A SEMI-INFINITE CONSTRUCTION OF UNITARY $N=2$ MODULES

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ABSTRACT. We show that each unitary representation of the $N=2$ superVirasoro algebra can be realized in terms of “collective excitations” over a filled Dirac sea of fermionic operators satisfying a generalized exclusion principle. These are semi-infinite forms in the modes of one of the fermionic currents. The constraints imposed on the fermionic operators have a counterpart in the form of a model one-dimensional lattice system, studying which allows us to prove the existence of a remarkable monomial basis in the semi-infinite space. This leads to a Rogers–Ramanujan-like character formula. We construct the $N=2$ action on the semi-infinite space using a filtration by finite-dimensional subspaces (the structure of which is related to the supernomial coefficients); the main technical tool is provided by the dual functional realization. As an application, we identify the coinvariants with the dual to a space of meromorphic functions on products of punctured Riemann surfaces with a prescribed behaviour on multiple diagonals. For products of punctured CP$^1$, such spaces are related to the unitary $N=2$ fusion algebra, for which we also give an independent derivation.

1. Introduction

We construct a new, semi-infinite, realization of unitary representations of the $N=2$ superconformal algebra, in which every state in the module is a “collective excitation” over a filled Dirac sea of fermionic operators satisfying a nontrivial exclusion principle. The $N=2$ superVirasoro algebra is spanned by Virasoro generators $L_n$, Heisenberg generators $H_n$, and the modes of two fermionic...
currents $Q_n$ and $G_n$, $n \in \mathbb{Z}$. The semi-infinite realization takes a considerably different input: a given unitary $N=2$ module is spanned by the action of a single fermionic current $G_n$, $n \in \mathbb{Z}$, subject to the conditions $S^p_a = 0$, where, for a fixed positive integer $p$,

$$S^p_a = \sum_{i_0 < \cdots < i_{p-2}} \left( \prod_{m < n} (i_m - i_n) \right) G_{i_0} \cdots G_{i_{p-2}}, \quad a \in \mathbb{Z}. \tag{1.1}$$

Informally, the semi-infinite construction can be rephrased by saying that no other $N=2$ generators except $G_n$ are needed to span a unitary $N=2$ representation. The apparent “mismatch in the number of the degrees of freedom” is resolved because the module is spanned by semi-infinite forms in $(G_n)_{n \in \mathbb{Z}}$. With the “exclusion principle” $S^p_a = 0$ imposed in addition to the standard Pauli principle for fermions, it is a nontrivial result (a character identity) that there are precisely as many semi-infinite forms as there are states in a unitary $N=2$ module. Moreover, the semi-infinite forms carry a representation of the $N=2$ algebra with the central charge $3(1 - \frac{2}{p})$; although eventually related to the structure of the right-hand side of (1.1), this algebra action is highly non-obvious from the conditions $S^p_a = 0$ imposed on the semi-infinite forms.

In physical terms, semi-infinite realizations [1]–[6] of representations of infinite-dimensional algebras are collective effects of a “quasiparticle” type: the representation space is filled by excitations over a Dirac sea of operators satisfying an exclusion principle. The resulting nonstandard realizations of representations can be viewed from the utilitarian standpoint as a particular quasiparticle basis in the space of states. In general, it is in no way obvious from the construction that these quasiparticle states exactly fill an irreducible representation of any algebra. But in the cases where this is so, calculating the character in two ways (in accordance with the quasiparticle picture and in the standard basis) gives a nontrivial identity of the type of generalized Rogers–Ramanujan identities [7], [8]. These identities (often called the Rogers–Ramanujan–Gordon–Andrews identities), originally motivated by combinatorial correspondences, were investigated by different methods, in particular with regard to their relation to conformal field theory [11]–[18]. Semi-infinite constructions of representations thus give a representation-theory interpretation of a number of nontrivial combinatorial phenomena [19], [I]. The corresponding character formulas can be interpreted as a result of the simultaneous existence and interaction of particles of several types. Formulas of this type for the partition function on the torus are related to the thermodynamic Bethe ansatz [20]–[25] and were also investigated in [26]–[28].

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1 We recall the classical statement (see [7]): “The partitions of an integer $n$ for which the difference between any two parts is not smaller than two are equinumerous to the partitions of $n$ into parts $\equiv 1$ or 4 (mod 5).” Relations between the Rogers–Ramanujan identities and conformal field theory characters go back to [9], [10].

2 The standard logic is as follows [I]. Consider, for example, two types of bosons (of “colors” 1 and 2); without the interaction between the bosons, the partition function is $1/(q)_\infty^3$, which equals $\sum_{n_1 \geq 0} \sum_{n_2 \geq 0} q^{n_1+n_2}/(q)_{n_1}(q)_{n_2}$ by the Durfee formula (we use the standard notation in Eq. (1.7)). The “semi-infinite” character formulas have a similar form, however the exponent in the numerator acquires a quadratic form in $n_1, n_2, \ldots$, which is interpreted as an interaction between the different types of quasiparticles.
In semi-infinite constructions, the representation space (rigorously defined as an inductive limit) is generated by semi-infinite forms, or the products
\[(1.2)\quad V_{\alpha_1}^{(s_1)} V_{\alpha_2}^{(s_2)} \cdots V_{\alpha_n}^{(s_n)} \cdots .\]
The subscripts (“modes”) take an infinite number of values, for example \(\alpha_i \in \mathbb{Z}\), and it can be assumed as a rule that \(\alpha_1 \leq \alpha_2 \leq \cdots\). The superscripts distinguish a finite number \((s_1, s_2, \ldots \in \{1, 2, \ldots, M\})\) of “types” of elements (all elements are the same type in the simplest case). It is assumed that starting with some number \(\iota\), the sequence \((s_i, \alpha_i)_{i \geq \iota}\) is periodic, i.e., the shift \(\alpha_i \mapsto \alpha_i + \nu\) maps the sequence into itself for some \(\nu\). An essential point is that the semi-infinite forms are then considered modulo some identifications, whose role amounts to the possibility of expressing a semi-infinite form with a sequence \((\alpha_i)\) that is “too dense” through a linear combination of “thinned out” semi-infinite forms.

In the known constructions of the semi-infinite type, the elements \(V^{(s)}\) are either some vertex operators for a given infinite-dimensional algebra or some currents taking values in the algebra. Thus, in the semi-infinite construction of [1], [29] for an affine algebra \(\hat{a}\), the elements \(V^{(s)}\) are a part of the currents with the values in the nilpotent subalgebra of \(a\); a description of the same space in terms of different operators is given in [2], where the semi-infinite construction is intermediate (in a certain sense) between the case where the operators \(V^{(s)}\) are currents in the algebra and the case where they are vertex operators acting between different modules. Semi-infinite constructions for \(\hat{\mathfrak{sl}}(2)\) modules where \(V^{(s)}\) are vertex operators arise [30], [3] from the Haldane–Shastry spin chains [31], [32] and the related Calogero model with spin [33]. The corresponding quasiparticle basis is obtained as a conformal limit (taken in the neighborhood of the antiferromagnetic state) of the space of states in the Haldane–Shastry model; this limit produces a direct sum of the two integrable level-1 \(\hat{\mathfrak{sl}}(2)\) representations. A decomposition into irreducible representations of the Yangian then leads to a new basis in level-1 representations that can be written using spin-1/2 \(\hat{\mathfrak{sl}}(2)\) vertex operators, which in the context of the Haldane–Shastry model are interpreted as the creation operators for spinon excitations, quasiparticles with the spin 1/2 and with a half-integer statistics [34], [30], [3], [35]. In the cases where \(V^{(s)}_{\alpha}\) are fermionic operators (for example, for the \(N = 2\) superconformal algebra), the semi-infinite construction can be viewed as a generalization to the “interacting” fermions of the infinite-wedge representation, which is a classical tool in investigating a number of problems in representation theory and beyond. The “interaction” here is understood not quite literally but in the sense that the semi-infinite forms are considered modulo identifications, i.e., satisfy a set of relations (as a result, in contrast to the infinite-wedge construction, semi-infinite constructions give irreducible representations or finite direct sums thereof).

Semi-infinite spaces can be investigated using filtrations by some subspaces that can be conveniently studied. For example, one of the filtrations involved in the semi-infinite construction in [1] for the level-\(k\) vacuum representation of the \(\hat{\mathfrak{sl}}(2)\) algebra consists of finite-dimensional spaces \(M^{+}[i]\),...
which as vector spaces are isomorphic to the $(\ell+1)$-multiple product $\mathbb{C}^{k+1} \otimes \cdots \otimes \mathbb{C}^{k+1}$ [31]. This is similar to what is observed in the corner transfer matrix method, where the space of states is a semi-infinite tensor product of finite-dimensional spaces and its “approximations” have the form of the above tensor product. We recall that the results obtained in integrable statistical mechanics models indicate an intimate relation between the space of states of the corresponding lattice model and a representation of some infinite-dimensional algebra [38]–[43]. Although the correspondence is not straightforward by far, it is very interesting to investigate the relations of semi-infinite constructions of representations to exactly integrable statistical mechanics models (this requires a $q$-deformation of the algebra action on the semi-infinite space).

For the $N=2$ super-Virasoro (superconformal) algebra, the possibility of constructing the semi-infinite realization of its unitary modules was pointed out in [50], and we now develop the recipe sketched there. We also note the interest in the $N=2$ supersymmetry precisely originating in investigations of the generalized Rogers–Ramanujan–Gordon–Andrews identities [11], [13]. As noted above, “reducing the number of generators” in the semi-infinite investigations of the generalized Rogers–Ramanujan–Gordon–Andrews identities [11], [13]. As noted above, “reducing the number of generators” in the semi-infinite $N=2$ realization results in that only the $\mathcal{G}_n$ modes are needed for generating the entire module. This counterintuitive statement is summarized in Theorem 1.1 below (its proof starts with Theorem 3.2 and is finished in Theorem 5.5).

Let $p \geq 3$ be a fixed positive integer, and let $G(p)$ be the algebra generated by anticommuting elements $(\mathcal{G}_n)_{n \in \mathbb{Z}}$ and an invertible operator $U$ such that $U \mathcal{G}_n U^{-1} = \mathcal{G}_{n+1}$, with $\mathcal{G}_n$ satisfying the constraints $S_a^p = 0 \ (a \in \mathbb{Z})$ with $S_a^p$ defined in (1.1) [1].

**Theorem 1.1.** Let $W$ be the representation of $G(p)$ induced from the trivial one-dimensional representation of the Grassmann algebra with the generators $(\mathcal{G}_n)_{n \geq 0}$ (on the vacuum vector $|0\rangle$). Let $r$ be a fixed integer such that $1 \leq r \leq p-1$ and let $C_r$ be the $G(p)$-submodule in $W$ generated from the vector $\mathcal{G}_{-r} \cdots \mathcal{G}_{-1}|0\rangle$ and $V_{r,p}$ the submodule generated from the set of vectors

\begin{equation}
\mathcal{G}_{a-p+1} \cdots \mathcal{G}_{a-r+1} \mathcal{G}_{a-r} \cdots \mathcal{G}_{a-1} |\alpha\rangle - |\alpha-p\rangle, \quad \alpha \in \mathbb{Z},
\end{equation}

where $|\alpha\rangle = U^\alpha|0\rangle$. The quotient space $W/(V_{r,p} + C_r)$ is a representation of the $N=2$ algebra with the central charge

\begin{equation}
c = 3 \left(1 - \frac{2}{p}\right)
\end{equation}

and, moreover, is isomorphic to a direct sum of unitary $N=2$ representations,$^3$

\begin{equation}
W(r,p) \equiv \frac{W}{V_{r,p} + C_r} \simeq \bigoplus_{\theta=0}^{p-1} \mathcal{R}_{r,p,\theta}.
\end{equation}

$^3$We consider only graded representations of $G(p)$ of the form $W = \bigoplus_{i \geq 0} W_i$ with $\mathcal{G}_n W_i \subset W_{i+n}$. Although the expression $S_a^p$ contains infinite sums, its action on such representations is well defined; following the standard abuse of terminology, we mean that the “algebra generated by $\mathcal{G}_n$” involves, in particular, infinite combinations with a well-defined action on graded spaces of the above form.

$^4$We recall (see Sec. 2.3) that unitary $N=2$ representations $\mathcal{R}_{r,p,\theta}$ with central charge [14] are labeled by a pair of integers $(r, \theta)$ such that $0 \leq \theta \leq r - 1$ and $1 \leq r \leq p - 1$. It can be assumed that $0 \leq \theta \leq p - 1$ with each representation then labeled twice and the summation in (1.3) taken over the spectral flow orbit (see Sec. 2.4).
The “semi-infiniteness” is here hidden in the relations $|\alpha\rangle = \mathcal{G}_{\alpha+1} \cdots \mathcal{G}_{\alpha+p-1} \mathcal{G}_{\alpha+p-1} \cdots \mathcal{G}_{\alpha+p-1} \times |\alpha+p\rangle$ imposed via taking the quotient; they can be applied recursively, leading to a representation of $W(r,p)$ as the space of semi-infinite forms in $(\mathcal{G}_n)_{n \in \mathbb{Z}}$ satisfying a set of relations (of which the most important ones follow from the conditions $S^\alpha_\alpha = 0$). At the level of characters, we have a combinatorial corollary of Theorem 1.1: taking the characters of the representations whose isomorphism is established in Sec. 5.6, we obtain

**Corollary 1.2.** For $k \in \mathbb{N}$, $1 \leq r \leq k+1$, and $0 \leq \theta \leq k+1$, there is the identity

$$
\sum_{N_1 \geq \cdots \geq N_k \in \mathbb{Z}} \frac{\prod_{m=1}^{k} N_m \frac{1}{q} \left( \sum_{m=r}^{k} N_m - \theta \sum_{m=1}^{k} N_m + \frac{1}{2} \sum_{m=1}^{k} N_m^2 + \sum_{1 \leq m \leq n \leq k} N_m N_n \right)}{(q)_{N_1-N_2}(q)_{N_2-N_3} \cdots (q)_{N_{k-1}-N_k}(q)_{\infty}} = \prod_{n \geq 1} \frac{1 + z q^n}{1 - q^n} \prod_{n \geq 1} \frac{1 + z^{-1} q^n}{1 - q^n}.
$$

This involves the standard notation

$$(q)_n = (1-q) \cdots (1-q^n).$$

It is useful to note that the integer parameter $k$ is $p-2$ in terms of the parameter in Theorem 1.1 (regarding the quasiparticle interpretation of the left-hand side of (1.6), see footnote 2).

Although isomorphism (1.3) and therefore identity (1.6) are eventually derived from the structure of the right-hand side of (1.1), the relation between (1.1) and (1.5) is far from obvious. In this paper, we describe in detail the methods used in the construction and proofs. The problems solved in what follows can be put into a general perspective of semi-infinite constructions:

- choosing the elements whereby the space is generated and formulating a system of conditions (constraints) on these elements (for the $N=2$ algebra, these are the fermions $\mathcal{G}_n$ and conditions (1.1) respectively);
- counting the number of states remaining in each grade after imposing the conditions, i.e., finding the character (the main complication here consists in taking the constraints into account, i.e., in working with a space that is not freely generated);
- constructing an appropriate—monomial—basis in the semi-infinite space, typically by a certain “thinning out” procedure (we note that in the set of all semi-infinite forms, there are linear dependences resulting from the imposed constraints);
- constructing the algebra action on the semi-infinite forms built from the elements satisfying the chosen constraints (an obvious complication occurs in the case where, as with our construction,
the semi-infinite forms constructed from the modes of a “small” number of currents must carry a representation of the entire algebra);
- decomposing the semi-infinite space into a (direct, if possible) sum of representations of the basic algebra;
- finding representations of some other algebraic structures on subspaces of the semi-infinite space.

Realizing this program (where we do not consider the last item) requires using a combination of different means. We now describe them in more detail for the $N=2$ superconformal algebra, essentially following the contents of this paper although in an order somewhat different from the order of sections (these methods are also interesting because the problems that we solve in constructing the semi-infinite realization are typical of a number of semi-infinite constructions beyond the present one (cf. the spinon basis construction in [3]).)

**Inductive limit (Sec. 3.1):** The semi-infinite space $W_{r,p;\theta}$ that is eventually shown to be isomorphic to the unitary $N=2$ representation $\mathcal{K}_{r,p;\theta}$ is defined as the inductive limit of the spaces $W_{r,p;\theta}(\iota)$ generated, as $\iota$ increases, from progressively more twisted highest-weight states, namely, from those states on which the progressively larger part of the generators $G_n$ become creation operators as $\iota \to \infty$, with “only” the $(G_n)_{n \geq \iota p + \theta}$ generators remaining the annihilation operators.

**The dual space (Sec. 4.1):** A key role in studying the inductive limit is played by the dual space, which can be realized using “polynomials in an infinite number of variables” (more precisely, polynomial differential forms) [4]. The dual to the quotient space in Eq. (1.5) is a subspace of polynomials satisfying certain conditions on $(p-1)$-multiple diagonals $x_1 = \cdots = x_{p-1}$ and at zero. Specifically, we investigate the space

\begin{equation}
W_{r,p;0}(0)^* \subset \mathbb{C} \oplus \mathbb{C}[x] dx \oplus \mathbb{C}\langle x_1, x_2 \rangle dx_1 dx_1 \oplus \mathbb{C}\langle x_1, x_2, x_3 \rangle dx_1 dx_2 dx_3 \oplus \cdots ,
\end{equation}

where $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ are antisymmetric polynomials in $n$ variables and the space $W_{r,p;0}(0)^*$ is singled out by the vanishing conditions on multiple diagonals and at zero. The character of this space can be evaluated by introducing a filtration on each $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ induced by the lexicographic order on partitions of $n$.

**The characters (Sec. 4.2):** The sum over partitions gives a formula of the Rogers–Ramanujan–Gordon–Andrews type for the character of $W_{r,p;0}(0)$, and after applying $N=2$ algebra automorphisms, also for all the spaces $W_{r,p;\theta}(\iota)$, $\iota \in \mathbb{Z}$, involved in the inductive limit. A remarkable property of these character formulas is that they admit the limit as $\iota \to \infty$. This limit is a candidate for the character of the semi-infinite space $W_{r,p;\theta} = \lim_{\iota \to \infty} W_{r,p;\theta}(\iota)$, but proving that the limit is the character requires establishing that all the mappings $W_{r,p;\theta}(\iota) \to W_{r,p;\theta}(\iota + 1)$ used in constructing the inductive limit are embeddings (and the limit of the characters is hence equal to the character of the inductive limit).
The thin basis (Sec. 3.2): The above statement regarding the embeddings follows from the existence of a remarkable monomial basis consisting of the states in which the modes $G_n$ are “thinned out” in accordance with the procedure described in Theorem 1.3 below. By the sequence of occupation numbers associated with a semi-infinite form $G_{i_1} \ldots G_{i_n} \ldots$ ($i_1 < \cdots < i_n < \cdots$), we mean the sequence $(\gamma(n))$ of elements labeled by integers $n \geq i_1$ that are equal to zero except for $\gamma(i_1) = \gamma(i_2) = \cdots = 1$. The construction of the special monomial basis, called the thin basis, is given in the following theorem (in Sec. 3.2, we prove an equivalent statement in Lemma 3.3).

**Theorem 1.3.** There exists a basis in $W_{r,p;\theta}$ whose elements are semi-infinite forms $G_{i_1} G_{i_2} \ldots$ satisfying the following conditions:

1. for any $n$, $i_{n+p-2} - i_n \geq p$ (hence, any segment $[n, n+p-1] \subset \mathbb{Z}$ of the length $p$ can contain at most $p-2$ nonzero occupation numbers).
2. For $n \gg 1$, the sequence of occupation numbers $\alpha(n)$ is periodic with the period $p$ and exactly $p-2$ occurrences of 1 per period and is

\[
\begin{array}{c}
\underbrace{1, \ldots, 1, 0}_{r-1}, \underbrace{1, \ldots, 1, 0}_{p-r-1}, \ldots, \underbrace{1, \ldots, 1, 0}_{r-1}, \underbrace{1, \ldots, 1, 0}_{p-r-1}, \ldots
\end{array}
\] (1.9)

**Lattices, crosses, and recursions** (Sec. 4.3): Construction of the thin basis corresponds to a model statistical system on a semi-infinite one-dimensional lattice. Each semi-infinite form $G_{i_1} \ldots G_{i_n} \ldots$ can be represented as a configuration of crosses on the lattice, for example

\[
\times \circ \times \circ \circ \circ \times \circ \circ \times \ldots
\] (1.10)

(a site with a cross corresponds to the occupation number 1, others to 0). The “enumeration” of thin basis elements then becomes the problem of finding the partition function of all configurations of crosses with any $p$ consecutive lattice sites carrying at most $p-2$ crosses. These partition functions can be analyzed using a version of the so-called Andrews–Schur method, which consists in establishing recursive relations and subsequently taking the limit as the “finitization parameter” goes to infinity. This allows us to show that the semi-infinite forms described in Theorem 1.3 indeed constitute a basis in the semi-infinite space.

**The algebra action on the semi-infinite space** (Sec. 5): It can be directly shown that the derived characters of the semi-infinite spaces $W_{r,p;\theta}$ coincide with the characters of the corresponding unitary $N = 2$ representations $\mathfrak{r}_{r,p;\theta}$; from the existing mapping $W_{r,p;\theta} \rightarrow \mathfrak{r}_{r,p;\theta}$, it is then easy to deduce that $W_{r,p;\theta}$ is in fact isomorphic to $\mathfrak{r}_{r,p;\theta}$, which in turn implies the existence of the $N = 2$ algebra action on $W_{r,p;\theta}$. However, the explicit form of this action then remains unknown. We choose a more “conceptual” approach and directly construct the $N = 2$ action on the semi-infinite space $W_{r,p;\theta}$. The definition of the semi-infinite space does not suggest that this space carries an action of the generators $Q_n$, $L_n$, and $H_n$, with $n \in \mathbb{Z}$.
satisfying algebra (2.1); the existence of this action is extremely sensitive to the imposed constraints \( S^p_n = 0 \). The methods for constructing the action developed here can also be useful in investigating other semi-infinite spaces where constraints of a different form can be compatible with the action of another algebra. These methods are as follows:

A. **The positive filtration** (Sec. 5.1): In constructing the \( N = 2 \) algebra action on the semi-infinite space, we use a filtration of \( W_{r,p;\theta} \) by finite-dimensional spaces called the *positive filtration*. For affine algebras, similar subspaces are known as *Demazure modules* and have been studied from different standpoints (see [44]–[46] and the bibliography therein).

B. **Differential operators** (Sec. 5.2): On the above finite-dimensional subspaces, we construct the action of a part of the \( N = 2 \) algebra generators via differential operators (in finitely many Grassmann variables \( G_n, \ 0 \leq n \leq N \)). More precisely, differential operators a priori act on a freely generated space, and we must therefore show that their action is compatible with taking the quotient with respect to the relations induced by the basic conditions (1.1) on the subspace.

C. **The dual picture** (Sec. 5.3): To verify compatibility of the differential operator action with relations (1.1), we again use the functional realization. The operators dual to the differential operators become the “homology-type” differentials on polynomials, which considerably simplifies the statements that must be proved.

D. **Gluing the action from pieces** (Sec. 5.4): The action of the \( N = 2 \) generators on a vector of the semi-infinite space must not depend on the filtration term to which this vector is viewed to belong. We prove this independence as well as the independence of the constructed action from any arbitrariness involved in the construction. We actually construct the action of only a part of the \( N = 2 \) generators that *generate the entire algebra*, and we then prove that these generate precisely the \( N = 2 \) superconformal algebra.

E. **Isomorphism with the unitary representation** (Sec. 5.6): As soon as it is established that the semi-infinite space \( W_{r,p;\theta} \) is a module over the \( N = 2 \) algebra, it is easy to verify that the mapping \( W_{r,p;\theta} \to K_{r,p;\theta} \) into the unitary representation is an isomorphism of \( N = 2 \) modules, which completes the semi-infinite construction.

The semi-infinite construction of unitary \( N = 2 \) modules is closely related to a similar construction of the unitary \( \widehat{\mathfrak{sl}}(2) \) modules [1]. In particular, the method used in constructing the \( N = 2 \) algebra action on the semi-infinite space also allows us to define the \( \widehat{\mathfrak{sl}}(2) \) algebra action on the corresponding semi-infinite space (Sec. 5.3). This involves the Demazure modules; the corresponding characters are related to the generalized Pascal triangles [11] and supernomial coefficients [17], and in Sec. 6.1, we also consider a combinatorial construction of bases in the \( N = 2 \) “Demazure” subspaces (cf. [14]–[16]). As another aspect of the relation between \( \widehat{\mathfrak{sl}}(2) \) and \( N = 2 \) structures, we consider the correspondence between the modular functors [29] (Sec. 2.3) and use it in relating the (unitary) \( \widehat{\mathfrak{sl}}(2) \) and \( N = 2 \) fusion rules. The modular functor can be described somewhat more explicitly in
the semi-infinite realization (see Sec. 6.2), because the representation is generated by only a part of the currents, which allows identifying the space dual to coinvariants with the space of meromorphic functions on products of punctured Riemann surfaces, with the functions required to possess a prescribed behavior on multiple diagonals (which is a counterpart of the relations $S^p_n = 0$) and at the punctures. In some cases, the dimensions of these functional spaces can be evaluated directly; for the products of $\mathbb{CP}^1$, on the other hand, these dimensions follow from the fusion rules for the $N=2$ algebra. The unitary $N=2$ fusion rules have been obtained from the Verlinde conjecture [48], but we give an independent derivation based on acting on the $\hat{s}\ell(2)$ fusion rules with the functor that realizes the equivalence of categories [49], [50], see Sec. 6.3.

2. The $N=2$ algebra

In this section, we recall the main facts pertaining to the $N=2$ superconformal algebra and motivate the semi-infinite construction. The reader may wish to go directly to Sec. 2.4 and use Secs. 2.1, 2.2, and (partly) 2.3 for reference.

2.1. The $N=2$ algebra and the spectral flow. The $N=2$ superconformal algebra is generated by the bosonic operators $L_n$ (Virasoro algebra generators) and $H_n$ (Heisenberg algebra) and the fermionic operators $G_n$ and $Q_n$. We consider the algebra in the basis where the nonvanishing commutation relations are given by

$$
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n}, & [H_m, H_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[L_m, G_n] &= (m - n)G_{m+n}, & [H_m, G_n] &= 0, \\
[L_m, Q_n] &= -nQ_{m+n}, & [H_m, Q_n] &= -Q_{m+n}, \\
[G_m, Q_n] &= 2L_{m+n} - 2nH_{m+n} + \frac{c}{3}(m^2 + m)\delta_{m+n,0},
\end{align*}
$$

(2.1)

where $m, n \in \mathbb{Z}$ and the bracket $[\cdot, \cdot]$ denotes the supercommutator. The central charge $c \neq 3$ can be conveniently parametrized as $c = 3(1 - 2/p)$ with $p \in \mathbb{C} \setminus \{0\}$.

The algebra automorphisms include the group $\mathbb{Z}$ of automorphisms $U_\theta, \theta \in \mathbb{Z}$, called the spectral flow [51]. In the basis chosen in (2.1), the spectral flow acts as

$$
\begin{align*}
U_\theta : & \quad L_n \mapsto L_n + \theta H_n + \frac{c}{6}(\theta^2 + \theta)\delta_{n,0}, & H_n \mapsto H_n + \frac{c}{3}\theta\delta_{n,0}, & \theta \in \mathbb{Z}, \\
& \quad Q_n \mapsto Q_{n-\theta}, & G_n \mapsto G_{n+\theta}
\end{align*}
$$

(2.2)

The action of the spectral flow on an $N=2$ module gives a nonisomorphic representation in general. The modules transformed by the spectral flow are called the twisted modules. We use the notation

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6 This equivalence is “modulo” the spectral flows, and this is why the fusion algebras for the $\hat{s}\ell(2)$ and $N=2$ theories are not identical; however, the equivalence statement is sufficiently powerful to derive the $N=2$ fusion algebra from the $\hat{s}\ell(2)$ fusion. The correspondence between $\hat{s}\ell(2)$ and $N=2$ fusion algebras can be considered in the framework of the correspondence between $\hat{s}\ell(2)$ and $N=2$ modular functors in Sec. 2.3, which extends the equivalence of representation categories.
$\mathfrak{D}_{\bullet\theta} = U_{\theta}\mathfrak{D}_{\bullet}$, which is also applied to modules carrying other labels, for example, $\mathfrak{R}_{r,p;\theta} = U_{\theta}\mathfrak{R}_{r,p}$ for the unitary representations considered in what follows. The parameter $\theta$ is called the twist. We omit zero twist from the notation.

The character of an $N=2$ module $\mathfrak{D}$ is defined as
\begin{equation}
\omega_{\mathfrak{D}}(z, q) = \text{Tr}_{\mathfrak{D}}(z^{H_0} q^{L_0}),
\end{equation}
where taking the trace over $\mathfrak{D}$ involves a sesquilinear form $\left[52\right]$ that can be found in $\left[50\right]$ in our current notation. Under the spectral flow action, characters transform as
\begin{equation}
\omega_{U_{\theta}\mathfrak{D}}(z, q) = z^{-\hat{\vartheta} \theta} q^{(\theta^2 - \theta)} \omega_{\mathfrak{D}}(z q^{-\theta}, q).
\end{equation}

We define the twisted highest-weight vector $|h, p; \theta\rangle$ as a state satisfying the annihilation conditions
\begin{equation}
Q_{-\theta+m}|h, p; \theta\rangle = G_{\theta+m}|h, p; \theta\rangle = L_{m+1}|h, p; \theta\rangle = H_{m+1}|h, p; \theta\rangle = 0, \quad m \in \mathbb{N}_0
\end{equation}
and also the conditions
\begin{align}
\left( H_0 + \frac{c}{3} \theta \right) |h, p; \theta\rangle &= h |h, p; \theta\rangle, \\
\left( L_0 + \theta H_0 + \frac{c}{6} (\theta^2 + \theta) \right) |h, p; \theta\rangle &= 0
\end{align}
(where the second one follows from the annihilation conditions).

Anticipating some of what follows, we note that the semi-infinite construction also involves twisted states on which, however, only the action of the $G_n$ operators is defined such that the same vanishing conditions as for the corresponding states in the unitary module are satisfied, but the action of the remaining generators must be reconstructed (such that all the relations that hold in the unitary module are satisfied).

### 2.2. Unitary representations of the $N=2$ algebra $\left[51\right]$. The unitary $N=2$ representations $\left[52\right]$ are periodic under the spectral flow with the period $p$; that is, the spectral flow $U_p$ produces an isomorphic representation,
\begin{equation}
\mathfrak{R}_{r,p;\theta+p} \approx \mathfrak{R}_{r,p;\theta}.
\end{equation}
For unitary representations, the twist parameter $\theta$ can therefore be considered modulo $p$. Moreover, for a given $p$, there are only $p(p-1)/2$ nonisomorphic unitary representations $\mathfrak{R}_{r,p;\theta}$, which can be labeled by $0 \leq \theta \leq r-1$ and $1 \leq r \leq p-1$, because there are the isomorphisms of $N=2$ modules
\begin{equation}
\mathfrak{R}_{r,p;\theta+r} \approx \mathfrak{R}_{r,p;\theta}, \quad 1 \leq r \leq p-1, \quad \theta \in \mathbb{Z}_p.
\end{equation}
Representations with zero twist are denoted by $\mathfrak{R}_{r,p} \equiv \mathfrak{R}_{r,p;0}$ for brevity.

The characters $\omega_{\mathfrak{R}_{r,p;\theta}}$ of the unitary $N=2$ representations $\mathfrak{R}_{r,p;\theta}$ in the notation in $\left[51\right]$ (see also $\left[53\right]–\left[55\right]$) are given by
\begin{equation}
\omega_{\mathfrak{R}_{r,p;\theta}}(z, q) = z^{\frac{2p-r+1}{p} - \theta} q^{\frac{\theta r - \theta^2 + \theta}{p}} \frac{\eta(q^p)^3}{\eta(q)^3} \frac{\vartheta_{1,0}(z, q) \vartheta_{1,1}(q^r, q^p)}{\vartheta_{1,0}(z q^{-\theta}, q^p) \vartheta_{1,0}(z q^{r-\theta}, q^p)}.
\end{equation}
2.3. Equivalence of categories and related issues. A linear combination of the unitary \( N = 2 \) characters belonging to the same spectral flow orbit gives the characters
\[
\chi_{r,k} = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2} (m^2 - m)} z^{-m} = \prod_{m \geq 0} \left( 1 - q^m \right) \prod_{m \geq 1} \left( 1 - z q^m \right) \prod_{m \geq 1} \left( 1 - q^m \right),
\]
(2.11)

These characters (where we can set \( \theta = 0 \) for simplicity, because the behavior of characters under the spectral flow transformations is determined by (2.4)) can also be expanded with respect to the theta functions \( \theta_{1,0}(\cdot, q^p, q^{p-2}) \) as
\[
\omega_{r,p}(z, q) = z \frac{1 + 2 \theta}{p} \theta \left( 1 - \frac{2}{p} (\theta^2 - \theta) \frac{1 - \theta - q^{p-2}}{2} \right) \times \sum_{a=0}^{p-3} z^a q^{\frac{1}{2} (a^2 - (2\theta + 1)a)} \prod_{m \geq 0} \left( 1 + z q^m \right) \prod_{m \geq 1} \left( 1 + q^m \right) \prod_{m \geq 1} \left( 1 - q^m \right).
\]
(2.12)

where the string functions \( C_{r,p}^{a}(q) \) depend only on \( q \).

These characters (where we can set \( \theta = 0 \) for simplicity, because the behavior of characters under the spectral flow transformations is determined by (2.4)) can also be expanded with respect to the theta functions \( \theta_{1,0}(\cdot, q^p, q^{p-2}) \) as
\[
\omega_{r,p}(z, q) = z \frac{1 + 2 \theta}{p} \theta \left( 1 - \frac{2}{p} (\theta^2 - \theta) \frac{1 - \theta - q^{p-2}}{2} \right) \times \sum_{a=0}^{p-3} z^a q^{\frac{1}{2} (a^2 - (2\theta + 1)a)} \prod_{m \geq 0} \left( 1 + z q^m \right) \prod_{m \geq 1} \left( 1 + q^m \right) \prod_{m \geq 1} \left( 1 - q^m \right).
\]
(2.12)

where the string functions \( C_{r,p}^{a}(q) \) depend only on \( q \).

### 2.3. Equivalence of categories and related issues

A linear combination of the unitary \( N = 2 \) characters belonging to the same spectral flow orbit gives the characters \( \chi_{r,k} \) of the unitary \( \hat{s\ell}(2) \) representations \( L_{r,k} \),
\[
\chi_{r,p-2}(z, q) \theta_{1,0}(z, q) = q^{r-1} z^{p-2} \prod_{a=0}^{p-1} \omega_{r,p}(q y^{-a}, q) y^{-a} z^{-a q^{\frac{1}{2} a - 2 - r-2}} \prod_{a=0}^{p-1} \theta_{1,0}(z, q^{2p-2} q^{-r-2}).
\]
(2.14)

This follows from of the isomorphism of \( N = 2 \) modules [19], [50]
\[
L_{r,k} \otimes \Omega \approx \bigoplus_{\theta = 0}^{k+1} R_{r,k+2,\theta} \otimes \left( \bigoplus_{\nu \in \sqrt{2(k+2)} \mathbb{Z}} H_{\frac{1}{2} (\theta - \nu)} \right),
\]
(2.15)

where \( \Omega \) is the free-fermion module and \( H_{\frac{1}{2} (\theta - \nu)} \) are Fock spaces (the sum over the lattice \( \sqrt{2(k+2)} \mathbb{Z} \)
defines a vertex operator algebra). Equation (2.13) allows us to establish the functorial correspondence
\[
L_{r,k+2,\bullet} \approx R_{r,k+2,\bullet}
\]
(2.16)

between the spectral flow orbits of each algebra. This is an equivalence between categories whose objects are spectral flow orbits [19]; in other words, this is the equivalence of representation categories of the algebras obtained by adding the spectral flow operator to the universal enveloping algebra (of the \( \hat{s\ell}(2) \) and the \( N = 2 \) algebras respectively).

Expanding on the equivalence of categories, we now describe the correspondence between modular functors for the \( \hat{s\ell}(2) \) and \( N = 2 \) theories. This correspondence is interesting, in particular, because the \( \hat{s\ell}(2) \) modular functor has the well-known geometric interpretation in terms of the moduli spaces
of $SL_2$ vector bundles, whereas no a priori geometric interpretation is known for the $N=2$ modular functor.

For a given Riemann surface $\mathcal{E}$, the modular functor $\text{MOD}_{\hat{\mathfrak{sl}}(2)}(k, \mathcal{E})$ of the level-$k$ $\hat{\mathfrak{sl}}(2)$ Wess–Zumino–Witten theory and the modular functor $\text{MOD}_{N=2}(p, \mathcal{E})$ of the theory based on the $N=2$ algebra with central charge (1.4) are related by
\begin{equation}
\text{MOD}_{N=2}(p, \mathcal{E}) = \text{Coinv}_{H^1(\mathcal{E}, \mathbb{Z}_2) / V_+} \left( \text{Inv}_{V_+} \left( \text{MOD}_{\hat{\mathfrak{sl}}(2)}(p-2, \mathcal{E}) \otimes \text{MOD}_{\text{free}}(p, \mathcal{E}) \right) \right),
\end{equation}
where $\text{MOD}_{\text{free}}(p, \mathcal{E})$ is the modular functor of the free theory whose vertex operator algebra is associated with the lattice $\sqrt{2p} \mathbb{Z}$ (see (2.13)). The $\hat{\mathfrak{sl}}(2)$ modular functor $\text{MOD}_{\hat{\mathfrak{sl}}(2)}(k, \mathcal{E})$ carries a projective action $\mathfrak{O}(\cdot)$ of the group $H^1(\mathcal{E}, \mathbb{Z}_2)$ such that $\mathfrak{O}(\alpha \beta) = \mathfrak{O}(\alpha) \mathfrak{O}(\beta) (-1)^{k(\alpha, \beta)}$ for the cycles $\alpha$ and $\beta$ with the intersection form $\langle \alpha, \beta \rangle$ or, in other words, there is the action of the Heisenberg group $\Gamma$ with the center $\mathbb{Z}_2$ such that $\Gamma / \mathbb{Z}_2 = H^1(\mathcal{E}, \mathbb{Z}_2)$. The modular functor of the free theory, which is the space of level-$2p$ theta functions on the Jacobian of $\mathcal{E}$, is an irreducible representation of the Heisenberg group obtained as the central extension of $H^1(\mathcal{E}, \mathbb{Z}_{2p}) \simeq \mathbb{Z}_{2p}^g$ with the help of $\mathbb{Z}_{2p}$, where $g$ is the genus of $\mathcal{E}$. It is therefore a projective representation of the subgroup $H^1(\mathcal{E}, \mathbb{Z}_2) \subset H^1(\mathcal{E}, \mathbb{Z}_{2p})$ (with the embedding induced by $\mathbb{Z}_2 \to \mathbb{Z}_{2p}$). Therefore, the group $\Gamma$ acts on the tensor product $\text{MOD}_{\hat{\mathfrak{sl}}(2)}(p-2, \mathcal{E}) \otimes \text{MOD}_{\text{free}}(p, \mathcal{E})$. Moreover, the center then acts trivially, and the tensor product becomes a representation of the $H^1(\mathcal{E}, \mathbb{Z}_2)$ group. In this group, which is isomorphic to $\mathbb{Z}_2^g$, one must choose the maximal $\langle \cdot, \cdot \rangle$-isotropic subspace $V_+$ and take invariants with respect to $V_+$ in the tensor product. The invariants carry the action of the quotient $\mathbb{Z}_2^g / V_+$, and one must then take coinvariants with respect to this action. This explains the notation used in (2.17).

2.4. Motivation for the semi-infinite construction. The semi-infinite construction of the unitary $N=2$ representations can be motivated by the following observations. In the vacuum representation, we consider the decoupling condition for the singular vector $Q_{1-p} Q_{2-p} \ldots Q_{-1}|0, p, 0\rangle$. The corresponding field
\begin{equation}
\partial^{p-2} Q(z) \ldots \partial Q(z) Q(z)
\end{equation}
(with $\partial = \partial / \partial z$ and $Q(z) = \sum_{n \in \mathbb{Z}} Q_n z^{-n-1}$) has vanishing correlation functions with all the fields in the corresponding “minimal” model (cf. [54]). The identity $\partial^{p-2} Q(z) \ldots \partial Q(z) Q(z) = 0$ therefore holds in unitary representations as an “operator” equality in the sense that it is satisfied when acting on any vector of any unitary representation. Similarly, the “symmetric” relation $\partial^{p-2} G(z) \ldots \partial G(z) G(z) = 0$ also holds for $G(z) = \sum_{n \in \mathbb{Z}} G_n z^{-n-2}$. Any of these relations can be chosen as the starting point of the semi-infinite construction. Remarkably, this suffices for reconstructing the entire representation. We choose the $G$-relation (which is a matter of convention
and/or application of \( N=2 \) algebra automorphisms). Therefore, we define
\[
S^p(z) = \partial^{p-2}G(z) \ldots \partial G(z)G(z), \quad G(z) = \sum_{n \in \mathbb{Z}} G_n z^{-n-2},
\]
and rewrite the condition \( S^p(z) = 0 \) in terms of the modes \( G_n \) as \( S^p_0 = 0 \) (see (1.1)). For example, for \( p = 3 \), the first several relations satisfied on the vacuum vector are given by
\[
G_{-2}G_{-1} = 0, \quad G_{-3}G_{-1} = 0, \quad 3G_{-4}G_{-1} + G_{-3}G_{-2} = 0,
4G_{-5}G_{-1} + 2G_{-4}G_{-2} = 0, \quad 5G_{-6}G_{-1} + 3G_{-5}G_{-2} + G_{-4}G_{-3} = 0, \quad \ldots.
\]

In addition, the highest-weight vector \(| r, p; \theta \rangle_{irr} \) of a given unitary representation \( R_{r,p;\theta} \) satisfies another vanishing condition, \( G_{\theta-r} \ldots G_{\theta-1} | r, p; \theta \rangle_{irr} = 0 \), which is the decoupling condition for another singular vector. Next, the state \( G_{\theta-p+1} \ldots G_{\theta-r}G_{\theta-r+1} \ldots G_{\theta-1} | r, p; \theta \rangle_{irr} \) satisfies the twisted highest-weight conditions (2.23) with the twist \( \vartheta = \theta - p \). Acting on this state with the operators \( G_{-p+\theta-p+1} \ldots G_{-p+\theta-r}G_{-p+\theta-r+1} \ldots G_{-p+\theta-1} \), we obtain a twisted highest-weight state with the twist \( \vartheta = \theta - 2p \), etc. Acting on \(| r, p; \theta \rangle_{irr} \) with the \( Q_n \) modes, we similarly obtain twisted highest-weight states with the twists \( \theta + p, \theta + 2p, \ldots \). We label these extremal states as \(| r, p; \theta | \rangle \), where the number \( \iota \in \mathbb{Z} \) determines the twist via \( \vartheta = \theta + \iota p \). Explicitly, the charge–level coordinates of \(| r, p; \theta | \rangle \) are read off from
\[
L_0 | r, p; \theta | \rangle = \left( \frac{\theta - 1}{p} + \frac{1}{2} \frac{p - 2}{p} \vartheta^2 - \theta - \frac{p}{2} + \iota^2 \frac{p}{2} - \iota r + \iota (p - 2) \theta \right) | r, p; \theta | \rangle,
\]
\[
H_0 | r, p; \theta | \rangle = \left( - \frac{r - 1}{p} - \frac{p - 2}{p} \vartheta - (p - 2) \iota \right) | r, p; \theta | \rangle.
\]
These states satisfy, in particular, the conditions
\[
G_n | r, p; \theta | \rangle = 0 \quad \text{for} \quad n \geq \iota p + \theta,
\]
\[
G_{\iota p+\theta-r} \ldots G_{\iota p+\theta-1} | r, p; \theta | \rangle = 0,
G_{\iota p+\theta-r+1} \ldots G_{\iota p+\theta-1} | r, p; \theta | \rangle = | r, p; \theta | \rangle \quad \text{for} \quad \iota \geq 1,
\]
where \( \iota \in \mathbb{Z} \). In the charge–level coordinates on the plane, we represent the states \( \ldots G_{-3}G_{-2}G_{-1} | r, p; 0 \rangle \) with arrows (see Fig. 11) or somewhat more schematically, by replacing several consecutive arrows with sections of parabolas; states (2.23) are then the cusps where the sections of parabolas join (half of the cusps correspond to the mode \( G_{\iota p+\theta-r} \) omitted in (2.23) and the other half are the states \(| r, p; \theta | \rangle \)).

The idea behind the semi-infinite construction is to “generate the module from the state \(| r, p; \infty \rangle \)” located infinitely far in the bottom right in Fig. 1, using only the \( G_n \) modes; this amounts to introducing semi-infinite forms in the fermionic operators \( (G_i)_{i \in \mathbb{Z}} \). We see in what follows that any unitary representation \( R_{r,p;\theta} \) can be realized via this semi-infinite construction.
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numbers \( \alpha \)

denote the quotient of the Grassmann algebra of the operators \( \mathcal{G}_n \), \( n \in \mathbb{Z} \), over the ideal \( S^p \) generated by \( (S^n_a)_{a \in \mathbb{Z}} \), see Eq. (1.1) (eventually, we identify \( \mathcal{G}_n \) with the corresponding \( N=2 \) generators).

3. The semi-infinite space \( \mathcal{W}_{r,p,\theta} \)

We fix an integer \( p \geq 3 \) and also \( r \) and \( \theta \) such that \( 1 \leq r \leq p-1 \) and \( 0 \leq \theta \leq r-1 \). We let \( g(p) \)
denote the quotient of the Grassmann algebra of the operators \( \mathcal{G}_n \), \( n \in \mathbb{Z} \), over the ideal \( S^p \) generated by \( (S^n_a)_{a \in \mathbb{Z}} \), see Eq. (1.1) (eventually, we identify \( \mathcal{G}_n \) with the corresponding \( N=2 \) generators).

3.1. The inductive limit. The semi-infinite space \( \mathcal{W}_{r,p,\theta} \) is spanned by polynomials in \( \mathcal{G}_n \in g(p) \), \( n \in \mathbb{Z} \), acting on the states

\[
(\lvert r, p; \theta|\iota\rangle_{\infty/2})_{\iota \in \mathbb{Z}},
\]

such that

\[
\mathcal{G}_n\lvert r, p; \theta|\iota\rangle_{\infty/2} = 0, \quad n \geq \iota p + \theta,
\]

\[
\mathcal{G}_{\iota p + \theta - r} \cdots \mathcal{G}_{\iota p + \theta - 1}| r, p; \theta|\iota\rangle_{\infty/2} = 0,
\]

\[
\mathcal{G}_{\iota p + \theta - p + 1} \cdots \mathcal{G}_{\iota p + \theta - r - 1}\mathcal{G}_{\iota p + \theta - r + 1} \cdots \mathcal{G}_{\iota p + \theta - 1}| r, p; \theta|\iota\rangle_{\infty/2} = | r, p; \theta|\iota - 1\rangle_{\infty/2}.
\]

Therefore, \( \mathcal{W}_{r,p,\theta} \) is the linear span of states of the form \( \mathcal{G}_{i_1} \mathcal{G}_{i_2} \cdots \mathcal{G}_{i_m} | r, p; \theta|\iota\rangle_{\infty/2} \) considered modulo the ideal \( S^p \). We call the elements of \( \mathcal{W}_{r,p,\theta} \) the semi-infinite forms; we also, somewhat loosely, use the term “semi-infinite form” for polynomials in \( \mathcal{G}_n \) acting on \( | r, p; \theta|\iota\rangle_{\infty/2} \), rather than for their images in the quotient. We write \( \mathcal{W}_{r,p} \) instead of \( \mathcal{W}_{r,p,\theta} \). Obviously, \( \mathcal{W}_{r,p,\theta} \) is a module over \( g(p) \).

Remark 3.1. In less formal terms, the state \( | r, p; \theta|\iota\rangle_{\infty/2} \) can be viewed as the semi-infinite product

\[
\mathcal{G}_{\iota p + \theta + 1} \cdots \mathcal{G}_{\iota p + \theta - p - r - 2} \mathcal{G}_{\iota p + \theta + p - r - 1} \cdots \mathcal{G}_{\iota p + \theta - 1} \times
\]

\[
\times \mathcal{G}_{\iota p + \theta + p + 1} \cdots \mathcal{G}_{\iota p + \theta + 2p - r - 1} \mathcal{G}_{\iota p + \theta + 2p - r} \cdots \mathcal{G}_{\iota p + \theta + 2p - 1} \cdots,
\]

where the associated sequence of occupation numbers is periodic with the period \( p \). The elements of \( \mathcal{W}_{r,p,\theta} \) are then those semi-infinite forms of \( (\mathcal{G}_i)_{i \in \mathbb{Z}} \) (modulo \( S^p \)) for which the sequence of occupation numbers \( \alpha(i) \) becomes periodic with the period \( p \) for \( i \gg 1 \).

We let \( \mathcal{W}_{r,p,\theta}(\iota) \) denote the space generated by \( (\mathcal{G}_n) \in g(p) \), \( n \in \mathbb{Z} \), from the state \( | r, p; \theta|\iota\rangle \) satisfying the same annihilation conditions as those in (3.2) and (3.3). There are the mappings

\[
\cdots \rightarrow \mathcal{W}_{r,p,\theta}(-2) \rightarrow \mathcal{W}_{r,p,\theta}(-1) \rightarrow \mathcal{W}_{r,p,\theta}(0) \rightarrow \mathcal{W}_{r,p,\theta}(1) \rightarrow \cdots,
\]
induced by mapping the vacuum vectors as

\[ (3.7) \quad \langle r, p; \theta | \iota - 1 \rangle \mapsto g_{\mu + \theta - p + 2} \cdots g_{\mu + \theta - r - 1} g_{\mu + \theta - r + 1} \cdots g_{\mu + \theta - 1} | r, p; \theta | \iota, \quad \iota \in \mathbb{Z}. \]

Mappings (3.6) commute with the action of \( G_n \). The mappings \( W_{r,p;\theta}(\iota) \) \( \rightarrow W_{r,p;\theta} \) induced by

\[ (3.8) \quad W_{r,p;\theta}(\iota) \rightarrow W_{r,p;\theta}(\iota + 1) \]

are embeddings, i.e., the space \( W_{r,p;\theta} \) admits the filtration

\[ (3.10) \quad \cdots \subset W_{r,p;\theta}(-2) \subset W_{r,p;\theta}(-1) \subset W_{r,p;\theta}(0) \subset W_{r,p;\theta}(1) \subset \cdots. \]

This follows from the existence of a remarkable monomial basis in \( W_{r,p;\theta}(\iota) \) that agrees with the mappings \( W_{r,p;\theta}(\iota) \rightarrow W_{r,p;\theta}(\iota + 1) \), as we discuss momentarily.

### 3.2. The thin basis.

In each space \( W_{r,p;\theta}(\iota) \), we construct a monomial basis, which we call the thin basis because it consists of semi-infinite forms where the modes \( G_n \) are “thinned out” as explained in Theorem 1.3. We reformulate the desired result as follows (Theorem 1.3 is reproduced by writing \( |r, p; \theta| \iota \rangle_{\infty/2} \) as the semi-infinite product (3.3)).

**Lemma 3.3.** The set of states \( g_{i_m} \cdots g_{i_2} g_{i_1} | r, p; \theta| \iota \rangle_{\infty/2}, \) for which \( \iota p + \theta > i_1 > i_2 > \cdots > i_m, \)

\( i_a - i_{a+p-2} \geq p \) for any \( a \), and \( \iota p + \theta - i_r > r \), constitute a basis in the space \( W_{r,p;\theta}(\iota) \).
We refer to the thin basis elements as *thin monomials*. Theorem 3.2 now follows from the observation that in terms of thin monomials, the mappings $W_{r,p;\theta}(t) \to W_{r,p;\theta}(t+1)$ are implemented by
\[(3.11) \quad G_{i_m} \cdots G_{i_2} G_{i_1} |r, p; \theta| t \rangle_{\infty/2} \mapsto G_{i_m} \cdots G_{i_2} G_{i_1} \times
\times G_{i_p + \theta + 1} \cdots G_{i_p + \theta + p - r - 1} G_{i_p + \theta + p - r} \cdots G_{i_p + \theta + p - 1} |r, p; \theta| t + 1 \rangle_{\infty/2}.
\]
Indeed, once the indices $i_m, \ldots, i_1$ satisfy the conditions of Lemma 3.3, the indices $i_m, \ldots, i_1, \ i_p + \theta + 1, \ldots, \ i_p + \theta + p - r - 1, \ i_p + \theta + p - r + 1, \ldots, \ i_p + \theta + p - 1$ also satisfy these conditions.

**Proof** of Lemma 3.3 consists of two parts, the first of which is simple, but the second requires certain effort. We briefly describe the first part and then focus on the second.

The first part of the proof amounts to asserting that any semi-infinite form can be rewritten as a linear combination of the states $G_{i_m} G_{i_{m-1}} \cdots G_{i_1} |r, p; \theta| t \rangle_{\infty/2}$ satisfying the conditions of the Lemma. This follows from the fact that if any monomial in $G_n$ contains a combination of modes violating the conditions of the lemma, then in accordance with the relations $S^n_0 = 0$, this monomial can be expressed through a linear combination of other monomials *each of which is lexicographically smaller than the original one*. The new monomials, obviously, can also involve combinations of modes violating the conditions of the lemma, however the argument regarding the lexicographic ordering allows developing an iteration procedure. It converges after a finite number of steps (and thus gives a linear combination of monomials satisfying the conditions of the Lemma) because

1. the space $\overline{W_{r,p;\theta}(t)}$ is bigraded via $G_{i_m} G_{i_{m-1}} \cdots G_{i_1} |r, p; \theta| t \rangle_{\infty/2} \mapsto (m, i_1 + i_2 + \cdots + i_m);
2. each graded component is a finite-dimensional space;
3. the procedure of expressing a monomial through a linear combination of others using the relations $S^n_0 = 0$ preserves the bigrading.

In the second part of the proof, it remains to show that the thin monomials are linearly independent. This follows by comparing the characters. The calculation of characters is rather involved, and we give it in Sec. 4. The proof is completed by the statement of Lemma 4.2 in Sec. 4.3.

**Examples.** We give several examples of the transformation to the thin basis.

**A.** $p = 3, \ r = 1$. Directly eliminating dense combinations of modes leads to $G_{-15} G_{-14} G_{-12} G_{-13} G_{-4} \times G_{-3} |1, 3; 0| 0 \rangle = 3393 G_{-21} |1, 3; 0| -5 \rangle + 4185 G_{-26} G_{-15} |1, 3; 0| -4 \rangle + 5697 G_{-19} G_{-16} |1, 3; 0| -4 \rangle + 1755 \times G_{-19} G_{-15} G_{-12} |1, 3; 0| -3 \rangle + 315 G_{-18} G_{-15} G_{-12} G_{-9} |1, 3; 0| -2 \rangle.$

**B.** $p = 4, \ r = 1$. As a simple example, we have $G_{-5} G_{-4} G_{-2} |1, 4; 0| 0 \rangle = -2 G_{-6} |1, 4; 0| -1 \rangle$. Similarly, $G_{-5} G_{-4} G_{-3} |1, 4; 0| 0 \rangle = -10 G_{-7} |1, 4; 0| -1 \rangle - 8 G_{-6} G_{-4} G_{-2} |1, 4; 0| 0 \rangle.$
4. Functional realization and characters

We use the realization of the (graded-)dual space to the semi-infinite space in terms of polynomial differential forms. This functional realization is a powerful tool in studying properties of semi-infinite spaces.

4.1. The functional realization of $W_{r, p}(0)^*$. We first note that for any space generated from a vacuum $|\rangle$ by fermionic generators $G_{\leq -1}$, the graded-dual space can be identified with differential forms in some variables $x_1, x_2, \ldots$ as follows. In each graded component with a given charge (the number of $G_*$) $n \geq 1$, we arrange all the states into a generating function $G(x_1) \ldots G(x_n)|\rangle$, where

$$G(x) = \sum_{m \leq -1} G_m x^{m-1}. \quad (4.1)$$

The states $G_{i_1} \ldots G_{i_n}|\rangle$ are reproduced by taking the integrals

$$\oint dx_1 x_1^{i_1} \ldots \oint dx_n x_n^{i_n} G(x_1) \ldots G(x_n)|\rangle. \quad (4.2)$$

Any functional $\langle \ell |$ on the charge-$n$ subspace is determined by all of its values,

$$\langle \ell | G(x_1) \ldots G(x_n) |\rangle = f_\ell(x_1, \ldots, x_n) dx_1 \ldots dx_n, \quad (4.3)$$

where $f_\ell$ is an antisymmetric polynomial in $x_1, \ldots, x_n$ (and the product of the differentials is symmetric). Therefore, the space dual to $W(0)$, the space freely generated by the modes $G_{n \leq -1}$, can be identified with polynomial differential forms in $x_1, x_2, \ldots$,

$$W(0)^* = \mathbb{C} \oplus \mathbb{C}[x] dx \oplus \mathbb{C}[x_1, x_2] dx_1 dx_2 \oplus \mathbb{C}[x_1, x_2, x_3] dx_1 dx_2 dx_3 \oplus \cdots, \quad (4.4)$$

where $\mathbb{C}(x_1, \ldots, x_n)$ are antisymmetric polynomials.

For quotient spaces, the functional realization of the corresponding dual space is given by a subspace in the space of polynomials. Let $W(0)_n$ denote the charge $n$ subspace and $W(0)_n^*$ its dual, i.e., the subspace of polynomials in $n$ variables in $W(0)^*$. In the dual language, taking the quotient with respect to the ideal generated by $S_0^p$ (see (1.2)) corresponds to restricting to those antisymmetric polynomials $f(x_1, \ldots, x_n)$ for which

$$\frac{\partial^{p-2}}{\partial x_{i_1}^{p-2}} \frac{\partial^{p-3}}{\partial x_{i_2}^{p-3}} \cdots \frac{\partial}{\partial x_{i_{p-2}}^{p-3}} f(x_1, x_2, \ldots) \bigg|_{x_1 = x_2 = \cdots = x_{i_{p-2}} = x_{i_{p-1}}} = 0. \quad (4.5)$$

\footnote{Whether $G(x)$ is viewed as a 1-differential or, for example, a 2-differential, is a matter of convention; however, this convention must agree with the choice of the functional spaces involved in the functional realization. With the choice made in the text, all functional spaces are polynomials $\mathbb{C}[x_1, \ldots, x_n]$, rather than $(x_1 \ldots x_n)\mathbb{C}[x_1, \ldots, x_n]$ with some positive or negative $\nu$.}
Another condition is read off from Eq. (3.3) (for \( \theta = \iota = 0 \)) as
\[
(4.6) \quad \frac{\partial^{r-1}}{\partial x_{i_1}^{r-1}} \frac{\partial^{r-2}}{\partial x_{i_2}^{r-2}} \cdots \frac{\partial}{\partial x_{i_{r-1}}^{r-1}} f(x_1, x_2, \ldots) \bigg|_{x_1=x_2=\cdots=x_{i_r}=0} = 0.
\]
Let \( W_{r,p}(0)^* \) denote the space of antisymmetric polynomials \( f \) satisfying these conditions.

Because any antisymmetric polynomial can be represented as
\[
(4.7) \quad f(x_1, x_2, \ldots) = \Delta(x_1, x_2, \ldots) \phi(x_1, x_2, \ldots), \quad \Delta(x_1, x_2, \ldots) = \prod_{1 \leq i < j} (x_i - x_j),
\]
with a symmetric polynomial \( \phi(x_1, x_2, \ldots) \), conditions (4.6) and (4.3) become the following restrictions on symmetric polynomials:

**P1.** \( \phi(0, \ldots, 0, x_{r+1}, \ldots, x_n) = 0; \)

**P2.** \( \phi(x, x, \ldots, x, x_p, x_{p+1}, \ldots, x_n) = 0. \)

**Example.** For \( p = 3 \), condition [P2] can be easily solved as
\[
(4.8) \quad f(x_1, \ldots, x_n) dx_1 \ldots dx_n = \prod_{i < j} (x_i - x_j)^3 \varphi(x_1, \ldots, x_n) dx_1 \ldots dx_n,
\]
where \( \varphi \) is a symmetric polynomial. For \( r = 1 \), condition [P1] is immediately solved by \( \varphi(x_1, \ldots, x_n) = x_1 \ldots x_n \overline{\varphi}(x_1, x_2, \ldots, x_n) \), whence it follows that the space \( W_{1,3}(0)^* \) consists of the differential forms
\[
(4.9) \quad \prod_{i < j} (x_i - x_j)^3 x_1 \ldots x_n \overline{\varphi}(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n,
\]
where \( \varphi \) are arbitrary symmetric polynomials. Such an explicit description, however, is not available for \( p \geq 4 \).

**4.2. Characters of \( W_{r,p; \iota}(\iota) \).** The functional realization allows us to calculate the characters of \( W_{r,p; \iota}(\iota) \). The idea of [1] consists in finding, for each \( n \geq 1 \), the character of the space \( W_{r,p}(0)^* = W_{r,p}(0)^* \cap \mathbb{C}(x_1, \ldots, x_n) \), which coincides with the character of \( W_{r,p}(0)_n \) (the subspace in \( W_{r,p}(0) \) generated by \( n \) modes \( G_i \)); then
\[
(4.10) \quad \text{char} W_{r,p}(0)(z, q) = \sum_{n \geq 0} z^n \text{char} W_{r,p}(0)_n(q).
\]
The dependence on both \( \iota \) and \( \theta \) can be reconstructed in accordance with the spectral flow.

**4.2.1. Example:** \( p = 3 \). The functional model of \( W_{1,3}(0)^* \) is given by (1.9), and the space of symmetric polynomials in \( n \) variables is algebraically generated by the elementary symmetric polynomials; the contribution of these polynomials to the character of \( W_{1,3}(0)^* \) is therefore \( 1/(q)_n \). Next, the product \( \prod_{i < j} (x_i - x_j)^3 \) contributes \( q^{3n(n-1)/2} \) to the character, the differentials \( dx_1 \ldots dx_n \)

contribute $q^n$, and another $q^n$ comes from $x_1 \ldots x_n$. Therefore,

\begin{equation}
\text{char } W_{1,3}(0) = \text{char } W_{1,3}(0)^* = \sum_{n \geq 0} \frac{z^n q^{\frac{3n^2 + n}{2}}}{(q)_n}.
\end{equation}

Applying the spectral flow transform in accordance with (2.4), we find

\begin{equation}
\text{char } W_{1,3}(t)(z, q) = z^{-t} q^{\frac{3t-4}{2}} \text{char } W_{1,3}(0)(zq^{-3t}, q) = \sum_{n \geq 0} \frac{z^n q^{\frac{3n^2 + n}{2}}}{(q)_n} = \sum_{n \geq -t} \frac{z^n q^{\frac{3n^2 + n}{2}}}{(q)_{n+d}}.
\end{equation}

This expression admits the limit

\begin{equation}
\lim_{t \to \infty} \text{char } W_{1,3}(t)(z, q) = \sum_{n=-\infty}^{+\infty} \frac{z^n q^{\frac{3n^2 + n}{2}}}{\prod_{m \geq 1} (1 - q^m)} = q^{-\frac{1}{2}} \vartheta_{1,0}(zq^{-1}, q^3),
\end{equation}

which coincides with (2.10). Much more work is needed to show this remarkable coincidence in the general case.

4.2.2. The general case: $p \geq 4$.

**Lemma 4.1.** For $p \geq 3$ and $1 \leq r \leq p - 1$, the character of the space $W_{r,p}(0)$ is given by

\begin{equation}
\text{char } W_{r,p}(0)(z, q) = \sum_{n \geq 0} \sum_{N_1 \geq \cdots \geq N_{p-2} \geq 0 \atop N_1 + \cdots + N_{p-2} = n} \frac{z^n q^{\frac{3n^2 + n}{2}}}{(q)_{N_1-N_2}(q)_{N_2-N_3} \cdots (q)_{N_{p-3}-N_{p-2}}(q)_{N_{p-2}}}.
\end{equation}

**Proof.** In the functional realization of $W_{r,p}(0)^*$, we consider all the partitions of the set $\{x_1, \ldots, x_n\}$ and introduce the lexicographic ordering on the partition (i.e., we set $(r_1, r_2, \ldots, r_k) \prec (r'_1, r'_2, \ldots, r'_k)$ if $r_i < r'_i$ for the first pair $(r_i, r'_i)$ such that $r_i \neq r'_i$).

We fix a partition of $n$ written as $\{1, \ldots, n\} = M_1 \cup \cdots \cup M_\ell$ and let $|M_\alpha| = r_\alpha$. We say that a polynomial $\varphi(x_1, \ldots, x_n)$ vanishes on this partition if it vanishes whenever all the variables $x_{i_\alpha}$ for $i_\alpha \in M_\alpha$ take the same value, $x_{i_\alpha} = a_\alpha$ for all $\alpha = 1, \ldots, \ell$. We write this as $\varphi(a_1; \ldots; a_\ell) = 0$.

The character of $W_{r,p}(0)^*$ can be written as

\begin{equation}
\text{char } W_{r,p}(0)(z, q) = z^{-\frac{p-1}{p}} \sum_{n \geq 0} z^n q^{\frac{3n^2-n}{2}} Z_{r,p}^n(q),
\end{equation}

where the factor $q^{(n^2-n)/2}$ corresponds to $\Delta(x_1, \ldots, x_n)$ in (1.7) and $Z_{r,p}^n(q)$ is the partition function of symmetric polynomials in $n$ variables that vanish on the partition $\hat{\rho} = (p-1, 1, 1, \ldots, 1)$.

For a partition $P$, we consider the set of symmetric polynomials that vanish on every partition $P' \succ P$. The lexicographic order on partitions then induces a filtration on symmetric polynomials in $n$ variables. If $Z_{r,p}^n(P)(q)$ is the partition function of the associated graded factor $Gr_P$, we have

\begin{equation}
Z_{r,p}^n(q) = \sum_{\{P|P \prec \hat{\rho}\}} Z_{r,p}^n(P)(q).
\end{equation}
To find $Z_{r,p}^n(q)$, we first consider the case where $r = p + 1$; condition $\mathbb{P}_1$ is then a consequence of $\mathbb{P}_2$. For a partition $P$ with the parts $M_1, \ldots, M_\ell$ (and with $|M_\alpha| = r_\alpha$), the graded factor $\text{Gr}_P$ is spanned by polynomials $\varphi(a) \equiv \varphi(a_1; \ldots; a_\ell)$ satisfying the conditions

\[ \text{(mult): } \varphi(a) = 0 \text{ if } a_\alpha = a_\beta, \text{ with the multiplicity of the zero equal to } 2 \min(r_\alpha, r_\beta), \]

\[ \text{(sym): } \varphi(\ldots; a_\alpha; \ldots; a_\beta; \ldots) = \varphi(\ldots; a_\beta; \ldots; a_\alpha; \ldots) \text{ whenever } r_\alpha = r_\beta. \]

Recalling the differentials from (4.3), we can therefore see that the partition function $Z_{p-1,p}^n(q)$ of the graded factor coincides with the partition function of the space spanned by

\[ (4.17) \prod_{1 \leq \alpha \leq \beta \leq \ell} (a_\alpha - a_\beta)^{2r_\beta} \cdot \varphi(a) \cdot \prod_{r = 1}^\ell (da_\alpha)^r, \]

where a polynomial $\varphi$ satisfies symmetry requirement (sym). Let $\nu_m$ be the number of parts $M_\alpha$ such that $r_\alpha = m$. The contribution of the chosen partition $P$ to the partition function is then

\[ (4.18) \frac{q^{\sum_{\alpha < 0.5} 2r_\beta + \sum_\alpha r_\alpha}}{\prod_{j \geq 1} (q)^{\nu_j}} = \frac{q^{\sum_{\beta = 1}^\ell (2\beta - 1)r_\beta}}{\prod_{j \geq 1} (q)^{\nu_j}} = \frac{q^{\sum_{m=1}^{p-1} N_m^2}}{\prod_{m=1}^{p-1} (q)^{N_m}}, \]

where $N_m = \nu_m + \nu_{m+1} + \cdots + \nu_\ell$ are the elements of the partition $P$ transposed. Therefore,

\[ (4.19) Z_{p-1,p}^n(q) = \sum_{N_1 \geq \cdots \geq N_{p-2} \geq 0} \frac{q^{\sum_{m=1}^{p-1} N_m^2}}{\prod_{m=1}^{p-1} (q)^{N_m^2}}. \]

If $r \neq p - 1$, we must additionally account for condition $\mathbb{P}_1$ which amounts to taking symmetric polynomials $\varphi(a)$ satisfying $\varphi(a)|_{a_1 = a_2 = \cdots = a_\ell = 0} = 0$. For this, we recall the filtration $\mathbb{P}_1^n \subset \cdots \subset \mathbb{P}_r^n$ on the space $\mathbb{P}_r^n$ of symmetric polynomials in $n$ variables satisfying condition $\mathbb{P}_1$,

\[ (4.20) \mathbb{P}_r^n = \sigma_1 \mathbb{C}[\sigma_1, \ldots, \sigma_n] + \sigma_{n-1} \mathbb{C}[\sigma_1, \ldots, \sigma_{n-1}] + \cdots + \sigma_{n-r+1} \mathbb{C}[\sigma_1, \ldots, \sigma_{n-r+1}], \]

where $\sigma_1 = x_1 + \cdots + x_n, \ldots, \sigma_n = x_1 \cdots x_n$ are the elementary symmetric polynomials in $n$ variables. In the corresponding graded factor, the explicit $\sigma_i$ factors result in additionally multiplying partition function (4.18) by $q^{\sum_{m=1}^{p-1} N_m}$. Thus,

\[ (4.21) Z_{r,p}^n(q) = \sum_{N_1 \geq \cdots \geq N_{p-2} \geq 0} \frac{q^{\sum_{m=1}^{p-1} N_m^2}}{\prod_{m=1}^{p-1} (q)^{N_m^2} q^{\sum_{m=1}^{p-1} N_m}}. \]

Inserting (4.21) in (4.15), we obtain (4.14).

Using relation (2.4) with $\theta = p_\ell$, we find the characters of the spectral-flow transformed spaces as

\[ (4.22) \text{char } W_{r,p}(t)(z, q) = z^{-(p-2)t} q^{-\frac{1}{2}(p^2 - 1)} \text{char } W_{r,p}(0)(z q^{-p_\ell}, q) \]

\[ = q^{-\frac{1}{2}(p^2 - 1)} \sum_{n \geq -(p-2)t} (N_1 \geq \cdots \geq N_{p-2} \geq 0) \sum_{N_1 + \cdots + N_{p-2} = n + (p-2)t} q^{-p_\ell n + (p-2)t} z^{n - r - 1} \]
contributing exactly configurations of crosses on \( p \) consecutive lattice sites carry at most \( p - 2 \) crosses.

We first consider the same restriction on configurations on a lattice with a finite number of sites \( L \), with the sites labeled by \( 1 \leq i \leq L \). Each configuration of crosses at the sites \( (i_1, \ldots, i_m) \) contributes \( z^m q^{i_1+\ldots+i_m} \) to the partition function. Let \( \omega_{L,r,p}(z,q) \) be the partition function of all the configurations of crosses on \( L \) sites satisfying the conditions that

- **C1.** there are no more than \( r - 1 \) crosses at the sites \( 1, \ldots, r \);
- **C2.** on any \( p \) consecutive sites, there are no more than \( p - 2 \) crosses.

For \( r = p - 1 \), the first condition follows from the second. The configurations of crosses encoding the thin basis elements are then recovered in the \( L \to \infty \) limit. Remarkably, we have the following lemma.

**Lemma 4.2.** The partition function of the configurations of crosses satisfying conditions **C1** and **C2** on a semi-infinite lattice is equal to the character \( \text{char} W_{r,p}(0)(z,q) \) defined in Lemma 4.1.

\[
\omega_{L,r,p}(z,q) = z^{-\frac{r-1}{p}} \lim_{L \to \infty} \omega_{L,r,p}(z,q) = \text{char} W_{r,p}(0)(z,q).
\]

**Proof.** The idea of the proof is as follows. Expanding both sides of (4.23) in powers of \( z \) gives two systems of functions, each of which is completely determined by a set of recursive relations and the appropriate “initial values.” These two sets of recursive relations and the initial values are identical, which implies (4.23). We now proceed with the details.

The partition function of the configurations of crosses satisfies the recursive relation

\[
\omega_{L,r,p}(z,q) = \sum_{j=0}^{r-1} z^j q^{\frac{j(j+1)}{2}} \omega_{L-j-1,p-j-1,p}(zq^{j+1},q).
\]

Indeed, condition **C1** selects those configurations that are the disjoint union \( \bigcup_{j=0}^{r-1} \) of configurations with exactly \( j \) occupied sites (and the site \( j + 1 \) free). For each \( j \), cutting off these occupied sites
and the adjacent free site leaves configurations of crosses on $L - j - 1$ sites; these configurations satisfy a “boundary condition” that keeps track of the crosses at the sites $1, \ldots, j$ on the original lattice: by $C_2$ there can be no more than $p - j - 2$ crosses in the beginning of the sublattice. The overall factor $z^j q^{(j+1)/2}$ in (4.24) is precisely the contribution of the crosses at $1, \ldots, j$, and the “spectral flow” transformation $z \mapsto zq^{j+1}$ in the argument accounts for relabeling the sites on the sublattice. This shows (4.24).

The initial conditions for the recursive are set on lattices with $L \leq p - 2$, where condition $C_2$ does not apply, and therefore,

$$\omega_{L,r,p}(z, q) = \chi_{L-1,p}(zq, q) + \sum_{j=1}^{r-1} z^j q^{j(j+1)/2} \chi_{L-j-1,p}(zq^{j+1}, q),$$

where $L \leq p - 2$ and

$$\chi_{\ell,p}(z, q) = \sum_{i=0}^{\ell} z^i q^{i(i+1)/2} \begin{bmatrix} \ell \\ i \end{bmatrix}_q = (-qz)_\ell \quad \text{for} \quad 0 \leq \ell \leq p - 2$$

(and $\chi_{0,p}(z, q) = 1$). With these initial conditions, recursive relations (4.24) completely determine $\omega_{L,r,p}(z, q)$.

In the limit as $L \to \infty$, we expand the partition function in powers of $z$ as

$$\lim_{L \to \infty} \omega_{L,r,p}(z, q) = \sum_{n \geq 0} z^n q^{(n-1)/2} B^n_{r,p}(q).$$

It then follows from (4.24) that

$$B^n_{r,p}(q) = \sum_{j=0}^{r-1} q^n B_{p-j-1,p}^m(q)$$

(for $n < r - 1$, the summation on the right-hand side goes from 0 to $n$; it is convenient to set $B^n_{r,p}(q) = 0$ for $n < 0$, which allows Eq. (4.28) to be used in all cases). These can be rewritten as recursive relations expressing $B^n_{r,p}(q)$ through $B^m_{r,p}(q)$ with $m < n$. The initial values for these recursive relations are given by $B^m_{r,p}$ for $m < p - 1$, where condition $C_2$ does not apply. We already saw in (4.25) and (4.26) how to evaluate $\omega_{L,r,p}(z, q)$ on a lattice where condition $C_2$ is not imposed, and it only remains to take the $L \to \infty$ limit of (4.25). Let $P$ denote the projector on the space of polynomials in $z$ of the order $< p - 1$. We then have

$$\sum_{m=0}^{p-2} z^m q^{m(n-1)/2} B^m_{r,p}(q) = P \sum_{j=0}^{r-1} z^j q^{j(j+1)/2} (-zq^{j+2})_\infty =$$

$$= P \sum_{j=0}^{r-1} z^j q^{j(j+1)/2} \left( 1 + \sum_{n \geq 1} \frac{(zq^{j+2})^n q^{n(n-1)/2}}{(1-q) \ldots (1-q^n)} \right) =$$

$$= P \sum_{n \geq 0} z^n q^{n(n-1)/2} \sum_{j=0}^{r-1} q^{2n-j} \sum_{j \leq n} \frac{1}{(1-q) \ldots (1-q^{n-j})}. $$
The inner sum in the right-hand side therefore defines the initial values $B_{n}^{n}(q)$ for $n \leq p - 2$.

Turning to the character of $W_{r,p}(0)(z, q)$, we recall expansion (4.13), where, as we have seen,

\[ Z_{r,p}^{n}(q) = \sum_{n_{1},..,n_{p-2} \geq 0} \frac{q^{N_{1}^{2}+..+N_{p-2}}}{(q)_{n_{1}}..(q)_{n_{p-2}}} \]

with $N_{i} = n_{i} + \cdots + n_{p-2}$; these expressions satisfy the relation [7]

\[ Z_{r,p}^{n}(q) = \sum_{j=0}^{r-1} q^{n} Z_{r-j, p}^{n-j}(q). \]

Indeed, using the notation in [4], Eq. (4.30) becomes $Z_{i,p}^{n}(q) = R_{p-1,i}^{n}(q)$, where the generating functions $\sum_{n \geq 0} x^{n} R_{\kappa,i}^{n}(q) = R_{\kappa,i}(x; q) = J_{\kappa,i}(0; x; q)$ are known to satisfy the identities

\begin{align*}
R_{\kappa,i}(x; q) - R_{\kappa,i-1}(x; q) &= (xq)^{i-1} R_{\kappa,\kappa-i+1}(xq; q) \quad \text{for} \quad 1 \leq i \leq \kappa, \\
R_{\kappa,0}(x; q) &= 0.
\end{align*}

Summing these relations (with $\kappa = p - 1$) over $i = 1, \ldots, r - 1$, we obtain (4.31).

To complete the proof, it remains to find the initial values for this recursive, namely, $Z_{r,p}^{m}(q)$ for $m < \kappa$. We have [4]

\[ R_{\kappa,i}(x; q) = \sum_{n \geq 0} x^{n} q^{n+k+n^{2}+n^{-m}} \frac{(1 - x^{i}q^{2n+i})(-1)^{n}q^{n(n-1)}}{(q)_{n}(q^{n+1})}. \]

The coefficients entering the expansion in powers of $x^{j}$ for $j \leq \kappa - 1$ follow only from the term with $n = 0$ in this sum, and therefore,

\begin{align*}
\sum_{n=0}^{\kappa-1} x^{n} R_{\kappa,i}^{n}(q) &= P \frac{1 - x^{i}q^{i}}{(xq)_{\infty}} = P \frac{1 - x^{i}q^{i}}{(1-xq)} \cdot \frac{1}{(1-xq)(1-xq^{2})} = \\
&= P \sum_{m=0}^{i-1} x^{m} q^{m} \left( 1 + \sum_{n \geq 1} \frac{q^{2n}x^{n}}{(1-q)\cdots(1-q^{n})} \right) = \\
&= P \sum_{n \geq 0} x^{n} \sum_{j=0}^{i-1} \frac{q^{2n-j}}{(1-q)\cdots(1-q^{n-j})},
\end{align*}

where the inner sum on the right-hand side determines $R_{\kappa,i}^{n}$ for $n < \kappa$. These are seen to be the same as $B_{n}^{n}$ (obviously, there is no dependence on $\kappa = p - 1$ because it arises only in the higher $B_{r,\kappa}^{n}$ and $Z_{r,\kappa}^{n}$ due to conditions [C2] and [P2]).

We can therefore see that $Z_{r,p}^{m}(q)$ satisfy $Z_{r,p}^{m}(q) = B_{r,p}^{m}(q)$ for $m < p$; by the recursive relations, this implies that $Z_{r,p}^{n}(q) = B_{r,p}^{n}(q)$ for all $n$ and hence Eq. (4.23).

The coincidence of the characters shows that there are no linear relations among thin monomials. This completes the proof of Lemma 3.3 and hence of Theorem 3.2.
4.4. The character of the semi-infinite space. Theorem 3.2 has an important corollary.

Theorem 4.3. The character of the semi-infinite space \( W_{r,p;θ} \) is given by

\[
\text{char} W_{r,p;θ}(z, q) = z^{1 - r + 2\theta \frac{p}{p} - θ} q^{\frac{1}{2} \frac{p^2}{2} - θ - 1 - r \frac{p}{p}} \times \\
\times \sum_{N_1 ≥ ... ≥ N_{p-2} ∈ \mathbb{Z}} (N_1N_2(q)N_2N_3 ... (q)N_{p-3}N_{p-2}(q)) \times \\
\times q^{\frac{1}{2} (\sum_{m=1}^{p-2} N_m - \sum_{m=1}^{r-1} N_m + \frac{3}{2} \sum_{m=1}^{p-2} N_m^2 + \sum_{1 ≤ m ≤ p-2} N_m N_{m'}}.
\]

Proof. It is a direct consequence of Theorem 3.2 that

\[
\text{char} W_{r,p;θ}(z, q) = \lim_{τ → ∞} \text{char} W_{r,p;θ}(τ)(z, q) = \\
= \lim_{τ → ∞} z^{-(p-2)θ} q^{\frac{1}{2} (p^2 - 1)} \text{char} W_{r,p;θ}(0)(zq^{-p}, q),
\]

where we applied the spectral flow transform formula (2.3) in the last equality. Next, a remarkable property of Eq. (4.22) is that it has a well-defined limit as \( τ → ∞ \),

\[
\lim_{τ → ∞} \text{char} W_{r,p;θ}(τ)(z, q) = z^{1 - r + 2\theta \frac{p}{p} - θ} q^{\frac{1}{2} \frac{p^2}{2} - θ - 1 - r \frac{p}{p}} \times \\
\times \sum_{n = -∞}^{+∞} \sum_{N_1 ≥ ... ≥ N_{p-2} ∈ \mathbb{Z}} \sum_{N_1 + ... + N_{p-2} = n} q^{\frac{1}{2} n^2 - θ n + \sum_{m=1}^{p-2} N_m^2 + \frac{1}{2} \sum_{m=1}^{p-2} N_m} \\
\times (q)N_1N_2(q)N_2N_3 ... (q)N_{p-3}N_{p-2}(q) \times
\]

(where the \( θ \) dependence is reconstructed in accordance with the spectral flow).

Remark 4.4. The existence of the thin basis has the following implication for the construction of antisymmetric polynomials satisfying the basic conditions (4.3). Let \( C(x_1, ..., x_n) \) be polynomials in \( C(x_1, ..., x_n) \) satisfying (4.3). There is an associative multiplication on antisymmetric polynomials: for \( f ∈ C(x_1, ..., x_n) \) and \( g ∈ C(x_1, ..., x_n') \), we define \( f * g ∈ C(x_1, ..., x_{n+n'}) \) as

\[
(f * g)(x_1, ..., x_{n+n'}) = \text{Alt}_{x_1, ..., x_{n+n'}} \left( f(x_1, ..., x_n)g(x_{n+1}, ..., x_{n+n'}) \prod_{1 ≤ i ≤ n, n+1 ≤ i' ≤ n+n'} (x_i - x_{i'}) \right),
\]

where \( \text{Alt} \) means alternation. For \( n = 2 \) and \( n' = 1 \), for example,

\[
(f * g)(x_1, x_2, x_3) = f(x_1, x_2)g(x_3)(x_1 - x_3)(x_2 - x_3) + \\
+ f(x_2, x_3)g(x_1)(x_2 - x_1)(x_3 - x_1) - f(x_1, x_3)g(x_2)(x_1 - x_2)(x_3 - x_2).
\]

Now, if \( f_1 ∈ C(x_1, ..., x_{n_1})^{(p_1)} \) and \( f_2 ∈ C(x_1, ..., x_{n_2})^{(p_2)} \), it is easy to see that the polynomial \( f_1 * f_2 \) is in \( C(x_1, ..., x_{n_1+n_2})^{(p_1+p_2-2)} \), i.e.,

\[
* : C(x_1, ..., x_{n_1})^{(p_1)} × C(x_1, ..., x_{n_2})^{(p_2)} → C(x_1, ..., x_{n_1+n_2})^{(p_1+p_2-2)}.
\]
As a consequence of the thin-basis lemma, we have a stronger statement: the mapping
\[(4.42) \quad \ast : \bigoplus_{n_1, n_2 \geq 1} \mathbb{C}\langle x_1, \ldots, x_{n_1}\rangle^{(p_1)} \times \mathbb{C}\langle x_1, \ldots, x_{n_2}\rangle^{(p_2)} \to \mathbb{C}\langle x_1, \ldots, x_n\rangle^{(p_1+p_2-2)} \]
is an epimorphism. This construction is useful because for \(p = 3\), the polynomials satisfying (4.3) are known explicitly (see (I.9)).

4.5. Semi-infinite construction for the \(\hat{s}\ell(2)\) algebra \([1]\). The level-\(k\) \(\hat{s}\ell(2)\) algebra
\[(4.43) \quad [h_i, e_n] = e_{n+i}, \quad [h_i, f_n] = -f_{n+i}, \quad [e_i, f_j] = 2h_{i+j} + k\delta_{i+j,0}, \quad [h_i, h_j] = \frac{1}{2}k\delta_{i+j,0}
\]
admits a semi-infinite realization of any unitary representation \(\hat{\mathcal{L}}_{r,k}\) (where \(1 \leq r \leq k + 1\)). The key elements of the construction are the semi-infinite forms in commuting elements \((f_n)_{n \in \mathbb{Z}}\) satisfying the constraints following from the conditions
\[(4.44) \quad f(z)^{k+1} = 0 \quad \text{for} \quad f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n-1}.
\]
The semi-infinite space is generated by \((f_n)_{n \in \mathbb{Z}}\) from the vectors \(|r, k|\rangle_{\hat{s}\ell(2)}\) such that \(f_{2i+1}|r, k|\rangle_{\hat{s}\ell(2)} = 0, \quad i \geq 1, \) and
\[(4.45) \quad (f_{2i})^r|r, k|\rangle_{\hat{s}\ell(2)} = 0,
(4.46) \quad (f_{2i-1})^{k-r+1}(f_{2i})^{r-1}|r, k|\rangle_{\hat{s}\ell(2)} = |r, k|\rangle_{\hat{s}\ell(2)}(r - 1).\]

In more formal terms, there is a theorem parallel to Theorem \([1]\). Let \(F(k)\) be the algebra generated by the elements \((f_n)_{n \in \mathbb{Z}}\) modulo the relations following (4.44) and by an invertible operator \(U\) such that \(Uf_nU^{-1} = f_{n+1}\).

**Theorem 4.5.** Let \(M\) be the representation of the algebra \(F(k)\) induced from the trivial one-dimensional representation of the algebra of \((f_n)_{n \geq 0}\) (on the vacuum vector \(|0\rangle\)). Let \(C_r\) (\(1 \leq r \leq k + 1\)) be the \(F(k)\)-submodule generated from the vector \(f_{-1}^r|0\rangle\), and \(N_{r,k}\) the \(F(k)\)-submodule generated from the set of vectors
\[(4.47) \quad f_{\alpha-2}^{k-r+1}f_{\alpha-1}^{r-1}|\alpha\rangle = |\alpha - 2\rangle, \quad \alpha \in \mathbb{Z},
\]
where \(|\alpha\rangle = U^\alpha|0\rangle\). The quotient space \(M/(N_{r,k} + C_r)\) is a representation of the \(\hat{s}\ell(2)_k\) algebra and, moreover, is isomorphic to a direct sum of unitary \(\hat{s}\ell(2)_k\) representations,
\[(4.48) \quad \mathcal{M}(r, k) \equiv \frac{M}{N_{r,k} + C_r} = \bigoplus_{\theta = 0}^1 \mathcal{L}_{r,k;\theta} = \mathcal{L}_{r,k} \oplus \mathcal{L}_{k+2-r,k}.
\]

In the semi-infinite \(\hat{s}\ell(2)_k\) space, there also exists a monomial basis consisting of “thin” monomials; the above proof applies with minimal modifications. The construction of the \(N = 2\) algebra action on the semi-infinite space can be easily carried over to the case of the \(\hat{s}\ell(2)\) algebra (see Sec. 5.3).
A priori, the conditions imposed on the semi-infinite construction do not suggest that the space is 
a representation of any algebra; for the constraints (1.1), however, this representation can be found.

**Theorem 5.1.** The semi-infinite space $W_{r,p;\theta}$ is a module over the $N=2$ algebra.

The problem with constructing the $N=2$ action on $W_{r,p;\theta}$ is nontrivial because we must define the 
action of $Q_n$, $L_n$, and $H_n$ on the states $G_{i_1}G_{i_2}\cdots G_{i_m}|r,p;\theta|\iota\rangle_{\infty/2}$ constructed only from $G_n$ and show that this action can be pushed forward to the quotient with respect to the ideal $S^p$ generated by $S^p_a$. The action of the $N=2$ algebra on $W_{r,p;\theta}$ is defined in several steps. The main tool here is the positive filtration on $W_{r,p;\theta}$ by finite-dimensional subspaces $W_{r,p;\theta}^\iota$ (similar to the Demazure modules, see [14]–[40]) that allows rewriting any element of $W_{r,p;\theta}$ as a linear combination of semi-infinite forms involving only nonnegative modes $G_{n\geq 0}$. The next problem consists in taking the quotient, i.e., in verifying that the action is independent of the chosen representative of a state written in terms of nonnegative $G$-modes. In Sec. 5.2, we define the action of a part of the $N=2$ generators using differential operators acting on finite-dimensional subspaces whose quotients are the subspaces in the positive filtration. To prove that these differential operators can be pushed forward to the quotient, we take the dual space and use the functional realization (Sec. 5.3). In Sec. 5.4, we then show that the action of the entire $N=2$ algebra on the entire semi-infinite space can be obtained by consistently gluing together the “partial” actions on the subspaces. In particular, this gives the action of $Q_{n\geq 0}$, together with $G_{n\leq 0}$, which act on the semi-infinite space by definition, these generate the entire $N=2$ algebra, and it only remains to verify that their action on $W_{r,p;\theta}$ is precisely the $N=2$ algebra action. We show that the necessary relations are satisfied on any vector from the semi-infinite space.

5.1. The positive filtration. Let $W_{r,p;\theta}^\iota \subset W_{r,p;\theta}$ be the subspace generated from the extremal state $|r,p;\theta|\iota\rangle_{\infty/2}$ by $G_{\geq 0} \in g(p)$. Relations (3.4) with $\iota \geq 1$ determine the sequence of embeddings

\[(5.1) \quad W_{r,p;\theta}^\iota[0] \subset \cdots \subset W_{r,p;\theta}^\iota[\iota] \subset W_{r,p;\theta}^\iota[\iota+1] \subset \cdots .\]

**Lemma 5.2.** Sequence (5.1) is a filtration on the space $W_{r,p;\theta}$.

Therefore, each state in $W_{r,p;\theta}$ can be represented as a linear combination of monomials involving only nonnegative modes $G_{n\geq 0}$.

**Proof.** Abusing the terminology, we say a “semi-infinite form $G_{i_m}\cdots G_{i_1}|r,p;\theta|\iota\rangle_{\infty/2}” meaning in fact its representative in the freely generated space whose quotient is $W_{r,p;\theta}$. The statement of the lemma is that each semi-infinite form has a representative expressed only through nonnegative modes $G_{n\geq 0}$, i.e., a representative of each state $G_{i_m}\cdots G_{i_1}|r,p;\theta|\iota\rangle_{\infty/2} \in W_{r,p;\theta}$ can be chosen from some $W_{r,p;\theta}^\iota[\iota']$. If $\iota < 0$, Eq. (3.4) allows us to rewrite the state as $G_{i_m}\cdots G_{i_1}$.
Remark 5.3. The choice of the modes $G_{i_{a}}$ is a matter of convention; for any $\mu \geq 0$, there exists a similar filtration in terms of the spaces generated by $G_{i \geq \mu}$. We denote it as

$$ \mathcal{W}_{r,p,\theta}^{(\mu)}[t] \subset \cdots \subset \mathcal{W}_{r,p,\theta}[\mu+1] \subset \mathcal{W}_{r,p,\theta}[\mu] \subset \cdots. $$

In this notation, the above positive filtration corresponds to $\mu = 0$, $\mathcal{W}_{r,p,\theta}[t] \equiv \mathcal{W}_{r,p,\theta}^{(0)}[t]$ (see Fig. 3). These filtrations are crucial for constructing the $N=2$ action.

We give several examples of rewriting semi-infinite forms in terms of positive modes. We recall that the $|r,p;\theta|_{i}\rangle$ states are defined in (3.13)–(3.4). For $p = 3$ and $r = 1$, we have $|1,3;0|_{-1} = (1/24)G_{1}G_{2}G_{3}G_{4}|1,3;0\rangle - (3/40)G_{1}G_{2}G_{3}G_{6}G_{8}|1,3;0\rangle$. Complexity of the expressions involving only
positive modes grows very fast, as, for example, we see from

\begin{equation}
(5.4) \quad G_{-4}|1,3;00\rangle = -\frac{6}{125} G_1 G_2 G_3 G_4 G_5 |1,3;04\rangle - \frac{818}{17525} G_1 G_2 G_3 G_5 G_7 |1,3;04\rangle + \\
\quad + \frac{1}{2475} G_1 G_2 G_4 G_6 G_0 |1,3;04\rangle - \frac{600}{17525} G_1 G_2 G_3 G_4 G_6 G_12 |1,3;05\rangle + \frac{1248}{1575175} G_1 G_2 G_3 G_5 G_9 G_{11} |1,3;05\rangle + \\
\quad + \frac{342}{2275} G_1 G_2 G_3 G_4 G_6 G_9 G_{11} |1,3;05\rangle + \frac{24760}{175175} G_1 G_2 G_3 G_6 G_9 G_{10} |1,3;05\rangle + \frac{21}{3575} G_1 G_2 G_3 G_5 G_6 G_{0} |1,3;05\rangle - \\
\quad - \frac{10}{3185} G_1 G_3 G_4 G_6 G_0 |1,3;05\rangle + \frac{4808}{175175} G_1 G_2 G_3 G_6 G_9 G_7 |1,3;05\rangle - \frac{3}{3185} G_1 G_2 G_3 G_5 G_6 G_9 |1,3;05\rangle - \\
\quad - \frac{6}{2275} G_1 G_2 G_3 G_6 G_7 G_6 |1,3;05\rangle + \frac{9280}{980} G_1 G_2 G_3 G_5 G_6 G_9 G_{12} G_{15} |1,3;06\rangle - \frac{17140}{980} G_1 G_2 G_3 G_5 G_6 G_{9+12} G_{14} |1,3;06\rangle - \\
\quad - \frac{17}{1960} G_1 G_3 G_5 G_6 G_9 G_{11} G_{12} |1,3;06\rangle + \frac{9280}{980} G_1 G_2 G_3 G_5 G_6 G_9 G_{10} G_{12} |1,3;06\rangle.
\end{equation}

We thus obtain states belonging to $W_{1,3,0}^+(\ell = 6)$.

### 5.2. Differential operators for generators on subspaces

To prove Theorem 5.1, we first construct the action of a part of the $N=2$ generators on each space $W_{r,p,\theta}^{(\mu)}[\ell]$ involved in (5.3) (see Fig. 2 for $\mu = 0$). In $W_{r,p,\theta}^{(\mu)}[\ell]$, we define the action of the operators $Q_{-\mu}, Q_{\mu+1}, \ldots, Q_{p+\theta-1}$, $L_1, L_2, \ldots, H_1, H_2, \ldots$, and $G_r, G_{\mu+1}, \ldots, G_{p+\theta-1}$ (the latter act tautologically), which eventually becomes a part of the $N=2$ algebra action on the entire semi-infinite space $W_{r,p,\theta}$.

As the first step, we “standardize” the spaces by applying the spectral flow mapping each $W_{r,p,\theta}^{(\mu)}[\ell]$ into the space $V_{r,p}^N$ generated by $G_{-1}, \ldots, G_{-N}$. Let $V(N)$ (where $N$ is a positive integer) be the subspace in $W(0)$ generated by $G_{-1}, G_{-2}, \ldots, G_{-N}$ from the corresponding vacuum vector $|\rangle$. As in Sec. 4.1, $W(0)$ denotes the freely generated space, and $W_{r,p}(0)$ is the quotient of $W(0)$ with respect to the subspace $I_{r,p}|\rangle$, where $I_{r,p}$ is the ideal generated by the elements

\begin{equation}
(5.5) \quad S_a = \sum_{n_0 \leq \cdots \leq n_{p-2}, a = \sum_{n_0 + \cdots + n_{p-2} = a}} \left( \prod_{i < j} (n_i - n_j) \right) G_{n_0} \cdots G_{n_{p-2}}, \quad a = -\frac{p(p-1)}{2}, -\frac{p(p-1)}{2} - 1, \ldots,
\end{equation}

and by the element $S_r = G_{r-r+1} \cdots G_{-1}$. We define $V_{r,p}^N$ as the subspace in $W_{r,p}(0)$ generated by the modes $G_{-1}, \ldots, G_{-N}$ from the vacuum vector $|\rangle \equiv |r,p;00\rangle$. Therefore, $V_{r,p}^N = V(N)/I_{r,p}(N)$, where $I_{r,p}(N) = V(N) \cap I_{r,p}$ (and we write $I_{r,p} \equiv I_{r,p}|\rangle$), or equivalently,

\begin{equation}
(5.6) \quad 0 \longrightarrow I_{r,p} \longrightarrow V(0) \longrightarrow W_{r,p}(0) \longrightarrow 0
\end{equation}

We let $\partial_n$ denote the operator $\partial/\partial G_n$ acting on the freely generated space $W(0)$ and its subspaces. Although the operators $\partial_n$ certainly do not act on the quotient space $V_{r,p}^N$, there are differential operators constructed from $\partial_n$ and $G_m$ that do, and these are a part of the $N=2$ generators.

**Lemma 5.4. The differential operators**

\begin{equation}
(5.7) \quad Q_{\ell}^{(r,p,N)} = q^{(r,p,N)} \delta_{\ell,N} \partial_{-N} + \sum_{n,m = -N}^{n+m = -N} (m-n) G_{\ell+n+m} \partial_n \partial_m, \quad \ell = N, N+1, \ldots, 2N-1,
\end{equation}
\[ L_n^{(r,p,N)} = \sum_{m=-N}^{n-1} (n-m) \mathcal{G}_{n+m} \partial_m + l^{(r,p,N)} \delta_{n,0}, \quad n \geq 0, \]
\[ H_n^{(r,p,N)} = \sum_{m=-N}^{-1} \mathcal{G}_{n+m} \partial_m + h^{(r,p,N)} \delta_{n,0}, \]

where (for an arbitrary \( h^{(r,p,N)} \))
\[ l^{(r,p,N)} = \frac{p-2}{p} N + Nh^{(r,p,N)} - Np - r - 1, \]
\[ q^{(r,p,N)} = \frac{p-2}{p} (N^2 + N) - 2Np - 1 - r \]

have a well-defined action on the space \( V^N_{r,p} \). Together with \( \mathcal{G}_\ell \) for \( \ell \geq -N \), these operators satisfy the corresponding commutation relations in (2.1).

The spaces \( W^+_{r,p;\theta}[\ell] \) involved in the positive filtration are related to \( V^N_{r,p} \) with \( N = \theta \) by spectral flow transformations, namely,
\[ U_{-r-p-\theta} W^+_{r,p;\theta}[\ell] \simeq V^{r+p+\theta}_{r,p}. \]
These isomorphisms allow us to carry the operator action over to the above spaces \( W^+_{r,p;\theta}[\ell] \).

The main complication in proving Lemma 5.4 is rooted in the fact that the ideal \( I_{r,p}^{(N)} \subset V(N) \) is not described explicitly (cf. the remarks before Theorem 3.2). The idea of the (quite lengthy) proof of Lemma 5.4 is expressed in a more compact form in Sec. 5.5 for the \( \hat{sl}(2) \) algebra.

5.3. Duals to differential operators on functional spaces. To prove Lemma 5.4, we first note that \( q^{(r,p,N)} \) is fixed by the requirement that \( Q^{(r,p,N)}_N \) preserve the relations \( \partial^N_a = 0 \) (which can be seen by directly applying the differential operators); the form of \( l^{(r,p,N)} \) is chosen such that the commutator \( [Q^{(r,p,N)}_N, Q^{(r,p,N)}_{-N}] \) is the same as in the \( N=2 \) algebra. However, \( h^{(r,p,N)} \) is still arbitrary; moreover, the central term arising in the last commutator can be absorbed in \( L_0^{(r,p,N)} \) and \( H_0^{(r,p,N)} \), and therefore, the \( N=2 \) central charge is still undetermined. Now, the operators \( \mathcal{G}_\ell^{(r,p,N)} \), \( \mathcal{Q}_\ell^{(r,p,N)} \), \( L_\ell^{(r,p,N)} \), and \( H_\ell^{(r,p,N)} \) preserve the subspace \( V(N) \), and it must be verified that they also preserve the ideal \( J_{r,p} \). We replace this with the dual statement and apply the functional realization in Sec. 4.1.
Dualizing (5.6), we obtain

\[ 0 \longrightarrow W_{r,p}(0)^* \longrightarrow W(0)^* \longrightarrow \mathcal{J}_{r,p}^* \longrightarrow 0 \]

(5.12)

\[ 0 \longrightarrow V_{r,p}^N \longrightarrow V(N)^* \longrightarrow \mathcal{J}_{r,p}(N)^* \longrightarrow 0 \]

The property of the operators to have a well-defined action on the quotient then reformulates as the condition for their duals to preserve the subspace \( W_{r,p}(0)^* \) of antisymmetric polynomials satisfying conditions (4.5) and (4.6) modulo the subspace \( \mathcal{J}(N) \) spanned by the monomials \( x_1^{i_1} \ldots x_n^{i_n} \) with \( i_j \geq N \) for at least one \( j \) (more precisely, modulo the intersection of this subspace with the space of skew-symmetric polynomials).

As before, we let \( \mathbb{C}\langle x_1, \ldots, x_M \rangle^{(p)} \) denote polynomials in \( \mathbb{C}\langle x_1, \ldots, x_M \rangle \) satisfying condition (4.5). We now establish that for \( f \in \mathbb{C}\langle x_1, \ldots, x_M \rangle^{(p)} \),

\[ \mathcal{K}_{n}^{(r,p,N)} f \in \mathbb{C}\langle x_1, \ldots, x_M \rangle^{(p)}, \quad \mathcal{K}_n = \mathcal{H}_n \text{ or } \mathcal{L}_n, \]

while \( \mathcal{Q}_\ell \) preserve the conditions modulo \( \mathcal{J}(N) \),

\[ \mathcal{Q}_\ell^{(r,p,N)} f \in \mathbb{C}\langle x_1, \ldots, x_{M+1} \rangle^{(p)} + \mathcal{J}(N), \]

(5.14)

and hence the expression

\[ \frac{\partial}{\partial x_2} \frac{\partial^2}{\partial x_3^2} \ldots \frac{\partial^{p-2}}{\partial x_{p-1}^{p-2}} \left( \mathcal{Q}_\ell^{(r,p,N)} f \right) (x_1, \ldots, x_{M+1}) \bigg|_{x_1=x_2=\ldots=x_{p-1}} \]

vanishes modulo \( \mathcal{J}(N - p + 2) \). For this, we derive a suitable form of the dual operators.

The operators dual to \( \mathcal{G}_\ell \) act between the \( M \)-variable subspaces in \( W(0)^* \) as

\[ \mathbb{C}\langle x_1, \ldots, x_{M-1} \rangle \ dx_1 \ldots dx_{M-1} \xrightarrow{\mathcal{G}_\ell^*} \mathbb{C}\langle x_1, \ldots, x_M \rangle \ dx_1 \ldots dx_M, \]

(5.16)

\[ \left( \mathcal{Q}_{\ell}^{(r,p,N)} f \right) (x_1, \ldots, x_{M-1}) = -\frac{1}{(\ell - 1)!} \left. \frac{\partial^{\ell-1}}{\partial x_M^{\ell-1}} f(x_1, \ldots, x_M) \right|_{x_M=0}. \]

(5.17)

The duals to \( \partial_\ell = \partial/\partial \mathcal{G}_{-\ell} \) are the pairwise anticommuting differentials

\[ \mathbb{C}\langle x_1, \ldots, x_M \rangle \ dx_1 \ldots dx_M \xrightarrow{\partial_{\ell}} \mathbb{C}\langle x_1, \ldots, x_{M+1} \rangle \ dx_1 \ldots dx_{M+1}, \]

acting as

\[ (\partial_{\ell} f)(x_1, \ldots, x_{M+1}) = \sum_{j=1}^{M+1} (-1)^{M+1-j} f([x_j]^{(M+1)})(1) \]

(5.18)

\[ (\partial_{-\ell}^* f)(x_1, \ldots, x_{M+1}) = \sum_{j=1}^{M+1} (-1)^{M+1-j} f([x_j]^{(M+1)})(1) \]

(5.19)
where we use the notation
\[(5.20) \quad [x]^{(M)} = (x_1, \ldots, x_M), \quad [x]^{(M)}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M),\]
\[(5.21) \quad [x]^{(M)}_{ij} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_M), \quad 1 \leq i < j \leq M.\]

For \(f \in \mathbb{C}(x_1, \ldots, x_M)\), we then have, for example,
\[(5.22) \quad (\partial^*_m \partial^*_n f)(x_1, \ldots, x_{M+2}) = \sum_{1 \leq i < j \leq M+2} (-1)^{i+j+1} f([x]^{(M+2)}_{ij})(x_i^{m-1} x_j^{n-1} - x_i^{m-1} x_j^{n-1}).\]

The dual action of \(H_n\) and \(L_n\) is given by
\[(5.23) \quad H^*_n f = (p_n + h^{(r,p,N)}_n \delta_{n,0}) \cdot f,\]
\[(5.24) \quad (L^*_n f)(x_1, \ldots, x_M) = \left( (2n+1)p_n(x_1, \ldots, x_M) + \sum_{j=1}^M x_j^{n+1} \frac{\partial}{\partial x_j} \right) f(x_1, \ldots, x_M),\]

where \(g \cdot h\) denotes the “pointwise” multiplication (of a symmetric and an antisymmetric polynomial) and we introduce the symmetric polynomials
\[(5.25) \quad p_r(x_1, \ldots, x_M) = x_1^r + \cdots + x_M^r.\]

It is obvious that the operators \(H_n\) with \(n \geq 0\) map \(W_{r,p}(0)^*\) into itself. For \(L_n\), this statement is verified as follows. We split the summation \(5.24\) as
\[(5.26) \quad \sum_{j=1}^M \frac{\partial}{\partial x_j} f(x_1, \ldots, x_M) = \left( \sum_{j=1}^{p-1} + \sum_{j=p}^M \right) \frac{\partial}{\partial x_j} f(x_1, \ldots, x_M).\]

Each term in the second sum in the right-hand side of \(5.26\) preserves the conditions
\[(5.27) \quad \frac{\partial}{\partial x_2} \frac{\partial^2}{\partial x_3^2} \cdots \frac{\partial^{p-2}}{\partial x_{p-1}^{p-2}} f \Big|_{x_1=x_2=\cdots=x_{p-1}} = 0.\]

To the first sum, we apply the operator
\[(5.28) \quad \frac{\partial}{\partial x_2} \frac{\partial^2}{\partial x_3^2} \cdots \frac{\partial^{p-2}}{\partial x_{p-1}^{p-2}},\]
commute it with
\[(5.29) \quad \sum_{j=1}^{p-1} x_j^{n+1} \frac{\partial}{\partial x_j}\]
and restrict the expression obtained to the \((p-1)\)-diagonal \(x_1 = x_2 = \cdots = x_{p-1} = x\), after which the sum
\[(5.30) \quad \sum_{j=1}^{p-1} x_j^{n+1} \frac{\partial}{\partial x_j} = x^{n+1} \sum_{j=1}^{p-1} \frac{\partial}{\partial x_j}.\]
becomes the derivative along the diagonal, and hence,

\[ \sum_{j=1}^{p-1} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_2} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^{p-2}}{\partial x_{p-1}^{p-2}} f \bigg|_{x_1=x_2=\cdots=x_{p-1}} = 0. \]

Property (5.13) is thus established for \( L_n \).

Some more work is required with the \( Q^{(r,p,N)^*}_\ell \) operators. They act in the same direction as the differentials in (5.18); using (5.17) and (5.22), we find

\[ (5.32) \quad (Q^{(r,p,N)^*}_\ell f)(x_1, \ldots, x_{M+1}) = q^{(r,p,N)} \delta_{\ell,N} \sum_{j=1}^{M+1} (-1)^{M+1-j} f([x]^{(M+1)}_j) x_j^{\ell-1} + \]

\[ + \sum_{1 \leq i < j \leq M+1} (-1)^{i+j} \sum_{m,n=1}^{N} \left( x_j \frac{\partial}{\partial x_j} - x_i \frac{\partial}{\partial x_i} \right) \left( x_i^{n-1} x_j^{m-1} + x_i^{m-1} x_j^{n-1} \right) \frac{D_{M+m-\ell-1} f([x]^{(M+1)}_j)}{D_{M}^{n+m-\ell-1}} \bigg|_{x_1=x_2=\cdots=x_{M+1}} \]

where \( D_{\ell} f(\xi_1, \ldots, \xi_M) = \frac{d}{\partial t} f(\xi_1, \ldots, \xi_{i-1}, \xi_i + t, \xi_{i+1}, \ldots, \xi_M) \bigg|_{t=0} \). Since we are only interested in terms modulo \( J(N) \), we can bring the last equation to a much more tractable form by adding suitable terms from \( J(N) \). For \( \ell \geq N \) as in the condition of the lemma, the Taylor series can be completed by terms from \( J(N) \), and \( Q^{(r,p,N)^*}_\ell \) can therefore be redefined modulo \( J(N) \) to (omitting the \((M+1)^{\text{st}}\) superscript for brevity)

\[ (Q^{(r,p,N)^*}_\ell f)(x_1, \ldots, x_{M+1}) = q^{(r,p,N)} \delta_{\ell,N} \sum_{j=1}^{M+1} (-1)^{M+1-j} f([x]^{(M+1)}_j) x_j^{\ell-1} + \]

\[ + \sum_{1 \leq i < j \leq M+1} (-1)^{i+j} \left( x_j \frac{\partial}{\partial x_j} - x_i \frac{\partial}{\partial x_i} \right) \left( F_{\ell}(x_i, x_j) \left( f([x]_{ij}, x_i) + f([x]_{ij}, x_j) \right) \right), \]

where

\[ F_{\ell}(x, y) = \sum_{n=0}^{\ell-1} x^n y^{\ell-1-n} = \frac{x^\ell - y^\ell}{x - y}. \]

Note that for antisymmetric polynomials, \( f([x]^{(M+1)}_{ij}, x_i) = (-1)^{M-i} f([x]^{(M+1)}_j) \) and \( f([x]^{(M+1)}_{ij}, x_j) = (-1)^{M+1-j} f([x]^{(M+1)}_i) \) for \( i < j \). Further, when acting with the derivatives on the sum of two \( f \) polynomials, we obtain

\[ F_{\ell}(x_i, x_j) \left( x_j \frac{\partial}{\partial x_j} - x_i \frac{\partial}{\partial x_i} \right) \left( f([x]_{ij}, x_i) + f([x]_{ij}, x_j) \right) = \]

\[ = (x_j^\ell - x_i^\ell) \sum_{n=1}^{M+i} \frac{(-1)^{M+i}}{n!} (x_j - x_i)^{n-1} \frac{\partial^n i[f]}{\partial x_i^n}([x]_j) \in J(N), \]

where \( i[f](y_1, \ldots, y_M) = y_i D_{\ell} f(y_1, \ldots, y_M) \). Thus, \( Q^{(r,p,N)^*}_\ell f \) can be redefined modulo \( J(N) \) as

\[ (5.34) \quad (Q^{(r,p,N)^*}_\ell f)(x_1, \ldots, x_{M+1}) = \sum_{j=1}^{M+1} (-1)^{M+j} f([x]_j) \left( \sum_{i=1}^{M+1} A_{\ell}(x_j, x_i) - q^{(r,p,N)} \delta_{\ell,N} x_j^{\ell-1} \right) \]


with
\[
A_\ell(x_j, x_i) = \sum_{m=1}^{\ell} (\ell + 1 - 2m)x_j^{\ell-m}x_i^{m-1} = \frac{(\ell - 1)(x_j^{\ell+1} - x_i^{\ell+1}) - (\ell + 1)(x_j^{\ell}x_i - x_i^{\ell}x_j)}{(x_j - x_i)^2}
\]
(5.35)
\[
= (x_j - x_i)B_\ell(x_j, x_i), \quad B_\ell(x_j, x_i) = \sum_{m=1}^{\ell-1} m(\ell - m)x_j^{\ell-m-1}x_i^{m-1}.
\]

This can be expressed in a more “invariant” form as follows. We define the differential \(d_m\) to be \(\partial_{-m-1}\) up to a sign,
\[
(d_m f)(x_1, \ldots, x_{M+1}) = \sum_{j=1}^{M+1} (-1)^j x_j^m f([x]_j), \quad m \geq 0.
\]
(5.36)

Up to a similar conventional sign factor \((-1)^{M+1}\) (which is inessential for the vanishing property we want to show), we finally rewrite (5.34) as
\[
Q_{\ell}(r,p,N)^* f = \sum_{m=1}^{\ell} (\ell + 1 - 2m)p_{m-1} \cdot d_{\ell-m} f - q^{(r,p,N)}\delta_{\ell,N} d_{\ell-1} f.
\]
(5.37)

Equation (5.14) can now be shown by induction on the number of variables \(M\). Assuming that (5.14) holds for all \(f \in \mathbb{C}(x_1, \ldots, x_{M-1})^{(p)}\), we take a polynomial \(f \in \mathbb{C}(x_1, \ldots, x_M)^{(p)}\) and consider
\[
G_m Q_{\ell}(r,p,N)^* f = -Q_{\ell}(r,p,N)^* (G_m f) + L_{m+\ell}^* f - 2\ell H_{m+\ell}^* f + \frac{p-2}{p}(m^2 + m)\delta_{m+\ell,0} f,
\]
(5.38)

where the right-hand side follows from the explicit expressions (most easily, in the form of differential operators). In (5.38), we already know that \(L_{m+\ell}^*\) and \(H_{m+\ell}^*\) map \(\mathbb{C}(x_1, \ldots, x_{M})^{(p)}\) into itself. Moreover, we see from (5.14) that \(G_m f\) is a polynomial in \(M - 1\) variables and, moreover, it belongs to \(\mathbb{C}(x_1, \ldots, x_{M-1})^{(p)}\). By the induction hypothesis, therefore, \(Q_{\ell}(r,p,N)^* (G_m f) \in \mathbb{C}(x_1, \ldots, x_{M})^{(p)} + \mathcal{J}(N)\); thus, the same is true for \(G_m Q_{\ell}(r,p,N)^* f\). This in turn tells us much about \(Q_{\ell}(r,p,N)^* f\): the first \(N - 1\) terms in its Taylor expansion in \(x_{M+1}\) are in \(\mathcal{J}(N) + \mathbb{C}(x_1, \ldots, x_M)^{(p)}\); the remaining terms do not interest us, however, because they certainly are in \(\mathcal{J}(N)\). It only remains to prove the induction base, namely that \(Q_{\ell}(r,p,N)^* f \in \mathbb{C}(x_1, \ldots, x_p)^{(p)} + \mathcal{J}(N)\) for \(f \in \mathbb{C}(x_1, \ldots, x_{p-1})^{(p)}\), which follows from the explicit form (5.37).

Finally, condition (4.10) on polynomials is easiest to consider in terms of the symmetric polynomial corresponding to a given antisymmetric one (see (4.7)); this condition states the vanishing of the polynomial at zero, which can be reformulated in terms of filtration (4.20). It is then easy to directly verify that the operators preserve these vanishing conditions.

5.4. The algebra action on \(W_{r,p;\theta}\) by gluing the pieces together. To complete the proof of Theorem 5.1, we construct the action of the \(N=2\) generators on the entire semi-infinite space \(W_{r,p;\theta}\).
by gluing together the actions constructed on each of the spaces involved in filtration (3.3). This is done in several steps.

Because the dependence on the spectral flow parameter \( \theta \) does not affect the general structure of the results, we temporarily set \( \theta = 0 \) to simplify the formulas. We therefore consider the semi-infinite spaces \( W_{r,p} \equiv W_{r,p;0} \), with the dependence on \( \theta \) to be reconstructed in accordance with the spectral flow. We write \( |r, p; t\rangle_{\infty/2} \) for \( |r, p; 0\rangle_{\infty/2} \).

5.4.1. For a given semi-infinite form \( |x\rangle \in W_{r,p} \) with the charge–level bigrade \( (h, \ell) \), we choose \( \iota \) and \( \mu < \iota p - 1 \) such that all the states in this bigrade lie in \( W_{r,p}^{(\mu)}[t] \). Wishing to define the state \( \mathcal{O}_n|x\rangle \) for \( \mathcal{O} = \mathcal{L}, \mathcal{H}, \text{or} \mathcal{Q} \), we also require that the space \( W_{r,p}^{(\mu)}[\iota \mu] \) contain all the states in the corresponding bigrade (given by \( (h, \ell - a) \) for \( \mathcal{O} = \mathcal{L} \) or \( \mathcal{H} \) and \( (h - 1, \ell - a) \) for \( \mathcal{Q} \). This can be always achieved by increasing \( \iota \) or \( \mu \).

Similarly to (5.11), the spectral flow transform can be used to map the state \( |r, p; t\rangle_{\infty/2} \) into \( |r, p; 0\rangle_{\infty/2} \) and all the subspaces \( W_{r,p}^{(\mu)}[\iota \mu] \) into the corresponding spaces \( V_{r,p}^{N} \) in Sec. 5.2.

\[
U_{-\iota p}W_{r,p}^{(\mu)}[\iota \mu] \simeq V_{r,p}^{\mu - \iota \iota \mu}.
\]

In \( V_{r,p}^{\mu - \iota \iota \mu} \), we apply the operator \( U_{-\iota p}\mathcal{O}_n U_{-\iota p} \) to \( U_{-\iota p}|x\rangle \) using the differential operators in Lemma 5.4, and then use the spectral flow to transform the result back into \( W_{r,p}^{(\mu)}[\iota \mu] \). We thus set

\[
\mathcal{O}_n|x\rangle = U_{\iota p}\left( (U_{-\iota p}\mathcal{O}_n U_{-\iota p})^{(\mu - \iota \iota \mu)} U_{-\iota p}|x\rangle\right),
\]

where

\[
\mathcal{O}_n = \mathcal{L}_n, \quad n \geq 0; \quad \mathcal{O}_n = \mathcal{H}_n, \quad n \geq 0; \quad \mathcal{O}_n = \mathcal{Q}_n, \quad n \geq -\mu,
\]

and the superscript in (5.40) means that the corresponding differential operator from Lemma 5.4 is applied.

The operators in Eqs. (5.7) and (5.8) depend on \( N = \iota p - \mu \) and involve an additional parameter \( h^{(r,p,N)} \). Another free parameter arises when the spectral flow is applied: while mapping semi-infinite forms involves only the relations \( U_{\theta} G_n U_{-\theta} = G_{n+\theta} \) and \( U_{-\iota p}|r, p; t\rangle_{\infty/2} = |r, p; 0\rangle_{\infty/2} \), which are a part of the definition of the semi-infinite space, it is understood that the operator \( U_{-\iota p}\mathcal{O}_n U_{-\iota p} \) is evaluated using the \( N = 2 \) spectral flow formula (2.2). This gives rise to the parameter \( c \), which is also free at this stage.

Remarkably, all the free parameters are uniquely fixed by the consistency requirements. Moreover, it can then be proved that the action defined above is independent of the chosen \( \iota \) and \( \mu \). This is shown in Secs. 5.4.2 and 5.4.3; in Secs. 5.4.4 and 5.4.5, we then establish that the action of the generators constructed defines precisely the \( N = 2 \) algebra.

5.4.2. Using Eqs. (2.2) and (3.9), we compare the eigenvalues of \( \mathcal{H}_0 \) on the states \( |r, p; t\rangle_{\infty/2} \) and \( |r, p; t - 1\rangle_{\infty/2} \). In view of Eqs. (3.4) (see also Fig. 2), these eigenvalues differ by \( p - 2 \). On the other
hand,

\[ U_p \mathcal{H}_0 |r, p| \ell \rangle_{\infty/2} = U_p \mathcal{H}_0 U_{-p} |r, p| \ell \rangle_{\infty/2} = \left( \mathcal{H}_0 + \frac{c}{3} p \right) |r, p| \ell \rangle_{\infty/2}, \]

which gives central charge (1.4) expressed through the parameter \( p \) in the basic relation (1.1).

A similar argument applied to \( \mathcal{L}_0 \) gives

\[ U_p \mathcal{L}_0 |r, p| \ell \rangle_{\infty/2} = U_p \mathcal{L}_0 U_{-p} |r, p| \ell \rangle_{\infty/2} = \left( \mathcal{L}_0 + p \mathcal{H}_0 + \frac{c}{6} (p^2 + p) \right) |r, p| \ell \rangle_{\infty/2}, \]

where we already know \( c \). On the other hand, (minus) the difference between the eigenvalues of \( \mathcal{L}_0 \) on the states \( |r, p| \ell - 1 \rangle_{\infty/2} \) and \( |r, p| \ell \rangle_{\infty/2} \) is given by

\[ \left( [\ell p - 1] + \cdots + \left( (\ell - 1) p + 1 \right) \right) - [\ell p - r] = \ell p (p - 1) - \frac{1}{2} p(p - 1) - \ell p + r, \]

whence it follows that the parameter \( h^{(r, p, N)} \) in Lemma 5.4 is given by

\[ h^{(r, p, N)} = -\frac{r - 1}{p}. \]

With this \( h^{(r, p, N)} \) in (5.3), it follows that \( l^{(r, p, N)} = 0 \).

Restoring the dependence on \( \theta \), we now have the relation

\[ \mathcal{H}_0 |r, p; \theta \rangle_{\infty/2} = \left( -\frac{r - 1}{p} - \frac{p - 2}{p} \theta - (p - 2) \ell \right) |r, p; \theta \rangle_{\infty/2} \]

holding in \( \mathcal{W}_{r, p, \theta} \). This gives the same eigenvalues as in the unitary module (given by Eq. (2.22)), and the same is easily seen to be true for the eigenvalues of \( \mathcal{L}_0 \) on \( |r, p; \theta \rangle_{\infty/2} \) (given by Eq. (2.23)).

5.4.3. In addition to \( \mathcal{L}_0 \) and \( \mathcal{H}_0 \), the action of all the operators \( \mathcal{L}_{\geq 0} \) and \( \mathcal{H}_{\geq 0} \) is carried over from \( \mathcal{V}_{r, p}^{\mu - \mu} \) to \( \mathcal{W}_{r, p} \). Similarly, in each \( \mathcal{W}_{r, p}^{(\mu)}[\ell] \), we obtain the action of the operators \( \mathcal{Q}_n \) for \( n \geq -\mu \).

A priori, they depend on the space \( \mathcal{V}_{r, p}^{\mu - \mu} \) in which the differential operators are applied. Now, as noted above, Eq. (5.10) follows from the condition that the operator \( \mathcal{Q}_{N}^{(r, p, N)} \) preserve the relations \( S_a = 0 \). It is remarkable that for this \( q^{(r, p, N)} \) and with \( h^{(r, p, N)} \) chosen in (5.43), we can rewrite Eq. (5.7) as

\[ \mathcal{Q}_{\ell}^{(r, p, N)} = 2 \sum_{n=-N}^{-1} \left( \mathcal{L}_{n+\ell}^{(r, p, N)} - \ell \mathcal{H}_{n+\ell}^{(r, p, N)} + \frac{1}{2} \frac{p - 2}{p} (\ell^2 - \ell) \delta_{\ell+n,0} \right) \partial_n. \]

Moreover, for \( h^{(r, p, N)} \) of form (5.45) (and with \( l^{(r, p, N)} = 0 \)), the right-hand side of (5.47) depends on \( \ell \) only through the summation limits. This implies that the differential operators in Lemma 5.4 commute with the embeddings \( \mathcal{V}_{r, p}^{N-1} \to \mathcal{V}_{r, p}^{N} \). Therefore, for \( n \geq 0 \), the operators \( \mathcal{L}_n, \mathcal{H}_n, \text{ and } \mathcal{Q}_{-\mu+n} \) acting in \( \mathcal{W}_{r, p} \) as defined in (5.40) are independent of \( \ell \) and \( \mu \).

5.4.4. We have seen that for any \( \mu \in \mathbb{Z} \), the operators \( \mathcal{H}_{\geq 0} \), \( \mathcal{L}_{\geq 0} \), \( \mathcal{G}_{\geq \mu} \), and \( \mathcal{Q}_{\geq -\mu} \) are well-defined on each state in \( \mathcal{W}_{r, p;\theta} \); moreover, these operators satisfy the \( N=2 \) commutation relations. Hence, there is a family of subalgebras of the \( N=2 \) algebra acting on the semi-infinite space \( \mathcal{W}_{r, p;\theta} \). It remains to define the action of \( \mathcal{H}_{<0} \) and \( \mathcal{L}_{<0} \) and show that all the \( N=2 \) commutation relations are satisfied for the operators \( \mathcal{H}_n, \mathcal{L}_n, \mathcal{G}_n, \text{ and } \mathcal{Q}_n \) with \( n \in \mathbb{Z} \).
We still have not found the commutators $[Q_m, G_n]$ with $m + n \leq -1$, which do not follow from the argument based on the spectral flow. We find them by “solving the Jacobi identities.” We first show that the operators $Q_n$ and $G_n, \ n \in \mathbb{Z}$, satisfy the relations

\begin{align}
[Q_m, [G_n, Q_\ell]] &= 2(m - n)G_{m+n+\ell}, \\
[Q_\ell, [Q_m, G_n]] &= 2(\ell - m)Q_{\ell+m+n}.
\end{align}

Starting with (5.49), we set

\begin{equation}
\mathcal{X}(\ell, m, n) = [Q_\ell, [Q_m, G_n]] - 2(\ell - m)Q_{\ell+m+n}
\end{equation}

and show that this expression acts by zero on any state $|\alpha\rangle \in W_{r,p,\theta}$. It suffices to consider monomial states $|\alpha\rangle$. For given $\ell, m$, and $n$, there exists a positive integer $\mu$ such that $n \geq \mu$ and $\ell, m \geq -\mu$. We choose any such $\mu$ and use filtration (5.3). Let $|\alpha\rangle \in W_{r,p,\theta}[\ell_0]$. This means that the state is represented as $|\alpha\rangle = G_{a_1} \ldots G_{a_\nu}|r, p; \theta|\ell_0\rangle_{\infty/2}$, where $a_j \geq \mu$. We let $a$ denote one of $a_j$.

Because $a + \ell \geq 0$, $a + m \geq 0$, and $a + \ell + m + n \geq 0$, the commutation relations found so far allow us to evaluate

\begin{equation}
[Q_a, \mathcal{X}(\ell, m, n)] = 2[L_{a+\ell} - \ell H_{a+\ell}, [Q_m, G_n]] - 2[Q_\ell, [L_{a+m} - m H_{a+m}, G_n]] - \\
4(\ell - m)(L_{a+\ell+m+n} - (\ell + m + n)H_{a+\ell+m+n}).
\end{equation}

Using the already established commutation relations again, we then have

\begin{equation}
[Q_a, \mathcal{X}(\ell, m, n)] = 2(\ell - m)[Q_{a+\ell+m}, G_n] + 2(a - n)[Q_m, G_{a+\ell+m}] + \\
+ 2(a - n)[Q_\ell, G_{a+m+n}] - 4(\ell - m)(L_{a+\ell+m+n} - (\ell + m + n)H_{a+\ell+m+n}).
\end{equation}

We also obtain $a + \ell + m \geq -\mu$, $a + \ell + n \geq \mu$, and $a + m + n \geq \mu$, and once again applying the already known commutation relations, we see that the right-hand side of Eq. (5.52) vanishes.

The action of $\mathcal{X}(\ell, m, n)$ on the state $|\alpha\rangle = G_{a_1} \ldots G_{a_\nu}|r, p; \theta|\ell_0\rangle_{\infty/2}$ is thus given by $(-1)^r G_{a_1} \ldots \times G_{a_\nu} \mathcal{X}(\ell, m, n)|r, p; \theta|\ell_0\rangle_{\infty/2}$. But it can be assumed (possibly at the expense of increasing $\ell_0$) that $\mathcal{X}(\ell, m, n)|r, p; \theta|\ell_{\infty/2} = 0$, because the operator $\mathcal{X}(\ell, m, n)$ maps each state $|r, p; \theta|\ell_{\infty/2}$ with $\ell \gg 0$ into a bigrade containing no states. Therefore, $\mathcal{X}(\ell, m, n) = 0$ in $W_{r,p,\theta}$.

To prove relations (5.48), we denote $\mathcal{Y}(m, n, \ell) = [G_m, [G_n, Q_\ell]] - 2(m - n)G_{m+n+\ell}$ and following (5.51) and (5.52) mutatis mutandis, directly verify that the commutators $[Q_a, \mathcal{Y}(m, n, \ell)]$ and $[Q_a, \mathcal{Y}(m, n, \ell)]$ vanish for all $a$ satisfying the conditions

\begin{equation}
a + \ell \geq 0, \quad a + n \geq 0, \quad a + m \geq 0, \quad a + \ell + m + n \geq 0.
\end{equation}

Next, examining the gradings shows that $\mathcal{Y}(m, n, \ell)|r, p; \theta|\ell_{\infty/2} = 0$ for $\ell' \ll -1$. However, this does not directly imply that $\mathcal{Y}(m, n, \ell)$ vanishes on any state from the semi-infinite space (compared to the case with $\mathcal{X}(m, n, \ell)$, the problem is with the apparent asymmetry of the semi-infinite construction with respect to the $G_n$ and $Q_n$ generators). It remains to be shown that $\mathcal{Y}(m, n, \ell)|r, p; \theta|\ell_{\infty/2} = 0$
for \( \ell \gg 1 \), because any state in the semi-infinite space can be generated by the modes \( \mathcal{G}_a \) from \( |r, p; \theta|\ell\rangle_{\infty/2} \) with a sufficiently large \( \ell \).

It suffices to show that for any \( \ell \gg 1 \), the state \( |r, p; \theta|\ell\rangle_{\infty/2} \) can be obtained by acting with the modes \( \mathcal{G}_a \) and \( \mathcal{Q}_a \) satisfying conditions (5.53) on the state \( |r, p; \theta'|\ell'\rangle_{\infty/2} \) with some \( \ell' \) for which \( \ell + m + n \geq p\ell' + \theta \). Let \( M \) denote the minimal integer \( a \) satisfying (5.53). We fix \( \ell \) such that \( p\ell \geq M \). We then have

\[
(5.54) \quad \mathcal{G}_{p+\theta+1} \cdots \mathcal{G}_{p+\theta+p-r-1} \mathcal{Q}_{M+\mu} \cdots \mathcal{Q}_{M+1} \mathcal{Q}_{M} |r, p; \theta|\ell'\rangle_{\infty/2} = F |r, p; \theta|\ell\rangle_{\infty/2}
\]

for

\[
(5.55) \quad \ell' = -(p-1)\ell - \theta - M - p + r + 1,
\]

\[
(5.56) \quad \mu = (p-2)(p\ell + \theta + M + p - r - 1) + p - r - 2,
\]

which follows because both sides of (5.54) belong to the same eigenspace of \( \mathcal{H}_0 \) and \( \mathcal{L}_0 \) and this eigenspace is a 1-dimensional subspace in \( \mathcal{W}_{r,p,\theta} \). If we show that \( F \neq 0 \) in Eq. (5.54), we can conclude that \( \mathcal{Y}(m, n, \ell) |r, p; \theta|\ell\rangle_{\infty/2} = 0 \) for \( \ell \gg 1 \), because it has already been shown that \( \mathcal{Y}(m, n, \ell) \) commutes with all \( \mathcal{G}_a \) and \( \mathcal{Q}_a \) for \( a \geq M \) and \( \mathcal{Y}(m, n, \ell) |r, p; \theta|\ell\rangle_{\infty/2} = 0 \).

To verify that the coefficient \( F \) is nonvanishing, it is easiest to use the mapping \( \mathcal{W}_{r,p,\theta} \to \mathfrak{R}_{r,p,\theta} \) to the unitary \( N=2 \) module, under which each state (3.1) goes into the corresponding extremal state \( |r, p; \theta|\ell\rangle \in \mathfrak{R}_{r,p,\theta} \) (see Sec. 2.4). In \( \mathfrak{R}_{r,p,\theta} \), the image of (5.54) is satisfied with a nonvanishing \( F \). Therefore, the coefficient \( F \) in (5.54) cannot vanish, which implies that \( \left[ \mathcal{G}_m, \left[ \mathcal{G}_n, \mathcal{Q}_\ell \right] \right] = 2(m - n)\mathcal{G}_{m+n+\ell} \) in the semi-infinite space.

**5.4.5.** Equations (5.48) and (5.49) imply all the commutation relations (2.1). Indeed, for \( a < 0 \), we can define the operators

\[
(5.57) \quad \mathcal{L}_a^{m,n} = \frac{1}{2(m-n)}(m[\mathcal{G}_{n+a}, \mathcal{Q}_{-m}] - n[\mathcal{G}_{m+a}, \mathcal{Q}_{-n}]), \quad m \neq n,
\]

\[
(5.58) \quad \mathcal{H}_a^{m,i,n} = \frac{1}{2m}(m[\mathcal{G}_{m+a}, \mathcal{Q}_{-n}] - 2\mathcal{L}_a^{i,n}), \quad m \neq 0.
\]

Let \( A \) be the algebra generated by \( \mathcal{Q}_n \) and \( \mathcal{G}_n \) satisfying relations (5.48) and (5.49). It follows from (5.48) and (5.49) that for any fixed \( m, i, n \in \mathbb{Z} \), the operators

\[
\mathcal{L}_a^{m,n}, \quad \mathcal{H}_a^{m,i,n}, \quad a < 0,
\]

\[
\mathcal{L}_j, \quad \mathcal{H}_j, \quad j \geq 0,
\]

\[
\mathcal{Q}_j, \quad \mathcal{G}_j, \quad j \in \mathbb{Z},
\]

satisfy commutation relations (2.1). In particular, the operators \( \mathcal{L}_a^{m,n} - \mathcal{L}_a^{m',n'} \) and \( \mathcal{H}_a^{m,i,n} - \mathcal{H}_a^{m',i',n'} \) commute with \( \mathcal{Q}_j \) and \( \mathcal{G}_\ell \) and generate a commutative ideal \( Z \) in \( A \); moreover, \( Z \) is in the center of \( A \), and the quotient of \( A \) with respect to \( Z \) coincides with the \( N=2 \) algebra. Thus, \( A \) is a central
extension of the \( N=2 \) algebra of the form
\[
\mathcal{G}_m, Q_n = 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2 + m)\delta_{m+n,0} + f_{n,m},
\]
where \( f_{n,m} \neq 0 \) only for \( n < 0 \) and \( m < 0 \). In view of the Jacobi identities, \( f_{n,m} = 0 \), and therefore, \( Z = 0 \) and the operators \( \mathcal{L}_a^m \) and \( \mathcal{H}_a^{m,n} \) are independent of \( m, i, \) and \( n \).

This completes the proof of Theorem 5.1. This sufficiently strong result leads to the statement of Theorem 1.1. Before considering that theorem, however, we consider a similar construction on the \( \hat{\mathfrak{sl}}(2) \) semi-infinite space.

5.5. The \( \hat{\mathfrak{sl}}(2) \) action on the semi-infinite space. As for the \( N=2 \) algebra, we consider the filtration of the \( \hat{\mathfrak{sl}}(2) \) semi-infinite space by finite-dimensional subspaces
\[
M_{r,k}^+ [0] \to M_{r,k}^+ [1] \to \cdots \to M_{r,k}^+ [\ell] \to \cdots,
\]
where \( M_{r,k}^+ [\ell] \) is generated by \( f_0, f_1, \ldots, f_2 \), from \( |r, k| \) \( \hat{\mathfrak{sl}}(2) \) (up to a spectral flow transform, \( M_{r,k}^+ [\ell] \) are the Demazure modules [44]–[46]). On each subspace \( M_{r,k}^+ [\ell] \), we can define a part of the \( \hat{\mathfrak{sl}}(2) \) generators as differential operators with respect to \( f_0, f_1, \ldots, f_2 \); after the appropriate spectral flow transform, we obtain differential operators with respect to \( f_{-1}, f_{-2}, \ldots, f_{-2+1} \). It must be shown that they preserve the ideal generated by the elements
\[
\sum_{-1 \leq i_1, \ldots, i_{k+1} \leq -2+1} f_{i_1} \cdots f_{i_{k+1}} = 0, \quad a = -k-1, -k-2, \ldots,
\]
\[
\sum_{i_1, \ldots, i_{k+1} = a} f_{i_1} \cdots f_{i_{k+1}} = 0,
\]
where \( \partial_m = \partial/\partial f_m \).

We now go over to the dual formulation, where the problem, similarly to Sec. 5.3, essentially reduces to finding the action of the generators on the space of symmetric polynomials. The dual generators are given by
\[
(f^*_\ell \phi)(x_1, x_2, \ldots, x_{M-1}) = \left. \frac{1}{(\ell - 1)!} \partial x_M^{\ell-1} \phi(x_1, x_2, \ldots, x_M) \right|_{x_M=0},
\]
\[
(h^*_\ell \phi)(x_1, x_2, \ldots, x_M) = \sum_{i=1}^{M} x_i^\ell \phi(x_1, x_2, \ldots, x_M),
\]
\begin{equation}
(c_1^N \phi)(x_0, x_1, x_2, \ldots, x_M) = - \sum_{0 \leq i < j \leq M} F_\ell(x_i, x_j) \left( \phi([x]_i) + \phi([x]_j) \right) + kN \delta_\ell N \sum_{0 \leq i \leq M} x_i^{N-1} \phi([x]_i)
\end{equation}

with $F_\ell$ defined in (5.33).

As for the $N=2$ algebra, the formula for $c_1^N$ was obtained by adding some terms from the ideal $\mathfrak{J}(N)$ generated by the monomials $x_1^{i_1} \cdots x_n^{i_n}$ in which $i_j \geq N$ for at least one $j$; in the algebra of symmetric polynomials, this ideal is generated by the polynomials $p_j$ with $j \geq N$ (see (5.25)).

It must be established that the action of $c_1^N$ does not violate the vanishing conditions on $(k+1)$-diagonals,

\begin{equation}
\phi \left( x, x, \ldots, x, x_{k+2}, \ldots \right) = 0,
\end{equation}

or, more precisely, that the polynomial $c_1^N \phi$ satisfies these conditions modulo the ideal $\mathfrak{J}(N)$. To calculate the right-hand side of (5.66) for $x_0 = x_1 = \cdots = x_k$, we split the double sum in (5.66) as

\begin{equation}
\sum_{0 \leq i < j \leq M} F_\ell(x_i, x_j) \left( \phi([x]_i) + \phi([x]_j) \right) = \left( \sum_{0 \leq i < j \leq k} + \sum_{0 \leq i \leq k < j} + \sum_{k < i < j \leq M} \right) F_\ell(x_i, x_j) \left( \phi([x]_i) + \phi([x]_j) \right).
\end{equation}

The third sum is readily seen to vanish on the diagonal in view of (5.67). Inserting $x_0 = x_1 = \cdots = x_k = x$ into the second sum, we obtain terms of the form

\begin{equation}
\frac{x^\ell - x_j^\ell}{x - x_j} \phi \left( \underbrace{x, x, \ldots, x}_{k+1}, x_{k+1}, \ldots \right).
\end{equation}

Again in view of (5.67), however, $x - x_j$ is a divisor of the polynomial $\phi \left( \underbrace{x, x, \ldots, x}_{k+1}, \ldots \right)$, and therefore each of these terms is in $\mathfrak{J}(N)$ (we recall that $\ell \geq N$).

It remains to consider the first sum in (5.68). In this case, $F_\ell(x, x) = \ell x^{\ell-1}$. Each term in the sum lies in the ideal for $\ell > N$, but for $\ell = N$, we have the terms involving $x^{\ell-1}$, which is not in the ideal. However, these terms can be summed up into the expression

\begin{equation}
-2N \frac{k(k+1)}{2} x^{N-1} \phi \left( \underbrace{x, x, \ldots, x}_{k+1}, x_{k+1}, \ldots \right),
\end{equation}

which is precisely canceled by the term

\begin{equation}
+k(k + 1)N \delta_{N,N} x^{N-1} \phi \left( \underbrace{x, x, \ldots, x}_{k+1}, \ldots \right),
\end{equation}

which arises from the second term in the formula for $c_1^N$, Eq. (5.66). This finishes the proof that operators (5.64)–(5.66) preserve relations (5.67) and therefore have a well-defined action on $M_{r,k}[u]^*$. 
Hence, the action is well defined on $M_{r,k}^+[\iota]$ in that it preserves the ideal generated by the left-hand sides of (5.61).

We note that the key role in the above cancellation is played by the coefficient $k$ in front of the second term in (5.60). The same coefficient becomes the level of the $\hat{s}\ell(2)$ representation thereby constructed. A combination of these facts ensures the existence of the $\hat{s}\ell(2)$ action.

The demonstration of the $\hat{s}\ell(2)$ action on the semi-infinite space is now completed following the same strategy as in Sec. 5.4. One first verifies that the action of positive modes defined on different subspaces $M_{r,k}^+[\iota]$ agrees with the embeddings in (5.60). The spectral flow then allows one to define the action of negative modes (which again involves the argument that along with (5.60), there exist similar filtrations with $M_{r,k}^+[\iota]$ replaced by the spaces $M_{r,k}^{[\iota]}[\iota]$ generated by the modes $f_n$ with $n \geq \mu$).

To complete the proof, it only remains to verify the Serre relations. For this, one first establishes that the left-hand sides of the Serre relations commute with all the modes $f_n$ for $n \geq -\mu$, and because the modes $f_n$ generate the entire semi-infinite space, it then follows that the Serre relations are satisfied once they are satisfied on a particular extremal state, which can be verified directly. The Serre relations guarantee that the algebra action constructed is the action of the $\hat{s}\ell(2)$ algebra.

5.6. The isomorphism with the representation $R_{r,p\theta}$. To continue with the $N=2$ algebra story, it remains to show that the semi-infinite space, which we now know to be an $N=2$ module, is isomorphic to a unitary $N=2$ representation.

There is a mapping

\[ W_{r,p;\theta} \rightarrow R_{r,p;\theta}, \]

obtained by identifying each state (3.1) with the corresponding extremal state in $R_{r,p;\theta}$ defined in Sec. 2.4 (and obviously, identifying each $G_n$ with the corresponding $N=2$ algebra generator acting in the unitary representation). Obviously, these mappings commute with the action of $G_n$ and $U_{\pm p}$.

**Theorem 5.5.** Mapping (5.72) is an isomorphism of $N=2$ modules.

**Proof.** We have the $N=2$ module $W_{r,p;\theta}$, in which relations (1.1) are satisfied. In this case, it is easiest to use the equivalence of the $N=2$ and $\hat{s}\ell(2)$ representation categories \[ [19], [50].\] Applying the functor to $W_{r,p;\theta}$, we obtain an $\hat{s}\ell(2)$-module from the category $O$ (more precisely, the spectral-flow orbit, whose length is equal to 2 on integrable representations), in which conditions (4.44) are satisfied. Any such $\hat{s}\ell(2)$-module is a direct sum of integrable (unitary) modules,

\[ F(W_{r,p}) = \bigoplus_{\alpha} L(\alpha). \]

---

\[ ^8 \text{We recall that an } \hat{s}\ell(2) \text{ module is integrable if it decomposes into a sum of finite-dimensional representations with respect to any of its } \hat{s}\ell(2) \text{ subalgebras generated by } e_i \text{ and } f_{-i} \text{ (it actually suffices to establish this decomposition for two such subalgebras); it is this property that follows from } (4.44) \ [36]. \]
Applying the inverse functor, we obtain another sum of representations,
\[ W_{r,p,\theta} = \bigoplus_{\beta} R_{(\beta)}, \]
where each module in the right-hand side is necessarily a unitary \( N = 2 \) representation, i.e., some \( R_{r',p,\theta'} \), and the statement of the theorem follows by comparing the eigenvalues of \( L_0 \) and \( H_0 \).

It follows from Theorem 5.5 that the expression in (4.36) coincides with the corresponding unitary \( N = 2 \) character. Comparing this with the known expression for the same character (see Sec. 2.2), we obtain the combinatorial identity in the corollary of Theorem 1.1 (see Sec. 1).

As another corollary, the “semi-infinite” expression for the unitary \( N = 2 \) characters gives a formula for the string functions \( C_{r,p}(q) \) read off from representing the \( N = 2 \) characters as in (2.13). To obtain this representation, we rewrite Eq. (4.38) (where we can set \( \theta = 0 \)) by splitting the summation over \( n \) as
\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{\ell \in \mathbb{Z}} \sum_{a=0}^{p-3} f((p-2)\ell + a). \]

We then shift each of the summation variables \( N_m \) as \( N_m \mapsto N_m + \ell \), which does not change the factors \( (q)_{N_m-N_{m+1}} \). The summation over \( \ell \) can therefore be performed, with the result that the “semi-infinite” formula for the character takes the form (2.13) with the string function
\[ C_{r,p}^a(q) = \sum_{N_1 \geq \ldots \geq N_p \in \mathbb{Z}} \frac{q^{\sum_{m=1}^{p-2} N_m + \sum_{m=1}^{p-2} N_m}}{(q)_{N_1-N_2}(q)_{N_2-N_3} \ldots (q)_{N_{p-3}-N_{p-2}} (q)_{N}}. \]

6. Some related constructions

6.1. Positive bases: paths and the generalized Pascal triangles. Filtration (5.1) allows us to rewrite a representative of any state in the semi-infinite space \( W_{r,p,\theta} \) through only nonnegative modes \( G_{n \geq 0} \). The next interesting problem is to construct a basis in each term of this filtration. We construct such bases in the finite-dimensional subspaces \( M_{r,k}^+[\iota] \) (see (5.60)) of the \( \hat{sl}(2) \) semi-infinite space using some kind of a “Demazure induction” (cf. [14]–[16]). The combination of bases in the corresponding \( M_{r,k}^+[\iota] \) spaces gives a basis in the \( N = 2 \) space \( W_{r,p,\theta}^+[\iota] \).

6.1.1. Positive bases in unitary \( \hat{sl}(2) \) modules. We consider the filtration by finite-dimensional subspaces in Eq. (5.60). Basis vectors in \( M_{r,k}^+[\iota] \) are in a 1 : 1 correspondence with paths on a rectangular lattice whose construction we now explain. We label the horizontal lines of the lattice by \( \ell_0, \ell_1, \ldots \) such that \( \ell_0 \) corresponds to the bottom. On each line \( \ell_m \), we label the sites as \( \ell_{m,n} \) with \( n \geq 0 \). We assign the site \( \ell_{0,0} \) the symbol \( [k+1, \ldots, k+1, r] \). A path ending at \( \ell_{m,n} \) is a connected sequence of links between \( \ell_{0,0} \) and \( \ell_{m,n} \) (see Fig. 3), with each site along the path assigned an \([\ldots]\) symbol as follows. If the site \( \ell_{m,n} \) on the path is already assigned a symbol \([a_1, \ldots, a_p]\),
where \( a_1 \geq a_2 \geq \cdots \geq a_\nu \), the path can be continued from this site via one of the following two steps whenever the corresponding conditions are satisfied:

1. Moving to the site \( \ell_{m+1,n} \) and assigning it the symbol \([a_2, \ldots, a_\nu]\) (provided \( \nu \geq 2 \)).
2. Moving to the site \( \ell_{m,n+1} \) and assigning it the symbol \([a_1, \ldots, a_{\lambda-1}, a_{\lambda-1}, a_\nu] \) if \( a_\lambda \geq 2 \), where \( \lambda \) is defined by the conditions \( a_1 = \cdots = a_\lambda > a_{\lambda+1} \) and \( \lambda = \nu \) if \( a_1 = \cdots = a_\nu \geq 2 \).

We let \( \blacktriangleleft \) and \( \blacktriangleright \) denote the link created by taking the respective steps 1 and 2. More precisely, we write \( \blacktriangleright_m \) for each \( \blacktriangleright \) link between any two sites on \( \ell_m \). We note that different paths assign different \([\ldots]\) symbols to the same site.

A path ending at \( \ell_{m,n} \) is said to be admissible if it cannot be continued by step 1 from \( \ell_{m,n} \) (it may or may not be continued via step 2).

Basis vectors in \( M_{r,k}^+[\iota] \) are in a 1 : 1 correspondence with the admissible paths. The path consisting of only the \( \blacktriangleleft \) links corresponds to the twisted highest-weight vector \(|r,k|_{\hat{sl}(2)}\rangle \) (see (4.45) and (4.46)). For any other admissible path, let \( \blacktriangleright_{j_0}, \blacktriangleright_{j_1}, \ldots, \blacktriangleright_{j_K} \) (where \( 0 \leq j_0 \leq \cdots \leq j_K \)) be its \( \blacktriangleright \) links. The corresponding basis vector is given by

\[
(6.1) \quad f_{j_0}f_{j_1}\cdots f_{j_K}|r,k|_{\hat{sl}(2)}\rangle \in M_{r,k}^+[\iota].
\]

This construction is based on the fact that \( M_{r,k}^+[\iota] \simeq Gr\left( \mathbb{C}^{k+1} \otimes \cdots \otimes \mathbb{C}^{k+1} \otimes \mathbb{C}^r \right) \equiv [k+1, \ldots, k+1, r] \),

\[
(6.2) \quad M_{r,k}^+[\iota] \simeq Gr\left( \mathbb{C}^{k+1} \otimes \cdots \otimes \mathbb{C}^{k+1} \otimes \mathbb{C}^r \right) \equiv [k+1, \ldots, k+1, r],
\]

where \( Gr \) means taking the graded object associated with a filtration existing on the tensor product. Traveling along the paths then corresponds to “traveling” through the tensor product factors. The last formula can also be viewed as an explanation of the square-bracket notation. For the representations with \( r = 1 \), the last tensor factor and, correspondingly, ‘1’ in \([k+1, \ldots, k+1, 1]\) can be dropped. Indeed, it is easy to see that replacing the starting symbol \([\underbrace{k+1, \ldots, k+1}_{\iota+1}, 1]\) with \([\underbrace{k+1, \ldots, k+1}_{\iota+1}]\) does not change the resulting vectors (6.1).

**Figure 3.** A path on the lattice. The starting point is assigned the symbol \([k+1, \ldots, k+1, r]\), and the sites along the path are assigned \([\ldots]\) symbols in accordance with the two rules. These rules determine the possible ways of continuing the path.
Figure 4. The labels on paths for $k = 2$, $r = 2$, and $\iota = 1$. Each cluster of $[]$-labels is attached to a lattice site and the respective ▲ or ▶ symbol shows by which step this label was obtained from a preceding one. All the admissible paths can be drawn by connecting the labels following the ▲ and ▶ directions (all the paths start at the bottom-left corner).

In Fig. 4, we consider the example where $k = 2$, $r = 2$, and $\iota = 1$, and the origin of paths $\ell_{0,0}$ is therefore assigned $[3,3,2]$. The basis vectors read off from the admissible paths in accordance with (6.1) are given by

$$
\begin{align*}
&f_2 f_1 f_2 f_1^3 f_0^2 f_1^2 f_0^5 \\
f_1 &f_0 f_2 f_0 f_1 f_0 f_1 f_0 \\
f_0 &f_1 f_2 f_0 f_1 f_0 f_1 f_0 \\
f_0 f_1 &f_0 f_1 f_0 f_1 f_0 f_1 \\
f_0 &f_1 f_2 f_0 f_1 f_0 f_1 f_0 \\
f_2 &f_0 f_1 f_0 f_1 f_0 f_1 f_0 \\

\end{align*}
$$

(6.3)

acting on the twisted highest-weight state $|2,2,1\rangle_{\hat{s}\ell(2)}$.

6.1.2. “Positive” characters. The space $M_{r,k}[\iota]$ is graded by the number of modes $f_i$ applied to the twisted highest-weight vector,

$$
M_{r,k}[\iota] = \bigoplus_{j=0}^{r+2k} M_{r,k}[\iota; j],
$$

(6.4)

where each $M_{r,k}[\iota; j]$ is generated from $|r, k|\iota\rangle_{\hat{s}\ell(2)}$ by precisely $j$ operators $f_\bullet$. The dimensions of $M_{r,k}[\iota; j]$ are arranged into a generalized Pascal triangle (cf. [11]).

In the generalized Pascal triangle labeled by $k$ and $r$, the top row consists of $r$ units, and each element in the $i$th row is the sum of $k+1$ elements of the $(i-1)$th row: for even $k$, the sum includes the element above the chosen one and $(k+2)/2 - 1$ of its neighbours on each side; for odd $k$, it runs over $(k+1)/2$ elements “north-west” and $(k+1)/2$ elements “north-east” of the chosen one. For $k = 1$, both cases $r = 1$ and $r = 2$ reduce to the standard Pascal triangles (starting with 1 and 1, 1, 2, 3, ...).
in the top row respectively); for $k = 2$, the triangles are shown in Fig. 5. The dimension of $M_{r,k}^+[\iota ; j]$ is read of from the $j$th entry of the $2\iota$th row in the Pascal triangle with the parameters $k$ and $r$.

Each space $M_{r,k}^+[\iota ; j]$ is graded by the sum of modes of $f_i$. The character of $M_{r,k}^+[\iota]$ can be written through the $q$-supernomial coefficients $(L_k)_q$, which are defined by the generating functions (6.5)

$$T_{L_k}(z, q) = \sum_{a=0}^{\infty} z^a \binom{L_k}{a}_q$$

$$= \sum_{L_1 \geq N_1, L_2 + N_1 \geq N_2, \ldots, L_k + N_{k-1} \geq N_k \geq 0} z^{\sum_{i=1}^{k} N_i q^{\sum_{j=1}^{i-1} N_i + 1} ((\sum_{j=1}^{i} L_j) - N_i)} \times$$

$$\times \left[ \binom{L_1}{N_1}_q \left[ \binom{L_2 + N_1}{N_2}_q \left[ \binom{L_3 + N_2}{N_3}_q \cdots \binom{L_k + N_{k-1}}{N_k}_q \right] \right] \right],$$

where $L_k = (L_1, L_2, \ldots, L_k)$ is a $k$-dimensional vector with nonnegative integer entries and we use the standard notation

$$\binom{n}{m}_q = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}}, & n \geq m \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

The supernomial coefficients are generating functions of partitions admitting Durfee dissection with the defects $(L_1, L_2, \ldots, L_k)$.

The character of $M_{r,k}^+[\iota]$ is given by (6.6)

$$\text{char } M_{r,k}^+[\iota](z, q) = z^{-k_1 - r + 1} q^{k_k + \iota (r-1)} T_{L_k}(z, q^{-1}),$$

where the $k$-dimensional vector $L_k$ is chosen as $L_k = (2\iota, 0, 0, \ldots, 0, 1, 0, \ldots, 0)$. After some algebra, Eq. (6.7) can be rewritten through $q$-binomial coefficients as

$$\text{char } M_{r,k}^+[\iota](z, q) = \sum_{N_1, \ldots, N_k} z^{N_1 + N_2 + \cdots + N_k} q^{\sum_{m=1}^{k} N_m^2 + \sum_{i=k+2-r}^{k} N_m \times}$$

$$\times \left[ \binom{L_1}{N_1}_q \left[ \binom{L_2 + N_1}{N_2}_q \left[ \binom{L_3 + N_2}{N_3}_q \cdots \binom{L_k + N_{k-1}}{N_k}_q \right] \right] \right],$$

where $L_k = (L_1, L_2, \ldots, L_k)$ is a $k$-dimensional vector with nonnegative integer entries and we use the standard notation

$$\binom{n}{m}_q = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}}, & n \geq m \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$
There exists an isomorphism \( W \) induced by mapping the basis elements of vector spaces. This gives a basis in the generalized Pascal triangles (see an example in Fig. 6). These dimensions are given by the generalized Fibonacci numbers \( F_{p+\theta} \) satisfying the defining relations \( F_i^{p+\theta} = F_{i-1}^{p+\theta} + F_{i-2}^{p+\theta} + \cdots + F_{i-(p-1)}^{p+\theta} \).

We conjecture a character formula for \( W_{r, p; \theta}[N] \). We normalize the analogue of (6.8) such that the state \( |r, p; \theta|0\rangle \) is in the grade (0, 0).

**Figure 6.** Knight moves. The dimensions of the spaces \( W^{+}_{2,4,1}[1; j] \) are selected in the positions connected by knight moves in the generalized Pascal triangle.
Conjecture 6.1. The characters of the subspaces involved in the positive filtration of unitary \( N=2 \) representations are given by

\[
\text{char } W_{r,p;\theta}^+ \alpha(z, q) = z^{-(p-2)\nu - r + 1} q^{\frac{(p-2)(\nu+1)}{2} - (p_r - r - \theta(p-2))} S_{L_p}(z, q^{-1}),
\]

where

\[
S_{L_p}(z, q) = \sum_{a=0}^{\infty} z^a q^{\frac{a^2-4a}{2}} \left( L_p - (a, 0, \ldots, 0) \right)_q,
\]

with \( L_p = \left( p\nu + \theta, 0, 0, \ldots, 0, 1, 0, \ldots, 0 \right). \)

These characters can be rewritten as

\[
\text{char } W_{r,p;\theta}^+ \alpha(z, q) = \sum_{N_1, N_2, \ldots, N_{p-2}} z^n q^{\frac{n^2}{2} - \theta n + \sum_{m=1}^{p-2} N_m^2 + \sum_{m=r}^{p-2} N_m} \times
\]

\[
\times \left[ \begin{array}{c} 2l + \theta - r + 1 - n \\ l + N_1 \end{array} \right]_{q} \left[ \begin{array}{c} l + N_2 \\ l + N_1 \end{array} \right]_{q} \cdots \left[ \begin{array}{c} l + N_{p-r-1} \\ l + N_{p-r} \end{array} \right]_{q} \times
\]

\[
\times \left[ \begin{array}{c} l + N_{p-r-1} + 1 \\ l + N_{p-r} + 1 \end{array} \right]_{q} \left[ \begin{array}{c} l + N_{p-r} + 1 \\ l + N_{p-r+1} + 1 \end{array} \right]_{q} \cdots \left[ \begin{array}{c} l + N_{p-3} + 1 \\ l + N_{p-2} + 1 \end{array} \right]_{q},
\]

where \( n = \sum_{m=1}^{p-2} N_m. \) It is easy to verify that this expression has the correct limit as \( \nu \to \infty, \)

\[
\lim_{\nu \to \infty} \left( z^{1 - \frac{1}{p}} q^{\frac{1}{2}(1 - \frac{1}{p})(\theta^2 - \theta) + \frac{\theta - 1}{p}} \right) \text{char } W_{r,p;\theta}^+ \alpha(z, q) = \text{char } W_{r,p;\theta}(z, q),
\]

where the character char \( W_{r,p;\theta}(z, q) \) is given by (4.36).

6.2. The \( N=2 \) modular functor and functions on Riemann surfaces. The representation of the unitary modules via semi-infinite forms implies a relation between the \( N=2 \) modular functor and the spaces of skew-symmetric functions with prescribed singularities on Cartesian powers of a genus-\( g \) Riemann surface.

6.2.1. \( N=2 \) correlation functions in the semi-infinite picture. Let \( \mathcal{E}_n^g \) be a genus-\( g \) Riemann surface with \( n \) marked points \( P_1, \ldots, P_n \) and let \( (N=2)^{\text{out}} \) denote the algebra generated by the part of the \( N=2 \) currents \( \mathcal{G}(z), \mathcal{Q}(z), \mathcal{H}(z), \) and \( \mathcal{L}(z) \) that is holomorphic outside the points \( P_1, \ldots, P_n. \) One then defines the space of coinvariants

\[
\mathcal{Z} = \frac{\mathcal{R}_{r_1,p;\theta_1} \otimes \mathcal{R}_{r_2,p;\theta_2} \otimes \cdots \otimes \mathcal{R}_{r_n,p;\theta_n}}{(N=2)^{\text{out}} \mathcal{R}_{r_1,p;\theta_1} \otimes \mathcal{R}_{r_2,p;\theta_2} \otimes \cdots \otimes \mathcal{R}_{r_n,p;\theta_n}},
\]

where \( \mathcal{R}_{r_i,p;\theta_i} \) are unitary representations and \( \mathcal{R}_{r_1,p;\theta_1} \otimes \mathcal{R}_{r_2,p;\theta_2} \otimes \cdots \otimes \mathcal{R}_{r_n,p;\theta_n} \) is a representation of the algebra \( (N=2)_{P_1} \oplus \cdots \oplus (N=2)_{P_n}. \) Similarly to [29], this space of coinvariants can be shown to be isomorphic to the space of coinvariants with respect to the algebra generated by \( \mathcal{G}(z) \) and \( \mathcal{H}_0, \)

\[
\mathcal{Z} = \left( \frac{\mathcal{R}_{r_1,p;\theta_1} \otimes \mathcal{R}_{r_2,p;\theta_2} \otimes \cdots \otimes \mathcal{R}_{r_n,p;\theta_n}}{g^{\text{out}} \mathcal{R}_{r_1,p;\theta_1} \otimes \mathcal{R}_{r_2,p;\theta_2} \otimes \cdots \otimes \mathcal{R}_{r_n,p;\theta_n}} \right)^0,
\]
where \( g^{\text{out}} \) is the algebra generated by the part of the \( G(z) \) current that is holomorphic outside \( P_1, \ldots, P_n \) and \( (\cdot)^0 \) denotes the restriction to the zero-charge component (the zero-charge restriction comes from taking the coinvariants with respect to \( H_0 \)).

For a given set of representations \( \mathfrak{R}_{r_1,p_1 \theta_1}, \ldots, \mathfrak{R}_{r_n,p_n \theta_n} \) placed at the points \( P_1, \ldots, P_n \) on the Riemann surface and for fixed \( m \) and \( t_1, \ldots, t_n \), each linear functional \( \langle F \rangle : \mathfrak{N} \to \mathbb{C} \) defines the function of of \( (x_1, \ldots, x_m) \in \mathcal{G}_n^g \times \cdots \times \mathcal{G}_n^g \) given by

\[
\langle F | G(x_1) G(x_2) \ldots G(x_m) | \Phi_{r_1, \theta_1}(P_1) \otimes \Phi_{r_2, \theta_2}(P_2) \otimes \cdots \otimes \Phi_{r_n, \theta_n}(P_n) \rangle^g_p,
\]

where \( \Phi_{r, \theta} \) are the operators corresponding to extremal vectors \( (3.1) \). In accordance with the chosen conformal dimension of \( G \), expression (6.18) is a 2-differential in each variable. We rewrite this expression in local coordinates by separating the correlation function,

\[
F_{r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n}^{p, g, t_1, \ldots, t_n}(x_1, x_2, \ldots, x_m | P_1, P_2, \ldots, P_n)(dx_1)^2 (dx_2)^2 \cdots (dx_m)^2 =
\]

\[
\langle F | G(x_1) G(x_2) \ldots G(x_m) | \Phi_{r_1, \theta_1}(P_1) \otimes \Phi_{r_2, \theta_2}(P_2) \otimes \cdots \otimes \Phi_{r_n, \theta_n}(P_n) \rangle^g_p.
\]

The restriction to the zero charge component takes the form of the constraint

\[
mp + n - \left( r_1 + (p - 2)(\theta_1 + p_1) \right) - \cdots - \left( r_n + (p - 2)(\theta_n + p_n) \right) = 0.
\]

It follows that the functions \( F_{r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n}^{p, g, t_1, \ldots, t_n} \) defined in (6.19) are antisymmetric in \( x_1, \ldots, x_m \), are regular on \( \mathcal{G}_n^g \times \cdots \times \mathcal{G}_n^g \) except at the points \( P_i \), and possess the following properties.

1. On each \((p - 1)\)-diagonal \( x_{i_1} = x_{i_2} = \cdots = x_{i_p - 1} \), one has

\[
\frac{\partial^{p-2}}{\partial x_{i_1}^{p-2}} \frac{\partial^{p-3}}{\partial x_{i_2}^{p-3}} \cdots \frac{\partial}{\partial x_{i_p-2}} F_{r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n}^{p, g, t_1, \ldots, t_n} |_{x_{i_1}=x_{i_2}=\cdots=x_{i_p-1}} = 0.
\]

2. For each \( a \) such that \( a < r_j \), the function

\[
\frac{\partial^{a-1}}{\partial x_{i_1}^{a-1}} \frac{\partial^{a-2}}{\partial x_{i_2}^{a-2}} \cdots \frac{\partial}{\partial x_{i_a}^{a-2}} F_{r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n}^{p, g, t_1, \ldots, t_n}
\]

vanishes at \( x_{i_1} = x_{i_2} = \cdots = x_{i_a} = x \to P_j \) with an order not less than

\[
-a(\theta_j + \lambda_j p) + \frac{a(a - 3)}{2}.
\]

For each \( a \) such that \( r_j \leq a \leq p - 2 \), the function

\[
\frac{\partial^a}{\partial x_{i_a}^a} \frac{\partial^{r+1}}{\partial x_{i_{r+1}}^{r+1}} \frac{\partial^r}{\partial x_{i_r}^r} \frac{\partial^{r-2}}{\partial x_{i_{r-2}}^{r-2}} \cdots \frac{\partial^{r-1}}{\partial x_{i_2}^{r-1}} F_{r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n}^{p, g, t_1, \ldots, t_n}
\]

vanishes at \( x_{i_1} = x_{i_2} = \cdots = x_{i_a} = x \to P_j \) with the order not less than

\[
-a(\theta_j + \lambda_j p) + \frac{a(a - 4r_j + 3)}{2} - (r_j + 1)(r_j - 1)
\]

(negative-order zeros are poles). Condition (6.21) follows from the vanishing of (1.1), and (6.23) and (6.25) from relations (3.2)–(3.4).
We let $F_{r_1,r_2,...,r_n;\theta_1,\theta_2,...,\theta_n}(m)$ denote the space of all functions satisfying these properties; with Eq. (6.20) assumed to be satisfied, the notation is somewhat redundant; however, it is useful to keep $m$ as a free parameter and determine some other label from (6.20).

**Theorem 6.2.** For sufficiently large $m$, the assignment

\[
(F) \mapsto F_{r_1,r_2,...,r_n;\theta_1,\theta_2,...,\theta_n}(\cdot,\cdot,\cdot|P_1,P_2,\ldots,P_n)
\]

defined in (6.19) establishes an isomorphism between $Z^*$ and $F_{r_1,r_2,...,r_n;\theta_1,\theta_2,...,\theta_n}(m)$. For these $m$, the dimensions $d_{r_1,r_2,...,r_n;\theta_1,\theta_2,...,\theta_n} = \dim F_{r_1,r_2,...,r_n;\theta_1,\theta_2,...,\theta_n}(m)$ are independent of $m$ and $\iota_1,\ldots,\iota_n$.

The semi-infinite construction is essential in proving that all functions with the properties described above are the $N=2$ correlation functions. We do not give the proof here and consider only two examples for illustration. One example is a simple enumeration of functions on a single-punctured elliptic curve with the vacuum representation. The dimension of the space of these functions gives the number of primary fields in the corresponding minimal model. For $p=4$, we explicitly construct a basis in the functional space. The second example is with a three-punctured Riemann sphere, where the dimensions of the corresponding functional spaces are related to the $N=2$ fusion algebra; we digress in Sec 6.3 to derive the unitary $N=2$ fusion algebra from the $\widehat{\mathfrak{sl}}(2)$ unitary fusion algebra.

**6.2.2. Tori with one marked point for small $p$.** For $p=3$, we consider the vacuum module associated with a point on a torus (at the origin in the covering complex plane). Setting $n=1$ and $r_1=1$ in condition (5.20) then gives $\iota=m$. The independence from $m$ already occurs starting with $m=1$, and the space $F_{1,0}^{3,1}(1)$ (omitting the $\iota$ superscript for brevity) consists of functions on the torus with a pole at zero of an order $\leq 3$. We then have $d_{1,0}^{3,1} = 3$, and a basis in the space of such functions is

\[
1, \quad \varphi(x), \quad \varphi'(x),
\]

where

\[
\varphi(x) = \frac{1}{x^2} + \sum_{\omega \in \Lambda} \left( \frac{1}{(x-\omega)^2} - \frac{1}{\omega^2} \right)
\]

is the Weierstrass function, $\Lambda = \{m\omega_1 + n\omega_2 | m,n \in \mathbb{Z}\}$, $\omega_1,\omega_2 \in \mathbb{C}$, and $\Im(\omega_1/\omega_2) > 0$.

It is instructive to explicitly verify that the same dimension is also obtained for $m=2$, i.e., to verify that $\dim F_{1,0}^{3,1}(2) = d_{1,0}^{3,1} = 3$, where the calculation is entirely different because the basic condition (1) (see (6.21)) applies in this case. The corresponding space $F_{1,0}^{3,1}(2)$ consists of antisymmetric functions of two variables $x_1$ and $x_2$ with a pole of an order $\leq 6$ as $x_1 = x_2 \to 0$ and such that $(\partial f(x_1,x_2)/\partial x_1)|_{x_1=x_2} = 0$. This space has a basis

\[
f_1(x_1,x_2) = \varphi(x_1)^3 - 3\varphi(x_1)\varphi(x_2) + 3\varphi(x_1)\varphi(x_2)^2 - \varphi(x_2)^3,
\]

\[
f_2(x_1,x_2) = -g_2\varphi'(x_1) - \frac{1}{2}g_1\varphi(x_1)\varphi'(x_1) + \varphi(x_1)^3\varphi'(x_1) -
\]
\[
- \frac{3}{2} g_1 \phi(x_2) \phi'(x_1) + 3 \phi(x_2)^3 \phi'(x_1) + g_2 \phi'(x_2) + \frac{3}{2} g_1 \phi(x_1) \phi'(x_2) - 3 \phi(x_1)^3 \phi'(x_2) + \frac{1}{2} g_1 \phi(x_2) \phi'(x_2) - \phi(x_2)^3 \phi'(x_2),
\]

\[
f_3(x_1, x_2) = g_2 \phi(x_1) + g_1 \phi(x_1)^2 - 2 \phi(x_2)^3 \phi(x_2) - g_1 \phi(x_2)^2 + 2 \phi(x_1) \phi'(x_1) \phi'(x_2) - \phi(x_2) \phi'(x_1) \phi'(x_2),
\]

where \( g_1 = 30 \sum_{\omega \in \Lambda} \frac{1}{\omega^i} \) and \( g_2 = 140 \sum_{\omega \in \Lambda} \frac{1}{\omega^i} \). This recovers the dimension \( d_{1,0}^{3,1} = 3 \).

For \( p = 4 \), condition (6.20) becomes \( 2 \ell = m \). The first “sufficiently large” value of \( m \) is already \( m = 2 \), with \( \ell = 1 \). The corresponding space \( \mathcal{F}_{1,0}^{4,1}(2) \) consists of antisymmetric functions of two variables with a pole of an order \( \leq 4 \) at zero and such that the function \( (\partial^2 f(x_1, x_2) / \partial x_2^2)|_{x_1 = x_2 = x} \) has a pole of the order \( \leq 9 \) as \( x \to 0 \). A basis in the space of such functions can be chosen as

\[
(6.29) \quad f_1 = \phi(x_1) - \phi(x_2), \quad f_2 = \phi'(x_1) - \phi'(x_2), \quad f_3 = \phi(x_1)^2 - \phi(x_2)^2,
\]

\[
f_4 = \phi(x_1) \phi'(x_2) - \phi(x_2) \phi'(x_1), \quad f_5 = \phi(x_1)^2 \phi'(x_2) - \phi(x_2)^2 \phi'(x_1), \quad f_6 = \phi(x_1) \phi(x_2)^2 - \phi(x_2) \phi(x_1)^2,
\]

and we have \( d_{1,0}^{4,1} = 6 \).

These examples show that the dimensions of functional spaces coincide with the number of primary fields in the respective minimal model (i.e., with the dimensions of the modular functor for the torus).

### 6.3. The unitary \( N=2 \) fusion algebra

We consider the \( N=2 \) fusion algebra and then discuss its consequences for the functional spaces. The unitary \( N=2 \) fusion rules were derived in \[18\] from the Verlinde hypothesis. We give an alternative derivation, which starts with the \( \hat{\mathfrak{sl}}(2) \) fusion algebra; this derivation does not rely on the Verlinde theorem statement for the \( N=2 \) algebra, and the coincidence with the result in \[18\] may have an independent interest.

The fusion algebra \( \mathfrak{F}_{\hat{\mathfrak{sl}}(2)}(k) \) of the unitary level-\( k \) \( \hat{\mathfrak{sl}}(2) \) representations \( \mathfrak{L}_{r,k} \) is given by \[37\]

\[
(6.30) \quad \mathfrak{L}_{r_1,k} \otimes \mathfrak{L}_{r_2,k} = \bigoplus_{r_3=|r_1-r_2|+1 \atop \text{step}=2} \mathfrak{L}_{r_3,k}, \quad 1 \leq r_i \leq k + 1.
\]

It is easy to apply the method of \[49\] to (6.30): tensoring the \( \hat{\mathfrak{sl}}(2) \) modules with the module \( \Omega \) over free fermions and applying (2.13), we use \( \hat{\mathfrak{sl}}(2) \) fusion rules and then collect the terms on the right-hand side so as to identify some \( \mathfrak{R}_{r',p;\theta'} \) representations. This gives the \( N=2 \) fusion algebra

\[
(6.31) \quad \mathfrak{R}_{r_1,\theta_1} \otimes \mathfrak{R}_{r_2,\theta_2} = \bigoplus_{r_3=|r_1-r_2|+1 \atop \text{step}=2} \mathfrak{R}_{r_3,p;\theta_1+\theta_2+\delta(1-r_1-r_2+r_3)}, \quad 1 \leq r_1, r_2 \leq p - 1, \ \theta_1, \theta_2 \in \mathbb{Z}_p.
\]
Remark 6.3 (the spectral flow). In the fusion algebra, $K_{1;0}$ is the identity and $\Theta = K_{p-1;0}$ is the spectral flow operator acting on representations in accordance with (6.31) as

$$K_{p-1;0} \otimes K_{2;p} = K_{p-2;p;\theta+1} = K_{r2;p;\theta+1}.\quad (6.32)$$

We also have

$$K_{p-1;0} \otimes \ldots \otimes K_{p-1;0} = K_{1;p;0} \equiv 1, \quad \text{as must be the case with the spectral flow on unitary representations.}$$

The above fusion algebra is only for the Neveu–Schwarz sector. To extend the fusion to the Ramond sector, it suffices to add a single element $K_{1;p;1/2}$ such that

$$K_{1;p;1/2} \otimes \ldots \otimes K_{1;p;1/2} = K_{r,p;\theta+1/2}.\quad (6.33)$$

This relation completely determines the fusion involving Ramond sector representations. Moreover, the fusion can be formally extended to any other “sector” with the fractional twists $\theta = \beta/\alpha$ by adding a single field $K_{1;p;1/\alpha}$ such that

$$K_{1;p;1/\alpha} \otimes \ldots \otimes K_{1;p;1/\alpha} = K_{r,p;\theta+1/\alpha}.\quad (6.34)$$

Remark 6.4. Fusion rules (6.31) mean that the three-point function is nonvanishing if and only if

$$r + r' + r'' - 2\theta - 2\theta' - 2\theta'' - 3 = 0,$$

$$|r' - r''| < r < r' + r'', \quad r + r' + r'' < 2p, \quad r + r' + r'' \equiv 1 \mod 2.\quad (6.35)$$

This agrees with the result in [48] under the Neveu–Schwarz sector correspondence

$$k = r - \theta - \frac{1}{2}, \quad j = \theta + \frac{1}{2}, \quad 1 \leq r \leq p - 1, \quad 0 \leq \theta \leq r - 1$$

between our parameterization of the unitary $\mathcal{N}=2$ modules and the parameterization used in [55], [48].

We now show how the correspondence between the $\hat{sl}(2)$ and $\mathcal{N}=2$ fusion algebras derived above fits into the general scheme between the modular functors expressed by (2.17). In (2.17), we take $\mathcal{E}$ to be a torus and recall that there is a basis in the modular functor on the torus whose elements are given by unitary representation characters (such bases, more precisely, depend on a cycle on the torus). The fusion algebra is then an operation on the modular functor for the torus.

It turns out that the correspondence in Eq. (2.17) agrees with natural structures on the modular functors, in the present case, with the fusion algebra. With the modular functors for the torus identified with the respective fusion algebras $\mathfrak{F}_a(\kappa)$ for $a = \hat{sl}(2)$ and $\mathcal{N}=2$, Eq. (2.17) becomes

$$\mathfrak{F}_{\mathcal{N}=2}(p) = \text{Coinv}_{(R_1 \otimes P_1)} \left( \text{Inv}_{(R_2 \otimes P_2)} \left( \mathfrak{F}_{sl(2)}(p-2) \otimes \mathfrak{F}_{\text{free}}(p) \right) \right),\quad (6.37)$$

where the invariants and coinvariants are taken with respect to the subalgebras generated by the elements $R_1$, $R_2$, $P_1$, and $P_2$ explicitly constructed in what follows. The algebra $\mathfrak{F}_{\text{free}}(p)$, which is
The tensor product $\hat{s}\ell(2)(p-2) \otimes \hat{s}\text{free}(p)$ for $p = 4$ (the $\hat{s}\ell(2)$ level $k = 2$). The dots represent $L_{r,2} \otimes \nu^i$ for $r = 1, 2, 3$ and $i = 0, \ldots, 7$ (with $i$ labeling columns). The solid dots are those for which $r + i - 1$ is even. The fusion algebra for the algebra of vertex operators associated with the lattice $\sqrt{2p} \mathbb{Z}$ (see (2.13)), is isomorphic to the group algebra of the cyclic group $\mathbb{Z}_{2p}$ (as a linear space, it is represented by theta-functions of the level $2p = 2(k + 2)$, as can be seen from Eq. (2.14) for the characters). It carries an action of the Heisenberg group associated with half-periods, and we let $P_1$ and $P_2$ denote the corresponding generators (such that $P_1 P_2 = (-1)^p P_2 P_1$): if $\nu$, with $\nu^{2p} = 1$, is a generator of $\mathbb{Z}_{2p}$, we have

\begin{align}
P_1(\nu^i) &= \nu^{i+p}, \\
P_2(\nu^i) &= (-1)^i \nu^i.
\end{align}

Using the chosen basis, the Heisenberg group action on the $\hat{s}\ell(2)$ modular functor can be explicitly described as

\begin{align}
R_1(L_{r,k}) &= L_{k-r+2,k}, \\
R_2(L_{r,k}) &= (-1)^{r-1} L_{r,k}
\end{align}

(we recall that basis elements are identified with unitary $\hat{s}\ell(2)$ representations). We note that $R_2$ is an automorphism of the fusion algebra and $R_1$ is the spectral flow transform realized via fusion, $R_1(L_{r,k}) = L_{k+1,k} \otimes L_{r,k}$. It has an important property that (omitting the fusion operation sign) $(L_{k+1,k}a)(L_{k+1,k}b) = ab$ for any $a, b \in \mathcal{F}_{\hat{s}\ell(2)}(k)$. Taking the diagonal action of the Heisenberg group, we now see that $R_1 \otimes P_1$ indeed commutes with $R_2 \otimes P_2$ (i.e., the central element acts identically).

In the tensor product $\mathcal{F}_{\hat{s}\ell(2)}(p-2) \otimes \mathcal{F}_{\text{free}}(p)$, we then take the invariants with respect to $R_2 \otimes P_2$, i.e., restrict to the elements of the form $L_{r,k} \otimes \nu^i$ with even $r + i - 1$ (see Fig. 7). The construction is then completed by taking coinvariants with respect to the action of $R_1 \otimes P_1$.

We now label the elements of the space constructed on the right-hand side of (6.37) as

\begin{equation}
K_{r,p;\theta} = L_{r,p-2} \otimes \nu^{2\theta - r + 1} \mod R_1 \otimes P_1
\end{equation}

and identify $K_{r,p;\theta}$ with the corresponding generator of the $N=2$ fusion algebra. The identification under the action of $R_1 \otimes P_1$ means that (2.9) is satisfied by construction; the periodicity under the spectral flow transform by $p$ is also obvious. Moreover, calculating

\begin{equation}
(L_{r,1,k} \otimes \nu^{\theta_1}) \circ \mathbb{N} = (L_{r,2,k} \otimes \nu^{\theta_2}) = (L_{r,1,p-2} \otimes L_{r,2,p-2}) \otimes \nu^{\theta_1 + \theta_2}
\end{equation}
and using (6.30), we obtain the unitary $N = 2$ fusion algebra (6.31). Therefore, the relation between the $\hat{sl}(2)$ and $N = 2$ fusion algebras can be considered as a particular case of the general relation (2.17) that extends the equivalence of categories [50], [49] to modular functors.

Returning to the functional spaces introduced in Sec. 6.2, we reformulate the $N = 2$ fusion algebra as a statement on the dimensions of functional spaces. The structure constants of the fusion algebra coincide with the dimensions of the modular functor on the three-punctured $\mathbb{CP}^1$. Recalling that this modular functor is related to the functional spaces via Theorem 6.2, we obtain (omitting the $\iota$ superscripts for brevity)

**Proposition 6.5.** For $m \geq p - 1$, the dimensions of the spaces $\mathcal{F}^{p,0}_{r,r',r'',\theta,\theta'}(m)$ and $\mathcal{F}^{p,0}_{r,r',\theta,\theta'}(m)$ are given by

$$
d^{p,0}_{r,r',r'',\theta,\theta'} = \dim \mathcal{F}^{p,0}_{r,r',r'',\theta,\theta'}(m) = \begin{cases} 1, & \text{conditions (6.33) are satisfied}, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
d^{p,0}_{r',\theta,\theta'} = \dim \mathcal{F}^{p,0}_{r',\theta,\theta'}(m) = \begin{cases} 1, & r' = r \quad \text{and} \quad \theta' = r - \theta - 1, \\ 0, & \text{otherwise}, \end{cases}
$$

where $1 \leq r, r', r'' \leq p - 1$ and $0 \leq \theta, \theta', \theta'' \leq r - 1$.

### 7. Concluding remarks

Possibly the most conspicuous feature of semi-infinite constructions is the asymmetry between the upper-triangular and lower-triangular generators, i.e., for the $N = 2$ algebra, between the fermions $G$ and $Q$: starting with a module generated by the modes $G_n$ subject to the conditions $\partial^{p-2} G(z) \ldots \partial G(z) G(z) = 0$, we could then (with much effort) reconstruct the action of the other algebra generators, including $Q_m$. However, the relation $\partial^{p-2} Q(z) \ldots \partial Q(z) Q(z) = 0$ is certainly satisfied for the current constructed from the $Q_m$ generators; it is interesting to investigate consequences of this relation for the corresponding functional spaces.

We expect that the methods developed above for the $N = 2$ superconformal algebra can also be useful in other semi-infinite constructions. In similar constructs, depending on the chosen constraints (the analogue of the relations $S^2 = 0$) a module structure can be found on the semi-infinite space. Even in the case with only one (bosonic or fermionic) field satisfying the constraints, interesting representations of $W$ algebras can thus be obtained. Nontrivial relations between $\hat{sl}(2|1)$ and $N=2$ [58] and $\hat{sl}(2)$ [59] representation theories allow us to expect interesting semi-infinite constructions for a class of $\hat{sl}(2|1)$ representations.

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