A Note on Distinguishing Trees with the Chromatic Symmetric Function

Logan Crew\textsuperscript{*}

June 9, 2021

Abstract

For a tree $T$, consider its smallest subtree $T^*$ containing all vertices of degree at least 3. Then the remaining edges of $T$ lie on disjoint paths each with one endpoint on $T^*$. We show that the chromatic symmetric function of $T$ determines the size of $T^*$, and the multiset of the lengths of these incident paths. In particular, this generalizes a proof of Martin, Morin, and Wagner that the chromatic symmetric function distinguishes spiders.

1 Introduction

The chromatic symmetric function $X_G$ of a graph $G$ was introduced by Stanley in the 1990s \cite{Stanley}, and has since been very well-studied due to its connections to other areas of mathematics, most notably algebraic geometry \cite{AlgebraicGeometry, AlgebraicGeometry2}, and knot theory \cite{KnotTheory, KnotTheory2}. One of the central unanswered questions driving research into the chromatic symmetric function is whether it distinguishes non-isomorphic trees. It has been verified computationally that the chromatic symmetric function distinguishes trees of up to 29 vertices \cite{ComputationalVerification}, and many partial results have been discovered \cite{PartialResults1, PartialResults2}. In a notable work, Martin, Morin, and Wagner \cite{Martin} showed that the chromatic symmetric function of a tree contains the bivariate subtree polynomial of Chaudhary and Gordon \cite{ChaudharyGordon}, and from this they were able to prove that certain infinite families of trees (spiders and some caterpillars) are distinguished by the chromatic symmetric function.

In this work, we show that the bivariate subtree polynomial provides some more general information, namely the size of the smallest subtree that contains all vertices of degree at least 3, and the lengths of the paths (which must be the rest of the tree) that protrude from it. In particular, this generalizes the proof from \cite{Martin} that spiders are distinguished by the chromatic symmetric function.

2 Background

A graph $G = (V, E)$ consists of a vertex set $V$ and an edge multiset $E$ where the elements of $E$ are (unordered) pairs of (not necessarily distinct) elements of $V$. A simple graph is a graph $G = (V, E)$ in which $E$ is a set with no repeated elements; all graphs in this paper are simple. If $\{v_1, v_2\}$ is an edge, we will write it as $v_1v_2 = v_2v_1$. The vertices $v_1$ and $v_2$ are the endpoints of the edge $v_1v_2$. We will use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph $G$, respectively.

\textsuperscript{*}Department of Combinatorics & Optimization, University of Waterloo, Waterloo, ON, N2L 3E9.

Emails: lcrew@uwaterloo.ca,
For a vertex \( v \in V(G) \), its degree \( d(v) \) is the number of times \( v \) occurs as an endpoint of an edge in \( E(G) \). If the vertices of \( G \) are labelled \( v_1, \ldots, v_n \) such that \( d(v_1) \geq \cdots \geq d(v_n) \), the degree sequence of \( G \) is the tuple \( (d(v_1), \ldots, d(v_n)) \).

A subgraph of a graph \( G \) is a graph \( G' = (V', E') \) where \( V' \subseteq V \) and \( E' \subseteq E|_{V'} \), where \( E|_{V'} \) is the set of edges with both endpoints in \( V' \). An induced subgraph of \( G \) is a graph \( G' = (V', E|_{V'}) \) with \( V' \subseteq V \).

A path in a graph \( G \) is a nonempty sequence of edges \( v_1v_2, v_2v_3, \ldots, v_k \) such that there exists a path in \( G \) containing all vertices \( v_i \) for all \( i \). The vertices \( v_1 \) and \( v_k \) are the endpoints of the path. A cycle in a graph is a nonempty sequence of distinct edges \( v_1v_2, v_2v_3, \ldots, v_kv_1 \) such that \( v_i \neq v_j \) for all \( i \neq j \).

A graph \( G \) is connected if for every pair of vertices \( v_1 \) and \( v_2 \) of \( G \) there is a path in \( G \) with \( v_1 \) and \( v_2 \) as its endpoints. The connected components of \( G \) are the maximal induced subgraphs of \( G \) which are connected.

A tree is a connected graph containing no cycles. It is well-known that every tree with \( n \) vertices has \( n - 1 \) edges. A connected induced subgraph of a tree with at least one edge is called one of its subtrees. A vertex of degree 1 in a tree is called a leaf.

For a tree \( T \), its bivariate subtree polynomial \( S_T(q,r) \), introduced by Chaudhary and Gordon [3], is defined by

\[
S_T(q,r) = \sum_{\text{subtrees } S \text{ of } T} q^{|E(S)|} r^{|L(S)|}
\]

where \( L(S) \) is the set of leaves of \( S \) (which need not also be leaves in \( T \)).

If \( G = (V(G), E(G)) \) be a graph, a map \( \kappa : V(G) \to \mathbb{Z}_+ \) is called a coloring of \( G \). This coloring is called proper if \( \kappa(v_1) \neq \kappa(v_2) \) for all \( v_1, v_2 \) such that there exists an edge \( e = v_1v_2 \) in \( E(G) \). The chromatic symmetric function \( X_G \) of \( G \) is defined as [10]

\[
X_G(x_1, x_2, \ldots) = \sum_{\kappa \text{ proper}} \prod_{v \in V(G)} x_{\kappa(v)}
\]

where the sum ranges over all proper colorings \( \kappa \) of \( G \).

Martin, Morin, and Wagner [7] proved the following two results which we will require:

**Theorem 1.** For any tree \( T \), its chromatic symmetric function \( X_T \) determines its bivariate subtree polynomial \( S_T \).

**Lemma 2.** The bivariate subtree polynomial of a tree \( T \) determines its degree sequence.

## 3 Main Result

For a tree \( T \) we define its trunk \( T^0 \) to be the smallest connected induced subgraph that contains all vertices of degree \( \geq 3 \). For every leaf \( l \), we define its twig \( l \) to be the longest path \( P \) in \( T \) containing \( l \) such that every interior vertex (non-endpoint) of \( P \) has degree \( 2 \) in \( T \). We call a path of \( T \) a twig if it is a twig for one of its leaves. Thus, we may view any tree \( T \) as the union of its trunk with its twigs, as shown in Figure [1]

**Theorem 3.** From the bivariate subtree polynomial of a tree \( T \) (and thus from its chromatic symmetric function \( X_T \) by Theorem [7]), we can recover

- The size of \( T^0 \).
- The lengths of all twigs of \( T \).
Proof. First, using Lemma 2, we know the degree sequence of $T$, and thus we know how many leaves (and equivalently twigs) $T$ has; let $a$ be this number.

Note that the only way a subtree $S$ of $T$ can have exactly $a$ leaves is if it contains all vertices of degree at least 3. For suppose otherwise, that we have such a subtree $S$ with $a$ leaves that doesn’t contain some particular vertex $x$ of degree $\geq 3$. Then there is exactly one path connecting $x$ to $S$; in addition to the edge starting this path at $x$, there are at least 2 further edges coming out of $x$, each of which leads to at least one leaf of $T$. Hence the connected component of $T \setminus S$ containing $x$ has at least two leaves of $T$, and so $T$ has at least $a + 1$ leaves, a contradiction. Thus, a subtree of $T$ with $a$ leaves contains $T^o$. Furthermore, such a subtree $S$ additionally contains every edge of $T$ adjacent to some vertex of degree at least 3; for if $S$ does not contain some such edge adjacent to such a vertex $v$, then adding this edge increases the number of leaves of $S$ unless $v$ is itself a leaf of $S$, but then since $v$ has degree at least 3 in $T$, there are at least two distinct edges we can add to $S$ adjacent to $v$ to increase the number of leaves of $S$.

Thus, every subtree of $T$ with $a$ leaves contains the subtree that includes all of $T^o$, plus one edge from every twig. Since clearly all such trees contain $U$, as well as one additional edge from some twig of $T$ with more than one edge. Thus, there will be $a - t_1$ such trees, so since we know $a$, we may compute $t_1$.

Continuing inductively, we assume that for a fixed $k$ we know $t_1, \ldots, t_k$, and we therefore know that the remaining $a - \sum_{i=1}^{k} t_i$ twigs have length at least $k + 1$. From the subtree polynomial, we determine the number $S(k)$ of subtrees of $T$ with $v + k + 1$ vertices and $a$ leaves. Clearly all such trees contain $U$, as well as $k + 1$ additional vertices from twigs. We split into two kinds of trees: those that take all $k + 1$ vertices from a single twig, and those that do not. We may count the latter in terms of $t_1, \ldots, t_k, a$, since we are guaranteed to be able to take at least $k$ vertices in addition to the initial one in $U$ from the twigs of undetermined length; let this number be $f(t_1, \ldots, t_k, a)$. For the former, note that we have already taken one vertex from each twig in $U$, so we may only take
an additional $k + 1$ vertices from a twig of length at least $k + 2$. Thus, we have

$$S(k) = f(t_1, \ldots, t_k, a) + \sum_{i=k+2}^{|V(G)|} t_i$$

and so we may derive $\sum_{i=k+2}^{|V(G)|} t_i$. Thus, we have all the necessary information to compute

$$a - \sum_{i=1}^k t_i - \sum_{i=k+2}^{|V(G)|} t_i = t_{k+1}.$$  

This process may be repeated to find all twig lengths.

4 Acknowledgments

The author would like to thank Sophie Spirkl for helpful comments.

References

[1] Jos´e Aliste-Prieto, Anna de Mier, and Jos´e Zamora. On t rees with the same restricted U-polynomial and the Prouhet–Tarry–Escott problem. *Discrete Mathematics*, 340(6):1435–1441, 2017.

[2] Patrick Brosnan and Timothy Y Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Advances in Mathematics*, 329:955–1001, 2018.

[3] Sharad Chaudhary and Gary Gordon. Tutte polynomials for trees. *Journal of Graph Theory*, 15(3):317–331, 1991.

[4] SV Chmutov, SV Duzhin, and SK Lando. Vassiliev knot invariants iii. Forest algebra and weighted graphs. *Advances in Soviet Mathematics*, 21:135–145, 1994.

[5] Sam Heil and Caleb Ji. On an algorithm for comparing the chromatic symmetric functions of trees. *arXiv preprint arXiv:1801.07363*, 2018.

[6] Martin Loebl and Jean-S´ebastien Sereni. Isomorphism of weighted trees and Stanley’s conjecture for caterpillars. *Annales de l’institut Henri Poincaré D*, 6.3:357–384, 2019.

[7] Jeremy L Martin, Matthew Morin, and Jennifer D Wagner. On distinguishing trees by their chromatic symmetric functions. *Journal of Combinatorial Theory, Series A*, 115(2):237–253, 2008.

[8] Steven D Noble and Dominic JA Welsh. A weighted graph polynomial from chromatic invariants of knots. In *Annales de l’institut Fourier*, volume 49, pages 1057–1087, 1999.

[9] Radmila Sazdanovic and Martha Yip. A categorification of the chromatic symmetric function. *Journal of Combinatorial Theory, Series A*, 154:218–246, 2018.

[10] Richard P Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166–194, 1995.