EXISTENCE OF NONCONTRACTIBLE PERIODIC ORBITS OF
HAMILTONIAN SYSTEM SEPARATING TWO LAGRANGIAN
TORI ON $T^*\mathbb{T}^n$ WITH APPLICATION TO NON CONVEX
HAMILTONIAN SYSTEMS

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Abstract. In this paper, we show the existence of non contractible periodic
orbits in Hamiltonian systems defined on $T^*\mathbb{T}^n$ separating two Lagrangian tori
under certain cone assumption. Our result answers a question of Polterovich in
[P] in a sharp way. As an application, we find periodic orbits of almost all the
homotopy types on a dense set of energy level in Lorentzian type mechanical
Hamiltonian systems defined on $T^*\mathbb{T}^2$. This solves a problem of Arnold in [A].

1. Introduction

In a recent paper [P], Polterovich proved the existence of invariant measure $\mu$ for
Hamiltonian systems in the following setting.

Consider a symplectic manifold $(M, \omega)$ and a pair of compact subsets $X, Y \subset M$
with the following properties:

(P1) $Y$ cannot be displaced from $X$ by any Hamiltonian diffeomorphism $\theta$: $\theta(Y) \cap X \neq \emptyset$ for every $\theta \in \text{Ham}(M, \omega)$.

(P2) There exists a path $\{\phi_t\}, t \in [0, 1]$, $\phi_0 = \text{id}$ of symplectomorphisms so that $\phi_1$ displaces $Y$ from $X$: $\phi_1(Y) \cap X = \emptyset$.

We put $X' := \phi_1(Y)$ and $a := \text{Flux}(\{\phi_1\}) = \int_0^1 [i_{\phi_t}, \omega] dt$, then the main theorem in
[P] states as follows.

Theorem 1 (Theorem 1.1 of [P]). For every $F \in C^\infty(M, \mathbb{R})$ with

$$F|_X \leq 0, \quad F|_{X'} \geq 1,$$

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the Hamiltonian flow $\psi_t$ of $F$ possesses an invariant probability measure $\mu$ with
\begin{equation}
\langle a, \rho(\mu) \rangle \geq 1
\end{equation}
where $\rho(\mu)$ is the rotation vector of $\mu$.

A similar result is obtained in \cite{V} as Proposition 5.10 using a different approach.

It is natural to ask if the invariant measures are supported on periodic orbits. In
the same paper \cite{P}, the author asks the following question:

Can one, under assumptions of Theorem \cite{P} deduce existence of a closed orbit of
the Hamiltonian flow so that the corresponding rotation vector satisfies inequality
\begin{equation}
(1.1)
\end{equation}

Finding periodic orbit is an important theme in symplectic dynamics. As remarked
in \cite{G}, “there is a general principle in symplectic dynamics that a compactly sup-
ported function with sufficiently large variation must have fast non-trivial peri-
odic orbits or even one-periodic orbits if the function is constant near its max-
imum” (recall for instance the Hofer–Zehnder capacity and etc). However, this
principle is not correct in full generality. There is a famous counter-example of
Zehnder (see Example 2.1 of \cite{P}). Periodic orbit does not exist for the mani-
fold $(\mathbb{T}^4 = \mathbb{R}^4 / \mathbb{Z}^4, \omega = dp_1 \wedge dq_1 + \gamma dp_2 \wedge dq_1 + dp_2 \wedge dq_2)$ where $\gamma$ is irrational,
$F(p, q) = \sin 2(\pi p_1)$, and $X = \{p = 0\}$, $X' = \{p = (1/2, 0)\}$ two Lagrangian tori.

So we specialize to the case
\begin{equation}
M = T^*T^n, \quad \omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i
\end{equation}

and $X$ being the zero section $\{p = 0\}$, $X'$ another Lagrangian torus corresponding to $\{p = p^* \neq 0\}$, and ask for the existence of noncontractible periodic orbits. However,
there is an immediate counterexample given by the Hamiltonian
\begin{equation}
F(p, q) = \frac{\langle \alpha, p \rangle}{\langle \alpha, p^* \rangle}
\end{equation}

where $\alpha \in \mathbb{R}^n$ is completely irrational and $\langle \alpha, p^* \rangle \neq 0$. Any composition $\sigma \circ F$ has
no periodic orbits, where $\sigma : \mathbb{R} \to \mathbb{R}$ smooth. We choose $\sigma(x) = 0$ for $x \leq 0$, $\sigma = 1$
for $x \geq 1$ and monotone, then multiply $\sigma \circ F$ by a compactly supported function
$\eta(p) : \mathbb{R}^n \to \mathbb{R}$ that decays sufficiently slowly outside a big ball. We can make
the resulting Hamiltonian system $\eta(p) \cdot \sigma \circ F(p, q)$ have no noncontractible periodic
orbits of period 1.

In this paper, we show the existence of noncontractible periodic orbits of Hamilton-
ian systems in the following setting. Consider the symplectic manifold $(T^*T^n, \omega_0)$.
Consider two Lagrangian tori of $T^*T^n$: $X$ being the zero section $\{p = 0\}$ and $X'$
the section $\{p = p^*\}$, where $p^* = (p_1^*, \ldots, p_n^*) \in \mathbb{R}^n \setminus \{0\}$ with $p_i^* > 0$, $i = 1, \ldots, n
is a constant vector. We choose some large $R \gg \|p^*\|$ (where $\| \cdot \|$ is the Euclidean norm) and denote by $RT^*T^n$ the open set 

$$RT^*T^n := \{ (p,q) \in T^*T^n \mid \|p\| < R \}.$$ 

For a nonempty closed set $V$ in the complement of $p^*$, we consider time-periodic Hamiltonians $H(p,q,t)$ in the following set 

$$\mathcal{H}_c(RT^*T^n; V, p^*) := \{ H(p,q,t) \in C_\infty^{\mathrm{cpt}}(RT^*T^n \times T^1, \mathbb{R}) \mid H(p^*, q, t) \geq c, \text{ and } H(p, q, t) = 0, \text{ for } p \in V \},$$ 

where we use $\mathrm{cpt}$ to mean compactly supported. In this paper, the set $V$ is either $\mathbb{R}^n \setminus \mathcal{C}$ in the following Theorem 2 or $W$ in Theorem 3.

Our first result is the following

**Theorem 2.** Consider a cone $\mathcal{C}$ positively spanned by linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$, 

$$\mathcal{C} = \text{span}_+ \{v_1, v_2, \ldots, v_n\} := \left\{ \sum_{i=1}^{n} c_i v_i \mid c_i > 0, \forall i = 1, 2, \ldots, n \right\}.$$ 

Denote by $A = (v_1, \ldots, v_n)$ the matrix formed by $v_i$, $i = 1, \ldots, n$ as column vectors. Then for any point $p^*$ lying in the interior of the cone $\mathcal{C}$, for all $H(p,q,t) \in \mathcal{H}_c(RT^*T^n; \mathbb{R}^n \setminus \mathcal{C}, p^*)$, any homology class $\alpha \in H_1(T^n, \mathbb{Z}) \setminus \{0\}$ satisfying 

$$\langle p^*, \alpha \rangle \leq c, \text{ and } \alpha \in \mathcal{C}^* := \text{span}_+ \left\{ (A^T)^{-1} e_1, (A^T)^{-1} e_2, \ldots, (A^T)^{-1} e_n \right\},$$ 

where $e_i$, $i = 1, \ldots, n$ are the standard basis vectors of $\mathbb{R}^n$, there exists a periodic orbit of $H$ in the homology class $\alpha$ of period 1.

A positively spanned cone cannot contain any lines, so the cone $\mathcal{C}$ in the assumption is necessary to rule out the counterexample (1.2). As the angle at the tip of the cone $\mathcal{C}$ becomes more obtuse, the set of homology classes admitting periodic orbits becomes smaller. See Figure 1 for the picture of the cone $\mathcal{C}$ and its dual cone $\mathcal{C}^*$ in the two dimensional case (we choose $v_1 = (1,3)$, $v_2 = (3,1)$). Our Theorem 2 is sharp in view of the counterexample (1.2).

We get the next theorem when we choose $A$ to be identity in Theorem 2. For simplicity of notations, we introduce the following closed set 

$$W := \{ p \in \mathbb{R}^n \mid p_i \leq 0, \text{ for some } i = 1, \ldots, n \}.$$ 

**Theorem 3.** For all $H(p,q,t) \in \mathcal{H}_c(RT^*T^n; W, p^*)$, $p^* \in \mathbb{R}^n \setminus W$, and homology class $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_1(T^n, \mathbb{Z}) \setminus \{0\}$ satisfying 

$$\langle \alpha, p^* \rangle \leq c, \text{ and } \alpha_i > 0, \forall i = 1, \ldots, n,$$ 

(1.4)
there exists a 1-periodic orbit of the Hamiltonian flow of $H$ in the homology class $\alpha$.

The main body of the paper is devoted to proving Theorem 3. Our other theorems are derived from Theorem 3.

A closely related result is the following Theorem B of [BPS]. To state the theorem, we first define the symplectic action as

$$A_H(x) = \int_0^1 (H(x(t),t) - \lambda(\dot{x}(t))) \, dt, \quad \text{for } x \in C^\infty(T^1, RT^* T^n),$$

where $\lambda = pdq$ is the Liouville 1-form.

**Theorem 4** (Theorem B of [BPS]). For every compactly supported smooth Hamiltonian function $H \in C^\infty_c(RT^* T^n \times T^1, \mathbb{R})$ with $R = 1$ and every $e \in \mathbb{Z}^n$ such that

$$\|e\| \leq c := \inf_{(q,t) \in T^n \times T^1} H(0, q, t),$$

the Hamiltonian system has a periodic solution $x(t)$ in the homotopy class $e$ with action $A_H(x) \geq c$.

**Remark 1.** The cut-off $R$ is only to guarantee compactness when constructing Floer theory. This is reminiscent of the setting of Theorem 4. However, our result is not an immediate consequence of Theorem 4. Suppose we choose $R$ large enough and consider Hamiltonians $H$ whose oscillation (Hofer norm) is much smaller than $R$. If we rescale the fiber by $p \mapsto p/R$, correspondingly we should rescale the Hamiltonian $H \mapsto H/R$. The oscillation of the rescaled Hamiltonian is not large enough to produce noncontractible 1-periodic orbits applying Theorem 4. Our Theorem 3 gives plenty of 1-periodic orbits whose homology class satisfying (1.4). We will see in Lemma 3.1 that the periodic orbits that we find are not produced by the cut-off but by the oscillation on $W$ and $p^*$. 

**Figure 1.** The cone $C$ and the dual cone $C^*$
Theorem 5. Consider $H(p, q): T^*\mathbb{T}^n \to \mathbb{R}$ an autonomous Hamiltonian in $\mathcal{H}_c(RT^*\mathbb{T}^n; W, p^*)$. Then for each nontrivial homology class $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_1(\mathbb{T}^n, \mathbb{Z})$, there exists a dense subset $S_\alpha \subset (0, \min_p H(p, q))$ with the property that for each $s \in S_\alpha$, the level set $\{H = s\}$ contains a closed orbit (not necessarily period 1) in the class $\alpha$.

As an application of Theorem 3, we answer a question of Arnold in the following Theorem 6. For the problem, see Section 1.8 of [A], where Arnold asked for the existence of periodic orbits of the non convex system $H$. Consider Hamiltonian system of the form

$$H(p, q) = p_1 p_2 + V(q_1, q_2), \quad (p, q) \in T^*\mathbb{T}^2, \quad V(q) \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{R}).$$

Denote by $M := \max_{q \in \mathbb{T}^2} V(q) - \min_{q \in \mathbb{T}^2} V(q)$. For each homology class $\alpha \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$, $\alpha_1 > 0$, $\alpha_2 > 0$, there exists a dense subset $S_\alpha$ of $(M, \infty)$ such that for each $s \in S_\alpha$, there exists a periodic orbit lying on the energy level $\{H = s\}$ and with homology class $\alpha$.

The idea is to notice that the function $p_1 p_2$ is positive in the interior of the first quadrant and is zero on the boundary. After proper scaling and translation of the Hamiltonian to handle the bounded perturbation $V$, then composing it with $\sigma$ that we used in the paragraph of (1.2), we get a modified Hamiltonian to which Theorem 3 is applicable. We obtain a periodic orbit lying on the energy level of the modified Hamiltonian, which is also a periodic orbit of the Hamiltonian (1.6). See Section 5.3 for more details.

Remark 2. * Our system (1.6) is equivalent to Arnold’s original one up to linear symplectic transformation (see Section 5.3).

• If we want homology class $\alpha \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$, with $\alpha_1 < 0$, $\alpha_2 < 0$, we make the coordinates change

$$\alpha \mapsto -\alpha, \quad (p, q) \mapsto (-p, -q).$$

• If we want homology class $\alpha \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$, with $\alpha_1 < 0$, $\alpha_2 > 0$, we make the coordinates change

$$(\alpha_1, \alpha_2) \mapsto (-\alpha_1, \alpha_2), \quad (p_1, p_2, q_1, q_2) \mapsto (p_1, -p_2, q_1, -q_2), \quad H \mapsto -H.$$
If we want homology class $\alpha \in H_1(T^2, \mathbb{Z}) \setminus \{0\}$, with $\alpha_1 > 0$, $\alpha_2 < 0$, we make the coordinates change

$$(\alpha_1, \alpha_2) \mapsto (\alpha_1, -\alpha_2), \quad (p_1, p_2, q_1, q_2) \mapsto (-p_1, p_2, -q_1, q_2), \quad H \mapsto -H.$$ 

Then apply the above Theorem 6.

It is interesting to notice that the inequalities (1.1) in [P, V] go in the opposite direction as ours (1.4). (In our case, if we rescale the energy oscillation from $c$ to 1, the corresponding time rescaling will take the rotation vector $\alpha$ to $\alpha/c$). The invariant measure $\mu$ found in [V] verifies the equality (see Proposition 5.10 and Corollary 5.8 of [V])

$$(1.7) \quad \mathcal{A}(\mu) = \alpha(p^*) - \langle p^*, \rho(\mu) \rangle,$$

where $\alpha(p^*)$ is Mather’s $\alpha$ function, which can be considered as energy $c$, and $\mathcal{A}$ is the symplectic action (see [V] and our definition in (1.5)). On the other hand, in our proof, we always guarantee our periodic orbits satisfy the inequality $\mathcal{A} \leq c - \langle p^*, \alpha \rangle$ (see Section 1.2). In Mather theory for positive definite Lagrangian system [M], Equation (1.7) implies that $\mu$ is action minimizing. So we may think that the invariant measures found by [P, V] resemble the action minimizing measure of Mather. However, in our case, strict inequality may happen. Notice our action carries a negative sign compared to Mather’s action. This shows that our periodic orbits may not be action minimizing in Mather’s setting. It seems highly nontrivial that on the critical energy level when equality holds, invariant measure can be supported on periodic orbits.

Let us now review the literature briefly. The existence of certain periodic orbits in Hamiltonian systems is part of the story of Weinstein conjecture. Please refer to [G] for a review. We focus on results mostly relevant to ours. In [HV], the authors prove the existence of periodic orbits for Hamiltonians separating neighbourhoods of two points on $\mathbb{C}P^n$ using J-holomorphic curve techniques. Using the method of [HV], Gatien and Lalonde [GL] showed the existence of noncontractible periodic orbits for compactly supported Hamiltonians separating two Lagrangian tori on $T^*K$ where $K$ is the Klein bottle as well as the case when $\|p^*\|$ is sufficiently small for $T^*T^n$. In [L], Y.-J. Lee generalized the result of [GL] by introducing a Gromov-Witten type invariant. Notice $T^*T^n$ is exactly a case when the invariant of [L] vanishes, so that we have counterexample (1.2) and the Gromov-Witten invariant approach does not work in our setting. On the other hand, there is a Floer theoretical approach developed in [BPS], the authors obtain several results including Theorem 4. Their results are further generalized by [W, SW] to general manifolds $T^*M$ where $M$ is closed.

The method in our proof is to implement the machinery of [BPS]. We will show in the following sections that the method of [BPS] goes through.
The paper is organized as follows. In Section 2, we set up the machinery of Floer homology. This part follows mainly from [BPS] with some variations following [W]. Our new contribution is Lemma 2.5. We define the filtered Floer homology group in Section 2.1 and the inverse and direct limits of the groups in Section 2.2 induced by the monotone homotopies of Hamiltonians. We introduce exhausting sequences in Section 2.3 which would reduce the computation of the Floer homology group for any Hamiltonian to that for an exhausting sequence. In Section 2.4, we introduce a BPS type capacity which is suitable to find periodic orbits and another homological relative capacity which is accessible to computation and bound the other capacity. Next, in Section 2.5, we introduce the Morse-Bott theory which would be used to compute the Floer homology group of the exhausting sequence. In Section 3, we prove Theorem 3. In this section, we construct a family of profile functions as an exhausting sequence and study their first and second order derivatives carefully. We use Morse-Bott theory to compute the Floer homology group for the profile functions. Finally in Section 5, we prove Theorem 2, 5 and 6.

2. Floer homology

In this section, we set up the framework of [BPS]. Since we specialize to the manifold $T^*\mathbb{T}^n$, we get some simplification in the presentation.

2.1. Floer theory and spectral invariants.

2.1.1. Symplectic actions. We consider the standard symplectic form $\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$ and the Liouville 1-form $\lambda = \sum_{i=1}^{n} p_i dq_i$ such that $d\lambda = \omega_0$.

In this section, we define Floer homology for functions in $C^\infty_{cpt}(RT^*\mathbb{T}^n \times \mathbb{T}^1, \mathbb{R})$. We denote by $\mathcal{L}^{\mathbb{T}^n} := C^\infty(\mathbb{T}^1, RT^*\mathbb{T}^n)$ the space of free loops of $RT^*\mathbb{T}^n$. For each $x \in \mathcal{L}^{\mathbb{T}^n}$, we can associate to it a free homotopy class $[x] \in \pi_1(\mathbb{T}^n) = H_1(\mathbb{T}^n, \mathbb{Z})$. Given a homotopy class $\alpha \in \pi_1(\mathbb{T}^n)$ we write the loop space

$$\mathcal{L}_\alpha^{\mathbb{T}^n} := \{x \in \mathcal{L}^{\mathbb{T}^n} \mid [x] = \alpha\}.$$

Each $H(p,q,t) \in C^\infty_{cpt}(RT^*\mathbb{T}^n \times \mathbb{T}^1, \mathbb{R})$ determines a time-periodic Hamiltonian vector field $X_H$ through the relation

$$i(X_H)\omega_0 = -dH.$$

The space of 1-periodic solutions of the Hamiltonian equation representing a class $\alpha$ is denoted by

$$\mathcal{P}(H, \alpha) := \{x \in \mathcal{L}_\alpha^{\mathbb{T}^n} \mid \dot{x} = X_H(x(t))\}.$$
Elements of $\mathcal{P}(H,\alpha)$ are the critical points of the symplectic action $A_H(x)$ \[1.5\] for $x \in L_\alpha T^n$. 

2.1.2. Action spectrum and periodic orbits. The action spectrum is defined as 

$$ S(H;\alpha) = A_H(\mathcal{P}(H;\alpha)) = \{A_H(x) \mid x \in L_\alpha T^n, \ x(t) = X_H(x(t))\}. $$

Consider $-\infty \leq a < b \leq \infty$ and denote by $\mathcal{P}^{[a,b)}(H;\alpha)$ the set of 1-periodic solutions of the Hamiltonian system $H$ representing the class $\alpha$ and with action lying in the interval $[a,b)$:

$$ \mathcal{P}^{[a,b)}(H;\alpha) := \mathcal{P}_b(H;\alpha) \setminus \mathcal{P}_a(H;\alpha), \quad \mathcal{P}^a(H;\alpha) := \{x \in \mathcal{P}(H;\alpha) \mid A_H(x) < a\}. $$

We need to assume the $H \in C^\infty_{cpt}(RT^*T^n \times T^1, \mathbb{R})$ under consideration satisfies the following nondegeneracy condition:

$$ (\star) \text{ } \forall \ a, b \notin S(H;\alpha) \text{ and every } 1 \text{-periodic orbit } x \in \mathcal{P}(H;\alpha) \text{ is nondegenerate in the sense that the derivative } d\phi^1_H(x(0)) \text{ of the time-1 map } \phi^1_H \text{ does not have 1 in its spectrum.} $$

This nondegeneracy condition can be achieved by perturbing $H$ near each periodic orbit (see Section 2.1 of \cite{W}).

2.1.3. Floer homology group. We next define the Floer homology group $HF^{(a,b)}$ with $\mathbb{Z}_2$ coefficient as the homology of the chain complex $CF^{(a,b)}(H;\alpha)$ over $\mathbb{Z}_2$ which is generated by the 1-periodic orbits in $\mathcal{P}^{(a,b)}(H;\alpha)$, where we define

$$ CF^{(a,b)}(H;\alpha) := CF^b(H;\alpha) / CF^a(H;\alpha), \quad CF^a(H;\alpha) := \bigoplus_{x \in \mathcal{P}^a(H;\alpha)} \mathbb{Z}_2 x. $$

2.1.4. The boundary operator, energy. To define the boundary operator, we consider the perturbed Cauchy-Riemann equation

$$ \partial_s u + J_0(\partial_t u - X_H(u)) = 0. \tag{2.1} $$

For a smooth solution $u(s,t) : \mathbb{R}^1 \times \mathbb{T}^1 \to T^*T^n$ of (2.1), we define its energy as

$$ E(u) := \int_{\mathbb{T}^1} \int_{\mathbb{R}} \|\partial_s u\|^2 \, ds \, dt. $$

If $u$ is a finite energy solution of (2.1), then the limits exist

$$ \lim_{s \to \pm\infty} u(s,t) = x^\pm(t), \quad \lim_{s \to \pm\infty} \partial_s u(s,t) = 0 \tag{2.2} $$

and are uniform in $t$. Moreover, we have $x^\pm \in \mathcal{P}(H;\alpha)$ and the energy identity

$$ E(u) = A_H(x^-) - A_H(x^+). \tag{2.3} $$
This energy identity, the exactness of \( \omega_0 \) imply the space of finite energy solutions of (2.1) is compact with respect to compact-open topology. Namely, only the splitting into a finite sequence of adjacent Floer connecting orbits can occur in the limit.

2.1.5. **Compactness and nondegeneracy issues.** Throughout the paper, we fix the almost complex structure to be the standard one \( J_0 = \begin{bmatrix} 0 & -\text{id}_n \\ \text{id}_n & 0 \end{bmatrix} \). In order to guarantee the linearized operator of (2.1) to be surjective, we do not perturb the almost complex structure. Instead, we perturb the Hamiltonian \( H \) in an arbitrarily small neighborhood \( U \) of the image of \( u \) in \( (\mathbb{R} - \varepsilon)T^n \times T^1 \) where \( \varepsilon \) is chosen to be so small that \( H \in \mathcal{C}^\infty_{\text{cpt}}(\mathbb{R} - \varepsilon)T^n \times T^1, \mathbb{R} \). We can choose the perturbation to vanish up to second order along the orbits \( x^\pm \) (see Section 2.1 of [W] and Theorem 5.1 (ii) of [PHS]). Namely, there exists a small neighborhood \( \mathcal{U} \) of zero in \( \mathcal{C}^\infty_{\text{cpt}}(U, \mathbb{R}) \) and a subset \( \mathcal{U}_{\text{reg}} \) of regular perturbations of Baire’s second category such that the linearized operator of (2.1) is surjective, for all \( u \) solving equation (2.1) for the Hamiltonian \( H + h \) with \( h \in \mathcal{U}_{\text{reg}} \), and satisfying (2.2) with \( x^\pm \in \mathcal{P}(a,b)(H + h; \alpha) \).

2.1.6. **Moduli space and Floer homology.** For every \( H + h \) with \( h \in \mathcal{U}_{\text{reg}} \) and every pair of periodic orbits \( x^\pm \in \mathcal{P}(H; \alpha) \) the space \( \mathcal{M}(x^-, x^+; H, J_0; \alpha) \) of solutions of (2.1) with boundary conditions (2.2) is a smooth manifold whose dimension near a solution \( u \) of (2.1) and (2.2) is the difference of the Conley–Zehnder indices \( \mu_{\text{CZ}} \) of \( x^- \) and \( x^+ \) (relative to \( u \)). We denote the subspace of solutions of index one by \( \mathcal{M}^1(x^-, x^+; H, J_0; \alpha) \). It follows from Section 2.1.5 that the quotient \( \mathcal{M}^1(x^-, x^+; H, J_0; \alpha)/\mathbb{R} \) (modulo time shift) is a finite set for every pair of periodic orbits \( x^\pm \in \mathcal{P}(H; \alpha) \) with Conley–Zehnder index difference being 1. The Floer boundary operator \( \partial^H \) on the chain complex \( \text{CF}^b(H; \alpha) \) is defined as

\[
\partial^H x := \sum_{y \in \mathcal{P}(H; \alpha)} \#(\mathcal{M}^1(x, y; H, J_0; \alpha)/\mathbb{R}) y
\]

for every \( x \in \mathcal{P}(H; \alpha) \) with \( \mu_{\text{CZ}}(y) = \mu_{\text{CZ}}(x) + 1 \). The energy identity (2.3) shows that \( \text{CF}^a(H; \alpha) \) is a subcomplex, i.e. it is invariant under \( \partial^H \). We therefore get a boundary operator \( [\partial^H] \) on the quotient complex \( \text{CF}^{a,b}(H; \alpha) \). We finally define the homology of the quotient complex as

\[
\text{HF}^{a,b}(H, \alpha) := \frac{\ker([\partial^H]: \text{CF}^{a,b}(H; \alpha) \to \text{CF}^{a,b}(H; \alpha))}{\text{im}([\partial^H]: \text{CF}^{a,b}(H; \alpha) \to \text{CF}^{a,b}(H; \alpha))}.
\]

2.1.7. **Homotopic invariance.** The above homology group \( \text{HF}^{a,b}(H, \alpha) \) is defined for a fixed Hamiltonian. When we have a smooth homotopy of Hamiltonians \( H_s : s \in \mathbb{R} \)
with $H_s = H_0$ when $s \leq 0$ and $H_s = H_1$ when $s \geq 1$, we consider the following Cauchy-Riemann equation
\begin{equation}
\partial_s u + J_0 \partial_t u - \nabla H_s(u, t) = 0.
\end{equation}
The smooth solutions $u : \mathbb{R} \times \mathbb{T}^1 \to RT^*\mathbb{T}^n$ of (2.4) is a connecting orbit between two periodic orbits with the same Conley-Zehnder index. Namely we have uniformly in $t \in \mathbb{T}^1$ the limits
\begin{equation}
\lim_{s \to -\infty} u(s, t) = z_0(t), \quad \lim_{s \to +\infty} u(s, t) = z_1(t), \quad \lim_{s \to \pm\infty} \partial_s u(s, t) = 0
\end{equation}
where $z_i(t) \in \mathcal{P}(H_i, \alpha), i = 0, 1$ and $\mu_{CZ}(z_0) = \mu_{CZ}(z_1)$. We have the energy identity
\begin{equation}
E(u) = A_{H_0}(z_0) - A_{H_1}(z_1) - \int_{-\infty}^{\infty} \int_{\mathbb{T}^1} (\partial_s H_s)(u(s, t), t) dt ds.
\end{equation}
Similar to Section 2.1.5 we can find a second category subset of regular homotopies among all homotopies such that the linearized operator of (2.4) is surjective, for all elements $u$ of the moduli spaces $\mathcal{M}(z_0, z_1; H_s, J_0; \alpha)$ (see also Section 2.1 of [W]). Solution of (2.4) defines a Floer chain map from $\text{CF}(H_0, \alpha)$ to $\text{CF}(H_1, \alpha)$.

2.1.8. Monotone homotopy. Next, we define
\begin{equation}
\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha) := \{ H \in C^\infty_c(RT^*\mathbb{T}^n \times \mathbb{T}^1, \mathbb{R}) \mid a, b \notin S(H, \alpha) \}.
\end{equation}
Suppose there are two Hamiltonians $H_0, H_1 \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ satisfy
\begin{equation}
H_0(p, q, t) \geq H_1(p, q, t)
\end{equation}
for all $(p, q, t) \in RT^*\mathbb{T}^n \times \mathbb{T}^1$ as well as being nondegenerate in the sense of $(\ast)$. Then there exists a homotopy $s \mapsto H_s$ from $H_0$ to $H_1$ such that $\partial_s H_s \leq 0$. We call such a homotopy monotone. Every monotone homotopy $s \mapsto H_s$ induces a natural monotone homomorphism
\begin{equation}
\sigma_{H_1, H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \to \text{HF}^{[a,b]}(H_1; \alpha),
\end{equation}
which is independent of the choice of the monotone homotopy of Hamiltonians used to define it. We have the composition rule
\begin{equation}
\sigma_{H_2, H_1} \circ \sigma_{H_1, H_0} = \sigma_{H_2, H_0},
\end{equation}
whenever $H_0, H_1, H_2 \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ satisfy $H_0 \geq H_1 \geq H_2$, and $\sigma_{H H} = \text{id}$ for every $H \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$.

To make the monotone homomorphism $\sigma_{H_1, H_0}$ an isomorphism, we need the following proposition.
Proposition 2.1 (Proposition 4.5.1 of [BPS]). Let $-\infty \leq a < b \leq \infty$, $\alpha \in \pi_1(T^*\mathbb{T}^n)$ be a nontrivial homotopy class, and $H_0, H_1 \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ be such that $H_0 \geq H_1$. Suppose that there exists a monotone homotopy $\{H_s\}_{0 \leq s \leq 1}$ from $H_0$ to $H_1$ such that $H_s \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ for every $s \in [0, 1]$. Then $\sigma_{H_1H_0} : \text{HF}^{(a,b)}(H_0; \alpha) \to \text{HF}^{(a,b)}(H_1; \alpha)$ is an isomorphism.

2.2. Direct and inverse limits.

2.2.1. Partial order on $C^\infty_c(\mathbb{R}^n \times \mathbb{T}^1, \mathbb{R})$. We introduce a partial order on $C^\infty_c(\mathbb{R}^n \times \mathbb{T}^1, \mathbb{R})$ by

$$H_0 \preceq H_1 \iff H_0(p, q, t) \geq H_1(p, q, t) \quad \forall (p, q, t) \in RT^*\mathbb{T}^n \times \mathbb{T}^1.$$ 

The monotone homomorphisms $\sigma_{H_1H_0}$ of Section 2.1.8 give rise to a partially ordered system $(\text{HF}, \sigma)$ of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ defined in (2.7). By definition, this means that HF assigns to each $H \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ the $\mathbb{Z}_2$-vector space $\text{HF}^{(a,b)}(H; \alpha)$, and $\sigma$ assigns to all elements $H_0 \preceq H_1$ of $\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha)$ the monotone homomorphism $\sigma_{H_1H_0}$ subject to composition rule (2.9).

2.2.2. Inverse limit on $\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; W; \alpha)$. We restrict the partially ordered system $(\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha), \preceq)$ to a partially ordered system $(\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; W; \alpha), \preceq)$ where we define

$$(2.10) \quad \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; W; \alpha) := \left\{ H \in \mathcal{H}^{a,b}(RT^*\mathbb{T}^n; \alpha) \cap C^\infty_c((\mathbb{R}^n \setminus W) \times \mathbb{T}^n \times \mathbb{T}^1, \mathbb{R}) \right\},$$

We abbreviate $\mathcal{H}^{a,b}(RT^*\mathbb{T}^n; W; \alpha)$ as $\mathcal{H}^{a,b}(W)$.

The partial order set $(\mathcal{H}^{a,b}(W), \preceq)$ is downward directed: For all $H_1, H_2 \in \mathcal{H}^{a,b}(W)$ there exists $H_0 \in \mathcal{H}^{a,b}(W)$ such that $H_0 \preceq H_1$ and $H_0 \preceq H_2$. The functor $(\text{HF}, \sigma)$ is called an inverse system of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}^{a,b}(W)$. Its inverse limit (the absolute symplectic homology) is defined by

$$\mathcal{SH}^{(a,b)}(RT^*\mathbb{T}^n; W; \alpha) := \lim_{\leftarrow} \text{HF}^{(a,b)}(H; \alpha)$$

$$:= \left\{ \{a_H\}_{H \in \mathcal{H}^{a,b}(W)} \in \prod_{H \in \mathcal{H}^{a,b}(W)} \text{HF}^{(a,b)}(H; \alpha) \mid H_0 \leq H_1 \Rightarrow \sigma_{H_1H_0}(a_{H_0}) = a_{H_1} \right\}.$$

For $H \in \mathcal{H}^{a,b}(W)$, let

$$(2.11) \quad \pi_H : \mathcal{SH}^{(a,b)}(RT^*\mathbb{T}^n; W; \alpha) \to \text{HF}^{(a,b)}(H; \alpha)$$

be the projection to the component corresponding to $H$. It holds $\pi_{H_1} = \sigma_{H_1H_0} \circ \pi_{H_0}$, whenever $H_0 \preceq H_1$. 


2.2.3. Direct limit on $\mathcal{H}^{a,b}_c(W)$. To define relative symplectic homology, fix $c > 0$ and consider the subset
\begin{equation}
\mathcal{H}^{a,b}_c(R^*\mathbb{T}^n; W, p^*; \alpha) := \left\{ H \in \mathcal{H}^{a,b}_c(R^*\mathbb{T}^n; W; \alpha) \mid H(p^*, q, t) > c \right\}.
\end{equation}

We abbreviate it as $\mathcal{H}^{a,b}_c(W, p^*)$.

This set is upward directed: For all $H_0, H_1 \in \mathcal{H}^{a,b}_c(W, p^*)$ there exists $H_2 \in \mathcal{H}^{a,b}_c(W, p^*)$ such that $H_0 \preceq H_2$ and $H_1 \preceq H_2$. The functor $(HF, \sigma)$ is called a direct system of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}^{a,b}_c(W, p^*)$. Its direct limit is defined to be the quotient
\begin{equation}
\text{Sh}^{(a, b); c}(R^*\mathbb{T}^n; W, p^*; \alpha) := \lim_{H \in \mathcal{H}^{a,b}_c(W, p^*)} \text{HF}^{(a, b)}(H; \alpha)
\end{equation}

where $(H_0, a_0) \sim (H_1, a_1)$ iff there exists $H_2 \in \mathcal{H}^{a,b}_c(W, p^*)$ such that $H_0 \preceq H_2$, $H_1 \preceq H_2$ and $\sigma_{H_1, H_2}(a_0) = \sigma_{H_1, H_2}(a_1)$. This is an equivalence relation, since $\mathcal{H}^{a,b}_c(W, p^*)$ is upward directed. The direct limit is a $\mathbb{Z}_2$-vector space with the operations
\begin{equation}
k[H_0, a_0] := [H_0, ka_0], \quad [H_0, a_0] + [H_1, a_1] := [H_0, \sigma_{H_1, H_0}(a_0) + \sigma_{H_2, H_1}(a_1)],
\end{equation}

for all $k \in \mathbb{Z}_2$ and $H_2 \in \mathcal{H}^{a,b}_c(W, p^*)$ such that $H_0 \preceq H_2$ and $H_1 \preceq H_2$. For $H \in \mathcal{H}^{a,b}_c(W, p^*)$ define the homomorphism
\begin{equation}
\iota_H : \text{HF}^{(a, b)}(H; \alpha) \to \text{Sh}^{(a, b); c}(R^*\mathbb{T}^n; W, p^*; \alpha), \quad a_H \mapsto [H, a_H].
\end{equation}

It satisfies $\iota_{H_0} = \iota_{H_1} \circ \sigma_{H_1, H_0}$, whenever $H_0 \preceq H_1$.

2.3. Exhausting sequence. To compute direct and inverse limits we introduce the notion of exhausting sequences following [BPS]. Let $(G, \sigma)$ be a partially ordered system of $R$-modules over $(I, \preceq)$ and denote $\mathbb{Z}^+ := \{ \nu \in \mathbb{Z} \mid \nu > 0 \}$. A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^+}$ is called upward exhausting for $(G, \sigma)$ iff the following holds

- For every $\nu \in \mathbb{Z}^+$ we have $i_\nu \leq i_{\nu+1}$ and $\sigma_{i_\nu, i_{\nu+1}} : G_{i_\nu} \to G_{i_{\nu+1}}$ is an isomorphism.
- For every $i \in I$ there exists a $\nu \in \mathbb{Z}^+$ such that $i \preceq i_\nu$.

A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^-}$ is called downward exhausting for $(G, \sigma)$ iff the following holds

- For every $\nu \in \mathbb{Z}^-$ we have $i_{\nu-1} \leq i_\nu$ and $\sigma_{i_\nu, i_{\nu-1}} : G_{i_{\nu-1}} \to G_{i_\nu}$ is an isomorphism.
- For every $i \in I$ there exists a $\nu \in \mathbb{Z}^-$ such that $i_\nu \preceq i$.

We use exhausting sequences to simplify computations of direct and inverse limits.
Lemma 2.2 (Lemma 4.7.1 of [BPS]). Let \((G, \sigma)\) be a partially ordered system of \(R\)-modules over \((I, \preceq)\).

1. If \(\{i_\nu\}_{\nu \in \mathbb{Z}^+}\) is an upward exhausting sequence for \((G, \sigma)\) then \((I, \preceq)\) is upward directed and the homomorphism \(\iota_{i_\nu} : G_{i_\nu} \to \varinjlim G\) is an isomorphism for every \(\nu \in \mathbb{Z}^+\).

2. If \(\{i_\nu\}_{\nu \in \mathbb{Z}^-}\) is a downward exhausting sequence for \((G, \sigma)\) then \((I, \preceq)\) is downward directed and the homomorphism \(\pi_{i_\nu} : \varprojlim G \to G_{i_\nu}\) is an isomorphism for every \(\nu \in \mathbb{Z}^-\).

2.4. Capacities.

2.4.1. Symplectic homology. We cite the following proposition from [BPS] about the existence of a homomorphism between absolute and relative symplectic homologies which factors through Floer homology.

Proposition 2.3 (Proposition 4.8.1 of [BPS]). Let \(\alpha \in \pi_1(\mathbb{T}^n)\) be a nontrivial homotopy class and suppose that \(-\infty \leq a < b \leq \infty\). Then, for every \(c \in \mathbb{R}\), there exists a unique homomorphism

\[
T_{a,b}^{(a,b);c} : \text{SH}^{[a,b)}(RT^*\mathbb{T}^n; W; \alpha) \to \text{SH}^{[a,b);c}(RT^*\mathbb{T}^n; W, p^*; \alpha)
\]

such that for any two Hamiltonian functions \(H_0, H_1 \in \mathcal{H}^{a,b}_{c}(RT^*\mathbb{T}^n; W, p^*; \alpha)\) with \(H_0 \geq H_1\) the following diagram commutes:

\[
\begin{array}{ccc}
\text{SH}^{[a,b)}(RT^*\mathbb{T}^n; W; \alpha) & \xrightarrow{T_{a,b}^{(a,b);c}} & \text{SH}^{[a,b);c}(RT^*\mathbb{T}^n; W, p^*; \alpha) \\
\downarrow{\pi_{H_0}} & & \downarrow{\iota_{H_1}} \\
\text{HF}^{[a,b)}(H_0; \alpha) & \xrightarrow{\sigma_{H_1, H_0}} & \text{HF}^{[a,b)}(H_1; \alpha)
\end{array}
\]

Here

\[
\pi_{H_0} : \text{SH}^{[a,b)}(RT^*\mathbb{T}^n; W; \alpha) \to \text{HF}^{[a,b)}(H_0; \alpha)
\]

and

\[
\iota_{H_1} : \text{HF}^{[a,b)}(H_1; \alpha) \to \text{SH}^{[a,b);c}(RT^*\mathbb{T}^n; W, p^*; \alpha)
\]

are the canonical homomorphisms introduced in Section 2.2.2 and 2.2.3. In particular, since \(\sigma_{H, H} = \text{id}\) for every \(H \in \mathcal{H}^{a,b}_{c}(RT^*\mathbb{T}^n; W, p^*; \alpha)\), we have

\[
\begin{array}{ccc}
\text{SH}^{[a,b)}(RT^*\mathbb{T}^n; W; \alpha) & \xrightarrow{T_{a,b}^{(a,b);c}} & \text{SH}^{[a,b);c}(RT^*\mathbb{T}^n; W, p^*; \alpha) \\
\downarrow{\pi_H} & & \downarrow{\iota_H} \\
\text{HF}^{[a,b)}(H; \alpha) & & \text{HF}^{[a,b)}(H; \alpha)
\end{array}
\]
2.4.2. The homological relative capacity. Following [BPS] we define two capacities. For every nontrivial homotopy class \( \alpha \in \pi_1(\mathbb{T}^n) \) and every real number \( c > 0 \) we define the set

\[
\mathcal{A}_c(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha) := \{ a \in \mathbb{R} \mid \text{The homomorphism } T_{\alpha}^{[a, \infty); c} \text{ does not vanish} \}.
\]

The homological relative capacity of the triple \((\text{RT}^*\mathbb{T}^n; W, p^*)\) is the function

\[
\tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*): \pi_1(\mathbb{T}^n) \times [-\infty, \infty) \to [0, \infty]
\]

which assigns to the class \( \alpha \in \pi_1(\mathbb{T}^n) \) and the following number for \( a \geq -\infty \)

\[
(2.14) \quad \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) := \inf \{ c > 0 \mid \sup \mathcal{A}_c(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha) > a \}.
\]

Here we use the convention that \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \). For \( a = -\infty \) we abbreviate

\[
\tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha) := \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, -\infty) = \inf \{ c > 0 \mid \mathcal{A}_c(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha) \neq \emptyset \}.
\]

2.4.3. A relative symplectic capacity. We define the BPS type relative symplectic capacity by

\[
(2.15) \quad C(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) := \inf \{ c > 0 \mid \forall H \in \mathcal{S}(\text{RT}^*\mathbb{T}^n; W, p^*), \exists x \in \mathcal{S}(\mathcal{H}; \alpha) \text{ such that } \mathcal{A}_H(x) \geq a \}.
\]

We get the existence of periodic orbits provided we bound \( C(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \) from above. The capacity \( \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \) is computable and does bound \( C(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \), as said by the next proposition.

**Proposition 2.4** (Proposition 4.9.1 of [BPS]). Let \( \alpha \in \pi_1(\mathbb{T}^n) \) and \( a \in \mathbb{R} \). If \( \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) < \infty \) then every Hamiltonian \( H \in \mathcal{S}(\text{RT}^*\mathbb{T}^n; W, p^*) \) with \( c \geq \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \) has a 1-periodic orbit in the homotopy class \( \alpha \) with action \( \mathcal{A}_H(x) \geq a \). In particular,

\[
\tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \geq C(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a).
\]

The proof of this proposition is a word by word translation of that of Proposition 4.9.1 of [BPS]. We remark here that the function class \( \mathcal{S}_{c,b}(W, p^*) \) (2.12) (with \( b = \infty \)) in the definition of \( \tilde{C}(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \) (2.14) differs from the function class \( \mathcal{S}_c(\text{RT}^*\mathbb{T}^n; W, p^*) \) (1.3) in the definition of \( C(\text{RT}^*\mathbb{T}^n; W, p^*; \alpha, a) \) (2.15) by the strict inequality “\( > c \)”. Functions in \( \mathcal{S}_c(\text{RT}^*\mathbb{T}^n; W, p^*) \) can be approximated by that in \( \mathcal{S}_{c,b}(W, p^*) \). See the proof of Proposition 4.9.1 of [BPS] for the approximation argument.
2.5. Morse-Bott theory in Floer homology. We need to use Morse-Bott theory to compute Floer homology. We first give the definition of Morse-Bott manifolds (Section 5.2 of [BPS]).

**Definition 1.** A subset $P \subset \mathcal{P}(H; \alpha)$ is called a Morse-Bott manifold of periodic orbits if the set $C_0 := \{x(0) \mid x \in P\}$ is a compact submanifold of a symplectic manifold $M$ and $T_{x_0}C_0 = \text{Ker}(D\psi_1(x_0)-\text{id})$ for every $x_0 \in C_0$, where $\psi_1$ is the time-1 map of the Hamiltonian flow induced by the Hamiltonian $H(p, q, t) \in C_\infty^0(M, \mathbb{R})$.

For a compactly supported Hamiltonian system $H(p)$ defined on $(RT^*T^n, \omega_0)$ and depending only on variables in the fibers, the set $\left\{(p, q) \mid p = p_0, \quad \dot{q} = \frac{\partial H}{\partial p}(p_0) \in \mathbb{Z}^n \setminus \{0\}\right\}$ is foliated into invariant tori labeled by frequencies $\left\{\frac{\partial H}{\partial p}(p) \mid p \in \mathbb{R}^n\right\}$ according to Liouville-Arnold theorem. If we consider a torus corresponding to frequency $\dot{q} = \frac{\partial H}{\partial p}(p_0) \in \mathbb{Z}^n \setminus \{0\}$. This is an invariant torus foliated by periodic orbits of period 1. We pick any point $q(0)$ in the torus as initial condition to solve our Hamiltonian equation, the resulting periodic orbit lies completely on the torus. We have the following easy criteria to determine when such a torus is a Morse-Bott manifold.

**Lemma 2.5.** For a Hamiltonian system $H(p)$ defined on $(T^*T^n, \omega_0)$ and depending only on variables in the fibers, the set

$$P = \left\{(p, q) \mid p = p_0, \quad \dot{q} = \frac{\partial H}{\partial p}(p_0) \in \mathbb{Z}^n \setminus \{0\}\right\}$$

is a Morse-Bott manifold of periodic orbits for $H(p)$ iff

$$\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)(p_0) \neq 0, \quad i, j = 1, 2, \ldots n.$$  

**Proof.** The Hamiltonian equations are $\dot{q} = \frac{\partial H}{\partial p}(p), \quad \dot{p} = 0$. The linearized equation has the following form

$$\frac{d}{dt} \begin{bmatrix} \delta q \\ \delta p \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial^2 H}{\partial p^2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta q \\ \delta p \end{bmatrix}$$

where $(\delta p, \delta q) \in T_{(p,q)}(T^*T^n)$. This equation can be integrated explicitly, whose fundamental solution at time 1 is

$$D\psi_1 = \exp\begin{bmatrix} 0 & \frac{\partial^2 H}{\partial p^2} \\ 0 & 0 \end{bmatrix} = \text{id}_{2n} + \begin{bmatrix} 0 & \frac{\partial^2 H}{\partial p^2} \\ 0 & 0 \end{bmatrix}.$$
According to Definition 1, we only need to check for \((p_0, q_0) \in P\)

\[T_{(p_0, q_0)} P = \text{Ker}(D \psi_1(p_0, q_0) - \text{id})\]

On the one hand, the set \(P\) in consideration is an \(n\)-torus \(P = \{ (p, q) \mid p = p_0, q \in \mathbb{T}^n \}\), whose tangent space at \((p, q)\) is \(T_{(p, q)} P = \{ (\delta p, \delta q) \mid \delta p = 0, \delta q \in \mathbb{R}^n \}\). On the other hand we have

\[
\text{Ker}(D \psi_1(p_0, q_0) - \text{id}) = \text{Ker}\begin{bmatrix}
0 & \frac{\partial^2 H}{\partial p^2}(p_0) \\
0 & 0
\end{bmatrix} = \left( \text{Ker}\frac{\partial^2 H}{\partial p^2}(p_0), \delta q_0 \in \mathbb{R}^n \right).
\]

These tell us that to guarantee \(P\) is a Morse-Bott manifold, we need and only need to have that the matrix \(\frac{\partial^2 H}{\partial p^2}\) is nondegenerate at \(p_0\). □

Next, we cite the following theorem of Pozniak from [BPS] in order to compute Floer homology using Morse-Bott manifold.

**Theorem 7** (Theorem 5.2.2 of [BPS]). Let \(-\infty \leq a < b \leq \infty\), \(\alpha \in \pi_1(M)\), and \(H \in \mathcal{H}^{a,b}(M; \alpha)\). Suppose that the set \(P := \{ x \in \mathcal{P}(H; \alpha) \mid a < A_H(x) < b \}\) is a connected Morse–Bott manifold of periodic orbits. Then \(HF^{[a,b]}(H; \alpha) \cong H_*(P; \mathbb{Z}_2)\).

### 3. Construction of the profile functions

In this section, we prove Theorem 3. The idea is to construct a family of profile functions \(H_s(p)\), \(s \in \mathbb{R}\) that is both upward and downward exhausting. We will show that for \(a\) satisfying \(0 \leq a \leq c - \langle p^*, \alpha \rangle\), all the homology groups \(HF^{[a,\infty]}(H_s, \alpha)\) are isomorphic to each other and nonvanishing as \(s\) varies. The main result of this section is Lemma 3.1.

We have the following list of requirements for the family of profile functions \(H_s \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R})\).

- On the \(\mathcal{C}^0\) level: \(H_s(p)\) is both upward and downward exhausting, i.e. for each

\[
H \in \overline{\mathcal{H}_c}(\mathbb{R}^n; W, p^*) := \{ H(p, q, t) \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{T}^1 \cap (\mathbb{R}^n \setminus W) \times \mathbb{T}^n \times \mathbb{T}^1), \mathbb{R} \mid H(p^*, q, t) > c \},
\]

there exist \(s < s'\) such that \(H_{s'} < H < H_s\). Functions in \(\mathcal{H}_c\) can be approximated by functions in \(\mathcal{H}_c\) defined here.

- On the \(\mathcal{C}^1\) level: there exists a unique \(p_s\) such that \(\frac{\partial H_s(p_s)}{\partial p} = \alpha\), the homology class in Theorem 3 and the action of the corresponding periodic orbit is greater than \(a \in [0, c - \langle p^*, \alpha \rangle]\).
3.1. **Profile functions.** We first construct two families of profile functions for \( s \geq 1 \) and \( s \leq -1 \). Then we construct a homotopy from \( s = 1 \) to \( s = -1 \).

3.1.1. *A model function in one dimensional case.* Consider the function \( e^{-\frac{|x|^2}{2\delta}} \) where \( \delta \) is sufficiently small. The second order derivative vanishes at the point \( x = \pm \sqrt{\delta} \) (the turning points) and the first order derivative at \( x = \pm \sqrt{\delta} \) is \( \mp \frac{1}{\sqrt{\delta}} e^{-\frac{1}{2}} \). The value of the function at \( x = \pm \sqrt{\delta} \) is \( e^{-1/2} \approx 0.61 \).

We define a \( \mathcal{C}^1 \) function \( u \) as follows. We consider one copy of \( e^{-\frac{|x|^2}{2\delta}} \) and one copy \( -e^{-\frac{|x|^2}{2\delta}} + 1 \). After shifting horizontally the first function properly we use a piece of straight line of slope \( \frac{1}{\sqrt{\delta}} e^{-\frac{1}{2}} \) to join their turning points smoothly. The explicit expression of this function is given as follows (see (A) of Figure 2).

\[
(3.2) \quad u(x) = \begin{cases} 
1, & \text{if } x \geq (4 - e^{1/2}) \sqrt{\delta} := b, \\
-\frac{|x-b|^2}{2\delta}, & \text{if } x \in [b - \sqrt{\delta}, b], \\
e^{-1/2} \sqrt{\delta} x + 1 - 2e^{-1/2}, & \text{if } x \in [\sqrt{\delta}, b - \sqrt{\delta}], \\
e^{-\frac{|x|^2}{2\delta}} + 1, & \text{if } x \in [0, \sqrt{\delta}], \\
0, & \text{if } x \leq 0.
\end{cases}
\]
The function $u$ is $\mathcal{C}^\infty$ everywhere except at the two turning points $x = b - \sqrt{\delta}$, $\sqrt{\delta}$ where $u''$ is discontinuous as well as the two points $x = 0$, $b$ where $u''$ is discontinuous. We smoothen $u$ in a $\delta^{3/2}$ neighborhoods of the four points to get a function in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, which is still denoted by $u$. The smoothing can be done as follows. We use a partition of unity to localize in a $\delta^{3/2}$ neighbourhood of each point. The second derivative $u''$ decreases to zero continuously in a neighborhood of $\sqrt{\delta}$ and decreases from zero to negative continuously in a neighborhood of $b - \sqrt{\delta}$. In neighbourhoods of $b - \sqrt{\delta}$ and $\sqrt{\delta}$, we convolute $u''$ with a nonnegative compactly supported $\mathcal{C}^\infty$ approximating Dirac-$\delta$ function. At $x = b$, the second derivative $u''$ jumps from $-\delta^{-1}$ to zero. We join the two pieces smoothly such that $u''(x) = 0$ for $x \geq b - \frac{1}{3}\delta^{3/2}$ and $u''(x) \leq 0$ for $x \leq b$. Similar for $x = 0$. We also multiply a scalar to $u$ if necessary, such that it takes values from 0 to 1. The following are satisfied by the smoothed function $u$:

- $u(x) = 0$ for $x \leq \frac{1}{3}\delta^{3/2}$ and $u(x) = 1$ for $x \geq b - \frac{1}{3}\delta^{3/2}$.
- $u''$ does not change sign in the above four neighbourhoods so that $u'$ is monotone there.

Finally, we define

$$\delta_s := \frac{\delta}{2|s|}, \quad b_s := \frac{b}{2|s|} = (4 - e^{1/2})\sqrt{\delta_s}, \quad \forall s \in \mathbb{R}.$$ 

In the function $u$ we replace the parameters $\delta$ by $\delta_s$ and $b$ by $b_s$ to get a function called $u_s(x)$. So we have $u_s(x) = u_{-s}(x)$ and $u(x) = u_0(x)$.

### 3.1.2. Profile functions when $s \geq 1$.

For $s \geq 1$, our profile function is defined as a proper shift and rescaling of the products of the function $u$.

We define

$$F_s(p) = (c + 2^s) \prod_{i=1}^n u_s \left( \frac{p_i}{p_s} \right), \quad s \geq 1. \tag{3.3}$$

So this function is equal to 0 on $W$ and $c + 2^s$ for $p_i \geq p_s b_s$ for all $i = 1, \ldots, n$. See the upper two curves in (B) of Figure 2.

### 3.1.3. Profile functions when $s \leq -1$.

We define for $s \leq -1$,

$$F_s(p) = \max \left\{ (c + 2^{-s})u_s \left( b_s - \sqrt{\sum_{i=1}^n \left| \frac{p_i}{p_s} - 1 \right|^2} - 2^{-s} + 2^s - \frac{2^{-s} + 2^s}{c + 2^{-s}} F_{-s}(p) \right) \right\}. \tag{3.4}$$

When $p$ is close to $p^*$, $F_s$ takes the former expression in (3.4). The value at $p^*$ is $F_s(p^*) = c + 2^s$ and the function quickly decays to $-2^{-s} + 2^s$. When $p$ is close to $W$, ...
$F_s$ takes the latter expression in (3.4). Except the two cases, we have $F_s = -2^{-s} + 2^s$.

See the lower two curves in (B) of Figure 2.

3.1.4. Homotopy from $s = 1$ to $s = -1$. From $s = 1$ to $s = 0$, we use a shift

$$F_s(p) = F_1 \left( p - (1 - s) \left( 1 - \frac{b_1}{2} \right) p^* \right) - (1 - s) \cdot 3/4, \quad s \in [0, 1].$$

In the horizontal direction, the translation moves the point $b_1 p^*$ to $p^*$. In the vertical direction, the function moves down by $3/4$.

Now we see that $F_0(p) = F_{-1}(p)$, where

$$p \in W \bigcup \left\{ p \mid \delta_{-1}^{3/2} < \sqrt{\sum_{i=1}^{n} \left| \frac{p_i}{p_i^*} - 1 \right|^2} \leq \sqrt{\delta_{-1}}, \quad p_i < p_i^*, \quad \text{for all } i = 1, \ldots, n \right\}$$

because of $e^{-x^2} \cdot e^{-y^2} = e^{-x^2-y^2}$, where the LHS is for $F_0$ and the RHS is for $F_{-1}$.

Next, from $s = 0$ to $s = -1$, we use a linear homotopy

$$F_s(p) = -sF_{-1}(p) + (1 + s)F_0(p), \quad s \in [-1, 0].$$

To summarize the above, we have defined a function ((B) of Figure 2)

(3.5)

$$F_s(p) = \begin{cases} 
(c + 2^s) \prod_{i=1}^{n} u_s \left( \frac{p_i}{p_i^*} \right), & \text{if } s \geq 1, \\
F_1 \left( p - (1 - s) \left( 1 - \frac{b_1}{2} \right) p^* \right) - (1 - s) \cdot 3/4, & \text{if } s \in [0, 1], \\
-sF_{-1}(p) + (1 + s)F_0(p), & \text{if } s \in [-1, 0], \\
\max \left\{ (c + 2^{-s})u_s \left( b_s - \sqrt{\sum_{i=1}^{n} \left| \frac{p_i}{p_i^*} - 1 \right|^2} \right) - 2^{-s} + 2^s, \\
\frac{-2^{-s} + 2^s}{c + 2^{-s}} F_{-s}(p) \right\}, & \text{if } s \leq -1.
\end{cases}$$

Notice at the points $s = -1, 0, 1$, the homotopy $F_s$ is not smooth. We smoothen $F_s$ as a function of $s$ in neighbourhoods of the three points by localizing to a $\varepsilon$ neighborhood of each point using a partition of unity then convolute an approximating Dirac-$\delta$ function of $s$ only, which is $C^\infty$ and compactly supported.

3.1.5. The cut-off. Finally, we need to cut off $F_s$ properly to make it in $C^\infty_{opt}(RT^n \times \mathbb{T}^1, \mathbb{R})$. We define

(3.6)

$$w_s(x) = u_s((1 - b_s) - |x|),$$
which is zero for $|x| \geq 1 - b_s$ and 1 for $|x| \leq 1 - 2b_s$. Our profile function is defined to be

$$H_s(p) = F_s(p) \cdot w_s \left( \frac{1}{R} \|p\| \right).$$

The function $w_s \left( \frac{1}{R} \|p\| \right)$ is radially symmetric and its radial derivatives are all nonpositive.

It is easy to check that $H_s$ is an exhausting sequence for Hamiltonians in $\widehat{\mathcal{H}}_c(R^{*}T^n; W, p^*)$. Namely for $\forall H \in \widehat{\mathcal{H}}_c(R^{*}T^n; W, p^*)$, there exist $s > s'$ such that $H_s < H < H_{s'}$.

### 3.1.6. Location of Morse-Bott manifolds.

In this section, we find the $p_s$ satisfying $\frac{\partial H_s}{\partial p}(p_s) = \alpha$, where $\alpha$ satisfies the assumption of Theorem 3. The heuristics are simple. In the function $u$, if we want to solve $u'(x) = a \geq 0$ where $a$ is independent of $\delta$. For small $\delta$, we have two roots lying in the intervals $(b - \sqrt{\delta}, b)$, $(0, \sqrt{\delta})$. We do not expect to find solutions in the linear part of $u$ since $u'$ is either 0 or $O((1/\sqrt{\delta})$ there.

**Lemma 3.1.** Consider $\alpha \in H_1(T^n, \mathbb{Z}) \setminus \{0\}$ as in Theorem 3. For each $s$ there is a unique solution $p_s^+$ of $\frac{\partial H_s}{\partial p}(p) = \alpha$ satisfying the following as $\delta \to 0$:

- For $s \geq 1 + \varepsilon$, the point $p_s^+$ satisfies
  $$\frac{p_{s,i}^+}{p_i^*} \in \left( b_s - \sqrt{\delta_s}, b_s \right), \quad H_s(p_s^+) = c + 2^s - O(\delta)2^{-3s},$$

- For $s \leq -1 - \varepsilon$, the point $p_s^+$ satisfies
  $$\sqrt{n} \left( \frac{1}{p_i^*} - 1 \right)^2 < \sqrt{\delta_s}, \quad p_{s,i}^+ \leq p_i^*, \quad \forall i, \quad H_s(p_s^+) = c + 2^s - O(\delta)2^{3s},$$

where $O(\delta)$ in the above two cases are positive and independent of $s$.

For $s \in [-1 - \varepsilon, 1 + \varepsilon]$, the solution $p_s^+$ stays arbitrarily close to the following cases without smoothing with respect to $s$.

- As $s$ goes from 1 to 0, the solution $p_s^+$ moves to $p_{s-1}^+ + p^*$ with constant speed.
- For $-1 \leq s \leq 0$, we have $p_s^+ = p_{s-1}^+ = p_1^+ + p^*$.

If there is any other solution denoted by $p_s^-$, it must satisfy

$$H_s(p_s^-) - \langle p_s^-, \alpha \rangle < 0.$$
Proof. We first forget about the cut-off \( w_s \) and consider only \( F_s \). Close to the end of the proof, we study the effect of \( w_s \). We also forget about the smoothing with respect to \( s \) when working with \( p^+_s \), since once we have a solution of \( \frac{\partial H_s}{\partial p} = \alpha \) for the function \( H_s \) without smoothing with respect to \( s \), we get a solution of the smoothed one using implicit function theorem. The nondegeneracy condition \( \det \frac{\partial^2 H_s}{\partial p^2} (p^+_s) \neq 0 \) is given by the next Lemma 3.2.

Step 1, existence and uniqueness of \( p^+_s \).

Substep 1.1, the case \( s \geq 1 \).

In the proof, we define \( y_i = \frac{p_i}{p^+_s}, \ i = 1, \ldots, n \). We consider first \( s \geq 1 \) and \( y_i - b_s \in (-\sqrt{s}, 0) \). We introduce a new function

\[
\begin{align*}
    f_s(y) := F_s(p) = (c+2^s) \prod_{i=1}^{n} u_s(y_i) = (c+2^s) \exp \left( -\sum_{i} \frac{|y_i - b_s|^2}{2\delta_s} \right), \quad y_i \in (b_s - \sqrt{s}, b_s).
\end{align*}
\]

We also forget about the smoothing when defining \( u \) for a moment for the simplicity of notations and study it in the next paragraph. Consider level sets of \( f_s(y) = (c+2^s) \cdot C \) where \( C \in [e^{-n/2}, 1] \). We get a sphere \( \sum_i |y_i - b_s|^2 = 2\delta_s(-\ln C) \) for each \( C \), whose radius ranges from 0 to \( \sqrt{n}\delta_s \). Next consider

\[
\begin{align*}
    \frac{\partial f_s}{\partial y_i} = f_s \cdot (\ln u_s(y_i))' = -\frac{y_i - b_s}{\delta_s} C(c + 2^s)
\end{align*}
\]

evaluated on each level set \( C(c+2^s) \). When \( y \) moves on the sphere, the unit vector \( \frac{1}{\sum_i |y_i - b_s|^2} (b_s - y_1, \ldots, b_s - y_n) \) achieves any vector of the portion of \( S^{n-1} \) lying in the first quadrant since \( y_i < b_s \). Moreover the modulus

\[
\begin{align*}
    \left\| \frac{\partial f_s}{\partial y} \right\| = C(c + 2^s) \sqrt{\sum_{i} \frac{|y_i - b_s|^2}{\delta_s^2}} = C(c + 2^s) \sqrt{\frac{2(-\ln C)}{\delta_s}},
\end{align*}
\]

ranges from 0 to \( e^{-1/2}(c + 2^s)\sqrt{1/\delta_s} \) and is monotone with respect to \( C \) when \( C \in [e^{-1/2}, 1] \) since we have \((C\sqrt{-\ln C})' = -2\ln C - 1 \quad 2\sqrt{-\ln C} \). This shows that the image of the map \( \frac{\partial f_s}{\partial y} \) covers the first quadrant part of a ball of radius \( e^{-1/2}(c+2^s)\sqrt{1/\delta_s} = e^{-1/2}(c+2^s)2^s\sqrt{1/\delta} \) centered at the origin. Moreover we have that \( \frac{\partial f_s}{\partial y} \) is one-to-one in the domain \( \{ y : \sum_i |y_i - b_s|^2 < \delta_s \} \), since \( f_s(p) \) is monotone in the radial direction centered at the point \( b_s(1, \ldots, 1) \). Therefore, for \( \alpha \in H_1(T^n, \mathbb{Z}) \) in Theorem 3, if \( \delta \) is small enough, we can always find a unique preimage \( p^+_s \) of \( \alpha \) under the map.
\[ \frac{\partial F}{\partial p}(p) : \mathbb{R}^n \to \mathbb{R}^n \] for \( y \) in the region \( \{ y : \sum_i |y_i - b_s|^2 < \delta_s \} \). We do not expect to find any root of \( \frac{\partial F}{\partial p}(p) = \alpha \) in the domain \( \{ y : n\delta_s < \sum_i |y_i - b_s|^2 \leq \delta_s \} \) since \[ \| \frac{\partial f}{\partial y} \| \geq c^{-n/2}(c + 2^s)^{\sqrt{n/\delta_s}} \| \alpha \| \] there.

Substep 1.2, the smoothing.

In the definition of \( u_s \), within a \( \delta_s^{3/2} \) neighbourhood of \( b_s \) where the smoothing takes effect, \( u'_s \) goes monotonically from 0 to \( \frac{b_s - x}{\delta_s} \exp\left( -\frac{|x - b_s|^2}{2\delta_s} \right) \) which is bounded by \( \sqrt{\delta_s} \), so that the partial derivatives of \( f_s \) are bounded by \( O(\sqrt{\delta}) \) when \( c + 2^s \) is considered, which cannot be \( \alpha_i > 0 \), \( \forall i \) for \( \delta \) small enough. The smoothing in a \( \delta_s^{3/2} \) neighbourhood of \( b_s - \sqrt{\delta_s} \) does not create new root either since \( u'_s \geq \text{const.} \delta_s^{-1/2} \), which is too large to be \( \alpha/(c + 2^s) \).

Substep 1.3, the cases \( s \leq -1 \) and \( s \in [-1, 1] \).

To show the existence and uniqueness of the \( p_s^+ \) in the second bullet point in the lemma, we apply the same argument to the former expression in (3.4) after “max” when \( s \leq -1 \) by considering the function obtained from \( f_s \) with \( b_s \) replaced by 1. The solution \( p_s^+ \) is again unique since we require \( \alpha_i > 0 \), which forces \( y_i - 1 < 0 \) (see (3.9) with \( b_s \) replaced by 1). The smoothing does not produce new roots for the same reason as the previous substep. Moreover, the latter function in (3.4) after “max” does not create any root since all of its partial derivatives is nonpositive due to the negative factor \( \frac{2^{-s} + 2^s}{c + 2^s} \), which cannot be \( \alpha_i \).

The homotopy from \( s = 1 \) to \( s = 0 \) does not change the derivative of \( F_s \) up to a translation of \( p \). Moreover, the linear homotopy from \( s = 0 \) to \( s = -1 \) does not produce new solutions of \( \frac{\partial F}{\partial p}(p) = \alpha \).

Step 2, the value \( F_s(p_s^+) \).

Next, we evaluate \( F_s(p_s^+) \). It follows from (3.9) that at \( p_s^+ \) we have

(3.10) \[ b_s - y_i = \delta_s \alpha p_s^i / f_s. \]

We plug this back to the expression of \( F_s \) (3.8) to get that as \( \delta \to 0 \) and \( s \geq 1 \),

\[ F_s(p_s^+) = (c + 2^s)(1 - O(\delta_s)(c + 2^s)^{-2}) = (c + 2^s) - O(\delta)2^{-3s}. \]

For \( s \leq -1 \) with \( b_s \) replaced by 1 in (3.8), we have

\[ F_s(p_s^+) + 2^{-s} - 2^s = (c + 2^{-s})(1 - O(\delta_s)(c + 2^{-s})^{-2}) = (c + 2^{-s}) - O(\delta)2^{3s}. \]

This completes the proof of the \( p_s^+ \) part statement.
Step 3, the inequality satisfied by $p_s^-$. 

Substep 3.1, the case $s \geq 1$.

When $s \geq 1$, we consider possible roots $p_s^-$ with $y_j \notin (b_s - \sqrt{\delta_s} - \delta_s^{3/2}, b_s)$ for some $j$. Again consider the function $f_s$ (3.8). We have the calculation

$$\frac{\partial f_s}{\partial y_j} = \frac{u_s'(y_j)}{u_s(y_j)} f_s(y) = p_s^i \alpha_j.$$

We then get

$$F_s(p^-_s) - \langle p^-_s, \alpha \rangle = f_s(y) - \sum_i y_i p^-_i \alpha_i < f_s(y) - y_j p_j^i \alpha_j \leq f_s(y) \frac{u_s(y_j) - y_j u_s'(y_j)}{u_s(y_j)}.$$

Notice $u_s(y_j) = \int_0^{y_j} u'(t) dt < y_j u_s'(y_j)$ since $u_s'$ is monotone and $u_s'' \geq 0$ in the domain of $y_j$ under consideration. This shows $F_s(p^-_s) - \langle p^-_s, \alpha \rangle < 0$.

Substep 3.2, the cases $s \leq -1$ as well as $s \in [-1, 1]$.

Next we consider the case of $s \leq -1$. Possible roots $p_s^-$ must have $y_j - 1 \notin (-\sqrt{\delta_s}, 0)$ for some $j$ so that $\|y - 1^s\| \geq \sqrt{\delta_s}$ where $1^s := (1, 1, \ldots, 1) \in \mathbb{R}^n$. We need to invoke the former expression of (3.4) after “max” (the other one is excluded in Substep 1.3),

$$f_s(y) = F_s(p) = (c + 2^{-s}) u_s(b_s - \|y - 1^s\|) - 2^{-s} + 2^s.$$

We denote by $x_s = b_s - \|y - 1^s\|$ and take derivative directly we get

$$\frac{\partial f_s}{\partial y_i} = (c + 2^{-s}) u'_s(x_s) \frac{(-y_i + 1)}{\|y - 1^s\|} = p^*_i \alpha_i, \ \forall i.$$

This gives us

$$\| \frac{\partial f_s}{\partial y} \| = (c + 2^{-s}) u'_s(x_s) = \sqrt{\sum_i (p^*_i \alpha_i)^2} = O(1), \text{ as } \delta \to 0.$$

This shows that $x_s$ can only be in $[\delta_s^{3/2}, \sqrt{\delta_s} - \delta_s^{3/2}]$ if $x_s \notin [b_s - \sqrt{\delta_s}, b_s]$, otherwise $\| \frac{\partial f_s}{\partial y} \| = O(\delta_s^{-1/2})$ or $O(\delta_s^{1/2})$. We next invoke the fourth one in (3.2) to get

$$f_s(y) = (c + 2^{-s}) \left( -\exp \left( \frac{|x|^2}{2 \delta_s} \right) + 1 \right) - 2^{-s} + 2^s, \ \text{for } x_s \in [\delta_s^{3/2}, \sqrt{\delta_s} - \delta_s^{3/2}].$$

Taking derivative directly we get

$$\frac{\partial f_s}{\partial y} = (c + 2^{-s}) \exp \left( \frac{|x|^2}{2 \delta_s} \right) \frac{x}{\delta_s} (-y + 1^s).$$

Since the exponential term is bounded from below by $e^{-1/2}$, from $\| \frac{\partial f_s}{\partial y} \| = O(1)$ as $\delta \to 0$, we get $x_s = O(\delta_s)/(c + 2^{-s}), \ \forall i$. Plugging this back to $f_s$ we get for $\delta$ small
and all $s \leq -1$ that
\[ F_s(p^-) \simeq O(\delta)/((c + 2^{-s}) - 2^{-s} + 2^s = O(\delta)2^{\delta s} - 2^{-s} + 2^s \leq -1 < 0. \]

We always have $p^- \in \mathbb{R}^n \setminus W$ so that $\langle p^-, \alpha \rangle > 0$ hence $F_s - \langle p^-, \alpha \rangle < 0$.

The case $s \in [-1, 1]$ is obtained by the above two cases $s = \pm 1$. The smoothing with respect to $s$ can be made such that the smoothed $F_s$ is sufficiently close to the nonsmoothed one in $C^1(R^{T^*T^n} \times T^1, \mathbb{R})$ norm, so that the deviation of the root $p^-$ from the non-smoothed case is also sufficiently small using implicit function theorem. We get $F_s(p^-) - \langle p^-, \alpha \rangle < 0$ for the smoothed $F_s$ with $s \in [-1 - \varepsilon, 1 + \varepsilon]$.

**Step 4, the cut-off $w$.**

Finally, let us show that the cut-off $w_s$ does not create any root of the equation $\frac{\partial H_s}{\partial p} = \alpha$. We only need to consider the region where $p \in \mathbb{R}^n \setminus W$ and $\|p\|/R \simeq 1$.

First consider when $F_s(p) \geq 0$, hence $s \geq -1$. We have
\[ \frac{\partial H_s}{\partial p} = \frac{\partial F_s}{\partial p} w_s + F_s w'_s \frac{p}{\|p\|R}. \]

Since we have $\|p\|/R \simeq 1$, there must be at least one $j$ such that $p_j > b_s p^*_j$, so that $u'_s(p_j/p^*_j) = 0$ hence $\frac{\partial F_s}{\partial p_j} = 0$. Therefore we have $\frac{\partial H_s}{\partial p_j} \leq 0$ for this $j$, since all the entries of $F_s w'_s \frac{p}{\|p\|R}$ are nonpositive. However, we require $\alpha_i > 0$.

Next consider the case $F_s \leq 0$. We have $F_s - \langle p, \alpha \rangle < 0$ since $F_s < 0$ and all the entries of $p$ and $\alpha$ are positive. This completes the proof. \(\square\)

### 3.1.7. Hessian is nondegenerate.

In this section, we show that the condition in Lemma 2.5 $\det \left( \frac{\partial^2 H}{\partial p^2} \right) (p^+_s) \neq 0$ is satisfied at the point $p^+_s$ in Lemma 3.1. We have the following lemma.

**Lemma 3.2.** Consider $\alpha \in H_1(T^n, \mathbb{Z})$ satisfying the assumption of Theorem 3 and the point $p^+_s$ in Lemma 3.1 solving the equation $\frac{\partial H_s}{\partial p}(p) = \alpha$. Then we have for $\delta$ small enough and fixed, the matrix $\frac{\partial^2 H_s}{\partial p^2}(p^+_s)$ is negative definite $\forall s$.

**Proof.** We know from Lemma 3.1 that $p^+_s$ is not close to the boundary of $\{\|p\| \leq R\}$ so we have $H_s = F_s$. We forget about the smoothing with respect to $s$ for a moment. As in the proof of Lemma 3.1, we consider the function (3.8). Once we get $\frac{\partial^2 f_s}{\partial y_i \partial y_j}(y)$
we multiply the matrix \( \text{diag}\left\{ \frac{1}{p_i^*} \right\} \) to both the left and right of it to get \( \frac{\partial^2 F_s}{\partial p_i \partial p_j} (p^+_s) \), which does not change the signature. We have

\[
\frac{\partial^2 f_s}{\partial y_i \partial y_j} (y) = f_s \cdot \begin{cases} 
  u'_s(y_i)u'_s(y_j), & i \neq j; \\
  u'_s(y_i), & i = j.
\end{cases}
\]

We can rewrite \( \frac{\partial^2 f_s}{\partial y_i \partial y_j} (y) \) as a matrix form

\[
\frac{\partial^2 f_s}{\partial y_i \partial y_j} (y) = f_s \cdot (\Lambda + V \otimes V), \quad \text{where}
\]

\[
\Lambda = \text{diag}\left\{ \left( \frac{u''_s(y_i)}{u'_s(y_i)} - \left( \frac{u'_s(y_i)}{u_s(p_i)} \right)^2 \right) \right\} = \text{diag}\{ (\ln u_s(y_i))'' \} = -\frac{1}{\delta_s} \text{id},
\]

\[
V_i = \frac{u'_s(y_i)}{u_s(y_i)} = (\ln u_s(y_i))' = -\frac{y_i - b_s}{\delta_s}.
\]

Using (3.10), we see that the matrix \( V \otimes V \) is \( O(1) \) as \( \delta \to 0 \) where \( O(1) \) does not depend on \( s \). As a result the Hess \( F_s \) is diagonally dominant and negative definite.

The case \( s \leq -1 \) follows the same line of argument. The case \( F_s \) with \( s \in [-1, 1] \) is only a translation of \( F_1 \), which does not change the Hessian. When the smoothing with respect to \( s \) is taken into account, we have the same calculation as (3.12) except that we need to convolute with an approximating Dirac-\( \delta \) in the \( s \) variable. We make the smoothed \( F_s \) be sufficiently close to the nonsmoothed one in \( C^1(RT^n \times T^1, \mathbb{R}) \) norm so that the deviation of the root \( p^+_s \) from the non smoothed case is \( \ll \delta \) using implicit function theorem, which implies \( V \otimes V = O(1) \) for \( |s| < 2 \) using the calculation of \( V_i \) in (3.12), so that Hess\( F_s \) is still diagonally dominant. \( \square \)

4. Proof of Theorem 3

In this section, we proof Theorem 3 using Lemma 3.1 and the machinery set up in Section 2.

4.1. Computation of the action. We obtain Morse-Bott manifolds corresponding to \( p^+_s \) denoted by \( P^+_s \). These Morse-Bott manifolds are Lagrangian tori \( \mathbb{T}^n \). Along each periodic orbit \( x \subset P^+_s \) we evaluate the action

\[
A_{H_s}(x) = \int_0^1 H_s - \langle p, \dot{q} \rangle dt.
\]
For our profile function $H_s$, we have $\dot{q} = \frac{\partial H_s}{\partial p} = \alpha$ and $p_i > 0$ for all $i$. We have the following four cases.

- **Case 1, the action of $P_s^+$ when $s \geq 1 + \varepsilon$.**
  The value of the profile function $H_s(p_s^+)$ is obtained in Lemma 3.1 and $p_s^+ \to 0$ as $\delta \to 0$ or $s \to \infty$. The action is estimated as
  $$A_{H_s}(P_s^+) = c + 2s - O(\delta)2^{-3s} - \langle p_s^+, \alpha \rangle \to c + 2s, \quad \delta \to 0.$$  

- **Case 2, the action of $P_s^+$ when $s \leq -1 - \varepsilon$.**
  The value of the profile function $H_s(p_s^+)$ is also obtained in Lemma 3.1, and $p_s^+ \to p^*$ as $\delta \to 0$ or $s \to \infty$. The action is estimated as
  $$A_{H_s}(P_s^+) = c + 2s - O(\delta)2^{3s} - \langle p_s^+, \alpha \rangle \to c + 2s - \langle p^*, \alpha \rangle, \quad \delta \to 0.$$  

- **Case 3, the action of $P_s^+$ when $-1 - \varepsilon \leq s \leq 1 + \varepsilon$.**
  Consider first $F_s$ in (3.5) without smoothing with respect to $s$. As $s$ goes from 1 to 0, the point $p_s^+$ moves from a neighbourhood of 0 to a neighbourhood of $p^*$ with linear speed, so we get the action is
  $$A_{H_s}(P_s^+) = c + \frac{1}{2} - O(\delta) - \langle p_s^+, \alpha \rangle \to c + \frac{1}{2} - (1 - s)\langle p^*, \alpha \rangle, \quad \delta \to 0.$$  

  When $s$ goes from 0 to $-1$, the linear homotopy does not influence a neighbourhood of $p_s^+$, so the action is the same as $s = 0$ case. The smoothing of $F_s$ with respect to $s$ around the points $\pm 1, 0$ add only an error to the action that can be made as small as we wish using implicit function theorem.

  The $O$ term in the above cases are positive.

- **Case 4, the action of $P_s^-$.**
  Using the last statement in Lemma 3.1 we get $F_s(p_s^-) - \langle p_s^-, \alpha \rangle < 0$, so we get for $p_s^-$, the action satisfies $A_{H_s}(P_s^-) < 0$.

### 4.2. Proof of the main theorem.

**Proof of the main Theorem.** There are 4 steps.

**Step 1.** If $0 \leq a \leq c - \langle p^*, \alpha \rangle$, then $\text{SH}([0, \infty); (T^s \mathbb{T}^n; \alpha) \cong H_s(\mathbb{T}^n; \mathbb{Z}_2)}$ for $\forall s \in \mathbb{R}$.

Moreover, the homomorphism
$$\pi_s : \text{SH}([0, \infty); (T^s \mathbb{T}^n; \alpha) \to \text{HF}([0, \infty); (H_s; \alpha)$$

is an isomorphism whenever $H_s(p^*) > c$.

We use our action calculation in Section 4.1. Notice that component-wisely $p_s^+ < p_i^*$, $\forall s \in \mathbb{R}$, $\forall i$ we have
$$\langle p_s^+, \alpha \rangle < \langle p^*, \alpha \rangle.$$
This means that when \( c - \langle p_s^+, \alpha \rangle > c - \langle p^*, \alpha \rangle \geq a \), we have

\[
A_{H_s}(P_s^-) < 0 \leq a \leq c - \langle p^*, \alpha \rangle < A_{H_s}(P_s^+), \quad \forall s \in \mathbb{R}
\]

when \( a \) satisfies \( 0 \leq a \leq c - \langle p^*, \alpha \rangle \). Hence, by Theorem 7, \( HF^{[a, \infty)}(H_s; \alpha) \cong H_s(\mathbb{T}^n; \mathbb{Z}_2) \) since the Morse-Bott manifold \( P_s^+ \) is a torus, and by Proposition 2.1 the monotone homomorphism

\[
\sigma_{F_s}: HF^{[a, \infty)}(H_{s_0}; \alpha) \rightarrow HF^{[a, \infty)}(H_{s_1}; \alpha)
\]

is an isomorphism. Hence Step 1 follows from Lemma 2.2 (ii).

**Step 2.** If \( a > c - \langle p^*, \alpha \rangle \), then \( SH^{[a, \infty);c}(T^*\mathbb{T}^n; \alpha) = 0 \) for \( s \ll -1 \).

For \( s \ll -1 \), using the Step 2 in Section 4.1 we get that both the sets \( P_s^\pm \) have actions less than \( a \) for \( -s \) sufficiently large. Hence \( HF^{[a, \infty)}(H_s; \alpha) = 0 \) for \( -s \) sufficiently large. Hence Step 2 follows from Lemma 2.2 (i).

**Step 3.** If \( 0 \leq a \leq c - \langle p^*, \alpha \rangle \), then \( SH^{[a, \infty);c}(T^*\mathbb{T}^n; \alpha) \cong H_s(\mathbb{T}^n; \mathbb{Z}_2) \). Moreover, the homomorphism

\[
\iota_s : HF^{[a, \infty)}(H_s; \alpha) \rightarrow SH^{[a, \infty);c}(T^*\mathbb{T}^n; \alpha)
\]

is an isomorphism for \( s \ll -1 \).

We use the calculation in Case 1 of Section 4.1 again to obtain the same inequality (4.1) in Step 1. By Theorem 7, we have \( HF^{[a, \infty)}(H_s; \alpha) \cong H_s(P_s^+; \mathbb{Z}_2) \). By Proposition 2.1, the monotone homomorphism \( \sigma_{H_s} : HF^{[a, \infty)}(H_{s_0}; \alpha) \rightarrow HF^{[a, \infty)}(H_{s_1}; \alpha) \) is an isomorphism. Step 3 follows now from Lemma 2.2 (i).

**Step 4.** If \( 0 \leq a \leq c - \langle p^*, \alpha \rangle \) then the homomorphism

\[
T_{\alpha}^{[a, \infty);c} : SH^{[a, \infty)}(T^*\mathbb{T}^n; \alpha) \rightarrow SH^{[a, \infty);c}(T^*\mathbb{T}^n; \alpha)
\]

is an isomorphism.

According to its definition, \( H_s(p^*) > c \) for every \( s \). Hence, by Step 1, \( \pi_s \) is an isomorphism for every \( s \in \mathbb{R} \). Moreover, by Step 3, \( \iota_s \) is an isomorphism for every \( s \).

By Proposition 2.3, \( T_{\alpha}^{[a, \infty);c} = \iota_s \circ \pi_s \) for every \( s \). Hence \( T_{\alpha}^{[a, \infty);c} \) is an isomorphism.

According to the definition of \( \tilde{C}(RT^*\mathbb{T}^n; W, p^*; \alpha) \) in (2.14), we get that

\[
\tilde{C}(RT^*\mathbb{T}^n; W, p^*; \alpha) = a + \langle p^*, \alpha \rangle.
\]

According to Proposition 2.4, we get

\[
C(RT^*\mathbb{T}^n; W, p^*; \alpha) \leq \tilde{C}(RT^*\mathbb{T}^n; W, p^*; \alpha) = a + \langle p^*, \alpha \rangle < \infty.
\]

This implies that periodic orbits exist as claimed in Theorem 3. The proof is complete.
5. Proof of Theorem 2, 5 and 6

5.1. More general type of wedges. In this section, we prove Theorem 2.

Proof of Theorem 2

Let us consider more general wedges than $W$. Suppose the cone $C$ is positively spanned by vectors $v_1, \ldots, v_n$ assumed in the statement. We put these vectors together as columns of a matrix $A$. This matrix $A$ transforms the first quadrant to the given cone $C$. We suppose $p = AP$ and correspondingly $q = (A^T)^{-1}Q$, so that the transformation $(P,Q) \mapsto (p,q)$ is symplectic.

We start with Hamiltonian systems $H(p,q) \in \mathcal{H}_c(\mathbb{R}^n; T^n; \mathbb{C}, p^*)$ where $q^*$ lies in the interior of the cone $C$ and $W$ in (1.3) is replaced by $\mathbb{R}^n \setminus C$. After the symplectic transformation above induced by $A$, we get a Hamiltonian system $h(P,Q) = H(AP, (A^T)^{-1}Q) \in \mathcal{H}_c(\mathbb{R}^n; W, p^*)$ for some large $R'$. Consider homology class $\alpha \in H_1(T^n, \mathbb{Z})$ for the $(p,q)$ coordinates. So the homology class corresponding for $(P,Q)$ is $A^T\alpha$.

We need to assume that all the components of $A^T\alpha$ are positive in order to apply Theorem 3. This means $A^T\alpha \in \text{span}_+ \{e_1, \ldots, e_n\}$. This is equivalent to saying that $\alpha \in \text{span}_+ \{(A^T)^{-1}e_1, \ldots, (A^T)^{-1}e_n\}$. This completes the proof.

5.2. Dense existence. In this section we prove Theorem 5. The argument follows that of Theorem 3.4.1 of [BPS].

Proof of Theorem 5

We show for each $a,b$ satisfying $\min_q H(p^*, q) > b > a > 0$, there exists $s \in (a, b)$ such that the level set $\{H = s\}$ carries a closed orbit in the class $\alpha$. Define a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\sigma(r) = 0$, for $r \leq 0$,
- $\sigma(r) = 1$, for $r \geq 1$,
- $\sigma'(r) > 0$, for $0 < r < 1$.

Then we get a compactly supported function $F := c\sigma \left( \frac{H - a}{b - a} \right) \in \mathcal{H}_c(\mathbb{R}^n; W, p^*)$.

We apply Theorem 3 to get that $F$ has a 1-periodic orbit $x$ in the class $\alpha$. The orbit $x$ lies on a level set of $\{F = \rho\}$ where $\rho \in (0, c)$. Since $\sigma$ takes the interval $(a, b)$
injectively to \((0, c)\), there exists \(s \in (a, b)\) such that \(\{F = \rho\} = \{H = s\}\), hence \(x\) lies on the level set \(\{H = s\}\).

5.3. **Arnold’s problem.** In this section, we show Theorem 6.\footnote{Theorem 6}

**Proof of Theorem 6.** Our Hamiltonian system (1.6) is related to Arnold’s original one through the following symplectic transformation

\[
\begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & -1 \\
  1 & 1
\end{bmatrix}
\begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix},
\begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & -1 \\
  1 & 1
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix}.
\]

Suppose we want to find periodic orbits in an homology class \(\alpha\) with \(\alpha_i > 0, \forall i = 1, 2, \ldots, n\). We can always find \(p^* \in \mathbb{R}^n \setminus W\) and \(a, b\) such that

\[
\tag{5.1}
H(p^*, q) = p_1^* p_2^* + V(q) \geq p_1^* p_2^* - M > b > a > M.
\]

We choose \(c = \langle p^*, \alpha \rangle\) and define a Hamiltonian function using \(\sigma\) in Section 5.2

\[
F(p, q) = \begin{cases}
    c \cdot \sigma \left( \frac{H(p, q) - a}{b - a} \right) \cdot w_0(\|p\|/R), & p \in \mathbb{R}^n \setminus W, \\
    0, & p \in W.
\end{cases}
\]

where \(w_0(\|p\|/R)\) is the cut-off function introduced in (3.6) with \(s = 0\). We see easily \(F \in \mathcal{H}_c(RT^n; W, p^*)\) using (5.1).

We apply Theorem 3 to \(F\) to get that there exists a periodic orbit of \(F\) in the homology class \(\alpha\) with period one. Let us assume for a moment that the periodic orbit is not created by \(w_0 \neq 1\), namely \(\|p\|/R\) is not close to 1.

We get a periodic orbit on the energy level \(\{H = s\}\) where \(s \in (a, b)\). Since \(b > a\) can be arbitrary numbers greater than \(\langle p^*, \alpha \rangle\). We also get dense existence. Namely, there exists a dense subset \(S_\alpha\) of \((M, p_1^* p_2^* - M)\), such that for each \(s \in S_\alpha\), the energy level \(\{H = s\}\) contains a periodic orbit with homology class \(\alpha\). The argument can be done for any \(p^* \in \mathbb{R}^n \setminus W\) satisfying (5.1), so we get dense existence in the set of energy levels \((M, \infty)\). Once \(p^*\) is chosen, we need to choose \(R\) much larger than \(p^*\).

Finally, we show that the periodic orbit is not created by \(w_0 \neq 1\). We assume \(\|p\|/R \simeq 1\). We only need to consider \(p \in \mathbb{R}^n \setminus W\), since \(F(p, q) = 0\) when \(p \in W\).

We have the Hamiltonian equations

\[
\tag{5.2}
\begin{align*}
\dot{p} &= -\frac{\partial F}{\partial q} = -\frac{c}{b - a} \sigma' \cdot w_0 \cdot \frac{\partial V}{\partial q}, \\
\dot{q} &= \frac{\partial F}{\partial p} = \frac{c}{b - a} \sigma' \cdot w_0 \cdot \left( \frac{\partial H}{\partial p} \right) + c \sigma \cdot w' \cdot \frac{p}{\|p\|/R}.
\end{align*}
\]

Consider first \(p_1 p_2 > b + M\), then \(H(p, q) > b\) so that

\[
\sigma \left( \frac{H(p, q) - a}{b - a} \right) = 1 \quad \text{and} \quad \sigma' = 0.
\]

So we get \(\dot{p} = 0\), and \(\dot{q}\) have nonpositive entries since \(p \in \mathbb{R}^n \setminus W\) and
\( w' \leq 0 \). In this case, the homology class of a periodic orbit of \( F \) cannot be \( \alpha \) whose entries are positive. Notice once a periodic orbit enters the region \( \{ p_1 p_2 > b + M \} \), it always stays there because of \( \dot{p} = 0 \) and the periodicity.

It remains to consider a periodic orbit with \( p_1 p_2 \leq b + M \) during time 1. When \( \|p\|/R \approx 1 \), since \( \dot{p} \) is bounded, we must have for all time either

\[
p_1 \leq 2(b + M)/R, \quad p_2 \geq R/2, \quad \text{or} \quad p_2 \leq 2(b + M)/R, \quad p_1 \geq R/2.
\]

In the \( \dot{q} \) equation of (5.2), the factor \( \frac{c}{b-a} \sigma' \cdot w_0 \) in front of \( \frac{\partial H}{\partial p} = (p_2, p_1)^T \) is bounded, and the second term has nonpositive entries. For large enough \( R \), either \( p_1 \) or \( p_2 \) is close to zero. However, since we assume \( \alpha \in H_1(\mathbb{T}^n, \mathbb{Z}) \), \( \alpha_1 > 0 \), \( \alpha_2 > 0 \), a 1-periodic orbit of \( F \) with \( p_1 p_2 \leq b + M \) and \( \|p\|/R \approx 1 \) cannot have homology class \( \alpha \) for \( R \) large enough. This completes the proof.

\[\square\]

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