Fully adaptive proximal extrapolated gradient method for monotone variational inequalities

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Abstract The paper presents a fully adaptive proximal extrapolated gradient method for monotone variational inequalities. The proposed method uses fully non-monotonic and adaptive step sizes, that are computed using two previous iterates as an approximation of the locally Lipschitz constant without running a linesearch. Thus, it has almost the same low computational cost as classic proximal gradient algorithm, each iteration requires only one evaluation of a monotone mapping and a proximal operator. The method exhibits an ergodic $O(1/N)$ convergence rate and $R$-linear rate under a strong monotonicity assumption of the mapping. Applying the method to unconstrained optimization and fixed point problems, it is sufficient for convergence of iterates that the step sizes are estimated only by the local curvature of mapping, without any constraints on step size’s increasing rate. The numerical experiments illustrate the improvements in efficiency from the low computational cost and fully non-monotonic and adaptive step sizes.

Keywords Variational inequality · proximal gradient method · convex optimization · fully adaptive step size

Mathematics Subject Classification (2000) 47J20 · 65C10 · 65C15 · 90C33

1 Introduction

Let $\mathcal{H}$ be a finite dimensional vector space equipped with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. We consider the variational inequality (VI) problem:

\[
\text{find } x^* \in \mathcal{H} \text{ s.t. } \langle F(x^*), y - x^* \rangle + g(y) - g(x^*) \geq 0, \forall y \in \mathcal{H},
\]

where $F : \mathcal{H} \to \mathcal{H}$ is an operator and $g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function. We use $\text{dom } g$ to represent the domain of $g$, defined by $\text{dom } g := \{x \in \mathcal{H} : g(x) < \infty\}$. For a continuously differentiable and convex function $f : \mathcal{H} \to \mathbb{R}$ with its gradient denoted by $\nabla f = F$, then VI (1) is equivalent to optimization problem

\[
\min_{x \in \mathcal{H}} f(x) + g(x).
\]
Let $C$ be a closed and convex subset of $\mathcal{H}$. Let $l_C$ be the indicator function of the set $C$, that is, $l_C(x) = 0$ if $x \in C$ and $\infty$ otherwise. When $g(x) = l_C(x)$, VI (1) reduces to

$$\text{find } x^* \in C \text{ s.t. } (F(x^*), y - x^*) \geq 0, \forall y \in \mathcal{H}.$$  

(3)

Another important instance is the convex-concave saddle point problem:

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} \ g_1(x) + K(x, y) - g_2(y),$$

(4)

where $K : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a smooth convex-concave function, $g_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g_2 : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions. By writing down the first-order optimality condition, it is easy to see that problem (4) can be reformulated as (1) with $F$ and $g$ defined as

$$z = (x, y), \quad F(z) = \left[ \nabla_x K(x, y) \quad -\nabla_y K(x, y) \right], \quad g(z) = g_1(x) + g_2(y).$$

(5)

Problem (1) and its special cases (3) and (5) have wide applications in disciplines including mechanics, signal and image processing, and economics [2, 13, 21, 33, 41], to cite a few. Throughout the paper, the solution set $\mathcal{S}$ of problem (1) is assumed to be nonempty, and the following assumptions hold:

- **(A1)** $F$ is monotone, i.e.,

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

- **(A2)** $F$ is locally Lipschitz continuous;

- **(A3)** $g$ is a proper lower semicontinuous convex function.

To solve VI (1) and its special reformulations, many efficient methods have been proposed, for instance, extragradient method [18, 26], alternating direction method of multipliers (ADMM) [2, 5, 15], proximal (projected) gradient method [8, 16, 22, 25, 40, 43] and its accelerated and generalized versions [24, 34]. In this paper, we would concentrate on the most simple case of these approaches: proximal gradient (PG) method. Under the assumption that $F = \nabla f$ is $L$-Lipschitz continuous, that is, there exists $L > 0$ such that

$$\| F(x) - F(y) \| \leq L \| x - y \|, \quad \forall x, y \in \mathcal{H},$$

the iterative scheme of the classical PG method for problem (1) reads

$$x_{n+1} = \text{Prox}_{\lambda g}(x_n - \lambda F(x_n)), \quad (6)$$

where $\lambda$ is some positive number and can be viewed as a step size, and the proximal operator $\text{Prox}_{\lambda g} : \mathcal{H} \rightarrow \mathcal{H}$ is well defined in Section 2.

To establish convergence of the iteration (6), it often requires the restrictive assumptions that $F$ is $1/L$-cocoercive 1 and $\lambda \in [0, \frac{1}{2L}]$. Cocoercivity of an operator is a strictly strong property than Lipschitz continuity. To overcome this drawback, Tseng [43] modified the iteration (6) and proposed the following forward-backward-forward (FBF) method involving one proximal operator and two values of $F$ per iteration:

$$y_n = \text{Prox}_{\lambda g}(x_n - \lambda F(x_n)), \quad x_{n+1} = y_n + \lambda (F(x_n) - F(y_n)),$$

where $\lambda \in [0, \frac{1}{2L}]$. Since then, Tseng’s method has attracted a lot of interests due to its simplicity and generality, see [3, 4, 32] for more details.

To accelerate proximal gradient method in the spirit of Nesterov’s extrapolation techniques [34, 35], the inertial extrapolation has been conducted, whose basic idea is to make full use of

1 An operator $F$ is called $l$-cocoercive, for $l > 0$, if $\langle Fx - Fy, x - y \rangle \geq l \| Fx - Fy \|^2, \forall x, y \in \mathcal{H}$. In particular, if $f$ is a convex function and its gradient $\nabla f$ is $L$-Lipschitz, then $\nabla f$ is $1/L$-cocoercive.
that it is sufficient for convergence of iterates that the step sizes are estimated only by the local fixed point problems, propose a fully adaptive extrapolated gradient method (FAEG) and find assumption of the mapping holds. We apply this method to the unconstrained optimization and prove that FAPEG converges with ergodic rate $O(1/N)$.

Recently, some adaptive algorithms were present in [6,45] when $F$ is locally Lipschitz continuous, which do not require a linesearch to be run, and their step sizes are computed using current information about the iterates:

$$
\lambda_{n+1} = \min \left\{ \phi_n \lambda_n, \frac{\alpha\|y_{n+1} - y_n\|}{\|F y_{n+1} - F y_n\|} \right\}, \quad \text{if } F y_{n+1} - F y_n \neq 0,
$$

where $\lambda_n = 1$ or $\phi_n \geq 1$ but $\phi_n \to 1(n \to +\infty)$. Namely, the step size is monotonically decreasing or can break away from overdependence on the initial point, but it would have to be decreasing in the end for getting convergence, as the convergence is resulted from $\frac{\Delta n}{\lambda_{n+1}} \to 1(n \to +\infty)$.

In [29], Malitsky proposed the first fully adaptive forward-backward splitting method for variational inequalities, called golden ratio algorithm. This method used a convex combination of all previous iterates rather than the extrapolation method with $y_n = x_n + \delta(x_n - x_{n-1})$ for some $\delta > 0$. Numerically, the golden ratio algorithm can robustly work even for some highly non-monotone/nonconvex problems. Moreover, the step sizes are allowed to increase from iteration to iteration. These features inspired us to get an efficient PEG with adaptive step sizes. But the challenge we have to face is to design a completely new strategy for generating step sizes when the extrapolation is used, due to completely different iterative scheme in PEG from the golden ratio algorithm. Determining extrapolation parameter $\delta_0$ by two previous step sizes (instead of a fixed $\delta$) and designing a proper and allowable rate of step size to increase, we have achieved our goal and obtained adaptive PEG.

**Contribution.** We propose an efficient PEG with fully adaptive step sizes (FAPEG for short), rather than with linesearch to be run. By fully adaptive we mean that the step sizes are generated fully according to the local Lipschitz information of $F$, and allowed to increase at each iteration, without a constraint $\frac{\Delta n}{\lambda_{n+1}} \to 1(n \to +\infty)$. Each iteration of the method needs only one evaluation of the proximal operator and one value of $F$. What’s more, the allowable rate of step size to increase is adaptively depended to the ratio of two previous step sizes, unlike the golden ratio algorithm [29], where one needs to make a choice to balance the size of step and its allowable increasing rate. To our knowledge, it is the first fully adaptive PEG method with these properties. Moreover, we prove that FAPEG converges with ergodic rate $\mathcal{O}(1/N)$ and $R$-linear rate if a strong monotonicity assumption of the mapping holds. We apply this method to the unconstrained optimization and fixed point problems, propose a fully adaptive extrapolated gradient method (FAEG) and find that it is sufficient for convergence of iterates that the step sizes are estimated only by the local...
curvature of $F$, without any constraints on step size's increasing rate. Because of the more relaxed or unconstrained step size's increasing rate, numerical results for several problems illustrate the performance of FAPEG and FAEG is superior to the existing algorithms with non-monotonic step sizes.

The paper is organized as follows. In Section 2, we provide some useful facts and notations. In Section 3, we introduce our algorithm in detail and explore its convergence. Section 4 devotes to convergence properties of fully adaptive PEG for solving some special problems. Firstly, we consider unconstrained optimization problem and fixed point problem, and introduce an adaptive extrapolated gradient method. No need for step sizes's increasing rate, no information about the function except for the gradients. Then, we focus on the problem with linear $F$ and $g = 0$ from the perspective of fixed point theory. Numerical experiments on the problems tested in the literatures are provided and analyzed in Section 5. We finally conclude our paper in Section 6.

2 Preliminaries

In this section, we introduce some notations and facts on the well-known properties of the proximal operator, variational inequality and Young’s inequality, which are used for the sequel convergence analyses.

The proximal operator $\text{Prox}_{\lambda g} : \mathcal{H} \to \mathcal{H}$ with $\text{Prox}_{\lambda g}(x) = (I + \lambda \partial g)^{-1}(x)$, $\lambda > 0$, $x \in \mathcal{H}$, is defined by

$$\text{Prox}_{\lambda g}(x) := \arg\min_{y \in \mathcal{H}} \left\{ g(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}, \quad \forall x \in \mathcal{H}, \lambda > 0.$$  

Setting

$$\Phi(x, y) := \langle F(x), y - x \rangle + g(y) - g(x),$$  

it is clear that problem (1) is equivalent to finding $x^* \in \mathcal{H}$ such that $\Phi(x^*, y) \geq 0$ for all $y \in \mathcal{H}$.

**Fact 1** [1] Let $g : \mathcal{H} \to (-\infty, +\infty]$ be a convex function, $\lambda > 0$ and $x \in \mathcal{H}$. Then $p = \text{Prox}_{\lambda g}(x)$ if and only if

$$\langle p - x, y - p \rangle \geq \lambda [g(p) - g(y)], \quad \forall y \in \mathcal{H}.$$  

**Fact 2** Let $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}$ be two nonnegative real sequences and $\exists N > 0$ such that

$$a_{n+1} \leq a_n - b_n, \quad \forall n > N.$$  

Then $\{a_n\}_{n \in \mathbb{N}}$ is convergent and $\lim_{n \to \infty} b_n = 0$.

**Fact 3** (Young’s inequality) For all $a, b \geq 0$ and $\varepsilon > 0$, we have

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$  

**Fact 4** Let $\delta \in [0, +\infty]$ and $u, v \in \mathcal{H}$. Then

$$\|\delta u - \delta v\|^2 = (\delta + 1)\|u\|^2 - \delta\|v\|^2 + \delta(\delta + 1)\|u - v\|^2.$$  

The following identity (cosine rule) appears in many times and we will use it for simplicity of convergence analyses. For all $x, y, z \in \mathcal{H},$

$$\langle x - y, x - z \rangle = \frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x - z\|^2 - \frac{1}{2}\|y - z\|^2.$$  

(9)
3 Fully Adaptive Proximal Extrapolated Gradient Method

Now, we state our fully adaptive proximal extrapolated gradient (FAPEG) method.

Algorithm 1 (Fully adaptive PEG method for solving (1))

Step 0. Choose $x_0 = y_0 \in H$, $\lambda_0 > 0$ and $\hat{\lambda} > 0$. Take $\delta_0 = 1$, $\alpha = \zeta(\sqrt{2} - 1)$ with $\zeta \in ]0, 1]$. Set $n = 0$.

Step 1. Compute

$$x_{n+1} = \text{Prox}_{\lambda_n g}(x_n - \lambda_n F(y_n)).$$

If $x_{n+1} = x_n = y_n$, then stop: $x_{n+1}$ is a solution.

Step 2. Compute

$$y_{n+1} = x_{n+1} + \delta_n(x_{n+1} - x_n),$$

$$\rho_n = \sqrt{1 + \delta_n},$$

and

$$\lambda_{n+1} = \min \left\{ \rho_n \lambda_n, \frac{\alpha \|y_{n+1} - y_n\|}{\|Fy_{n+1} - Fy_n\|}, \hat{\lambda} \right\},$$

(10) if $Fy_{n+1} - Fy_n \neq 0$, otherwise.

Step 3. Update $\delta_{n+1} = \frac{\lambda_{n+1}}{\lambda_n}$. Set $n \leftarrow n + 1$ and return to step 1.

Remark 1 Since one have to compute $Fy_{n+1}$ for the next iteration, the extra cost in FAPEG is to compute $\|y_{n+1} - y_n\|$ and $\|Fy_{n+1} - Fy_n\|$, but this is cheap and comparable with the cost of a vector-vector. Hence, the cost per iteration of FAPEG is almost the same as the classical PG method (6) for all cases, mainly to compute one proximal operator and one monotone mapping.

Remark 2 The constant $\hat{\lambda}$ in FAPEG is given only to ensure the upper boundedness of $\{\lambda_n\}_{n \in \mathbb{N}}$. Hence, it makes sense to choose $\hat{\lambda}$ quite large.

Lemma 1 If the sequence $\{y_n\}_{n \in \mathbb{N}}$ generated by FAPEG is bounded then both $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\delta_n\}_{n \in \mathbb{N}}$ are bounded and separated from 0.

Proof. By Remark 2, $\{\lambda_n\}_{n \in \mathbb{N}}$ is upper bounded. As $\{y_n\}_{n \in \mathbb{N}}$ is bounded, recall that $F$ is a locally Lipschitz continuous mapping from assumption (A2), there exists some large enough $\hat{L} > 0$ such that $\|Fy_{n+1} - Fy_n\| \leq \hat{L}\|y_{n+1} - y_n\|$ for all $n \geq 1$. Then we deduce

$$\frac{\alpha \|y_{n+1} - y_n\|}{\|Fy_{n+1} - Fy_n\|} \geq \frac{\alpha \|y_{n+1} - y_n\|}{\hat{L}\|y_{n+1} - y_n\|} = \frac{\alpha}{\hat{L}},$$

for $Fy_{n+1} - Fy_n \neq 0$, which implies $\{\lambda_n\}_{n \in \mathbb{N}}$ is separated from 0 and has a lower bound $\min\{\frac{\alpha}{\hat{L}}, \lambda_0\}$, as $\lambda_{n+1} = \lambda_n$ when $Fy_{n+1} - Fy_n = 0$. The claim that $\{\delta_n\}_{n \in \mathbb{N}}$ is bounded and separated from 0 follows $\delta_n = \frac{\lambda_n}{\lambda_{n-1}}$ immediately. 

Notice that from Lemma 1, we have $\delta_n > 0$, then $\rho_n > 1$ and $\rho_n \leq \frac{1 + \sqrt{2}}{2}$ as $\delta_0 = 1$, namely, the step $\lambda_n$ is allowed to increase from iteration to iteration, so FAPEG is not sensitive to the initial point, unlike the monotonically increasing cases presented in [44, 45]. Moreover, the allowable rate $\rho_n = \sqrt{1 + \delta_n}$ of step sizes to increase in FAPEG is depended dynamically to the ratio $\delta_n$ of two previous step sizes, and it is more relaxed and beneficial for the numerical experiment than that used in the golden ratio algorithm [29], which is depended to a given $\phi$ (If $\phi = 3/2$ as in [29], $\rho = \frac{1}{\phi} + \frac{1}{\phi^2} = 10/9$) and strict to increase step sizes efficiently. As it will be illustrated in Section 5, this can be important for numerical efficiency.
3.1 Convergence Analysis

This section devotes to convergence properties of FAPEG, by using proximal inequality and Young’s inequality. We next give basic lemmas about the iterations generated by FAPEG, which play a crucial role in proving the main convergence results.

Lemma 2 Let \(\{x_n\}_{n \in \mathbb{N}}\) and \(\{y_n\}_{n \in \mathbb{N}}\) be two sequences generated by FAPEG. For any \(x \in \mathcal{H}\), we have

\[
\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \|[y_n - x_n] + [x_{n+1} - y_n]\|^2 + 2\alpha\|y_n - y_{n-1}\|\|x_{n+1} - y_n\| - 2\lambda_n\|(1 + \delta_n)g(x_n) - \delta_n g(x_{n-1}) - g(x)\| - 2\lambda_n\langle F(y_n), y_n - x \rangle.
\]

(11)

Proof. Followed by \(x_{n+1} = \text{Prox}_{\lambda_n g}(x_n - \lambda_n F(y_n))\) and Fact 1, we have

\[
\langle x_{n+1} + \lambda_n F(y_n), x - x_{n+1} \rangle \geq \lambda_n\langle g(x_{n+1}) - g(x)\rangle, \quad \forall x \in \mathcal{H},
\]

(12)

which shows

\[
\langle x_n - x_{n-1} + \lambda_n - 1 F(y_{n-1}), x - x_n \rangle \geq \lambda_n\langle g(x_n) - g(x)\rangle, \quad \forall x \in \mathcal{H}.
\]

Substituting \(x = x_{n+1}\) and \(x = x_{n-1}\) into the above inequality consecutively, we obtain

\[
\langle x_n - x_{n-1} + \lambda_n - 1 F(y_{n-1}), x_{n+1} - x_n \rangle \geq \lambda_n - 1\langle g(x_n) - g(x_{n+1})\rangle.
\]

(13)

\[
\langle x_n - x_{n-1} + \lambda_n - 1 F(y_{n-1}), x_{n-1} - x_n \rangle \geq \lambda_n - 1\langle g(x_n) - g(x_{n-1})\rangle.
\]

(14)

Multiplying (14) by \(\delta_n\) and then adding it to (13), which by \(y_n = x_n + \delta_n(x_n - x_{n-1})\) yields

\[
\langle x_n - x_{n-1} + \lambda_n - 1 F(y_{n-1}), x_{n+1} - y_n \rangle \geq \lambda_n - 1\langle (1 + \delta_n)g(x_n) - g(x_{n+1}) - \delta_n g(x_{n-1})\rangle.
\]

(15)

Multiplying (15) by \(\frac{\lambda_n}{\lambda_n - 1}\) and using \(\frac{\lambda_n}{\lambda_n - 1} = \delta_n\) and \(y_n = x_n + \delta_n(x_n - x_{n-1})\) again, we get

\[
\langle y_n - x_n + \lambda_n F(y_{n-1}), x_{n+1} - y_n \rangle \geq \lambda_n\langle (1 + \delta_n)g(x_n) - g(x_{n+1}) - \delta_n g(x_{n-1})\rangle.
\]

(16)

Then, adding (12) to (16) gives us

\[
\langle x_{n+1} - x_n, x - x_{n+1} \rangle + \langle y_n - x_n, x_{n+1} - y_n \rangle + \lambda_n\langle F(y_n) - F(y_{n-1}), y_n - x_{n+1} \rangle \geq \lambda_n\langle (1 + \delta_n)g(x_n) - g(x)\rangle - \delta_n g(x_{n-1})\rangle + \lambda_n\langle F(y_n), y_n - x \rangle.
\]

Finally, using (9), the updating of \(\lambda_n\) and Cauchy-Schwarz inequality, we obtain

\[
\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 - \|[y_n - x_n] + [x_{n+1} - y_n]\|^2 + 2\alpha\|y_n - y_{n-1}\|\|x_{n+1} - y_n\| - 2\lambda_n\|(1 + \delta_n)g(x_n) - \delta_n g(x_{n-1}) - g(x)\| - 2\lambda_n\langle F(y_n), y_n - x \rangle.
\]

The proof is completed. \qed
Lemma 3 Let \( \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \) be two sequences generated by FAPEG and \( \bar{x} \in S \) (the solution set of problem (1)). Then, we have

\[
\|x_{n+1} - \bar{x}\|^2 + 2\lambda_n (1 + \delta_n) \Phi(\bar{x}, x_n) \leq \|x_n - \bar{x}\|^2 + 2\lambda_{n-1} (1 + \delta_{n-1}) \Phi(\bar{x}, x_{n-1}) + (\sqrt{2} + 1)\alpha - 1 \|x_n - y_n\|^2 + (\sqrt{2} - 1) \|x_{n+1} - y_n\|^2 + \alpha \|x_n - y_{n-1}\|^2,
\]

where the function \( \Phi(\cdot, \cdot) \) is defined as in (8).

Proof. Using Fact 3 with \( \epsilon = \sqrt{2} \), we have

\[
2\|y_n - y_{n-1}\|y_n - x_{n+1}\| \leq \frac{1}{\sqrt{2}} \|y_n - y_{n-1}\|^2 + \sqrt{2}\|x_{n+1} - y_n\|^2.
\]

Meanwhile by Fact 3 with \( \epsilon = \sqrt{2} - 1 \), we deduce

\[
\|y_n - y_{n-1}\|^2 = \|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2 + 2(y_n - x_n, x_n - y_{n-1}) \leq \|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2 + 2\|y_n - x_n\||x_n - y_{n-1}| \leq (2 + \sqrt{2})\|y_n - x_n\|^2 + \sqrt{2}\|x_n - y_{n-1}\|^2.
\]

Combining the above inequalities yields

\[
2\|y_n - y_{n-1}\|y_n - x_{n+1}\| \leq \left((\sqrt{2} + 1)\|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2 + \sqrt{2}\|x_{n+1} - y_n\|^2\right).
\]

In addition, the monotonicity of \( F \) implies

\[
\lambda_n(F(y_n), y_n - \bar{x}) \geq \lambda_n(F(\bar{x}), y_n - \bar{x}) = \lambda_n[(1 + \delta_n)(F(\bar{x}), x_n - \bar{x}) - \delta_n(F(\bar{x}), x_{n-1} - \bar{x})].
\]

Substituting (17) and (18) into (11) with \( x = \bar{x} \), we deduce by the aids of \( \Phi(\cdot, \cdot) \) in (8) that

\[
\|x_{n+1} - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - \|x_{n+1} - x_n\|^2 + (\sqrt{2} + 1)\alpha \|y_n - x_n\|^2 + \alpha \|x_n - y_{n-1}\|^2 + \sqrt{2}\alpha \|x_{n+1} - y_n\|^2 + (\|x_{n+1} - x_n\|^2 - \|x_n - y_{n-1}\|^2 - \|x_{n+1} - y_n\|^2) - 2\lambda_n[(1 + \delta_n)\Phi(\bar{x}, x_n) - \delta_n\Phi(\bar{x}, x_{n-1})].
\]

Note that \( \Phi(\bar{x}, x_{n-1}) \geq 0 \) for any \( \bar{x} \in S \), by \( \delta_n = \frac{\lambda_n}{\lambda_n - 1} \) and \( \lambda_n \leq \rho_n - 1 \lambda_n - 1 \), we have

\[
\lambda_n\delta_n = \frac{\lambda_n^2}{\lambda_n - 1} \leq \rho_n^2 \lambda_n - 1 = \lambda_n - 1 (1 + \delta_{n-1}),
\]

then

\[
\|x_{n+1} - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 + (\sqrt{2} + 1)\alpha - 1 \|x_n - y_n\|^2 + (\sqrt{2} - 1) \|x_{n+1} - y_n\|^2 + \alpha \|x_n - y_{n-1}\|^2 + 2\lambda_n(1 + \delta_n)\Phi(\bar{x}, x_n) + 2\lambda_n(1 + \delta_{n-1})\Phi(\bar{x}, x_{n-1}).
\]

This completes the proof. \[ \square \]

Note that in the proof of Lemma 3, for a given \( \lambda_n - 1 \), the next step size \( \lambda_n \) needs to satisfy \( \lambda_n\delta_n \leq \lambda_n - 1 (1 + \delta_{n-1}) \), which implies \( \lambda_n \leq \lambda_n - 1 (1 + \delta_{n-1}) \). Namely, the allowable rate \( \rho_n \) of step sizes to increase is chosen not arbitrary, but as the largest value that satisfies \( \lambda_n\delta_n \leq \lambda_n - 1 (1 + \delta_{n-1}) \) in order to establish convergence.

By Lemma 3 and some transpositions, we have the following results directly.
Therefore, \( \lim \) in (21) and (22) for all \( \alpha = \zeta(\sqrt{2} - 1) \), we have

\[
a_{n+1} \leq a_n - b_n, \quad n \geq 1,
\]

with

\[
a_n = \|x_n - \bar{x}\|^2 + 2\lambda_{n-1}(1 + \delta_{n-1})\Phi(\bar{x}, x_{n-1}) + \zeta(\sqrt{2} - 1)\|x_n - y_{n-1}\|^2,
\]

\[
b_n = (1 - \zeta)\|x_{n+1} - y_n\|^2 + \|x_n - y_n\|^2.
\]

Below we state and prove our main convergence result of FAPEG.

**Theorem 1** Let \( \{x_n\}_{n \in \mathbb{N}} \) be the sequence generated by FAPEG. Then, \( \{x_n\}_{n \in \mathbb{N}} \) converges to a solution of problem (1).

Proof. By \( \zeta \in [0, 1] \) and \( \Phi(\bar{x}, \cdot) \geq 0 \) for any \( \bar{x} \in S \), we deduce that \( a_n \geq 0 \) and \( b_n \geq 0 \) defined in (21) and (22) for all \( n \geq 1 \). Using Lemma 4 and Fact 2 gives that \( \{a_n\}_{n \in \mathbb{N}} \) is convergent and \( \lim_{n \to \infty} b_n = 0 \). Then, we have

\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|x_n - y_n\| = 0.
\]

By \( \|x_{n+1} - x_n\| = \frac{1}{\lambda_n} \|x_{n+1} - y_{n+1}\| \) and \( \delta_n > 0 \), we also have that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

In what follows, we prove the sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to a solution of problem (1). Due to \( \|x_n - \bar{x}\|^2 \leq a_n, \{x_n\}_{n \in \mathbb{N}} \) is bounded. For any cluster \( x^* \in \mathcal{H} \) of \( \{x_n\}_{n \in \mathbb{N}} \), there exists a subsequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) that converges to \( x^* \). It is obvious that \( \{y_{n_k}\}_{k \in \mathbb{N}} \) also converges to \( x^* \). Next we verify that \( x^* \in \mathcal{S} \). Applying Fact 1, we deduce

\[
\left\langle \frac{x_{n_k+1} - x_{n_k}}{\lambda_{n_k}} + F(y_{n_k}), x - x_{n_k+1} \right\rangle \geq g(x_{n_k+1}) - g(x), \quad \forall x \in \mathcal{H}.
\]

Letting \( k \to \infty \) in (23) and using the facts \( \lim_{k \to \infty} \|x_{n_k+1} - x_{n_k}\| = 0 \), \( g(x) \) is lower semicontinuous and \( \lambda_n > 0 \), we obtain

\[
(F(x^*), x - x^*) \geq \lim_{k \to \infty} \inf g(x_{n_k+1}) - g(x) \geq g(x^*) - g(x), \quad \forall x \in \mathcal{H},
\]

which confirms \( x^* \in \mathcal{S} \).

Finally, we prove that \( x_n \to x^* \). We take \( \bar{x} = x^* \) in the definition of \( a_n \) and label as \( a^*_n \). Notice that \( \{\lambda_n\}_{n \in \mathbb{N}} \) is bounded and \( \Phi(x^*, \cdot) \) is continuous from (A3), we observe

\[
\lim_{n \to \infty} a^*_n = \lim_{k \to \infty} a^*_{n_k+1}
\]

\[
= \lim_{k \to \infty} \left( \|x_{n_k+1} - x^*\|^2 + 2\lambda_{n_k}(1 + \delta_{n_k})\Phi(x^*, x_{n_k}) + \zeta(\sqrt{2} - 1)\|x_{n_k+1} - y_{n_k}\|^2 \right)
\]

\[
= 0.
\]

Therefore, \( \lim_{k \to \infty} \|x_n - x^*\| = 0 \), which completes the proof. \( \square \)
3.2 Ergodic Convergence Rate

For monotone VI, it is known from [37] that an $O(1/N)$ rate of convergence exhibited by many algorithm [32,36,37] is optimal. In this subsection, we investigate the ergodic convergence rate of the sequence $\{y_n\}_{n\in\mathbb{N}}$ and prove the same results for our algorithm.

From [13] and [28, Lemma 2.12], $x^* \in S$ if and only if $x^* \in \text{dom } g$ and
\[
\max_{x \in \text{dom } g} \Phi(x, x^*) := \langle F(x), x^* - x \rangle + g(x^*) - g(x) = 0.
\]

The following theorem shows that the above criteria can be used to find $x^*$ under a desired accuracy.

**Theorem 2** Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be generated by FAPEG. For any $N \geq 2$, we define
\[
\hat{\lambda}_N = \frac{1}{N} \sum_{l=1}^{N} \lambda_l + \delta_1 \lambda_1 \quad \text{and} \quad \hat{x}_N = \frac{1}{\hat{\lambda}_N} \left( \sum_{l=2}^{N} \lambda_l y_l + (1 + \delta_1) \lambda_1 x_1 \right),
\]
then $\hat{x}_N \in \text{dom } g$ and
\[
\Phi(x, \hat{x}_N) \leq \frac{||x_1 - \bar{x}||^2 + \delta_1 \lambda_1 \Phi(\bar{x}, x_0) + \alpha ||x_1 - y_0||^2}{2\lambda_N}, \quad \forall x \in \mathcal{H}.
\]

**Proof.** First of all, we have by (19) that
\[
2\lambda_n(1 + \delta_n)\Phi(\bar{x}, x_n) - 2\lambda_n \delta_n \Phi(\bar{x}, x_{n-1}) \\
\leq ||x_n - \bar{x}||^2 - ||x_{n+1} - \bar{x}||^2 + \left(\sqrt{2} + 1\right) \alpha - 1 \right) ||x_n - y_n||^2 \\
+ \alpha ||x_n - y_{n-1}||^2 - \left(1 - \sqrt{2}\alpha \right) ||x_{n+1} - y_n||^2,
\]
for any $n \geq 1$. Summing from 1 to $N$, by $\alpha = \zeta (\sqrt{2} - 1)$ and $\zeta \in ]0,1[$ we obtain
\[
2 \left( \lambda_N(1 + \delta_N)\Phi(\bar{x}, x_N) + \sum_{l=1}^{N-1} [\lambda_l (1 + \delta_l) - \lambda_{l+1} \delta_{l+1}] \Phi(\bar{x}, x_l) \right) \\
\leq ||x_1 - \bar{x}||^2 - ||x_{N+1} - \bar{x}||^2 - \sum_{l=1}^{N} [1 - (\sqrt{2} + 1)\alpha] ||x_l - y_l||^2 + \alpha ||x_1 - y_0||^2 \\
- \sum_{l=1}^{N-1} \left[1 - (\sqrt{2} + 1)\alpha \right] ||x_{l+1} - y_l||^2 - \left[1 - \sqrt{2}\alpha \right] ||x_{N+1} - y_N||^2 + \delta_1 \lambda_1 \Phi(\bar{x}, x_0) \\
\leq ||x_1 - \bar{x}||^2 + \alpha ||x_1 - y_0||^2 + \delta_1 \lambda_1 \Phi(\bar{x}, x_0),
\]
Note that the function $\Phi(\bar{x}, \cdot)$ is convex and $\lambda_l (1 + \delta_l) - \lambda_{l+1} \delta_{l+1} \geq 0$ for any $1 \leq l \leq N - 1$. Applying the Jensen’s inequality and taking
\[
\lambda_N(1 + \delta_N) + \sum_{l=1}^{N-1} [\lambda_l (1 + \delta_l) - \lambda_{l+1} \delta_{l+1}] = \sum_{l=1}^{N} \lambda_l + \delta_1 \lambda_1
\]
to account, we have
\[
2 \left( \sum_{l=1}^{N} \lambda_l + \delta_1 \lambda_1 \right) \Phi(\bar{x}, \hat{x}_N) \leq ||x_1 - \bar{x}||^2 + \alpha ||x_1 - y_0||^2 + \delta_1 \lambda_1 \Phi(\bar{x}, x_0),
\]
where
\[
\hat{\lambda}_N \hat{x}_N = \lambda_N(1 + \delta_N)x_N + \sum_{l=1}^{N-1} [\lambda_l (1 + \delta_l) - \lambda_{l+1} \delta_{l+1}]x_l = \sum_{l=2}^{N} \lambda_l y_l + (1 + \delta_1) \lambda_1 x_1.
\]
Evidently, $\dot{x}_N \in \text{dom} \, g$ which ends the proof. \hfill $\square$

Notice that $\{\lambda_n\}_{n \in \mathbb{N}}$ has a lower bound $\tau := \min \{\frac{\alpha}{2}, \lambda_0\}$ from Lemma 1. Then we get $\dot{x}_N \to \infty$ as $N \to \infty$. This implies FAPEG has the same $O(1/N)$ convergence rate for ergodic sequence $\{\dot{x}_N\}_{N \in \mathbb{N}}$.

3.3 Linear Convergence

In this section, we establish $\mathcal{R}$-linear convergence of the sequence generated by the FAPEG method with $\rho_n < \sqrt{1 + \delta_n}$, under a strong monotonicity assumption of the mapping $F$, namely,

$$\langle F(x) - F(y), x - y \rangle \geq m \parallel x - y \parallel^2, \quad \forall x, y \in \mathcal{H},$$

for some $m > 0$.

**Theorem 3** Assume that $\{x_n\}$ is generated by FAPEG with $\rho_n < \sqrt{1 + \delta_n}$ under condition (24), then it converges to the unique solution of (1), at least linearly.

Proof. It is clear that, in this setting, problem (1) has a unique solution, which we denote by $\bar{x}$. By Fact 4, we have

$$\parallel y_n - \bar{x} \parallel^2 = (\delta_n + 1) \parallel x_n - \bar{x} \parallel^2 - \delta_n \parallel x_{n-1} - \bar{x} \parallel^2 + (\delta_n + 1) \delta_n \parallel x_n - x_{n-1} \parallel^2$$

$$\geq (\delta_n + 1) \parallel x_n - \bar{x} \parallel^2 - \delta_n \parallel x_{n-1} - \bar{x} \parallel^2.$$

Using strong monotonicity of $F$ (in place of monotonicity) in (18) and propagating the resulting inequality through the proof of Lemma 3 gives the inequality

$$\parallel x_{n+1} - \bar{x} \parallel^2 \leq \parallel x_n - \bar{x} \parallel^2 + \left[ (\sqrt{2} + 1) \alpha - 1 \right] \parallel x_n - y_n \parallel^2$$

$$+ \left( \sqrt{2} \alpha - 1 \right) \parallel x_{n+1} - y_n \parallel^2 + \alpha \parallel x_n - y_{n-1} \parallel^2$$

$$- 2 \lambda_n [(1 + \delta_n) \Phi(\bar{x}, x_n) - \delta_n \Phi(\bar{x}, x_{n-1})]$$

$$- 2 \lambda_n \delta_n (\delta_n + 1) m \parallel x_n - \bar{x} \parallel^2 + 2 \lambda_n \delta_n m \parallel x_{n-1} - \bar{x} \parallel^2$$

$$\leq \parallel x_n - \bar{x} \parallel^2 + \left( \sqrt{2} \alpha - 1 \right) \parallel x_{n+1} - y_n \parallel^2 + \alpha \parallel x_n - y_{n-1} \parallel^2$$

$$- 2 \lambda_n [(1 + \delta_n) \Phi(\bar{x}, x_n) - \delta_n \Phi(\bar{x}, x_{n-1})]$$

$$- 2 \lambda_n \delta_n (\delta_n + 1) m \parallel x_n - \bar{x} \parallel^2 + 2 \lambda_n \delta_n m \parallel x_{n-1} - \bar{x} \parallel^2,$$

where the second inequality is from $(\sqrt{2} + 1) \alpha - 1 < 0$. For any $n \geq 1$, let

$$a_n = \parallel x_n - \bar{x} \parallel^2,$$

$$b_n = 2 \lambda_n [(1 + \delta_n) \Phi(\bar{x}, x_n) + (1 - \sqrt{2} \alpha) \parallel x_{n+1} - y_n \parallel^2,$$

and

$$\beta_n = \max \left\{ \frac{\alpha}{1 - \sqrt{2} \alpha}, \frac{\delta_n \lambda_n}{(\delta_n - 1) \lambda_n - 1} \right\},$$

$$\eta_n = 2 \lambda_n m (\delta_n + 1),$$

$$r_n = \frac{\delta_n}{\delta_n + 1}.$$

Then, we deduce

$$a_{n+1} + b_{n+1} \leq [1 - 2 \lambda_n m (\delta_n + 1)] a_n + 2 \lambda_n m \delta_n a_{n-1} + \beta_n b_n$$

$$= (1 - \eta_n) a_n + \eta_n r_n a_{n-1} + \beta_n b_n, \quad \forall n \geq 1.$$
Recall that from Lemma 1, we observe $\eta > 0$, and the supremum of $r_n$, denoted by $r$, satisfies $r \in [0, 1]$. Since $0 < \alpha < \sqrt{2} - 1$ and $\rho_n < \sqrt{1 + \delta_n}$, we deduce $\frac{\nu_n}{1 - \sqrt{\delta_n}} < 1$ and $\delta_n \lambda_n < (\delta_n - 1) \lambda_{n-1}$, which means the supremum denoted by $\beta$ of $\beta_n$ satisfies $\beta \in [0, 1]$. 

For any positive number $\mu$ and positive sequence $\{\nu_n\}_{n \in \mathbb{N}}$, the formula (25) is equivalent to
\[
\begin{aligned}
a_{n+1} + \mu a_n + b_{n+1}
&\leq (1 - \eta_n + \mu)a_n + r \eta_n a_{n-1} + \beta b_n \\
&= \nu_n (a_n + \mu a_{n-1}) + \beta b_n + (1 - \eta_n + \mu - \nu_n) a_n + (r \eta_n - \mu \nu_n) a_{n-1}, \quad \forall n \geq 1.
\end{aligned}
\]
(26)
As in [27, 45], let $1 - \eta_n + \mu - \nu_n = 0$ and $r \eta_n - \mu \nu_n = 0$, we consider the following function
\[
f(x) = \frac{1 - x + \sqrt{(x - 1)^2 + 4rx}}{2}, \quad x \in [0, +\infty].
\]
Since for any $x \in [0, +\infty]$, we have
\[
2\sqrt{(x - 1)^2 + 4rx} f'(x) = \frac{4r(x - 1)}{\sqrt{(x - 1)^2 + 4rx + (x - 1) + 2r}} < 0,
\]
then $f$ is decreasing on $[0, +\infty]$ and
\[
0 < \nu_n = f(\eta_n) < f(0) = 1, \quad \forall n \geq 1.
\]
Let $\gamma = \max \left\{ \max_{1 \leq i \leq n} \nu_i, \beta \right\}$, clearly $0 < \gamma < 1$. Using (26), we conclude
\[
a_{n+1} + \mu a_n + b_{n+1} \leq \nu_n (a_n + \mu a_{n-1}) + \beta b_n \leq \gamma (a_n + \mu a_{n-1} + b_n), \quad \forall n \geq 1.
\]
This implies that
\[
\|x_{n+1} - \bar{x}\|^2 = a_{n+1} \leq a_{n+1} + \mu a_n + b_{n+1} \leq \gamma^n (a_1 + \mu a_0 + b_1), \quad \forall n \geq 1.
\]
This together with $a_1 + \mu a_0 + b_1 > 0$ completes the proof. 

4 Further Discussions on Special Cases

In this section, we apply classic or adaptive proximal extrapolated gradient method to some special cases of (1), and explore their convergence. It is noteworthy that, one rule is sufficient to automate extrapolated gradient descent: don’t overstep the local curvature, for the adaptive extrapolated gradient method to solve the unconstrained optimization and fixed point problems. In the other words, the step sizes can be completely determined by the local curvature of $F$.

4.1 Unconstrained optimization and fixed point problems

The basic unconstrained optimization problem is
\[
\min_{x \in \mathbb{R}^n} f(x),
\]
where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and differentiable function. Under assumption that the gradient $\nabla f$ is $L$-Lipschitz continuous, one can show that the classic gradient descent method with
\[
x_{n+1} = x_n - \lambda \nabla f(x_n), \quad \lambda \in [0, 2/L],
\]
converges to an optimal solution $x^* \in S$ (the solution set $S$) [39]. Moreover, with $\lambda = 1/L$ the convergence rate [12] is $f(x_n) - f(x^*) \leq \frac{L^2 \|x_0 - x^*\|^2}{2(2n+1)}$, and this bound is not improvable.
In practical applications, many interesting functions are smooth (or its gradient is Lipschitz) locally, but not globally. In $\mathbb{R}$, they include $exp(x)$, $log(x)$, $tan(x)$, $x^p (p > 1)$, etc. This section thus devotes to locally smooth functions. Note that $f$ is convex and differentiable, the mapping $F := \nabla f$ is monotone. Moreover, recall $F(x^*) = 0$, we deduce
\[ \langle F(x), x - x^* \rangle \geq 0, \forall x \in \mathbb{R}^n, \forall x^* \in \mathcal{S}. \] (28)
Our FAPEG can apply to problem (27), and in this case can be relaxed as following.

**Algorithm 2 (Fully adaptive extrapolated gradient method (FAEG) for solving (27))**

**Step 0.** Choose $x_0 = y_0 \in \mathcal{H}$, $\lambda_0 > 0$ and $\hat{\lambda} > 0$. Take $\delta_0 = 1$, $\alpha = \zeta(\sqrt{2} - 1)$ with $\zeta \in [0, 1]$. Set $n = 0$.

**Step 1.** Compute
\[ x_{n+1} = x_n - \lambda_n F(y_n). \]
If $x_{n+1} = x_n = y_n$, then stop: $x_{n+1}$ is a solution.

**Step 2.** Compute
\[ y_{n+1} = x_{n+1} + \delta_n (x_{n+1} - x_n), \]
and
\[ \lambda_{n+1} = \begin{cases} \min \left\{ \frac{\alpha \|y_{n+1} - y_n\|}{\|Fy_{n+1} - Fy_n\|}, \frac{\hat{\lambda}}{\lambda_n} \right\}, & \text{if } Fy_{n+1} - Fy_n \neq 0, \\ \frac{\lambda_n}{\lambda_n + \delta_n}, & \text{otherwise}, \end{cases} \]
(29)

**Step 3.** Update $\delta_{n+1} = \frac{\lambda_{n+1}}{\lambda_n}$. Set $n \leftarrow n + 1$ and return to step 1.

Comparing with FAPEG (Algorithm 1), the step sizes in Algorithm 2 are estimated only by the local curvature of $\nabla f$, without any constraints on its increasing rate, which will lead to better numerical efficiency, see Section 5. Choose $x = x^* \in \mathcal{S}$ in Lemma 2, by (28) we deduce $\langle F(y_n), y_n - x^* \rangle \geq 0$, then $g \equiv 0$ and the proof of Lemma 2 yield the following results.

**Lemma 5** Let $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$ be two sequences generated by FAEG and $x^* \in \mathcal{S}$. Then, we have
\[ \|x_{n+1} - x^*\|^2 + \alpha \|x_{n+1} - y_n\|^2 \leq \|x_n - x^*\|^2 + \alpha \|x_n - y_{n-1}\|^2 - \left[1 - (\sqrt{2} + 1)\alpha\right] (\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2). \] (30)

From inequality (30) together with $\alpha = \zeta(\sqrt{2} - 1)$, we obtain that the Lyapunov energy, the left-hand side of (30), is decreasing. This gives us boundedness of $\{x_n\}_{n \in \mathbb{N}}$, by which the convergence can be proved with easy.

For an operator $T : \mathbb{R}^n \to \mathbb{R}^n$, the fixed point equation $x = Tx$ is equivalent to the equation $F(x) = 0$ with $F = I - T$. By Fix $T$ we denote the fixed point set of the operator $T$. From [29, Section 3], in the case $\mathcal{S} = \text{Fix } T$, the condition (28) is equivalent with
\[ \langle x - Tx, x - x^* \rangle \geq 0, \forall x \in \mathbb{R}^n, \forall x^* \in \text{Fix } T. \]
The operator $T$ is demi-contractive if the inequality above holds. It is known that demi-contractive operator is a more general class of the operators, than pseudo-contractive, quasi-nonexpansive and nonexpansive operator. Actually, $T$ is demi-contractive if and only if $F$ satisfies (28).

From the observation above, one can apply the FAEG to find a fixed point of an operator $T$. In this case, we have $g \equiv 0$ and $F = I - T$. 

...
4.2 Linear $F$

In this section, we focus on a particular case of (1) with $F \neq 0$ be a bounded linear operator with operator norm $L = \|F\|$, due to its beneficial structure to present efficient algorithms and wide applications. We study convergence properties of PEG (7) for solving this special problem from the perspective of fixed point theory.

We present two sources where the VI (1) with $F$ be linear naturally arise. The first example is a simpler problem of composite minimization

$$\min_x f(x) + g(x) := \frac{1}{2}\|Ax - b\|^2 + g(x), \quad (31)$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then the gradient $\nabla f = A^*(Ax - b)$ is linear. Problem (31) is equivalent to (1) with $F$ = $\nabla f$. If $g(x) = \mu \|x\|_1$ ($\mu > 0$), problem (31) is a classic LASSO problem.

Another important source is the convex-concave saddle point problem (4) with $K(x, y) = \langle Qx, y \rangle$, where $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a bounded linear operator, $Q^*$ denotes the adjoint of the operator $Q$; $f^*$ denotes the Legendre-Fenchel conjugate of the function $f$, and $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions. By writing down the first-order optimality condition, problem (4) can be reformulated as (1) with $F$ and $g$ defined as

$$z = (x, y), \quad F(z) = \begin{bmatrix} Q^*y \\ - Qx \end{bmatrix}, \quad g(z) = g(x) + f^*(y).$$

Obviously, the operator $F$ above is linear as well.

For the linear operator $F$, the scheme of PEG can be simply described as

$$x_{n+1} = \text{Prox}_{\lambda g}[x_n - (1 + \delta)\lambda F(x_n) + \delta \lambda F(x_{n-1})],$$

for fixed $\lambda > 0$ and $\delta > 0$. If $\delta = 1$, the scheme (32) is identified with a special forward-backward splitting method introduced in [30] for the case that $F$ is linear.

By introducing the auxiliary variable $u_{n+1} := F(x_n)$, the scheme (32) can be expressed as the fixed point iteration in $\mathcal{H} \times \mathcal{H}$ given by

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \end{pmatrix} = M_g \circ \tilde{F} \begin{pmatrix} x_n \\ u_n \end{pmatrix},$$

with

$$M_g = \begin{bmatrix} \text{Prox}_{\lambda g} & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} I - (1 + \delta)\lambda F & \lambda \delta I \\ F & 0 \end{bmatrix}, \quad \lambda > 0, \quad \delta > 0. \quad (33)$$

To deduce convergence from the perspective of fixed point theory, $M_g$ is firmly nonexpansive, but it is not clear what properties the operator $M_g \circ \tilde{F}$ posses as the linear operator $\tilde{F}$ is not necessarily nonexpansive. Thus, we aim at finding proper $\delta$ and $\lambda$ so that $\tilde{F}$ is nonexpansive.

Let $r(A)$ be the spectral radius of linear operator $A$. If $r(\tilde{F}) < 1$, $\tilde{F}$ is nonexpansive then the scheme (32) will be convergent. In sequel, we make a further observation on the sufficient condition satisfying $r(\tilde{F}) < 1$.

**Lemma 6** For the linear operator $\tilde{F}$ defined in (33), when

$$0 < \lambda < \frac{\kappa(\delta)}{L}, \quad \delta > \frac{1}{2},$$

with $\kappa(\delta) = \sqrt{\frac{24 - 1}{2^2(23 + 1)}}$, we have $r(\tilde{F}) < 1$. 

Proof. Let $\rho \in \mathbb{C}$ be an eigenvalue of $\tilde{F}$ with the corresponding eigenvector $z = (z_1, z_2)^T \in \mathbb{C}$. Any pair $(\rho, z)$ of eigenvalue and eigenvector of $\tilde{F}$ fulfills

$$z_1 - (1 + \delta)\lambda Fz_1 + \lambda\delta z_2 = \rho z_1,$$

$$Fz_1 = \rho z_2.$$  \hfill (35) \hfill (36)

From (36), we get $z_2 = \frac{\rho}{\|\rho\|^2} Fz_1$. Substituting it into (35) yields

$$\left[ (1 + \delta) - \frac{\delta \rho}{\|\rho\|^2} \right] \lambda Fz_1 = (1 - \rho)z_1.$$  \hfill (37)

Note that for a real, linear, and monotone map $M$, and a complex vector $a = b + ic$, it holds that $\langle Ma, a \rangle = \langle Mb, b \rangle + \langle Mc, c \rangle + i\langle (M^T - M)b, c \rangle$ and thus, $\text{Re}(\langle Ma, a \rangle) \geq 0$. This together with the monotonicity of $F$ shows that $\text{Re}\left(\left[ (1 + \delta) - \frac{\delta \rho}{\|\rho\|^2} \right] (1 - \rho)\right) \geq 0$.

Denoting $\rho := x + iy \in \mathbb{C}$, the expression above reads as

$$[(1 + \delta)(x^2 + y^2) - \delta x](1 - x) - \delta y^2 \geq 0,$$

so

$$[(1 + \delta)x - 1]y^2 \leq x(1 - x)[(1 + \delta)x - \delta].$$  \hfill (38)

Recall the condition $r(\tilde{F}) < 1$, we have $|\rho| < 1$, i.e., $x^2 + y^2 < 1$. Thus, we deduce if

$$0 \leq x \leq \frac{\delta}{1 + \delta} \text{ or } \frac{1}{2\delta} < x < 1, \text{ if } \frac{1}{2} < \delta < 1,$$

$$x^2 + y^2 \leq x < 1 \text{ or } x = \frac{1}{2}(y^2 \leq \frac{1}{4}), \text{ if } \delta = 1,$$

$$0 \leq x < \frac{1}{2\delta} \text{ or } \frac{\delta}{1 + \delta} \leq x < 1, \text{ if } \delta > 1,$$

see Fig. 1, the conditions $|\rho| < 1$ and (38) hold.

Moreover, from (37), we observe

$$\|\left[ (1 + \delta) - \frac{\delta \rho}{\|\rho\|^2} \right] \lambda F\| = |1 - \rho|.$$ 

By $\rho = x + iy$, we have

$$\lambda^2\|F\|^2[(x^2 + y^2)(1 + \delta) - \delta x]^2 + \delta^2 y^2] = (x^2 + y^2)^2[(1 - x)^2 + y^2].$$

Thus using (38), we get

$$\lambda^2\|F\|^2 < \frac{1}{(1 + \delta)^2} \frac{x(1 - x)[2(1 + \delta)x - \delta - 1]^2}{[(1 + \delta)x - \delta][(1 + \delta)x - 1]^2},$$

combining $\|F\| = L$ and the conditions (39)-(41) on $\delta$ and $x$ gives

$$\lambda < \frac{1}{L} \sqrt{\frac{2\delta - 1}{\delta^2(\delta + 1)}}, \text{ if } \delta > \frac{1}{2}.$$ 

This completes the proof. \hfill \square
The region of \((x,y)\) satisfying (38) and \(x^2 + y^2 < 1\) with different \(\delta\).

**Fig. 1**

**Theorem 4** Let \(\{x_n\}\) be a sequence generated by (32) with \(\lambda < \frac{\kappa(\delta)}{L}\) and \(\delta > 1/2\), then it converges to a solution of (1) with linear \(F\).

Consider a special problem

\[ F(x) = 0 \quad \text{with} \quad F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \] (42)

Note that zero is the unique solution to this problem and that the operator \(F\) is 1-Lipschitz. This is a classical example of a monotone inclusion, where the forward-backward method fails. If \(\delta \leq 1/2\) or \(\lambda \geq \kappa(\delta)\) with \(\delta > 1/2\), we observe \(r(\tilde{F}) \geq 1\) with software Matlab, see Fig. 2. In addition, the smallest value of \(r(\tilde{F})\) is \(\sqrt{2}/2\), obtained at \(\delta = 1\) and \(\lambda = 1/2\) (Note that smallest value does not occur for the largest possible stepsize).

Recall that from Lemma 6, \(0 < \lambda < \frac{\kappa(\delta)}{L}\) with \(\delta > 1/2\) is a sufficient condition satisfying \(r(\tilde{F}) < 1\). Thus by the simple example (42), we observe that for the PEG iterative scheme (32), \(\frac{\kappa(\delta)}{L}\) is the optimal upper bound of the step size \(\lambda\).

Applying FAPEG to (42), we deduce that \(\delta_n \equiv 1\) and \(\lambda_n \equiv \lambda_{n-1} = \lambda \in ]0, \sqrt{2} - 1[\), FAPEG can be expressed as

\[
\begin{bmatrix}
  x_{n+1} \\
  u_{n+1}
\end{bmatrix}
= \tilde{F}
\begin{bmatrix}
  x_n \\
  u_n
\end{bmatrix}
= \begin{bmatrix}
  I - 2\lambda F & I \\
  F & 0
\end{bmatrix}
\begin{bmatrix}
  x_n \\
  u_n
\end{bmatrix}. \] (43)
Performing the golden ratio algorithm to solve (42), we deduce $\lambda_n \equiv \lambda_{n-1} = \lambda \in [0, \frac{\varphi}{2}]$ with $\varphi \in [0, \frac{1+\sqrt{5}}{2}]$, the golden ratio algorithm can be expressed as

$$
\begin{pmatrix}
\hat{z}_n \\
\hat{z}_{n+1}
\end{pmatrix} = R \begin{pmatrix}
\hat{z}_{n-1} \\
\hat{z}_n
\end{pmatrix} = \begin{bmatrix}
\frac{1}{2}I & \frac{\varphi-1}{\varphi}I \\
\frac{1}{2}I & \frac{\varphi-1}{\varphi}I - \lambda F
\end{bmatrix} \begin{pmatrix}
\hat{z}_{n-1} \\
\hat{z}_n
\end{pmatrix}.
$$

– **FAPEG.** From [30], the eigenvalues of $\tilde{F}$ in (43) are given by $\frac{1}{2} \pm \frac{1}{2}i\sqrt{8\lambda^2 - 1 - 4\lambda\sqrt{1 - 4\lambda^2}}$. By choosing the stepsize $\lambda$ close to $\sqrt{2} - 1$, we deduce that $r(\tilde{F}) \approx 0.8832$, the rate of convergence of $\{x_n\}$ can be made close to 0.8832.

– **golden ratio algorithm.** When $\varphi = 1.5$ and $\lambda = \varphi/2$ as in [29], the spectral radius of $R$, $r(R) \approx 0.8953$.

Similarly, for problem (42) with $F = I_2$, we have $r(\tilde{F}) \approx 0.7351$ and $r(R) \approx 0.8431$, so we conclude from $r(\tilde{F}) < r(R)$ that FAPEG is faster than the golden ratio algorithm for these particular problem, in terms of the number of iterations.

5 Numerical Experiments

We present numerical results to demonstrate the computational performance of FAPEG (Algorithm 1) and FAEG (Algorithm 2) with $\alpha = 0.414$ for solving some minimization problems. Although our methods do not have any restriction on the initial step size, we generate $\lambda_0$ as in [28], except

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\[1,2\] All codes are available at [http://www.escience.cn/people/changxiaokai/Codes.html](http://www.escience.cn/people/changxiaokai/Codes.html)
for some special cases. Choose any $y_{-1}$ in a small neighbourhood of the starting point $y_0$ and take

$$
\lambda_0 = \frac{\alpha \|y_{-1} - y_0\|}{\|Fx(y_{-1}) - Fx(y_0)\|},
$$

(44)

The following state-of-the-art algorithms are compared to investigate the computational efficiency:

- Proximal gradient method (PGM), with fixed step $\frac{1}{2}$;
- FISTA with fixed step $\frac{1}{T}$;
- fully explicit algorithm Golden Ratio Algorithm (EGRAAL) [29] with $\phi = 1.5$;
- Proximal extrapolated gradient methods [28, Algorithm 2] (PEG-L), with line search and $\alpha = 0.41, \sigma = 0.7$.

We denote the random number generator by seed for generating data again in Python 3.8. All experiments are performed on an Intel(R) Core(TM) i5-4590 CPU@ 3.30 GHz PC with 8GB of RAM running on 64-bit Windows operating system.

5.1 Convex feasibility problem

Given a number of closed convex sets $C_i \in H$, $i = 1, \cdots, m$, the convex feasibility problem (CFP) aims to find a point in their intersection: $x \in \cap_{i=1}^{m} C_i$. The CFP is very general and allows one to represent many practical problems in this form. We apply FAPEG and FAEG to CFP, by using VI (1) with

$$
g \equiv 0 \quad \text{and} \quad F = I - T,
$$

where $T = \frac{1}{m}(P_{C_1} + \cdots + P_{C_m})$. Its main advantage is that it can be easily implemented on a parallel computing architecture. For large-scale problems it is often much faster in practice due to parallelization and more efficient ways of computing $Tx$. One can look at the iteration $x_{k+1} = Tx_k$ as an application of the Krasnoselskii-Mann (KM) scheme for the firmly-nonexpansive operator $T$.

**Problem 1 (Intersection of balls.)** Now we consider a synthetic nonlinear feasibility problem. We have to find a point in $x \in \cap_{i=1}^{m} C_i$, where $C_i = B(c_i, r_i)$, a closed ball with a center $c_i \in \mathbb{R}^n$ and a radius $r_i > 0$.

The projection onto $C_i$ is simple: $P_{C_i}x$ equals to $\frac{x - c_i}{\|x - c_i\|^2} c_i$ if $\|x - c_i\| > r_i$ and $x$ otherwise. Thus, again computing $Tx = \frac{1}{m} \sum_{i=1}^{m} P_{C_i}x$ can be done in parallel very efficiently. We run two scenarios as in [29]: with $n = 1000$, $m = 2000$ and with $n = 2000$, $m = 1000$. Each coordinate of $c_i \in \mathbb{R}^n$ is drawn from $N(0, 100)$. Then we set $r_i = \|c_i\| + 1$ that ensures that zero belongs to the intersection of $C_i$. The starting point was chosen as the average of all centers: $x_0 = \frac{1}{m} \sum_{i=1}^{m} c_i$ and $\lambda_0 = 1$ for FAPEG. For FAEG, we generate $\lambda_0$ by (44).

Since the cost of iteration is essentially the same: EGRAAL, KM, FAPEG and FAEG is approximately the same, we show only how the residual $\|Fx_n\| = \|Tx_n - x_n\|$ is changing w.r.t. the number of iterations. To show the efficiency, we plot the result when $\text{seed} = 1$ from each of the above scenarios. Figs. 3 and 5 depict the behavior of residual $\|Fx_n\|$ and step sizes generated by FAPEG. Figs. 4 and 6 show the behavior of ratio $\frac{\lambda_n}{\lambda_{n-1}}$ from EGRAAL, FAPEG and FAEG. As one can see, the differences of residual $\|Fx_n\|$ and ratio $\frac{\lambda_n}{\lambda_{n-1}}$ are significant between EGRAAL, FAPEG and FAEG.

**Problem 2 (Tomography reconstruction)** The tomography reconstruction problem is to obtain a slice image of an object from a set of projections (sinogram). It is mathematically an instance of a linear inverse problem

$$Ax = \hat{b},
$$

(45)
\[ x_{k+1} = T x_k \]

\[ \| F x_n \| \]

\[ \lambda_n = \| F x_n \| \]

Fig. 3 The behavior of residual \( \| F x_n \| \) from different methods and the step sizes \( \lambda_n \) from FAEG for problem (1) with \( n = 2000, m = 1000 \) and seed = 1.

Fig. 4 The behavior of ratio \( \frac{\lambda_n}{\lambda_{n-1}} \) from EGRAAL, FAPEG and FAEG for problem (1) with \( n = 2000, m = 1000 \) and seed = 1.

Fig. 5 The behavior of residual \( \| F x_n \| \) from different methods and the step sizes \( \lambda_n \) from FAEG for problem (1) with \( n = 1000, m = 2000 \) and seed = 1.

where \( x \in \mathbb{R}^n \) is the unknown image, \( A \in \mathbb{R}^{m \times n} \) is the projection matrix, and \( b \in \mathbb{R}^m \) is the given sinogram. In practice, however, \( b \) is contaminated by some noise \( \varepsilon \in \mathbb{R}^m \), so we observe only \( \hat{b} = b + \varepsilon \).

Clearly, we can formulate this linear inverse problem as a convex feasibility problem to find a point \( x \in \cap_{i=1}^m C_i \) with \( C_i = \{ x : \langle a_i, x \rangle = b_i \} \). As a particular problem, we test and reconstruct the Shepp-Logan phantom image \( 256 \times 256 \) (thus, \( x \in \mathbb{R}^n \) with \( n = 2^{16} \)) from the far less measurements \( m = 2^{15} \). We generate the matrix \( A \in \mathbb{R}^{m \times n} \) from the scikit-learn library as in [29] and define
Fully adaptive proximal extrapolated gradient method

Fig. 6 The behavior of ratio $\frac{\lambda_n}{\lambda_{n-1}}$ from EGRAAL, FAPEG and FAEG for problem (1) with $n = 1000$, $m = 2000$ and seed = 1.

$b = Ax + \varepsilon$, where $\varepsilon \in \mathbb{R}^m$ is a random vector, whose entries are drawn from $\mathcal{N}(0, 1)$. The starting point was chosen as $x_0 = (0, \cdots, 0)$ and $\lambda_0 = 1$ for FAPEG. For FAEG, we generate $\lambda_0$ by (44).

We compare FAEG for solving (45), with EGRAAL and KM scheme: $x_{n+1} = Tx_n$ with $T = \frac{1}{m} \sum_{i=1}^{m} P_{C_i}$. In Fig. 7, we report how the residual $\|Fx_n\|$ is changing w.r.t. the number of iterations. Recall that the CPU time of three methods is almost the same, so one can reliably state that in this case the adaptive step sizes are efficient for the PEG.

Moreover, we can observe from Figs. 4, 6 and 8 that, the ratio $\frac{\lambda_n}{\lambda_{n-1}}$ from EGRAAL are $10/9$ for the majority of iterations due to $\rho = 10/9$, which implies the allowable rate $\rho_n$ of step sizes to increase in FAPEG is relaxed than $\rho$ used in EGRAAL, and then explains why FAPEG and FAEG often perform better than EGRAAL.

5.2 Sparse logistic regression

The sparse logistic regression problem is popular in machine learning applications where one aims to find a linear classifier. Let $(h_i, l_i) \in \mathbb{R}^n \times \{\pm 1\}$, $i = 1, \cdots, m$ be the training set, where $h_i \in \mathbb{R}^n$ is the feature vector of each data sample, and $l_i$ is the binary label. The formulation of sparse logistic regression reads

$$\min_{x \in \mathbb{R}^n} J(x) := \mu \|x\|_1 + \sum_{i=1}^{m} \log(1 + e^{l_i h_i^T x}),$$

(46)
where $\mu > 0$.

We known that stochastic methods seem to be more competitive if the size of the problem is quite large. Our motivation here is not to propose the best method for (46) but to demonstrate the performance of FAPEG on some real-world problems. Let $K_{ij} = -1, h_{ij}$ and set $\tilde{f}(y) = \sum_{i=1}^{m} \log(1 + \exp(y_i))$. Then the objective in (46) is $J(x) = f(x) + g(x)$ with $g(x) = \mu \|x\|_1$ and $f(x) = f(Kx)$. As $f$ is separable, it is easy to derive that $L_{\nabla f} = \frac{1}{4}$. Thus, $L_{\nabla f} = \frac{1}{4}\|K^T K\|$.

The data set from LIBSVM2 is considered, and we take two popular datasets: $a9a$ with $m = 32,561$, $n = 16,281$ and $real-sim$ with $m = 72,309$, $n = 20,958$. For both datasets, we set $\mu = 0.005\|A^T b\|_{\infty}$. We run all methods for sufficiently many iterations and compute the energy $J(x_n)$ in each iteration. If after $n$ iterations the residual was small enough: $\|x_n - \text{Prox}_{\lambda_n g}(x_n - \nabla f(x_n))\| \leq 10^{-6}$, we choose the smallest energy value among all methods and set it to $J^*$. In Fig. 9 we show how the energy residual $J(x_n) - J^*$ is changing w.r.t. the iterations. Since the dimensions in both problems are quite large, the CPU time for all methods is approximately the same. An explanation for such a good performance of FAPEG is of course that for this problem the global Lipschitz constant of $\nabla f$ is too conservative and the allowable rate $\rho_n$ of step sizes to increase in FAPEG is more relaxed.

![Fig. 8](image1.png)  
(a) EGRAAL.  
(b) FAEG.

**Fig. 8** The behavior of ratio $\frac{J(x_n)}{J_{n-1}}$ from EGRAAL and FAEG for problem (2).

![Fig. 9](image2.png)  
(a) a9a.  
(b) real-sim.

**Fig. 9** The energy residual $J(x_n) - J^*$ from different methods.

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3 https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
Fully adaptive proximal extrapolated gradient method

Fig. 10 The behavior of ratio $\frac{\lambda_n}{\lambda_{n-1}}$ from EGRAAL and FAPEG for the sparse logistic regression problem with data “a9a”.

5.3 HpHard problem

The third problem is HpHard problem, considered as in [28, 45]. Let $F(x) = Mx + q$ with $M = NN^T + S + D$ and $q \in \mathbb{R}^m$, where $N$, $D$ and $S \in \mathbb{R}^{m \times m}$, $S$ is a skew-symmetric matrix, every entry of $N$ and $S$ is uniformly generated from $(-5, 5)$. The matrix $D$ is diagonal and its diagonal entry is uniformly generated from $(0, 0.3)$. Every entry of $q$ is uniformly generated from $(-500, 0)$.

The feasible set is $C = \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = m\}$ and $g(x) = l_C(x)$.

Since solutions of (1) coincide with zeros of the residual function

$$r(x, y) := \|y - \text{Prox}_{\lambda_n g}(x - \lambda F(y))\| + \|x - y\|,$$

for some positive number $\lambda_n$, and $r_n := r(x_n, y_n) = \|x_{n+1} - y_n\| + \|x_n - y_n\| = 0$ implies $x_{n+1} = x_n = y_n$, thus we use $r_n < \epsilon$ with given $\epsilon = 10^{-10}$ to terminate our algorithms.

Fig. 11 Comparison of different methods for solving HpHard problem with seed = 1 and $m = 1000$.

In Fig. 11, we report how the residual $r_n$ is changing w.r.t. the computing time (Time) measured in seconds, as PEG-L needs to perform linesearch. The results presented in Fig. 11 show the adaptive step sizes proposed are efficient for the PEG, and PEG-L performs better than EGRAAL for this problem, though PEG-L may compute extra proximal operator and $F$.

From the behavior of ratio $\frac{\lambda_n}{\lambda_{n-1}}$ shown in Fig. 12, PEG-L is a little similar with FAPEG. While for EGRAAL, as for other problems the ratios $\frac{\lambda_n}{\lambda_{n-1}}$ are $10/9$ for the majority of iterations. This may be a main reason why PEG-L performs better than fully adaptive EGRAAL for this problem.
6 Conclusions and Perspectives

Without assuming the Lipschitz continuity of the mapping and running a linesearch, we have proposed the simple proximal extrapolated gradient method with fully non-monotonic and adaptive step sizes in this paper. A number of experiments illustrate that the adaptive strategy proposed is more efficient due to its moderate constraint, and the improvement can be resulted from the low computational cost and adaptive step sizes estimated by local Lipschitz constant.

We now present some possible directions for future research, which we personally consider to be interesting and important.

**Nonmonotone case.** Many scientific, engineering and economic areas involve the optimization of complex, nonlinear and possibly nonmonotone operators. A great interest for us is to explore theoretical guarantees of the proposed method in the nonmonotone settings.

**Extensions to other cases.** For other methods that need to estimate the Lipschitz constant, for instance, classic forward-backward splitting, three-operator splitting [10], so far it not clear how to derive fully adaptive step size. Based on our scheme it will be in particular interesting to do so, since our scheme

\[
\lambda_{n+1} = \begin{cases} 
\min \left\{ \rho_n \lambda_n, \frac{\alpha \|y_{n+1} - y_n\|}{\|Fy_{n+1} - Fy_n\|} \right\}, & \text{if } Fy_{n+1} - Fy_n \neq 0, \\
\lambda_n, & \text{otherwise},
\end{cases}
\]

uses \(\rho_n\) to determine the increasing of step, \(\alpha\) (or variable \(\alpha_n\)) to control its size with the help of local Lipschitz information.

**Stochastic settings.** For large-scale problems, computing \(F(y_n)\) becomes prohibitively expensive. For this reason, the stochastic VI methods that compute \(F(y_n)\) approximately can be advantageous over their deterministic counterparts, as it was shown in [19]. In particular, it is interesting to derive a stochastic and adaptive extrapolated gradient method.

**Improving the estimation of step size.** Obviously, the value of step size is estimated mainly by \(\frac{\alpha \|y_{n+1} - y_n\|}{\|Fy_{n+1} - Fy_n\|}\), the bound \(\alpha < \sqrt{2} - 1\) makes the step sizes smaller in case when the Lipschitz constant of the operator does not change too much, especially for the FAEG. It is interesting to study whether one can improve this estimation part.

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