Analytical and Numerical Study of the Aharonov–Bohm Effect in $3D$ and $4D$ Abelian Higgs Model

M.N. Chernodub, F.V. Gubarev and M.I. Polikarpov

ITEP, B.Cheremushkinskaya 25, Moscow, 117259, Russia

Abstract

We discuss the Aharonov–Bohm effect in three and four dimensional non-compact lattice Abelian Higgs model. We show analytically that this effect leads to the long-range Coulomb interaction of the charged particles, which is confining in three dimensions. The Aharonov–Bohm effect is found in numerical calculations in $3D$ Abelian Higgs model.

1 Introduction

It is well known that the four dimensional Abelian Higgs model has classical solutions called Abrikosov–Nielsen–Olesen strings [1]. These strings carry quantized magnetic flux and the wave function of the charged particle which is scattered on this string acquires additional phase. The shift in the phase is the physical effect which is the field–theoretical analog of the quantum–mechanical Aharonov–Bohm effect [2]: strings play the role of solenoids, which scatter charged particles. Topological long-range Aharonov–Bohm interaction between the strings and particles was discussed in papers [3, 4, 5].

The three dimensional Abelian Higgs model has the particle–like solutions usually called Abrikosov vortices. Analogously to the four dimensional case, the charged particle which is scattered on the vortex acquire the additional Aharonov–Bohm phase.

In Section 2 we study the interaction of the charged particles in the non-compact lattice Abelian Higgs model. We show that the Aharonov–Bohm effect gives rise to the long–range Coulomb–like interaction between test particles. In the three dimensional case the induced potential is confining, since the Coulomb interaction rises logarithmically.

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In Section 3 we present the results of the numerical calculations in the 3D lattice Abelian Higgs model with the non-compact gauge field. Our numerical results show the existence of the Aharonov–Bohm effect in this model.

2 Potential Induced by Aharonov–Bohm Effect

The partition function of the $D$ dimensional non-compact Abelian Higgs model on the lattice is (we are using the formalism of the differential forms on the lattice, see [5, 6] for the brief introduction):

$$Z = \int_{-\infty}^{+\infty} DA \int_{-\pi}^{+\pi} D\varphi \sum_{l(c_1) \in \mathbb{Z} Z} \exp \left\{ -\beta \|dA\|^2 - \gamma \|d\varphi + 2\pi l - NA\|^2 \right\},$$

where $A$ is the non-compact gauge field, $\varphi$ is the phase of the Higgs field and $l$ is the integer-valued one-form. For simplicity we consider the limit of the infinite Higgs boson mass, then the radial part of the Higgs field is frozen and we use the Villain form of the action.

One can rewrite [5] the integral (1) as the sum over the closed vortex ($D = 3$) or string ($D = 4$) trajectories using the analogue of Berezinski–Kosterlitz–Thouless (BKT) transformation [7]:

$$Z \propto Z^{BKT} = \text{const.} \sum_{*\sigma(\tau^{D-2}) \in \mathbb{Z}^*} \exp \left\{ -4\pi^2 \gamma \left( (*\sigma, (\Delta + m^2)^{-1} d^*\sigma) \right) \right\},$$

where $m^2 = N^2 \gamma / \beta$ is the classical mass of the vector boson $A$. Closed currents $*\sigma$ which are defined on the dual lattice represent the vortex trajectory ($D = 3$) or the string world sheet ($D = 4$). It can be easily seen from eq.(2) that the currents $*\sigma$ interact with each other through the Yukawa forces.

Using the same transformation for the quantum average of the Wilson loop $W_M(C) = \exp\{i M(A, j_c)\}$ leads to the following formula [4, 5, 8]:

$$< W_M(C) >_N = \frac{1}{Z^{BKT}} \sum_{*\sigma(\tau^{D-2}) \in \mathbb{Z}^*} \exp \left\{ -4\pi^2 \gamma \left( (*\sigma, (\Delta + m^2)^{-1} d^*\sigma) \right) \right\}$$

$$\quad - \frac{M^2}{4\gamma} (j_c, (\Delta + m^2)^{-1} j_c) - 2\pi i \frac{M}{N} \left( *j_c, (\Delta + m^2)^{-1} d^*\sigma \right) + 2\pi i \frac{M}{N} \text{IL}( *\sigma, j_c).$$

The first three terms in this expression are short-range Yukawa forces between defects and particles. The last long-range term has the topological origin: $\text{IL}( *\sigma, j_c)$ is the linking number between the world trajectories of the defects $*\sigma$ and the Wilson loop $j_c$:

$$\text{IL}( *\sigma, j_c) = ( *j_c, (\Delta^{-1} d^*\sigma) ).$$
In three (four) dimensions the trajectory of the vortex (string) \( \ast \sigma \) is a closed loop (surface) and the linking number \( IL \) is equal to the number of points at which the loop \( j_C \) intersects the two (three) dimensional volume bounded by the loop (surface) \( \ast \sigma \). The equation (3) is the lattice analogue of the Gauss formula for the linking number. This topological interaction corresponds to the Aharonov–Bohm effect in the field theory [3, 4, 5, 8].

In the limit

\[ N^2 \gamma \gg 1 \gg \beta, \]

the partition function is:

\[ Z_{0}^{BKT} = \sum_{\ast \sigma, (c_{D-2}) \in \mathbb{Z}} \exp \left\{ - \frac{4\pi^2 \beta}{N^2} \| \ast \sigma \|^2 \right\}. \]

In the corresponding three (four) dimensional continuum theory the term \( \| \ast \sigma \|^2 \) is proportional to the length (area) of the trajectory \( \ast \sigma \), therefore in the limit (5) the vortices (strings) are free. The quantum average (3) of the Wilson loop in the limit (5) is:

\[ <W_M(C)>_N = \frac{1}{Z_{0}^{BKT}} \sum_{\ast \sigma, (c_{D-2}) \in \mathbb{Z}} \exp \left\{ - \frac{4\pi^2 \beta}{N^2} \| \ast \sigma \|^2 + 2\pi i \frac{M}{N} IL (\ast \sigma, j_C) \right\}. \]

This formula describes the Aharonov–Bohm interaction of the free vortices (strings) carrying the flux \( \frac{2\pi}{N} \) with the test particle of the charge \( M \). Therefore the interaction between the charged particles is due to Aharonov–Bohm effect only.

In order to get the explicit expression for \( <W_M(C)>_N \) it is convenient to rewrite eq.(7) in the dual representation [8]:

\[ Z_{0}^{BKT} \propto Z_{0}^{dual} = \sum_{l(c_1) \in \mathbb{Z}} \exp \left\{ - \frac{N^2}{4\beta} \| l \|^2 \right\}. \]

The quantum average (8) in the dual representation is:

\[ <W_M(C)>_N = \frac{1}{Z_{0}^{dual}} \sum_{l(c_1) \in \mathbb{Z}} \exp \left\{ - \frac{N^2}{4\beta} \left( l - \frac{M}{N} j_C, \Delta^{-1} \left( l - \frac{M}{N} j_C \right) \right) \right\}. \]
Using the saddle point approximation \((N^2/\beta \gg 1)\), we get in the leading order:

\[
< W_M(C) >_N = \text{const.} \exp \left\{ -\kappa^{(0)}(M, N; \beta) \left( j_C, \Delta^{-1}j_C \right) \right\},
\]

(10)

where

\[
\kappa^{(0)}(M, N; \beta) = \frac{q^2 N^2}{4\beta},
\]

(11)

and

\[
q = \min_{K \in \mathbb{Z}} \left| \frac{M}{N} - K \right|,
\]

(12)

\(q\) is the distance between the ratio \(M/N\) and a nearest integer number. Expressions (10–12) depend on the fractional part of \(M/N\), this is the consequence of the Aharonov–Bohm effect. Interaction of the testing charges is absent if \(q = 0\) (\(M/N\) is integer), this corresponds to the complete screening of the charge \(M\) by the charge \(N\) of the Higgs bosons.

Let us consider the product of two Polyakov lines: \(W_M(C) = L^+_M(0) \cdot L_M(R)\). Then \((j_C, \Delta^{-1}j_C) = 2T \Delta^{-1}_{(D-1)}(R)\), where \(\Delta^{-1}_{(D)}(R)\) is the \(D\)-dimensional massless lattice propagator, and (11) is reduced to:

\[
< L^+_M(0) L_M(R) >_N = \text{const.} \exp \left\{ -2\kappa^{(0)}(M, N; \beta) T \Delta^{-1}_{(2)}(R) \right\}.
\]

(13)

At large \(R\) the propagator \(\Delta^{-1}_{(2)}(R)\) rises logarithmically: \(\Delta^{-1}_{(2)}(R) = \frac{c_3}{2} \ln R + \ldots\), and the propagator \(\Delta^{-1}_{(3)}(R)\) is inversely proportional to \(R\): \(\Delta^{-1}_{(3)}(R) = \frac{c_4}{2} R^{-1} + \ldots\), where \(c_3\) and \(c_4\) are some numerical constants. Thus, the Aharonov–Bohm effect in three and four dimensional Abelian Higgs model gives rise to the following long–range potentials:

\[
V_{D=3}(R) = c_3 \kappa^{(0)}(M, N; \beta) \cdot \ln R + O \left( R^{-1} \right),
\]

(14)

\[
V_{D=4}(R) = c_4 \kappa^{(0)}(M, N; \beta) \cdot \frac{1}{R} + O \left( R^{-2} \right).
\]

(15)

A 3D vortex model in the continuum was discussed in paper [9] and it was shown that the potential has the form \(V^c(R) \approx \text{const.} \cdot q^2 \psi_0 \ln \frac{R}{R_0}\), where \(\psi_0\) is proportional to the vortex condensate, and \(R_0\) is of the order of the vortex width. This expression and our lattice result (14) are in the agreement.

Note, that in paper [10] the confinement of the fractionally charged particles in the two dimensional Abelian Higgs model was found. Using the methods which are described in this Section it can be easily shown that the confinement in this model is due to the Aharonov–Bohm effect in two dimensions.
3 Numerical Calculations

Now we present the results of the numerical calculation of the potential between the test particles with the charge \( M \) in the three dimensional Abelian Higgs model, the charge of the Higgs boson is \( N = 6 \). The action of the model is chosen in the Wilson form:

\[
S[A, \varphi] = \beta ||dA||^2 - \gamma \sum_l \cos (d\varphi + NA)_l.
\]

In our calculations we use the standard Monte–Carlo method. The simulations are performed on the lattice of the size \( L^2 \times L_t, L = 16, L_t = 2, 4 \) for the charges \( M = 1, \ldots, N \).

We fit the numerical data for \( \langle L_M^+(0)L_M(R) \rangle_N \) by the formula:

\[
\ln \langle L_M^+(0)L_M(R) \rangle_N = 2 \kappa_{\text{num}} \cdot T \cdot \Delta_{(2)}(R) + C_{\text{num}},
\]

where \( \kappa_{\text{num}} \) and \( C_{\text{num}} \) are numerical fitting parameters. It turns out that the numerical data for \( \kappa_{\text{num}}(M) \) are well described by the formula (16).

![Graph](image)

Fig. 1: The ratios \( \zeta(M) \).

We present on the Fig. 1 the ratios \( \zeta(M) = \kappa_{\text{num}}(M)/\kappa^{(0)}(M) \) for \( \beta = 0.96, \gamma = 2.4, L_t = 2; \beta = 1.92, \gamma = 4.8, L_t = 4 \) and \( M = 1, \ldots, 5 \), where \( \kappa^{(0)} \) is the
semiclassical contribution to $\kappa$, eq.\([1]\). It is seen that $\zeta$ for different parameters of the model does not depend on the charge $M$ (if $M/N$ is not integer). This result is very interesting since it shows that the coefficient $\kappa^{num}$ is proportional to $q^2$. This fact means that the interaction of the tested particles is due to the Aharonov–Bohm effect even in the region of the parameters where the asymptotic expressions \([10,12]\) are not valid\([1]\). The difference of $\zeta$ from unity can be explained as the renormalization of $\kappa^{(0)}$ by quantum corrections.

The coefficient $\kappa^{num}$ for $M = N = 6$ is zero within the numerical errors. This result is in agreement with Aharonov–Bohm nature of the induced potential: the test charge $M$ is screened by the charge $N$ of the Higgs bosons, therefore the Aharonov–Bohm effect is absent and the induced potential $V(R)$ has no long–range terms.

### Conclusion and Acknowledgments

In this talk we show analytically and numerically that the Aharonov–Bohm interaction between the topological defects and the charged particles induce the long–range interaction between the particles. The induced interaction is of Coulomb type in the cases when the topological defects are free vortices ($D = 3$) and non–interacting strings ($D = 4$). Due to the long–ranged nature of the induced potential the Aharonov–Bohm effect may play a role in the dynamics of colour confinement in nonabelian gauge theories \([11]\).

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\(^1\)Since the usual Coulomb interaction of the test particles with the charge $M$ is proportional to $M^2$ (and not to $q^2$), the proportionality of $\kappa^{num}$ to $q^2$ also means that the Coulomb interaction of the tested particles is small at the considered values of the parameters of the model.
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