A PROOF FOR A CONJECTURE ON THE REGULARITY OF BINOMIAL EDGE IDEALS

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Abstract. In this paper we introduce the concept of clique disjoint edge sets in graphs. Then, for a graph $G$, we define the invariant $\eta(G)$ as the maximum size of a clique disjoint edge set in $G$. We show that the regularity of the binomial edge ideal of $G$ is bounded above by $\eta(G)$. This, in particular, settles a conjecture on the regularity of binomial edge ideals in full generality.

1. Introduction

Let $G$ be a graph on the vertex set $[n]$ and the edge set $E(G)$. Let also $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring over a field $\mathbb{K}$. Then, the binomial edge ideal associated to $G$, denoted by $J_G$, is the ideal in $S$ generated by all the quadratic binomials of the form $f_{ij} = x_i y_j - x_j y_i$, where $\{i, j\} \in E(G)$ and $i < j$. This class of ideals were introduced in [7] and [16], as a natural generalization of determinantal ideals, as well as the ideals generated by the adjacent 2-minors of a $2 \times n$-matrix of indeterminates.

In the meantime, many researchers have studied the algebraic properties and homological invariants of binomial edge ideals. A main goal is to understand how the invariants and properties of the ideal and the underlying graph are related, see e.g. [1, 2, 3, 4, 5, 6, 11, 14, 15, 18, 19, 20, 21] for some efforts in this direction.

One of the most interesting homological invariants associated to binomial edge ideals that has attracted much attention is the Castelnuovo-Mumford regularity, (or regularity for simplicity), namely,

$$\text{reg } S/J_G = \max \{ j - i : \beta_{i,j}(S/J_G) \neq 0 \}.$$

In [19], the second and third authors of the present paper, characterized the graphs $G$ for which $\text{reg } S/J_G = 1$. They also gave a characterization of the graphs $G$ with $\text{reg } S/J_G = 2$, in [21]. Another important result about the regularity of this class of binomial ideals appeared in [15], where the authors showed that

$$\mathcal{L}(G) \leq \text{reg } S/J_G \leq n - 1,$$

where $\mathcal{L}(G)$ denotes the sum of the lengths of longest induced paths of connected components of $G$. Recently, the upper bound $n - 1$ has been slightly improved in [5]. In [15] the authors additionally conjectured that $\text{reg } S/J_G \leq n - 2$, if $G$ is not $P_n$, the path on $n$ vertices. Later, in [11], this conjecture was proved by the
second and the third authors of this paper. On the other hand, in [19] it was shown that \( \text{reg} \, S/J_G \leq c(G) \), for the so-called closed graphs (also known as proper interval graphs), where \( c(G) \) denotes the number of maximal cliques of \( G \). Afterwards, in 2013, the following conjecture regarding the regularity of binomial edge ideals was posed by the second and third authors of this paper, (see [20, page 12] and [11, Conjecture A]).

**Conjecture 1.1.** Let \( G \) be a graph. Then

\[
\text{reg} \, S/J_G \leq c(G).
\]

Recall that a chordal graph is a graph with no induced cycle of length greater than 3. In [6], Ene and Zarojanu verified Conjecture 1.1 for a class of chordal graphs, called block graphs (i.e. chordal graphs in which any two maximal cliques intersect in at most one vertex). In [8] the conjecture was proved for the so-called fan graphs of complete graphs, another subclass of chordal graphs. Afterwards in [17], and later independently in [13], the authors verified Conjecture 1.1 for all chordal graphs. Very recently, in [9] the conjecture was proved for \( P_4 \)-free graphs.

In this paper first we supply a general upper bound for the regularity of binomial edge ideals. This bound indeed is based on a new concept that we call it compatible maps. In fact, such maps are defined from the set of all graphs to the set of non-negative integers that admit some specific properties. We also introduce the notion of clique disjoint edge set in graphs. Then, we associate to each graph \( G \), a graphical invariant denoted by \( \eta(G) \), which is defined as the maximum size of a clique disjoint edge set in \( G \). This enables us to provide a good combinatorial candidate of a compatible map which, in turn, yields a combinatorial upper bound for the regularity of binomial edge ideals. Then, in particular, we settle Conjecture 1.1 in full generality. Furthermore, we compare some of the known bounds for the regularity of binomial edge ideals in some examples. In particular, we give an infinite family \( \{G_n\}_{n=1}^{\infty} \) of graphs with

\[
\lim_{n \to \infty} (c(G_n) - \eta(G_n)) = \infty.
\]

Finally, a natural question regarding the regularity of binomial edge ideals will be posed.

Throughout the paper, all graphs are assumed to be simple (i.e. with no direction, loops and multiple edges).

2. Upper bounds for the regularity of binomial edge ideals

In this section we first introduce the concept of compatible maps from the set of all graphs to the set of non-negative integers. Then, we investigate about the regularity of binomial edge ideals considering this new concept. We also introduce the concept of clique disjoint edge sets in graphs to provide a combinatorial compatible map. This, in particular, enables us to prove Conjecture 1.1 in full generality.

In the following, for a graph \( G \) and \( T \subseteq V(G) \), we use the notation \( G - T \), for the induced subgraph of \( G \) on the vertex set \( V(G) \setminus T \). In particular, if \( T = \{v\} \), we use the notation \( G - v \) instead of \( G - \{v\} \), for simplicity. Moreover, we say that
is a free vertex of $G$, if the induced subgraph of $G$ on the vertex set $N_G(v)$ is a complete graph. Also, we set $\hat{G} = G - \mathcal{I}s(G)$, where $\mathcal{I}s(G)$ denotes the set of isolated vertices of $G$. Moreover, by $K_t$ we mean the complete graph on $t$ vertices, for every $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $G$ be a graph on $V(G) = [n]$ and $v \in [n]$. Associated to the vertex $v$, there is a graph, denoted by $G_v$, with the vertex set $V(G)$ and the edge set

$$E(G) \cup \{\{u, w\} : \{u, w\} \subseteq N_G(v)\},$$

where $N_G(v)$ denotes the set of neighbours of the vertex $v$ in $G$.

Now, in the following definition, we introduce certain maps from the set of all graphs to the set of non-negative integers $\mathbb{N}_0$. This enables us to obtain a general upper bound for the regularity of binomial edge ideals.

**Definition 2.1.** Let $\mathcal{G}$ be the set of all graphs. We call a map $\varphi : \mathcal{G} \rightarrow \mathbb{N}_0$, compatible, if it satisfies the following conditions:

(a) $\varphi(\hat{G}) \leq \varphi(G)$, for every $G \in \mathcal{G}$;

(b) if $G = \bigcup_{i=1}^t K_{n_i}$, where $n_i \geq 2$ for every $1 \leq i \leq t$, then $\varphi(G) \geq t$;

(c) if $G \neq \bigcup_{i=1}^t K_{n_i}$, then there exists $v \in V(G)$ such that

1. $\varphi(G - v) \leq \varphi(G)$, and

2. $\varphi(G_v) < \varphi(G)$.

We use the following lemma from [12]. In the following, $iv(G)$ denotes the number of non-free vertices of a graph $G$.

**Lemma 2.2.** [12, Lemma 3.4] Let $G$ be a graph and $v$ be a non-free vertex of $G$. Then, $\max\{iv(G_v), iv(G - v), iv(G_v - v)\} < iv(G)$.

We also need to fix a notation from [7] that will be used in the next theorem. Let $G$ be a graph on $[n]$ and $T \subseteq [n]$. Assume that $G_1, \ldots, G_{c_G(T)}$ are the connected components of $G - T$. Let $\hat{G}_1, \ldots, \hat{G}_{c_G(T)}$ be the complete graphs on the vertex sets $V(G_1), \ldots, V(G_{c_G(T)})$, respectively. Now, by $P_T(G)$ we mean the prime ideal

$$P_T(G) = (x_i, y_i)_{i \in T} + J_{\hat{G}_1} + \cdots + J_{\hat{G}_{c_G(T)}},$$

in the polynomial ring $S$.

Now, we are ready to state our first main theorem that establishes a general upper bound for the regularity of binomial edge ideals.

**Theorem 2.3.** Let $G$ be a graph on $[n]$ and $\varphi$ be a compatible map. Then

$$\text{reg } S/J_G \leq \varphi(G).$$

**Proof.** We prove the assertion by induction on $iv(G)$. If $iv(G) = 0$, then $G$ is a disjoint union of complete graphs. Let $\hat{G} = \bigcup_{i=1}^t K_{n_i}$, where $n_i \geq 2$ for every $1 \leq i \leq t$. It is well-known that $\text{reg } S/J_G = \text{reg } \hat{S}/J_{\hat{G}}$, where $\hat{S} = \mathbb{K}[x_i, y_i : i \in [n] \setminus \mathcal{I}s(G)]$. By [19, Theorem 2.1], we have $\text{reg } \hat{S}/J_{\hat{G}} = t$. On the other hand, we have $t \leq \varphi(\hat{G}) \leq \varphi(G)$, by Definition 2.1, parts (a) and (b). Therefore, in this case the assertion holds.
Now, we assume that \( iv(G) > 0 \). Let \( v \in [n] \) be the desired vertex for \( \varphi \) in condition (c) in Definition 2.1.

Let \( Q_1 = \bigcap_{T \subseteq [n], \ v \not\in T} P_T(G) \) and \( Q_2 = \bigcap_{v \in [n]} P_T(G) \). We have that \( Q_1 = J_{G_v}, Q_2 = (x_v, y_v) + J_{G-v} \) and also \( Q_1 + Q_2 = (x_v, y_v) + J_{G_v-v} \), see [4, Proof of Theorem 1.1] and [17, Proof of Theorem 3.5]. Therefore, the short exact sequence

\[
0 \rightarrow \frac{S}{J_G} \rightarrow \frac{S}{J_{G_v}} \oplus \frac{S_v}{J_{G-v}} \rightarrow \frac{S_v}{J_{G_v-v}} \rightarrow 0,
\]

is induced, where \( S_v = \mathbb{K}[x_i, y_i : i \in [n] \setminus \{v\}] \).

Now, the well-known regularity lemma implies that

(1) \( \text{reg} S/J_G \leq \max\{\text{reg} S/J_{G_v}, \text{reg} S_v/J_{G-v}, \text{reg} S_v/J_{G_v-v} + 1\} \).

By Lemma 2.2 and by the induction hypothesis, we get

(2) \( \text{reg} S/J_{G_v} \leq \varphi(G_v) < \varphi(G) \),

and

(3) \( \text{reg} S_v/J_{G-v} \leq \varphi(G-v) \leq \varphi(G) \).

Since \( G_v-v \) is an induced subgraph of \( G_v \), by [20, Proposition 8, part (b)] we have \( \text{reg} S_v/J_{G_v-v} \leq \text{reg} S/J_{G_v} \), and hence by (2) we get

(4) \( \text{reg} S_v/J_{G_v-v} < \varphi(G) \).

Therefore, the result follows by (1), (2), (3) and (4). \( \square \)

Next, we are going to provide a combinatorial compatible map. For this purpose, we assign a graphical invariant to a graph \( G \), denoted by \( \eta(G) \).

**Definition 2.4.** Let \( G \) be a graph and \( \mathcal{H} \subseteq E(G) \) with the property that no two elements of \( \mathcal{H} \) belong to a clique of \( G \). Then, we call the set \( \mathcal{H} \), a **clique disjoint edge set** in \( G \).

Moreover, we set

\[
\eta(G) := \max\{|\mathcal{H}| : \mathcal{H} \text{ is a clique disjoint edge set in } G\}.
\]

Now, in the next theorem, we provide a compatible map given by \( \eta(G) \).

**Theorem 2.5.** The map \( \eta : G \rightarrow \mathbb{N}_0 \) is compatible.

**Proof.** Let \( G \in \mathcal{G} \). It is clear that \( \eta(\widehat{G}) = \eta(G) \). Moreover, if \( G = \bigcup_{i=1}^t K_{n_i} \), where \( n_i \geq 2 \) for every \( 1 \leq i \leq t \), then we have that \( \eta(G) = t \). Therefore, it is enough to see that \( \eta \) satisfies condition (c) of Definition 2.1.

Assume that \( G \) is not a disjoint union of complete graphs. Therefore, there exists \( v \in V(G) \) such that \( v \) is not a free vertex of \( G \). We first observe that \( \eta(G-v) \leq \eta(G) \).

This indeed follows from the fact that every clique disjoint edge set in \( G-v \) is also a clique disjoint edge set in \( G \), since \( G-v \) is an induced subgraph of \( G \).

Now assume that \( \eta(G_v) = |\mathcal{H}| \), where \( \mathcal{H} = \{e_1, \ldots, e_{\eta(G_v)}\} \) is a clique disjoint edge set in \( G_v \). We consider the following cases:
First assume that $v \in \bigcup_{e_i \in \mathcal{H}} e_i$. Without loss of generality assume that $v \in e_1$. Note that $v \notin e_j$, for every $2 \leq j \leq \eta(G_v)$. Indeed, assume on the contrary that $v \in e_j$, for some $2 \leq j \leq \eta(G_v)$. Then, the edges $e_1$ and $e_j$ belong to a clique of $G_v$, a contradiction.

On the other hand, we have that $\mathcal{H} \setminus \{e_1\} \subseteq E(G)$. Indeed, otherwise assume that $e_j = \{u_j, w_j\} \notin E(G)$ for some $2 \leq j \leq \eta(G_v)$. Therefore, we have that $\{v, u_j\} \in E(G)$ and $\{v, w_j\} \in E(G)$. This implies that $e_1$ and $e_j$ belong to a clique of $G_v$, which is a contradiction. Also, since $v$ is not a free vertex of $G$, there exist vertices $\alpha$ and $\beta$ of $G$ such that $\{\alpha, \beta\} \subseteq N_G(v)$ and $\{\alpha, \beta\} \notin E(G)$. Now, it is observed that $\mathcal{H}' = \{\{v, \alpha\}, \{v, \beta\}, e_2, \ldots, e_{\eta(G_v)}\}$ is a clique disjoint edge set in $G$. Indeed, otherwise assume that either $\{v, \alpha\}$ and $\{v, \beta\}$ and $e_j$ belong to a clique of $G$ for some $2 \leq j, j' \leq \eta(G_v)$. Then, $e_1$ and $e_j$ or $e_1$ and $e_j'$ belong to a clique of $G_v$, a contradiction. Also, we have that $\{\{v, \alpha\}, \{v, \beta\}\} \cap \{e_2, \ldots, e_{\eta(G_v)}\} = \emptyset$, since $v \notin e_j$, for every $2 \leq j \leq \eta(G_v)$. This implies that $|\mathcal{H}'| = \eta(G_v) + 1$. Therefore, in this case we have that $\eta(G) \geq \eta(G_v) + 1$, as desired.

Next assume that $v \notin \bigcup_{e_i \in \mathcal{H}} e_i$. Now, if there exists $j = 1, \ldots, \eta(G_v)$ with $e_j = \{u_j, w_j\} \notin E(G)$, then with the same argument as used in the previous case, one could see that $\mathcal{H}' = (\mathcal{H} \setminus \{e_j\}) \cup \{\{v, u_j\}, \{v, w_j\}\}$ is a clique disjoint edge set in $G$ with $|\mathcal{H}'| = \eta(G_v) + 1$. This implies that $\eta(G) \geq \eta(G_v) + 1$. So, we may assume that $\mathcal{H} \subseteq E(G)$. Since $v$ is not a free vertex of $G$, there exist vertices $\alpha, \beta \in N_G(v)$ such that $\{\alpha, \beta\} \notin E(G)$. Notice that if for each $1 \leq i \leq \eta(G_v)$ the edges $e_i$ and $\{v, \alpha\}$ do not belong to a clique of $G$, then $\mathcal{H}_\alpha = \mathcal{H} \cup \{\{v, \alpha\}\}$ is a clique disjoint edge set in $G$. Similarly, if for each $1 \leq i \leq \eta(G_v)$ the edges $e_i$ and $\{v, \beta\}$ do not belong to a clique of $G$, then $\mathcal{H}_\beta = \mathcal{H} \cup \{\{v, \beta\}\}$ is a clique disjoint edge set in $G$. Thus, we get the desired result. Therefore, we assume that $e_i$ and $\{v, \alpha\}$ belong to a clique of $G$ and also $e_j$ and $\{v, \beta\}$ belong to a clique of $G$ for some $e_i, e_j \in \mathcal{H}$. This implies that $i = j$, otherwise $e_i$ and $e_j$ belong to a clique of $G_v$, which is a contradiction. Now, it is seen that

$$\mathcal{H}'' = (\mathcal{H} \setminus \{e_i\}) \cup \{\{v, \alpha\}, \{v, \beta\}\}$$

is a clique disjoint edge set in $G$ with $|\mathcal{H}''| = \eta(G_v) + 1$, and hence $\eta(G_v) < \eta(G)$, as desired. $\square$

Now, combining Theorem 2.3 and Theorem 2.5 we obtain:

**Corollary 2.6.** Let $G$ be a graph on $[n]$. Then

$$\text{reg} S/J_G \leq \eta(G).$$

We would like to remark that the above upper bound for the regularity could be sharp. For instance, let $G$ be the graph illustrated in Figure 1. Then, $\eta(G) = 4$. Also, $\text{reg} S/J_G = 4$ by [10, Proposition 3.8].
On the other hand, there are graphs $G$ for which $\reg S/J_G < \eta(G)$. For example, let $G$ be the closed graph illustrated in Figure 2 with $\eta(G) = 4$ and $\mathcal{L}(G) = 3$, where $\mathcal{L}(G)$ is the length of a longest induced path of $G$. Then, by [6, Theorem 2.2] we have $\reg S/J_G = 3$. In addition, $G - v$ is a closed graph too, with $\reg S_v/J_{G-v} = \mathcal{L}(G - v) = \eta(G - v) = 3$, for every vertex $v$ of $G$.

![Figure 2](image.png)

**Figure 2.** A closed graph $G$ with $\reg S/J_G = \mathcal{L}(G) < \eta(G)$.

It is clear that $\eta(G) \leq c(G)$, for every graph $G$. Therefore, as a consequence of Corollary 2.6 we get the following upper bound for the regularity of binomial edge ideals, which settles Conjecture 1.1 affirmatively.

**Corollary 2.7.** Let $G$ be a graph on $[n]$. Then

$$\reg S/J_G \leq c(G).$$

Note that there are some graphs $G$ for which $\reg S/J_G$ attains the upper bound $\eta(G)$ with $\eta(G) < c(G)$. For example, the graph $G_1$ depicted in Figure 3 has this property. Indeed, we have that $c(G_1) = 4$. Moreover, it is easily seen that $\mathcal{L}(G_1) = \eta(G_1) = 3$. Therefore, we have $\reg S/J_G = 3$, since the upper bound given in Corollary 2.6 coincides with the lower bound given in [15, Theorem 1.1].

Furthermore, we would like to construct an infinite family $\{G_n\}_{n=1}^\infty$ of graphs to show that the difference between the upper bounds $\eta(G_n)$ and $c(G_n)$ could be big enough for sufficiently large values of $n$. For this aim, let $G_1$ be the left side graph depicted in Figure 3. We follow the pictorial pattern illuminated in Figure 3 to obtain the graph $G_n$ for every $n \geq 2$, by replacing any triangle of $G_{n-1}$ by a copy of $G_1$. It is observed that $c(G_n) = 4^n$ and $\eta(G_n) \leq 3 \times 4^{n-1}$. Indeed, the latter inequality follows from the facts that $\eta(G_1) = 3$ and also the graph $G_n$ is covered by $4^{n-1}$ copies of $G_1$, for every $n \geq 2$. Thus,

$$\lim_{n \to \infty} (c(G_n) - \eta(G_n)) = \infty.$$
Finally, we would like to end this paper with asking a natural question if there is an explicit combinatorial characterization of graphs $G$ with $\mathcal{L}(G) = \eta(G)$. Notice that finding such characterization yields a precise formula for the regularity of the desired class of graphs. It is worth mentioning here that a characterization of chordal graphs $G$ with $\mathcal{L}(G) = c(G)$ was given in [17, Theorem 4.2]. Such graphs are called \textit{strong interval} graphs, which is clear that they satisfy the equality $\mathcal{L}(G) = \eta(G)$ as well.

\textbf{Acknowledgments:} The authors would like to thank the Institute for Research in Fundamental Sciences (IPM) for financial support. The research of the second author was in part supported by a grant from IPM (No. 99130113). The research of the third author was in part supported by a grant from IPM (No. 99050211).

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