TILING $n \times m$ RECTANGLES WITH $1 \times 1$ AND $s \times s$ SQUARES

RICHARD J. MATHAR

Abstract. We consider tilings of a rectangle which is $n$ units wide and $m$ units long by non-overlapping $1 \times 1$ squares and $s \times s$ squares. Bivariate generating functions are computed with the Transfer Matrix Method for moderately large but fixed widths $n$ as a function of the parameter $m$ and of the number of $s \times s$ squares in the rectangle.

1. Definitions

We consider the combinatorial problem of placing non-overlapping squares of shape $s \times s$ into rectangles of shape $n \times m$. Comparing the areas $nm$ of the hosting rectangle and the area $s^2$ of the individual square we find a trivial upper limit for the number $k$ of $s \times s$ squares that fit into the rectangle:

$$0 \leq k \leq \frac{nm}{s^2}.$$  \hfill (1)

The area in the rectangle that is not covered by the $s \times s$ squares is tiled with $1 \times 1$ squares (monomers), of which there are $nm - ks^2$.

The manuscript is basically an industrial scale evaluation of Heubach’s tilings [1].

Definition 1. $T_{n \times m}(s,k)$ is the number of ways of tiling the $n \times m$ rectangle with $k$ non-overlapping squares of shape $s \times s$ and with $nm - ks^2$ unit squares. Distributions obtained by flipping or rotating the rectangle are considered distinct and counted with multiplicity.

A basic example of such counting on commensurate grids are the tilings of Figure 1, which shows all variants of distributing two $2 \times 2$ squares on a $3 \times 5$ board. The total number of geometries that does not resolve how many squares fill the rectangle is:

Definition 2. The number of ways of tiling the $n \times m$ rectangle with $1 \times 1$ and $s \times s$ squares is

$$T_{n \times m}(s) \equiv \sum_{k=0}^{\lfloor nm/s^2 \rfloor} T_{n \times m}(s,k).$$  \hfill (2)

Bivariate ordinary generating functions will be noted as follows:

Definition 3.

$$\sum_{m \geq 0} \sum_{k \geq 0} T_{n \times m}(s,k)z^mt^k \equiv T_{n}(s,z,t).$$  \hfill (3)
2. Symmetries

There is always one way of filling the rectangle with monomers only:

\[ T_{n \times m}(s, 0) = 1. \]

If only one \( s \times s \) square is to be placed, it can be rooted at any of the \( n - s + 1 \) vertical and \( m - s + 1 \) horizontal grid points:

\[ T_{n \times m}(s, 1) = (n - s + 1)(m - s + 1), \quad m, n \geq s. \]

Rotating the rectangle (and rotating the embedded squares with it) does not change the count; it is symmetric with respect to interchange of \( n \) and \( m \):

\[ T_{n \times m}(s, k) = T_{m \times n}(s, k). \]

If the rectangle has a width and height that are integer multiples \( is \) and \( js \) of the square edge \( s \), there is one configuration with full coverage by the \( s \times s \) squares, not using interstitial monomers:

\[ T_{is \times js}(s, ij) = 1, \quad i, j \geq 1. \]

If the width or height are too small, only the \( 1 \times 1 \) squares fit in:

\[ T_{n \times m}(s, k) = \delta_{0, k}, \quad (s > m \vee s > n). \]

If the width equals the square size, \( s = n \), there is an obvious bijection to filling a line with monomers and straight \( s \)-ominos, equal to the number of compositions of \( m \) into 1’s and \( k \) \( s \)'s:

\[ T_{s \times m}(s, k) = \binom{m - (s - 1)k}{k}. \]
3. Transfer Matrix Technique

The bivariate generating functions are constructed with a variant of the Transfer Matrix technique specialized to the tiling combinatorics [2]. The growth front of incrementally adding $s \times s$ squares or leaving free space (that is, adding $1 \times 1$ squares) is encoded in an integer vector of $n$ entries, one per “lane.” These vectors are states in the directed graph of all possible fronts, and we construct all states that are reachable starting from a vector of all-zeros by repeatedly/recursively trying to attach a set of squares to the front to reach the next state. The only difference to the earlier strategy with univariate generating functions [2] is that a transition between two states does not only introduce a factor $z$ to indicate that the base line is rolled up by one unit, but also another factor $t^k$ where $k$ is the number of blocks of width $s$ that recede to the back of the front line.

The implementation of this is put into concrete in the appendix. A Java program constructs all the reachable states, counts them to get the size of the Transfer Matrix, fills the matrix with the factors of either zero (if the row state is not reachable directly from the column state) or $zt^k$, and writes a Maple program that actually solves the linear system of equations to get the head element of the inverse. The mere limitation is the patience needed to execute the Maple program if the Transfer Matrix (node count in the digraph) exceeds a dimension of, say, 350.

4. Results

The results are tabulated in the following format of a 4-dimensional table: Each line contains $s$, then $n$, then $m$, then a colon, then a sequence of $T_{n \times m}(s,k)$ for $k = 0, 1, 2, \ldots$ and finally another colon and the row sum (2). Univariate generating functions of the row sums are obtained by inserting $t = 1$ into the bivariate generating function (3).

4.1. $2 \times 2$ Squares. Putting $2 \times 2$ squares into rectangles yields:

\begin{verbatim}
2 1 1: 1 : 1
2 2 1: 1 : 1
2 2 2: 1 1 : 2
2 3 1: 1 : 1
2 3 2: 1 2 : 3
2 3 3: 1 4 0 : 5
2 4 1: 1 0 : 1
2 4 2: 1 3 1 : 5
2 4 3: 1 6 4 0 : 11
2 4 4: 1 9 16 8 1 : 35
2 5 1: 1 0 : 1
2 5 2: 1 4 3 : 8
2 5 3: 1 8 12 0 : 21
2 5 4: 1 12 37 34 9 0 : 93
2 5 5: 1 16 78 140 79 0 : 314
\end{verbatim}

The row sums are tabulated in [3, A245013]. If the rectangle is only 2 units wide, the problem is equivalent to counting monomer-dimer coverings of a stripe, and the Fibonacci numbers appear as row sums [3, A011973]:

\begin{verbatim}
2 2 1: 1 : 1
2 2 2: 1 1 : 2
2 2 3: 1 2 : 3
\end{verbatim}
The generating function is

\[
T_2(2, z, t) = \frac{1}{1 - z - z^2t}.
\]

If the rectangle is 3 units wide, each of the $2 \times 2$ squares has one more place to slide sideways, $T_{3 \times m}(2, k) = 2^k T_{2 \times m}(2, k)$ [3, A128099]:

\[
\begin{align*}
2 \ 3 \ 1: & \quad 1 \\
2 \ 3 \ 2: & \quad 1 \ 2 \\
2 \ 3 \ 3: & \quad 1 \ 4 \ 0 \\
2 \ 3 \ 4: & \quad 1 \ 6 \ 4 \ 0 \\
2 \ 3 \ 5: & \quad 1 \ 8 \ 12 \ 0 \\
2 \ 3 \ 6: & \quad 1 \ 10 \ 24 \ 8 \\
2 \ 3 \ 7: & \quad 1 \ 12 \ 40 \ 32 \\
2 \ 3 \ 8: & \quad 1 \ 14 \ 60 \ 80 \\
2 \ 3 \ 9: & \quad 1 \ 16 \ 84 \ 160 \\
2 \ 3 \ 10: & \quad 1 \ 18 \ 112 \ 280 \\
\end{align*}
\]

The row sums are [3, A001045]

\[
T_{3 \times m}(2) = 2^{n+1} + (-1)^n.
\]

The generating function is

\[
T_3(2, z, t) = \frac{1}{1 - z - 2z^2t}.
\]

If the rectangle is 4 units wide, there are options to stack the $2 \times 2$ squares [3, A128101]:

\[
\begin{align*}
2 \ 4 \ 1: & \quad 1 \\
2 \ 4 \ 2: & \quad 1 \ 3 \ 1 \\
2 \ 4 \ 3: & \quad 1 \ 6 \ 4 \ 0 \\
2 \ 4 \ 4: & \quad 1 \ 9 \ 16 \ 8 \\
2 \ 4 \ 5: & \quad 1 \ 12 \ 37 \ 34 \ 9 \\
2 \ 4 \ 6: & \quad 1 \ 15 \ 67 \ 105 \ 65 \\
2 \ 4 \ 7: & \quad 1 \ 18 \ 106 \ 248 \\
2 \ 4 \ 8: & \quad 1 \ 21 \ 154 \ 490 \\
2 \ 4 \ 9: & \quad 1 \ 24 \ 211 \ 858 \\
2 \ 4 \ 10: & \quad 1 \ 27 \ 277 \ 1379 \\
2 \ 4 \ 11: & \quad 1 \ 30 \ 352 \ 2080 \\
2 \ 4 \ 12: & \quad 1 \ 33 \ 436 \ 2988 \\
\end{align*}
\]

The generating function is

\[
T_4(2, z, t) = (-z*t+1) / (-z*t+2*z^2*t+z^3*t^2+z^3*t^3+1-z^2*t^2),
\]

and for the row sums [3, A054854]

\[
T_4(2, z, 1) = (1-z) / (-2*z^2 +2*z^3 -3 +z^-2 +z^-3 +1). \]

If the rectangle is 5 units wide [3, A054855],

\[
\begin{align*}
2 \ 5 \ 1: & \quad 1 \\
2 \ 5 \ 2: & \quad 1 \ 4 \ 3 \\
2 \ 5 \ 3: & \quad 1 \ 8 \ 12 \\
2 \ 5 \ 4: & \quad 1 \ 12 \ 37 \ 34 \ 9.
\end{align*}
\]
with generating function
\[ T_5(2,z,t) = \frac{(-z^2t^2 - zt + 1)}{(-3z^2t^2 - zt + 1)} \]
and with row sums
\[ T_5(2,z,1) = \frac{(-z^2 - z + 1)}{(-7z^2 - 2z + 2z^3 + 3z^4 + 1)}. \]

If the rectangle is 6 units wide \[3, A063650\]

with generating function
\[ T_6(2,z,t) = \frac{(-z^5t^7 + z^4t^6 + 3z^3t^4 + 2z^3t^3 - 2z^2t^3 - 2z^2t^2 - 2zt + 1)}{(24z^8t^{10} + 8z^8t^9 + 24z^7t^8 - 62z^6t^9 - 48z^6t^8 - 38z^5t^7 - 42z^5t^6 - 42z^4t^5 + 42z^4t^4 - 5t^5z^5 + 5z^4t^6 + 30z^3t^7 + 16z^3t^6 + 4zt^8 + 4zt^7 - 11zt^6 - 3t^2z^3 - 3t^2z^2 + 2zt + 1 - z)}, \]
and with row sums
\[ T_6(2,z,1) = \frac{(-4z^6 - 5z^5 - 2zt + 1)}{(32z^6 - 17zt^6 - 85z^5 - 16z^4 + 8zt^3 + 8zt^2 + z^2)}, \]

If it is 7 units wide

with generating function
\[ T_7(2,z,t) = \frac{(-6z^6t^9 + 4z^5t^8 - 6z^5t^7 + 12z^4t^6 - 4z^4t^5 + 3z^3t^4 - 3z^2t^3 - 3z^2t^2 + 2zt + 1)}{(24z^2t^{12} + 8z^2t^{11} + 24z^1t^{10} + 18z^1t^9 - 62z^0t^8 + 48z^0t^7 - 38z^5t^6 - 42z^5t^5 - 42z^4t^4 + 42z^4t^3 - 5t^5z^5 + 5z^4t^6 + 30z^3t^7 + 16z^3t^6 + 4zt^8 + 4zt^7 - 11zt^6 - 3t^2z^3 - 3t^2z^2 + 2zt + 1 - z)}, \]
and
\[ T_7(2,z,1) = \frac{(-6z^6 + 17zt^2 + 2) + 9z^3 - 6z^5)}{(32z^6 + 42z^5 - 116z^4 - 85z^3 + 138z^2 - 30z - 1)}. \]
with generating functions

\[ T_8(2,z,t) = \frac{(1 -85z^9t^{16} -53z^8t^{14} +49t^8z^4 +12z^6t^{11} +206t^{12}z^7 -10z^9t^{17} +2z^6t^{12} -45z^8t^{13} +43z^7t^{13} -16z^8t^{15} +271z^7t^{11} +11z^4t^5 +5t^3z^3 +6z^3t^6 -2z^t^2 -3zt^4 +40z^3t^4 +33z^10t^{10} +12zt^{11}t^{20} -6z^12t^{22} -2zt^4 -189z^5t^8 +12z^4t^7 +37z^3t^5 -7zt^{2}t^{2} -z^8t^{16} +5zt^{10}t^{19} -6z^5t^5 -9z^9t^{15} +39z^10t^{17} +28z^11t^{19} -8zt^{12}t^{21} +75t^{10}z^7 +t^{12}z^8 -85zt^{14}z^9 -2zt^{12}t^{20} +12zt^{18}z^{11} -104zt^{5}z^{7} +17zt^{10}t^{16} -4zt^6 +8zt^{4}t^{4})/(1 +592z^9t^{16} -105z^6t^{10} -63z^8t^{14} +8z^7t^{14} +192t^{2}z^4 +4z^3t^6 +935t^{12}z^7 -3z^8t^{17} -118zt^{9}t^{12} +2zt^{6}t^{12} -20zt^{8}t^{13} +50z^7t^{8} +154z^7t^{13} -44z^8t^{15} +2152z^7t^{11} +207z^4t^5 -426zt^{9}z^6 -23zt^{2}t^{3} +17zt^{3}z^3 +6z^3t^{6} -z -2zt^{2} +3zt^{7} +8zt^{3}t^{4} -88zt^{6}t^{7} -16zt^{12}t^{18} +2zt^{13}t^{20} +110zt^{14}z^{10} +7zt^{16}z^{11} +10zt^{11}t^{21} -5zt^{12}t^{23} +203zt^{10}t^{18} +157zt^{11}t^{20} -69zt^{12}t^{22} -2zt^{9}t^{18} -3zt^{2}t^{4} -607zt^{5}z^{8} +41zt^{4}t^{7} +2zt^{4}t^{7} +60zt^{3}z^{5} -20zt^{2}t^{2} +2zt^{3}z^{2} -4zt^2 +3z^8t^{16} +27zt^{10}t^{19} +zt^{10}t^{20} -12z^5t^{5} +192z^9t^{15} +64zt^{10}t^{17} +75zt^{11}t^{19} +192zt^{12}t^{21} -12z^t^{13}t^{24} +4zt^{14}t^{2} -6zt^{15} +4zt^{4}t^{7} +3zt^{5}t^{4} +44t^{25}z^{14} -100zt^{23}z^{13} +1612zt^{10}z^{7} -41zt^{12}z^{8} +248zt^{14}z^{9} -329zt^{12}t^{20} +54zt^{14}t^{24} +1208zt^{18}z^{11} -720zt^{5}z^{7} -194zt^{13}t^{22} +611zt^{10}t^{16} -219zt^{6}t^{8} -284zt^{6}z^{8} -206zt^{11}z^{8} +67zt^{11}t^{17} -88zt^{13}t^{21} +268zt^{15}t^{10} +16zt^{23}z^{14} -178zt^{12}t^{19} -922zt^{9}t^{13} +519zt^{9}z^{7} +68zt^{4}t^{4} -13zt^{6}t^{6} -22zt^{5}t^{5})}{1 -16zt^{12} +74zt^{4} +59zt^{7} -114zt^{8} -336zt^{9} +31zt^{6} +52zt^{11} +94zt^{10} +88zt^{3} -22zt^{2} -378zt^{5} -5zt^{3} +12zt^{14} -870zt^{12} +514zt^{4} +5430zt^{7} -704zt^{8} -6175zt^{9} -845zt^{6} +2810zt^{11} +171zt^{3} -50zt^{2} -1800zt^{5} -6zt^{3} -392zt^{13}}\]

A subset of these results where \( n = m \) collects the of ways of placing \( 2 \times 2 \) squares

\[
\begin{align*}
T_8(2,z,1) &= (1 -16z^{12} +74z^{4} +59z^{7} -114z^{8} -336z^{9} -31z^{6} +52z^{11} +94z^{10} +88z^{3} -22z^{2} -378z^{5} -5z^{3} +12z^{14} -870z^{12} +514z^{4} +5430z^{7} -704z^{8} -6175z^{9} -845z^{6} +2810z^{11} +171z^{3} -50z^{2} -1800z^{5} -6z^{3} -392z^{13})
\end{align*}
\]

A subset of these results where \( n = m \) collects the of ways of placing \( 2 \times 2 \) squares

into other squares \([3, A193580,A063443]::

\[
\begin{align*}
2.1.1 & : 1 \\
2.2.1 & : 1 2 \\
2.3.3 & : 1 4 0 : 5 \\
2.4.4 & : 1 9 16 8 : 1 35 \\
2.5.5 & : 1 1 6 7 8 140 7 9 0 0 0 : 314 \\
2.6.6 & : 1 25 228 964 1987 1974 978 242 27 1 : 6427 \\
2.7.7 & : 1 36 520 3920 16834 42368 62266 51504 21792 3600 0 0 0 : 202841 \\
2.8.8 & : 1 49 1020 11860 85275 397014 1220928 2484382 3324193 2882737 \backslash \\
& 1601292 569818 129657 18389 1520 64 1 : 12727570 \\
\text{4.2. } 3 \times 3 \text{ Squares. Placing } 3 \times 3 \text{ squares into rectangles yields:}
\end{align*}
\]

\[
\begin{align*}
3.1.1 & : 1 \\
3.2.1 & : 1 \\
\end{align*}
\]
If the rectangle is only 3 units wide, the problem is equivalent to tiling a $1 \times m$ board with monomers and straight trimers, see (9) and [3, A102547]:

The row sums are [3, A000930]

The generating function is

If the rectangle is 4 or 5 units wide, each square has one or two more places to go:

If the rectangle is 4 or 5 units wide, each square has one or two more places to go:

$T_3(3, z, t) = \frac{1}{1 - z - z^3 t}$;

$T_4(3, z, t) = \frac{1}{1 - z - 2z^3 t};$; $T_5(3, z, t) = \frac{1}{1 - z - 3z^3 t}$. 
More generally one may account for the additional freedom with a factor \( n - s + 1 \) for each of the \( k \) squares if the width remains smaller than twice the square’s size:

\[
T_{n \times m}(s, k) = (n - s + 1)^k T_{n \times m}(s, k), \quad s \leq n < 2s.
\]

This is echoed in the generating function (3):

\[
T_n(s, z, t) = \frac{1}{1 - z - (n - s + 1)z^s t}, \quad s \leq n < 2s.
\]

If the rectangle is at least twice as wide as the square, \( n = 2s \), squares may be stacked along the short direction:

\[
\begin{array}{llllll}
3 & 6 & 1: & 1 & : & 1 \\
3 & 6 & 2: & 1 & 0 & : & 1 \\
3 & 6 & 3: & 1 & 4 & 1 & : & 6 \\
3 & 6 & 4: & 1 & 8 & 4 & : & 13 \\
3 & 6 & 5: & 1 & 12 & 9 & 0 & : & 22 \\
3 & 6 & 6: & 1 & 16 & 30 & 12 & 1 & : & 60 \\
3 & 6 & 7: & 1 & 20 & 67 & 50 & 9 & : & 147 \\
3 & 6 & 8: & 1 & 24 & 120 & 128 & 36 & 0 & : & 309 \\
3 & 6 & 9: & 1 & 28 & 189 & 310 & 166 & 26 & 1 & : & 721 \\
3 & 6 & 10: & 1 & 32 & 274 & 660 & 561 & 176 & 16 & : & 1720 \\
3 & 6 & 11: & 1 & 36 & 375 & 1242 & 1461 & 672 & 100 & 0 & : & 3887 \\
3 & 6 & 12: & 1 & 40 & 492 & 2120 & 3362 & 2236 & 600 & 48 & 1 & : & 8900 \\
3 & 6 & 13: & 1 & 44 & 625 & 3358 & 7016 & 6480 & 2721 & 470 & 25 & : & 20740 \\
3 & 6 & 14: & 1 & 48 & 774 & 5020 & 13431 & 16296 & 9438 & 2472 & 225 & 0 & : & 47705 \\
3 & 6 & 15: & 1 & 52 & 939 & 7170 & 23871 & 36880 & 28220 & 10582 & 1713 & 80 & 1 & : & 109509 \\
\end{array}
\]

with generating function

\[
T_6(3,z,t) = \frac{-z^2 t + 1 - z^3 t^2}{-3z^3 t - z^4 t^2 + 1 - z^2 t - 2z^3 t^2 - z^5 t^3 + z^6 t^4 + 2z^5 t^2 + 2z^6 t^3},
\]

and the associated generating function of the row sums

\[
T_6(3,z,1) = \frac{-z^2 + 1 - z^3}{-5z^3 - z^4 + 1 - z^2 + 3z^5 + 3z^6}.
\]

If the rectangle is 7 units wide

\[
\begin{array}{llllll}
3 & 7 & 1: & 1 & : & 1 \\
3 & 7 & 2: & 1 & 0 & : & 1 \\
3 & 7 & 3: & 1 & 5 & 3 & : & 9 \\
3 & 7 & 4: & 1 & 10 & 12 & 0 & : & 23 \\
3 & 7 & 5: & 1 & 15 & 27 & 0 & : & 43 \\
3 & 7 & 6: & 1 & 20 & 67 & 50 & 9 & : & 147 \\
3 & 7 & 7: & 1 & 25 & 132 & 200 & 79 & 0 & : & 437 \\
3 & 7 & 8: & 1 & 30 & 222 & 500 & 314 & 0 & 0 & : & 1067 \\
3 & 7 & 9: & 1 & 35 & 337 & 1075 & 1179 & 333 & 27 & 0 & : & 2987 \\
3 & 7 & 10: & 1 & 40 & 477 & 2050 & 3469 & 2160 & 408 & 0 & : & 8605 \\
3 & 7 & 11: & 1 & 45 & 642 & 3550 & 8309 & 7998 & 2508 & 0 & 0 & : & 23053 \\
3 & 7 & 12: & 1 & 50 & 832 & 5700 & 17449 & 23936 & 13018 & 1820 & 81 & 0 & : & 62887 \\
3 & 7 & 13: & 1 & 55 & 1047 & 8625 & 33264 & 61599 & 52089 & 17218 & 1847 & 0 & 0 & : & 175745 \\
3 & 7 & 14: & 1 & 60 & 1287 & 12450 & 58754 & 140112 & 165540 & 87852 & 16147 & 0 & 0 & : & 482203 \\
\end{array}
\]

with generating function

\[
T_7(3,z,t) = (z^6 t^4 - z^5 t^3 - 3z^3 t^2 - z^4 t^2 + z^2 + 1)/( -3z^9 t^6 + 3z^8 t^5 + 4z^7 t^4 + 10z^6 t^4 + z^7 t^3 + 13z^5 t^3 + 2z^4 t^3 + 2z^5 t^2 - 4z^4 t^2 - 6z^3 t^2 + 2z^2 t^2 - z^2 - 1).
\]

and generating function
T_7(3,z,1) = \( \frac{z^6 - z^5 - 3z^3 - z^4 - z^2 + 1}{-4z^9 + 4z^8 + 5z^7 + 12z^6 + 2z^5 - 4z^4 - 10z^3 - z^2 - z + 1} \)

for the row sums. If the rectangle is 8 units wide

\[
\begin{align*}
3 & 8 1: 1 & 1 \\
3 & 8 2: 1 & 0 & 1 \\
3 & 8 3: 1 & 6 & 6 & 13 \\
3 & 8 4: 1 & 12 & 24 & 0 & 37 \\
3 & 8 5: 1 & 18 & 54 & 0 & 0 & 73 \\
3 & 8 6: 1 & 24 & 120 & 128 & 36 & 0 & 309 \\
3 & 8 7: 1 & 30 & 222 & 500 & 314 & 0 & 1067 \\
3 & 8 8: 1 & 36 & 360 & 1232 & 1246 & 0 & 0 & 2875 \\
\end{align*}
\]

with generating function

\[
T_8(3,z,t) = \frac{-z^8t^5 + 2z^7t^4 + z^5t^3 - 6z^3t^2 + 1 + 5z^6t^4 - 2zt^2 + 6z^7t^3 + 7z^9t^6 - 2z^2t + 5z^6t^4 + 4z^6t^4 + t^3z^7 + 4z^6t^4 + 6z^6t^4 + 13z^5t^3 + z^5t^2 - 7z^4t^2 - 12z^3t^2 - 2z^2t + 1 - z}{6z^{12}t^8 + 6z^{11}t^7 - 13z^{10}t^6 - 31z^9t^6 - 7z^9t^5 - 7z^8t^5 + 2z^8t^4 + 13z^7t^4 + 41z^6t^4 + t^3z^7 + 14z^5t^3 + z^5t^2 - 7z^4t^2 - 12z^3t^2 - 2z^2t + 1 - z}
\]

with row sums

\[
T_8(3,z,1) = \frac{-z^8 + 2z^7 + z^5 - 6z^3 + 1 + 6z^6 - 2z^2 - z^4 - z^9}{6z^{12} + 6z^{11} - 13z^{10} - 38z^9 - 5z^8 + 14z^7 + 45z^6 + 14z^5 - 7z^4 - 16z^3 - 2z^2 + 1 - z}
\]

If the rectangle is 9 units wide

\[
\begin{align*}
3 & 9 0: 1 & 1 \\
3 & 9 1: 1 & 0 & 1 \\
3 & 9 2: 1 & 0 & 0 & 1 \\
3 & 9 3: 1 & 7 & 10 & 1 & 19 \\
3 & 9 4: 1 & 14 & 40 & 8 & 63 \\
3 & 9 5: 1 & 21 & 90 & 27 & 0 & 139 \\
3 & 9 6: 1 & 28 & 189 & 310 & 166 & 26 & 1 & 721 \\
3 & 9 7: 1 & 35 & 337 & 1075 & 1179 & 333 & 27 & 0 & 2987 \\
3 & 9 8: 1 & 42 & 534 & 2540 & 4316 & 1740 & 216 & 0 & 9389 \\
3 & 9 9: 1 & 49 & 780 & 5048 & 13211 & 11984 & 4526 & 758 & 51 & 1 & 36409 \\
3 & 9 10: 1 & 56 & 1075 & 8942 & 33356 & 53062 & 37007 & 11116 & 1444 & 64 & 0 & 146123 \\
\end{align*}
\]

with generating function

\[
T_9(3,z,t) = \frac{(1 + 2z^8t^4 + 6z^10t^6 - 12z^3t^2 - 2z^2t + 5t^10z^14 - 5z^3t^3 - 2z^4t^3 + 10z^6t^6 - t^14z^18 + 40z^6t^6 - 7z^15t^15 - 19z^11t^9 + 32z^12t^10 - 50z^10t^8 - 53z^7t^7 - 2z^9t^6 + 23z^13t^10 + 6z^14t^11 + 13t^5z^8 + 12t^4z^7 - 4z^12t^9 - 32t^8z^11 + z^17t^16 - z^21t^19 + 3z^18t^17 + z^19t^17 - z^20t^17 + 7z^18t^16 + z^20t^18 + 5z^12t^12 - 19z^15t^14 - 57z^9t^8 - 29t^7z^10 + 12z^7t^6 - 10z^9t^9 + 12z^13t^11 + 13z^14t^12 - 3z^10t^9 + 8z^11t^10 + 45z^12t^11 - 5z^14t^13 - 2z^13t^12 - 4z^4t^2 + 40z^7t^5 + 8z^8t^6 + 32z^6t^4 + 44z^9t^5 - 7z^4t^3 + 5z^5t^4 - 7z^8t^7 + 2zt^3z^6 + 5t^14z^17 + t^13z^17 - 14t^13z^15 + t^14z^16 + t^15z^18 - 4t^13z^16 + 2t^11z^15 - 4z^16t^15 - 4z^13t^9 - 4z^9t^11t^7 - t^8z^12)/(1 + 14z^8t^4 - 110z^10t^6 - 22z^3t^2 + 34z^9t^14 - 2z^2t + 264t^10z^14 - 6z^3t^3 - 5z^3t^3 + 52t^8z^13 + 29zt^11z^16 + 13z^5t^3 + 15z^6t^6 - 19zt^{14}z^18 + 6zt^{10}z^{15} - z^2z^7t^4 + 3z^2t^21t^20 - z^7t^21 - 13z^2z^{21}t^16 - 5z^21t^21 + 7z^21t^16 + 7z^23t^19 + 3z^16t^15 - 7z^20t^15 + 96z^6t^7 + 6z^15t^{15} + 34z^11t^9 + 526z^12t^{10} - 11z^10t^8 + z^18t^15 - 38t^12z^17 - 435z^9t^7 - 2z^5t^2 - 282z^9t^6 + 446z^13t^10}
\]
+190*z^14*t^11 +70*t^5*z^8 +84*t^4*z^7 +417*z^12*t^9 -120*t^8*z^11 -36*z^20*t^16 -z^22*t^20 -51*z^21*t^18 +6*z^17*t^16 -29*z^21*t^19 +29*z^18*t^17 -3*z^19*t^17 +21*z^20*t^17 +153*z^18*t^16 -2*z^20*t^18 +15*z^12*t^12 -99*z^15*t^14 -z^20*t^19 +z^24*t^22 -15*t^13*z^18 -176*z^9*t^8 -431*t^7*z^10 +z^19*t^14 +27*z^7*t^6 -20*z^9*t^9 +142*z^6*t^4 -38*z^12*t^7 -23*z^11*t^6 +4*z^7*t^3 -8*z^8*t^6 +142*z^13*t^11 -3*z^10*t^9 +15*z^11*t^10 +175*z^12*t^15 -81*z^17*t^15 +254*z^13*t^9 -147*z^11*t^7 -1 +z^19 -25*z^20 -158*z^17 +868*z^13 +260*z^12 +119132920 52175594 12725724 1828210 152908 6884 144 1 : 461938624

More row sums are in [3, A140304].

The geometries of placing squares into squares of twice the edge length, \( n = m = 2s \), are with (5)

\[
T_{2s \times 2s}(s, 1) = (s + 1)^2,
\]

with (7)

\[
T_{2s \times 2s}(s, 4) = 1,
\]

and otherwise counted by considering the few number of constellations where all squares touch the bigger square:

\[
T_{2s \times 2s}(s, 2) = 2s(s + 2), \quad T_{2s \times 2s}(s, 3) = 4s.
\]
If the rectangle is 4 units wide, the problem is equivalent to placing 1-ominoes and straight tetrominos on a line, see (9) and [3, A180184]:

If the rectangle is 4 units wide, the problem is equivalent to placing 1-ominoes and straight tetrominos on a line, see (9) and [3, A180184]:

The generating function is

\[ T_4(4, z, t) = \frac{1}{1 - z - z^4 t} \]

\[ T_n(4, z, t) \] in the range \( n = 5 \ldots 7 \) are given by increasing the factor in front of the \( t \) in the denominator, see (18) and (19). The row sums are given by (14) and [3, A003269]. If the rectangle is 8 units wide,
with generating function
\[ T_8(4,z,t) = \frac{(-z^2 t + 1 + z^6 t^3 - z^4 t^2 - z^3 t)}{2 z^7 t^3 + 3 z^7 t^2 - z - z^5 t^2 - 4 z^4 t - z^2 t + 1 + 2 z^6 t^3 - 2 z^4 t^2 - z^10 t^5 + z^8 t^4 + 3 z^6 t^2 - 3 z^10 t^4 + 3 z^8 t^3} \]
and associated generating function of the row sums
\[ T_8(4,z,1) = \frac{(-z^2 + 1 + z^6 - z^4 - z^3)}{5 z^7 - z - z^5 - 6 z^4 - z^2 + 1 + 5 z^6 - 4 z^10 + 4 z^8}. \]

If the rectangle is 9 units wide,
\[ T_9(4,z,t) = \frac{(-z^7 t^3 - 4 z^4 t^2 - z^6 t^2 - 2 z^5 t^2 + 4 z^8 t^4 + 3 z^9 t^4 - z^10 t^5 + z^6 t^3 - z^3 t - z^2 t + 1 - z^12 t^6)}{(1 + 16 z^8 t^4 + 7 z^8 t^3 + 6 z^6 t^3 - z^2 t + 5 z^7 t^3 + 2 z^11 t^5 + z^6 t^2 - z + 13 z^9 t^4 - 7 z^4 t^2 - 4 z^5 t + 5 z^4 t + 3 z^3 t - 13 z^12 t^6 - 10 z^10 t^5 - 8 z^12 t^5 + 2 z^14 t^6 - 10 z^13 t^6 + 3 z^14 t^7 + 3 z^16 t^8 - 6 z^13 t^5 + 2 z^16 t^7 + 2 z^11 t^4 + 2 z^10 t^3)} \]
with row sums
\[ T_9(4,z,1) = \frac{(-z^7 - 4 z^4 + 2 z^6 - 2 z^5 + 4 z^8 + 3 z^9 - z^10 - z^3 - z^2 + 1 + z^12)}{(1 - z - z^2 - 12 z^4 - 4 z^5 + 7 z^6 + 23 z^7 + 17 z^9 + 6 z^7 + 4 z^11 - 8 z^10 - 21 z^12 + 5 z^14 - 16 z^13 + 5 z^16)}. \]

In overview, this is the number of ways of placing 4 × 4 squares into other squares:
### 4.4. $5 \times 5$ Squares

Placing $5 \times 5$ squares into rectangles yields:

| $m$ | $n$ | $C(m,n)$ | $T(m,n)$ | $A(m,n)$ | $R(m,n)$ | $S(m,n)$ |
|-----|-----|-----------|----------|----------|----------|----------|
| 5   | 1   | 1         | 1        | 1        | 1        | 1        |
| 5   | 2   | 1         | 1        | 1        | 1        | 1        |
| 5   | 3   | 1         | 1        | 1        | 1        | 1        |
| 5   | 4   | 1         | 1        | 1        | 1        | 1        |
| 5   | 5   | 1         | 1        | 1        | 1        | 1        |

If the rectangle is 5 units wide, the problem is equivalent to placing 1-ominos and straight pentominoes on a line, see (9), (16) and [3, A003520]:

If the rectangle is 5 units wide, the problem is equivalent to placing 1-ominos and straight pentominoes on a line, see (9), (16) and [3, A003520]:
If the rectangle is 6 to 9 units wide, the counts are described by (18) and (19). If it is 10 units wide,

with generating function

\[ T_{10}(5,z,t) = \frac{-2z^5t^2 -z^3t +1 +z^{10}t^4 +z^8t^3 -z^6t^2 -z^4t}{-z^{15}t^6 -4z^{15}t^5 -t^5z^{13} -4z^{13}t^4 +2z^{11}t^4 +3z^{10}t^4 +4z^{11}t^3 +8z^{10}t^3 +2z^9t^3 +2z^8t^3 +4z^9t^2 +4z^8t^2 -z^7t^2 -z^6t^2 -3z^5t^2 -5z^5t -z^3t +1 -z} \]

and for row sums

\[ T_{10}(5,z,1) = \frac{-2z^5 -z^3 +1 +z^{10} +z^8 -z^6 -z^4}{-5z^{15} -5z^{13} +6z^{11} +11z^{10} +6z^9 +6z^8 -z^7 -z^6 -8z^5 -z^3 +1 -z}. \]

If the rectangle is 11 units wide,
with generating function
\[ T_{11}(5,z,t) = \frac{(11z^{10}t^4 + 2z^8t^3 - z^8t^2 - z^{10}t^3 + z^{20}t^8 + 4z^{12}t^4 - 6z^{15}t^6 - 3z^{9}t^3 - z^4t + 2z^{13}t^5 - 4z^{16}t^6 + 7z^{11}t^4 - 2z^7t^2 + 1 - 6z^5t^2 - z^3t - z^{18}t^7 + 3z^{14}t^5)/(1 - 4z^{13}t^5 - 6z^5t - 9z^5t^2 - 4z^7t^2 + 16z^{10}t^3 + 14z^14t^5 + 28z^{11}t^4 - 4z^6t^2 + 29z^{10}t^4 + 4z^9t^3 - 39z^{15}t^6 - 3z^3t + 5z^{8}t^3 - 27z^{15}t^5 + 3z^{15}t^4 + 9z^{14}t^4 - 31z^{16}t^4 - 19z^{16}t^6 + 19z^{20}t^8 - 6z^7t^2 + 29z^{10}t^4 + 4z^9t^3 - 3z^{25}t^8 - 13zt^{25}t^8 + 16z^{17}t^6 + 13z^21t^8 + 2z^8t^2 + 15z^{12}t^4 - 3z^{25}t^9 + 12z^{21}t^7 + 2z^9t^2 + 3z^{13}t^3 - 12z^{17}t^5)}{1 - z - 4z^{13}t^5 - 6z^5t - 9z^5t^2 - 4z^7t^2 + 16z^{10}t^3 + 14z^14t^5 + 28z^{11}t^4 - 4z^6t^2 + 29z^{10}t^4 + 4z^9t^3 - 39z^{15}t^6 - 3z^3t + 5z^{8}t^3 - 27z^{15}t^5 + 3z^{15}t^4 + 9z^{14}t^4 - 31z^{16}t^4 - 19z^{16}t^6 + 19z^{20}t^8 - 6z^7t^2 + 29z^{10}t^4 + 4z^9t^3 - 3z^{25}t^8 - 13zt^{25}t^8 + 16z^{17}t^6 + 13z^21t^8 + 2z^8t^2 + 15z^{12}t^4 - 3z^{25}t^9 + 12z^{21}t^7 + 2z^9t^2 + 3z^{13}t^3 - 12z^{17}t^5). \]

and with row sums
\[ T_{11}(5,z,1) = \frac{(10z^{10} + z^8 + z^{12} - 6z^{15} - 3z^9 - z^4 + 2z^{13} - 4z^6 - 4z^{11} + 7z^{11} - 2z^7 + 1 - 6z^5 - z^3 - z^{18} + 3z^{14})/(1 + 37z^{20} - 6z^25 - z + 6z^{23} - 19z^{19} - 13z^{18} + 23z^{14} + 7z^8 + 21z^{12} - 63z^{15} + 6z^9 - 4z^6 - 52z^{16} + 45z^{10} + 39z^{11} - 4z^7 - 15z^5 - z^3 - 28z^{17} + 25z^{21})}{1 + 37z^{20} - 6z^25 - z + 6z^{23} - 19z^{19} - 13z^{18} + 23z^{14} + 7z^8 + 21z^{12} - 63z^{15} + 6z^9 - 4z^6 - 52z^{16} + 45z^{10} + 39z^{11} - 4z^7 - 15z^5 - z^3 - 28z^{17} + 25z^{21}). \]

Here is the number of ways of placing $5 \times 5$ squares into other squares:

\begin{align*}
5 & 1 1: 1 \ 1 \\
5 & 2 2: 1 \ 1 \\
5 & 3 3: 1 \ 1 \\
5 & 4 4: 1 \ 1 \\
5 & 5 5: 1 \ 1 \\
5 & 6 6: 1 \ 4 \ 5 \\
5 & 7 7: 1 \ 9 \ 10 \\
5 & 8 8: 1 \ 16 \ 0 \ 17 \\
5 & 9 9: 1 \ 25 \ 0 \ 0 \ 26 \\
5 & 10 10: 1 \ 36 \ 70 \ 20 \ 1 \ 128 \\
5 & 11 11: 1 \ 49 \ 276 \ 320 \ 79 \ 725 \\
5 & 12 12: 1 \ 64 \ 696 \ 1904 \ 1246 \ 0 \ 3911 \\
5 & 13 13: 1 \ 81 \ 1420 \ 7200 \ 9550 \ 0 \ 0 \ 18252 \\
5 & 14 14: 1 \ 100 \ 2550 \ 20900 \ 48175 \ 0 \ 0 \ 0 \ 71726 \\
5 & 15 15: 1 \ 121 \ 4200 \ 52140 \ 214680 \ 153190 \ 37040 \ 3476 \ 117 \ 0 \ 464966 \\
5 & 16 16: 1 \ 144 \ 6496 \ 115104 \ 790396 \ 1729976 \ 1479306 \ 532572 \ 78192 \ 3600 \ 0 \ 4735787 \\
\end{align*}

4.5. $6 \times 6$ Squares. If the rectangle is 6 units wide, the problem is equivalent to placing 1-ominos and straight 6-ominos on a line, see (9) and [3, A005708]:

\begin{align*}
6 & 1 1: 1 \ 1 \\
6 & 2 2: 1 \ 1 \\
6 & 3 3: 1 \ 1 \\
6 & 4 4: 1 \ 1 \\
6 & 5 5: 1 \ 1 \\
6 & 6 6: 1 \ 1 \ 2 \\
6 & 7 7: 1 \ 2 \ 3 \\
6 & 8 8: 1 \ 3 \ 4 \\
6 & 9 9: 1 \ 4 \ 5 \\
6 & 10 10: 1 \ 5 \ 6 \\
\end{align*}
If the rectangle is 7 to 11 units wide, the counts are described by (18) and (19). If the rectangle is 12 units wide,

with generating function

\[
T_{12}(6,z,t) = \frac{-2z^6t^2 - z^4t + 2z^9t^3 + 1 - z^3t + z^{12}t^4 + 2z^{10}t^3 - z^{15}t^5 - z^7t^2 - z^5t}{1 + z^{21}t^7 + 3z^9t^3 - z^3t + 2z^{13}t^4 - z^8t^2 + 10z^{12}t^3 - 5z^{18}t^5 + 5z^{13}t^3 + 3z^{11}t^3 - z^{18}t^6 + 5z^{11}t^2 + 5z^{21}t^6 - 3z^{15}t^5 - 10z^{15}t^4 + 5z^9t^2 - z^7t^2 - 3z^6t^2 + 5z^{10}t^2 - 6z^6t - 3z^{16}t^5 + 3z^{10}t^3 + 3z^{12}t^4 - 10z^{16}t^4}{1 + 8z^{10} + 6z^{21} - 13z^{16} - 6z^{18} - z - z^3 + 8z^9 - z^8 + 7z^{13} - 9z^{10} - 13z^5 + 8z^{11} - z^7 + 13z^{12}}.
\]

and with row sums

\[
T_{12}(6,z,1) = \frac{-2z^6 - z^4 + 2z^9 + 1 - z^3 + z^{12} + 2z^{10} - z^{15} - z^7 - z^5}{1 + 8z^{10} + 6z^{21} - 13z^{16} - 6z^{18} - z - z^3 + 8z^9 - z^8 + 7z^{13} - 9z^{10} - 13z^5 + 8z^{11} - z^7 + 13z^{12}}.
\]

If the rectangle is 13 units wide,
with generating function
\[ T_{13}(6,z,t) = \frac{(1 + 6z^{-21}t^7 + 7z^{-24}t^8 - z^{-27}t^9 - 18z^{-19}t^6 - z^{-30}t^{10} + 2z^{-22}t^7 + 13z^{-14}t^4 + 6z^{-25}t^8 + 16z^{-13}t^4 - 2z^{-16}t^5 + 2z^{-17}t^5 - 11z^{-20}t^6 - 17z^{-18}t^6 - 11z^{-15}t^5 - 5z^{-12}t^3 + 6z^{-15}t^4 - t^{-2}z^{10} - 2t^{-2}z^9 - z^{-3}t + 6z^{-9}t^3 - z^{-5}t - 7z^{-6}t^2 - 5z^{-7}t^2 - z^{-4}t + 17z^{-12}t^4 - 3z^{-11}t^3 - 3z^{-8}t^2 + 5z^{-10}t^3)/(1 + 4t^3z^{16} - z + 39z^{21}t^7 + 58z^{24}t^8 - 19z^{-27}t^9 - 81z^{-19}t^6 - 22z^{-30}t^{10} - 24z^{-27}t^8 - 64z^{-19}t^5 + 68z^{-24}t^7 - 28z^{-30}t^9 + 8z^{-22}t^6 + 12z^{-14}t^3 + 72z^{-25}t^7 + 14z^{-22}t^7 + 30z^{-14}t^4 + 67z^{-25}t^8 + 40z^{-13}t^4 - 26z^{-16}t^5 + 11z^{-17}t^5 - 66z^{-20}t^6 + 15z^{-21}t^6 - 68z^{-18}t^6 - 29z^{-15}t^5 + 26z^{-12}t^3 - 5z^{-15}t^4 + 3t^{-2}z^9 - 7z^{-6}t - 3t^{-4}t^2 + 12t^{-4}z^{-17} - 8t^{-6}t^2 - 4z^{-2}z^7 - 8t^{-6}t + 9z^{-9}t^3 - 63z^{-18}t^5 + 19z^{-13}t^3 - 10z^{-6}t^2 + 39z^{-26}t^8 + 44z^{-26}t^7 - 24t^{-5}t^2 + 52t^{-5}t^2 + 4^{-2}z^{-7}t^2 + 4z^{-3}t^7 + 11 + 4z^{-3}t^3 + 10 - 11t^{-1}t^4 + 12t^{-4}z^{-16} + 4z^{-18}t^4 + 4z^{-3}t^3 + 10 - 7z^{-28}t^9 - 38z^{-12}t^4 + 7z^{-11}t^3 + 3t^{-2}z^9 - 19z^{-31}t^10 - 24z^{-31}t^9 + 6z^{-15}t^3 - 4z^{-8}t^2 + 8z^{-10}t^3) \]

and with row sums
\[
T_{13}(6,z,t) = (1 - 3z^{-11} - 3z^{-8} + 6z^{-21} + 7z^{-24} - 18z^{-19} + 2z^{-22} + 13z^{-14} + 6z^{-25} + 16z^{-13} - 2z^{-16} + 2z^{-17} - 11z^{-20} - 17z^{-18} - 5z^{-15} + 16z^{-12} + 4z^{-10} + 4z^{-9} - z^{-3} + z^{-5} - 7z^{-6} - 5z^{-7} - z^{-4} - z^{-30})/(1 - z + 10z^{-11} - 4z^{-8} + 5z^{21} + 126z^{-24} - 43z^{-27} - 145z^{-19} + 22z^{-22} + 42z^{-14} + 139z^{-25} + 59z^{-13} - 33z^{-16} + 23z^{-17} - 118z^{-20} - 127z^{-18} - 26z^{-15} + 64z^{-12} + 11z^{-10} + 12z^{-9} - 3^{-2}z^{-6} - 4z^{-7} - 50z^{-30} + 7z^{-36} - 15z^{-28} - 12z^{-23} + 83z^{-26} + 7z^{-33} - 43z^{-31}) .
\]

This is the number of ways of placing 6 \times 6 squares into other squares:

| Size   | Count | Count | Count | Count | Count | Count | Count | Count | Count | Count | Count | Count | Count | Count |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 6 1 1  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 2 2  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 3 3  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 4 4  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 5 5  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 6 6  | 1 1   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 7 7  | 1 4   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 8 8  | 1 9   |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 6 9 9  | 1     | 16    | 0     | 17    |       |       |       |       |       |       |       |       |       |       |
| 6 10 10| 1     | 25    | 0     | 26    |       |       |       |       |       |       |       |       |       |       |
| 6 11 11| 1     | 36    | 0     | 0     | 37    |       |       |       |       |       |       |       |       |       |
| 6 12 12| 1     | 49    | 96    | 24    | 1     | 171   |       |       |       |       |       |       |       |       |
| 6 13 13| 1     | 64    | 366   | 380   | 79    | 890   |       |       |       |       |       |       |       |       |
| 6 14 14| 1     | 81    | 900   | 2240  | 1246  | 0     | 4468  |       |       |       |       |       |       |       |
| 6 15 15| 1     | 100   | 1800  | 8400  | 9550  | 0     | 0     | 19851 |       |       |       |       |       |       |
| 6 16 16| 1     | 121   | 3180  | 24200 | 48175 | 0     | 0     | 0     | 75677 |       |       |       |       |       |
| 6 17 17| 1     | 144   | 5166  | 58604 | 184681| 0     | 0     | 0     | 0     | 248596 |       |       |       |       |
| 6 18 18| 1     | 169   | 7986  | 127484| 648301| 410086| 82594 | 6146   | 159   | 1     | 128283 |       |       |
| 6 19 19| 1     | 196   | 11520 | 253920| 2016380| 4105216| 3013174 | 885104 | 102656 | 3600  | 0     | 10391767 |       |       |
4.6. $7 \times 7$ Squares. If the rectangle is 7 units wide, the problem is equivalent to placing 1-ominoes and straight 7-ominoes on a line, see (9) and [3, A005709]:

- $7\times1$: 1
- $7\times2$: 1
- $7\times3$: 1
- $7\times4$: 1
- $7\times5$: 1
- $7\times6$: 1
- $7\times7$: 1
- $7\times8$: 1
- $7\times9$: 1
- $7\times10$: 1
- $7\times11$: 1
- $7\times12$: 1
- $7\times13$: 1
- $7\times14$: 1
- $7\times15$: 1
- $7\times16$: 1
- $7\times17$: 1
- $7\times18$: 1
- $7\times19$: 1
- $7\times20$: 1
- $7\times21$: 1
- $7\times22$: 1

This is the number of ways of placing $7 \times 7$ squares into other squares:

- $7\times1$: 1
- $7\times2$: 1
- $7\times3$: 1
- $7\times4$: 1
- $7\times5$: 1
- $7\times6$: 1
- $7\times7$: 1
- $7\times8$: 1
- $7\times9$: 1
- $7\times10$: 1
- $7\times11$: 1
- $7\times12$: 1
- $7\times13$: 1
- $7\times14$: 1
- $7\times15$: 1
- $7\times16$: 1
- $7\times17$: 1
- $7\times18$: 1
- $7\times19$: 1
- $7\times20$: 1
- $7\times21$: 1
- $7\times22$: 1

4.7. $8 \times 8$ and larger Squares. This is the number of ways of placing $8 \times 8$ squares into other squares:

- $8\times1$: 1
- $8\times2$: 1
- $8\times3$: 1
- $8\times4$: 1
- $8\times5$: 1
This is the number of ways of placing 9 \times 9 squares into other squares:

1 1: 1
2 2: 1
3 3: 1
4 4: 1
5 5: 1
6 6: 1
7 7: 1
8 8: 1
9 9: 1
10 10: 1
11 11: 1
12 12: 1
13 13: 1
14 14: 1
15 15: 1
16 16: 1
17 17: 1
18 18: 1
19 19: 1
20 20: 1
21 21: 1
22 22: 1
23 23: 1
24 24: 1
25 25: 1

This is the number of ways of placing 10 \times 10 squares into other squares:

1 1: 1
2 2: 1
3 3: 1
The data base constructed above allows some extrapolations while $s > 1$:

**Conjecture 1.**

(24) $T_{2s \times (2s+1)}(s, 2) = 1 + 10s + 4s^2$.

(25) $T_{2s \times (2s+1)}(s, 3) = 2 + 16s$.

(26) $T_{2s \times (2s+1)}(s, 4) = 9$.

While $s > 2$:

**Conjecture 2.**

(27) $T_{2s \times (2s+2)}(s, 2) = 3 + 18s + 7s^2$.

(28) $T_{2s \times (2s+2)}(s, 3) = 8 + 40s$.

(29) $T_{2s \times (2s+2)}(s, 4) = 36$. 
Appendix A. Java Program for a Maple Generator

The two Java programs Hei.java and Tmat.java in the anc subdirectory emit a Maple program that in principle generates the inverse of $1 - M(z,t)$, where $1$ is the unit matrix and $M$ the transfer matrix. Since we are only interested in the top left element of the inverse—which is the bivariate generating function—, effectively only a linear system of equations is set up. The programs are compiled with

```
javac -cp . *.java
```

and the output of the main program of Tmat can be directly piped into Maple:

```
java -cp . Tmat s n | maple -q
```

The two command line options $s$ and $n$ are the edge length of the squares and the width of the rectangle. The programs are licensed under the LGPLv3.

References

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Königstuhl 17, 69117 Heidelberg, Germany