The Sojourn Time Problem for a $p$-Adic Random Walk and its Applications

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Received June 6, 2022; in final form, June 24, 2022; accepted June 24, 2022

Abstract—We consider the problem of the distribution of the sojourn time in a compact set $\mathbb{Z}_p$ in the case of a $p$-adic random walk. We rely on the results of our previous studies of the distribution of the first return time for a $p$-adic random walk and the results of Takacs on the study of the sojourn time problem for a wide class of random processes. For a $p$-adic random walk we find the mean sojourn time of the trajectory in $\mathbb{Z}_p$ and the asymptotics as $t \to \infty$ of arbitrary moments of the distribution of the sojourn time in $\mathbb{Z}_p$. We also discuss some possible applications of our results to the modeling of relaxation processes related to the conformational dynamics of protein.

DOI: 10.1134/S207004662204001X

Key words: $p$-adic mathematical physics, $p$-adic random walk, sojourn time problem, $p$-adic models of conformational dynamics.

1. INTRODUCTION

For a wide class of random processes, the problem of finding the sojourn time of the trajectory of a process in an arbitrary subset was solved by Takacs [1–3] and is currently widely used in various applications of the theory of probability and random processes (see, for example, [4–6]). In the present study, we apply these results to a random walk on the field of $p$-adic numbers $\mathbb{Q}_p$ (a $p$-adic random walk).

The analysis of $p$-adic random processes attracts keen interest in relation to the study of processes and phenomena that have an explicit or hidden ultrametric structure in various fields of physics, biology, and informatics: spin glasses, proteins, cognitive processes, genetic code, biological evolution, classification, taxonomy, and other fields (see, for example, [7–11]). It is worth special mention that a $p$-adic random walk provides an adequate mechanism to describe the dynamics of conformational rearrangement of protein molecules in the native state. The problem, considered in the present paper, of the sojourn time of the trajectory of a random walk in a given domain is directly related to such processes. It is known that ultrametric modeling of the conformational dynamics of protein is based on the description of the dynamics of a system as a random process in the ultrametric state space of the system, where probabilistic transitions between states are defined by the ultrametric distance between them. From the mathematical viewpoint, such a dynamics is described by the equation of $p$-adic random walk with the Vladimirov operator [8] with possible reaction terms. Similar models admitting exact analytic solutions were considered in [12–20] as models of ultrametric random walk on energy landscapes of protein molecules, models of reaction of myoglobin rebinding to small ligands, models of molecular nano-machine prototypes, etc. In paper [16] devoted to the description of experiments on the spectral diffusion in proteins, the variation in the absorption frequency of a chromophore marker bound to the active center of protein is associated with the average number of returns of the protein trajectory.

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to a given domain in the conformational space that is directly related to the sojourn time of the trajectory in the given domain of conformations. Therefore, in the general case, the problem of the sojourn time of protein in a given conformational domain is directly related to the description of processes in protein that can occur only if the protein is in a certain conformational subspace.

In this paper, following our previous studies, we consider a $p$-adic random walk as a Markov process $\xi (t, \omega) : R_+^1 \times \Omega \rightarrow \mathbb{Q}_p (\Omega$ is a set of elementary events with a sigma algebra $\Sigma$ and a probability measure $P$) with the transition function $f(y, t|x, 0) \equiv f(y - x, t)$. Below we will omit the argument $\omega$ and write $\xi (t) \equiv \xi (t, \omega)$. The function $f(x, t)$ is the fundamental solution of the equation of $p$-adic random walk with the Vladimirov operator

$$\frac{\partial}{\partial t} f(x, t) = - \frac{1}{\Gamma_p (-\alpha)} \int_{\mathbb{Q}_p} d_y y f(y, t) - f(x, t) \frac{1}{|y - x|_{\mathbb{Q}_p}^{\alpha + 1}}, \quad (1.1)$$

where $\Gamma_p (-\alpha) = \frac{1 - p^{-\alpha - 1}}{1 - p^\alpha}$ is the $p$-adic gamma function and the integration is performed with respect to the Haar measure on the field $\mathbb{Q}_p$ [8]. Since the solution $f_{r,a}(x, t)$ of equation (1.1) with the initial condition in an arbitrary ball $f_{r,a}(x, 0) = \Omega(|x - a|_{\mathbb{Q}_p}^{-r}) \equiv \begin{cases} 1, |x - a|_{\mathbb{Q}_p} \leq p^r, \\
0, |x - a|_{\mathbb{Q}_p} > p^r. \end{cases}$ is scale- and translation-invariant, i.e., $f_{r,a}(x, t) = f_{0,0}(p^{-r} (x - a), p^{-or} t)$, we will assume without loss of generality that the initial distribution of the random process $\xi (t)$ has the form

$$\varphi(x, 0) = \Omega(|x|_{\mathbb{Q}_p}) \quad (1.2)$$

In this case, the densities of the distribution function $\varphi(x, t)$ and the transition function $f(y - x, t)$ of the process $\xi (t)$ are, respectively,

$$\varphi(x, t) = \int_{\mathbb{Q}_p} \Omega(|k|_{\mathbb{Q}_p}) \exp \left(-|k|_{\mathbb{Q}_p}^\alpha t\right) \chi (-kx) d_p k,$$

$$f(y - x, t) = \int_{\mathbb{Q}_p} \exp \left(-|k|_{\mathbb{Q}_p}^\alpha t\right) \chi (-k(y - x)) d_p k,$$

where $\chi(x)$ is an additive character of the field $\mathbb{Q}_p$ [8].

For the process $\xi (t)$, consider random processes $I_{\mathbb{Z}_p}(t)$ and $I_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t)$, defined as

$$I_{\mathbb{Z}_p}(t) = \begin{cases} 1, \xi(t) \in \mathbb{Z}_p \\
0, \xi(t) \in \mathbb{Q}_p \setminus \mathbb{Z}_p \end{cases}, \quad I_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) = \begin{cases} 1, \xi(t) \in \mathbb{Q}_p \setminus \mathbb{Z}_p \\
0, \xi(t) \in \mathbb{Z}_p \end{cases},$$

as well as random processes $\theta_{\mathbb{Z}_p}(t)$ and $\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t)$:

$$\theta_{\mathbb{Z}_p}(t) = \int_0^t I_{\mathbb{Z}_p}(t') \, dt, \quad (1.3)$$

$$\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) = \int_0^t I_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t') \, dt. \quad (1.4)$$

We will call the processes (1.3) and (1.4) the sojourn time of the trajectory in $\mathbb{Z}_p$ and $\mathbb{Q}_p \setminus \mathbb{Z}_p$ in the time interval from 0 to $t$. The goal of the present work is the study of the processes (1.3) and (1.4). In solving
this problem, we use the results of our paper [17] devoted to the solution of the problem of first return to the domain \( \mathbb{Z}_p \), as well as the results of [1–3] devoted to the problem of the sojourn time of a trajectory of a wide class of random processes in an arbitrary subset.

The paper is organized as follows. In Section 2, we formulate the main results of solving the first return problem that we obtained in [17] and use in the present study. In Section 3, we adapt the main theorem on the sojourn time, which was proved in [1–3], to the case of a \( p \)-adic random walk. In Section 4, for a \( p \)-adic random walk we find the mean sojourn time of the trajectory in \( \mathbb{Z}_p \) and asymptotics as \( t \to \infty \) of the moments of the distribution of the sojourn time in \( \mathbb{Z}_p \). In the concluding section, we discuss some possible applications of our results to modeling relaxation processes related to the conformational dynamics of protein.

2. FIRST RETURN TO THE DOMAIN \( \mathbb{Z}_p \)

Our solution of the problem of the sojourn time of the trajectory of a \( p \)-adic random walk is based on the results we obtained when solving the problem of the distribution of the first return time to the domain \( \mathbb{Z}_p \) for the process \( \xi(t) \) in [17]. In this section, we present the main results of that paper, which will be needed in what follows.

The first return time of the trajectory of a random process \( \xi(t) \) to the domain \( \mathbb{Z}_p \) is a random variable \( \tau_{Z_p} \in \mathbb{R}_+ \) defined as

\[
\tau_{Z_p} = \inf \{ t > 0 : \exists t' : 0 < t' < t, \xi(t') \in \mathbb{Q}_p \setminus \mathbb{Z}_p, \xi(t) \in \mathbb{Z}_p \}.
\]

In general, the first return time \( \tau_{Z_p} \) is a random variable in an extended sense and can take values on extended nonnegative number axis \( \mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\} \). A process \( \xi(t) \) is said to be recurrent if \( \tau_{Z_p} \) exists and its values lie in \( \mathbb{R}_+ \) with probability 1. If \( \tau_{Z_p} \) exists and takes values in \( \mathbb{R}_+ \) with probability \( q < 1 \), then the process \( \xi(t) \) is said to be nonrecurrent. For a recurrent process, the distribution density \( f(t) \) of \( \tau_{Z_p} \) is normalized to 1. For a nonrecurrent process, \( \int_0^\infty f(t) dt = q \).

**Theorem 1.** The distribution density \( f(t) \) of the first return time \( \tau_{Z_p} \) satisfies the inhomogeneous Volterra equation

\[
v(t) = \int_0^t v(t - \tau)f(\tau)d\tau + f(t), \tag{2.1}
\]

where

\[
v(t) = -\frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \frac{\varphi(x,t)}{|x|_p^{\alpha+1}} dx
\]

is the probability of transition in unit time from \( \mathbb{Q}_p \setminus \mathbb{Z}_p \) to \( \mathbb{Z}_p \) at time \( t \).

The proof of this theorem is given in [17].

Denote by

\[
J(t) = \int_{\mathbb{Z}_p} \varphi(x,t) d_{\mathbb{Z}_p}x
\]

the probability to find the trajectory in \( \mathbb{Z}_p \) at time \( t \) (we will call it the survival probability). Then (1.1) implies

\[
\frac{dJ(t)}{dt} = v(t) - B_\alpha J(t), \tag{2.2}
\]

where

\[
B_\alpha = -\frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \frac{1}{|x|_p^{\alpha+1}} dx = \frac{1 - p^{-1}}{1 - p^{-\alpha-1}}.
\]
Denoting the Laplace transforms of the functions \( v(t) \), \( f(t) \), and \( J(t) \), respectively, by \( \hat{v}(s) \), \( \hat{f}(s) \), and \( \hat{J}(s) \), from (2.1) we obtain

\[
\hat{f}(s) = \frac{\hat{v}(s)}{1 + \hat{v}(s)} \tag{2.3}
\]

In terms of Laplace transforms, (2.2) has the form

\[
\hat{v}(s) = (B_\alpha + s)J(s) - 1, \tag{2.4}
\]

where \( \hat{J}(s) \) is the Laplace transform of \( J(t) \):

\[
\hat{J}(s) = \left(1-p^{-1}\right)\sum_{n=0}^{\infty} p^{-n} \frac{p^{-\alpha n}}{s + p^{-\alpha n}}.
\]

It follows from the representation (2.4) that

\[
\hat{f}(s) = 1 - \frac{1}{(B_\alpha + s)\hat{J}(s)}. \tag{2.5}
\]

Asymptotic estimates for the density of the distribution function \( f(t) \) of the first return time are given by the following theorem, which was proved in [17]:

**Theorem 2.** The function \( f(t) \) has the following asymptotic estimates as \( t \to \infty \):

\[
\begin{align*}
A_1 t^{-\frac{2o-1}{\alpha}} (1 + o(1)) & \leq f(t) \leq B_1 t^{-\frac{2o-1}{\alpha}} (1 + o(1)) \quad \text{for } \alpha > 1, \\
A_2 t^{-\frac{1}{\alpha}} (1 + o(1)) & \leq f(t) \leq B_2 t^{-\frac{1}{\alpha}} (1 + o(1)) \quad \text{for } \alpha < 1, \\
A_3 \frac{t^{-1}}{\log t} (1 + o(1)) & \leq f(t) \leq B_3 \frac{t^{-1}}{\log t} (1 + o(1)) \quad \text{for } \alpha = 1,
\end{align*}
\]

where \( o(1) \to 0 \) as \( t \to \infty \) and \( A_i, B_i, i = 1, 2, 3 \) are some functions of \( \alpha \) and \( p \).

Note that the distribution function \( f(t) \) of the first return time has the following property. For \( \alpha \geq 1 \), we have \( \int_0^{\infty} f(t) \, dt = \hat{f}(0) = 1 \), which means that the trajectory returns with probability 1 to the domain \( \mathbb{Z}_p \) in finite time. Hence, for \( \alpha \geq 1 \), the random walk is recurrent. For \( 0 < \alpha < 1 \), we have

\[
\int_0^{\infty} f(t) \, dt = \hat{f}(0) = \frac{p}{p^\alpha} \left( \frac{p^\alpha - 1}{p - 1} \right)^2 = C_\alpha < 1,
\]

which means that, with probability \( 1 - C_\alpha \), the trajectory does not return to the domain \( \mathbb{Z}_p \) in finite time. Thus, for \( \alpha < 1 \), the random walk is nonrecurrent.

For \( \alpha \geq 1 \) we can represent the process of random walk on \( \mathbb{Q}_p \) as an infinite sequence of events consisting in successive sojourns of the trajectory in the sets \( \mathbb{Z}_p, \mathbb{Q}_p \setminus \mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Q}_p \setminus \mathbb{Z}_p, \ldots \) with random independent sojourn times \( \zeta_1, \eta_1, \zeta_2, \eta_2, \ldots \), where \( \zeta_1, \zeta_2, \ldots \), just as \( \eta_1, \eta_2, \ldots \), have the same distribution. In this case, the distribution function \( \zeta_i \) and its density are given by

\[
G(t) = \Theta(t) \left(1 - \exp\left(-B_\alpha t\right)\right), \quad g(t) = \frac{dG(t)}{dt} = B_\alpha \exp\left(-B_\alpha t\right), \tag{2.6}
\]

respectively. In terms of Laplace transforms, the functions (2.6) are rewritten as

\[
\hat{G}(s) = \frac{1}{s} \hat{g}(s), \quad \hat{g}(s) = \frac{B_\alpha}{s + B_\alpha}. \tag{2.7}
\]

The distribution function \( \eta_i \) is

\[
H(t) = \int_0^t h(t) \, dt.
\]
and is explicitly defined by the Laplace transform \( \hat{H}(s) = \frac{1}{s} \hat{h}(s) \), where \( \hat{h}(s) \) is the Laplace transform of the distribution function density \( h(t) \), which, by the convolution relations

\[
f(t) = \int_0^t g(t-\theta) h(\theta) d\theta
\]

is expressed as

\[
\hat{h}(s) = \frac{\hat{f}(s)}{g(s)} = B^{-1}_\alpha \left( s + B\alpha - \frac{1}{J(s)} \right). \tag{2.8}
\]

The sum of random variables \( \alpha_n = \sum_{i=1}^{n} \zeta_i \) is the total sojourn time in the set \( \mathbb{Z}_p \) provided that the trajectory hits \( \mathbb{Z}_p \) precisely \( n \) times. In this case, the distribution function \( G_n(t) \) of the sum \( \alpha_n \) is determined by the Laplace transform

\[
\hat{G}_n(s) = \frac{1}{s} \left( \frac{B\alpha}{s + B\alpha} \right)^n, \tag{2.9}
\]

whose inverse is

\[
G_n(t) = 1 - \sum_{m=0}^{n-1} \frac{B\alpha}{m!} t^m \exp(-B\alpha t). \tag{2.10}
\]

In the case of a recurrent random walk \( (\alpha \geq 1) \), the sum of random variables \( t_{Q_p \setminus Z_p, n} = \sum_{i=1}^{n} \eta_i \) is the total sojourn time in the set \( Q_p \setminus Z_p \) provided that the trajectory hits \( Q_p \setminus Z_p \) precisely \( n \) times. In this case, the distribution function \( H_n(t) \) of the sum \( t_{Q_p \setminus Z_p, n} \) is determined by the Laplace transform

\[
\hat{H}_n(s) = \frac{1}{s} \hat{h}_n(s), \tag{2.11}
\]

where the Laplace transform \( \hat{h}_n(s) \) of the density \( h_n(t) \) is given by

\[
\hat{h}_n(s) = B^{-n}_\alpha \left( s + B\alpha - \frac{1}{J(s)} \right)^n. \tag{2.12}
\]

For \( \alpha < 1 \) we can also represent the process of a random walk on \( Q_p \) as a successive alternate sojourn of the trajectory in the sets \( \mathbb{Z}_p, Q_p \setminus \mathbb{Z}_p, Z_p, Q_p \setminus Z_p, \ldots \) with independent random sojourn times, defined by the sequence \( \zeta_1, \eta_1, \zeta_2, \eta_2, \ldots \). In this case, we should assume that \( \eta_1, \eta_2, \ldots \) take values on the extended nonnegative number axis \( \mathbb{R}_+ = \mathbb{R}_+ \cup \{ +\infty \} \). Note that, for any specific implementation of this process, the sequence \( \zeta_1, \eta_1, \zeta_2, \eta_2, \ldots \) is truncated at some number \( i = k \) such that the value of \( \eta_k \) on the extended number axis is equal to \( +\infty \). In this case, the distribution function \( G_n(t) \) is also determined by the formula (2.10), and the distribution function \( H(t) \) is

\[
H(t) = \begin{cases} 
\int_0^t h(t) dt, & t < +\infty \\
1, & t = +\infty
\end{cases},
\]

where \( h(t) \) is defined by the Laplace transform (2.8). Note that in this case the probability of exit of the trajectory from \( Q_p \setminus Z_p \) is different from 1 and equals

\[
\int_0^{+\infty} h(t) dt = \hat{h}(0) = \hat{f}(0) = C\alpha < 1 \text{ with } \alpha < 1.
\]
When $\alpha < 1$, we can also introduce a random variable $\beta_n$ — the total sojourn time in the set $Q_p \setminus \mathbb{Z}_p$ provided that the trajectory hits $Q_p \setminus \mathbb{Z}_p$ precisely $n$ times, taking values on the extended nonnegative number axis $\mathbb{R}_+$. Note that in this case $\beta_n$ cannot be represented as a sum of independent random variables $\eta_i$. The distribution function $H_n (t)$ of the random variable $\beta_n$ is equal to

$$H_n (t) = \begin{cases} \int_0^t h_n (t) \, dt, & t < +\infty \\ 1, & t = +\infty \end{cases},$$

where $h_n (t)$ is defined by its Laplace transform (2.12). Note that $\int_0^\infty h_n (t) \, dt = C_\alpha^n < 1$.

3. DISTRIBUTION OF THE SOJOURN TIME IN THE DOMAIN $\mathbb{Z}_p$

Denote the integral distribution functions of the processes $(1.3)$ and $(1.4)$ by $\Phi_{\mathbb{Z}_p} (\theta, t)$ and $\Phi_{Q_p \setminus \mathbb{Z}_p} (\theta, t)$, respectively, and their densities, by $\phi_{\mathbb{Z}_p} (\theta, t)$ and $\phi_{Q_p \setminus \mathbb{Z}_p} (\theta, t)$, respectively. Since

$$\theta_{\mathbb{Z}_p} (t) + \theta_{Q_p \setminus \mathbb{Z}_p} (t) = t,$$

it follows that $\phi_{\mathbb{Z}_p} (\theta, t) = \phi_{Q_p \setminus \mathbb{Z}_p} (t - \theta, t)$ and the functions $\Phi_{\mathbb{Z}_p} (\theta, t)$ and $\Phi_{Q_p \setminus \mathbb{Z}_p} (\theta, t)$ are related by

$$\Phi_{\mathbb{Z}_p} (\theta, t) = 1 - \Phi_{Q_p \setminus \mathbb{Z}_p} (t - \theta, t).$$

**Theorem 3.** The distribution functions $\Phi_{Q_p \setminus \mathbb{Z}_p} (\theta, t)$ and $\Phi_{\mathbb{Z}_p} (\theta, t)$ are given by

$$\Phi_{Q_p \setminus \mathbb{Z}_p} (\theta, t) = \sum_{n=0}^{\infty} H_n (\theta) (G_n (t - \theta) - G_{n+1} (t - \theta)), \quad (3.3)$$

$$\Phi_{\mathbb{Z}_p} (\theta, t) = 1 - \sum_{n=0}^{\infty} H_n (t - \theta) (G_n (\theta) - G_{n+1} (\theta)). \quad (3.4)$$

where $G_n (t) (n > 0)$ is defined by formula (2.10), $H_n (t) (n > 0)$ is defined by the Laplace transforms (2.11)–(2.12), and for $n = 0$ we set $G_0 (t) \equiv 1$ and $H_0 (t) = \Theta (t)$.

**Proof.** This theorem was proved in the general case in [1–3] by two methods. In view of the importance of this result, we reproduce one of the proofs in full, with its adaptation to our case of a $p$-adic random walk. The proof given here covers the cases of recurrent ($\alpha \geq 1$) and nonrecurrent ($\alpha < 1$) random walks.

Let us fix $t$. We also fix a number $\theta$, $0 \leq \theta < t$, and consider the event $\theta_{Q_p \setminus \mathbb{Z}_p} (t) \leq \theta$. Next, consider the equation

$$\theta_{\mathbb{Z}_p} (u) = t - \theta \quad (3.5)$$

for $u$. Denote by $\tau = \tau (t - \theta)$ the random variable that is the smallest of all the solutions $u \in [0, +\infty)$ satisfying the equation (3.5). For fixed $t$ and $\theta$, $\tau$ is a random variable. Let us determine the existence condition of $\tau$.

For a recurrent random walk ($\alpha \geq 1$), the solution $\tau$ exists always. One can verify this by the following arguments. Consider a specific implementation of a random process and suppose that the trajectory visits $\mathbb{Z}_p$ precisely $n + 1$ times during the time from 0 to $u$. Denote $t_{\mathbb{Z}_p, n} = \xi_1 + \xi_2 + \cdots + \xi_n$ and $t_{Q_p \setminus \mathbb{Z}_p, n} = \eta_1 + \eta_2 + \cdots + \eta_n$. In this case, $t_{\mathbb{Z}_p, n} + \Delta t = t - \theta$, where $\Delta t = \zeta_{n+1} (\text{if the system is in $\mathbb{Z}_p$ at time $u$})$ or $\Delta t = \zeta_{n+1} (\text{if the system is in $Q_p \setminus \mathbb{Z}_p$ at time $u$})$. In the first case, the solution of equation (3.5) is unique, and $\tau = u$. In the second case, equation (3.5) has an infinite number of solutions that differ by the time of the last sojourn in $Q_p \setminus \mathbb{Z}_p$, and then $\tau$ is the smallest of all solutions $\pi$ satisfying equation (3.5), where $\theta_{\mathbb{Z}_p} (\tau) = t - \theta$, $\xi (\tau) \in \mathbb{Z}_p$. Thus, $\tau$ is a current instant of time if the trajectory visits $\mathbb{Z}_p$ the $(n + 1)$th time (in the first case) or the time of the first transition $\mathbb{Z}_p \to Q_p \setminus \mathbb{Z}_p$ after the system visited $\mathbb{Z}_p$ the $(n + 1)$th time (in the second case). It is obvious that $\theta_{Q_p \setminus \mathbb{Z}_p} (\tau) = t_{Q_p \setminus \mathbb{Z}_p, n}$ and $t_{\mathbb{Z}_p, n} < \theta_{\mathbb{Z}_p} (\tau) \leq t_{\mathbb{Z}_p, n+1}$.
For a nonrecurrent random walk \((\alpha < 1)\), an event is possible in which the trajectory hits the set \(\mathbb{Q}_p \setminus \mathbb{Z}_p\) \(n\) times and then does not leave it in finite time. In this case, for sufficiently small \(\theta\), equation (3.5) may not have a solution for \(u\) in the domain \(u \in [0, +\infty)\). Then, for any \(u \in [0, +\infty)\) we have \(\theta_{\mathbb{Z}_p}(u) < t - \theta\). Then, by (3.1), it follows that \(\theta_{\mathbb{Z}_p}(u) \leq \theta_{\mathbb{Z}_p}(t) + \theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) - \theta\); for \(u = t\), this implies \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) > \theta\). If \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta\), then \(\theta_{\mathbb{Z}_p}(t) > t - \theta\), and the random variable \(\tau\) exists.

Thus, the variable \(\tau\) always exists in the probability subspace defined by the condition \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta\).

Next, taking into account the equalities \(\theta_{\mathbb{Z}_p}(\tau) = t - \theta\) and \(\theta_{\mathbb{Z}_p}(t) + \theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) = t\) and the fact that \(\theta_{\mathbb{Z}_p}(t)\) and \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t)\) are nondecreasing functions of \(t\), we have a chain of equivalent events:

\[
\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta \iff \theta_{\mathbb{Z}_p}(\tau) \leq \theta_{\mathbb{Z}_p}(t) \iff \tau \leq t \iff \theta_{\mathbb{Z}_p}(\tau) + \theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq t \iff \theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq \theta.
\]

Therefore, the probabilities of the events \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta\) and \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq \theta\) are identical:

\[
P[\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta] = P[\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq \theta].
\]

Notice that, for a fixed \(n\), the equality \(\theta_B(\tau) = t_{\mathbb{Q}_p \setminus \mathbb{Z}_p,n}\) holds provided that \(\tau_{\mathbb{Z}_p,n} < t - \theta \leq \tau_{\mathbb{Z}_p,n+1}\). Thus, the event \(\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq \theta\) is decomposed into a sum of \(n = 0, 1, 2, \ldots\) independent events \((\tau_{\mathbb{Q}_p \setminus \mathbb{Z}_p,n} \leq \theta) \cup (\tau_{\mathbb{Z}_p,n} < t - \theta \leq \tau_{\mathbb{Z}_p,n+1})\). Then

\[
P[\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(t) \leq \theta] = P[\theta_{\mathbb{Q}_p \setminus \mathbb{Z}_p}(\tau) \leq \theta] = \sum_{n=0}^{\infty} P[(\tau_{\mathbb{Q}_p \setminus \mathbb{Z}_p,n} \leq \theta) \cup (\tau_{\mathbb{Z}_p,n} < t - \theta \leq \tau_{\mathbb{Z}_p,n+1})];
\]

this implies (3.3). Then (3.2) implies (3.4), which proves Theorem 3.

4. THE MEAN SOJOURN TIME IN \(\mathbb{Z}_p\) AND THE ASYMPTOTICS AS \(t \to \infty\) OF THE MOMENTS OF THE DISTRIBUTION OF THE SOJOURN TIME IN \(\mathbb{Z}_p\)

**Theorem 4.** The mean sojourn time of a trajectory in \(\mathbb{Z}_p\) is defined by

\[
\langle \theta \rangle (t) = \int_{0}^{t} J(t') \, dt'.
\]  

**Proof.** Applying formula (3.4) and integrating by parts with regard to \(\Phi_{\mathbb{Z}_p}(0,t) = 0\) and \(\Phi_{\mathbb{Z}_p}(t,t) = 1\), we obtain

\[
\langle \theta \rangle (t) = \int_{0}^{t} \theta \, d\Phi_{\mathbb{Z}_p}(\theta,t) = \sum_{n=0}^{\infty} \int_{0}^{t} H_{n}(t-\theta)(G_{n}(\theta)-G_{n+1}(\theta)) \, d\theta.
\]  

Take the Laplace transform of (4.2). Then, taking into account (2.9) and (2.12), we obtain

\[
\langle \hat{\theta} \rangle (s) = \sum_{n=0}^{\infty} \hat{H}_{n}(s)\left(\hat{G}_{n}(s)-\hat{G}_{n+1}(s)\right)
\]

\[
= \frac{1}{s} \sum_{n=0}^{\infty} \left(s+B_{\alpha}\right)^{-n} \left(\frac{1}{s+B_{\alpha}}\right)^{n} \frac{1}{s+B_{\alpha}}
\]

\[
= \frac{1}{s} \frac{1}{s+B_{\alpha}} - \left(\frac{1}{s+B_{\alpha}} - \frac{1}{J(s)}\right) = \frac{1}{s} J(s).
\]

\(p\)-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS  Vol. 14 No. 4  2022
Passing to the Laplace originals, we obtain (4.1), which proves the theorem.

Next, we establish the asymptotic behavior of the moments \( \langle \theta^\beta \rangle (t) \) as \( t \to \infty \) for \( \beta > 0 \). To this end, we need several theorems.

**Theorem 5.** Let

\[
\Psi (\theta, t) \equiv \sum_{n=0}^{\infty} \tilde{H}_n (\theta) (G_n (t - \theta) - G_{n+1} (t - \theta)),
\]

where \( G_n (t) \) is the \( n \) fold iterated convolution of some distribution function \( G (t) \) that has a finite first moment \( \tilde{\alpha} \) and a finite second moment \( \tilde{\beta} \), and \( \tilde{H}_n (\theta) \) is the \( n \) fold iterated convolution of some distribution function \( \tilde{H} (\theta) \) such that

\[
B = \lim_{t \to \infty} \left( 1 - \tilde{H} (t) \right) t^\gamma
\]

is a finite quantity for some \( 0 < \gamma < 1 \), \( G_0 (t) \equiv 1 \), and \( H_0 (t) = \Theta (t) \). Then, for any \( x > 0 \),

\[
\lim_{t \to \infty} \Psi (t - xt^\gamma, t) = F_\gamma \left( \left( \frac{\tilde{\alpha}}{Bx} \right)^{\frac{1}{\gamma}} \right),
\]

where \( F_\gamma (y) \) is the integral function of stable distribution having the characteristic function

\[
\tilde{f}_\gamma (k) = \exp \left( - |k|^{\gamma} \left( \cos \left( \frac{\pi \gamma}{2} \right) - i \sin \left( \frac{\pi \gamma}{2} \right) \text{sign} (k) \right) \Gamma (1 - \gamma) \right).
\]

The proof of Theorem 5 is given in [21].

It follows from Theorem 5 that for the function

\[
\tilde{\Psi} (\theta, t) = 1 - \Psi (t - \theta, t)
\]

we have

\[
\lim_{t \to \infty} \tilde{\Psi} (xt^\gamma, t) = 1 - F_\gamma \left( \left( \frac{1}{B_{\alpha} Bx} \right)^{\frac{1}{\gamma}} \right). \tag{4.5}
\]

Since the function \( \tilde{\Psi} (xt^\gamma, t) = 1 - \Psi (t - xt^\gamma, t) \) is a probability measure by its meaning, it is bounded for \( x > 0 \). Therefore, the limit (4.5) converges uniformly in \( x \), and we have

\[
\frac{d}{dx} \lim_{t \to \infty} \tilde{\Psi} (xt^\gamma, t) = \lim_{t \to \infty} t^\gamma \tilde{\psi} (xt^\gamma, t) = f_\gamma \left( \left( \frac{1}{B_{\alpha} Bx} \right)^{\frac{1}{\gamma}} \right) \frac{1}{\gamma} \left( \frac{1}{B_{\alpha} Bx} \right)^{\frac{1}{\gamma}} - \frac{1}{\gamma - 1} \frac{1}{x^{\frac{\gamma - 1}{\gamma}}} , \tag{4.6}
\]

where \( f_\gamma (t) \) is the density of the function of stable distribution with the characteristic function (4.4) and \( \tilde{\psi} (\theta, t) = \frac{d}{d\theta} \tilde{\Psi} (\theta, t) \).

The distribution (2.6) has finite first two moments, namely, \( \langle t_{z_p,1} \rangle = \frac{1}{B_{\alpha}} \equiv \tilde{\alpha} \) and \( \langle t_{z_p,1}^2 \rangle = \frac{1}{B_{\alpha}^2} \equiv \tilde{\beta} \). In addition, for \( \alpha \geq 1 \), Theorem 2 implies that

\[
C_1 \leq (1 - H (t)) t^\gamma \leq C_2, \ \alpha > 1, \ \gamma = \frac{\alpha - 1}{\alpha}, \tag{4.7}
\]

where \( C_i, \ i = 1, 2, \) are some functions of \( \alpha \) and \( p \). This gives us right to apply formula (4.6) for the function \( \phi_{z_p} (xt^\gamma, t) \). Namely, taking into account (4.7) and (4.5), we obtain the following equation also for \( \alpha > 1 \) also for \( \alpha > 1 \) for \( x > 0 \):
Proof. Denoting where \( a = \Gamma(1 - \gamma) \), we write

\[
f_{\gamma}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ikt) \exp\left(-|k|^\gamma \left(\cos\left(\frac{\pi \gamma}{2}\right) - i \sin\left(\frac{\pi \gamma}{2}\right) \text{sign}(k)\right) a\right) ds
\]

\[
= \frac{1}{\pi} \text{Re} \int_0^{+\infty} \exp(-ikt) \exp\left(-a \exp\left(-i\frac{\pi \gamma}{2}\right) k^\gamma\right) dk.
\]

Applying the Cauchy theorem, we can represent this integral in the form

\[
\oint_C \exp(-itz) \exp\left(-\exp\left(-i\frac{\pi \gamma}{2}\right) z^\gamma\right) dz,
\]

where the contour \( C \) is bounded by the quarters of two circles with \( \text{Re}z > 0, \text{Im}z < 0 \) and radii \( r \) and \( R \), respectively, and two segments \([r,R]\) on the real and imaginary axes. Passing to the limit as \( r \to 0 \) and \( R \to \infty \) and applying the Jordan lemma, we obtain

\[
\int_0^{+\infty} \exp(-ikt) \exp\left(-a \exp\left(-i\frac{\pi \gamma}{2}\right) k^\gamma\right) dk = -i \int_0^{+\infty} \exp(-kt) \exp\left(-a \exp\left(-i\pi \gamma\right) k^\gamma\right) dk;
\]

hence,

\[
f_{\gamma}(t) = -\frac{1}{\pi} \text{Re} i \int_0^{+\infty} \exp(-kt) \exp\left(-a \exp\left(-i\pi \gamma\right) k^\gamma\right) dk.
\]
Expanding the exponent in a series and changing the variable $kt = x$, we obtain

$$f_\gamma(t) = -\frac{1}{\pi t} \text{Rei} \sum_{n=0}^{\infty} (-1)^n \frac{a^n \exp(-ni\pi\gamma)}{n!} t^{-n\gamma} \int_0^{+\infty} \exp(-x)x^{n\gamma}dx$$

$$= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n \sin(n\pi\gamma)}{n!} \Gamma(n\gamma + 1) t^{-n\gamma-1},$$

which implies the assertion of the lemma.

**Theorem 6.** For $\alpha > 1$ and $t \to \infty$,

$$D_{\min} t^{\frac{\alpha-1}{\alpha}} (1 + o(1)) \leq \left\langle \theta^3 \right\rangle(t) \leq D_{\max} t^{\frac{\alpha-1}{\alpha}} (1 + o(1)), \quad \beta < \frac{\alpha}{\alpha - 1}, \quad (4.11)$$

$$\tilde{D}_{\min} t^{\beta - \frac{1}{\alpha}} (1 + o(1)) \leq \left\langle \theta^3 \right\rangle(t) \leq \tilde{D}_{\max} t^{\beta - \frac{1}{\alpha}} (1 + o(1)), \quad \beta \geq \frac{\alpha}{\alpha - 1}, \quad (4.12)$$

where $D_{\min}$, $D_{\max}$, $\tilde{D}_{\min}$, $\tilde{D}_{\max}$ are some functions of $\alpha$ and $\beta$.

**Proof.** Denote $\frac{\alpha-1}{\alpha} \equiv \gamma$. Consider

$$\left\langle \theta^3 \right\rangle(t) = \int_0^t \theta^3 \frac{d\varphi_{\eta_p}(\theta, t)}{d\theta} d\theta = \int_0^t \theta^3 \varphi_{\eta_p}(\theta, t) d\theta.$$  

Let us change the variable $\theta = xt^\gamma$:

$$\left\langle \theta^3 \right\rangle(t) = t^{(\beta+1)\gamma} \int_0^{t^{1-\gamma}} x^\beta \varphi_{\eta_p}(xt^\gamma, t) dx.$$  

By (4.8),

$$M_{\min}(t) (1 + o(1)) \leq \left\langle \theta^3 \right\rangle(t) \leq M_{\max}(t) (1 + o(1)),$$

$$M_{\min}(t) = \frac{1}{\gamma} \left( \frac{1}{B_\alpha C_i} \right)^{\frac{1}{\gamma}} t^{\beta \gamma} \int_0^{t^{1-\gamma}} x^{\beta-1} x^{-\frac{1}{\gamma}} \min_{i=1,2} \left\{ \left( \frac{1}{B_\alpha C_i} \right)^{\frac{1}{\gamma}} f_\gamma \left( \left( \frac{1}{B_\alpha C_i x} \right)^{\frac{1}{\gamma}} \right) \right\} dx, \quad (4.13)$$

$$M_{\max}(t) = \frac{1}{\gamma} t^{\beta \gamma} \int_0^{t^{1-\gamma}} x^{\beta-1} x^{-\frac{1}{\gamma}} \max_{i=1,2} \left\{ \left( \frac{1}{B_\alpha C_i} \right)^{\frac{1}{\gamma}} f_\gamma \left( \left( \frac{1}{B_\alpha C_i x} \right)^{\frac{1}{\gamma}} \right) \right\} dx. \quad (4.14)$$

By Lemma 1 we have $f_\gamma \left( \left( \frac{1}{B_\alpha C_i x} \right)^{\frac{1}{\gamma}} \right) = A_i x^{\frac{\gamma+1}{\gamma}} (1 + O(x))$ as $x \to 0$; therefore, the integrands in (4.13) and (4.14) behave as $x^\beta$ as $x \to 0$. Since $f_\gamma(0)$ is finite, as $x \to \infty$, the integrand behaves as $x^{\beta-1 - \frac{1}{\gamma}} = x^{\beta-1 - \frac{\alpha}{\alpha - 1}}$. Therefore, for $i = 1, 2$, the integral

$$\int_0^{\infty} x^{\beta-1} x^{-\frac{1}{\gamma}} f_\gamma \left( \left( \frac{1}{B_\alpha C_i x} \right)^{\frac{1}{\gamma}} \right) dx$$

converges for $\beta < \frac{\alpha}{\alpha - 1}$, and in this case we have, as $t \to \infty$, 

$$-\frac{1}{\pi t} \text{Rei} \sum_{n=0}^{\infty} (-1)^n \frac{a^n \exp(-ni\pi\gamma)}{n!} t^{-n\gamma} \int_0^{+\infty} \exp(-x)x^{n\gamma}dx$$

$$= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n \sin(n\pi\gamma)}{n!} \Gamma(n\gamma + 1) t^{-n\gamma-1},$$

which implies the assertion of the lemma.
proteins were determined by the variation of the characteristic width of the spectral hole in organic matrices (see [22–25] and references cited therein). The properties of spectral diffusion in proteins were determined by the variation of the characteristic width of the spectral hole \( \sigma_\nu \) as a function of time \( t \) since the burnout time (the "observation" time). In addition, we have investigated the dependence of the characteristic width of the spectral hole \( \sigma_\nu \) on the interval between the preparation time of the sample and the burnout time of the spectral hole (the so-called "ageing" time \( t_a \)) for a fixed observation time \( t \). The main results obtained in these experiments are as follows. First, in almost all observation times, the spectral hole has the shape of a Gaussian curve. Second, the native structure is formulated in Theorem 6. Here we discuss one of possible applications of this result.

For a random variable \( X \), which proves the theorem.

\[ M_{\min} (t) = D_{\min} t^{\beta \gamma} = D_{\min} t^{\frac{\alpha - 1}{\alpha} \beta}, \]
\[ M_{\max} (t) = D_{\max} t^{\beta \gamma} = D_{\max} t^{\frac{\alpha - 1}{\alpha} \beta}, \]

for some \( D_{\min} \) and \( D_{\max} \). To find the asymptotics as \( t \to \infty \) in the case of \( \beta \geq \frac{\alpha}{\alpha - 1} \), we also should take into account the contribution of the upper limit of the integrals in (4.13) and (4.14)

\[ t^{\beta \gamma} \int_0^{t^{1-\gamma}} x^{\beta - 1} x^{\frac{1}{\gamma}} f_\gamma \left( \left( \frac{1}{B_\alpha C_\beta x} \right)^\gamma \right) dx \]
\[ = t^{\beta \gamma} \int_\epsilon^{t^{1-\gamma}} x^{\beta - 1} \frac{1}{\gamma} f_\gamma (0) dx (1 + o (1)) \]
\[ = \frac{1}{\beta - \frac{\alpha}{\gamma}} t^{\beta \gamma} t^{\frac{\beta - 1}{\gamma} (1-\gamma)} f_\gamma (0) dx (1 + o (1)) \]
\[ = \frac{1}{\beta - \frac{\alpha}{\gamma}} t^{\beta - \frac{1}{\gamma} f_\gamma (0) dx (1 + o (1))}, \]

which proves the theorem.

5. CONCLUSIONS AND POSSIBLE APPLICATIONS

The main result of this work is the asymptotic behavior as \( t \to \infty \) of the moments of the distribution of a random variable — the sojourn time of the trajectory in \( \mathbb{Z}_p \) for a \( p \)-adic random walk for \( \alpha > 1 \). This result is formulated in Theorem 6. Here we discuss one of possible applications of this result.

As already mentioned in the Introduction, one of the most important applications of the process of \( p \)-adic random walk is modeling the conformational dynamics of protein molecules in the native state. Of particular interest are processes in protein that can occur only if the protein is in a certain domain of the conformational space of states. It is known that, in many experiments, the fluctuation dynamics of a protein molecule can be investigated by following the random variation of the optical absorption frequency of a chromophore marker located in the active center of the protein. This frequency of optical absorption of the marker is sensitive to the spatial arrangement of the surrounding atoms. In experiments, one often applies the method of "frequency coloring" of a small part of markers in a sample by inducing an irreversible photochemical transition in it by pulsed monochromatic pumping. As a result, in the absorption spectrum of a sample, this part of markers turns out to be “dark,” and a spectral hole with the characteristic width on the order of the absorption bandwidth of the marker is formed in the absorption spectrum. This technique is called the method of spectral hole burnout. When the absorption frequency of the markers located in individual protein molecules of a sample randomly changes due to the rearrangements of their atomic surrounding, the spectral hole broadens. This phenomenon is called spectral diffusion. By the change in the width of the spectral hole with time, one can judge of the fluctuation dynamics of the structure under study.

Spectral diffusion was studied at low temperatures for various globular proteins embedded in different organic matrices (see [22–25] and references cited therein). The properties of spectral diffusion in proteins were determined by the variation of the characteristic width of the spectral hole \( \sigma_\nu \) as a function of time \( t \) since the burnout time (the “observation” time). In addition, we have investigated the dependence of the characteristic width of the spectral hole \( \sigma_\nu \) on the interval between the preparation time of the sample and the burnout time of the spectral hole (the so-called “ageing” time \( t_a \)) for a fixed observation time \( t \). The main results obtained in these experiments are as follows. First, in almost all observation times, the spectral hole has the shape of a Gaussian curve. Second, the native
proteins exhibit a power-law broadening of the spectral hole by the law $\sigma_\nu(t) \sim t^b$ with the characteristic exponent $b = 0.27 \pm 0.03$. Third, the exponent $b$ is almost independent of the temperature $T$ of a sample in the temperature interval from 0.1 to 4.2 K. Moreover, in the experiments, we also observed a power-law dependence $\sigma_\nu(t_a) \sim t_a^{-c}$ of the spectral hole width on the ageing time of the sample with the characteristic exponent $c = 0.07 \pm 0.01$ (for a fixed observation time and a fixed temperature of 4.2 K.). Nevertheless, the results of the experiments do not allow us to make a conclusion about the dependence of the exponent $c$ on temperature.

To describe the dependence of the characteristic width of the spectral hole on the observation time $t$, the authors of the experimental studies [22–25] proposed a model of “superposition” of two random processes. This model is based on a Wiener random process on the absorption frequency axis in which the role of time is played by the maximum deviation of the coordinate of another Wiener random process. In spite of the fact that exact mathematical implementation of such a process was not presented in these works, the authors showed that the root-mean-square deviation of the absorption frequency in such a process should depend on time as $\sigma(t) \sim t^{0.25}$. Note also that this model carries no information about the dynamics of conformational rearrangements of the surrounding of a chromophore marker and does not display the characteristic features of the ultrametric kinetics of protein. The dependence of the characteristic width of the spectral hole on ageing time is not described in this model either. In [16], the authors attempted to describe the dependence of the characteristic width of the spectral hole on the observation time and ageing time using the ultrametric concept of conformational dynamics of protein. For a fixed value of the temperature parameter $\alpha$, they could correlate the power law of $\sigma_\nu$ as a function of the observation time $t$ and ageing time $t_a$ with experiment. Nevertheless, this model is not completely consistent because, first, it contains a number of hard-to-interpret assumptions and, second, it cannot explain the independence of the exponent $b$ in the broadening law $\sigma_\nu(t) \sim t^b$ on temperature. Thus, at present there is no completely consistent model describing experiments on spectral diffusion; this fact stimulates further investigations in this direction.

Here we make an attempt to present another model for a possible description of experiments of this kind. The main assumption of the model is that the conformational dynamics of protein is determined by a random walk on the ultrametric conformational space of quasiequilibrium macromolecules, which is identified with the field of $p$-adic numbers $\mathbb{Q}_p$. The distribution function of proteins over the space of conformational states is subject to equation (1.1), where $\alpha = \frac{T_0}{T}$ is a temperature parameter, $T$ is temperature, $T_0$ is a parameter describing a temperature scale. In this case, the variation of the absorption frequency of the chromophore marker occurs if and only if the protein is in the conformational subspace described by a ring of $p$-adic integers $\mathbb{Z}_p$. When the protein is outside the conformational domain $\mathbb{Z}_p$, no variation of the absorption frequency of the chromophore marker occurs. To implement such a model, one can describe the variation of the absorption frequency of the chromophore marker by some non-Markovian Gaussian random process $\nu(\theta)$ ($\theta$ is the time variable) on the frequency axis. We impose the self-similarity condition $f(\nu, t) = t^{-h}f(\nu t^{-h}, 1)$ with $h \neq 1$ on the distribution function of such a process $f(\nu, t)$ and require the finiteness of the second moment (see, for example, [26, 27]). It follows from these conditions that the variance of such a process depends on time $\theta$ as $\langle \nu^2(\theta) \rangle_\nu = D\theta^{2h}$ for some parameter $D$. Below we assume that this process is possible if and only if the protein states belong to the conformational domain $\mathbb{Z}_p$. Thus, the dependence of the frequency $\nu(t)$ on the real time $t$ is described by the superposition of two random processes: the process $\nu(\theta)$ and the process $\theta(t)$ — the sojourn time of protein in the conformational domain $\mathbb{Z}_p$, i.e., $\nu(t) \equiv \nu(\theta(t))$. In this case, the width of the spectral hole $\sigma(t)$ is the root-mean-square deviation of the trajectory of the process from its initial position, i.e., $\sigma(t) = \left(\left(\langle \nu^2(\theta(t)) \rangle_\nu \right)_{\theta}^{\frac{1}{2}}\right)_{\theta}^{\frac{1}{2}} = \left(\left(\langle D\theta^{2h}(t) \rangle_\theta \right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$. For large times $t$ and low temperatures ($\alpha > 1$), according to Theorem 6, we have $\sigma(t) \sim t^{\frac{(\alpha-1)h}{\alpha}}$. In the limit as $\alpha \to \infty$ (at ultra-low temperatures), the exponent $\sigma(t)$ does not depend on $\alpha$ and is equal to $h$, which gives agreement with experiment for $h = b$. Note that this model automatically guarantees the Gaussian shape of the spectral hole. In addition, this makes it possible to trace the “fine” temperature dependence of the exponent in the experimentally established variation law of the spectral hole width. In the case of ageing (i.e., when between the preparation time of the sample and burnout time of the spectral hole $t_a$)
passes, the sojourn time of the protein in the conformational subspace \( \mathbb{Z}_p \) starting from \( t_a \) is \( \theta (t_a + t) - \theta (t_a) \). In this case \( \sigma (t, t_a) = \left( \left\langle \nu^2 (\theta (t_a + t) - \theta (t_a)) \right\rangle_\theta \right)^{\frac{1}{2}} = \left( D \left\langle (\theta (t_a + t) - \theta (t_a))^{2h} \right\rangle_\theta \right)^{\frac{1}{2}}. \) Our preliminary estimates of this expression (the details of which we omit here) show that \( \sigma (t, t_a) \sim t_a^\alpha \) for \( t_a \gg t \). In this case, the coincidence with the experimental value of the exponent \( c \) occurs at the values \( \alpha = \frac{b}{c} \) of the temperature parameter, which allows one to establish a correlation between \( \alpha \) and \( T \) and thus determine the value of the parameter \( T_0 \) which defines the temperature scale. In conclusion, note that a more detailed analysis of the model is of undoubted interest, which, in turn, requires its precise mathematical implementation. We hope to implement this investigation in a separate publication in the nearest future.

**FUNDING**

The study was supported in part by the Ministry of Education and Science of Russia by State assignment to educational and research institutions under project FSSS-2020-0014.

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