Nilpotent extensions of blocks

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1 Introduction

1.1 The nilpotent blocks over an algebraically closed field of characteristic $p > 0$ were introduced in [2] as a translation for blocks of the well-known Frobenius Criterion on $p$-nilpotency for finite groups. They correspond to the simplest situation with respect to the so-called fusion inside a defect group, and the structure of the source algebras of the nilpotent blocks determined in [9] confirms that these blocks represent indeed the easiest possible situation.

1.2 However, when the field of coefficients is not algebraically closed, together with Fan Yun we have seen in [3] that, in the general situation, the structure of the source algebra of a block which, after a suitable scalar extension, decomposes in a sum of nilpotent blocks — a structure that we determine in [3] — need not be so simple.

1.3 At that time, we already knew some examples of a similar fact in group extensions, namely that a non-nilpotent block of a normal subgroup $H$ of a finite group $G$ may decompose in a sum of nilpotent blocks of $G$. In this case, we also have been able to describe the source algebra structure, which is quite similar to (but easier than) the structure described in [3]. With a big delay, we explain this result here.

1.4 Actually, this phenomenon is perhaps better described by saying that a normal sub-block of a nilpotent block need not be nilpotent. However, the normal sub-blocks of nilpotent blocks are quite special: they are basically Morita equivalent [15, §7] to the corresponding block of their inertial subgroup. Then, as a matter of fact, a normal sub-block of such a block still fulfills the same condition.

1.5 Thus, let us call inertial block any block of a finite group that is basically Morita equivalent [15, §7] to the corresponding block of its inertial subgroup; as a matter of fact, in [12, Corollaire 3.6] we already exhibit a large family of inertial blocks; see also [14, Appendix]. The main purpose of this paper is to prove that a normal sub-block of an inertial block is again an inertial block. Since a nilpotent block is basically Morita equivalent to its defect group [9, Theorem 1.6 and (1.8.1)], and the corresponding block of its inertial subgroup is also nilpotent, a nilpotent block is, in particular, an inertial block and thus, our main result applies.
2 Quoted results and inertial blocks

2.1 Throughout this paper $p$ is a fixed prime number, $k$ an algebraically closed field of characteristic $p$ and $O$ a complete discrete valuation ring of characteristic zero having the residue field $k$. Let $G$ be a finite group; following Green [5], a $G$-algebra $A$ is a torsion-free $O$-algebra $A$ of finite $O$-rank endowed with a $G$-action; we say that $A$ is primitive if the unity element is primitive in $A^G$. A $G$-algebra homomorphism from $A$ to another $G$-algebra $A'$ is a not necessarily unitary $G$-algebra homomorphism $f : A \rightarrow A'$ compatible with the $G$-actions. We say that $f$ is an embedding whenever

$$\text{Ker}(f) = \{0\} \quad \text{and} \quad \text{Im}(f) = f(1_A)A'f(1_A)$$

and that $f$ is a strict semicovering if $f$ is unitary, the radical $J(A)$ of $A$ contains Ker$(f)$ and, for any $p$-subgroup $P$ of $G$, $J(A^P)$ contains $f(J(A^P))$ and $f(i)$ is primitive in $A'^P$ for any primitive idempotent $i$ of $A^P$ [6, §3].

2.2 Recall that, for any subgroup $H$ of $G$, a point $\alpha$ of $H$ on $A$ is an $(A^H)^*$-conjugacy class of primitive idempotents of $A^H$ and the pair $H_\alpha$ is a pointed group on $A$ [7, 1.1]; if $H = \{1\}$, we simply say that $\alpha$ is a point of $A$. For any $i \in \alpha$, $iA_i$ has an evident structure of $H$-algebra and we denote by $A_\alpha$ one of these mutually $(A^H)^*$-conjugate $H$-algebras and by $A(H_\alpha)$ the simple quotient of $A^H$ determined by $\alpha$; we call multiplicity of $\alpha$ the square root of the dimension of $A(H_\alpha)$. If $f : A \rightarrow A'$ is a $G$-algebra homomorphism and $\alpha'$ a point of $H$ on $A'$, we call multiplicity $m(f)_{\alpha'}^\alpha$ of $f$ at $(\alpha, \alpha')$ the dimension of the image of $f(i)A^H i'$ in $A'(H_{\alpha'})$ for $i \in \alpha$ and $i' \in \alpha'$; we still consider the $H$-algebra $A'_\alpha = f(i)A'f(i)$ together with the unitary $H$-algebra homomorphism induced by $f$ and the embedding of $H$-algebras

$$A_\alpha \rightarrow A'_\alpha \leftarrow A'_{\alpha'}$$

2.2.1. A second pointed group $K_\beta$ on $A$ is contained in $H_\alpha$ if $K \subset H$ and, for any $i \in \alpha$, there is $j \in \beta$ such that [7, 1.1]

$$ij = j = ji$$

2.2.2; then, it is clear that the $(A^K)^*$-conjugation induces $K$-algebra embeddings

$$f_\beta^\alpha : A_\beta \rightarrow \text{Res}_K^H(A_\alpha)$$

2.2.3.

2.3 Following Broué, for any $p$-subgroup $P$ of $G$ we consider the Brauer quotient and the Brauer homomorphism [1, 1.2]

$$\text{Br}_P^A : A^P \rightarrow A(P) = A^P / \sum_Q A_Q^P$$

2.3.1, where $Q$ runs over the set of proper subgroups of $P$, and call local any point $\gamma$ of $P$ on $A$ not contained in Ker$(\text{Br}_P^A)$ [7, 1.1]. Recall that a local pointed group $P_\gamma$ contained in $H_\alpha$ is maximal if and only if Br$_P(\alpha) \subset A(P_\gamma)_P$.
fulfilling subgroup $H$ has a structure of a group homomorphism $\rho: G \rightarrow A^*$, moreover, the maximal local pointed groups $P_\gamma$ contained in $H_\alpha$ — called the defect pointed groups of $H_\alpha$ — are mutually $H$-conjugate [7, Theorem 1.2].

2.4 Let us say that $A$ is a $p$-permutation $G$-algebra if a Sylow $p$-subgroup of $G$ stabilizes a basis of $A$ [1, 1.1]. In this case, recall that if $P$ is a $p$-subgroup of $G$ and $Q$ a normal subgroup of $P$ then the corresponding Brauer homomorphisms induce a $k$-algebra isomorphism [1, Proposition 1.5]

$$(A(Q))(P/Q) \cong A(P)$$  \hspace{1cm} \text{(2.4.1)}$$

moreover, choosing a point $\alpha$ of $G$ on $A$, we call Brauer $(\alpha, G)$-pair any pair $(P, e_A)$ formed by a $p$-subgroup $P$ of $G$ such that $\text{Br}_P^A(\alpha) \neq \{0\}$ and by a primitive idempotent $e_A$ of the center $Z(A(P))$ of $A(P)$ such that

$$e_A \cdot \text{Br}_P^A(\alpha) \neq \{0\}$$  \hspace{1cm} \text{(2.4.2)}$$

note that any local pointed group $Q_\delta$ on $A$ contained in $G_\alpha$ determines a Brauer $(\alpha, G)$-pair $(Q, f_A)$ fulfilling $f_A \cdot \text{Br}_P^A(\delta) \neq \{0\}$.

2.5 Then, it follows from Theorem 1.8 in [1] that the inclusion between the local pointed groups on $A$ induces an inclusion between the Brauer $(\alpha, G)$-pairs; explicitly, if $(P, e_A)$ and $(Q, f_A)$ are two Brauer $(\alpha, G)$-pairs then we have

$$(Q, f_A) \subset (P, e_A)$$  \hspace{1cm} \text{(2.5.1)}$$

whenever there are local pointed groups $P_\gamma$ and $Q_\delta$ on $A$ fulfilling

$$Q_\delta \subset P_\gamma \subset G_\alpha \quad , \quad f_A \cdot \text{Br}_P^A(\delta) \neq \{0\} \quad \text{and} \quad e_A \cdot \text{Br}_P^A(\gamma) \neq \{0\}$$  \hspace{1cm} \text{(2.5.2)}$$

Actually, according to the same result, for any $p$-subgroup $P$ of $G$, any primitive idempotent $e_A$ of $Z(A(P))$ fulfilling $e_A \cdot \text{Br}_P^A(\alpha) \neq \{0\}$ and any subgroup $Q$ of $P$, there is a unique primitive idempotent $f_A$ of $Z(A(Q))$ fulfilling

$$e_A \cdot \text{Br}_P^A(\alpha) \neq \{0\} \quad \text{and} \quad (Q, f_A) \subset (P, e_A)$$  \hspace{1cm} \text{(2.5.3)}$$

Once again, the maximal Brauer $(\alpha, G)$-pairs are pairwise $G$-conjugate [1, Theorem 1.14].

2.6 Here, we are specially interested in the $G$-algebras $A$ endowed with a group homomorphism $\rho: G \rightarrow A^*$ inducing the action of $G$ on $A$, called $G$-interior algebras; in this case, for any pointed group $H_\alpha$ on $A$, $A_\alpha = iAi$ has a structure of $H$-interior algebra mapping $y \in H$ on $\rho(y)i = ip(y)$; moreover, setting $x \cdot a \cdot y = \rho(x) a \rho(y)$ for any $a \in A$ and any $x, y \in G$, a $G$-interior algebra homomorphism from $A$ to another $G$-interior algebra $A'$ is a $G$-algebra homomorphism $f: A \rightarrow A'$ fulfilling

$$f(x \cdot a \cdot y) = x \cdot f(a) \cdot y$$  \hspace{1cm} \text{(2.6.1)}$$
2.7 In particular, if $H$ and $K$ are two pointed groups on $A$, we say that an injective group homomorphism $\varphi : K \to H$ is an $A$-fusion from $K$ to $H$ whenever there is a $K$-interior algebra embedding

$$f_\varphi : A \beta \longrightarrow \text{Res}_K^H(A)$$

such that the inclusion $A \beta \subset A$ and the composition of $f_\varphi$ with the inclusion $A \alpha \subset A$ are $A^*$-conjugate; we denote by $F_A(K, H)$ the set of $H$-conjugacy classes of $A$-fusions from $K$ to $H$ and, as usual, we write $F_A(H)$ instead of $F_A(H, H)$. If $A = iA$ for $i \in \alpha$, it follows from [8, Corollary 2.13] that we have a group homomorphism

$$F_A(H) \longrightarrow N_A^*(H \cdot i)/H \cdot (A H)^*$$

and if $H$ is a $p$-group then we consider the $k^*$-group $\hat{F}_A(H)$ defined by the pull-back

$$\begin{align*}
F_A(H) & \longrightarrow N_A^*(H \cdot i)/H \cdot (A H)^* \\
\uparrow & \uparrow \\
\hat{F}_A(H) & \longrightarrow N_A^*(H \cdot i)/H \cdot (i + J(A H))^*
\end{align*}$$

2.8 Recall that, for any subgroup $H$ of $G$ and any $H$-interior algebra $B$, the induced $G$-interior algebra is the induced bimodule

$$\text{Ind}^G_H(B) = kG \otimes_{kH} B \otimes_{kH} kG$$

endowed with the distributive product defined by the formula

$$(x \otimes b \otimes y)(x' \otimes b' \otimes y') = \begin{cases} 
  x \otimes y' \otimes b \otimes x' & \text{if } yx' \in H \\
  0 & \text{otherwise}
\end{cases}$$

where $x, y, x', y' \in G$ and $b, b' \in B$, and with the structural homomorphism

$$G \longrightarrow \text{Ind}^G_H(B)$$

mapping $x \in G$ on the element

$$\sum_y xy \otimes 1_B \otimes y^{-1} = \sum_y y \otimes 1_B \otimes y^{-1}x$$

where $y \in G$ runs over a set of representatives for $G/H$.

2.9 Obviously, the group algebra $OG$ is a $p$-permutation $G$-interior algebra and, for any primitive idempotent $b$ of $Z(OG)$ — called an $O$-block of $G$ — the conjugacy class $\alpha = \{b\}$ is a point of $G$ on $OG$. Moreover, for any $p$-subgroup $P$ of $G$, the Brauer homomorphism $\text{Br}_P = Br_P^G$ induces a $k$-algebra isomorphism [10, 2.8.4]

$$kC_G(P) \cong \langle OG \rangle(P)$$
thus, up to identification throughout this isomorphism, in a Brauer \((\{b\}, G)\)-pair \((P, e)\) as defined above — called Brauer \((b, G)\)-pair from now on — \(e\) is nothing but a \(k\)-block of \(C_G(P)\) such that \(e \text{Br}_P(b) \neq 0\). Setting

\[
\bar{C}_G(P) = C_G(P)/Z(P)
\]

recall that the image \(\bar{e}\) of \(e\) in \(k\bar{C}_G(P)\) is a \(k\)-block of \(\bar{C}_G(P)\) and that the Brauer First Main Theorem affirms that \((P, e)\) is maximal if and only if the \(k\)-algebra \(k\bar{C}_G(P)\bar{e}\) is simple and the inertial quotient

\[
E = N_G(P, e)/P\cdot C_G(P)
\]

is a \(p'\)-group [17, Theorem 10.14].

2.10 For any \(p\)-subgroup \(P\) of \(G\) and any subgroup \(H\) of \(N_G(P)\) containing \(P\cdot C_G(P)\), we have

\[
\text{Br}_P((OG)^H) = (OG)(P)^H
\]

and therefore any \(k\)-block \(e\) of \(C_G(P)\) determines a unique point \(\beta\) of \(H\) on \(OG\) (cf. 2.2) such that \(H_\beta\) contains \(P_\gamma\) for a local point \(\gamma\) of \(P\) on \(OG\) fulfilling [9, Lemma 3.9]

\[
e \cdot \text{Br}_P(\gamma) \neq \{0\}
\]

Recall that, if \(Q\) is a subgroup of \(P\) such that \(C_G(Q) \subset H\) then the \(k\)-blocks of \(C_G(Q) = C_H(Q)\) determined by \((P, e)\) from \(G\) and from \(H\) coincide [1, Theorem 1.8]. Note that if \(P\) is normal in \(G\) then the kernel of the obvious \(k\)-algebra homomorphism \(kG \rightarrow k(G/P)\) is contained in the radical \(J(kG)\) and contains \(\text{Ker}(\text{Br}_P)\); thus, in this case, isomorphism 2.9.1 implies that any point of \(P\) on \(kG\) is local.

2.11 Moreover, for any local pointed group \(P_\gamma\) on \(OG\), the action of \(N_G(P_\gamma)\) on the simple algebra \((OG)(P_\gamma)\) (cf. 2.2) determines a central \(k^*\)-extension or, equivalently, a \(k^*\)-group \(\hat{N}_G(P_\gamma)\) [10, §5] and it is clear that the Brauer homomorphism \(\text{Br}_P\) determines a \(N_G(P_\gamma)\)-stable injective group homomorphism from \(C_G(P)\) to \(\hat{N}_G(P_\gamma)\). Then, up to a suitable identification, we set

\[
E_G(P_\gamma) = N_G(P_\gamma)/P\cdot C_G(P) \quad \text{and} \quad \hat{E}_G(P_\gamma) = \hat{N}_G(P_\gamma)/P\cdot C_G(P)
\]

recall that from [8, Theorem 3.1] and [10, Proposition 6.12] we obtain a canonical \(k^*\)-group isomorphism (cf. 2.7.3)

\[
\hat{E}_G(P_\gamma)^* \cong \hat{F}_{OG}(P_\gamma)
\]
2.12 In particular, a maximal local pointed group \( P_\gamma \) on \( OGb \) determines a \( k \)-block \( e \) of \( G(P) \), which is still a \( k \)-block of the group

\[
N = N_G(P_\gamma) = N_G(P, e)
\]

— called the inertial subgroup of \( b \) — and also determines a unique point \( \nu \) of \( N \) on \( OGb \) such that \( P_\gamma \subset N_\nu \) (cf. 2.10); obviously, we have \( E = E_G(P_\gamma) \) (cf. 2.9.3). \( P_\gamma \) is still a defect pointed group of \( N_\nu \) and \((P, e)\) is a maximal Brauer \((\hat{e}, N)\)-pair, where \( \hat{e} \) denotes the \( O \)-block of \( N \) lifting \( e \). As above, \( N \) acts on the simple \( k \)-algebra (cf. 2.9)

\[
k\bar{C}_G(P_\gamma) \cong (OG)(P_\gamma)
\]

and therefore we get \( k^\ast \)-groups \( \hat{N} \) and \( \hat{E} \) of \( \hat{E}_G(P_\gamma) \).

2.13 Moreover, since \( E \) is a \( p' \)-group, it follows from [17, Lemma 14.10] that the short exact sequence

\[
1 \rightarrow P/Z(P) \rightarrow N/C_G(P) \rightarrow E \rightarrow 1
\]

splits and that all the splittings are conjugate to each other; thus, any splitting determines an action of \( E \) on \( P \) and it is easily checked that the semidirect products

\[
L = P \rtimes E \quad \text{and} \quad \hat{L} = P \rtimes \hat{E}
\]

do not depend on our choice. At this point, it follows from [10, Proposition 14.6] that the source algebra of the block \( \hat{e} \) of \( N \) is isomorphic to the \( P \)-interior algebra \( \mathcal{O}_\gamma \hat{L} \), and therefore it follows from [3, Proposition 4.10] that the multiplication in \( OGb \) by a suitable idempotent \( \ell \in \nu \) determines an injective unitary \( P \)-interior algebra homomorphism

\[
\mathcal{O}_\gamma \hat{L} \rightarrow (OG)_\gamma
\]

2.14 On the other hand, a Dade \( P \)-algebra over \( \mathcal{O} \) is a \( p' \)-permutation \( P \)-algebra \( S \) which is a full matrix algebra over \( \mathcal{O} \) and fulfills \( S(P) \neq \{ 0 \} \) [11, 1.3]. For any subgroup \( Q \) of \( P \), setting \( \tilde{N}_P(Q) = N_P(Q)/Q \) we have (cf. 2.4.1)

\[
(S(Q))(\tilde{N}_P(Q)) \cong S(N_P(Q))
\]

and therefore \( \text{Res}_Q^S(S) \) is a Dade \( Q \)-algebra; moreover, it follows from [11, 1.8] that the Brauer quotient \( S(Q) \) is a Dade \( \tilde{N}_P(Q) \)-algebra over \( k \); thus, \( Q \) has a unique local point on \( S \). In particular, if \( S \) is primitive (cf. 2.1) then \( S(P) \cong k \) and therefore we have

\[
\dim(S) \equiv 1 \pmod{p}
\]

so that the action of \( P \) on \( S \) can be lifted to a unique group homomorphism from \( P \) to the kernel of the determinant \( \det_S \) over \( S \); at this point, it follows from [11, 3.13] that the action of \( P \) on \( S \) always can be lifted to a well-determined \( P \)-interior algebra structure for \( S \).
2.15 Recall that a block $b$ of $G$ is called nilpotent whenever the quotients $N_G(Q,f)/C_G(Q)$ are $p$-groups for all the Brauer $(b,G)$-pairs $(Q,f)$ [2, Definition 1.1]; by the main result in [9], the block $b$ is nilpotent if and only if, for a maximal local pointed group $P_\gamma$ on $O_Gb$, $P$ stabilizes a unitary primitive Dade $P$-subalgebra $S$ of $(O_Gb)_\gamma$ fulfilling

$$ (O_Gb)_\gamma = SP \cong S \otimes O \hat{P} $$

where we denote by $SP$ the obvious $O$-algebra $\bigoplus_{u \in P} Su$ and, for the right-hand isomorphism, we consider the well-determined $P$-interior algebra structure for $S$.

2.16 Now, with the notation in 2.12 above, we say that the block $b$ of $G$ is inertial if it is basically Morita equivalent [15, 7.3] to the corresponding block $\hat{b}$ of the inertial subgroup $N$ of $b$ or, equivalently, if there is a primitive Dade $P$-algebra $S$ such that we have a $P$-interior algebra embedding [15, Theorem 6.9 and Corollary 7.4]

$$ (O_G)_{\gamma} \rightarrow S \otimes O_{\hat{L}} $$

Note that, in this case, in fact we have a $P$-interior algebra isomorphism

$$ (O_G)_{\gamma} \cong S \otimes O_{\hat{L}} $$

and the Dade $P$-algebra $S$ is uniquely determined; indeed, the uniqueness of $S$ follows from [19, Lemma 4.5] and it is easily checked that

$$ (S \otimes O_{\hat{L}})(P) \cong S(P) \otimes_k (O_{\hat{L}})(P) \cong kZ(P) $$

and that the kernel of the Brauer homomorphism $Br_P^{S \otimes O_{\hat{L}}}$ is contained in the radical of $S \otimes O_{\hat{L}}$, so that this $P$-interior algebra is also primitive.

3 Normal sub-blocks of inertial blocks

3.1 Let $G$ be a finite group, $b$ an $O$-block of $G$ and $(P,e)$ a maximal Brauer $(b,G)$-pair (cf. 2.9). Let us say that an $O$-block $c$ of a normal subgroup $H$ of $G$ is a normal sub-block of $b$ if we have $eb \neq 0$; we are interested in the relationship between the source algebras of $b$ and $c$, specially in the case where $b$ is inertial.

3.2 Note that we have $bTr_G^G(c) = b$ where $G_c$ denotes the stabilizer of $c$ in $G$; since we know that $eBr_P(b) \neq 0$ (cf. 2.9), up to modifying our choice of $(P,e)$ we may assume that $P$ stabilizes $c$; then, considering the $G$-stable semisimple $k$-subalgebra $\sum_x k \cdot bc^x$ of $kG$, where $x \in G$ runs over a set of representatives for $G/G_c$, it follows from [19, Proposition 3.5] that $bc$ is an
O-block of $G_c$ and that $P$ remains a defect $p$-subgroup of this block, and then from [19, Proposition 3.2] that we have

$$OGb \cong \text{Ind}_{G_c}^G(OG_cbc)$$

so that the source algebras of the $O$-block $b$ of $G$ and of the block $bc$ of $G_c$ are isomorphic.

3.3 Thus, from now on we assume that $G$ fixes $c$, so that we have $bc = b$. Then, note that $\alpha = \{c\}$ is a point of $G$ on $OH$ (cf. 2.2), so that, choosing a block $e^H$ of $C_H(P)$ such that $e^He \neq 0$, $(P, e^H)$ is a Brauer $(\alpha, G)$-pair (cf. 2.4 and 2.9.1) and it follows from the proof of [18, Proposition 15.9] that we may choose a maximal Brauer $(c, H)$-pair $(Q, f^H)$ fulfilling

$$(Q, f^H) \subset (P, e^H), \quad Q = H \cap P \quad \text{and} \quad e\text{Br}_P(f^H) \neq 0$$

3.4 Since we have $e\text{Br}_P(f^H) \neq 0$ and $f^H$ is $P$-stable, from the obvious commutative diagram

$$
\begin{align*}
(OH)(Q) \quad &\longrightarrow \quad (OG)(Q) \\
\cup \quad &\cup \\
(OH)(Q)^P \quad &\longrightarrow \quad (OG)(Q)^P \\
\downarrow \quad &\downarrow \\
(OH)(P) \quad &\longrightarrow \quad (OG)(P)
\end{align*}
$$

we get a local point $\delta^c$ of $Q$ on $OG$ such that the multiplicity $m_{\delta}^c$ of the inclusion $(OH)^Q \subset (OG)^Q$ at $\delta^c$ (cf. 2.2) is not zero and $Q_{\delta^c}$ is contained in $P_{\gamma^c}$; similarly, we get a local point $\gamma$ of $P$ on $OH$ fulfilling

$$m_{\gamma}^c \neq 0 \quad \text{and} \quad Q_{\delta} \subset P_{\gamma}$$

At this point, the following commutative diagram (cf. 2.2.1)

$$
\begin{align*}
\text{Res}_{Q}^P(OH)_{\gamma} \quad &\longrightarrow \quad \text{Res}_{Q}^P(OG)_{\gamma} \\
(\uparrow) \quad &\quad (\uparrow) \\
(\overline{OH})_{\delta} \quad &\longrightarrow \quad (\overline{OG})_{\delta} \quad \text{Res}_{Q}^P(OG)_{\gamma^c} \\
(\uparrow) \quad &\quad (\uparrow)
\end{align*}
$$

where all the $Q$-interior algebra homomorphisms but the horizontal ones are embeddings, already provides some relationship between the source algebras of $b$ and $c$ (cf. 2.2).
3.5 If $R_ε$ is a local pointed group on $OH$, we set
\[ C_G(R_ε) = C_G(R) \cap N_G(R_ε) \quad \text{and} \quad E_G(R_ε) = N_G(R_ε)/R \cdot C_G(R_ε) \]
and denote by $b(ε)$ the block of $C_H(R)$ determined by $ε$, and by $\bar{b}(ε)$ the image of $b(ε)$ in $kC_H(R) = k(C_H(R)/Z(R))$; recall that we have a canonical $\bar{C}_G(R)$-interior algebra isomorphism [19, Proposition 3.2]
\[ k\bar{C}_G(R)\text{Tr}_{\bar{C}_G(R_ε)}^\epsilon(\bar{b}(ε)) \cong \text{Ind}_{\bar{C}_G(R_ε)}^{\bar{C}_G(R)}(k\bar{C}_G(R_ε)\bar{b}(ε)) \]
Moreover, note that if $ε^a$ is a local point of $R$ on $OG$ such that $m_{ε^a}^G \neq 0$ then we have
\[ E_G(R_{ε^a}) \subset E_G(R_ε) \]
indeed, the restriction to $C_H(R)$ of a simple $kC_G(R)$-module determined by $ε^a$ is semisimple (cf. 2.9.1) and therefore $C_G(R)$ acts transitively on the set of local points $ε'$ of $R$ on $OH$ such that $m_{ε'}^G \neq 0$, so that we have
\[ N_G(R_{ε')} \subset C_G(R) \cdot N_G(R_ε) \]
Then, we also define $E_H(R_{ε^a}) = N_H(R_{ε^a})/R \cdot C_H(R)$.

3.6 Since $(Q, f^H)$ is a maximal Brauer $(c, H)$-pair, we have (cf. 2.12.2)
\[ k\bar{C}_H(Q)f^H \cong (OH)(Q_δ) \]
and, according to the very definition of the $k^*$-group $\hat{N}_G(Q_δ)$, we also have a $k^*$-group homomorphism
\[ \hat{N}_G(Q_δ) \longrightarrow (k\bar{C}_H(Q)f^H)^* \]
then, denoting by $\hat{C}_G(Q_δ)$ the corresponding $k^*$-subgroup of $\hat{N}_G(Q_δ)$ and setting
\[ Z = C_G(Q_δ)/C_H(Q) \quad \text{and} \quad \hat{Z} = \hat{C}_G(Q_δ)/C_H(Q) \]
it follows from [19, Theorem 3.7] that we have a canonical $\hat{C}_G(Q_δ)$-interior algebra isomorphism
\[ k\hat{C}_G(Q_δ)f^H \cong k\bar{C}_H(Q)f^H \otimes_k (k_*/\hat{Z})^c \]
Now, this isomorphism and the corresponding isomorphism 3.5.2 determine a $k$-algebra isomorphism
\[ Z(k\hat{C}_G(Q))\text{Tr}_{\hat{C}_G(Q_δ)}^{\bar{Q}}(f^H) \cong Z(k_*/\hat{Z}) \]
and induce a bijection between the set of local points $δ^a$ of $Q$ on $OGb$ such that $m_{δ^a}^G \neq 0$ and the set of points of the $k$-algebra $(k_*/\hat{Z})^c \hat{b}_δ$ where we denote by $\text{Br}_Q(b)$ the image of $\text{Br}_Q(b)$ in $k\bar{C}_G(Q)$ and by $\hat{b}_δ$ the image of $\hat{\text{Br}}_Q(b)\text{Tr}_{\hat{C}_G(Q_δ)}^{\bar{Q}}(f^H)$ in the right-hand member of isomorphism 3.6.5.
Proposition 3.7 With the the notation above, the idempotent \( \hat{b}_b \) is primitive in \( Z(k, \hat{Z})^{E_G(Q_\delta)} \). In particular, if \( E_G(Q_\delta) \) acts trivially on \( \hat{Z} \) then \( P_{\gamma, \alpha} \) contains \( Q_\delta \alpha \) for any local point \( \delta \) of \( Q \) on \( \text{OG} \) such that \( \text{m}_3^G \neq 0 \).

Proof: Since \( Q = H \cap P \), for any \( a \in (OG)^P \) it is easily checked that

\[
\text{Br}_Q(\text{Tr}_P^G(a)) = \text{Tr}_P^{N_G(Q)}(\text{Br}_Q(a)) \tag{3.7.1}
\]

and, in particular, we have \( \text{Br}_Q((OG)^P) \cong kC_G(Q)^{N_G(Q)} \) (cf. 2.9.1); consequently, since the idempotent \( b \in (OG)^P \) is primitive in \( Z(OG) \), setting \( E_G(Q) = N_G(Q)/Q.C_G(Q) \), \( \text{Br}_Q(b) \) is still primitive in \( [17, \text{Proposition 3.23}] \)

\[
kC_G(Q)^{N_G(Q)} = Z(kC_G(Q))^{E_G(Q)} \tag{3.7.2}
\]

which amounts to saying that \( N_G(Q) \) acts transitively over the set of \( k \)-blocks of \( C_G(Q) \) involved in \( \text{Br}_Q(b) \); hence, since any \( k \)-block of \( C_G(Q) \) maps on a \( k \)-block of \( \hat{C}_G(Q) \) (cf. 2.9), \( \text{Br}_Q(b) \) is also primitive in \( Z(k\hat{C}_G(Q))^{E_G(Q)} \) and then, it suffices to apply isomorphism 3.6.5.

On the other hand, identifying \((OG)(Q)\) with \( kC_G(Q) \) (cf. 2.9.1), it is easily checked that \( \text{Br}_Q((OG)^P) = kC_G(Q)^P \) and therefore, for any \( i \in \gamma^\alpha \), the idempotent \( \text{Br}_Q(i) \) is primitive in \( kC_G(Q)^P \) [17, Proposition 3.23]; thus, since the canonical \( P \)-algebra homomorphism \( kC_G(Q) \to k\hat{C}_G(Q) \) is a strict semicovering [16, Theorem 2.9], it follows from [6, Proposition 3.15] that the image \( \text{Br}_Q(i) \) of \( \text{Br}_Q(i) \) in \( k\hat{C}_G(Q)^P \) remains a primitive idempotent and that, denoting by \( \gamma^\alpha \) the point of \( P \) on \( k\hat{C}_G(Q) \) determined by \( \text{Br}_Q(i) \), \( P_{\gamma, \alpha} \) remains a maximal local pointed group on the \( N_G(Q) \)-algebra \( k\hat{C}_G(Q) \).

Moreover, since \( P \) fixes \( f_\alpha^H \) (cf. 3.3), we may choose \( i \in \gamma^\alpha \) fulfilling \( \text{Br}_Q(i) = \text{Br}_Q(i)f_\alpha^H \); in this case, it follows from isomorphism 3.5.2 and from [19, Proposition 3.5] that \( \text{Br}_Q(i) \) is a primitive idempotent of \( (k\hat{C}_G(Q_\delta)f_\alpha^H)^P \) and that \( P_{\gamma, \alpha} \) is also a maximal local pointed group on the \( N_G(Q_\delta) \)-algebra \( k\hat{C}_G(Q_\delta)f_\alpha^H \).

But, it follows from isomorphism 3.6.4 that we have

\[
(kC_G(Q_\delta)f_\alpha^H)(P) \cong (k\hat{C}_H(Q)f_\alpha^H)(P) \otimes_k (k, \hat{Z})^\gamma(P) \tag{3.7.3}
\]

and therefore, since evidently \( ib = i \), \( P_{\gamma, \alpha} \) determines a maximal local pointed group \( P_{\gamma, \alpha} \) on \((k, \hat{Z})^\gamma \hat{b}_\delta \) [9, Theorem 5.3 and Proposition 5.9]. Moreover, if \( E_G(Q_\delta) \) acts trivially on \( \hat{Z} \) then \( \hat{b}_\delta \) is a block of \( \hat{Z} \) and all the maximal local pointed groups on the \( N_G(Q_\delta) \)-algebra \((k, \hat{Z})^\gamma \hat{b}_\delta \) are mutually \( N_G(Q_\delta) \)-conjugate (cf. 2.5); in this case, since \( N_G(Q_\delta) \) acts trivially on the set of points
of the $k$-algebra $(k_\hat{\mathbb{Z}})^\circ \hat{b}_\delta$, $P_{\hat{\mathbb{Z}}}$ contains $Q_{\hat{\mathbb{Z}}}$ for any local point $\hat{\delta}^\mathbb{Z}$ of $Q$ on $(k_\hat{\mathbb{Z}})^\circ \hat{b}_\delta$, which amounts to saying that any idempotent $\hat{\delta} \in \hat{\mathbb{Z}}$ has a nontrivial image in all the simple quotients of $(k_\hat{\mathbb{Z}})^\circ \hat{b}_\delta$ (cf. 2.2.2), and the last statement follows from 3.6.

**Proposition 3.8** Let $\hat{\delta}^\mathbb{Z}$ be a local point of $Q$ on $\mathfrak{OG}$ such that $m_{\hat{\delta}^\mathbb{Z}} \neq 0$. The commutator in $\hat{N}_G(Q_{\delta})/\mathbb{Z} \cdot C_H(Q)$ induces a group homomorphism

$$\varpi : F \rightarrow \text{Hom}(\mathbb{Z}, k^*)$$

and $\text{Ker}(\varpi)$ is contained in $E_H(Q_{\delta^\mathbb{Z}})$. In particular, $E_H(Q_{\delta^\mathbb{Z}})$ is normal in $F$, $F/E_H(Q_{\delta^\mathbb{Z}})$ is an Abelian $p'$-group and, denoting by $\hat{K}^\mathbb{Z}$ and $\hat{K}^\mathbb{Z}_{\delta^\mathbb{Z}}$ the respective converse images in $\hat{C}_G(\mathbb{Q})$ of the fixed points of $F$ and $E_H(Q_{\delta^\mathbb{Z}})$ over $\hat{\mathbb{Z}}$, we have the exact sequence

$$1 \rightarrow \hat{K}^\mathbb{Z} \rightarrow \hat{K}^\mathbb{Z}_{\delta^\mathbb{Z}} \rightarrow \text{Hom}(F/E_H(Q_{\delta^\mathbb{Z}}), k^*) \rightarrow 1$$

**Proof:** It is quite clear that $F$ and $Z$ are normal subgroups of the quotient $N_G(Q_{\delta})/QC_H(Q)$ and therefore their converse images $\hat{F}$ and $\hat{Z}$ in the quotient $\hat{N}_G(Q_{\delta})/QC_H(Q)$ still normalize each other; but, since we have

$$N_H(Q_{\delta}) \cap C_G(Q_{\delta}) = C_H(Q)$$

their commutator is contained in $k^*$; hence, identifying $\text{Hom}(\mathbb{Z}, k^*)$ with the group of the automorphisms of the $k^*$-group $\hat{F}$ which act trivially on $Z$, we easily get homomorphism 3.8.1.

In particular, $\text{Ker}(\varpi)$ acts trivially on the $k^*$-group $\hat{\mathbb{Z}}$ and therefore, since its action is compatible with the bijection in 3.6 above, it is contained in $E_H(Q_{\delta^\mathbb{Z}})$; hence, since the $p'$-group $\text{Hom}(\mathbb{Z}, k^*)$ is Abelian, $E_H(Q_{\delta^\mathbb{Z}})$ is normal in $E_H(Q_{\delta})$ (cf. 3.5.3) and $F/E_H(Q_{\delta^\mathbb{Z}})$ is Abelian.

Symmetrically, the commutator in $\hat{N}_G(Q_{\delta})/QC_H(Q)$ also induces surjective group homomorphisms

$$\hat{C}_G(Q_{\delta}) \rightarrow \text{Hom}(F/\text{Ker}(\varpi), k^*)$$

$$\hat{C}_G(Q_{\delta}) \rightarrow \text{Hom}(E_H(Q_{\delta^\mathbb{Z}})/\text{Ker}(\varpi), k^*)$$

and it is quite clear that the kernels respectively coincide with $\hat{K}^\mathbb{Z}$ and $\hat{K}^\mathbb{Z}_{\delta^\mathbb{Z}}$; consequently, the kernel of the surjective group homomorphism

$$\hat{C}_G(Q_{\delta})/\hat{K}^\mathbb{Z} \rightarrow \hat{C}_G(Q_{\delta})/\hat{K}^\mathbb{Z}_{\delta^\mathbb{Z}}$$

is canonically isomorphic to $\text{Hom}(F/E_H(Q_{\delta^\mathbb{Z}}), k^*)$. We are done.
3.9 Assume that \( b \) is an inertial block of \( G \) or, equivalently, that there is a primitive Dade \( P \)-algebra \( S \) such that, with the notation in 2.13 above, we have a \( P \)-interior algebra isomorphism
\[
(OG)_{\gamma_G} \cong S \otimes_O O \hat{L} \tag{3.9.1}
\]
where we consider \( S \) endowed with the unique \( P \)-interior algebra structure fulfilling \( \det_S(P) = \{1\} \) (cf. 2.14). In this case, it follows from [6, Lemma 1.17] and [8, proposition 2.14 and Theorem 3.1] that (cf. 2.8)
\[
E = F_O G(P_{\gamma_G}) = F_S(P_{\{1\}}) \cap F_{O \hat{L}}(P_{\{1\}}) \tag{3.9.2}
\]
and, in particular, that \( S \) is \( E \)-stable \([8, Proposition 2.18]\). Moreover, since we have a \( P \)-interior algebra embedding (cf. 2.14)
\[
O \rightarrow \text{End}_O(S) \cong S \otimes_O S \tag{3.9.3}
\]
we still have a \( P \)-interior algebra embedding
\[
O \hat{L} \rightarrow S^0 \otimes_O (OG)_{\gamma_G} \tag{3.9.4}
\]

3.10 Conversely, always with the notation in 2.13, assume that \( S \) is an \( E \)-stable Dade \( P \)-algebra or, equivalently, that \( E \) is contained in \( F_S(P_\pi) \) where \( \pi \) denotes the unique local point of \( P \) on \( S \) (cf. 2.14); since we have [9, Proposition 5.9]
\[
F_S(P_\pi) \cap F_O G(P_{\gamma_G}) \subset F_S \otimes_O O G(P_{\pi \times \gamma_G}) \tag{3.10.1}
\]
where \( \pi \times \gamma_G \) denotes the local point of \( P \) on \( S^0 \otimes_O O G \) determined by \( \pi \) and \( \gamma_G \) [9, Proposition 5.6], and we still have [18, Theorem 9.21]
\[
\hat{F}_S(P_\pi) \cong k^* \times F_S(P_\pi) \tag{3.10.2}
\]
it follows from [9, proposition 5.11] that the \( k^* \)-group \( \hat{E} \) is isomorphic to a \( k^* \)-subgroup of \( \hat{F}_S \otimes_O O G(P_{\pi \times \gamma_G}) \); then, since \( E \) is a \( p' \)-group, it follows from [10, Proposition 7.4] that there is an injective unitary \( P \)-interior algebra homomorphism
\[
O \hat{L} \rightarrow (S^0 \otimes_O O G)_{\pi \times \gamma_G} \tag{3.10.3}
\]
and, in particular, we have
\[
|P||E| \leq \text{rank}_O(S^0 \otimes_O O G)_{\pi \times \gamma_G} \tag{3.10.4}
\]

**Proposition 3.11** With the notation above, the block \( b \) is inertial if and only if there is an \( E \)-stable Dade \( P \)-algebra \( S \) such that
\[
\text{rank}_O(S^0 \otimes_O O G)_{\pi \times \gamma_G} = |P||E| \tag{3.11.1}
\]

**Proof:** If \( b \) is inertial then the equality 3.11.1 follows from the existence of embedding 3.9.4.
Conversely, we claim that if equality 3.11.1 holds then the corresponding homomorphism 3.10.3 is an isomorphism; indeed, since this homomorphism is injective and we have \( \text{rank}_\mathcal{O}(\mathcal{O}, \bar{L}) = |P||E| \), it suffices to prove that the reduction to \( k \) of homomorphism 3.10.3 remains injective; but, according to [10, 2.1], it also follows from [10, Proposition 7.4] that, setting \( kS = k \otimes_\mathcal{O} S \), there is an injective unitary \( P \)-interior algebra homomorphism

\[
k \sigma \bar{L} \longrightarrow (k S^\sigma \otimes_k kG)_{\pi \times \gamma^G}
\]

where \( \tilde{\pi} \) and \( \tilde{\gamma}^G \) denote the respective images of \( \pi \) and \( \gamma^G \) in \( kS^\sigma \) and \( kG \), which is a conjugate of the reduction to \( k \) of homomorphism 3.10.3.

Now, embedding 3.9.3 and the structural embedding

\[
(S^\sigma \otimes_\mathcal{O} OG)_{\pi \times \gamma^G} \longrightarrow S^\sigma \otimes_\mathcal{O} (OG)_{\gamma^G}
\]

determine \( P \)-interior algebra embeddings

\[
\begin{align*}
\overline{\|} & \quad \uparrow \\
S \otimes_\mathcal{O} (S^\sigma \otimes_\mathcal{O} OG)_{\pi \times \gamma^G} & \longrightarrow S \otimes_\mathcal{O} S^\sigma \otimes_\mathcal{O} (OG)_{\gamma^G} \\
S \otimes_\mathcal{O} \mathcal{O}_* \bar{L} & \quad (OG)_{\gamma^G}
\end{align*}
\]

thus, since \( P \) has a unique local point on \( S \otimes S^\sigma \otimes_\mathcal{O} (OG)_{\gamma^G} \) [9, Theorem 5.3], we get a \( P \)-interior algebra embedding

\[
(OG)_{\gamma^G} \longrightarrow S \otimes_\mathcal{O} \mathcal{O}_* \bar{L}
\]

which proves that \( b \) is inertial. We are done.

3.12 With the notation above, assume that the block \( b \) is inertial; then, denoting by \( \chi \) the unique local point of \( Q \) on \( S \) (cf. 2.14) and by \( \delta^G \) a local point of \( Q \) on \( \mathcal{O}Gb \) such that \( m_\delta^G \neq 0 \), there is a unique local point \( \delta^L \) of \( Q \) on \( \mathcal{O}_* \bar{L} \) such that isomorphism 3.9.1 induces a \( Q \)-interior algebra embedding [9, Proposition 5.6]

\[
(OG)_{\delta^G} \longrightarrow S_\chi \otimes_\mathcal{O} (\mathcal{O}, \bar{L})_{\delta^L}
\]

but, the image of \( Q \) in \( (S_\chi)^* \) need not be contained in the kernel of the corresponding determinant map. Note that, as above, it follows from this embedding and from [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] that

\[
E_G(Q_{\delta^G}) = F_{OG}(Q_{\delta^G}) = F_S(Q_\chi) \cap F_{\mathcal{O}_* \bar{L}}(Q_{\delta^L})
\]

so that the Dade \( Q \)-algebra \( S_\chi \) is \( E_G(Q_{\delta^G}) \)-stable; as in 2.13 above, let us consider the corresponding semidirect products

\[
M = Q \rtimes F \quad \text{and} \quad \hat{M} = Q \rtimes \hat{F}
\]

We are ready to state our main result.
Theorem 3.13  With the notation above, assume that the block $b$ of $G$ is inertial. Then, there is a $Q$-interior algebra isomorphism

\[(OH)_δ \cong S_χ \otimes_O O_\delta \hat{M} \quad 3.13.1\]

and, in particular, the block $c$ of $H$ is inertial too.

Proof: We argue by induction on $|G/H|$; in particular, if $H'$ is a proper normal subgroup of $G$ which properly contains $H$, it suffices to choose a block $c'$ of $H'$ fulfilling $c'b \neq 0$ to get $c'c \neq 0$ and the induction hypothesis successively proves that the block $c'$ of $H'$ is inertial and then that the block $c$ is inertial too; moreover, setting $Q' = H' \cap P$, the corresponding Dade $Q'$-algebra comes from $S$ and therefore the final Dade $Q$-algebra also comes from $S$. Consequently, since $G$ fixes $c$, it follows from the Frattini argument that we have (cf. 2.3)

\[G = H \cdot N_G(Q_δ) \quad 3.13.2\]

and therefore we may assume that either $C_G(Q_δ) \subset H$ or $G = H \cdot C_G(Q_δ)$.

Firstly assume that $C_G(Q_δ) \subset H$; in this case, it follows from [18, Proposition 15.10] that $b = c$; moreover, since $C_G(Q_δ) = C_H(Q)$, it follows from 3.6 above that $Q$ has a unique local point $δ^G$ on $OGb$ such that $m_{δ^G} \neq 0$, and from isomorphism 3.6.4 that we have

\[(OH)(Q_δ) \cong k\hat{C}_H(Q)\hat{f}'' \cong k\hat{C}_G(Q_δ)\hat{f}'' \quad 3.13.3;\]

in particular, $N_G(Q_δ)$ normalizes $Q_δ^G$ and therefore the inclusion 3.5.3 becomes an equality

\[E_G(Q_δ^G) = E_G(Q_δ) \quad 3.13.4;\]

thus, since $F$ is obviously contained in $E_G(Q_δ)$, $S_χ$ is $F$-stable too. Consequently, according to Proposition 3.11, it suffices to prove that

\[\text{rank}_O(S_χ \otimes_O OH)_{χ × δ} = |Q||F| \quad 3.13.5.\]

As in 3.12 above, the $P$-interior algebra embedding 3.9.4 induces a $Q$-interior algebra embedding [9, Theorem 5.3]

\[(O, L)_{δL} \rightarrow S_χ^O \otimes_O (OG)_{δG} \quad 3.13.6\]

and it suffices to apply again [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] to get

\[E_L(Q_δ^L) = F_{O, L}(Q_δ^L) = F_S(Q_χ) \cap F_{OG}(Q_δ^G) \quad 3.13.7,\]

so that we obtain

\[E_L(Q_δ^L) = E_G(Q_δ^G) \subset F_S(Q_χ) \quad 3.13.8.\]
In particular, it follows from [8, Proposition 2.12] that for any \(x \in N_G(Q_\delta)\) there is \(s_x \in (S_\chi)^*\) fulfilling

\[s_x \cdot u = u \cdot s_x\]  \hspace{1cm} 3.13.9

for any \(u \in Q\), and therefore, choosing a set of representatives \(X \subset N_G(Q_\delta)\) for \(G/H\) (cf. 3.13.2), we get an \(OQ\)-bimodule direct sum decomposition

\[S_\chi^O \otimes OG = \bigoplus_{x \in X} (s_x \otimes x)(S_\chi^O \otimes OH)\]  \hspace{1cm} 3.13.10.

But, for any \(x \in N_G(Q_\delta)\), the element \(s_x \otimes x\) normalizes the image of \(Q\) in \(S_\chi^O \otimes OH\) and it is clear that it also normalizes the local point \(\chi \times \delta\) of \(Q\) on this \(Q\)-interior algebra; more precisely, if \(S_\chi = \ell S\ell\) for \(\ell \in \chi\) and \((OH)_{\delta} = j(OH)j\) for \(j \in \delta\), there is \(j' \in \chi \times \delta\) such that [9, Proposition 5.6]

\[j'(\ell \otimes j) = j' = (\ell \otimes j)j'\]  \hspace{1cm} 3.13.11;

thus, for any \(x \in N_G(Q_\delta)\) the idempotent \(j'^*(s_x \otimes x)\) still belongs to \(\chi \times \delta\) and therefore there is an invertible element \(a_x\) in \((S_\chi^O \otimes OH)^Q\) fulfilling

\[j'^*(s_x \otimes x) = j'^*(a_x)\]  \hspace{1cm} 3.13.12,

so that we get the new \(OQ\)-bimodule direct sum decomposition

\[j'(S_\chi^O \otimes OG) j' = \bigoplus_{x \in X} (s_x \otimes x)(a_x)^{-1}j'(S_\chi^O \otimes OH) j'\]  \hspace{1cm} 3.13.13.

Moreover, the equality in 3.13.8 forces the group \(E_G(Q_\delta) = E_G(Q_\delta^O)\) to have a normal Sylow \(p\)-subgroup and therefore, since we are assuming that \(C_G(Q_\delta) \subset H\), it follows from equality 3.13.2 that the quotient \(G/H\) also has a normal Sylow \(p\)-subgroup. At this point, arguing by induction, we may assume that \(G/H\) is either a \(p\)-group or a \(p'\)-group.

Firstly assume that \(G/H\) is a \(p\)-group or, equivalently, that \(G = H \cdot P\) [9, Lemma 3.10]; in this case, it follows from [6, Proposition 6.2] that the inclusion homomorphism \(OH \to OG\) is a strict semicovering of \(Q\)-interior algebras (cf. 2.1) and, in particular, we have \(\delta \subset \delta^O\) since \(m_\delta^O \neq 0\); similarly, since for any subgroup \(R\) of \(Q\) we have [9, Proposition 5.6]

\[(S^O \otimes OH)(R) \cong S(R)^O \otimes_k (OH)(R)\]
\[(S^O \otimes OG)(R) \cong S(R)^O \otimes_k (OG)(R)\]  \hspace{1cm} 3.13.14,
it follows from [6, Theorem 3.16] that the corresponding \( Q \)-interior algebra homomorphism \( S^p \otimes_O H \to S^p \otimes_O G \) is also a strict semicovering and, in particular, we have \( \chi \times \delta \subseteq \chi \times \delta^\circ \), so that \( j' \) belongs to \( \chi \times \delta^\circ \).

But, since \( Q_\delta^\circ \subseteq P_\gamma^\circ \) (cf. 3.4.4), it is easily checked that \( Q_{\chi \times \delta^\circ} \subseteq P_{\pi \times \gamma^\circ} \), where as above \( \pi \) is the unique local point of \( P \) on \( S \), and therefore we get the \( Q \)-interior algebra embedding (cf. embeddings 2.2.3 and 3.9.4)

\[
(S^p \otimes_O G)_{\chi \times \delta^\circ} \longrightarrow \text{Res}^P_Q(S^p \otimes_O G)_{\pi \times \gamma^\circ} \cong \text{Res}^P_Q(O_* \hat{L})
\]

in particular, it follows from equality 3.13.13 that we have

\[
|X| \operatorname{rank}_O (S^p_\chi \otimes_O H)_{\chi \times \delta} \leq |L|
\]

Moreover, we have \( |X| = |G/H| = |P/Q| \) and, since \( C_P(Q) \subseteq Q \), it follows from [4, Ch. 5, Theorem 3.4] that a subgroup of \( E \subseteq L \) which centralizes \( Q = H \cap P \) still centralizes \( P \), so that \( E \) acts faithfully on \( Q \); in particular, \( \delta^k \) is the unique local point of \( Q \) on \( O_* \hat{L} \) (actually, we have \( \delta^k = \{1_{O_* \hat{L}}\} \)) and therefore, since (cf. 3.13.4 and 3.13.8)

\[
E_L(Q_{\delta^k}) = E_G(Q_{\delta^\circ}) = E_G(Q_{\delta}) \supset F
\]

and \( E_G(Q_{\delta})/F \) is a \( p' \)-group, the \( p' \)-group \( E \) is actually isomorphic to \( F \).

Consequently, it follows from the inequalities 3.10.4 and 3.13.16 that

\[
|F||Q| \leq \operatorname{rank}_O (S^p_\chi \otimes_O H)_{\chi \times \delta} \leq |L|/|X| = |F||Q|
\]

which forces equality 3.13.6.

Secondly assume that \( G/H \) is a \( p' \)-group; in this case, we have \( Q = P \), \( \delta = \gamma \) and \( \delta^\circ = \gamma^\circ \); in particular, since we are assuming that

\[
C_G(Q_{\delta}) \subseteq H \quad \text{and} \quad E_G(Q_{\delta^\circ}) = E_G(Q_{\delta})
\]

we actually get

\[
|X| = |G/H| = |E_G(P_{\gamma^\circ})|/|E_H(Q_{\delta})| = |E|/|F|
\]

Moreover, we claim that, as above, the idempotent \( j' \) remains primitive in \( (S \otimes_O G)^P \) \(^\dagger\), so that it belongs to \( \pi \times \gamma^\circ \); indeed, setting

\[
A' = j'(S^p \otimes_O G)j' \quad \text{and} \quad B' = j'(S^p \otimes_O H)j'
\]

let \( i' \) be a primitive idempotent of \( A'^P \) such that \( \text{Br}_P(i') \neq 0 \); in particular, \( i' \) belongs to \( \pi \times \gamma^\circ \) and we may assume that

\[
i'A'i' = (S^p \otimes_O G)_{\pi \times \gamma^\circ} \cong O_* \hat{L}
\]

\(^\dagger\) The corresponding argument has been forgotten in [18] at the end of the proof of Proposition 15.19!
It is clear that the multiplication by $B'$ on the left and the action of $P$ by conjugation endows $A'$ with a $B'P$-module structure and, since the idempotent $j'$ is primitive in $B'P$, equality 3.13.13 provides a direct sum decomposition of $A'$ in indecomposable $B'P$-modules. More explicitly, note that $B'$ is an indecomposable $B'P$-module since we have $\text{End}_{B'P}(B') = B'P$; but, for any $x \in X$, the invertible element
\[
a'_x = (s_x \otimes x)(a_x)^{-1}j'
\]
of $A'$ together with the action of $x$ on $P$ determine an automorphism $g_x$ of $B'P$; thus, equality 3.13.13 provides the following direct sum decomposition on indecomposable $B'P$-modules
\[
A' \cong \bigoplus_{x \in X} \text{Res}_{g_x}(B')
\]
Moreover, we claim that the $B'P$-modules $\text{Res}_{g_x}(B')$ and $\text{Res}_{g_x}(B')$ for $x, x' \in X$ are isomorphic if and only if $x = x'$; indeed, a $B'P$-module isomorphism
\[
\text{Res}_{g_x}(B') \cong \text{Res}_{g_{x'}}(B')
\]
is necessarily determined by the multiplication on the right by an invertible element $b'$ of $B'$ fulfilling
\[
(xux^{-1})b' = b'(x'ux'^{-1})
\]
or, equivalently, $(u\cdot j')b' = u^{xx^{-1}}j'$ for any $u \in P$, which amounts to saying that the automorphism of $P$ determined by the conjugation by $x'^{-1}$ is a $B'$-fusion from $P_\gamma$ to $P_\gamma$ [8, Proposition 2.12]; but, once again from [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] we get
\[
F_{A'}(P_\gamma, \sigma) = E_G(P_\gamma) = E
\quad \text{and} \quad
F_{B'}(P_\gamma) = E_H(P_\gamma)
\]
hence our claim now follows from qualities 3.13.20.

On the other hand, it is clear that $A'i'$ is a direct summand of the $B'P$-module $A'$ and therefore there is $x \in X$ such that $\text{Res}_{g_x}(B')$ is a direct summand of the $B'P$-module $A'i'$; but, it follows from [8, Proposition 2.14] that we have
\[
F_{i'A'i'}(P_\gamma, \sigma) = F_{A'}(P_\gamma, \sigma) = E
\]
and therefore, once again applying [8, Proposition 2.12], for any $y \in N_G(P_\gamma, \sigma)$ there is an invertible element $c'_y$ in $A'$ fulfilling
\[
c'_y(u\cdot i')(c'_y)^{-1} = yuy^{-1}i'
\]
for any \( u \in P \); then, for any \( x' \in X \), it is clear that \( A' \ell' = A' \ell' c_{z-1} \) has a direct summand isomorphic to \( \text{Res}_{z,x'}(B') \), which forces the equality of the \( O \)-ranks of \( A' \ell' \) and \( A' \ell' c_{z-1} \), so that \( A' \ell' = A' \ell' c_{z-1} \), which proves our claim. Consequently, it follows from the equalities 3.13.13 and 3.13.20 that

\[
\text{rank}_O (S^\circ \otimes O H)_{\chi \times \delta} = |L|/|X| = |F||Q| \tag{3.13.30}
\]

so that equality 3.13.6 holds.

From now on, we assume that \( H \cdot C_G(Q_\delta) = G \); in particular, \( C_G(Q) \) stabilizes \( \delta \), so we have \( E_G(Q_\delta) = E_H(Q_\delta) = F \) and we can choose the set of representatives \( X \) for \( G/H \) contained in \( C_G(Q) \), so that this time we get the \( OQ \)-bimodule direct sum decomposition

\[
S^\circ \otimes O G = \bigoplus_{x \in X} (1_\delta \otimes x)(S^\circ \otimes O H) \tag{3.13.31}
\]

Since any \( z \in C_G(Q) \) stabilizes \( \delta \), choosing again \( \ell \in \chi \), \( j \in \delta \) and \( j' \in \chi \times \delta \) such that \([9, \text{Proposition 5.6}]

\[
j'(\ell \otimes j) = j' = (\ell \otimes j)j' \tag{3.13.32}
\]

there is an invertible element \( a_z \in (O H)^Q \) fulfilling \( j^z = j^a_z \); consequently, with the notation above, from these choices and equality 3.13.31 we have

\[
A' = \bigoplus_{x \in X} (1_\delta \otimes x(a_z)^{-1}) B' \tag{3.13.33}
\]

As in Proposition 3.8, denote by \( \hat{K}^\delta \) the converse image in \( \hat{C}_G(Q) \) of the fixed points of \( F \) in \( \hat{Z} \) and by \( K^\delta \) the \( k^* \)-quotient \( \hat{K}^\delta/k^* \) of \( \hat{K}^\delta \); since \( \hat{K}^\delta \) is a normal \( k^* \)-subgroup of \( \hat{C}_G(Q) \), \( H \cdot K^\delta \) is a normal subgroup of \( G \) and therefore, arguing by induction, we may assume that it coincides with \( H \) or with \( G \).

Firstly assume that \( H \cdot K^\delta = G \); in this case, since we have \( K^\delta = C_G(Q) \), \( F \) acts trivially on \( \hat{Z} \) and we have \( \hat{F} = \hat{E}_H(Q_\delta) \) for any local point \( \delta^G \) of \( Q \) on \( O G b \) such that \( m^G_\delta \neq 0 \), so that \( S^\chi_\delta \) is \( F \)-stable (cf. 3.12.2); consequently, according to Proposition 3.11, once again it suffices to prove that

\[
\text{rank}_O (S^\circ \otimes O H)_{\chi \times \delta} = |Q||F| \tag{3.13.34}
\]

For any \( z \in C_G(Q) \), the element \( z(a_z)^{-1} \) stabilizes \( j(O H)j = (O H)_\delta \), and actually it induces a \( Q \)-interior algebra automorphism \( q_z \) of the source algebra \( (O H)_\delta \); but, symmetrically, \( C_G(Q) \) acts trivially on \([8, \text{Proposition 2.14 and Theorem 3.1}]

\[
\hat{F} = \hat{E}_H(Q_\delta)^* \cong \hat{F}_{(O H)_\delta}(Q_\delta) \tag{3.13.35}
\]
hence, it follows from \[10, \text{Proposition 14.9}\] that \(g_x\) is an inner automorphism and therefore, up to modifying our choice of \(a_x\), we may assume that \(z(a_x)^{-1}\) centralizes \((\mathcal{O}H)_\delta\); then, for any \(x \in X\) the element \(1_x \otimes x(a_x)^{-1}\) centralizes 
\[B' = j'(S^0 \otimes_{\mathcal{O}} \mathcal{O}H)j'\]
and therefore, denoting by \(C\) the centralizer of \(B'\) in \(A'\), it follows from equality \(3.13.33\) that we have 
\[A' = C \otimes_{Z(B')} B'\]
in particular, we get \(A'^Q = C \otimes_{Z(B')} B'^Q\) which induces a \(k\)-algebra isomorphism \([10, 14.5.1]\)
\[A'(Q) \cong C \otimes_{Z(B')} kZ(Q)\]
and then it follows from isomorphism \(3.6.4\) that 
\[k \otimes_{Z(B')} C \cong (k, \hat{Z})^\alpha\]

At this point, for any local point \(\delta^\alpha\) of \(Q\) on \(\mathcal{O}Gb\) such that \(m_\delta^\alpha \neq 0\), it follows from Proposition \(3.7\) that \(Q_{\delta^\alpha} \subset P_{\gamma^\alpha}\), so that \(Q_{\chi \times \delta^\alpha} \subset P_{\pi \times \gamma^\alpha}\) \([9, \text{Proposition 5.6}]\) and therefore \(\chi \times \delta^\alpha\) is also a local point of \(Q\) on the \(P\)-interior algebra (cf. embedding \(3.9.4\))
\[(S^0 \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi \times \gamma^\alpha} \cong \mathcal{O}_s \hat{L}\]
actually, since \(N_G(P)\) normalizes \(Q = H \cap P\), \(Q\) is normal in \(L\) and therefore all the points of \(Q\) on \(\mathcal{O}_s \hat{L}\) are local (cf. \(2.10\)). In conclusion, since \(\{1_x\}\) is the unique point of \(P\) on \(\mathcal{O}_s \hat{L}\), isomorphism \(3.13.40\) induces a bijective correspondence between the sets of local points of \(Q\) on 
\[j'(S^0 \otimes_{\mathcal{O}} \mathcal{O}Gb)j' = A'(1 \otimes b)\]
and on \(\mathcal{O}_s \hat{L}\); moreover, note that if two local points \(\chi \times \delta^\alpha\) and \(\chi \times \varepsilon^\alpha\) of \(Q\) on the left-hand member of \(3.13.40\) correspond to two local points \(\delta^\alpha\) and \(\varepsilon^\alpha\) of \(Q\) on \(\mathcal{O}_s \hat{L}\), choosing suitable \(j^\alpha \in \delta^\alpha\), \(k^\alpha \in \varepsilon^\alpha\), \(j^\alpha \in \delta^\alpha\) and \(k^\alpha \in \varepsilon^\alpha\), from isomorphism \(3.13.40\) we still get an \(OQ\)-bimodule isomorphism
\[j^\alpha A'k^\alpha \cong j^\alpha (\mathcal{O}_s \hat{L})k^\alpha\]
Consequently, since we have \(A'^Q = C \otimes_{Z(B')} B'^Q\) and \(C\) is a free \(\mathcal{Z}(B')\)-module, for suitable primitive idempotents \(j^\alpha\) and \(k^\alpha\) of \(C\) we have (cf. \(3.13.37\) and \(3.13.38\))
\[
\dim (k \otimes_{Z(B')} (j^\alpha Ck^\alpha)) \text{rank}_\mathcal{O}(B') = \text{rank}_\mathcal{O}(j^\alpha (\mathcal{O}_s \hat{L})k^\alpha)
\]
\[
\dim (k \otimes_{Z(B')} (j^\alpha Ck^\alpha)) = \text{rank}_{kZ(Q)}(j^\alpha (\mathcal{O}_s \hat{L})k^\alpha)(Q)
\]
3.13.43;
thus, since the respective multiplicities (cf. 2.2) of points $\hat{\delta}^G$ and $\text{Br}_Q^G(\hat{\delta}^G)$ of $Q$ on $O_\ast \hat{L}$ and on $(O_\ast \hat{L})(Q) \cong k_\ast C_L(Q)$ coincide each other, we finally get

$$|L| = \text{rank}_O(O_\ast \hat{L}) = |\hat{C}_L(Q)| \text{rank}_O(B')$$ 3.13.44.

But, according to 3.5.4, $N_G(P_{\gamma^G})$ normalizes $\gamma$ which determines $f^H$ (cf. 3.3.1) and therefore $\gamma$ determines the unique local point $\delta$ of $Q$ on $O_H$ associated with $f^H$; thus, $N_G(P_{\gamma^G})$ is contained in $N_G(Q_{\delta})$ which acts trivially on $\hat{Z}$, and therefore $N_G(P_{\gamma^G})$ stabilizes all the local points $\delta^G$ of $Q$ on $O_Gb$ fulfilling $m_{\delta^G} \neq 0$ (cf. 3.6); hence, it follows from isomorphism 3.13.40 above that, denoting by $\delta^G$ the point of $Q$ on $O_\ast \hat{L}$ determined by $\delta^G$, $L$ normalizes $Q_{\delta^G}$; in particular, we have

$$F = E_G(Q_{\delta}) = E_G(Q_{\delta^G}) = F(OG)_{\gamma^G}(Q_{\delta^G})$$

$$= E_L(Q_{\delta^G}) = L/Q \cdot C_L(Q)$$ 3.13.45

and therefore from equality 3.13.44 we get

$$|F||Q| = |L||\hat{C}_L(Q)| = \text{rank}_O(B')$$ 3.13.46,

which proves that $c$ is inertial.

Finally, assume that $K^\delta = C_H(Q)$; in this case, since the commutator in $\hat{N}_G(Q_{\delta})/Q \cdot C_H(Q)$ induces a group isomorphism

$$\hat{C}_G(Q_{\delta})/\hat{K}^\delta \cong \text{Hom}(F/\text{Ker}(\varpi), k^*)$$ 3.13.47,

the quotient $G/H$ is an Abelian $p'$-group and, in particular, we have $P = Q$. But, since with our choices above we still have (cf. 3.13.33)

$$(OG)_{\delta} = j(OG)j = \bigoplus_{x \in X} x(a_x)^{-1}(OH)_{\delta}$$ 3.13.48

where the element $x(a_x)^{-1}$ determines a $Q$-interior algebra automorphism of $(OH)_{\delta}$, it suffices to consider the $k^*$-group

$$\hat{U} = \bigcup_{x \in X} x(a_x)^{-1}((OH)_{\delta}^Q)^*$$ 3.13.49

to get the $Q$-interior algebra $(OG)_{\delta}$ as the crossed product [3, 1.6]

$$(OG)_{\delta} \cong (OH)_{\delta} \otimes ((OH)_{\delta}^Q)^*, \hat{U}$$ 3.13.50.
Then, since $G/H$ is a $p'$-group, denoting by $U$ the $k^*$-quotient of $\hat{U}$ it follows from [10, Proposition 4.6] that the exact sequence
\[
1 \longrightarrow j + J((OH)_\delta^Q) \longrightarrow U \longrightarrow G/H \longrightarrow 1
\]
is split and therefore, for a suitable central $k^*$-extension $\widehat{G/H}$ of $G/H$, we still get an evident $Q$-interior algebra isomorphism
\[
(OG)_\delta \cong (OH)_\delta \otimes_{k^*} \widehat{G/H}
\]
at this point, it suffices to compute the Brauer quotients at $Q$ of both members to get
\[
k \otimes_{kZ(Q)} (OG)_\delta(Q) \cong k\hat{G/H}
\]
and therefore, comparing this $k$-algebra isomorphism with isomorphism 3.6.4, we obtain a $Q$-interior algebra isomorphism
\[
(OG)_\delta \cong (OH)_\delta \otimes_{k^*} \hat{Z}^o
\]
for a suitable action of $Z$ over $(OH)_\delta$ defined, up to inner automorphisms of the $Q$-interior algebra $(OH)_\delta$, by the group homomorphism
\[
Z \longrightarrow \operatorname{Aut}_{k^*}(\hat{E}_H(Q_\delta))
\]
induced by the commutator in $\hat{N}_G(Q_\delta)\big/\hat{Q}C_H(Q)$ [10, Proposition 14.9].

Similarly, considering the trivial action of $Z$ over $S$, we also obtain the $Q$-interior algebra isomorphism
\[
S^o \otimes_O (OG)_\delta \cong (S^o \otimes_O (OH)_\delta) \otimes_{k^*} \hat{Z}^o
\]
since $\chi \times \delta$ is the unique local point of $Q$ on $S^o \otimes_O (OH)_\delta$, we have $j^{\chi} = j^{\delta z}$ for a suitable inverse element $b_z$ in $(S^o \otimes_O (OH)_\delta)^Q$; hence, arguing as above, we finally obtain a $Q$-interior algebra isomorphism
\[
(S^o \otimes_O OG)_{\chi \times \delta} \cong (S^o \otimes_O OH)_{\chi \times \delta} \otimes_{k^*} \hat{Z}^o
\]
Moreover, since the $k$-algebra $k\hat{Z}$ is now semisimple, for any pair of primitive idempotents $i$ and $i'$ of $O, \hat{Z}$ we have $ii'O, \hat{Z})i' = O$ or $\{0\}$; now, since $O, \hat{Z}$ is a unitary $O$-subalgebra of $(S^o \otimes_O OG)_{\chi \times \delta} \subset S^o \otimes_O OG$, in the first case from isomorphism 3.13.56 we get
\[
\operatorname{rank}_O(ii'(S^o \otimes_O OG)i') \leq \operatorname{rank}_O(S^o \otimes_O OH)_{\chi \times \delta}
\]
whereas in the second case we simply get $i(S^o \otimes_O OG)i' = \{0\}$; hence, since isomorphism 3.13.57 implies that
\[
\operatorname{rank}_O(S^o \otimes_O OG)_{\chi \times \delta} = \operatorname{rank}_O(S^o \otimes_O OH)_{\chi \times \delta} |Z|
\]
all the inequalities 3.13.58 are actually equalities and, in particular, we get (cf. embedding 3.9.4)

$$|L| = \text{rank}_O(S^\circ \otimes O G)_{\pi \times \gamma^G} = \text{rank}_O(S^\circ \otimes O H)_{\chi \times \delta} \quad 3.13.60$$

since $P = Q$ and $\pi \times \gamma^G = \chi \times \delta^G$ (cf. 3.4). Consequently, according to Proposition 3.11, it suffices to prove that $S$ is $F$-stable.

On the other hand, it follows from Proposition 3.7 that $F$ acts transitively over the set of primitive idempotents of $Z(k, \hat{Z}) b_3$; but, since $k, \hat{Z}$ is semisimple, this set is canonically isomorphic to the set of points of this $k$-algebra (cf. 2.2), so that $F$ acts transitively over the set of local points $\delta^G$ of $Q$ on $\mathcal{O}Gb$ fulfilling $m_\delta^G \neq 0$ (cf. 3.6). More precisely, choosing $\delta^G = \gamma^G$ and denoting by $\hat{K}^G$ the converse image in $\hat{G}(Q)$ of the fixed points of $E_H(Q_\delta^G)$ in $\hat{Z}$ and by $K^G$ the $k^*$-quotient of $\hat{K}^G$, as above $H \cdot K^G$ is a normal subgroup of $G$ and therefore, arguing by induction, we may assume that either $C_H(Q) = K^G$ or $G = H \cdot K^G$.

In the first case, it follows from Proposition 3.8 that

$$F = E_H(Q_\delta^G) \subset E_G(Q_\delta^G) = E \quad 3.13.61$$

so that $S$ is indeed $F$-stable (cf. 3.9). In the second case, since we have (cf. Proposition 3.8)

$$F/E_H(Q_\delta^G) \cong K^G/K^G \cong G/H \cong Z \quad 3.13.62,$$

the number of points of $\mathcal{O}, \hat{Z}$ coincides with its $O$-rank which forces the $k^*$-group isomorphism $\hat{Z} \cong k^* \times Z$; in particular, isomorphism 3.13.54 becomes the $Q$-interior algebra isomorphism

$$(\mathcal{O}G)_\delta \cong (\mathcal{O}H)_\delta Z = \bigoplus_{z \in Z} (\mathcal{O}H)_\delta^z \quad 3.13.63$$

and therefore we have $(\mathcal{O}G)_\delta^Z \cong (\mathcal{O}H)_\delta^Z Z$.

Thus, since $Q = P$, we may assume that the image $i$ of $\frac{1}{|Z|} \sum_{z \in Z} z$ in $(\mathcal{O}G)_\delta \subset \mathcal{O}G$ belongs to $\delta^G = \gamma^G$ and then we get (cf. 3.9.1)

$$S \otimes \mathcal{O} L \cong i(\mathcal{O}G)i \cong (\mathcal{O}H)_\delta^Z \quad 3.13.64.$$ 

But, it follows from [10, Proposition 7.4] that there is a unique $j + J((\mathcal{O}H)_\delta^Z)$-conjugacy class of $k^*$-group homomorphisms

$$\hat{\alpha} : Q \times \hat{F} \rightarrow ((\mathcal{O}H)_\delta)^* \quad 3.13.65$$

mapping $u \in Q$ on $u \cdot j$; then, since $Z$ is a $p^j$-group, it follows from [3, Lemma 3.3 and Proposition 3.5] that we can choose $\alpha$ in such a way that $Z$ normalizes $\alpha(\hat{F})$ and then we have $[Z, \alpha(\hat{F})] \subset k^*$. In this case, $\alpha(\hat{F})$ stabilizes $(\mathcal{O}H)_Z^Z$ and, as a matter of fact, this $O$-algebra contains $\alpha(\hat{F})^Z = \hat{E}$. 

Consequently, throughout isomorphisms 3.13.64, $F$ acts on the $Q$-interior algebra $S \otimes O, \hat{L}$ and therefore it acts on the quotient

$$S \otimes O, \hat{L} / J(S \otimes O, \hat{L}) \cong S \otimes O, k, \hat{E}$$ 3.13.66;

but, $k, \hat{E}$ has a trivial $Q$-interior algebra structure and it is semisimple, so that we have

$$k, \hat{E} \cong \prod_\theta (k, \hat{E})(\theta)$$ 3.13.67,

where $\theta$ runs over the set of points of $k, \hat{E}$, and then any tensor product $T_\theta = S \otimes_O (k, \hat{E})(\theta)$ is a Dade $Q$-algebra over $k$ similar to $S \otimes_O k$ [11, 1.5]; hence, $F$ permutes these Dade $Q$-algebras which amounts to saying that it fixes the similarity class of $S \otimes_O k$; finally, it follows from [11, 1.5.2] that $S$ is also $F$-stable. We are done.

4 Normal sub-blocks of nilpotent blocks

4.1 With the notation of section 3, assume now that the block $b$ of $G$ is nilpotent; since we already know that $(OG)_{\gamma} \cong S \otimes O OP$ for a suitable Dade $P$-algebra $S$ [9, Main Theorem], the block $b$ is also inertial and therefore we already have proved that the normal sub-block $c$ of $H$ is inertial too; let us show with the following example — as a matter of fact, the example which has motivated this note — that the block $c$ need not be nilpotent.

Example 4.2 Let $\mathfrak{F}$ be a finite field of characteristic different from $p, q$ the cardinal of $\mathfrak{F}$ and $\mathfrak{E}$ a field extension of $\mathfrak{F}$ of degree $n \neq 1$; denoting by $\Phi_n$ the $n$-th cyclotomic polynomial, assume that $p$ divides $\Phi_n(q)$ but not $q - 1$, that $\Phi_n(q)$ and $q - 1$ have a nontrivial common divisor $r$ — which has to be a prime number† — and that $n$ is a power of $r$. For instance, the triple $(p, q, n)$ could be $(3, 5, 2)$, $(5, 3, 4)$, $(7, 4, 3)$ ...

Set $G = GL_{\mathfrak{F}}(\mathfrak{E})$ and $H = SL_{\mathfrak{E}}(\mathfrak{E})$, and respectively denote by $T$ and by $W$ the images in $G$ of the multiplicative group of $\mathfrak{E}$ and of the Galois group of the extension $\mathfrak{E}/\mathfrak{F}$; since $p$ does not divide $q - 1$, $T \cap H$ contains the Sylow $p$-subgroup $P$ of $T$ and, since $p$ divides $\Phi_n(q)$, we have

$$C_G(P) = T \quad \text{and} \quad N_G(P) = T \rtimes W$$ 4.2.1;

consequently, since $W$ acts regularly on the set of generators of a Sylow $r$-subgroup of $T$, a generator $\varphi$ of the Sylow $r$-subgroup of $\text{Hom}(T, \mathbb{C}^*)$ determines a local point $\gamma$ of $P$ on $OG$ such that

$$N_G(P_\gamma) = T = C_G(P)$$ 4.2.2

† We thank Marc Cabanes for this remark.
and, by the Brauer First Main Theorem, \( P_\gamma \) is a defect pointed group of a block \( b \) of \( G \) which, according to [13, Proposition 5.2], is nilpotent by equality 4.2.2.

On the other hand, since \( r \) divides \( q - 1 \), the restriction \( \psi \) of \( \varphi \) to the intersection \( T \cap H = C_H(P) \) has an order strictly smaller than \( \varphi \) and therefore, since we clearly have

\[
N_H(P)/C_H(P) \cong W
\]  

4.2.3,
\( r \) divides \( |N_H(P_\delta)/C_H(P)| \) where \( \delta \) denotes the local point of \( P \) on \( OH \) determined by \( \psi \); once again by the Brauer First Main Theorem, \( P_\delta \) is a defect pointed group of a block \( c \) of \( H \), which is clearly a normal sub-block of the block \( b \) of \( G \) and it is not nilpotent since \( r \) divides \( |N_H(P_\delta)/C_H(P)| \).

**Corollary 4.3** A block \( c \) of a finite group \( H \) is a normal sub-block of a nilpotent block \( b \) of a finite group \( G \) only if it is inertial and has an Abelian inertial quotient.

**Proof:** We already have proved that \( c \) has to be inertial. For the second statement, we borrow the notation of Proposition 3.8; on the one hand, since the block \( b \) is nilpotent, we know that \( E_G(Q_{\delta G}) \) is a \( p \)-group; on the other hand, it follows from this proposition that \( E_H(Q_{\delta G}) \) is a normal subgroup of \( F \) and that \( F/E_H(Q_{\delta G}) \) is Abelian; since the inertial quotient \( F \) is a \( p' \)-group, we have \( E_H(Q_{\delta G}) = \{1\} \) and \( F \) is Abelian. We are done.

**Remark 4.4** Conversely, if \( P \) is a finite \( p \)-group and \( E \) a finite Abelian \( p' \)-group acting faithfully on \( P \), the unique block of \( \hat{L} = P \rtimes \hat{E} \) for any central \( k^* \)-extension of \( E \) is a normal sub-block of a nilpotent block of a finite group obtained as follows. Setting

\[
Z = \text{Hom}(E, k^*)
\]

4.4.1,

it is clear that \( Z \) acts faithfully on \( \hat{E} \) fixing the \( k^* \)-quotient \( E \); thus, the semidirect product \( \hat{E} \rtimes Z \) still acts on \( P \) and we finally consider the semidirect product

\[
\hat{M} = P \rtimes (\hat{E} \rtimes Z) = \hat{L} \rtimes Z
\]  

4.4.2.

Then, we clearly have

\[
(\mathcal{O}_* M)(P) \cong k((Z(P) \times Z)
\]  

4.4.3

and therefore any group homomorphism \( \varepsilon : Z \to k^* \) determines a local point of \( P \) on \( \mathcal{O}_* \hat{M} \) — still noted \( \varepsilon \); but \( E \) acts on \( kZ \), regularly permuting the set of its points; hence, we get

\[
N_M(P_\varepsilon) = k^* \times P \times Z
\]  

4.4.4

and therefore \( P_\varepsilon \) is a defect pointed group of the nilpotent block \( \{1_{\mathcal{O}_* \hat{M}}\} \) of \( \hat{M} \).
References

[1] Michel Broué and Lluis Puig, *Characters and Local Structure in G-algebras*, Journal of Algebra, 63(1980), 306-317.

[2] Michel Broué and Lluis Puig, *A Frobenius theorem for blocks*, Inventiones math., 56(1980), 117-128.

[3] Yun Fan and Lluis Puig, *On blocks with nilpotent coefficient extensions*, Algebras and Representation Theory, 1(1998), 27-73 and Publisher revised form, 2(1999), 209.

[4] Daniel Gorenstein, *“Finite groups”* Harper’s Series, 1968, Harper and Row.

[5] James Green, *Some remarks on defect groups*, Math. Zeit., 107(1968), 133-150.

[6] Burkhard Külshammer and Lluis Puig, *Extensions of nilpotent blocks*, Inventiones math., 102(1990), 17-71.

[7] Lluis Puig, *Pointed groups and construction of characters*, Math. Zeit. 176(1981), 265-292.

[8] Lluis Puig, *Local fusions in block source algebras*, Journal of Algebra, 104(1986), 358-369.

[9] Lluis Puig, *Nilpotent blocks and their source algebras*, Inventiones math., 93(1988), 77-116.

[10] Lluis Puig, *Pointed groups and construction of modules*, Journal of Algebra, 116(1988), 7-129.

[11] Lluis Puig, *Affirmative answer to a question of Feit*, Journal of Algebra, 131(1990), 513-526.

[12] Lluis Puig, *Algèbres de source de certains blocks des groupes de Chevalley*, in *“Représentations linéaires des groupes finis”*, Astérisque, 181-182 (1990), Soc. Math. de France

[13] Lluis Puig, *Une correspondance de modules pour les blocks à groupes de défaut abéliens*, Geometriae Dedicata, 37(1991), 9-43.

[14] Lluis Puig, *On Joanna Scopes’ Criterion of equivalence for blocks of symmetric groups*, Algebra Colloq., 1(1994), 25-55.

[15] Lluis Puig, *“On the Morita and Rickard equivalences between Brauer blocks”*, Progress in Math., 178(1999), Birkhäuser, Basel.

[16] Lluis Puig, *Source algebras of p-central group extensions*, Journal of Algebra, 235(2001), 359-398.

[17] Lluis Puig, *“Blocks of Finite Groups”*, Springer Monographs in Mathematics, 2002, Springer-Verlag, Berlin, Barcelona.

[18] Lluis Puig, *“Frobenius categories versus Brauer blocks”*, Progress in Math., 274(2009), Birkhäuser, Basel.

[19] Lluis Puig, *Block Source Algebras in p-Solvable Groups*, Michigan Math. J. 58(2009), 323-328