FREE PRODUCT-LIKE PERMANENCE RESULTS FOR SOME COARSE INVARIANTS

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ABSTRACT. We extend a result of Antolín and Dreesen to show that certain graph products of groups over infinite graphs preserve finite asymptotic dimension. We also extend their result to Yu’s property A. Finally, we prove some elementary permanence properties of Dranishnikov’s asymptotic property C and the straight finite decomposition complexity of Dranishnikov and Zarichnyi.

1. INTRODUCTION

The asymptotic dimension of a metric space was introduced by Gromov [Gro93] in his study of large-scale invariants of finitely generated groups. It is the large-scale analog of covering dimension in topology. Although interesting in its own right, asymptotic dimension gained the interest of the larger mathematical community following the work of G. Yu [Yu98]. Yu showed that a finitely generated group with finite asymptotic dimension satisfies the famous Novikov higher signature conjecture. This generated interest in determining whether the asymptotic dimension of various groups and classes of groups is finite. Although later work of Yu and others (e.g. [Yu00, KY12], among others) has refined the technology to determine whether a group satisfies the Novikov (or related) conjectures, there is still a great deal of interest in this simple large-scale invariant.

The asymptotic dimension is a coarse invariant. Every finitely generated group with a fixed finite generating set can be endowed with a left-invariant proper metric (the word metric) corresponding to that generating set. Any other choice of finite generating set gives rise to a metric space that is quasi-isometric to the group in the original metric. Thus, the large-scale structure of the metric space associated to the group is invariant of the choice of finite generating set. This means that the large-scale invariants associated to the group in a word metric are invariants of the group and not invariants of the specific metric obtained by the choice of a generating set. This also makes the class of finitely generated groups a natural one for the study of asymptotic invariants.

It is natural to consider left-invariant proper metrics on other countable groups and to wonder to what extent they are large-scale invariants. Dranishnikov-Smith [DS06] and Smith [Smi06] considered the asymptotic dimension of arbitrary countable groups in proper left-invariant metrics. In particular, Smith showed that all left-invariant proper metrics on a countable group are coarsely equivalent. This

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increased the class of groups for which asymptotic invariants were natural to consider. On the other hand, proper left-invariant metrics are not always natural, e.g. \( \mathbb{Q} \) with its usual metric is not proper.

The class of groups for which the finiteness of asymptotic dimension is known is vast and contains (among many others) hyperbolic groups [Roe05], nilpotent groups [BD06], solvable groups with rational Hirsch length [DS06], Coxeter groups [DJ99], mapping class groups [BBF10], and groups admitting a proper isometric action on finite dimensional CAT(0)-cube complexes [Wri12]. Moreover this class is closed under the operations of (finite) direct product, free products with amalgamation, and group extensions [BD06]. Recently, Antolín and Dreesen [AD13] computed a formula for the asymptotic dimension of a graph product of groups using results of Dranishnikov [Dra08] and Green [Gre90]. One of the goals of this paper is to extend this result to certain graph products over infinite graphs: Theorem 3.4. Because these graphs are infinite, the techniques of Antolín and Dreesen are not applicable. Instead we exploit the structure of these graph products to explicitly construct the covers from the definition of asymptotic dimension at each scale \( R \).

Dranishnikov, Keesling and Uspenskij [DKU98] showed that \( \text{asdim} \mathbb{Z}^n = n \), so any group that contains a copy of \( \mathbb{Z}^n \) for each \( n \) will necessarily have infinite asymptotic dimension. Such groups are not difficult to construct (for example, see [Roe03]). For groups and spaces whose asymptotic dimension may be infinite, one can consider other dimension-like coarse invariants, such as asymptotic property C, finite decomposition complexity, straight finite decomposition complexity, or property A. For metric spaces with bounded geometry, finite asymptotic dimension implies both asymptotic property C and finite decomposition complexity. Both of these notions imply straight finite decomposition complexity. Finally, spaces with straight finite decomposition complexity have Yu’s property A. See [Gol13] for a nice summary of these implications.

The second goal of this paper is to apply the techniques of [AD13] to point out that graph products of groups with property A have property A. Although, we would like to extend this result to asymptotic property C, it is not clear that it does extend. In particular, the standard approach to such properties breaks down completely in the case of property C. We were able to show that asymptotic property C is preserved by certain infinite unions (Theorem 4.2) and free products (Theorem 4.4), but we cannot show that it is preserved by amalgams over non-trivial groups and direct products. If amalgams and direct products could be shown to preserve asymptotic property C, then the techniques of [AD13] could be applied to show that graph products preserve it. In light of [PP09], it is conceivable that there is some space (or even a group) with asymptotic property C whose square does not have asymptotic property C.

The final goal of the paper is to prove some permanence results for straight finite decomposition complexity along the lines of those shown in [GTY13].

The paper is organized as follows. In the next section we recall some of basic facts and give precise definitions. In Section 3 we state and prove the main theorem concerning infinite graph products and asymptotic dimension. We also show that property A is preserved by finite graph products. In the fourth section, we prove that asymptotic property C is preserved by free products and certain infinite unions. We end this section with some open questions concerning asymptotic property C.
The permanence properties of straight finite decomposition complexity appear in the final section.

2. Preliminary notions and definitions

Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. A function \(f : X \to Y\) is called \textit{uniformly expansive} if there is a non-decreasing \(\rho_2 : [0,\infty) \to [0,\infty)\) such that
\[
d(f(x), f(x')) \leq \rho_2(d(x, x')).
\]
The function \(f : X \to Y\) is called \textit{effectively proper} if there is some proper, non-decreasing \(\rho_1 : [0,\infty) \to [0,\infty)\) such that
\[
\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')).
\]
The function \(f : X \to Y\) is called a \textit{coarsely uniform embedding} if there exist functions \(\rho_1, \rho_2 : [0,\infty) \to [0,\infty)\) such that,
\[
\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x'))
\]
and \(\rho_1 \to \infty\). The spaces \(X\) and \(Y\) are said to be \textit{coarsely equivalent} if there is a coarsely uniform embedding of \(X\) to \(Y\) and there is some \(R > 0\) so that \(Y = N_R(f(X))\). When the \(\rho_i\) can be taken to be linear, \(f\) is called a \textit{quasi-isometric embedding} and the corresponding equivalence is \textit{quasi-isometry}.

Let \(R > 0\) be a (large) real number. A collection \(U\) of subsets of the metric space \(X\) is said to be \(R\)-\textit{discrete} if there is a uniform bound on the diameter of the sets in \(U\) and if, whenever \(U \neq U'\) are sets in \(U\), then \(d(U, U') > R\), where \(d(U, U') = \inf \{d(x, x') | x \in U, x' \in U'\}\). We will often refer to such families as being \textit{uniformly bounded} and \(R\)-\textit{disjoint}. Gromov \cite{Gro93} describes this situation by saying that \(\cup_{U \in \mathcal{U}} U\) is 0-dimensional on \(R\)-scale.

We say that the \textit{asymptotic dimension} of the metric space \(X\) does not exceed \(n\) and write \(\text{asdim} X \leq n\) if for each (large) \(R > 0\), \(X\) can be written as a union of \(n + 1\) sets with dimension 0 at scale \(R\). There are several other useful formulations of the definition (see \cite{BD08}) but we shall content ourselves with this one.

In \cite{Yu00}, G. Yu defined property A for discrete metric spaces as a generalization of amenability of groups. A discrete metric space \(X\) has \textit{property A} if for any \(r > 0\) and any \(\varepsilon > 0\), there is a collection of finite subsets \(\{A_x\}_{x \in X}\) where \(A_x \subset X \times \mathbb{N}\), so that

\begin{enumerate}
  \item \((x, 1) \in A_x\) for each \(x \in X\);
  \item for every pair \(x, y\) in \(X\) with \(d(x, y) < r\), \(\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon\); and
  \item there is some \(R\) so that if \((y, n) \in A_x\), then \(d(x, y) \leq R\).
\end{enumerate}

Dranishnikov defined the asymptotic analog of Haver’s property C for metric spaces \cite{Dra00}. We say that a metric space \(X\) has \textit{asymptotic property C} if for any given number sequence \(R_1 \leq R_2 \leq R_3 \leq \cdots\) there exist uniformly bounded families \(\mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^n\) so that \(\mathcal{U}^t\) is \(R_t\)-disjoint and so that \(\cup_{t=1}^n \mathcal{U}^t\) covers \(X\). It is clear that a metric space with finite asymptotic dimension will have asymptotic property C. Dranishnikov showed that a discrete metric space with bounded geometry and asymptotic property C also has property A \cite[Theorem 7.11]{Dra00}.

Guentner, Tessera and Yu \cite{GT13, GT12} defined another coarse invariant of groups that is applicable when the asymptotic dimension is infinite: finite decomposition complexity. Following this, Dranishnikov and Zarichnyi defined a related
notion: straight finite decomposition complexity. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be families of metric spaces. For a positive \( R \), we say that \( \mathcal{X} \) is \( R \)-decomposable over \( \mathcal{Y} \) and write \( \mathcal{X} \xrightarrow{\mathcal{Y}} \mathcal{Y} \) if for any \( X \in \mathcal{X} \) one can write

\[
X = Y^0 \cup Y^1 \text{ where } Y^i = \bigsqcup_{R \text{-disjoint}} Y^{ij}, \text{ for } i=0,1,
\]

where the sets \( Y^{ij} \in \mathcal{Y} \) and the notation indicates that distinct elements of the disjoint union are actually \( R \)-separated.

We begin by describing the metric decomposition game for \( X \). In this game two players take turns. First, Player 1 asserts a number \( R_1 \). Player 2 responds by finding a metric family \( Y^1 \) and a \( R_1 \)-decomposition of \( \{X\} \) over \( Y^1 \). Then, Player 1 selects a number \( R_2 \) and Player 2 again finds a family \( Y^2 \) and an \( R_2 \)-decomposition of \( Y^1 \) over \( Y^2 \). Player 2 wins if the game ends in finitely many steps with a family that consists of uniformly bounded subsets. The metric space \( X \) is said to have finite composition complexity or FDC, if there is a winning strategy for Player 2 in the metric decomposition game for \( X \), [2].

The metric space \( X \) has straight finite decomposition complexity SFDC if for every sequence \( R_1 \leq R_2 \leq \cdots \) there exists an \( n \) and metric families \( Y^i \) (\( i = 1, 2, \ldots, n \)) so that \( X \xrightarrow{Y^1} Y^1, Y^{i-1} \xrightarrow{R_i} Y^i \) for \( i = 2, 3, \ldots, n \), and such that \( Y^n \) is uniformly bounded.

A finitely generated group with generating set \( S = S^{-1} \) can be endowed with a left-invariant metric called the word metric by taking \( d_S(g, h) = \|g^{-1}h\|_S \), where the norm \( \| \cdot \|_S \) is zero at the identity and otherwise is the length of a shortest \( S \)-word that presents \( \gamma \). It is easy to see that if \( S \) and \( S' \) are finite generating sets on the finitely generated group \( \Gamma \), then the metric spaces \( (\Gamma, d_S) \) and \( (\Gamma, d_{S'}) \) are quasi-isometric.

The situation for non-finitely generated groups is less clear. Ideally, one would like to endow any countable group with a metric structure that is an invariant of coarse isometry. In [3], J. Smith showed that on a countable group any two left-invariant, proper metrics are coarsely equivalent. Moreover he shows that a weight function (defined below) on a countable group induces a left-invariant, proper metric. By a weight function on a generating set \( S \), one may wish to endow any countable group with a metric structure that is an invariant of coarse isometry. In [3], J. Smith showed that on a countable group any two left-invariant, proper metrics are coarsely equivalent. Moreover he shows that a weight function (defined below) on a countable group induces a left-invariant, proper metric. By a weight function on a generating set \( S = S^{-1} \) for a group, we mean a function \( w : S \to \mathbb{R}_+ \) for which

1. if \( w(s) = 0 \) then \( s = e \);
2. \( w(s) = w(s^{-1}) \); and
3. for each \( N \in \mathbb{N} \), \( w^{-1}([0, N]) \) is finite.

One then defines a norm by \( \| \gamma \| = \inf \{ \sum w(s_i) \mid x = s_1s_2 \cdots s_n \} \), where the norm of the identity is defined to be 0 (i.e., it is presented by the empty product).

Let \( \Gamma \) be an undirected graph without loops or multiple edges. Let \( E(\Gamma) \) be the set of vertices and edges of \( \Gamma \), respectively. Suppose that \( \mathcal{G} = \{G_v\} \) is a collection of groups indexed by the elements of \( E(\Gamma) \). The graph product \( \Gamma \mathcal{G} \) of the collection \( \mathcal{G} \) over the graph \( \Gamma \) is defined to be the free product of the \( G_v \) with the additional relations that whenever \( \{v, v'\} \) is an edge in \( \Gamma \), then \( gg' = g'g \) for all \( g \in G_v \) and \( g' \in G_{v'} \). Thus, if \( E(\Gamma) = \emptyset \), \( \Gamma \mathcal{G} = *G_v \). If \( \Gamma \) is the complete graph on \( n \) vertices, we obtain the direct product \( \Gamma \mathcal{G} = G_{v_1} \times \cdots \times G_{v_n} \). Graph products were introduced in Green’s thesis [4].

We will often refer to a word in a graph product (or free product) as being expressed in syllables. We say that \( g = g_1 \cdots g_t \) is an expression of \( g \) in syllables
if each $g_i$ is reduced and non-trivial, each $g_i$ belongs to some single vertex group, and no two consecutive $g_i$ and $g_{i+1}$ belong to the same vertex group.

3. Asymptotic dimension of graph products

In this section we extend the result of Antolín and Dreesen concerning asymptotic dimension of graph products of groups in two directions. First, we extend the asymptotic dimension result to include certain infinite graphs. Second, we show that one can replace finite asymptotic dimension everywhere with property A and arrive at the corresponding conclusion.

Let $Γ$ be a countable graph. Define a weight function on $w : V(Γ) → \mathbb{N}$ by taking any one-to-one correspondence between $V(Γ)$ and $\mathbb{N}$. For each vertex $v \in V(Γ)$, let $G_v$ be a finitely generated group, with generating set $S_v$. We insist that the set $S_v$ be closed under inverses and not contain the identity element. Define a weight function $w : \sqcup S_v → \mathbb{N}$ by $w(s) = w(v)$, where $s \in S_v$ is a generator. Clearly, $w$ is a weight function.

Next, suppose that $r > 0$ is given. Define a graph $Γ_r$ by setting the vertex set of $Γ_r$ equal to $w^{-1}([0, r])$. The edge set of $Γ_r$ contains precisely those edges in $Γ$ for which both vertices are also in $Γ_r$. Let $g ∈ ΓΣ$ be an element. We will say that a reduced word $g_1 \cdots g_k$ is a presentation of $g$ in $Γ_r$-standard form if

(1) $g = g_1 \cdots g_k$ with each $g_i$ a reduced syllable and

(2) Whenever $g = h_1 \cdots h_k$ is a reduced word in reduced syllables presenting $g$ we have

$$\max \{|i | g_i \notin Γ_rΣ\} \geq \max \{|i | h_i \notin Γ_rΣ\}.$$ 

This second condition amounts to saying that each $Γ_r$ syllable is commuted as far to the right of the word as possible. Call an element $x$ of $ΓΣ$ permissible if the standard form of $x$ does not end with a non-trivial element of $Γ_rΣ$. In other words, $x$ is permissible if no reduced word that presents the element $x$ can be made to end with any non-trivial syllable $Γ_r$. In this way, we will consider the identity to be permissible.

Lemma 3.1. Let $ΓΣ$ be a graph product of finitely generated groups $Σ = \{G_v\}$ with the metric described above. Let $r > 0$ be given and take $Γ_r$ as above. Then, each element of $ΓΣ$ can be written in the form $xb$, where $x$ is permissible and $b ∈ Γ_rΣ$. Moreover, if $x \neq x'$ are permissible, then $d(xb, x'b') > r$.

Proof. First, we check that each element has such a form. To this end, let $g ∈ ΓΣ$ be given and write $g = g_1 \cdots g_t$ as an expression in syllables. We proceed by induction on the number of syllables $t$. If $t = 1$, then either $g_1$ is in $Γ_rΣ$ or not. In the first case, it can be written as $xg_1$, where $x = e$. In the latter case, $x = g_1$ is permissible.

Suppose now that every word of syllable length at most $t − 1$ can be written in the form $xb$ with $x$ permissible and $b ∈ Γ_rΣ$. Then, consider $g = g_1 \cdots g_t$. Since $g_1 \cdots g_{t−1}$ has syllable length shorter than $t$ it can be written in the form $xb$. Therefore, express $x$ and $b$ in syllables so that we have $g = x_1 \cdots x_pb_{p+1} \cdots b_{t−1}g_t$. If $g_t$ itself is in $Γ_rΣ$, then this word is already in permissible form.

Suppose therefore, that $g_t \notin Γ_rΣ$. If it commutes with $b_{t−1}$, then we can write $b_{t−1}g_t = g_tb_{t−1}$ and therefore we have $g = x_1 \cdots b_{t−1}g_t = x_1 \cdots g_tb_{t−1}$. Now, since its length is less than $t$, the element $x_1 \cdots g_t$ can be written as some $x'b'$ in permissible form. But, then $g = x'b'b_{t−1}$ is a permissible presentation of $g$. 


Finally, we consider the case in which \( g \) does not commute with \( b t^{-1} \). If any rearrangement of this word allows \( g \) to commute past a syllable, then we apply the argument of the preceding paragraph to obtain a word in permissible form. Otherwise, \( x = g \) is already permissible.

Now, we show the disjointness condition holds. Suppose that \( x \) and \( x' \) are distinct, but permissible. Then, write \( x = x' = z \) for some \( z \in \Gamma G \). Observe that \( z \not\in \Gamma_r \), as, if it were, then \( xz \) would be a presentation of \( x' \) that ends with a non-trivial element of \( \Gamma_r \), which is not allowed. Thus, \( z \) must contain some element that is not in \( \Gamma_r \). Hence it contains a generator \( s \) from a group with weight \( > r \).

Thus, \( d(xb, x'b') = \| b^{-1}zb' \| \geq \| s \| > r \). □

For a graph \( \Gamma \), we recall that the clique number \( \omega(\Gamma) \) is the maximum number of vertices in a clique in \( \Gamma \); i.e., the size of the largest set of vertices for which each pair is connected by an edge in \( \Gamma \).

**Theorem 3.2** ([AD13 Theorem 6.3]). Let \( \Gamma \) be a finite simplicial graph and let \( \mathcal{G} \) be a family of finitely generated groups indexed by vertices of \( V(\Gamma) \). Let \( G = \Gamma \mathcal{G} \). Let \( C \) be the collection of subsets of \( V(\Gamma) \) spanning a complete graph. Then

\[
\text{asdim} G \leq \max_{C \in C} \sum_{v \in C} \max(1, \text{asdim} G_v).
\]

For our present purposes, we need a slightly weaker result that we state as a corollary.

**Corollary 3.3.** Let \( \Gamma \) be a finite simplicial graph with \( \omega(\Gamma) \leq k \) and let \( \mathcal{G} \) be a collection of finitely generated groups indexed by \( v \in V(\Gamma) \) such that \( 0 < \text{asdim} G_v \leq n \) for all \( v \in V(\Gamma) \). Then, \( \text{asdim} \Gamma \mathcal{G} \leq nk \).

**Proof.** We have that \( \max(1, \text{asdim} G_v) = \text{asdim} G_v \) for each \( v \). Also, there is at least one \( C \in C \) with \( \omega(\Gamma) \) elements. Thus,

\[
\text{asdim} G \leq \max_{C \in C} \sum_{v \in C} \max(1, \text{asdim} G_v) \leq \omega(\Gamma) \max_{v \in V(\Gamma)} \{ \text{asdim} G_v \} \leq kn.
\]

Really, all that is necessary for the preceding proof to work is that at least one of the \( G_v \) should be infinite, forcing \( n > 0 \). If all \( G_v \) are finite, then \( \text{asdim} G \leq k \) instead of the estimate given above, which would be \( 0 = nk \).

**Theorem 3.4.** Let \( \Gamma \) be a countable graph with clique number \( \omega(\Gamma) \leq k \). Suppose that \( \{G_v\}_{v \in V(\Gamma)} \) is a collection of finitely generated groups with \( 0 < \text{asdim} G_v \leq n \) for all \( v \in V(\Gamma) \). Then, in a left-invariant proper metric, \( \text{asdim} \Gamma \mathcal{G} \leq nk \).

**Proof.** For a given \( r > 0 \) we will construct a cover by \( nk + 1 \) uniformly bounded, \( r \)-disjoint families of subsets of \( \Gamma \mathcal{G} \). Since \( \Gamma \mathcal{G} \) is a countable group that is not finitely generated, we endow it with a metric arising from a weight function \( \bar{w} : V(\Gamma) \to \mathbb{N} \) as described above.

Define a subgraph \( \Gamma_r \) of \( \Gamma \) by setting \( V(\Gamma_r) = \bar{w}^{-1}([0, r]) \) and by defining an edge between two vertices of \( \Gamma_r \) if and only if there is an edge between these vertices in \( \Gamma \). By Corollary 3.3 we know that \( \text{asdim} \Gamma_r \mathcal{G} \leq nk \). Thus, there is a cover by \( nk + 1 \) \( r \)-disjoint families of uniformly bounded sets, say \( \mathcal{U}^0, \mathcal{U}^1, \ldots, \mathcal{U}^{nk} \). Let \( P \subset \Gamma \mathcal{G} \) denote the set of all \( \Gamma_r \)-permissible elements.
For each $i$ define the collection $\{xU \mid x \in P, U \in \mathcal{U}^i\}$. We claim that for each $i$, the collection is $r$-disjoint and uniformly bounded. Moreover, we claim that the union of these collections covers $\Gamma \mathfrak{S}$.

Since the metric on $\Gamma \mathfrak{S}$ is left-invariant, we know that $d(xu, xu') = d(u, u')$, for all $xu$ and $xu'$ in $xU$. Since $\text{diam}(U)$ is uniformly bounded, we have that $\text{diam}(xU)$ is also uniformly bounded.

Next, suppose that $xU$ and $x'U'$ are distinct sets, where $U, U' \in \mathcal{U}^i$. If $x = x'$, then we have $d(xU, x'U') = d(xU, xU') = d(U, U')$, and since these sets must be different (yet still in the same family $\mathcal{U}^i$), they are at least $r$-disjoint. If $x \neq x'$, then by the previous lemma $d(xu, x'u') > r$ and so these two families are $r$-disjoint.

Finally, we show that the collection of all such families covers $\Gamma \mathfrak{S}$. To this end, let $g \in \Gamma \mathfrak{S}$ be given. Then, by the lemma $g = xb$, where $x \in P$ and $b \in \Gamma, \mathfrak{S}$. Thus, there is some $i$ and some $U \in \mathcal{U}^i$ so that $b \in U$. Thus, $g \in xU$, as required. \hfill $\square$

The following result and proof follow are similar to [AD13] Theorem 6.3.

**Corollary 3.5.** Let $\Gamma$ be a finite graph. If all the $G_v$ have property A, $\Gamma \mathfrak{S}$ has property $A$.

**Proof.** We proceed by induction on $|V(\Gamma)|$. We note that if $|V(\Gamma)| = 1$, then $\Gamma \mathfrak{S} = G_v$, which is assumed to have property $A$.

Now we suppose that $|V(\Gamma)| = n > 1$ and also that the theorem holds for graphs with fewer than $n$ vertices.

Then let $v \in V(\Gamma)$ be any vertex, and put $A = \{v\} \cup \text{lk}(v), B = \Gamma - \{v\}, C = \text{lk}(v)$. Then, by [Gre90] we have that $\Gamma \mathfrak{S} = G_A *_{G_C} G_B$.

Now, we have two cases. In the first case, $A = \Gamma$. Then, since $v$ is connected to each vertex of $C$ and this encompasses all vertices of $\Gamma$, we have that $\Gamma \mathfrak{S} = G_v \times G_C$. Now $G_v$ has property A by assumption. Since $|V(C)| < |V(\Gamma)|$ the induction hypothesis implies that $G_C$ has property $A$. Since property $A$ is preserved by direct products ([Yu00]), $\Gamma \mathfrak{S}$ has property $A$.

In the second case, where $A \neq \Gamma$, we have then that $|V(A)| < |V(\Gamma)|$. By definition, we have that $|V(B)| < |V(\Gamma)|$. And so, by our induction hypothesis, $G_A$ and $G_B$ both have property $A$. Since amalgamated free products preserve property $A$ ([Dyk04][Tu01][Bel03]), we conclude that $\Gamma \mathfrak{S}$ has property $A$. \hfill $\square$

**Question 3.6.** Let $\Gamma$ be a countably infinite graph and suppose that all $G_v \in \mathfrak{S}$ have property $A$. Then in a proper, left-invariant metric, does $\Gamma \mathfrak{S}$ have property $A$?

4. **Asymptotic Property C**

The goal of this section is to show that asymptotic property C is preserved by some infinite unions and free products.

We consider the case where $X$ can be expressed as a union of a collection of spaces with uniform property C with the additional property that for each $r > 0$ there is a “core” space such that removing this core from the families leaves the families $r$-disjoint. We begin by stating some results from [BD01].

Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a metric space $X$. Let $V \in \mathcal{V}$ and $d > 0$. Let $\mathcal{N}_d(V; \mathcal{U})$ be the set of all $U \in \mathcal{U}$ such that $d(V, U) \leq d$. The $d$-saturated union of $\mathcal{V}$ in $\mathcal{U}$ is the set $\mathcal{V} \cup_d \mathcal{U} = \{\mathcal{N}_d(V, \mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(V, U) > d \ \forall \ V \in \mathcal{V}\}$. Note that (in general) $\mathcal{V} \cup_d \mathcal{U} \neq \mathcal{U} \cup_d \mathcal{V}$ and that $\emptyset \cup_d \mathcal{U} = \mathcal{U} = \mathcal{U} \cup_d \emptyset$. 

Suppose that $\bigcup_i U_i$ is a collection of subsets that is uniformly property $C$. Suppose further that for each $r > 0$, the free product $X \ast Y$ above proposition, $W \ast Y$ is $d$-disjoint and uniformly bounded. Then, $\bigcup_i U_i$ is $d$-disjoint and uniformly bounded.

We will say that the family $X_\alpha$ satisfies asymptotic property $C$ uniformly in $\alpha$ if for every sequence $R_1 < R_2 < \cdots$ there exist $B_1 < B_2 < \cdots$ so that for each $\alpha$ there exist families $U_\alpha^i$ of $R_i$-disjoint, $B_i$-bounded families $(i = 1, \ldots, n)$ so that $\bigcup_{i=1}^n U_\alpha^i$ covers $X_\alpha$.

**Theorem 4.2.** Suppose that $X = \bigcup_\alpha X_\alpha$ is a countable union of spaces that have uniform property $C$. Suppose further that for each $r > 0$ there is a $Y_r \subset X$ so that $Y_r$ has asymptotic property $C$ and such that the family $\{X_\alpha - Y_r\}$ is $r$-disjoint. Then, $X$ has asymptotic property $C$.

**Proof.** Let $d_1 < d_2 < \cdots$ be a sequence of positive numbers. For each $\alpha$, choose families $U_\alpha^i$ of $d_i$-disjoint, $R_i$-bounded sets, $i = 1, 2, \ldots, n$. Since $R_i$ are upper bounds on diameters, we may take them to be increasing and insist that $R_i \geq d_i$. Put $r = 5R_n$. Take $Y_r$ as in the statement of the theorem.

Let $V^1, V^2, \ldots, V^k$ be $5R_i$-disjoint, $B_i$-bounded families of sets whose union covers $Y_r$.

Let $U_\alpha^i_0$ denote the restriction of $U_\alpha^i$ to $X_\alpha - Y_r$. Next, put $\overline{U^i} = \bigcup_\alpha U_\alpha^i$. Note that $\overline{U^i}$ is $R_i$-bounded and $d_i$ disjoint. Finally, set $W^i = V^i \cup \overline{U^i}$, for $i = 1, 2, \ldots, \max\{k, n\}$. Here, we take $V^i = \emptyset$ or $U^i = \emptyset$ if $i > k$ or $i > n$, respectively. Thus, in these cases, we have $W^i = \overline{U^i}$ or $W^i = V^i$, respectively. By the above proposition, $W^i$ is $d_i$-disjoint and uniformly bounded. It is clear that this collection covers $X$.

**Proposition 4.3.** Let $A$ and $B$ be countable groups equipped with left-invariant proper metrics. Give $A \ast B$ the corresponding metric generated by the norms of $A$ and $B$. Let $r > 0$ be given and take let $Y_r = \{g \in A \ast B \mid \max\{\|g_1\| \mid g_1 \in A \cup B\} \leq r\}$. Then, $\mathrm{asdim} Y_r = 1$.

**Proof.** We define a tree $T$ with $V(T) = Y_r$, where edges connect vertices $x$ and $y$ if and only if either $y = x\alpha$ for some $\alpha \in A$ where $x$ ends with a non-trivial element of $B$ or $y = xb$ for some $b \in B$ where $x$ ends with some non-trivial element of $A$. We also connect the vertex $e$ to all elements of $A$ and $B$ that are in $Y_r$.

The identity map $(Y_r, d) \to (T, d_e)$ from $Y_r$ in the metric it inherits from $A \ast B$ to the tree $T$ in the edge-length metric is then a quasi-isometry. In particular,

$$\frac{1}{r}d(g, h) \leq d_e(g, h) \leq 2d(g, h).$$

Thus, $Y_r$ has the same asymptotic dimension as the tree $T$. Since a tree containing an infinite discrete geodesic ray has asymptotic dimension 1, we conclude that $Y_r$ has asymptotic dimension 1.

**Theorem 4.4.** Let $A$ and $B$ be countable groups with left-invariant proper metrics. Equip the free product $G = A \ast B$ with the natural left-invariant proper metric it inherits from $A$ and $B$. If $A$ and $B$ have asymptotic property $C$, then $G$ has asymptotic property $C$.

**Proof.** Let $R_1 < R_2 < \cdots$ be a given sequence of positive real numbers. Suppose that $U^1, U^2, \ldots, U^n$ is a collection of uniformly bounded subsets of $A$ such that each
Let $\mathcal{U}^i$ be $R_i$-disjoint and whose union covers $A$. Similarly, take $R_i$-disjoint uniformly bounded families $\mathcal{V}^i$ ($i = n + 1, n + 2, \ldots, n + k$) of subsets of $B$ whose union covers $B$. Put $r = R_{n+k}$ and take $Y_r \subset G$. Since $\operatorname{asdim} Y_r = 1$, take $\mathcal{Y}^{n+k+1}$ and $\mathcal{Y}^{n+k+2}$ to be uniformly bounded families of $R_{n+k+2}$-disjoint sets whose union covers $Y_r$. Let

$$
\overline{\mathcal{U}} = \bigcup_{i=1}^n \overline{\mathcal{U}^i}, \quad \overline{\mathcal{V}} = \bigcup_{i=n+1}^{n+k} \overline{\mathcal{V}^i}, \quad \text{and} \quad \mathcal{Y} = \mathcal{Y}^{n+k+1} \cup \mathcal{Y}^{n+k+2}
$$

For every $g \in G$, write $g = g_1g_2 \cdots g_t$, where each $g_i$ is a reduced element in $A \cup B$ and no consecutive elements belong to the same group. Let $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ denote the restrictions of $\mathcal{U}$ and $\mathcal{V}$ to $A - B_r(e)$ and $B - B_r(e)$, respectively.

Finally, let $X = X_r$ be the subset of $G$ of elements whose standard presentation ends in some $g \in A \cup B$ with $\|g\| > r$.

Define families $\mathcal{W}^i$ as follows: $\mathcal{W}^i = \{gU \mid g \in G, U \in \overline{\mathcal{U}^i}\}$ for $i = 1, 2, \ldots, n$. For $i = n + 1, n + 2, \ldots, n + k$, define $\mathcal{W}^i = \{gV \mid g \in G, V \in \overline{\mathcal{V}^i}\}$. Finally, for $i = n + k + 1, n + k + 2$, take $\mathcal{W}^i = \{xY \mid x \in X, Y \in \mathcal{Y}^i\}$. We show that the union of these families covers $G$, that each family is uniformly bounded, and that any distinct elements of the same family $\mathcal{W}^i$ are $R_i$ disjoint.

To see that it covers, we write $g$ in syllable form, $g = g_1 \cdots g_t$. If $g \in X_r$, then we see that $g_t$ is longer than $r$. Since $g_t$ is in either $A$ or $B$, there is some element of a cover that contains it. Thus, we find $g \in g_1 \cdots g_{t-1}W$, where $W$ is the element of $\overline{\mathcal{U}}$ or $\overline{\mathcal{V}}$ containing $g_t$. Suppose now that $g \notin X_r$. Then, we can write $g$ as $g_1 \cdots g_y$, where $y \in Y_r$ and $g_t \notin B_r(e)$ (or else $g_t$ is empty). Since $y \in Y_r$ there is some element $Y$ of $\mathcal{Y}$ containing $y$ and therefore $g_1 \cdots g_y \in g_1 \cdots g_y Y$. Thus, our collection covers.

That the families are uniformly bounded follows from the left invariance of the metric.

Finally, we check the disjointness conditions. To this end, suppose that we are given two elements of the same family, $\mathcal{W}^i$. If $i \leq n$, then we are considering $xa$ and $x'a'$, where $a$ and $a'$ are in some $U \in \overline{\mathcal{U}^i}$. If $x \neq x'$, then $d(xa, x'a') = \|x^{-1}x'x'a'\| \geq \|a\| + \|a'\| \geq 2r > R_i$. Suppose therefore that $x = x'$. Then, we are considering the pair $xa$ and $xa'$. But, $d(xa, xa') = d(a, a')$ which is at least $R_i$ when $a$ and $a'$ are not in the same element of $\mathcal{U}^i$. A similar proof holds for the other forms of $\mathcal{W}$.

A free product-like construction of pointed metric spaces is defined in [BD01]. From the above theorem, we see that this construction will pass to free products of proper discrete metric spaces with bounded geometry.

**Corollary 4.5.** Let $Z = X \ast Y$ be the free product of the proper, discrete metric spaces with bounded geometry. If $X$ and $Y$ have asymptotic property C, then so does $Z$.

**Question 4.6.** Is asymptotic property C preserved by amalgamated products?

**Question 4.7.** Is asymptotic property C preserved by direct products?

If the answers to the previous two questions are both yes, then it would immediately follow that the following question also has a positive answer.

**Question 4.8.** Let $\Gamma$ be a finite graph. If all the $G_v$ have asymptotic property C, does $\Theta \Gamma$ have asymptotic property C?

If the answer to that question is yes, one could additionally ask the following.
Question 4.9. Let \( \Gamma \) be a countably infinite graph with bounded clique number. Suppose that all \( G_n \) have asymptotic property \( C \). Then, in a proper, left-invariant metric, does \( G \) have asymptotic property \( C \)?

5. Straight Finite Decomposition Complexity

The goal of this section is to apply the techniques of Guentner, Tessera and Yu \([\text{GTY13}, \text{GTY12}]\) to the notion of straight finite decomposition complexity defined by Dranishnikov and Zarichnyi \([\text{DZ13}]\). It is shown in \([\text{DZ13}]\) that sFDC is a coarse invariant, is preserved by finite unions, and is preserved by some infinite unions (analogous to our theorem above about property \( C \)). We extend these results to show that sFDC is preserved by fiberings and conclude that it is preserved by amalgamated products and graph products.

We begin by recalling some of the results from \([\text{DZ13}]\).

**Theorem 5.1** \((\text{[DZ13] Theorem 3.1})\). If \( f : X \to Y \) is a coarse equivalence and if \( Y \) has sFDC, then so does \( X \).

We include a proof for the reader’s convenience and also because we will use the same technique to prove our fibering theorem.

**Proof.** Let \( f : X \to Y \) be uniformly expansive and effectively proper. Suppose that \( \rho : [0, \infty) \to [0, \infty) \) is an increasing function for which \( d(f(x), f(x')) \leq \rho(d(x, x')) \) for all \( x \) and \( x' \) in \( X \).

Let \( R_1 < R_2 < \cdots \) be given and set \( S_i = \rho(R_i) \) for each \( i \). By way of notation, put \( \{Y\} = V^0 \). Then, since \( Y \) has sFDC, there is some \( m \in \mathbb{N} \) and metric families \( V^1, V^2, \ldots, V^m \) so that \( V^0 \xrightarrow{S_1} V^1 \xrightarrow{S_2} V^2 \xrightarrow{S_3} \cdots \xrightarrow{S_m} V^m \) with \( V^m \) bounded. According to \([\text{GTY13}, \text{Lemma 3.1.1}]\), if \( V^i - \xrightarrow{i} V^i \) then \( f\left(V^i\right) - \xrightarrow{S} f\left(V^i\right) \).

More explicitly, write \( Y = V^0_0 \cup V^1_1 \), where

\[ V^1_1 = \bigcup_{S_1 \text{-disjoint}} V^1_{i_j}, \]

and \( V^1_{i_j} \in V^1 \). Then \( X = f^{-1}(Y) = f^{-1}(V^0_0) \cup f^{-1}(V^1_1) \), with

\[ f^{-1}(V^1_1) = \bigcup_{R_1 \text{-disjoint}} f^{-1}(V^1_{i_j}). \]

Then, for each \( V \in V^1 \), write \( V = V^2 \cup V^2 \where

\[ V^2 = \bigcup_{S_2 \text{-disjoint}} V^2_{i_j}, \]

and \( V^2_{i_j} \in V^2 \). Then, as above, obtain an \( R_2 \)-decomposition of \( f^{-1}(V^1) \) over \( f^{-1}(V^2) \). We continue in this way until we eventually find an \( R_m \)-decomposition of \( f^{-1}(V^{m-1}) \) over \( f^{-1}(V^m) \). Since \( f \) is effectively proper and \( V^m \) is bounded, we apply \([\text{GTY13}, \text{Lemma 3.1.2}]\) to conclude that \( f^{-1}(V^m) \) is bounded, as required. \( \square \)

Next, we obtain a version of \([\text{GTY13}, \text{Theorem 3.1.4}]\) for straight finite decomposition complexity.

**Theorem 5.2.** Let \( X \) and \( Y \) be metric spaces and let \( f : X \to Y \) be a uniformly expansive map. Assume that \( Y \) has sFDC and that for every bounded family \( \mathcal{V} \) in \( Y \), the inverse image \( f^{-1}(\mathcal{V}) \) has sFDC. Then, \( X \) has sFDC.
Proof. Let $R_1 < R_2 < \cdots$ be given. Since $Y$ has straight finite decomposition complexity, and since $f$ is uniformly expansive, we take $S_i = \rho(R_i)$ as in the previous theorem to find families $V^1, V^2, \ldots, V^m$ so that $V^i \xrightarrow{S_i} V^i$ and for which $V^m$ is bounded. Then, as before, we pull these families back to $X$ to obtain $f^{-1}(V^m) \xrightarrow{R_i} f^{-1}(V^i)$. Since we assume that $f^{-1}(V^m)$ has straight finite decomposition complexity, we take the sequence $R_{m+1}, R_{m+2}, \ldots$ and find $n$ and families $U^{m+1}, U^{m+2}, \ldots, U^{m+n}$ so that $U^{m+j-1} \xrightarrow{R_{m+j}} U^{m+j}$ with $U^{m+n}$ bounded. Then, with $U^i = f^{-1}(V^i)$ for $i = 1, 2, \ldots, m$ we have $U^{i-1} \xrightarrow{R_i} U^i$ for all $i = 1, 2, \ldots, m+n$ as required.

\[ \blacksquare \]

Proposition 5.3. Let $G$ be a countable group expressed as a union of subgroups $G = \cup G_i$, where each $G_i$ has straight finite decomposition complexity. Then, $G$ has straight finite decomposition complexity.

Proof. We equip $G$ with a proper left-invariant metric. Let $R_1 < R_2 < \cdots$ be given. Since the metric is proper, there is some $G_i$ that contains $B_{R_1}(e)$. Then, the decomposition of $G$ into cosets of $G_i$ is $R_1$-disjoint and each coset is isometric to $G_i$, which is assumed to have sFDC. \[ \blacksquare \]

The fibering theorem and the fact that the map $g \mapsto g.x$ for a group acting by isometries on a metric space is uniformly expansive [GTY13, Lemma 3.2.2] immediately imply:

Proposition 5.4. Let $G$ be a countable group acting on a metric space $X$ with straight finite decomposition complexity. If there is a $x_0 \in X$ so that for every $R > 0$ the $R$-coarse stabilizer of $x_0$ has straight finite decomposition complexity, then $G$ has straight finite decomposition complexity. \[ \blacksquare \]

Corollary 5.5. The following results easily follow from this theorem.

1. sFDC is closed under group extensions.
2. sFDC is closed under free products with amalgamation and HNN extensions.
3. sFDC is closed under finite graph products.
4. FDC is closed under finite graph products.

Proof. \hspace{1cm} (1) Suppose that $1 \to K \to G \xrightarrow{\phi} H \to 1$ is an exact sequence of countable groups with $H$ and $K$ both having straight finite decomposition complexity. Let $G$ act on $H$ by the rule $g.h = \phi(g)h$. The $R$-coarse stabilizer is coarsely equivalent to $K$, so it has sFDC. Thus, by the theorem, $G$ has sFDC.

(2) This follows from the Bass-Serre theory of graphs of groups. More precisely, if $G$ is an amalgamated product (or HNN extension), then there is a tree $T$ and an action of $G$ on that $T$ by isometries with vertex stabilizers isomorphic to the factors of the amalgam. The coarse stabilizers of the action will therefore have sFDC and so $G$ itself will.

(3) This follows from parts (1) and (2) using the technique of Corollary 3.5 or AD13.

(4) This is immediate from the results of GTY13 using the technique of Corollary 3.5 or AD13. \[ \blacksquare \]
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