THE FIRST COHOMOLOGY OF LIE SUPERALGEBRA $\tilde{P}(2)$ WITH COEFFICIENTS IN KAC MODULES

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Abstract. Over a field of characteristic $p > 2$, firstly, the structure of Kac modules of Lie superalgebra $\tilde{P}(2)$ and the weight space decompositions are given. Secondly, the weight-derivations of $\tilde{P}(2)$ to its Kac modules are computed. Finally, the first cohomology of $\tilde{P}(2)$ with coefficients in Kac modules is determined.

Keywords: Lie superalgebras, Kac modules, derivation, cohomology

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1. INTRODUCTION

It is well known that the representation is of great importance in the study of both Lie algebras and Lie superalgebras. In the present, the representation of Lie algebras has a relatively complete system. Lie superalgebras are the natural generalization of Lie algebras and can be divided into modular Lie superalgebras and non-modular Lie superalgebras according to the different characteristics of basic fields. Since the 1970s, many important research achievements have been get in the representations of non-modular Lie superalgebras, such as [2, 3, 5]. So many researchers began to focus on the representation of modular Lie superalgebras [4, 1, 5].

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The cohomology groups are helpful for the representations of modular Lie superalgebras. First Leits and Fuks calculated the cohomology of classical Lie superalgebras with coefficients in trivial modules in 1984. Then the following results on the Lie superalgebras cohomology are gradually plentiful and substantial, include some discussion on nontrivial modules, for example [8, 10, 11, 13]. But very little seems to be known about the study of modular Lie superalgebras cohomology with coefficients in nontrivial or adjoint modules except [13, 7, 6, 16].

In representation theories of Lie superalgebras, the Kac modules are of significance. In 2007, Su Yucai and Zhang Ruibin calculated the first and second cohomology of $\mathfrak{sl}_m|n$ and $\mathfrak{osp}_{2|2n}$ to finite-dimensional irreducible modules and Kac modules[10]. In 2010, Shu Bin and Zhang Chaowen studied Witt superalgebras and defined its Kac modules[11]. In 2020, Wang Shujuan and Liu Wende studied the first cohomology of $\mathfrak{sl}_2|1$ with coefficients in $\chi$-reduced Kac modules and simple modules [13].

In the present article, over a field of characteristic $p > 2$, we describe firstly the structure of Kac modules of Lie superalgebra $\tilde{\mathfrak{p}}(2)$ and compute the weight space decompositions of $\tilde{\mathfrak{p}}(2)$ and its Kac modules relative to a fixed Cartan subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{p}}(2)$. Then the work under consideration is reduced to computing the weight-derivations, which preserve the $\mathfrak{h}^*$-gradings, of $\tilde{\mathfrak{p}}(2)$ to these weight spaces. Finally, the first cohomology of Lie superalgebra $\tilde{\mathfrak{p}}(2)$ with coefficients in Kac modules are obtained by means of the fact that each derivation of a finite dimensional Lie superalgebras to its module is equal to a weight-derivation module an inner derivation.

2. preliminaries

Throughout the paper, all vector spaces are over a field $F$ of characteristic $p > 2$ and finite-dimensional. Let $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ be the two-element field. The symbol $|x|$ or $zd(x)$ denotes the $\mathbb{Z}_2$-degree or $Z$-degree of a $\mathbb{Z}_2$-homogeneous or $Z$-homogeneous element $x$ respectively. And we said that element $x$ is even or odd, if $|x|$ is $\bar{0}$ or $\bar{1}$.

The set of all $\mathbb{Z}_2$-homogeneous elements in the $\mathbb{Z}_2$-degree space $V$ is represented by $\text{hg}(V)$. Write $\langle v_1, \ldots, v_k \rangle$ for the vector space spanned by $v_1, \ldots, v_k$ over $F$. In the $\mathbb{Z}_2$-degree vector space $\langle v_1, \ldots, v_m \mid v_{m+1}, \ldots, v_{m+k} \rangle$, we assume that $|v_i| = 0$ and $|v_{m+j}| = 1$, where $i = 1, \ldots, m$, $j = 1, \ldots, k$.

Let $M$ be a Lie superalgebra $\mathfrak{g}$-module. Recall that a $\mathbb{Z}_2$-homogeneous linear mapping $\varphi$ is a derivation of parity $|\varphi|$ of $\mathfrak{g}$ to $\mathfrak{g}$-module $M$ provided that

$$\varphi([x, y]) = (-1)^{|\varphi||x|}x\varphi(y) - (-1)^{|y||\varphi|+|x|}y\varphi(x), \text{ for all } x, y \in \text{hg}(\mathfrak{g}).$$

A derivation $\varphi$ of $\mathfrak{g}$ to $M$ is said to be inner determined by $v$ if there exists $v \in \text{hg}(\mathfrak{g})$ such that $\varphi(x) = (-1)^{|x||v|}xv$ for any $x \in \text{hg}(\mathfrak{g})$, record as $\mathfrak{D}_v$. Otherwise, $\varphi$ is called an outer derivation. Let $\mathfrak{h}$ be a Cartan subalgebra in the even part of $\mathfrak{g}$. Suppose
that $g$ and $M$ possess weight space decompositions with respect to $h$: $g = \oplus_{\alpha \in h^*} g_\alpha$ and $M = \oplus_{\alpha \in h^*} M_\alpha$. A derivation $\varphi$ of $g$ to $M$ is called a weight-derivation relative to $h$ if $\varphi(g_\alpha) \subseteq M_\alpha$, for all $\alpha \in h^*$. Let $\text{Der}(g, M)$ denote the vector space spanned by all the $\mathbb{Z}_2$-homogeneous derivations of $g$ to $M$. Write $\text{Ider}(g, M)$ for the vector space spanned by all inner derivations. The first cohomology of $g$ with coefficients in $M$ is the quotient module:

$$\text{H}^1(g, M) = \text{Der}(g, M)/\text{Ider}(g, M).$$

**Definition 2.1.** Let $g = g_{-1} \oplus g_0 \oplus g_{+1}$ be a restricted Lie superalgebra with $\mathbb{Z}$-grading. Suppose that $M(\lambda)$ is a simple finite-dimensional $g_0$-module with the highest weight $\lambda$, and $g_{+1}M(\lambda) = 0$. Regarding $M(\lambda)$ as $(g_0 \oplus g_{+1})$-module, we call the induced module $K(\lambda) = \text{U}(g) \otimes \text{U}(g_0 \oplus g_{+1})M(\lambda)$ restricted Kac module of $g$, where $\text{U}(g)$ is the enveloping algebra of Lie superalgebra $g$.

Note that $K(\lambda) \cong \text{U}(g_{-1}) \otimes \mathbb{F} M(\lambda)$ as a vector space.

**3. THE STRUCTURE OF KAC MODULES**

Lie superalgebra $\tilde{P}(2)$ is defined as follows:

$$\tilde{P}(2) := \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in gl(2, 2) \bigg| B = B^T, C = -C^T \right\}.$$ 

From now on, write $g$ for $\tilde{P}(2)$ and $e_{ij}$ for the $4 \times 4$ matrices which has 1 in the position $(i, j)$ and 0 elsewhere. Let

$$\gamma = e_{41} - e_{32}, \quad h_1 = e_{33} - e_{11}, \quad h_2 = e_{44} - e_{22}, \quad \alpha = e_{43} - e_{12}, \quad \beta = e_{34} - e_{21}.$$ 

Note that $g$ possesses a $\mathbb{Z}$-grading structure $g = g_{-1} \oplus g_0 \oplus g_{+1}$, then the following elements form the basis of $g$:

$$g_{-1} = \langle \gamma \rangle, \quad g_0 = \langle h_1, h_2, \alpha, \beta \rangle, \quad g_{+1} = \langle e_{13}, e_{24}, e_{14} + e_{23} \rangle.$$ 

Fix the standard Cartan subalgebra $h$ of $g_0$ spanned by $h_1$ and $h_2$. Let $\varepsilon_1, \varepsilon_2 \in h^*$ make $\varepsilon_i(h_j) = \delta_{ij}, \ i, j = 1, 2$. Then the roots and the root-vectors for $g$ can be obtained in the Table 3.1 below.

| Table 3.1: Roots and root-vectors for $g$ |
|------------------------------------------|
| Roots | $\theta$ | $-2\varepsilon_1$ | $-\varepsilon_1 - \varepsilon_2$ | $-\varepsilon_1 + \varepsilon_2$ | $-2\varepsilon_2$ | $\varepsilon_1 - \varepsilon_2$ | $\varepsilon_1 + \varepsilon_2$ |
| Root-vectors | $h_1, h_2$ | $e_{13}$ | $e_{14} + e_{23}$ | $\alpha$ | $e_{24}$ | $\beta$ | $\gamma$ |
Let \( \lambda = a \varepsilon_1 + b \varepsilon_2 \), where \( a, b \in \mathbb{F}_p \). Suppose \( v_0 \) satisfies: \( h_i v_0 = \lambda(h_i)v_0 \), \( \alpha v_0 = 0 \).

Inductively define \( v_k = \beta^k v_0 \), where \( k \in \mathbb{Z} \), and note that \( \beta^{[n]} = 0 \), then \( \beta^p v_0 = 0 \), that is, \( v_p = 0 \). Therefore, we get that \( v_0, v_1, \ldots, v_{p-1} \) are nonzero and linearly independent. Then \( V := \langle v_0, v_1, \ldots, v_{p-1} \rangle \) is a \( p \)-dimensional module of \( \mathfrak{g}_0 \).

For any \( c \in \mathbb{F}_p \), let \( \Phi(c) \in \{0, 1, \ldots, p - 1\} \subseteq \mathbb{Z} \) such that \( \Phi(c) \equiv c \pmod{p} \). Also due to

\[
h_1 v_k = (a + k)v_k, \quad h_2 v_k = (b - k)v_k,
\]

\[
\alpha v_k = k(b - a - k + 1)v_{k-1}, \quad \beta v_k = v_{k+1}, \quad 0 \leq k \leq p - 1.
\]

It is easy to know \( W := \langle v_{\Phi(b-a+1)}, \ldots, v_{p-1} \rangle \) is a maximal submodule of \( V \). Hence \( M(\lambda) := V/W \) becomes a simple module of \( \mathfrak{g}_0 \).

Without confusion, we write the images of the elements of \( V \) in \( M(\lambda) \) still by the elements of \( V \) itself. Thus

\[
M(\lambda) = \langle v_0, v_1, \ldots, v_{\Phi(b-a)} \rangle.
\]

Let \( g_{+1} M(\lambda) = 0 \), \( M(\lambda) \) be regarded as the simple module of \( \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \). By Definition 2.1

\[
K(\lambda) = \langle 1 \otimes v_k, \gamma \otimes v_k, k = 0, 1, \ldots, \Phi(b-a) \rangle.
\]

It is easy to see that \( |1 \otimes v_k| = 0 \), \( |\gamma \otimes v_k| = 1 \). The module action of \( \mathfrak{g} \) on \( K(\lambda) \) is given below:

\[
h_1(1 \otimes v_k) = (a + k) \otimes v_k, \quad h_2(1 \otimes v_k) = (b - k) \otimes v_k,
\]

\[
h_1(\gamma \otimes v_k) = (a + k + 1) \otimes v_k, \quad h_2(\gamma \otimes v_k) = (b - k + 1) \otimes v_k,
\]

\[
\alpha(1 \otimes v_k) = k(b - a - k + 1) \otimes v_{k-1}, \quad \alpha(\gamma \otimes v_k) = k(b - a - k + 1) \gamma \otimes v_{k-1},
\]

\[
\beta(1 \otimes v_k) = \begin{cases} 1 \otimes v_{k+1}, & 0 \leq k < \Phi(b-a), \\ 0, & k = \Phi(b-a), \end{cases}
\]

\[
\beta(\gamma \otimes v_k) = \begin{cases} \gamma \otimes v_{k+1}, & 0 \leq k < \Phi(b-a), \\ 0, & k = \Phi(b-a), \end{cases}
\]

\[
e_{13}(1 \otimes v_k) = e_{24}(1 \otimes v_k) = (e_{14} + e_{23})(1 \otimes v_k) = 0,
\]

\[
\gamma(1 \otimes v_k) = \gamma \otimes v_k, \quad \gamma(\gamma \otimes v_k) = 0,
\]

\[
e_{13}(\gamma \otimes v_k) = k(b - a - k + 1) \otimes v_{k-1}, \quad (e_{14} + e_{23})(\gamma \otimes v_k) = (b - a - 2k) \otimes v_k,
\]

\[
e_{24}(\gamma \otimes v_k) = \begin{cases} -1 \otimes v_{k+1}, & 0 \leq k < \Phi(b-a), \\ 0, & k = \Phi(b-a). \end{cases}
\]

We list the weight-vector of \( K(\lambda) \) relative to \( h \), where \( a, b \in \mathbb{F}_p \) (see Table 3.2).

| Table 3.2: Weights and weight-vectors of \( K(\lambda) \) |
|-----------------------------------------------|
| Weights | \((a + k)\varepsilon_1 + (b - k)\varepsilon_2\) | \((a + k + 1)\varepsilon_1 + (b - k + 1)\varepsilon_2\) |
| Weight-vectors | \(1 \otimes v_k\) | \(\gamma \otimes v_k\) |
4. Target-weight spaces of $K(\lambda)$

The following fact will simplify the computation of the first cohomology $H^1(g, K(\lambda))$.

**Lemma 4.1.** Each derivation of finite dimensional Lie superalgebra $g$ to $g$-module $M$ is equal to a weight-derivation module an inner derivation.

In view of Lemma 4.1 and the definition of the first cohomology, it is sufficient to compute the weight-derivations of $g$ to $K(\lambda)$ relative to $h$, if we want to consider the first cohomology of $g$ to $K(\lambda)$. However, the target-weight spaces should be given before calculating the weight-derivation. This section aims to solve this problem.

Let $c, d \in \mathbb{Z}, c, d := \{x \in \mathbb{Z} | c \leq x \leq d\}$.

**Lemma 4.2.** The relationships about $\Phi(b), \Phi(b + 1), \Phi(b + 2)$ and $\Phi(2b + 2)$ are as follows:

(i) $\Phi(b) \leq \Phi(2b + 2) \iff \Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 2\}$,
(ii) $\Phi(b + 1) \leq \Phi(2b + 2) \iff \Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 1\}$,
(iii) $\Phi(b + 2) \leq \Phi(2b + 2) \iff \Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 2\}$.

**Proof.** Since $\Phi(b) \in 0, p - 1$, the scopes of $\Phi(b), \Phi(b + 1), \Phi(b + 2)$ and $\Phi(2b + 2)$ can be obtained (see Table 4.1).

| $\Phi(b)$ | $0$ | $1$ | $\ldots$ | $\frac{p - 3}{2}$ | $\frac{p - 1}{2}$ | $\ldots$ | $p - 3$ | $p - 2$ | $p - 1$ |
|----------|-----|-----|---------|------------------|------------------|---------|---------|---------|---------|
| $\Phi(b + 1)$ | $1$ | $2$ | $\ldots$ | $\frac{p - 3}{2}$ | $\frac{p - 1}{2}$ | $\ldots$ | $p - 2$ | $p - 1$ | $0$ |
| $\Phi(b + 2)$ | $2$ | $3$ | $\ldots$ | $\frac{p - 3}{2}$ | $\frac{p - 1}{2}$ | $\ldots$ | $p - 1$ | $0$ | $1$ |
| $\Phi(2b + 2)$ | $2$ | $4$ | $\ldots$ | $p - 1$ | $1$ | $\ldots$ | $p - 4$ | $p - 2$ | $0$ |

It can be seen from Table 4.1 that

$$\Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 2\}, if \Phi(b) \leq \Phi(2b + 2),$$

$$\Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 1\}, if \Phi(b + 1) \leq \Phi(2b + 2),$$

$$\Phi(b) \in 0, \frac{p - 3}{2} \cup \{p - 2\}, if \Phi(b + 2) \leq \Phi(2b + 2).$$

□

By the similar methods, we can proof the following Lemma 4.3–Lemma 4.5.

**Lemma 4.3.** The relationships about $\Phi(b), \Phi(b - 1), \Phi(b + 1)$ and $\Phi(2b)$ are as follows:

(i) $\Phi(b) \leq \Phi(2b) \iff \Phi(b) \in 0, \frac{p - 1}{2}$,
Lemma 4.4. The relationships about $\Phi(b+1), \Phi(b+2), \Phi(b+3)$ and $\Phi(2b+4)$ are as follows:

(i) $\Phi(b+1) \leq \Phi(2b+4) \iff \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-3\}$,
(ii) $\Phi(b+2) \leq \Phi(2b+4) \iff \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-2\}$,
(iii) $\Phi(b+3) \leq \Phi(2b+4) \iff \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-3\}$.

Lemma 4.5. The relationships about $\Phi(b-1)$ and $\Phi(2b-2)$ are as follows:

$\Phi(b-1) \leq \Phi(2b-2) \iff \Phi(b) \in 1, \frac{p-3}{2}$.

The above work is all in preparation for finding the target-weight spaces of $K(\lambda)$.

Proposition 4.1. The target-weight spaces of $K(\lambda)$ are

(4.1)

$$K(\lambda)_{(-2,0)} = \begin{cases} 
\langle 1 \otimes v_{\Phi(b)} \mid 0 \rangle, & \Phi(b) \in 0, \frac{p-3}{2} \cup \{p-2\} \text{ and } a+b = -2, \\
\langle 0 \mid \gamma \otimes v_{\Phi(b+1)} \rangle, & \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-3\} \text{ and } a+b = -4, \\
\langle 0 \mid 0 \rangle, & \text{otherwise.}
\end{cases}$$

(4.2)

$$K(\lambda)_{(-1,-1)} = \begin{cases} 
\langle 1 \otimes v_{\Phi(b+1)} \mid 0 \rangle, & \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1\} \text{ and } a+b = -2, \\
\langle 0 \mid \gamma \otimes v_{\Phi(b+2)} \rangle, & \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-2\} \text{ and } a+b = -4, \\
\langle 0 \mid 0 \rangle, & \text{otherwise.}
\end{cases}$$

(4.3)

$$K(\lambda)_{(-1,1)} = \begin{cases} 
\langle 1 \otimes v_{\Phi(b-1)} \mid 0 \rangle, & \Phi(b) \in 1, \frac{p-1}{2} \cup \{p-1\} \text{ and } a+b = 0, \\
\langle 0 \mid \gamma \otimes v_{\Phi(b)} \rangle, & \Phi(b) \in 0, \frac{p-3}{2} \cup \{p-2\} \text{ and } a+b = -2, \\
\langle 0 \mid 0 \rangle, & \text{otherwise.}
\end{cases}$$

(4.4)

$$K(\lambda)_{(0,-2)} = \begin{cases} 
\langle 1 \otimes v_{\Phi(b+2)} \mid 0 \rangle, & \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-2\} \text{ and } a+b = -2, \\
\langle 0 \mid \gamma \otimes v_{\Phi(b+3)} \rangle, & \Phi(b) \in 0, \frac{p-5}{2} \cup \{p-1, p-3\} \text{ and } a+b = -4, \\
\langle 0 \mid 0 \rangle, & \text{otherwise.}
\end{cases}$$
\[ K(\lambda)_{(1,-1)} = \begin{cases} 
(1 \otimes v_{\Phi(b+1)} | 0), & \Phi(b) \in 1, \frac{p-1}{2} \cup \{p-1\} and a + b = 0, \\
(0 | \gamma \otimes v_{\Phi(b+2)}), & \Phi(b) \in 0, \frac{p-3}{2} \cup \{p-2\} and a + b = -2, \\
(0 | 0), & otherwise.
\end{cases} \]

\[ K(\lambda)_{(1,1)} = \begin{cases} 
(1 \otimes v_{\Phi(b-1)} | 0), & \Phi(b) \in 1, \frac{p+1}{2} and a + b = 2, \\
(0 | \gamma \otimes v_{\Phi(b)}), & \Phi(b) \in 0, \frac{p-1}{2} \cup \{p-1\} and a + b = -2, \\
(0 | 0), & otherwise.
\end{cases} \]

\[ K(\lambda)_{(0,0)} = \begin{cases} 
(1 \otimes v_{\Phi(b)} | 0), & \Phi(b) \in 0, \frac{p-1}{2} \cup \{p-1\} and a + b = 0, \\
(0 | \gamma \otimes v_{\Phi(b+1)}), & \Phi(b) \in 0, \frac{p-3}{2} \cup \{p-2\} and a + b = -2, \\
(0 | 0), & otherwise.
\end{cases} \]

Proof. We take (4.6) and (4.7) as examples to prove Proposition 4.1.1 Equations (4.1)-(4.5) can be proved similarly. By Table 2, the weights of \(1 \otimes v_k\) and \(\gamma \otimes v_k\) are 
\((a + k, b - k)\) and \((a + k + 1, b - k + 1)\) respectively. In order to prove (4.6), we need the weights of \(1 \otimes v_k\) and \(\gamma \otimes v_k\) to be \((1, 1)\). There are the following two cases.

Case 1: Let \((a + k, b - k) = (1, 1)\). We have \(a + b = 2\), so that \(b - a = 2b - 2\), thus \(\Phi(b - a) = \Phi(2b - 2)\). The following conclusions can be drawn:
Subcase 1.1: \(a + b \neq 2\). \(1 \otimes v_k\)'s weight is not \((1, 1)\), \(0 \leq k \leq \Phi(b - a)\).
Subcase 1.2: \(a + b = 2\). By Lemma 4.3, we have:
Subcase 1.2.1: When \(\Phi(b) \in 1, \frac{p+1}{2}, \{p-1\} \cup \{0\}\), \(1 \otimes v_k\)'s weight is not \((1, 1)\), \(0 \leq k \leq \Phi(b - a)\).
Subcase 1.2.2: When \(\Phi(b) \in \frac{p+1}{2}, \{p-1\} \cup \{0\}\), \(1 \otimes v_k\)'s weight is \((1, 1)\).

Case 2: Let \((a + k + 1, b - k + 1) = (1, 1)\). We have \(a + b = 0\), so that \(b - a = 2b\), thus \(\Phi(b - a) = \Phi(2b)\). The following conclusions can be drawn:
Subcase 2.1: \(a + b \neq 0\). \(\gamma \otimes v_k\)'s weight is not \((1, 1)\), \(0 \leq k \leq \Phi(b - a)\).
Subcase 2.2: \(a + b = 0\). By Lemma 4.3 (i), we have:
Subcase 2.2.1: When \(\Phi(b) \in \frac{p+1}{2}, \{p-1\} \cup \{0\}\), \(\gamma \otimes v_k\)'s weight is not \((1, 1)\), \(0 \leq k \leq \Phi(b - a)\).
Subcase 2.2.2: When \(\Phi(b) \in 0, \frac{p-1}{2}\), \(\gamma \otimes v_k\)'s weight is \((1, 1)\).

Analogously, in order to prove (4.7), we need the weights of \(1 \otimes v_k\) and \(\gamma \otimes v_k\) to be \((0, 0)\). There are the following two cases.
Therefore, the following formula holds:

\[
\{ g \} \introduce \text{four outer derivations. Consider the linear mappings of } b
\]

Subcase 3.2: \( a + b = 0 \). By Lemma 4.3 (i), we have:

Subcase 3.2.1: When \( \Phi(\cdot) \in v_k \), \( 1 \otimes v_k \)'s weight is not \((0, 0)\), \( 0 \leq k \leq \Phi(b - a) \).

Subcase 3.2.2: When \( \Phi(\cdot) = 0, p - 1 \), \( 1 \otimes v_k \)'s weight is \((0, 0)\).

Subcase 3.1: \( a + b = -2 \).

Subcase 4.1: \( a + b = -2 \).

Subcase 4.2: \( a + b = -2 \). By Lemma 4.2 (ii), we have:

Subcase 4.2.1: When \( \Phi(\cdot) \in v_k \), \( 1 \otimes v_k \)'s weight is not \((0, 0)\), \( 0 \leq k \leq \Phi(b - a) \).

Subcase 4.2.2: When \( \Phi(\cdot) = 0, p - 1 \cup \{ -1 \} \), \( 1 \otimes v_k \)'s weight is \((0, 0)\).

In summary, the proofs of (1.6) and (1.7) are completed. \( \square \)

5. \( H^1(g, K(\lambda)) \)

The following lemma will simplify the calculation of the first cohomology.

**Lemma 5.1.** Suppose that \( \phi \) is a weight-derivation of \( g \) to \( K(\lambda) \), we have \( x \phi(h_i) = 0 \), \( i = 1, 2 \), for all \( x \in g \).

**Proof.** We assume that \( x \in g_\alpha \), where \( \alpha \in h^\ast \), then \( h_i x = \alpha(h_i) x \), \( i = 1, 2 \). Since \( \phi \) is a weight-derivation, we get \( h_i \phi(x) = \alpha(h_i) \phi(x) \). By the definition of derivation, the following formula holds:

\[
\alpha(h_i) \phi(x) = \phi(\alpha(h_i) x) = \phi([h_i, x]) = h_i \phi(x) - (-1)^{|\alpha||x|} x \phi(h_i)
\]

\[
= \alpha(h_i) \phi(x) - (-1)^{|\alpha||x|} x \phi(h_i),
\]

Therefore, \( x \phi(h_i) = 0 \), \( i = 1, 2 \). \( \square \)

Notice that the weight-mappings from \( g \) to \( K(\lambda) \) are zero mapping for \( \Phi(a + b) \notin \{ 0, 2, p - 2, p - 4 \} \) from Proposition 4.1.

Before computing the first cohomology of \( g \) with coefficients in \( K(\lambda) \), we first introduce four outer derivations. Consider the linear mappings of \( g \) to \( K(\lambda) \).

If \( a + b = -2 \) and \( \Phi(b) = p - 2 \), we define \( \varphi_1, \varphi_2 \), such that

\[
\varphi_1: \alpha \mapsto \gamma \otimes v_{p-2}, \quad \epsilon_{13} \mapsto -1 \otimes v_{p-2};
\]

\[
\varphi_2: \beta \mapsto \gamma \otimes v_0, \quad \epsilon_{24} \mapsto 1 \otimes v_0.
\]

If \( a + b = -2 \) and \( \Phi(b) = p - 1 \), we define \( \varphi_3 \), such that

\[
\varphi_3: h_i \mapsto \gamma \otimes v_0, \quad i = 1, 2.
\]
If \( a + b = -4 \) and \( \Phi(b) = p - 1 \), we define \( \varphi_4 \), such that

\[
\varphi_4 : e_{13} \mapsto 2\gamma \otimes v_0, \quad e_{24} \mapsto \gamma \otimes v_2, \quad e_{14} + e_{23} \mapsto -2\gamma \otimes v_1.
\]

Here we take the convention that, the element of \( \text{Hom}(g, K(\lambda)) \) vanishes on the standard basis elements of \( g \) which do not appear. For example \( \varphi_1(h_1) = 0 \), the same below.

**Lemma 5.2.** Each \( \varphi_k \) is both an outer derivation and a weight-derivation for \( k = 1, 2, 3, 4 \).

**Proof.** By the definition of derivation and Proposition 4.1, it is obvious that \( \varphi_k \) is a derivation and weight-derivation for \( k = 1, 2, 3, 4 \). Suppose conversely \( \varphi_k \) is a nonzero inner derivation given by \( v \in K(\lambda) \). By the definition of weight-derivation, the weight of \( v \) is \((0,0)\). For \( \varphi_1, \varphi_2, \varphi_4 \), we know \( v = 0 \) by Proposition 4.1 (1)-(5), contradictorily. Hence \( \varphi_1, \varphi_2, \varphi_4 \) are outer derivations. For \( \varphi_3 \), we may assume \( v = e\gamma \otimes v_0 \) by Proposition 4.1 (7), where \( e \in \mathbb{F} \). According to the definition of inner derivation,

\[
\mathcal{D}_v(h_1) = h_1(e\gamma \otimes v_0) = 0 \neq \gamma \otimes v_0.
\]

Contradictorily. So \( \varphi_3 \) is outer. \( \square \)

Below, we compute \( H^1(g, K(\lambda)) \). By Lemma 4.1 we only need to compute the weight-derivations of \( g \) to \( K(\lambda) \).

**Proposition 5.1.**

\[
H^1(g, K(\lambda)) = \begin{cases}
\mathbb{F}\varphi_1 + \mathbb{F}\varphi_2, & a + b = -2 \text{ and } \Phi(b) = p - 2, \\
\mathbb{F}\varphi_3, & a + b = -2 \text{ and } \Phi(b) = p - 1, \\
\mathbb{F}\varphi_4, & a + b = -4 \text{ and } \Phi(b) = p - 1, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** According to the range of \( a + b \) and Proposition 4.1 we proof Proposition 5.1 for the following four cases. Note in advance that the coefficients \( m_i \) set below are in \( \mathbb{F} \), \( i = 1, \cdots, 7 \). Let \( \varphi \) be a weight-derivation of \( g \) to \( K(\lambda) \) in each of the following cases.

Case 1: \( a + b = -2 \). The following conclusions can be drawn.
Subcase 1.1: $\Phi(b) \in \overline{\mathbb{R}_{\geq 0}}$. By Proposition 4.1 (1)-(5) and (7), we may assume $\varphi$:

$h_1 \mapsto m_1 \gamma \otimes \Phi(b+1)$,
$h_2 \mapsto m_2 \gamma \otimes \Phi(b+1)$,
$\alpha \mapsto m_3 \gamma \otimes \Phi(b)$,
$\beta \mapsto m_4 \gamma \otimes \Phi(b+2)$,
$e_{13} \mapsto m_5 \otimes \Phi(b)$,
$e_{24} \mapsto m_6 \otimes \Phi(b+2)$,
$e_{14} + e_{23} \mapsto m_7 \otimes \Phi(b+2)$.

Obviously, $|\varphi| = 1$. Thus, from the Lemma 5.1 and the definition of derivation, $\varphi$ is a weight-derivation and the following equations hold:

$$
\begin{align*}
\alpha \varphi(h_i) &= m_i \alpha(\gamma \otimes \Phi(b+1)) = 0, \ i = 1, 2, \\
\varphi([\alpha, \beta]) &= \alpha \varphi(\beta) - \beta \varphi(\alpha), \\
\varphi([e_{13}, \gamma]) &= -e_{13} \varphi(\gamma) - \gamma \varphi(e_{13}), \\
\varphi([e_{24}, \gamma]) &= -e_{24} \varphi(\gamma) - \gamma \varphi(e_{24}), \\
\varphi([e_{14} + e_{23}, \gamma]) &= -(e_{14} + e_{23}) \varphi(\gamma) - \gamma \varphi(e_{14} + e_{23}).
\end{align*}
$$

That is,

$$
\begin{align*}
m_i(b + 1)(b + 2)\gamma \otimes \Phi(b) &= 0, \ i = 1, 2, \\
|m_3(b + 2)(b + 1) - m_3 \gamma \otimes \Phi(b+1)| &= 0, \\
m_3 \gamma \otimes \Phi(b) &= -m_5 \gamma \otimes \Phi(b), \\
-m_4 \gamma \otimes \Phi(b+2) &= -m_6 \gamma \otimes \Phi(b+2), \\
-m_7 \gamma \otimes \Phi(b+1) &= 0.
\end{align*}
$$

By solving above equations, we have

$$
\begin{align*}
m_1 &= m_2 = 0, \\
m_3 &= m_4(b + 1)(b + 2), \\
m_5 &= -m_4(b + 1)(b + 2), \\
m_6 &= m_4, \\
m_7 &= 0.
\end{align*}
$$

It is easy to verify that $\varphi = m_4 \mathcal{D} \gamma \otimes \Phi(b+1)$, that is, $\varphi$ is inner.
Subcase 1.2: $\Phi(b) = p - 2$. By Proposition 4.1 (1) and (3)-(5), we may assume $\varphi$:

\[
\begin{align*}
\alpha &\mapsto m_3 \gamma \otimes v_{p-2}, \\
\beta &\mapsto m_4 \gamma \otimes v_0, \\
e_{13} &\mapsto m_5 \otimes v_{p-2}, \\
e_{24} &\mapsto m_6 \otimes v_0.
\end{align*}
\]

Obviously, $|\varphi| = \bar{1}$. Thus, it can be known by the definition of derivation that $\varphi$ is a weight-derivation and the following equations hold:

\[
\begin{align*}
\varphi([e_{13}, \gamma]) &= -e_{13} \varphi(\gamma) - \gamma \psi(e_{13}), \\
\varphi([e_{24}, \gamma]) &= -e_{24} \varphi(\gamma) - \gamma \psi(e_{24}).
\end{align*}
\]

It shows that

\[
\begin{align*}
m_3 \gamma \otimes v_{p-2} &= -m_5 \gamma \otimes v_{p-2}, \\
-m_4 \gamma \otimes v_0 &= -m_6 \gamma \otimes v_0.
\end{align*}
\]

We get

\[
\begin{align*}
m_5 &= -m_3, \\
m_6 &= m_4.
\end{align*}
\]

From Lemma 5.2 it is easy to see that $\varphi = m_3 \varphi_1 + m_4 \varphi_2$, and $\varphi_1, \varphi_2$ are outer derivations.

Subcase 1.3: $\Phi(b) = p - 1$. By Proposition 4.1 (2) and (7), we may suppose $\varphi$:

\[
\begin{align*}
h_1 &\mapsto m_1 \gamma \otimes v_0, \\
h_2 &\mapsto m_2 \gamma \otimes v_0, \\
e_{14} + e_{23} &\mapsto m_7 \otimes v_0.
\end{align*}
\]

Obviously, $|\varphi| = \bar{1}$. According to the definition of derivation, we have the following equations:

\[
\begin{align*}
\varphi([\alpha, \beta]) &= 0, \\
\varphi([\alpha, e_{24}]) &= 0.
\end{align*}
\]

We obtain

\[
\begin{align*}
(m_2 - m_1) \gamma \otimes v_0 &= 0, \\
-m_7 \otimes v_0 &= 0.
\end{align*}
\]

Write $m_1, m_2$ for $m, m \in F$. From Lemma 5.2 it is easy to know that $\varphi = m \varphi_3$, and $\varphi$ is an outer derivation when $m \neq 0$.

Case 2: $a + b = -4$. The following conclusions can be drawn.
Subcase 2.1: $\Phi(b) \in [0, \frac{p-2}{2}]$. By Proposition 4.1 (1), (2) and (4), we may suppose \( \varphi \):
\[
\begin{align*}
  e_{13} &\mapsto m_5 \gamma \otimes v_{\Phi(b+1)}, \\
  e_{24} &\mapsto m_6 \gamma \otimes v_{\Phi(b+3)}, \\
  e_{14} + e_{23} &\mapsto m_7 \gamma \otimes v_{\Phi(b+2)}.
\end{align*}
\]
Obviously, $|\varphi| = \bar{0}$. The following equations hold from the definition of derivation,
\[
\begin{align*}
  \varphi([\alpha, e_{13}]) &= \alpha \varphi(e_{13}), \\
  \varphi([\alpha, e_{24}]) &= \alpha \varphi(e_{24}), \\
  \varphi([\alpha, e_{14} + e_{23}]) &= \alpha \varphi(e_{14} + e_{23}).
\end{align*}
\]
Then,
\[
\begin{align*}
  \begin{cases}
    m_5(b+1)(b+4)\gamma \otimes v_{\Phi(b)} = 0, \\
    -m_7 \gamma \otimes v_{\Phi(b+2)} = m_6(b+2)(b+3)\gamma \otimes v_{\Phi(b+2)}, \\
    -2m_5 \gamma \otimes v_{\Phi(b+1)} = m_7(b+2)(b+3)\gamma \otimes v_{\Phi(b+1)}.
  \end{cases}
  \quad \Rightarrow \\
  \begin{cases}
    m_5 = 0, \\
    m_6 = 0, \\
    m_7 = 0.
  \end{cases}
\end{align*}
\]
Therefore, $\varphi = 0$.

Subcase 2.2: $\Phi(b) = p - 1$. By Proposition 4.1 (1), (2) and (4), we may assume $\varphi$:
\[
\begin{align*}
  e_{13} &\mapsto m_5 \gamma \otimes v_0, \\
  e_{24} &\mapsto m_6 \gamma \otimes v_2, \\
  e_{14} + e_{23} &\mapsto m_7 \gamma \otimes v_1.
\end{align*}
\]
Obviously, $|\varphi| = \bar{0}$. These equations are obtained by the definition of derivation,
\[
\begin{align*}
  \begin{cases}
    \varphi([\alpha, e_{24}]) &= \alpha \varphi(e_{24}), \\
    \varphi([\alpha, e_{14} + e_{23}]) &= \alpha \varphi(e_{14} + e_{23}).
  \end{cases}
\end{align*}
\]
It follows that
\[
\begin{align*}
  \begin{cases}
    -m_7 \gamma \otimes v_1 = 2m_6 \gamma \otimes v_1, \\
    -2m_5 \gamma \otimes v_0 = 2m_7 \gamma \otimes v_0.
  \end{cases}
  \quad \Rightarrow \\
  \begin{cases}
    m_5 = 2m_6, \\
    m_7 = -2m_6.
  \end{cases}
\end{align*}
\]
From Lemma 5.2 it is easy to see that $\varphi = m_6 \varphi_4$, and $\varphi$ is an outer derivation when $m_6 \neq 0$.

Subcase 2.3: $\Phi(b) = p - 3$. By Proposition 4.1 (1) and (4), we may suppose $\varphi$:
\[
\begin{align*}
  e_{13} &\mapsto m_5 \gamma \otimes v_{p-2}, \\
  e_{24} &\mapsto m_6 \gamma \otimes v_0.
\end{align*}
\]
Obviously, $|\varphi| = 0$. We obtain the following equations by definition of derivation:

$$
\begin{align*}
\varphi([\alpha, e_{13}]) &= \alpha \varphi(e_{13}), \\
\varphi([\beta, e_{24}]) &= \beta \varphi(e_{24}).
\end{align*}
$$

It implies that

$$
\begin{align*}
\begin{cases}
m_5(p - 2) \gamma \otimes v_{p-3} = 0, \\
m_6 \gamma \otimes v_1 = 0.
\end{cases}
\end{align*}
\Rightarrow
\begin{align*}
m_5 = 0, \\
m_6 = 0.
\end{align*}
$$

Consequently, $\varphi = 0$.

Subcase 2.4: $\Phi(b) = p - 2$. By Proposition 4.1 (2), we may assume $\varphi$:

$$
e_{14} + e_{23} \mapsto m_7 \gamma \otimes v_0.
$$

Obviously, $|\varphi| = 0$. We have

$$
-m_7 \gamma \otimes v_0 = \varphi([\beta, e_{13}]) = 0.
$$

Comparing the coefficients gives $m_7 = 0$, so $\varphi = 0$.

Case 3: $a + b = 0$. The following conclusions can be drawn.

Subcase 3.1: $\Phi(b) \in 1, \frac{b-1}{2}$. By Proposition 4.1 (3) and (5)-(7), we may assume $\varphi$:

$$
h_1 \mapsto m_1 \otimes v_{\Phi(b)}, \\
h_2 \mapsto m_2 \otimes v_{\Phi(b)}, \\
\alpha \mapsto m_3 \otimes v_{\Phi(b-1)}, \\
\beta \mapsto m_4 \otimes v_{\Phi(b+1)}, \\
\gamma \mapsto m_8 \gamma \otimes v_{\Phi(b)}.
$$

Obviously, $|\varphi| = 0$. According to Lemma 5.1 and the definition of derivation we have the following equations:

$$
\begin{align*}
\begin{cases}
\alpha \varphi(h_i) = m_i \alpha(1 \otimes v_{\Phi(b)}) = 0, & i = 1, 2, \\
\varphi([e_{13}, \gamma]) = e_{13} \varphi(\gamma), \\
\varphi([e_{24}, \gamma]) = e_{24} \varphi(\gamma).
\end{cases}
\end{align*}
$$

Then,

$$
\begin{align*}
\begin{cases}
m_i b(b + 1) \otimes v_{\Phi(b-1)} = 0, & i = 1, 2, \\
m_3 \otimes v_{\Phi(b-1)} = m_8 b(b + 1) \otimes v_{\Phi(b-1)}, \\
-a_4 \otimes v_{\Phi(b+1)} = -m_8 \otimes v_{\Phi(b+1)},
\end{cases}
\Rightarrow
\begin{align*}
m_1 = m_2 = 0, \\
m_3 = b(b + 1)m_8, \\
m_4 = m_8.
\end{align*}
$$
Easy to verify $\varphi = m_8 \mathcal{D}_{1 \otimes v_{\Phi(b)}}$, that is, $\varphi$ is inner.

Subcase 3.2: $\Phi(b) = 0$. By Proposition 4.1 (6) and (7), we may suppose $\varphi$:

\[
\begin{align*}
    h_1 &\mapsto m_1 \otimes v_0, \\
    h_2 &\mapsto m_2 \otimes v_0, \\
    \gamma &\mapsto m_8 \gamma \otimes v_0.
\end{align*}
\]

Obviously, $|\varphi| = 0$. Thus the following equation holds from the Lemma 5.1:

\[
\gamma \varphi(h_i) = m_i \gamma \otimes v_0 = 0, \quad i = 1, 2.
\]

Comparing the coefficients gives $m_1 = m_2 = 0$. By calculation we find that $m_8$ is arbitrary, so it is easy to verify $\varphi = m_8 \mathcal{D}_{1 \otimes v_0}$, that is, $\varphi$ is inner.

Subcase 3.3: $\Phi(b) = p - 1$. By Proposition 4.1 (3) and (5), we may suppose $\varphi$:

\[
\begin{align*}
    \alpha &\mapsto m_3 \otimes v_{p-2}, \\
    \beta &\mapsto m_4 \otimes v_0.
\end{align*}
\]

Obviously, $|\varphi| = 0$. We obtain the following equations by definition of derivation:

\[
m_3 \otimes v_{p-2} = \varphi([e_{13}, \gamma]) = 0, \\
-m_4 \otimes v_0 = \varphi([e_{24}, \gamma]) = 0.
\]

Hence $m_3 = m_4 = 0$, and $\varphi = 0$.

Case 4: $a + b = 2$ and $\Phi(b) \in \frac{p+1}{2}$. By Proposition 4.1 (6), we may suppose $\varphi$:

\[
\gamma \mapsto m_8 \otimes v_{\Phi(b-1)}.
\]

Obviously, $|\varphi| = 1$. We get the equation

\[
0 = \varphi([\gamma, \gamma]) = -2 \varphi(\gamma) = -2m_8 \gamma \otimes v_{\Phi(b-1)} = -2m_8 \gamma \otimes v_{\Phi(b-1)}.
\]

Therefore, $m_8 = 0$, and $\varphi = 0$.

In summary, we get this proposition by the definition of the first cohomology. □

**Theorem 5.1.**

\[
\dim(H^1(\mathfrak{g}, K(\lambda))) = \begin{cases} 
2, & \text{if } a + b = -2 \text{ and } \Phi(b) = p - 2, \\
1, & \text{if } a + b = -2 \text{ or } -4 \text{ and } \Phi(b) = p - 1, \\
0, & \text{otherwise},
\end{cases}
\]

where $a, b \in \mathbb{F}_p$, $\Phi(b) \in \{0, 1, \ldots, p - 1\}$. 

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Proof. By Proposition [6.1] it suffices to prove $\varphi_1$ and $\varphi_2$ are linearly independent modulo the inner derivation space $\text{Ider}(g, K(\lambda))$ when $a + b = -2$ and $\Phi(b) = p - 2$. Suppose that $0 \neq t \in K(\lambda)$, such that $\varphi_1$ and $\varphi_2$ are linearly dependent modulo $\text{Ider}(g, K(\lambda), K(\lambda))$. Because $\varphi_1$ and $\varphi_2$ are weight-derivations, then $t \in K(\lambda, 0)$. By Proposition 4.1 (7), we know $t = 0$, contradictorily. Other cases, it is obvious from Proposition 5.1. □

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