ONE-RELATOR SASAKIAN GROUPS

BY

INDRANIL BISWAS and MAHAN MJ (Mumbai)

Abstract. We prove that any one-relator group $G$ is the fundamental group of a compact Sasakian manifold if and only if $G$ is either finite cyclic or isomorphic to the fundamental group of a compact Riemann surface of genus $g > 0$ with at most one orbifold point of order $n \geq 1$. We also classify all groups of deficiency at least 2 that are also the fundamental group of some compact Sasakian manifold.

1. Introduction. Fundamental groups of compact Sasakian manifolds are called Sasakian groups. While the Kähler groups have been studied for long (see [ABCKT] and references therein), the study of their odd-dimensional relatives, viz. Sasakian groups, has been started relatively recently [Ch], [BM2]. In this paper, we extend the results of [BM1], [Ko] to Sasakian groups by analyzing one-relator Sasakian groups and Sasakian groups of deficiency greater than one.

We prove the following generalization of the corresponding statement for Kähler groups [Ko] (see Theorem 3.3 below):

**Theorem 1.1.** Let $G$ be a Sasakian group with $\text{def}(G) > 1$. Then $G$ must be an orbifold surface group with genus greater than 1.

The following theorem classifies one-relator Sasakian groups (see Theorem 4.10 below):

**Theorem 1.2.** Let $G$ be an infinite one-relator group. Then $G$ is Sasakian if and only if it is isomorphic to

$$\langle a_1, b_1, \ldots, a_i, b_i, \ldots, a_g, b_g \mid \left(\prod_{i=1}^{g} [a_i, b_i]\right)^n \rangle,$$

where $g$ and $n$ are some positive integers.

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We observe that each of the groups
\[ \left\langle a_1, b_1, \ldots, a_g, b_g \mid \left( \prod_{i=1}^{g} [a_i, b_i] \right)^n \right\rangle, \quad g, n > 0, \]
can in fact be realized as the fundamental group of a closed Sasakian manifold as follows. It was shown in [BM1] that these groups are fundamental groups of smooth projective varieties. Since the fundamental group of a smooth projective variety is also a Sasakian group [Ch, Proposition 1.2], the above groups are also Sasakian groups.

It was shown in [Ch] that all finite groups are Sasakian. Since the only finite one-relator groups are finite cyclic groups, it follows that finite one-relator Sasakian groups are precisely the finite cyclic groups.

A useful tool we establish along the way is the following generalization of the corresponding fact for Kähler groups in [ABCKT, Ch. 4] (see Theorem 3.1 below):

**Theorem 1.3.** If $G$ is a Sasakian group that satisfies $\beta_1^2(G) > 0$, then $G$ is virtually a surface group.

In Theorem 1.3 $\beta_1^2(G)$ denotes the first $l^2$ Betti number of $G$.

### 2. Sasakian groups of positive deficiency.

We refer to [BoGa] for the definition and basic properties of Sasakian manifolds. The Sasakian structure on any Sasakian manifold $M$ can be perturbed in order to make the Sasakian structure quasi-regular [Ru, OV, Theorem 1.2]. In particular, every Sasakian group is the fundamental group of some closed quasi-regular Sasakian manifold. In view of this, henceforth all Sasakian manifolds considered will be assumed to be quasi-regular.

Let $M$ be a quasi-regular closed Sasakian manifold. The group $U(1)$ acts on $M$; let
\[ \alpha : M \times U(1) \to M \]
be the action map. The Kähler orbifold base $M/U(1)$ of $M$ will be denoted by $B$ [Ru, BoGa, Theorem 7.1.3]. Let
\[ f : M \to M/U(1) = B \]
be the quotient map. Let
\[ G = \pi_1(M) \]
be the fundamental group. For notational convenience we will omit mentioning the base point for any fundamental group.

Take a regular point $x_0 \in B$. This means that the action of $U(1)$ on the fiber $f^{-1}(x_0)$ is free. Let $i : f^{-1}(x_0) \to M$ be the inclusion map of the regular fiber. The map $U(1) \to f^{-1}(x_0)$ defined by $\lambda \to x_1 \lambda$, where $x_1 \in f^{-1}(x_0)$
is any given point, is a diffeomorphism. The image \( i_\ast \pi_1(f^{-1}(x_0)) = i_\ast \mathbb{Z} \) is a (finite or infinite) cyclic group, which we shall denote by \( C \); to clarify, \( C \) may be the trivial group. Let 
\[
Q := \pi_{1}^{\text{orb}}(B)
\]
be the orbifold fundamental group of \( B \) regarded as the orbifold quotient of \( M \) by the action of \( U(1) \). Then there is a short exact sequence 
\[
1 \to C \to G = \pi_1(M) \to Q \to 1.
\]

**Lemma 2.1** ([BlGo], [Fu], [Ta], [Ch]). For the group \( G \) in (2.3), the first Betti number \( b_1(G) \) is even.

Note that from Lemma 2.1 it follows that a non-trivial free group with finitely many generators cannot be a Sasakian group.

The *deficiency* of a finitely presented group \( \Gamma \) is the maximum of \( n - r \) taken over all possible finite presentations of \( \Gamma \), where \( n \) and \( r \) are the numbers of generators and relations respectively.

**Corollary 2.2.** If \( G \) in (2.3) is of positive deficiency, then \( b_1(G) \geq 2 \). If \( G \) in (2.3) is an infinite one-relator group, then \( b_1(G) \geq 2 \).

**Proof.** Since \( G \) has positive deficiency, we get \( b_1(G) \geq 1 \). From Lemma 2.1 it follows that \( b_1(G) \geq 2 \).

Since infinite one-relator groups have positive deficiency, the second statement follows from the first.

We shall denote the first \( l^2 \) Betti number of a group \( H \) by \( \beta_1^2(H) \). As Kotschick did in [Ko], we shall make essential use of the following theorem due to Gaboriau and Lück.

**Theorem 2.3** ([Lü], [Ga]). If \( 1 \to N \to H \to Q \to 1 \) is a short exact sequence of infinite finitely generated groups with \( H \) finitely presented, then \( \beta_1^2(H) = 0 \).

**Corollary 2.4.** Assume that \( G \) in (2.3) is of positive deficiency (for example, it is an infinite one-relator group). If \( C \) in (2.3) is infinite, then \( \beta_1^2(G) = 0 \).

**Proof.** In view of Theorem 2.3 it suffices to show that \( Q \) in (2.3) is infinite. Towards a contradiction, suppose \( Q \) is finite. Then \( G \) has a subgroup \( C = \mathbb{Z} \) of finite index in it. Let \( N \) be the finite covering of \( M \) such that \( C = \pi_1(N) \subset \pi_1(M) = G \). So \( N \) is a quasi-regular closed Sasakian manifold with fundamental group \( \mathbb{Z} \). This contradicts Lemma 2.1.

We shall also need the following fundamental inequality:

**Lemma 2.5** ([Gr]). Let \( \text{def}(H) \) denote the deficiency of \( H \). Then 
\[
\text{def}(H) - 1 \leq \beta_1^2(H).
\]
Combining Corollary 2.4 and Lemma 2.5 leads to:

**Lemma 2.6.** Assume that $G$ in (2.3) is of deficiency at least 2. Then the cyclic group $C$ in (2.3) is finite.

*Proof.* Since $G$ is of deficiency at least 2, Lemma 2.5 says that $\beta_2^2(G) \geq 1$. On the other hand, if $C$ is infinite, then $\beta_1^2(C) = 0$ by Corollary 2.4. Hence the cyclic group $C$ in (2.3) is finite. ■

The following lemma is proved in [Ko].

**Lemma 2.7 ([Ko, Lemma 3]).** If $C$ in (2.3) is finite, then $Q$ in (2.3) satisfies the condition

$$\beta_1^2(Q) = p \cdot \beta_1^2(G),$$

where $p$ is the order of $C$.

**Lemma 2.8.** Assume that $G$ in (2.3) is of deficiency at least 2. Then $Q$ in (2.3) is the fundamental group of a smooth projective orbifold with $\beta_1^2(Q) > 0$.

*Proof.* Since $M$ in (2.2) is a closed quasi-regular Sasakian manifold, the quotient $B$ in (2.2) is a projective orbifold.

Since $G$ is of deficiency at least 2, Lemma 2.6 implies that $C$ in (2.3) is finite. In view of Lemma 2.5, the deficiency assumption also implies that $\beta_2^2(G) > 0$. Hence $\beta_1^2(Q) > 0$ by Lemma 2.7. ■

3. Sasakian groups with deficiency greater than 1

3.1. From Sasakian groups to virtual surface groups. A group $H$ is said to be virtually a surface group if some finite index subgroup of $H$ is the fundamental group of a closed surface of positive first Betti number. The following theorem was proved in [Gr], [ABR], [ABCKT, p. 47, Theorem 4.1] for Kähler groups.

**Theorem 3.1.** If $G$ in (2.3) satisfies the condition $\beta_1^2(G) > 0$, then $G$ is virtually a surface group.

*Proof.* Let $C, G, Q$ be as in (2.3). As we saw in the proof of Corollary 2.4, if $Q$ is finite then $C$ is finite. On the other hand, if $Q$ is infinite, then Theorem 2.3 implies that $C$ is finite. Therefore, we conclude that $C$ is finite.

Hence, by Lemma 2.7 we have $\beta_1^2(Q) > 0$. Note that $Q$ equals the orbifold fundamental group of the orbifold $B$ in (2.2).

Consider the Sasakian metric $g_M$ on the quasi-regular Sasakian manifold $M$ in (2.2). In the proof of [ABCKT, p. 47, Theorem 4.1] substitute $(M, g_M)$ in place of the Kähler metric. Then it is straightforward to check that all results in [ABCKT] Sections 4.2 and 4.3 remain valid.
Let 
\[ \psi : \tilde{M} \to M \]
be the universal covering of the quasi-regular Sasakian manifold \( M \) in (2.2). Consider the action of the fundamental group \( \pi_1(M) \) on \( \tilde{M} \). Set 
\[ \widehat{M} := \tilde{M}/C, \]
where \( C \subset \pi_1(M) = G \) is the subgroup in (2.3). Let 
(3.1) 
\[ p_0 : \widehat{M} \to M \]
be the natural projection.

The action of \( U(1) \) on \( M \) in (2.1) canonically lifts to an action of \( U(1) \) on \( \widehat{M} \). To prove this, let 
\[ \chi = \alpha \circ (p_0 \times \text{Id}_{U(1)}) : \widehat{M} \times U(1) \to M \]
be the composition, where \( \alpha \) is the map in (2.1). The image of the homomorphism 
\[ \chi_* : \pi_1(\widehat{M} \times U(1)) \to \pi_1(M) \]
is clearly the subgroup \( C \subset G = \pi_1(M) \). From this it follows that the action of \( U(1) \) on \( M \) canonically lifts to an action of \( U(1) \) on \( \widehat{M} \). Let 
(3.2) 
\[ \varphi : \widehat{M} \to \hat{B} := \widehat{M}/U(1) \]
be the orbifold quotient for this action of \( U(1) \) on \( \widehat{M} \).

So we have a commutative diagram 
(3.3) 
\[
\begin{array}{ccc}
\widehat{M} & \xrightarrow{p_0} & M \\
\downarrow{\varphi} & & \downarrow{f} \\
\hat{B} & \xrightarrow{q} & B 
\end{array}
\]
where \( p_0, \varphi \) and \( f \) are the maps in (3.1), (3.2) and (2.2) respectively. The map \( q \) in (3.3) is an étale Galois covering, in the orbifold category, with Galois group \( Q = G/C \).

Following the proof of Theorem 4.14 in [ABCKT, Section 4.4, p. 53] we see that there are a proper holomorphic map to the unit disk 
(3.4) 
\[ h : \hat{B} \to \mathbb{D}^2 := \{ z \in \mathbb{C} \mid |z| < 1 \} \]
and a homomorphism \( \rho : Q \to \text{Aut}(\mathbb{D}^2) = \text{PSL}(2, \mathbb{R}) \) such that

- the fibers of \( h \) are connected,
- \( h \) is \( Q \)-equivariant for the action of \( Q \) on \( \mathbb{D}^2 \) given by \( \rho \) and the Galois action of \( Q \) on \( \hat{B} \), and
- the homomorphism \( h^* : \mathcal{H}^1_{(2)}(\mathbb{D}^2) \to \mathcal{H}^1_{(2)}(\hat{B}) \) corresponding to \( h \) is an isomorphism.
Consider the complex one-dimensional orbifold $O = \mathbb{D}^2/\rho(Q)$ with orbifold fundamental group $Q_1 = \rho(Q)$. Since $h$ in (3.4) is $Q$-equivariant, we conclude that

- $h$ induces a holomorphic (orbifold) map $h_1 : B \to O$ inducing a surjective homomorphism $h_{1*} : Q \to Q_1$ of orbifold fundamental groups, and
- the fibers of $h_1$ are compact and connected, because $h$ is proper with connected fibers.

Consequently, there exists a short exact sequence

$$1 \to K \to Q \to Q_1 \to 1$$

(3.5) given by $\rho = h_{1*}$. It was observed at the beginning of the proof that $\beta^2_1(Q) > 0$. Also, note that $Q_1$ is an infinite group, because $B$, being compact, does not have any non-constant map to a finite quotient of $\mathbb{D}^2$. In view of these facts, by Theorem 2.3, the group $K$ in (3.5) must be finite.

Since $O$ is a compact complex one-dimensional orbifold (it is compact because $B$ has a non-constant map to it), we conclude that $Q_1$ is virtually a surface group. Passing to a further finite index subgroup $Q_2$ of $Q_1$ if necessary, we may assume that $Q_2$ is a surface group acting trivially on the finite normal subgroup $K$ in (3.5). Hence $Q$ is virtually a surface group.

Finally, since $C$ is finite, as pointed out at the beginning of the proof, the same argument applied to $G$ now shows that $G$ is virtually a surface group.

By Theorem 3.1 and Lemma 2.5, we immediately have:

**Proposition 3.2.** If $G$ in (2.3) satisfies the condition $\text{def}(G) > 1$, then $G$ is virtually a surface group.

### 3.2. From virtual surface groups to orbifold groups

In this subsection, we classify virtual surface groups of positive deficiency. Let $G$ be a virtual surface group; equivalently, there exists an exact sequence

$$1 \to \pi_1(S) \to G \to F \to 1,$$

(3.6) where $S$ is a compact surface of positive genus, and $F$ is a finite group acting by automorphisms on $\pi_1(S)$.

**Theorem 3.3.** Assume that $G$ in (2.3) satisfies the condition $\text{def}(G) > 1$. Then $G$ must be an orbifold surface group with genus greater than 1.

**Proof.** By Proposition 3.2, $G$ is a virtual surface group.

**Euclidean case.** Suppose that the genus of the surface $S$ in (3.6) for $G = G$ is one. Then there exists a two-dimensional crystallographic group $H$ such that

$$G = H \times F_1,$$
where $F_1$ is the subgroup of $F$ in (3.6) acting trivially on $\pi_1(S)$. The deficiency of any crystallographic group other than $\mathbb{Z} \times \mathbb{Z}$, and the Klein bottle group, is non-negative. The deficiency of any finite group is also non-negative. Since $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group have deficiency exactly 1, and need at least two generators, it follows that if $F_1$ is non-trivial, then $\text{def}(G) \geq 0$. Thus, $F_1$ is trivial, and we have one of the two possibilities for $G$:

- $\mathbb{Z} \times \mathbb{Z}$,
- the Klein bottle group.

Since the Klein bottle group has first Betti number 1, it cannot be Sasakian (see Lemma 2.1). Finally, $\mathbb{Z} \times \mathbb{Z}$ has deficiency 1 and is ruled out by the hypothesis.

**Hyperbolic case.** Next, suppose that the genus of $S$ in (3.6) is greater than 1. By the Nielsen realization theorem [Ke], there exists a hyperbolic orbifold $O$ with $\pi_1(O) = H$ such that

$$G = H \times B_1$$

with $B_1$ finite. As in the genus one case, it follows that $B_1$ is trivial. If $O$ is non-orientable, then $b_1(O)$ is odd, forcing $O$ to be an orientable hyperbolic orbifold (see Lemma 2.1 again). Further, from the assumption that the deficiency is greater than 1 it follows that the genus of $O$ is greater than 1.

### 4. One-relator groups

#### 4.1. One-relator Sasakian groups are projective orbifold groups.

Murasugi has described in detail the centers of one-relator groups:

**Theorem 4.1 ([Mu, Theorems 1, 2]).** Let $\mathcal{G} = \langle x_1, \ldots, x_k \mid w^n \rangle$ be a one-relator group with $n \geq 1$. If $k \geq 3$, then the center $Z(\mathcal{G})$ is trivial. If $\mathcal{G}$ is not abelian with $k = 2$, and $Z(\mathcal{G})$ is not trivial, then $Z(\mathcal{G})$ is infinite cyclic.

**Theorem 4.2 ([KS, p. 219]).** Let $\mathcal{G}$ be a one-relator group having a (non-trivial) finitely presented normal subgroup $H$ of infinite index. Then $\mathcal{G}$ is torsion-free and has two generators. Further, $\mathcal{G}$ is an infinite cyclic or infinite dihedral extension of a finitely generated free group $N$ (meaning $N \subset \mathcal{G}$) satisfying the following:

- $H \subset N$ if $H$ is not cyclic, and
- $H \cap N$ is trivial if $H$ is cyclic.

**Proposition 4.3.** Assume that $G$ in (2.3) is an infinite one-relator group. Then

- $C$ in (2.3) is trivial, and
- $G = Q$ is a projective orbifold group.
Proof. By Theorem 4.1, $C$ is trivial if $\text{def}(G) > 1$. In that case, $G = Q$ is a projective orbifold group.

On the other hand, if $G$ is abelian, then $G = \mathbb{Z} \times \mathbb{Z}$ and $G$ is a projective orbifold group, as it is the fundamental group of an elliptic curve.

Therefore, assume that $\text{def}(G) = 1$ and $G$ is non-abelian.

Since $\text{def}(G) = 1$, the minimum number of generators of the one-relator group $G$, which we shall denote by $k$, is $2$.

Since $C$ is contained in the center $Z(G)$ of $G$, if $Z(G)$ is trivial, then the proposition follows. So we assume that $Z(G)$ is non-trivial.

By Theorem 4.1, $C = \mathbb{Z}$. Hence, by Theorem 4.2, $G = N \times \mathbb{Z}$, where $N = F_r$ is free of rank $r$ greater than one; we note that if $N$ is free of rank 1, then $G = \mathbb{Z} \times \mathbb{Z}$ and $C \neq \mathbb{Z}$.

Next, since $G$ is one-relator, we have $b_2(G) \leq 1$. On the other hand,

$$b_2(N \times \mathbb{Z}) = r > 1,$$

so we get a contradiction. Thus, if $k = 2$, then $Z(G)$ is trivial, forcing $C$ to be trivial. Consequently, $G = Q$ is a projective orbifold group. ■

We shall need the following result.

PROPOSITION 4.4 ([FKS, Theorem 1]). Let $G = \langle x_1, \ldots, x_k | w^n \rangle$ be a one-relator group with $n \geq 1$. If $k = 1$, then $G$ is torsion-free. Else, every torsion element in $G$ is conjugate to a power of $w$, and the subgroup generated by torsion elements in $G$ is the free product of the conjugates of $w$.

4.2. From orbifold projective groups to projective groups. In this subsection we assume that $G$ in (2.3) is a one-relator Sasakian group

$$G = \langle x_1, \ldots, x_k | w^n \rangle.$$

If $\text{def}(G) > 1$, Theorem 3.3 shows that $G = \pi_1(O)$, where $O$ is a hyperbolic orbifold of genus greater than 1.

Therefore, we assume that $\text{def}(G) = 1$, so $k = 2$.

Since $k = 2$, Proposition 4.3 further shows that

$$G = \pi_1(B), \hspace{1cm} (4.1)$$

where $B$ is the projective orbifold in (2.2).

It can be shown that $\dim_C B > 1$. Indeed, if $\dim_C B = 1$, then $M$ in (2.2) is a 3-dimensional Seifert-fibered manifold. Since $G = \pi_1(M)$ is infinite, it follows that the center of $G$ is infinite cyclic [Ho, Ch. 12]. But this contradicts Proposition 4.3. So we conclude that $\dim_C B > 1$.

Structure of singularities. We analyze the locus of the points of the orbifold $B$ in (2.2) with non-trivial inertia group. Such examples exist even when, at the global level, the Sasakian manifold is simply connected [CMST]
Let $\mathcal{S}$ be a connected component of this locus of $B$. Let $U_S$ denote a regular neighborhood of $S$ in $B$.

**Lemma 4.5.** Let $2m$ denote the real dimension of $B$. There exists a lens space $L$ of dimension $2l - 1$, with $l \geq 2$, such that the topological space for the orbifold $U_S$ is homeomorphic to $cL \times D_{2m-2l}$, where $cL$ denotes the cone on $L$ and $D_{2m-2l}$ denotes a ball of real dimension $2m - 2l$.

**Proof.** Recall $B = M/U(1)$ with $M$ being a smooth manifold. All the isotropy subgroups for the action of $U(1)$ on $M$ are finite cyclic groups. The local structure of $B$ follows from this (see [BoGa, Theorem 4.7.7]).

Let $i_S : U_S \to B$ be the inclusion map between orbifolds, and let $i_{S,*}$ denote the induced homomorphism between orbifold fundamental groups.

**Lemma 4.6.** Suppose $i_{S,*} (\pi_{1}^{\text{orb}}(U_S))$ is trivial for all components $S$. Then the group $G$ is projective. In particular, if $G$ is torsion-free, it is projective.

**Proof.** We may blow up the orbifold $B$ and obtain another orbifold $\varpi : B' \to B$ such that the underlying topological space for $B'$ is a smooth projective variety. We note that the underlying topological space for the projective orbifold $B''$ is a smooth projective variety if the locus in $B''$ of points with non-trivial inertia is a normal crossing divisor. In view of the given condition, we may choose $B'$ such that $\pi_1(B)$ coincides with the fundamental group of the underlying topological space for $B'$. But the underlying topological space for $B'$ is a smooth projective variety, so $\pi_1(B)$ is a projective group.

If $G$ is torsion-free, then $i_{S,*} (\pi_1(B))$ must be trivial for all components $S$. The conclusion of the lemma follows.

**Lemma 4.7.** Suppose $i_{S,*} (\pi_{1}^{\text{orb}}(U_S))$ is non-trivial for some $S$. Then any torsion element of $G$ has a non-trivial power that is conjugate to an element of $i_{S,*} (\pi_{1}^{\text{orb}}(U_S))$.

**Proof.** This is an immediate consequence of Proposition 4.4.

We next analyze the case where the underlying topological space for $B$ is smooth.

**Lemma 4.8.** Suppose that the underlying topological space for the projective orbifold $B$ is smooth. Then $G$ is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold $V_1$ with at most one cone-point. Furthermore, $V_1$ is orientable.

**Proof.** Let

\[ G = \langle x_1, \ldots, x_k \mid w^n \rangle, \quad n > 1. \]

Then, replacing the orbifold $B$ by its underlying topological space $\mathcal{B}$, we note by Proposition 4.4 and Lemma 4.7 that $\pi_1(\mathcal{B})$ is the quotient of $\pi_1(B)$ by
some powers of \( w \). Hence \( G_1 = \pi_1(B) \) is of the form
\[
G_1 = \langle x_1, \ldots, x_k \mid w^r \rangle, \quad r \geq 1.
\]

Since \( B \) is a smooth projective variety, \( G_1 \) is a one-relator Kähler group. Hence, by the classification of one-relator Kähler groups [BM1] (see also [Ko]), \( G_1 \) is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold \( V \) with at most one cone-point \( y_0 \). If \( r = 1 \), then \( G_1 \) is torsion-free. Else the loop that goes around \( y_0 \) represents the conjugacy class of \( w \). Replacing \( r \) by \( n \), we note that \( G \) is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold \( V_1 \), where \( V_1 \) differs from \( V \) only in the order of the cone-point \( y_0 \). Thus, \( G \) is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold \( V \) with at most one cone-point. \( \blacksquare \)

Lemma 4.5 furnishes the local structure of singularities of \( B \). We now describe the blow-up. We first note that a lens space \( cL \) on a lens space \( L \) of dimension \( 2k - 1 \) with fundamental group \( \mathbb{Z}/q\mathbb{Z} \) can be realized as the topological boundary of the total space \( E_q \) of a twisted line bundle (more precisely, the \( q \)th tensor power of the tautological line bundle) over \( \mathbb{C}P^{k-1} \). Hence, the blow-up of \( cL \) may be obtained by removing an open neighborhood of the cone point \( y_0 \in cL \) and attaching a copy of \( E_q \) along the resulting boundary \( L \). Let \( \text{BU}(L) \) denote the blow-up of \( cL \), and let \( \text{ED}(L) \subset \text{BU}(L) \) be the exceptional divisor, which is homeomorphic to \( \mathbb{C}P^{k-1} \). Then we note the following two properties of \( \text{BU}(L) \):

**Remark 4.9.**

(1) If \( cL \) has an orbifold cone-point of order \( q \) then \( \text{BU}(L) \) is also an orbifold so that \( \text{ED}(L) \) is an orbifold locus of ramification order \( q \).

(2) The underlying topological space of \( \text{BU}(L) \) is a smooth manifold.

Next, we describe the local structure of a singularity of \( B \) as obtained in Lemma 4.5 along with the \( U(1) \) bundle over it. Let \( E_S \) denote the circle bundle over \( U_S \) induced by the inclusion of \( U_S \) into \( B \). Thus, \( E_S \subset M \). Then we have the local structure of the Sasakian manifold \( M \) given by the following commutative diagram:

\[
\begin{array}{ccc}
D^{2k} \times U(1) & \longrightarrow & E_S \\
\downarrow & & \downarrow \\
D^{2k} & \longrightarrow & U_S = cL \\
\end{array}
\]

**Theorem 4.10.** Let
\[
G = \langle x_1, \ldots, x_n \mid w^k \rangle, \quad k > 1,
\]
be a one-relator Sasakian group. Then \( G \) is isomorphic to the fundamental
group of a (real) two-dimensional compact orbifold $V$ with at most one cone-point. Further, the underlying manifold of $V$ is orientable.

Proof. As discussed at the beginning of Section 4.2 it remains to deal with the case $n = 2$. If $k = 1$, then $G$ is torsion-free by Proposition 4.4 and the hypothesis of Lemma 4.6 is satisfied, forcing $G$ to be the fundamental group of a smooth projective variety. The main theorem of [BM1] now furnishes the result.

Else, the local structure of singularities is given by Lemma 4.5. Blowing up and using Remark 4.9 we obtain an orbifold $B_1$ satisfying the following:

- The underlying topological space $B_1$ of $B_1$ is a smooth projective variety.
- The orbifold locus of $B_1$ is a divisor.
- Any loop that goes around a component of the divisor represents an element of $\pi_1^{\text{orb}}(B) = G$ that is conjugate to $w^r$ for some $r$ that divides $k$.

Hence, by Lemma 4.8, $G_1 = \pi_1(B_1)$ is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold $V$ with at most one cone-point. The rest of the argument is a replica of the last part of the proof of Lemma 4.8 forcing $G$ to be isomorphic to the fundamental group of a (real) two-dimensional compact orbifold $V_1$, where $V_1$ differs from $V$ only in the order of the cone-point.

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Indranil Biswas, Mahan Mj
School of Mathematics
Tata Institute of Fundamental Research
1 Homi Bhabha Road
Mumbai 400005, India
E-mail: indranil@math.tifr.res.in
mahan@math.tifr.res.in