Spectral radius and the 2-power of Hamilton cycles
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Abstract: Let $G$ be a graph of order $n$ and spectral radius be the largest eigenvalue of its adjacency matrix, denoted by $\mu(G)$. In this paper, we determine the unique graph with maximum spectral radius among all graphs of order $n$ without containing the 2-power of a Hamilton cycle.

Keywords: 2-power of graphs; Hamilton cycle; Spectral radius.

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1 Introduction

We will start with introducing some background information that will lead to our main results. Some important previously established facts will also be presented.

1.1 Background

Let $G = (V(G), E(G))$ be a simple graph and $\mu(G)$ be the largest eigenvalue of its adjacency matrix, where $V(G)$ and $E(G)$ are vertex set and edge set, respectively. For $u, v \in V(G)$, the distance between $u$ and $v$ is defined to be the number of edges of a shortest path from $u$ to $v$. We write $d(G)$ for $|E(G)|$, $\delta(G)$ stands for the degree of the vertex $u$ in $G$, and $\delta(G)$ stands for the minimum degree in $G$. We use $K_n, S_n, P_n$ and $C_n$ to denote the complete graph, the star, the path and the cycle of order $n$, respectively. Note that $C_n$ is also called a Hamilton cycle of $G$ with order $n$. The join of two disjoint graphs $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex of $G_1$ to each vertex of $G_2$. For odd number $n$, we use $F_n$ to stand for a union of $\binom{n}{3}$ triangles sharing a single common vertex. $K_{a,b}$ denotes the complete bipartite graph with vertex partition sets of sizes $a$ and $b$. Let $S_{n,k}$ be the graph obtained by joining each vertex of $K_k$ to $n-k$ isolated vertices. A wheel $W_n$ is the graph obtained by joining $C_{n-1}$ with an additional vertex. Let $G + e$ denote the graph obtained from $G$ by adding an edge $e$ between a pair of non-adjacent vertices. Let $G^−$ denote the set of graphs obtained from $G$ by deleting any edge and $G^+$ denote the set of graphs obtained from $G$ by adding a new vertex and joining it to any one vertex of $G$. If there is only one non-isomorphic graph in $G^−$ or $G^+$, then we also use $G^−$ or $G^+$ to denote this unique graph. Also, $G^{+k}$ denotes the set of graphs obtained from $G$ by adding a new vertex and joining it to any $k$ vertices of $G$. Let $\overline{G}$ denote the complement graph of $G$, where any two vertices in $\overline{G}$ are adjacent if they are not adjacent in $G$. The $k$-power of a graph $G$, denoted by $G^k$, is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in $G$ is at most $k$. Let $H$ be any subgraph of $G$. Then $G \setminus E(H)$ denotes the graph obtained from $G$ by deleting edges of $H$. We adopt the notation and terminologies in [1, 14, 31, 37, 38] except as stated otherwise.

Let $\mathcal{K}(n,t)$ be the set of graphs on $n$ vertices with $t$ edges. Let $M_n$ be the $2n$-vertex graph on $n$ independent edges. Let $S_n^*$ be the graph obtained from $S_n$ by adding a new vertex and joining it to a leaf of $S_n$. Let $T_{a,b,c}$ stand for a $T$-shaped tree defined as a tree with a single vertex $u$ of degree 3 such that $T_{a,b,c} - u = P_a \cup P_b \cup P_c$ ($a \leq b \leq c$). Similarly, $T_{a,b,c,d}$ stands for a $T$-shaped tree defined as a tree with a single vertex $u$ of degree 4 such that $T_{a,b,c,d} - u = P_a \cup P_b \cup P_c \cup P_d$ ($a \leq b \leq c \leq d$) and $T_{a,b,c,d,e}$ can be defined in the same way. Let $D_n$ be the double-snake of order $n$, depicted in Figure 1. Let $\mathcal{L}_n$ be a set of graphs with $n \geq 9$ and the graphs in $\mathcal{L}_n$ are depicted in Figure 2.

For an integer $k \geq 0$, the $k$-closure of the graph $G$ is a graph obtained from $G$ by successively joining pairs of nonadjacent vertices whose degree sum is at least $k$ (in the resulting graph at each stage) until

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no such pair remains \[3, 39\]. Write \(C_k(G)\) for the \(k\)-closure of \(G\). Note that \(d_{C_k(G)}(u) + d_{C_k(G)}(v) \leq k - 1\) for any pair of nonadjacent vertices \(u\) and \(v\) in \(C_k(G)\).

For short, we omit the isolated vertices of graphs in this paper. For example, in the case \(n = 6\), \(M_2\) should be \(M_2 \cup K_3\) with minimum degree one (without considering isolated vertices). Similarly, \(S_{n-4} \cup K_3\) should be \(S_{n-4} \cup K_3 \cup K_1\) with minimum degree one.

A well known theorem of Dirac [8] states that if \(G\) is a graph on \(n\) vertices with \(\delta(G) \geq \frac{n}{2}\), then \(G\) contains a hamiltonian cycle. In 1963, Posá (see also in \([9]\) by Erdős) conjectured that if \(\delta(G) \geq \frac{2}{3}n\), then \(G\) contains a \(C_n^2\). Later in 1974, Seymour [36] generalized Posá’s conjecture and conjectured that if \(\delta(G) \geq \frac{1}{k+1}n\), then \(G\) contains a \(C_n^k\). Using the Regularity Lemma and Blow-up Lemma, Komlós, Sárközy and Szemerédi [22] proved Seymour’s conjecture in asymptotic form, then in [23, 24] they proved both conjectures for \(n \geq n_0\). Later Levitt, Sárközy and Szemerédi [25] presented another proof (and a general method) that avoids the use of the Regularity Lemma and thus the resulting \(n_0\) is much smaller.

Fan and Haggkvist [10] proved that if \(\delta(G) \geq \frac{5}{7}n\), then \(G\) contains a \(C_n^2\). Fan and Kierstead [11] showed that for every \(\epsilon \geq 0\), there exists a constant \(c\), such that if \(\delta(G) \geq (\frac{2}{3} + \epsilon)n + c\), then \(G\) contains a \(C_n^2\). Next they [12] showed that if \(\delta(G) \geq \frac{2n-1}{3}\), then \(G\) contains a \(P_n^2\). For more results about the existence of \(C_n^k\) in graphs, we refer the reader to [13, 24].

A central problem of extremal graph theory is as following: for a given graph \(H\), what is the maximum number of edges of an \(H\)-free graph of order \(n\)? This question and its extensions are called Turán type problems (for example, [20, 34]). Moreover, much attention has been paid to spectral Turán type problems, i.e., what is the maximum (signless Laplacian, \(p\)-Laplacian) spectral radius of an \(H\)-free graph of
order \( n \) (see for example, 15 14 35 58)? In recent years, some attention has been given to the relations between (signless Laplacian, \( p \)-Laplacian) spectral radius and cycles of fixed length and particularly Hamilton cycle; see 3 15 10 17 90 52 55. Yuan 20 determined the maximum number of edges of a graph without containing the 2-power of a Hamilton cycle which extends a well-known theorem of Ore 34 concerning the maximum number of edges of a graph without containing a Hamilton cycle. Motivated by 20 directly, it is worth to focus on the spectral Turán type problems, i.e., what is the maximum (signless Laplacian, \( p \)-Laplacian) spectral radius of a \( C_n^2 \)-free graph of order \( n \)? In this paper, we determine the unique graph with maximum spectral radius among all graphs of order \( n \) without containing the 2-power of a Hamilton cycle. On the other hand, in this paper, we also consider the relationship between the complement of a graph \( G \) without containing the 2-power of a Hamilton cycle and its spectral radius \( \mu(G) \), one may refer to 15 18 38. Other related results can be found in 2 6 7 4 14 19 31 37 20 27 28 29.

1.2 Main results

In this article, we determine the unique graph with maximum spectral radius among all graphs of order \( n \) without containing the 2-power of a Hamilton cycle. This extends a theorem of Long-Tu Yuan 20 in 2021 concerning the maximum number of edges of an \( n \)-vertex graph without containing the 2-power of a Hamilton cycle. We will establish the following theorems.

**Theorem 1.1.** Let \( G \) be graph on \( n \geq 18 \) vertices. If \( e(G) \geq \frac{n^2-3n+3}{2} \), then \( G \) contains the 2-power of a Hamilton cycle unless \( G \) is a subgraph of \( Y_n \), where \( Y_n \) is \( K_n \setminus E(S_{n-3}) \), \( K_n \setminus E(S_{n-4} \cup S_4) \) or \( K_n \setminus E(S_{n-4} \cup K_3) \).

**Theorem 1.2.** Let \( G \) be graph on \( n \geq 18 \) vertices. If \( \mu(G) \geq \mu(K_n \setminus E(S_{n-3})) \), then \( G \) contains the 2-power of a Hamilton cycle unless \( G = K_n \setminus E(S_{n-3}) \).

**Theorem 1.3.** Let \( G \) be graph on \( n \geq 18 \) vertices. If \( \mu(G) \leq \sqrt{n-5} \), then \( G \) contains the 2-power of a Hamilton cycle unless \( C_n(G) = K_n \).

1.3 Preliminaries

In this subsection, we first recall some important known results and then present a few technical lemmas which will be used in the proofs of our main results.

**Theorem 1.4 (20).** Let \( G \) be an \( n \)-vertex graph with \( n \geq 6 \) and without containing the 2-power of a Hamilton cycle. Then we have \( e(G) \leq \begin{cases} 25, & n = 8; \\ 49, & n = 11; \\ C_{n-1}^2 + 3, & \text{otherwise.} \end{cases} \) Moreover, the equality holds if and only if \( G = K_n \setminus E(F) \) with \( F \in \mathcal{H}_n \), where \( \mathcal{H}_n \) is a family of graphs with \( n \geq 6 \) as follows:

\[
\mathcal{H}_n = \begin{cases} S_3, & n = 6; \\ S_4, K_3, & n = 7; \\ K_3, & n = 8; \\ K_4, S_6, & n = 9; \\ K_4, S_7, & n = 10; \\ K_4, & n = 11; \\ S_9, & n = 12; \\ S_{10}, & n = 13; \\ S_{11}, K_5, & n = 14; \\ S_{n-3}, & n \geq 15. \end{cases}
\]

**Proposition 1.5 (20).** Let \( n \geq 6 \). If \( C_{n-1}^2 \) contains a copy of \( F \) with \( n - 1 \) vertices, then

(i) \( C_n^2 \) contains each graph in \( F^+ \) as a subgraph.

(ii) Let \( \left\lceil \frac{n-1}{4} \right\rceil - 1 = t \geq 1 \). Then \( C_n^2 \) contains each graph in \( F^+t \) as a subgraph.
Lemma 1.6 (24). Let \( n \geq 6 \). If \( n \neq 8, 11 \), then \( C_n^2 \) contains each copy of \( F \in K(n, n - 4) \backslash H_n \). If \( n = 8, 11 \), then \( C_n^2 \) contains each copy of \( F \in K(n, n - 5) \backslash H_n \).

The following fact is obvious.

Fact 1.7. Let \( n \geq 6 \). If \( \frac{n}{4} \in \mathbb{Z} \), then \( C_n^2 \) contains a copy of \( K_{\frac{n}{4}} \) but not \( K_{\frac{n}{4} + 1} \). If \( \frac{n}{4} \notin \mathbb{Z} \), then \( C_n^2 \) contains a copy of \( K_{\frac{n}{4} + 1} \).

Lemma 1.8 (19). Let \( G \) be a simple connected graph of order \( n \) with \( m \) edges. The spectral radius \( \mu(G) \) satisfies \( \mu(G) \leq \sqrt{2m - n + 1} \) with equality if and only if \( G \) is isomorphic to \( S_n \) or \( K_n \).

Theorem 1.9 (15). Let \( G \) be a graph of order \( n \) and spectral radius \( \mu(G) \). If \( \mu(G) > n - 2 \), then \( G \) contains a Hamilton cycle unless \( G = K_{n - 1} + e \). Let \( \mu(G) \) be the spectral radius of the complement graph \( \overline{G} \). If \( \mu(G) \leq \sqrt{n - 2} \), then \( G \) contains a Hamilton cycle unless \( G = K_{n - 1} + e \).

Lemma 1.10 (18). Let \( G \) be a graph of order \( n \). Then the spectral radius \( \mu(G) \) of \( G \) satisfies \( \mu(G) \geq \frac{1}{n} \sum_{u \in V(G)} d_G^2(u) \) with equality if and only if each component of \( G \) is an \( r \)-regular graph or an \((r_1, r_2)\)-biregular graph, where \( r^2 = \frac{1}{n} \sum_{u \in V(G)} d_G^2(u) \) and \( r_1, r_2 \) satisfy \( r_1 r_2 = r^2 \).

Lemma 1.11. Let \( 6 \leq n \leq 15 \). Then \( C_n^2 \) contains each copy of \( F \in K(n, n - 3) \) unless \( F \) contains one of \( \mathcal{F}_n \) as a subgraph, where \( \mathcal{F}_n \) is a family of graphs with \( 6 \leq n \leq 15 \) as follows:

\[
\mathcal{F}_n = \begin{cases}
S_3, & n = 6; \\
S_4, C_4, K_3, & n = 7; \\
K_3, S_5, & n = 8; \\
K_4, S_6, F_5, & n = 9; \\
K_4, S_7, S_5, & n = 10; \\
K_4, S_8, & n = 11; \\
K_5, S_9, & n = 12; \\
S_{10}, K_5, & n = 13; \\
S_{11}, K_5, & n = 14; \\
S_{12}, & n = 15.
\end{cases}
\]

Proof. Let \( n = 6 \). Then \( C_6^2 = M_3 \) and \( K(6, 2) = \{M_2, S_3\} \). Hence it is easy to see that \( K(6, 3) = \{M_3, S_3 \cup K_2, P_4, S_5, K_3\} \). Then obviously the lemma holds for \( n = 6 \).

Let \( n = 7 \). Then \( C_7^2 = C_7 \) and \( K(7, 3) = \{M_3, S_3 \cup K_2, P_4, K_3, S_4\} \). Hence it is easy to see that \( K(7, 4) = \{M_2 \cup S_3, P_4 \cup K_2, 2S_3, P_5, K_3 \cup K_2, C_4, T_{1,1,2}, S_4 \cup K_2, K_3^+; S_5\} \). Then obviously the lemma holds for \( n = 7 \).

Let \( n = 8 \). Then clearly \( C_8^2 \) does not contain a copy of \( K_3 \). It is easy to see that \( K(8, 4) = \{M_4, M_2 \cup S_3, P_4 \cup K_2, 2S_3, S_4 \cup K_2, P_5, K_4^+, C_4, S_5, K_3 \cup K_2, T_{1,1,2}\} \) and \( K(8, 5) = \{M_2 \cup P_4, M_2 \cup S_4, S_4 \cup S_4, P_5 \cup K_2, T_{1,1,2} \cup K_2, C_4^+, P_4 \cup S_3, P_6, T_{1,2,2}, 2S_3 \cup K_2, T_{1,1,3}, D_6, C_4 \cup K_2, C_5\} \cup \mathcal{F}_8' \). Let \( \mathcal{F}_8' = \{T_{1,1,1,2}, K_4^+ \cup K_2, K_3 \cup S_3, K_3 \cup M_2, S_5 \cup K_2, S_6, S_5', K_4', G_2, G_3, G_4\} \), where \( G_2, G_3 \) and \( G_4 \) are obtained from \( K_3^+ \) by adding a new vertex and joining it to a vertex of \( K_3^+ \) with degree one, two and three, respectively. It is straightforward to check that \( C_8^2 \) contains each graph in \( K(8, 5) \backslash \mathcal{F}_8' \) and the graphs in \( \mathcal{F}_8' \) contains a copy of either \( K_3 \) or \( S_5 \). So the lemma holds for \( n = 8 \).

Let \( n = 9 \). Let \( t = \left\lceil \frac{a+b}{4} \right\rceil - 1 = \left\lceil \frac{a+b}{4} \right\rceil - 1 = 1 \). Regardless of isolated vertices of \( F \), we consider the following two cases:

(a.1) \( \delta(F) \geq t + 2 = 2 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil = 6 \).

(b.1) \( \delta(F) = 1 \). Then by Proposition 1.5(i), we only need to consider \( F \in \mathcal{F}_8' \).

In case of (a.1) the graphs with \( \delta(F) \geq 2 \) and \( \epsilon(F) = 6 \) are \( C_6, 2K_3, K_4, F_5, K_2,3 \) and \( C_5 + e \). A simple observation shows that \( C_9^2 \) contains \( C_6, 2K_3, K_4, F_5, K_2,3 \) and \( C_5 + e \) as subgraphs. Now the graphs in (b.1) are the graphs obtained from a graph belonging to \( \mathcal{F}_8' \) by adding an isolated vertex \( v \) and an arbitrary edge.
In case of (a.4), the graphs with $\delta(F) \geq t + 1 = 3$. Then the number of non-isolated vertices of $F$ is at most \(\left\lfloor \frac{2(n-3)}{(n-2)} \right\rfloor = 4\).

(b.2) $\delta(F) = 2$. Then by Proposition 1.8(iii), we only need to consider $F \in \{K_4^-, S_6\}^{+2}$.

(c.2) $\delta(F) = 1$. Then by Proposition 1.8(i), we only need to consider the graphs obtained from a graph in $K(9,6)$ which contains one of graph in $F_9$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

Clearly, there is no graph in (a.2). As for the graphs in (b.2), we only need to consider $F \in \{K_4^-, S_6\}^{+2}$ because of $\delta(F) = 2$ without considering isolated vertices. A simple observation shows that $\overline{C_{10}}^2$ contains each graph in $\{K_4^-, S_5, S_2\}$ as a subgraph. Clearly, we know that $\overline{C_{10}}^2$ contains a copy of $K_4^-$ but not $K_4$ by Fact 1.7. Thus it is not hard to show that $\overline{C_{10}}^2$ contains each graph in (c.2) except $K_4 \cup K_2, K_4^+, S_2, S_7 \cup K_2, S_8, S_7 + e$. Thus we are done for $n = 10$. Moreover, $\overline{C_{10}}^2$ contains each graph in $K(10,6) \setminus \{K_4, S_7\}$ as a subgraph by Lemma 1.6.

Let $n = 11$. Let $t = \left\lceil \frac{n-1}{2} \right\rceil - 1 = \left\lceil \frac{11-1}{2} \right\rceil - 1 = 2$. Note that $\overline{C_{11}}^2$ contains each graph in $K(11,7) \setminus \{K_4, K_4 \cup K_2, S_8\}$ by Lemma 1.6. Regardless of isolated vertices of $F$, we consider the following cases:

(a.3) $\delta(F) \geq t + 1 = 3$. Then the number of non-isolated vertices of $F$ is at most \(\left\lfloor \frac{2(n-3)}{(n-2)} \right\rfloor = 5\).

(b.3) $\delta(F) = 2$. Then by Proposition 1.8(ii), we only need to consider $F \in \{K_4, S_7\}^{+2}$.

(c.3) $\delta(F) = 1$. Then by Proposition 1.8(i), we only need to consider the graphs obtained from a graph in $K(10,7)$ which contains one of graph in $F_{10}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.3) the unique graph with $\delta(F) \geq 3$ and $e(F) = 8$ is $W_5$ and $\overline{C_{11}}^2$ contains a copy of $W_5$. As for the graphs in (b.3), we only need to consider $F \in \{K_4\}^{+2} = K_4^+$ because of $\delta(F) = 2$.

Clearly, we know that $\overline{C_{11}}^2$ contains a copy of $K_4^-$ but not $K_4$ by Fact 1.7 and it is easy to check that $\overline{C_{11}}^2$ contains a copy of $S_{5,2}$, but it is not hard to show that $\overline{C_{11}}^2$ contains each graph in (c.3) except $S_5, S_8 \cup K_2, S_9, S_8 + e, K_4 \cup M_2, K_4 \cup S_3, K_4^+ \cup K_2, G_7, G_8$ and $G_9$, where $G_7, G_8$ and $G_9$ are obtained from $K_4^+$ by adding a new vertex and joining it to a vertex of $K_4^+$ with degree one, three and four respectively. Thus we are done for $n = 11$.

Let $n = 12$. Let $t = \left\lceil \frac{n-1}{2} \right\rceil - 1 = \left\lceil \frac{12-1}{2} \right\rceil - 1 = 2$. Regardless of isolated vertices of $F$, we consider the following cases:

(a.4) $\delta(F) \geq t + 1 = 3$. Then the number of non-isolated vertices of $F$ is at most \(\left\lfloor \frac{2(n-3)}{(n-2)} \right\rfloor = 6\).

(b.4) $\delta(F) = 2$. Then by Proposition 1.8(ii), we only need to consider $F \in \{K_4^+, K_4 \cup K_2, S_8\}^{+2}$.

(c.4) $\delta(F) = 1$. Then by Proposition 1.8(i), we only need to consider the graphs obtained from a graph in $K(11,8)$ which contains one of graph in $F_{11}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.4), the graphs with $\delta(F) \geq 3$ and $e(F) = 9$ are $K_5^-, K_3, K_3$ and $G_{10}$, where $G_{10}$ is obtained from two vertex disjoint copies of $K_5$ and joining three independent edges between them. Clearly, we know that $\overline{C_{12}}^2$ contains a copy of $K_4$ but not $K_5^-$ by Fact 1.7 and it is easy to check that $\overline{C_{12}}^2$ contains copies of $K_3$ and $G_{10}$. As for the graphs in (b.4), we only need to consider $F \in \{K_4^+, K_4 \cup K_2\}^{+2}$ because of $\delta(F) = 2$ and also it is not hard to see that $\overline{C_{12}}^2$ contains each graph in $\{K_4, K_2 \cup K_4\}^{+2}$.

Thus it is not hard to show that $\overline{C_{12}}^2$ contains each graph in (c.4) except $S_9, S_9 \cup K_2, S_{10}$ and $S_9 + e$. Thus we are done for $n = 12$. Moreover, $\overline{C_{12}}^2$ contains each graph in $K(12,8) \setminus \{S_9\}$ as a subgraph by Lemma 1.6.
Let $n = 13$. Let $t = \left\lceil \frac{n-1}{4} \right\rceil - 1 = \left\lceil \frac{13-1}{4} \right\rceil - 1 = 2$. Regardless of isolated vertices of $F$, we consider the following three cases:

(a.5) $\delta(F) \geq t + 1 = 3$. Then the number of non-isolated vertices of $F$ is at most $\left\lceil \frac{2(n-3)}{4} \right\rceil = 6$.

(b.5) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_9\}^{12}$. 

(c.5) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph in $\mathcal{K}(12,9)$ which contains one of graph in $\mathcal{F}_{12}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.5), the graphs with $\delta(F) \geq 3$ and $e(F) = 10$ are $K_5, W_6, G_{11}, G_{12}$ and $G_{13}$, where $G_{11}, G_{12}$ and $G_{13}$ are depicted in Figure 3. Clearly, we know that $\overline{C_{13}^2}$ contains a copy of $K_5^2$ but not $K_5$ by Fact 1.7 and it is not hard to check that $\overline{C_{13}^2}$ contains copies of $W_6, G_{11}, G_{12}$ and $G_{13}$. As for the graphs in (b.5), there is no graph in (b.5) because of $\delta(F) = 2$. Thus it is not hard to show that $\overline{C_{13}^2}$ contains each graph in (c.5) except $S_9^*, S_{10} \cup K_2, S_{11}$ and $S_{10} + e$. Thus we are done for $n = 13$. Moreover, $\overline{C_{13}^2}$ contains each graph in $\mathcal{K}(13,9) \setminus \{S_{10}\}$ as a subgraph by Lemma 1.6 and obviously $\overline{C_{13}^2}$ contains each graph in $\mathcal{K}(13,8)$ as a subgraph.

Figure 3: $G_{11}, G_{12}$ and $G_{13}$.

Let $n = 14$. Let $t = \left\lceil \frac{n-1}{4} \right\rceil - 1 = \left\lceil \frac{14-1}{4} \right\rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.6) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lceil \frac{2(n-3)}{4} \right\rceil = 5$.

(b.6) $\delta(F) = 3$. Then by Proposition 1.5(ii), $\overline{C_{14}^2}$ contains each copy of $F \in \mathcal{K}(13,8)^{+3}$. 

(c.6) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{10}\}^{+2}$. 

(d.6) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph in $\mathcal{K}(13,10)$ which contains one of graph in $\mathcal{F}_{13}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.6) there does not exist a graph with $\delta(F) \geq 4$ and $e(F) = 11$. As for the graphs in (c.6), there is no graph in (c.6) because of $\delta(F) = 2$. Clearly, we know that $\overline{C_{14}^2}$ contains a copy of $K_5^2$ but not $K_5$ by Fact 1.7. Thus it is not hard to show that $\overline{C_{14}^2}$ contains each graph in (d.6) except $K_5^2, K_5 \cup K_2, S_{11}^*, S_{11} \cup K_2, S_{12}$ and $S_{11} + e$. Thus we are done for $n = 14$. Moreover, $\overline{C_{14}^2}$ contains each graph in $\mathcal{K}(14,10) \setminus \{S_{11}, K_5\}$ as a subgraph by Lemma 1.6 and obviously $\overline{C_{14}^2}$ contains each graph in $\mathcal{K}(14,9)$ as a subgraph.

Let $n = 15$. Let $t = \left\lceil \frac{n-1}{4} \right\rceil - 1 = \left\lceil \frac{15-1}{4} \right\rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.7) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lceil \frac{2(n-3)}{4} \right\rceil = 6$.

(b.7) $\delta(F) = 3$. Then by Proposition 1.5(ii), $\overline{C_{15}^2}$ contains each copy of $F \in \mathcal{K}(14,9)^{+3}$. 

(c.7) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{11}, K_5\}^{+2}$. 

(d.7) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph in $\mathcal{K}(14,11)$ which contains one of graph in $\mathcal{F}_{14}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.7) the unique graph with $\delta(F) \geq 4$ and $e(F) = 12$ is $K_2 \lor C_4$ and it is easy to check that
Theorem 1.12. Let \( n \geq 15 \). Then \( \overline{C_n^2} \) contains each copy of \( F \in \mathcal{K}(n, n-3) \) unless \( F \) contains \( S_{n-3} \) as a subgraph.

Proof. We prove the lemma by induction on \( n \). For \( n = 15 \), the lemma follows from Lemma 1.14. Suppose it is true for \( n - 1 \). Then by induction hypothesis, \( \overline{C_{n-1}^2} \) contains each copy of \( F \in \mathcal{K}(n-1, n-4) \) unless \( F \) contains \( S_{n-4} \) as a subgraph. Moreover, \( \overline{C_{n-1}^2} \) contains each graph in \( \mathcal{K}(n-1, n-5) \setminus \{ S_{n-4} \} \) and each graph in \( \mathcal{K}(n-1, n-6) \) by Lemma 1.6.

It is now sufficient to show that \( \overline{C_n^2} \) contains each copy of \( F \in \mathcal{K}(n, n-3) \) unless \( F \) contains \( S_{n-3} \) as a subgraph. Let \( t = \left\lceil \frac{n-4}{3} \right\rceil - 1 \). Regardless of isolated vertices of \( F \), we consider the following two cases:

(a) \( 1 \leq \delta(F) \leq t \). Applying Proposition 1.4, it is easy to see that \( \overline{C_n^2} \) contains each copy of \( F \) unless \( F \in \{ S_{n-3} \cup K_2, S_{n-3}^*, S_{n-3} + e, S_{n-2} \} \), in which each graph contains \( S_{n-3} \) as a subgraph.

(b) \( \delta(F) \geq t + 1 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil \).

We know that \( \overline{C_n^2} \) contains a copy of \( K_4^* \) by Fact 1.7. Now, if \( n \geq 22 \), then we have \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil \leq \left\lceil \frac{n}{3} \right\rceil \).

Thus \( \overline{C_n^2} \) contains a copy of \( F \) for \( n \geq 22 \). Let \( n = 16 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil = 6 \) and \( e(F) = 13 \). By Fact 1.7, \( \overline{C_n^2} \) contains a copy of \( K_6^* \) which has 14 edges and one can easily check that \( \overline{C_{16}^2} \) contains a copy of \( F \). Let \( n = 17 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil = 7 \) and \( e(F) = 15 \). Actually \( F \) must be one of \( \{ K_6^*, G_{14}, G_{15} \} \) because \( \delta(F) \geq \left\lceil \frac{n-4}{4} \right\rceil = 4 \), where \( G_{14} \) and \( G_{15} \) are depicted in Figure 4. Clearly, we know that \( \overline{C_{17}^2} \) contains a copy of \( K_6^* \) by Fact 1.7 and it is not hard to check that \( \overline{C_{17}^2} \) contains copies of \( G_{14} \) and \( G_{15} \). So \( \overline{C_{17}^2} \) contains a copy of \( F \). Let \( n = 18 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil = 6 \) and \( e(F) = 15 \). Thus \( F = K_6 \) and \( \overline{C_{18}^2} \) contains a copy of \( K_6 \) by Fact 1.7. Let \( n = 19 \) or \( n = 20 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(19-3)}{t+1} \right\rceil = \left\lceil \frac{2(20-3)}{t+1} \right\rceil = 6 \) and \( e(F) = 16 \) or 17. Clearly, such \( F \) does not exist. Let \( n = 21 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-3)}{t+1} \right\rceil = 7 \) and \( e(F) = 18 \). By Fact 1.7, \( \overline{C_{21}^2} \) contains a copy of \( K_7 \) which has 21 edges and one can easily check that \( \overline{C_{21}^2} \) contains a copy of \( F \).

This completes the proof. \( \square \)

Figure 4: \( G_{14} \) and \( G_{15} \).
Lemma 1.13. Let $6 \leq n \leq 18$. Then $\overline{C_6^*}$ contains each copy of $F \in K(n, n-2) \backslash \{S_{n-4} \cup S_4, S_{n-4} \cup K_3\}$ unless $F$ contains one of $\mathcal{E}_n$ as a subgraph, where $\mathcal{E}_n$ is a family of graphs with $6 \leq n \leq 18$ as follows:

$$
\mathcal{E}_n = \begin{cases}
S_3, & n = 6; \\
S_4, C_4, K_3, C_5, & n = 7; \\
K_3, S_5, K_{2,3}, & n = 8; \\
K_4^-, S_6, F_5, & n = 9; \\
K_4, S_7, S_{5,2}, W_5, G_1, & n = 10; \\
K_4, S_8, S_{6,2}, & n = 11; \\
K_5^-, S_9, & n = 12; \\
S_{10}, K_5, & n = 13; \\
S_{11}, K_5, & n = 14; \\
S_{12}, & n = 15; \\
S_{13}, & n = 16; \\
S_{14}, K_6, & n = 17; \\
S_{15}, & n = 18;
\end{cases}
$$

where $G_1$ is depicted in Figure 5.

![Figure 5: G1](image)

Proof. Let $n = 6$. Then $\overline{C_6^*} = M_3$ and $K(6, 3) = \{M_3, S_3 \cup K_2, P_4, S_4, K_3\}$. Hence it is easy to see that each graph in $K(6, 4)$ contains a copy of $S_3$. Then the lemma holds for $n = 6$.

Let $n = 7$. Then $\overline{C_7^*} = C_7$ and $K(7, 4) = \{M_2 \cup S_3, P_4 \cup K_2, 2S_3, P_6, K_3 \cup K_2, C_4, T_{1,1,2}, S_4 \cup K_2, K_3^+, S_5\}$. Clearly, now we only need to consider $\{M_2 \cup S_3, P_4 \cup K_2 \cup K_1, 2S_3 \cup K_1, P_6 \cup K_1, K_3 \cup K_2, 16\}$ and it is not hard to show that $\overline{C_7^*}$ contains copies of the above-mentioned graphs except $T_{1,1,3}, T_{1,2,2}, G_2, G_3, C_4^+, C_5, S_4 \cup S_5, D_6, K_3 \cup S_3, C_4 \cup K_2, K_3^+ \cup K_2, K_3 \cup M_2$ and $T_{1,2,2} \cup K_2$. Then obviously the lemma holds for $n = 7$.

Let $n = 8$. Then clearly $\overline{C_8^*}$ contains $K_3$ and $K_{2,3}$ but contains copies of $D_6$ and $D_7$. By Lemma 11 we know that $K(8, 5) = \{M_2 \cup P_4, M_2 \cup S_4, S_3 \cup S_4 \cup K_1, P_5 \cup K_2 \cup K_1, T_{1,1,2} \cup K_1, C_4^+ \cup K_3, P_4 \cup S_3 \cup K_1, P_6 \cup K_2, T_{1,2,2} \cup K_2, 2S_3 \cup K_2, T_{1,1,3} \cup K_2, D_6 \cup K_2, C_4 \cup K_2 \cup K_2, C_6 \cup K_3 \cup F_8^+\}$. It is straightforward to check that $\overline{C_8^*}$ contains copies of the above-mentioned graphs except $T_{1,1,1,2} \cup K_2, T_{1,1,2,2}, T_{1,1,1,3}, G_{16}, G_{17}, S_5^+ \cup K_2, S_5 \cup S_3, 2S_4, K_{2,3}, K_3 \cup S_4, K_3 \cup P_4, K_3 \cup S_3 \cup K_2, K_3^+ \cup M_2, K_3^+ \cup S_3, G_5, G_6, K_2 \cup K_2, G_3 \cup K_2, G_4 \cup K_2, G_18, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, C_5 + e$ and $K_3 \cup K_2$, where $G_{18} - G_{23}$ are depicted in Figure 6, $G_{16}$ is obtained from $D_6$ by adding a new vertex and joining it to a vertex of $D_6$ with degree three and $G_{17}$ is obtained from $C_4^+$ by adding a new vertex and joining it to a vertex of $C_4^+$ with degree three. So the lemma holds for $n = 8$.

For $n \geq 9$, it is clear that the graphs in $K(n, n-2)$ containing one of $S_{n-3}, S_{n-4} \cup S_4$ and $S_{n-4} \cup K_3$ as a subgraph form the set $\mathcal{E}_n$.

Let $n = 9$. Let $t = \left\lceil \frac{n+1}{4} \right\rceil - 1 = \left\lceil \frac{9+1}{4} \right\rceil - 1 = 1$. Regardless of isolated vertices of $F$, we consider the following two cases:
(a.1) $\delta(F) \geq t + 1 = 2$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{5} \right\rfloor = 7$.

(b.1) $\delta(F) = 1$. Then by Proposition 1.3(i), we only need to consider the graphs obtained from a graph $E$ in $\mathcal{K}(8, 6)$ which contains one of graphs in $\mathcal{E}_9 \cup \{2S_4\}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.1) the graphs with $\delta(F) \geq 2$ and $e(F) = 7$ are $S_{5,2}, K_1 \vee P_4, G_{24}, G_{25}, G_{26}, G_{27}, G_{28}, G_{29}, K_3$ and $K_3 \cup K_4$, where $G_{24} - G_{29}$ are depicted in Figure 7. A simple observation shows that $\overrightarrow{G}$ contains each graph of $K_{2,3}, C_5, C_6, C_7$ and $K_3$ as a subgraph but does not contain copies of $K_4^-$ and $F_5$. Thus it is easy to check that $\overrightarrow{G}$ contains each graph in (a.1) except $S_{5,2}, K_1 \vee P_4$ and $G_{24}$ which obviously contain a copy of $K_4^-$. As for the graphs in (b.1), note that if the graph $E$ contains one of graphs in $\mathcal{E}_9$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $\overrightarrow{G}$ contains each graph left in (b.1) except those graphs in $\mathcal{E}_9 \cup \{G_{30}, G_{31}, S_{7}^+, S_7 \cup K_2, S_8, S_7 + \epsilon\}$, where $G_{30}$ is obtained from $F_5$ by adding a new vertex and joining it to a vertex of $F_5$ with degree four and $G_{31}$ is obtained from $G_6$ by adding a new vertex and joining it to a vertex of $G_6$ with degree four. So we are done for $n = 9$. Moreover, $\overrightarrow{G}$ contains each copy of $F \in \mathcal{K}(9, 6)$ unless $F$ contains one of graphs in $\mathcal{F}_9$ as a subgraph by Lemma 1.11.

Let $n = 10$. Let $t = \left\lceil \frac{n-1}{4} \right\rceil - 1 = \left\lfloor \frac{n-1}{4} \right\rfloor - 1 = 2$. Regardless of isolated vertices of $F$, we consider the following three cases:

(a.2) $\delta(F) \geq t + 1 = 3$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{7} \right\rfloor = 5$.

(b.2) $\delta(F) = 2$. Then by Proposition 1.3(ii), we only need to consider the graphs obtained from a graph $D$ in $\mathcal{K}(9, 6)$ which contains one of graphs in $\mathcal{F}_9$ as a subgraph by adding an isolated vertex $v$ and two arbitrary edges incident with $v$.

(c.2) $\delta(F) = 1$. Then by Proposition 1.3(i), we only need to consider the graphs obtained from a graph $E$ in $\mathcal{K}(9, 7)$ which contains one of graphs in $\mathcal{E}_9 \cup \{S_5 \cup S_4, S_5 \cup K_3\}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.2) the unique graph with $\delta(F) \geq 3$ and $e(F) = 8$ is $W_5$ which belongs to $\mathcal{E}_{10}$. Thus there is no graph in (a.2). A simple observation shows that $\overrightarrow{G}$ contains each graph of $K_4^-$ and $F_5$ as a subgraph but does not contain copies of $K_4, S_{5,2}, G_1$ and $W_5$. As for the graphs in (b.2), note that if the graph $D$ contains a copy of $S_6$, then we do not need to consider $D^+$ because of $\delta(F) = 2$. Moreover, we do not need to consider the graph $D = K_4$ belonging to $\mathcal{E}_{10}$. Thus it is easy to check that $\overrightarrow{G}$ contains each graph left in (b.2) except $G_4$ which belongs to $\mathcal{E}_{10}$. As for the graphs in (c.2), note that if the graph $E$ contains one of graphs in $\mathcal{E}_{10}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $\overrightarrow{G}$ contains each graph left in (c.2) except those graphs in $\mathcal{E}_{10} \setminus \{S_8^+, S_8 \cup K_2, S_9, S_8 + \epsilon\}$. So we are done for $n = 10$. Moreover, $\overrightarrow{G}$ contains each
copy of \( F \in \mathcal{K}(10,7) \) unless \( F \) contains one of graphs in \( \mathcal{F}_{10} \) as a subgraph by Lemma 1.11.

Let \( n = 11. \) Let \( t = \left\lceil \frac{n-1}{4} \right\rceil = 1 = \left\lceil \frac{11-1}{4} \right\rceil - 1 = 2. \) Regardless of isolated vertices of \( F, \) we consider the following three cases:

(a.3) \( \delta(F) \geq t + 1 = 3. \) Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-2)}{2t+1} \right\rceil = 6. \)

(b.3) \( \delta(F) = 2. \) Then by Proposition 1.5 ii, we only need to consider the graphs obtained from a graph \( D \in \mathcal{K}(10,7) \) which contains one of graphs in \( \mathcal{F}_{10} \) as a subgraph by adding an isolated vertex \( v \) and two arbitrary edges incident with \( v. \)

(c.3) \( \delta(F) = 1. \) Then by Proposition 1.5 i, we only need to consider the graphs obtained from a graph \( E \in \mathcal{K}(10,8) \) which contains one of graphs in \( \mathcal{E}_{10} \cup \{ S_6 \cup S_4, S_6 \cup K_3 \} \) as a subgraph by adding an isolated vertex \( v \) and an arbitrary edge incident with \( v. \)

In case of (a.3) the graphs with \( \delta(F) \geq 3 \) and \( e(F) = 9 \) are \( K_5^-, K_{3,3} \) and \( G_{10}. \) A simple observation shows that \( C_7^{11} \) contains each graph of \( K_5^-, K_{3,3}, G_{10}, S_6, W_5 \) and \( G_1 \) as a subgraph but does not contain copies of \( K_4 \) and \( S_6. \) As for the graphs in (b.3), note that if the graph \( D \) contains a copy of \( S_t, \) then we do not need to consider \( D^{+2} \) because of \( \delta(F) = 2. \) Moreover, we do not need to consider the graph \( D \) which contains a copy of \( K_4 \) belonging to \( \mathcal{E}_{11}, \) i.e. \( D \) is either \( K_4^+ \) or \( K_4 \cup K_2 \) under the circumstance. Thus it is easy to check that \( C_7^{11} \) contains each graph left in (b.3) except \( S_6, \) which belongs to \( \mathcal{E}_{11}. \) As for the graphs in (c.3), note that if the graph \( E \) contains one of graphs in \( \mathcal{E}_{11} \) as a subgraph, then we do not need to consider \( E^+. \) Without considering the above situations, it is not hard to show that \( C_7^{11} \) contains each graph left in (c.3) except those graphs in \( \mathcal{L}_{11} \cup \{ S_5, S_9, K_2, S_{10}, S_9 + e \}. \) So we are done for \( n = 11. \) Moreover, \( C_7^{12} \) contains each copy of \( F \in \mathcal{K}(11,8) \) unless \( F \) contains one of graphs in \( \mathcal{F}_{11} \) as a subgraph by Lemma 1.11.

Let \( n = 12. \) Let \( t = \left\lceil \frac{n-1}{4} \right\rceil = 1 = \left\lceil \frac{12-1}{4} \right\rceil - 1 = 2. \) Regardless of isolated vertices of \( F, \) we consider the following three cases:

(a.4) \( \delta(F) \geq t + 1 = 3. \) Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-2)}{2t+1} \right\rceil = 6. \)

(b.4) \( \delta(F) = 2. \) Then by Proposition 1.5 ii, we only need to consider the graphs obtained from a graph \( D \) in \( \mathcal{K}(11,8) \) which contains one of graphs in \( \mathcal{F}_{11} \) as a subgraph by adding an isolated vertex \( v \) and two arbitrary edges incident with \( v. \)

(c.4) \( \delta(F) = 1. \) Then by Proposition 1.5 i, we only need to consider the graphs obtained from a graph \( E \) in \( \mathcal{K}(11,9) \) which contains one of graphs in \( \mathcal{E}_{11} \cup \{ S_7, S_4, S_7 \cup K_3 \} \) as a subgraph by adding an isolated vertex \( v \) and an arbitrary edge incident with \( v. \)

In case of (a.4) the graphs with \( \delta(F) \geq 3 \) and \( e(F) = 10 \) are \( K_5, W_6, G_{11}, G_{12} \) and \( G_{13}. \) A simple observation shows that \( C_7^{12} \) contains each graph of \( K_5, W_6, S_6, G_{11}, G_{12} \) and \( G_{13} \) as a subgraph but does not contain a copy of \( K_5^- \). As for the graphs in (b.4), note that if the graph \( D \) contains a copy of \( S_5, \) then we do not need to consider \( D^{+2} \) because of \( \delta(F) = 2. \) Thus it is easy to check that \( C_7^{12} \) contains each graph left in (b.4) except those graphs in \( \mathcal{L}_{12} \cup \{ S_5, S_{10}, K_2, S_{11}, S_{10} + e \}. \) So we are done for \( n = 12. \) Moreover, \( C_7^{13} \) contains each copy of \( F \in \mathcal{K}(12,9) \) unless \( F \) contains one of graphs in \( \mathcal{F}_{12} \) as a subgraph by Lemma 1.11.

Let \( n = 13. \) Let \( t = \left\lceil \frac{n-1}{4} \right\rceil = 1 = \left\lceil \frac{13-1}{4} \right\rceil - 1 = 2. \) Regardless of isolated vertices of \( F, \) we consider the following three cases:

(a.5) \( \delta(F) \geq t + 1 = 3. \) Then the number of non-isolated vertices of \( F \) is at most \( \left\lceil \frac{2(n-2)}{2t+1} \right\rceil = 7. \)

(b.5) \( \delta(F) = 2. \) Then by Proposition 1.5 ii, we only need to consider the graphs obtained from a graph \( D \) in \( \mathcal{K}(12,9) \) which contains one of graphs in \( \mathcal{F}_{12} \) as a subgraph by adding an isolated vertex \( v \) and two arbitrary edges incident with \( v. \)

(c.5) \( \delta(F) = 1. \) Then by Proposition 1.5 i, we only need to consider the graphs obtained from a graph \( E \) in \( \mathcal{K}(12,10) \) which contains one of graphs in \( \mathcal{E}_{12} \cup \{ S_8, S_4, S_8 \cup K_3 \} \) as a subgraph by adding an isolated vertex \( v \) and an arbitrary edge incident with \( v. \)

In case of (a.5) the graphs with \( \delta(F) \geq 3 \) and \( e(F) = 11 \) are \( G_{32} - G_{40}, \) which are depicted in Figure 8. A simple observation shows that \( C_7^{13} \) contains each graph of \( K_5^-, G_{32} - G_{40} \) and \( S_6 + e \) as a subgraph.
but does not contain a copy of $K_5$. As for the graphs in (b.5), note that if the graph $D$ contains a copy of $S_9$, then we do not need to consider $D^{+2}$ because of $\delta(F) = 2$. Thus it is easy to check that $C_{13}^t$ contains each graph left in (b.5). As for the graphs in (c.5), note that if the graph $E$ contains one of graphs in $E_{13}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $C_{13}^t$ contains each graph left in (c.5) except those graphs in $L_{13}(\{S_{11}^*, S_{11} \cup K_2, S_{12}, S_{11} + e\})$. So we are done for $n = 13$. Moreover, $C_{13}^t$ contains each copy of $F \in K(13, 9) \setminus \{S_{10}\}$ as a subgraph by Lemma 1.6 and $C_{13}^t$ contains each copy of $F \in K(13, 10)$ unless $F$ contains one of graphs in $F_{13}$ as a subgraph by Lemma 1.11.

![Figure 8: $G_{32} - G_{40}$](image)

Let $n = 14$. Let $t = \lceil \frac{n-1}{4} \rceil - 1 = \lceil \frac{14-1}{4} \rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.6) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{4} \right\rfloor = 6$.

(b.6) $\delta(F) = 3$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{10}\}^{+3}$.

(c.6) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider the graphs obtained from a graph $D$ in $K(13, 10)$ which contains one of graphs in $F_{13}$ as a subgraph by adding an isolated vertex $v$ and two arbitrary edges incident with $v$.

(d.6) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph $E$ in $K(13, 11)$ which contains one of graphs in $E_{13}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.6) the unique graph with $\delta(F) \geq 4$ and $e(F) = 12$ is $K_2 \vee C_4$. A simple observation shows that $C_{14}^t$ contains copies of $K_5^-$ and $K_2 \vee C_4$ but does not contain a copy of $K_5$. As for the graphs in (b.6), there is no graph in (b.6) because of $\delta(F) = 3$. As for the graphs in (c.6), note that if the graph $D$ contains a copy of $S_{10}$, then we do not need to consider $D^{+2}$ because of $\delta(F) = 2$. Moreover, we do not need to consider the graph $D$ which contains a copy of $K_5$ belonging to $E_{14}$.

Thus, there is no graph in (c.6). As for the graphs in (d.6), note that if the graph $E$ contains one of graphs in $E_{14}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $C_{14}^t$ contains each graph left in (d.6) except those graphs in $L_{14}(\{S_{12}^*, S_{12} \cup K_2, S_{13}, S_{12} + e\})$. So we are done for $n = 14$. Moreover, $C_{14}^t$ contains each graph in $K(14, 10) \setminus \{S_{11}, K_5\}$ as a subgraph by Lemma 1.6 and $C_{14}^t$ contains each copy of $F \in K(14, 11)$ unless $F$ contains one of graphs in $F_{14}$ as a subgraph by Lemma 1.11.

Let $n = 15$. Let $t = \lceil \frac{n-1}{4} \rceil - 1 = \lceil \frac{15-1}{4} \rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.7) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{4} \right\rfloor = 6$.

(b.7) $\delta(F) = 3$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{11}, K_5\}^{+3}$.

(c.7) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider the graphs obtained from a graph $D$ in $K(14, 11)$ which contains one of graphs in $E_{14}$ as a subgraph by adding an isolated vertex $v$ and two arbitrary edges incident with $v$.

(d.7) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph $E$ in $K(14, 12)$ which contains one of graphs in $E_{14}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.7) the unique graph with $\delta(F) \geq 4$ and $e(F) = 13$ is $K_2 \vee K_4^-$. A simple observation shows that $C_{15}^t$ contains copies of $K_5$ and $K_2 \vee K_4^-$ but does not contain a copy of $K_5^-$. As for the graphs in (b.7), we only need to consider $F \in \{K_5\}^{+3}$ because of $\delta(F) = 3$. As for the graphs in (c.7), note that if
the graph $D$ contains a copy of $S_{11}$, then we do not need to consider $D^{+2}$ because of $\delta(F) = 2$. Thus it is easy to check that $C_{15}^2$ contains each graph left in (b.7) as well as (c.7). As for the graphs in (d.7), note that if the graph $E$ contains one of graphs in $E_{15}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $C_{15}^2$ contains each graph left in (d.7) except those graphs in $L_{15} \setminus \{S_{13}, S_{13} \cup K_2, S_{14}, S_{13} + e\}$. So we are done for $n = 15$. Moreover, $C_{15}^2$ contains each graph in $K(15,11) \setminus \{S_{12}\}$ as a subgraph by Lemma 1.10 and $C_{15}^2$ contains each copy of $F \in K(15,12)$ unless $F$ contains one of graphs in $F_{15}$ as a subgraph by Lemma 1.11.

Let $n = 16$. Let $t = \lceil \frac{16}{4} \rceil - 1 = \lceil \frac{16-1}{4} \rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.8) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{t+1} \right\rfloor = 7$.

(b.8) $\delta(F) = 3$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{12}\}^+$.

(c.8) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider the graphs obtained from a graph $D$ in $K(15,12)$ which contains one of graphs in $F_{15}$ as a subgraph by adding an isolated vertex $v$ and two arbitrary edges incident with $v$.

(d.8) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph $E$ in $K(15,13)$ which contains one of graphs in $E_{15} \cup \{S_{11} \cup S_4, S_{11} \cup K_3\}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.8) the graphs with $\delta(F) \geq 4$ and $e(F) = 14$ are $K_6, G_{14}$ and $G_{15}$. A simple observation shows that $C_{16}^2$ contains copies of $K_6, G_{14}$ and $G_{15}$ but does not contain a copy of $K_6$. As for the graphs in (b.8), there is no graph in (b.8) because of $\delta(F) = 3$. As for the graphs in (c.8), note that if the graph $D$ contains a copy of $S_{12}$, then we do not need to consider $D^{+2}$ because of $\delta(F) = 2$. Thus there is no graph in (c.8). As for the graphs in (d.8), note that if the graph $E$ contains one of graphs in $E_{16}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $C_{16}^2$ contains each graph left in (d.8) except those graphs in $L_{16} \setminus \{S_{14}, S_{14} \cup K_2, S_{15}, S_{14} + e\}$. So we are done for $n = 16$. Moreover, $C_{16}^2$ contains each graph in $K(16,12) \setminus \{S_{13}\}$ as a subgraph by Lemma 1.10 and $C_{16}^2$ contains each copy of $F \in K(16,13)$ unless $F$ contains $S_{13}$ as a subgraph by Lemma 1.12.

Let $n = 17$. Let $t = \lceil \frac{17}{4} \rceil - 1 = \lceil \frac{17-1}{4} \rceil - 1 = 3$. Regardless of isolated vertices of $F$, we consider the following four cases:

(a.9) $\delta(F) \geq t + 1 = 4$. Then the number of non-isolated vertices of $F$ is at most $\left\lfloor \frac{2(n-2)}{t+1} \right\rfloor = 7$.

(b.9) $\delta(F) = 3$. Then by Proposition 1.5(ii), we only need to consider $F \in \{S_{13}\}^+$.

(c.9) $\delta(F) = 2$. Then by Proposition 1.5(ii), we only need to consider the graphs obtained from a graph $D$ in $K(16,13)$ which contains $S_{13}$ as a subgraph by adding an isolated vertex $v$ and two arbitrary edges incident with $v$.

(d.9) $\delta(F) = 1$. Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph $E$ in $K(16,14)$ which contains one of graphs in $E_{16} \cup \{S_{12} \cup S_4, S_{12} \cup K_3\}$ as a subgraph by adding an isolated vertex $v$ and an arbitrary edge incident with $v$.

In case of (a.9) the graphs with $\delta(F) \geq 4$ and $e(F) = 15$ are $G_{10} \lor K_1, K_3 \lor P_4, K_7 \lor C_5, K_5, G_{14} + G_{42}$ and $G_{43}$, where $G_{14} - G_{43}$ are depicted in Figure 9. A simple observation shows that $C_{17}^2$ contains copies of $G_{10} \lor K_1, G_{41}, G_{42}, K_3 \lor P_4, K_2 \lor C_5, K_5$ and $G_{43}$ but does not contain a copy of $K_6$. As for the graphs in (b.9), there is no graph in (b.9) because of $\delta(F) = 3$. As for the graphs in (c.9), note that if the graph $D$ contains a copy of $S_{13}$, then we do not need to consider $D^{+2}$ because of $\delta(F) = 2$. Thus there is no graph in (c.9). As for the graphs in (d.9), note that if the graph $E$ contains one of graphs in $E_{17}$ as a subgraph, then we do not need to consider $E^+$. Thus, without considering the above situations, it is not hard to show that $C_{17}^2$ contains each graph left in (d.9) except those graphs in $L_{17} \setminus \{S_{14}, S_{14} \cup K_2, S_{16}, S_{15} + e\}$. So we are done for $n = 17$. Moreover, $C_{17}^2$ contains each graph in $K(17,13) \setminus \{S_{14}\}$ as a subgraph by Lemma 1.10 and $C_{17}^2$ contains each copy of $F \in K(17,14)$ unless $F$ contains $S_{14}$ as a subgraph by Lemma 1.12.

Obviously, $C_{17}^2$ contains each graph in $K(17,12)$ as a subgraph.

Let $n = 18$. Let $t = \lceil \frac{18}{4} \rceil - 1 = \lceil \frac{18-1}{4} \rceil - 1 = 4$. Regardless of isolated vertices of $F$, we consider the following five cases:
In case of (a.10) there does not exist a graph with \( \delta \) isolated vertex \( v \). Let \( \delta \) (e.10), note that if the graph \( F \) in (e.10), consider the following two cases:

(b.10) \( \delta(F) = 4 \). Then by Proposition 1.5(ii), \( \overline{C_{18}} \) contains each copy of \( F \in K(17, 12)^{+4} \).

(c.10) \( \delta(F) = 3 \). Then by Proposition 1.5(ii), we only need to consider \( F \in \{ S_{14} \}^{+3} \).

(d.10) \( \delta(F) = 2 \). Then by Proposition 1.5(ii), we only need to consider the graphs obtained from a graph \( D \) in \( K(17, 14) \) which contains \( S_{14} \) as a subgraph by adding an isolated vertex \( v \) and two arbitrary edges incident with \( v \).

(e.10) \( \delta(F) = 1 \). Then by Proposition 1.5(i), we only need to consider the graphs obtained from a graph \( E \) in \( K(17, 15) \) which contains one of graphs in \( \mathcal{E}_{17} \cup \{ S_{13}, S_{4}, S_{13} \cup K_{3} \} \) as a subgraph by adding an isolated vertex \( v \) and an arbitrary edge incident with \( v \).

In case of (a.10) there does not exist a graph with \( \delta(F) \geq 5 \) and \( e(F) = 16 \). A simple observation shows that \( \overline{C_{18}} \) contains copies of \( K_{6} \cup K_{2} \) and \( K_{6}^{+} \). As for the graphs in (c.10), there is no graph in (c.10) because of \( \delta(F) = 3 \). As for the graphs in (d.10), note that if the graph \( D \) contains a copy of \( S_{14} \), then we do not need to consider \( D^{+2} \) because of \( \delta(F) = 2 \). Thus there is no graph in (d.10). As for the graphs in (e.10), note that if the graph \( E \) contains one of graphs in \( \mathcal{E}_{18} \) as a subgraph, then we do not need to consider \( E^{+} \). Thus, without considering the above situations, it is not hard to show that \( \overline{C_{18}} \) contains each graph left in (e.10) except those graphs in \( \mathcal{L}_{18} \setminus \{ S_{16}, S_{16} \cup K_{2}, S_{17}, S_{16} + e \} \). So we are done for \( n = 18 \).

\[ \square \]

**Lemma 1.14.** Let \( n \geq 18 \). Then \( \overline{C_{n}} \) contains each copy of \( F \in K(n, n - 2) \setminus \{ S_{n-4} \cup S_{4}, S_{n-4} \cup K_{3} \} \) unless \( F \) contains \( S_{n-3} \) as a subgraph.

**Proof.** We prove the lemma by induction on \( n \). For \( n = 18 \), the lemma follows from Lemma 1.13. Suppose it is true for \( n - 1 \). Then by induction hypothesis, \( \overline{C_{n-1}} \) contains each copy of \( F \in K(n - 1, n - 3) \setminus \{ S_{n-5} \cup S_{4}, S_{n-5} \cup K_{3} \} \) unless \( F \) contains \( S_{n-4} \) as a subgraph. Moreover, \( \overline{C_{n-1}} \) contains each graph in \( K(n - 1, n - 5) \setminus \{ S_{n-4} \} \) and contains each graph in \( K(n - 1, n - 6) \) by Lemma 1.6. And \( \overline{C_{n-1}} \) contains each copy of \( F \in K(n - 1, n - 4) \) unless \( F \) contains \( S_{n-4} \) as a subgraph by Lemma 1.12.

It is now sufficient to show that \( \overline{C_{n}} \) contains each copy of \( F \in K(n, n - 2) \setminus \{ S_{n-4} \cup S_{4}, S_{n-4} \cup K_{3} \} \) unless \( F \) contains \( S_{n-3} \) as a subgraph. Let \( t = \left\lceil \frac{n - 1}{2} \right\rceil - 1 \). Regardless of isolated vertices of \( F \), we consider the following two cases:

(a) \( 1 \leq \delta(F) \leq t \). Applying Proposition 1.5, it is easy to see that \( \overline{C_{n}} \) contains each copy of \( F \) unless \( F \in \mathcal{L}_{n} \setminus \{ S_{n-3} \cup K_{2}, S_{n-2}^{+}, S_{n-2} + e, S_{n-1} \} \), in which each graph \( F \) contains either \( S_{n-3} \) as a subgraph or \( F \in \{ S_{n-4} \cup S_{4}, S_{n-4} \cup K_{3} \} \).

(b) \( \delta(F) \geq t + 1 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor \).

We know that \( \overline{C_{n}} \) contains a copy of \( K_{\frac{n}{2}} \), by Fact 1.7. Now, if \( n \geq 22 \), then we have \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor \leq \left\lfloor \frac{n}{3} \right\rfloor \).

Thus \( \overline{C_{n}} \) contains a copy of \( F \) for \( n \geq 22 \). Let \( n = 19 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor = 6 \) and \( e(F) = 17 \). Clearly, such \( F \) does not exist. Let \( n = 20 \). Then the number of non-isolated vertices of \( F \) is at most \( \left\lfloor \frac{2(n-2)}{3} \right\rfloor = 7 \) and \( e(F) = 18 \). Actually \( F \) must be the graph

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![Figure 9: Graphs G41, G42 and G43.](image-url)
Lemma 1.14, we have that by the Perron-Frobenius theorem. From Theorem 1.9, we have that obviously

\[ \mu(n-1) = \nu(n-1) = n-2 \]

Combining the Perron-Frobenius theorem and Theorem 1.1, we have that \( G \) contains the 2-power of a Hamilton cycle or

\[ Y_n \] contains \( G \) as a subgraph.

This completes the proof.

Proof of Theorem 1.2: Note that

\[ \mu(G) \geq \mu(K_n \setminus E(S_{n-3})) > \mu(K_{n-1}) = n-2 \]

by the Perron-Frobenius theorem. From Theorem 1.1, we have that \( G \) contains a Hamilton cycle or \( G = K_{n-1} + e \). Since \( \mu(K_{n-1} + e) < \mu(K_n \setminus E(S_{n-3})) \), we have \( G \) contains a Hamilton cycle and obviously \( G \) is connected.

Let \( m = e(G) \). From Lemma 1.8, we have \( \mu(G) \leq n - n + 1 \) with equality if and only if \( G = K_n \) or \( K_n \). Note that \( \mu(S_n) = \sqrt{n-1} < n-2 \), then if the equality holds, then \( G = K_n \) and in this case the proof is done. If the strict inequality holds, then \( 2m - n + 1 > (n-2)^2 \) and so \( m > \frac{n^2 - 3n + 3}{2} \). By Theorem 1.1, we have that \( G \) contains the 2-power of a Hamilton cycle or \( Y_n \) contains \( G \) as a subgraph, where \( Y_n \) is a family of \( K_n \setminus E(S_{n-3}), K_n \setminus E(S_{n-4} \cup S_4) \) and \( K_n \setminus E(S_{n-4} \cup K_3) \). One can easily check that

\[ \mu(K_n \setminus E(S_{n-4} \cup S_4)) < \mu(K_n \setminus E(S_{n-3})) \]

and

\[ \mu(K_n \setminus E(S_{n-4} \cup K_3)) < \mu(K_n \setminus E(S_{n-3})). \]

Combining the Perron-Frobenius theorem and \( \mu(G) \geq \mu(K_n \setminus E(S_{n-3})) \), we have that \( G \) contains the 2-power of a Hamilton cycle or \( G = K_n \setminus E(S_{n-3}) \). This completes the proof.

Proof of Theorem 1.3: We prove by contradiction. For short, let \( H = C_n(G) \). Assume that \( G \) does not contain the 2-power of a Hamilton cycle \( C_2^2 \). Note that \( \mu(G) \leq \sqrt{n-3} < \sqrt{n-2} \) and \( G \) contains a copy of \( \overline{H} \). By the Perron-Frobenius theorem, we obtain \( \mu(\overline{H}) \leq \mu(G) < \sqrt{n-2} \). From Theorem 1.1, we have that \( G \) contains a Hamilton cycle and obviously \( G \) is connected because \( G \neq K_{n-1} + e \) in this case

\[ \mu(K_{n+1} + e) = \mu(S_{n-1}) = \sqrt{n-2} > \sqrt{n-3}. \]

Now we consider the following two cases:

(i) If \( C_n^2 \) does not contain a copy of \( \overline{H} \), we have that \( e(\overline{H}) \geq n - 4 \) by Theorem 1.4. Now the main
property of \( C_n(G) = H \) gives \( d_H(u) + d_H(v) \leq n - 1 \) for every pair of nonadjacent vertices \( u \) and \( v \) of \( H \); thus,
\[
d_{\overline{H}}(u) + d_{\overline{H}}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq n - 1
\]
for every edge \( uv \in E(\overline{H}) \). Summing these inequalities for all edges \( uv \in E(\overline{H}) \), we obtain
\[
\sum_{uv \in E(\overline{H})} (d_{\overline{H}}(u) + d_{\overline{H}}(v)) \geq (n - 1)e(\overline{H})
\]
and since each term \( d_{\overline{H}}(u) \) appears in the left-hand sum precisely \( d_{\overline{H}}(u) \) times, we see that
\[
\sum_{v \in V(\overline{H})} d^2_{\overline{H}}(v) = \sum_{uv \in E(\overline{H})} (d_{\overline{H}}(u) + d_{\overline{H}}(v)) \geq (n - 1)e(\overline{H}).
\]
By Lemma 1.10 we have
\[
n\mu^2(\overline{H}) \geq \sum_{v \in V(\overline{H})} d^2_{\overline{H}}(v) \geq (n - 1)e(\overline{H}).
\]
Since \( \overline{H} \subseteq \overline{G} \), we have \( \mu(\overline{H}) \leq \mu(\overline{G}) \leq \sqrt{n - 5} \) and so,
\[
n(n - 5) \geq n\mu^2(\overline{G}) \geq n\mu^2(\overline{H}) \geq (n - 1)e(\overline{H}).
\]
This easily gives
\[
e(\overline{H}) \leq \frac{n(n - 5)}{n - 1} < n - 4, n \geq 18,
\]
a contradiction.

(ii) From (i), we must have that \( \overline{C_n^2} \) contains a copy of \( \overline{H} \) but \( \overline{C_n^2} \) does not contain a copy of \( \overline{G} \). In this case our aim is to prove \( H = K_n \). Assume that there exists an edge \( uv \in E(\overline{G}) \) satisfying \( d_{\overline{H}}(u) + d_{\overline{H}}(v) \geq n - 1 \).

Now we consider \( G \) with the smallest spectral radius satisfying that \( C_n^2 \) does not contain a copy of \( G \). Firstly, we consider three graphs \( H_1, H_2 \) and \( H_3 \) with \( n \) vertices and \( n - 2 \) edges, which are depicted in Figure 10. Let \( g(\lambda), h(\lambda) \) and \( f(\lambda) \) be the characteristic polynomial of \( H_1, H_2 \) and \( H_3 \) respectively. One can easily know that
\[
g(\lambda) = \lambda^5 - (b + 2c + a + 1)\lambda^3 - 2c\lambda^2 + (ab + ac + bc)\lambda
\]
and
\[
h(\lambda) = \lambda^5 - (b + 2c + a + 1)\lambda^3 - (2c - 2)\lambda^2 + (ab + ac + bc + 2c - 1)\lambda,
\]
thus \( g(\lambda) - h(\lambda) = -\lambda(2c - 1 + 2\lambda) \). Note that \( \mu(G) \) is the largest root of the characteristic polynomial of \( G \). Thus it is obvious that \( \mu(H_2) < \mu(H_1) \). Then it is clearly that \( G \) with the smallest spectral radius satisfying that \( C_n^2 \) does not contain a copy of \( G \) is exactly \( H_3 \). Similarly, we have \( f(\lambda) = \lambda^4 + (-b - 1 - a)\lambda^2 + ab \) with \( a + b + 2 = n - 1 \) and at this time
\[
\mu(G) = \mu(H_3) = \sqrt[n]{2 + 2(n - 3) + 2\sqrt{2(n - 3) + 1 + (a - b)^2}}.
\]
To simplify the proof, we denote \( H_3 \) by \( T_{a,b} \) relying on \( a \) and \( b \) satisfying \( a + b + 2 = n - 1 \). By Lemma 1.14 we have that \( G = T_{n-5,2} \) and
\[
\mu(T_{n-5,2}) = \frac{\sqrt{2n - 4 + 2\sqrt{n^2 - 12n + 44}}}{2} > \sqrt{n - 4} > \sqrt{n - 5},
\]
a contradiction. Consequently, we have \( d_G(u) + d_G(v) \geq n \) for every pair of nonadjacent vertices \( u \) and \( v \) of \( G \). Then \( H = C_n(G) = K_n \).

This completes the proof. ■

In this paper, we consider the spectral conditions for a graph containing \( C_n^2 \) as a subgraph. What can we say for the problems for \( C_n^3 \)? Furthermore, what happens if the minimum degree is fixed? We leave them for further research. We will also consider maximum signless laplacian (\( p \)-Laplacian) spectral radius version of the problem among the same family of graphs in the near future.
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