COUNTING NUMERICAL SEMIGROUPS BY GENUS AND EVEN GAPS VIA KUNZ-COORDINATE VECTORS

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Abstract. We construct a one-to-one correspondence between a subset of numerical semigroups with genus \( g \) and \( \gamma \) even gaps and the integer points of a rational polytope. In particular, we give an overview to apply this correspondence to try to decide if the sequence \((n_g)\) is increasing, where \( n_g \) denotes the number of numerical semigroups with genus \( g \).

1. Introduction

A numerical semigroup \( S \) is a subset of \( \mathbb{N}_0 \) such that \( 0 \in S \), it is closed under addition and the set \( G(S) := \mathbb{N}_0 \setminus S \), the set of gaps of \( S \), is finite. The number of elements \( g = g(S) \) of \( G(S) \) is called the genus of \( S \) and the first non-zero element in \( S \) is called the multiplicity of \( S \). If \( S \) is a numerical semigroup with genus \( g \) then one can ensure that all gaps of \( S \) belongs to \([1, 2g] \); in particular, \( \{2g + i : i \in \mathbb{N}_0\} \subseteq S \) and the number of numerical semigroups with genus \( g \), denoted by \( n_g \), is finite. Some excellent references for the background on numerical semigroups are the books [5] and [7].

Throughout this paper, we keep the notation proposed by Bernardini and Torres [1]: the set of numerical semigroups with genus \( g \) is denoted by \( S_g \) and has \( n_g \) elements and the the set of numerical semigroups with genus \( g \) and \( \gamma \) even gaps is denoted by \( S_{\gamma}(g) \) and has \( N_{\gamma}(g) \) elements.

In this paper we use the quite useful parametrization

\[
\mathbf{x}_g : S_{\gamma}(g) \rightarrow S_{\gamma}, S \mapsto S/2,
\]

where \( S/2 := \{s \in \mathbb{N}_0 : 2s \in S\} \).

Naturally, the set \( S_{\gamma}(g) \) and the map \( \mathbf{x}_g \) can be generalized. Let \( d > 1 \) be an integer. The set of numerical semigroups with genus \( g \) and \( \gamma \) gaps which are congruent to 0 modulo \( d \) is denoted by \( S_{(d, \gamma)}(g) \). There is a natural parametrization given by

\[
\mathbf{x}_{gd} : S_{(d, \gamma)}(g) \rightarrow S_{\gamma}, S \mapsto S/d,
\]

where \( S/d := \{s \in \mathbb{N}_0 : ds \in S\} \). This concept appears in [8], for instance.

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In this paper, we obtain a one-to-one correspondence between the set $x_g^{-1}(T)$ and the integer points of a rational polytope.

As an application of this correspondence, we give a new approach to compute the numbers $N_\gamma(g)$. Our main goal is finding a new direction to discuss the following question.

\[(1.2) \quad \text{Is it true that } n_g \leq n_{g+1}, \text{ for all } g?\]

The first few elements of the sequence $(n_g)$ are 1, 1, 2, 4, 7, 12, 23, 39, 67. Kaplan [6] wrote a nice survey on this problem and one can find information of these numbers in Sloane’s On-line Encyclopedia of Integer Sequences [10].

Bras-Amorós [3] conjectured remarkable properties on the behaviour of the sequence $(n_g)$:

1. $\lim_{g \to \infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1$;
2. $\lim_{g \to \infty} \frac{n_{g+1}}{n_g} = \varphi := \frac{1+\sqrt{5}}{2}$;
3. $n_{g+2} \geq n_{g+1} + n_g$ for any $g$.

Zhai [12] proved that $\lim_{g \to \infty} n_g \varphi^{-g}$ is a constant. As a consequence, it confirms that items (1) and (2) hold true. However, item (3) is still an open problem; even a weaker version, proposed at (1.2), is an open question. Zhai’s result also ensures that $n_g < n_{g+1}$ for large enough $g$. Fromentin and Hivert [4] verified that $n_g < n_{g+1}$ also holds true for $g \leq 67$.

Torres [11] proved that $S_\gamma(g) \neq \emptyset$ if, and only if, $2g \geq 3\gamma$. Hence,

\[(1.3) \quad n_g = \sum_{\gamma=0}^{\lfloor 2g/3 \rfloor} N_\gamma(g).\]

In order to work on Question (1.2), Bernardini and Torres [1] tried to understand the effect of the even gaps on a numerical semigroup. By using the so-called $t$-translation, they proved that $N_\gamma(g) = N_\gamma(3\gamma)$ for $g \geq 3\gamma$ and also $N_\gamma(g) < N_\gamma(3\gamma)$ for $g < 3\gamma$. Although numerical evidence points out that $N_\gamma(g) \leq N_\gamma(g+1)$ holds true for all $g$ and $\gamma$, their methods could not compare numbers $N_\gamma(g_1)$ and $N_\gamma(g_2)$, with $3\gamma/2 \leq g_1 < g_2 < 3\gamma$. Notice that if $N_\gamma(g_1) \leq N_\gamma(g_2)$, for $3\gamma/2 \leq g_1 < g_2 < 3\gamma$ then $n_g < n_{g+1}$, for all $g$.

2. Apéry set and Kunz-coordinate vector

Let $S$ be a numerical semigroup and $n \in S$. The Apéry set of $S$ (with respect to $n$) is the set $Ap(S, n) = \{s \in S : s - n \notin S\}$. If $n = 1$, then $S = \mathbb{N}_0$ and $Ap(\mathbb{N}_0, 1) = \{0\}$. If $n > 1$, then a there are $w_1, \ldots, w_{n-1} \in \mathbb{N}$ such that $Ap(S, n) = \{0, w_1, \ldots, w_{n-1}\}$, where $w_i = \min\{s \in S : s \equiv i \pmod{n}\}$. 
Proposition 2.1. Let $S$ be a numerical semigroup with multiplicity $m$ and $Ap(S, m) = \{0, w_1, \ldots, w_{m-1}\}$. Then

$$S = \langle m, w_1, w_2, \ldots, w_{m-1} \rangle.$$

Proof. It is clear that $m \in \langle m, w_1, w_2, \ldots, w_{m-1} \rangle$. For $s \in S$, $s \neq m$, there is $i \in \{1, \ldots, m - 1\}$ such that $s = mk + i$. By minimality of $w_i$, there is $k \in \mathbb{N}_0$ such that $s = w_i + km \in \langle m, w_1, w_2, \ldots, w_{m-1} \rangle$. On the other hand, $m, w_1, \ldots, w_{m-1} \in S$. □

Let $S$ be a numerical semigroup, $n \in S$ and consider $Ap(S, n) = \{0, w_1, \ldots, w_{n-1}\}$. There are $e_1, \ldots, e_{n-1} \in \mathbb{N}$ such that $w_i = ne_i + i$, for each $i \in \{1, \ldots, n - 1\}$. The vector $(e_1, \ldots, e_{n-1}) \in \mathbb{N}_{n-1}$ is called the Kunz-coordinate vector of $S$ (with respect to $n$). In particular, if $m$ is the multiplicity of $S$, then the Kunz-coordinate vector of $S$ (with respect to $m$) is in $\mathbb{N}^{m-1}$. This concept appears in [2], for instance.

A natural task is finding conditions for a vector $(x_1, \ldots, x_{m-1}) \in \mathbb{N}^{m-1}$ to be a Kunz-coordinate vector (with respect to the multiplicity $m$ of $S$) of some numerical semigroup $S$ with multiplicity $m$. The following examples illustrate the general method, which is presented in Proposition 2.4.

Example 2.2. Numerical semigroups with multiplicity 2 are $\langle 2, 2e_1 + 1 \rangle$, where $e_1 \in \mathbb{N}$.

There is a one-to-one correspondence between the set of numerical semigroups with multiplicity 2 and the set of positive integers given by $\langle 2, 2e_1 + 1 \rangle \mapsto e_1$.

Example 2.3. Let $S = \langle 3, 3e_1 + 1, 3e_2 + 2 \rangle$ be a numerical semigroup with multiplicity 3 and genus $g$, where $e_1, e_2 \in \mathbb{N}$. By minimality of $w_1 = 3e_1 + 1$ and $w_2 = 3e_2 + 2$, $(e_1, e_2)$ satisfies

$$\begin{cases} (3e_1 + 1) + (3e_1 + 1) \geq 3e_2 + 2 \\ (3e_2 + 2) + (3e_2 + 2) \geq 3e_1 + 1. \end{cases}$$

The set of gaps of $S$ has $e_1 + e_2$ elements, since $G(S) = \{3n_1 + 1 : 0 \leq n_1 < e_1\} \cup \{3n_2 + 2 : 0 \leq n_2 < e_2\}$. Thus, $e_1 + e_2 = g$. On the other hand, if $(e_1, e_2) \in \mathbb{N}^2$ is such that $2e_1 \geq e_2$, $2e_2 + 1 \geq e_1$ and $e_1 + e_2 = g$, then $\langle 3, 3e_1 + 1, 3e_2 + 2 \rangle$ is a numerical semigroup with multiplicity $m$ and genus $g$.

Hence, there is a one-to-one correspondence between the set of numerical semigroups with multiplicity 3 and the vectors of $\mathbb{N}^2$ which are solutions of

$$\begin{cases} 2X_1 \geq X_2 \\ 2X_2 + 1 \geq X_1 \\ X_1 + X_2 = g. \end{cases}$$
In order to give a characterization of numerical semigroups with fixed multiplicity and fixed genus, the main idea is generalizing Example 2.3. The following is a result due to Rosales et al. [8].

**Proposition 2.4.** There is a one-to-one correspondence between the set of numerical semigroups with multiplicity \( m \) and genus \( g \) and the positive integer solutions of the system of inequalities

\[
\begin{align*}
X_i + X_j & \geq X_{i+j}, \quad \text{for } 1 \leq i \leq j \leq m - 1; i + j < m; \\
X_i + X_j + 1 & \geq X_{i+j-m}, \quad \text{for } 1 \leq i \leq j \leq m - 1; i + j > m \\
\sum_{k=1}^{m-1} X_k & = g.
\end{align*}
\]

Let \( S = (m, w_1, \ldots, w_{m-1}) \) be a numerical semigroup with multiplicity \( m \) and genus \( g \), where \( w_i = me_i + i \). The main idea of the proof is using the minimality of \( w_1, \ldots, w_{m-1} \) and observing that \( w_i + w_j \equiv i + j \pmod{m} \) and \( G(S) = \bigcup_{i=1}^{m-1} \{ mn_i + i : 0 \leq n_i < e_i \} \). For a full proof, see [8].

#### 3. The main result and an application to a counting problem

In [1], the calculation of \( N_{g}(g) \) was given by

\[(3.1) \quad N_{g}(g) = \sum_{T \in S_{g}} \#x^{-1}_{g}(T).\]

In this section, we present a new way for computing those numbers. In order to do this, we fix the multiplicity of \( T \in S_{g} \).

First of all, we obtain a relation between the genus and the multiplicity of a numerical semigroup.

**Proposition 3.1.** Let \( S \) be a numerical semigroup with genus \( g \) and multiplicity \( m \). Then \( m \leq g + 1 \).

**Proof.** If a numerical semigroup \( S \) has multiplicity \( m \) and genus \( g \) with \( m \geq g + 2 \), then the number of gaps of \( S \) would be, at least, \( g + 1 \) and it is a contradiction. Hence \( m \leq g + 1 \).

\[\square\]

**Remark 3.2.** The bound obtained in Proposition 3.1 is sharp, since \( \{0, g + 1, \ldots\} \) has genus \( g \) has multiplicity \( g + 1 \).
If $\gamma = 0$, then $S_0 = \{N_0\}$ and $x_0^{-1}(N_0) = \{(2, 2g + 1)\}$. Hence, $N_0(g) = 1$, for all $g$. If $\gamma > 0$, we divide the set $S_\gamma$ into the subsets $S_\gamma^m := \{S : g(S) = \gamma \text{ and } m(S) = m\}$, where $m \in [2, \gamma + 1] \cap \mathbb{Z}$. We can write

$$S_\gamma = \bigcup_{m=2}^{\gamma+1} S_\gamma^m.$$ (3.2)

Putting (3.1) and (3.2) together, we obtain

$$N_\gamma(g) = \sum_{m=2}^{\gamma+1} \sum_{T \in S_\gamma^m} \#x_g^{-1}(T).$$

Thus, it is important to give a characterization for $T \in S_\gamma^m$. We can describe $T$ by its Apéry set (with respect to its multiplicity $m$) and write

$$T = \langle m, me_1 + 1, me_2 + 2, \ldots, me_{m-1} + (m - 1) \rangle,$$

where $me_i + i = \min\{s \in S : s \equiv i \pmod{m}\}$.

The next result characterizes all numerical semigroups of $x_g^{-1}(T)$, for $T \in S_\gamma^m$. It is a consequence of Proposition 2.4.

**Theorem 3.3.** Let $T = \langle m, me_1 + 1, \ldots, me_{m-1} + (m - 1) \rangle \in S_\gamma^m$. A numerical semigroup $S$ belongs to $x_g^{-1}(T)$ if, and only if,

$$S = \langle 2m, 2me_1 + 2, \ldots, 2me_{m-1} + (2m - 2), 2mk_1 + 1, 2mk_3 + 3, \ldots, 2mk_{2m-1} + (2m - 1) \rangle,$$

where $(k_1, k_3, \ldots, k_{2m-1}) \in \mathbb{N}_0^m$ satisfies the system

$$
\begin{align*}
\{ \text{(*)} \} & \begin{cases} X_{2i-1} + e_j \geq X_{2(i+j)-1}, & \text{for } 1 \leq i \leq m; 1 \leq j \leq m - 1; i + j \leq m; \\
X_{2i-1} + e_j + 1 \geq X_{2(i+j-m)-1}, & \text{for } 1 \leq i \leq m; 1 \leq j \leq m - 1; i + j > m; \\
* & * \end{cases} \\
\{ \text{(**)} \} & \begin{cases} X_{2i-1} + X_{2j-1} \geq e_{i+j-1}, & \text{for } 1 \leq i \leq j \leq m; i + j \leq m; \\
X_{2i-1} + X_{2j-1} + 1 \geq e_{i+j-1-m}, & \text{for } 1 \leq i \leq j \leq m; i + j \geq m + 2; \\
\sum_{i=1}^m X_{2i-1} = g - \gamma, & \end{cases}
\end{align*}
$$

**Proof.** The even numbers $2m, 2me_1 + 2, \ldots, 2me_{m-1} + 2me_{m-1} + 2(m - 1)$ belongs to $Ap(2m, S)$. Let $2mk_1 + 1, 2mk_3 + 3, \ldots, 2mk_{2m-1} + (2m - 1)$ be the odd numbers of $Ap(2m, S)$. Thus, $(e_1, k_1, e_2, k_3, \ldots, e_{m-1}, k_{2m-1}) \in \mathbb{N}_0^{m-1}$ is the Kunz-coordinate vector of $S$ (with respect to $2m$).

Now, we apply Proposition 2.4. Inequalities given in (*)& come from sums of an odd element of $Ap(2m, S)$ with an even element of $Ap(2m, S)$, while inequalities given in (**)) come from sums of two odd elements of $Ap(2m, S)$. Since $(e_1, \ldots, e_{m-1})$ is the Kunz-coordinate.
vector of $T$ (with respect to $m$), then the sum of two even elements of $Ap(2m, S)$ belongs to $S$. Finally, last equality comes from the fact that $S$ has $g - \gamma$ odd gaps. \hfill \Box

**Remark 3.4.** Some of the numbers $k_i$ can be zero. Hence, it is possible that the multiplicity of $S$ is not $2m$.

**Example 3.5.** Let $T = \langle 2, 2\gamma + 1 \rangle \in S_\gamma^2$, with $\gamma \in \mathbb{N}$. Theorem 3.3 ensures that if $S \in x_g^{-1}(T)$, then

$$S = \langle 4, 4\gamma + 2, 4k_1 + 1, 4k_3 + 3 \rangle,$$

where $(k_1, k_3) \in \mathbb{N}_0^2$ satisties

$$(\#) \begin{cases} (\ast) \{-\gamma - 1 \leq X_3 - X_1 \leq \gamma \\
(\ast\ast) \{ X_1 + X_1 \geq \gamma \\
X_3 + X_3 + 1 \geq \gamma \\
X_1 + X_3 = g - \gamma. \end{cases}$$

The region in $\mathbb{R}^2$ given by inequalities $(\ast)$ and $(\ast\ast)$ is the following hatched figure:

The set of integer points of that region is in one-to-one correspondence with the set \{ $S \in S_\gamma(g) : S/2$ has multiplicity 2 \}.

If $g$ is fixed, then the set of points that satisfies the system $(\#)$ is a polytope (a line segment). We are interested in the set of integer points of this polytope. The following figure shows examples for some values of $g$. Each integer point represents a numerical semigroup of the set \{ $S \in S_\gamma(g) : S/2$ has multiplicity 2 \}. 

![Graphical representation of the region and its correspondence with the set of semigroups.](image-url)
Let \( N_\gamma^m(g) = \sum_{T \in S_\gamma^m} \# x_g^{-1}(T) \). After some computations, we obtain

\[
N_\gamma^2(g) = \begin{cases} 
0, & \text{if } g < 2\gamma \\
 k + 1, & \text{if } g = 2\gamma + k \text{ and } k \in \{0, 1, \ldots, \gamma - 1\} \\
\gamma + 1, & \text{if } g \geq 3\gamma.
\end{cases}
\]

In particular, \( N_\gamma^2(g) \leq N_\gamma^2(g + 1) \). We leave the following open question.

(3.3) Let \( \gamma \in \mathbb{N} \) and \( m \in [2, \gamma + 1] \cap \mathbb{Z} \). Is it true that \( N_\gamma^m(g) \leq N_\gamma^m(g + 1) \), for all \( g \)?

A positive answer to Question (3.3) implies a positive answer to Question (1.2).

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