A splitting/polynomial chaos expansion approach for stochastic evolution equations

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Abstract. In this paper, we combine deterministic splitting methods with a polynomial chaos expansion method for solving stochastic parabolic evolution problems. The stochastic differential equation is reduced to a system of deterministic equations that we solve efficiently by splitting methods. The method can be applied to a wide class of problems where the related stochastic processes are given uniquely in terms of stochastic polynomials. A comprehensive convergence analysis is provided and numerical experiments validate our approach.

1. Introduction

Splitting methods are numerical methods for solving differential equations, both ordinary and partial differential equations (PDEs), involving operators that are decomposable into a sum of (differential) operators. These methods are used to improve the speed of calculation for problems involving decomposable operators and in particular to solve multidimensional PDEs by reducing them to a sum of one-dimensional problems [15]. Splitting methods have been successfully applied to many types of PDEs, e.g., [21, 22]. Exponential splitting methods are applied in cases where the explicit solution of a splitted equation can be computed. Such computations often rely on applying fast Fourier techniques; see for instance [48]. Resolvent splitting is used in cases where the splitted equation cannot be solved explicitly [23, 44]; here we consider this type of methods.

There are also many results in the literature about the approximation of solutions of stochastic partial differential equations (SPDEs) using splitting methods; see e.g., [4, 5, 7, 10, 14, 17, 18] and references therein. In [17], a splitting method for nonlinear stochastic equations of Schrödinger type is proposed. There, the authors approximate the solution of the problem by a sequence of solutions of two types of equations: one without stochastic term and other containing only the stochastic term. They prove that an appropriate combination of the solutions of these equations converges strongly to
the solution of the original problem. Exponential integrators for nonlinear Schrödinger equations with white noise dispersion were proposed in [10]. For a stochastic incompressible time-dependent Stokes equation, different time-splitting methods were studied in [7]. In [4], the convergence of a Douglas–Rachford-type splitting algorithm is presented for general SPDEs driven by linear multiplicative noise. In this work, a splitting/polynomial chaos expansion is considered for stochastic evolution equations. Our approach has not been considered in the literature for solving these types of SPDEs so far.

We consider inhomogeneous stochastic parabolic evolution equations of the form

\[ \frac{\partial}{\partial t} u(t, x, \omega) = L u(t, x, \omega) + G(t, x, \omega), \quad t \in (0, T], \quad x \in X, \quad \omega \in \Omega \]

\[ u(0, x, \omega) = u^0(x, \omega) \]  

(1)

with a differential operator \( L \) that can be split into the sum of two differential operators \( L = A + B \) acting on Hilbert space-valued stochastic processes over an underlying probability space \( (\Omega, \mathcal{F}, P) \), and a non-Gaussian noise term \( G \); see e.g., [26,33,36–38]. The term \( G \) can be a general type of noise, spatial, temporal or spatial-temporal noise. The existence of a random parameter \( \omega \) is due to uncertainties coming from initial conditions and a random force term. Therefore, the solution is considered to be a Hilbert space-valued stochastic process. Throughout the paper, we are going to study equations of the form (1) where random inputs are uniformly distributed.

Deterministic inhomogeneous parabolic evolution equations that correspond to (1) of the form

\[ \frac{\partial}{\partial t} u(t, x, y) = (A + B) u(t, x, y) + f(t, x, y), \quad t \in (0, T], \quad (x, y) \in (0, 1)^2 \]

\[ u(0, x, y) = u^0(x, y) \]

(2)

subject to homogeneous Dirichlet boundary conditions were studied in [15]. Here \( Au = \partial_x (a(x, y) \partial_x u) \), \( Bu = \partial_y (b(x, y) \partial_y u) \) and \( L = A + B \) denote a strongly elliptic differential operator with sufficiently smooth coefficients \( a \) and \( b \), and \( f \) is a general type of inhomogeneity. The authors provide a rigorous convergence analysis of the exponential Lie and Strang splitting applied to (2).

Linear stochastic evolution equations of the form

\[ d u(t) = (M u(t) + f(t)) \, dt + (C u(t) + g(t)) \, dB_t, \quad t \in (0, T], \]

\[ u(0) = u^0, \]

(3)

where \( M \) and \( C \) are differential operators acting on Hilbert space-valued stochastic processes, \( \{B_t\}_{t \geq 0} \) is a cylindrical Brownian motion on a given probability space \( (\Omega, \mathcal{F}, P) \) and \( f \) and \( g \) are deterministic functions that were studied in [38].

The authors solved the stochastic evolution problem (3) involving Gaussian noise in an appropriate weighted Wiener chaos space. Note that Eq. (3) can be written in an equivalent form
\[
\frac{d}{dt} u(t) = \mathcal{M} u(t) + f(t) + (C u(t) + g(t)) \diamond W_t, \quad t \in (0, T]
\]
\[
u(0) = u^0,
\]
where \( W_t \) is a (temporal) white noise and \( \diamond \) represents the Wick product (stochastic convolution) within the white noise analysis. Also, a representation of a Skorokhod integral in terms of an operator convolution-type Wick product is proven in \([36]\) and a convolution-type noise in stochastic evolution equations is considered. We recall that the Skorokhod integral is an extension of the Itô integral for non-adapted processes. It can be regarded as the adjoint operator of the Malliavin derivative in the \( L^2 \)-sense \([25,26]\).

Stochastic processes with finite second moments on white noise spaces can be represented in series expansion form in terms of a family of orthogonal stochastic polynomials. The classes of orthogonal polynomials are chosen depending on the underlying probability measure \([25,26]\). Namely, the Askey scheme of hypergeometric orthogonal polynomials and the Sheffer system \([46,47]\) can be used to define several discrete and continuous distribution types \([50]\). For example, in the case of the Gaussian measure, the orthogonal basis of the space of random variables with finite second moments is constructed by the use of the Hermite polynomials. We consider problems with non-Gaussian random inputs. The noise term is considered to be uniformly distributed. It is known that in order to obtain a square integrable solution of \((3)\) with deterministic initial condition, it is enough to assume that the operator \( \mathcal{M} - \frac{1}{2} CC^* \) is uniformly elliptic and that the stochastic part (the noise term) is sufficiently regular; see e.g., \([13]\). In this work, the assumptions on the input data for problem \((1)\) will be set such that the existence of a square integrable solution is always established. We do not consider solutions which are generalized stochastic processes as in \([36,38]\), since our focus is on numerical treatment.

Our approach is general enough to be applied to problems with additive noise, problems involving multiplicative noise and problems with convolution-type noise \([36]\). For instance, with this approach the heat equation with random potential, the heat equation in random (inhomogeneous and anisotropic) media and the Langevin equation can be solved. If \((1)\) does have a sufficiently regular solution, this solution can be projected on an orthonormal basis in some Hilbert space, resulting in a system of equations for the corresponding Fourier coefficients. Thus, we use the so-called polynomial chaos method or the chaos expansion method and define the solution of \((1)\) as a formal Fourier series with the coefficients computed by solving the corresponding system of deterministic PDEs \([38]\). With this method, the deterministic part of a solution is separated from its random part. Particularly, in the case of Gaussian noise, the orthonormal basis of stochastic polynomials involves the Hermite polynomials and in the case when the noise term is uniformly distributed, the orthonormal basis involves the Legendre polynomials \([46]\). By construction, the solution is strong in the probabilistic sense. It is uniquely determined by the coefficients, free terms, initial condition and the noise term. The coefficients in the Fourier series are uniquely
determined by Eq. (1) and are computed by solving (numerically) the corresponding lower-triangular system of deterministic parabolic equations. The polynomial chaos method has been successfully applied for solving general classes of SPDEs. The list of references is long, here we mention just a few [26,36,41,43]. In [33–35], this approach has been recently applied to the stochastic optimal regulator control problem [19]. Linear SPDEs with multiplicative arbitrary driving noise are studied in [42], and a distribution-free Skorokhod–Malliavin calculus framework based on generalized stochastic polynomial chaos expansion is proposed. For linear SPDEs, the corresponding deterministic system is independent of the type of randomness involved, and so are the first and second moments. Numerical treatment of such distribution-free SPDEs based on the stochastic polynomial chaos approach is recently presented in [9].

Practical application of the Wiener polynomial chaos involves two truncations, truncation with respect to the number of the random variables and truncation with respect to the order of the orthogonal Askey polynomials used (in the particular case considered, the Legendre polynomials); see e.g. [27].

Recent results on the existence and uniqueness of the classical (square integrable) solution for stochastic heat equations driven by Gaussian noise for various covariance structures were presented in [28]. The numerical treatment based on the truncation of the polynomial chaos expansion approach is well understood for SPDEs driven by Gaussian randomness [26]. In case of Gaussian disturbances, an estimate for the truncation error was derived in [39]. The authors used the multiple Itô integral formula [30] to show that the mean square error converges exponentially with respect to the expansion order and linearly with respect to the number of random variables included, but grows exponentially as time increases. A numerical scheme for stochastic differential equations based on the Wiener–Itô chaos expansion was proposed in [20]. The authors applied Malliavin calculus in order to derive an explicit upper bound for the square integrable approximation error. Numerical simulations were performed for nonlinear problems and elliptic SPDEs with random coefficients in [49] to validate the efficiency of Wick–Malliavin approximation for polynomial nonlinearity and small noise. In [50], the authors demonstrated numerically that the exponential convergence rate for the Wiener–Askey polynomial chaos is realized when an appropriate stochastic polynomial basis is chosen depending on the distribution of random input.

The paper is organized as follows. In Sect. 2, we introduce the notation and basic concepts used in the following sections. In Sect. 3, we present the splitting/polynomial chaos expansion approach and provide a complete convergence analysis. Finally, in Sect. 4 we validate our approach with a numerical experiment.

2. Preliminaries

In this section, we briefly recall polynomial chaos representations of random variables and stochastic processes. We first recall the Wiener–Itô chaos expansion for the Gaussian case and then present the way how general polynomial chaos expansion
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representations are derived. Particular emphasis is then given to Legendre polynomials and the corresponding Wiener–Legendre expansion, and to the Karhunen–Loève expansion.

We recall that a set of continuous polynomials \( \{p_m(x)\}_{m \in \mathbb{N}_0}, x \in \mathbb{R} \) is orthogonal with respect to the weighting function \( w \) if

\[
\int_{-\infty}^{\infty} p_m(x) p_n(x) w(x) dx = d_m \delta_{m,n} \quad \text{for all } m, n \geq 0.
\]

Here, \( \delta_{m,n} \) denotes the Kronecker delta. The set of polynomials is called orthonormal if the constants \( d_m \) are equal to 1 for all \( m \in \mathbb{N}_0 \); see [8].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with the probability measure \( \mathbb{P} \) and let \( L^2(\mathbb{P}) = L^2(\Omega, \mathcal{F}, \mathbb{P}) \) denote the space of random variables with finite second moments. Then, the space \( L^2(\mathbb{P}) \) is a Hilbert space. The scalar product of two random variables \( F, G \in L^2(\mathbb{P}) \) is given by

\[
(F(\omega), G(\omega))_{L^2(\mathbb{P})} = \mathbb{E}(F(\omega)G(\omega)),
\]

where \( \mathbb{E} \) denotes the expectation with respect to the measure \( \mathbb{P} \). In particular, in the following, we denote by \( \mu \) the Gaussian measure and by \( \nu \) the uniform measure. We also note that \((\Omega, \mathcal{F}, \mu)\) is the Gaussian white noise probability space, where the existence of the Gaussian measure \( \mu \) is guaranteed by the Bochner–Minlos theorem [26].

Let \( \mathcal{I} = (\mathbb{N}_0^\mathbb{N})_c \) be the set of sequences of non-negative integers which have only finitely many nonzero components \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0, \ldots) \), \( \alpha_i \in \mathbb{N}_0 \), \( i = 1, 2, \ldots, m, m \in \mathbb{N} \). Particularly, \((0, 0, \ldots)\) is the zero vector. We denote by \( \epsilon^{(k)} = (0, 0, 0, 0, 1, 0, \ldots) \), \( k \in \mathbb{N} \) the \( k \)th unit vector. The length of \( \alpha \in \mathcal{I} \) is the sum of its components \( |\alpha| = \sum_{k=1}^{\infty} k \alpha_k \) and \( \alpha! = \prod_{i \in \mathbb{N}} \alpha_i! \). Further, we denote \( (2\mathbb{N})^\alpha = \prod_{k=1}^{\infty} (2k)^{\alpha_k} \) and use \( \sum_{\alpha \in \mathcal{I}} (2\mathbb{N})^{-q\alpha} < \infty \) if \( q > 1 \); see [26].

2.1. Wiener–Itô chaos expansion representation

First, we briefly recall the main results from the Wiener–Itô chaos expansion. Let \( L^2(\mu) \) be the Hilbert space of random variables with finite second moments on the Gaussian white noise probability space \((\Omega, \mathcal{F}, \mu)\). Here \( \mathcal{F} \) can be considered to be a sigma algebra generated by a Brownian motion, i.e., a Gaussian process.

Let \( \{h_n\}_{n \in \mathbb{N}_0} \) be the Hermite polynomials given through the recursion

\[
\begin{align*}
h_0(x) & = 1, \\
h_1(x) & = x, \\
h_{n+1}(x) & = x h_n(x) + nh_{n-1}(x) \quad \text{for } n \geq 2, \ x \in \mathbb{R}.
\end{align*}
\]

Define the \( \alpha \)th Fourier–Hermite polynomial as the product

\[
H_\alpha(\omega) = \prod_{i \in \mathbb{N}} h_{\alpha_i}(\eta_i(\omega)),
\]
represented in terms of the Hermite polynomials evaluated at appropriate components of the sequence \( \eta = (\eta_1, \eta_2, \ldots) \) of independent Gaussian variables with zero mean and unit variance. Especially,

\[
H_{(0,0,...)}(\omega) = \prod_{i\in\mathbb{N}} h_0(\eta_i(\omega)) = 1 \quad \text{and} \\
H_{\varepsilon(k)}(\omega) = h_1(\eta_k(\omega)) \prod_{i \neq k, i \in \mathbb{N}} h_0(\eta_i(\omega)) = \eta_k(\omega), \quad k \in \mathbb{N}.
\]

**Theorem 1.** (Wiener–Itô chaos expansion theorem, [26]) Each square integrable random variable \( F \in L^2(\mu) \) can be uniquely represented in the form

\[
F(\omega) = \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}(\omega),
\]

where \( f_{\alpha} \in \mathbb{R} \) for \( \alpha \in \mathcal{I} \). Moreover, it holds

\[
\|F\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{I}} f_{\alpha}^2 \|H_{\alpha}\|_{L^2(\mu)}^2 < \infty.
\]

The family of stochastic polynomials \( \{H_{\alpha}\}_{\alpha \in \mathcal{I}} \) forms an orthogonal basis of \( L^2(\mu) \) such that

\[
\mathbb{E}(H_{\alpha} H_{\beta}) = \alpha! \delta_{\alpha\beta},
\]

for all \( \alpha, \beta \in \mathcal{I} \); see [26]. Here \( \delta_{\alpha\beta} \) denotes the Kronecker delta. Thus, the sequence of coefficients in (7) is a sequence of real numbers obtained by \( f_{\alpha} = \frac{1}{\alpha!} \mathbb{E}(F H_{\alpha}) \), \( \alpha \in \mathcal{I} \). Moreover, due to (9), the convergence condition (8) reads

\[
\|F\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{I}} f_{\alpha}^2 \alpha! < \infty.
\]

The following also holds:

\[
\mathbb{E}(H_{(0,0,...)}) = 1 \quad \text{and} \quad \mathbb{E}(H_{\alpha}) = 0 \quad \text{for } |\alpha| > 0.
\]

Property (9) is a consequence of the orthogonality of the Hermite polynomials

\[
\int_{\mathbb{R}} h_n(x) h_m(x) \, d\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_n(x) h_m(x) e^{-x^2/2} \, dx = n! \delta_{m,n}
\]

for all \( m, n \in \mathbb{N} \). In the previous orthogonality relation, the weighting function \( w(x) \) is of the form \( w(x) = e^{-x^2/2}, x \in \mathbb{R} \).

It follows from (9) that the variance is

\[
\text{Var}(H_{\alpha}(\omega)) = \mathbb{E}(H_{\alpha}^2(\omega)) = \alpha! \quad \text{for } |\alpha| > 0.
\]

By combining (6) and \( \mathbb{E}(H_{\varepsilon(k)}(\omega)) = 0 \), we obtain \( \text{Var}(H_{\varepsilon(k)}(\omega)) = \varepsilon^{(k)!} = 1, k \in \mathbb{N} \), which is in compliance with (10).
Remark 1. Note that the Fourier–Hermite polynomials can be defined in terms of a sequence of uncorrelated Gaussian variables \( \{\eta_k\}_{k \in \mathbb{N}} \). However, in the Gaussian case this is equivalent to assume that \( \{\eta_k\}_{k \in \mathbb{N}} \) are independent Gaussian random variables.

Remark 2. An alternative statement of Theorem 1 gives the chaos expansion decomposition of a random variable in terms of multiple Itô integrals [26]. Namely, for every \( F \in L^2(\mu) \) there exists a unique family of symmetric functions \( f_n \in \hat{L}^2(\mathbb{R}^n), \quad n \in \mathbb{N} \) such that it holds

\[
F(\omega) = \sum_{n=0}^{\infty} I_n(f_n) = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n(f_n)
\]

and

\[
\| F \|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \| f_n \|^2_{L^2(\mathbb{R}^n)} < \infty.
\]

In [30] Itô proved that for \( \alpha = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots) \in \mathcal{I} \) the \( \alpha \)th Fourier–Hermite polynomial can be written as

\[
H_\alpha(\omega) = \int_{\mathbb{R}^{|\alpha|}} h_\alpha^{\otimes \alpha}(t) \, dB_t^{\otimes |\alpha|}(\omega),
\]

where \( \{h_k\}_{k \in \mathbb{N}} \) are the Hermite functions and \( h_\alpha^{\otimes \alpha} = h_1^{\otimes \alpha_1} \otimes \cdots \otimes h_m^{\otimes \alpha_m} \) is the symmetrized tensor product with factors \( h_1, \ldots, h_m \) each \( h_k \) being taken \( \alpha_k \) times. Thus, the connection between the two chaos expansion theorems is given by \( f_n = \sum_{|\alpha|=n} c_\alpha h_\alpha^{\otimes \alpha} \).

We also recall that in white noise analysis in the Gaussian case, the elements of the sequence \( \{\eta_k\}_{k \in \mathbb{N}} \) can be represented as Itô integrals of the first order of deterministic square integrable functions. Then norms in \( L^2(\mu) \) are established by the Itô isometry; see [26].

In [50], it was shown that the initial construction of the Wiener chaos which corresponds to the Gaussian measure and Hermite polynomials can be extended also to other types of measures, where instead of the Hermite polynomials other classes of orthogonal polynomials from the Askey scheme of hypergeometric orthogonal polynomials are used [2,46]. For example, the Gamma distribution corresponds to the Laguerre polynomials and thus to the Wiener–Laguerre chaos, while the Beta distribution is related to the Jacobi polynomials and thus to the Wiener–Jacobi chaos, etc. Moreover, in [46] it was proven that the optimal exponential convergence rate for each Wiener–Askey chaos can be realized.

In this paper, we deal with stochastic evolution problems with non-Gaussian random inputs which are uniformly distributed. From the Askey scheme of orthogonal polynomials, it follows that the uniform distribution, as a special case of the Beta distribution, corresponds to the special class of the Jacobi polynomials, the Legendre polynomials [1]. Therefore, we are going to work with the Wiener–Legendre polynomial chaos.
2.2. Polynomial chaos representation

Based on the construction and properties of Wiener–Itô chaos expansion in the Gaussian case that we presented in Sect. 2.1, recently Mikulevicius and Rozovskii introduced a distribution-free stochastic analysis [42]. In this context, the driving noise term is defined as

\[ N_t(\omega) = \sum_{k=1}^{\infty} m_k(t) \theta_k(\omega) \]  \hspace{1cm} (11)

and the corresponding stochastic process is given by

\[ N_t(\omega) = \sum_{k=1}^{\infty} \left( \int_0^t m_k(s) \, ds \right) \theta_k(\omega), \]

where \( \Theta = \{ \theta_k \}_{k \in \mathbb{N}} \) is a sequence of uncorrelated random variables with zero mean and unit variance and \( \{ m_k \}_{k \in \mathbb{N}} \) is a complete orthonormal basis in \( L^2([0, T]) \). The distribution-free term arises from the fact that each random variable can be of any distribution, and they do not necessarily need to be independent. Note that the chaos expansion forms of the noise term and the process are similar to the representations of the corresponding white noise process and the Brownian motion in the Gaussian case, respectively,

\[ W_t(\omega) = \sum_{k=1}^{\infty} h_k(t) \eta_k(\omega) \quad \text{and} \quad B_t(\omega) = \sum_{k=1}^{\infty} \left( \int_0^t h_k(s) \, ds \right) \eta_k(\omega), \]

where \( \{ h_k \}_{k \in \mathbb{N}} \) are Hermite functions and \( \{ \eta_k \}_{k \in \mathbb{N}} \) are uncorrelated Gaussian random variables with zero mean and unit variance. For stochastic polynomial expansion, the following two assumptions are required:

(s1) For each finite dimensional random vector \((\theta_1, \ldots, \theta_d)\), the moment generating function \(E(\exp(a_1\theta_1 + \cdots + a_d\theta_d))\) exists for all \((a_1, \ldots, a_d) \in \mathbb{R}^d\) in some neighborhood of zero.

(s2) There exists an orthogonalization \(\{ O_\alpha \}_{\alpha \in \mathcal{I}}\) of the system of polynomials \(\{ \theta^\alpha \}_{\alpha \in \mathcal{I}} = \{ \prod_{k \in \mathbb{N}} \theta_{\alpha_k} \}_{\alpha \in \mathcal{I}}\) such that for each \( n \in \mathbb{N} \), the set \(\{ O_\alpha \}_{|\alpha| \leq n}\) spans the same linear subspace \(\mathcal{H}_n\) as \(\{ \theta^k \}_{k \leq n}\) and for each \(|\alpha| = n + 1\)

\[ O_\alpha(\omega) = \theta^\alpha - \text{Projection}_{\mathcal{H}_n} \theta^\alpha. \]

Note that assumption (s1) implies that all \( \theta_k \) have all moments. Also, if every \( \theta_k \) is bounded, then (s1) holds. The basis functions are then rescaled stochastic polynomials \( P_\alpha = a_\alpha O_\alpha \) such that

\[ E(P_\alpha(\omega) P_\beta(\omega)) = \alpha! \delta_{\alpha,\beta}, \]  \hspace{1cm} (12)

so we obtain the relation as in the Gaussian case (10). Then, \( P_{(0,0,\ldots)}(\omega) = 1 \) and

\[ P_{\theta^k}(\omega) = \theta_k(\omega), \quad k \geq 1. \]  \hspace{1cm} (13)
The authors proved in [42] that \( \{P_\alpha\}_{\alpha \in \mathcal{I}} \) form an orthogonal polynomial basis in \( L^2(\Omega, \sigma(\Theta), \mathbb{P}) \).

**Theorem 2.** (Polynomial chaos expansion theorem,[42]) Let the assumptions (s1) and (s2) hold. Then, each \( F \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \), where \( \mathcal{F} = \sigma(\Theta) \), can be represented in the chaos expansion form

\[
F(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha P_\alpha(\omega) \quad (14)
\]

where \( f_\alpha = \frac{1}{\alpha!} \mathbb{E}(F P_\alpha) \). Moreover it holds

\[
\|F\|_{L^2(\mathbb{P})}^2 = \mathbb{E}(F^2) = \sum_{\alpha \in \mathcal{I}} \alpha! f_\alpha^2 < \infty.
\]

For the special case where \( \{\theta_k\}_{k \in \mathbb{N}} \) are independent and identically distributed (i.i.d.) random variables, if there exists an orthogonal set of polynomials \( \{p_n\}_{n \in \mathbb{N}} \) such that \( \mathbb{E} (p_n(\theta_k) p_m(\theta_k)) = n! \delta_{m,n} \), and the moment generating function of \( \theta_k \) is finite at zero, the polynomial chaos basis is given by

\[
P_\alpha(\omega) = \prod_{k=1}^{\infty} p_{\alpha_k}(\theta_k), \quad \alpha \in \mathcal{I}. \quad (15)
\]

Note that the polynomials \( \{p_n\}_{n \in \mathbb{N}} \) from the Askey scheme of hypergeometric orthogonal polynomials and the Sheffer system correspond to several discrete and continuous distribution types [50]. For example, if \( \theta_k \) are i.i.d. random variables with Gamma distribution then \( p_k \) are the Laguerre polynomials and the corresponding \( P_\alpha \) are called the Fourier–Laguerre polynomials, if \( \theta_k \) are i.i.d. with Poisson distribution, the corresponding polynomials \( p_k \) are Charlier polynomials and \( P_\alpha \) are the Fourier–Charlier polynomials, and if \( \theta_k \) are uniformly distributed then \( p_k \) are Legendre polynomials and \( P_\alpha \) are called the Fourier–Legendre polynomials and are denoted by \( L_\alpha \).

### 2.3. Wiener–Legendre chaos representation

Denote by \( \{l_n(x)\}_{n \in \mathbb{N}_0} \) the Legendre polynomials on \([-1, 1]\). These polynomials are defined by the recursion

\[
\begin{align*}
l_0(x) &= 1, \\
l_1(x) &= x, \\
(n + 1) l_{n+1}(x) &= (2n + 1) x l_n(x) - n l_{n-1}(x) \quad \text{for} \ n \geq 1.
\end{align*}
\]

They can be also obtained from Rodrigues’ formula [1,46]

\[
l_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.
\]

The Legendre polynomials satisfy the second-order differential equation \( (1 - x^2) l''_n(x) - 2x l'_n(x) + n(n+1)l_n(x) = 0 \), which appears in physics when solving the Laplace
equation in spherical coordinates [46], and can be rewritten as an eigenvalue problem
\[
\frac{d}{dx} \left( (1-x^2) \frac{d}{dx} l_n(x) \right) = -\lambda_n l_n(x), \quad \text{with } \lambda_n = n(n+1) \text{ being the eigenvalues and } l_n \text{ the corresponding eigenfunctions [1]. These polynomials are orthogonal and it holds}
\[
\int_{-1}^{1} l_m(x) l_n(x) \, dx = \frac{2}{2n+1} \delta_{m,n}, \quad m, n \in \mathbb{N}_0. \tag{17}
\]
The previous property (17) is equivalent to the orthogonality relation with respect to the uniform measure on \([-1, 1]\), i.e., the measure with the constant weighting function \(w(x) = \frac{1}{2}\). This is seen by rewriting (17) in the form (5) as
\[
\int_{-1}^{1} l_m(x) l_n(x) \frac{1}{2} \, dx = \frac{1}{2n+1} \delta_{m,n}, \quad m, n \in \mathbb{N}_0. \tag{18}
\]
In this section, we consider square integrable random variables and stochastic processes on a white noise probability space \((\Omega, \mathcal{F}, \nu)\) with the measure \(\nu\) generated by the uniform distribution. We define the \(\alpha\)th Fourier–Legendre polynomial as the product
\[
L_{\alpha}(\omega) = \prod_{i \in \mathbb{N}} \sqrt{(2\alpha_i + 1) \cdot \alpha_i^!} \cdot l_{\alpha_i} \left( \frac{\xi_i(\omega)}{\sqrt{3}} \right), \quad \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}, \tag{19}
\]
where \(\{l_n\}_{n \in \mathbb{N}_0}\) are the Legendre polynomials and \(\xi = (\xi_1, \xi_2, \ldots)\) is a sequence of independent uniformly distributed random variables on \([-\sqrt{3}, \sqrt{3}]\) with zero mean and unit variance, i.e., for the each component \(\xi_i, i \in \mathbb{N}\) we have \(E(\xi_i) = 0\) and \(\text{Var}(\xi_i) = \mathbb{E}(\xi_i^2) = 1\). Note that the product in (19) is finite since each \(\alpha \in \mathcal{I}\) has only finitely many nonzero components. Particularly,
\[
L_{(0,0,\ldots)}(\omega) = 1 \quad \text{and} \quad L_{\varepsilon(\omega)}(\omega) = \xi_k(\omega) \quad \text{for } k \in \mathbb{N}. \tag{20}
\]
We also have
\[
E(L_{(0,0,\ldots)}(\omega)) = 1 \quad \text{and} \quad E(L_{\alpha}(\omega)) = 0 \quad \text{for } |\alpha| > 0, \tag{21}
\]
since \(\xi(\omega)\) has zero mean. Moreover, from the orthogonality (17) of the Legendre polynomials we obtain that the family of the Fourier–Legendre polynomials \(\{L_{\alpha}\}_{\alpha \in \mathcal{I}}\) is also orthogonal and
\[
E(L_{\alpha} L_{\beta}) = E(L_{\alpha}^2) \cdot \delta_{\alpha,\beta} = \alpha! \cdot \delta_{\alpha\beta} \tag{22}
\]
for all \(\alpha, \beta \in \mathcal{I}\). The scaling factor \(\sqrt{(2\alpha_i + 1) \cdot \alpha_i^!}\) is included in the definition of the Fourier–Legendre polynomials (19) in order to obtain the same values for the variances \(\text{Var}(L_{\alpha}(\omega))\) as the variances \(\text{Var}(H_{\alpha}(\omega))\) in the Gaussian case (10). Clearly, \(\text{Var}(L_{\alpha}(\omega)) = E(L_{\alpha}^2(\omega)) = \alpha!\) for \(|\alpha| > 0\).

Now, we formulate a special form of Theorem 2 for this case.
Theorem 3. (Wiener–Legendre chaos expansion theorem) Each random variable \( F \in L^2(\nu) \) can be uniquely represented in the form

\[
F(\omega) = \sum_{\alpha \in I} f_\alpha L_\alpha(\omega),
\]

where

\[
f_\alpha = \frac{1}{\mathbb{E}(L_\alpha^2)} \mathbb{E}(FL_\alpha) = \frac{1}{\alpha!} \mathbb{E}(FL_\alpha), \quad \alpha \in I
\]
is the corresponding sequence of real coefficients. Moreover, it holds

\[
\|F\|_{L^2(\nu)}^2 = \sum_{\alpha \in I} f_\alpha^2 \mathbb{E}(L_\alpha^2) = \sum_{\alpha \in I} f_\alpha^2 \alpha! < \infty.
\]

Remark 3. We note here that the chaos representation (23) of a random variable with finite second moment with respect to the underlying probability measure \( \nu \) can be extended to square integrable stochastic processes, where a family of real numbers \( f_\alpha \) is replaced by an appropriate family of functions with values in a certain Banach space \( X \). Particularly, an \( X \)-valued square integrable process \( u \) can be represented as

\[
u = \sum_{\alpha \in I} u_\alpha L_\alpha, \quad (24)
\]

where the coefficients satisfy \( u_\alpha \in X, \alpha \in I \) and the convergence condition

\[
\sum_{\alpha \in I} \|u_\alpha\|_X^2 \mathbb{E}(L_\alpha^2) = \sum_{\alpha \in I} \|u_\alpha\|_X^2 \alpha! < \infty
\]
holds true. Henceforth, we denote by \( \mathbb{D}(X) \) the set of square integrable \( X \)-valued stochastic processes and write

\[
\mathbb{D}(X) = \{ u = \sum_{\alpha \in I} u_\alpha L_\alpha \mid u_\alpha \in X, \text{s.t. } \sum_{\alpha \in I} \|u_\alpha\|_X^2 \alpha! < \infty \}.
\]

In this context, the notation \( u \in \mathbb{D}(C([0, T], X)) \) means that the coefficients of the square integrable process \( u \) given in the form

\[
u(t, x, \omega) = \sum_{\alpha \in I} u_\alpha(t, x) L_\alpha(\omega), \quad t \in [0, T], \ x \in X, \ \omega \in \Omega \quad (25)
\]
satisfy \( u_\alpha \in C([0, T], X) \) for all \( \alpha \in I \). Additionally, the estimate

\[
\sum_{\alpha \in I} \|u_\alpha\|_{C([0, T], X)}^2 \mathbb{E}(L_\alpha^2) = \sum_{\alpha \in I} \sup_{t \in [0, T]} \|u_\alpha(t)\|_X^2 \alpha! < \infty
\]
holds. Similarly, a process \( u \in \mathbb{D}(C^1([0, T], X)) \) can be represented in the form (25) with coefficients \( u_\alpha \in C^1([0, T], X) \) for all \( \alpha \in I \) such that it holds

\[
\sum_{\alpha \in I} \|u_\alpha\|_{C^1([0, T], X)}^2 \mathbb{E}(L_\alpha^2) = \sum_{\alpha \in I} \left( \sup_{t \in [0, T]} \|u_\alpha(t)\|_X^2 + \sup_{t \in [0, T]} \frac{d}{dt} u_\alpha(t) \right)^2 \alpha! < \infty.
\]
2.4. Karhunen–Loève expansion

The Karhunen–Loève expansion gives a way to represent a stochastic process as an infinite linear combination of orthogonal functions on a bounded interval. It is used to represent spatially varying random inputs in stochastic models. Various applications of the Karhunen–Loève expansion can be found in uncertainty propagation through dynamical systems with random parameter functions [12,16,32].

Theorem 4. (Karhunen–Loève expansion theorem, [16]) Let \( v(x, \omega) \) be a spatially varying square integrable random field defined over the spatial domain \( D \) and a given probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with mean \( \tilde{v}(x) \) and continuous covariance function \( C_v(x_1, x_2) \). Then, \( v(x, \omega) \) can be represented in the form

\[
v(x, \omega) = \tilde{v}(x) + \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} e_k(x) Z_k(\omega),
\]

(26)

where \( \lambda_k \) and \( e_k \), \( k \in \mathbb{N} \) are the eigenvalues and eigenfunctions of the covariance function, i.e., they solve the integral equation

\[
\int_D C_v(x_1, x_2) e_k(x_2) \, dx_2 = \lambda_k e_k(x_1), \quad x_1 \in D, \ k \in \mathbb{N},
\]

(27)

and \( Z_k \) are uncorrelated zero mean random variables that have unit variance.

For some particular covariance functions \( C_v \), the eigenpairs \( (\lambda_k, e_k)_{k \in \mathbb{N}} \) are known a priori, and the eigenvalues \( \lambda_k \) decay as \( k \) increases. In general, the eigenvalues and eigenvectors of the covariance function have to be calculated numerically, i.e., by solving a discrete version of (27). This constitutes the bottleneck of the method as it requires a large number of calculations. The most commonly used correlation function for stochastic processes is the exponential function, also known as the Bessel correlation function. Its Karhunen–Loève expansion can be solved analytically [16]. In the one-dimensional case, it takes the form \( e^{-|x_1 - x_2|} \) with \( x_1, x_2 \in D \). In our numerical example in Section 4, this form of the correlation function is considered.

In practical applications, the series are truncated, i.e., the random field is approximated by

\[
\hat{v}(x, \omega) = \tilde{v}(x) + \sum_{k=1}^{n} \sqrt{\lambda_k} e_k(x) Z_k(\omega),
\]

(28)

which is the finite representation with the minimal mean square error over all such finite representations.

Remark 4. Comparing the representation (26) with the form (23), we note that the random field \( v \) has the structure of the Wiener–Askey polynomial chaos up to order one. If \( \{\xi_k\}_{k \in \mathbb{N}} \) is an independent and identically distributed sequence of uniform random variables over \( [-\sqrt{3}, \sqrt{3}] \), then \( \{\xi_k\}_{k \in \mathbb{N}} \) is the sequence of uncorrelated random variables. A non-Gaussian spatial noise process \( v \) generated by the uniform distribution,
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i.e., of the form (11), can be represented as

\[ v(x, \omega) = \mathbb{E}(v(x, \omega)) + \sum_{k \in \mathbb{N}} v_{\ell(k)}(x) L_{\ell(k)}(\omega) \]

\[ = \mathbb{E}(v(x, \omega)) + \sum_{k \in \mathbb{N}} v_{\ell(k)}(x) \xi_k(\omega), \]  

(29)

where the property (15), i.e., its version (20) for the Fourier–Legendre polynomials, is used. Moreover, the random field \( v \) has the representation of the form (26). By comparing them, we obtain \( \bar{v}(x) = \mathbb{E}(v(x, \omega)) \), \( v_{\ell(k)}(x) = \sqrt{\lambda_k} e_k(x) \), where \( \lambda_k \) and \( e_k \) are the eigenvalues and eigenfunctions of the covariance function \( C_v \), and \( Z_k(\omega) = \xi_k(\omega) \) is a sequence of uncorrelated uniformly distributed zero mean random variables that have unit variance. The truncated version \( \tilde{v} \) of the representation (29) is given by

\[ \tilde{v}(x, \omega) = \bar{v}(x) + \sum_{k=1}^{n} v_{\ell(k)}(x) L_{\ell(k)}(\omega). \]  

(30)

There, \( n \) corresponds to the finite number of random variables of the sequence \( Z = (Z_1, Z_2, \ldots, Z_n) \) that are applied in the approximation. This is used in Sect. 4.

More details on methods based on stochastic polynomial representations can be found, for example, in [3, 11, 16, 32, 42, 50, 51].

3. Splitting methods for SPDEs

In this section, we introduce a new numerical method which combines the Wiener–Askey polynomial chaos expansion [50] with deterministic splitting methods [15]. The method is then applied to problem (3) with non-Gaussian random inputs. First, we are going to state a theorem on the existence and uniqueness of the solution of (1). Then, we recall some convergence results of splitting methods in the deterministic setting. Finally, we provide a convergence analysis of our approach which is the main result of this section. Through this section, we denote \( \mathcal{L} = A + B \).

3.1. Existence and uniqueness of the solution

Recall that a solution of the considered stochastic evolution problem (1) belongs to the space of square integrable stochastic processes whose coefficients are continuously differentiable deterministic functions with values in \( X \). Let \( L(X) \) denote the space of linear and bounded operators on \( X \).

**Definition 1.** A process \( u \) is a (classical) solution of (1) if \( u \in \mathbb{D}(C([0, T], X)) \cap \mathbb{D}(C^1((0, T], X)) \) and if \( u \) satisfies (1).

Let the following assumptions hold:
(A1) Let $\mathcal{L}$ be a coordinatewise operator defined on some domain $\mathcal{D}(\mathcal{L})$ dense in $X$, i.e.,

$$\mathcal{L} u = \sum_{\alpha \in \mathcal{I}} \mathcal{L}(u_\alpha) L_\alpha$$

for $u$ of the form (25). Moreover, let $\mathcal{L}$ be the infinitesimal generator of a $C_0$ semigroup $(S_t)_{t \geq 0}$ of type $(M, w)$, i.e.,

$$\|S_t\|_{\mathcal{L}(X)} \leq M e^{wt}, \quad t \geq 0$$

for some $M > 0$ and $w \in \mathbb{R}$.

(A2) Let $u^0 \in \mathcal{D}(X)$ and $\mathcal{L}u^0 \in \mathcal{D}(X)$, i.e.,

$$\sum_{\alpha \in \mathcal{I}} \|u^0_\alpha\|_X^2 \alpha! < \infty \quad \text{and} \quad \sum_{\alpha \in \mathcal{I}} \|\mathcal{L}u^0_\alpha\|_X^2 \alpha! < \infty.$$

(A3) The noise process is given in the form $G(t, x, \omega) = \sum_{\alpha \in \mathcal{I}} g_\alpha(t, x) L_\alpha \in \mathcal{D}(C^1([0, T], X))$, i.e., it holds

$$\sum_{\alpha \in \mathcal{I}} \|g_\alpha\|^2_{C^1([0, T], X)} \alpha! < \infty.$$

We note here that the derivative is a coordinatewise operator, i.e., for a process $u \in \mathcal{D}(C^1([0, T], X))$ it holds

$$\frac{d}{dt} u(t, \omega) = \frac{d}{dt} \left( \sum_{\alpha \in \mathcal{I}} u_\alpha(t) L_\alpha(\omega) \right) = \sum_{\alpha \in \mathcal{I}} \left( \frac{d}{dt} u_\alpha(t) \right) L_\alpha(\omega).$$

**Theorem 5.** (Existence and uniqueness of the solution) If the assumptions (A1)-(A3) hold, then the stochastic Cauchy problem

$$u_t(t, \omega) = \mathcal{L} u(t, \omega) + G(t, \omega), \quad u(0, \omega) = u^0(\omega)$$  \hspace{1cm} (31)

has a unique solution

$$u(t, \omega) = \sum_{\alpha \in \mathcal{I}} \left( S_t u^0_\alpha + \int_0^t S_{t-s} g_\alpha(s) \, ds \right) L_\alpha(\omega)$$  \hspace{1cm} (32)

in $\mathcal{D}(C^1([0, T], X))$.

**Proof.** We present the main steps of the proof. We are looking for a solution in chaos representation form

$$u(t, \omega) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t) L_\alpha(\omega).$$
Then, by applying the chaos expansion method, the stochastic Eq. (31) is transformed to the infinite system of deterministic problems

\[
\frac{d}{dt} u_\alpha(t) = \mathcal{L} u_\alpha(t) + g_\alpha(t),
\]

\[u_\alpha(0) = u_\alpha^0\]  

(33)

for all \(\alpha \in \mathcal{I}\) that can be solved in parallel. Since \(g_\alpha \in C^1([0, T], X)\), the inhomogeneous initial value problem (33) has a solution \(u_\alpha(t) \in C^1((0, T], X)\) for all \(\alpha \in \mathcal{I}\). Moreover, the solution \(u_\alpha\) is given by

\[u_\alpha(t) = S_t u_\alpha^0 + \int_0^t S_{t-s} g_\alpha(s) \, ds, \quad t \in [0, T];\]  

(34)

see [45]. Thus, for all fixed \(\alpha \in \mathcal{I}\) the solution \(u_\alpha(t)\) exists for all \(t \in [0, T]\), and it is a unique classical solution on the whole interval \([0, T]\). Also,

\[\frac{d}{dt} u_\alpha(t) = S_t \mathcal{L} u_\alpha^0 + \int_0^t S_{t-s} \frac{d}{ds} g_\alpha(s) \, ds + S_t g_\alpha(0), \quad \alpha \in \mathcal{I}, \quad t \in [0, T].\]

Moreover, the series \(\sum_{\alpha \in \mathcal{I}} u_\alpha(t) L_\alpha\) converges in \(\mathbb{D}(C^1([0, T], X))\). Namely, from the assumptions (A1)-(A3) we obtain

\[
\sum_{\alpha \in \mathcal{I}} \|u_\alpha\|_{C^1([0, T], X)}^2 \mathbb{E}(L_\alpha^2) = \sum_{\alpha \in \mathcal{I}} (\sup_{t \in [0, T]} \|u_\alpha(t)\|_X^2 + \sup_{t \in [0, T]} \|\frac{d}{dt} u_\alpha(t)\|_X^2) \mathbb{E}(L_\alpha^2) 
\leq c \sum_{\alpha \in \mathcal{I}} \left(\|u_\alpha^0\|_X^2 + \|\mathcal{L} u_\alpha^0\|_X^2 + \|g_\alpha\|_{C^1([0, T], X)}^2\right) \alpha! < \infty,
\]

where \(c = c(M, w, T) = \max\{3M^2 e^{2wT}, \frac{3M^2}{2w} (e^{2wT} - 1)\}\) is a constant depending on \(M, w\) and \(T\). Namely, from

\[
\sum_{\alpha \in \mathcal{I}} \sup_{t \in [0, T]} \|u_\alpha(t)\|_X^2 \mathbb{E}(L_\alpha^2) 
\leq 2 \sum_{\alpha \in \mathcal{I}} \sup_{t \in [0, T]} \left(\|S_t\|_{L(X)}^2 \|u_\alpha^0\|_X^2 + \int_0^t \|S_{t-s}\|_{L(X)}^2 \|g_\alpha(s)\|_X^2 ds\right) \alpha! 
\leq \sum_{\alpha \in \mathcal{I}} \left(2M^2 e^{2wT} \|u_\alpha^0\|_X^2 + \frac{M^2}{w} (e^{2wT} - 1) \sup_{t \in [0, T]} \|g_\alpha(t)\|_X^2\right) \alpha!
\]

and

\[
\sum_{\alpha \in \mathcal{I}} \sup_{t \in [0, T]} \|\frac{d}{dt} u_\alpha(t)\|_X^2 \mathbb{E}(L_\alpha^2) 
\leq 3 \sum_{\alpha \in \mathcal{I}} \sup_{t \in [0, T]} \left(\|S_t\|_{L(X)}^2 \|\mathcal{L} u_\alpha^0\|_X^2 + \int_0^t \|S_{t-s}\|_{L(X)}^2 \|\frac{d}{ds} g_\alpha(s)\|_X^2 ds \right) + \|S_t\|_{L(X)}^2 \|g_\alpha(0)\|_X^2 \alpha! 
\leq \sum_{\alpha \in \mathcal{I}} \left(3M^2 e^{2wT} \left(\|\mathcal{L} u_\alpha^0\|_X^2 + \|g_\alpha(0)\|_X^2\right) + \frac{3M^2}{2w} (e^{2wT} - 1) \sup_{t \in [0, T]} \|\frac{d}{dt} g_\alpha(t)\|_X^2\right) \alpha!\]
we conclude \[ \sum_{\alpha \in I} \| u_\alpha \|_{C^1([0,T],X)}^2 \alpha! < \infty. \]

\[ \square \]

**Remark 5.** If an operator \( A \) is the infinitesimal generator of a \( C_0 \) semigroup and \( B \) is a bounded operator, then the operator \( \mathcal{L} = A + B \) is also the infinitesimal generator of a \( C_0 \) semigroup and Theorem 5 holds. In particular, Theorem 5 also holds for analytic semigroups. Additionally, Theorem 5 can be formulated for a general class of orthogonal polynomials \( P_\alpha, \alpha \in I \) given in Section 2.2, since the system (33) of deterministic problems is independent of the type of noise arising in the stochastic Cauchy problem (31).

### 3.2. Splitting methods for deterministic problems

We briefly recall the convergence of two operator resolvent splitting methods: resolvent Lie splitting (a first-order method) and trapezoidal resolvent splitting (a second-order method).

Resolvent splitting methods for the time integration of abstract evolution equations were studied in [23]. The convergence properties of splitting methods for inhomogeneous evolution equations were analyzed in [44]. Other splitting methods were also considered in the literature. For example, exponential splitting methods for homogeneous problems with unbounded operators were presented in [21,22]. The inhomogeneous case was studied in [15]. Error bounds for exponential operator splittings were further discussed in [31].

#### 3.2.1. Analytic setting

Let \( X \) be an arbitrary Hilbert space with the norm \( \| \cdot \| \) and inner product \((\cdot, \cdot)\). In the following sections, an unbounded operator \( G : D(G) \subset X \to X \) is denoted by \((D(G), G)\), where \( D(G) \) is the domain of \( G \) in \( X \). Further, \( \rho(G) \) denotes the resolvent of \( G \). For \( t \in [0, T] \), we consider the inhomogeneous evolution equation

\[ \frac{d}{dt} u(t) = \mathcal{L} u(t) + g(t) = A u(t) + B u(t) + g(t), \quad u(0) = u^0, \]

where \((D(L), L), (D(A), A)\) and \((D(B), B)\) are linear unbounded operators in \( X \) such that \( D(L) \subset D(A) \cap D(B) \) and \( g : [0, T] \to X \). We recall the main results from [23,44].

Let the following assumptions hold:

(a1) The operators \((D(L), L), (D(A), A)\) and \((D(B), B)\) are maximal dissipative and densely defined in \( X \).

(a2) \( D(L^2) \subset D(AB) \)

(a3) Let \( 0 \in \rho(L) \), let \( L^{-1} g(t) \in D(AB) \) for all \( t \in [0, T] \) and

\[ \max_{0 \leq t \leq T} \| A B L^{-1} g(t) \| \leq c \]
with a moderate constant $c$.

Recall that an operator $(D(G), G)$ on a Hilbert space $X$ is maximal dissipative in $X$ if and only if $\Re (Gx, x) \leq 0$ for all $x \in D(G)$ and range $(I - G) = X$.

Since we assumed that $X$ is a Hilbert space, every maximal dissipative operator in $X$ is densely defined. The assumption $(a1)$ is equivalent to claiming that the operators generate $C_0$ semigroups of contractions on $X$ and the related resolvents are all nonexpansive on $X$; see [45]. Additionally, from $(a1)$ the following estimates hold

$$
\| (I - hA)^{-1} \| \leq 1 \quad \text{and} \quad \| (I - hB)^{-1} \| \leq 1 \quad \text{for all} \quad h \geq 0.
$$

We recall briefly the results from regularity theory for analytic semigroups needed in the following sections.

**Theorem 6.** ([40]) Let $L$ be the generator of an analytic semigroup and let the data of problem (35) satisfy

$$
u^0 \in D(L), \quad g \in C^0([0, T], X) \tag{36}
$$

for some $\theta > 0$. Then, the exact solution of problem (35) is given by the variation of constants formula

$$
u(t) = e^{tL}\nu^0 + \int_0^t e^{(t-\tau)L} g(\tau) \, d\tau, \quad 0 \leq t \leq T. \tag{37}
$$

It possesses the regularity

$$
u \in C^1([0, T], X) \cap C([0, T], D(L)).
$$

The same regularity is obtained if $g$ is only continuous but has a slightly improved spatial regularity; see [40, Corollary 4.3.9].

**Theorem 7.** ([44]) Let $L$ be the generator of an analytic semigroup. Under the further assumptions

$$
u^0 \in D(L), \quad Lu^0 + g(0) \in D(L), \quad g \in C^{1+\theta}([0, T], X) \tag{38}
$$

for some $\theta > 0$, the solution (37) of the evolution Eq. (35) possesses the improved regularity

$$
u \in C^2([0, T], X) \cap C^1([0, T], D(L)). \tag{39}
$$

In the following, we present two deterministic resolvent splitting methods [29], the resolvent Lie splitting and the resolvent trapezoidal splitting, that were both applied to inhomogeneous evolution Eq. (35) in [44].
3.2.2. Resolvent Lie splitting

The exact solution of the evolution Eq. (35) is given by the variation of constants formula (37). Then, at time $t_{n+1} = t_n + h$, with a positive step size $h$, the solution can be written as

$$u(t_{n+1}) = e^{hL}u(t_n) + \int_0^h e^{(h-s)L} g(t_n + s) \, ds.$$  

After expanding $g(t_n + s)$ in Taylor form, we obtain

$$u(t_{n+1}) = e^{hL}u(t_n) + \int_0^h e^{(h-s)L} \left(g(t_n) + sg'(t_n) + \int_{t_n}^{t_n+s} (t_n + s - \tau) g''(\tau) d\tau\right) ds;$$

see [44].

For resolvent Lie splitting, the numerical solution of (35) at time $t_{n+1}$ is denoted by $u^{n+1}$ and it is given by

$$u^{n+1} = (I - hB)^{-1}(I - hA)^{-1}(u^n + h g(t_n)).$$  

(40)

**Theorem 8.** (Resolvent Lie splitting, [44]) Let the assumptions (a1), (a2) and (a3) be fulfilled and let the solution satisfy (39). Then the resolvent Lie splitting (40) is first-order convergent, i.e., the global error satisfies the bound

$$\|u(t_n) - u^n\| \leq Ch, \quad 0 \leq t_n \leq T$$  

with a constant $C$ that can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$.

**Remark 6.** The constant $C$ in (41) depends on derivatives of the solution $u$ and on $AB^{-1}L^{-1}g(t)$, which are uniformly bounded on $[0, T]$ due to the assumptions of Theorem 8. A detailed proof is given in [44].

In particular, for a homogeneous evolution problem ($g = 0$) the global error (41) can be estimated as

$$\|u(t_n) - u^n\| \leq ch \left( \|u^0\| + \|L^2 u^0\| + \|L^2 u^0\| \right),$$

where the positive constant $c$ is independent on $n$ and $h$; see [23].

We note that the full-order convergence of Lie resolvent splitting only requires additional smoothness in space of the inhomogeneity $g$.

3.2.3. The trapezoidal splitting

For a trapezoidal splitting method, the numerical solution of (35) at time $t_{n+1} = t_n + h$ with a positive time step size $h$ is given by

$$u^{n+1} = \left(I - \frac{h}{2}B\right)^{-1}\left(I - \frac{h}{2}A\right)^{-1}\left(\left(I + \frac{h}{2}A\right)\left(I + \frac{h}{2}B\right)u^n + \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right)\right)$$  

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with \( u^0 = u(0) \).

As we are considering a second-order method, we need more regularity of the solution. For analytic semigroups, this requirement can be expressed in terms of the data.

The following modification of the assumption \((a3)\) is needed:

\[(a4) \text{ Let } 0 \in \rho(\mathcal L), \text{ let } \mathcal L^{-1}g'(t) \in \mathcal D(AB) \text{ for all } t \in [0, T] \text{ and} \]

\[
\max_{0 \leq t \leq T} \|AB\mathcal L^{-1}g'(t)\| \leq c
\]

with a moderate constant \( c \).

Since we assumed \( X \) to be a Hilbert space, it follows from assumption \((a1)\) that the estimates

\[
\|(I + hA)(I - hA)^{-1}\| \leq 1 \quad \text{and} \quad \|(I + hB)(I - hB)^{-1}\| \leq 1
\]

hold for all \( h > 0 \).

**Theorem 9.** ([44]) Let \( \mathcal L \) be the generator of an analytic semigroup. If

\[
g \in C^{2+\theta}([0, T], X),
\]

\[
u^0 \in \mathcal D(\mathcal L), \quad \mathcal L\nu^0 + g(0) \in \mathcal D(\mathcal L), \quad \mathcal L^2\nu^0 + \mathcal Lg(0) + g'(0) \in \mathcal D(\mathcal L)
\]

for some \( \theta > 0 \), then the exact solution \((37)\) of the inhomogeneous evolution Eq. \((35)\) satisfies

\[
u \in C^3([0, T], X) \cap C^2([0, T], D(\mathcal L)).
\]

**Theorem 10.** (The trapezoidal splitting method, [44]) Let the assumptions \((a1), (a2)\) and \((a4)\) be fulfilled and let the solution satisfy \((44)\). Then the trapezoidal splitting method \((42)\) is second-order convergent, i.e., the global error satisfies the bound

\[
\|\nu(t_n) - \nu^n\| \leq Ch^2, \quad 0 \leq t_n \leq T
\]

with a constant \( C \) that can be chosen uniformly on \([0, T]\) and, in particular, independently of \( n \) and \( h \).

**Remark 7.** The constant \( C \) in \((45)\) depends on derivatives of the solution \( u \) and on \( AB\mathcal L^{-1}g'(t) \), which are uniformly bounded on \([0, T]\) due to the assumptions of Theorem 10. More details are given in [44].

3.3. Convergence analysis

In order to solve problem \((31)\) numerically, we approximate the solution \( u \) by the truncated chaos representation form

\[
\tilde u = \sum_{\alpha \in \mathcal I_{m,K}} u_{\alpha} L_{\alpha}.
\]
where $I_{m,K} = \{ \alpha \in I \, | \, \alpha = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots), |\alpha| \leq K \}$. Here, $K \in \mathbb{N}$ is the highest degree of the employed Legendre polynomials and $m \in \mathbb{N}$ is the number of random variables we want to use in the approximation (46). The $m$-dimensional random vector $\xi = (\xi_1, \ldots, \xi_m)$ has independent and identically distributed components $\xi_i \sim U([-1, 1])$ for $i = 1, \ldots, m$. The choice of $m$ and $K$ influences the accuracy of the approximation. They can be chosen so that the norm of the approximation remainder $u - \tilde{u}$ is smaller than a given tolerance. The sum in (46) has
\[ P = \frac{(m + K)!}{m! K!} \] terms, which means that $P$ coefficients of the solution will be computed. Thus, only the first $P$ equations of system (33) are solved and in this way the approximation of the solution is obtained. The global error of the proposed numerical scheme depends on the error generated by the truncation of the chaos expansion and the error of the discretization method. Also, the statistics $\mathbb{E}(\tilde{u})$ and $\text{Var}(\tilde{u})$ of the approximated solution can be calculated in terms of the obtained discretized coefficients. For more details on the truncation (46); see for instance [50]. In the following, we consider the two numerical resolvent splitting methods, Lie splitting and trapezoidal splitting, and provide an error analysis for both of them.

**Theorem 11.** (Error generated by the truncation of the Wiener–Legendre chaos expansion) Let $\tilde{u}$ denote the truncated chaos representation of the solution $u$ of the stochastic evolution problem (31) given in the form (46). Let the assumptions (A1)-(A3) hold. Then, $\tilde{u}$ approximates the solution $u$ and the approximation error satisfies the a priori bound
\[ \|u - \tilde{u}\|_{C^1([0,T],X)}^2 \leq c \sum_{\alpha \in I \setminus I_{m,K}} \left( \|u_0^\alpha\|_X^2 + \|\mathcal{L}u_0^\alpha\|_X^2 + \|g_\alpha\|_{C^1([0,T],X)}^2 \right) \mathbb{E}(L_\alpha^2) < \infty, \] where $c = c(M, w, T)$ is a constant depending on $M$, $w$ and $T$.

**Proof.** The approximation error due to the elimination of the higher order components of the Wiener–Legendre chaos expansion and the truncation of the noise term is obtained by
\[ \|u - \tilde{u}\|_{C^0([0,T],X)}^2 = \sum_{\alpha \in I \setminus I_{m,K}} u_\alpha L_\alpha \|\mathbb{E}(L_\alpha^2) \] which is finite by the assumptions (A1)-(A3). In the last estimate, we employed the bound derived in the proof of Theorem 5. Here $c$ is a constant depending on $M$, $w$ and $T$. \qed
Remark 8. We note that Theorem 11 can also be formulated for the general class of polynomials defined in Section 2.2.

Remark 9. Note that we can characterize the approximation error appearing in Theorem 11 by imposing slightly stronger assumptions on the data. In applications, the subset $\mathcal{I}_{m,K} \subset \mathcal{I}$ is chosen in such a way that the approximation error $u - \tilde{u}$ is smaller than a given tolerance. Thus, in view of the assumptions (A1)-(A3), we assume that there exists $q > 1$ such that

$$C_\alpha \leq c \cdot \frac{(2N)^{-q\alpha}}{\alpha!} \quad (49)$$

holds for $\alpha \in \mathcal{I}$, where $C_\alpha = \|u_\alpha^0\|^2_X + \|Lu_\alpha^0\|^2_X + \|g_\alpha\|^2_{C^1([0,T],X)}$. Then, the error bound is of the form

$$\|u - \tilde{u}\|_{D(C^1([0,T],X))}^2 \leq c \sum_{\alpha \in \mathcal{I}\setminus \mathcal{I}_{m,K}} C_\alpha \alpha!$$

$$\leq c \sum_{\alpha \in \mathcal{I}\setminus \mathcal{I}_{m,K}} (2N)^{-q\alpha}$$

$$= c \left( \sum_{i=m+1}^{\infty} \sum_{j=1}^{K} \sum_{\alpha \in \mathcal{I}_{i,j}} (2N)^{-q\alpha} + \sum_{j=K+1}^{\infty} \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{I}_{i,j}} (2N)^{-q\alpha} \right)$$

$$\leq c \left( e^{\frac{2}{q-1}} \cdot \frac{q}{(q-1)} \cdot Kq^{-1} + e^{\frac{1}{2q-1}(q-1)} \cdot \frac{1}{2q(m+1)(q-1)} \right).$$

The bound in the last step in (50) is proven in [6].

In order to apply the splitting methods in the setting of [44], we are going to consider the analytic case. We replace the assumption (A1) with the assumption:

(B1) Let $(A, D(A))$, $(B, D(B))$ and $(\mathcal{L}, D(\mathcal{L}))$ be coordinatewise operators that generate analytic semigroups of contractions on $X$. Let $D(\mathcal{L}^2) \subseteq D(AB)$.

Further, for the case of the resolvent Lie splitting we replace the assumptions (A2) and (A3) by:

(B2) The noise process given by

$$G = \sum_{\alpha \in \mathcal{I}} g_\alpha L_\alpha \quad (51)$$

belongs to $\mathbb{D}(C^{1+\theta}([0,T],X))$ for some $\theta > 0$, i.e.,

$$\sum_{\alpha \in \mathcal{I}} \|g_\alpha\|^2_{C^{1+\theta}([0,T],X)} \mathbb{E}(L^2_\alpha) < \infty \quad (52)$$

holds.

(B3) Let $u^0 \in \mathbb{D}(D(\mathcal{L}))$ and $Lu^0 + G(0) \in \mathbb{D}(D(\mathcal{L}))$, i.e.,

$$\sum_{\alpha \in \mathcal{I}} \|u_\alpha^0\|^2_{D(\mathcal{L})} \mathbb{E}(L^2_\alpha) < \infty \quad \text{and} \quad \sum_{\alpha \in \mathcal{I}} \|Lu_\alpha^0 + g_\alpha(0)\|^2_{D(\mathcal{L})} \mathbb{E}(L^2_\alpha) < \infty.$$
(B4) Let $0 \in \sigma(L)$, let $L^{-1}G(t) \in \mathbb{D}(\mathcal{D}(AB))$ for all $t \in [0, T]$ and let the coefficients $g_\alpha$ of $G$, given by (51), satisfy the estimate

$$\max_{0 \leq t \leq T} \|ABL^{-1}g_\alpha(t)\| \leq d_\alpha, \quad 0 \leq t \leq T$$

with a moderate constant $d_\alpha$ for each $\alpha \in \mathcal{I}$.

Note that under these assumptions the existence theorem, Theorem 5, still holds. Particularly, for the resolvent Lie splitting it reads:

**Theorem 12.** Let $L$ be the generator of an analytic semigroup. Under the assumptions (B2) and (B3), the solution (32) of the stochastic evolution problem (1) possesses the improved regularity

$$u \in \mathbb{D}(C^2([0, T], X)) \cap \mathbb{D}(C^1([0, T], D(L))).$$

**Proof.** By the method of chaos expansion, the stochastic evolution problem (1) transforms to the system of deterministic problems (33). From (B2) and (B3), it follows that $u_0^\alpha$ and $g_\alpha$ for each $\alpha \in \mathcal{I}$ satisfy the assumptions (38). After applying Theorem 7, we obtain the improved regularity $u_\alpha \in C^2([0, T], X) \cap C^1([0, T], D(L)), \alpha \in \mathcal{I}$. □

**Theorem 13.** (Discretization error, the resolvent Lie splitting) Let the assumptions (B1)–(B4) be fulfilled. Let $\tilde{u}$ denote the truncated chaos representation of the solution $u$ of the stochastic evolution problem (31) given in the form (46). Further, let $\tilde{u}_{\text{dis}}$ be the square integrable process

$$\tilde{u}_{\text{dis}}^n = \sum_{\alpha \in \mathcal{I}_{m,K}} u_{\alpha,\text{dis}}^n L_\alpha,$$

where its coefficients $u_{\alpha,\text{dis}}^n, \alpha \in \mathcal{I}_{m,K}$ are the numerical approximations of $u_\alpha$ for $\alpha \in \mathcal{I}_{m,K}$ at time $t_n = nh$ with a positive step size $h$. Then the following error bound, obtained by the Lie splitting (40)

$$\|u_\alpha(t_n) - u_{\alpha,\text{dis}}^n\|_X \leq c_\alpha \cdot h, \quad \alpha \in \mathcal{I}_{m,K}$$

holds, where the constant $c_\alpha$ can be chosen uniformly on $[0, T]$ and, in particular, independently of $n$ and $h$. Moreover, the difference $\tilde{u}(t_n) - \tilde{u}_{\text{dis}}^n$ can be estimated uniformly on $0 \leq t_n \leq T$ by the a priori bound

$$\|\tilde{u}(t_n) - \tilde{u}_{\text{dis}}^n\|^2_{\mathbb{D}(X)} \leq \sum_{\alpha \in \mathcal{I}_{m,K}} \|u_\alpha(t_n) - u_{\alpha,\text{dis}}^n\|^2_X \mathbb{E}(L_\alpha^2) \leq h^2 \sum_{\alpha \in \mathcal{I}_{m,K}} c_\alpha^2 \alpha! < \infty.$$

**Proof.** The coefficients $u_\alpha$, for each $\alpha \in \mathcal{I}_{m,K}$ are the exact solutions of the deterministic initial value problems (33) and $u_{\alpha,\text{dis}}^n$ are their numerical approximations obtained
by the resolvent Lie splitting (40). Moreover, \( u_\alpha \) satisfy the assumptions (38) for all \( \alpha \in \mathcal{I} \). Thus, we can apply Theorem 8 to each initial value problem (33) and obtain the global estimate (41), i.e., \( \| u_\alpha(t_n) - u_{\alpha,\text{dis}}^n \|_X \leq c_\alpha \cdot h \) for each \( \alpha \in \mathcal{I}_{m,K} \).

From Parseval’s identity and the orthogonality of the polynomial basis \( \{ L_\alpha \} \), and using the error (55) for the resolvent Lie splitting, we obtain

\[
\| \tilde{u}(t_n) - \tilde{u}_{\text{dis}}^n \|_{\mathcal{D}(X)}^2 = \| \sum_{\alpha \in \mathcal{I}_{m,K}} u_\alpha(t_n) L_\alpha - \sum_{\alpha \in \mathcal{I}_{m,K}} u_{\alpha,\text{dis}}^n L_\alpha \|_{\mathcal{D}(X)}^2 \\
= \sum_{\alpha \in \mathcal{I}_{m,K}} \| u_\alpha(t_n) - u_{\alpha,\text{dis}}^n \|_X^2 \alpha!
\leq h^2 \sum_{\alpha \in \mathcal{I}_{m,K}} c_\alpha^2 \alpha! < \infty,
\]

since the sum is finite. This leads to the desired result. \( \square \)

**Remark 10.** By studying the proof of Theorem 8 given in [44], we get a better insight into the structure of the constant \( c_\alpha \). Namely, it can be represented as a product of two constants. The first one depends on the operators, i.e., it depends on \( M, w \) and \( T \), and it is independent of \( n, h \) and \( \alpha \) (see also Remark 6), while the second one depends only on the data.

If we impose stronger assumptions on the initial data, we can say more about the second constant. Assume that (49), given in Remark 9, and the following modification of (B4)

\[
\max_{0 \leq t \leq T} \| AB \mathcal{L}^{-1} g_\alpha(t) \| \leq c \cdot \frac{(2N)^{-q\alpha}}{\sqrt{\alpha!}}, \quad 0 \leq t \leq T
\]
hold for some \( q > 1 \) and all \( \alpha \). Here the constant \( c \) depends only on \( M, w \) and \( T \). Then we obtain the following estimate of the discretization error for the resolvent Lie splitting

\[
\| \tilde{u}(t_n) - \tilde{u}_{\text{dis}}^n \|_{\mathcal{D}(X)}^2 = \sum_{\alpha \in \mathcal{I}_{m,K}} \| u_\alpha(t_n) - u_{\alpha,\text{dis}}^n \|_X^2 \alpha!
\leq h^2 c^2 \sum_{\alpha \in \mathcal{I}_{m,K}} (2N)^{-2q\alpha} = h^2 c^2 A_\alpha < \infty.
\]

Since each \( \alpha \in \mathcal{I}_{m,K} \) is of the form \( \alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots) \) such that its length \( |\alpha| \leq K \), then for \( q > 1 \) the constant \( A_\alpha \) can be estimated in the following way

\[
A_\alpha = \sum_{\alpha \in \mathcal{I}_{m,K}} (2N)^{-2q\alpha} = \sum_{j=0}^{K} \sum_{|\alpha|=j} (2N)^{-2q\alpha} = \sum_{j=0}^{K} \sum_{|\alpha|=j} \left( \prod_{i=1}^{m} 2^{\alpha_i} i^{\alpha_i} \right)^{-2q}
\leq \sum_{j=0}^{K} 2^{-2qj} \cdot 2^{jm} \leq 1 - 2^{(m-2q)(K+1)} \quad \frac{1}{1 - 2^{m-2q}}.
\]
In the last part, we used the combinatorial result that a number \( n \in \mathbb{N} \) can be written as a sum of \( k \) numbers, i.e., \( n = x_1 + \cdots + x_k, x_i \in \{0, 1, \ldots, n\} \) in \( (n+k-1) = (n+k-1) \) different ways.

In the case of the trapezoidal resolvent splitting, we need the following additional assumptions:

(B5) The noise process \( G \) given by (51) belongs to \( \mathbb{D}(C^{2+\theta}([0, T], X)) \) for some \( \theta > 0 \).

(B6) Let \( \mathcal{L}^2u^0 + \mathcal{L}G(0) + G'(0) \in \mathbb{D}(\mathcal{D}(\mathcal{L})) \), i.e.,

\[
\sum_{\alpha \in \mathcal{I}} \|\mathcal{L}^2u^0_\alpha + \mathcal{L}g_\alpha(0) + g'_\alpha(0)\|^2_{\mathcal{D}(\mathcal{L})} \mathbb{E}(\mathcal{L}^2_\alpha) < \infty.
\]

(B7) Let \( 0 \in \rho(\mathcal{L}) \), let \( \mathcal{L}^{-1}G'(t) \in \mathbb{D}(\mathcal{D}(AB)) \) for all \( t \in [0, T] \) and let the coefficients \( g_\alpha \) of \( G \) given by (51), satisfy the estimate

\[
\max_{0 \leq t \leq T} \|AB\mathcal{L}^{-1}g'_\alpha(t)\| \leq d_\alpha
\]

with a moderate constant \( d_\alpha \) for each \( \alpha \in \mathcal{I} \).

**Theorem 14.** Let \( \mathcal{L} \) be the generator of an analytic semigroup. Under the assumptions (B3), (B5) and (B6), the solution (32) of the stochastic evolution problem (1) has the improved regularity

\[
u \in \mathbb{D}(C^3([0, T], X)) \cap \mathbb{D}(C^2([0, T], \mathcal{D}(\mathcal{L}))). \tag{57}
\]

**Proof.** The method of chaos expansion transforms the stochastic evolution problem (1) to the system of deterministic problems (33). From (B3), (B5) and (B6), it follows that \( u^0_\alpha \) and \( g_\alpha \) for each \( \alpha \in \mathcal{I} \) satisfy the assumptions (43). Then, the improved regularity \( u_\alpha \in C^3([0, T], X) \cap C^2([0, T], \mathcal{D}(\mathcal{L})) \), for \( \alpha \in \mathcal{I} \) follows from Theorem 9. \( \square \)

**Theorem 15.** (Discretization error, the trapezoidal resolvent splitting) Let the assumptions (B1), (B3) and (B5)–(B7) be fulfilled. Let \( \tilde{u} \) denote the truncated chaos representation of the solution \( u \) of the stochastic evolution problem (31) given in the form (46). Further, let \( \tilde{u}^{n}_{dis} \) be the square integrable process given in the form (54), where its coefficients \( u^{n}_{\alpha,dis} \), \( \alpha \in \mathcal{I}_{m,K} \) are the numerical approximations of \( u_\alpha \) for \( \alpha \in \mathcal{I}_{m,K} \) at time \( t_n = nh \) with a positive step size \( h \). Then, the following error bound, obtained by the trapezoidal splitting (42)

\[
\|u_\alpha(t_n) - u^{n}_{\alpha,dis}\| \leq c_\alpha h^2, \quad \alpha \in \mathcal{I}_{m,K} \tag{58}
\]

holds, where the constant \( c_\alpha \) can be chosen uniformly on \([0, T]\) and, in particular, independently of \( n \) and \( h \). Moreover, the difference \( \tilde{u}(t_n) - \tilde{u}^{n}_{dis} \) can be estimated uniformly on \( 0 \leq t_n \leq T \) by the a priori bound

\[
\|\tilde{u}(t_n) - \tilde{u}^{n}_{dis}\|^2_{\mathbb{D}(X)} \leq h^4 \sum_{\alpha \in \mathcal{I}_{m,K}} c^2_\alpha \alpha! < \infty.
\]
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Proof. From the assumptions, it follows that the coefficients $u_0^\alpha$ and $g_\alpha$ satisfy (43) for each $\alpha \in I_{m,K}$. We apply the trapezoidal resolvent splitting (42) in order to obtain the approximation $u_{\alpha, dis}^n$ of the exact solution $u_\alpha(t_n)$ evaluated at $t_n$ of the initial value problem (33) for each $\alpha \in I_{m,K}$. Thus, by Theorem 10 we obtain the global error estimate (45), i.e., the bound (58) holds for each $\alpha \in I_{m,K}$. This then leads to

$$
\| \tilde{u}(t_n) - \tilde{u}_{\alpha, dis}^n \|_{D(X)}^2 = \sum_{\alpha \in I_{m,K}} \| u_\alpha(t_n) - u_{\alpha, dis}^n \|_{X}^2 \alpha! \\
\leq h^4 \sum_{\alpha \in I_{m,K}} c_\alpha^2 \alpha! 
$$

which is finite. □

Remark 11. From the proof of Theorem 10 presented in [44], we again obtained that the constant $c_\alpha$ is composed of two separate parts. First part depends on the operators, i.e., it is a constant that depends only on $M, w$ and $T$, and it is independent of $n, h$ and $\alpha$ (see also Remark 7), while the second part is data dependent, i.e., it is a constant given in terms of $\alpha$.

Under stronger assumptions on the initial data, we can investigate how the constant $c_\alpha$ depends on $\alpha$. Assume that the condition (49) and the following modification of (B7)

$$
\max_{0 \leq t \leq T} \| AB \mathcal{L}^{-1} g'_\alpha(t) \| \leq c \cdot \left( \frac{2N}{q^2} \right)^{-\frac{q^2}{2}} \sqrt{\alpha} 
$$

hold for some $q > 1$ and all $\alpha$, with $c$ depending only on $M, w$ and $T$. Then we obtain the following estimate of the discretization error for the trapezoidal resolvent splitting

$$
\| \tilde{u}(t_n) - \tilde{u}_{\alpha, dis}^n \|_{D(X)} = \sum_{\alpha \in I_{m,K}} \| u_\alpha(t_n) - u_{\alpha, dis}^n \|_{X}^2 \alpha! \leq h^4 c_\alpha^2 A_\alpha < \infty, 
$$

where $A_\alpha$ is given in (56).

The full error estimates of the Wiener–Legendre chaos expansion combined with the two splitting methods are given in the following theorem.

Theorem 16. (Full error estimate)

(1) Let the assumptions of Theorem 13 hold. Then, the full error estimate of the Wiener–Legendre chaos expansion combined with the resolvent Lie splitting satisfies the following bound

$$
\| u(t_n) - \tilde{u}_{\alpha, dis}^n \|_{D(X)}^2 \leq 2c \sum_{\alpha \in I_{\alpha, \alpha, K}} \left( \| u_0^\alpha \|_X^2 + \| \mathcal{L} u_0^\alpha \|_X^2 + \| g_\alpha \|_{C^1([0,T],X)}^2 \right) \alpha! \\
+ 2 h^2 \sum_{\alpha \in I_{m,K}} c_\alpha^2 \alpha!. 
$$

(59)
(2) Let the assumptions of Theorem 15 hold. Then, the full error estimate of the Wiener–Legendre chaos expansion combined with the trapezoidal resolvent splitting satisfies the bound
\[
\|u(t_n) - \tilde{u}_{\text{dis}}^n\|_{D(X)}^2 \leq 2c \sum_{\alpha \in \mathcal{I} \setminus \mathcal{I}_m, K} \left( \|u_0^0\|_X^2 + \|\mathcal{L}u_0^0\|_X^2 + \|g_\alpha\|_{C^1([0,T],X)}^2 \right) \alpha! + 2h^4 \sum_{\alpha \in \mathcal{I}_{m,K}} c_\alpha^2 \alpha!.
\]

Proof. The full error estimate reads
\[
\|u(t_n) - \tilde{u}_{\text{dis}}^n\|_{D(X)}^2 = \sum_{\alpha \in \mathcal{I}} \|u_\alpha(t_n) - \tilde{u}_{\alpha,\text{dis}}^n\|_{D(X)}^2 \leq 2 \sum_{\alpha \in \mathcal{I} \setminus \mathcal{I}_m, K} \|u_\alpha(t_n)\|_X^2 \mathbb{E}(L_\alpha^2) + 2 \sum_{\alpha \in \mathcal{I}_{m,K}} \|\tilde{u}_\alpha(t_n) - u_{\alpha,\text{dis}}^n\|_X^2 \mathbb{E}(L_\alpha^2)
\]
by the triangle inequality and the orthogonality property (22). We apply Theorem 11 to the first term. In the case of the resolvent Lie splitting, the estimate (16) follows after applying Theorem 13, while in case of the trapezoidal resolvent splitting, Theorem 15 leads to the desired estimate (16).

In both cases, the constant $c_\alpha$ is a product of two constants, one that depends only on the operators and other that depends on the initial data. However, both are independent on $h$ and $n$. \qed

Remark 12. Under those additional assumptions that we imposed in Remark 10, and respectively in Remark 11, the full error for the resolvent Lie splitting can be estimated in the following way
\[
\|u(t_n) - \tilde{u}_{\text{dis}}^n\|_{D(X)}^2 \leq 2c \sum_{\alpha \in \mathcal{I} \setminus \mathcal{I}_m, K} (2N)^{-q_\alpha} + 2h^2 \sum_{\alpha \in \mathcal{I}_{m,K}} (2N)^{-q_\alpha}
\]
while the full error bound for the trapezoidal resolvent splitting has the form
\[
\|u(t_n) - \tilde{u}_{\text{dis}}^n\|_{D(X)}^2 \leq 2c \sum_{\alpha \in \mathcal{I} \setminus \mathcal{I}_m, K} (2N)^{-q_\alpha} + 2h^4 \sum_{\alpha \in \mathcal{I}_{m,K}} (2N)^{-q_\alpha}.
\]

4. Numerical results

In this section, we validate the proposed method and the convergence analysis presented in the previous section. For this purpose, we consider the two-dimensional problem
\[
    u_t = \mathcal{L}u + v + 1, \quad u(0) = 0, \quad u|_{\partial D} = 0,
\]
(61)
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where \( u = u(t, x, y) \) is the sought-after function, the operator \( \mathcal{L} \) is defined by
\[
\mathcal{L}u = (A + B)u = (au_x)_x + (bu_y)_y
\]
over the spatial domain \( D = [-1, 1]^2 \) with state variables \( x \) and \( y \), the spatial non-Gaussian (uniform) noise \( v \) given in the form (26) and \( t \in [0, T] \) for some \( T > 0 \). This problem is an example of the problem class (1) with zero initial and boundary conditions. The solution \( u \) of the considered problem (61) is given in its polynomial chaos representation (25) and approximated by a truncated expansion (46) in terms of Fourier–Legendre polynomials. The truncation procedure is explained in detail in Sect. 3.3.

Consider the set of multiindices \( \mathcal{I}_{m,K} \subset \mathcal{I} \), i.e.,
\[
\mathcal{I}_{m,K} = \{ \alpha \in \mathcal{I} \mid \alpha = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots), |\alpha| \leq K \}.
\]
In this section, elements \( \alpha \in \mathcal{I}_{m,K} \) will be denoted as \( m \)-tuples \( \alpha = (\alpha_1, \ldots, \alpha_m) \), omitting the components \( \alpha_j = 0, j \geq m + 1 \).

Moreover, we set
\[
\varepsilon^{(k)} = (\varepsilon^{(k)}_1, \ldots, \varepsilon^{(k)}_m), \quad \varepsilon^{(k)}_j = \delta_{kj}.
\]
For fixed \( m \in \mathbb{N} \), we consider an index function
\[
K_m : \mathcal{I}_{m,K} \rightarrow \{0, 1, \ldots, P - 1\}
\]
which enumerates multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{I}_{m,K} \). The function \( K_m \) is a bijection and each \( \alpha \in \mathcal{I}_{m,K} \) corresponds to a unique \( K_m(\alpha) = n \in \{0, 1, \ldots, P - 1\} \). Here \( P \) is the number of terms in the truncated solution \( \tilde{u} \) to be computed, given by (47).

For our purpose, we define the function \( K_m \) by
\[
K_m(0, 0, \ldots, 0, 0) = 0,
\]
\[
K_m(\varepsilon^{(k)}) = k \quad \text{for} \quad 1 \leq k \leq m,
\]
\[
K_m(\varepsilon^{(k)} + \varepsilon^{(\ell)}) = m + (m - 1) + \cdots + (m - k + 1) + \ell \quad \text{for} \quad 1 \leq k \leq \ell \leq m,
\]
\[
\ldots
\]
\[
K_m(0, 0, \ldots, 0, K) = P - 1.
\]

We use the index function \( K_m \) to enumerate the Fourier–Legendre polynomials \( L_\alpha \) for each \( \alpha \in \mathcal{I}_{m,K} \). Thus, we denote by \( \{\Phi_n\}_{n \in \{0, 1, \ldots, P - 1\}} \) the ordered Fourier–Legendre polynomials
\[
\Phi_n(\omega) = \Phi_{K_m(\alpha)}(\omega) = L_\alpha(\omega)
\]
for \( n = K_m(\alpha), \alpha \in \mathcal{I}_{m,K} \), where we use the definition (19) of the Fourier–Legendre polynomials.

In this way, we ordered \( \{L_\alpha\}_{\alpha \in \mathcal{I}_{m,K}} \) and will further on work with the set of ordered polynomials \( \{\Phi_n\}_{n \in \{0, 1, \ldots, P - 1\}} \) because of its simpler notation.
For example, following the just introduced notation, we have $\Phi_0(\omega) = L_{(0,0,\ldots,0)}(\omega)$ $= 1$ and

$$
\Phi_k(\omega) = L_{e(k)}(\omega) = \xi_k(\omega) \text{ for } 1 \leq k \leq m.
$$

Also, by applying the definition of the Legendre polynomials (16) we have

$$
\Phi_{m+1}(\omega) = L_{(2,0,\ldots,0)}(\omega) = l_2\left(\frac{\xi_1(\omega)}{\sqrt{3}}(\omega)\right) = \frac{1}{2} \xi_1(\omega) - \frac{1}{2},
$$
as well as

$$
\Phi_{m+2}(\omega) = L_{(1,1,0,\ldots,0)}(\omega) = l_1\left(\frac{\xi_1(\omega)}{\sqrt{3}}(\omega)\right) l_1\left(\frac{\xi_2(\omega)}{\sqrt{3}}(\omega)\right) = \frac{1}{3} \xi_1(\omega)\xi_2(\omega).
$$

Moreover, it holds

$$
\Phi_{P-1}(\omega) = L_{(0,0,\ldots,0,K)}(\omega) = l_K\left(\frac{\xi_n(\omega)}{\sqrt{3}}(\omega)\right).
$$

In the next step, we represent the solution $u$ of problem (61) by its truncated polynomial chaos expansion (46) and the noise term by its representation (30). Inserting the representations in (61) gives

$$
\sum_{\alpha \in I_{m,K}} (u_{\alpha})_t L_{\alpha} = \sum_{\alpha \in I_{m,K}} \mathcal{L}u_{\alpha} L_{\alpha} + \bar{v} + 1 + \sum_{j=1}^{m} \sqrt{\lambda_j} e_j Z_j.
$$

We recall that $\bar{v}$ is the mean value (the expectation) of the random field $v$. Further, the eigenvectors $e_j$ that correspond to the eigenvalues $\lambda_j$, $j = 1, \ldots, m$ are obtained from the spectral decomposition of the covariance function of the random field via solution of the homogeneous Fredholm integral equation of the second kind (27). It is assumed that $Z_j$ are uncorrelated uniform random variables. By performing a Galerkin projection, we obtain

$$
\sum_{\alpha \in I_{m,K}} (u_{\alpha})_t \mathbb{E}\left( L_{\alpha} L_{\beta} \right) = \sum_{\alpha \in I_{m,K}} \mathcal{L}u_{\alpha} \mathbb{E}\left( L_{\alpha} L_{\beta} \right) + (\bar{v} + 1) \mathbb{E}(L_{\beta}) + \sum_{j=1}^{m} \sqrt{\lambda_j} e_j \mathbb{E}(Z_j L_{\beta})
$$

for $\beta \in I_{m,K}$. Then, by applying the properties of the Fourier–Legendre polynomials (21) and (22) we obtain a system of deterministic Eq. (33). Particularly,

(i) for $|\alpha| = 0$:

$$
(u_{(0,0,\ldots,0)})_t = \mathcal{L}u_{(0,0,\ldots,0)} + \bar{v} + 1, \quad u_{(0,0,\ldots,0)}(0) = 0, \quad u_{(0,0,\ldots,0)}|_{\partial D} = 0 \quad (62)
$$
(ii) for \(|\alpha| = 1\), i.e., \(\alpha = \varepsilon^{(k)}\), \(1 \leq k \leq m\):
\[
(u_{\varepsilon^{(k)}})_t = \mathcal{L}u_{\varepsilon^{(k)}} + \sqrt{\lambda_k} e_k, \quad u_{\varepsilon^{(k)}}(0) = 0, \quad u_{\varepsilon^{(k)}}|_{\partial D} = 0
\]  
(63)

(iii) for \(|\alpha| > 1\):
\[
(u_{\alpha})_t = \mathcal{L}u_{\alpha}, \quad u_{\alpha}(0) = 0, \quad u_{\alpha}|_{\partial D} = 0.
\]  
(64)

From (64), we clearly deduce that \(u_{\alpha} \equiv 0\) for \(|\alpha| > 1\). In the calculations, we also used \(\mathbb{E}(L_{(0,0,\ldots,0)} Z_j) = \mathbb{E}(Z_j) = 0\) for \(j \geq 1\) and
\[
\mathbb{E}(Z_j L_{\beta}) = \mathbb{E}(l_1(Z_j) L_{\beta}) = \mathbb{E}(L_{\varepsilon^{(j)}} L_{\beta}) = \delta_{\beta,\varepsilon^{(j)}} \mathbb{E}(L_{\varepsilon^{(j)}}^2) = \delta_{\beta,\varepsilon^{(j)}}.
\]

This particularly implies
\[
\sum_{j=1}^{m} \sqrt{\lambda_j} e_j \mathbb{E}(Z_j L_{\varepsilon^{(k)}}) = \sqrt{\lambda_k} e_k \quad \text{for} \quad 1 \leq k \leq m,
\]
which is used in Eq. (63).

The obtained system (62), (63) and (64) can be represented in terms of the index function \(K_m\), i.e., in the form
\[
(u_n)_t = \mathcal{L}u_n + g_n, \quad u_n(0) = 0, \quad u_n|_{\partial D} = 0
\]  
(65)
for \(0 \leq n \leq P - 1\), where each \(n\) corresponds to an \(\alpha \in \mathcal{I}_{m,K}\). Each equation in (65) has the form of an inhomogeneous deterministic initial value problem, where the inhomogeneities \(g_n\) are given by: \(g_0 = \tilde{v} + 1\) and \(g_n = \sqrt{\lambda_n} e_n\) for \(1 \leq n \leq m\) and \(g_n = 0\) for \(m < n \leq P - 1\).

One way to approximate numerically a problem of the form
\[
u_t = (A + B)u + g, \quad u(0) = u^0, \quad u|_{\partial D} = 0
\]
with \(D = [-1, 1]^2\) is to define a grid consisting of \(N \times N\) equidistant computational points and define the discrete operators \(A_s\) and \(B_s\) by
\[
(A_s u_{\text{dis}})_{i,j} = \frac{1}{2s} \left( \frac{d}{dx} a_{i,j} (u_{\text{dis}}_{i+1,j} - u_{\text{dis}}_{i-1,j}) \right) + \frac{1}{s^2} \left( a_{i,j} (u_{\text{dis}}_{i+1,j} - 2u_{\text{dis}}_{i,j} + u_{\text{dis}}_{i-1,j}) \right),
\]
\[
(B_s u_{\text{dis}})_{i,j} = \frac{1}{2s} \left( \frac{d}{dy} b_{i,j} (u_{\text{dis}}_{i,j+1} - u_{\text{dis}}_{i,j-1}) \right) + \frac{1}{s^2} \left( b_{i,j} (u_{\text{dis}}_{i,j+1} - 2u_{\text{dis}}_{i,j} + u_{\text{dis}}_{i,j-1}) \right),
\]
where
\[
\frac{d}{dx} a_{i,j} = \frac{d}{dx} a(is, js), \quad \text{and} \quad a_{i,j} = a(is, js),
\]
\[
\frac{d}{dy} b_{i,j} = \frac{d}{dy} b(is, js), \quad \text{and} \quad b_{i,j} = b(is, js)
\]
for \(i, j = 1, \ldots, N\) and \(s = 2/(N+1)\). Due to the homogeneous Dirichlet boundary conditions, we have:
\[
u_{\text{dis}}_{0,j} = u_{\text{dis}}_{N+1,j} = u_{\text{dis}}_{i,0} = u_{\text{dis}}_{i,N+1} = 0
\]
for all $i, j = 0, \ldots, N + 1$. By setting $L_s = A_s + B_s$, we obtain the discretized problem

$$\frac{d}{dt} u^{\text{dis}} = L_s u^{\text{dis}} + g_s(t), \quad u^{\text{dis}}(0) = 0,$$

where $g_s$ denotes the discretization of the inhomogeneity $g$.

Note that the number $P = \frac{(m+K)!}{m!K!}$ of partial differential equations one has to solve in (65) increases fast due to the factorials occurring in (47). Since $g_n = 0$ for all $m < n \leq P - 1$, $u_n = 0$ is consequently the solution of the $n$th partial differential equation of (65). Therefore, we only have to solve the first $m + 1$ partial differential equations instead of all $P$. Further, we see that the solution does not depend on the highest degree $K$ of the $m$-dimensional Wiener–Legendre polynomials.

Let $u^n_n$ denote the numerical solution $u_n$ at time $t_n = hn$ and $g^n_n$ the function $g_n$ evaluated at time $t_n$. By setting

$$u^{n+1}_n = (I - hA_s)^{-1}(I - hB_s)^{-1}(u^n_n + h g^n_n)$$

the Lie resolvent splitting method is defined; see (40).

The trapezoidal splitting method is given by

$$u^{n+1}_n = \left( I - \frac{h}{2}B_s \right)^{-1} \left( I - \frac{h}{2}A_s \right)^{-1} \left[ \left( I + \frac{h}{2}A_s \right) \left( I + \frac{h}{2}B_s \right) u^n_n + \frac{h}{2}(g^n_n + g^{n+1}_n) \right],$$

see (42).

In our numerical experiment, we consider (61) with constant coefficients $a(x, y) = b(x,y) = 1$ for all $(x, y) \in D = [-1, 1]^2$ and set $T = 1$. Note that for some $n \in \{0, \ldots, m\}$ the inhomogeneities $g_n$ might be incompatible with the boundary conditions at the corners of the spatial domain $D$. Such an incompatibility results in order reduction; see [24]. This in particular leads to large errors near the corners of $D$. To overcome this problem, we apply the modified Lie resolvent splitting [24] in this situation.

For $n \in \{0, \ldots, m\}$, let $u_n$ be the solution of the partial differential equation (65).

Let $I = \{1, 2, 3, 4\}$ be the set of indices of the corners of the spatial domain $D$. They are enumerated from 1 to 4 counter-clockwise starting from the corner with coordinates $(-1, -1)$. Suppose that the inhomogeneity $g_n$ does not vanish at the corners $I_n \subset I$. Let $g_{n,i}(t)$ denote the value of the function $g_n$ at corner $i \in I_n$ and time $t \geq 0$. For $g_{n,i}(0) \neq 0$, we set

$$f_i = p_i g_n(0)/g_{n,i}(0),$$

where the polynomials $p_i$ are given by

$$p_1 = \frac{1}{4}(x - 1)(y - 1), \quad p_2 = -\frac{1}{4}(x + 1)(y - 1),$$

$$p_3 = \frac{1}{4}(x + 1)(y + 1), \quad p_4 = -\frac{1}{4}(x - 1)(y + 1).$$
These four polynomials form a partition of unity.

Let \( v_i \) be the solution of the stationary problem

\[
\mathcal{L} v_i = f_i \quad \text{in } D, \quad v_i|_{\partial D} = 0,
\]

for \( i \in I_n \). Note that \( v_i \) can be computed once and for all.

Then, let

\[
\tilde{g}_n(t) = g_n(t) + \sum_{i \in I_n} g'_{n,i}(t) v_i - g_{n,i}(t) f_i, \quad \tilde{u}_{n,0} = u_n(0) + \sum_{i \in I_n} g_{n,i}(0) v_i
\]

and apply the resolvent Lie splitting to the problem

\[
\begin{align*}
(\tilde{u}_n)_t &= \mathcal{L}\tilde{u}_n(t) + \tilde{g}_n(t), & \tilde{u}_n(0) = \tilde{u}_{n,0}, & \tilde{u}_n|_{\partial D} = 0.
\end{align*}
\]

By setting

\[
u_{n,\text{mod}}^n = \tilde{u}_n^n - \sum_{i \in I_n} g_{n,i}(nh) v_i \quad \text{for } n \in \mathbb{N}, \tag{67}
\]

we obtain the modified splitting scheme. Note that in our case \( g'_{n,i}(t) = 0 \) for all \( i \in I_n \) and for all \( n = 0, \ldots, m \) since none of the inhomogeneities \( g_n \) is time dependent.

In the implementation, the set \( I_n \) for \( n = 0, \ldots, m \) is constructed by checking the values of the inhomogeneities \( g_n \) at the corners, i.e.,

\[
I_n = \{ i \in \{1, 2, 3, 4\} \mid |g_{n,i}(0)| \geq \text{TOL} \}
\]

for a user chosen tolerance \( \text{TOL} \). If \( I_p = \emptyset \), the standard Lie resolvent splitting given in (66) is applied.

In the following, we consider problem (61) with the spatial noise term \( v \) given by (26) with the Bessel covariance function

\[
C_v(x, y) = \exp\{-\|x - y\|^2\}.
\]

The reference solution \( u^\text{ref}_n \) at time \( t \) is calculated according to

\[
u^\text{ref}_n(t) = \exp(t \mathcal{L})u_n(0) + t\varphi_1(t \mathcal{L}) g_n,
\]

where \( \varphi_1(z) = \frac{\exp(z) - 1}{z} \) and \( \exp(\cdot) \) denote the matrix exponential. This means that for each component we have an exact solution \( u^\text{ref}_n \). In all the examples shown, we fix the highest degree of ordered Fourier–Legendre polynomials to \( K = 3 \) and use a maximal number of \( m = 120 \) random variables in the truncated chaos expansion. Thus, there are \( P = \binom{123}{3} \) terms in the truncated solution. The mean of the random field \( v \) is chosen to be \( \bar{v} = 0 \). If not stated explicitly, we fix the number of computational points to \( N \times N = 40 \times 40 \).

Figure 1 illustrates the impact of the modification of the Lie resolvent splitting method. The figure shows the pointwise error of the numerical solution at time \( T = 1 \), i.e., \( |u_0(T) - u^\text{ref}_0(T)| \) over the spatial domain \( D = [-1, 1]^2 \) when calculated with the
Figure 1. Pointwise error of $u_0$ over the domain $\mathcal{D} = [-1, 1]^2$ for the Lie splitting (left) and the modified Lie splitting (right).

Lie splitting and the modified Lie splitting given in (66) and (67), respectively. The pointwise error of the solution $u_0$ is not only reduced at all the four corners of the domain $\mathcal{D}$ but also approximately decreases by an order of magnitude.

Figure 2 shows the discrete $L^2$ error of $u_n$, $n = 0, \ldots, 7$ calculated with different time step sizes $h$. The time step sizes are set to $h_q = 2^q$ for $q = -13, \ldots, -4$. The blue line denotes the error of the modified Lie splitting scheme of order 1. The red line and the green line illustrate the error of the Crank–Nicolson scheme and the trapezoidal splitting method, both of order two. The black dashed lines have slope 1 and 2, respectively. We see that for each $m$, the order plots confirm the respective orders of the methods which can be derived from theory.

The empirical variance $\text{Var}(u)$ of $u$ is given by

$$\text{Var}(u) = \mathbb{E} (u - \mathbb{E}(u))^2 = \sum_{n=1}^{P} u_n^2 \mathbb{E}(\Phi_n^2),$$

where we used the linearity of the expectation $\mathbb{E}$ and the orthogonality of the Fourier–Legendre polynomials. Furthermore, since $u_\alpha \equiv 0$ for $|\alpha| > 1$, i.e., $u_n \equiv 0$ for $n > m$, the number of non-zero summands in the sum is $m$ and since $\mathbb{E}(\Phi_n^2) = 1$ for $1 < n \leq m$, $\text{Var}(u)$ reduces to

$$\text{Var}(u) = \sum_{n=1}^{m} u_n^2.$$
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Figure 2. Order plots for the first eight different solutions $u_n$, $n = 0, \ldots, 7$ computed with the correspondent methods. The dashed lines have slope 1 and 2, respectively.

Figure 3 shows the discrete $L^2$ error of the empirical variance of $u$ at time $T = 1$ where the summation is truncated at different $m$. The time step $h$ used for the calculations is $h = 2^{-10}$. Here, we clearly see the superiority of the methods of order two compared to the modified Lie splitting for which the numerical approximation error prevails over the error induced by the truncation of the sum.

Finally, we report the computational work which is needed to solve the system of partial differential equations given in (62)–(64). Table 1 summarizes the computational time needed to obtain one solution of the system of partial differential equations as a function of the number of spatial grid points $N \times N = 2^k \times 2^k$ for $k = 2, 3, \ldots, 7$. The highest number of grid points we are able to use (16384) is quite low due to the fact that the calculation of the eigenvalues and eigenfunctions of the integral equation given in (27) requires the storage of a dense matrix of the size $N^2 \times N^2$. We clearly see that the Crank–Nicolson method is by far the slowest. Both splitting methods perform approximately the same for smaller $N$, while for $N = 2^7$, Lie splitting starts to clearly outperform trapezoidal splitting in terms of computational time.
Figure 3. Discrete $L^2$ error of $\text{Var}(u)$ for different number of variables $m$ used in the expansion. The employed methods are: Crank–Nicolson (CN), modified Lie splitting (MLSPL) and trapezoidal splitting (TSPL)

Table 1. Average computational time (in s) for the calculation of one solution $u_m$ for different degrees of freedom $N$, i.e., the number of computational points used in the discretization of $D$ and the operator $\mathcal{L}$

| $N \times N$ | CN [s]   | MLSPL [s] | TSPL [s] |
|---------------|----------|-----------|----------|
| 4 $\times$ 4  | 0.0133   | 0.0373    | 0.0530   |
| 8 $\times$ 8  | 0.0214   | 0.0241    | 0.0379   |
| 16 $\times$ 16| 0.1005   | 0.1008    | 0.0948   |
| 32 $\times$ 32| 0.4098   | 0.3652    | 0.4047   |
| 64 $\times$ 64| 2.7620   | 1.8051    | 1.8237   |
| 128 $\times$ 128| 41.1284 | 9.3921    | 13.5091  |

The employed methods are: Crank–Nicolson (CN), modified Lie splitting (MLSPL) and trapezoidal splitting (TSPL)

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