Stability of Non-Abelian Black Holes and Catastrophe Theory

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Abstract

Two types of self-gravitating particle solutions found in several theories with non-Abelian fields are smoothly connected by a family of non-trivial black holes. There exists a maximum point of the black hole entropy, where the stability of solutions changes. This criterion is universal, and the changes in stability follow from a catastrophe-theoretic analysis of the potential function defined by black hole entropy.

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After Bartnik and McKinnon discovered a non-trivial particle-like structure (BM particle) in the Einstein-Yang-Mills theory, a variety of self-gravitating structures with non-Abelian fields have been found. Besides the BM particle, researches have discovered the colored black hole, the Skyrmion or the Skyrme black hole, the monopole or the black hole in monopole (monopole black hole), the particle solution with massive Proca field (Procaon) or the Proca black hole, and the sphaleron or sphaleron black holes.

One of the most important questions about these self-gravitating non-Abelian structures is, are they stable? The BM particle and the colored black hole are unstable against radial perturbations, while both the Skyrmion and the monopole, and the corresponding black hole solutions, are stable. The sphaleron and its black hole solution may be unstable because of their topological structure. Are there any common properties in those non-Abelian structures? Can we find any universal understanding for them? Answering these questions is the main purpose of the present paper. We will soon show that there is a universal picture for self-gravitating non-Abelian structures that incorporates these black hole solutions, and that accounts for their stability properties via a catastrophe-theoretic analysis of the black hole entropy, \( S = \text{the area of event horizon}/4 \) regarded as a potential function.

We have re-analyzed 5 models, which are listed in Table 1. Some known results concerning these models are also summarized in the table. Remarkably, except for the colored black hole and the monopole black hole, all solutions share the following properties:

1. There are two particle-like solutions. One corresponds to the known particle solution without gravity and the other has properties similar to those of the BM particle.

2. Two branches of black hole solutions, which leave from two particles, bifurcate at some critical horizon radius. Beyond this critical point, where the black hole has a maximum mass and a maximum entropy, there exists no non-trivial structure. The upper branch in Fig. 1 has larger entropy than that of the lower branch. Hence, we shall call each of them high- and low-entropy branches, respectively. The low-entropy branch is similar to the colored black hole solution, and the high-entropy branch approaches the Schwarzschild black hole in the “low energy” limit. Here, the “low energy” means that the mass of the particle is much smaller than the Planck mass \( m_P \equiv G^{-1/2} \). It is realized in the limit as \( \mu \rightarrow 0 \),
where \( \mu \) is a mass of the relevant non-Aberian field, e.g., \( \mu = g \Phi_0 \) (the vacuum expectation value of the Higgs field) for the Einstein-Yang-Mills- Higgs system, or \( \mu = g_S f_S \) (two coupling constants of Skyrme field) for the Eisntein-Skyrme system. (Note that \( g_S^2 = 4\pi g^2 \) in our notation). On the other hand, in the limit of “high energy”, no solution exists. It disappears around \( \mu \sim m_P/g \). (3) The high-entropy branch is stable (except for the sphaleron solution, but see later), while the low-entropy branch is unstable citeDHS,LM,SZ,HDS. (4) The specific heat in the high-entropy branch is always negative, while the specific heat in the low-entropy branch changes its sign a few times[6].

In order to obtain a universal picture with the properties (1) - (4) above, we have reanalyzed the 5 models listed in Table 1 and found the following new results[10]: (5) Fixing the horizon radius \( r_H \), there are two black hole solutions with different masses. Those two branches are bifurcated at some critical radius. In the mass-radius \((M-r_H)\) plane, the solution curve has a cusp at this critical point \( C \). (see the figure 1). The stability changes at this cusp, that is, the high-entropy branch is stable while the low-entropy one is unstable against radial perturbations. (6) If we draw the solution curve in the three dimensional space of the mass \( M \), the entropy of the black hole \( S \), and the field strength at the horizon \( B_H \equiv (\text{Tr} F^2)^{1/2}|_{\text{horizon}} \), it becomes smooth. (see the figure 2). Here, the expression \( B_H \) has been used because only the radial component of magnetic part of non-Abelian field is finite at the horizon. Only the projection onto the \( M-S \) plane (and then onto the \( M-r_H \) plane) provides a cusp. The cusp, at which the stability changes, corresponds to a turning point in the three dimensional picture, where the black hole entropy takes the maximum value.

The appearance of a cusp as a critical point of stability is often discussed in catastrophe theory[17, 18] and in its application to astrophysics [19, 20]. In the present case, if we regard \( S, M \) and \( B_H \) as a potential function, a control parameter, and a generalized coordinate, respectively, we may apply catastrophe theory to the present stability problem as follows. In catastrophe theory, solutions are regarded as extremal points on the Whitney surface, \( S = S(M, B_H) \), when the control parameter \( M \) is fixed. If a solution is a maximal point, then that solution is stable because its entropy is maximal. On the other hand, if it is a minimal point, then it is unstable. At the maximum entropy, the solution turns out to be an inflection point, beyond which there is no extremal point, i.e., there is no black hole solution[10].
We may wonder what happens with the sphaleron black hole, because both its high- and low-entropy branches are unstable for topological reasons. Is this consistent with our interpretation of stability via catastrophe theory? When we discuss stability, in general there are many modes to be investigated. A general argument about the instability of the sphaleron is based on a topological analysis\[12\], which does not choose any specific mode. On the other hand, when we discuss stability change using catastrophe theory, we focus on some specific mode. For the sphaleron without gravity, the stability analysis with a spherically symmetric ansatz was done\[21\]. It was explicitly shown that there is only one unstable mode. Although no analysis has been made, so far, for the case of the gravitating sphaleron or the sphaleron black hole, we guess that it may be stable in the high-entropy branch against radial perturbations except for one unstable mode corresponding to the above. In the low-entropy branch, some of stable modes become unstable. The sphaleron black hole picks up at least one more unstable mode beyond the critical point. In this sense, we argue that the high-entropy branch is “stable” while the low-entropy one is unstable. If this is so, then catastrophe theory accounts correctly for the stability of black holes even in the sphaleronic case.

From the above discussion, we see that we can classify non-Abelian black holes into two types, (A) and (B):

(A) High-entropy “neutral” types
The high-entropy branch is “stable”. The field strength at the horizon\(B_H\) is still small as well as the black hole is globally neutral. The black hole is approximately neutral. We may adopt the following picture for this type of black hole. The non-Abelian structure may be approximated as a uniform vacuum energy density \(\rho_{\text{vac}}\) with a sphere whose radius is the Compton wave length of the massive non-Abelian field. As for the black hole solution, the horizon must exist in the region of uniform vacuum energy. Otherwise, non-trivial non-Abelian structure is swallowed by the black hole, resulting in a trivial Schwarzschild solution. This explains why there is an upper bound on the mass or horizon radius for this non-trivial solution. From our picture, the high-entropy “neutral” black hole near the horizon is approximated by the Schwarzschild-de Sitter spacetime with the cosmological constant\(=8\pi G\rho_{\text{vac}}\). In the limit of “low energy”, the solution approaches the Schwarzschild black hole. The negative specific heat is also consistent with that of the Schwarzschild or Schwarzschild-de Sitter spacetime. The self-gravitating particle
approaches the known particle solution in a Minkowski background[3, 12]. Such a particle can exist without gravity.

(B) Low-entropy “locally charged” types

The low-entropy branch is unstable. The structure of this type of black hole is quite similar to the colored black hole. Although the black hole is globally neutral, $B_H$ does not vanish at the horizon. Its value is rather large. An effective charge appears near the black hole horizon. Furthermore, in the “low energy” limit, the solution approaches the colored black hole[4, 6]. The non-trivial structure in this case is due to the kinetic term of non-Abelian gauge field, $\text{Tr} F^2$. Gravity must play an essential role in this non-trivial structure, because the BM particle cannot exist without gravity and the mass scale is about $m_{BM} \sim m_P/g$, which is almost independent of $\mu$ (or $\Phi_0, f_S$). Just as we found strange behavior in the specific heat of a colored black hole citeTM, we find a few times changes of its sign (see Table 1).

As for the excited state, i.e., higher-node solutions, we find that a similar cusp exists[6, 16]. We expect that when the solution goes beyond this cusp (the maximum entropy point), another instability will appear. Since the colored black hole has $n$ unstable modes for an $n$-node solution citeSZ,SW, we expect that the high-entropy branch of $n$-node solutions has $(n - 1)$ unstable modes while the low-entropy branch has $n$ unstable modes.

We have, so far, discussed all known non-Abelian structures except for the monopole black hole and the colored black hole, and have presented a universal picture. Although the colored black hole does not always have all the properties we have discussed above, it should be included in our universal picture. Because the colored black hole and the Schwarzschild black hole are obtained exactly as “low energy” limits of the low- and high-entropy branches, respectively. The only exceptional solution is the monopole or the monopole black hole, which is globally charged.

It should be stressed, however, that although the monopole black hole has different properties from the types (A) and (B) above and shows more complicated behaviours[8, 1, 10], the catastrophe theory is again applied to the stability analysis[23]. Depending on the parameters $g, \lambda$, and $\Phi_0$ in the Einstein-Yang-Mills-Higgs system, there seem to be the following two cases citeLNW,BFM,AB,TMMT:

(I) The mass of the monopole black hole increases monotonically as entropy increases and the
solution eventually reaches at a bifurcation point $B$ with the RN black hole branch. No cusp appears. The monopole black hole is stable, while the RN solution becomes unstable beyond this bifurcation point $B$\(^5\).

(II) For some range of parameters, the solution curve of the monopole black holes has a cusp $C$ in the $M$-$S$ plane\(^1\), where the black hole has the maximum entropy. There are two solutions with the same horizon radius (the same entropy) but different masses just as with the other type of non-trivial black holes. When the radius gets small in the second (low-entropy) branch, the solution either may merge to the RN black hole at a bifurcation point $B$ or might reach to another particle solution (BFM particle citeBFM). We guess that the second low-entropy branch is unstable while the first high-entropy branch is stable (see \(^1\)). The RN black hole is stable before the bifurcation point $B$, but it becomes unstable beyond $B$.

All these behaviors (I) and (II), including the stability of RN black hole, follow easily from catastrophe theory, if the entropy is regarded as the potential function\(^2\). The entropy $S$ with a fixed mass $M$ is maximal for the stable branch but becomes minimal for the unstable branch\(^2\).

In this letter, we have re-analyzed the known non-Abelian black holes, as well as the corresponding the self-gravitating particle-like solutions. We find a universal picture: The globally neutral solutions are classified into two types depending on whether they are almost neutral or locally charged. The “neutral” type (high-entropy branch) is similar to the Schwarzschild-de Sitter solution and “stable” (see the previous discussion for sphaleron black holes). The “locally charged” type (low-entropy branch) is like the colored black hole and unstable. Its specific heat changes the sign a few times with respect to the mass. When those two types coincide, the entropy becomes maximum. Catastrophe theory can be applied to analyze the stability of these black holes. One stable mode in the high-entropy branch becomes unstable beyond the bifurcation point. This may be true also for the sphaleron black hole. As for the globally charged black hole (monopole black hole), we can also apply catastrophe theory to the stability analysis, although the behavior of the solutions is more complicated.

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[24] As Aichelburg and Bizon claimed citeAB, the mass is not always a good indicator to analyze the stability.
## Table 1

**Table Caption**

**Table 1:** The properties of 5 models including non-Abelian fields. See text about the meaning of “stable” for the sphaleron black hole. BFM particle means one of two non-trivial solutions found in [9], which is more massive than the usual monopole. $C_{\varphi}$ denotes how many times the sign of the specific heat changes in the branch. The Reissner-Nordström black holes in Einstein-Maxwell and Einstein-Yang-Mills-Higgs systems are listed as references. In order to define parameters in the theories such as a gauge coupling constant $g$, we show the Lagrangians of non-Abelian fields and the potentials of Higgs fields.

| Black holes | non-Abelian fields | Higgs fields | particles | black hole modes | mass (entropy) | $C_{\varphi}$ |
|-------------|---------------------|--------------|-----------|-----------------|---------------|-------------|
| 1. colored BH | Yang-Mills field ($SU(2)$) | — | BM particle | 1(unstable) | $m < \infty$ | 2 |
| | $-\frac{1}{16\pi} \text{Tr} F^2$, $F = dA + gA \wedge A$ | | | | | |
| 2. Skyrme BH | Skyrme field ($SU(2) \times SU(2)$) | — | BM type | 1(stable) | finite (high) | 0 |
| | $-\frac{1}{2} \text{Tr} F^2 - \frac{1}{4} \text{Tr} A^2$, $F = -dA$ | | | | | |
| 3. Proca BH | massive Yang-Mills (Proca) field | — | Procaon | 1(unstable) | finite (low) | 0 |
| | $-\frac{1}{4\pi} \left[ \frac{1}{4} \text{Tr} F^2 - \frac{1}{2} \mu^2 \text{Tr} A^2 \right]$ | | | | | |
| 4. sphaleron BH | Yang-Mills field ($SU(2)$) | (complex doublet) | BM type | 1(stable) | finite (high) | 0 |
| | $-\frac{1}{16\pi} \text{Tr} F^2$ | | | | | |
| | $-\lambda (\Phi^\dagger \Phi - \Phi_0^2)^2$ | | | | | |
| 5. monopole BH | Yang-Mills field ($SU(2)$) | (real triplet) | BFM particle | 1(stable) | finite (low) | 0 |
| | $-\frac{1}{16\pi} \text{Tr} F^2$ | | | | | |
| | $-\frac{1}{4\pi} (P h^2 - \Phi_0^2)^2$ | | | | | |
| Reissner-Nordström BH | electromagnetic field ($U(1)$) | — | — | 1(stable) | $m < \infty$ | 1 |
| | Yang-Mills field ($SU(2)$) | (real triplet) | — | 1(stable or unstable) | $m < \infty$ | 1 |

**Figure Captions**

**Figure 1:** The mass-horizon radius diagrams for (a) the Skyrme black hole with $f_S/m_P = (i) 0.01$, (ii) 0.02, and (iii) 0.03, (b) the Proca black hole with $\mu/gm_P = (i) 0.05$, (ii) 0.10, and (iii) 0.15, and (c) the sphaleron black hole with $\lambda = 0.125$ and $\Phi_0/m_P = (i) 0.1$, (ii) 0.2, and (iii) 0.3. $C$ is a cusp, where the black hole has a maximum entropy. Beyond its entropy there is no non-Abelian black hole. The Schwarzschild black hole (the dot-dashed line) and the colored black hole (the dotted line) are also shown as references.

**Figure 2:** The solution curve in the three dimensional space of $(M, B_H, S)$ and its projections onto each two dimensional planes for a Skyrme black hole with $f_S/m_P = 0.02$. The cusp $C$ in $M$-$S$ plane is a critical point for stability. For the fixed control parameter $M$, two solutions are at extremal points on the Whitney surface; the maximal one is stable, while the minimal one is unstable. Beyond the critical point $C$, there is no extremal point, i.e., no non-Abelian black hole.
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