Entanglement and perfect quantum error correction

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Abstract

The entanglement of formation gives a necessary and sufficient condition for the existence of a perfect quantum error correction procedure.

1 Quantum error correction

Suppose a composite quantum system \(RQ\) is initially in a pure joint input state \(|\Psi^{RQ}\rangle\). The subsystem \(Q\) undergoes a dynamical evolution described by a trace-preserving, completely positive (CP) map \(\mathcal{E}\). The joint output state is therefore

\[
\rho^{RQ'} = I \otimes \mathcal{E} \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right).
\]

(1)

This situation describes the transmission of “quantum information” (the entanglement between \(R\) and \(Q\)) via a noisy quantum channel. For example, imagine that \(RQ\) is a quantum computing device. The overall state of the device is entangled. Subsystem \(Q\) is imperfectly isolated from the environment, and thus experiences noise and distortion given by \(\mathcal{E}\). The problem of sending entanglement through a channel in this way is closely related to other tasks of quantum information transfer, such as the transmission of an unknown quantum state of \(Q\) \[\Pi\].
We are interested in the question of whether the original input state $|\Psi^{RQ}\rangle$ can be restored by some possible operation on $Q$ alone. Such a restoring operation is called a “quantum error correction” procedure [2]. We say that perfect quantum error correction is possible when there exists a trace preserving CP map $\mathcal{D}$ on $Q$ such that

$$|\Psi^{RQ}\rangle \langle \Psi^{RQ}| = I \otimes \mathcal{D}(\rho^{RQ}). \tag{2}$$

If no such $\mathcal{D}$ exists, we may still be able to do approximate quantum error correction, in which case we restore $\rho^{RQ}$ to a state close to the original. (“Close” here is usually defined in terms of the fidelity or some equivalent measure.) In this paper, we will mostly be concerned with the question of perfect (unit fidelity) error correction.

We first note that, if the input state of $RQ$ is a product state, then it is always possible to restore the input state by means of an operation on $Q$. Thus, the problem of error correction is only non-trivial when $|\Psi^{RQ}\rangle$ is entangled. The entanglement of a pure state of $RQ$ is measured by the entropy $S_{Q}$ of the subsystem $Q$:

$$S_{Q} = -\text{Tr} \rho^{Q} \log \rho^{Q}. \tag{3}$$

Of course, since $RQ$ is in a pure state, then $S_{Q} = S^{R}$.

The output state $\rho^{RQ'}$ is generally not pure. Schumacher and Nielsen [3] defined the “coherent information” to be

$$I = S^{Q'} - S^{RQ'}. \tag{4}$$

This quantity has a number of significant properties. It is positive only if the output state $\rho^{RQ'}$ is entangled. Furthermore, it cannot be increased by any operation on $Q$ alone. Since the initial coherent information is just $S^{Q}$, this means that $I \leq S^{Q}$ after the action of $\mathcal{E}$. Furthermore, any loss of $I$ due to the action of $\mathcal{E}$ is irreversible, i.e., cannot be reversed by any subsequent evolution of $Q$. It follows that $I = S^{Q}$ is a necessary condition for the existence of a perfect quantum error correction operation $\mathcal{D}$.

In [3], it is shown that the condition $I = S^{Q}$ is also sufficient for the existence of such an operation. In outline, we imagine a larger quantum system $RQE$ that includes the environment $E$ with which $Q$ interacts. The initial state of the environment is a pure state $|0^{E}\rangle$, and the interaction of $Q$ and $E$ is described by the unitary operator $U^{QE}$.

(Since the operation $\mathcal{E}$
is a trace-preserving CP map, it must always be realizable in this way as a unitary evolution on a larger system.) The condition $I = S^Q$ implies that the output state of the subsystem $RE$ is a product state. From this product structure, a perfect error correction procedure can be constructed. In short, the lack of any correlation between $R$ and $E$ after the evolution is sufficient to permit the restoration of the original $QE$ state by an error-correction operation $D$.

2 Entanglement of formation

The coherent information $I$ is a measure of the entanglement of $Q$ with $R$ after it has undergone its noisy evolution. There are, however, many other ways to measure the entanglement of the output state $\rho^{RQ'}$. One of the most fundamental is the “entanglement of formation” [4], denoted $E$. The entanglement of formation of a pure state $|\psi^{AB}\rangle$ is just $E = S^A$, the entropy of one of the subsystems. A mixed state $\rho^{AB}$ has an entanglement of formation

$$E = \min \sum_k p_k E_k$$

(5)

where $E_k$ is the entanglement of the pure state $|\phi^{AB}_k\rangle$ and the minimum is taken over all pure-state ensembles such that $\rho^{AB} = \sum_k p_k |\phi^{AB}_k\rangle\langle\phi^{AB}_k|$. $E$ has the property that it cannot be increased by local quantum operations on, or the exchange of classical information between, the two subsystems.

The entanglement of formation is related to the “entanglement resources” necessary to create the quantum state. However, to make this connection sharp one must define an asymptotic entanglement of formation

$$E_\infty(\rho^{AB}) = \lim_{n \to \infty} \frac{1}{n} E((\rho^{AB})^\otimes n)$$

(6)

$E_\infty$ is the asymptotic number of maximally entangled qubit pairs needed to create the state $\rho^{AB}$ by local operations and classical communication—that is, for large $n$, about $nE_\infty$ pairs are required to make $n$ copies of the state $\rho^{AB}$. Though the definitions of $E_\infty$ and $E$ are distinct, and we can see that $E_\infty \leq E$, it is not known whether or not these are actually equal in general [4]. We will here use the “single system” definition of the entanglement of formation $E$, since we are not primarily concerned with asymptotic questions.
The coherent information $I$ and the entanglement of formation $E$ of the output state $\rho^{RQ'}$ satisfy $I \leq E$. To see this, suppose we have an ensemble of $RQ$ states such that $\rho^{RQ'} = \sum_k p_k \rho_k^{RQ}$, then

$$S^{Q'} - \sum_k p_k S_k^Q \leq S^{RQ'} - \sum_k p_k S_k^{RQ}. \quad (7)$$

(This follows from the strong subadditivity of the entropy functional [1].)

For any ensemble of pure states, $S_k^{RQ} = 0$ and so

$$S^{Q'} - S^{RQ'} \leq \sum_k p_k S_k^Q. \quad (8)$$

If we choose the pure state ensemble that minimizes the right-hand side, we obtain $I \leq E$.

For the input pure state of $RQ$, both $E$ and $I$ are equal to $S^Q$. For the output state, the condition that $E = S^Q$ is weaker than the condition that $I = S^Q$, since we can have $I < E$. Thus, $E = S^Q$ is a necessary condition for the existence of a perfect quantum error correction operation $D$. Remarkably, it turns out that this is also a sufficient condition. We now show this. Suppose that $E = S^Q$ for our output state $\rho^{RQ'}$. Our argument is based on three facts.

**Fact 1: Concavity of the entropy.** Suppose we write a mixed state as an ensemble of states: $\rho = \sum_k p_k \rho_k$. Then

$$S \geq \sum_k p_k S_k \quad (9)$$

with equality if and only if $\rho_k = \rho$ for all $k$ with $p_k > 0$. In our context, the condition that $E = S^Q$ means that

$$0 = S^{R'} - \min \sum_k p_k S_k^{R'}, \quad (10)$$

where the minimum is taken over all pure state ensembles for $\rho^{RQ'}$. Equation (9) then tells us that

$$0 = S^{R'} - \sum_k p_k S_k^{R'} \quad (11)$$

for any pure state ensemble for $\rho^{RQ'}$, and therefore all of the elements of such an ensemble have $\rho_k^R = \rho^{R'}$. Since we can consider any mixed state to
be made up of pure states, this is also true for $\rho^{RQ'}$ ensembles that include mixed states.

**Fact 2: Choice of ensemble is choice of ancilla measurement.** Hughston, Jozsa and Wootters [8] give a useful characterization of all the pure state ensembles that can lead to a particular density operator $\rho^A$ for a system $A$. We “purify” the state by envisioning a pure state $|\psi^{AB}\rangle$ of a larger composite system $AB$ such that $\rho^A = \text{Tr}_B |\psi^{AB}\rangle\langle\psi^{AB}|$. A measurement on system $B$ will lead to an ensemble of relative states of $A$. In [8] it is shown that, given a purification $|\psi^{AB}\rangle$ of $\rho^A$, we can realize any ensemble for $\rho^A$ as an ensemble of relative states for some measurement on $B$. In other words, the choice of $\rho^A$ ensemble is exactly the same as the choice of measurement on the purifying system $B$.

In our context, we can include the environment system $E$ as before, with the whole system $RQE$ in the pure state $|\Psi^{RQE'}\rangle$. $E$ purifies $RQ$, so an ensemble of $RQ$ states corresponds to a measurement on $E$. From Fact 1, we know that every element of an ensemble for $\rho^{RQ'}$ yields the same state $\rho^R$ on $R$ alone. Thus, for any possible outcome of any measurement on $E$, the relative state of $R$ will be $\rho^R$. This means that the probabilities of the outcomes of possible $R$-measurements are unaffected by the particular outcomes of an $E$-measurement.

**Fact 3: No correlation implies product state.** Quantum state tomography [2] allows the reconstruction of a quantum state $\rho$ from the outcome distributions of a finite number of possible measurements on the quantum system. This procedure, when applied to a composite quantum system $AB$, has two important features. First, it is sufficient to consider only product measurements of $A$ and $B$ to do tomography of the joint state. Second, if no statistical correlations appear between the outcomes of the $A$ and $B$ measurements, the resulting joint state must be a product state $\rho^A \otimes \sigma^B$. Thus, a necessary and sufficient condition for $A$ and $B$ to be in a product state is that no correlations arise in any product measurement of the systems.

Since we have shown that $E = S^Q$ implies no statistical correlations between $E$-measurements and $R$-measurements on the output state, we can conclude that the output state of the subsystem $RE$ is a product state $\rho^{RE} \otimes \sigma^{E'}$. Given such a product state, we can apply the procedure in [3] to give an explicit error correction operation $D$ that will restore the input state $|\Psi^{RQ}\rangle$ of $RQ$ with perfect fidelity. Therefore, perfect quantum error correction is possible if and only if $E = S^Q$. 

5
3 Intrinsic expressions for $I$ and $E$

Both the coherent information $I$ and the entanglement of formation $E$ are “intrinsic” quantities to the system $Q$—that is, they can be expressed entirely in terms of the input state $\rho^Q$ of $Q$ alone and the trace-preserving CP map $\mathcal{E}$ that describes $Q$’s dynamics. First, we note that the map $\mathcal{E}$ can be given an “operator sum” representation [2]:

$$\mathcal{E} (\rho^Q) = \sum_k A_k \rho^Q A_k^\dagger,$$

(12)

where the $A_k$ operators satisfy $\sum_k A_k^\dagger A_k = 1$. A given $\mathcal{E}$ always has many different operator sum representations. Suppose we have a unitary matrix $V_{kl}$, and define some operators $B_k$ as linear combinations of the $A_k$’s:

$$B_k = \sum_l V_{kl} A_l.$$

(13)

Then the $B_k$’s give an alternate operator sum representation for $\mathcal{E}$.

The operator sum representation is closely related to the unitary representation for $\mathcal{E}$, in which $\mathcal{E}$ is given via unitary evolution on a larger system that includes the environment $E$. Once again, $E$ is taken to be initially in a pure state $|0^E\rangle$, and the interaction of $Q$ and $E$ is given by the unitary operator $U^{QE}$. Let $|k^E\rangle$ be a basis of $E$ states, and define the operator $A_k$ on $Q$ by the “partial inner product”

$$A_k |\psi^Q\rangle = \langle k^E | U^{QE} |\psi^Q0^E\rangle,$$

(14)

where $|\psi^Q0^E\rangle$ is shorthand for $|\psi^Q\rangle \otimes |0^E\rangle$. We can use the $|k^E\rangle$ basis to do a partial trace over the $E$ system, so that

$$\mathcal{E} (\rho^Q) = \text{Tr}_E \left[ U^{QE} (\rho^Q \otimes |0^E\rangle \langle 0^E|) U^{QE\dagger} \right]$$

$$= \sum_k \langle k^E | U^{QE} (\rho^Q \otimes |0^E\rangle \langle 0^E|) U^{QE\dagger} |k^E\rangle$$

$$= \sum_k A_k \rho^Q A_k^\dagger.$$

(15)

The unitary freedom in the operator sum representation is the same as the freedom to choose a basis for the environment system $E$. 

6
The operator sum representation of $\mathcal{E}$ gives the output state $\rho^{Q'} = \mathcal{E}(\rho^Q)$ as an ensemble of $Q$ states. If we let
\[ p_k = \text{Tr} A_k \rho^Q A_k^\dagger \]
\[ \rho_k^Q = \frac{1}{p_k} (A_k \rho^Q A_k^\dagger), \]
then $\rho^{Q'} = \sum_k p_k \rho_k^Q$. Different operator sum representations yield different ensembles for the same output state.

The entanglement of formation $E$ of the $\rho^{RQ'}$ state can be written
\[ E = \min \sum_k p_k S_k^Q \]
where $S_k^Q$ is the entropy of $\rho_k^Q$ (as defined above) and the minimum is taken over all operator sum representations for $\mathcal{E}$. In a similar way, the coherent information $I$ can be written
\[ I = S^{Q'} - \min H(\tilde{\mathcal{p}}) \]
where $H(\tilde{\mathcal{p}}) = -\sum_k p_k \log p_k$ and the minimum is once again taken over all operator sum representations. We can see why this is true by appealing to a unitary representation. The $p_k$'s are the diagonal entries of the output density matrix for the environment $E$, and $S_{E'} = \min H(\tilde{\mathcal{p}})$ (where we minimize over basis states). Since the global state of $RQE$ is pure, $S_{E'} = S_{RQ'}$.

4 Generalization

We pointed out that both $I$ and $E$ were measures of entanglement of the state $\rho^{RQ'}$, and that $I = E = S^Q$ for the input pure state $|\Psi^{RQ'}\rangle$. We now consider other possible measures of the entanglement of $\rho^{RQ'}$. Suppose $M$ is such a measure, and that it satisfies the following conditions:

1. $M = S^Q$ when $RQ$ is in a pure state.
2. $M$ is additive if we have many copies of $\rho^{RQ'}$; that is,
\[ M \left( (\rho^{RQ'})^\otimes n \right) = n M \left( \rho^{RQ'} \right). \]
3. $M$ does not increase on average under local operations on, or classical communication between, $R$ and $Q$.

Coherent information satisfies (1) and (2) but not (3); the asymptotic entanglement of formation $E_\infty$ satisfies all three; it is not known whether the “single system” entanglement of formation $E$ satisfies (2) (for this is exactly the question of whether $E = E_\infty$). Conditions (1)–(3) are similar to those discussed in [2].

We will now show that perfect quantum error correction is possible if and only if $M = S^Q$ for the output state $\rho^{RQ'}$.

“Only if” is easy to see. Initially, $M = S^Q$. If $M$ decreases under the action of $E$ on $Q$, then this loss cannot be made up by any error correction procedure, which must be a local operation on $Q$. Thus, the original state can be restored only if $M = S^Q$ after $E$ acts.

To show that $M = S^Q$ is sufficient to allow perfect error correction, we will show that $M \leq E$. Imagine that we begin with $nE_\infty$ maximally entangled qubit pairs, for which $M_n = nE_\infty$. We know that, if $n$ is large, we can use these pairs to make about $n$ copies of our state $\rho^{RQ'}$ by local operations and classical communication. Since $M$ cannot increase in this process, $nM \leq M_n$, and so $M \leq E_\infty$. But we have seen that $E_\infty \leq E$, so $M \leq E$.

We know that $E \leq S^Q$. Thus, if $M = S^Q$ then $E = S^Q$. As we have seen, this is sufficient to guarantee the existence of a perfect error correction operation $D$ for $Q$. $M = S^Q$ is therefore both necessary and sufficient for the existence of $D$.

Remarkably, inequivalent entanglement measures lead to equivalent conditions for perfect quantum error correction. The coherent information $I$, the entanglement of formation $E$ (or its asymptotic form $E_\infty$), and entanglement measures $M$ satisfying our properties all share the feature that they are conserved by the evolution $E$ on $Q$ only when that evolution produces no correlations between $R$ and $E$.

5 Remarks

We have assumed that $Q$ may interact with environment, while $R$ remains untouched. Suppose instead that both $Q$ and $R$ independently interact with separate parts of the environment, so that

$$\rho^{RQ'} = E^R \otimes E^Q \left| \Psi^{RQ} \right\rangle \left\langle \Psi^{RQ} \right| .$$

(20)
We say in this case that perfect quantum error correction is possible if the original state of $RQ$ can be restored by local operations and classical communication. It turns out that this can be done if and only if $E = S^Q$; furthermore, if error correction is possible at all, then no classical communication between $R$ and $Q$ is necessary.

Once again, $E = S^Q$ is plainly a necessary condition, and we must show that it is also sufficient. Suppose $E = S^Q$ after the operation $\mathcal{E}^R \otimes \mathcal{E}^Q$. We can imagine that this operation occurs in two stages:

$$
\rho^{RQ'} = (\mathcal{E}^R \otimes I^Q) \circ (I^R \otimes \mathcal{E}^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right). \tag{21}
$$

After the first stage, in which $I^R \otimes \mathcal{E}^Q$ acts, we must have $E = S^Q$. Therefore, at this stage there exists an operation $\mathcal{D}^Q$ on $Q$ that can accomplish perfect error correction. That is,

$$
|\Psi^{RQ}\rangle \langle \Psi^{RQ}| = (I^R \otimes \mathcal{D}^Q) \circ (I^R \otimes \mathcal{E}^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right). \tag{22}
$$

Alternately, we note that

$$
\rho^{RQ'} = (I^R \otimes \mathcal{E}^Q) \circ (\mathcal{E}^R \otimes I^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right), \tag{23}
$$

in which case $E = S^Q = S^R$ after the first operation, and an error correction operation $\mathcal{D}^R$ exists at this stage:

$$
|\Psi^{RQ}\rangle \langle \Psi^{RQ}| = (\mathcal{D}^R \otimes I^Q) \circ (\mathcal{E}^R \otimes I^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right). \tag{24}
$$

Now we can see that $\mathcal{D}^R \otimes \mathcal{D}^Q$ will correct the complete operation:

$$
(\mathcal{D}^R \otimes \mathcal{D}^Q) \circ (\mathcal{E}^R \otimes \mathcal{E}^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right)
= (I^R \otimes \mathcal{D}^Q) \circ \left( \mathcal{D}^R \otimes I^Q \circ (I^R \otimes \mathcal{E}^Q) \circ (\mathcal{E}^R \otimes I^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right) \right)
= (I^R \otimes \mathcal{D}^Q) \circ \left( \mathcal{D}^R \otimes I^Q \circ (\mathcal{E}^R \otimes I^Q) \left( |\Psi^{RQ}\rangle \langle \Psi^{RQ}| \right) \right)
= |\Psi^{RQ}\rangle \langle \Psi^{RQ}|. \tag{25}
$$

Thus, $E = S^Q$ is a necessary and sufficient condition for local correction of the quantum state, even if both subsystems have experienced independent noisy evolutions.

Throughout this paper, we have focused our attention on the issue of perfect error correction. What about approximate error correction? We have
elsewhere shown that, if the loss of coherent information is small, then an operation $D$ exists that will nearly restore the original state $|\Psi^{RQ}\rangle$. To be precise, if $S^Q - I < \epsilon$, then there exists an operation $D$ on $Q$ that will restore the input state with fidelity $F > 1 - 2\sqrt{\epsilon}$. Is there an analogous theorem for the entanglement of formation $E$? That is, suppose $S^Q - E < \epsilon$. With what fidelity can error correction be performed? This and many other questions remain unresolved.

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