Phase diagram and symmetry breaking of SU(4) spin-orbital chain in a generalized external field

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Abstract

The ground state phases of a one-dimensional SU(4) spin-orbital Hamiltonian in a generalized external field are studied on the basis of Bethe-ansatz solution. Introducing three Landé g factors for spin, orbital and their products in the SU(4) Zeeman term, we discuss systematically the various symmetry breaking. The magnetization versus external field are obtained by solving Bethe-ansatz equations numerically. The phase diagrams corresponding to distinct residual symmetries are given by means of both numerical and analytical methods.

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I. INTRODUCTION

There have been much interest in the study of spin models with orbital degeneracy due to experimental progress related to many transition-metal and rare-earth compounds such as LaMnO$_3$, CeB$_6$, and perovskite lattice, as in KCuF$_3$. Those systems involve orbital degree of freedom in addition to spin ones. Almost three decades ago, Kugel and Khomskii had pointed out the possibility of orbital excitations in these systems. As a model system, it exhibits some fascinating physical features which is lack without orbital degree of freedom. The isotropic case of spin system with orbital degeneracy was shown to have an enlarged SU(4) symmetry, and one dimensional model is known to exactly solvable. Materials related to spin-orbital systems in one dimension include quasi-one-dimensional tetrahis-dimethylamino-ethylene (TDAE)-C$_{60}$, artificial quantum dot arrays, and degenerate chains in Na$_2$Ti$_2$Sb$_2$O and Na$_2$V$_2$O$_5$ compounds. It is therefore worthwhile to systematically study the features of one dimensional model. Theoretical studies has found the strong interplay of orbital and spin degrees of freedom in the excitations spectra. It has been noticed that the presence of orbital may results in various interesting magnetic properties. Applying a conventional magnetic field, the spin orbital chain with SU(4) symmetry is shown to reduce to a model with orbital SU(2) symmetry in the ground state. Recently, we showed that the magnetization process becomes more complicated if taking account of the contribution from orbital sector. We have explained that the competition between spin and orbital degree of freedom leads to an orbital anti-polarization phase. However, the external field we introduced in ref is not the most general one for SU(4) systems.

In this paper, we study a SU(4) spin-orbital chain in the presence of a generalized external field on the basis of its Bethe ansatz solution. Our paper is organized as follows. In Sec. we introduce the Bethe-ansatz solution and the Zeeman like term which is going to be added to the original SU(4) Hamiltonian. In Sec. we give some useful remarks on the quantum number configurations for the ground state in the presence of external field that is characterized by three parameters. We also simply demonstrate the thermodynamic limit of the Bethe-ansatz equation and briefly present the dress energy description of ground state in the presence of external field. In Sec. we study the magnetization properties of a Hamiltonian in the regime with one-parameter symmetry breaking. In Sec. we study...
both magnetization and the phase diagram in the regimes with two-parameter symmetry breaking. Various phases and the quantum phase transitions are obtained by both numerical calculation and analytical formulation. Concerning to various phases we present detailed explanation in terms of group theory. Sec. VI gives a brief summary.

II. THE MODEL AND ITS SOLUTION

We start from the following Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{N} \left[ 2 T_j \cdot T_{j+1} + \frac{1}{2} \right] \left( 2 S_j \cdot S_{j+1} + \frac{1}{2} \right) - 1 \right].$$

(1)

where $S_j$ and $T_j$ denote respectively spin and orbital operators at site $j$, both are generators of SU(2) group characterizing the spin and orbital degree of freedom of outer shell electrons in some transitional metal oxides at the insulating regime. The coupling constant is set to unit for simplicity. It has been pointed that the above Hamiltonian possesses an enlarged SU(4) symmetry\[1\] rather than SU(2)×SU(2) symmetry.

The four states that carry out the fundamental representation of SU(4) group is denoted by

$$|\uparrow\rangle = |1/2, -1/2\rangle, \quad |\bar{\uparrow}\rangle = |1/2, 1/2\rangle,$$

$$|\downarrow\rangle = |-1/2, 1/2\rangle, \quad |\bar{\downarrow}\rangle = |-1/2, -1/2\rangle$$

(2)

These bases are labelled by the eigenvalues of $S^z$ and $T^z$, \textit{i.e.}, $|S^z, T^z\rangle$. As the $su(4)$ Lie algebra is of rank 3, there exists third generator $2S^z T^z$ which possesses simultaneous eigenvalue together with $S^z$ and $T^z$. For convenience, we denote this new generator by $U^z$ hereafter. In the terminology of group theory, however, the quadruplet can also be labelled by the weight vectors which is defined by eigenvalues of $O^z_1, O^z_2, O^z_3$ that constitute the Cartan subalgebra of $su(4)$ Lie algebra. Here we adopt the Chevalley basis because the physical quantities can be conveniently expressed in this basis.

The eigenvalues of $S^z, T^z, U^z$ as well as that of $O^z_1, O^z_2, O^z_3$ are given in Table I, the relation between these two basis reads\[2,\]

$$S^z = O^z_1 + 2O^z_2 + O^z_3,$$

$$T^z = O^z_1 + O^z_3,$$

$$U^z = O^z_1 - O^z_3.$$

(3)
TABLE I: The eigenvalue of $S^z$, $T^z$, $U^z$ and $z$-component of $O_1^z, O_2^z, O_3^z$ for the four basis states (Eq. 2).

| State | $S^z$ | $T^z$ | $U^z$ | $O_1^z$ | $O_2^z$ | $O_3^z$ |
|-------|-------|-------|-------|--------|--------|--------|
| $|↑\rangle$ | 1/2   | 1/2   | 1/2   | 0      | 0      |        |
| $|\uparrow\rangle$ | 1/2   | -1/2  | -1/2  | -1/2   | 1/2    | 0      |
| $|\downarrow\rangle$ | -1/2  | 1/2   | -1/2  | 0      | -1/2   | 1/2    |
| $|\downarrow\rangle$ | -1/2  | -1/2  | 1/2   | 0      | 0      | -1/2   |

The present model has been solved by Bethe-ansatz method, its energy spectrum is given by

$$E_0(M, M', M'') = -\sum_{a=1}^{M} \frac{1}{1/4 + \lambda_a^2},$$

where the $\lambda$'s are solutions of the following coupled transcendental equations

$$2\pi I_a = N\theta_{-1/2}(\lambda_a) + \sum_{a'=1}^{M'} \theta_1(\lambda_a - \lambda_{a'})$$

$$+ \sum_{b=1}^{M'} \theta_{-1/2}(\lambda_a - \mu_b),$$

$$2\pi J_b = \sum_{a=1}^{M} \theta_{-1/2}(\mu_b - \lambda_a) + \sum_{b'=1}^{M'} \theta_1(\mu_b - \mu_{b'})$$

$$+ \sum_{c=1}^{M''} \theta_{-1/2}(\mu_b - \nu_c),$$

$$2\pi K_c = \sum_{b=1}^{M'} \theta_{-1/2}(\nu_c - \mu_b) + \sum_{c'=1}^{M''} \theta_1(\nu_c - \nu_{c'}),$$

where $\theta_a(x) = -2\tan^{-1}(x/\alpha)$. The $\lambda, \mu$ and $\nu$ are rapidities related to the three generators of the Cartan subalgebra of the $su(4)$ Lie algebra. The quantum numbers $\{I_a, J_b, K_c\}$ specify a state in which there are $N - M$ number of sites in the state $|\uparrow\rangle$, $M - M'$ in $|\uparrow\rangle$, $M' - M''$ in $|\downarrow\rangle$, and $M''$ in $|\downarrow\rangle$. Hence the $z$-component of total spin, orbital and $U^z$ are obtained as $S^z_{\text{tot}} = N/2 - M'$, $T^z_{\text{tot}} = N/2 - M + M' - M''$, and $U^z_{\text{tot}} = N/2 - M + M''$.

In present SU(4) model, a three-parameter external field $(h_1, h_2, h_3)$ can be introduced
to write out a most general Zeeman-like energy.

\[ H_{\text{zee}} = \sum_{m} h_m O_{m}^{z}. \]  

(6)

For more clear physics implication, we re-choose the parameters to write the effective magnetization \( M_z \),

\[ M^z = g_s S_{\text{tot}}^z + g_t T_{\text{tot}}^z + g_u U_{\text{tot}}^z. \]  

(7)

where \( g_s, g_t, g_u \) are generalized Landé g factors for \( S^z, T^z \) and \( U^z \) respectively. Eq. (7) can be expressed in terms of the number of rapidities,

\[ M^z = \frac{N}{2}(g_s + g_t + g_u) - M(g_t + g_u) \]
\[ -M'(g_s - g_t) - M''(g_t - g_u), \]  

(8)

Because the Zeeman-like term commutes with the SU(4) Hamiltonian (1), the energy spectrum in the presence of external field are simply related to the energy spectrum in the absence of external field,

\[ E(h, M, M', M'') = E_0(M, M', M'') - hM^z, \]  

(9)

where \( E_0(M, M', M'') \) is determined by eqs. (5). Obviously, the application of the external field with different magnitude just brings about various level crossings.

In terms of \( O_1, O_2, O_3 \), the magnetization (7) becomes

\[ M^z = (g_s + g_t + g_u)O_1^z + 2g_s O_2^z + (g_s + g_t - g_u)O_3^z. \]  

(10)

which breaks SU(4) symmetry down to various lower symmetries depending on the distinct regions in the parameter space.

III. THE GROUND STATE CONFIGURATION

Based on the Bethe ansatz solution of the model, we first give the quantum number description of the ground state, which is useful for numerical method. We also give the dress energy description for the ground state and propose the conditions to determine quantum phase transitions, which is useful for analytic study.
It has been known \cite{1,2} that the ground state of the Hamiltonian is a SU(4) singlet for the case of $N = 4n$. The configuration of the quantum number for ground state $\{I_a, J_b, K_c\}$ ($a = 1, 2, ..., 3n; b = 1, 2, ..., 2n; c = 1, 2, ..., n$) are consecutive integers (or half integers) arranging symmetrically around the zero. In the presence of magnetic field, however, the Zeeman term brings about level crossings and the state with $M = 3n, M' = 2n, M'' = n$ is no longer the ground state. Therefore the numbers $M, M', M''$ for the lowest energy state are related to the magnitude of the applied external field.

In order to solve the Bethe ansatz equation numerically, we need to determine the possible configuration of quantum numbers for give values $M, M'$ and $M''$. The property of Young tableau requires that Max($M$) = $3N/4$, Max($M'$) = $N/2$, Max($M''$) = $N/4$, and $N - M \geq M - M' \geq M' - M'' \geq M''$ for a given $N$. Then one is able to analyze the change of energy level for each state, which determines the true ground state for a given external field. One can also calculates the magnetization by Eq. \ref{eq:magnetization}.

In the thermodynamic limit, the energy is expressed in terms of densities of the rapidities,

$$E/N = -\frac{\hbar}{2}(g_s + g_t + g_u)$$

$$+ \int_{-\lambda_0}^{\lambda_0} \sigma(\lambda)[-2\pi K_{1/2}(\lambda) + (g_t + g_u)h]d\lambda$$

$$+ (g_s - g_t)h \int_{-\mu_0}^{\mu_0} \omega(\mu)d\mu$$

$$+ (g_t - g_u)h \int_{-\nu_0}^{\nu_0} \tau(\nu)d\nu$$

\hfill (11)
These densities satisfy the following coupled integral equations

\[
\sigma(\lambda) = K_{1/2}(\lambda) - \int_{-\lambda_0}^{\lambda_0} K_1(\lambda - \lambda')\sigma(\lambda')d\lambda' \\
+ \int_{-\mu_0}^{\mu_0} K_{1/2}(\lambda - \mu)\omega(\mu)d\mu,
\]
\[
\omega(\mu) = \int_{-\lambda_0}^{\lambda_0} K_{1/2}(\mu - \lambda)\sigma(\lambda)d\lambda \\
- \int_{-\mu_0}^{\mu_0} K_1(\mu - \mu')\omega(\mu')d\mu' \\
+ \int_{-\nu_0}^{\nu_0} K_{1/2}(\mu - \nu)\tau(\nu)d\nu,
\]
\[
\tau(\nu) = \int_{-\mu_0}^{\mu_0} K_{1/2}(\nu - \mu)\omega(\mu)d\mu \\
- \int_{-\nu_0}^{\nu_0} K_1(\nu - \nu')\tau(\nu')d\nu'.
\tag{12}
\]

where \(K_n(x) = \pi^{-1}n/(n^2 + x^2)\) and \(\lambda_0, \mu_0\) and \(\nu_0\) are determined by

\[
\int_{-\lambda_0}^{\lambda_0} \sigma(\lambda) = \frac{M}{N}, \\
\int_{-\mu_0}^{\mu_0} \omega(\mu) = \frac{M'}{N}, \\
\int_{-\nu_0}^{\nu_0} \tau(\nu) = \frac{M''}{N}.
\tag{13}
\]

It is more convenient to introduce dress energy \[16\]. The iteration of eq. (12) gives rise to

\[
\varepsilon(\lambda) = -2\pi K_{1/2}(\lambda) + (g_t + g_u)h \\
- \int_{-\lambda_0}^{\lambda_0} K_1(\lambda - \lambda')\varepsilon(\lambda')d\lambda' \\
+ \int_{-\mu_0}^{\mu_0} K_{1/2}(\lambda - \mu)\zeta(\mu)d\mu,
\]
\[
\zeta(\mu) = (g_s - g_t)h + \int_{-\lambda_0}^{\lambda_0} K_{1/2}(\mu - \lambda)\varepsilon(\lambda)d\lambda \\
- \int_{-\mu_0}^{\mu_0} K_1(\mu - \mu')\zeta(\mu')d\mu' \\
+ \int_{-\nu_0}^{\nu_0} K_{1/2}(\mu - \nu)\xi(\nu)d\nu,
\]
\[
\xi(\nu) = (g_t - g_u)h + \int_{-\mu_0}^{\mu_0} K_{1/2}(\nu - \mu)\zeta(\mu)d\mu \\
- \int_{-\nu_0}^{\nu_0} K_1(\nu - \nu')\xi(\nu')d\nu'.
\tag{14}
\]
where $\varepsilon, \zeta,$ and $\xi$ are the dress energies in $\lambda, \mu$ and $\nu$ sectors respectively. It is worthwhile to point out that the dress energy is also the thermal potentials at zero temperature, i.e., 
\[
\exp(\varepsilon/T) = \rho^h/\rho, \quad \exp(\zeta/T) = \sigma^h/\sigma, \quad \text{and} \quad \exp(\xi/T) = \omega^h/\omega.
\]
The thermal Bethe-ansatz equation in the zero-temperature limit turns to eq. (14). In terms of dress energy, the energy (13) is simplified to
\[
E/N = -\frac{\hbar}{2}(g_s + g_t + g_u) + \int_{-\lambda_0}^{\lambda_0} K_{1/2}(\lambda)\varepsilon(\lambda)d\lambda. \quad (15)
\]
Apparently, the ground state is a quasi-Dirac sea where the states of negative dress energy, $\varepsilon(\lambda) < 0$, $\zeta(\mu) < 0$, $\xi(\nu) = 0$, are fully occupied. The Fermi points of the three rapidities are determined by
\[
\varepsilon(\lambda_0) = 0, \quad \zeta(\mu_0) = 0, \quad \xi(\nu_0) = 0. \quad (16)
\]
The system will be magnetized if the applied field enhance the dress energy, because it makes the corresponding Fermi points decline. The quantum phase transition occurs when any of the Fermi points shrinks to zero. As a result, the critical values of the external field are solved by
\[
\varepsilon(0)|_{h = h_c} = 0, \quad \zeta(0)|_{h = h'_c} = 0, \quad \xi(0)|_{h = h''_c} = 0 \quad (17)
\]
These conditions together with eq. (14) enable one to calculate those critical values.

IV. REGIMES WITH ONE-PARAMETER SYMMETRY BREAKING

The application of external field makes the SU(4) symmetry break down to various regimes with different residual symmetry. In this section, we shall discuss the simplest cases of single parameter hierarchy. There are three special directions in the weight space of SU(4). If the external field is supplied along those directions, i.e., either $h_1$, $h_2$ and $h_3$ in eq. (6) is not vanished, a partial breaking of a SU(2) to U(1) will take place. Let us consider them respectively.
A. Residual SU(3)×U(1) symmetry

If $g_s = 0, g_t = -g_u > 0$, the Zeeman interaction becomes

$$\mathcal{M}^z = 2g_t O_3^z = g_t M' - 2g_t M''.$$  \hspace{1cm} (18)

The occurrence of operator $O_3^z$ makes the Hamiltonian noncommutable with $O_3^\pm$. Thus a SU(2) subgroup generated by $O_3^z$, $O_3^+$, $O_3^-$ is broken down to U(1). Analyzing the level crossing from Eq. (9), we shown the magnetization curve in Fig. 1.

Because the terms $g_t + g_u$ and $g_s - g_t$ in the first two equations of eqs. (14) are non-positive, the external field can not enhance the two dress energies $\varepsilon(\lambda)$ and $\zeta(\mu)$, the Fermi points in both $\lambda$ and $\mu$ sectors are fixed. On the contrary, the Zeeman term has a positive contribution to $\xi(\nu)$, and its two Fermi points will decline when the external field increases. Although it is a SU(4) singlet labelled by Young tableau $[n^4]$ in the absence of external field, the ground state possess a residual SU(3)×U(1) symmetry in the presence of the aforementioned one parameter external field at small magnitude, which corresponds to phase IV labelled by four-row Young tableau. In this regime are there still three type of rapidities that solve the Bethe-ansatz equation. The U(1) is generated by $O_3^z$, while the SU(3) is generated by the following eight operators

$$O_1^z = \frac{1}{2}(T^z + U^z), \quad O_2^z = \frac{1}{2}(S^z - T^z),$$

$$O_1^+ = (\frac{1}{2} + S^z)T^+, \quad O_2^+ = S^+T^-, \quad O_1^- = (\frac{1}{2} + S^z)T^-, \quad O_2^- = S^-T^+,$$

$$O_{1+2}^+ = S^+(\frac{1}{2} + T^z), \quad O_{1+2}^- = S^-(\frac{1}{2} + T^z).$$  \hspace{1cm} (19)

There exists a critical field when those two Fermi points shrink to zero, the rapidity $\nu$ disappears in the Bethe-ansatz equation. Thus a quantum phase transition occurs at the critical field which separate two phases, we call phase IV and phase III.

The magnetization process can be clearly illustrated by the evolution of Young tableau,
which shows the evolution of Young tableau in magnetization process, i.e., from SU(4) singlet to SU(3)×U(1) states and then to a SU(3) singlet. Physically, we have $M^z/N = 0$ at zero external field due to there are a quart of the total sites being respectively in the states $|\uparrow\rangle$, $|\uparrow\rangle$, $|\downarrow\rangle$ and $|\downarrow\rangle$. Turning on the external field leads to the spin-orbital flipping, $|\downarrow\rangle \rightarrow |\uparrow\rangle$, $|\downarrow\rangle \rightarrow |\uparrow\rangle$ and $|\downarrow\rangle \rightarrow |\downarrow\rangle$, which result in non-vanishing magnetization. The SU(3)×U(1) symmetry makes the above three flipping processes favor to occur simultaneously. When the external field succeeds a critical value, it goes into phase III where all the states $|\downarrow\rangle$ have been flipped over. In this phase the $z$-component of total spin and total orbital keep positive constant $S^z/N = T^z/N = 1/6$ while that of $U^z$ keeps a negative constant $U^z/N = -1/6$. Consequently, the magnetization reaches a saturate value $M^z/N = 2/3$ and the ground state becomes the SU(3) singlet regardless of the magnitude of the external field in phase III.

B. Residual SU(2)×SU(2) symmetry

Applying the external field along the direction of the second simple root of $su(4)$ Lie algebra, we will have a symmetry breaking from SU(4) to SU(2)×U(1)×SU(2) for ground state. This is realized by the choice of Landé g factors $g_u = 0, g_s = -g_t$, which makes the magnetization to be

$$M^z = 2g_s O^z_2$$

$$= g_s M - 2g_s M' + g_s M''.$$ (20)

Those two SU(2) are generated respectively by

$$\left\{ \frac{1}{2}(T^z + U^z), \left( \frac{1}{2} + S^z \right) T^\pm \right\},$$

$$\left\{ \frac{1}{2}(T^z - U^z), \left( \frac{1}{2} - S^z \right) T^\pm \right\}. \quad (21)$$

As $g_s - g_t$ in the second equation of (14) is positive but both $g_t + g_u$ and $g_t - g_u$ in the first and third equation are negative, the external field makes the Fermi points in $\mu$ sector to shrink. The critical value of the external field when quantum phase transition occurs is determined by $\zeta(0)|_{h_c} = 0$. This critical point separates two different phases, we call phase IV and phase II.
The magnetization curve is shown in Fig. 1. It can also be illustrated by the evolution of the Young tableau,

![Young Tableau Diagram]

The spin-orbital flipping process caused by the applied external field has several characteristics. During the flipping process, $|\downarrow\rangle$ and $|\downarrow\rangle$ flips simultaneously into $|\uparrow\rangle$ and $|\uparrow\rangle$ pairs, which makes four-row Young tableau reduce to the two-row Young tableau when across the critical field. Apparently, the eigenvalues of both $T^z$ and $U^z$ do not change during the magnetization process. Only $S^z$ contributes to the magnetization $M^z$. The total spin is completely polarized (i.e., $M^z$ is saturated) once the phase IV transits to phase II. The phase that Yamashita et al. discussed agrees with this special case.

C. Residual $U(1) \times SU(3)$ symmetry

If $g_s = 0$ and $g_t = g_u$, the magnetization becomes

\[
\mathcal{M}^z = 2g_t O^z_f \\
= g_t(N - 2M + M').
\]

which implies that the external field was applied along the first simple root of $su(4)$ Lie algebra. This gives rise to a symmetry breaking down to $U(1) \times SU(3)$ for ground state. For the sake of saving space, we omitted the operators that generate those symmetry. Because this parameter choice implies that $g_t + g_u$ in $\lambda$ sector positive but the $g$ factor terms in both $\mu$ and $\nu$ sectors are non-positive, the quantum phase transition is only related to $\lambda$ sector. The critical value is determined by $\varepsilon(0)|_{\text{hc}} = 0$. This critical point separates the system into two phases, phase IV and phase I. The magnetization process is shown in Fig. 1.

It is helpful to illustrate this process by the evolution of Young tableau,

![Young Tableau Diagram]

From Fig 1 we see that the spin, orbital as well as $U^z$ are polarized simultaneously versus external field. The above Young tableau indicates that the states $|\uparrow\rangle, |\downarrow\rangle, |\downarrow\rangle$ change to the
state $|↑⟩$ simultaneously because they carries out a SU(3) representation and the system possess SU(3) symmetry. After the system is fully polarized, the magnetization reach the maximum value $M^z/N = 1/2$. Then the residual symmetry of the ground state is only of U(1).

V. REGIMES WITH TWO-PARAMETER SYMMETRY BREAKING

In the previous section, we discussed the simplest case where merely one SU(2) subgroup symmetry is broken. In the following, we will consider the regimes with two-parameter symmetry breaking, which involves more SU(2) subgroup.

A. Residual $U(1) \times U(1) \times SU(2)$ symmetry

Under the restriction $g_u = g_s + g_t$, the magnetization becomes,

$$M^z = 2(g_s + g_t)O_1^z + 2g_sO_2^z$$

which indicates that the residual symmetry of ground state is $U(1) \times U(1) \times SU(2)$, in which the SU(2) is generated by $O_3^z = (T^z - U^z)/2$ and $O_3^\pm = T^\pm (1/2 - S^z)$.

The magnetization curves for different $g_t/g_s$ is plotted in Fig. 2 and the phase diagram in terms of $g_t/g_s$ versus $h$ is given in Fig. 3. On the one hand, because the eigenvalue of $T^z$ equals that of $U^z$ for both states $|↑⟩$ and $|↑⟩$, the flipping process occurred in phase II contributes to both magnetization of $T^z$ and $U^z$ equivalently. On the other hand, the flipping from $|↓⟩$ and $|↓⟩$ occurs simultaneously in the phase IV due to the SU(2) symmetry. As a results, the magnetization of $T^z$ and $U^z$ are expected to be the same in the whole process, which can be seen from our numerical calculation in Fig. 2.

For $g_t/g_s < 0.5$, there exists three distinct phases, denoted by IV, II and I respectively according to the number of the rows of Young tableau. The SU(2) symmetry makes the states $|↓⟩$ and $|↓⟩$ flip simultaneously when the external field increases. This makes the four-row Young tableau turn to two-row Young tableau directly, hence the phase III labelled by three-row Young tableau will not take place. The boundary between phase IV and phase II is determined from $\zeta(0) = 0$ and $\xi(0) = 0$ together,

$$g_t/g_s = \frac{1}{2} + \frac{1}{2h}K_{1/2}(0) * \varepsilon(0),$$

where
where $\varepsilon$ can be computed from eqs. (14) numerically. For sufficient large external field, all $S$, $T$ and $U$ are frozen to the z-direction, which brings about the occurrence of phase I. The boundary between phase I and phase II is determined by $\varepsilon(0) = 0$, i.e.,

$$
g_t/g_s = \frac{1}{h} - \frac{1}{2}.  \tag{25}$$

This can also be derived from the competition between the states related to the Young tableau $[N - 1, 1]$ and $[N]$.

The asymptotic behavior at large $h$ is $g_t/g_s = -1/2$, which implies that the phase I will never occur as long as $g_t/g_s < -1/2$. The magnetization process in the region $-1/2 < g_t/g_s < 1/2$ can be illustrated by the following evolution of Young tableau,

![Young tableau evolution](image)

The boundary between phase IV and phase I is determined by $\varepsilon(0) = 0$, $\zeta(0) = 0$ and $\xi(0)$ together,

$$
g_t/g_s = \frac{3}{2h} - 1.  \tag{26}$$

The common solution of Eqs. (24-26) gives $h = 1$ and $g_t/g_s = 1/2$, which is a three-phase-coexist point.

### B. Residual U(1) × SU(2) × U(1) symmetry

If the external field along the direction of the second simple root is quenched but those along the other directions are kept, we will have symmetry breaking down to $U(1) \times SU(2) \times U(1)$. Such kind of symmetry breaking is caused by Zeeman term of the following magnetization

$$
\mathcal{M}^z = (g_t + g_u)O_1 + (g_t - g_u)O_3. \tag{27}
$$

The magnetization curves with different $g_u$ is plotted in Fig. and the phase diagram in terms of $g_u/g_t$ versus $h$ is given in Fig. For $g_u/g_t < 1/2$, there exists three phases denoted by IV, III and I respectively.
The boundary between phases IV and III is determined from $\xi = 0$, which can be solved numerically. For sufficient large external field, the magnetization is saturated reaching phase I. This phase transition occurs at

$$g_u/g_t = \frac{2}{h} - \frac{1}{2}$$

(28)

Thus if $g_u/g_t < -1/2$, the phase I will never occur regardless of the magnitude of external field. So in the region $-1/2 < g_u/g_t < 1/2$, the magnetization process can be illustrated by Young tableau

![Young tableau](image)

In phase III, the length of the second row and that of the third row in their corresponding Young tableau is always equal due to the SU(2) symmetry. Thus the probability of pure spin-flipping $|\downarrow\rangle \rightarrow |\uparrow\rangle$ and pure orbital-flipping $|\downarrow\rangle \rightarrow |\downarrow\rangle$ is the same. Additionally, the process $|\uparrow\rangle \rightarrow |\downarrow\rangle$ contributes the same for the magnetization of $S^z$ and $T^z$. These properties result in the same magnetization curves of $S^z$ and $T^z$ shown in Fig. 4. Flipping over the state $|\downarrow\rangle$ which has positive eigenvalue of $U^z$ brings about a negative magnetization of $U^z$ in phase IV.

The phase I will never occur for $g_u/g_t < -1/2$, as is similar to the case of SU(3)$\times$U(1). Actually, it recovers the case of residual SU(3)$\times$U(1) symmetry at $g_u/g_t = -1$.

If $g_u/g_t > 1/2$, we can see from Fig. 5 that the phase IV transits into phase I directly along with the increase of external field. The boundary which separates these two phase is determined by

$$g_u/g_t = \frac{3}{h} - 1.$$  

(29)

Obviously, the point at $g_u/g_t = 0.5$ and $h = 2$ is a three-phase-coexist point. The magnetization properties in the region of $g_u/g_t > 1/2$ is similar to the case of U(1)$\times$SU(3), particularly, a larger U(1)$\times$SU(3) symmetry remains for $g_u/g_t = 1$. 
C. Residual SU(2) × U(1) × U(1) symmetry

The third case of two-parameter symmetry breaking is produced by the Zeeman term with the restriction to the the Landé g factor \( g_u = -g_s - g_t \). The magnetization now reads

\[
\mathcal{M}^z = 2g_s O^z_2 + 2(g_s + g_t) O^z_3.
\]

which breaks the SU(4) symmetry down to SU(2) × U(1) × U(1).

The magnetization curves for different \( g_t/g_s \) is plotted in Fig. 6 and phase diagram in terms of \( g_t/g_s \) versus \( h \) is given in Fig. 7. In the region of \( g_t/g_s > -1/2 \), there exist three phases denoted by IV, III, and II respectively. The final state is characterized by two rows Young tableau at sufficient large external field due to the SU(2) symmetry. The boundary separates phase III and II is determined by \( \zeta(0) = 0 \), which reads

\[
g_t/g_s = \frac{1}{2} - \frac{\ln 2}{h}.
\]

Obviously if \( g_t/g_s > 1/2 \), the phase II will never occur regardless the magnitude of external field, and the magnetization process is similar to the case of SU(3) × U(1).

In the region \(-1/2 < g_t/g_s < 1/2\), the magnetization process can be illustrated by the following Young tableau,

\[
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
\bullet & \bullet & \bullet & \bullet \\
\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\]

Fig. 6 shows that the magnetization process of \( S^z, T^z \) and \( U^z \) are quite different. In phase IV, the flippings from \( |\downarrow\rangle \) to other three states possess positive contribution to \( S^z \) and \( T^z \), but negative contribution to \( U^z \). In phase III, \( S^z \) undergoes polarization continually, while \( U^z \) undergoes polarization but \( T^z \) undergoes anti-polarization. Since in the final phase, both eigenvalues are zero and the spin magnetization are saturated, this phase will not change for any further increase of the external field.

The point of \( g_t/g_s = 0.5, \ h = \ln 2 \) is a three phase coexistence point. It can be seen that there exist two phases merely for \( g_u/g_t < -1/2 \), and phase IV transits into phase II directly. Actually, the residual symmetry of SU(2) × U(1) × SU(2) is restored when \( g_t/g_s = -1 \).
VI. BRIEF SUMMARY

In above, we studied the magnetization properties of a SU(4) spin-orbital chain in the presence of a generalized external field that has three parameters due to the Cartan subalgebra of $su(4)$ Lie algebra has three generators. Those three parameters are re-chosen so that to relate them with the spin $S$, orbital $T$ and their product $U$. We called them three Landé g factors. Then all possible symmetry breaking and the corresponding magnetization process induced by that external field are studied respectively. The ground state phase diagram caused by the competition of quantum fluctuation and Zeeman-like effect are studied by solving the Bethe-ansatz equations numerically. The phase transition boundaries derived by studying the dress energy equations analytically. The features of various phases and transitions between them are explained in detailed in terms of group theory analysis.

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[1] Y. Q. Li, M. Ma, D. N. Shi and F. C. Zhang, Phys. Rev. Lett 81, 3527 (1998).
[2] Y. Q. Li, M. Ma, D. N. Shi and F. C. Zhang, Phys. Rev. B 60, 12781 (1999).
[3] A. Joshi, M. Ma, F. Mila, D. N. Shi and F. C. Zhang, Phys. Rev. B 60, 6584 (1999).
[4] Y. Tokura and N. Nagaosa 2000 Science 288 462
[5] A. M. Oleś, L. F. Feiner and J. Zaanen, Phys. Rev. B 61, 6257 (2000).
[6] C. Itoi, S. Qin and I. Affleck, Phys. Rev. B 61, 6747 (2000).
[7] Y. L. Lee and Y. W. Lee, Phys. Rev. B 61 6765 (2000).
[8] Y. Yamashita, N. Shibata, and K. Ueda, Phys. Rev. B 61, 4012 (2000).
[9] S. J. Gu, and Y. Q. Li, Phys. Rev. B 66, 092404 (2002).
[10] K. I. Kugel and D. I. Khomskii, Zh. Eksp. Theor. Fiz. 64, 1429 (1973) [Sov. Phys. JETP 37, 725 (1973)].
[11] B. Sutherland, Phys. Rev. B 12, 3795 (1975); Phys. Rev. Lett. 20, 98 (1968).
[12] D. P. Arovas and A. Auerbach, Phys. Rev. B 52, 10114(1995).
[13] A. Onufriev and J. B. Marston, Phys. Rev. B 59, 12573 (1999).
[14] E. Axtell, T. Ozawa, S. Kauzlarich and R. R. P. Singh, J. Solid. Chem. 134, 423 (1997).
FIG. 1: The magnetization $M_z$, $S_z$, $T_z$, $U_z$ of the system in with (1) $g_s = 0, g_t = -g_u$; (2) $g_u = 0, g_s = g_t$; and (3) $g_s = 0, g_t = g_u$.

[15] S. K. Pati, R. R. P. Singh and D. I. Khomskii, Phys. Rev. Lett 81, 5406 (1998).

[16] H. Frahm, V. E. Korepin, Phys. Rev. B 42, 10553 (1990).
FIG. 2: The magnetization $M_z, S_z, T_z, U_z$ of the system with $g_s = 2, g_u = g_s + g_t$, and $g_t = 2.0, 1.5, 1.0, 0.5, 0.0, -0.5, -1.0$.

FIG. 3: The phase diagram of $g_t/g_s$ versus $h$ with $g_u = g_s + g_t$ and residual symmetry $U(1) \times U(1) \times SU(2)$.
FIG. 4: The magnetization $M_z, S_z, T_z, U_z$ of the system with $g_s = 0, g_t = 2$, and $g_u = 2.0, 1.5, 1.0, 0.5, 0.0, -0.5, -1.0$.

FIG. 5: The phase diagram of $g_u/g_t$ versus $h$ with $g_s = 0$ and residual symmetry $U(1) \times SU(2) \times U(1)$.

FIG. 6: The magnetization $M_z, S_z, T_z, U_z$ of the system with $g_s = 2, g_u = -g_s - g_t$, and $g_t = 2.0, 1.5, 1.0, 0.5, 0.0, -0.5, -1.0$. 
FIG. 7: The phase diagram of $g_t/g_s$ versus $h$ with $g_u = -g_s - g_t$. and residual symmetry $SU(2) \times U(1) \times U(1)$