Interrelations between different canonical descriptions of dissipative systems

D Schuch\(^1\), J Guerrero\(^2\), F F López-Ruiz\(^3,5\) and V Aldaya\(^4\)

\(^1\) Institut für Theoretische Physik, J.W. Goethe-Universität Frankfurt am Main, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany
\(^2\) Departamento de Matemática Aplicada, Universidad de Murcia, Campus de Espinardo, E-30100 Murcia, Spain
\(^3\) Departamento de Física Aplicada, Universidad de Cádiz, Campus de Puerto Real, E-11510 Puerto Real, Cádiz, Spain
\(^4\) Instituto de Astrofísica de Andalucía, IAA-CSIC, Apartado Postal 3004, E-18080 Granada, Spain

E-mail: Schuch@em.uni-frankfurt.de, juguerre@um.es, paco.lopezruiz@uca.es and valdaya@iaa.es

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Abstract
There are many approaches for the description of dissipative systems coupled to some kind of environment. This environment can be described in different ways; only effective models are being considered here. In the Bateman model, the environment is represented by one additional degree of freedom and the corresponding momentum. In two other canonical approaches, no environmental degree of freedom appears explicitly, but the canonical variables are connected with the physical ones via non-canonical transformations. The link between the Bateman approach and those without additional variables is achieved via comparison with a canonical approach using expanding coordinates, as, in this case, both Hamiltonians are constants of motion. This leads to constraints that allow for the elimination of the additional degree of freedom in the Bateman approach. These constraints are not unique. Several choices are studied explicitly, and the consequences for the physical interpretation of the additional variable in the Bateman model are discussed.

Keywords: classical mechanics, dissipation, constraints, Caldirola Kanai, Bateman, expanding coordinates

1. Introduction

Realistic physical systems are not isolated but in contact with some kind of environment, causing phenomena like irreversibility of the time-evolution and dissipation of energy. These kinds of effects can be described by phenomenological equations of motion like the Langevin equation with a linear velocity-dependent friction force. However, this does not fit into the conventional Lagrangian or Hamiltonian formalism of classical mechanics, where the canonical variables are the physical position and momentum or related with them via canonical transformations, and the Hamiltonian function is the sum of kinetic and potential energies. Attempts to obtain the afore-mentioned friction force by adding a kind of ‘friction potential’ to the Hamiltonian have not been successful (at least not at the classical level). Other attempts to include friction effects into the Hamiltonian formalism exist, but different prices must be paid for this purpose.

In the conventional system-plus-reservoir approach, the system of interest is coupled to an environment with many (in the limit, infinitely many) degrees of freedom (e.g., coupled linearly to a bath of harmonic oscillators \([1]\)), where the system and the environment together are considered to be a closed Hamiltonian system. Via averaging over the environmental degrees of freedom and other procedures (for details see, e.g., \([2]\)), an equation of motion for the system of interest including a friction force can finally be obtained. One drawback in employing this method is the large number of environmental degrees of freedom that must be considered in the beginning (though they are eliminated in the end). This leads to large, cumbersome, and expensive calculations. In its quantized version, this approach is usually applied to the...
density matrix, causing the computational effort to scale at least quadratically with the number of degrees of freedom and, in the case of the Caldeira–Leggett model [1], can also lead to unphysical negative probabilities.

The number of environmental degrees of freedom is drastically reduced to one in an approach by Bateman [3] describing the damped harmonic oscillator. In order to be able to apply the canonical formalism, the phase-space dimension must be doubled to obtain a kind of effective description. The new degree of freedom can be considered as a collective one for the bath, which absorbs energy dissipated by the damped oscillator. The variable of the dual system that fulfills a time-reversed equation with an acceleration force of the same magnitude as the friction force of the Langevin equation, but with a different sign, looks like a position variable. Its relation to, and interpretation in terms of, physical position and momentum (or velocity), particularly when linked to other canonical approaches, is investigated in this work.

After the rediscovery of the Bateman dual Hamiltonian by Morse and Feshbach [4] and Bopp [5], it also has been studied with respect to various different features in recent years. So squeezed states for the Bateman Hamiltonian were considered in [6] and [7], and a quantum field theoretical approach was used by Vitiello et al. [8]. One author also tried to apply the dual approach as a dissipative quantum model of the brain [9]. Quantization using Feynman’s path integral method was discussed by Blasone and Fizba [10, 11], and the Bateman system was also studied by the same authors and Vitiello [11, 12] as a toy model for ’t Hooft’s proposal of a deterministic version of quantum mechanics [13]. More recently, together with Scardigli, these authors considered a composite system of two classical Bateman oscillators as a particle in an effective magnetic field [14]. Complex eigenvalues of the quantized version of Bateman’s Hamiltonian in connection with resonances and two-dimensional parabolic potential barriers are discussed in [15, 16]. Also, the Wigner function for the Bateman system on noncommutative phase space [17] and the inclusion of a time-dependent external force [18] have been studied. The Bateman approach (as well as the one of Caldirola–Kanai (CK), that is being considered subsequently) is also discussed in an attempt to reformulate a dissipative system in terms of an infinite number of non-dissipative ones [21]. A different method for the description of dissipative systems, which seems to have some advantages in the high-energy regime, has been compared with the Bateman approach [22] and shown to be equivalent to it locally. Finally, in a rather recent paper [23] by Bender et al the Bateman Hamiltonian enlarged by a quadratic term in the two dual coordinates is studied as a model for two coupled optical resonators. This shows, despite the age of Bateman’s approach, that there is still considerable interest in, and potential applicability of, this model.

Another frequently applied approach for the description of dissipative systems that does not take into account the individual degrees of freedom of the environment is the one of Caldirola [19] and Kanai [20]. Actually, no environmental degree of freedom appears explicitly in this approach; only the effect of the environment on the system of interest is taken into account. This method is a formal canonical one that again leads to an equation of motion with the same damping force, but now derived from a Lagrangian or Hamiltonian that contains no additional friction terms, only a multiplying factor. The corresponding Hamiltonian, however, no longer represents the energy of the system and is also not a constant of motion. The most serious point of criticism usually raised against this approach is its apparent violation of the uncertainty principle in its quantized form that can be obtained via canonical quantization. This criticism can be refuted if the relation between the canonical variables, and quantities depending on them, and the usual physical variables is properly taken into account (for details see [24]). In the following, however, only the classical version is being considered.

The final approach to be mentioned in more detail here uses an exponentially expanding coordinate system [25, 26]. The canonical position and momentum variables of this approach, as in the case of Caldirola–Kanai (CK), are connected with the physical position and momentum via a noncanonical transformation. In this case, however, the Hamiltonian is a constant of motion and, for certain initial conditions, even represents the initial energy of the physical system that is dissipated during its time-evolution. It formally looks like that of an undamped harmonic oscillator, only with shifted frequency. Therefore, all known results from the undamped oscillator can be used, and the corresponding results for the damped case are obtained via the non-canonical transformation from the canonical to the physical system. In particular, after canonical quantization, no problems occur with the uncertainty principle [26]. This approach is connected with the one of CK via a canonical transformation [24, 27] (however, with an explicitly time-dependent generating function).

There are further similar canonical approaches using modified Lagrange and Hamilton functions for the system of interest, like the one by Lemos [28], that also has a conserved Hamiltonian. But these approaches are related to the one in expanding coordinates (and, therefore, also to the one of CK) via canonical transformations, and are not considered further in this work (for details, see also [29]).

It has been shown by Sun and Yu [30, 31] that it is possible to get to the CK Hamiltonian from the Caldeira–Leggett model, thus demonstrating a kind of physical equivalence of the two methods. On the other hand, group-theoretical arguments have been used to link the CK approach to the one by Bateman [32]. In this paper it is shown explicitly how the Bateman approach can be related to the canonical one using expanding coordinates. For this purpose the variables of the dual system must be eliminated by imposing some constraints; it is shown how this can be expressed in terms of physical position and velocity of the damped system. We take advantage of the circumstance that both Hamiltonians are constants of motion. The transition to the CK system is then achieved simply via a time-dependent canonical transformation.

The discussion is restricted to a one-dimensional system, in particular the damped harmonic oscillator (where the
damped free motion can be obtained in the limit \( \omega \to 0 \), and essentially to the classical case.

After an outline of the Bateman model follows a short presentation of the approach using the expanding coordinates and the one of CK as well as their interrelation. To find the connection with the Bateman approach, the variables of the dual system are removed by imposing constraints, which can be done in different ways. Some examples and their consequences are discussed in detail, and conclusions drawn at the end.

2. The Bateman approach

The Bateman Hamiltonian \( H_B \), expressed in terms of the position variables \( x \) and \( y \) and the corresponding canonical momenta \( p_x \) and \( p_y \), has the form

\[
H_B = \frac{1}{m} p_x p_x + \frac{\gamma}{2} (y p_y - x p_x) + \frac{m}{2} \left( \alpha^2 - \frac{\gamma^2}{4} \right) y = H_\Omega + D
\]

with \( D = \frac{\gamma}{2} (y p_y - x p_x) \). The Poisson brackets of \( H_B \) with \( D \) as well as with \( H_\Omega \) vanish, so both are constants of motion (in the quantized version, the corresponding three operators commute).

The Hamiltonian equations of motion are

\[
\frac{\partial H_B}{\partial p_x} = \frac{1}{m} p_x - \frac{\gamma}{2} x = \dot{x},
\]

\[
\frac{\partial H_B}{\partial p_y} = \frac{1}{m} p_y + \frac{\gamma}{2} y = \dot{y}
\]

\[
\frac{\partial H_B}{\partial x} = -\frac{\gamma}{2} p_x + m \left( \alpha^2 - \frac{\gamma^2}{4} \right) y = -\dot{p}_x,
\]

\[
\frac{\partial H_B}{\partial y} = \frac{\gamma}{2} p_y + m \left( \alpha^2 - \frac{\gamma^2}{4} \right) x = -\dot{p}_y,
\]

where, from (2), \( p_x \) and \( p_y \) can be expressed as

\[
p_x = m \left( \dot{x} + \frac{\gamma}{2} x \right)
\]

\[
p_y = m \left( \dot{y} - \frac{\gamma}{2} y \right)
\]

From there, and with the help of equation (3), the equations of motion for \( x \) and \( y \) can be obtained as

\[
\ddot{x} + \gamma \dot{x} + \alpha^2 x = 0
\]

\[
\ddot{y} - \gamma \dot{y} + \alpha^2 y = 0
\]

Equation (6) is just the equation for the damped harmonic oscillator with friction force \(-\gamma \dot{x}\), whereas, in the time-reversed equation for \( y \), the accelerating force \(+\gamma \dot{y}\) occurs.

From equations (6), (7) and (4), (5), it is clear that the \((x, p_x, y, p_y)\) space splits into two invariant subspaces: the one of variables \((x, p_x)\) undergoing a damped oscillator motion, and the one of variables \((y, p_y)\) with time-reversed (accelerated) behaviour. Using the equations of motion, it can also be shown that

\[
\frac{d}{dt} H_B = 0,
\]

i.e., \( H_B \) is a dynamical invariant which, in a first naive attempt, could be interpreted in the way that the energy dissipated by the damped system is gained by the accelerated one. Rewritten in terms of \( x, y \) and the corresponding velocities \( \dot{x} \) and \( \dot{y} \), the terms depending on the friction (or acceleration) coefficient \( \gamma \) cancel out (although the Lagrangian does contain terms in \( \gamma \)), and it remains

\[
H_B = m \left( \dot{y} y + \alpha^2 x y \right).
\]

In fact, the individual energies, and their change in time for both systems, written in terms of the velocities, take the form

\[
\frac{d}{dt} E_x = -\gamma m x^2,
\]

\[
E_x = \frac{m}{2} x^2 + \frac{m}{2} \alpha^2 x^2,
\]

\[
\frac{d}{dt} E_y = +\gamma m y^2.
\]

So, the sum of \( E_x \) and \( E_y \) would be constant and (apart from another constant term) could be equal to \( H_B \) if

\[
\frac{d}{dt} (E_x + E_y) = \gamma m (y^2 - x^2) = 0,
\]

which is fulfilled only for \( y = \pm x \); so \( y \) and \( x \) could differ, at most, by a constant, and \( H_B \), as given in (9) (again apart from a constant term), would turn into \( H_B \to m(\dot{x}^2 + \alpha^2 x^2) \), i.e., the energy of two undamped harmonic oscillators.

However, \( \dot{y} \), derived from the solution of equation (7), differs from \( \dot{x} \), derived from the solution of equation (6), by more than just its sign; so one has to be careful with this simple picture of energy transfer between the \( x \) and \( y \) systems. This is rather clear if we notice that both degrees of freedom in the Bateman system (regardless of their physical interpretation) are so involved that both Hamiltonians for \( x \) and \( y \) are harmonic oscillator Hamiltonians for \( x \) and \( y \), and \( H_B \) is an interaction term. One can rotate the phase space in order to obtain a new system of two oscillators with opposite signs coupled through an interaction term (see, e.g., [14], equations (21–23)), where again the energy of the whole system is conserved. It is transferred from one of the transformed oscillators to the other, but none of the coordinates of the rotated oscillators represents the physical position variable. When constraints that are imposed on the systems are considered, it becomes
even more obvious later on that y is not just a simple position coordinate like x in this model.

3. Effective canonical description of dissipative systems in expanding coordinates and in the CK approach

Now, briefly, two approaches are presented that are able to describe the damped harmonic oscillator in the framework of Hamiltonian mechanics using only one canonical position and momentum variable. These variables, however, are connected with physical position and momentum via non-canonical transformations. In the following, canonical variables and corresponding Hamiltonians will be characterised by a hat.

3.1. Exponentially expanding coordinate system

The Hamiltonian \( \hat{H}_{\text{exp}} \) depends on a coordinate \( \hat{Q} \) that, in comparison with the physical position variable \( x \), expands exponentially. The corresponding canonical momentum \( \hat{P} \) displays a similar behaviour, i.e.,

\[
\hat{H}_{\text{exp}} = \frac{1}{2m} \hat{P}^2 + \frac{m}{2} \left( \alpha^2 - \frac{\gamma^2}{4} \right) \hat{Q}^2
\]  

(15)

with

\[
\hat{Q} = x e^{\hat{t}},
\]

\[
\hat{P} = m \hat{Q} = m \left( \hat{x} + \frac{\gamma}{2} \hat{x} \right) e^{\hat{t}},
\]

and

\[
\frac{\partial \hat{H}_{\text{exp}}}{\partial \hat{P}} = \frac{1}{m} \hat{P} = \hat{Q},
\]

\[
\frac{\partial \hat{H}_{\text{exp}}}{\partial \hat{Q}} = m \Omega^2 \hat{Q} = -\hat{P}.
\]  

(16)

Hamiltionian (15) looks like that of an undamped harmonic oscillator with shifted frequency \( \Omega = (\alpha^2 - \gamma^2/4)^{1/2} \), and the corresponding equation of motion for \( \hat{Q} \) is consequently

\[
\ddot{\hat{Q}} + \Omega^2 \hat{Q} = 0.
\]  

(17)

Expressed in terms of the physical position variable \( x \), equation (6) is regained, including the friction force. Obviously, also

\[
\frac{d}{dt} \hat{H}_{\text{exp}} = 0
\]  

(18)

is valid and too, can be confirmed by rewriting \( \hat{H}_{\text{exp}} \) in terms of \( x \) and \( \hat{x} \) as

\[
\hat{H}_{\text{exp}} = \frac{m}{2} \left[ \dot{x}^2 + \gamma \dot{x} x + \alpha^2 x^2 \right] e^{\hat{t}} = \text{const},
\]  

(19)

which for \( x_0 = 0 \) or \( \dot{x}_0 = 0 \) even represents the initial energy of the system. It is interesting to note that in general \( \hat{H}_{\text{exp}} \), written in terms of \( x \) and \( \dot{x} \), coincides with a conserved quantity for the damped harmonic oscillator already considered in the literature (see, for instance, the expression for \( I_s \) in equation (19) of [33]). In the context of the CK description presented below, the eigenstates of the quantum operator corresponding to this invariant \( \hat{H}_{\text{exp}} \) are known as loss-energy states (see [34]), although the fact that \( \hat{H}_{\text{exp}} \) is constant allows us to find the quantum operator that represents it in a broader context.

3.2. CK approach

In the CK approach, the position variable remains unchanged, \( \hat{x} = x \), whereas only the canonical momentum shows an exponential expansion, i.e.,

\[
\hat{H}_{\text{CK}} = \frac{1}{2m} \hat{P}^2 e^{-\gamma \hat{t}} + \frac{m}{2} \alpha^2 \hat{x}^2 e^{\gamma \hat{t}}
\]  

(20)

with

\[
\dot{x} = x, \quad \hat{P} = p \ e^{\gamma \hat{t}} = m \hat{x} \ e^{\gamma \hat{t}}.
\]  

(21)

From the Hamiltonian equations of motion

\[
\frac{\partial \hat{H}_{\text{CK}}}{\partial \hat{P}} = \frac{1}{m} \hat{P} e^{-\gamma \hat{t}} = \dot{x},
\]

\[
\frac{\partial \hat{H}_{\text{CK}}}{\partial \hat{Q}} = m \alpha^2 \hat{x} e^{\gamma \hat{t}} = -\hat{P}
\]  

(22)

it again follows that the physical position variable obeys equation (6). Expressed in terms of \( x \) and \( \hat{x} \), \( \hat{H}_{\text{CK}} \) now takes the form

\[
\hat{H}_{\text{CK}} \equiv \frac{m}{2} \left[ \dot{x}^2 + \alpha^2 \hat{x}^2 \right] e^{\gamma \hat{t}} = E(t) e^{\gamma \hat{t}} \neq \text{const}.
\]  

(23)

The canonical variables of this approach are connected with the ones of \( \hat{H}_{\text{exp}} \), via

\[
\hat{x} = \hat{Q} e^{-\gamma \hat{t}} \quad \text{or} \quad \hat{Q} = \hat{x} e^{\gamma \hat{t}},
\]

\[
\hat{P} = \hat{P} e^{-\gamma \hat{t}} \quad \text{or} \quad \hat{P} = \hat{P} e^{\gamma \hat{t}} + m \frac{\gamma}{2} \hat{Q} e^{\gamma \hat{t}}
\]  

(24)

(25)

where the explicitly time-dependent generating function for the canonical transformation between the two systems is given by

\[
\hat{F}_s (\hat{x}, \hat{P}, t) = \hat{x} \hat{P} e^{\gamma \hat{t}} - \frac{m}{4} \gamma \hat{x}^2 e^{\gamma \hat{t}}.
\]  

(26)

4. Linking the Bateman approach with \( \hat{H}_{\text{exp}} \)

In order to connect the Bateman approach with the canonical approaches presented in section 3, it will be stipulated that

(1) the equation of motion (6) for the position variable of the dissipative system is the same as the equation of motion for the position variable (including the friction force) in the two canonical approaches when these are expressed in terms of the physical position variable \( x \).
(2) The Bateman Hamiltonian represents a constant of motion with the dimension of an energy.

In order to connect the two descriptions of a dissipative system we may assume that the conserved quantity \( H_B \) is identical to the conserved quantity \( \dot{H}_{\text{exp}} \) and impose some constraints so that the dual variable \( y \) and the corresponding momentum are eliminated.

As the constraints are obtained via comparison with Hamiltonians on the formal canonical level, the notation in terms of physical position and momentum variables.

So, the Bateman Hamiltonian is now written as

\[
H_B = \frac{1}{m} \dot{\hat{p}} \hat{\dot{p}} + \frac{Y}{2} (\dot{\hat{y}} \hat{\dot{p}} - \dot{\hat{p}} \dot{\hat{y}}) + m \left( \omega^2 - \frac{\gamma^2}{4} \right) \ddot{\hat{y}} \tag{27}
\]

which must be compared with \( \dot{H}_{\text{exp}} \), as given in (19), where \( \ddot{x} = \dot{x} \) is valid, since \( x \) fulfills equation (6) for the physical position variable. From equation (4), it follows that \( \ddot{\hat{p}} = m \left( \ddot{\hat{x}} + \frac{\gamma}{2} \dot{\hat{x}} \right) = m \left( \ddot{x} + \frac{\gamma}{2} \dot{x} \right) \), so none of the product terms of \( \ddot{x} \) and \( \ddot{\hat{p}} \) with one of the other variables \( \dot{y} \) and \( \ddot{\hat{p}} \) in (27) contains the exponential factor \( e^{\alpha t} \) that is common in equation (19).

Following the prescription outlined above, we equate \( \dot{H}_B \) (in the form of equation (9)) with \( \dot{H}_{\text{exp}} \):

\[
\frac{m}{2} e^{\alpha t} \left[ \dot{x}^2 + \gamma \dot{x} x + \omega^2 x^2 \right] = \ddot{\hat{p}} \ddot{x} + m \frac{\gamma}{2} \dot{\hat{x}} \dot{\hat{y}} + m \omega^2 \dot{x} \dot{y}, \tag{28}
\]

which is only possible if \( \dot{\hat{y}} \) and \( \ddot{\hat{p}} \) are expressed in terms of \( \dot{x} \) and \( \ddot{x} \).

For this purpose the ansatz

\[
\ddot{\hat{p}} = e^{\alpha t} (a \dddot{x} + b \dot{x}) \quad \text{and} \quad \dot{\hat{y}} = e^{\alpha t} (c \dddot{x} + d \dot{x}) \tag{29}
\]

is inserted into (28), and the coefficients of \( \dddot{x} \), \( \ddot{x} \), and \( \dot{x} \) terms are equated, leading to

\[
d = \frac{1}{2} \tag{30}
\]

\[
a = \frac{m}{2} (1 - \gamma c) \tag{31}
\]

\[
b = m \left( \frac{\gamma}{4} - \omega^2 c \right) \tag{32}
\]

where \( a \), \( b \), and \( c \) still have to be determined. As only two equations (31) and (32) are given, one parameter is still free to be chosen. Note that expressing (29) in terms of canonical variables,

\[
\dot{y} = e^{\alpha t} (c \dddot{x} + d \dot{x})
\]

\[
= e^{\alpha t} \left( \frac{c}{m} \ddot{\hat{p}} + \left( d - \frac{\gamma}{2} c \right) \dot{x} \right)
\]

\[
\ddot{\hat{p}} = e^{\alpha t} (a \dddot{x} + b \dot{x})
\]

\[
= e^{\alpha t} \left( \frac{a}{m} \ddot{\hat{p}} + \left( b - \frac{\gamma}{2} a \right) \dot{x} \right) \tag{33}
\]

an explicit time-dependent character of the contraints shows up, although they are compatible with the equations of motion; that is, the total time derivative of the constraints is zero. The explicit dependence on time of the constraints is traced back to the fact that they have non-vanishing Poisson brackets with the Hamiltonian.

In the following, a more detailed discussion of the cases (1) \( c = 0 \), (2) \( a = 0 \), and (3) \( b = 0 \) is given.

Generally, one parameter can be eliminated, leaving a condition for the relation of, e.g., \( a \) and \( b \) that has to be fulfilled for any choice of \( c \), etc.,

\[
a = \frac{1}{2 \omega^2} (m \Omega^2 + \gamma b) \quad \text{or} \quad \frac{b}{\gamma} = \frac{\omega^2 a}{m} - \frac{m}{2} \Omega^2. \tag{34}
\]

### 4.1. The case \( c = 0 \)

For this choice of \( c \) it follows that \( a = \frac{m}{2} \) and \( b = m \frac{\gamma}{4} \). In the following discussions, from the Bateman system \( \ddot{x} = \dot{x} = \hat{Q} e^{-\frac{\gamma}{4} t} \) and \( \ddot{\hat{p}} = m (\ddot{x} + \frac{\gamma}{2} \dot{x}) = \hat{P} e^{-\frac{\gamma}{4} t} \) is always valid; only \( \dot{y} \) and \( \ddot{\hat{p}} \) expressed in terms of \( \dot{x} \) and \( \ddot{x} \) are changing. Therefore, \( \dot{\hat{y}} \) and \( \ddot{\hat{p}} \) are supplied with a second subscript, indicating which parameter has been set equal to zero.

So, in this case, one obtains

\[
\dot{\hat{y}} = \frac{1}{2} \dddot{x} e^{\alpha t} = \frac{1}{2} \hat{Q} e^{\alpha t} \tag{35}
\]

\[
\ddot{\hat{p}}_{,c} = \frac{m}{2} \left( \dddot{x} + \frac{\gamma}{2} \dot{x} \right) e^{\alpha t} = \frac{1}{2} \hat{P} e^{\alpha t}. \tag{36}
\]
Inserting this into $\hat{H}_b$ (equation (27)) turns it into

$$\begin{align*}
\hat{H}_{b,c} &= \frac{1}{m} \hat{p}_{\lambda,c} \hat{p}_x + m \left( \omega^2 - \frac{\gamma^2}{4} \right) \hat{p}_x \\
&= H_\Omega \\
\end{align*}$$

since

$$D = \frac{\gamma}{2} \left( \dot{\hat{p}}_x - \dot{\hat{p}}_{\lambda,c} \right) = 0.$$  

(38)

$\hat{H}_{b,c}$ in (37) when expressed in terms of $x$ and $\dot{x}$ is identical to $\hat{H}_{exp}$ as given in (19). However, $\hat{H}_{b,c}$ is no longer a Hamiltonian that provides the correct equations of motion, the reason being that the constraints contain an explicit time-dependence.

On the other hand, $\hat{x}$ as defined in (35) now fulfills the equation of motion for $x$, i.e.,

$$\ddot{x} - \gamma \dot{x} + \omega^2 x = \frac{1}{2} \left( \ddot{x} + \gamma \dot{x} + \omega^2 x \right) e^{\eta} = 0.$$  

(39)

4.2. The case $a = 0$

For this choice of $a$ it follows that $c = \frac{1}{\gamma}$ and $b = -\frac{m}{\gamma^2} \left( \omega^2 - \frac{\gamma^2}{4} \right)$. The canonical variables that are still missing attain in this case the values

$$\begin{align*}
\dot{\hat{p}}_x &= \frac{1}{\gamma} \left( \ddot{x} + \frac{\gamma}{2} x \right) e^{\eta} \\
&= \frac{1}{m\gamma} \hat{p}_x e^{\eta} \\
\hat{p}_{\lambda,a} &= -\frac{m}{\gamma} \left( \omega^2 - \frac{\gamma^2}{4} \right) x e^{\eta} \\
&= -\frac{m}{\gamma} \Omega^2 \hat{Q} e^{\eta}. \\
\end{align*}$$  

(40)

(41)

Inserted into $\hat{H}_{b,c}$, this now yields

$$\begin{align*}
\hat{H}_{b,a} &= \frac{\gamma}{2} \left( \dot{\hat{p}}_x - \dot{\hat{p}}_{\lambda,a} \right) = D \\
\end{align*}$$

(42)

$\hat{H}_\Omega = \frac{1}{m} \hat{p}_{\lambda,a} \hat{p}_x + m \left( \omega^2 - \frac{\gamma^2}{4} \right) \hat{p}_x$, i.e., just the opposite situation to the case $c = 0$.

Again, $\hat{H}_{b,a}$ is no longer a proper Hamiltonian function that provides the correct equations of motion (see the comments in the previous case).

In this case, the equation of motion for $\hat{y}_b$ leads to

$$\begin{align*}
\ddot{\hat{y}}_b - \gamma \dot{\hat{y}}_b + \omega^2 \hat{y}_b &= \frac{\gamma}{\omega^2} \frac{d}{dt} \left( \ddot{x} + \gamma \dot{x} + \omega^2 x \right) \\
&\quad + \frac{1}{2} \left( \ddot{x} + \gamma \dot{x} + \omega^2 x \right) e^{\eta} \\
&= 0. \\
\end{align*}$$  

(44)

4.3. The case $b = 0$

Now one obtains $c = \frac{\gamma}{\omega^2}$ and $a = \frac{m}{\omega^2} \left( \omega^2 - \frac{\gamma^2}{4} \right)$, leading to

$$\begin{align*}
\dot{\hat{y}}_b &= \left( \frac{\gamma}{\omega^2} \ddot{x} + \frac{\gamma}{2} \dot{x} \right) e^{\eta} \\
&= \frac{1}{2a\omega^2} \left( \frac{\gamma}{\omega^2} \ddot{\hat{p}}_x + \Omega^2 \hat{Q} \right) e^{\eta} \\
\dot{\hat{p}}_{a,b} &= \frac{m}{2a\omega^2} \left( \omega^2 - \frac{\gamma^2}{4} \right) \dot{x} e^{\eta} \\
&= \frac{m\Omega^2}{2a\omega^2} \left( \frac{1}{m} \hat{p} - \frac{\gamma}{2} \hat{Q} \right) e^{\eta}. \\
\end{align*}$$  

(45)

(46)

Comparison with $\hat{H}_b$ (equations (27) or (19)) shows that now

$$\begin{align*}
\hat{H}_\Omega &= \frac{1}{m} \hat{p}_{a,b} \hat{p}_x + m \left( \omega^2 - \frac{\gamma^2}{4} \right) \\
\times \hat{y}_b &= \left( 1 - \frac{\gamma^2}{4 \omega^2} \right) \hat{H}_b = \frac{\Omega^2}{a\omega} \hat{H}_b \\
\end{align*}$$

(47)

$$D = \frac{\gamma}{2} \left( \dot{\hat{p}}_x - \dot{\hat{p}}_{a,b} \right) = -\frac{\gamma^2}{4} \hat{H}_b$$

(48)

is valid.

The equation of motion for $\hat{y}_b$ now takes the form

$$\begin{align*}
\ddot{\hat{y}}_b - \gamma \dot{\hat{y}}_b + \omega^2 \hat{y}_b &= \frac{\gamma^2}{\omega^2} \frac{d}{dt} \left( \ddot{x} + \gamma \dot{x} + \omega^2 x \right) \\
&\quad + \frac{1}{2} \left( \ddot{x} + \gamma \dot{x} + \omega^2 x \right) e^{\eta} \\
&= 0. \\
\end{align*}$$  

(49)

5. Quantum mechanical interpretation

Although the discussion in this paper is essentially restricted to the classical case, in the following an outline is given of how the results obtained in the last section can be interpreted in a quantum mechanical context.

The dual variables $\hat{y}$ and $\hat{p}_x$ can be expressed, as shown in sections 4.1 and 4.2, entirely in terms of $x$ and $\dot{x}$ (or $p$), or $\hat{Q}$ and $\hat{P}$, respectively. Because the connection between the Bateman Hamiltonian and the one in expanding coordinates or the CK one has been shown at the canonical level, $\hat{Q}$ and $\hat{P}$ seem to be the most appropriate variables. In these variables, the Hamiltonians $\hat{H}_{b,c}$ (equation (37)), $\hat{H}_{b,a}$ (equation (42)), and the sum of $H_\Omega$ (equation (47)) and $D$ (equation (48)) all take the form of Hamiltonian (15).

As we are working at the canonical level, quantization can be achieved in the usual way, e.g., in position space by keeping the position operator $Q_{op} = \hat{Q}$ as a $c$-number and replacing the momentum operator by $\hat{P}_{op} = \frac{\hbar}{i} \frac{\partial}{\partial Q_{op}}$. The resulting
canonical SE then has the form
\[
\dot{i\hbar} \frac{\partial \hat{\Psi}_{\exp}(\hat{Q}, t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial (\hat{Q})^2} + \left( \frac{\hbar}{m} \frac{\partial}{\partial \hat{Q}} - \frac{\gamma}{4} \right)^2 \right\} \hat{\Psi}_{\exp}(\hat{Q}, t).
\]
(50)

The exact analytic solution in the form of a Gaussian wave packet can be written as
\[
\hat{\Psi}_{\exp}(\hat{Q}, t) = \hat{N}_{\exp}(t) \times \exp \left\{ i \left[ \hat{\nu}_{\exp}(t) \hat{Q}^2 + \frac{1}{\hbar} \langle \hat{P} \rangle \hat{Q} + \hat{K}_{\exp}(t) \right] \right\}
\]
(51)
with \( \hat{Q} = \hat{Q} - \langle \hat{Q} \rangle \), \( \langle \hat{P} \rangle = m \langle \hat{Q} \rangle \) and \( \hat{\nu}_{\exp} \) as a complex function of time. The normalization function \( \hat{N}_{\exp}(t) \) and the phase factor \( \hat{K}_{\exp}(t) \) are purely time-dependent and not relevant for the equations of motion determining the evolution of the maximum and width of the wave packet, which are the two parameters completely describing the Gaussian function.

Inserting equation (51) into (50) provides the equation of motion for the maximum as
\[
\frac{d^2}{dt^2} \langle \hat{Q} \rangle + \left( \alpha^2 - \frac{\gamma^2}{4} \right) \langle \hat{Q} \rangle = 0,
\]
(52)
or, bearing in mind the fact that the mean values follow the classical equations of motion at the physical as well as at the canonical level, this can also be expressed in terms of the physical position variable as
\[
\frac{d^2}{dt^2} \langle x \rangle + \gamma \frac{d}{dt} \langle x \rangle + \alpha^2 \langle x \rangle = 0,
\]
(53)
i.e., a damped harmonic oscillator, in agreement with equation (6).

The equation of motion for the wave packet width depends on the complex variable \( \hat{\nu}_{\exp}(t) \) and can be expressed in terms of the slightly modified variable \( \hat{C}_{\exp} = \frac{\hbar}{m} \hat{\nu}_{\exp} \) in the form of the (complex) Riccati equation
\[
\dot{\hat{C}}_{\exp} + \hat{C}_{\exp}^2 + \left( \alpha^2 - \frac{\gamma^2}{4} \right) = 0,
\]
(54)
where the imaginary part is connected with the mean square deviation of position (or position uncertainty) via \( \dot{\hat{C}}_{\exp} = \frac{\hbar}{2m} \langle \hat{Q}^2 \rangle \) with \( \langle \hat{Q}^2 \rangle = \langle \hat{Q}^2 \rangle - \langle \hat{\nu}_{\exp} \rangle \). So far, the classical equation of motion for the position, expressed in physical variables, i.e., equation (53), coincides with our expectations. There are different ways of connecting the canonical Riccati equation (54) with the equation of motion describing the dynamics of the wave packet width (and thus the tunnelling currents) of the dissipative system at the physical level.

As the connection between dissipative canonical wave packet solutions of the CK model and corresponding solutions at the physical level have already been established [24], we first want to show the connection between the quantized versions of the expanding system and the CK model, where a subsequent non-unitary transformation supplies the solution at the physical level. Additionally, we confirm this procedure by showing that the same result can also be obtained in one (non-unitary) step, going directly from the expanding canonical system to the physical one.

The connection between the wave functions in the different descriptions can be found using Schrödinger’s original definition [35] of the wave function via the action function \( S \), according to \( S = \frac{i}{\hbar} \ln \Psi \). The generating function \( F_{\gamma}(\hat{x}, \hat{P}, t) \) connecting the CK approach and the expanding system has already been given in equation (26); the corresponding connection between the action functions then has the form
\[
\dot{S}_{\exp} = \dot{S}_{\text{CK}} + m \frac{\gamma}{4} \hat{C}^2 e^{\gamma t},
\]
(55)
which translates into the relation between the wave functions as
\[
\hat{\Psi}_{\text{CK}} = \exp \left\{ \frac{-im}{2\hbar} \frac{\gamma}{4} \hat{C}^2 e^{\gamma t} \right\} \hat{\Psi}_{\exp},
\]
(56)
where \( \hat{\Psi}_{\text{CK}}(x, t) \) is the solution of the time-dependent Schrödinger equation obtained via canonical quantization of the Hamiltonian (20). Because the transition between the canonical variables (\( \hat{\xi} = x, \hat{\eta} = px \)) and the physical variables (\( x,p \)) is given by a non-canonical transformation, the transition between the wave functions \( \hat{\Psi}_{\text{CK}}(x, t) \) and \( \Psi(x, t) \), where \( \Psi(x, t) \) fulfils a nonlinear modification of the Schrödinger equation at the physical level [24, 36], is given by a non-unitary transformation according to
\[
\ln \Psi(x, t) = e^{-\gamma t} \ln \hat{\Psi}_{\text{CK}}(x, t).
\]
(57)
(For details, see [24].)

The direct transformation of the solution from the expanding system to the physical one would have been obtained, in agreement with (56), via
\[
\ln \Psi(x, t) = e^{-\gamma t} \ln \hat{\Psi}_{\exp}(x, t) = \frac{-im}{2\hbar} \frac{\gamma}{4} \hat{C}^2.
\]
(58)
These transformations require that the Riccati equation determining the wave packet width in the CK system be connected with equation (54) via
\[
\dot{\hat{C}}_{\text{CK}} + \hat{C}_{\text{CK}}^2 e^{\gamma t} + \alpha^2 e^{\gamma t} = 0
\]
(59)
which is in agreement with the Riccati equation obtained from the quantized Hamiltonian (20), i.e.,
\[
\dot{\hat{C}}_{\exp} + \hat{C}_{\exp}^2 e^{-\gamma t} + \alpha^2 e^{-\gamma t} = 0
\]
(60)
and the corresponding Riccati equation at the physical level,
\[
\dot{\hat{C}} + \gamma C + C^2 + \alpha^2 = 0,
\]
(61)
where \( C_I = \frac{\hbar}{2m(c^2)} \) is now related to the physical wave packet width or position uncertainty. It should be pointed out that, due to the non-unitary transformation (57), \( C(t) \) and \( \dot{C}_{\text{CK}}(t) \)
are related via

\[ C(t) = \hat{C}_{\text{CK}}(t) e^{-i\gamma t}. \]  

(62)

Since \( C_{\text{B}}^{-1} \propto \langle \hat{x}^2 \rangle \), i.e., \( \langle \hat{x}^2 \rangle_{\text{CK}} = \langle \hat{x}^2 \rangle e^{-i\gamma t} \) (with \( \hat{x} = x \)), this led to the unphysical interpretation that, with increasing time, the position uncertainty is shrinking to a \( \delta \)-function, i.e., leading to localization of the quantum system.

The non-unitary transformation (57) shows that this is not the case, as physical and canonical levels must strictly be distinguished, even if the variables are partially identical.

Finally, because the two levels are connected via a non-unitary transformation, the wave function at the physical level is no longer normalized if the one at the canonical level was before the transformation. This can be taken into account by adjusting the (possibly TD) normalization factor of the physical wave function (see also [24]).

6. Conclusions

There are several approaches for the description of dissipative systems taking into account the environment in an effective way while still conforming to the conventional Hamiltonian formalism. In the case of the damped harmonic oscillator, they lead to an equation of motion for the damped system, including a linear velocity-dependent friction force. For the Bateman Hamiltonian, the environment is substituted by one additional variable and the corresponding momentum. It has been shown in our analysis that this variable, fulfilling a formal equation of motion with an accelerating force, is not just a position variable of a separate system that absorbs the energy dissipated by the damped system. In fact, the naïve idea that the Bateman system represents a degree of freedom interacting with a one-dimensional thermal bath is not quite appropriate: the system is so intricate that the Hamiltonian is not written as the sum of the two individual Hamiltonians plus an interaction term. However, after imposing suitable constraints (which equate \( \hat{H}_{\text{B}} \) and \( \hat{H}_{\text{exp}} \)), we arrive at the expanding coordinates system that does describe the damped harmonic oscillator and whose variables are connected with physical position and momentum via a non-canonical transformation.

More precisely, the link between the Bateman approach and the ones using canonical Hamiltonians with only one variable and the corresponding momentum can be achieved via an approach using an exponentially expanding coordinate system, because, in this case, the Hamiltonian is a constant of motion and can be compared with the constant Hamiltonian of the Bateman model.

It emerges that the constraints are not uniquely defined, as there are more parameters than equations for their determination. Several illustrative examples for the choice of the constraints are discussed in detail. In general, the Hamiltonian \( \hat{H}_{\text{B}} \), after imposing the constraints, is no longer a Hamiltonian in the sense that it would provide correct equations of motion, as the constraints contain an explicit time-dependence.

The relation between the variables of the Bateman system and the ones in expanding coordinates can be given explicitly in terms of the (physical) position and velocity of the damped system; the connection with the CK model can finally be achieved via the time-dependent canonical transformation between this model and the one using the expanding coordinates. For this purpose, \( \hat{F}_2 \) (see equation (26)) can be written in terms of \( \hat{Q} \) and \( \hat{p} \) as \( \hat{F}_2(\hat{Q}, \hat{p}, t) = \hat{Q} \hat{p} e^{-i\gamma t} + m^2 \hat{Q}^2 \) with \( \dot{\hat{F}}_2(\hat{Q}, \hat{p}, t) = -\frac{\gamma}{2} \hat{Q} \hat{p} e^{-i\gamma t} \pm \frac{\gamma}{2} \hat{Q} \hat{p} e^{-i\gamma t} \pm \frac{\gamma}{2} m x \hat{p} e^{-i\gamma t} \), so \( \hat{H}_{\text{CK}} = \hat{H}_{\text{exp}} + \frac{\gamma}{2} \hat{F}_2(\hat{Q}, \hat{p}, t) \) is valid. A different way to link the time-dependent constraints in a time-dependent canonical transformation to get directly from the Bateman Hamiltonian to the CK Hamiltonian has been shown in [14]. Both approaches to describe the dissipative system, the one in expanding coordinates and the one by CK, can be related with a description in terms of the physical position and momentum variables (as expressed in equation (6) with the friction force) via (different) non-canonical transformations according to equations (16) and (21), respectively.

At the canonical level, a quantum mechanical version can be obtained in the usual way by canonical quantization. In order to link the different canonical approaches, the canonical transformations have to be replaced by corresponding unitary transformations. The wave functions are related to the action function in the same way as originally defined by Schrödinger. Finally, the link to the physical level can be provided by non-unitary transformations followed by the necessary normalization.

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