Crossover from Fermi to Non-Fermi Liquid in Two-Dimensional Interacting Fermions

Ken YOKOYAMA

Department of Physics, Tokyo University, Tokyo 113

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Self-energy at zero temperature is investigated up to the third-order of interaction using one-patch model in two dimensions, whose interaction process corresponds to $g_4$-process of g-ology model in one dimension. The self-energy $\Sigma^L(k, \epsilon)$ diverges at $\epsilon = \xi_k$, and the contribution from the particle-hole process in third-order self-energy diagrams has a stronger divergence compared to the one from the particle-particle process. This implies that the loop-cancellation in the forward scattering is insufficient due to the effect of the warping of the Fermi surface. The strong energy dependence of the self-energy in the vicinity of $\epsilon = \xi_k$ implies the existence of the crossover from Fermi to non-Fermi liquid behavior as the momentum becomes away from the Fermi momentum, and this crossover is enhanced as interaction becomes stronger.

KEYWORDS: two dimensional electron gas, forward scattering, crossover from Fermi to non-Fermi liquid, loop-cancellation

The low-energy excitation of the interacting fermions with short-range force is established as Tomonaga-Luttinger (TL) liquid in one dimension and Fermi liquid in three dimensions, while as for two dimensions it is still in controversy. In case of one dimension, forward scattering processes, which are denoted $g_{\parallel}$- and $g_{\perp}$-process in $g$-ology model, leads to TL liquid.\cite{1} The main features of TL liquid are the following two things: vanishing jump of momentum distribution at Fermi momentum and spin-charge separation. The $g_{\parallel}$-process is related to the former, and the $g_{\perp}$-process to the latter in the following sense. Taking account of only $g_{\parallel}$- or $g_{\perp}$-process, the velocity of spin and charge excitations, $v_s$ and $v_\sigma$, respectively, become different, which indicates the existence of spin-charge separation and results in the two-peak structure of spectral-weight.\cite{2,3} On the other hand, if we consider only $g_{\perp}$-process (here $g_{\parallel}$-process related term cancels if we neglect momentum dependence of the coupling constant), the parameters $K_\sigma$ and $K_\parallel$, which equal to 1 for free fermions and characterize anomalous power-laws of various correlation functions, deviate from 1. This also leads to vanishing jump of momentum distribution at Fermi surface.

In case of two dimensions, it was suggested that anomalous behavior of forward scattering phase-shift leads to non-Fermi liquid even at weak-coupling quite similarly to TL liquid.\cite{4,5} But in this stage, there is no theory which confirms this possibility. There is also no signal of non-Fermi liquid state from many-body perturbation approach in two dimensions.\cite{6-17} In the following we investigate this possibility from perturbation theory in detail.

To start with, we consider the following correspondence between the model in one and two dimensions. We obtain low-energy effective theories by integrating out degrees of freedom of electrons far from Fermi points (or surface), which is so-called the elimination of fast modes.\cite{10,18} In one dimension, the low-energy effective theory is $g$-ology model, and there are two branches corresponding to two Fermi points. In case of two dimensions, the low-energy effective theory has only degrees of freedom of electrons in a thin shell with thickness $\Lambda$ around the Fermi surface. Since momenta of electrons are allowed only within the thin shell, interaction processes are extremely restricted; only three kinds of processes shown in fig. 1, i.e., forward, exchange and Cooper scatterings, are allowed (here we neglect Umklapp process).\cite{10,18} Dividing the thin shell around the Fermi surface to many small patches of the size $\Lambda \times \Lambda$, we obtain the following low-energy effective action for zero temperature:

$$S \equiv S_0 + S_{\text{forward}} + S_{\text{exchange}} + S_{\text{Cooper}}$$ (1a)

$$S_0 = \sum_\sigma \int_{k_\epsilon} Z_k^{-1}(i\epsilon - \xi_k) c_\sigma^\dagger c_\sigma$$ (1b)

$$S_{\text{forward}} = -\frac{1}{2} \sum_{ij} \sum_{\sigma\sigma'} \int_{k_\epsilon} \int_{k'_{\epsilon'}} \int_{q_\omega}$$

* E-mail: yokoyama@watson.phys.s.u-tokyo.ac.jp
Fig. 1. Three kinds of interaction processes. (a) Forward scattering, \((k\sigma, k'\sigma' \rightarrow k\sigma, k'\sigma')\). (b) Exchange scattering, \((k\sigma, k' - \sigma \rightarrow k' - \sigma, k\sigma')\). (c) Cooper scattering, \((k\sigma, -k\sigma' \rightarrow k'\sigma, -k'\sigma')\).

\[ S_{\text{exchange}} = -\frac{1}{2} \sum_{i \neq j} \sum_{\sigma} \int_{k_{el}} \int_{k_{el}'} \int_{q_{\omega}} g_{F} \sigma_{k'k}(q) c_{k+q,\sigma}^{\dagger} c_{k'-q,\sigma'} c_{k',\sigma'\sigma} \]  

\[ S_{\text{Cooper}} = -\frac{1}{2} \sum_{i \neq j} \sum_{\sigma} \int_{k_{el}} \int_{k_{el}'} \int_{q_{\omega}} g_{C} \sigma_{k'k}(q) c_{k+q,\sigma}^{\dagger} c_{k'-q,\sigma'} c_{k',\sigma'\sigma} \]  

\[ S_{\text{Cooper}} = -\frac{1}{2} \sum_{i \neq j} \sum_{\sigma} \int_{k_{el}} \int_{k_{el}'} \int_{q_{\omega}} g_{C} \sigma_{k'k}(q) c_{k+q,\sigma}^{\dagger} c_{k'-q,\sigma'} c_{k',\sigma'\sigma} \]  

where

\[ \int_{k_{el}} \int_{k_{el}'} \int_{q_{\omega}} = \int \frac{d^{2}k}{(2\pi)^{2}} \int_{-\infty}^{\infty} d\omega \]  

and \(c_{k,\sigma}^{\dagger}\) and \(c_{k,\sigma}\) are Grassmann variables for fermion with momentum and energy \(k = (k, i\omega)\) and spin \(\sigma\), and \(g_{F}, g_{E}\) and \(g_{C}\) are coupling constants for forward, exchange and Cooper processes. Denoting patches of the size \(\Lambda \times \Lambda\) as \(\Lambda_{j}\) (\(j\) is an index, which is the integration for momenta \(k, k'\) in eqs. (1c), (1d) and (1e) are performed in the region where \(k, k+q \in \Lambda_{j}\) and \(k', k'-q \in \Lambda_{j}\) are satisfied. \(S_{\text{Cooper}}\) is a term related to Cooper instability in case of attractive interaction. Regarding patches in two dimensions as analogs of branches in one dimension, we can make correspondence from gology model to the model given by the action of the form eq. (1).

Namely, \(S_{\text{forward}}\) corresponds to \(g_{2}\) and \(g_{4}\)-terms, and \(S_{\text{exchange}}\) to \(g_{1}\)-term in gology model. As for \(S_{\text{forward}}\), the case \(i = j\) and \(i \neq j\) correspond to \(g_{2}\) and \(g_{4}\)-terms, respectively.\(^{10, 18}\)

In one dimension, forward scatterings, i.e., \(g_{2}\) and \(g_{4}\)-processes, lead to TL liquid, and the question is what character the low-energy excitations have in the presence of the term \(S_{\text{forward}}\) in eq. (1) in two dimensions. In the following, we consider the simplest case, in which there exists only \(i = j\) term in \(S_{\text{forward}}\), and calculate the self-energy up to the third order of interaction. In one dimension, this process leads to the spin-charge separation. Following the procedure shown graphically in fig. 2, we introduce the model described by an action of the form \(S = S_{0} + S_{I}\), where

\[ S_{0} = \sum_{\sigma} \int_{k_{el}} Z^{-1} \left( i \epsilon - \xi_{k} \right) c_{k,\sigma}^{\dagger} c_{k,\sigma} \]  

and

\[ S_{I} = -\sum_{\sigma} \int_{k_{el}} \int_{k_{el}'} \int_{q_{\omega}} c_{k+q,\sigma}^{\dagger} c_{k'-q,\sigma'} c_{k',\sigma'\sigma} \]  

Assuming \(g_{F} \sigma_{k'k}(q)\) is an analytic function in the vicinity of \(k = k'\) and \(q = 0\), and neglecting \(k, k'\) and \(q\)-dependences of \(g_{F} \sigma_{k'k}(q)\) in a patch, we have replaced \(g_{F} \sigma_{k'k}(q)\) to a constant \(U\) in eq. (4). The renormalization factor \(Z_{k}\) is also replaced to the constant \(Z\) in eq. (3). The origin of momentum and \(k_{x}\) and \(k_{y}\)-axes are taken as shown in fig. 2, and momentum cut-offs are introduced as \(|k_{x}|, |k_{y}| < \Lambda\). We approximate the energy dispersion as\(^{19}\)

\[ \xi_{k} = v k_{x} + A \frac{k_{y}^{2}}{2} \]  

Furthermore, we define cut-off energies \(\epsilon_{x}\) and \(\epsilon_{y}\) and the constant \(\bar{U}\) for later convenience as

\[ \epsilon_{x} = v \Lambda, \quad \epsilon_{y} = A \Lambda^{2}, \quad \bar{U} = U \Lambda^{2} \]  

Using this model, we evaluate the contributions of the diagram shown in figs. 3(a) and 3(b), which we denote \(\Sigma_{pp}^{R}(k, \epsilon)\) and \(\Sigma_{pp}^{R}(k, \epsilon)\), respectively, where \(k = (k, 0)\).

Firstly, we consider the contribution of the diagram in fig. 3(a), \(\Sigma_{pp}^{R}(k, \epsilon)\). \(\Sigma_{pp}(k, \epsilon + i\delta)\) is expressed as

\[ \Sigma_{pp}(k, \epsilon) = \bar{U}^{3} \int_{q_{\omega}} \left[ sgn(\omega) \Im \left[ K^{R}(q, \omega) \right] \right]^{2} G_{q-k}(\omega - \epsilon) \]  

\[ + sgn(\omega - \epsilon) \left[ K^{R}(q, \omega) \right]^{2} \Im G_{q-k}(\omega - \epsilon) \]  

Here \(K^{R}(q, \omega)\) is a particle-particle correlation function
defined as

\[ K^R(q, \omega) \equiv \int_{k_x} \text{sgn}(x) G^R_{q-k}(\omega - x) \text{Im} G^R_k(x), \quad (8) \]

which is expressed approximately for \( |\omega|, |vq_x|, |Aq_x^2| \ll \hat{\epsilon}_x, \hat{\epsilon}_y \) as

\[
K^R(q, \omega) \approx \begin{cases} 
K_0 + \frac{iZ^2\omega}{4\pi v A^{1/2}(\omega - vq_x - Aq_x^2/4)^{1/2}} & (\omega - vq_x - Aq_x^2/4 > 0) \\
K_0 + \frac{Z^2\omega}{4\pi v A^{1/2}(\omega + vq_x + Aq_x^2/4)^{1/2}} & (\omega - vq_x - Aq_x^2/4 < 0),
\end{cases} \tag{9}
\]

where

\[
K_0 \equiv \lim_{q \to 2k_F} \lim_{w \to 0} K^R(q, w) = \frac{Z^2A}{2\pi^2v}. \tag{10}
\]

Here the sequence of the limiting procedure is important reflecting the singularity of the particle-particle correlation in the vicinity of \( q = 2k_F \). We extract the term which contains singular part of self-energy \( \Sigma^R_{pp}(k, \epsilon) \) in \( k \) and \( \epsilon \), which is denoted as \( \Sigma^R_{pp}(k, \epsilon) \) and defined as

\[
\Sigma^R_{pp}(k, \epsilon) = -U^3 \int \frac{d^2q}{(2\pi)^2} \int_0^\pi \frac{d\omega}{\pi} [K^R(q, \omega)]^2 \text{Im} G^R_{q-k}(\omega - \epsilon). \tag{11}
\]

The analytic part of the self-energy, i.e., \( \Sigma^R_{pp}(k, \epsilon) - \Sigma^R_{pp}(k, \epsilon) \), is considered to be related to various renormalizations, and whose effect can be absorbed into the renormalizations of the constants \( v, A \) and \( Z \). Substituting eq. (9) to eq. (11), we obtain

\[
\Sigma^R_{pp}(k, \epsilon) = \begin{cases} 
c_1\epsilon + \frac{iZ^2\overline{U}^3\tilde{K}_0\epsilon^2 \log |\epsilon - vK|}{8\pi^2\tilde{\epsilon}_x^2\tilde{\epsilon}_y} & (\epsilon > vK) \\
c_1\epsilon - \frac{Z^2\overline{U}^3\epsilon^3}{96\pi^2\tilde{\epsilon}_x^2\tilde{\epsilon}_y (\epsilon - vK)^{3/2}} & (\epsilon < vK), \tag{12}
\end{cases}
\]

where

\[
c_1 = \frac{U^3}{4\pi^2} \int d^2q \left[K^R(q, \omega)\right]^2 \text{Im} G^R_q(\omega) \bigg|_{\omega=0}, \tag{13}
\]

and \( \tilde{K}_0 = K_0A^{-2} \). The effect of the first term in eq. (12), \( c_1\epsilon \), can be also absorbed into the renormalizations of the constants \( v, A \) and \( Z \).

Secondly, we consider the contribution of the diagram fig. 3(b), which is denoted as \( \Sigma^R_{ph}(k, \epsilon) \) and expressed as

\[
\Sigma^R_{ph}(k, \epsilon) = U^3 \int_{q_k} \left[\text{sgn}(\omega) \text{Im} \left[\chi^R(q, \omega)\right]^2 G^R_{q+k}(\omega + \epsilon) + \text{sgn}(\omega + \epsilon) \left[\chi^A(q, \omega)\right]^2 \text{Im} G^R_{q+k}(\omega + \epsilon)\right]. \tag{14}
\]

Here \( \chi^R(q, \omega) \) is the particle-hole correlation function defined as

\[
\chi^R(q, \omega) = \int_{k_x} \left[\text{sgn}(x) G^R_{q+k}(\omega + x) \text{Im} G^R_k(x) + \text{sgn}(\omega + x) G^R_k(x) \text{Im} G^R_{q+k}(\omega + x)\right]. \tag{15}
\]

Substituting the energy dispersion given by eq. (5), we obtain the expressions for \( \chi^R(q, \omega) \) in case \( |\omega|, |vq_x|, |Aq_x^2| \ll \hat{\epsilon}_x, \hat{\epsilon}_y \) as

\[
\chi^R(q, \omega) = \begin{cases} 
\chi_0 - \frac{Z^2\omega}{4\pi^2vAq_y} \log \left(\frac{AAq_y - \omega + vq_x}{-AAq_y - \omega + vq_x}\right) & (|\omega - vq_x| < AA|q_y|) \\
\chi_0 - \frac{Z^2\omega}{4\pi^2vAq_y} \log \left(\frac{AAq_y - \omega + vq_x}{AAq_y + \omega - vq_x}\right) + i\pi \text{sgn}|q_y| & (|\omega - vq_x| > AA|q_y|). \tag{16}
\end{cases}
\]

Here \( \chi_0 \) is defined as

\[
\chi_0 \equiv \lim_{q \to 0 \omega \to 0} \chi^R(q, \omega) = -\frac{Z^2A}{2\pi^2v}, \tag{17}
\]

which is proportional to the density of state at the Fermi energy. In the same way as in the case of the evaluation of \( \Sigma^R_{pp}(k, \epsilon) \), we obtain the singular part of \( \Sigma^R_{ph}(k, \epsilon) \) by

\[
\Sigma^R_{ph'}(k, \epsilon)
\]
\[= U^3 \int \frac{d^2 q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} [\chi^R(q, \omega)]^2 \text{Im} G^R_{q+k}(\omega + \epsilon), \tag{18}\]

which is evaluated from eqs. (16) and (18) in case of \(\epsilon > v k\), for example, as follows;

\[\Sigma^R_{ph}(k, \epsilon) \simeq c_2 \epsilon k^3 \frac{\epsilon^2}{8\pi^2 v^2 A} \left\{ \log \frac{\Lambda A q_y + (\epsilon - v k)}{\Lambda A q_y - (\epsilon - v k)} - i\pi \right\} \]

\[\simeq c_2 \epsilon + \frac{iU^3 Z^5 \chi_0^2 \epsilon^2}{8\pi^2 v^2 A} \log \frac{\Lambda A^2}{|\epsilon - v k|} + \frac{i(\log 2)U^2 Z^3 \chi_0^2 \epsilon^2}{24\pi^5 v^3 A} \frac{\epsilon^3}{|\epsilon - v k|}, \tag{19}\]

where

\[c_2 \equiv \frac{U^3}{4\pi^3} \int d^2 q [\chi^R(q, \omega)]^2 \text{Im} G^R_{q}(\omega) \bigg|_{\omega = 0}. \tag{20}\]

Evaluating \(\Sigma^R_{ph}(k, \epsilon)\) for \(\epsilon < v k\) in the same way, we obtain the final result as

\[\Sigma^R_{ph}(k, \epsilon) \simeq c_2 \epsilon + \frac{iU^5 Z^3 \chi_0^2 \epsilon^2}{8\pi^3 \epsilon^3} \log \frac{\Lambda A}{|\epsilon - v k|} + \frac{i(\log 2)U^2 Z^3 \chi_0^2 \epsilon^2}{24\pi^5 v^3 A} \frac{\epsilon^3}{|\epsilon - v k|}, \tag{21}\]

where \(\chi_0 = \chi_0 \Lambda^{-2}\).

As for second-order term, we can obtain the singular part in the same way as follows;

\[\Sigma^R_{2nd}(k, \epsilon) \simeq c_3 \epsilon - \frac{iU^2 Z^3 \chi_0^2}{16\pi^3 v^2 A} \log \frac{\Lambda A^2}{|\epsilon - v k|}, \tag{22}\]

where

\[c_3 \equiv \frac{U^2}{4\pi^3} \int d^2 q K^R(q, \omega) \text{Im} G^R_{q}(\omega) \bigg|_{\omega = 0}. \tag{23}\]

From eqs. (12) and (21), we have to notice that there exists a region where \(\text{Im} \Sigma^R_{ph}(k, \epsilon)\) has a positive value, although \(\text{Im} \Sigma^R_{ph}(k, \epsilon)\) has to be negative-definite. This is due to the reason that we have considered only up to the third order. Eqs. (12), (21) and (22) implies the possibility that the spectral-weight \(\pi^{-1} |\text{Im} G^R_{q}(k, \epsilon)|\) has a two-peak structure in the vicinity of \(\epsilon = \tilde{\epsilon}_k\) due to the divergence of the self-energy at \(\epsilon = \tilde{\epsilon}_k\). The singular part of the self-energy is relevant only in the vicinity of \(\epsilon = \tilde{\epsilon}_k\), and we can define the width \(\Delta(k)\) of the structure of the spectral-weight in this vicinity as \(\Delta(k) = |\Sigma^R(k, v k + \Delta(k))|\). Evaluating \(\Delta(k)\) for \(\Sigma_{ph}\) and \(\Sigma_{2nd}\), which we denote \(\Delta_{pp}(k), \Delta_{ph}(k)\) and \(\Delta_{2nd}(k)\), respectively, we obtain

\[\Delta_{pp}(k) \simeq (Z^5/2U^3 \epsilon^3) |v k|^{3/2} \]
the two third-order diagrams do not cancel. This implies that the loop-cancellation is insufficient, and the residual contribution is relevant in the vicinity of $\epsilon = v_k$. This can be considered as follows: although for completely flat Fermi surface, the exact loop-cancellation holds even in two-dimension, there is a warping of the Fermi surface characterized by the energy scale $\tilde{\epsilon}_y$ in general. In case of $|\epsilon|, |v_k| \gg \tilde{\epsilon}_y$, the loop-cancellation is a good approximation, and the self-energy can be expanded by small parameters $\tilde{\epsilon}_y/|\epsilon|, |v_k|$. If we consider the low-energy limit, however, the effect of warping becomes crucially important; $\tilde{\epsilon}_y/|\epsilon|, |v_k|$ is no more a small parameter in case of $|\epsilon|, |v_k| \ll \tilde{\epsilon}_y$. This implies that the loop-cancellation is not suitable as a starting point to consider the low-energy excitation in two-dimension even for forward scattering model. Especially, the effect of the residual term is noticeable in the vicinity of $\epsilon = v_k$.

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