac{\rho_v^{(2)}(\sigma)}{\sigma + Q^2} d\sigma,$$

$$\mathcal{Q}_v^{(2)}(s) = \int_0^{\infty} \frac{\rho_v^{(2)}(\sigma)}{\sigma} d\sigma,$$

from which we derive ($\mathcal{D} \equiv d^k/dv^k$)

$$\mathcal{A}_v^{(2)}(L) = \mathcal{A}_v^{(1)}(L) + c_1 \mathcal{D} \mathcal{A}_v^{(1)}(L) + c_1^2 \mathcal{D}^2 \mathcal{A}_v^{(1)}(L),$$  \hspace{1cm} (36)

where we have expanded $a_v^{(2)}(L)$ in terms of $a_v^{(1)}(L)$. The above expression can be generalized to any real index $\nu$, in both the Euclidean and the Minkowski space, to obtain

$$\begin{bmatrix}
\mathcal{A}_v^{(2)} \\
\mathcal{Q}_v^{(2)}
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}_v^{(1)} \\
\mathcal{Q}_v^{(1)}
\end{bmatrix} + c_1 \mathcal{D} \begin{bmatrix}
\mathcal{A}_v^{(1)} \\
\mathcal{Q}_v^{(1)}
\end{bmatrix}$$

$$+ c_1^2 \mathcal{D}^2 \begin{bmatrix}
\mathcal{A}_v^{(1)} \\
\mathcal{Q}_v^{(1)}
\end{bmatrix} + \mathcal{O}(\mathcal{D}^3 \mathcal{A}_v^{(1)}).$$  \hspace{1cm} (37)

Though these expressions are not exact, the achieved accuracy is of the order of about 1% down to the value $L = L_0 = 0$ and, hence, they are sufficient for most practical QCD applications. A more accurate expression for the two-loop Minkowski coupling which includes the contribution of the next higher order $\mathcal{O}(\mathcal{D}^3 \mathcal{Q}_v^{(1)})$ is given in \([17]\). Moreover, one can find there the analogous three-loop expression with a similarly good quality of convergence.

The particular advantage of FAPT is the possibility to perform the analytization analytization of powers of the running coupling multiplied by logarithms of $L$ and its powers, e.g., $[(a_v^{(2)}(L)^n L^n)]_{\text{in}} = \mathcal{L}_v^{(2)}(L)$. As we emphasized before, such terms appear already in NLO calculations of typical hadronic amplitudes, like form factors, or when taking into account evolution effects.
Table 2 displays in approximate form the analytization of such expressions. Their usefulness will become evident below, when we will consider the influence of such terms in the calculation of the factorized part of the pion’s electromagnetic form factor.

8. SPACELIKE PION’S ELECTROMAGNETIC FORM FACTOR IN FAPT

Having set up the calculational tools of FAPT, we now detail concrete applications, starting with an example in the spacelike region. Consider again Eq. (18)—this time including into the analytization procedure also the logarithmic term (labeled by the acronym KS [23] in order to indicate the analytization of the amplitude as a whole): \( \ln(Q^2/\mu_F^2) = \ln(\lambda_R Q^2/\Lambda^2) - \ln(\lambda_R \mu_F^2/\Lambda^2) \). Then, we find \( (\bar{x} = 1 - x) )

\[
\begin{align*}
[Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{KS}}^\text{An} & = \frac{\Delta_1^{(2)}(\lambda_R Q^2)}{4\pi} \left[ C^{(2)}_{\text{FAPT}}(x, y; \lambda_R, \mu_F^2) \right] \\
& + \frac{\Delta_2^{(2)}(\lambda_R Q^2)}{4\pi} [C^{(2)}_{\text{FAPT}}(x, y; \lambda_R, \mu_F^2) (6 + 2\ln(\bar{x} y))] ,
\end{align*}
\]

with

\[
\Delta_2^{(2)}(Q^2) = \Delta_2^{(2)}(Q^2) - \Delta_2^{(2)}(Q^2) \ln[Q^2/\Lambda^2] \quad (39)
\]

encoding the deviation from the result obtained with the Maximal Analytization procedure and where

\[
\begin{align*}
\Delta_2^{(2)}(Q^2) & = \left[ (\alpha_s^{\text{LO}}(Q^2))^2 \ln \left( \frac{Q^2}{\Lambda^2} \right) \right]_{\text{KS}}^\text{An} \\
& = 4\pi \left[ \frac{(\alpha_s^{\text{LO}}(Q^2))^2}{b_0} - \frac{(\alpha_s^{\text{LO}}(Q^2))^2}{b_0} \right]_{\text{KS}}^\text{An} .
\end{align*}
\]

Carrying out the KS analytization, we get the approximate expression

\[
\begin{align*}
\mathcal{L}_{2,1}^{\text{app}(2)}(Q^2) & = \frac{4\pi}{b_0} \left[ A_1^{(2)}(Q^2) + c_1 \frac{4\pi}{b_0} f_\Delta(Q^2) \right] \\
& = \frac{4\pi}{b_0} \left[ A_1^{(2)}(Q^2) + c_1 \frac{4\pi}{b_0} f_\Delta(Q^2) \right]
\end{align*}
\]

(41)

which can be further improved by substituting the \( c_1 \)-term in Eq. (41) by its two-loop expression.

\[ Q_{2,1}^{\text{imp}(2)}(L) = \mathcal{A}_1^{(2)}(L) + c_1 \mathcal{A}_2^{(2)}(L) \]—see [24],

where [cf. Table 2, \( a_{\nu}^{\pi,2}(L) \)]

\[
f_\Delta(Q^2) = \sum_{n \geq 2} \left[ \psi(2) \zeta(-n - 1) - \frac{d\zeta(-n - 1)}{dn} \right] \times \frac{\left( -\ln(Q^2/\Lambda^2) \right)^n}{\Gamma(n + 1)}
\]

and \( \zeta(x) \) is the Riemann Zeta-function. The main effect of including the logarithmic term into the analytization procedure of \( F_{\text{Fact}}^{\text{KS}}(Q^2) \) is to suppress its sensitivity on the variation of the factorization scale to a minimum. It was shown in [21] that varying the factorization scale from 1 GeV^2 to 10 GeV^2, the form factor changes by a mere 1.5 percent reaching for a (hypothetical) factorization scale of 50 GeV^2 just the level of about 2.5 percent. Note here that it is possible to treat this problem in another way, namely, to fix \( \mu^2 = Q^2 \) from the very beginning and then use FAPT to correctly account for the evolution factors \( E^{\text{LO}}_\nu(Q^2) \sim \alpha_s^{\nu}(Q^2) \) in the Gegenbauer coefficients \( a_\nu(Q^2) \), with \( \nu_\nu \) being the corresponding fractional evolution exponents. This type of calculation was performed in [24] and it was shown that the difference between the two approaches, i.e., FAPT with fixed \( \mu^2 = 5.76 \text{ GeV}^2 \) and FAPT with the renormalization scale fixed at \( \mu^2 = Q^2 \), is of the order of 1.5%.

To conclude, using FAPT, the dependence of a perturbatively calculated QCD quantity on all perturbative scheme and scale settings is diminished already at the NLO level.

9. HIGGS BOSON DECAY INTO A \( b\bar{b} \) PAIR USING FAPT IN MINKOWSKI SPACE

The second FAPT application deals with the Higgs boson decay into a \( b\bar{b} \) pair in the timelike region. We will present here only the main results and refer for the technical details to [17, 25]. A dedicated analysis of the uncertainties involved in the calculation of the decay width of the Higgs boson into bottom quarks was recently given by Kataev and Kim [3].

Consider now the decay of a scalar Higgs boson to a \( b\bar{b} \) pair in terms of the quantity \( R_5 \) at the four-loop level from which one can then obtain the width \( \Gamma(H \rightarrow b\bar{b}) \). Note that the application of FAPT in Minkowski space is not plagued by Landau singularities. However, the analytic continuation from the spacelike to the timelike region induces so-called “kinematical” \( \pi^2 \) terms that may be comparable in magnitude with the higher-order expansion coefficients and have, therefore, to be summed. The first step in our approach is
the correlator of two scalar currents $J^\xi_b = \bar{\Psi}_b \Psi_\xi_b$ for bottom quarks with mass $m_b$, coupled to the scalar Higgs boson with mass $M_H$, and where $Q^2 = -q^2$, i.e.,

$$\Pi(Q^2) = (4\pi^2)i \int dx e^{iqx} \langle 0 | T[J^\xi_b(x) J^\xi_b(0)] | 0 \rangle. \quad (43)$$

Then, $R_S(s) = \text{Im} \Pi(-s - i\varepsilon)/(2\pi s)$ and one can express the decay width in terms of $R_S$:

$$\Gamma(H \to b\bar{b}) = \frac{G_F}{4\sqrt{2\pi}} M_H m_b^4 M_R(s = M_H^2). \quad (44)$$

The quantity $R_S$ is obtained from the Adler function $D$ via analytic continuation into the Euclidean space using the transformation $A_M$ (or, equivalently, the integral transformation $\tilde{R}$), as shown in Fig. 4. The reason for this detour is that in the Euclidean region perturbation theory works. Hence, we write

$$\tilde{D}_S(Q^2; \mu^2) = 3 m_b^2(Q^2) \left[ 1 + \sum_{n \geq 1} d_n(Q^2/\mu^2) a^\nu_n(\mu^2) \right]. \quad (45)$$

The second step is to calculate

$$\tilde{R}_S(s) \equiv \tilde{R}_S(Q^2 = s, \mu^2 = s) = 3 m_b^2(s) \left[ 1 + \sum_{n \geq 1} r_n a^\nu_n(s) \right] \quad (46)$$

using the techniques developed by Gorishnii et al. (1990) [26] and Chetyrkin et al. (1997) [27].

As already mentioned, the coefficients $r_n$ contain characteristic $\pi^2$ terms originating from the integral transformation $\tilde{R}$ of the powers of the logarithms entering $D_S$. The latter is related to $\tilde{R}_S(s, s)$ by means of a dispersion relation. Notice that these logarithms have two different sources: one is the running of $a_\xi$ in $\tilde{D}_S$, while the other is related to the evolution of the heavy-quark mass $m_b^2(Q^2)$. As a result, the coefficients $r_n$ in (46) are connected to the coefficients $d_n$ in $\tilde{D}_S$ (calculable in Euclidean space) and to a combination of the mass anomalous dimension $\gamma_1$ and the $\beta$-function coefficients $b_i$ multiplied by $\pi^2$ powers. Appealing to the explicit results given in [17], we remark that the total amount of these terms is of the order of the coefficient $d_1$. FAPT can by construction account for these terms to all orders of the perturbative expansion. This is because in FAPT one does not need to expand the renormalization factors into a truncated series of logarithms; instead one can transform them by means of the $A_M$-operation “as a whole”.

The running mass in the $l$-loop approximation, $m_{l\ell}$, can be cast in terms of the renormalization-group invariant quantity $\tilde{m}_{l\ell}$ to read

$$m_{l\ell}(Q^2) = \tilde{m}_{l\ell}^2(a_\xi(Q^2)) \frac{v_0}{f_{l\ell}(a_\xi(Q^2))}, \quad (47)$$

where $v_0 = 2\gamma_0/b_0$ and the expansion of $f_{l\ell}(x)$ at the three-loop order is given by

$$f_{l\ell}(a_\xi) = 1 + a_\xi \frac{b_1}{2} \left( \frac{\gamma_1 - \gamma_0}{b_0} \right) + a_\xi \frac{b_2}{16 b_0^2} \left[ \frac{\gamma_0 - \gamma_1 + 2 \left( \frac{\gamma_0 - \gamma_1}{b_0} \right)^2}{b_1} \right] + \mathcal{O}(a_\xi^3). \quad (48)$$

Expanding the running mass in a power series according to

$$m_{l\ell}^2(Q^2) = \tilde{m}_{l\ell}^2(a_\xi(Q^2)) \left[ 1 + \sum_{m \geq 1} \Delta_m(a_\xi(Q^2))^{m + \nu_0} \right] \quad (49)$$

and setting $\mu^2 = Q^2$, we find

$$\left[ 3 m_b^2 \sum_{l=1}^{l-1} \tilde{D}_S^{(l)} \right] = \left( a_\xi^{(l)}(Q^2) \right)^{v_0} + \sum_{n \geq 1} d_n(a_\xi^{(l)}(Q^2))^{n + v_0} \quad (50)$$

with

$$\Delta_m^{(l)} = e_m^{(l)} + \sum_{k \geq 1} d_k e_m^{(l-k)}. \quad (51)$$

Note that we have separated the mass-evolution multiloop effects (third term of Eq. (50)) from the original series expansion of $D$ (truncated at $n = l$). These contributions are represented by the second term on the RHS of Eq. (50). In practice, for $Q \geq 2$ GeV, i.e., for $a_\xi < 0.4$, the truncation at $m = l + 4$ of the summation in (49) is sufficient, given that the truncation error is less than 1 percent.

By applying the analytization operation $A_M$ to the quantity $\tilde{D}_S^{(l)}(Q^2)$, we finally obtain

$$\tilde{R}_S^{(l)}_{MFAPT} = A_M\left[ D_S^{(l)} \right] \quad (52)$$

$$= 3 m_b^2 \left[ a_\xi^{(l)} + \sum_{n \geq 1} d_n a_\xi^{(l)} + \sum_{m \geq 1} \Delta_m^{(l)} a_\xi^{(l)} \right].$$
The long-dashed curve shows the predictions obtained from Eq. (52) for \( N_f = 5 \) (solid blue line), displayed for different loop orders: \( l = 2 \) (left panel, \( \Lambda_{N_f} = 263 \, \text{MeV} \)) and \( l = 3 \) (right panel with \( \Lambda_{N_f} = 261 \, \text{MeV} \)). The value of \( \Lambda_{N_f} = 5 \) = 111 MeV in all cases corresponds to \( \frac{1}{2} \sum_{l,s} (s = \frac{m^2}{2}, \, N_f = 5) = 0.120 \).

The results for \( \tilde{R}_S(M_H^2) \) as a function of the Higgs mass \( M_H \), calculated with different perturbative approaches in the \( \overline{\text{MS}} \) scheme, are shown in Fig. 6. The long-dashed curve shows the predictions obtained in [27] using standard fixed-order perturbative QCD at the \( l = 4 \) loop level of the expansion. The solid curve next to it represents the FAPT prediction (cf. (52)), including in the second sum all evolution effects up to \( m = l + 4 \) and fixing the active flavor number to \( N_f = 5 \). On the other hand, the contributions of the higher-loop renormalization-group dependent terms are accumulated in the coefficients \( \Delta^{(l)}_m \) by means of the parameters \( \gamma_0 \) and \( b_0 \). Obviously, the standard perturbative QCD approach and FAPT yield similar predictions for this observable, starting with the two-loop running. The reason for the slightly larger FAPT prediction lies in the fact that the coefficients \( \alpha_n \) contain by means of the index \( \nu_0 \) the resummed contribution of an infinite series of \( \pi^2 \)-terms (as well as all \( \gamma_0 \) and \( b_0 \) terms) that renders them ultimately smaller than the corresponding powers of the standard coupling. Finally, the dotted green curve about 8% below the other lines, in both panels of Fig. 6, illustrates the estimate obtained with the “naive non-Abelianization” and an optimized power-series expansion that makes use of the “contour-improved integration technique” [28].

The above Higgs decay results can be further extended by resumming the whole nonpower series [25], whereas the case of the three-loop running coupling was elaborated in detail in [29]. This way, one may extract more reliable predictions for the width \( \Gamma_{H \rightarrow b\bar{b}} \) in terms of the Higgs-boson mass \( M_H \). This is indeed possible by employing an appropriate generating function to determine the next two higher expansion coefficients reaching at the truncation level of \( \text{N}^3\text{LO} \) an accuracy of the order of 1%, which is even better than the 2% uncertainty involved in the \( \tilde{m}_s(\tilde{m}_b^2) \) estimates. The predictions for the one-loop and the two-loop width are illustrated in Fig. 7 vs. the mass of the Higgs boson for which the Penin–Steinhauser value \( \tilde{m}_s(\tilde{m}_b^2) = 4.35 \pm 0.07 \, \text{GeV} \) [30] was used. Adopting instead the RG-effective \( b \)-quark mass \( \tilde{m}_s(\tilde{m}_b^2) = 4.19 \pm 0.05 \, \text{GeV} \), calculated by Kühn and Steinhauser [31], one obtains results somewhat reduced by about 7%.

\[
[a_\gamma(s)]_\infty = \alpha^{(l)}_\gamma(s) \equiv \left( \frac{4}{b_0} \right)^\gamma \alpha^{(l)}_\gamma(s).
\]
been implemented in the form of the Mathematica package FAPT.m [32]. Yet FAPT is not a complete formalism. We still do not really know how to consider Sudakov gluon resummation within FAPT because under the analytization requirement resummation does not mean exponentiation. This important issue has yet to be understood and developed. Also the application of FAPT to compute power corrections to the spacelike pion’s LO electromagnetic form factor and the inclusive Drell–Yan cross section, started in [23], has so far not been exploited beyond the leading power and for other hadronic quantities.

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