Abstract. Information Theoretic analysis of the periods of a hyperelliptic curve provides more information about the well-known but abstract relationship between the branch points and the periods. Here one constructs a canonical homology basis for a hyperelliptic curve that shows that its periods must satisfy certain constraints and defines an open set in the Siegel upper half space that cannot contain any period matrices of hyperelliptic curves.

Introduction

For decades now researchers have used algebraic curves to address questions in coding and cryptography. This raises interesting questions about the curves themselves based on Information Theoretic considerations. In one direction, suppose that Alice wants to tell Bob about a compact curve (i.e., Riemann Surface) with genus $g \geq 2$. By Torelli’s Theorem, she need only send him the period matrix, transmitting $O(g^2)$ complex numbers. The curve she is describing only depends on $O(g)$ parameters, the dimension of the moduli space being $3g - 3$, so there is a lot of redundancy in her message. The period matrix is sparse in the sense of [D], and should therefore be compressible.

The perspective that the period matrix is a compressible signal is the central idea of the Information–Theoretic Schottky Problem. The attempt to apply ideas from Information Theory and Compressed Sensing [D] to the Schottky problem has led to many interesting experiments, conjectures, and theorems [W12].

These questions only make sense when the genus is large. While considering them don’t think of a compact surface of genus 3 or 4 or so; the more likely model is below:
The result described here is purely analytical, rather than computational; however, it was inspired by an attempt to implement ideas in blind Compressed Sensing, as described in [GE].

**Period Matrices.**

For an introduction to periods see [GH].

The standard constructions for the period matrix of a compact Riemann Surface (henceforth *curve*) $X$ of positive genus $g$ go as follows. Choose a symplectic basis $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ for the singular homology $H_1(X, \mathbb{Z})$; this means that the intersections $\alpha_i \cdot \alpha_j$, $\beta_i \cdot \beta_j$, and $\alpha_i \cdot \beta_j$ are zero when $i \neq j$, and $\alpha_i \cdot \beta_j = 1$. Then choose a basis $\omega_1, \ldots, \omega_g$ for the space $H^{1,0}(X)$ of holomorphic differentials on $X$, normalized so $\int_{\alpha_i} \omega_j = \delta_{ij}$, the Dirac delta. The matrix $\Omega_{ij} := \int_{\beta_i} \omega_j$ is the *period matrix*; Riemann proved that it is symmetric with positive definite imaginary part. The torus $\mathbb{C}^g/[I\Omega]$ is the *Jacobian* of $X$ ($I$ is the $n \times n$ identity matrix). By Torelli’s Theorem, the Jacobian determines all of the properties of $X$. In practice deciding which properties apply is seldom successful (but see [W07]).

The normalized period matrix, whose left half is the $g \times g$ identity, is symmetric with positive-definite imaginary part, and the space of such matrices forms the *Siegel upper half-space* $\mathcal{H}_g$. Its dimension is $g(g + 1)/2$, while the moduli space of curves of degree $g$ has dimension $3g - 3$. Distinguishing the period matrices from arbitrary elements of $\mathcal{H}_g$ is the *Schottky Problem*. See [G] for details on the problem and some of its previous solutions.
Information-Theoretic or statistical analysis of the periods depends on constructing a set of real numbers from the periods; typical choices are magnitude-squared and argument. The primary tool is then to sort the modified periods in descending or, in the extreme, non-decreasing order. It is possible to construct

For example, the magnitudes of the periods of the Fermat curve of degree 9 have the distribution in Figure 1. This data was generated using Maple’s PeriodMatrix routine [De], then sorted using Microsoft Excel. The process is further explained below.

![Fig. 2: Periods of the Fermat Curve](image)

It is possible to construct an Abelian variety with any non-decreasing distribution by putting the periods into a symmetric matrix and adjusting so the imaginary part is positive-definite. In numerical experiments distributions from curves always have the concavity suggested in Figure 1.

**Hyperelliptic Curves.**

Hyperelliptic Curves admit a degree 2 cover of the Riemann Sphere; if the genus is $g$ one has an equation

$$y^2 = \prod_{i=0}^{g}(x - P_i)(x - Q_i) = f(x)$$

the curve ramifies at the $P$’s and $Q$’s. (These are assumed to be distinct.)

It is well-known [FK] that the branch points are holomorphic functions of the periods; this is an abstract result and there is no information about how the distribution of the periods affects the placement of the branch points. See [R] for an indication of the difficulties
involved. Nor are there many results on finding properties of a curve from its period matrix, which Torelli’s Theorem suggests is possible; [W07] is one example.

The main result is:

**Theorem.** It is not possible to construct a hyperelliptic curve with an arbitrary period distribution.

The proof is by showing that it is impossible for a hyperelliptic curve to have all of its periods equal. This is a fairly simple argument based on careful choice of a canonical homology basis and examination of the integrals that define the periods.

Note that this result is consistent with earlier numerical results [W12] that suggest that the periods of hyperelliptic curve tend to have little variance in their arguments, that is, the periods are clustered near a line in the complex plane.

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**Period Distributions**

The matrix structure of the periods is only relevant here in that it has to have a positive–definite imaginary part. But because a permutation of the bases of $H_1(X, \mathbb{Z})$ or $H^{1,0}(X)$ permutes the matrix entries they are not intrinsic so one works with lists of periods.

A commonly–used strategy for signal compression is to expand the signal in some kind of series (such as a Fourier series) and use only the terms with the “biggest” coefficients. This has also long been a strategy with Singular Value Decompositions; see [GD]. (This is distinct from the noise–reduction strategy which uses only the lower–frequency terms.) It is not widely appreciated that E. Lorenz used this technique to introduce Ordinary Differential Equation approximations to a set of Partial Differential Equations in the first papers on Chaos [L].

When thinking of periods as a distribution or signal, then, the first step is to create a set of real numbers from the periods. Various methods have been used: the modulus, the squared–modules, and the argument. Now sort the real versions of the periods in descending order to make a list $p_1, p_2, \ldots, p_{g(g-1)/2}$, and plot $(n, p_n)$.

If the periods were uniformly distributed the plot would be a straight line, but numerical experiments suggest that it is concave up.

A symmetric matrix with positive–definite imaginary part is the period matrix of an abelian variety: the theta–function constructed from it converges. The Schottky Problem comes into play by noting that any random set of periods, perhaps adjusted by a few factors of the form $e^{i\phi}$ (where $\phi$ is a real number) to make the imaginary part of the matrix positive–definite, defines an abelian variety. In particular, it is possible to define an abelian variety all of whose periods have the same modulus, but, conjecturally, this is not the Jacobian of a Riemann Surface.

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**The Canonical Basis**

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The analysis of the periods uses a specific canonical homology basis, which is best explained pictorially for genus 3. The branch points are labelled $P_i, Q_i$ for $i = 1, 2, 3$. It is convenient to have one branch point at $\infty$.

![Canonical Homology Basis](image)

**Fig. 3: Canonical Homology Basis**

Figure 3 omits the orientations of the cycles for clarity. The double cover of the Riemann sphere is evident, and the “tubes” represent the effect of crossing a cut conceptually, not literally.

Figure 4 details the cuts in the target plane of the hyperelliptic projection. Dashed portions of the cycles are on opposite sheets, so an evident intersection between a dashed cycle and a solid cycle is not an actual intersection.
Branch Point Constraints

To determine the periods one integrates the cohomology basis over each of the cycles in the canonical homology basis. Refer to Figure 4: there are three cycles, each surrounding the pairs $P_i, Q_i$. Call these $\alpha_i$.

The rest of the argument is quite simple: in the picture, there is a homology between $\alpha_1 + \alpha_2$ and $\alpha_3$, so for any holomorphic 1–form $\omega$

$$\int_{\alpha_1} \omega + \int_{\alpha_2} \omega = \int_{\alpha_3} \omega;$$

similar statements hold in general for genus larger than 3. But if the periods are equal then, in this case, one of the integrals is zero, which is absurd.

Conclusion

The significance of this result is in the consequence, not the proof of the theorem. It identifies a locus in the moduli space of hyperelliptic surfaces which are not jacobians, thus determining a partial solution to the Schottky problem. Also significant is the information-theoretic perspective that led to this result.

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