An approach to a fuzzy problem using the fuzzy Laplace transform under the generalized differentiability

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Abstract
In this paper, the solutions of a second-order fuzzy initial value problem are studied by the fuzzy Laplace transform under the generalized differentiability. An example is solved. Finally, conclusions are given.

Keywords
Second-order fuzzy differential equation, Fuzzy initial value problem, Fuzzy Laplace transform.

AMS Subject Classification
03E72, 34A07, 44A10

1 Introduction

There are many approaches to define fuzzy derivative. The first one is Hukuhara derivative [1–3]. But Hukuhara derivative has a drawback. Solution becomes fuzzier as time goes. Thus, fuzzy solution behaves differently from the crisp solution. The second one is generalized Hukuhara derivative [4–8]. The third one is Zadeh’s extension principle [9]. Another approach is differential inclusion [10].

Fuzzy Laplace transform is useful to solve fuzzy differential equation. It is found the solution of the fuzzy differential equation satisfying the initial condition by fuzzy Laplace transform directly. Allahviranloo and Barkhordari Ahmadi first introduced fuzzy Laplace transform [11]. In many papers, solution of fuzzy differential equation was studied by fuzzy Laplace transform [12–14].

In this paper, we investigate the solutions of the problem

\[ \beta u''(t) + \delta u'(t) = [0]^\alpha, \ t > 0 \]  

by the fuzzy Laplace transform under the concept of generalized differentiability, where \( \beta, \delta > 0, [0]^\alpha = [-1 + \alpha, 1 - \alpha] \), \( Q \), and \( W \) are symmetric triangular fuzzy numbers with supports \( [q_\alpha, \bar{q}_\alpha], [w_\alpha, \bar{w}_\alpha] \) and the \( \alpha \)-level sets of \( Q, W \) are

\[ Q^\alpha = [Q_\alpha, \bar{Q}_\alpha] = \left[q + \left(\frac{q-q}{2}\right)\alpha, q - \left(\frac{q-q}{2}\right)\alpha\right], \]  

\[ W^\alpha = [W_\alpha, \bar{W}_\alpha] = \left[w + \left(\frac{w-w}{2}\right)\alpha, w - \left(\frac{w-w}{2}\right)\alpha\right]. \]

This paper is organized as in section 2 Preliminaries, in section 3 Main Results, in section 4 Conclusion.

2. Preliminaries

Definition 2.1. [15] A fuzzy number is a mapping \( u : \mathbb{R} \rightarrow [0, 1] \) satisfying the properties \( \{x \in \mathbb{R} \mid u(x) > 0\} \) is compact, \( u \) is normal, \( u \) is convex fuzzy set, \( u \) is upper semi-continuous on \( \mathbb{R} \).

Let \( \mathbb{R}_F \) show the set of all fuzzy numbers.

Definition 2.2. [4] Let be \( u \in \mathbb{R}_F \). The \( \alpha \)-level set of \( u \) is

\[ [u]^\alpha = [u_\alpha, \bar{u}_\alpha] = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}, 0 < \alpha \leq 1. \]
If $\alpha = 0$, 

$$|u|^0 = cl \{ supp u \} = cl \{ x \in \mathbb{R} \mid u(x) > 0 \}.$$ 

Remark 2.3. [4] The parametric form $[u_{\alpha}, \pi_{\alpha}]$ of a fuzzy number satisfying the following requirements is a valid $\alpha$-level set.

- $u_{\alpha}$ is left-continuous monotonic increasing (non-decreasing) bounded on $[0, 1]$.
- $\pi_{\alpha}$ is left-continuous monotonic decreasing (non-increasing) bounded on $[0, 1]$.
- $u_{\alpha}$ and $\pi_{\alpha}$ are right-continuous for $\alpha = 0$.
- $u_{\alpha} \leq \pi_{\alpha}$, $0 \leq \alpha \leq 1$.

Definition 2.4. [15] The $\alpha$-level set of symmetric triangular fuzzy number $Q$ with support $[q, \bar{q}]$ is 

$$[Q]^\alpha = [Q_{\alpha}, \bar{Q}_{\alpha}] = \left[ q + \left( \frac{\bar{q} - q}{2} \right) \alpha, \bar{q} - \left( \frac{\bar{q} - q}{2} \right) \alpha \right].$$

Definition 2.5. [4] Let be $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, $u + v$ and $\lambda u$ are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0, 1]$. $[u]^\alpha + [v]^\alpha$ and $\lambda [u]^\alpha$ mean the usual addition of two intervals (subsets) of $\mathbb{R}$ and the usual product between a scalar and a subset of $\mathbb{R}$, respectively.

Definition 2.6. [4, 16] Let be $u, v \in \mathbb{R}_F$. If $u = v + w$ such that there exists $w \in \mathbb{R}_F$, $w$ is the Hukuhara difference of $u$ and $v$, it is denoted as $w = u - v$.

Definition 2.7. [4, 17] Let be $f : [a, b] \to \mathbb{R}_F$ and $x_0 \in [a, b]$. If there exists $f^\prime (x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0) , f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^\prime (x_0),$$

$f$ is Hukuhara differentiable at $x_0$.

Definition 2.8. [4] Let be $f : [a, b] \to \mathbb{R}_F$ and $x_0 \in [a, b]$. If there exists $f^\prime (x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0) , f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^\prime (x_0),$$

$f$ is (1)-differentiable at $x_0$. If there exists $f^\prime (x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0 + h) , f(x_0 + h) \ominus f(x_0)$ and the limits hold

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^\prime (x_0),$$

$f$ is (2)-differentiable.

Theorem 2.9. [7] Let $f : [a, b] \to \mathbb{R}_F$ be fuzzy function and denote $[f(x)]^\alpha = \left[ f_{\alpha}(x), \overline{f}_{\alpha}(x) \right]$, for each $\alpha \in [0, 1]$.

(i) If the function $f$ is (1)-differentiable, the lower function $f_{\alpha}$ and the upper function $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ f_{\alpha}^\prime(x), \overline{f}_{\alpha}^\prime(x) \right]$.

(ii) If the function $f$ is (2)-differentiable, the lower function $f_{\alpha}$ and the upper function $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ \overline{f}_{\alpha}^\prime(x), \overline{f}_{\alpha}^\prime(x) \right]$.

Theorem 2.10. [7] Let $f^\prime : [a, b] \to \mathbb{R}_F$ be fuzzy function and denote $\left[ f(x) \right]^\alpha = \left[ f_{\alpha}(x), \overline{f}_{\alpha}(x) \right]$, for each $\alpha \in [0, 1]$, the function $f$ is (1)-differentiable or (2)-differentiable.

(i) If $f$ and $f^\prime$ are (1)-differentiable, $f_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ f_{\alpha}^\prime(x), \overline{f}_{\alpha}^\prime(x) \right]$.

(ii) If $f$ is (1)-differentiable and $f^\prime$ is (2)-differentiable, $f_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ \overline{f}_{\alpha}^\prime(x), f_{\alpha}^\prime(x) \right]$.

(iii) If $f$ is (2)-differentiable and $f^\prime$ is (1)-differentiable, $f_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ f_{\alpha}^\prime(x), \overline{f}_{\alpha}^\prime(x) \right]$.

(iv) If $f$ and $f^\prime$ are (2)-differentiable, $f_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable, $\left[ f^\prime(x) \right]^\alpha = \left[ \overline{f}_{\alpha}^\prime(x), \overline{f}_{\alpha}^\prime(x) \right]$.

Definition 2.11. [12, 13] Let $f : [a, b] \to \mathbb{R}_F$ be fuzzy function. The fuzzy Laplace transform of $f$ is

$$F(s) = L(f(t)) = \int_{0}^{\infty} e^{-st} f(t) dt = \left[ \lim_{t \to \infty} \int_{0}^{t} e^{-st} f(t) dt, \lim_{t \to 0} \int_{t}^{\infty} e^{-st} f(t) dt \right].$$

$$F(s, \alpha) = L \left( \left[ f(t) \right]^\alpha \right) = \left[ L \left( f_{\alpha}(t) \right), L \left( \overline{f}_{\alpha}(t) \right) \right],$$

$$L \left( f_{\alpha}(t) \right) = \int_{0}^{\infty} e^{-st} f_{\alpha}(t) dt = \lim_{\tau \to \infty} \int_{0}^{\tau} e^{-st} f_{\alpha}(t) dt,$$

$$L \left( \overline{f}_{\alpha}(t) \right) = \int_{0}^{\infty} e^{-st} \overline{f}_{\alpha}(t) dt = \lim_{\tau \to \infty} \int_{0}^{\tau} e^{-st} \overline{f}_{\alpha}(t) dt.$$
Theorem 2.13. [12, 13] Suppose that \( f \) and \( \bar{f} \) are continuous fuzzy-valued functions on \([0, \infty)\) and exponential order \( \alpha \) and that \( f'' \) is piecewise continuous fuzzy-valued function on \([0, \infty)\).

If \( f \) and \( \bar{f} \) are (1)-differentiable,
\[
L \left( f''(t) \right) = s^2 L(f(t)) \ominus s f(0) \ominus \bar{f}'(0),
\]
if \( f \) is (1)-differentiable and \( \bar{f} \) is (2)-differentiable,
\[
L \left( f''(t) \right) = -f'(0) \ominus (-s^2) L(f(t)) - s f(0),
\]
if \( f \) is (2)-differentiable and \( \bar{f} \) is (1)-differentiable,
\[
L \left( f''(t) \right) = s f(0) \ominus (-s^2) L(f(t)) \ominus \bar{f}'(0),
\]
if \( f \) and \( \bar{f} \) are (2)-differentiable,
\[
L \left( f''(t) \right) = s^2 L(f(t)) \ominus s f(0) - \bar{f}'(0).
\]

Theorem 2.14. [11, 13] Let be \( f(t) \), \( g(t) \) continuous fuzzy-valued functions and \( c_1 \) and \( c_2 \) constants, then
\[
L(c_1 f(t) + c_2 g(t)) = (c_1 L(f(t))) + (c_2 L(g(t))).
\]

### 3. Main Results

In this section, we research the solutions of the problem (1.1)-(1.2) by fuzzy Laplace transform under the concept of generalized differentiability. In this paper, (i) solution means that \( u \) is (i)-differentiable, \( u' \) is (i)-differentiable, i=1,2.

i) (1,1) solution of the problem:
If \( u \) and \( u' \) are (1)-differentiable, since
\[
L([0]^a) = \beta \left( s^2 L(u(t)) \ominus su(0) \ominus u'(0) \right)
+ \delta (sL(u(t)) \ominus u(0)),
\]
we have the equations
\[
L(-1 + \alpha) = \beta s^2 L(u_\alpha(t)) - \beta su_\alpha(0) - \beta u'_\alpha(0)
+ \delta sL(u_\alpha(t)) - \delta u_\alpha(0),
\]
\[
L(1 - \alpha) = \beta s^2 L(\bar{u}_\alpha(t)) - \beta s\bar{u}_\alpha(0) - \beta \bar{u}'_\alpha(0)
+ \delta sL(\bar{u}_\alpha(t)) - \delta \bar{u}_\alpha(0).
\]

Using the initial values (1.2), we get
\[
L(u_\alpha(t)) (\beta s^2 + \delta s) = \frac{-1 + \alpha}{s} + \beta W_\alpha + (\beta s + \delta) \bar{Q}_\alpha,
\]
\[
L(\bar{u}_\alpha(t)) (\beta s^2 + \delta s) = \frac{1 - \alpha}{s} + \beta W_\alpha + (\beta s + \delta) \bar{Q}_\alpha.
\]
From this, we obtain
\[
L(u_\alpha(t)) = \frac{-1 + \alpha}{\beta s^3 + \delta s^2} + \frac{\beta W_\alpha}{\beta s^3 + \delta s} + \frac{\bar{Q}_\alpha}{s}.
\]

Now, taking the inverse Laplace transform of the above equations, (1.1) solution is obtained as
\[
u_\alpha(t) = \left( \frac{-1 + \alpha}{\delta} \right) \left( t + \frac{\beta}{\delta} \left( e^{-\frac{s}{\beta}} - 1 \right) \right)
+ \beta W_\alpha \left( 1 - e^{-\frac{s}{\beta}} \right) + Q_\alpha,
\]
\[
\overline{u}_\alpha(t) = \left( \frac{1 - \alpha}{\delta} \right) \left( t + \frac{\beta}{\delta} \left( e^{-\frac{s}{\beta}} - 1 \right) \right)
+ \beta W_\alpha \left( 1 - e^{-\frac{s}{\beta}} \right) + \overline{Q}_\alpha.
\]

\[[u(t)]^a = [u_\alpha(t), \bar{u}_\alpha(t)].\]

ii) (1,2) solution of the problem:
If \( u \) is (1)-differentiable and \( u' \) is (2)-differentiable, we have the equations
\[
L([0]^a) = \beta \left( -u'(0) \ominus (-s^2) L(u(t)) \ominus su(0) \right)
+ \delta (sL(u(t)) \ominus u(0)),
\]
\[
L(-1 + \alpha) = -\beta su_\alpha(0) + \beta s^2 L(\bar{u}_\alpha(t)) - \beta s\bar{u}_\alpha(0)
+ \delta sL(\overline{u}_\alpha(t)) - \delta \bar{u}_\alpha(0),
\]
\[
L(1 - \alpha) = -\beta \bar{u}'_\alpha(0) + \beta s^2 L(\bar{u}_\alpha(t)) - \beta su_\alpha(0)
+ \delta sL(\bar{u}_\alpha(t)) - \delta \bar{u}_\alpha(0).
\]

From this, we obtain the equations
\[
\beta s^2 L(\bar{u}_\alpha(t)) + \delta sL(u_\alpha(t)) = \frac{-1 + \alpha}{s} + \beta s\overline{Q}_\alpha + \beta W_\alpha + \delta Q_\alpha.
\]
\[
(3.1)
\]
\[
\beta s^2 L(u_\alpha(t)) + \delta sL(\bar{u}_\alpha(t)) = \frac{1 - \alpha}{s} + \beta sQ_\alpha + \beta W_\alpha + \delta \overline{Q}_\alpha.
\]
\[
(3.2)
\]

If \( L(\bar{u}_\alpha(t)) \) in the equation (3.1) is replaced by the equation (3.2), we have
\[
L(u_\alpha(t)) = \left( 1 - \alpha \right) \left( \frac{\beta s + \delta}{s^2 (\beta^2 s^2 - \delta^2)} \right)
- \frac{\beta \delta W_\alpha}{s (\beta^2 s^2 - \delta^2)} + \frac{\bar{Q}_\alpha}{s} + \frac{\beta^2 W_\alpha}{\beta^2 s^2 - \delta^2},
\]
\[
(3.3)
\]
Taking inverse Laplace transform of the equation (3.3), the lower solution is obtained as

\[ u_\alpha (t) = (1 - \alpha) \left( \frac{\beta}{\delta^2} \left( \frac{e^{\delta t} + e^{-\delta t}}{2} - 1 \right) + \frac{1}{\delta} \right) \left( \frac{\beta \left( e^{\delta t} - e^{-\delta t} \right)}{2 \delta} - t \right) - \frac{\beta W_\alpha}{\delta} \left( \frac{e^{\delta t} + e^{-\delta t}}{2} - 1 \right) + \frac{\beta W_\alpha}{2\delta} \left( e^{\delta t} - e^{-\delta t} \right) + Q_\alpha. \]

Similarly, the upper solution is obtained as

\[ \bar{u}_\alpha (t) = (-1 + \alpha) \left( \frac{\beta}{\delta^2} \left( \frac{e^{\delta t} + e^{-\delta t}}{2} - 1 \right) + \frac{1}{\delta} \right) \left( \frac{\beta \left( e^{\delta t} - e^{-\delta t} \right)}{2 \delta} - t \right) - \frac{\beta W_\alpha}{\delta} \left( \frac{e^{\delta t} + e^{-\delta t}}{2} - 1 \right) + \frac{\beta W_\alpha}{2\delta} \left( e^{\delta t} - e^{-\delta t} \right) + Q_\alpha. \]

iii) (2,1) solution of the problem:
If \( u \) is (2)-differentiable and \( u' \) is (1)-differentiable, since

\[ L ([0]) = \beta \left( -su(0) + sL(u(t)) + u'(0) \right) + \delta (-u(0) \otimes (-sL(u(t)))) \]

we have the equations

\[ L(-1 + \alpha) = -\beta s\bar{u}_\alpha (0) + \beta sL(\bar{u}_\alpha (t)) - \beta u'_\alpha (0) - \delta \bar{u}_\alpha (0) + \delta sL(\bar{u}_\alpha (t)), \]

\[ L(1 - \alpha) = -\beta s u_\alpha (0) + \beta sL(u_\alpha (t)) - \beta u'_\alpha (0) - \delta u_\alpha (0) + \delta sL(u_\alpha (t)). \]

That is,

\[ L(u_\alpha (t)) = \frac{1 - \alpha}{\beta s^3 + \delta s^2} + \frac{\beta W_\alpha}{\beta s^2 + \delta s} + \frac{(\beta s + \delta) Q_\alpha}{\beta s^2 + \delta s}, \]

\[ L(\bar{u}_\alpha (t)) = \frac{-1 + \alpha}{\beta s^3 + \delta s^2} + \frac{\beta W_\alpha}{\beta s^2 + \delta s} + \frac{(\beta s + \delta) Q_\alpha}{\beta s^2 + \delta s}. \]

From this, (2.1) solution is obtained as

\[ u_\alpha (t) = \left( \frac{1 - \alpha}{\delta} \right) \left( t + \frac{\beta}{\delta} \left( e^{\delta t} - e^{-\delta t} \right) ight) + \frac{\beta W_\alpha}{\delta} \left( e^{\delta t} - e^{-\delta t} \right) + Q_\alpha, \]

\[ \bar{u}_\alpha (t) = \left( \frac{-1 + \alpha}{\delta} \right) \left( t + \frac{\beta}{\delta} \left( e^{\delta t} - e^{-\delta t} \right) - 1 \right) + \frac{\beta W_\alpha}{\delta} \left( e^{\delta t} - e^{-\delta t} \right) + Q_\alpha. \]

iv) (2,2) solution of the problem:
If \( u \) and \( u' \) are (2)-differentiable, since

\[ L ([0]) = \beta \left( s^2 L(u(t)) \otimes su(0) - u'(0) \right) + \delta (-u(0) \otimes (-sL(u(t)))) \]

the equations

\[ L(-1 + \alpha) = \beta s^2 L(u_\alpha (t)) - \beta su_\alpha (0) - \beta u'_\alpha (0) - \delta \bar{u}_\alpha (0) + \delta sL(\bar{u}_\alpha (t)), \]

\[ L(1 - \alpha) = \beta s^2 L(\bar{u}_\alpha (t)) - \beta su_\alpha (0) - \beta u'_\alpha (0) - \delta u_\alpha (0) + \delta sL(u_\alpha (t)) \]

are obtained. These yield

\[ \beta s^2 L(u_\alpha (t)) + \delta sL(u_\alpha (t)) = \frac{-1 + \alpha}{s} + \beta s Q_\alpha + \beta W_\alpha + \delta Q_\alpha, \]

(3.4)

\[ \beta s^2 L(\bar{u}_\alpha (t)) + \delta sL(\bar{u}_\alpha (t)) = \frac{1 - \alpha}{s} + \beta s Q_\alpha + \beta W_\alpha + \delta Q_\alpha, \]

(3.5)

If \( L(\bar{u}_\alpha (t)) \) in the equation (3.5) is replaced by the equation (3.4), we have

\[ L(u_\alpha (t)) = \left( -1 + \alpha \right) \left( \frac{\beta s + \delta}{s^2 \left( \beta s^2 - \delta^2 \right)} \right) \left( \frac{\beta s^2 + \delta s^2 - \beta s^3}{(\beta s^2 + \delta s)^2} + \frac{Q_\alpha}{s} + \frac{\beta W_\alpha}{\beta s^2 s^2 - \delta^2} \right), \]

(3.6)

Taking inverse Laplace transform of the equation (3.6), the lower solution is obtained as

\[ u_\alpha (t) = \left( -1 + \alpha \right) \left( \frac{\beta}{\delta} \left( e^{\delta t} + e^{-\delta t} \right) - 1 \right) + \frac{1}{\delta} \left( \frac{\beta \left( e^{\delta t} - e^{-\delta t} \right)}{2 \delta} - t \right) + \frac{\beta W_\alpha}{\delta} \left( e^{\delta t} + e^{-\delta t} \right) - 1 + \frac{\beta W_\alpha}{2\delta} \left( e^{\delta t} - e^{-\delta t} \right) + Q_\alpha. \]
Similarly, the upper solution is obtained as:

\[ u\alpha (t) = (1 - \alpha) \left( \frac{\beta}{\delta^2} \left( e^{\frac{\delta}{\beta} t} + e^{-\frac{\delta}{\beta} t} \right) \right) \]

\[ + \frac{1}{\delta} \left( \frac{\beta}{\delta} \left( e^{\frac{\delta}{\beta} t} - e^{-\frac{\delta}{\beta} t} \right) - t \right) \]

\[ - \frac{\beta W_\alpha}{\delta} \left( e^{\frac{\delta}{\beta} t} + e^{-\frac{\delta}{\beta} t} \right) - 1 \]

\[ + \frac{\beta W_\alpha}{2\delta} \left( e^{\frac{\delta}{\beta} t} - e^{-\frac{\delta}{\beta} t} \right) + \Omega_\alpha. \]

**Example 3.1.** Consider the solutions of the problem:

\[ u''(t) + u'(t) = [0]^\alpha, \quad t > 0, \]

\[ u(0) = [0]^\alpha, \quad u'(0) = [1]^\alpha \]

by fuzzy Laplace transform, where

\[ [0]^\alpha = [-1 + \alpha, 1 - \alpha], \quad [1]^\alpha = [\alpha, 2 - \alpha]. \]

(1,1) solution is a valid fuzzy function. If

\[ \frac{\partial u\alpha (t)}{\partial \alpha} \geq 0, \quad \frac{\partial u\alpha (t)}{\partial \alpha} \leq 0, \quad u\alpha (t) \leq u\alpha (t), \]

\[ u\alpha'(t) \leq u\alpha'(t), \quad u\alpha'(t) \leq u\alpha'(t), \]

(1,2) solution is a valid fuzzy function. For (1,1) solution, since

\[ \frac{\partial u\alpha (t)}{\partial \alpha} = t + 1 > 0, \quad \frac{\partial u\alpha (t)}{\partial \alpha} = -t - 1 < 0, \]

\[ \pi\alpha_a (t) - u\alpha_a (t) = 2(1 - \alpha)(t + 1) \geq 0, \]

\[ \pi\alpha''_a (t) - u\alpha''_a (t) = 2(1 - \alpha) \geq 0, \]

(1,1) solution is a valid fuzzy function. Similarly, (1,2) solution is a valid fuzzy function. Also, for (1,1) and (1,2) solutions, since

\[ u_1(t) = 1 - e^{-t}, \]

\[ u_1(t) - u\alpha_a (t) = (1 - \alpha)(t + 1) = u\alpha_a (t) - u_1(t), \]

(1,1) and (1,2) solutions are symmetric triangular fuzzy numbers for any \( t > 0 \) time. (2,1) solution is

\[ u\alpha (t) = (1 - \alpha)(t + e^{-t} - 2) + 2(1 - \alpha) \left( e^{t} - e^{-t} \right) \]

\[ = t - e^{-t} + \alpha(1 - t), \]

\[ \pi\alpha_a (t) = (1 - \alpha)(t + e^{-t} - 2) + \alpha(1 - e^{-t}) \]

\[ = 2 - t - e^{-t} + \alpha(1 - t), \]

\[ [u(t)]^\alpha = [u\alpha_a (t), \pi\alpha_a (t)]. \]

(2,2) solution is

\[ u\alpha (t) = (1 - \alpha)(t + e^{-t} - 2) - \alpha \left( e^{t} + e^{-t} \right) \]

\[ = t - e^{-t} + \alpha(1 - t), \]

\[ \pi\alpha_a (t) = (1 - \alpha)(t + e^{-t} - 2) - \alpha \left( e^{t} + e^{-t} \right) \]

\[ = 2 - t - e^{-t} + \alpha(1 - t), \]

\[ [u(t)]^\alpha = [u\alpha_a (t), \pi\alpha_a (t)]. \]
(2.1) solution is a valid fuzzy function. If

\[ \frac{\partial u_\alpha(t)}{\partial \alpha} \geq 0, \quad \frac{\partial \pi_\alpha(t)}{\partial \alpha} \leq 0, \quad u_\alpha(t) \leq \pi_\alpha(t), \]

(2.2) solution is a valid fuzzy function. For (2,1) solution, since

\[ \frac{\partial u_\alpha(t)}{\partial \alpha} = 1 - t, \quad \frac{\partial \pi_\alpha(t)}{\partial \alpha} = t - 1, \]

\[ u_\alpha(t) - \pi_\alpha(t) = 2(1 - \alpha)(1 - t), \]

if \( t \leq 1 \), we have \( \frac{\partial u_\alpha(t)}{\partial \alpha} \geq 0, \frac{\partial \pi_\alpha(t)}{\partial \alpha} \leq 0, \quad u_\alpha(t) \leq \pi_\alpha(t) \). Also, since

\[ u_\alpha(t) - \pi_\alpha(t) = 2(1 - \alpha) \geq 0, \quad u_\alpha''(t) - \pi_\alpha''(t) = 0, \]

(2,1) solution is a valid fuzzy function for \( t \leq 1 \). Similarly, (2,2) solution is a valid fuzzy function for \( t \leq 1 \). In addition, for (2,1) and (2,2) solutions, since

\[ u_1(t) = 1 - e^{-t} = \pi_1(t), \]

\[ u_1(t) - \pi_1(t) = (\alpha - 1)(t - 1) = \pi_\alpha(t) - \pi_\alpha(t), \]

(2,1) and (2,2) solutions are symmetric triangular fuzzy number for any \( t > 0 \) time.

4. Conclusion

In this paper, we study the solutions of a second-order fuzzy initial value problem using the fuzzy Laplace transform under the generalized differentiability. We use symmetric triangular fuzzy number, Hukuhara difference, the properties of fuzzy Laplace transform and fuzzy arithmetic. We solve an example related to the problem. We obtain that (1,1) and (1,2) solutions are valid fuzzy functions and (2,1) and (2,2) solutions are valid fuzzy functions for \( t \leq 1 \). Also, we obtain that all of the solutions are symmetric triangular fuzzy numbers for any \( t > 0 \) time.

References

[1] J. J. Buckley, T. Feuring, Fuzzy differential equations, *Fuzzy Sets and Systems*, 110(1)(2000), 43–54.
[2] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24(3)(1987), 301–317.
[3] H. Gültekin, N. Altunışık, On solution of two-point fuzzy boundary value problems, *The Bulletin of Society for Mathematical Services and Standards*, 11(2014), 31–39.
[4] A. Khastan, J. J. Nieto, A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis*, 72(9-10) (2010), 3583–3593.
[5] B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151(3)(2005), 581–599.
[6] B. Bede, I. J. Rudas, A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, *Information Sciences*, 177(7)(2007), 1648–1662.
[7] A. Khastan, F. Bahrami, K. Ivaz, New results on multiple solutions for nth-order fuzzy differential equations under generalized differentiability, *Boundary Value Problems*, (2009), 1–13.
[8] H. Gültekin Çiiltik, The relationship between the solutions according to the noniterative method and the generalized differentiability of the fuzzy boundary value problem, *Malaya Journal of Matematik*, 6(4) (2018), 781–787.
[9] M. Oberguggenberger, S. Pittschmann, Differential equations with fuzzy parameters, *Mathematical and Computer Modelling of Dynamical Systems*, 5(3)(1999), 181–202.
[10] E. Hûllermeier, An approach to modelling and simulation of uncertain dynamical systems, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5(2)(1997), 117–137.
[11] T. Allahviranloo, M. Barkhordari Ahmadi, Fuzzy Laplace transforms, *Soft Computing*, 14(3)(2010), 235–243.
[12] S. Salahshour, T. Allahviranloo, Applications of fuzzy Laplace transforms, *Soft Computing*, 17(1)(2013), 145–158.
[13] K. R. Patel, N. B. Desai, Solution of variable coefficient fuzzy differential equations by fuzzy Laplace transform, *International Journal on Recent and Innovation Trends in Computing and Communication*, 5(6)(2017), 927–942.
K. R. Patel, N. B. Desai, Solution of fuzzy initial value problems by fuzzy Laplace transform, *Kalpa Publications in Computing*, 2(2017), 25–37.

H.-K. Liu, Comparison results of two-point fuzzy boundary value problems, *International Journal of Computational and Mathematical Sciences*, 5(1)(2011), 1–7.

M. L. Puri, D. A. Ralescu, Differentials of fuzzy functions, *Journal of Mathematical Analysis and Applications*, 91(2)(1983), 552–558.

B. Bede, Note on “Numerical solutions of fuzzy differential equations by predictor-corrector method”, *Information Sciences*, 178(7)(2008), 1917–1922.