Nuclear Shell Model Calculations of Neutralino-Nucleus Cross Sections for $^{29}$Si and $^{73}$Ge

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Abstract. We present the results of detailed nuclear shell model calculations of the spin-dependent elastic cross section for neutralinos scattering from $^{29}$Si and $^{73}$Ge. The calculations were performed in large model spaces which adequately describe the configuration mixing in these two nuclei. As tests of the computed nuclear wave functions, we have calculated several nuclear observables and compared them with the measured values and found good agreement. In the limit of zero momentum transfer, we find scattering matrix elements in agreement with previous estimates for $^{29}$Si but significantly different than previous work for $^{73}$Ge. A modest quenching, in accord with shell model studies of other heavy nuclei, has been included to bring agreement between the measured and calculated values of the magnetic moment for $^{73}$Ge. Even with this quenching, the calculated scattering rate is roughly a factor of 2 higher than the best previous estimates; without quenching, the rate is a factor of 4 higher. This implies a higher sensitivity for germanium dark matter detectors. We also investigate the role of finite momentum transfer upon the scattering response for both nuclei and find that this can significantly change the expected rates. We close with a brief discussion of the effects of some of the non-nuclear uncertainties upon the matrix elements.
I. Introduction

A host of astronomical evidence points to the existence of large amounts of dark matter in the Universe [1]. Despite the overwhelming amount of evidence for this dark matter’s existence, its exact nature remains a mystery. Numerous candidates have been proposed. These include both ordinary baryonic and non-baryonic matter [2]. Among the best motivated, and hence highly favored, of non-baryonic candidates is the lightest supersymmetric particle (LSP). Experimental and theoretical considerations suggest that the LSP is a neutralino, $\tilde{\chi}$, made up of a linear combination of the photino, Z-ino, and 2 higgsinos (or equivalently, the B-ino, neutral W-ino, and 2 higgsinos)

$$\tilde{\chi} = Z_1 \tilde{B} + Z_2 \tilde{W}_3 + Z_3 \tilde{H}_1 + Z_4 \tilde{H}_2.$$  

The neutralino is an ideal dark matter candidate. The motivation for its existence arises naturally in modern theories of particle physics [3], not as an ad hoc solution to the dark matter problem. For a very large region of supersymmetric parameter space, neutralinos provide densities sufficient to account for the mass-energy density of the Universe [4]. The $\tilde{\chi}$ also possesses the virtue of potential detectability. Its detection may be possible in at least two ways: indirectly, through the products of $\tilde{\chi}\tilde{\chi}$ annihilation from capture in the Sun or Earth [5], or directly, via elastic neutralino-nucleus, $\tilde{\chi}N$, scattering in a detector [6,7]. In either case, the $\tilde{\chi}N$ elastic scattering cross section is an essential ingredient. In this paper we discuss detailed nuclear structure calculations relevant to $\tilde{\chi}N$ scattering for two important elements, silicon and germanium.

Neutralino-nucleus scattering is governed by physics at several energy scales. The mass and the mixing of the $\tilde{\chi}$, and hence its interactions with quarks, are fixed near the electroweak scale. The spin-dependent interactions of the $\tilde{\chi}$ with protons and neutrons are determined by the distribution of quark spin within the nucleon, which depends upon physics at the QCD scale. There are also spin-independent contributions which depend upon the quark content of the nucleons. At the modest momentum transfers available to dark matter particles ($O(\text{MeV})$) the $\tilde{\chi}$ interacts with the entire nucleus, not individual nucleons within it. Thus, nuclear structure plays an important role in determining the strength of the $\tilde{\chi}N$ cross section. The uncertainties inherent in the electroweak and QCD-scale physics can be parameterized in different ways. The electroweak parameterization entails choosing the exact composition and mass of the $\tilde{\chi}$ which, in turn, is determined by parameters in a Lagrangian of a Supersymmetric model. At the QCD level there are currently two competing possibilities for the coupling of protons and neutrons to the $\tilde{\chi}$. The currently favored parameterization depends upon the measured quark spin content of the proton determined by the European Muon Collaboration (EMC) [8]. In contrast to
these values of the spin content are those derived from the Naive Quark Model (NQM). Experiments are being carried out which will hopefully clarify this issue. In this paper, we will investigate the effects of both the EMC and the NQM estimates, as well as the uncertainties in their determination.

At the nuclear level, several attempts have been made to improve the cross section calculations. Initial investigations into spin-dependent $\chi N$ scattering made use of the independent single particle shell model (ISPSM) in which the character of the nucleus is completely determined by a single unpaired valence nucleon [6,9,10]. Subsequent studies [11-15] have revealed a number of inadequacies of the ISPSM and improved upon its estimates of the scattering matrix element. Engel and Vogel [11] used the odd group model (OGM) and mirror pair $\beta-\text{decays}$, in an extended odd group model (EOGM), to show that the ISPSM can be off by a large factor when nuclei are far from closed shells (see also ref. [12]). Pacheco and Strottman [13] reached the same conclusion by performing detailed nuclear shell model calculations for several light nuclei. The odd group and shell model treatments obtain good agreement for light nuclei but, as the atomic mass increases, nuclear configuration mixing produces cancellations not considered in the OGM. This is most clearly evident in $^{35}\text{Cl}$ (the largest nucleus considered in ref. [13]) where the OGM and shell model estimates of the matrix elements differ by roughly a factor of 3 (see Table 1). For comparison, in Table 1 we have also included our calculation of the spin in $^{35}\text{Cl}$ using the same interaction and model space as discussed for $^{29}\text{Si}$ in Section IIa. It is reassuring to note that the two shell model calculations are in close agreement despite the use of different interactions. This effect has also been seen in $^{93}\text{Nb}$ [14] revealing the need for full shell model calculations of heavier nuclei.

The need for more detailed calculations of the spin content of heavier nuclei is exacerbated by the fact that at least two of the most promising detector programs currently under way are based upon $^{73}\text{Ge}$ and $^{29}\text{Si}$ [15]. Both of these nuclei are likely to have large amounts of configuration mixing, making OGM estimates suspect. Furthermore, both $^{73}\text{Ge}$ and $^{29}\text{Si}$ require very large model spaces. This is why the calculations were not carried out in ref. [13]. In fact, in a recent review [7] the statement was made that, “...until a reliable calculation in $^{73}\text{Ge}$ has been carried out, the description of the nuclear physics of dark matter detection will be incomplete.”

With these last comments as motivation, we have used the nuclear shell model code CRUNCHER [16] to perform large basis calculations of $^{73}\text{Ge}$ and $^{29}\text{Si}$ with the aim of providing accurate cross sections for determining event rates for spin-dependent scattering in the Ge and Si detectors currently under development.

All of the work discussed above has been in the zero momentum transfer limit. As the
experimental lower limit to the $\tilde{\chi}$ mass continues to increase, finite momentum transfer needs to be considered. This is especially true for heavier nuclei, such as $^{73}\text{Ge}$ [7]. Unfortunately, phenomenological models, such as the OGM and the Interacting Boson Fermion Model (IBFM) [17] which have been used in the $q = 0$ limit cannot be easily extended to finite momentum transfer. Furthermore, the ISPSM has been shown to be a poor approximation at finite $q$ [7,14,18]. Fortunately, incorporating finite momentum transfer into a full nuclear shell model calculation is straightforward. The formalism is presented in [7]. It is a simple extension of that used in semi-leptonic weak and electromagnetic interactions with nuclei [19]. In Section IV, we show that finite momentum transfer is important for both $^{29}\text{Si}$ and $^{73}\text{Ge}$.

II. The Zero Momentum Transfer Limit

Neutralinos in the halo of our Galaxy can be characterized by a mean virial velocity of, $v \simeq \langle v \rangle \simeq 300\text{km/sec} = 10^{-3}c$. The maximum characteristic momentum transfer in $\tilde{\chi}N$ scattering will be $q_{max} = 2M_r v$ where $M_r$ is the reduced mass of the $\tilde{\chi}N$ system. If the product $q_{max}R$ is small ($\ll 1$), where $R$ is the nuclear size, the matrix element for spin-dependent $\tilde{\chi}N$ scattering reduces to a very simple form [7,11]

$$\mathcal{M} = A\langle N|a_pS_p + a_nS_n|N \rangle \cdot s_{\tilde{\chi}}$$

where

$$S_i = \sum_k s_i(k), \quad i = p, n$$

is the total nuclear spin operator, $k$ is a sum over all nucleons, and $a_p$, $a_n$ are $\tilde{\chi}$-nucleon coupling constants which depend upon the quark spin-distribution within the nucleons and on the composition of the $\tilde{\chi}$, see e.g. eqs. (12) and (13) for the specific cases of the $\tilde{B}$ and $\tilde{\gamma}$. Much of the uncertainties arising from electroweak and QCD scale physics are encompassed by $a_p$ and $a_n$. The normalization $A$ involves the coupling constants, masses of the exchanged bosons and various LSP mixing parameters that have no effect upon the nuclear matrix element. To maintain contact with the previous literature [4,6,10,11,13], we note that eq. (2) has often been written as

$$\mathcal{M} = A\lambda\langle N|J|N \rangle \cdot s_{\tilde{\chi}}$$

with

$$\lambda = \frac{\langle N|a_pS_p + a_nS_n|N \rangle}{\langle N|J|N \rangle} = \frac{\langle N|(a_pS_p + a_nS_n) \cdot J|N \rangle}{J(J+1)}. \quad (4b)$$

Examples of the full $\tilde{\chi}N$ cross section can be found in the appendix and in refs. [4,7,10,11].

Calculations of the matrix element in eq. (2) are straightforward, although computationally intensive, in the nuclear shell model.
Our approach to modeling the nuclei $^{29}$Si and $^{73}$Ge is quite straightforward in principle. Using a reasonable two-body interaction Hamiltonian we calculate the nuclear wave functions in an appropriate model space. As checks on these wave functions we compute the excited state energy level spectrum, the magnetic moments, and the single particle spectroscopic factors for each nucleus and compare them with experimentally measured quantities. Once reasonable agreement between the calculations and measurements is obtained the ground state wave functions are used to calculate the $\chi N$ nuclear matrix element, $\langle N|a_pS_p + a_nS_n|N \rangle$ of eq. (2). (We present our results using the convention that all angular momentum operators are evaluated in their $z$-projection in the maximal $M_J$ state, e.g. $\langle S \rangle \equiv \langle N|S|N \rangle = \langle J, M_J = J|S_z|J, M_J = J \rangle$.) These wave functions are also used to compute the more complicated finite momentum transfer matrix elements. In practice these steps may be quite time consuming. In the following two subsections we detail the steps entailed in calculating the nuclear matrix elements for spin-dependent $\chi N$ scattering with $^{29}$Si and $^{73}$Ge.

IIa. $^{29}$Si

For the nucleus $^{29}$Si, the calculation of the nuclear wave functions and matrix elements is almost as straightforward as presented above, because a well defined and tested interaction in a well defined model space exists [20]. This greatly facilitates the calculation. In fact, after the completion of these calculations, it was brought to our attention that the calculation of $^{29}$Si’s spin had already been done, using the same interaction and model space, but in a different context [21]. In spite of this, we will detail our calculation for two reasons. First, the calculation of the $^{29}$Si matrix element will serve as a useful comparison for the accuracy of the calculation of $^{73}$Ge’s matrix element. The $^{29}$Si computation will help to establish just how reliably one can use the shell model to determine the relevant observables. Second, we will show in a subsequent section that finite momentum transfers are important for Si and this calculation does not exist in the literature.

For our calculation of the wave functions for $^{29}$Si we have used the USD interaction of Wildenthal [20] in a full sd shell model space. This interaction has been meticulously developed and tested over many years and is known to accurately reproduce numerous nuclear observables for sd shell nuclei. Our computations were performed using the Lanczos method m-scheme nuclear shell model code CRUNCHER [16] and its auxiliary codes. The m-scheme basis for $^{29}$Si in this model space has a dimension of 80115 Slater determinants. The calculation of the wave functions takes roughly a day and a half of time on a dedicated SPARC station.

In Fig. 1 we present the calculated vs. the measured energy spectrum [22] for the 10
lowest eigenstates of $^{29}\text{Si}$. It is clear that good overall agreement is achieved.

A test of more relevance to the $\tilde{\chi}N$ scattering cross section is the comparison between the predicted and measured ground-state magnetic moment for $^{29}\text{Si}$. This is because of the similarity of the relevant operators. The magnetic moment, $\mu$, is given by:

$$\mu = \langle N|g_n^s S_n + g_n^l L_n + g_p^s S_p + g_p^l L_p|N \rangle$$

(5)

where the $S_i$ are the total spin operators (in their $z$ projection) from eq. (3), and the $L_i$ are the analogous orbital angular momentum operators. The free particle $g$-factors are given by: $g_n^s = -3.826$, $g_n^l = 0$, $g_p^s = 5.586$, and $g_p^l = 1$ (in nuclear magnetons). There is ample evidence in shell model calculations for many nuclei that these $g$-factors are quenched. In $^{28}\text{Si}$ this quenching is not necessary and is, in this context, counterproductive. We will address this issue again in a later section (quenching seems to be necessary to get $\mu$ “right” in $^{73}\text{Ge}$). Using our wave functions we find $\mu_{\text{calc}} = -0.50$, in agreement with the previous calculation [21]. This is to be compared with the measured value of $\mu_{\text{exp}} = -0.555$, which is good agreement by shell model standards. We also note that the ISPSM gives a value of $\mu_{\text{ISPSPM}} = -1.91$, in strong disagreement with experiment.

As a final check on the accuracy of our wave functions, we have calculated the spectroscopic factors for a number of one nucleon transfer reactions and compared them to the experimentally determined values. Spectroscopic factors $S(J; jJ_0)$ are defined in [23] as

$$S(J; jJ_0) = |\langle \psi(J,M) | \Psi(j,J_0; J,M) \rangle|^2$$

(6)

where $\Psi(j,J_0; J,M)$ is the state obtained by coupling an odd nucleon with angular momentum $j$ to the core with angular momentum $J_0$ to obtain total angular momentum $J$. $\psi(J,M)$ is the true nuclear wave function. As an example, we compute $S$ for the reaction $^{28}\text{Si} + d \rightarrow ^{29}\text{Si} + p$ going from the ground state of $^{28}\text{Si}$, $J = 0$, to the ground state of $^{29}\text{Si}$, $J = 1/2$, in the ISPSM. Since the incoming neutron has spin 1/2 the only amplitude which can contribute (in the ISPSM) is that in which the neutron is in a $L = 0$ state. Thus, we have $S(1/2; 1/2, 0) = 1$. (Note, experimental data is generally presented as $(2J+1)S(J; j, J_0)$ and we will adhere to that convention.) In a full shell model calculation (and real nuclei), configuration mixing complicates this simple picture.

To calculate the $S(J; j, J_0)$ for $^{29}\text{Si}$, we have computed the ground state wave function for $^{28}\text{Si}$ using the same Hamiltonian and model space. As detailed in ref. [16] we then act on this wave function with a neutron creation operator and take the scalar product of the result with the relevant $^{29}\text{Si}$ wave function. The most important calculation is, of course, the ground state to ground state transition since it most directly probes the nuclear configurations which can interact with the $\tilde{\chi}$. As an additional check, we have
calculated the spectroscopic factors to the first two excited states as well. The results of these calculations are presented in Table 2 along with an unweighted average of the comparable experimental data from Endt [24]. We also present the ISPSM results. Table 2 reveals the good agreement between the full shell model calculation and experiment as well as the inadequacy of the ISPSM. This provides confidence in our wave functions and our calculations of the $\tilde{\chi}N$ matrix elements.

Once we have suitable wave functions, the nuclear matrix element in eq. (2) ($\langle N|a_p S_p + a_n S_n |N\rangle$) is easily calculated for any set of $a_p$ and $a_n$. Here we compute the spins in eq. (2), and then multiply them by the desired $a_i$ to find the total matrix element. In Table 3 we present the results of this exercise using our wave functions. Our results agree with those found in ref. [21]. In Table 3 we also present the results found in the ISPSM, the OGM, and the extended OGM approach. It is immediately apparent that the ISPSM is a very poor approximation to the actual nuclear configuration. Configuration mixing causes the total angular momentum to be shared between the nuclear spin and orbital angular momentum, contrary to the assumptions of the ISPSM (the total angular momentum resides in the spin of the $2S_{1/2}$ neutron). It is also apparent that the OGM treatments and the shell model calculations are in fair agreement as to the properties of $^{29}\text{Si}$. The important point here is that the protons contribute negligibly to the relevant nuclear properties, the key assumption of the OGM. We will find, however, that this assumption is violated for $^{73}\text{Ge}$.

IIb. $^{73}\text{Ge}$

The availability of a well developed and tested interaction in the $sd$ shell made the calculation of the $^{29}\text{Si}$ ground state wave function relatively simple. For our calculation of the ground state of $^{73}\text{Ge}$ we did not have the luxury of a good (in the sense that it has been phenomenologically determined and extensively tested) Hamiltonian and there is not a well defined model space. Hence there is more uncertainty in determining a good nuclear state which accurately reproduces the measured observables. There is some evidence that $^{73}\text{Ge}$ may be deformed [7] meaning that a large model space may be required. The extent to which our wave functions can be trusted will be revealed by how well they reproduce observable phenomena. Our approach has been to choose fairly large model spaces which should include most of the relevant excitations and apply a fairly simple but well motivated interaction to the nucleons within this space. We then examine the ordering and spacing of the excited state energy levels. The single particle energies of the interaction are then modified and the procedure is repeated. Thus, by a process of iteration, we have been able to reproduce the low lying energy levels in $^{73}\text{Ge}$. This procedure has been previously used with some success in this mass region [25]. As before, once a suitable fit to the energy
spectrum is obtained, the magnetic moment and spectroscopic factors are calculated as additional checks on the wave function. When we are convinced that we have reasonable wave functions, we then calculate the spin distribution. We now present this procedure in some detail.

In our study of $^{73}\text{Ge}$, we have chosen to use the PMMA interaction [26]. This interaction is an analytic approximation to the Kallio and Kolltvit [27] interaction, which is a reasonable approximation to a full $G$-matrix calculation. This interaction has proven to be both adequate and tractable in shell model calculations. As discussed above, the single particle energies were varied in order to match the energy spectrum of low lying excited states. Using this interaction we have investigated two different model spaces. The first, which we will refer to as the small space, has a m-scheme basis dimension of 24731 Slater determinants. Despite the fairly large size of the bases, rather severe truncations in the space have been enacted. The small space is the smallest in which we could obtain agreement with the spectrum. The large basis dimension was determined by computer time and memory constraints. Each iteration (of which there were many) consumed roughly 4 days of SPARC time and several hundred Mbytes of memory.

Each of these model spaces is “customized” to $^{73}\text{Ge}$ and requires some explanation. In standard notation the small space may be described by: $((1f_{5/2}, 2p_{3/2}, 2p_{1/2})^{16} (1g_{9/2})^{1} (2d_{5/2}, 1g_{7/2})^{0} + (1f_{5/2}, 2p_{3/2}, 2p_{1/2})^{16} (1g_{9/2})^{0} (2d_{5/2}, 1g_{7/2})^{1} + (1f_{5/2}, 2p_{3/2}, 2p_{1/2})^{15} (1g_{9/2})^{2} (2d_{5/2}, 1g_{7/2})^{0})$ In words: the 16 particles in the $1f_{5/2}, 2p_{3/2},$ and $2p_{1/2}$ shells may “move” freely about in these shells, there is also 1 particle (a neutron) in the $1g_{9/2}$; additionally, excitations of the $1g_{9/2}$ neutron into the $2d_{5/2}$ or $1g_{7/2}$ shells is allowed, or 1 particle from the $1f_{5/2}, 2p_{3/2},$ and $2p_{1/2}$ shells may be excited into the $1g_{9/2}$ orbital (there it can pair with the already present neutron). In the same notation, the large model space allows configurations given by: $((1f_{5/2}, 2p_{3/2})^{14} (2p_{1/2}, 1g_{9/2})^{3} (2d_{5/2}, 1g_{7/2})^{0} + (1f_{5/2}, 2p_{3/2})^{14} (2p_{1/2}, 1g_{9/2})^{2} (2d_{5/2}, 1g_{7/2})^{1} + (1f_{5/2}, 2p_{3/2})^{13} (2p_{1/2}, 1g_{9/2})^{4} (2d_{5/2}, 1g_{7/2})^{0})$. The major difference here is that it is now possible to have up to 4 particles in the $1g_{9/2}$ orbital.

In Fig. 2 we show the excited state energy spectrum of the first 10 calculated states compared to the measured spectrum in the same energy range for both the large and the small spaces. In the small space we were able to iterate until essentially perfect agreement between theory and experiment was obtained for the lowest lying five excited states. There is then a large region where there are many observed states but no calculated states. This is because our limited model space is not large enough to contain all of the 2-particle 1-hole excitations necessary to describe such states. The omission of these seniority 3
configurations, from the ground state, is probably not too serious. At excitation energies \( \gtrsim 1 \text{ MeV} \) there are again several calculated states which seem to have analogs in the measured spectrum of states.

With the greater number of available excitations in the large space one would expect to start describing states in the spectrum between 0.5 MeV and 1 MeV, and this is observed for the 10 lowest lying states, as demonstrated in the right side of Fig. 2. In the large space it was impractical and unnecessary to iterate until ‘perfect’ agreement was obtained for the lowest energy states (compare the scales of Figs. 1 and 2 and the relative errors). We did iterate until the small changes in the single particle energies needed to move the states no longer affected the spin matrix elements significantly. Figure 2 reveals reasonable agreement between our low lying calculated states and the observed distribution of states giving us reasonable assurance that we have obtained a good description of the \(^{73}\text{Ge}\) ground state. We now apply our other tests in the hope of confirming this statement.

As previously noted, the most relevant test for the spin matrix elements is the comparison of the magnetic moments. Here our calculations do an acceptable job but some quenching of the \( g \)-factors is required to obtain complete agreement with experiment. All other attempts to calculate the matrix elements have relied heavily upon quenching as well: the IBFM [17] and the OGM [7,11] (which can be viewed as a well motivated prescription for quenching the ISPSM value of \( \mu \)). In fact, the quenching required by our wave functions is significantly less than in either of the above methods. We will discuss the issue of quenching in a subsequent section. The measured magnetic moment of \(^{73}\text{Ge}\) is \( \mu_{\text{exp}} = -0.879 \) which is significantly different from the ISPSM value of \( \mu_{\text{ISPSM}} = -1.91 \). The values we calculate (without quenching) are: \( \mu_{ss} = -1.468 \) for the small space and \( \mu_{ls} = -1.239 \) for the large space, a major improvement over the ISPSM, especially for the large space. The main reason for the decrease in \( \mu \) can be seen in Table 3. Due to configuration mixing, the protons significantly contribute to the total angular momentum, especially through their orbital part. Referring to eq. (5) it is obvious that a large value of \( \langle L_p \rangle \) can have a significant effect on \( \mu \). This large value for \( L_p \) is one of the major results of the present work and the primary reason that our answers will differ from those of previous analyses. While one might wish for better agreement between the measured and calculated magnetic moments, by typical shell model standards [28,29], reasonable agreement has been achieved. Thus we continue to have confidence in our results.

The final test we have performed for \(^{73}\text{Ge}\) is to compare the calculated and measured spectroscopic factors, eq. (6), for the reaction \(^{72}\text{Ge} + d \rightarrow ^{73}\text{Ge} + p \) [30]. As with the magnetic moment comparison, we find quite reasonable results, albeit not without some level of ambiguity. We present our results, the experimental data, and the ISPSM
values for the three lowest lying states in Table 4. For the ground state to ground state reaction, labeled 1g9/2 in Table 4, good agreement is obtained between the computed and experimental value for both model spaces. Since this is the reaction which directly tests the state we are interested in, this is quite encouraging. The small space obviously does not allow enough excitations to correctly describe the excited state transitions. The large space does a much better job on the excited states but if one were interested in their properties, a better description might be desirable. Fortunately, we are only interested in the ground state of 73Ge which seem to be adequately described in these model spaces.

After examining the comparisons of the experimental and calculated values of the excited state spectra, magnetic moments, and spectroscopic factors we believe that we have obtained a reasonable description of the 73Ge ground state. The case for this is not unassailable but our wave functions do include pieces of physics which other analyses have neglected and they do reproduce a number of observables with good accuracy. Having established the viability of our calculation of the ground state wave function of 73Ge, we now discuss 73Ge–˜χ scattering.

The primary results of this section have already been presented in Table 3, which shows our results for the spin and orbital pieces of total angular momentum (in their z-projection). We compare our results to those found in the ISPSM, the OGM, and the IBFM. We will confine our comments henceforth to the large space calculation since it seems to be a better representation of the ground state. The small space serves primarily to illustrate the sensitivity to configuration mixing. The fact that an increase in the size of the model space leads to an improvement in the magnetic moment is especially heartening. Table 3 shows that these effects are small but finite. As a final comment regarding Table 3 before we discuss its contents, we point out that: there are two rows for the IBFM, in the first row the required quenching should be applied to ap and an in the matrix elements (this corresponds to quenching the g-factors for µ), in the second the same quenching has, instead, been applied to the spins [7] (here the free particle g-factors would be used for the magnetic moment). These two approaches are equivalent but help to highlight the different results obtained in the IBFM and shell model approaches.

The most interesting feature of Table 3 is the large contribution of the proton’s orbital angular momentum found in our calculations. This large value of ⟨Lp⟩ results in significant differences in the spin matrix elements found in this investigation vs. those found in the OGM and IBFM approaches. The proton’s orbital angular momentum makes a major contribution to the magnetic moment which is completely neglected in the OGM and not adequately represented in the IBFM. Surprisingly (and coincidentally), our results for the spin tend to agree with those found in the ISPSM. A large contribution to the angular
momentum by the protons violates the assumptions of the OGM. This explains the factor of 2 difference in $\langle S_n \rangle$ between the OGM and our approach. (In a shell model treatment of $^{93}$Nb, a similar result was found, except there it is the neutrons carrying a large amount of orbital angular momentum which, once again, invalidates the OGM [14].) The IBFM wave functions do not possess a large proton contribution to the angular momentum of the nucleus and hence produce an unquenched magnetic moment of $\mu_{IBFM} = -1.785$, similar to the single particle model. This is to be contrasted with the shell model value of $\mu = -1.239$. Thus, in the IBFM a very large quenching factor of 0.523 must be applied to the spin matrix elements (i.e. either $a_i$ or $\langle S_i \rangle$ as is shown in Table 3) in order to match the measured magnetic moment of $\mu_{exp} = -0.879$. If the same procedure is applied to the shell model matrix elements (a procedure we do not recommend, see the next section) a quenching factor of only 0.792 is required. This is considerably less quenching of the spin than in the IBFM. If we apply this quenching factor to the large space spins in Table 3 (in the same manner as was done to create the second IBFM row), we find $\langle S_n \rangle = 0.371$ and $\langle S_p \rangle = 0.009$, still quite different from the OGM and the IBFM (and now the ISPSM) results. Thus, we find that a full shell model calculation using realistic wave functions obtains significantly different values for the spin matrix elements than previous analyses. In fact, our results could result in up to a four-fold increase in the $\tilde{\chi}N$ scattering rate over the estimates in previous work. This implies a much greater sensitivity for germanium dark matter detectors. We are now ready to investigate the scattering response at finite momentum transfer but first we address the issue of quenching in more detail.

III. Quenching of the Spin Matrix Elements

The concept of quenching of the magnetic moment $g$-factors and Gamow-Teller strength function in shell model calculations has a long history. What is clear is that quenching is almost universally required once nuclei are filled beyond the p-shell. To quote from a thorough investigation of the M1 (magnetic moment) operator in the $sd$ model space [28], “There are several reasons why even ‘perfect’ shell-model calculations of electromagnetic matrix elements based on the free-nucleon characteristics of the neutron and proton should differ from the corresponding experimental values. In reality, the nuclear wave functions must be more complicated than those of the theoretical model we use. Real nuclear states must involve nucleonic degrees of freedom beyond the $sd$-shell space. In addition, non-nucleonic degrees of freedom which involve delta isobars and mesons may be important in the observed phenomena.” This ‘excluded’ physics also plays a role in weak interaction calculations [31]. The preferred method for dealing with these inadequacies of the shell model is to quench the relevant $g$-factors and so construct an effective magnetic moment
operator. In the \textit{sd} [28] and \textit{fp} [29] shells, systematic studies have been made to determine the values of modified \textit{g}-factors which give the best fit to the magnetic moment for a large number of nuclei in each shell. In the \textit{sd} shell study, the operator \((\chi^{(2)} \otimes s)^{(1)}\) was shown to make a significant contribution to the effective magnetic moment operator making comparisons between \(\mu\) and the \(\tilde{\chi}N\) matrix element somewhat more ambiguous. We can use these optimized \textit{g}-factors to estimate the amount of quenching we expect for our calculations of \(\tilde{\chi}N\) scattering on \(^{29}\text{Si}\) and \(^{73}\text{Ge}\).

As mentioned above, quenching in the \textit{sd}-shell with the USD interaction has been thoroughly investigated and an optimum set of quenched \textit{g}-factors has been found [28]. Following Table 3 of [28] the optimized \textit{g}-factors for \(^{29}\text{Si}\) should be: \(g_n^s = -3.24, g_p^s = 4.75, g_n^l = -0.091,\) and \(g_p^l = 1.129\). We note that these spin \textit{g}-values are consistent with an overall spin quenching factor of \(\sim 0.85\). However, we see that it is not only the spin \textit{g}-factors which are modified. Using these values of the \textit{g}-factors in eq. (5) leads to the following magnetic moment \(\mu(\text{quenched}) = -0.45\), slightly worse than obtained with no quenching. This is not surprising, since the above values of the \textit{g}'s were obtained as a global fit to many \textit{sd}-shell nuclei besides \(^{29}\text{Si}\). Since the wave function for \(^{29}\text{Si}\) already matches the measured magnetic moment it is not surprising that quenching damages the agreement. In view of this we do not advocate any quenching for the \(^{29}\text{Si}\) matrix elements.

As with all other aspects of this investigation, things become considerably more complicated when we consider \(^{73}\text{Ge}\). Due to the complexity of the nuclei in this mass region, no systematic studies of quenching have been done. The nearest relevant study of which we are aware involves nuclei in the lower \textit{fp} shell with \(41 \leq A \leq 49\) [29]. In this work, three different interactions are compared and three very different results emerge for the quenched \textit{g}-factors. Additionally, all three estimates of the optimized \textit{g}-factors result in improved agreement between the measured and calculated magnetic moment of \(^{73}\text{Ge}\). For purposes of comparison, in Table 5 we show the free particle \textit{g}-factors, the 3 sets of quenched \textit{g}-factors from ref. [29], and the resultant magnetic moment of \(^{73}\text{Ge}\) obtained using the large space wave function. The key point of Table 5 is that changes in one of the \textit{g}'s can be compensated for by changes in another and the choice of the best set is dependent upon the interaction chosen. Table 5 reveals that the effective \textit{g}-factors from the KB1 [29] interaction produce the best fit to \(\mu_{\text{exp}}\) for \(^{73}\text{Ge}\). These \textit{g}-factors result from an almost exclusive quenching of the \textit{isovector} part of the spin (isovector quenching factor = 0.86).

There is one additional piece of information which may be used to investigate the required quenching, namely Gamow-Teller (GT) \(\beta\)-decays. As with the magnetic moment operator, the GT operator \((\propto g_{A\tau_{\pm}}S)\) requires quenching when comparing shell model GT strengths to the measured values [31]. Since the GT operator is directly related to
the isovector part of the spin, it is a further probe of the quenching of the isovector spin. Several studies have shown that the free nucleon value of $g_A = 1.25$ is on average reduced to a value of $g_A \approx 1$ in heavy nuclei. This corresponds to an isovector spin quenching factor of 0.8. This is in reasonable accord with the isovector quenching of the KB1 interaction discussed above and the similar quenching factor found in the $sd$ shell near $A = 28$, $(4.00/4.76 = 0.84)$ [28]. Finally, we note that this GT quenching plays a role in the mirror pair analysis of Engel and Vogel (through their ratio $R$) [11]. The issue of quenching is thus quite general and quite troublesome when dealing with spin matrix elements of heavy nuclei.

The calculated magnetic moment for $^{73}$Ge with free particle $g$-factors already shows significant improvement in agreement with measured values over previous treatments due simply to the inclusion of configuration mixing. If exact agreement is desired, an effective operator, with quenched $g$-factors must be invoked. If quenching is invoked, we advocate quenching only the isovector spin part. Following this prescription a quenching of the isovector spin $g$-factor by a factor of 0.833 produces a calculated magnetic moment (large space) of $\mu = -0.879$, in perfect agreement with experiment. This quenching factor is in agreement with the expectations of GT decay in heavy nuclei. In addition, it is consistent with the isovector quenching found in the $sd$ [28] and $fp$ [29] shells when fitting to magnetic moment data. By following this prescription we follow the path with greatest motivation and obtain agreement with experiment by quenching only one number, the isovector component of the spin. Using this quenching factor we find effective spin $g$-factors of $g_n^{(eff)} = -3.04 = 0.795 g_n^{(free)}$ and $g_p^{(eff)} = 4.80 = 0.859 g_p^{(free)}$. In the zero momentum transfer limit this quenching is equivalent to the large space matrix elements becoming $\langle S_n \rangle = 0.372$ and $\langle S_p \rangle = 0.009$ (when combined with the free particle $g$-factors). These values should be viewed as lower limits to the spin-matrix elements. These values remain quite different from any of the previous estimates performed in the ISPSM, OGM, or IBFM.

In this section we have shown that a simple quenching of the isovector-spin matrix element by a factor of 0.833 produces agreement with the measured magnetic moment data for $^{73}$Ge. A quenching factor of this type, and roughly this magnitude, could have been predicted based upon previous shell model studies [28,29,31]. Given the amount of evidence which supports this quenching, we feel that the quenched values for the spin matrix elements are the ones most likely to be correct. Hence in calculating the $^{73}$Ge–$N$ scattering cross section, one should use the values in the last row of Table 3 (labeled “large,quenched”) and the normal $a_n$ and $a_p$ or use the values in the row above that with a quenching factor of 0.833 applied to the isovector combination of $a_n$ and $a_p$ while the
isoscalar combination remains unchanged. [To be more explicit: Calculate \( a_1 = a_p - a_n \) and \( a_0 = a_p + a_n \) where nothing has yet been quenched. Using these values of \( a_0 \) and \( a_1 \) recalculate \( a_p \) and \( a_n \) but now quench the isovector piece, \( a_p = \frac{1}{2}(a_0 + 0.833a_1) \) and \( a_n = \frac{1}{2}(a_0 - 0.833a_1) \). These quenched values of the \( a_i \)'s may now be combined with the values of the spin in second to last line of Table 3 to calculate the total matrix element.]

Having addressed the issue of quenching, we may now consider \( \tilde{\chi} \) scattering with \(^{29}\text{Si}\) and \(^{73}\text{Ge}\) at finite momentum transfer. For simplicity, at finite momentum transfer, any quenching will be applied to \( a_n \) and \( a_p \), as described in the square brackets above or in the appendix, and not to the spin matrix elements.

IV. Finite Momentum Transfers

When the LSP was first proposed as a viable dark matter candidate, its preferred mass was between 5 and 10 GeV [32]. With a mass of this order and a typical galactic halo velocity \((v \approx 10^{-3}c)\), the neutralino’s total momentum \((q \sim M_r v \sim 10 \text{ MeV})\) was small compared to the inverse of the nuclear size \((1/R \sim 1/1 \text{ fm} \sim 200 \text{ MeV})\) and the zero momentum transfer limit was appropriate for studies of \( \tilde{\chi}N \) scattering. Since then, experiments at accelerators have pushed the allowed \( \tilde{\chi} \) mass, \( m_{\tilde{\chi}} \), to larger values (there are ways around this if some of the theoretical assumptions are relaxed [33]), and it has been shown that heavy \( \tilde{\chi} \)'s are just as viable as a dark matter candidate as the lighter ones [4,34]. As \( m_{\tilde{\chi}} \) becomes larger than a few 10's of GeV the product \( qR \) starts to become non-negligible and finite momentum transfer must be considered for heavier nuclei. (The maximum allowed momentum transfer is \( q_{\text{max}} = 2M_r v \).) In ref. [7] a simple set of rules is given for when finite momentum transfers are not important: i.e. the product \( q_{\text{max}}R < 1 \) if (a) \( A < 28 \) or (b) \( m_{\tilde{\chi}} < (1/(1.2A^{4/3} + 100)) \text{ GeV} \). Thus, we see that finite momentum transfers may be important in \(^{29}\text{Si}\) and are quite likely to be important for \(^{73}\text{Ge}\). In this section we will show that these effects are important for both \(^{29}\text{Si}\) and \(^{73}\text{Ge}\). For spin-independent scattering these effects have been considered in refs. [9,4]. For spin-dependent scattering a simple approximation for finite momentum transfer was applied in [12].

The formalism for elastic \( \tilde{\chi}N \) scattering at all momentum transfers has been developed in refs. [7,18]. It is a straightforward extension of the formalism developed for the study of weak and electromagnetic semi-leptonic interactions in nuclei [19]. The differential \( \tilde{\chi}N \) cross section is given by:

\[
\frac{d\sigma}{dq^2} = \frac{8G_F^2}{(2J+1)v^2}S(q),
\]

where \( S(q) \) is the spin structure function

\[
S(q) = \sum_{L \text{ odd}} (|\langle N||T_{L}^{el5}(q)||N\rangle|^2 + |\langle N||L_{L}^{5}(q)||N\rangle|^2).
\]
\( \mathcal{T}^{el5}(q) \) and \( L^5(q) \) are the transverse electric and longitudinal multipole projections of the axial vector current operator \([19]\). For their explicit form in the \( \tilde{\chi} \) context, see \([7,18]\). The reduced matrix elements of the multipoles in eq. (8) are easily evaluated in the harmonic oscillator basis in the nuclear shell model \([19]\). In the limit of zero momentum transfer \( S(q) \) reduces to \([7,18]\)

\[
S(0) = \frac{1}{4\pi} |\langle N|| \sum_i \frac{1}{2} (a_0 + a_1 \tau^i_3) \sigma_i ||N\rangle|^2
\]

or, equivalently

\[
S(0) = \frac{1}{4\pi} |(a_0 + a_1) \langle N||S_p||N\rangle + (a_0 - a_1) \langle N||S_n||N\rangle|^2
\]

where \( a_0 \) (\( a_1 \)) is the isoscalar (isovector) projection of the \( a_n \) and \( a_p \) of eq. (2) \( (a_0 = a_n + a_p \) and \( a_1 = a_p - a_n \)). To make the connection with the zero momentum transfer case even clearer, we can “unreduce” the matrix elements in eq. (9b) and rewrite it as

\[
S(0) = \frac{1}{\pi} f(N) |a_p \langle N||S_p||N\rangle + a_n \langle N||S_n||N\rangle|^2
\]

where the matrix elements in eq. (9c) now correspond to the entries in Table 3. The factor \( f(N) \) accounts for the Wigner 3-\( J \) function which relates the reduced and unreduced values of the matrix elements and has the values \( f^{(29)Si} = 6 \) and \( f^{(73)Ge} = 12.222 \). Thus, eq. (8) reduces to the correct expression in this limit. As an aid in discussing the most general neutralino state, \( S(q) \) may be split into a pure isoscalar piece, \( S_{00} \), a pure isovector piece, \( S_{11} \), and an interference term, \( S_{01} \), in the following way:

\[
S(q) = a_0^2 S_{00}(q) + a_1^2 S_{11}(q) + a_0 a_1 S_{01}(q).
\]

Two factors contribute to the maximum allowed momentum transfer. As \( m_{\tilde{\chi}} \) becomes much greater than the nuclear mass, \( m_N \), the reduced mass asymptotes to \( M_r \to m_N \). Also, the \( \tilde{\chi} \) ’s have a Maxwellian velocity distribution in the halo and some will possess velocities significantly greater than \( \langle v \rangle \approx 10^{-3}c \). A maximum velocity of \( v_{max} \approx 2\langle v \rangle \) implies maximum momentum transfers of \( q_{max}^{(29)Si} \approx 108 \text{ MeV} \) and \( q_{max}^{(73)Ge} \approx 227 \text{ MeV} \). Neither of these values is small compared to the inverse nuclear size. In a harmonic oscillator basis, the fiducial nuclear size is set by the oscillator parameter, \( b = 1 \text{ fm} A^{1/6} = (1/197.327 \text{ MeV}) A^{1/6} \) \([23]\). Hence we have values \( b^{(29)Si} \approx 1.75 \text{ fm} \) and \( b^{(73)Ge} \approx 2.04 \text{ fm} \). These values result in the products: \( (q_{max} b)^{Si} \approx 0.96 \) and \( (q_{max} b)^{Ge} \approx 2.80 \). The effects of non-zero momentum transfer might play an important role in these nuclei (depending upon the \( \tilde{\chi} \) mass). The nuclear multipole matrix elements in eq. (8) are most easily evaluated and discussed in terms of the quantity \( y \equiv (bq/2)^2 \). We will use both \( y \) and
$q$ in our discussion. The corresponding maximum values of $y$ are: $y_{\text{max}}(^{29}\text{Si}) = 0.23$ and $y_{\text{max}}(^{73}\text{Ge}) = 1.96$.

Having determined the relevant range of momentum transfers to investigate, we may now discuss the results obtained for the spin structure function, $S(q)$ of eqs. (8)-(10), using our wave functions for $^{29}\text{Si}$ and the large space wave functions for $^{73}\text{Ge}$. In Figs. 3 and 4 we display the functions $S_{00}(y), S_{11}(y)$, and $S_{01}(y)$ for $^{29}\text{Si}$ and $^{73}\text{Ge}$ respectively. With these functions it is a simple matter to calculate $S(\tilde{\chi})$ for any $\tilde{\chi}$ once the relevant $a_0$ and $a_1$ are known. As an aid in this endevour, we have made simple exponential fits to $S(y)$ for both $^{29}\text{Si}$ and $^{73}\text{Ge}$. We find that for $^{29}\text{Si}$

$$S_{^{29}\text{Si}}^{\text{fit}}(y) = 0.00818(a_0^2e^{-4.428y} + 1.06a_1^2e^{-6.264y} - 2.06a_0a_1e^{-5.413y})$$  \hspace{1cm} (11a)

provides a highly accurate fit to $S(y)$ for $y < 0.15$. Beyond this value of $y$ eq. (11a) begins to seriously underestimate the value of $S(y)$. The equivalent function for $^{73}\text{Ge}$ is

$$S_{^{73}\text{Ge}}^{\text{fit}}(y) = 0.20313(1.102a_0^2e^{-7.468y} + a_1^2e^{-8.856y} - 2.099a_0a_1e^{-8.191y}),$$  \hspace{1cm} (11b)

valid for $y < 0.2$. These two fits should give reasonable values for $S(q)$ in their region of validity for most neutralinos. We note that, if we assume $q = M_r v$, $y = 0.1$ corresponds to $m_{\tilde{\chi}} \simeq 600$ GeV for $^{73}\text{Ge}$, already at the upper end of interesting $\tilde{\chi}$ masses. Thus, eqs. (11) lack of validity at high $y$ ($q$) should not be a serious impediment to their use. A more serious problem with the fits in eq. (11) occurs when $a_p \gg a_n$ (or equivalently, when $a_0 \simeq a_1$). In this regime of $\tilde{\chi}$ parameter space, there is a near cancellation of $S_{00}$ and $S_{11}$ against $S_{01}$ and the fits do not accurately reproduce the full result. This is a problem for the NQM values of $a_n$ and $a_p$ for a pure $\tilde{B}$, for example (these are given below). For a true representation of $S(q)$ in this case, one must use the full shell model results as presented in Figs. 3 and 4 for all $y$. With that warning, the fits in eq. (11) should be quite useful for determining $S(q)$ for the great majority of $\tilde{\chi}$ cases. Quenching is implemented in eqs. (11) by making the substitution $a_1 \rightarrow 0.833a_1$.

In Figs. 5 and 6 we display $S(q)$ for a pure $\tilde{B}$ ($Z_1 = 1, Z_2 = Z_3 = Z_4 = 0$). Theoretical arguments lead one to believe that an almost pure $\tilde{B}$ (purity $> 99\%$) is the most favored $\tilde{\chi}$ composition once $m_{\tilde{\chi}}$ becomes large enough that finite momentum effects become important [34,35]. (The symmetric or anti-symmetric linear combinations of $\tilde{H}_1$ and $\tilde{H}_2$ are also favored configurations.) For a pure $\tilde{B}$ the $\tilde{\chi}$-nucleon couplings are determined by [4]

$$a_{p(n)} = \left\{ \frac{1}{2\cos^2 \theta_W} \sum_{u,d,s} \Delta q((T_{3L} - e_q)^2 + e_q^2) \right\} \zeta_q$$  \hspace{1cm} (12)

$$\zeta_q = -2\frac{m_W^2}{m_q^2 - m_{\tilde{\chi}}^2} \sin^2 \theta_W$$
where \( \theta_W \) is the weak angle, \( T_{3L} \) is the third component of weak isospin, \( e_q \) is the charge of the quark, and \( \Delta q \) is the fraction of the nucleon’s spin carried by each flavor of quark (see the appendix). The sum is over the up, down, and strange quarks in the nucleon. The term in the curly brackets has often been referred to as \( a_{p(n)} \) in much of the previous literature [10,11,13]. Following ref. [36] the EMC values for the proton \( \Delta q \)'s are: \( \Delta u = 0.78, \Delta d = -0.50, \) and \( \Delta s = -0.16, \) all with an uncertainty of \( \pm 0.08. \) Inserting these values into eq. (12) gives

\[
a_p = 0.179\zeta_q \quad \text{and} \quad a_n = -0.097\zeta_q, \quad \text{or equivalently,} \quad a_0 = 0.082\zeta_q \quad \text{and} \quad a_1 = 0.276\zeta_q. \]

The NQM values of the spin content are [37] \( \Delta u = 0.93, \Delta d = -0.33, \) and \( \Delta s = 0.0 \) resulting in \( a_p = 0.254\zeta_q \) and \( a_n = -0.017\zeta_q \) \( (a_0 = 0.237\zeta_q, a_1 = 0.271\zeta_q). \)

In Fig. 5 we show \( S(q) \) for \(^{29}\text{Si}\) using the shell model calculations discussed in Section IIa. The solid line is computed using the EMC values of \( a_{p(n)} \) and the dashed line results from using the NQM values. For \(^{29}\text{Si}\), the response is dominated by the contribution from neutrons. Hence, the small value of \( a_n \) in the NQM leads to a severe depression in the calculated matrix element. In Fig. 6a we plot \( S(q) \) for \(^{73}\text{Ge}\) using the same set of \( a \)'s. Here we see similar behavior but with a much larger amplitude due to the larger amount of spin carried by the neutrons in \(^{73}\text{Ge}\) \( \{((S_n)_{Ge}/(S_n)_{Si})^2 = (0.47/0.13)^2 \approx 13\}. \) Clearly \(^{73}\text{Ge}\) is a better choice as a detector for the pure \( \tilde{B} \) case. Next, in Fig. 6b we present \( S(q) \) for \(^{73}\text{Ge}\) where we have quenched the isovector coupling constant in accordance with the discussion of Section III, \( a_1' = 0.833a_1 = 0.230. \) The quenching results in an overall decrease in the amplitude of \( S(q) \) for the EMC cross section and a slight increase in the NQM counterpart.

Finally, in Fig. 7 we present \( S(q) \) for a photino scattering with \(^{73}\text{Ge}\). A photino massive enough to make finite momentum effects relevant is highly unlikely [34,35]. However, Fig. 7 is useful, both because a pure photino has often been considered in the literature, and because it illustrates the effects upon \( S(q) \) of changing \( a_n \) and \( a_p \). For a pure photino, the nucleon couplings are

\[
a_{p(n)} = \left\{ \sum_{u,d,s} \Delta q e_q^2 \right\} \zeta_q, \quad (13)
\]

resulting in: \( a_p = 0.273\zeta_q \) and \( a_n = -0.153\zeta_q \) for the EMC values of the spin content and \( a_p = 0.376\zeta_q \) and \( a_n = -0.043\zeta_q \) for the NQM estimates. Now that we have presented our results, we may compare them to other work.

In Fig. 8 we compare \( S(q) \) for our \(^{29}\text{Si}\) result with the shell model (solid line), the single particle model (dashed line), and the single particle model quenched so that \( S(0) \) agrees with the OGM prediction (dash-dotted line). This last is not self-consistent but it is the only way to extend the OGM to finite momentum transfers [12]. Two things are immediately apparent from Fig. 8. First, the unquenched ISPSM grossly over-predicts the
value of \( S(q) \) for all \( q \). Second, the ISPSM with OGM quenching seems to do a very good job of predicting \( S(q) \). We quench by reducing \( a_n \) by the factor \( 0.15/0.50 = 0.3 \). There is a slight difference in their \( q = 0 \) normalization that is maintained throughout (see Table 1). This mismatch would be exacerbated by using the value of \( \langle S_n \rangle \approx 0.2 \) of the EOGM [7,11]. There is also a very slight difference between the shapes of the ISPSM and the shell model results but this appears rather trivial.

Figure 9 is the \(^{73}\text{Ge}\) counterpart to Fig. 8. Here, \( S(q) \) is plotted for: the large space shell model results with no quenching (solid line); the large space shell model result with an isovector quenching factor of 0.833 (long dashed line); the ISPSM with and without a similar isovector quenching (dotted and short dashed lines respectively); and the ISPSM quenched to agree with the OGM at \( q = 0 \) (dot-dashed line). The most relevant comparison is between the shell model results, quenched or unquenched, and the ISPSM results quenched to agree with the OGM at \( q = 0 \) \( (a_n \rightarrow (0.23/0.5)a_n) \). We see that there is a very large discrepancy of either a factor \( \sim 2 \) or \( \sim 4 \) for the \( \tilde{B} \) case. Configuration mixing has made a major difference in this case because it has allowed for a large contribution to \( \langle L_p \rangle \), see Table 1. The quenched IBFM result, when combined with an ISPSM response function, is in accord with the OGM + ISPSM result.

A final issue which bears mentioning, in both the zero and finite momentum transfer regimes, is the uncertainties in the nucleon coupling constants, \( a_n \) and \( a_p \). So far, we have concentrated upon our attempts to improve the estimates of the cross section by generating more realistic nuclear wave functions. We have already demonstrated, in Figs. 5 and 6, the large differences in the matrix element which result from the EMC vs. the NQM estimates of the proton’s spin content. Here we wish to point out that even within the context of the EMC result, large differences in the rate are allowed. By varying the EMC values of the \( \Delta q \)'s within the error bars we can vary \( S(q) \) as shown in Fig. 10. Here the solid line is the same as the solid line in Fig. 9 and the other lines are obtained by varying \( \Delta u, \Delta d, \) and \( \Delta s \) within the allowed EMC errors. A huge range of values is allowed for \( S(q) \). In fact, the uncertainty due to the \( \Delta q \)'s is far larger than that arising due to nuclear processes. Hopefully, new experiments will eventually determine the \( \Delta q \)'s with better precision. These new values, when coupled with the new picture of the nuclear structure presented here, will allow for a more accurate and complete description of \( \tilde{\chi}N \) scattering for any neutralino.

In this section we have demonstrated that finite momentum transfers can have rather sizable effects on the \( \tilde{\chi}N \) scattering cross section and hence on the scattering rate. In most cases, heavier \( \tilde{\chi} \)'s will have suppressed cross sections. This, coupled with the fact that it takes fewer heavy \( \tilde{\chi} \)'s than light ones to account for the dark matter, implies that it will
be more difficult to detect neutralinos via the direct detection technique if they are very heavy.

V. Discussion and Conclusions

In this paper we have presented detailed nuclear shell model calculations of spin-dependent elastic $\check{\chi}N$ scattering from $^{29}$Si and $^{73}$Ge. Major $\check{\chi}$ dark matter detection programs based upon these two nuclei are well under way [15]. Our results reveal both the strengths and the weaknesses of previous, more phenomenological, analyses [7]. In both nuclei the ISPSM is found to be a poor approximation to the true nuclear configuration. In $^{29}$Si at zero momentum transfer, the OGM finds results which are in accord with those of our calculations. This shows that, for this nucleus, the OGM adequately represents the configuration mixing. This is not the case in $^{73}$Ge. In this nucleus, the shell model result is in severe disagreement with the OGM and quenched IBFM results. The main difference comes from the large amount of orbital angular momentum carried by the protons. This large value for $\langle L_p \rangle$ violates a key assumption of the OGM and is not found in the IBFM. The larger value of $\langle L_p \rangle$ in our calculation leads to a large improvement in the calculated vs. measured magnetic moment of $^{73}$Ge but still does not lead to complete agreement with free particle $g$-factors. Thus we have discussed the difficult issue of quenching as it applies to $^{73}$Ge. Even with quenching included our results still differ from those of the OGM and IBFM.

Unlike the OGM or the IBFM, it is straightforward to extend the shell model results to finite momentum transfer in a consistent manner. We find that finite momentum transfer will play an important role in determining the $\check{\chi}N$ scattering rate for very heavy neutralinos. Here again, we have compared the shell model results against the OGM combined with the ISPSM form factor. This latter approach seems to work well for $^{29}$Si but significant disagreement is found for $^{73}$Ge.

The major concern regarding our results is the validity of the ground state wave functions we have generated. To help allay that concern, we have used our wave functions to calculate a number of nuclear observables and compared the results to the measured values. In $^{29}$Si excellent agreement was found for the magnetic moment, the excited state energy spectrum, and the first 3 spectroscopic factors. This was not surprising since we were able to use a well tested interaction in a well defined model space. These calculations in $^{29}$Si serve another useful purpose. The disagreement between the calculations and measurements in $^{29}$Si serve to calibrate how accurately one can calculate nuclear properties in the shell model. Thus, the magnitude of the “errors” in the good calculation of silicon will help to establish how believable our calculation of $^{73}$Ge is.
In $^{73}$Ge the picture is more complicated. No phenomenological interaction has been developed for this region of the periodic table and fairly severe truncations to the model space have to be imposed to obtain manageable dimensions. Despite these obstacles, our best calculated ground state wave function for $^{73}$Ge seems to be a good match when compared to experiment. We have obtained good agreement with the low lying excited state energy spectrum. The ground state to ground state spectroscopic factor is also in fair agreement with experiment. The spectroscopic factors for transitions to excited states reveal that our model space does not contain enough configurations to completely describe the excited states, although the agreement between theory and experiment is not unreasonable. We reiterate that our model space was designed to describe the ground state and for that it does an acceptable job.

The final comparison is between magnetic moments. Using the large space wave function a significant improvement over the ISPSM or IBFM estimates of $\mu$ is found. Unfortunately, perfect agreement with experiment is not obtained but this is not unexpected or particularly damaging. Some sort of quenching of both $\mu$ and the GT spin matrix elements is almost universally required in shell model calculations of heavy nuclei. We find that an isovector spin quenching factor of 0.833 results in agreement with the measured value of the magnetic moment. This method and amount of quenching are consistent with previous studies of both GT and $\mu$ quenching and is therefore well motivated and simple. It is not manifestly obvious that quenching is needed in $^{73}$Ge–$\tilde{\chi}$ scattering but, if so, we are confident that the correct answer lies in the range between the quenched and unquenched values of our calculation. In light of the above comparisons we feel that we have obtained an adequate description of the ground state structure of $^{73}$Ge and that any calculation in a larger model space or with a different interaction will confirm our results. The disagreement between calculation and measurement in each case is somewhat larger than for $^{29}$Si but is not so large as to cause us to doubt the result. The case is not airtight but 3 different lines of argument all indicate that the large space wave function is a better representation of the real nucleus than has previously been available.

As a summary and “User’s Guide” we have collected the formulae, references, and suggested quenching methods in the appendix. Formulae for the spin-dependent neutralino cross section and the event detection rate in a bolometric detector are also given there.

In Section IV we described the formalism for $\tilde{\chi}N$ scattering at finite momentum transfer. The cross section is given by eq. (7) and all of the nuclear physics is contained in the spin structure function, $S(q)$, given by eqs. (8)–(10). $S(q)$ may be decomposed into a pure isovector piece $S_{11}$, a pure isoscalar piece $S_{00}$, and an interference term $S_{01}$, see eq. (10) (we will refer to these as $S_{ij}$ where $i, j = 0, 1$). By taking suitable linear combinations
of $a_n$ and $a_p$, the isoscalar and isovector nucleon-$\tilde{\chi}$ coupling constants, $a_0$ and $a_1$, can be found for any composition neutralino. In Figs. 3 and 4 we presented the $S_{ij}$ for $^{29}\text{Si}$ and $^{73}\text{Ge}$ respectively. Along with $a_0$ and $a_1$ these plots can be used to compute the terms $a_i a_j S_{ij}$ which go into eq. (10) to find $S(q)$ for any given $\tilde{\chi}$. As an alternative to this we have provided convenient fits to $S(q)$ for both nuclei in eq. (11). These fits are quite accurate, for most $\tilde{\chi}$ compositions, in the regime $y \lesssim 0.15$. As noted in section IV, there are certain combinations of $a_0$ and $a_1$ for which eq. (11) is a poor approximation. With these words as a guide and using the formulae in the appendix, it should be straightforward to incorporate this work into calculations of the $\tilde{\chi}N$ scattering rate.

To help place this work in a larger context, we close by briefly discussing one final issue. This is a brief comparison of the spin dependent vs. the spin independent scattering. In addition to the spin dependent scattering discussed at length in this paper, there is spin independent scattering which may produce competitive or larger scattering rates [4,7]. The spin independent piece is coherent, and thus, the cross section scales as (atomic number)$^2$. So, for heavy nuclei or light Higgs bosons this piece may dominate depending upon the mass and composition of the neutralino. Engel [7,18] has shown that the spin-independent form factor, corresponding to $S(q)$, falls off much faster at large $q$ than does $S(q)$ itself. Thus, for very large $m_{\tilde{\chi}}$ spin dependent scattering is likely to dominate. Furthermore, for the pure $\tilde{\chi}$ states (e.g. a pure $\tilde{B}$), the coherent piece is suppressed (note, however, that recent work by Nojiri and Drees [38] may change this conclusion). Therefore spin-dependent scattering will probably dominate for these pure states. Outside of these two regimes it is not clear which piece, spin dependent or independent, will dominate the cross section and hence the rate.

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Appendix: User’s Guide

In this appendix we collect the formulae necessary to convert our results for the nuclear matrix elements into usable cross sections and interaction rates for dark matter detectors.
We give only the spin-dependent cross section. To get the total cross section the spin-independent cross section [4,5,39,38] must be added to the cross sections below. We follow the notation and conventions of references [4] and [5] with only slight modifications. In the limit of zero momentum transfer the cross section for spin-dependent elastic $\tilde{\chi}N$ scattering is

$$\sigma_{SD} = \frac{24m_\tilde{\chi}^2 m_N^2 G_F^2}{\pi(m_\tilde{\chi} + m_N)^2} 4^3 \Lambda^2 J(J + 1)$$

(A1)

where $m_N$ is the nuclear mass, $m_\tilde{\chi}$ is the neutralino mass, $J$ is the total nuclear spin, $G_F$ is the Fermi constant, and

$$\Lambda = \frac{\langle N|a_p S_p + a_n S_n|N\rangle}{J} = \frac{a_p\langle S_p\rangle}{J} + \frac{a_n\langle S_n\rangle}{J}.$$  

(A2)

The spin averages $\langle S_p \rangle$ and $\langle S_n \rangle$ are given in Table 3 (but see below concerning quenching before using them), while $a_p$ and $a_n$ contain the details of the supersymmetric model, as well as the quark spin content of the proton and neutron:

$$a_{p(n)} = \sum_{u,d,s} A'_q \Delta q,$$

(A3)

$$A'_q = \frac{1}{2} T_{3L}^q (Z_3^2 - Z_4^2) - x_q^2 \left\{ \frac{2m_q^2 d_q^2}{4m_w^2} + [T_{3L}^q Z_2 - \tan \theta_W (T_{3L}^q - e_q) Z_1]^2 + \tan^2 \theta_W e_q^2 Z_1^2 \right\},$$

(A4)

and

$$x_q^2 = \frac{m_W^2}{m_q^2 - m_\tilde{\chi}^2}.$$  

(A5)

The various quantities used in the definition of $A'_q$ involve the neutralino couplings and mass matrix, and can be found in ref [4]. For the special cases of photinos and B-inos, the $a_{p(n)}$ are given in eqs. (12) and (13). The values $\Delta q$ of the spin content of the nucleons are not well known. Some commonly used values are given in the text just after eq. (12). The $\Delta q$ for the neutron can be found from the $\Delta q$ for the proton by switching $\Delta u$ and $\Delta d$. (We note, for completeness, that new data on the deuteron’s spin structure function has very recently become available. A new analysis incorporating this data along with the EMC results yields values of $\Delta u = 0.81$, $\Delta d = -0.45$, and $\Delta s = -0.12$; all with an uncertainty of $\pm 0.03$ [41].)

For $^{29}$Si, we suggest using the values of $\langle S_n \rangle$ and $\langle S_p \rangle$ from the “Shell Model” row of Table 3 with no quenching. For $^{73}$Ge, we advocate quenching the isovector piece of the nucleon-$\tilde{\chi}$ coupling constants $a_p$ and $a_n$. Hence one should not use eq. (2). To quench the isovector piece of $a_p$ and $a_n$ one first computes $a_p$ and $a_n$ from eq. (A3) and then constructs the isovector and isoscalar coupling constants: $a_1 = a_p - a_n$ and $a_0 = a_p + a_n$. The isovector piece is then multiplied by the quenching factor, $a'_1 = 0.833 a_1$. In the zero
momentum transfer limit, the $a_p$ and $a_n$ are then recalculated using the quenched value $a'_1$. These values are then used in eq. (A2) in place of those calculated directly from eq. (A3). In other words to quench by a factor $Q (= 0.833)$ in eq. (A2) make the replacement:

$$a_p \implies a'_p = \frac{1}{2}[a_p(1 + Q) + a_n(1 - Q)] = 0.917a_p + 0.0835a_n$$

$$a_n \implies a'_n = \frac{1}{2}[a_p(1 - Q) + a_n(1 + Q)] = 0.0835a_p + 0.917a_n. \quad (A6)$$

In this scheme one uses the values of $\langle S_n \rangle$ and $\langle S_p \rangle$ from the “Shell Model (large)” row of Table 3. (not the “quenched” row.)

To make contact with previous work, we note that the ISPSM (single particle shell model) can be obtained from eq. (A1) by the simple substitution

$$\Lambda^2 \implies a_{p(n)}^2 \lambda^2, \quad (A7)$$

where in the ISPSM only one of either $a_n$ or $a_p$ contributes and

$$\lambda = \frac{1}{2} \left[ 1 + \frac{s_p(n)(s_p(n) + 1) - l_p(n)(l_p(n) + 1)}{J(J + 1)} \right] = \frac{s_p(n)}{J} \quad (A8)$$

where $J$ is the total nuclear angular momentum ($J = s_p(n) + l_p(n)$), $s_p(n)$ and $l_p(n)$ are the unpaired proton or neutron spin and orbital angular momentum respectively. For example, in the ISPSM, $^{73}$Ge is modeled as a neutron with $l_n = 4$ and $J = \frac{9}{2}$, while $^{29}$Si is modeled as a neutron with $l_n = 0$, and $J = \frac{1}{2}$.

When the neutralino mass is over a few tens of GeV, it becomes important to include the effect of finite momentum transfer. Eqs. (7-11) in the text show how to include this effect. To implement the suggested quenching scheme for $^{73}$Ge simply make the replacement eq. (A6) (or equivalently $a_1 \to a'_1 = 0.833a_1$) in eqs. (7-11). Note in particular that eq. (9c) can be written:

$$S(0) = \frac{2J + 1}{\pi} \Lambda^2 J(J + 1). \quad (A9)$$

Since we work in the $z$-projection of the maximally stretched state, the function $f(N)$ introduced in eq. (9c) has the form $f(N) = f(J) = (2J + 1)(J + 1)/J$.

The rate per unit detector mass with which neutralinos interact with a detector is given by [9,4,7]

$$R = \frac{\rho}{m_N m_X} \int v dv f(v) \int_0^{4M_r^2v^2} dq \frac{d\sigma}{dq}, \quad (A10)$$

where, $m_N$ is the mass of the nucleus, $M_r = m_N m_X/(m_N + m_X)$, $d\sigma/dq^2$ is given in eq. (7), $\rho \approx 0.4$ GeV / cm$^3$ is the halo density (assumed to consist entirely of neutralino’s), $f(v)$ is the velocity distribution of the halo, and $v_0 \approx 220$ km s$^{-1}$ is the circular velocity.
in the halo, which sets the scale for the halo gravitational potential. For simplicity one can choose a Maxwellian distribution of velocities $f(v) = 4v^2 \exp\left(-v^2/v_0^2\right)/(v_0^3\sqrt{\pi})$, or one can follow [9,4,40] and include the effects of the Sun and Earth motion, as well as the effect of a detector threshold.

Integrating eq. (A10) one gets

$$R = \frac{2}{\sqrt{\pi}} \eta_v \frac{\rho}{\mu \tilde{m} N} v_0 \sigma_{SD} , \quad (A11)$$

where [9] $\eta_v \approx 1.3$ gives a correction for the speed of the Sun through the halo, and the effect of integrating over $q^2$ is summarized by

$$\eta_c = \sigma_{SD}^{-1} \int dq^2 \frac{d\sigma}{dq^2}.$$ 

In the limit of zero momentum transfer, the $q^2$ integral in eq. (A1) becomes $\sigma_{SD}$ and $\eta_c \to 1$. Note again that to find the total event rate one must include the spin-independent piece of the cross section which is not considered in this paper.

In order to evaluate $\eta_c$ one may use the fits to $S(q)$ given in eq. (11). This will give an approximation which should be good for most neutralinos (but see the cautions in the text). We re-write eq. (11) as

$$S^{fit}(y) = \sum_{i=1}^{3} A_i e^{-B_i y},$$

where the $A_i$ and $B_i$ are given in eq. (11), and $y = (b q/2)^2$. For $^{73}$Ge, $b \approx 2.04$ fm, and for $^{29}$Si, $b \approx 1.75$ fm. Note now $\sum A_i = S^{fit}(0)$. Then

$$\eta_c = \left\{ \frac{\sum_i A_i}{\alpha_i \left( \frac{1}{2} + p^2 + \frac{pe-p^2}{\sqrt{\pi} \text{erf} p} \right) } \right\} / \sum_i A_i , \quad (A12)$$

where

$$\beta_i = \frac{\exp\left(-\frac{p^2 \alpha_i}{1+\alpha_i} \frac{p}{\sqrt{1+\alpha_i}} \text{erf} p \right)}{\sqrt{1+\alpha_i} \text{erf} p} , \quad p = v_{SUN}/v_0 \approx 1, \quad \alpha_i = B_i b^2 v_0^2 M_r^2 .$$

In the limit $p = 0$ (the limit of a stationary Sun or isotropic velocity distribution)

$$\eta_c \to \sum_i A_i \alpha_i^{-1} [1 - 1/(1 + \alpha_i)] / \sum_i A_i .$$
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Table 1. Summary of the calculated spin of $^{35}$Cl.

|               | $\langle S_n \rangle$ | $\langle S_p \rangle$ |
|---------------|------------------------|------------------------|
| ISPSM        | 0                      | -0.3                   |
| OGM [7,11]   | 0                      | -0.15                  |
| EOGM [7,11]  | 0.014                  | -0.094                 |
| Shell Model  | -0.011                 | -0.059                 |
| This Work    | -0.0088                | -0.051                 |

Table 2. Summary of the spectroscopic factors, $S(J; jJ_0)$ for $^{29}$Si. The experimental values are an unweighted average of the data followed by (in parentheses) the spread in the data, as represented by the low and high measurements, from ref. [24].

| Orbital      | $J^\pi$ | $(2J + 1)S_{Exp.}$ | $(2J + 1)S_{Calc.}$ | $(2J + 1)S_{ISP SM}$ |
|--------------|---------|--------------------|---------------------|----------------------|
| $2s_{1/2}$   | $1^+$   | 1.02 (0.74–1.5)    | 0.90                | 2                    |
| $1d_{3/2}$   | $3^+$   | 3.62 (2.1–5.2)    | 2.65                | 4                    |
| $1d_{5/2}$   | $5^+$   | 1.14 (0.53–1.8)    | 0.85                | 6                    |
Table 3. The value of the (z-projection) of the nuclear spin and orbital angular momentum matrix elements and the predicted magnetic moments, $\mu$, (using free particle $g$-factors) for $^{29}$Si and $^{73}$Ge. The last row contains our values of the large space shell model calculation with an isovector quenching factor of 0.833 applied to the spin matrix elements of the row above. If the magnetic moment has been used as input it is necessarily the measured value and has been enclosed in parentheses.

|        | $\langle S_n \rangle$ | $\langle S_p \rangle$ | $\langle L_n \rangle$ | $\langle L_{\pi} \rangle$ | $\mu$      |
|--------|-----------------------|------------------------|------------------------|-------------------------|------------|
| **$^{29}$Si** |                       |                        |                        |                         |            |
| ISPSM  | 0.5                   | 0                      | 0                      | 0                       | -1.91      |
| OGM    | 0.15                  | 0                      | 0.35                   | 0                       | (-0.555)   |
| EOGM   | 0.204                 | 0.054                  | ?                      | ?                       | (-0.555)   |
| Shell Model | 0.13                  | -0.002                 | 0.37                   | 0.019                   | -0.50      |
| **$^{73}$Ge** |                       |                        |                        |                         |            |
| ISPSM  | 0.5                   | 0                      | 4                      | 0                       | -1.91      |
| OGM    | 0.23                  | 0                      | 4.27                   | 0                       | (-0.879)   |
| IBFM   | 0.469                 | -0.009                 | 3.981                  | 0.060                   | -1.785     |
| IBFM(quenched) | 0.245                 | -0.005                 | ?                      | ?                       | (-0.879)   |
| Shell Model (small) | 0.496                 | 0.005                  | 3.596                  | 0.40                    | -1.468     |
| Shell Model (large) | 0.468                 | 0.011                  | 3.529                  | 0.491                   | -1.239     |
| Shell Model (large,quenched) | 0.372                 | 0.009                  | 3.529                  | 0.491                   | (-0.879)   |

Table 4. Summary of the spectroscopic factors, $S(J; jJ_0)$ for $^{73}$Ge. The experimental values are an unweighted average of the data followed by (in parentheses) the spread in the data, as represented by the low and high measurements, from ref. [30].

| Orbital  | $J^\pi$ | $(2J + 1)S_{Exp.}$ | $(2J + 1)S_{Calc.}$ | $(2J + 1)S_{ISPSM}$ |
|----------|---------|--------------------|---------------------|---------------------|
| **Large Space** |         |                    |                     |                     |
| $1g_{9/2}$ | $9^+/2$ | 6.16 (5.04–7.35)   | 3.96                | 10                  |
| $2d_{5/2}$ | $5^+/2$ | 0.29 (0.28–0.30)   | 0.74                | 6                   |
| $2p_{1/2}$ | $1^-/2$ | 0.67 (0.52–0.78)   | $3.3 \times 10^{-4}$ | 2                   |
| **Small Space** |         |                    |                     |                     |
| $1g_{9/2}$ | $9^+/2$ | 6.16 (5.04–7.35)   | 5.91                | 10                  |
| $2d_{5/2}$ | $5^+/2$ | 0.29 (0.28–0.30)   | 3.55                | 6                   |
| $2p_{1/2}$ | $1^-/2$ | 0.67 (0.52–0.78)   | $\sim 0$            | 2                   |
Table 5. The sets of quenched $g$-factors used in ref. [29] and their effect upon the magnetic moment of $^{73}$Ge using the large space wave functions.

| Interaction | $g_n^s$ | $g_p^s$ | $g_n^l$ | $g_p^l$ | $\mu$ |
|-------------|---------|---------|---------|---------|-------|
| Free        | -3.826  | 5.586   | 0       | 1       | -1.239|
| FPD6        | -3.41   | 6.75    | 0.05    | 0.71    | -0.995|
| FPMI3       | -2.53   | 5.87    | -0.11   | 0.87    | -1.079|
| KB1         | -3.19   | 4.87    | 0       | 1       | -0.948|
| Experiment  |         |         |         |         | -0.879|
Figure Captions

Figure 1 The calculated (left) and measured [22] (right) excited state energy spectrum of $^{29}$Si. The ground state has $J^\pi = \frac{1}{2}^+$. The dashed lines connect states with the same $J^\pi$. The $J^\pi$ values for most of the states have been included for reference.

Figure 2 The calculated small space (left), measured [22] (center), and calculated large space (right) energy spectra of $^{73}$Ge. The ground state has $J^\pi = \frac{9}{2}^+$. Where we have felt comfortable with the identifications, we have connected states with identical $J^\pi$’s by dashed lines. The $J^\pi$’s for some of the lower lying states have been noted on the plot.

Figure 3 The calculated functions $S_{00}$, $S_{11}$, and $S_{01}$ from eq. (10) for $^{29}$Si as a function of $y$. The solid line = $S_{00}$, the short dashed line = $S_{11}$, and the long dashed line = $S_{01}$.

Figure 4 The calculated functions $S_{00}$, $S_{11}$, and $S_{01}$ from eq. (10) for $^{73}$Ge (large space) as a function of $y$. The solid line = $S_{00}$, the short dashed line = $S_{11}$, and the long dashed line = $S_{01}$.

Figure 5 The spin structure function, $S(q)$, for a pure $\tilde{B}$ scattering with $^{29}$Si. The solid line is derived using the EMC values for $a_{n(p)}$; the dashed line uses the NQM estimates of these couplings.

Figure 6 A) The spin structure function, $S(q)$, for a pure $\tilde{B}$ scattering with $^{73}$Ge (large space). The solid line is derived using the EMC values for $a_{n(p)}$; the dashed line uses the NQM estimates for these couplings. B) The same as (A) but with an isovector quenching factor of 0.833 applied to enforce agreement with the measured magnetic moment at $q = 0$ ($a_1 \rightarrow 0.833a_1$).

Figure 7 The spin structure function, $S(q)$, for a pure $\tilde{\gamma}$ scattering with $^{73}$Ge (large space calculation). The solid line uses the EMC values for $a_{n(p)}$, the long dashed line uses these values but with the isovector piece quenched by a factor of 0.833, and the short dashed line uses the NQM estimates for $a_{n(p)}$.

Figure 8 The spin structure function, $S(q)$, for a pure $\tilde{B}$ (EMC couplings) scattering with $^{29}$Si for three different models. The solid line is the shell model result and corresponds to the solid line in Fig. 5. The dashed line is the ISPSM result. The dot-dashed line is the ISPSM structure function with $a_n$ quenched so that $S(0)$ agrees with the OGM result.

Figure 9 Several different models of $S(q)$ for a pure $\tilde{B}$ scattering with $^{73}$Ge: solid line = large space shell model result with no quenching (= solid line in Fig. 6A), long-dashed line = large space shell model result with isovector quenching (= solid line in Fig. 6B), short-dashed line = ISPSM result, dotted line = ISPSM result with isovector quenching, and the dot-dashed line = ISPSM result quenched to agree with the OGM at $q = 0$.

Figure 10 $S(q)$ for a pure $\tilde{B}$, with EMC couplings, scattering with $^{73}$Ge (large space calculation).
Here we have varied the $\Delta u$, $\Delta d$, and $\Delta s$ of the EMC result within the allowed errors ($\pm 0.08$) in deriving $a_{n(p)}$. The uncertainty due to the $\Delta q$’s is currently much larger than the nuclear uncertainty; compare with figure 9.