QCD Sum Rules and $1/N_c$ expansion: On the low energy dominance and separation of scattering backgrounds

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In multiquark correlator analyses with $1/N_c$ classifications, it is possible to separate the scattering background and to justify the factorization of condensates, which allows us to achieve an isolated peak saturation in the QCD sum rules for multiquark currents. Then we can extract leading $N_c$ properties of the ground state. An application to the $\sigma$ meson is demonstrated.

1 QCD sum rules for multiquark correlators: Difficulties

The multiquark states have been intriguing subjects of researches to study the infrared aspects of QCD beyond those manifested in the usual hadron spectra. One of the conventional approaches from QCD is to analyze correlators of multiquark currents, which is related to the hadronic spectral functions through the dispersion relation. Compared to analyses for usual meson and baryon correlators, the multiquark cases are expected to have the following difficulties: (i) The complications associated with the many-body system, which make the calculation of correlators much involved. (ii) The decays into usual hadrons, whose backgrounds contaminate the signal from multiquark states. We will show that the use of $1/N_c$ expansion \[1\ [2\] greatly reduces both difficulties, and, more importantly, enables to make logically clear statements \[3\].

We perform the correlator analyses using the Borel transformed QCD sum rules (QSR) for the hadronic correlators $\int dxe^{iqx} \langle T\hat{J}(x)\hat{J}^\dagger(0)\rangle$ [4]:

$$\sum_n C_n(M^2;\mu^2)(\hat{O}_n(\mu^2)) = \int_0^\infty ds \frac{e^{-s/M^2}}{\pi} \frac{1}{s} \text{Im}\Pi^h(s),$$  \hspace{1cm} \text{(1)}
where LHS is calculated by the operator product expansion (OPE) and RHS is the integral of the hadronic spectrum. Here $M$ characterizes the scale of momentum flow between interpolating fields, and $\mu$ is factorization scale separating high energy calculable part and low energy condensate part.

Usually we are interested in the ground state properties of RHS in eq. (1), and they can be investigated by taking small $M^2$ due to the exponential factor $e^{-s/M^2}$. But we can not take arbitrarily small $M^2$ value since the OPE is basically expansion of $\langle \hat{O}_d \rangle / M^d \sim (\Lambda_{\text{QCD}} / M)^d$ and is violated in small $M^2$ region. Thus we must take intermediate region where the integral is well-saturated by ground state contribution and the OPE is well-converged. These are quantified as: pole contribution $\geq 50\%$ of the total [5, 6] (for upperbound $M^2_{\text{max}}$) and highest dimension terms $\leq 10\%$ of whole OPE (for lowerbound $M^2_{\text{min}}$). Practically, without these $M^2$ constraints, we will be stuck with the pseudo-peak artifacts leading to wrong conclusions, as discussed in [6].

Keeping these constraints in mind, let us consider how the first difficulty (i) is manifested in the QSR analyses. The multiquark correlators involve many quark lines, sharing only small fraction of the momentum with a scale relevant to the multiquark states. As a result, we have to incorporate more non-perturbative diagrams in the OPE calculations than usual hadron analyses, to include relevant low energy correlations. Otherwise, we can not find the $M^2$ window due to the lack of sufficient low energy contributions.

Here we shall add one remark about the low energy dominance. In the case of multiquark correlator analyses, it is frequently said that the pole-dominance is spoiled by large continuum (or high energy part of spectrum) contributions peculiar to multiquark correlators [6]. This is a somewhat misleading expression. A real problem in the early studies is the lack of low energy correlation. In fact, in certain tetraquark correlators, inclusion of higher dimension terms enables to take sufficiently small $M^2$ value, achieving the low energy dominance not much worse than usual meson sum rules, and even better than baryon sum rules [6]. Thus we conclude that a relevant problem is numerical ambiguities in values of higher dimension condensates.

The second difficulty (ii) peculiar to multiquark cases arises as follows: even if we take into account the sufficient low energy correlations, they might be contributions from scattering backgrounds which are ground states in most of multiquark correlators. This prevents us from investigating the multiquark states of our interests.

Both difficulties seem profound. However, as we will show, leading $N_c$ properties of multiquark states are tractable and, in principle, the corrections to them can be treated step by step with identifying origins of contaminations.
2 Factorization of condensates and separation of background based on $1/N_c$

In this section, we discuss how useful $1/N_c$ expansion is in the application of the QSR. The first merit is presented in the evaluation of the higher dimension condensates. It is well-known that in large $N_c$ they can be factorized into the products of known condensates, $\langle \bar{q}q \rangle$, $\langle G^2 \rangle$, and $\langle \bar{q}g_\sigma Gq \rangle$. Thus we can extend the OPE calculations to sufficiently higher dimension terms needed to include relevant low energy contributions. Here we must make one assumption on the $N_c$ scaling of these low dimension condensates. Since $\langle O \rangle = (\langle \bar{q}q \rangle, \langle \alpha_s G^2 \rangle, \langle \bar{q}g_\sigma Gq \rangle)$ are known to be $O(N_c)$, we simply assume $\langle O \rangle|_{N_c} = \langle O \rangle N_c/3$.

Next, we discuss how to separate the scattering background from the multiquark correlators. The point is that $N_c$ counting of the quark-gluon graphs is directly related to the qualitative picture of hadronic states. We will show that with taking appropriate interpolating fields, the multiquark state and background can be assigned in the different order of $N_c$, and their mixing occurs only in the higher order of $1/N_c$ than those studied in this work. Then we can concentrate on the quark-gluon graphs which are saturated by the isolated poles which reflect leading $N_c$ properties of multiquark states.

To make discussions concrete, we consider the $\sigma$ meson with $I = J = 0$ as a candidate of the tetraquark (4q) state. The 4q operators with the $\sigma$ quantum number are given (assuming the ideal mixing for the $\sigma$ meson) by $J_{MM}(x) = \sum_{F=1}^{3} J_F^M(x) J_{F1}^M(x)$ as products of meson operators $J_F^M = \bar{q} \tau_F \Gamma_M q$, where Dirac matrix $\Gamma_M = (1, \gamma_\mu)$ and $\tau_F$ ($F = 1, 2, 3$) are the Pauli matrices acting on $q = (u, d)^T$.

Let us start the $1/N_c$ classifications of quark-gluon graphs in the 2-point correlators, graphs (a)-(c) in Fig.1 where planar-gluon lines are not explicitly drawn. The leading $N_c$ diagrams start from Fig.1a), which include only 2 planar loops and thus reflect irrelevant free 2-meson scattering states. This indicates that the studies of 4q components require systematic steps beyond the leading $N_c$. Thus we must proceed to the next leading order of $1/N_c$, $O(N_c)$ diagrams which could include the multiquark states.

![Figure 1: $O(N_c^2)$ and $O(N_c)$ quark-gluon diagrams for 2 and 3 point correlators.](image)
For systematic classifications, it is useful to first investigate the overlap strength of
the operator \( J_{M M} \) with hadronic states by \( \frac{1}{N_c} \), then to use them in the classification of subleading diagrams of 2-point correlators. For this purpose, we consider the 3-point correlator among \( J_{M M} \) and two separated \( J_{M'} \) (Fig.1 d-f). Simple calculations indicate that the overlap strength of the 4q field with 2 meson states is
\[
\langle 0 | J_{M M} | M' M' \rangle = O(N_c) \delta_{M M'} + O(1) + \ldots \text{ since the leading order diagrams are } O(N_c^2) \text{ for } M = M' (M \neq M') \text{ and the overlap strength of } J'_{M} \text{ with 2q meson state } |M' \rangle \text{ is } O(N_c^{1/2}).
\]

Now we can classify the hadronic states in 2-point correlators \( \langle J_{M M} J_{M' M'} \rangle \) based on \( \frac{1}{N_c} \) (See, Fig.1 a-c): (a) If \( M = M' \), \( O(N_c^2) \) quark-gluon graphs include only the free 2M scattering states in the region \( E \geq 2m_M \). If \( M \neq M' \), the contributions from these quark-gluon diagrams vanish, indicating the absence of free 2 meson scattering states. (b) \( O(N_c) \) graphs include the 2M or 2M' scattering and could include 4q poles.

Note that if \( M, M' \) are different from pseudoscalar or axial vector, \( 2\pi \) scattering intermediate states are not included up to \( O(N_c) \) diagrams. Interactions are needed to transfer initial states into the \( 2\pi \) intermediate states, but such interactions are suppressed by \( \frac{1}{N_c} \). Then the resonance peaks (if exist) below \( 2m_M \) are isolated and have no width since the decay channels are absent at this order. Therefore, now we can reduce the \( \sigma \) spectrum in the 4q correlator into peak(s) plus continuum if we retain only diagrams up to \( O(N_c) \) for the appropriate currents. We will study these cases.

This separate investigation of the \( O(N_c^2) \) and \( O(N_c) \) parts enables to perform the step by step analyses for each hadronic state. We relate the OPE, term by term of \( \frac{1}{N_c} \), to the integral of the hadronic spectral function through the dispersion relation:

\[
\Pi_{N_c}^{\text{ope}}(-Q^2) = \int_0^\infty ds \frac{\pi \text{Im}\Pi_{N_c}^h(s)}{s + Q^2} (n = 2, 1). \tag{2}
\]

The ground state in the reduced \( O(N_c) \) spectra \( \text{Im}\Pi_{N_c}^h(s) \) can be described as the sharp peak due to the absence of the decay channels. Applying the usual quark-hadron duality to the higher excited states, \( \pi \text{Im}\Pi_{N_c}^h(s) = \lambda^2 \delta(s - m_h^2) + \theta(s - s_{th}) \pi \text{Im}\Pi_{N_c}^{\text{ope}}(s) \), and taking the moment of Borel transform of Eq.(2), we can express the effective mass as

\[
m_h^2(M^2; s_{th}) = \frac{\int_{s_{th}}^{s_{th} + M^2} ds e^{-s/M^2} \text{Im}\Pi_{N_c}^{\text{ope}}(s)}{\int_{s_{th}}^{s_{th} + M^2} ds e^{-s/M^2} \text{Im}\Pi_{N_c}^{\text{ope}}(s)}. \tag{3}
\]

\( s_{th} \) can be uniquely fixed to satisfy the least sensitivity [4] of the expression [3] against the variation of \( M \), since the physical peak should not depend on the artificial expansion parameter \( M \). This criterion is justified only when the peak is very narrow, and our
1/$N_c$ reduction of spectra is essential for its application to allow the QSR framework to determine all physical parameters ($m_h, \lambda, s_{th}$) in a self-contained way.

### 3 Borel analyses for the reduced hadron spectra

In this section, we show the results of Borel analyses. We concentrate on the $O(N_c)$ results of the off diagonal correlator $\langle J_{VV} J^\dagger_{SS} \rangle$, whose leading order is $O(N_c)$ thus without the factorization violations at the $O(N_c)$ OPE. The OPE is carried out up to dimension 12. The numerical analyses are performed with the values for $N_c = 3$ case, $\alpha_s(1\text{GeV}) = 0.4$, $\langle \alpha_s G^2 / \pi \rangle = (0.33 \text{ GeV})^4$, $\langle q\bar{q} \rangle = -(0.25 \pm 0.03 \text{ GeV})^3$, and $m_0^2 = \langle q\bar{q} \sigma G q \rangle / \langle q\bar{q} \rangle = (0.8 \pm 0.1) \text{ GeV}^2$, respectively. Most results shown below will use the central value.

First we show in the left panel of Fig.2 the results of the large $N_c$ 2q correlators (expanded up to dimension 6) for the vector meson as a reference and the scalar meson as the 2q state in the $\sigma$ meson. The downarrow (upper arrow) indicates the values of $M_{\text{min}}^2$ ($M_{\text{max}}^2(s_{th})$). Following the $E_{th}$ ($\equiv \sqrt{s_{th}}$) fixing criterion, we fix $E_{th}$ to 1.0 (1.4) GeV for vector (scalar) mesons, and determine the mass as 0.65 (1.10) GeV.

Now we turn to the $O(N_c)$ part of the 4q correlator, $\langle J_{VV} J^\dagger_{SS} \rangle$, to investigate the possibility of 4q states. Shown in the middle panel of Fig.2 are the effective masses for $E_{th}$=1.0, 1.2, and 1.4 GeV. We select the $E_{th}$=1.2 GeV case and evaluate the mass as $\sim 0.90 \text{ GeV}$, which is obviously lower than that of the 2q scalar meson case, $\sim 1.10 \text{ GeV}$ in large $N_c$ limit, and thus is considered as the mass of 4q state. Although these absolute values depend on the details of condensate values, the inequality $m_\rho < m_{4q} < m_S$ is insensitive to them. In the right panel of Fig.2, we show the $\langle q\bar{q} \rangle$ and $m_0^2$ dependence of 2q vector, scalar meson masses, and of the 4q mass deduced from $\langle J_{VV} J^\dagger_{SS} \rangle$.

![Figure 2](image.png)

Figure 2: The large $N_c$ 2q correlator results for the scalar and vector mesons (left), the $O(N_c)$ part of the 4q correlators (middle), and their mass relation for the various condensate values (right).
We have also investigated both $O(N_c^2)$ and $O(N_c)$ parts of $\langle J_{SS} J_{SS}^\dagger \rangle$ ($\langle J_{VV} J_{VV}^\dagger \rangle$). As for $O(N_c^2)$ part, low energy contribution is almost absent below two meson thresholds. More precisely speaking, acceptable effective mass plots can be obtained only if we take much larger $E_{th}$ value than the two meson thresholds, giving large effective masses. This indicates that $O(N_c^2)$ part includes only free scattering states consistently with $1/N_c$ arguments. As for $O(N_c)$ part, $\langle J_{SS} J_{SS}^\dagger \rangle$ ($\langle J_{VV} J_{VV}^\dagger \rangle$) give similar values $\sim 0.9$ (0.8) GeV, as $\langle J_{VV} J_{SS}^\dagger \rangle$ case, despite of the possible factorization violation in the formers.

We have also investigated 4q correlators with $I=2$. They do not allow us to find any stable effective mass plots or any reasonably wide $M^2$ window. This would indicate the absence of 4q pole in $I=2$ channel, consistenly with experimental observations.

In conclusion, it is found that the $1/N_c$ expansion is very useful to analyze the leading $N_c$ properties of multiquark states with separating scattering backgrounds. The QSR results show consistencies with expectations from $1/N_c$ arguments, and further provide the prediction on 4q components in the $\sigma$ meson without $\pi\pi$ correlations. The leading $N_c$ analyses for multiquark correlators may be also useful to distinguish two meson molecules and tetraquarks, which is discussed for the recently found charmed mesons (X,Y,Z). Since initial states are controled by interpolating fields, and $1/N_c$ countings give observations on intermediate states, the combination of them enables to separate the molecular type contributions in the spectral functions. Further discussions will be made elsewhere.

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