The induced interaction in a Fermi gas with a BEC-BCS crossover

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Abstract

We study the effect of the induced interaction on the superfluid transition temperature of a Fermi gas with a BEC-BCS crossover. The Gorkov-Melik-Barkhudarov theory about the induced interaction is extended from the BCS side to the entire crossover, and the pairing fluctuation is treated in the approach by Nozières and Schmitt-Rink. At unitarity, the induced interaction reduces the transition temperature by about twenty percent. In the BCS limit, the transition temperature is reduced by a factor about 2.22, as found by Gorkov and Melik-Barkhudarov. Our result shows that the effect of the induced interaction is important both on the BCS side and in the unitary region.

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I. INTRODUCTION

One of the most important developments in experiments on ultra-cold atoms is the observation of BEC-BCS crossover [1, 2, 3, 4, 5, 6, 7, 8] which was originally predicated for strongly-coupled superconductors [9, 10]. In experiments on ultra-cold atoms, the interaction between atoms can be tuned by the technique of Feshbach resonance, and the system can evolve smoothly from a Bardeen-Cooper-Schrieffer (BCS) pairing state to a Bose-Einstein condensation (BEC) state of diatomic molecules. In the BCS limit where the interaction is weakly attractive, the atoms are paired into a BCS state below a critical temperature, very similar to electrons in conventional superconductors. In the BEC limit where the interaction is weakly repulsive, tightly-bound diatomic molecules are formed, and a molecular BEC state appears below a critical temperature. Near the resonance, in the unitary region [11, 12] where the size of the scattering length is much larger than the inter-particle spacing, the system is strongly correlated in both the normal and superfluid states.

The BEC-BCS crossover can be qualitatively explained by a mean-field BCS theory [10]. In this theory, the size of atom pairs decreases as the system goes from the BCS side to the BEC side. In the BEC limit, the pair size is so small that atom pairs become diatomic molecules. However, the mean-field theory predicts an exponentially-divergent superfluid transition temperature in the BEC limit [13], which is against the result from the theory about an ideal Bose gas. Nozières and Schmitt-Rink (NSR) [14] first pointed out that the pairing fluctuation must be taken into account to obtain the correct superfluid transition temperature $T_c$ of the BEC-BCS crossover. The pairing fluctuation is especially important in the BEC limit where nearly all atoms become thermal molecules at $T_c$. Following NSR’s pioneer work, many theoretical studies have focused on improving NSR’s method and extending their analysis to the broken symmetry state [13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], which was recently reviewed in Ref. [25, 26].

However in the BCS limit, a different type of fluctuation is important. Gorkov and Melik-Barkhudarov (GMB) [27] found that there is a modification to the pairing interaction due to the many-body medium, referred to as the induced interaction [28]. The induced interaction suppresses pairing considerably and reduces the superfluid transition temperature $T_c$ by a factor about 2.22 with respect to the mean-field $T_c$ in the BCS limit. The fluctuation considered by GMB is in the particle-hole channel, different from that in the particle-particle
FIG. 1: Critical temperatures of BEC-BCS crossover. The solid line is our result after taking into account the induced interaction in the NSR approach, the dashed line is the original NSR result, the dotted line is the result from BCS mean-field theory, and the dot-dashed line is GMB’s result given by Eq. (9). These theoretical results are compared with the data from QMC simulations [31, 32] shown in symbols.

channel considered by NSR. In the BCS limit the NSR fluctuation is much less important than the GMB fluctuation. When the system moves from the BCS side towards the BEC side, the GMB fluctuation becomes weaker and the NSR fluctuation becomes stronger. In the BEC limit, the NSR fluctuation is dominant. In an accurate description of the BEC-BCS crossover, both GMB and NSR fluctuations should be treated properly, which has not been addressed except in two recent renormalization-group studies [29, 30].

In this work, we present our result about the induced interaction in the whole BEC-BCS crossover. First we extend the GMB theory about the induced interaction from the BCS limit to the strongly-interacting region. Then we consider the induced interaction in the NSR framework and compute $T_c$ for the entire BEC-BCS crossover. Our main result is shown in Fig. 1. Compared with the original NSR result, the superfluid transition temperature $T_c$ is reduced considerably both on the BCS side and in the unitary region. In the BCS limit, we recover the GMB result. At unitarity, the critical temperature $T_c$ is found to be $T_c = 0.178T_F$, close to the results from Quantum Monte Carlo (QMC) simulations [31, 32], and about 20% smaller than the NSR result. Our result shows that induced interaction plays an important role in the unitary region and on the BCS side. Discussions and conclusions
are given in the end.

II. THE INDUCED INTERACTION

A Fermi gas with a wide Feshbach resonance can be described by a single-channel model,

\[ \mathcal{H} = -\sum_{\sigma} \frac{\hbar^2}{2m} \psi_\sigma \nabla^2 \psi_\sigma + g \psi_\uparrow \psi_\uparrow \psi_\downarrow \psi_\downarrow, \tag{1} \]

where the coupling constant is given by \( g = 4\pi \hbar^2 a_s/m \), \( a_s \) is the scattering length, and \( \psi_\sigma \) is the field operator for spin component \( \sigma \). In this work, we consider only the homogeneous spin-balanced case where the densities of spin-\( \uparrow \) and spin-\( \downarrow \) atoms are the same.

In the BCS limit when the interaction is weakly attractive, Gorkov and Melik-Barkhudarov (GMB) \[27\] showed that in the particle-hole channel there is a correction to the pairing interaction from the many-body background, given by the Feynman diagram shown in Fig. 2(a). Other diagrams of the same order are not as important. For example, the diagram in Fig. 2(b) corresponds to an effective interaction between atoms with the same spin component, which is strongly suppressed at low temperatures.

Beyond the BCS limit, when the interaction is strong, higher order diagrams are important. We generalize the GMB approximation by considering all the diagrams of the same general type as the GMB diagram shown in Fig. 2(a) and summing all these diagrams together as shown in Fig. 2(c), and obtain the induced interaction in the normal state given by

\[ U_{\text{ind}}(p_1, p_2; p_3, p_4) = -\frac{g^2 \chi(p_1 - p_4)}{1 + g \chi(p_1 - p_4)}, \tag{2} \]

where \( p_i = (k_i, \omega_i) \) is a vector in the space of wave-vector and frequency, and \( \omega_i = (2l + 1)\pi/(\hbar \beta) \) is the Matsubara frequency of a fermion, \( \beta = 1/(k_B T) \). The total momentum and energy are conserved in the scattering, \( p_1 + p_2 = p_3 + p_4 \). The function \( \chi \) is taken as the polarization function of a non-interacting Fermi gas with the same chemical potential \( \mu \), given by

\[ \chi(p') = \frac{1}{\hbar^2 \beta V} \sum_p G_0(p)G_0(p + p') \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{f_k - f_{k+k'}}{i\hbar \Omega_l + \epsilon_k - \epsilon_{k+k'}}, \tag{3} \]
where \( p' = (k', \Omega_l) \), \( \Omega_l = 2l\pi / (\hbar \beta) \) is the Matsubara frequency of a boson, \( V \) is the volume, \( f_k = 1/[1 + \exp(\beta \xi_k)] \) is the Fermi distribution function, \( \xi_k = \epsilon_k - \mu \), and \( \epsilon_k = \hbar^2 k^2 / 2m \).

The Green’s function of a non-interaction Fermi gas \( G_0(p) \) is given by \( G_0(p) = \hbar / (i\hbar \omega_l - \xi_k) \).

Including the induced interaction, the effective interaction between two atoms with different spin components is given by

\[
U_{\text{tot}}(p_1, p_2; p_3, p_4) = g + U_{\text{ind}}(p_1, p_2; p_3, p_4) = \frac{g}{1 + g\chi(p_1 - p_4)}. \tag{4}
\]

Although the effective interaction is a function of transferred momentum and frequency, at low temperatures only its s-wave part plays an important role on pairing. As in GMB’s work, we approximate this s-wave component \( g' \) by averaging the polarization function

\[
g' = \frac{g}{1 + g\langle \chi \rangle}. \tag{5}
\]

When \( \mu > 0 \), the average of the polarization function \( \langle \chi \rangle \) is obtained by setting the frequencies and total momentum to zero and taking all the initial and final states of atoms from the Fermi surface, i.e. \( k_1 = -k_2, k_3 = -k_4, k_1 = k_2 = k_3 = k_4 = k_F \), which yields

\[
\langle \chi \rangle = \frac{m}{4\pi^2 \hbar^2} \int_{-1}^{1} \cos \theta \int_{0}^{\infty} dk \frac{k}{k'} f_k \ln \left| \frac{k' - 2k}{k' + 2k} \right|, \tag{6}
\]

where \( k' = |k_1 - k_4| = k_F \sqrt{2(1 + \cos \theta)} \), \( k_F \) is the Fermi wavevector, and \( \theta \) is the angle between \( k_1 \) and \( k_3 \). When the chemical potential \( \mu \) turns negative on the BEC side, the
Fermi surface disappears, and the average $\langle \chi \rangle$ is taken at zero frequency and in the limit that all the momentum go to zero, $k_1 = k_2 = k_3 = k_4 \to 0$, same as $k' \to 0$ limit of Eq. (6),

$$\langle \chi \rangle = -\frac{m}{2\pi^2 \hbar^2} \int_0^\infty dk f_k.$$  

(7)

In both cases, the function $\langle \chi \rangle$ is always negative and monotonically decreasing with the increase in the chemical potential $\mu$.

In the BCS limit, the critical temperature $T_c$ is much less than the Fermi temperature $T_F$. Near $T_c$, one obtains

$$\langle \chi \rangle \approx -\frac{\ln(4e)}{3} N(\epsilon_F),$$

where $N(\epsilon_F) = mk_F/(2\pi^2 \hbar^2)$ is the density of states for one spin species at Fermi energy. The effective s-wave interaction $g'$ is approximately given by

$$\frac{1}{g'} \approx \frac{1}{g} - \frac{\ln(4e)}{3} N(\epsilon_F).$$  

(8)

With the effective pairing interaction $g'$, the GMB result of $T_c$ can be obtained,

$$T_c^{(\text{GMB})} \approx \left(\frac{2}{e}\right)^{7/3} \frac{\gamma}{\pi} T_F e^{\pi/2k_F a_s} \approx 0.28 T_F e^{\pi/2k_F a_s},$$

where $\gamma = e^c$, $c$ is the Euler constant. The GMB result $T_c^{(\text{GMB})}$ is smaller by a factor of $(4e)^{1/3} \approx 2.22$ than the mean-field $T_c$.

### III. THE T-MATRIX AND CORRECTION TO DENSITY

To obtain the critical temperature for the whole BEC-BCS crossover, we compute the $T$-matrix, as shown in Fig. 3

$$t(p') = \frac{g'}{1 + g' \chi_p(p')} = \frac{1}{1/g + \langle \chi \rangle + \chi_p(p')},$$

(10)

where the pair susceptibility $\chi_p(p')$ in particle-particle channel is given by

$$\chi_p(p') = \frac{1}{\hbar^2 \beta V} \sum_p G_0(p) G_0(p' - p)$$

$$= \int \frac{d^3 k}{(2\pi)^3} \frac{f_k + f_{k' - k} - 1}{i\hbar \Omega_l - \xi_k - \xi_{k' - k}}.$$  

(11)

Comparing with the conventional $T$-matrix approach, we have replaced the coupling constant $g$ by the effective s-wave interaction $g'$ due to the induced interaction.
FIG. 3: Diagrams of the $T$-matrix. The wiggled line represents the effective s-wave interaction $g'$. 

According to Thouless criterion, the superfluid instability at $T_c$ is due to the divergence of $t(p' = 0)$ which is equivalent to

$$\frac{m}{4\hbar^2 \pi a_s} + \langle \chi \rangle = \int \frac{d^3 k}{(2\pi)^3} \left( \frac{2f_k - 1}{2\xi_k} + \frac{1}{2\xi_k} \right),$$  \hspace{1cm} (12)

where the last term on the right-hand side is the counter term due to vacuum renormalization. Comparing the $T_c$-equation given by Eq. (12) with that in the BCS mean-field theory, the effect of the induced interaction is equivalent to making the scattering length larger. In the BCS limit, the induced interaction leads to a reduction of $T_c$ from the mean-field result, as given in Eq. (9). In the BEC limit, the effect of the induced interaction is negligible.

As Nozières and Schmitt-Rink pointed out [14], pairing fluctuations in the particle-particle channel are important especially on the BEC side. In the NSR theory, the total atom density includes not only the fermion density in the mean-field approximation, but also contributions from fluctuations of molecular fields. In the BEC limit, near $T_c$, the fluctuation contribution is dominant and the critical temperature $T_c$ is given by the BEC temperature of an ideal Bose gas. One way to take into account the NSR effect is to calculate the Hartree self-energy generated by the $T$-matrix and its contribution to density [21]. With our $T$-matrix given by Eq. (10) which includes the induced interaction, the self energy is given by

$$\Sigma(p) = \frac{1}{\hbar^2 \beta V} \sum_{p'} t(p') G_0(p' - p).$$  \hspace{1cm} (13)

To the first order, the Dyson’s equation is given by

$$G(p) = G_0(p) + G_0(p) \Sigma(p) G_0(p),$$

and the particle density is given by

$$n = \frac{2}{\hbar \beta V} \sum_p G(p)e^{-i\omega_0^0} = n_f + \Delta n,$$  \hspace{1cm} (14)

where the mean-field density is given by

$$n_f = 2 \int f_k \frac{d^3 k}{(2\pi)^3}.$$
and the fluctuation contribution $\Delta n$ is given by

$$
\Delta n = \frac{2}{\hbar^3(\beta V)^2} \sum_{p'} \sum_{p} G_0^2(p) G_0(p'-p)t(p').
$$

(15)

If we omit the $\langle \chi \rangle$ term due to the induced interaction in the $T$-matrix given by Eq. (10), the density equation given by Eq. (15) is the same as that in the NSR theory.

IV. THE SUPERFLUID TRANSITION TEMPERATURE

The superfluid transition temperature $T^c$ as a function of the total atom density $n$ can be solved from the two coupled equations (12) and (14). In the BCS limit, since $n_f \gg \Delta n$, the GMB result about $T_c$ given by Eq. (9) can be recovered. In the BEC limit, at $T_c$, the mean-field density is negligible, $\Delta n \gg n_f$, the $T$-matrix is proportional to the propagator of noninteracting molecules, and the density of total atoms is approximately given by density of molecules. Thus the transition temperature in the BEC limit is given by the condensation temperature of an ideal Bose gas, $T_{c(BEC)} = 0.218 T_F$.

Our numeric result of the critical temperature for entire crossover is shown in Fig. 1. As expected, in the BCS limit, it agrees with GMB theory; in the BEC limit, it recovers the condensation temperature of ideal molecules. At unitarity, we obtain $T_c = 0.178 T_F$, which is close to the QMC result $T_c = 0.15(1) T_F$ [31, 32]. In comparison, the results from other theoretical studies are $T_c = 0.222 T_F$ in the original NSR theory, $T_c = 0.160 T_F$ in a full self-consistent NSR treatment [23], $T_c = 0.26 T_F$ in pseudogap crossover theory [21], $T_c = 0.264 T_F$ [29] and $T_c = 0.13 T_F$ [30] in renormalization group studies. Compared with the original NSR result, our critical temperature is about 20% lower, implying that the induced interaction still plays an important role in the unitary region. The chemical potential at $T_c$ in our results is $\mu(T_c) = 0.598 T_F$, higher than QMC results, $\mu(T_c) = 0.493(14)$ [31] and $\mu(T_c) = 0.43(1)$ [32]. Our results can probably be improved by self-consistently taking into account the self energy in the computations of the induced interaction and $T$-matrix. This issue will be addressed in our further studies.

As reported in previous works [13, 14, 18, 20, 23], we also find that the critical temperature reaches a maximum on the BEC side, as shown in Fig. 1. Compared with the original NSR result, the position of this peak is further away from the resonance due to the induced interaction. Our results show that the peak is located at $1/k_F a_s = 0.437$, and $T^{peak} =$
0.231T_F, close to the QMC estimation of the peak position \(1/k_F a_s \geq 0.474(8)\) and \(T_{\text{peak}} \geq 0.252(15)T_F\) [31].

The effect of the fluctuation in the particle-hole channel on the superfluid transition temperature was also studied in the renormalization group approach for a two-channel model mostly in the wide resonance case [29] and for a single-channel model [30]. In the BCS limit, the superfluid transition temperature was found in agreement with the GMB result in Ref. [29], and smaller than the GMB result in Ref. [30] due to the simplification in the momentum dependence of the interaction vertex. At unitarity, the superfluid transition temperature was found to be \(T_c = 0.264T_F\) [29] and \(T_c = 0.13T_F\) [30], while we obtain \(T_c = 0.178T_F\) and the QMC result is \(T_c = 0.15(1)T_F\) [31, 32]. On the BEC side when \(k_F a_s = 0.5\), the superfluid transition temperature was found to be \(T_c \approx 0.25T_F\) [29], and our result shows \(T_c \approx 0.22T_F\). These quantitative differences may be resolved in future studies with better theoretical treatments.

V. CONCLUSION

In conclusion, the effect of the induced interaction due to the many-body medium is studied in a Fermi gas with the BEC-BCS crossover. The GMB theory is extended from the BCS limit to the entire crossover. With the induced interaction considered, the superfluid transition temperature \(T_c\) is computed for the entire crossover in the NSR framework. The induced interaction reduces the critical temperature \(T_c\) considerably on the BCS side and in the unitary region. Our results of \(T_c = 0.178T_F\) at unitarity and the \(T_c\)-peak location are in reasonable agreements with results from quantum Monte Carlo simulations. Our results show that the effect of the induced interaction is important both in the unitary region and on the BCS side. We would like to thank T.-L. Ho for helpful discussions. This work is supported by NSFC under Grant No. 10674007, and by Chinese MOST under grant number 2006CB921402.

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