RICCI CURVATURE AND MONOPOLE CLASSES ON 3-MANIFOLDS

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Abstract. We prove an $L^2$-estimate of Ricci curvature in terms of harmonic 1-forms on a closed oriented Riemannian 3-manifold admitting a solution of any rescaled Seiberg-Witten equations. We also give a necessary condition to be a monopole class on some special connected sums.

1. Introduction

A second cohomology class is called a monopole class if it arises as the first Chern class of a Spin$^c$ structure for which the Seiberg-Witten equations admit a solution for every choice of a Riemannian metric. It is well-known by LeBrun [9, 10, 12] that the existence of a monopole class gives various curvature estimates of a Riemannian 4-manifold. These immediately give corresponding estimates on 3-manifolds by using the dimensional reduction.

Theorem 1.1 ([19]). Let $(M, g)$ be a smooth closed oriented Riemannian 3-manifold with $b_1(M) \geq 1$. Suppose that it admits a solution of the Seiberg-Witten equations for a Spin$^c$ structure $\mathfrak{s}$. Then

$$\int_M (s_-)_g^2 \, d\mu_g \geq \frac{16\pi^2 |c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M [\omega]^2_g \, d\mu_g},$$

where $(s_-)_g$ is $\min(s_g, 0)$ at each point and $s_g$ is the scalar curvature of $g$. Furthermore if the Seiberg-Witten invariant of $\mathfrak{s}$ is nonzero, then for a nonzero element $\omega$ in $H^1_{DR}(M)$

$$\int_M |r_g|^2 \, d\mu_g \geq \frac{8\pi^2 |c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M [\omega]^2_g \, d\mu_g},$$

where $r_g$ is the Ricci curvature of $g$.
Here, the Seiberg-Witten invariant in case of $b_1(M) = 1$ means that of the chamber for arbitrarily small perturbations. We conjectured that the above Ricci curvature estimate still holds true when $c_1(s)$ is a monopole class.

In this article, we show that it holds true if $c_1(s)$ is a strong monopole class meaning that it is the first Chern class of a Spin$^c$ structure of $M$ which admits a solution of the rescaled Seiberg-Witten equations of $s$ for any rescaling and any Riemannian metric (See Theorem 3.2).

Following LeBrun [13], we define the $f$-rescaled Seiberg-Witten equations on a 3-manifold $M$ to be

\[ \begin{cases} D_A \Phi = 0 \\ F_A = (\Phi \otimes \Phi^* - |\Phi|^2 \text{Id}) f, \end{cases} \]

where the rescaling factor $f$ is a positive smooth function on $M$. An obvious but important fact is that if the Seiberg-Witten invariant of $s$ is nonzero, then $c_1(s)$ is a strong monopole class. But it is not known yet whether every monopole class is a strong monopole class.

In general, it is very difficult to find a monopole class which has zero Seiberg-Witten invariant. Connected sums of 4-manifolds with $b_1^+ > 1$ are good candidates. In dimension 4, Bauer and Furuta [2, 1] devised a new refined invariant of Seiberg-Witten moduli space to prove the existence of a monopole class on some connected sums of Kähler surfaces. But it seems that no 3-dimensional example has been found yet. In the final section, we apply our curvature estimates to find a necessary condition to be a monopole class on some special connected sums.

For a brief introduction to the Seiberg-Witten theory, the readers are referred to [16, 20].

### 2. The Yamabe problem for modified scalar curvature

Let $(X, g)$ be a smooth closed oriented Riemannian 4-manifold and $W_+$ be the self-dual Weyl curvature. By the modified scalar curvature we mean

\[ \mathcal{S} \equiv s - \sqrt{\text{tr}(W_+)} . \]

We will denote the set of $C^{2,\alpha}$ metrics for $\alpha \in (0, 1)$ conformal to $g$ by $[g]$. Assume that there exists a metric in $[g]$ with nonpositive \( \int_X \mathcal{S} \, d\mu \). Then as observed by Gursky [6] and LeBrun [11], the standard proof of the Yamabe problem [14] proves that there exists a $C^{2,\alpha}$ metric in $[g]$ such that $\mathcal{S}$ is a nonpositive constant. As in the Yamabe problem, it is a “minimizer” realizing

\[ \mathcal{Y}(X, [g]) \equiv \inf_{\tilde{g} \in [g]} \frac{\int_X \mathcal{S}_{\tilde{g}} \, d\mu_{\tilde{g}}}{(\text{Vol}_{\tilde{g}})^{\frac{2}{n}}} \]

We also have:
Lemma 2.1. For \( r \in [2, \infty] \),
\[
\mathcal{Y}(X, [g]) = - \inf_{g \in [g]} \left( \int_X |\mathcal{G}_g|^r \, d\mu_g \right)^{\frac{1}{r}} \left( \text{Vol}_g \right)^{\frac{1}{2} - \frac{1}{r}},
\]
where the infimum is realized only by the minimizer which is unique up to a constant multiplication.

Proof. The \( L^\infty \) case is an immediate consequence of the other cases because of the inequality
\[
||\mathcal{G}_g||_{L^\infty} (\text{Vol}_g)^{\frac{1}{2}} \geq \left( \int_X |\mathcal{G}_g|^r \, d\mu_g \right)^{\frac{1}{r}} \left( \text{Vol}_g \right)^{\frac{1}{2} - \frac{1}{r}},
\]
whose equality is attained only when \( \mathcal{G}_g \) is constant.

For \( r \in [2, \infty) \), we will use the technique of Besson, Courtois, and Gallot [3]. Let \( g \) be a minimizer. Let \( \tilde{g} = u^2 g \), where \( u : X \to \mathbb{R}^+ \) is a \( C^2 \) function. Note that \( u \) satisfies the modified Yamabe equation
\[
\mathcal{G}_g u^3 = \mathcal{G}_g u + 6 \Delta_g u.
\]
Therefore
\[
\left( \int_X |\mathcal{G}_g|^r \, d\mu_g \right)^{\frac{1}{r}} \left( \text{Vol}_g \right)^{\frac{1}{2} - \frac{1}{r}} = \left( \int_X |\mathcal{G}_g|^r u^4 \, d\mu_g \right)^{\frac{1}{r}} \left( \int_X u^4 \, d\mu_g \right)^{\frac{1}{r} - \frac{1}{2}} \geq \frac{\int_X -\mathcal{G}_g u^2 \, d\mu_g}{(\int_X d\mu_g)^{\frac{1}{2}}} \geq \frac{\int_X -\mathcal{G}_g u^2 \, d\mu_g}{\text{Vol}_g^{\frac{1}{2}}}
\]
\[
= \frac{\int_X -\mathcal{G}_g + 6 \frac{1}{u} d^* du \, d\mu_g}{\text{Vol}_g^{\frac{1}{2}}} \geq \frac{\int_X -\mathcal{G}_g \, d\mu_g}{\text{Vol}_g^{\frac{1}{2}}},
\]
where the first inequality is an application of the Hölder inequality, and the equality holds if and only if \( u \) is a positive constant. It also follows that any minimizer is a constant multiple of \( g \). \( \square \)

3. Ricci curvature estimate

Let us start with the following lemma:

Lemma 3.1. Let \((M, g)\) be a smooth closed oriented Riemannian 3-manifold and \( \mathfrak{s} \) be a Spin\(^c\) structure on it. If it admits a solution for a rescaled Seiberg-Witten equations, then any \( C^{2, \alpha} \)-metric \( \tilde{g} \in [g] \) also has a solution of the rescaled Seiberg-Witten equations for \( \mathfrak{s} \).
Proof. We claim that if \((A, \Psi)\) is a solution with respect to \(g\), then \((A, e^{-\varphi} \Psi)\) is a solution with respect to \(\tilde{g} = e^{2\varphi} g\). Mapping an orthonormal frame \(\{e_1, e_2, e_3\}\) of \(g\) to an orthonormal frame \(\{e^{-\varphi} e_1, e^{-\varphi} e_2, e^{-\varphi} e_3\}\) of \(\tilde{g}\) gives a global isomorphism of two orthonormal frame bundles and hence a global isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry.

For a proof of the \(\text{Spin}^c\) Dirac equation, one is referred to [8], and the curvature equation is immediate from the fact that

\[
|F_A|_{\tilde{g}} = e^{-2\varphi}|F_A|_g = e^{-2\varphi} f|\Psi|_g^2 = f|e^{-\varphi} \Psi|_{\tilde{g}}^2,
\]

where \(f\) is the rescaling factor.

\[\square\]

**Theorem 3.2.** Let \((M, g)\) be a smooth closed oriented Riemannian 3-manifold with \(b_1(M) \geq 1\) and \(s\) be a \(\text{Spin}^c\) structure on it. Suppose that it admits a solution for the rescaled Seiberg-Witten equations for any rescaling. Then for any smooth metric \(\tilde{g}\) conformal to \(g\) and any nonzero \(\omega \in H^1_{DR}(M)\),

\[
\int_M |r_\tilde{g}|^2 d\mu_{\tilde{g}} \geq \frac{8\pi^2 |c_1(\tilde{g}) \cup [\omega]|^2}{\int_M |\omega|^2_{\tilde{g}} d\mu_{\tilde{g}}},
\]

and the equality holds if and only if \((M, \tilde{g})\) is a Riemannian submersion onto \(S^1\) with totally geodesic fiber isometric to a compact oriented surface of genus \(\geq 1\) with a non-positive constant curvature metric whose volume form is a multiple of \(*\omega\) for \(\tilde{g}\)-harmonic \(\omega\), and \([c_1(\tilde{g})]\) is a multiple of \([*\omega]\) in \(H^2_{DR}(M)\), where \(*\) denotes the Hodge star with respect to \(\tilde{g}\).

Proof. In order to prove the inequality, we may assume \(c_1(\tilde{g}) \neq 0 \in H^2(M, \mathbb{R})\).

Then the Seiberg-Witten equations have an irreducible solution for any metric in \([g]\), implying that there cannot exist a metric in \([g]\) with nonnegative scalar curvature, and hence there exists a smooth metric in \([g]\) with negative scalar curvature.

For notational convenience, let \(g\) be any smooth metric in \([g]\). Consider the product metric \(g + dt^2\) on \(M \times S^1\), where \(t \in [0, 1]\) is a global coordinate of \(S^1\).

By the previous section, there exists a \(C^{2,\alpha}\) metric \(\tilde{g} \in [g + dt^2]\) which minimizes \(\mathcal{H}(M \times S^1, [g + dt^2])\) satisfying that \(s - \sqrt{6}|W_+|\) is a negative constant.

**Lemma 3.3.** \(\tilde{g}\) is invariant under the translation along \(S^1\)-direction.

Proof. Let \(\tilde{g} = (g + dt^2)\psi\) for a positive smooth function \(\psi\) on \(M \times S^1\), and we will show \(\psi(x, t) = \psi(x, t + c)\) for \((x, t) \in M \times S^1\) for any \(c\).

Since \((g + dt^2)\psi(x, t + c)\) is also a minimizer, by Lemma 2.1 there exists a smooth positive function \(\varphi\) on \(S^1\) such that

\[
\psi(x, t + c) = \varphi(c)\psi(x, t)
\]

for any \((x, t)\). For any \((x, t)\),

\[
\int_{M \times S^1} \psi(x, t) \, d\mu_{g + dt^2} = \int_{M \times S^1} \psi(x, t + c) \, d\mu_{g + dt^2}
\]
\[ = \varphi(c) \int_{M \times S^1} \psi(x, t) \, d\mu_g + dt^2, \]

where the first equality is due to the translation invariance of \( dt^2 \). Since \( \psi > 0 \), we conclude that \( \varphi(c) = 1 \) for any \( c \). \( \square \)

We write the metric \( \hat{g} \) as the warped form \( h + f^2 dt^2 \) for \( f : M \to \mathbb{R}^+ \) where \( h \) is the metric \( f^2 g \) on \( M \). Let \( \{ e_1, e_2, e_3, e_4 = \frac{\partial}{\partial t} \} \) be a local orthonormal frame \( M \times S^1 \) with respect to \( h + dt^2 \), and \( \{ \omega^i \mid i = 1, \ldots, 4 \} \) its dual coframe. Recall the first Cartan’s structure equations:

\[ d\omega^i = -\omega^i_j \wedge \omega^j, \]

where \( \omega^i_j \) are the connection 1-forms of \( h + dt^2 \). Obviously \( \omega^i_4 \) are all zero for all \( i \). Take an orthonormal coframe of \( \hat{g} \) as \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4 \} \) and apply the first Cartan’s structure equations to \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4 \} \), then one can see that the connection 1-forms \( \hat{\omega}^i_j \) of \( \hat{g} \) are given by

\[ \hat{\omega}^i_j = \omega^i_j \quad \text{for} \ i, j = 1, 2, 3, \]

\[ \hat{\omega}^4_j = \frac{\partial f}{\partial e_j} \omega^4 \quad \text{for} \ j = 1, 2, 3. \]

Let \( (A, \Phi) \) be a solution of the \( \frac{1}{f} \)-rescaled Seiberg-Witten equations for \( s \) on \( (M, h) \), whose existence is guaranteed by Lemma 3.1. Then it is a translation-invariant solution of the Seiberg-Witten equations for \( s \) on \( (M \times S^1, h + dt^2) \).

We claim that \( (A, \hat{\Phi}) \) is a solution of the unrescaled Seiberg-Witten equations for \( s \) on \( (M \times S^1, h + dt^2) \). Let’s denote the objects of \( \hat{g} = h + f^2 dt^2 \) corresponding to that of \( h + dt^2 \) by \( \hat{\cdot} \). The \( \text{Spin}^c \) Dirac equation reads

\[ \hat{D}_A(f^{-\frac{1}{2}} \Phi) \]

\[ = \sum_{i=1}^{4} \hat{e}_i \nabla_{\hat{e}_i} (f^{-\frac{1}{2}} \Phi) \]

\[ = \sum_{i=1}^{4} \hat{e}_i \left( \frac{\partial}{\partial e_i} (f^{-\frac{1}{2}} \Phi) + \frac{1}{2} \left( \sum_{j<k} \hat{\omega}^k_j (\hat{e}_i) \hat{e}_j \hat{e}_k + A(\hat{e}_i)) f^{-\frac{1}{2}} \Phi \right) \right) \]

\[ = \sum_{i=1}^{3} e_i \left( \frac{\partial}{\partial e_i} (f^{-\frac{1}{2}} \Phi) + \frac{1}{2} \left( \sum_{j<k \leq 3} \omega^k_j (e_i) e_j e_k + A(e_i)) f^{-\frac{1}{2}} \Phi \right) \right) + \frac{\partial f}{\partial e_4} (\omega^4 (e_4) e_4) f^{-\frac{1}{2}} \Phi \]

\[ = \sum_{i=1}^{3} e_i \left( - \frac{f^{-\frac{1}{2}}}{2} \frac{\partial f}{\partial e_i} \Phi + f^{-\frac{1}{2}} \frac{\partial \Phi}{\partial e_i} + \frac{1}{2} \left( \sum_{j<k \leq 3} \omega^k_j (e_i) e_j e_k + A(e_i)) f^{-\frac{1}{2}} \Phi \right) \right) \]
\[ + \frac{1}{2} \left( \sum_{j=1}^{3} \frac{\partial f}{\partial e_j} f^j \right) f^{-\frac{1}{2} \Phi} \]
\[ = f^{-\frac{1}{2}} D_A \Phi \]
\[ = 0, \]
and the curvature equation reads
\[ F^+_A = \frac{1}{2} (F_A + (\ast_h F_A) \wedge \omega^4) \]
\[ \simeq \frac{1}{2} (F_A + (\ast_h F_A) \wedge \omega^4) \]
\[ = \frac{1}{f} (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id}) \]
\[ = (f^{-\frac{1}{2}} \Phi) \otimes (f^{-\frac{1}{2}} \Phi)^* - \frac{|f^{-\frac{1}{2}} \Phi|^2}{2} \text{Id}, \]
where the equivalence in the second line means the identification as an endomorphism of the plus spinor bundle (Mapping an orthonormal frame \( \{e_1, \ldots, e_4\} \) of \( h + dt^2 \) to an orthonormal frame \( \{e_1, \ldots, e_3, \check{e}_4\} \) of \( h + f^2 dt^2 \) gives a global isomorphism of two orthonormal frame bundles, inducing a global isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry).

Lemma 3.4.
\[ \int_{M \times S^1} \left( \frac{2}{3} s_\gamma + dt^2 - 2\sqrt{\frac{2}{3}} |W_+|_{|g + dt^2|^2} \right)^2 d\mu_\gamma + dt^2 \geq 32\pi^2 ((\pi^*c_1)^+)^2, \]
where \((\pi^*c_1)^+\) is the self-dual harmonic part of \(\pi^*c_1\) with respect to \(g + dt^2\), and \(\pi : M \times S^1 \rightarrow M\) is the projection map.

Proof. This immediately follows from LeBrun’s method of Theorem 2.2 in [11]. First by using Lemma 2.1,
\[ \int_{M \times S^1} \left( \frac{2}{3} s_\gamma + dt^2 - 2\sqrt{\frac{2}{3}} |W_+|_{|g + dt^2|^2} \right)^2 d\mu_\gamma + dt^2 \]
\[ \geq \int_{M \times S^1} \left( \frac{2}{3} s_\gamma - 2\sqrt{\frac{2}{3}} |W_+|_\gamma \right)^2 d\mu_\gamma, \]
and the right hand side is equal to
\[ (3.1) \quad \left( \int_{M \times S^1} d\mu_\gamma \right)^\frac{1}{2} \left( \int_{M \times S^1} \left( \frac{2}{3} s_\gamma - 2\sqrt{\frac{2}{3}} |W_+|_\gamma \right)^2 d\mu_\gamma \right)^\frac{1}{2}, \]
because \(\check{\gamma}\) has constant \(\frac{2}{3} s - 2\sqrt{\frac{2}{3}} |W_+|_\gamma\). Now we use the fact that \((M \times S^1, \check{\gamma})\) admits a solution of the unrescaled Seiberg-Witten equations for \(s\). Combining its Weitzenböck formula with the Weitzenböck formula for the self-dual harmonic 2-forms, we conclude that (3.1) is greater than or equal to \(32\pi^2 ((\pi^*c_1)^+)^2\). □
Now using the above lemma, we get
\[
\int_M |r_g|^2 d\mu_g = \int_{M \times S^1} |r_{g+dt^2}|^2 d\mu_{g+dt^2}
\]
\[
= 8 \int_{M \times S^1} \left( \frac{s_{g+dt^2}}{24} + \frac{1}{2} |W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2}
\]
\[-8\pi^2 (2\chi + 3\tau)(M \times S^1) \geq \frac{1}{2} \int_{M \times S^1} \left( \frac{2}{3} s_{g+dt^2} - 2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2} \geq 0
\]
(3.2)
\[
\geq \frac{1}{2} \int_{M \times S^1} \left( \frac{2}{3} s_{g+dt^2} - 2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2} \geq 16\pi^2 (\pi^* d_1)^2
\]
(3.3)
\[
\geq \frac{8\pi^2 c_1 \cup [\omega]^2}{\int_M |W_+|^2 d\mu_g}
\]
(3.4)
where the second equality is due to the 4-dimensional Chern-Gauss-Bonnet theorem, and the first inequality is simple applications of Hölder inequality which was proved in LeBrun [12].

**Lemma 3.5.** The equality of the theorem statement holds if and only if \((M \times S^1, g + dt^2)\) is a Kähler manifold of non-positive constant scalar curvature with the Kähler form a multiple of \(*\omega + \omega \wedge dt\) for harmonic \(\omega\), and \([c_1(\omega)]\) is a multiple of \([\omega]\) in \(H^2_{DR}(M)\).

**Proof.** Let’s first consider the case when \([c_1] \neq 0 \in H^2_{DR}(M)\). It is shown in [12] that both equalities in (3.2) and (3.3) hold if and only if \(g + dt^2\) is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of \((\pi^* c_1)^+\). The equality in (3.4) holds if and only if \(\omega = \omega^h = \pi^* c_1^h\), where \((\cdot)^h\) denotes the \(g\)-harmonic part.

When \([c_1] = 0 \in H^2_{DR}(M)\), the equality implies that the metric is Ricci-flat (In fact, it’s a flat manifold \(T^3/\Gamma\), because the dimension is 3). By the Weitzenböck formula for 1-forms, \(\omega^h\) is a nonzero parallel 1-form. Then \(*\omega^h + \omega^h \wedge dt\) is a nonzero parallel 2-form on \((M \times S^1, g + dt^2)\), and hence a Kähler form with the obvious complex structure compatible with the orientation. Conversely suppose that \((M \times S^1, g + dt^2)\) is scalar-flat Kähler. Since a Kähler curvature is a (symmetric) section of \(\wedge^{1,1} \otimes \wedge^{1,1}\), on any scalar-flat Kähler surface the Riemann curvature restricted to self-dual two forms must be zero, and hence so is \(W_+\). Then by the 4-dimensional Chern-Gauss-Bonnet theorem,
\[
\int_{M \times S^1} |r_{g+dt^2}|^2 d\mu_{g+dt^2} = \int_{M \times S^1} \left( \frac{1}{3} (s_{g+dt^2})^2 + 4|W_+|_{g+dt^2}^2 \right) d\mu_{g+dt^2} \geq 0
\]
If the equality holds, we have a parallel splitting of $TM$ by $\omega$ and $*\omega$, each of which gives a manifold of constant scalar curvature of dimension 1 and 2 respectively by the above lemma. Thus the universal cover of $(M, g)$ is isometric to $H^2 \times \mathbb{R}^1$ or $\mathbb{R}^3$. In the first case, $M$ is a quotient by a discrete subgroup of $PSL(2, \mathbb{R}) \times \mathbb{Z}$, and in the second case, $M \times S^1$ is a complex torus or a bielliptic surface by the Enriques-Kodaira classification. Therefore $(M, g)$ is obtained by identifying two boundaries of $\Sigma \times [0, 1]$ by an orientation-preserving isometry of a compact Riemann surface $(\Sigma, g_c)$ of genus $\geq 1$ with a constant curvature metric $g_c$. Then $(M, g)$ is locally a Riemannian submersion onto $S^1$ with totally geodesic fiber $(\Sigma, g_c)$.

Conversely, suppose that $(M, g)$ is such an oriented Riemannian submersion $\pi : M \to S^1$. The metric being locally a product, $(M \times S^1, g + dt^2)$ is a Kähler manifold with an obvious complex structure and a Kähler form $d\Omega + \pi^* ds \wedge dt$ where $d\Omega$ is the volume form of $(\Sigma, g_c)$ and $ds$ is the volume form of the base. The scalar curvature of $g$ is just the constant scalar curvature of $g_c$ and the first Chern class of $\Sigma$ is a multiple of $d\Omega$ which is equal to $*\pi^* ds$. By the above lemma, $(M, g)$ attains the equality, thereby completing the proof.

Remark. As seen in the proof, the condition that $(M, g)$ admits a solution for the rescaled Seiberg-Witten equations for any rescaling is superfluous. It is enough to suppose that $(M \times S^1, \tilde{g})$ has a solution for the unrescaled Seiberg-Witten equations.

In particular, if $(M, g)$ has an isometric $G$-action which can be lifted to $\mathfrak{s}$ for a compact Lie group $G$, then it is enough for $M$ to have solutions of the Seiberg-Witten equations for any $G$-invariant metric. In this case, $c_1(\mathfrak{s})$ is called a $G$-monopole class, and a $G$-monopole class sometimes exists even when the ordinary Seiberg-Witten invariant vanishes. This will be dealt with in our forthcoming paper ([21]).

Remark. It is well-known that on a compact Kähler surface $(X, g, \varpi)$ with $b_1^+(X) = 1$, the Seiberg-Witten invariant of its canonical Spin$^c$ structure for a small perturbation and a Riemannian metric $\tilde{g} \in [g]$ is equal to $\pm 1$, if $\deg(c_1(X)) := [c_1(X)] \cdot [\varpi]$ is negative (See [16]), and also true for any perturbation and Riemannian metric if $c_1(X)$ is a torsion by using the celebrated Taubes’s theorem [22] and the wall crossing formula [4, 15].

Now let’s discuss some immediate implications of the above theorem. First, by taking $\omega$ to be $*c_1^h(\mathfrak{s})$ where $(-)^h$ denotes the harmonic part, we have

$$\int_M |r_3|^2 d\mu_{g} \geq 8\pi^2 \int_M |c_1^h(\mathfrak{s})|^2 d\mu_{g}.$$ 

More interestingly we can get a lower bound of $L^2$-norm of a harmonic 1-form on $M$: 

Corollary 3.6. Under the same hypothesis as Theorem 3.2, if $\tilde{g}$ is not \textit{nat},
\[
\left( \int_M |\omega|^2 d\mu_{\tilde{g}} \right)^{\frac{1}{2}} \geq \frac{2\sqrt{2\pi}|\alpha \cup [\omega]|}{(\int_M |r_{\tilde{g}}|^{2d\mu_{\tilde{g}}})^{\frac{1}{2}}},
\]
where $\alpha$ is a convex combination of any two such $c_1(\mathfrak{s})$’s.

4. Monopole classes on connected sums

Our curvature estimates provide an easy toolkit in the study of a closed 3-manifold $M$ with a non-torsion monopole. In [19], we derived the inevitability of collapsing when such a manifold has zero Yamabe invariant which implies the existence of a sequence of unit-volume Riemannian metrics $\{g_i\}$ on $M$ satisfying
\[
\inf_i \int_M s^2_{g_i} d\mu_{g_i} = 0.
\]
We also found a necessary condition to be a monopole class in a specific example as follows:

**Proposition 4.1 ([19]).** Let $M$ be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and $N$ be a closed oriented 3-manifold with $b_1(N) = 0$. Then the rational part of a monopole class, if any, of $M \# N$ is of the form $m[F]$ for an integer $m$ satisfying $|m| \leq |\chi(F)|$, where $\chi(F)$ is the Euler characteristic of the fiber $F$.

(Warning: In the statement of Theorem 1.4 of [19], $b_1(N) = 0$ is missing by mistake.)

We give a generalization of this to connected sums:

**Proposition 4.2.** Let $M_i$ for $i = 1, \ldots, n$ be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and $N$ be any closed oriented 3-manifold. Then the rational part of a monopole class, if any, of $M_1 \# \cdots \# M_n \# N$ is of the form
\[
\beta + \sum_{i=1}^{n} m_i [F_i]
\]
for $\beta \in H^2(N, \mathbb{Z})$ and an integer $m_i$ satisfying $|m_i| \leq -\chi(F_i)$, where $\chi(F_i)$ is the Euler characteristic of the fiber $F_i$ in $M_i$.

**Proof.** Let $\alpha$ be a monopole class of $X = M_1 \# \cdots \# M_n \# N$.

First if any $F_i$ is a 2-sphere, then the only possibility for $M_i$ is $S^1 \times S^2$. Letting $[\omega]$ be the Poincaré-dual of $F_i$, we only have to show that it pairs zero with $\alpha$. Let $0 < \varepsilon \ll 1$.

Take a metric of positive scalar curvature on the $M_i$. For the connected sum, we take a small ball on $M_i$, and a representative $\omega$ of $[\omega]$ to be supported outside of that ball. Then perform the Gromov-Lawson type surgery [5, 17, 18] on it keeping the positivity of scalar curvature to get a compact manifold $M'_i$ with a cylindrical end. And then contract it small enough so that
\[
\int_{M'_i} |\omega|^2 d\mu \leq \varepsilon.
\]
On the other part of $X$, we put any metric such that it satisfies
\[(4.1) \quad \int (s_-)^2 d\mu < 1,\]
and perform the Gromov-Lawson surgery such that the cylindrical end matches with that of the above-made $M'_i$ while still satisfying (4.1).

After gluing these two pieces, we have that
\[
\int_X |\omega|^2 d\mu \leq \varepsilon \quad \text{and} \quad \int_X (s_-)^2 d\mu < 1.
\]
Applying Theorem 1.1, we get
\[
|4\pi \alpha \cup [\omega]|^2 < \varepsilon,
\]
which proves $\alpha \cup [\omega] = 0$.

Secondly, let’s consider the case of $M_i$ with $\chi(F_i) \leq 0$. Let $[\omega] \in H^1(M_i, \mathbb{R})$. By the Mayer-Vietoris principle, $H^1(M_i, \mathbb{R})$ is generated by
\[
\pi_i^* dt, \quad \text{and} \quad \{[\sigma] \in H^1(F_i, \mathbb{R}) | f_i^*[\sigma] = [\sigma]\},
\]
where $\pi_i : M_i \to S^1$ is the projection map and $f_i$ is the monodromy diffeomorphism.

When $[\omega]$ is one of the latter ones, we have to show that it pairs zero with $\alpha$. We can express $\omega$ as
\[
\frac{1}{d_i} \sum_{n=1}^{d_i} (f_i^n)^* \sigma
\]
for such $\sigma$ satisfying $[f_i^* \sigma] = [\sigma]$, where $d_i$ is the order of $f_i$.

By taking a $f_i$-invariant metric on $F_i$, we can put a locally-product metric on $M_i$ such that $\pi_i$ is a Riemannian submersion with totally geodesic fibers onto a circle of radius $\varepsilon$ and
\[(4.2) \quad \int (s_-)^2 d\mu < \varepsilon.
\]
We can take a small simply-connected open set $B$ in $F_i$, which is invariant under $f_i$, and take a representative $\sigma$ of the above $[\sigma] \in H^1(F_i, \mathbb{R})$ to be supported outside of $B$.

For the connected sum, we perform the Gromov-Lawson surgery on $B \times I(\frac{\varepsilon}{10}) \subset M_i$ where $I(\frac{\varepsilon}{10})$ is the interval of length $\frac{\varepsilon}{10}$ to get $M'_i$ with a cylindrical end while still satisfying (4.2). On the other part of $X$, as before we put a metric with a cylindrical end isometric to that of this $M'_i$ while satisfying (4.1).

After gluing two pieces, we have
\[
\int_X |\omega|^2 d\mu \leq C\varepsilon \quad \text{and} \quad \int_X (s_-)^2 d\mu < 1 + \varepsilon
\]
for a constant $C > 0$. Hence by Theorem 1.1
\[
|4\pi \alpha \cup [\omega]|^2 < C\varepsilon(1 + \varepsilon),
\]
proving $\alpha \cup [\omega] = 0$.

Finally when $\omega = \pi^* dt$, the adjunction inequality on a 3-manifold $X$, which can be proved in the same way as 4-manifolds ([7]) by considering $X \times S^1$ gives

$$|\langle \alpha, [F_i] \rangle| \leq -\chi(F_i),$$

completing the proof. \qed

**Remark.** As noted, $M_i \times S^1$ for $M_i$ as above admits a Kähler metric of constant scalar curvature, and each $M_i$ admits a $d_i$-fold covering space which is $F_i \times S^1$.

For $M_i$ with $F_i = S^2$, one can use the argument of gluing of moduli spaces of Seiberg-Witten equations along cylindrical ends to prove $4\pi\alpha \cup dt = 0$.

It seems plausible to conjecture:

**Conjecture 4.3.** Let $N_i$ for $i = 1, \ldots, n$ be a closed oriented 3-manifold. Then (the rational part of) any monopole class of $N_1 \# \cdots \# N_n$ is expressed as $\sum_i \alpha_i$, where $\alpha_i$ is a monopole class of $N_i$.

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