Non-trivial class of the mixed $U(\sigma+\mu)$-vector solitons.

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Abstract

There has been found an exact solution of the mixed problem for Shrödinger's compact $U(m)$-vector nonlinear model with an arbitrary sign of the coupling constant. It is shown, that in case of $m \geq 3$ there is a new class of solutions - mixed $U(\sigma+\mu)$-vector solitons with "inelastic" (changing the form without the energy loss) interaction at $\sigma > 1$ and strict elastic - at $\sigma = 1$. They correspond to the color complexes consisting of $\sigma$-bright and $\mu$-dark solitons ($\sigma + \mu = m$) and they can exist both in self-focusing and defocusing medias. The universal $N$-soliton formula for the attraction and repulsion cases has been obtained by the method of Hirota for the first time.

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0. The evolutionary system of coupled nonlinear Shrödinger equations (NLS-m)

\[ i\dot{\psi}_j = \sum_{k=1}^{m} a_{jk} |\psi_k|^2 \psi_j, \quad j = 1, m \]  \hspace{1cm} (1)

\[ \psi_j \in \mathbb{C}, \quad \dot{\psi}_j = \partial_\zeta + ic_j \partial_{\zeta \xi}; \quad a_{jk}, c_j \in \mathbb{R} \]

arises within the weak coupling limits in various nonrelativistic models of the nonlinear field theory. The integrability conditions and the exact solutions of NLS are of a broad practical interest (nonlinear optics, plasma, ferromagnetism, hydrodynamics, Bose-Einstein atomic condensates etc.[1-6]) together with an academic one. A strict mathematical derivation of the two connected parabolic motion equations that are equivalent to the system (1) when $m = 2$ and $c_1 = c_2$ is given in [7] where the self-influence

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of different polarization waves in the nonlinear media of a tensor response has been studied for the first time. At (1) $c_j$ parameters determine the dispersion, and matrix coefficients $a_{jk}$ when $j \neq k$ and $j = k$, determine nonlinear interaction and self-action of $\psi_j$ fields correspondingly. Subject to the implication of the variable $\xi$ and the sign of the parameters product $\text{sign}(c_j a_{jk}) = \varepsilon_k$ the system (1) at a classical level describes a spatial or time evolution of the $m$-component field in the nonlinear cube-medium\[7,8]\; at the quantum one - Bose-gas with $m$-color degrees of freedom \[9,10\] with attractive ($\varepsilon_k > 0$) or repulsive ($\varepsilon_k < 0$) point interaction.

The exact integrable cases of the system (1), by Liouville implication, are highly limited and require satisfaction of the rigid conditions in space of the controlled parameters $c_j, a_{jk}$. On the bases of Zakharov’s theorems about the additional motion invariants one may show (the proof will be given in a separate work), that when the conditions are met $a_{jk} = \pm a_{kk}, c_j = \pm c$ the system (1) admits representation of a zero curvature (Lax-pair) and is embeddable into the scheme of the inverse scattering method (ISM). The integrable reductions of NLS-$m$ arising in this case form the family of solitons vector models with a unitary $U(m)$ and pseudounitary $U(m,n)$ group of symmetry. The known exact solutions of the given family of models, for example, those of the $U(2)$-vector model of Manakov \[8\] ($L_0 = L_1 = L_2, \sigma = 2, \mu = 0$)

$$i\dot{L}_0 \psi_j^* = \varepsilon(|\psi_2| - |\psi_2|^2)\psi_j^*, \; j = 1, 2$$

(2)

are "single-color" multisolitons: bright solitons ($\psi_{1,2} \sim \text{sech}\alpha$) in a self-focusing medium ($\varepsilon > 0$) \[8\] and dark ones ($\psi_{1,2} \sim \tanh\beta$) in a defocusing medium ($\varepsilon < 0$) \[11,12\]. Bright-vector \[8\] and dark-vector \[11,12\] soliton solutions that are constructed on the base of the traditional boundary problems of the $U(1)$-scalar bright ($\psi(\pm\infty) = 0$) and dark ($\psi(\pm\infty) = \rho \exp(i\Theta)$) solitons \[5,13\] may be classed as trivial ones in the given sense. The exact solutions of the mixed vector solitons in the pseudo-Euclidean $U(1,1)$-model \[9\] and in the Euclidean $U(1+1)$-model \[2\] with a defocusing ($\varepsilon < 0$) nonlinearity have demonstrated \[12\] that they also interact (as $U(1)$-scalar solitons) in the elastic (trivial) fashion.

For the first time it was shown in the given work that in the family of $U(m)$-vector nonlinear models of Shrödinger (integrable reductions of the system (1))

$$i\dot{L}_0 \psi_j^* = \varepsilon \sum_{k=1}^{m} |\psi_k|^2 \psi_j^*, \; j = 1, m$$

(3)

with the boundary conditions of the mixed density

$$\begin{align*}
\psi_\sigma(\zeta, \xi) \big|_{|\xi| \to \infty} &\to 0, \\
\psi_\mu(\zeta, \xi) \big|_{|\xi| \to \infty} &\to \rho_\mu e^{i\Theta_\mu},
\end{align*}$$

(4)

at the $m \geq 3$, there exist a class of exact solutions - "color" $U(\sigma+\mu)$-vector solitons with a nontrivial interaction (intermode interchange). The conditions (4) imply that every degree of freedom $\psi_n(1 \leq n \leq m)$ in the system (3) has its vacuum (condensate) of zero $\rho_n = 0$ or finite $\rho_n \neq 0$
density \( \rho_n^2 \) with an asymptotic phase \( \Theta_n \). In this case \( m = \sigma + \mu \), and in the rest \( \sigma \) and \( \mu \), take on the arbitrary values \( 1, 2, \ldots, m \).

\( N \)-soliton formula of the \( m \)-component system (3-4) depends definitely from the medium character (\( \text{sign} \, \varepsilon = \pm 1 \)) and in this sense it is a universal one for both self-focusing (attractive, +1) and defocusing (repulsive, −1) media of cube-linearity.

ISM in case of the system (3-4) faces the necessity of analysis of the \( (m + 1) \)-sheet Riemann surfaces and thereupon the \( N \)-soliton solution is obtained by a more economical (in a mathematical sense) method of Hirota [14]. It is shown that an elastic (trivial) interaction of the mixed (bright and dark) \( U(1 + 1) \)-vector solitons [9,12] is a consequence of the \( N \)-soliton solution factorability for the system (3-4) in a particular case of \( m = 2 \). In the general position situation the \( N \)-soliton solution is not factorizable and the interaction of the color multisolitons has the inelastic nature (changing the form at the energy conservation). There have been found special cases when the interchange between nonlinear modes does not arise and the \( N \)-soliton scattering turns out to be a factorizable one.

1. Let’s introduce the functions of Hirota

\[
G_j = H \psi_j, \quad G_j \in \mathbb{C}, \quad H \in \mathbb{R}, \quad j = \overline{1, m}
\]

and make a transition \( \left( \hat{L}_j \rightarrow \hat{D}_j \right) \) from the linear operators \( \hat{L} \) to the bilinear \( \hat{D} \) ones that are defined hereinafter as

\[
\hat{D} \left( U \cdot V \right) = (\hat{D} U)V - U(\hat{D} V).
\]

Hereinafter variables \( \zeta \) and \( \xi \) at (1), (3-4) will be attributed the sense of time \( t \) and coordinate \( x \).

Taking into account the scaling changes \( |\varepsilon| = 2, c_j = c(>0), x \rightarrow x \sqrt{c}, \) the system (1) (due to the concept of Hirota) forms a bilinear family of the compact \( U(m) \)-symmetry

\[
\hat{D}_1 G_j \cdot H = 0, \quad j = \overline{1, m} \\
\hat{D}_2 H \cdot H = 2\delta \sum_{k} |G_k|^2, \\
(\hat{D}_1 = i\hat{D}_t + \hat{D}_2, \quad \hat{D}_2 = \hat{D}_x^2 - \lambda).
\]

where \( \lambda \in \mathbb{R} \) is an arbitrary parameter which will be defined lower; \( \delta = \text{sign} \, \varepsilon \).

In the formalism of the bilinear operators for the functions \( G_j \) and \( H \) there is a representation in a series form due to the formal parameter \( \varepsilon \). Let’s chose this representation in such a way that it was coordinated with the non-trivial boundary conditions (4), i.e.:

\[
G_j = \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} (g_{0\mu} g_{2\nu}^\mu \delta_{j\mu} + \varepsilon g_{0\sigma} g_{2\nu+1}^\sigma \delta_{j\sigma}) \\
H = \sum_{\nu=0}^{\infty} \varepsilon^{2\nu} h_{2\nu}; \quad g_0^\mu = h_0 = g_{0\sigma} = 1, \\
\delta_{\alpha\beta} = \text{Kronecker’s symbol}, \quad j = \overline{1, m}.
\]

3
It is obvious that functions $G_j$ determine $m$-component field ($m=\sigma+\mu$) in an arbitrary combination $\sigma$ and $\mu$ (for example, $\sigma$-bright solitons and $\mu$-dark ones). We will get $N$-soliton solutions of the bilinear system (5) following the standard scheme of Hirota ($R \sim G_j, H; \ j = 1, m$).

$$R = R_0 \xrightarrow{\delta^0} R_1 \xrightarrow{\delta^1} R_2 \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{N-1}} R_N.$$ (7)

Let’s lay the vacuum solution $g_{0\mu} = \rho_{\mu} \exp(i\Theta_{\mu})$. $\Theta_{\mu} = k_{\mu}x - (k_{\mu}^2 + \lambda)t$ in a zero order in $\varepsilon$. One can determine $\lambda = -2\delta \sum_{\mu} \rho_{\mu}^2$ from (5). In the physics of optical solitons the sign function $\delta = \text{sign} \varepsilon$ defines a self-focusing ($\delta = +1$) or defocusing ($\delta = -1$) character of the nonlinear medium. We have the following in a zero order in $\varepsilon$:

$$g^{(j)} = \sum_{n=1}^{N} \gamma_{n}^{(j)} \exp(\eta_{n}), \ \eta_{n} = \zeta_{n}x + i(\zeta_{n}^2 + 2\delta \sum_{\mu} \rho_{\mu}^2)t,$$

where $\gamma_{n}^{(j)}$ and $\zeta_{n}$ are arbitrary complex parameters. In a one-soliton ($N = 1$) sector series (6) in the scheme (7) stop in the second order in $\varepsilon$, in a two-soliton ($N = 2$) sector they stop in the fourth order in $\varepsilon$ etc.

The solutions of the system (5) describing the propagation dynamics of $N$-solitons in two $\sigma$ and $\mu$-sectors of $U(m)$-vector $\psi$-space (fixed on the 6th order in $\varepsilon$) is represented as:

$$H\psi_{\sigma} = \sum_{n=1}^{N} \dot{\eta}_{n} (\varepsilon^{1} \gamma_{n}^{\sigma} + \varepsilon^{3} \hat{a}_{nij}^{\sigma} + \varepsilon^{5} \sum_{lm} \hat{a}_{nijlm}^{\sigma} \hat{\eta}_{m} \hat{\eta}_{m} + \ldots)$$ (8)

$$H\psi_{\mu} = g_{0\mu} (\varepsilon^{0} + \sum_{ij} \dot{\eta}_{ij} \varepsilon^{2} \hat{a}_{ij}^{\mu} + \sum_{lm} \dot{\eta}_{lm} \varepsilon^{4} \hat{a}_{lmij}^{\mu} + \varepsilon^{6} \sum_{qr} \hat{a}_{ijlmqr} \hat{\eta}_{q} \hat{\eta}_{r} \hat{\eta}_{r} + \ldots)$$ (9)

$$H = \varepsilon^{0} + \sum_{ij} \dot{\eta}_{ij} \varepsilon^{2} \hat{a}_{ij} + \sum_{lm} \dot{\eta}_{lm} \varepsilon^{4} \hat{a}_{lmij} + \varepsilon^{6} \sum_{qr} \hat{a}_{ijlmqr} \hat{\eta}_{q} \hat{\eta}_{r} \hat{\eta}_{r} + \ldots$$
Here $\varepsilon = 1$ (R.Hirota), \( \tilde{\eta}_n = \exp(\eta_n) \):

\[ a_{n}^\mu = \frac{\zeta_{ni}}{\zeta_{nj}\zeta_{ij}}(\gamma_{ni}^\mu \tilde{a}_{ij} - \gamma_{nj}^\mu \tilde{a}_{nj}), \]

\[ a_{ijlm} = \frac{\zeta_{il}}{\zeta_{ij}\zeta_{jm}\zeta_{lm}}(\tilde{a}_{ij} \tilde{a}_{lm} - \tilde{a}_{im} \tilde{a}_{lj}), \]

\[ a_{n}^\mu_{ijlm} = \frac{\zeta_{il}^\mu}{\zeta_{nj}\zeta_{nm}} \gamma_{ijlm}^\mu \left\{ \sum_{m=1}^{\infty} \rho_{l}\rho_{n} \right\}, \]

\[ a_{\cdots ij\cdots}^\mu = \tilde{z}_{ij}^\mu a_{\cdots ij\cdots} - \tilde{z}_{j\mu} = \tilde{z}_{j\mu}^{\ast}, z_{j\mu} = \zeta - i k_{\mu}, \quad (10) \]

\[ \tilde{a}_{ij} = \frac{\sum_{\sigma=1}^{\infty} \gamma_{ij}^\sigma \tilde{z}_{j\sigma}}{\zeta_{ij}(\delta + \sum_{\mu=1}^{\infty} \rho_{l}\rho_{n} / \zeta_{j\mu}\tilde{z}_{j\mu})}, \quad a_{ij} = \frac{\tilde{a}_{ij}}{\zeta_{ij}}. \]

It is quite evident from here that the two-soliton \((N = 2)\) solution stops at the 4th order in \( \varepsilon \). At the same time one can assure that the formulas (8-9) correspond to the exact three-soliton \((N = 3)\) solution of the system under discussion (5). The solutions of the higher order are not given here because of their awkwardness.

2. One-soliton \((N=1)\) solution of Shrödinger’s mixed \( U(\sigma+\mu)\)-vector nonlinear model (3-4) from (8-10) takes the form:

\[ \begin{pmatrix} \psi_{(\sigma)} \\ \psi_{(\mu)} \end{pmatrix} = H^{-1} \begin{pmatrix} \gamma_{n}^{\sigma} e^{\eta_{1}} \\ g_{n}(1 + a_{1n}^{\sigma} e^{\eta_{1} + \eta_{1}}) \end{pmatrix} \quad (11) \]

where Hirota function is \( H = 1 + a_{1n} e^{\eta_{1} + \eta_{1}} \),

\[ a_{11}^{-1} = c_{11}^{2}(\delta + \sum_{\mu=1}^{\infty} \rho_{l}^{2} |z_{1\mu}|^{-2}) / \sum_{\sigma=1}^{\infty} |\gamma_{1}^\sigma|^{2}, \]

\[ a_{11}^{\mu} = \tilde{z}_{11}^{\mu} a_{11}, \quad \tilde{z}_{11}^{\mu} = - \exp(2i\phi_{1\mu}), \]

\[ \phi_{1\mu} = \arctan((\text{Im} \zeta_{1} - k_{\mu}) / \text{Re} \zeta_{1}). \]

As one can see the \( U(\sigma+\mu)\)-vector soliton of the mixed color (11) is a dynamics-and-topological formation and in particular cases coincides with known earlier one-color bright-vector \( (\psi_{(\mu)} = 0, \delta = + 1) \) [8] and dark-vector \( (\psi_{(\sigma)} = 0, \delta = - 1) \) [11,12] solitons. However, there is a principally new thing, i.e. the fact what the exact solution (11), unlike the vector solitons [8,11,12] has its place in the system (3) both in the cases of attraction (self-focusing, \( \delta = + 1 \)) and repulsion (defocusing, \( \delta = - 1 \)). Besides that, one should mention that a universal color \( U(\sigma+\mu)\)-vector soliton (11) can be in several states determined by its dynamics-and-topological nature. It is convenient to interpret these states in the language of particles. Let’s denote the admissible isotopic states of the color \( U(\sigma+\mu)\)-vector soliton
by the symbol \{\sigma, \mu\}, where \sigma + \mu = m. Then by analogy with a quantum chromodynamics, the state with a mixed color \{\sigma \neq 0, \mu \neq 0\} may be supposed to be "aromatic" and the one with a mixing miss \{(0, \mu), \{\sigma, 0\}\} - "no aromatic" (one-color). In the given analogy the \(U(m)\)-vector soliton of a mixed color has the intrinsic structure and the existence of different states is natural for such a compound particle. For example, in case of Shrödinger's \(U(5)\)-vector nonlinear model the solution (11) for one \{3,2\} of the 4 admissible (\{1,4\}, \{2,3\}, \{3,2\}, \{4,1\}) aromatic states of the color \(U(3+2)\)-vector solitons has the form:

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
(\psi_3 \psi_4 \psi_5)^t
\end{pmatrix}
= 
\begin{pmatrix}
A_1(\tanh X + i \tan \phi_1) e^{i \Theta_1} \\
A_2(\tanh X + i \tan \phi_2) e^{i \Theta_2} \\
(B_3 B_4 B_5)^t \text{sech} X e^{i \Theta}
\end{pmatrix}.
\tag{12}
\]

\[A_{\mu} = \rho_{\mu} \cos \phi_{\mu}, \Theta_{\mu} = k_{\mu} x - (k_{\mu}^2 - 2\delta \sum_{\mu=1}^2 \rho_{\mu}^2) t + \phi_{\mu},\]

\[\phi_{\mu} = \arctan((v - 2k_{\mu})/u); \quad 2X = u(x - vt - x_0),\]

\[B_{\sigma} = \gamma^\sigma \left(\sum_{\mu=1}^2 A_{\mu}^2 + \delta \sum_{\mu=1}^2 |\gamma^\sigma|^2\right)^{1/2},\]

\[2\Theta = vx + (u^2 - v^2 + 8\delta \sum_{\mu=1}^2 \rho_{\mu}^2) t/2;\]

\[\mu = 1, 2; \quad \sigma = 3, 4, 5;\]

where \(u = 2\text{Re} \zeta_1\) and \(v = 2\text{Im} \zeta_1\) are soliton's reverse width and velocity. It is clear that a mixed \(U(5)\)-vector soliton (12) consists of two dark (\(\psi_1, \psi_2\)) and three bright (\(\psi_3, \psi_4, \psi_5\)) components (nonlinear modes). It is also obvious that the number of all the \(U(5)\)-vector soliton admissible states is equal to 6, however two of them are no aromatic (one-color) vector solitons: a bright-vector one \(\{5,0\}\) in case of self-focusing medium \((\delta = +1)\) and a dark-vector one \(\{0,5\}\) in case of defocusing medium \((\delta = -1)\). The change of the medium refraction index \(\Delta n^2 \sim |\psi_1|^2 + |\psi_2|^2 + ... + |\psi_5|^2\) induced by the \(\psi_1, \ldots, \psi_5\) components interaction may be calculated by the forward substitution of an explicit solution (12). However, there comes a universal formula from the bilinear system of equations (5) for the whole family of Shrödinger's \(U(m)\)-vector nonlinear models,

\[
\Delta n^2 = \sum_{\mu} \rho_{\mu}^2 + \delta \frac{d}{dx} \ln H, \tag{13}
\]

which permits to determine the value of \(\Delta n^2\) of the Hirota unified function . So, for example, in case of consideration of the above \(U(5)\)-model \(\mu = \overline{1,2}, H = 1 + a_{11} e^{n_1 + \phi_1}\). There will be derived from the (13)

\[
\Delta n^2 = \rho_1^2 + \rho_2^2 \pm \left(\frac{u^2}{4}\right) \text{sech}^2 \left[\frac{u(x - vt - x_0)}{2}\right]
\]

for the self (de) focusing (+(-)) medium.

One should mention that presence of vacuum-condensate in the system with a finite density \(\rho_{\mu}^2\) in the defocusing \((\delta = -1)\) media imposes a natural limitation on the color vector soliton characteristics: \(u^2 \leq 4 \sum_{\mu} \rho_{\mu}^2 \cos^2 \phi_{\mu}\).
Nevertheless as far as the number of the admissible aromatic states of the color $V(m)$-vector soliton is equal to $(m-1)$, the detection of exactly the same states in the multimode optical systems may turn out an event, which is more probable than one-color states whose number is equal to two.

3. Two-soliton ($N=2$) solution and dynamics of the mixed color multisoliton interaction.

Let’s show that the interaction of the color multisols (8,9) in Shrödinger’s mixed $V(m)$-vector nonlinear model (3,4) is nontrivial (changing a form without any energy loss) at $m \geq 3$ and there has a place the intermode interchange (energy) that is proportional to the nonlinear modes intensity of solitons. Let’s study for this goal (without community limitation) asymptotic ($t \to \pm \infty$) behavior of the color multisols (8,9) at $N=2$.

Two-soliton ($N=2$) solution from (8,9) takes the form ($\sigma+\mu=m$):

$$
\left(\psi_{(\sigma)} \psi_{(\mu)}\right) = H^{-1} \left(\gamma_1 e^{\eta_{11}t + \eta_{12}x} + \sum_{i,j}^2 a_{ij} e^{\eta_{ij}t + \eta_{ij}\gamma_{ij}}\right),
$$

where Hirota function

$$
H = (1 + \sum_{i,j}^2 a_{ij} e^{\eta_{ij}t} + a_{1122} e^{\eta_{11}t + \eta_{22}x}).
$$

Let $v_1 > v_2 (\text{Im} \xi_1 > \text{Im} \xi_2)$, where $v_n$ is $S_n^-$-soliton velocity in $j$ mode ($j=1,2,\ldots,\mu,\mu+1,\ldots,\mu+\sigma$). Solution (14) at $t \to \pm \infty$ on the paths $\xi_n = x - v_n t$ of separate solitons fall into the sum of the free one-soliton solution of the type (12):

$$
\psi_{(j)}(x,t) \bigg|_{t \to \pm \infty} = \sum_n C_{j}^{n\pm} S_{(j)}^n(x - v_{n} t, x_{0n}^\pm) e^{i \phi_{nj}}
$$

Here $C_{j}^{n\pm}$-amplitude, $S_{(j)}^n$-envelope of $j$ mode of $n$ soliton $n = 1, 2; j = \sigma, \mu$:

$$
S_{(\sigma)}^n = \frac{1}{2} \left(\psi_{(\sigma)}^n + \psi_{(\sigma)}^n\right), \quad S_{(\mu)}^n = \frac{1}{2} \left(\psi_{(\mu)}^n + \psi_{(\mu)}^n\right), \quad Y_{n}^{\pm} = \text{tanh}(x - v_{n} t - x_{0n}^\pm)/2,
$$

$$
\phi_{nm} = \text{arctan}((v_n - 2k_m)/u_n), \quad u_n = 2\text{Re} \xi_n, \quad v_n = 2\text{Im} \xi_n.
$$

Amplitudes $C_{j}^{n\pm}$ of $S_{(j)}^n$ solitons before ($-$) and after ($+$) the interaction are connected by the relations $C_{j}^{n\pm} = S_{j}^n C_{j}^{n\pm}$, where $S$-special matrix, converting asymptotic at $t \to -\infty$ in the asymptotic at $t \to +\infty$:

$$
S_1^\sigma = \tilde{C}_{12} (1 - s_1 \gamma_{12}) (1 - s_1 s_2)^{1/2}, \quad 2C_1^- = \gamma_1 (a_{11})^{-1/2}, \quad \tilde{S}_1^\mu = e^{(2\phi_{2\mu} - \eta)},
$$

$$
C_1^+ = \rho_{\mu} \cos \phi_{1\mu} e^{(\phi_{1\mu} + \eta)}, \quad \tilde{S}_2^\sigma = \tilde{C}_{21} (1 - s_2 \gamma_{21})^{-1} (1 - s_1 s_2)^{1/2}, \quad 2C_2^- = a_{21} (a_{1122} \gamma_{21})^{1/2}, \quad \tilde{S}_2^\mu = e^{-(2\phi_{2\mu} - \eta)},
$$

$$
C_2^+ = \rho_{\mu} \cos \phi_{2\mu} e^{(2\phi_{1\mu} + \phi_{2\mu})},
$$

$$
\gamma_1^\sigma = \frac{\gamma_1}{\gamma_2}, \quad s_1 = \frac{\gamma_{12}}{\gamma_2}, \quad s_2 = \frac{\gamma_{21}}{\gamma_2}, \quad |\tilde{C}_{12}| = |\tilde{C}_{21}| = 1.
$$
Hence it is quite clear that solitons velocities $v_n$-motion invariants, and phases $X_{kn}^n$ and amplitudes $C_{kn}^n$ are not such. As $|S_\mu^m| \neq 1$ and $|S_\mu^m| = 1$ in the situation of a general position, it is quite obvious that the interaction of the mixed color $U(\sigma + \mu)$-vector solitons (14) has a nontrivial (inelastic) character. As a result of such an interaction between nonlinear modes there arises the intensity interchange $\sim |S_\mu^m|^2$. The intermode exchange initiates energy redistribution in the components (nonlinear modes) of the color vector solitons. However, the given interchange phenomena in the $\sigma$- and $\mu$- modes have their peculiarities: $\sigma$-modes interchange by the finite energy $\sim |S_\mu^m|^2$ and maintain the sign; $\mu$- modes maintain energy ($|S_\mu^m|^2 = 1$), but change their polarity and attain an additional phase jump $\sim (2 \phi_{nm} + \pi)$ as an interaction result. It is clear that $\mu$- modes interact only with different values of phases.

Asymptotic analysis shows in general that the exchange between the components of a separate color soliton is not arbitrary (chaotic) but is correlated with the commensurable changes in the components of all the other solitons. A nontrivial interaction between the color multisolitons (14) and the admissible scenarios of the intermode exchange in the system (3,4) are regulated by the laws of conservation: a) $\sum_j^{m} |C_j^m|^2 = \sum_j^{m} |C_j^{m+1}|^2$ -total intensity of a separate soliton $S_j^n$ and b) $\sum_n \sum_j^{m} |C_j^n|^2 = \sum_n \sum_j^{m} |C_j^{n+1}|^2$ - full intensity of all the solitons before (–) and after (+) interaction. Validity of the given laws may be easily seen from the asymptotic formulas (16). Besides this, shifts of their inertia centers $\Delta X_n = X_{kn}^{+} - X_{km}$, $\Delta X_n = (-1)^{n+1} \zeta_{nm}^{-1} \ln \chi$, where

$$\chi = \frac{\zeta_{12} - \zeta_{21}}{\zeta_{12} + \zeta_{21}} \sqrt{1 + \frac{\zeta_{12} \zeta_{21}}{\zeta_{12} + \zeta_{21}} (1 - \epsilon)},$$

$$\epsilon = \frac{\delta + \sum_{\mu} \sum_{\sigma} \rho_{\mu}/|z_{1\mu}|^2 \delta + \sum_{\mu} \sum_{\sigma} \rho_{\mu}/|z_{2\mu}|^2 \sum_{\nu} \sum_{\sigma} \gamma_{\nu}^2 \gamma_{\sigma}^2 \gamma_{\nu}^2 \gamma_{\sigma}^2}{\beta + \sum_{\mu} \sum_{\sigma} \rho_{\mu}/|z_{1\mu}|^2 \sum_{\sigma} \rho_{\mu}/|z_{2\mu}|^2 \sum_{\nu} \sum_{\sigma} \gamma_{\nu}^2 \gamma_{\sigma}^2 \gamma_{\nu}^2 \gamma_{\sigma}^2}$$

(17)

arising as a result of solitons interaction obey the condition of Sudzuki-Zakharov-Shabat (the law of solitons center of inertia conservation):

$$\zeta_{11} \Delta X_1 + \zeta_{22} \Delta X_2 = 0.$$  

The last one is a consequence of the value conservation $I_{tot} = \int \sum_n |\psi_n|^2 dx$ in time $t$. The laws of conservation and strict formulas that are mentioned above permit to determine the interchange "kinetics" and admissible scenarios of the intermode switches in the system of the color multisolitons (14) by the quantitative fashion. However, let’s pay attention to the factor $\epsilon$ in (17).

Due to the apparent many particle effects in $\epsilon$ the shifts of the inertia centers of the color solitons $\Delta X_n$ don’t accord with a conventional feature of factorization that is traditional for the ordinary solitons. Thus, $\epsilon$ (in a concentrated form) points at a complex nature of interaction of the mixed $U(\sigma + \mu)$-vector solitons (14). In the situation of a general position (solitons parameters $\gamma_\nu$, $\zeta_{nm}$, $(z_{j\mu})$, vacuum densities $\rho_{j\mu}$ and component number $\sigma + \mu = m$ - arbitrary), the shifts $\Delta X_n$ of solitons because of $\epsilon$ can’t be presented in a two-particle form, $N$-soliton scattering doesn’t
come to the pair one and the interaction of mixed $U(\sigma+\mu)$-vector solitons at $\sigma\geq2$ is nontrivial (changing the form at the energy conservation). In a particular case of the linear dependence of the parameters $\gamma_{i}^{\sigma}\gamma_{j}^{\mu}-\gamma_{i}^{\mu}\gamma_{j}^{\sigma}=0$ their influence on $\epsilon$ disappear, the vacuum contribution are balanced and the center shifts $\Delta X_{n}$ admit two-particle representation. Consequently, the $N$-soliton solution is factorized and the soliton interaction becomes elastic ($|S_{n}^{a}|=1, j=\sigma, \mu$). Besides this, from the (17) directly results that in a special case of Manakov’s mixed $U(2)$-model [12] $\sigma=\mu=1$ the interaction of the mixed $U(1+1)$-vector solitons is strictly elastic. In all the other cases at $m\geq3$ color $U(\sigma+\mu)$-vector solitons ($m=\sigma+\mu$) interact by the nontrivial fashion and there is an energetic interchange between their nonlinear modes that is consistent with the above-mentioned laws of conservation.

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