Topological phases in ring resonators: recent progress and future prospects

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Topological photonics has emerged as a novel paradigm for the design of electromagnetic systems from microwaves to nanophotonics. Studies to date have largely focused on the demonstration of fundamental concepts, such as non-reciprocity and waveguiding protected against fabrication disorder. Moving forward, there is a pressing need to identify applications where topological designs can lead to useful improvements in device performance. Here we review applications of topological photonics to ring resonator-based systems, including one- and two-dimensional resonator arrays, and dynamically-modulated resonators. We evaluate potential applications such as quantum light generation, disorder-robust delay lines, and optical isolation, as well as future research directions and open problems that need to be addressed.

Keywords: Topological photonics; Silicon photonics; Ring resonator; Coupled resonator optical waveguide; Optical isolator

I. INTRODUCTION

Demand for miniaturised optical components such as waveguides and lenses that can be incorporated into compact photonic devices is pushing fabrication techniques to their limits. Continued progress will require new approaches to minimise the detrimental influence of fabrication imperfections and disorder. Topological photonics is a young sub-field of photonics which seeks to address this challenge using novel design approaches inspired by exotic electronic condensed matter materials such as topological insulators1,2. Loosely speaking, topological systems provide a systematic way to create disorder-robust modes or observables using a collection of imperfect components or modes. For example, certain classes of “topologically nontrivial” systems exhibit special edge modes which can propagate reliably without backscattering even in the presence of strongly scattering defects, forming the basis for superior optical waveguides.

One natural setting where this robustness can potentially be useful is in the design of integrated photonic circuits3, where the strong light confinement brings sensitivity to nanometer-scale fabrication imperfections. However, there is not yet any disruptive killer application where topological photonic devices have achieved superior performance compared to mature design paradigms, despite growing interest in topological photonics since seminal works published in 20084,5. To help bridge this gap, several reviews have been published recently, some providing comprehensive surveys of topological photonic systems1,2,6,7, and others focusing on specific applications such as incorporating topological concepts into active devices such as lasers8, nanophotonics9, non-reciprocal devices10, and nonlinear optical processes11.

The aim of this brief review is to complement these recent surveys with a concise introduction to applications of ideas from topological photonics to optical ring resonator-based systems. Ring resonators are a versatile and important ingredient of integrated photonic circuits, as they can be used as compact filters, sensors, and delay lines, and as a means of enhancing nonlinear optical effects12. However, active tuning is typically required to compensate for resonance shifts induced by various perturbations, increasing device complexity and energy consumption13,14. We will discuss some of the ways in which topological designs may lead to superior devices with improved reliability. We will focus on topological systems formed by coupling together multiple resonators to form a lattice, or by considering coupling between multiple resonances of a single ring using external modulation or nonlinearity. In both cases the lattice formed by the coupled modes can be designed to have topological band structures and achieve protection against certain kinds of disorder.

The outline of this article is as follows: Sec. II starts with a brief review of the basic concepts underlying topological photonics and ring resonators. Next, we review the implementation of topological photonics using arrays of coupled ring resonators in Sec. III. Sec. IV discusses how we can use dynamic modulation as a novel degree of freedom for the engineering of topological effects in ring resonators. We discuss future research directions and promising potential applications of topological ring resonators in Sec. V, before concluding with Sec. VI.

II. BACKGROUND

A. Topological photonics

The key idea underlying topological photonics is the bulk-boundary correspondence, which states that the
topological properties of the photonic band structure of a bulk periodic medium can be related to the appearance of robust modes localized to edges or domain walls of the system. These “topologically-protected” edge modes have very different properties compared to conventional defect modes. For example, as they arise due to the topological properties of the bulk photonic band structure, they are robust against certain classes of local perturbations at the edge or domain wall and can only be removed by large perturbations capable of closing the bulk band gap. Thus, topology allows us to create special robust boundary modes protected by a higher-dimensional bulk.

For the purposes of this review, two classes of topological modes are of interest. One-dimensional media can exhibit topologically-protected modes localized to the ends of the system. These modes have their frequencies pinned to the middle of the band gap, even in the presence of disorder, as long as certain symmetries are preserved. So in this case, topology provides a systematic way to create localized defect modes at a specific frequency.

The second important class of topological modes are edge states of two-dimensional topological systems. These exhibit either unidirectional or spin-controlled propagation along the edge. The former appear in topological systems with broken time-reversal symmetry (known as quantum Hall topological phases), while in photonics the latter require a combination of time-reversal symmetry and some other internal symmetry protecting a spin-like degree of freedom (known as quantum spin-Hall topological phases). In both cases the resulting topological edge states are of interest as a means of creating disorder-robust optical waveguides.

The generic approach to create a topological system is to start with a simple periodic medium exhibiting some degeneracy in its photonic band structure, and then break one of the symmetries protecting the degeneracy to open a band gap. Breaking the symmetry in the right way will create a topologically nontrivial band gap hosting protected edge modes. Fig. 1 shows a simple one-dimensional example of this idea, where one can reduce the translation symmetry of a simple one-dimensional periodic lattice by staggering its site positions to create a lattice of dimers. Breaking the symmetry opens a mini-gap the lattice’s band structure, analogous to the Su-Schrieffer-Heeger tight binding model for electron transport in polymers. There are two inequivalent ways to dimerize a finite lattice, corresponding to two inequivalent topological phases. Protected mid-gap end modes emerge if the dimers are broken at the ends of the system, corresponding to the non-trivial phase.

To create two-dimensional topological phases there are two common approaches: using synthetic gauge fields or by perturbing honeycomb lattices. Synthetic gauge field refers to complex (direction-dependent) coupling between different sites of the photonic lattice, arising in systems with non-reciprocity or broken time-reversal symmetry. Complex coupling is formally equivalent to the effect of an electromagnetic vector potential on electron transport. Using a suitable position-dependent synthetic gauge field allows one to create an effective magnetic field for light, resulting in an analogue of the quantum Hall topological phase. The second approach is based on honeycomb lattices, which exhibit Dirac point degeneracies at the corners of its Brillouin zone. Weak symmetry-breaking perturbations are capable of lifting the degeneracy to create a topological band gap. One can break either time reversal symmetry to create the quantum Hall phase, or other internal symmetries (e.g. related to sublattice or polarization degrees of freedom) to create quantum spin Hall phases.

It is important to stress that these topological photonic systems are only analogous to the topological tight bind-
ing models used to describe electronic condensed matter systems. Thus, while the electronic quantum Hall phase exhibits a Hall conductivity precisely quantized to 1 part in $10^9$, in photonics various effects such as material absorption, out-of-plane scattering, and imperfect symmetries mean that the topological protection is only approximate, so the edge modes are only protected against certain classes of perturbations. Thus, it is crucial to identify systems where topology provides protection against the most significant sources of disorder. Most studies to date rely on deliberately-introduced defects to demonstrate topological protection.

Other classes of topological models beyond the above gapped topological insulating phases are attracting increasing attention. For example, higher-order topological phases can give rise to modes localized to the corners of two-dimensional systems\(^\text{20}\). Three-dimensional Weyl topological phases exhibit protected degeneracies in their bulk band structure\(^\text{21,22}\). Non-Hermitian topological phases can emerge in systems with structured gain or loss\(^\text{23–26}\). For further discussion on photonic topological phases we direct the reader to Refs. 1, 2, 6, and 7.

B. Ring resonators

Ring resonator generally refers to any optical waveguide forming a closed loop, regardless of its size or shape\(^\text{12}\). Fig. 2 presents some examples of ring resonators, including micrometer-scale integrated optical resonators, millimeter-scale spoof plasmon resonators, and kilometer-scale fiber loops. Microwaves and fiber loops provide a convenient setting for studying novel design approaches as they are easier to fabricate, and in some cases can be assembled using off-the-shelf optical components, while the integrated photonic circuits are of more interest for potential applications due to their compactness and scalability.

Resonances occur whenever the propagation phase accumulated over a round trip forms a multiple of $2\pi$. Key characteristics of ring resonators are their resonance width, free spectral range (FSR; the spacing between neighbouring resonances), quality factor (resonance frequency divided by width), and finesse (free spectral range divided by resonance width). The resonance width is dictated both by the intrinsic losses due to waveguide bending, scattering losses due surface roughness, absorption, and extrinsic losses introduced by coupling the resonator to external waveguides.

The free spectral range is inversely proportional to the round trip path length. In systems with small FSR such as long fiber loops one typically studies the propagation dynamics in the time domain. For on-chip signal processing applications it is desirable to have a large free spectral range, exceeding the signal bandwidth, demanding a high refractive index contrast to minimise waveguide bending losses. The resulting strong light confinement in turn makes the ring’s resonances highly sensitive to local perturbations to the refractive index, which can be both a strength and a weakness. For example, the sensitivity to perturbations allows ring resonators to be employed as highly compact and efficient sensors and optical switches\(^\text{27–30}\). On the other hand, for spectral filtering applications active tuning is typically to keep the resonance fixed at the desired frequency\(^\text{31}\).

A high quality factor is desirable for nonlinear optics applications. Typical nonlinear optical effects present in integrated silicon photonic circuits include thermal nonlinearity; the Kerr effect, two photon absorption, free carrier generation, and free carrier dispersion\(^\text{12}\). Of these, the thermal nonlinearity is usually dominates, resulting in a bistable response as the input frequency is tuned\(^\text{35}\). The differing time scales of the thermal and carrier-induced nonlinear responses and lead to complex pulsating dynamics\(^\text{36}\). Ultra-fast Kerr nonlinearities are employed for frequency mixing applications, where the relatively uniformly spacing of the ring resonances are ideal for frequency comb generation\(^\text{37–41}\).

The strong dispersion close to resonance allows ring resonators to be used to delay and store optical signals. For single rings there is a trade-off between the delay time and the operating bandwidth (their product is a fixed constant). Larger delays for a fixed bandwidth

![Image](https://example.com/image.png)
can be achieved using arrays of coupled rings, known as coupled resonator optical waveguides (CROWs). In integrated photonics, however, disorder leads to misalignment of the individual rings, severely degrading the CROW performance. Therefore one requires either clever designs that are robust against fabrication variations or active tuning to compensate for the disorder.

The thermo-optic effect is the most common way to tune integrated photonic ring resonators, using micro-heaters placed on top of the individual rings. While thermal tuners offer a large tuning range, they are slow (operating on the µs scale), have poor energy efficiency, require careful design to minimise cross-talk between different tuners, and inevitably introduce sensitivity to environmental temperature fluctuations. Electro-optic tuning can operate much faster (sub ns) and with greater energy efficiency, however the tuning range is smaller and additional absorption losses are introduced.

There is growing interest in methods to improve the reliability of ring resonator-based devices, such as creating temperature-insensitive resonators by combining materials with opposite thermo-optic coefficients, and introducing tunable backscattering to cancel out backscattering caused by fabrication imperfections via destructive interference. For an in-depth discussion of the physics and applications of photonic ring resonators we recommend Refs. 12, 44, and 54.

III. TOPOLOGICAL COUPLED RESONATOR LATTICES

Arrays of coupled ring resonators provide a flexible platform for implementing topological lattice models. In weakly coupled arrays light propagation is governed by effective tight binding Hamiltonians which describe the evanescent coupling of light between neighboring resonators. The magnitude of the coupling coefficients can be controlled simply by varying the separation between the resonators. Furthermore, coupling resonant rings via anti-resonant links allows one to tune the phase of the coupling. One can effectively break time-reversal symmetry by considering modes with a fixed “spin” (clockwise or anti-clockwise circulation direction) and neglecting backscattering within the rings. On the other hand, taking this backscattering into account introduces an in-plane effective magnetic field. Together, these ingredients enable the realization of a wide variety of topological tight binding models in one and two dimensions.

Studies of topological ring resonator lattices began with the seminal theoretical work of Hafezi et al., which showed that one can effectively break time-reversal symmetry by exciting modes with a particular circulation direction and then use asymmetric link rings to implement an analogue of the quantum Hall lattice model. The asymmetric link rings result in a phase difference between the two coupling directions, equivalent to a vector potential in the tight binding Hamiltonian. By making this hopping phase inhomogeneous (e.g. proportional to the y coordinate) one can create a vector potential formally equivalent to an out-of-plane effective magnetic field, which implements a lattice model of the quantum Hall effect, as shown in Fig. 3(A,B). The topology of the quantum Hall lattice is characterized by the quantized Chern number, which determines the number of chiral backscattering-protected states at the edge of the lattice. Note that as the underlying system obeys time reversal symmetry the opposite spin hosts counter-propagating edge states (forming a quantum spin Hall phase), and hence the protection against backscattering only holds as long as the two spins remain decoupled.

Following this proposal the first experiment was reported in 2013. In contrast to the ideal lossless case,
in practice the individual rings exhibit losses (e.g. due to roughness of the waveguide walls, absorption). This sets an upper limit on the propagation length of the topological edge states, even though they may be protected against backscattering induced by disorder in the rings’ resonant frequencies. These losses were harnessed in the experiment to directly image the propagation of the topological edge states by measuring the light scattered out of the device plane. Fig. 3(C) shows the device and an image of the topological edge states, which reliably travel from the input port to an output port. In contrast the bulk states exhibited Anderson localization due to the strong intrinsic disorder present in the system. Subsequent experiments measured the delay times through several devices, showing indeed that the topological edge states preserve ballistic light transport with low device-to-device fluctuations in the photon delay times, whereas regular CROWs exhibit strong fluctuations due to the disorder-induced scattering\(^57\). The quantum Hall lattice model was also implemented using silicon nitride ring resonators, however the propagation distance of the topological edge states was limited by the stronger intrinsic losses\(^58\).

Other topological tight binding models have also been studied using ring resonator lattices. A higher order topological phase exhibiting protected corner states was demonstrated in 2019\(^59\). Next-nearest neighbour coupling was used to implement an analogue of the Haldane model\(^19\), which exhibits quantum Hall edge states even in the absence of a net effective magnetic flux\(^60,61\). Zhu et al. have proposed a honeycomb lattice design hosting topological edge states which co-exist with a nearly flat bulk band\(^62\). It is also possible to shrink these two-dimensional lattices down to quasi-one-dimensional delay lines, which maintain some resistance against disorder\(^63\).

In 2018 the ring resonator platform was used to implement one- and two-dimensional topological laser models by embedding a quantum well gain medium into the resonators\(^64-67\). One-dimensional experiments were carried out using the Su-Schrieffer-Heeger lattice, created by staggering the separation between neighbouring rings. Pumping one of two sublattices comprising the array induced lasing of its mid-gap topological edge states\(^64,65\). Two-dimensional lasing experiments utilised the quantum Hall lattice, where a pump localized to the lattice edges induced lasing in its chiral edge states\(^66,67\). In both cases, the potential advantage of the topological approach is the ability to induce lasing in collective array modes that are protected against certain types of disorder in the individual rings. For further information on topological lasing, we recommend the recent reviews Refs. 8 and 11.

One advantage of the silicon photonics platform is the ability to implement actively-tunable devices, for example by incorporating thermo-optic phase shifters to tune the resonant frequencies of the individual rings. Mittal et al.\(^68\) employed tunable phase shifters at the edge of the quantum Hall lattice to directly measure the topological winding number of the edge states. Similar tuning of the effective magnetic flux in ring-shaped lattices enables the observation of the Hofstadter butterfly via the lattice’s scattering resonances\(^69,70\). There are recent proposals to implement phase shifters throughout the entire lattice in order to tune its topological properties, thereby enabling one to switch the topological edge states on or off or re-route them between different output ports\(^71-73\). Zhao et al. demonstrated controllable re-routing of topological states in the quantum Hall resonator lattice using structured bulk gain\(^73\).

The above studies of topological resonator lattices focused on the weak coupling limit described by tight binding models. However, topological phases can also arise in the strongly coupled lattices, without requiring external modulation or anti-resonant link rings\(^73-78\). These “anomalous Floquet” phases are not predicted by the tight binding approximation and only emerge when con-
sidering the full transfer matrix description of the light coupling between neighbouring rings. A ring resonator lattice implementing an anomalous Floquet topological phase was first demonstrated in 2016 using spoof plasmons at microwave frequencies\textsuperscript{32}. Similar anomalous edge states can emerge in weakly coupled arrays with gain and loss, where the transfer matrix description is essential to account for the growth or decay of the optical field as it circulates through each ring\textsuperscript{79}.

The anomalous Floquet topological phase was scaled up to telecom wavelengths by Afzal et al. using the silicon photonic resonator lattice shown in Fig. \textsuperscript{474}. The main distinguishing feature compared to previous observations of quantum Hall edge states is the presence of edge states in all of the array’s band gaps. One advantage of the strongly-coupled anomalous Floquet phases is that their bulk bands and edge states can have bandwidths comparable to the rings’ free spectral range, in contrast to tight binding lattices which are typically restricted to small bandwidths.

**IV. TOPOLOGY OF DYNAMICALLY-MODULATED RESONATORS**

Resonators undergoing the dynamic modulation of the refractive index provide a flexible way to construct effective lattice models described by time-dependent tight binding Hamiltonians. Such systems break time-reversal symmetry and provide an important platform for implementing various topological phenomena. This specific subject was started with the pioneering paper by Fang et al.\textsuperscript{80}, who showed that in photonic systems where the refractive index is harmonically modulated, the modulation phase actually gives rise to an effective gauge potential for photons. Based on this idea, a photonic Aharonov-Bohm interferometer was proposed as a design for an optical isolator.

A subsequent work by Fang et al. proposed a scheme for generating an effective magnetic field for photons\textsuperscript{18}, based on spatially inhomogeneous modulation phases. The effective magnetic field for photons breaks time-reversal symmetry and can be used to induce nontrivial quantum Hall phases in two-dimensional resonator lattices. Topologically-protected one-way edge modes can be excited in this dynamically-modulated resonator lattice, which are robust against defects [see Fig. 5]. In contrast to the lattices discussed in the previous section, these topological modes are robust against spin-flipping disorders.

Experiments based on these ideas were implemented in a variety of platforms. The first proof-of-principle demonstration of a photonic Aharonov-Bohm interferometer used an electrical network at radio frequencies\textsuperscript{82}. In 2014, the photonic Aharonov-Bohm effect has been demonstrated at visible wavelengths by utilizing an effective gauge potential induced by photon-phonon interactions\textsuperscript{83}. Later in the same year, the presence of an effective gauge field has been constructed by using the on-chip silicon photonic technology, where the refractive index of the silicon coupled waveguides was modulated by an applied voltage\textsuperscript{84}. These works are important proofs-of-concepts for photonic gauge potentials induced by dynamic modulation.

These proposals for creating effective gauge potentials in dynamically-modulated resonators have triggered many follow-up studies on the manipulation of light via modulation phases. For example, spatially inhomogeneous distribution of modulation phases in two-dimensional resonator lattices can be used to control the flow of light\textsuperscript{85–88}. A spatially homogeneous but time-dependent distribution of modulation phases in three dimensions has also been considered, which results in propagation analogous to the dynamics of electrons in the presence of a time-dependent electric field\textsuperscript{89}. By temporally modulating the effective electric field, one can time reverse the propagation of one-way quantum Hall edge states\textsuperscript{90}.

In the above studies the modulations under consideration were treated weakly, so that the dynamics satisfy the rotating-wave approximation. Topological phase transitions have also been studied in the ultrastrong coupling regime, where the rotating wave approximation fails. In the ultrastrong coupling regime the topological edge modes have been shown to exhibit larger bandwidth and less susceptibility to losses\textsuperscript{91}. On the other hand, experimental efforts are still ongoing to achieve ultrastrong coupling using ring resonators. As an experimental proof of concept, light guiding by an effective gauge potential\textsuperscript{87} has been demonstrated in tilted waveguide arrays\textsuperscript{92}. Moreover, lithium niobate microring resonators have been coupled and modulated by external microwave excitation, which leads to an effective photonic molecule\textsuperscript{83}. Various platforms have therefore been shown as potential candidates for exploring resonators.
under strong dynamic modulation.

Dynamically-modulated resonators also provide an important platform for studying topological physics in real space. It turns out that they also provide a unique platform to explore higher-dimensional topological physics in lower dimensional physical systems, by incorporating synthetic dimensions in photonics\(^5\). Inspired by earlier works of synthetic dimensions in lattice systems\(^1\), resonators supporting multiple degenerate modes with different orbital angular momentum (OAM) have been used to simulate the topological physics, where the synthetic dimension is constructed by coupling modes with different OAM using a pair of spatial light modulators\(^\text{97–100}\). On the other hand, dynamically-modulated ring resonators with the modulation frequency close to the resonators’ free-spectral-range (FSR) naturally gives rise to a synthetic dimension along the frequency axis of light\(^\text{81,101}\). Using this idea, two-dimensional topologically-protected one-way edge states have been proposed using one-dimensional resonator lattices [see Fig. 5(C)]. Such edge modes convert the frequency of light unidirectionally towards higher (or lower) frequency components as shown in Fig. 5(D), which could form the basis for a topological frequency converter\(^\text{81}\). The four-dimensional quantum Hall effect can also be studied using this approach, by combining a three-dimensional resonator lattice with a fourth synthetic frequency dimension\(^\text{101}\).

Synthetic dimensions in dynamically-modulated resonators also provide a platform for exploring novel topological phases that are difficult to implement using pure spatial lattices. For example, using synthetic dimensions it is possible to implement three-dimensional Weyl\(^\text{102,103}\) and topological insulating phases\(^\text{104}\) using two-dimensional arrays of rings. In two-layer two-dimensional ring lattices, higher-order topological phases exhibiting corner states have also been designed\(^\text{105}\). Based on the scheme of creating topological system in a one-dimensional array of ring resonators, a mode-locked topological insulator laser in synthetic dimensions has been suggested, which triggers potential applications for developing active photonic devices\(^\text{106}\).

One significant advantage of synthetic dimensions implemented using dynamically modulated resonators is the ability to flexibly control the connectivity of the couplings in the synthetic space, which is difficult to achieve in real space lattices\(^\text{107–109}\). For example, one can introduce long-range couplings along the synthetic frequency dimension by using modulation frequencies that are multiples of the FSR, enabling emulation of the two-dimensional Haldane model using three rings\(^\text{110}\). Moreover, in a single resonator, one can combine two internal degrees of freedom of light such as frequency and OAM to construct a two-dimensional synthetic lattice\(^\text{111}\). In such synthetic lattices, the effective magnetic field can be naturally introduced through the additional coupling waveguides, thereby creating topologically-protected one way edge states. This may enable the robust manipulation of entanglement between multiple degrees of freedom of light.

Besides topological physics, many other interesting analogies with quantum and condensed matter physics can be demonstrated using dynamically-modulated resonators, including Bloch oscillations\(^\text{112,113}\), parity-time symmetric systems\(^\text{114,115}\), and flatband lattices\(^\text{116,117}\). Furthermore, the creation of local nonlinearity in the synthetic frequency dimension are under study, which could significantly broaden the range of Hamiltonians involving local interactions that can be considered in the photonic synthetic space in dynamically-modulated resonators\(^\text{118}\).

We conclude this Section by discussing some recent experimental demonstrations of synthetic dimensions in photonics. The first photonic topological insulator in a synthetic dimension was demonstrated using an array of multimode waveguides, where modulation of the refractive index along the waveguide axis played the role of the dynamic modulation\(^\text{120}\). The dynamically-modulated resonator has also been implemented in the fiber-based ring experiments incorporating commercial electro-optic modulators\(^\text{121,122}\), where band structures associated with one-dimensional synthetic lattices along the frequency axis of light have been measured\(^\text{123}\). Based on this experimental setup, one can use the clockwise/counter-
clockwise mode in a single ring as another degree of freedom to construct a synthetic Hall ladder with two independent physical synthetic dimensions [see Fig. 6]. An effective magnetic flux was generated in the experiment and signatures of topological chiral one-way edge modes were observed. Recently, integrated lithium niobate resonators under dynamic modulation gives another potential experimental platform to construct synthetic dimensions and explore topological photonics in a synthetic space, which shall be potentially significant for on-chip applications.

V. FUTURE DIRECTIONS

Having reviewed some of the seminal works on implementing topological effects using ring resonators, we now discuss some promising directions for future research, including fundamental studies of topological phenomena and practical problems which must be solved to make topological ring resonators viable for device applications.

Most studies of topological ring resonators to date have focused on Hermitian topological lattice models, in which strictly speaking the topological protection only holds in the absence of any gain or loss. The study of non-Hermitian topological phases induced by appropriately-structured gain or loss is a topic attracting enormous interest nowadays. Non-Hermitian coupling, which can induce novel non-Hermitian topological phases, can be implemented in coupled resonator lattices either by introducing asymmetric backscattering to the site rings or adding gain or loss into the links to induce a hopping direction-dependent amplification or attenuation. The latter has been implemented using variable-gain amplifiers in a microwave network and coupled fiber loops. Experiments with microring resonators remain limited to topological laser experiments based on adding gain to existing Hermitian topological phases, making this an interesting direction for further studies. Can we draw ideas from non-Hermitian topological phases to exploit or minimize the scattering losses present in integrated photonic ring resonators?

Ring resonators also provide an ideal platform for studying nonlinear topological systems, due to the enhancement of nonlinear effects provided by high quality factor microresonators. Recent experiments have harnessed nonlinearities to generate frequency combs from single resonators. It will be interesting consider topological band structure effects in this context. The high flexibility in controlling the inter-ring coupling in resonator lattices also provides an opportunity to implement exotic forms of nonlinearity, such as models with nonlinear coupling. Systems with nonlinear coupling can exhibit nonlinearity-induced topological transitions, which were so far limited to electronic circuit experiments. Fiber loops are another promising platform for exploring nonlinear effects due to their long accessible propagation lengths and ability to compensate for losses using fiber amplifiers.

Recently, several studies have proposed the use of topological modes supported by domain walls of topological photonic crystals as a means of constructing novel classes of ring resonators. Light confinement typically weaker than the standard approach based on integrated photonic ridge waveguides, meaning larger resonator sizes are required. A potential benefit of topological photonic crystal-based ring resonators is their ability to support sharp corners without bending losses. However, it remains to be seen whether they will be competitive with existing ring resonators. For example, topological photonic crystals have been shown to exhibit large losses (>-100 dB/cm) compared to conventional photonic crystal waveguides (5 dB/cm), due to out-of-plane scattering losses.

Research on topological photonics has so far largely focused on the fundamental science and demonstration of novel topological effects. There remains a large gap between these studies and potential applications which needs to be bridged. While new kinds of topological phenomena such as higher order topological phases continue to attract fundamental interest, the need for more challenging ingredients such as high dimensional lattices or protecting symmetries makes any useful applications a far off prospect at this stage. Moving forward, what is needed is better optimization of existing topological designs to make them more competitive with standard components, moving from a paradigm of demonstrating topological robustness by deliberately introducing defects (as is the case in most experiments utilising photonic crystals, waveguide arrays, and metamaterials), to one where there is topological protection against the actual imperfections which limit the performance of real devices. Topological ring resonator lattices using silicon photonic devices are noteworthy as they are perhaps the only platform to date in which the topology imbues protection against intrinsic disorder, i.e. the misalignment in the rings’ resonant frequencies.

Many applications of ring resonators employ small systems consisting of up to a few coupled rings. Generally for topological protection to hold, we need to have a bulk, requiring a large system size. Intuitively, it is the presence of a bulk which allows signals to route around imperfections on the edge. So another important direction is to determine how to use topological ideas to improve the performance of small systems of a few coupled resonators. This is a direction where concepts such as synthetic dimensions will likely play a key role.

Finally, one of the most exciting potential near-term applications of topological resonator lattices is as reliable delay lines or light sources in large scale quantum photonic circuits. For example, topological edge states may be useful as disorder-robust delay lines for entangled states of light. In 2018, Mittal et al. demonstrated experimentally the generation of correlated photon pairs via spontaneous four wave mixing in a topological edge mode. They observed better reproducibility of the
 photon spectral statistics over several devices compared to regular CROWs, which is promising for the scaling up and mass production of quantum photonic circuits. Very recently this idea was generalized to dual pump spontaneous four wave mixing, which allows one to tune the resulting two photon correlations by changing the pump frequencies.\textsuperscript{143}

VI. CONCLUSION

We have provided an overview of how ring resonators provide a highly flexible platform for studying topological band structure effects in photonicics. Ring resonators have not only enabled the implementation of seminal topological lattice models from condensed matter physics, but have also been used for some of the first observations of novel topological phases in any platform, such as higher order topological corner states. There is now strong theoretical and experimental evidence that topological ideas may be useful for designing superior delay lines or frequency-converters in integrated photonic circuits. As the basic concepts are now well-established, future research will need to shift focus towards optimization of existing topological ring resonator systems to improve their performance and make their figures of merit more competitive with conventional ring resonator-based components.

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REFERENCES

1. T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, “Topological photonics,” Rev. Mod. Phys. 91, 015006 (2019).
2. A. B. Khanikaev and G. Shvets, “Two-dimensional topological photonics,” Nature Photonics 11, 763–773 (2017).
3. Y. Wu, C. Li, X. Hu, Y. Ao, Y. Zhao, and Q. Gong, “Applications of topological photonics in integrated photonic devices,” Advanced Optical Materials 5, 1700357 (2017).
4. F. D. M. Haldane and S. Raghu, “Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry,” Phys. Rev. Lett. 100, 013904 (2008).
5. Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, “Reflection-free one-way edge modes in a gyromagnetic photonic crystal,” Phys. Rev. Lett. 100, 013905 (2008).
6. L. Yuan, Q. Lin, M. Xiao, and S. Fan, “Synthetic dimension in photonics,” Optica 5, 1396–1405 (2018).
7. T. Ozawa and H. M. Price, “Topological quantum matter in synthetic dimensions,” Nature Reviews Physics 1, 349–357 (2019).
8. Y. Ota, K. Takata, T. Ozawa, A. Amo, Z. Jia, B. Kante, M. Notomi, Y. Arakawa, and S. Iwamoto, “Active topological photonics,” Nanophotonics 9, 547–567 (2020).
9. M. S. Rider, S. J. Palmer, S. R. Pocock, X. Xiao, P. Arroyo Huidobro, and V. Giannini, “A perspective on topological 4D nanophotonics: Current status and future challenges,” Journal of Applied Physics 125, 120901 (2019).
10. W. Chen, D. Leykam, Y. Chong, and L. Yang, “Nonreciprocity in synthetic photonic materials with nonlinearity,” MRS Bulletin 43, 443451 (2018).
11. D. Smirnova, D. Leykam, Y. Chong, and Y. Kvishar, “Nonlinear topological photonics,” Applied Physics Reviews 7, 021306 (2020).
12. W. Bogartts, P. De Heyn, T. Van Vaerenbergh, K. De Vos, S. Kumar Selvaraja, T. Claes, P. Dumon, P. Bienstman, D. Van Thourhout, and R. Baets, “Silicon microring resonators,” Laser & Photonics Reviews 6, 47–73 (2012).
13. Z. Zhou, B. Yin, Q. Deng, X. Li, and J. Cui, “Lowering the energy consumption in silicon photonic devices and systems,” Photon. Res. 3, B28–B46 (2015).
14. D. Thomson, A. Zilkie, J. E. Bowers, T. Komijanovic, G. T. Reed, L. Vivien, D. Marris-Morini, E. Cassan, L. Virot, J.-M. Fedéi, J.-M. Hartmann, J. H. Schmid, D.-X. Xu, F. Boeuf, P. O’Brien, G. Z. Mashanovich, and M. Nedeljkovic, “Roadmap on silicon photonics,” Journal of Optics 18, 073003 (2016).
15. D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, “Quantized Hall conductance in a two-dimensional periodic potential,” Phys. Rev. Lett. 49, 405–408 (1982).
16. C. L. Kane and E. J. Mele, “Quantum spin Hall effect in graphene,” Phys. Rev. Lett. 95, 226801 (2005).
17. W. P. Su, J. R. Schrieffer, and A. J. Heeger, “Solitons in polyacetylene,” Phys. Rev. Lett. 42, 1698–1701 (1979).
18. K. Fang, Z. Yu, and S. Fan, “Realizing effective magnetic field for photons by controlling the phase of dynamic modulation,” Nature Photonics 6, 782–787 (2012).
19. F. D. M. Haldane, “Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the parity anomaly,” Phys. Rev. Lett. 61, 2015–2018 (1988).
20. W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, “Quantized electric multipole insulators,” Science 357, 61–66 (2017).
21. B. Yan and C. Felser, “Topological materials: Weyl semimetals,” Annual Review of Condensed Matter Physics 8, 337–354 (2017).
22. L. Lu, L. Fu, J. D. Joannopoulos, and M. Soljačić, “Weyl points and line nodes in gyroid photonic crystals,” Nature Photonics 7, 294–299 (2013).
23. V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres, “Topological states of non-Hermitian systems,” Eur. Phys. J. Special Topics 227, 1295–1308 (2018).
24. L. F. Foa Torres, “Perspective on topological states of non-Hermitian lattices,” Journal of Physics: Materials 3, 014002 (2019).
25. E. J. Bergholtz, J. C. Budich, and F. K. Kunst, “Exceptional topology of non-Hermitian systems,” arXiv:1912.10048 (2019).
26. Y. Ashida, Z. Gong, and M. Ueda, “Non-Hermitian physics,” arXiv:2006.01837 (2020).
27. A. M. Armani and K. J. Vahala, “Heavy water detection using ultra-high-Q microresonators,” Opt. Lett. 31, 1896–1898 (2006).
28. Y. Sun and X. Fan, “Optical ring resonators for biochemical and chemical sensing,” Analytical and Bioanalytical Chemistry 399, 205–211 (2011).
29. H. Xu, M. Hafezi, J. Fan, J. M. Taylor, G. F. Strouse, and Z. Ahmed, “Ultra-sensitive chip-based photonic temperature sensor using ring resonator structures,” Opt. Express 22, 3098–3104 (2014).
30. H. Jayathilaka, H. Shoman, R. Beeck, N. A. F. Jaeger, L. Chrostowski, and S. Shekhar, “Automatic configuration and wavelength locking of coupled silicon ring resonators,” Journal of Lightwave Technology 36, 210–218 (2018).
A. Bisianov, M. Wimmer, U. Peschel, and O. A. Egorov, “Stability of topologically protected edge states in nonlinear fiber loops,” Phys. Rev. A 100, 063830 (2019).

V. R. Almeida and M. Lipson, “Optical bistability on a silicon chip,” Opt. Lett. 29, 2387–2389 (2004).

G. Priem, P. Dumon, W. Bogaerts, D. V. Thourhout, G. Morthier, and R. Baets, “Optical bistability and pulsating behaviour in silicon-on-insulator ring resonator structures,” Opt. Express 13, 9623–9628 (2005).

T. J. Kippenberg, A. L. Gaeta, M. Lipson, and M. L. Gorodetsky, “Dissipative Kerr solitons in optical microresonators,” Science 361, eaan8083 (2018).

J. Vasco and V. Savona, “Slow-light frequency combs and dissipative Kerr solitons in coupled-cavity waveguides,” Phys. Rev. Applied 12, 064065 (2019).

T. E. Drake, T. C. Briles, J. R. Stone, D. T. Spencer, D. R. Carlson, D. D. Hickstein, Q. Li, D. Westly, K. Srinivasan, S. A. Diddams, and S. B. Papp, “Teraerhtz-rate Kerr-microresonator optical clockwork,” Phys. Rev. X 9, 031023 (2019).

L. Chang, W. Xie, H. Shu, Q.-F. Yang, B. Shen, A. Boes, J. D. Peters, W. Jin, C. Xiang, S. Liu, G. Mollé, S.-F. Yu, X. Wang, K. Srinivasan, S. B. Papp, K. Vahala, and J. E. Bowers, “Ultra-efficient frequency comb generation in AlGaAs-on-insulator microresonators,” Nature Communications 11, 1331 (2020).

J. Riemersberger, A. Lukashchuk, M. Karpov, W. Weng, E. Lucas, J. Liu, and T. J. Kippenberg, “Massively parallel coherent laser ranging using a soliton microcomb,” Nature 581, 164–170 (2020).

A. Yariv, Y. Xu, R. K. Lee, and A. Scherer, “Coupled-resonator optical waveguide: a proposal and analysis,” Opt. Lett. 24, 711–713 (1999).

A. Canciamilla, M. Torregiani, C. Ferrari, F. Morichetti, R. M. D. L. Rue, A. Samarelli, M. Sorel, and A. Meloni, “Silicon coupled-ring resonator structures for slow light applications: potential, impairments and ultimate limits,” Journal of Optics 12, 104008 (2010).

F. Morichetti, C. Ferrari, A. Canciamilla, and A. Meloni, “The first decade of coupled resonator optical waveguides: bringing slow light to applications,” Laser & Photonics Reviews 6, 74–96 (2012).

H. Takesue, N. Matsuda, E. Kuramochi, W. J. Munro, and M. Notomi, “An on-chip coupled resonator optical waveguide single-photon buffer,” Nature Communications 4, 2725 (2013).

M. L. Cooper, G. Gupta, M. A. Schneider, W. M. J. Green, S. Assefa, F. Xia, Y. A. Vlasov, and S. Mookherjea, “Statistics of light transport in 25-ring silicon coupled-resonator optical waveguides,” Opt. Express 18, 26505–26516 (2010).

B. Ouyang, Y.-R. Yang, W. Bogaerts, and J. Caro, “Silicon ring resonators with a free spectral range robust to fabrication variations,” Opt. Express 27, 38698–38707 (2019).

S. Sokolov, J. Lian, E. Yiace, S. Combré, A. D. Rossi, and A. P. Moesk, “Tuning out disorder-induced localization in nanophotonic cavity arrays,” Opt. Express 25, 4598–4606 (2017).

M. Jacques, A. Samani, E. El-Fiky, D. Patel, Z. Xing, and D. V. Plant, “Optimization of thermo-optic phase-shifter design and mitigation of thermal crosstalk on the SOI platform,” Opt. Express 27, 10456–10471 (2019).

B. Guha, B. B. C. Kyotoku, and M. Lipson, “CMOS-compatible athermal silicon microring resonators,” Opt. Express 18, 3487–3493 (2010).

E. Timurdogan, C. M. Sorace-Agaskar, J. Sun, E. Shah Hosseini, A. Biberman, and M. R. Watts, “An ultralow power athermal silicon modulator,” Nature Communications 5, 4008 (2014).

B. Guha, J. Cardenas, and M. Lipson, “Athermal silicon microring resonators with titanium oxide cladding,” Opt. Express 21, 26557–26563 (2013).

A. Li and W. Bogaerts, “Fundamental suppression of backscattering in silicon microrings,” Opt. Express 25, 2092–2099 (2017).

A. Li and W. Bogaerts, “Using backscattering and backcoupling in silicon ring resonators as a new degree of design freedom,” Laser & Photonics Reviews 13, 1800214 (2019).

M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, “Robust optical delay lines with topological protection,” Nature Physics 7, 907–912 (2011).

M. Hafezi, S. Mittal, J. Fan, A. Migdall, and J. M. Taylor, “Imaging topological edge states in silicon photonics,” Nature Photonics 7, 1001 (2013).

S. Mittal, J. Fan, S. Faez, A. Migdall, J. M. Taylor, and M. Hafezi, “Topologically robust transport of photons in a synthetic gauge field,” Phys. Rev. Lett. 113, 087403 (2014).

C. Yin, Y. Chen, X. Jiang, Y. Zhang, Z. Shao, P. Xu, and S. Yu, “Realizing topological edge states in a silicon-oxide microresonator and photonic integrated circuit,” Opt. Lett. 41, 4791–4794 (2016).

S. Mittal, V. V. Orre, G. Zhu, M. A. Gorlach, A. Podudubny, and M. Hafezi, “Photonic quadrupole topological phases,” Nature Photonics 13, 692–696 (2019).

D. Leykam, S. Mittal, M. Hafezi, and Y. D. Chong, “Reconfigurable topological phases in next-nearest-neighbor coupled resonator lattices,” Phys. Rev. Lett. 121, 023901 (2018).

S. Mittal, V. V. Orre, D. Leykam, Y. D. Chong, and M. Hafezi, “Photonic anomalous quantum Hall effect,” Phys. Rev. Lett. 123, 043201 (2019).

X.-Y. Zhu, S. K. Gupta, X.-C. Sun, C. He, G.-X. Li, J.-H. Jiang, X.-P. Liu, M.-H. Lu, and Y.-F. Chen, “Z2 topological edge state in honeycomb lattice of coupled resonant optical waveguides with a flat band,” Opt. Express 26, 24307–24317 (2018).

J. Han, C. Gneiting, and D. Leykam, “Helical transport in coupled resonator waveguides,” Phys. Rev. B 99, 224201 (2019).

M. Parto, S. Wittek, H. Hodaee, G. Harari, M. A. Bandres, J. Ren, M. C. Rechtsman, M. Segev, D. N. Christodoulides, and M. Khajaviakh, “Edge-mode lasing in 1D topological active arrays,” Phys. Rev. Lett. 120, 113901 (2018).

H. Zhao, P. Xiao, M. H. Teimourpour, S. Malzard, R. El-Ganainy, H. Schomerus, and L. Feng, “Topological hybrid silicon microlasers,” Nature Communications 9, 981 (2018).

M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajaviakh, “Topological insulator laser: Experiments,” Science 359, eaar4005 (2018).

G. Harari, M. A. Bandres, Y. Lumer, M. C. Rechtsman, Y. D. Chong, M. Khajaviakh, D. N. Christodoulides, and M. Segev, “Topological insulator laser: Theory,” Science 359, eaar4003 (2018).

S. Mittal, S. Ganesan, J. Fan, A. Vaezi, and M. Hafezi, “Measurement of topological invariants in a 2D photonic system,” Nature Communications 7, 1349–1356 (2016).

Y. Ao, X. Hu, C. Li, Y. You, and Q. Gong, “Topological properties of coupled resonator array based on accurate band structure,” Phys. Rev. Materials 2, 105201 (2018).

T. J. Zimmerling and V. Van, “Generation of Hofstadter’s butterfly spectrum using circular arrays of microring resonators,” Opt. Lett. 45, 714–717 (2020).

Z. A. Kudyshev, A. V. Kildishev, A. Boltasseva, and V. M. Shalasev, “Photonic topological phase transition on demand,” Nano Letters 18, 1349–1356 (2019).

Z. A. Kudyshev, A. V. Kildishev, A. Boltasseva, and V. M. Shalasev, “Tuning topology of photonic systems with transparent conducting oxides,” ACS Photonics 6, 1922–1930 (2019).
H. Zhao, X. Qiao, T. Wu, B. Midya, S. Longhi, and L. Feng, “Non-Hermitian topological light steering,” Science 365, 1163–1166 (2019).

S. Afaal, T. J. Zimmerling, Y. Ren, D. Perron, and V. Van, “Realization of anomalous Floquet insulators in strongly coupled nanophotonic lattices,” Phys. Rev. Lett. 124, 253901 (2020).

G. Q. Liang and Y. D. Chong, “Optical resonator analog of a two-dimensional topological insulator,” Phys. Rev. Lett. 110, 203904 (2013).

M. Pasek and Y. D. Chong, “Network models of photonic Floquet topological insulators,” Phys. Rev. B 89, 075113 (2014).

T. Shi, H. J. Kimble, and J. I. Cirac, “Topological phenomena in classical optical networks,” Proceedings of the National Academy of Sciences 114, E8967–E8976 (2017).

S. Afaal and V. Van, “Topological phases and the bulk-edge correspondence in 2D photonic microring resonator lattices,” Opt. Express 26, 14567–14577 (2018).

Y. Ao, X. Hu, Y. You, C. Lu, Y. Fu, X. Wang, and Q. Gong, “Topological phase transition in the non-Hermitian coupled resonator array,” Phys. Rev. Lett. 125, 013902 (2020).

K. Fang, Z. Yu, and S. Fan, “Photonic Aharonov-Bohm effect based on dynamic modulation,” Phys. Rev. Lett. 108, 135301 (2012).

L. Yuan, W. Shi, and S. Fan, “Photonic gauge potential in a system with a synthetic frequency dimension,” Optica Letters 41, 741–744 (2016).

K. Fang, Z. Yu, and S. Fan, “Experimental demonstration of a photonic Aharonov-Bohm effect at radio frequencies,” Phys. Rev. B 87, 060301(R) (2013).

E. Li, B. J. Eggleton, K. Fang, and S. Fan, “Photonic Aharonov-Bohm effect in photonon interactions,” Nature Communications 5, 3225 (2014).

L. Yuan, W. Shi, and S. Fan, “Photonic gauge potential in a system with a synthetic frequency dimension,” Optica Letters 41, 741–744 (2016).

K. Fang and S. Fan, “Controlling the flow of light using the inhomogeneous effective gauge field that emerges from dynamic modulation,” Phys. Rev. Lett. 111, 203901 (2013).

K. Fang, Z. Yu, and S. Fan, “Photonic de Haas-van Alphen effect,” Optics Express 21, 18216–18224 (2013).

Q. Lin and S. Fan, “Light guiding by effective gauge field for photons,” Phys. Rev. X 4, 031031 (2014).

M. Minakov and V. Savona, “Haldane quantum Hall effect for light in a dynamically modulated array of resonators,” Optica 3, 200–206 (2016).

L. Yuan and S. Fan, “Three-dimensional dynamic localization of light from a time-dependent effective gauge field for photons,” Phys. Rev. Lett. 114, 243901 (2015).

L. Yuan, M. Xiao, and S. Fan, “Time reversal of a wave packet with temporal modulation of gauge potential,” Phys. Rev. B 94, 140305(R) (2016).

L. Yuan and S. Fan, “Topologically nontrivial Floquet band structure in a system undergoing photonic transitions in the ultrastrong-coupling regime,” Phys. Rev. A 92, 053822 (2015).

Y. Lumer, M. A. Bandres, M. Heinrich, L. J. Maczewsky, H. Herzig-Sheinfux, A. Szameit, and M. Segev, “Light guiding by artificial gauge fields,” Nature Photonics 13, 329–345 (2019).

M. Zhang, C. Wang, Y. Hu, A. Sham-Ansari, T. Ren, S. Fan, and M. Lončar, “Electronically programmable photonic molecule,” Nature Photonics 13, 36–40 (2019).

D. I. Tsomokos, S. Ashhab, and F. Nori, “Using superconducting qubit circuits to engineer exotic lattice systems,” Phys. Rev. A 82, 052311 (2010).

O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, “Quantum simulation of an extra dimension,” Phys. Rev. Lett. 108, 133901 (2012).

D. Jukić and H. Buljan, “Four-dimensional photonic lattices and discrete tesseract solitons,” Phys. Rev. A 87, 013814 (2013).

X.-W. Luo, X. Zhou, C.-F. Li, J.-S. Xu, G.-C. Guo, and Z.-W. Zhou, “Quantum simulation of 2D topological physics in a 1D array of optical cavities,” Nature Communications 6, 7704 (2015).

B. Y. Sun, X. W. Luo, M. Gong, G. C. Guo, and Z. W. Zhou, “Weyl semimetal phases and implementation in degenerate optical cavities,” Phys. Rev. A 96, 013857 (2017).

X.-F. Zhou, X.-W. Luo, S. Wang, G.-C. Guo, X. Zhou, H. Pu, and Z.-W. Zhou, “Dynamically manipulating topological physics and edge modes in a single degenerate optical cavity,” Phys. Rev. Lett. 118, 083603 (2017).

T. Ozawa, H. M. Price, N. Goldman, O. Zilberberg, and I. Carusotto, “Synthetic dimensions in integrated photonics: From optical isolation to four-dimensional quantum Hall physics,” Phys. Rev. A 93, 043827 (2016).

Q. Lin, M. Xiao, L. Yuan, and S. Fan, “Photonic Weyl point in a two-dimensional resonator lattice with a synthetic frequency dimension,” Nature Communications 7, 13731 (2016).

Y. Zhang and Y. Zhu, “Generation of Weyl points in coupled optical microdisk-resonator arrays via external modulation,” Phys. Rev. B 96, 014302 (2017).

Q. Lin, X.-Q. Sun, M. Xiao, S.-C. Zhang, and S. Fan, “A three-dimensional photonic topological insulator using a two-dimensional ring resonator lattice with a synthetic frequency dimension,” Science Advances 4, eaat2774 (2018).

A. Dutt, M. Minakov, and S. Fan, “Higher-order topological insulators in synthetic dimensions,” arXiv:1911.11310 (2019).

Z. Yang, E. Lustig, G. Harari, Y. Plotnik, Y. Lumer, M. A. Bandres, and M. Segev, “Mode-locked topological insulator using synthetic dimensions,” Phys. Rev. X 10, 011059 (2020).

A. Schwartz and B. Fischer, “Laser mode hyper-combs,” Optics Express 21, 6196–6204 (2013).

B. A. Bell, K. Wang, A. S. Solntsev, D. N. Neshev, A. A. Sukhorukov, and B. J. Eggleton, “Spectral photonic lattices with complex long-range coupling,” Optica 4, 1433–1436 (2017).

J. G. Titchener, B. Bell, K. Wang, A. S. Solntsev, B. J. Eggleton, and A. A. Sukhorukov, “Synthetic photonic lattice for single-shot reconstruction of frequency combs,” APL Photonics 5, 030805 (2020).

L. Yuan, M. Xiao, Q. Lin, and S. Fan, “Synthetic space with arbitrary dimensions in a few rings undergoing dynamic modulation,” Phys. Rev. B 97, 104105 (2018).

L. Yuan, Q. Lin, A. Zhang, M. Xiao, X. Chen, and S. Fan, “Photonic gauge potential in one cavity with synthetic frequency and orbital angular momentum dimensions,” Phys. Rev. Lett. 122, 083903 (2019).

S. Longhi, “Dynamic localization and Bloch oscillations in the spectrum of a frequency mode-locked laser,” Optics Letters 30, 786–788 (2005).

L. Yuan and S. Fan, “Bloch oscillation and unidirectional translation of frequency in a dynamically modulated ring resonator,” Optica 3, 1014–1018 (2016).

S. Longhi, “PT-symmetric mode-locking,” Optics Letters 41, 4518–4521 (2016).

L. Yuan, Q. Lin, M. Xiao, A. Dutt, and S. Fan, “Pulse shortening in an actively mode-locked laser with parity-time symmetry,” APL Photonics 3, 086103 (2018).

S. Longhi, “Aharonov-Bohm photonic cages in waveguide and coupled resonator lattices by synthetic magnetic fields,” Opt. Lett. 39, 5892–5895 (2014).

D. Yu, L. Yuan, and X. Chen, “Isolated photonic flatband with the effective magnetic flux in a synthetic space including the frequency dimension,” arXiv:2004.12542 (2020).

L. Yuan, A. Dutt, M. Qin, S. Fan, and X. Chen, “Creating locally interacting Hamiltonians in the synthetic frequency dimension for photons,” arXiv:1909.12466 (2019).
119 A. Dutt, Q. Lin, L. Yuan, M. Minkov, M. Xiao, and S. Fan, “A single photonic cavity with two independent physical synthetic dimensions,” Science 367, 59–64 (2020).

120 E. Lustig, S. Weimann, Y. Plotnik, Y. Lumer, M. A. Bandres, A. Szameit, and M. Segev, “Photonic topological insulator in synthetic dimensions,” Nature 567, 356–360 (2019).

121 C. Chen, X. Ding, J. Qin, Y. He, Y.-H. Luo, M.-C. Chen, C. Liu, X.-L. Wang, W.-J. Zhang, H. Li, L.-X. You, Z. Wang, D.-W. Wang, B. C. Sanders, C.-Y. Lu, and J.-W. Pan, “Observation of topologically protected edge states in a photonic two-dimensional quantum walk,” Phys. Rev. Lett. 121, 100502 (2018).

122 H. Chalabi, S. Barik, S. Mittal, T. E. Murphy, M. Hafezi, and E. Waks, “Synthetic gauge field for two-dimensional time-multiplexed quantum random walks,” Phys. Rev. Lett. 123, 150503 (2019).

123 A. Dutt, M. Minkov, Q. Lin, L. Yuan, D. A. B. Miller, and S. Fan, “Experimental band structure spectroscopy along a synthetic dimension,” Nature Communications 10, 3122 (2019).

124 C. Reimer, Y. Hu, A. Shams-Ansari, M. Zhang, and M. Loncar, “High-dimensional frequency crystals and quantum walks in electro-optic microcombs,” arXiv:1909.01303 (2019).

125 S. Malzard, C. Poli, and H. Schomerus, “Topologically protected defect states in open photonic systems with non-Hermitian charge-conjugation and parity-time symmetry,” Phys. Rev. Lett. 115, 200402 (2015).

126 S. Malzard and H. Schomerus, “Bulk and edge-state arcs in non-Hermitian coupled-resonator arrays,” Phys. Rev. A 98, 033807 (2018).

127 S. Longhi, D. Gatti, and G. Della Valle, “Non-Hermitian transparency and one-way transport in low-dimensional lattices by an imaginary gauge field,” Phys. Rev. B 92, 094204 (2015).

128 D. Leykam, S. Flach, and Y. D. Chong, “Flat bands in lattices with non-Hermitian coupling,” Phys. Rev. B 96, 064305 (2017).

129 W. Hu, H. Wang, P. P. Shum, and Y. D. Chong, “Exceptional points in a non-Hermitian topological pump,” Phys. Rev. B 95, 184306 (2017).

130 S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, and A. Szameit, “Topological funneling of light,” Science 368, 311–314 (2020).

131 M. Menotti, B. Morrison, K. Tan, Z. Vernon, J. E. Sipe, and M. Liscidini, “Nonlinear coupling of linearly uncoupled resonators,” Phys. Rev. Lett. 122, 013904 (2019).

132 Y. Hadad, A. B. Khanikaev, and A. Alù, “Self-induced topological transitions and edge states supported by nonlinear staggered potentials,” Phys. Rev. B 93, 155112 (2016).

133 Y. Hadad, J. C. Soric, A. B. Khanikaev, and A. Alù, “Self-induced topological protection in nonlinear circuit arrays,” Nature Electronics 1, 178–182 (2018).

134 Y. Yang and Z. H. Hang, “Topological whispering gallery modes in two-dimensional photonic crystal cavities,” Opt. Express 26, 21235–21241 (2018).

135 D. Smirnova, S. Kruk, D. Leykam, E. Melik-Gaykazyan, D.-Y. Choi, and Y. Kivshar, “Third-harmonic generation in photonic topological metasurfaces,” Phys. Rev. Lett. 123, 103901 (2019).

136 M. Jalali Mehrabad, A. P. Foster, R. Dost, E. Clarke, P. K. Patil, I. Farrer, J. Heffernan, M. S. Skolnick, and L. R. Wilson, “A semiconductor topological photonic ring resonator,” Applied Physics Letters 116, 061102 (2020).

137 S. Barik, A. Karasahin, S. Mittal, E. Waks, and M. Hafezi, “Chiral quantum optics using a topological resonator,” Phys. Rev. B 101, 205303 (2020).

138 E. Sauer, J. P. Vasco, and S. Hughes, “Theory of intrinsic propagation losses in topological edge states of planar photonic crystals,” arXiv:2005.12828 (2020).

139 S. Mittal, V. V. Orre, and M. Hafezi, “Topologically robust transport of entangled photons in a 2D photonic system,” Opt. Express 24, 15631–15641 (2016).

140 M. C. Rechtsman, Y. Lumer, Y. Plotnik, A. Perez-Leija, A. Szameit, and M. Segev, “Topological protection of photonic path entanglement,” Optica 3, 925–930 (2016).

141 J. Han, A. A. Sukhorukov, and D. Leykam, “Disorder-protected quantum state transmission through helical coupled-resonator waveguides,” arXiv:2006.04467 (2020).

142 S. Mittal, E. A. Goldschmidt, and M. Hafezi, “A topological source of quantum light,” Nature 561, 502–506 (2018).

143 V. V. Orre, S. Mittal, E. A. Goldschmidt, and M. Hafezi, “Tunable quantum interference using a topological source of indistinguishable photon pairs,” arXiv:2006.03084 (2020).