$L^p$ Boundedness of Commutators of Riesz Transforms Associated to Schrödinger Operator

Zihua Guo, Pengtao Li, and Lizhong Peng
LMAM, School of Mathematical Sciences, Peking University
Beijing 100871, China
E-mail: zihuaguo@math.pku.edu.cn, liptao@163.com, lzpeng@pku.edu.cn

Abstract: In this paper we consider $L^p$ boundedness of some commutators of Riesz transforms associated to Schrödinger operator $P = -\Delta + V(x)$ on $\mathbb{R}^n$, $n \geq 3$. We assume that $V(x)$ is non-zero, nonnegative, and belongs to $B_q$ for some $q \geq n/2$. Let $T_1 = (-\Delta + V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta + V)^{-1/2}\nabla$. We obtain that $[b, T_j]$ ($j = 1, 2, 3$) are bounded operators on $L^p(\mathbb{R}^n)$ when $p$ ranges in a interval, where $b \in \text{BMO}(\mathbb{R}^n)$. Note that the kernel of $T_j$ ($j = 1, 2, 3$) has no smoothness.

Keywords: Commutator, BMO, Smoothness, Boundedness, Riesz transforms associated to Schrödinger operators

2000 MS Classification: 47B38, 42B25, 35Q40

1 Introduction

Let $P = -\Delta + V(x)$ be the Schrödinger differential operator on $\mathbb{R}^n$, $n \geq 3$. Throughout the paper we will assume that $V(x)$ is a non-zero, nonnegative potential, and belongs to $B_q$ for some $q > n/2$. Let $T_j$ ($j = 1, 2, 3$) be the Riesz transforms associated to Schrödinger operators, namely, $T_1 = (-\Delta + V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta + V)^{-1/2}\nabla$. $L^p$ boundedness of $T_j$ ($j = 1, 2, 3$) was widely studied([7], [8]). In this paper, we will discuss the $L^p$ boundedness of the commutator operators $[b, T_j] = bT_j - T_j b$ ($j = 1, 2, 3$), where $b \in \text{BMO}(\mathbb{R}^n)$.

A nonnegative locally $L^q$ integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q$ ($1 < q < \infty$), if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^q dx\right)^{\frac{1}{q}} \leq C \left(\frac{1}{|B|} \int_B V dx\right)$$

holds for every ball $B$ in $\mathbb{R}^n$.

Remark 1.1. By Hölder inequality we can get that $B_{q_1} \subset B_{q_2}$, for $q_1 \geq q_2 > 1$. One remarkable feature about the $B_q$ class is that, if $V \in B_q$ for some $q > 1$, then there exists $\epsilon > 0$, which depends only on $n$ and the constant $C$ in (1), such that $V \in B_{q_1+\epsilon}$ (2). It’s also well known that, if $V \in B_q$, $q > 1$, then $V(x)dx$ is a doubling measure, namely for any $r > 0$, $x \in \mathbb{R}^n$,

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy.$$  

*Research supported by NNSF of China No.10471002, RFDP of China No: 20060001010.
It was proved that if $V \in B_n$, then $T_3$ is a Calderón-Zygmund operator ([7]). According to the classical result of R. Coifman, R. Rochberg, and G. Weiss ([1]), $[b, T_3]$ is bounded on $L^p (1 < p < \infty)$ in this case. So we restrict ourselves to the case that $V \in B_q (n/2 < q < n)$, when considering $[b, T_3]$.

We recall that an operator $T$ taking $C_c^\infty (\mathbb{R}^n)$ into $L^1_{loc} (\mathbb{R}^n)$ is called a Calderón-Zygmund operator if

(a) $T$ extends to a bounded linear operator on $L^2 (\mathbb{R}^n)$,

(b) there exists a kernel $K$ such that for every $f \in L^\infty (\mathbb{R}^n)$,

$$
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \quad a.e. \text{ on } \{\text{supp } f\}^c,
$$

(c) the kernel $K(x, y)$ satisfies the Calderón-Zygmund estimate

$$
|K(x, y)| \leq \frac{C}{|x-y|^n};
$$

$$
|K(x+h, y) - K(x, y)| \leq \frac{C|h|^\delta}{|x-y|^{n+\delta}}; \quad (4)
$$

$$
|K(x, y+h) - K(x, y)| \leq \frac{C|h|^\delta}{|x-y|^{n+\delta}}; \quad (5)
$$

for $x, y \in \mathbb{R}^n, |h| < \frac{|x-y|}{2}$ and for some $\delta > 0$.

If $T$ is a Calderón-Zygmund operator, $b \in \text{BMO}$, the boundedness on every $L^p (1 < p < \infty)$ of $[b, T]$ was first discovered by Coifman, Rochberg and Weiss ([1]). Later, Strömberg [4] gave a simple proof, adopting the idea of relating commutators with the sharp maximal operator of Fefferman and Stein. In both proof, the smoothness of the kernel ([1]) plays a key role. However, in our problem the kernel has no smoothness of this kind due to $V$. This difficulty can be overcome by our basic idea. We discover that the kernels have some other kind of smoothness.

**Definition 1.2.** $K(x, y)$ is said to satisfy $H(m)$ for some $m \geq 1$, if there exists a constant $C > 0$, such that, $\forall \ l > 0, \ x, x_0 \in \mathbb{R}^n$ with $|x - x_0| \leq l$, then

$$
\sum_{k=5}^{\infty} k(2^k l)^{n/m} \left( \int_{2^k t \leq |y-x_0| < 2^{k+1} l} |K(x, y) - K(x_0, y)|^m dy \right)^{1/m} < C,
$$

(6)

where $1/m' = 1 - 1/m$.

This kind of smoothness was not new. We find that the case $m = 1$ was given by Meyer([5]). It’s easily seen that if $K(x, y)$ satisfies ([4]), then $K(x, y)$ satisfies $H(m)$ for every $m \geq 1$. By Hölder inequality we can get that if $K(x, y)$ satisfies $H(m)$ for some $m \geq 1$, then $K(x, y)$ satisfies $H(t)$ for $1 \leq t \leq m$. We now list some results concerning $L^p$ boundedness of $T_j$ ($j = 1, 2, 3$), and refer the readers to [7] for further details. We will adopt the notation $1/p' = 1 - 1/p$ for $p \geq 1$ throughout the paper.

**Theorem 1.3** (Theorem 3.1, [7]). Suppose $V \in B_q$ and $q \geq n/2$. Then, for $q' \leq p \leq \infty$,

$$
\left\|(-\Delta + V)^{-1}Vf\right\|_p \leq C_p\|f\|_p.
$$

**Theorem 1.4** (Theorem 5.10, [7]). Suppose $V \in B_q$ and $q \geq n/2$. Then, for $(2q)' \leq p \leq \infty$,

$$
\left\|(-\Delta + V)^{-1/2}V^{1/2}f\right\|_p \leq C_p\|f\|_p.
$$
Theorem 1.5 (Theorem 0.5, [7]). Suppose $V \in B_q$ and $\frac{n}{2} \leq q < n$. Let $(1/p_0) = (1/q) - (1/n)$. Then, for $p_0' \leq p < \infty$,
\[
\left\| (-\Delta + V)^{-1/2} \nabla f \right\|_p \leq C_p \| f \|_p.
\]

The basic idea in [7] is that, to exploit a pointwise estimate of the kernel and the comparison to the kernel of classical Riesz transform. Generally, it is based on the following two basic facts.

(i) If $q < \infty$.

(ii) If $p < \infty$.

(iii) If $p_0' \leq p < \infty$.

Corollary 1.7. Suppose $V \in B_q$ and $q \geq n/2$. Then
\[
\left\| [b, T^*_1] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p,
\]

and
\[
\left\| [b, T^*_2] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p,
\]

and
\[
\left\| [b, T^*_3] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p,
\]

and
\[
\left\| [b, T^*_4] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p,
\]

We know that $T^*_1 = V(-\Delta + V)^{-1}$, $T^*_2 = V^{1/2}(-\Delta + V)^{-1/2}$, and $T^*_3 = -\nabla(-\Delta + V)^{-1/2}$. By duality we can easily get that
\[
\left\| [b, T^*_1] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p, \quad 1 < p \leq q,
\]

\[
\left\| [b, T^*_2] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p, \quad 1 < p \leq 2q,
\]

\[
\left\| [b, T^*_3] f \right\|_p \leq C_p \| b \|_{BMO} \| f \|_p, \quad 1 < p \leq p_0.
\]

From Theorem 1 (i), we can get the result concerning second order Riesz transform. Let $T_4 = (-\Delta + V)^{-1} \nabla^2$, then $T^*_4 = \nabla^2(-\Delta + V)^{-1}$. Indeed, $T_4 = (-\Delta + V)^{-1} \nabla^2 = \nabla(-\Delta + V)^{-1} \nabla = (I - (-\Delta + V) V) \frac{\nabla}{2 \nabla} = (I - T_1) \frac{\nabla}{2 \nabla}$. We have
For classical Riesz transform, the converse problem was also considered in \cite{1}. This implies a new characterization of $\textbf{BMO}$. In this paper we also discuss the converse problem. Namely, if $[b,T_3]$ is bounded on $L^2$, do we have $b \in \textbf{BMO}$? The answer is negative for general $V \in B_q$. It is due to that, for some good $V$, the kernel of $T_3$ is better than that of Riesz transform, which makes that the commutator can absorb mild singularity. We give a counterexample for $V \equiv 1$. On the other hand, if imposing some integrability condition on $V$, we can have the converse.

Throughout this paper, unless otherwise indicated, we will use $C$ and $c$ to denote constants, which are not necessarily the same at each occurrence. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$, such that $c \leq A/B \leq C$.

The paper is organized as following. In Section 2, we will give the estimates of the kernels $K_j(j = 1, 2, 3)$ of the operators $T_j$. The proof of Theorem 1 is stated in Section 3. In Section 4, we discuss the converse problem.

2 Estimate of the kernels

This section is devoted to give the estimate of the kernels associated to $T_j$ ($j = 1, 2, 3$) and denoted by $K_j(x, y)$ ($j = 1, 2, 3$) respectively. Let $\Gamma(x, y, \tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x) + i\tau)$, $\tau \in \mathbb{R}$, and $\Gamma_0(x, y, \tau)$ for the operator $-\Delta + i\tau$, $\tau \in \mathbb{R}$. Clearly, $\Gamma(x, y, \tau) = \Gamma(y, x, -\tau)$.

For $x \in \mathbb{R}^n$, the function $m(x, V)$ is defined by

$$\frac{1}{m(x, V)} = \sup \{r > 0 : \frac{1}{r^{n-2}} \int B(x, r)V(y)dy \leq 1 \}.$$ 

The function $m(x, V)$ reflects the scale of $V(x)$ essentially, but behaves better. It is deeply studied in \cite{7}, and will play a crucial role in our proof. We list some properties of $m(x, V)$ here, and their proof can be found in \cite{7}.

**Lemma 2.1** (Lemma 1.4, \cite{7}). Assume $V \in B_q$ for some $q > n/2$, then there exist $C > 0$, $c > 0$, $k_0 > 0$, such that, for any $x$, $y$ in $\mathbb{R}^n$, and $0 < r < R < \infty$,

(a) $0 < m(x, V) < \infty$,

(b) If $h = \frac{1}{m(x, V)}$, then $\frac{1}{h^{n-2}} \int B(x, h)V(y)dy = 1$,

(c) $m(x, V) \sim m(y, V)$, if $|x - y| \leq \frac{C}{m(x, V)}$,

(d) $m(y, V) \leq C \{1 + |x - y|m(x, V)\}^{k_0} m(x, V)$,

(e) $m(y, V) \geq cm(x, V) \{1 + |x - y|m(x, V)\}^{-k_0/(1+k_0)}$,

(f) $c\{1 + |x - y|m(y, V)\}^{1/(k_0+1)} \leq 1 + |x - y|m(x, V) \leq C\{1 + |x - y|m(y, V)\}^{k_0+1}$,

(g) $\frac{1}{r^{n-2}} \int B(x, r)V(y)dy \leq C(\frac{R}{r})^{n/q-2} \cdot \frac{1}{r^{n-2}} \int B(x, R)V(y)dy$.

Estimating the kernels mainly relies on functional calculus and a pointwise estimate of $\Gamma(x, y, \tau)$ that was given in \cite{7}.

**Theorem 2.2** (Theorem 2.7, \cite{7}). Suppose $V \in B_{n/2}$. Then, for any $x, y \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, and integer $k > 0$,

$$\Gamma(x, y, \tau) \leq \frac{C_k}{\{1 + |\tau|^{1/2}|x - y|\}^k \{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n-2}}.$$ 

where $C_k$ is a constant independent of $x, y, \tau$. 

4
The next lemma is used to control the integration of $V$ on a ball.

**Lemma 2.3.** Suppose $V \in B_q$ for some $q > n/2$. Let $N > \log_2 C_0 + 1$, where $C_0$ is the constant in (2). Then for any $x_0 \in \mathbb{R}^n$, $R > 0$,

$$\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0, R)} V(\xi)d\xi \leq CR^{n-2}.$$  

**Proof.** There exists an integer $j_0 \in \mathbb{Z}$ such that $2^{j_0}R \leq \frac{1}{m(x_0, V)} < 2^{j_0+1}R$. We will discuss in following two cases.

Case 1: $j_0 < 0$. By (2), Lemma 2.3, and (b) of Lemma 2.1, we can get

$$\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0, R)} V(\xi)d\xi \leq \frac{1}{(2^{j_0})^N} \int_{B(x_0, R)} V(\xi)d\xi \leq \frac{1}{(2^{j_0})^N} C_0^{j_0} (2^{j_0}R)^{n-2} \leq R^{n-2} \quad (since \ N > \log_2 C_0).$$

Case 2: $j_0 \geq 0$. By (b) and (g) of Lemma 2.1 we can get

$$\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0, R)} V(\xi)d\xi \leq \int_{B(x_0, R)} V(\xi)d\xi \leq \frac{C}{R^{n-2}} \int_{B(x_0, R)} V(\xi)d\xi \leq R^{n-2}.$$  

This completes the proof of Lemma 2.3. \qed

Before giving the estimate of the kernels, we still needs one lemma, which is proved in [7].

**Lemma 2.4** (Lemma 4.6, [7]). Suppose $V \in B_{q_0}$, $q_0 > 1$. Assume that $-\Delta u + (V(x) + i\tau)u = 0$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^n$, $R > 0$. Then

(a) for $x \in B(x_0, R)$,

$$|\nabla u(x)| \leq C \sup_{B(x_0, 2R)} |u| \cdot \int_{B(x_0, 2R)} \frac{V(y)}{|x - y|^{n-1}}dy + \frac{C}{R^{n+1}} \int_{B(x_0, 2R)} |u(y)|dy,$$

(b) if $(n/2) < q_0 < n$, let $(1/t) = (1/q_0) - (1/n)$, $k_0 > \log_2 C_0 + 1$

$$\left(\int_{B(x_0, R)} |\nabla u|^tdx\right)^{1/t} \leq CR^{(n/q_0) - 2}\{1 + Rm(x_0, V)\}^{k_0} \sup_{B(x_0, 2R)} |u|.$$  

Now we are ready to give the estimate of the kernels.
Lemma 2.5. Suppose \( V \in B_q \) for some \( q > n/2 \). Then, there exists \( \delta > 0 \) and for any integer \( k > 0, 0 < h < |x - y|/16 \),

\[
|K_1(x, y)| \leq \frac{C_k}{\{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n - 2}} V(y),
\]

\[
|K_1(x + h, y) - K_1(x, y)| \leq \frac{C_k}{\{1 + m(x, V)|x - y|\}^k} \cdot \frac{|h|^\delta}{|x - y|^{n - 2 + \delta}} V(y).
\]

Lemma 2.6. Suppose \( V \in B_q \) for some \( q > n/2 \). Then, there exists \( \delta > 0 \) and for any integer \( k > 0, 0 < h < |x - y|/16 \),

\[
|K_2(x, y)| \leq \frac{C_k}{\{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n - 1}} V(y)^{1/2},
\]

\[
|K_2(x + h, y) - K_2(x, y)| \leq \frac{C_k}{\{1 + m(y, V)|x - y|\}^k} \cdot \frac{|h|^\delta}{|x - y|^{n - 1 + \delta}} V(y)^{1/2}.
\]

Lemma 2.7. Suppose \( V \in B_q \) for some \( n/2 < q < n \). Then, there exists \( \delta > 0 \) and for any integer \( k > 0, 0 < h < |x - y|/16 \),

\[
|K_3(x, y)| \leq \frac{C_k}{\{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n - 1}} \left( \int_{B(y, |x - y|)} \frac{V(\xi)}{|y - \xi|^{n - 1}} d\xi + \frac{1}{|x - y|} \right),
\]

\[
|K_3(x + h, y) - K_3(x, y)| \leq \frac{C_k}{\{1 + m(y, V)|x - y|\}^k} \cdot \frac{|h|^\delta}{|x - y|^{n - 1 + \delta}} \left( \int_{B(y, |x - y|)} \frac{V(\xi)}{|y - \xi|^{n - 1}} d\xi + \frac{1}{|x - y|} \right) .
\]

Remark 2.8. If \( V \in B_q \), then \( 1 \) follows immediately from \( 13 \). This can tell us how the kernel behave when \( V \) changes. However, we don't have similar result about the smoothness with respect to the second variable.

Proof of Lemma 2.7. We easily know that \( K_1(x, y) = \Gamma(x, y, 0)V(y) \). It immediately follows from Theorem 2.2 that, for any \( x, y \in \mathbb{R}^n \),

\[
|K_1(x, y)| \leq \frac{C_k}{\{1 + m(x, V)|x - y|\}^k} \cdot \frac{1}{|x - y|^{n - 2}} V(y).
\]

For \( 9 \), fix \( x, y \in \mathbb{R}^n \), and fix \( n/2 < q_0 < \min(n, q) \), then we know \( V \in B_{q_0} \). Let \( R = \frac{|x - y|}{8} \), \( 1/t = 1/q_0 - 1/n \), then \( \delta = 1 - n/t > 0 \) and for any \( 0 < h < R^2 \), it follows from the embedding theorem of Morrey (see \( 3 \)) that

\[
|K_1(x + h, y) - K_1(x, y)|
\]
\[
\leq |\Gamma(x + h, y, 0) - \Gamma(x, y, 0)|V(y)
\]
\[
\leq C|h|^{1-(n/t)} \left( \int_{B(x, R)} |\nabla_2 \Gamma(z, y, 0)|^t dz \right)^{1/t} V(y).
\]
and then using Lemma 2.4 we have

\[
|K_1(x + h, y) - K_1(x, y)| \\
\leq C|h|^{1 - (n/t)} R^{(n/q_0) - 2} \left\{ 1 + Rm(x, V) \right\}^{k_0} \sup_{z \in B(x, 2R)} |\Gamma(z, y, 0)| V(y) \\
\leq C\frac{|h|^\delta}{R^\delta} \left\{ 1 + Rm(x, V) \right\}^{k_0} \sup_{z \in B(x, 2R)} |\Gamma(z, y, 0)| V(y) \\
\leq C\frac{|h|^\delta}{R^\delta} \left\{ 1 + Rm(x, V) \right\}^{k_0} \sup_{z \in B(x, 2R)} \frac{C_1}{1 + m(y, V)|z - y|} \frac{C_1}{|z - y|^{n-2}} V(y) \\
\leq C_k \frac{|h|^\delta}{|x - y|^\delta} \frac{1}{\left\{ 1 + m(x, V)|x - y| \right\}^k} \frac{1}{|x - y|^{n-2}} V(y) \quad (k_1 \text{ large}).
\]

where we used (f) of Lemma 2.1 in the last inequality.

\[\square\]

**Proof of Lemma 2.6.** By functional calculus, we may write

\[(-\Delta + V)^{-1/2} = -\frac{1}{2\pi} \int_\mathbb{R} (-i\tau)^{-1/2} (-\Delta + V + i\tau)^{-1} d\tau,
\]

then we know that

\[K_2(x, y) = -\frac{1}{2\pi} \int_\mathbb{R} (-i\tau)^{-1/2} \Gamma(x, y, \tau) d\tau V(y)^{1/2}.
\]

In order to estimate the integration, we claim that: For \(k > 2\), then

\[\int_\mathbb{R} |\tau|^{-1/2} \left\{ 1 + |\tau|^{1/2} |x - y| \right\}^{-k} d\tau \leq \frac{C_k}{|x - y|}.
\]

(15)

In fact, we have

\[
\int_\mathbb{R} |\tau|^{-1/2} \left\{ 1 + |\tau|^{1/2} |x - y| \right\}^{-k} d\tau \\
= \left( \int_{|\tau| \leq |x - y|^{-2}} + \int_{|\tau| \geq |x - y|^{-2}} \right) |\tau|^{-1/2} \left\{ 1 + |\tau|^{1/2} |x - y| \right\}^{-k} d\tau \\
\leq \int_{|\tau| \leq |x - y|^{-2}} |\tau|^{-1/2} d\tau + \int_{|\tau| \geq |x - y|^{-2}} |\tau|^{(-k-1)/2} |x - y|^{-k} d\tau \\
\leq \frac{C_k}{|x - y|}.
\]

From Theorem 2.2 and the estimate (15), we immediately get (10). For (11), fix \(x, y \in \mathbb{R}^n\), and fix \(n/2 < q_0 < \min(n, q)\), then we know \(V \in B_{q_0}\). Let \(R = \frac{|x - y|}{8}\), \(1/t = 1/q_0 - 1/n\), then \(\delta = 1 - n/t > 0\) and for any \(0 < h < \frac{R}{2}\), we have

\[|K_2(x + h, y) - K_2(x, y)| \leq \frac{1}{2\pi} \int_\mathbb{R} |\tau|^{-1/2} |\Gamma(x + h, y, \tau) - \Gamma(x, y, \tau)| d\tau V(y)^{1/2}.
\]

(16)
Similarly, it follows from the embedding theorem of Morrey and Lemma 2.4 that

\[
|\Gamma(x+h, y, \tau) - \Gamma(x, y, \tau)| \leq C|h|^{1-(n/t)} \left( \int_{B(x, R)} |\nabla_x \Gamma(z, y, \tau)|^t dz \right)^{1/t} 
\]

\[
\leq C|h|^{1-(n/t)} R^{(n/q)-2} \{1 + Rm(x, V)\}^{k_0} \sup_{z \in B(x, 2R)} |\Gamma(z, y, \tau)| 
\]

\[
\leq C \frac{|h|^\frac{\delta}{R^3}}{R^3} \{1 + Rm(x, V)\}^{k_0} \sup_{z \in B(x, 2R)} |\Gamma(z, y, \tau)| 
\]

\[
\leq C \frac{|h|^\frac{\delta}{R^3}}{R^3} \{1 + Rm(x, V)\}^{k_0} \sup_{z \in B(x, 2R)} C_k \frac{1 + |\tau|^{1/2}|z - y|}{\{1 + m(y, V)|z - y|\}^k} \cdot \frac{1}{|z - y|^{n-2}}. 
\]

Hence, insert this to (16), it follows from the estimate (15) that

\[
|K_2(x + h, y) - K_2(x, y)| \leq C_k \frac{|h|^\frac{\delta}{\delta} \{1 + m(y, V)|x - y|\}^k V(y)^{1/2}. 
\]

\[\square\]

**Proof of Lemma 2.4.** By partial integral, we know that

\[
K_3(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-1/2} \nabla_y \Gamma(x, y, \tau) d\tau. \tag{17}
\]

Fix \( x, y \in \mathbb{R}^n \), Let \( R = \frac{|x-y|}{8} \), \( 1/t = 1/q - 1/n \), \( \delta = n/q - 2 > 0 \), and for any \( 0 < h < \frac{R}{2} \), we have

\[
|K_3(x + h, y) - K_3(x, y)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\tau|^{-1/2} |\nabla_y \Gamma(x + h, y, \tau) - \nabla_y \Gamma(x, y, \tau)| d\tau. \tag{18}
\]

Similarly, it follows from the imbedding theorem of Morrey and Lemma 2.4 that

\[
|\nabla_y \Gamma(x + h, y, \tau) - \nabla_y \Gamma(x, y, \tau)| \leq C|h|^{1-(n/t)} \left( \int_{B(x, R)} |\nabla_x \nabla_y \Gamma(z, y, \tau)|^t dz \right)^{1/t} 
\]

\[
\leq C|h|^{1-(n/t)} R^{(n/q)-2} \{1 + Rm(x, V)\}^{k_0} \sup_{z \in B(x, 2R)} |\nabla_y \Gamma(z, y, \tau)|. 
\]

Since \( \Gamma(z, y, \tau) = \Gamma(y, z, -\tau) \), then \( \nabla_y \Gamma(z, y, \tau) = \nabla_x \Gamma(y, z, -\tau) \). It follows from (a) of Lemma 2.4 that

\[
\sup_{z \in B(x, 2R)} |\nabla_y \Gamma(z, y, \tau)| \leq \sup_{z \in B(x, 2R)} |\nabla_x \Gamma(y, z, -\tau)| 
\]

\[
\leq \sup_{z \in B(x, 2R)} \left\{ \sup_{\eta \in B(y, |y-z|/4)} |\Gamma(\eta, z, -\tau)| \cdot \int_{B(y, |y-z|/2)} \frac{V(\xi)}{|y - \xi|^n-1} d\xi + \frac{C}{|y - z|^{n+1}} \int_{B(y, |y-z|/2)} \Gamma(\xi, z, -\tau) d\xi \right\}. 
\]

\[8\]
Using the fact that $|\eta - z| \sim |y - z|$, $|\xi - z| \sim |y - z|$ and $|x - y| \sim |y - z|$, choosing $k_1$ sufficiently large, it follows from Theorem 2.2 and (f) of Lemma 2.1 that

$$
\sup_{z \in B(x, 2R)} |\nabla y \Gamma(z, y, \tau)| \leq \sup_{z \in B(x, 2R)} \frac{C_{k_1}}{(1 + |\tau|^{1/2}|y - z|)^{k_1} \{1 + m(z, V)|y - z|\}^{k_1}} \cdot \frac{1}{|y - z|^{n-2}} \int_{B(y, |x - y|)} \frac{V(\xi)}{|y - \xi|^{n-1}} d\xi
$$

From the estimate \ref{15} and \ref{20}, we immediately get \ref{12}. Inserting \ref{20} to \ref{19}, we get that

$$
|\nabla y \Gamma(x + h, y, \tau) - \nabla y \Gamma(x, y, \tau)| \leq \frac{C_k}{|x - y|^\delta} \frac{1}{C_k} \cdot \left( \frac{1 - |x - y|^{n-2}}{|x - \xi|^{n-1}} \int_{B(y, |x - y|)} \frac{V(\xi)}{|y - \xi|^{n-1}} d\xi + \frac{1}{|x - y|^n} \right).
$$

Inserting \ref{21} to \ref{18}, we get from the estimate \ref{15} that

$$
K_3(x + h, y) - K_3(x, y) \leq \frac{C_k}{|x - y|^\delta} \frac{1}{C_k} \cdot \left( \frac{1 - |x - y|^{n-2}}{|x - \xi|^{n-1}} \int_{B(y, |x - y|)} \frac{V(\xi)}{|y - \xi|^{n-1}} d\xi + \frac{1}{|x - y|^n} \right).
$$

\[ \blacksquare \]

3 Proof of main results

We first discuss the problem for general operator $Tf(x) = \int K(x, y)f(y)dy$. Later, we will specialize to $T_j$ ($j = 1, 2, 3$).

**Proposition 3.1.** Let $m > 1$, suppose $T$ is bounded on $L^p$ for every $p \in (m', \infty)$, and $K$ satisfies $H(m)$, then $\forall \ b \in \text{BMO}$, $[b, T]$ is bounded on $L^p$ for every $p \in (m', \infty)$, and

$$
\| [b, T]f \|_p \leq C_p \| b \|_{\text{BMO}} \| f \|_p.
$$

We adopt the idea of Strömberg (see [9]). Recall that the sharp function of Fefferman-Stein is defined by

$$
M^s f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B|dy,
$$

where $f_B = \frac{1}{|B|} \int_B f(y)dy$, and the supremum is taken on all balls $B$ with $x \in B$. 
Recall that $\text{BMO}$ is defined by

$$\text{BMO}(\mathbb{R}^n) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \| f \|_{\text{BMO}} = \| M^s f \|_\infty < \infty \}.$$  \hspace{1cm} (23)

Two basic facts about $\text{BMO}$ may be in order. We use $2^k B$ to denote the ball with the same center as $B$ but with $2^k$ times radius.

$$|f_{2^k B} - f_B| \leq C(k+1) \| f \|_{\text{BMO}}, \text{ for } k > 0.$$  \hspace{1cm} (24)

The second one is due to John-Nirenberg.

$$\| f \|_{\text{BMO}} \sim \sup_B \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{1/p}, \text{ for any } p > 1.$$  \hspace{1cm} (25)

Proposition 3.1 follows immediately from the following lemma and a theorem of Fefferman-Stein on sharp function.

**Lemma 3.2.** Let $T$ satisfies the same condition in Proposition 3.1 Then $\forall s > m'$, there exists constant $C_s > 0$, such that $\forall f \in L^1_{\text{loc}}, b \in \text{BMO}$

$$M^s([b, T] f)(x) \leq C_s \| b \|_{\text{BMO}} \{ M_s(T f)(x) + M_s(f)(x) \},$$  \hspace{1cm} (26)

where $M_s(f) = M(|f|^s)^{1/s}$ and $M$ is Hardy-Littlewood maximal function.

**Proof.** Fix $s > m'$, $f \in L^1_{\text{loc}}$, $x \in \mathbb{R}^n$, and fix a ball $I = B(x_0, l)$ with $x \in I$. We only need to control $J = \frac{1}{|I|} \int_I |[b, T] f(y) - ([b, T] f)_I| dy$ by the right side of (26). Let $f = f_1 + f_2$, where $f_1 = f \chi_{32I}$, $f_2 = f - f_1$. Then $[b, T] f = [b - b_I, T] f = (b - b_I) T f - T(b - b_I) f_1 - T(b - b_I) f_2 \triangleq A_1 f + A_2 f + A_3 f$, and we get

$$J \leq \frac{1}{|I|} \int_I |A_1 f(y) - (A_1 f)_I| dy + \frac{1}{|I|} \int_I |A_2 f(y) - (A_2 f)_I| dy + \frac{1}{|I|} \int_I |A_3 f(y) - (A_3 f)_I| dy \triangleq J_1 + J_2 + J_3.$$

Step 1. First we consider $J_1$. By Hölder inequality and (25),

$$J_1 \leq \frac{2}{|I|} \int_I |A_1 f(y)| dy \leq \frac{2}{|I|} \int_I |(b - b_I) T f(y)| dy \leq 2 \left( \frac{1}{|I|} \int_I |(b - b_I)|^s dy \right)^{\frac{1}{s}} \left( \frac{1}{|I|} \int_I |T f(y)|^s dy \right)^{\frac{1}{s}} \leq 2 \| b \|_{\text{BMO}} M_s(T f)(x).$$
Step 2. Second we consider $J_2$. Fix $s_1$ such that $s > s_1 > m'$, and let $s_2 = \frac{s_1}{s_2}$, then we have

$$J_2 \leq 2 \left( \frac{1}{|I|} \int_I |A_2 f(y)|dy \right)^{\frac{1}{s_1}}$$

$$\leq 2 \left( \frac{1}{|I|} \int_I |A_2 f(y)|^{s_1} dy \right)^{\frac{1}{s_1}}$$

$$\leq 2 \left( \frac{1}{|I|} \int_{32I} |(b - b_I) f(y)|^{s_1} dy \right)^{\frac{1}{s_1}}$$

$$\leq C \left( \frac{1}{|32I|} \int_{32I} |b - b_I|^{s_2} dy \right)^{\frac{1}{s_2}} \left( \frac{1}{|32I|} \int_{32I} |f(y)|^{s_2} dy \right)^{\frac{1}{s_2}}$$

$$\leq C \|b\|_{BMO} M_s(f)(x).$$

Step 3. Last we consider $J_3$. Set $c_I = \int_{|z - x_0| > 32l} K(x_0, z)(b(z) - b_I)f(z)dz$, then we have that

$$J_3 \leq 2 \left( \frac{1}{|I|} \int_I \left| \int_{|z - x_0| \leq 32l} \{K(y, z) - K(x_0, z)\}(b(z) - b_I)f(z)dz \right| dy \right)^{\frac{1}{s_1}}$$

$$\leq 2 \left( \frac{1}{|I|} \int_I \left| \int_{|z - x_0| \leq 32l} \{K(y, z) - K(x_0, z)\}(b(z) - b_I)f(z)dz \right| dy \right)^{\frac{1}{s_1}}$$

$$\leq 2 \int_I \sum_{k=5}^{\infty} \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} \left| \{K(y, z) - K(x_0, z)\}(b(z) - b_I)f(z)dz \right| dy.$$

From Hölder’s inequality, we get

$$J_3 \leq 2 \left( \frac{1}{|I|} \int_I \sum_{k=5}^{\infty} \left( \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} |K(y, z) - K(x_0, z)|^m dz \right)^{1/m} \right. \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} \left| (b(z) - b_I)f(z) \right|^{m'} dz \right)^{1/m'}$$

$$\leq 2 \int_I \sum_{k=5}^{\infty} \left( \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} |K(y, z) - K(x_0, z)|^m dz \right)^{1/m} \left( \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} \left| (b(z) - b_I)f(z) \right|^{m'} dz \right)^{1/m'}$$

$$\leq C \sup_{k \geq 5} \frac{1}{(2k)^{n/m'}L^m_k} \left( \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} \left| (b(z) - b_I)f(z) \right|^{m'} dz \right)^{1/m'}$$

$$\leq C \sup_{k \geq 5} \frac{1}{k} \left( \frac{1}{(2k)^{n/m'}L^m_k} \int_{2^{k-1}l \leq |z - x_0| < 2^{k+1}l} \left| (b(z) - b_I)f(z) \right|^{m'} dz \right)^{1/m'}$$

$$\leq C \sup_{k \geq 5} \frac{1}{k} (k + 2) \|b\|_{BMO} M_s f(x) \quad \text{(by (24))}$$

$$\leq C \|b\|_{BMO} M_s f(x).$$

This completes the proof of lemma 3.2.
Proof of Theorem 1.6. Now we begin to prove Theorem 1.6. Considering Remark 1.1, we can assume \( q > \frac{n}{2}, q' < p \). We first prove (i). By Proposition 3.1 and Theorem 1.3, it suffices to prove that \( K_1 \) satisfies \( H(q) \) (see (6)). From (9), we have
\[
\left( \int_{2^k t \leq |y-x_0| < 2^{k+1} t} |K_1(x,y) - K_1(x_0, y)|^q dy \right)^{1/q} \leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N \int_{B(x_0,2^{k+l})} V(\xi) d\xi^{1/(2q)}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N 2^{k-2q} \int_{B(x_0,2^{k+l})} V(\xi) d\xi^{1/2}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left( 2^k l \right)^{-n/2 + (n-2)/2}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left( 2^k l \right)^{-n/2q}
\]

Thus, we can get
\[
\sum_{k=5}^{\infty} k(2^k l) \frac{1}{q} \left( \int_{2^k t \leq |y-x_0| < 2^{k+1} t} |K_1(x,y) - K_1(x_0, y)|^q dy \right)^{1/q} \leq \sum_{k=5}^{\infty} C k \frac{1}{(2^k l)^{n-1+\delta}} \leq C.
\]

For the proof of (ii).— It suffices to prove that \( K_2 \) satisfies \( H(2q) \). From (11), we have
\[
\left( \int_{2^k t \leq |y-x_0| < 2^{k+1} t} |K_2(x,y) - K_2(x_0, y)|^{2q} dy \right)^{1/(2q)} \leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N \int_{B(x_0,2^{k+l})} V(\xi) d\xi^{1/(2q)}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N 2^{k-2q} \int_{B(x_0,2^{k+l})} V(\xi) d\xi^{1/2}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left( 2^k l \right)^{-n/2 + (n-2)/2}
\]
\[
\leq C \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left( 2^k l \right)^{-n/2q}
\]

hence, we get
\[
\sum_{k=5}^{\infty} k(2^k l) \frac{1}{2^q} \left( \int_{2^k t \leq |y-x_0| < 2^{k+1} t} |K_2(x,y) - K_2(x_0, y)|^{2q} dy \right)^{1/(2q)} \leq \sum_{k=5}^{\infty} C k \frac{1}{(2^k l)^{n-1+\delta}} \leq C.
\]
Last, we prove (iii).— It suffices to prove that \( K_3 \) satisfies \( H(p_0) \). From [13], we have

\[
\left( \int_{2^k l \leq |y-x_0| < 2^{k+1} l} |K_3(x, y) - K_3(x_0, y)|^{p_0} dy \right)^{1/p_0} \\
\leq C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N \left\| \int \frac{V(\xi) \chi_{B(x_0, 2^k l)}}{|y-\xi|^{n-1}} d\xi \right\|_{L_{p_0}^\delta} + \frac{l^\delta}{(2^k l)^{(n/p_0') + \delta}} \\
\leq C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} \left\{ 1 + m(x_0, V) 2^k l \right\}^N \int_{B(x_0, 2^k l)} V(\xi) d\xi + \frac{l^\delta}{(2^k l)^{(n/p_0') + \delta}} \\
\leq C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} (2^k l)^{n/q-2} + \frac{l^\delta}{(2^k l)^{(n/p_0') + \delta}} \\
\leq C \frac{l^\delta}{(2^k l)^{(n/p_0') + \delta}},
\]

therefore, we get

\[
\sum_{k=5}^\infty k (2^k l)^{p_0} \left( \int_{2^k l \leq |y-x_0| < 2^{k+1} l} |K_3(x, y) - K_3(x_0, y)|^{p_0} dy \right)^{1/p_0} \\
\leq \sum_{k=5}^\infty C k \frac{l^\delta}{(2^k l)^{\delta}} \leq C.
\]

\[\square\]

### 4 The Converse Result

This section is devoted to the converse problem. Recall that \( T_3 = \nabla(-\Delta + V)^{-1/2} \) is the Riesz transform associated to Schrödinger operator. A natural problem is that whether the converse holds. Namely, if \([b, T_3]\) is bounded on \( L^2 \), do we have \( b \in \text{BMO} \)? This is quite subtle. If \( V \equiv 0 \), it reduces to the classical Riesz transform. However, for general \( V \in B_q \), the converse fails. Considering \( V \equiv 1 \), which is in \( B_q \) for every \( q > 1 \), we have the following,

**Theorem 4.1.** There exist a function \( b \notin \text{BMO} \), such that \([b, T_3]\) is bounded on \( L^2 \).

**Proof.** Consider \( b = x_j \), we know that \( b \notin \text{BMO} \). We have that,

\[
[b, T_3] f = x_j \nabla(-\Delta + 1)^{-1/2} f - \nabla(-\Delta + 1)^{-1/2} (x_j f).
\]

From Plancherel equality, we can get

\[
\| [b, T_3] f \|_2 = \left\| \partial_j \left( \frac{\xi}{(1 + \xi^2)^{1/2}} \hat{f} \right) - \frac{\xi}{(1 + \xi^2)^{1/2}} \partial_j \hat{f} \right\|_2 \\
= \left\| \partial_j \left( \frac{\xi}{(1 + \xi^2)^{1/2}} \right) \hat{f} \right\|_2 \\
\leq \| f \|_2.
\]

\[\square\]
The converse example in Theorem 2 implies that the assumption $V \in B_q$ is too weak, it can not guarantee the function $b \in BMO$. However if we assume $V$ satisfies some additional conditions, for example, if $V$ is $L^p$ integrable, then the converse could be true. Let $T_3' = (-\Delta)^{1/2}(-\Delta + V)^{-1/2}$, then from $T_3' = (-\Delta)^{-1/2}\nabla \cdot T_3$, we know the results above also hold with $T_3$ replaced by $T_3'$.

**Theorem 4.2.** If $[b, T_3], [b, T_3']$ and $V^{1/2}(-\Delta)^{-1/2}$ is bounded on $L^2$, then $b \in BMO$.

**Proof.** From $[b, T_3], [b, T_3']$ is bounded on $L^2$, and

$$[b, T_3] = [b, \nabla(-\Delta)^{-1/2}T_3'] = [b, \nabla(-\Delta)^{-1/2}]T_3' + \nabla(-\Delta)^{-1/2}[b, T_3],$$

we have $[b, \nabla(-\Delta)^{-1/2}]T_3'$ is bounded on $L^2$.

We claim that, $[b, \nabla(-\Delta)^{-1/2}]$ is bounded on $L^2$, which implies the theorem from the well known theorem of Coifman, Rochberg and Weiss. It suffices to prove that $T_3'$ has a converse bounded on $L^2$. Note that $T_3'^{-1} = (-\Delta + V)^{1/2}(-\Delta)^{-1/2}$, and

$$T_3'^{-1}f = (-\Delta + V)^{1/2}(-\Delta)^{-1/2}f = (-\Delta + V)^{-1/2}(-\Delta + V)(-\Delta)^{-1/2}f = (-\Delta + V)^{-1/2}(-\Delta)^{1/2}f + (-\Delta + V)^{-1/2}V^{1/2}V^{1/2}(-\Delta)^{-1/2}f,$$

Therefore, by using $V^{1/2}(-\Delta)^{-1/2}$ is bounded on $L^2$, we can easily get the conclusion of Theorem 3.

**Corollary 4.3.** If $[b, T_3], [b, T_3']$ is bounded on $L^2$, and $V \in L^{n/2} \cap B_q$ for $q > n/2$, then $b \in BMO$.

**Proof.** we only need to prove that $V^{1/2}(-\Delta)^{-1/2}$ is bounded on $L^2$. This follows directly from Hölder inequality and fractional integration that,

$$\left\|V^{1/2}(-\Delta)^{-1/2}f\right\|_2 \leq C \left\|V^{1/2}\right\|_n \left\|(-\Delta)^{-1/2}f\right\|_{2n/(n-2)} \leq C \left\|V\right\|_{n/2} \left\|f\right\|_2.$$

**References**

[1] R.Coifman, R.Rochberg, and G.Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611-635.

[2] F.Gehring, The $L^p$-integrability of the Partial Derivatives of a Quasi-conformal Mapping, Acta Math.,130(1973),265-277.

[3] D.Gilberg and N.Trunderg, Elliptic Partial Differential Equations of Second Order, Second Ed., Springer Verlag, 1983.

[4] S.Janson, Mean oscillation and commutators of singular integral operators, Ark.Mat. 16 (1978), 263-270.

[5] Y.Meyer, La Minimalité de le espace de Besov $B_{1,1}$ et la continuité des operateurs definis par des integrales singulieres, Monografias de Matematicas,Vol.4,Univ.Autonoma de Madrid,1985.

[6] A.Torchinsky, Real-variable methods in harmonic analysis, Academic Press, 1986.

[7] Z.Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier, 45, 2 (1995), 513-546.

[8] J.Zhong, Harmonic analysis for some Schrödinger type operators, Ph.D.Thesis, Princeton University, 1993.