DOUBLE-LOOP ALGEBRAS AND THE FOCK SPACE

M. Varagnolo and E. Vasserot

Introduction. The main motivation of this article comes from physics: it is related to the Yangian symmetry in conformal field theory and the spinons basis. In a few words, it has been recently noticed that level one representations of the affine Lie algebra \( \widehat{\mathfrak{sl}}_n \) admit an action of a quantum group, the Yangian of type \( A_n^{(1)} \). A quantized version of this statement says that the Fermionic Fock space admits two different actions of the quantized enveloping algebra of \( \widehat{\mathfrak{sl}}_n \). The first one is a \( q \)-deformation of the well-known level-one representation of the affine Lie algebra (see [H], [KMS]). When \( q \) is one this representation may be viewed as a particular case of the Borel-Weil theorem for loop groups (see [PS]). The second one is a level-zero action arising from solvable lattices models (more precisely the Calogero-Sutherland and the Haldane-Shastry models, see [JKKMP], [TU], and the references therein). Quite remarkably these two constructions can be glued together to get a representation of a new object (introduced in [GKV] and [VV]): a toroidal quantum group, i.e. a two parameters deformation of the universal extension of the Lie algebra \( \mathfrak{sl}_n[x^\pm 1, y^\pm 1] \). The aim of this note is three-fold. First we define a representation of the quantized toroidal algebra, \( \mathcal{U} \), on the Fock space generalizing the two actions of the affine quantum group previously known. For that purpose we first construct an action on the space \( \Lambda^m(\mathbb{C}^n[z^\pm 1]) \) for any positive integer \( m \) by means of the Schur-type duality between \( \mathcal{U} \) and Cherednik’s double affine Hecke algebra established in [VV], then we explain how to perform the limit \( m \to \infty \). The second purpose of this article is to explain to which extend this representation can be viewed in geometrical terms, by means of correspondences on infinite flags manifolds. A complete geometric picture would require equivariant K-theory of some infinite dimensional variety. The correct definition of such K-groups will be done in another work (see [GKV] and [GG] for related works). We will mainly concentrate here on the algebraic aspects. At last, an essential point in the Fock space representation that we consider is that it involves some polynomial difference operators. It is due to the fact, proved in section 13, that the classical toroidal algebra, i.e. the specialization to \( q = 1 \) of the toroidal algebra, is isomorphic to the enveloping algebra of the universal central extension of a current Lie algebra over a quantum torus.

Y. Saito, K. Takemura and D. Uglov have obtained similar results in [STU]. The computations in the proof of the formulas (12.7-8) (formula (6.16) in [STU]), not written in the first version of our paper, are different but rely on the same results from [TU] and [VV].
Fix $q \in \mathbb{C}^\times$. The toroidal Hecke algebra of type $\mathfrak{gl}_m$, $\mathcal{H}_m$, is the unital associative algebra over $\mathbb{C}[x^\pm]$ with the generators $T_i^\pm, X_j^\pm, Y_j^\pm, i = 1, 2, ..., m - 1, j = 1, 2, ..., m$ and the relations

$$T_i T_i^{-1} = T_i^{-1} T_i = 1, \quad (T_i + q^{-1}) (T_i - q) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_i T_j = T_j T_i \quad \text{if } |i - j| > 1,$$

$$X_0 Y_i = x Y_i X_0, \quad X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i,$$

$$X_j T_i = T_i X_j, \quad Y_j T_i = T_i Y_j, \quad \text{if } j \neq i, i + 1$$

where $X_0 = X_1 X_2 \cdots X_m$. The algebra $\mathcal{H}_m$ has been introduced previously by Cherednik to prove the Macdonald conjectures (see [C1]). Let us mention that we have fixed the central element $x$ in a different way than in [C1]. Put $Q = X_1 T_1 \cdots T_{m-1} \in \mathcal{H}_m$. Then, $T_i^\pm, X_j^\pm, Q^\pm, i = 1, 2, ..., m - 1, j = 1, 2, ..., m$, is a system of generators of $\mathcal{H}_m$. Besides, for any $i = 1, 2, ..., m - 1$ a direct computation gives $Q Y_i Q^{-1} = Y_{i+1}$, and $Q Y_m Q^{-1} = x Y_1$. Indeed we have

**Proposition 1.** The toroidal Hecke algebra $\mathcal{H}_m$ admits a presentation in terms of generators $T_i^\pm, Y_j^\pm, Q^\pm, i = 1, 2, ..., m - 1, j = 1, 2, ..., m$, with relations

$$T_i T_i^{-1} = T_i^{-1} T_i = 1, \quad (T_i + q^{-1}) (T_i - q) = 0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_i T_j = T_j T_i \quad \text{if } |i - j| > 1,$$

$$Y_i Y_j = Y_j Y_i, \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1},$$

$$Y_j T_i = T_i Y_j, \quad \text{if } j \neq i, i + 1$$

$Q T_i Q^{-1} = T_i$ \quad (1 < i < m), \quad $Q^2 T_{m-1} Q^{-2} = T_1,$

$Q Y_i Q^{-1} = Y_{i+1} \quad (1 \leq i \leq m - 1), \quad Q Y_m Q^{-1} = x Y_1.$

**Remarks.** 1.1. Given a permutation $w \in \mathfrak{S}_m$ and a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ in terms of the simple transpositions $s_1, s_2, ..., s_{m-1}$, set as usual $T_w = T_{i_1} T_{i_2} \cdots T_{i_k} \in \mathcal{H}_m$. In particular $T_w$ is independent of the choice of the reduced decomposition. Moreover, given a $m$-tuple of integers $a = (a_1, a_2, ..., a_m)$, denote by $X^a$ and $Y^a$ the corresponding monomials in the $X_i$’s and $Y_i$’s. It is known that the $X^a Y^b T_w$’s form a basis of $\mathcal{H}_m$.

1.2. One consequence of the existence of the basis of monomials above is that the subalgebra of $\mathcal{H}_m$ generated by the $T_i$’s and the $X_i$’s is isomorphic to the affine Hecke algebra of type $\mathfrak{gl}_m$. Let denote by $\mathcal{H}_m$ this subalgebra. Similarly, the subalgebra $\mathcal{H}_m \subset \mathcal{H}_m$ generated by the $T_i$’s alone is isomorphic to the finite Hecke algebra of type $\mathfrak{gl}_m$. 

The maps
\[ \omega : \quad T_i, Q, Y_i, x, q \mapsto -T_i^{-1}, (-q)^{-m-1}Q, Y_i^{-1}, x^{-1}, q, \]
\[ \gamma : \quad T_i, X_i, Y_i, x, q \mapsto T_i, Y_i^{-1}, X_i^{-1}, x, q, \]
extend uniquely to an involution and an anti-involution of \( \tilde{H}_m \). Let us remark that, if \( S_m, A_m \in \tilde{H}_m \) are the symmetrizer and the antisymmetrizer of \( \tilde{H}_m \), that is to say
\[ S_m = \sum_{w \in \mathcal{S}_m} q^{l(w)} T_w \quad \text{and} \quad A_m = \sum_{w \in \mathcal{S}_m} (-q)^{l(w)} T_w, \]
where \( l : \mathcal{S}_m \to \mathbb{N} \) is the length, then \( \omega(S_m) = q^{m(m-1)} A_m \).

2. Fix \( p \in \mathbb{C}^\times \) and set \( R_m = \mathbb{C}[z_1^\pm, z_2^\pm, \ldots, z_m^\pm] \). Consider the following operators in \( \text{End} (R_m) \):
\[ t_{i,j} = (1 + s_{i,j}) \frac{q^{-1}z_i - qz_j}{z_i - z_j} - q^{-1}, \quad 1 \leq i, j \leq m, \]
\[ x_i = t_{i-1,i} s_{i-1,i} \cdots t_{1,i} s_{1,i} D_i s_{i,m} t_{i,m}^{-1} \cdots s_{i,i+1} t_{i,i+1}^{-1}, \quad i = 1, 2, \ldots, m, \]
\[ y_i = z_i^{-1}, \quad i = 1, 2, \ldots, m, \]
where \( s_{i,j} \) acts on Laurent polynomials by permuting \( z_i \) and \( z_j \), and \( D_i \) is the difference operator such that \((D_i f)(z_1, z_2, \ldots, z_m) = f(z_1, \ldots, qz_i, \ldots, z_m)\). The following result was first noticed by Cherednik.

**Proposition 2.** The map
\[ T_i \mapsto t_{i,i+1}, \quad Y_i \mapsto y_i, \quad Q \mapsto D_1 s_{1,m} s_{1,m-1} \cdots s_{1,2}, \quad x \mapsto p, \]
extends to a representation of \( \tilde{H}_m \) in \( R_m \). Moreover, in this representation \( X_i \) acts as \( x_i \).

**Remark 2.** Let us recall that \( \tilde{H}_m \subset \hat{H}_m \) is the subalgebra generated by the \( T_i \)'s and the \( X_i \)'s. The trivial module of \( \hat{H}_m \) is the one-dimensional representation such that \( T_i \) acts by \( q \) and \( X_i \) by \( q^{2i-m-1} \). Then \( R_m \) is the \( \hat{H}_m \)-module induced from the trivial representation of \( \tilde{H}_m \). In other words, if \( I_m \subset \hat{H}_m \) is the left ideal generated by the \( T_i - q \)'s and the \( X_i - q^{2i-m-1} \)'s, then \( R_m \) is identified with the quotient \( \hat{H}_m/I_m \) where \( \hat{H}_m \) acts by left translations.

3. Fix \( d \in \mathbb{C}^\times \) and an integer \( n \geq 3 \). The toroidal quantum group of type \( \mathfrak{sl}_n, \hat{U} \), is the complex unital associative algebra generated by \( e_{i,k}, f_{i,k}, h_{i,l}, k_i^{\pm 1} \), where \( i = 0, 1, \ldots, n-1, k \in \mathbb{Z}, l \in \mathbb{Z}^\times \), and the central elements \( c^{\pm 1} \). The relations are expressed in term of the formal series
\[ e_i(z) = \sum_{k \in \mathbb{Z}} e_{i,k} z^{-k}, \quad f_i(z) = \sum_{k \in \mathbb{Z}} f_{i,k} z^{-k}, \]
and \( k_i^{\pm}(z) = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{k \geq 1} h_{i,\pm k} z^{\pm k} \right) \), as follows
\[ k_i \cdot k_i^{-1} = c \cdot c^{-1} = 1, \quad [k_i^{\pm}(z), k_j^{\pm}(w)] = 0, \]
\[
\theta_{-a_{ij}}(c^2 d^{-m_{ij} w} z^{-1}) \cdot k_i^\pm(w) = \theta_{a_{ij}}(c^{-2} d^{-m_{ij} w} z^{-1}) \cdot k_j^\pm(w) \cdot k_i^\pm(z),
\]
\[
k_i^\pm(z) \cdot e_j(w) = \theta_{a_{ij}}(c^{-1} d^{-m_{ij}} w z^{\pm 1}) \cdot e_j(w) \cdot k_i^\pm(z),
\]
\[
k_i^\pm(z) \cdot f_j(w) = \theta_{-a_{ij}}(cd^{-m_{ij}} w^{\pm 1} z^{\pm 1}) \cdot f_j(w) \cdot k_i^\pm(z),
\]
\[
(q - q^{-1})[e_i(z), f_j(w)] = \delta(i = j) \left( \epsilon(c^{-2} \cdot z/w) \cdot k_i^+(c \cdot w) - \epsilon(c^2 \cdot z/w) \cdot k_i^-(c \cdot z) \right),
\]
\[
(d^{m_{ij}} - q^{-m_{ij}} w) \cdot e_i(z) \cdot e_j(w) = (q^{a_{ij}} d^{m_{ij}} z - w) \cdot e_j(w) \cdot e_i(z),
\]
\[
(q^{a_{ij}} d^{m_{ij}} z - w) \cdot f_i(z) \cdot f_j(w) = (d^{m_{ij}} z - q^{a_{ij}} w) \cdot f_j(w) \cdot f_i(z),
\]
\[
\{e_i(z_1) \cdot e_i(z_2) \cdot e_j(w) - (q + q^{-1}) \cdot e_i(z_1) \cdot e_j(w) \cdot e_i(z_2) + e_j(w) \cdot e_i(z_1) \cdot e_i(z_2)\}
\]
\[
+ \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,
\]
\[
\{f_i(z_1) \cdot f_i(z_2) \cdot f_j(w) - (q + q^{-1}) \cdot f_i(z_1) \cdot f_j(w) \cdot f_i(z_2) + f_j(w) \cdot f_i(z_1) \cdot f_i(z_2)\}
\]
\[
+ \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,
\]
\[
[e_i(z), e_j(w)] = [f_i(z), f_j(w)] = 0 \quad \text{if } a_{ij} = 0,
\]
where \(\epsilon(z) = \sum_{n=0}^{\infty} z^n, \; \theta_m(z) \in C[[z]]\) is the expansion of \(q^{m/z-q^{-m}}\), and \(a_{ij}, \; m_{ij}\),

are the entries of the following \(n \times n\)-matrices

\[
A = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 2 & -1 \\
-1 & 0 & \cdots & 1 & -2
\end{pmatrix}, \quad
M = \begin{pmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

Let \(\hat{U}\) be the quantized enveloping algebra of \(\widehat{sl}_n\), i.e. the algebra generated by \(e_i, f_i, k_i^{\pm 1}\) with \(i = 0, 1, ..., n-1\) modulo the Kac-Moody type relations

\[
k_i \cdot k_i^{\pm 1} = 1, \quad k_i \cdot k_j = k_j \cdot k_i,
\]
\[
k_i \cdot e_j = q^{a_{ij}} e_j \cdot k_i, \quad k_i \cdot f_j = q^{-a_{ij}} f_j \cdot k_i,
\]
\[
[e_i, f_j] = \delta(i = j) \frac{k_i - k_i^{-1}}{q - q^{-1}},
\]

and, if \(i \neq j\),

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0,
\]

where

\[
e_i^{(k)} = e_i^k/[k]!, \quad f_i^{(k)} = f_i^k/[k]!,
\]
\[
[k] = \frac{q^k-q^{-k}}{q-q^{-1}}, \quad [k]! = [k][k-1] \cdots [1].
\]
Let us recall that $\hat{U}$ admits another presentation, the Drinfeld new presentation (see [D]), similar to the presentation of $\hat{U}$ above. The isomorphism between the two presentations of $\hat{U}$ is announced in [D] and proved in [B].

As indicated in [GKV] the algebra $\hat{U}$ contains two remarkable subalgebras, $\hat{U}_h$ and $\hat{U}_v$, both isomorphic to a quotient of $\hat{U}$. The first one, the horizontal subalgebra, is generated by $e_{i0}, f_{i0}, k^\pm_1$, with $i = 0, 1, ... , n - 1$. These elements satisfy the above relations. The second one, the vertical subalgebra, is generated by $d^k e_{i,k}, d^k h_{i,k}, k^\pm_1$, where $i = 1, 2, ... , n - 1, k \in \mathbb{Z}$ and $l \in \mathbb{Z}^\times$. These elements satisfy the relations of the new presentation of $\hat{U}$. Fix $e_i = e_{i,0}$ and $f_i = f_{i,0}$, for any $i = 0, 1, ... , n - 1$. It is convenient to fix an additional triple of elements $e_n, f_n$ and $k^\pm_n$ such that $\hat{U}_v$ is generated by $e_i, f_i, k^\pm_i$, with $i = 1, 2, ... , n$, satisfying the previous Kac-Moody type relations.

4. For any complex vector space $V$ and any formal variable $\zeta$ denote by $V[\zeta^{\pm1}]$ the tensor product $V \otimes \mathbb{C}[\zeta^{\pm1}]$. Let $v_1, v_2, ... , v_n$ be a basis of $\mathbb{C}^n$. Set $v_{i + nk} = v_i \zeta^{-k}$ for all $i = 1, 2, ... , n$ and all $k \in \mathbb{Z}$. The vectors $v_i, i \in \mathbb{Z}$, form a basis of $\mathbb{C}^n[\zeta^{\pm1}]$. Given $k \in \mathbb{Z}$, write $k = n \bar{k} + \overline{\tau}$, where $\bar{k}$ is a certain integer and $\overline{\tau} \in \{1, 2, ... , n\}$. The space $\mathbb{C}^n[\zeta^{\pm1}]$ is endowed with a representation of the quantized enveloping algebra of $\mathfrak{sl}_n$, $\hat{U}$, such that the Kac-Moody generators act as

\[
\begin{align*}
e_i(v_j) &= \delta(\overline{j} = i + 1) v_{j - 1}, \\
f_i(v_j) &= \delta(\overline{j} = i) v_{j + 1}, \\
k_i(v_j) &= q^{\delta(\overline{j - i}) - \delta(\overline{i + j})} v_j,
\end{align*}
\]

for all $j \in \mathbb{Z}$ and $i = 0, 1, 2, ... , n - 1$, where $\delta(P)$ is 1 if the statement $P$ is true and 0 otherwise.

5. Consider now the tensor product $\bigotimes^m \mathbb{C}^n[\zeta^{\pm1}] = (\mathbb{C}^n)^{\otimes m}[\zeta_1^{\pm1}, ... , \zeta_m^{\pm1}]$. The monomials in the $v_i$’s are parametrized by $m$-tuple of integers. Such a $m$-tuple can be viewed as a function $j: \mathbb{Z} \to \mathbb{Z}$ such that $j(k + m) = j(k) + n$ for all $k$: the map $j$ is simply identified with the $m$-tuple $(j_1, j_2, ... , j_m) = (j(1), j(2), ... , j(m))$. Let $P_m$ be the set of all such functions. Then, $\bigotimes^m \mathbb{C}^n[\zeta^{\pm1}]$ is endowed with a $\hat{U}$ action generalizing the representation in section 4 as follows (see [GRV]): if $j \in P_m$,

\[
\begin{align*}
e_i(v_j) &= q^{-\delta j^{-1}(i)} \sum_{k \in j^{-1}(i+1)} q^{2\delta \{l \in j^{-1}(i) \mid l \leq k\}} v_k^+, \\
f_i(v_j) &= q^{-\delta j^{-1}(i+1)} \sum_{k \in j^{-1}(i)} q^{2\delta \{l \in j^{-1}(i+1) \mid l \leq k\}} v_k^-, \\
k_i(v_j) &= q^{\delta j^{-1}(i) - \delta j^{-1}(i+1)} v_j,
\end{align*}
\]

where $i = 0, 1, ... , n - 1, v_j = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_m}$ and $j_k^\pm$ is the function associated to the $m$-tuple $(j_1, j_2, ... , j_k \pm 1, ... , j_m)$. This action commutes with the action of $H_m$ such that:

- $T_k, k = 1, 2, ... , m - 1$, is represented by $\tau_{k,k+1}$, where $\tau_{k,l}$ is the automorphism
of $\bigotimes^m \mathbb{C}^n[\zeta^\pm 1]$ which acts on the $k$-th and $l$-th components as, $\forall i, j \in \mathbb{Z}$,

$$v_{ij} = v_i \otimes v_j \mapsto \begin{cases} q v_{ij} & \text{if } i = j, \\ q^{-1}v_{ji} & \text{if } i < j, \\ q v_{ji} + (q - q^{-1})v_{ij} & \text{if } i > j, \end{cases}$$

and which acts trivially on the other components.

- $Q$ is represented by $q : v_{i_1 i_2 \ldots i_m} \mapsto v_{i_m, i_1 \ldots i_{m-1}} = v_{i_m, i_1 \ldots i_{m-1}} \zeta_1$.

We do not prove here that the operators above satisfy the relations of $\hat{U}$ and $\hat{H}_m$ since it will follow immediately from the results in section 7 or the results in section 6.

6. Geometrically the formulas in the previous section may be viewed as follows. Suppose that $q$ is a prime power and let $F$ be the field with $q^2$ elements. Denote by $K = F((z))$ the field of Laurent power series. A lattice in $K^m$ is a free $F[z]$ submodule of $K^m$ of rank $m$. Let $B$ be the set of complete periodic flags, i.e. of sequences of lattices $L = (L_i)_{i \in \mathbb{Z}}$ such that

$$L_i \subset L_{i+1}, \quad \dim_{F}(L_{i+1}/L_i) = 1 \quad \text{and} \quad L_{i+m} = L_i \cdot z^{-1}.$$ 

Similarly $B^n$ is the set of $n$-steps periodic flags in $K^m$, i.e. of sequences of lattices $L^n = (L^n_i)_{i \in \mathbb{Z}}$ such that

$$L^n_i \subset L^n_{i+1}, \quad \text{and} \quad L^n_{i+m} = L^n_i \cdot z^{-1}.$$ 

The group $GL_m(K)$ acts on $B$ and $B^n$ in a natural way. The orbits of the diagonal action of $GL_m(K)$ on $B^n \times B$ are parametrized by $P_m$. If $e_1, e_2, \ldots, e_m \in K^m$ is the canonical basis and $e_{i+m} = e_i \cdot z^{-1}$ for all $k \in \mathbb{Z}$, we associate to $j \in P_m$ the orbit, say $O_j$, of the pair $(L^n_j, L)$ where

$$L^n_{j,i} = \prod_{j(j) \leq i} F e_j \quad \text{and} \quad L_i = \prod_{j \leq i} F e_j.$$ 

Let $C_{GL_m(K)}[B \times B]$ and $C_{GL_m(K)}[B^n \times B^n]$ be the convolution algebras of invariant complex functions supported on a finite number of orbits. It is well known that $C_{GL_m(K)}[B \times B]$ is isomorphic to $\hat{H}_m$ (see [IM]). Similarly if $i = 0, 1, \ldots, n - 1$ let $m_i, \chi_i^\pm, \chi_i^0 \in C_{GL_m(K)}[B^n \times B^n]$ be such that

- $m_i(L', L) = \dim(L_i/L_0), \quad \forall L, L' \in B^n,$
- $\chi_i^\pm$ is the characteristic function of the set
  $$\{(L^\pm, L^\mp) \in B^n \times B^n \mid L^-_j \subset L^+_j \quad \text{and} \quad \dim(L^+_j/L^-_j) = \delta(\overline{\tau} = \overline{j}), \quad \forall j \in \mathbb{Z} \},$$
- $\chi_i^0$ is the characteristic function of the diagonal in $B^n \times B^n$.

Then the map

$$e_i \mapsto q^{m_{i-1} - m_i} \chi_i^+, \quad f_i \mapsto q^{m_i - m_{i+1} - m_i} \chi_i^- \quad \text{and} \quad k_i \mapsto q^{2m_i - m_{i-1} - m_{i+1}} \chi_i^0,$$
Thus, there is an isomorphism. The convolution product
\[ \hat{U} \to \mathbb{C}_{GL_m}([\mathcal{B}^n \times \mathcal{B}]) \]. This statement is
stated without a proof in [GV; Theorem 9.2]. For the convenience of the reader a
proof is given in the appendix. Let us mention however that this computation is
nothing but an adaptation of the non affine case proved in [BLM]. As a consequence,
the convolution product induces an action of \( \hat{U} \) and \( \hat{H}_m \) on \( \mathbb{C}_{GL_m}([\mathcal{B}^n \times \mathcal{B}]) \).

**Proposition 6.** The isomorphism of vector spaces \( \bigotimes^m \mathbb{C}^n [\xi^\pm 1] \to \mathbb{C}_{GL_m}([\mathcal{B}^n \times \mathcal{B}] \) mapping \( v_j \) to the characteristic function of the orbit \( O_j \) is an isomorphism of
\( \hat{U} \times \hat{H}_m \)-modules between the representation of section 5 and the representation by convolution.

**Proof.** Put \( G = GL_m(\mathbb{K}) \) and let \( B \subset G \) be the Iwahori subgroup, i.e. the subgroup
of matrices mapping each \( e_i \) to a linear combination of the type
\[ \sum_{j \leq i} a_{ij} e_j \quad \text{with} \quad a_{ij} \in \mathbb{F}, \quad a_{ii} \neq 0. \]

Let us first compute the \( \hat{H}_m \)-action. In order to simplify the notations we fix \( m = 2 \).
For any \( a, b \in \mathbb{Z} \) let \( L(a, b) \subset \mathbb{K}^2 \) be the lattice with basis \( (e_{1+2a}, e_{2+2b}) \), i.e.
\[ L(a, b) = \left( \prod_{k \leq a} \mathbb{F} e_{1+2k} \right) \oplus \left( \prod_{k \leq b} \mathbb{F} e_{2+2k} \right). \]

Let us recall that the element \( L \in \mathcal{B} \) is the sequence of lattices such that \( L_0 = L(-1, -1) \), \( L_1 = L(0, -1) \) and \( L_{i+2} = L_i \cdot z^{-1} \) for any \( i \in \mathbb{Z} \). In particular,
\[ O_j \cap (\mathcal{B}^n \times \{ L \}) = (B \cdot L^n) \times \{ L \}. \]

By definition, the isomorphism \( \hat{H}_m \sim \mathbb{C}_G[\mathcal{B} \times \mathcal{B}] \) maps \( T_i \) to \( q^{-1} \) times the characteristic function of the \( G \)-orbit
\[ G \cdot (L', L) \subset \mathcal{B} \times \mathcal{B}, \]
where \( L'_0 = L_0 \) and \( L'_1 = L(-1, 0) \). For any \( t \in \mathbb{F} \) fix \( \phi_t \in G \) such that \( \phi_t(e_1) = t e_1 + e_2 \) and \( \phi_t(e_2) = e_1 \). The map
\[ \mathbb{F} \to (G \cdot (L', L)) \cap (\mathcal{B} \times \{ L \}), \quad t \mapsto (\phi_t(L), L) \]
is an isomorphism. The convolution product
\[ \ast : \mathbb{C}_G[\mathcal{B}^n \times \mathcal{B}] \otimes \mathbb{C}_G[\mathcal{B} \times \mathcal{B}] \to \mathbb{C}_G[\mathcal{B}^n \times \mathcal{B}] \]
is defined as
\[ f \ast g (L^n_k, L) = \sum_{L'' \in \mathcal{B}} f(L^n_k, L'') \cdot g(L'', L). \]

Thus,
\[ T_i(v_j) = q^{-1} \sum_{k \in P_2} n(j, k) v_k, \quad \text{where} \quad n(j, k) = \sharp \{ t \in \mathbb{F} \mid \phi_t^{-1}(L^n_k) \in B \cdot L^n_j \}. \]
Suppose first that \( j_1 = j_2 \). In this case \( L_j^n \) is a sequence of lattices of the type \( L(a, a) \) where \( a \in \mathbb{Z} \). Thus, \( L_j^n \) is fixed by \( B \) and by \( \phi_t \) for all \( t \) and \( T_1(v_j) = q v_j \).

Suppose that \( j_1 < j_2 \). Then \( L_j^n \) is a sequence of lattices of the type \( L(a, b) \) with \( a \geq b \) and the inequality is strict for at least one lattice in the sequence. The only possibility to get \( L_k^n \in \phi_t(B \cdot L_j^n) \) for some \( k \in \mathcal{P}_m \) and some \( t \in \mathbb{F} \) is that \( t = 0 \) and, then, necessarily \( k = (j_2, j_1) \). Thus, \( T_1(v_j) = q^{-1} v_{j_2j_1} \).

If \( j_1 > j_2 \) the formula for \( T_1(v_j) \) follows from the previous case and the relation \((T_1 + q^{-1})(T_1 - q)\).

As for the \( Q \) it is immediate that \( \vartheta \) is the convolution product by the characteristic function of the \( G \)-orbit of the pair \((L''_n, L)\), where \( L''_i = L_{i+1} \) for all \( i \in \mathbb{Z} \). Let us now compute the \( \mathcal{U} \)-action. The integer \( m \) is no longer supposed to be 2. We consider the convolution product

\[
\ast : C_G[\mathcal{B}^n \times \mathcal{B}^n] \otimes C_G[\mathcal{B}^n \times \mathcal{B}] \rightarrow C_G[\mathcal{B}^n \times \mathcal{B}].
\]

Given \( j, k \in \mathcal{P}_m^0 \) let \( \chi_{j,k} \in C_G[\mathcal{B}^n \times \mathcal{B}^n] \) be the characteristic function of the \( G \)-orbit of the pair \((L_j^n, L_k^n)\). By definition the map \( \mathcal{U} \rightarrow C_G[\mathcal{B}^n \times \mathcal{B}^n] \) send \( e_i, f_i \) and \( k_i \) respectively to

\[
\begin{align*}
\sum_{j \in \mathcal{P}_m^0} q^{-2j^{-1}(i)} \delta(j^{-1}(i + 1) \neq 0) \chi_{j+1,j}, \\
\sum_{j \in \mathcal{P}_m^0} q^{-2j^{-1}(i+1)} \delta(j^{-1}(i) \neq 0) \chi_{j,j}, \\
\sum_{j \in \mathcal{P}_m^0} q^{2j^{-1}(i)-2j^{-1}(i+1)} \chi_{j,j},
\end{align*}
\]

where \( s = \max j^{-1}(i) \) and \( i = 0, 1, \ldots, n - 1 \). Suppose first that \( j \in \mathcal{P}_m^0 \). Then

\[
\mathcal{O}_j \cap (\mathcal{B}^n \times \{L\}) = \{(L_j^n, L)\}.
\]

If \( j^{-1}(i + 1) \neq 0 \) then

\[
(G \cdot (L_{j+1}^n, L_j^n)) \cap (\mathcal{B}^n \times \{L_j^n\}) = \{L_{j+1}^n \mid k \in j^{-1}(i + 1) \} \times \{L_j^n\},
\]

and, thus,

\[
e_i(v_j) = q^{-2j^{-1}(i)} \sum_{k \in j^{-1}(i+1)} v_{j,k}^{-1}.
\]

The operator

\[
v_j \mapsto q^{-2j^{-1}(i)} \sum_{k \in j^{-1}(i+1)} q^{2s(l \in j^{-1}(i) \mid l > k)} v_{j,k}^{-1}, \quad \forall j \in \mathcal{P}_m,
\]

commutes to \( \mathcal{H}_m \). When \( j \in \mathcal{P}_m^0 \) it is precisely the action of \( e_i \) written above. Thus it coincides with \( e_i \). The proof is similar for \( f_i \) and \( k_i \).

\[\square\]

7. The affine quantum group \( \mathcal{U} \) is known to admit the structure of a Hopf algebra whose coproduct \( \Delta \) is such that for any \( i = 0, 1, \ldots, n - 1 \)

\[
\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i.
\]
As a consequence, $\mathbf{U}$ acts on $\bigotimes^m \mathbb{C}^n[\xi^{\pm 1}]$ by iterating the action of $e_i, f_i, k_i$ on $\mathbb{C}^n[\xi^{\pm 1}]$ given in section 4. Let us call this representation the tensor representation of $\mathbf{U}$. It is important to notice that the resulting representation of $\mathbf{U}$ is isomorphic to the geometric one given in section 5. The purpose of this section is to write explicitly such an isomorphism. In particular it will follows that the formulas in section 5 do define a representation of $\mathbf{U}$. Let us first prove the following technical result. According to the proposition 1 the action of $\mathbf{H}_m$ on $\bigotimes^m \mathbb{C}^n[\xi^{\pm 1}]$ described in section 5 restricts to a representation of the ring $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}]$.

Lemma 7. The space $\bigotimes^m \mathbb{C}^n[\xi^{\pm 1}]$ is a free module over the ring $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}]$, with basis the monomials $v_j$ such that $j$ belongs to the set

$$P_m^1 = \{ j \in P_m \mid 1 \leq j_1, j_2, \ldots, j_m \leq n \}.$$  

Proof. Let us consider $q$ as a formal variable. For any $j \in P_m$ we write $j = n\underline{j} + \underline{j}$, with $\underline{j} \in P_m$ and $\underline{j} \in P_m$. Let $F$ be the free $\mathbb{C}[q^{\pm 1}]$-module with basis $P_m$. Let $p$ be the map

$$p : F \rightarrow \bigoplus_{j \in P_m} \mathbb{C}[q^{\pm 1}] v_j, \quad j \mapsto X^{-\underline{j}}(v_{\underline{j}}).$$

The map $p$ is surjective. Namely for all $s, t, \underline{j}$, there exists a monomial $x$ in the $T_i^{\pm 1}$'s such that $\sigma_s x v_j = x(v_j)$. Thus, for all $\underline{j}$ there is a monomial $x$ in $Q^{\pm 1}$ and the $T_i^{\pm 1}$'s such that $v_{\underline{j}} = x(v_j)$. The surjectivity follows since the $X^a T_m$'s form a basis of $\mathbf{H}_m$. As for the injectivity, the kernel of $p$ is free and vanishes when $q = 1$. Thus we are done.

Remark 7.1. As a consequence if $P_m^0 = \{ j \in P_m \mid 1 \leq j_1 \leq j_2 \leq \ldots \leq j_m \leq n \}$, then $\bigotimes^m \mathbb{C}^n[\xi^{\pm 1}] = \sum_{j \in P_m^0} \mathbf{H}_m \cdot v_j$.

Let $\Psi$ be the unique $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}]$-linear automorphism of $\bigotimes^m \mathbb{C}^n[\xi^{\pm 1}]$ such that

$$\Psi(v_j) = q^{s(1 \leq s < t \leq m \mid j_s < j_t)} v_j, \quad \forall j \in P_m^1,$$

and let $\hat{\Phi}$ be the linear isomorphism

$$\hat{\Phi} : \bigotimes^m \mathbb{C}^n[X^{\pm 1}] \rightarrow \bigotimes^m \mathbb{C}^n[\xi^{\pm 1}], \quad X^a \otimes v_j \mapsto X^a \cdot \Psi(v_j), \quad \forall j \in P_m^1.$$

The following result is stated without a proof in [GRV] and the analogue in the finite case is given in [GL].

Proposition 7. For any $i = 0, 1, \ldots, n-1$, the operators $\hat{\Phi}^{-1} e_i \circ \hat{\Phi}$, $\hat{\Phi}^{-1} f_i \circ \hat{\Phi}$, and $\hat{\Phi}^{-1} k_i \circ \hat{\Phi}$ on $\bigotimes^m \mathbb{C}^n[X^{\pm 1}]$ do coincide with $\Delta^{m-1}(e_i)$, $\Delta^{m-1}(f_i)$, and $\Delta^{m-1}(k_i)$. Moreover for all $j \in P_m^1$ and all $P \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}]$,

$$\hat{\Phi}^{-1} T_k \circ \hat{\Phi}(P v_j) = \begin{cases} 
q P^{s_k} v_j + (q^{-1} - q) \frac{X_{k+1} P - P^{s_k}}{X_k - X_{k+1}} v_j & \text{if } j_k = j_{k+1}, \\
P^{s_k} \sigma_k v_j + (q^{-1} - q) \frac{X_k P - P^{s_k}}{X_{k+1} - X_k} v_j & \text{if } j_k < j_{k+1}, \\
P^{s_k} \sigma_k v_j + (q^{-1} - q) \frac{X_{k+1} P - X_k P^{s_k}}{X_k - X_{k+1}} v_j & \text{if } j_k > j_{k+1}.
\end{cases}$$
where $\sigma_k$ stands for $\sigma_{k,k+1}$ and $P^{s_k}$ is $P$ with $X_k$ and $X_{k+1}$ exchanged.

**Proof.** Since $T_k$ commutes with any polynomial symmetric in the variables $X_k$ and $X_{k+1}$, we get

\[
2T_k(P v_j) = T_k((P + P^{s_k}) + Q(X_k - X_{k+1})) v_j,
\]

\[
= (P + P^{s_k}) T_k v_j + QT_k(X_k - X_{k+1}) v_j,
\]

where $Q$ is a symmetric polynomial in $X_k$ and $X_{k+1}$. Now

\[
T_k X_k = X_{k+1} T_k + (q^{-1} - q) X_{k+1},
\]

\[-T_k X_{k+1} = (q^{-1} - q) X_{k+1} - X_k T_k,
\]

from which we get

\[
T_k(P v_j) = P^{s_k} T_k v_j + (q^{-1} - q) \frac{X_{k+1}(P - P^{s_k})}{X_k - X_{k+1}} v_j.
\]

Thus

\[
\dot{\Phi}^{-1} \circ T_k \circ \dot{\Phi}(P v_j) = q^{s_{\{1 \leq s < \ell \leq m | j_s < j_{s+1}\}}} P^{s_k} \Psi^{-1}(T_k v_j) + (q^{-1} - q) \frac{X_{k+1}(P - P^{s_k})}{X_k - X_{k+1}} v_j
\]

and the formulas follow from an easy case by case computation, according to the value of $T_k v_j$ as in section 5. As for $\dot{U}$, let consider the case of $e_i$, $i = 0, 1, ..., n - 1$, since the other cases are quite similar. Then

\[
\dot{\Phi}^{-1} \circ e_i \circ \dot{\Phi}(P v_j) =
\]

\[
= q^{s_{\{1 \leq s < \ell \leq m | j_s < j_{s+1}\}}} \sum_{k \in j^{-1}(i+1)} q^{2 \sharp \{l \in j^{-1}(i) | l > k\}} P \Psi^{-1}(v_{j_k}),
\]

\[
= \sum_{k \in j^{-1}(i+1)} q^{2 \sharp \{l \in j^{-1}(i) | l > k\} - \sharp \{l \in j^{-1}(i+1) | l > k\} + \sharp \{l \in j^{-1}(i) | l < k\}} P v_{j_k},
\]

\[
= \sum_{k \in j^{-1}(i+1)} m \\sum_{k=1}^m (1 \otimes \cdots \otimes e_i \otimes k_i \otimes \cdots \otimes k_i \otimes \cdots \otimes k_i) v_j,
\]

\[
= (\Delta^{m-1} e_i) v_j.
\]

\[\square\]

**Remark 7.2.** The action of the $T_i$'s on $\bigotimes^m \mathbb{C}[X^{\pm 1}]$ given above and the product by the $X_i$'s determine a representation of $\mathbb{H}_m$ on the tensor module, introduced for the first time in [GRV].

8. Let now consider the K-theoretic analogue of the previous construction in the same way as in [GRV], [GKV]. Fix another set of formal variables $z_1^{\pm 1}, ..., z_m^{\pm 1}$. The
purpose of this section is to explain how the actions of $\tilde{H}_m$ and $\tilde{U}$ on $\bigotimes^m \mathbb{C}[\zeta^{\pm 1}]$ defined in section 5 can be induced to commuting representations of $\tilde{H}_m$ and $\tilde{U}$ on the space $V_m = (\mathbb{C}^n)^{\otimes m} [\zeta_1^{\pm 1}, ..., \zeta_m^{\pm 1}, z_1^{\pm 1}, ..., z_m^{\pm 1}]$. The formulas for the induced action of $\tilde{H}_m$ may be taken from [C2] for instance. The generators $T_i, Y_i, Q$ act as follows: for any $P \in R_m$ and $v \in \bigotimes^m \mathbb{C}[\zeta^{\pm 1}]$,

\begin{align*}
T_i(v \cdot P) &= (\tau_i, i+1(v) - qv) \cdot s_{i, i+1}(P) + v \cdot t_{i, i+1}(P), \\
Y_i(v \cdot P) &= v \cdot P z_i^{-1}, \\
Q(v \cdot P) &= \vartheta(v) \cdot D_1 s_{1, m}s_{1, m-1} \cdots s_{1, 2}(P),
\end{align*}

and the central element $x$ goes to a fixed $p \in \mathbb{C}^\times$. The operators $t_{i, i+1}, s_{i, i+1}$ are defined in section 2 and $\tau_i, i+1, \vartheta$ are defined in section 5. As for the action of $\tilde{U}$ on $V_m$ let $e_i, f_i, k_i \in \text{End} \tilde{H}_m(V_m) (i = 0, 1, 2, ..., n - 1)$ be as in section 5 and consider an additional triple of operators $e_n, f_n, k_n \in \text{End} \tilde{H}_m(V_m)$ such that for all $j \in F_m$,

\begin{align*}
e_n(v_j) &= q^{2j-1}(n-1) p^{1/n} \sum_{k \in j-1(1)} q^{2k-1-m} Y_k^{-1} \cdot v_j \cdot \zeta_k^{-1}, \\
f_n(v_j) &= q^{2j-1}(1+1) p^{-1/n} \sum_{k \in j-1(1)} q^{n-2k+1} Y_k \cdot v_j \cdot \zeta_k, \\
k_n(v_j) &= q^{2j-1}(n-2j-1) p^{1/n} \cdot v_j.
\end{align*}

where $p^{1/n}$ is a fixed $n$-th root of $p$.

**Proposition 8.** The operators $e_i, f_i, k_i, i = 0, 1, 2, ..., n - 1$, (resp. $i = 1, 2, ..., n$) define an action of $\tilde{U}$ on $V_m$ commuting with $\tilde{H}_m$. Moreover these two actions of $\tilde{U}$ can be glued in a representation of $\tilde{U}$ commuting to $\tilde{H}_m$ if $d = q^{1/p^{1/n}}$.

Once again we do not prove this proposition here since it follows immediately from the proof of proposition 9.

9. Let us recall that $V_m = (\mathbb{C}^n)^{\otimes m}[\zeta_1^{\pm 1}, ..., \zeta_m^{\pm 1}, z_1^{\pm 1}, ..., z_m^{\pm 1}]$. In section 8 we have defined an action of $\tilde{H}_m$ and $\tilde{U}$ on $V_m$. The Schur duality is an equivalence of categories between finite dimensional representations of the symmetric group and of the linear group. It has been generalized to quantum groups by Jimbo, Drinfeld and Cherednik. In [VV] we proved a similar duality between $\tilde{H}_m$ and $\tilde{U}$. Since the $\tilde{U}$-action on $V_m$ commutes to $\tilde{H}_m$, to any right ideal $J \subset \tilde{H}_m$ we can associate a representation of the quantized toroidal algebra on the quotient $(J \cdot V_m) \backslash V_m$. Let consider the right $\tilde{H}_m$-module $J \backslash \tilde{H}_m$. The purpose of this section is to prove that the Schur dual of $J \backslash \tilde{H}_m$, as defined in [VV], is isomorphic to $(J \cdot V_m) \backslash V_m$. By definition, the underlying vector space of the Schur dual of $J \backslash \tilde{H}_m$ is

$$(J \backslash \tilde{H}_m) \otimes \tilde{H}_m (\mathbb{C}^n)^{\otimes m}.$$ 

In order to describe the action of $\tilde{U}$ on this space set

$$e_\theta(v_j) = \delta(j = n)v_1, \quad f_\theta(v_j) = \delta(j = 1)v_n, \quad k_\theta(v_j) = q^{\delta(j = 1)-\delta(j = n)}v_j,$$
and $f_{\theta,l} = 1^{\otimes l-1} \otimes f_{\theta} \otimes (k_{\theta}^{-1})^{\otimes m-l}$, $e_{\theta,l} = k_{\theta}^{\otimes l-1} \otimes e_{\theta} \otimes 1^{\otimes m-l}$, for all $j = 1, 2, ..., n$ and $l = 1, 2, ..., m$. Then for all $j \in \mathcal{P}_m^1$ and $x \in J \backslash \mathcal{H}_m$,

$$e_0(x \otimes v_j) = \sum_{l=1}^{m} x \cdot X_l \otimes f_{\theta,l}(v_j), \quad f_0(x \otimes v_j) = \sum_{l=1}^{m} x \cdot X_l^{-1} \otimes e_{\theta,l}(v_j),$$

$$e_n(x \otimes v_j) = q^{-l} \cdot p^{l/n} \sum_{l=1}^{m} x \cdot Y_l^{-1} \otimes f_{\theta,l}(v_j), \quad f_n(x \otimes v_j) = q^{-l} \cdot p^{l/n} \sum_{l=1}^{m} x \cdot Y_l \otimes e_{\theta,l}(v_j),$$

and, if $i = 1, 2, ..., n - 1$,

$$e_i = 1 \otimes \Delta^{m-1} e_i, \quad f_i = 1 \otimes \Delta^{m-1} f_i$$

(see [VV] and [CP] for more details). In the particular case $J = \{0\}$, the Schur dual of the right regular module $\mathcal{H}_m$ is endowed with an action of $\mathcal{H}_m$ by left translations which commutes to $\mathcal{H}_m$.

**Proposition 9.** Fix $d = q^{-l} \cdot p^{l/n}$. The $\mathcal{U} \times \mathcal{H}_m$-module $V_m$ is isomorphic to the Schur dual of the right regular representation of $\mathcal{H}_m$. As a consequence the quotient $(J \cdot V_m) \backslash V_m$ is isomorphic to the Schur dual of $\mathcal{H}_m$ for any right ideal $J \subset \mathcal{H}_m$.

**Proof.** The proposition 7 implies that the bijection

$$\hat{\varphi} : \mathcal{H}_m \otimes \mathcal{H}_m (\mathbb{C}^n)^{\otimes m} \rightarrow \mathcal{H}_m \otimes \mathbb{C}^n[X^{\pm 1}],$$

intertwines the tensor representation of $\mathcal{U}$ and $\mathcal{H}_m$ given in section 7, and the convolution action of $\mathcal{U}$ and $\mathcal{H}_m$ described in section 5. This isomorphism extends to a bijection $\tilde{\varphi}$

$$\tilde{\varphi} : \tilde{\mathcal{H}}_m \otimes \mathcal{H}_m (\mathbb{C}^n)^{\otimes m} \rightarrow V_m, \quad Y^b X^a \otimes v \mapsto Y^b X^a \cdot \Psi(v), \quad \forall v \in (\mathbb{C}^n)^{\otimes m},$$

where $\tilde{\mathcal{H}}_m$ acts on $V_m$ as in section 8. This morphism is well defined because the monomials $Y^b X^a$ form a basis of $\mathcal{H}_m$ as a right $\mathcal{H}_m$-module. Now,

1. $V_m$ is the $\mathcal{H}_m$-module induced from the $\mathcal{H}_m$-module $\mathcal{H}_m \otimes \mathbb{C}^n[X^{\pm 1}]$ (see section 8),

2. $\tilde{\mathcal{H}}_m \otimes \mathcal{H}_m (\mathbb{C}^n)^{\otimes m}$ is isomorphic to the $\mathcal{H}_m$-module $\mathcal{H}_m \otimes \mathcal{H}_m (\mathbb{C}^n)^{\otimes m}$,

3. $\tilde{\varphi}$ is an isomorphism of $\mathcal{H}_m$-modules.

Thus $\tilde{\varphi}$ is an isomorphism of $\mathcal{H}_m$-modules. As for the $\tilde{\mathcal{U}}$-action let us do a direct computation. Since $\tilde{\varphi}$ commutes with the left action of $\mathcal{H}_m$ it suffices to prove that for all $j \in \mathcal{P}_m^0$ and all $i = 0, 1, ..., n$,

$$\tilde{\varphi} \circ e_i(1 \otimes v_j) = e_i \circ \tilde{\varphi}(1 \otimes v_j) \quad \text{and} \quad \tilde{\varphi} \circ f_i(1 \otimes v_j) = f_i \circ \tilde{\varphi}(1 \otimes v_j).$$

If $i \neq n$ this has already been proved in the proposition 7. As for the remaining
cases, we get
\[ \Phi \circ e_n(1 \otimes v_j) = q^{-1}p^{1/n} \sum_{l=1}^{m} \Phi(Y_l^{-1} \otimes f_{\theta,l}(v_j)), \]
\[ = q^{-1}p^{1/n} \sum_{l=1}^{m} Y_l^{-1} \cdot \Psi(f_{\theta,l}(v_j)), \]
\[ = q^{-1}p^{1/n} \sum_{l \in J_1} q^{2l-1}(n-2l^{-1}+l)Y_l^{-1} \cdot \Psi(v_l^{-1}l^{-1}), \]
\[ = q^{2l-1}(n-2l^{-1}+\sum_{s<j} q) \sum_{l \in J_1} p^{1/n} q^{2l-1-l}Y_l^{-1} \cdot v_l^{-1}l^{-1}, \]
\[ = e_n \circ \Phi(1 \otimes v_j). \]
The computation for \( f_n \circ \Phi \) is similar. \( \square \)

10. In the remaining three sections we fix \( d = q^{-1}p^{1/n} \). Let consider the right ideal \( \gamma \omega(I_m) \subset \tilde{H}_m \), where \( I_m \) is the ideal defined in section 2. Let \( \wedge^m \) be the quotient \((\gamma \omega(I_m) \cdot V_m) \setminus V_m \). Recall that \( \tilde{H}_m \) acts on the tensor module\[ \bigotimes^m \mathbb{C}^n[X^{\pm 1}] = (\mathbb{C}^n)^{\otimes m}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}] \]as in the proposition 7. Then, set (see [KMS])
\[ \Omega = \sum_{i=1}^{m-1} \ker (T_i - q) \subset \bigotimes^m \mathbb{C}^n[X^{\pm 1}]. \]

**Lemma 10.** The space \( \wedge^m \) is isomorphic to the quotient \( \bigotimes^m \mathbb{C}^n[X^{\pm 1}]/\Omega \).

**Proof.** By definition, \( \wedge^m \) is isomorphic to
\[ (\gamma \omega(I_m) \cdot \tilde{H}_m) \otimes H_m (\mathbb{C}^n)^{\otimes m}. \]
The left ideal \( I_m \) is generated by the \( T_i - q \)'s and by \( Q - 1 \). Thus \( \gamma \omega(I_m) \) is the right ideal of \( \tilde{H}_m \) generated by the \( T_i + q^{-1} \)'s and the \( Y_i - q^{2i-2} \)'s. Since the monomials \( Y^b X^a T_w, k \in \mathbb{Z}, a, b \in \mathbb{Z}^m, w \in \mathfrak{S}_m \), form a basis of \( \tilde{H}_m \), the map\[ (\gamma \omega(I_m) + Y^b X^a T_w) \otimes v \mapsto q^{2s} \sum_{i=1}^{m-1} X^a (T_w v), \]
for all \( k \in \mathbb{Z}, a \in \mathbb{Z}^m \) and \( v \in (\mathbb{C}^n)^{\otimes m} \), is an isomorphism from the space of \( q \)-wedges to the quotient of \( \bigotimes^m \mathbb{C}^n[X^{\pm 1}] \) by the right ideal generated by the \( T_i + q^{-1} \)'s. We are thus reduced to prove that
\[ \sum_{i=1}^{m-1} \ker (T_i - q) = \sum_{i=1}^{m-1} \im (T_i + q^{-1}) \]
in \( \bigotimes^m \mathbb{C}^n[X^{\pm 1}] \). One inclusion follows from the relation \( (T_i - q)(T_i + q^{-1}) = 0 \). The equality can be proved by using the formulas in section 5 since
\[ (\tau_{1,2} - q)(v_{ij}) = \begin{cases} 0 & \text{if } i = j, \\ q^{-1}v_{ji} - qv_{ij} & \text{if } i < j, \\ qv_{ji} - q^{-1}v_{ij} & \text{if } i > j, \end{cases} \]
and 
\[(\tau_{1,2} + q^{-1})(v_{ij}) = \begin{cases} 
(q + q^{-1})v_{ij} & \text{if } i = j, \\
q^{-1}(v_{ji} + v_{ij}) & \text{if } i < j, \\
q(v_{ji} + v_{ij}) & \text{if } i > j,
\end{cases}\]
for all \(i, j \in \mathbb{Z}\) (where, as in section 4, \(v_{i+nk} = v_i \zeta^{-k}\) for all \(k \in \mathbb{Z}\) and \(i = 1, 2, \ldots, n\)).

\[\square\]

According to [KMS], elements of \(\wedge^m\) are called \(q\)-wedges. From now on it will be more convenient to view the \(q\)-wedges space as the quotient \(\bigotimes^n \mathbb{C}[X^{\pm 1}]/\Omega\). Let \(\wedge : \bigotimes^n \mathbb{C}[X^{\pm 1}] \to \wedge^m\) be the projection. For all \(j \in \mathcal{P}_m\) we denote indifferently by 
\[\wedge v_j \quad \text{or} \quad v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_m}\]
the projection of the monomial \(v_j = v_{j_1} v_{j_2} \cdots v_{j_m}\) in the \(q\)-wedges space. From section 9 the algebra \(\hat{U}\) acts on \(\wedge^m\). Let \(\hat{x}_i\) be the image of \(\omega(X_i)\) in \(\text{End}(\mathbb{R}_m)\), where \(X_i\) acts on \(\mathbb{R}_m\) by the operator \(x_i\) defined in section 2.

**Theorem 10.** The action of the generators of \(\hat{U}\) on \(\wedge^m\) is given as follows: \(\forall j \in \mathcal{P}_m^1, \forall P(X) \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \forall i = 1, 2, \ldots, n - 1,\)
\[
\begin{align*}
\mathbf{e}_i \cdot \wedge P(X^{-1}) v_j &= \wedge (P(X^{-1}) \Delta^{m-1} (e_i) \cdot v_j), \\
\mathbf{f}_i \cdot \wedge P(X^{-1}) v_j &= \wedge (P(X^{-1}) \Delta^{m-1} (f_i) \cdot v_j), \\
\mathbf{k}_i \cdot \wedge P(X^{-1}) v_j &= \wedge (P(X^{-1}) \Delta^{m-1} (k_i) \cdot v_j), \\
\mathbf{e}_0 \cdot \wedge P(X^{-1}) v_j &= \sum_{l=1}^{m} \wedge (X_l P(X^{-1}) f_{\theta, l} \cdot v_j), \\
\mathbf{f}_0 \cdot \wedge P(X^{-1}) v_j &= \sum_{l=1}^{m} \wedge (X_l^{-1} P(X^{-1}) e_{\theta, l} \cdot v_j), \\
\mathbf{k}_0 \cdot \wedge P(X^{-1}) v_j &= \wedge (P(X^{-1}) k_{\theta}^{-1} \otimes l \cdot v_j), \\
\mathbf{e}_n \cdot \wedge P(X^{-1}) v_j &= q^{-1} p^{1/n} \sum_{l=1}^{m} \wedge (\hat{x}_l(P)(X^{-1}) f_{\theta, l} \cdot v_j), \\
\mathbf{f}_n \cdot \wedge P(X^{-1}) v_j &= q p^{-1/n} \sum_{l=1}^{m} \wedge (\hat{x}_l^{-1}(P)(X^{-1}) e_{\theta, l} \cdot v_j), \\
\mathbf{k}_n \cdot \wedge P(X^{-1}) v_j &= \wedge (P(X^{-1}) k_{\theta}^{-1} \otimes l \cdot v_j).
\end{align*}
\]

**Proof.** According to the isomorphism in the proposition above and the description of the Schur dual recalled in section 9, we must prove that \(P(X^{-1}) Y_{l}^{-1} - \hat{x}_l(P)(X^{-1}) \in \gamma \omega(I_m)\) for all \(l\) and \(P\). By definition of \(\hat{x}_l\) we have \(\omega(X_i) P(Y^{-1}) - \hat{x}_l(P)(Y^{-1}) \in I_m\). Thus
\[
\omega \gamma (P(X^{-1}) Y_{l}^{-1} - \hat{x}_l(P)(X^{-1})) = \omega(X_i) P(Y^{-1}) - \hat{x}_l(P)(Y^{-1}) \in I_m.
\]
Remark 10. A direct computation gives

\[ \hat{x}_l = q^{m-1} t_{l-1,i} s_{l-1,i} t_{l-2,i} s_{l-2,i} \cdots t_{1,i} s_{1,i} D_i s_{1,m} t_{1,m} \cdots s_{1,l+1} t_{1,l+1}. \]

In particular \( \hat{x}_l = D_l \) if \( q \) is one.

Let us notice that it is possible to write explicit formulas for the action of all the Drinfeld generators on the tensor module. We will use these formulas in the proof of theorem 12. Given \( 1 \leq i < j \leq m \), set

\[ T_{i,j} = T_i T_{i+1} \cdots T_j \quad \text{and} \quad T_{j,i} = T_j T_{j-1} \cdots T_i. \]

Proposition 10. [VV; Theorem 3.3] For any \( j \in \mathcal{P}_m \) and \( i = 1, 2, ..., n - 1 \), set \( s = \max j^{-1}(i) \). Then,

\[ f_i(w) \cdot \wedge P(X^{-1}) v_j = q^{1-\delta^{-1}(i)} \sum_{k \in j^{-1}(i)} q^{s-k} \wedge P(X^{-1}) T_{k,s-1} e(q^{n} p^{j/n} w \gamma) v_{j+1}^+, \]

if \( j^{-1}(i) \neq \emptyset, 0 \) else, and

\[ e_i(w) \cdot \wedge P(X^{-1}) v_j = q^{1-\delta^{-1}(i+1)} \sum_{k \in j^{-1}(i+1)} q^{s-1} \wedge P(X^{-1}) T_{k,s+1} e(q^{n} p^{j/n} w \gamma) v_{j+1}, \]

if \( j^{-1}(i+1) \neq \emptyset, 0 \) else.

The shift in the powers of \( q \) in [VV] is due to a different normalization of the \( T_i \)'s.

11. According to [KMS] set \( u_j = X^{-1} u_j \) for all \( j \in \mathcal{P}_m \). The Fock space, \( \wedge^{\infty/2} \), is the linear span of semi-infinite monomials

\[ \wedge u_j = u_{j_1} \wedge u_{j_2} \wedge u_{j_3} \wedge u_{j_4} \wedge \cdots \]

where \( j_1 > j_2 > j_3 > j_4 > ... \) and \( j_{k+1} = j_k - 1 \) for \( k >> 1 \). It splits as a direct sum of an infinite number of sectors, \( \wedge_{(e)}^{\infty/2} \), parametrized by an integer \( e \in \mathbb{Z} \). Let denote by \( e = (e_1, e_2, e_3, ...) \) the infinite sequence such that \( e_k = e - k + 1 \) for any \( k \in \mathbb{Z} \). Then, \( \wedge_{(e)}^{\infty/2} \) is the linear span of semi-infinite monomials \( \wedge u_j \) such that \( j_1 > j_2 > j_3 > j_4 > ... \) and \( j_k = e_k \) for \( k >> 1 \). In particular the sector \( \wedge_{(e)}^{\infty/2} \) contains the element

\[ |e\rangle = u_{e_1} \wedge u_{e_2} \wedge u_{e_3} \wedge u_{e_4} \wedge \cdots \]

called the vacuum vector. The space \( \wedge^{\infty/2}_{(e)} \) is \( \mathbb{N} \)-graded. Recall that the height of an infinite sequence \( j = (j_1, j_2, j_3, ...) \) with only a finite number of non-zero terms is defined as \( |j| = \sum j_k \). Then, if \( j_1 > j_2 > j_3 > j_4 > ... \) and \( j_k = e_k \) for \( k >> 1 \), set

\[ \deg(\wedge u_j) = |j - e|. \]
The degree of the vacuum vector is zero. We denote by $\Lambda^{\infty/2, k}_{(e)} \subset \Lambda^{\infty/2}_{(e)}$ the homogeneous component of degree $k$.

12. Fix $e \in \mathbb{Z}$, $p \in \mathbb{C}^\times$ and a generic $q \in \mathbb{C}^\times$. As before, $m$ is a non-negative integer. We want to construct a representation of $\hat{U}$ on the Fock space as a limit when $m \to \infty$ of the representation of $\hat{U}$ on $\Lambda^m$. It is proved in [KMS] that the normally ordered $q$-wedges, i.e. the $\wedge u_j$'s such that $j_k \geq e_k$ for all $k$, form a basis of $\Lambda^m$. Put $e_m = (e_1, e_2, \ldots, e_m)$. Let $\Lambda^m_{(e)} \subset \Lambda^m$ be the linear span of the normally ordered $q$-wedges $\wedge u_j$ such that $j_k \geq e_k$ for all $k$. The space $\Lambda^m_{(e)}$ admits a grading similar to the Fock space grading:

$$\deg(\wedge u_j) = |j - e_m|, \quad \forall j \in P^m_{no} \text{ s.t. } j \geq e_m.$$

Let $\Lambda^{m,k}_{(e)} \subset \Lambda^m_{(e)}$ be the component of degree $k$ and set

$$P^{no,k}_m = \{ j \in P^m_{no} \mid |j - e_m| = k \}.$$

Let $\pi^{m,k}_{(e)} : \Lambda^{m+n,k}_{(e)} \to \Lambda^{m,k}_{(e)}$ be the projections

$$\wedge u_j \mapsto \begin{cases} 
\wedge u_{j_1 j_2 \cdots j_m} & \text{if } j_k = e_k \quad \forall k, l \in [m+1, m+n], \\
0 & \text{else},
\end{cases}$$

for all $j \in P^{no,k}_m$.

**Proposition 12.** [TU] For any $m, k \in \mathbb{N}$,

1. $\Lambda^{m,k}_{(e)} \subset \Lambda^m_{(e)}$ is $\hat{U}_q$-stable.

2. If $\overline{m} = \overline{e}$ then $\pi^{m,k}_{(e)} \in \text{Hom}_{\hat{U}}(\Lambda^{m+n,k}_{(e)}, \Lambda^{m,k}_{(e)})$. If moreover $m \geq k$ then $\pi^{m,k}_{(e)}$ is invertible.

3. Suppose that $\overline{m} = \overline{e}$ and $m \geq k$. Given $s \in \mathbb{N}$ and $j = (i, l) \in P^{no,k}_{m+s}$ with $i \in P^m_{no}$ and $l \in P^s_{no}$,

$$\begin{cases} 
\hat{l}_m > \hat{l}_{m+1} = \cdots = \hat{l}_{m+s} \\
1 \leq r \leq m
\end{cases} \implies \wedge(X^{-1} Y^{\pm 1}_{r-1} Y^{\pm 1}_{r} Y^{\pm 1}_{r+1} \cdots Y^{\pm 1}_{m+s}) = \wedge(X^{-1} Y^{\pm 1}_r X^{-1} Y^{\pm 1}_{r+1} \cdots Y^{\pm 1}_{m+s}).$$

**Proof.** The statement (12.1) follows from [TU; Proposition 4 and (4.28)] and (12.2) follows from [TU; Proposition 5 and 6]. The formula (12.3) follows from [TU; (4.40)], from $m \geq k$ and from the fact that both terms in the equality have degree $k$. The shift in the powers of $q$ in [TU; (4.40)] is due to a different normalization of the $Y_i$'s.

As a consequence, if $\overline{m} = \overline{e}$ and $m \geq k$, the map

$$\Lambda^{m,k}_{(e)} \to \Lambda^{\infty/2,k}_{(e)}, \quad v \mapsto v \wedge (e_1 + m),$$
is a linear isomorphism and $\wedge_{(e)}^{\infty/2,k}$ inherits a $\check{U}_v$-action from $\wedge_{(e)}^{m,k}$ which is independent of the choice of such an $m$. For any $v \in \wedge_{(e)}^{m,k}$ and any $i = 0, 1, ..., n$ set

$$e_i(v \wedge |e_{1+m}>) = e_i(v) \wedge k_i |e_{1+m} > + v \wedge e_i |e_{1+m} >,$$

$$f_i(v \wedge |e_{1+m}>) = f_i(v) \wedge |e_{1+m} > + k_i^{-1}(v) \wedge f_i |e_{1+m} >,$$

$$k_i(v \wedge |e_{1+m}>) = k_i(v) \wedge k_i |e_{1+m} >,$$

where

$$e_i |e_{1+m} > = 0, \quad f_i |e_{1+m} > = \delta(i = 0) u_{e_m} \wedge |e_{2+m} >, \quad k_i |e_{1+m} > = \phi^{-\delta(i = 0)} |e_{1+m} >.$$

Thus $e_i, f_i, k_i$, $i = 1, 2, ..., n$, are precisely the generators of the action of $\check{U}_v$ on $\wedge_{(e)}^{\infty/2,k}$.

**Theorem 12.** The formulas above define a representation of $\check{U}$ on each sector of the Fock space.

**Proof.** Let define $\phi_\infty$ as the linear automorphism of the Fock space $\wedge u_j \mapsto \wedge u_{1+j}$, for all normally ordered $q$-wedges $\wedge u_j \in \wedge_{(e)}^{\infty/2}$. Let us prove that

$$e_i(w) = \phi^{-1}_{\infty} \circ e_{i+1}(p^{1/n} w) \circ \phi_\infty, \quad f_i(w) = \phi^{-1}_{\infty} \circ f_{i+1}(p^{1/n} w) \circ \phi_\infty,$$

$$k_i^\pm(w) = \phi^{-1}_{\infty} \circ k_i^\pm (p^{1/n} w) \circ \phi_\infty, \quad \forall i = 1, 2, ..., n - 2,$$

$$e_{n-1}(w) = \phi^{-2}_{\infty} \circ e_1(p^{2/n} w) \circ \phi_\infty^2, \quad f_{n-1}(w) = \phi^{-2}_{\infty} \circ f_1(p^{2/n} w) \circ \phi_\infty^2,$$

$$k_{n-1}^\pm(w) = \phi^{-2}_{\infty} \circ k_1^\pm (p^{2/n} w) \circ \phi_\infty^2.$$

The algebra $\check{U}_v$ acts on each sector. Moreover, the map

$$e_{i,k} \mapsto a^k e_{i+1,k}, \quad f_{i,k} \mapsto a^k f_{i+1,k}, \quad h_{i,k} \mapsto a^k h_{i+1,k},$$

(where the index $n + 1$ stands for 0) extends to an automorphism of $\check{U}$ for any $a \in \mathbb{C}^\times$. As a consequence of (12.4) and (12.5), setting

$$e_0(w) = \phi^{-1}_{\infty} \circ e_1(p^{1/n} w) \circ \phi_\infty, \quad f_0(w) = \phi^{-1}_{\infty} \circ f_1(p^{1/n} w) \circ \phi_\infty,$$

we will get an action of $\check{U}$ on the Fock space. Since the operators $e_0, f_0$ and $k_0^\pm$ coincide with the degree zero Fourier components of the formal series $e_0(w), f_0(w)$ and $k_0^\pm(w)$ respectively, we will be done. For any $m \in \mathbb{N}$, consider the map

$$\phi_m : \wedge_{(e)}^m \to \wedge_{(e+1)}^{m+1}, \quad \wedge u_j \mapsto \wedge u_{1+j} \wedge u_{e_m}, \quad \forall j \in P_m.$$

Set $h = \tau + (n + 1)k$ and $\wedge_{(e)}^{\infty/2,\leq h} = \bigoplus_{i \leq h} \wedge_{(e)}^{\infty/2,i}$. Let us observe that

$$\phi_m \leq h, \quad \forall u \in \wedge_{(e)}^{\infty/2,k}.$$
The maps $\phi_\infty$ and $\phi_m$ do not preserve the degree. However, if $\underline{m} = \underline{r}$ and $\underline{m} \geq \underline{h}$, we get the following commutative diagram
\[
\begin{array}{ccc}
\Lambda_{\infty}\{e\}^{2, k} & \xrightarrow{\sim} & \Lambda_{\infty}\{m\}^{k} \\
\phi_\infty \downarrow & & \downarrow \phi_m \\
\Lambda_{\infty}\{e\}^{2, \leq h} & \xrightarrow{\sim} & \Lambda_{\infty}\{m+1\}^{\leq h}
\end{array}
\]
where the horizontal arrows are the projections on the first $m$ (resp. $m+1$) components. Since the horizontal maps commute to $\mathcal{U}_v$, it suffices to prove (12.4), (12.5) on $\Lambda_{\infty}\{m, k\}$ with respect to $\phi_m$. As a consequence of the theorem 10 and [VV, proposition 3.4], the map
\[
\Lambda^m \longrightarrow \Lambda^m, \quad \forall u_j \mapsto \wedge u_{j+1}, \quad \forall j \in \mathcal{P}_{m+1}^n,
\]
intertwines $e_i(w), f_i(w), k^\pm_{i+1}(w)$ and $e_{i+1}(w), f_{i+1}(w), k^\pm_{i+1}(w)$ for all $i$. For the relations (12.4) we are thus reduced to prove that for any $j = (j_1, j_2, \ldots, j_m+1) \in \mathcal{P}_{m+1}^{n, k}$, if $i = (i_1, i_2, \ldots, i_m)$ and $\overline{j}_{m+1} = n$ then
\[
e_i(-1)(\wedge u_j) = e_{i_1-1}(\wedge u_{j_1}) \wedge u_{j_2} \wedge u_{j_3} \wedge \ldots \wedge u_{j_m}.
\]
for all $i = 1, 2, \ldots, n-2$. Let us prove the first equality, the second being quite similar. Let $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ and $k \in \mathcal{P}_{m}^0$ be such that $\wedge u_i = \wedge P v_k$. In order to simplify the notations, we omit the symbol $\otimes$ when no confusion is possible. Put $a = -\frac{\mathcal{Z}}{\mathcal{J}+1}$. Then,
\[
ea_i(-1)(\wedge u_j) = \wedge P X_{m+1}^a e_{i_1-1}(u_k v_n),
\]
from the formula in proposition 10. Then, use (12.3) to conclude. As for the relations (12.5), we have to prove the following formula : for any $j = (j_1, j_2, \ldots, j_{m+1}) \in \mathcal{P}_{m+2}^{n, k}$ set $i = (i_1, i_2, \ldots, i_{m})$. If $\overline{j}_{m+1} = n-1, \overline{j}_{m+2} = n$, and $\overline{\mathcal{j}}_{m+1} = \overline{\mathcal{j}}_{m+2}$, then
\[
e_{n-1, -1}(\wedge u_j) = e_{n-1, -1}(\wedge u_i) \wedge u_{j_{m+1}} \wedge u_{j_{m+2}},
\]
\[
f_{n-1, 1}(\wedge u_j) = f_{n-1, 1}(\wedge u_i) \wedge u_{j_{m+1}} \wedge u_{j_{m+2}},
\]
Let us prove (12.8) for instance. Let $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$, $k, l \in \mathbb{N}$ and $k \in \mathcal{P}_{m-l-k}^0$ be such that
\[
\wedge u_i = \wedge P v_{k} v_{n-l-1}^l v_{n-k}^l \wedge k^{-1}\{n-1, n\} = \emptyset.
\]
Put $a = -\frac{\mathcal{Z}}{\mathcal{J}+2}$. Then,
\[
f_{n-1, 1}(\wedge u_j) = \wedge P X_{m+1}^a X_{m+2}^a f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n-1} v_{n}),
\]
\[
= \wedge P X_{m+1}^a X_{m+2}^a T_{m, m-k+1} f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n} v_{n-1} v_{n}),
\]
\[
= \wedge P X_{m+1}^a X_{m+2}^a T_{m, m-k+1} v_{k} f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n} v_{n-1} v_{n}) v_{n-k}^l v_{n} v_{n-1} v_{n},
\]
\[
= \wedge P X_{m+1}^a X_{m+2}^a f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n}) v_{n-k}^l v_{n} v_{n-1} v_{n},
\]
\[
= \wedge P X_{m+1}^a X_{m+2}^a f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n}) v_{n-k}^l v_{n} v_{n-1} v_{n},
\]
\[
= \wedge P X_{m+1}^a X_{m+2}^a f_{n-1, 1}(v_{k} v_{n-l-1}^l v_{n-k}^l v_{n}) v_{n-k}^l v_{n} v_{n-1} v_{n}.
\]
The equalities (12.9), (12.10) and (12.12) are immediate from the formulas in propositions 7 and 10, while the equality (12.13) follows from (12.3). As for the equality (12.11), first note that

\[ f_{n-1,1}(v_{n-1}^{l+1}) = q^{-n}p^{1-n/n} \sum_{k=1}^{l+1} q^{1-k} T_{k,l} Y_{l+1} v_n \]

\[ = q^{-l-n}p^{1-n/n} Y_{l+1} v_n + q^{-n}p^{1-n/n} \sum_{k=1}^{l} q^{1-k} T_{k,l} Y_{l} v_n \]

\[ = q^{-l-n}p^{1-n/n} Y_{l+1} v_n + q^{-n}p^{1-n/n} \sum_{k=1}^{l} q^{1-k} T_{k,l-1} Y_{l} v_{n-1} v_n + \]

\[ + q^{-n}p^{1-n/n} \sum_{k=1}^{l} q^{1-k} (q^{-1}) T_{k,l} Y_{l} v_{n-1} v_n, \]

\[ = A v_n + f_{n-1,1}(v_{n-1}^{l}) v_{n-1}, \]

where \( A \) is an expression in the first \( l \) components. In this computation we have used the proposition 10, the equalities

\[ T_l = T_l^{-1} + q - q^{-1} \quad \text{and} \quad T_l^{-1} Y_{l+1}^{-1} = Y_{l+1}^{-1} T_l, \]

and the formulas for the \( T_l \)'s given in proposition 7. Now, in order to get (12.11), it is enough to observe that

\[ \wedge P X_{m+1}^a X_{m+2}^a T_{m,m-k+1} v_k A v_n^{k+2} \]

vanishes (see [KMS, Lemma 2.2]). □

13. Let us now consider the classical case, i.e. \( q = 1 \). Let \( \mathbf{A} = \mathbb{C}[z^{\pm 1}, D^{\pm 1}] \) be the algebra of polynomial difference operators in one variable \( z \). Suppose that \( p \in \mathbb{C}^\times \) is generic. Let us recall that \( z \) and \( D \) satisfy the commutation relation \( D z = p z D \). The algebra of matrices with coefficients in \( \mathbf{A} \) is denoted by \( \mathfrak{gl}_n(\mathbf{A}) \). It may be viewed as a Lie algebra with the usual commutator. Let \( \mathfrak{sl}_n(\mathbf{A}) \subset \mathfrak{gl}_n(\mathbf{A}) \) be the derived Lie subalgebra, i.e. \( \mathfrak{sl}_n(\mathbf{A}) = [\mathfrak{gl}_n(\mathbf{A}), \mathfrak{gl}_n(\mathbf{A})] \). It is known that \( \mathfrak{sl}_n(\mathbf{A}) \subset \mathfrak{gl}_n(\mathbf{A}) \) is the subset of matrices with trace in \([\mathbf{A}, \mathbf{A}]\). Since \( \mathfrak{sl}_n(\mathbf{A}) \) is perfect, it admits a universal central extension (see [G] and [KL] for more details). Let \( \mathfrak{sl}_{n,d} \) be the complex Lie algebra generated by \( e_{i,k}, f_{i,k}, h_{i,k} \), where \( i = 0, 1, \ldots, n-1, \)

\( k \in \mathbb{Z} \), and a central element \( c \), modulo the relations

\[ [h_{i,k}, h_{j,l}] = d^{k,l} a_{i,j} e_{i,j}, \]

\[ [h_{i,k}, e_{j,l}] = d^{k,l} a_{i,j} e_{j,k+l}, \quad [h_{i,k}, f_{j,l}] = -d^{k,l} a_{i,j} f_{j,k+l}, \]

\[ d^{-m_{i,j}} [e_{i+k+1,j}, e_{j,l}] - [e_{i,k}, e_{j,l+1}] = d^{-m_{i,j}} [f_{i+k+1,j}, f_{j,l}] - [f_{i,k}, f_{j,l+1}] = 0, \]

\[ [e_{i,k}, f_{j,l}] = d^{k,l} a_{i,j} (h_{i,k+l} + k\delta(k = -l)c), \]

\[ \text{ad}_{c}^{-1} a_{i,j} (e_{j,k}) = 0 \quad \text{if} \quad i \neq j. \]

The algebra \( \mathfrak{U}_{|q=1} \) is the enveloping algebra of \( \mathfrak{sl}_{n,d} \). It is proved in [MRY] that if \( d = 1 \) then \( \mathfrak{sl}_{n,d} \) is isomorphic to the universal central extension, denoted \( \hat{\mathfrak{sl}}_n \), of \( \mathfrak{sl}_n[x^{\pm 1}, y^{\pm 1}] \). It is proved in [K] that this Lie algebra can be described as follows:
set \( B = \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \) and \( \Omega_B = B \, dx \oplus B \, dy, \) then \( \mathfrak{s}l_n = (\mathfrak{sl}_n \otimes B) \oplus (\Omega_B/dB), \) with the bracket such that \( \Omega_B/dB \) is central and
\[
[a \otimes f, b \otimes g] = [a, b] \otimes fg + (ab) (df)g, \quad \forall a, b \in \mathfrak{s}l_n, \quad \forall f, g \in B,
\]
where \((\cdot | \cdot)\) is the normalized Killing form of \( \mathfrak{s}l_n, \) i.e. \((a|b)\) is the trace of \( ab, \) and \( df \) is the differential of \( f. \) The following result explains why toroidal algebras can be represented by difference operators, for instance as in the previous sections. As usual, we denote by \( E_{ab}, a, b = 1, 2, \ldots, n, \) the elementary matrices of \( \mathfrak{gl}_n. \)

**Theorem 13.1.** Set \( d = p^{1/n}. \) The map
\[
c \mapsto 0,
\]
\[
e_{i,k} \mapsto p^{ki/n} E_{i,i+1} \otimes D^{-k}, \quad e_{0,0} \mapsto E_{n1} \otimes z,
\]
\[
f_{i,k} \mapsto p^{ki/n} E_{i+1,i} \otimes D^{-k}, \quad f_{0,0} \mapsto E_{1n} \otimes z^{-1},
\]
\[
h_{i,k} \mapsto p^{ki/n} (E_{ii} - E_{i+1,i+1}) \otimes D^{-k}, \quad h_{0,0} \mapsto (E_{nn} - E_{11}) \otimes 1,
\]
where \( k \in \mathbb{Z} \) and \( i \neq 0, \) extends uniquely to a Lie algebra homomorphism \( \pi : \mathfrak{s}l_{n,d} \to \mathfrak{s}l_n(A) \) such that \( (\mathfrak{s}l_{n,d}, \pi) \) is the universal central extension of \( \mathfrak{s}l_n(A). \)

**Proof.** Let \( g(z) \) be the invertible element of \( \mathfrak{gl}_n(A) \) defined as
\[
g(z)(v_i) = z^{-\delta(i=n)} v_{i+1}, \quad \forall i = 1, 2, \ldots, n.
\]
The conjugation by \( g(z) \) is an automorphism of the associative algebra \( \mathfrak{gl}_n(A) \) preserving the Lie subalgebra \( \mathfrak{s}l_n(A). \) As for the \( e_{i,k} \)’s a direct computation gives
\[
g(z) \circ (E_{i,i+1} \otimes D^{-k}) \circ g(z)^{-1} = E_{i+1,i+2} \otimes D^{-k},
\]
\[
g(z)^2 \circ (E_{n-1,n} \otimes D^{-k}) \circ g(z)^{-2} = p^{-k} E_{12} \otimes D^{-k},
\]
if \( i = 1, 2, \ldots, n - 2. \) Similar formulas hold for the images of \( f_{i,k} \) and \( h_{i,k}. \) The argument in the beginning of the proof of the theorem 12 implies that the map \( \pi \) such that
\[
e_{i,k} \mapsto p^{ki/n} E_{i,i+1} \otimes D^{-k},
\]
\[
f_{i,k} \mapsto p^{ki/n} E_{i+1,i} \otimes D^{-k},
\]
\[
h_{i,k} \mapsto p^{ki/n} (E_{ii} - E_{i+1,i+1}) \otimes D^{-k},
\]
if \( k \in \mathbb{Z}, \) \( i \neq 0, \) and
\[
e_{0,k} \mapsto p^{k} g(z)(E_{n-1,n} \otimes D^{-k}) g(z)^{-1} = E_{n1} \otimes zD^{-k},
\]
\[
f_{0,k} \mapsto p^{k} g(z)(E_{n,n-1} \otimes D^{-k}) g(z)^{-1} = E_{1n} \otimes D^{-k} z^{-1},
\]
\[
h_{0,k} \mapsto p^{k} g(z)((E_{n-1,n-1} - E_{nn}) \otimes D^{-k}) g(z)^{-1} = (p^k E_{nn} - E_{11}) \otimes D^{-k},
\]
is a morphism of Lie algebras $\tilde{\mathfrak{sl}}_{n,d} \to \mathfrak{sl}_n(A)$. As for the surjectivity of $\pi$ let first note that
\[ [A, A] = \bigoplus_{(l,k) \neq (0,0)} \mathbb{C} \cdot z^k D^l. \]
Then, as a vector space, $\mathfrak{sl}_n(A)$ is the sum of $\mathfrak{sl}_n \otimes A$ and the subspace of $\mathfrak{gl}_n(A)$ of diagonal matrices with coefficients in $[A, A]$. The horizontal Lie algebra (i.e. the Lie subalgebra generated by $e_{i,0}, f_{i,0}, h_{i,0}$ with $i = 0, 1, ..., n - 1$) and the vertical Lie algebra (generated by $e_{i,k}, f_{i,k}, h_{i,k}$ with $i \neq 0$) are isomorphic to $\hat{\mathfrak{sl}}_n$, the affine Lie algebra of type $A_{n-1}^{(1)}$; the projection of $\hat{\mathfrak{sl}}_n$ onto each of them preserves the $\mathbb{Z}$-gradation and thus the kernel is trivial. As a consequence, the elements $E_{ij} \otimes z^k$, $E_{ij} \otimes D^l$, with $i \neq j$, and $(E_{ii} - E_{i+1i+1}) \otimes z^k$, $(E_{ii} - E_{i+1i+1}) \otimes D^l$ are in the image of $\pi$. Moreover, since $p$ is generic, we have
\[ z^k D^l = \frac{1}{1 - p^k} [z^k, D^l] \quad k, l \neq 0 \]
and so $E_{ij} \otimes z^k D^l$, where $i \neq j$, and $(E_{ii} - E_{i+1i+1}) \otimes z^k D^l$ belong to the image of $\pi$ too. Thus we need only to prove that $E_{ii} \otimes z^k D^l \in \text{Im} (\pi)$, for $(k, l) \neq (0, 0)$. In the case $k, l \neq 0$ this follows from
\[ [E_{ii+1} \otimes z^k, E_{i+1i} \otimes D^l] = z^k D^l (E_{ii} - p^k E_{i+1i+1}), \]
\[ [E_{ii+1} \otimes D^l, E_{i+1i} \otimes z^k] = z^k D^l (p^k E_{ii} - E_{i+1i+1}), \]
and from the fact that $p$ is not a root of unity. In the other cases use:
\[ z^k = \frac{1}{1 - p^{-k}} [D, D^{-1} z^k], \quad D^l = \frac{1}{1 - p^{-l}} [z, z^{-1} D^l]. \]
Both algebras $\tilde{\mathfrak{sl}}_{n,d}$ and $\mathfrak{gl}_n(A)$ are graded by $\mathbb{Z} \times Q$, where $Q$ is the root lattice of $\hat{\mathfrak{sl}}_n$. Set
\[ \deg(e_{i,k}) = (k, \alpha_i), \quad \deg(f_{i,k}) = (k, -\alpha_i), \]
\[ \deg(h_{i,k}) = (k, 0), \quad \deg(e) = (0, 0), \]
and
\[ \deg(E_{j,j+1} \otimes z^l D^{-k}) = (k, l \delta + \alpha_j), \quad \deg(E_{j+1,j} \otimes z^l D^{-k}) = (k, l \delta - \alpha_j), \]
\[ \deg(E_{jj} \otimes z^l D^{-k}) = (k, l \delta), \]
where $\alpha_0, \alpha_1, ..., \alpha_{n-1}$ are the simple roots of $\hat{\mathfrak{sl}}_n$, $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ and $i = 0, 1, ..., n - 1$. The map $\pi$ is graded. The subspace of $\mathfrak{sl}_n(A)$ of degree $(k, \alpha)$ is one-dimensional if $\alpha$ is a real root of $\hat{\mathfrak{sl}}_n$ and zero-dimensional if $\alpha$ is non-zero and is not a root of $\hat{\mathfrak{sl}}_n$. On the other hand the Lie algebra $\tilde{\mathfrak{sl}}_{n,d}$ may be viewed as an integrable module over the horizontal Lie subalgebra which is isomorphic to $\hat{\mathfrak{sl}}_n$. Thus one can prove as in [MRY] that the subspace of $\tilde{\mathfrak{sl}}_{n,d}$ of degree $(k, \alpha)$ is one-dimensional if $\alpha$ is a real root of $\hat{\mathfrak{sl}}_n$ and zero-dimensional if $\alpha$ is non-zero and is not a root of $\hat{\mathfrak{sl}}_n$. It follows that the degree of an element of $\text{Ker} \pi$ is in $\mathbb{Z} \times (\mathbb{Z} \cdot \delta)$ and that $(\tilde{\mathfrak{sl}}_{n,d}, \pi)$ is a central extension of $\mathfrak{sl}_n(A)$. The Lie algebra $\tilde{\mathfrak{sl}}_{n,d}$ is perfect.
A similar theorem holds in the case $p \to 1$. Let $\mathfrak{sl}_{n,\bar{o}}$ be the complex Lie algebra generated by $e_{i,k}, f_{i,k}, h_{i,k}$, where $i = 0, 1, \ldots, n - 1$, $k \in \mathbb{N}$, modulo the relations

$$[h_i(z), h_j(w)] = 0,$$

$$[h_i(z), e_j(w)] = \frac{a_{ij}}{w - z + m_{ij}}(e_j(z) - e_j(w)),$$

$$[h_i(z), f_j(w)] = -\frac{a_{ij}}{w - z + m_{ij}}(f_j(z) - f_j(w)),$$

$$(z - w - m_{ij})[e_i(z), e_j(w)] = (z - w - m_{ij})[f_i(z), f_j(w)] = 0,$$

$$[e_i(z), f_j(w)] = \frac{\delta(i = j)}{w - z}(h_i(z) - h_i(w)),$$

$$\text{ad}^{1-a_{ij}}_{e_{i,0}}(e_j(z)) = \text{ad}^{1-a_{ij}}_{f_{i,0}}(f_j(z)) = 0 \quad \text{if} \quad i \neq j,$$

where $e_i(z) = \sum_{k \in \mathbb{N}} e_{i,k} z^{-k-1}$, $f_i(z) = \sum_{k \in \mathbb{N}} f_{i,k} z^{-k}$ and $h_i(z) = \sum_{k \in \mathbb{N}} h_{i,k} z^{-k}$.

Put $\partial = \frac{d}{dz}$.

**Theorem 13.2.** The map

$$e_{i,k} \mapsto E_{i,i+1} \otimes (\partial - i/n)^k, \quad e_{0,0} \mapsto E_{n1} \otimes z,$$

$$f_{i,k} \mapsto E_{i+1,i} \otimes (\partial - i/n)^k, \quad f_{0,0} \mapsto E_{1n} \otimes z^{-1},$$

$$h_{i,k} \mapsto (E_{ii} - E_{i+1,i+1}) \otimes (\partial - i/n)^k, \quad h_{0,0} \mapsto (E_{nn} - E_{11}) \otimes 1,$$

where $k \in \mathbb{Z}$ and $i \neq 0$, extends uniquely to a Lie algebra homomorphism $\pi : \mathfrak{sl}_{n,\bar{o}} \to \mathfrak{sl}_n(\mathbb{C}[z^{\pm 1}, \partial])$ such that $(\mathfrak{sl}_{n,\bar{o}}, \pi)$ is the universal central extension of $\mathfrak{sl}_n(\mathbb{C}[z^{\pm 1}, \partial])$.

**Remark 13.** The algebras $\mathfrak{sl}_{n,d}$ and $\mathfrak{sl}_{n,\bar{o}}$ admit a presentation similar to the double-loop presentation of $\mathfrak{sl}_n$. Let us recall it. Fix a complex unital associative algebra $A$. Set $Id = \sum_{i=1}^n E_{ii} \in \mathfrak{gl}_n$ and

$$[a, b]_+ = ab + ba - \frac{2}{n} (a(b) Id), \quad \forall a, b \in \mathfrak{sl}_n,$$

$$[f, g]_+ = fg + gf, \quad \forall f, g \in A.$$
Let $I \subset A \otimes A$ be the linear span of the elements

$$f \otimes g + g \otimes f \quad \text{and} \quad fg \otimes h - f \otimes gh - g \otimes hf$$

for all $f, g, h \in A$. Denote by $\langle \cdot | \cdot \rangle : A \otimes A \to A \otimes A/I$ the projection. The first cyclic homology group $HC_1(A)$ is the kernel of the map

$$\langle A | A \rangle \longrightarrow [A, A], \quad \langle f | g \rangle \mapsto [f, g].$$

As a vector space, $\mathfrak{sl}_n(A)$ is the direct sum of $\mathfrak{sl}_n \otimes A$ and $\text{Id} \otimes [A, A]$. The bracket on $\mathfrak{sl}_n(A)$ is such that

$$[a \otimes f, b \otimes g] = \frac{1}{n}(a | b)\text{Id} \otimes [f, g] + \frac{1}{2}[a, b] \otimes [f, g]_+ + \frac{1}{2}[a, b]_+ \otimes [f, g],$$

$$[\text{Id} \otimes f, a \otimes g] = [a \otimes f, \text{Id} \otimes g] = a \otimes [f, g],$$

where $a, b \in \mathfrak{sl}_n$ and $f, g \in A$. Similarly the universal central extension of $\mathfrak{sl}_n(A)$ is the direct sum of $\mathfrak{sl}_n \otimes A$ and $\langle A | A \rangle$ with the bracket

$$[a \otimes f, b \otimes g] = \frac{1}{n}(a | b)(f | g) + \frac{1}{2}[a, b] \otimes [f, g]_+ + \frac{1}{2}[a, b]_+ \otimes [f, g],$$

$$\langle \langle f | g \rangle, \langle f' | g' \rangle \rangle = \langle [f, g] | [f', g'] \rangle,$$

$$\langle [f | g], a \otimes h \rangle = a \otimes [[f, g], h].$$

In particular, the center is isomorphic to $HC_1(A)$ (see [BGK, 359-360] for more details).

**Appendix.** Recall that $q$ is a prime power and $\mathbb{F}$ is the field with $q^2$ elements. Denote by $\mathbb{K} = \mathbb{F}((z))$ the field of Laurent power series and by $B^n$ the set of $n$-steps periodic flags (see section 6). Let $C_{GL_m(\mathbb{K})}[B^n \times B^n]$ be the convolution algebra of invariant complex functions supported on a finite number of $GL_m(\mathbb{K})$-orbits, where the convolution product

$$\star : C_{GL_m(\mathbb{K})}[B^n \times B^n] \otimes C_{GL_m(\mathbb{K})}[B^n \times B^n] \to C_{GL_m(\mathbb{K})}[B^n \times B^n]$$

is defined as

$$f \star g(L'', L) = \sum_{L' \in B^n} f(L'', L') \cdot g(L', L).$$

If $i = 0, 1, ..., n - 1$ let $m_i, \chi_i^+ \chi_0^0, \chi_i \in C_{GL_m(\mathbb{K})}[B^n \times B^n]$ be such that

- $m_i(L', L) = \dim(L_i/L_0), \quad \forall L, L' \in B^n,$
- $\chi_i^+$ is the characteristic function of the set
  $$\{(L^+, L^-) \in B^n \times B^n | L^- \subset L^+_j \quad \text{and} \quad \dim(L^+_j/L^-_j) = \delta(j = j), \quad \forall j \in \mathbb{Z} \},$$
- $\chi_0^0$ is the characteristic function of the diagonal in $B^n \times B^n.$
Proposition. The map
\[ e_i \mapsto q^{m_i - m_j} \chi_i^+, \quad f_i \mapsto q^{m_i - m_j} \chi_i^-, \quad k_i \mapsto q^{2m_i - m_j - m_i + 1} \chi^0, \]
extends to an algebra homomorphism \( \hat{U} \to \mathbb{C}_{GL_m}([B^\times B^n]). \)

Proof. We have to prove the \( q \)-deformed Kac-Moody relations written in section 3. The relations
\[ k_i \cdot k_i^{-1} = 1, \quad k_i \cdot k_j = k_j \cdot k_i, \]
\[ k_i \cdot e_j = q^{a_{ij}} e_j \cdot k_i, \quad k_i \cdot f_j = q^{-a_{ij}} f_j \cdot k_i, \]
are immediate. As for
\[ [e_i, f_j] = \delta(i = j) \frac{k_i - k_i^{-1}}{q - q^{-1}}, \]
let first remark that
\[ q^{m_i - m_j} \chi_i^+ \cdot q^{m_j - m_j + 1} \chi_j^- = q^{m_i - m_j + m_j - m_j + 1 + \delta(i = j)} - \delta(i = j + 1) \chi_i^+ \cdot \chi_j^-, \]
\[ q^{m_j - m_j + 1} \chi_j^- \cdot q^{m_i - m_j} \chi_i^+ = q^{m_i - m_j + m_j - m_j + 1 + \delta(i = j)} - \delta(i = j + 1) \chi_j^- \cdot \chi_i^+. \]
If \( i \neq j \) then \( \chi_i^+ \cdot \chi_j^- = \chi_j^- \cdot \chi_i^+ \) is the characteristic function of the set of pairs \( (L', L) \) such that
\[ L'_k = L_k \quad \text{if} \quad \bar{k} \neq \bar{i}, \quad L'_j \subset L_j, \quad L_i \subset L'_i, \quad \dim(L_j/L'_j) = \dim(L'_i/L_i) = 1. \]
Thus,
\[ [e_i, f_j] = \delta(i = j) q^{m_i - m_j + 1 + \chi_i^+ \cdot \chi_j^- - \chi_j^- \cdot \chi_i^+}, \]
\[ = \delta(i = j) q^{m_i - m_j + 1 + (q^{2m_j - 1} - q^{2m_j - 1})} = q^{2} - 1 \chi^0, \]
where the last equality simply comes from \( \mathbb{Z}(\mathbb{F}[k]) = 1 + q^2 + \ldots + q^{2k} \). Since the \( e_i, f_j \) are locally nilpotent and since the \( k_i \) are semisimple, the Serre relations follow from general theory of \( \hat{U} \). \( \square \)

Acknowledgements. The authors are grateful to V. Ginzburg for stimulating discussions.

References

[B] Beck, J.: Braid group action and quantum affine algebras. Comm. Math. Phys., 165 (1994), 555-568.

[BGK] Berman, S., Gao, Y., Krylyuk, Y.S.: Quantum tori and the structure of elliptic quasi-simple Lie algebras. J. Funct. Anal., 135 (1996), 339-389.

[BLM] Beilinson, A., Lusztig, G., MacPherson, R.: A geometric setting for quantum groups. Duke Math. J., 61 (1990), 655-675.
[C1] Cherednik, I.: Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald’s operators. *Int. Math. Res. Notices*, 6 (1992), 171-179.

[C2] Cherednik, I.: Induced representations of double affine Hecke algebras and applications. *Math. Res. Lett.*, 1 (1994), 319-337.

[CP] Chari, V., Pressley, A.: Quantum affine algebras and affine Hecke algebras. *qalg-preprint*, 9501003.

[D] Drinfeld, V.: A new realization of Yangians and quantized affine algebras. *Soviet. Math. Dokl.*, 36 (1988), 212-216.

[G] Garland, H.: The arithmetic theory of loop groups. *Publ. I.H.E.S.*, 52 (1980), 5-136.

[GG] Grojnowski, I., Garland, H.: ’Affine’ Hecke algebras associated to Kac-Moody groups. *Preprint*, (1995).

[GKV] Ginzburg, V., Kapranov, M., Vasserot, E.: Langlands reciprocity for algebraic surfaces. *Math. Res. Lett.*, 2 (1995), 147-160.

[GL] Grojnowski, I., Lusztig, G.: On bases of irreducible representations of quantum $GL_n$. *Contemp. Math.*, 139 (1992).

[GRV] Ginzburg, V., Reshetikhin, N., Vasserot, E.: Quantum groups and flag varieties. *Contemp. Math.*, 175 (1994), 101-130.

[GV] Ginzburg, V., Vasserot, E.: Langlands reciprocity for affine quantum groups of type $A_n$. *Internat. Math. Res. Notices*, 3 (1993), 67-85.

[H] Hayashi, T.: $Q$-analogues of Clifford and Weyl algebras - spinor and oscillator representations of quantum enveloping algebras. *Comm. Math. Phys.*, 127 (1990), 129-144.

[IM] Iwahori, N., Matsumoto, H.: On some Bruhat decompositions and the structure of Hecke rings of $p$-adic Chevalley groups. *Pub. I.H.E.S.*, 25 (1965), 5-48.

[JKKMP] Jimbo, M., Kedem, R., Konno, H., Miwa, T., Petersen, J.: Level-0 structure of level-1 $U_q(^{\hat{\text{sl}}}_2)$-modules and Mac-Donald polynomials. *J. Phys.*, A28 (1995), 5589.

[K] Kassel, C.: Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra. *J. Pure Appl. Algebra*, 34 (1985), 265-275.

[KL] Kassel, C., Loday, J.-L.: Extensions centrales d’algèbres de Lie. *Ann. Inst. Fourier, Grenoble*, 32(4) (1982), 119-142.

[KMS] Kashiwara, M., Miwa, T., Stern, E.: Decomposition of $q$-deformed Fock space. *Selecta Mathematica, New Series*, 1 (1995), 787.

[MRY] Moody, R.V., Rao, S.E., Yokonuma, T.: Toroidal Lie algebras and vertex representations. *Geom. Dedicata*, 35 (1990), 283-307.

[PS] Pressley, A., Segal, G.: Loop groups. *Oxford Mathematical Monographs*, 1986.

[STU] Saito, Y., Takemura, K., Uglov, D.: Toroidal actions on level 1 modules of $U_q(^{\hat{\text{sl}}}_n)$. *qalg-preprint*, 9702024.
[TU] Takemura, K., Uglov, D.: Level-0 action of $U_q(\hat{sl}_n)$ on the $q$-deformed Fock spaces. qalg-preprint, 9607031.

[VV] Varagnolo, M., Vasserot, E.: Schur duality in the toroidal setting. Comm. Math. Phys., 182 (1996), 469-484.

Michela Varagnolo
Dipartimento di Matematica
Università di Tor Vergata
via della Ricerca Scientifica
00133 Roma
Italy
email: varagnol@axp.mat.utovrm.it

Eric Vasserot
Département de Mathématiques
Université de Cergy-Pontoise
2 Av. A. Chauvin
95302 Cergy-Pontoise Cedex
France
email: vasserot@math.pst.u-cerny.fr