Anisotropic models of dark energy dominated Universe

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Abstract

In this paper, we investigate the role of dynamical dark energy in the evolution of Universe within the framework of a spatially homogeneous Bianchi type-I space-time filled with non-interacting matter and dark energy fluids. We utilize a special law of variation for the average Hubble’s parameter in Bianchi-I space-time that leads to power-law and exponential-law cosmologies. Also, it yields constant value of deceleration parameter given by \( q = n - 1 \), which provides accelerating models of the Universe for \( n < 1 \) and decelerating ones for \( n > 1 \). In the present work, we emphasize that decelerating models can be described by the usual matter fluid, while dynamics of accelerating Universe can be described by considering some exotic type of matter such as the dark energy. We study the role of dark energy with variable equation of state parameter in the evolution of power-law and exponential-law cosmological models. The analysis of the models reveals that the present acceleration, isotropy and flatness of the Universe are the natural consequences of dark energy. The equation of state parameter of dark energy is found to be consistent with the seven year WMAP observations. The analysis of the lookback time, proper distance, luminosity distance, angular diameter distance and event horizon is also carried out for the power-law and exponential-law models.

Keywords: Bianchi-I space-time · Average Hubble’s parameter · Accelerating Universe · Dark energy

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1 Introduction

The late time acceleration of the Universe is confirmed by different data sets of complimentary nature such as type Ia supernovae (SN Ia), baryon oscillations, galaxy clustering, cosmic microwave background (CMB) and weak lensing [1]. The cause of accelerated expansion of the Universe has been attributed to some unknown form of energy named as the dark energy (DE). It is a hypothetical form of energy that permeates all of space and tends to increase the rate of expansion of the Universe [2]. The most recent WMAP observations indicate that DE accounts for 72% of the total mass energy of the Universe [3]. Many cosmologists believe that the simplest candidate for the DE is the cosmological constant (Λ) or vacuum energy since it fits the observational data well. During the cosmological evolution, the Λ-term has the constant energy density and pressure \( p^{(\text{de})} = -\rho^{(\text{de})} \), where the superscript \( (\text{de}) \) stands for DE. However, one has the reason to dislike the cosmological constant since it always suffers from the theoretical problems such as the “fine-tuning” and “cosmic coincidence” puzzles [2]. That is why, the different forms of dynamically changing DE with an effective equation of state (EoS), \( w^{(\text{de})} = p^{(\text{de})}/\rho^{(\text{de})} < -1/3 \), have been proposed in the literature. Other possible forms of DE include quintessence \( (w^{(\text{de})} > -1) \) [5], phantom \( (w^{(\text{de})} < -1) \) [6] etc. The observational constraints on the EoS parameter have been found in various

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works in literature. For instance, the seven year WMAP results put \( w_{de} \) in the range \(-1.55 < w_{de} < -0.7 \) (see, Jarosik et al. [7]).

Even though the Universe on a large scale, appears isotropic and homogeneous at the present time, there are no observational data that guarantee in an epoch prior to the recombination. The sorts of matter fields in the early Universe are uncertain. In the early stages of evolution, the Universe could not have had such a smoothed out picture because near the big bang singularity neither the assumption of spherical symmetry nor of isotropy can be strictly valid. Therefore, to describe the evolution of Universe right from the big bang, it appears appropriate to suppose a geometry that is more general than just the isotropic and homogeneous FRW geometry. In this regard, Bianchi type-I space-time is of fundamental importance since it provides the requisite framework. A spatially homogeneous and anisotropic Bianchi type-I model is considered as the simplest generalization of the FRW flat model. It is described by the line element (in units \( c = 1 \))

\[
ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2,
\]

where \( A, B \) and \( C \) are the metric functions or directional scale factors of cosmic time \( t \). If any two of the directional scale factors are equal and third one is different (say, \( A \neq B = C \)), the space-time is said to be axially symmetric or locally rotationally symmetric (LRS). In case \( A \neq B \neq C \), the space-time is totally anisotropic.

Following Berman [8], Kumar and Singh [9] proposed a special law of variation for the average Hubble’s parameter in Bianchi-I space-time, which yields a constant value of deceleration parameter (DP). The law reads as

\[
H = D(ABC)^{\frac{n}{3}} = Da^{-n},
\]

where \( D > 0, n \geq 0 \) are constants and \( a = (ABC)^{\frac{1}{3}} \) is the average scale factor of the Bianchi-I space-time (1).

Such a law of variation for Hubble’s parameter is not inconsistent with the observations and is also approximately valid for slowly time-varying DP models. The law provides explicit forms of scale factors governing the Bianchi-I Universe and facilitates to describe accelerating as well as decelerating modes of evolution of the Universe. Models with constant DP have been extensively studied in the literature in different contexts (see, Kumar and Singh [9] and references therein). Most of the models with constant DP have been studied by considering perfect fluid or ordinary matter in the Universe. But the ordinary matter is not enough to describe the dynamics of an accelerating Universe as mentioned earlier. This motivates the researchers to consider the models of the Universe filled with some exotic type of matter such as the DE along with the usual matter fluid. Recently Akarsu and Kilinc [10, 11, 12], Yadav et al. [13, 14, 15], Pradhan et al. [16, 17], Adhav et al. [18, 19, 20] and Kumar et al. [21, 22, 23] have studied the role of DE with variable EoS parameter in cosmological models evolving with constant DP.

In this paper, we have considered non-interacting matter fluid and DE energy components with constant DP within the framework of the Bianchi-I space-time (1). The EoS parameter of DE has been allowed to vary with time. The paper is organized as follows. In Sect. 2, the field equations have been presented. Sect. 3 deals with the solutions of the field equations and physical behavior of the model. Sect. 4 is devoted to the study of some observational quantities such as lookback time, proper distance, luminosity distance, angular diameter distance and event horizon for the derived models. Finally, concluding remarks have been given in Sect. 5.

## 2 Field equations

The Einstein’s field equations in case of a mixture of perfect fluid and DE components, in the units \( 8 \pi G = c = 1 \), read as

\[
R_{ij} - \frac{1}{2} g_{ij} R = -T_{ij},
\]
where $T_{ij} = T_{ij}^{(m)} + T_{ij}^{(de)}$ is the overall energy momentum tensor with $T_{ij}^{(m)}$ and $T_{ij}^{(de)}$ as the energy momentum tensors of matter and DE, respectively. These are given by

$$T_{ij}^{(m)} = \text{diag} \left[ -\rho^{(m)}, p^{(m)}, p^{(m)}, p^{(m)} \right]$$

$$= \text{diag} \left[ -1, w^{(m)}, w^{(m)}, w^{(m)} \right] \rho^{(m)}$$

and

$$T_{ij}^{(de)} = \text{diag} \left[ -\rho^{(de)}, p^{(de)}, p^{(de)}, p^{(de)} \right]$$

$$= \text{diag} \left[ -1, w^{(de)}, w^{(de)}, w^{(de)} \right] \rho^{(de)}$$

where $\rho^{(m)}$ and $p^{(m)}$ are, respectively the energy density and pressure of the matter fluid component while $w^{(m)} = p^{(m)} / \rho^{(m)}$ is its EoS parameter. Similarly, $\rho^{(de)}$ and $p^{(de)}$ are, respectively the energy density and pressure of the DE component while $w^{(de)} = p^{(de)} / \rho^{(de)}$ is the corresponding EoS parameter.

In a comoving coordinate system, the field equations (3), for the Bianchi-I space-time (1), in case of (4) and (5), read as

$$\ddot{B} + \frac{\dot{B} \dot{C}}{BC} = -w^{(m)} \rho^{(m)} - w^{(de)} \rho^{(de)},$$

$$\ddot{C} + \frac{\dot{A} \dot{B}}{AB} = -w^{(m)} \rho^{(m)} - w^{(de)} \rho^{(de)},$$

$$\ddot{A} + \frac{\dot{B} \dot{C}}{BC} + \frac{\dot{C} \dot{A}}{CA} = \rho^{(m)} + \rho^{(de)}.$$  

Here the over dot denotes derivative with respect to $t$.

The energy conservation equation $T_{ij;j} = 0$ yields

$$\dot{\rho}^{(m)} + 3(1 + w^{(m)}) \rho^{(m)} H + \dot{\rho}^{(de)} + 3(1 + w^{(de)}) \rho^{(de)} H = 0,$$  

where $H = \dot{a} / a$.

### 3 Solution of Field Equations

Subtracting (6) from (7), (6) from (8), (7) from (8) and taking second integral of each, we get the following three relations respectively:

$$\frac{A}{B} = d_1 \exp \left( x_1 \int a^{-3} dt \right),$$

$$\frac{A}{C} = d_2 \exp \left( x_2 \int a^{-3} dt \right),$$

$$\frac{B}{C} = d_3 \exp \left( x_3 \int a^{-3} dt \right),$$

where $d_1, x_1, d_2, x_2, d_3$ and $x_3$ are constants of integration.
From (11)-(13), the metric functions can be explicitly written as

\[ A(t) = a_1 a \exp \left( b_1 \int a^{-3} dt \right), \]  

(14)

\[ B(t) = a_2 a \exp \left( b_2 \int a^{-3} dt \right), \]  

(15)

\[ C(t) = a_3 a \exp \left( b_3 \int a^{-3} dt \right), \]  

(16)

where

\[ a_1 = \sqrt[3]{d_1 d_2}, a_2 = \sqrt[3]{d_1^{-1} d_3}, a_3 = \sqrt[3]{d_2 d_3}^{-1}, \]

\[ b_1 = \frac{x_1 + x_2}{3}, b_2 = \frac{x_3 - x_1}{3}, b_3 = \frac{-(x_2 + x_3)}{3}. \]

These constants satisfy the following two relations:

\[ a_1 a_2 a_3 = 1, b_1 + b_2 + b_3 = 0. \]  

(17)

The field equations (6)-(9) involve seven unknown variables, viz., \( A, B, C, w^{(m)}, w^{(de)}, \rho^{(m)} \) and \( \rho^{(de)} \). Therefore, to find a deterministic solution of the equations with the law (2), we need two suitable assumptions connecting the unknown variables.

Following Akarsu and Kilinc [10], we assume that the matter fluid and DE components interact minimally. Therefore, the energy momentum tensors of the two sources may be conserved separately.

The energy conservation equation \( T^{(m)}_{ij} \dot{=} 0 \), of the matter fluid leads to

\[ \dot{\rho}^{(m)} + 3(1 + w^{(m)}) \rho^{(m)} H = 0. \]  

(18)

Therefore, (10) gives

\[ \dot{\rho}^{(de)} + 3(1 + w^{(de)}) \rho^{(de)} H = 0, \]  

(19)

the energy conservation equation of DE component.

Next, we assume that the EoS parameter of the matter fluid to be a constant, that is,

\[ w^{(m)} = \frac{p^{(m)}}{\rho^{(m)}} = \text{const.}, \]  

(20)

while \( w^{(de)} \) has been allowed to vary with time since the current cosmological data from SN Ia, CMB and large scale structures mildly favor dynamically evolving DE crossing the phantom divide line (PDL) \( (w^{(de)} = -1) \) [10].

In view of (20), integration of (18) leads to

\[ \rho^{(m)} = c_0 a^{-3(1 + w^{(m)})}, \]  

(21)

where \( c_0 \) is a positive constant of integration.

In the following subsections, we discuss the DE cosmology for \( n \neq 0 \) and \( n = 0 \) by using the law (2).
3.1 DE Cosmology for \( n \neq 0 \)

In this case, integration of (2) leads to

\[
a(t) = (nDt + c_1)^{\frac{1}{n}},
\]

where \( c_1 \) is a constant of integration.

Inserting (22) into (14)-(16), we have

\[
A(t) = a_1(nDt + c_1)^{\frac{1}{n}} \exp \left[ \frac{b_1}{D(n - 3)}(nDt + c_1)^{\frac{n-3}{n}} \right],
\]

\[
B(t) = a_2(nDt + c_1)^{\frac{1}{n}} \exp \left[ \frac{b_2}{D(n - 3)}(nDt + c_1)^{\frac{n-3}{n}} \right],
\]

\[
C(t) = a_3(nDt + c_1)^{\frac{1}{n}} \exp \left[ \frac{b_3}{D(n - 3)}(nDt + c_1)^{\frac{n-3}{n}} \right],
\]

provided \( n \neq 3 \).

Therefore, the model (1) becomes

\[
ds^2 = -dt^2 + T^2 \left( a_1^2 e^{kb_1 T^\frac{n-3}{n}} dx^2 + a_2^2 e^{kb_2 T^\frac{n-3}{n}} dy^2 + a_3^2 e^{kb_3 T^\frac{n-3}{n}} dz^2 \right),
\]

where \( T = nDt + c_1 \) and \( k = \frac{2}{D(n-3)} \).

The Hubble parameter \( (H) \), energy density \( (\rho^{(m)}) \) of perfect fluid, DE density \( (\rho^{(de)}) \) and EoS parameter \( (w^{(de)}) \) of DE, for the model (26) are found to be

\[
H = D(nDt + c_1)^{-1},
\]

\[
\rho^{(m)} = c_0(nDt + c_1)^{-\frac{3(1+w^{(m)})}{n}},
\]

\[
\rho^{(de)} = 3D^2(nDt + c_1)^{-2} - \beta_0(nDt + c_1)^{\frac{6}{n}} - c_0(nDt + c_1)^{-\frac{3(1+w^{(m)})}{n}},
\]

\[
w^{(de)} = \frac{1}{\rho^{(de)}} \left[ (2n - 3)D^2(nDt + c_1)^{-2} - \beta_0(nDt + c_1)^{\frac{6}{n}} - c_0w^{(m)}(nDt + c_1)^{-\frac{3(1+w^{(m)})}{n}} \right],
\]

where \( \beta_0 = b_1^2 + b_2^2 + b_1 b_2 \).

The above solutions satisfy the equation (19) identically, as expected.

The spatial volume \( (V) \) and expansion scalar \( (\theta) \) of the model read as

\[
V = a^3 = (nDt + c_1)^{\frac{3}{n}},
\]

\[
\theta = 3H = 3(nDt + c_1)^{-1}.
\]

The anisotropy parameter \( (A) \) and shear scalar \( (\sigma) \) of the model are given by

\[
\dot{A} = \frac{1}{9H^2} \left[ \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right)^2 + \left( \frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right)^2 \right] = \frac{2\beta_0}{3D^2(nDt + c_1)^{-\frac{2(3-n)}{n}}},
\]
\[ \sigma^2 = \frac{3}{2} \dot{A}H^2 = \beta_0(nDt + c_1)^{\frac{3}{n}}. \] (34)

The density parameter \( \Omega^{(m)} \) of perfect fluid and the density parameter \( \Omega^{(de)} \) of DE are as follows:

\[ \Omega^{(m)} = \frac{\rho^{(m)}}{3H^2} = \frac{c_0}{3D^2}(nDt + c_1)\dot{\beta_0}^{\frac{2n-3(1+w^{(m)})}{n}}, \] (35)

\[ \Omega^{(de)} = \frac{\rho^{(de)}}{3H^2} = 1 - \frac{\beta_0}{D^2}(nDt + c_1)^{\frac{2(3-n)}{n}} - \frac{c_0}{3D^2}(nDt + c_1)^{\frac{2n-3(1+w^{(m)})}{n}}. \] (36)

The value of DP \((q)\) is found to be

\[ q = \frac{\ddot{a}}{a^2} = n - 1, \] (37)

which is a constant. A positive sign of \( q \), i.e., \( n > 1 \) corresponds to the standard decelerating model whereas the negative sign of \( q \), i.e., \( 0 < n < 1 \) indicates acceleration. The expansion of the Universe at a constant rate corresponds to \( n = 1 \), i.e., \( q = 0 \). Also, recent observations of SN Ia \([24\text{-}31]\) reveal that the present Universe is accelerating and value of DP lies somewhere in the range \(-1 < q < 0\). In a recent paper, Kumar \([32]\) found DP in the same range for the power-law cosmology by using latest data from SN Ia and \( H(z) \) observations. It follows that in the derived model, one can choose the values of DP consistent with the observations.

![Figure 1: Scale factors vs time with \( b_1 = 0.3, b_2 = 3.3, b_3 = -3.6, D = 2, n = 0.5 \) and other constants taken as unity.](image)

We observe that at \( t = -c_1/nD \), the spatial volume vanishes while all other parameters diverge. Therefore, the model has a big bang singularity at \( t = -c_1/nD \), which can be shifted to \( t = 0 \) by choosing \( c_1 = 0 \). The singularity is point type as the directional scale factors \( A(t), B(t) \) and \( C(t) \) vanish at the initial moment. The cosmological evolution in Bianchi-I space-time is expansionary, since \( A(t), B(t) \) and \( C(t) \) monotonically increase with the cosmic time as may be observed from Fig.1. Also, we see that the scale factors start growing at different rates in the beginning. This confirms the anisotropy of the derived model in the early Universe. But later on, the growth of scale factors takes place at the same rate. So the Universe expands at same rate in the three spatial directions, i.e., the Universe becomes isotropic. Since the present-day Universe is isotropic, so the derived model is consistent with the observations.
The parameters $H$, $\rho^{(m)}$, $\rho^{(de)}$, $\theta$ and $\sigma^2$ start off with extremely large values, and continue to decrease with the expansion of the Universe. The anisotropy parameter $\tilde{A}$ also decreases with the cosmic evolution provided $n < 3$. This shows that anisotropy of the model goes off during the cosmic evolution. The spatial volume grows with the cosmic time. The EoS parameter $w^{(de)}$ of DE asymptotically approaches $\frac{2n}{3} - 1$ provided $n < 3$. Fig. 2 depicts the variation of $w^{(de)}$ during the cosmic evolution for some selected values of constants. We observe that $w^{(de)}$ starts in phantom region ($w^{(de)} < -1$), crosses the PDL ($w^{(de)} = -1$) and finally varies in the quintessence region ($w^{(de)} > -1$). Also, seven year WMAP observations predict that $-1.55 < w_{de} < -0.7$ (see, Jarosik et al. [7]). Therefore, the derived model predicts the variation of $w^{(de)}$ consistent with the seven year WMAP observations (see, Fig. 2).

Adding (36) and (37), we get the overall density parameter

$$\Omega = \Omega^{(m)} + \Omega^{(de)} = 1 - \frac{\beta_0}{D^2} (nDt + c_1)^{\frac{2(3-n)}{n}} = 1 - \frac{3}{2} \tilde{A}. \quad (38)$$

Fig. 3 demonstrates the behavior of density parameters in the evolution of Bianchi-I Universe for selected values of constants. We observe that initially the ordinary matter density dominates the Universe. But later on, the DE dominates the evolution, which is probably responsible for the accelerated expansion of the present-day Universe. The overall density parameter curve becomes straight after the dominance of DE. Thus, DE may be a possible cause of flatness of the Universe.

### 3.2 DE Cosmology for $n = 0$

In this case, integration of (2) yields

$$a(t) = c_2 e^{Dt}, \quad (39)$$

where $c_2$ is a positive constant of integration.
Figure 3: Density parameters vs $t$ with $n = 0.05$, $D = 2.4$, $c_0 = 12$, $b_0 = c_1 = 1$, $w^{(m)} = 0$.

The metric functions, therefore, read as

$$A(t) = a_1c_2 \exp \left( Dt - \frac{b_1}{3Dc_2^3}e^{-3Dt} \right),$$  \hspace{1cm} (40)

$$B(t) = a_2c_2 \exp \left( Dt - \frac{b_2}{3Dc_2^3}e^{-3Dt} \right),$$  \hspace{1cm} (41)

$$C(t) = a_3c_2 \exp \left( Dt - \frac{b_3}{3Dc_2^3}e^{-3Dt} \right).$$  \hspace{1cm} (42)

Hence, the model (11) becomes

$$ds^2 = -dt^2 + 2e^{2Dt} \left( a_1^2e^{hb_1}e^{-3Dt} \, dx^2 + a_2^2e^{hb_2}e^{-3Dt} \, dy^2 + a_3^2e^{hb_3}e^{-3Dt} \, dz^2 \right).$$  \hspace{1cm} (43)

where $h = \frac{-2}{3Dc_2^3}$.

The Hubble parameter, energy density of perfect fluid, DE density and EoS parameter of DE, for the model (43) are obtained as

$$H = D,$$  \hspace{1cm} (44)

$$\rho^{(m)} = c_0c_2^{-3(1+w^{(m)})}e^{-3D(1+w^{(m)})t}$$ \hspace{1cm} (45)

$$\rho^{(de)} = 3D^2 - \beta_0c_2^{-6}e^{-6Dt} - c_0c_2^{-3(1+w^{(m)})}e^{-3D(1+w^{(m)})t},$$ \hspace{1cm} (46)

$$w^{(de)} = \frac{1}{\rho^{(de)}} \left[ -3D^2 - \beta_0c_2^{-6}e^{-6Dt} - c_0c_2^{-3(1+w^{(m)})}w^{(m)}e^{-3D(1+w^{(m)})t} \right].$$ \hspace{1cm} (47)

The above solutions satisfy the equation (19) identically, as expected.

The spatial volume and expansion scalar of the model read as

$$V = c_2^3e^{3Dt},$$ \hspace{1cm} (48)
$\theta = 3D.$ \hfill (49)

The anisotropy parameter and shear scalar of the model are given by

$$A = \frac{2\beta_0 c^{-6}}{3D^2} e^{-6Dt},$$ \hfill (50)

$$\sigma^2 = \beta_0 c^{-6} e^{-6Dt}.$$ \hfill (51)

The density parameters of perfect fluid and DE are as follows:

$$\Omega^{(m)} = \frac{c_0 c_2^{-3(1+w^{(m)})}}{3D^2} e^{-3D(1+w^{(m)})t},$$ \hfill (52)

$$\Omega^{(de)} = 1 - \frac{\beta_0 c_2^{-6}}{3D^2} e^{-6Dt} - \frac{c_0 c_2^{-3(1+w^{(m)})}}{3D^2} e^{-3D(1+w^{(m)})t}.$$ \hfill (53)

The DP is given by

$$q = -1.$$ \hfill (54)

Recent observations of SN Ia ([24]-[32]) suggest that the Universe is accelerating in its present state of evolution. It is believed that the way Universe is accelerating presently; it will expand at the fastest possible rate in future and forever. For $n = 0$, we get $q = -1$; incidentally this value of DP leads to $dH/dt = 0$, which implies the greatest value of Hubble’s parameter and the fastest rate of expansion of the Universe. Therefore, the derived model can be utilized to describe the dynamics of the late time evolution of the actual Universe. So, in what follows, we emphasize upon the late time behavior of the derived model.

Fig. 4 shows the behavior of the EoS parameter $w^{(de)}$ of DE. We observe that $w^{(de)}$ varies within the phantom region and never crosses the PDL. For sufficiently large times, $w^{(de)} \approx -1$. Therefore, late time dynamics of the Universe can be described by DE represented by cosmological constant.

Further, at late times, we have

$$\rho^{(m)} \approx 0, \rho^{(de)} \approx 3D^2$$

and

$$A \approx 0.$$ 

This shows that the ordinary matter density becomes negligible whereas the accelerated expansion of the Universe continues with non-zero and constant DE density at late times, as predicted by the observations. Also, the Universe remains isotropic at late times. Fig. 5 illustrates the behavior of the density parameters during the evolution of the Universe in the derived model. It is observed that the DE component dominates the Universe at late times and $\Omega \approx 1$. 

9
4 Kinematics Tests

We now study the consistency of the models, discussed in the last section, with the observational parameters. We know that as we look back from our position at $r = 0$, $t = t_0$ to some object located at $r_1$, we deal with some past moment of time $t_1 < t_0$. Since $r_1$ and $t_1$ are not directly measurable, we measure some physical parameters such as redshift, proper distance, luminosity distance, angular diameter etc. The various cosmic measures sensitively depend on the parameters of the models, and therefore the physical properties of the distant objects are also influenced by these parameters [33, 34]. In what follows, we discuss some important cosmic measures such as lookback time, proper distance, luminosity...
distance, angular diameter distance and event horizon for the models, discussed in the cases $n \neq 0$ and $n = 0$, respectively.

### 4.1 Lookback Time-Redshift

The lookback time is the radiation travel time, taken by a photon emitted by a source at an instant $t_z$ in the past, to reach an observer at the present epoch $t_0$ ($z = 0$). For a given redshift $z$, the scale function $a(t_z)$ is related to $a(t_0)$ by

$$1 + z = \frac{a(t_0)}{a(t_z)}. \quad (55)$$

Therefore, the lookback time is exactly the time interval $\Delta t = t_0 - t_z$, required by the Universe to evolve between these two values of the scale factor. An object at $z = 1$ emitted its light when the Universe was half its present scale ($a = 0.5a_0$). How long ago the light was emitted (the lookback time), it depends on the dynamics of the Universe.

Substituting (22) into (55), we get

$$t_z = t_0(1 + z)^{-n}, \quad (56)$$

which can be expressed as

$$H_0(t_0 - t_z) = \frac{1}{n} \left[ 1 - (1 + z)^{-n} \right], \quad (n \neq 0) \quad (57)$$

where $H_0$ is the Hubble’s constant in km s$^{-1}$ Mpc$^{-1}$, and its value at present is 72 km s$^{-1}$ Mpc$^{-1}$. However, note that $H_0$ is dimensionally similar to the reciprocal of time. The reciprocal of Hubble’s constant is called Hubble time $T_H$:

$$T_H = H_0^{-1},$$

where $T_H$ is expressed in s, and $H_0$ in s$^{-1}$. If $H_0$ is expressed in km s$^{-1}$ Mpc$^{-1}$, and $T_H$ in gigayears (1 Gyr = 1 billion years = $10^9$ years), then $T_H = 977.8 H_0^{-1}$. It is important to note that the age of the Universe can be less than or greater than the Hubble time, depending on the values of $w^{(m)}$ and $w^{(de)}$.

Taking the limit $z \to \infty$ in (57), the present age of the Universe (the extrapolated time back to the bang) is

$$t_0 = \frac{1}{n} H_0^{-1}, \quad (58)$$

which, for $n = 3/2$, reduces to the expression of the standard dust model.

The lookback time, in terms of the DP, may be obtained as

$$H_0(t_0 - t_z) = \frac{1}{1 + q} \left[ 1 - (1 + z)^{-1+q} \right]. \quad (59)$$

For small $z$, equation (59) gives

$$H_0(t_0 - t_z) = z - \left(1 + \frac{q}{2}\right) z^2 + \ldots. \quad (60)$$

Also for $q = \frac{1}{2}$, equation (59) yields

$$H_0(t_0 - t_z) = \frac{2}{3} \left[ 1 - (1 + z)^{-1/2} \right], \quad (61)$$

which is used to describe the lookback time in Einstein-de Sitter Universe. In the limit $z \to \infty$, we obtain $\Delta t = (2/3)T_H$.

For $n = 0$, i.e., $q = -1$, we find the following expression for the lookback time:

$$H_0(t_0 - t_z) = \ln(1 + z). \quad (62)$$
For small \( z \), we have

\[ H_0(t_0 - t_z) = z - \frac{1}{2}z^2. \]  \hfill (63)

In Fig. 6, we plot the lookback time as a function of the redshift \( z \) for \( H_0 = 100 \text{ km s}^{-1}\text{ Mpc}^{-1} \), and selected values of \( q \). We observe that the lookback time is larger for negative values of \( q \). Thus, in an accelerating Universe, the light from a distant object to reach an observer at the present epoch, would take larger time in comparison to the decelerating one. It is maximum in case of \( q = -1 \).

### 4.2 Proper Distance-Redshift

If an observer at \( r = 0 \) and \( t = t_0 \) receives the light emitted by a source at any instant \( t \) located at \( r = r_1 \) with redshift \( z \), then the proper distance between the source and observer is given by

\[ d(z) = r_1(z)a(t_0), \]  \hfill (64)

where the radial coordinate distance \( r_1(z) \) is given by

\[ r_1(z) = \int_t^{t_0} \frac{dt}{a(t)} = \frac{1}{(n-1)H_0a(t_0)} \left[ 1 - (1 + z)^{1-n} \right], \]  \hfill (65)

provided \( n \neq 1 \).

Therefore, equation (64) leads to

\[ d(z) = \frac{1}{(n-1)H_0} \left[ 1 - (1 + z)^{1-n} \right]. \]  \hfill (66)

In terms of \( q \), the proper distance can be written as

\[ d(z) = \frac{1}{qH_0} \left[ 1 - (1 + z)^{-q} \right]. \]  \hfill (67)

For small values of \( z \), we get

\[ d(z) = \frac{1}{H_0} \left[ z - \frac{1}{2}(1+q)z^2 + ....... \right]. \]  \hfill (68)
Taking the limit as \( z \to \infty \) in equation (67), we get the maximum proper distance as

\[
d(z = \infty) = \frac{H_0^{-1}}{q}.
\]  

Substituting \( q = \frac{1}{2} \) into (67), we get the following proper distance for Einstein-de Sitter Universe:

\[
d(z) = \frac{2}{H_0} \left[ 1 - (1 + z)^{-\frac{1}{2}} \right].
\]

For \( n = 0 \), we find

\[
d(z) = \frac{1}{H_0} z.
\]

In Fig. 7, we show the proper distance as the function of redshift for \( H_0 = 100 \, \text{km s}^{-1} \, \text{Mpc}^{-1} \), and selected values of \( q \). We observe that the proper distance is larger in case of the accelerating models of the Universe. The proper distance curves become almost straight as \( q \) gets values closer to \( -1 \). There is a linear relationship between proper distance and redshift for \( q = -1 \).

### 4.3 Luminosity Distance-Redshift

The luminosity distance of a light source is defined as

\[
d_L = \left( \frac{L}{4\pi\ell} \right)^{\frac{1}{2}} = r_1(z)a(t_0)(1 + z),
\]

where \( L \) and \( \ell \) are absolute and apparent luminosities, respectively. \( r_1(z) \) is the radial coordinate distance of the object at light emission, and is given by equation (65). In view of (64), the equation (72) takes the form

\[
d_L = d(z)(1 + z).
\]
Substituting (66) into (73), the luminosity distance-redshift relation reads as

$$d_\ell = \frac{1}{(n-1)H_0} [(1+z) - (1+z)^{2-n}].$$  \(74\)

In terms of \(q\), the luminosity distance-redshift relation can be written as

$$d_\ell = \frac{1}{qH_0} [(1+z) - (1+z)^{1-q}].$$  \(75\)

Expanding (75) for small redshifts, after some algebra, we find

$$d_\ell = \frac{1}{H_0} \left[ z + \frac{1}{2} (1-q) z^2 + ........ \right].$$  \(76\)

The luminosity distance for the Einstein-de Sitter Universe \((q = \frac{1}{2})\) is given by

$$d_\ell = \frac{2}{H_0} \left[ (1+z) - (1+z)^{\frac{3}{2}} \right].$$  \(77\)

For \(q = 1\), \(H_0d_\ell = z\), which shows linear relationship between luminosity distance and redshift. For \(n = 0\), i.e., \(q = -1\), we find

$$d_\ell = \frac{1}{H_0} (z + z^2).$$  \(78\)

![Figure 8: Luminosity distance \((d_\ell)\) versus redshift \(z\).](image)

Luminosity distance as a function of the redshift is shown in Fig. 8 for \(H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}\), and selected values of \(q\). We observe that all curves start with the linear Hubble law \((H_0d_\ell = z)\) for small \(z\), but then only the curve for \(q = 1\) shows linearity all the way. We also note that for the same redshift, the luminosity distance is larger for smaller values of \(q\). The rate of increase of \(d_\ell\) is fastest in case of \(q = -1\). All the curves merge for small \(z\).
4.4 Angular Diameter Distance-Redshift

The angular diameter of a light source of proper distance \( d(z) \) at \( r = r_1 \) and \( t = t_1 \), observed at \( r = 0 \) and \( t = t_0 \), is given by

\[
\delta = \frac{d(z)}{r_1 a(t_1)} = \frac{d(z)(1 + z)^2}{d_\ell}.
\]  
(79)

The angular diameter distance \( d_A \) is defined as the ratio of the source diameter to its angular diameter, and therefore

\[
d_A = \frac{d(z)}{\delta} = r_1 a(t_1) = d_\ell(1 + z)^{-2}.
\]  
(80)

Inserting (74) into (80), we find

\[
d_A = 1 \left(\frac{n-1}{n-1}H_0\right) \left[(1 + z)^{-1} - (1 + z)^{-n}\right].
\]  
(81)

In terms of \( q \), the angular diameter distance can be written as

\[
d_A = \frac{1}{qH_0} \left[(1 + z)^{-1} - (1 + z)^{-(1+q)}\right].
\]  
(82)

For the Einstein-de Sitter Universe, we get

\[
d_A = \frac{2}{H_0} \left[(1 + z)^{-1} - (1 + z)^{-\frac{3}{2}}\right].
\]  
(83)

The extremum value of angular diameter distance occurs at \( z_c = (1 + q)^{\frac{1}{q}} - 1 \). Therefore, for the Einstein-de Sitter Universe, \( d_A \) is extremum at \( z_c = \frac{5}{4} \).

For \( n = 0 \), i.e., \( q = -1 \), we find

\[
d_A = \frac{1}{H_0} \left(\frac{z}{1 + z}\right).
\]  
(84)

In Fig. 9, we show the angular diameter distance versus redshift for selected values of \( q \) in the units of \( H_0^{-1} \). We observe that each of the angular diameter distance curve has one point of maxima. For all models, the angular diameter distance increases with increasing \( z \), reaches its maximum value at a given \( z_c \), and eventually begins to decrease. Usually \( d_A \) has a maximum (or minimum) for some \( z = z_c \).
4.5 Event Horizon

The event horizon \( r_E \) is the limit of the proper distance such that a light signal sent by a distant source at \( r = r_1, \ t = t_1 \) will never reach an observer at \( r = 0, \ t = t_0 \) if \( r_1 > r_E \). In other words, the two observers having a proper distance greater than \( r_E \), can never communicate each other. Mathematically, the event horizon is given by

\[
    r_E = a(t_0) \int_{t_0}^{\infty} \frac{dt}{a(t)}. \tag{85}
\]

Substituting (39) into (85), we get

\[
    r_E = \frac{1}{(1 - n)H_0}. \tag{86}
\]

For \( n = 0 \), i.e., \( q = -1 \), we find

\[
    r_E = \frac{1}{H_0}. \tag{87}
\]

We observe that an event horizon exists in the models where \( n < 1 \) or \( q < 0 \). It follows that event horizon does not exist in the Einstein-de Sitter Universe. Equation (87) gives the event horizon, where no observer beyond a proper distance \( r_E \) at \( t = t_0 \) can communicate with another observer.

5 Concluding Remarks

We have presented singular \((n \neq 0)\) and non-singular \((n = 0)\) Bianchi type-I models of Universe by taking into account dynamical DE along with the usual matter fluid content. The models describe shearing, non-rotating and expanding Universe. The anisotropy of the Universe goes off during the cosmic evolution. The analysis of the solutions reveals that the Universe achieves isotropy and flatness after the dominance of DE. Thus, one may hope that DE would be a plausible explanation to the unsolved problems of cosmology. The values of EoS parameter of DE have been found in the range predicted by seven year WMAP observations. Thus the derived models are consistent with the WMAP observations and hence are physically viable.

The explicit expressions for the observational parameters such as the lookback time, proper distance, luminosity distance, angular diameter distance and event horizon have been obtained in the power-law and exponential-law cosmologies. The graphical analysis reveals that the lookback time, proper distance, luminosity distance and angular diameter distance have larger values in accelerating models of the Universe. The maximum values have been observed in the case \( q = -1 \). The event horizon exists in the accelerating models of the Universe. It may be noted that the analytical expressions of the observational parameters in power-law and exponential-law models have been discussed in many works [35, 36, 37]. However, in addition the present work presents the graphical analysis of the parameters, and demonstrates their behavior in accelerating models in contrast with the decelerating models.

It is worthwhile to mention that although any constant DP model is not capable of representing the whole or a long period of history of the Universe at one go, it can still be considered as an approximation for representing a particular period of the history of the Universe, for instance, the vicinity of the present time of the Universe [23, 32]. Thus the models presented in this work can be utilized for this purpose. For instance, the non-singular model (exponential-law model) successfully describes the late time evolution of accelerating Universe with DE and puts forward cosmological constant as a candidate of DE.

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