A NON EXPLICIT COUNTEREXAMPLE TO A PROBLEM OF QUASI-NORMALITY

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Abstract. In 1986, S.Y. Li and H. Xie proved the following theorem: Let \( k \geq 2 \) and let \( \mathcal{F} \) be a family of functions meromorphic in some domain \( D \), all of whose zeros are of multiplicity at least \( k \). Then \( \mathcal{F} \) is normal if and only if the family \( \mathcal{F} = \left\{ \frac{f^{(k)}}{f^k} : f \in \mathcal{F} \right\} \) is locally uniformly bounded in \( D \).

Here we give, in the case \( k = 2 \), a counterexample to show that if the condition on the multiplicities of the zeros is omitted, then the local uniform boundedness of \( \mathcal{F}_2 \) does not imply even quasi-normality. In addition, we give a simpler proof for the Li-Xie theorem that does not use Nevanlinna’s Theory which was used in the original proof.

1. Introduction

Marty’s Theorem characterizes normality by using the first derivative and it has an obvious geometrical meaning.

H.L. Royden, [3], extended one direction of Marty’s Theorem and proved

**Theorem 1.** Let \( \mathcal{F} \) be a family of meromorphic functions in \( D \), with the property that for each compact set \( K \subset D \), there is a positive increasing function \( h_K \) such that

\[
|f'(z)| \leq h_K(|f(z)|)
\]

for all \( f \in \mathcal{F} \) and \( z \in K \). Then \( \mathcal{F} \) is normal in \( D \).

This result was extended further in various directions. In [1] and [11], \( h_K \) is limited to only 5 values. In [4] Thm.2, \( h_K \) is replaced by a nonnegative function that needs to be bounded in a neighborhood of some \( x_0, 0 \leq x_0 < \infty \). Then, in [7] it was shown that it is enough that \( h_K \) be finite only in a single point \( x_0, x_0 > 0 < \infty \). Moreover, in [4] Thm.3, this result is extended further to higher derivatives, i.e., \( f' \) is replaced by \( |f^{(\ell)}(z)| \leq h_K(|f(z)|), f \in \mathcal{F}, z \in K, \) where \( \ell \geq 2 \) and the members of \( \mathcal{F} \) have zeros

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of multiplicity $\geq l$. The following generalization of Marty’s Theorem also deals with higher derivatives.

**Theorem 2.** Let $\mathcal{F}$ be a family of functions meromorphic on $D$ such that each $f \in \mathcal{F}$ has zeros only of multiplicity $\geq k$. Then $\mathcal{F}$ is normal in $D$ if and only if the family

$$
\mathcal{F}_k = \left\{ \frac{f^{(k)}}{1 + |f|^{k+1}} : f \in \mathcal{F} \right\}
$$

is locally uniformly bounded in $D$.

The direction $(\Rightarrow)$ in Theorem 2 is true even without the assumption that the zeros of each $f \in \mathcal{F}$ are of multiplicity at least $k$. In Section 2, we give a simpler proof for Theorem 2 without using Nevanlinna’s Theory. The condition on the multiplicities of $f \in \mathcal{F}$ is essential in the direction $(\Leftarrow)$. Indeed, let $\tilde{\mathcal{F}}_k$ be the family of all polynomials of degree at most $k-1$ in some domain $D \subset \mathbb{C}$. Then $\frac{f^{(k)}}{1 + |f|^{k+1}} = 0$ for each $f \in \tilde{\mathcal{F}}_k$, but $\tilde{\mathcal{F}}_k$ is not normal in $D$. However, $\tilde{\mathcal{F}}_k$ is a quasi-normal family in $D$ (of order $k-1$). The question that naturally arises is whether the condition (2) implies quasi-normality.

The conjecture that (2) implies quasi-normality (without the assumption on the multiplicities of the zeros) gets support also from another direction.

First let us set some notation. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$. We write $f_n \xrightarrow{\text{X}} f$ on $D$ to indicate that the sequence $\{f_n\}$ converges to $f$ in the spherical metric uniformly on compact subsets of $D$ and $f_n \Rightarrow f$ on $D$ if the convergence is in the Euclidean metric.

Let us recall the well-known result of L. Zalcman.

**Lemma 1** (Zalcman’s Lemma). A family $\mathcal{F}$ of functions meromorphic in some domain $D$ is not normal at $z_0 \in D$ if and only if there exist points $z_n$ in $D$, $z_n \to z_0$; numbers $\rho_n \to 0^+$, and functions $f_n \in \mathcal{F}$ such that

$$
f_n(z_n + \rho_n \zeta) \xrightarrow{\text{X}} g(\zeta) \quad \text{in} \quad \mathbb{C},
$$

where $g$ is a nonconstant meromorphic function in $\mathbb{C}$.

Now, suppose that $g$ is a limit function from (3), and we have some $C > 0$ and $r > 0$ such that

$$
\frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \leq C \quad \text{for every} \quad z \in \Delta(z_0, r) \quad \text{and} \quad n \in \mathbb{N}.
$$
Let us denote the poles of $g$ (if any) by $P_g$. Then

$$f_n(z_n + \rho_n \zeta) \Rightarrow g(\zeta) \quad \text{on} \quad \mathbb{C} \setminus P_g.$$  

(Here we substitute “$\Rightarrow$” by “$\Rightarrow$” since in every compact subset of $\mathbb{C} \setminus P_g$, $f_n(z_n + \rho_n \zeta)$ is holomorphic for large enough $n$).

Differentiating (5) $k$ times given

$$\rho_n^k f_n^{(k)}(z_n + \rho_n \zeta) \Rightarrow g^{(k)}(\zeta) \quad \text{in} \quad \mathbb{C} \setminus P_g.$$  

But then by (3) and (4), we get that $g^{(k)} \equiv 0$ in $\mathbb{C} \setminus P_g$ and so $g^{(k)} \equiv 0$ in $\mathbb{C}$. This implies that $g$ is a polynomial of degree at most $k - 1$. Hence, we get that the collection of all limit functions obtained by (3) is a quasi-normal family.

However, it turns out that without the condition on the multiplicities of the zeros, the family $\mathcal{F}$ of Theorem 2 is not quasi-normal.

We suffice to construct a detailed counterexample for the case $k = 2$. This is the content of Section 3.

2. Proof of Theorem 2

Assume first that $\mathcal{F}$ is locally uniformly bounded in $D$, and suppose by negation that $\mathcal{F}_k$ is not normal at some $z_0 \in D$. Then similarly to (3) we get the existence of $f_n, z_n, \rho_n$ and $g$ such that $f_n(z_n + \rho_n \zeta) \Rightarrow g(\zeta)$ in $\mathbb{C}$. With the same reasoning, we deduce that $g$ is a polynomial of degree at most $k - 1$. But now according to the condition on the multiplicities of the zeros of each $f_n$, we get that the zeros of $g$ also must be of multiplicity at least $k$. This implies that $g$ has no zeros and thus $g$ is a constant function, a contradiction.

For the proof of the opposite direction, we need the following lemma.

**Lemma 2.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of meromorphic functions in a domain $D$, satisfying $f_n \overset{X}{\Rightarrow} \infty$ in $D$. Then for every $\ell \in \mathbb{N}$, $\frac{f_n^{(\ell)}(z)}{f_n^{(\ell)}(z)} \Rightarrow 0$ in $D$.

**Proof.** We apply induction. Since $\frac{1}{f_n(z)} \Rightarrow 0$ in $D$, we can differentiate it and obtain that $\frac{f_n^{(\ell)}(z)}{f_n^{(\ell)}(z)} \Rightarrow 0$ in $D$, and this proves the case $\ell = 1$. 

Assume that the lemma holds for \( m \leq \ell \). We prove it now for the case \( m = \ell + 1 \). We have \( \frac{f(0)}{f_n(z)} \to 0 \) in \( D \), and hence, since \( f_n(z) \to \infty \) in \( D \), also \( \frac{f(0)}{f_n(z)} \to 0 \) in \( D \). Differentiating the last convergence gives

\[
\frac{f_n^{(\ell+1)}(z)}{f_n^{\ell+2}} - (\ell + 2) \frac{f_n' f_n^{(\ell)}}{f_n^2} f_n^{(\ell+1)}(z) \to 0 \quad \text{in} \quad D.
\]

The induction assumption for \( m = 1 \) and \( m = \ell \) implies that the right term in the left hand above converges uniformly to 0 on compacta of \( D \), and thus also \( \frac{f_n^{(\ell+1)}(z)}{f_n^{\ell+2}(z)} \to 0 \) in \( D \), as required.

Let us prove now the opposite direction of Theorem 2. Assume that \( \mathcal{F} \) is normal in \( D \), and suppose by negation that \( \mathcal{F}_k \) is not locally uniformly bounded in any neighborhood of some \( z_0 \in D \). Thus, there exist functions \( f_n \in \mathcal{F} \), and points \( z_n \to z_0 \) such that

\[
\frac{f_n^{(k)}(z_n)}{1 + |f_n^{k+1}(z_n)|} \to \infty \quad \text{as} \quad n \to \infty.
\]

By the normality of \( \mathcal{F} \), \( \{f_n\}_{n=1}^{\infty} \) has a subsequence that, without loss of generality, we also denote by \( \{f_n\}_{n=1}^{\infty} \), such that \( f_n \rightharpoonup f \) in \( D \).

We separate now into cases according to the nature of \( f \).

**Case 1.1** \( f(z_0) \in \mathbb{C} \).

For small enough \( r > 0 \), \( f_n^{(k)}(z) \Rightarrow f^{(k)}(z) \) in \( \Delta(z_0, r) \), and also \( 1 + |f_n^{k+1}(z)| \Rightarrow 1 + |f(z)|^{k+1} \) in \( \Delta(z_0, r) \). Since \( 1 + |f_n(z)|^{k+1} \geq 1 \), we get that \( \frac{f_n^{(k)}(z)}{1 + |f_n(z)|^{k+1}} \Rightarrow \frac{f^{(k)}(z)}{1 + |f(z)|^{k+1}} \) in \( \Delta(z_0, r) \), a contradiction to (6).

**Case 1.2** \( f(z_0) = \infty \).

Here, for small enough \( r > 0 \), \( f \) is holomorphic in \( \Delta'(z_0, r) \) and in addition \( |f_n(z)| \geq 2 \) and \( |f(z)| \geq 2 \) for large enough \( n \). Thus \( \frac{f(z)}{1 + f(z)^{k+1}} \) are holomorphic in \( \Delta(z_0, r) \) for large enough \( n \). We then get by the maximum principle that

\[
\frac{f_n^{(k)}(z)}{1 + f_n(z)^{k+1}} \Rightarrow \frac{f^{(k)}(z)}{1 + f(z)^{k+1}} \quad \text{in} \quad \Delta(z_0, r)
\]

and then for large enough \( n \),

\[
\max_{|z-z_0| \leq r/2} \frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \leq \max_{|z-z_0| \leq r/2} \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} \leq \max_{|z-z_0| \leq r/2} \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} + 1.
\]

The last expression is a positive constant, that does not depend on \( n \) and this is a contradiction to (6).
Case 2 $f = \infty$.

In this case, we get by Lemma 2 that $\frac{f^{(k)}(z)}{f_n(z)^{p+1}} \to 0$ in $D$, and this is a contradiction to (6).

3. Constructing the counterexample

We construct a sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$, such that for every $n \geq 1$ and $z \in \Delta(0, 2), \frac{|f''(z)|}{1+|f'(z)|} \leq 1$ and $\{f_n\}_{n=1}^{\infty}$ is not quasi-normal in $\Delta(0, 2)$.

Let $g_n(z) = z^n - 1, n \geq 1$. The zeros of $g_n$ are all simple, $g_n(z^{(n)}_\ell) = 0, 0 \leq \ell \leq n - 1$, where $z^{(n)}_\ell$ is the $\ell$-th root of unity of order $n$. Define for every $n \geq 1$, $h_n = g_n e^{p_n}$, where $p_n$ is a polynomial to be determined. We have $h'_n = (g'_n + g_n p'_n)e^{p_n}$, and $g'_n(z^{(n)}_\ell) \neq 0, 0 \leq \ell \leq n - 1$. We want that

$$\text{(7)} \quad p'_n(z^{(n)}_\ell) = -g''_n(z^{(n)}_\ell)/2g'_n(z^{(n)}_\ell), \quad 0 \leq \ell \leq n - 1$$

to get that $h''_n(z^{(n)}_\ell) = 0$.

We have

$$h^{(3)}_n = e^{p_n} (g^{(3)}_n + 3g''_n p_n + 3g'_n p'_n + g_n p''_n + 3g'_n p''_n + 3g''_n p''_n + g_n p''_n)$$

We want that

$$\text{(8)} \quad p''_n(z^{(n)}_\ell) = -(g^{(3)}_n + 3g''_n p'_n + 3g'_n p''_n)/3g'_n \bigg|_{z = z^{(n)}_\ell}, \quad 0 \leq \ell \leq n - 1$$

to get $h^{(3)}_n(z^{(n)}_\ell) = 0$.

Observe that when (7) is satisfied to determine $p'_n(z^{(n)}_\ell)$, then as in (7), condition (8) is in fact a condition that depends only on the values of $g_n$ and its derivatives at the points $z^{(n)}_\ell, 0 \leq \ell \leq n - 1$.

We have

$$h^{(4)}_n = e^{p_n} (g^{(4)}_n + 4g^{(3)}_n p'_n + 6g''_n p''_n + 4g'_n p^{(3)}_n + g_n p^{(4)}_n + 6g''_n p''_n + 12g'_n p''_n + 3g''_n p''_n + 3g''_n p''_n + g_n p''_n)$$

we want that

$$\text{(9)} \quad p^{(3)}_n(z^{(n)}_\ell) = -(g^{(4)}_n + 4g^{(3)}_n p'_n + 6g''_n p''_n + 6g''_n p''_n + 12g'_n p''_n + 4g'_n p^{(3)}_n) / 4g'_n \bigg|_{z = z^{(n)}_\ell}, \quad 0 \leq \ell \leq n - 1$$
to get $h_n^{(4)}(z^{(n)}_\ell) = 0$. Observe that when (7) and (8) are satisfied to determine $p_n'(z^{(n)}_\ell)$ and $p_n''(z^{(n)}_\ell)$, then also (9) is in fact a condition that depends only on the values of $g_n$ and its derivatives at the points $z^{(n)}_\ell$, $0 \leq \ell \leq n - 1$. By the theory of interpolation [5, p. 52], for every $n \geq 1$ the conditions (7), (8) and (9) can be achieved with a polynomial $p_n$ of degree at most $4n - 1$.

Now, by our construction, for every $n \geq 1$, $h_n''$ has a zero of multiplicity at least 3 at each point $z^{(n)}_\ell$, $0 \leq \ell \leq n - 1$, and so $\frac{h_n''}{h_n}$ is holomorphic (in fact, entire) in $\Delta(0, 2)$. Thus we have $\max_{z \in \Delta(0, 2)} |h_n''(z)/h_n^{3}(z)| = c_n > 0$.

Define now for every $n \geq 1$, $f_n := a_n \cdot h_n$, where $|a_n|$ is a large enough constant such that $|a_n^2| / |z|^2 \leq 1$ and such that every subsequence of $\{f_n\}_{n=1}^\infty$ is not normal at any point of $\partial \Delta = \{z : |z| = 1\}$. In fact, we can take $|a_n|$ to be so large such that $f_n \to \infty$ locally uniformly in $\mathbb{C} \setminus \partial \Delta$.

Now, for $z = z^{(n)}_\ell$, $0 \leq \ell \leq n - 1$, $f_n''(z^{(n)}_\ell) = 0$ and thus the left hand side of (2) is zero. If $z \neq z^{(n)}_\ell$, $z \in \Delta(0, 2)$, then $f_n(z) \neq 0$ and

$$\frac{|f_n''(z)|}{1 + |f_n(z)|^3} \leq \frac{|f_n''(z)|}{|f_n(z)|^3} = \frac{1}{|a_n|^2} \frac{|h_n''(z)|}{|h_n(z)|^3} \leq \frac{c_n}{|a_n|^2} \leq 1$$

and (2) is satisfied (uniformly in $\Delta(0, 2)$). This completes the proof that $\{f_n\}_{n=1}^\infty$ has the desired properties to be a counterexample.

4. Some Remarks

**Remark 1.** We have not obtained an explicit formula for $f_n$, and this explains the title of this paper.

**Remark 2.** We have shown in fact a stronger counterexample: The condition that $\left\{f_n'' : f \in \mathcal{F}\right\}$ is locally uniformly bounded does not imply quasi-normality of the family $\mathcal{F}$.

**Remark 3.** An interesting open problem is to find a differential inequality (maybe of the sort that was mentioned in this paper) that implies quasi-normality and does not imply normality.

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