Canonical basis linearity regions arising from quiver representations

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Abstract

In this paper we show that there is a link between the combinatorics of the canonical basis of a quantized enveloping algebra and the monomial bases of the second author [21] arising from representations of quivers. We prove that some reparametrization functions of the canonical basis, arising from the link between Lusztig’s approach to the canonical basis and the string parametrization of the canonical basis, are given on a large region of linearity by linear functions arising from these monomial basis for a quantized enveloping algebra.

Keywords: Quantum group, Lie algebra, canonical basis, parametrization functions, monomial basis, representations of quivers, degenerations, piecewise-linear functions.

1 Introduction

Let $U = U_q(g)$ be the quantum group associated to a semisimple Lie algebra $g$ of rank $n$. The negative part $U^-$ of $U$ has a canonical basis $B$ with favourable properties (see Kashiwara [5] and Lusztig [13, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight $U$-modules.

The reparametrization functions of the canonical basis studied in this paper arise from considering two parametrizations of the canonical basis, both dependent on the choice of a reduced expression for the longest element $w_0$ in the corresponding Weyl group compatible with a quiver. The first parametrization arises from correspondence between the canonical basis and a basis of PBW type,
and the second is the string parametrization (see [4, §2], the end of Section 2 in [18] and [8]). The reparametrization functions are useful in relating these two parametrizations, and in [5] they were shown to have applications to the description of the tensor product multiplicities of simple \( g \)-modules in an approach involving totally positive varieties. They were also shown to be closely connected with the description of the canonical basis in [6] and [17].

Our approach to these functions differs from that of Berenstein and Zelevinsky [5] in that we make the linearity of these functions on certain regions explicit, and that we show how they are linked with the representation theory of quivers. In particular, we show that, for reduced expressions compatible with quivers, they can be described, on certain cones (which we call degeneration cones), as linear functions arising from the monomial basis of \( U^- \) of the second author [21], which was found using properties of quiver representations, and in particular, degenerations of representations.

We show that the degeneration cones contain the Lusztig cones of [14]. Thus the reparametrization functions mentioned above, when restricted to Lusztig cones, can be described using quiver theory. Lusztig cones have been shown to have many interpretations and connections. For example, they are used to describe regular functions on a reduced real double Bruhat cell of the corresponding algebraic group [22], they have links with primitive elements in the dual canonical basis (this can be seen using [4]) and therefore with the representation theory of affine Hecke algebras [8], they are known to correspond to regions of linearity of the Lusztig reparametrization functions and tight monomials (see [4]), and they can be described using the homological algebra of representations of quivers [4].

The structure of the paper is as follows. In §2 we recall the parametrizations of the canonical basis we will need, and in §3 we recall the connection between reduced expressions for the longest word in the Weyl group and quivers. In §4 we recall the monomial basis and corresponding linear functions defined in [21], as well as showing that the functions we will use are invertible. In §5 we recall the Lusztig cones of [14], and in §6 we define the degeneration cones referred to above. In §7 we define a set of cones in the PBW parametrization, which we show in §9 correspond the Lusztig cones in appropriate string parametrizations; in §8 we show that each cone in §7 is contained in the corresponding degeneration cone. Finally, in section §10 we show that the linear functions arising in [21] coincide with the reparametrization functions on the degeneration cones, using the description of the operators of Kashiwara given in [18].

## 2 Parametrizations of the canonical basis

Let \( g \) be the simple Lie algebra over \( \mathbb{C} \) of type \( A_n \) and \( U \) be the quantized enveloping algebra of \( g \). Then \( U \) is a \( \mathbb{Q}(v) \)-algebra generated by the elements \( E_i, F_i, K_\mu, i \in \{1,2,\ldots,n\}, \mu \in \mathbb{Q} \), the root lattice of \( g \). Let \( U^+ \) be the subalgebra generated by the \( E_i \) and \( U^- \) the subalgebra generated by
the $F_i$.

Let $W$ be the Weyl group of $g$. It has a unique element $w_0$ of maximal length. For each reduced expression $i$ for $w_0$ there are two parametrizations of the canonical basis $B$ for $U^-$. The first arises from Lusztig’s approach to the canonical basis [3, §14.4.6], and the second arises from Kashiwara’s approach [5].

**Lusztig’s Approach**

There is an $\mathbb{Q}$-algebra automorphism of $U$ which takes each $E_i$ to $F_i$, $F_i$ to $E_i$, $K_\mu$ to $K_{-\mu}$ and $v$ to $v^{-1}$. We use this automorphism to transfer Lusztig’s definition of the canonical basis in [10, §3] to $U^-$.

Let $T_i$, $i = 1, 2, \ldots, n$, be the automorphism of $U$ as in [12, §1.3] given by:

$$T_i(E_j) = \begin{cases} 
-F_j K_j, & \text{if } i = j, \\
E_j, & \text{if } |i - j| > 1, \\
-E_i E_j + v^{-1} E_j E_i & \text{if } |i - j| = 1
\end{cases}$$

$$T_i(F_j) = \begin{cases} 
-K_j^{-1} E_j, & \text{if } i = j, \\
F_j, & \text{if } |i - j| > 1, \\
-F_j F_i + v F_i F_j & \text{if } |i - j| = 1
\end{cases}$$

$$T_i(K_\mu) = K_{-\langle \mu, \alpha_i \rangle h_i}, \text{ for } \mu \in \mathbb{Q},$$

where the $\alpha_i$ are the simple roots and the $h_i$ are the simple coroots of $g$.

For each $i$, let $r_i$ be the automorphism of $U$ which fixes $E_j$ and $F_j$ for $j = i$ or $|i - j| > 1$ and fixes $K_\mu$ for all $\mu$, and which takes $E_j$ to $-E_j$ and $F_j$ to $-F_j$ if $|i - j| = 1$. Let $T_i'' = T_i r_i$ be the automorphism of $U$ as in [3, §37.1.3]. Let $c \in \mathbb{N}^N$, where $N = \ell(w_0)$, and $i$ be a reduced expression for $w_0$. Let

$$F_i^c := F_i^{(c_1)} T_{i_1}'' T_{i_2}'' \cdots T_{i_{\ell(c_2)}}'' T_{i_{\ell(c_2)} - 1}'' \cdots T_{i_{\ell(c_N)} - 1}''(F_i^{(c_N)}).$$

Define $B_i = \{ F_i^c : c \in \mathbb{N}^N \}$. Then $B_i$ is the basis of PBW-type corresponding to the reduced expression $i$. Note that, if the reduced expression $i$ is adapted to a quiver in the sense of [10], then this basis can also be constructed using the Hall algebra approach of [19]. Let $^-$ be the $\mathbb{Q}$-algebra automorphism from $U$ to $U$ taking $E_i$ to $E_i$, $F_i$ to $F_i$, and $K_\mu$ to $K_{-\mu}$, for each $i \in [1, n]$ and $\mu \in \mathbb{Q}$, and $v$ to $v^{-1}$. Lusztig proves the following result in [10, §§2.3, 3.2].
Theorem 2.1 (Lusztig)
The \( \mathbb{Z}[v]-\)span \( \mathcal{L} \) of \( B_i \) is independent of \( i \). Let \( \pi : \mathcal{L} \to \mathcal{L}/v\mathcal{L} \) be the natural projection. The image \( \pi(B_i) \) is also independent of \( i \); we denote it by \( B \). The restriction of \( \pi \) to \( \mathcal{L} \cap \overline{\mathcal{L}} \) is an isomorphism of \( \mathbb{Z} \)-modules \( \pi_i : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/v\mathcal{L} \). Also \( B = \pi_1^{-1}(B) \) is a \( \mathbb{Q}(v) \)-basis of \( U^{-} \), which is the canonical basis of \( U^{-} \).

Lusztig’s theorem provides us with a parametrization of \( B \), dependent on \( i \). If \( b \in B \), we write \( \phi_i(b) = c \), where \( c \in \mathbb{N}^N \) satisfies \( b \equiv F_i \mod v\mathcal{L} \). Note that \( \phi_i \) is a bijection.

For reduced expressions \( j \) and \( j' \) for \( w_0 \), Lusztig defines in [10, §2.6] a reparametrization function \( R_j^j : \mathbb{N}^N \to \mathbb{N}^N \). This function was shown by Lusztig to be piecewise linear and its regions of linearity were shown to have significance for the canonical basis, in the sense that elements \( b \) of the canonical basis with \( \phi_j(b) \) in the same region of linearity of \( R_j^j \) often have similar form. For example, this can be seen from the explicit descriptions of the canonical basis of type \( A_2 \) and \( A_3 \), as computed by Lusztig in [10] and by Xi in [24], respectively. More evidence for the importance of these regions is their connection with the multiplicativity properties of dual canonical bases.

The string parametrization

Let \( \tilde{E}_i \) and \( \tilde{F}_i \) be the Kashiwara operators on \( U^{-} \) as defined in [2, §3.5]. Let \( A \subseteq \mathbb{Q}(v) \) be the subring of elements regular at \( v = 0 \), and let \( \mathcal{L}' \) be the \( A \)-lattice spanned by arbitrary products \( \tilde{F}_{j_1} \tilde{F}_{j_2} \cdots \tilde{F}_{j_m} \cdot 1 \) in \( U^{-} \). We denote the set of all such elements by \( S \). The following results were proved by Kashiwara in [2].

Theorem 2.2 (Kashiwara)
(i) Let \( \pi' : \mathcal{L}' \to \mathcal{L}'/v\mathcal{L}' \) be the natural projection, and let \( B' = \pi'(S) \). Then \( B' \) is a \( \mathbb{Q} \)-basis of \( \mathcal{L}'/v\mathcal{L}' \) (the crystal basis).
(ii) Furthermore, \( \tilde{E}_i \) and \( \tilde{F}_i \) each preserve \( \mathcal{L}' \) and thus act on \( \mathcal{L}'/v\mathcal{L}' \). They satisfy \( \tilde{E}_i(B') \subseteq B' \cup \{0\} \) and \( \tilde{F}_i(B') \subseteq B' \). Also for \( b, b' \in B' \) we have \( \tilde{F}_i b = b' \), if and only if \( \tilde{E}_i b' = b \).
(iii) For each \( b \in B' \), there is a unique element \( \tilde{b} \in \mathcal{L}' \cap \overline{\mathcal{L}'} \) such that \( \pi'(\tilde{b}) = b \). The set of elements \( \{ \tilde{b} : b \in B' \} \) forms a basis of \( U^{-} \), the global crystal basis of \( U^{-} \).

It was shown by Lusztig [2, 2.3] that the global crystal basis of Kashiwara coincides with the canonical basis of \( U^{-} \).

There is a parametrization of \( B \) arising from Kashiwara’s approach, again dependent on a reduced expression \( i \) for \( w_0 \). Let \( i = (i_1, i_2, \ldots, i_k) \) and \( b \in B \). Let \( a_1 \) be maximal such that \( \tilde{E}_{i_{a_1}} b \not\equiv 0 \mod v\mathcal{L}' \); let \( a_2 \) be maximal such that \( \tilde{E}_{i_{a_2}} \tilde{E}_{i_{a_1}} b \not\equiv 0 \mod v\mathcal{L}' \), and so on, so that \( a_N \) is maximal such that
\[
\tilde{E}_{i_{a_N}} \tilde{E}_{i_{a_{N-1}}} \cdots \tilde{E}_{i_{a_2}} \tilde{E}_{i_{a_1}} b \not\equiv 0 \mod v\mathcal{L}'.
\]
Let \(a = (a_1, a_2, \ldots, a_N)\). We write \(\psi_i(b) = a\). This is the crystal string of \(b\) — see [1, §2] and the end of Section 2 in [8]; see also [5]. It is known that \(\psi_i(b)\) uniquely determines \(b \in B\) (see [18, §2.5]). We have \(b \equiv F_{i_1}^{a_1} F_{i_2}^{a_2} \cdots F_{i_N}^{a_N} \cdot 1 \mod vL'\). The image of \(\psi_1\) is a cone which appears in [4]. We shall call this the string cone \(X_{st}(i) = \psi_1(B)\).

We next consider a function which links the string parametrization with the parametrization arising from Lusztig’s approach. For reduced expressions \(i\) and \(j\) for \(w_0\), consider the maps

\[
X_{st}(i) \longrightarrow B \rightarrow \mathbb{N}^N
\]

We define \(S_i^j = \phi_j^{-1} : X_{st}(i) \rightarrow \mathbb{N}^N\), a reparametrization function. This function has appeared in, for example, [4].

**Remark on Notation:**

We shall use the following notation convention. A cone \(C\) which is to be regarded as a subset of \(X_{st}\), i.e. to be thought of as a set of strings, will be given the subscript “st” to denote this, thus \(C_{st}\). A cone \(C\) which is to be regarded as a subset of \(\mathbb{N}^N\), and is to be regarded as a set of PBW parameters for the canonical basis, will be denoted with the subscript “PBW”, thus \(C_{PBW}\). In each case, it will be clear from the context which reduced expression for \(w_0\) is being used.

### 3 Reduced Expressions Compatible with Quivers

We now fix the type of \(g\) once and for all to be type \(A_n\). However, a lot of the results we use hold in a greater generality, and we also believe that the results proved here also hold in greater generality (at least for the simply-laced Dynkin case). In this section we shall recall an explicit description of reduced expressions compatible with quivers in type \(A_n\), which we shall need in studying such expressions. We shall write a reduced expression \(s_{i_1} s_{i_2} \cdots s_{i_N}\) for \(w_0\) as the \(N\)-tuple \(i = (i_1, i_2, \ldots, i_N)\). Given two such reduced expressions \(i\) and \(i'\), we write \(i \sim i'\) if there is a sequence of commutations (of the form \(s_i s_j = s_j s_i\) with \(|i - j| > 1\)) which, when applied to \(i\), give \(i'\). This is an equivalence relation on the set of reduced expressions for \(w_0\), and the equivalence classes are called commutation classes.

Let \(Q = (Q_0, Q_1, s, t)\) be a quiver, i.e. a finite oriented graph with set of vertices \(Q_0\), set of arrows \(Q_1\), and maps \(s, t : Q_1 \rightarrow Q_0\) associating to each arrow its source and target, respectively. If \(Q\) is a quiver and \(i\) is a sink in \(Q\) (i.e. all the arrows incident with \(i\) have \(i\) as target), we denote by \(s_i(Q)\) the quiver with all of the arrows incident with \(i\) reversed. If \(i\) is a reduced expression for \(w_0\), we say (following Lusztig [10]) that \(i\) is compatible with \(Q\) if \(i_1\) is a sink in \(Q\), \(i_2\) is a sink in \(s_{i_1}(Q)\), \(i_3\) is a sink in \(s_{i_2} s_{i_1}(Q)\), \ldots, \(i_N\) is a sink in \(s_{i_{N-1}} s_{i_{N-2}} \cdots s_{i_1}(Q)\). It is known that if \(Q\) is any quiver of
type $A_n$, then there is always at least one reduced expression $i$ compatible with $Q$, and that the set of reduced expressions compatible with $Q$ is the commutation class of $i$.

Berenstein, Fomin and Zelevinsky give a nice description of a reduced expression compatible with a given quiver $Q$ in type $A_n$ in [3, §4.4.3]. Suppose that $Q$ is such a quiver. Number the edges of the quiver from 1 to $n-1$, starting from the left hand end. Berenstein, Fomin and Zelevinsky construct an arrangement as follows. Consider a square in the plane, with horizontal and vertical sides. We will draw $n+1$ ‘pseudo-lines’ in this square. Put $n+1$ points onto the left-hand edge of the square, equally spaced, numbered 1 to $n+1$ from top to bottom, so that 1 and $n+1$ are at the corners. Do the same for the right-hand edge, but number the points from bottom to top. Line$_h$ will join point $h$ on the left with point $h$ on the right. For $h = 1, n+1$, Line$_h$ will be a diagonal of the square. For $h \in [2, n]$, Line$_h$ will be a union of two line segments of slopes $\pi/4$ and $-\pi/4$. There are therefore precisely two possibilities for Line$_h$. If edge $h-1$ in $Q$ is oriented to the left, the left segment has positive slope, while the right one has negative slope; if edge $h-1$ is oriented to the right, it goes the other way round.

**Example:** Berenstein, Fomin and Zelevinsky give the following example. Consider the case $A_5$, with $Q$ given by the quiver with vertices 1, 2, 3, 4, 5 and arrows $1 \rightarrow 2$, $2 \leftarrow 3$, $3 \rightarrow 4$, and $4 \leftarrow 5$. (We shall denote such a quiver by $RLRL$, where an $R$ (respectively, $L$) denotes an edge oriented to the right (respectively, left).) The Berenstein-Fomin-Zelevinsky arrangement for this quiver is shown in Figure 1.

![Figure 1: The Berenstein, Fomin and Zelevinsky diagram for the quiver $RLRL$](image)

If $i$ is a reduced expression for $w_0$ in type $A_n$, then the chamber diagram for $i$ is given by a set of pseudolines numbered from 1 to $n$. Two sets of points numbered 1 to $n$ are arranged on the vertical lines of a square as in the above picture. Underneath the square, the simple reflections in $i$ are written from left to right. The $i$th pseudoline then links the point marked $i$ on the left with the
point marked \( i \) on the right, in such a way that immediately above the simple reflection \( i_j \) from \( i \) pseudolines \( i_j \) and \( i_j + 1 \) cross. See \[3, 1.4\] for details. Berenstein, Fomin and Zelevinsky prove the following result.

**Proposition 3.1** (Berenstein, Fomin and Zelevinsky) A reduced expression \( i \) for \( w_0 \) is compatible with the quiver \( Q \) if and only if its chamber diagram is isotopic to the arrangement defined above corresponding to \( Q \).

**Proof:** see \[3, §4.4.3\]. □

Since each simple reflection appearing in such a reduced expression \( i \) corresponds to a crossing of two pseudo-lines in the diagram, each pseudo-line in the diagram (consisting of a part of positive slope and a part of negative slope) gives rise to some of the simple reflections appearing in \( i \); each simple reflection appears twice in this way as two lines crossing correspond to a simple reflection. We can remove this duplication by counting, for each line, only the simple reflections which arise during the part of the line which is of positive slope. In this way, each edge of the quiver, which corresponds to a line in the diagram, gives rise to some of the simple reflections in the reduced expression \( i \); we include also the line from the bottom left of the diagram to the top right. Each simple reflection arises for a unique edge of the quiver (or comes from the extra line).

Number the edges of \( Q \) from 1 to \( n - 1 \), starting from the left. Suppose that the edges \( l_1, l_2, \ldots, l_a \) all point to the left, and that edges \( r_1, r_2, \ldots, r_b \) all point to the right, and that every edge is one of these, where \( l_1 < l_2 < \cdots < l_a \) and \( r_1 < r_2 < \cdots < r_b \). For \( m \in \mathbb{N} \), denote by \((m \downarrow 1)\) the sequence \( m, m-1, \ldots, 2, 1 \). Then it is easy to see that the above construction shows that the reduced expression

\[
i(Q) = (l_1 \downarrow 1)(l_2 \downarrow 1) \cdots (l_a \downarrow 1)(n \downarrow 1)(n \downarrow n + 1 - r_b)(n \downarrow n + 1 - r_{b-1}) \cdots (n \downarrow n + 1 - r_1)
\]

is compatible with \( Q \); it follows that the reduced expressions compatible with \( Q \) are precisely those commutation equivalent to \( i(Q) \).

### 4 Monomial Bases arising from Representations of Quivers

Let \( Q = (Q_0, Q_1, s, t) \) be a quiver. We assume \( Q \) to be of Dynkin type, which means that the unoriented graph \( \Delta \) underlying \( Q \) is a disjoint union of Dynkin diagrams of type \( A, D, E \). Let \( k \) be an arbitrary field. Then we can form the path algebra \( kQ \) of \( Q \) over \( k \), which has the paths in \( Q \) as a \( k \)-basis (including an ‘empty’ path for each vertex \( i \in Q_0 \)), and multiplication of paths given by concatenation if possible, and zero otherwise. This is a finite dimensional \( k \)-algebra since \( Q \), being of Dynkin type, has no oriented cycles. We form the category \( \text{mod}kQ \) of finite-dimensional
representations of $kQ$. The isoclasses of simple objects $S_i$ in $\text{mod} \ kQ$ correspond bijectively to the vertices $i \in Q_0$ of $Q$. Let $\mathbb{N}Q$ be the free abelian semigroup spanned by elements $\alpha_i$ for $i \in Q_0$; it can be identified with the positive root lattice of the root system $R$ of type $\Delta$. For a representation $M \in \text{mod} \ kQ$, we denote by $d_i$ for $i \in Q_0$ the Jordan-Hölder multiplicity of the simple $S_i$ in $M$. This allows us to define a map $\dim$ from the set of isoclasses in $\text{mod} \ kQ$ to $\mathbb{N}Q$ by $\dim(M) = \sum_{i \in Q_0} d_i \alpha_i$.

The fundamental result in the theory of Dynkin quivers is:

**Theorem 4.1 (Gabriel)**

The map $\dim$ induces a bijection between isoclasses of indecomposable objects $X_\alpha$ in $\text{mod} kQ$ and positive roots $\alpha \in R^+ \subset \mathbb{N}Q$ for type $\Delta$.

The Auslander-Reiten quiver of the algebra $kQ$ is defined as the oriented graph having the isoclasses of indecomposable representations of $kQ$ as vertices, and arrows corresponding to irreducible maps (i.e. morphisms in $\text{mod} kQ$ between indecomposable objects which cannot be factored into a composition of non-split maps); see [1] for details. An explicit construction will be given in section 4.

The following definition was first introduced in [21]:

**Definition 4.2** A partition $R^+ = I_1 \cup \ldots \cup I_s$ of $R^+$ into disjoint subsets $I_k$ is called directed if

(i) $\text{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0$ for all $\alpha, \beta$ in the same part $I_k$,
(ii) $\text{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0$ and $\text{Hom}_{kQ}(X_\beta, X_\alpha) = 0$ if $\alpha \in I_k$, $\beta \in I_l$, where $1 \leq k < l \leq s$.

The existence of (several!) directed partitions of a given quiver $Q$ can be seen using Auslander-Reiten theory (see [21]): if $\text{Hom}_{kQ}(U, V) \neq 0$ (resp. $\text{Ext}^1_{kQ}(V, U) \neq 0$) for indecomposables $U, V \in \text{mod} kQ$, then there exists a path (resp. a proper path) from $[U]$ to $[V]$ in the Auslander-Reiten quiver of $kQ$.

But since this graph is directed, we can enumerate the isoclasses of indecomposables in $\text{mod} kQ$ as $[U_1], \ldots, [U_{\nu}]$, such that $\text{Hom}_{kQ}(U_p, U_q) = 0$ for $p > q$, and $\text{Ext}^1_{kQ}(U_p, U_q) = 0$ for $p \leq q$. Define roots $\alpha^p$ by $[U_p] = [X_{\alpha^p}]$. By definition, the partition $R^+ = \{\alpha^1\} \cup \ldots \cup \{\alpha^s\}$ is directed. All other directed partitions can be constructed by coarsening such a partition.

Fix a directed partition $R^+ = I_1 \cup \ldots \cup I_s$ from now on. We will associate to it a sequence $i = i_1 \ldots i_\ell$, as well as a function $D$ from the set of isoclasses in $\text{mod} kQ$ to $\mathbb{N}^s$. Enumerate the vertices $Q_0$ of $Q$ as $Q_0 = \{1, \ldots, n\}$ such that the existence of an arrow $i \rightarrow j$ in $Q$ implies $i < j$. For each $p = 1, \ldots, s$, write the subset of $Q_0$ consisting of all vertices $i$ such that $\alpha_i$ appears with non-zero coefficient in some root $\alpha \in I_p$ as $\{i_1^p, \ldots, i_{\nu^p}^p\}$, increasing with respect to the above defined ordering on $Q_0$. Then the sequence $i$ is defined as

$$i = i_1^1 \ldots i_{\ell_1}^1 \ldots i_1^2 \ldots i_{\ell_2}^2 \ldots i_1^s \ldots i_{\ell_s}^s.$$
Given an isoclass \([M]\) in mod\(kQ\), we can write \(M = \bigoplus_{\alpha \in R^+} X_{\alpha}^c\), using Gabriel’s Theorem and Krull-Schmidt. Write \(s \in \alpha\) to indicate that the simple root \(\alpha_s\) appears in the expression for \(\alpha\) as a sum of simple roots. Let \(D(M)\) be the tuple \(a = (a_j)_{j=1,2,\ldots,k}\) defined as follows. If \(\alpha^j\) lies in \(I_p\), then

\[
a_j = \sum_{\alpha \in I_p, \ i_j \in \alpha} c_{\alpha}.
\]

This defines a function \(D\) from the set of isoclasses in mod\(kQ\) to \(\mathbb{N}^N\), which is obviously additive, i.e. \(D(M \oplus N) = D(M) + D(N)\). Identifying the set of isoclasses in mod\(kQ\) with \(NR^+\) via \(\bigoplus_{\alpha} X_{\alpha}^c \mapsto \sum_{\alpha} c_{\alpha} \alpha\), we thus get a linear function \(D : \mathbb{Z}R^+ \to \mathbb{Z}^l\). The main result of this paper is that, for suitable choices of \(Q\) and the directed partition, the function \(D\) coincides with one of the reparametrization functions \((S_i^j)^{-1}\) on an explicitly described region.

The original use of the function \(D\) lies in the following theorem (see [21]):

**Theorem 4.3 (Reineke)** Writing \(i = i_1 \ldots i_t\) and \(D = (D_1, \ldots, D_t)\), the set

\[
\{F_{i_{i_1}}^{D_1(c)}, \ldots, F_{i_{i_t}}^{D_t(c)} \in U^- : c \in NR^+\}
\]

is a basis for \(U^-\).

Moreover, these monomial bases for \(U^-\) have good properties with respect to base change: the base change coefficients to a PBW basis (resp. to the canonical basis) form upper unitriangular matrices with respect to a certain ordering (namely, related to the degeneration ordering on quiver representations), and these coefficients have representation theoretic (resp. geometric) interpretations (see [21], [22]).

The function \(D\) is not invertible in general, but it is for special directed partitions, called regular in [21]. We now introduce a certain class of directed partitions for quivers of type \(A\), and compute the inverse of \(D\) in these special cases.

We thus assume that \(Q\) is of type \(A_n\), and we fix once and for all the reduced expression for \(w_0\) given by \(k = (1,2,1,3,2,1,\ldots,n,n-1,\ldots,1)\). This is one of the most regular reduced expressions, and is compatible with the quiver \(Q_k\) given in Figure 2. We conjecture that our results hold, with similar proofs, if this quiver is replaced by an arbitrary Dynkin quiver of type \(A_n\), but the Kashiwara operators are in this general case more difficult to describe combinatorially. In the special case we are considering, it is possible to handle this combinatorics. See [19] for a description of the combinatorics of Kashiwara operators on canonical basis elements using parametrizations of the canonical basis arising from PBW-bases of \(U^-\) corresponding to reduced expressions compatible with arbitrary quivers.

We can construct the Auslander-Reiten quiver of our quiver \(Q\) in the following way (see [1]). Let \(NQ\) be the quiver with vertices \(N \times \{1,2,\ldots,n\}\). Whenever there is an arrow \(i \to j\) in \(Q\), we draw
one arrow \((z, i) \to (z, j)\) and one arrow \((z, j) \to (z + 1, i)\) for each \(z \in \mathbb{N}\). Define \(A(Q)\) to be the full subquiver of \(\mathbb{N}Q\) consisting of all vertices \((z, i)\) such that \(1 \leq z \leq (h + a_i - b_i)/2\) where, for each \(i \in \{1, 2, \ldots, n\}\), \(a_i\) (respectively \(b_i\)) is the number of arrows in the unoriented path from \(i\) to \(\sigma(i)\) that are directed towards \(i\) (respectively \(\sigma(i)\)). Here, \(\sigma\) is the unique permutation of the vertices of \(Q\) such that \(w_0(\alpha_i) = -\alpha(\sigma(i))\), and \(h\) is the Coxeter number. Then \(A(Q)\) is the Auslander-Reiten quiver of \(Q\).

A reduced expression \(i\) for \(w_0\) defines an ordering on the set \(\Phi^+\) of positive roots of the root system associated to \(W\). We write \(\alpha^j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})\) for \(j = 1, 2, \ldots, N\). Then \(\Phi^+ = \{\alpha^1, \alpha^2, \ldots, \alpha^N\}\). For \(c = (c_1, c_2, \ldots, c_N) \in \mathbb{Z}^N\), write \(c_{\alpha^j} = c_j\). If \(\alpha = \alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j\) with \(i < j\), we also write \(c_{ij}\) for \(c_{\alpha_{i,j}}\).

Let \(\alpha^1, \alpha^2, \ldots, \alpha^N\) be the ordering induced on \(\Phi^+\) by \(k\). We can write an element of \(\mathbb{N}^N\) as \(c = (c_{ij})\), where \(1 \leq i \leq j \leq n\), using the above. The canonical basis is parametrized, via the Lusztig parametrization \(\phi_k : B \to \mathbb{N}^N\). We can write an element of \(\mathbb{N}^N\) as an array based on the Auslander-Reiten quiver for \(Q_k\). We write \(c_{ij}\) in place of the module corresponding to the positive root \(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j\), with \(c_{11}, c_{22}, \ldots, c_{nn}\) on the first row, \(c_{12}, c_{23}, \ldots, c_{n-1,n}\) along the second, interspersing the first row elements, and so on, until \(c_{1,n}\) on the last row. For example, when \(n = 5\), see Figure 3.

Suppose that \(Q'\) is another quiver of type \(A_n\). We can define a directed partition of \(A(Q)\) in the following way. Fix \(z \in \mathbb{N}\), and set \(z_1 = z\). Let \(v_1 := (z_1, 1)\) be a vertex of \(\mathbb{N}Q\). For \(i = 2 \ldots n\), define a vertex \(v_i = (z_i, i)\) of \(\mathbb{N}Q\) inductively, as follows. If \(i - 1 \to i\) is an arrow in \(Q'\), then let \(v_i\) be the head of the unique arrow with source \(v_{i-1}\). If \(i - 1 \leftarrow i\) is an arrow in \(Q'\), then let \(v_i\) be the source of the unique arrow with head \(v_{i-1}\). Let \(S_z = (v_1, v_2, \ldots, v_n) \subseteq \mathbb{N}Q\). Then it is clear that \(\mathbb{N}Q = \cup_{z \in \mathbb{N}} S_z\). It follows that \(A(Q) = \cup_{z \in \mathbb{N}} (S_z \cap A(Q))\); note that this decomposition must be finite. It is clear from the construction that this is a directed partition of \(A(Q)\). Let \(T_z = S_z \cap A(Q)\). We call each \(S_z\) a slice of \(\mathbb{N}Q\). If \(T_z\) is non-empty, we call it a slice of \(A(Q)\).

**Example:** Consider the case \(A_5\), with \(Q' = RLRL\). The corresponding decomposition of \(A(Q)\) into slices is given in Figure 4. Each vertex of \(A(Q)\) is denoted by a number, indicating the number of the slice it lies in.

![Figure 2: The quiver compatible with k](image-url)

![Figure 3: Array of elements of N^{15}](image-url)
Let us now specialise to the case where $Q = Q_k$ is the quiver with arrows $i \leftarrow i + 1$ for $i = 1, 2, \ldots, n - 1$, compatible with $k$. Let $Q'$ be an arbitrary quiver. Note that we can identify $(i, j)$, for $1 \leq i \leq j \leq n$, with the indecomposable module with dimension vector $\alpha_i + \cdots + \alpha_j$, so each such pair lies in a corresponding slice $T_z$. If $v = (z, a) \in A(Q)$, let $i(v), j(v)$ be such that the dimension vector of the module at vertex $v$ is $\alpha_{i(v)} + \cdots + \alpha_{j(v)}$.

Recall the function $D : \mathbb{Z}^R^+ \to \mathbb{Z}^k$ associated in section 4 to a quiver and a directed partition. Consider the special case where $Q = Q_k$, and where the directed partition is associated to another quiver $Q'$ of type $A$ as above. In this case it is easy to see that the sequence $i$ is given by

$$i(Q') = (l_1 \downarrow 1)(l_2 \downarrow 1) \cdots (l_a \downarrow 1)(n \downarrow 1)(n \downarrow n + 1 - r_b)(n \downarrow n + 1 - r_{b-1}) \cdots (n \downarrow n + 1 - r_1),$$

a reduced expression for the longest word in the Weyl group compatible with $Q'$. Here, each of the bracketed parts of $i(Q')$ arises from a part of the directed partition associated to $Q'$.

In this special case, we can easily prove the invertibility of $D$.

**Lemma 4.4** Suppose that $Q = Q_k$, and that $D$ is the function associated to the directed partition of $A(Q)$ corresponding to another quiver $Q'$ of type $A$. Then the function $D$ is invertible with inverse function $E = D^{-1}$. Moreover, the components $c_\alpha$ of $c = (c_\alpha) = E(a)$ for some $a \in \mathbb{N}^N$ and $\alpha \in R^+$ are of the form $c_\alpha = a_k - a_t$ or $c_\alpha = a_k$.

**Proof:** Recall from Section 4 that we can write

$$i = (i_1^1 \cdots i_1^t \cdots i_s^1 \cdots i_s^t),$$

and that $D(c)$ is the tuple $a = (a_j)_{j=1, 2, \ldots, k}$ such that if the root $\alpha_j$ lies in $I_p$, then

$$a_j = \sum_{\alpha \in I_p, \ i_j \in \alpha} c_\alpha.$$  

In our case, it is easily seen from the definition of the directed partition that we can write

$$I_p = \{\alpha_u^p, \ldots, \alpha_v^p\},$$

such that the root $\alpha_u^p$ is of length $u$, and $\alpha_u^p = \alpha_{u-1}^p + \alpha_{j_u}$ for some simple root $\alpha_{j_u}$ (we formally set $\alpha_0^p = 0$). It follows that $\alpha_u^p = \alpha_{j_1} + \cdots + \alpha_{j_u}$, which in particular implies $t_u' = t_u$. We also see that

$$\{i_1^p, \ldots, i_u^p\} = \{j_1, \ldots, j_u\};$$

Figure 4: Slice structure of $A(Q_k)$ corresponding to the quiver RLRL
denote by $\sigma$ the permutation defined by $j_{\sigma(u)} = i^p_u$. Then we can compute $a^p_u$ as:

$$a^p_u = \sum_{\alpha \in I_p, \ i_u \in \alpha} c_{\alpha} = c_{\alpha_{u, u}} + \ldots + c_{\alpha_{t_p}}.$$

It follows that

$$c_{\alpha_{u, u}} = a^p_{\sigma^{-1}u} - a^p_{\sigma^{-1}(u+1)}$$ if $u \neq t_p$, and $c_{\alpha_{u, u}} = a^p_{\sigma^{-1}u}$ otherwise.

This proves the claimed properties of $D$.

**Remark on Notation**

In the sequel, we shall consider directed partitions in $A(Q)$ (so that, in the above, $Q = Q_k$), arising from an arbitrary quiver $Q$ of type $A_n$ (which was denoted $Q'$ above).

## 5 The Lusztig cones

Lusztig [14] introduced certain regions which, in low rank, give rise to canonical basis elements of a particularly simple form. The Lusztig cone corresponding to a reduced expression $i$ for $w_0$ is defined to be the set of points $a \in \mathbb{N}^N$ satisfying the following inequalities:

(*) For every pair $s, s' \in [1, k]$ with $s < s'$, $i_s = i_{s'} = i$ and $i_p \neq i$ whenever $s < p < s'$, we have

$$\left( \sum_p a_p \right) - a_s - a_{s'} \geq 0,$$

where the sum is over all $p$ with $s < p < s'$ such that $i_p$ is joined to $i$ in the Dynkin diagram. We shall denote this cone by $L_{st}(i)$ (as we shall regard it as a set of strings of $B$ in direction $i$).

It was shown by Lusztig [14] that, in type $A_n$, if $a \in L_{st}(i)$ then the monomial $F_{i_1}^{(a_1)}F_{i_2}^{(a_2)} \ldots F_{i_N}^{(a_N)}$ lies in the canonical basis $B$, provided $n = 1, 2, 3$. The first author [16] showed that this remains true if $n = 4$, but it is false for $n \geq 5$ by [16], [20]. The Lusztig cones have been studied in the papers [6], [15] and [17] in type $A$ for every reduced expression $i$ for the longest word, and have also been studied by Bedard in [8] for arbitrary finite (simply-laced) type for reduced expressions compatible with a quiver whose underlying graph is the Dynkin diagram. Bedard describes these vectors using the Auslander-Reiten quiver of the quiver and homological algebra, showing they are closely connected to the representation theory of the quiver.

## 6 The Degeneration Cones

We define the cone $C_{PBW}(Q) \subseteq \mathbb{N}^N$ corresponding to a quiver $Q$ to be the set of points $c = (c_{ij})$ satisfying the inequalities (C1) and (C2) below. We define the degeneration cone corresponding to
Define a *component* of \( Q \) to be a maximal full subgraph \( X \) of \( Q \) subject to the condition that all of the arrows of \( X \) point in the same direction. Call \( X \) a *left* (respectively, *right*) component of \( Q \) if its arrows all point to the left (respectively, right). If \( X \) is a component of \( Q \) (left or right), let \( S_z(X) \) (respectively, \( T_z(X) \)) denote the part of the slice \( S_z \) (respectively, \( T_z \)) corresponding to \( X \).

**Example:** Consider the example with \( Q = RLRL \) given above. Then \( Q \) has 4 components, each containing one edge. Two are right components, and two are left components. For each component \( X \), we indicate the subsets of slices, \( T_z(X) \), in \( A(Q) \); see Figure 5. In each case, the numbers \( z \) denote elements of the sets \( T_z(X) \), and the empty circles denote elements not in any subset \( T_z(X) \).

Firstly, let \( X \) be a left component of \( Q \), and suppose \( T_z, T_{z+1} \) are consecutive slices in \( A(Q) \) such that \( S_z(X) = T_z(X) \) and \( S_{z+1}(X) = T_{z+1}(X) \) (i.e. the sets \( S_z(X) \) and \( S_{z+1}(X) \) of \( NQ \) are contained entirely inside \( A(Q) \)). We can order \( T_z(X) \) linearly by the second entry of the vertices appearing in it (recall from Section 4 that each vertex is regarded as a pair \((r, i)\), where \( r \in \mathbb{N} \) and \( i \) is a vertex of \( Q \), i.e. an element of \( \{1, 2, \ldots, n\} \)). Write \( T_z(X) = \{x_1, x_2, \ldots x_k\} \). Similarly, we can write \( T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\} \). For \( t = 1 \ldots k \), let \( i_t = i(x_t) \), and let \( j_t = j(x_t) \). Note that \( i(y_t) = i_t + 1 \) and \( j(y_t) = j_t + 1 \). Then inequalities (C1) are:

\[
\sum_{r=a}^{k} c_{i_r, j_r} \geq \sum_{r=a}^{k} c_{i_{r+1}, j_{r+1}} \text{ for } a = 1 \ldots k.
\]

Secondly, let \( X \) be a right component of \( Q \), and suppose \( T_z, T_{z+1} \) are consecutive slices in \( A(Q) \). We can order \( T_z(X) \) linearly by the second component of the vertices appearing in it; write \( T_z(X) = \{x_1, x_2, \ldots x_k\} \). Similarly, we can write \( T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\} \), where \( k \geq l \), since \( X \) is a right component. For \( t = 1 \ldots k \), let \( i_t = i(x_t) \), and let \( j_t = j(x_t) \). Note that \( i(y_t) = i_t + 1 \) and \( j(y_t) = j_t + 1 \).
for $t = 1 \ldots l$. Then inequalities (C2) are:

(C2) $c_{i_r,j_r} \geq c_{i_{r+1},j_{r+1}}$ for $r = 1 \ldots l - 1$.

The cone $C_{PBW}(Q)$ is defined to be the set of $(c_{ij}) \in \mathbb{N}^n$ satisfying all of the inequalities (C1) and (C2).

**Example:** In our running example (see the start of this section), the inequalities (C1) and (C2) are given as follows.

Component 1: $c_{11} \geq c_{22} \geq c_{33} \geq c_{44} \geq c_{55}$.
Component 2: $c_{13} \geq c_{24} \geq c_{35}$ and $c_{13} + c_{23} \geq c_{24} + c_{34} \geq c_{35} + c_{45}$.
Components 3 and 4 give rise to no inequalities in this example.

7 The PBW-version of the Lusztig cones

We now define a second cone $L_{PBW}(Q)$. It will be seen later that this cone is the image, under the reparametrization map $S_{(L)}$, of the Lusztig cone $L_{st}(Q)$. We'll then see that $C_{PBW}(Q)$ (which will be seen to be the PBW-version of the degeneration cone) contains the PBW-version of the Lusztig cone $L_{PBW}(Q)$.

Firstly, let $X$ be a left component of $Q$, and suppose $T_z, T_{z+1}$ are consecutive slices in $A(Q)$. We can order $T_z(X)$ linearly by the second entry of the vertices appearing in it; write $T_z(X) = \{x_1, x_2, \ldots, x_k\}$. Similarly, we can write $T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\}$, where $k \leq l$ (since $X$ is a left component). For $t = 1 \ldots k$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$ for $1 \leq t \leq k$. Then the defining inequalities of $L_{PBW}(Q)$ are

(L1) If $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$ then $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_{r+1},j_{r+1}}$.

(L2) $c_{i_r,j_r} \leq c_{i_{r+1},j_{r+1}}$ for $r = 2 \ldots k - 1$.

Secondly, let $X$ be a right component of $Q$, and suppose $T_z, T_{z+1}$ are consecutive slices in $A(Q)$. We can order $T_z(X)$ linearly by the second entry of the vertices appearing in it; write $T_z(X) = \{x_1, x_2, \ldots, x_k\}$. Similarly, we can write $T_{z+1}(X) = \{y_1, y_2, \ldots, y_l\}$, where $k \geq l$ (since $X$ is a right component). For $t = 1 \ldots l$, let $i_t = i(x_t)$, and let $j_t = j(x_t)$. Note that $i(y_t) = i_t + 1$ and $j(y_t) = j_t + 1$ for $1 \leq t \leq l$. Then the defining inequalities of $L_{PBW}(Q)$ are

(L3) If $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$ then $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=1}^{k} c_{i_{r+1},j_{r+1}}$.

(L4) $c_{i_r,j_r} \geq c_{i_{r+1},j_{r+1}}$ for $r = 2 \ldots l - 1$. 

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Finally, if $X$ is a left component and is the leftmost component of $Q$, then we have:

(L5) $c_{i_1,j_1} \leq c_{i_1+1,j_1+1}$,

and if $X$ is a right component and is the leftmost component of $Q$, then we have:

(L6) $c_{i_1,j_1} \geq c_{i_1+1,j_1+1}$.

8 The relationship between the Lusztig cones and the degeneration cones in the PBW-parametrization

We show in this section, that, for an arbitrary quiver $Q$ of type $A_n$, we have $L_{PBW}(Q) \subseteq C_{PBW}(Q)$.

Firstly, we show that certain inequalities hold in $L_{PBW}$. Note that we will show in the next sections that $L_{PBW}(Q)$ is the set of PBW-parameters of the Lusztig cone $L_{st}(Q)$.

Lemma 8.1 Suppose $c \in L_{PBW}(Q)$. Then the following inequalities hold:

If $X$ is a left component of $Q$, then (with the notation above),

(I) $c_{i_1,j_1} \leq c_{i_1+1,j_1+1}$, and
(II) $c_{i_k,j_k} \geq c_{i_k+1,j_k+1}$.

If $X$ is a right component, then

(III) $c_{i_1,j_1} \geq c_{i_1+1,j_1+1}$, and
(IV) $c_{i_k,j_k} \leq c_{i_k+1,j_k+1}$.

The inequalities (III) and (IV) only hold if they make sense — i.e. if $k = l$ (which is the case when $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$).

Proof: Let $c \in L_{PBW}(Q)$. Suppose first that the leftmost component $X_1$ of $Q$ is a left component. The inequality (I) for $X_1$ is the defining inequality (L5) of $L_{PBW}(Q)$. Suppose that $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$. Then we have, from (L1) and (L2), that:

(a) $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(b) $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 2 \ldots k - 1$. 

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Since (I) and (b) hold, if (II) were false, (a) would be false — a contradiction. Therefore (II) holds for $X_1$.

A similar argument shows that the inequalities (III) and (IV) hold for $X_1$ in the case when $X_1$ is a right component.

Suppose we are still in the case where $X_1$ is a left component. Let $X_2$ be the component immediately to the right of $X_1$ (if it exists); this must be a right component. Note that the inequality (II) for $X_1$ is inequality (III) for $X_2$. We can then argue as above for $X_2$ to deduce (IV) for $X_2$.

We can argue similarly in the case where $X_1$ is a right component. In this way we can deduce the relevant inequalities for all components of $Q$, by induction on the number of the component, starting from the left, and we are done. \[\square\]

**Proposition 8.2** Let $Q$ be an arbitrary quiver of type $A_n$, and let $L_{PBW}(Q)$ and $C_{PBW}(Q)$ be the cones as above. Then $L_{PBW}(Q) \subseteq C_{PBW}(Q)$.

**Proof:** Let $c \in L_{PBW}(Q)$. Suppose that $X$ is any left component of $Q$, and that $S_z(X) = T_z(X)$ and $S_{z+1}(X) = T_{z+1}(X)$. The following inequalities hold:

(c) $\sum_{r=1}^{k} c_{i_r,j_r} \geq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, and
(d) $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 1 \ldots k-1$ and $c_k,j_k \geq c_{k+1,j_{k+1}}$.

Inequality (c) comes from (L1), and inequalities (d) come from (L2) and Lemma 8.1(II). We show that the inequalities (C1) all hold. We already have the inequality $c_{i_k,j_k} \geq c_{i_{k+1},j_{k+1}}$, from (d). If we also had $c_{i_{k-1},j_{k-1}} + c_{i_k,j_k} \leq c_{i_{k-1}+1,j_{k-1}+1} + c_{i_k+1,j_k+1}$, then this, together with the inequalities $c_{i_r,j_r} \leq c_{i_r+1,j_r+1}$ for $r = 1, \ldots, k-2$ would give $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=1}^{k} c_{i_r+1,j_r+1}$, contradicting (c). Similarly, if we had $\sum_{r=a}^{k} c_{i_r,j_r} \leq \sum_{r=a}^{k} c_{i_{r+1},j_{r+1}}$, for some $1 \leq a \leq k-2$, the inequalities in (d) would give us $\sum_{r=1}^{k} c_{i_r,j_r} \leq \sum_{r=a}^{k} c_{i_{r+1},j_{r+1}}$, contradicting (c). Hence $\sum_{r=a}^{k} c_{i_r,j_r} \geq \sum_{r=a}^{k} c_{i_{r+1},j_{r+1}}$, for any $1 \leq a \leq k-1$, and we see that the defining inequalities (C1) of $C_{PBW}(Q)$ are satisfied.

Suppose next that $X$ is any right component of $Q$. Then the following inequalities hold, using (L4) and Lemma 8.1(III).

(e) $c_{i_r,j_r} \geq c_{i_r+1,j_r+1}$ for $r = 1 \ldots l-1$.

In this case, (e) contains all of the defining inequalities (C2) of $C_{PBW}(Q)$. We thus see that $L_{PBW}(Q) \subseteq C_{PBW}(Q)$. \[\square\]
The image of the Lusztig cone $L_{st}(Q)$ under $E$ is $L_{PBW}(Q)$

We show in this section that the image under $E$ of the Lusztig cone $L_{st}(Q)$ is indeed $L_{PBW}(Q)$. It will follow from results in the next section that $S_k^*(L_{st}(Q)) = L_{PBW}(Q)$, since we shall see that $S_k^*(Q)$ and $E$ are identical on $L_{st}(Q)$.

We suppose that $a = (a_1, a_2, \ldots, a_N) \in L_{st}(Q)$, and let $c = (c_{ij}) = E(a)$. We will show that $a \in L_{st}(Q)$ if and only if $c \in L_{PBW}(Q)$. We translate the linear inequalities defining $L_{st}(Q)$ using the linear function $E$, and show that they become the defining inequalities of $L_{PBW}(Q)$. We also show that $a_i \geq 0$ for all $i$ if and only if $c_{ij} \geq 0$ for all $1 \leq i \leq j \leq n$. Since $E$ is a linear function, it will follow that $E(L_{st}(Q)) = L_{PBW}(Q)$.

Recall that:

\[ i(Q) = (l_1 \searrow 1)(l_2 \searrow 1) \cdots (l_a \searrow 1)(n \searrow n+1-r_b)(n \searrow n+1-r_{b-1}) \cdots (n \searrow n+1-r_1) \]

is compatible with $Q$ (see the end of Section 3).

The inequalities defining $L_{st}(Q)$ arise from pairs of equal simple reflections occurring in $i(Q)$. Note that, for the first $a+1$ factors appearing in the above, each factor is always contained in the one immediately to the right, and that, for the last $b+1$ factors, each factor is always contained in the one immediately to the left. It is clear that the defining inequalities for $L_{st}(Q)$ always arise from such pairs of factors. To make the notation clearer, let us define $l_{a+1} = r_{b+1} = n$.

Let us consider such a pair, $(l_p \searrow 1)(l_{p+1} \searrow 1)$, and suppose that $1 \leq s \leq l_p$, so $s$ occurs both in $(l_p \searrow 1)$ and in $(l_{p+1} \searrow 1)$. Suppose $s$ appears in position $t$ of $i$ in the first factor; it then must appear in the second factor — suppose that this is in position $u$ in the second factor. Let us first assume that $s > 1$. The defining inequality of $L_{st}(Q)$ arising from this pair of $s$'s is:

\[ a_{t+1} + a_{u-1} \geq a_t + a_u. \] (2)

We can rewrite this as:

\[ a_t - a_{t+1} \leq a_{u-1} - a_u. \] (3)

Since $D(c) = a$, we have (from equation (3) in Section 3):

\[ a_t = \sum_{\alpha \in T_{n+1-p}, \ s \in \alpha} c_\alpha \]

\[ a_{t+1} = \sum_{\alpha \in T_{n+1-p}, \ s-1 \in \alpha} c_\alpha \]

\[ a_{u-1} = \sum_{\alpha \in T_{n-p}, \ s+1 \in \alpha} c_\alpha \]

\[ a_u = \sum_{\alpha \in T_{n-p}, \ s \in \alpha} c_\alpha \]
Thus, the defining inequalities for $\text{LE}$

By the description of the coordinates of the function $\text{LE}$, we get precisely the defining inequalities for $\text{LE}$, all of the form $\alpha + \beta$ where $\alpha = \beta + \gamma$, for some simple root $\gamma$. It follows that $c_{ij} \geq 0$ for all $1 \leq i \leq j \leq n$.

We have proved:

**Theorem 9.1** $E(L_{st}(Q)) = L_{PBW}(Q)$. 

Example: We return to our running example, with $Q = RLRL$ in type $A_5$. Let $a \in \mathbb{N}^N$. Then $E(a) = (c) = (c_{ij})$ where $c$ is given by the diagram in Figure 6.

$$
\begin{array}{cccccc}
   a_2 - a_1 & a_5 - a_4 & a_9 - a_8 & a_{13} - a_{12} & a_{15} \\
   a_1 & a_4 - a_6 & a_8 - a_{10} & a_{12} - a_{14} \\
   a_6 - a_3 & a_{10} - a_7 & a_{14} \\
   a_3 & a_7 - a_{11} \\
   a_{11}
\end{array}
$$

Figure 6: The function $S_{i(Q)}^k(a)$

We give below the correspondence between the defining inequalities of $L_{st}(Q)$ and those of $L_{PBW}(Q)$. Recall that $i(Q) = (2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, 5, 4, 3, 5)$.

| Inequality of $L_{st}(Q)$ | Inequality of $L_{PBW}(Q)$ |
|---------------------------|---------------------------|
| $a_2 + a_4 \geq a_1 + a_5$ | $c_{11} \geq c_{22}$ |
| $a_5 \geq a_2 + a_6$ | $c_{22} + c_{23} \geq c_{11} + c_{12}$ |
| $a_4 + a_7 \geq a_3 + a_8$ | $c_{23} + c_{13} \geq c_{34} + c_{24}$ |
| $a_5 + a_8 \geq a_4 + a_9$ | $c_{22} \geq c_{33}$ |
| $a_6 + a_9 \geq a_5 + a_{10}$ | $c_{33} + c_{34} \geq c_{22} + c_{23}$ |
| $a_{10} \geq a_6 + a_{11}$ | $c_{24} + c_{25} \geq c_{13} + c_{14}$ |
| $a_8 \geq a_7 + a_{12}$ | $c_{34} + c_{24} \geq c_{45} + c_{35}$ |
| $a_9 + a_{12} \geq a_8 + a_{13}$ | $c_{33} \geq c_{44}$ |
| $a_{10} + a_{13} \geq a_9 + a_{14}$ | $c_{44} + c_{45} \geq c_{33} + c_{34}$ |
| $a_{13} \geq a_{12} + a_{15}$ | $c_{44} \geq c_{55}$ |

10 Description of the function $S_{i(Q)}^k$

We now show that the function $S = S_{i(Q)}^k$ coincides with the function $E = D^{-1}$ arising from representations of quivers, as described in Section 4, on the degeneration cone $C_{st}(Q)$. We first of
all note that each edge of \( Q \) corresponds in a natural way to a slice of \( A(Q_k) \); such a slice generates the factor of \( i(Q) \) corresponding to this edge — see Section 3.

We suppose that \( a \in C_{st}(Q) = D(C_{PBW}(Q)) \) (the degeneration cone). We will show that \( a \in X_{st}(Q) \) (the string cone), and that \( S(a) = E(a) \). It then follows that, as \( E \) and \( S \) are bijective, \( S^{-1}(c) = a = D(c) \) for any \( c \in C_{PBW}(Q) \). This is the result we would like to prove, and this section is mainly devoted to achieving this aim. In the following analysis, \( a \) shall denote an arbitrary element of \( C_{st}(Q) \), and \( e = E(a) \).

In applying \( S^k \) to \( a = (a_{ij}) \), we need to compute \( c \) such that \( \tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_k} \cdot 1 \equiv F^c_k \mod v \mathcal{L} \), where \( i = i(Q) \). Recall (see the end of Section 3) that
\[
i(Q) = (a_{11})(a_{12}) \cdots (a_{a \gamma 1})(n \gamma 1)(n \gamma n + 1 - r_b)(n \gamma n + 1 - r_{b-1}) \cdots (n \gamma n + 1 - r_1).
\]

Let \( m \) be an edge of \( Q \). Then in the Berenstein-Fomin-Zelevinsky arrangement corresponding to \( Q \) (see, for example, Figure 3), the line corresponding to \( m \) passes first through the lines corresponding to \( R \)'s to the left of \( m \), starting from the right, then the line from top left to bottom right, followed by the \( L \)'s to the left of \( E \) in \( Q \), from left to right. This is how the reduced expression \( i(Q) \) is built up. Thus we apply the product
\[
\tilde{F}(l_1) \tilde{F}(l_2) \cdots \tilde{F}(l_a) \tilde{F}(n) \tilde{F}(r_b) \tilde{F}(r_{b-1}) \cdots \tilde{F}(r_1)
\]
\[\text{(4)}\] to 1 = \( F^0_k \) where \( 0 \) denotes the zero vector, and \( \tilde{F}(m) \) is a monomial of Kashiwara operators defined as follows. Given \( m \in [1, n] \), let \( d \) be maximal so that \( l_d < m \) and let \( e \) be maximal so that \( r_e < m \). Then we have that \( d + e = m - 1 \), so \( m - e - d = 1 \), and \( n - e - d = n - m + m - e - d = n + 1 - m \).

Then, if edge \( m \) is an \( L \), we have:
\[
\tilde{F}(m) = \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{m-1} \cdots \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{m-e-1} \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{m-e-2} \cdots \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{m-e-d},
\]
and if edge \( m \) is an \( R \), or \( m = n \), we have:
\[
\tilde{F}(m) = \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{n-1} \cdots \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{n-e-1} \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{n-e-2} \cdots \tilde{F}^{\alpha_{1r}+\alpha_{e-1}+1,m+1}_{n-e-d}.
\]

We can regard the vector \( c \in \mathbb{N}^N \) as the vector \((c_\alpha)_{\alpha \in \Phi^+} \) (see section 4.3). Each root \( \alpha \in \Phi^+ \) will lie in a slice \( T_z \) of \( \Phi^+ \). Given \( z \in \mathbb{N} \), let \( c(z) \) denote the vector \( c \) but with \( c_\alpha \) set to zero for every \( \alpha \) lying in a slice \( T_z \) with \( z' < z \); thus \( c(n + 1) = 0 \) and \( c(1) = c \).

For \( i = 1, 2, \ldots, n \), let \( \bar{P}_i \) denote the \( i \)th product appearing in (4). Thus, for \( i = 1, 2, \ldots, a, \)
\[
\bar{P}_i = \tilde{F}(l_i), \quad \bar{P}_{a+1} = \tilde{F}(n), \quad \text{and for } i = a + 2, \ldots, n, \quad \bar{P}_i = \tilde{F}(r_{a+2-i+b}), \quad \text{and we have}
\]
\[
\bar{P}_1 \bar{P}_2 \cdots \bar{P}_n = \tilde{F}(l_1) \tilde{F}(l_2) \cdots \tilde{F}(l_a) \tilde{F}(n) \tilde{F}(r_b) \tilde{F}(r_{b-1}) \cdots \tilde{F}(r_1).
\]

We will show that, for \( z = 1, 2, \ldots, n, \)
\[
\bar{P}_z \cdot F^c_k(z + 1) \equiv F^c_k(z) \mod v \mathcal{L}, \quad \text{from which it will follow that}
\]
\[
\bar{P}_1 \bar{P}_2 \cdots \bar{P}_n \cdot 1 \equiv F^c_k.
\]
as required.

We also need to show that \( a \in X_{st}(Q) \). We will use the following definiton: A monomial action \( \bar{F}^{b_1}_{j_1} \bar{F}^{b_2}_{j_2} \cdots \bar{F}^{b_t}_{j_t} \cdot \bar{F}^x_k \) is said to satisfy (STRING) provided that

\[
\bar{E}_j \cdot \bar{F}^{b_{u+1}}_{j_{u+1}} \bar{F}^{b_{u+2}}_{j_{u+2}} \cdots \bar{F}^{b_t}_{j_t} \cdot \bar{F}^x_k \equiv 0 \mod vL,
\]

for \( u = 1, 2, \ldots, t \). We will also use the description of the action of the \( \bar{F}_i \)'s on a PBW basis modulo \( vL \) as proved by the second author in [13]:

**Proposition 10.1** (Reineke) Suppose \( c = (c_{ij}) \in \mathbb{N}^N \). For each \( 1 \leq i \leq j \leq n \), define

\[
f_{ij} = \sum_{k=1}^{i} c_{k,j} - \sum_{k=1}^{i-1} c_{k,j-1}.
\]

Let \( i_0 \) be maximal so that \( f_{i_0,j} = \max_i f_{ij} \). Then \( \bar{F}_j \) increases \( c_{i_0,j} \) by 1, decreases \( c_{i_0,j-1} \) by 1 (unless \( i_0 = j \), when this latter effect does not occur), and leaves the other \( c_{ij} \)'s unchanged.

It is easy to see, using Theorem 2.2(ii), that this implies the following description of the \( \bar{E}_i \)'s:

**Proposition 10.2** Suppose \( c = (c_{ij}) \in \mathbb{N}^N \). For each \( 1 \leq i \leq j \leq n \), define

\[
f_{ij} = \sum_{k=1}^{i} c_{k,j} - \sum_{k=1}^{i-1} c_{k,j-1}.
\]

Let \( i_0 \) be minimal so that \( f_{i_0,j} = \max_i f_{ij} \). Then \( \bar{E}_j \) decreases \( c_{i_0,j} \) by 1, increases \( c_{i_0,j-1} \) by 1 (unless \( i_0 = j \), when this latter effect does not occur), and leaves the other \( c_{ij} \)'s unchanged, except, if \( c_{i_0,j} = 0 \), then it acts as zero modulo \( qL \).

We will need the following technical Lemmas, describing the action of \( \bar{F}_i \) and \( \bar{E}_i \) in certain circumstances:

**Lemma 10.3** Fix \( 1 \leq i \leq j \leq n \) and \( s \in \mathbb{N} \). Suppose the following hold for a triangle \( (c_{kl}) \in \mathbb{N}^N \):

(a) For \( k = 1 \ldots i - 1 \), we have \( \sum_{l=k+1}^i c_{lj} \geq \sum_{l=k}^{i-1} c_{l,j-1} \).
(b) For \( k = i + 1 \ldots j \), we have \( s \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^k c_{ij} \).

Then \( \bar{F}^s_j \) acts on \( (c_{kl}) \) by increasing \( c_{ij} \) by \( s \), decreasing \( c_{i,j-1} \) by \( s \), and leaving the other \( c_{kl} \)'s unchanged.
Proof: For $t = 0 \ldots s - 1$, define a new triangle $c^t_{kl}$ by

$$c^t_{kl} = \begin{cases} 
  c_{ij} + t, & k = i, l = j \\
  c_{i,j-1} - t, & k = i, l = j - 1 \\
  c_{kl}, & \text{otherwise}
\end{cases}$$

The Lemma clearly holds if for all $t = 0 \ldots i - 1$, the index $i_0$ of Proposition 10.1 equals $i$. This translates into the following inequalities for the $f_{kl}$'s of Proposition 10.1:

$$f_{kj} \leq f_{ij} \text{ for } k = 1 \ldots i - 1, \quad f_{kj} < f_{ij} \text{ for } k = i + 1 \ldots j.$$ 

Using the definitions of $(c^t_{kl})$ and $f_{kl}$, this translates into the following conditions:

$$t \geq \sum_{l=k}^{i-1} c_{l,j-1} - \sum_{l=k+1}^{i} c_{lj} \text{ for } t = 0 \ldots s - 1, k = 1 \ldots i - 1,$$

which is equivalent to condition (a), and

$$t < \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj} \text{ for } t = 0 \ldots s - 1, k = i + 1 \ldots j,$$

which is equivalent to condition (b). $\square$

Lemma 10.4 Fix $1 \leq i \leq j \leq n$ and $s \in \mathbb{N}$. Suppose the following hold for a triangle $(c_{kl}) \in \mathbb{N}^N$:

(a) For $k = 1 \ldots i$, we have $c_{kj} = 0$.
(b) For $k = i + 1 \ldots j$, we have $s \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{lj}$.

Then for $\tilde{s} := s + \sum_{k=1}^{i-1} c_{k,j-1}$, we have:

$$\tilde{F}^s_j(c_{kl})_{kl} = \begin{cases} 
  0, & l = j - 1, k < i \\
  c_{k,j-1}, & l = j, k < i \\
  c_{i,j-1} - s, & l = j - 1, k = i \\
  s, & l = j, k = i \\
  c_{kl}, & \text{otherwise}
\end{cases}$$

Proof: We proceed by induction on $i$. If $i = 1$, the claimed statement follows directly from Lemma 10.3. For arbitrary $i$, we set $i' = i - 1, s' = c_{i-1,j-1}$ and claim that we can apply the Lemma --
which we assume to be already true for \( i' \) — with the same triangle \((c_{kl})\), but with \((i, j, s)\) replaced by \((i', j, s')\). We thus have to show that the assumptions of the Lemma are satisfied:

Condition (a) is satisfied trivially. For \( k = i + 1 \ldots j \), condition (b) gives \( \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{l,j} \geq s \geq 0 \), which (using \( c_{ij} = 0 \) by condition (a)) means

\[
\sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{l,j} \geq c_{i-1,j-1} - c_{i,j} = s'.
\]

For \( k = i' + 1 = i \), the desired condition is \( s' \leq c_{i-1,j-1} - c_{ij} \), which clearly holds since \( c_{ij} = 0 \).

Thus, we can apply the Lemma to \((i', j, s')\). Setting \((c'_{kl}) = \tilde{F}_j^s(c_{kl})\), we find:

\[
c'_{kl} = \begin{cases}
0, & l = j - 1, k < i' \\
c_{k,j-1}, & l = j, k < i' \\
c_{i',j-1} - s, & l = j - 1, k = i' \\
s, & l = j, k = i' \\
c_{kl}, & \text{otherwise}
\end{cases} = \begin{cases}
0, & l = j - 1, k < i \\
c_{k,j-1}, & l = j, k < i \\
c_{kl}, & \text{otherwise}.
\end{cases}
\]

Since \( s' = \sum_{k=1}^{i-1} c_{k,j-1} \), it thus remains to show that \( \tilde{F}_j^s \) acts on \((c'_{kl})\) by decreasing \( c_{i,j-1} \) by \( s \), increasing \( c_{ij} = 0 \) by \( s \), and leaving the rest of \((c'_{kl})\) unchanged. To prove this, we only have to check the assumptions of Lemma \([10.3]\) for \((c'_{kl})\): for condition (a) of Lemma \([10.3]\), this is trivial since \( c'_{k,j-1} = 0 \) for \( k < i \); for condition (b) of Lemma \([10.3]\) we just use condition (b) of the present Lemma. Applying Lemma \([10.3]\), we see that we are done. \(\square\)

**Lemma 10.5** Fix \( 1 \leq i \leq j \leq n \) and \( s \in \mathbb{N} \). Suppose the following hold for a triangle \((c_{kl}) \in \mathbb{N}^N:\)

(a) For \( k = 1 \ldots i \), we have \( c_{kj} = 0 \).

(b) For \( k = i + 1 \ldots j \), we have \( 0 \leq \sum_{l=i}^{k-1} c_{l,j-1} - \sum_{l=i+1}^{k} c_{l,j} \).

Then we have \( \tilde{E}_j(c_{kl}) = 0 \).

**Proof:** For each \( 1 \leq p \leq j \leq n \), define

\[
f_{pj} = \sum_{k=1}^{p} c_{kj} - \sum_{k=1}^{p-1} c_{k,j-1},
\]

as in Proposition \([10.2]\). By assumption (a), \( f_{pj} = -\sum_{k=1}^{p-1} c_{k,j-1} \), for \( p = 1, 2, \ldots i \), and, using both
We start by considering the action of the monomial \( \tilde{p} \) for

\[
f_{pj} = \sum_{k=1}^{p} c_{k,j} - \sum_{k=1}^{p-1} c_{k,j-1}
\]

for \( p = i + 1, i + 2, \ldots, j \). Hence if \( p_0 \) is minimal so that \( f_{p_0 j} = \max_p f_{pj} \), we must have \( p_0 \leq i \). It follows from (a) that \( c_{p_0 j} = 0 \), so we conclude that \( \tilde{E}_j \) acts as zero modulo \( q \mathcal{L} \). \( \square \)

We start by considering the action of the monomial \( \tilde{F}(r_p) \) (where \( 1 \leq p \leq b + 1 \) — recall that \( r_{b+1} = n \)) on \( c(n + 2 - p) \). Recall that, for \( p = 1, 2, \ldots, b + 1 \), we have

\[
\tilde{F}(r_p) = \tilde{F}_{n-p+1}^{a_{1}}, \ldots, \tilde{F}_{n-p+1}^{a_{n+2-p}} \tilde{F}_{n-p}^{d_{1}} \tilde{F}_{n-p}^{d_{2}} \ldots \tilde{F}_{n-p}^{d_{b+2-n}} \tilde{F}_{n-p}^{c_{1}} \tilde{F}_{n-p}^{c_{2}} \ldots \tilde{F}_{n-p}^{c_{n+2-p}}.
\]

The initial part of the computation is reasonably easy:

\textbf{Lemma 10.6} Suppose that \( 1 \leq p \leq b \). Let \( d \in \mathbb{N}^N \) be the triangle given by

\[
d_{n+1-p, n+1-p} = a_{1}, r_{p+1} - a_{1} r_{p+1},
\]

\[
d_{n-p, n+1-p} = a_{1} r_{p+1} - a_{2} r_{p+1},
\]

\[
d_{n-p, 1, n+1-p} = a_{2} r_{p+1} - a_{3} r_{p+1},
\]

\[\vdots\]

\[
d_{n+2-p, d, n+1-p} = a_{d-1} r_{p+1} - a_{d} r_{p+1},
\]

\[
d_{n+3-p, d, n+1-p} = a_{d} r_{p+1}.
\]

Then we have

\[
\tilde{F}_{n-p+1}^{a_{1}} \ldots \tilde{F}_{n-p}^{a_{n+2-p}} \tilde{F}_{n-p}^{d_{1}} \ldots \tilde{F}_{n-p}^{d_{b+2-n}} \tilde{F}_{n-p}^{c_{1}} \ldots \tilde{F}_{n-p}^{c_{n+2-p}} \equiv \tilde{F}_{n-p}^{c_{n+2-p} + d}.
\]

Furthermore, the action

\[
\tilde{F}_{n-p+1}^{a_{1}} \ldots \tilde{F}_{n-p}^{a_{n+2-p}} \tilde{F}_{n-p}^{d_{1}} \ldots \tilde{F}_{n-p}^{d_{b+2-n}} \tilde{F}_{n-p}^{c_{1}} \ldots \tilde{F}_{n-p}^{c_{n+2-p}}
\]

satisfies (STRING).

\textbf{Proof:} We first note that we have \( a_{1, r+1} \geq a_{2, r+1} \geq \ldots \geq a_{b+1, r+1} \). These follow from the fact that \( a = D(c) \), since we know that all the coordinates of \( c \) are nonnegative: we use equation (5) in Section 4. Let \( l_0 = 0 \). We know that, for \( t = 0, 1, \ldots, d - 1 \),

\[
a_{l_1, r+1} = \sum_{a \in T_{n+1-p, n-p+1-t} \subseteq \alpha} c_{a},
\]

(5)
and

$$a_{t+1,r+1} = \sum_{\alpha \in T_{n+1-p,n-p-1}\alpha} c_\alpha. \quad (6)$$

Let $t \in \{0, 1, \ldots, d-1\}$. Since $n-p+1-t \leq n-p+1$, it follows that if $\alpha_{n-p-t}$ appears in a root of slice $p$, so does $\alpha_{n-p+1-t}$. Thus, by equations (6) and (3), we have $a_{t+1,r+1} \geq a_{t+1,r+1+1}$. As required.

Let $x = c(n+2-p)$. It is easy to see, using Proposition 10.4, that $\tilde{F}_{n+1-p,1}^{a_{n+1,p+1}}$ sets $x_{n+1-p,n+1-r-1}$ to be $a_{n+1,p+1}$, and leaves the other $x_{ij}$ unchanged. Similarly, it is easy to see, using Proposition 10.4, that $\tilde{F}_{n+1-p,1}$ acts as zero on $x$. In both cases this is because $x_{k,n+1-r-1} = 0$ for $k = 1, 2, \ldots n+1-r-1$, and $x_{k,n-r} = 0$ for $k = 1, 2, \ldots n-r$, so that all of the $f_{ij}$ in Proposition 10.4 or Proposition 10.2 are zero.

Next, $\tilde{F}_{n+1-p,1}^{a_{n+2-2-p}+1}$ sets $x_{n+2-2-p,n+2-r} = 0$ and sets $x_{n+1-r,n+2-r} = a_{n+1,p+1}$ and $a_{n+1,p+1}$ to be $a_{n+1,p+1} = a_{n+1,p+1}$. The rest of the $F_i$'s act in a similar way, until we finally get the vector described in the Lemma. It is also easy to see that the given monomial action satisfies (STRING) using Proposition 10.2. □.

The next part of the computation, which involves computing $\tilde{F}_{n+2-2-p+1}^{a_{n+2-2-p}+1} \bullet \tilde{F}_{n+2-2-p+1}^{a_{n+2-2-p}+1} \bullet \tilde{F}_{n+2-2-p+1}^{a_{n+2-2-p}+1}$, is more involved. We need to consider what happens step-by-step, and to understand what is happening in detail along each slice $T_z = T_{n+1-p}^{A(Q_k)}$. Let $T_z = \{\beta_1, \beta_2, \ldots \beta_{k_z}\}$, where the $k_z$ roots in $T_z$ are listed according to increasing height. Thus $\beta_{k_1} = \alpha_{n+1-p}$.

Given $q$ such that $1 \leq q \leq p-1$, we define a vector $c(p,q)$ as follows. (In order to simplify notation, we use the numbering arising from the edges of $Q$ oriented to the right, rather than the usual slice numbering.) If $\alpha$ belongs to a slice numbered $n+2-p$ or greater (i.e. corresponding to one of the right-oriented edges numbered $r_1, r_2, \ldots, r_{p-1}$), then $c_\alpha(p,q) = c_\alpha$. We also set, for $i = 1, 2, \ldots, r_q-1$, $c_{\beta_{r_q}}(p,q) = a_{i+1,r_q+1} - a_{i+1,r_q+1}$, and set $c_{\beta_{r_q}}(p,q) = a_{i+1,r_q+1} - a_{i+1,r_q+1}$, where $l_i$ is the number of the edge of the first $L$ to the right of edge $r_q$. Suppose that $\beta_{r_q} = \alpha_{ij}$. Then we also set $c_{ij}(p,q) = a_{i+1,r_q+1} - a_{i+1,r_q+1}$. We write $c(p,0)$ for the vector $c(n+2-p)+d$ appearing in Lemma 10.4. The vector $c(p,q-1)$ (which appears in the Lemma below) can be visualised as in Figure 7 (note that the case displayed is where edge $r_q-1$ is oriented to the right). We have:

**Lemma 10.7** Suppose that $1 \leq p \leq b$ and that $1 \leq q \leq p-1$. We have that

(a) $\tilde{F}_{n+1-p+q}^{a_{n+1,p+1}} \bullet F_{k}^{c(p,q-1)} \equiv F_{k}^{c(p,q)} \mod vL$, and that

(b) $\tilde{F}_{n+1-p+q}^{a_{n+1,p+1}} \bullet F_{k}^{c(p,q-1)} \equiv 0 \mod vL$. 

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Figure 7: The vector $c(p, q - 1)$. 
Proof: Suppose that $\beta_{z,r_q-1} = \alpha_{ij}$. It is easy to check that $i = n + 1 - p - (t - 1)$ and $j = n + 1 - p + q - 1$, where $l_t$ is the number of the edge of the first $L$ to the right of edge $r_q - 1$. Let $s = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1}$, (Note that these notations were used for $c(p,q)$ in the above, rather than $c(p,q-1)$).

We apply Lemma 10.4 with the pair $i,j$ as above. We need to check the assumptions of the Lemma. It is clear from the definition of $c(p,q-1)$ that $c_{k,j+1}(p,q-1) = 0$ for $k = 1,2,\ldots,i$. Suppose that $i + 1 \leq k \leq j + 1$. We consider

$$\sum_{l=i}^{k-1} c_{l,j}(p,q-1) - \sum_{l=i+1}^{k} c_{l,j+1}(p,q-1).$$

This is a sum of terms, each of which is the sum of elements on one side on an inequality (C1) or (C2) (see section 3), minus the corresponding sum of elements on the other side of the inequality.

Since the vector $c \in CPBW(Q)$ satisfies inequalities (C1) and (C2), we have that:

$$\sum_{l=i}^{k-1} c_{l,j} - \sum_{l=i+1}^{k} c_{l,j+1} \geq 0$$

(7)

(indeed, the inequalities of $CPBW(Q)$ are designed to ensure this holds). By the definition of $c(p,q-1)$, it is clear that for $l = i + 1,\ldots,j$, $c_{l,j} = c_{l,j}(p,q-1)$ and that for $l = i + 1,\ldots,k$, $c_{l,j+1} = c_{l,j+1}(p,q-1)$. Furthermore, $c_{i,j}(p,q-1) = a_{r_q-1+1,r_p+1} - a_{l_t+1,r_p+1} = s + (a_{r_q-1+1,r_p+1} - a_{r_q+1,r_p+1}) = s + c_{i,j}$. It follows from Equation (7) that

$$\sum_{l=i}^{k-1} c_{l,j}(p,q-1) - \sum_{l=i+1}^{k} c_{l,j+1}(p,q-1) \geq s$$

(8)

as required in Lemma 10.4. Thus the conditions of the Lemma are all satisfied, and it applies. Let $\hat{s} = s + \sum_{k=1}^{l-1} c_{k,j} = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1} + a_{l_t+1,r_p+1} - a_{l_t+1,r_p+1} + a_{l_t+1,r_p+1} - a_{l_t+1,r_p+1} + \cdots + a_{l_t+1,r_p+1} - a_{l_t+1,r_p+1} + a_{l_t+1,r_p+1} + a_{l_t+1,r_p+1} = a_{r_q+1,r_p+1}$. Then it is easy to check that Lemma 10.4 tells us that $F_{n+1-q+1}^{a_{r_q+1,r_p+1}} \cdot F_k^{c(p,q-1)} \equiv F_k^{c(p,q)} \mod vL$, giving us (a) above. This can easily be visualised in Figure 2, the value of $c_{i,j}(p,q-1)$ is displayed at $A$ in the diagram, and this is replaced by $a_{r_q+1,r_p+1} - a_{r_q+1,r_p+1}$; the value of $c_{i,j+1}(p,q-1)$ (which is zero) is displayed at $B$ in the diagram, and this is replaced by $s = a_{r_q+1,r_p+1} - a_{l_t+1,r_p+1}$. Finally, the values on the diagonal below and to the left of $A$ are all moved one step down and to the right (to places where the value is currently zero), forming the diagonal below and to the left of $B$. The new values in the diagonal below $A$ are all zero.

In order to show (b), we apply Lemma 10.3 to $c(p,q-1)$, again with the pair $i,j+1$. It is clear that the conditions of this Lemma are satisfied by $c(p,q-1)$, since they are same as those in Lemma 10.4 but with $s$ taken to be zero. The Lemma tells us that $F_{n+1-q+1}^{a_{r_q+1,r_p+1}} c(p,q-1) \equiv 0 \mod vL$, giving us (b) above. $\square$

We can now prove our main result:
Theorem 10.8 Let \( c \in C_{PBW}(Q) \). Then \( (S_{i(Q)}^k)^{-1}(c) = D(c) \).

**Proof:** Let \( a \in C_{st}(Q) = D(C_{PBW}(Q)) \), and suppose \( 1 \leq p \leq b \). Lemma 10.6 tells us that
\[
F_{n-p+1}^{a_1, m_1 + 1} \ldots F_{n-p}^{a_{r_i}, m_{r_i} + 1} \equiv F_{k}^{c(p, 0)}. 
\]
Lemma 10.7(a) tells us that, for \( 1 \leq q \leq p - 1 \), we have \( F_{n+1-p+1}^{a_{r_q+1}, m_{r_q} + 1} \cdot F_{k}^{c(p, q-1)} \equiv F_{k}^{c(p, q)} \mod vL \). Repeated application of this second result (for \( q = 1, 2, \ldots, p-1 \)) tells us that \( F_{k}^{c(n + 2 - p)} \equiv F_{k}^{c(n+1)} \mod vL \), for \( p = 1, 2, \ldots, b \). It follows that \( F_{k}^{c(n)} \cdot F_{k}^{c(r_b)} \cdot F_{k}^{c(r_{b-1})} \cdot \ldots \cdot F_{k}^{c(r_1)} \cdot F_{k}^{c(e(1))} \equiv F_{k}^{c(n+1)} \mod vL \). A proof similar to the one given above can be used to show that \( F_{k}^{c(n)} \cdot F_{k}^{c(e(1))} \equiv F_{k}^{c(n+1)} \mod vL \), for \( i = 1, 2, \ldots, a \). It then follows that \( F(l_1) \cdot \ldots \cdot F(l_a) \cdot F(n) \cdot F(r_b) \cdot F(r_{b-1}) \cdot \ldots \cdot F(r_1) \cdot F_{k}^{c(e(1))} \equiv F_{k}^{c(n+1)} \mod vL \), as required. It also follows from Lemmas 10.6 and 10.7 (and a similar proof for the monomial \( l_1 \cdot \ldots \cdot l_a \)) that the monomial action \( F(l_1) \cdot \ldots \cdot F(l_a) \cdot F(n) \cdot F(r_b) \cdot F(r_{b-1}) \cdot \ldots \cdot F(r_1) \cdot F_{k}^{c(e(1))} \equiv F_{k}^{c(n+1)} \mod vL \), satisfies (STRING), from which it follows that \( a \in X_{st}(Q) \).

We thus have that, for all \( a \in C_{st}(Q) = D(C_{PBW}(Q)) \), \( S(a) = E(a) \) and \( a \in X_{st}(Q) \). It follows that for all \( c \in C_{PBW}(Q) \), \( (S_{i(Q)}^k)^{-1}(c) = D(c) \), as required. □

We conjecture that this theorem holds in the case when \( k \) is replaced by any reduced expression for \( w_0 \) compatible with a quiver.

We have the corollary:

**Corollary 10.9** \( (S_{i(Q)}^k)^{-1}(C_{PBW}(Q)) = C_{st}(Q) \).

**Proof:** This follows from the Theorem and the definition of the degeneration cone \( C_{st}(Q) = D(C_{PBW}(Q)) \). □

Finally, since (by Theorem 9.1.), \( E(L_{st}(Q)) = L_{PBW}(Q) \) and since \( L_{PBW}(Q) \subseteq C_{PBW}(Q) \), we have, by Theorem 10.8, that \( S \) and \( E \) coincide on \( L_{st}(Q) \). It follows that:

**Theorem 10.10** \( S_{i(Q)}^k(L_{st}(Q)) = L_{PBW}(Q) \).

**Example:** We return to our running example, with \( Q = RLRL \) in type \( A_5 \). By Theorem 10.8, for \( a \) such that \( E(a) \in C_{PBW}(Q) \), we have \( S_{i(Q)}^k(a) = (c) = (c_{ij}) \) where \( c_{ij} \) is given in Figure 1.

Using the description of \( D \), we have, for \( c \in L_{PBW}(Q) \), that \( (S_{i(Q)}^k)^{-1}(c) = a \), where \( a = (c_{12}, c_{11} + c_{12}, c_{14}, c_{23} + c_{13} + c_{14}, c_{22} + c_{23} + c_{13} + c_{14}, c_{13} + c_{14}, c_{25} + c_{15}, c_{34} + c_{24} + c_{25} + c_{15}, c_{33} + c_{34} + c_{24} + c_{25} + c_{15}, c_{24} + c_{25} + c_{15}, c_{45} + c_{35}, c_{44} + c_{45} + c_{35}, c_{35}, c_{55}) \).

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