SPECTRAL PROPERTIES OF 4-DIMENSIONAL
COMPACT FLAT MANIFOLDS

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ABSTRACT. We study the spectral properties of a large class of compact flat Riemannian manifolds of dimension 4, namely, those whose corresponding Bieberbach groups have the canonical lattice as translation lattice. By using the explicit expression of the heat trace of the Laplacian acting on p-forms, we determine all p-isospectral and L-isospectral pairs and we show that in this class of manifolds, isospectrality on functions and isospectrality on p-forms for all values of p are equivalent to each other. The list shows for any p, 1 ≤ p ≤ 3, many p-isospectral pairs that are not isospectral on functions and have different lengths of closed geodesics. We also determine all length isospectral pairs (i.e. with the same length multiplicities), showing that there are two weak length isospectral pairs that are not length isospectral, and many pairs, p-isospectral for all p and not length isospectral.

Introduction

If M is a compact Riemannian n-manifold, let Spec_p(M) denote the p-spectrum of M, that is, the collection of eigenvalues, counted with multiplicities, of the Laplacian acting on smooth p-forms. If Spec_p(M) = Spec_p(M'), then M and M' are said to be p-isospectral. In the function case, i.e. if p = 0, one says that M and M' are isospectral. The set of all lengths of closed geodesics of M (resp. lengths with multiplicities) is called the weak length spectrum or L-spectrum (resp. length spectrum or [L]-spectrum) of M.

In [6, 7] multiplicity formulas for the eigenvalues of the Laplacian acting on natural vector bundles over compact flat manifolds were given, together with isospectrality criteria, that allow to give a variety of examples and counterexamples. In the multiplicity formulas for eigenvalues for the Laplacian acting on p-forms, certain traces appear. For flat manifolds of diagonal type (see Section 1) these are given by integral values of Krawtchouk polynomials, hence one can use their integral zeros to construct examples of p-isospectral manifolds that are not 0-isospectral. In [8] different types of length spectra are studied and p-isospectrality is compared with length isospectrality. Some of the examples in [6, 7, 8] occur already in dimension 4 and, given that there is a full classification of the corresponding Bieberbach groups in this case ([1]), it seems natural to study the spectral properties of flat manifolds for n = 4. This is the main goal of the present paper.

It is known that if two flat manifolds are isospectral, the corresponding covering tori must be isospectral to each other ([9]). It is thus natural, when comparing spectra, to consider flat manifolds both having the same covering torus. In this paper we will restrict ourselves to manifolds covered by the standard torus, that is, the corresponding Bieberbach groups have the canonical (cubic) lattice of translations. We will be left with 54 out of the 74 diffeomorphism classes of four dimensional manifolds and we shall see that some of the diffeomorphism classes allow several different isometry classes. If we include representatives for all these isometry
classes, then we end with a total of 79 different isometry classes of Bieberbach 4-manifolds, covered by the standard 4-dimensional torus.

An outline of the paper is as follows. In Section 1 we briefly review the main facts on Bieberbach groups and some results from [6, 7, 8]. We also give a table with the values of the Krawtchouk polynomials $K^p_j(j)$ for $0 \leq p, j \leq 4$ (see (1.3)).

In Section 2 we give a full set of representatives for the isometry classes in each diffeomorphism class, and compute the ingredients for the $p$-heat traces and the lengths of closed geodesics for all Bieberbach groups in our class. At the end of the section, we use formula (1.1) to give explicit computations of multiplicities of eigenvalues in the particular case of two flat manifolds with holonomy group $D_4$. These turn out to be $p$-isospectral if and only if $p = 1, 3$. Furthermore they do not have the same lengths of closed geodesics. This is the first example with these properties for flat manifolds having nonabelian holonomy groups.

In Section 3 we give expressions for the $p$-heat traces $Z^p_\Gamma(s)$ in the context of this paper and then rewrite $Z^p_\Gamma(s)$ in terms of polynomial expressions that are simpler than the zeta functions, and still encode the same spectral information.

In the last section we give all pairs of Sunada-isospectral, $p$-isospectral and $L$-isospectral flat manifolds of dimension 4 in our context, by comparison of the zeta functions $Z^p_\Gamma(s)$. We will show that all 0-isospectral pairs are actually $L$-isospectral, $p$-isospectral for all $p$ and have the same holonomy representation. Moreover, the non-diffeomorphic pairs occur only in the class of Bieberbach manifolds of diagonal type. On the other hand, our list will reveal a large number of $p$-isospectral pairs for either $p = 1$ and $p = 3$, or for $p = 2$, that are not isospectral on functions and have different lengths of closed geodesics. Also, we shall see that there exist several $L$-isospectral pairs that are not isospectral. Finally, we take into account length multiplicities, finding all $[L]$-isospectral pairs, i.e. having the same lengths of closed geodesics and the same multiplicities for each length. The list obtained shows, already in dimension 4, that non-diffeomorphic Bieberbach manifolds that are $p$-isospectral for all $0 \leq p \leq 4$, are often not $[L]$-isospectral (4 pairs out of 9).

1. Preliminaries

We shall first recall some standard facts on flat Riemannian manifolds (see [3]). A discrete, cocompact subgroup $\Gamma$ of the isometry group of $\mathbb{R}^n$, $I(\mathbb{R}^n)$, is called a crystallographic group. If furthermore, $\Gamma$ is torsion-free, then $\Gamma$ is said to be a Bieberbach group. Such $\Gamma$ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group $\Gamma$. Any such manifold arises in this way. Any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely as $\gamma = BL_b$, with $B \in O(n)$, $b \in \mathbb{R}^n$ and $L_b$ denotes translation by $b$. The translations in $\Gamma$ form a normal maximal abelian subgroup of finite index, identified with $\Lambda$, a lattice in $\mathbb{R}^n$ which is $B$-stable for each $BL_b \in \Gamma$. The quotient $F := \Lambda \backslash \Gamma$ is called the holonomy group of $\Gamma$ and gives the linear holonomy group of the Riemannian manifold $M_\Gamma$. The action of $F$ on $\Lambda$ defines an integral representation of $F$, usually called the holonomy representation.

If $BL_b$ is in $\Gamma$, denote $n_B := \dim \ker(B - \text{Id})$. The torsion-free condition on $\Gamma$ implies that for any $BL_b \in \Gamma$ we have $n_B > 0$ and furthermore, $b_+ := p_B(b) \neq 0$, where $p_B$ denotes the orthogonal projection onto $\ker(B - \text{Id})$.

We now recall from [6, 7] some facts on the spectrum of Laplace operators on vector bundles over flat manifolds. If $\tau$ is an irreducible representation of $K = O(n)$ and $G = I(\mathbb{R}^n)$ we form the vector bundle $E_\tau$ over $G/K \simeq \mathbb{R}^n$ associated to $\tau$ and consider the corresponding bundle $\Gamma \backslash E_\tau$ over $\Gamma \backslash \mathbb{R}^n = M_\Gamma$. Let $-\Delta_\tau$ be the connection Laplacian on this bundle. For any nonnegative real number $\mu$, let $\Lambda^*_\mu = \{ \lambda \in \Lambda^* : \|\lambda\|^2 = \mu \}$, where $\Lambda^*$ denotes the dual lattice of $\Lambda$. In [7],
Theorem 2.1, it is shown that the multiplicity of the eigenvalue $4\pi^2\mu$ of $-\Delta_\tau$ is given by
\begin{equation}
\label{mult_eq}
d_{\tau,\mu}(\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in BL_b \in \Lambda \Gamma} \text{tr}(\tau(B)) e_{\mu,\gamma}
\end{equation}
where $e_{\mu,\gamma} = \sum_{v \in \Lambda^*_\mu : Bv = v} e^{-2\pi i v \cdot b}$. In the case when $\tau = \tau_p$, the $p$-exterior representation of $O(n)$, we shall write $\text{tr}_p(B)$ and $d_{p,\mu}(\Gamma)$ in place of $\text{tr}_\tau(B)$ and $d_{\tau_p,\mu}(\Gamma)$ respectively. If $p = 0$ we have $\Delta_{\tau_0} = \Delta$, the standard Laplacian on functions.

For a special class of flat manifolds the terms in formula (1.1) can be made more explicit. A Bieberbach group $\Gamma$ is said to be of diagonal type if there exists an orthonormal $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ of the lattice $\Lambda$ such that for any element $BL_b \in \Gamma$, $Be_i = \pm e_i$ for $1 \leq i \leq n$ (see [7], Definition 1.3). Similarly, $M_\Gamma$ is said to be of diagonal type, if $\Gamma$ is so. We note that it may be assumed that the lattice $\Lambda$ of $\Gamma$ is the canonical lattice.

These manifolds have, in particular, holonomy group $F \simeq \mathbb{Z}_2^r$, for some $1 \leq r \leq n - 1$. After conjugation by a translation we may assume furthermore that $b \in \frac{1}{2} \Lambda$, for any $BL_b \in \Gamma$ (see [7], Lemma 1.4). In this case, the terms $e_{\mu,\gamma}$ in the multiplicity formula (1.1) become sums of $1$'s and $-1$'s. Moreover, the traces $\text{tr}_p(B)$ are given by integral values of the so called Krawtchouk polynomials $K_p^n(x)$ (see [6], Remark 3.6, and also [7]; see [4] for more information on Krawtchouk polynomials). Namely, we have:
\begin{equation}
\label{tr_p_eq}
\text{tr}_p(B) = K_p^n(n - n_B), \quad \text{where } K_p^n(x) := \sum_{t=0}^p (-1)^t \binom{x}{t} \binom{n-x}{p-t}.
\end{equation}

For further use, we now give a table with the values of the Krawtchouk polynomials $K_p^n(j)$ for $0 \leq p, j \leq 4$.

| $p$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $K_0^2(0)$ | 1 | 4 | 6 | 4 | 1 |
| $K_1^2(1)$ | 1 | 2 | 0 | -2 | -1 |
| $K_2^2(2)$ | 1 | 0 | -2 | 0 | 1 |
| $K_3^2(3)$ | 1 | -2 | 0 | 2 | -1 |
| $K_4^2(4)$ | 1 | -4 | 6 | -4 | 1 |

The next lemma gives some auxiliary facts on the Krawtchouk polynomials.

**Lemma 1.1** (See [4], p. 76). If $1 \leq k, j \leq n$ we have
\begin{enumerate}
\item $K_k^n(j) = (-1)^j K_{n-k}^n(j) = (-1)^k K_k^n(n-j)$. Hence if $n$ even, then $K_{k}^n(j) = 0$ for $j$ odd and $K_{k}^n(j) = 0$, for $k$ odd.
\item $\binom{n}{j} K_k^n(j) = \binom{n}{k} K_j^n(k)$. This implies that $K_k^n(j) = 0$ if and only if $K_j^n(k) = 0$.
\end{enumerate}

If $\Gamma$ is of diagonal type and $BL_b \in \Gamma$, we have $n_B = \dim(\mathbb{R}^n)^B = |\{1 \leq i \leq n : Be_i = e_i\}|$. We set
\begin{equation}
\label{sunada_num_eq}
n_B\left(\frac{1}{2}\right) := |\{1 \leq i \leq n : Be_i = e_i \quad \text{and} \quad b \cdot e_i \equiv \frac{1}{2} \mod \mathbb{Z}\}|.
\end{equation}

If $0 \leq t \leq d \leq n$, the Sunada numbers for $\Gamma$ are defined by
\begin{equation}
\label{sunada_num_eq_1}
c_{d,t}(\Gamma) := |\{ BL_b \in \Gamma : n_B = d \quad \text{and} \quad n_B\left(\frac{1}{2}\right) = t\}|.$
\end{equation}
We note that by Lemma 1.1 in [8], one has that \( n_B(\frac{1}{2}) > 0 \) for any \( \gamma = BL_b \in \Gamma, \gamma \not\in \Lambda \).

**Remark 1.2.** In [7] it is shown that the equality of the Sunada numbers \( c_{d,t}(\Gamma) = c_{d,t}(\Gamma') \) for every \( d,t \), is equivalent to have that \( M_\Gamma \) and \( M_{\Gamma'} \) verify the conditions in Sunada’s theorem. In this case one says that \( M_\Gamma \) and \( M_{\Gamma'} \) are Sunada isospectral (see [7], Definition 3.2, Theorem 3.3, and the discussion following it). In particular this implies that \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral for all \( 0 \leq p \leq n \).

For \( \Gamma \) a Bieberbach group, \( \tau \) a finite dimensional representation of \( O(n) \) and \( Re(s) > 0 \), we consider the heat trace zeta function

\[
Z^\Gamma_\tau(s) := \sum_{\lambda \in Spec(M)} e^{-\lambda s} = \sum_{\mu \geq 0} d_{\tau,\mu}(\Gamma) e^{-4\pi^2\mu s}.
\]

This series is uniformly convergent for \( Re(s) > \varepsilon \), for any \( \varepsilon > 0 \). In the case when \( \tau = \tau_p \), for some \( 0 \leq p \leq n \), we write \( Z^\Gamma_\tau(s) = Z^\Gamma_{\tau_p}(s) \).

The lengths of closed geodesics in \( M_\Gamma \) are the numbers \( \|b_+ + \lambda_+\| \) with \( BL_b \in \Gamma \) running through a full set of representatives of \( F = \Lambda \setminus \Gamma \) and \( \lambda \in \Lambda \). Recall that \( b_+ = p_B(b) \), where \( p_B \) denotes the orthogonal projection onto \( ker(B - Id) \).

We will make use of the next result from [8].

**Theorem 1.3.** (i) In the notation above, we have

\[
Z^\Gamma_\tau(s) = \frac{1}{|\Gamma|} \sum_{BL_b \in F} \frac{tr \tau(B)}{vol(\Lambda B)} (4\pi s)^{-\frac{n}{2}} \sum_{\lambda_+ \in p_B(\Lambda)} e^{-\frac{\|b_+ + \lambda_+\|^2}{4s}}.
\]

\( Spec(M_\Gamma) \) determines the lengths of closed geodesics of \( M_\Gamma \) and the numbers \( n_B; Spec_\tau(M_\Gamma) \) (in particular \( Spec_p(M_\Gamma) \), for any fixed \( p \geq 0 \)) determines the spectrum of the torus \( T_\Lambda = \Lambda \setminus \mathbb{R}^n \) and the cardinality of \( F \).

(ii) If \( \Gamma \) is a Bieberbach group of diagonal type with \( F \cong \mathbb{Z}^r_2 \) then we have:

\[
Z^\Gamma_\tau_p(s) = \frac{1}{2} \sum_{d=1}^n \sum_{\lambda_+ \in \mathbb{Z}^d} K_p^n(n - d) (4\pi s)^{-\frac{d}{2}} \sum_{t=0}^d \sum_{c_{d,t}(\Gamma) \theta_{d,t}(\frac{1}{4\pi s})}
\]

where, for \( Re(s) > 0 \),

\[
\theta_{d,t}(s) = \sum_{(m_1, \ldots, m_d) \in \mathbb{Z}^d} e^{-s(\sum_{j=1}^d (\frac{1}{4} + m_j)^2 + \sum_{j=t+1}^d m_j^2)}.
\]

Furthermore, \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral if and only if

\[
K_p^n(n - d) c_{d,t}(\Gamma) = K_p^n(n - d) c_{d,t}(\Gamma')
\]

for each \( 1 \leq t \leq d \leq n \). In particular, if \( c_{d,t}(\Gamma) = c_{d,t}(\Gamma') \) for every \( d,t \), then \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral for all \( p \). If, for \( p \) fixed, \( K_p^n(x) \) has no integral roots and \( M_\Gamma \) and \( M_{\Gamma'} \) are \( p \)-isospectral then they are Sunada isospectral, hence \( q \)-isospectral for all \( 0 \leq q \leq n \). In particular, for groups of diagonal type, isospectral implies Sunada isospectral.

**Remark 1.4.** In [8] it is shown by asymptotic methods that the formula in (i) of the theorem implies that two isospectral flat manifolds must be both orientable or both nonorientable. This is not true for \( p \)-isospectral pairs (see [6]).
2. Bieberbach Groups with canonical translation lattice

In [1] a complete list of isomorphism classes of the crystallographic groups of dimension 4 is given. This list contains 4783 groups out of which 74 are Bieberbach groups. Among these Bieberbach groups there are 54 which allow the canonical lattice as lattice of translations and 20 that do not.

In this section we shall briefly explain how to obtain the Bieberbach groups from the data given in Tables 1C and 2C in [1] and how to present them in the notation given in Section 1. These will be given in tables containing the matrices $B_i$ and the corresponding translation vectors $b_i$ and some additional information. For convenience, in the last table we give the Sunada numbers for those groups $\Gamma$ having diagonal holonomy representation.

For simplicity, we shall denote by $1, \ldots, 74$, the 74 Bieberbach groups in Table 1C, p. 408 of [1], giving a full set of representatives for the isomorphism classes. Groups having the same holonomy representation (i.e. the same $\mathbb{Z}$-class) are put together into families $\mathcal{F}_1, \ldots, \mathcal{F}_8$.

In the notation in [1], the $\mathbb{Z}$-class is indicated by the first 3 numbers, for instance, in the group $2=2/1/1$ below, by $2/1/1$. The full list is the following: $1=1/1/1; 2=2/1/1/2; 3=2/1/2; 4=2/2/1/2; 5=3/1/1/2; 6=3/1/2/2; \mathcal{F}_1=\{1, 8, 10, 11\}=\{1/1/6, 7, 10, 11, 13\}$; $12 = 4/1/2/4; \mathcal{F}_2 = \{13, 14, 15\} = \{4/1/3, 4, 11, 12\}; 16 = 4/1/4/7; 17 = 4/1/6/4; \mathcal{F}_3 = \{18, 19, 20, 21\} = \{4/2/1/8, 11, 12, 16\}; 22 = 4/1/2/3/4; 23 = 4/3/1/6; \mathcal{F}_4 = \{24, 25, 26, 27\} = \{5/1/2/7, 8, 9, 10\}; 28 = 5/1/3/6; 29 = 5/1/4/6; 30 = 5/1/6/6; 31 = 5/1/7/4; 32 = 5/1/10/4; \mathcal{F}_5 = \{33, 34, 35, 36, 37, 38, 39, 40, 41, 42\} = \{6/1/1/41, 45, 49, 63, 64, 66, 81, 82, 83, 92\}; \mathcal{F}_6 = \{43, 44\} = \{6/2/1/27, 50\}; 45 = 7/2/1/2; 46 = 7/2/2/2; 47 = 8/1/1/2; 48 = 8/1/2/2; 49 = 9/1/2/2; 50 = 12/1/2/2; 51 = 12/1/3/2; 52 = 12/1/4/2; 53 = 12/1/6/2; 54 = 12/3/4/6; 55 = 12/3/10/5; 56 = 12/3/11/1; \mathcal{F}_7 = \{57, 58\} = \{13/1/8, 11\}; 59 = 13/1/3/8; \mathcal{F}_8 = \{60, 61, 62\} = \{13/4/1, 14, 20, 23\}; 63 = 13/4/4/11; 64 = 14/1/1/2; 65 = 14/1/3/2; 66 = 14/2/3/2; 67 = 14/3/1/4; 68 = 14/3/5/4; 69 = 14/3/6/4; 70 = 15/1/1/10; 71 = 15/4/1/10; 72 = 24/1/2/4; 73 = 24/1/4/4; 74 = 25/1/1/10.

In Table 1C in [1] we find, for each group $\Gamma$, matrices $A_1, \ldots, A_k$ and translation vectors $a_1, \ldots, a_k$ together with a (Bravais) type of lattice. The matrices are not orthogonal in general, however they are orthogonal with respect to some suitable inner product. In Table 2C in [1], there is a list of real symmetric matrices $G = (g_{ij})$ (Gram matrices) corresponding to the types of lattices given in Table 1C. The entries of these matrices $G$ are parameters representing the scalar products of the $\mathbb{Z}$-basis vectors of a lattice of translations $\Lambda$ of any group having this lattice type.

More precisely, Table 1C gives a crystallographic group $\Gamma_1 = \langle A_1 L_{a_1}, \ldots, A_k L_{a_k}, L_{\Lambda_1} \rangle$ where $\Lambda_1 = Z v_1 \oplus \cdots \oplus Z v_4$ is a lattice in $\mathbb{R}^4$ with $A_i \Lambda_1 = \Lambda_1$ for $1 \leq i \leq 4$. As mentioned before, Table 2C gives a symmetric matrix $G = (g_{ij})$, where $g_{ij} = w_i \cdot w_j$ and $\Lambda_2 = Z w_1 \oplus \cdots \oplus Z w_4$, that gives rise to a group $\Gamma_2 = X \Gamma_1 X^{-1}$ for some unimodular matrix $X$. Such $X \in SL_4(\mathbb{Z})$ and its inverse $X^{-1}$ are given in Table 2C, together with the matrix $G$. Now, the matrices $B_1 := X A_1 X^{-1}, \ldots, B_k := X A_k X^{-1}$ obtained in this way become orthogonal with respect to the basis $\{w_1, \ldots, w_4\}$ of $\Lambda_2$. We will take $\Lambda_2 = \mathbb{Z}^4$. We list the matrices $X \neq Id$ at the end of the section.

We remark that there is often a difference between the vectors $a_i$ in Table 1C and those used in this paper. This is so because, as explained in Section 1, any element $\gamma \in \Gamma$ decomposes uniquely as $\gamma = BL_b$, with $B \in O(n)$ and $b \in \mathbb{R}^n$, and so $\gamma$ acts on $x \in \mathbb{R}^n$ by $\gamma(x) = Bx + Bb$. However, in [1] the elements of $\Gamma$ are written in the form $\gamma = (A, a)$ with action $\gamma(x) = Ax + a$, so $(A, a)$ corresponds to $AL_{A^{-1}a}$ in our notation. We still have to conjugate $(A, a)$ by the corresponding matrix $X$. In this way, from the original data $(A, a)$ we obtain $BL_b = XAX^{-1}L_{X^{-1}A^{-1}a}$.

In this paper we are interested in groups having the canonical lattice as lattice of translations. Therefore, we need to look for the types of lattices having Gram matrices $G$ whose parameters
admit the identity matrix $G = Id$. For example, for the lattices of type X/IV (see [1], p. 275) we have

$$G = \begin{pmatrix} b & 0 & c \\ a & 0 & -b \\ 0 & c & -2c \end{pmatrix}$$

and we see that $G \neq Id$ for any possible choice of $a, b, c$. Thus, a Bieberbach group having type X/IV does not admit the canonical lattice as translation lattice, and hence the groups 52, 59, 63 are not in the class considered.

Taking into account the full list of Gram matrices in Table 2C in [1] one can check that the 4-dimensional Bieberbach groups admitting the canonical lattice as lattice of translations are: 1, 2, 3, 4, 5, 6, $F_1 = \{7, 8, 9, 10, 11\}$, 12, $F_2 = \{13, 14, 15\}$, $F_3 = \{18, 19, 20, 21\}$, 22, 23, $F_4 = \{24, 25, 26, 27\}$, 28, 29, $F_5 = \{33, 34, 35, 36, 37, 38, 39, 40, 41, 42\}$, $F_6 = \{43, 44\}$, 45, 47, 50, 51, 54, 56, $F_7 = \{57, 58\}$, $F_8 = \{60, 61, 62\}$, 64, 67, 72 and 74.

We shall present the above 54 groups in tables, together with 25 additional groups to give a complete set of representatives of the isometry classes of groups in our class. Each one of the groups in question lies in one table and groups having the same holonomy representation are put together in the same table. Also, for convenience, all manifolds having holonomy group isomorphic to $\mathbb{Z}_2$ are put together in the same table (Table 1). Finally, the groups numbered 72 and 74 are not included in the tables, for simplicity. They can not come up in any $p$-isospectral pair because they are the only groups with holonomy groups of order 12 and 24 respectively. The tables show the non trivial elements in the holonomy group, together with the corresponding translation vectors and some additional information that will allow to obtain explicit expressions for the zeta functions, and to determine the $L$-spectra of the associated manifolds.

The tables are organized in the following manner. Suppose that $\Gamma = (B_1 L_{b_1}, \ldots, B_k L_{b_k}, \Lambda)$. In the first row of the table, labeled $B$, we give the non trivial matrix elements of the holonomy group $F$. These matrices $B \in O(4, \mathbb{R}) \cap GL_4(\mathbb{Z})$ are written in the form $d(C_1, C_2), d(C_1, C_2, C_3)$ or $d(C_1, C_2, C_3, C_4)$ where, for instance, $d(C_1, C_2)$ indicates a matrix with $C_1, C_2$ lying diagonally in $B$ and similarly in the other cases. Here $C_i, 1 \leq i \leq 4$, will be one of the following matrices: $\pm 1, \pm I, \pm J, \pm T, \pm T^t$ or $K$ with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\bar{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. $$

The second group of entries, corresponds to the translation vectors $b$. Here we include one row for each group in the family, denoting with primes the different isometry classes in the same diffeomorphism class.

In the table we also indicate a set of generators over $\mathbb{Z}$ of the space fixed by $B$, i.e. $\Lambda^B = \{v \in \Lambda : Bv = v\}$, the numbers $n_B$, $vol(\Lambda^B)$ and the $p$-traces $tr_p(B)$. In general, these traces are given by integer values of Krawtchouk polynomials, $K^p_n(j)$, but there are some exceptions. In these cases all the values of the $p$-traces, for $0 \leq p \leq 4$, are listed. The components of the vectors $b$ that are not in $\Lambda^B$ are put in between parentheses in the tables. In the next section we will use these tables to compute, for each group $\Gamma$, the corresponding $p$-heat trace zeta function from which one can read off spectral information on $M_\Gamma$.

Given a group $k$, there may be several different isometry classes of Bieberbach groups in the isomorphism class of $k$, to be denoted by $k', k''$ and so on (see for instance, Table 2.1). One shows that all such classes can be obtained by conjugation of the matrices in the point group $F$ of $k$, by some $C \in GL_4(\mathbb{Z}) \cap Z(F) \smallsetminus O(4)$, where $Z(F)$ denotes the centralizer of $F$. 


The following is a list of representatives for the isometry classes of Bieberbach groups having canonical lattice of translations. The corresponding classes in the notation in [1] are given in between parentheses.

**Table 2.1:** 2 (2/1/1), 3 (2/1/2), 4 (2/2/1), 5 (3/1/1), 6 (3/1/2) \((F \simeq \mathbb{Z}_2)\).

| # | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| B | \(d(I, 1, -1)\) | \(d(I, J)\) | \(d(-I, -1, 1)\) | \(d(-I, I)\) | \(d(1, J, -1)\) |
| \(b\) | \(\frac{e_1}{2}\) | \(\frac{e_1}{2}\) | \(\frac{e_1}{2}\) | \(\frac{e_1}{2}\) | \(\frac{e_1}{2}\) |
| \(b'\) | \(\frac{e_2+e_3}{2}\) | \(\frac{e_1+e_2}{2}\) | \(\frac{e_1+e_2}{2}\) | \(\frac{e_3+e_4}{2}\) | \(\frac{e_1+e_2+e_3}{2}\) |
| \(b''\) | \(\frac{e_1+e_2+e_3}{2}\) | \(\frac{e_1+e_2+e_4}{2}\) | \(\frac{e_1+e_2+e_3+e_4}{2}\) | \(\frac{e_1+e_2+e_3+e_4}{2}\) | \(\frac{e_1+e_2+e_3+e_4}{2}\) |
| \(\Lambda^B\) | \(e_1, e_2, e_3\) | \(e_1, e_2, e_3 + e_4\) | \(e_4\) | \(e_3, e_4\) | \(e_1, e_2 + e_3\) |
| \(n_B\) | 3 | 3 | 1 | 2 | 2 |
| \(vol(\Lambda^B)\) | 1 | \(\sqrt{2}\) | 1 | 1 | \(\sqrt{2}\) |
| \(tr_p(B)\) | \(K_p^4(1)\) | \(K_p^4(1)\) | \(K_p^4(3)\) | \(K_p^4(2)\) | \(K_p^4(2)\) |
| \(\beta_1, \beta_2\) | 3, 3 | 3, 3 | 1, 3 | 2, 2 | 2, 2 |
| Orientable | no | no | yes | yes | yes |
| Diagonal-type | yes | no | yes | yes | no |

**Table 2.2:** 47 (8/1/1) \((F \simeq \mathbb{Z}_3, \text{non-diagonal, orientable, } \beta_1 = 2, \beta_2 = 1)\)

| | | \(\gamma_1\) | \(\gamma_1^2\) |
|---|---|---|---|
| B | \(d(1, T)\) | \(d(1, T')\) |
| \(b_{47}\) | \(\frac{e_1}{3}\) | \(\frac{e_1}{3}\) |
| \(b'_{47}\) | \(\frac{e_1+e_2+e_3+e_4}{3}\) | \(\frac{e_1+e_2+e_3+e_4}{3}\) |
| \(\Lambda^B\) | \(e_1, e_2 + e_3 + e_4\) | \(e_1, e_2 + e_3 + e_4\) |
| \(n_B\) | 2 | 2 |
| \(vol(\Lambda^B)\) | \(\sqrt{3}\) | \(\sqrt{3}\) |
| \(tr_p(B)\) | 1 1 0 1 1 | 1 1 0 1 1 |
Table 2.3: $F_1 = \{7, 8, 9, 10, 11\}$ (4/1) ($F \simeq \mathbb{Z}_2^2$, diagonal, non-orientable, $\beta_1 = 2, \beta_2 = 1$)

| $B$     | $\gamma_1$ | $\gamma_2$ | $\gamma_1 \gamma_2$ |
|---------|-------------|-------------|----------------------|
| $b_7$   | $\frac{e_3}{2}$ | $\frac{e_2}{2}$ | $\frac{e_2}{2}$ |
| $b_7'$  | $\frac{e_3}{2}$ | $\frac{e_1 + e_2}{2}$ | $\frac{e_1 + e_2}{2}$ |
| $b_8$   | $\frac{e_3}{2}$ | $\frac{e_2 + e_4}{2}$ | $\frac{e_2 + (e_1 + e_4)}{2}$ |
| $b_8'$  | $\frac{e_3}{2}$ | $\frac{e_1 + e_2 + e_4}{2}$ | $\frac{e_1 + e_2 + (e_1 + e_4)}{2}$ |
| $b_9$   | $\frac{e_3}{2}$ | $\frac{e_1}{2}$ | $\frac{e_1}{2}$ |
| $b_9'$  | $\frac{e_3}{2}$ | $\frac{e_1 + e_2}{2}$ | $\frac{e_1 + e_2}{2}$ |
| $b_{10}$| $\frac{e_3}{2}$ | $\frac{e_1 + e_4}{2}$ | $\frac{e_1 + e_2 + (e_1 + e_4)}{2}$ |
| $b_{10}'$ | $\frac{e_3 + e_2}{2}$ | $\frac{e_1 + e_4}{2}$ | $\frac{e_1 + (e_4)}{2}$ |
| $b_{10''}$ | $\frac{e_3 + e_2}{2}$ | $\frac{e_1 + e_4}{2}$ | $\frac{e_1 + e_2 + (e_1 + e_4)}{2}$ |
| $b_{11}$ | $\frac{e_3 + e_2}{2}$ | $\frac{e_1 + e_4}{2}$ | $\frac{e_1 + (e_4)}{2}$ |
| $b_{11'}$ | $\frac{e_3 + e_2}{2}$ | $\frac{e_1 + e_4}{2}$ | $\frac{e_1 + (e_4)}{2}$ |

| $\Lambda^B$ | $n_B$ | $\text{vol}(\Lambda^B)$ | $tr_p(B)$ |
|-------------|------|----------------|----------|
| $e_1, e_2, e_3$ | $3$ | $1$ | $K_p^4(1)$ |
| $e_1, e_2, e_4$ | $3$ | $1$ | $K_p^4(1)$ |
| $e_1, e_2$ | $2$ | $1$ | $K_p^4(2)$ |

Table 2.4: $12$ (4/1/2) ($F \simeq \mathbb{Z}_2^2$, non-diagonal, non-orientable, $\beta_1 = 2, \beta_2 = 1$)

| $B$     | $\gamma_1$ | $\gamma_2$ | $\gamma_1 \gamma_2$ |
|---------|-------------|-------------|----------------------|
| $b_{12}$ | $\frac{e_3}{2}$ | $\frac{e_1}{2}$ | $\frac{e_1 + e_2}{2}$ |
| $b_{12}'$ | $\frac{e_1 + e_2}{2}$ | $\frac{e_1}{2}$ | $\frac{e_2}{2}$ |

| $\Lambda^B$ | $n_B$ | $\text{vol}(\Lambda^B)$ | $tr_p(B)$ |
|-------------|------|----------------|----------|
| $e_1, e_2, e_3 + e_4$ | $3$ | $\sqrt{2}$ | $K_p^4(1)$ |
| $e_1, e_2, e_3 - e_4$ | $3$ | $\sqrt{2}$ | $K_p^4(1)$ |
| $e_1, e_2$ | $2$ | $1$ | $K_p^4(2)$ |
Table 2.5: $\mathcal{F}_2 = \{13, 14, 15\}$ (4/1/3) ($F \simeq \mathbb{Z}_2^2$, non-diagonal, non-orientable, $\beta_1 = 1, \beta_2 = 0$)

|   | $\gamma_1$ | $\gamma_2$ | $\gamma_1 \gamma_2$ |
|---|------------|------------|---------------------|
| $B$ | $d(1, I, -1)$ | $d(1, J, 1)$ | $d(1, J, -1)$ |
| $b_{13}$ | $\epsilon_{1/2}$ | $\epsilon_{1/2}$ | $\epsilon_{1/(e_1)}$ |
| $b_{13}'$ | $\frac{e_1 + e_2 + e_3}{2}$ | $\frac{e_2}{2}$ | $\frac{e_1 + e_2 + e_3 + e_4}{2}$ |
| $b_{14}$ | $\frac{e_2}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ |
| $b_{14}'$ | $\frac{e_2}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ |
| $b_{15}$ | $\frac{e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ |
| $b_{15}'$ | $\frac{e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ | $\frac{e_1 + e_2 + e_3}{2}$ |
| $\Lambda^B$ | $e_1, e_2, e_3$ | $e_1, e_2 + e_3, e_4$ | $e_1, e_2 + e_3$ |
| $n_B$ | 3 | 3 | 2 |
| $\text{vol}(\Lambda^B)$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ |
| $tr_p(B)$ | $K^4_p(1)$ | $K^4_p(1)$ | $K^4_p(2)$ |

Table 2.6: $\mathcal{F}_3 = \{18, 19, 20, 21\}$ (4/2/1) ($F \simeq \mathbb{Z}_2^2$, diagonal, non-orientable, $\beta_1 = 1, \beta_2 = 1$)

|   | $\gamma_1$ | $\gamma_2$ | $\gamma_1 \gamma_2$ |
|---|------------|------------|---------------------|
| $B$ | $d(I, -1, 1)$ | $d(-I, -1, 1)$ | $d(-I, I)$ |
| $b_{18}$ | $\frac{e_4}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{e_1}{2}$ |
| $b_{19}$ | $\frac{e_2}{2}$ | $\frac{e_4}{2}$ | $\frac{(e_2) + e_4}{2}$ |
| $b_{19}'$ | $\frac{e_1 + e_2}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{(e_1 + e_2) + e_3 + e_4}{2}$ |
| $b_{20}$ | $\frac{e_2}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{(e_1 + e_2) + e_3 + e_4}{2}$ |
| $b_{20}'$ | $\frac{e_2}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{(e_1 + e_2) + e_3 + e_4}{2}$ |
| $b_{21}$ | $\frac{e_2 + e_4}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{(e_2) + e_3}{2}$ |
| $b_{21}'$ | $\frac{e_2 + e_4}{2}$ | $\frac{(e_3) + e_4}{2}$ | $\frac{(e_2) + e_3}{2}$ |
| $\Lambda^B$ | $e_1, e_2, e_4$ | $e_4$ | $e_3, e_4$ |
| $n_B$ | 3 | 1 | 2 |
| $\text{vol}(\Lambda^B)$ | 1 | 1 | 1 |
| $tr_p(B)$ | $K^4_p(1)$ | $K^4_p(3)$ | $K^4_p(2)$ |
Table 2.7: 22 (4/2/3) \((F \simeq \mathbb{Z}_2^2, \text{non-diagonal, non-orientable}, \beta_1 = 1, \beta_2 = 0)\)

| \(B\) | \(\gamma_1\) | \(\gamma_2\) | \(\gamma_1\gamma_2\) |
|------|------------|------------|---------------------|
| \(b_{22}\) | \(\frac{e_1}{2}\) | \(\frac{e_4}{2}\) | \(\frac{(e_1)+e_4}{2}\) |
| \(b_{22}'\) | \(\frac{e_3}{2}\) | \(\frac{e_1}{2}\) | \(e_1+e_2+e_3\) |
| \(\Lambda^B\) | \(e_1, e_2 + e_3, e_4\) | \(e_4\) | \(e_2 - e_3, e_4\) |
| \(n_B\) | 3 | 1 | 2 |
| \(vol(\Lambda^B)\) | \(\sqrt{2}\) | 1 | \(\sqrt{2}\) |
| \(tr_p(B)\) | \(K_p^4(1)\) | \(K_p^4(3)\) | \(K_p^4(2)\) |

Table 2.8: 23 (4/3/1) \((F \simeq \mathbb{Z}_2^2, \text{diagonal, non-orientable}, \beta_1 = 0, \beta_2 = 1)\)

| \(B\) | \(\gamma_1\) | \(\gamma_2\) | \(\gamma_1\gamma_2\) |
|------|------------|------------|---------------------|
| \(b_{23}\) | \(\frac{e_3}{2}\) | \(\frac{(e_2)+e_4}{2}\) | \(\frac{e_1+(e_3+e_4)}{2}\) |
| \(b_{23}'\) | \(\frac{e_3}{2}\) | \(\frac{(e_1+e_3)+e_4}{2}\) | \(\frac{e_1+e_2+(e_3+e_4)}{2}\) |
| \(\Lambda^B\) | \(e_3\) | \(e_4\) | \(e_1, e_2\) |
| \(n_B\) | 1 | 1 | 2 |
| \(vol(\Lambda^B)\) | 1 | 1 | 1 |
| \(tr_p(B)\) | \(K_p^4(3)\) | \(K_p^4(3)\) | \(K_p^4(2)\) |

Table 2.9: \(F_4 = \{24, 25, 26, 27\}\) (5/1/2) \((F \simeq \mathbb{Z}_2^2, \text{diagonal, orientable}, \beta_1 = 1, \beta_2 = 0)\)

| \(B\) | \(\gamma_1\) | \(\gamma_2\) | \(\gamma_1\gamma_2\) |
|------|------------|------------|---------------------|
| \(b_{24}\) | \(\frac{e_3}{2}\) | \(\frac{(e_2)+e_4}{2}\) | \(\frac{e_2}{2}\) |
| \(b_{25}\) | \(\frac{e_3}{2}\) | \(\frac{e_1+(e_2)}{2}\) | \(\frac{(e_1)+e_2+e_4}{2}\) |
| \(b_{26}\) | \(\frac{e_3}{2}\) | \(\frac{e_1}{2}\) | \(\frac{(e_1)+e_2+(e_3)}{2}\) |
| \(b_{27}\) | \(\frac{e_3}{2}\) | \(\frac{e_1+(e_2)+e_4}{2}\) | \(\frac{(e_1)+e_2+(e_3)+e_4}{2}\) |
| \(\Lambda^B\) | \(e_3, e_4\) | \(e_1, e_4\) | \(e_2, e_4\) |
| \(n_B\) | 2 | 2 | 2 |
| \(vol(\Lambda^B)\) | 1 | 1 | 1 |
| \(tr_p(B)\) | \(K_p^4(2)\) | \(K_p^4(2)\) | \(K_p^4(2)\) |
Table 2.10: 28 (5/1/3) \((F \simeq \mathbb{Z}_2^2, \text{non-diagonal, orientable, } \beta_1 = 1, \beta_2 = 0)\)

| \(B\)  | \(\gamma_1\) | \(\gamma_2\) | \(\gamma_1 \gamma_2\) |
|------|------|------|------|
| \(b_{28}\) | \(d(1, -J, -1)\) | \(d(1, J, -1)\) | \(d(1, -I, 1)\) |
| \(\Lambda B\) | \(\frac{e_1}{2}\) | \(\frac{e_1 + e_4}{2}\) | \(\frac{e_4}{2}\) |
| \(n_B\) | \(e_1, e_2 - e_3\) | \(e_1, e_2 + e_3\) | \(e_1, e_4\) |
| \(\text{vol}(\Lambda B)\) | \(2\) | \(2\) | \(2\) |
| \(\text{tr}_p(B)\) | \(K^4_p(2)\) | \(K^4_p(2)\) | \(K^4_p(2)\) |

Table 2.11: 29 (5/1/4) \((F \simeq \mathbb{Z}_2^2, \text{non-diagonal, orientable, } \beta_1 = 1, \beta_2 = 0)\)

| \(B\)  | \(\gamma_1\) | \(\gamma_2\) | \(\gamma_1 \gamma_2\) |
|------|------|------|------|
| \(b_{29}\) | \(d(-1, J, 1)\) | \(d(1, J, -1)\) | \(d(-1, 1, -1)\) |
| \(b_{29}'\) | \(\frac{(-e_2 + e_3 + e_4)}{2}\) | \(\frac{e_1}{2}\) | \(\frac{(e_1) + e_2 + e_3 + (e_4)}{2}\) |
| \(\Lambda B\) | \(2\) | \(2\) | \(2\) |
| \(n_B\) | \(e_2 + e_3, e_4\) | \(e_1, e_2 + e_3\) | \(e_1, e_3\) |
| \(\text{vol}(\Lambda B)\) | \(\sqrt{2}\) | \(\sqrt{2}\) | \(1\) |
| \(\text{tr}_p(B)\) | \(K^4_p(2)\) | \(K^4_p(2)\) | \(K^4_p(2)\) |

Table 2.12: 45 (7/2/1) \((F \simeq \mathbb{Z}_4, \text{non-diagonal, orientable, } \beta_1 = 2, \beta_2 = 1)\)

| \(B\)  | \(\gamma_1\) | \(\gamma_1^2\) | \(\gamma_1^3\) |
|------|------|------|------|
| \(b_{45}\) | \(d(I, -\bar{J})\) | \(d(I, -I)\) | \(d(I, \bar{J})\) |
| \(b_{45}'\) | \(\frac{e_1}{4}\) | \(\frac{e_2}{2}\) | \(\frac{3e_1}{4}\) |
| \(\Lambda B\) | \(\frac{e_1 + e_2}{4}\) | \(\frac{e_1 + e_2}{2}\) | \(\frac{3e_1 + 3e_2}{4}\) |
| \(n_B\) | \(2\) | \(2\) | \(2\) |
| \(\text{vol}(\Lambda B)\) | \(1\) | \(1\) | \(1\) |
| \(\text{tr}_p(B)\) | \(1\ 2\ 2\ 2\ 1\) | \(K^4_p(2)\) | \(1\ 2\ 2\ 2\ 1\) |
Table 2.13: 50  (12/1/2) ($F \simeq \mathbb{Z}_4$, non-diagonal, non-orientable, $\beta_1 = 1, \beta_2 = 0$)

| $B$  | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ |
|------|--------------|--------------|--------------|
| $b_{50}$ | $d(-1,1,-\bar{J})$ | $d(I,-I)$ | $d(-1,1,\bar{J})$ |

| $\Lambda^B$ | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ |
|--------------|--------------|--------------|--------------|
| $n_B$ | $e_2$ | $e_1,e_2$ | $e_2$ |
| $\text{vol}(\Lambda^B)$ | 1 | 2 | 1 |

| $tr_p(B)$ | 1 0 0 0 -1 | $K_p^4(2)$ | 1 0 0 0 -1 |

Table 2.14: 51  (12/1/3) ($F \simeq \mathbb{Z}_4$, non-diagonal, non-orientable, $\beta_1 = 0, \beta_2 = 1$)

| $B$  | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ |
|------|--------------|--------------|--------------|
| $b_{51}$ | $d(J,-\bar{J})$ | $d(I,-I)$ | $d(J,\bar{J})$ |

| $\Lambda^B$ | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ |
|--------------|--------------|--------------|--------------|
| $n_B$ | $e_1 + e_2$ | $e_1,e_2$ | $e_1 + e_2$ |
| $\text{vol}(\Lambda^B)$ | $\sqrt{2}$ | 1 | $\sqrt{2}$ |

| $tr_p(B)$ | 1 0 0 0 -1 | $K_p^4(2)$ | 1 0 0 0 -1 |

Table 2.15: 64  (14/1/1) ($F \simeq \mathbb{Z}_6$, non-diagonal, non-orientable, $\beta_1 = 1, \beta_2 = 0$)

| $B$  | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ | $\gamma_1^4$ | $\gamma_1^5$ |
|------|--------------|--------------|--------------|--------------|--------------|
| $b_{64}$ | $d(1,-T)$ | $d(1,T^t)$ | $d(1,-1,-1,-1)$ | $d(1,T)$ | $d(1,-T^t)$ |

| $\Lambda^B$ | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ | $\gamma_1^4$ | $\gamma_1^5$ |
|--------------|--------------|--------------|--------------|--------------|--------------|
| $n_B$ | $e_1$ | $e_1,e_2+e_3+e_4$ | $e_1$ | $e_1,e_2+e_3+e_4$ | $e_1$ |
| $\text{vol}(\Lambda^B)$ | 1 | $\sqrt{3}$ | 1 | $\sqrt{3}$ | 1 |

| $tr_p(B)$ | 1 1 0 -1 1 | 1 1 0 1 1 | $K_p^4(3)$ | 1 1 0 1 1 | 1 1 0 -1 1 |

Table 2.16: 67  (14/3/1) ($F \simeq D_3$, non-diagonal, orientable, $\beta_1 = 0, \beta_2 = 0$)

| $B$  | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ | $\gamma_1^4$ | $\gamma_1^5$ |
|------|--------------|--------------|--------------|--------------|--------------|
| $b_{67}$ | $d(1,T)$ | $d(1,T^t)$ | $d(-1,J,1)$ | $d(-1,1,J)$ | $d(-1,K)$ |

| $\Lambda^B$ | $\gamma_1$  | $\gamma_1^2$ | $\gamma_1^3$ | $\gamma_1^4$ | $\gamma_1^5$ |
|--------------|--------------|--------------|--------------|--------------|--------------|
| $n_B$ | $e_1,e_2+e_3+e_4$ | $e_1,e_2+e_3+e_4$ | $e_2,e_3+e_4$ | $e_3,e_2+e_4$ | $e_3,e_2+e_4$ |
| $\text{vol}(\Lambda^B)$ | $\sqrt{3}$ | $\sqrt{3}$ | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2}$ |

| $tr_p(B)$ | 1 1 0 1 1 | 1 1 0 1 1 | $K_p^4(2)$ | $K_p^4(2)$ | $K_p^4(2)$ |
Table 2.17: \( F_5 = \{33, 34, 35, 36, 37, 38, 39, 40, 41, 42\} (6/1/1) (F \simeq \mathbb{Z}_2^3, \text{diagonal, non-orientable}, \beta_1 = 1, \beta_2 = 0)\).

| \( B \) | \( \gamma_1 \) | \( \gamma_2 \) | \( \gamma_3 \) | \( \gamma_1 \gamma_2 \) | \( \gamma_1 \gamma_3 \) | \( \gamma_2 \gamma_3 \) | \( \gamma_1 \gamma_2 \gamma_3 \) |
|---|---|---|---|---|---|---|---|
| \( b_{33} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{34} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{35} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{36} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{37} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{38} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{39} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{40} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{41} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{42} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |

| \( \Lambda^B \) | \( e_2, e_3, e_4 \) | \( e_1, e_2, e_4 \) | \( e_3, e_4 \) | \( e_2, e_4 \) | \( e_1, e_3, e_4 \) | \( e_4 \) | \( e_1, e_4 \) |
| \( n_B \) | 3 | 3 | 2 | 2 | 3 | 3 | 1 |
| \( \text{vol}(\Lambda^B) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \text{tr}_p(B) \) | \( K_3^3(1) \) | \( K_3^3(1) \) | \( K_3^3(2) \) | \( K_3^3(2) \) | \( K_3^3(1) \) | \( K_3^3(3) \) | \( K_3^3(2) \) |

Table 2.18: \( F_6 = \{43, 44\} (6/2/1) (F \simeq \mathbb{Z}_2^3, \text{diagonal, non-orientable}, \beta_1 = 0, \beta_2 = 0)\).

| \( B \) | \( \gamma_1 \) | \( \gamma_2 \) | \( \gamma_3 \) | \( \gamma_1 \gamma_2 \) | \( \gamma_1 \gamma_3 \) | \( \gamma_2 \gamma_3 \) | \( \gamma_1 \gamma_2 \gamma_3 \) |
|---|---|---|---|---|---|---|---|
| \( b_{43} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |
| \( b_{44} \) | \( e_1/2 \) | \( e_2/2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1+e_3 \) | \( e_1+e_2 \) | \( e_1 + (e_2+e_3) \) |

| \( \Lambda^B \) | \( e_1, e_2, e_3 \) | \( e_2, e_4 \) | \( e_3, e_4 \) | \( e_2, e_4 \) | \( e_1, e_3, e_4 \) | \( e_4 \) | \( e_1, e_4 \) |
| \( n_B \) | 3 | 1 | 2 | 2 | 1 | 1 | 2 |
| \( \text{vol}(\Lambda^B) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \text{tr}_p(B) \) | \( K_3^3(1) \) | \( K_3^3(3) \) | \( K_3^3(2) \) | \( K_3^3(2) \) | \( K_3^3(3) \) | \( K_3^3(3) \) | \( K_3^3(2) \) |

Table 2.19: \( F_7 = \{57, 58\} (13/1/1) (F \simeq \mathbb{Z}_2 \times \mathbb{Z}_4, \text{non-diagonal, non-orientable}, \beta_1 = 1, \beta_2 = 1)\).

| \( B \) | \( \gamma_1 \) | \( \gamma_1^2 \) | \( \gamma_1^3 \) | \( \gamma_2 \) | \( \gamma_1 \gamma_2 \) | \( \gamma_1^2 \gamma_2 \) | \( \gamma_1^3 \gamma_2 \) |
|---|---|---|---|---|---|---|---|
| \( b_{57} \) | \( e_2 \) | \( e_2 \) | \( 3e_2 \) | \( e_2 \) | \( e_2 \) | \( e_2 \) | \( e_2 \) |
| \( b_{58} \) | \( e_2 \) | \( e_2 \) | \( 3e_2 \) | \( -e_2 \) | \( -e_2 \) | \( -e_2 \) | \( -e_2 \) |

| \( \Lambda^B \) | \( e_2 \) | \( e_1, e_2 \) | \( e_2, e_3, e_4 \) | \( e_2, e_3, e_4 \) | \( e_1, e_2 \) | \( e_2 \) | \( e_1, e_2 \) |
| \( n_B \) | 1 | 2 | 1 | 3 | 2 | 1 | 2 |
| \( \text{vol}(\Lambda^B) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \text{tr}_p(B) \) | 1 0 0 0 -1 | 1 0 0 0 -1 | \( K_3^3(2) \) | \( K_3^3(1) \) | 1 2 2 2 1 | \( K_3^3(3) \) | 1 2 2 2 1 |
Table 2.20: 54 (12/3/4) (F ≃ D₄, non-diagonal, non-orientable, β₁ = 0, β₂ = 0)

|       | γ₁  | γ₁² | γ₁³ | γ₂  | γ₁γ₂ | γ₁²γ₂ | γ₁³γ₂ |
|-------|-----|-----|-----|-----|------|--------|--------|
| B     | (d(J, i), d(I, -I) | d(J, -J) | d(J, -I) | d(I, J) | d(J, 1, -1) | d(I, J) | d(J, 1, -1) | d(I, -J) |
| b₁₂₄ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ |

Table 2.21: 56 (12/4/3) (F ≃ D₄, non-diagonal, non-orientable, β₁ = 0, β₂ = 0)

|       | γ₁  | γ₁² | γ₁³ | γ₂  | γ₁γ₂ | γ₁²γ₂ | γ₁³γ₂ |
|-------|-----|-----|-----|-----|------|--------|--------|
| B     | (d(J, i), d(I, -I) | d(J, -J) | d(I, -J) | d(-I, J) | d(1, -I) | d(-I, J) | d(-I, -I) |
| b₁₂₆ | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) | e₂ + (e₂² + e₁) |

Table 2.22: F₈ = {60, 61, 62} (13/4/1) (F ≃ D₄, non-diagonal, orientable, β₁ = 1, β₂ = 0)

|       | γ₁  | γ₁² | γ₁³ | γ₂  | γ₁γ₂ | γ₁²γ₂ | γ₁³γ₂ |
|-------|-----|-----|-----|-----|------|--------|--------|
| B     | (d(J, i), d(I, -I) | d(J, -J) | d(I, J) | d(I, -I) | d(1, -I) | d(1, J) | d(1, J) |
| b₆₀   | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ |
| b₆₁   | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ |
| b₆₂   | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ | e₁ + e₂ |

Example 2.1 (A 1-isospectral pair with holonomy group D₄). As an application of the methods in this section, we will use formula (1.1) to carry out explicit calculations of multiplicities of eigenvalues for the groups 60 and 61, both having holonomy groups isomorphic to D₄. As a consequence, we shall see that 60, 61 are 1-isospectral and 3-isospectral but they are not p-isospectral for p ≠ 1, 3. Since the manifolds associated to the groups 60, 61 are orientable, 1-isospectral and 3-isospectral are equivalent in this case.

According to formula (1.1) we have:

\[
(2.1) \quad d_{p, \mu}(\Gamma) = \frac{1}{8} (\frac{4}{p}) |\Lambda| + \frac{1}{8} \sum_{\gamma = B_\mu \in \Lambda \setminus \Gamma} tr_p(B) e_{\mu, \gamma}(\Gamma)
\]

where \(\Lambda = \{\lambda \in \Lambda : ||\lambda||^2 = \mu\}\). For the elements \(\gamma = \gamma_1^2, \gamma_2, \gamma_1\gamma_2, \gamma_1^2\gamma_2, \gamma_1^3\gamma_2\), we have that \(tr_p(B) = K_p^4(2)\) (see Table 2.22 above), so for \(p = 1\) or \(p = 3\) these traces vanish (see (1.3)). Thus, using that \(tr_p(B_1) = tr_p(B_1^2) = 2\) for \(p = 1, 3\), we get

\[
(2.2) \quad d_{1, \mu}(\Gamma) = d_{3, \mu}(\Gamma) = \frac{1}{2} |\Lambda| + \frac{1}{2} e_{\mu, \gamma_1}(\Gamma) + \frac{1}{2} e_{\mu, \gamma_2}(\Gamma).
\]
In order to check 1-isospectrality it suffices to show that 
\(e_{\mu,\gamma_1}(\Gamma_{60}) = e_{\mu,\gamma_1}(\Gamma_{61})\) and 
\(e_{\mu,\gamma_1}(\Gamma_{60}) = e_{\mu,\gamma_1}(\Gamma_{61})\). We have:

\[
e_{\mu,\gamma_1}(\Gamma_{60}) = \sum_{v \in \Lambda_\mu} e^{-2\pi i v \cdot e_1} = \sum_{v \in \mathbb{Z} e_1 \oplus \mathbb{Z} e_2, \|v\|^2 = \mu} e^{-\frac{\pi i}{2} v \cdot e_1}
\]

\[
e_{\mu,\gamma_1}(\Gamma_{61}) = \sum_{v \in \Lambda_\mu, B_1 v = v} e^{-2\pi i v \cdot (\frac{1}{2} + e_1)} = \sum_{v \in \mathbb{Z} e_1 \oplus \mathbb{Z} e_2, \|v\|^2 = \mu} e^{-\frac{\pi i}{2} v \cdot e_1} e^{-\pi i v \cdot e_1}.
\]

Since \(e^{-\pi i v \cdot e_1} = 1\) for \(v \in \mathbb{Z} e_1 \oplus \mathbb{Z} e_2\), we have that \(e_{\mu,\gamma_1}(\Gamma_{60}) = e_{\mu,\gamma_1}(\Gamma_{61})\). Similarly, we get that \(e_{\mu,\gamma_1}(\Gamma_{60}) = e_{\mu,\gamma_1}(\Gamma_{61})\).

Observe that we can read off this fact from the rows of the vectors \(b\) in Table 2.22, since we know that the orthogonal projection of the vectors in between parentheses onto the space fixed by the corresponding matrix \(B\), is zero. In this way we have proved that the manifolds \(M_{60}, M_{61}\) obtained from the groups \(60, 61\) are 1-isospectral and hence 3-isospectral.

We now show that they can not be 0, 2 nor 4-isospectral. Take \(\mu = 1\). The formula for the multiplicities for \(p = 0, 2, 4\) in this particular case is:

\[
d_{p,1} = \frac{1}{8} \binom{4}{p}|\Lambda_1| + \frac{1}{8} \text{tr}_p(B_1) (e_{1,\gamma_1} + e_{1,\gamma_3}) + \frac{1}{8} K^p_4(2) (e_{1,\gamma_1^2} + e_{1,\gamma_2} + e_{1,\gamma_1 \gamma_2} + e_{1,\gamma_1^2 \gamma_2} + e_{1,\gamma_2 \gamma_2}).
\]

Computing the corresponding \(e_{1,\gamma_i}\)'s we obtain

| \(\gamma\) | \(\gamma_1\) | \(\gamma_3\) | \(\gamma_1 \gamma_2\) | \(\gamma_1^2 \gamma_2\) | \(\gamma_3^2 \gamma_2\) |
| --- | --- | --- | --- | --- | --- |
| \(e_{1,\gamma}(\Gamma_{60})\) | 0 | 2 | -2 | 0 | -4 |
| \(e_{1,\gamma}(\Gamma_{61})\) | 0 | 2 | -2 | -4 | -4 |

By this, and taking into account that \(\Lambda_1 = \{\pm e_1; \pm e_2; \pm e_3; \pm e_4\}\), we get

\[
d_{p,1}(\Gamma_{60}) = \binom{4}{p} \text{tr}_p(B_1) - \frac{1}{2} K^p_4(2)
\]

\[
d_{p,1}(\Gamma_{61}) = \binom{4}{p} + \frac{1}{2} \text{tr}_p(B_1) - \frac{3}{2} K^p_4(2).
\]

Since \(K^p_4(2) \neq 0\) for \(p = 0, 2, 4\) (see (1.3)), it is clear that \(d_{p,1}(\Gamma_{60}) \neq d_{p,1}(\Gamma_{61})\) and this shows that \(M_{60}\) and \(M_{61}\) can not be \(p\)-isospectral for \(p = 0, 2, 4\). For example, for \(p = 0, 4\), we have \(d_{p,1}(\Gamma_{60}) = 1\) and \(d_{p,1}(\Gamma_{61}) = 0\), while for \(p = 2\) we have \(d_{2,1}(\Gamma_{60}) = 6\) and \(d_{2,1}(\Gamma_{61}) = 4\).

Appendix. Here we list the matrices \(X \neq \text{Id}\) and \(C\), corresponding to the groups described in the tables in this section.

\[
X_3 = X_{12} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
X_6 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
X_{13} = X_{14} = X_{15} = X_{22} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
X_{28} = X_{29} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
X_{47} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
C'_2 = C'_{29} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C''_2 = C''_{29} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_3 = C'_{3} = C'_9 = C'_{10} = C'_{11} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_3 = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C''_3 = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_{3} = C'_{47} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_5 = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_{6} = C'_{13} = C'_{14} = C'_{15} = C'_{22} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
C'_7 = C'_{8} = C''_{10} = C''_{12} = C'_9 = C'_{20} = C'_1 = C'_{21} = C'_3 = C'_{45} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\]
Finally, we shall give here a list with the Sunada numbers $c_{d,t}$’s for those groups having diagonal holonomy representation. We only give the values for $c_{1,1}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{3,3}$, since $c_{1,0} = c_{2,0} = c_{3,0} = 0, c_{1,0} = 1$ and $c_{4,1} = c_{4,2} = c_{4,3} = c_{4,4} = 0$ for any 4-dimensional Bieberbach group of diagonal type.

| $\Gamma_\#$ | $c_{1,1}$ | $c_{2,1}$ | $c_{2,2}$ | $c_{3,1}$ | $c_{3,2}$ | $c_{3,3}$ |
|------------|----------|----------|----------|----------|----------|----------|
| 2          | 0        | 0        | 0        | 0        | 0        | 0        |
| 2'         | 0        | 0        | 0        | 0        | 0        | 0        |
| 4          | 0        | 1        | 0        | 0        | 0        | 0        |
| 5          | 0        | 0        | 1        | 0        | 0        | 0        |
| 7          | 0        | 1        | 0        | 2        | 0        | 0        |
| 7'         | 0        | 0        | 1        | 1        | 1        | 0        |
| 9          | 0        | 0        | 1        | 2        | 0        | 0        |
| 9'         | 0        | 1        | 0        | 1        | 1        | 0        |
| 10         | 0        | 0        | 1        | 1        | 1        | 0        |
| 10'        | 0        | 1        | 0        | 1        | 0        | 1        |
| 11         | 0        | 0        | 1        | 0        | 0        | 0        |
| 11'        | 0        | 0        | 1        | 0        | 1        | 1        |
| 18         | 1        | 1        | 0        | 0        | 0        | 0        |
| 19         | 1        | 1        | 0        | 1        | 0        | 0        |
| 19'        | 1        | 1        | 0        | 1        | 0        | 0        |
| 20         | 1        | 0        | 1        | 0        | 0        | 0        |
| 20'        | 1        | 0        | 0        | 0        | 0        | 0        |
| 21         | 1        | 1        | 0        | 0        | 0        | 0        |
| 21'        | 1        | 1        | 0        | 0        | 0        | 0        |
| 23         | 2        | 1        | 0        | 0        | 0        | 0        |
| 23'        | 2        | 0        | 1        | 0        | 0        | 0        |
| 24         | 0        | 3        | 0        | 0        | 0        | 0        |
| 25         | 0        | 2        | 1        | 0        | 0        | 0        |
| 26         | 0        | 3        | 0        | 0        | 0        | 0        |
| 27         | 0        | 1        | 2        | 0        | 0        | 0        |
| 33         | 1        | 2        | 1        | 3        | 0        | 0        |
| 34         | 1        | 1        | 2        | 2        | 1        | 0        |
| 35         | 1        | 3        | 0        | 1        | 1        | 1        |
| 36         | 1        | 3        | 0        | 2        | 1        | 0        |
| 37         | 1        | 2        | 1        | 2        | 0        | 1        |
| 38         | 1        | 1        | 2        | 2        | 1        | 0        |
| 39         | 1        | 2        | 1        | 1        | 2        | 0        |
| 40         | 1        | 3        | 3        | 1        | 1        | 1        |
| 41         | 1        | 2        | 1        | 1        | 2        | 0        |
| 42         | 1        | 3        | 0        | 0        | 3        | 0        |
| 43         | 3        | 2        | 1        | 1        | 0        | 0        |
| 44         | 3        | 3        | 0        | 0        | 1        | 0        |

3. Zeta functions and $p$-heat trace polynomials

In this section we will show that the $p$-heat trace $Z^T_p(s)$ of an $n$-dimensional compact flat manifold $M_F$ can be viewed as a polynomial of degree $n$ in a finite number of algebraically independent functions $x(s), y(s), z_1(s), \ldots, z_m(s)$, for $s > 0$. We let the dimension $n$ of $M_F$ be arbitrary, until further notice.

We begin by recalling some facts on the $L$-spectrum of a Bieberbach manifold $\Gamma$. For $\gamma = BL_b$, denote by $p_B : \mathbb{R}^n \to (\mathbb{R}^n)^B$ the orthogonal projection and put $b_+ = p_B(b)$. The $L$-spectrum of $M_F$ is given by $Spec_L(M_F) = \{||b_+ + \lambda_+|| : BL_b \in \Gamma\}$ with $BL_b$ running through a full set of representatives of $F$ and $\lambda \in \Lambda$ (see Section 1). Note that for the $L$-spectrum, only the components of $b$ not between parentheses in the tables count. In this section, to each element $B$ we will associate a monomial in several variables and will express the corresponding zeta functions in terms of these monomials, thus getting polynomial expressions for the $p$-heat traces in which the $L$-spectrum information is encoded. We will give a complete list of these polynomials from which one can recover the $L$-spectrum of the corresponding manifold.

By Theorem 1.3, for a compact flat manifold $M_F = \Gamma \setminus \mathbb{R}^n$ with translation lattice $\Lambda$ and holonomy group $F$ we have the following expression for the $p$-heat traces:

$$Z^T_p(s) = \frac{1}{|\Gamma|} \sum_{BL_b \in \Lambda \setminus \Gamma} \frac{\text{tr}_p(B)}{\text{vol}(\Lambda^B)} (4\pi s)^{-\frac{n_B}{2}} \sum_{\lambda_+ \in p_B(\Lambda)} e^{-\frac{||b_+ + \lambda_+||^2}{4s}}. $$

and if $\Gamma$ is of diagonal type, with $F \simeq \mathbb{Z}^T_2$, we can write:

$$Z^T_p(s) = \frac{1}{2^r} \sum_{d=1}^n K^n_p(n - d) (4\pi s)^{-\frac{d}{2}} \sum_{t=0}^d c_{d,t}(\Gamma) \theta_{d,t}(\frac{1}{4s})$$
where, for $Re(s) > 0$,

$$\theta_{d,t}(s) = e^{-s((\sum_{j=1}^{t}(\frac{1}{2} + m_j)^2 + \sum_{j=t+1}^{d} m_j)^2)}.$$  

The theta functions $\theta_{d,t}$ have simple expressions in terms of $\theta_0$ and $\theta_1$, where

$$(3.3) \quad \theta_0(s) := \theta_{1,0}(s) = \sum_{m \in \mathbb{Z}} e^{-sm^2}, \quad \theta_1(s) := \theta_{1,1}(s) = \sum_{m \in \mathbb{Z}} e^{-s(\frac{1}{2} + m)^2}, \quad \text{for } Re(s) > 0.$$  

We define the functions

$$x(s) := \frac{\theta_0\left(\frac{1}{4s}\right)}{\sqrt{4\pi s}} \quad \text{and} \quad y(s) := \frac{\theta_1\left(\frac{1}{4s}\right)}{\sqrt{4\pi s}}.$$  

**Lemma 3.1.** For $Re(s) > 0$ we have $\theta_{d,t}(s) = \theta_0^{d-t}(s) \theta_1^{t}(s)$. Furthermore, if $a_{d,t} \in \mathbb{C}$ and $\sum_{0 \leq t \leq d} a_{d,t} x(s)^{d-t} y(s)^t = 0$, then $a_{d,t} = 0$ for any $0 \leq d \leq t$.

**Proof.** Decomposing $\mathbb{Z}^d$ as $\mathbb{Z}^{d-t} \oplus \mathbb{Z}^t$ we get

$$\theta_{d,t}(s) = \sum_{(m_1, \ldots, m_d) \in \mathbb{Z}^{d-t}} e^{-s\sum_{j=1}^{t}(\frac{1}{2} + m_j)^2} \sum_{(m_1, \ldots, m_{d-t}) \in \mathbb{Z}^{d-t}} e^{-s\sum_{j=t+1}^{d} m_j^2} = \theta_0^{d-t}(s) \theta_1^{t}(s).$$  

To prove the second assertion, we note that, as $s \to +\infty$,

$$\theta_0^{d-t}(s) \sim 1, \quad \theta_1^{t}(s) \sim 2^t e^{-\frac{t}{4}}.$$  

Now, if $u = 1/4s$ then, as $s \downarrow 0$,

$$\sum_{0 \leq t \leq d} a_{d,t} x(s)^{d-t} y(s)^t = \sum_{0 \leq t \leq d} a_{d,t} \left(\frac{u}{\pi}\right)^{d/2} \theta_0(u)^{d-t} \theta_1(u)^t$$

$$\sim \sum_{0 \leq t \leq d} a_{d,t} 2^t \left(\frac{u}{\pi}\right)^{d/2} e^{-\frac{tu}{4}}.$$  

Now if not all $a_{d,t} = 0$, let $t_m$ be minimal with the property that some $a_{d,t_m} \neq 0$ and let $d_M$ be maximal among the $d$ so that $a_{d,t_m} \neq 0$. Multiplying the previous expression by $u^{-d_M/2} e^{\frac{tu}{4}}$ and letting $u \to \infty$ we have that all terms, except the one corresponding to $d = d_M, t = t_m$, tend to zero. Thus, we get that $a_{d_M,t_m} 2^t \pi^{d/2} = 0$, a contradiction.  

**Proposition 3.2.** Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type with canonical translation lattice $\Lambda$ and holonomy group $F$. Then $Z_\Gamma^\Lambda(s) \in \mathbb{Q}[\mathbb{Z}[x(s), y(s)]]$ has degree $n$, no independent term, and the coefficient of $x(s)^k$ is 0 (resp. 1) for all $k = 1, \ldots, n-1$ (resp. $k = n$).

**Proof.** By (3.2) and using the notation $Z_{Id}(s) := (4\pi s)^{-n/2} \sum_{\lambda \in \Lambda} e^{-\|\lambda\|^2/4s}$ we can write

$$Z_\Gamma^\Lambda(s) = \left(\frac{\theta}{|F|}\right) Z_{Id}(s) + \frac{1}{|F|} \sum_{d=1}^{n-1} K_p^n(n-d) \sum_{t=0}^{d} c_{d,t}(\Gamma) \frac{\theta_{d,t}\left(\frac{1}{4s}\right)}{(4\pi s)^{\frac{d}{2}}}.$$  

But

$$Z_{Id}(s) = (4\pi s)^{-n/2} \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} e^{-\left(\sum_{i=1}^{n} m_i^2\right)/4s} = \theta_{n,0}\left(\frac{1}{4s}\right) \frac{1}{(\sqrt{4\pi s})^n} = x(s)^n.$$  

and similarly,

$$(3.4) \quad \frac{\theta_{d,t}\left(\frac{1}{4s}\right)}{(4\pi s)^{d/2}} = \frac{\theta_0^{d-t}\left(\frac{1}{4s}\right) \theta_1^{t}\left(\frac{1}{4s}\right)}{(\sqrt{4\pi s})^{d-t} (\sqrt{4\pi s})^t} = x(s)^{d-t} y(s)^t$$
Thus
\[ |F| Z_p^\Gamma(s) = \binom{n}{p} x(s)^n + \sum_{d=1}^{n-1} K_p^n(n-d) \sum_{t=0}^{d} c_{d,t}(\Gamma) x(s)^{d-t} y(s)^t. \]

Since all numbers \( \binom{n}{p} \), \( K_p^n(n-d) \) and \( c_{d,t}(\Gamma) \) are integers, we are done. \( \square \)

In the case of groups with holonomy representation that is not of diagonal type, one has an analogue of (3.2) but the expression is more complicated. Since \( \Lambda \) is the canonical lattice, \( \Lambda = \Lambda' \). Furthermore, we have that \( 1 \leq n_B \leq n \) and \( n_B = n \) if and only if \( B = \text{Id} \). Thus, adding over \( \gamma = BL_b \) with fixed \( n_B = d \), we may write:
\[
Z_p^\Gamma(s) = \frac{\binom{n}{p}}{|\Gamma|^s} Z_{\text{Id}}(s) + \frac{1}{|\Gamma|^s} \sum_{d=1}^{n} (4\pi s)^{-\frac{d}{2}} \sum_{\gamma = BL_b \in \Lambda \setminus \Gamma \atop n_B = d} \frac{\text{tr}_p(B)}{\text{vol}(\Lambda B)} Z_p^{\Gamma}(s)
\]

where \( Z_\gamma(s) := \sum_{\gamma \in \mathbb{G}(\Lambda)} e^{-\frac{||\lambda_+ + b_+||^2}{4s}} \).

**Remark 3.3.** Let \( \Gamma \) a Bieberbach group of even (resp. odd) dimension \( n \) with diagonal holonomy representation and canonical lattice of translations \( \Lambda \). If \( M_\Gamma \) is the associated compact flat manifold then \( M_\Gamma \) is orientable if there is an only if \( Z_p^\Gamma(x(s), y(s)) \) has only even (resp. odd) degree terms for all \( 0 \leq p \leq n \), that is, \( c_{d,t}(\Gamma) = 0 \) for all \( d = 1, 3, \ldots, n-1 \) (resp. \( d = 0, 2, \ldots, n \)) and for all \( t \).

Indeed, let \( r : \Gamma \to O(n) \) be the projection \( r(BL_b) = B \). It is known that \( M_\Gamma \) is orientable if and only if \( r(\gamma) \in SO(n) \) for all \( \gamma \in \Gamma \), that is, \( n_B \in 2\mathbb{Z} \) (resp. \( n_B \in 2\mathbb{Z} + 1 \)) for all \( B \in F \) since \( B \) is a diagonal matrix for each \( BL_b \in \Gamma \). Now, using the expression (3.5) with canonical \( \Lambda \) and diagonal matrices, and comparing it with the expression (3.2), we see that:
\[
\sum_{\gamma = BL_b \in \Lambda \setminus \Gamma \atop n_B = d} \text{tr}_p(B)(4\pi s)^{-\frac{d}{2}} Z_p^{\Gamma}(s) = \sum_{1 \leq t \leq d} K_p^n(n-d)c_{d,t}(\Gamma) x(s)^{d-t} y(s)^t
\]

are the terms of degree \( d \), and this says that \( M_\Gamma \) is orientable if and only if \( Z_p^\Gamma(x(s), y(s)) \) has only even (resp. odd) degree terms. Now since
\[
\text{Coef}(x(s)^{d-t} y(s)^t) = K_p^n(n-d) c_{d,t}(\Gamma)
\]
and \( K_p^n(n-d) \neq 0 \) in general, it follows that \( M_\Gamma \) is orientable if and only if \( c_{d,t}(\Gamma) = 0 \) for each \( d = 1, 3, \ldots, n-1 \) if \( n \) is even (resp. \( d = 2, 4, \ldots, n-1 \) if \( n \) is odd), and for all \( t \).

**Dimension 4.** We now look at the special case of dimension 4. From Section 2 we know that the information of the \( L \)-spectrum is encoded in the terms \( Z_p^\Gamma(x(s), y(s)) \) of expression (3.2) we just have to look at the \( \theta_{d,t} \)'s, or as we have seen, at the monomials \( x(s)^{d-t} y(s)^t \). In general, however, if \( \Gamma \) is not of diagonal type, the squared lengths of the closed geodesics, \( ||\lambda_+ + b_+||^2 \), have summands different from \( m^2 \) or \( (m + \frac{1}{2})^2 \). For \( \text{Re}(s) > 0 \) and \( r = \frac{1}{4\pi s}, 0 \leq r < 1 \), we define the function
\[
\phi_{d,r}(s) := \sum_{(m_1, \ldots, m_d) \in \mathbb{Z}^d} e^{-s(m_1 + \frac{1}{2}m_2 + \cdots + m_d)^2} - m_d + r \]
and we set
\[
z_{d,r}(s) := \frac{\phi_{d,r}(\frac{1}{4\pi s})}{\sqrt{4\pi s}}.
\]

**Remark 3.4.** Note that \( \phi_{1,0}(z) = \theta_0(z) \) and \( \phi_{1,1/2}(z) = \theta_1(z) \) so we could just use the \( \phi_{d,r} \)'s, instead of the \( \theta_{d,t} \)'s, to express the zeta functions as polynomials.
Proposition 3.5. Let $\Gamma$ be a 4-dimensional Bieberbach group with canonical lattice $\Lambda$ and holonomy group $F$. Let $K = \mathbb{Z}(\sqrt{2}, \sqrt{3})$. Then $Z_\nu^\Gamma(s)$ is given by a polynomial expression with coefficients in $K$ in the variables $x(s), y(s), z_{1,r}(s), z_{2,r}(s), z_{3,r}(s)$, where $r = k/|F|$ and $0 \leq k \leq |F| - 1$.

Proof. The proposition follows by case by case verification, directly from the tables in Section 2. We need only check those groups with non-diagonal holonomy representation. We will carry out the verification for just one group to illustrate the method.

Take the group numbered 29 (see Table 2.11). For the first element $B_1$ we have $b_1 = -e_2 + e_3 + e_4$ and $\Lambda^B_1$ has the basis $\{e_2, e_3, e_4\}$. Since the vector $-e_2 + e_3$ is orthogonal to $e_2 + e_3$, it appears in the table in between parentheses. Thus $(b_1)_+ = \frac{1}{2}e_4$. Take $\lambda = m_1 e_1 + \cdots + m_4 e_4$ in $\Lambda$. The orthogonal projection of $\lambda$ onto the subspace generated by $e_2 + e_3$ is given by:

$$
\left((m_1 e_1 + m_2 e_2 + m_3 e_3 + m_4 e_4) \cdot \frac{(e_2+e_3)}{\sqrt{2}}\right) \frac{e_2+e_3}{\sqrt{2}} = \frac{(m_2+m_3)(e_2+e_3)}{\sqrt{2}},
$$

thus $b_1 + \lambda_+ = \frac{(m_2+m_3)}{\sqrt{2}} (e_2+e_3) + (m_4 + \frac{1}{2}) e_4$. In this way, for $\gamma_1 = B_1 L_{b_1}$, we have:

$$
Z_{\gamma_1}^{29}(s) = \sum_{\lambda_+ \in \Lambda^B_1} \sum_{(m_1,m_2,m_3) \in \mathbb{Z}^3} e^{\frac{|b_1+\lambda_+|^2}{4s}} = \sum_{(m_1,m_2,m_3) \in \mathbb{Z}^3} - \left\{ \frac{(m_1+m_2)^2+(m_3+\frac{1}{2})^2}{4s} \right\} e^{\frac{(m_1+m_2)^2}{4s}} e^{(m_3+\frac{1}{2})^2 \frac{1}{4s}} = \phi_{2,0}(\frac{1}{13}) \theta_1(\frac{1}{13}).
$$

Doing the same for $\gamma_2 = B_2 L_{b_2}$ and $\gamma_3 = \gamma_1 \gamma_2$ we get $Z_{\gamma_2}^{29}(s) = \phi_{2,0}(\frac{1}{15}) \theta_1(\frac{1}{15})$ and $Z_{\gamma_3}^{29}(s) = \theta_{2,2}(\frac{1}{15}) = \theta_1^2(\frac{1}{15})$. Now, using formula (3.5) we get for this case

$$
Z_{\nu}^{29}(s) = \frac{1}{4} Z_{\lambda_1}(s) + \frac{K_+^{(2)}}{2} \frac{1}{4\pi s} \left( \sqrt{2} Z_{\gamma_1}^{29}(s) + \sqrt{2} Z_{\gamma_2}^{29}(s) + Z_{\gamma_3}^{29}(s) \right)
$$

$$
= \frac{1}{4} x(s)^4 + \frac{K_+^{(2)}}{2} \frac{1}{4\pi s} \left( \sqrt{2} \phi_{2,0}(\frac{1}{15}) \theta_1(\frac{1}{15}) + \frac{1}{15} \right)
$$

$$
= \frac{1}{4} x(s)^4 + \frac{K_+^{(2)}}{2} \left( \sqrt{2} y(s) z_{2,0}(s) + (y(s)^2) \right)
$$

By repeating this argument for each group of non-diagonal type the proposition follows. \[\square\]

We note that these polynomials carry the $L$-spectrum information only in the variables $x(s), y(s), z_{i,r}(s), 1 \leq i \leq 3$. From Propositions 3.2 and 3.5 the following result is clear:

Corollary 3.6. $M_\Gamma$ and $M_{\Gamma'}$ are $L$-isospectral if and only if the nonzero monomials appearing in $Z_{\Gamma'}(s)$ and $Z_{\Gamma'}(s)$ are the same. In particular, if $\Gamma, \Gamma'$ are of diagonal type then: $M_\Gamma$ and $M_{\Gamma'}$ are $L$-isospectral if and only if for each fixed $1 \leq i \leq 3$ we have

$$
\{(t, d) : c_{d,t}(\Gamma) \neq 0 \} = \{(t, d) : c_{d,t}(\Gamma') \neq 0 \}.
$$

We now give the list of $p$-heat trace polynomials for the groups in the class considered in this paper, except for 72, 74, which have holonomy groups of orders 12 and 24 respectively and can not be $p$-isospectral to any other Bieberbach group.
Theorem 3.7. In the notation above we have:

\[ F \simeq \{ \text{Id} \} : \]
\[ Z^4_p(s) = (\frac{1}{p}) x(s)^4. \]

\[ F \simeq \mathbb{Z}_2 : \]
\[ 2Z^2_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s). \]
\[ 2Z^2_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)y(s)^2. \]
\[ 2Z^2_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) y(s)^3. \]
\[ 2Z^3_p(s) = (\frac{1}{p}) x(s)^4 + \frac{\sqrt{7}}{2} K^4_p(1) x(s)y(s)z_{20}(s) = 2Z^3_p(s). \]
\[ 2Z^3_p(s) = (\frac{1}{p}) x(s)^4 + \frac{\sqrt{7}}{2} K^4_p(1) y(s)^2 z_{20}(s) = 2Z^3_p(s). \]
\[ 2Z^3_p(s) = (\frac{1}{p}) x(s)^4 + \frac{\sqrt{7}}{2} K^4_p(1) y(s)z_{20}(s) = 2Z^3_p(s). \]
\[ 2Z^5_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(2) x(s)y(s). \]
\[ 2Z^5_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(2) y(s)^2. \]
\[ 2Z^6_p(s) = (\frac{1}{p}) x(s)^4 + \frac{\sqrt{7}}{2} K^4_p(2) y(s)z_{20}(s) = 2Z^6_p(s). \]

\[ F \simeq \mathbb{Z}_3 : \]
\[ 3Z^4_p(4)(s) = (\frac{1}{p}) x(s)^4 + \frac{\sqrt{3}}{3} tr_p(B) (z_{1,1/3}(s)z_{3,0}(s) + z_{1,2/3}(s)z_{3,0}(s)) = 3Z^4_p(4)(s). \]

\[ F \simeq \mathbb{Z}_2^2 : \]
\[ 4Z^7_p(s) = (\frac{1}{p}) x(s)^4 + 2K^4_p(1) x(s)^2 y(s) + K^4_p(2) x(s)y(s). \]
\[ 4Z^7_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + x(s)y(s)^2 + K^4_p(2) y(s)^2 = 4Z^{10}(s). \]
\[ 4Z^8_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + x(s)y(s)^2) + K^4_p(2) x(s)y(s) = 4Z^{9}(s). \]
\[ 4Z^8_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + y(s)^2 + K^4_p(2) y(s)^2. \]
\[ 4Z^9_p(s) = (\frac{1}{p}) x(s)^4 + 2K^4_p(1) x(s)^2 y(s) + K^4_p(2) y(s)^2. \]
\[ 4Z^9_p(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + y(s)^3 + K^4_p(2) x(s)y(s). \]
\[ 4Z^{10}(s) = (\frac{1}{p}) x(s)^4 + 2K^4_p(1) x(s)y(s)^2 + K^4_p(2) x(s)y(s). \]
\[ 4Z^{11}(s) = (\frac{1}{p}) x(s)^4 + 2K^4_p(1) x(s)y(s)^2 + K^4_p(2) y(s)^2. \]
\[ 4Z^{11}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + y(s)^2) + K^4_p(2) x(s)y(s). \]
\[ 4Z^{12}(s) = (\frac{1}{p}) x(s)^4 + \sqrt{2} K^4_p(1) x(s)y(s)z_{20}(s) + K^4_p(2) y(s)^2. \]
\[ 4Z^{12}(s) = (\frac{1}{p}) x(s)^4 + \sqrt{2} K^4_p(1) (x(s)^2 y(s) + x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s). \]
\[ 4Z^{13}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + \sqrt{2} x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s). \]
\[ 4Z^{13}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + \sqrt{2} x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s). \]
\[ 4Z^{14}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + \sqrt{2} K^4_p(2) x(s)y(s) + \sqrt{2} K^4_p(2) y(s)z_{20}(s) = 4Z^{14}(s). \]
\[ 4Z^{14}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + \sqrt{2} x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s). \]
\[ 4Z^{15}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + \sqrt{2} x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s) = 4Z^{15}(s). \]
\[ 4Z^{15}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) (x(s)^2 y(s) + \sqrt{2} x(s)y(s)z_{20}(s)) + \sqrt{2} K^4_p(2) y(s)z_{20}(s). \]
\[ 4Z^{16}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + K^4_p(2) x(s)y(s) + K^4_p(3) y(s) = 4Z^{16}(s). \]
\[ 4Z^{16}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + K^4_p(2) x(s)y(s) + K^4_p(3) y(s) = 4Z^{16}(s). \]
\[ 4Z^{16}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + K^4_p(2) x(s)y(s) + K^4_p(3) y(s) = 4Z^{16}(s). \]
\[ 4Z^{16}(s) = (\frac{1}{p}) x(s)^4 + K^4_p(1) x(s)^2 y(s) + K^4_p(2) x(s)y(s) + K^4_p(3) y(s) = 4Z^{16}(s). \]
$4Z_p^{24}(s) = \left(\frac{s}{p}\right) x(s)^4 + 3K_p^4(2) x(s) y(s) = 4Z_p^{26}(s)$.

$4Z_p^{25}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (2x(s) y(s) + y(s)^2)$.

$4Z_p^{27}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + 2y(s)^2)$.

$4Z_p^{28}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + \sqrt{2} y(s) z_{2,0}(s))$.

$4Z_p^{29}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (y(s)^2 + \sqrt{2} y(s) z_{2,0}(s))$.

$4Z_p^{30}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + \sqrt{2} y(s) z_{2,0}(s))$.

$4Z_p^{31}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + \frac{\sqrt{2}}{s} z_{2,0}(s) + \frac{\sqrt{2}}{s} z_{2,1/2}(s))$.

$F \simeq \mathbb{Z}_4$:

$4Z_p^{45}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) x(s) y(s) + tr_p(B) x(s) (z_{1,1/4}(s) + z_{1,3/4}(s))$.

$4Z_p^{47}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (y(s)^2 + tr_p(B) (z_{1,1/4}(s)^2 + 3z_{3,4}(s)^2)$.

$4Z_p^{50}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) x(s) y(s) + tr_p(B) (z_{1,1/4}(s) + z_{1,3/4}(s))$.

$4Z_p^{51}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + 2y(s)^2 + \sqrt{2} tr_p(B) z_{2,1/2}(s))$.

$F \simeq \mathbb{Z}_6$:

$6Z_p^{64}(s) = \left(\frac{s}{p}\right) x(s)^4 + \frac{\sqrt{3}}{s} tr_p(B^2) (z_{1,1/3}(s) z_{2,0}(s) + z_{1,2/3}(s) z_{2,0}(s)) + K_p^4(3) (y(s) + tr_p(B) (z_{1,1/6}(s) + z_{1,5/6}(s))$.

$F \simeq D_4$:

$6Z_p^{67}(s) = \left(\frac{s}{p}\right) x(s)^4 + 3\frac{\sqrt{7}}{s} K_p^4(2) (y(s) z_{2,0}(s) + \frac{\sqrt{7}}{s} tr_p(B) (z_{1,1/3}(s) + z_{1,2/3}(s)) z_{2,0}(s)$.

$F \simeq \mathbb{Z}_3^2$:

$8Z_p^{35}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) x(s)^2 y(s) + K_p^4(2) (x(s) y(s) + y(s)^2) + K_p^4(3) (y(s)$.

$8Z_p^{34}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (2x(s) y(s) + x(s) y(s)^2) + K_p^4(2) (x(s) y(s) + 2y(s)^2 + K_p^4(3) (y(s)$.

$8Z_p^{30}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s) + x(s) y(s)^2 + y(s)^3) + 3K_p^4(2) x(s) y(s) + K_p^4(3) (y(s)$.

$8Z_p^{36}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (2x(s)^2 y(s) + x(s) y(s)^2) + 3K_p^4(2) (x(s) y(s) + K_p^4(3) (y(s)$.

$8Z_p^{37}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (2x(s)^2 y(s) + y(s)^3) + K_p^4(2) (2x(s) y(s) + y(s)^2 + K_p^4(3) (y(s)$.

$8Z_p^{38}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s)^2 y(s) + 2x(s) y(s)^2) + K_p^4(2) (2x(s) y(s) + y(s)^2 + K_p^4(3) (y(s)$.

$8Z_p^{39}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s)^2 y(s) + 2x(s) y(s)^2) + K_p^4(2) (2x(s) y(s) + y(s)^2 + K_p^4(3) (y(s)$.

$8Z_p^{40}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s)^2 y(s) + 2x(s) y(s)^2) + K_p^4(2) (2x(s) y(s) + y(s)^2 + K_p^4(3) (y(s)$.

$8Z_p^{41}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s)^2) + 3K_p^4(2) x(s) y(s) + K_p^4(3) (y(s)$.

$8Z_p^{42}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s)^2) + 3K_p^4(2) x(s) y(s) + K_p^4(3) (y(s)$.

$8Z_p^{43}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s)^2) + K_p^4(2) (2x(s) y(s) + y(s)^2) + 3K_p^4(3) (y(s)$.

$8Z_p^{44}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s)^2) + 3K_p^4(2) x(s) y(s) + 3K_p^4(3) (y(s)$.

$F \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$:

$8Z_p^{55}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(1) (x(s) y(s)^2) + K_p^4(2) (2x(s) y(s) + tr_p(B_1) (z_{1,1/4}(s) + z_{1,3/4}(s) + tr_p(B_1 B_2) x(s) (z_{1,1/4}(s) + z_{1,3/4}(s) + K_p^4(3) (y(s) = 8Z_p^{58}(s)$.

$F \simeq D_4$:

$8Z_p^{54}(s) = \left(\frac{s}{p}\right) x(s)^4 + \sqrt{2} K_p^4(1) (x(s) y(s) z_{2,0}(s) + K_p^4(2) (y(s)^2 + \sqrt{2} y(s) z_{2,0}(s) + \sqrt{2} tr_p(B_1) z_{2,1/2}(s)$.

$8Z_p^{56}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (y(s) x(s) + \sqrt{2} z_{2,0}(s)) + tr_p(B_1) (z_{1,1/4}(s) + z_{1,3/4}(s) + 2K_p^4(3) (y(s$.

$8Z_p^{60}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (3x(s) y(s) + \sqrt{2} y(s) z_{2,0}(s) + tr_p(B_1) x(s) (z_{1,1/4}(s) + z_{1,3/4}(s))$.

$8Z_p^{61}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (x(s) y(s) + 2y(s)^2 + \sqrt{2} y(s) z_{2,0}(s)) + tr_p(B_1) x(s) (z_{1,1/4}(s) + z_{1,3/4}(s))$.

$8Z_p^{62}(s) = \left(\frac{s}{p}\right) x(s)^4 + K_p^4(2) (3x(s) y(s) + \sqrt{2} y(s) z_{2,0}(s)) + tr_p(B_1) y(s) (z_{1,1/4}(s) + z_{1,3/4}(s))$. 

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Remark 3.8. (a) With a similar asymptotic argument as the one used in Lemma 3.1, one can show that the functions $x(s), y(s), z_{i,k}(s)$ for $1 \leq i \leq 3, 0 \leq k \leq |F| - 1$, appearing in the previous theorem are algebraically independent. This fact will be used in the proof of the main results in the following section.

(b) Let $\Gamma$ be a 4-dimensional Bieberbach group with canonical lattice and holonomy group $F$. We note that one can read off some properties of $\Gamma$ from the expression of the zeta function. For instance: (i) $\Gamma$ has diagonal holonomy representation if and only if $Z_p^\Gamma(s) \in \mathbb{Z}[x(s), y(s)]$. (ii) $M_\Gamma$ is orientable if and only if for every $i, j, k$ with $i + j + k$ odd, the coefficient of $x(s)^iy(s)^jz(s)^k$ in $Z_0^\Gamma(s)$ is zero.

4. ISOSPECTRAL CLASSES

In this section we shall use the information on the $p$-heat trace polynomials in the previous sections to determine all isospectral classes for the different type of spectra for all compact flat manifolds of dimension 4 in the class considered. We will thus find all pairs that are isospectral, $p$-isospectral, $L$-isospectral or $[L]$-isospectral in this case.

Theorem 4.1 ($p$-isospectrality). Let $\Gamma, \Gamma'$ be two 4-dimensional Bieberbach groups with canonical translation lattice. Then:

(i) If $p = 0, 4$: $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral if and only if $\{\Gamma, \Gamma'\}$ is is of the following:

\begin{align*}
3, 3' \} & : \{6, 6' \} \ (F \simeq \mathbb{Z}_2), \{47, 47' \} \ (F \simeq \mathbb{Z}_3), \{7', 10 \}, \{8, 9' \}, \{14, 14' \}, \\
15, 15' \} \ & : \{18, 19 \}, \{19', 21 \}, \{22, 22' \}, \{24, 26 \} \ (F \simeq \mathbb{Z}_2^2), \text{ or } \{34, 38 \}, \{35, 40 \}, \\
39, 41 \} \ & : \{F \simeq \mathbb{Z}_3^2 \}, \{57, 58 \} \ (F \simeq \mathbb{Z}_2 \times \mathbb{Z}_4).
\end{align*}

(ii) If $p = 1, 3$: $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral if and only if $\Gamma, \Gamma'$ both belong to one of the following families:

\begin{align*}
\{3, 3' \} & : \{6, 6' \} \ (F \simeq \mathbb{Z}_2), \{47, 47' \} \ (F \simeq \mathbb{Z}_3), \{7, 9 \}, \\
\{7', 8, 9', 10 \}, \{8', 10' \}, \{10'' , 10', \{14, 14' \}, \{15, 15' \}, \{18, 19, 20 \}, \{19', 20', 21 \}, \\
22, 22' \} \ & : \{23, 23' \} \ (F \simeq \mathbb{Z}_2^2), \{24, 25, 26, 27, 28, 29, 29', 50, 51 \} \ (F \simeq \mathbb{Z}_2^2 \times \mathbb{Z}_4), \{34, 36, 38 \}, \{39, 41 \}, \\
\{39, 40 \} \ & : \{F \simeq \mathbb{Z}_2^2 \}, \{57, 58 \} \ (F \simeq \mathbb{Z}_2 \times \mathbb{Z}_4) \text{ or } \{60, 61 \} \ (F \simeq \mathbb{Z}_4).
\end{align*}

(iii) If $p = 2$: $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral if and only if $\Gamma, \Gamma'$ both belong to one of the following families:

\begin{align*}
\{2, 2', 2'' \} & : \{3, 3' \} \ (F \simeq \mathbb{Z}_2), \{47, 47' \} \ (F \simeq \mathbb{Z}_3), \\
\{7, 8, 9', 10', 10'', 12', 18, 19, 19', 21, 21', 23, 50 \}, \{7', 8', 9, 10, 11, 12, 20, 20', 23', \\
31 \} \ & : \{F \simeq \mathbb{Z}_2^2 \times \mathbb{Z}_4), \{33, 37, 39, 41, 43 \}, \{34, 38 \}, \{35, 36, 40, 42, 44 \} \ (F \simeq \mathbb{Z}_2^3), \{57, 58 \} \ (F \simeq \mathbb{Z}_2 \times \mathbb{Z}_4).
\end{align*}

Proof. By Theorem 1.3, if $M_\Gamma$ and $M_{\Gamma'}$ are $p$-isospectral then $|F| = |F'|$. Therefore when looking for $p$-isospectrality we only need to compare pairs both with $F \simeq \mathbb{Z}_2$, $F \simeq \mathbb{Z}_3$, $F \simeq \mathbb{Z}_2^2$ or $\mathbb{Z}_4$, $F \simeq \mathbb{Z}_2^3 \times \mathbb{Z}_4$ or $D_4$ or else $F \simeq \mathbb{Z}_6$ or $D_3$. Also, for each $p, \beta_p$ gives the multiplicity of the eigenvalue 0 of $\Delta_p$, hence we need only consider pairs having the same $\beta_p$. Take $|F| = 2$. Here the groups are $2, 2', 2'', 3, 3', 3'', 4, 5, 5', 6, 6'$. Looking at the corresponding $p$-heat trace polynomials in Section 3, we see that they are all different from each other, hence they cannot be pairwise isospectral. However, the vanishing of the Krawtchouk polynomials for some values of $p$ yields some $p$-isospectrality for $1 \leq p \leq 3$. For instance, for $p = 1$ or 3 we know that $K^4_p(2) = 0$, so $Z^5_p(s) = Z^5_p(s) = Z^6_p(s) = 2x(s)^4$. Thus $5, 5'$, $6$ and $6'$ are 1-isospectral and 3-isospectral. In the same way, $K^4_p(1) = K^4_p(3) = 0$ for $p = 2$ and so $Z_2(s) = 3x(s)^4$ for all groups $2, 2', 2'', 3, 3', 3'', 4$, hence these groups are 2-isospectral to each other.

In the cases of the remaining groups, i.e. those with $|F| = 3, 4, 6$ and 8, if $p = 0$, using the complete list of the $p$-heat trace polynomials given in Section 3, we easily see that only the pairs $\{3, 3' \}, \{3', 3'' \}, \{6, 6' \}, \{47, 47' \}, \{7', 10 \}, \{8, 9' \}, \{14, 14' \}, \{15, 15' \}, \{18, 19 \}, \{19', 21 \}, \{22, 22' \}, \{24, 26 \}, \{34, 38 \}, \{35, 40 \}, \{39, 41 \}, \{57, 58 \}$ have the same $p$-heat
trace expressions for $p = 0$. Actually they have the same $p$-heat trace for all $p$, thus they are $p$-isospectral for $0 \leq p \leq 4$.

In the case of the $p$-spectrum for $1 \leq p \leq 3$, again by comparison of the $p$-heat traces one obtains the $p$-isospectral sets asserted in the theorem. \hfill \Box

**Corollary 4.2.** Let $\Gamma, \Gamma'$ be as in Theorem 4.1. Then $M_\Gamma$ and $M_{\Gamma'}$ are isospectral if and only if they are $p$-isospectral for $0 \leq p \leq 4$.

**Remark 4.3.** (i) Observe that the groups in each isospectral pair in Theorem 4.1 have the same holonomy representation.

(ii) The assertion in the corollary fails to be true for general flat manifolds. Examples of 0-isospectral and not 1-isospectral flat manifolds in dimension $n \geq 6$ are given in [6]. In [2] two nonhomeomorphic flat 3-manifolds that are isospectral on functions are given, showing this is the only such pair. These manifolds have different holonomy groups: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively and one can check they are not isospectral on 1-forms by comparison of the $p$-heat traces.

(iii) The theorem also shows that isospectrality on functions is much less common than $p$-isospectrality for $1 \leq p \leq 3$. Some facts can be observed. For $p = 1, 3$, one of the $p$-isospectral pairs, $\{60, 61\}$, involves two groups with non-abelian holonomy group $D_4$. Also, in the $p$-isospectral set $\{24, 25, 26, 27, 28, 29, 29', 50, 51\}$ five different holonomy representations are present. Here, the first seven groups have holonomy $F \simeq \mathbb{Z}_4^2$ and give orientable manifolds, while the last two groups have holonomy $F \simeq \mathbb{Z}_4$ and produce non-orientable manifolds. Relative to 2-isospectral classes, again some manifolds are orientable and others not, and different holonomy groups occur.

We now turn into the study of length spectra for the flat 4-manifolds considered in this paper. It is a classical result that in the case of flat tori the eigenvalue spectrum and the length spectrum determine each other. This is not true for general Riemannian manifolds, even in the flat case. There exist 4-dimensional flat manifolds (namely $8, 9'$ in our notation) which are isospectral but such that some multiplicities of closed geodesics are different (see [8], Ex. 3.4, up to isometry; see also the references therein for earlier examples in the case of nilmanifolds). In particular, this shows that in general, the eigenvalue spectrum does not determine the length spectrum. In the converse direction, in [8] an example was given, for $n = 13$, of $[L]$-isospectral flat manifolds which are not isospectral on functions.

The study of $[L]$-isospectrality is delicate since multiplicities must be taken into account. We recall that the multiplicity of a length $l$ is the number of conjugacy classes in $\Gamma$ having length $l$. We shall see that already in the context in this paper, most $L$-isospectral pairs fail to be $[L]$-isospectral. The next result describes the situation.

**Theorem 4.4 (L-isospectrality).** Let $\Gamma, \Gamma'$ be two 4-dimensional Bieberbach groups with canonical translation lattice.

(i) The $L$-isospectral sets are the following: $\{3, 3''\}, \{3', 3'''\}, \{6, 6'\}, \{47, 47'\}, \{7', 10\}, \{8, 9'\}, \{14, 14'\}, \{15, 15'\}, \{18, 19\}, \{19', 21\}, \{22, 22'\}, \{24, 26\}, \{34, 38, 39, 41\}, \{35, 40\}, \{57, 58\}$ and, in addition, the sets $\{25, 27\}, \{33, 43\}, \{42, 44\}$.

(ii) The $[L]$-isospectral pairs are $\{3, 3''\}, \{3', 3'''\}, \{6, 6'\}, \{47, 47'\}, \{7', 10\}, \{14, 14'\}, \{15, 15'\}, \{18, 19\}, \{19', 21\}, \{22, 22'\}, \{24, 26\},$ and $\{57, 58\}$.

**Proof.** (i) All isospectral pairs in (i) of Theorem 4.1 are $L$-isospectral. As we have seen in Section 3, the information on the $L$-spectrum of $\Gamma$ depends only on the monomials occurring in the 0-heat trace. Since for the $L$-spectrum we do not look at multiplicities, $\Gamma, \Gamma'$ will be $L$-isospectral if and only if the same monomials occur in the 0-heat trace polynomials (see corollary 3.6).
Checking the list of p-heat traces we see, for instance, that \(4Z_p^{25}(s) = \binom{n}{p} x(s)^4 + K_p^4(2) \left( y(s)^2 + 2x(s)y(s) \right) \) and \(4Z_p^{27}(s) = \binom{n}{p} x(s)^4 + K_p^2(2) \left( x(s)y(s) + 2y(s)^2 \right) \), thus 25 and 27 are L-isospectral. In the same way we see that \(\{33, 43\}, \{42, 44\}\) and \(\{34, 38, 39, 41\}\) are L-isospectral.

(ii) Since \([L]-isospectrality\) implies \(L\)-isospectrality, we must only consider the sets given in (i).

In [8], the length spectrum of flat manifolds is studied and a criterion to prove \([L]\)-isospectrality is given (see Proposition 3.1). This criterion depends on the existence of a bijection \(\Phi\) with certain properties between partitions \(P\) and \(P'\) of \(F\) and \(F'\) respectively. Also, in [8] the criterion is applied to show \([L]\)-isospectrality of some pairs, including the pair \(\{24, 26\}\) (see Example 3.3, up to isomorphism).

We shall sketch the verification for the pair \(\{3, 3''\}\) (see Table 2.1). We have that \(\Gamma \) (resp. \(\Gamma''\)) is generated by \(\gamma = d(I, J)L_b\) (resp. \(\gamma'' = d(I, J)L_{b'}\)) and \(L_{\Lambda}, \) where \(b = \frac{e+b}{2}\), (resp. \(b'' = \frac{e+b}{2}\)).

The elements of \(\Gamma \) (resp. \(\Gamma''\)) are either of the form \(\gamma L_\lambda\) (resp. \(\gamma'' L_\lambda\)) or \(L_\lambda\) with \(\lambda = m_1e_1 + m_2e_2 + m_3e_3 + m_4e_4 \in \Lambda, \) i.e. \(m_i \in \mathbb{Z}\).

We define a correspondence \(\phi\) from \(\Gamma\) to \(\Gamma''\) such that \(L_\lambda \mapsto L_\lambda\) and \(\gamma L_\lambda \mapsto \gamma'' L_\lambda L_{-e_3}\). We note that both of these elements have squared length \((m_1 + \frac{1}{2})^2 + m_2^2 + \frac{1}{2}(m_3 + m_4)^2\).

If we use the relation (4.1) below we see that \(\gamma L_\lambda \sim \gamma L_{\lambda L_{m(e_3-e_4)}}\) and similarly \(\gamma'' L_\lambda \sim \gamma'' L_{\lambda L_{m(e_3-e_4)}}\), for any \(m \in \mathbb{Z}\). We note that (4.2) does not introduce new conjugacy relations among the elements of \(\Gamma\) and \(\Gamma''\), since the holonomy group is cyclic.

It is now easy to see that the map \(\phi\) induces the bijection \(\Phi = \text{Id}\) from \(F\) to \(F'' = F\), satisfying all the conditions in Proposition 3.1 of [8], hence \(\{3, 3''\}\) are \([L]\)-isospectral.

The \([L]\)-isospectrality of the remaining pairs in (ii) of the theorem, can be proved similarly, with \(\Phi = \text{Id}\) as the bijection. We shall omit this verification.

To conclude the proof of the theorem we must show that the remaining \(L\)-isospectral sets, \(\{8, 9'\}, \{25, 27\}, \{33, 43\}, \{34, 38, 39, 41\}, \{35, 40\}, \{42, 44\}\), are not \([L]\)-isospectral.

The pair \(\{8, 9'\}\) was shown not to be \([L]\)-isospectral in [8] (up to isometry, see Example 3.4). We now discuss in some detail the case \(\{25, 27\}\).

If \(l\) is the length of a closed geodesic in \(M_\Gamma\), the multiplicity is given by \(m_{\Gamma}(l) = \sum_{\gamma \in \Lambda \setminus \Gamma} m_{\gamma}(l)\) where

\[
m_{\gamma}(l) = \# \{ [\gamma L_\lambda] \in [\Gamma] : \lambda \in \Lambda, \ell(\gamma L_\lambda) = l \}.
\]

In order to compute these multiplicities it is necessary to have a parametrization of the \(\Gamma\)-conjugacy classes in \(\Gamma\). This is complicated in general but it simplifies for Bieberbach groups of diagonal type. From [8] we have the following relations. For general \(\Gamma\), if \(\gamma_i = B_i L_{b_i}, \gamma_j = B_j L_{b_j} \in \Gamma\) and \(\lambda, \mu \in \Lambda\), the conjugations \(\gamma_i L_\lambda \gamma_i^{-1}\) and \(\gamma_i L_\lambda L_{(B_i^{-1} - \text{Id})\mu}\) give the relations:

\[
(4.1) \quad L_\lambda \sim L_{B_i \lambda} \quad \text{and} \quad \gamma_i L_\lambda \sim \gamma_i L_{\lambda L_{(B_i^{-1} - \text{Id})\mu}}
\]

Furthermore, if \(F\) is abelian, the conjugation \(\gamma_j(\gamma_i L_\lambda)\gamma_j^{-1}\), with \(i \neq j\), gives the relation

\[
(4.2) \quad \gamma_i L_\lambda \sim \gamma_i L_{\eta_{ij}(\lambda)} \quad \text{where} \quad \eta_{ij}(\lambda) := B_j \lambda + (B_j - \text{Id}) b_i + B_j (B_j - \text{Id}) b_j.
\]

The next table gives the elements \(\gamma_1 = B_1 b_1, \gamma_2 = B_2 L b_2, \gamma_{12} = B_{12} b_{12}\) in \(25, 27\), in column notation, i.e. showing in the columns, for each \(B L b \in \Gamma\), the diagonal entries of \(B\) together with the coordinates of the corresponding translation vector \(b\) (see [6]).

| \(B_1\) | \(b_{12}^{d_1}\) | \(b_1^{d_1}\) | \(B_2\) | \(b_{12}^{d_2}\) | \(b_2^{d_2}\) | \(B_1 B_2\) | \(b_{12}^{d_{12}}\) | \(b_2^{d_{12}}\) |
|---|---|---|---|---|---|---|---|---|
| -1 | 1 | 1/2 | 1/2 | -1 | 1/2 | 1/2 |
| -1 | -1 | 1/2 | 1/2 | 1 | 1/2 | 1/2 |
| 1 | 1/2 | -1 | -1 | 1 | 1/2 | 1/2 |
| 1 | 1/2 | 1 | 1/2 | 1 | 1/2 | 1/2 |

The squared lengths of closed geodesics corresponding to elements not in \(\Lambda\) are given by
Thus $m_3^2 + (m_4 + \frac{1}{2})^2$, $(m_1 + \frac{1}{2})^2 + m_4^2$, $(m_2 + \frac{1}{2})^2 + (m_4 + \frac{1}{2})^2$, with $m_i \in \mathbb{Z}$.

The minimal length in both cases is $l = \frac{1}{2}$. We shall show that this length has different multiplicities for 25 and 27. In 25 the elements with $l = \frac{1}{2}$ have the form:

$B_1 L_{\frac{\lambda_1}{2}}$ where $\lambda_1 = m_1 e_1 + m_2 e_2 - m_4 e_4$ with $m_1, m_2 \in \mathbb{Z}$, $m_4 \in \{0, 1\}$

$B_2 L_{\frac{\lambda_2}{2}}$ where $\lambda_2 = -n_1 e_1 + n_2 e_2 + n_3 e_3$ with $n_2, n_3 \in \mathbb{Z}$, $n_1 \in \{0, 1\}$.

In 27 the elements with $l = \frac{1}{2}$ have the form:

$B_1 L_{\frac{\lambda_1'}{2}}$ where $\lambda_1' = r_1 e_1 + r_2 e_2 - r_3 e_3$ with $r_1, r_2 \in \mathbb{Z}$, $r_3 \in \{0, 1\}$.

It follows from (4.1) that $m_1, m_2, n_2, n_3, r_1, r_2$ can be taken mod (2).

Using the relations (4.2) we have:

$\gamma_1 L_{m_1 e_1 + m_2 e_2 + m_4 e_4} \sim \gamma_1 L_{(m_1 - 1)e_1 - (m_2 + 1)e_3 + m_4 e_4}$ in 25

$\gamma_2 L_{n_1 e_1 + n_2 e_2 + n_3 e_3} \sim \gamma_2 L_{-(n_1 + 1)e_1 - (n_2 + 1)e_2 + n_3 e_3}$ in 25

$\gamma_1 L_{r_1 e_1 + r_2 e_2 - r_3 e_3} \sim \gamma_1 L_{(r_1 - 1)e_1 - (r_2 + 1)e_3 - r_3 e_3}$ in 27.

These three relations divide by 2 the number of relations, and there are no other relations. Thus $m_{25}(\frac{1}{2}) = 4 + 4 = 8$ while $m_{27}(\frac{1}{2}) = 4$. This shows that 25, 27 do not have the same multiplicities.

The verification of non $|L|$-isospectrality in the remaining cases can be done similarly as for 25, 27 (by comparing the multiplicities of small lengths) and will be omitted.

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