Stress-energy of a quantized scalar field in static wormhole spacetimes

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An analytical approximation of \( \langle T_{\mu\nu} \rangle \) for a quantized scalar field in a static spherically symmetric spacetime with a topology \( S^2 \times R^2 \) is obtained. The gravitational background is assumed slowly varying. The scalar field is assumed to be both massive and massless, with an arbitrary coupling \( \xi \) to the scalar curvature and in a zero temperature vacuum state. It is demonstrated that for some values of curvature coupling the stress-energy has the properties needed to support the wormhole geometry.

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I. INTRODUCTION

In recent years Lorentzian wormholes have been of some interest as nontrivial geometric and physical objects. The topology of such spacetimes is \( R^2 \times S^2 \). The characteristic feature of wormhole geometry is the existence of a sphere of minimal square which cannot be contracted to the point. The possibility of the existence of a static spherically symmetrical traversable wormhole as a topology-nontrivial solution of the Einstein equations was first studied by Morris and Thorne \[1\]. They found that the matter which threads the wormhole throat should have unusual properties. In particular, the radial tension of the matter should exceed its density both locally at the throat \[1\] and integrally along the radial direction \[2\]. As an example of such matter they suggested the Casimir vacuum between conducting spherical plates. Violations of the energy conditions in static wormholes have been analyzed in detail by a number of authors (see, for example, \[3\]). Recently Barceló and Visser \[4,5\] demonstrated that a scalar field with a positive curvature coupling violates all the standard energy conditions. They found the entire branch of traversable wormhole solutions for gravity plus a nonminimally coupled massless scalar field. Solutions describing primordial wormholes are obtained for \( N = 4 \) SU(N) super Yang-Mills theory in \[6\]. The possibility of the existence of wormhole solutions has been investigated in Brans-Dicke theory \[7–9 \], higher derivative gravity \[10\], and Gauss-Bonnet theory \[11\].

Within the framework of general relativity as an example of the matter that supports the wormhole geometry one can consider the vacuum of quantized fields. This approach gives the possibility of describing the wormhole metric as a self-consistent solution of the semiclassical Einstein field equations \[12,13\]. The principal problem encountered in this method is the impossibility of calculating the functional dependence of the stress-energy tensor of quantized fields on the metric tensor. Nevertheless, in some cases approximate expressions can be obtained \[14–29\]. In above mentioned work \[12,13\] the approximations of Frolov-Zel’nikov \[21\] and Anderson-Hiscock-Samuel \[29\] were used. But as noted in \[31\] in spacetime that is a direct product of the Minkowski plane and a two-dimensional sphere of fixed radius these approximations are not applicable. The approximate expressions for the electromagnetic and neutrino stress-energy tensor obtained by Khatsymovsky \[26,27\] do not have this deficiency. In this work the results were obtained by the WKB method, and, as the zero order of the expansions was used, the exact solutions of the mode equations were obtained in spacetime with the metric \( ds^2 = -f_0 dt^2 + \frac{dr^2}{r_0^2} + r_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) \) \( (f_0 \) and \( r_0 \) are constants), i.e., in the above mentioned spacetime. It is well known that the vacuum expectation value of the stress-energy tensor of quantized fields depends on the topology of spacetime. In this connection the approach of Khatsymovsky seems more natural in wormhole spacetimes.

In this paper an approximate expression for the stress-energy tensor of a quantized scalar field in static spherically symmetric spacetimes with topology \( R^2 \times S^2 \) is calculated. The Anderson-Hiscock-Samuel approach \[29\] is used but the zeroth order of the WKB solution of the mode equation is redefined by analogy with \[26,27\] and mode sums are computed exactly as in \[30,31\].

The units \( \hbar = c = 1 \) are used throughout the paper.
II. AN UNRENORMALIZED EXPRESSION FOR $\langle T_{\mu}^{\nu} \rangle$

In this section the Euclidean space approach is used to calculate an unrenormalized expression for $\langle T_{\mu}^{\nu} \rangle$ for a scalar field in a static spherically symmetric spacetime with topology $S^2 \times R^2$. The metric for a general static spherically symmetric spacetime when continued analytically into Euclidean space can be written as

$$ds^2 = f(\rho)d\tau^2 + dr^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2).$$  \hspace{1cm} (1)

We assume that $-\infty < \rho < \infty$, $f$ and $r$ are arbitrary functions of $\rho$, and $\tau$ is the Euclidean time ($\tau = it$, where $t$ is the coordinate corresponding to the timelike Killing vector which always exists in a static spacetime).

$\langle T_{\mu}^{\nu} \rangle$ of a quantized scalar field $\phi$ can be computed using the method of point splitting from the Euclidean Green’s function $G_E(x, \tilde{x})$ as follows \cite{29}

$$\langle T_{\mu}^{\nu} \rangle_{\text{unren}} = (1/2 - \xi) (g^{\mu\alpha} G_{E;\tilde{x}_\nu} + g^\beta G_{E;\tilde{\alpha} \beta}) + (2\xi - 1/2) \delta^\mu_\nu g^{\sigma \tilde{\alpha}} G_{E;\sigma \tilde{\alpha}} - \xi G_{E;\mu \nu} \nonumber + g^{\mu \tilde{\alpha}} g^\beta G_{E;\tilde{\alpha} \beta}) + 2\xi \delta^\mu_\nu (m^2 + \xi R) G_E + \xi (R^\mu_\nu - \delta^\mu_\nu R/2) G_E - \delta^\mu_\nu m^2 G_E/2, \hspace{1cm} (2)$$

where $m$ is the mass of the scalar field, $\xi$ is its coupling to the scalar curvature $R$, and $g^{\alpha \beta}$ is the bivector of parallel transport of a vector at $\tilde{x}$ to one at $x$.

In \cite{29} the form of $G_E(x, \tilde{x})$ was derived for a scalar field in a static spherically symmetric spacetime when the field is in the zero temperature vacuum state defined with respect to the timelike Killing vector

$$G_E(x, \tilde{x}) = \frac{1}{4\pi^2} \int_0^\infty d\omega \cos[\omega(\tau - \tilde{\tau})] \sum_{l=0}^\infty (2l + 1) P_l(\cos \gamma) C_{\omega l} p_{\omega l}(\rho_\ast) q_{\omega l}(\rho_\ast), \hspace{1cm} (3)$$

where $P_l$ is a Legendre polynomial, $\gamma = \cos \theta \cos \tilde{\theta} + \sin \theta \sin \tilde{\theta} \cos(\varphi - \tilde{\varphi})$, $C_{\omega l}$ is a normalization constant, $\rho_\ast$ and $\rho_\ast$ represent the lesser and greater of $\rho$ and $\tilde{\rho}$, respectively, and the modes $p_{\omega l}(\rho)$ and $q_{\omega l}(\rho)$ obey the equation

$$\left\{ \frac{d^2}{d\rho^2} + \left[ \frac{1}{2f} \frac{df}{d\rho} + \frac{1}{r^2} \frac{dr^2}{d\rho} \right] \frac{d}{d\rho} - \left[ \omega^2 + \frac{l(l + 1)}{\rho^2} + m^2 + \xi R \right] \right\} \left\{ p_{\omega l} \right\} = 0. \hspace{1cm} (4)$$

They also satisfy the Wronskian condition

$$C_{\omega l} \left[ p_{\omega l} \frac{d q_{\omega l}}{d\rho} - q_{\omega l} \frac{d p_{\omega l}}{d\rho} \right] = -\frac{1}{r f^{1/2}}. \hspace{1cm} (5)$$

After the substitution

$$p_{\omega l} = \frac{1}{\sqrt{2r^2 W}} \exp \left\{ \int_\rho^0 W f^{-1/2} d\rho \right\}, \hspace{1cm} (6)$$

$$q_{\omega l} = \frac{1}{\sqrt{2r^2 W}} \exp \left\{ - \int_\rho^0 W f^{-1/2} d\rho \right\},$$

it is easy to see that the Wronskian condition (5) is obeyed if

$$C_{\omega l} = 1 \hspace{1cm} (7)$$

and the mode equation (4) gives the following equation for $W(\rho)$:

$$W^2 = \omega^2 + \frac{f}{r^2} \left[ (l + 1) + 2\xi + m^2 r^2 \right] + \frac{f'(W^2)^2}{8 W^2} + \frac{f}{4} \frac{(W^2)^\prime}{W^2} - \frac{5f}{16} \frac{(W^2)^2}{W^4} + V, \hspace{1cm} (8)$$

where

$$V = f \left( \frac{(r^2)^\prime}{2r^2} + \frac{f'(r^2)}{4r^2} - \frac{(r^2)^2}{4r^4} \right) + \xi f \left( -\frac{f''}{f} - 2 \frac{(r^2)^\prime}{r^2} + \frac{f'^2}{2f^2} + \frac{(r^2)^2}{2r^4} - \frac{f'(r^2)}{r^2} \right). \hspace{1cm} (9)$$

The prime denotes the derivative with respect to $\rho$.

Equation (8) can be solved iteratively when the metric functions $f(\rho)$ and $r^2(\rho)$ are varying slowly, that is,
\[ \lambda_{WKB} = \frac{L_*}{L} \ll 1, \quad (10) \]

where

\[ L_* = \left[ m^2 + \frac{2\xi}{r^2} \right]^{-1/2}, \quad (11) \]

and \( L \) is a characteristic scale of variation of the metric functions:

\[ L^{-1} = \max \left\{ \left| \ln(f r^2) \right|', \left| \ln(f r^2) \right|''^{1/2}, \left| \ln(f r^2) \right|'''^{1/3}, \ldots \right\}. \quad (12) \]

The zeroth-order WKB solution of Eq. (8) corresponds to neglecting terms with derivatives in this equation

\[ W^2 = U_0, \quad (13) \]

where

\[ U_0 = \omega^2 + \frac{f}{r^2} \left( \frac{l + \frac{1}{2}}{2} \right)^2 + \frac{f}{r^2} \mu^2, \]

\[ \mu^2 = m^2 r^2 + 2\xi - \frac{1}{4}. \quad (14) \]

Let us stress that \( U_0 \) is the exact solution of Eq. (8) in a spacetime with metric \( ds^2 = f_0 dr^2 + \rho^2 + r_0^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \), where \( f_0 \) and \( r_0 \) are constants.

Below as in Ref. [31] it is assumed that

\[ \mu^2 > 0 \quad (15) \]

or

\[ r^2 > \frac{1 - 8\xi}{4m^2}. \quad (16) \]

The second-order solution is

\[ W^2 = U_0 + \frac{f' U_0'}{8 U_0} + \frac{f U_0''}{4 U_0} - \frac{5f U_0'^2}{16 U_0^2}. \quad (17) \]

Now we can rewrite expression (3) using expressions (8) and (9) and then suppose \( \rho = \tilde{\rho}, \theta = \tilde{\theta}, \phi = \tilde{\phi} \). The superficial divergences in the sums over \( l \) that appear in this case can be removed as in Refs. [32,39].

\[ \langle T^t_{\mu} \rangle_{\text{unren}} = \left\{ \frac{1}{2} g^{tt} - \frac{\xi}{f} - \frac{\xi f (g^{tt})^2 - \xi (g^{\tilde{\rho}\tilde{\rho}})^2}{r^2 \partial^2 \xi} B_1 + \left( 2\xi - \frac{1}{2} \right) g^{\tilde{\rho}\tilde{\rho}} B_2 + \frac{1}{r^2} \right\} \]

\[ + \left[ \frac{f'}{2f} \frac{(g^{tt})^2}{r^2} + f \left( \frac{f'}{2f} - \frac{(r^2)'}{r^2} \right) (g^{\tilde{\rho}\tilde{\rho}})^2 \right] B_4 \]

\[ + \xi R^t_{\mu} B_1 + i\xi f g^{tt} \sqrt{\frac{f}{r^2}} \partial \xi B_1 + i\xi \left[ 2g^{\tilde{\rho}\tilde{\rho}} + 2g^{tt} g^{\tilde{\rho}\tilde{\rho}} \right] \sqrt{\frac{f}{r^2}} \partial \xi B_4, \quad (18) \]

\[ \langle T^\rho_{\tilde{\rho}} \rangle_{\text{unren}} = \left\{ \left[ 2\xi - \frac{1}{2} \right] g^{\tilde{\rho}\tilde{\rho}} + \frac{\xi}{f} + \frac{\xi f (g^{\tilde{\rho}\tilde{\rho}})^2}{r^2 \partial^2 \xi} B_1 + \frac{1}{2} g^{\tilde{\rho}\tilde{\rho}} B_2 \right\} \]

\[ + \frac{1}{r^2} \left[ \xi - \xi (g^{\tilde{\rho}\tilde{\rho}})^2 - \frac{1}{2} \right] B_3 + \left[ \xi \left( \frac{f'}{2f} + \frac{(r^2)'}{r^2} \right) (1 + (g^{\tilde{\rho}\tilde{\rho}})^2) + \xi f' (g^{\tilde{\rho}\tilde{\rho}})^2 \right] B_4 \]

\[ + \left[ \xi (1 + (g^{\tilde{\rho}\tilde{\rho}})^2) \left( \frac{1}{4r^2} - m^2 - \xi R \right) - \left( 2\xi - \frac{1}{2} \right) \right] \frac{1}{4r^2} + \left( 2\xi - \frac{1}{2} \right) m^2 + \xi R \]

\[ + \xi R^\rho_{\tilde{\rho}} B_1 + i\xi f g^{\tilde{\rho}\tilde{\rho}} g^{\tilde{\rho}\tilde{\rho}} \sqrt{\frac{f}{r^2}} \partial \xi B_1 + i\xi \left[ 2g^{\tilde{\rho}\tilde{\rho}} + 2g^{tt} g^{\tilde{\rho}\tilde{\rho}} \right] \sqrt{\frac{f}{r^2}} \partial \xi B_4, \quad (19) \]
\[
\langle T^\theta \rangle_{\text{uncen}} = \left(2\xi - \frac{1}{2}\right) g^{\theta i} \frac{f}{r^2} \frac{\partial}{\partial \xi} B_1 + \left(2\xi - \frac{1}{2}\right) g^{\rho \theta} B_2 + 2\xi \frac{\partial}{\partial \xi} B_3 - \frac{\xi (r^2)^{\prime \prime}}{2 r^2} B_4 \\
+ \left[ \frac{2\xi}{r^2} + \left(2\xi - \frac{1}{2}\right) (m^2 + \xi R) + \xi B_0 \right] B_1 + i 2 \left(2\xi - \frac{1}{2}\right) g^{\rho \theta} \sqrt{\frac{f}{r^2}} \frac{\partial}{\partial \xi} B_4,
\]

where

\[
B_1 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^2} \left[ \sqrt{\frac{f}{r^2}} \left(\frac{l+1/2}{W} - 1\right) \right],
\]

\[
B_2 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^4} \left[ - \left( l + \frac{1}{2} \right) \frac{r^2 W}{2f} \left(\frac{f}{r^2}\right)' \right] + \left( l + \frac{1}{2} \right) \frac{r^2 W}{2f} \left(\frac{f}{r^2}\right) + \left( l + \frac{1}{2} \right)^2 + \left( \frac{(u^2 + \mu^2)}{4} \right) \]

\[
+ \frac{r^2}{2f} V - \frac{(r^2)^{\prime \prime}}{2f} + \frac{r^2 f'}{16f} \left( \frac{f}{r^2} \right)' \left( \frac{r^2}{f} \right)' - \frac{(r^2)^{\prime \prime}}{4} \left( \frac{f}{r^2} \right) + \frac{r^2}{8} \left( \frac{r^2}{f} \right)'' \left( \frac{r^2}{f} \right) - \frac{7r^2}{32} \left( \frac{f}{r^2} \right)^{\prime \prime} \left( \frac{r^2}{f} \right)^{-2},
\]

\[
B_3 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^2} \left[ \sqrt{\frac{f}{r^2}} \left(\frac{l+1/2}{W} - \left( l + \frac{1}{2} \right)^2 + \frac{(u^2 + \mu^2)}{2} + \frac{r^4}{8f} \left( \frac{f}{r^2} \right)'' \right) \right]

+ \frac{r^4 f'}{16f^2} \left( \frac{f}{r^2} \right) - \frac{5r^4}{32f} \left( \frac{f}{r^2} \right)^{\prime \prime} + \frac{r^2}{2f} V, \]

\[
B_4 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^4} \left[ \left( l + \frac{1}{2} \right) \frac{(r^2)^{\prime \prime}}{4r^2} \sqrt{\frac{f}{r^2}} \left( \frac{f}{r^2} \right) - \left( l + \frac{1}{2} \right) \frac{|\epsilon|}{4} \sqrt{\frac{f}{r^2}} \left( \frac{f}{r^2} \right) \right]

+ \frac{(r^2)^{\prime \prime}}{4|\epsilon|} + \frac{|\epsilon| f'}{4f},
\]

\[
\epsilon = \sqrt{\frac{f}{r^2}(r - \hat{\tau})}.
\]

III. USE OF THE WKB APPROXIMATION IN EVALUATING \(\langle T^\nu \rangle\)

The approximate (of second WKB order) expressions for the quantities \(B_1, B_2, B_3,\) and \(B_4\) are obtained by substituting the WKB expansion of \(W^2\) [Eq. (8)] into Eqs. (21)-(24):

\[
B_1 = \frac{1}{4\pi^2} \left\{ \frac{1}{r^2} S_0^0(\epsilon, \mu) - \frac{V}{2f} S_1^0(\epsilon, \mu) - \frac{r^2}{16f^2} \left[ f' \left( \frac{f}{r^2} \right) \right]'' + \frac{2f}{r^2} \left( \frac{f}{r^2} \right)'' \right\}

+ \frac{5r^4}{32f^2} \left( \frac{f}{r^2} \right)^{\prime \prime} S_3^0(\epsilon, \mu) + \frac{5r^4}{16f^3} f \left( \frac{f}{r^2} \right) \left( \frac{f}{r^2} \right)' S_3^0(\epsilon, \mu)

+ \frac{5r^4}{32f^3} f \left( \frac{f}{r^2} \right)^{\prime \prime} S_3^0(\epsilon, \mu) + O \left( \frac{r^2}{L^4} \right),
\]
\[
B_2 = \frac{1}{4\pi^2} \left\{ \frac{1}{r^2} S_{-1}^{0}(\epsilon, \mu) - \frac{1}{4f r^6} \left[ - (r^2)^2 f + 2V r^4 \right] S_{0}^{0}(\epsilon, \mu) \right. \\
- \frac{1}{16 f^2 r^2} \left[ 2f r^2 \left( \frac{f}{r^2} \right)^{'''} - 4f \left( r^2 \right)^2 \left( \frac{f}{r^2} \right)^{'} + r^2 f' \left( \frac{f}{r^2} \right)^{''} \right] S_{1}^{0}(\epsilon, \mu) \\
- \frac{1}{16 f^2 r^2} \left[ r^2 f' \left( \mu^2 f \right)^{'} + 2f r^2 \left( \mu^2 f \right)^{''} - 4f \left( r^2 \right)^2 \left( \mu^2 f \right)^{'} \right] S_{1}^{0}(\epsilon, \mu) \\
+ \frac{7r^2}{32 f^2} \left( \frac{f}{r^2} \right)^{'} S_{2}^{0}(\epsilon, \mu) + \frac{7r^2}{16 f^2} \left( \frac{f}{r^2} \right)^{'} \left( \mu^2 f \right)^{'} S_{1}^{0}(\epsilon, \mu) \\
+ \frac{7r^2}{32 f^2} \left( \mu^2 f \right)^{'} S_{2}^{0}(\epsilon, \mu) \right\} + O \left( \frac{1}{L^4} \right), 
\tag{27}
\]

\[
B_3 = \frac{1}{4\pi^2} \left\{ \frac{1}{r^2} S_{1}^{0}(\epsilon, \mu) - \frac{V}{2f} S_{1}^{0}(\epsilon, \mu) - \frac{r^2}{16 f^2} \left[ f^{'} \left( \frac{f}{r^2} \right)^{''} \right] \\
+ 2f \left( \frac{f}{r^2} \right)^{''} \left[ S_{2}^{0}(\epsilon, \mu) - \frac{r^2}{16 f^2} \left[ 2f \left( \mu^2 f \right)^{''} \right] \left( \mu^2 f \right)^{'} \right] S_{2}^{0}(\epsilon, \mu) \\
+ \frac{5r^4}{32 f^2} \left( \frac{f}{r^2} \right)^{'} S_{3}^{0}(\epsilon, \mu) + \frac{5r^4}{16 f^2} f \left( \frac{f}{r^2} \right)^{'} \left( \mu^2 f \right)^{'} S_{3}^{0}(\epsilon, \mu) \\
+ \frac{5r^4}{32 f^2} f \left( \mu^2 f \right)^{'} S_{3}^{0}(\epsilon, \mu) \right\} + O \left( \frac{r^2}{L^4} \right), 
\tag{28}
\]

\[
B_4 = \frac{1}{4\pi^2} \left\{ - \frac{(r^2)^2}{2f^4} S_{0}^{0}(\epsilon, \mu) - \frac{1}{4f} \left( \frac{f}{r^2} \right)^{'} S_{1}^{0}(\epsilon, \mu) \\
- \frac{1}{4f} \left( \mu^2 f \right)^{'} S_{1}^{0}(\epsilon, \mu) \right\} + O \left( \frac{1}{L^4} \right). 
\tag{29}
\]

The mode sums and integrals in these expressions are of the form

\[
S_m^0(\epsilon, \mu) = \int_0^\infty du \cos(\epsilon u) \sum_{l=0}^\infty \left\{ \frac{(l + 1/2)^{2m+1}}{u^2 + \mu^2 + (l + 1/2)^2} \right. - \text{subtraction terms} \left. \right\}, 
\tag{30}
\]

where \( m \) and \( n \) are integers, \( m \geq 0 \) and \( n \geq -1 \). The subtraction terms for the sum over \( l \) can be obtained by expanding the function that is summed in inverse powers of \( l \) and truncating at \( O(l^0) \). Such subtracting corresponds to removing the superficial divergences in the sums over \( l \) discussed above:

\[
S_{0}^{0}(\epsilon, \mu) = \int_0^\infty du \cos(\epsilon u) \sum_{l=0}^\infty \left\{ \frac{(l + 1/2)^3}{\sqrt{u^2 + \mu^2 + (l + 1/2)^2}} - \left( l + \frac{1}{2} \right)^2 + \frac{u^2 + \mu^2}{2} \right\}, 
\tag{31}
\]

\[
S_{-1}^{0}(\epsilon, \mu) = \int_0^\infty du \cos(\epsilon u) \sum_{l=0}^\infty \left\{ \left( l + \frac{1}{2} \right) \sqrt{u^2 + \mu^2 + (l + 1/2)^2} - \left( l + \frac{1}{2} \right)^2 - \frac{u^2 + \mu^2}{2} \right\}, 
\tag{32}
\]

\[
S_{m}^{0}(\epsilon, \mu) = \int_0^\infty du \cos(\epsilon u) \sum_{l=0}^\infty \left\{ \frac{(l + 1/2)^{2m+1}}{\left[ u^2 + \mu^2 + (l + 1/2)^2 \right]^{(2m+1)/2}} - 1 \right\}. 
\tag{33}
\]
For other quantities $S^n_m(\varepsilon, \mu)$ there are no subtraction terms. The details of calculations of $S^n_m(\varepsilon, \mu)$ in the limit $\varepsilon \to 0$ are discussed in Appendix A:

\[
S_0^1(\varepsilon, \mu) = \frac{4}{\varepsilon^2} - \frac{\mu^2}{\varepsilon^2} + \left( \frac{7}{960} - \frac{\mu^4}{4} \right) \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) + \frac{3}{16} \mu^4 + \mu^4 I_2(\mu) + O \left( \varepsilon^2 \ln |\varepsilon| \right),
\]

\[
S_{-1}^0(\varepsilon, \mu) = -\frac{2}{\varepsilon^4} - \frac{\mu^2}{\varepsilon^4} \left( \frac{\mu^2}{2} - \frac{1}{24} \right) + \left( \frac{\mu^4}{8} - \frac{\mu^2}{48} + \frac{7}{1920} \right) \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) - \frac{3}{32} \mu^4
+ \frac{\mu^2}{96} - \frac{7}{3840} - \frac{\mu^4}{2} I_1(\mu) + \frac{\mu^4}{2} I_2(\mu) + O \left( \varepsilon^2 \ln |\varepsilon| \right),
\]

\[
S_0^0(\varepsilon, \mu) = \frac{1}{\varepsilon^2} + \left( \frac{\mu^2}{2} - \frac{1}{24} \right) \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) - \frac{\mu^2}{4} - \mu^2 I_1(\mu) + \varepsilon^2 \left\{ -\frac{5}{64} \mu^4 + \frac{\mu^2}{96} \right.
- \frac{7}{3840} + \frac{\mu^4}{16} - \frac{\mu^2}{96} + \frac{7}{3840} \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) - \frac{\mu^4}{4} I_1(\mu) + \frac{\mu^4}{4} I_2(\mu)
+ O \left( \varepsilon^4 \ln |\varepsilon| \right),
\]

\[
S^n_0(\varepsilon, \mu) = \left( \sum_{k=0}^{n} \frac{(-1)^{n+k} n!}{k!(n-k)!(2n-2k-1)} \right) \left\{ -\frac{1}{\varepsilon^2} - \frac{\mu^2}{2} \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) + \frac{\mu^2}{4} \right.
- \frac{\mu^2}{(2n-1)!!} \left( \frac{d}{d\mu} \right)^n \left( \mu^{2n} I_1(\mu) \right) + O \left( \varepsilon^2 \ln |\varepsilon| \right) \} (n \geq 1),
\]

\[
S^n_1(\varepsilon, \mu) = -\left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) + \mu \frac{d}{d\mu} I_0(\mu) + \varepsilon^2 \left\{ -\frac{\mu^2}{4} + \frac{1}{48} \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) \right.
+ \frac{\mu^2}{4} - \frac{1}{96} + \frac{\mu^2}{2} I_1(\mu)
+ O \left( \varepsilon^4 \ln |\varepsilon| \right),
\]

\[
S^n_{n+1}(\varepsilon, \mu) = \frac{1}{(2n+1)!!} \left\{ -2^n n! \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) + \mu^2 \left( \frac{\partial}{\mu \partial \mu} \right)^{n+1} (\mu^{2n} I_0(\mu))
+ \varepsilon^2 \left\{ -\mu^2 2^{n-2} n! \left( C + \frac{1}{2} \ln \left| \frac{\mu^2 \varepsilon^2}{4} \right| \right) + \mu^2 2^{n-2} n! + \frac{\mu^2}{2} \left( \frac{\partial}{\mu \partial \mu} \right)^n (\mu^{2n} I_1(\mu)) \right\} \right.
+ O \left( \varepsilon^4 \ln |\varepsilon| \right) \} (n \geq 1),
\]

\[
S^n_m(\varepsilon, \mu) = \frac{(2m+1)!!}{(2n+1)!!} \left( -\frac{\partial}{\mu \partial \mu} \right)^{n-m-1} S^n_{m+1}(\varepsilon, \mu) \quad (m \geq 0, \, n \geq m + 2),
\]

where

\[
I_n(\mu) = \int_0^\infty \frac{x^{2n-1} \ln |1 - x^2|}{1 + e^{2\mu x}} dx.
\]

Substitution of these expressions into Eqs. (21)-(24) and then into Eqs. (15)-(24) gives nontrivial components of $\langle T^\mu_\nu \rangle_{unren}$.

The renormalization of $\langle T^\mu_\nu \rangle$ is achieved by subtracting from $\langle T^\mu_\nu \rangle_{unren}$ the counterterms $\langle T^\mu_\nu \rangle_{DS}$ and then letting $\tilde{\tau} \to \tau$:

\[
\langle T^\mu_\nu \rangle_{ren} = \lim_{\tilde{\tau} \to \tau} \left[ \langle T^\mu_\nu \rangle_{unren} - \langle T^\mu_\nu \rangle_{DS} \right].
\]

The expression for $\langle T^\mu_\nu \rangle_{DS}$ are displayed in Ref. [33] and are determined by $\sigma$, one half the square of the distance between the points $x$ and $\tilde{x}$ along the shortest geodesic connecting them, and $\sigma^\mu$, covariant derivative of $\sigma$. For the metric $\bar{g}^{\mu\nu}$ the calculations of these quantities and $\bar{g}^{\mu\nu}$ are similar to those in Ref. [29]:

6
\begin{align}
\sigma^i &= (t - \bar{t}) + \frac{f^2}{24f}(t - \bar{t})^3 + O ((t - \bar{t})^5), \\
\sigma^\nu &= -\frac{f'}{4}(t - \bar{t})^2 + O ((t - \bar{t})^4), \\
\sigma^0 &= \sigma^\theta = 0, \\
\sigma &= \frac{1}{2}g_{\mu\nu}\sigma^\mu\sigma^\nu, \\
g^{\bar{t}i} &= \frac{g^{\bar{t}\bar{t}}}{f} = -\frac{1}{f} - \frac{f^2}{8f^2}(t - \bar{t})^2 + O ((t - \bar{t})^4), \\
g^{\bar{t}\bar{t}} &= -g^{\bar{t}i}(t - \bar{t}) + O ((t - \bar{t})^3) \\
\end{align}

(43)

and

\begin{align}
\langle T^i_{\bar{t}} \rangle_{DS} &= \frac{1}{4\pi^2} \left( -\frac{6}{r^4} - \frac{1}{r^2}\xi^2 \right) - \left( \frac{m^2}{2} + \frac{1}{6r^2} - \frac{\xi}{2r^2} + \left( \frac{2\xi f''}{r^2} + \frac{(r^2)''}{24r^4} - \frac{(r^2)''}{6r^2} + \frac{2\xi f'^r}{r^2f} \right) \right) + \left\{ \frac{m^4}{8} + \left( \frac{1}{12r^2} - \frac{\xi}{2r^2} \right)m^2 - \frac{\xi^2}{24r^4} + \frac{\xi}{6r^4} - \frac{1}{60r^4} + \left( \frac{(r^2)''}{12r^2} - \frac{\xi}{2r^2} + \frac{(r^2)''}{8r^4} \right)m^2 + \left( \frac{3\xi^2 (r^2)''}{4r^6} + \frac{\xi (r^2)'^r}{4r^6} + \frac{(r^2)'^r}{4r^2} \right) \right\} \left( C + \frac{1}{2} \ln \left( \frac{m_{\text{Dyson}}}r \right) \right) - \frac{m^4}{32} + \left( \frac{1}{124r^2} - \frac{\xi}{24r^4} \right)m^2 - \frac{\xi^2}{6r^4} + \frac{\xi}{60r^4} + \left( \frac{(r^2)''}{48r^6} - \frac{\xi}{4r^6} \right)m^2 - \frac{\xi^2 (r^2)'^r}{4r^6} + \frac{5\xi (r^2)''}{96r^2f^2} + \frac{f^{r2}}{2r^2f^2} - \frac{(r^2)''}{4r^4f^2} + \frac{\xi f''}{r^4f^2} + \frac{f''}{24r^4f^2} + \frac{\xi (r^2)'^r}{6r^2f^2} + \frac{1}{L^2} \right) \right), \\
\end{align}

(45)

\begin{align}
\langle T^\nu_{\bar{t}} \rangle_{DS} &= \frac{1}{4\pi^2} \left( \frac{2}{r^2} - \frac{1}{r^2}\xi^2 \right) - \left( \frac{m^2}{2} + \frac{\xi}{2r^2} - \frac{1}{6r^2} \right) + \left( \frac{5f'^2}{12r^2} + \frac{(r^2)''}{24r^4} - \frac{(r^2)''}{12r^2f^2} \right) \right) + \left\{ \frac{m^4}{8} + \left( \frac{1}{12r^2} - \frac{\xi}{2r^2} \right)m^2 - \frac{\xi^2}{2r^4} + \frac{\xi}{6r^4} - \frac{1}{60r^4} + \left( \frac{(r^2)''}{12r^2} + \frac{\xi}{2r^2} - \frac{(r^2)''}{8r^4} \right)m^2 + \left( \frac{3\xi^2 (r^2)'^r}{4r^6} - \frac{\xi^2 (r^2)'^r}{4r^6} + \frac{(r^2)'^r}{4r^2} \right) \right\} \left( C + \frac{1}{2} \ln \left( \frac{m_{\text{Dyson}}}r \right) \right) + \frac{3m^4}{32} + \left( \frac{\xi}{4r^4} - \frac{1}{24r^2} \right)m^2 + \left( \frac{(r^2)''}{96r^4} - \frac{\xi^2 (r^2)'^r}{4r^6} + \frac{5\xi f'^2}{8r^2f^2} + \frac{f'^2}{48r^2f^2} \right)m^2 - \frac{5f'^2}{288r^2f^2} + \frac{f'^2}{48r^4f^2} + \frac{\xi (r^2)''}{24r^2f^2} - \frac{5\xi f'^2}{48r^2f^2} - \frac{5\xi f'^2}{24r^2f^2} - \frac{f''}{24r^4f^2} + \frac{\xi (r^2)''}{6r^2f^2} + \frac{1}{L^2} \right) \right), \\
\end{align}

(46)

\begin{align}
\langle T^0_{\bar{t}} \rangle_{DS} &= \frac{1}{4\pi^2} \left( \frac{2}{r^2} - \frac{1}{r^2}\xi^2 \right) - \left( \frac{m^2}{2} + \frac{\xi (r^2)'^r}{4r^4} + \frac{(r^2)''}{12r^2} \right) + \left( \frac{5f'^2}{12r^2} - \frac{f'^2}{24r^4} - \frac{f'^2}{12r^2f^2} \right) \right) + \left\{ \frac{m^4}{8} + \left( \frac{\xi^2}{2r^4} + \frac{\xi}{6r^4} - \frac{1}{60r^4} \right)m^2 - \frac{\xi (r^2)''}{2r^2f^2} - \frac{\xi f''}{2r^4f^2} + \frac{f''}{24r^4f^2} \right\} + \left( \frac{(r^2)''}{24r^2f^2} - \frac{\xi (r^2)''}{2r^2f^2} + \frac{f''}{2r^4f^2} \right) \right), \\
\end{align}

(47)
The expressions (48) and (49) may be simplified in the cases
\[ L \text{const} \langle \text{or}\text{response to a finite renormalization of the coefficients of terms in the gravitational Lagrangian and must be fixed} \]
C of nontrivial components of
\[ \langle T_{\mu}^{\nu} \rangle_{\text{ren}} \]
Let us stress that the renormalized expectation value of the stress-energy tensor (48), (49) are
\[ \text{have the form} \]
\[ \text{The procedure described above gives a second-order WKB approximation of} \langle T_{\mu}^{\nu} \rangle_{\text{ren}}. \text{The zeroth-order expressions of nontrivial components of} \langle T_{\mu}^{\nu} \rangle_{\text{ren}} \text{have the form} \]
\[ \langle T_{i}^{i} \rangle_{\text{ren}} = \langle T_{\rho}^{\rho} \rangle_{\text{ren}} = \frac{1}{4 \pi^{2}} \left( \frac{m^{2}}{2 r^{2}} \left( \frac{\xi}{8} - \frac{1}{8} \right) + \frac{1}{r^{4}} \left( \frac{79}{7680} - \frac{11}{96} \xi + \frac{3}{8} \xi^{2} \right) \right) \]
\[ + \left[ - \frac{m^{4}}{8} + \frac{m^{2}}{2 r^{2}} \left( \frac{1}{6} - \xi \right) + \frac{1}{r^{4}} \left( - \frac{1}{60} + \frac{1}{6} \xi - \frac{1}{2} \xi^{2} \right) \right] \ln \sqrt{\frac{m^{2}}{m_{\text{DS}} r^{2}}} \]
\[ + \left[ \frac{m^{4}}{2} + 2 m^{2} \left( \xi - \frac{1}{8} \right) + \frac{2}{r^{4}} \left( \xi - \frac{1}{8} \right)^{2} \right] I_{1}(\mu) - I_{2}(\mu) \right) \right) \}, \]
\[ \langle T_{\theta}^{\theta} \rangle_{\text{ren}} = \langle T_{\phi}^{\phi} \rangle_{\text{ren}} = \frac{1}{4 \pi^{2}} \left( \frac{m^{2}}{2 r^{2}} \left( \frac{\xi}{8} - \frac{1}{8} \right) + \frac{1}{r^{4}} \left( - \frac{1}{60} + \frac{1}{6} \xi - \frac{1}{2} \xi^{2} \right) \right) \ln \sqrt{\frac{m^{2}}{m_{\text{DS}} r^{2}}} \]
\[ + \left[ \frac{m^{4}}{2} + 2 m^{2} \left( \xi - \frac{1}{8} \right) + \frac{2}{r^{4}} \left( \xi - \frac{1}{8} \right)^{2} \right] I_{1}(\mu) \]
\[ + \left[ \frac{m^{4}}{2} + 2 m^{2} \left( \xi - \frac{1}{8} \right) + \frac{2}{r^{4}} \left( \xi - \frac{1}{8} \right)^{2} \right] I_{2}(\mu) \right) \}. \]
The second-order expressions of \( \langle T_{\mu}^{\nu} \rangle_{\text{ren}} \) are given in Appendix B.
Let us stress that the renormalized expectation value of the stress-energy tensor (48), (49) are exact if \( f(\rho) = f_{0} = \text{const} \) and \( r(\rho) = r_{0} = \text{const} \); i.e., in the spacetime with metric
\[ ds^{2} = -f_{0} dt^{2} + d\rho^{2} + r_{0}^{2}(d\theta^{2} + \sin^{2} \theta d\varphi^{2}). \]
The expressions (48) and (49) may be simplified in the cases \( L^{2} \gg r^{2}, m^{2} r^{2} \ll 1 \) and \( L^{2} \gg r^{2}, m^{2} r^{2} \gg 1. \)

A. The case \( L^{2} \gg r^{2}, m^{2} r^{2} \ll 1 \)

In this case
\[ \langle T_{i}^{i} \rangle_{\text{ren}} = \langle T_{\rho}^{\rho} \rangle_{\text{ren}} = \frac{1}{4 \pi^{2} r^{4}} \left( \frac{m^{2}}{2 r^{2}} \left( \frac{\xi}{8} - \frac{1}{8} \right) + \frac{1}{r^{4}} \left( \frac{79}{7680} - \frac{11}{96} \xi + \frac{3}{8} \xi^{2} \right) \right) \]
\[ + \left( \frac{2 \xi^{2}}{r^{4}} - \frac{1}{2} \xi + \frac{1}{32} \right) \left( I_{1} \left( 2 \xi - \frac{1}{4} \right) - I_{2} \left( 2 \xi - \frac{1}{4} \right) \right) \right) \]
\[ + O \left( \frac{m^{2}}{r^{2}} \right) + O \left( \frac{1}{L^{2} r^{2}} \right), \]
\[ (51) \]
\[
\langle T_{\theta\theta}^6 \rangle_{\text{ren}} = \langle T_{\varphi\varphi}^6 \rangle_{\text{ren}} = \frac{1}{4\pi^2 r^4} \left\{ \left( -\frac{1}{8} \xi^2 + \frac{1}{32} \xi - \frac{1}{512} \right) + \left( \frac{1}{2} \xi^2 - \frac{1}{6} \xi + \frac{1}{60} \right) \ln \sqrt{\frac{8 \xi - 1}{4 m_{DS}^2 r^2}} \right. \\
+ \left( -2 \xi^2 + \frac{1}{2} \xi - \frac{1}{32} \right) \left[ I_1 \left( 2 \xi - \frac{1}{4} \right) - I_2 \left( 2 \xi - \frac{1}{4} \right) \right] \left. \right\} + O \left( \frac{m^2}{r^2} \right) + O \left( \frac{1}{L^2 r^2} \right). \quad (52)
\]

In particular, if \( \xi = 1/6 \),

\[
I_1(\sqrt{3}/6) \approx -0.05962, \quad I_2(\sqrt{3}/6) \approx 0.50385
\]

and

\[
\langle T^\mu_\nu \rangle_{\text{ren}} = \frac{1}{4\pi^2 r^4} \left[ -0.00171 + \frac{1}{360} \ln \frac{1}{m_{DS} |r|} \right] + O \left( \frac{m^2}{r^2} \right) + O \left( \frac{1}{L^2 r^2} \right). \quad (55)
\]

**B. The case \( L^2 \gg r^2, m^2 r^2 \gg 1 \)**

In this case \( m_{DS} = m \). Therefore

\[
\ln \sqrt{\frac{\mu^2}{m^2 r^2}} = \frac{8 \xi - 1}{8 m^2 r^2} - \frac{(8 \xi - 1)^2}{64 m^4 r^4} + \frac{(8 \xi - 1)^3}{384 m^6 r^6} + O \left( \frac{1}{m^8 r^8} \right), \quad (56)
\]

\[
I_1(\mu) = -\frac{7}{1920} m^4 r^4 + \frac{1}{m^6 r^6} \left[ \frac{7}{480} \left( \xi - \frac{1}{8} \right) - \frac{31}{32256} \right] + O \left( \frac{1}{m^8 r^8} \right),
\]

\[
I_2(\mu) = -\frac{31}{16128} m^6 r^6 + O \left( \frac{1}{m^8 r^8} \right)
\]

and

\[
\langle T^\mu_\nu \rangle_{\text{ren}} = \left( \begin{array}{cccc}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{array} \right) \left[ \frac{1}{4\pi^2 m^2 r^6} \left( \frac{\xi^3}{6} - \frac{\xi^2}{12} + \frac{\xi}{60} - \frac{1}{630} \right) \right] + O \left( \frac{1}{m^4 r^8} \right) + O \left( \frac{1}{m^2 L^2 r^4} \right). \quad (58)
\]

As was demonstrated by Morris and Thorne [1] the radial tension \( \tau \) at the wormhole throat must be positive. In the considered case \( \tau = -\langle T^\rho_\varphi \rangle_{\text{ren}} > 0 \) for

\[
\xi > \left( \frac{9800 + 735 \sqrt{205}}{210} \right)^{1/3} - \frac{7}{6(9800 + 735 \sqrt{205})^{1/3}} + \frac{1}{6} \approx 0.2538. \quad (59)
\]

As an example of a metric function \( r^2(\rho) \) satisfying the condition \( L^2 \gg r^2, m^2 r^2 \gg 1 \) one can consider a function growing as a logarithm:

\[
d s^2 = -d t^2 + d \rho^2 + r_0^2 \left[ 1 + \alpha^2 \ln \left( 1 + \frac{\rho^2}{\rho_0^2} \right) \right] \left( d \theta^2 + \sin^2 \theta d \phi^2 \right), \quad (60)
\]

where \( r_0, \alpha, \) and \( \rho_0 \) are constants. The conditions \( L^2 \gg r^2 \) and \( r^2 m^2 \gg 1 \) gives \( \alpha^2 r_0^2/\rho_0^2 \ll 1 \) and \( r_0^2 m^2 \gg 1 \), respectively.
IV. CONCLUSION

We have obtained an analytical approximation of the stress-energy tensor of quantized scalar fields in static spherically symmetric spacetimes with topology $S^2 \times R^2$. For some values of coupling to the scalar curvature $\xi$ the stress-energy tensor obtained has the needed exotic (in the sense of Morris and Thorne [1]) properties to support a static wormhole. All three approximations \((12), (13), (14), (15)\) and \((18)\) for \(T^{\mu \nu}_{\text{ren}}\) are conserved, i.e.,

\[
\langle T^{\mu \nu}_{\text{ren}; \mu} \rangle = 0,
\]

and for the conformally invariant field the approximation for \(\langle T^{\mu \nu}_{\text{ren}} \rangle\) has a trace equal to the trace anomaly. These approximations are determined by the correlation between three length scales: \(L (12), r (1)\) and the Compton length \(1/m\). The use of the expression obtained for \(\langle T^{\mu \nu}_{\text{ren}} \rangle\) as a source term in the semiclassical Einstein field equations \(G^{\mu \nu}_{\text{ren}} = 8\pi G \langle T^{\mu \nu}_{\text{ren}} \rangle\) demands some accuracy. First of all in these equations a new length scale \(\lambda_{\text{ren}}(1)\) and the next orders are needed for calculation of these quantities.

In conclusion let us note that in the case \(L^2 \lesssim r^2\) the WKB parameter $\lambda_{\text{WKB}}$ [Eq. (10)] coincides with the small parameter \(1/(mL)\) of the DeWitt-Schwinger expansion of \(\langle T^{\mu \nu} \rangle\) and with the small parameter used in \([29]\) to obtain an analytical approximation for \(\langle T^{\mu \nu}_{\text{ren}} \rangle\). In this case as the first and second WKB orders of \(\langle T^{\mu \nu}_{\text{ren}} \rangle\) vanish [see Eqs. \((31)-(33)\)] and the next orders are needed for calculation of these quantities.

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APPENDIX A

The sums over \(l\) in expression \((30)\) can be evaluated by using the following method:

\[
J_n = 2i \sum_{l=0}^{n} f(l+1/2) = -2\pi i \sum_{l=0}^{n} \text{res}_{z=l+1/2} [f(z) \tan(\pi z)] = -\int_{C} f(z) \tan(\pi z) dz, \tag{A1}
\]

where \(C\) is a closed contour which surrounds a region on the complex plane containing the poles of \(\tan(\pi z): z = l+1/2\) for \(l = 0, 1, 2, ..., n\) . It is supposed that contour \(C\) is passed anticlockwise, \(f(z)\) is a holomorphic function inside this contour, and

\[
|f(x+iy)| < \epsilon(x)e^{\alpha|y|}, \quad 0 < \alpha < 2\pi, \tag{A2}
\]

where

\[
\epsilon(x) \to 0 \quad \text{for} \quad x \to +\infty. \tag{A3}
\]

Let us choose the contour so that

\[
J_n = \left\{ \int_{q-i}^{q+ih} + \int_{q+1-i}^{q+i+1} \right\} f(z) \tan(\pi z) dz - \left\{ \int_{q+i}^{q+ih} + \int_{q+i+1}^{q+i+1} \right\} f(z) \tan(\pi z) dz = 2i \int_{q}^{q+i+1} f(x) dx - \int_{q}^{q-i} [f(z) - f(z + n + 1)] [\tan(\pi z) + i] dz - \int_{q-i}^{q+i+1} f(z) [\tan(\pi z + i)] dz + \int_{q}^{q+i} [f(z) - f(z + n + 1)] [\tan(\pi z) - i] dz - \int_{q+i}^{q+i+1} f(z) [\tan(\pi z) - i] dz, \tag{A4}
\]
where $-1/2 < q < 1/2$ and $h > 0$. However,

$$J_\neq = \left| \int_{q-i}^{q+n+1} f(z)[\tan(\pi z + i)]dz \right| \leq (n + 1) \max_{q \leq x \leq q+n+1} |f(x - ih)||\tan[\pi(x - ih)] + i|$$

$$< (n + 1) \max_{q \leq x \leq q+n+1} \epsilon(x) \frac{2e^{nh}}{(e^{2\pi h} - 1)},$$

$$J_\neq = \left| \int_{q+i}^{q+n+1+i} f(z)[\tan(\pi z) - i]dz \right| \leq (n + 1) \max_{q \leq x \leq q+n+1} |f(x + ih)||\tan[\pi(x + ih)] - i|$$

$$< (n + 1) \max_{q \leq x \leq q+n+1} \epsilon(x) \frac{2e^{nh}}{(e^{2\pi h} - 1)},$$

therefore $J_\neq \to 0, J_\neq \to 0$ if $h \to \infty$, and in this case

$$J_n = 2i \int_q^{q+n+1} f(x)dx - \int_q^{q-i\infty} [f(z) - f(z + n + 1)][\tan(\pi z) + i]dz$$

$$+ \int_q^{q+i\infty} [f(z) - f(z + n + 1)][\tan(\pi z) - i]dz,$$

(A5)

If we let $n \to \infty$ then

$$\left| \int_q^{q-i\infty} f(z + n + 1)[\tan(\pi z) + i]dz \right| \leq \epsilon(q + n + 1) \int_0^{\infty} e^{ay} \left| e^{-i\pi q} \frac{2e^{-i\pi q}}{e^{-i\pi q} + e^{-2\pi y}e^{i\pi q}} \right| dy$$

$$\leq \text{const} \times (q + n + 1) \int_0^{\infty} e^{(a-2\pi)y} dy \to 0 \quad (n \to \infty),$$

(A7)

$$\left| \int_q^{q+i\infty} f(z + n + 1)[\tan(\pi z) - i]dz \right| \leq \epsilon(q + n + 1) \int_0^{\infty} e^{ay} \left| e^{i\pi q} \frac{2e^{i\pi q}}{e^{i\pi q} + e^{2\pi y}e^{-i\pi q}} \right| dy$$

$$\leq \text{const} \epsilon(q + n + 1) \int_0^{\infty} e^{(a-2\pi)y} dy \to 0, \quad (n \to \infty)$$

(A8)

and

$$J_\infty = 2i \sum_{l=0}^{\infty} f(l + 1/2) = 2i \int_q^{\infty} f(x)dx - \int_q^{q-i\infty} f(z)[\tan(\pi z) + i]dz$$

$$+ \int_q^{q+i\infty} f(z)[\tan(\pi z) - i]dz$$

(A9)

or

$$\sum_{l=0}^{\infty} f(l + 1/2) = \int_q^{\infty} f(x)dx + \int_q^{q-i\infty} \frac{f(z)}{1 + e^{2i\pi z}} dz - \int_q^{q+i\infty} \frac{f(z)}{1 + e^{-2i\pi z}} dz,$$

(A10)

where $-1/2 < q < 1/2$, $f(z)$ is a holomorphic function for $Rez > q$, and $f(z)$ satisfies condition (A2).

Using the last expression we can calculate the sums in Eq. (3):

$$\sum_{l=0}^{\infty} \left\{ \frac{(l + 1/2)^3}{\sqrt{u^2 + \mu^2} + (l + 1/2)^2} - \left( l + \frac{1}{2} \right)^2 + \frac{u^2 + \mu^2}{2} \right\} = \lim_{q \to 0} \left\{ \int_q^{\infty} \left[ \frac{x^3}{\sqrt{u^2 + \mu^2} + x^2} \right] dx \right.\right.$$
\[
\sum_{l=0}^{\infty} \left\{ (l + \frac{1}{2}) \sqrt{u^2 + \mu^2 + (l + 1/2)^2} - \frac{u^2 + \mu^2}{2} \right\} = \lim_{q \to 0} \left\{ \int_{q}^{\infty} x \sqrt{u^2 + \mu^2 + x^2} \right\} = \lim_{q \to 0} \left\{ \int_{q}^{\infty} \frac{x^2}{(1 + e^{2\pi x})} \right\} = -\left( \frac{u^2 + \mu^2}{2} \right)^{3/2} \frac{1}{4}.
\]
\[+2\int_{0}^{q} \frac{u^2 + \mu^2 - x^2}{(1 + e^{2\pi x})} dx, \quad (A12)\]

\[
\sum_{l=0}^{\infty} \left\{ \frac{(l + 1/2)^{2m+1}}{u^2 + \mu^2 + (l + 1/2)^2} \right\} = \lim_{q \to 0} \left\{ \int_{q}^{\infty} \frac{x^{2m+1}}{(u^2 + \mu^2 + x^2)^{2m+1/2} - 1} dx \right\} + \int_{q-i\infty}^{q+i\infty} \frac{z^{2m+1}}{(u^2 + \mu^2 + z^2)^{2m+1/2} - 1} dz - \int_{q+i\infty}^{\infty} \frac{z^{2m+1}}{(u^2 + \mu^2 + z^2)^{2m+1/2}} dz - \left( \text{terms of this integral that diverge in the limit } \delta \to 0 \right) = \sum_{k=0}^{m} \frac{(-1)^{m+k} m!}{k!(m-k)!} \frac{\sqrt{u^2 + \mu^2}}{2m - 2k - 1} + \frac{2}{(2m - 1)!} \int_{0}^{\sqrt{u^2 + \mu^2}} \frac{xdx}{\sqrt{u^2 + \mu^2 - x^2}} \left( \frac{d}{xdx} \right)^{m+1} \frac{x^{2m}}{1 + e^{2\pi x}} \quad (m \geq 0), \quad (A13)\]

\[
\sum_{l=0}^{\infty} \left\{ (l + 1/2)^{2m+1} \right\} = \lim_{q \to 0} \left\{ \int_{q}^{\infty} \frac{x^{2m+1}}{(u^2 + \mu^2 + x^2)^{2m+3/2}} dx \right\} = \lim_{q \to 0} \left\{ \int_{q}^{\infty} \frac{z^{2m+1}}{(u^2 + \mu^2 + z^2)^{2m+3/2}} (1 + e^{2\pi z}) dz \right\} = \frac{2^{m} m!}{(2m + 1)!} \sqrt{u^2 + \mu^2} + \frac{2}{(2m + 1)!} \int_{0}^{\sqrt{u^2 + \mu^2}} \frac{xdx}{\sqrt{u^2 + \mu^2 - x^2}} \left( \frac{d}{xdx} \right)^{m+1} \frac{x^{2m}}{1 + e^{2\pi x}} \quad (m \geq 0), \quad (A14)\]

\[
\sum_{l=0}^{\infty} \left\{ (l + 1/2)^{2m+1} \right\} = \frac{(2m + 1)!}{(2n - 1)!} \left( -\frac{\partial}{\mu \partial \mu} \right)^{m} \sum_{l=0}^{\infty} \left\{ (l + 1/2)^{2m+1} \right\} = \frac{2(-1)^{m}}{(2n - 1)!} \left( -\frac{\partial}{\mu \partial \mu} \right) \int_{0}^{\sqrt{u^2 + \mu^2}} \frac{xdx}{\sqrt{u^2 + \mu^2 - x^2}} \left( \frac{d}{xdx} \right)^{m+1} \frac{x^{2m}}{1 + e^{2\pi x}} \quad (m \geq m + 2). \quad (A15)\]
Now we can calculate the integrals over \(u\) in (30). Note that these expressions must be expanded in powers of \(\varepsilon\):

\[
\int_0^\infty \cos(\varepsilon u) \sqrt{u^2 + \mu^2} du = \left( -\frac{d^2}{d\varepsilon^2} + \mu^2 \right) \int_0^\infty \cos(\varepsilon u) \sqrt{u^2 + \mu^2} du = \left( -\frac{d^2}{d\varepsilon^2} + \mu^2 \right) K_0(\varepsilon \mu)
\]

\[
= -\frac{\mu}{\varepsilon} K_1(\varepsilon \mu) = \left( -\frac{\mu^2}{2} - \frac{\mu^4}{16} \varepsilon^2 \right) \left( C + \frac{1}{2} \ln \left( \frac{(\varepsilon \mu)^2}{4} \right) \right) - \frac{1}{\varepsilon^2} + \frac{\mu^2}{4} + \frac{5}{64} \mu^4 \varepsilon^2 + O(\varepsilon^4 \ln |\varepsilon|), \tag{A16}
\]

where \(K_n(x)\) is Macdonald's function and \(C\) is Euler's constant.

The second type of integral over \(u\) has the form

\[
I_+ = \int_0^\infty du \cos(\varepsilon u) \int_0^{\frac{1}{\varepsilon}} f(x)(u^2 + \mu^2 - x^2)^{1/2} \, dx.
\]  
(A18)

Changing the order of integration over \(u\) and \(x\) gives

\[
I_+ = \int_0^\infty dx f(x) \int_0^{\infty} du \frac{\cos(\varepsilon u)}{\sqrt{u^2 + \mu^2 - x^2}} + \int_0^\infty dx f(x) \int_0^{\infty} du \frac{\cos(\varepsilon u)}{\sqrt{x^2 - \mu^2}} \tag{A19}
\]

Since

\[
\int_0^\infty \frac{\cos(\varepsilon u) du}{\sqrt{u^2 + \mu^2 - x^2}} = K_0(\varepsilon \sqrt{\mu^2 - x^2})
\]

\[
= \left( 1 + \frac{\varepsilon^2 (\mu^2 - x^2)}{4} \right) \left( C + \frac{1}{2} \ln \left( \frac{\varepsilon^2 (\mu^2 - x^2)}{4} \right) \right) - \frac{\varepsilon^2 (\mu^2 - x^2)}{4} + O(\varepsilon^4 \ln |\varepsilon|), \tag{A20}
\]

\[
\int_0^\infty \frac{\cos(\varepsilon u) du}{\sqrt{x^2 - \mu^2}} = \frac{\pi}{2} N_0(\varepsilon \sqrt{x^2 - \mu^2})
\]

\[
= \left( 1 - \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right) \left( C + \frac{1}{2} \ln \left( \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right) \right) - \frac{\varepsilon^2 (x^2 - \mu^2)}{4} + O(\varepsilon^4 \ln |\varepsilon|), \tag{A21}
\]

where \(N_0(x)\) is Neumann’s function, one can obtain

\[
I_- = \int_0^\infty dx f(x) \left\{ \left( -1 + \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right) \left( C + \frac{1}{2} \ln \left( \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right) \right) - \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right\} + O(\varepsilon^4 \ln |\varepsilon|), \tag{A22}
\]

\[
I_+ = -\frac{\partial^2}{\partial \varepsilon^2} I_- + \int_0^\infty dx f(x) (\mu^2 - x^2) \left\{ -\frac{1}{\varepsilon} + \frac{1}{2} \ln \left( \frac{\varepsilon^2 (x^2 - \mu^2)}{4} \right) \right\} + O(\varepsilon^2 \ln |\varepsilon|) \tag{A23}
\]

If we also take into consideration
\[\int_0^\infty \frac{x^3}{1 + e^{2\pi x}} \, dx = \frac{7}{1920}, \quad \int_0^\infty \frac{x^3}{1 + e^{2\pi x}} \, dx = \frac{1}{48} \quad \text{ (A24)}\]

\[\frac{\partial}{\partial \mu} f(\mu x) = \frac{x}{\partial \mu} f(\mu x), \quad \text{ (A25)}\]

the resulting expressions for \(S_{\mu}^m(\varepsilon, \mu)\) can be presented as Eqs. (34)-(40).

**APPENDIX B**

\[\langle T^\mu_i \rangle_{\text{ren}} = \frac{1}{4\pi^2} \left\{ \left( \frac{\xi}{8r^2} - \frac{1}{64r^2} \right) m^2 + \frac{3\xi^2}{8r^4} - \frac{11\xi}{96r^4} + \frac{79}{7680r^4} + \left[ \frac{m_k^2}{2} + \left( \frac{2\xi}{r^2} - \frac{1}{4r^2} \right) \right] m^2 \right. \]

\[- \frac{\xi}{2r^4} + \frac{2\xi^2}{r^4} + \frac{1}{32r^4} \left. \right\} I_1(\mu) + \left[ \frac{m_\parallel^4}{8} + \left( \frac{1}{12r^2} - \frac{\xi}{2r^2} \right) m^2 + \left( \frac{\xi}{6r^4} - \frac{1}{60r^4} \right) \right] \]

\[- \frac{\xi^2}{2r^4} \right\} \ln \sqrt{\frac{\mu^2}{m_{\text{D8}}^2 r^2}} + \left[ f_{\xi}(r^4) \right]_{12r^2 f - f''} + \frac{1}{2304f^2 \mu^2} - \frac{f''}{152f^2 \mu^2} - \frac{288f^2 \mu^2}{152f^2 \mu^2} \right] m^2 + \left[ f'' + \frac{f''}{4608f^2 \mu^4} + \frac{(r^2)''}{48r^4} \right. \]

\[
\left[ (r^2)'' + \frac{64f^2 \xi^2(r^2)'}{2304f^2 \mu^4} + \frac{(r^2)''}{48r^4} \right] + \left[ \left( \frac{1}{2r^2} \right)^2 + \frac{1}{12r^2} \right] \]
\[
\begin{align*}
\langle T_{\mu}^\nu \rangle_{\text{ren}} &= \frac{1}{4r^2} \left\{ \left( \frac{\xi}{8r^2} - \frac{1}{64r^2} \right) m^2 + \frac{79}{7680r^4} - \frac{11\xi}{96r^4} + \frac{3\xi^2}{8r^4} + \left[ \frac{m^4}{2} + \left( \frac{2\xi}{r^2} - \frac{1}{4r^2} \right) m^2 \right] + \frac{2\xi^2}{r^4} \right\} I_1(\mu) + \left[ \frac{m^4}{8} + \left( \frac{1}{12r^2} - \frac{\xi}{2r^2} \right) m^2 - \frac{1}{60r^2} + \frac{\xi}{6r^4} - \frac{\xi^2}{2r^4} \right] \ln \sqrt{\frac{\mu^2}{m_{\text{DS}}r^2}} \\
+ \left[ \frac{f''}{1152\mu^2} - \frac{f''(r^2)'^2}{48f^2} + \frac{f''}{2304f^2\mu^2} \right] m^2 - \frac{9216f^2r^2m^2}{48f^2} + \frac{f''(r^2)\xi}{6f^4r^4} \\
- \frac{f''(r^2)'^2}{48f^2} + \frac{f''(r^2)'^2}{2304f^2\mu^2} + \frac{5f''(r^2)'\xi}{2304f^2r^2\mu^2} + \frac{5f''(r^2)'^2}{2304f^2r^2\mu^2} + \frac{f''^2\xi}{4f^4r^4} \\
+ \frac{f''(r^2)'^2}{48f^2} - \frac{f''(r^2)'^2}{32f^2} + \frac{f''(r^2)'^2}{192f^2r^2} + \frac{32f^2}{288r^6} + \frac{5f''(r^2)'\xi}{1152f^2r^2\mu^2} + \frac{5f''(r^2)'^2}{2304f^2r^2\mu^2} \\
+ \left[ \frac{f''(r^2)'^2}{48f^2} + \frac{f''(r^2)'^2}{192f^2r^2} - \frac{3f''(r^2)\xi}{128f^2r^2\mu^2} + \frac{3f''(r^2)'^2}{2304f^2r^2\mu^2} \right] \ln \sqrt{\frac{\mu^2}{m_{\text{DS}}r^2}} + \left[ \frac{3f''(r^2)\xi^2}{128f^2r^2\mu^2} - \frac{3f''(r^2)'^2}{2304f^2r^2\mu^2} \right] \\
\right.
\end{align*}
\]
\[
\langle \mathcal{T}_q \rangle_{\text{ren}} = \langle \mathcal{T}_e \rangle_{\text{ren}} = \left\{ \frac{\xi}{(8\pi)^2} \left[ \frac{m^4}{32\pi^2} + \frac{2\xi}{2\pi q^2} - \frac{1}{4\pi^2} \right] \ln \sqrt{\frac{\mu^2}{m_{\text{DSS}}^2}} + \left[ \frac{f^2}{16\pi^2} - \frac{f'(r')^2}{48\pi^2} - \frac{f''}{48\pi^2} + \frac{f'''}{1152\pi^2} \right] \right\}
\]

(B2)
\[ \begin{align*}
&\xi^2(r^2)^2 - \frac{24 r^6}{2 r^2} - \frac{5 \xi (r^2)' f'}{384 f^4} + \frac{f' \xi^2 (r^2)''}{124 f^4} - \frac{\xi^3 f' (r^2)'}{64 f^4} \left( \frac{\partial}{\partial \mu} \right)^2 \mu^2 f_1(\mu) + \left( \frac{r^2 f''}{16 f} \right)^2 \\
&- \frac{\xi r^2 f''}{4 f} - \frac{5 r^2 f'^2}{16 r^2 f} + \frac{\xi r^2 f'^2}{16 r^2 f} \left( \frac{\xi f''}{4 f} - \frac{(r^2)^2}{32 r^4} - \frac{f''}{64 f} - \frac{\xi^2 f''}{64 r^2} + \frac{3 \xi f'^2}{2 r^2} + \frac{\xi^2 f'(r^2)'}{4 f^2} + \frac{3 \xi (r^2)'^2}{8 r^4} + \frac{5 \xi^2 f'^2}{4 f^2} - \frac{\xi^2 (r^2)'}{4 r^4} \right) \\
&+ \frac{3 \xi f' (r^2)'}{16 r^2 f} - \frac{\xi^2 f'(r^2)'}{2 r^2 f} - \frac{\xi (r^2)''}{8 r^4 f} + \frac{5 \xi^2 f' (r^2)'}{4 f^2} - \frac{\xi^2 (r^2)'}{4 r^4} - \frac{7 f'^2}{16 f} \\
&- \frac{3 \xi^2 (r^2)'^2}{16 r^2 f} + \frac{33 \xi^2 f'^2}{164 r^2 f} - \frac{\xi f''}{164 r^4 f} - \frac{\xi f'(r^2)'}{4 r^4 f} - \frac{27 \xi (r^2)'^2}{8 r^4} + \frac{5 \xi^2 f'^2}{4 r^4 f} + \frac{5 (r^2)' f}{512 f^2} + \frac{7 (r^2)' f'}{1024 r^2 f^2} \\
&+ \xi^2 f'' \left( \frac{5 (r^2)' f}{160 f} - \frac{5 r^2 f'^2}{8 r^4 f} - \frac{\xi (r^2)'^2}{32 f} + \frac{5 \xi^2 f'(r^2)'}{8 r^2 f} + \frac{15 \xi (r^2)' f'}{16 f^2} \right) \left( \frac{\xi f'(r^2)'}{32 f} - \frac{\xi (r^2)'^2}{32 f} - \frac{\xi^3 (r^2)'}{8 r^4 f} - \frac{15 \xi^3 f'(r^2)'}{4 r^2 f^2} \right) \left( \frac{\xi f'(r^2)'}{32 f} - \frac{\xi (r^2)'^2}{32 f} - \frac{\xi^3 (r^2)'}{8 r^4 f} - \frac{15 \xi^3 f'(r^2)'}{4 r^2 f^2} \right) \\
&+ \frac{5 \xi^2 f'^2}{4 r^4 f} \left( \frac{5 \xi^2 f'^2}{8 r^4 f} - \frac{\xi^2 (r^2)'}{8 r^4 f} + \frac{5 \xi^2 f'^2}{4 r^4 f} + \frac{5 (r^2)' f}{512 f^2} + \frac{5 (r^2)' f'}{1024 r^2 f^2} \right) \left( \frac{5 \xi^2 f'^2}{8 r^4 f} - \frac{\xi^2 (r^2)'}{8 r^4 f} + \frac{5 \xi^2 f'^2}{4 r^4 f} + \frac{5 (r^2)' f}{512 f^2} + \frac{5 (r^2)' f'}{1024 r^2 f^2} \right) \\
&+ \frac{15 \xi^2 (r^2)'^2}{4 r^2 f} - \frac{15 \xi^2 f'(r^2)'}{32 r^4 f} + \frac{15 \xi^2 (r^2)'^2}{4 r^2 f} - \frac{5 \xi^2 (r^2)'^2}{8 r^4 f} + \frac{5 \xi^2 (r^2)' f'}{512 f^2} + \frac{5 \xi^2 (r^2)' f'}{1024 r^2 f^2} \\
&- \frac{15 \xi^2 (r^2)'^2}{16 r^6} + \frac{15 \xi^2 f'(r^2)'}{128 r^4 f} + \frac{15 \xi^2 (r^2)'^2}{16 r^6} - \frac{5 \xi^2 (r^2)' f'}{128 r^4 f} + \frac{15 \xi^2 f'(r^2)'}{128 r^4 f} - \frac{15 \xi^2 f'(r^2)'}{1024 r^2 f^2} \\
&+ \frac{5 \xi f'(r^2)'}{8 r^2 f} + \frac{5 \xi^2 f'(r^2)'}{8 r^2 f} + \frac{5 \xi^2 (r^2)' f'}{512 r^4 f} - \frac{15 \xi f'(r^2)'}{64 r^4 f} + \frac{15 \xi^3 f'(r^2)'}{16 r^4 f} - \frac{5 \xi^3 f'(r^2)'}{1024 r^2 f^2} \\
&+ \frac{5 \xi^2 f'^2}{16 r^2 f} + \frac{5 \xi^2 f'^2}{16 r^2 f} - \frac{5 \xi (r^2)' f'}{16 r^2 f} + \frac{10 \xi^3 (r^2)'^2}{4 r^2 f} + \frac{5 \xi^3 (r^2)'^2}{16 r^2 f} + \frac{5 \xi^3 (r^2)' f'}{1024 r^2 f^2} \\
&+ \frac{5 \xi^2 f'^2}{8 r^4 f} + \frac{5 \xi^2 f'^2}{8 r^4 f} - \frac{5 \xi (r^2)' f'}{8 r^4 f} + \frac{10 \xi^3 (r^2)'^2}{4 r^2 f} + \frac{5 \xi^3 (r^2)' f'}{1024 r^2 f^2} + \frac{5 \xi^2 f'^2}{8 r^4 f} + \frac{5 \xi^2 f'^2}{8 r^4 f} - \frac{5 \xi (r^2)' f'}{8 r^4 f} + \frac{10 \xi^3 (r^2)'^2}{4 r^2 f}
\end{align*}\]

\[ (B3) \]
