An algorithm for the orthogonal decomposition of financial return data

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ABSTRACT
We present an algorithm for the decomposition of periodic financial return data into orthogonal factors of expected return and “systemic”, “productive”, and “nonproductive” risk. Generally, when the number of funds does not exceed the number of periods, the expected return of a portfolio is an affine function of its productive risk.

Key Words: portfolio selection, mean-variance analysis, principal components of risk
Preface

This is a paper about our \texttt{rtndecomp} algorithm, an algorithm for decomposing financial return data into expected returns and principal components of risk. A complete listing of the algorithm appears in Appendix B. Section 5 describes exactly what the algorithm does. The rest of the paper is background—more or less.

The paper is accompanied by three ancillary text files:

- \texttt{rtndecomp.m} – The GNU Octave function.
- \texttt{GPLv3.txt} – The GNU General Public License governing the use of the \texttt{rtndecomp.m} code.
- \texttt{AdjustedClosingPrices_2010-2011.csv} – The adjusted closing prices, in tab-separated-value (spreadsheet) format, of 22 iShares exchange traded funds on the 505 market days from 2009-12-31 to 2011-12-30 inclusive. These prices are normalized at 100.000 on 2010-12-31. This means that the proportions in a notional portfolio $p = (p_1, \ldots, p_{22})$ represent the proportions of the 22 securities in an actual investment portfolio at the close of 2010-12-31. The security proportions in the same investment portfolio are typically different at the close of any one of the other 504 market days under consideration.

Section 6, \textbf{Examples of output}, illustrates the application of the algorithm to real world data. All computations in this section are based on the adjusted closing prices in “AdjustedClosingPrices_2010-2011.csv.”
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1 The standard model

We start with a synopsis of the “standard mean-variance portfolio selection model” ([Markowitz(1987), pp. 3–5]) for ex post return data.

Given an $M \times n$ matrix $R = [r_1, \ldots, r_n]$ of successive periodic returns ($M$ returns for each of $n$ securities), an investor is to choose the proportions $p = [p_1, \ldots, p_n]^T$ invested in each security, the proportions being subject to the constraints $p_j \geq 0$ ($j = 1, \ldots, n$), $\sum_{j=1}^{n} p_j = 1$. We assume that the periodic returns, $r_p \in \mathbb{R}^M$, of the corresponding investment portfolio satisfy the linear hypothesis

$$r_p = \sum_{j=1}^{n} r_j p_j = R p.$$  

(1)

We also assume that expected periodic return is a linear function of periodic return or, in other words, the expected periodic return of security $j$ is given by $e_j = \omega^T r_j$ for $j = 1, \ldots, n$. Here the weight vector $\omega \in \mathbb{R}^M$ should satisfy $\omega_i > 0$ ($i = 1, \ldots, M$) and $\sum_{i=1}^{M} \omega_i = 1$.

Under these assumptions, the expected periodic return of the investment portfolio corresponding to $p$ is

$$e_p = \sum_{j=1}^{n} e_j p_j = E p,$$  

(2)

with $E = [e_1, \ldots, e_n] = \omega^T R$, and the variance of portfolio return is

$$v_p = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk} p_j p_k = p^T V p,$$  

(3)

where the $n \times n$ covariance matrix $V = [v_{jk}]$ is given by

$$v_{jk} = \sum_{i=1}^{M} \omega_i z_{ij} z_{ik} \quad (j, k = 1, \ldots, n),$$  

(4)

the deviation or “risk” vectors $z_j \in \mathbb{R}^M$ ($j = 1, \ldots, n$) being defined by

$$z_j = r_j - 1_M e_j,$$  

(5)

with $1_M \in \mathbb{R}^M$ representing the constant return vector of all 1’s.

Caveat. If the periodic returns in $R$ are normalized linear returns, then the normalized linear returns of each investment portfolio in the $n$ securities satisfy the linear hypothesis (1) with respect to some $p = [p_1, \ldots, p_n]^T$, and all of the above statements follow ([Norton(2011)]). More typically, when compound periodic returns are used, the linear hypothesis cannot be satisfied by any $p$, and the arguments of this paper do not apply.
2 Geometry

We will consider the ex post standard model from a geometric standpoint. The a priori weights, $\omega$, of section 1 induce a Euclidean metric on the space of consecutive periodic returns, $\mathbb{R}^M$:

$$\langle x, y \rangle_\omega = \sum_{i=1}^{M} \omega_i x_i y_i, \quad \|x\|_\omega = \sqrt{\langle x, x \rangle_\omega}, \quad \text{for } x, y \in \mathbb{R}^M. \quad (6)$$

Two return vectors $x$ and $y$ are orthogonal (perpendicular to each other) if $\langle x, y \rangle_\omega = 0$. The vector of all ones, $1_M$, is a unit vector in this Euclidean space since $\sum_{i=1}^{M} \omega_i = 1$. The expected return axis, the $E$-axis, points in the $1_M$-direction. The $E$-coordinate of any periodic return vector $r \in \mathbb{R}^M$,

$$e = \langle 1_M, r \rangle_\omega = \sum_{i=1}^{M} \omega_i r_i, \quad (7)$$

is its expected return.

Each periodic return vector, $r$, has an orthogonal decomposition into its (scalar) expected-return component, $e$, and its (vector) risk component,

$$z = r - 1_M e, \quad (8)$$

with expected return zero. The standard deviation of periodic return is simply the length or norm of the risk component,

$$\sigma(r) = \|z\|_\omega, \quad (9)$$

and the variance of periodic return is its square norm,

$$\nu(r) = \|z\|^2_\omega. \quad (10)$$

The covariance matrix $V = [v_{jk}]$ of (4) is the Gram matrix of inner products of the security risk vectors $Z = [z_1, \ldots, z_n]$ of (5):

$$v_{jk} = \langle z_j, z_j \rangle_\omega \quad (j, k = 1, \ldots, n). \quad (11)$$

3 Linear subspaces and flats

We are concerned with notional portfolios in $n$ specific securities. The return vectors of these portfolios lie in the the linear subspace $\mathcal{L}(R)$ of $\mathbb{R}^M$ spanned by the return
vectors, $R = [r_1, \ldots, r_n]$, of the individual securities:

$$\mathcal{L}(R) = \{ \sum_{j=1}^{n} r_j t_j : t_j \in \mathbb{R} \}.$$  

The risk components of portfolio return vectors lie in the linear subspace $\mathcal{L}(Z)$ of $\mathbb{R}^M$ spanned by the risk components, $Z = [z_1, \ldots, z_n]$, of the $r_j$:

$$\mathcal{L}(Z) = \{ \sum_{j=1}^{n} z_j t_j : t_j \in \mathbb{R} \}.$$  

Since the proportions of the securities in a notional portfolio must sum to 1, portfolio return vectors and their risk components are contained in the flats (affine subspaces of $\mathbb{R}^M$) defined by

$$\mathcal{F}(R) = \{ \sum_{j=1}^{n} r_j t_j : \sum_{j=1}^{n} t_j = 1 \}$$

and

$$\mathcal{F}(Z) = \{ \sum_{j=1}^{n} z_j t_j : \sum_{j=1}^{n} t_j = 1 \},$$

respectively. We will refer to these as the $R$- and $Z$-flats.

Finally, we will be concerned with differences in periodic return vectors, and the corresponding differences in their risk components, from one notional portfolio to another. Such difference vectors reside in the tangent spaces

$$\mathcal{T}(R) = \{ \sum_{j=1}^{n} r_j t_j : \sum_{j=1}^{n} t_j = 0 \}$$

and

$$\mathcal{T}(Z) = \{ \sum_{j=1}^{n} z_j t_j : \sum_{j=1}^{n} t_j = 0 \}$$

of the $R$- and $Z$-flats.

**Proposition 1.** If $1_M \notin \mathcal{T}(R)$, then the risk component mapping $r \mapsto z$ defined by (7) and (8) restricts to a linear isomorphism of $\mathcal{T}(R)$ onto $\mathcal{T}(Z)$.  


Proof. The mapping from $\mathcal{I}(R)$ onto $\mathcal{I}(Z)$ can be expresses as $\Delta r \mapsto \Delta z$ with

$$\Delta r = \Delta z + 1_M \sum_{j=1}^n e_j t_j, \quad \Delta z = \sum_{j=1}^n z_j t_j, \quad \text{and} \quad \sum_{j=1}^n t_j = 0.$$ 

To show that this mapping is a linear isomorphism, we need to show that $\Delta r = 0_M$ whenever $\Delta z = 0_M$. But, if $\Delta z = 0_M$, then $\Delta r = 1_M \sum_{j=1}^n e_j t_j$. And then, since $1_M \notin \mathcal{I}(R)$, $\sum_{j=1}^n e_j t_j = 0$ and $\Delta r = 0_M$. \qed

Corollary. If $1_M \notin \mathcal{I}(R)$, then

$$\sum_{j=1}^n z_j t_j \mapsto \sum_{j=1}^n r_j t_j \quad \text{for} \quad \sum_{j=1}^n t_j = 1 \quad (12)$$

is a well-defined mapping of $\mathcal{I}(Z)$ onto $\mathcal{I}(R)$. It is the inverse of the risk component mapping from $\mathcal{I}(R)$ onto $\mathcal{I}(Z)$.

4 Components of portfolio risk

The total variance of return of the periodic returns in $R = [r_1, \ldots, r_n]$ is the sum of the variances of return of the individual securities:

$$v_T = \sum_{j=1}^n v_{jj} = \sum_{j=1}^n \|z_j\|^2_\omega. \quad (13)$$

This is a measure of the volatility of the return data as a whole, of the spread of the periodic returns in $R = [r_1, \ldots, r_n]$ away from their expected values $E = [e_1, \ldots, e_n]$.

Given a unit risk vector $u \in \mathcal{L}(Z)$, the variance of return of security $j$ in the $u$-direction is the square of the $u$-coordinate of its risk vector, $(u, z_j)_\omega^2$. The total variance of return in the $u$-direction is the sum of the $u$-directional variances:

$$v_u = \sum_{j=1}^n (u, z_j)_\omega^2. \quad (14)$$

If $\mathcal{U} \subset \mathcal{L}(Z)$ is an orthonormal basis for $\mathcal{L}(Z)$ (a pairwise-orthogonal set of unit vectors that span $\mathcal{L}(Z)$), then

$$\|z_j\|^2_\omega = \sum_{u \in \mathcal{U}} (u, z_j)_\omega^2 \quad (j = 1, \ldots, n). \quad (15)$$
Consequently
\[ v_T = \sum_{j=1}^{n} \|z_j\|^2_\omega = \sum_{j=1}^{n} \sum_{u \in \mathcal{U}} \langle u, z_j \rangle^2_\omega = \sum_{u \in \mathcal{U}} \sum_{j=1}^{n} \langle u, z_j \rangle^2_\omega = \sum_{u \in \mathcal{U}} v_u. \]  \hspace{1cm} (16)

For principal component analysis ([Wikipedia(2011)]) one attempts to choose the orthogonal basis \( \mathcal{U} \) (orthogonal coordinate system if you will) so that the sum of \( u \)-directional-total-variances on the right side of (16) decomposes or “explains” the total variance, \( v_T \), in a particularly meaningful way. We are aiming for such a decomposition of the total variance of return in this paper. Our idea of a “particularly meaningful way” will be defined in this section.

4.1 Systemic risk

Lemma. Let \( z_0 \) denote the point in the \( Z \)-flat that is closest to the origin:
\[ \|z_0\|_\omega = \min \{ \|z\|_\omega : z \in \mathcal{F}(Z) \}. \]
Then
\[ \langle z_0, z - z_0 \rangle_\omega = 0 \quad \text{for all} \quad z \in \mathcal{F}(Z). \]  \hspace{1cm} (17)

Proof. Given \( z \in \mathcal{F}(Z), z(t) = z_0 + (z - z_0)t \) is in \( \mathcal{F}(Z) \) for all \( t \in \mathbb{R} \). By definition \( \|z(t)\|^2_\omega \) achieves its minimum value of \( \|z_0\|^2_\omega \) at \( t = 0 \). Consequently
\[ \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \|z(t)\|^2_\omega = \langle z_0, z - z_0 \rangle_\omega = 0. \]

Proposition 2. Let \( f_0 = \|z_0\|_\omega \). Then
\[ v_{jk} = f_0^2 + \hat{v}_{jk} \quad \text{with} \quad \hat{v}_{jk} = \langle z_j - z_0, z_k - z_0 \rangle_\omega \quad (j, k = 1, \ldots, n). \]  \hspace{1cm} (18)

Proof.
\[ v_{jk} = \langle z_j, z_k \rangle_\omega \]  \hspace{1cm} (11)
\[ = \langle (z_j - z_0) + z_0, (z_k - z_0) + z_0 \rangle_\omega \]
\[ = \langle z_j - z_0, z_k - z_0 \rangle_\omega + \langle z_0, z_0 \rangle_\omega \]
\[ + \langle z_j - z_0, z_0 \rangle_\omega + \langle z_0, z_k - z_0 \rangle_\omega \]  \hspace{1cm} (bilinear expansion)
\[ = \langle z_j - z_0, z_k - z_0 \rangle_\omega + \langle z_0, z_0 \rangle_\omega + 0 + 0 \]  \hspace{1cm} (by the lemma)
\[ = \hat{v}_{jk} + f_0^2. \]
Corollary 1.

\[ v_T = n f_0^2 + \hat{v}_T \text{ where } \hat{v}_T = \sum_{j=1}^{n} \hat{v}_{jj} = \sum_{j=1}^{n} \|z_j - z_0\|_\omega^2. \]  

(19)

This follows from (13) and (18). Then, by rewriting (18) in matrix form, we see that

Corollary 2.

\[ V = f_0^2 + \hat{V} \text{ where } \hat{V} = [\hat{v}_{jk}] \ (j, k = 1, \ldots, n). \]  

(20)

Here we adopt the convention that the sum of a scalar and a matrix is the original matrix with the scalar added to its every coefficient.

We refer to \( f_0 \) as the systemic portfolio risk, \( f_0^2 \) is the systemic portfolio variance, and \( nf_0^2 \) is the total systemic variance of the system. The variance of return of any notional portfolio \( \mathbf{p} \) decomposes into its systemic and nonsystemic parts:

\[ v_p = \mathbf{p}^T V \mathbf{p} = f_0^2 + \mathbf{p}^T \hat{V} \mathbf{p}. \]  

(21)

The second equation follows from (20) and \( \sum_{j=1}^{n} p_j = 1 \).

A minimum-variance portfolio is a notional portfolio \( \mathbf{p} \) whose variance is less than or equal to the variance of any other notional portfolio \( \mathbf{q} \) with the same expected return. Minimum-variance portfolios play a crucial role in Markowitz’s mean-variance analysis ([Markowitz(1987)]). By (21)

\[ \mathbf{p}^T V \mathbf{p} \leq \mathbf{q}^T V \mathbf{q} \text{ if and only if } \mathbf{p}^T \hat{V} \mathbf{p} \leq \mathbf{q}^T \hat{V} \mathbf{q} \]

for notional portfolios \( \mathbf{p} \) and \( \mathbf{q} \). Consequently, the collection of all minimum-variance portfolios is completely determined by the singular, nonsystemic covariance matrix \( \hat{V} \) and the expected return matrix \( E \).

If \( f_0 = \|z_0\|_\omega \neq 0 \), we will take \( \mathbf{u}_0 = z_0/f_0 \) to be the first vector in our orthonormal basis \( \mathcal{U} \) for \( \mathcal{L}(Z) \). This is the direction of systemic risk. Every vector \( \mathbf{z} \in \mathcal{F}(Z) \) has the same \( \mathbf{u}_0 \)-coordinate, \( \langle \mathbf{u}_0, \mathbf{z} \rangle_\omega = f_0 \), as can be seen from the expansion

\begin{align*}
\langle \mathbf{u}_0, \mathbf{z} \rangle_\omega &= \langle \mathbf{u}_0, z_0 + (z - z_0) \rangle_\omega \\
&= \langle \mathbf{u}_0, z_0 \rangle_\omega + \langle \mathbf{u}_0, z - z_0 \rangle_\omega \quad \text{(linear expansion)} \\
&= \langle \mathbf{u}_0, \mathbf{u}_0 f_0 \rangle_\omega + 0 \quad \text{(definition of } \mathbf{u}_0 \text{ and (17))} \\
&= f_0.
\end{align*}

We will assume until further notice that \( 1_M \notin \mathcal{F}(R) \). Then the mapping \( \mathbf{z} \mapsto \mathbf{r} \) from the \( Z \)-flat onto the \( R \)-flat is well-defined by the corollary to Proposition 1, and \( \mathbf{z}_0 \in \mathcal{F}(Z) \) is the risk component of a unique \( \mathbf{r}_0 \in \mathcal{F}(R) \). We will refer to

\[ e_0 = \langle 1_M, \mathbf{r}_0 \rangle_\omega \]  

(22)
as the **systemic portfolio return** of our system. Note that \( e_0 \) may not be the expected return of any notional portfolio \( p \), all of whose coefficients must be nonnegative.

### 4.2 Productive risk

Equation (17) shows that the tangent space \( T(Z) \) is the orthogonal complement of \( z_0 \) in \( L(Z) \). Indeed \( T(Z) \) is spanned by the difference vectors \( z_j - z_0 \) \((j = 1, \ldots, n)\), and the nonsystemic covariance matrix, \( \hat{V} \), is the Gram matrix of these difference vectors. We will select the remaining orthonormal basis vectors \( u_i \) \((i = 1, \ldots, m; m < n)\) from \( T(Z) \). Then the total nonsystemic variance \( \hat{v}_T \) of (19) will decompose as the sum of the squares of the \( u_i \)-coordinates of the \( z_j - z_0 \),

\[
\hat{v}_T = \sum_{j=1}^{n} \|z_j - z_0\|^2_\omega = \sum_{i=1}^{m} \sum_{j=1}^{n} (u_i, z_j - z_0)_{\omega}^2 = \sum_{i=1}^{m} \hat{v}_{u_i}, \tag{23}
\]

and \( \hat{V} \) will factor as \( \hat{V} = F^T F \), with the coefficients of the \( m \times n \) factor matrix \( F \) given by

\[
f_{ij} = (u_i, z_j - z_0)_{\omega} \quad (i = 1, \ldots, m; j = 1, \ldots, n). \tag{24}
\]

We will continue to assume that \( 1_M \notin T(R) \), so that \( r \mapsto z \) is a bijection of \( T(R) \) onto \( T(Z) \), and further suppose that the \( n \) securities do not all have the same expected return. Under these assumptions the orthogonal projection of \( 1_M \) onto \( T(R) \) is neither \( 0_M \) nor \( 1_M \) itself.

Figure 2: The \((u_1, 1_M)\)-plane

Let \( v_1 \in T(R) \) denote the unit vector in the direction of the orthogonal projection of \( 1_M \) onto \( T(R) \), as shown in Figure 2. Then \( v_1 \) is the direction of steepest increase of expected return in the \( R \)-flat. Changes in expected return depend only on changes of periodic return in the \( v_1 \) direction in the sense that

\[
\Delta e = <1_M, \Delta r >_\omega = <1_M, v_1 >_\omega < v_1, \Delta r >_\omega \tag{25}
\]

for all \( \Delta r \in T(R) \).

Now set

\[
u_1 = \frac{v_1 - 1_M(1_M, v_1)_{\omega}}{\|v_1 - 1_M(1_M, v_1)_{\omega}\|_{\omega}} \in T(Z) \tag{26}
\]
so that
\[ \mathbf{v}_1 = \mathbf{u}_1 \cos \phi + \mathbf{1}_M \sin \phi \quad \text{with} \quad 0 < \phi < \frac{\pi}{2} \] (27)
as shown in Figure 2.

**Proposition 3.** Let
\[ \Delta e = \langle \mathbf{1}_M, \Delta r \rangle_{\omega}, \]
\[ \Delta z = \Delta r - \mathbf{1}_M \Delta e, \]
for \( \Delta r \in \mathcal{F}(R) \). Then
\[ \Delta e = e_F \langle \mathbf{u}_1, \Delta z \rangle_{\omega}, \] (28)
with the \( \mathbf{u}_1 \) of (26) and \( e_F = \tan \phi \) as in Figure 2.

**Proof.** The result follows from (25) and
\[ \langle \mathbf{1}_M, \mathbf{v}_1 \rangle_{\omega} = \sin \phi, \]
\[ \langle \mathbf{v}_1, \Delta r \rangle_{\omega} = \langle \mathbf{u}_1, \Delta r \rangle_{\omega} \cos \phi + \langle \mathbf{1}_M, \Delta r \rangle_{\omega} \sin \phi \]
\[ = \langle \mathbf{u}_1, \Delta z \rangle_{\omega} \cos \phi + \Delta e \sin \phi. \]

Under the assumption \( \mathbf{1}_M \notin \mathcal{F}(R) \), the change in expected return, \( \Delta e \), from one notional portfolio to another depends only the change in risk component, \( \Delta z \), of the respective return vectors. This is a consequence of Proposition 1. Proposition 3 now shows that such a change in expected return depends only on the change in the risk component in the \( \mathbf{u}_1 \)-direction. For this reason we refer to the \( \mathbf{u}_1 \)-direction of the \( Z \)-flat as the **direction of productive risk**. Changes in portfolio risk vectors in directions orthogonal to the \( \mathbf{u}_1 \)-direction have no effect on expected reward. Such changes are **nonproductive** in this sense.

**Corollary 1.**
\[ e = e_0 + e_F \langle \mathbf{u}_1, z \rangle_{\omega} \quad \text{for all} \quad r = z + \mathbf{1}_M e \in \mathcal{F}(R). \] (29)

**Proof.** Set \( \Delta r = r - r_0 \) in Proposition 3. Then
\[ e - e_0 = e_F \langle \mathbf{u}_1, z - z_0 \rangle_{\omega} \]
\[ = e_F (\langle \mathbf{u}_1, z \rangle_{\omega} - \langle \mathbf{u}_1, z_0 \rangle_{\omega}) \]
\[ = e_F \langle \mathbf{u}_1, z_0 \rangle_{\omega}. \]
Here \( \langle \mathbf{u}_1, z_0 \rangle_{\omega} = 0 \) since \( \mathbf{u}_1 \in \mathcal{F}(Z) \) and \( z_0 \) is orthogonal to \( \mathcal{F}(Z) \). \( \square \)
Corollary 2.  

\[ e_0 = e_e - e_F \langle u_1, z_e \rangle_\omega \]  \hspace{1cm} (30)

for any convenient \( r_* = z_* + 1_M e_* \in \mathcal{F}(R) \).

Let us now define the productive risk of the system, \( \tau_1 \), as 

\[ \tau_1 = \sqrt{\sum_{j=1}^{n} \langle u_1, z_j - z_0 \rangle^2_\omega} = \sqrt{\sum_{j=1}^{n} \langle u_1, z_j \rangle^2_\omega} \]  \hspace{1cm} (31)

with \( \tau_1^2 \) being the productive variance. We include the middle, \( z_0 \) expression in this definition to emphasize that the productive risk is coming from the tangent space \( \mathcal{T}(Z) \), which is spanned by the \( z_j - z_0 \). The middle expression collapses to the last expression because \( \langle u_1, z_0 \rangle_\omega = 0 \).

4.3 Nonproductive risk

Each notional portfolio \( p \) has a corresponding risk vector \( z = Zp \). By (29) the expected return of the portfolio is completely determined by the \( u_1 \) coordinate of \( z \) and the parameters \( e_0 \) and \( e_F \). However, the sum of the systemic and productive variances, \( f_0^2 + \langle u_1, z \rangle^2_\omega \), is just a part of the portfolio variance. The remaining variance is nonproductive, having no effect on the expected return of the portfolio.

**Definition 1.** We now define the principal nonproductive risks, \( \tau_i > 0 \), and the corresponding principal directions of nonproductive risk, \( u_i \in \mathcal{T}(Z) \) (\( \|u_i\|_\omega = 1 \)), for \( i = 2, \ldots, m \), where \( m \) is the dimension of \( \mathcal{T}(Z) \), the rank of \( \hat{V} \). The definition proceeds by induction:

for \( i = 2, \ldots, m \),

\[ \tau_i^2 = \sum_{j=1}^{n} \langle u_i, z_j \rangle^2_\omega \]

\[ = \max \left\{ \sum_{j=1}^{n} \langle u, z_j \rangle^2_\omega : u \in \mathcal{T}(Z), \|u\|_\omega = 1, \langle u_k, u \rangle_\omega = 0 \ (k = 1, \ldots, i - 1) \right\} \]

**Remark.** The \( \tau_i = \sqrt{\tau_i^2} \) are uniquely determined, and, in the generic case, when \( \tau_2 > \tau_3 > \ldots > \tau_m \), the principal directions of nonproductive risk are unique up to multiplication by \(-1\). We will assume this case to simplify the discussion. The \texttt{rtndecomp} algorithm presented in Appendix B makes no such assumption.
Corollary 1. The total nonsystemic variance can be decomposed into its productive and nonproductive parts as
\[ \hat{v}_T = \tau_1^2 + \sum_{i=2}^{m} \tau_i^2 \] (32)

Corollary 2. The nonsystemic covariance matrix \( \hat{V} \) factors as \( \hat{V} = F^T F \), where the coefficients of the \( m \times n \) factor matrix \( F \) are given by
\[ f_{ij} = \langle u_i, z_j - z_0 \rangle_\omega = \langle u_i, z_j \rangle_\omega \quad (i = 1, \ldots, m; j = 1, \ldots, n) \] (33)

Corollary 3. The variance of return of any notional portfolio \( p \) can be decomposed into its systemic, productive, and nonproductive parts as
\[ v_p = f_0^2 + \left( \sum_{j=1}^{n} f_{1j} p_j \right)^2 + \sum_{i=2}^{m} \left( \sum_{j=1}^{n} f_{ij} p_j \right)^2, \] (34)
with the \( f_{ij} \) of (33).

These three corollaries of Definition 1 follow immediately from the preceding discussion. We refer to the factor matrix \( F \) of Corollary 2 as the nonsystemic risk matrix.

4.4 Mean-variance analysis

Let
\[ \Phi : \mathbb{R}^n \to \mathbb{R}^2, \quad p \mapsto (e_p, v_p), \]
denote the mean-variance mapping defined by (2) and (3). In view of the preceding discussion \( \Phi \) can be factored as
\[ \Phi : \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \to \mathbb{R}^2, \quad p \mapsto f_p \mapsto (e_p, v_p), \] (35)
with
\[ e_p = e_0 + e_F x_p, \] (36)
\[ v_p = f_0^2 + \| f_p \|^2 = f_0^2 + x_p^2 + \| y_p \|^2, \] (37)
where
\[ f_p = \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} F(1,:) \\ F(2:m,:) \end{bmatrix} p = F p. \] (38)
Here \( F(1,:) \) and \( F(2:m,:) \) denote the productive and nonproductive rows of the \( m \times n \) nonsystemic risk matrix \( F \), respectively.

We are primarily interested in the image, \( \Phi(\Delta) \), of the notional portfolio simplex
\[ \Delta = \{ p \in \mathbb{R}^n : \sum_{j=1}^{n} p_j = 1 \text{ and } p_j \geq 0 \text{ for } j = 1, \ldots, n \}. \]
[Markowitz(1987)] refers to this image as the obtainable EV set.
**Remark.** We apologize for the reuse of notation here. We have been using $\Delta$ to indicate a difference vector. Now $\Delta$ is the standard $(n - 1)$-simplex in $\mathbb{R}^n$. In the future we hope the meaning of $\Delta$ will be clear by its context.

Example 1 and the corresponding Figure 3 illustrate the factorization (35)–(37) in the $n = 3, m = 2$ case.

**Example 1.**

$$F = \begin{bmatrix} -4 & 2 & 4 \\ 2 & -2 & 3 \end{bmatrix} \quad \text{and} \quad \hat{V} = F^T F = \begin{bmatrix} 20 & -12 & -10 \\ -12 & 8 & 2 \\ -10 & 2 & 25 \end{bmatrix}.$$  

**Figure 3: Factorization of the mean-variance mapping $\Phi$**

In Figure 3 the set of minimum-variance portfolios is specified by the piecewise linear path $APQC$ though the simplex $\Delta$. The portfolios $P = 50\% A + 50\% B$ and $Q = 60\% B + 40\% C$. 

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are called *corner portfolios* for obvious reasons. Equation (37) implies that the portfolio \( E \) of absolute minimum variance corresponds to \((x, y) = (0, 0)\). Since \( P \) and \( Q \) have \((x, y)\)-representations (-1, 0) and (2.8, 0), respectively, we must have

\[
E = \frac{14}{19} P + \frac{5}{19} Q.
\]

The parameters \( e_0, e_F \geq 0 \), and \( f_0 \geq 0 \) are inconsequential. The set of minimum-variance portfolios is independent of these parameters.

Portfolios on the piecewise linear path \( EQC \) in \( \Delta \) are *efficient*: besides having minimum variance for their expected return, they have maximum expected return for their variance.

The \( x \)- and \( y \)-axes through the upper-left-hand simplex of Figure 3 are the preimages of the respective axes on the lower-left \( xy \)-plane via the mapping \( p \mapsto (x, y) = F p \). The image axes are perpendicular to each other, but the preimage axes are not. The preimage of the \( x \)-axis is the *critical line* of the mapping \( \Phi|\{\sum_{j=1}^3 p_j = 1\} \). The derivative of \( \Phi|\{\sum_{j=1}^3 p_j = 1\} \) has rank 1 along this line and rank 2 everywhere else. In effect \( \Phi \) folds the \( \sum_{j=1}^3 p_j = 1 \) plane over this critical line.

Returning to the general situation let us point out that the factorization of the mean variance mapping in (35)–(37) leads to a natural, geometric characterization of minimum-variance portfolios. First note that the portfolio simplex \( \Delta \) is mapped onto a convex polytope \( P = F(\Delta) \) in \((x, y)\)-space \( \mathbb{R}^m \) due to the linearity of \( p \mapsto F p \). If the \( x \)-axis (the \( u_1 \)-axis) passes through this polytope, then every point in the intersection of the \( x \)-axis and the polytope is the image of a minimum-variance portfolio \( p \)—simply because \( y_p = 0 \) and \( v_p \) can’t get any smaller than \( v_p = f_0^2 + x_p^2 \). More generally, let \( x_{\text{min}} \) and \( x_{\text{max}} \) be the minimum and maximum values of the coefficients in the “productive” row, \( X = F(1,:) \), of \( F \). Given \( x_* \) between \( x_{\text{min}} \) and \( x_{\text{max}} \), suppose \( y_* \in \mathbb{R}^{m-1} \) satisfies

\[
\|y_*\| = \min \{\|y\| : (x_*, y) \in P \cap \{x = x_*\}\}.
\]  

Then any portfolio \( p \in \Delta \) that \( F \) maps onto \((x_*, y_*)\) (and there is at least one) is a minimum-variance portfolio.

### 4.5 Relaxing assumptions

Since Section 4.2 we have been assuming that the \( n \) given securities do not all have the same expected return; however there is no problem if the returns are identical. Then there is no productive risk, all nonsystemic risk is nonproductive: \( u_1, \ldots, u_m \) are the principal directions of nonproductive risk. This case is signalled by \( e_F = 0 \), and (29) still holds with \( e_0 \) being the common expected return.
The case when $1_M$ parallels the R-flat, when $1_M \in \mathcal{F}(R)$, is more problematic. This situation typically arises when there are more securities than periods. Then there is no unambiguous systemic return, $e_0$, and no well-defined gradient of expected return, $g = u_1 e_F \in \mathcal{F}(Z)$.

To handle the $1_M \in \mathcal{F}(R)$ case we anchor ourselves at the mean risk component, $\bar{z} = \frac{1}{n} \sum_{j=1}^{n} z_j$, with mean expected return, $\bar{e} = \frac{1}{n} \sum_{j=1}^{n} e_j$. Our approximate gradient, $g = \sum_{j=1}^{n} (z_j - \bar{z}) g_j$, is the least-squares solution of (28) in the form

$$e_k - \bar{e} = \sum_{j=1}^{n} (z_k - \bar{z}, z_j - \bar{z}) \omega g_j \quad (k = 1, \ldots, n).$$

(40)

We use this $g$ to define the parameters

$$e_F = \|g\|_{\omega}, \quad u_1 = g/e_F, \quad e_0 = \bar{e} - \langle g, \bar{z} \rangle_{\omega},$$

(41)

for the approximate version of (29):

$$e \approx e_0 + e_F (u_1, z)_{\omega}. \quad (42)$$

Note that the definition of $e_0$ is essentially (30) with $e_* = \bar{e}$ and $z_* = \bar{z}$.

The remaining orthogonal directions of risk are defined inductively by Definition 1, and the corollaries of that definition continue to hold.

### 4.6 Scaling output

Up to this point expected returns, $e_0$ and $E = [e_1, \ldots, e_n]$, and risk coefficients, $f_0$ and $F = [f_{ij}] \ (i = 1, \ldots, m; j = 1, \ldots, n)$, have been measured in the same percent-per-period units. While days or weeks may be used for computational purposes, annualized, percent-per-year output is usually preferred to daily or weekly percentages.

To compensate for this preference we add a periods-per-unit-of-time parameter $\rho$ to the periodic returns $R$ and weights $\omega$ required by our algorithm. Then, at the end of the computations, percent-per-period expected returns and risks are scaled to percent-per-unit-of-time units as follows:

| percent per-unit-of-time | percent per-period |
|--------------------------|--------------------|
| $E$                      | $\rho \times E$   |
| $e_0$                    | $\rho \times e_0$ |
| $e_F$                    | $\sqrt{\rho} \times e_F$ |
| $F$                      | $\sqrt{\rho} \times F$ |
| $f_0$                    | $\sqrt{\rho} \times f_0$ |
The idea behind this scaling is statistical. Assume, for example, that daily returns are independent random variables from one market-day to the next and there are (typically) $\rho = 252$ market-days per year. The annual return is the sum of $\rho$ daily returns; so the expected value of annual return is $\rho$ times the daily expected value. This accounts for the $\rho$ multipliers above. The variance of annual return is $\rho$ times the daily variance due to the independence assumption, but risk or standard deviation is the square root of variance; consequently $\sqrt{\rho}$ is the appropriate multiplier of $F$ and $f_0$. Finally $e_F$ is the rate of change of expected return to risk; so $\sqrt{\rho} = \rho/\sqrt{\rho}$ is the appropriate multiplier.

5 The rtndecomp function – arguments and relationships

The GNU Octave listing of the rtndecomp function appears in Appendix B. In this section we give the function header and describe its arguments. The relationships between the output arguments were derived in the last section.

function: \[E, F, f_0, e_0, e_F\] = rtndecomp \((R, \omega, \rho)\)

purpose To decompose financial return data into orthogonal risk-factors.

input
- \(R\) – \(M \times n\) matrix of periodic returns.
- \(\omega\) – \(M\)-vector of positive weights or a scalar.
  If \(\omega\) is a scalar or if \(R\) is the only input argument, then \(\omega\) defaults to $\omega_i = 1/M$ for $i = 1, \ldots, M$.
- \(\rho\) – periods per unit time. ($\rho \geq 1$, default: $\rho = 1$)
  e.g., $\rho = 252$ market-days per year.

output
- \(E\) – \(1 \times n\) matrix of expected returns.
- \(F\) – \(m \times n\) matrix of risk coefficients.
  \(\text{rank}(F) = m\) unless \(F = \text{zeros}(1, n)\).
- \(f_0\) – systemic risk. ($f_0 \geq 0$)
- \(e_0\) – systemic expected return.
- \(e_F\) – expected return per unit of productive risk.
  ($e_F \geq 0$; if $e_F = 0$ there is no productive risk)

global output
eflag = true if and only if a nonzero, constant $M$-vector is parallel to the returns flat, $\mathcal{F}(R)$, or, said another way, if and only if $1_M \in \mathcal{T}(R)$.

relationships
1) $E = \rho \omega^T R$.
2) The $n \times n$ covariance of returns matrix, $V$, is given by 
   \[ V = Z^T \text{diag}(\omega \rho) Z, \] 
   where $Z = R - 1_M \omega^T R$.
3) $V = f_0^2 + F^T F$. (the scalar $f_0^2$ is added to each coefficient of $F^T F$)
4) $E = e_0 + e_F X$ unless eflag is true.
   Here $X = F(1,1:n)$ denotes the first row of $F$, and, in the equation, $e_0$ is added to each coefficient of $e_F X$. When eflag is true, the equation is an approximation, but 
   \[ \text{mean}(E) = e_0 + e_F \text{mean}(X) \] 
   remains true.
5) $v_T = \sum_{j=1}^n V(j,j) = nf_0^2 + \tau_1^2 + \sum_{i=2}^m \tau_i^2$, with $\tau_i = \sqrt{F(i,1:n)F(i,1:n)^T}$ for $i = 1, \ldots, m$. The right-hand side of the equation for $v_T$ represents the decomposition of total variance into systemic, productive, and nonproductive parts (though $\tau_1^2$ is nonproductive if $e_F = 0$).
6) $\tau_2 \geq \ldots \geq \tau_m > 0$. (principal nonproductive risks)
7) $F(2:m,1:n)F(2:m,1:n)^T = \text{diag}(\tau_2^2, \ldots, \tau_m^2)$.
8) If $e_F = 0$ and $F \neq [0, \ldots, 0]$, then $\tau_1 \geq \ldots \geq \tau_m > 0$ and $F(1:m,1:n)F(1:m,1:n)^T = \text{diag}(\tau_1^2, \ldots, \tau_m^2)$.

6 Examples of output

6.1 Five large ETFs – 2010

In our first example we apply the rttdecomp algorithm (listed in Appendix B) to 2010 daily returns from the five iShares exchange traded funds (ETFs)

1. IEF – iShares Barclays 7-10 Year Treasury Bond Fund
2. IWB – iShares Russell 1000 Index Fund
3. IWM – iShares Russell 2000 Index Fund
4. EFA – iShares MSCI EAFE Index Fund
5. EEM – iShares MSCI Emerging Markets Index Fund

Figure 4 shows the growth of these securities in 2010. These are plots of adjusted closing prices against time. The adjusted closing prices are normalized at 100 on 2010-12-31; thus notional portfolios specify the closing proportions of actual investment.
portfolios at the end of 2010 ([Norton(2011)]). If ex post analysis were to deem a certain notional portfolio $\mathbf{p}$ as optimal, one would buy

$$ s_j = 100 \frac{p_j}{a_j} $$

shares of fund $j$, per $100$ invested, to invest in the optimal portfolio at the end of 2010. Here $a_j$ is the 2010-12-31 closing price of fund $j$ ($j = 1, \ldots, n; \ n = 5$).

Figure 4: 2010 adjusted closing prices of five large ETFs
prices normalized at 100 on 2010-12-31

6.1.1 Algorithm input

The graphs in Figure 4 correspond to a $253 \times 5$ matrix $A$ of adjusted closing prices for these 5 ETFs on the 253 market days from 2009-12-31 thru 2010-12-31 inclusive. The normalized, linear, daily returns for these funds are given by the $252 \times 5$ matrix $R = \Delta A$, where $\Delta$ is the $252 \times 253$ difference operator

$$ \Delta = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ -1 & 1 \end{bmatrix}. $$

We won’t use all of $R$ in this example, just the last $M = 200$ rows, just the last 200 market-day returns—which correspond to the colored-background portion of Figure 4.
In addition to the $M \times n$ return matrix $R$, the \texttt{rtndecomp} algorithm requires an $M$-vector of weights, $\omega$, and a scaling factor, $\rho$, that specifies the number of periods per unit time. We use $\rho = 252$ market-days per year throughout this paper.

To see how the weights affect output we will consider two systems of weights: gray, uniform weights, where each of the $M = 200$ market days has the same importance, $\omega_i = 1/200$, and the more colorful \textit{late-heavy} weight system pictured in Figure 5.

The colors of the late-heavy system correspond to the colored regions of Figure 4. The returns of the last 30 market days of 2010 are weighted by $1/140$ each; returns for the first 70 days count half as much as these, or $1/280$ per day; and the weights for the middle 100 days,

$$\omega_{i+70} = \frac{1 + i/101}{280} \quad \text{for} \quad i = 1, \ldots, 100,$$

increase uniformly between these two extremes. The sum of these late-heavy weights is the sum of the yellow, green, and blue areas in Figure 5:

$$70 \times \frac{1}{280} + 100 \times \frac{1}{2} \times \left( \frac{1}{280} + \frac{1}{140} \right) + 30 \times \frac{1}{140} = 1,$$

as required.

The idea behind late-heavy weighting is simple. Think of investing at the end of a 200 market-day period. The recent performance of a group of securities may be more important than their performance further back in the past—as a predictor of their near-future performance. Thus an investment analysis might weight the recent performance more heavily.

### 6.1.2 Algorithm output

The output arguments of the \texttt{rtndecomp} function

$$[E, F, f_0, e_0, e_F] = \texttt{rtndecomp}(R, \omega, \rho) \quad (44)$$

were described in the Section 5. Tables 1 and 2 show the output when the function is applied to the $200 \times 5$ matrix of daily returns described above. The late-heavy weights of Figure 5 were used for Table 1 and uniform weights for Table 2.
Table 1: Decomposition of return data – 5 large ETF universe last 200 market-days of 2010 – late-heavy weights

| fund | IEF | IWB | IWM | EFA | EEM | $\hat{V}_T$ |
|------|-----|-----|-----|-----|-----|-----------|
| $E$  | 2.86| 18.77| 27.12| 13.84| 21.79| $364$ 24.8% |
|      | $-2.79$ | 7.80 | 13.36 | 4.52 | 9.81 |           |
| $F$  | 4.14 | $-11.81$ | $-13.48$ | $-19.57$ | $-16.81$ | $1004$ 68.4% |
|      | $2.30$ | $-2.15$ | $-5.06$ | 0.01 | 6.12 | $73$ 5.0% |
|      | $-4.14$ | 2.02 | $-2.26$ | $-0.89$ | 0.40 | $27$ 1.9% |

| $\hat{V}_T$ | 47 | 209 | 391 | 404 | 416 | 1468 100% |
|             | 3.2% | 14.2% | 26.6% | 27.5% | 28.4% | 100% |

with $f_0 = 5.07$, $e_0 = 7.05$, and $e_F = 1.502$

Table 2: Decomposition of return data – 5 large ETF universe last 200 market-days of 2010 – uniform weights

| fund | IEF | IWB | IWM | EFA | EEM | $\hat{V}_T$ |
|------|-----|-----|-----|-----|-----|-----------|
| $E$  | 7.79 | 11.62 | 17.47 | 8.12 | 17.27 | $123$ 7.4% |
|      | 0.00 | 3.02 | 7.62 | 0.27 | 7.46 |           |
| $F$  | 5.41 | $-15.08$ | $-18.63$ | $-20.92$ | $-19.15$ | $1408$ 85.3% |
|      | $-1.74$ | 2.31 | 6.80 | $-2.94$ | $-5.71$ | $96$ 5.8% |
|      | $-3.33$ | 2.09 | $-1.48$ | $-2.17$ | 1.22 | $24$ 1.4% |

| $\hat{V}_T$ | 43 | 246 | 453 | 451 | 456 | 1651 100% |
|             | 2.6% | 14.9% | 27.5% | 27.3% | 27.7% | 100% |

with $f_0 = 4.86$, $e_0 = 7.78$, and $e_F = 1.271$

The $E, e_0, F,$ and $f_0$ coefficients in these tables are in percent-per-year units, the slope $e_F$ is unitless, and the nonsystemic variance totals in the $\hat{V}_T$ sections are in percent-per-year-squared units.

As described in Section 4, the coefficients of each risk matrix $F$ are the coordinates of the nonsystemic risk components of the individual securities with respect to an orthonormal basis, $\{u_1, \ldots, u_m\}$, for the tangent space of the $Z$-flat. The sum of the squares of these coefficients is the total nonsystemic variance of the system. The $\hat{V}_T$-row of either table shows how this nonsystemic variance is distributed among the individual funds. As one might expect, the nonsystemic variance of the bond fund, IEF, is substantially less than that of any of the equity funds—under either system of weights.

The green row, $X = [x_1, \ldots, x_n] = F(1, :)$, of each $F$ contains the productive risk coefficients, the $u_1$-coordinates, of the security risk vectors. Changes in productive
risk from one notional portfolio to another produce corresponding changes in expected return. If there is no change in productive risk, there is no change in expected return.

The orange rows, \( F(2:m,:) \), of each \( F \) contain the risk coefficients in the principal directions of nonproductive risk. These rows are pairwise orthogonal (up to roundoff error). The first orange row, \( Y = [y_1, \ldots, y_n] = F(2,:) \), represents the most significant or major direction of nonproductive risk. It contains the \( u_2 \)-coordinates of the security risk vectors. The \( \hat{V}_T \)-column of each table shows how the total nonsystemic variance is decomposed into productive and nonproductive components.

In addition to its nonsystemic variance each fund has a \textit{systemic variance} of \( f_0^2 \) so that the \textit{total variance} of the system is decomposed into its systemic, productive, and nonproductive parts as

\[
v_T = n f_0^2 + \sum_{j=1}^{n} f_{1j}^2 + \sum_{i=2}^{m} \sum_{j=1}^{n} f_{ij}^2.
\] (45)

Table 3 shows the total variance decompositions corresponding to Tables 1 and 2.

|                      | late-heavy weights | uniform weights |
|----------------------|-------------------|-----------------|
| systemic variance \((n f_0^2)\) | 128 \(\text{8.0\%}\) | 118 \(\text{6.7\%}\) |
| productive variance \((\sum x^2)\) | 364 \(\text{22.8\%}\) | 123 \(\text{7.0\%}\) |
| major nonproductive variance \((\sum y^2)\) | 1004 \(\text{62.9\%}\) | 1408 \(\text{79.6\%}\) |
| other nonproductive variance | 100 \(\text{6.3\%}\) | 120 \(\text{6.8\%}\) |
| total variance        | 1596 \(\text{100.0\%}\) | 1769 \(\text{100.0\%}\) |

6.1.3 The \textit{XE}-plane

We will let

\[
x = Xp = F(1,:)p = \sum_{j=1}^{n} f_{1j}p_j
\] (46)

denote the productive risk coordinate of \( z = \sum_{j=1}^{n} z_jp_j \) and

\[
e = E_p = \sum_{j=1}^{n} e_jp_j
\] (47)

be the corresponding expected return coordinate. Figure 6 shows the graphs of

\[
e = e_0 + e_F x,
\] (48)
corresponding to the late-heavy and uniform weight systems. The plotted security points realize the \( X \) and \( E \) rows of Tables 1 and 2. The grid scale is \( 5 \times 5 \).

These are 2-dimensional slices of the 6-dimensional \( ZE \)-spaces, \( \mathcal{L}(Z) \times \mathbb{R} \), corresponding to the two weight systems we are considering. Each periodic return vector, \( \mathbf{r} = \mathbf{z} + \mathbf{1}_M \mathbf{e} \), corresponds to a point \( (\mathbf{z}, \mathbf{e}) \) in \( ZE \)-space. Figure 6 shows the orthogonal projection, \( (x, e) \), \( x = \langle \mathbf{u}_1, \mathbf{z} \rangle_\omega \), of these points onto the respective \( XE \)-planes.

In each picture the blue segment connecting the security points is the projection of the portfolio polytope in \( ZE \)-space onto the \( XE \)-plane. The line through this blue segment is the projection of the entire \( R \)-flat.

Figure 6 shows how the relationship between productive risk and expected return can vary with weight system. Productive risk accounts for 22.8% of the total variance under the late-heavy system, but only 7.0% under the uniform system. This difference shows up in the extra width of the late-heavy picture.

Looking at the vertical spread of Figure 6 one notices that the expected returns of the stock funds are higher and the bond fund return lower under the late-heavy system. This is because the negative returns of the stock funds in the first 70 days of the 200 market-day sample count more in the uniform system, and the negative returns of the bond fund in the last 30 days count more in the late-heavy system. (Figure 4)

6.1.4 The \texttt{eflag} flag

When the matrix equation \( E = e_0 + e_F X \) holds exactly, the global variable \texttt{eflag} is 0 or \texttt{false}. This is typically the case when there are many more periods than funds, as in the five-fund, 200-market-day examples just considered. Figure 6 displays this relationship graphically.

Rather than increase the number of funds to illustrate the \texttt{eflag = true} condition let us decrease the number of periods from \( M = 200 \) market days to \( M = 3 \) quarters. Now \( \rho = 4 \) (quarters per year), and we will use late-heavy weights, \( \omega = [\omega_1, \omega_2, \omega_3]^T \), comparable to those in Figure 5.

Here is the complete data.
Table 4: Quarter-ending adjusted closing prices

| date      | IEF   | IWB   | IWM   | EFA   | EEM   |
|-----------|-------|-------|-------|-------|-------|
| 2010-03-31| 92.925| 91.157| 85.766| 93.619| 87.102|
| 2010-06-30| 100.196| 80.605| 77.316| 79.141| 77.686|
| 2010-09-30| 104.498| 89.958| 85.884| 93.452| 93.194|
| 2010-12-31| 100.000| 100.000| 100.000| 100.000| 100.000|

Table 5: Quarterly returns and late-heavy weights

| quarter | IEF  | IWB  | IWM  | EFA  | EEM  | weights |
|---------|------|------|------|------|------|---------|
| 2       | 7.271| −10.552| −8.450| −14.478| −9.416| 2/9     |
| 3       | 4.302| 9.353| 8.568| 14.311| 15.508| 3/9     |
| 4       | −4.498| 10.042| 14.116| 6.548| 6.806| 4/9     |

Consider the return vector matrix, \( R = [r_1, \ldots, r_5] \), in Table 5. It is easy to see that the return-flat tangent space

\[
\mathcal{T}(R) = \left\{ \sum_{j=1}^{5} r_j t_j : \sum_{j=1}^{5} t_j = 0 \right\}
\]

is all of \( \mathbb{R}^3 \); in particular, the constant return vector, \( \mathbf{1}_3 \), is contained in \( \mathcal{T}(R) \). This is the \texttt{eflag = true} condition implying that equation (48) is not exact.

The \texttt{rtndecomp} output corresponding to the data of Table 5 is shown in Table 6. The yellow row shows the approximate \( e \)-values that result from applying equation (48) to the \( x \)-values of the green, productive risk row.

Table 6: Decomposition of return data – 5 large ETF universe last three quarters of 2010 – late-heavy weights

| fund | IEF   | IWB   | IWM   | EFA   | EEM   | \( \hat{V}_T \) |
|------|-------|-------|-------|-------|-------|-----------------|
| \( E \) | 4.20  | 20.94 | 29.01 | 17.85 | 24.41 |                  |
| \( \text{approx} \) | 4.84  | 24.15 | 26.64 | 21.75 | 19.03 |                  |
| \( F \) | −10.13| 12.42 | 15.34 | 9.63  | 6.45  | 626 42.2%       |
|       | −0.78 | −11.44| −8.37 | −19.06| −17.07| 856 57.8%       |
| \( \hat{V}_T \) | 103   | 285   | 305   | 456   | 333   | 1483 100%       |

with \( f_0 = 0.005 \), \( e_0 = 13.51 \), and \( e_F = 0.856 \)
Figure 7 shows the projection, \((x, y, e) \mapsto (x, e)\), of the Table 6 data onto the \(XE\)-plane. Here \(y\) is the nonproductive risk variable represented by the orange row of \(F\). The blue polygon is the image of the portfolio polyhedron.

The line through the blue polygon is the graph of the (approximate) expected return function \(e = e_0 + e_F x\) (48). The \texttt{rtndecomp} algorithm guarantees that this graph passes through the mean \(XE\)-point, \((\bar{x}, \bar{e}) = (6.74, 19.28)\) in this example.

Figure 7 is comparable to the late-heavy side of Figure 6. The \(e\)-values have the same order in both pictures, but the \(x\)-values of the middle funds, EFA, IWB, and EEM, are permuted from one picture to the other.

6.1.5 The \(XY\)- and \(EV\)-planes

Let us now return to the late-heavy weight output in Table 1. Given a portfolio \(p\) let

\[
\begin{align*}
x &= x_p = Xp = F(1,:p), \\
y &= y_p = Yp = F(2,:p),
\end{align*}
\]

denote the productive and major nonproductive risk coordinates of \(p\) (or really of \(z_p = Zp\)), respectively, and let

\[
y = y_p = F(2:m,:p), \quad m = 4,
\]

denote the full vector of nonproductive risk corresponding to \(p\). As noted in Section 4.4,

\[
\begin{align*}
e &= e_p = e_0 + e_F x_p, \\
v &= v_p = f_0^2 + x_{p}^2 + \|y_p\|^2,
\end{align*}
\]

Figure 8 shows the images, \((X,Y)(\Delta)\) and \(\Phi(\Delta)\), of the portfolio simplex \(\Delta\) in the \(XY\)- and \(EV\)-planes, respectively. The \(x\) and \(y\) grid lines are 5 units apart. Since the \((x,y)\)-tuples are coordinate vectors with respect to an orthonormal basis, the \(XY\)-image is the perpendicular projection of the four-dimensional polytope, \(F(\Delta)\), onto the \(XY\)-plane.

In Figure 8 the \(y\)-coordinates of the securities are actually the negatives of the \(Y\) coefficients in Table 1. This sign change makes the comparison of the \(XY\) and \(EV\)
images more natural, but it has no effect on our analysis—a principal direction of nonproductive risk is, at most, determined up to a reflection through the origin. On the other hand, it is important to note that the stock funds are all in the first quadrant and the bond fund, IEF, is in the third quadrant of XY-side of Figure 8. This corresponds to the fact that the stock funds are positively correlated with each other and negatively correlated with the bond fund. This is also why the bond fund, IEF, is a component of every minimum-variance portfolio other than single security portfolio of maximum expected return, IWM.

The solid black path through either image corresponds to the set of minimum-variance portfolios. As noted at the end of Section 4.4, a minimum-variance portfolio at a particular $x = x^*$ must minimize the value $\|y\|$ on the polytope $F(\Delta) \cap \{x = x^*\}$. Points on the dotted path in either image approximate this criterion. They correspond to portfolios that minimize $|y|$ rather that $\|y\|$.

The point $E$ in either figure is the image of the portfolio $p_E$ of absolute minimum variance. $p_E$ is an efficient portfolio. The solid black path to the right of $E$ is the image of the other efficient portfolios. These efficient portfolios, in total, make up the piecewise linear path in $\Delta$ that goes from $p_E$ through the “corner portfolios” of Table 7 to IWM. The corner portfolios show up as the corners above the $x$-axis in the $XY$-image of the minimum-variance path.
Table 7: Optimal portfolio paths – 5 large ETF universe last 200 days of 2010 – late-heavy weights

|                | minimum-||y|| (efficient) path | minimum-\|y\| path |
|----------------|---------------------------------|-------------------|
|                | PE corner portfolios IWM        | xE corner portf portf IWM |
| IEF            | 0.690 0.618 0.550 0             | 0.777 0.765 0      |
| IWB            | 0.310 0.382 0 0                | 0 0 0             |
| IWM            | 0 0 0.450 1.000               | 0.187 0.235 1.000 |
| EFA            | 0 0 0 0                      | 0.036 0 0         |
| EEM            | 0 0 0 0                      | 0 0 0             |
| x              | 0.49 1.26 4.47 13.36         | 0.49 1.00 13.36    |
| e              | 7.79 8.94 13.78 27.12        | 7.79 8.56 27.12   |
| σ              | 5.69 5.88 8.48 20.41         | 6.33 6.38 20.41   |
| avg e          | 17.45                         | 17.45             |
| rms σ          | 12.70                         | 12.74             |

Table 7 also shows the x_E- and corner portfolios of the minimum-\|y\| path over the efficient x-range from x_E = 0.49 to x_max = 13.36. The minimum-\|y\| path and the efficient path are exactly the same from x = 4.47 to x_max, but the paths differ between x_E and x = 4.47, the most substantial σ-differences occurring near x_E.

The average value of e over the two portfolio paths in Table 7 is just the average of the end values, 7.79 and 27.12. On the other hand, the average variance,

\[
\text{avg } v = \frac{1}{x_{\text{max}} - x_E} \int_{x_E}^{x_{\text{max}}} v(x) \, dx,
\]

and the root-mean-square risk, \( \text{rms } \sigma = \sqrt{\text{avg } v} \), depend on the whole path.

Remark. Throughout this paper we use Markowitz’s Critical Line Algorithm as described in [Niedermayer and Niedermayer(2006)] to compute minimum-variance paths through portfolio simplices.

6.2 Eighteen emerging markets ETFs – 2010

Now let us consider a larger universe of securities—the 18 iShares emerging markets ETFs that existed throughout 2010

1. BKF – iShares MSCI BRIC Index Fund
2. ECH – iShares MSCI Chile Investable Market Index Fund
3. EEM – iShares MSCI Emerging Markets Index Fund
4. EMIF – iShares S&P Emerging Markets Infrastructure Index Fund
5. **EPU** – iShares MSCI All Peru Capped Index Fund  
6. **ESR** – iShares MSCI Emerging Markets Eastern Europe Index Fund  
7. **EWM** – iShares MSCI Malaysia Index Fund  
8. **EWT** – iShares MSCI Taiwan Index Fund  
9. **EWW** – iShares MSCI Mexico Investable Market Index Fund  
10. **EWY** – iShares MSCI South Korea Index Fund  
11. **EWZ** – iShares MSCI Brazil Index Fund  
12. **EZA** – iShares MSCI South Africa Index Fund  
13. **FCHI** – iShares FTSE China (HK Listed) Index Fund  
14. **FXI** – iShares FTSE China 25 Index Fund  
15. **ILF** – iShares S&P Latin America 40 Index Fund  
16. **INDY** – iShares S&P India Nifty 50 Index Fund  
17. **THD** – iShares MSCI Thailand Investable Market Index Fund  
18. **TUR** – iShares MSCI Turkey Investable Market Index Fund

Our input to rtndecomp will be the $200 \times 18$ matrix $R = [r_{ij}]$ of normalized, linear, daily returns for the 18 emerging markets funds listed above, over the last 200 market days of 2010. The returns are normalized on 2010-12-31—they are daily adjusted-closing-price differences divided by 2010-12-31 adjusted closing prices. We will stick to the late-heavy weights $\omega$ of Figure 5 and use $\rho = 252$ market-days per year.

The $17 \times 18$ risk matrix $F$ corresponding to this example is not displayed, but Table 8 summarizes how the total variance of return is decomposed by $f_0$ and $F$, and Figure 9 shows the $XY$ and $EV$ planar representations of the rtndecomp output.

|                          |        |        |
|--------------------------|--------|--------|
| systemic variance $(n f_0^2)$ | 1131   | 13.9%  |
| productive variance $(\sum x^2)$ | 1090   | 13.4%  |
| major nonproductive variance $(\sum y^2)$ | 4427   | 54.3%  |
| other nonproductive variance | 1512   | 18.5%  |
| total variance            | 8160   | 100.0% |

As in Section 6.1.5, $E$ is the image of the efficient portfolio, $\mathbf{p}_E$, of absolute minimum variance. All efficient portfolios from this 18 ETF universe are made up of the four funds circled in red. These four funds have the least risk ($\sigma$) of the eighteen, and their expected returns are among the highest. This is an unusual situation—where risk and return seem to be inversely related.

As in Section 6.1.5, the solid black path in either picture corresponds to the set of minimum-variance portfolios. The path of minimum-$|y|$ portfolios is dashed. The faint interior lines are two-security-portfolio paths.
The efficient, minimum-\(\|y\|\) portfolio at \(x = x_{\text{ECH}} = -3.33\),
\[
p = 41.5\% \text{ ECH} + 24.0\% \text{ EPU} + 29.3\% \text{ EWM} + 5.1\% \text{ EWT},
\]
has expected return \(e = 39.97\) and risk \(\sigma = 12.71\). On the other hand, the single security \text{ECH} is the minimum-\(|y|\) portfolio at this value of \(x\) (and \(e\)), but the risk of \text{ECH} is \(\sigma_{\text{ECH}} = 14.46\), or 13.79\% more than the efficient value. Apparently, in this case, the minimum-\(|y|\) portfolio is not a good approximation of the minimum-\(\|y\|\) portfolio. This is apparent in Figure 9.

6.2.1 The efficient four

Let us restrict our attention to the four emerging market funds, \text{ECH}, \text{EPU}, \text{EPU}, \text{EPU}, that make up the efficient portfolios of Figure 9. Figure 10 shows how these funds grew in 2010, and Table 9 shows the output of \texttt{rtddecomp} restricted to their returns over the last 200 market days of 2010.
Figure 10: 2010 adjusted closing prices of four emerging market ETFs
prices normalized at 100 on 2010-12-31

Table 9: Decomposition of return data – 4 emerging market ETFs
last 200 market-days of 2010 – late-heavy weights

| fund | ECH | EPU | EWM | EWT | \( \hat{V}_T \) |
|------|-----|-----|-----|-----|---------------|
| E    | 39.97 | 52.07 | 30.71 | 36.20 |               |
| F    | 1.45  | 8.51  | -3.94| -0.74 | 91 31.0%      |
|      | -5.98 | 5.01  | 3.04 | 7.55  | 127 43.6%     |
|      | -3.44 | 3.58  | 3.04 | -6.33 | 74 25.4%      |
| \( \hat{V}_T \) | 50 110 34 98 |               | 292 100% |
|      | 17.0% | 37.8% | 11.7% | 33.5% | 100%          |

with \( f_0 = 12.62 \), \( e_0 = 37.47 \), and \( e_F = 1.717 \)

The \( XY \) and \( EV \) planar representations of Table 9 are shown in Figure 11.

The minimum-\( ||y|| \) portfolio path in Table 10 is efficient in either the current four-fund universe or in the original eighteen-emerging-market-fund universe. On the other hand the minimum \( |y| \)-path is based on the four-fund \( XY \) representation in Figure 11. This minimum-\( |y| \) path is an extremely close approximation of the efficient path, with the maximum \( \sigma \)-difference of less than 0.2% occurring at \( x = x_E = 0 \).
Figure 11: Planar representations of return data – 4 emerging markets ETFs last 200 market-days of 2010 – late-heavy weights

\[ e = e_0 + e_F x \]
\[ v = f_0^2 + x^2 + \| y \|^2 \]

\[ P_E = 39.5\% \text{ ECH} + 13.0\% \text{ EPU} + 41.6\% \text{ EWM} + 5.9\% \text{ EWT} \]

Table 10: Optimal portfolio paths – 4 emerging market ETF universe last 200 days of 2010 – late-heavy weights

|                  | minimum-\(\| y \|\) (efficient) path | minimum-\(| y |\) path |
|------------------|-------------------------------------|------------------------|
|                  | \( P_E \) | corner portfolios | EPU | \( x_E \) | corner portfolios | EPU |
| ECH              | 0.395     | 0.464 0.455 0    | 0   | 0.371 | 0.456 0   |
| EPU              | 0.130     | 0.503 0.545 1.000| 0   | 0.156 | 0.544 1.000|
| EWM              | 0.416     | 0 0 0            | 0   | 0.473 | 0 0       |
| EWT              | 0.059     | 0.033 0 0       | 0   | 0    | 0 0        |
| \( x \)          | 0.00      | 4.93 5.30 8.51   | 0.00 | 5.29 8.51 |
| \( e \)          | 37.47     | 45.94 46.57 52.07| 37.47 | 46.55 52.07|
| \( \sigma \)     | 12.62     | 13.55 13.70 16.42| 12.64 | 13.69 16.42|
| avg \( e \)      | 44.77     |                  | 44.77 |
| rms \( \sigma \) | 13.73−    |                  | 13.73+ |
6.3 Relative risk decomposition

The risk, $\sigma$, of an individual fund or portfolio depends only on its periodic returns. However, the systemic, productive, and nonproductive components of this risk depend on the universe of funds in which the fund or portfolio resides. Table 11 illustrates this dependence with a fund and a portfolio from the universes we have considered. In this table each total risk, $\sigma$, is the square root of the sum of the squares of its four component risks.

Table 11: Decomposition of risk relative to fund universe
last 200 days of 2010 – late-heavy weights

|                     | decomposition of EEM | decomposition of PE |
|---------------------|----------------------|--------------------|
| expected return ($e$) | 21.79 21.79          | 37.47 37.47        |
| systemic risk ($f_0$) | 7.93   5.07          | 7.93   12.62       |
| productive risk ($|x|$) | 8.30   9.81          | 4.01   0.00        |
| major nonproductive risk ($|y|$) | 17.37 16.81         | 7.83   0.00        |
| other nonproductive risk | 2.96   6.13         | 4.37   0.00        |
| total risk ($\sigma$) | 21.03 21.03          | 12.62 12.62        |

6.4 2011 results

So far we have restricted our examples to the last 200 market days of 2010 with adjusted closing prices and returns normalized at the closing prices of that year. Late-heavy weights have been emphasized with the idea that a strong performance in the latter part of 2010 should carry over into 2011.

This did not turn out to be the case. Figures 12 and 13 show how the five large ETFs of Section 6.1 and the four emerging market ETFs of Section 6.2.1 performed over 2011. These are graphs of daily adjusted closing prices. Again the prices have been normalized at 100 on 2010-12-31 so that notional portfolio proportions correspond to investment portfolio proportions at 2010 closing prices.
Tables 12 and 13 show the output of \texttt{rtndecomp} as applied to this data. Here we have used the full 252 markets-days of returns with uniform weighting. Now the
expected returns are the total returns of the respective securities over the whole of 2011.

Table 12: Decomposition of return data – 5 large ETF universe
the 252 market-days of 2011 – uniform weights

| fund | IEF | IWB | IWM | EFA | EEM |
|------|-----|-----|-----|-----|-----|
| E    | 15.65 | 1.24 | -4.43 | -12.23 | -18.79 |
| F    | 1.59 | -6.29 | -9.39 | -13.66 | -17.26 | 615 | 20.8% |
|      | -6.91 | 21.05 | 27.09 | 22.92 | 21.30 | 2203 | 74.6% |
|      | -1.80 | 0.21 | -5.70 | 7.11 | -1.19 | 88 | 3.0% |
|      | -1.84 | 1.27 | -3.35 | -2.36 | 4.96 | 46 | 1.6% |
| \(\hat{V}_T\) | 57 | 484 | 866 | 768 | 777 | 2952 | 100% |

with \(f_0 = 4.62\), \(e_0 = 12.74\), and \(e_F = 1.827\)

Table 13: Decomposition of return data – 4 emerging market ETF universe
the 252 market-days of 2011 – uniform weights

| fund | ECH | EPU | EWM | EWT |
|------|-----|-----|-----|-----|
| E    | -26.45 | -21.78 | -2.60 | -21.78 |
| F    | -10.85 | -8.00 | 3.70 | -8.01 | 260 | 28.8% |
|      | 4.02 | -19.66 | 3.58 | 7.08 | 466 | 51.7% |
|      | 9.09 | -1.70 | -0.84 | -9.46 | 176 | 19.5% |
| \(\hat{V}_T\) | 217 | 453 | 27 | 204 | 901 | 100% |

with \(f_0 = 19.88\), \(e_0 = -8.66\), and \(e_F = 1.639\)

It is quite easy to show that the expected expected returns in Tables 12 and 13 are, in fact, total returns for the year. In the general the expected return of a security is given by

\[
e = \rho \sum_{i=1}^{M} \omega_i r_i
\]  

(49)

with the \(r_i (i = 1, \ldots, M)\) being the successive periodic returns. When \(\rho = M\), \(\omega_i = 1/M\), and \(r_i = (a_i - a_{i-1})/a_0\), with the \(a_i (i = 0, 1, \ldots, M)\) being successive adjusted closing prices for the security, equation (49) simplifies to

\[
e = \frac{a_M}{a_0} - 1,
\]  

(50)

which is, essentially by the definition of adjusted closing prices ([Norton(2010)]), the total return of the security over the \(M\) periods.
7 Summary

We have described an orthogonal decomposition of the space of ex post periodic returns for a given universe of securities. The risk space, which is orthogonal to the expected return axis, is decomposed into systemic, productive, and principal nonproductive dimensions. Our `rtndecomp` function accomplishes this decomposition. A technical discussion and listing of this algorithm is given in the appendix.

In the examples of Section 6 we have emphasized the two-dimensional XY-projection of periodic return data. The minimum-$|y|$ path through the portfolio simplex is easily obtained from the XY-projection of the data. The minimum-$|y|$ path can very closely approximate the minimum-variance path for a small universe of securities.

In the future we hope to develop a minimum-variance algorithm of the form

$$
\text{function: } P = \text{minvar}(E, F)
$$

$$
\text{with } [E, F] = \text{rtndecomp}(R, \omega, \rho).
$$

Here $P = [p_1, \ldots, p_n]$ would contain successive corner portfolios of the minimum-variance path through $\Delta$ and include single security portfolios at either end.

Given $[E, F] = \text{rtndecomp}(R, \omega, \rho)$ and such a `minvar` function, the minimum-$|y|$ path would be given by $P_2 = \text{minvar}(E, F(1:2,:))$. More generally, the piecewise-linear paths through $\Delta$ determined by the portfolios in $P_k = \text{minvar}(E, F(1:k,:))$ ($k = 2, \ldots, m$) would approximate the minimum-variance path $P = P_m$ with successively better approximations; moreover each portfolio in $P_k$ would contain at most $k$ securities.
A The flow of the rtndecomp algorithm

Section B gives the complete GNU Octave listing of the rtndecomp algorithm. In this section we describe the ideas behind specific sections of the listing. Certain simplifying assumptions have been made in the interest of clarity. For example we only consider $\rho = 1$ periods per unit time. Scaling for different $\rho$, e.g. $\rho = 252$ market-days per year, occurs at the end of the algorithm, as described in Section 4.6.

This section is arranged in blocks. Each block summarizes a specific section of the rtndecomp code.

**Initial setup** (rtndecomp: 105 – 111)

\[
E = \omega^T R; \quad \# \quad \omega > 0_M, \quad \omega^T 1_M = \sum \omega = 1.
\]

\[
\beta = \sqrt{\omega}; \quad \# \quad \|\beta\|^2 = \beta^T \beta = 1.
\]

\[
Z = \text{diag}(\beta) R - \beta E; \quad \# \quad \beta^T Z = 0, \quad V = Z^T Z \text{ (covariance matrix)}.
\]

The linear isometry

\[
r \mapsto x = \text{diag}(\beta)r, \quad \beta = \sqrt{\omega}
\]

converts the $\omega$-metric into the standard, sum-of-squares-metric for $\mathbb{R}^M: \langle r, r \rangle_\omega = x^T x$. The risk vectors $z_j$ in the $M \times n$ matrix $Z = [z_1, \ldots, z_n]$ are the isometric images of the risk vectors $z_j$ of (5); now the covariance matrix $V = [v_{jk}]$ of (4) is given by $V = [z_j^T z_k] = Z^T Z$.

**QR factorization of Z** (rtndecomp: 113 – 121)

\[
Z = QF; \quad \# \quad \text{compact QR factorization}.
\]

(now \(F \sim [z_1, z_2, \ldots, z_n]\))

The Octave code for the QR factorization actually reads

\[
[Q, F, J] = \text{qr}(Z, 0);
\]

The output consists of an $M \times n$ matrix $Q$ with orthonormal columns, an upper-triangular, $n \times n$ matrix $F$, and a permutation, $J$, of the index sequence $[1, \ldots, n]$. These matrices satisfy $Z(:, J) = QF$.

This is QR factorization with column pivoting. At the end of the actual rtndecomp algorithm, $F$-columns are returned to the initial order with the replacement $F := F(:, J^{-1})$. In this description we assume that $M \geq n$ and that $Z$ is not rank deficient. Thus we can skip column pivoting and start with the QR factorization $Z = QF$.

The columns of $Q$ make up an orthonormal basis for the range of $Z$ in $\mathbb{R}^M$. The columns $f_1, \ldots, f_n$ of the upper-triangular $F$ are the coordinate vectors of the $z_1, \ldots, z_n$ in $Z$ with respect to this basis. We signify this situation with the notation $F \sim [z_1, \ldots, z_n]$. It simply says that “$F$ is the matrix of coordinate vectors for
[\mathbf{z}_1, \ldots, \mathbf{z}_n] \text{ with respect to some orthonormal basis for the range of } [\mathbf{z}_1, \ldots, \mathbf{z}_n].” Then, regardless of the orthonormal basis \( Q \), \( \mathbf{z}_j^T \mathbf{z}_k = f_j^T Q^T Q f_k = f_j^T f_k \) for \( j, k = 1, \ldots, n \), since \( Q^T Q = I_n \).

**Z-flat tangent space** (rtndecomp: 158 – 163)

\[ F(1, j) := F(1, j) - F(1, 1); \quad (j = 2, \ldots, n) \]

[now \( F \sim [\mathbf{z}_1, \mathbf{z}_2 - \mathbf{z}_1, \ldots, \mathbf{z}_n - \mathbf{z}_1] \)]

The matrix \( F \) is still upper triangular, but now the \( \mathbf{z} \)-vectors corresponding to the second through the last columns of \( F \) span the \( Z \)-flat tangent space, \( \mathcal{T}(Z) \).

**Hessenberg QR via Givens** (rtndecomp: 165 – 175)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\pm f_0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} = F \quad \text{(A.1)}
\]

(still \( F \sim [\mathbf{z}_1, \mathbf{z}_2 - \mathbf{z}_1, \ldots, \mathbf{z}_n - \mathbf{z}_1] \))

Here we apply a sequence of \( n - 1 \) Givens rotations, \( G_1, \ldots, G_{n-1} \), to zero the subdiagonal elements of the upper Hessenburg submatrix \( F(:, 2 : n) \). Then

\[ F := G_{n-1}^T \cdots G_1^T F \quad \text{and} \quad Q := Q G_1 \cdots G_{n-1} \]

with \( QF = [\mathbf{z}_1, \mathbf{z}_2 - \mathbf{z}_1, \ldots, \mathbf{z}_n - \mathbf{z}_1] \) at either end of the sequence. This process is described in Section 5.2.4 of [Golub and Van Loan(1989)].

Looking at the last row of the final \( F \) in (A.1) we see that \( \mathbf{q}_n \) is orthogonal to the \( Z \)-flat tangent space: \( \mathbf{q}_n^T (\mathbf{z}_j - \mathbf{z}_1) = 0 \) for \( j = 2, \ldots, n \). It follows that \( \mathbf{z}_0 = \mathbf{q}_n f_{n1} \) is the point on the \( Z \)-flat that is closest to origin, with \( f_0 = |f_{n1}| = ||\mathbf{z}_0|| \) being the systemic risk of the system.

**Extract systemic risk** \( f_0 \) (rtndecomp: 177 – 189)

\[ f_0 = |f_{n1}|; \]

discard the last row of \( F \):

34
\[ F = \begin{bmatrix} x & x & x & x \\ x & 0 & x & x \\ x & 0 & 0 & x \\ x & 0 & 0 & 0 \\ \pm f_0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x & x \\ x & 0 & x & x \\ x & 0 & 0 & x \\ x & 0 & 0 & 0 \end{bmatrix} = F \]

\( m = n - 1; \quad (m = \text{the number of rows of } F) \)

(now \( F \sim [z_1 - z_0, z_2 - z_1, \ldots, z_n - z_1] \))

Here the last column of the previous \( Q \) is discarded so that the remaining \( m = n - 1 \) columns form an orthonormal basis for the \( Z \)-flat tangent space, \( \mathcal{T}(Z) \).

**Z-flat gradient of expected return** (rtndecomp: 204 – 223)

\( g^T F(:,2:n) = E(2:n) - e_1; \quad \# \text{ eflag = false if solution is exact.} \)

\( e_0 = e_1 - g^T F(:,1); \quad e_F = \|g\|; \quad \# \text{ when eflag = false.} \)

If expected return is an affine function of (vector) risk, then

\[ [e_2 - e_1, \ldots, e_n - e_1] = g^T [z_2 - z_1, \ldots, z_n - z_1] \]

can be solved exactly for \( g \in \mathcal{T}(Z) \). This will be the case (and eflag will be false) unless \( 1_M \) is parallel to the \( R \)-flat (Proposition 1).

In this discussion we will assume that \( 1_M \) is not parallel to the \( R \)-flat. Then \( u_1 = g/\|g\| \in \mathcal{T}(Z) \) is direction of steepest increase in expected return, and

\[ e = e_0 + e_F u_1^T z \]  \hspace{1cm} (A.2)

holds for all \( z \) in the \( Z \)-flat, where \( e_0 = e_1 - g^T z_1 \) and \( e_F = \|g\| \).

In this block we solve for the coordinates of \( g \) with respect to the current orthonormal basis, \( Q = [q_1, \ldots, q_m] \), for the tangent space \( \mathcal{T}(Z) \). Then \( e_0 \) and \( e_F \) are computed from the coordinate representation. In the following block, \( g \in \mathbb{R}^m \) denotes the coordinate vector corresponding to \( g \in \mathcal{T}(Z) \).

**Householder reflection** (rtndecomp: 225 – 235)

\[ H = I_m - \beta vv^T; \quad \beta = 2/\|v\|^2; \quad Hg = \delta_1 \|g\|; \]

\( F := HF; \)

(still \( F \sim [z_1 - z_0, z_2 - z_1, \ldots, z_n - z_1] \))

\( F(:,j) := F(:,j) + F(:,1); \quad (j = 2, \ldots, n) \)

(now \( F \sim [z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0] \))

The first basis vector, \( q_1 \), of the current basis \( Q \) has coordinate vector \( \delta_1 \), the first column of the \( m \times m \) identity matrix \( I_m \). If \( H \) is the Househoulder reflection of \( \mathbb{R}^m \) that maps \( g \) to \( \delta_1 \|g\| \), then the first column of \( QH \) is the direction of maximum
increase in expected return in $J(Z)$. Thus, after the replacements $Q := QH$ and $F := HF$, we still have $F \sim [z_1 - z_0, z_2 - z_1, \ldots, z_n - z_1]$, but now $e_F$ times the first coordinates of $F$ produces the corresponding changes in expected return:

$$e_F F(1, :) = [e_1 - e_0, e_2 - e_1, \ldots, e_n - e_1].$$

Then the replacements $F(:, j) := F(:, j) + F(:, 1)$ $(j = 2, \ldots, n)$ result in

$$F \sim [z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0]$$

and

$$E = [e_1, \ldots, e_n] = e_0 + e_F F(1, :)$$

with $V = nf_0^2 + F^T F$.

**Principal components of nonproductive risk** (rtndecom: 237 – 243)

$$F(2 : m, :) = U \Sigma V^T; \quad \Sigma = \text{diag} (\tau_2, \ldots, \tau_m); \quad F(2 : m, :) := \Sigma V^T;$$

(final $F \sim [z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0]$)

Entering this block the first row of $F$ consists of the coordinates of $[z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0]$ in the productive risk direction. The remaining $m - 1$ rows represent the nonproductive risks.

The principal components of nonproductive risk are computed with the code

$$[U, S, V] = \text{svd}(F(2 : m, :) , 0);$$

The output matrices $U$ and $V$ (not to be confused with the covariance matrix $V$) have orthonormal columns and dimensions $(m - 1) \times (m - 1)$ and $n \times (m - 1)$, respectively. The principal nonproductive risks, $\tau_2 \geq \ldots \geq \tau_m > 0$, of Definition 1 are contained in the diagonal matrix $\Sigma$ ( = $S$ in the rtndecomp code). Now the replacements

$$F(2 : m, :) := \Sigma V^T, \quad Q(:, 2 : m) := Q(:, 2 : m) U$$

maintain $F \sim [z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0]$, but organize the last $m - 1$ rows of $F$ in the principal directions of nonproductive risk, with row norms $\|F(i, :)\| = \tau_i$ for $i = 2, \ldots, m$.

In this discussion we have described how the basis matrix $Q$ changes from one block to another. It is always the case that $Q^T Q = I$, though the size of the identity matrix diminishes from $n \times n$ as the algorithm progresses. The actual rtndecomp algorithm makes no attempt to keep track of the changing $Q$. 


B The Octave algorithm

# Function: [E, F, f0, e0, eF] = rtndecomp(rtns, wgts, pput)
#
# Purpose
# To decompose financial return data into orthogonal
# "systemic", "productive", and "nonproductive"
# risk-factors.
# Input
# rtns -- M by n matrix of periodic returns.
# wgts -- M by 1 vector of positive weights or a scalar,
#   default: wgts = ones(M, 1) / M if wgts is a
#   scalar or rtns is the only input.
# pput -- periods per unit time. default: pput = 1.
# Output
# E -- 1 by n matrix of expected returns.
# F -- m by n matrix of risk coefficients.
# rank(F) = m unless F = zeros(1, n).
# f0 -- systemic risk (nonnegative).
# e0 -- systemic expected return.
# eF -- return per unit change in the F(1,:)--direction.
# (eF >= 0)
# Global output
# eflag -- true iff the constant return vector, ones(M, 1),
#   parallels the returns flat or, said another way,
#   ones(M, 1) = rtns * x
#   for some n-vector x with sum(x) = 0.
# Variables and relationships
# The 1 x n expected return matrix is
#   E = (wgts * pput)' * rtns.
# The n x n covariance of return matrix is
#   V = Z' * diag(wgts * pput) * Z,
# where Z is the M x n matrix of risk vectors
#   Z = rtns - ones(M, 1) * E / pput.
# The output variables satisfy
# 1) V = f0^2 + F' * F,
# 2) E = e0 + eF * F(1,:) unless eflag is true;
#   then this relationship is approximate. However
#   mean(E) = mean(e0 + eF * F(1,:)) is always true.
# We refer to norm(F(1,:)) as the productive risk
# when eF is nonzero.
# 3) The row norms,
#   tau(i) = norm(F(i,:)) (i = i0, .., m),
are the principal nonproductive risks, where

\[ i0 = 2 \text{ if } eF > 0, \text{ and } i0 = 1 \text{ if } eF = 0. \]

The nonproductive risks \( \tau(i) \) decrease with

increasing \( i \), and the corresponding row vectors

are pairwise orthogonal:

\[ F(i, :) \cdot F(j, :) = 0 \ (i0 \leq i < j \leq m). \]

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# details.

function [E, F, f0, e0, eF] = rtndecomp(rtns, wgts, pput)

if !((nargin >= 1 || nargin <= 3)
    usage("[E,F,f0,e0,eF]=rtndecomp(rtns,wgts,pput)");
endif

[M, n] = size(rtns);
global eflag;
eps0 = eps * 1e2;  # precision

### check/set wgts & pput
if nargin == 1 || isscalar(wgts)
    wgts = ones(M, 1) / M;
else
    if !isvector(wgts)
        error("wgts must be a scalar or a vector");
    endif
    if length(wgts) != M
        error("wgts must have length M");
    endif
endif
if any(wgts <= 0)
    error("wgts must be positive");
endif
if rows(wgts) == 1
    wgts = wgts';
endif
wgts /= sum(wgts);
endif
if nargin == 3
    if pput < 1
        error("g must be 1 or greater");
    endif
else
    pput = 1;
endif
sqrtpput = sqrt(pput);

## expected return matrix E and "risk" vectors Z for pput = 1
E = wgts' * rtns;
if nargout <= 1;
    E *= pput;
    return;
endif
sqrtwgts = sqrt(wgts);
Z = diag(sqrtwgts) * rtns;
Z -= sqrtwgts * E;  #% V = Z' * Z = covariance matrix
sigZ = norm(Z, "fro");  # square root of total variance

## QR factorization of Z with column pivoting
[Q, F, J] = qr(Z, 0);  # Q is not used
m = rows(F);
epsZ = eps0 * sigZ;
while m > 1 && norm(F(m, m : n)) <= epsZ
    F(m, :) = [ ];
    m -= 1;
endwhile
Jinv = [1 : n] * eye(n)(:, J)';

## finish up when m = 1 and all coefficients of F are the same
if n == 1
    f0 = abs(F);
    if pput > 1
        E *= pput; f0 *= sqrtpput;
    endif
    e0 = E; eF = 0; F = 0; eflag = 0;
return;
elseif m == 1
    Fmin = min(F); Fmax = max(F);
    if Fmax - Fmin < eps0 * max(abs(Fmin), abs(Fmax))
        e0 = mean(E);
        f0 = abs((Fmin + Fmax)/2);
        eF = 0;
        F = zeros(1, n);
        ee = sum(E.*E);
        eflag = ee - n * e0 * e0 > eps0 * ee;
        if pput > 1
            E *= pput; e0 *= pput; f0 *= sqrtpput;
        endif
        return;
    endif
endif
endif

## At this point the columns of F are the coordinate vectors
# of the columns of Y = [y1, y2, ..., yn] = Z(:, J) with
# respect an orthonormal basis for the linear space, L(Z),
# spanned by the columns of Y or Z. We represent this
# situation with the notation
#   F ~ [y1, y2, ..., yn].
# As the algorithm progresses the orthonomal basis for L(Z)
# changes. We do not keep track of the changing basis, but
# we do continue to note the elements of L(Z) represented by
# the coordinate-vector columns of F.

## E-flat & Z-flat tangent spaces
e1 = E(J(1)); # base expected return
y1 = Z(:, J(1)); # base risk vector
nm1 = n - 1;
B = E(J(2 : n)) - e1; # differential expected returns
F(1, 2 : n) = F(1, 1); # differential risks

## QR decomposition of Z-flat tangent space by Givens
# rotations
for j = 2 : m
    jm1 = j - 1;
    [cs, sn] = givens(F(jm1, j), F(j, j));
    GV = [cs, sn; -sn, cs]; # Givens rotation
    F(jm1 : j, 1) = GV * F(jm1 : j, 1);
    F(jm1 : j, j : n) = GV * F(jm1 : j, j : n);
    F(j, j) = 0;
endfor
# $F \sim [y_1, y_2 - y_1, \ldots, y_n - y_1]$

## Systemic risk $f_0$

If $\|F(m, m + 1 : n)\| \leq \epsilon Z$

### $\text{the Z-flat tangent space has dimension } m - 1$

$$f_0 = \text{abs}(F(m, 1));$$

$m = 1;$

Else

### $\text{the Z-flat tangent space has dimension } m$

$$f_0 = 0;$$

Endif

# $F \sim [y_1 - y_0, y_2 - y_1, \ldots, y_n - y_1]$

where $y_0 = z_0$ is the point in the Z-flat that is closest to the origin.

## Check for constant $E$

$E_{\text{min}} = \min(E);$ $E_{\text{max}} = \max(E);$

If $E_{\text{max}} - E_{\text{min}} < \epsilon_0 \times \max(|E_{\text{min}}|, |E_{\text{max}}|)$

### $E$ is constant: finish up

$$F(:, 2 : n) += F(:, 1) \times \text{ones}(1, n - 1);$$

# $F \sim [y_1 - y_0, y_2 - y_0, \ldots, y_n - y_0]$

$[U, S, V] = \text{svd}(F(:, Jinv), 0);$  

$F = S \times V'$;

$E = \text{pput};$ $e_0 = \text{mean}(E);$  

$e_F = 0;$ $f_0 = \text{sqrtpput};$ $F = \text{sqrtpput};$

Return;

Endif

## $Z$-flat direction of maximum increase

### In expected return

$g = (B / F(:, 2 : n))'$;  

### $E$-gradient if eflag

$$e_{\text{flag}} = \text{norm}(g' \times F(:, 2 : n) - B) / \text{norm}(B) > \epsilon_0;$$

If $!e_{\text{flag}}$ # $E = e_0 + e_F \times F(1, :)$ will be exact

$$e_0 = e_1 - g' \times F(:, 1);$$

$e_F = \text{norm}(g);$  

# $F \sim [y_1 - y_0, y_2 - y_1, \ldots, y_n - y_1]$

Else # $E = e_0 + e_F \times F(1, :)$ will be approximate

$$F(:, 2 : n) += F(:, 1) \times \text{ones}(1, n - 1);$$

# $F \sim [y_1 - y_0, y_2 - y_0, \ldots, y_n - y_0]$

$$v = \text{mean}(F')';$$  

### $v = y_{\text{mean}} - y_0$

$G = F - v \times \text{ones}(1, n);$  

# $G \sim [y_1 - y_{\text{mean}}, y_2 - y_{\text{mean}}, \ldots, y_n - y_{\text{mean}}]$

$e_{\text{mean}} = \text{mean}(E);$  

$C = E(J) - e_{\text{mean}};$
\begin{verbatim}
223  g = (C / G)';  # E-gradient if !eflag
224  e0 = emean - g' * v;
225  eF = norm(g);
226  endif
227
228  ## the Householder reflection, \( Hg = (x, 0, \ldots, 0) \)
229  # puts the productive risk coordinates in \( F(1, :) \)
230  [v, b] = housh(g, 1, 0);
231  signx = sign((g - (b * v) * (v' * g))(1));
232  F = F - (b * v) * (v' * F);
233  F(1, :) *= signx;
234  if ~eflag
235    # \( F \sim [y1 - y0, y2 - y1, \ldots, yn - y1] \)
236    F(:, 2 : n) += F(:, 1) * ones(1, n - 1);
237  endif
238  # \( F \sim [y1 - y0, y2 - y0, \ldots, yn - y0] \)
239
240  ## put the principal components of nonproductive risk
241  # in \( F(2 : m, :) \)
242  if m > 1
243    [U, S, V] = svd(F(2 : m, :), 0);
244    F(2 : m, :) = S * V';
245  endif
246  # \( F \sim [y1 - y0, y2 - y0, \ldots, yn - y0] \)
247
248  ## unpermute columns of \( F \)
249  F = F(:, Jinv);
250  # \( F \sim [z1 - z0, z2 - z0, \ldots, zn - z0] \)
251
252  ## scale these results if pput != 1
253  if pput != 1
254    E *= pput; e0 *= pput;
255    eF *= sqrtpput; f0 *= sqrtpput; F *= sqrtpput;
256  endif
257
258 endfunction
\end{verbatim}
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