CROSSING CHANGE ALTERNATING KNOTS

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Abstract. In this paper we define Crossing Change Alternating Knots (CCA knots) and their generalization: $k$-CCA knots.

Definition 0.1. Let be given a diagram $D$ of a knot (or link). In $D$ we make a crossing change in every crossing separately, and the rest of the crossings remain unchanged. From the diagram $D$ with $n$ crossings we obtain $n$ new diagrams, each with a single crossing changed, and the corresponding $n$ knots (or links). A diagram $D$ is called Crossing Change Alternating (shortly, CCA) if all the knots (links) obtained by the crossing changes are alternating. A knot (or link) $K$ is CCA if it has at least one CCA diagram.

It is clear that a CCA knot (or link) could be alternating, or non-alternating.

If an alternating knot has a minimal CCA diagram, all its minimal diagrams are CCA (according to Tait Flyping Theorem). A large class of CCA knots and links are rational knots and links.

In the case of alternating knots (links) it is sufficient to find one minimal diagram which is CCA, and all its minimal diagrams will be CCA. However, this is not true for non-alternating knots (links): a non-alternating knot (link) can have two different minimal diagrams, where one is CCA, and the other is not. For example, the minimal diagram $(2 1, 2) (3, -2)$ (Fig. 1a) of the knot 10$_{150}$ is not CCA, but its another minimal diagram $8^* - 2 : 20 : -1 -1$ is CCA.

Figure 1. (a) The minimal not CCA diagram $(2 1, 2) (3, -2)$ of the knot 10$_{150}$; (b) the minimal CCA diagram $8^* - 2 : 20 : -1 -1$ of the same knot.

Moreover, CCA-property is not necessarily realized on minimal diagrams. For example, all minimal diagrams of the knot 10$_{151}$ (given in Conway notation as $(2 1, 2) (21, -2)$) are not CCA, but its non-minimal diagram $6^* 2 - 1 - 1.2 : 20$ is CCA, so the knot 10$_{151}$ is a CCA knot without minimal CCA diagrams.

Thanks to the last example, proving that some knot is CCA is very difficult, because we need to check all diagrams, and not just the minimal ones. As the obstruction for a knot to be CCA we can use the alternation number. The alternation number of a link $L$, denoted by $\text{alt}(L)$, is the minimal number of crossing changes needed to deform $L$ into an alternating link, where the minimum is taken over all diagrams of $L$ [1]. According to T. Abe [2], for
Figure 2. (a) The minimal not CCA diagram $2 1 2 1$ of the knot $10_{151}$. All its minimal diagrams are not CCA; (b) non-minimal CCA diagram $6^* 2 - 1 - 12 : 20$ of the same knot.

Every knot $K$ alternation number satisfies the inequality $|s(K) - (-\sigma(K))| \leq alt(K)$, where $s(K)$ is the Rasmussen signature of $K$, and $-\sigma(K)$ the negative signature of $K$. It is clear that any knot $K$ with $alt(K) > 1$ cannot be CCA. T. Abe [2] proved that for every torus knot $T_{p,q}$ ($2 \leq p < q$)

1. $alt(T_{p,q}) = 0 \iff p = 2$;
2. $alt(T_{p,q}) = 1 \iff (p, q) = (3, 4)$ or $(3, 5)$;
3. $alt(T_{p,q}) \geq 2 \iff$ otherwise.

Hence, we know that there is an infinite number of knots that are not CCA. However, the obstruction $|s(K) - (-\sigma(K))| \leq alt(K)$ is not strong enough for many knots for which we suspect that are not CCA. E.g., for every alternating knot the Rasmussen signature and signature coincide, and there are many alternating knots which are candidates for knots that are not CCA. Such candidates are all alternating knots with a minimal diagram which is not CCA.

Definition 1 can be generalized in order to define $k$-CCA knots:

**Definition 0.2.** Let be given a diagram $D$ of a knot (or link). In $D$ we make $k$ crossing changes in each subset of crossings consisting from $k$ crossings ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$), and the rest of the crossings remain unchanged. From the diagram $D$ with $n$ crossings we obtain $\binom{n}{k}$ new diagrams, each with $k$ crossings changed, and the corresponding $\binom{n}{k}$ knots (or links). A diagram $D$ is called $k$-Crossing Change Alternating (shortly, $k$-CCA) if all the knots (links) obtained by the crossing changes are alternating. A knot (or link) $K$ is $k$-CCA if it has at least one $k$-CCA diagram.

In the same way as before, we expect that there exist knots that are $k$-CCA, but without a minimal $k$-CCA diagram, so it will be very difficult to conclude that some knot is $k$-CCA or not. As the obstruction for a knot to be $k$-CCA we can use the same obstruction as before. However, based on the computations on minimal diagrams, there will be many candidates for knots that are not $k$-CCA, for which will be very difficult to show that they are not $k$-CCA. Making computations only on the minimal diagrams, we can conclude that, e.g., the diagram $21,21,2^+$ of the knot $9_{28}$ is not 1-CCA, it is 2-CCA, and not 3-, 4-, nor 5-CCA. On the other hand, the minimal diagram $21,21,2^-$ of the knot $8_{20}$ is 1-, 2-, and 4-CCA, but is not 3-CCA. After checking all minimal diagrams of this knot, we can conclude that none of them is 3-CCA, but we are not able to say that the knot $21,21,2^-$ is 3-CCA or not, because we need to check all its non-minimal diagrams, and for this knot the mentioned obstruction based on alternating number is not helpful.
1. CCA knots with $n \leq 12$ crossings

All computations in this paper are made in the program \textit{LinKnot}. The first table contains alternating CCA knots with $n \leq 12$ crossings with minimal CCA diagrams.
| 3₁ | 3₂ | 3₃ | 3₄ | 3₅ | 3₆ | 3₇ | 3₈ | 3₉ |
|----|----|----|----|----|----|----|----|----|
| 4₅ | 4₆ | 4₇ | 4₈ | 4₉ | 4₁₀ | 4₁₁ | 4₁₂ | 4₁₃ |
| 5₁ | 5₂ | 5₃ | 5₄ | 5₅ | 5₆ | 5₇ | 5₈ | 5₉ |
| 6₁ | 6₂ | 6₃ | 6₄ | 6₅ | 6₆ | 6₇ | 6₈ | 6₉ |
| 7₁ | 7₂ | 7₃ | 7₄ | 7₅ | 7₆ | 7₇ | 7₈ | 7₉ |
| 8₁ | 8₂ | 8₃ | 8₄ | 8₅ | 8₆ | 8₇ | 8₈ | 8₉ |
| 9₁ | 9₂ | 9₃ | 9₄ | 9₅ | 9₆ | 9₇ | 9₈ | 9₉ |
| 10₁ | 10₂ | 10₃ | 10₄ | 10₅ | 10₆ | 10₇ | 10₈ | 10₉ |

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The second table contains non-alternating CCA knots with the minimal CCA diagram.

| Knot | Description | CCA Diagram |
|------|-------------|-------------|
| 819  | 3,3,−2      | \(\{8\},\{6,12,14,16,4,10\}\) |
| 820  | 3,2,1,−2    | \(\{8\},\{4,8,12,14,16,6,10\}\) |
| 821  | 2,1,2,1,−2  | \(\{8\},\{4,8,12,2,14,16,6,10\}\) |
| 942  | 2,2,3,−2    | \(\{9\},\{4,8,18,14,2,16,6,10,12\}\) |
| 943  | 2,1,1,3,−2  | \(\{9\},\{4,8,10,14,2,16,18,6,12\}\) |
| 944  | 2,2,2,1,−2  | \(\{9\},\{4,8,12,2,16,6,18,10,14\}\) |
| 945  | 2,1,1,2,1,−2| \(\{9\},\{4,8,10,16,2,14,18,6,12\}\) |
| 946  | 3,3,−3      | \(\{9\},\{8,12,16,14,18,−4,−2,6,10\}\) |
| 947  | 8*−20       | \(\{9\},\{6,8,10,16,14,−18,4,2,−12\}\) |
| 948  | 2,1,2,1,−3  | \(\{9\},\{4,10,−14,−12,16,2,−6,18,8\}\) |
| 949  | −20 : −20 : −20 | \(\{9\},\{6,−10,−14,12,−16,−2,18,−4,−8\}\) |
| 10150 | 6*−2:2:2:2:2:0 | \(\{10\},\{6,10,16,20,14,2,−18,4,8,−12\}\) |
| K11n8 | 6*2:2:1:0:−3:0 | \(\{11\},\{4,8,16,20,2,−18,6,22,−12,−10,14\}\) |
| K11n115 | 6*2:−3:2:2:0 | \(\{11\},\{6,12,16,22,−18,−20,2,4,8,−10,14\}\) |
| K11n123 | 6*−3:2:2:2:0 | \(\{11\},\{6,10,16,22,18,2,−20,8,4,−14,−12\}\) |
| K11n124 | 6*2:−3:2:2:2:2:0 | \(\{11\},\{6,10,16,22,18,2,−20,8,4,−14,−12\}\) |
| K11n143 | 6*−2:2:−2:2:0:2:0 | \(\{11\},\{6,12,−16,22,−18,2,20,−4,−8,14,10\}\) |
| K11n157 | 9*−3 | \(\{11\},\{6,18,16,12,4,2,−20,−22,10,8,−14\}\) |
| K12n147 | −2 − 1 − 1,2,1,1,2,1 | \(\{12\},\{4,14,18,16,−12,−22,2,24,20,6,−10,−8\}\) |

The knot \(10_{151} = (21,2)(21,−2)\) (Fig. 2) is the example of a CCA knot without minimal CCA diagram. Its non-minimal diagram \(6*2−1−1.2:20\) with the DT code \(\{11\},\{−4,10,16,20,2,−22,18,8,12,6,14\}\) is CCA.

For all other knots with \(n\le12\) crossings we don’t know are they CCA or not. If they are, they can have only non-minimal CCA diagrams.

For all knots \(K\) up to \(n=11\) crossings T. Abe \([2]\) proved that \(alt(K)\le1\). Hence, we need to consider only knots with \(n=12\) crossings.

We computed alternation numbers from all minimal diagrams, and concluded that all knots \(K\) with \(n=12\) crossings have the alternation number \(alt(K)\le1\), except the knots \(K12n426, K12n706, K12n801, K12n835, K12n838, K12n888\) with the alternation number 2.

## References

[1] A. Kawauchi, On alternation numbers of links. [http://www.sci.osaka-cu.ac.jp/~kawauchi/altnumber.pdf](http://www.sci.osaka-cu.ac.jp/~kawauchi/altnumber.pdf)

[2] T. Abe, An estimation of the alternation number of a torus knot, J. Knot Theory Ramifications 18, 3 (2009) 363-379.

[3] S. V. Jablan, R. Sazdanović, *LinKnot- Knot Theory by Computer*. World Scientific, New Jersey, London, Singapore, 2007, [http://math.ict.edu.rs/](http://math.ict.edu.rs/)

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