Passive Polarimetric Multistatic Radar Detection of Moving Targets

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Abstract

We study the exploitation of polarimetric diversity in passive multistatic radar for detecting moving targets. Unlike conventional isotropic models in which targets are modeled as a collection of uniform spheres, we model targets as a collection of dipole antennas with unknown directions. We consider a multistatic configuration in which each receiver is equipped with a pair of orthogonally polarized antennas, one directed to a scene of interest collecting target-path signal and another one having a direct line-of-sight to a transmitter-of-opportunity collecting direct-path signal. We formulate the detection of moving target problem in a generalized likelihood ratio test framework under the assumption that direct-path signal is available. We show that the result can be reduced to the case in which the direct-path signal is absent. We present a method for estimating the dipole moments of targets. Extensive numerical simulations show the performance of both the detection and the dipole estimation tasks with and without polarimetric diversity.

Index Terms

passive radar, polarimetry, multistatic, moving targets

I. INTRODUCTION

A. Overview

Passive radar senses the environment using transmitters of opportunity such as digital television stations, wideband cell-phone towers, radio transmissions, satellites, etc. The past decade has seen a sharp rise in the availability of such transmitters and with it a growing research interest in passive radar systems [1]–[12].

The polarimetric diversity in the context of passive radar, refers to each receiver being equipped with a pair of linear and orthogonally polarized antennas. Polarimetric diversity in passive radar is important for the following reasons:

• In conventional passive radars, sources, scatterers and receivers are typically assumed to be isotropic and polarimetric diversity is not considered [2], [3], [13]–[20]. Under the isotropy assumption, the vector wave equation simplifies to a scalar one. However, for the multistatic configuration the directionality/polarization of electromagnetic (EM) waves from the scatterers and antennas become increasingly important and the isotropy assumption may no longer hold.

• Polarization diversity provides an additional antenna at each receiver that is polarized in an orthogonal direction. This means that even if the signal received is weak at one antenna, a relatively strong signal is received at the other ensuring the effectiveness of spatial diversity offered by the multistatic configuration.

• Polarimetric radar can provide additional information that is not available in conventional non-polarimetric radars, including polarization characteristics of targets that can aid in classification and recognition tasks [21]–[26].

These reasons motivate the exploitation of polarimetric diversity in passive radar to produce better detection, imaging and classification performance than that of conventional non-polarimetric radar.

In this paper, we consider a multistatic configuration in which each receiver is equipped with a pair of polarimetrically diverse antennas, one of which has a direct line-of-sight to a transmitter of opportunity collecting...
direct-path signal and another one is directed to a scene of interest collecting target-path signal. We derive a target model and a received signal model for the scattered field from moving targets in a polarimetrically diverse configuration starting from first principle. Our derivation is based on the vector wave equation, using the dyadic Green’s function and dyadic reflectivity taking into account the linear motion of a moving target. We reduce the dyadic reflectivity to a spatially distributed dipole target model. This model is derived from the eigendecomposition of the target dyad. This decomposition induces a three colocated dipoles for the target at each spatial location with each eigenvector interpreted as a dipole moment. To reduce the size of the problem, we consider only the dominant eigenvalue/eigenvector - effectively modeling a scatterer as a single dipole with unknown dipole direction and reflectivity. We then model the receive and transmit antennas as dipole antennas to arrive at the final form of the received signal model.

We assume that the transmitted waveform and transmit antenna polarization state is unknown and formulate the moving target detection problem as a generalized likelihood ratio test. We derive the test statistic for both the case in which the direct-path signal is available and one in which it is not. We present a method of estimating target dipole direction and hence its polarization state. In [27], we analyze the performance gains in moving target detection due to polarimetric diversity in passive multistatic radar. Extensive numerical simulations show that polarimetric diversity provides superior detection performance than that of non-diverse case. Additionally, it provides information on anisotropic characteristic of scatterers that can be used for target recognition tasks.

B. Related Work

The exploitation of polarimetry in radar application has its origins as far back as the mid 20th century [28]–[32]. However traditionally, radar polarimetry has been applied to active monostatic systems, as in polarimetric Synthetic Aperture Radar (SAR) and polarimetric SAR interferometry [24], [25], [33], [34]. In [35] multistatic polarimetric active radar has been studied for imaging of moving targets. In this work, the multistatic system exhibits polarimetric diversity in both transmitter and receiver which are both modeled as dipole antennas. The imaging scheme, however, does not extract polarization information about the target scene and consists of weighted sum of correlations between the received signal and scaled and delayed versions of the transmitted waveforms. In contrast, we study multistatic passive radar systems in which a priori knowledge on the transmitted waveforms or the polarimetric state of the transmitters of opportunity is not available. Furthermore, our objective is not only to estimate the reflectivity but also the polarization state of targets.

The dipole target model was first studied in [36]. In [33], this model is utilized and a filtered-backprojection type imaging scheme for monostatic active polarimetric SAR was developed. [33] differs from our work in that it considers imaging of static targets using an active, monostatic configuration. In addition, it constrains target dipole moments to lie on a flat ground plane. We do not assume such constraints on the dipole moment direction.

Multistatic passive radar detection problems have been studied in [2]–[4], [15]–[17]. These works do not consider polarimetric diversity nor the estimation of target’s polarization state. In [2], spatially resolved detection of stationary scatterers is studied for multiple scattering environment. This is extended to moving targets in [3], [4]. In these papers, the authors make explicit isotropic assumptions for the target, transmitters and receivers. Our GLRT-based detection scheme is similar to the one presented in [15], [16]. However, in addition to our target and data models being different, we derive our test-statistic by considering the full continuous data and noise processes. We also relax the assumption that all noise processes have a common variance.

Passive polarimetry has been explored in remote sensing applications in the optical and infrared spectrum [37]. In acoustic imaging, ambient polarized acoustic noise field is used to recover the scattering matrix [38]. In [39], [40], polarimetric diversity for passive radar detection (PCL) is studied. In this work, they experimentally assess the effectiveness of polarimetric diversity in mitigating interference on the direct-path signal. Not only is their target, data model and test-statistic different from present work, but they do not consider estimation of the target’s polarization state. In addition, we consider the case in which direct-path signal is unavailable. In [41], we studied multistatic polarimetric passive radar for a single or widely spatially separated stationary targets without direct-path signals. In the present work, we consider ground moving targets and derive spatially resolved target detection scheme along with estimation of its polarization state. In addition, we consider two cases: when the direct-path signal is available at each receiver and when no such signal is available.
C. Organization of the Paper

The paper is organized as follows: in Section II we derive the model of scattered vector field from a moving target. In Section III, we derive the dipole target model from the eigendecomposition of dyadic permittivity. The polarimetric data model is derived in Section IV. In Section V, we use the polarimetric data model to derive the target detection and target dipole moment estimation problems. In Section VI, we present numerical simulations to demonstrate the performance of the detection and estimation methods. Section VII concludes the paper.

II. INCIDENT AND SCATTERED FIELD MODELS

A. Geometric Configuration of Multistatic Passive Radar

Fig. 1 depicts a typical configuration of a distributed passive radar that is of interest in this paper. We assume that there are $M$ stationary receivers distributed about a scene of interest and a single transmitter of opportunity. The location of the $k$-th receiver is denoted as $\mathbf{a}_k \in \mathbb{R}^3$ and the location of the transmitter of opportunity is denoted as $\mathbf{a}_t \in \mathbb{R}^3$. We assume that each receiver exhibits polarimetric diversity. In other words, each receiver consists of a pair of orthogonal dipole antennas receiving scattered signals from the scene of interest. Furthermore, we consider a case in which at each receiver, the direct-path signal and target-path signal is separated. This separation can be achieved, for instance, by employing a beamforming technique on an array of receive antennas at each receiver location as in [15]. The direct-path signal is also referred to as the reference channel and the target-path signal is referred to as the surveillance channel in the literature [15], [20]. We assume that the transmitter consists of a single dipole antenna and that the dipole direction of the transmitter is not known.

![Fig. 1: Distributed Polarimetric Passive Radar Scenario. $\mathbf{a}_k$ is the $k$-th receiver location and $\mathbf{a}_t$ is the transmitter location.](image)

B. Scattered Field Model with Dyadic Reflectivity

Since we are interested in modeling the directionality of EM waves scattered from the target of interest, we begin with vector wave equation. We assume that the medium is reciprocal and lossless and is characterized by constant real-valued permeability, $\mu_0$ and spatially varying real-valued dyadic permittivity, $\bar{\varepsilon}: \mathbb{R}^3 \rightarrow S^3$ where $S^3$ is the space of $3 \times 3$ real symmetric matrices [42]. The vector wave equation is then [42]

$$\nabla \times \nabla \times \mathbf{E} = \omega^2 \mu_0 \bar{\varepsilon} \mathbf{E} + i\omega \mu_0 \mathbf{J}$$

(1)

where $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the source term and $\omega$ is the temporal frequency.

**Assumption 1.** We assume that the background medium is homogeneous and isotropic with scalar permittivity $\varepsilon_0 \in \mathbb{R}$.

Under Assumption 1, we model $\bar{\varepsilon}$ as a perturbation of $\varepsilon_0$, i.e.,

$$\bar{\varepsilon}(\mathbf{x}) = \varepsilon_0 \mathbf{I}_3 + \bar{\varepsilon}_{sc}(\mathbf{x})$$

(2)

$^1$Direct-path signal is the signal received directly from the transmitter without any background scattering.

$^2$Target-path travels from the transmitter to the target and then scatters from the target and travels to the receiver.
where $I_3$ is a $3 \times 3$ identity matrix. We assume that $\tilde{\varepsilon}_{sc}$ is compactly supported.

We can then rewrite (1) as

$$\nabla \times \nabla \times E - k^2 E = \omega^2 \mu_0 \tilde{\varepsilon}_{sc} E + i \omega \mu_0 J$$

(3)

where $k = \omega \sqrt{\mu_0 \varepsilon_0}$ is the wavenumber. Here, we note that $E = E^{in} + E^{sc}$ where $E^{in}$ is the incident field originating from the source $J$ and $E^{sc}$ is the scattered field. Using this fact, we can split (3) into two vector wave equations as

$$\nabla \times \nabla \times E^i - k^2 E^i = i \omega \mu_0 J$$

(4)

and

$$\nabla \times \nabla \times E^{sc} - k^2 E^{sc} = \omega^2 \mu_0 \tilde{\varepsilon}_{sc} E.$$  

(5)

The solutions to the vector wave equations in (4) and (5) are given by [42]

$$E^{in}(x, \omega) = i \omega \mu_0 \int \hat{G}(x, x', \omega) J(x', \omega) dx'$$

(6)

and

$$E^{sc}(y, \omega) = \omega^2 \mu_0 \int \hat{G}(y, x, \omega) \tilde{\varepsilon}_{sc}(x) E(x, \omega) dx$$

(7)

where $\hat{G}(y, x, \omega)$ is the dyadic Green’s function$^3$ given by [33], [42]

$$\hat{G}(y, x, \omega) = \left( I + \frac{\nabla \nabla}{k^2} \right) \frac{e^{i \omega |y - x|/c_0}}{4\pi |y - x|}.$$  

(8)

(7) is the vector version of the well-known Lippmann-Schwinger equation [43]. We assume a weak surface scattering and make the Born approximation to linearize (7), arriving at

$$E^{sc}(y, \omega) \approx \omega^2 \mu_0 \int \hat{G}(y, x, \omega) \tilde{\varepsilon}_{sc}(x) E^{in}(x, \omega) dx.$$  

(9)

C. Scattered and Incident Field From a Moving Target

In the previous section, we derived the general form of the scattered EM vector field. In this section, we derive the model for the scattered field from a moving scatterer based on (7). We begin with the following assumption.

**Assumption 2.** We assume that the target at $x \in \mathbb{R}^3$ at time $t = 0$ is moving with constant velocity $v_x \in \mathbb{R}^3$. Let $z = x + v_xt$ be location of the target at time $t$.

From (7), we express the scattered field from the moving target in time domain as

$$E^{sc}(y, t) \approx \int G(y, x + v \tau, t - \tau) Q(x, \nu)$$

$$\partial^2 E^{in}(x + v \tau, \tau) d\tau dx d\nu$$

(10)

where $G(y, x, t)$ is the inverse Fourier transform of (8) and

$$Q(x, \nu) = \mu_0 \tilde{\varepsilon}_{sc}(x) \delta(\nu - v_x).$$  

(11)

The model in (10) is similar to the one derived in [3] for the scalar scattered field from a moving target. We have extended this to the vector field case using the solution to the vector wave equation. This model captures anisotropic scattering from targets in multistatic configurations. In addition, consideration of the vector scattered field allows us to model the polarimetry of the scattered field.

We assume that the ground topography $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is known. Let a target be initially located at $x = [x, \psi(x)]^T$ and moving with velocity $v_x = [v_{x}, \nabla x \psi(x) \cdot v_x]^T$ where $x, v_x \in \mathbb{R}^2$. Thus, we can write the reflectivity function, $Q$, that depends on $x, v \in \mathbb{R}^3$ in terms of $\tilde{Q}$ that depends on $x, \nu \in \mathbb{R}^2$, as follows:

$$Q(x, v) = \tilde{Q}(x, v) \delta(x_3 - \psi(x)) \delta(v_3 - \nabla x \psi(x) \cdot v_x)$$

(12)

$^3$In [5], $c_0$ is the wave speed in free-space and $\nabla \nabla$ is the Hessian operator.
where
\[ \tilde{Q}(\mathbf{x}, \mathbf{v}) = \mu_0 \tilde{\varepsilon}_{\text{sc}}(\mathbf{x}) \delta(\mathbf{v} - \mathbf{v}_x). \] (13)
We refer to \( \tilde{Q} \) as the dyadic phase space reflectivity function. Note that \( \tilde{\varepsilon}_{\text{sc}} \) is equal to the dyadic permittivity, \( \tilde{\varepsilon}_{\text{sc}} \), with the third spatial variable \( x_3 \) constrained to be \( x_3 = \psi(\mathbf{x}) \). We approximate (13) as
\[ \tilde{Q}(\mathbf{x}, \mathbf{v}) \approx \mu_0 \tilde{\varepsilon}_{\text{sc}}(\mathbf{x}) \varphi(\mathbf{v} - \mathbf{v}_x) \] (14)
where \( \varphi \) is a smooth function approximating \( \delta \), such as a Gaussian with a small variance. Inserting (12) into (10) and integrating out the third component of \( \mathbf{x} \) and \( \mathbf{v} \), we get
\[ \mathcal{E}^{\text{sc}}(\mathbf{y}, t) \approx \int G(\mathbf{y}, \mathbf{x} + \mathbf{v} \tau, t - \tau) \tilde{Q}(\mathbf{x}, \mathbf{v}) \] (15)
\[ \partial^2_x \mathcal{E}^{\text{in}}(\mathbf{x} + \mathbf{v} \tau, \tau) d\tau d\mathbf{x} d\mathbf{v}. \]
For a typical ground moving target, the distance covered by the moving target while the transmitted signal travels from the transmitter to the target and then from the target to the receiver is much smaller than the distance from the target to the receiver. Under this assumption, we make the following first-order approximation:
\[ |\mathbf{x} + \mathbf{v} t - \mathbf{y}| \approx |\mathbf{x} - \mathbf{y}| + (\mathbf{x} - \mathbf{y}) \cdot \mathbf{v} t \] (16)
where \( \hat{x} \) denotes the unit vector in the direction of \( x \). Plugging (16) into the phase in (8) we have
\[ G(\mathbf{x} + \mathbf{v} t, \mathbf{y}, t) \approx \int \left( I + \nabla \nabla \right) \frac{e^{i\omega |\mathbf{y} - \mathbf{x}|/c_0}}{4\pi |\mathbf{y} - \mathbf{x}|} \] (17)
e\left(\omega + (\mathbf{x} - \mathbf{y}) \cdot \mathbf{v} / c_0\right) t \, d\omega.
Thus, plugging (17) into the inverse Fourier transform of (6), we have
\[ \mathcal{E}^{\text{in}}(\mathbf{x} + \mathbf{v} \tau, \tau) \approx \mathcal{E}^{\text{in}}(\mathbf{x}, (1 + (\mathbf{x} - \mathbf{y}) \cdot \mathbf{v} / c_0) \tau). \] (18)
Now, using (18) and (17) in (15), the scattered field at the \( k \)-th receiver becomes
\[ \mathcal{E}^{\text{sc}}(\mathbf{a}_k^r, t) = \int G(\mathbf{a}_k^r, \mathbf{x}, t - \alpha_k^r(\mathbf{x}, \mathbf{v}) \tau) \tilde{Q}(\mathbf{x}, \mathbf{v}) \] (19)
\[ \partial^2_x \mathcal{E}^{\text{in}}(\mathbf{x}, \alpha_k^r(\mathbf{x}, \mathbf{v}) \tau) d\tau d\mathbf{x} d\mathbf{v} \]
where
\[ \alpha_k^r(\mathbf{x}, \mathbf{v}) = 1 - \frac{(\mathbf{x} - \mathbf{a}_k^r) \cdot \mathbf{v}}{c_0} \] (20)
\[ \alpha^l(\mathbf{x}, \mathbf{v}) = 1 + \frac{(\mathbf{x} - \mathbf{a}_k^r) \cdot \mathbf{v}}{c_0}. \] (21)
Making the change of variables, \( \tau' = \alpha_k^l(\mathbf{x}, \mathbf{v}) \tau \) we have
\[ \mathcal{E}^{\text{sc}}(\mathbf{a}_k^r, t) = \int G(\mathbf{a}_k^r, \mathbf{x}, t - \tau') \tilde{Q}(\mathbf{x}, \mathbf{v}) \alpha_k^r(\mathbf{x}, \mathbf{v}) \] (22)
\[ \partial^2_{\tau'} \mathcal{E}^{\text{in}}(\mathbf{x}, \alpha_k(\mathbf{x}, \mathbf{v}) \tau') d\tau' d\mathbf{x} d\mathbf{v} \]
where
\[ \alpha_k(\mathbf{x}, \mathbf{v}) = \frac{\alpha^l(\mathbf{x}, \mathbf{v})}{\alpha_k^r(\mathbf{x}, \mathbf{v})}. \] (23)
In temporal Fourier domain, (22) becomes
\[ \mathcal{E}^{\text{sc}}(\mathbf{a}_k^r, \omega) = \int G(\mathbf{a}_k^r, \mathbf{x}, \omega) \frac{\omega^2}{\alpha_k(\mathbf{x}, \mathbf{v})^3} \tilde{Q}(\mathbf{x}, \mathbf{v}) \] (24)
\[ \mathcal{E}^{\text{in}} \left( \mathbf{x}, \frac{\omega}{\alpha_k(\mathbf{x}, \mathbf{v})} \right) d\mathbf{x} d\mathbf{v}. \]
\(^4\text{In (17), we made the approximation } |\mathbf{x} + \mathbf{v} t - \mathbf{y}| \approx |\mathbf{x} - \mathbf{y}| \text{ for the denominator.}\)
For any realistic ground moving target, we can safely assume that \(|v| \ll c_0\). Thus, we make the approximation that \(\alpha_k^3 \approx 1\) so that (24) becomes:

\[
\mathcal{E}^{sc}(\mathbf{a}_k^r, \omega) = \omega^2 \int G(\mathbf{a}_k^r, \mathbf{x}, \omega) \mathcal{Q}(\mathbf{x}, \mathbf{v}) \mathcal{E}^{\text{in}}(\mathbf{x}, \frac{\omega}{\alpha_k(\mathbf{x}, \mathbf{v})}) d\mathbf{x} d\mathbf{v}.\]  

(25)

III. DIPOLE TARGET MODEL

By reciprocity and the fact that \(\mathcal{E}_{sc}\) is assumed to be real-valued, \(\mathcal{E}_{sc}(\mathbf{x})\) admits an eigendecomposition. Let \(\rho_1, \rho_2, \rho_3\) be the eigenvalues of \(\mathcal{E}_{sc}\) and \(e_1, e_2, e_3\) be the corresponding eigenvectors. Then, the \(\mathcal{Q}\mathcal{E}^{\text{in}}\) term in (25) can be written as:

\[
\mathcal{Q}\mathcal{E}^{\text{in}} = \sum_{i=1}^{3} \varphi \mu_0 \rho_i (e_i^T \mathcal{E}^{\text{in}}) e_i.\]  

(26)

Note that we have suppressed \(\mathbf{x}\) and \(\mathbf{v}\) dependencies for \(\rho_i, e_i, \) and \(\varphi\) for notational simplicity. The decomposition in (26) can be interpreted as a source with 3 colocated dipole antennas, each with dipole moment \(e_i\) [36].

**Assumption 3.** We approximate (26) with the largest eigenvalue and the corresponding eigenvector.

Let \(\rho\) be the largest eigenvalue and \(e_{sc}\) be the corresponding eigenvector. Then,

\[
\mathcal{E}_{sc}(\mathbf{x}) \approx \rho(\mathbf{x})e_{sc}(\mathbf{x})e_{sc}^T(\mathbf{x}).\]  

(27)

This approximation is equivalent to modeling the scatterer at \(x \in \mathbb{R}^3\) by a short dipole antenna. Under this model, an extended target becomes a collection of dipole antennas. Such a dipole model for scatterers was first studied by Gustafsson in [36] and more recently by Voccola et. al. in [33], [34]. Fig. 2 illustrates the idea of the dipole target model.

Fig. 2: Dipole model of target. Target is modeled by a collection of dipole antennas in lieu of traditional model as collection of points scatterers.

IV. POLARIMETRIC DATA MODEL

In the previous sections, we derived the models for the incident and the scattered fields from a moving target and the dipole target model approximation. In this section, we utilize these models to derive a model for the signal received by polarimetrically diverse dipole receivers due to a signal originating from a dipole transmitter and scattered by a moving target.

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\(^3\)By Cauchy-Schwarz we have that \(\mathbf{x} \cdot \mathbf{v} \leq |\mathbf{v}|\. Thus, from (20) and (21), we have that \((\alpha_k^r)^3 \leq 1 + O\left(\frac{|\mathbf{v}|}{c_0}\right)\) and similar for \(\alpha^t\). Along with the assumption that \(|\mathbf{v}| \ll c_0\) we have \(\alpha_k^3 \approx 1\).
A. Model for the Incident Field from a Short Dipole Antenna

Let $e^i$ denote the dipole moment of the transmit antenna. Under the assumption that the length of the antenna is small compared to the distance from the antenna to the scene, we can approximate (6) as (see [33], [34], [41])

$$E^{in}(x, \omega) = e^{i \omega |x-a^t|/c_0} A_t(x, \omega) \hat{p}(\omega) r^\perp_t(x)$$  

where $\hat{p}(\omega)$ is the Fourier transform of the transmitted waveform $p(t)$, $A_t$ is transmitter related terms including the antenna beam patterns and the geometric spreading factors and

$$r^\perp_t(x) = -(\hat{x} - \hat{a}^t) \times (\hat{x} - \hat{a}^t) \times e^t.$$  

Again, $\hat{x}$ denotes the unit vector in the direction of $x$ and $r^\perp_t$ is a triple vector cross product term that expresses the vector $e^t$ projected onto the plane with normal $(\hat{x} - \hat{a}^t)$.

B. The Model for the Ideal Target-Path Signal

Let $e^i_{k,H}$ and $e^i_{k,V}$ denote the dipole moments of the antennas at the $k$-th receiver where $H$ and $V$ denote the horizontal and vertical dipole moments, respectively. Under the assumption that the lengths of the dipole antennas are small compared to the distance from the target to the receiver, the received signal at the $k$-th receiver can be modeled as [33], [34], [41]

$$d_k^{0,TP}(\omega) = \omega^2 \int e^{i \omega \phi_k(x,v) / c_0} A_t \left( x, \frac{\omega}{\alpha_k(x,v)} \right) \hat{p} \left( \frac{\omega}{\alpha_k(x,v)} \right) q_e(x,v) q_e(x,v) dx dv,$$

where

$$q_e(x,v) = \rho(x) \varphi(v - v_x) e_{sc}(x) \langle r^\perp_t(x), e_{sc}(x) \rangle$$  

$$\phi_k(x,v) = |x - a_k^v| + \frac{|x - a_k^t|}{\alpha_k(x,v)}$$  

$$A^v_k(x,\omega) = [A_{r,k,H}(x,\omega) r^\perp_{r,k,H}(x), A_{r,k,V}(x,\omega) r^\perp_{r,k,V}(x)]^T$$  

$$r^\perp_{r,k,p}(x) = -(\hat{x} - \hat{a}_k^v) \times (\hat{x} - \hat{a}_k^t) \times e^v_{k,p} \quad p \in \{H, V\}.$$  

Fig. [3] illustrates the process of receiving the scattered signal from a dipole transmitter, dipole target, and dipole receiver. Note that the received signal depends on $r^\perp_t$, $r^\perp_{r,k,p}$, and $e_{sc}$. More specifically, the strengths of the signals are directly proportional to the scalar product between $r^\perp_t$ and $e_{sc}$ and $r^\perp_{r,k,p}$ and $e_{sc}$. This implies that the dipole-target model captures the anisotropic scattering, because $r^\perp_{r,k,p}$ depends on the look direction $(\hat{x} - \hat{a}_k^v)$. If $e_{sc}$ is roughly parallel to $(\hat{x} - \hat{a}_k^v)$, the received signal becomes weak. Conversely, if $e_{sc}$ is roughly orthogonal to $(\hat{x} - \hat{a}_k^v)$, the received signal becomes strong.

The $A_{r,k,p}$ are terms related to the receive antennas including the antenna beam patterns and the geometric spreading factors. We assume $A_t$ and $A_{r,k,p}$, are slowly varying functions of frequency, $\omega$. Furthermore, we assume a narrowband (w.r.t. the center frequency, $\omega_0$) transmitted waveform having frequency support $\omega \in \Omega_0 := [\omega_0 - B/2, \omega_0 + B/2]$. Let $p(t) = \hat{p}(t) e^{i \omega_0 t}$ where $\hat{p}(t)$ is a slowly varying function of $t$ such that the approximation $\hat{p}(\alpha_k t) \approx \hat{p}(t)$ holds. Thus, $\hat{p} \left( \frac{\omega}{\alpha_k} \right) \approx \alpha_k \hat{p}(\omega)$. Since $\hat{p}(\omega) = \hat{p}(\omega - \omega_0)$, we have $\hat{p} \left( \frac{\omega}{\alpha_k} \right) \approx \alpha_k \hat{p}(\omega - \alpha_k \omega_0)$ with bandlimited support $\omega \in \Omega_k := \{\alpha_k \omega_0 - B/2, \alpha_k \omega_0 + B/2\}$, $k = 1, \ldots, M$.

**Assumption 4.** We assume that $A_t$ and $A_{r,k,p}$ do not vary much within $\Omega_k$, and approximate $A_t$ and $A_{r,k,p}$ as constants w.r.t. $\omega$.

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6We justify this assumption by noting that most sources of opportunities are narrowband and antenna beam patterns are close to uniform w.r.t. $\omega$ within the band.
Fig. 3: An illustration of the transmission of the electric field from a dipole transmitter, scattered by a dipole target, and then received by a dipole receiver. Note that the strength of the scattered signal not only depends on the orientation of the dipole moments of the receiver, transmitter, and scatterer, but also the look directions from the transmitter to the scatterer and from the scatterer to the receiver. This implies that the strength of the scattered signal depends on the direction of the underlying field. Hence, the dipole model captures anisotropic scattering.

With this assumption, (30) becomes

$$d_k^{0,TP}(\omega) \approx \frac{1}{\omega_0} \int e^{i(\omega + \alpha_k\omega_0)p_k(x,v)/c_0} A_t(x) \tilde{p}(\omega) A^r_k(x) \tilde{q}_k(x,v) dx dv,$$

where we make the substitution $$\omega = \omega' - \alpha_k\omega_0$$, $$\omega' \in \Omega_k$$ so that $$\omega \in \Omega_B := [-\frac{B}{2}, \frac{B}{2}]$$. Furthermore, we make the approximation for the scalar multiple $$(\omega + \alpha_k\omega_0)^2 \approx \omega_0^2$$ under the assumption that $$|v| \ll c_0$$ and $$B \ll c_0$$.

C. Model for the Ideal Direct-Path Signal

The direct-path signal received at the k-th receiver is proportional to the scalar product of the incident field given in (28) with the dipole moment of the receive antenna, and does not include the target related terms present in the target-path signal defined in (35). The direct-path signal is also scaled by terms related to the receiver denoted by $$A_{r,k,H}^{DP}$$ and $$A_{r,k,V}^{DP}$$.

At the k-th receiver, the transmitter related term, $$A_t(x)$$ in (28) is evaluated at the receiver location, $$x = a_k$$. Let $$A_{k,DP}^{dp} = A_t(a_k), k = 1, \ldots, M$$. Then, the ideal, noise-free direct-path signal at the k-th receiver becomes

$$d_k^{0,DP}(\omega) = e^{i(\omega + \alpha_k\omega_0)|a_k|} A_{k,DP}^{dp} e^{i\omega_0|a_k|} A^r_k \hat{p}(\omega), \quad \omega \in \Omega_B$$

where

$$A_{k,DP} = [A_{r,k,H}^{DP} r_{r,k,H}(a^l), A_{r,k,V}^{DP} r_{r,k,V}(a^l)]^T.$$  

Note that in (36), we use the vector identity $$a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$$ to make the substitution $$r_{r,k,H}^l(a_k^p) \cdot e^r_{k,p} = r_{r,k,H}^l(a^l_p) \cdot e^r_k$$ where $$p \in \{H, V\}$$.
D. Model for the Noisy Received Signals

(35) and (36) represent ideal, noise-free received signal models. In practice, however, all received signals are corrupted by noise. Thus, we model target-path and direct-path received signals as

\[ d_{k}^{TP}(\omega) = d_{k}^{0,TP}(\omega) + n_{k}^{TP}(\omega) \]  \hspace{1cm} (38)
\[ d_{k}^{DP}(\omega) = d_{k}^{0,DP} + n_{k}^{DP}(\omega). \]  \hspace{1cm} (39)

where

\[ n_{k}^{TP}(\omega) = [n_{k,H}(\omega), n_{k,V}(\omega)]^{T} \]  \hspace{1cm} (40)
and

\[ n_{k}^{DP}(\omega) = [n_{k,H}(\omega), n_{k,V}(\omega)]^{T} \]  \hspace{1cm} (41)

are the random noise processes present in the target-path and direct-path signals, respectively, at the \( k \)-th receive antennas.

**Assumption 5.** We assume that for all \( \omega, \omega' \), \( n_{k}^{TP}(\omega) \) is independent of \( n_{l}^{TP}(\omega') \) for \( k \neq l \) and that \( n_{k,H}(\omega) \) is independent of \( n_{k,V}(\omega') \). Similarly, for all \( \omega, \omega' \), \( n_{k}^{DP}(\omega) \) is independent of \( n_{l}^{DP}(\omega') \) for \( k \neq l \) and that \( n_{k,H}(\omega) \) is independent of \( n_{k,V}(\omega') \). Furthermore, \( n_{k}^{TP}(\omega) \) is independent of \( n_{l}^{DP}(\omega') \) for all \( k, l, \omega, \omega' \). Finally, we assume that all noise processes are zero-mean Gaussian white noise processes, i.e., the auto-covariance functions \( E[n_{k,p}^{TP}(\omega)(n_{k,p}^{TP}(\omega'))^{*}] = (\sigma_{k,p}^{TP})^{2}\delta(\omega - \omega') \) and \( E[n_{k,p}^{DP}(\omega)(n_{k,p}^{DP}(\omega'))^{*}] = (\sigma_{k,p}^{DP})^{2}\delta(\omega - \omega') \) for \( k = 1, \ldots, M \) and \( p \in \{H, V\} \).

We now consider the second-order statistics of all measurements. These statistics can be succinctly expressed as matrix-valued covariance functions by first considering the following random vectors:

\[ \mathbf{n}^{TP}(\omega) = [n_{1,H}(\omega), n_{1,V}(\omega), \ldots, n_{M,H}(\omega), n_{M,V}(\omega)]^{T} \]  \hspace{1cm} (42)
and

\[ \mathbf{n}^{DP}(\omega) = [n_{1,H}(\omega), n_{1,V}(\omega), \ldots, n_{M,H}(\omega), n_{M,V}(\omega)]^{T} \]  \hspace{1cm} (43)

Next, we define the following matrix valued covariance functions

\[ \mathbf{K}^{TP}(\omega, \omega') = E[\mathbf{n}^{TP}(\omega)(\mathbf{n}^{TP}(\omega'))^{H}] \]  \hspace{1cm} (44)
and

\[ \mathbf{K}^{DP}(\omega, \omega') = E[\mathbf{n}^{DP}(\omega)(\mathbf{n}^{DP}(\omega'))^{H}]. \]  \hspace{1cm} (45)

By Assumption 5 on the noise processes, \( \mathbf{K}^{TP}(\omega, \omega') \) and \( \mathbf{K}^{DP}(\omega, \omega') \) can be expressed as

\[ \mathbf{K}^{TP}(\omega, \omega') = \Sigma^{TP}\delta(\omega - \omega') \]  \hspace{1cm} (46)
and

\[ \mathbf{K}^{DP}(\omega, \omega') = \Sigma^{DP}\delta(\omega - \omega'). \]  \hspace{1cm} (47)

where \( \Sigma^{TP} \) and \( \Sigma^{DP} \) are diagonal matrices that do not depend on \( \omega \).

Finally, we assume that \( \Sigma^{TP} \) and \( \Sigma^{DP} \) are non-singular. Since \( \Sigma^{TP} \) and \( \Sigma^{DP} \) are both diagonal matrices, this assumption is satisfied so long as no \( n_{k,p}^{TP}(\omega) \) or \( n_{k,p}^{DP}(\omega) \) is identically zero.

V. TARGET DETECTION AND TARGET DIPOLE ESTIMATION

We first set up a binary hypothesis test and next address it via generalized likelihood ratio test (GLRT). The solution to the GLRT leads to both the test-statistic for the detection task as well as a method for estimating the dipole moment of the target.
A. Target Detection

We consider two detection problems: one where the direct-path signal is available and another problem where it is not. We present a unified framework to address both problems.

For the rest of this section we assume that we have a single point target. For multiple targets, if we assume that scattered signal from each target can be separated in time, i.e., the targets are sparsely distributed in space so that their respective ranges to each receiver are far apart, then we can apply time-domain windows to the data and treat each return separately as single point targets.

For a point target located at \( \mathbf{x}^* \) and moving with velocity \( \mathbf{v}^* \), \( \dot{\mathbf{q}}_e(\mathbf{x}, \mathbf{v}) = \rho^* \mathbf{e}_s^* \delta(\mathbf{x} - \mathbf{x}^*) \delta(\mathbf{v} - \mathbf{v}^*) \) where \( \rho^* \) and \( \mathbf{e}_s^* \) are constants. Thus, (35) becomes

\[
\mathbf{d}_{k,T}^{TP}(\omega) = \omega_0^2 \mathbf{e}(\omega + \omega_0^2 \mathbf{x}^*, \mathbf{v}^*) / c_0 A_1 \hat{\rho}(\omega) \mathbf{A}_k^T \dot{\mathbf{q}}_e^* \tag{48}
\]

where \( \dot{\mathbf{q}}_e^* = \beta^* \rho^* \mathbf{e}_s^* \) and \( \beta^* = \varphi(0)(\mathbf{r}^T_1(\mathbf{x}^*), \mathbf{e}_s^*) \).

We hypothesize that the point target is located at \( \mathbf{x}_h \) moving with velocity \( \mathbf{v}_h \) and let \( \bar{\mathbf{x}}_h = (\mathbf{x}_h, \mathbf{v}_h) \). We consider the vectorized measurements

\[
\mathbf{d}^{TP} = [\mathbf{d}_1^{TP}, \mathbf{d}_2^{TP}, \ldots, \mathbf{d}_M^{TP}]^T \tag{49}
\]

and

\[
\mathbf{d}^{DP} = [\mathbf{d}_1^{DP}, \mathbf{d}_2^{DP}, \ldots, \mathbf{d}_M^{DP}]^T. \tag{50}
\]

Now, let

\[
\mathbf{D}^{TP} = \text{diag}[e^{ij(\omega + \alpha_k(\bar{\mathbf{x}}_h))}] I_2, \quad k = 1, \ldots, M 
\]

and

\[
\mathbf{D}^{DP} = \text{diag}[e^{ij(\omega + \alpha_k)}] a_s^* - a_s'^* / c_0 I_2, \quad k = 1, \ldots, M 
\]

where \( \text{diag} \) denotes a block diagonal matrix with \( I_2 \) being a \( 2 \times 2 \) identity matrix.

Let

\[
\mathbf{s}^{DP} = \left[ A_{1,1}^{DP}(\mathbf{A}_1^{DP} \mathbf{e}_s^T) \mathbf{T}, A_{2,1}^{DP}(\mathbf{A}_2^{DP} \mathbf{e}_s^T) \mathbf{T}, \ldots, A_{M,1}^{DP}(\mathbf{A}_M^{DP} \mathbf{e}_s^T) \mathbf{T} \right]^T, \tag{53}
\]

and

\[
\mathbf{r}_e = \omega_0^2 A_1 [(\mathbf{A}_1 \dot{\mathbf{q}}_e^* T), (\mathbf{A}_2 \dot{\mathbf{q}}_e^* T), \ldots, (\mathbf{A}_M \dot{\mathbf{q}}_e^* T)]^T. \tag{54}
\]

In addition to target detection, we are interested in estimating \( \mathbf{r}_e \) up to a scaling factor to eventually estimate the target dipole, \( \mathbf{e}_s \). Note that the transmitter term \( A_t \) is included in \( \mathbf{r}_e \). However, since \( A_t \) is merely a scaling term, it neither affects the solution to the target detection problem nor estimation of the target dipole.

Given (36), (48), (51), (52), (53), and (54), for each hypothesized location and velocity pair, \( \bar{\mathbf{x}}_h = (\mathbf{x}_h, \mathbf{v}_h) \), we address the detection problem as the following test of hypothesis:

\[
\mathcal{H}_1 : \mathbf{d}^{TP}(\omega|\bar{\mathbf{x}}_h) = \hat{\rho}\mathbf{D}^{TP}\mathbf{r}_e + \mathbf{n}^{TP}(\omega)
\]

\[
\mathbf{d}^{DP}(\omega) = \hat{\rho}\mathbf{D}^{DP}\mathbf{s}^{DP} + \mathbf{n}^{DP}(\omega) \tag{55}
\]

\[
\mathcal{H}_0 : \mathbf{d}(\omega|\bar{\mathbf{x}}_h) = \mathbf{n}^{TP}(\omega)
\]

\[
\mathbf{d}^{DP}(\omega) = \hat{\rho}\mathbf{D}^{DP}\mathbf{s}^{DP} + \mathbf{n}^{DP}(\omega).
\]

In (55), we assume that direct-path received signals are available. If they are not, we can omit \( \mathbf{d}^{DP}(\omega) \) in the formulation or, equivalently, set them equal to zero.

Since both \( \hat{\rho} \) and \( \mathbf{r}_e \) are unknown, we address the hypothesis test in (55) within the GLRT framework and write the test-statistic as follows:

\[
\lambda(\bar{\mathbf{x}}) = \max_{\mathcal{H}_1, \mathcal{H}_0} \frac{\max_{\hat{\rho}, \mathbf{r}_e} \mathbf{P}_{\mathcal{H}_1|\bar{\mathbf{x}}}}{\max_{\hat{\rho}, \mathbf{r}_e} \mathbf{P}_{\mathcal{H}_0|\bar{\mathbf{x}}}} \tag{56}
\]

where \( P_{\mathcal{H}_1|\bar{\mathbf{x}}} \) and \( P_{\mathcal{H}_0|\bar{\mathbf{x}}} \) are probability measures for the stochastic processes under \( \mathcal{H}_1 \) and \( \mathcal{H}_0 \), respectively, given \( \bar{\mathbf{x}} \). Thus, the GLRT is the ratio of the maximum likelihood estimates (MLEs) of \( \hat{\rho} \) and \( \mathbf{r}_e \) using the Radon-Nicodym derivatives of \( P_{\mathcal{H}_1|\bar{\mathbf{x}}} \) and \( P_{\mathcal{H}_0|\bar{\mathbf{x}}} \). Under the noise assumption in Assumption 5, we have Gaussian pdfs for each \( \omega \) under \( \mathcal{H}_1 \) and \( \mathcal{H}_0 \).
The solution to (56) is given in Theorem 1.

**Theorem 1** (Direct-path signal available.). Suppose $\hat{p}$ is continuous in $\Omega_B$ and $|\rho^*| > 0$. Let

\[
\hat{d} = [(d_{TP})^T, (d_{DP})^T]^T, \\
\hat{D} = \begin{bmatrix} D_{TP} & 0 \\ 0 & D_{DP} \end{bmatrix},
\]

and

\[
\hat{\Sigma} = \begin{bmatrix} \Sigma_{TP} & 0 \\ 0 & \Sigma_{DP} \end{bmatrix}.
\]

Furthermore, let

\[
\hat{Q}_1 = \int \hat{D}^H \hat{d} \hat{D} d\omega,
\]

and

\[
\hat{Q}_0 = \int (D_{DP})^H d_{DP} (d_{DP})^H D_{DP} d\omega.
\]

Then, under Assumption 5 the optimal test statistics to the GLRT problem (56) is

\[
\lambda(\Omega) = \lambda_{\max}(\hat{\Sigma}^{-1/2} \hat{Q}_1 \hat{\Sigma}^{-1/2}) - \lambda_{\max}((\Sigma_{DP})^{-1/2} \hat{Q}_0 ((\Sigma_{DP})^{-1/2}).
\]

**Proof.** See Appendix A.

(62)

If the direct-path signal is unavailable, e.g. because it is not collected, we set $d_{DP} \equiv 0$ in the binary hypothesis problem (55). The solution to this modified problem is given by the following corollary to Theorem 1.

**Corollary 1** (Direct-path signal is not available). By letting $d_{DP} \equiv 0$, the test statistic (62) becomes

\[
\lambda(\Omega) = \lambda_{\max}\left((\Sigma_{TP})^{-1/2} R (\Sigma_{TP})^{-1/2}\right).
\]

where

\[
R = \int (D_{TP})^H d_{TP} (d_{TP})^H D_{TP} d\omega.
\]

**Proof.** See Appendix D.

(63)

**B. Estimation of Dipole Direction**

Let $w$ be the eigenvector of $\hat{\Sigma}^{-1/2} \hat{Q}_1 \hat{\Sigma}^{-1/2}$ corresponding to its largest eigenvalue. By (78), we see that $w$ is the estimate of $\Sigma^{-1/2} \sigma$ up to a scalar. Now, since $\hat{\Sigma}$ is block diagonal,

\[
\hat{\Sigma}^{-1/2} = \begin{bmatrix} \Sigma^{-1/2} & 0 \\ 0 & (\Sigma_{DP})^{-1/2} \end{bmatrix}.
\]

Then, by (65) and (72), $w$ can be written as

\[
w = [w_1^T, w_2^T]^T.
\]

where $w_1$ is the estimate of $\Sigma^{-1/2} \sigma_e$ up to a scalar. Hence, by the definition of $\sigma_e$ in (54), it follows that $e_{sc}$ can be approximated by unit vector in the direction of $A_1^+ (\Sigma_{TP})^{-1/2} w_1$ where $A_1^+$ is the Moore-Penrose pseudoinverse of

\[
A_r = [([A_1^T, (A_2^T, \ldots, (A_M^T)]^T).
\]

(66)

When no direct-path signal is available, by Corollary 1 we can restrict our attention to eigenvector corresponding to the maximum eigenvalue of $(\Sigma_{TP})^{-1/2} R (\Sigma_{TP})^{-1/2}$. Let $u$ be that eigenvector. Then, by similar reasoning as above, $u$ is the estimate of $\Sigma^{-1/2} \sigma_e$. Thus, we can again approximate $e_{sc}$ by the unit vector in the direction of $A_1^+ (\Sigma_{TP})^{-1/2} u$. 


Fig. 4: An illustration of the configuration of the antennas and the scene. The transmitter is located at position (15, 15, 5) km. The receivers are positioned around a circle of radius 10 km.

VI. NUMERICAL SIMULATIONS

We evaluate the performance of the GLRT-based spatially resolved detection and dipole moment estimation using simulated data. Specifically, we compare the performance between the set up with and without polarimetric diversity. The effect of polarimetric diversity depends on the spatial location of the antennas. We further evaluate the effect of spatial diversity on the detection performance with and without polarimetric diversity. We also compare the performance due to polarimetric diversity with and without direct-path signal.

A. The Configuration for Simulations

The configuration we use in all simulation is as follows: A single transmitter is located at (15, 15, 5) km from the scene center transmitting an 8 MHz bandwidth signal. The scene is a 400 × 400 m flat ground topography. All receivers lie on a circle of radius 10 km from the scene’s center on the ground plane, all at 5 km above the scene.

The transmitted waveforms are linearly polarized in a single direction with center frequency of 2 GHz. The dipole moment of the transmit antenna is parallel to the ground plane in all cases. Two linearly polarized dipole antennas, labeled $H$ and $V$ are placed at each receiver. The dipole moment of each receiver’s $H$-polarized antenna is parallel to the ground and points in the direction $(\sin \theta_n, -\cos \theta_n, 0)$, where $\theta_n$ is the azimuth angle of the $n$-th receiver location relative to the $x$-axis. For the $V$-polarized receive antennas, the dipole moment directions all point upwards in the $z$-axis direction, i.e. $(0, 0, 1)$. Only the $H$-polarized receive antennas are used for the case in which there is no polarization diversity present in the receivers. Fig. 4 illustrates the simulation setup.

Fig. 5 shows an example test-statistic image for 3 point targets. The plot also shows the dipole moment directions of each target (solid blue line at each target) and its estimated dipole moment (yellow dashed line at each target). Average target-path signal SNR and average direct-path signal SNR were both kept at 0 dB, and 6 receivers were used with 256 samples.

B. Description of the Experiements

The numerical simulations presented in this section is categorized into two cases:

1) The effect of polarization diversity on target detection task.
2) The effect of polarization diversity on target dipole moment estimation.

We study the detection task via probability of detection as our metric. We present three subsets of simulated experiments for the detection task.

1) No spatial diversity: Single receiver case with direct-path signals.
2) Limited spatial diversity: Two receiver case with and without direct-path signals.
3) Full spatial diversity: Six receivers surrounding the scene.
In first two scenarios, we limit the spatial diversity to focus on the effect of polarization diversity. The first scenario presents a case with single receiver in which we compare case with polarization diversity and without where direct-path signal is available in both cases. In order to compare with the case without direct-path signal, in the second scenario we place two receivers about the scene. Lastly, we examine the case with full spatial diversity and evaluate the detection performance therein. In the first two scenarios, objective is to show polarization diversity can improve detection performance when spatial diversity is limited. The third scenario, we aim to show that polarization diversity can help to fully exploit spatial diversity.

C. Target Detection Performance

To evaluate the target detection performance using the GLRT-based approach stated in Theorem 1, we numerically estimate the probability of detection $P_d$ under constant false alarm rate (CFAR) of $0.001$. The threshold values that achieve the CFAR are determined numerically using 10,000 realizations of noise under the null hypothesis, $H_0$. The $\mathbf{\pi}$ is held constant for each experiment. The noise processes added to each received signal are all assumed to have common variances of $(\sigma_{TP})^2$ for target-path signals and $(\sigma_{DP})^2$ for direct-path signals. In other words, we set $\Sigma^{TP} = (\sigma_{TP})^2 I$ and $\Sigma^{DP} = (\sigma_{DP})^2 I$ where $I$ is the identity matrix. The variances are determined by the average signal SNRs.

1) Single receiver with direct-path signal: We begin our investigation with the simplest scenario of a single receiver with direct-path signal available. In this scenario, we only compare cases with direct-path signal between polarization diversity and no polarization diversity. We look at two cases, with high direct-path signal SNR (10 dB) and with low direct-path signal SNR (-30 dB). The receiver was placed so that $H$-polarized and $V$-polarized target-path signals had roughly equivalent SNRs. The simulation results are shown in Fig. 6.

In Fig. 6 we see improvement in detection performance for polarimetrically diverse case over no polarimetric diversity, with both high direct-path SNR and low direct-path SNR. The improvement is similar in both cases. The high direct-path SNR primarily functions to shift the probability of detection graph. It also changes the slope of the probability of detection graph so that the degradation in probability of detection as function of target-path signal SNR is more gradual. Using fixed $P_d = 0.9$ as a figure of merit, we see a 2.63 dB improvement in received signal SNR. If we fix $P_d = 0.5$ for case without polarimetric diversity, we see that there is improvement of 0.4 in probability of detection with polarimetric diversity for high direct-path SNR case and improvement of 0.5 in probability of detection for low direct-path SNR case.

2) Two receiver with and without direct-path signal: In this subsection, we present simulations with limited spatial diversity, with two receivers. By having two receivers, we can in edition compare scenarios between case with direct-path signals available and case without. We again look at case with high direct-path SNR (10 dB) and low direct-path SNR (-30 dB). Fig. 7 shows the result.
(a) Direct-path SNR : 10 dB
(b) Direct-path SNR : -30 dB

Fig. 6: The probability of detection ($P_d$) vs. target-path signal SNR ($SNR_H$) for a single receiver scenario. Blue line labeled POL is with polarization diversity. The red dashed line labeled NOPOL is without polarization diversity (with only $H$-polarized signal being used for detection). $P_d$ was estimated using 10,000 realizations of noise under CFAR of 0.001. In both cases, at fixed $P_d = 0.9$, there is an improvement of 2.63 dB in signal SNR.

(a) Average direct-path signal SNR : 10 dB
(b) Average direct-path signal SNR : -30 dB

Fig. 7: The probability of detection ($P_d$) vs. average target-path signal SNR ($SNR_{AVG}$) for a two receiver scenario. Blue line labeled DP-POL is with polarization diversity and direct-path signals available. The red dashed line labeled DP-NOPOL is without polarization diversity (with only $H$-polarized signal being used for detection) and direct-path signals available. Green line labeled POL is with polarization diversity and without direct-path signals available. Magenta line labeled NOPOL is without polarization diversity and without direct-path signals available. $P_d$ was estimated using 10,000 realizations of noise under CFAR of 0.001. At fixed $P_d = 0.9$, we see 3.31 dB improvement for high average direct-path SNR in average target-path signal SNRs with polarization diversity over without. With low average direct-path SNR, the improvement is 3.38 dB. Without direct-path, the improvement is 3.39 dB.

We see improvement in detection performance for polarimetrically diverse case over no polarimetric diversity, with both high direct-path SNR and low direct-path SNR. The improvement is again similar in both cases. For fixed $P_d = 0.9$ as a figure of merit, we see a 3.31 and 3.38 dB improvement in average received signal SNR for case with high average direct-path SNR and low average direct-path SNR, respectively. Without direct-path, the improvement is 3.39 dB. If we fix $P_d = 0.5$ for case without polarimetric diversity, we see that there is improvement of about 0.5 in probability of detection with polarimetric diversity for both high direct-path SNR, low direct-path SNR, and without direct-path signal cases. Furthermore, we see that with low direct-path SNR, the detection performance
Fig. 8: The probability of detection ($P_d$) vs. average target-path signal SNR ($SNR_{AVG}$) for a two receiver scenario. Blue line labeled DP-POL is with polarization diversity and direct-path signals available. The red dashed line labeled DP-NOPOL is without polarization diversity (with only $H$-polarized signal being used for detection) and direct-path signals available. Green line labeled POL is with polarization diversity and without direct-path signals available. Magenta line labeled NOPOL is without polarization diversity and without direct-path signals available. $P_d$ was estimated using 10,000 realizations of noise under CFAR of 0.001. At fixed $P_d = 0.9$, we see 1.54 dB improvement for high average direct-path SNR in average target-path signal SNRs with polarization diversity over without. With low average direct-path SNR, the improvement is 1.90 dB. For the middle case, the improvement is 1.89 dB. Without direct-path, the improvement is 1.77 dB.

3) Full spatial diversity: 6 receivers: The final scenario we examined was with full spatial diversity using 6 receivers equally spaced on the circle around the scene. For this experiment, we show results for 3 different average direct-path signal SNRs. Fig. 8 shows the results of the probability of detection the 3 different average direct-path SNR cases. Comparing Fig. 8a and Fig. 8, we see that the improvement in the probability of detection graph is minimal with growing high direct-path SNRs.

For fixed $P_d = 0.9$ as a figure of merit, we see a 1.59 dB improvement in average received signal SNR is observed for high average direct-path SNR (10 dB), 1.89 dB improvement with average direct-path SNR at 0 dB and 1.90 dB for low average direct-path SNR (−30 dB). Without direct-path, the improvement is 1.77 dB. Fixing $P_d = 0.5$ for case without polarimetric diversity, we see that there is improvement of about 0.27 in probability of detection with polarimetric diversity for 10 dB average direct-path SNR, 0.31 for 0 dB average direct-path SNR, and 0.5 for −30 dB average direct-path SNR. For case without direct-path the improvement in $P_d$ is 0.45.
D. Numerical Simulation of Target Dipole Moment Estimation

We also examine the performance of dipole moment estimation. We again use 6 receivers equally spaced on a circle about the center of the scene. The performance criteria used is the angle between the true dipole moment and the estimated one. Specifically, we use 1,000 realizations of noise and use the sample average of the estimated angles, labeled $\Delta \phi$, in radians, between the true dipole moment and the estimated one as the final performance criteria. We repeated this process for various target-path noise levels. Fig. 9 shows the resulting graph of $\Delta \phi$ vs. average target-path signal SNRs ($SNR_{AVG}$). Note that non-diverse cases completely fail to accurately estimate the dipole moment. Namely, $\Delta \phi$ is close to $\pi/2$ for all $SNR_{AVG}$ values. This is due to the fact that for the non-diverse case, estimated dipole moment all have a dominant component along the $z$-axis. The $z$-axis component of the target dipole moment cannot be estimated accurately since it lies in the orthogonal complement of the subspace spanned by the dipole moments of the $H$-polarized receive antennas as they all lie in a plane parallel to the ground plane. Furthermore, in the case of dipole moment estimation, high level of direct-path noise degrades the estimation performance to the extent that having only the target-path signal results in better estimates. This is due to the fact that, although the direct-path signals do not contain information about the dipole moment of the target directly, the estimation of $\hat{s}$, i.e. the eigenvector, does depend on the direct-path signals.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we derived from first principles, a novel polarimetric data model for moving target that takes into account anisotropic scattering. This model represents a target as a spatially distributed collection of dipole antennas. We considered a bistatic scenario with a single transmitter of opportunity equipped with a dipole antenna with unknown dipole moment direction and multiple receivers equipped with a pairs of orthogonally polarized dipole antennas. We address the target detection with and without direct-path signal in a unified GLRT framework and derived a method to estimate the dipole moment of the target. The detection test-statistic is given in terms of the maximum eigenvalues of whitened data correlation matrices. For the case with the direct-path signal, the test-statistic is the difference between the maximum eigenvalues of full data correlation matrix (with both target-path and direct-path received signals) and the correlation matrix for the direct-path signals only. When direct-path signals are not available, the test-statistic reduces to the maximum eigenvalue of the target-path correlation matrix.
The dipole moment estimation was derived from the eigenvector associated with the maximum eigenvalue of the data correlation matrix. This estimation method falls out naturally from the solution to a generalized eigenvalue problem from the GLRT framework.

In addition, through series of numerical simulations, we show that polarimetric diversity helps in both the detection and dipole estimation tasks. Specifically, numerical simulations show that polarimetric diversity improves the probability of detection under constant false alarm rate. The improvement was observed both with limited spatial diversity and full spatial diversity. For the dipole estimation, we further show that, without diversity, full 3-dimensional estimation of the dipole moment is untenable due to non-trivial null space. We further showed that availability of direct-path signal generally improves both the detection and the estimation tasks.

In the companion paper [27], we provide a mathematical analysis of the detection performance. For further extension of this work, we may add structured noise such as clutter and examine the detection test-statistics under such scenario. An alternative to GLRT could also be employed for the detection task, such as a Bayesian framework.

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A. Proof of Theorem \[ \mathcal{H}_1 \]

Define the inner product
\[
\langle f, g \rangle_K = \int g^H K f d\omega d\omega' = \int g^H \Sigma^{-1} f d\omega
\]  
(68)

and the associated norm as
\[
\|f\|_K = \sqrt{\langle f, f \rangle_K} = \sqrt{\int f^H \Sigma^{-1} f d\omega}.
\]  
(69)

Under the Gaussian noise assumptions in Section \[ IV-D \] we have the following log-likelihood function under \( \mathcal{H}_1 \)
\[
\ell_{\mathcal{H}_1}(\hat{p}, s^{DP}, r_e) = -\frac{1}{2} \left( \|d - \hat{p}D r_e\|_{K_{TP}}^2 + \|d^{DP} - \hat{p}D^{DP} s^{DP}\|_{K_{DP}}^2 \right),
\]  
(70)

and under \( \mathcal{H}_0 \)
\[
\ell_{\mathcal{H}_0}(\hat{p}, s^{DP}) = -\frac{1}{2} \left( \|d\|_{K_{TP}}^2 + \|d^{DP} - \hat{p}D^{DP} s^{DP}\|_{K_{DP}}^2 \right).
\]  
(71)

The \( \langle 56 \rangle \) implies that we can then solve the MLE problem by maximizing \( \langle 70 \rangle \) w.r.t. \((\hat{p}, s^{DP}, r_e)\) and maximizing \( \langle 71 \rangle \) w.r.t. \((\hat{p}, s^{DP})\) and taking their difference.

Let
\[
\tilde{s} = [r_e^T, (s^{DP})^T]^T
\]  
(72)

and
\[
\tilde{K}(\omega, \omega') = \Sigma \delta(\omega - \omega').
\]  
(73)
By definition of Gâteaux derivative we have that

\[
\ell_{\mathcal{H}_1}(\hat{p}, \hat{s}) = -\frac{1}{2} \left( \left\| \tilde{d} - \hat{p} \tilde{D}s \right\|_K^2 \right). \tag{74}
\]

We maximize (74) first over \( \hat{p} \) and find \( \hat{p} \) in terms of \( \hat{s} \), then maximize it over \( \hat{s} \). By maximizing over \( \hat{p} \), we arrive at a solution given by the following Lemma.

**Lemma 1.** Given (74),

\[
\hat{p}^* = \operatorname{argmax}_{\hat{p}} \ell_{\mathcal{H}_1}(\hat{p}, \hat{s}) = \frac{\hat{s}^{H} \hat{\Sigma}^{-1} \hat{D}^{H} \hat{d}}{\hat{s}^{H} \hat{\Sigma}^{-1} \hat{s}}. \tag{75}
\]

**Proof.** See Appendix B. \qed

Using Lemma 1 we plug (75) into (74) and we arrive at

\[
\max_{\hat{p}} \ell_{\mathcal{H}_1}(\hat{p}, \hat{s}) = \frac{1}{2} \left( J(\hat{s}) - \left\| \tilde{d} \right\|_K^2 \right) \tag{76}
\]

where

\[
J(\hat{s}) = \int \left\| \hat{d}^{H} \hat{D} \hat{\Sigma}^{-1} \hat{s} \right\|_{\hat{s}^{H} \hat{\Sigma}^{-1} \hat{s}}^{-1} d\omega. \tag{77}
\]

In Appendix C it is shown that maximizing (77) w.r.t. \( \hat{s} \) is equivalent to solving the following generalized eigenvalue problem:

\[
\hat{\Sigma}^{-1} Q_1 \hat{\Sigma}^{-1} \hat{s} = J(\hat{s}) \hat{\Sigma}^{-1} \hat{s}. \tag{78}
\]

The solution to (78) is given by \( \lambda_{\max}(\hat{\Sigma}^{-1/2} Q_1 \hat{\Sigma}^{-1/2}) \). Therefore,

\[
\max_{\hat{p}, \hat{s}} \ell_{\mathcal{H}_1}(\hat{p}, \hat{s}) = \frac{1}{2} \left( \lambda_{\max}(\hat{\Sigma}^{-1/2} Q_1 \hat{\Sigma}^{-1/2}) - \left\| \tilde{d} \right\|_K^2 \right). \tag{79}
\]

We can follow similar procedure to solve maximization of (77) and arrive at

\[
\max_{\hat{p}, \hat{s}_{DP}} \ell_{\mathcal{H}_0}(\hat{p}, \hat{s}_{DP}) = \frac{1}{2} \left( \lambda_{\max}(\hat{\Sigma}_{DP})^{-1/2} Q_0(\hat{\Sigma}_{DP})^{-1/2} \right. \left. - \left\| \tilde{d} \right\|_K^2 \right). \tag{80}
\]

Noting that common scalar term \( 1/2 \) is irrelevant, we arrive at our conclusion that

\[
\lambda(\hat{x}) = \lambda_{\max}(\hat{\Sigma}^{-1/2} Q_1 \hat{\Sigma}^{-1/2}) - \lambda_{\max}(\hat{\Sigma}_{DP})^{-1/2} Q_0(\hat{\Sigma}_{DP})^{-1/2}. \tag{81}
\]

**B. Proof of Lemma 7**

Since (74) is Fréchet differentiable we use Gâteaux derivative to differentiate (74) w.r.t. \( \hat{p} \). First note that

\[
\argmax_{\hat{p}} \ell_{\mathcal{H}_1}(\hat{p}, \hat{s}) = \argmax_{\hat{p}} 2 \Re \{ \langle \tilde{d}, \hat{p} \tilde{D}s \rangle_K \} - \left\| \hat{p} \tilde{D}s \right\|_K^2. \tag{82}
\]

By definition of Gâteaux derivative we have that

\[
D \left( \left\| \hat{p} \tilde{D}s \right\|_K^2 \right) (h) = D(\langle \tilde{D}\hat{s}\hat{p}, \tilde{D}\hat{s}\hat{p} \rangle_K)(h) \tag{83}
\]

\[
= 2 \Re \{ \langle \tilde{D}\hat{s}\hat{p}, \tilde{D}\hat{s}\hat{h} \rangle_K \} \tag{84}
\]

\[
= 2 \Re \left\{ \int \tilde{s}^{H} \tilde{D}^{H} \hat{\Sigma}^{-1} \tilde{D} \hat{s} \hat{p}^{*} d\omega \right\} \tag{85}
\]

\[
= 2 \Re \left\{ \int \tilde{s}^{H} \hat{\Sigma}^{-1} \tilde{D} \hat{s} \hat{p}^{*} d\omega \right\} \tag{86}
\]

\[
= 2 \Re \{ \langle \tilde{s}\hat{p}, \hat{s}\hat{h} \rangle_K \} \tag{87}
\]

\[
= 2 \Re \{ \langle \tilde{s}, \hat{s} \hat{h} \rangle_K \} \tag{88}
\]

\[
= 2 \Re \{ \langle \tilde{s}, \hat{h} \rangle_K \} \tag{89}
\]
where, in the last two lines, we used the fact that diagonal matrices commute and that \( \hat{\mathbf{D}} \) is unitary for every \( \omega \). By similar computation, we have that

\[
D(\langle \mathbf{d}, \hat{p} \hat{\mathbf{D}} \bar{s} \rangle)(h) = \langle \mathbf{d}, h \hat{\mathbf{D}} \bar{s} \rangle_{\bar{\mathbf{K}}} = \langle \hat{\mathbf{D}}^H \mathbf{d}, h \bar{s} \rangle_{\bar{\mathbf{K}}}
\]

\[
= \int \bar{s}^H \hat{\Sigma}^{-1} \hat{\mathbf{D}}^H dh^* d\omega. \tag{88}
\]

Thus, we have

\[
D \ell(\hat{p}, \bar{s}) = \text{Re}\{\langle \hat{\mathbf{D}}^H \mathbf{d}, \bar{s} \rangle_{\bar{\mathbf{K}}} - \langle \bar{s} \hat{p}, \bar{s} \rangle_{\bar{\mathbf{K}}}\}. \tag{89}
\]

Setting (89) equal to zero implies the desired result.

\[C. \text{ Derivation of (78)}\]

Note that \( \bar{s} \) is not a function of \( \omega \). Under the assumption that \( |\rho|^* > 0 \), the integrand of \( J(\bar{s}) \) is Lipschitz w.r.t. \( \bar{s} \). Furthermore, \( \hat{\mathbf{D}} \) is continuous w.r.t. \( \omega \) and so is \( \hat{\mathbf{d}} \) given that \( \hat{p} \) is continuous w.r.t. \( \omega \). Since the domain of integration, \( \Omega_B \), is compact, and the integrand continuous w.r.t. \( \omega \), the integrand is Lipschitz w.r.t. \( \omega \). Thus, by Lebesgue Dominated Convergence theorem we can interchange the integral and the gradient operator arriving at

\[
\nabla_{\bar{s}} J(\bar{s}) = \int \nabla_{\bar{s}} \left( \frac{|\hat{d}^H \hat{\mathbf{D}} \hat{\Sigma}^{-1} \bar{s}|^2}{\bar{s}^H \hat{\Sigma}^{-1} \bar{s}} \right) d\omega. \tag{90}
\]

We now use the product rule to evaluate the gradient of the integrand. First, we have that

\[
\nabla_{\bar{s}} |\hat{d}^H \hat{\mathbf{D}} \hat{\Sigma}^{-1} \bar{s}|^2 = 2 \hat{\Sigma}^{-1} \hat{\mathbf{D}}^H \hat{d} \hat{\mathbf{D}} \hat{\Sigma}^{-1} \bar{s}. \tag{91}
\]

Next, we have by the chain rule,

\[
\nabla_{\bar{s}} \frac{1}{\bar{s}^H \hat{\Sigma}^{-1} \bar{s}} = -\frac{2 \hat{\Sigma}^{-1} \bar{s}}{(\bar{s}^H \hat{\Sigma}^{-1} \bar{s})^2}. \tag{92}
\]

Using (91) and (92), we have

\[
= 2 \int \left( \frac{\hat{\Sigma}^{-1} \hat{\mathbf{D}}^H \hat{d} \hat{\Sigma}^{-1} \bar{s}}{\bar{s}^H \hat{\Sigma}^{-1} \bar{s}} - \frac{|\hat{d}^H \hat{\mathbf{D}} \hat{\Sigma}^{-1} \bar{s}|^2}{(\bar{s}^H \hat{\Sigma}^{-1} \bar{s})^2} \right) d\omega \tag{93}
\]

\[
= \frac{2}{\bar{s}^H \hat{\Sigma}^{-1} \bar{s}} \left( \hat{\Sigma}^{-1} Q_1 \hat{\Sigma}^{-1} - J(\bar{s}) \hat{\Sigma}^{-1} \right) \bar{s}.
\]

Plugging in (93) into (90) and setting \( \nabla_{\bar{s}} J = 0 \) we arrive at the generalized eigenvalue problem in (78).

\[D. \text{ Proof of Corollary (7)}\]

By setting \( \mathbf{d}^{DP} \equiv \mathbf{0} \), we have that \( \hat{\mathbf{D}}^{DP} = \mathbf{0} \). This leads to

\[
\hat{\Sigma} = \begin{bmatrix} \Sigma^{TP} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{94}
\]

Note that \( \hat{\Sigma} \) is no longer invertible. We substitute \( \hat{\Sigma}^{-1} \) by its Moore-Penrose pseudoinverse,

\[
\hat{\Sigma}^+ = \begin{bmatrix} (\Sigma^{TP})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{95}
\]

This leads to replacing \( \hat{\Sigma}^{-1/2} Q_1 \hat{\Sigma}^{-1/2} \) by

\[
(\hat{\Sigma}^+)^{1/2} Q_1 (\hat{\Sigma}^+)^{1/2} \tag{96}
\]
Let
\[ R = \int (D^{TP})^H d^{TP} (d^{TP})^H d^{TP} d\omega. \] 

(97)

Then,
\[ (\hat{\Sigma}^+)^{1/2} Q_1 (\hat{\Sigma}^+)^{1/2} = \begin{bmatrix} (\Sigma^{TP})^{-1/2} R (\Sigma^{TP})^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} \] 

(98)

We also have that \( Q_0 = 0 \) and \( (\Sigma^{DP})^+ = 0 \). This leads to the test statistic \((62)\) in Theorem \([1]\) being equivalent to \((63)\).