Constraints on interacting dynamical dark energy from the cosmological equation of state

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Abstract:

We consider a generic description of interacting dynamical Dark Energy, characterized by an equation of state with a time dependent coefficient $w(t)$, and which may interact with both radiation and matter. Without referring to any particular cosmological model, we find a differential equation which must be satisfied by $w(t)$ and involving the function $Q(t)$ which describes the interaction between Dark Energy and the other cosmological fluids. The relation we find may serve as a test or constraint for various models of interacting dynamical Dark Energy.
1 Introduction

The accelerating expansion of our Universe \cite{1,2} is well described by a cosmological constant in the Einstein-Hilbert action of General Relativity, paradoxically introduced by Einstein \cite{3} in order to have a static Universe. Current observations do not allow, up to now, to claim that the description of Dark Energy (DE) in terms of a cosmological constant is not correct. There are a few well known good reasons to be unsatisfied with the description in terms of a cosmological constant, but the known difficulties, or seemingly unnatural coincidences \cite{4}, can be solved invoking very peculiar initial conditions. In the hopeful wait of an experimental conclusive evidence, theorists since long time provided us with a variety of alternative models for DE \cite{5}, with the request that any cosmological model should reproduce an Universe which, at our epoch, is almost perfectly flat and filled by matter and DE in the ratio of about 3/7, where the DE is effectively approximated by a constant. The number of candidates for DE models is exceeded only by the hypothesis on the nature of Dark Matter, which is the sign that the problem remains unsolved. In Literature alternatives can be found, like, for instance, the quintessence models \cite{6,7}, where the role of the cosmological constant is played by scalar potentials, suitably parametrized to get the desired behavior, and the K-essence models \cite{8,9,10}, likewise built in terms of scalar fields, where the accelerated expansion of the Universe is driven by the kinetic term. Both quintessence and K-essence models belong to the wider category of modified theories of gravity, whose purpose is to extend their range of validity to large, galactic, scales. In the most general case, any dynamical, as opposed to constant, model for DE may interact with all the components of the cosmological perfect fluid in terms of which is written the energy momentum tensor appearing at the right hand side of the Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \ T_{\mu\nu}. \]  

(1.1)

The coupling could be minimal, through the metric dependent invariant measure, or non minimal, with direct and non trivial coupling with gravity, like in the scalar-tensor theories, or by means of direct interactions with baryonic matter, and/or with neutrinos, and/or with Dark Matter \cite{11,12,13,14,15,16}.

In this paper we keep a very general perspective. Without referring to any particular DE model, we assume only that DE is realized by means of a perfect fluid satisfying an Equation of State (EoS) with a time dependent \(w\)-coefficient

\[ p_{\text{DE}} = w(t) \rho_{\text{DE}} \]  

(1.2)

and that DE interacts non minimally with any cosmological component. The interactions result in broken covariant conservation laws of the energy
momentum tensors of the single cosmological components

$$\nabla_\mu (T_i)_\mu = (Q_i)_\nu \quad i=\text{matter, radiation, DE} \quad (1.3)$$

keeping the total energy momentum tensor conserved. The aim of this paper is to give a criterion to select amongst different models of interacting dynamical DE, assuming only the validity of the Friedmann equations for the scale factor appearing in the Robertson-Walker metric. This subject has been faced following different strategies [17, 18, 19, 20, 21, 22, 23, 24], all of which need some kind of assumptions, on the phenomenological form of the interactions $Q_i$, or on the choice of the potential in the quintessence models, for instance. In this paper, we try to be as much general as possible, adopting a model independent cosmographic approach (see [25] for an updated review).

In order to reach this goal, in Section 2 we relate the time derivative of the DE EoS coefficient $w(t)$ to the interactions $Q(t)$, by means of the kinematic variables associated to the scale factor $a(t)$: the Hubble parameter $H(t)$, the deceleration $q(t)$ and the jerk $j(t)$, sometimes called statefinder $r(t)$-parameter [26]. In Section 3 we evaluate the measurable quantities which are involved, comparing their experimental values with the ones expected in the ΛCDM model. In the concluding Section 4 we summarize and discuss our results.

## 2 Constraints on interacting dynamical Dark Energy

The energy momentum tensor for a cosmological perfect fluid is:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \quad , \quad (2.1)$$

where $U_\mu$ is the fluid four-velocity, $\rho$ is the rest-frame energy density and $p$ is the isotropic rest-frame pressure. The EoS relates pressure and energy density and its general form is:

$$p = p(\rho) \quad , \quad (2.2)$$

whose simplest case is represented by the linear relation

$$p = w\rho \quad , \quad (2.3)$$

where $w$ is a coefficient not depending from the energy density $\rho$. We consider here the more general EoS (2.2), whose Taylor expansion around the energy density at the present epoch $\rho_0 = \rho(t)|_{t=t_0}$ is

$$p(\rho) = p_0 + \kappa_0(\rho - \rho_0) + \mathcal{O}[(\rho - \rho_0)^2] \quad , \quad (2.4)$$

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where \( p_0 = p(\rho_0) \) and \( \kappa_0 = \left. \frac{\delta p}{\delta \rho} \right|_{t=t_0} \).

The aim is to express the first two coefficients of the above expansion in terms of the scale factor \( a(t) \) appearing in the Robertson - Walker metric

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

(2.5)

where \( k \) is a constant parameter related to the spatial curvature: \( k = 0, k > 0 \) and \( k < 0 \) for flat, closed and open Universes, respectively.

More precisely, we would like to write \( p_0 \) and \( \kappa_0 \) in terms of the kinematic variables related to the dimensionless time derivatives of \( a(t) \), namely the Hubble parameter \( H(t) \), the deceleration \( q(t) \) and the jerk \( j(t) \), which are observables quantities, thus making this approach independent from a particular cosmological model. The kinematic variables are defined as follows

\[
H = \frac{\dot{a}}{a}, \quad q = -\frac{\ddot{a}}{aH^2}, \quad j = \frac{\dddot{a}}{aH^3}.
\]

(2.6, 2.7, 2.8)

It is customary to suppose that the cosmological fluid is an incoherent mixture of the three forms of canonical fluids \( (i = 1 \text{ matter}, i = 2 \text{ radiation}, \ i = 3 \text{ DE represented by a cosmological constant } \Lambda) \) plus, following a standard notation [27], the spatial curvature contribution \( (i = 4) \), each satisfying the linear EoS

\[
p_i = w_i \rho_i.
\]

(2.9)

According this standard picture, the values of \( w_i \) are displayed in Table 1.

| \( i \) | \( w_i \) |
|---|---|
| 1: matter | 0 |
| 2: radiation | 1/3 |
| 3: DE/\( \Lambda \) | -1 |
| 4: curvature | -1/3 |

Table 1. EoS coefficients.

Consequently, the EoS [23] takes the form

\[
\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} w_i \rho_i.
\]

(2.10)

Once we define the density parameters

\[
\Omega_i = \frac{8\pi G}{3H^2} \rho_i,
\]

(2.11)

\[\text{From now on, } \mathcal{O}_0 \equiv \mathcal{O}(t)|_{t=t_0}, \text{ for any observable } \mathcal{O}(t), \text{ where } t_0 \text{ stands for the present epoch.}\]
the Friedmann equation can be written

\[ \sum_i \Omega_i = 1 . \] (2.12)

We denote with \( \overline{O} \) the average of generic physical observables \( \mathcal{O}_i \) weighted by the density parameters \( \Omega_i \) of each fluid:

\[ \overline{O} \equiv \frac{4}{\Omega} \sum_i \mathcal{O}_i \Omega_i . \] (2.13)

Consequently, from the EoS (2.3) we have

\[ w = \frac{p}{\rho} = \frac{\sum_i p_i}{\sum_i \rho_i} = \frac{\sum_i w_i \rho_i}{\sum_i \rho_i} = \frac{\sum_i w_i \Omega_i}{\sum_i \Omega_i} = \sum_i w_i \Omega_i = \overline{w} . \] (2.14)

In this paper we introduce the following two generalizations with respect to the standard approach:

1. We allow a time dependence of the EoS coefficients \( w_i \) appearing in (2.9)

\[ p_i = w_i(t) \rho_i . \] (2.15)

Even though we are mostly interested in physical situations where only the DE fluid may have a time dependent \( w_3(t) \), for the moment we take a more general attitude. The known scalar quintessence model for DE is an example of DE fluid with a time dependent EoS coefficient, but we point out that in this paper we do not necessarily limit ourselves to this particular case.

2. The energy momentum tensor (2.1) is given by the sum of the different components of the perfect cosmological fluid listed in Table 1. We give the possibility to each component to break the conservation law:

\[ \nabla_{\mu}(T_{i\nu})^{\mu} = (Q_{i\nu}) , \] (2.16)

keeping the total energy momentum tensor conserved, which implies a constraint on the breakings

\[ \nabla_{\mu}T_{\nu}^{\mu} = 0 \Rightarrow \sum_i (Q_{i\nu}) = 0 . \] (2.17)

In most cases, only the matter and DE components of the energy momentum tensor possibly display a breaking of the conservation law in the late Universe, not the radiation nor the curvature contributions. Again, for the moment we stay on general grounds, and the breakings \( Q_i \), which, because of the constraint (2.17) must be at least two, physically correspond to interactions between the cosmological components fluids. Examples of non-vanishing DE interactions are given in [17, 18, 19, 20, 21, 22, 23, 24].
Deriving both sides of the Friedmann equation (2.12) with respect to time, we have
\[ \sum_i \dot{\Omega}_i = 0 \Rightarrow \sum_i \frac{d}{dt} \left( \frac{\rho_i}{H^2} \right) = 0 \, . \] (2.18)

To calculate \( \dot{\rho}_i \), we use the covariant conservation of the energy momentum tensor (2.1), The \( \nu = 0 \) component of (2.16) gives
\[ \dot{\rho}_i = -3H(\rho_i + p_i) + Q_i \, , \] (2.19)
where we defined
\[ Q_i \equiv -(Q_i)_0 = +(Q_i)^0 \, . \] (2.20)

On the other hand
\[ \dot{H} = \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 = -H^2(1 + q) \, , \] (2.21)
where we used the definition (2.7) of the deceleration \( q(t) \).

Coming back to Eq. (2.18), we can now write
\[ 0 = \sum_i \left( \frac{\dot{\rho}_i}{H^2} - \frac{2}{H^2} \rho_i \dot{H} \right) \]
\[ = \sum_i \left( -3H(1 + w_i)\rho_i + \frac{Q_i}{H^2} + \frac{2}{H^2} \rho_i H^2(1 + q) \right) \] (2.23)
\[ = -3H(1 + \bar{w}) + 2H(1 + q) \, , \] (2.24)
where we used (2.19), (2.9) and (2.21), and we used the definition of weighted average (2.13) for the quantities \( w_i(t) \) and the constraint (2.17) on the breakings \( Q_i \).

Therefore, using (2.14), the following relation holds
\[ \bar{w} = w = \frac{2q - 1}{3} \, , \] (2.25)
which, in particular, relates the first coefficient of the Taylor expansion of the EoS (2.4) to the deceleration \( q(t) \), since
\[ p_0 = w_0 \rho_0 = \frac{2q_0 - 1}{3} \rho_0 \, . \] (2.26)

Notice that the relation (2.25) has a general validity, since it holds also for \( \dot{w}_i \neq 0 \) and \( Q_i \neq 0 \).
Let us now consider $\kappa_0$, the second coefficient of the EoS Taylor expansion (2.4):

$$\kappa = \frac{dp}{d\rho} = \frac{\dot{p}}{\dot{\rho}} = \frac{\sum_i \dot{p}_i}{\sum_i \dot{\rho}_i} = \frac{\sum_i (w_i \dot{\rho}_i + \dot{w}_i \rho_i)}{\sum_i (\dot{\rho}_i + Q_i)}$$

$$= \frac{\sum_i \{w_i [-3H(1 + w_i)\rho_i + Q_i] + \dot{w}_i \rho_i\}}{\sum_i((-3H)(1 + w_i)\rho_i + Q_i]}$$

$$= \frac{\sum_i((-3H)(w_i + w_i^2)\rho_i + w_i Q_i + w_i \rho_i)}{\sum_i((-3H)(1 + w_i)\rho_i)}$$

$$= \frac{w + \overline{w^2}}{1 + w} \frac{8\pi G \sum_i w_i Q_i}{3H^3} \frac{\overline{w}}{1 + w} - \frac{3H}{1 + w} \kappa_0,$$  \hspace{1cm} (2.27)

where we took into account (2.19), (2.17) and (2.13). We need $\dot{w}$, i.e. the weighted average of the time derivatives of the EoS coefficients $w_i(t)$, which vanish in the ΛCDM model. To obtain it, we look for an expression for the time derivative of the weighted average $\overline{w}$:

$$\dot{\overline{w}} = \sum_i \frac{d}{dt} (w_i \Omega_i) = \dot{\overline{w}} + \frac{8\pi G}{3} \sum_i w_i \frac{d}{dt} \left( \frac{\rho_i}{H^2} \right)$$

$$= \dot{\overline{w}} + \frac{8\pi G}{3} \sum_i \left( \frac{\dot{\rho}_i}{H^2} - \frac{\rho_i}{H^3} \frac{2}{H} \dot{H} \right)$$

$$= \dot{\overline{w}} + \frac{8\pi G}{3} \sum_i \left[ (-3H)(1 + w_i)\rho_i + Q_i \right] + \rho_i \frac{2}{H} (1 + q) \right]$$

$$= \dot{\overline{w}} - 3H (\overline{w} + \overline{w^2}) + \frac{8\pi G}{3H^2} \sum_i w_i Q_i + 3H \overline{w} (1 + w) \overline{w},$$  \hspace{1cm} (2.28)

where, in the last row, we used (2.25) to eliminate the deceleration $q$ in favor of $\overline{w}$. Introducing the variance of the values $w_i$

$$\sigma_w^2 = \overline{w^2} - \overline{w^2},$$  \hspace{1cm} (2.29)

we get

$$\overline{w} = \dot{\overline{w}} + 3H \sigma_w^2 - \frac{8\pi G}{3H^2} \sum_i w_i Q_i.$$  \hspace{1cm} (2.30)

It is easily seen that, using in (2.27) the above expression (2.30) for $\overline{w}$ and the definitions of the deceleration $q(t)$ (2.7) and of the jerk $j(t)$ (2.8), we finally get

$$\kappa = \frac{dp}{d\rho} = \frac{j - 1}{3(1 + q)},$$  \hspace{1cm} (2.31)

which, as $w(t)$ (2.25), is an universal quantity, whose expression is valid whether $\dot{w}_i \neq 0$ and $Q_i \neq 0$ or not.
Let us take for a moment Eq. (2.30) at $\dot{w}_i = Q_i = 0$, which is the standard case we are generalizing in this paper. For the variance (2.29) we get

$$\sigma^2_w = -\frac{\bar{w}}{3H} = \frac{2}{9}[j - q(1 + 2q)] ,$$

(2.32)

where $\bar{w}(t)$ in (2.25) and the definition (2.28) of the jerk $j(t)$ have been used. The above expression for $\sigma^2_w$ tells us how the weighted accuracy on the estimate of the $w_i$, assumed to be constant, evolves in time, driven by the time dependence of the cosmological parameters $\Omega_i$ only. In general, it is not allowed to deduce, as done in [28], that the right hand side of (2.32) is non-negative, since the weights $\Omega_i$ present in $\sigma^2_w = \sum_i (w_i - \bar{w})^2 \Omega_i$ may be negative. Indeed, while $\Omega_1$ and $\Omega_2$ are certainly non-negative functions of time, since they are related to matter and radiation energy density respectively, the density parameters $\Omega_3$ and $\Omega_4$, which refer to DE and curvature, might, in principle, have any sign. What we can state, is that, at our epoch, $\Omega_1^{(0)} \simeq 0.3$, $\Omega_2^{(0)} \simeq 0$ and $\Omega_3^{(0)} \simeq 0.7$ [1], and, consequently, that the Universe, in excellent approximation, is spatially flat $k \simeq 0$. Therefore, at our epoch, but not at any time, the right hand side of (2.32) is non-negative

$$j_0 \geq q_0(1 + 2q_0) .$$

(2.33)

Eq. (2.33) is a constraint which must be satisfied, at $t = t_0$, by the kinematical variables related to the time derivatives of the scale factor $a_0$, namely the deceleration $q_0$ and the jerk $j_0$.

In case of non vanishing $\dot{w}_i$ and $Q_i$, we have the more general relation (2.30), which relates the possible time dependent $w_i(t)$ appearing in the EoS (2.9) of the cosmological fluids (2.3) to their corresponding interactions (2.16). At the present epoch $t = t_0$ we have:

$$\bar{w} + 3H\sigma^2_w \bigg|_{t=t_0} \equiv K_0 = \sum_i \left(\dot{w}_i \Omega_i + \frac{8\pi G}{3H^2} w_i Q_i \right) \bigg|_{t=t_0} ,$$

(2.34)

where $K_0$ is a physical observable, depending on measurable quantities (density parameters and kinematic variables), and is numerically evaluated in Section 3, from the most recent observational data [29].

Making the reasonable assumption that only the DE component of the cosmological perfect fluid may have an EoS of the form (2.3) with $\dot{w}_3(t) \neq 0$, and observing that the relevant interactions are the ones involving DE, which translates into $Q_3 \neq 0$, the relation (2.31) at $t = t_0$ reduces to

$$\dot{w}_3 \Omega_3 + \frac{8\pi G}{3H^2} w_3 Q_3 \bigg|_{t=t_0} = K_0 ,$$

(2.35)

where we used the fact that, for matter, $w_1$ strictly vanishes: $w_1 = 0$. 

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Since the values at our epoch of the DE density parameter $\Omega_3^{(0)}$, of the coefficient of the DE EoS $w_3^{(0)}$, of the Hubble constant $H_0$ and of the quantity $K_0$, are known, the relation (2.35) represents a constraint on the possible theoretical models of interacting dynamical DE, in particular on the time dependence of the DE EoS coefficient $\dot{w}_3^{(0)}$ and on the DE interaction $Q_3^{(0)}$

$$w_3(t) = w_3^{(0)} + \dot{w}_3^{(0)}(t-t_0) + O(t^2)$$ \hspace{1cm} (2.36)

$$Q_3(t) = Q_3^{(0)} + O(t) \hspace{1cm} (2.37)$$

which, according to (2.35), are not independent one from each other.

3 Comparison with $\Lambda$CDM model and observational results

The aim of this paper is to put constraints, mainly by means of the relation (2.35), on the possible models of DE, with particular concern on the DE EoS and on the interaction DE-matter. In order to be able to make a comparison, it is useful to summarize what is predicted by the Standard Model of cosmology, and to take a quick look to the present phenomenology. After the observational evidence from supernovae for an accelerating Universe and a cosmological constant [1], we know that our Universe is filled by DE and (mostly dark) matter:

$$\Omega_1 \simeq 0.3 \ ; \ \Omega_3 \simeq 0.7 \ ,$$ \hspace{1cm} (3.1)

where we used the notations adopted in this paper, according to which the subscripts 1 and 3 stand for matter and DE, respectively. As immediate consequence of the Friedmann equation, our Universe is almost flat

$$k \simeq 0 \ ,$$ \hspace{1cm} (3.2)

at our epoch the radiation contribution to the whole cosmological perfect fluid being highly suppressed:

$$\Omega_2 \simeq 0 \ .$$ \hspace{1cm} (3.3)

The $\Lambda$CDM model well describes this scenario, where the DE is realized by means of a cosmological constant $\Lambda$. The $\Lambda$CDM situation, including the EoS coefficients $w_i$ of the single cosmological fluids, is summarized in the following Table 2:

|       | i=1: matter | i=2: radiation | i=3: DE | i=4: curvature |
|-------|-------------|----------------|---------|----------------|
| $\Omega_i$ | 0.3        | 0.0            | 0.7     | 0.0            |
| $w_i$     | 0.0         | 1/3            | -1.0    | -1/3           |

**Table 2.** $\Lambda$CDM coefficients.
In the ΛCDM model the only EoS coefficient which survives is \( w_3 \). Its value \( (w_3 = -1) \) corresponds to the contribution to the cosmological fluid coming from the cosmological constant, whose presence modifies the vacuum Einstein equations as follows

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}R = 0 ,
\]

which can be considered as the vacuum contribution to the energy momentum tensor

\[
(T_3)_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu} ,
\]

where \( G \) is the Newton constant. The energy momentum tensor (3.5) is of the form (2.1) provided that

\[
p_3 = -p_3 = -\frac{\Lambda}{8\pi G} ,
\]

which justifies the fact that, according to the ΛCDM model, the DE EoS coefficient \( w_3 \) must be equal to minus one.

According to the ΛCDM model, the jerk variable (2.8) should be constant, and in particular

\[
\Lambda_{\text{CDM}} \Rightarrow j(t) = 1 .
\]

This can be seen in many ways. In the particular framework of this paper, let us consider the expression (2.32) for the variance of the EoS coefficient \( w \), which holds for \( \dot{w}_i = Q_i = 0 \) and hence true in the ΛCDM model:

\[
j = \frac{9}{2}\sigma_w^2 + q(2q + 1) .
\]

From the definition of weighted average (2.13), of variance (2.29) and using the fact that in the ΛCDM model the only non-vanishing EoS coefficient is the DE one, we have

\[
\sigma_w^2 = w_3^2\Omega_3(1 - \Omega_3) .
\]

On the other hand, from \( \bar{w} \) in (2.25), we get the following relation for the deceleration parameter \( q(t) \):

\[
q = \frac{3}{2}w_3\Omega_3 + \frac{1}{2} .
\]

Using (3.9) and (3.10), we get the following relation for the jerk parameter

\[
j = 1 + \frac{9}{2}w_3(1 + w_3) ,
\]

which is equal to one if the DE is described by a cosmological constant \( w_3 = -1 \). Therefore, the jerk parameter is a model independent, kinematic observable valuable to test the ΛCDM model, since any deviation from \( j = 1 \)
is a signal of alternative descriptions. We shall comment on the experimental situation later in this Section.

Let us now consider $K_0$ defined by the left hand side of (2.34)

$$K_0 \equiv \frac{\dot{\rho}}{\rho} + 3H^2 \sigma \left|_{t=t_0} \right. . \quad (3.12)$$

According to the $\Lambda$CDM model, the DE has a constant EoS $w$-coefficient, and does not interact, hence, from the right hand side of (2.34), $K_0$ should vanish identically:

$$\Lambda \text{CDM} \Rightarrow K_0 = 0 . \quad (3.13)$$

From its definition (3.12), it is easy to check that $K_0$ can be written in terms of measurable quantities as follows:

$$\frac{K_0}{H_0} = 3(1 - \Omega_1) - 2 \left[ j_0 - q_0(1 + 2q_0) \right] . \quad (3.14)$$

We point out that, analogously to the case concerning the jerk parameter $j(t) \neq 1$, a non-vanishing value for $K_0$ would be the certain signal for a deviation from the $\Lambda$CDM model. But even a $K_0$ compatible with zero would not represent a confirmation of $\Lambda$CDM. Both cases $K_0 = 0$ and $K_0 \neq 0$, in fact, could be realized by means of an interacting dynamical DE, with $\dot{w}_3 \neq 0$ and/or $Q_3 \neq 0$. Once again, our point of view is to test and constrain possible models of interacting dynamical DE, assuming that $\Lambda$CDM is a model which well describes, “only” in an effective way, the observations on the Universe so far.

Finally, we give out some numbers. Concerning the deceleration $q_0$ and the jerk $j_0$, evaluated at our epoch, we report in Table 3 four maximum likelihood values, with their 68% confidence intervals, corresponding to different combinations of data, taken at low redshifts (the details of the analysis can be found in [29]):

|   | $a$       | $b$       | $c$       | $d$       |
|---|-----------|-----------|-----------|-----------|
| $q_0$ | $-0.644 \pm 0.223$ | $-0.6401 \pm 0.187$ | $-0.930 \pm 0.218$ | $-1.2037 \pm 0.175$ |
| $j_0$ | $1.961 \pm 0.926$  | $1.946 \pm 0.871$  | $3.369 \pm 1.270$  | $5.423 \pm 1.497$  |

Table 3. Deceleration and jerk.

According to the $d$-dataset in Table 3, which contains all the others (see [29] for details), it is apparent that the $\Lambda$CDM value $j_0 = 1$ is incompatible with data, at 3.06σ confidence limit.

Concerning $K_0$, it is convenient to consider the dimensionless quantity $K_0/H_0$, in order to get rid of the well known tension existing on the Hubble constant [30, 31, 32, 33, 34, 35]. A rough and preliminary estimate, which takes into account the values of $q_0$ and $j_0$ given above and the value of the matter
density parameter $\Omega_1$, which, according to the latest SN Ia measurements from the Pantheon Catalogue \cite{36}, is

$$\Omega_1 = 0.298 \pm 0.022 ,$$

(3.15)
gives the four values listed in Table 4:

| $K_0$      | $a$          | $b$          | $c$          | $d$          |
|------------|--------------|--------------|--------------|--------------|
| $-0.516 \pm 0.685$ | $-0.512 \pm 0.622$ | $-1.026 \pm 0.931$ | $-1.817 \pm 1.091$ |

Table 4. Estimates of $K_0$.

All the above values of $K_0$ are compatible with the $\Lambda$CDM value $K_0 = 0$, within $3\sigma$. Therefore this is not a conclusive result, neither in favor nor in contrast to the $\Lambda$CDM model. A more accurate, constrained analysis will be done following the Bayesian methods in cosmology \cite{37}, but for the moment our aim is just to give an estimate of the right hand side of our result (2.35), by means of observable quantities.

4 Conclusions

The points where the $\Lambda$CDM model creaks are more and more. An example of these weaknesses is the well known tension on the measurements of the Hubble constants $H_0$. The value given by the Planck collaboration \cite{30,31} in the framework of the $\Lambda$CDM model is incompatible with other, model independent, estimates \cite{32,33}. The inconsistencies become milder if a dynamical DE is invoked \cite{38}. Therefore, there are strong motivations to investigate models of dynamical DE which, in the most general case, displays an EoS with a time dependent coefficient $w_{DE}(t)$ and which may interact, in principle, with matter and/or radiation through a (partial) breaking $Q_{DE}(t)$ of the covariant conservation of the energy momentum tensor. In this paper we found the relation (2.34), which must be satisfied also by any model of interacting dynamical DE

$$\Omega_{DE} \dot{w}_{DE} + \frac{8\pi G}{3H^2} Q_{DE} w_{DE} = K ,$$

(4.1)

where $K(t)$ is expressed in terms of measurable quantities. The above equation is a differential equation for the DE EoS parameter $w_{DE}(t)$ with time dependent coefficients, one of which is the interaction $Q_{DE}(t)$. It must hold at any time, in particular must be satisfied by phantom DE models, crossing the $\Lambda$CDM point $w = -1$ in both directions. The analysis which led to (4.1) is model independent, in the sense that we only assumed that the scale factor $a(t)$ appearing in the Robertson-Walker metric (2.5) obeys the Friedmann equation (2.12) and that the quantities involved in $K(t)$ are the
density parameters and the kinematic variables, hence are directly measurable. According to the ΛCDM model, $w_{DE} = -1$, $j = 1$ and $K = 0$. Any deviation from these values must be interpreted as a failure of the ΛCDM model. Low-redshift data show that, at present time, $j_0 \neq 1$, while, according to a rough estimate, $K_0$ is compatible with zero, but a more accurate evaluation is in progress. Finally, the relation \( [\text{[1]}] \) can be read in several ways, depending whether the DE is interacting or not \( (Q_{DE} \neq 0 \text{ or } Q_{DE} = 0) \). It is important to emphasize this point because previous attempts to get informations on the Dark Sector rely on particular assumptions. Our result, together with a careful analysis of the available observational data, may provide a model independent description of the Dark Sector, as well as a constraint for generic parametrizations of the EoS coefficients $w_{DE}$ and of the interactions $Q_{DE}(t)$.

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