Measurement compatibility in Bell nonlocality tests

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(Dated: June 26, 2018)

Noncompatibility of observables, or measurements, is one of the key features of quantum mechanics, related, among others, to Heisenberg’s uncertainty relations and Bell nonlocality. In this manuscript we show, however, that even though noncompatible measurements are necessary for the violation of any Bell inequality, novel, relevant Bell-like inequalities may be obtained if compatibility relations are assumed between the local measurements of one (or more) of the parties. Hence, compatibility of measurements is not necessarily a drawback and may, on the other hand, be useful for the detection of Bell nonlocality.

Introduction – Quantum theory is fundamentally distinct from any classical theory of physics, a fact that is well known and accepted nowadays. Even though there is no consensus regarding a physical principle that explains such departure, the distinction between quantum and classical mechanics is clear at the level of their mathematical formalisms. For instance, two of the most interesting nonclassical features present in quantum theory are entanglement and incompatibility between measurements.

The fact that there are measurements, or observables, that are incompatible, whose outcomes cannot be jointly predicted for any quantum state, is one of the key ingredients behind some of the most astonishing phenomena related to nonclassicality, such as Bell nonlocality [1, 2] and Bell-Kochen-Specker contextuality [3, 4]. Both arose from investigations regarding the completeness of quantum theory [5] and are related to stronger-than-classical correlations between outcomes of measurements performed on quantum systems. The corresponding theories, though, have been developed independently, in a sense, and each presents its own particular features. In this manuscript we focus our attention in the former, although the reader may notice that there will be elements of the later.

The paradigmatic example of Bell nonlocality of quantum systems takes place in a bipartite measurement scenario, where two characters, Alice and Bob, are able to perform two possible dichotomic measurements on their respective share – each in its own laboratory – of a previously prepared joint system. Denoting by $A_x \in \{ \pm 1 \}$ the outcome of measurement $x \in \{0, 1\}$ of Alice, and by $B_y \in \{ \pm 1 \}$ the outcome of measurement $y \in \{0, 1\}$ of Bob, any so-called local hidden variable (LHV) theory of joint probabilities that govern the behaviour of the measurement devices will lead to mean values that necessarily obey the Clauser-Horne-Shimony-Holt (CHSH) [6] inequality

$$
(A_0(B_0 + B_1) + A_1(B_0 - B_1)) \leq 2. 
$$

If the measurements are performed on quantum systems, however, the inequality does not need to be respected, and a violation of up to the value of $2\sqrt{2}$ [7] may be observed. This shows that quantum theory is incompatible with LHV theories, an observation first made by Bell [1].

Two necessary conditions for Bell nonlocality to manifest in quantum systems are (i) entanglement, in the state of the shared system; and (ii) incompatibility between the measurements, in each party. Curiously, neither (i) [8] nor (ii) [9] is a sufficient condition. Regarding (i), it became an important question in the field (for both fundamental and practical reasons) to identify which entangled states could ultimately lead to Bell nonlocality. On one hand, there has been an effort to obtain examples of local entangled states, those that never lead to nonlocal correlations [10–14]. On the other hand, several nonstandard measurement scenarios have been proposed where even local entangled states can lead to Bell nonlocality, and concepts like hidden-nonlocality [15–19] and activation of nonlocality [20–23] have been defined.

In this manuscript we propose a novel measurement scenario where we hack condition (ii) and explicitly assume the existence of subsets of compatible measurements of (at least) one of the parties. We show how this assumption drastically changes the measurement scenario and leads to a new range of Bell-like inequalities.
that are potential new tools to improve on condition (i), leading to new examples of entangled states that have not been known to be nonlocal. We present an example, in one of the simplest scenarios of this approach, where, without the compatibility assumption, the only relevant Bell inequality is CHSH \cite{chsh}, and, with the compatibility assumption, 26 new Bell-like inequalities arise. We explicitly show that one of these inequalities reveals Bell nonlocality in a family of quantum states for a region of parameters where the CHSH inequality is not violated. Also, there is numerical evidence that, in part of this region, the states do not violate the I\textsubscript{3322} inequality \cite{3322} either.

**Preliminaries** – Consider a scenario where two parties, Alice and Bob, are able to perform different measurements on their subsystems of a shared physical system. The measurements are implemented by black-boxes, of which only classical inputs and outputs are available to the users. The inputs and outputs of Alice’s box are labeled \( x \in \mathcal{X} \) and \( a \in \mathcal{A} \), respectively, and the inputs and outputs of Bob’s box are labeled \( y \in \mathcal{Y} \) and \( b \in \mathcal{B} \), respectively. The best description of the boxes is given by the joint probabilities \( p(a, b|x, y) \) of the parties to observe outputs \( a \) and \( b \), on the condition that inputs \( x \) and \( y \) are chosen, respectively. The collection of probabilities \( p(a, b|x, y) \) for all \( a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{X}, y \in \mathcal{Y} \) is referred as the behaviour or the empirical model of the box, and will be denoted \( \rho \).

A behaviour is said to be no-signalling if the choice of input of one of the parties cannot influence the marginal probability distribution of outcomes of the other, i.e., it satisfies the following no-signalling conditions:

\[
p(a|x) = \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y'), \tag{2a}
\]

\[
p(b|y) = \sum_a p(a, b|x, y) = \sum_a p(a, b|x', y). \tag{2b}
\]

More restrictedly, a behaviour is said to be local if there exist a variable \( \lambda \), a probability distribution \( q(\lambda) \), and probability distributions \( p(a|x, \lambda) \) and \( p(b|y, \lambda) \) such that

\[
p(a, b|x, y) = \sum_\lambda q(\lambda)p(a|x, \lambda)p(b|y, \lambda). \tag{3}
\]

It can be shown that the probability distributions \( p(a|x, \lambda) \) and \( p(b|y, \lambda) \) can be made deterministic without loss of generality.

If the boxes perform measurements on quantum systems, the probabilities are given by Born’s rule:

\[
p(a, b|x, y) = \text{tr} \left( \rho P_{a|x} \otimes Q_{b|y} \right), \tag{4}
\]

where \( \rho \) is the density operator that describes the state of the joint system and \( P_{a|x} \) and \( Q_{b|y} \) are, in general, POVM effects. It is well known that, in general Bell scenarios, the set of local behaviours is strictly contained in the set of quantum behaviours, which, in its turn, is strictly contained in the set of no-signalling behaviours.

Now, consider the single black box of Bob, with inputs \( y \in \mathcal{Y} \), and outputs \( b \in \mathcal{B} \). Suppose, however, that some measurements are compatible; each set of compatible measurements defines a context, \( y \in \mathcal{Y} \), and let \( \mathcal{C} = \{y\} \) denote the set of possible contexts of the scenario. Compatible measurements can be jointly performed, and a consistent joint probability distribution of the outcomes can be defined. Let \( b \) denote the ordered outcomes of the measurements in a context \( y \). Then, the behaviour of this single box is best described by the probabilities \( p(b|y) \). Noting that an individual measurement can appear in more than one context, it is usual to assume that the marginal behaviour of each individual measurement \( y \in \mathcal{Y} \) to be well defined, regardless of the context. This leads to the so-called no-disturbance conditions:

\[
p(b|y) = \sum_{b, y' \neq y} p(b|y) = \sum_{b, y' \neq y} p(b'|y'), \tag{5}
\]

for all \( b \) in \( \mathcal{B} \), for all \( y \in \mathcal{Y} \), and for all \( y, y' \in \mathcal{C} \) such that \( y \in \mathcal{Y} \cap y' \).

Suppose, now, a bipartite scenario, as considered previously, but let Bob be able to perform joint measurements according to given compatibility rules that lead to a set of contexts \( \mathcal{C} \). Then, the joint behaviour of the boxes will be given by probabilities \( p(a, b|x, y) \), for all \( a \in \mathcal{A}, b \in \mathcal{B}^{\otimes y}, x \in \mathcal{X}, y \in \mathcal{Y} \). We assume the behaviour to be no-signalling, and the marginal, local behaviour of Bob’s box to obey the no-disturbance conditions.

We define a behaviour to be local in this scenario if there are a variable \( \lambda \), a probability distribution \( q(\lambda) \), and probability distributions \( p(a|x, \lambda) \) and \( p(b|y, \lambda) \) such that

\[
p(a, b|x, y) = \sum_\lambda q(\lambda)p(a|x, \lambda)p(b|y, \lambda). \tag{6}
\]

Here, \( p(a|x, \lambda) \) can be assumed to be deterministic probability distributions. This is due to the fact that the set of marginal behaviours \( P_A = \{p(a|x)\} \) is a convex set with finitely many extremal points, i.e., a polytope, whose vertices are exactly the deterministic distributions that suffice in the definition; let \( P_A \) denote this polytope. The set of marginal behaviours \( P_B = \{p(b|y)\} \) is also a polytope, but the vertices are not necessarily deterministic: they are the vertices of the no-disturbance polytope of Bob’s local measurement scenario. Let \( P_{AB} \) denote this polytope. It is easy to see, then, that, in this definition, the joint behaviour \( p = \{p(a, b|x, y)\} \) will be a convex combination of a finite set of points, so the set of \( p \) will also be a polytope, \( P_{AB} \); its vertices are all possible ‘products’ between the vertices of \( P_A \) and \( P_B \).

Now, given that the set of local behaviours is a polytope whose vertices are known (provided the vertices of \( P_B \) are known), we can change its representation by
means of specialised software, such as ports [27] or panda [1], and obtain the inequalities associated to its facets. Such inequalities will be Bell-like inequalities whose violation certify Bell nonlocality, in the sense that these correlations can not be explained locally, by Eq. (6).

**Application** — Consider a bipartite scenario where Alice is able to perform 2 dichotomic measurements, so $\mathcal{A} = \mathcal{X} = \{0, 1\}$, and Bob is able to perform 4 dichotomic measurements, $\mathcal{B} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1, 2, 3\}$, where, also, the measurements of Bob are assumed to be compatible according to the contexts $\mathcal{C} = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}\}$. In this scenario, a behaviour certify Bell nonlocality, in the sense that these correlations can be retrieved as

$$\langle A_x B_y \rangle = p(a + b_1 + b_2 = 0|x, y) - p(a + b_1 + b_2 = 1|x, y), \quad (7)$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{C}$, where we assume the sums are modulo 2 and $(b_1, b_2)$ being the respective outcomes of the $\mathcal{B}$ measurements $(y_1, y_2) = y$. ‘Marginal’ correlators $\langle A_x \rangle$, $\langle B_y \rangle$, $\langle A_x B_y \rangle$, for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $y \in \mathcal{C}$, are analogously defined with the corresponding marginal probability distributions. The behaviours will, then, be vectors $\vec{c} \in \mathbb{R}^d$, where each component is a correlator. It is easy to check that there are exactly 26 correlators in total, so $d = 26$; and, given the correlators, all the 64 probabilities can be retrieved as

$$p(a, b|x, y) = \frac{1}{8} [1 + (1)^a \langle A_x \rangle + (1)^b \langle B_y \rangle + (1)^a (1)^b \langle A_x B_y \rangle + (1)^a (1)^b \langle A_x \rangle \langle B_y \rangle] \quad (8)$$

Now, we want to characterise the facets of the local, no-disturbance polytope of the scenario, $\mathcal{P}_{AB}$. The first step is to obtain all the extremal points of Bob’s no-disturbance polytope. In all scenarios where the compatibility relations among dichotomic measurements are either of the form $\langle B_y \rangle = \pm 1$ and $\langle A_x B_y \rangle = \pm (\langle A_x \rangle \langle B_y \rangle)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{C}$, or of the form $\langle B_y \rangle = 0$, for all $y \in \mathcal{Y}$, or of the form $\langle B_{(y_1, y_2)} \rangle = \cdots = - \langle B_{(y_1, y_2)} \rangle = 1$, with an odd number of $-1$ components. Then, the extremal points of the local, no-disturbing polytope will be behaviours whose bipartite correlators are of the form

$$\langle A_x B_y \rangle = \langle A_x \rangle \langle B_y \rangle, \quad (11a)$$
$$\langle A_x B_y \rangle = \langle A_x \rangle \langle B_y \rangle, \quad (11b)$$

where $\langle A_x \rangle \in \{\pm 1\}$ and the behaviour of Bob’s box is given by either Eqs. (16) or Eqs. (17), for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $y \in \mathcal{C}$.

Having all the extremal points, we used panda to obtain the facets of the local, no-disturbance polytope. We found 26 classes of inequalities, all of which are given in the Supplementary Material. This result should be contrasted to the fact that, in standard bipartite Bell scenarios where no assumption regarding compatibility is made, if the number of measurements of one of the parties is 2 and they are dichotomic, the only Bell inequality, up to relabellings, is the CHSH inequality, as has been proven by Pironio in Ref. [24].

Actually, the compatibility relations we assume makes our scenario similar to a tripartite Bell scenario, if we assign measurements $B_0$ and $B_2$ to one party and $B_1$ and $B_3$ to another. For this reason, some of the inequalities we obtain are equivalent to Sliwa’s inequalities [7] (see discussion in the Supplementary Material). Note, however, that, had we assumed another compatibility structure for Bob’s measurements, e.g., if $G$ was a pentagon instead of a square, then it would not be possible to relate the scenario to any usual Bell scenario.

Among the 26 inequalities we obtain, one has the form

$$2 \langle B_0 \rangle + \langle (1 - B_0)(A_0 B_1 + B_3) + A_1 (B_1 - B_3) \rangle \leq 2. \quad (12)$$

Note that the term in square brackets corresponds to the left-hand side of a CHSH inequality between Alice and measurements 1 and 3 of Bob. In order to study the quantum violation of inequality (12), it is convenient to define observables

$$A_x = P_{0|x} - P_{1|x}, \quad (13a)$$
$$B_y = Q_{0|y} - Q_{1|y}. \quad (13b)$$

where $P_{a|x}$ and $Q_{b|y}$ are projectors associated to outcomes $a$ and $b$ of measurements $x$ and $y$, respectively, so the correlators will be evaluated as $\langle A_x B_y \rangle = \text{tr}(\rho A_x \otimes B_y)$, where $B_y = B_{y_1} B_{y_2}$, and $[B_{y_1}, B_{y_2}] = 0$ for all $y \in \mathcal{C}$.

Inequality (12) is equivalent to the class #4 of Sliwa [7]. For quantum systems, it is maximally violated up to the value $4\sqrt{2} - 2$, attained by a two-qubit maximally entangled state embedded in $\mathbb{C}^2 \otimes \mathbb{C}^4$ [5]. We now show that this inequality can certify the nonlocality of bipartite quantum states that do not violate the CHSH inequality.

Consider the following two-parameter family of two-qubit states

$$\rho(a, w) = w |\psi(a)\rangle\langle\psi(a)| + (1 - w) |00\rangle\langle00|, \quad (14a)$$
where
\[ |\psi(\alpha)\rangle = \sqrt{\alpha} |01\rangle + \sqrt{1-\alpha} |10\rangle. \] (14b)

This family is known to include the two-qubit states with highest entanglement (as quantified by negativity and concurrence) that do not violate the CHSH inequality [34]. We, then, perform a see-saw optimisation, embedding the states in \( \mathbb{C}^2 \otimes \mathbb{C}^4 \) in order to impose the compatibility relations among the measurements (details in the Supplementary Material), and search for the lowest value of \( w \) such that the inequality is violated, for each \( \alpha \). The results are displayed in Fig. 1, where we also plot the critical values of \( w \) as a function of \( \alpha \) for the CHSH inequality, provided by means of the Horodecki criterium [35], and upper bounds on the critical values of \( w \), obtained by means of a see-saw optimisation, for the \( I_{3322} \) inequality [25] – a relevant Bell inequality in the scenario where Alice and Bob perform three dichotomic measurements each –, given by the expression
\[
-\langle A_1 \rangle - \langle A_2 \rangle - \langle B_1 \rangle - \langle B_2 \rangle - \langle A_1 B_1 \rangle - \langle A_2 B_1 \rangle - \langle A_1 B_2 \rangle - \langle A_2 B_2 \rangle + \langle A_3 B_2 \rangle - \langle A_1 B_3 \rangle + \langle A_2 B_3 \rangle \leq 4.
\] (15)

In fact, the state \( \rho (0.80, 0.85) \) in family (23a) was the example considered in Ref. [25] of a state that does not violate the CHSH inequality, that, however, violates \( I_{3322} \). In Ref. [36], the authors show that, for \( \alpha = 0.80 \), inequality \( I_{3322} \) is violated for \( w \geq 0.837 \), in excellent agreement with the value 0.838 we obtain, corroborating with the precision of our lower bounds.

Discussion –

Although the simple example we consider is sufficient evidence of the potential of the approach we introduce, it is only the first step in a novel direction for the study of Bell nonlocality. In principle, one could assume a plethora of more intricate local compatibility structures in scenarios with any number of parties, leading to a whole new range of new Bell-like inequalities.

The compatibility structure we consider can be realised – with some loss of generality – in a tripartite scenario, where each party is able to perform two dichotomic measurements. An advantage of the tripartite implementation is that one does not need to assume the compatibility relations; they would naturally hold due to space-like separation of the parties, implying that the test would be \textit{device-independent}. Also, note that the locality assumption in tripartite scenarios is more restrictive than the condition we demand in our scenario, since each \( p (b|y, \lambda) \) is required to obey the non-disturbance condition (5) in the later, as opposed to strict locality, in the former. This shows that our local, locally non-disturbing polytope is strictly larger than the corresponding tripartite local polytope; more explicitly, notice that any vertex whose marginal behaviour \( p (b|y) \) obeys Eqs. (17) is not tripartite-local.

One interesting fact, however, that is discussed in more detail in the Supplementary Material, is that some of the inequalities we obtain are isomorphic to the tripartite inequalities obtained by Sliwa [7], including Inequality (12). This equivalence, together with the results presented in this manuscript, prove that there are multipartite Bell inequalities that are useful to witness the Bell nonlocality of bipartite quantum states in a subtler way than just merging parts.

Two other scenarios that demand comparison are the ones obtained when we consider joint measurements of \( B_y \) and \( B_{y+1} \) (addition modulo 4) as new measurements \( B_y' \). In the first case, if we consider \( B_y = B_y + B_{y+1} \) (addition modulo 2), Bob will have four mutually incompatible dichotomic measurements, and Reference [24] shows that the only relevant inequalities for describing the local polytope are those obtained with the local polytope in CHSH family. Hence, our inequalities can show nonlocal behaviour not revealed when such coarse grained version is considered. The specific Inequality (12), for example, could never be written in such scenario, since correlators like \( \langle B_y \rangle \) or \( \langle A_z B_y \rangle \) cannot be written as functions of the probabilities of the outcomes of \( B_y' \). In the second case, if we consider \( B_y'' = (B_y, B_{y+1}) \), then Bob will have four mutually incompatible four-outcome measurements. Once more, Ref. [24] implies that only CHSH inequalities are relevant for such a scenario, while the extra correlations coming from each \( B_y \) being an element of \( B_y' \) and \( B_y'' \) also would make it a somehow special realisation of this.
Bell scenario (with such additional constraints).

Also, it is worth mentioning that somewhat similar scenarios have been previously considered, with different focuses and assumptions. In Ref. [18], the authors present a formalism to study nonlocality in sequential measurement scenarios, mainly focused on the proper consequences due to the causal structures underlying the sequences of measurements. In Ref. [37], the authors argue, considering a bipartite scenario where one of the parties is able to perform sequential measurements, that local contextuality may lead to Bell nonlocality, although the definition of locality adopted is somehow intricate and seems to implicitly assume local noncontextuality.

Conclusion — In this manuscript we present a novel approach to Bell nonlocality, an approach that takes into account the possibility that one (or more) of the parties is able to perform joint measurements according to given compatibility rules. We provide a precise definition of locality, or, more specifically, of local behaviours in these scenarios, and show, by means of a simple but elegant example, that this definition may lead to new, interesting Bell-like inequalities that may provide advantages over known Bell inequalities in witnessing the nonlocality of quantum states. We discuss in some detail a simple example where such advantage appears; in particular, the family of Horodecki states (23a) shows nonlocal behaviour for parameters where neither CHSH, nor $I_{3322}$ could witness it.

Acknowledgements — R. R. would like to thank Jean-Daniel Bancal and Marcos César de Oliveira. This work used computing resources and assistance of the John David Rogers Computing Center (CCJDR) in the Institute of Physics “Gleb Wataghin”, University of Campinas. The authors acknowledge support from the Brazilian agencies CNPq and FAPEPEX. This work is part of the Brazilian National Institute for Science and Technology on Quantum Information.

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Measurement compatibility in Bell nonlocality tests
Supplementary Material

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INEQUALITIES

In the scenario we have considered, Alice is able to perform two dichotomic measurements, and Bob is able to perform four dichotomic measurements; Bob’s measurements, however, can be jointly performed according to compatibility rules provided by the contexts \( C = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}\} \).

Using the representation of correlators, we have listed all extremal points of the polytope of behaviours that are local, according to the definition we provided in the main text, such that Bob’s marginal behaviours respect the no-disturbance conditions. Bob’s extremal marginal behaviours belong to either one of two distinct classes:

(i) noncontextual behaviours: are behaviours of the form

\[
\langle B_y \rangle = \pm 1 \tag{16a}
\]

\[
\langle B_y \rangle = \prod_{y \in Y} \langle B_{y'} \rangle \tag{16b}
\]

for all \( y \in Y \) and \( y \in C \);

(ii) contextual, no-disturbing behaviours: are behaviours of the form

\[
\langle B_y \rangle = 0, \forall y \in Y; \tag{17a}
\]

\[
\langle B_{(y_1, y_2)} \rangle = \cdots = -\langle B_{(y_1, y_n)} \rangle = 1, \tag{17b}
\]

with an odd number of negative signs in Eq. (17b).

Then, the extremal points of the local, locally no-disturbing polytope will be behaviours whose bipartite correlators are of the form

\[
\langle A_x B_y \rangle = \langle A_x \rangle \langle B_y \rangle \tag{18a}
\]

\[
\langle A_x B_y \rangle = \langle A_x \rangle \langle B_y \rangle \tag{18b}
\]

where \( \langle A_x \rangle \in \{\pm 1\} \) and the behaviour of Bob’s box is given by either Eqs. (16) or Eqs. (17), for all \( x \in X, y \in Y \) and \( y \in C \).

Having listed all the extremal points of the the local, no-disturbance polytope, we used the software \texttt{panda} [1] to change the representation of the polytope, and obtained 26 inequalities, up to relabelings that respect the local compatibility rules. These inequalities are listed in Table I. This table should be read as follows: each row represents an inequality, labeled by the number in the first column. Each column, then, has the coefficient of the correlator represented in the heading, where the measurements \( x, y_1 \) and \( y_2 \) are the corresponding numbers in the second row. The second to last column refers to the local bound \( \beta_L \) of each inequality. In the last column, we list the quantum maxima \( \beta_Q \) (exact up to the given precision) of each inequality. The maxima were upper bounded by means of the Navascués-Pironio-Acín [2] hierarchy of semi-definite programs that outer-approximate the set of quantum
correlations, implemented in python with the aid of the NCPOL2SDPA [3] library. The values listed correspond to the third level of the hierarchy, and the optimisations were performed with the MOSEK [4] solver. We have also computed lower bounds on the maxima by means of a see-saw optimisation, detailed later in this Supplementary Material. The lower and upper bounds on quantum maxima obtained differ by less than $5 \times 10^{-4}$ for all inequalities; on average, they differ by $3 \times 10^{-5}$. For the 15 inequalities that are equivalent to Sliwa’s inequalities, our results are in perfect agreement with those of 5, 6, where, among other results, quantum maxima are computed and analysed for all Sliwa’s inequalities.

As an example, consider inequality 25. It is

$$\langle A_0B_0 \rangle + \langle A_0B_2 \rangle + \langle A_1B_0 \rangle + \langle A_1B_2 \rangle + 2 \langle A_0B_0B_1 \rangle +$$

$$\langle A_0B_2B_3 \rangle - \langle A_0B_3B_0 \rangle - 2 \langle A_1B_0B_1 \rangle + \langle A_1B_2B_3 \rangle - \langle A_1B_3B_0 \rangle \leq_L 4 \leq_Q 5.6568 \quad (19)$$

**RELATION TO SLIWA’S INEQUALITIES**

Note that the particular scenario we consider share some similarity with a tripartite Bell scenario, where each party is able to perform two dichotomic measurements; the correspondence becomes explicit if one considers $B_0$ and $B_2$ as the possible measurements of a second party, while $B_1$ and $B_3$ are the possible choices of a third party. This Bell scenario has been studied by Sliwa [7], who obtained 46 distinct classes of Bell inequalities by assuming full locality between the three parties. Should have we considered only noncontextual marginal behaviours of Bob, (16), the inequalities obtained would be all equivalent to Sliwa’s 46 inequalities. However, by including points of the form (17) (i) we obtain inequalities that are not equivalent to Sliwa’s; and (ii) any violation of the inequalities is a certification of bipartite nonlocality, a fact that would not be true otherwise, since stronger-than-classical (contextual) correlations in the marginal behaviour of Bob could lead to violations of the inequalities obtained solely via (16).

Sliwa’s inequalities are, as discussed, related to a polytope that is contained in the local, no-disturbance polytope we characterise. It would not be surprising, then, if some of the facets of the polytope were equivalent to the inequalities of Sliwa, and this is exactly what we observe. For the 15 inequalities we obtained that are equivalent to an inequality of Sliwa, we provide in the second column of table I the number referring to the enumeration in Ref. [7].

**SEE-SA W OPTIMISATION**

To compute lower bounds on the maximum quantum violation of the 26 inequalities we study, as well as the bounds on the critical parameters of the family of quantum states we present in the main text, we implemented variations of an optimisation algorithm known as see-saw iteration, introduced by Werner and Wolf in Ref. [8]. Our implementation follows the steps described in section II.B.3 of Ref. [9], with minor adjustments.

For standard Bell inequalities where the parties perform dichotomic measurements, the algorithm is based on the idea that, if the quantum state and the measurements of all but one of the parties are fixed, then optimisation over the measurements of the remaining party can be carried out explicitly. Consider, for clarity, a bipartite scenario; extensions to multipartite ones are straightforward. Let the operator associated to a given Bell inequality be

$$\beta = \sum_{x=1}^{m_A} \sum_{y=1}^{m_B} \sum_{a=-1}^{1} \sum_{b=-1}^{1} c_{x,y}^{a,b} Q_{a|x} \otimes R_{b|y}, \quad (20)$$

where $x$ ($y$) labels the choice among the $m_A$ ($m_B$) possible measurements of party $A$ ($B$), a ($b$) labels the possible outcomes, $Q_{a|x}$ ($R_{b|y}$) is the measurement operator associated to outcome $a$ ($b$) of measurement $x$ ($y$), and $c_{x,y}^{a,b}$ are the respective coefficients that define the inequality. Then, the quantum average value of the inequality can be written, as a function of the state $\rho$ and the measurement operators, as

$$S_Q(\rho, \{Q_{a|x}\}, \{R_{b|y}\}) = \sum_{b,y} \text{tr} (\rho R_{b|y} R_{b|y}), \quad (21a)$$

where

$$\rho_{R_{b|y}} = \sum_{a,x} c_{x,y}^{a,b} \text{tr}_A (\rho (Q_{a|x} \otimes 1)), \quad (21b)$$
| #  | # S | $\langle A_x \rangle$ | $\langle B_y \rangle$ | $\langle A_x B_y \rangle$ | $\langle B_{y1} B_{y2} \rangle$ | $\langle A_x B_{y1} B_{y2} \rangle$ | $\beta_L$ | $\beta_Q$ |
|----|----|-----------------|-----------------|-----------------|-----------------|-----------------|--------|--------|
| 1  | 1  | 1 0 1 1 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | -1 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 1 | 1.000 |
| 2  | -2 | 0 0 1 1 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 3  | -2 | 0 0 1 0 1 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 4  | -2 | 0 0 0 0 0 0 0 0 | 2 1 0 0 1 0 0 0 | -1 -1 1 1 1 1 1 1 | -1 -1 1 1 1 1 1 1 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 5  | -2 | 0 0 0 0 0 0 0 0 | 1 1 1 1 1 1 1 1 | -1 -1 -1 -1 -1 -1 -1 -1 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 6  | -2 | 0 0 0 0 0 0 0 0 | 0 0 0 0 2 2 0 0 | -1 -1 -1 -1 -1 -1 -1 -1 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 7  | -1 | 1 1 0 1 0 0 1 0 | 1 2 0 1 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 8  | -1 | 1 1 0 1 0 0 1 0 | 1 2 0 1 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 9  | -1 | 1 1 0 0 0 0 2 2 | 1 1 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 10 | -1 | 1 0 0 0 0 2 1 1 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 11 | 17 | 1 0 0 0 0 1 1 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 12 | 6  | 0 1 1 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 3 | 4.656 |
| 13 | -1 | 0 1 1 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 5 | 7.012 |
| 14 | -1 | 0 1 1 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 5 | 6.656 |
| 15 | 4  | 0 0 2 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 5 | 6.656 |
| 16 | 19 | 0 0 0 1 0 0 1 2 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.782 |
| 17 | 18 | 0 0 0 1 0 0 1 2 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.753 |
| 18 | 15 | 0 0 0 0 0 0 2 2 | 2 2 2 2 2 2 2 2 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 6.000 |
| 19 | 14 | 0 0 0 0 0 0 2 1 | 1 0 1 0 1 0 1 2 | 1 1 1 1 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 20 | 12 | 0 0 0 0 0 0 2 1 | 1 0 1 0 1 0 1 2 | 1 1 1 1 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 21 | 14 | 0 0 0 0 0 0 2 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 22 | 13 | 0 0 0 0 0 0 2 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 23 | 11 | 0 0 0 0 0 0 2 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 24 | 10 | 0 0 0 0 0 0 1 1 | 1 1 1 1 1 1 1 1 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 4.000 |
| 25 | 9  | 0 0 0 0 0 0 1 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 4 | 5.656 |
| 26 | 3  | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | -1 -1 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 2 | 2.828 |

**TABLE I.** All the 26 classes of inequalities that are facets of the local, no-disturbing polytope of the measurement scenario we introduce, displayed as coefficients of correlators. Some of the inequalities are equivalent to Sliwa’s inequalities [7]; the corresponding class in [7] is displayed in the second column, #S. The second-to-last column displays the local bounds of the inequalities, and the last column displays their respective maximal quantum violations, exact, up to the given precision.
where tr\(_A(\cdot)\) denotes the partial trace over subsystem \(A\). For fixed \(\rho\) and \(Q_{a|x}\), \(S_Q\) is a linear function of \(R_{b|y}\). And, since \(R_{1,y} = 1 - R_{-1|y}\), we have

\[
\sum_{b=\pm 1} \text{tr} \left( \rho_{R_b|y} R_{b|y} \right) = \text{tr} \left( (\rho R_{+1|y} - \rho R_{-1|y}) R_{+1|y} \right) + \text{tr} \left( \rho R_{-1|y} \right). \tag{22}
\]

This expression can be optimised by setting \(R_{+1|y}\) equal to the projector onto the positive subspace of \(\rho R_{+1|y} - \rho R_{-1|y}\). This procedure can, then, be iterated, so optimisation can be carried over all measurements of all parties. If desired, the quantum state can be optimised over, in an even simpler fashion: in any step, the optimal quantum state can be taken as a pure state given by an eigenvector of \(\beta\) associated to its maximal eigenvalue.

Note that the first step of the see-saw algorithm already requires a choice of state and measurements, so they should be randomly generated in the beginning of the process. Although it is clear that the algorithm will converge after a sufficient number of steps, one cannot guarantee that it will converge to the global maximum of the problem. Any solution, however, is a lower bound to the optimal solution, so it is recommended to restart the algorithm with as many random ‘seeds’ as feasible.

The scenario we consider, as discussed in the previous section, is similar to a tripartite scenario where each of the three parties is able to perform two dichotomic measurements. Our implementation makes use of this similarity, assuming measurements \(B_1\) and \(B_3\) are implemented by a third party. On one hand, this assumption guarantees the compatibility relations assumed in the scenario; on the other hand, it leads to loss of generality. This is one more reason (despite the fact that the see-saw does not necessarily converge to the global maximum) that advocates against optimality of the bounds computed via this method.

Our goal was to compute upper bounds on the critical values of \(w\), as a function of \(\alpha\), such that the states

\[
\rho = w |\psi(\alpha)\rangle\langle\psi(\alpha)| + (1 - w) |00\rangle\langle00|, \tag{23a}
\]

where

\[
|\psi(\alpha)\rangle = \sqrt{\alpha} |01\rangle + \sqrt{1 - \alpha} |10\rangle, \tag{23b}
\]

violate inequality \#15 (we have numerical evidence that this is the best inequality among the ones we listed to witness the nonlocality of such states). The code was implemented in MATLAB, with the aid of the QETLAB [10] library. We suppose a system with local Hilbert spaces \(\mathcal{H}_A = \mathbb{C}^2\) and \(\mathcal{H}_B = \mathbb{C}^4\), where the states (23a) are embedded trivially, with \(B_i = B_i \otimes \mathbb{I}_2\), for \(i \in \{0,2\}\), and \(B_j = \mathbb{I}_2 \otimes B_j\), for \(j \in \{1,3\}\), where \(B_i\) acts in \(\mathbb{C}^2\) and \(\mathbb{I}_2\) is the identity operator in the same space. Then, for each of 100 values of \(\alpha\) equally spaced in the interval \([1/2, 1]\), we start with \(w = 3/4\) and run the see-saw with at most 500 random ‘seeds’ – projective measurements for all parties, and a local unitary \(U\) acting on \(\mathbb{C}^4\) that we apply to state, so it is not always fixed in the same basis as the virtual parties \(B\) and \(C\) are divided. The process is iterated 8 times for different values of \(w\), that is updated according to a bissection scheme: if a violation of the inequality is obtained in iteration \(i\), the value of \(w\) is updated to \(w - 2^{-(i+2)}\); if, after all seeds, no violation is obtained, the value of \(w\) is updated to \(w + 2^{-(i+2)}\).

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