Local interior estimates for the Euler-Einstein system

Taxiarchis Papakostas\textsuperscript{1} and Demetrios Pliakis\textsuperscript{2}

\textsuperscript{1} Department of Natural Resources and Environment, Technological Educational Institute of Crete, Chania GR 73103
\textsuperscript{2} Department of Electronics, Technological Educational Institute of Crete, Chania GR 73103

Abstract.

We consider the spacetime of a perfect fluid described by the Einstein-Euler equations. We assume given spatial growth for the fluid and spacetime initial data and estimate the temporal evolution of the spatial growth rates through the wave equations derived for the fluid and spacetime data.

1. Introduction

We study an asymptotically flat spacetime filled with perfect fluid inside an initial geodesic ball. We assume a spatial local growth condition for the fluid and the initial slice of spacetime. We estimate the dynamic variation of these growth rates under the assumption of the existence of a local maximal foliation. Here we sketch briefly their derivation. These estimates facilitate the numerical solution of the Einstein-Euler equations that describe such gravitational systems.

Specifically, we consider initial data for the spacetime and the perfect fluid, consisting of first and second fundamental forms, lapse function, fluid velocity and density:

\[(\Sigma_0, \gamma_0, k_0, \phi_0, \psi_0), \quad (v(0), \log(\rho_0 + p_0)).\]

The fluid is assumed to evolve adiabatically and posses an equation of state: \(p = f(\rho, s)\) We assume that \(\Sigma_0\) is asymptotically flat: the fluid is contained inside a ball \(S^2_r = \partial B^3_r\):

\[v(\Sigma_0 \setminus B^3_{12}) = 0, \quad \rho_0(\Sigma_0 \setminus B^3_{12}) = 0.\]

We assume initial local growth estimates inside \(B^3_r\) after a decomposition of this ball in domains:

\[B^3_r = \bigcup_{j=1}^N B^3_{ij}.\]

For cut-offs \(\text{supp}(\zeta) \subset B^3_{ij}\) we introduce the quantities:

\[\varrho^i(\varphi; \zeta; \Sigma_0) = \frac{D^i(\varphi; \zeta; \Sigma_0)}{D^0(\varphi; \zeta; \Sigma_0)}, \quad D^i(\varphi; \zeta; \Sigma_0) = \int_{\Sigma_0} \zeta^2 |\nabla^i \varphi|^2,\]

and

\[\gamma_i[\varphi] = \sup_{\zeta \in S} \varrho^i(\varphi; \zeta; \Sigma_0).\]

In this paper we estimate the time evolution of the preceding constants \(\gamma_i[\varphi]\) for

\[\varphi = \phi_0, \quad |k_0|, \quad |v_0|, \quad \psi_0, \quad \sigma_0 = (\nabla_j v^j)(0), \quad |\sigma| = \left| \frac{1}{2} (\nabla_l v_m - \nabla_m v_l)(0) \right|.\]
assuming as given the dynamical estimates for the last two quantities. We will combine these estimates with the dynamical construction of the decomposition through the derivation of the estimates of the bump function and its derivatives through the various estimates.

These estimates will be combined with an iterative solution of the nonlinear hyperbolic equations describing the system.

Plan of the paper We present here mainly the energy estimates and appeal to the results in [9] on the Nash-DeGiorgi-Moser tecnique for the $L^\infty - L^p$ estimates on domains of the above form.

2. The deformation of the metric on the time slice

The weightfunction $\theta$ is constructed to measure the deformation of the foliation of the Euclidean ball by concentric spheres to a foliation by deformed spheres with the appearance of bumps. This deformation will be defined using a cover of the unit sphere as follows. On each time slice $\Sigma_t$ - diffeomorphic to $\mathbb{R}^3$ endowed with the metric $\gamma$ we consider the deformation of the foliation by Euclidean balls though the collection of bumps $\{B_{r,\ell}, \theta_\ell\}_{\ell=1,r<R}^N$ where

$$S^2 = \bigcup_{\ell=1}^N B_\ell$$

$$\theta_\ell : [0, R] \times B_\ell \to \mathbb{R}, \quad [0, R] \times B_\ell = B_{R,\ell}, \quad B_{r,\ell} = B_{R,\ell} \cap S^2_r$$

The foliation is deformed so that each leaf is deformed through the graphs of the function $\theta_\ell$ over $B_{r,\ell}$. Specifically, we have that the metric on $B_R$ assumes the following form:

$$\gamma_0 = a^2 dr^2 + \hat{\gamma}$$

The bump function are regularized on each patch $B_{r,\ell}$ by the method of harmonic approximation that is described in [9].

3. Assumptions and Results

We make the following assumptions:

- The vorticity $\omega$ and divergence $\sigma$ of the velocity field $\nabla$ satisfy estimates of the form

$$\int_{\Sigma_t} \zeta^2(|\nabla \omega|^2 + |\nabla \sigma|^2) \leq \varphi(t) \int_{\Sigma_t} \zeta^2|\nabla|^2.$$

- The bump and lapse functions $\theta, \alpha$ respectively are assumed to satisfy bounds of the form, for $k = 1, 2, 3$:

$$\int_{B_j} \zeta^2(|\nabla^k \theta|^2) \leq (\varphi(t))^k \int_{B_j} \zeta^2|\theta|^2$$

$$\int_{B_j} \zeta^2(|\nabla^k \alpha|^2) \leq (\alpha(t))^k \int_{B_j} \zeta^2 \alpha^2$$

- We assume that the equation of state relating pressure and density satisfies estimates of the form:

$$p = f(\rho) \sim \rho^\gamma, \gamma \geq 1$$

- We introduce the following localized energy on our data:

$$E(\zeta; D)(t) = \int_{\Sigma_t} \zeta^2 \left( |\text{Ric}|^2 + |\nabla k|^2 + |\nabla \phi|^2 + |\nabla \sigma|^2 + |\nabla \psi|^2 + |\omega|^2 + (\sigma)^2 + \alpha^2 + |\nabla^3 \theta|^2 \right)$$

where the data of our system are

$$D = [(\phi, \gamma, k); (v, \psi, \omega, \sigma); (\alpha, \theta)]$$
Furthermore, we assume that the functions localizing in the domains $B_j$ satisfy estimates of the form:

$$v(\zeta) = \pm|v|\beta_j\zeta(1-\zeta), \quad \zeta_0 = \pm\beta_j\zeta(1-\zeta)$$

where $\zeta(r, \vartheta, t) = \ell(r)\ell \left(\frac{|p(\vartheta)|}{r}\right) \ell(t)$ and $\ell(t) = 1$ if $\frac{T}{2} < t < 1$ and 0 if $0 < t < \frac{1}{4}$ or $T > \frac{3}{2}$.

We introduce then the domains $B_j \subset \Sigma_t$

$$B_j(\theta, \epsilon) = \{x \in \Sigma_t / (1 - \epsilon + \epsilon^{j+1})\theta \leq |\theta(x)| \leq (1 + \epsilon^{j+1})\theta\}$$

We prove then for $t \in [0, T)-T$ computed explicitly that the following estimates hold:

$$\sup_{B_j} a \leq P(\eta, \gamma) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} \psi \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\omega| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\sigma| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\phi| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\nabla \phi| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\nabla^2 \phi| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |k| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

$$\sup_{B_j} |\nabla k| \leq P(\eta, \varphi_j) t^{\kappa_0} E(D)(0)$$

while for $0 \leq t < T$ we have that:

$$\sup_{B_j} \varphi \leq \varphi_j \sim \theta^{m+3}t^6$$

$$\sup_{B_j} \alpha \leq \alpha_j \sim \theta^{m+3}t^6$$

All these exponents are explicitly calculated. In particular the lifespan is determined through the estimates:

$$|\nabla \theta| \geq (1 + \epsilon)|\nabla|, \quad |\nabla| \leq \epsilon(1 - \eta^2), \quad \eta^2 = f'(\rho)$$

The velocity estimate asserts the hyperbolic nature of the pressure wave equation.

### 4. The energy estimates

The wave equation for the Lorentzian metric $g$ and source function $S$:

$$\Box_g \psi = S$$

where

$$g = -\lambda^2 dt + \gamma$$

We introduce the vector fields:

$$T = \frac{\lambda}{\epsilon} \frac{\partial}{\partial t}, \quad L = T + W, \quad L = T - W$$

for a vector field $W$ tangential to the time slices and constructed in the next paragraph. The energy momentum tensor, then, for the wave equation reads:

$$T_{\mu\nu} = \psi\mu\psi\nu - \frac{1}{2}g_{\mu\nu}\psi^\lambda\psi_\lambda$$
with divergence
\[ T_{\mu\nu} = \psi^{\mu} S \]

The associated current
\[ (X) J^\mu = T^\mu_\nu X_\nu = X(\psi) \phi^\mu - \frac{1}{2} X^\mu_\lambda \phi^\lambda \psi_X \]
obyes the equation:
\[ (X) J^\mu \phi^\mu = (X) \pi_{\mu\nu} T^\mu_\nu + S \cdot X(\psi) \]
where the deformation tensor refers to:
\[ (X) \pi_{\mu\nu} = \frac{1}{2} (\nabla_\mu X_\nu + \nabla_\nu X_\mu) \]

5. Construction of the vector fields
The vector field \( W \) introduced above is the gradient vector field
\[ W = \frac{\nabla \theta}{|\nabla \theta|^2} \]
for the function \( \theta \) defined on the time slice.

Domains We denote by \( C_u, C_y \), the level sets of the functions \( u, y \) and by \( \Sigma_t \) the time slices. We denote the region formed by the level sets of \( t, u, y \):
\[ D(u_0, 2u_0; t) = \{(t', x)/0 \leq t' \leq t, u_0 \geq -u_0, \text{ and } y \leq 2u_0\} \]
Inside this set it is contained the cylinder \( C_{[0,t],R} \). Specifically let \( R \) be a regular value of \( \theta(t', \cdot), t' \in [0, t] \), then:
\[ C_{[0,t],R} = \{(t', x)/\theta(t', x) \leq R\} \]

Local optical functions We introduce the solutions of the eikonal equations \( u, y \):
\[ L(u) = 0, \quad L(y) = 0 \]
The solution of these equations proceeds as follows. First we introduce the coordinates \( s, \vartheta \) as
\[ W(s) = 1, \quad W(\vartheta_1) = W(\vartheta_2) = 0 \]
then we perform the local change of time variable
\[ \tau = \alpha(t, s, \vartheta) - s \]
that we select as solution
\[ \lambda \alpha_t = \alpha_s \]
through the flow out of the characteristic equation
\[ \frac{ds}{d\tau} = -\frac{1}{\lambda} \]
with initial value \( s = 0, \tau = 0 \) and \( \alpha(\tau, c(\tau)) = \tau \). Finally the optical function is
\[ u = \tau - s, \quad u = \tau + s. \]
The integral identities. For a cut-off function $\zeta$ localizing in $D(-u_0, 2u_0; t)$ we have:

\[
\int_{\Sigma_t} \zeta^2 T(X, T) = \int_{\Sigma_0} \zeta^2 T(X, L) + \int_{C_u} \zeta^2 T(X, L) + \int_{\Sigma_0} \zeta^2 T(X, T) + \int_{\Sigma_{u_0}} \zeta^2 T(X, T) + \int_{C_{-u_0}} \zeta^2 T(X, T) + \int_{D(-u_0, u_0; t)} T^{\mu\nu}(X) \pi_{\mu\nu} + S(X) (\phi^\nu - X(\zeta) \phi^\nu \phi^\nu) \zeta.
\]

We calculate that:

\[
T(T, T) = \frac{1}{2} (\lambda^2 \phi^2 + |\nabla \psi|^2) = \frac{1}{2} \varepsilon
\]

\[
T(L, L) = |L(\psi)|^2 = \lambda^2 |\nabla \phi|^2 + W(\psi)^2 + 2\lambda \psi W(\psi)
\]

\[
T(L, L) = |L(\psi)|^2 = \lambda^2 |\nabla \psi|^2 + W(\phi)^2 - 2\lambda \psi W(\psi)
\]

\[
T(T, L) = \frac{1}{2} (\lambda^2 \phi^2 + |\nabla \psi|^2) + \lambda \psi W(\psi)
\]

\[
T(T, L) = \frac{1}{2} (\lambda^2 \phi^2 + |\nabla \psi|^2) - \lambda \psi W(\psi)
\]

These terms are majorised by $\varepsilon$. Similarly we have that

\[
T^{\mu\nu}(T) \pi_{\mu\nu} = (\lambda \psi_t \lambda_t - 4 \nabla \lambda \cdot \nabla \phi) \lambda^2 \phi^2 - 4 (\nabla \psi \cdot \nabla \zeta) \lambda \zeta \leq 4 \zeta^2 \varepsilon + 4 \zeta^2 \lambda^4 (\lambda^2 + |\nabla \lambda|^2).
\]

Localized energy. If we select as $X = T$ then the above identity suggests for the energy and the cutoff function selected so that we restrict in the interior or that the integrals on the $C, C$ vanish then:

\[
E^\lambda \psi(t) = \int_{\Sigma_t} \zeta^2 \psi, \quad \varepsilon^\lambda = (\lambda^2 \phi^2 + |\nabla \psi|^2),
\]

that

\[
E^\lambda \psi(t) \leq E^\lambda \psi(0) + \int_{D(-u_0, 2u_0; t)} \zeta^2 \psi + I + II,
\]

where

\[
I \leq 2 \int_{D(-u_0, 2u_0; t)} \zeta^2 \lambda^4 (\lambda^2 + |\nabla \lambda|^2),
\]

and

\[
II = \int_{D(-u_0, 2u_0; t)} \zeta^2 \lambda S \psi_t.
\]

The coarea formula suggests that:

\[
\int_{D(-u_0, 2u_0; t)} \zeta^2 \varepsilon = \int_0^t dt' \left( \int_{\Sigma_{t'}} \lambda \zeta^2 \varepsilon d\sigma \right) \leq \int_0^t \left( \sup_{\Sigma_{t'}} \lambda \right) E(t') dt'
\]

The term $I$ assumes a majorisation of the form:

\[
I \leq \int_0^t \psi(t')
\]
for a function $v$ that will be determined while $I$ will assume a majorisation of the form due to the structure of $S$ as

$$I \leq \int D(-u_0, 2u_0; t) \zeta^2 \lambda S \psi_t \leq \int_0^t \gamma(t')E(t'),$$

where $\gamma$ is a function that will be determined in every case.

$$E^\lambda_\psi (t) \leq E^\lambda_\psi (0) + \int_0^t \gamma(t')E^\lambda_\psi (t') + v(t').$$

Therefore Gronwall’s inequality we obtain:

$$E^\lambda_\psi (t) \leq E(0)e^{\int_0^t \gamma(t')dt'} + \int_0^t v(t')e^{\int_0^t \gamma(t'')dt''} dt'.$$

Energy radiated away We select $X = L, L$, and in the domain $\mathcal{R}(-u_0, 2u_0; t)$:

$$\mathcal{R}(-u_0, 2u_0; t) = \{(t', x)/0 \leq t' \leq t, -2u_0 \leq u \leq -u_0 \text{ or } 2u_0 \leq u \geq 4u_0\}$$

we have that:

$$\int_0^t \gamma(t')e^{\int_0^t \gamma(t'')dt''} dt' = \gamma(0) + \int_0^t \gamma(t')E^\lambda_\psi (t') + v(t').$$

Therefore we obtain similar estimates as before:

$$E^\lambda_\psi (t) \leq E(0)e^{\int_0^t \gamma(t')dt'} + \int_0^t v(t')e^{\int_0^t \gamma(t'')dt''} dt'.$$

6. The Euler-Einstein system

The system is described by the Einstein equations for the energy momentum tensor:

$$T_{\mu \nu} = (\rho + p)u_\mu u_\nu + pg_{\mu \nu}$$

Einstein equations then

$$R_{\mu \nu} - \frac{R}{2}g_{\mu \nu} = T_{\mu \nu}$$

lead the Euler equations through Bianchi identity:

$$\nabla^\nu T_{\mu \nu} = 0$$

Space-time We will use the space-time decomposition that requires

$$g = -\phi^2 dt^2 + \gamma$$

for the lapse function function $\phi$ and the induced Riemannian metric on the slices $\mathcal{H}_t$ denoted by $\gamma(t)$. The second fundamental form of the slices is $k(t)$ and we have the first variation identity

$$\partial_t \gamma = -2\phi k \text{ (FV)}$$
The second variation formula and the Gauss-Codazzi (GC) and Gauss equations under the assumption of maximal slicing, i.e. tr(k) = 0 are written as:

\[
\partial_t k_{ij} = -\nabla_i \nabla_j \phi + \left( R_{ij} - 2k_{im}k_{jm} - \frac{1}{2}R\gamma_{ij} - 2T_{ij} \right) \phi \quad \text{(SV)}
\]

\[
\nabla_i k_{jm} - \nabla_j k_{im} = R_{m0ij} \quad \text{(GC)}
\]

\[
R_{ij} - k_{il}k_{lj} = R_{i0j0} + R_{ij} \quad \text{(G)}
\]

Taking the trace of \((G)\) we obtain that:

\[
R - |k|^2 = T_{00} \Rightarrow |k|^2 + (p + \rho)\beta^2 \phi^2 = \phi^2 p + R
\]

We observe that

\[
|k|^2 + p|u|^2 + \rho \beta = R \Rightarrow R \geq p|u|^2, |k|^2, \rho \beta
\]

The trace of \((GC)\) gives us that:

\[
\text{div}(k) = 2T_{j0} = 2\tilde{p}\beta v_j
\]

The \textit{lapse function}  The trace of \((SV)\) suggests that

\[
\phi \Delta_\gamma \phi + \left( |k|^2 + (2(p + \rho)\beta + (p - \rho))\phi^2 \right) \phi^2 = 0
\]

This suggests that

\[
\Delta_\gamma \phi \leq 0
\]

implying that \(\phi \geq 1\) since this contradicts the existence of areas where \(\phi \leq 1\) due to the fact that \(\Delta_\gamma \phi \leq 0\) even then. This gives the integral inequality for \(\phi, \zeta:\)

\[
\int_{\Sigma_t} \zeta^2 |\nabla \phi|^2 = \int_{\Sigma_t} \zeta^2 \left[ |k|^2 \phi^2 + ((p + \rho)\beta + (p - \rho)) \phi^4 + \phi \nabla \zeta^2 \cdot \nabla \phi \right]
\]

Therefore we obtain

\[
\int_{\Sigma_t} |\nabla (\zeta \phi)|^2 \leq \left( \int_{\Sigma_t} (\beta^5 + |k|^{10} + (p + \rho)^5) \right)^{1/5} \left( \int_{\Sigma_t} \zeta^2 \phi^5 \right)^{4/5}
\]

Similarly we obtain for the time derivative of the lapse function that:

\[
\int_{\Sigma_t} |\nabla (\zeta \phi_t)|^2 \leq C_2 \left( \int_{\Sigma_t} (\beta^5 + k^{10} + (p + \rho)^5) \right)^{1/5} \left( \int_{\Sigma_t} \zeta^2 \phi^5 \right)^{1/5}
\]

since \(\phi > 1\).

\textit{Estimates for} \(|\nabla^2 \phi|\)  We use the formula:

\[
\int_{\Sigma_t} \zeta |\nabla^2 \phi|^2 \leq C \left[ \int_{\Sigma_t} (\zeta^2 + |\nabla \zeta|^2) \left( |k|^2 |\nabla k|^2 + (1 + \eta^2)^2 |\nabla \rho|^2 + (p + \rho)^2 |\nabla \beta|^2 \right) + (\rho + p)^{3/2} \beta^3 + |k|^3 \right]
\]
7. The acoustical metric

The continuity equation could written in the form for the $\psi = \log(\rho + p)$:

\[ \tilde{g}^{\mu\nu} \nabla_\nu \psi = -\eta^{-2} (v^\mu \nabla_\mu v^\nu + v^\mu \sigma), \]

where

\[ \tilde{g}^{\mu\nu} = g^{\mu\nu} + \eta^{-2} u^\mu v^\nu, \quad \sigma = \nabla_\nu v^\nu \]

We readily derive form this equation that:

\[ |v_t^0| \leq \frac{\phi}{2\beta} (\eta^2 \phi^2 + \beta^2) \sqrt{\varepsilon^2 \chi + |\phi_t| + \beta \phi |\nabla \phi| + \phi |\nabla v|}, \]

and also

\[ |v_t| \leq \phi (\eta^2 \phi^2 + \beta^2) \sqrt{\varepsilon^2 \chi + \frac{\phi}{2\beta} |\nabla \phi| + \frac{\phi}{2\beta} |\nabla \phi| + \phi (|\nabla v| + |\nabla \beta|} \]

We introduce the change of variables along the time-slice generated by the flow of the fluid velocity:

\[ \frac{dX^j}{ds} = -\frac{\phi \beta}{\delta} v^j(x, t), \quad X^j(0, x, t) = x, \quad \beta = \sqrt{1 + |v|^2}, \quad \delta = 1 - \eta^2 \]

The metric is simplified as

\[ \tilde{g} = -\chi^2 dt^2 + \tilde{\gamma}, \quad \tilde{\gamma} = X^* \gamma \]

where

\[ \chi^2 = \phi^2 \left( 1 - \frac{\Upsilon^2}{\delta} \right) \leq \phi^2, \quad \Upsilon = |v| \geq \sqrt{2} |v| \]

provided that

\[ 0 \leq |v|^2 \leq \frac{2\delta}{\sqrt{1 + 4\delta} + 1} \]

otherwise the acoustical metric then switches to Riemannian for large pressure and low velocities. Accordingly we have a Poisson equation. We introduce the acoustical connection which is related to the connection of $g$ through the relation of the Christoffel symbols:

\[ \tilde{\Gamma}^\nu_{\nu\kappa} = \Gamma^\nu_{\nu\kappa} + F^\nu_{\nu\kappa}, \]

where

\[ F^\mu_{\nu\kappa} = \frac{1}{\eta^2(1 - \eta^2)} \left[ u^\mu u_\nu u_\kappa + u^\mu S_{\kappa\nu} - u^\mu (u_\nu \nabla_\kappa \eta + u_\kappa \nabla_\nu \eta) - \frac{1}{2} v^\mu (v^\rho \nabla_\rho)(v_\kappa v_\nu) \right] + \frac{1}{\eta^2} [\Omega^\mu_{\kappa\nu} + \Omega^\nu_{\kappa\mu}]. \]

We bound this as

\[ |F| \leq \frac{1}{2\eta^3} \left[ |S| |v| + 2 |v|^2 |\nabla \eta| + \frac{\beta^4}{2\phi^2} (\eta^2 \sqrt{\varepsilon} + |\sigma|) + \frac{2}{\eta^2} |\Omega| \right], \]

if $\varepsilon^2 \chi = \chi^2 v^2 + |\nabla v|^2$. The wave equation for the acoustical metric reads as:

\[ \Box \tilde{g} \psi = S(v, \eta), \]

where

\[ S(v, \eta) = \eta^{-2} \left[ (\nabla_\mu \zeta + \nabla_\mu \log(\chi \tilde{\sigma} \gamma)) v^\nu \nabla_\nu v^\mu - \nabla_\mu v^\nu \nabla_\nu \nabla_\mu v^\mu - v^\nu \nabla_\nu \sigma + v^\nu v^\mu \nabla_\mu (\log(\chi \tilde{\sigma} \gamma)) + R\Gamma_{\mu\nu} v^\mu v^\nu \right], \]

where

\[ \tilde{\sigma}(x) = \det(DX)(X^{-1}(x)) \]

We will apply the general method for the Lorentzian metric

\[ -\chi^2 dt^2 + \tilde{\gamma} \]

where

\[ \tilde{\gamma} = X^* \gamma \]
The integral identities For a cut-off function $\zeta$ localizing in $D(-u_0, 2u_0; t)$ we have:

\[
\int_{\Sigma_t} \zeta^2 T(X, T) = \int_{\Sigma_{2u_0}} \zeta^2 T(X, L) + \int_{C_{-u_0}} \zeta^2 T(X, L) + \int_{\Sigma_0} \zeta^2 T(X, T) + \int_{D(-u_0, u_0; t)} T^{\mu\nu}(X) \pi_{\mu\nu} + S \chi (\psi) + (2X(\psi)\psi^\nu\zeta_\nu - X(\zeta)\psi^\nu\psi_\nu) \zeta,
\]

We calculate that:

\[
T(T, T) = \frac{1}{2} (\chi^2 \psi_\xi^2 + |\nabla \psi|^2) = \frac{1}{2} \varepsilon, \\
T(L, L) = |L(\psi)|^2 = \chi^2|\nabla \psi|^2 + W(\psi)^2 + 2\chi \psi W(\psi), \\
T(L, L) = |L(\psi)|^2 = \chi^2|\nabla \psi|^2 + W(\psi)^2 - 2\chi \psi W(\psi), \\
T(L, L) = |L(\psi)|^2 = \chi^2|\nabla \psi|^2 - W(\psi)^2,
\]

These terms are majorised by $\varepsilon^\psi$. Similarly we have that

\[
T^{\mu\nu}(T) \pi_{\mu\nu} = (\chi \psi \lambda_4 - 4\nabla X \cdot \nabla \psi) \chi^2 \psi \zeta^2 - 4(\nabla^2 \psi \cdot \nabla \zeta) \chi \zeta \leq 4\zeta^2 \varepsilon^\psi + 4\zeta^2 \chi^4 (\chi^2 + |\nabla \chi|^2),
\]

Interior Energy estimates If we select as $X = T$ then the above identity suggests for the energy and the cutoff function selected so that we restrict in the interior or that the integrals on the $C, \Sigma$ vanish then:

\[
E^\psi(\chi, \psi) = \int_{\Sigma_t} \zeta^2 \varepsilon^\psi, \\
E^\psi(\chi, \psi) \leq E^\psi(0) + \int_{D(-u_0, 2u_0; t)} \zeta^2 \varepsilon^\psi + I + II,
\]

where

\[
I \leq 2 \int_{D(-u_0, 2u_0; t)} \zeta^2 \chi^4 (\chi^2 + |\nabla \chi|^2),
\]

and

\[
II = \int_{D(-u_0, 2u_0; t)} \chi \zeta^2 S \psi_t,
\]

The coarea formula suggests that

\[
\int_{D(-u_0, 2u_0; t)} \zeta^2 \varepsilon^\psi = \int_0^t \left( \int_{\Sigma_\tau} \zeta^2 \varepsilon^\psi d\sigma \right) \leq \int_0^t \left( \sup_{\Sigma_\tau} \phi \right) E^\psi(\chi, \psi)(t'),
\]

We estimate provided that $\Upsilon \leq \varepsilon \delta, \varepsilon \leq \frac{1}{20}$

\[
\chi^4 (\chi^2 + |\nabla \chi|^2) \leq \chi^4 [(1 - \varepsilon)(\phi^2 + |\nabla \psi|^2) + \chi^2 (\psi^2 + |\nabla \Upsilon|^2) + \varepsilon^2 \eta^2 + |\nabla \eta|^2] \leq 2\beta ((1 + \phi^2)|\nabla \psi|^2 + \sigma^2) + 2\beta^2 \phi^2 \eta^2 \varepsilon^\psi,
\]
and also if for the equation of state $p = f(\rho)$

$$ \eta^2 + |\nabla \eta|^2 \leq e^{2\psi} f''\varepsilon $$

Similarly we compute

$$ \mathcal{Y}_t^2 + |\nabla \mathcal{Y}|^2 \leq 2\beta (|v_t|^2 + |\nabla v|^2) \leq 2\beta \left( (1 + \phi^2)|\nabla v|^2 + \beta^2 \phi^2 \eta^2 |\nabla \psi|^2 + \sigma^2 \right), $$

In conclusion we have that setting

$$ \nu_f(t) = \int_{\Sigma_t} \zeta^2 |\nabla v|^2 $$

we bound

$$ I \leq 2 \int_0^t \sup_{\Sigma_{t'}} \left[ \phi^6 \left( e^{2\psi} f'' + \phi \beta^2 \eta^2 \right) \right] E(t') dt' + \int_0^t \sup_{\Sigma_{t'}} \left( \beta \phi_7 \right) \nu_f(t') $$

The term $II$ is majorised as follows

$$ \int_{D(-u_0,2u_0;t)} \chi \zeta^2 \mathcal{S}_\psi t = \int_{D(-u_0,2u_0;t)} \frac{\chi}{\eta^2} \zeta^2 \psi_t \left[ \tilde{\nabla}_\mu Z + \tilde{\nabla}_\mu \log(\chi \bar{\sigma} \gamma) \right] v^\rho \tilde{\nabla}_\nu v^\mu - $$

$$ - \int_{D(-u_0,2u_0,t)} \frac{\chi}{\eta^2} \zeta^2 \psi_t \left[ \tilde{\nabla}_\mu v^\nu \tilde{\nabla}_\nu v^\mu + v^\nu \nabla_\nu \sigma - v^\nu \tilde{\nabla}_\mu (\log(\chi \bar{\sigma} \gamma)) - \tilde{\mathbf{R}} c_{\mu \nu} v^\mu v^\nu \right] $$

We estimate

$$ |\nabla \log(\chi \bar{\sigma} \gamma)| \leq c \left| \frac{\phi_8}{\phi} \right| + \phi \left( \phi^2 + \beta^2 \right) \sqrt{\varepsilon} \psi + \frac{\phi_4}{2\beta} |\nabla \phi| + \frac{\phi_5}{2\beta} |\sigma| + \phi (|\omega| + |\nabla \beta|). $$

We get after integration by parts, $L^\infty - L^2$ and Cauchy-Schwarz that:

$$ II \leq \int_0^t E(t') + \nu_{II}(t') dt', $$

where

$$ \nu_{II}(t) = \int_{\Sigma_t} \frac{\phi^2 \zeta^2}{\eta^2} \left[ (|\nabla Z|^2 + |\nabla \log(\chi \bar{\sigma} \gamma)|^2) \beta^2 |\nabla v|^2 + |\tilde{\mathbf{R}} c|^2 \beta^2 + \eta^2 \sigma^4 + \frac{e^{2\psi} f'' - \beta^2 \sigma^2 \eta^4}{\eta^2} \right]. $$

We arrive at the inequality:

$$ E(t) \leq E(0) + \int_0^t \left( \gamma(t') E(t') + \nu(t') \right) dt', $$

where

$$ \gamma = \sup_{\Sigma_{t'}} \left[ \phi^6 \left( e^{2\psi} f'' + \phi \beta^2 \eta^2 \right) \right] + 1, $$

$$ \nu = \sup_{\Sigma_{t'}} \left( \beta \phi_7 \nu_1 \right) + \nu_{II}. $$

Therefore Gronwall’s inequality we obtain:

$$ E(t) \leq e^{\int_0^t \gamma(t') dt'} \left[ E(0) + \int_0^t e^{-\int_0^{t'} \gamma(t'') dt''} \nu(t') dt' \right]. $$

**Energy radiated away** Integrating in the exterior region then

$$ E(t) \leq e^{\int_0^t \gamma(t') dt'} \left[ E(0) + \int_0^t e^{-\int_0^{t'} \gamma(t'') dt''} \nu(t') dt' \right]. $$
The wave equation for the second fundamental form. Moreover the second variation equation provides through the equation for the Ricci curvarture the wave equation for the second fundamental form, [1]:

$$\phi \square_j k = S$$

where

$$S_{ij} = S_{00,ij} + S_{01,ij} + S_{10,ij} + S_{11,ij} + S_{2,ij} + S_{3,ij},$$

and we have that

$$S_{00,ij} = -4\phi \left[ k_i l_j k_{jm} - \frac{1}{2} R^l_{ijmn}k_{ml} - \frac{3}{4} (R_{mlk_j}k^m_j + R_{jmk_i}k^m_i) + \frac{1}{2} R_{kij} \right] - \gamma^{rs} \left[ \Gamma^m_{rs} \Gamma^n_{jl} k_{il} + \Gamma^m_{rs} \Gamma^n_{jl} k_{jl} \right],$$

$$S_{01,ij} = -4\phi \left[ (p + \rho)(k_i l_j v_j + k_j l_i v_i) + p (\gamma_{ij} k^l_{i} + \gamma_{il} k^l_{j}) \right],$$

$$S_{10,ij} = 2(\nabla_i \phi)(\nabla_i k^j + \nabla_j k^i) - 3\nabla_i \phi \nabla^i k_{ij} + \phi^{-1} k_{ij} |\nabla \phi|^2 + \nabla_m \log \phi \cdot \nabla_m k_{ij} + \gamma^{rm} \Gamma^s_{rmi} \nabla_r k_{ij},$$

$$S_{11,ij} = \phi^{-1} (k_i l_j \phi + k_j l_i \phi) \nabla^i \phi - 3\phi^{-2} \phi_l (Ric_{ij} - 2k_{ij} k^n_j - \frac{1}{2}(R + 4\rho) \gamma_{ij} - 2(\rho + p) v_i v_j,$n

$$S_{2,ij} = -[k_{jr} \nabla_i \nabla^r \phi + k_{ir} \nabla_j \nabla^r \phi - k_{ij} (|\phi|^2 + (2(p + \rho) \beta + (p - \rho) \phi^2) |\phi|^2) + 3\phi^{-3} \phi_t \nabla_i \nabla_j \phi,$n

$$= \partial_t T_{ij} + \nabla_i (\phi^2 T_{j}^0) + \nabla_j (\phi^2 T_{i}^0),$$

We use the energy density:

$$\varepsilon_\phi^k = \phi^2 |k|_i^2 + |\nabla k|^2,$$

and the cutoff function \( \zeta \) with support contained in \( B_j(\theta, \epsilon) \). Moreover we restrict in the interior- the integrals over \( C, C \) vanish then:

$$E^{\phi, k}(t) = \int_{\Sigma t} \zeta^2 \varepsilon_\phi^k, $$

Therefore we

$$E^{\phi, k}(t) \leq E^{\phi, k}(0) + \int \int \zeta^2 \varepsilon_\phi^k + I + II,$$

where I, II are given as in the general case and we majorise below. The coarea formula suggests that:

$$\int \int \zeta^2 \varepsilon = \int^t_0 dt \left( \int_{\Sigma_{t'}} \phi \zeta^2 \varepsilon d\sigma \right) \leq \int^t_0 \left( \sup_{\Sigma_{t'}} \phi \right) E.$$
where
\[ \mu_1 = \left( |Rm| + e^\psi \beta \right) \phi^2 + (|\nabla Rm| + |\nabla \phi|^2 + |Rmi|) + (1 + c e^\psi) |\nabla \phi|^3, \]
\[ \mu_2 = \phi |\nabla \phi| \left( |k|^2 + |\nabla Rm||k| + \phi |\nabla \phi| + \beta c e^\psi + |Rmi| \right). \]

The terms involving the Riemann tensor are majorised through the constraint equations as for the Ricci tensor
\[ |\text{Ric}| \leq c_3 [e^\psi \beta^2 + |k|^2], \]
\[ |\nabla \text{Ric}| \leq c_4 \left[ e^\psi |\nabla \psi| + e^\psi |\nabla \beta| + |\nabla k| \right] \leq c_4 \left[ e^\psi \left( \varepsilon^\psi \chi \right)^{1/2} + |\kappa_1| + \left( \varepsilon^k \phi \right)^{1/2} \right]. \]

while the Riemann tensor in three dimensions is estimated through Ricci and scalar curvature as
\[ |Rm| \leq c_5 [e^\psi \beta^2 + |k|^2], \]
\[ |\nabla Rm| \leq c_6 \left[ e^\psi (|\nabla \psi| + |\nabla \beta|) + |\nabla k| \right] \leq c_6 \left[ e^\psi \left( \varepsilon^\psi \chi \right)^{1/2} + |\kappa_1| + \left( \varepsilon^k \phi \right)^{1/2} \right]. \]

Now we use Sobolev inequality and the \( L^\infty - L^2 \) estimates to get that:
\[ \left| \int_{\Sigma_t} S \cdot k \xi^2 \phi \right| \leq c_{\gamma}(t) E(t) + v_{\gamma}(t), \]

if we have set that:
\[ \gamma(t) = \left[ \int_{\Sigma_t} \left( \xi^2 + |
abla \xi|^2 \right) \left( \phi^2 |
abla \phi|^2 |k|^4 + |
abla Rm|^2 |k|^2 + (1 + \beta c e^\psi + |Rm|^2 + \phi^4 |
abla \phi|^4 \right) \right]^{1/2} + \left( \int_{\Sigma_t} \left( \xi^2 + |
abla \xi|^2 \right) \mu_l^6 / 5 \right)^{5/6}. \]

and
\[ v_{\gamma}(t) = \int_{\Sigma_t} \left( |Rm| + e^\psi (\beta + \phi^2) \right) \phi^4 |
abla \phi|^2. \]

Then using the estimates for the lapse function we get for
\[ E(t) \leq E(0) + \int_0^t \left( \gamma(t') E(t') + v(t') \right). \]

Therefore Gronwall’s inequality we obtain:
\[ E(t) \leq E(0) e^{-\int_0^t \gamma(t') dt'} + \int_0^t v(t') e^{-\int_0^t \gamma(t'') dt''} dt'. \]

**Slab estimate for the lapse function** We start with the propagation equation:
\[ \frac{d \Phi}{dt} = 2 \int_{\Sigma_t} \xi^2 \phi \phi_t \leq \int_{\Sigma_t} \xi^2 \left( \phi^2 + \phi_t^2 \right) \leq \gamma(t) \Phi(t) + v(t), \]

if
\[ \Phi = \int_{\Sigma_t} \xi^2 \phi^2, \]
\[ \gamma(t) = c \left( \int_{\Sigma_t} \left( \beta^{5/4} + |k|^4 + (p + \rho)^5 \right) \right)^{2/5}, \]

and
\[ v(t) = \left( \int_{\Sigma_t} \left( \beta^{5} + k^{10} + (p + \rho)^5 \right) \right)^{2/5}. \]

Therefore we have that
\[ \int_{\Sigma_t} \xi^2 \phi^2 \leq e^{\int_0^t \gamma(t') dt'} \int_{\Sigma_0} \xi^2 \phi^2 + \int_0^t v(t') e^{\int_0^{t'} \gamma(t'') dt''} dt'. \]
8. Estimates for fluids

Fluid currents The equation of motion for the fluid reads that:

$$\nabla_\nu T^{\mu\nu} = 0.$$  

We introduce the current:

$$(X) J^\mu = T^{\mu\nu} X_\nu = (\rho + p)(X_\nu v^\nu)v^\mu + \rho X^\mu.$$

In particular if $X = v$ then

$$(v) J^\mu = -\rho v^\mu.$$  

We compute that

$$\nabla_\nu \left( \zeta^2 (X) J^\nu \right) = \zeta^2 (X) \pi_{\mu\nu} T^{\mu\nu} + 2\zeta (X) J^\nu \nabla_\nu \zeta,$$

then in the domain spanned by $\Sigma_0, \Sigma_t, C_u, C_\mu$ we obtain that:

$$\int_{\Sigma_t} \zeta^2 T(X, T) = \int_{\Sigma_0} \zeta^2 T(X, T) - \int_{C_{-mu}} \zeta^2 (T(X, L)) - \int_{C_{2mu}} \zeta^2 (T(X, L)) +$$

$$+ \int_{D(-u_0,2u_0; t)} \zeta^2 (X) \pi_{\mu\nu} T^{\mu\nu} + 2\zeta (X) J^\nu \cdot \nabla_\nu \zeta.$$

We calculate

$$T(T, T) = (\rho + p)\beta^2 - p = \beta^2\rho + p|v|^2 \geq |v|^2(\rho + p),$$

and

$$T(T, L) = \phi(\rho + p)\beta \left[ 1 + \frac{v(\theta)}{v(\theta)} \right] - \phi(\rho + p)\beta \left[ 1 - \frac{|v|}{v(\theta)} \right],$$

and

$$|T(T, L)| \leq \phi(\rho + p)\beta \left[ 1 + \frac{|v|}{v(\theta)} \right],$$

We will assume an estimate

$$|\nabla \theta| \geq (1 + \epsilon)|v|.$$

and derive the conditions that assert its validity.

Velocity estimates Select $X = \alpha^2 v^i \partial_j$ then we have that:

$$\int_{\Sigma_t} \zeta^2 e^{\psi} \beta |v|^2 \alpha^2 \frac{1}{\phi} = \int_{\Sigma_0} \zeta^2 e^{\psi} \beta |v|^2 \alpha^2 \frac{1}{\phi} + \int_{C_{-mu}} \alpha^2 \zeta^2 (T(v, L)) + \int_{C_{2mu}} \alpha^2 \zeta^2 (T(v, L))$$

$$+ \int_{D(-u_0,2u_0; t)} \zeta^2 e^{\psi} \left[ |v|^2 v(\alpha^2) - \alpha^2 \left( \sigma + \frac{1 + 2\eta}{1 + \eta^2} v(\psi) \right) \right] - 2\zeta^2 e^{\psi} \left[ |v|^2 \left( v(\zeta) + \frac{\beta}{\phi} \zeta_0 \right) + pe^{-\psi} \beta^2 v(\zeta) \right].$$

The coarea formula implies that

$$\int_{D(-u_0,2u_0; t)} \zeta^2 \rho \alpha v(\alpha) = \int_0^t \left( \sup_{\Sigma_{\alpha}} \phi \right) \Gamma_0(t'),$$

where

$$\Gamma_0(t) = \int_{\Sigma_t} \zeta^2 \rho \alpha v(\alpha) \leq 2 \int_{\Sigma_t} \zeta^2 \beta \rho \alpha (\phi^2 \alpha^2 + |\nabla \alpha|^2)^{\frac{3}{2}}.$$
The velocity Selecting $\alpha = 1$ or $X = v$ and dropping the lateral integrals after selecting $\zeta$ so that the integrals on the conical surfaces are annihilated then we obtain:

$$
\int_{\Sigma_t} \frac{\zeta^2 e^\psi \beta |v|^2}{\phi} = \int_{\Sigma_0} \frac{\zeta^2 e^\psi \beta |v|^2}{\phi} - \iint_{D(-u_0,2u_0; t)} \zeta^2 e^\psi \left[ (\sigma + \frac{(1+2\eta^2)}{1+\eta^2} v(\psi)) - 2\zeta e^\psi \left( |v|^2 (\pi(\zeta) + \frac{\beta}{\phi} \zeta_0) + pe^{-\psi} \beta^2 v(\zeta) \right) \right].
$$

Therefore we set

$$
\kappa_0(t) = \int_{\Sigma_t} \frac{\zeta^2 e^\psi \beta |v|^2}{\phi}.
$$

The coarea formula and Cauchy-Schwarz

$$
\kappa_0(t) \leq \kappa_0(0) + 4 \int_0^t \gamma_0(t') \kappa_0(t') + v_0(t'),
$$

where

$$
\gamma_0(t) = \left( \sup_{\Sigma_t} \phi \right)^2 (1 + \beta_j) + \varphi(t),
$$

$$
v_0(t) = \int_{\Sigma_t} \zeta^2 e^\psi \left[ \frac{\phi^2}{\beta^2 \chi^2} \left( \eta^2 \phi^2 + \frac{\beta^2}{\phi^2} + 1 \right) e^\psi + \frac{\phi^2}{\beta^2} + \phi^2 |\nabla \phi|^2 \right].
$$

Then Gronwall’s lemma suggests that:

$$
\kappa_0(t) \leq \kappa_0(0)e \int_0^t \gamma_0(t') dt' + \int_0^t v_0(t')e \int_0^{t'} \gamma_0(t'') dt'' dt'.
$$

The velocity gradient Similarly selecting $\alpha = |\nabla v|^2$, $X = |\nabla v|^2 v$ and $\zeta$ as before then:

$$
\int_{\Sigma_t} \frac{\zeta^2 e^\psi \beta |v|^2 |\nabla v|^2}{\phi} = \int_{\Sigma_0} \frac{\zeta^2 e^\psi \beta |v|^2 |\nabla v|^2}{\phi} + \iint_{D(-u_0,2u_0; t)} \zeta^2 e^\psi \left[ |v|^2 (|\nabla v|^2 - |\nabla v|^2 \left( \sigma + \frac{(1+2\eta^2)}{1+\eta^2} v(\psi) \right)) \right] + 2\zeta |\nabla v|^2 e^\psi \left( |v|^2 (\pi(\zeta) + \frac{\beta}{\phi} \zeta_0) + pe^{-\psi} \beta^2 v(\zeta) \right).
$$

The first term far more intriguing we compute with Hölder and Sobolev inequalities that

$$
\int_{\Sigma_t} \zeta^2 e^\psi \left[ |\nabla|^2 |\nabla^2 v| \right] \leq c \left( 1 + \frac{1}{\epsilon} \right) \varphi(t)^5 \kappa_0(t)^{7/2} + \varphi(t)^{1/2} \int_{\Sigma_t} \zeta^\psi.
$$

The other terms contribute that:

$$
\kappa_1(t) = \int_{\Sigma_t} \frac{\zeta^2 e^\psi \beta |\nabla v|^2 |v|^2}{\phi}.
$$

Therefore

$$
\kappa_1(t) \leq \kappa_1(0) + \int_0^t (\gamma_1 \kappa_1(t') + v_1) dt'
$$

and for

$$
\gamma_1(t) = \left( \sup_{\Sigma_t} \phi \right)^2 (1 + \beta_j) + \varphi(t),
$$

$$
v_1(t) = c \left( 1 + \frac{1}{\epsilon} \right) \varphi(t)^5 \kappa_0(t)^{5/2} + \varphi(t)^{1/2} \int_{\Sigma_t} \zeta^\psi.
$$

Then Gronwall’s lemma suggests that:

$$
\kappa_1(t) \leq \kappa_1(0)e \int_0^t \gamma_1(t') dt' + \int_0^t v_1(t')e \int_0^{t'} \gamma_1(t'') dt'' dt'.
$$
The compressibility Selecting $\alpha = \sigma = \text{div}_\Sigma_t(v)$, $X = \sigma^2 v$ then:

$$\int_{\Sigma_t} \zeta^2 e^\psi |v|^2 |\sigma|^2 \frac{\phi}{\phi} = \int_{\Sigma_0} \zeta^2 e^\psi |v|^2 |\sigma|^2 \frac{\phi}{\phi} +$$

$$\iiint_{D (-u_0, 2u_0; t)} \zeta^2 e^\psi \left[ |v|^2 v(\sigma^2) - \left( \alpha + \frac{(1 + 2R^2)}{1 + \eta^2} v(\psi) \right) \right] + 2\zeta \sigma^2 e^\psi \left[ |v|^2 \left( v(\zeta) + \frac{\beta}{\phi} \zeta_0 \right) + pe^{-\psi} \beta^2 v(\zeta) \right].$$

Therefore we set

$$\kappa_{10}(t) = \int_{\Sigma_t} \zeta^2 e^\psi |v|^2 |\sigma|^2 \frac{\phi}{\phi}.$$

Similarly as before the first term is the most delicate we get

$$\int_{\Sigma_t} \zeta^2 e^\psi |v|^2 |\sigma| |\nabla \sigma| \leq c \left( \varphi(t)^{2+\kappa_0(t)} + \varphi(t)^{1/2} \int_{\Sigma_t} \zeta^2 e^\psi \varepsilon \chi \right).$$

Therefore

$$\kappa_{10}(t) \leq \kappa_{10}(0) + \int_0^t \left( \gamma_{10} \kappa_{10}(t') + \nu_{10}(t') \right) dt'.$$

Then Gronwall’s lemma suggests that:

$$\kappa_{10}(t) \leq \kappa_{10}(0) e^\int_0^t \gamma(t') dt' + \int_0^t \nu(t') e^\int_0^{t'} \gamma(t'') dt'' dt'.$$

The velocity estimates on the slice We recall the following identity from [3]:

$$\int_{\Sigma_t} |\nabla U|^2 = \int_{\Sigma_t} \left( \frac{1}{2} |A(U)|^2 + |D(U)|^2 \right) - \int_{\Sigma_t} \left( 2p + 1 \right) \text{Ric}_{mn} U^m U^n - \frac{R}{2} |U|^2 +$$

$$+ 2p \int_{\Sigma_t} T(U) \cdot R \cdot U - \frac{p}{2} R |T(U)|^2.$$

where

$$A(U)_{a_1 \ldots a_p bc} = \nabla_c U_{a_1 \ldots a_p b} - \nabla_b U_{a_1 \ldots a_p c}$$

$$D(U)_{a_1 \ldots a_p} = \nabla^m U_{a_1 \ldots a_p m}$$

$$T(U)_{a_1 \ldots a_p} = g^{bc} U_{a_1 \ldots a_p}$$

$$T(U) \cdot \text{Ric} \cdot U = T(U)_{a_1 \ldots a_p} \cdot \text{Ric}_{mn} \cdot U^{a_1 \ldots a_p} u_{mn}$$

We will use this for the projection of the velocity on the slice

$$v^\mu = \nu^\mu - \frac{\beta}{\phi} \delta^\mu_0$$

then we have for $\omega = *A(\overline{v})$, $\sigma = D(\overline{v})$

$$\int_{\Sigma_t} |\nabla (\overline{v})|^2 = \int_{\Sigma_t} \frac{1}{2} |\omega|^2 + |D(\overline{v})|^2 - \text{Ric}(\overline{v}, \overline{v}) + 2\zeta \nabla \overline{v} \cdot \nabla \overline{\zeta} + |\overline{v}|^2 |\nabla \overline{\zeta}|^2.$$
9. Bump function estimates

The required estimates on the bumps $\theta_\ell$ we will drop the subscript $\ell$ follow from estimates on the second fundamental form - which provide estimates for the hessian of $|\nabla^2 \theta|$ and in turn of $\theta, |\nabla \theta|$. We verify readily that the traceless second fundamental form $\hat{\mu}$ and the mean curvature $h$ of the deformed spheres is related to the hessian of the bump function for positive constants $c_1, c_2$:

$$c_1 \frac{|\nabla^2 \theta|}{1 + |\nabla \theta|^2} \leq \hat{h} \leq c_2 \frac{|\nabla^2 \theta|}{1 + |\nabla \theta|^2}, \quad h = \Delta \theta$$

The first, second variation equations and Gauss-Codazzi equations involve $h_{AB}, h, \hat{h}_{AB} = h_{AB} - \frac{1}{2} h \delta_{AB}$ and read as

$$\frac{d\hat{h}_{AB}}{dr} = -a h, \quad \frac{dh}{dr} = -a \left( \frac{1}{2} h^2 + \hat{|h|^2} + R_{NN} \right) - \Delta a,$$

$$\frac{dh_{AB}}{dr} = -a \left( h h_{AB} + S_{AB} \right) - \nabla_A \nabla_B a + \frac{1}{2} \delta_{AB} \Delta a,$$

$$\Delta h_{AB} = \frac{1}{2} \nabla_A h + P_A,$$

where $P_A = \text{Ric}_{AN}, S_{AB}$ is the restriction of Ric on the leaves. These provide with the integral estimates for $H = \frac{1}{2} (\hat{h}^2 + h^2)$:

$$\frac{d}{dr} \int_{B_r} \zeta^2 H^2 + \int_{B_r} \zeta^2 (4a - 1 - |\text{Ric}|) h H \leq c \int_{B_r} \zeta^2 \left( \frac{1}{4} |\nabla a|^2 + \frac{1}{2} |\nabla h|^2 \right)$$

We assume now bounds of the form:

$$\int_{B_j} \zeta^2 |\nabla a|^2 \leq \gamma_a \int_{B_j} \zeta^2 a^2$$

and

$$\int_{B_j} \zeta^2 |\nabla h|^2 \leq \gamma_h \int_{B_j} \zeta^2 h^2$$

and obtain after integration that

$$\left( \int_{B_j} \zeta^2 H^2 \right) (r) \leq \left( \int_{B_j} \zeta^2 H^2 \right) (0) e^{\int_0^r g(r') dr'} + \left( \int_{B_j} \zeta^2 d^2 e^{\int_0^r g(r') dr''} \right)$$

where

$$g(r) = \frac{1}{2} \sup_{B_j} |a - 1 - |\text{Ric}| - \gamma_h|, \quad a(r) = \int_{B_j} \zeta^2 |\nabla a|^2$$

Appendix A. Harmonic Approximation

We approximate harmonically functions $f$ solving the boundary value problem:

$$\Delta \tilde{f} = 0, \quad \tilde{f}|_{\partial \Omega_{[0,1]} } = f$$

We will use then the initial form of this $\tilde{f}$. We recall Lojasiewicz inequality for analytic functions near their critical value $\tilde{f}_c$:

$$|\nabla \tilde{f}| > c_2 |\tilde{f} - \tilde{f}_c|^{1-\mu}$$

for some suitable rational $\mu < 1$. 
Appendix B. Generalized Hardy inequality

We present here the basic ingredient for the derivation of the local estimates: the generalized Hardy’s inequality. In the estimates of [9] we use as homogeneous polynomial the initial form of the harmonic approximation of the weight \( \beta \), \( P = \tilde{\beta}_0 \).

Let then \( P \) be a homogeneous polynomial of degree \( m \). Then we introduce the quantities

\[
H^0(P) = P^{-\frac{2}{m}}, \quad H^1(P) = \frac{\|\nabla P\|^2}{P^2}, \quad H^2(P) = \left| \frac{\Delta P}{P} \right|
\]

If \( f \in C_0^\infty(\{x \in \mathbb{R}^3 \mid \{P = 0\}\}) \) then

\[
\int_{I_3} H^j(P) f^2 \leq c_j(P) \int_{I_3} |\nabla f|^2
\]

We start with Lagrange identity

\[
\frac{\|\nabla P\|^2}{P^2} = \frac{|x|^2 \|\nabla P\|^2}{|x|^2 P^2} = \frac{(x \cdot \nabla P)^2}{P^2} - \frac{1}{r^2 R^2} \sum_{i<j} \left( x_i \frac{\partial R}{\partial x_j} - x_j \frac{\partial R}{\partial x_i} \right)^2
\]

and we recall that

\[
\Omega_{ij} = \frac{1}{r} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)
\]

generate the rotation group \( SO(3, \mathbb{R}) \) and \( \mathbb{R}^3 \) and are tangent to \( S^2 \). We introduce coordinates in the conical sector

\[
K_1 = \{ x \in \mathbb{R}^3 / |x_3| \geq \frac{e}{1 + e} |x| \}
\]

then

\[
x_1 = \frac{r \xi_1}{\sqrt{1 + |\xi|^2}}, \quad x_2 = \frac{r \xi_2}{\sqrt{1 + |\xi|^2}}, \quad x_3 = \frac{r}{\sqrt{1 + |\xi|^2}}, \quad \xi = (\xi_1, \xi_2)
\]

Notice that \( \|\nabla P\|^2 \) is a homogeneous polynomial of degree \( 2m - 2 \) and

\[
\|\nabla P\|^2 = r^{2(m-1)} \left( m^2 R^2 + r^2 |\mathcal{R}|^2 \right) = P^2 \left( \frac{m^2}{r^2} + \frac{|\mathcal{R}|}{R} \right)
\]

where

\[
\mathcal{R} = \mathcal{G} - (\eta \cdot \nabla \mathcal{G}) \eta
\]

We set that

\[
P(x) = r^m R(\xi), \quad R(x) = \frac{1}{(1 + |\xi|^2)^{\frac{m}{2}}} \tilde{R}(\xi)
\]

that:

\[
\left| \frac{\nabla P}{P} \right|^2 = \frac{m^2}{r^2} + |\xi|^2 |\mathcal{R}|^2 + (1 + |\xi|^2) (E(R))^2, \quad E(R) = \xi \cdot \mathcal{R}
\]

Lojasiewicz inequality suggests that for rational numbers \( \ell_0 = 1, \ell_1, \cdots, \ell_N \), positive constants \( c_0, \ldots, c_N, \epsilon_0 < 1, \ldots, \epsilon_N < 1 \), and critical values \( R_0, \ldots, R_N \):

\[
|\mathcal{R}| \geq \sum_{j=1}^N c_j |R - R_j|^{1-\ell_j}
\]

if we localize in neighbourhoods:

\[
|R| \geq \epsilon_j |R_j|
\]
then
\[ |\nabla R| \geq \sum_{j=0}^N c_j (1 - \epsilon_j)^{1-\ell_j} |R|^{1-\ell_j} \]

For finite number of points \( \eta_1, \ldots, \eta_N \) and for natural numbers \( \mu_1, \ldots, \mu_N, \mu_j \leq m - 1 \):
\[ \chi_j^2 |\nabla R|^2 \geq c_j |\eta - \eta_j|^{2\mu_j} \]

We compute that
\[ \int_{I_3} P^{-\frac{2}{m}} f^2 = \int_{I_3} \frac{P^{2(1 - \frac{1}{m})}}{|\nabla P|^2} H^1(P) f^2 \]

and observe that
\[ M(\varepsilon) = \frac{P^{2(1 - \frac{1}{m})}}{|\nabla P|^2} \]

is a real analytic function homogeneous of degree 0. The preceding estimates combined with Young’s inequality for \( q = m\ell_j, p = \frac{q}{q-1} \) assert that
\[ M(\varepsilon) \leq \sum_{j=1}^N \frac{\chi_j^2}{(pm^2)^{\frac{1}{r}} (c_j q) \frac{r}{q}} \frac{\partial^2}{\partial x^2} \]

Therefore we reduce the inequality to the one for \( H^1(P) \) thanks to the \( r^2 \) factor for the volume in spherical coordinates. Integration by parts reduces the inequality for \( H^2(P) \) to the one for \( H^1(P) \). Applying Lojasiewicz inequality on the round sphere we deduce that:
\[ \chi_j |R| \leq \frac{\chi_j}{c_j} |\nabla R|^{1-\ell_j} \]

and hence we have that:
\[ \chi_j \left| \frac{\nabla R}{R} \right|^2 \leq |\eta - \eta_j|^{2\mu_j} \]

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