Gacs Algorithmic Complexity on Hilbert Spaces
and
Some of its Applications

By
Samad Khabbazi Oskouei

Under
Supervisions of
Ahmad Shafiei Deh Abad
and
Fabio Benatti

A thesis submitted to the
School of Mathematics, Statistics and Computer Science
In partial fulfillment of the requirements for
The degree of Doctor of Philosophy in
Applied Mathematics

April 2015
بنام خدا
صارحلصه دفاعی رساله دکتری
دانشکده ریاضی آمار و علوم کامپیوتر - پردیس علوم

با عناية به آنین نامه آموزشی دوره دکتری و مصوبه 119/2028788780288972920 الیه،
رئیس دانشگاه
شماره دانشجویی: 1420/21

در تاریخ: 12/04/2026

 panties

در محل دانشگاه ریاضی آمار و علوم کامپیوتر با عنوان: "پیچیدگی اکتومپتی" دارد.

راهنمای آن با حضور هیات داوران تشکل شد و براساس کیفیت رساله، ارائه دفاعه و تحویل پایان دو،
ریاست مورد نظر می‌باشد.

دکتر عباس نورذری دانلی


| رتبه | مرتبه دانشگاهی | مرتبه خانوادگی | نام و نام خانوادگی | مشخصات هیئت داوران |ردیف |
|------|----------------|----------------|------------------|---------------------|------|
| 1    | دانشگاه تربیت ایرانیا | دانشگاه تربیت ایرانیا | دکتر احمد شیخی | استاد ریاضیات دوم | Dr. Fabio Benatti |
| 2    | دانشگاه تربیت ایرانیا | دانشگاه تربیت ایرانیا | دکتر وحید حسینی | داور خارجی | |
| 3    | دانشگاه صنعتی شریف | دانشگاه صنعتی شریف | دکتر علی رضایی | داور داخلی | |
| 4    | دانشگاه صنعتی شریف | دانشگاه صنعتی شریف | دکتر مجید علی زاده | داور داخلی | |
| 5    | دانشگاه تهران | دانشگاه تهران | دکتر محمدرضا اسدی | نامنهاده کمیته تحقیقات تکمیلی دانشگاه ریاضی آمار و علوم کامپیوتر | |

همچنین، دکتر عباس نورذری دانلی، کمیته تحقیقات تکمیلی دانشگاه ریاضی آمار و علوم کامپیوتر

توضیحات:

دکتر عباس نورذری دانلی

تاریخ: 12/04/2026

امضاء

امضاء
Abstract

We extend the notion of Gacs quantum algorithmic entropy, originally formulated for finitely many qubits, to infinite dimensional quantum spin chains and investigate the relation of this extension with two quantum dynamical entropies that have been proposed in recent years. Further, we prove an extension of Brudno’s theorem in quantum spin chains with shift dynamics.
Acknowledgements

I would like to express my special appreciation and thanks to my advisors Professor Ahmad Shafiei Deh Abad and Professor Fabio Benatti, you have been a tremendous mentor for me. I would like to thank you for encouraging my research and for allowing me to grow as a research scientist. Your advice on both research as well as on my career have been priceless. I would also like to thank my committee members, professor Vahid Karimipour, Dr. Ali Rezakhani, Dr. Majid Alizade and Dr. Mohammad Bager Asadi for serving as my committee members even at hardship. I also want to thank you for letting my defense be an enjoyable moment, and for your brilliant comments and suggestions, thanks to you.

A special thanks to my family. Words cannot express how grateful I am to my mother for all of the sacrifices that you’ve made on my behalf. Your prayer for me was what sustained me thus far. I would like also express appreciation to my beloved wife Elham who spent sleepless nights with and was always my support in the moments when there was no one to answer my queries.

At the end, I would like the support this thesis by the STEP programme of the Abdus Salam ICTP of Trieste.
Contents

Introduction .......................................................... iii

1 Programs and Computable Functions .............................. 1
   1.1 A Programming Language .................................. 2
   1.2 Kolmogorov Complexity and Semi-Computable Functions ... 8
   1.3 Relation between Algorithmic Complexity and Thermodynamics . . . . 13

2 Classical and Quantum Entropy .................................. 17
   2.1 Classical dynamical systems ................................. 17
   2.2 Classical Spin Chains and Algebraic Formulation .......... 22
   2.3 Quantum Dynamical Systems ................................. 26
      2.3.1 Quantum Spin Chains .................................. 26
      2.3.2 AF entropy ............................................. 29
      2.3.3 CNT Entropy .......................................... 33
      2.3.4 Relation Between CNT and AF Entropies in Quantum Spin Chains . . . . 35

3 Semi-computable States and Semi-computable Density Matrices .... 38
   3.1 Universal Semi-computable Semi-density Matrices on Infinite Separable Hilbert spaces . ........................................... 38
Introduction

Computability theory: Algorithms and computational techniques started to be studied at least since the Babylonians and later by Euclid (c. 330 B.C). But, only around the 20ies, mathematicians could successfully formalize these concepts which then applied to modern computers. The subject ripened with Godel’s Incompleteness Theorem 1931 which solved with NO the first Hilbert problem:

*Can a formal system prove its own consistency?*

In addition, the result opened new ways to solve the second Hilbert problem, namely the decision problem. Indeed, Godel invented the theory of primitive recursive functions and later extended it to general recursive functions in 1934. On the other hand, formal definitions of computability were given in the mid-1930’s by Kleene and Turing. However, a major breakthrough in computability theory occurred in 1936 with Turing’s work on recursion theory [44]. Since then, computability had not only a fundamental role in computer science but also many applications to logic, algebra, analysis, and , nowadays, it plays a role in as different fields as physics and economy.

Kolmogorov complexity theory: it was born as an attempt to answering questions of the following kind

*When can mathematical objects such as finite or infinite binary strings be termed random?*

and

*Given two binary strings, how can one decide which one of them is more random than another?*
Clearly, the issue at stake here is how to measure the randomness of strings? The measures which are used in computability theory and algorithmic information theory should explore the relationships among three fundamental aspects: 1) relative computability, as measured by notions such as Turing reducibility; 2) information content, as measured by notions such as Kolmogorov complexity; 3) randomness of individual objects, as first successfully defined by Martin-Löf [31] (but prefigured by others, dating back at least to the work of von Mises [47]). In this thesis, we will focus on the second aspect: informational string content and descriptional complexity. Let us consider two sequences such as

10101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101
that any universal Turing machine that runs them outputs the target string.

**Ergodic theory:** Ergodic theory goes back to Boltzmann and Gibbs. It provides a successful mathematical framework for the description of dynamical systems. It gives a probabilistic approach to dynamics that is useful to investigate statical properties of the evolution of a mechanical system over long time scales. Further, ergodic theory [25] explains why in thermodynamical systems mean values of observables coincide with time averages and why trajectories in ergodic systems fill the phase-space densely. The KS-entropy, introduced by Kolmogorov and developed by Sinai [10], defined on classical dynamical systems has provided a link among different fields of mathematics and physics. In fact, in the light of the first theorem of Shannon [18], the KS entropy gives the maximal compression rate of the information emitted by ergodic information sources. A theorem of Pesin [30] relates it to the positive Lyapounov exponents and thus to the exponential amplification of initial small errors, in other words, to classical chaos. Finally, a theorem of Brudno [15] links the KS entropy to the compressibility of classical trajectories by means of computer programs, namely to their Kolmogorov complexity. In fact, Brudno’s theorem establishes relations among all the above mentioned issues.

Since Quantum Mechanics teaches us that the basic structure of the world is non-commutative and because of the fast development of quantum computation and information theory, it has become important to extend such classical notions to quantum dynamical systems.

**Quantum mechanics:** Quantum mechanics developed by Heisenberg, Schrodinger, Dirac and others in the 20ies describes the behavior of elementary, particles, atoms and molecules.
In quantum mechanics, dynamical systems are described by a Hilbert space, whose vectors provide their physical states, and Hamiltonian self-adjoint operators that generate their dynamics. By the Stone–von Neumann uniqueness theorem [13], this Hilbert space description is suitable for the systems with finite number of freedoms. Since with infinitely many degrees of freedom, for example one dimensional lattice $\mathbb{Z}$ the sites carrying each $d$-level spins, we do not have this properties then considering the $C^*$-algebra of observables is more convenient.

**Quantum dynamical entropies:** Since there are different approaches to the information content in quantum systems, there are as well several extensions of the KS-entropy to quantum dynamical systems. One aim of these extensions is to classify quantum dynamical systems as done in classical dynamical systems by the KS-entropy. Recently, they have been used in quantum information theory in relations to the capacity of quantum channels and quantum algorithmic complexity. Two quantum dynamical entropies, one proposed by Connes, Narnhofer and Thirring (CNT entropy) [1] and the other one introduced by Alicki and Fannes (AF entropy) [2], have been more used than the others. The two entropies are defined differently from each other, and they may exhibit a different behavior on a same quantum dynamical systems.

**Quantum Turing machines** A fundamental goal in computer technology is to construct computer devices with high speed and low prices, what has implied a steady increase in miniaturization. In view of this fact, information processing under quantum mechanical rules is becoming a concrete and substantial issue [14]. The first suggestion of quantum computers was given by Feynman who predicted that quantum computers might provide more efficient computing devices than classical (probabilistic) computers. Once these ad-
vantages have been demonstrated by the first quantum algorithms, quantum computation and quantum information theory started blossoming [24, 38, 37].

In view of the importance of classical Turing machines for the development of classical computability theory, it soon became important to extend these notions to the quantum realm: quantum Turing machines (QTM) and universal quantum Turing machines (UQTM) were thus introduced in [20].

The subsequent question was how to reformulate the notion of algorithmic complexity in a way that it could be used for quantum systems, too. Several proposals have been put forward that reflect different points of view. However, all of them have the same basic intuitive idea that complexity should characterize properties of systems that are difficult to describe. They can roughly be summarized as follows:

1. Qbit quantum complexity: one may decide to describe quantum states by means of other quantum states that are processed by UQTMs [9]: the corresponding complexity will be denoted by QC^q.

2. Bit quantum complexity: one may decide to define the complexity of quantum states using classical [45] programs run by UQTMs which is denoted by QC^c.

3. Quantum circuit complexity: one may choose to relate the complexity of a quantum state to the complexity of the (classical) description of the quantum circuits that operatively construct the state [33, 34]. The corresponding complexity will be denoted by QC_{net}.

4. Gacs complexity: one may extend the notion of universal probability and define a quantum universal semi-density matrix [22, 7] and then, mimicking the classical approach, introduce a quantum complexity as minus its logarithm. This thesis is
based on exactly this latter train of ideas

**Thesis subject:** The recent developments in quantum mechanics that, together with the birth of the so-called quantum computation theory, have also led to the development of a broad quantum information theory, have spurred the attempt to extend the concept of algorithmic complexity to the quantum realm. As we have seen, there exist different proposals of quantum algorithmic complexity that, while agreeing on quantum states as description targets, differ on how their description should be achieved.

In all cases a useful guide to sort out the various quantum extensions of algorithmic complexity is provided by the relations between the classical algorithmic complexity and the Shannon entropy. Even when not pretending to exactly reproducing them in a non-commutative context, it is nevertheless important to clarify the connections, if any, between the quantum algorithmic complexities and the von Neumann entropy or related concepts. In particular, in the classical setting a theorem of Brudno [15] states that almost every trajectory of an ergodic classical system has an algorithmic complexity rate which equals the Shannon entropy rate, the latter being also known as Kolmogorov-Sinai, or dynamical entropy. Inequivalent quantum extensions of the Kolmogorov-Sinai entropy have also been proposed [1, 2, 39, 46].

In [6], a relation was established between the quantum algorithmic complexity as formulated in [21], that we shall refer to as Gacs complexity (entropy) in the following, and the quantum dynamical entropies of the shift automorphism on quantum spin chains as formulated by Connes, Narnhofer and Thirring (CNT-entropy) [1] and by Alicki and Fannes (AF-entropy) [2]. A quantum spin chain is a one-dimensional lattice with $d$-level quantum systems at each site and the lattice translations or shift-automorphisms are the simplest
possible dynamics. For ergodic translation invariant states $\omega$ on quantum spin chains, the CNT-entropy equals the von Neumann entropy density $s(\omega)$, while the AF-entropy equals $s(\omega) + \log d$.

In [6], the extra term $\log d$ is given an informational interpretation in terms of the Gacs complexity per spin in the Alicki-Fannes construction. There, the limit rate is obtained starting from increasingly large, but finite-dimensional sub-chains and using the formulation in [21] that concerns arbitrary, but finite number of spins. As a consequence of the construction of the complexity rate from below, that is from finite dimensional sub-algebras to the infinite dimensional spin-chain, a constraint had to be imposed in [6] on the growth of the classical complexity of finite-size density matrices; namely, that it be slower than the size of the sub-chain. Instead, in this thesis, we construct a Gacs complexity quantity starting directly from the infinite dimensional quantum spin chain. The resulting complexity is equivalent to the finite dimensional one when restricted to finite portions of the chain, but allows us to remove the unnecessary limitation mentioned above. As a result, we report an instance of quantum spin-chain with finite Alicki-Fannes entropy equalling the Gacs complexity rate, while finite-size density matrices have Kolmogorov complexities diverging faster than $n$.

Further, an extension of Brudno’s theorem using the Gacs complexities are mentioned in this thesis. One way to extend it is to reformulate the lower Gacs complexity in classical dynamical systems and then reformulate Brudno’s theorem using it in quantum spin chains. Another way that we extend it is by the help of a generalization of the classical Shannon-MacMillan theorem, or quantum Shannon-MacMillan theorem, [11] in ergodic quantum spin chains with shift dynamics. The two proposals are mentioned in this thesis, the first one is just formulated in the classical case, where it reduces to a short
proof of the Brudno’s theorem. While the proof of Brudno’s theorem in quantum spin chain results from the second one. Namely, the rate of lower Gacs complexity of minimal projections which are dominated by a sequence of projections with high probability is equal to the von-Neumann entropy rate of the state.

The organization of the thesis is as follows:

**Chapter 1:** We shortly introduce computability theory based on a specific programming language [19]. We also briefly describe Turing machines and define the Kolmogorov complexity with an attached thermodynamical interpretation.

**Chapter 2:** We explain how classical and quantum dynamical systems can be given a unifying algebraic description as commutative and non-commutative $C^*$-algebras, respectively. Then, we introduce the two quantum dynamical entropies, CNT and AF, which are extensions of the classical KS-entropy, their relations and properties with applications to quantum spin chains. Most of the material in this chapter is taken from [6].

**Chapter 3:** We first extend the concepts of semi-computable semi-density matrices, Gacs entropy to infinite dimensional separable Hilbert space, and the apply them to quantum spin chains.

**Chapter 4:** We introduce classical version of Gacs complexity; then, using the semi-computability concept, we will give a short proof a restricted version of Brudno’s theorem.

**Chapter 5:** This final chapter is entirely devoted to the extension of Brudno’s theorem to the case of the shift dynamics on quantum spin chains.
Chapter 1

Programs and Computable Functions

In this chapter, we first introduce computability theory which plays an important role in computer science: it will be done by means of a specific programming language. So, we introduce necessary concepts and tools such as computable functions and partial computable. Further, we review the notion of Turing machine which is the simplest mathematical model of computing device.

Complexity in computer science is usually either computational or descriptional, where the first one refers to the number of needed computational steps in a given program and the second one measures the amount of information in a program. In the following, we shall concentrate on the latter case. Finally, we will consider a thermodynamical application of Kolmogorov complexity to an oversimplified model of computing device which shows the relations between data compression and energy cost.
1.1 A Programming Language

We are going to introduce computability theory based on a specific programming language $P$.

This consists of the letters:

$$X_1, X_2, \ldots, X_n,$$

which will be called input variables with values in $\mathbb{N} \cup \{0\}$. The output variable will be denoted by $Y$. In most programs, we also need local variables $Z_1, Z_2, \ldots, Z_k$. Moreover, $P$ contains the following instructions.

1. $V \rightarrow V + 1$: Increase by 1 the value of the variable $V$.

2. $V \rightarrow V - 1$: If the value of $V$ is zero leave it unchanged; otherwise decrease it by 1.

3. IF $V \neq 0$ GOTO L: If the value of $V$ is nonzero, perform the instruction with label L; otherwise proceed to the next instruction in the list.

The labels

$$A_1, B_1, C_1, E_1, A_2, B_2, C_2, E_2, A_3, \ldots,$$  \hspace{1cm} (1.1.1)

are used to indicate a specific instruction of a program, a program $P$ being a finite list of above instructions.

A program can halt in ways: in the first one, there are no more instructions after the last one in the list which constitutes the program. In the second case, an instruction labeled $L$ is to be executed, while, there is no instruction with that label $L$ in the program; we usually denote the label $L$ with the letter $E$. 
Example 1.1.1. The following program computes the function \( f(x, y) = x + y \).

\[
Y \to X_1 \\
Z_1 \to X_2 \\
[ B ] \quad \text{If } Z_1 \neq 0 \text{ GOTO } A \\
\quad \text{GOTO } E \\
[ A ] \quad Z_1 \to Z_1 - 1 \\
\quad Y \to Y + 1 \\
\quad \text{GOTO } B
\]

where GOTO E is an abbreviation for

\[
Z_2 \to Z_2 + 1 \\
\quad \text{IF } Z_2 \neq 0 \text{ GOTO } E.
\]

Moreover, since there is no label E, the command GOTO E forces the program to halt. Of course, the symbols \( X_1, X_2 \) denote input variables, \( Z_1 \) a local variable, \( Y \) the output variable, while \( A, B, E, L \) are labels.

We will show that programs can be assigned natural numbers in a specific way called Godel numbering. Namely, we will show that there exists a one-to-one correspondence between \( \mathbb{N} \) and the set of all programs in a programming language \( P \). The corresponding number of each program \( p \) is denoted by \( \#(p) \). In such a way, the program can be retrieved from its number:
Let the variables be listed as follows:

\[ Y, X_1, Z_1, X_2, Z_2, X_3, Z_3, \ldots, \]

and the labels be listed as in Example 1.1.1. The number assigned to a given variable is its position number in the above list. For example: \( \#(Z_1) = 3 \). The number assigned to a given label is also its position number. Now, let \( I \) be an instruction of the program \( p \).

Let's define for any \( x, y \in \mathbb{Z}, < x, y > = 2^y(2y + 1) - 1 \). Then, the number assigned to \( I \) is defined by

\[ \#(I) = < a, < b, c >, \]

where

- if \( I \) is labeled \( L \) then \( a = \#(L) \); otherwise 0.

- if the variable \( V \) is used in \( I \), then \( c = \#(V) - 1 \).

- if \( I \) contains one of the following statements

\[ v \rightarrow v, \quad v \rightarrow v + 1, \quad v \rightarrow v - 1, \]

then \( b = 0, 1 \) or 2, respectively.

- if the statement

\[ \text{If } V \neq 0 \text{ GOTO } L, \]

is used in \( I \) then \( b = \#(L) + 2 \).

The number of the program \( p \) consisting of the instructions \( I_1, I_2, \ldots, I_k \) is defined by

\[ \#(p) = \lfloor \#(I_1), \#(I_2), \ldots, \#(I_k) \rfloor := 2^{\#(I_1)} \cdot 3^{\#(I_2)} \cdot \ldots \cdot p_k^{\#(I_k)} - 1, \]
where \( p_k \) is the \( k \)-th prime number. A program with the number \( n \) will be denoted by \( p_n \).

**Example 1.1.2.**

\[
\text{[A]} \quad X \to X + 1 \\
\text{IF } X \neq 0 \text{ GOTO A}
\]

*The program contains two instructions, which will be called \( I_1 \) and \( I_2 \), respectively. Instruction \( I_1 \) is labeled by \( A \) thus \( a = \#(A) = 1 \), \( b = 1 \) and \( c = \#(X) - 1 = 1 \); therefore, \( \#(I_1) = 21 \). Since \( I_2 \) is unlabeled,

\[
\#(I_2) = \langle 0, < 3, 1 > \rangle = 46.
\]

Finally, \( \#(p) = [21, 46] = 2^{21} \cdot 3^{46} - 1 \).

**Definition 1.1.1.** A function \( f : \mathbb{N}^n \to \mathbb{N} \) is called partially computable if there exists a program \( p \), which for each \( (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \), halts on input \( (X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \) if and only if \( f(x_1, x_2, \ldots, x_n) \) is defined and its output \( Y \) is equal to \( f(x_1, x_2, \ldots, x_n) \).

By the Gödel numbering, the set of all programs of a programming language is enumerated and hence the set of all partially computable functions is also enumerated.

The function \( f : \mathbb{N}^n \to \mathbb{N} \) computed by a program \( p_k \) is denoted by \( \phi_k \). Since one program may halt or not on an input value, partially computable functions may be not defined on certain values.

**Definition 1.1.2.** A program \( p \) is called computable if it halts on each input value.

It is important that there is no enumeration for computable functions. Indeed, the set \( \{ n : \phi_n \text{ is a computable function} \} \) is not computable.
There are different mathematical models for computability theory. One of them is the so-called Turing machine \cite{44}. These models are all equivalent. Indeed, each program in any programming language can be simulated in other programming languages. More precisely,

**Church-Turing thesis 1.1.1.** A function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is effectively computable if and only if \( f \) is partial recursive if and only if it is Turing computable, where effectively computable means that for a given function there exists a brief way or an algorithm to compute it for input numbers.

Different models reinforce our intuition regarding what is computable. Alan Turings in 1936 introduced a mathematical model of a computing device that mechanically works on a tape which is specially used to operate as a CPU inside a computer. More precisely,

A Turing machine \( T \) consists of a two-sided infinite tape, subdivided into square cells, and a reading/writing head. To describe a Turing program for \( T \), one needs programming symbols as follows:

1. Only one of the tape symbols \( S_0, S_1, \ldots, S_n \) can be written on each tape cell. We usually assume that \( S_0 = 0 \) called "blank", and \( S_1 = 1 \). The set of tape symbols \( S_1, S_2, \ldots, S_n \) is called an alphabet set, where only a finite number of them is allowed to be written on the tape while the remaining cells are "blank". In the following the alphabet consists of "1".

2. \( T \) consists of a finite list of internal states \( q_0, q_1, \ldots, q_s \). These states specify the state of the reading head before any given program step.

3. The action symbols are \( L \) (move left one cell ), \( R \) (move right one cell), \( 1 \) (print 1) and \( 0 \) (erase the current cell) which are used by a program to tell the reading head
what to do in relation to its current cell.

A program for $T$ is a finite list of instructions, called quadruples, $q_iSAq_j$. The meaning of this symbol is as follows: "if $T$ is in state $q_i$ reading tape symbol $S$, then perform action $A$ and pass into new internal state $q_j$". To input $n \in \mathbb{N}$ on the tape, we write $n + 1$ 1’s on the tape and set the reading head in starting state $q_0$ reading the leftmost 1. If there is no applicable quadruple in $T$ then $T$ halts and the output of the program is the remaining number of 1’s on the tape.

A function is called Turing computable if there exists a Turing program that computes it. Now using Godel numbering, let $\phi_0, \phi_1, \phi_2, \ldots$ be the enumeration for partially computable functions from $\mathbb{N}$ into $\mathbb{N}$.

**Definition 1.1.3.** The partially computable function $f(x, y) = \phi_x(y)$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$ is called universal: it which simulates any partially computable function $\phi_n$ from $\mathbb{N}$ to $\mathbb{N}$ for a given number $n \in \mathbb{N}$.

In the real world, a standard (classical) computer or quantum computer cannot execute a program for infinite long time. For this reason, we introduce the following useful notion of decidable predicate:

**Definition 1.1.4.** The predicate $STP^{(n)}(x_1, x_2, \ldots, x_n, y, t)$ is defined as follows:

$$STP^{(n)}(x_1, x_2, \ldots, x_n, y, t) \iff \text{Program number } y \text{ halts after } t \text{ or fewer steps on inputs } x_1, x_2, \ldots, x_n$$

$$\iff \text{There is a computation of program number } y \text{ of length } \leq t + 1, \text{beginning with inputs } x_1, x_2, \ldots, x_n$$

where $x_1, x_2, \ldots, x_n$ are input variables of the program.
The set \( \bigcup_{n \geq 0} \{0, 1\}^n \) of all binary strings of finite length will be denoted by \( \{0, 1\}^* \) or \( \Omega_2 \). The map \( \text{str} = \{0, 1\}^* \to \mathbb{N} \) where \( \text{str}(a_0a_1 \ldots a_n) = 2^{n+1} - 1 + \sum_{k=0}^{n} a_k 2^k \), \( a_i = 0, 1 \), for each \( 0 \leq i \leq n \), defines a one-to-one correspondence between \( \{0, 1\}^* \) and \( \mathbb{N} \).

**Definition 1.1.5.** A function \( f : \{0, 1\}^* \to \{0, 1\}^* \) is called (partially computable) computable if the function \( x \circ f \circ x^{-1} : \mathbb{N} \to \mathbb{N} \) is (partially computable) computable.

Let \( x \) and \( y \) be two elements of \( \{0, 1\}^* \). We say that \( x \) is a prefix of \( y \) if there exists an element \( z \in \{0, 1\}^* \) such that \( xz = y \) where \( xz \) means a concatenation of \( x \) and \( z \).

**Definition 1.1.6.** A subset \( S \subseteq \{0, 1\}^* \) is called prefix-free if no element of \( S \) is a prefix of another elements.

**Definition 1.1.7.** A partially computable function is called prefix-free if its domain is a prefix-free subset of \( \{0, 1\}^* \).

It has been shown [23] that there exist prefix-free universal functions capable to simulate all other prefix-free functions.

**Example 1.1.3.**

\[
A^* = \{1 \underbrace{i_1i_2 \ldots i_n}_{n \text{ times}} \mid i_1i_2 \ldots i_n \in \Omega_2\},
\]

is clearly a prefix-free set. The function \( f : \Omega_2 \to \Omega_2, \ f(x) = x, \) if \( x \in A^* \), otherwise undefined, is a prefix-free function with domain \( A^* \).

### 1.2 Kolmogorov Complexity and Semi-Computable Functions

Algorithmic complexity theory was developed by Kolmogorov, Solomonoff and Chaitin in order to measure the information content of a binary string. It is based on the fact that
regular strings, such as a piece of text, have short descriptions. Consider for example the
two strings
\[ s := 111111111111111, \]
\[ t := 1001101111000010. \]

One way to describe the string \( s \) which is a repetition of the bit 1, is \textbf{print 1 n times}. But, there is no pattern underlying the string \( t \). Therefore, the length of a program that
describes it is longer than the number of its bits, and the length of the description of the
string \( s \) is clearly shorter than the length of the description of the string \( t \).

We are going to define the Kolmogorov complexity.

One attributes to a binary string \( i^{(n)} = i_1 i_2 \cdots i_n \in \{0,1\}^n \) of length \( n \) a complexity
\( C(i^{(n)}) \) measured by the length of any shortest program \( p^* \) (another binary string of length
\( \ell(p^*) \)) in the domain of a binary universal partially computable function or equivalently, the
shortest program for a Universal Turing Machine (UTM) \( U \), with output \( i^{(n)} \),
\[ C(i^{(n)}) = \min \{ \ell(p) : U[p] = i^{(n)} \}. \tag{1.2.1} \]

The prefix property means that if \( U \) halts on a program \( p \) it does not continue to read on
if another program \( q \) is appended to \( p \); in other words, no halting program can be used as
prefix to a halting program.

\textit{Remark} 1.2.1. The main properties of a prefix-free set is the Kraft inequality, the im-
portant inequality in coding theory, with many relevant consequences. Furthermore, by
relations (1.2.9) and (1.2.10), the rate of the prefix-free Kolmogorov complexity is equal to
the rate of the Kolmogorov complexity. Therefore, we will consider prefix-free Kolmogorov
complexity in this thesis.

Properties of the classical Kolmogorov complexity

1. If \( \mathcal{U} \) is a universal computer (prefix-free computer) then for every computer (prefix-
   free computer \( Q \)) \( P \) there exists constants \( c_p > 0 \) (\( k_q > 0 \)) such that for all string
   \( i \in \Omega_2 \),
   \[
   C_{\mathcal{U}}(i) \leq C_P(i) + c_p, \tag{1.2.3}
   \]
   and
   \[
   K_{\mathcal{U}}(i) \leq K_Q(i) + k_q, \tag{1.2.4}
   \]
   where the constants \( c_p \) and \( k_q \) do not depend on \( i \).

2. The number of all strings \( i \) with complexity \( C(i) < c \) satisfies the following inequality
   \[
   \# \{i | i \in \Omega : C_{\mathcal{U}}(i) < c\} < 2^c. \tag{1.2.5}
   \]
   Thus, there are no more than \( 2^c \) string \( i \) with complexity \( C(i) < c \).

3. Universal probability of a binary string \( i \) is defined by
   \[
   P_{\mathcal{U}}(i) = \sum_{(p, U(p) = i)} 2^{-l(p)}, \tag{1.2.6}
   \]
   where \( \mathcal{U} \) is a universal prefix-free Turing machine. It is shown that for every program

10
there exists a constant numbers \( c_p > 0 \) such that for all string \( i \in \Omega \)

\[
P_U(i) \leq c_p \cdot P_P(i), \tag{1.2.7}
\]

where the constant \( c_p \) does not depend on the string \( i \). In addition, it is proved that for a constant \( c > 0 \),

\[
K(i) \doteq - \log P_U(i), \tag{1.2.8}
\]

where \( c \) does not depend on \( i \). The symbol \( \doteq \) means that there exist constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
K(i) \leq - \log P_U(i) + c_1, \quad - \log P_U(i) \leq K(i) + c_2,
\]

for each any binary string \( i \).

4. For any string \( i^{(n)} \),

\[
C(i^{(n)}) \leq K(i^{(n)}), \tag{1.2.9}
\]

and if \( p \) is a program such that \( C(i^{(n)}) = \ell(p) \), then it follows that

\[
K(i^{(n)}) \leq C(i^{(n)}) + 2 \log \ell(p) + c_p \leq C(i^{(n)}) + 2 \log n + c_p. \tag{1.2.10}
\]

5. Unfortunately, algorithmic complexity or Kolmogorov complexity is not computable; therefore, there is no effective way to compute \( P_U \). But, it can be approximated within arbitrary precision.

Let \( h : \mathbb{N} \times \mathbb{N} \to \mathbb{R} \) be a function. Then, for each \( n \), \( h_n : \mathbb{N} \to \mathbb{R} \) is defined as follows

\[
h_n(x) = h(n, x).
\]

**Definition 1.2.1.** A function \( g : \mathbb{N} \to \mathbb{R} \) is called lower semi-computable if there exists a
computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that the sequence $f_n$ is an increasing sequence and $\lim_{n \to \infty} f_n = g$.

**Definition 1.2.2.** A function $\mu : \mathbb{N} \to \mathbb{R}$ is called a (semi-computable) semi-measure if it is a positive semi-computable function such that $\Sigma_x \mu(x) \leq 1$.

**Definition 1.2.3.** A function $h : \mathbb{N} \to \mathbb{R}$ is called upper semi-computable if $-h$ is lower semi-computable and it will be called computable if it is lower and upper semi-computable.

**Definition 1.2.4.** A semi-computable semi-measure $\mu$ is called universal if for any semi-computable semi-measure $\nu$ there exists a constant $c_\nu > 0$ such that for each $x \in \mathbb{N}$, $c_\nu \nu(x) \leq \mu(x)$.

The existence of a universal semi-measure is proved by Levin [49]:

**Theorem 1.2.1.** There is a semi-computable semi-measure $\mu$ with the property that for any other semi-computable semi-measure $\nu$ there is a constant $c_\nu > 0$ such that for all $x \in \mathbb{N}$ we have $c_\nu \nu(x) \leq \mu(x)$.

Levin [49] is also proved a relation between prefix Kolmogorov complexity and universal semi-measure as follows

**Theorem 1.2.2.** (Levin’s Coding Theorem) We have $K(x) \overset{\text{def}}{=} -\log \mu(x)$, for all $x \in \Omega_2$.

**Theorem 1.2.3.** Any semi-computable function $\psi : \mathbb{N} \to \mathbb{Q}$ can be represented by a computable function from $\mathbb{N}$ into $\mathbb{N}$.

**Proof.** Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be a computable function and increasing with respect to the first argument $n$, and such that

$$\psi(x) = \lim_{n \to \infty} f(n, x) \quad \text{for all} \quad x \in \mathbb{N}.$$
Let \( \phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Q} \) be defined by \( \phi(n, x) = f(n, x) \) if \( x \leq n \), otherwise 0. Then, \( \phi_n \) is an increasing sequence of computable functions and \( \lim_{n \to \infty} \phi_n = \psi \). Let \( \Phi : \mathbb{N} \to \mathbb{N} \) be defined as follows: \( \Phi(n) = 2^{\alpha(\phi(n, 0))} \times 3^{\alpha(\phi(n, 1))} \times \cdots \times p_n^{\alpha(\phi(n, n))} \), where \( p_n \) is the \( n \)-th prime number and \( \alpha : \mathbb{Q} \to \mathbb{N} \) is an injection. Then, \( \Phi \) is a computable function.

Now, \( \psi \) can be defined by \( \Phi \) as follows:

\[
\psi(n) = \alpha^{-1}(\Phi(n)_n), \quad \text{where} \quad \Phi(n)_n = \alpha(\phi(n, n)).
\]

In this way, we will represent all necessary semi-computable quantities that appear in the following like semi-computable semi-measures, semi-computable Hilbert space vectors and semi-density matrices by computable functions on \( \mathbb{N} \).

### 1.3 Relation between Algorithmic Complexity and Thermodynamics

One nice application of algorithmic complexity concerns the relations between computation and thermodynamics. Since computation is a physical process, its thermodynamical cost is certainly important. A usual question in computation theory is about which processes can be performed reversibly and which ones are necessarily irreversible. Rolf Landauer and Charles Bennett have been shown that any thermodynamically irreversible computer operation should be logically irreversible. For instance, data erasure is an example of irreversible process as it eliminates irretrievable information.

Let us consider a cubic box of volume \( V \) containing a gas molecule with a freely moving piston which can be used to locate the molecule on the left side of the box; this molecule

\[ ^{1} \text{An instance of such an injection is the map } \iota' \circ \iota \text{ defined in section 3.1} \]
position can be identified as a bit 1.

The flip operation which transforms bit 1 (molecule confined to the left half of the box) into bit 0 (molecule confined in the right half of the box) can be performed reversibly by slowly rotating the box around its vertical axis and thus exchanging the two halves of the box.

The compression of the piston that confines the molecule to one half of the box, can be performed isothermically, without changing the temperature of the box. After that, one allows the piston slowly return to the initial state. Correspondingly, there occurs a loss of information due to the doubling of the volume, that the molecule can occupy: this amounts to erasing one bit of information.

Now, we can construct a computer working with only two bits per tape cell which can be operated in analogy with the thermodynamical box depicted above. Let us consider the simple program to add the two bits and save its result in the memory.

\[
\begin{align*}
0 \oplus 0 &= 0, & 0 \oplus 1 &= 1, & 1 \oplus 0 &= 1, & 1 \oplus 1 &= 0.
\end{align*}
\]
Since the results of two operations $0\ 1$ and $1\ 0$ are the same $0\ 1$, the operation is not logically reversible and hence it is not thermodynamically reversible. But, we can use more tape cells to solve this difficulty, which writes sum of the two bits in different part of the memory, using the additional bits to save the inputs and the outputs. Therefore, we can construct a logically reversible computer operation. For instance in the sum of two bits, the first two memory cell store the inputs and the second two memory cell which are initially zeros save the outputs as bits.

\[
\begin{align*}
0\ 0\ 0\ 0 & \rightarrow \ 0\ 0\ 0\ 0 \\
0\ 1\ 0\ 0 & \rightarrow \ 0\ 1\ 0\ 1 \\
1\ 0\ 0\ 0 & \rightarrow \ 1\ 0\ 0\ 1 \\
1\ 1\ 0\ 0 & \rightarrow \ 1\ 1\ 1\ 0 \\
\end{align*}
\]

Now, we identify the free energy, namely energy that can be transformed into expendible work, and free memory.

The problem is that, if we want to operate reversibly by storing extra information, the free memory will soon become saturated and demand data erasure. This process consumes free energy by generating heat: one would then try to compress as much as possible the garbage data before erasing them.

If $T$ is the temperature at which the computation is performed, the heat generated, equivalently the free energy consumed, by erasing one bit of information is given by

\[
\Delta S = \frac{\Delta Q}{T},
\]

where $\Delta S = \kappa \log 2$, $\kappa = 1.38 \times 10^{-23} J/K$. Therefore, when 1 or 0 is written in a bit the amount of free energy.

Suppose the garbage data occupying the free memory correspond to the string $i^{(n)}$. The
best way to compress it is to use a program $p^*$ with shortest length such that $U(p^*) = i^{(n)}$ where $U$ is a universal Turing machine.

Now, the minimal free energy consumption amounts to $\Delta_{\text{opt}} F = -\kappa T C(i^{(n)})$ which is a lower bound to $\Delta F = -n\kappa T \log 2$, where $n$ is the length of $i^{(n)}$. 
Chapter 2

Classical and Quantum Entropy

In this chapter, we introduce the Kolmogorov-Sinai entropy for classical dynamical systems with the aid of symbolic models. Symbolic models will then be associated to the algebraic description of classical spin chains. This will lead us to the introduction of quantum spin chains and of two quantum dynamical entropies, that of Connes, Narnhofer and Thirring (CNT) and that of Alicki and Fannes (AF).

2.1 Classical dynamical systems

Classical dynamical systems can be defined as abstract mathematical objects in terms of triples \((\chi, T, \nu)\) where

1. \(\chi\) is a phase-space; namely, \(\chi\) is a measure space endowed with a \(\sigma\)-algebra \(\Sigma\) of measurable sets.

2. \(T\) is a measurable map such that for any \(A \in \Sigma \Rightarrow T^{-1}(A) \in \Sigma\).

3. \(\nu\) is a \(T\)-Invariant probability measure on \(\chi\); namely, \(\nu(\chi) = 1\) and \(\nu \circ T^{-1} = \nu\).

A reversible dynamical system is a dynamical system such that for the discrete time evolution \(T, T^{-1}\) is also measurable such that, \(\nu \circ T = \nu\) and if \(A \in \Sigma\) then \(T(A) \in \Sigma\).
Definition 2.1.1. Let \((\chi, T, \nu)\) be a classical dynamical system. A finite measurable partition \(\mathcal{P}\) of \(\chi\) is a finite set of disjoint measurable subsets \(P_1, P_1, \ldots, P_n\) of \(\chi\) such that \(\chi = \bigcup_{i=1}^{n} P_i\). The elements \(P_i\) of \(\mathcal{P}\) are usually called atoms.

Composition of two partitions \(\mathcal{P}\) and \(\mathcal{Q}\) are also a partition \(\mathcal{P} \lor \mathcal{Q} = \{ P_k \cap Q_l | P_k \in \mathcal{P}, Q_l \in \mathcal{Q} \}\).

One way to study continuous phase-spaces with discrete time dynamics is by discretizing the continuous phase-space using a finite partition, a process called coarse-graining. Firstly, we introduce the meaning of trajectory in a dynamical system \((\chi, T, \nu)\) with discrete time evolution \(T\).

In general, for a given element \(x \in \chi\), the trajectory of \(x\) is defined as set \(\{ T^k x \}\) where \(k \in \mathbb{Z}\). Indeed, it shows the position of an element on phase-space after after \(k\) time-steps. Then, one defines a coarse-grained trajectory issuing from \(x\) by using finite partitions.

Definition 2.1.2. Let \((\chi, T, \nu)\) be a dynamical system with the finite measurable partition \(\mathcal{P}\) of \(\chi\) with \(p\) elements. The coarse-grained trajectory through \(x \in \chi\) dependent on partition \(\mathcal{P}\) is defined by the string \(\tilde{\Omega}_{i}^{p} \ni i(x) := i_1 i_2 i_3 \ldots \) where \(T^k(x) \in P_{i_k}\). By varying \(x \in \chi\), the set of such strings will be denoted by \(\tilde{\Omega}_{p}^{\mathbb{Z}}\) where \(\tilde{\Omega}_{p}^{\mathbb{Z}} \subseteq \Omega_{p}^{\mathbb{Z}}\). Therefore, for a phase point \(x \in \chi\), the trajectory \(\{ T^k x \}_{k \in \mathbb{Z}}\) can be encoded by a string dependent on a specified finite measurable partition of phase-space.

For a given dynamical system and a finite measurable partition \(\mathcal{P}\), the symbolic dynamical system \((\tilde{\Omega}_{p}^{\mathbb{Z}}, T, \nu_{\mathcal{P}})\), is defined as follows

1. The \(\sigma\)-algebra of measurable sets is generated by cylinders consisting a cylinder
consists of all strings whose elements have fixed values in chosen intervals:

\[ C_{i_l}^{[l]} = \{ i \in \Omega_p : i_l = i \}, \quad C_{i_j i_{j+1} \ldots i_k}^{[j,k]} = \{ i \in \Omega_p : i_{j+l} = i_{j+l}, l = 0, 1, \ldots, k-j \}. \]

2. \( T_\sigma \) is a left shift dynamics along strings on \( \Omega_p \). In other words, for a string \( i \in \Omega_p \),

\( (T_\sigma(i))_j = i_{j+1} \).

3. The probability measure \( \nu_p \) is defined by \( \nu_P(i^{(n)}) = \nu(P_i^{(n)}) \), which

\[ P_i^{(n)} := P_{i_0} \bigcap T^{-1}(P_{i_1}) \bigcap \cdots \bigcap T^{-n+1}(P_{i_{n-1}}). \]  

(2.1.1)

Remark 2.1.1. It is straightforward to see that, in the symbolic dynamical system \((\tilde{\Omega}_p, T_\sigma, \nu_P)\), the invariance condition \( \nu \circ T^{-1} = \nu \) is equivalent to

\[ \sum_{i=1}^{p} \nu_P(i_1 i_2 \ldots i_n) = \nu_P(i_1 \ldots i_n), \]  

(2.1.2)

Notice that the invariance condition is different from the compatibility condition

\[ \sum_{i_{n-1}}^{p} \nu_P(i_1 i_2 \ldots i_n) = \nu_P(i_1 i_2 \ldots i_{n-1}), \]  

(2.1.3)

which must hold for all probability measures \( \nu \).

The Kolmogorov-Sinai (KS) entropy of classical dynamical systems is, roughly speaking, the highest Shannon entropy rate for all its symbolic models. Indeed, let

\[ \mathcal{P}^{(n)} := \{ P_i^{(n)} | i^{(n)} = i_0 i_1 \ldots i_n, i_j \in I_p \} \]
be a refinement of the partition $\mathcal{P}$. The entropy of $\mathcal{P}^{(n)}$ is measured by the Shannon entropy of the probability distribution $\{\nu(\mathcal{P}_i^{(n)})\}_{\Omega_p^{(n)}}$,

$$H_\nu(\mathcal{P}^{(n)}) := -\sum_{\Omega_p^{(n)}} \nu(\mathcal{P}_i^{(n)}) \log \nu(\mathcal{P}_i^{(n)}).$$

(2.1.4)

Now, KS entropy associated with $\nu, T, \mathcal{P}$ is defined as the shannon entropy rate

$$h_{KS}^\nu(T, \mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} H_\nu(\mathcal{P}^{(n)}) = \inf_{n} \frac{1}{n} H_\nu(\mathcal{P}^{(n)}).$$

(2.1.5)

Now, by taking sup over all partitions, one can get a definition independent of partitions.

**Definition 2.1.3.** The KS entropy of the classical dynamical system $(\chi, T, \nu)$ is defined by

$$h_{KS}^\nu(T) := \sup_{\mathcal{P}} h_{KS}^\nu(T, \mathcal{P}),$$

(2.1.6)

where the sup is taken over all finite measurable partitions $\mathcal{P}$.

**Remark 2.1.2.** It is not easy to compute sup in the KS entropy definition. But, by the Kolmogorov-Sinai Theorem [10], if there exists a generating partition $\mathcal{P}$, then

$$h_{KS}^\nu(T_\sigma) = h_{KS}^\nu(T, \mathcal{P}),$$

where a generating partition is a finite partition such that the set of refined partitions $\mathcal{P}^{(n)}$ for all $n \in \mathbb{N}$, generates the $\sigma$-algebra $\Sigma$ of phase space $\chi$.

The following simple example shows us the computation of the KS-entropy for the Bernoulli shift dynamics.

**Example 2.1.1.** *(Bernoulli shifts)* Let us consider a shift dynamical system $(\Omega_2, T_\sigma, \nu)$
where the measure \( \nu \) is locally defined as follows

\[
\nu(C^{[j,k]}_{i_j i_{j+1} \ldots i_k}) = p^{(k-j+1)}(i_j i_{j+1} \ldots i_k),
\]

where

\[
p^{(n)}(i_1 \ldots i_n) = \prod_{j=1}^{n} p(i_j), \quad p(i) \geq 0, \quad \sum_{i=1}^{d} p(i) = 1.
\]

On the other hand, \( \mathcal{P} := \{C_j^{(0)}\}_{j=1}^{p} \) is a generating partition for the \( \sigma \)-algebra of cylinders.

Therefore,

\[
h_{KS}^{\nu}(T_\sigma) = \frac{1}{n} \sum_{s=0}^{n-1} \nu(\psi \phi \circ T^s) = - \sum_{i=1}^{p} p(i) \log p(i) = H_{\nu}(\mathcal{P}).
\]

Ergodic theory developed in [25] explains when and why mean values of observables coincide with their time-averages why trajectories in ergodic systems fill the phase-space densely.

**Definition 2.1.4.** A dynamical system \((\chi, T, \nu)\) is called ergodic if for every \( \psi, \phi \in L^2_\nu(\chi) \),

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t-1} \nu(\psi(\phi \circ T^s)) = \nu(\psi)\nu(\phi).
\]

The quantity \( C(x, \mathcal{P}) := \lim \sup_n \frac{1}{n} (\min_{i^{(n)}} C(i^{(n)}, \mathcal{P})) \), where \( C(i^{(n)}, \mathcal{P}) := C(i^{(n)}) \), is called the complexity of a point \( x \in \chi \) with respect to a finite measurable partition \( \mathcal{P} \).

The quantity \( C(x) := \sup_{\mathcal{P}} C(i^{(n)}, \mathcal{P}) \) is called the complexity of the trajectory of \( x \in \chi \).

The two following theorems proved by Brudno [15], shows a relation between compression of data and the Kolmogorov complexity. Actually, it sets a relation between different subjects in mathematics, computer science and physics.

**Theorem 2.1.2.** In a binary ergodic source \((\Omega_2, T_\sigma, \pi)\), with entropy rate \( h_{KS}^{\sigma}(T_\sigma) \), we
have
\[
\lim_{n \to \infty} \frac{1}{n} C(\varepsilon^{(n)}) = h_{\pi}(T_{\sigma}),
\]
(2.1.8)

for almost all \( \varepsilon^{(n)} \in \Omega_2 \) with respect to \( \pi \).

**Theorem 2.1.3.** Let \((\chi, T, \nu)\) be an ergodic dynamical system and \(\mathcal{P}\) be a finite measurable partition of \(\chi\); then
\[
C(x, \mathcal{P}) = h_{\sigma}^{KS}(T_{\sigma}, \mathcal{P}) \quad \nu - a.e.
\]
(2.1.9)

If \(\mathcal{P}\) is a generating partition then,
\[
C(x, \mathcal{P}) = h_{\sigma}^{KS}(T_{\sigma}) \quad \nu - a.e.
\]
(2.1.10)

### 2.2 Classical Spin Chains and Algebraic Formulation

In many cases, it proves useful to investigate classical dynamical systems using algebraic tools. Namely, instead of working with phase-space trajectories, one considers observables (suitable functions over the phase-space) and their time-evolution. In other words, to a given dynamical system \((\chi, T, \nu)\), where \(\chi\) is a compact metric space, one can associate a \(C^*\)-algebraic triplet \((C(\chi), \Theta_T, \omega_\nu)\) \footnote{\(C(\chi)\) is the Banach algebra \(\ast\)-algebra (with identity) of continuous complex value functions on \(\chi\) endowed with the uniform topology given by the sup norm.} and a von Neumann triplet \((L^\infty_{\nu}(\chi), \Theta_T, \omega_\nu)\) where state \(\omega_\nu\) and automorphism \(\Theta_T\) are defined as follows:
\[
\omega_\nu(f) = \int_{\chi} d\nu(x) f(x),
\]
(2.2.1)
\[
\Theta_T(f) = f \circ T,
\]
(2.2.2)

for all \(f \in C(\chi)\) or \(L^\infty_{\nu}(\chi)\).
**Example 2.2.1.** (Koopmann-von Numann formalism\(^2\)) Let \((\chi, T, \nu)\) be a dynamical system. The Koopmann-von Neumann unitary operator \(U_T\) is defined as follows

\[(U_T\psi)(x) = \psi(Tx),\]

for any \(\psi \in L^2_\nu(\chi)\) and \(x \in \chi\). Let define \(<f|g> = \int_\chi \overline{f(x)}g(x)dx\) be the scalar product of any \(f, g \in L^2_\nu(\chi)\). The automorphism \(\Theta_T\) is implemented by \(U_T\) as follows

\[<x|U_T f U_T^\dagger \psi > = f(Tx) <Tx|U_T^\dagger \psi > = f(Tx) <T^{-1} \circ Tx|\psi >\]

\[= <x|\Theta_T(f) \psi >,\]

for any \(f \in C(\chi)\). Of course, the state \(\omega_\nu\) is defined like the above definition.

Now, we introduce some definitions here as follows

**Definition 2.2.1.** A positive operator \(\rho\) of Hilbert space \(\mathbb{H}\) is called density matrix if \(\text{Tr}(\rho) = 1\).

**Definition 2.2.2.** For a given density matrix \(\rho\) with spectral decomposition \(\sum_i \lambda_i |\lambda_i><\lambda_i|\), the von Neumann entropy is defined as follows

\[S(\rho) = -\text{Tr}(\rho \log \rho) = - \sum_i \lambda_i \log \lambda_i.\]

In addition, relative entropy for given two density matrices \(\rho\) and \(\sigma\) is given by

\[S(\rho, \sigma) = \text{Tr}(\rho (\log \rho - \log \sigma)).\]

\(^2\)The previous one is a technique which allows one to reformulate classical dynamical systems in terms of Hilbert spaces and unitary time-evolutions, as one does with quantum mechanical systems where one encounters the following basic concepts.
It is useful to look at symbolic models of classical dynamical systems as classical spin chains.

A classical spin chain is the mathematical way of modeling a classical ferromagnet as a one-dimensional lattice $\mathbb{Z}$ whose sites support identical classical spins capable of assuming $p$ possible states. In this case, to each site corresponds an algebra of $p \times p$ diagonal matrices over $\mathbb{C}$ which is denoted by $D_p(\mathbb{C})$.

The diagonal matrices $P_j$ whose elements are all zero but for the $jj$-th entry which is equal to 1, constitute a set of generating projections $P_j$, $1 \leq j \leq p$, for the algebra $D_p(\mathbb{C})$. Thus, an element $D$ of $D_p(\mathbb{C})$ is of the form $\sum_{j=1}^{p} d_j P_j$, where $d_j$'s are complex numbers.

The spin algebra of $n$ particles located at the lattice sites $-n \leq j \leq n$ will be denoted by $D_{[-n,n]} := \bigotimes_{j=-n}^{n} (D_p(\mathbb{C}))_j$ where $(D_p(\mathbb{C}))_j = D_p(\mathbb{C})$, for each $-n \leq j \leq n$. Indeed, each element of that algebra is a $p^n \times p^n$ matrix of the form

$$D_{[-n,n]}^{(2n+1)} := \sum_{i^{(n)} \in \Omega_{2n+1}} d(i^{(n)}) P_{i^{(n)}}^{[-n,n]}$$

where $d(i^{(n)})$'s are complex numbers and $P_{i^{(n)}}^{[-n,n]} := P_{i-n} \otimes P_{i-n+1} \otimes \ldots \otimes P_{i+n}$ are projectors.

Let us consider the symbolic dynamical system $(\Omega_p, T, \nu)$, that is a shift dynamical system over two-sided infinite sequences of symbols from an alphabet with $p$ elements. The $C^*$-algebraic triplet $(D_\mathbb{Z}, \Theta, \omega)$ associated with the symbolic dynamical system $(\Omega_p^\mathbb{Z}, T, \nu)$ as outlined before is indeed a classical spin chain.

- Let us define the commutative algebra $D_\mathbb{Z} := \bigcup_{n \in \mathbb{N}} D_{[-n,n]}^{\text{uniform}}$, inductively extended from local algebras by a method which is known as $C^*$-inductive limit \[43\].
• $\Theta_\sigma$ is an algebraic automorphisms

$$\Theta_\sigma(\mathbb{I}_{-n-1} \otimes A \otimes \mathbb{I}_{[n+1]} = \mathbb{I}_{-n} \otimes A \otimes \mathbb{I}_{[n+2]}, \quad (2.2.3)$$

for each $A \in \mathcal{D}_{[-n,n]}$. Therefore,

$$\Theta_\sigma(\mathcal{D}_{[-n,n]}) = \mathcal{D}_{[-n+1,n+1]}.$$

(2.2.4)

Indeed, $\mathcal{D}_{[-n,n]}$ is embedded in $\mathcal{D}_\mathbb{Z}$ by the map $A \mapsto \mathbb{I}_{-n-1} \otimes A \otimes \mathbb{I}_{[n+1]}$, for $A \in \mathcal{D}_{[-n,n]}$, and from now on, we identify $\mathcal{D}_{[-n,n]}$ with $\mathbb{I}_{-n-1} \otimes \mathcal{D}_{[-n,n]} \otimes \mathbb{I}_{[n+1]}$.

• Let us consider the local density matrix

$$\rho_\nu^{(n)}(\mathbb{I}^{(n)}) := \sum_{\mathbb{I}^{(n)} \in \Omega^{(n)}} \nu(\mathbb{I}^{(n)}) P^{[0,n-1]}_{\mathbb{I}^{(n)}}, \quad (2.2.5)$$

on $\mathcal{D}_{[-n,n]}$. Then, the global density matrix $\rho_\nu$ is defined as $\lim_{n \to \infty} \rho_\nu^{(n)}$. Furthermore, a global state is defined by

$$\omega_\nu(A) := \text{Tr}_{\mathcal{D}_{[-n,n]}}(A \rho_\nu^{(n)}) \quad \forall A \in \mathcal{D}_{[-n,n]}, \quad (2.2.6)$$

With the notations of (2.3.1), the KS-entropy for classical spin dynamics computes by the following relation:

$$h_\nu^{KS}(T) = s(\omega) := \lim_{n \to \infty} \frac{1}{n} S(\omega \mid \mathcal{D}_{[-n,n]}), \quad (2.2.7)$$

where $S(\omega \mid \mathcal{D}_{[-n,n]}) = S(\rho_\nu^{(n)})$ and $s(\omega)$ is called the von Neumann entropy rate.
2.3 Quantum Dynamical Systems

Quantum dynamical systems, are in general introduced as non-commutative algebraic triplets.

**Definition 2.3.1.** A quantum dynamical system is a triplet \((\mathcal{A}, \Theta, \omega)\) where \(\mathcal{A}\) is a \(C^*\)-algebra with identity \(1\), and

- the dynamics \(\Theta\) corresponds to a group of automorphisms \(\Theta_t : \mathcal{A} \to \mathcal{A}, t \in G\), which
  
  \[ G = \mathbb{R} \text{ or } G = \mathbb{Z}, \text{ and, for any } t, s \in G, \Theta_t \circ \Theta_s = \Theta_s \circ \Theta_t = \Theta_{t+s}. \]

- The state \(\omega : \mathcal{A} \to \mathbb{C}\) is a normalized, positive, \(\Theta\)-invariant expectation, namely
  
  \[ \omega \circ \Theta_t = \omega \text{ for all } t \in G. \]

Classical spin chains are particular cases of quantum dynamical systems, where their associated \(C^*\)-algebras are commutative.

2.3.1 Quantum Spin Chains

A quantum spin chain is the \(C^*\)-algebra that arises from the norm completion of local quantum spin algebras of the tensor product form

\[
M_{[-n,n]} = M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \otimes \cdots M_d(\mathbb{C}) = M_d^{\otimes 2n+1} = M_{d^{2n+1}}(\mathbb{C}). \tag{2.3.1}
\]

The interpretation is straightforward: one is dealing with a one-dimensional lattice each site of which supports a \(d\)-level quantum system (or \(d\)-dimensional spin). In the norm-topology (the norm is the one which coincides with the standard matrix-norm on each local algebra) the limit \(n \to +\infty\) of the nested sequence \(M_{[-n,n]}\) gives rise to the norm-complete infinite dimensional algebra

\[
\mathcal{M} := \lim_p M_{[-p,p-1]}, \tag{2.3.2}
\]
that describes an infinite quantum spin lattice, that is a quantum spin chain. In the
following we shall consider $d = 2$, namely a chain of 2-level quantum spins, or spin 1/2
particles, or in the modern jargon, qubits.

Any local spin operator, say $A \in M_{[-n,n]}$, is naturally embedded into $M$ as

$$M_{[-n,n]} \ni A \mapsto 1_{-n-1} \otimes A \otimes 1_{n+1} \in M,$$  \hspace{1cm} (2.3.3)

where $1_{-n-1}$ stands for the infinite tensor products of $2 \times 2$ identity matrices up to
site $-n - 1$, while $1_{n+1}$ stands for the infinite tensor product of infinitely many identity
matrices from site $n + 1$ onwards. In this way, the local algebras are sub-algebras of the
infinite one sharing a same identity operator.

The simplest dynamics on such quantum spin chains is given by the right shift

$$\Theta[M_{[-n,n]}] = M_{[-n+1,n+1]} \quad \text{and} \quad \Theta[1_{-n-1} \otimes A \otimes 1_{n+1}] = 1_{-n} \otimes A \otimes 1_{n+2}.$$ \hspace{1cm} (2.3.4)

Any state $\omega$ on $M$ is a positive, normalized linear functional whose restrictions to the local
sub-algebras are density matrices $\rho_{[-n,n]}$, namely positive matrices in $M_{[-n,n]}(\mathbb{C})$ such that

$$\text{Tr}_{[-n,n]} \rho_{[-n,n]} = 1:$$

$$M_{[-n,n]} \ni A \mapsto \omega(A) = \text{Tr}_{[-n,n]} \left( \rho^{(n)} A \right).$$ \hspace{1cm} (2.3.5)

The degree of mixedness of such density matrices is measured by the von Neumann entropy

$$S(\rho_{[-n,n]}) = -\text{Tr}_{[-n,n]}(\rho_{[-n,n]} \log \rho_{[-n,n]}) = - \sum_j r_j^{[-n,n]} \log r_j^{[-n,n]},$$ \hspace{1cm} (2.3.6)

where $0 \leq r_j^{[-n,n]} \leq 1$, $\sum_j r_j^{[-n,n]} = 1$, are the eigenvalues of $\rho_{[-n,n]}$. Notice that the von
Neumann entropy is nothing but the Shannon entropy of the spectrum of $\rho_{[-n,n]}$ which
indeed amounts to a discrete probability distribution.
In the above expressions $\text{Tr}_{[-n,n]}$ stands for the trace computed with respect to any orthonormal basis of the Hilbert space $H_{[-n,n]} = (\mathbb{C}^2)^{\otimes 2n+1}$ onto which $A$ linearly acts. Let $|i\rangle \in \mathbb{C}^2$, $i = 0, 1$, be a chosen orthonormal basis in $\mathbb{C}^2$; then, a natural orthonormal basis in $H_{[-n,n]}$ will consist of tensor products of single spin basis vectors:

$$|i_{[-n,n]}\rangle = \bigotimes_{j=-n}^n |i_j\rangle = |i_{-n}i_{-n+1}\cdots i_n\rangle,$$

namely its elements are indexed by binary strings $i_{[-n,n]} \in \{0, 1\}^{2n+1}$. By going to the limit of an infinite chain, a corresponding representation Hilbert space is generated by orthonormal vectors again denoted by $|i_{[-n,n]}\rangle$ where $n$ arbitrarily varies and every $i_{[-n,n]}$ is now a binary sequence in $\{0, 1\}^Z$ where all $i_k \not\in [-n, n]$ are chosen equal to 0. We shall denote by $i$ such binary strings, by $\Omega$ their set and by $|i\rangle$ the corresponding orthonormal vectors which form the so-called standard basis of $H$.

Remark 2.3.1. While all representations of finite size quantum spin chains are unitarily equivalent to the Fock representation \[42\], what we are considering here is just one of the infinitely many inequivalent Hilbert space representations for the genuinely infinite quantum spin chain. Indeed, the representation Hilbert space we are considering is a particular case of the so-called GNS construction \[42\]: it is created acting with finitely many spin flips $|0\rangle \mapsto |1\rangle$ on the GNS cyclic state represented by all spins being in the state $|0\rangle$. By choosing $i_k = 1$ outside any finite interval $[-n, n]$ one gets another representation of the same algebra $\mathcal{M}$. However, the new representation is inequivalent to the previous one as there is no unitary operator mapping one Hilbert space into the other. Such a unitary operator should indeed flip infinitely many spins.
From (2.3.5), a compatibility relation immediately follows; namely,

$$\omega(A \otimes \mathbf{I}_n) = \text{Tr}_{[0,n]}(A \otimes \mathbf{I}_n) = \omega(A) = \text{Tr}_{[0,n-1]}(\rho_{[0,n-1]} A) \quad \forall A \in M_{2^n}(\mathbb{C}),$$

so that

$$\text{Tr}_n \rho_{[0,n]} = \rho_{[0,n-1]}.$$  \hfill (2.3.8)

On the other hand, if

$$\omega(\mathbf{1}_0 \otimes A) = \text{Tr}_{[0,n]}(\rho_{[0,n]} \mathbf{1}_0 \otimes A) = \omega(A) = \text{Tr}_{[0,n-1]}(\rho_{[0,n-1]} A) \quad \forall A \in M_{2^n}(\mathbb{C}),$$

namely, if $\omega$ is a transationally invariant state, then

$$\rho_{[0,n-1]} = \text{Tr}_0 \rho_{[0,n]} \quad \forall n.$$  \hfill (2.3.9)

To any translationally invariant state $\omega$ on a quantum spin chain there is associated a well-defined von Neumann entropy rate (see for instance [4]):

$$s(\omega) = \lim_{n \to +\infty} \frac{1}{n} S(\rho_{[0,n-1]}) = -\lim_{n \to +\infty} \frac{1}{n} \text{Tr}_{[0,n-1]} \left( \rho_{[0,n-1]} \log \rho_{[0,n-1]} \right).$$  \hfill (2.3.10)

### 2.3.2 AF entropy

The AF or AFL entropy developed by Alicki, Fannes and Lindblad [2, 29] is an extension of the concept of KS entropy in classical dynamical systems to discrete-time non-commutative quantum dynamical systems. The construction of the AF entropy is based on the notion of quantum partitions of unity. These later together with the dynamics give rise, similarly to the classical case, to quantum symbolic models of quantum dynamical systems. By means of the von Neumann entropy, one then defines the AF entropy of a
quantum dynamical system as the optimal von Neumann entropy rate over all its quantum symbolic models. Let \((\mathcal{A}, \Theta, \omega)\) be a quantum dynamical system.

**Definition 2.3.2.** A finite collection of operators \(\mathcal{Z} = \{Z_i\}_{i=1}^{\mathcal{Z}}\), where \(Z_i \in \mathcal{A}\) is called an operational partition of unity (OPU) if

\[
\sum_{i=1}^{\mathcal{Z}} Z_i^\dagger Z_i = \mathbb{1}, \quad (2.3.11)
\]

where \(\mathcal{Z}\) is the cardinality of \(\mathcal{Z}\).

- The refinement of two partitions \(\mathcal{Z}_1 = \{Z_{1i}\}_{i=1}^{\mathcal{Z}_1}\) and \(\mathcal{Z}_2 = \{Z_{2j}\}_{j=1}^{\mathcal{Z}_2}\), is defined naturally by

\[
\mathcal{Z}_1 \circ \mathcal{Z}_2 := \{Z_{1i}Z_{2j}\}_{i,j=1}^{\mathcal{Z}_1,\mathcal{Z}_2},
\]

which is also an OPU. Moreover, time-evaluation of an OPU \(\mathcal{Z} = \{Z_i\}_{i=1}^{\mathcal{Z}}\) at time \(t = k \in \mathbb{Z}\) under the dynamics \(\Theta\) is OPU which is defined by

\[
\mathcal{Z} := \Theta^k(\mathcal{Z}) = \{\Theta^k(Z_i)\}_{i=1}^{\mathcal{Z}}. \quad (2.3.12)
\]

- Let \(Z_{i(n)} := \Theta^{n-1}(Z_{n-1}) \ldots \Theta(Z_{i1})Z_{i0}\). Clearly the set \(\mathcal{Z}^{(n)} := \{Z_{i(n)}\}_{i(n) \in \Omega_{\mathcal{Z}}^{(n)}}\) is again an OPU. Now, for \(\mathcal{Z} = \{Z_j\}_{j=1}^{\mathcal{Z}}\), one can define a \(|\mathcal{Z}| \times |\mathcal{Z}|\) density matrix \(\rho[\mathcal{Z}]\) as follows

\[
M_{\mathcal{Z}}(\mathbb{C}) \ni \rho[\mathcal{Z}] := \sum_{i,j=1}^{\mathcal{Z}} |z_i|^\dagger z_j |\omega(Z_j^\dagger Z_i), \quad (2.3.13)
\]

where \(\{|z_i|\}_{i=1}^{\mathcal{Z}}\) is a fixed orthonormal basis in the finite dimensional Hilbert space \(\mathbb{C}^{\mathcal{Z}}\). Moreover, the density matrix associated with \(\mathcal{Z}^{(n)}\) has the form

\[
M_{\mathcal{Z}}(\mathbb{C})^{\otimes n} \ni \rho[\mathcal{Z}^{(n)}] := \sum_{i^{(n)},j^{(n)} \in \Omega_{\mathcal{Z}}^{(n)}} |z_{i^{(n)}|}^\dagger z_{j^{(n)}|} |\omega(Z_{j^{(n)}}^\dagger Z_{i^{(n)}}), \quad (2.3.14)
\]
where

$$|z_{i(n)}> := |z_{i_1}> \otimes |z_{i_2}> \otimes \ldots |z_{i_n}> .$$

The translation invariance $\omega \circ \Theta = \omega$ and the compatibility relation are expressed by

$$Tr_{(1)}(\rho[Z^{n+1}]) = Tr_{(n+1)}(\rho[Z^{n+1}]) = \rho[Z^{(n)}]. \quad (2.3.15)$$

Thus the family $\rho[Z^{(n)}]$ denoted by $\rho^{(n)}$ in section 2.3.1, $n \in \mathbb{N}$ gives a state $\omega_Z$ over $A_Z := \mathcal{M}$, where $\mathcal{M}$ is defined in 2.3.1. For a given quantum dynamical system $(\mathcal{A}, \Theta, \omega)$ and a chosen OPU in a suitable subalgebra $A_0$, the AF-entropy is constructed over their associated quantum symbolic system $(A_Z, \Theta_\sigma, \omega_Z)$, or quantum spin dynamics with right shift dynamics, together with given OPU. We restrict ourselves to subalgebra $A_0$ because in general the mean von Neumann entropy of $(A_Z, \Theta_\sigma, \omega_Z)$, with the translation invariance $\Theta$, may not exist.

**Definition 2.3.3.** Let $A_0 \subseteq \mathcal{A}$ be a $\Theta$-invariant subalgebra and let $Z \subseteq A_0$ be an OPU. Let us define

$$h^{AFL}_\omega(\Theta, Z) := \limsup_{n \to \infty} \frac{1}{n} S(\rho[Z^{(n)}]), \quad (2.3.16)$$

where $S(\rho[Z^{(n)}])$ is the von Neumann entropy of the density matrix associated with the OPU $Z^{(n)}$. The AF entropy of the quantum dynamical system $(\mathcal{A}, \Theta, \omega)$ is defined by

$$h^{AFL}_\omega(\Theta) := \sup_{Z \subseteq A_0} h^{AFL}_\omega(\Theta, Z). \quad (2.3.17)$$

**Remark 2.3.2.** The $\limsup$ in (2.3.16) has to be used for the sequence of density matrices $\rho[Z^{(n)}]$ is not a stationary one [2, 3]. In fact, while consistency holds as tracing $\rho[Z^{(n)}]$ over the $n$-th factor yields the density matrix corresponding to the first $n - 1$ factors,
\[ \operatorname{Tr}_n \rho[Z^{(n)}] = \rho[Z^{(n-1)}], \text{ stationarity does not; indeed, in general, } \operatorname{Tr} \rho[Z^{(n)}] \neq \rho[Z^{(n-1)}]. \]

**Example 2.3.1.** As a concrete example consider a set of 4 matrix units \( U_{ij} \in M_2(\mathbb{C}) \) such that \( U_{ij}^\dagger = U_{ji}, U_{ij}U_{kl} = \delta_{jk}U_{il} \) and \( \sum_{i=1}^2 U_{ii} = 2 \). Dividing them by \( \sqrt{2} \) one gets a partition of unit

\[ \mathcal{U} = \left\{ \frac{U_{ij}}{\sqrt{2}} \right\}_{i,j=1,2} \in M_2(\mathbb{C}), \]

the simplest choice being

\[ U_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_{12} = U_{21}^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

The refined partition that results after \( n \) applications of the right shift is

\[ \mathcal{U}^{(n)} = \left\{ \frac{U_{i(n)j(n)}}{2^{n/2}} \right\}, \quad U_{i(n)j(n)} = U_{i0j0} \otimes U_{i1j1} \otimes \cdots U_{i_{n-1}j_{n-1}} \in M_2^\otimes n(\mathbb{C}) = M_{[0,n-1]} \, . \]

(2.3.18)

The associated density matrices \( \rho[\mathcal{U}^{(n)}] \in M_4^\otimes (\mathbb{C}) \) have entries and von Neumann entropy given by

\[ \frac{1}{2^n} \operatorname{Tr} \left( \rho^{(n)} U_{i(n)j(n)}^\dagger U_{k(n)\ell(n)} \right) = \frac{1}{2^n} \operatorname{Tr} \left( \rho^{(n)} U_{j0i0} U_{k0i0} \otimes U_{j1i1} U_{k1i1} \otimes \cdots \right) = \frac{1}{2^n} \operatorname{Tr} \left( \rho^{(n)} \delta_{i0k0} U_{j0\ell0} \otimes \delta_{i1k1} U_{j1\ell1} \otimes \cdots \right) = \frac{1}{2^n} \rho^{(n)} \]

\[ S \left( \rho[\mathcal{U}^{(n)}] \right) = S(\rho^{(n)}) + n \, . \]

(2.3.19) (2.3.20) (2.3.21)

The last equality in (2.3.19) comes from the fact that \( \operatorname{Tr} \left( \rho^{(n)} U_{i(n)j(n)} \right) \) are the matrix elements of \( \rho^{(n)} \) with respect to the orthonormal basis defined by the choice of matrix...
units. Entropy rate and the Alicki-Fannes entropy then result

\[ h^{AF}_\omega(\Theta) = h^{AF}_\omega(\Theta, U) = \limsup_{n \to \infty} \frac{1}{n} S(\rho[U^{(n)}]) = s(\omega) + 1. \]  

(2.3.22)

Properties of the AF entropy

• When a quantum dynamical system \((A, \Theta, \omega)\) reduces to a classical dynamical system \((\chi, T, \nu)\), the AF entropy of the triplet \((M, \Theta_T, \omega_\nu)\) is

\[ h^{AF}_\omega(\Theta_T, M) = h^{KS}_\nu(T), \]  

(2.3.23)

where \(M := L^\infty(\chi)\) and \(h^{KS}_\nu(T)\) is the Kolmogorov-Sinai entropy.

• Let \((A_\mathbb{Z}, \omega)\) be a quantum spin chain with single site matrix algebra \(M_d(\mathbb{C})\). The AF entropy with respect to every local subalgebra \(A_{[p,q]} \subseteq A_0\) is given by

\[ h^{AF}_\omega(\Theta_\sigma) = s(\omega) + \log d, \]  

(2.3.24)

where the dynamics is the right-shift \(\Theta_\sigma\) over \(A_\mathbb{Z}\), and \(s(\omega)\) is the mean von Neumann entropy of the translation-invariant \(\omega\) (see Section 2.3.4).

2.3.3 CNT Entropy

The CNT entropy introduced by Connes, Naranhofer and Thirling [1] is a generalization of the KS-entropy to quantum dynamical systems which is based on convex decompositions of the state \(\omega\).

Definition 2.3.4. Let \(M\) and \(A\) be two \(C^*\)-algebras. A linear map \(\gamma : M \to A\) is called completely positive if \(\gamma \otimes id_n : M \otimes Mat_n(\mathbb{C}) \to A \otimes Mat_n(\mathbb{C})\) is a positive operator for each \(n \in \mathbb{N}\).
Each state $\omega$ is the form $\omega(A) = \text{Tr}(A\rho)$ for a unique positive element, or density matrix, $\rho$. The entropy of the state $\omega$ is defined the von Neumann entropy of the associated density matrix.

Let us consider a convex decomposition

$$\omega = \sum_{i^{(n)} \in I^{(n)}} \lambda_{i^{(n)}}^{(n)} \omega_{i^{(n)}}, \quad I^{(n)} = I_1 \times I_2 \times \ldots \times I_n,$$

where $\lambda_{i^{(n)}}^{(n)}$ are positive weights and $I_j$’s are generic index sets. The marginal density matrices arising from this decomposition is denoted by $\omega = \sum_{i_j \in I_j} \lambda_{i_j}^{(n)} \omega_{i_j}^{(n)}$, $j = 1, 2, \ldots, n$,

where

$$\omega_{i_j}^{(n)} = \sum_{i^{(n)}, i_j \text{ fixed}} \frac{\lambda_{i^{(n)}}^{(n)}}{\lambda_{i_j}^{(n)}} \omega_{i^{(n)}}^{(n)}, \quad \lambda_{i_j}^{(n)} = \sum_{i^{(n)}, i_j \text{ fixed}} \lambda_{i^{(n)}}^{(n)}.$$

Let $\Lambda^{(n)} = \{\lambda_{i^{(n)}}^{(n)}\}$ and $\Lambda_j = \{\lambda_{i_j}^{(n)}\}$ be probability distributions associated with the scalar products in (2.3.25) and (2.3.26).

**Definition 2.3.5.** Let $\mathcal{A}$ be a $C^*$-algebra endowed with a state $\omega$. Let $\gamma_i : M_i \subset \mathcal{A}$, $i = 1, 2, \ldots, n$ be CPU maps from finite dimensional $C^*$-algebras into $\mathcal{A}$. Their entropy with respect to $\omega$ is:

$$H_{\omega}(\gamma_1, \gamma_2, \ldots, \gamma_n) := \sup_{\omega = \sum_{i^{(n)}} \lambda_{i^{(n)}}^{(n)} \omega_{i^{(n)}}} \left\{ H(\Lambda^{(n)}) - \sum_{j=1}^{n} H(\Lambda_j) + \sum_{j=1}^{n} \sum_{i_j \in I_j} \lambda_{i_j}^{(n)} S(\omega_{i_j}^{(n)} \circ \gamma_j, \omega \circ \gamma_j) \right\},$$

where $\omega \circ \gamma_j$ is a state over $M$ and $H$ is the Shannon entropy.

The CNT entropy rate for a completely positive map $\gamma : M \rightarrow \mathcal{A}$ where $M$ is a finite dimensional $C^*$-subalgebra of $\mathcal{A}$, is defined as follows

$$h_{\omega}^{CNT}(\Theta, \gamma) = \lim_{n \to \infty} \frac{1}{n} H_{\omega}(\gamma, \Theta \circ \gamma, \ldots, \Theta^{n-1} \circ \gamma). \quad (2.3.27)$$
The existence of the above limit is shown in [1]. The CNT dynamical entropy is defined by

\[ h^{CNT}_\omega(\Theta) = \sup_{\gamma} h^{CNT}_{\omega}(\Theta, \gamma). \] (2.3.28)

It is proved in [1] that in \(d\)-level quantum spin chains with shift dynamics,

\[ h^{CNT}_\omega(\Theta) = s(\omega). \] (2.3.29)

**Example 2.3.2.** Let us consider the quantum spin chain \((\mathcal{M}, \Theta_\sigma, \omega)\) with right shift dynamics, where the state \(\omega\) is defined using the density matrix \(\rho^{(n)} = \rho \otimes \cdots \otimes \rho\). Then, we have

\[ h^{CNT}_\omega(\Theta) = s(\omega) = \lim_{n \to \infty} \frac{S(\rho^{(n)})}{n} = \lim_{n \to \infty} \frac{nS(\rho)}{n} = S(\rho). \]

### 2.3.4 Relation Between CNT and AF Entropies in Quantum Spin Chains

In this section we show that from physical point of view in 2-level quantum spin chains with shift dynamics \(h^{AF}_\omega(\Theta) = h^{CNT}_\omega(\Theta) + 1\) [5].

Consider a two level spin chain \(\mathcal{M}_2\) where \(h^{CNT}_\omega(\Theta) = s(\omega)\) and \(h^{AF}_\omega(\Theta_\sigma) = s(\omega) + 1\).

The origin of the difference by 1 = \(\log 2\) between the AF-entropy and the entropy rate (which is equal to the CNT-entropy) lies in that the AF-entropy accounts for measurement-like disturbances on the state of the quantum chain. In quantum mechanics generic measurement processes on a system in a state described by density matrix \(\rho\) are identified by partitions of unity \(Z = \{Z_i\}\) and the state is changed by the measurement process as follows:

\[ \rho \mapsto \sum_i Z_i \rho Z_i^\dagger. \] (2.3.30)
Suppose

\[ M_2^n(\mathbb{C}) \ni \rho^{(n)} = \sum_i r^{(n)}_i |r^{(n)}_i\rangle \langle r^{(n)}_i| \]  

(2.3.31)
is the spectral decomposition of a local state for \( n \) qubits described by the local algebra \( M_{[0,n-1]} \); any such mixed state can be purified, that is transformed into a projector, by coupling \( M_{[0,n-1]} \) to itself and by doubling \( \rho^{(n)} := \rho_{[0,n-1]} \) into

\[ \mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n} \ni |\sqrt{\rho^{(n)}}\rangle = \sum_i \sqrt{r^{(n)}_i} |r^{(n)}_i\rangle \otimes |r^{(n)}_i\rangle . \]  

(2.3.32)

Given the refined partition of unity \( U^{(n)} \) in (2.3.18), one further amplifies the Hilbert space from \( \mathbb{C}^{4^n} \) to \( \mathbb{C}^{4^n} \otimes \mathbb{C}^{4^n} \) and constructs the following vector state

\[ \mathbb{C}^{4^n} \otimes \mathbb{C}^{4^n} \ni |\Psi[U^{(n)}]\rangle = \sum_i \sum_{(k^{(n)}\ell^{(n)})} \sqrt{r^{(n)}_i} U_{k^{(n)}\ell^{(n)}} |r^{(n)}_i\rangle \otimes |r^{(n)}_i\rangle \otimes |k^{(n)}\ell^{(n)}\rangle , \]  

(2.3.33)

where the vectors \( |k^{(n)}\ell^{(n)}\rangle \) indexed by pairs of binary strings in \( \Omega_2^n \) form an auxiliary orthonormal basis in \( \mathbb{C}^{4^n} \) of cardinality \( 2^n \times 2^n \).

One thus sees that \( |\Psi[U^{(n)}]\rangle \) is the vector state of a three-partite system consisting of the \( n \) qubits, system \( I \), a copy of the latter, system \( II \), and a copy of the first two, system \( III \). From the projection \( P = |\Psi[U^{(n)}]\rangle \langle \Psi[U^{(n)}]| \), by tracing over the first two systems, respectively over the last one, one obtains the following marginal states on \( M_{[0,n-1]} \otimes M_{[0,n-1]} \),

\[ \text{Tr}_{I,II}(P) = \rho[U^{(n)}] , \quad \text{respectively} \]  

(2.3.34)

\[ \text{Tr}_{III}(P) = \sum_{(k^{(n)}\ell^{(n)})} U_{k^{(n)}\ell^{(n)}} \otimes 1 |\sqrt{\rho^{(n)}}\rangle \langle \sqrt{\rho^{(n)}}| U_{k^{(n)}\ell^{(n)}}^\dagger \otimes 1 = R[U^{(n)}] . \]  

(2.3.35)

Since the latter states are marginal density matrices of a pure state, they have the same
spectrum and thus the same von Neumann entropy (see for instance [4])

\[ S\left(\rho[U^{(n)}]\right) = S\left(R[U^{(n)}]\right) = S(\rho^{(n)}) + n . \]

Thence, the entropy associated to \( \omega \) and to the partition of unity \( U^{(n)} \), that is \( \rho[U^{(n)}] \), is also the entropy of the state \( R[U^{(n)}] \) which results from the action of the POVM \( \{U_{k^{(n)}}^\dagger \otimes 1\} \) on the purified state \(|\sqrt{\rho^{(n)}}\rangle\langle \sqrt{\rho^{(n)}}|\).
Chapter 3

Semi-computable States and Semi-computable Density Matrices

In this chapter, we will look at quantum mechanical tools as Hilbert space vectors, density matrices and generic linear operators on them from the point of view of computability theory. This is necessary in order to introduce the concept of Gacs complexity which is based on a quantum extension of the classical notion of universal semi-measure devised for finite-dimensional quantum systems to infinite dimensional separable Hilbert spaces. We shall then use Gacs complexity to present a Brudno’s like relation for quantum spin chains.

3.1 Universal Semi-computable Semi-density Matrices on Infinite Separable Hilbert spaces

We start by fixing the necessary notations and symbols.

1. Let the set $Q'$ be defined as follows

$$Q' = \{(\varepsilon, p, q) \in \{0, 1\} \times \mathbb{N}_+ \times \mathbb{N}_+ | p \text{ and } q \text{ are coprime}\} \cup \{(0, 0, 0)\}.$$
The mapping $\iota: Q \to Q'$ defined by

$$\iota(0) = (0,0,0), \quad \iota\left(\frac{p}{q}\right) = (0,p,q), \quad \iota\left(-\frac{p}{q}\right) = (1,p,q)$$

is bijective and the mapping $\iota': Q' \to \mathbb{N}$ defined by $(\varepsilon,p,q) \to <\varepsilon, <p,q>>$, where $<x,y> = 2^x(2y+1)-1$, is injective. We can identify $Q$ with the subset $\iota' \circ \iota(Q)$ of $\mathbb{N}$. Similarly, any finite dimensional rational matrix will be represented by a natural number.

2. With reference to the indexing of the standard basis in [2.3.7], we shall consider the set of all functions from $\mathbb{Z}$ into the set $\{0,1\}$ with finite support and denote it by $\Omega$.

Let $i \in \Omega$ and let $\theta: \mathbb{Z} \to \mathbb{Z}$ be the left shift $\theta(n) = n - 1$. Then $\theta$ induces the map $(\theta(i))_n = i_{n+1}$ on $\Omega$. The restriction of $i$ to the subinterval $I$ will be denoted by $i_I$.

Furthermore, let $p, q \in \mathbb{Z}$ and $p \leq q$. Assume that the support of $i \in \Omega$ is contained in the interval $[p,q] \subset \mathbb{Z}$. Then, $i = 0][_{[p,q]}[0]$, where $0] = i_{p-1}$ and $0 = i_{q+1}$ are infinite sequences of 0’s.

3. The map $\Omega \to \mathbb{N} \times \mathbb{N}$ that associates to $i \in \Omega$ the integers

$$(x = \sum_{k<0} i_k 2^{-k}, \quad y = \sum_{k \geq 0} i_k 2^k)$$

is bijective. Therefore, the following two maps

$$\Omega \ni i \mapsto \eta(i) = <x,y>, \quad \nu(i) = y - sign(x)[(x,y) + 1]/2 + y], \quad (3.1.1)$$

where $sign(x) = 0$ if $x = 0$ otherwise $sign(x) = 1$, are bijections between $\Omega$ and $\mathbb{N}$,
respectively \( Z \). Then, the inverse mapping

\[
\zeta : (i, j) \mapsto \nu^{-1}(\eta(j) - \text{sign}(\eta(i))[(<\eta(i), \eta(j) > + 1)/2 + \eta(j)])
\]  

(3.1.2)

identifies \( \Omega \times \Omega \) with \( \Omega \).

4. Let \( \Sigma \) be the power set of \( \Omega \). For \( A \in \Sigma \), let \( \mu(A) = \#(A) \). Given the measure space

\((\Omega, \Sigma, \mu)\), by the identification of \( \Omega \) with \( Z \), the Hilbert space \( L^2(\Omega, \Sigma, \mu) \) consists of the square-summable functions \( f : \mathbb{Z} \rightarrow \mathbb{C} \), i.e. \( \sum_{x \in \mathbb{Z}} |f(x)|^2 < \infty \). For any \( i \in \Omega \), consider the function \( \delta_i \) defined by

\[
\delta_i(i) = 1, \quad \delta_i(j) = 0 \quad \forall \ j \neq i.
\]

The set of these functions which is in one-to-one correspondence with \( \Omega \) is a Hilbert basis for \( L^2(\Omega, \Sigma, \mu) \) and for each \( i \in \Omega \), \( \delta_i \) will be denoted by \( |i> \). Therefore, the representation Hilbert space \( \mathbb{H} \) for the quantum spin chain is isomorphic to \( L^2(\Omega, \Sigma, \mu) \).

5. The mapping \( \zeta \) identifies \( \mathbb{H} \otimes \mathbb{H} \) with \( \mathbb{H} \). Furthermore, the set of all elements \( i \in \Omega \) with support included in \([-n, n] \) will be denoted by \( \Omega_{[-n, n]} \). The subspaces of \( L^2(\Omega, \Sigma, \mu) \) generated by \( \Omega_{[-n, n]} \), namely \( L^2(\Omega_{[-n, n]}) \), are isomorphic to the local quantum spin Hilbert spaces \( \mathbb{H}_{[-n, n]} = \mathbb{C}^{\otimes 2n+1} \). The corresponding orthogonal projections from \( \mathbb{H} \) onto \( \mathbb{H}_{[-n, n]} \) will be denoted by \( P_n \), and the canonical injection from \( \mathbb{H}_{[-n, n]} \) into \( \mathbb{H} \) will be denoted by \( i_n \). In the following we will identify \( \mathbb{H}_{[-n, n]} \) with the subspace \( i_n(\mathbb{H}_{[-n, n]}) \) of the Hilbert space \( \mathbb{H} \).

6. For a linear operator \( T \) on \( \mathbb{H} \), \( P_n TP_n \) will be denoted by \( T_n \).
**Definition 3.1.1.**

1. A vector $|\psi\rangle = \sum_{i \in \Omega} a_i |i\rangle \in \mathbb{H}$ will be termed elementary if of its expansion coefficients $a_i$ with respect to the fixed orthonormal basis $\{|i\rangle\}$ only a finite number is not zero and those are algebraic numbers.

2. A state $|\psi\rangle = \sum_{i \in \Omega} a_i |i\rangle \in \mathbb{H}$ where $a_i \in \mathbb{R}$, will be termed semi-computable if there exist a computable sequence of elementary vectors $|\psi_n\rangle = \sum_{i \in \mathbb{N}} a_{n,i} |i\rangle$ and a computable function $k : \mathbb{N} \to \mathbb{Q}$, such that $\lim_{n \to \infty} k_n = 0$, and for each $n, |a_i - a_{n,i}| \leq k_n$. Since the set of all computable functions is countable, the set of all semi-computable elements of $\mathbb{H}$ is countable.

3. A linear operator $M_{2^{n+1}}(\mathbb{C}) \ni T : \mathbb{H}[-n,n] \to \mathbb{H}[-n,n]$, will be called elementary if the real and imaginary parts of all of its matrix entries are rational numbers. It follows that the elementary operators can be numbered.

4. The linear operator $T : \mathbb{H} \to \mathbb{H}$, is a semi-density matrix if $T$ is positive and $0 \leq \text{Tr}(T) \leq 1$.

5. Let $n_1, n_2 \in \mathbb{N}$ and $n_1 \leq n_2$. Let $T_j : \mathbb{H}[-n_j, n_j] \to \mathbb{H}[-n_j, n_j]$, $j = 1, 2$, be two linear operators: $T_2$ will be said to be quasi-greater than $T_1$, $T_1 \leq_q T_2$, if $P_{n_1} T_2 P_{n_1} - T_1 \geq 0$, where $P_{n_1}$ is the canonical projection from $\mathbb{H}_{n_2}$ to $\mathbb{H}_{n_1}$. A sequence of linear operators $T_n : \mathbb{H}[-n,n] \to \mathbb{H}[-n,n]$ will be called quasi-increasing if for all $n \geq 1$, $T_{n+1} \geq_q T_n$.

**Lemma 3.1.1.** Each elementary state can be identified by a natural number.

**Proof.** The complex number $z$ is said algebraic number if there are integer numbers $x_0, \ldots, x_n$, not all zero, such that $p(z) = x_0 z^n + x_1 z^{n-1} + \ldots + x_{n-1} z + x_n = 0$. 

41
Now, we arrange the roots of any polynomial \( p(z) = 0 \) by the lexicographical order as \((z_0, \ldots, z_n)\). Let’s define

\[
w(z_i) = 2^n 3^i \cdots p_{n+2}^{x_i} p_{n+3}^{z_i},
\]

where \( x_j' = f(x_j), f : \mathbb{Z} \to \mathbb{N} \) is one-to-one and surjective function.

Let \( |\psi\rangle = \sum_{i \in \Omega} a_i |i\rangle \in \mathbb{H} \) be an elementary state, where \( a_i \) is algebraic number. We also define

\[
w'(|\psi\rangle) = 2^n 3^{w(a_0)} \cdots p_{n+2}^{w(a_n)},
\]

where \( n \) is the smallest number such that \( a_i = 0 \), for \( i \notin [-n, n] \). Therefore, each state can be identified by a natural number.

**Proposition 3.1.2.** Let \( T_n \) be a quasi-increasing sequence of semi-density matrices on \( \mathbb{H} \). Then \( \lim_{n \to \infty} T_n \) converges in the trace-norm to a semi-density matrix.

**Proof.** Since the sequence \( T_n \) is quasi-increasing, \( (\text{Tr}(T_n)) \) is an increasing sequence and since for every \( n \), \( \text{Tr}(T_n) \leq 1 \), the sequence converges in trace-norm, \( \|X\|_{tr} = \text{Tr}\sqrt{X^*X} \) to an operator \( T \) in the Banach space \( T(\mathbb{H}) \) of trace-class operators on \( \mathbb{H} \), moreover

\[
\text{Tr}(T) = \lim_{n \to +\infty} \text{Tr}(T_n) = \lim_{n \to +\infty} \|T_n\|_{tr} = \|T\|_{tr} \leq 1.
\]

Therefore, \( T \) must be positive; otherwise, if \( T \) had negative eigenvalues then \( \|T\|_{tr} > \text{Tr}(T) \) and this would contradict the previous equality.

Now, we give the definition of semi-computable semi-density matrices.

**Definition 3.1.2.** A linear operator \( T \) on \( \mathbb{H} \) is a semi-computable semi-density matrix, if there exists a computable quasi-increasing sequence of elementary semi-density matrices \( T_n \in B(\mathbb{H}_{[-n,n]}) \subseteq B(\mathbb{H}) \) such that \( \lim_{n \to \infty} ||T - T_n||_{tr} = 0 \).
The following lemma gives us a method for checking the positivity of a matrix.

A polynomial $P \in \mathbb{C}[x]$ of degree $n$ is called of type $\Pi$ if it has the following form:

$$P(x) = \sum_{0 \leq k \leq n} (-1)^k \lambda_k x^{n-k}, \text{ and } \lambda_0 = 1.$$ 

**Lemma 3.1.3.** Let $P \in \mathbb{C}[x]$ be of type $\Pi$. Assume that all solutions of the equation $P(x) = 0$ are real. Then these solutions are all positive.

**Proof.** Consider the following system of equations:

$$\sum_{1 \leq k_1 \leq n} \lambda_{k_1} = a_1, \quad \sum_{1 \leq k_1 < k_2 \leq n} \lambda_{k_1} \lambda_{k_2} = a_2, \quad \sum_{1 \leq k \leq l \leq n} \lambda_k \lambda_l = a_3, \ldots, \lambda_1 \lambda_2 \ldots \lambda_n = a_n,$$

where $a_i \geq 0$ for any $1 \leq i \leq n$. To prove the lemma it is sufficient to prove that under the above conditions, all $\lambda_i$'s are positive. Assume that $\lambda_n$ is negative. From the above system we obtain the following one:

$$\sum_{1 \leq k_1 \leq n-1} \lambda_{k_1} = a_1 - \lambda_n,$$

$$\sum_{1 \leq k_1 < k_2 \leq n-1} \lambda_{k_1} \lambda_{k_2} = a_2 + \lambda_n(a_1 - \lambda_n),$$

$$\lambda_1 \lambda_2 \ldots \lambda_{n-1} = a_{n-1} - \lambda_n \sum_{1 \leq k_1 < k_2 < \ldots < k_{n-2} \leq n-1} \lambda_{k_1} \lambda_{k_2} \ldots \lambda_{k_{n-2}}.$$ 

All right hand sides are positive. Therefore, $\lambda_1 \lambda_2 \ldots \lambda_{n-1}$ is positive. But $\lambda_n$ is negative and $\lambda_1 \lambda_2 \ldots \lambda_{n-1} \lambda_n$ is positive. This is a contradiction. \qed

**Theorem 3.1.4.** The set of all semi-computable semi-density matrices on $\mathbb{H}$ can be enumerated.

**Proof.** Let $\phi_0, \phi_1, \ldots, \phi_n, \ldots$ be the standard enumeration of all partially computable functions on $\mathbb{N}$. For $n \in \mathbb{N}$, we change $\phi_n$ into $\psi_n$ which represents a semi-computable semi-
density matrix $ρ_n$ on $H$. Let $ψ_n(0) = 0$. Assume that $ψ_n(x)$ is defined for $0 \leq x \leq t - 1$ and $z$ is the smallest integer number such that $ψ_n(t - 1) = φ_n(z)$. To define $ψ_n(t)$, assume that there is a least integer number $x_0$, $0 \leq x_0 \leq t$, greater than $z$, satisfying the relation $STP^{(1)}(x_0, n, t) = 1$ and $φ_n(x_0)$ can be interpreted as an elementary semi-density matrix $ρ_n(t)$ strictly quasi-greater than $ρ_n(t - 1)$. Then we set $ψ_n(t) = φ_n(x_0)$. Otherwise, $ψ_n(t) = ψ_n(t - 1)$. Clearly, $ψ_n$ is a computable function and by Theorem 3.1.2, $\lim_{t \to \infty} ρ_n(t)$ is a semi-computable semi-density matrix. Conversely to each semi-computable semi-density matrix on $H$ there corresponds a computable function $ψ : \mathbb{N} \to \mathbb{N}$ of the above form.

**Theorem 3.1.5.** Let $S$ and $T$ be semi-density matrices on $H$, and let $T$ be invertible. If $S \leq T$, then $\sqrt{S}(\log S)\sqrt{S} \leq \sqrt{S}(\log T)\sqrt{S}$.

**Proof.** For $0 < t \in \mathbb{R}$, both $t + S$ and $t + T$ are invertible and $(t + S)^{-1} \geq (t + T)^{-1}$. Therefore,

$$\sqrt{S} \left( \int_0^\infty \left( \frac{1}{t + S} - \frac{1}{t + T} \right) dt \right) \sqrt{S} = \sqrt{S} (\log (t + S) - \log (t + T)) \sqrt{S} \mid_{0}^{+\infty} \geq 0.$$

But

$$\sqrt{S} \log (t + S)\sqrt{S} = \sum_{0}^{\infty} \lambda_i \log (t + \lambda_i) |φ_i > < φ_i|,$$

where, $λ_i$’s are eigenvalues of $S$ with associated eigenvectors $|φ_i >$. Since by convention $0 \log 0 = 0$, the operators $\sqrt{S} \log (S)\sqrt{S} = \sum_{0}^{\infty} \lambda_i \log (λ_i) |φ_i > < φ_i| \leq 0$ is well defined.

On the other hand for $t \neq 0$,

$$\sqrt{S} (\log (t + S) - \log (t + T)) \sqrt{S} = \sqrt{S} (\log (1 + S/t) - \log (1 + T/t)) \sqrt{S}.$$
Therefore,

\[
0 \leq \sqrt{S} \left( \log (t + S) - \log (t + T) \right) \sqrt{S} \bigg|_{t=0}^{+\infty}
\]

\[
= \sqrt{S} \left( \log T - \log S \right) \sqrt{S} + \lim_{t \to \infty} \sqrt{S} \left( \log \left( 1 + S/t \right) - \log \left( 1 + T/t \right) \right) \sqrt{S}
\]

\[
= \sqrt{S} \left( \log T - \log S \right) \sqrt{S}.
\]

\[\square\]

**Definition 3.1.3.** A semi-computable semi-density matrix \(\hat{\mu}\) is called universal if for any semi-computable semi-density matrix \(\hat{\mu}\) there exists a constant \(C_\rho > 0\) such that \(C_\rho \rho \leq \hat{\mu}\).

The existences of a universal semi-density matrix in finite dimensional Hilbert spaces and its applications to algorithmic complexity is proved in [49]. Based on the preceding discussion, we are now able to show that universal semi-densities exist in infinite dimensional separable Hilbert space, and that they are related to each other by a universality condition.

**Theorem 3.1.6.** There exists a universal semi-computable semi-density matrix on any infinite dimensional, separable Hilbert space \(\mathbb{H}\).

**Proof.** Let \(\hat{\mu}_0, \hat{\mu}_1, \ldots, \hat{\mu}_n, \ldots\) be the enumeration of all semi-computable semi-density matrices and set

\[
\hat{\mu} = \sum_{k \geq 0} 2^{-k} \hat{\mu}_k . \tag{3.1.3}
\]

Clearly, \(\hat{\mu}\) is a semi-computable semi-density matrix, and for each semi-computable semi-density matrix \(\hat{\mu}_k\) we have \(2^{-k} \hat{\mu}_k \leq \hat{\mu}\). Therefore, \(\hat{\mu}\) is a universal semi-computable semi-density matrix. \(\square\)
3.2 Semi-computable operators

Let $T$ be an bounded operator on $\mathbb{H}$. Then, $T$ can be written as $T = (T_1 + iT_2)/2$ where $T_1 = (T + T^\dagger)/2$ and $T_2 = (T - T^\dagger)/2i$ are self-adjoint operators. Moreover, each self-adjoint operator $T_1, T_2 \in \mathbb{H}$ can be written as $T_i = T_{i1} - T_{i2}$, $i = 1, 2$, where $T_{ij}$, $i, j = 1, 2$, are positive operators and $T_{ij}/\|T_{ij}\| \leq I$. Indeed, $T_{i1} = (|A| + A)/2$ and $T_{i2} = (|A| - A)/2$, $i = 1, 2$ [12].

Now, let $T$ be a positive linear operator $\leq I$. Then $T_n = P_nTP_n$ is called elementary if all of its matrix elements are rational numbers.

A mapping $\phi : \mathbb{N} \to \mathbb{N}$ is interpreted as a semi-computable linear operator $T$ from $\mathbb{H}$ into $\mathbb{H}$ if for each $n \in \mathbb{N}$, $\phi(n)$ has the form $\phi(n) = <\lambda, <\phi_1(n), \phi_2(n)>, <\phi'_1(n), \phi'_2(n)>$, where $\lambda$ is an integer number independent of $n$ and $\phi_1(n)$, $\phi_2(n)$, $\phi'_1(n)$, and $\phi'_2(n)$ can be interpreted as elementary positive operators $T_{1n}$, $T_{2n}$, $T'_{1n}$ and $T'_{2n}$ all less than or equal to $\lambda I$ and the sequences $T_{1n}$, $T_{2n}$, $T'_{1n}$ and $T'_{2n}$ are all quasi-increasing and $T = \lim_{n \to \infty}(T_{1n} - T_{2n})/2 + (T'_{1n} - T'_{2n})/2$. If for each $n$, $(T_{1n} - T_{2n}) = 0$, or $(T'_{1n} - T'_{2n}) = 0$, $T$ is a semi-computable self-adjoint operator, and if for each $n$, three of four operators $T_{1n}$, $T_{2n}$, $T'_{1n}$ and $T'_{2n}$, are zero, then $T$ is a semi-computable bounded positive operator.

**Definition 3.2.1.** With the above notations $T \in B(\mathbb{H})$ is called a semi-computable semi-unitary operator if for each $n$,

$$T_nT_n^\dagger \leq I \quad \text{and} \quad T_n^\dagger T_n \leq I,$$

where, $T_n = (T_{1n} - T_{2n})/2 + (T'_{1n} - T'_{2n})/2$.

**Lemma 3.2.1.** Let $T$ and $S$ be semi-computable semi-unitary operators. Then

1. $T \circ S$ is also a semi-computable semi-unitary operator.
2. $T^\dagger$ is also a semi-computable semi-unitary operator.

Proof. Since $T$ and $S$ are semi-computable semi-unitary operators then they are constructed by sequences $T_n$ and $S_n$ convergent in trace-norm to $T$ and $S$. For each $n \in \mathbb{N}$, we have

$$T_n = (T_{1n} - T_{2n})/2 + (T'_{1n} - T'_{2n})/2,$$

$$S_n = (S_{1n} - S_{2n})/2 + (S'_{1n} - S'_{2n})/2,$$

where $T_{1n}$, $T_{2n}$, $T'_{1n}$, $T'_{2n}$, $S_{1n}$, $S_{2n}$, $S'_{1n}$ and $S'_{2n}$ are elementary operators.

It is clear that multiplications and adjoint of elementary operators are also elementary. Therefore, $T_n^\dagger$ and $T_n \circ S_n$ are constructed from elementary operators and hence $T^\dagger$ and $T \circ S$ are also semi-computable semi-unitary operators.

### 3.3 Lower and Upper Gacs Complexities

In this section with the help of a universal semi-computable semi-density matrix we will give the lower and upper Gacs algorithmic complexities in an infinite dimensional separable Hilbert space.

**Definition 3.3.1.** Let $\rho$ be a semi-computable semi-density matrix on the Hilbert space $\mathbb{H}$. The lower and upper Gacs algorithmic complexities are defined by

$$\underline{H}(\rho) = -\log \text{Tr}(\rho \hat{\mu}),$$

and

$$\overline{H}(\rho) = -\text{Tr}(\rho \log \hat{\mu}).$$

By the Levin’s theorem [1.2.2] we have $K(x)^{\dagger} \equiv \mu(x)$, $x \in \mathbb{N}$. Now, it is natural to
define a like relation in $\mathbb{H}$ for $\hat{\mu}$. Since, $\hat{\mu}$ is a positive operator less that $I$, we define

$$\kappa = -\log \hat{\mu}.$$  \hfill (3.3.3)

**Theorem 3.3.1.** Let $f$ be a convex function on an interval $[a, b]$ containing all the eigenvalues of positive operator $A$, then for all density matrices $\rho$ such that $\text{tr}(\rho f(A)) < \infty$,

$$f(\text{Tr}(\rho A)) \leq \text{Tr}(\rho f(A)).$$  \hfill (3.3.4)

**Proof.** Let us consider the spectral decomposition $\rho = \sum_i r_i |r_i><r_i|$. Since $f$ is a convex function, by [13] for each $i$,

$$f(<r_i|A|r_i>) \leq <r_i|f(A)|r_i>.$$  

By taking summation over all $i$, we have

$$f(\text{Tr}(\rho A)) = f(\sum_i r_i <r_i|A|r_i>)$$

$$\leq \sum_i r_i f(<r_i|A|r_i>)$$

$$\leq \sum_i r_i <r_i|f(A)|r_i>$$

$$\leq \text{Tr}(\rho f(A)).$$

We deduce that $f(\text{Tr}(\rho A)) \leq \text{Tr}(\rho f(A))$. \hfill $\Box$

**Corollary 3.3.2.** $-\log x$ is a convex function for $x > 0$, then $H(\rho) \leq \overline{H}(\rho)$ for each density matrix $\rho \in \mathbb{H}$.

**Remark 3.3.1.** Both complexities can be infinite. Indeed, let $|u_n><u_n|$ be a eigenvector
of $\mu$ in the spectral decomposition of it. Now,

$$H(\langle u_n | u_n \rangle) = -\log \text{Tr}(\langle u_n | u_n | \hat{\mu} \rangle) = -\log \langle u_n | \hat{\mu} | u_n \rangle = -\log r_n,$$

where $r_n$ is an eigenvalue that can be made as small as one likes. Since, $\text{Tr}(\hat{\mu}) \leq 1$ and hence $\sum_n r_n \leq 1$. Therefore, by the Corollary 3.3.2 $H(\langle u_n | u_n \rangle)$ can be also infinite.

**Theorem 3.3.3.** Let $|\hat{i}^{(n)}\rangle$, $\hat{i}^{(n)} \in \Omega_2$, be an orthonormal basis for the Hilbert space $\mathbb{H}$. Then, we have

$$\overline{H}(|\hat{i}^{(n)}\rangle) = H(|\hat{i}^{(n)}\rangle) \pm K(|\hat{i}^{(n)}\rangle),$$

where $K$ is the Kolmogorov complexity.

**Proof.** Let’s define the function $f(i^{(n)}) = \langle i^{(n)} | \hat{\mu} | i^{(n)} \rangle$, which is semi-computable and $\sum_{i^{(n)}} f(i^{(n)}) \leq 1$. Therefore, by the universality of $\mu$, there exists a constant number $c > 0$ such that $cf(i^{(n)}) \leq \mu(i^{(n)})$. Thus,

$$-\log \mu(i^{(n)}) \leq -\log f(i^{(n)}) - \log c \Rightarrow K(i^{(n)}) \lesssim H(i^{(n)}).$$

On the other hand, the semi-density matrix $\rho = \sum_{i^{(n)}} \mu |i^{(n)}\rangle \langle i^{(n)}|$ is semi-computable and hence $\rho \leq \hat{\mu}$. Therefore,

$$K(i^{(n)}) = \langle i^{(n)} | -\log \rho |i^{(n)}\rangle \geq \langle i^{(n)} | -\log \hat{\mu} |i^{(n)}\rangle = \overline{H}(i^{(n)}).$$

From Corollary 3.3.2 we have

$$K(i^{(n)}) \overset{\pm}{=} H(i^{(n)}) \overset{\pm}{=} \overline{H}(i^{(n)}).$$
The next property is related to composite systems. Indeed, let $X$ and $Y$ be two physical systems and $\mathbb{H}_X$ and $\mathbb{H}_Y$ be their related Hilbert spaces. Then, $\mathbb{H}_{XY} := \mathbb{H}_X \otimes \mathbb{H}_Y$ is the Hilbert space system associated to $XY$. Let $\rho_{XY}$ be a density matrix on $\mathbb{H}_{XY}$. Then $\rho_X = \text{Tr}_Y(\rho)$ and $\rho_Y = \text{Tr}_X(\rho)$ are called marginal density matrices for $\mathbb{H}_X$ and $\mathbb{H}_Y$, respectively. The subadditivity relation \[ S(\rho_{XY}) \leq S(\rho_X) + S(\rho_Y). \] (3.3.5)

The lower and upper Gacs complexities have also subadditivity properties.

Let $\preceq$ denote inequality to within an additive constant, and $\ast$ inequality to within a multiplicative constant.

**Theorem 3.3.4.** Let $XY$ be a composite system of two subsystems $X$ and $Y$. Let $\hat{\mu}_{XY}$, $\hat{\mu}_X$ and $\hat{\mu}_Y$ be associated universal semi-density matrices. Then,

$$\hat{\mu}_X \otimes \hat{\mu}_Y \ast \hat{\mu}_{XY}. \quad (3.3.6)$$

Moreover, for each $\rho \in \mathbb{H}_X$ and $\sigma \in \mathbb{H}_Y$,

$$\mathcal{H}(\rho \otimes \sigma) \preceq \mathcal{H}(\rho) + \mathcal{H}(\sigma), \quad (3.3.7)$$

and

$$\mathcal{H}(\rho \otimes \sigma) \ast \mathcal{H}(\rho) + \mathcal{H}(\sigma). \quad (3.3.8)$$

**Proof.** It is clear that $\text{Tr}(\hat{\mu}_X \otimes \hat{\mu}_Y) \leq 1$. Since, $\hat{\mu}_X$ and $\hat{\mu}_Y$ are universal semi-density matrices then there exist two increasing sequence of semi-computable semi-density matrices converging to them, respectively. Therefore, the tensor product of the two sequences\footnote{If $A_n$ and $B_n$ are two operators in $\mathbb{H}_X$ and $\mathbb{H}_Y$, then $A_n \otimes B_n$ is a sequence in $\mathbb{H}_{XY}$} is
also an increasing sequence\footnote{If $A,B,C,D$ are positive bounded operators with $A \leq B$ and $C \leq D$ then $A \otimes C \leq B \otimes D$ \cite{III}.} converging to $\hat{\mu}_X \otimes \hat{\mu}_Y$. Thus, $\hat{\mu}_X \otimes \hat{\mu}_Y$ is a semi-computable semi-density matrix on $\mathbb{H}_{XY}$. By the universality of $\hat{\mu}_{XY}$, there exists a constant $c > 0$ such that

$$c\hat{\mu}_X \otimes \hat{\mu}_Y \leq \hat{\mu}_{XY}.$$ 

The proof of the two next parts follows from the following equality:

$$\log \hat{\mu}_X \otimes \hat{\mu}_Y = \log \hat{\mu}_X \otimes 1 + \log 1 \otimes \hat{\mu}_Y.$$ 

\[\square\]

In the classical Kolmogorov complexity, we have monotonicity property $K(x) \hat{+} K(x, y)$ where $K(x, y) := K(<x, y>)$, for $x, y \in \mathbb{N}$. This property is also true for the Gacs algorithmic complexities.

**Theorem 3.3.5.**

$$\text{Tr}_Y \hat{\mu}_{XY} \hat{=} \hat{\mu}_X, \quad (3.3.9)$$

Moreover, for $\rho \in \mathbb{H}_X$ and $\sigma \in \mathbb{H}_Y$,

$$\overline{H}(\rho) \hat{+} \overline{H}(\rho \otimes \sigma), \quad (3.3.10)$$

and

$$\underline{H}(\rho) \hat{+} \underline{H}(\rho \otimes \sigma). \quad (3.3.11)$$

**Proof.** Let us define $\rho_X = \text{Tr}_Y \hat{\mu}_{XY}$. It is clear that $\rho_X$ is a semi-density matrix. On the other hand, there exists a sequence of semi-computable semi-density matrices $\rho_{XY}^{(n)}$ such that $\rho_{XY}^{(n)} \nearrow \hat{\mu}_{XY}$. Thus, we have $\text{Tr}_Y(\rho_{XY}^{(n)}) \nearrow \rho_X$. Therefore, $\rho_X$ is a semi-computable
semi-density matrix on $H_X$. By the universality of $\hat{\mu}_X$, there exists a constant $c > 0$ such that $c\rho_X \leq \hat{\mu}_X$.

Now, let’s define the density matrix $\sigma_{XY} = \hat{\mu}_X \otimes |\psi><\psi|$, $|\psi\rangle \in H_Y$, $||\psi|| = 1$, where $|\psi><\psi|$ is a fixed semi-computable density matrix. Like the proof of Theorem 3.3.4, $\sigma_{XY}$ is a semi-computable semi-density matrix. Therefore, there exists a constant $c' > 0$ such that $c'\sigma_{XY} \leq \hat{\mu}_{XY}$. Then,

$$c'\hat{\mu}_X \leq \text{Tr}_Y \hat{\mu}_{XY} = \rho_X.$$ 

Thus,

$$\text{Tr}_Y \hat{\mu}_{XY} = \hat{\mu}_X.$$ 

Now, let $|\psi><\psi|$ and $\rho$ be density matrices, where $\sum_i r_i |\phi_i><\phi_i|$ is the spectral decomposition of $\rho$. Then, we have

$$\text{Tr}(\rho \otimes |\psi><\psi|\hat{\mu}_{XY}) = \sum_i r_i <\phi_i|\psi \hat{\mu}_{XY} \phi_i|\psi >$$

$$\leq \sum_i r_i <\phi_i|\text{Tr}_Y \hat{\mu}_{XY} \phi_i >$$

$$= \text{Tr}(\rho \hat{\mu}_X).$$

Finally, let $\sigma$ be a density matrix on $H_Y$ with the spectral decomposition $\sum_j s_j |\psi_j><\psi_j|$. Then, we have

$$\text{Tr}(\rho \otimes \sigma \hat{\mu}_{XY}) = \sum_j s_j \text{Tr}(\rho \otimes |\psi_j><\psi_j|)$$

$$\leq \text{Tr}(\rho \hat{\mu}_X) \leq \text{Tr}(\rho \hat{\mu}_X)$$

$\square$
It is important to know whether the evolution of a quantum dynamical system has effects on the Gacs complexities or not. In the following theorem we show that when time evolution is an elementary unitary operator then modulo a constant number, the Gacs complexities is invariant.

**Theorem 3.3.6.** Let $U$ be any elementary unitary operator. Then, for any semi-density matrix $\rho \in \mathbb{H}$,

$$H(U \rho U^\dagger) \pm H(\rho), \quad H(U \rho U^\dagger) \pm H(\rho).$$

(3.3.12)

**Proof.** Since $U$ is an elementary unitary operator, then $U \hat{\mu} U^\dagger$ and $U^\dagger \hat{\mu} U$ are semi-computable semi-density matrices and hence there are constants $c_{U \hat{\mu} U^\dagger}$ and $c_{U^\dagger \hat{\mu} U} > 0$ such that

$$c_{U \hat{\mu} U^\dagger} U \hat{\mu} U^\dagger \leq \hat{\mu}, \quad c_{U^\dagger \hat{\mu} U} U^\dagger \hat{\mu} U \leq \hat{\mu}.$$

From the second one, we have

$$c_{U^\dagger \hat{\mu} U} U^\dagger \hat{\mu} U \leq U \hat{\mu} U^\dagger.$$

Therefore, $U \hat{\mu} U^\dagger$ is also a universal semi-measure and thus the result follows. \qed

**Theorem 3.3.7.** Let $P \neq 0$ be a lower semi-computable projection with $d = \text{Tr} P < \infty$. then,

$$H(\rho)^{\dagger} < \log d - \log(\text{Tr} P).$$

(3.3.13)

**Proof.** Let $\rho$ be the semi-computable semi-density matrix $P/d$. Then, there exists a constant $c_{\rho} > 0$ such that $c_{\rho} P/d \leq \hat{\mu}$.

$$H(\rho) = -\log \text{Tr}(\rho \hat{\mu})^{\dagger} < -\log \text{Tr}(\rho(P/d))^{\dagger} \equiv \log d - \log \text{Tr}(\rho P).$$

\qed
Let us consider the spectral decomposition of $\hat{\mu} = \sum_i u_i |u_i><u_i|$ where $u_1 \geq u_2 \geq \ldots$

Let $E_k = \sum_{i=1}^{k} |u_i><u_i|$ be a projection on $\mathbb{H}$. The following theorem gives a lower bound of the Gacs algorithmic complexities.

**Theorem 3.3.8.** *(Lower bounds).* Let $\rho$ be a semi-density matrix and let $\lambda > 1$. If $\mathcal{H}(\rho) < k$, then

$$\text{Tr}(\rho E_{2\lambda k}) > 1 - 1/\lambda.$$ 

In addition, if $\mathcal{H}(\rho) < k$ then

$$\text{Tr}(\rho E_{2\lambda k}) > 2^{-k}(1 - 1/\lambda),$$

where $E_{2\lambda k} := E_{\lfloor 2\lambda k \rfloor}$.

**Proof.** Let us consider the spectral decomposition of $\hat{\mu} = \sum_i u_i |u_i><u_i|$ where $u_1 \geq u_2 \geq \ldots$. By the assumption $\mathcal{H}(\rho) < k$. Therefore, we have

$$\sum_i -\log u_i < u_i |\rho| u_i > < k.$$ 

Now, let $m$ be the first $i$ with $u_i \leq 2^{-\lambda k}$. Since $\sum_i u_i \leq 1$, $m \leq 2^{\lambda k}$. In addition,

$$\lambda k \sum_{i \geq m} < u_i |\rho| u_i > \leq \sum_{i \geq m} -\log u_i < u_i |\rho| u_i > \leq \mathcal{H}(\rho \hat{\mu}) < k.$$ 

Therefore, $\sum_{i \geq m} < u_i |\rho| u_i > < 1/\lambda$.

By the assumption $\mathcal{H}(\rho) \leq k$, we have

$$-\log \sum_i u_i < u_i |\rho| u_i > \leq k \Rightarrow \sum_i u_i < u_i |\rho| u_i > \geq 2^{-k}.$$
Let $m$ be the first $i$ with $u_i < 2^{-k}/\lambda$. Since, $\sum_i u_i \leq 1$ we have $m \leq 2^k \lambda$. Therefore,

$$\sum_{i \geq m} u_i < u_i |\rho| u_i \geq \frac{2^{-k}}{\lambda} \sum_{i \geq m} < u_i |\rho| u_i \geq \frac{2^{-k}}{\lambda}.$$

Now,

$$\text{Tr}(\rho E_m) = \sum_{i \leq m} < u_i |\rho| u_i \geq \sum_{i \leq m} u_i < u_i |\rho| u_i \geq 2^{-k} - \sum_{i \geq m} u_i < u_i |\rho| u_i \geq 2^{-k}(1 - \frac{1}{\lambda}).$$

\[\square\]

### 3.4 Applications of Upper Gacs Complexity

**Definition 3.4.1.** Since Theorem 3.1.6 establishes the existence of a universal semi-density matrix for an infinite dimensional quantum spin chain, we take (3.3.3) with $\hat{\mu}$ as in (3.1.3) as the complexity operator of a quantum spin chain and (3.3.2) as the Gacs entropy of any density matrix $\rho$ associated with the chain.

Notice that the complexity operator of the quantum chain assigns the following Gacs entropy to a local density matrix $\rho_{[-n,n]}$ on $\mathbb{H}_{[-n,n]}$:

$$\overline{H}(\rho_{[-n,n]}) = \text{Tr}(\rho_{[-n,n]} P_n \kappa P_n),$$

(3.4.1)

where $P_n$ projects the Hilbert space $\mathbb{H}$ on which $\hat{\mu}$ acts onto the finite dimensional Hilbert space $\mathbb{H}_{[-n,n]}$ on which $\rho_{[-n,n]}$ acts.

On the other hand, one could consider the restriction $\hat{\mu}(n) = P_n \hat{\mu} P_n$ of the universal density matrix $\hat{\mu}$ to $\mathbb{H}_{[-n,n]}$; the natural guess is that $\hat{\mu}(n)$ might indeed be a universal semi-computable semi-density matrix for the local spin algebra $M_{[-n,n]}$.

That is indeed so is proved in the next Lemma. Then, given a local spin algebra $M_{[-n,n]}$.
we obtain the original finite dimensional formulation of [21]. Indeed, given \( \hat{\mu}^{(n)} = P_n \hat{\mu} P_n \) its complexity operator will be

\[
\kappa^{(n)} = -\log \hat{\mu}^{(n)},
\]

(3.4.2)

and, given a state \( \rho^{(n)} = \rho_{[-n,n]} \) on \( \mathbb{H}_{[-n,n]} \), its Gacs algorithmic entropy will be

\[
\overline{H}^{(n)}(\rho^{(n)}) = -\text{Tr}(\rho^{(n)} \log \hat{\mu}^{(n)}),
\]

(3.4.3)

where the trace is computed on \( \mathbb{H}_{[-n,n]} \).

**Lemma 3.4.1.** Let \( T \) be a universal semi-computable semi-density matrix which is the limit of a computable quasi-increasing sequence of elementary semi-density matrices \( T_n \).

Then, for each \( k, P_k T P_k \) is a universal semi-computable semi-density matrix on \( \mathbb{H}_{[-k,k]} \).

**Proof.** Clearly, the sequence \( P_k T_n P_k, n \geq k, \) is a computable quasi-increasing sequence of elementary semi-density matrices; moreover,

\[
\lim_{n \to \infty} P_k T_n P_k = P_k T P_k.
\]

Since \( T \) is a universal semi-computable semi-density matrix, for each semi-computable semi-density matrix \( R_k \) on \( \mathbb{H}_{[-k,k]} \), there exists a positive constant \( C_k \) such that

\[
T - C_k R_k \geq 0 \implies P_k T P_k - C_k R_k \geq 0.
\]

Based on the infinite dimensional formulation of the complexity operator, we can now study the Gacs algorithmic complexity per site of translation invariant states of quantum spin chains and relate it to their von Neumann entropy rate and AF-entropy.
Theorem 3.4.2. Let $\rho^{(n)} \in B^{+}_{1}(\mathbb{H}_{[-n,n]})$ be a computable sequence of semi-computable density matrices giving rise to a shift-invariant state $\omega$ on the quantum spin chain $\mathcal{M}$.

Then
\[ \lim_{n \to \infty} \frac{H^{(n)}(\rho^{(n)})}{2n + 1} = \lim_{n \to \infty} \frac{H(\rho^{(n)})}{2n + 1} = s(\omega) , \] (3.4.4)

where $s(\omega)$ is the von Neumann entropy rate in (2.3.10). Also, with reference to the Alicki-Fannes entropy and the density matrices $R[U^{(n)}]$ on the doubled local sub-algebras $M_{[-n,n]} \otimes M_{[-n,n]}$ in (2.3.35), it holds that
\[ \lim_{n \to \infty} \frac{H^{(n)}(R[U^{(n)}])}{2n + 1} = \lim_{n \to \infty} \frac{H(R[U^{(n)}])}{2n + 1} = s(\omega) + 1 . \] (3.4.5)

Proof. By normalizing $\hat{\mu}^{(n)}$ with $\text{Tr}(\hat{\mu}^{(n)}) \leq 1$ and using that for any two density matrices $\rho_1, \rho_2$ invertible, $\text{Tr}(\rho_1(\log \rho_1 - \log \rho_2)) \geq 0$, one estimates
\[ S(\rho^{(n)}) \leq -\text{Tr} \left( \rho^{(n)}(\log \hat{\mu}^{(n)} - \log \text{Tr}(\hat{\mu}^{(n)})) \right) \leq H^{(n)}(\rho^{(n)}). \]

Analogously, $S(\rho^{(n)}) \leq H(\rho^{(n)})$. Observe that $\hat{\mu}$ on $\mathbb{H}$ and $\hat{\mu}^{(n)}$ on $\mathbb{H}_{[-n,n]}$ for each $n$ are invertible.

Let $\rho = \sum_{n \geq 2} \rho^{(n)}/n(\log n)^2$. Then, $\rho$ is a semi-computable semi-density matrix. So, there exists $p \in \mathbb{N}$ such that $\rho \leq 2^p \hat{\mu}$. Because of the operator monotonicity of the logarithm, one estimates
\[ S(\rho^{(n)}) \leq H(\rho^{(n)}) = -\text{Tr} \left( \rho^{(n)} \log \hat{\mu} \right) \leq p - \text{Tr}(\rho^{(n)} \log \rho) \]
\[ \leq S(\rho^{(n)}) + p + \log n + 2 \log \log n. \]
Since \( p \) is independent of \( n \), then clearly we have

\[
\lim_{n \to \infty} \frac{H(\rho^{(n)})}{2n + 1} = s(\omega).
\]

On the other hand, \( \rho^{(n)} \leq 2^n n (\log n)^2 \hat{\mu} \) and hence

\[
S(\rho^{(n)}) \leq H^{(n)}(\rho^{(n)}) = -\text{Tr}_{[-n,n]}(\rho^{(n)} \log \hat{\mu}^{(n)})
\]

\[
= -\text{Tr}_{[-n,n]}(\rho^{(n)} \log P_n \hat{\mu} P_n)
\]

\[
\leq -\text{Tr}_{[-n,n]}(\rho^{(n)} \log P_n \rho^{(n)} P_n) + p + \log n + 2 \log \log n
\]

\[
\leq S(\rho^{(n)}) + p + \log n + 2 \log \log n,
\]

(3.4.6)

where \( p \) is independent of \( n \), then

\[
\lim_{n \to \infty} \frac{H^{(n)}(\rho^{(n)})}{2n + 1} = s(\omega).
\]

The relations in (3.4.5) can be proved in the same way, once one extends the construction of a universal semi-computable semi-density matrix to the case of the \( C^* \)-algebra arising from the inductive limit of the nested net of double local sub-algebras \( M_{[-n,n]} \otimes M_{[-n,n]} \).

This can be done by means of the map in (3.4.5).

In [6], both the above relations have been proved under the condition that the Kolmogorov complexity rates

\[
\lim_{n \to \infty} \frac{\kappa(\rho^{(n)})}{2n + 1} = 0 = \lim_{n \to \infty} \frac{\kappa(U^{(n)})}{2n + 1}.
\]

(3.4.7)

This restriction is not necessary; indeed, by constructing, as done before, an infinite dimensional universal semi-computable semi-density matrix, one can control all universal
semi-computable semi-density matrices of the local sub-algebras of the quantum chains, independently of \( n \).

The following example indeed shows an instance of quantum spin chain which does not satisfy the conditions \( (3.4.7) \) and nevertheless fulfils the conclusions of Theorem \( 3.4.2 \).

**Example 3.4.3.** Let \( P_0 \) and \( P_1 \) be two orthogonal projections in \( M_2(\mathbb{C}) \) and let \( P_{i(n)} = \bigotimes_{j=0}^{n-1} P_{i_j} \) denote the orthogonal projections obtained by tensor products. Let the starting one site density matrix be \( \rho_{\{0\}} = \frac{P_0 + P_1}{2} \) and assume that \( \rho^{(n)} = \rho_{[0,n-1]} \) be defined such that its complexity \( K(\rho^{(n)}) \geq n^2 \). We now recursively construct \( \rho^{(n+1)} \) so that, on one hand the family of density matrices satisfies the compatibility and translation invariant conditions \( (2.3.8) \) and \( (2.3.9) \), whence

\[
\lim_{n \to \infty} \frac{S(\rho^{(n)})}{n} = s(\omega) < +\infty ,
\]

and, on the other hand, so that \( K(\rho^{(n+1)}) \geq (n + 1)^2 \), whence

\[
\lim_{n \to \infty} \frac{K(\rho^{(n)})}{n} = +\infty .
\]

Write \( \rho^{(n)} = \sum_{\tilde{i}(n)} a^{\{0\}(n)}_{\tilde{i}(n)} P_{\tilde{i}(n)} \). Then, the conditions \( (2.3.8) \) and \( (2.3.9) \) yield

\[
\text{Tr}_{\{0\}} \rho^{(n+1)} = \text{Tr}_{\{n+1\}} \rho^{(n+1)} = \rho^{(n)} ,
\]

whence

\[
\sum_{\tilde{i}(n) \in \Omega_2^{(n)}} (a^{\{0\}(n)}_{\tilde{i}(n)} + a^{\{1\}(n)}_{\tilde{i}(n)}) P_{\tilde{i}(n)} = \sum_{\tilde{i}(n) \in \Omega_2^{(n)}} (a^{\{0\}(n)}_{\tilde{i}(n)} + a^{\{1\}(n)}_{\tilde{i}(n)} P_{\tilde{i}(n)} = \sum_{\tilde{i}(n) \in \Omega_2^{(n)}} a^{\{\tilde{i}\}(n)}_{\tilde{i}(n)} P_{\tilde{i}(n)} .
\]
Then, because of the orthogonality of the projections $P_i^{(n)}$, it follows that

$$a_0i^{(n-2)}0 + a_{0i}^{(n-2)}1 = a_0i^{(n-2)}$$
$$a_0i^{(n-2)}1 + a_{1i}^{(n-2)}1 = a_1i^{(n-2)}1$$
$$a_{1i}^{(n-2)}1 + a_{1i}^{(n-2)}0 = a_{1i}^{(n-2)}$$
$$a_{1i}^{(n-2)}0 + a_{0i}^{(n-2)}0 = a_{1i}^{(n-2)}0$$

for any of the $2^{n-2}$ strings $i^{(n-2)} \in \Omega_2^{n-2}$. In this way, the system of $2^n$ equations can be subdivided into $2^{n-2}$ sub-systems of 4 equations each. Let us focus upon the system above defined by the string $i^{(n-2)}$; the values at the right hand side have been chosen at step $n-1$. They are positive, with all the others they sum up to 1. Without loss of generality, we may assume they are in decreasing order: $a_{0i}^{(n-2)} \geq a_{1i}^{(n-2)}1 \geq a_{1i}^{(n-2)} \geq a_{1i}^{(n-2)}0 > 0$.

We can now choose $a_{1i}^{(n-2)}1 = x_{i}^{(n-2)}$, a positive real number such that $x_{i}^{(n-2)} \leq a_{1i}^{(n-2)}$ with Kolmogorov complexity $K(x_{i}^{(n-2)}) \geq n^2$. Then,

$$a_{1i}^{(n-2)}0 = a_{1i}^{(n-2)} - x_{i}^{(n-2)}; \; a_{0i}^{(n-2)}1 = a_{1i}^{(n-2)}1 - x_{i}^{(n-2)}; \; a_{0i}^{(n-2)}0 = a_{0i}^{(n-2)} - a_{1i}^{(n-2)}1 + x_{i}^{(n-2)}.$$

Therefore, the coefficients at step $n$ are positive, the sum of all of them is 1 and they satisfy the desired condition on the increase of the algorithmic complexity of $\rho^{(n)}$. 
Chapter 4

The Classical Gacs Algorithmic Complexity

In this chapter we apply Gacs complexity to classical dynamical systems. Here, we assume that the probability measure of the symbolic dynamical system associated with a given dynamical system and the considered finite measurable partition are semi-computable. Of course, the probability measure condition forces semi-computable probability measures to be computable. We will also prove a version of the Brudno’s theorem based on a given universal semi-measure.

4.1 Gacs algorithmic complexity in classical dynamical systems

Definition 4.1.1. Let \((\chi, T, \nu)\) be a dynamical system. let \(\mathcal{P}\) be a finite measurable partition of \(\chi\). The associated symbolic dynamical system \((\Omega_\mathcal{P}, T_\sigma, \nu_\mathcal{P})\) (see section 2.1) is called a semi-computable symbolic dynamical system if \(\nu_\mathcal{P}\) as a function of \(\Omega_2\) into \(\mathbb{R}\) is a semi-computable probability measure.

Notice that since \(\sum_{f^{(\alpha)} \in \Omega_2} \nu_\mathcal{P}(f^{(\alpha)}) = 1\), the semi-computable \(\nu_\mathcal{P}\) is computable.
Hence, we can always take \( \nu_P \) computable.

**Remark 4.1.1.** We mention that \( \nu_P \) is not a measure from \( \Omega_p \) into \( \mathbb{R} \). Because,

\[
\sum_{i^{(n)} \in \Omega_p} \nu_P(i^{(n)}) = \sum_{n=1}^{\infty} 1 = \infty.
\]

The definition of classical Gacs algorithmic complexity mimics the construction of the KS entropy. Indeed, we define the Gacs algorithmic complexity for a given semi-computable symbolic dynamical system \((\Omega_p, T_\sigma, \nu_P)\) as follows

\[
G(T, P^{(n)}) = -\sum_{i^{(n)} \in \Omega_2^{(n)}} \nu_P(i^{(n)}) \log \mu(i^{(n)}),
\]

(4.1.1)

where \( \mu \) is a universal semi-computable semi-measure on \( \Omega_p \). We can interpret this definition as giving the information content of the semi-computable probability measure contained in a universal semi-measure. On the other hand, \( \mu(i^{(n)}) > 0 \), so that \( G(T, P^{(n)}) \) is a finite quantity.

The rate of Gacs algorithmic complexity is naturally defined as

\[
G(T, P) = \limsup_{n \to \infty} \frac{1}{n} G(T, P^{(n)}).
\]

(4.1.2)

**Definition 4.1.2.** Let \((\chi, T, \nu)\) be a dynamical system. The rate of Gacs algorithmic complexity is

\[
G(T) := \sup_{\nu_P} G(T, \mathcal{P}),
\]

(4.1.3)

where \( \mathcal{P} \) is a finite measurable partition such that \( \nu_P \) is computable.

**Remark 4.1.2.** In general the sup in the above definition is computed over all finite measurable partitions. However, in order to use semi-universal semi-computable measures, we
restrict ourselves to computable finite measurable partitions cases.

Now, we encounter the following natural question: Is there any relation between the Gacs algorithmic complexity and KS entropy in ergodic classical dynamical systems? We are going to provide the answer.

**Theorem 4.1.1.** Let \((\chi, T, \nu)\) be a semi-computable dynamical system, then

\[
G_\nu(T) \leq h_\nu^{KS}(T). \tag{4.1.4}
\]

**Proof.** Let \(\mathcal{P}\) be a finite measurable partition of \(\chi\) such that \(\nu_P\) is a computable. Since \(\nu_P\) cannot be a measure on \(\Omega_p\), we consider a semi-computable semi-measure \(f\) on \(\Omega_p\), defined as follows

\[
f(i^{(n)}) = \frac{1}{\sum_{n=1}^{\infty} \delta(n) \nu_P(i^{(n)})}, \quad i^{(n)} \in \Omega_p,
\]

where \(\delta(n) = \frac{1}{n \log^2 n}\). Then, there exists a constant \(c_{\nu_P} > 0\), depending on \(\nu_P\), such that for any \(i^{(n)} \in \Omega_p\),

\[
c_{\nu_P} \delta(n) \nu_P(i^{(n)}) \leq \mu(i^{(n)}).
\]

Thus,

\[- \sum_{i^{(n)} \in \Omega^{(n)}_2} \nu_P(i^{(n)}) \log \mu(i^{(n)}) \leq - \sum_{i^{(n)} \in \Omega^{(n)}_2} \nu_P(i^{(n)}) \log \nu_P(i^{(n)}) - \log c_P - \log \delta(n).
\]

Therefore,

\[G(T, \mathcal{P}) \leq h_\nu^{KS}(T, \mathcal{P}) \leq h_\nu^{KS}(T).
\]

Now, we take the sup over all computable \(\nu_P\). Then,

\[G(T) \leq h_\nu^{KS}(T).
\]
Theorem 4.1.2. Let \((\Omega_p, T_\sigma, \nu)\) be a binary ergodic dynamical system where \(\nu\) is computable. Then,

\[
G(T) = h^K_\nu(T_\sigma), \quad \nu - a.e. \tag{4.1.5}
\]

Proof. Let \(\mathcal{P}\) be a finite measurable partition of \(\Omega_p\). Let \(\nu^{(n)}_\mathcal{P}\) be its related probability measure where \(\nu^{(n)}_\mathcal{P}(i^{(n)}) = \nu(C_{i_0,i_1,...,i_{n-1}}^{[0,n-1]})\). It is clear that \(\sum_{i^{(n)} \in \Omega_p^{(n)}} \nu^{(n)}_\mathcal{P}(i^{(n)}) = 1\). Notice that, \(\nu_\mathcal{P}\) is \(\nu^{(n)}_\mathcal{P}\) for each \(n \in \mathbb{N}\). By Theorem 1.2.1 and Inequality (1.2.10) for all \(x \in \mathbb{N}\), we have,

\[
-\log c_1 + C(x) \leq -\log \mu(x) \leq K(x) + \log c \leq C(i^{(n)}) + 2 \log n + \log c_2, \tag{4.1.6}
\]

where \(c_1 > 0\) and \(c_2 > 0\) are constant numbers. Since we can represent each finite length binary string \(i \in \Omega_p\) by an integer number, by applying the theorem 4.1.6, we obtain

\[
G(T, \mathcal{P}^{(n)}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i^{(n)} \in \Omega_p^{(n)}} \nu_\mathcal{P}(i^{(n)})C(i^{(n)}). \tag{4.1.7}
\]

By Brudno’s theorem 2.1.8 for \(\epsilon > 0\) there is an integer number \(N\) such that for any \(N \ni n \geq N\),

\[
\frac{1}{n}C(i^{(n)}) \geq h^K_\nu(T_\sigma) - \epsilon.
\]

Therefore, by Theorem 4.1.1,

\[
h^K_\nu(T_\sigma) \geq G(T) \geq G(T, \mathcal{P}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i^{(n)} \in \Omega_p^{(n)}} \nu_\mathcal{P}(i^{(n)})C(i^{(n)}) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{i^{(n)} \in \Omega_p^{(n)}} \nu(i^{(n)})(h^K_\nu(T_\sigma) - \epsilon) \geq h^K_\nu(T_\sigma) - \epsilon.
\]
Thus,

\[ h^{KS}_\nu(T_\sigma) = G(T) = k(i) = c(i) \quad \nu - a.e, \]

where \( k \) and \( c \) are rate of the prefix Kolmogorov and Kolmogorov complexities, respectively, which are defined as follows:

\[
k(i) = \lim_{n \to \infty} \frac{K(i^{(n)})}{n}, \quad c(i) = \lim_{n \to \infty} \frac{C(i^{(n)})}{n} \]

Now, the question is: Can we give a short proof for the classical Brudno theorem in ergodic semi-computable cases? In the following theorem, we will give a short proof for a ergodic source dynamical systems.

**Theorem 4.1.3.** Let \((\Omega_2, T_\sigma, \nu)\) be a semi-computable binary ergodic source with KS entropy rate \( h^{KS}_\nu(T_\sigma) \). Then,

\[
\lim_{n \to \infty} -\frac{\log \mu(i^{(n)})}{n} = h^{KS}_\nu(T_\sigma), \quad \nu - a.e,
\]

for almost all \( i \in \Omega_p \) with respect to \( \nu \).

**Proof.**

**Part 1:** Let us consider the function \( f \) from \( \Omega_2 \) into \( \mathbb{R} \) as follows,

\[
f(i^{(n)}) = \frac{1}{\sum_{n=1}^{\infty} n^{-2}} \frac{1}{n^2} \nu(i^{(n)}),
\]

where \( \sum_{n=1}^{\infty} n^{-2} = \pi^2/6 \). It is straight word to check that the function \( f \) is a measure. Since the probability measure \( \nu \) is a computable measure and hence \( f \) is also a computable probability distribution. Then, by universality of the semi-measure \( \mu \) there exists constant
number $c > 0$ such that
\[
\frac{6c}{\nu^2 n^2} \nu(i^{(n)}) \leq cf(i^{(n)}) \leq \mu(i^{(n)}).
\]

Therefore,
\[
\limsup_{n \to \infty} - \frac{\log \mu(i^{(n)})}{n} \leq \limsup_{n \to \infty} - \frac{\log \nu(i^{(n)})}{n} \leq h_{\nu}^{KS}(T_\sigma), \quad \nu - a.e,
\]
where we used the Shannon-Mc Millan-Breiman theorem [10] for the second inequality.

**part 2:** The proof of inverse inequality is exactly like the Brudno’s theorem.

From the Asymptotic Equipartition Property (AEP) and Shannon-Mc Millan-Breiman theorem [10], we know that for the set $A_{\epsilon}^{(n)} = \{i^{(n)} \in \Omega_2^{(n)} | 2^{-n(h_{\nu}^{KS}(T_\sigma)+\epsilon)} \leq \nu(i^{(n)}) \leq 2^{-n(h_{\nu}^{KS}(T_\sigma)-\epsilon)}\}$,

\[
\text{Prob}(A_{\epsilon}^{(n)}) \approx 1 \quad \text{and} \quad (1-\epsilon)2^{n(h_{\nu}^{KS}(T_\sigma)-\epsilon)} < \#(A_{\epsilon}^{(n)}) < 2^{-n(h_{\nu}^{KS}(T_\sigma)+\epsilon)}.
\]

By Theorem [1.2.2] and Inequality (1.2.5) we have
\[
\#\{i^{(n)} : \mu(i^{(n)}) \geq 2^{-c' + \log \delta(n) + \log c} \} \leq 2^{c'} - 1.
\]

Therefore,
\[
\#\{i^{(n)} : \mu(i^{(n)}) \geq 2^{-c'} \} \leq 2^{c'} + \alpha - 1, \quad (4.1.8)
\]
where $\alpha = - \log \delta(n) - \log c' > 0$. we define the subset of $A_{\epsilon}^{(n)} \subseteq \Omega_2$ as follows
\[
A_{\epsilon}^{(n)} = \{i^{(n)} \in A_{\epsilon}^{(n)} | \mu(i^{(n)}) \geq 2^{-n(H_{\nu} - 2c)}\}.
\]

(4.1.9)

This means that each element $i^{(n)} \in A_{\epsilon}^{(n)}$ is the initial prefix of length $n$, of some strings
in \( \Omega_2 \). Then,

\[
\nu(\hat{A}_\epsilon^{(n)}) = \nu(\{i^{(n)} | \mu(i^{(n)}) \geq 2^{-n(h_\nu - 2\epsilon)}, i^{(n)} \in A_\epsilon^{(n)}\}) \leq \#(\hat{A}_\epsilon^{(n)}) \cdot \max_{i^{(n)} \in \Omega_2^{(n)}} \nu(i^{(n)}) \\
\leq 2^{n(h_\nu - 2\epsilon) + \alpha + 1} \cdot 2^{-n(h_\nu - \epsilon)} = 2^{-n\epsilon + \alpha + 1}.
\]

We know that there is some strings \( i^{(n)} \notin A_\epsilon^{(n)} \) such that \( \mu(i^{(n)}) \geq 2^{-n(h_\nu - 2\epsilon)} \), so let

\[
\hat{A}_\epsilon^{(k)} = \{i^{(n)} | \mu(i^{(k)}) \geq 2^{-k(h_\nu - 2\epsilon)}, i^{(n)} \in (\hat{A}_\epsilon^{(k)})^c\},
\]

where \( (\hat{A}_\epsilon^{(k)})^c = \Omega_2 \setminus \hat{A}_\epsilon^{(k)} \).

Let \( B_\epsilon^{(n)} = \bigcup_{k \geq n} \hat{A}_\epsilon^{(k)} \) then \( \nu(B_\epsilon^{(n)}) \leq \nu(\bigcup_{k \geq n} \hat{A}_\epsilon^{(k)})^c = 1 - \nu(\bigcap_{k \geq n} \hat{A}_\epsilon^{(k)}) \). Therefore,

\[
\nu\left(\bigcup_{k \geq n} \{\hat{A}_\epsilon^{(k)} \cup \hat{A}_\epsilon^{(k)}\}\right) \leq \nu\left(\bigcup_{k \geq n} \hat{A}_\epsilon^{(k)}\right) + \nu(B_\epsilon^{(k)}) \\
\leq \sum_{k \geq n} 2^{-k\epsilon + \alpha + 1} + \nu(B_\epsilon^{(k)}) \leq \frac{2^{-k\epsilon + \alpha + 1}}{1 - 2^{-\epsilon}} + 1 - \nu(\bigcap_{k \geq n} \hat{A}_\epsilon^{(k)}) \quad (4.1.10)
\]

Now, let \( i_1, i_2, \ldots \in \Omega_2 \) be a binary sequence whose initial prefixes are typical for \( k \geq n \), namely \( i_1, i_2, \ldots, i_k \in A_\epsilon^{(k)} \). Then \( i \in \bigcap_{k \geq n} \hat{A}_\epsilon^{(n)} \).

It is clear that \( \lim_{n \to \infty} \nu(\bigcap_{k \geq n} \hat{A}_\epsilon^{(n)}) = 1 \). Therefore,

\[
\liminf_{n \to \infty} \frac{-\log \mu(i^{(n)})}{n} \geq h_\nu^{KS}(T_\sigma) - \epsilon, \quad \nu - a.e,
\]
Chapter 5

Brudno’s Theorem in Quantum Spin Chains with Shift Dynamics

In this chapter we investigate the extensions of the classical Brudno’s theorem to quantum spin chains with right-shift dynamics using the quantum Shannon-MacMillan theorem.

5.1 Extension of Brudno’s Theorem

In the classical case systems, Brudno proved a relation between ergodicity theory and Kolmogorov complexity [15]. It is natural to ask ourselves that what is the extension of this theorem in quantum dynamical systems? To extend this theorem, we should extend the meaning of Kolmogorov Complexity and $KS$-entropy from the classical dynamical systems to the quantum cases.

In this thesis, we focus on the Gacs extension of Kolmogorov complexity and on AF and CNT extensions of the $KS$-entropy. Now, what about is the generalization of Brudno’s theorem? The first step is to define the concept of trajectory in quantum systems. Unfortunately, the definition of trajectory as defined in symbolic dynamical system using
partitions in quantum systems is not easy. Therefore, we will proceed without using trajectories. Our method is independent of the partition of unity used in the definition of the $AF$-entropy.

Our method is used the notion of semi-computability which is described in chapter 3. Since, the space of the Fermionic algebras using the Jordan-Wigner transformation is infinite tensor product of $d$-level matrices and semi-computability is defined on infinite dimensional Hilbert spaces, then, it will be an appropriate method to investigate the dynamics of Fermionic particles. In the Bosonic case, the semi-computability concept should be extended to $C^*$-algebras which is another problem and we don’t consider in this thesis.

Therefore, we proceed to extend the Brudno’s theorem based on semi-computability concept in quantum spin chains with shift dynamics. The quantum Shannon-MacMillan theorem for translation invariant ergodic quantum spin systems on $\mathbb{Z}$ lattice is formulated in [11]. Now, we want to investigate a version of the Brudno theorem using the quantum Shannon-MacMillan theorem. Here, we use projections instead of ”almost every” in Brudno theorem.

Before going further, we give a definition of quantum ergodic theory which is based on the algebraic formalism.

**Definition 5.1.1.** For a given quantum dynamical system $(\mathcal{A}, \Theta, \omega)$, ergodicity corresponds to the behavior of the discrete time-average of two-point correlation functions and is defined by

$$
\lim_{n \to \infty} \frac{1}{2T + 1} \sum_{t=-T}^{T} \omega(A^\dagger \Theta_t(B)C) = \omega(AC)\omega(B),
$$

(5.1.1)

where $A, B, C \in \mathcal{A}$ and $t \in \mathbb{Z}$. 

69
The quantum Shannon-MacMillan theorem is as follows [11]:

**Theorem 5.1.1.** Assume that \((A_Z, \Theta, \omega)\), with \(A = M_d(\mathbb{C})\) as a site algebra, is an ergodic quantum spin-chain with mean entropy \(s(\omega)\). Then, for all \(\delta > 0\) there exists \(N_\delta \in \mathbb{N}\) such that for all \(n \geq N_\delta\), there is an orthogonal projection \(p_n(\delta) \in A_n\) such that

1. \(\omega(p_n(\delta)) = \text{Tr}_n(\rho(p_n(\delta))) \geq 1 - \delta\),

2. for all minimal projections \(0 \neq p_n \in A_n\) dominated by \(p_n(\delta)\), \((p \leq p_n(\delta))\) \((1 - \delta)2^{n(s(\omega) + \delta)} < \omega(p_n(\delta)) < 2^{n(s(\omega) + \delta)}\),

3. \(2^{n(s(\omega) + \delta)} < \text{Tr}_n(p_n(\delta)) < 2^{n(s(\omega) + \delta)}\).

In other words, in ergodic quantum dynamical systems with shift dynamics, there is a sequence of projections, with high probability, such that for any sequence of minimal projectors dominated by them, the rate of lower Gacs complexity of them is equal to the von Neumann entropy rate \(s(\omega)\).

In the following definition the density matrices \(\rho^{(n)}\) are semi-computable. Then, there exists a sequence of elementary matrices \(\rho^{(n)}_m\) such that \(\rho^{(n)}_m \nearrow \rho^{(n)}\) in the trace-norm. By chapter 3 each elementary matrix \(\rho^{(n)}_m\) corresponds to a natural number \(a_{nm}\).

**Definition 5.1.2.** A faithful state \(\omega\) on \(A_Z\) is called a semi-computable (computable) state if the associated local density matrices \(\rho^{(n)}\) on \((M_d(\mathbb{C}))^\otimes n \subseteq A_Z\) are semi-computable (computable) semi-density matrices and the function \((m, n) \rightarrow a_{nm}\) from \(\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\) is computable, \(\rho^{(n)}_m \nearrow \rho^{(n)}\) and \(\text{rank}\rho^{(n)}_m = n\).

An important question is: are the eigenvalues of a semi-computable semi-density matrix \(\rho^{(n)}\) semi-computables [27]?
Theorem 5.1.2. Let $T$ be a compact positive operator $n B(ℋ)$ with $\dim(ℋ) < \infty$, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ listed in decreasing order tending to 0. Therefore,

$$\lambda_k = \max_{\dim V = k} \min_{v \in V - \{0\}} \frac{\langle v | T | v \rangle}{\|v\|^2},$$

and

$$\lambda_k = \max_{\dim V = k-1} \min_{v \in V \perp \{0\}} \frac{\langle v | T | v \rangle}{\|v\|^2}.$$

In both cases, $V$ runs over subspaces of $ℋ$ of the stated dimension, and in the first case it is assumed that $V \subseteq \text{Ker}(T) \perp$. Moreover, If $T_n$ is a sequence of positive compact operators such that $T_n \to T$ in norm topology on $B(ℋ)$, then $T$ is a positive compact operator such that

$$\lim_{n \to \infty} \lambda_k(T_n) = \lambda_k(T),$$

where $\lambda_k(T_n)$'s are eigenvalues of the $T_n$ in decreasing listed order, for each $n \in \mathbb{N}$.

Let $\rho_n$ be a sequence of semi-computable semi-density matrices where $\rho_n \to \rho$. Therefore, $\rho_n$'s are compact operators. Then,

$$\lim_{n \to \infty} \lambda_k(\rho_n) = \lambda_k(\rho).$$

On the other hand for a given semi-computable semi-density matrix $\rho_n$, there exists a sequence $\rho_{mn}$ of elementary matrices such that $\rho_{mn} \to \rho_n$. Therefore, the eigenvalues of $\rho_n$ can be considered as limit of the eigenvalues of $\rho_{mn}$ and the eigenvectors of $\rho_n$ are also semi-computable.

Let $U$ be a semi-computable semi-unitary operator. The operator $U^\dagger \hat{\mu} U$ may not be

\[\text{The operator } T \text{ is called compact if there exists a sequence of operators } T_n \text{ with } \dim \text{Im}(T_n) < \infty \text{ for all } n, \text{ and } \lim_{n \to \infty} T_n = T; \text{ in the norm topology on } B(ℋ).\]
a semi-computable semi-density matrix and hence it may not be a universal semi-density matrix.

Lemma 5.1.3. Let \((A, \sigma, \omega)\) be a quantum spin chain, with \(A = M_d(\mathbb{C})\) as its site-algebras and \(\omega\) a semi-computable faithful state. Let \(\rho\) be a associated density matrix to \(\omega\) and \(\rho^{(n)} = \text{Tr}_{-n} \hat{n} \rho\). Let’s define \(U\) be a unitary operator with \(U_n |\hat{n}_{\mu}^{(n)} \rangle = |\hat{r}_{\mu}^{(n)} \rangle\), for \(\hat{\mu}^{(n)} \in \Omega_2^{(n)}\), where \(\hat{\mu} = \sum_{\hat{\mu}^{(n)}} \mu_{\hat{\mu}^{(n)}} |\mu_{\hat{\mu}^{(n)}} \rangle \rangle < \mu_{\hat{\mu}^{(n)}} |\rangle\rangle \) and \(\rho^{(n)} = \sum_{\hat{\mu}^{(n)} \in \Omega_2^{(n)}} r_{\hat{\mu}^{(n)}} |r_{\hat{\mu}^{(n)}} \rangle \rangle < r_{\hat{\mu}^{(n)}} |\rangle\). We have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma^{(n)} \hat{\mu}^{(n)}) = \limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma^{(n)} U_n \hat{\mu}^{(n)} U_n^\dagger),
\]

for any density matrix \(\sigma^{(n)} \in A_2^{(n)}\).

Proof. By 3.4 elements of the sequence \(\hat{\mu}^{(n)} = P_n \hat{\mu} P_n\) are universal semi-density matrices on \((M_d(\mathbb{C}))^{\otimes n}\), where \(P_n\) is a projection from \(\mathbb{H}\) to \(\mathbb{H}^n\).

Let us consider the spectral decompositions \(\hat{\mu}^{(n)}\) and \(\rho^{(n)}\), respectively as follows:

\[
\sum_{\hat{\mu}^{(n)} \in \Omega_2^{(n)}} \mu_{\hat{\mu}^{(n)}} |\mu_{\hat{\mu}^{(n)}} \rangle \rangle < \mu_{\hat{\mu}^{(n)}} |\rangle\rangle,
\]

\[
\sum_{\hat{\mu}^{(n)} \in \Omega_2^{(n)}} r_{\hat{\mu}^{(n)}} |r_{\hat{\mu}^{(n)}} \rangle \rangle < r_{\hat{\mu}^{(n)}} |\rangle\rangle.
\]

Let us define \(U_n |\hat{\mu}^{(n)} \rangle = |\hat{r}_{\hat{\mu}^{(n)}} \rangle\).

Because \(\rho^{(n)}\) and \(\hat{\mu}^{(n)}\) are semi-computable density matrices and hence there exist computable sequences of elementary matrices \(\rho_m^{(n)}\) and \(\hat{\mu}_m^{(n)}\) such that \(\rho_m^{(n)} \not\geq \rho^{(n)}\) and \(\hat{\mu}_m^{(n)} \not\geq \hat{\mu}^{(n)}\), respectively. Moreover, ranks of the \(\rho_m^{(n)}\) and \(\hat{\mu}_m^{(n)}\) are equal to \(n\), for each \(n \in \mathbb{N}\). Therefore, the operator \(U_{mn}\) defined by \(U_{mn} \hat{\mu}_m^{(n)} U_{mn} = \rho_m^{(n)}\) is an elementary unitary operator and the function \(m \to U_{mn}\) is computable.

On the other hand, \(\hat{\mu}\) is a semi-computable density matrix and hence there exits a
sequence of elementary operators $\hat{\mu}_k$ such that $\hat{\mu}_k \nearrow \hat{\mu}$ in trace-norm. But, $U_{mn}^{\dagger} \hat{\mu}_k U_{mn}$ is a computable sequence of elementary which convergence increasingly to $U_{mn}^{\dagger} \hat{\mu} U_{mn}$ and thus $U_{mn}^{\dagger} \hat{\mu} U_{mn}$ is a semi-computable semi-density matrix.

Let us consider the following operator

$$\hat{K}_m = \sum_n \frac{1}{n \log^2 n} U_{mn}^{\dagger} \hat{\mu} U_{mn}.$$ 

Using Theorem 3.3.6 it is clear that $\hat{K}_m$ is a semi-computable semi-density matrix. Therefore, there exists a constant $c_m > 0$ such that

$$c_m \frac{1}{n \log^2 n} U_{mn}^{\dagger} \hat{\mu} U_{mn} \leq c_m \hat{K}_m \leq \hat{\mu}.$$ 

Now, we have

$$\hat{\mu} \leq \frac{1}{c_m} n \log^2 n U_{mn} \hat{\mu} U_{mn}^{\dagger}.$$ 

Let $\sigma^{(n)}$ be a semi-density matrix on $(M_d(\mathbb{C}))^\otimes n$. Then,

$$\limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma^{(n)} \hat{\mu}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\frac{1}{c_m} n \log^2 n \sigma^{(n)} U_{mn} \hat{\mu} U_{mn}^{\dagger}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma^{(n)} U_{mn} \hat{\mu} U_{mn}^{\dagger}).$$
On the other hand,

\[
\left| \text{Tr}(\sigma(n)U_{mn}\hat{\mu}U_{mn}^\dagger - \sigma(n)U_{n}\hat{\mu}U_{n}^\dagger) \right| \leq \|\sigma(n)\| \left| \text{Tr} \left( U_{mn}\hat{\mu}U_{mn}^\dagger - U_{n}\hat{\mu}U_{n}^\dagger \right) \right|
\]

\[
\leq \text{Tr} \left( U_{mn}\hat{\mu}U_{mn}^\dagger - U_{n}\hat{\mu}U_{n}^\dagger \right)
\]

\[
\leq \text{Tr} \left( U_{mn}\hat{\mu}U_{mn}^\dagger + U_{mn}\hat{\mu}U_{n}^\dagger - U_{mn}\hat{\mu}U_{n}^\dagger - U_{n}\hat{\mu}U_{n}^\dagger \right)
\]

\[
\leq \|U_{mn}\hat{\mu}\| \left| \text{Tr} \left( U_{mn}^\dagger - U_{n}^\dagger \right) \right| + \|\hat{\mu}U_{n}^\dagger\| \left| \text{Tr} \left( U_{mn} - U_{n} \right) \right|
\]

\[
\leq \text{Tr} \left( U_{mn}^\dagger - U_{n}^\dagger \right) + \text{Tr} \left( U_{mn} - U_{n} \right)
\]

\[
\leq \epsilon + \epsilon.
\]

Therefore,

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma(n)\hat{\mu}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \text{Tr}(\sigma(n)U_{n}\hat{\mu}U_{n}^\dagger).
\]

With this method we can also prove the other hand of the above inequality. \(\square\)

In the following theorem we prove the extension of the Brudno’s theorem in quantum dynamical systems with shift dynamics. Of course, the projections defined in [11] are replaced by new projections which satisfy all the needed properties.

**Theorem 5.1.4.** Let \((A_Z, \Theta_\alpha, \omega)\), with \(A = M_d(\mathbb{C})\) as a site algebra, be an ergodic quantum spin-chain with mean entropy \(s(\omega)\) where \(\omega\) is faithful and semi-computable. Then, for any \(\epsilon > 0\), there exists a sequence of projections \(p_n(\epsilon) \in A_n\) and a number \(N_\epsilon \in \mathbb{N}\) such that for any \(n \geq N_\epsilon\), we have,

1. \(\omega(p_n(\epsilon)) = \text{Tr}(\rho(n)p_n(\epsilon)) > 1 - \epsilon\),
2. for any minimal projection $0 \neq p_n \in \mathcal{A}_n$ dominated by $p_n(\epsilon)$ ($p_n \leq p_n(\epsilon)$), we have

$$2^{-n(s(\omega)+\epsilon)} \leq \omega(p_n) \leq 2^{-n(s(\omega)-\epsilon)}.$$

3. $(1 - 2^{-n\epsilon})2^{n(s(\omega)-\epsilon)+\alpha_n} < \text{Tr}_n(p_n(\epsilon)) < 2^{n(s(\omega)+\epsilon)}$.

4. $\lim_{n \to \infty} \frac{1}{n} \log \text{Tr}(\hat{\mu} p_n) = s(\omega)$,

where $\lim_{n \to \infty} \frac{\alpha_n}{n} = 0$.

Proof. Let $\rho^{(n)}$ be a local density matrix on the local algebra $\mathcal{A}_n = M_{[0,n]}$ such that $\omega(A) = \text{Tr}_{[0,n]}(\rho^{(n)} A), A \in \mathcal{A}_n$. Let $\sum_i r_i^{(n)} |r_i^{(n)} \rangle \langle r_i^{(n)}|$ be the spectral decomposition of $\rho^{(n)}$, $n \in \mathbb{N}$, which is sorted decreasingly in accordance to eigenvalues. We also define two sets as follow:

$$A'_v^{(n)} = \{ \bar{l} \in \Omega_2^{(n)} | 2^{-n(s(\omega)+\epsilon)} \leq r_i^{(n)} \leq 2^{-n(s(\omega)-\epsilon)} \},$$

and

$$B'_v^{(n)} = \{ i \in \Omega_2 : \mu^{(n)}(i^{(n)}) < 2^{-n(s(\omega)-2\epsilon)}, i^{(n)} \text{ is the initial prefix of string } i \},$$

where $\bar{l}$ is the binary expansion of the number $l$. According to 4.1.8

$$\#(B'_v^{(n)}) \leq 2^{n(s(\omega)-2\epsilon)+\alpha_n},$$

where $\alpha_n > 0$ is a constant number and $\lim_{n \to \infty} \frac{\alpha_n}{n} = 0$. Now, we define a sequence of projections $p_n(\epsilon)$ on the GNS representation of $\mathcal{A}_n$ by

$$p_n(\epsilon) = \sum_{i^{(n)} \in A'_v^{(n)} \cap B'_v^{(n)}} |r_i^{(n)} \rangle \langle r_i^{(n)}|.$$
Let $p_n \leq p_n(\epsilon)$ be a minimal projection on $A_n$. Then, its representation is as follows

$$p_n = |\psi_n \rangle \langle \psi_n| = \sum_{i(n) \in A_n(\epsilon) \cap B_n(\epsilon)} c_i(n)|r_i(n)\rangle \text{ where } \sum_{i(n) \in A_n(\epsilon) \cap B_n(\epsilon)} |c_i(n)|^2 = 1$$

In the definition of $p_n(\epsilon)$, we restrict ourselves to the set $A_n^c(\epsilon)$ which is smaller than $A_n(\epsilon)$ in 4.1.8.

Proof of 1:

$$\text{Tr}(\rho^{(n)} p_n(\epsilon)) \geq \sum_{i(n) \in A_n(\epsilon)} r_i(n) - \sum_{i(n) \in A_n(\epsilon) \setminus B_n(\epsilon)} r_i(n) \geq 1 - \epsilon - \sum_{i(n) \in A_n(\epsilon) \setminus B_n(\epsilon)} 2^{-n(s(\omega) - \epsilon)} \geq 1 - \epsilon - 2^{-n(s(\omega) - \epsilon)} \#(B_n^c)^c \geq 1 - \epsilon - 2^{-n(s(\omega) - \epsilon)} 2^{n(s(\omega) - 2\epsilon) + \alpha_n} \geq 1 - \epsilon - 2^{-n\epsilon + \alpha_n} \geq 1 - \frac{3\epsilon}{2}.$$

In the fifth inequality above, it is clear that $2^{-n\epsilon + \alpha_n} \to 0$.

Proof of 2:

According to Theorem 5.1.1 we have

$$\omega(p_n) = \sum_{i(n) \in A_n(\epsilon) \cap B_n(\epsilon)} r_i(n) |c_i(n)|^2 \leq 2^{-n(s(\omega) - \epsilon)} \sum_{i(n) \in A_n(\epsilon) \cap B_n(\epsilon)} |c_i(n)|^2 \leq 2^{-n(s(\omega) - \epsilon)}.$$
\[ \omega(p_n) = \sum_{(n) \in A \cap B} \epsilon_i | < r_i (n) | \psi_i (n) > |^2 \]

\[ \geq 2^{-n(s(\omega)+\epsilon)} \sum_{(n) \in A \cap B} | \epsilon_i |^2 \]

\[ \geq 2^{-n(s(\omega)+\epsilon)}. \]

**Proof of 3:**

\[ \text{Tr}_n(p_n(\epsilon)) \leq \sum_{(n) \in A \cap B} 1 \]

\[ \leq \sum_{(n) \in A} 1 \]

\[ \leq 2^n(s(\omega)+\epsilon). \]

We also have

\[ \text{Tr}_n(p_n(\epsilon)) \geq \sum_{(n) \in A \cap B} 1 \]

\[ \geq \sum_{(n) \in A} 1 - \sum_{(n) \in A \setminus B} 1 \]

\[ \geq 2^n[s(\omega)-\epsilon] - 2^n[s(\omega)-2\epsilon]+\alpha_n \]

\[ \geq (1 - 2^{-n\epsilon})2^n(s(\omega)-\epsilon)+\alpha_n. \]

**Proof of 4:** Since the quantum system is semi-computable, thus the density matrix

\[ \eta = \sum_{n=2}^{\infty} \delta(n)\rho^{(n)}, \text{ where } \omega(A) = \text{Tr}(\rho^{(n)}A), \]

is also a semi-computable semi-density matrix and hence there exists a constant number
$c > 0$ such that $c \delta(n) \rho^{(n)} \leq c \eta \leq \hat{\mu}$, for all $n \in \mathbb{N}$. Therefore,

$$\limsup_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{\mu} p_n) \leq \limsup_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\rho^{(n)} p_n) \leq \limsup_{n \to \infty} -\frac{1}{n} \log \omega(p_n) \leq \limsup_{n \to \infty} -\frac{1}{n} \log \left( 1 - 2^{-n} \right) 2^{-n(s(\omega) + \epsilon + \alpha_n)} \leq s(\omega) + \epsilon + \alpha_n.$$

Thus,

$$\limsup_{n \to \infty} -\frac{\text{Tr}(\hat{\mu} p_n)}{n} \leq s(\omega) + \epsilon.$$

Since $\omega$ is a faithful state then the number of eigenvectors of $\rho^{(n)} = \rho |A_n$ is exactly $2^n$.

Hence, the operator $\hat{T}^{(n)} = \sum_{i(n) \in \Omega_2^{(n)}} \mu(i^{(n)}) | r^{(n)}_{i^{(n)}} > < r^{(n)}_{i^{(n)}} |$ is a semi-computable semi-density matrix. Therefore, there exists a constant number $c_T > 0$ such that $c_T \hat{T} \leq \hat{\mu}$.

Let us define the linear map $U_n |\mu(i^{(n)}) > = | r^{(n)}_{i^{(n)}} >$, where $\sum_{i^{(n)}} \mu(i^{(n)}) |\mu(i^{(n)}) > < \mu(i^{(n)}) |$ is the spectral decomposition of $\hat{\mu}$. Now, by Lemma 5.1.3 we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{\mu} p_n) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \text{Tr}(U_n \hat{\mu}^{(n)} |U_n^{-1} p_n) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{T}^{(n)} p_n) \geq \liminf_{n \to \infty} -\frac{1}{n} \log \left( \sum_{i^{(n)} \in \Omega_2^{(n)}} \mu(i^{(n)}) < r^{(n)}_{i^{(n)}} | p_n | r^{(n)}_{i^{(n)} >} \right) \geq \liminf_{n \to \infty} -\frac{1}{n} \log (2^{-n(s(\omega) - 2\epsilon)} \sum_{i^{(n)} \in \Omega_2^{(n)}} < r^{(n)}_{i^{(n)}} | p_n | r^{(n)}_{i^{(n)} >} \right) \geq s(\omega) - 2\epsilon.$$

The main important quantum correlation is entanglement which dose’t holds in classical dynamical systems.
Now, we say that the density matrix $\rho$ on the Hilbert space $\mathcal{H}_{XY} := \mathbb{H}_X \otimes \mathbb{H}_Y$ associated with a composite system with the two subsystems $X$ and $Y$ is separable if

$$\rho = \sum_{(i_1, i_2) \in I_1 \times I_2} \lambda_{i_1 i_2} \rho_{i_1}^1 \otimes \rho_{i_2}^2, \quad \lambda_{i_1 i_2} \geq 0, \quad \sum_{(i_1, i_2) \in I_1 \times I_2} \lambda_{i_1 i_2} = 1.$$ 

The density matrix $\rho$ is called entangled if it is not a separable state.

For example, the density matrix $\rho = |\psi \rangle \langle \psi|$, $|\psi \rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ on the Hilbert space $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is entanglement. Indeed, we cannot write $|\psi \rangle = |a \rangle \langle b|$ where $|a \rangle$ and $|b \rangle$ are states on $M_2(\mathbb{C})$.

In the following theorem, we will prove that entanglement in pure states dose not change the von Neumann entropy rate. In this case, we consider the product of two universal semi-measures which is not in general a universal semi-measure, instead of, a universal semi-measure on space of tensor product of two Hilbert spaces related to the GNS representation. Then, we show that the Gacs entropy rate is also equal to two times von Neumann entropy rate. Thus, it shows us that the entanglement dose not exceed of the lower Gacs entropy rate. Indeed, we know that entanglement is a quantum correlation and when we consider a many number of spins in large scale in the classical dynamical systems, or thermodynamical limit, the following theorem tells us that the effects of entanglement and pure states are equal.

**Theorem 5.1.5.** Let $((\mathcal{A}_Z)_{XY}, \Theta'_\sigma, \omega_{XY})$ be a composite quantum spin chain consisting of two ergodic spin chains $((\mathcal{A}_Z)_X, \Theta_\sigma, \omega)$ and $((\mathcal{A}_Z)_Y, \Theta_\sigma, \omega)$ with the same mean entropy $s(\omega)$ and faithful state $\omega$ where $\omega_{XY} = \omega \otimes \omega$ and $\Theta'_\sigma = \Theta_\sigma \otimes \Theta_\sigma$. Let $\rho_{[-n,n]}$ be a semi-computable semi-density matrix on both $\mathbb{H}_{XY}$ and consider the universal semi-measures $\hat{\mu}_X$ and $\hat{\mu}_Y$ on $\mathbb{H}_{XY}$. Then, there is a sequence of projectors $p_{2n}(\epsilon) \in \mathbb{H}_{2n} \subseteq \mathbb{H}_{XY}$ such
that for a sequence of minimal density matrices $\sigma^{(2n)} \leq p_{2n}(\epsilon)$ on $H_{XY}$, one has:

$$\lim_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{\mu}_X \otimes \hat{\mu}_Y \sigma^{(2n)}) = s(\omega).$$ \quad (5.1.2)

**Proof.** It is clear that the sequence of projectors $(p_X)_n(\epsilon) \otimes (p_Y)_n(\epsilon)$ satisfy the conditions 1, 2, 3 of Theorem 5.1.4, where $(p_X)_n(\epsilon)$ and $(p_Y)_n(\epsilon)$ are projections related to the mentioned conditions on $H_X$ and $H_Y$, respectively. Now, consider the sequence of minimal projections $\sigma^{(2n)} = |\psi^{(2n)}\rangle \langle \psi^{(2n)}| \leq p_{Xn} \otimes p_{Yn}$. According to the proof of Theorem 5.1.4 we can write $|\psi^{(2n)}\rangle$ as follows:

$$|\psi^{(2n)}\rangle = \sum_{i(n) j(n) \in A_{\epsilon}^{(n)} \cap B_{\epsilon}^{(n)}} a_{i(n)j(n)} |r_{i(n)} s_{j(n)}\rangle,$$

where $A_{\epsilon}^{(n)}$ and $B_{\epsilon}^{(n)}$ are defined in Theorem 5.1.4. The remaining of the proof is like that of Theorem 5.1.4.

$$\limsup_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\sigma^{(2n)} \hat{\mu}_X \otimes \hat{\mu}_Y) \leq$$

$$\leq \limsup_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\sigma^{(2n)} \hat{\mu}_X \otimes \hat{\mu}_Y)$$

$$\leq \limsup_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\sigma^{(2n)} \rho_X \otimes \rho_Y)$$

$$\leq \limsup_{n \to \infty} -\frac{1}{n} \log <\psi^{(2n)}| \rho_X \otimes \rho_Y |\psi^{(2n)}>$$

$$\leq \limsup_{n \to \infty} -\frac{1}{n} \log \sum_{i(n) j(n)} r_{i(n)} s_{j(n)} |a_{i(n)j(n)}|^2$$

$$\leq \limsup_{n \to \infty} -\frac{1}{n} \log \left( \sum_{i(n) j(n) \in A_{\epsilon}^{(n)} \cap B_{\epsilon}^{(n)}} |a_{i(n)j(n)}|^2 2^{-2n(s(\omega)+\epsilon)} \right)$$

$$\leq 2s(\omega) + 2\epsilon,$$
and

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{\mu}_X \otimes \hat{\mu}_Y \sigma^{2n}) \geq \\
\geq \liminf_{n \to \infty} -\frac{1}{n} \log \text{Tr}(\hat{T}_X \otimes \hat{T}_Y \sigma^{2n}) \\
\geq \liminf_{n \to \infty} -\frac{1}{n} \log \left( \sum_{i^{(n)}, j^{(n)} \in \Omega_2} \mu(i^{(n)})\mu(j^{(n)}) < r_{i^{(n)}}s_{j^{(n)}}|\sigma^{2n}|r_{i^{(n)}}r_{j^{(n)}} > \right) \\
\geq \liminf_{n \to \infty} -\frac{1}{n} \log \left( \sum_{i^{(n)}, j^{(n)} \in A^{(n)} \cap B^{(n)}} 2^{-2n(s(\omega)-2\epsilon)} < r_{i^{(n)}}s_{j^{(n)}}|\sigma^{2n}|r_{i^{(n)}}r_{j^{(n)}} > \right) \\
\geq 2s(\omega) - 4\epsilon.
\]

In the first inequality, we use Lemma 5.1.4
Conclusion

In this work we have extended the notions of computability, semi-computability, semi-computable vector states, and semi-computable density matrices to infinite dimensional Hilbert spaces. These extensions are necessary to describe algorithmically by classical Turing machines quantum systems with infinitely many degrees of freedom. In this paper we have applied them to the discussion, from a computer science point of view, of the complexity of quantum spin chains with the shift dynamics.

In classical information theory, Brudno has proved a relation between the Kolmogorov-Sinai dynamical entropy of ergodic time-evolutions and the algorithmic complexity per unit time step of all almost trajectories. In quantum information theory there are different extensions of both the Kolmogrov-Solomonoff-Chatin algorithmic complexity and of the Kolmogorov-Sinai dynamical entropy: their possible relations can be found in [4].

The techniques developed in this thesis have been applied to quantum spin chains. They allowed us to show that the Gacs algorithmic entropy per site of translation invariant states is equal to the von Neumann entropy rate. This could be done by removing an unnecessary condition in a previous proof of the same relations [6].

One proposal to extend the Brudno’s theorem is to consider the classical version of the concepts of the Gacs complexities based on semi-computable semi-measure functions using the classical Brudno’s theorem. The essential obstacle to extend the Beoudno’s theorem
based on associated symbolic dynamical systems is that we have no appropriate meaning of trajectory in the associated symbolic dynamical systems. But, we have given a short proof of the Brudno’s theorem in classical dynamical systems.

At the end, we have shown an extension of the Brudno’s theorem using the quantum Shannon-Mac Millan theorem which is directly derived without using the classical Brudno’s theorem, where ”almost every for all trajectories” in the classical case is replaced by a sequence of high probabilities projections. Furthermore, it has shown that entanglement and pure density matrices have the same role in the thermodynamic limit. Roughly speaking, rate of the log of the trace of the tensor product of universal semi-density matrices, associated to Hilbert spaces of subsystems, times density matrices pure or entangled, are equal to rate of von-Numann entropy of the state.
Bibliography

[1] W. Thirring, A. Connes, H. Narnhofer. Dynamical entropy of $c^*$ algebras and von Neumann algebras. *Comm. Math. Phys.*, 112(4):691–719, 1987.

[2] R. Alicki and M. Fannes. Defining quantum dynamical entropy. *Lett. Math. Phys.*, 32:75–82, 1994.

[3] R. Alicki and M. Fannes. *Quantum Dynamical Systems*. Oxford University Press, Cambridge, UK, 2001.

[4] Fabio Benatti. *Dynamics, Information and Complexity in Quantum Systems*. Theoretical and Mathematical Physics, Springer.

[5] Fabio Benatti. Gacs quantum complexity and quantum entropy.

[6] Fabio Benatti. Quantum dynamical entropy and gacs algorithmic entropy. *Entropy*, 14:1259–1273, 2012.

[7] Fabio Benatti, Samad Khabbazi Oskouei, and Ahmad Shafiei Deh Abad. Gacs quantum algorithmic entropy in infinite dimensional hilbert spaces. *Journal of Mathematical Physics*, 55:082205, 2014.

[8] C. Bennett. The thermodynamics of computation. *Int. J. Th. Phys.*, 21:905–940, 1982.
[9] A. Berthiaume, W. van Dam, and S. Laplante. Quantum kolmogorov complexity. *J. Compute. System Sci*, 63:201–221, 2001.

[10] P. Billingsley. *Ergodic Theory and Information*. J. Wiley, New York, 1965.

[11] Igor Bjelaković, Tyll Krger, Rainer Siegmund-schultze, and Arleta Szkoła. The shannon-mcmillan theorem for ergodic quantum lattice systems. *invent.math*, 155:203–222, 2004.

[12] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. Springer, second edition.

[13] Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2*. Springer, second edition.

[14] S.L. Braunstein. *Quantum Computing*. Wiley-VHC, Weinheim, 1999.

[15] A. A. Brudno. Entropy and the complexity of the trajectories of a dynamical system. *Trans. Moscow Math. Soc*, 2:127–151, 1983.

[16] Cristian Calude. *Information and Randomness: An Algorithmic Perspective*. Springer, Berlin, second edition, 2002.

[17] J.G. Chaitin. On the length of programs for computing finite binary sequences. *J. Assoc. Comp. Mach*, 13:547–569, 1966.

[18] M. Cover and Joy A. Thomas. *Elements of information theory*. J. Wiley, second edition.

[19] M. Davis, R. Sigal, and E.J Weyuker. *Computability, complexity, and languages: Fundamentals of theoretical computer science*. Academic Press, San Diego, CA, 1994.
[20] U. Vazirani E. Bernstein. Quantum complexity theory. *SIAM Journal on Computing*, 26:1411–1473, 1997.

[21] Peter Gacs. Quantum algorithmic entropy. *J. Phys. A*, 34:6859–6880, 2001.

[22] Peter Gacs. Lecture notes on descriptional complexity and randomness. Technical report, Boston University, 2010.

[23] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.

[24] J. Gruska. *Quantum Computing*. Mc Graw Hill, London, 1999.

[25] A.Y. Khinchin. *Mathematical Foundations of Statical Mechanics*. Dover, New York, 1949.

[26] A. Kolmogorov. Three approaches to the quantitative definition of information. *Problems of Information Transmission*, 1:1–7, 1965.

[27] E. Kowalski. Spectral theory in hilbert spaces. ETH Zürich, 2009.

[28] R. Landauer. Irreversibility and heat generation in the computing process. *IBM J. Research*, 3:183, 1961.

[29] G. Lindblad. Dynamical entropy for quantum systems: Lec. notes math. 1988.

[30] R. Mane. *Ergodic theory and differential dynamics*. Springer, Berlin.

[31] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[32] M. Li and P.M.P. Vitany. *Introduction to Kolmogrov Complexity and its applications*. Springer, New York, NY, USA, 2008.
[33] C.E. Mora and H.J. Briegel. Algorithmic complexity and entanglement of quantum states. *Phys. Rev. Lett.*, 95:200503–200507, 2005.

[34] C.E. Mora and H.J. Briegel. Algorithmic complexity of quantum states. *Int. J. Quantum Information*, 4:417, 2006.

[35] M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, UK, 2011.

[36] Andre Nies. *Computability and Randomness*. Oxford University Press, 2009.

[37] Masanori Ohya and Igor Volovich. *Mathematical Foundations of Quantum Information and Computation and Its Applications to Nano-and-Bio-Systems*. Springer, 2011.

[38] M. Mosca P. Kaye, R. Laflamme. *An Introduction to Quantum Computing*. Oxford University Press, Oxford UK, 2007.

[39] W. Slomczynski and K. Zyczkowski. Quantum chaos: an entropy approach. *J. Math. Phys*, 35:5674–5701, 1994.

[40] R. Solomonoff. A formal theory of inductive inference. *Inform. Contr*, 7:224–254, 1964.

[41] Jan Stochel. Seminormality of operators from their tensor product. *Proceedings of the American Mathematical Society*, 124:135–140, 1996.

[42] Franco Strocchi. *Symmetry Breaking*. Lecture Notes in Physics 643, Springer, 2005.

[43] M. Takesaki. *Theory of Operator Algebras III*. Springer-Verlag, Berlin, 2003.
[44] A. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematics Society*, 42:230–265, 1937.

[45] P. Vitanyi. Quantum kolmogorov complexity based on classical descriptions. *IEEE Trans. Inf. Th*, 47:2464–2479, 2001.

[46] D. Voiculescu. Dynamical approximation entropies and topological entropy in operator algebras. *Comm. Math. Phys*, 170:249–281, 1995.

[47] R. vonMises. Grundlagen derwahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, 5:52–99, 1919.

[48] Alfred Wehrl. General properties of entropy. *Reviews of Modern Physics*, 50:221–260, 1978.

[49] A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. *Russian Math. Surveys*, page 11, 1970.