INFRARED RENORMALIZATION IN NON-RELATIVISTIC QED
AND SCALING CRITICALITY

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ABSTRACT. We consider a spin-$\frac{1}{2}$ electron in a translation-invariant model of non-relativistic Quantum Electrodynamics (QED). Let $H(p, \sigma)$ denote the fiber Hamiltonian corresponding to the conserved total momentum $p \in \mathbb{R}^3$ of the Pauli electron and the photon field, regularized by a fixed ultraviolet cutoff in the interaction term, and an infrared regularization parametrized by $0 < \sigma \ll 1$ which we ultimately remove by taking $\sigma \searrow 0$. For $|p| < \frac{1}{2}$, all $\sigma > 0$, and all values of the fine-structure constant $\alpha < \alpha_0$, with $\alpha_0 \ll 1$ sufficiently small and independent of $\sigma$, we prove the existence of a ground state eigenvalue of multiplicity two at the bottom of the essential spectrum. Moreover, we prove that the renormalized electron mass satisfies $1 < m_{\text{ren}}(p, \sigma) < 1 + \alpha$, uniformly in $\sigma \geq 0$, in units where the bare mass has the value 1, and we prove the existence of the renormalized mass in the limit $\sigma \searrow 0$. Our analysis uses the isospectral renormalization group method of Bach-Fröhlich-Sigal introduced in [1, 2] and further developed in [3, 4]. The limit $\sigma \searrow 0$ determines a scaling-critical renormalization group problem of endpoint type, in which the interaction is strictly marginal (of scale-independent size). The main achievement of this paper is the development of a method that provides rigorous control of the renormalization of a strictly marginal quantum field theory characterized by a non-trivial scaling limit. The key ingredients entering this analysis include a hierarchy of exact algebraic cancelation identities exploiting the spatial and gauge symmetries of the model, and a combination of the isospectral renormalization group method with the strong induction principle.

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1. Introduction

In this paper, we give a solution to the problem of infrared mass renormalization in non-relativistic Quantum Electrodynamics (QED), the mathematical theory of non-relativistic quantum mechanical matter (electrons, positrons) interacting with the quantized electromagnetic radiation field (light, photons).

We consider a Pauli electron of spin $\frac{1}{2}$ in a translation-invariant model of non-relativistic QED in $\mathbb{R}^3$. To make it mathematically well-defined, we regularization the Hamiltonian with a fixed ultraviolet (high frequency) cutoff in the interaction term which eliminates the interaction of the electron with photons of high energy, and an infrared (low frequency) regularization parametrized by $0 < \sigma \ll 1$ which we ultimately remove by letting $\sigma \downarrow 0$. Our aim is to characterize the particle spectrum of the Hamiltonian $H(\sigma)$ of the regularized model, to prove bounds on
the infrared renormalized mass that are uniform in $\sigma \geq 0$, and to establish its existence in the limit $\sigma \searrow 0$.

Since the model is translation invariant, we can study the fiber Hamiltonians $H(p, \sigma)$ separately for different values of the conserved momentum $p \in \mathbb{R}^3$. Our key aim is to control the regularity of the infimum, $E(p, \sigma)$, of the spectrum of $H(p, \sigma)$ as a function of $|p|$, in the limit $\sigma \searrow 0$ ($E(p, \sigma)$ is a radial function of $p$). For $|p| < \frac{1}{3}$ and $\sigma > 0$, we prove that $E(p, \sigma)$ is an eigenvalue of multiplicity two at the bottom of the essential spectrum of $H(p, \sigma)$ (see also [18] for the degeneracy of the ground state energy). All our results hold for sufficiently small values of the fine-structure constant $\alpha < \alpha_0$, where $\alpha_0$ is independent of $\sigma \geq 0$.

We derive uniform upper and lower bounds on the renormalized electron mass

$$m_{\text{ren}}(\vec{p}, \sigma) = \frac{1}{\partial_{|p|}^2 E(\vec{p}, \sigma)} \quad (1.1)$$

of the form (the "bare mass" for $\alpha = 0$ has the value 1 in our units)

$$1 < m_{\text{ren}}(\vec{p}, \sigma) < 1 + c_0 \alpha, \quad (1.2)$$

for $\alpha < \alpha_0$, where the constant $c_0$ is independent of $\sigma \geq 0$. Moreover, we prove the existence of the renormalized mass

$$m_{\text{ren}}(\vec{p}) := \lim_{\sigma \searrow 0} m_{\text{ren}}(\vec{p}, \sigma) \quad (1.3)$$

at fixed $\vec{p}$ with $0 \leq |\vec{p}| < \frac{1}{3}$, and of the joint limit

$$m_{\text{ren}}(0) = \lim_{(\vec{p}, \sigma) \to (0, 0)} m_{\text{ren}}(\vec{p}, \sigma). \quad (1.4)$$

The estimate (1.2) plays a quintessential rôle in the construction of infraparticle scattering theory and various related problems, [9, 24]. The Fourier multiplication operator $e^{-itE(\vec{p}, \sigma)}$ controls the dispersive behavior of the free time evolution of infraparticle states, and for stationary phase estimates, control of the Hessian of $E(\vec{p}, \sigma)$, as obtained from (1.1) and (1.2), is crucial. Due to the absence of a gap separating $E(\vec{p}, \sigma)$ from the essential spectrum, conventional perturbation theoretic approaches unavoidably produce divergent results in the limit $\sigma \searrow 0$. This is a manifestation of the infrared problem in non-relativistic QED. For a discussion of the infrared problem in the operator-algebraic context, we refer to [9] and the references therein (see also the remarks in Section 3).

In the joint work [4] with V. Bach, J. Fröhlich, and I.M. Sigal, analogous results are proven for the spin 0 model, including bounds of the form (1.2), for $0 < |\vec{p}| < \frac{1}{3}$ and $\sigma > 0$, but for $\alpha < \alpha_0(\sigma)$ where $\alpha_0(\sigma) \searrow 0$ as $\sigma \searrow 0$. For $|\vec{p}| = 0$ and under the hypothesis that the limits

$$\lim_{\sigma \searrow 0, \vec{p} \to 0} m_{\text{ren}}(\vec{p}, \sigma) = \lim_{\vec{p} \to 0, \sigma \searrow 0} m_{\text{ren}}(\vec{p}, \sigma) \quad (1.5)$$

commute, bounds of the form (1.2) on $m_{\text{ren}}(0, \sigma)$ are proven in [4] for $\alpha < \alpha_0$ (independent of $\sigma$) which are uniform in $\sigma \geq 0$. In particular, an explicit, finite, convergent algorithm is constructed in [4] that determines $m_{\text{ren}}(0, 0)$ to any given precision, with rigorous error bounds. It is immediately clear that (1.4) supplies [4] with the condition (1.5). The present work is in many aspects a continuation of the analysis of [4], and some familiarity with [4] might be helpful for its reading.
The analysis in [4] is based on the isospectral renormalization group method, and shows for $0 < |\vec{p}| < \frac{1}{3}$ that, in the subcritical case $\sigma > 0$ and for the type of regularization used in [4], the interaction is driven to zero by scaling, at an exponential, $\sigma$-dependent rate under repeated applications of the renormalization map; the renormalization group problem is of is irrelevant type. In contrast, the case $\sigma = 0$ is a problem of endpoint type in which the interaction and the free Hamiltonian $H(\vec{p},0)$ exhibit the same behavior under scaling. In the context of renormalization group theory, this defines a marginal problem, and a priori, the following three scenarios are possible: (1) The size of the interaction grows polynomially in the number of applications of the renormalization map; the problem is marginally relevant. (2) The size of the interaction decreases polynomially in the number of applications of the renormalization map; the problem is marginally irrelevant. (3) The size of the interaction neither diminishes nor increases under repeated applications of the renormalization map; the problem is strictly marginal.

As we prove in the present work, the problem of infrared renormalization in non-relativistic QED in the endpoint case $\sigma = 0$ is of strictly marginal type, i.e. the size of the interaction is scale-independent. A main goal of the present work is to extend the isospectral renormalization group method of [3, 4], based on the smooth Feshbach map, to models in quantum field theory which are strictly marginal. To prove uniform boundedness of the interaction, we invoke the strong induction principle, and combine it with composition identities satisfied by the smooth Feshbach map. Moreover, our method involves the use of hierarchies of non-perturbative identities originating from spatial and gauge symmetries of the model, which are used to control the precise cancellations of terms in certain infinite sums.

The isospectral renormalization group produces a convergent series expansion of $E(\vec{p},0)$ and $m_{\text{ren}}(\vec{p})$ in powers of $\alpha$ in which the coefficients are $\alpha$-dependent, and divergent as $\alpha \to 0$ (see also [23]). However, we emphasize that $E(\vec{p},0)$ and $m_{\text{ren}}(\vec{p})$ do not exist as ordinary power series in $\alpha$ (with $\alpha$-independent coefficients), and are thus inaccessible to more conventional methods of perturbation theory.

The contents of this paper are further developments of the work conducted in [8], which is available online, but unpublished. All results and methods of [8] are here fundamentally improved, optimized and extended. Some of the main differences comprise: (1) The term of order $O(\alpha)$ in the uniform upper bound on the renormalized mass (1.2) is optimal in powers of the fine structure constant $\alpha$. In [8], the corresponding bound is of the form $O(\alpha^\delta)$, for some $\delta > 0$. (2) In contrast to [8], we include the electron spin here. It is therefore necessary in our analysis to prove that the Zeeman term (which involves the magnetic field operator) in $H(\vec{p},\sigma)$ is, in renormalization group terminology, an irrelevant operator (it scales to zero). The inclusion of electron spin has the consequence that the generalized Wick kernels in the present work are matrix-valued (in [8], they are scalar). For their analysis, the spatial symmetries of the model enter in a more significant way in our proofs than in [8]. (3) The existence of the renormalized mass in the limit $\sigma \to 0$ is proven here, but not in [8]. (4) Most proofs are new or significantly improved.

A detailed introduction to the problem of infrared mass renormalization in the context of the isospectral renormalization group method is given in [4]. The uniform bounds on the infrared renormalized mass have important applications, for example
in infraparticle scattering theory [9, 24], in certain approaches to the problem of enhanced binding [10] (see also [15, 19] for enhanced binding), and as noted above, in algorithmic schemes for the computation of the renormalized mass [4]. Moreover, our results are used in [5].

In the present work, the ultraviolet cutoff $\Lambda$ is fixed. Some important results related to the asymptotics of ground state energies, binding, and thermodynamic limits are established in [13, 20, 21, 22] for arbitrary values of $\alpha$ and $\Lambda$, and without infrared cutoff. For some recent works discussing the problem of ultraviolet mass renormalization, which is not being addressed here, we refer to [14, 16, 17]. For a survey of recent developments in the mathematical study of non-relativistic QED, we refer to [25].

Notations. We use units in which the velocity of light $c$, Planck’s constant $\hbar$, and the bare electron mass $m$ have the values $c = \hbar = m = 1$. The letters $C$ or $c$ will denote various constants whose values may change from one estimate to another. $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the bounded linear operators $\mathcal{H}_1 \to \mathcal{H}_2$ on Banach spaces $\mathcal{H}_i$. $I_c(\alpha) \subset \mathbb{R}$ is the closed interval $[a - c, a + c]$, and $I_c = I_c(0)$.

$\vec{v} = (v_1, v_2, v_3)$ denotes a vector in $\mathbb{R}^3$. $\vec{v} \cdot \vec{v}'$ denotes the Euclidean scalar product, and $\vec{v}^2 \equiv \vec{v} \cdot \vec{v} \equiv |\vec{v}|^2$. $B_r(\vec{x}) \subset \mathbb{R}^3$ is the closed ball of radius $r$ centered at $\vec{x} \in \mathbb{R}^3$, and $B_r \equiv B_r(\vec{0})$. $\vec{v} = (v_0, \vec{v})$ denotes a vector in $\mathbb{R}^4$. $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ denotes the vector of Pauli matrices (cf. (2.15)).

2. Definition of the Model

We consider a translation invariant model of non-relativistic QED in $\mathbb{R}^3$ that describes a freely propagating, non-relativistic, spin $\frac{1}{2}$ Pauli electron interacting with the quantized electromagnetic radiation field.

The electron Hilbert space is given by

$$\mathcal{H}_{el} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

where the factor $\mathbb{C}^2$ accounts for the electron spin.

The Hilbert space accounting for degrees of freedom of the quantized electromagnetic field is given by the photon Fock space

$$\mathfrak{F}(L^2(\mathbb{R}^3)) = \bigoplus_{n \geq 0} \mathfrak{F}_n(L^2(\mathbb{R}^3)),$$

$$\mathfrak{F}_n(L^2(\mathbb{R}^3)) = \text{Sym}_n \left( L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \right)^\otimes n,$$

where the factors $\mathbb{C}^2$ accommodate the polarization of the photon in the Coulomb gauge, and $\text{Sym}_n$ fully symmetrizes the $n$ factors in the tensor product. A vector $\Phi \in \mathfrak{F}$ is a sequence

$$\Phi = (\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n)}, \ldots), \quad \Phi^{(n)} \in \mathfrak{F}_n,$$
where \( \Phi^{(n)} = \Phi^{(n)}(k_1, \lambda_1, \ldots, k_n, \lambda_n) \) is symmetric in all \( n \) variables \((k_j, \lambda_j)\). \( k_j \in \mathbb{R}^3 \) is the momentum, and \( \lambda_j \in \{+,-\} \) labels the two possible polarizations of the \( j \)-th photon. For brevity, let \( \mathfrak{F} \equiv \mathfrak{F}(L^2(X)) \), \( \mathfrak{F}_n \equiv \mathfrak{F}_n(L^2(X)) \) if \( X = \mathbb{R}^3 \). The scalar product on \( \mathfrak{F} \) is given by 

\[
\langle \Phi_1, \Phi_2 \rangle = \sum_{n \geq 0} \langle \Phi^{(n)}_1, \Phi^{(n)}_2 \rangle \mathfrak{F}_n.
\]

For \( \lambda \in \{+,-\} \) and \( f \in L^2(\mathbb{R}^3) \), we introduce annihilation operators 

\[
a(f, \lambda) : \mathfrak{F}_n \to \mathfrak{F}_{n-1},
\]

with

\[
(a(f, \lambda)\Phi)^{(n-1)}(\bar{k}_1, \lambda_1, \ldots, \bar{k}_{n-1}, \lambda_{n-1}) = \sqrt{n} \int d^3\bar{k}_n f(\bar{k}_n) \Phi^{(n)}(\bar{k}_1, \lambda_1, \ldots, \bar{k}_{n-1}, \lambda_{n-1})
\]

and creation operators

\[
a^*(f, \lambda) : \mathfrak{F}_n \to \mathfrak{F}_{n+1} \text{ with } a^*(f, \lambda) = (a(f, \lambda))^* \]

which satisfy the canonical commutation relations

\[
[a(f, \lambda), a^*(g, \lambda')] = (f, g)_{L^2} \delta_{\lambda, \lambda'}
\]

\[
[a^2(f, \lambda), a^2(g, \lambda')] = 0, \quad f, g \in L^2(\mathbb{R}^3),
\]

where \( a^2 \) denotes either \( a \) or \( a^* \). The Fock vacuum

\[
\Omega_f = (1, 0, 0, \ldots) \in \mathfrak{F}
\]

is the unique unit vector satisfying

\[
a(f, \lambda) \Omega_f = 0
\]

for all \( f \in L^2(\mathbb{R}^3) \).

Since \( a(f, \lambda) \) is antilinear and \( a^*(f, \lambda) \) is linear in \( f \), one can define operator-valued distributions \( a^2(\bar{k}, \lambda) \) with

\[
a(f, \lambda) = \int_{\mathbb{R}^3} d^3\bar{k} f(\bar{k}) a(\bar{k}, \lambda), \quad a^*(f, \lambda) = \int_{\mathbb{R}^3} d^3\bar{k} f(\bar{k}) a^*(\bar{k}, \lambda),
\]

satisfying

\[
[a(\bar{k}', \lambda'), a^*(\bar{k}, \lambda)] = \delta_{\lambda, \lambda'} \delta(\bar{k} - \bar{k}')
\]

\[
[a^2(\bar{k}', \lambda'), a^2(\bar{k}, \lambda)] = 0
\]

for all \( \bar{k}, \bar{k}' \in \mathbb{R}^3 \) and \( \lambda, \lambda' \in \{+,-\} \), and

\[
a(\bar{k}, \lambda) \Omega_f = 0
\]

for all \( \bar{k}, \lambda \).

We introduce the notation

\[
K := (\bar{k}, \lambda) \in \mathbb{R}^3 \times \{+,-\}, \quad \int dK := \sum_{\lambda = \pm} \int_{\mathbb{R}^3} d^3\bar{k},
\]

for pairs of photon momenta and polarization labels.
The Hamiltonian and the momentum operator of the free photon field are respectively given by
\[ H_f = \int dK \, k \, a^*(K) a(K) \]
\[ \vec{P}_f = \int dK \, \vec{k} \, a^*(K) a(K), \] (2.12)
and are selfadjoint operators on \( \mathfrak{g} \).

The Hilbert space of states for the full system is given by the tensor product Hilbert space
\[ H = \mathcal{H}_{el} \otimes \mathfrak{g}. \] (2.13)

The Hamiltonian of non-relativistic QED for the coupled system comprising the electron and the quantized radiation field is given by
\[ H(\sigma) = \frac{1}{2} \left( i \nabla \otimes 1_f - \sqrt{\alpha} \tilde{A}_\sigma(\vec{x}) \right)^2 + \sqrt{\alpha} \vec{\tau} \cdot \tilde{B}_\sigma(\vec{x}) + 1_{el} \otimes H_f, \] (2.14)
where \( \vec{\tau} = (\tau_1, \tau_2, \tau_3) \) denotes the vector of Pauli matrices
\[ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (2.15)
and \( \sqrt{\alpha} \) is the bare electron charge, with \( \alpha > 0 \) being the fine-structure constant.

The operators
\[ \tilde{A}_\sigma(\vec{x}) = \int \frac{dK}{|k|} \kappa_\sigma(|\vec{k}|) \left( \vec{\varepsilon}(K)e^{2\pi i \vec{k} \cdot \vec{x}} \otimes a(K) + h.c. \right) \]
\[ \tilde{B}_\sigma(\vec{x}) = \int \frac{dK}{|k|} \kappa_\sigma(|\vec{k}|) \left( i\vec{k} \wedge \vec{\varepsilon}(K)e^{2\pi i \vec{k} \cdot \vec{x}} \otimes a(K) + h.c. \right) \] (2.16)
stand for the quantized electromagnetic vector potential and the magnetic field operator. In agreement with the Coulomb gauge condition, the polarization vectors \( \vec{\varepsilon}(\vec{n}_k, +), \vec{\varepsilon}(\vec{n}_k, -) \in \mathbb{R}^3 \) (with \(|\vec{\varepsilon}(K)| = 1\)) form an orthonormal basis together with \( \vec{n}_k := \frac{\vec{k}}{|k|} \) in \( \mathbb{R}^3 \), for every \( k \in \mathbb{R}^3 \setminus \{0\} \).

The function \( \kappa_\sigma \) implements an ultraviolet cutoff (comparable to the electron rest energy \( mc^2 = 1 \) in our units) and an infrared regularization parametrized by \( 0 < \sigma \ll 1 \). For technical reasons specific to our methods, we require that \( \kappa_\sigma \) is non-zero for \( 0 < x < \Lambda \), where we can assume \( \Lambda = 1 \) for the ultraviolet cutoff. The infrared regularization used in \( [4] \) has the form \( \kappa_\sigma(x) = \chi(x < 1)x^\sigma \), and softens the singularity of the photon form factor to \(|\vec{k}|^{-1/2+\sigma}\).

To study properties of the strictly marginal model in the scaling critical case \( \sigma \searrow 0 \), it is more convenient to use an infrared regularization where \( \kappa_\sigma(x) = 1 \) for \( x > \sigma \). For definiteness, we choose
\[ \kappa_\sigma(x) := \begin{cases} (x/\sigma)^K & \text{for } x < \sigma \\ 1 & \text{for } x \in [\sigma, \frac{1}{2}], \text{ and } C^\infty \text{ on } (\frac{1}{2}, 1) \\ 0 & \text{for } x > 1 \end{cases}, \] (2.17)
where $\sigma > 0$ is arbitrarily small, and which we will send to zero in the end. The exponent $K > 0$ is arbitrary. We will for simplicity assume that $K = 1$, but everything discussed here can be easily adapted to any $K > 0$, or to any $\kappa_\sigma$ which is smooth and monotonic on $[0, \sigma]$, with $\kappa_\sigma(0) = 0$ and $\kappa_\sigma(\sigma) = 1$.

The operator of the total momentum of the electron and the quantized electromagnetic field is given by

$$\vec{P}_{\text{tot}} = i\nabla_x \otimes 1_f + 1_{el} \otimes \vec{P}_f.$$ 

(2.18)

The model is translation invariant, $[\vec{P}_{\text{tot}}, H(\sigma)] = 0$. We write

$$H = \int_{\mathbb{R}^3} d^3\vec{p} \mathcal{H}_{\vec{p}}$$

(2.19)

in direct integral decomposition, where

$$\mathcal{H}_{\vec{p}} \cong \mathbb{C}^2 \otimes \mathfrak{F}$$

(2.20)

denotes the fiber Hilbert space associated to the conserved total momentum $\vec{p} \in \mathbb{R}^3$.

The fibers $\mathcal{H}_{\vec{p}}$ are invariant under $e^{-itH(\sigma)}$. It thus suffices to study the restriction of $H(\sigma)$ to $\mathcal{H}_{\vec{p}}$.

$$H(\vec{p}, \sigma) := \frac{1}{2} (\vec{p} - \vec{P}_f - \sqrt{\alpha} \vec{A}_\sigma) \cdot (\vec{p} - \vec{P}_f - \sqrt{\alpha} \vec{A}_\sigma) + \sqrt{\alpha} \vec{A}_\sigma \cdot \vec{B}_\sigma + H_f.$$ 

(2.21)

where

$$\vec{A}_\sigma := \vec{A}_\sigma(\vec{0})$$

$$\vec{B}_\sigma := i(\vec{P}_f \wedge \vec{A}_\sigma + \vec{A}_\sigma \wedge \vec{P}_f).$$

$H(\vec{p}, \sigma)$ is the fiber Hamiltonian corresponding to the conserved momentum $\vec{p}$.

3. Main Theorem

The main results of this paper characterize the infimum, $E(\vec{p}, \sigma)$, of the spectrum of $H(\vec{p}, \sigma)$, for values $\alpha < \alpha_0$ where $\alpha_0 \ll 1$ is effective, small, and independent of $\sigma$. For $0 \leq |\vec{p}| < \frac{1}{\sqrt{3}}$, we prove that $E(\vec{p}, \sigma)$ is an eigenvalue of multiplicity two at the bottom of the essential spectrum of $H(\vec{p}, \sigma)$, for any $\sigma > 0$. In particular, we prove upper and lower bounds on the renormalized electron mass which are uniform in $\sigma \geq 0$, and we establish the existence of the renormalized mass in the limit $\sigma \to 0$.

Using the results of this paper, it is shown in [9] that when $\sigma = 0$, $H(\vec{p}, 0)$ has no ground state in $\mathbb{C}^2 \otimes \mathfrak{F}$ if $|\vec{p}| > 0$; see Theorem 3.2 below, which quotes the main results of [9].

**Theorem 3.1.** For $0 \leq |\vec{p}| < \frac{1}{\sqrt{3}}$, there exists a constant $\alpha_0 > 0$ (independent of $\sigma$) such that for all $0 < \alpha < \alpha_0$, the following hold:

- **A.** For any $\sigma > 0$,

$$E(\vec{p}, \sigma) := \inf \text{spec}_{\mathbb{C}^2 \otimes \mathfrak{F}} H(\vec{p}, \sigma)$$

(3.1)

is an eigenvalue of multiplicity two at the bottom of the essential spectrum of $H(\vec{p}, \sigma)$. 

• There exists a constant $c_0$ independent of $\sigma$ and $\alpha$ such that
\[ 1 - c_0\alpha < \partial^2_{|\vec{p}|} E(\vec{p}, \sigma) < 1, \]
and
\[ \left| \nabla_\vec{p} E(\vec{p}, \sigma) - \frac{|\vec{p}|^2}{2} \right| < c_0 |\vec{p}|, \]
\[ \left| E(p, \sigma) - \frac{|\vec{p}|^2}{2} - \frac{\alpha}{2} \langle \Omega_f, A^2_\sigma \Omega_f \rangle \right| < c_0 |\vec{p}|^2/2. \]

• The renormalized electron mass
\[ m_{\text{ren}}(\vec{p}, \sigma) = \frac{1}{\partial^2_{|\vec{p}|} E(\vec{p}, \sigma)} \]
is bounded by
\[ 1 < m_{\text{ren}}(\vec{p}, \sigma) < 1 + c_0 \alpha, \]
uniformly in $\sigma \geq 0$.

• For every $\vec{p}$ with $0 \leq |\vec{p}| < 1/3$, there exists a sequence $\{\sigma_n\}_{n \in \mathbb{N}}$ converging to zero such that
\[ m_{\text{ren}}(\vec{p}) := \lim_{n \to \infty} m_{\text{ren}}(\vec{p}, \sigma_n) \]
exists. Moreover,
\[ \tilde{m}_{\text{ren}}(\vec{0}) := \lim_{\sigma \searrow 0} m_{\text{ren}}(\vec{0}, \sigma) \]
exists (for the proof, see [4]), and
\[ m_{\text{ren}}(\vec{0}) = \lim_{|\vec{p}| \searrow 0} m_{\text{ren}}(\vec{p}) = \tilde{m}_{\text{ren}}(\vec{0}), \]
i.e., the limits $|\vec{p}| \searrow 0$ and $\sigma \searrow 0$ commute.

3.1. Remarks.

(1) For the proof of Theorem 3.1, we can invoke many constructions and results from [3], and especially from [4]. As noted in the introductory section, it is established in [4] that there exists $\alpha_0(\sigma) > 0$ for any $\sigma > 0$ such that for all $\alpha < \alpha_0(\sigma)$, the statements $A$ - $C$ of Theorem 3.1 hold. However, the bound derived in [4] is such that $\alpha_0(\sigma) \searrow 0$ as $\sigma \searrow 0$. The key purpose of the present paper is to prove $\sigma$-independent estimates.

For $|\vec{p}| = 0$, bounds of the form (3.5) are proved in [4] for $\alpha < \alpha_0$, with $\alpha_0$ independent of $\sigma$, under the assumption that the limits
\[ \lim_{\sigma \to 0} \lim_{|\vec{p}| \to 0} \partial^2_{|\vec{p}|} E(\vec{p}, \sigma) = \lim_{|\vec{p}| \to 0} \lim_{\sigma \to 0} \partial^2_{|\vec{p}|} E(\vec{p}, \sigma) \]
commute. In particular, an explicit, finite, and convergent algorithm is constructed in [4] which determines $m_{\text{ren}}(\vec{0}, 0)$ to any arbitrary given precision, with rigorous error bounds. It is clear that (3.8) supplies [4] with the condition (3.9).

(2) The uniform bounds (3.2) have important applications in the construction of dressed one-electron states in the operator-algebraic framework, and in infraparticle
scattering theory. Let $\mathfrak{A}_\rho := B(\mathbb{C}^2 \otimes \mathfrak{F}_\rho)$ denote the $C^*$-algebra of bounded operators on the Fock space $\mathfrak{F}_\rho(L^2(\mathbb{R}^3 \setminus B_\rho))$. Then, we define the $C^*$-algebra

$$\mathfrak{A} := \bigvee_{\rho > 0} \mathfrak{A}_\rho, \quad (3.10)$$

where the closure is taken with respect to the operator norm. Ground state eigenvectors belonging to $E(\vec{p}, \sigma)$ are parametrized by $\vec{u} \in S^2$,

$$\Psi_a(\vec{p}, \sigma) \in \mathbb{C}^2 \otimes \mathfrak{F} \quad \text{with} \quad \langle \Psi_a(\vec{p}, \sigma), \mathcal{F} \Psi_a(\vec{p}, \sigma) \rangle = \vec{u} \quad (3.11)$$

and $\|\Psi_a(\vec{p}, \sigma)\|_{\mathbb{C}^2 \otimes \mathcal{B}} = 1$. For fixed $\vec{u} \in S^2$, $\Psi_a(\vec{p}, \sigma)$ defines a normalized, positive state

$$\omega_{\vec{p}, \sigma}(A) := \langle \Psi_a(\vec{p}, \sigma), A \Psi_a(\vec{p}, \sigma) \rangle, \quad A \in \mathfrak{A}, \quad (3.12)$$

on $\mathfrak{A}$, referred to as a dressed one-electron state or an infraparticle state. In physical terms, it accounts for an electron in a bound state with an infinite number of low frequency (soft) photons of small total energy.

The following results are proved in [9].

**Theorem 3.2.** (C-Fröhlich, [9]) Assume Theorem 3.1. Then, with $\Psi_a(\vec{p}, \sigma)$ and $\omega_{\vec{p}, \sigma}$ as defined above, the following hold independently of $\vec{u} \in S^2$:

- **Let**

  $$N_f = \int dK a^*(K) a(K) \quad (3.13)$$

  denote the photon number operator. Then,

  $$-c\alpha + c' \alpha |\nabla E(\vec{p}, \sigma)|^2 \log \frac{1}{\sigma} + \langle \Psi_a(\vec{p}, \sigma), N_f \Psi_a(\vec{p}, \sigma) \rangle \leq C \alpha + C' \alpha |\nabla E(\vec{p}, \sigma)|^2 \log \frac{1}{\sigma}, \quad (3.14)$$

  for positive constants $c, C, c' < C'$, and $r_+ := \max\{r, 0\}$. That is, the expected photon number in the ground state diverges logarithmically in the limit $\sigma \searrow 0$ if $|p| > 0$ (since $\vec{p} \neq 0 \Leftrightarrow \nabla E(\vec{p}, \sigma) \neq 0$).

- Every sequence $\{\omega_{\vec{p}, \sigma_n}\}$ with $\sigma_n \searrow 0$ as $n \to \infty$ possesses a subsequence $\{\omega_{\vec{p}, \sigma_{n_j}}\}$ which converges weak-$^*$ to a state $\omega_{\vec{p}}$ on $\mathfrak{A}$ as $j \to \infty$. The state $\omega_{\vec{p}}$ restricted to $\mathfrak{A}_\rho$ is normal for any $\rho > 0$.

- The state $\omega_{\vec{p}, \sigma}$ satisfies

  $$\int dK \left| \omega_{\vec{p}, \sigma}(a(K)^* a(K)) - \omega_{\vec{p}, \sigma}(a(K))^2 \right| \leq c \alpha, \quad (3.15)$$

  uniformly in $\sigma \geq 0$.

- Let $\pi_{\vec{p}}$ denote the representation of $\mathfrak{A}$, $\mathcal{H}_{\omega_{\vec{p}}}$ the Hilbert space, and $\Omega_{\vec{p}} \in \mathcal{H}_{\omega_{\vec{p}}}$ the cyclic vector corresponding to $(\omega_{\vec{p}}, \mathfrak{A})$ by the GNS construction, (with $\omega_{\vec{p}}(A) = \langle \Omega_{\vec{p}}, \pi_{\vec{p}}(A) \Omega_{\vec{p}} \rangle$, for all $A \in \mathfrak{A}$). Moreover, let

  $$v_{\vec{p}, \sigma, \lambda}(k) := -\sqrt{\alpha} (\vec{k}, \lambda) \cdot \nabla E(\vec{p}, \sigma) \frac{k_s(|\vec{k}|)}{|\vec{k}|^2} \frac{1}{|\vec{k}| - \vec{k} \cdot \nabla E(\vec{p}, \sigma)}, \quad (3.16)$$

  and

  $$v_{\vec{p}, \lambda}(\vec{k}) := \lim_{\sigma \searrow 0} v_{\vec{p}, \sigma, \lambda}(\vec{k}). \quad (3.17)$$
Then, $\pi_{\vec{p}}$ is quasi-equivalent to $\pi_{\text{Fock}} \circ \alpha_{\vec{p}}$ (where $\pi_{\text{Fock}}$ is the Fock representation of $\mathfrak{A}$), and $\alpha_{\vec{p}}$ is the $\ast$-automorphism of $\mathfrak{A}$ determined by

$$\alpha_{\vec{p}}(a^+_k(k)) = a^+_k(k) + v^*_{\vec{p},\lambda}(k).$$

(3.18)

- The following relations between the Fock representation and $\pi_{\vec{p}}$ hold:
  1. If $\vec{p} = \vec{0}$
     $$\lim_{\sigma \to 0} \omega_{\vec{0},\sigma}(N_f) < c\alpha,$$
     (3.19)
     and $\pi_{\vec{0}}$ is (quasi-)equivalent to $\pi_{\text{Fock}}$.
  2. If $\vec{p} \neq \vec{0}$, $\pi_{\vec{p}}$ is unitarily inequivalent to the Fock representation, and
     $$\lim_{\sigma \to 0} \omega_{\vec{p},\sigma}(N_f) = \infty,$$
     (3.20)
     but $\omega_{\vec{p}}$ has a "local Fock property":
     (a) For every $\rho > 0$, the restriction of $\omega_{\vec{p}}$ to $\mathfrak{A}_{\rho}$ determines a GNS representation which is quasi-equivalent to the Fock representation.
     (b) For every bounded region $B$ in physical $\vec{x}$-space, the restriction of $\omega_{\vec{p}}$ to the local algebra $\mathfrak{A}(B)$ determines a GNS representation which is quasi-equivalent to the Fock representation of $\mathfrak{A}(B)$.

A key ingredient in the proof is the uniform bound (3.2) on the renormalized electron mass.

Theorem 3.2 provides a crucial ingredient (the correct coherent transformation in the construction of the scattering state) for the construction of infraparticle scattering states in non-relativistic QED, extending recent results of Pizzo [24] for the Nelson model, see [9].

(3) Theorem 3.1 can be straightforwardly extended to Nelson’s model. It is defined on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F}_{\text{bos}}, \quad \mathfrak{F}_{\text{bos}} := \bigoplus_{n \geq 0} (L^2(\mathbb{R}^3))^\otimes_n,$$

(3.21)

with $\mathfrak{F}_{\text{bos}}$ a Fock space of scalar bosons. Introducing creation- and annihilation operators $a^*_k(\vec{k})$, the Nelson Hamiltonian is given by

$$H_{\text{Nelson}}(\sigma) = -\frac{1}{2} \Delta_{\vec{x}} \otimes 1_{\text{bos}} + 1 \otimes H_{\text{bos}} + g \int_{\mathbb{R}^3} d^3\vec{k} v_\sigma(\vec{k}) (e^{-i(\vec{k},\vec{x})} \otimes a^*(\vec{k}) + e^{i(\vec{k},\vec{x})} \otimes a(\vec{k})),
$$

(3.22)

where $v_\sigma(\vec{k}) := \frac{\kappa(\vec{k})}{|\vec{k}|^\frac{3}{2}}$ and where $g$ is a small coupling constant.

$$H_{\text{bos}} = \int dK |\vec{k}| a^*(K) a(K) \quad \text{and} \quad \vec{P}_{\text{bos}} = \int dK \vec{k} a^*(K) a(K)$$

are the Hamiltonian and momentum operator of the free boson field. Due to translation invariance, it again suffices to consider the restriction of $H_{\text{Nelson}}(\sigma)$ to a fiber Hilbert space $\mathcal{H}_{\vec{p}}$ corresponding to the conserved total momentum $\vec{p} \in \mathbb{R}^3$,

$$H_{\text{Nelson}}(\vec{p},\sigma) = \frac{1}{2} (\vec{p} - \vec{P}_{\text{bos}})^2 + H_{\text{bos}} + ga^*(v_\sigma) + ga(v_\sigma),
$$

(3.23)
Applying a Bogoliubov transformation,

\[ H_{\text{Nelson}}(\vec{p}, \sigma) \rightarrow \alpha_{\text{Bog}, \sigma}(H_{\text{Nelson}}(\vec{p}, \sigma)) , \]

which acts on creation- and annihilation operators by way of

\[ \alpha_{\text{Bog}, \sigma}(a^\dagger(\vec{k})) = a^\dagger(\vec{k}) - \frac{v_\sigma(\vec{k})}{|\vec{k}|} , \]

the Nelson Hamiltonian at fixed conserved total momentum \( \vec{p} \) is transformed into

\[ H_{\text{B-N}}(\vec{p}, \sigma) = \frac{1}{2}(\vec{p} - \vec{P}_{\text{bos}} - ga(\vec{\omega}_\sigma) - ga^*(\vec{\omega}_\sigma))^2 + H_{\text{bos}} , \]

where \( \vec{\omega}_\sigma(\vec{k}) := v_\sigma(\vec{k})\frac{\vec{k}}{|\vec{k}|} \) is a radially directed, vector-valued function in the boson momentum space.

The Bogoliubov transformation can be implemented by

\[ \alpha_{\text{Bog}, \sigma}(A) = U_{\text{Bog}, \sigma} AU_{\text{Bog}, \sigma}^* , \]

where \( U_{\text{Bog}, \sigma} \) is unitary if \( \sigma > 0 \), but which is not unitary in the limit \( \sigma \downarrow 0 \).

The Nelson model admits soft boson sum rules that are very similar to the soft photon sum rules introduced in section 7.7 for the QED model (although the Nelson model has no gauge symmetry), \[4\]. The only difference is that the photon polarization vector \( \vec{\varepsilon}(K) \) appearing in (4.29), (4.30), (4.31) is replaced by the radial unit vector \( \frac{\vec{k}}{|\vec{k}|} \). The results of this paper can be straightforwardly extended to the Nelson model.

(4) We remark that the upper bound \(|\vec{p}| < \frac{1}{3}\) in Theorem 3.1 has a purely technical origin. While we make no attempt to optimize it here, we note that it cannot be improved beyond a critical value \( p_c < 1 \). It is expected that the eigenvalue \( E(\vec{p}, \sigma) \) dissolves into the continuous spectrum as \(|\vec{p}| \nearrow p_c \), while a resonance appears (a phenomenon similar to Cherenkov radiation). An analysis of this problem is beyond the scope of this work.

(5) Due to the absence of creation or annihilation of particles (electrons) or antiparticles (positrons) in non-relativistic QED, there is, in contrast to relativistic QED (see \[6\]), no renormalization of the finestructure constant \( \alpha \).

(6) Our proof uses the isospectral, operator-theoretic renormalization group method pioneered by V. Bach, J. Fröhlich, and I. M. Sigal in \[1\], \[2\], and further developed in \[3\], \[4\]. We apply and further extend the formulation based on the "smooth Feshbach map" of \[3\], \[4\]. In the limit \( \sigma \downarrow 0 \), the interaction in \( \tilde{H}(\vec{p}, \sigma) \) is purely marginal. The main goal in the present work is the development of a method to control the size of the interaction in a purely marginal theory.

4. Wick ordering and symmetries

In this section, we discuss three properties of \( H(\vec{p}, \sigma) \) which play a crucial rôle in a more general context later:
The fiber Hamiltonian $H(\vec{p}, \sigma)$ can be written in \textit{generalized Wick ordered normal form}, i.e., as a sum of Wick-ordered (all creation-operators stand on the left of all annihilation operators) monomials of creation- and annihilation operators which are characterized by \textit{operator-valued} integral kernels (referred to as \textit{generalized Wick kernels}).

$H(\vec{p}, \sigma)$ is symmetric under rotations and reflections with respect to a plane perpendicular to $\vec{p}$ containing the origin. We observe that the non-interacting Hamiltonian in $H(\vec{p}, \sigma)$ is a scalar a multiple of $1_2$. We prove in Lemma 4.1 below that any Mat$(2 \times 2, \mathbb{C})$-valued generalized Wick monomial $f[H_f, \vec{P}_f]$ of degree zero that admits these symmetries is necessarily a multiple of $1_2$.

Moreover, $H(\vec{p}, \sigma)$ admits \textit{soft photon sum rules}, which are a generalization of the differential \textit{Ward-Takahashi identities} of QED. Those are hierarchies of non-perturbative, exact identities which originate from $U(1)$ gauge invariance.

4.1. \textbf{Generalized Wick ordered normal form.} The \textit{generalized Wick ordered normal form} of the fiber Hamiltonian $H(\vec{p}, \sigma)$ is given by

$$H(\vec{p}, \sigma) = E[\vec{p}] + T[H_f, \vec{P}_f, \vec{p}] + W_1 + W_2,$$

where

$$E[\vec{p}] := \frac{\vec{p}^2}{2} + \frac{\alpha}{2} \left\langle \Omega_f, \hat{A}^2 \Omega_f \right\rangle \in \mathbb{R}_+.$$  \hspace{1cm} (4.1)

The \textit{free Hamiltonian}

$$T[H_f, \vec{P}_f, \vec{p}] := H_f - \vec{p} \cdot \vec{P}_f + \frac{\vec{p}^2}{2}$$ \hspace{1cm} (4.2)

commutes with $H_f, \vec{P}_f$. The \textit{interaction Hamiltonian} is a sum

$$W_L = \sum_{M+N=L} W_{M,N}, \quad W_{M,N} = W^*_{N,M}, \quad L = 1, 2,$$

where the operators $W_{M,N}$ are the following \textit{generalized Wick monomials}:

$$W_{0,1} = \int \frac{dK \kappa_{\sigma}(|\vec{k}|)}{|\vec{k}|^{1/2}} w_{0,1}[H_f, \vec{P}_f, \vec{p}; K] a(K) = W^*_{1,0}.$$ \hspace{1cm} (4.3)

The integral kernel

$$w_{0,1}[H_f, \vec{P}_f, \vec{p}; K] := -\sqrt{\alpha} (\vec{p} - \vec{P}_f) \cdot \vec{c}(K) + \sqrt{\alpha} \vec{c} \cdot (i \vec{k} \wedge \vec{c}(K))$$ \hspace{1cm} (4.4)

is a Mat$(2 \times 2, \mathbb{C})$-valued operator-function of $K$, which commutes with $H_f, \vec{P}_f$. We shall refer to it as the \textit{generalized Wick kernel} of order $(0,1)$, and $w_{1,0} = w^*_{0,1}$. Furthermore, we have the Wick monomials

$$W_{1,1} = \int \frac{dK dK' \kappa_{\sigma}(|\vec{k}|) \kappa_{\sigma}(|\vec{k}'|)}{(|\vec{k}| |\vec{k}'|)^{1/2}} a^*(K) w_{1,1}[H_f, \vec{P}_f; K, K'] a(K'),$$

$$W_{0,2} = \int \frac{dK dK' \kappa_{\sigma}(|\vec{k}|) \kappa_{\sigma}(|\vec{k}'|)}{(|\vec{k}| |\vec{k}'|)^{1/2}} w_{0,2}[H_f, \vec{P}_f; K, K'] a(K) a(K'),$$

$$W_{2,0} = \int \frac{dK dK' \kappa_{\sigma}(|\vec{k}|) \kappa_{\sigma}(|\vec{k}'|)}{(|\vec{k}| |\vec{k}'|)^{1/2}} a^*(K) a^*(K') w_{0,2}[H_f, \vec{P}_f; K, K'],$$ \hspace{1cm} (4.5)
with generalized Wick kernels

\[
\begin{align*}
    w_{1,1}[H_f, \vec{P}_f; K, K'] &= 2\alpha \varepsilon(K) \cdot \varepsilon(K') , \\
    w_{0,2}[H_f, \vec{P}_f; K, K'] &= \alpha \varepsilon(K) \cdot \varepsilon(K') , \\
    w_{2,0}[H_f, \vec{P}_f; K, K'] &= \alpha \varepsilon(K) \cdot \varepsilon(K') ,
\end{align*}
\]

(4.8)
of orders (1,1), (0,2), and (2,0), respectively. In case of \( H(\vec{p}, \sigma) \), the number of Wick monomials is evidently finite; we will later study classes of Hamiltonians where the interaction part is a norm-convergent, infinite series of Wick monomials.

4.2. Rotation and reflection invariance. We let \( U_R^\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \) denote the unitary representation of \( SO(3) \) defined by

\[
(U_R^\Phi)_{n}(\vec{k}_1, \lambda_1, \cdots, \vec{k}_n, \lambda_n) = \Phi_n(R\vec{k}_1, \lambda_1, \cdots, R\vec{k}_n, \lambda_n) , \quad R \in SO(3) .
\]

(4.9)

We denote the representation \( SU(2) \to SO(3) \) by \( R^\bullet : h \mapsto R_h \), and

\[
\text{Ad}_{U_{R_h}^\mathfrak{g}}[A] = U_{R_h}^\mathfrak{g} A(U_{R_h}^\mathfrak{g})^* ,
\]

(4.10)

with \( A \) defined on \( \mathfrak{g} \). Then, clearly,

\[
\begin{align*}
    \text{Ad}_{U_{R_h}^\mathfrak{g}}[H_f] &= H_f , & \text{Ad}_{U_{R_h}^\mathfrak{g}}[\vec{p}] &= \vec{p} , & \text{Ad}_{U_{R_h}^\mathfrak{g}}[\vec{P}_f] &= R_h \vec{P}_f , \\
    \text{Ad}_{U_{R_h}^\mathfrak{g}}[\vec{p} \wedge \vec{P}_f] &= R_h(\vec{p} \wedge \vec{P}_f) , & \text{Ad}_{U_{R_h}^\mathfrak{g}}[\vec{A}_\sigma] &= R_h \vec{A}_\sigma .
\end{align*}
\]

(4.11)

Moreover, conjugating the vector of Pauli matrices \( \vec{\tau} = (\tau_1, \tau_2, \tau_3) \) by \( h \) yields

\[
h \vec{\tau} h^* = R_h \vec{\tau} ,
\]

and

\[
U_h := h \otimes U_{R_h}^\mathfrak{g}
\]

(4.12)
defines a unitary representation of \( SU(2) \) on \( \mathbb{C}^2 \otimes \mathfrak{g} \).

It is easy to see that

\[
U_h H(R_h \vec{p}, \tau) U_h^* = H(\vec{p}, \sigma) ,
\]

(4.13)
i.e. \( H(\vec{p}, \sigma) \) is rotation invariant.

Let \( \vec{n}_\vec{p} = \frac{\vec{p}}{||\vec{p}||} \). We consider the unitary reflection operator on \( \mathfrak{g} \)

\[
U_{\text{ref},\vec{p}}^\mathfrak{g} := \exp \left[ \frac{i\pi}{2} \int_{\mathbb{R}^3 \times \{+,-\}} dK \left( a^*(\vec{k}, \lambda) a(\vec{k}, \lambda) - a^*(\vec{k}, \lambda) a(M_{\vec{p}}\vec{k}, \lambda) \right) \right] ,
\]

(4.14)

where \( M_{\vec{p}}\vec{k} := -k ||\vec{n}_\vec{p} + \vec{k} \perp \), with \( k || := \vec{k} \cdot \vec{n}_\vec{p} \) and \( \vec{k} \perp := (\vec{k} - k || \vec{n}_\vec{p}) \). Clearly, \( M_{\vec{p}}^2 = 1 \). We point out the similarity of (4.14) to the parity inversion operator in relativistic QED, see for instance [6].

One can straightforwardly verify that

\[
U_{\text{ref},\vec{p}}^\mathfrak{g} a^x(\vec{k}, \lambda) (U_{\text{ref},\vec{p}}^\mathfrak{g})^* = a^x(M_{\vec{p}}\vec{k}, \lambda) ,
\]

(4.15)

and correspondingly with \( \vec{k} \) and \( M_{\vec{p}}\vec{k} \) exchanged. Hence, \( H(\vec{p}, \sigma) \) is invariant under reflection with respect to a plane perpendicular to \( \vec{p} \) containing the origin.
Under conjugation by $U^\delta_{ref,\vec{p}}$
\[
\text{Ad}_{U^\delta_{ref,\vec{p}}} [H_f] = H_f, \quad \text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{p}] = \vec{p}, \quad \text{Ad}_{U^\delta_{ref,\vec{p}}} [P_f^\parallel] = -P_f^\parallel
\]
\[
\text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{A}_f^\parallel] = -\vec{A}_f^\parallel, \quad \text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{P}_f^\parallel] = \vec{P}_f^\parallel, \quad \text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{A}_f^\perp] = \vec{A}_f^\perp,
\]
\[
\text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{p} \wedge \vec{P}_f] = \text{Ad}_{U^\delta_{ref,\vec{p}}} [\vec{p} \wedge \vec{P}_f^\parallel] = \vec{p} \wedge \vec{P}_f^\parallel = \vec{p} \wedge \vec{P}_f
\]
while under conjugation by $\tau^\parallel = \vec{\tau} \cdot \vec{n}_\vec{p}$,
\[
\tau^\parallel \to \tau^\parallel, \quad \tau^\perp \to -\tau^\perp,
\]
where
\[
\tau^\perp = \tau - \text{diag}(n_{\vec{p}}) \cdot \vec{\tau}.
\]

For
\[
U_{\text{ref,}\vec{p}} := \tau^\parallel \otimes U^\delta_{\text{ref,}\vec{p}},
\]

it follows that
\[
U_{\text{ref,}\vec{p}} H(-\vec{p},\tau) U^*_{\text{ref,}\vec{p}} = H(\vec{p},\tau),
\]
i.e. $H(\vec{p},\tau)$ is reflection invariant.

An important ingredient in our analysis is the fact that any reflection and rotation invariant $\text{Mat}(2 \times 2, \mathbb{C})$-valued function of $H_f, \vec{P}_f$ and $\vec{p}$ is a scalar operator (i.e. a multiple of $1_2$).

**Lemma 4.1.** Let $A$ denote a $\text{Mat}(2 \times 2, \mathbb{C})$-valued Borel function of $H_f, \vec{P}_f, \vec{p}$, satisfying
\[
U_h A(H_f, \vec{P}_f, R_h \vec{p}) U^*_h = A(H_f, \vec{P}_f, \vec{p}) \quad \text{for all } h \in SU(2).
\]
Then,
\[
A(H_f, \vec{P}_f, \vec{p}) = a_0(H_f, \vec{P}_f, \vec{p}) 1_2,
\]
for a Borel function $a_0 : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$, where
\[
U_h a_0(H_f, \vec{P}_f, R_h \vec{p}) U^*_h = a_0(H_f, \vec{P}_f, \vec{p}) \quad \text{for all } h \in SU(2).
\]
Hence, $A(H_f, \vec{P}_f, \vec{p})$ transforms according to the trivial representation of $SU(2)$.

**Proof.** Representing $A \in \text{Mat}(2 \times 2, \mathbb{C})$ in the basis of Pauli matrices $\{\tau_0, \tau_1, \tau_2, \tau_3\}$,
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2} \tau_0 + \frac{a_{11} - a_{22}}{2} \tau_3
\]
\[
+ \frac{a_{12} + a_{21}}{2} \tau_1 + \frac{a_{12} - a_{21}}{2i} \tau_2,
\]
we write
\[
A = a_0 \tau_0 + \vec{a} \cdot \vec{\tau},
\]
with $\vec{a} = (a_1, a_2, a_3)$, and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$. We will refer to $a_0$ as the scalar, and $\vec{a}$ as the vector part of $A$ (which are in general $\mathbb{C}$-valued).
Since
\[ U_h A(H_f, \vec{P}_f, R_h \vec{p}) U_h^* = a_0(H_f, R_h \vec{P}_f, R_h \vec{p}) \tau_0 + \vec{a}(H_f, R_h \vec{P}_f, R_h \vec{p}) \cdot (R_h \vec{\tau}) , \] (4.25)
assumption (4.19) implies that
\[ a_0(H_f, R_h \vec{P}_f, R_h \vec{p}) = a_0(H_f, \vec{P}_f, \vec{p}) \]
\[ \vec{a}(H_f, R_h \vec{P}_f, R_h \vec{p}) = R_h \vec{a}(H_f, \vec{P}_f, \vec{p}) . \] (4.26)
We write \( \vec{a} \) in the basis \( \vec{p}, \vec{P}_f, \vec{p} \wedge \vec{P}_f \),
\[ \vec{a}(H_f, \vec{P}_f, \vec{p}) = b_1(H_f, \vec{P}_f, \vec{p}) \vec{p} + b_2(H_f, \vec{P}_f, \vec{p}) \vec{P}_f + b_3(H_f, \vec{P}_f, \vec{p}) \vec{p} \wedge \vec{P}_f , \] (4.27)
where
\[ b_j(H_f, R_h \vec{P}_f, R_h \vec{p}) = b_j(H_f, \vec{P}_f, \vec{p}) , \ j = 1, 2, 3 , \] (4.28)
are scalar functions of \( H_f, \vec{P}_f, \vec{p} \). By rotation invariance, \( a_0, b_1, b_2, b_3 \) are functions of the rotation invariant combinations \( \vec{p}^2, \vec{p} \cdot \vec{P}_f, \vec{P}_f^2 \) only. Hence,
\[ U_{\text{ref}, \vec{p}}^3 a_0(H_f, \vec{P}_f, -\vec{p}) (U_{\text{ref}, \vec{p}}^3)^* = a_0(H_f, \vec{P}_f, \vec{p}) , \]
and likewise for \( b_j \). However, \( \vec{\varphi} \cdot \vec{p}, \vec{\varphi} \cdot \vec{P}_f = \vec{\varphi}^\parallel \cdot \vec{P}_f^\parallel + \vec{\varphi}^\perp \cdot \vec{P}_f^\perp \), and \( \vec{\varphi} \cdot (\vec{p} \wedge \vec{P}_f) = \vec{\varphi}^\perp \cdot (\vec{p} \wedge \vec{P}_f) \) change their signs under conjugation by \( U_{\text{ref}, \vec{p}} \), see (4.16) and (4.17). Therefore, the conditions (4.19) and (4.20) can only be simultaneously satisfied if \( b_1 = b_2 = b_3 = 0 \).

4.3. Gauge invariance and soft photon sum rules. An important property of the model under consideration is that on all levels of the renormalization group analysis, the corresponding effective Hamiltonians (introduced in Section 6) satisfy soft photon sum rules, which can be considered as a generalization of the differential Ward-Takahashi identities in QED. For the fiber Hamiltonian \( H(\vec{p}, \sigma) \), they correspond to the following relations.

Let \( \vec{n} \in \mathbb{R}^3, |\vec{n}| = 1 \). It is easy to see that
\[ \sqrt{\alpha} \vec{e}(\vec{n}, \lambda) \cdot \nabla_{\vec{\rho}_j} T(\vec{P}, \vec{p}) = - \lim_{x \to 0} w_{0,1}[\vec{P}; (x\vec{n}, \lambda)] \]
\[ = - \lim_{x \to 0} w_{1,0}[\vec{P}; (x\vec{n}, \lambda)] \] (4.29)
holds for any choice of \( \vec{n} \). Furthermore,
\[ \sqrt{\alpha} \vec{e}(\vec{n}, \lambda) \cdot \nabla_{\vec{\rho}_j} w_{0,1}[\vec{P}, \vec{p}; \vec{K}] = -2 \lim_{x \to 0} \vec{w}_{0,1}[\vec{P}; (x\vec{n}, \lambda), \vec{K}] \]
\[ = - \lim_{x \to 0} w_{1,1}[\vec{P}; (x\vec{n}, \lambda), \vec{K}] , \] (4.30)
and likewise,
\[ \sqrt{\alpha} \vec{e}(\vec{n}, \lambda) \cdot \nabla_{\vec{\rho}_j} w_{1,0}[\vec{P}, \vec{p}; \vec{K}] = -2 \lim_{x \to 0} \vec{w}_{2,0}[\vec{P}; (x\vec{n}, \lambda), \vec{K}] \]
\[ = - \lim_{x \to 0} w_{1,1}[\vec{P}; (x\vec{n}, \lambda), \vec{K}] . \] (4.31)
(4.29), (4.30) and (4.31) correspond to the soft photon sum rules on the most basic level.
4.4. Organization of the proof. For an introductory exposition of the isospectral renormalization group method, and a discussion of problems connected to infrared mass renormalization in non-relativistic QED, we refer to [3, 4]. The proof of Theorem 3.1 is essentially organized as follows:

- In Section 5, we introduce the isospectral smooth Feshbach map, and recall some of its key properties from [3, 4].
- In Section 6, we introduce effective Hamiltonians belonging to a subclass of the bounded operators on the reduced Hilbert space $\mathcal{H}_{\text{red}} = \mathbb{C}^2 \otimes \mathbb{1}[H_f < 1]$, which are reflection and rotation symmetric, and satisfy soft photon sum rules. Moreover, we introduce a Banach space of generalized Wick kernels $\mathbb{W}^>_{\infty}$ which parametrize the effective Hamiltonians.
- In Section 7, we define an isospectral renormalization map $R_\rho$ on a polydisc $\Omega \subset \mathbb{W}^>_{\infty}$, given by the composition of the smooth Feshbach map with a rescaling transformation, and a renormalization of a spectral parameter.

We then state the main technical results of this work:

- Theorem 7.12 asserts that $R_\rho$ is codimension-3 contractive on $\Omega$, and that it is marginal on a subspace of dimension 3 (after explicitly projecting out a one-dimensional subspace of relevant perturbations). However, no control on the growth of the marginal interactions under repeated applications of $R_\rho$ is provided at this point. $R_\rho$ is shown to preserve reflection and rotation symmetry, and the soft photon sum rules.
- We introduce a strong induction assumption $s\text{Ind}[n]$ which asserts that the marginal interactions admit an $n$-independent upper bound after $n$ applications of $R_\rho$. Theorem 7.13 asserts that $s\text{Ind}[n - 1]$ implies $s\text{Ind}[n]$ for any $n$.

- In Section 8 we prove Theorem 7.12. We use the soft photon sum rules to reduce the number of a priori independent marginal operators, and the spatial symmetries of the model to prove that the operators originating from the Zeeman term in $H(\vec{p}, \sigma)$ (the term proportional to the magnetic field operator $\vec{B}_\sigma$) are irrelevant.

- In Section 9 we prove Theorem 7.13. To establish the strong induction step $s\text{Ind}[n - 1] \Rightarrow s\text{Ind}[n]$, we combine Theorem 7.12 with composition identities satisfied by the smooth Feshbach map.

- In Section 10 we prove the uniform bounds on the renormalized mass asserted in Theorem 3.1. This is accomplished by relating $m_{\text{ren}}(\vec{p}, \sigma)$ to the renormalization group flow of one of the operators contained in the effective Hamiltonians.

- In Section 11 we prove the existence of the renormalized mass in the limit $\sigma \searrow 0$ for $\vec{p}$ with $0 \leq |\vec{p}| < \frac{1}{\Lambda}$.

For the proof of Theorem 3.1 we will invoke many constructions and results from [3, 4].
5. The Smooth Feshbach Map

In this section, we introduce the smooth Feshbach map and the associated intertwining maps, mostly quoting results from [3, 4].

5.1. Definition of the smooth Feshbach map. Let $\mathcal{H}$ be a separable Hilbert space, and let $0 \leq \chi \leq 1$ denote a positive, selfadjoint operator on $\mathcal{H}$. Introducing $\bar{\chi} := \sqrt{1 - \chi^2}$, we obtain the partition of unity $\chi^2 + \bar{\chi}^2 = 1$ on $\mathcal{H}$.

We let $P_\chi, P_{\bar{\chi}}$ denote the orthoprojectors associated to the subspaces $\text{Ran}(\chi), \text{Ran}(\bar{\chi}) \subset \mathcal{H}$, respectively, and let $P_{\perp \chi} = 1 - P_\chi$ and $P_{\perp \bar{\chi}} = 1 - P_{\bar{\chi}}$, their respective complements. It is clear that the spaces $\text{Ran}(\chi)$ and $\text{Ran}(\bar{\chi})$ are mutually complementary if and only if $\chi$ is a projector.

**Definition 5.1.** A pair of closed operators $(H, \tau)$ acting on $\mathcal{H}$ is a Feshbach pair corresponding to $\chi$ if:

- The domains of $H$ and $\tau$ coincide, and are invariant under $\chi$ and $\bar{\chi}$. Moreover, $[\chi, \tau] = 0 = [\bar{\chi}, \tau]$.
- Let
  \[ H_{\bar{\chi}} := \tau + \bar{\chi}\omega \bar{\chi}, \omega := H - \tau. \]  
  (5.1)
  The operators $\tau, H_{\bar{\chi}}$ are bounded invertible on $\text{Ran}(\bar{\chi})$.
- Let
  \[ \tilde{R} := H_{\bar{\chi}}^{-1} \quad \text{on} \quad \text{Ran}(\chi), \]  
  (5.2)
  and let $H_{\bar{\chi}} = U|H_{\bar{\chi}}|$ denote the polar decomposition of $H_{\bar{\chi}}$ on $\text{Ran}(\bar{\chi})$. Then,
  \[ \|\tilde{R}\|_{\mathcal{H} \rightarrow \mathcal{H}} < \infty, \]  
  \[ \|\tilde{R}^\dagger U^{-1} \bar{\chi}\omega \bar{\chi}\|_{\text{Ran}(\chi) \rightarrow \mathcal{H}}, \quad \|\chi \omega \bar{\chi} \tilde{R}^\dagger \|_{\mathcal{H} \rightarrow \text{Ran}(\chi)} < \infty. \]  
  (5.3)

We write
\[ \mathfrak{FP}(\mathcal{H}, \chi) \]  
for the set of Feshbach pairs on $\mathcal{H}$ corresponding to $\chi$.

The smooth Feshbach map is defined by
\[ F_\chi : \mathfrak{FP}(\mathcal{H}, \chi) \rightarrow \mathcal{L}(\mathcal{H}) \]  
\[ (H, \tau) \mapsto \tau + \chi \omega \chi - \chi \omega \bar{\chi} \tilde{R}^\dagger \bar{\chi} \omega \chi, \]  
(5.5)
where $\mathcal{L}(\mathcal{H})$ denotes the linear operators $\mathcal{H} \rightarrow \mathcal{H}$. Furthermore, we introduce the intertwining maps
\[ Q_\chi : \mathfrak{FP}(\mathcal{H}, \chi) \rightarrow B(\text{Ran}(\chi), \mathcal{H}) \]  
\[ (H, \tau) \mapsto \chi - \bar{\chi} \tilde{R} \bar{\chi} \omega \chi \]  
(5.6)
\[ Q^2_\chi : \mathfrak{FP}(\mathcal{H}, \chi) \rightarrow B(\mathcal{H}, \text{Ran}(\chi)) \]  
\[ (H, \tau) \mapsto \chi - \chi \omega \bar{\chi} \tilde{R} \bar{\chi}. \]  
(5.6)
We note that the mutually complementary subspaces \( \text{Ran}(\chi), \text{Ran}(\chi)^\perp \subseteq \mathcal{H} \) are invariant under \( F_\chi(H,\tau) \). On \( \text{Ran}(\chi)^\perp, F_\chi(H,\tau) \) equals \( \tau \), while it is a bounded operator on \( \text{Ran}(\chi) \).

5.2. Isospectrality. The smooth Feshbach map, combined with the intertwining operators, implements a non-linear, isospectral correspondence between closed operators on \( \mathcal{H} \) and ones on the Hilbert subspace \( \text{Ran}(\chi) \subseteq \mathcal{H} \), according to the following main theorem.

**Theorem 5.2.** (Feshbach isospectrality theorem) Let \((H,\tau) \in \mathfrak{F}(\mathcal{H},\chi)\). Then, the following hold:

- The operator \( H \) is bounded invertible on \( H \) if and only if \( F_\chi(H,\tau) \) is bounded invertible on \( \text{Ran}(\chi) \subseteq H \). If \( H \) is invertible,
  \[
  F_\chi(H,\tau)^{-1} = \chi H^{-1} \chi + \bar{\chi} \tau^{-1} \bar{\chi} \tag{5.7}
  \]
  and
  \[
  H^{-1} = Q_\chi(H,\tau) F_\chi(H,\tau)^{-1} Q_\chi^*(H,\tau) + \bar{\chi} \bar{R} \bar{\chi} \tag{5.8}
  \]
- Let \( \psi \in \mathcal{H} \). Then, \( H\psi = 0 \) on \( \mathcal{H} \) if and only if \( F_\chi(H,\tau)\chi\psi = 0 \) on \( \text{Ran}(\chi) \subseteq \mathcal{H} \).
- Let \( \phi \in \text{Ran}(\chi) \). Then, \( F_\chi(H,\tau)\zeta = 0 \) on \( \text{Ran}(\chi) \subseteq \mathcal{H} \) if and only if \( HQ_\chi(H,\tau)\zeta = 0 \) on \( \mathcal{H} \).

We furthermore quote the following lemma from [4].

**Lemma 5.3.** Let \((H,\tau) \in \mathfrak{F}(\mathcal{H},\chi)\). Then, the following identities hold.

\[
\chi F_\chi(H,\tau) = HQ_\chi(H,\tau)
\]
and
\[
F_\chi(H,\tau)\chi = Q_\chi^5(H,\tau) H \tag{5.9}
\]

\[
Q_\chi^5(H,\tau)HQ_\chi(H,\tau) = F_\chi(H,\tau) - F_\chi(H,\tau)\bar{\chi}\tau^{-1}\bar{\chi}F_\chi(H,\tau) \tag{5.10}
\]

5.3. Derivations. Consider a Hilbert space \( \mathcal{H} \) with a dense subspace \( \mathcal{D} \subseteq \mathcal{H} \), and let \( \mathcal{L}(\mathcal{D},\mathcal{H}) \) denote the space of linear (not necessarily bounded) operators from \( \mathcal{D} \) to \( \mathcal{H} \).

A derivation \( \partial \) is a linear map \( \text{Dom}(\partial) \to \mathcal{L}(\mathcal{D},\mathcal{H}) \), defined on a subspace \( \text{Dom}(\partial) \subseteq \mathcal{L}(\mathcal{D},\mathcal{H}) \), which obeys Leibnitz’ rule. That is, for \( A,B \in \text{Dom}(\partial), \text{Ran}(B) \subseteq \mathcal{D}, \text{and } A,B \in \text{Dom}(\partial), \)

\[
\partial[AB] = \partial[A]B + A\partial[B] \ .
\]

Let \((H,\tau) \in \mathfrak{F}(\mathcal{H},\chi)\), and assume that \( H,\tau \in \mathcal{L}(\mathcal{D},\mathcal{H}) \), where \( \mathcal{D} := \text{Dom}(H) = \text{Dom}(\tau) \) and that \( H,\tau,\chi,\bar{\chi} \) and the composition of operators in the definition of \( F_\chi(H,\tau) \) are contained in \( \text{Dom}(\partial) \).

**Theorem 5.4.** Assume that \( \partial[\bar{\chi}], \bar{\chi} \) are bounded operators which leave \( \mathcal{D} \) invariant, and which commute with \( \tau \) and with one another. Then, under the assumptions...
stated above, and writing \( Q^{(b)} = Q^{(d)}_\chi(H, \tau) \),

\[
\partial[F_\chi(H, \tau)] = \partial[\tau] + \chi\omega\hat{\chi} \partial[\tau] \hat{R}_\chi \omega \chi + Q^2 \partial[\omega]Q \\
+ \partial[\chi]\hat{H}Q + Q^2 \hat{H} \partial[\chi] \\
- 2\chi\omega(\tau^{-1}\partial[\chi] - \hat{R}_\chi \omega \tau^{-1}\partial[\chi]) \tau \hat{R}_\chi \omega \chi .
\]

(5.11)

and

\[
\partial[Q] = -\hat{\chi} \hat{R}_\chi \partial[H]Q \\
\partial[Q^2] = -Q^2 \partial[H] \hat{\chi} \hat{R}_\chi .
\]

(5.12)

In particular \( \text{(5.11)} \) reduces to

\[
\partial[F_\chi(H, \tau)] = Q^2 \partial[H]Q .
\]

(5.13)
in the special case where

\[
[\partial[\chi], \hat{\chi}] = 0 = \partial[\tau]
\]

(5.14)
is satisfied.

5.4. Composition identities. For two subsequent applications of the smooth Feshbach map, the following concatenation rule holds.

**Theorem 5.5.** Let \( 0 \leq \chi_1, \chi_2 \leq 1 \) be a pair of mutually commuting, selfadjoint operators, and \( \hat{\chi}_j := (1 - \chi_j^2)^{1/2} \). We assume that \( \chi_1 \chi_2 = \chi_2 \chi_1 = \chi_2 \), such that \( \text{Ran}(\chi_2) \subseteq \text{Ran}(\chi_1) \subseteq \mathcal{H} \). Let

\[
(H, \tau_1) \in \mathfrak{F}\mathfrak{P}(\mathcal{H}, \chi_1) \\
(H, \tau_2) \in \mathfrak{F}\mathfrak{P}(\mathcal{H}, \chi_2) \\
(F_1, \tau_{12}) \in \mathfrak{F}\mathfrak{P}(\text{Ran}(\chi_1), \chi_2)
\]

(5.15)

with \( F_1 := F_{\chi_1}(H, \tau_1) \), where \( \tau_1, \tau_{12} \) commute with \( \chi_j, \hat{\chi}_j \).

Then,

\[
F_{\chi_2}(H, \tau_2) = F_{\chi_2}(F_1, \tau_{12}) , \\
Q_{\chi_2}(H, \tau_2) = Q_{\chi_1}(H, \tau_1) Q_{\chi_2}(F_1, \tau_{12}) , \\
Q_{\chi_2}^\#(H, \tau_2) = Q_{\chi_1}^\#(F_1, \tau_{12}) Q_{\chi_2}^\#(H, \tau_1) ,
\]

(5.16)

if and only if \( \tau_2 = \tau_{12} \). Furthermore,

\[
A Q_{\chi_2}(H, \tau_2) = A Q_{\chi_2}(F_1, \tau_{12}) \\
Q_{\chi_2}^\#(H, \tau_2) A = Q_{\chi_2}^\#(F_1, \tau_{12}) A ,
\]

(5.17)

for all operators \( A \) acting on \( \mathcal{H} \) that satisfy \( A \hat{\chi}_1 = \hat{\chi}_1 A = 0 \).

5.5. Grouping of overlap terms. The Feshbach pairs \( (H, \tau) \in \mathfrak{F}\mathfrak{P}(\mathcal{H}, \chi) \) considered in this paper have the property that \( H = T + W \) with \( T \neq \tau, [T, \chi] = 0 = [T, \tau] \) and \( [W, \chi], [W, \tau] \neq 0 \), and where the operator \( W \) has a small relative bound with respect to \( T \).

For the resolvent expansions in powers of \( W \) instead of \( \omega = H - \tau = T - \tau + W \) (which is in general not small), we regroup the terms in the smooth Feshbach map to manifestly separate the contributions from \( T \) and \( W \) contained in \( \omega \). For this purpose, we introduce the operator \( \Upsilon_\chi(T, \tau) \) in \( \text{(5.19)} \). Notably, it differs from
Lemma 5.6. Let \((H, \tau) \in \mathfrak{Ψ}(H, \chi)\), and assume that \(H = T + W\), where \([T, \chi] = [T, \tau] = 0\). Let

\[ T' := T - \tau \quad \text{and} \quad \bar{R}_0(T, \tau) := (\tau + \bar{\chi}T')^{-1} \tag{5.18} \]
on Ran(\bar{\chi}). Moreover, let

\[ \Upsilon_{\chi}(T, \tau) := 1 - \bar{\chi}T'\bar{R}_0 = P_x^* + P_x^\tau \bar{R}_0 \tag{5.19} \]
on Ran(\chi), where Ran(\Upsilon_{\chi}(T, \tau) - 1) = Ran(\bar{\chi}T'), and where \(\Upsilon_{\chi}(T, \tau)\) commutes with \(\tau, \chi, \bar{\chi}\) and \(T\). Then,

\[ F_{\chi}(H, \tau) = \tau + \chi T'\Upsilon_{\chi}(T, \tau)\chi \tag{5.20} \]
and in particular, \(\Upsilon_{\chi} \equiv 1\) if and only if \(\tau = T\).

Moreover,

\[ \Upsilon_{\chi}(T_1, \tau_1) - \Upsilon_{\chi}(T_2, \tau_2) = \bar{\chi}^2(T_2 - T_1)\bar{R}_0(T_2, \tau_2)\Upsilon_{\chi}(T_1, \tau_1) \tag{5.21} \]

\[ - \bar{\chi}^2(\tau_2 - \tau_1)T_1\bar{R}(T_1, \tau_1)\bar{R}(T_2, \tau_2) \]
where \((T_i, \tau_i), i = 1, 2\), satisfy the same assumptions as \((T, \tau)\).

Proof. We only verify (5.21); all other statements were proved in [4]. Let \(\bar{R}_{0,i} := \bar{R}(T_i, \tau_i)\). We have

\[ \Upsilon_{\chi}(T_1, \tau_1) - \Upsilon_{\chi}(T_2, \tau_2) = \bar{\chi}^2(T_2 - T_1)\bar{R}_{0,2} - \bar{\chi}^2T_1(\bar{R}_{0,1} - \bar{R}_{0,2}) \]

\[ = \bar{\chi}^2(T_2 - T_1)(\bar{R}_{0,2} - \bar{\chi}^2T_1\bar{R}_{0,1}\bar{R}_{0,2}) \]

\[ - \bar{\chi}^2T_1\bar{R}_{0,1}\bar{R}_{0,2}(\tau_2 - \tau_1) \]

\[ = \bar{\chi}^2(T_2 - T_1)\bar{R}_{0,2}\Upsilon_{\chi}(T_1, \tau_1) - \bar{\chi}^2T_1\bar{R}_{0,1}\bar{R}_{0,2}(\tau_2 - \tau_1) \tag{5.22} \]

using

\[ \bar{R}_{0,1} - \bar{R}_{0,2} = \bar{R}_{0,1}\bar{R}_{0,2}(\tau_2 + \bar{\chi}^2T_2' - (\tau_1 + \bar{\chi}^2T_1')) \tag{5.23} \]

\[ = \bar{R}_{0,1}\bar{R}_{0,2}(\tau_2 - \tau_1 + \bar{\chi}^2(T_2 - T_1)) \]

where \(T_i' = T_i - \tau_i\). This establishes (5.21). \(\square\)

6. ISOSPECTRAL RENORMALIZATION GROUP: EFFECTIVE HAMILTONIANS

In this section, we introduce a space of effective Hamiltonians. While the basic constructions are similar or equal to those in [4], some significant modifications will be formulated in later sections.

We introduce the "reduced" Hilbert space

\[ \mathcal{H}_{red} := \mathbb{C}^2 \otimes \mathbf{1}(H_f < 1)\mathfrak{Ψ} \subset \mathbb{C}^2 \otimes \mathfrak{Ψ}, \tag{6.1} \]

and choose a smooth cutoff function

\[ \chi_1[x] := \sin\left[\frac{\pi}{2}\Theta(x)\right] \tag{6.2} \]
on \([0,1]\), with
\[
\Theta \in C_0^\infty([0,1]; [0,1])
\]
and
\[
\Theta = 1 \text{ on } [0, \frac{3}{4}].
\]
Together with
\[
\tilde{\chi}_1[x] := \sqrt{1 - \chi_1^2[x]},
\]
we obtain the selfadjoint cutoff operators \(\chi_1[H_f]\) and \(\tilde{\chi}_1[H_f]\) on \(\mathcal{H}_{red}\) (and on \(\tilde{\mathcal{F}}\)).

We introduce the notation
\[
P := (H_f, \vec{p}_f),
\]
with associated spectral variables
\[
X := (X_0, \vec{X}) \in [0,1] \times B_1.
\]
We introduce a class of bounded operators on \(\mathcal{H}_{red}\), referred to as effective Hamiltonians, of the form
\[
H = E[p] \chi_1^2[H_f] + T[P; \vec{p}] + \chi_1[H_f]W[p] \chi_1[H_f],
\]
parametrized by the conserved momentum \(\vec{p} \in \mathbb{R}^3\). \(E[p] \in \mathbb{R}\) is a spectral parameter. The operator \(T[P; \vec{p}]\) is referred to as the free, or the non-interacting term in the effective Hamiltonian, and the function \(T[\cdot; \vec{p}] : [0,1] \times B_1 \rightarrow \mathbb{R}\) has the form
\[
T[X; \vec{p}] = X_0 + T'[X; \vec{p}], \quad T'[X; \vec{p}] = \chi_1^2[X_0] \tilde{T}[X; \vec{p}],
\]
with
\[
\partial_{X_0}^a \tilde{T}[0; \vec{p}] = 0 \quad \text{and} \quad \partial_{X_0}^a T'[0; \vec{p}] = 0, \quad a = 0, 1.
\]
Clearly, \(T[0; \vec{p}] = 0\), and \(T[P; \vec{p}]\) commutes with every component of \(P\). The detailed list of assumptions imposed on \(\tilde{T}[X; \vec{p}]\) is presented in Section 7.3.

The operator \(W\) in the effective Hamiltonian is referred to as its interaction term,
\[
W = \sum_{M+N=1}^\infty W_{M,N},
\]
where the operator \(W_{M,N}\) is a generalized Wick monomial of degree \((M, N)\) of the form
\[
W_{M,N} \equiv W[w_{M,N}] = \int d\mu_\sigma(K^{(M,N)}) a^*(K^{(M)}) w_{M,N} [P; \vec{p}; K^{(M,N)}] a^*(\bar{K}^{(N)}),
\]
where we introduce the notation (recalling that $K = (\vec{k}, \lambda) \in B_1 \times \{+,-\}$)

$$
K^{(M)} := (K_1, \ldots, K_M) \\
\vec{K}^{(N)} := (\vec{K}_1, \ldots, \vec{K}_N) \\
K^{(M,N)} := (K^{(M)}, \vec{K}^{(N)}) \\
a^\sigma(K^{(M)}) := \prod_{i=1}^M a^\sigma(K_i) \\
\Sigma[\vec{k}^{(N)}] := k_1 + \cdots + k_N
$$

for $M, N \geq 0$, and $a^\sigma = a$ or $a^*$. The integration measure $d\mu_\sigma$ on $(B_1 \times \{+,-\})^{M+N}$ is given by

$$
d\mu_\sigma(K^{(M,N)}) := \prod_{i=1}^M \prod_{j=1}^N \frac{dK_i}{|k_i|^{1/2}} \frac{d\vec{K}_j}{|\vec{k}_j|^{1/2}},
$$

We note that hereby, the cutoff function $\kappa_\sigma$ is incorporated into the integration measures $d\mu_\sigma$ if $\sigma \leq 1$, and absorbed into the generalized Wick kernels $w_{M,N}$ if $\sigma > 1$. Moreover, we note that for $\sigma > 1$ and $|k| \leq 1$, we have $\kappa_\sigma(|k|) = \frac{|k|}{\sigma}$ with $\kappa_\sigma$ given in (2.17).

For $M + N \geq 1$, the generalized Wick kernels $w_{M,N}$ are Mat$(2 \times 2, \mathbb{C})$-valued functions of $X, K^{(M,N)}$, and $\vec{p}$, of the form

$$
w_{M,N} := w^0_{M,N} 1_2 + \vec{\tau} \cdot \vec{w}_{M,N}
$$

in the basis of Pauli matrices $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$.

We shall refer to

$$
w^0_{M,N} \quad \text{and} \quad \vec{w}_{M,N} := (w^1_{M,N}, w^2_{M,N}, w^3_{M,N})
$$

as the scalar, and the vector component of $w_{M,N}$, respectively. Every component of $w_{M,N}$ is separately fully symmetric with respect to $K_1, \ldots, K_M$ and $\vec{K}_1, \ldots, \vec{K}_N$.

For $M + N = 0$,

$$
w_{0,0} = w^0_{0,0} 1_2 \quad (\vec{w}_{0,0} \equiv 0)
$$

is assumed to be purely scalar.

6.1. The Banach space of generalized Wick kernels. We recall that

$$
\mathcal{X} = (X_0, \vec{X}) \in [0,1] \times B_1
$$
denotes the quadruple of spectral variables corresponding to \( \mathcal{P} = (H_f, \mathcal{P}_f) \). Let 
\[
\mathcal{a} := (a_0, \mathcal{a}) , \quad \mathcal{a} := (a_1, a_2, a_3) 
\]
with \( a_i \in \mathbb{N}_0 \),
\[
\mathcal{a} := (a_0, \mathcal{a}), \quad \mathcal{a} := (a_1, a_2, a_3) 
\]
\[
|\mathcal{a}| := \sum_{j=1}^{3} a_j , \quad |\mathcal{a}| := |a_0| + |\mathcal{a}| 
\]
(6.18)

\[
\partial_{\mathcal{X}}^a := \sum_{j=0}^{3} \partial_{X_j}^{a_j} 
\]
\[
\nabla_{\mathcal{X}}^a := \sum_{j=1}^{3} \partial_{X_j}^{a_j} . 
\]
(6.19)

For \( M = N = 0 \), we introduce the norms
\[
\|w_{0,0}\|_{0,0} := \sup_{|X| \leq X_0 \in \mathcal{I}} |w_{0,0}| 
\]
(6.21)
and
\[
\|w_{0,0}\|_{0,0} := \sum_{0 \leq |w| \leq 2} \|\partial_{\mathcal{X}}^w w_{0,0}\|_{0,0} + \sum_{|\mathcal{a}| = 0,1} \|\partial_{\mathcal{P}}^a \partial_{\mathcal{X}}^a w_{0,0}\|_{0,0} 
\]
(6.22)
(by definition, the vector part of \( w_{0,0} \) is zero). Writing
\[
\|A\|_{\text{Mat}(2 \times 2, \mathbb{C})} := \sqrt{\text{Tr} A^* A} , 
\]
(6.23)
we define
\[
\|w_{M,N}\|_{M,N} := \left(2\pi^{\frac{1}{2}}\right)^{M+N} \sup_{|X| \leq X_0 \in \mathcal{I}} \sup_{K^{(M,N)}} \|w_{M,N} [X; K^{(M,N)}]\|_{\text{Mat}(2 \times 2, \mathbb{C})} 
\]
and
\[
\|w_{M,N}\|_{M,N} := \sum_{0 \leq |w| \leq 2} \|\partial_{\mathcal{X}}^w w_{M,N}\|_{M,N} + \sum_{|\mathcal{a}| = 0,1} \|\partial_{\mathcal{P}}^a \partial_{\mathcal{X}}^a w_{M,N}\|_{M,N} 
\]
\[
+ \sum_{a=0,1} \sup_{(\bar{k}, \lambda) \in K^{(M,N)}} \|\partial_{\mathcal{P}}^a \partial_{\mathcal{K}} \partial_{\mathcal{X}}^{\bar{w}} w_{M,N}\|_{M,N} 
\]
(6.24)
for \( M + N \geq 1 \).

We note the following differences from [4]:

- The kernels \( w_{M,N} \) in [4] are scalar, and the norms used in [4] do not contain second order derivatives with respect to \( X_0 \), or mixed derivatives in \( |\mathcal{P}|, |\mathcal{K}| \).
- In [4], different norms are introduced for \( |\mathcal{P}| = 0 \), and for \( |\mathcal{P}| > 0 \).
- In [4], the infrared regularization \( \kappa_\sigma \) is attributed to the generalized Wick kernels \( w_{M,N} \), and not to the integration measure \( d\mu_\sigma \). Therefore, the corresponding norm in [4] depends on \( \sigma \), while here, it does not.
We define the Banach spaces

\[ W_{0,0}^\sharp := \{ w_{0,0} \mid \|w_{0,0}\|_{0,0}^\sharp < \infty \} \]

and

\[ W_{M,N}^\sharp := \{ w_{M,N} \mid \|w_{M,N}\|_{M,N}^\sharp < \infty \} \quad (6.25) \]

of generalized Wick kernels of degree \((M, N)\) with \(M + N \geq 0\).

**Lemma 6.1.** Let \(M, N \in \mathbb{N}_0\), and \(M + N \geq 1\). Let \(\|w_{M,N}\|_{M,N} < \infty\), and \(W_{M,N} := W_{M,N}[w_{M,N}]\) as in \((6.17)\). Then, the operator norm \(\| \cdot \|_{op}\) of \(W_{M,N}\) on \(H_{\text{red}}\) is bounded by

\[
\|W_{M,N}\|_{op} \leq \left( \frac{1}{M} \right)^{\frac{M}{2}} \left( \frac{1}{N} \right)^{\frac{N}{2}} \|w_{M,N}\|_{M,N} \cdot
\]

\(P^\perp_{\Omega_f} := 1 - \langle \Omega_f | \Omega_f \rangle\) is the projection onto the complement of the subspace spanned by the Fock vacuum in \(\mathcal{F}\).

The proof is given in \([3]\).

In order to accommodate infinite sums of Wick monomials, we define the spaces

\[ W_k^\sharp := \bigoplus_{M+N=k} W_{M,N}^\sharp \quad (6.26) \]

for \(k \geq 1\) and

\[ w_k := (w_{M,N})_{M+N=k} \quad (6.27) \]

with

\[ \|w_k\|^\sharp_{\xi} := \xi^{-k} \sum_{M+N=k} \|w_{M,N}\|_{M,N}^\sharp \quad (6.28) \]

Moreover, for \(0 < \xi < 1\), we introduce the Banach space

\[ W_k^\sharp := \bigoplus_{m \geq k} W_m^\sharp \quad , \quad k \geq 1 \quad , \quad (6.29) \]

of sequences of generalized Wick kernels

\[ w_k := (w_m)_{m \geq k} \quad (6.30) \]

for which

\[ \|w_k\|^\sharp_{\xi} := \sum_{m \geq k} \|w_m\|^\sharp_{\xi} \quad (6.31) \]

is finite.

In the case \(M = N = 0\) (recalling again that \(w_{0,0} = w_{0,0}^0 1_2\) is scalar),

\[
w_{0,0}[X; \vec{p}] = w_{0,0}^0[0; \vec{p}] + (w_{0,0}[X; \vec{p}] - w_{0,0}[0; \vec{p}]) \quad (6.32)
\]

induces the decomposition

\[ \mathcal{M}_{0,0} = \mathbb{R} \oplus \mathcal{W}^\sharp \]
with
\[ T^\sharp := \left\{ T \colon \bigcup_{r \in [0,1]} \{ r \} \times B_r \to \mathbb{R} \mid \| T \|_T^b < \infty, \ T[0; p] = 0, \ T[X_0, RX; p] = T[X; R^{-1}p] \ \forall R \in O(3) \right\}, \]
and
\[ \| T \|_T^b := \| T \|_{0,0}^b. \] (6.33)
The pair \((T^\sharp, \| \cdot \|_T^b)\) is a Banach space.

We introduce the Banach space
\[ \mathcal{M}^\sharp_{\geq 0} := \mathbb{R} \oplus T^\sharp \oplus \mathcal{M}^\sharp_{\geq 1} \] (6.34)
endowed with the norm
\[ \| w \|_\xi^b := \sum_{a=1}^{\infty} |\partial_{\tilde{p}} a| w_{a,0}[\tilde{p}; \tilde{p}]| + \| T \|_T^b + \| w_{\geq 1} \|_{\xi}^b \] (6.35)
for \( w \in \mathcal{M}^\sharp_{\geq 0} \). To a sequence of generalized Wick kernels
\[ w := (E, T, \{ w_{M,N} \}_{M+N \geq 1}) \in \mathcal{M}^\sharp_{\geq 0}, \]
we associate the effective Hamiltonian
\[ H[w] = E[\tilde{p}] \chi^2[H_f] + T[P; \tilde{p}] + \chi_1[H_f] W[w] \chi_1[H_f] \] (6.36)
with
\[ W[w] := \sum_{M+N \geq 1} W_{M,N} [w_{M,N}], \] (6.37)
which is of the form \( (6.7) \).

The following result corresponds to Theorem 3.3 of [3].

**Lemma 6.2.** The map
\[ H : \mathcal{M}^\sharp_{\geq 0} \to B(\mathcal{H}_{\text{red}}) \]
\[ w \mapsto H[w] = (6.36) \] (6.38)
is an injective embedding of \( \mathcal{M}^\sharp_{\geq 0} \) into the bounded operators on \( \mathcal{H}_{\text{red}} \).

Moreover,
\[ \| H[w] \|_{\text{op}} \leq \| w \|_{\xi}^b \] (6.39)
for \( 0 < \xi < 1 \) and \( w \in \mathcal{M}^\sharp_{\geq 0} \), and more generally,
\[ \| H[w_{\geq k}] \|_{\text{op}} \leq \xi^k \| w_{\geq k} \|_{\xi}^b \] (6.40)
for \( w_{\geq k} \in \mathcal{M}^\sharp_{\geq k} \).
7. Isospectral renormalization group: Renormalization map

In this section, we introduce the isospectral renormalization map. While the structure of the exposition is similar as in [3, 4], the constructions themselves are significantly more subtle. The strictly marginal type of the problem under consideration now enters the constructions in a very essential manner (see also Remarks 7.3 and 7.4 below).

7.1. Definition of the isospectral renormalization map. We consider families of effective Hamiltonians parametrized by \( w[r] \) which depend differentiably on a real-valued spectral parameter \( r \in I_{\overline{100}} := [-\overline{100}, \overline{100}] \). Let \( W_{\geq 0} \) denote the Banach space of \( W_{\geq 0} \)-valued differentiable functions on the interval \( I_{\overline{100}} \), endowed with the norm

\[
\| T \|_T := \sup_{r \in I_{\overline{100}}} \| T[r] \|_T^2 , \quad \| w \|_\xi := \sup_{r \in I_{\overline{100}}} \| w[r] \|_\xi^2 , \quad (7.1)
\]

where

\[
\| T[r] \|_T^2 := \| T[r] \|_T^b + \| \partial_r T[r] \|_{0,0} \quad (7.2)
\]

and

\[
\| w \|_\xi^2 := \sum_{a+b=0,1} \sup_{r \in I_{\overline{100}}} \| \partial^{a}_{\vec{p}} \partial^{b}_r w_{0,0}[r; \xi; \vec{p}] \|_T^2 + \| T \|_T^2 + \| w \|_{\geq 1}^2 , \quad (7.3)
\]

with

\[
\| w_k[r] \|_\xi^2 := \| w_k[r] \|_\xi^2 + \xi^{-k} \sum_{M+N=k} \| \partial_r w_{M,N}[r] \|_{M,N} \quad (7.4)
\]

\[
\| w_{\geq k}[r] \|_\xi^2 := \sum_{j \geq k} \| w_j[r] \|_\xi^2 .
\]

The statements of Lemma 6.2 also hold in the case where \( \| \cdot \|_\xi^2 \) is replaced by \( \| \cdot \|_\xi^2 \), as can be easily seen. Let \( H(W_{\geq 0}) \) denote the Banach space of differentiable families of effective Hamiltonians

\[
I_{\overline{100}} \to H(W_{\geq 0}) , \quad r \mapsto H[w[r]] , \quad (7.5)
\]

in \( B(H_{\text{red}}) \).

Remark 7.1. The spectral parameter \( r \) is chosen real, and not in \( \mathbb{C} \) as in [4], due to certain technical advantages related to strict marginality, see Remark 7.3 below.

Remark 7.2. Henceforth, we will frequently omit \( \vec{p} \) from the notation; it is always understood that the effective Hamiltonians depend on \( \vec{p} \).

Let \( w \in W_{\geq 0} \) and let \( 0 < \rho \leq \frac{1}{2} \), which will be fixed later. The renormalization transformation \( R_\rho \) is defined by the composition of the following three operations:

(F) The smooth Feshbach map \( F_{\chi_{\rho}[H_f]} \) is applied to the Feshbach pair

\[
(H[w[r]], \alpha[r] H_f) \in \mathfrak{F}(H_{\text{red}}, \chi_{\rho}[H_f]) .
\]
Thereby, the degrees of freedom in the range of photon field energies in $[\rho, 1]$ are "eliminated (decimated)". The scalar $\alpha[r] \in \mathbb{R}$ with $|\alpha[r] - 1| \ll 1$ is determined by the implicit condition (7.7) below, and

$$\chi_\rho[H_f] := \chi_1[H_f/\rho],$$

where $\chi_1$ is defined in (6.2).

(S) A unitary rescaling transformation $S_\rho$ with

$$S_\rho[A] = \alpha[r]^{-1} \rho^{-1} \Gamma_\rho A \Gamma_\rho^*,$$

where $\Gamma_\rho$ implements unitary dilation by a factor $\rho$ on $\mathfrak{F}$ (see Section 7.1.2 for detailed definitions).

(E) A transformation $E_\rho$ of the spectral parameter $r \in I_{\frac{1}{10}}$ in $w[r]$.

7.1.1. The operation (F). The detailed constructions are presented in Sections 7.2 and 7.3. The main steps can be summarized as follows.

We first verify for $w$ in a polydisc $\Omega(\varepsilon, \delta, \eta, \lambda, \sigma) \subset \mathfrak{M}_{\mathbb{S}}$ (the definition is given in Section 7.3), $|r| < \frac{1}{10}$, and $|\alpha - 1| < \eta$ sufficiently small, that

$$(H[w[r]], \alpha H_f) \in \mathfrak{F}(\mathcal{H}_{red}, \chi_\rho[H_f]),$$

i.e. $(H[w[r]], \alpha H_f)$ is a Feshbach pair corresponding to $\chi_\rho[H_f]$.

We then choose the coefficient $\alpha$ in $\tau = \alpha H_f$ to be given by the unique solution of the implicit equation

$$\alpha[r] = \left\langle \Omega_f, \partial_{H_f} F_{\chi_\rho[H_f]}(H[w[r]], \alpha[r] H_f) \Omega_f \right\rangle$$

(7.7)

for $\alpha[r]$, with $|\alpha[r] - 1| \ll \eta$. The existence and uniqueness of this solution are proved in Proposition 7.6. The correct choice of $\alpha[r]$ is crucial for the convergence of the renormalization group recursion in later sections. The reasons are outlined in Remark 7.3 below.

7.1.2. The operation (S). The rescaling transformation $S_\rho$ is obtained from unitarily scaling the photon momenta by a factor $\rho$, followed by multiplication with a scalar factor $(\alpha[r] \rho)^{-1}$,

$$S_\rho[A] = \frac{1}{\alpha[r] \rho} \Gamma_\rho A \Gamma_\rho^*,$$

(7.8)

where $\Gamma_\rho$ is the unitary dilation operator on $\mathfrak{F}$. It satisfies

$$S_\rho[a^*(\tilde{K}^{(M)})a(\tilde{K}^{(N)})] = \alpha[r]^{-1} \rho^{-1-M-N} \tilde{a}^*(\rho^{-1}K^{(M)})a(\rho^{-1}K^{(N)}),$$

(7.9)

where we write

$$\rho^{-1}K := (\rho^{-1}k, \lambda), \quad \rho^{-1}K^{(M)} := (\rho^{-1}K_1, \ldots, \rho^{-1}K_M),$$

(7.10)

for $K \in \mathbb{R}^3 \times \{+, -, 0\}$.

To determine the action of rescaling on the generalized Wick kernels, we first observe that under the scaling $\tilde{k}_i \to \rho \tilde{k}_i$, $\tilde{k}_j \to \rho \tilde{k}_j$, the integration measures $d\mu_\sigma(K^{(M+N)})$ in (6.13) produce a factor $\rho^{5(M+N)}$. The cutoff function is modified by $\kappa_\sigma \to \kappa_{\rho^{-1}\sigma}$.
As a convention, we attribute the scaling factors $\rho^{\frac{k}{2}(M+N)}$ from the integration measures to $w_{M,N}$. In addition, when $\sigma > 1$, the cutoff function $\kappa_\sigma$, given by $\kappa_\sigma(|k|) = \frac{|k|}{\sigma}$, is absorbed into the generalized Wick kernel $w_{M,N}$.

Then, restricted to $H[\mathfrak{M}_{\geq 0}] \subset \mathcal{B}[\mathcal{H}_{\text{red}}]$, $S_\rho$ induces a rescaling map $s_\rho$ on $\mathfrak{M}_{\geq 0}$ by

$$S_\rho[H[w]] =: H[s_\rho[w]] =: H[(s_\rho[w_{M,N}])_{M,N \geq 0}],$$

where

$$s_\rho[w_{M,N}][X; K^{(M,N)}] = \left\{ \begin{array}{ll}
\alpha[r]^{-1} \rho^{M+N-1} w_{M,N}[\rho X; \rho K^{(M,N)}] & \text{if } \sigma \leq 1 \\
\alpha[r]^{-1} \rho^{2(M+N)-1} w_{M,N}[\rho X; \rho K^{(M,N)}] & \text{if } \sigma > 1 .
\end{array} \right.$$  

(7.12)

The powers of $\rho$ are obtained as follows. A factor $\rho^{\frac{k}{2}(M+N)}$ enters from the scaling of the integration measure $d\mu_\sigma(K^{(M,N)})$. For $\sigma > 1$, an additional factor $\rho^{M+N}$ enters from the scaling of $\kappa_\sigma(|k|) = \frac{|k|}{\sigma}$ (one factor $\rho$ for each of the $M + N$ momentum variables; if $K \neq 1$ in (7.17), the factor is $\rho^{K(M+N)}$). In addition, there is a factor $\rho^{-\frac{k}{2}(M+N)}$ from the unitary scaling of $M$ creation- and $N$ annihilation operators, see (7.9). Finally, an overall factor $\rho^{-1}$ is produced by multiplicative factor $(\alpha[r]\rho)^{-1}$ in the definition of $S_\rho$.

This implies the bounds

$$\|s_\rho[w_{M,N}]\|_{M,N} \leq |\alpha[r]|^{-1} \rho^{M+N-1}\|w_{M,N}\|_{M,N} \quad \text{if } \sigma \leq 1 .$$

(7.13)

and

$$\|s_\rho[w_{M,N}]\|_{M,N} \leq |\alpha[r]|^{-1} \rho^{2(M+N)-1}\|w_{M,N}\|_{M,N} \quad \text{if } \sigma > 1 .$$

(7.14)

Thus, if $\sigma \leq 1$, all $\|w_{M,N}\|_{M,N}$ with $M + N \geq 2$ are contracted by a factor $\leq \rho$; they are therefore irrelevant in the renormalization group terminology. The generalized Wick kernels with $M + N = 1$ do not scale with any positive power of $\rho$; this property is referred to as marginality.

In the case $\sigma > 1$, $\|w_{M,N}\|_{M,N}$ is contracted by a factor $\leq \rho$, for all $M + N \geq 1$; that is, all generalized Wick kernels are irrelevant if $\sigma > 1$. When we speak of marginal interactions, it is understood that we refer to the case $\sigma < 1$.

7.1.3. The operation $\mathcal{E}$. Given $w \in \mathfrak{M}_{\geq 0}$ with $E[r] := w_{0,0}[r; \mathfrak{Q}]$, we define

$$\mathcal{U}[w] := \left\{ r \in I_{\text{red}} \mid |E[r]| \leq \frac{\rho}{100} \right\} ,$$

(7.15)

and consider the map

$$E_\rho : \mathcal{U}[w] \to I_{\text{red}} , \quad r \mapsto (\alpha[r]\rho)^{-1} E[r] .$$

(7.16)

$E_\rho$ is a bijection, and $\mathcal{U}[w]$ is close to the interval $I_{\text{red}}$, provided that $w$ is close to the non-interacting theory defined in Section 7.2 below.
7.1.4. The renormalization transformation. Composing the rescaling transformation \( S_\rho \), the transformation of the spectral parameter \( E_\rho \), and the smooth Feshbach map, we now define the renormalization transformation \( R_\rho \).

We recall from Lemma 6.2 that the map \( H : \mathfrak{m} \mapsto H[\mathfrak{m}] \) injectively embeds \( \mathfrak{m}_{\geq 0} \) into the bounded operators on \( \mathcal{H}_{\text{red}} \). \( \text{Dom}(R_\rho) \), the domain of \( R_\rho \), is defined by those elements \( \mathfrak{m} \in \mathfrak{m}_{\geq 0} \) for which

\[
R_\rho^H[H[\mathfrak{m}]] : = S_\rho \left[ F_{\lambda[H]}(H[\mathfrak{m}], \alpha[r]H) \right]
\]

is well-defined and in the domain of \( H^{-1} \), where \( \hat{r} \in I_{\text{red}} \) and

\[
r = E_\rho^{-1}[\hat{r}] \in I_{\text{red}}.
\]

The real number \( \alpha[r] \in 1 + D_\eta \) is defined by (7.7). The map \( R_\rho^H : B(\mathcal{H}_{\text{red}}) \to B(\mathcal{H}_{\text{red}}) \) is referred to as the renormalization map acting on operators.

Remark 7.3. The definition (7.7) of \( \alpha[r] \in 1 + D_\eta \) ensures that no operator proportional to \( H_f \chi_1^2[H_f] \) is generated by \( R_\rho \). We note that with an additional term of the form \( cH_f \chi_1^2[H_f] \) in the non-interacting part \( T \) of the effective Hamiltonian, it cannot be ruled out that \( |c| \) becomes large under repeated applications of \( R_\rho \). Once \( |c| \geq 1 \), the operator \( H_f + cH_f \chi_1^2[H_f] \) may develop spurious zero spectrum in the vicinity of \( H_f = 1 \) which strongly complicates the analysis. This phenomenon is suppressed by the choice of \( \alpha[r] \) stated in (7.7).

Remark 7.4. Using the factor \( \alpha[r]^{-1} \) in \( S_\rho \), the coefficient of the operator \( H_f \) in \( H[R_\rho[\mathfrak{m}]] \) is normalized to have the value 1 (as in \( H[\mathfrak{m}] \)). In the absence of this factor, the coefficient of \( H_f \) increases to \( O(\log \frac{1}{\sigma_0}) \) under repeated applications of the renormalization map \( \sigma_0 \) is the infrared cutoff in the fiber Hamiltonian \( H(\hat{p}, \sigma_0) \).

In contrast, it is not necessary in \( \mathbf{[4]} \) to keep the coefficient of \( H_f \) fixed because there, its smallness is ensured by the \( \sigma_0 \)-dependent bounds on the fine-structure constant (in contrast, the results proved here hold for \( \alpha < \alpha_0 \) with \( \alpha_0 \) independent of \( \sigma_0 \)).

Given \( R_\rho^H \), we define the renormalization map acting on generalized Wick kernels

\[
\mathcal{R}_\rho := H^{-1} \circ R_\rho^H \circ H
\]

on \( \text{Dom}(R_\rho) \subset \mathfrak{m}_{\geq 0} \). It is shown in Section 7.3 that the intersection of the domain and range of \( \mathcal{R}_\rho \) contains a family of polydiscs.

7.2. Choice of a reference theory. We compare \( \mathfrak{m} \in \text{Dom}(R_\rho) \) to a reference family of non-interacting theories parametrized by \( \mathfrak{m}_0^{(\bar{p}, \lambda)} \in \text{Dom}(R_\rho) \), which we introduce here (of the same form as in \( \mathbf{[4]} \)). A central task of our analysis is to prove that \( \| \mathfrak{m} - \mathfrak{m}_0^{(\bar{p}, \lambda)} \|_\xi \) remains small under iterations of \( R_\rho \).

We choose comparison kernels of the form

\[
\mathfrak{m}_0^{(\bar{p}, \lambda)}[r] = E[r] \oplus T_0^{(\bar{p}, \lambda)}[r; X] \oplus \Omega_{\geq 1},
\]

where

\[
T_0^{(\bar{p}, \lambda)}[r;\mathcal{P}] = H_f + \chi_1^2[H_f] Y^{(\bar{p}, \lambda)}[\mathcal{P}](r \hat{p} P + \lambda P_r^2),
\]
with
\[ \Upsilon^{(\bar{\rho}, \lambda)}_{\chi_1}[r; \underline{P}] := \Upsilon_{\chi_1[H_f]}(E[r] \chi^2_1[H_f] + T_0^{(\bar{\rho}, \lambda)}[\underline{P}], H_f) , \] (7.22)
see [5.19].

Therefore,
\[ H^{[\underline{P}]_{(\bar{\rho}, \lambda)}[r]} = H_f + \chi^2_1[H_f](E[r] - |\bar{p}|P_f^\parallel + \lambda P_f^2) \]
\[ - \frac{(|\bar{p}|P_f^\parallel - \lambda P_f^2)^2 \chi^2_1[H_f]}{H_f + \chi^2_1[H_f](E[r] - |\bar{p}|P_f^\parallel + \lambda P_f^2)} \] (7.23)
In the limit \( E[r], \lambda \to 0 \), the operator
\[ \lim_{r \to 0} \lim_{\lambda \to 0} H^{[\underline{P}]_{(\bar{\rho}, \lambda)}[r]} = H_f - |\bar{p}|P_f^\parallel \chi^2_1[H_f] - \frac{(|\bar{p}|P_f^\parallel)^2 \chi^2_1[H_f]}{H_f - \chi^2_1[H_f]|\bar{p}|P_f^\parallel} \] (7.24)
defines a fixed point of the renormalization transformation \( \mathcal{R}_p \).

7.3. Detailed structure of \( T \). It is necessary to impose more detailed requirements on the structure of \( T \) than those formulated in [6.8].

We recall from [6.5] that
\[ T[r; \underline{P}; \bar{p}] = H_f + \chi^2_1[H_f] \overline{T}[r; \underline{P}; \bar{p}] . \] (7.25)
We require that \( \overline{T} \) has the form
\[ \overline{T}[r; \underline{P}] = (\beta[r; \bar{p}]P_f^\parallel + \lambda_T P_f^2 + \delta T[r; \underline{P}; \bar{p}]) \overline{\Upsilon}[r; \underline{P}; \bar{p}] , \] (7.26)
where:
- The scalar \( \beta[r; \bar{p}] \in \mathbb{R} \) in \( r \in I_{\underline{\text{fin}}} \) and \( \bar{p} \), with
\[ |\beta[r; \bar{p}] + \bar{p}|, |\partial_{\bar{p}} \beta[r; \bar{p}] + 1| \ll 1 \] for all \( r \in I_{\underline{\text{fin}}} \). (7.27)
- The parameter \( \lambda_T \) is a real number independent of \( r \) and \( \bar{p} \), and \( 0 \leq \lambda_T \leq \frac{1}{2} \).
- The operator \( \overline{\Upsilon}[r; \underline{P}; \bar{p}] \) is close to \( \Upsilon^{(\bar{\rho}, \lambda)}_{\chi_1}[r; \underline{P}; \bar{p}] \),
\[ ||\overline{\Upsilon} - \Upsilon^{(\bar{\rho}, \lambda)}_{\chi_1}||_{\text{T}} \ll 1. \] (7.28)
Moreover,
\[ \chi_{\rho}[H_f] \overline{\Upsilon}[r; \underline{P}; \bar{p}] = 0 . \] (7.29)
- The function \( \delta T[r; X; \bar{p}] \) satisfies
\[ \frac{\partial \overline{\Upsilon}}{\partial X} \bigg|_{X=0} \delta T[r; X] = 0 \quad \text{for} \ 0 \leq |a| \leq 1 , \]
\[ ||\delta T||_T \ll 1 . \] (7.30)
It is a small error term, and \( O(|X|^2) \) in the limit \( |X| \to 0 \).
7.4. The domain of $\mathcal{R}_\rho$. We next prove that the domain of $\mathcal{R}_\rho$ contains a polydisc of the form

$$\mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma) := \left\{ w = (E, T, w_\geq 1) \in \mathbb{M}_{\geq 0} \left| \begin{array}{c} \| w_1 \|_\xi < \eta, \\ \| w_\geq 2 \|_\xi < \varepsilon, \end{array} \right. \right\} \quad \text{with } T \text{ as in (7.25) } \sim (7.30),$$

where

$$\lambda_T = \lambda, \quad \| \delta T \|_\mathcal{X} < \delta, \quad \| \tilde{Y} - \Upsilon_{\chi_0}(\bar{p}; \lambda) \|_\mathcal{X} < K_\Theta \delta,$$

and for $a = 0, 1$,

$$\sup_{r \in I} \left| \beta \right| < \frac{\delta}{2}, \quad (7.35)$$

for parameters

$$0 \leq |\hat{p}| < \frac{1}{3}, \quad 0 \leq \varepsilon \leq \eta \ll 1 \quad 0 \leq \delta \ll 1 \quad 0 \leq \lambda \leq \frac{1}{2}, \quad (7.37)$$

with

$$\xi = \frac{1}{10} \quad (7.38)$$

fixed. The constant $K_\Theta > 2$ only depends on the smooth cutoff function $\Theta$ introduced in (6.2), and is determined in (8.76) below. The parameter $\varepsilon$ measures the size of the projection of the polydisc to a codimension 3 subspace of irrelevant perturbations, and is referred to as an irrelevant parameter. On the other hand, $\delta$ and $\eta$ measure the projection of the polydisc to a dimension 3 subspace of operators which are strictly marginal in the limit $\sigma \searrow 0$, and are therefore referred to as marginal parameters.

We remark that $w \in \mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma)$ implies that

$$\| T - T^{(\bar{p}; \lambda)} \|_\mathcal{X} < K_\Theta \delta, \quad (7.39)$$

where the constant $K_\Theta'$ only depends on $\Theta$. This is discussed in detail in Section 8.4.1 below.

Accordingly, one can verify that

$$\left\{ w \in \mathbb{M}_{\geq 0} \left| \| w - w^{(\bar{p}; \lambda)} \|_\xi \leq \eta \right\} \subset \mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma) \subset \left\{ w \in \mathbb{M}_{\geq 0} \left| \| w - w^{(\bar{p}; \lambda)} \|_\xi \leq 2\delta + 2\eta \right\} \right.,$$

see [3]. Hence, $\mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma)$ is comparable to an $(\delta, \varepsilon, \eta)$-ball around $w^{(\bar{p}; \lambda)}$. 
Lemma 7.5. Let $0 < \xi < 1$, $\sigma > 0$ and $0 < \rho < \frac{1}{2}$. Then,
\[ I_{\rho} \subseteq U[w] \subseteq I_{\frac{\rho}{200}} \] (7.40)
for all $w \in \Omega(\varepsilon, \delta, \eta, \lambda, \sigma)$ with $\eta < \frac{\rho}{200}$, and
\[ |\rho \alpha[r; \vec{p}] \partial_r E_\rho[r] - 1| \leq 2 \eta, \] (7.41)
for all $r \in U[w]$. Then, $E_\rho : U[w] \rightarrow I_{\frac{\rho}{100}}$ is a bijection.

Proof. By definition of $U(\varepsilon, \delta, \eta, \lambda, \sigma)$, we have
\[ |E[r] - r| < \eta, \] and since $r \in U[w] = \{ r \in I_{\frac{\rho}{100}} | |E[r]| < \frac{\rho}{100} \}$, one infers that
\[ |r - |E[r]|| < |E[r] - r| < \eta. \] (7.42)
Hence, (7.40) holds for $\eta < \frac{\rho}{200}$.

To prove (7.41), we note that
\[ \sup_{r \in U[w]} |\partial_r (E[r] - r)| \leq \sup_{r \in I_{\frac{\rho}{100}}} |\partial_r (E[r] - r)| \leq \eta \] (7.43)
from the definition of $U(\varepsilon, \delta, \eta, \lambda, \sigma)$. Using Proposition 7.6 below, we find
\[ |\rho \alpha[r; \vec{p}] \partial_r E_\rho[r] - 1| \leq \frac{\partial_r \alpha[r; \vec{p}]}{\alpha[r; \vec{p}]} |E[r]| + |\partial_r E_\rho[r] - 1| \]
\[ \leq c\eta^2 + \eta < 2\eta \] (7.44)
for $\eta$ sufficiently small. \qed

Proposition 7.6. Assume that $0 \leq |\vec{p}| < \frac{1}{2}$, $0 < \rho < \frac{1}{2\alpha}$, $0 < \xi < 1$, and $w \in \Omega(\varepsilon, \delta, \eta, \lambda, \sigma)$. Then,
\[ (H[w][r; \vec{p}], \alpha H_f) \in \mathfrak{H}(\mathcal{H}_{\text{red}}, \chi_\rho[H_f]) \] (7.45)
for $r \in I_{\frac{\rho}{100}}$. That is, $(H[w][r; \vec{p}], \alpha H_f)$ defines a Feshbach pair corresponding to $\chi_\rho[H_f]$, for all $\alpha \in 1 + D_\eta$, and all $r \in U[w]$.

Moreover, there is a unique solution $\alpha[r; \vec{p}]$ of
\[ \alpha[r; \vec{p}] = \left\langle \Omega_f, \partial H_f F_{\chi_\rho[H_f]}(H[w][r; \vec{p}], \alpha[r; \vec{p}]H_f)\Omega_f \right\rangle \] (7.46)
which satisfies
\[ |\alpha[r; \vec{p}] - 1| < \frac{c\eta^2}{\rho^3} \]
\[ |\partial_r \alpha[r; \vec{p}]|, |\partial_{\vec{p}} \alpha[r; \vec{p}]| < \frac{c\eta^2}{\rho^3} \] (7.47)
and in particular,
\[ \frac{|\partial_{\vec{p}} \alpha[r; \vec{p}]|}{\alpha[r; \vec{p}]} < \frac{c\eta^2}{\rho^3}. \] (7.48)
The constants are independent of $\rho, \eta$. 
Proof. To verify (7.45) for $r \in I_{100}$, |$\tilde{p}$| < $\frac{1}{3}$, and all $\alpha \in 1 + D_\eta$, one can straightforwardly adopt the corresponding results from [4].

To prove that (7.46) has a unique solution in $1 + D_\eta$, we note first that

$$F_{\chi_\rho[H_f]}(H[w;\tilde{p}], \alpha[r;\tilde{p}]H_f) = E[r]\chi_\rho^2[H_f] + (I) + (II)$$

(7.49)

with

$$(I) := \alpha[r;\tilde{p}]H_f + \chi_\rho^2[H_f] \left( T[r;\mathcal{P};\tilde{p}] - \alpha[r;\tilde{p}]H_f \right)$$

(7.50)

and

$$(II) := \chi_\rho[H_f] \gamma_\rho W \chi_\rho[H_f] - \chi_\rho[H_f] \gamma_\rho W \chi_\rho[H_f] \tilde{R}[w;\tilde{p}] \tilde{R}[w;\tilde{p}] W \chi_\rho[H_f] \right) \chi_\rho[H_f] ,$$

where we introduce the abbreviated notations

$$W = W[r;\tilde{p}] \equiv W[w;\tilde{p}] = \sum_{M+N \geq 1} W_{M,N}[w;\tilde{p}]$$

$$\gamma_\rho = \gamma_\rho[r;\mathcal{P};\tilde{p}] \equiv \gamma_\rho(T[r;\mathcal{P};\tilde{p}], \alpha[r;\tilde{p}]H_f) .$$

(7.52)

For (I), we note that

$$T[r;\mathcal{P};\tilde{p}] - \alpha[r;\tilde{p}]H_f = (1 - \alpha[r;\tilde{p}]H_f) + T'[r;\mathcal{P};\tilde{p}] ,$$

(7.53)

where

$$\langle \Omega_f , \partial_{H_f}T'[r;\mathcal{P};\tilde{p}]\Omega_f \rangle = 0$$

(7.54)

(see the definition of $T$ in (6.8)). Therefore,

$$\langle \Omega_f , \partial_{H_f}(I)\Omega_f \rangle = 1 .$$

(7.55)

Next, we consider (II). Using

$$\partial_{H_f}\chi_\rho[H_f] \Omega_f = 0$$

$$\partial_{H_f}\gamma_\rho[r;\mathcal{P};\tilde{p}] \Omega_f = 0$$

$$\langle \Omega_f , \partial_{H_f}W[r;\tilde{p}]\Omega_f \rangle = 0 ,$$

(7.56)

we get

$$\langle \Omega_f , \partial_{H_f}(II)\Omega_f \rangle = - \langle \Omega_f , \partial_{H_f}(W \tilde{R}[w;\tilde{p}] \tilde{R}[w;\tilde{p}] W)\Omega_f \rangle .$$

(7.57)

From Lemma 7.17 below, we find

$$\|\partial_{H_f} W[r;\tilde{p}]\|_{op} \leq \|w\|_{\infty} < \eta < \varepsilon < 2\eta$$

$$\|\partial_{H_f} \tilde{R}[w;\tilde{p}]\|_{op} \leq \frac{C_\omega}{\rho^{1+a}}$$

$$\|\partial_{H_f} \chi_\rho[H_f]\|_{op} \leq \frac{C_\omega}{}\rho$$

(7.58)

for $a = 0, 1$. Indeed, let

$$\tilde{R}_0[r;\mathcal{P};\tilde{p}] = \left( \alpha[r;\tilde{p}]H_f + \chi_\rho[H_f] \tilde{T}[r;\mathcal{P};\tilde{p}] \right)^{-1},$$

(7.59)

denote the free resolvent on $\text{Ran}(\tilde{\chi}_\rho[H_f])$, and $\tilde{T}[r;\mathcal{P};\tilde{p}] = T[r;\mathcal{P};\tilde{p}] - \alpha[r;\tilde{p}]H_f$.

From the resolvent identity

$$\tilde{R}[w;\tilde{p}] = R_0[r;\mathcal{P};\tilde{p}] - \tilde{R}_0[r;\mathcal{P};\tilde{p}] \tilde{\chi}_\rho \chi_1[H_f] W \chi_1 \tilde{\chi}_\rho[H_f] \tilde{R}[w;\tilde{p}]$$

(7.60)
and Lemma 7.7, we find
\[ \| \tilde{R} \|_{op} \leq (1 - \| \tilde{R}_0 \|_{op})^{-1} \| W \|_{op} \| \tilde{R}_0 \|_{op} \leq \frac{C_\phi}{\rho} . \]  \hspace{1cm} (7.61)

Moreover,
\[ \| \partial_{H_j} \tilde{R} \|_{op} \leq (\| \partial_{H_j} \tilde{R}_0 \|_{op} + \| \partial_{H_j} W \|_{op}) \sum_{L \geq 1} L (\| W \|_{op} \| \tilde{R}_0 \|_{op})^{L-1} \]
\[ \leq \left( \frac{C_\phi}{\rho^2} + \eta \right) \sum_{L \geq 1} L \left( \frac{C_\phi \eta}{\rho} \right)^{L-1} \leq \frac{2C_\phi}{\rho^2} \]  \hspace{1cm} (7.62)
by Lemma 7.7 and \( \eta \ll \rho \).

Consequently, one finds
\[ |a[r; \tilde{p}] - 1| = |\Omega_f, \partial_{H_j}(II) \Omega_f| \leq \frac{c\eta^2}{\rho^2} \]  \hspace{1cm} (7.63)
for a constant independent of \( \rho \). This implies that r.h.s. of (7.60) = 1 + \( O(\eta^2) \) for \( a[r; \tilde{p}] \in 1 + D_\eta \). Consequently, there exists a solution of (7.60) in \( 1 + D_\eta \).

To prove uniqueness, we note that only \( \tilde{R}[w][r; \tilde{p}] \) in (7.57) depends on \( a[r; \tilde{p}] \).

Similarly as in (7.62), one finds
\[ \| \partial_{H_j}^a \partial_{[r; \tilde{p}]} b \tilde{R}[w][r; \tilde{p}] \|_{op} \leq \frac{10C_\phi}{\rho^4} , \quad a, b = 0, 1 \]  \hspace{1cm} (7.64)
and a straightforward calculation shows that
\[ \sup_{a \in 1 + D_\eta} \| \partial_{[r; \tilde{p}]} \partial_{\alpha} (\Omega_f, H_{x_r}(II) \Omega_f) \|_{op} \leq \frac{c\eta^2}{\rho^3} \ll 1 \]  \hspace{1cm} (7.65)
for a constant \( c \) independent of \( \rho \). This implies that (7.46) has a unique solution.

The estimates in (6.1-4) are obtained from
\[ \partial_r a[r; \tilde{p}] = \Omega_f, \partial_r (II) \Omega_f \]
\[ \partial_{[r; \tilde{p}]} a[r; \tilde{p}] = \Omega_f, \partial_{[r; \tilde{p}]} (II) \Omega_f \]  \hspace{1cm} (7.66)
and a straightforward calculation using
\[ \| \partial_{a}^b \partial_{[r; \tilde{p}]}^b \partial_{H_j} W \|_{op} \leq c \| w \|_{\infty} \| \xi \| \leq c (\eta + \varepsilon) \leq c' \eta \]
\[ \| \partial_{a}^b \partial_{[r; \tilde{p}]}^b \partial_{H_j} \tilde{R}[w][r; \tilde{p}] \|_{op} \leq \frac{c}{\rho^3} \]
\[ \| \partial_{a}^b \partial_{[r; \tilde{p}]}^b \partial_{H_j} \chi_{\rho}[H_j] \|_{op} = 0 \]  \hspace{1cm} (7.67)
for \( a + b = 1 \) and \( c = 0, 1 \), similarly as in (7.62) and (7.64).

Lemma 7.7. Assume that \( w \in \mathfrak{U}(\varepsilon, \delta, \eta, \lambda, \sigma) \) and \( r \in I^{(\rho)} \). There is a constant \( C_\phi \) only depending on the smooth cutoff function \( \Theta \) in (6.2) such that
\[ \| \partial_r \tilde{R}_0 \|_{op} + \sum_{0 \leq |\alpha| \leq 2} \rho^{1+\| \alpha \|} \| \partial_{\alpha}^\rho \tilde{R}_0 \|_{op} + \sum_{0 \leq |\alpha| \leq 1} \rho^{2+\| \alpha \|} \| \partial_{[r; \tilde{p}]} \partial_{\alpha}^\rho \tilde{R}_0 \|_{op} \leq C_\phi \]  \hspace{1cm} (7.68)
Moreover,
\[ \| \partial_r W[w] \|_{op} + \sum_{0 \leq |\alpha| \leq 2} \| \partial_{\alpha}^\rho W[w] \|_{op} + \sum_{0 \leq |\alpha| \leq 1} \| \partial_{[r; \tilde{p}]} \partial_{\alpha}^\rho W[w] \|_{op} \leq \eta + \varepsilon \]  \hspace{1cm} (7.69)
Proof. For \( \varepsilon, \delta, \eta, \lambda, \sigma \) in \( \mathcal{U} \), we have

\[
|\tilde{R}_0[r; \mathcal{P}]| < cH_f^{-1}
\]

on \( \text{Ran}(\tilde{\varrho}_\rho[H_f]) \) (see (4)), and thus in particular \( \|R_0\|_{op} \leq \frac{\rho}{\rho^*} \). The estimate (7.68) follows from the fact that the left hand side can be bounded by

\[
\text{l.h.s. of } (7.68) \leq \left( \sum_{0 \leq |\varpi| \leq 2} \rho^{1+|\varpi|}\|\tilde{R}_0\|_{op}^{1+|\varpi|} + \sum_{0 \leq |\varpi| \leq 1} \rho^{2+|\varpi|}\|\tilde{R}_0\|_{op}^{2+|\varpi|} \right)\|T\|_\infty
\]

\[
\leq c_\Theta\|T\|_\infty
\]

(7.71)

where the constant \( c_\Theta \) only depends on the smooth cutoff function \( \Theta \), and where \( \|T\|_\infty < c \) follows from the definition of \( \mathcal{U} \).

The estimate (7.69) is an immediate consequence of Lemma 5.6 and the definition of \( \mathcal{U} \).

\[ \square \]

7.5. Generalized Wick ordering. The next step in determining \( \hat{w} = \mathcal{R}_\rho[w] \) consists of finding the generalized Wick ordered normal form of the right hand side of (7.17) (we suppress \( \hat{p} \) in the notation).

We note that

\[
\|\mathcal{Y}_\rho[r; \mathcal{P}]\|_{op} \leq c,
\]

(7.72)

where the constant is independent of \( \rho \), and that

\[
\|W[r]\|_{op} \leq \eta + \varepsilon < 2\eta
\]

(7.73)

for \( \varepsilon, \delta, \eta, \lambda, \sigma \) (recalling that \( \varepsilon < \eta \) by (7.37)).

Recalling the expression for the smooth Feshbach map given in Lemma 5.6, the resolvent expansion in powers of \( W[r] \) yields

\[
F_{\chi_\rho[H_f]}(H[w[r]], \alpha[r]H_f) = E[r] \chi_\rho^2[H_f] + A_0 + \sum_{L=1}^{\infty} (-1)^{L-1} A_L
\]

(7.74)

where

\[
A_0 := \alpha[r]H_f + \chi_\rho^2[H_f] \mathcal{Y}_\rho[r; \mathcal{P}] (T[r; \mathcal{P}] - \alpha[r]H_f)
\]

(7.75)

and

\[
A_L := \chi_\rho[H_f] \mathcal{Y}_\rho[r; \mathcal{P}]
\]

\[
\left[ W[w[r]] \chi_\rho^2[H_f] \chi_\rho[H_f] \tilde{R}_0[r, \mathcal{P}] \right]^{L-1} W[w[r]] \mathcal{Y}_\rho[r; \mathcal{P}] \chi_\rho[H_f].
\]

(7.76)

From (7.73),

\[
\|A_L\|_{op} < C_\Theta L^L \rho^{-L+1} \eta^L.
\]

Hence, the series \( \sum_{L=1}^{\infty} (-1)^{L-1} A_L \) is norm convergent when \( \eta \) is sufficiently small.

We introduce the operators

\[
W_{m+n+q}^{m+n+q}[w] := P_{red} \int_{B_{\rho}^{-1+q}} d\mu_\sigma(Q^{(p,q)}) \alpha^*(Q^{(p)})
\]

\[
u_{m+n+q}^{m+n+q}[\mathcal{P} + \mathcal{X}; Q^{(p)}, K^{(m)}; \bar{Q}^{(q)}, \bar{K}^{(n)}]a(\bar{Q}^{(q)}) P_{r}(\mathcal{A}^{(q)}).
\]

(7.78)
The generalized Wick ordered form of the $L$-th term in the resolvent expansion \([7.74]\) is given as follows.

**Lemma 7.8.** For \(w = (w_{M,N})_{M+N \geq 1} \in \mathcal{M}_{\geq 1}^\mathbb{R}, \) let \(W_{M,N} := W_{M,N}[w_{M,N}], \) \(W = \sum_{M+N \geq 1} W_{M,N}, \) and let \(F_0, \ldots, F_L \in \mathcal{W}_{0,0}. \) Moreover, let \(S_M \) denote the \(M\)-th symmetric group. Then,

\[
F_0 W F_1 W \cdots W F_{L-1} W F_L = H[\bar{w}],
\]

where \(\bar{w} = (\bar{w}_{M,N})_{M+N \geq 0} \in \mathcal{M}^\mathbb{R}_0\) is determined by the symmetrization with respect to \(K^{(M)}\) and \(\bar{K}^{(N)}, \)

\[
\bar{w}_{M,N}[X; K^{(M,N)}] = \text{Sym}_{M,N} \bar{w}'_{M,N}[X; K^{(M,N)}], \tag{7.79}
\]

with

\[
\text{Sym}_{M,N} \bar{w}'_{M,N}[X; K^{(M,N)}] = \frac{1}{M!N!} \sum_{\pi \in S_M} \sum_{\bar{\pi} \in S_N} w_{M,N}[X; K_{\pi(1)}, \ldots, K_{\pi(M)}; \bar{K}_{\bar{\pi}(1)}, \ldots, \bar{K}_{\bar{\pi}(N)}],
\]

and

\[
\bar{w}'_{M,N}[X; K^{(M,N)}] = \sum_{m_1+\cdots+m_L=M} \sum_{n_1+\cdots+n_L=N} \prod_{\ell=1}^L \binom{m_\ell + p_\ell}{p_\ell} \binom{n_\ell + q_\ell}{q_\ell} \langle \Omega_f, F_0[X + \bar{X}_0] \bar{W}_1[X + \bar{X}_1; K^{(m_1,n_1)}_{1,\ell}] \cdots F_L[X + \bar{X}_L; K^{(m_L,n_L)}_{L,\ell}] \rangle.
\]

Here we are using the definitions

\[
\bar{W}_1[X + \bar{X}_1; K^{(m,\ell, n)}_{1,\ell}] := \frac{W_{p_\ell,q_\ell}[X + \bar{X}_1; K^{(m,\ell, n)}_{1,\ell}]}{w_{p_\ell,q_\ell}[X + \bar{X}_1; K^{(m,\ell, n)}_{1,\ell}]}, \tag{7.81}
\]

\[
K^{(M,N)} = (K^{(m_1,n_1)}_{1,\ell}, \ldots, K^{(m_L,n_L)}_{L,\ell}), \quad K^{(m,\ell, n)}_{1,\ell} := (K^{(m,\ell)}, \bar{K}^{(n)}_{\ell}),
\]

and

\[
X_{\ell} := \sum K^{(n_1)}_{\ell} + \cdots + \sum K^{(n_{\ell-1})}_{\ell} + \sum K^{(n_{\ell+1})}_{\ell+1} + \cdots + \sum K^{(n_L)}_{L}, \quad \bar{X}_{\ell} := \sum K^{(n_1)}_{\ell} + \cdots + \sum K^{(n_{\ell-1})}_{\ell} + \sum K^{(n_{\ell+1})}_{\ell+1} + \cdots + \sum K^{(n_L)}_{L}, \tag{7.82}
\]

where \(\sum K^{(n_j)}_{j} \) is defined in \([6.12]\).

Next, we apply rescaling, and transform the spectral parameter, thus obtaining

\[
H[\bar{w}[\bar{r}]] = \rho_H[H[w[r]]] = S_{\rho}(F_{\chi_{\rho}}[H]](H[w[r]], \alpha[r]H_f)) \tag{7.83}
\]

(see \([7.17]\)) with \(r = E_{\rho}^{-1}[\bar{r}]. \)

The renormalized generalized Wick kernels \(\bar{w}[\bar{r}]\) have the following explicit form.
Lemma 7.9. Let \( \hat{r} \in I_{M,N} \), and \( r := E^{-1}_\rho [\hat{r}] \in U[w] \). Then, one obtains \( \hat{W} = (\hat{w}_{M,N})_{M+N \geq 1} \) with

\[
\hat{w}_{M,N}[\hat{r}; X] = \rho^{M+N-1} \frac{1}{a_{[r]}} \text{Sym}_{M,N} \sum_{L=1}^\infty (-1)^{L-1} \sum_{m_1+\ldots+m_L=M} \sum_{n_1+\ldots+n_L=N} \prod_{\ell=1}^L \left( \begin{array}{c} m_\ell + p_\ell \\ n_\ell + q_\ell \end{array} \right) (\hat{X}_\rho)^{m_1,n_1} \cdots (\hat{X}_\rho)^{m_L,n_L} \hat{W}_L[r; \rho(X + X_{L-1})] \hat{W}_L[r; \rho(X + X_L)] \cdots (\hat{X}_\rho)^{m_1,n_1} \hat{W}_L[r; \rho(X + X_0)] \cdots \hat{W}_L[r; \rho(X + X_0)] \hat{W}_L[r; \rho(X + X_0)]
\]

for \( M + N \geq 1 \), and

\[
\hat{w}_{0,0}[\hat{r}; X] = \frac{1}{a_{[E^{-1}_\rho[\hat{r}]]}} \left\{ \mathcal{R}_\rho [E[\cdot] \oplus w_{0,0} \oplus \Omega_1] \right\} (7.85)
\]

for \( M = N = 0 \).

The statements of Lemmata 7.9 and 7.8 are purely algebraic, and the proofs can be adopted straightforwardly from 3.4.

7.6. Spatial symmetries. We shall require that the effective Hamiltonians possess the spatial symmetries of the fiber Hamiltonian \( H(\vec{p}, \sigma) \) in Section 4.2.

Definition 7.10. Let the operators \( U_h \) and \( U_{\text{ref}, \vec{p}} \) be defined as in Section 4.2. We say that the effective Hamiltonian \( H = H[\vec{p}, \vec{p}] \in \mathcal{B}(\mathcal{H}_{\text{red}}) \) in (7.7) satisfies property \( \text{Sym}[\vec{p}] \) if

\[
U_h H[\vec{p}, R_h \vec{p}] U_h^* = H[\vec{p}, \vec{p}]
\]

for all \( h \in SU(2) \), and

\[
U_{\text{ref}, \vec{p}} H[\vec{p}, -\vec{p}] U_{\text{ref}, \vec{p}}^* = H[\vec{p}, \vec{p}].
\]

(Invariance under rotations and under reflections with respect to a plane orthogonal to \( \vec{p} \).)
7.7. Soft photon sum rules. The generalized Wick kernels $w_{M,N}$ are all mutually linked by a hierarchy of non-perturbative identities, referred to as the soft photon sum rules. For the scalar model, which neglects the spin of the electron, they were introduced in [4]. For the model including the spin of the electron, the generalized Wick kernels $w_{M,N}$ are Mat$(2 \times 2, \mathbb{C})$-valued (for $M + N \geq 1$; we recall that $w_{0,0}$ is scalar), but the formal expressions for the identities remain unchanged. For our construction, the quintessential property of the soft photon sum rules is the fact that they are preserved by the renormalization map, see Section 8.2.

**Definition 7.11.** Let $\vec{n} \in \mathbb{R}^3$, $|\vec{n}| = 1$, be an arbitrary unit vector, and let $\vec{\varepsilon}(\vec{n},\lambda)$ denote the photon polarization vector orthonormal to $\vec{n}$ labeled by the polarization index $\lambda$. We say that the sequence of generalized Wick kernels $w \in \mathcal{W}_{\geq 0}$ satisfies the soft photon sum rules $\text{SR}[\sigma]$ if the identity

$$\sqrt{\alpha} \vec{\varepsilon}(\vec{n},\lambda) \cdot \nabla_X w_{M,N}[X; K^{(M,N)}]$$

$$= \mu(\sigma)(M + 1) \lim_{x \to 0} w_{M+1,N}[X; K^{(M+1,N)}]_{K_{M+1}=(x\vec{n},\lambda)}$$

$$= \mu(\sigma)(N + 1) \lim_{x \to 0} w_{M,N+1}[X; K^{(M,N+1)}]_{K_{N+1}=(x\vec{n},\lambda)}$$

holds for all $M,N \geq 0$, and any choice of $\vec{n}$. The factor $\mu(\sigma)$ is given by $\mu(\sigma) = 1$ if $\sigma \leq 1$, and satisfies $\mu(\rho^{-1}\sigma) = \rho \mu(\sigma)$ if $\sigma > 1$, for $0 < \rho < 1$.

We remark that for $K \neq 1$ in (2.17), one would have $\mu(\rho^{-1}\sigma) = \rho K \mu(\sigma)$ instead.

The recursive application of (7.86), rooted at $M,N = 0$, and in the order indicated by

```
  ...  \uparrow ...
  \uparrow  \quad \uparrow \\
  w_{2,0}    w_{1,0}   ...
  \downarrow  \quad \downarrow \\
 w_{0,0}    w_{1,1}   ...
  \downarrow  \quad \downarrow \\
  w_{0,1}    w_{1,1}   ...
  \downarrow  \quad \downarrow \\
  w_{0,2}    w_{0,2}   ...
  \downarrow  \quad \downarrow \\
  ...  \uparrow ...
```

links all generalized Wick kernels to one another.

In QED, the soft photon sum rules can be interpreted as a generalization of the differential Ward-Takahashi identities. A more detailed discussion is given in [4].

7.8. Codimension-3 contractivity of $\mathcal{R}_\rho$ on a polydisc. Let

$$\mathcal{U}^{(\text{sym})}(\varepsilon, \delta, \eta, \lambda, \sigma) := \{ w \in \mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma) \mid w \text{ satisfies } \text{SR}[\sigma] \}$$

and the symmetries $\text{Sym}[\rho]$.
denote the subset of elements in the polydisc $\Omega(\varepsilon, \delta, \eta, \lambda, \sigma)$ (defined in Section 7.4), which are rotation and reflection symmetric according to Definition 7.10, and which satisfy the soft photon sum rules (7.86).

Our first main result states that the renormalization map is codimension-3 contractive on sufficiently small polydiscs of this type.

**Theorem 7.12.** The renormalization map $R_\rho$ is codimension-3 contractive on the polydisc $\Omega^{(\text{sym})}(\varepsilon, \delta, \eta, \lambda, \sigma)$:

Assume that $0 \leq |\vec{p}| < \frac{1}{4}$, and let $\rho$ and $\xi$ be given as in (7.105). Then, there exist constants $\varepsilon_0, \delta_0$ (small and independent of $\sigma$) such that for all $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq \delta \leq \delta_0 + 2\varepsilon_0$,

$$R_\rho : \Omega^{(\text{sym})}(\varepsilon, \delta, \eta, \lambda, \sigma) \to \Omega^{(\text{sym})}(\hat{\varepsilon}, \hat{\delta}, \hat{\eta}, \hat{\lambda}, \hat{\sigma}),$$

(7.89)

where

$$\begin{align*}
\hat{\varepsilon} &\leq \frac{\varepsilon}{4} + \frac{\eta}{4} \\
\hat{\delta} &\leq \delta + \frac{\eta}{2} \\
\hat{\eta} &= 10 C_{\Theta}^2 \sqrt{\alpha} \xi^{-1} (1 + |\vec{p}| + \hat{\delta}) + \frac{\eta}{2} \\
\hat{\lambda} &= \rho \lambda \\
\hat{\sigma} &= \rho^{-1} \sigma
\end{align*}$$

and

$$\begin{align*}
\hat{\varepsilon} &\leq \frac{\varepsilon}{4} + \frac{\eta}{4} \\
\hat{\delta} &\leq \delta + \frac{\eta}{2} \\
\hat{\eta} &= \frac{\eta}{2} \\
\hat{\lambda} &= \rho \lambda \\
\hat{\sigma} &= \rho^{-1} \sigma
\end{align*}$$

(7.90) if $\sigma \leq 1$

(7.91) if $\sigma > 1$.

The constant $C_{\Theta}$ is the same as in Lemma 7.7.

The parameter $\varepsilon$ measures the projection of the polydisc along the codimension 3 subspace of irrelevant interactions (which are contracted by a factor $\leq \frac{1}{2}$ under application of $R_\rho$), while $\delta$ and $\eta$ measure its projection to a dimension 3 center manifold of marginal perturbations. With every application of $R_\rho$, the infrared cutoff parametrized by $\sigma$ is scaled by a factor $\rho^{-1}$. The interaction kernels $w_1$ behave like strictly marginal operators, i.e. their size remains constant under repeated applications of $R_\rho$, as long as $\sigma < 1$, see Section 7.1.2. The main new techniques in this paper address this regime, i.e., $\sigma \leq 1$. When $\sigma > 1$, the kernels $w_1$ behave like irrelevant operators, i.e. their size converges to zero at an exponential rate under repeated applications of $R_\rho$. In this case, they can then be completely controlled with the results of [1].

**7.9. Strong induction argument.** The upper bounds provided by (7.90) are clearly insufficient to control the growth of $\delta$ and $\eta$ under repeated applications of the renormalization map. The next main step in the analysis is to prove that nevertheless, the size of $\delta$ and $\eta$ does not increase under any number of applications of $R_\rho$. To achieve this result, we let $\delta_n, \eta_n$ denote the constants in the above bounds after $n$ iterations of $R_\rho$, and invoke the strong induction principle. This means that for every step $n \to n + 1$, we study the entire orbit $w^{(k)} \in \Omega^{(\text{sym})}(\varepsilon_k, \delta_k, \eta_k, \lambda_k, \sigma_k)$,
for $0 \leq k \leq n$, with initial condition $w^{(0)}(0) \in \Sigma^{(\text{sym})}(\epsilon_0, \delta_0, \eta_0, \lambda_0, \sigma_0)$ (provided by the "first Feshbach decimation step", see Section 9.1 and [4], where $\sigma_0$ denotes the initial infrared cutoff in the fiber Hamiltonian $H(\vec{p}, \sigma_0)$).

We make the following key observations:

- By (7.90), uniform boundedness of $\delta_n$ in $n$ automatically implies uniform boundedness of $\eta_n$.
- After $n$ applications of the renormalization map, one arrives at $\sigma_n = \rho^{-n}\sigma_0$. Thus, $\sigma_n \leq 1$ if $n \leq N(\sigma_0)$, and $\sigma_n > 1$ if $n > N(\sigma_0)$ for
  \[ N(\sigma_0) = \left\lceil \frac{\log \frac{1}{\sigma_0}}{\log \frac{1}{\rho}} \right\rceil. \]  
  \[ (7.92) \]
  Hence, (7.90) and (7.91) imply, under the condition that $\delta_n$ is uniformly bounded in $n$, that the interaction $w_1$ undergoes a transition from strictly marginal behavior to irrelevant behavior at $n = N(\sigma_0)$. This means that the upper bound in $\|w_1\|_{\xi} < \eta_n$ is essentially independent of $n$ in the regime $n \leq N(\sigma_0)$, but in the regime $n > N(\sigma_0)$, $\eta_n$ decreases by a factor at least $\frac{1}{2}$ under every application of $R_{\rho}$.
- In the regime $n > N(\sigma_0)$, it can be easily inferred from the estimates (7.91) in Theorem 7.8 that $\delta_n < \delta^{N(\sigma_0)} + 2\eta_{N(\sigma_0)}$ uniformly in $n$, and that $\epsilon_n$, $\eta_n < \frac{2(n-N(\sigma_0)) + \eta_{N(\sigma_0)}}{\sigma_0}$ converge to zero at a $\sigma_0$-independent exponential rate as $n \to \infty$. For these large values of $n$, it is not necessary to invoke Theorem 7.13.

The key goal of this part of the analysis is to prove that $\delta_n$ is uniformly bounded in $n$ and $\sigma_0$. The main result can be stated as follows.

**Theorem 7.13.** Let $\sigma_0 \ll 1$ (the infrared cutoff in the fiber Hamiltonian $H(\vec{p}, \sigma_0)$) be arbitrary but fixed. Invoking Theorem 7.12, we assume that $R_{\rho}$ is codimension 3 contractive on the polydisc $\Sigma^{(\text{sym})}(\epsilon_0, \delta_0 + 2\eta_0, \eta_0, \lambda_0, \sigma_0)$ for $\epsilon_0 < \eta_0 < c\sqrt{\alpha} \ll \rho^3$ sufficiently small, $\delta_0 < \alpha c$, and $\lambda_0 = \frac{1}{2}$.

Let $w^{(0)} \in \Sigma^{(\text{sym})}(\epsilon_0, \delta_0, \eta_0, \lambda_0, \sigma_0)$. By sInd[$n$], we denote the strong induction assumption that for $0 \leq k \leq n$, one has

\[ w^{(k)} \in \Sigma^{(\text{sym})}(\epsilon_k, \delta_k, \eta_k, \lambda_k, \sigma_k), \]  

where

\[ w^{(k)} = R_{\rho}[w^{(k-1)}] \quad \text{for} \quad 1 \leq k \leq n, \]  

and

\[
\begin{align*}
\epsilon_k & \leq \eta_k < c\sqrt{\alpha} \\
\delta_k & \leq C_0\alpha \\
\eta_k & \leq 20 C_0^2 \sqrt{\alpha} \xi^{-1}(1 + |\vec{p}| + C_0\alpha) \\
\lambda_k & = \rho^{-k}\lambda_0, \quad \lambda_0 = \frac{1}{2} \\
\sigma_k & = \rho^{-k}\sigma_0.
\end{align*}
\]  

(7.95)
Then, for \( \alpha < \alpha_0 \) with \( \alpha_0 \) sufficiently small (independent of \( \sigma_0 \)) and any \( n \geq 0 \), \( s_{\text{Ind}}[n] \) implies \( s_{\text{Ind}}[n+1] \). The constant \( C_0 \) is independent of \( n, \alpha, \) and \( \sigma_0 \), and is determined in Proposition 9.6.

From Theorem 7.13, we obtain the desired uniform bounds on \( \delta_n \) with respect to \( n \), and also on \( \varepsilon_n, \eta_n \).

Theorem 7.13 is the key tool that allows us to establish pure marginality of the interaction. For its proof, we use (9.26), which is a version of the identity of Lemma 9.1 in [4]; it allows to "collapse" intermediate scales between effective Hamiltonians on non-successive scales.

The marginal operators in the model considered here are given by \( H_f \) and \( \beta[r; \rho \rho_0] P^0_f \) in \( T \) (see (7.25) and (7.26)), and \( w_1 = (w_{10}, w_{01}) \) (counted as only one marginal direction because \( w_{10} = w_{01}^* \)). The key application of the soft photon sum rules is given in the proof of Theorem 7.12; they are used to relate \( \beta[r; \rho \rho_0] \) to \( w_1 \) (by gauge invariance), whereby the number of independent marginal operators is reduced from three to two.

By definition of the renormalization map, the coefficient of the operator \( H_f \) in \( T \) has the constant value 1. It thus remains to prove that the size of \( \beta[r; \rho \rho_0] \) is independent of \( n \). A main difficulty here is that the marginal operators \( H_f \) and \( \beta[r; \rho \rho_0] P^0_f \) in \( T \) are not related via gauge invariance, and it is at this point where the identity (9.26) mentioned above enters.

8. PROOF OF THEOREM 7.12

In this section, we prove the codimension-3 contractivity of \( R_\rho \) asserted in Theorem 7.12. For details omitted here, we refer to the proof of Theorem 6.6 in [4].

8.1. Wick ordering. We adopt the following notation from [4]. For fixed \( L \in \mathbb{N} \), let

\[
m, p, n, q := (m_1, p_1, n_1, q_1, \ldots, m_L, p_L, n_L, q_L) \in \mathbb{N}^4 L_0^L \quad (8.1)
\]

and

\[
M := |m| = m_1 + \cdots + m_L, \quad N := |n| = n_1 + \cdots + n_L \quad (8.2)
\]

and we recall the definitions (7.81) and (7.82). We let

\[
\tilde{V}_{m,p,n,q}^{(L)}[w | X; K^{(M,N)}] := \langle \Omega_f, \prod_{\ell = 1}^L \{ \tilde{W}_\ell[r; \rho(X + \tilde{X}_0); \rho K^{(m_\ell, n_\ell)}] F_\ell[X] \} \Omega_f \rangle, \quad (8.3)
\]

and

\[
V_{m,p,n,q}^{(L)}[w | X; K^{(M,N)}] := F_0[X] \tilde{V}_{m,p,n,q}^{(L)}[w | X; K^{(M,N)}] \quad (8.4)
\]

where

\[
F_0[X] := \Upsilon_\rho[r; \rho(X + \tilde{X}_0)] \, , \quad F_L[X] := \Upsilon_\rho[r; \rho(X + \tilde{X}_L)] \quad (8.5)
\]
and

\[ F_\ell [X] := \frac{(\chi_0^2 \chi_0^2) [H_f + \rho(X_0 + \tilde{X}, 0)]}{E[r] + \alpha[r] H_f + \rho(X_0 + \tilde{X}, 0) + \chi_0^2 T[r; P + \rho(\tilde{X} + \tilde{X})]} \]

(8.6)

for \( \ell = 1, \ldots, L - 1 \), with \( T[r; \tilde{X}] = X_0 + \chi_0^2 [X_0] \tilde{T}[r; \tilde{X}] \).

Then, for \( \tilde{r} \in I_{\tilde{r}} \) and \( r := E_{\tilde{r}}^{-1} \tilde{r} \),

\[ \tilde{w}_{M,N}[\tilde{r}; \tilde{X}; K^{M,N}] = \frac{1}{\alpha[r; \tilde{r}]} \tilde{w}_{M,N}[\tilde{r}; \tilde{X}; K^{M,N}], \]

(8.7)

where

\[ \tilde{w}_{M,N}[\tilde{r}; \tilde{X}; K^{M,N}] = \rho^{M+N-1} \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{m_1 + \cdots + m_L = M \atop n_1 + \cdots + n_L = N} \left[ \prod_{l=1}^{L} \left( \begin{array}{c} m_l + p_l \\ p_l \end{array} \right) \left( \begin{array}{c} n_l + q_l \\ q_l \end{array} \right) \right] V^{(L)}_{m_1, p_1, n_1, q_1} \tilde{w}^{\tilde{r}}_{M,N}[\tilde{r}; \tilde{X}; K^{M,N}], \]

(8.8)

where the factors \( \rho^{M+N-1} \) are due to the rescaling transformation, see (7.12).

**Lemma 8.1.** For \( L \geq 1 \) fixed, and \( m, p, n, q \in \mathbb{N}_0^L \) with \( |m| = M \) and \( |n| = N \), one has \( V^{(L)}_{m, p, n, q} \in \mathbb{M}_{M,N}^2 \). Furthermore,

\[ \| F_0 \|_\mathcal{I}, \| F_L \|_\mathcal{I} < C_\Theta \]

(8.9)

and

\[ \| F_\ell \|_\mathcal{I} < \frac{C_\Theta}{\rho}, \]

(8.10)

where the constant \( C_\Theta \) is the same as in Lemma 7.7 (it depends only on the choice of the smooth cutoff function \( \Theta \) in (6.3)). Moreover,

\[ \rho^{M+N-1} \| \partial_{\tilde{r}}^a \partial_{\tilde{r}}^a V^{(L)}_{m_1, p_1, n_1, q_1} \|_{M,N}, \rho^{M+N-1} \| \partial_{\tilde{r}}^{a} V^{(L)}_{m_1, p_1, n_1, q_1} \|_{M,N} \]

(8.11)

\[ \leq (L + 1)^2 C_\Theta^{L+1} \rho^{M+N+1+a-L} \prod_{l=1}^{L} \frac{\| w_{m_l + p_l, n_l + q_l} \|_{m_l + p_l, n_l + q_l}}{p_l^{p_l/2} q_l^{q_l/2}}, \]

for \( a = 0, 1 \) and any \( \tilde{k} \in k^{(M,N)} \). Furthermore,

\[ \rho^{M+N-1} \| \partial_{\tilde{r}}^a V^{(L)}_{m_1, p_1, n_1, q_1} \|_{M,N} \]

(8.12)

\[ \leq (L + 1)^2 C_\Theta^{L+2} \rho^{M+N+|\tilde{a}|} \prod_{l=1}^{L} \frac{\| w_{m_l + p_l, n_l + q_l} \|_{m_l + p_l, n_l + q_l}}{p_l^{p_l/2} q_l^{q_l/2}}, \]

for \( 1 \leq |\tilde{a}| \leq 2 \). For \( |\tilde{a}| = 1 \),

\[ \rho^{M+N-1} \| \partial_{\tilde{r}}^a V^{(L)}_{m_1, p_1, n_1, q_1} \|_{M,N} \]

(8.13)

\[ \leq (L + 1)^2 C_\Theta^{L+2} \rho^{M+N+|\tilde{a}|} \prod_{l=1}^{L} \frac{\| w_{m_l + p_l, n_l + q_l} \|_{m_l + p_l, n_l + q_l}}{p_l^{p_l/2} q_l^{q_l/2}}, \]
Consequently,

\[ \rho^{M+N-1} \| V^{(L)}_{m,p,n,q} \|_{M,N}^2 \leq (L + 1)^2 C_\Theta^{L+2} \rho^{M+N-L} \prod_{l=1}^{L} \frac{\| w_{m_l+p_l,n_l+q_l}[r] \|_2^2}{p_l^{p_l/2} q_l^{q_l/2}}, \]  

(8.14)

using the convention \( p^0 = 1 \) for \( p = 0 \).

**Proof.** We only demonstrate the argument for the term involving a derivative in \( \hat{r} \) because it has no counterpart in [4] (where derivatives in the spectral parameter are controlled by analyticity). For the other cases, we refer to [4].

We have for \( r = E_r^{-1}[\hat{r}] \)

\[
\partial_r V^{(L)}_{m,p,n,q}[w|X;K^{(M,N)}] = \sum_{j=0}^{L} \left\langle \Omega_f \left[ \prod_{\ell=1}^{j-1} F_{\ell-1}[X] \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] \right] \right. \\
\left. \left( \partial_r F_j[X] \right) \left[ \prod_{\ell=j+1}^{L} \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] F_\ell[X] \right] \Omega_f \right\rangle \\
+ \sum_{j=1}^{L} \left\langle \Omega_f \left[ F_0[X] \left[ \prod_{\ell=1}^{j-1} \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] F_\ell[X] \right] \right. \\
\left. \left( \partial_r \tilde{W}_j[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] \right) \right. \\
\left. \left[ \prod_{\ell=j+1}^{L} F_\ell[X] \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] F_\ell[X] \right] \Omega_f \right\rangle.
\]  

(8.15)

(8.16)

Using (8.15) and (8.16) to bound \( \| F_\ell \|_\text{op} \) and \( \| \partial_r F_\ell \|_\text{op} \),

\[
\| (8.15) \|_{M,N} \leq \rho(1 + cn) \sum_{j=0}^{L} \| \partial_r F_j \|_\text{op} \prod_{\ell \neq j}^{L} \| F_\ell[X] \|_\text{op} \\
\prod_{\ell=1}^{L} \| \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] \|_\text{op} \leq (L + 1)^{L+1} \rho^{-L+2} \prod_{\ell=1}^{L} \| \tilde{W}_\ell[r; \rho(X + X_\ell); \rho K^{(m_\ell,n_\ell)}_\ell] \|_\text{op}.
\]  

(8.17)
and

\[
\| (8.10) \|_{M,N} \leq \rho \left[ \prod_{\ell=0}^{L} \| P_{\ell} |X\|_{op} \right] \left\{ \sum_{j=1}^{L} \| \partial_{r} \tilde{W}_{j} [r; \rho(X + X_{j}); \rho K_{j}^{(m_{j}, n_{j})}] \|_{op} \right. \\
\prod_{\ell=1, \ell \neq j} L \| \tilde{W}_{\ell} [r; \rho(X + X_{j}); \rho K_{\ell}^{(m_{\ell}, n_{\ell})}] \|_{op} \bigg\} \\
\leq L C_{\Theta}^{L+1} \rho^{-L+2} \left\{ \sum_{j=1}^{L} \| W_{m_{j}, n_{j}}^{p_{j}, q_{j}} [\partial_{r} w[r]; \rho(X + X_{j}); \rho K_{j}^{(m_{j}, n_{j})}] \|_{op} \right. \\
\prod_{\ell=1, \ell \neq j} L \| \tilde{W}_{\ell} [r; \rho(X + X_{j}); \rho K_{\ell}^{(m_{\ell}, n_{\ell})}] \|_{op} \bigg\}. \tag{8.18}
\]

Here, we used that for \( r = E_{\rho}^{-1}[\bar{r}] \),

\[ | \partial_{r} f[r] | \leq \rho (1 + c\eta) \| (\partial_{r} f)[r] \| , \tag{8.19}\]

see (8.33) below. The factor \((1 + c\eta) < 2\) has been absorbed into the definition of the constant \( C_{\Theta} \).

The remaining cases can be adapted straightforwardly from the proof of Lemma 7.1 in [4].

8.2. Preservation of the soft photon sum rules.

**Lemma 8.2.** The renormalization map preserves the soft photon sum rules,

\[
\mathcal{R}_{\rho} : \mathcal{SR} [\sigma] \rightarrow \mathcal{SR} [\rho^{-1} \sigma] \tag{8.20}
\]

where \( \mathcal{SR} [\sigma] \) is defined in (7.86). That is, given \( w \in \mathcal{U}^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma) \), which satisfies \( \mathcal{SR} [\sigma] \), it follows that \( \hat{w} = \mathcal{R}_{\rho} [w] \) satisfies \( \mathcal{SR} [\rho^{-1} \sigma] \).

**Proof.** It is proved in [4] for the scalar model (zero electron spin) that the renormalization map \( \mathcal{R}_{\rho} \) preserves the soft photon sum rules. The argument is purely algebraic, and it applies equally to the spin \( \frac{1}{2} \) model. The fact that the generalized Wick kernels are here complex \( 2 \times 2 \) matrices, and not scalars, does not affect the proof.

8.3. Preservation of the symmetries. In this section, we prove that the symmetries of the fiber Hamiltonian \( H(\vec{p}, \sigma) \) described in Section 4.2 are inherited by the effective Hamiltonians, in the sense of Definition 7.10, and preserved by the renormalization map.

**Lemma 8.3.** Assume that \( w \in \mathcal{U}^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma) \), and that

\[
U H[\hat{w}[r; R\vec{p}]] U^{*} = H[\hat{\hat{w}}[r; \vec{p}]] , \tag{8.21}\]

where \( U \) stands either for \( U_{h} \) or for \( U_{ref} \), and \( R \) denotes either \( R_{h} \) or \( -1 \) in the notation of Definition 7.10. Then,

\[
U H[\hat{\hat{w}}[r; R\vec{p}]] U^{*} = H[\hat{\hat{w}}[\hat{\bar{r}}; \vec{p}]] \tag{8.22}\]

for \( \hat{w} = \mathcal{R}_{\rho} [w] \), with \( \hat{\bar{r}} = E_{\rho} [\bar{r}] \).
Proof. Let for brevity
\[ \omega[r; \bar{p}] := H[w[r; R\bar{p}]] - \alpha[r; \bar{p}]H_f, \]  
(8.23)
where \( \alpha[r; \bar{p}] \) is defined in \( (7.7) \), and
\[ \bar{R}[r; \bar{p}] := (\alpha[r; \bar{p}]H_f + \bar{\chi}_\rho[H_f]\omega[r; \bar{p}]\bar{\chi}_\rho[H_f])^{-1} \]  
(8.24)
on \( \text{Ran}[\bar{\chi}_\rho[H_f]] \). From
\[ UF[H_f]U^* = f[H_f], \]  
(8.25)
for any Borel function \( f \), we find
\begin{align*}
UF_{\chi_\rho}[H_f](H[w[r; R\bar{p}]]) \alpha[r; R\bar{p}]HF_f)^U^* &= U\left(\alpha[r; R\bar{p}]HF_f + \chi_\rho[H_f]\omega[r; R\bar{p}]\chi_\rho[H_f]\right)U^*
- \chi_\rho[H_f]\omega[r; R\bar{p}]\bar{\chi}_\rho[H_f]\bar{R}[R\bar{p}]\bar{\chi}_\rho[H_f]\omega[r; R\bar{p}]\chi_\rho[H_f])U^*
= \alpha[r; R\bar{p}]HF_f + \chi_\rho[H_f]U\omega[r; R\bar{p}]U^*\chi_\rho[H_f]
- \chi_\rho[H_f]U\omega[r; R\bar{p}U^*\bar{\chi}_\rho[H_f]U\omega[r; R\bar{p}]U^*\bar{\chi}_\rho[H_f]
= F_{\chi_\rho}[H_f]/(UH[w[r; R\bar{p}]U^*, \alpha[r; R\bar{p}]HF_f). \]  
(8.26)
Therefore,
\[ UHR_{\rho}[H[w[r; R\bar{p}]]]U^* = U H \rho \circ S_{\rho}[F_{\chi_\rho}[H_f](H[w[r; R\bar{p}]]), \alpha[r; R\bar{p}]HF_f)]U^*
= E_{\rho} \circ S_{\rho}[U F_{\chi_\rho}[H_f](H[w[r; R\bar{p}]], \alpha[r; R\bar{p}]HF_f)]U^*]
= E_{\rho} \circ S_{\rho}[F_{\chi_\rho}[H_f](UH[w[r; R\bar{p}]U^*, \alpha[r; R\bar{p}]HF_f)]
= R_{\rho}[UH[w[r; R\bar{p}]]U^*], \]  
(8.27)
which implies that given \( (8.24) \),
\[ UHR_{\rho}[H[w[r; R\bar{p}]]]U^* = R_{\rho}[H[w[r; \bar{p}]]] \]  
(8.28)
or likewise,
\[ UH[R_{\rho}[w[\hat{r}; R\bar{p}]]]U^* = H[R_{\rho}[\hat{r}; R\bar{p}]] \]  
(8.29)
with \( \hat{r} = E_{\rho}[r] \).

This implies that \( \mathcal{R}_{\rho} \) preserves rotation and reflection symmetry.

\[ \]  
8.4. Codimension three contractivity. We now come to the core of the proof of Theorem \( (7.12) \) and verify that
\[ \mathcal{R}_{\rho} : \Omega^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma) \rightarrow \Omega^{(sym)}(\hat{\varepsilon}, \hat{\delta}, \hat{\eta}, \hat{\lambda}, \hat{\sigma}) \]  
with \( (7.90) \) and \( (7.91) \). This implies that \( \mathcal{R}_{\rho} \) is contractive on a codimension three subspace of \( \Omega^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma) \) at a contraction rate which is independent of \( \sigma \).

Our proof is organized as follows.

First, we show that from application of \( \mathcal{R}_{\rho} \), the kernels \( \bar{w}_{>2} \) are contracted by a factor \( \leq \frac{1}{2} \) by pure scaling, for \( \rho \) sufficiently small. This implies that they belong to a codimension-3 subspace of \( \Omega^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma) \) of irrelevant perturbations.

To control the marginal kernels \( \bar{w}_{4} = (w_{0,1}, w_{1,0}) \), we invoke the soft photon sum rules, and relate \( w_{4} \) to the coefficient of the marginal operator \( P_{f} \parallel \) in the
non-interacting Hamiltonian \( T \). Hereby, we can reduce the number of independent marginal operators by one, and it remains to control the renormalization of marginal operators in \( T \) under \( \mathcal{R}_\rho \). This is the main topic of Section 9 below.

In [4], a similar argument has been used in the special case \(|\vec{p}| = 0\), to determine the renormalized electron mass for \(|\vec{p}| = 0\) in the limit \( \sigma \downarrow 0 \).

We now give the detailed proof of Theorem 7.12.

8.4.1. Bounds on \( \hat{E} \) and \( \hat{T} \). We begin with \( M + N = 0 \), and first discuss the renormalization of \( E[r] \) (see (8.30) in the definition of the polydisc \( \Omega(\varepsilon, \delta, \eta, \lambda, \sigma) \)). Let

\[
E = E^{-1}[\vec{r}] \in \mathcal{I}_{\frac{1}{200}} \quad \text{for} \quad \hat{r} \in \mathcal{I}_{\frac{1}{100}} , \tag{8.30}
\]

where \( E_\rho[r] = \frac{1}{\alpha(r)} E[r] \) (see Lemma 7.20). Let us first prove

\[
\sup_{\hat{r} \in \mathcal{I}_{\frac{1}{100}}} \{ |\partial_{\vec{r}^2} \alpha| E^{-1}[\vec{r}; \vec{p}]|, |\partial_{\vec{r}^2} \alpha| E^{-1}[\vec{r}; \vec{p}]| \} \leq \frac{c \eta^2}{\rho^3} \tag{8.31}
\]

(note that in contrast to Proposition 7.6, the argument is here \( E^{-1}[\vec{r}] \)) and

\[
\sup_{\hat{r} \in \mathcal{I}_{\frac{1}{100}}} |\partial_{\vec{r}^2} E^{-1}[\vec{r}]| < c \eta , \tag{8.32}
\]

\[
|\partial_{\vec{r}^2} E^{-1}[\vec{r}] - \rho| < c \rho \eta . \tag{8.33}
\]

To this end, we recall that

\[
\sup_{r \in \mathcal{I}_{\frac{1}{100}}} \{ |\alpha[r; \vec{p}] - 1|, |\partial_{\vec{r}^2} \alpha| |r; \vec{p}|, |\partial_{\vec{r}^2} \alpha| |r; \vec{p}| \} \leq \frac{c \eta^2}{\rho^3} \tag{8.34}
\]

from Proposition 7.6 and

\[
\sup_{|r| < \frac{1}{200}} \{ |E[r] - r|, |\partial_r E[r] - 1|, |\partial_{\vec{r}^2} E[r]| \} < \eta \tag{8.35}
\]

from the definition of the polydisc in Section 7.4.

The estimate (8.31) follows from

\[
\sup_{|\vec{r}| < \frac{1}{200}} |\partial_{\vec{r}^2} \alpha| E^{-1}[\vec{r}; \vec{p}]| \leq \sup_{|r| < \frac{1}{200}} |\partial_{\vec{r}^2} \alpha| |r; \vec{p}| + \left( \sup_{|r| < \frac{1}{200}} |\partial_r \alpha| |r; \vec{p}| \right) \sup_{|\vec{r}| < \frac{1}{100}} |\partial_{\vec{r}^2} E^{-1}[\vec{r}]| \tag{8.36}
\]

and

\[
\sup_{|\vec{r}| < \frac{1}{100}} |\partial_{\vec{r}^2} \alpha| E^{-1}[\vec{r}; \vec{p}]| \leq \left( \sup_{|r| < \frac{1}{200}} |\partial_r \alpha| |r; \vec{p}| \right) \sup_{|\vec{r}| < \frac{1}{100}} |\partial_{\vec{r}^2} E^{-1}[\vec{r}]| , \tag{8.37}
\]

and from using (8.34), (8.31).

To prove (8.32), we observe that \( \partial_{\vec{r}^2} E_\rho[E^{-1}[\vec{r}]] = \partial_{\vec{r}^2} \hat{r} = 0 \) implies that

\[
\sup_{|\vec{r}| < \frac{1}{100}} |\partial_{\vec{r}^2} E^{-1}[\vec{r}]| \leq \sup_{|r| < \frac{1}{200}} \frac{1}{|\partial_r E_\rho[r]|} |(\partial_{\vec{r}^2} E_\rho)[r]| . \tag{8.38}
\]
We have
\[\sup_{r \in I} |(\partial_r E_\rho)[r]| \leq \sup_{r \in I} \frac{1}{|\alpha| r^\rho} \left( |\partial_r \alpha[r]| + |\partial_r E[r]| \right) \leq \frac{2}{\rho} \left( \frac{\eta^2 |r|}{\rho^3} + \eta \right) \leq \frac{3\eta}{\rho}, \tag{8.39}\]

for \(\eta \ll \rho\). On the other hand,
\[|((\partial_r E_\rho)[r]| \geq \frac{1}{|\alpha| r^\rho} \left( |\partial_r E[r]| - \frac{|\partial_r \alpha[r]|}{|\alpha| r^\rho} |E[r]| \right) \geq \frac{1 - c' \eta}{(1 + c \eta^2) \rho}. \tag{8.40}\]

One thus obtains (8.32).

The estimate (8.33) follows immediately from
\[|\partial_r E_\rho^{-1}[\hat{r}]| = \frac{1}{|(\partial_r E_\rho) E_\rho^{-1}[\hat{r}]|}, \tag{8.41}\]

together with
\[|\partial_r E_\rho)[r]| \leq \frac{1}{|\alpha| r^\rho} \left( |\partial_r E[r]| + \frac{|\partial_r \alpha[r]|}{|\alpha| r^\rho} |E[r]| \right) \leq \frac{1 + c' \eta}{(1 - c \eta^2) \rho}, \tag{8.42}\]

and (8.40).

Let
\[\tilde{\mathcal{E}}[\hat{r}; \vec{p}] := \tilde{w}_{0,0}[r; \hat{\Omega}; \vec{p}] = \hat{r} + \alpha[r]^{-1} \rho^{-1} \Delta \tilde{w}_{0,0}[E_{\rho}^{-1}[\hat{r}]; \hat{\Omega}; \vec{p}], \tag{8.43}\]

where
\[\Delta \tilde{w}_{0,0}[\hat{r}; \hat{X}; \vec{p}] := \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{p_1, q_1, \ldots, p_L, q_L, p_{\ell}, q_{\ell} \geq 1} \frac{\tilde{V}_{\hat{r}, 0, p, q}^{(L)}[r; \hat{X}; \vec{p}]}{p_{\ell}^{p_{\ell}/2} q_{\ell}^{q_{\ell}/2}}, \tag{8.44}\]

see (8.3).

By the arguments presented in Section 8.3, the operator \(\Delta \tilde{w}_{0,0}[\hat{r}; \hat{X}; \vec{p}]\) is rotation- and reflection symmetric. Thus, by Lemma 4.1 it is a scalar (its vector part is identically zero).

We note that
\[\rho^{-1} \|\tilde{V}_{0,0,0,p,q}^{(L)}\|_{0,0}^2 \leq \frac{10}{(L + 1)^2} C_{\hat{\Omega}} \rho^{-L} \prod_{\ell=1}^{\infty} \frac{\|w_{p_{\ell}, q_{\ell}}[r]\|_{p_{\ell}/2}^{p_{\ell}/2} q_{\ell}/2}{p_{\ell}^{p_{\ell}/2} q_{\ell}^{q_{\ell}/2}}, \tag{8.45}\]
which corresponds to the bounds in Lemma 8.1 for $V_{0,p,0,q}^{(L)}$, but with one power of $C_\Theta$ less here because there is a factor $F_0$ less. Hence, we find
\[
\rho^{-1} \| \Delta \tilde{w}_{0,0} \|_\Pi \leq \rho^{-1} \sup_{\tilde{r} \in \mathbb{T}^{L+1}} \sum_{L \geq 2} p_{L+q_{L}} \left[ \sup_{X} \partial_T \tilde{V}_{0,p,0,q}^{(L)}[X] \right] + \sum_{|\alpha|=1} \sup_{X} \partial_{|\alpha|} \tilde{V}_{0,p,0,q}^{(L)}[X] \right] 
+ \sum_{0 \leq |\alpha| \leq 2} \sup_{X} \partial_{\alpha} \tilde{V}_{0,p,0,q}^{(L)}[X] \right]
\leq C_\Theta \sum_{L=2}^{\infty} (L + 1)^2 \left( \frac{C_\Theta}{\rho} \right)^{L} \left( \sum_{p+q \geq 1} \sup_{r \in \mathbb{T}^{L+1}} \| w_{p,q} [r] \|_{p,q} \right)^{L}
\leq C_\Theta \sum_{L=2}^{\infty} (L + 1)^2 \left( \frac{C_\Theta}{\rho} \right)^{L} \left( \sum_{p+q \geq 1} \xi^{-p-q} \sup_{r \in \mathbb{T}^{L+1}} \| w_{p,q} [r] \|_{p,q} \right)^{L}
\leq C_\Theta \sum_{L=2}^{\infty} (L + 1)^2 \left( \frac{C_\Theta}{\rho} \right)^{L} \left( \| \tilde{w}_{0,0} \|_{\xi} \right)^{L}
\leq 12 C_\Theta \left( \frac{C_\Theta}{\rho} \| \tilde{w}_{0,0} \|_{\xi} \right)^{2} \leq \frac{C_\Theta^2}{\rho^2} (2\eta)^2 \leq \frac{\eta}{10},
\]
for
\[
\eta \ll \rho^3 \quad \xi \leq \frac{1}{4}
\]
(see also 8.103).

Thus, 8.43 yields
\[
\sup_{\tilde{r} \in \mathbb{T}^{L+1}} | \tilde{E}[\tilde{r}; \tilde{p}] - \tilde{r} | \leq \sup_{|r|,|\tilde{r}| \leq \frac{1}{2^{L+1}}} \| \alpha [r; \tilde{p}] \|^{-1} \| \Delta \tilde{w}_{0,0} [r; \mathbb{P} = \emptyset; \tilde{p}] | \leq (1 + c\eta^2) \| \Delta \tilde{w}_{0,0} \|_\Pi \leq \eta \quad (8.48)
\]
for $a = 0, 1$ and 8.47. Moreover,
\[
\sup_{\tilde{r} \in \mathbb{T}^{L+1}} | \partial_{|\alpha|} \tilde{E}[\tilde{r}; \tilde{p}] | \leq \sup_{\tilde{r} \in \mathbb{T}^{L+1}} (1 + c\eta^2) \| \partial_{|\alpha|} \Delta \tilde{w}_{0,0} [\tilde{r}; E_{\rho}^{-1}[\tilde{r}]; \mathbb{P} = \emptyset; \tilde{p}] | + \sup_{\tilde{r} \in \mathbb{T}^{L+1}} (1 + c\eta^2) \| \partial_{|\alpha|} \Delta \tilde{w}_{0,0} [\tilde{r}; E_{\rho}^{-1}[\tilde{r}]; \mathbb{P} = \emptyset; \tilde{p}] | \| \partial_{|\alpha|} E_{\rho}^{-1}[\tilde{r}] | \leq (1 + c\eta^2) \| \Delta \tilde{w}_{0,0} \|_\Pi (1 + c\eta) \leq \frac{\eta}{2}
\]
(8.49)

This completes the discussion of the renormalization of $E[r]$.

Next, we discuss $T[r; \mathbb{P}; \tilde{p}]$, and determine the renormalized expressions for the conditions 7.32 - 7.33 in the definition of $\mathcal{U}(\varepsilon, \delta, \eta, \lambda, \sigma)$. To this end, we again
let \( r = E^{-1}_\rho [\hat{r}] \), and consider
\[
\hat{w}_{0,0}[\hat{r}; \hat{X}; \hat{p}] = \alpha[r]^{-1} \left[ \alpha[r; \hat{p}] H_f + E[r] \chi^2_1[H_f] \right]
+ \chi^2_1[H_f] \left[ (1 - \alpha[r]) H_f + \beta[r; \hat{p}] P_{\parallel} + \rho \lambda \hat{P}_f^2 \right.
\left. + \rho^{-1} \Delta \tilde{w}_{0,0}[r; \hat{X}; \hat{p}] \right] + \Delta \tilde{w}_{0,0}[r; \hat{X}; \hat{p}] \right]
\]
\[
= \hat{E}[\hat{r}] \chi^2_1[H_f] + \hat{T}[\hat{r}; \hat{X}; \hat{p}] F_0[r; \hat{X}; \hat{p}] \tag{8.50}
\]
with
\[
\hat{T}[\hat{r}; \hat{X}; \hat{p}] = H_f + \chi^2_1[H_f] \left( \hat{\beta}[\hat{r}; \hat{p}] P_{\parallel} + \hat{\lambda} \hat{P}_f^2 \right.
\left. + \hat{\delta} T[\hat{r}; \hat{X}; \hat{p}] \right) F_0[r; \hat{X}; \hat{p}] \tag{8.51}
\]
The terms in (8.51) are determined by the Taylor expansion of \( \Delta \tilde{w}_{0,0} \) in \( \hat{X} \) up to a quadratic remainder term.

The operator \( \hat{T}[\hat{r}; \hat{X}; \hat{p}] \) is rotation- and reflection symmetric, and is therefore a scalar (see Lemma 4.1).

We note that there is no term proportional to \( H_f \) in the brackets in (8.51) because the defining condition for \( \alpha[r], 1 - \alpha[r] + \partial r \beta \Delta \tilde{w}_{0,0}[r; \hat{X}; \hat{p}] = 0 \),
\[
\tag{8.52}
\]
suppresses the creation of a term proportional to \( \chi^2_1[H_f] H_f \) by \( R_\rho \), see Proposition \ref{prop:supp}, Remark \ref{rem:supp}, and Remark \ref{rem:supp2}.

Furthermore,
\[
\hat{\beta}[\hat{r}; \hat{p}] = \alpha[r; \hat{p}]^{-1} \left( \beta[r; \hat{p}] + (\partial \chi_{\parallel} \Delta \tilde{w}_{0,0})[r; \hat{X}; \hat{p}] \right) \tag{8.53}
\]
and
\[
\hat{\lambda} = \rho \lambda. \tag{8.54}
\]
The operator
\[
\hat{\delta} T[\hat{r}; \hat{X}; \hat{p}] = \alpha[r]^{-1} \left[ \rho^{-1} \delta T[r; \rho \hat{X}; \hat{p}] + (1 - \alpha[r]) \rho \lambda \hat{P}_f^2 \right.
\left. + \rho^{-1} \Delta \tilde{w}_{0,0}[r; \hat{X}; \hat{p}] \right.
\left. - \rho^{-1} \sum_{0 \leq|\alpha| \leq 1} (\partial \hat{\rho} \Delta \tilde{w}_{0,0})[r; \hat{X}; \hat{p}] \right] \tag{8.55}
\]
is of order \( O(|\hat{X}|^2) \) as \( \hat{X} \to \hat{0} \), and contains the quadratic Taylor remainder term of \( \Delta \tilde{w}_{0,0} \).

We first recall that
\[
\sup_{\hat{r} \in I} |(\hat{\partial}_r \beta)[E^{-1}_\rho \hat{r}; \hat{p}]| \leq \eta \tag{8.56}
\]
from the definition of $\Omega^{(sym)}(\varepsilon, \delta, \eta, \lambda, \sigma)$. Therefore,
\[
\sup_{\vec{r} \in I} |\partial_{\vec{r}} \beta[E_p^{-1}[\vec{r}]; \vec{p}]| \leq \eta \sup_{\vec{r} \in I} |\partial_{\vec{r}} E_p^{-1}[\vec{r}]| \leq \rho \eta (1 + c\eta) \leq \frac{\eta}{9} \tag{8.57}
\]
from (8.33), and $\rho \leq \frac{1}{10}$ ($\rho$ is determined in (8.105) below). For $a = 0, 1$,
\[
|\partial_{\vec{p} \vec{r}}^a (\tilde{\beta}[\vec{r}; \vec{p}] - \beta[E_p^{-1}[\vec{r}]; \vec{p}])| = |\partial_{\vec{p} \vec{r}}^a ((\alpha[E_p^{-1}[\vec{r}]; \vec{p}]^{-1} - 1)\beta[E_p^{-1}[\vec{r}]; \vec{p}] + \partial_{\vec{p} \vec{r}}^a \Delta \bar{w}_{0,0}[E_p^{-1}[\vec{r}]; \vec{0}; \vec{p}]| < \frac{c\eta^2}{\rho^3} + \frac{\eta}{10} \tag{8.58}
\]
This follows straightforwardly using
\[
\sup_{\vec{r} \in I} |\partial_{\vec{p} \vec{r}}^a (\beta[\vec{r}; \vec{p}] + |\vec{p}|)| < \frac{\delta}{2}, \quad a = 0, 1, \tag{8.59}
\]
from the definition of the polydisc $\Pi(\varepsilon, \delta, \eta, \lambda, \sigma)$ (see Section 7.3), and
\[
\sup_{\vec{r} \in I} \left|\partial_{\vec{p} \vec{r}} (\beta[E_p^{-1}[\vec{r}]; \vec{p}] + |\vec{p}|)\right| = \sup_{\vec{r} \in I} \left[|\partial_{\vec{p} \vec{r}} \beta[E_p^{-1}[\vec{r}]; \vec{p}] + 1| + \left|\partial_{\vec{r}} \beta[E_p^{-1}[\vec{r}]; \vec{p}] \partial_{\vec{p} \vec{r}} E_p^{-1}[\vec{r}]\right|\right] < \frac{\delta}{2} + \frac{2\delta \eta}{\rho} \tag{8.60}
\]
with (8.39) and (8.56). Moreover,
\[
\sup_{\vec{r} \in I} \left|\partial_{\vec{p} \vec{r}}^a \partial_{\vec{r}} \Delta \bar{w}_{0,0}[E_p^{-1}[\vec{r}]; \vec{0}; \vec{p}]\right| < \frac{\eta}{10}
\]
\[
\sup_{\vec{r} \in I} \left|\partial_{\vec{r}} \Delta \bar{w}_{0,0}[E_p^{-1}[\vec{r}]; \vec{0}; \vec{p}]\right| < \frac{\eta}{10} \tag{8.61}
\]
from (8.40), and
\[
|\partial_{\vec{p} \vec{r}}^a (1 - \alpha[E_p^{-1}[\vec{r}])| < \frac{c\eta^2}{\rho^3}, \quad a = 0, 1, \tag{8.62}
\]
see (8.41). We thus find for $a = 0, 1$ that
\[
\sup_{\vec{r} \in I} \left|\partial_{\vec{r}} (\tilde{\beta}[\vec{r}; \vec{p}] + |\vec{p}|)\right| < \frac{\eta}{9} + \frac{\eta}{10} < \frac{\eta}{2}
\]
\[
\sup_{\vec{r} \in I} \left|\partial_{\vec{p} \vec{r}}^a (\tilde{\beta}[\vec{r}; \vec{p}] + |\vec{p}|)\right| < \frac{\delta}{2} + \frac{2\delta \eta}{\rho} + \frac{\eta}{10} + \frac{c\eta^2}{\rho^3} < \frac{\delta}{2} + \frac{\eta}{2} \tag{8.63}
\]
from (8.58) and (8.59), for $\delta, \eta \ll \rho^3$ sufficiently small.

Next, we discuss the renormalization of $\delta T$. Note that since
\[
(\partial_{\vec{r} \vec{r}} \delta T)[r; \vec{0}; \vec{p}] = 0, \tag{8.64}
\]
we have
\[ \sup_{|\chi| \leq X_0} |\frac{\partial}{\partial \bar{\mu}} \partial \bar{\nu} \hat{T}[E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]| \leq \sum_{|\bar{\mu}| = 2} |\frac{\partial}{\partial \bar{\mu}} \partial \bar{\nu} \hat{T}[E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]| \]
for \( 0 \leq |\nu| \leq 2 \) and \( b = 0, 1 \). Consequently,
\[ \| \hat{T} \|_\tau \leq 32 \left( \frac{|\partial_{\bar{\mu}} \alpha [E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]|}{|\alpha [E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]|} + 1 \right) \]
\[ \left[ \rho \| \hat{T} \|_\tau + (|\partial_{\bar{\mu}} \alpha [E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]| + |\alpha [E^{-1}_{\rho} | \bar{\nu}; \bar{\mu}]| - 1)\rho \lambda \| \hat{X}^2 \|_\tau \right] + \| \rho^{-1} \Delta \hat{w}_{0,0} \|_\tau \]
\[ \leq 32 \left( 1 + c\eta^2 \right) \rho \| \hat{T} \|_\tau + \frac{c\eta^2}{\rho^3} . \]
using (8.31). Since by assumption, \( \| \hat{T} \|_\tau < \delta \), we find
\[ \| \hat{T} \|_\tau < \frac{\delta}{2} + c\eta^2 \] (8.66)
for
\[ \rho \leq \frac{1}{100} , \] (8.67)
which is determined in (8.105) below. Therefore, \( \hat{\delta} \) is determined by (8.63).

To carry out the induction step for (7.34), we recall from (7.22) and (8.5) that
\[ Y_{\chi_1}^{(\bar{ \nu}_4 \bar{ \mu})} = \Gamma_\rho Y_{\chi_1[H_f]} \left( T_0^{(\bar{ \nu}_4 \bar{ \mu})} | r; \bar{ \nu}_4 ; H_f \right) \bigg|_{\text{Ran}(\chi_1[H_f])} \Gamma_\rho^* , \] (8.68)
where \( \Gamma_\rho \) is the unitary dilation operator, see (7.22) and (8.5).

We note that
\[ T_1 := T_0^{(\bar{ \nu}_4 \bar{ \mu})} | r; \bar{ \nu}_4 ; H_f \bigg|_{\text{Ran}(\chi_1[H_f])} = E[r] + |\bar{ \mu}| P_f^\| + \bar{ \lambda} P_f^2 \] (8.69)
and
\[ T_2 := \left( E[r] \chi_1[H_f] + T[r; \bar{ \nu}_4 ; \bar{ \mu}] \right) \bigg|_{\text{Ran}(\chi_1[H_f])} \]
\[ = E[r] + H_f + \beta[r] P_f^\| + \bar{ \lambda} P_f^2 + \delta T[r; \bar{ \nu}_4 ; \bar{ \mu}] \] (8.70)
using
\[ \chi_\rho[H_f] \chi_1[H_f] = \chi_\rho[H_f] \] and \( \chi_\rho[H_f] \bar{T}[r; \bar{ \nu}_4 ; \bar{ \mu}] = 0 , \] (8.71)
see (7.29). It is clear that
\[ \chi_\rho[H_f] F_0[| r; \bar{ \nu}_4 ; \bar{ \mu} | = 0 , \] (8.72)
by the definition (8.5). Moreover, using (5.21) and \( \tau_1 := H_f, \tau_2 := \alpha[r] H_f \),
\[ \| F_0 - Y_{\chi_1}^{(\bar{ \nu}_4 \bar{ \mu})} \|_\tau \leq \| Y_{\chi_1[H_f]}(T_1, H_f) - Y_{\chi_1[H_f]}(T_2, \alpha[r] H_f) \|_\tau \]
\[ \leq \| \chi_\rho[H_f] R_0(T_2, \tau_1) Y_{\chi_1[H_f]}(T_2, \tau_1) \|_\tau \| T_2 - T_1 \|_\tau \] (8.73)
\[ + \| \chi_\rho[H_f] T_1 R_0(T_1, \tau_1) \|_\tau \| \tau_2 - \tau_1 \|_\tau \]
where $\| \cdot \|_{\mathcal{T}, \rho}$ is defined as $\| \cdot \|_\mathcal{T}$ in (6.33), but with the supremum over $X_0 \in [0, 1]$ replaced by the supremum over $X_0 \in [0, \rho]$. We find

$$\| \bar{\chi}_\rho[H_f] \tilde{R}_0(T_2, \tau_1) Y_{\bar{\chi}_\rho[H_f]}(T_2, \tau_1) \|_{\mathcal{T}, \rho} \leq \frac{C_{\Theta}}{\rho}$$

(8.74)

and

$$\| \bar{\chi}_\rho[H_f] T_1 \tilde{R}_0(T_1, \tau_1) \|_{\mathcal{T}, \rho} \leq C\Theta .$$

(8.75)

Therefore

$$\| F_0 - \Gamma^{(\bar{\rho}, \bar{\lambda})}_\chi \|_{\mathcal{T}} \leq \frac{C_{\Theta}}{\rho} \left( \| (E[r] + X_0 + \beta[r]X\| + \tilde{\lambda}X^2 + \delta T[r; X; \bar{p}] \|_{\mathcal{T}} 
+ \| \alpha[r]X_0 - X_0 \|_{\mathcal{T}} \right)$$

$$\leq \frac{C_{\Theta}}{\rho} \left( (\| \alpha[r] - 1 \| X_0 \|_{\mathcal{T}} + \| (\beta[r] + |\bar{p}|)X\|_{\mathcal{T}} 
+ \| \delta T[r; \rho X; \bar{p}] \|_{\mathcal{T}} \right)$$

$$\leq \frac{C_{\Theta}}{\rho} \left( 4\delta + \frac{c\eta^2}{\rho^2} \right) \leq K_{\Theta} \hat{\delta}$$

(8.76)

for the choice (8.105) of $\rho$. The constant $K_{\Theta}$ only depends on the smooth cutoff function $\Theta$ in (6.2), and defines the value of the constant $K_{\Theta}$ in the definition of the polydisc $\Pi(\varepsilon, \delta, \eta, \lambda, \sigma)$, see (7.34). Moreover, $\hat{\delta}$ is as in (8.108).

In particular, we have

$$\| \tilde{T} - T_0^{(\bar{\rho}, \bar{\lambda})} \|_{\mathcal{T}} \leq \| (\bar{\beta} + |\bar{p}|)X\|_{\mathcal{T}} + \| \delta T \|_{\mathcal{T}} \| F_0 \|_{\mathcal{T}}$$

$$\leq \left( \| \tilde{T} \|_{\mathcal{C}} \right) \| F_0 \|_{\mathcal{T}}$$

< $K_{\Theta}'(\hat{\delta} + \hat{\eta})$, 

(8.77)

for a constant $K_{\Theta}'$ which only depends on $\Theta$.

8.4.2. Irrelevant kernels: Bounds on $\| \tilde{w}_{M,N} \|_\xi$. Recalling

$$\tilde{w}_{M,N}[\tilde{r}; X; x\tilde{n}, \lambda] = \frac{1}{\alpha[E_{\rho}^{-1}[\tilde{r}]; \tilde{p}]} \bar{w}_{M,N}[\tilde{r}; X; x\tilde{n}, \lambda]$$

(8.78)

from Lemma 7.8

$$\| \tilde{w}_{M,N} \|_{M,N}^2 \leq \left( 1 + \left( \frac{\partial \tilde{p}}{\alpha[E_{\rho}^{-1}[\tilde{r}]; \tilde{p}]} \right) \right) \frac{1}{\alpha[E_{\rho}^{-1}[\tilde{r}]; \tilde{p}]} \| \tilde{w}_{M,N} \|_{M,N}^2$$

$$\leq \left( 1 + \frac{c\eta^2}{\rho^2} \right) \| \tilde{w}_{M,N} \|_{M,N}^2 ,$$

(8.79)
by Proposition 7.6 Using Lemma 7.9 Lemma 8.1 and \((m+p) \leq 2^{m+p}\), we find

\[
\| \tilde{w}_{M,N}[\tilde{r}] \|_{M,N}^2 \leq \sum_{L=1}^{\infty} C_\phi^2 (L+1)^2 \left( \frac{C_\phi}{\rho} \right)^L (2\rho)^M + N \tag{8.80}
\]

\[
\sum_{m_1 + \ldots + m_L = M} \sum_{n_1 + \ldots + n_L = N} \prod_{\ell=1}^{L} \{ \left( \frac{2}{\sqrt{p_\ell}} \right)^{p_\ell} \left( \frac{2}{\sqrt{q_\ell}} \right)^{q_\ell} \| w_{M_\ell,N_\ell}[r]\|_{M_\ell,N_\ell}^2 \}
\]

where

\[
M_\ell := m_\ell + p_\ell \ , \ N_\ell := n_\ell + q_\ell \ . \tag{8.81}
\]

Summing over \(m, p, n, q\), we get

\[
\| \tilde{w}_{\geq 2}[\tilde{r}] \|_{\xi}^2 \leq 2 C_\phi^2 \rho^2 \sum_{M+N \geq 2} \xi^{-(M+N)} \| w_{M,N}[r]\|_{M,N}^2 \tag{8.82}
\]

\[
\sum_{m_1 + \ldots + m_L = M} \sum_{n_1 + \ldots + n_L = N} \prod_{\ell=1}^{L} \xi^{m_\ell+n_\ell} \sum_{p=0}^{M} \sum_{q=0}^{N} \left( \frac{2}{\sqrt{p}} \right)^{p} \left( \frac{2}{\sqrt{q}} \right)^{q} \xi^{-(m_\ell+p_\ell+n_\ell+q_\ell)} \| w_{M_\ell,N_\ell}[r]\|_{M_\ell,N_\ell}^2 \}
\]

and using the definition of the norm \( \| \cdot \|_{\xi}^2 \),

\[
\| \tilde{w}_{\geq 2}[\tilde{r}] \|_{\xi}^2 \leq 2 C_\phi^2 \rho^2 \sum_{M+N \geq 2} \xi^{-(M+N)} \| w_{M,N}[r]\|_{M,N}^2 \tag{8.83}
\]

\[
+ 2 C_\phi^2 \rho^2 \sum_{L=2}^{\infty} (L+1)^2 \left( \frac{C_\phi}{\rho} \right)^L \tag{8.84}
\]

\[
\left\{ \sum_{M+N \geq 1} \left( \sum_{p=0}^{M} \left( \frac{2}{\sqrt{p}} \right)^{p} \right) \left( \sum_{q=0}^{N} \left( \frac{2}{\sqrt{q}} \right)^{q} \right) \xi^{-(M+N)} \| w_{M,N}[r]\|_{M,N}^2 \right\}^L ,
\]

where \(8.83\) accounts for \(L = 1\), and \(8.84\) for the rest. Hence,

\[
\| \tilde{w}_{\geq 2} \|_{\xi} \leq 2 C_\phi^3 \rho A \| w_{\geq 2} \|_{\xi} + 2 C_\phi^2 \rho^2 \sum_{L=2}^{\infty} (L+1)^2 \left( \frac{C_\phi}{\rho} \right)^L A^{2L} (\| w_{\geq 1} \|_{\xi})^L
\]

with

\[
A := \sum_{p=0}^{\infty} \left( \frac{2}{\sqrt{p}} \right)^p \leq \sum_{p=0}^{\infty} (2\xi)^p = \frac{1}{1 - 2\xi} \leq 2 , \tag{8.85}
\]

assuming that \(\xi \leq \frac{1}{4}\).

Letting

\[
B := \frac{C_\phi}{\rho (1 - 2\xi)^2} \| w_{\geq 1} \|_{\xi} \leq \frac{4C_\phi}{\rho} \| w_{\geq 1} \|_{\xi} , \tag{8.86}
\]

\[
\]
We recall from (8.63) that 

$$\hat{\rho} < (i) \text{ The case } \sigma <$$

Since (8.92) only depends on 

$$a \parallel \xi \in L^{2} \text{, we have }$$

$$w_{\sigma} \in \mathbb{R}^{3},$$

Then

$$\lim_{x \rightarrow 0} \frac{w_{\sigma}}{\xi} \frac{\xi}{2} \leq \frac{\xi}{2}, \quad a = 0, 1.$$ 

Since (8.92) only depends on $$r$$ and $$\bar{p},$$ but neither on $$\bar{X}$$ nor $$|\bar{k}|,$$ we find

$$\sum_{a=0,1} \frac{\partial_{\bar{p}} w_{1,0}[\bar{r}]}{|\bar{p}|} \frac{\partial_{\bar{k}} w_{1,0}[\bar{r}] |}{w_{1,0}[\bar{r}] |}$$

$$\leq \sum_{a=0,1} \frac{\partial_{\bar{p}} w_{1,0}[\bar{r}]}{|\bar{p}|} \frac{\partial_{\bar{k}} w_{1,0}[\bar{r}] |}{w_{1,0}[\bar{r}] |} + \sum_{|a|=1} \frac{\partial_{\bar{p}}}{\partial_{\bar{p}}} w_{1,0}[\bar{r}] |$$

and

$$\eta \ll 1 \text{ sufficiently small.}$$

8.4.3. Marginal kernels: Bounds on \( ||\hat{w}||_{\xi}. \) In the case \( M + N = 1, \) we use the soft photon sum rules \( SR[\sigma]. \) That is, for any arbitrary unit vector \( \bar{n} \subset \mathbb{R}^{3}, \)

$$\lim_{x \rightarrow 0} w_{1,0}[\bar{r}; \bar{X}; x\bar{n}, \lambda] = \sqrt{\alpha} \mu(\sigma) \varepsilon(\bar{n}, \lambda) \cdot \nabla_{\bar{X}} T[\bar{r}; \bar{X}; \bar{p}], \quad (8.89)$$

and likewise for \( w_{0,1}. \) Since the soft photon sum rules are preserved by \( R_{\rho}, \) they imply that the renormalized quantities \( \hat{T}, \hat{w}_{1,0} \) and \( \hat{w}_{0,1} \) likewise satisfy

$$\lim_{x \rightarrow 0} \hat{w}_{1,0}[\bar{r}; \bar{X}; x\bar{n}, \lambda] = \sqrt{\alpha} \mu(\bar{\sigma}) \varepsilon(\bar{n}, \lambda) \cdot \nabla_{\bar{X}} \hat{T}[\bar{r}; \bar{X}; \bar{p}], \quad (8.90)$$

where \( \hat{r} = E_{\rho}[\bar{r}], \) \( \hat{\sigma} = \rho^{-1} \sigma, \) and \( \mu(\hat{\sigma}) = \rho^{\lambda(\sigma+1)} \mu(\sigma) \) (see (7.86)). With

$$\hat{\beta}[\hat{r}; \hat{p}] = \nabla_{\hat{X}} \hat{T}[\hat{r}; \hat{X} = 0; \hat{p}], \quad (8.91)$$

we have

$$\lim_{x \rightarrow 0} \hat{w}_{1,0}[\hat{r}; \hat{X}; x\bar{n}, \lambda] = \sqrt{\alpha} \mu(\hat{\sigma}) \varepsilon(\bar{n}, \lambda) \cdot \hat{\beta}[\hat{r}; \hat{p}], \quad (8.92)$$

For \( \sigma < 1, \) we have \( \mu(\sigma) = 1. \) For \( \sigma > 1, \) we have \( \mu(\hat{\sigma}) = \rho \mu(\sigma), \) and thus gain a factor \( \rho < \frac{1}{\delta} \) from the application of \( R_{\rho}. \)

(i) The case \( \sigma \leq 1. \) We recall from (8.63) that

$$\sup_{\bar{r} \in T} \left| \frac{\partial_{\bar{p}}}{\partial_{\bar{p}}} \left[ \frac{E_{\rho}^{-1}[\bar{r}]}{\bar{p}} \right] \right| \leq \frac{\eta}{2}$$

and

$$\left| \frac{\partial_{\bar{p}}}{\partial_{\bar{p}}} \left[ \hat{\beta}[\hat{r}; \hat{p}] + |\hat{p}| \right] \right| < \frac{\delta}{2} + \frac{\eta}{2}, \quad a = 0, 1.$$
by Taylor’s theorem. All terms involving derivatives in $X$ and $\vec{k}$ obtain a factor $\rho$ from rescaling, as shown below. Hence, (8.92) is the marginal part of $w_{1,0}$.

The key observation here is that by use of the soft photon sum rules, the marginal parts of $\hat{w}_{1,0}$ and $\hat{w}_{0,1}$ are entirely determined by $\hat{T}$. Moreover, since $T$ is scalar (it has no vector part), only the scalar components of $w_{1,0}$ and $w_{0,1}$ are marginal, while the vector parts scale like irrelevant operators. This implies that the term proportional to $\vec{B}_\sigma$ in the fiber Hamiltonian $H(\vec{p}, \sigma)$ in (8.91) is an irrelevant operator.

Using (8.93), we find that

$$\|\hat{w}_{1,0}\|_{1,0}^2 = \sum_{a=0,1} \|\partial_\vec{p}^a \hat{w}_{1,0}\|_{1,0} + \|\partial_{\vec{p}} \hat{w}_{1,0}\|_{1,0} + \sum_{1 \leq |a| \leq 2} \|\partial_{\vec{p}}^a \hat{w}_{1,0}\|_{1,0}$$

$$+ \sum_{|a|=1} \|\partial_{\vec{p}} \partial_{\vec{k}}^a \hat{w}_{1,0}\|_{1,0} + \sum_{a=0,1} \|\partial_{\vec{p}}^a \hat{w}_{1,0}\|_{1,0}$$

$$\leq \sqrt{\alpha}(1 + |\vec{p}| + \delta) + 2 \left( \sum_{1 \leq |a| \leq 2} \|\partial_{\vec{p}}^a \hat{w}_{1,0}\|_{1,0} + \|\partial_{\vec{p}} \hat{w}_{1,0}\|_{1,0} \right)$$

$$+ \sum_{|a|=1} \|\partial_{\vec{p}} \partial_{\vec{k}}^a \hat{w}_{1,0}\|_{1,0} + \sum_{a=0,1} \|\partial_{\vec{p}}^a \hat{w}_{1,0}\|_{1,0} \right). \quad (8.94)$$

To bound the sums in the bracket in (8.94), we note that similarly as in (8.79),

$$\|\partial_{\vec{p}}^a \partial_{\vec{y}} \hat{w}_{1,0}\|_{1,0} \leq (1 + c \eta) \|\partial_{\vec{p}}^a \partial_{\vec{y}} \hat{w}_{1,0}\|_{1,0} \quad (8.95)$$

for $Y = |\vec{k}|$ or a component of $X$.

The leading term in $\hat{w}_{1,0}$ corresponding to $L = 1$ (where $p$ and $q$ are zero) is given by

$$V_{1,0,0,0}^{(L=1)}[\hat{w}; X; K] = \left\{ \Omega_f, F_0[\hat{X} + \vec{k}] \hat{W}_1[r; \rho X; \rho K] F_1[\hat{X}] \Omega_f \right\} \quad (8.96)$$

so that

$$\|V_{1,0,0,0}^{(L=1)}[\hat{w}; X; K]\|_{1,0}^2 \leq \|w_{1,0}[r; \rho X; \rho K]\|_{1,0}^2 \|F_0\|_{F}^2 \|F_1\|_{F}^2, \quad (8.97)$$

by the Leibnitz rule, and recalling the definition of the norms $\| \cdot \|_F \equiv \| \cdot \|_{M,N}$ in (8.24) and $\| \cdot \|_2$ in (8.33). A similar calculation is explained in detail in [4]. By (8.90) and similar considerations as in (8.94),

$$\|w_{1,0}[r; \rho X; \rho K]\|_{1,0}^2 \leq \sqrt{\alpha}(1 + |\vec{p}| + \delta)$$

$$+ 2 \rho \left( \sum_{1 \leq |a| \leq 2} \|\partial_{\vec{p}}^a w_{1,0}\|_{1,0} + \sum_{|a|=1} \|\partial_{\vec{p}} \partial_{\vec{k}}^a w_{1,0}\|_{1,0} \right)$$

$$+ \|\partial_{\vec{p}} w_{1,0}\|_{1,0} + \sum_{a=0,1} \|\partial_{\vec{p}}^a \partial_{\vec{k}} w_{1,0}\|_{1,0} \right)$$

$$\leq \sqrt{\alpha}(1 + |\vec{p}| + \delta) + \frac{\sqrt{\alpha} \eta}{2} + 2 \rho |w_{1,0}|_{1,0}^2, \quad (8.98)$$

where the factor $\rho$ enters through the derivatives with respect to $X$, $\vec{r}$ and $|\vec{k}|$. Moreover,

$$\|F_0\|_{F}^2, \|F_1\|_{F}^2 \leq C_{\Theta}. \quad (8.99)$$
Consequently,
\[
\|V^{(L=1)}_{1,0,0,0}[\rho]\|_{1,0}^2 \leq C^2_{\Theta} \left( \sqrt{\alpha} (1 + |\vec{p}| + \hat{\sigma}) + \frac{\sqrt{\alpha} \eta}{2} + 2 \rho \|w_{1,0}\|_{1,0}^2 \right). \tag{8.100}
\]

The case for \(\hat{w}_{0,1}\) is identical.

The sum of terms contributing to \(\hat{w}_{1,0}\) for \(L \geq 2\) can be bounded by
\[
2 C^2_{\Theta} \sum_{L=2}^{\infty} (L+1)^2 \left( \frac{C_{\Theta}}{\rho} \right)^L (2\rho)^{M+N} \sum_{m_1 + \cdots + m_L = M} \sum_{n_1 + \cdots + n_L = N} \sum_{p_1 + \cdots + p_L = L; q_1 + \cdots + q_L = L} \prod_{\ell=1}^{L} \left\{ \left( \frac{2}{\sqrt{p_\ell}} \right)^{p_\ell} \left( \frac{2}{\sqrt{q_\ell}} \right)^{q_\ell} \|w_{M_\ell,N_\ell}[r]\|_{M_\ell,N_\ell}^{1/2} \right\} 
\leq 2 C^2_{\Theta} \rho \xi \sum_{L=2}^{\infty} (L+1)^2 B^L 
\leq 384 \frac{C^4_{\Theta}}{\rho} \xi \|w_{1}\|_{1}^2 \xi, \tag{8.101}
\]

with \(B\) defined in (8.86), similarly as in the discussion of (8.87).

In conclusion,
\[
\|\hat{w}_{1}\|_{\xi} \leq \xi^{-1} \|\hat{w}_{1,0}\|_{1,0} + \xi^{-1} \|\hat{w}_{0,1}\|_{0,1} 
\leq 10 C^2_{\Theta} \sqrt{\alpha} \xi^{-1} (1 + |\vec{p}| + \hat{\delta}) 
+ (10 C^2_{\Theta} + 1536 \frac{C^4_{\Theta} \xi}{\rho^2}) \rho \|w_{1}\|_{\xi} 
\leq 10 C^2_{\Theta} \sqrt{\alpha} \xi^{-1} (1 + |\vec{p}| + \hat{\delta}) + \frac{\eta}{2}, \tag{8.102}
\]

independently of \(\sigma\), for \(\xi \leq \frac{1}{4}\), with
\[
\eta \ll \rho^3, \quad \rho = \frac{1}{150 C^2_{\Theta}}, \tag{8.103}
\]
and using \(\|w_{1}\|_{\xi} < \eta + \varepsilon < 2\eta\).

(ii) The case \(\sigma \geq 1\). Combined with the additional scaling factor \(\rho\) from \(\mu(\hat{\sigma}) = \rho \mu(\sigma)\) in (8.92), the arguments used for \(M+N \geq 2\) in Section 8.4.2 straightforwardly imply
\[
\|\hat{w}_{1}\|_{\xi} \leq \frac{\eta}{2} \tag{8.104}
\]
if \(\sigma > 1\).

8.5. Concluding the proof of Theorem 7.12 For
\[
\rho = \min \left\{ \frac{1}{K_{\Theta}}, \frac{1}{20 C^2_{\Theta}}, \frac{1}{150 C^2_{\Theta}}, \frac{1}{100} \right\}, \quad \xi = \frac{1}{10}, \tag{8.105}
\]
(see (7.34), (8.47), (8.67), (8.88)) and
\[
\alpha, \eta, \delta \ll \rho^3, \tag{8.106}
\]
we conclude that
\[ R_\rho : \mathcal{U}^{(\text{sym})} (\varepsilon, \delta, \eta, \lambda, \sigma) \rightarrow \mathcal{U}^{(\text{sym})} (\tilde{\varepsilon}, \tilde{\delta}, \tilde{\eta}, \tilde{\lambda}, \tilde{\sigma}) \]  
(8.107)
with
\[
\tilde{\eta} = \begin{cases} 
10 C_0^2 \sqrt{\alpha} \xi^{-1} (1 + |\vec{p}| + \tilde{\delta}) + \eta/2 & \text{if } \sigma \leq 1 \\
\eta/2 & \text{if } \sigma > 1 
\end{cases} 
\]
\[ \tilde{\varepsilon} \leq \frac{\varepsilon}{4} + \frac{\eta}{4} \]
\[ \tilde{\delta} \leq \delta + \frac{\eta}{2} \]
\[ \tilde{\lambda} = \rho \lambda \]
\[ \tilde{\sigma} = \rho^{-1} \sigma . \]  
(8.108)
All constants only depend on the smooth cutoff function \( \Theta \) (given that \( \rho \) is fixed by (8.105)). This completes the proof of Theorem 7.12. 

9. Proof of Theorem 7.13

In this section, we establish the strong induction step
\[ \text{sInd}[n-1] \Rightarrow \text{sInd}[n] \]  
(9.1)
for \( n \geq 1 \) to prove Theorem 7.13. In order to verify (9.1), we combine Theorem 7.12 with algebraic identities satisfied by the smooth Feshbach map.

Let \( \sigma_0 \) denote the infrared cutoff in the original fiber Hamiltonian \( H(\vec{p}, \sigma_0) \), and
\[ N(\sigma_0) := \left\lceil \frac{\log \frac{1}{\sigma_0}}{\log \frac{1}{\rho}} \right\rceil . \]  
(9.2)
For the range of scales \( n \leq N(\sigma_0) \), one has \( \sigma_n = \rho^{-n} \sigma_0 \leq 1 \). As has been noted before, (7.90) in Theorem 7.12 is insufficient to control the growth of the parameters \( \delta_n \) and \( \eta_n \).

For the range of scales \( n > N(\sigma_0) \) where \( \sigma_n = \rho^{-n} \sigma_0 > 1 \), part (7.91) of Theorem 7.12 implies that \( \delta_n \) and \( \eta_n \) decay exponentially. Hence, given \( \text{sInd}[N(\sigma_0)] \), Theorem 7.12 immediately implies (9.1) for all \( n > N(\sigma_0) \).

9.1. Base case: The first decimation step. We associate the fiber Hamiltonian \( H(\vec{p}, \sigma_0) \) with the scale \(-1\). In the first decimation step, the spectrally shifted fiber Hamiltonian \( H(\vec{p}, \sigma_0) + E^{(-1)}[r_{-1}] \) (with \( r_{-1} \in I_{\text{inf}} (\frac{E}{2}) \)) is mapped to an element
\[ w^{(0)} \in \mathcal{U}^{(\text{sym})} (\varepsilon_0, \delta_0, \eta_0, \lambda_0, \sigma_0) . \]  
(9.3)
The parameters \( \varepsilon_0 < \eta_0 \) and \( \delta_0 \) are independent of \( \sigma_0 \), and satisfy
\[
\varepsilon_0 \leq \eta_0 < c \sqrt{\alpha} \]
\[ \delta_0 = C_0 \alpha \]
\[ \eta_0 = 10 C_0^2 \sqrt{\alpha} \xi^{-1} (1 + |\vec{p}| + C_0 \alpha) \]
\[ \lambda_0 = \frac{1}{2}, \]  
(9.4)
imposing
\[ \alpha \ll \rho^6 \] (9.5)
on the finestructure constant, see (8.105). These results are proved in Section 11 of [4].

9.2. Strong induction step. From here on, the parameters \( \rho \) and \( \xi \) are assigned the fixed values in (8.105).

The strong induction assumption \( \text{sInd}[n-1] \) states that for all \( 0 \leq k \leq n-1 \),
\[ w^{(k)} \in U^{(\text{sym})}(\varepsilon_k, \delta_k, \eta_k, \lambda_k, \sigma_k) \] (9.6)
with
\[ w^{(k)} = \mathcal{R}_\rho[w^{(k-1)}] \quad \text{for } 1 \leq k \leq n-1 , \] (9.7)
and
\[ \varepsilon_k \leq \eta_k, \quad \delta_k \leq C_0 \alpha, \quad \eta_k = 20 C_0^2 \sqrt{\alpha \xi^{-1}} (1 + |\vec{p}| + C_0 \alpha), \quad \lambda_k = \rho^k \lambda_0 \quad \text{with } \lambda_0 = \frac{1}{2}, \] (9.8)
\[ \sigma_k = \rho^{-k} \sigma_0 . \]
see Theorem 7.13. The constant \( C_0 \) is independent of \( n \) and \( \sigma_0 \), and will be determined in Proposition 9.6 below.

To prove Theorem 7.13 we assume \( \text{sInd}[n-1] \), and infer from Theorem 7.12 that
\[ w^{(n)} = \mathcal{R}_\rho[w^{(n-1)}] \in U^{(\text{sym})}(\varepsilon_n, \delta_n, \eta_n, \lambda_n, \sigma_n) , \] (9.9)
where
\[ \delta_n \leq C_0 \alpha + \frac{\eta_{n-1}}{2}, \quad \eta_n = 10 C_0^2 \sqrt{\alpha \xi^{-1}} (1 + |\vec{p}| + C_0 \alpha) + \frac{\eta_{n-1}}{2}, \] (9.10)
\[ \lambda_n = \rho^n \lambda_0, \quad \sigma_n = \rho^{-n} \sigma_0 . \]
and
\[ \varepsilon_n \leq \frac{\varepsilon_{n-1}}{4} + \frac{\eta_{n-1}}{4} \leq \frac{\eta_{n-1}}{2} \leq \eta_n . \] (9.11)
To establish \( \text{sInd}[n] \), and to determine the constant \( C_0 \), we use the following "bootstrap" argument: We assume that \( \text{sInd}[n-1] \) holds for an unspecified finite constant \( C_0 \). Using this assumption, we prove (in Propositions 9.4 and 9.5 below) that for all \( \alpha < \alpha_0 \) with \( C_0 \alpha_0 < 1 \) sufficiently small, there exists an explicitly computable constant \( C'_0 \) independent of \( n \) and \( \alpha \) such that
\[ \delta_k \leq C'_0 \alpha \] (9.12)
for all \( k \) with \( 0 \leq k \leq n \). Together with \( \text{sInd}[n-1] \) and Theorem 7.12 we then find
\[ \eta_k \leq 10 C_0^2 \sqrt{\alpha \xi^{-1}} (1 + |\vec{p}| + C'_0 \alpha) + \frac{\eta_{k-1}}{2} \] (9.13)
for \( 1 \leq k \leq n \), from which one infers
\[ \eta_n \leq 20 C_0^2 \sqrt{\alpha \xi^{-1}} (1 + |\vec{p}| + C'_0 \alpha) . \] (9.14)
This implies that in \( \text{sInd}[n] \), we can choose
\[
C_0 = C'_0 ,
\] (9.15)
and since \( n \) is arbitrary, this is valid for all \( n \). The details left out here are presented in Proposition 9.6.

Let \( r_n \in J_{1\over 100} \) denote the spectral parameter corresponding to \( w^{(n)}[r_n] \). The spectral parameters \( r_k \) associated to \( w^{(k)}[r_k] \), for \( 0 \leq k < n \), are recursively defined by
\[
J(k) : r_k \mapsto r_{k+1} = E^{(k)}_\rho [r_k] ,
\] (9.16)
see (7.16), and
\[
r_k = J^{-1}_{(k,n)}[r_n] := (J(k))^{-1} \circ \cdots \circ (J(n-1))^{-1}[r_n]
\] (9.17)
for \( -1 \leq k < n \). Furthermore, \( r_0 = J_{(-1)}[r_{-1}] \) is obtained in the first decimation step. For a detailed discussion of this part, we refer to [3, 4].

For notational convenience, we write
\[
\alpha^{(k)} := \alpha^{(k)}[r_k; \vec{p}]
\] and
\[
H^{(k)} := H[w^{(k)}[r_k; \vec{p}]].
\] For \( \vec{u} \in S^2 \), let \( \psi_{\vec{u}} \in \mathbb{C}^2 \) with \( \|\psi_{\vec{u}}\|_{\mathbb{C}^2} = 1 \), \( \langle \psi_{\vec{u}}, \vec{p} \psi_{\vec{u}} \rangle = \vec{u} \), and we define \( \Omega_{\vec{u}} := \psi_{\vec{u}} \otimes \Omega_f \).

To establish (9.12), we prove that the coefficient
\[
\beta^{(n)} = \beta^{(n)}[r_n; \vec{p}] := \langle \Omega_{\vec{u}}, \partial_{\vec{p}} \rangle H^{(n)} \Omega_{\vec{u}}
\] (9.21)
of the marginal operator \( P_f^\parallel \) in the non-interacting part of \( H[w^{(n)}[r_n]] \) satisfies
\[
| \beta^{(n)} + |\vec{p}| |, | \partial_{\vec{p}} \beta^{(n)} + 1 | < c\alpha ,
\] (9.22)
where the constant is independent of \( n \) and \( \sigma_0 \). This in turn directly implies (9.13) via the soft photon sum rules, as explained in Section 8.4.3.

To prove (9.22), we invoke the following identities which are provided by Lemma 15.2 in [4].

**Lemma 9.1.** For \( n > m \geq 0 \), let
\[
Q^{(m,n)} := Q^{(m)} \Gamma^{(m+1)}_\rho \cdots Q^{(n-1)} \Gamma^{(n)}_\rho ,
\] and
\[
Q^{\sharp(0,n)} := Q^{\sharp(0,n)} \cdots Q^{\sharp(n)} \Gamma^{\sharp(n+1)}_\rho \cdots Q^{\sharp(n)} \Gamma^{\sharp(n)}_\rho.
\] (9.23)
Then, the identities
\[
H_{(m)}Q_{(m,n)} = \left[ \prod_{k=m}^{n-1} \alpha(k) \right] \rho^{n-m+}(\Gamma^*_\rho)^{n-m+} \chi_1[H_f]H_{(n)} ,
\]
\[
Q^\sharp_{(m,n)} H_{(m)} = \left[ \prod_{k=m}^{n-1} \alpha(k) \right] \rho^{n-m+} H_{(n)} \chi_1[H_f](\Gamma^*_\rho)^{n-m+} ,
\] (9.25)
and
\[
Q^\sharp_{(m,n)} H_{(m)} Q_{(m,n)} = \left[ \prod_{k=m}^{n-1} \alpha(k) \right] \rho^{n-m+} \left[ H_{(n)} - H_{(n)} \bar{\chi}_1[H_f] H_f^{-1} \chi_1[H_f] H_{(n)} \right]
\] (9.26)
hold for all \( m \) with \(-1 \leq m < n \) and \( m_+ := \max\{m, 0\} \).

Some basic properties of the vectors \( Q_{(-1,n)} \Omega_{\vec{u}} \in \mathbb{C}^2 \otimes \mathfrak{g} \) and \( Q_{(m,n)} \Omega_{\vec{u}} \in H_{\text{red}} \) are summarized in the following proposition.

**Proposition 9.2.** Assume that \( \text{SInd}[n-1] \) holds for \( n \geq 0 \). Then,
\[
\left\langle \Omega_{\vec{u}}, Q_{(-1,n)}^2 Q_{(-1,n)} \Omega_{\vec{u}} \right\rangle = \left\| Q_{(-1,n)} \Omega_{\vec{u}} \right\|^2 = \left[ \prod_{l=1}^{n-1} \alpha(l) \right] (1 - \text{err}^{(1)}_n) ,
\] (9.27)
where \( \text{err}^{(1)}_n \) is defined in (9.34), and
\[
|\text{err}^{(1)}_n| , |\partial[\beta]\text{err}^{(1)}_n| < c \eta_n^2 .
\] (9.28)
In particular,
\[
\left| \prod_{k=1}^{n} \alpha(k) \right| < \exp \left[ c \sum_{k=0}^{n} \eta_k^2 \right] < \exp \left[ c \alpha \min\{n, N(\sigma_0)\} \right]
\] (9.29)
for constants \( c \) which are independent of \( n \).

Moreover,
\[
1 < \left\| Q_{(m,n)} \Omega_{\vec{u}} \right\| \leq \left\| Q_{(-1,n)} \Omega_{\vec{u}} \right\| ,
\] (9.30)
for any \( m \) with \(-1 \leq m < n \), and
\[
\left\| H_f^\sharp Q_{(-1,n)} \Omega_{\vec{u}} \right\|^2 < \frac{c \alpha}{s} \left\| Q_{(-1,n)} \Omega_{\vec{u}} \right\|^2 ,
\] (9.31)
for any \( 0 < s \leq 1 \), where the constant \( c \) is independent of \( n, \sigma_0, \alpha, \) and \( s \).

**Proof.** We first of all note that since the spectral parameters \( r_k, -1 \leq k \leq n \), are real-valued, \( Q_{(m,n)}^\sharp \) is the adjoint of \( Q_{(m,n)} \), and we immediately have
\[
\left\langle \Omega_{\vec{u}}, Q_{(m,n)}^\sharp Q_{(m,n)} \Omega_{\vec{u}} \right\rangle = \left\| Q_{(m,n)} \Omega_{\vec{u}} \right\|^2
\] (9.32)
for all \(-1 \leq m \leq n \).
The following result can be straightforwardly adopted from Lemma 15.3 in [4]. For $0 \leq |p| < \frac{1}{2}$, and any choice of $m$ with $-1 \leq m < n$, we have

\[
\left\langle \Omega_{\alpha}, \partial_{H_f} H_{(n)} \Omega_{\alpha} \right\rangle = \left[ \prod_{k=-1}^{n-1} \alpha_{(k)}^{-1} \right] \left\langle \Omega_{\alpha}, Q_{(-1,n)}^t \partial_{H_f} Q_{(-1,n)} \Omega_{\alpha} \right\rangle + \text{err}_n ,
\]

where the error term is defined by

\[
\text{err}_n^{(I)} = (I_1) + (I_2) + (II)
\]

with

\[
(I_1) = \left[ \prod_{j=-1}^{n-1} \alpha_{(j)}^{-1} \right] \left\langle \Omega_{\alpha}, Q_{(-1,n)}^t H_{(-1)} \partial_{H_f} Q_{(-1,n)} \Omega_{\alpha} \right\rangle
\]

\[
(I_2) = \rho^n \left[ \prod_{j=-1}^{n-1} \alpha_{(j)}^{-1} \right] \left\langle \Omega_{\alpha}, (\partial_{H_f} Q_{(-1,n)}^t) H_{(-1)} Q_{(-1,n)} \Omega_{\alpha} \right\rangle = (I_1)^* \]

and

\[
(II) = \left\langle \Omega_{\alpha}, \partial_{H_f} \left( H_{(n)} \chi_1 [H_f] H_f^{-1} \chi_1 [H_f] H_{(n)} \right) \Omega_{\alpha} \right\rangle .
\]

Using (9.25), and

\[
\partial_{H_f} \Gamma_{\rho} = \rho \Gamma_{\rho} \partial_{H_f} , \quad \partial_{H_f} \Gamma_{\rho}^* = \rho^{-1} \Gamma_{\rho}^* \partial_{H_f} ,
\]

we find

\[
(I_1) = \rho^n \left[ \prod_{j=-1}^{n-1} \alpha_{(j)}^{-1} \right] \left\langle \Omega_{\alpha}, H_{(n)} \chi_1 [H_f] \Gamma_{\rho}^n \partial_{H_f} Q_{(-1,n)} \Omega_{\alpha} \right\rangle
\]

\[
= \sum_{j=-1}^{n-1} \rho^{n-j} \left\langle \Omega_{\alpha}, H_{(n)} \Gamma_{\rho}^n \right\rangle \chi_1 [\rho^{-n} H_f] Q_{(-1,j-1)} (\partial_{H_f} Q_{(j)}) \Gamma_{\rho}^* Q_{(j+1,n)} \Omega_{\alpha} \right\rangle
\]

\[
= \rho \left\langle \Omega_{\alpha}, H_{(n)} \chi_1 [H_f] \Gamma_{\rho} \partial_{H_f} Q_{(n-1)} \Omega_{\alpha} \right\rangle .
\]

Here, we used

\[
\chi_1 [\rho^{-n} H_f] Q_{(-1,j-1)} = \chi_1 [\rho^{-n} H_f] Q_{(-1)0} \Gamma_{\rho}^n Q_{(1)} \Gamma_{\rho}^* \cdots \Gamma_{\rho}^* (j-1) \Gamma_{\rho}^* = \chi_1 [\rho^{-n} H_f] Q_{(0)} \Gamma_{\rho}^n Q_{(1)} \Gamma_{\rho}^* \cdots \Gamma_{\rho}^* (j-1) \Gamma_{\rho}^* = \cdots = (\Gamma_{\rho})^j \chi_1 [\rho^{-n+k} H_f] Q_{(k)} \Gamma_{\rho}^* \cdots \Gamma_{\rho}^* (j-1) \Gamma_{\rho}^* = (\Gamma_{\rho})^j \chi_1 [\rho^{-n+j} H_f] ,
\]

since for all $r > 1$,

\[
\chi_1 [\rho^{-r} H_f] Q_{(k)} = \chi_1 [\rho^{-r} H_f] (\chi_{\rho}[H_f] - \bar{\chi}_{\rho}[H_f] \bar{R}(k) \bar{\chi}_{\rho}[H_f] (H_{(k)} - \tau(k) H_f) \chi_{\rho}[H_f])
\]

\[
= \chi_1 [\rho^{-r} H_f] \chi_{\rho}[H_f]
\]

\[
= \chi_1 [\rho^{-r} H_f] ,
\]

(9.40)
see also (5.17), and Lemma 15.3 in [4]. By $s\text{Ind}[n - 1]$, we conclude that
\[ |(I_1)| \leq \|W(n)\|_{op}\|\partial_{H_f} Q_{(n-1)} \Omega_{\tilde{u}}\| \leq c\eta_n^2 \] (9.41)
for a constant which is independent of $n$ and $\sigma_0$ (the constant only depends on $\rho$, which is fixed by (8.105) in this part of the analysis). The term $(I_2)$ can be treated in the same way. Moreover, it is easy to see that
\[ |(II)| \leq c \left(\|W(n)\|_{op} + \|\partial_{H_f} W(n)\|_{op}\right)^2 \leq c\eta_n^2 . \] (9.42)
Thus, $\text{err}_n^{(1)}$ depends only on the effective Hamiltonian on the last scale $n$, and from $s\text{Ind}[n - 1]$ follows that
\[ |\text{err}_n^{(1)}| < c\eta_n^2 . \] (9.43)
We note that
\[ |\partial_{\sigma}\text{err}_n^{(1)}| < c\eta_n^2 \] (9.44)
is obtained from a similar analysis.

Since $\partial_{H_f} H_{(-1)} = 1$, we find
\[ \left\langle \Omega_{\tilde{u}} , Q_{(-1,n)}^r Q_{(-1,n)} \Omega_{\tilde{u}} \right\rangle = \left[ \prod_{i=1}^{n-1} \alpha(i) \right] (1 - \text{err}_n^{(1)}) , \] (9.45)
which establishes (9.27).

To prove (9.29), we recall from (7.41) that $|\alpha(k) - 1| < c\eta_k^2$, one gets
\[ \left| \prod_{k=1}^{N(\sigma_0)} \alpha(k) \right| < \exp \left[ N(\sigma_0) \sup_{0 \leq k \leq N(\sigma_0)} \eta_k^2 \right] \] (9.46)
and
\[ \left| \prod_{k=N(\sigma_0)+1}^{\infty} \alpha(k) \right| < \exp \left[ c\eta_{N(\sigma_0)}^2 \sum_{k > N(\sigma_0)} 2^{-2(k-N(\sigma_0))} \right] \] (9.47)
\[ < \exp \left[ c\eta_{N(\sigma_0)}^2 \right] . \]

Since by $s\text{Ind}[n - 1],$
\[ \sup_{0 \leq k < n} \eta_k < c\xi^{-1} \alpha , \] (9.48)
holds for $n = N(\sigma_0)$, and $\xi$ is independent of $\alpha$ and $\sigma_0$, the claim follows.

To prove (9.30), let $P_1[H_f] = \chi[H_f < 1]$. We have
\[ \|Q_{(m,n)} \Omega_{\tilde{u}}\|\|Q_{(-1,n)} \Omega_{\tilde{u}}\| = \| (\Gamma_{\rho}^*)^m P_1 [H_f] Q_{(m,n)} \Omega_{\tilde{u}}\|\|Q_{(-1,n)} \Omega_{\tilde{u}}\| \]
\[ \geq \left| \left\langle \Omega_{\tilde{u}}^r , Q_{(m,n)} \Gamma_{\rho}^* P_1 [H_f] Q_{(-1,n)} \Omega_{\tilde{u}} \right\rangle \right| \]
\[ \geq \left| \left\langle Q_{(m,n)} \Omega_{\tilde{u}} , \Gamma_{\rho}^m P_1 [H_f] Q_{(-1,n)} \Omega_{\tilde{u}} \right\rangle \right| \]
\[ = \left| \left\langle Q_{(m,n)} \Omega_{\tilde{u}} , \Gamma_{\rho}^m (\Gamma_{\rho}^*)^m P_1 [H_f] Q_{(m,n)} \Omega_{\tilde{u}} \right\rangle \right| \]
\[ = \|Q_{(m,n)} \Omega_{\tilde{u}}\|^2 . \] (9.49)
which follows from the same considerations as in (9.39) and (9.40). Thus,
\[ \|Q_{(m,n)}\Omega_{\tilde{a}}\| \leq \|Q_{(-1,n)}\Omega_{\tilde{a}}\| . \] (9.50)
Moreover, one easily sees that
\[ \langle \Omega_{\tilde{a}}, Q_{(m,n)}\Omega_{\tilde{a}} \rangle = \langle \Omega_{\tilde{a}}, \Omega_{\tilde{a}} \rangle = 1 , \] (9.51)
hence
\[ \|Q_{(m,n)}\Omega_{\tilde{a}}\| \geq 1 \] (9.52)
for any \(-1 \leq m < n \in \mathbb{N}_0\). This implies (9.30).

To prove (9.31), we observe that
\[
\|H^\tau_f Q_{(m,n)}\Omega_{\tilde{a}}\| \leq \|H^\tau_f \chi_{\rho}[H_f] \Gamma_{\rho}^* Q_{(m+1,n)}\Omega_{\tilde{a}}\|
+ \|H^\tau_f Q_{(m)} \Gamma_{\rho}^* Q_{(m+1,n)}\Omega_{\tilde{a}}\|
\] (9.53)
where for \(0 \leq m < n\),
\[ Q'_{(m)} := \tilde{\chi}_\rho[H_f]\tilde{R}(m)\tilde{\chi}_\rho[H_f](H_{(m)} - \tau_{(m)}H_f)\chi_{\rho}[H_f] , \] (9.54)
on \(\mathcal{H}_{red}\), and \(Q_{(m)} = \chi_{\rho}[H_f] - Q'_{(m)}\). For \(m = -1\),
\[ Q'_{(-1)} := \tilde{\chi}_1[H_f]\tilde{R}(-1)\tilde{\chi}_1[H_f](H_{(-1)} - \tau_{(-1)}H_f)\chi_{\rho}[H_f] \] (9.55)
on \(\mathbb{C}^2 \otimes \tilde{\mathbb{F}}\) and \(Q_{(-1)} = \chi_{\rho}[H_f] - Q'_{(-1)}\).

Next, we use the estimate
\[
\|Q'_{(m)} \Gamma_{\rho}^* Q_{(m+1)}\| < \frac{c(n_m + n_{m+1})}{\rho^3} < \frac{cn_m}{\rho^3} \] (9.56)
from Lemma 12.2 in [4]. It implies that
\[
\|H^\tau_f Q_{(m,n)}\Omega_{\tilde{a}}\| \leq \rho^\tau \|H^\tau_f Q_{(m+1,n)}\Omega_{\tilde{a}}\|
+ \frac{cn_m}{\rho^3} \|\chi_{\rho}[H_f] Q_{(m+2,n)}\Omega_{\tilde{a}}\|
\leq \rho^\tau \|H^\tau_f Q_{(m+1,n)}\Omega_{\tilde{a}}\|
+ \frac{cn_m}{\rho^3} (1 - cn_{m}) \|\chi_{\rho}[H_f] Q_{(-1,n)}\Omega_{\tilde{a}}\| \] (9.57)
using Lemma 9.2. Thus, iterating,
\[
\|H^\tau_f Q_{(0,n)}\| < \sup_{0 \leq m < n} \left\{ \frac{cn_m}{(1 - \rho^\tau)\rho^3 (1 - cn_{m})} \right\} \|\chi_{\rho}[H_f] Q_{(-1,n)}\Omega_{\tilde{a}}\|
< \frac{c\sqrt{\alpha}}{(1 - \rho^\tau)\rho^3} \|\chi_{\rho}[H_f] Q_{(-1,n)}\Omega_{\tilde{a}}\| , \] (9.58)
by use of \(\sup_{0 \leq m < n} \eta_m < c\sqrt{\alpha}\), which follows from sInd\(n\).

Hence, from \(1 - \rho^\tau \sim s\) as \(s \searrow 0\),
\[
\|H^\tau_f Q_{(-1,n)}\Omega_{\tilde{a}}\| \leq \|H^\tau_f Q_{(0,n)}\| + \|H^\tau_f Q'_{(-1)} Q_{(0,n)}\Omega_{\tilde{a}}\|
\leq \|H^\tau_f Q_{(0,n)}\| + \frac{c\sqrt{\alpha}}{s} \|Q_{(-1,n)}\Omega_{\tilde{a}}\| \] (9.59)
where the constant \( c \) is because then, (9.32) is available. In \([4]\), the fact that the interaction is irrelevant makes an application of (9.32) unnecessary.

**Remark 9.3.** A key reason we are using spectral parameters in \( \mathbb{R} \), as opposed to \( \mathbb{C} \) in \([3]\), is because then, (9.32) is available. In \([3]\), the fact that the interaction is irrelevant makes an application of (9.32) unnecessary.

**Proposition 9.4.** For \( n \geq 0 \)

\[
\beta_n = \frac{\langle \Omega_{\alpha}, Q_{(-1,n)}(\partial_p H_{(-1)})Q_{(-1,n)} \Omega_{\alpha} \rangle}{\langle \Omega_{\alpha}, Q_{(-1,n)}^2(\Omega_{\alpha} \rangle (1 - \text{err}_n^{(1)}) + \text{err}_n^{(2)},
\]

where \( \text{err}_n^{(2)} = O(\eta_n^2) \) is defined in (9.66).

For \( \alpha < \alpha_0 \) with \( C_0 \alpha \ll 1 \) sufficiently small (see (9.63)),

\[
| \beta_n | < c_0 \alpha
\]

where the constant \( c_0 \) is independent of \( n \) and \( \alpha \).

**Proof.** From (9.20), we find

\[
\beta_n = \langle \Omega_{\alpha}, H_{(n)} \Omega_{\alpha} \rangle = \left[ \prod_{k=-1}^{n} \alpha^{-1}_{(k)} \right] (\text{main}_n + \text{err}_n^{(2)})
\]

where

\[
\text{main}_n = \rho^{-n} \langle \Omega_{\alpha}, \partial_p Q_{(-1,n)}H_{(-1)}Q_{(-1,n)} \Omega_{\alpha} \rangle
\]

\[
\text{err}_n^{(2)} = -\langle \Omega_{\alpha}, \partial_p \chi[H_{(n)}]H_f^{-1} \chi[H_f]H_{(n)} \rangle \Omega_{\alpha} \rangle.
\]

It is easy to verify that

\[
|\text{err}_n^{(2)}|, |\partial_p \text{err}_n^{(2)}| < c_0 \eta_n^2.
\]
Proposition 9.5. Assume that
\[ n \]
\[ \text{Proposition 9.5.} \quad \text{Assume that} \]
\[ n \]
The factor \( \rho^{-n} \) is eliminated by pulling the differentiation operator \( \partial_{\bar{p}^j} \) through the \( n \) rescaling operators \( \Gamma_p \) in \( Q^2_{(-1,n)} \) from the left.

Using Proposition 9.2, we thus find
\[ \beta_{(n)} = \frac{\langle \Omega_{\bar{u}}, Q^2_{(-1,n)}(\partial_{\bar{p}^j} H (-1)) Q_{(-1,n)} \Omega_{\bar{u}} \rangle}{\langle \Omega_{\bar{u}}, Q^2_{(-1,n)} Q_{(-1,n)} \Omega_{\bar{u}} \rangle} (1 - err^{(1)}_n) + err^{(3)}_n, \]
where \( err^{(1)}_n \) is defined in (9.34). This establishes (9.63). From
\[ \langle \Omega_{\bar{u}}, Q^2_{(-1,n)}(\partial_{\bar{p}^j} H (-1)) Q_{(-1,n)} \Omega_{\bar{u}} \rangle \]
(1)
\[ \| \]
\[ \text{Lemma 7.8 in} \]
\[ \text{[4]} \]
Moreover, for \( n \) rescaling operators \( \Gamma_p \) in \( Q^2_{(-1,n)} \) from the left.

Using Proposition 9.2, we thus find
\[ \beta_{(n)} = \frac{\langle \Omega_{\bar{u}}, Q^2_{(-1,n)}(\partial_{\bar{p}^j} H (-1)) Q_{(-1,n)} \Omega_{\bar{u}} \rangle}{\langle \Omega_{\bar{u}}, Q^2_{(-1,n)} Q_{(-1,n)} \Omega_{\bar{u}} \rangle} (1 - err^{(1)}_n) + err^{(3)}_n, \]
where \( err^{(1)}_n \) is defined in (9.34). This establishes (9.63). From
\[ \lambda_{\bar{p}}^j H (-1) = \lambda_{\bar{p}}^j H (\bar{p}, \sigma_0) = -|\bar{p}| + P^\parallel + \sqrt{\alpha} A^\parallel, \]
(9.70)
one finds
\[ \beta_{(n)} = -|\bar{p}| + err^{(3)}_n \]
(9.71)
with
\[ err^{(3)}_n := |\bar{p}| err^{(1)}_n + err^{(2)}_n \]
(9.72)
This establishes (9.63).

Let \( \tilde{A}_\sigma = A^+_\sigma + \tilde{A}^-_\sigma \), where \( \tilde{A}^-_\sigma \) is the term involving annihilation operators. From the Schwarz inequality,
\[ \| \tilde{A}^-_\sigma Q_{(-1,n)} \Omega_{\bar{u}} \| < c \| \tilde{H}^\parallel_{\bar{v}} Q_{(-1,n)} \Omega_{\bar{u}} \|. \]
(9.73)
Moreover, \( |\tilde{P}^j| < H_{\bar{v}} \). Thus, using Proposition 9.2
\[ \| \tilde{A}^-_\sigma Q_{(-1,n)} \Omega_{\bar{u}} \| \]
(9.74)
uniformly in \( n \). \[ \square \]

Proposition 9.5. Assume that sInd\([n - 1]\) holds for \( n \geq 0 \). Then,
\[ \partial_{\bar{p}^j} \beta_{(n)} = \left[ -1 + 2 \frac{\langle \Omega_{\bar{u}}, (\partial_{\bar{p}^j} Q^2_{(-1,n)} H (-1))(\partial_{\bar{p}^j} Q_{(-1,n)} \Omega_{\bar{u}} \rangle)}{\langle \Omega_{\bar{u}}, Q^2_{(-1,n)} Q_{(-1,n)} \Omega_{\bar{u}} \rangle} \right] (1 - err^{(1)}_n) \]
+ err^{(3)}_n, \]
where \( |err^{(3)}_n| < c \eta^2_n \).

Moreover, for \( |\bar{p}| < \frac{1}{4} \) and \( C_0 \alpha \ll 1 \) sufficiently small (see (9.8)),
\[ |\partial_{\bar{p}^j} \beta_{(n)} + 1| < c_0 \alpha, \]
(9.76)
where the constant $c_0$ is independent of $n$ and $\alpha$.

Proof. To prove (9.75), let

$$\Psi_{(-1,n)} := Q_{(-1,n)} \Omega_{\bar{\alpha}}.$$  \hspace{1cm} (9.77)

We recall (9.65) whereby

$$\partial_{[\bar{\rho}]} \beta(n) = \partial_{[\bar{\rho}]} \left[ \frac{\langle \Psi_{(-1,n)}, (\partial_{[\bar{\rho}]} H_{(-1)}^{(1)}) \Psi_{(-1,n)} \rangle}{\langle \Psi_{(-1,n)}, \Psi_{(-1,n)} \rangle} (1 - \text{err}_n^{(1)}) + \text{err}_n^{(2)} \right]$$

$$= \left[ - \frac{\langle \partial_{[\bar{\rho}]} \Psi_{(-1,n)}, (\partial_{[\bar{\rho}]} H_{(-1)}) \Psi_{(-1,n)} \rangle}{\langle \Psi_{(-1,n)}, \Psi_{(-1,n)} \rangle} \right] (1 - \text{err}_n^{(1)})$$

$$+ \frac{\langle \Psi_{(-1,n)}, (\partial_{[\bar{\rho}]} H_{(-1)}) \partial_{[\bar{\rho}]} \Psi_{(-1,n)} \rangle}{\langle \Psi_{(-1,n)}, \Psi_{(-1,n)} \rangle} - 1 \right] (1 - \text{err}_n^{(1)})$$

using $\partial_{[\bar{\rho}]} \partial_{[\bar{\rho}]} H_{(-1)} = 1$. The error terms $\text{err}_n^{(1)}$ and $\text{err}_n^{(2)}$ are defined in (9.34) and (9.66), respectively.

From (9.25), we find

$$\left| \langle \partial_{[\bar{\rho}]} \Psi_{(-1,n)}, (\partial_{[\bar{\rho}]} H_{(-1)} \Psi_{(-1,n)}) \rangle \right|$$

$$\leq \rho^n \left[ \sum_{\ell = 1}^{n-1} \left| \partial_{[\bar{\rho}]} \alpha_{(\ell)} \right| \prod_{j \neq \ell}^{n-1} \left| \alpha_{(j)} \right| \right] \left| \langle \partial_{[\bar{\rho}]} \Psi_{(-1,n)}, \Gamma_{\rho}^{-n} \chi_1 [H_{f}] \partial_{[\bar{\rho}]} H_{(n)} \Omega_{\bar{\alpha}} \rangle \right|$$

$$+ \rho^n \left[ \prod_{j = 1}^{n-1} \left| \alpha_{(j)} \right| \right] \left| \langle \partial_{[\bar{\rho}]} \Psi_{(-1,n)}, \Gamma_{\rho}^{-n} \chi_1 [H_{f}] \partial_{[\bar{\rho}]} H_{(n)} \Omega_{\bar{\alpha}} \rangle \right|$$

(9.79)

using (9.29). By (7.47), we have

$$\sum_{\ell = 1}^{n-1} \left| \partial_{[\bar{\rho}]} \alpha_{(\ell)} \right| \prod_{j \neq \ell}^{n-1} \left| \alpha_{(j)} \right| \leq n \left[ \sup_{\ell} \left| \partial_{[\bar{\rho}]} \alpha_{(\ell)} \right| \right] \prod_{j = 1}^{n-1} (1 + c \eta_j^2)$$

$$\leq c n \eta_n^2 \exp \left( c \sum_{j = 1}^{n-1} \eta_j^2 \right)$$

$$< c n \alpha \exp(c \alpha n)$$

(9.80)
and using (6.17) (see also (9.94) and the subsequent discussion), we find
\[ (9.79) \quad \leq (1 + c \alpha n) \rho^n \exp(c \alpha^2 n) \sum_{a=0,1} \left\langle \partial_{[\bar{p}]} Q\langle n \rangle \Omega_{\bar{a}}, \chi_1[H_\bar{p}] \partial_{[\bar{p}]} H\langle n \rangle \Omega_{\bar{a}} \right\rangle \]
\[ \leq (1 + c \alpha n) \rho^n \exp(c \alpha^2 n) \left\| \partial_{[\bar{p}]} Q\langle n \rangle \Omega_{\bar{a}} \right\| \sum_{a=0,1} \left\| \partial_{[\bar{p}]} H\langle n \rangle \Omega_{\bar{a}} \right\| \]
\[ \leq c \eta_n^2 \]
for \( \rho \leq \frac{1}{100} \) (see (8.105)). Thus,
\[ \left\langle \Psi\langle -1, n \rangle, (\partial_{[\bar{p}]} H\langle -1 \rangle) \Psi\langle -1, n \rangle \right\rangle = -\left\langle \Psi\langle -1, n \rangle, H\langle -1 \rangle \partial_{[\bar{p}]} \Psi\langle -1, n \rangle \right\rangle + O(\eta_n^2). \]

Moreover, from
\[ H\langle -1 \rangle = T\langle -1 \rangle + W\langle -1 \rangle + J\langle -1, n \rangle[r_n] \]
(see (9.17) for the definition of \( J\langle m, n \rangle \)), and
\[ (\partial_{[\bar{p}]} H\langle -1 \rangle) = -\partial_{\bar{p} j} H\langle -1 \rangle + \partial_{[\bar{p}]} J\langle -1, n \rangle[r_n] \]
(from \( \partial_{[\bar{p}]} H\langle \bar{p}, \sigma_0 \rangle = -\partial_{\bar{p} j} H\langle \bar{p}, \sigma_0 \rangle \)), we find
\[ \partial_{[\bar{p}]} J\langle -1, n \rangle[r_n] = \frac{\left\langle \Psi\langle -1, n \rangle, (\partial_{\bar{p} j} H\langle -1 \rangle) \Psi\langle -1, n \rangle \right\rangle}{\left\langle \Psi\langle -1, n \rangle, \Psi\langle -1, n \rangle \right\rangle} + O(\eta_n^2) \]
(9.84)

(notating that \( \|\Psi\langle -1, n \rangle\| \geq 1 \), since \( \langle \Omega_{\bar{a}}, \Psi\langle -1, n \rangle \rangle = 1 \). Thus,
\[ \partial_{[\bar{p}]} \beta\langle n \rangle = \left[ -1 + 2 \frac{\left\langle \partial_{[\bar{p}]} \Psi\langle -1, n \rangle, H\langle -1 \rangle \partial_{[\bar{p}]} \Psi\langle -1, n \rangle \right\rangle}{\left\langle \Psi\langle -1, n \rangle, \Psi\langle -1, n \rangle \right\rangle} \right] (1 - \text{err}^{(1)}_n) + \text{err}^{(3)}_n \]
where
\[ \text{err}^{(3)}_n := -\frac{\left\langle \Psi\langle -1, n \rangle, (\partial_{\bar{p} j} H\langle -1 \rangle) \Psi\langle -1, n \rangle \right\rangle}{\left\langle \Psi\langle -1, n \rangle, \Psi\langle -1, n \rangle \right\rangle} \partial_{[\bar{p}]} \text{err}^{(1)}_n + \partial_{[\bar{p}]} \text{err}^{(2)}_n + O(\eta_n^2) \]
\[ = - (\beta\langle n \rangle - \text{err}^{(1)}_n) \frac{\partial_{[\bar{p}]} \text{err}^{(1)}_n}{1 - \text{err}^{(n)}_n} + \partial_{[\bar{p}]} \text{err}^{(2)}_n + O(\eta_n^2). \]

(9.85)
To estimate \( \text{err}^{(3)}_n \), we note that by (9.8) and (9.11) (which are based on \( \text{Ind}[n-1] \) and Theorem (4.22),
\[ |\beta\langle n \rangle - |\bar{p}|| \leq C_0 \alpha + \frac{\eta_{n-1}}{2} \leq C_0 \alpha + 10 C_0^2 \sqrt{\alpha} \xi^{-1} (1 + |\bar{p}| + C_0 \alpha) < c \sqrt{\alpha} \]
(9.86)
Hence, by Proposition (9.4) and (9.67),
\[ \text{err}^{(3)}_n \leq c \sqrt{\alpha} \frac{|\partial_{[\bar{p}]} \text{err}^{(1)}_n|}{1 - |\text{err}^{(1)}_n|} + |\partial_{[\bar{p}]} \text{err}^{(2)}_n| + c \eta_n^2 \]
\[ \leq \frac{c \sqrt{\alpha} \eta_n^2}{1 - c \alpha} + c \eta_n^2 \]
\[ \leq c \eta_n^2, \]
(9.87)
where the constants are independent of \( n \).
To prove (9.76), we use
\[ \partial_{\bar{\rho}} Q_{(m,n)} = (\partial_{\bar{\rho}} Q_{(m)}) \Gamma_\rho Q_{(m+1,n)} + Q_{(m)} \Gamma_\rho \partial_{\bar{\rho}} Q_{(m+1,n)} \] (9.88)
and
\[ H_{(m)} Q_{(m)} = \alpha_{(m)} \rho \Gamma_\rho \chi_1[H_j] H_{(m+1)} \] (9.89)

Clearly,
\[ \langle \Omega_{\bar{\alpha}}, (\partial_{\bar{\rho}} Q_{(-1,n)}) H_{(-1)} (\partial_{\bar{\rho}} Q_{(-1,n)}) \Omega_{\bar{\alpha}} \rangle = \sum_{-1 \leq j, k < n} \langle \Omega_{\bar{\alpha}}, Q_{(j+1,n)} \Gamma_\rho (\partial_{\bar{\rho}} Q_{(j)}) \rangle \]
\[ Q_{(-1,j-1)} H_{(-1)} Q_{(-1,k-1)} \]
\[ (\partial_{\bar{\rho}} Q_{(k)}) \Gamma_\rho Q_{(k+1,n)} \Omega_{\bar{\alpha}} \] (9.90)

Let us consider the case \( j \leq k \).

We first show that the terms with \( j < k \) vanish. One has
\[ Q_{(-1,j-1)} H_{(-1)} Q_{(-1,k-1)} = Q_{(-1,j-1)} H_{(-1)} Q_{(-1,l-1)} Q_{(l,k-1)} \]
\[ = \left[ \prod_{l=1}^{j-1} \alpha_{(l)} \right] \rho^{k-j} \left\{ H_{(j)} Q_{(j,k-1)} \right\} - H_{(j)} \chi_1[H_j] H_f^{-1} \chi_1[H_j] H_{(j,k-1)} \} \],
where the second term in the brackets vanishes unless \( j = k \), since
\[ \chi_1[H_j] H_{(j)} Q_{(j,k-1)} = \rho^{k-j} \chi_1[H_j] \chi_{\rho^{k-j}}[H_j] H_{(k-1)} \]
\[ = 0 \text{ if } j < k \] (9.92)
(because \( \chi_1[H_j] \chi_{\rho^l}[H_j] = 0 \) for all \( l > 0 \)).

Thus, assuming that \( j < k \), the corresponding term in (9.90) is given by
\[ \rho^j \langle \Omega_{\bar{\alpha}}, Q_{(j+1,n)} \Gamma_\rho (\partial_{\bar{\rho}} Q_{(j)}) \rangle \chi_{\rho^{k-j}}[H_j] H_{(k)} \]
\[ (\partial_{\bar{\rho}} Q_{(k)}) \Gamma_\rho Q_{(k+1,n)} \Omega_{\bar{\alpha}} \] (9.93)

However, for any \( l > 0 \),
\[ \partial_{\bar{\rho}} Q_{(j)} \chi_{\rho^l}[H_j] = 0 \],
(9.94)
since the kernel of \( \partial_{\bar{\rho}} Q_{(j)} \) in \( H_{red} \) is contained in \( \text{Ran}(\chi_{\rho}[H_j]) \).

We thus conclude that
\[ \langle \Omega_{\bar{\alpha}}, (\partial_{\bar{\rho}} Q_{(-1,n)}) H_{(-1)} (\partial_{\bar{\rho}} Q_{(-1,n)}) \Omega_{\bar{\alpha}} \rangle = \sum_{-1 \leq k < n} A_k \] (9.95)
Proof. We recall the discussion at the beginning of Section 9.2. We first assume and \((9.76)\) hold. Moreover, we assume that \(\alpha < \alpha_0\) (we recall that \(\rho\) has been fixed for this part of our analysis, see \((8.105)\)).

Then, Propositions 9.4 and 9.5 imply that there exists an explicitly computable constant \(c_0\) independent of \(n\) and \(\alpha\) such that

\[
\sum_{a=0,1} |\partial_{\beta(a)}(\beta(n) + |\tilde{p}|) | \leq 2 c_0 \alpha . \tag{9.103}
\]
Since $\delta_n$ is by definition an upper bound on the left hand side, we can choose
\[
\delta_n \leq 2c_0 \alpha,
\]
where $c_0$ is the same constant as in (9.64) and (9.76). Likewise, the same argument implies for all $0 \leq k \leq n$ that $\delta_k \leq 2c_0 \alpha$, for the given, unspecified choice of $C_0$.

By Theorem 7.12 this implies that
\[
\eta_k \leq 10C_0^2 \sqrt{\alpha} (1 + |\vec{p}| + 2c_0 \alpha) + \frac{\eta_{k-1}}{2}
\]
for all $1 \leq k \leq n$. Thus,
\[
\eta_n \leq \sum_{k=0}^{n} 10C_0^2 \sqrt{\alpha} (1 + |\vec{p}| + 2c_0 \alpha) 2^{-k}
\]
\[
= 20C_0^2 \sqrt{\alpha} (1 + |\vec{p}| + 2c_0 \alpha)
\]
This establishes $s_{\text{Ind}}[n]$ with
\[
C_0 = 2c_0.
\]
Since $n$ was arbitrary, and $c_0$ is independent of $n$, this implies that $s_{\text{Ind}}[n]$ holds for $C_0 = 2c_0$ and all $n$, provided that $\alpha \leq \alpha_0$ with $c_0 \alpha_0 \ll 1$ sufficiently small, and $\alpha_0$ independent of $\sigma_0$. \qed

\section*{10. Proof of Theorem 3.1}

The proof of Theorem 3.1 can be straightforwardly completed by use of Theorems 7.12 and 7.13. We will in fact demonstrate that as $n \to \infty$,
\[
\beta_n \to - \partial_{|\vec{p}|} E(\vec{p}, \sigma_0)
\]
and
\[
\partial_{|\vec{p}|} \beta_n \to - m_{\text{ren}}(\vec{p}, \sigma_0)^{-1},
\]
where the sequence of spectral parameters $(r_n)_{n \geq 0}$ in $\beta_n$ is chosen suitably.

We have proved in the previous sections that $s_{\text{Ind}}[n]$ holds for all $n$, and that
\[
|\beta_n + |\vec{p}| |, \quad |\partial_{|\vec{p}|} \beta_n + 1| < c_0 \alpha \quad (10.1)
\]
hold uniformly in $n$. But this implies for the renormalized infrared mass that
\[
|m_{\text{ren}}(\vec{p}, \sigma_0) - 1| < c_0 \alpha \quad (10.2)
\]
uniformly in $\sigma_0$.

For this part of the analysis, we will extensively apply constructions and results from [4] to abbreviate our discussion.

\subsection*{10.1. Reconstruction of the ground state.}

We determine the ground state eigenvalue $E(\vec{p}, \sigma_0)$ of $H(\vec{p}, \sigma_0)$ and its 2-dimensional eigenspace. This is accomplished by combining Theorem 7.12 and Theorem 7.13 with arguments from [4].

As proved in Section 9 the property $s_{\text{Ind}}[n]$ formulated in Theorem 7.13 holds for all $n \in \mathbb{N}_0$. This implies the following:
For \( n \leq N(\sigma_0) \), we have
\[
\delta_n \leq C_0 \alpha \\
\varepsilon_n \leq \eta_n \leq C_1 \sqrt{\alpha} \\
\sigma_n = \rho^{-n} \sigma_0 \\
\lambda_n = \rho^n \lambda_0
\] (10.3)
for constants \( C_0, C_1 \) independent of \( n, \sigma_0, \) and \( \alpha \).

For \( n > N(\sigma_0) \), we have
\[
\delta_n \leq C_0 \alpha \\
\varepsilon_n \leq \eta_n \leq 2^{-(n-N(\sigma_0))} C_1 \sqrt{\alpha} \\
\sigma_n = \rho^{-n} \sigma_0 \\
\lambda_n = \rho^n \lambda_0
\] (10.4)

We let
\[
E_{(n)}[r; \vec{p}] := w_{0,0}^{(n)}[r; \vec{p}]
\] (10.5)
and recall from Lemma 7.5 that
\[
J_{(n)} : U_{(n)} \to I_{100}, \quad r \mapsto (\alpha[r; \vec{p}] \rho)^{-1} E_{(n)}[r; \vec{p}],
\] (10.6)
where
\[
U_{(n)} := U_{\overline{w}(n)} = \{ r \in I_{100} \mid |E_{(n)}[r; \vec{p}]| \leq \frac{\rho}{100} \}.
\]
We define for \(-1 \leq m < n\)
\[
e_{(n,m)} := J_{(n)}^{-1} \circ \cdots \circ J_{(m)}^{-1} \,[0] \in \mathbb{R}.
\] (10.7)
By the same arguments as in the proof of Theorem 12.1 in [4],
\[
e_{(n,\infty)} := \lim_{m \to \infty} e_{(n,m)} \in \mathbb{R}
\] (10.8)
exists, and by construction,
\[
J_{(n)}[e_{(n,\infty)}] = e_{(n+1,\infty)}.
\] (10.9)
Moreover,
\[
|e_{(n,\infty)}| < 2^{-(n-N(\sigma_0))} \eta_0,
\] (10.10)
which tends to zero at an exponential rate as \( n \to 0 \).

Let
\[
\tilde{H}_{(n)} := H_{\overline{w}(n)}[e_{(n,\infty)}; \vec{p}] \\
\tilde{\alpha}_{(n)} := \alpha^{(n)}[e_{(n,\infty)}; \vec{p}] \\
\tilde{\beta}_{(n)} := \beta^{(n)}[e_{(n,\infty)}; \vec{p}]
\] (10.11)
and
\[
\tilde{Q}^{(1)}_{(n)} := Q_{x(H_f)}^{(1)}(\tilde{H}_{(n)}, \tilde{\alpha}_{(n)} H_f).
\] (10.12)
Moreover, for \(-1 \leq m < n\), let
\[
\tilde{Q}_{(m,n)} := \tilde{Q}_{(m)} \Gamma^*_\rho \tilde{Q}^{(1)} \cdots \Gamma^*_\rho \tilde{Q}_{(n-1)} \Gamma^*_\rho \\
\tilde{Q}^2_{(m,n)} := \Gamma^*_\rho \tilde{Q}^{2}_{(n-1)} \Gamma^*_\rho \cdots \tilde{Q}_{(1)} \Gamma^*_\rho \tilde{Q}_{(m)}.
\] (10.13)
and
\[
\tilde{Q}_{(-1,n)} := \tilde{Q}(-1)\tilde{Q}(0,n)
\]
\[
\tilde{Q}^\dagger_{(-1,n)} = \tilde{Q}^\dagger(0,n)\tilde{Q}^\dagger(-1).
\] (10.14)

We emphasize that as before, \( \tilde{Q}^\dagger_{(m,n)} \) is the adjoint of \( \tilde{Q}_{(m,n)} \); since the spectral parameters \( e(n,\infty) \) are real-valued.

**Proposition 10.1.** There exists a constant \( \alpha_0 > 0 \) (independent of \( \sigma_0 \)) such that for any \( \sigma_0 > 0 \), and all \( \alpha < \alpha_0 \), the infimum \( E(\tilde{p},\sigma_0) \) of the spectrum of \( H(\tilde{p},\sigma_0) \) is an eigenvalue of multiplicity two at the bottom of the essential spectrum. For \( \tilde{u} \in S^2 \), let \( \psi_{\tilde{u}} \in \mathbb{C}^2 \) with \( \|\psi_{\tilde{u}}\|_{\mathbb{C}^2} = 1 \), \( \langle \psi_{\tilde{u}}, \tilde{\tau} \psi_{\tilde{u}} \rangle = \tilde{u} \), and \( \Omega_{\tilde{u}} = \psi_{\tilde{u}} \otimes \Omega_f \).

Then, for any choice of \( \psi_{\tilde{u}} \), the strong limit
\[
\Psi_{\tilde{u}}(\tilde{p},\sigma_0) := s - \lim_{n \to \infty} \tilde{Q}_{(-1,n)} \Omega_{\tilde{u}}
\] (10.15)
exists in \( \mathbb{C}^2 \otimes \mathfrak{F} \). Under the normalization condition
\[
\langle \Omega_{\tilde{u}}, \Psi_{\tilde{u}}(\tilde{p},\sigma_0) \rangle_{\mathbb{C}^2 \otimes \mathfrak{F}} = 1,
\] (10.16)
we have
\[
\|\Psi_{\tilde{u}}(\tilde{p},\sigma_0)\|_{\mathbb{C}^2 \otimes \mathfrak{F}} \leq \exp(\alpha N(\sigma_0)) < \infty,
\] (10.17)
and
\[
H(\tilde{p},\sigma_0)\Psi_{\tilde{u}}(\tilde{p},\sigma_0) = E(\tilde{p},\sigma_0)\Psi_{\tilde{u}}(\tilde{p},\sigma_0),
\] (10.18)
with \( \langle \Psi_{\tilde{u}}(\tilde{p},\sigma_0), \tilde{\tau} \Psi_{\tilde{u}}(\tilde{p},\sigma_0) \rangle = \tilde{u} \). Hence, \( \Psi_{\tilde{u}}(\tilde{p},\sigma_0) \) is an element of the 2-dimensional eigenspace corresponding to the ground state eigenvalue \( E(\tilde{p},\sigma_0) \) at the infimum of the spectrum of \( H(\tilde{p},\sigma_0) \).

More generally, for any \( n \geq 0 \) and any choice of \( \psi_{\tilde{u}} \), the strong limit
\[
\bar{\Psi}_{(n,\infty)} := s - \lim_{m \to \infty} \tilde{Q}_{(n,m)} \Omega_{\tilde{u}}
\] (10.19)
extists in \( \mathcal{H}_{red} \), and
\[
\bar{H}_{(n)} \bar{\Psi}_{(n,\infty)} = 0.
\] (10.20)
The vector \( \bar{\Psi}_{(n,\infty)} \) belongs to the 2-dimensional eigenspace corresponding to the ground state eigenvalue 0 of \( \bar{H}_{(n)} = H(\bar{w}^{(n)}[e(n,\infty)]) \).

**Proof.** Proposition 10.1 corresponds to Theorem 12.1 in [4] (for spin 0 and \( \alpha < \alpha_0(\sigma_0) \)). Given (10.3) and (10.4), which are uniform in \( \sigma_0 \), we can straightforwardly adapt the proof of Theorem 12.1 in [4] to our situation. Accordingly, we shall omit some of the details in our exposition, and refer to [4] instead.

We first establish the existence of the strong limit
\[
\Psi_{\tilde{u}}(\tilde{p},\sigma_0) = s - \lim_{n \to \infty} \tilde{Q}_{(-1,n)} \Omega_{\tilde{u}}
\] (10.21)
in $\mathbb{C}^2 \otimes \mathfrak{F}$. To this end, we verify that the sequence $(\tilde{Q}_{(-1,n)} \Omega_{\tilde{u}})_{n \geq 0}$ is Cauchy in $\mathbb{C}^2 \otimes \mathfrak{F}$. We have for any $m > N(\sigma_0)$

$$\left\| \tilde{Q}_{(-1,m)} \Omega_{\tilde{u}} - \tilde{Q}_{(-1,m+1)} \Omega_{\tilde{u}} \right\|_{\mathbb{C}^2 \otimes \mathfrak{F}} \leq \left\| \tilde{Q}_{(-1,m)} \right\|_{\text{op}} \left\| \tilde{Q}(m) - \chi_\rho[H_f] \right\|_{\mathfrak{H}_{\text{red}}}$$

$$\leq \left\| \tilde{Q}_{(-1)} \tilde{Q}(0) \right\|_{\text{op}} \left\| \prod_{j=1}^{m-1} \tilde{Q}(j) \prod_{j=1}^{m-1} \tilde{Q}(j+1) \right\|_{\text{op}}$$

$$\leq \exp \left( c \sum_{j=1}^{m-1} \eta_j \right) \leq \exp \left( c \sum_{j=1}^{m-1} \eta_j \right) \leq 2^{-(m-N(\sigma_0))} \leq \exp \left( c \sqrt{\alpha} N(\sigma_0) \right) 2^{-N(\sigma_0)} \leq \exp \left( c \sqrt{\alpha} \alpha N(\sigma_0) \right) 2^{-N(\sigma_0)} \leq \exp \left( c \sqrt{\alpha} \alpha N(\sigma_0) \right) 2^{-(m-N(\sigma_0))},$$

see (9.56), (10.3), and (10.4), combined with $\left\| \tilde{Q}(m) - \chi_\rho[H_f] \right\|_{\mathfrak{H}_{\text{red}}} < c \eta_m$, see [4]. Therefore, for any $n > m,$

$$\left\| \tilde{Q}_{(-1,m)} \Omega_{\tilde{u}} - \tilde{Q}_{(-1,n)} \Omega_{\tilde{u}} \right\|_{\mathbb{C}^2 \otimes \mathfrak{F}} \leq \exp \left( c \sqrt{\alpha} \alpha N(\sigma_0) \right) 2^{-(m-N(\sigma_0))} \leq \exp \left( c \sqrt{\alpha} \alpha N(\sigma_0) \right) 2^{-(m-N(\sigma_0))}\left[10.23\right]$$

Since the upper bound converges to zero as $m \to \infty$, $(\tilde{Q}_{(-1,n)} \Omega_{\tilde{u}})_{n \geq 0}$ is a Cauchy sequence in $\mathbb{C}^2 \otimes \mathfrak{F}$. For a detailed exposition, we refer to [3] [4]. Moreover, we have

$$\lim_{n \to \infty} \left\| \tilde{Q}_{(-1,n)} \Omega_{\tilde{u}} \right\|_{\mathbb{C}^2 \otimes \mathfrak{F}} = \lim_{n \to \infty} \left[ \prod_{j=1}^{n} \tilde{\alpha}(n) \right] (1 - \tilde{c}_n^{(1)})$$

$$\leq \exp \left( c \alpha N(\sigma_0) \right) < \infty \left[10.24\right]$$

from Proposition [12] and

$$\left\langle \Omega_{\tilde{u}}, \tilde{Q}_{(-1,n)} \Omega_{\tilde{u}} \right\rangle_{\mathbb{C}^2 \otimes \mathfrak{F}} = 1. \left[10.25\right]$$

independently of $n$, which implies [10.17].

Furthermore,

$$\tilde{H}_{(-1)} \Psi_{\tilde{\varphi}}(\tilde{\rho}, \sigma_0) = 0 \left[10.26\right]$$

follows from an iterated application of [9.26], and is equivalent to [10.18].

Since the choice of $\tilde{\psi}_{\tilde{u}} \in \mathbb{C}^2$ was arbitrary, and $H_{(-1)} = H(\tilde{\rho}, \sigma_0) - E(\tilde{\rho}, \sigma_0)$ is independent of $\psi$, [10.18] implies that

$$\left\{ \Psi_{\tilde{\varphi}}(\tilde{\rho}, \sigma_0) \text{ as in } [10.15] \bigg| \psi_{\tilde{u}} \in \mathbb{C}^2, \left\langle \psi_{\tilde{u}}, \tilde{\rho} \psi_{\tilde{u}} \right\rangle = \tilde{u} \right\} \subset \mathbb{C}^2 \otimes \mathfrak{F} \left[10.27\right]$$

is the 2-dimensional eigenspace corresponding to $E(\tilde{\rho}, \sigma_0)$, the ground state eigenvalue of $H(\tilde{\rho}, \sigma_0)$, which borders without a gap to $\text{ess spec} H(\tilde{\rho}, \sigma_0)$. 
For general $n$, (10.19) and (10.20) follow from the same reasoning. Since $\widetilde{H}_n$ is independent of $\psi_n$, and (10.20) holds for any choice of $\psi_n$, the eigenspace associated to the eigenvalue $0$ at the bottom of the spectrum of $\widetilde{H}_n$ is given by $\{\tilde{\Psi}_n \in C^2, \|\psi_n\|_{C^2} = 1\} \subset \mathcal{H}_{\text{red}}$. For further details, we refer to [1].

10.2. Infrared mass renormalization. Finally, we establish the uniform bounds on the renormalized electron mass stated in Theorem 3.1. To this end, we use the Feynman-Hellman formula

$$\partial_{\beta^2} E(\beta, \sigma_0) = \frac{\langle \Psi \overset{\beta}{u}(\beta, \sigma_0), (\partial_{\beta^2} H(\beta, \sigma_0)) \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle}{\langle \Psi \overset{\beta}{u}(\beta, \sigma_0), \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle}, \quad (10.28)$$

and

$$\partial_{\beta^2} E(\beta, \sigma_0) = 1 - 2 \frac{\langle \partial_{\beta^2} \Psi \overset{\beta}{u}(\beta, \sigma_0), (H(\beta, \sigma_0) - E(\beta, \sigma_0)) \partial_{\beta} \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle}{\langle \Psi \overset{\beta}{u}(\beta, \sigma_0), \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle} < 1. \quad (10.29)$$

A detailed discussion of these formulas is given in [1].

**Proposition 10.2.** For $\alpha < \alpha_0$ with $\alpha_0$ sufficiently small, independently of $\sigma_0$, the renormalized electron mass is bounded by

$$1 < m_{\text{ren}}(\beta_0, \sigma_0) < 1 + c_0 \alpha, \quad (10.30)$$

with a constant $c_0$ independent of $\sigma_0 \geq 0$.

**Proof.** For $n \geq 0$, $\bar{u} \in S^2$, and any $\psi_n \in C^2$ with $\langle \psi_n, \frac{d}{d \bar{u}} \psi_n \rangle = \bar{u}$, we have that

$$\beta_n = \langle \Omega_n, \partial_{\beta^2} H_n \Omega_n \rangle, \quad (10.31)$$

with $\Omega_n = \psi_n \otimes \Omega_f$, satisfies

$$|\beta_n| < c_0 \alpha, \quad (10.32)$$

uniformly in $n$, where

$$\widetilde{H}_n = \frac{\langle \bar{Q}_{(-n)} \Omega_n, (\partial_{\beta^2} H_{(-n)} \Omega_{(-n)}) \bar{Q}_{(-n)} \Omega_n \rangle}{\langle \bar{Q}_{(-n)} \Omega_n, \bar{Q}_{(-n)} \Omega_n \rangle} (1 - \text{err}_{n}^{(1)}) + \text{err}_{n}^{(2)}, \quad (10.33)$$

We have

$$\tilde{\beta}_n = \frac{\langle \bar{Q}_{(-n)} \Omega_n, (\partial_{\beta^2} H_{(-n)} \Omega_{(-n)}) \bar{Q}_{(-n)} \Omega_n \rangle}{\langle \bar{Q}_{(-n)} \Omega_n, \bar{Q}_{(-n)} \Omega_n \rangle} (1 - \text{err}_{n}^{(1)}) + \text{err}_{n}^{(2)}, \quad (10.34)$$

where

$$\text{err}_{n}^{(j)} := \text{err}_{n}^{(j)} \big|_{r_n = e_{(n, \infty)}} , \quad j = 1, 2, 3, \quad (10.34)$$

and

$$|\text{err}_{n}^{(1)}|, |\text{err}_{n}^{(2)}| \leq c_0^2 \times 0 \quad (n \to \infty), \quad (10.35)$$

see (10.34) and Proposition 9.2. Hence, with $\Psi \overset{\beta}{u}(\beta, \sigma_0)$ as in (10.15), we find

$$\lim_{n \to \infty} \tilde{\beta}_n = - \frac{\langle \Psi \overset{\beta}{u}(\beta, \sigma_0), (\partial_{\beta^2} H(\beta, \sigma_0)) \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle}{\langle \Psi \overset{\beta}{u}(\beta, \sigma_0), \Psi \overset{\beta}{u}(\beta, \sigma_0) \rangle} = - \partial_{\beta^2} E(\beta, \sigma_0),$$

with $\beta(\sigma_0)$. This completes the proof.
which follows from
\[
\partial_{\tilde{p}\tilde{r}} H(\tilde{p}, \sigma_0) = - \partial_{\tilde{r}} H(\tilde{p}, \sigma_0) ,
\]
and the Feynman-Hellman formula (10.28). Hence, by Proposition 9.2,
\[
| \partial_{\tilde{p}\tilde{r}} E(\tilde{p}, \sigma_0) - |\tilde{p}| | < c\alpha ,
\]
uniformly in \( \sigma_0 \).

To estimate \( \partial^2_{\tilde{p}\tilde{r}} E(\tilde{p}, \sigma_0) \), we consider
\[
\partial_{\tilde{p}\tilde{r}} \tilde{\beta}_1(n) = (\partial_{\tilde{p}\tilde{r}} \beta_1(n)) \bigg|_{r_n = e(n, \infty)} + (\partial_{\tilde{r}} \beta_1(n)) \bigg|_{r_n = e(n, \infty)} \partial_{\tilde{r}} e(n, \infty) .
\]
We recall from Propositions 9.4 and 9.5 that
\[
\sup_{r_n \in I_{\tilde{r}}} | \partial_{\tilde{r}} \tilde{\beta}_1(n) + 1 | , \sup_{r_n \in I_{\tilde{r}}} | \partial_{\tilde{r}} \tilde{\beta}_1(n) | < c\alpha ,
\]
where \( c \) is independent of \( n \) and \( \alpha \).

Moreover, we observe that
\[
\lim_{n \to \infty} | \partial_{\tilde{p}\tilde{r}} e(n, \infty) | = 0 .
\]
To see this, we note that \( \tilde{H}(n) \Psi_{n, \infty} = 0 \) is equivalent to
\[
(\tilde{T}(n) + \tilde{W}(n)) \tilde{\Psi}_{n, \infty} = - e(n, \infty) \chi^2_f H_f \tilde{\Psi}_{n, \infty} ,
\]
which follows from the definition of \( \tilde{H}(n) \). Therefore,
\[
\partial_{\tilde{p}\tilde{r}} e(n, \infty) = - \frac{\langle \Psi_{n, \infty} , (\partial_{\tilde{p}\tilde{r}} (\tilde{T}(n) + \tilde{W}(n))) \tilde{\Psi}_{n, \infty} \rangle}{\langle \Psi_{n, \infty} , \chi^2_f H_f \tilde{\Psi}_{n, \infty} \rangle} .
\]
Using
\[
s - \lim_{n \to \infty} \tilde{\Psi}_{n, \infty} = \Omega_{\tilde{r}}
\]
and
\[
\lim_{n \to \infty} \langle \Psi_{n, \infty} , (\partial_{\tilde{p}\tilde{r}} (\tilde{T}(n) + \tilde{W}(n))) \tilde{\Psi}_{n, \infty} \rangle = \lim_{n \to \infty} \langle \Omega_{\tilde{r}} , (\partial_{\tilde{p}\tilde{r}} (\tilde{T}(n) + \tilde{W}(n))) \Omega_{\tilde{r}} \rangle = \lim_{n \to \infty} \langle \Omega_{\tilde{r}} , (\partial_{\tilde{p}\tilde{r}} \tilde{W}(n)) \Omega_{\tilde{r}} \rangle < \lim_{n \to \infty} || \partial_{\tilde{p}\tilde{r}} \tilde{W}(n) ||_{\text{op}}
\]
\[
\leq \lim_{n \to \infty} || \tilde{w}(n) ||_{\text{op}} \leq \lim_{n \to \infty} \eta_n = 0 ,
\]
for every fixed value of the infrared cutoff, \( \sigma_0 > 0 \). Therefore,
\[
\partial_{\tilde{p}\tilde{r}} \tilde{\beta}_1(n) = \left[ - 1 + 2 \frac{(\partial_{\tilde{p}\tilde{r}} \tilde{Q}_1(n, \sigma_0) - E(\tilde{p}, \sigma_0)) \partial_{\tilde{p}\tilde{r}} \tilde{Q}_1(n, \sigma_0) \Omega_{\tilde{r}}}{\langle \tilde{Q}_1(n, \sigma_0) \Omega_{\tilde{r}} , \tilde{Q}_1(n, \sigma_0) \Omega_{\tilde{r}} \rangle} \right] (1 - \tilde{\epsilon}_{\text{err}}^{(1)})
\]
\[
+ \epsilon_{\text{err}}^{(3)} + (\partial_{\tilde{r}} \tilde{\beta}_1(n)) \bigg|_{r_n = e(n, \infty)} \partial_{\tilde{r}} e(n, \infty) ,
\]
as follows from Proposition 9.5.
Hence, in combination with (10.39), we find
\[
\lim_{n \to \infty} \partial_{\| \vec{p} \|^2} \tilde{\beta}(n) = -1 + 2 \frac{\langle \partial_{\| \vec{p} \|^2} \Psi_{\vec{p}}(\vec{p}, \sigma_0), (H(\vec{p}, \sigma_0) - E(\vec{p}, \sigma_0))\partial_{\| \vec{p} \|^2} \Psi_{\vec{p}}(\vec{p}, \sigma_0) \rangle}{\langle \Psi_{\vec{p}}(\vec{p}, \sigma_0), \Psi_{\vec{p}}(\vec{p}, \sigma_0) \rangle}
\]
by (10.29), since
\[
| \tilde{\epsilon} \epsilon^{(3)}_{\sigma_n} | < \epsilon \eta_n^2 \searrow 0 \quad (n \to \infty). 
\]
We thus obtain
\[
1 - c_0 \alpha < \partial_{\| \vec{p} \|^2} E(\vec{p}, \sigma_0) < 1 ,
\]
uniformly in \( \sigma_0 \).

Thus, we obtain for the renormalized electron mass
\[
m_{\text{ren}}(\vec{p}, \sigma_0) = \frac{1}{\partial_{\| \vec{p} \|^2} E(\vec{p}, \sigma_0)}
\]
that
\[
1 < m_{\text{ren}}(\vec{p}, \sigma_0) < 1 + c_0 \alpha ,
\]
uniformly in \( \sigma_0 \geq 0 \).

This completes the proof of \( A - C \) in Theorem 3.1.

11. Existence of the renormalized mass

In this section, we prove the existence of the renormalized mass in the limit in which the infrared regularization is removed, in the form as stated in part \( D \) of Theorem 3.1. This will conclude the analysis of this paper.

11.1. The limit \( \sigma \searrow 0 \) for fixed \( \vec{p} \). Because of the bounds (10.49) which are uniform in \( \sigma_0 \) and \( \vec{p} \), we find that for every \( \vec{p} \) with \( 0 \leq |\vec{p}| < \frac{1}{3} \), there exists a sequence \( \{ \sigma_n \} \) with \( \sigma_n \searrow 0 \) as \( n \to \infty \) such that
\[
m_{\text{ren}}(\vec{p}) := \lim_{n \to \infty} m_{\text{ren}}(\vec{p}, \sigma_n)
\]
exists.

11.2. The joint limit \( (\vec{p}, \sigma) \to (\vec{0}, 0) \) for \( \sigma > 0 \). It is established in [4], that the limit
\[
\bar{m}_{\text{ren}}(\vec{0}) := \lim_{\sigma \to 0} m_{\text{ren}}(\vec{0}, \sigma)
\]
exists, and a convergent algorithm is constructed to compute it to any arbitrary given level of precision. In this subsection, we shall prove that the joint limit \( (\vec{p}, \sigma) \to (\vec{0}, 0) \) of \( m_{\text{ren}}(\vec{p}, \sigma) \) exists on
\[
D_N := \{ (\vec{p}, \sigma) \mid 0 \leq |\vec{p}| < \frac{1}{3}, \sigma \geq |\vec{p}|^N \}
\]
for \( \frac{\vec{p}}{|\vec{p}|} \) fixed, and \( 1 \ll N < \infty \) arbitrarily large, but finite. That is,
\[
\tilde{m}_{\text{ren}}(\vec{0}) = \lim_{D_N \ni (\vec{p}, \sigma) \to (\vec{0}, 0)} m_{\text{ren}}(\vec{p}, \sigma).
\]

To this end, we observe that
\[
E'''(\vec{p}, \sigma) = 6 \left( \Psi'_{\vec{u}}(\vec{p}, \sigma), (H'(\vec{p}, \sigma) - E'(\vec{p}, \sigma)) \Psi'_{\vec{u}}(\vec{p}, \sigma) \right),
\]
given that \( \|\Psi_{\vec{u}}(\vec{p}, \sigma)\| = 1 \), and where \((\cdot)'\) is shorthand for \( \partial_{\vec{p}}(\cdot) \). Using Lemma 11.1 below, combined with the Schwarz inequality and \( |E'(\vec{p}, \sigma)| < c \), one gets
\[
|E'''(\vec{p}, \sigma)| \leq c \log \frac{1}{\sigma},
\]
for a constant \( c \) independent of \( \vec{p} \). Accordingly, we find
\[
|E''(\vec{p}, \sigma) - E''(\vec{0}, \sigma)| \leq c |\vec{p}| \log \frac{1}{\sigma},
\]
and consequently, for all
\[
(\vec{p}, \sigma) \in D_{N, \delta} := \{ (\vec{p}, \sigma) \in D_N \mid |(\vec{p}, \sigma)| = \sqrt{|\vec{p}|^2 + \sigma^2} < \delta \},
\]
we have
\[
|E''(\vec{p}, \sigma) - E''(\vec{0}, \sigma)| \leq \sup_{(\vec{p}, \sigma) \in D_{N, \delta}} c |\vec{p}| \log \frac{1}{\sigma}
\]
for an arbitrary \( \eta > 0 \). Consequently,
\[
\lim_{D_N \ni (\vec{p}, \sigma) \to (\vec{0}, 0)} E''(\vec{p}, \sigma) = \lim_{\sigma \to 0} E''(\vec{0}, \sigma).
\]
But from the definition of the renormalized mass (10.48), this is equivalent to
\[
\lim_{D_N \ni (\vec{p}, \sigma) \to (\vec{0}, 0)} m_{\text{ren}}(\vec{p}, \sigma) = \lim_{\sigma \to 0} m_{\text{ren}}(\vec{0}, \sigma) = \tilde{m}_{\text{ren}}(\vec{0}),
\]
which is what we wanted to show.

**Lemma 11.1.** Assume that \( \|\Psi_{\vec{u}}(\vec{p}, \sigma)\| = 1 \). Then,
\[
\|H'(\vec{p}, \sigma) \Psi'_{\vec{u}}(\vec{p}, \sigma)\|^2, \|\Psi'_{\vec{u}}(\vec{p}, \sigma)\|^2 < c \log \frac{1}{\sigma}.
\]

**Proof.** The estimate
\[
\|\Psi'_{\vec{u}}(\vec{p}, \sigma)\|^2 < c \log \frac{1}{\sigma}
\]
follows from similar considerations as those explained in the proof of Proposition 9.5. We shall not repeat the detailed argument here.

To prove the second asserted estimate, we recall that
\[
H(\vec{p}, \sigma) = \frac{1}{2} (H'(\vec{p}, \sigma))^2 + \sqrt{\alpha} \vec{\tau} \cdot \vec{B}_\sigma + H_f.
\]
Since
\[
|\vec{B}_\sigma| \leq c \sqrt{1 + H_f} \leq c (1 + H_f),
\]
it is clear that
\[
H_f + \sqrt{\alpha} \vec{\tau} \cdot \vec{B}_\sigma \geq -c \sqrt{\alpha} + (1 - c' \sqrt{\alpha}) H_f,
\]
for a constant \( c' \).
where $1 - c' \sqrt{\alpha} > 0$. Therefore, we have

$$
\| H'(\vec{p}, \sigma) \Psi'_u(\vec{p}, \sigma) \|^2 \leq \langle \Psi'_u(\vec{p}, \sigma) , (H(\vec{p}, \sigma) - E(\vec{p}, \sigma)) \Psi'_u(\vec{p}, \sigma) \rangle 
\quad + (|E(\vec{p}, \sigma)| + c \sqrt{\alpha}) \| \Psi'_u(\vec{p}, \sigma) \|^2 
\leq |m_{\text{ren}}(\vec{p}, \sigma) - 1| + c \| \Psi'_u(\vec{p}, \sigma) \|^2 
\leq c \alpha + c' \log \frac{1}{\sigma},
$$

(11.15)

as claimed. \hfill \square

11.3. The limit $|\vec{p}| \to 0$ for $\sigma = 0$. Finally, we prove that for $\sigma = 0$, the limit $|\vec{p}| \to 0$ agrees with $\tilde{m}_{\text{ren}}(\vec{0})$, i.e., the order of taking the limits $|\vec{p}| \to 0$ and $\sigma \to 0$ can be reversed.

To this end, let $\{\vec{p}_j\}_{j \in \mathbb{N}}$ denote any sequence converging to $\vec{0}$ along a fixed direction $\frac{\vec{p}_j}{|\vec{p}_j|}$. For every $\vec{p}_j$, let $\{\sigma_n(\vec{p}_j)\}$ denote the sequence in (11.1) corresponding to $\vec{p} = \vec{p}_j$. From each such sequence, we may extract an element $\sigma_n(\vec{p}_j)$ such that the sequence $\{\sigma_n(\vec{p}_j)\}_{j \in \mathbb{N}}$ converges to 0 as $j \to 0$, in such a way that $0 < c_1 < \frac{|\vec{p}_j|}{\sigma_n(\vec{p}_j)} < c_2$ holds for all $j$.

Accordingly, we find that

$$
|m_{\text{ren}}(\vec{p}_j) - \bar{m}_{\text{ren}}(\vec{0})| \leq |m_{\text{ren}}(\vec{p}_j) - m_{\text{ren}}(\vec{p}_j, \sigma_n(\vec{p}_j))| 
\quad + |m_{\text{ren}}(\vec{p}_j, \sigma_n(\vec{p}_j)) - \bar{m}_{\text{ren}}(\vec{0})|
$$

(11.16)

Taking $j \to \infty$, it follows from the discussion in Section 11.1 that the first term on the right hand side converges to zero, and from the discussion in Section 11.2 that the second term also converges to zero. This implies

$$
\lim_{j \to \infty} m_{\text{ren}}(\vec{p}_j) = \bar{m}_{\text{ren}}(\vec{0}),
$$

(11.17)

for any sequence $\{\vec{p}_j\}$ converging to $\vec{0}$, with $\frac{\vec{p}_j}{|\vec{p}_j|}$ fixed.

This concludes the proof of part $D_\alpha$ of Theorem 3.1. \hfill \square

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