The hyperbolic positive energy theorem

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Abstract

We show that the causal-future-directed character of the energy-momentum vector of \( n \)-dimensional asymptotically hyperbolic Riemannian manifolds with spherical conformal infinity, \( n \geq 3 \), can be traced back to that of asymptotically Euclidean general-relativistic initial data sets satisfying the dominant energy condition.

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1 Introduction

An interesting global invariant of asymptotically hyperbolic general relativistic initial data sets is the energy-momentum vector \( \mathbf{m} \equiv (m_\mu) \) [8, 11, 30] (compare [1, 12]). It is known that \( \mathbf{m} \) is timelike future pointing under a spin condition [8, 9, 17, 24, 30]. The object of this work is to show how to remove this condition for asymptotically hyperbolic Riemannian manifolds with spherical conformal infinity.

Given an asymptotically hyperbolic manifold, under natural asymptotic conditions the total energy-momentum vector \( \mathbf{m} \) can be defined by integrals involving a mass aspect function over the conformal boundary at infinity. It has been
shown in [2] that the mass aspect function cannot be negative when, in dimensions $3 \leq n \leq 7$, the scalar curvature (also known as “Ricci scalar” in the physical literature) is bounded from below by $-n(n-1)$; compare [22] for a related result for manifolds with boundary when $n = 3$. The restriction on the dimension there arises from the regularity theory of constant-mean-curvature surfaces in Riemannian manifolds. Here we show how the above property of the mass aspect function can instead be inferred from positivity results for asymptotically Euclidean general relativistic initial data sets, in all dimensions.

For this we will need the following result from [23] and [20] (the latter reference uses the former to establish the rigidity part of the claim); conjecturally, the result also follows by reduction as in [15, 16, 28] from the time-symmetric case [26, 27, 29]:

**Theorem 1.1** Let $(M, g)$ be an asymptotically Euclidean Riemannian manifold, where $M$ is the union of a compact set and of an asymptotically flat region, of dimension $n \geq 3$. Suppose that the initial data set $(M, g, K)$ possesses a well defined energy-momentum vector $m$. If the dominant energy condition holds, then

$m$ is causal future pointing or vanishes;

furthermore, in the last case $(M, g, K)$ arises from a hypersurface in Minkowski spacetime.

In view of the hypotheses of Theorem 1.1, from now on the manifold $M$ will be assumed to be the union of a compact set and an asymptotic end diffeomorphic to $[0, \infty) \times S^{n-1}$. In other words, $M$ will be assumed to be a smooth connected non compact manifold with (only) one end diffeomorphic to $[0, \infty) \times S^{n-1}$.

**Remark 1.2** For some applications, such as studies of black holes, it would be desirable to consider more general settings. For instance, in the context of non-degenerate black holes one would like to allow initial data sets with a weakly outer trapped boundary. One would also like to include $(M, g)$’s which are complete with several ends, some of them not asymptotically flat when degenerate black holes occur. In view of our reduction procedure, any extension of Theorem 1.1 to such situations would immediately carry over to the hyperbolic setting; compare Remark 1.6 below.

Theorem 1.1, together with the perturbation results in [7] and the gluing constructions of [6] allows us to remove the restriction on dimension in the proof of positivity of asymptotically hyperbolic mass.

Indeed, let $(M, g)$ denote an $n$-dimensional Riemannian manifold with the just-explained topology, and let $\mathbb{H}^n$ denote $n$-dimensional hyperbolic space (with sectional curvatures normalised to $-1$). We use Theorem 1.1 to establish the following generalisation of a result proved in dimensions $3 \leq n \leq 7$ in [2]:

**Theorem 1.3** Consider an $n$-dimensional Riemannian manifold $(M, g)$, $n \geq 3$, either with

1. scalar curvature $R(g)$ satisfying

$$R(g) \geq -n(n - 1),$$
2. or equipped with a symmetric two-covariant tensor $K$ such that $(g,K)$ satisfies the dominant energy condition with cosmological constant $\Lambda = -n(n-1)/2$.

If $(M,g)$ contains a region isometric to the complement of a compact set in $\mathbb{H}^n$, then $(M,g)$ is isometrically diffeomorphic to $\mathbb{H}^n$.

In fact we provide a proof, alternative to that of [2], in all dimensions $n \geq 3$. The result is well known when $(M,g)$ is a spin manifold [3, 8, 25, 30].

A deformation calculation from [2] together with Theorem 1.3 gives immediately (a closely related and somewhat more general version can be found in Theorem 3.2 below):

**Theorem 1.4** Consider an asymptotically hyperbolic manifold $(M,g)$ of dimension $n \geq 3$ with spherical conformal infinity and well-defined mass aspect function $\mu$. If

$$R(g) \geq -n(n-1),$$

then $\mu$ cannot be negative.

Using the “exotic hyperbolic gluings” à la Carlotto-Schoen of [6], the deformation results of [7] and Theorem 1.4 leads to our main result here:

**Theorem 1.5** Consider an asymptotically hyperbolic manifold $(M,g)$ of dimension $n \geq 3$ with spherical conformal infinity and well-defined energy-momentum vector $m$. If

$$R(g) \geq -n(n-1),$$

then $m$ is timelike future-directed or vanishes; in the last case $M$ is isometrically diffeomorphic to hyperbolic space.

**Remark 1.6** In the asymptotically hyperbolic (AH) spin case, positivity of the mass can be established for manifolds which are complete with compact boundary with mean curvature satisfying $H \leq n-1$ (cf., e.g., [8]). Our method, for reducing the CMC AH case to an asymptotically Euclidean (AE) one (which uses the replacement of $K$ by $K - g$, compare (3.2)-(3.3)), transforms such boundaries in the AH regime to ones satisfying $H \leq 0$ in the AE framework, thus clarifying perhaps the somewhat mysterious AH condition $H \leq n-1$. □

**Remark 1.7** Our theorems apply without further due to general relativistic data sets for which the energy of matter fields is positive and $\text{tr}_g K$ vanishes, or in fact if

$$|K|^2_g - (\text{tr}_g K)^2 \geq 0.$$

The same proofs establish the equivalents of Theorems 1.4 and 1.5 for general relativistic initial data $(M,g,K)$ satisfying the dominant energy condition in cases where the extrinsic curvature tensor $K$ is supported away from the asymptotically hyperbolic boundary at infinity regardless of (1.3). A version of Theorem 1.4 which allows non-compactly support curvature tensors $K$ would immediately lead to a corresponding generalisation of Theorem 1.4. □
Remark 1.8 The heart of the proof of Theorem 1.5 involves perturbing the metric and proceeds through a contradiction argument. This establishes a weaker property of $m$, namely the causal future directed character of the energy-momentum, or its vanishing, and does not allow us to analyse the borderline cases. The stronger conclusion of Theorem 1.5 makes appeal to [19], where it is shown how to use the weaker conclusion to assert that $m$ is timelike or vanishing, and vanishes only for hyperbolic space.

2 Localised “Maskit” gluings

The aim of this section is to analyse what happens with the energy-momentum under the localised Maskit-type gluings of asymptotically hyperbolic manifolds of [6, Section 3.5] for spherical conformal boundaries. It is clear that the analysis below can be adapted to the conformal gluings of Isenberg, Lee and Stavrov [21] in the spherical case, and we will not discuss this case any further. On the other hand it is not obvious how to adapt our arguments to non-spherical conformal boundaries, it would be of interest to settle this.

The gluing construction relevant for our purposes proceeds as follows: Consider points $p_1, p_2$, lying on the conformal boundary of two asymptotically hyperbolic initial data sets $(M_1, g_1, K_1)$ and $(M_2, g_2, K_2)$ satisfying the dominant energy condition, with total energy-momentum vectors $m_1 \equiv (m^1_\mu)$ and $m_2 \equiv (m^2_\mu)$. We will assume that both $(M_1, g_2)$ and $(M_2, g_2)$ have spherical conformal infinity (by this we mean that the conformal class of the boundary metric is that of the canonical metric on the sphere), with the extrinsic curvature tensors asymptoting to zero, and with well-defined total energy-momentum, cf. [8, 10]. As shown in [6] for deformations of data sets preserving the vacuum condition or for scalar curvature deformations preserving an inequality, or in Appendix B below for deformations of data sets preserving the dominant energy condition, for all $\varepsilon > 0$ sufficiently small we can construct new initial data sets $(M_i, g_{i,\varepsilon}, K_{i,\varepsilon}), i = 1, 2$, such that the metrics are the hyperbolic metric in coordinate half-balls $\mathcal{U}_{\varepsilon}^i$ of radius $\varepsilon$ (for the “compactified” metric) around $p_1$ and $\mathcal{U}_{\varepsilon}^2$ around $p_2$, and the $K_{i,\varepsilon}$’s are zero there.

Now, both the construction above and the definition of energy-momentum of an asymptotically hyperbolic initial data set requires choosing a coordinate system in which the metric manifestly approaches the hyperbolic metric [8]. Let us denote by $\phi$ the choice of such a coordinate system. In the case of spherical conformal infinity such structures can be parameterised by the conformal group of $S^{n-1}$. Indeed, let us denote by $\phi$ a coordinate system $(r, \theta^A)$, with $\theta^A$ parameterising $S^{n-1}$, in which $g$ takes the form

$$g \to r \to \infty \quad \frac{dr^2}{1 + r^2} + r^2 \hat{h}_{AB}(\theta) d\theta^A d\theta^B =: b,$$

with $\hat{h}$ being the unit round metric on $S^{n-1}$, where the asymptotics is understood by requiring the $b$-norm $|g - b|_b$ of $g - b$ to decay to zero as one recedes to infinity. (To have a well defined energy-momentum one also needs fall-off rates, as well as derivative-decay conditions [8, 10], which will be implicitly assumed

$$\Box$$
whenever relevant). If \( \tilde{\phi} \) is another such coordinate system in which

\[
g \to r \to \infty \quad \frac{d\tilde{r}^2}{1 + \tilde{r}^2} + \tilde{r}^2 \hat{h}_{AB}(\tilde{\theta}) d\tilde{\theta}^A d\tilde{\theta}^B =: \hat{h},
\]

then there exists a conformal transformation \( \Lambda \) of \( (S^{n-1}, \hat{h}) \) so that \( \hat{h} \) is obtained from \( \hat{b} \) by applying \( \Lambda \) (cf., e.g., [8, 10]). We then write \( \tilde{\phi} = \Lambda \phi \).

We will write \((M, g, K, \phi)\) for an asymptotically hyperbolic initial data set \((M, g, K)\) with an asymptotic structure \(\phi\).

Choose, then, some asymptotic structures \(\phi_i\) on \((M_i, g_i, K_i)\) and let

\[
m_{\mu}^{i, \varepsilon} \equiv (m_{\mu}^{i, \varepsilon})
\]

denote the energy-momenta of \((M_i, g_i, \varepsilon, K_i, \phi_i)\). The construction in [6] guarantees that

\[
m_{\mu}^{i, \varepsilon} \to_{\varepsilon \to 0} m_{\mu}^i, \quad i = 1, 2.
\]

Remark 2.1 For our further purposes we need to complement (2.3) with a decay rate, which can be read-off from [6, Section 3.3]. We note that the background hyperbolic metric, denoted in the current work by \(b\), is denoted by \(\hat{b}\) there, and that we use here \(s\) instead of the symbol \(b\) used in [6] for the rate of radial decay of the glued metrics. Let us thus assume that

\[
|g - b|_b = O(r^{-\sigma}) \text{ with } \sigma > (n - 1)/2 + s
\]

for some \(s \in [n/2, (n + 1)/2]\). By Remark 3.6 and the comments at the end of Section 3.3 in [6], the \(b\)-norm of the correction \(h_\varepsilon\) to the metric arising from the gluing procedure is then of order \(o(\varepsilon^{\sigma-s}r^{-s})\). Now recall that the mass is the limit, as \(r\) tends to infinity, of an integral on a sphere of radius \(r\) of a quantity, say \(U\) (cf. (2.8) below). In this limit only the part of \(U\) linear in \(e_\varepsilon = g_\varepsilon - b = g - b + h_\varepsilon = e + h_\varepsilon\) matters. This implies (cf. the last, unnumbered, equation in the proof of [6, Theorem 3.7])

\[
m_{\mu}^{i, \varepsilon} - m_{\mu}^i = o(\varepsilon^{\sigma-s}), \quad i = 1, 2.
\]

When working in weighted Hölder spaces, the decay rate \(s > n/2\) is needed for the final mass to be well defined, while one needs \(\sigma \leq n\) for the initial mass to be non-zero. The resulting best approach rate \(\sigma - s\) is as close as desired to, but smaller than, \(n/2\). In our positivity proof below, the decay rate \(\sigma = n\) is obtained by invoking the density results of [14], independently of the decay rate of the metric under consideration. We will see that this suffices for the positivity theorem in space dimensions \(n \geq 5\).

A more careful analysis, which provides

\[
m_{\mu}^{i, \varepsilon} - m_{\mu}^i = o(\varepsilon^{n/2}), \quad i = 1, 2,
\]

is needed when \(n = 4\). To obtain this estimate one notes that all the gluing results in [6] can be traced back to Theorem 3.3 there, or an initial data version thereof, where the solution of the problem at hand is constructed in weighted Sobolev spaces. In view of the already mentioned results in [14], in [6, Theorem 3.3] we can take \(\sigma = n\), and choose \(b\) there (which coincides with \(s\) here).
Figure 2.1: A small spherical cap $D^1_\varepsilon \subset S^{n-1}$ (middle picture) is mapped to the upper-half sphere by the conformal transformation $\Lambda^1_\varepsilon$. The final metric coincides with the hyperbolic one throughout the upper half-ball. We assume the topology of conformal infinity to be spherical, but the interior does not have to be topologically trivial. Thus, the topology inside the final upper half-sphere is that of a half-ball, with the boundary of the lower-half sphere (including the equatorial hyperplane) bounding the remaining topology of $M_1$.

to be equal to $n/2$ (the mass still being well defined, the correction being in $H^1_{1,z-n/2}$ in the notation of [6]). This provides the desired decay rate (2.5); compare [6, Remark 3.4].

Let $D^i_\varepsilon$ denote the $(n - 1)$-dimensional ball centered at $p_i$, contained in the conformal boundary, obtained by intersecting $\mathcal{U}^i_\varepsilon$ with the conformal boundary; see the middle figure in Figure 2.1.

Let $\mathcal{H} \subset \mathbb{H}^n$ denote the equatorial plane in the Poincaré ball model. So $\mathcal{H}$ is a totally geodesic hypersurface which “cuts the hyperbolic space in half”, and its conformal completion intersects the conformal boundary of $\mathbb{H}^n$ at the equator. We denote by $\mathbb{H}^n_+ \mathcal{H}$ the part of $\mathbb{H}^n$ which lies “above” $\mathcal{H}$ and by $\mathbb{H}^n_− \mathcal{H}$ the part of $\mathbb{H}^n$ which lies “below” $\mathcal{H}$. In the half-space model, with coordinates $(z, w^i)$ so that the conformal boundary is given by $\{z = 0\}$ and the hyperbolic metric $b$ takes the form

$$b = \frac{dz^2 + (dw^1)^2 + \ldots + (dw^{n-1})^2}{z^2},$$

(2.6)

the hypersurface $\mathcal{H}$ can be taken to the hypersurface $\{z > 0, w^1 = 0\}$. The closure of $\mathcal{H}$ intersects then the conformal boundary at $\{z = 0, w^1 = 0\}$.

Recall that the connected component of the group of conformal isometries of a sphere $S^{n-1}$ is isomorphic to component of the identity of the Lorentz group in dimension $n + 1$, and the action $\Lambda \mathbf{m}$ of $\Lambda$ on the energy-momentum vector $\mathbf{m}$ is the standard action of the Lorentz group on $\mathbb{R}^{1,n}$.

Let us denote by $\Lambda^1_\varepsilon$ the conformal transformation of the conformal boundary which maps $D^i_\varepsilon$ to the upper half-sphere, cf. the right figure in Figure 2.1. In physics terminology, $\Lambda^1_\varepsilon$ is a boost in the direction of $−p_1$, when $S^{n-1}$ is viewed as a coordinate sphere in $\mathbb{R}^n$; this can be seen e.g. from [7, Section 3.4]. The velocity of the boost tends to the speed of light as $\varepsilon$ tends to zero.

After applying $\Lambda^1_\varepsilon$ we obtain a set $(M_1, g_1, z, K_1, \Lambda^1_\varepsilon \phi_1)$ with energy-momentum vector $\Lambda^1_\varepsilon \mathbf{m}_1$. $M_1$ contains a region, denoted by $\mathcal{V}^i_\varepsilon$ which, in the coordinate
system $\Lambda^1_1 \phi_1$ coincides with the “half-hyperbolic-space $H$. The boundary $\partial V_1^1$ corresponds in the local coordinates to $H$, is totally geodesic, and the metric $g_{1,\epsilon}$ is exactly hyperbolic throughout $\mathbb{H}^n_+$ and in a neighborhood of $H \approx \partial V_1^1$. Thus

$$\mathbb{H}^n_+ \approx V_1^1 \subset U_1^1 \subset M_1,$$

(2.7)

where $\approx$ in (2.7) means “isometrically diffeomorphic to”.

Letting $\mathbb{V}_\mu$ be an ON basis of the space of static KIDs, we have [18] (compare [4, Equation (IV.40)])

$$\Lambda^1_1 m^{1,\epsilon}_\mu = -\lim_{r \to \infty} \int_{S^{n-1}(r)} V_\mu Z^j (R^i j - \frac{R}{n} \delta^j_i) d\sigma_i,$$

(2.8)

where $R_{ij}$ is the Ricci tensor of the metric $g_{1,\epsilon}$, $R$ its trace, and we have ignored an overall dimension-dependent positive multiplicative factor. Further, $Z$ is the “dilation vector field”: if using the Poincaré-ball model, so that the hyperbolic metric $b$ takes the form

$$b = \frac{4}{(1 - |x|^2)^2} \delta,$$

(2.9)

where $\delta$ is the Euclidean metric, we have

$$Z \equiv Z^i \partial_i = \frac{(1 - |x|^2) x^i \partial_i}{1 + |x|^2}.$$

Finally, still using the Poincaré-ball model, the $S^{n-1}(r)$’s can be taken to be coordinate spheres of radius $1 - 1/r$ accumulating at the conformal boundary as $r$ tends to infinity.

Let $S^{n-1}_+(r)$ denote the part of the sphere which lies under the equator, and $S^{n-1}_-(r)$ the part above the equator. Since the trace-free part of the Ricci tensor of $g_{1,\epsilon}$ vanishes on $H_+$, only the integrals on $S^{n-1}_+(r)$ contribute:

$$\Lambda^1_2 m^{1,\epsilon}_\mu = -\lim_{r \to \infty} \int_{S^{n-1}_+(r)} V_\mu Z^j (R^i j - \frac{R}{n} \delta^j_i) d\sigma_i.$$

(2.10)

Similarly there exists a conformal transformation of the conformal boundary, which will be denoted by $\Lambda^2_2$, which maps $D^2$ to the lower half-sphere, cf. Figure 2.2. After applying this transformation, $M_2$ contains a region $V_2^2$ isometrically diffeomorphic to the half-hyperbolic-space $H_+$, as well as its totally geodesic boundary $H$, with the metric being exactly hyperbolic in a neighborhood of $H$, and throughout $V_2^2$. Analogously to (2.7) we have

$$\mathbb{H}^n_+ \approx V_2^2 \subset U_2^2 \subset M_2,$$

One also finds

$$\Lambda^2_2 m^{1,\epsilon}_\mu = -\lim_{r \to \infty} \int_{S^{n-1}_+(r)} V_\mu Z^j (R^i j - \frac{R}{n} \delta^j_i) d\sigma_i,$$

(2.11)

where $R_{ij}$ is now the Ricci tensor of the metric $g_{2,\epsilon}$.
Figure 2.2: A small spherical cap $D^2_\varepsilon \subset S^{n-1}$ (left picture) is mapped to the lower-half sphere by the conformal transformation $\Lambda^2_\varepsilon$. The final manifold, obtained by gluing together the manifold from the rightermost Figure 2.1 with the manifold from the middle figure here, has a metric coincides with $g_1,\varepsilon$ on the “lower half” of the final manifold and with $g_2,\varepsilon$ on the “upper half”, except for a strip near the equatorial hyperplane.

Since the metric equals the hyperbolic metric near $\mathcal{H}$ in both manifolds, and the extrinsic curvature tensor vanishes near $\mathcal{H}$ in both manifolds, we can construct a new family of asymptotically hyperbolic initial data sets $(M, g, K, \phi)$ by gluing together $M_1 \setminus \mathcal{H}^1_\varepsilon$ with $M_2 \setminus \mathcal{H}^2_\varepsilon$ along $\mathcal{H}$, with the obvious asymptotic structure $\phi$ and the obvious smooth initial data set $(g, K)$.

Denote by $m^\varepsilon \equiv (m^\varepsilon_\mu)$ the total energy-momentum vector of $(M, g, K, \phi)$. We have

$$m^\varepsilon_\mu = \lim_{r \to \infty} \int_{S^{n-1}(r)} V^i Z^j (R^i_{\mu j} - \frac{R}{n} \delta^i_j) d\sigma,$$

In other words,

$$m^\varepsilon = \Lambda^1_\varepsilon m^{1,\varepsilon} + \Lambda^2_\varepsilon m^{2,\varepsilon}. \quad (2.12)$$

Thus, the energy-momentum is additive under the above Maskit-type gluing, in a sense made precise by (2.13).

3 Positivity for asymptotically hyperbolic Riemannian manifolds

For our proofs we will in fact only need the following special case of Theorem 1.1:

**Theorem 3.1** Let $(M, g, K)$ be an initial data set satisfying the dominant energy condition so that outside of a compact set $g$ is flat and $K$ vanishes. Then $(M, g)$ can be isometrically embedded in Minkowski space-time so that $K$ is the second fundamental form of the embedded hypersurface.
Proof of Theorem 1.4: Consider an asymptotically hyperbolic manifold $(M, g)$ as in the statement of the theorem. As explained in [2], if the mass aspect function is negative one can construct on $M$ a new metric $g_1$ which satisfies
\[ R(g_1) \geq -n(n-1), \] (3.1)
and which coincides with the hyperbolic metric $\hat{g}$ in a neighborhood of the conformal boundary at infinity.

Now, $(M, g_1)$ can be viewed as a general relativistic initial data set $(M, g_1, K_1 \equiv 0)$ for Einstein equations with negative cosmological constant $\Lambda = -n(n-1)/2$, with matter fields with positive energy density $\rho_1 := R(g_1) + n(n-1)$, and with vanishing matter current $J_1 := 0$. But one can also view $(M, g_1)$ as an initial data set
\[ (M, g_2 := g_1, K_2 := -g_2) \] (3.2)
for Einstein equations with vanishing cosmological constant, with positive energy density $\rho_2 = \rho_1$, and with vanishing matter current $J_2 \equiv 0$. This follows immediately from the following trivial calculation
\[ R(g_2) \equiv R(g_1) = \kappa \rho + 2 \Lambda = \kappa \rho + |K_2|_{g_2}^2 - \text{tr}_{g_2}(K_2)^2. \] (3.3)

Let us denote by $\mathcal{V} \subset M$ the region of $M$ where $g_2$ is exactly hyperbolic. Then $(\mathcal{V}, g_2)$ can be embedded into $(n+1)$-dimensional Minkowski space-time $\mathbb{R}^{1,n}$ as a hyperboloid so that its second fundamental form coincides with $K_2$. By passing to a subset of $\mathcal{V}$ if necessary there exists $R > 0$ so that $\mathcal{V}$ can be identified with $\mathcal{V} := \{ t = -\sqrt{1+|\vec{x}|^2}, |\vec{x}| > R \}$. (3.4)

Let $f$ be any smooth function which coincides with $-\sqrt{1+|\vec{x}|^2}$ for $R \leq |\vec{x}| \leq R + 1$, which equals to a constant for large $|\vec{x}|$, and such that the graph of $f$ over the set $\{ |\vec{x}| \geq R \}$ is spacelike. Let $M_5$ denote this graph, let $g_3$ be the metric induced by the Minkowski metric on the graph of $f$ and let $K_3$ be the second fundamental form of the graph. Let $(M_4, g_4, K_4)$ be obtained by taking the union of $(M \setminus \mathcal{V}, g, 0)$ and $(\mathcal{V}, g_3, K_3)$ and identifying points at the joint boundary $|\vec{x}| = R$. Then $(M_4, g_4, K_4)$ is a smooth general relativistic initial data satisfying the dominant energy condition which is exactly flat at large distances. By Theorem 3.1 the data set $(M_4, g_4, K_4)$ arises from Minkowski space-time, in particular $M_4$ is spin. So $(M, g)$ is a spin manifold with timelike past-pointing total mass, which is not possible by [8].

Proof of Theorem 1.3: The proof of Theorem 1.4 using $g_1 := g$ provides a general relativistic initial data set $(M_4, g_4, K_4)$ satisfying the dominant energy condition which, at large distances, arises from a hypersurface in Minkowski space-time. Theorem 3.1 shows that $(M_4, g_4)$ can be embedded in Minkowski space-time. In particular $(M_4, g_4)$ is spin, and the result follows from, e.g., [8] in the Riemannian case, or [9] in the initial data setting.

Using a deformation of the metric as in [7, Corollary 1.4] when $n \geq 4$ (one can alternatively invoke [5, Theorem 1.2] when $n \geq 8$), and the fact that every three dimensional manifold is spin together with the usual Positive Energy Theorem [9], one obtains the following variation of Theorem 1.4:
Theorem 3.2 Let \((M, g)\), \(n \geq 3\), be a manifold with scalar curvature \(R[g] \geq -n(n-1)\) with a metric which is smoothly conformally compactifiable with spherical conformal infinity. Suppose that in the coordinates of (2.1) the metric \(g\) approaches the hyperbolic metric as \(O(r^{-n})\) and assume that \(R[g] + n(n-1) = O(r^{-n-1})\). Then the total energy-momentum vector of \((M, g)\) cannot be timelike past-pointing.

We are ready now to pass to the

**Proof of Theorem 1.5:** For further reference we will carry out the argument as far as possible (and thus not as far as the final conclusions) for general relativistic initial data sets satisfying the dominant energy condition, to make it clear how any generalisation of Theorem 1.4 to include the extrinsic curvature tensor \(K\) would provide a generalisation of Theorem 1.5.

Suppose that there exists an asymptotically hyperbolic initial data set \((M_1, g_1, K_1)\) with spacelike or null past-pointing energy-momentum vector \(m^1\) with respect to an asymptotic structure \(\phi_1\). If \(K_1 \equiv 0\), we can make a deformation of the metric as in [5, Proposition 6.2] to achieve that \([g - b_0]_{\phi_1} = O(r^{-n})\) in the coordinates of (2.1) while maintaining the space-like or past-directed causal character of the energy-momentum vector. In the general case, by approximation results in the spirit of those in [14] one should similarly be able to obtain that \([g - b_0]_{\phi_1} = O(r^{-n})\) in the coordinates of (2.1); alternatively one could just assume these fall-off rates.

Changing \(\phi_1\) (i.e., applying a conformal transformation at the conformal boundary) if necessary, we can choose the asymptotic structure \(\phi_1\) so that

\[
m^1 \equiv (m_0^1, \bar{m}^1) \in \mathbb{R} \times \mathbb{R}^n
\]  
(3.5)

satisfies

\[
m_0^1 < 0.\]
(3.6)

Choose \(p_1 \in S^{n-1}\) to be \(-\bar{m}^1\).

Let \((M_1, g_1, K_1, \phi_1)\) be a family as in Section 2, constructed by a gluing procedure which, in half-plane-model coordinates, replaces \(g_1\) by the hyperbolic metric and \(K_1\) by zero in a half-ball of radius \(\varepsilon\) centred at \(p_1\). Set

\[
(M_2, g_2, K_2, \phi_2) := (M_1, g_1, K_1, \phi_1).
\]

The maps \(\Lambda^1_\varepsilon\) of Section 2 act on \(m^1\) as boosts in the direction of \(\bar{m}^1\). Let \(R_\varepsilon\) be a rotation of the sphere by \(\pi\) around some arbitrarily chosen axis passing through the equator, thus \(R_\varepsilon\) maps the north pole to the south pole. We can choose \(\Lambda^2_\varepsilon\) to be \(R_\varepsilon \Lambda^1_\varepsilon\). Then the space-part of \(\Lambda^2_\varepsilon m^2 = \Lambda^2_\varepsilon m^1\) cancels out that of \(\Lambda^1_\varepsilon m^1\):

\[
\Lambda^1_\varepsilon m^1 + \Lambda^2_\varepsilon m^1 = \Lambda^1_\varepsilon m^1 + R_\varepsilon \Lambda^1_\varepsilon m^1 = \Lambda^1_\varepsilon (2m^1, \bar{0}).
\]
(3.7)

Hence

\[
m^\varepsilon = \Lambda^1_\varepsilon m^1 + \Lambda^2_\varepsilon m^2 = \Lambda^1_\varepsilon m^1 + R_\varepsilon \Lambda^1_\varepsilon m^1 = \Lambda^1_\varepsilon m^1 + R_\varepsilon m^1 - m^1 = \Lambda^1_\varepsilon (2m^1, \bar{0}) = (\Lambda^1_\varepsilon + R_\varepsilon \Lambda^1_\varepsilon)(m^1 + m^1 - m^1) = \Lambda^1_\varepsilon (2m^1, \bar{0}) + (1 + (\Lambda^1_\varepsilon)^{-1}R_\varepsilon \Lambda^1_\varepsilon)(m^1 - m^1)\]
(3.8)
We wish to show that \((\ast)\) goes to zero as \(\varepsilon\) goes to zero. For this, we need to analyse the matrix \((A_1^\varepsilon)^{-1}R_\varepsilon A_1^\varepsilon\). If we align the coordinate system so that \(\vec{m}^1\) points in the direction of the first coordinate axis, we can write

\[
A_1^\varepsilon = \begin{pmatrix}
\gamma_\varepsilon & -\gamma_\varepsilon v_\varepsilon & 0 & 0 \\
-\gamma_\varepsilon v_\varepsilon & \gamma_\varepsilon & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \text{Id}
\end{pmatrix}, \quad R_\varepsilon = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \text{Id}
\end{pmatrix},
\]

(3.9)

with \(v_\varepsilon \in (-1,1), \gamma_\varepsilon := (1 - v_\varepsilon^2)^{-1/2}\), and \(\text{Id}\) being the \((n-2)\times(n-2)\) identity matrix. Then \((A_1^\varepsilon)^{-1}\) coincides with \(A_1^\varepsilon\) except for changing \(v\) to its negative, and we find

\[
(A_1^\varepsilon)^{-1}R_\varepsilon A_1^\varepsilon = \begin{pmatrix}
\gamma_\varepsilon^2(1 + v_\varepsilon^2) & -2\gamma_\varepsilon^2 v_\varepsilon & 0 & 0 \\
2\gamma_\varepsilon^2 v_\varepsilon & -\gamma_\varepsilon^2(1 + v_\varepsilon^2) & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & \text{Id}
\end{pmatrix}.
\]

(3.10)

If we view the sphere \(S^{n-1}\) as embedded in \(\mathbb{R}^n\) with Euclidean coordinates \((x^i, x^n)\), with \(-\vec{m}^1\) the north pole and \(\vec{m}^1\) the south pole, then the Lorentz transformation \(A_1^\varepsilon\) of (3.9) corresponds to the conformal transformation (cf., e.g., [7, Section 3.4])

\[
S^{n-1} \ni (x^i, x^n) \mapsto \frac{1}{\gamma_\varepsilon(1 - v_\varepsilon x^n)}(x^i, \gamma_\varepsilon(x^n - v_\varepsilon)) \in S^{n-1}.
\]

(3.11)

If we denote by \(\cos \theta\) the angle off the axis passing through the south pole and the north pole, then \(v_\varepsilon\) has to be chosen so that \(\cos(\varepsilon)\) is mapped to \(\pi/2\). Equivalently, if \(n^i\) is a unit vector in Euclidean \(\mathbb{R}^{n-1}\), (3.11) should give

\[
S^{n-1} \ni (\sin(\varepsilon)n^i, \cos(\varepsilon)) \mapsto \frac{1}{\gamma_\varepsilon(1 - v_\varepsilon \cos(\varepsilon))}(n^i, 0) \in S^{n-1},
\]

(3.12)

i.e., \(v_\varepsilon = \cos(\varepsilon)\), hence

\[
\gamma = \frac{1}{\sin(\varepsilon)} \approx \varepsilon^{-1}
\]

(3.13)

for small \(\varepsilon\). Hence \((\ast)\) in (3.8) can be estimated as

\[
|\langle\ast\rangle| \equiv |(1 + (A_2^\varepsilon)^{-1}R_\varepsilon A_1^\varepsilon)(m^{1,\varepsilon} - m^1)| \\
\leq C(1 + \varepsilon^{-2})|m^{1,\varepsilon} - m^1| \leq C^2 \varepsilon^2 \varepsilon^{-2},
\]

(3.14)

where we used (2.4). So \((\ast)\) will tend to zero if \(n \geq 5\). The case \(n = 4\) follows in the same way using (2.5). This implies that for \(\varepsilon\) small enough the vector \(\vec{m}^\varepsilon\) is the image by a boost of a timelike past-pointing vector, hence timelike past pointing, which (in the case where \(K_1 \equiv 0\)) contradicts Theorem 3.2. We conclude that \(\vec{m}\) is causal future-pointing, or vanishes. \(\square\)

### A KIDs on the half-space model

The half-space model is very useful to carry-out the Maskit-type gluings, and the formulae in this Appendix are relevant when considering energy-momentum in this context.
We use coordinates \((w^i, z)\) as in (2.6). The non trivial Christoffel symbols of \(b\) are
\[
\Gamma^z_{zz} = -z^{-1}, \quad \Gamma^z_{ij} = z^{-1}\delta_{ij}, \quad \Gamma^k_{iz} = -z^{-1}\delta_i^k.
\]
The equation \(\nabla^2 N - Nb = 0\) is then equivalent to
\[
\partial^2 z N + z^{-1}\partial z N - z^{-2} N = 0,
\]
\[
\partial_i\partial_j N - z^{-1}\partial z N\delta_{ij} - z^{-2} N\delta_{ij} = 0.
\]
A calculation shows that the following functions provide an ON basis of the space of solution
\[
V^{(i)} := w^i z^{-1}, \quad V^{(n)} := \frac{|\vec{w}|^2 + z^2 - 1}{2z}, \quad V^{(0)} := \frac{|\vec{w}|^2 + z^2 + 1}{2z}.
\] (A.1)

Similarly, the Killing equations \(\nabla_a Y_b + \nabla_b Y_a = 0\) are equivalent to
\[
\partial_z Y_z + z^{-1}Y_z = 0,
\]
\[
\partial_i Y_i + \partial Y_i + 2z^{-1} Y_i = 0,
\]
\[
\partial_i Y_j + \partial_j Y_i - 2z^{-1} Y_i\delta_{ij} = 0.
\]
The solutions are spanned by
\[
Y_X := X_i dw^i / z^2,
\]
where is \(X\) a Killing one-form of the Euclidean space \(\mathbb{R}^{n-1}\) (providing a subspace of dimension \(n(n-1)/2\) of the set of solutions), and
\[
Y_{c,A} = (c + A^j w_j) dz / z + \left(-A_i / 2 + [A^j w_j w_i - \frac{1}{2} A_i (\sum (w^j)^2) + cw_i] / z^2\right) dw^i,
\]
with free constants \(c\) and \(A^i\) (associated, respectively, with dilations and inversions of \(\mathbb{R}^{n-1}\)).

**B** An “exotic hyperbolic gluing” preserving the dominant energy condition

In this appendix we show how to adapt the constructions in [6] to the modified constraint operator of [13]. This leads to a Carlotto-Schoen-type gluing procedure, in the asymptotically hyperbolic setting, which preserves the dominant-energy-condition (DEC) character of initial data sets. See [14] for some related results.

To proceed, it is convenient to start by pointing-out the correspondence of notations in [6] and in [13]:
\[
k \equiv K, \quad \pi \equiv K - \text{tr}_g(K) g, \quad \rho \equiv 2\mu - 2\Lambda, \quad 2(J_{\text{Corvino-Huang}})^i_j \equiv g^{ij} (J_{\text{here}})^j_i.
\]
The constraint operator \(\mathcal{C}(K, g)\) of [6] is closely related to the constraint map \(\Phi(g, \pi)\) of [13]. Inspired by the definition of the modified constraint map \(\Phi^W_{(g, \pi)}\)
in equation (2.3) of [13], we define the (recentred) modified constraint operator by
\[ C^W(K,g) := C(K + \delta K, g + \delta g) - C(K, g) - \frac{1}{2} (\delta g \cdot (J(K, g) + W), 0), \]
where \( W \) is a one form (which will be determined in the construction to follow) with decay and differentiability properties modelled on those of \( J \), while
\[ (\delta g \cdot Z)_i := (\delta g)_{ij} g^{jk} Z_k. \]

The linearisation of this modified constraint operator at \((0, 0)\) is
\[ P^W(K,g)(\delta K, \delta g) = P(K,g)(\delta K, \delta g) - \frac{1}{2} (\delta g \cdot (J(K, g) + W), 0), \]
where \( P(K,g) \) is the linearisation of the standard constraint operator at \((K,g)\). In particular when \( J(K, g) = 0 \), as is the case for vacuum initial data sets, it holds that
\[ P^0(K,g) = P(K,g). \]
Since the supplementary terms in the operators involved are of lower order, standard arguments show that the estimates for \( P^*(K,g) \) derived in [6] remain true for \((P^W(K,g))^* \) when \((K, g)\) is sufficiently close to a vacuum initial data set and \( W \) close to zero in the gluing region. In particular, given a constant \( \tau \), the estimates necessary to carry-out the gluing to the “hyperbolic data” \((\tau \delta g, \delta g)\), hold, which is the first step of the Maskit gluing of Section 2. Now, in [6] we work in spaces where the kernel of \( P^*(K,g) \) is trivial, and this continues to be the case for \((P^W(K,g))^* \) when \((K, g)\) is close to a vacuum initial data set and \( W \) close to zero, again in the gluing region. This allow us to carry-out the gluing, as follows:

First, given sufficiently small \((\delta J, \delta \rho)\) and \( W \), where “sufficiently small” is meant in the sense of weighted Sobolev spaces as in the corresponding theorems of [6], the equivalent of Theorem 3.1 of [13] provides a solution to the equation
\[ \mathcal{E}_W(K,g)(\delta K, \delta g) = (\delta J, \delta \rho), \tag{B.1} \]
so a solution to
\[ \mathcal{E}(K + \delta K, g + \delta g) = \mathcal{E}(K, g) + (\delta J, \delta \rho) - \frac{1}{2} (\delta g \cdot (J(K, g) + W), 0). \]
If \( \delta g \) is small enough, if \( W = \delta J \), and if we define
\[ \overline{\rho} = \rho(K + \delta K, g + \delta g), \quad \overline{J} = J(K + \delta K, g + \delta g), \quad \overline{g} = g + \delta g, \]
then by [13, Lemma 3.3],
\[ \overline{\rho} - |\overline{J}|_{\overline{g}} \geq \rho + \delta \rho - |J + \delta J|_g. \]
Note that this can be used to prove an equivalent of Corollary 3.4 of [13] in our context.

Now for the gluing of two initial data sets, we define
\[ g =: \chi g_1 + (1 - \chi) g_2, \quad K := \chi K_1 + (1 - \chi) K_2, \]
\((\delta J, \delta \rho) := \chi C(K_1, g_1) + (1 - \chi)C(K_2, g_2) - C(K, g) + (0, (\delta \rho)_{0})\).

The pointwise estimate (3.1) of [13] remains of course valid. The solution to equation (B.1) with \(W = \delta J\), obtained by invoking the implicit function theorem, will produce an “exotic hyperbolic gluing” which preserves the dominant energy condition, in the spirit of Theorem 3.7 of [13] but adapted to the setup of interest here, provided that we can construct a small smooth non-negative function \((\delta \rho)_{0}\) (corresponding to \(2\psi_0\) in [13]; recall that our \(J\) is half of theirs in the formulae here) such that

\[
(\delta \rho)_{0} \geq \chi(1 - \chi)(|g_1 - g_2|_{g_1}|J_1|_{g_1} + |g_1 - g_2|_{g_2}|J_2|_{g_2}),
\]

for all \((g_1, K_1), (g_2, K_2)\) close to an hyperbolic data on a fixed annulus. In the notations of [6], the function

\[
(\delta \rho)_{0} = c\chi(1 - \chi)z^\sigma
\]

with a sufficiently small constant \(c\), satisfies the desired inequality. When the two initial data set come from rescaled ones, such as \(g_\lambda\) of Section 3.3 in [6], one checks that the corresponding difference of metrics is of order \(O_{g_\lambda}(\lambda^\sigma z^\sigma)\) and the \(J\)’s are of the same order. We can then take \(c = c'\lambda^{\sigma'} (\sigma < \sigma' < 2\sigma)\) to guarantee that the correction \(h_\lambda\) has the right order \(o_{g_\lambda}(\lambda^\sigma z^\sigma)\), remaining compatible with (B.2) (compare the end of the proof of [6, Theorem 3.7]).

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