BLOCH TYPE SPACES ON THE UNIT BALL OF A HILBERT SPACE

ZHENGHUA XU

Abstract. In this article, we initiate the study of Bloch type spaces on the unit ball of a Hilbert space. As applications, the Hardy-Littlewood theorem in infinite-dimensional Hilbert spaces and characterizations of some holomorphic function spaces related to the Bloch type space are presented.

1. Introduction

The classical Bloch space of holomorphic functions on the unit disk $D$ of $C$ was extended to the higher dimension cases. In 1975, K.T. Hahn introduced the notion of Bloch functions on bounded homogeneous domains in $C^n$ using terminology from differential geometry [5]. In [15, 16], R.M. Timoney studied further Bloch functions on bounded homogeneous domains in terms of the Bergman metric. In [9], S.G. Krantz and D. Ma considered function theoretic and functional analytic properties of Bloch functions on strongly pseudoconvex domain. To have a more complete insight on the theory of the Bloch space in the finite dimensional space, see the book [21] by Zhu.

Recently, Bloch functions on the unit ball of an infinite-dimensional Hilbert space have been studied in [1]. In this article, we shall continue the study in [1] and consider Bloch type spaces on the unit ball of a Hilbert space. Especially, we give four norms of the Bloch type space and show their equivalences and present some other equivalent characterizations for Bloch functions from the geometric perspective which are the infinite-dimensional generalization of [15, Theorem 3.4].

Let $B$ be the unit ball of the complex Hilbert space $E$ and $\partial B$ be the unit sphere. The class of all holomorphic functions $f : B \to C$ is denoted by $H(B)$. Denote by Aut($B$) the group of all biholomorphic mappings of $B$ onto $B$. For $0 \leq \alpha < \infty$, let $H^\infty_{\alpha}$ be the space of holomorphic functions $f \in H(B)$ satisfying

$$\sup_{x \in B}(1 - \|x\|)^\alpha|f'(x)| < \infty.$$  

When $\alpha = 0$, we write $H^\infty$ for $H^\infty_0$.

The classical $\alpha$-Bloch space is the space of holomorphic functions $F : D \to C$ satisfying

$$\|F\|_{\alpha(D)} := \sup_{z \in D}(1 - |z|^2)^\alpha|F'(z)| < +\infty.$$  

Now we introduce four seminorms of the Bloch type spaces for $f \in H(B)$.

Denote

$$\|f\|_{1,\alpha} := \sup_{x \in B}(1 - \|x\|)^\alpha\|Df(x)\|.$$  

2010 Mathematics Subject Classification. Primary 32A18; Secondary 46E15.

Key words and phrases. Bloch type space, Lipschitz space, Hardy-Littlewood theorem, Hilbert space.

This work was supported by the NNSF of China (11071230).
Definition 2. Let $f$ be a holomorphic function on a domain $\Omega$ in $E$. A disk

$$D(z_0, r) = \{ z_0 \in \mathbb{C} : |z - z_0| < r \}$$

is called a schlicht disk in the range of $f$ if there exists a holomorphic function $g : \mathbb{D} \to \Omega$ such that $f \circ g(z) = z_0 + rz$. 

\[ \|f\|_{2, \alpha} := \sup_{x \in \mathbb{B}} (1 - |x|)^\alpha |Rf(x)|, \]

and

\[ \|f\|_{3, \alpha} := \sup_{y \in \partial \mathbb{B}} \|f_y\|_{B^\alpha(B)}, \]

where $f_y(z) = f(zy)$ for $z \in \mathbb{D}$ and \( \nabla f(x) = Df \circ \varphi_z(0) \) with $\varphi_z \in \text{Aut}(\mathbb{B})$.

Note that

\[ \|f\|_{4, \alpha} := \sup_{x \in \mathbb{B}} (1 - |x|)^{\alpha - 1} \|\overline{\nabla} f(x)\|, \]

we have

\[ \|f\|_{4, \alpha} = \sup_{x \in \mathbb{B}, w \neq 0} \frac{|Df(x)(w)(1 - |x|^2)|^\alpha}{\sqrt{(1 - |x|^2)^{\alpha}}}. \]

For $\alpha > 0$, denote

\[ B^\alpha = \{ f \in H(\mathbb{B}) : \|f\|_{1, \alpha} < +\infty \}, \]

and

\[ T_\alpha = \{ f \in H(\mathbb{B}) : \|f\|_{4, \alpha} < +\infty \}. \]

Equipped with the norm $\|f\|_\alpha = |f(0)| + \|f\|_{1, \alpha}$ for $f \in B^\alpha$, the Bolch type space $B^\alpha$ becomes a Banach space as usual. Denote the class of Bloch functions defined on $\mathbb{B}$ by $B$ instead of $B^1$. The little Bloch space will be denoted by $B_0$ which consists of functions $f \in B$ such that

\[ \lim_{\|x\| \to 1^-} \|\overline{\nabla} f(x)\| = 0. \]

Now our first main result can be described as follows:

**Theorem 1.** Let $\mathbb{B}$ be the unit ball of the complex Hilbert space $E$. Let $f$ be complex-valued holomorphic on $\mathbb{B}$. For $\alpha > 0$, the three seminorms $\|f\|_{1, \alpha}$, $\|f\|_{2, \alpha}$ and $\|f\|_{3, \alpha}$ are equivalent. Moreover, for $\dim E \geq 2$,

(i) If $0 < \alpha < \frac{1}{2}$, then $f \in T_\alpha$ iff $f$ is constant.

(ii) If $\alpha = \frac{1}{2}$, then $f \in T_{1/2}$ iff $|Df(x)(y)|$ is bounded for all $x \in \mathbb{B}$ and $y \in \partial \mathbb{B}$ with ${(x, y)} = 0$.

(iii) If $\alpha > \frac{1}{2}$, then the two seminorms $\|f\|_{1, \alpha}$ and $\|f\|_{4, \alpha}$ are equivalent.
Following [10], a holomorphic function \( f: B \to \mathbb{C} \) is said to be normal if
\[
M_f = \sup \left\{ \frac{(1 - \|x\|^2)\|Df(x)\|}{1 + |f(x)|^2} : x \in B \right\} < +\infty.
\]

With these two concepts in hand, our second main result can be established as follows.

**Theorem 3.** Let \( f \in H(B) \). Then the following conditions are equivalent:

(i) \( f \) is a Bloch function;

(ii) The radii of the schlicht disks in the range of \( f \) are bounded above;

(iii) The family \( \{f \circ g : g \in H(D, B)\} \)

is a family of Bloch functions with uniformly bounded Bloch norm;

(iv) The family \( \{f \circ g - f \circ g(0) : g \in H(D, B)\} \)

is a normal family in the sense of Montel;

(v) The family \( F_f := \{h = f \circ \phi - f \circ \phi(0) : \phi \in \text{Aut}(B)\} \)

is a family of normal functions such that \( M_h \) is uniformly bounded.

**Remark 4.** The Bloch space on bounded symmetric domains in arbitrary complex Banach spaces was considered in [3]. Although we treat only Bloch functions defined on the unit ball of a Hilbert space, all results in Theorem 3 can be proved in the more general setting of bounded symmetric domains.

The remaining part of this paper is organized as follows. Theorems 1 and 3 are proved in Section 2. In Section 3, the Hardy-Littlewood theorem in infinite-dimensional Hilbert spaces is established as an application of Theorem 1. In addition, we give some equivalent characterizations for holomorphic function spaces related to the Bloch type space which are the generalizations of main results in [2, 19] in infinite-dimensional spaces. Note that we need to overcome the restriction of dimension in generalizing main results in this article.

2. **Proof of Theorems 1 and 3**

In order to prove Theorem 1 we need to establish two lemmas.

**Lemma 5.** Let \( \alpha \geq 0 \) and \( f : B \to \mathbb{C} \) be a holomorphic function.

(i) If \( |f(x)| \leq (1 - \|x\|^2)^{-\alpha} \) for all \( x \in B \), then
\[
|Df(x)(y)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha - 1/2},
\]
for all \( x \in B, y \in \partial B \) with \( \langle x, y \rangle = 0 \).

(ii) If \( |Df(x)(y)| \leq (1 - \|x\|^2)^{-\alpha} \) for all \( x \in B, y \in \partial B \) with \( \langle x, y \rangle = 0 \), then
\[
|Rf(x)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha - 1/2}, \quad \forall x \in B.
\]

(iii) If \( |f(x)| \leq (1 - \|x\|^2)^{-\alpha} \) for all \( x \in B \), then
\[
|Rf(x)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha - 1}, \quad \forall x \in B.
\]

Where \( C(\alpha) \) is constant depending on \( \alpha \).
Proof. (i) Let $x = r x' \in \mathbb{B}$ be such that $x, y \in \partial \mathbb{B}$ with $(x, y) = 0$. Let us consider the holomorphic function $F : B^2 \to \mathbb{C}$ given by $F(z_1, z_2) = f(z_1 x' + z_2 y)$. By assumption, we get $|F(z_1, z_2)| \leq (1 - |z_1|^2 - |z_1|^2)^{-\alpha}$. By [13] Lemma 6.4.6, it holds that 

$$
\left| \frac{\partial F}{\partial z_2}(r, 0) \right| \leq C(\alpha)(1 - r^2)^{-\alpha - 1/2},
$$

that is

$$
|Df(x)(y)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha - 1/2}.
$$

(ii) Based on the results on $\mathbb{C}^2$ (cf. [13] Lemma 1(a)), we can obtain the desired estimates applying the same method as in (i). (iii) is just a corollary from (i) and (ii). □

**Lemma 6.** Let $\alpha \geq 0$ and $f : \mathbb{B} \to \mathbb{C}$ be a holomorphic function. If

$$
|DRf(x)(y)| \leq (1 - \|x\|^2)^{-\alpha - 1/2},
$$

for all $x \in \mathbb{B}$ and $y \in \partial \mathbb{B}$ with $(x, y) = 0$, then

$$
|Df(x)(y)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha},
$$

for all $x \in \mathbb{B}$ and $y \in \partial \mathbb{B}$ with $(x, y) = 0$.

Proof. For $f \in H(\mathbb{B})$, we rewrite it as $\sum_{n=0}^{\infty} P_n(x)$, where $P_n$ is an $n$-homogeneous polynomial, that is, the restriction to the diagonal of a continuous $n$-linear form on the $n$-fold space $E \times \cdots \times E$. Then $R P_n = n P_n$ and $D P_n$ is $(n - 1)$-homogeneous $(n \geq 1)$, so that

$$(D R P_n)(tx') = n(D P_n)(tx') = nt^{n-1}(D P_n)(x'),$$

which implies

$$
\int_0^r (D R P_n)(tx') dt = r^n(D P_n)(x') = n(D P_n)(tx'),
$$

which leads to

$$
r D f(x)(y) = \int_0^r (D R f)(tx')(y) dt.
$$

Let $x = r x'$ with $x' \in \partial \mathbb{B}$. For $r \in [1/2, 1)$, $y \in \partial \mathbb{B}$ with $(x, y) = 0$, we have

$$
|Df(x)(y)| \leq 2 \int_0^r |(D R f)(tx')(y)| dt \leq 2 \int_0^r (1 - t^2)^{-\alpha - 1/2} dt,
$$

then

$$(1 - \|x\|^2)^\alpha |Df(x)(y)| \leq 2 \int_0^r (1 - t)^{-1/2} dt = 4(1 - \sqrt{1 - r}) < 4,$$

for $\|x\| \in [1/2, 1)$, $y \in \partial \mathbb{B}$ with $(x, y) = 0$. Hence, by the maximum principle for holomorphic mappings in Banach spaces,

$$
|Df(x)(y)| \leq 4(\frac{4}{3})^\alpha
$$

for $\|x\| \in [0, 1/2)$, $y \in \partial \mathbb{B}$ with $(x, y) = 0$, as desired. □

We now are in a position to prove Theorem 1.

**Proof of Theorem 1.** Let $x \in \mathbb{B}$ be fixed. For any $y \in \partial \mathbb{B}$, by the projection theorem, we write $y = z_1 x + z_2 x_1$ with $x_1 \in \partial \mathbb{B}$ and $(x, x_1) = 0$. For $\|x\| \geq 1/2$, we have

$$
|Df(x)(y)| \leq 2 |R f(x)| + |Df(x)(x_1)|.
$$

Suppose that $\|f\|_{2, \alpha} = 1$, by Lemma 5 (i),

$$
|DR f(x)(x_1)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha - 1/2},
$$

for $\|x\| \in [0, 1/2)$, $y \in \partial \mathbb{B}$ with $(x, y) = 0$, as desired. □
so by Lemma 4

\[ |Df(x)(x_1)| \leq C(\alpha)(1 - \|x\|^2)^{-\alpha}. \]

Hence, for \( \|x\| \geq 1/2, \)

\[ \|Df(x)\| \leq (1 + C(\alpha))(1 - \|x\|^2)^{-\alpha}. \]

By the maximum principle for holomorphic mappings in Banach spaces, we get \( \|f\|_{2,\alpha} \leq \|f\|_{1,\alpha} \leq C(\alpha). \) Hence the two seminorms \( \| \cdot \|_{1,\alpha} \) and \( \| \cdot \|_{2,\alpha} \) are equivalent.

Notice that \( zf'(z) = Rf(zy) \) for any holomorphic \( f \) defined on \( \mathbb{B} \), we have that \( \|f\|_{2,\alpha} \leq \|f\|_{3,\alpha} \). For \( |z| \geq 1/2, \) we have

\[ (1 - |z|^2)^\alpha |f'(z)| \leq 2(1 - |z|^2)^\alpha |Rf(zy)|. \]

Hence

\[ \sup_{|z| \geq 1/2} (1 - |z|^2)^\alpha |f'(z)| \leq 2 \sup_{|x| \geq 1/2} (1 - |x|^2)^\alpha |Rf(x)| \leq 2\|f\|_{2,\alpha}. \]

Combining this with the maximum principle for holomorphic functions, one can show that

\[ \|f\|_{1,\alpha} \leq (\frac{1}{3})^{2\alpha}\|f\|_{2,\alpha}. \]

Hence, the two seminorms \( \| \cdot \|_{2,\alpha} \) and \( \| \cdot \|_{3,\alpha} \) are equivalent.

Note that \( \|f\|_{1,\alpha} = \|f\|_{4,\alpha} \) for \( \dim E = 1 \). Assume that \( \dim E \geq 2 \) in the sequel.

(i) Let \( 0 < \alpha < \frac{3}{2} \) and \( f \in T_\alpha \). It holds that

\[ |Df(x)(y)| \leq \|f\|_{4,\alpha}(1 - \|x\|^2)^{1/2 - \alpha}, \]

for all \( x \in \mathbb{B} \) and \( y \in \partial \mathbb{B} \) with \( \langle x, y \rangle = 0 \), which means \( \limsup_{x \to \partial \mathbb{B}} |Df(x)(y)| = 0 \) for all \( x \in \mathbb{B} \) and \( y \in E \) with \( \langle x, y \rangle = 0 \). In fact, there holds that \( g(x) = Df(x)(y) = 0 \) for all \( x \in \mathbb{B} \) and \( y \in E \) with \( \langle x, y \rangle = 0 \). For any fixed \( x \in \partial \mathbb{B} \) and \( y \in E \) such that \( \langle x, y \rangle = 0 \), denote the slice function \( h(z) = g(xz) \) on \( \mathbb{D} \) satisfying the property \( \limsup_{z \to \partial \mathbb{D}} |h(z)| = 0 \). Applying the maximum principle to the holomorphic mapping \( h \), we see that \( h(z) = 0 \) on \( \mathbb{D} \).

Let \( x \in \mathbb{B} \) be fixed. For every \( w \in \partial \mathbb{B} \), by the projection theorem, we can write \( w = zx + y \) with \( \langle x, y \rangle = 0 \) for some \( z \in \mathbb{C}, y \in \mathbb{B} \). Now we get

\[ |Df(x)(w)| \leq |Df(x)(zx)| + |Df(x)(y)| = |z||Rf(x)|, \]

so

\[ \|Df(x)\| = \sup_{w \in \partial \mathbb{B}} |Df(x)(w)| \leq \frac{1}{\|x\||Rf(x)|} \leq \|Df(x)\|. \]

Whence \( Df(x) \) and \( \overline{\mathbb{F}} \) are complex linear. Note that \( Df : \mathbb{B} \to E^* \) is also holomorphic and thus \( Df(x) = 0 \) on \( \mathbb{B} \), as desired.

(ii) Suppose that \( |Df(x)(y)| \leq C \) for all \( x \in \mathbb{B} \) and \( y \in \partial \mathbb{B} \) with \( \langle x, y \rangle = 0 \). Fix \( x \in \mathbb{B} \). Every \( w \in E \) can be decomposed as \( w = zx + y \) with \( \langle x, y \rangle = 0 \) for some \( z \in \mathbb{C}, \) then

\[ (1 - \|x\|^2)\|w\|^2 + |\langle w, x \rangle|^2 = |z|^2\|x\|^2 + (1 - \|x\|^2)\|y\|^2. \]

It holds that

\[ \frac{|Df(x)(w)|(1 - \|x\|^2)^{1/2}}{\sqrt{(1 - \|x\|^2)\|w\|^2 + |\langle w, x \rangle|^2}} \leq \frac{|z||Rf(x)|(1 - \|x\|^2)^{1/2}}{\sqrt{|z|^2\|x\|^2 + (1 - \|x\|^2)\|y\|^2}} + \frac{|Df(x)(y)|(1 - \|x\|^2)^{1/2}}{\sqrt{|z|^2\|x\|^2 + (1 - \|x\|^2)\|y\|^2}} \leq \frac{1}{\|x\||Rf(x)|(1 - \|x\|^2)^{1/2} + |Df(x)(\frac{y}{\|y\|})|. \]
With Lemma \[ 5 \text{ (ii)}, \] we can prove (ii) following the previous idea.

(iii) Taking the analogous arguments in \[ 11 \text{ Theorem 3.8} \] and based on the results in \[ 15 \text{ Lemma 4.8}, \] we can prove easily the desired results and omit it here. \[ \square \]

Remark 7. From the proof of Theorem \[ 11 \] we see that seminorms \( \| f \|_{2,\alpha} \) and \( \| f \|_{3,\alpha} \) are equivalent for any holomorphic function \( f \) defined on the unit ball of Banach spaces.

Proof of Theorem \[ 13 \] (i)\( \Rightarrow \) (ii) Suppose \( f \) is a Bloch function on \( \mathbb{B} \). Let \( D(z_0, r) \) be a schlicht disk in the range of \( f \). Then there exists a holomorphic function \( g : \mathbb{D} \to \mathbb{B} \) such that \( f \circ g(z) = z_0 + rz \).

Applying the Schwarz lemma for holomorphic functions (cf. \[ 4 \text{ p. 287} \]), we have
\[
\left( 1 - \| g(0) \|^2 \right) \| Dg(0) \|^2 + |\langle g(0), Dg(0) \rangle|^2 \leq 1.
\]

Let \( g(0) = x \) and \( Dg(0) = w \). Thus
\[
r = |Df(x)(w)| \leq \frac{|Df(x)(w)| (1 - \| x \|^2)}{\sqrt{(1 - \| x \|^2) \| w \|^2 + |\langle w, x \rangle|^2}} \leq \| \nabla f(x) \|,
\]
which shows that the radii of the schlicht disks in the range of \( f \) are bounded above by \( Q_f = \sup_{x \in \mathbb{B}} \| \nabla f(x) \| \).

(ii)\( \Rightarrow \) (i) Suppose the radii of the schlicht disks in the range of \( f \) are bounded above by \( R \). For any fixed \( y \in \partial \mathbb{B} \), define \( g : \mathbb{D} \to \mathbb{B} \) by \( g(z) = zy \). Fix \( x \in \mathbb{B} \). By Bloch’s theorem, the holomorphic function \( f \circ \varphi_x \circ g \) has a schlicht disk in its range of radius \( B|f \circ \varphi_x \circ g)'(0)| = B|Df \circ \varphi_x(0)(y)| \), where \( B \) denotes Bloch’s constant.

Therefore, by assumption, it follows that
\[
|Df \circ \varphi_x(0)(y)| \leq \frac{R}{B},
\]
for all \( x \in \mathbb{B} \) and \( y \in \partial \mathbb{B} \), thus \( f \) is a Bloch function, as desired.

(ii)\( \Leftrightarrow \) (iii) In the proofs above, we have
\[
R \leq Q_f \leq \frac{R}{B}.
\]

Note that the schlicht disks in the range of \( f \) are exactly those disks which are schlicht disks in the range of \( f \circ g \) for some \( g \in H(\mathbb{D}, \mathbb{B}) \). The desired result follows.

(iii)\( \Leftrightarrow \) (iv) Following the arguments in \[ 15 \text{ Theorem 3.4}, \] one can prove it and we omit details.

(v)\( \Rightarrow \) (i) Note that any \( h \in \mathcal{F}_f \) satisfies \( h(0) = 0 \). By hypothesis, we have
\[
\{ \| Df \circ \phi(0) \| : \phi \in \text{Aut}(\mathbb{B}) \} = \{ \| Dh(0) \| : h \in \mathcal{F}_f \}
\]
is bounded, as claimed.

(i) \( \Rightarrow \) (v) From the inequality
\[
\| Df \circ \phi(x) \| = \| Df \circ \phi \circ \varphi_x(0)(D\varphi_x(0))^{-1} \| \leq \frac{Q_f}{1 - \| x \|^2},
\]
it follows that the family \( \mathcal{F}_f \) is a family of normal functions such that \( M_h \) is uniformly bounded above by \( Q_f \). Now the proof of the theorem is complete. \[ \square \]
3. Applications

Hardy and Littlewood \[6\] gave a characterization of holomorphic Lipschitz space \(\Lambda_\alpha(D)\) of order \(\alpha \in (0, 1]\) on the unit open disk \(D\) of the complex plane \(\mathbb{C}\), which states that a holomorphic function \(f\) in \(D\) satisfies

\[
\sup_{z, w \in D, z \neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < +\infty
\]

if and only if

\[
(3.1) \quad \sup_{z \in D} (1 - |z|^2)^{1-\alpha} |f'(z)| < +\infty.
\]

In \[8\], Krantz extended the result to harmonic functions. As an application of Theorem 1, we first establish the Hardy-Littlewood theorem in the infinite-dimensional Hilbert space. Denote

\[
\text{Lip}_\alpha := \left\{ f \in H(B) : \sup_{x, y \in B, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha} < +\infty \right\}.
\]

Theorem 8. Let \(\alpha \in (0, 1]\). Then \(\text{Lip}_\alpha = B^{1-\alpha}\).

Proof. Suppose that \(f \in B^{1-\alpha}\), then \(f \in \text{Lip}_\alpha\) by \[17\] Lemma 3.6.

Conversely, let \(f \in \text{Lip}_\alpha\) and \(x \in \partial B\). Let us consider the holomorphic function \(F : D \to \mathbb{C}\) given by \(F(z) = f(zx)\), which is in \(\Lambda_\alpha(D)\). By the classical Hardy-Littlewood theorem, we have

\[
\sup_{z \in D} (1 - |z|^2)^{1-\alpha} |F'(z)| < +\infty,
\]

which implies that

\[
\sup_{x \in B} (1 - |x|^2)^{1-\alpha} |Rf(x)| < +\infty.
\]

From Theorem \[1\] it follows that \(f \in B^{1-\alpha}\). \(\square\)

Holland and Walsh \[7\] further considered Hardy-Littlewood theorem to the limit case \(\alpha = 0\) for holomorphic Bloch space and proved that a holomorphic function \(f\) in \(D\) satisfies

\[
\sup_{z \in B} (1 - |z|^2) |f'(z)| < +\infty
\]

if and only if

\[
(3.2) \quad \sup_{z, w \in B, z \neq w} \sqrt{(1 - |z|^2)(1 - |w|^2)} \left| \frac{f(z) - f(w)}{z - w} \right| < +\infty.
\]

The extensions to higher dimensions of the Holland-Walsh result as in \(3.2\) were obtained in \[11, 13\] for holomorphic functions in the unit ball of \(\mathbb{C}^n\). Later, Pavlović found that the Holland-Walsh result holds even for an arbitrary \(C^1\)-function defined on the unit ball of \(\mathbb{R}^n\) \[12\].

In \[19\] Zhao gave a characterization of holomorphic Bloch-type spaces on the unit ball of \(\mathbb{C}^n\).

Theorem 9. Let \(0 < \alpha \leq 2\). Let \(\lambda\) be any real number satisfying the following properties:

(i) \(0 \leq \lambda \leq \alpha\) if \(0 < \alpha < 1\);
(ii) \(0 < \lambda < 1\) if \(\alpha = 1\);
(iii) \(\alpha - 1 \leq \lambda \leq 1\) if \(1 < \alpha \leq 2\).
Then a holomorphic function \( f \) on open unit ball \( \mathbb{B}^n \) of \( \mathbb{C}^n \) is such that
\[
\sup_{z \in \mathbb{B}^n} (1 - |z|)^\alpha |\nabla f(z)| < +\infty
\]
if and only if
\[
\sup_{z, w \in \mathbb{B}^n, z \neq w} ((1 - |z|)^\lambda (1 - |w|)^{\alpha - \lambda}) \frac{|f(z) - f(w)|}{|z - w|} < +\infty.
\]

In [19] the author also offered some examples to show that the conditions on \( \alpha \) and \( \lambda \) in Theorem 9 cannot be improved. In fact, Theorem 9 does hold for \( \alpha = \lambda = 0 \) and hold for any \( C^1 \)-function defined on the unit ball of any infinite dimensional Hilbert space. Recently, Dai and Wang in [2] reveal the reason in theory that some equivalent characterizations of the Bloch type space require extra conditions for \( \alpha \). In present paper, we shall show the main results in [2] still hold in our setting.

Denote
\[
S_{\alpha, \lambda} := \left\{ f \in H(\mathbb{B}) : \sup_{x, y \in \mathbb{B}, x \neq y} (1 - \|x\|)^\lambda (1 - \|y\|)^{\alpha - \lambda} \frac{|f(x) - f(y)|}{\|x - y\|} < +\infty \right\}.
\]

**Theorem 10.**

(i) Let \( \alpha, \lambda \) be any real numbers satisfying the following properties:
(a) \( 0 < \lambda < \alpha - 1 \) if \( 1 < \alpha \leq 2 \);
(b) \( 0 < \lambda \leq \alpha/2 \) if \( \alpha > 2 \).
Then \( S_{\alpha, \lambda} = B^{\lambda+1} \).

(ii) Let \( \alpha, \lambda \) be any real numbers satisfying the following properties:
(a) \( 1 < \lambda < \alpha \) if \( 1 < \alpha \leq 2 \);
(b) \( \alpha/2 < \lambda \leq \alpha \) if \( \alpha > 2 \).
Then \( S_{\alpha, \lambda} = B^{\alpha - \lambda + 1} \).

(iii) Let \( \alpha \geq 1 \). Then \( S_{\alpha, \lambda} = H^\infty \) for \( \lambda = 0 \) or \( \lambda = \alpha \).

Note that the results (i) and (ii) in Theorem 10 does hold for all harmonic functions defined on the unit ball in \( \mathbb{R}^n \). From Theorems 9 and 10 the space \( S_{\alpha, \lambda} \) is described completely for all cases \( 0 \leq \lambda \leq \alpha \).

In order to prove Theorem 10 we first need generalise a result in [20] by Theorem 1.

**Lemma 11.** Let \( \alpha > 1 \). Then \( B^\alpha = H^\infty_{\alpha-1} \).

**Proof.** Let \( x \in \partial \mathbb{B} \) and \( f \in H^\infty_{\alpha-1} \) with \( \|f\|_{H^\infty_{\alpha-1}} = 1 \). Let us consider the holomorphic function \( F : \mathbb{D} \to \mathbb{C} \) given by \( F(z) = f(zx) \). Then \( (1 - |z|^2)^{\alpha - 1}|F(z)| \leq 1 \). By [20] Proposition 7, there exists some constant \( M > 0 \) such that
\[
(1 - |z|^2)^{\alpha}|F'(z)| \leq M, \; \forall z \in \mathbb{D},
\]
that is
\[
(1 - |z|^2)^{\alpha}|Df(zx)(x)| \leq M, \; \forall z \in \mathbb{D},
\]
which implies
\[
(1 - \|y\|^2)^{\alpha}|Rf(y)| \leq M, \; \forall y \in \mathbb{B}.
\]
Applying Theorem 1 if follows that \( f \in B^\alpha \).

Conversely, let \( f \in B^\alpha \), thus we have \( F \in B^\alpha \), which is also in \( H^\infty_{\alpha-1} \) by [20] Proposition 7 again. Consequently, we have \( f \in H^\infty_{\alpha-1} \). \( \Box \)
Proof of Theorem \[17\] By Lemma \[11\] we can prove (i) and (ii) as in \[2\] and omit its details. It is easy to check (iii) if we can prove that
\[
|f(x) - f(a)| \leq 2 \frac{\|x - a\|}{1 - \|x\|}, \quad \forall x, a \in \mathbb{B},
\]
for holomorphic functions \( f \in H^\infty \) with \( \|f\|_{H^\infty} = 1 \).

Let us show inequality \[3.3\] holds. From the Schwarz lemma for holomorphic functions, we have
\[
|f(x) - f(0)| \leq 2\|x\|, \quad \forall x \in \mathbb{B}.
\]
Applying this inequality to the holomorphic function \( f \circ \varphi_a \), it holds that
\[
|f(x) - f(a)| \leq 2\|\varphi_a(x)\| \leq 2 \frac{\|a - x\|}{1 - \langle x, a \rangle} \leq 2 \frac{\|x - a\|}{1 - \|x\|}, \quad \forall x, a \in \mathbb{B},
\]
as desired. \( \square \)

REFERENCES
1. O. Blasco, P. Galindo, A. Miralles, Bloch functions on the unit ball of an infinite dimensional Hilbert space, J. Funct. Anal. 267 (2014), no. 4, 1188-1204.
2. J. Dai, B. Wang, Characterizations of some function spaces associated with Bloch type spaces on the unit ball of \( \mathbb{C}^n \), J. Inequal. Appl. 2015, 2015:330, 10 pp.
3. F. Deng, C. Ouyang, Bloch spaces on bounded symmetric domains in complex Banach spaces, Sci. China Ser. A 49 (2006), no. 11, 1625-1632.
4. I. Graham, G. Kohr, Geometric function theory in one and higher dimensions, Monographs and Textbooks in Pure and Applied Mathematics, 255. Marcel Dekker, Inc., New York, 2003.
5. K.T. Hahn, Holomorphic mappings of the hyperbolic space in the complex Euclidean space and the Bloch theorem, Canad. J. Math 27 (1975), 446-458.
6. G.H. Hardy, J.E. Littlewood, Some properties of fractional integrals II, Math. Z. 34 (1932), no. 1, 403-439.
7. F. Holland, D. Walsh, Criteria for membership of Bloch space and its subspace, BMOA. Math. Ann. 273 (1986), no. 2, 317-335.
8. S.G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, Exposition. Math. 1 (1983), no. 3, 193-260.
9. S.G. Krantz, D. Ma, Bloch functions on strongly pseudoconvex domains, Indiana Univ. Math. J. 37 (1988), no. 1, 145-163.
10. O. Lehto, K.I. Virtanen, Boundary behaviour and normal meromorphic functions, Acta Math. 97 (1957), 47-65.
11. M. Nowak, Bloch space and Möbius invariant Besov spaces on the unit ball of \( \mathbb{C}^n \), Complex Variables Theory Appl. 44 (2001), no. 1, 1-12.
12. M. Pavlović, On the Holland-Walsh characterization of Bloch functions. Proc. Edinb. Math. Soc. (2) 51 (2008), no. 2, 439-441.
13. G. Ren, C. Tu, Bloch space in the unit ball of \( \mathbb{C}^n \). Proc. Amer. Math. Soc. 133 (2005), no. 3, 719-726.
14. W. Rudin, Function theory in the unit ball of \( \mathbb{C}^n \). Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
15. R.M. Timoney, Bloch functions in several complex variables I, Bull. Lond. Math. Soc. 12 (1980) 241-267.
16. R.M. Timoney, Bloch functions in several complex variables II, J. Reine Angew. Math. 319 (1980) 1-22.
17. Z. Xu, Strong Hardy-Littlewood theorems for holomorphic mappings in Banach spaces, preprint.
18. W. Yang, C. Ouyang, Exact location of \( \alpha \)-Bloch spaces in \( L^p_a \) and \( H^p \) of a complex unit ball, Rocky Mountain J. Math. 30 (2000), no. 3, 1151-1169.
19. R. Zhao, A characterization of Bloch-type spaces on the unit ball of \( \mathbb{C}^n \), J. Math. Anal. Appl. 330 (2007), no. 1, 291-297.
20. K. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 (1993), no. 3, 1143-1177.
21. K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Grad. Texts in Math., vol. 226, SpringerVerlag, New York, 2005.
Zhenghua Xu, Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
E-mail address: xzhengh@mail.ustc.edu.cn