VOLUME GROWTH AND PUNCTURE REPAIR IN CONFORMAL GEOMETRY

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Abstract. Suppose $M$ is a compact Riemannian manifold and $p \in M$ an arbitrary point. We employ estimates on the volume growth around $p$ to prove that the only conformal compactification of $M \setminus \{p\}$ is $M$ itself.

1. Introduction

Though this article is primarily concerned with conformal differential geometry in dimension $\geq 3$, the phenomenon we wish to describe also occurs in dimension 2 as follows.

Theorem 1. Suppose that $M$ is a compact connected Riemann surface and $p \in M$. Suppose that $N$ is a compact connected Riemann surface and $U \subset N$ an open subset such that $U \cong M \setminus \{p\}$ as Riemann surfaces. Then this isomorphism extends to $N \cong M$.

Stated more informally, there is no difference between the ‘punctured Riemann surface’ $M \setminus \{p\}$ and the ‘marked Riemann surface’ $(M, p)$. In fact, although we have stated the theorem in terms of compact Riemann surfaces, the result itself is local:

In this picture the punctured open disc is assumed to be conformally isomorphic to the open set $U$ (but nothing is supposed concerning the boundary $\partial U$ of $U$ in $N$). We may conclude that $N$ must be, in fact, be the disc and $U \hookrightarrow N$ the punctured disc, tautologically included.

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For simplicity, however, the results in this article will be formulated for compact manifolds, their local counterparts being left to the reader.

By a conformal manifold we shall mean a smooth manifold equipped with an equivalence class of Riemannian metrics $[g_{ab}]$ where the notion of equivalence is that $\hat{g}_{ab} = \Omega^2 g_{ab}$ for some positive smooth function $\Omega$.

**Theorem 2.** Suppose $M$ is a compact connected conformal manifold and $p \in M$. Suppose $N$ is a compact connected conformal manifold and $U \subset N$ an open subset such that $U \cong M \setminus \{p\}$ as conformal manifolds. Then this isomorphism extends to $N \cong M$.

Since an oriented conformal structure in 2 dimensions is the same as a complex structure, Theorem 2 generalises Theorem 1. It is well known, however, that conformal geometry in dimensions $\geq 3$ enjoys a greater rigidity than in 2 dimensions and so one expects a different proof. Such proofs of Theorem 2 (and beyond) can be found in [1]. In this article, however, we shall prove Theorem 2 by a method that also works (but much more easily so) in dimension 2.

## 2. Puncture repair in 2 dimensions

*Proof of Theorem 1.* With reference to picture (11), introducing polar coordinates $(r, \theta)$ on the disc and hence on $U$, we are confronted by a smooth positive function $\Omega(r, \theta)$ so that, if the $\eta_{ab}$ denotes the standard metric $dr^2 + r^2 d\theta^2$ on the disc, then the metric $\hat{\eta}_{ab} = \Omega^2 \eta_{ab}$ extends to $N$. If $\partial U \subset N$ contains two or more points, then the concentric curves $\{r = \epsilon\}$ as $\epsilon \downarrow 0$ have length bounded away from zero in the metric $\hat{\eta}_{ab}$. In other words, for some $\epsilon > 0$ and $\ell > 0$, we have

$$\int_0^{2\pi} \Omega(r, \theta) r \, d\theta \geq \ell, \quad \forall 0 < r < \epsilon.$$

By the Cauchy-Schwarz inequality, for any fixed $r$,

$$\left(\int_0^{2\pi} \Omega(r, \theta) \, d\theta\right)^2 \leq 2\pi \int_0^{2\pi} \Omega^2(r, \theta) \, d\theta$$

and it follows that

$$\int_0^{2\pi} \Omega^2 \, d\theta \geq \int_0^\epsilon \frac{1}{2\pi} \left(\int_0^{2\pi} \Omega \, d\theta\right)^2 r \, dr \geq \frac{1}{2\pi} \int_0^\epsilon \frac{\ell^2}{r} \, dr = \infty.$$

However, the integral on the left is the area of $\{0 < r < \epsilon\} \subseteq U$ in the metric $\hat{\eta}_{ab}$, which must be finite if $\hat{\eta}_{ab}$ is to extend smoothly to $N$. $\square$
3. Puncture repair in Euclidean $n$-space

In 2 dimensions, the local existence of isothermal coordinates implies that it is sufficient to repair only the unit disc in $\mathbb{R}^2$ with its standard metric $\eta_{ab}$. Such a normalisation is unavailable in higher dimensions.

Proof of Theorem 2 in flat space. With reference to (1), now viewed as a picture in $n$ dimensions, we shall suppose that the object on the left is a punctured ball in $\mathbb{R}^n$ with its standard Euclidean metric and aim to conclude, just as we did in case $n = 2$, that $\partial U \subset N$ is a single point. To do this, we replace polar coordinates by spherical coordinates $R > 0 \times \Sigma \ni (r, x)$ mapping into $\mathbb{R}^n \setminus \{0\}$, where

$$\Sigma = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ni \mathbb{R}^n \setminus \{0\}$$

is the unit $(n - 1)$-sphere and investigate the behaviour of a smooth positive function $\Omega = \Omega(r, x)$ defined for $r$ sufficiently small and having the property that the metric $\tilde{\eta}_{ab} = \Omega^2 \eta_{ab}$ extends to $N$. If $\partial U \subset N$ contains two or more points, then the concentric hypersurfaces $\{r = \epsilon\}$ as $\epsilon \downarrow 0$ have diameter bounded away from zero in the metric $\tilde{\eta}_{ab}$. In other words, for some $\epsilon > 0$ and $\ell > 0$, we have

$$\forall 0 < r < \epsilon, \quad \text{there are } \alpha, \beta \in \Sigma \quad \text{s.t. } \int^\beta_\alpha \Omega(r, x) r \geq \ell,$$

where the integral is along any path from $\alpha$ to $\beta$ on the unit sphere $\Sigma$ (with respect to the standard round metric on $\Sigma$).

**Lemma 1.** Suppose $\Omega : \Sigma \to \mathbb{R}_{>0}$ is smooth and there are two points $\alpha, \beta \in \Sigma$ such that $\int^\beta_\alpha \Omega \geq d > 0$ for all smooth paths on $\Sigma$ joining $\alpha$ to $\beta$. Then

$$\int_\Sigma \Omega^n \geq C_n d^n,$$

where $C_n$ is a universal constant, independent of the location of $\alpha, \beta$.

The proof of this lemma is given in an appendix. To finish the proof of our theorem, we compute the volume of the collar $\{0 < r < \epsilon\}$ with respect to the metric $\tilde{\eta}_{ab} = \Omega^2 \eta_{ab}$ as

$$\int^\epsilon_0 \int_\Sigma \Omega^n r^{n-1} \, dr \geq \int^\epsilon_0 \int_0^\ell \frac{\ell^n}{r} \, r^{n-1} \, dr = C_n \ell^n \int^\epsilon_0 \frac{\ell^n}{r} \, dr = \infty,$$

which should be finite if $\tilde{\eta}_{ab}$ is to extend smoothly to $N$. □

Remark. By stereographic projection, conformally repairing a puncture in Euclidean $\mathbb{R}^n$ is equivalent to conformally repairing a puncture in the round sphere $S^n$. It follows already that $S^n$ is the unique conformal compactification of $\mathbb{R}^n$. 

4. PUNCTURE REPAIR NEAR THE EUCLIDEAN METRIC

The estimates in the previous section are sufficiently robust that they apply for metrics sufficiently close to Euclidean. More specifically, suppose $g_{ab}(x) \, dx^a \, dx^b$ is a Riemannian metric on a punctured ball in $\mathbb{R}^n$ centred on the origin and such that, in standard Cartesian coordinates $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ with standard Euclidean metric $\eta_{ab} \, dx^a \, dx^b$,

- the volume form for $g_{ab}$ is the standard Euclidean one,
- the metrics $g_{ab}(x)$ and $\eta_{ab}$ satisfy

\begin{equation}
\|X\|_{g(x)} \leq 2\|X\|_{\eta} \leq 4\|X\|_{g(x)}, \quad \forall \text{ vectors } X
\end{equation}

for all $x$ near the origin, say for $\|x\|_{\eta} < \epsilon$.

Again working in spherical coordinates near the origin, if $\hat{g}_{ab} \equiv \Omega^2 g_{ab}$ on $U$ extends to $N$, then the concentric hypersurfaces \{r = \epsilon\} as $\epsilon \downarrow 0$ have diameter bounded away from zero in the metric $\hat{g}_{ab}$ and hence also in the commensurate metric $\hat{\eta}_{ab} \equiv \Omega^2 \eta_{ab}$. Therefore, according to the proof given in the previous section, the volume of the collar $\{0 < r < \epsilon\}$ with respect to the metric $\hat{\eta}_{ab}$ is infinite. But $\hat{\eta}_{ab}$ has the same volume form as $\hat{g}_{ab}$, which contradicts $\hat{g}_{ab}$ extending to $N$.

5. PUNCTURE REPAIR IN $n$ DIMENSIONS

Proof of Theorem 2 in general. We only need show that there are local coördinates on an arbitrary Riemannian manifold so that conditions (2) are satisfied. Certainly, we can arrange local coördinates so that $g_{ab}$ agrees with $\eta_{ab}$ at the origin. The volume form for $g_{ab}$ is then

$$F(x^1, x^2 \cdots, x^n) \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

for some smooth function $F$ with $F(0) = 1$, which can be absorbed by changing just the first coördinate. The condition that $g_{ab}$ and $\eta_{ab}$ are commensurate, as in (2), follows near the origin by continuity. \(\square\)

APPENDIX A. PROOF OF LEMMA 1

In fact, we shall prove the following minor generalisation.

Lemma 2. Let $\Sigma$ denote the unit $m$-sphere with its usual round metric. Suppose $\Omega : \Sigma \to \mathbb{R}_{>0}$ is smooth and there are two points $\alpha, \beta \in \Sigma$ such that $\int_{\alpha}^{\beta} \Omega \geq d > 0$ for all smooth paths on $\Sigma$ joining $\alpha$ to $\beta$. Then, for any $s > m$,

$$\int_{\Sigma} \Omega^s \geq C_{m,s} \, d^s,$$

where $C_{m,s}$ is a universal positive constant, independent of the location of $\alpha$ and $\beta$. 
**Proof.** We shall calculate using stereographic coördinates on $\Sigma$. Recall that the round metric on the unit $m$-sphere may be written as

$$
\left( \frac{2}{1 + (u^1)^2 + \cdots + (u^m)^2} \right)^2 \left( (du^1)^2 + \cdots + (du^m)^2 \right)
$$

in these coördinates. Translating the origin to $a \in \mathbb{R}^m$ gives

$$
\left( \frac{2}{1 + \|u+a\|^2} \right)^2 \left( (du^1)^2 + \cdots + (du^m)^2 \right)
$$

instead and we may use such a translation to suppose that $\alpha$ and $\beta$ are located at the origin and out at infinity in this stereographic projection. Let $v \in \mathbb{R}^m$ be a unit vector and consider the curve

$$(0, \infty) \ni \rho \mapsto \rho v$$

joining the origin to infinity. Then

$$d \leq \int_0^\infty \frac{2\Omega}{\int_0^\infty \frac{2\Omega}{1 + \|\rho v + a\|^2}} \frac{2\Omega}{1 + \|\rho v + a\|^2} \cdot \int_0^\infty \frac{2\Omega}{1 + \|\rho v + a\|^2} \cdot \int_0^\infty \frac{2\Omega}{1 + \|\rho v + a\|^2}$$

The Hölder inequality for conjugate exponents $s$ and $s/(s-1)$ implies that

$$\left( \int_0^\infty fg d\mu \right)^s \leq \left( \int_0^\infty f^s d\mu \right) \left( \int_0^\infty g^{s/(s-1)} d\mu \right)^{s-1}$$

and if we take

$$f = \Omega, \quad g = \frac{(1 + \|\rho v + a\|^2)^{m-1}}{\rho^{m-1}}, \quad d\mu = \frac{\rho^{m-1} d\rho}{(1 + \|\rho v + a\|^2)^m}$$

then we conclude that

$$\left( \int_0^\infty \frac{2\Omega}{1 + \|\rho v + a\|^2} \right)^s \leq A_v^{s-1} \int_0^\infty \frac{\Omega^s \rho^{m-1} d\rho}{(1 + \|\rho v + a\|^2)^m},$$

where

$$A_v \equiv \int_0^\infty \frac{d\rho}{\rho^{(m-1)/(s-1)}(1 + \|\rho v + a\|^2)^{(s-m)/(s-1)}}.$$
Recall that \( v \) is an arbitrarily chosen unit vector in \( \mathbb{R}^m \). If we integrate over all such vectors, then the left hand side of this inequality yields \( \int \Omega^s \) whereas we already know that
\[
\int_0^\infty \frac{\Omega d\rho}{1 + \|\rho v + a\|^2} \geq \frac{d}{2}.
\]
Therefore \( \int \Omega^s \geq (S_{m-1}/(2^s B^{s-1})) \, ds \), where \( S_{m-1} \) denotes the area of the unit \((m-1)\)-sphere. This is a bound of the required form. \( \square \)

**Lemma 3.** If \( p, q > 1 \) are conjugate exponents, \( 1/p + 1/q = 1 \), then
\[
\mathbb{R} \ni t \mapsto \int_0^\infty \frac{dx}{x^{1/p}(1 + (x + t)^2)^{1/q}}
\]
is bounded.

**Proof.** Firstly, note that \((q + 1)/2q > 1/2\), so
\[
\int_1^\infty \frac{dx}{(1 + (x + t)^2)^{(q+1)/2q}} \leq \int_{-\infty}^\infty \frac{dx}{(1 + x^2)^{(q+1)/2q}} < \infty.
\]
Since \( p \) and \( q \) are conjugate exponents, so are \( 2p - 1 \) and \((q + 1)/2\), and we may apply Young’s inequality with these exponents to conclude that
\[
\frac{1}{x^{1/p}(1 + (x + t)^2)^{1/q}} \leq \frac{1}{2p - 1} \frac{1}{x^{(2p-1)/p}} + \frac{2}{q + 1} \frac{1}{(1 + (x + t)^2)^{(q+1)/2q}}.
\]
Also, observe that
\[
\int_0^1 \frac{dx}{x^{1/p}} = \int_1^\infty \frac{dx}{x^{(2p-1)/p}} = \frac{p}{p - 1}.
\]
Therefore,
\[
\int_0^\infty \frac{dx}{x^{1/p}(1 + (x + t)^2)^{1/q}} \leq \int_0^1 \frac{dx}{x^{1/p}} + \int_1^\infty \frac{dx}{x^{1/p}(1 + (x + t)^2)^{1/q}}
\]
\[
= \frac{p}{p - 1} + \int_1^\infty \frac{dx}{x^{1/p}(1 + (x + t)^2)^{1/q}}
\]
whilst Young’s inequality shows that the second integral is bounded above by
\[
\frac{1}{2p - 1} \int_1^\infty \frac{dx}{x^{(2p-1)/p}} + \frac{2}{q + 1} \int_1^\infty \frac{dx}{(1 + (x + t)^2)^{(q+1)/2q}}.
\]
Assembling these various estimates gives
\[
\frac{2p^2}{(2p - 1)(p - 1)} + \frac{2}{q + 1} \int_{-\infty}^\infty \frac{dx}{(1 + x^2)^{(q+1)/2q}}
\]
as a bound on the original integral. \( \square \)
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