ON THE NUMBER OF LIMIT CYCLES OF A QUARTIC POLYNOMIAL SYSTEM

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Abstract. In this paper, we consider a quartic polynomial differential system with multiple parameters, and obtain the existence and number of limit cycles with the help of the Melnikov function under perturbation of polynomials of degree $n = 4$.

1. Introduction and main result. As we know, there have been many studies for the class of polynomial differential systems of the form

$$
\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y),
$$

(1.1)

where $P_n(x, y), Q_n(x, y)$ are polynomials of degree $n$. The second part of the Hilbert’s 16th problem is to ask the number of limit cycles of (1.1) and their position. The problem is still open up to now. However, many results have been obtained for some systems of lower degrees such as quadratic, cubic and quartic polynomial systems, see [3, 4, 1, 2, 6, 8, 9, 7, 5, 10].

Benterki and Llibre [1] investigated a quartic polynomial differential system of the form

$$
\dot{x} = -y + \sum_{s=1}^{6} \varepsilon^s \sum_{0 \leq i+j \leq 4} a_{ij}^{(s)} x^i y^j,
$$

$$
\dot{y} = x + ax^3 y + bxy^3 + \sum_{s=1}^{6} \varepsilon^s \sum_{0 \leq i+j \leq 4} b_{ij}^{(s)} x^i y^j,
$$

(1.2)

where $\varepsilon$ is a small parameter. Evidently, the unperturbed system of system (1.2) has the form

$$
\dot{x} = -y,
$$

$$
\dot{y} = x + ax^3 y + bxy^3.
$$

(1.3)
Note that (1.3) is invariant under the change \((x, t) \rightarrow (-x, t)\). This implies that the origin is a center for (1.3). The authors of [1] studied the limit cycles of system (1.2) bifurcated from the origin by using the averaging theory. The details are as follows.

For the quartic polynomial differential system (1.2), first they made the scaling \(x = \varepsilon X, y = \varepsilon Y\), to obtain

\[
\dot{X} = -Y + \sum_{s=1}^{6} \varepsilon^{s-1} \sum_{0 \leq i+j \leq 4} \varepsilon^{i+j} a_{ij}^{(s)} X^{i}Y^{j},
\]

\[
\dot{Y} = X + \varepsilon^{3}aX^{3}Y + \varepsilon^{3}bXY^{3} + \sum_{s=1}^{6} \varepsilon^{s-1} \sum_{0 \leq i+j \leq 4} \varepsilon^{i+j} b_{ij}^{(s)} X^{i}Y^{j}.
\]

Assume

\[
a_{00}^{(1)} = 0, b_{00}^{(1)} = 0.
\]

By making the change of polar coordinates \(X = r \cos \theta, Y = r \sin \theta\), (1.4) becomes

\[
\dot{r} = \varepsilon^{3}r \left( a_{00}^{(1)} \cos^{2} \theta \sin^{2} \theta + b_{00}^{(1)} \cos \theta \sin \theta + \sum_{i+j \leq 4} \varepsilon^{i+j} a_{ij}^{(1)} r^{i+j} \cos^{i} \theta \sin^{j} \theta \right)
\]

\[
+ \sum_{s=1}^{6} \varepsilon^{s-1} \sum_{0 \leq i+j \leq 4} \varepsilon^{i+j} a_{ij}^{(s)} r^{i+j} \cos^{i} \theta \sin^{j} \theta,
\]

\[
\dot{\theta} = 1 + \varepsilon^{3} \left( a_{00}^{(1)} \cos \theta \sin \theta + b_{00}^{(1)} \cos^{2} \theta \sin^{2} \theta \right) + \sum_{s=1}^{6} \varepsilon^{s-1} \sum_{0 \leq i+j \leq 4} \varepsilon^{i+j} a_{ij}^{(s)} r^{i+j} \cos^{i} \theta \sin^{j} \theta
\]

\[
+ \sum_{s=1}^{6} \varepsilon^{s-1} \sum_{0 \leq i+j \leq 4} \varepsilon^{i+j} a_{ij}^{(s)} r^{i+j} \cos^{i} \theta \sin^{j} \theta.
\]

The above equations yield the following \(2\pi\)-periodic equation

\[
\frac{dr}{d\theta} = \varepsilon F_{1}(\theta, r) + \varepsilon^{2} F_{2}(\theta, r) + \varepsilon^{3} F_{3}(\theta, r) + \varepsilon^{4} F_{4}(\theta, r)
\]

\[
+ \varepsilon^{5} F_{5}(\theta, r) + \varepsilon^{6} F_{6}(\theta, r) + \varepsilon^{7} R(\theta, r, \varepsilon),
\]

where the functions \(F_{1}, \ldots, F_{6}\) and \(R\) are all \(2\pi\)-periodic in \(\theta\) with

\[
F_{1}(\theta, r) = a_{00}^{(2)} \cos \theta + a_{01}^{(2)} \sin \theta + r(\theta_{00}^{(1)} \cos \theta + \theta_{01}^{(1)} \cos \theta \sin \theta + \theta_{10}^{(1)} \sin^{2} \theta),
\]

\[
F_{2}(\theta, r) = -a_{00}^{(1)} \cos \theta - a_{01}^{(1)} \sin \theta + r(\theta_{00}^{(1)} \cos \theta + \theta_{01}^{(1)} \cos \theta \sin \theta + \theta_{10}^{(1)} \sin^{2} \theta)
\]

\[
+ \cos \theta(\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)}) \sin \theta + \cos \theta \sin \theta(r\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)} + \theta_{02}^{(2)} + \theta_{12}^{(2)} + \theta_{20}^{(2)} + \theta_{21}^{(2)} + \theta_{22}^{(2)})
\]

\[
+ \cos \theta \sin \theta(\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)} + \theta_{02}^{(2)} + \theta_{12}^{(2)} + \theta_{20}^{(2)} + \theta_{21}^{(2)} + \theta_{22}^{(2)})
\]

\[
+ \cos \theta \sin \theta(\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)} + \theta_{02}^{(2)} + \theta_{12}^{(2)} + \theta_{20}^{(2)} + \theta_{21}^{(2)} + \theta_{22}^{(2)})
\]

\[
+ \cos \theta \sin \theta(\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)} + \theta_{02}^{(2)} + \theta_{12}^{(2)} + \theta_{20}^{(2)} + \theta_{21}^{(2)} + \theta_{22}^{(2)})
\]

\[
+ \cos \theta \sin \theta(\theta_{00}^{(2)} + \theta_{01}^{(2)} + \theta_{10}^{(2)} + \theta_{11}^{(2)} + \theta_{02}^{(2)} + \theta_{12}^{(2)} + \theta_{20}^{(2)} + \theta_{21}^{(2)} + \theta_{22}^{(2)})
\]

In order to study (1.7), they considered a more general system of the form than (1.7)

\[
x^{(i)} = \sum_{j=0}^{k} \varepsilon^{j} F_{i}(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon),
\]

where \(R : \mathbb{R} \times D \times (-\varepsilon_{0}, \varepsilon_{0}) \to \mathbb{R}\) and \(F_{i} : \mathbb{R} \times D \to \mathbb{R}\) for \(i = 0, 1, \ldots, k\), are continuous functions, and \(T\)-periodic in the first variable, with \(D\) an open interval
of \( \mathbb{R} \), and \( \varepsilon \) a small parameter. Let \( x(t, z, \varepsilon) \) denotes the solution of system (1.8) satisfying \( x(0, z, \varepsilon) = z \). Then the authors of [2] obtained for the displacement function of (1.8)

\[
x(T, z, \varepsilon) - z = \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \cdots,
\]

where

\[
f_1(z) = \int_0^T F_1(t, z) dt, \\
f_2(z) = \int_0^T (F_2(t, z) + \partial F_1(t, z) y_1(t, z)) dt, \\
f_3(z) = \int_0^T (F_3(t, z) + \partial F_2(t, z) y_1(t, z) + \frac{1}{2} \partial^2 F_1(t, z) y_1(t, z)^2 + \frac{1}{2} \partial^2 F_1(t, z) y_2(t, z)) dt, \\
f_4(z) = \int_0^T (F_4(t, z) + \partial F_3(t, z) y_1(t, z) + \frac{1}{2} \partial^2 F_2(t, z) y_1(t, z)^2 + \frac{1}{2} \partial^2 F_2(t, z) y_2(t, z) \\
+ \frac{1}{6} \partial^3 F_1(t, z) y_1(t, z)^3 + \frac{1}{6} \partial^3 F_1(t, z) y_3(t, z)) dt, \\
f_5(z) = \int_0^T (F_5(t, z) + \partial F_4(t, z) y_1(t, z) + \frac{1}{2} \partial^2 F_3(t, z) y_1(t, z)^2 + \frac{1}{2} \partial^2 F_3(t, z) y_2(t, z) \\
+ \frac{1}{6} \partial^2 F_2(t, z) y_3(t, z) + \frac{1}{6} \partial^2 F_1(t, z) y_1(t, z) y_3(t, z) + \frac{1}{8} \partial^2 F_1(t, z) y_2(t, z)^2 \\
+ \frac{1}{4} \partial^3 F_1(t, z) y_1(t, z)^2 y_2(t, z) + \frac{1}{24} \partial^3 F_1(t, z) y_1(t, z)^4 + \frac{1}{24} \partial F_1(t, z) y_4(t, z)) dt, \\
f_6(z) = \int_0^T (F_6(t, z) + \partial F_5(t, z) y_1(t, z) + \frac{1}{2} \partial^2 F_4(t, z) y_2(t, z) \\
+ \frac{1}{2} \partial^2 F_4(t, z) y_1(t, z)^2 + \frac{1}{6} \partial^2 F_3(t, z) y_3(t, z) + \frac{1}{6} \partial^2 F_3(t, z) y_2(t, z) y_1(t, z) \\
+ \frac{1}{4} \partial^2 F_2(t, z) y_1(t, z)^2 y_2(t, z) + \frac{1}{8} \partial^2 F_2(t, z) y_2(t, z)^2 + \frac{1}{24} \partial^4 F_1(t, z) y_1(t, z)^4 \\
+ \frac{1}{12} \partial F_1(t, z) y_5(t, z) + \frac{1}{24} \partial F_1(t, z) y_1(t, z) y_4(t, z) + \frac{1}{12} \partial^2 F_1(t, z) y_2(t, z) y_3(t, z) \\
+ \frac{1}{12} \partial^2 F_1(t, z) y_1(t, z)^2 y_3(t, z) + \frac{1}{12} \partial^2 F_1(t, z) y_1(t, z)^2 y_2(t, z) \\
+ \frac{1}{120} \partial^5 F_1(t, z) y_1(t, z)^5 dt,
\]

where

\[
y_1(t, z) = \int_0^t F_1(s, z) ds, \\
y_2(t, z) = \int_0^t (2F_2(s, z) + 2\partial F_1(s, z) y_1(s, z)) ds, \\
y_3(t, z) = \int_0^t (6F_3(s, z) + 6\partial F_2(s, z) y_1(s, z) + 3\partial^2 F_1(s, z) y_1(s, z)^2 \\
+ 3\partial F_1(s, z) y_2(s, z)) ds,
\]
The following is our main result. Theorem 1.1 \cite{1}. For $0 < \varepsilon \ll \lambda \ll 1$, and study the number of limit cycles with the help of the Melnikov function. The following is our main result.

\begin{align*}
y_4(t, z) &= \int_0^t (24F_4(s, z) + 24\partial F_3(s, z)y_1(s, z) + 12\partial^2 F_2(s, z)y_1(s, z)^2 \\
&+ 12\partial F_2(s, z)y_2(s, z) + 12\partial^2 F_1(s, z)y_1(s, z)y_2(s, z) + 4\partial^3 F_1(s, z)y_1(s, z)^3 \\
&+ 4\partial F_1(s, z)y_3(s, z))ds,
y_5(t, z) &= \int_0^t (120F_5(s, z) + 120\partial F_4(s, z)y_1(s, z) + 60\partial^2 F_3(s, z)y_1(s, z)^2 \\
&+ 60\partial F_3(s, z)y_2(s, z) + 60\partial^2 F_2(s, z)y_1(s, z)y_2(s, z) + 20\partial^3 F_2(s, z)y_1(s, z)^3 \\
&+ 20\partial F_2(s, z)y_3(s, z) + 20\partial^2 F_1(s, z)y_1(s, z)y_3(s, z) + 15\partial^3 F_1(s, z)y_2(s, z)^2 \\
&+ 30\partial^3 F_1(s, z)y_1(s, z)^2y_2(s, z) + 5\partial^4 F_1(s, z)y_1(s, z)^4 + 5\partial F_1(s, z)y_4(s, z))ds,
y_6(t, z) &= \int_0^t (720F_6(s, z) + 720\partial F_5(s, z)y_1(s, z) + 360\partial F_4(s, z)y_2(s, z)^2 \\
&+ 360\partial^2 F_4(s, z)y_1(s, z)^2 + 120\partial F_3(s, z)y_3(s, z) + 360\partial^2 F_3(s, z)y_1(s, z)y_2(s, z) \\
&+ 120\partial^3 F_3(s, z)y_1(s, z)^3 + 300\partial F_2(s, z)y_4(s, z) + 120\partial^2 F_2(s, z)y_1(s, z)y_2(s, z) \\
&+ 30\partial^4 F_2(s, z)y_1(s, z)^4 + 90\partial F_2(s, z)y_2(s, z)^2 + 180\partial^3 F_2(s, z)y_1(s, z)^2y_2(s, z) \\
&+ 60\partial F_1(s, z)y_5(s, z) + 30\partial^2 F_1(s, z)y_1(s, z)y_3(s, z) + 60\partial^2 F_1(s, z)y_2(s, z)^2y_3(s, z) \\
&+ 60\partial^3 F_1(s, z)y_1(s, z)^2y_3(s, z) + 60\partial^4 F_1(s, z)y_1(s, z)y_3(s, z)^2y_2(s, z) \\
&+ 90\partial^5 F_1(s, z)y_1(s, z)y_4(s, z)^2 + 60\partial F_1(s, z)y_5(s, z)y_3(s, z)^2)ds.
\end{align*}
Theorem 1.2. System (1.9) can have four limit cycles near the origin for $0 < \varepsilon \ll \lambda \ll 1$ by the first Melnikov function.

We note that Eq.(1.2) and Eq.(1.9) are both quartic systems. By comparing Theorem 1.1 and Theorem 1.2, we see that 2 more limit cycles appear in (1.9) by using our method.

2. A preliminary theorem. In this section, we introduce a preliminary theorem which will be used to prove the main result. Consider a near-Hamiltonian system of the form

$$\dot{x} = H_y(x, y, \lambda) + \varepsilon p(x, y, \varepsilon, \lambda),$$

$$\dot{y} = -H_x(x, y, \lambda) + \varepsilon q(x, y, \varepsilon, \lambda),$$

where $0 < \varepsilon \ll \lambda \ll 1$ and $H_y(x, y, \lambda), p(x, y, \varepsilon, \lambda), q(x, y, \varepsilon, \lambda)$ are $C^\infty$ functions. Then we can write for $\lambda$ small

$$H(x, y, \lambda) = \sum_{j=0}^{2} \lambda^j H_j(x, y) + O(\lambda^3),$$

$$p(x, y, 0, \lambda) = \sum_{j=0}^{2} \lambda^j p_j(x, y) + O(\lambda^3),$$

$$q(x, y, 0, \lambda) = \sum_{j=0}^{2} \lambda^j q_j(x, y) + O(\lambda^3).$$

Suppose the unperturbed system of (2.1) has a family of periodic orbits given by $L_{\lambda}(h) : H(x, y, \lambda) = h, h \in J_{\lambda}$, where $J_{\lambda}$ is an interval and $L_{\lambda}(h) \to L_0(h)$ as $\lambda \to 0$ for $h \in J_0 \equiv \lim_{\lambda \to 0} J_{\lambda}$. For Eq.(2.1), the first order Melnikov function is

$$M(h, \lambda) = \oint_{L_{\lambda}(h)} (q dx - p dy)|_{\varepsilon = 0}.$$

The function has the power series in $\lambda$

$$M(h, \lambda) = M_0(h) + \lambda M_1(h) + \cdots + \lambda^n M_n(h) + O(\lambda^{n+1}),$$

where

$$M_0(h) = \oint_{L_{\lambda}(h)} (q dx - p dy)|_{\varepsilon = \lambda = 0}.$$

The authors [3] obtained the formulas for $M_1$ and $M_2$ appearing in (2.3) as stated in the following theorem.

Theorem 2.1 ([3]). Let (2.2) hold. Then

(i) the function $M_1$ in (2.3) has the expression

$$M_1(h) = -\oint_{L_0(h)} H_1(x, y)[(p_0)_x + (q_0)_y]dt + \oint_{L_0(h)} q_1 dx - p_1 dy.$$

(ii) Suppose that there exist a region $G$ and $C^\infty$ functions $p^*(x, y)$ and $q^*(x, y)$ defined on $G$ such that for all $(x, y) \in G$

$$-H_1(x, y)[(p_0)_x + (q_0)_y] = (H_0)_x p^* + (H_0)_y q^*.$$
Then for $M_2$ in (2.3) we have
\[
M_2(h) = -\frac{1}{2} \oint_{L_0(h)} \varphi_1(x, y) dt - \oint_{L_0(h)} \varphi_2(x, y) dt + \oint_{L_0(h)} q_2 dx - p_2 dy, \tag{2.6}
\]
where $L_0(h) \subset G$ and
\[
\varphi_1 = (H_1 p^*)_x + (H_1 q^*)_y,
\varphi_2 = H_1 [(p_1)_x + (q_1)_y] + H_2 [(p_0)_x + (q_0)_y].
\]

The formulas for $M_1$ and $M_2$ can be used to find the number of zeros of the Melnikov function $M(h, \lambda)$ and estimate the number of limit cycles ([3]) for $0 < \varepsilon \ll \lambda \ll 1$.

3. The proof of the main result. Consider the unperturbed system of system (1.9)
\[
\dot{x} = -y, \quad \dot{y} = x + \lambda xy^3, \tag{3.1}
\]
which is integrable. One can see that system (3.1) has an integrating factor $\mu(y) = (1 + \lambda y^3)^{-1}$. Then system (1.9) for $1 + \lambda y^3 > 0$ is equivalent to a near-Hamiltonian system of the form
\[
\begin{align*}
\dot{x} &= \frac{-y}{1 + \lambda y^3} + \varepsilon p(x, y, \lambda), \\
\dot{y} &= x + \varepsilon q(x, y, \lambda),
\end{align*} \tag{3.2}
\]
where
\[
\begin{align*}
p(x, y, \lambda) &= \frac{1}{1 + \lambda y^3} \sum_{k=0}^{2} \lambda^k \sum_{0 \leq i+j \leq 4} a_{ij}^{(k)} x^i y^j, \\
q(x, y, \lambda) &= \frac{1}{1 + \lambda y^3} \sum_{k=0}^{2} \lambda^k \sum_{0 \leq i+j \leq 4} b_{ij}^{(k)} x^i y^j. \tag{3.3}
\end{align*}
\]
Clearly, the unperturbed system of (3.2) has a family of periodic orbits for $\lambda$ small given by
\[
L_\lambda(h) : H(x, y, \lambda) \equiv \frac{1}{2} x^2 + \int_{0}^{y} \frac{t}{1 + \lambda t^3} dt = h, \quad h \in J_\lambda = (0, h_\lambda), \tag{3.4}
\]
where
\[
h_\lambda = \int_{0}^{+\infty} \frac{y}{1 + \lambda y^3} dy \geq \int_{0}^{+\infty} \frac{y}{(1 + \sqrt{\lambda} y^2)^2} dy = \frac{1}{2\sqrt{\lambda}},
\]
for $\lambda > 0$. Let
\[
L_0(h) = \{ h, \frac{1}{2} x^2 + \frac{1}{2} y^2 = h, \quad h > 0 \}.
\]

Apparently, $L_\lambda(h) \rightarrow L_0(h)$ for $h > 0$ as $\lambda \rightarrow 0$.

Notice that for $0 < \lambda \ll 1$, we have
\[
\frac{1}{1 + \lambda y^3} = 1 - \lambda y^3 + \lambda^2 y^6 + O(\lambda^3). \tag{3.5}
\]
Then, by (3.3) and (3.5) we can write the functions \( p(x, y, \lambda) \) and \( q(x, y, \lambda) \) into the expansions

\[
p(x, y, \lambda) = (1 - \lambda y^3 + \lambda^2 y^6 + O(\lambda^3)) \sum_{k=0}^{2} \lambda^k \sum_{0 \leq i+j \leq 4} a_{ij}^{(k)} x^i y^j
\]

\[
= p_0(x, y) + \lambda p_1(x, y) + \lambda^2 p_2(x, y) + O(\lambda^3),
\]

\[
q(x, y, \lambda) = (1 - \lambda y^3 + \lambda^2 y^6 + O(\lambda^3)) \sum_{k=0}^{2} \lambda^k \sum_{0 \leq i+j \leq 4} b_{ij}^{(k)} x^i y^j
\]

\[
= q_0(x, y) + \lambda q_1(x, y) + \lambda^2 q_2(x, y) + O(\lambda^3),
\]

where

\[
p_0(x, y) = \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^j, q_0(x, y) = \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^j,
\]

\[
p_1(x, y) = -\sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^j,
\]

\[
q_1(x, y) = -\sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^j,
\]

\[
p_2(x, y) = \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+6} - \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} a_{ij}^{(2)} x^i y^j,
\]

\[
q_2(x, y) = \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^{j+6} - \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} b_{ij}^{(2)} x^i y^j.
\]

Further, we have

\[
H(x, y, \lambda) = \frac{1}{2} x^2 + \int_0^y [t - \lambda t^4 + \lambda^2 t^7 + O(\lambda^3)] dt
\]

\[
= H_0(x, y) + \lambda H_1(x, y) + \lambda^2 H_2(x, y) + O(\lambda^3),
\]

where

\[
H_0(x, y) = \frac{1}{2} x^2 + \frac{1}{2} y^2, \quad H_1(x, y) = -\frac{1}{5} y^5, \quad H_2(x, y) = \frac{1}{8} y^8.
\]

As we know (see [3]), Eq.(3.2) has a displacement function of the form for \( \varepsilon \) small

\[
d(h, \varepsilon, \lambda) = \varepsilon M(h, \lambda) + O(\varepsilon^2), \tag{3.6}
\]

where

\[
M(h, \lambda) = \int_{L_h} q dx - p dy,
\]

and \( p, q \) are given in (3.3). Since \( M(h, \lambda) \) is \( C^\infty \) for small \( \lambda \), it has an expansion below for \( \lambda \) small

\[
M(h, \lambda) = M_0(h) + \lambda M_1(h) + \lambda^2 M_2(h) + O(\lambda^3), \tag{3.7}
\]

where

\[
M_0(h) = M(h, 0) = \int_{L_0} q_0 dx - p_0 dy. \tag{3.8}
\]

Next, we will use Theorem 2.1 to compute \( M_0(h), M_1(h) \) and \( M_2(h) \).

For \( M_0 \) we first have
Lemma 3.1. For Eq.(3.2), $M_0(h)$ in (3.7) has the following expression

$$M_0(h) = h(A_0 + A_1 h),$$

where

$$A_0 = -2\pi(b_{01}^{(0)} + a_{10}^{(0)}),$$

$$A_1 = -\pi(3b_{03}^{(0)} + a_{12}^{(0)} + b_{21}^{(0)} + 3a_{30}^{(0)}).$$

Hence, $M_0(h) \equiv 0$ if and only if

$$a_{10}^{(0)} = -b_{01}^{(0)}, a_{12}^{(0)} = -b_{21}^{(0)} - 3a_{30}^{(0)}.$$  \hspace{1cm} (3.10)

Proof. By (3.8), we can obtain that

$$M_0(h) = \oint_{L_0(h)} \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^j dx - \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^j dy.$$ \hspace{1cm} (3.11)

Further, by polar coordinate transformation

$$\begin{align*}
x &= \sqrt{2h} \cos \theta, \\
y &= \sqrt{2h} \sin \theta,
\end{align*}$$

and noting

$$\begin{align*}
dx &= -\sqrt{2h} \sin \theta d\theta, \\
dy &= \sqrt{2h} \cos \theta d\theta,
\end{align*}$$

then (3.11) becomes

$$M_0(h) = \sum_{0 \leq i+j \leq 4} (\sqrt{2h})^{i+j+1} \int_0^{2\pi} [-b_{ij}^{(0)} \cos^i \theta \sin^j \theta - a_{ij}^{(0)} \cos^{i+1} \theta \sin^j \theta] d\theta$$

$$= \sum_{0 \leq i+j \leq 4} D_{ij} I_{ij}(h),$$

where

$$D_{00} = b_{00}^{(0)}, D_{01} = b_{01}^{(0)} + a_{10}^{(0)}, D_{02} = b_{02}^{(0)} + \frac{1}{2} a_{11}^{(0)},$$

$$D_{03} = b_{03}^{(0)} + \frac{1}{3} a_{12}^{(0)}, D_{04} = b_{04}^{(0)} + \frac{1}{4} a_{13}^{(0)}, D_{10} = b_{10}^{(0)},$$

$$D_{11} = b_{11}^{(0)} + 2a_{20}^{(0)}, D_{12} = b_{12}^{(0)} + a_{21}^{(0)}, D_{13} = b_{13}^{(0)} + \frac{2}{3} a_{22}^{(0)},$$

$$D_{20} = b_{20}^{(0)}, D_{21} = b_{21}^{(0)} + 3a_{30}^{(0)}, D_{22} = b_{22}^{(0)} + \frac{3}{2} a_{31}^{(0)},$$

$$D_{30} = b_{30}^{(0)}, D_{31} = b_{31}^{(0)} + 4a_{40}^{(0)}, D_{40} = b_{40}^{(0)},$$

and

$$I_{ij}(h) = -((\sqrt{2h})^{i+j+1} \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta).$$

(3.14)

It is direct that

$$I_{01}(h) = -2\pi h, I_{03}(h) = -3\pi h^2, I_{21}(h) = -\pi h^2,$$

$$I_{00}(h) = I_{02}(h) = I_{04}(h) = I_{10}(h) = I_{11}(h) = I_{12}(h) = 0,$$

$$I_{13}(h) = I_{20}(h) = I_{22}(h) = I_{30}(h) = I_{31}(h) = I_{40}(h) = 0.$$  \hspace{1cm} (3.15)
Inserting the values of the above integrals into (3.13) gives
\[ M_0(h) = h(A_0 + A_1 h), \]
where \( A_0 \) and \( A_1 \) satisfy (3.9). Clearly, \( M_0(h) \equiv 0 \) if and only if \( A_0 = A_1 = 0 \), which is equivalent to (3.10). This ends the proof. \( \square \)

As \( M_0 = 0 \), the following lemma gives the expression of \( M_1(h) \).

**Lemma 3.2.** Let (3.10) be satisfied. For Eq. (3.2), \( M_1(h) \) in (3.7) has the following expression
\[ M_1(h) = h(\bar{A}_0 + \bar{A}_1 h + \bar{A}_2 h^2 + \bar{A}_3 h^3), \]
where
\begin{align*}
\bar{A}_0 &= -2\pi(b_{01}^{(1)} + a_{10}^{(1)}), \\
\bar{A}_1 &= \pi(3b_{02}^{(0)} - 3b_{03}^{(1)} - a_{12}^{(1)} - b_{21}^{(1)} - 3a_{30}^{(1)}), \\
\bar{A}_2 &= \pi(3b_{02}^{(0)} + b_{20}^{(0)}), \\
\bar{A}_3 &= \pi(\frac{7}{4}b_{04}^{(0)} - \frac{1}{2}b_{13}^{(0)} + \frac{3}{4}b_{22}^{(0)} + \frac{3}{4}b_{40}^{(0)}).
\end{align*}

Then \( M_1(h) \equiv 0 \) if and only if
\begin{align*}
a_{10}^{(1)} &= b_{01}^{(1)} + a_{12}^{(1)} = 3b_{03}^{(1)} + b_{21}^{(1)} + 3a_{30}^{(1)} = 3b_{00}^{(0)}, \\
b_{20}^{(0)} &= -3b_{02}^{(0)} + a_{13}^{(0)} = \frac{7}{2}b_{04}^{(0)} + \frac{3}{2}b_{22}^{(0)} + \frac{3}{2}b_{40}^{(0)}. \quad (3.17)
\end{align*}

**Proof.** By Theorem 2.1, we have
\[ M_1(h) = M_{11}(h) - M_{12}(h), \]
where
\begin{align*}
M_{11}(h) &= \oint_{L_0(h)} q_1 dx - p_1 dy \\
&= \oint_{L_0(h)} \left( \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^j \right) dx + \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+3} \\
&\quad - \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^j dy, \quad (3.19)
\end{align*}

\begin{align*}
M_{12}(h) &= \oint_{L_0(h)} H_1(x, y) [(p_0)_x + (q_0)_y] dt \\
&= \oint_{L_0(h)} \left( \frac{1}{5} y^5 \right) [\sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^j] x + \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^j] dt.
\end{align*}

Similar to dealing with \( M_0(h) \), \( M_{11}(h) \) in (3.19) becomes
\begin{align*}
M_{11}(h) &= \sum_{0 \leq i+j \leq 4} \sum_{0 \leq i+j \leq 7} (\sqrt{2h})^{i+j+4} \int_0^{2\pi} \left[ b_{ij}^{(0)} \cos^i \theta \sin^{j+4} \theta + a_{ij}^{(0)} \cos^{i+1} \theta \sin^{j+3} \theta \right. \\
&\quad - b_{ij}^{(1)} \cos^i \theta \sin^{j+1} \theta - a_{ij}^{(1)} \cos^{i+1} \theta \sin^j \theta] d\theta \\
&\quad - \sum_{0 \leq i+j \leq 7} E_{ij} I_{ij}(h), \quad (3.20)
\end{align*}
where
\[ E_{00} = i^{(1)}_{00}, E_{01} = i^{(1)}_{01} + a^{(1)}_{10}, E_{02} = b^{(1)}_{02} + \frac{1}{2} a^{(1)}_{13}, E_{03} = b^{(1)}_{03} - \frac{1}{3} a^{(1)}_{12}, \]
\[ E_{04} = a^{(0)}_{11} - a^{(0)}_{10} - \frac{1}{3} a^{(1)}_{11}, E_{05} = -\frac{1}{2} a^{(1)}_{11} - b^{(0)}_{02}, E_{06} = -\frac{1}{3} a^{(0)}_{12} - b^{(0)}_{03}, \]
\[ E_{07} = -\frac{1}{3} a^{(0)}_{10} - b^{(0)}_{04}, E_{10} = b^{(1)}_{10}, E_{11} = b^{(1)}_{11} + 2 a^{(1)}_{20}, E_{12} = b^{(1)}_{12} + a^{(1)}_{12}. \]
\[ E_{13} = i^{(1)}_{13} - b^{(0)}_{10} + 2 a^{(1)}_{21}, E_{14} = -\frac{1}{2} a^{(0)}_{20} - b^{(0)}_{11}, E_{15} = -\frac{2}{5} a^{(0)}_{21} - b^{(0)}_{12}, \]
\[ E_{16} = -\frac{1}{3} a^{(0)}_{22} - b^{(0)}_{13}, E_{20} = b^{(1)}_{20}, E_{21} = b^{(1)}_{21} + 3 a^{(1)}_{30}, E_{22} = b^{(1)}_{22} + 3 a^{(0)}_{31}, \]
\[ E_{23} = -b^{(0)}_{20}, E_{24} = -\frac{3}{4} a^{(0)}_{30} - b^{(0)}_{21}, E_{25} = -\frac{3}{5} a^{(0)}_{31} - b^{(0)}_{22}, E_{30} = b^{(1)}_{30}, E_{31} = b^{(1)}_{31} + 4 a^{(1)}_{40}, \]
\[ E_{32} = 0, E_{33} = -b^{(0)}_{30}, E_{34} = -a^{(0)}_{40} - b^{(0)}_{31}, E_{40} = b^{(1)}_{40}, E_{41} = 0, E_{43} = b^{(0)}_{40}, \]
and \( I_{ij}(h) \) is defined by (3.14). Obviously, we can obtain the values in (3.20)
\[ I_{05}(h) = -5 \pi h^3, I_{23}(h) = -\pi h^3, I_{07}(h) = -\frac{35}{4} \pi h^4, \]
\[ I_{25}(h) = -\frac{5}{4} \pi h^4, I_{43}(h) = -\frac{3}{4} \pi h^4, I_{06}(h) = I_{14}(h) = 0. \] (3.21)
Inserting the values in (3.15) and (3.21) into (3.20), it follows that
\[ M_{11}(h) = -2 \pi (a^{(1)}_{10} + a^{(1)}_{11} + \pi (b^{(0)}_{00} - b^{(0)}_{03} - a^{(1)}_{12} - b^{(0)}_{21} - 3 a^{(1)}_{30})) h^2 + \pi (b^{(0)}_{04} + a^{(0)}_{11}) h^3 + \pi \left( \frac{35}{4} b^{(0)}_{04} + \frac{5}{4} b^{(0)}_{12} + \frac{5}{4} a^{(0)}_{13} + \frac{3}{4} b^{(0)}_{22} + \frac{1}{3} b^{(0)}_{31} \right) h^4. \] (3.22)
Note that along the curve \( L_0(h) \), \( dt = -\frac{1}{y} dx \). By using polar coordinate transformation (3.12), we have
\[ M_{12}(h) = \frac{1}{4} \sum_{0 \leq i+j \leq 7} K_{ij} I_{ij}(h), \] (3.23)
where
\[ K_{04} = -\frac{1}{5} (a^{(0)}_{10} + b^{(0)}_{10}), K_{05} = \frac{1}{5} (a^{(1)}_{11} + 2 b^{(0)}_{12}), K_{06} = -\frac{1}{5} (a^{(0)}_{12} + 3 b^{(0)}_{03}), \]
\[ K_{07} = -\frac{1}{5} (a^{(0)}_{13} + 4 b^{(0)}_{04}), K_{14} = -\frac{1}{5} (2 a^{(0)}_{20} + b^{(0)}_{21}), K_{15} = -\frac{1}{5} (2 a^{(0)}_{21} + 2 b^{(0)}_{12}), \]
\[ K_{16} = \frac{1}{5} (2 a^{(0)}_{22} + 3 b^{(0)}_{13}), K_{24} = -\frac{1}{5} (3 a^{(0)}_{30} + b^{(0)}_{21}), K_{25} = -\frac{1}{5} (3 a^{(0)}_{31} + 2 b^{(0)}_{22}), \]
\[ K_{34} = -\frac{1}{5} (4 a^{(0)}_{40} + b^{(0)}_{31}), \]
and \( I_{ij}(h) \) is defined by (3.14). Substituting the values of (3.15) and (3.21) into (3.23), one can find that
\[ M_{12}(h) = \pi (a^{(0)}_{11} + 2 b^{(0)}_{02}) h^3 + \pi (7 b^{(0)}_{04} + \frac{7}{4} a^{(0)}_{13} + \frac{1}{2} b^{(0)}_{22} + \frac{3}{4} a^{(0)}_{31}) h^4. \] (3.24)
Lemma 3.3. Let (3.10) and (3.17) hold, such that $M_0(h) = M_1(h) \equiv 0$. Then $M_2(h)$ in (3.7) has the following expression

$$M_2(h) = h(B_0 + B_1 h + B_2 h^2 + B_3 h^3 + B_4 h^4),$$

where

$$B_0 = -2\pi(b_{01}^{(2)} + a_{10}^{(2)}),$$

$$B_1 = -\pi(3b_{03}^{(2)} + a_{12}^{(2)} - 3b_{00}^{(1)} + b_{21}^{(2)} + 3a_{30}^{(2)}),$$

$$B_2 = \pi(3b_{02}^{(2)} + b_{20}^{(1)}),$$

$$B_3 = -\pi\left(\frac{341}{80}a_{10}^{(0)} - \frac{1}{2}a_{13}^{(1)} + \frac{161}{80}b_{01}^{(0)} + \frac{3}{4}b_{22}^{(1)} + \frac{3}{4}b_{40}^{(1)} + \frac{7}{4}b_{04}^{(1)}\right),$$

$$B_4 = -\pi\left(\frac{1827}{400}b_{03}^{(0)} - \frac{91}{400}b_{21}^{(0)} + \frac{749}{400}a_{12}^{(0)} + \frac{267}{400}a_{30}^{(0)}\right).$$

Proof. Let

$$p^* = \frac{1}{5}(2a_{20}^{(0)} + b_{11}^{(0)})y^5 + \frac{1}{5}(2a_{21}^{(0)} + b_{12}^{(0)})y^6 + \frac{1}{5}(2a_{22}^{(0)} + 3b_{13}^{(0)})y^7 + \frac{1}{5}(3a_{30}^{(0)} + b_{21}^{(0)})xy^5$$

$$+ \frac{1}{5}(3a_{31}^{(0)} + 2b_{22}^{(0)})xy^6 + \frac{1}{5}(4a_{40}^{(0)} + b_{31}^{(0)})x^2y^5,$$

$$q^* = \frac{1}{5}(a_{10}^{(0)} + b_{01}^{(0)})y^4 + \frac{1}{5}(a_{11}^{(0)} + 2b_{02}^{(0)})y^5 + \frac{1}{5}(a_{12}^{(0)} + 3b_{03}^{(0)})y^6 + \frac{1}{5}(a_{13}^{(0)} + 4b_{04}^{(0)})y^7.$$

Then we have

$$-\frac{1}{5}y^5[(p_0)_x + (q_0)_y] = -xp^* - yq^*,$$

which satisfies (2.5) of Theorem 2.1.

Hence, by (2.6), we have

$$M_2(h) = M_{21}(h) - M_{22}(h) - M_{23}(h),$$

where

$$M_{21}(h) = \int_{L_0(h)} q_2 dx - p_2 dy,$$

$$= \int_{L_0(h)} \left[ \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^{j+6} - \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} b_{ij}^{(2)} x^i y^j \right] dx,$$

$$- \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+6} - \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} a_{ij}^{(2)} x^i y^j dy,$$

$$M_{22}(h) = \frac{1}{2} \int_{L_0(h)} \left[ \frac{1}{5}y^5 p^* \right]_x + \left( \frac{1}{5}y^5 q^* \right) dy,$$

$$M_{23}(h) = \int_{L_0(h)} (f_1 + f_2) dt.$$
where

$$
\begin{align*}
   f_1 &= \frac{1}{5} y^5 [(p_1)_x + (q_1)_y] \\
   &= \frac{1}{5} y^5 \left( - \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^j x \right) \\
   &\quad + \left( - \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^{j+3} + \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^j y \right), \\
\end{align*}
$$

\begin{align}
   \text{(3.30)}
\end{align}

$$
\begin{align*}
   f_2 &= (-\frac{1}{4} y^8) [(p_0)_x + (q_0)_y] \\
   &= (-\frac{1}{4} y^8) \left( \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^j x + \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^j y \right). \\
\end{align*}
$$

\begin{align}
   \text{(3.31)}
\end{align}

For (3.27), one achieves that

$$
\begin{align*}
   M_{21}(h) &= \oint_{L_0(h)} q_2 dx - p_2 dy \\
   &= \oint_{L_0(h)} \sum_{0 \leq i+j \leq 4} b_{ij}^{(0)} x^i y^{j+6} dx - \sum_{0 \leq i+j \leq 4} a_{ij}^{(0)} x^i y^{j+6} dy \\
   &\quad + \oint_{L_0(h)} \left( - \sum_{0 \leq i+j \leq 4} b_{ij}^{(1)} x^i y^{j+3} dx + \sum_{0 \leq i+j \leq 4} a_{ij}^{(1)} x^i y^{j+3} dy \right) \\
   &\quad + \oint_{L_0(h)} \left( \sum_{0 \leq i+j \leq 4} b_{ij}^{(2)} x^i y^j dx - \sum_{0 \leq i+j \leq 4} a_{ij}^{(2)} x^i y^j dy \right). \\
\end{align*}
$$

\begin{align}
   \text{(3.32)}
\end{align}

Then, by making use of polar coordinate transformation (3.19), combining $M_1(h)$ in (3.27) and the above equation, we can obtain

$$
\begin{align*}
   M_{21}(h) &= \sum_{0 \leq i+j \leq 10} F_{ij} I_{ij}(h), \\
\end{align*}
$$

\begin{align}
   \text{(3.32)}
\end{align}

where

$$
\begin{align*}
   F_{00} &= b_{00}^{(2)}, F_{11} = a_{11}^{(2)} + e_{11}^{(2)}, F_{02} = b_{02}^{(2)} + \frac{1}{2} a_{11}^{(2)}, F_{10} = b_{03}^{(2)} + \frac{1}{3} a_{12}^{(2)} - b_{00}^{(1)}, \\
   F_{04} &= b_{04}^{(2)} + \frac{1}{4} a_{12}^{(2)} - \frac{3}{4} b_{01}^{(1)} - b_{00}^{(1)}, F_{05} = b_{05}^{(2)} + \frac{1}{3} b_{01}^{(1)} - \frac{1}{6} b_{00}^{(1)}, F_{06} = b_{06}^{(2)} + \frac{3}{10} b_{01}^{(1)} - \frac{1}{6} b_{00}^{(1)}, \\
   F_{07} &= \frac{1}{7} b_{07}^{(2)} + \frac{1}{3} b_{01}^{(1)} + b_{04}^{(2)} - b_{00}^{(1)}, F_{08} = \frac{1}{9} b_{02}^{(2)} + \frac{1}{5} b_{01}^{(1)}, F_{09} = b_{03}^{(2)} + \frac{1}{9} b_{01}^{(1)}, \\
   F_{010} &= b_{04}^{(2)} + \frac{1}{10} a_{10}^{(2)}, F_{10} = a_{10}^{(2)}, F_{11} = b_{11}^{(2)} + 2 a_{20}^{(2)}, F_{12} = b_{12}^{(2)} + a_{21}^{(2)}, \\
   F_{13} &= b_{13}^{(2)} + \frac{2}{3} a_{22}^{(2)} - b_{01}^{(1)}, F_{14} = -b_{14}^{(1)} - \frac{1}{2} b_{01}^{(1)} + \frac{3}{5} b_{11}^{(1)} + \frac{1}{3} b_{01}^{(1)} - \frac{3}{4} a_{22}^{(1)} - b_{13}^{(1)}, \\
   F_{17} &= b_{17}^{(2)} + \frac{2}{5} b_{20}^{(2)} + F_{18} = b_{18}^{(2)} + \frac{1}{2} a_{21}^{(2)}, F_{19} = b_{19}^{(2)} + \frac{3}{5} b_{22}^{(2)} + F_{20} = b_{20}^{(2)} + F_{21} = b_{21}^{(2)} + 3 a_{30}^{(2)}, \\
   F_{22} &= a_{22}^{(2)} + \frac{3}{7} a_{31}^{(2)}, F_{23} = -b_{23}^{(1)} - \frac{3}{4} a_{30}^{(1)}, F_{24} = b_{24}^{(1)} + \frac{3}{7} a_{31}^{(1)}, F_{25} = -b_{25}^{(2)} - \frac{3}{5} a_{32}^{(1)}, F_{26} = b_{26}^{(2)}, \\
   F_{27} &= b_{27}^{(2)} + \frac{3}{7} a_{30}^{(2)}, F_{28} = b_{28}^{(2)} + \frac{3}{8} a_{31}^{(2)}, F_{30} = b_{30}^{(2)} + b_{27}^{(2)} + 4 a_{40}^{(2)}, F_{32} = 0, F_{33} = -b_{33}^{(1)}, \\
   F_{34} &= -b_{34}^{(2)} - \frac{3}{7} a_{40}^{(2)} + F_{35} = 0, F_{36} = b_{36}^{(2)}, F_{37} = b_{37}^{(2)} + \frac{4}{7} a_{40}^{(2)}, F_{40} = b_{40}^{(2)}, F_{41} = F_{42} = 0, \\
   F_{43} &= -b_{43}^{(2)} + F_{44} = F_{45} = 0, F_{46} = b_{46}^{(0)}.
\end{align*}
$$
and $I_{ij}(h)$ is given in (3.14). Computing the integral values, it follows that

$$
I_{09}(h) = -\frac{63}{4} \pi h^5, I_{023}(h) = -\pi h^3, I_{25}(h) = -\frac{5}{4} \pi h^4, I_{27}(h) = -\frac{7}{4} \pi h^5,
$$

$$
I_{08}(h) = I_{0,10}(h) = I_{17}(h) = I_{18}(h) = I_{19}(h) = I_{26}(h) = 0,
$$

$$
I_{28}(h) = I_{36}(h) = I_{37}(h) = I_{46}(h) = 0. \tag{3.33}
$$

Under (3.32), by (3.15), (3.21) and (3.33), we obtain

$$
M_{21}(h) = -2\pi (b_{01}^{(2)} + a_{12}^{(2)}) h + \pi (3b_{03}^{(1)} - 3b_{03}^{(2)} - a_{12}^{(2)} - b_{21}^{(2)} - 3a_{30}^{(2)}) h^2 + (b_{20}^{(1)} + 5b_{02}^{(1)} + a_{11}^{(1)}) h^3 + \pi(\frac{35}{4}b_{04}^{(1)} + \frac{5}{4}b_{22}^{(1)} + \frac{5}{4}b_{13}^{(1)} + \frac{3}{4}b_{40}^{(1)} + \frac{3}{4}b_{31}^{(1)} - \frac{35}{4}b_{01}^{(0)}) h^5 - \frac{5}{4}a_{10}^{(0)} h^4 + \pi(-\frac{63}{4}b_{03}^{(0)} - \frac{7}{4}b_{21}^{(0)} - \frac{7}{4}a_{12}^{(0)} - \frac{3}{4}a_{30}^{(0)}) h^5. \tag{3.34}
$$

As before, by using polar coordinate transformation (3.19), (3.28) becomes

$$
M_{22}(h) = \sum_{7 \leq j \leq 10} G_{0j} I_{0j}(h) + \sum_{7 \leq j \leq 9} G_{1j} I_{1j}(h),
$$

where

$$
G_{07} = -\frac{9}{50} (a_{10}^{(0)} + b_{01}^{(0)}), G_{08} = \frac{1}{5} (a_{11}^{(0)} + 2b_{02}^{(0)}),
$$

$$
G_{09} = -\frac{1}{50} (11a_{12}^{(0)} + 33b_{03}^{(0)} + 3a_{30}^{(0)} + b_{21}^{(0)}),
$$

$$
G_{010} = -\frac{1}{50} (12a_{13}^{(0)} + 48b_{04}^{(0)} + 3a_{31}^{(0)} + 2b_{22}^{(0)}),
$$

$$
G_{17} = G_{18} = 0, G_{19} = -\frac{1}{25} (4a_{40}^{(0)} + b_{31}^{(0)}),
$$

and $I_{ij}(h)$ is defined by (3.14). Further, by (3.21), (3.33) and the above equations, we have

$$
M_{22}(h) = -\frac{63}{4} (a_{10}^{(0)} + b_{01}^{(0)}) \pi h^4 - (\frac{693}{200}a_{12}^{(0)} + \frac{2079}{200}b_{03}^{(0)} + \frac{189}{200}a_{30}^{(0)} + \frac{63}{200}b_{21}^{(0)}) \pi h^5. \tag{3.35}
$$

By (3.30) and (3.31), it follows that

$$
f_1 = \frac{1}{5} \left( \sum_{0 \leq i+j \leq 4} ia_{ij}^{(0)} x^{i-1} y^{j+8} + \sum_{0 \leq i+j \leq 4} ia_{ij}^{(1)} x^{i-1} y^{j+5} - \sum_{0 \leq i+j \leq 4} (j+3) b_{ij}^{(0)} x^i y^{j+7} \right)
$$

$$
+ \sum_{0 \leq i+j \leq 4} j b_{ij}^{(1)} x^i y^{j+4}, \tag{3.36}
$$

$$
f_2 = -\frac{1}{4} \left( \sum_{0 \leq i+j \leq 4} ia_{ij}^{(0)} x^{i-1} y^{j+8} + \sum_{0 \leq i+j \leq 4} j b_{ij}^{(0)} x^i y^{j+7} \right).
$$

Combining (3.29), (3.36), and by using polar coordinate transformation (3.19), $M_{23}(h)$ becomes

$$
M_{23}(h) = \sum_{0 \leq i+j \leq 10} R_{ij} I_{ij}(h), \tag{3.37}
$$
Proof of Theorem 1.2

I and linearly independent. For convenience, let \( h > 0 \) and it has at most four zeros in \( H \). Hence, by (3.26), (3.34), (3.35) and (3.38), one achieves finally that

\[
M_2 = h(B_0 + B_1 h + B_2 h^2 + B_3 h^3 + B_4 h^4),
\]

where \( B_0, B_1, B_2, B_3, B_4 \) satisfy (3.25). This finishes the proof. \( \square \)

Now we present the proof of Theorem 1.2.

**Proof of Theorem 1.2**

Obviously, under the conditions (3.10) and (3.17), \( M_2(h) \neq 0 \) and it has at most four zeros in \( h > 0 \). Next, we prove that \( B_0, B_1, B_2, B_3, B_4 \) are linearly independent. For convenience, let

\[
\begin{align*}
a^{(2)}_{10} &= a^{(2)}_{12} = b^{(1)}_{00} = b^{(2)}_{21} = a^{(2)}_{30} = b^{(1)}_{20} = a^{(1)}_{13} = b^{(0)}_{01} = b^{(1)}_{04} \\
&= b^{(1)}_{22} = b^{(1)}_{40} = a^{(0)}_{12} = b^{(0)}_{21} = a^{(0)}_{30} = 0.
\end{align*}
\]

Then (3.25) becomes

\[
\begin{align*}
B_0 &= -2\pi b^{(2)}_{01}, \\
B_1 &= -3\pi b^{(2)}_{03}, \\
B_2 &= 3\pi b^{(1)}_{02}, \\
B_3 &= -341 \pi a^{(0)}_{10}, \\
B_4 &= -1827 \pi b^{(0)}_{03}.
\end{align*}
\]
Therefore,
\[
\det \frac{\partial (B_0, B_1, B_2, B_3, B_4)}{\partial (\ell^{(2)}_{01}, \ell^{(2)}_{03}, \ell^{(1)}_{02}, a^{(0)}_{10}, b^{(0)}_{03})} = \frac{5607063}{16000} \pi^5.
\]
Hence, we can choose proper values of \( \ell^{(2)}_{01}, \ell^{(2)}_{03}, \ell^{(1)}_{02}, a^{(0)}_{10}, b^{(0)}_{03} \) such that
\[
0 < B_0 \ll -B_1 \ll B_2 \ll -B_3 < B_4,
\]
which ensures that \( M_2(h) \) has four simple zeros in \( h > 0 \) near \( h = 0 \). Thus, \( M(h, \lambda) \) can have four zeros in \( h > 0 \) near \( h = 0 \) as \( \lambda \) is small. Then, the displacement function (3.6) of Eq.(3.2) has also four zeros in \( h > 0 \) near \( h = 0 \) for \( 0 < \varepsilon \ll \lambda \ll 1 \). Hence, system (3.2) has four limit cycles near the origin for \( 0 < \varepsilon \ll \lambda \ll 1 \). This means that system (1.9) has precisely four limit cycles near the origin for \( 0 < \varepsilon \ll \lambda \ll 1 \). Also, since \( M_2(h) \) has at most four zeros in \( h > 0 \) as it is not zero identically, system (1.9) has at most four limit cycles in the finite plane for \( 0 < \varepsilon \ll \lambda \ll 1 \) if \( M_2(h) \neq 0 \). The conclusion is proved.

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