Three-dimensional $BF$ Theories and the Alexander–Conway Invariant of Knots

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Abstract

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We study 3-dimensional $BF$ theories and define observables related to knots and links. The quantum expectation values of these observables give the coefficients of the Alexander-Conway polynomial.
1 Introduction

After Witten [1] clarified the existing relation between Chern-Simons topological field theory and Jones link polynomials, a significant amount of research in this subject has been developed.

From [1] we learned that link polynomials with two variables (the so-called HOMFLY polynomials) can be recovered in the framework of Chern-Simons quantum field theory, by considering the expectation values of link observables, given (classically) by the product of the traces of the holonomies computed along the components of the given link.

More precisely the situation is as follows: the expectation values of link-observables in a Chern-Simons field theory with gauge group $SU(N)$, are related to the HOMFLY polynomial $P(l, m)$, provided that we require the normalization condition $P(l, m)(\emptyset) = 1$, where $\emptyset$ is the empty knot (and not the unknot) and provided (more fundamentally) that the two variables $m$ and $l$ are assigned some specific values depending on the integer $N$ and on the (quantized) coupling constant of the theory.

The Alexander-Conway polynomial $\Delta(z)$ is the specialization of the HOMFLY polynomial $P(l, m)$, characterized by the condition $l = 1$ and $m = z$. These conditions are incompatible with the condition arising from Chern-Simons field theory. This explains why we cannot recover the Alexander-Conway polynomial in the framework of Chern-Simons field theory.

Here we propose to consider a topological field theory in 3-dimensions, that differs, in some key features, from the Chern-Simons theory and that is related, as we shall see in this paper, to the Alexander-Conway polynomial. This field theory is called $BF$, from the symbols used to denote the fundamental fields of the theory: $F$ (or $F_A$) stands for the curvature of a connection $A$, while $B$ is an extra field which behaves, under gauge transformations, as the difference of two connections.

A completely similar field theory can be defined in four dimensions, by assuming that the field $B$ is a 2-form (instead of a 1-form) which behaves like the curvature of a connection. Like the Chern-Simons theory, $BF$ theories have been considered extensively in the literature ([3], [5]) both in 3 and in 4 dimensions.

What has not been attempted so far, (a part from [6]) was to relate these topological field theories with knot-invariants. Here we mean ordinary knots for 3-dimensional $BF$ theories and 2-knots for 4-dimensional $BF$ theories.
One of the reasons why this attempt has not been made, may well have to do with the problem of finding a reasonable link- (or knot-)observable. Let us elaborate a little bit on this point.

In a topological field theory of the $BF$ type, one has to compute functional integrals, by integrating over two basic fields: the connection $A$ and the $B$ field. This is a crucial difference with the Chern-Simons case, where the only field involved is the connection $A$. The link-observables should therefore contain not only the connection $A$, but also the field $B$. In a $BF$ theory expectation values of link-observables containing only the field $A$ (like the Wilson loop operator) or only the field $B$ appear to be trivial.

In this paper we propose a special kind of link-observable, which contains both the field $B$ and the field $A$ (through the holonomy). More precisely the observable is defined first by integrating along a knot $K$ the differential 1-form given at $y \in K$ by $\text{Hol}_y^x B(y) \text{Hol}_y^{x_0}$, where $x_0$ is a fixed point on $K$, and then by taking the trace of the exponential of the integral defined above (in a given representation).

In a similar way, we can associate to any link $L$, the product of the traces of the exponentials of the above integrals, computed over all the components of $L$.

In this way the knot and link observables of the $BF$ theory, share the same basic properties (including gauge invariance) with the Wilson line operators considered by Witten, but they depend on two different fields and moreover are expressed as traces of exponentials of integrals of differential forms along the knots, i.e. no further path-ordering is required besides the path-ordering encoded in the definition of the holonomy.

In this paper we are going to prove that the $BF$ theory gives directly the Alexander-Conway polynomial in a variable $z$ that is proportional to the coupling constant of the theory.

Differently from the Chern-Simons case, no resummation is needed in order to recover the knot-polynomials. Here the terms of the perturbative series are exactly the coefficients of the Alexander-Conway polynomial.

This is perfectly consistent with the fact that Feynman integrals, in a topological field theory with knots incorporated, define Vassil’ev invariants of the given knot, and that the only polynomial whose coefficients are Vassil’ev invariants of finite type, is the Alexander-Conway polynomial. In this way, the $BF$ theory is suggesting a way of computing (at least theoretically) the coefficients of the Alexander-Conway polynomial of a given
knot, as multiple integrals over copies of the knot. The kernels of these integrals are similar in nature, to the ones considered in defining linking numbers and the Arf invariant, but involve convolutions of higher orders (for a comparison, one may see the discussion on “Chinese characters” in [9]).

One of the relevant advantages of $BF$ theories is that they can be defined in any dimension. The 4-dimensional case has been discussed in a preliminary way, in [6]. Even in that case one can in principle define invariants of 2-knots, but the situation is far more complicated than the one considered in this paper.

As far as physical applications are concerned, this paper can be considered as a simplified model for a more complicated 4-dimensional theory. The obvious hope is that by considering topological field theories (i.e. diffeomorphism-invariant field theories), both in 3 and 4 dimensions, we can make some progress in understanding the subtle appearance of the metric in quantum gravity as the result of a broken phase of a topological field theory.

A different relation between quantum fields and the Alexander-Conway polynomial has been considered in [11].

2 BF-Theory in 3D: Action, Feynman Rules and Observables

We consider a compact Riemannian 3-manifold $M$ and a (trivial) principal $SU(N)$-bundle over $M$ with connection $A$ and curvature $F$. Our quantum field theory will include a field $B$, classically given by a $su(N)$-valued 1-form of the adjoint type, i.e. a 1-form given by the difference of two connections.

We define the $BF$-action as:

$$S_{BF} = -\frac{1}{2\pi} \int_M Tr(B \wedge F) = -\frac{1}{2\pi} \int_M d^3x \, e^{\mu\nu\rho} Tr[B_\mu(\partial_\nu A_\rho + A_\nu A_\rho)].$$  \hfill (1)

As is well known, under a gauge-transformation $g(x)$ the above fields transform as follows:

\begin{align*}
A(x) &\to g^{-1}(x)A(x)g(x) + g^{-1}(x)dg(x), \\
B(x) &\to g^{-1}(x)B(x)g(x) \\
F(x) &\to g^{-1}(x)F(x)g(x).
\end{align*}

(2)
It is then evident that the action (1) is gauge invariant. Moreover the action (1) is also diffeomorphism invariant. These invariance properties, combined with the fact that the Lagrangian does not depend on a background metric, can be summarized by saying that we are dealing with a topological quantum field theory (2, 3).

Moreover the action (1) is also invariant under the following special BF-transformation:

\[ B(x) \rightarrow B(x) + d_A \chi(x) \] (3)

where \( \chi \) is a \( su(N) \)-valued 0-form of the adjoint type (i.e. a 0-form that behaves like an infinitesimal gauge transformation) and \( d_A \) denotes the covariant derivative.

The Gibbs measure of our theory will be given by \( \exp\left[\left(-\frac{1}{k}\right)S_{BF}\right] \), where \( k \) is the coupling constant.

Differently from the Chern-Simons case, the gauge invariance of our action ensures that (the inverse of) the coupling constant should not necessarily be equal to \( \sqrt{-1} \) times an integer.

Also differently from the Chern-Simons case, we can rescale one of the fundamental fields of the theory with the coupling constant. In this way we are able to shift the coupling constant from the action into the observables\(^1\).

From now on we incorporate the coupling constant in the field \( B \), by setting \( kB_{\text{new}} = B_{\text{old}} \).

In the following we shall use the Lie-algebra notation \( A = \sum_a A^a T^a \) and \( B = \sum_a B^a T^a \), where \( T^a \) are the generators of \( su(N) \) satisfying the following relations:

\[
\begin{align*}
[T^a, T^b] &= f_{abc} T^c, \\
\text{Tr}(T^a T^b) &= -\frac{1}{2} \delta^{ab}.
\end{align*}
\] (4)

Here \( f_{abc} \) are the structure constants for \( su(N) \) and a summation over upper and lower indices is understood.

As is well know, the covariant quantization of this theory requires the introduction of a gauge fixing (e.g. the Landau gauge). The gauge fixing necessarily requires the introduction of a background metric. However, as shown in [3] the quantum action, given by the sum of the action (1) plus the gauge-fixing and the Faddeev–Popov terms has an energy-momentum tensor \( T_{\mu\nu} \) which can be written as a pure BRST variation: \( T_{\mu\nu} = [Q, t_{\mu\nu}] \),

\(^1\)Remark that a scalar times a \( B \)-field is still a \( B \)-field, while a scalar times a connection, is not a connection any more.
where $Q$ is the BRST charge and the explicit form of $t_{\mu\nu}$ is irrelevant for our purposes. Thus the expectation values of a gauge-invariant observable $O$, with a vanishing commutator with $Q$, computed on physical states, i.e. states annihilated by $Q$, turns out to be diffeomorphism invariant.

Since we are mainly interested in the perturbative treatment of (1), we shall choose $M = S^3$ as ambient space and work in a given chart. Namely we will be working locally in $\mathbb{R}^3$, where we will be able to choose a flat metric $g_{\mu\nu} = \delta_{\mu\nu}$.

The starting point for the perturbative quantization of (1) are the Feynman rules, which are given by the propagators:

$$\langle A^a_\mu(x)B^b_\nu(y) \rangle = \delta^{ab}\epsilon^\rho_{\mu\nu}(x - y)^\rho \left|\frac{x - y}{3}\right|$$

and the 3-vertex:

$$V = -\frac{1}{8\pi} \int d^3xe^{\mu\rho}f_{abc}B^a_\mu A^b_\nu A^c_\rho.$$ 

(5)

Notice that the previous expressions for the propagators and the vertex, imply the first important property of the perturbative expansion:

**P.1** The correlation functions $\langle A^sB^k \rangle$ are non trivial only if $k \geq s$

In [5] it is shown that perturbation theory, in the Landau gauge, is finite. This is due to the existence of an underlying supersymmetry, which is in turn a consequence of the invariance of the action under both transformations (2) and (3). In particular the two sets of Faddeev-Popov ghosts arising from the invariance of the action under transformations (2) and (3), contribute to the expectation values of a gauge-invariant observable exactly by cancelling the graphs obtained by contracting the fields $B$ and $A$ inside vertex insertions.

In other words in the perturbative expansion of the BF-theory, one does not need to consider Wick contractions inside vertex insertions. This imply the second important property of our perturbative expansion:

**P.2** The correlation functions $\langle A^sB^k \rangle$ are non trivial only if $s \geq k/2$.

The partition function of the $BF$ theory (without knots) has been shown to be related to the analytical Ray–Singer torsion [3].
When we incorporate knots, the partition function should then be related to the torsion of the **exterior of the knot**, which in turn is related to the Alexander-Conway polynomial [6]. This observation convinced us that we should look for a direct verification, in perturbation theory, of the claim that the partition function with knots incorporated, gives the Alexander-Conway polynomial.

Up to now nothing has been done in constructing explicitly the correlation functions for **$BF$** theories and discuss their topological interpretation, as it has been partly done for the Chern-Simons (CS) theory. In proving the above claim concerning the Alexander-Conway polynomial, we are filling this gap.

The first problem we face is the construction of the analogue of the Wilson line observable for the BF theories. Here we have two fundamental fields $A$ and $B$, so one must consider observables containing both of them.

This construction is realized in few steps. As in the case for the Wilson line operator we start by considering a fixed representation $R$ of $SU(N)$ and the holonomy operator associated to a path $\gamma$ connecting two point $x$ and $y$

$$\text{Hol}_{x}^{y}(A; \gamma) \equiv \mathcal{P} \exp \left[ \int_{\gamma(x,y)} dz^{\mu} A_{\mu}(z) \right], \quad (7)$$

where $\mathcal{P}$ denotes path-ordering. We then combine the matrices $\text{Hol}$ and $B$ (in the given representation $R$) and construct a matrix-valued 1-form:

$$G_{\mu}(y; \gamma, \gamma') \equiv \text{Hol}_{x_{0}}^{y}(A; \gamma)B_{\mu}(y)\text{Hol}_{y}^{x_{0}}(A; \gamma'), \quad (8)$$

where $x_{0}$ is an arbitrary fixed point in $S^3$ and $\gamma, \gamma'$ are two smooth curves connecting $x_{0}$ and $y$ and, respectively, $y$ and $x_{0}$. Under a gauge transform $g(x)$ we have the following transformation rules:

$$\text{Hol}_{x}^{y}(A; \gamma) \rightarrow g^{-1}(x)\text{Hol}_{x}^{y}(A; \gamma)g(y),$$

and

$$G(y; \gamma, \gamma') \rightarrow g^{-1}(x_{0})G(y; \gamma, \gamma')g(x_{0}).$$

We can now integrate the field $G$ along a knot $K$. We generally will choose the point $x_{0}$ to lie on $K$; with this choice, the quantity $\int_{K} G$ transforms under
a gauge transformation, exactly as \( \text{Hol}_{x_0}(A, K) \) (in the given representation \( R^{\mathbb{R}} \)).

We are now able to associate, for any given representation \( R \) of \( SU(N) \), (gauge-invariant) observables to any knot \( K \) in \( S^3 \). For instance we may consider, for any positive integer \( n \),

\[
\text{Tr}_R \left[ \oint_K G(y; \gamma, \gamma') \right]^n.
\]

More generally we will be interested in the “series” of the above observables, namely in the observable \( W_R(K; k) \) given by:

\[
W_R(K; k) := \text{Tr}_R \exp \left[ k \oint_K G(y; \gamma, \gamma') \right],
\]

(9)

where \( k \) is the coupling constant\(^3\). In principle the observable \( W_R(K; k) \) depends also on the choice of \( \gamma \) and \( \gamma' \), but, as we shall see later on, this dependency will not really matter (as far as \( \gamma \cup \gamma' \) and \( K \) are unlinked). Hence we will not include the paths \( \gamma \) and \( \gamma' \) among the variables on which our observables explicitly depend.

It is important to notice that the observable (9) is also invariant under the special transformations (3). In fact consider \( G \) defined as above and replace \( B \) with \( d_A \chi \) in the definition of \( G \). We assume here that \( \chi \) (which is of the same nature of an infinitesimal gauge transformation) vanishes at the fixed point \( x_0 \). So we have:

\[
\oint_K \text{Hol}_{x_0}^y(A; \gamma)d_A \chi \text{Hol}_{y_0}^y(A; \gamma') = \oint_K d(\text{Hol}_{x_0}^y(A; \gamma) \chi \text{Hol}_{y_0}^y(A; \gamma'))
\]

\[
= [\text{Hol}(A, \gamma \cup \gamma'), \chi(x_0)] = 0.
\]

In other words, when we set \( B_t = B + td_A \chi \), then we have:

\[
\frac{d}{dt} \bigg|_{t=0} W_R(K; k, t) = 0,
\]

\(^2\) This implies that the \( n \)-point function constructed by the \( G \)-fields is a gauge-singlet, i.e.

\[
\langle G(x_1) \ldots G(x_n) \rangle = f(x_1, \ldots, x_n) 1
\]

for a suitable scalar function \( f \). Here we omitted the dependency of the field \( G \) on the paths \( \gamma \)’s.

\(^3\) We should remember that we rescaled the \( B \)-field.

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where $W_R(K; k, t)$ has been obtained by replacing $B$ with $B_t$ in $W_R(K; k)$.

In principle we can associate to each point $y \in K$ a different pair of distinct paths $\gamma, \gamma'$ to be included in the observable (9). A natural choice consists instead of defining a knot $K_f$ infinitesimally close to the given knot $K$, but never intersecting $K$, so that $\gamma \cup \gamma' = K_f$. This choice will automatically eliminate the divergences produced by the propagator when it is evaluated at coincident points.

The knot $K_f$ is called a framing for the knot $K$. In local coordinates the equation for $K_f$ can be given as follows:

$$x^\mu(t) = y^\mu(t) + \epsilon n^\mu(t), \quad (\epsilon > 0, |n(t)| = 1),$$

where $y^\mu(t)$ is a parametrization of $K$ and $n^\mu(t)$ is a vector field normal to $K$. As far as the notation is concerned, we will write $G(K_f)$ to denote the dependency of the field $G$ on the framing $K_f$.

As we shall see in the next section, the expectation value of the observable (9) is invariant under a deformation of the framing $K_f$, provided that the knot $K$ is not intersected. Hence the only residual dependence on $K_f$ lies in the linking number $\ln(K_f; K)$ between the knot and its framing. We will consistently use the “standard framing”, namely we will consistently require $\ln(K_f; K) = 0$.

Our basic aim is to compute the normalized expectation value in perturbation theory:

$$\langle K \rangle_R(k) := \frac{\langle W_R(K; k) \rangle}{\langle W_R(\bigcirc; k) \rangle}, \quad (10)$$

where $\bigcirc$ denotes the unknot, and the expectation value is given by a functional integration with respect to the Gibbs measure given by $\exp(-S_{BF})$.

It is precisely (10) that gives the Alexander-Conway polynomial, as we are going to show.

### 3 Framing Invariance and Surgery: Preliminary Non-Perturbative Results

As we already mentioned, care should be exercised in dealing with the framing $K_f$. 

In quantum Chern-Simons theory one really finds invariants of framed knots. Hence it is perfectly natural to ask whether the same situation occurs here. This is not the case. The basic idea is that in the BF-theory we choose the standard framing at the very beginning, and we stick to this choice in all our calculations. The choice of the framing is part of the definition of the observables and tautologically the framing-independence is guaranteed at a non-perturbative level. In Chern-Simons theory, on the contrary, the framing-dependent regularization has to be assigned order by order in the perturbative calculation ([17],[8]).

In order to have a better understanding of the framing-independence of our quantum field theory, we consider the effect on our observable of a deformation of $K$ and $K_f$ localized at a given point $x$, with coordinates $x_{\mu}$ (see [7] for a related approach).

The following identities must be taken into account:

$$\delta \frac{\delta S}{\delta K^\mu(x)} \int_K G = \dot{K}^\nu(x) \Hol^x_{x_0} (d_A B)_{\mu\nu}(x) \Hol^x_{x_0},$$

$$\delta \frac{\delta S}{\delta K^\nu_f(x)} \Hol^x_{w}(K_f) = \dot{K}_f^\nu(x) \Hol^x_{w} F_{\mu\nu}(x) \Hol^x_{x}.$$  \quad (11)

$\dot{K}^\nu(x)$ and $\dot{K}_f^\nu(x)$ are the tangent vectors to the knot $K$ and, respectively, to its framing $K_f$.

The functional derivatives of the BF action are as follows:

$$-\frac{\delta S}{\delta B^a(x)} = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} F_{\nu\rho}^a(x),$$

$$-\frac{\delta S}{\delta A^a_\mu(x)} = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} (d_A B)^a_{\nu\rho}(x).$$  \quad (12)

In force of equations (12), we can represent $F$ and $d_A B$, appearing in the vacuum expectation value of any observable, as functional derivatives of the Gibbs measure $\exp(-S_{BF})$. In particular the variation with respect to $K$ ($K_f$) of the vacuum expectation value of the observable $W$, can be replaced by a functional derivative with respect to $A$ ($B$).

An integration by parts allows us to shift this derivative to the remaining
part of the observable. The scheme is as follows:

\[
\frac{\delta}{\delta K(x)} \rightarrow d_{AB}(x) \rightarrow \frac{\delta}{\delta A(x)},
\]

\[
\frac{\delta}{\delta K_f(x_f)} \rightarrow F(x) \rightarrow \frac{\delta}{\delta B(x)},
\]

Thus a deformation of \(K\) \((K_f)\) gives a contribution only if the functional derivative with respect to \(A\) \((B)\) is not zero. Since \(A\) \((B)\) lives on \(K_f\) \((K)\), this is possible only if the deformation of \(K\) path intersects \(K_f\), namely only if \(\ln(K;K_f)\) is changed. But we have to stick to the standard framing, \(i.e.\) no modification of \(\ln(K;K_f)\) is allowed. We conclude that \(\langle K \rangle_R(k)\) defines a true knot invariant.

As a preliminary non-perturbative computation we now derive a “surgery formula” for \(\langle K \rangle_R(k)\). For this purpose we need to recall some mathematical background concerning the “tangles” in knot theory.

A tangle is obtained by a link (knot) diagram by breaking two edges as in fig. 1. One can sum two tangles \(A\) and \(B\), as in fig. 2, by forming the tangle \(A + B\) in which the right outer strings of \(A\) and the left outer strings of \(B\) are joined in agreement with their orientation. One may recover a link diagram from a tangle in the two ways described in fig. 3. The two link diagrams above are denoted respectively by the symbols \(A^N\) and \(A^D\) where the superscript \(N\) and \(D\) stands for “numerator” and “denominator”. The terminology here is due to Conway \([12]\). It can be easily checked that if \(A^N\) \((A^D)\) is a knot diagram, then \(A^D\) \((A^N)\) is the diagram of a two-component link.

We know \([13]\) that when \(P\) is the two variable HOMFLY polynomial or any specialization of it (like the Alexander-Conway polynomial) then for any tangles \(A\) and \(B\) we have:

\[
P[(A + B)^D] = P[A^D]P[B^D],
\]

(14)

Here \(P\) is normalized so that \(P(\bigcirc) = 1\).

We shall now show that the condition (14) is actually satisfied by our normalized knot invariant \(\langle K \rangle_R(k)\), \(i.e.\) the knot invariant \(\langle W_R(K; k) \rangle\), divided by \(\langle W_R(\bigcirc; k) \rangle\).

Namely we want to show that in quantum field theory the following relation holds:

\[
\langle W_R((A + B)^D; k) \rangle \langle W_R(\bigcirc; k) \rangle = \langle W_R(A^D; k) \rangle \langle W_R(B^D; k) \rangle.
\]

(15)
We assume that $A^D$ and $B^D$ are knot-diagrams, or equivalently that $(A + B)^D$ is a knot-diagram.

The main ingredient in the proof is the cluster property of the vacuum expectation values, which allows to rewrite the l.h.s of equation (15) as:

$$\langle W_R((A + B)^D; k) \rangle \langle W_R(\bigodot; k) \rangle = \langle W_R((A + B)^D; k) W_R(\bigodot; k) \rangle$$

where $(A + B)^D$ and the unknot are supposed, for the purposes of quantum field theory, to be at an infinite distance. By using the diffeomorphism invariance of the BF theory we may move the unknot $\bigodot$ over $(A + B)^D$ as shown in fig.4. Furthermore we are free to move the companion of the unknot, denoted here by the symbol: $\bigodot_f$, independently of $\bigodot$, as far we keep $\bigodot_f$ and $\bigodot$ unlinked. Hence the l.h.s of equation (15) becomes:

$$\langle \text{Tr} \exp \left[ k \left( \int_{A'} + \int_{B'} \right) G(K_f = A_f' + B_f') \right] \times \text{Tr} \exp \left[ k \left( \int_{U_1} + \int_{U_2} \right) G(\bigodot_f = U_{1f} + U_{2f}) \right] \rangle.$$

Here $A'$ ($B'$) are the paths obtained by splicing together the strings of $A$ ($B$) which are not spliced in the sum $A + B$ and $U_{1,2}$ is a suitable decomposition of $\bigodot$ as shown in fig.5. By using again the diffeomorphism invariance, we take $A'$ and $B'$ infinitely apart, so that when $G$ is integrated over $A'$ ($B'$) we may neglect the contribution coming from $B_f'$ ($A_f'$) and replace it by $U_{1f}$ ($U_{2f}$). We can then repeat this operation for $\bigodot$ and the l.h.s. of equation (15) becomes:

$$\langle \text{Tr} \exp \left[ k \int_{A'} G(A_f' + U_{1f}) + k \int_{B'} G(U_{2f} + B_f') \right] \times \text{Tr} \exp \left[ k \int_{U_1} G(A_f' + U_{1f}) + k \int_{U_2} G(U_{2f} + B_f') \right] \rangle.$$

Since averages of $G$-fields are in a gauge-singlet representation, it follows that one is allowed to replace the traces with the dimensions of the representation. Moreover, we can freely commute two $G$-fields defined on two widely separated points. Both of these properties imply that one can treat the arguments of the exponentials as Abelian-like fields. Therefore the l.h.s.
of equation (15) becomes:

\[
\langle \text{Tr} \exp \left[ k \left( \int_{A'} + \int_{U_1} \right) G(K_f = A'_f + U_1f) \right] \times \text{Tr} \exp \left[ k \left( \int_{U_2} + \int_{B'} \right) G(K_f = U_2f + B'_f) \right] \rangle = \langle W_R(A^D; k) W_R(B^D; k) \rangle,
\]

where we have used the identities \( A' + U_1 = A^D \) and \( U_2 + B' = B^D \). By using again the diffeomorphism invariance and the cluster properties the l.h.s. of equation (15) finally becomes:

\[
\langle W_R(A^D; k) \rangle \langle W_R(B^D; k) \rangle,
\]

namely we have proved equation (15).

4 Perturbative Expansion

Let us consider the perturbative expansion of \( \langle W_R(K; k) \rangle \) in powers of \( k \). At the \( n - th \) order in \( k \) we have a product of \( n \) factors \( \int G \). Then, by taking into account the structure of the \( G \) operator in (8), we have to compute a correlation function with \( n \) \( B \)-fields and an arbitrary number of \( A \)-fields coming from the (path-ordered) expansions of the holonomies. Furthermore we have to take into account the properties \( \text{P.1} \) and \( \text{P.2} \) of the perturbative series, that we derived in section 2.

When fields are evaluated at coincident points, we could have, in principle, non-analytic correlation functions. But the structure of \( BF \) theories does not present this problem. In fact we have:

- one possible source of divergence given by the propagator \( \langle AB \rangle \). This divergence does not appear since \( B \)-fields live on the knot \( K \) while \( A \)-fields live on the framing \( K_f \) and \( K_f \) does not intersect \( K \).

- another possible source of divergence coming from a vertex insertion. Again this divergence does not appear since the structure of the observables implies that between two \( B \)-fields there exists always an \( A \)-field which forbids them to be evaluated at coincident points.
In order to calculate the terms of the perturbative expansion, we have to compute, according to the Wick theorem, the convolution of two and three point correlation functions of the form. Since we do not have contractions inside vertex insertions, these correlation functions are given by:

$$\langle A_\mu^a(x) B_\nu^b(y) \rangle = 4\pi l_{\mu\nu}(x, y) \delta^{ab},$$
$$\langle B_\rho^a(z) A_\nu^b(y) B_\mu^c(x) \rangle = (4\pi)^2 v_{\mu\nu\rho}(x, y, z) f^{abc},$$

where, in force of (5) and (6), $l$ and $v$ are explicitly given by

$$l_{\mu\nu}(x, y) = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} (x-y)^\rho, \quad (17)$$
$$v_{\mu\nu\rho}(x, y, z) = \epsilon^{\alpha\beta\gamma} \int_{S^3} d^3 w \ l_{\mu\alpha}(x, w) \ l_{\nu\beta}(y, w) \ l_{\rho\gamma}(z, w). \quad (18)$$

We will refer to (17) as to the $l-$kernel and to (18) as to the $v-$kernel. These kernels have the following obvious symmetries:

$$l_{\mu\nu}(x, y) = l_{\nu\mu}(y, x), \quad v_{\mu\nu\rho}(x, y, z) = v_{\nu\mu\rho}(y, z, x) = v_{\rho\mu\nu}(z, x, y), \quad (19)$$

Since the correlators (16) have to be integrated over the knot and its framing, it is useful to define the following loop-dependent kernels

$$l_K(s_1, s_2) := \hat{K}_\mu^\nu(s_1) \hat{K}_\nu^\rho(s_2) l_{\mu\nu}(K(s_1), K_f(s_2)), \quad v_K(s_1, s_2, s_3) := \hat{K}_\mu^\nu(s_1) \hat{K}_\nu^\rho(s_2) \hat{K}_\rho^\sigma(s_3) v_{\mu\nu\rho}(K(s_1), K_f(s_2), K(s_3)), \quad (20)$$

where $K(\cdot)$ and $K_f(\cdot) : [0, 1] \to S^3$ are parametrizations of the knot and of its framing. As an immediate consequence of the property (19) we have:

$$v_K(s_1, s_2, s_3) = -v_K(s_3, s_2, s_1), \quad (21)$$

Moreover, in the limit when the spacing between $K$ and $K_f$ goes to zero (without modifying the standard framing), we have also:

$$v_K(s_1, s_2, s_3) = v_K(s_2, s_3, s_1). \quad (22)$$

One of the main results of the present section is to prove the following peculiar feature of our theory:
The terms in the perturbative expansion, which contain an odd number of $B-$fields vanish.

Indeed at these orders in perturbation theory, we have:

1. amplitudes which directly allow a factorization of the term $L(K, K_f) := \int_0^1 ds_1 \int_0^1 ds_2 l_K(s_1, s_2)$. But this term is the (Gauss formula for the) linking number $\ln(K_f; K)$, that is identically zero, due to our choice of the standard framing.

2. amplitudes corresponding to Feynman graphs of the form depicted in fig.6. These amplitudes are computed by requiring $K_f$ to be kept apart from $K$ at a distance $\epsilon$. Then we send $\epsilon$ to zero. As we will show at the end of this section, this allows us to use the symmetries (22) and to show that also these amplitudes are identically zero.

Let us now analyze first the structure of terms of even order in the perturbative expansion.

The terms of order $k^{2n}$ include a set of graphs given only by convolution of $v-$kernels, i.e. containing exactly $n A$-fields as in fig.7. We shall call these graphs “V-graphs”. Their structure is of the form

$$W^V := \langle (B^2A)^n \rangle . \quad (23)$$

At the $2n$ th perturbative order, there exist other Feynman graphs, of the type

$$\langle B^{2n}A^{n+s} \rangle , \quad n \geq s > 0 \quad (24)$$

obtained by inserting (and Wick-contracting) $s A$-fields in the graphs (23). The insertion of one $A$-fields implies the replacement of a $v-$kernel in the graphs (23) with a pair of $l-$kernels.

Concerning the Lie algebra factors for the graphs of order $2n$, we have shown that, up to the fourth order, it is always given by $(c_v c_2(R))^n$. Here $R$ is the given representation of $SU(N)$, $c_2(R) \mathbf{1}$ is its Casimir operator (i.e. $\sum_a R(T^a)R(T^a)$) and $c_v$ is defined by:

$$f^{acd} f^{bcd} = c_v \delta^{ab}. \quad (25)$$
We conjecture that at any perturbative order \(2n\), all graphs have \((c_v c_2(R))^n\) as a common factor. That means that the true expansion parameter must be:

\[
z^2 := (4\pi k)^2 c_v c_2(R).
\]  

(26)

In order to justify the above conjecture, we notice that if there existed Feynman amplitudes with different Lie algebra factors, then we would be able to construct a multivariable knot-invariant. But this will be incompatible with the skein relation (that we will prove in section 6).

Hence (by taking into account property P.3), our perturbative expansion will look like:

\[
\langle W \rangle_R (K; k) = \dim(R) \sum_{n=0}^{\infty} w_{2n}(K) z^{2n},
\]  

(27)

for some suitable invariants \(w_{2n}(K)\) of the knot \(K\), with no residual dependence on the group itself. Here \(z\) is given as in (26) and \(\dim(R)\) is the dimension of the given representation.

We want now to construct explicitly the coefficients \(w_{2n}(K)\).

Let us first consider graphs of the type (23), i.e. graphs given by convolutions of \(n v\)-kernels. They can be of two types: connected and non-connected. The connected ones \(w^V_{2n}(K)\) are defined as;

\[
w^V_{2n}(K) \equiv \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \cdots \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \cdots \prod_{i=1}^n v_K(s_i, s_{i-1}, s_i).
\]  

(28)

Here we have set: \(s^0_2 \equiv s^n_3\). In equation (28) we can alternatively replace:

\[
\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \cdots \text{with} \quad \frac{1}{2} \int_0^1 ds_1 \int_0^{s_1} ds_3 \int_0^{s_3} ds_2.
\]

In equation (28), one can immediately notice that each \(v\)-kernel is “linked” to the next one, and so there is only one “chain” of “linked” \(v\)-kernels, this is exactly the meaning of the word “connected”. Non-connected \(V\)-graphs are \(V\)-graphs containing more than one chain of “linked” \(v\)-kernels.

Up to the fourth order in perturbation theory, the non-connected \(V\)-graphs do not appear and it is reasonable to expect that the same will be

\[\text{[
\text{Instead in the CS theory the analogous of the } w_{2n} \text{ coefficients have an explicit dependence on the group.}
\] }16
true at any order of perturbation theory. We will therefore consider only connected V-graphs, but it should be pointed out that this is not a serious restriction. In other words, all the arguments we developed and are going to develop, will work equally well (with minor modifications) even if non trivial non-connected V-graphs existed.

Now we are going to discuss a method that will enable us to produce all the terms of any given order of the perturbation series, from the V-graphs of the same order.

In order to do this we will use the following “rule” for Wick-contractions (denoted by over/underlines):

\[
\begin{align*}
B \ldots A' A \ldots B' &= B \ldots A \ A' \ldots B' + B \ldots [ A', A ] \ldots B' \\
\end{align*}
\]

When we sum the first term of the r.h.s. of (29) with the term which is equal to the l.h.s. of (29), but has two contractions interchanged, we obtain a contribution that does not require a path-ordering of \( A \) and \( A' \). By iterating this procedure we produce at the end linking numbers between \( K \) and \( K_f \) that are zero. The rule (29) has a nice diagrammatical interpretation as shown in fig.8.

The perturbative series is then constructed out of terms like the second part of the r.h.s. of (29). Each one of these terms can be obtained by a term which has one field \( A \) less (but has the same Lie algebra factor). Namely each one of these terms is obtained by replacing a \( v \)-kernel with two \( l \)-kernels. We will give then an analytic description of this procedure, by introducing an operator \( \mathcal{D} \) that changes a \( v \)-kernel into the products of two \( l \)-kernels.

Before doing so we would like to notice, as a side remark, that fig.8 has a strong resemblance with one of the rules considered in the computation of Vassil’ev knot invariant (of finite type) \[9\]. This is not surprising, since at any fixed order of perturbation theory, we expect that Feynman integrals of topological field theories (with knots incorporated) will produce Vassil’ev invariants \[5\].

\[5\] In CS theory the situation was considerably different, since a redefinition of the coupling constant \( k \rightarrow t = \exp(2\pi i N/k) \) was needed. This redefinition implied an infinite resummation of the Feynman graphs.
Let us now define the operator $\mathcal{D}$, that we mentioned above. The operator $\mathcal{D}$ is assumed to transform the term $v_K(s_1^i, s_2^{i-1}, s_3^i)$ in (28), according to the following rule:

$$\mathcal{D}v_K(s_1^i, s_2^{i-1}, s_3^i) \equiv$$

$$\frac{1}{2} \int_{s_2^i}^{s_1^i} d\tilde{s}_2 \, l_K(s_1^i, s_2^{i-1})l_K(s_3^i, \tilde{s}_2)$$

$$+ \frac{1}{2} \int_{s_3^i}^{s_2^i} d\tilde{s}_2 \, l_K(s_1^i, s_2^i)l_K(s_3^i, s_2^i).$$

(30)

Notice that both $v_K$ and $\mathcal{D}v_K$ change sign when we exchange $s_1^i$ with $s_3^i$.

So far we have defined the action of the operator $\mathcal{D}$ on a single $v-$kernel. But (28) is given, in general, by the convolution of many $v-$kernel. The action of the operator $\mathcal{D}$ on (28) is then completely defined by assuming that $\mathcal{D}$ satisfies the Leibniz rule.

In a similar way we can define the action of $\mathcal{D}$ on $w_{2n+1}(K)$, i.e. on the connected graph of order $2n+1$ given by the convolution of $n$ $v-$kernels and one $l-$kernel. This graph, which is of the type $\langle B^{2n+1}A^{n+1} \rangle$, is defined as:

$$w_{2n+1} = \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_3 \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_3 \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_3 \cdots$$

$$\times \int_0^1 ds_1^n \int_0^1 ds_2^n \int_0^1 ds_3^n l_K(s_1^0, s_2^n, s_3^n) \prod_{i=1}^{n} v_K(s_1^i, s_2^{i-1}, s_3^i).$$

(31)

In a completely similar way we can define the action of $\mathcal{D}$ on any kernel given by the convolution of an arbitrary number of $v-$kernels and $l-$kernels. The basic idea is always that any given $v-$kernel $v_k(x, y', z)$ is transformed into:

$$(\mathcal{D}v_K)(x, y, y', z) = \frac{1}{2} \int_{y}^{y'} d\bar{y} \, l_K(x, \bar{y})l_K(z, y') + \frac{1}{2} \int_{y'}^{x} d\bar{y} \, l_K(x, y')l_K(z, \bar{y}),$$

(32)

where the variable $y$ is the same variable appearing in the other $v-$ or $l-$kernel to which the given $v-$kernel is “linked”. A graphical description of the action of the operator $\mathcal{D}$ is given in fig.9.

Together with the operator $\mathcal{D}$, we can consider its exponential, i.e. the operator $\exp \mathcal{D}$, defined, in the standard way, as a power series. Notice that
by applying $\mathcal{D}$ a number of times greater than the number of $v-$kernels appearing in a given graph, one obtains zero. So the operator $\exp \mathcal{D}$ is given, for any assigned graph, only by a finite sum of powers of $\mathcal{D}$.

The perturbative expansion of our $BF$-theory satisfies the following rules:

1. all the terms of order $2n$ of the perturbative expansion are obtained by applying the operator $e^\mathcal{D}$ to the graph (28).

2. all the terms of order $2n + 1$ of the perturbative expansion are obtained by applying the operator $e^\mathcal{D}$ to the graph (31).

In order to prove the property $P.3$, it is sufficient to show that (31) is zero, thus implying that any other graph obtained from (31) by applying the operator $\mathcal{D}$ is also zero.

Due to the special symmetry (22), obtained in the limit when the spacing between the knot and its framing is sent to zero, we have in (31):

$$v_k(s_1^1, s_2^0, s_3^1) = v_k(s_2^0, s_3^1, s_1^1) = v_k(s_3^1, s_1^1, s_2^0)$$

and hence:

$$3w_{2n+1} = \frac{1}{2} \int_0^1 ds_1^0 \int_0^1 ds_2^0 \int_0^1 ds_3^0 \int_0^1 ds_3^1 \left[ \left( \int_{s_1^1}^{s_2^0} + \int_{s_1^1}^{s_3^1} \right) ds_2^1 \right]$$

$$\times \int_0^1 ds_1^2 \int_0^{s_1^1} ds_2^2 \int_0^{s_2^1} ds_2^2 \cdots \int_0^1 ds_1^n \int_0^{s_1^n} ds_2^n \int_0^{s_2^n} ds_3^n$$

$$\times l_K(s_1^0, s_2^n) \prod_{i=1}^n v_K(s_i^i, s_i^i, s_i^3) = 0.$$ (33)

5 Explicit Computations up to the fourth order

By taking into account (27), the normalized expectation value (11), will be given by the following formula

$$\langle K \rangle_R(k) = \sum_{n=0}^{\infty} a_{2n}(K) z^{2n},$$ (34)

Footnotes:

6 This fact is directly connected to property $P.1$ of $BF$ theory.

7 And to its non-connected counterparts, if they exist.

8 This proof applies equally well to any non-connected counterpart of (31).
where $z$ is defined in (26) and $a_{2n}(K)$ are to be determined as a function of the coefficients $w_{2n}(K)$ and the corresponding coefficients for the unknot $w_{2n}(\bigcirc)$.

More precisely we have:

$$
\begin{align*}
a_0(K) &= w_0(K), \\
a_2(K) &= w_2(K) - w_2(\bigcirc), \\
a_4(K) &= w_4(K) - w_2(\bigcirc) a_2(K) - w_4(\bigcirc), \\
\end{align*}
$$

\(\ldots\)

$$
\begin{align*}
a_{2n}(K) &= w_{2n}(K) - \sum_{i=1}^{n} w_{2i}(\bigcirc) a_{2n-2i}(K).
\end{align*}
$$

At order zero we have simply $w_0(K) = 1$. At the second order we have only two graphs as shown in fig.10. The V-graph (i.e. $\langle B^2 A \rangle$) is given by

$$
w^V_2(K) = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 v_K(s_1, s_2, s_3),
$$

while the graph $\langle B^2 A^2 \rangle$ is given by:

$$
\mathcal{D}w^V_2(K) = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 l_K(s_1, s_2) l_K(s_3, \bar{s}_2).
$$

The total second order term is then $w_2(K) = (1 + \mathcal{D})w^V_2(K)$. As a consequence, $w_2(K)$ is the same as the second order contribution of the perturbative expansion of the CS theory, so that we can use the results of [17] and obtain:

1. $w_2(\bigcirc) = -\frac{1}{24}$;
2. $a_2(k) = w_2(K) + \frac{1}{24}$

As was pointed out in [17], the term $a_2(K)$ is the so called Arf invariant, that is well known to be the second coefficient of the Alexander-Conway polynomial ([10]). The third order of the perturbative expansion is trivial, thanks to the results of the previous section. The structure of the corresponding graphs is shown in fig.11. We now consider the fourth order of the perturbative expansion. The coefficient $a_4(K)$, is obtained by considering the Feynman diagrams shown in fig.12. An explicit computation shows that
there exists only one V-graph (\( \langle B^4 A^2 \rangle \)), which is completely connected (see fig.12a), namely:

\[
\begin{align*}
\omega_V^4(K) &= \frac{1}{4} \int_0^1 ds_1 \int_0^1 ds_3 \int_0^1 dt_1 \int_0^1 dt_3 \int_{s_3}^{s_1} ds_2 \int_{t_3}^{t_1} dt_2 v_K(s_1, t_2, s_3) v_K(t_1, s_2, t_3).
\end{align*}
\]

(38)

The contribution corresponding to fig.12b is given by:

\[
\begin{align*}
\mathcal{D} \omega_V^4(K) &= \frac{1}{4} \int_0^1 ds_1 \int_0^1 ds_3 \int_0^1 dt_1 \int_0^1 dt_3 \int_{s_3}^{s_1} ds_2 \int_{t_3}^{t_1} dt_2 d\bar{s}_2 d\bar{t}_2
\end{align*}
\]

\[
\times [l_K(s_1, t_2)l_K(s_3, \bar{s}_2) v_K(t_1, s_2, t_3) + l_K(s_1, s_2)l_K(s_3, t_2) v_K(t_1, \bar{s}_2, t_3)],
\]

(39)

while the contribution of fig.12c is given by

\[
\begin{align*}
\frac{1}{2} \mathcal{D}^2 \omega_V^4(K) &= \frac{1}{8} \int_0^1 ds_1 \int_0^1 ds_3 \int_0^1 dt_1 \int_0^1 dt_3 \int_{s_3}^{s_1} ds_2 \int_{t_3}^{t_1} dt_2 \int_{\bar{t}_3}^{\bar{t}_1} \int_{\bar{s}_3}^{\bar{s}_1} d\bar{s}_2 d\bar{t}_2
\end{align*}
\]

\[
\times [l_K(s_1, t_2)l_K(s_3, \bar{s}_2)l_K(t_1, s_2)l_K(t_3, \bar{t}_2) + l_K(s_1, \bar{t}_2)l_K(s_3, \bar{s}_2)l_K(t_1, t_2)l_K(t_3, s_2)].
\]

(40)

The sum \((1 + \mathcal{D} + 1/2 \mathcal{D}^2) \omega_V^4(K)\) gives the term \(\omega_4(K)\). In order to obtain the normalized fourth order knot-invariant, one has to compute \(\omega_4(\bigcirc)\) for the unknot and apply the relations (33).

In principle (33) allows the computation of the coefficients of the Alexander-Conway polynomial at any order. We have in fact a close analytic expression for such coefficients. Actual computations may be difficult, but the situation is considerably simpler than the one in Chern-Simons theory. In CS theory, all the terms of the perturbative series were framing-dependent, and a close analytic expression for such terms was not available.

6 Skein Relation

In the previous sections we proved that the expectation values \(\langle K \rangle_R(k)\) of our knot-observables:

a) satisfy the (denominator) surgery formula which is common to all known link polynomials (including the Alexander-Conway polynomial)

b) are a power series (or a polynomial) in the variable: \(z := (4\pi k) \sqrt{c_0 c_2(R)}\).

The coefficients of these power series are knot-invariants
c) coincide, up to second order, with the Alexander-Conway Polynomial $\Delta_K(z)$

d) contain only terms of even order in $z$

We would like now to prove, in the framework of perturbation theory, our claim that $\langle K \rangle_R(k)$ coincides at all orders with the Alexander-Conway Polynomial.

Let us recall briefly the axiomatic definition of such polynomial.

For any link $L$, $\Delta_L(z)$ is a polynomial of one variable $z$ normalized so that $\Delta(\bigcirc) = 1$. It is a link-invariant satisfying the following skein relation:

$$\Delta_{K_+}(z) - \Delta_{K_-}(z) = z\Delta_{K_0}(z), \quad (41)$$

where $K_+, K_-, K_0$ are three oriented knots/links that are exactly the same except near a crossing point where they look like as in fig.13. The existence of such polynomial follows directly from the definition of the classical Alexander-Conway invariant [12]. We recall here some basic properties of the Alexander-Conway polynomial:

1. the polynomial satisfying the normalization condition and the skein relation defined above, is necessarily unique. Moreover any knot-invariant given by a (formal) power series satisfying the above skein relation, and the normalization condition, must necessarily be a polynomial

2. the above skein relation cannot be defined in terms of knots only. In fact, for any link $L_+$ with $s$ components, $L_-$ has also $s$ components, but $L_0$ has either $s + 1$ or $s - 1$ components

3. if a link $L$ is composed by two links separated by a sphere, then the polynomial $\Delta_L$ is zero,

4. if $H$ denotes the Hopf link (i.e. the link given by two linked circles, with linking number +1), then $\Delta_H(z) = z$

5. for any link $L$, equation (41) can be equivalently rewritten as:

$$a_{n+1}(L_+) - a_{n+1}(L_-) = a_n(L_0), \quad (42)$$

where we have set $\Delta_L(z) \equiv \sum_{n \geq 0} a_n(L)z^n$. 

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Further properties of the Alexander-Conway polynomial are

6. for a knot $K$, $\Delta_K(z)$ is an \textit{even} polynomial in $z$ \cite{10}

7. for a 2-component link $L$, $\Delta_L(z)$ is an \textit{odd} polynomial in $z$

8. 

\[ a_0(K) = \begin{cases} 1 & \text{if } K \text{ is a knot} \\ 0 & \text{otherwise} \end{cases} \]

\[ a_1(L) = \begin{cases} \ln(K^1; K^2) & \text{if } L = K^1 \cup K^2 \text{ is a two-component link} \\ 0 & \text{otherwise} \end{cases} \]

As far as the coefficient $a_2(K)$ (Arf-invariant) is concerned, we notice that we have $a_2(K_+) - a_2(K_-) = \ln(K_0^1; K_0^2)$, where $K_0^1$ and $K_0^2$ are the two components of $K_0$. We are going to recover directly this relation, in the framework of perturbation theory.

By property 2 above, we know that, if we want to recover a skein relation in the framework of perturbation theory, we have to define $< W_R(L; k) >$ for a link $L$ which is not necessarily a knot.

For any link $L$ with components $\{K_i\}_{i=1,2,\ldots,s}$ we first consider:

\[ W_R(L; k) \equiv \prod_i \text{Tr} \exp \left[ k \int_{K_i} G \right]. \]  

(43)

Instead of the expectation values $< W_R(L; k) >$ we will focus here on the \textit{connected correlation functions} $< W_R(L; k) >_c$.

The connected correlation functions can be defined inductively as follows (see \textit{e.g.} \cite{4}).

Let $L$ be a link with $s$ components and let $\mathcal{P}(s)$ the set of all the non-trivial partitions of the set $\{1, 2, \ldots, s\}$.

For any such partition $\sigma \in \mathcal{P}(s)$, represented by a multi-index $[\sigma_1, \sigma_2, \ldots, \sigma_{k_\sigma}]$, with

\[ \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_{k_\sigma} = \{1, 2, \ldots, s\}, \]

we consider the corresponding collection of links $L_{\sigma_1}, L_{\sigma_2}, \ldots, L_{\sigma_{k_\sigma}}$.

Then we set, for a knot $K$:

\[ < W_R(K; k) >_c \equiv < W_R(K; k) >, \]
and for a link $L$, with $s$ components:

$$< W_R(L; k) >_c \equiv < W_R(L; k) > - \sum_{\sigma \in P(s)} < W_R(L_{\sigma_1}; k) >_c \cdots < W_R(L_{\sigma_s}; k) >_c$$

The above definition implies:

- for any link $L = L_1 \cup L_2$ composed by two separate links $L_1$ and $L_2$, we have $< W_R(L; k) >_c = 0$. This is a direct consequence of the cluster property and is consistent with property 3 above.

- the (denominator) surgery law is satisfied. If $\mathcal{A}$ and $\mathcal{B}$ are two tangles, then we have:

$$\langle W_R((\mathcal{A} + \mathcal{B})^D; k) \rangle \langle W_R(\bigcirc; k) \rangle = \langle W_R(\mathcal{A}^D; k) \rangle \langle W_R(\mathcal{B}^D; k) \rangle.$$

The proof of the above identity is verbatim the same we have considered for the surgery of two knots. It can be easily seen that the same is true also for the connected correlation functions.

We are now in position to recover the skein relation (41) for our link-observables.

We start by considering one knot with a selected crossing $K = K_-$ and its switched counterpart $K_+$. The switching can be seen as the result of applying a singular deformation operator, namely:

$$\langle W_R(K_+; k) \rangle = \langle W_R(K_-; k) \rangle + \frac{\delta}{\delta v} \langle W_R(K_-; k) \rangle,$$

where $v$ is a vector (singular vector field) based at the given crossing point of the link. The singular deformation operator should in fact be applied to both the knot $K$ and the framing $K_f$. So we can set:

$$\frac{\delta}{\delta v} \equiv v_\mu \left( \frac{\delta}{\delta K^\mu} + \frac{\delta}{\delta K_f^\mu} \right).$$

In switching the crossing of both the knot and of the framing from a crossing of type $-$ to a crossing of type $+$, we are changing the framing of

---

9. This singular deformation operator is a close relative of the derivation introduced by Sossinsky for the computation of Vassil’ev invariants.
the original knot. In order to restore the standard framing, we need to twist
the pair \((K, Kf)\) as shown in fig.14.

We now use the integration by part techniques. From now on, we consider
the fundamental representation of \(SU(N)\). The generators \(T^a\) satisfy \([4]\) and
also the: Fierz identity:

\[
\sum_a T^a_{ij} T^a_{kl} = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).
\] (46)

When \(R\) is the fundamental representation, then \(c_2(R)\) will be more simply
denoted by \(c_2\). The form of the \(BF\)—action allows the following substitutions:

\[
\frac{\delta}{\delta A^a_{\mu}(x)} \rightarrow \frac{\delta}{\delta A^a_{\mu}(x)},
\]

\[
\frac{\delta}{\delta K^\rho_{\mu}(x)} \rightarrow \frac{\delta}{\delta B^\rho_{\mu}(x)},
\] (47)

where \(x\) is the point where the (singular) deformation is performed.

Next we apply the above operators (derivatives) to our fields. When we
consider the variation of the holonomy, with respect to the connection, we
obtain the same formula obtained in the Chern-Simons theory ([7]):

\[
\frac{\delta}{\delta A^a_{\mu}(x)} \text{Hol}^x_0 = \dot{K}^\rho_{\mu}(x) \text{Hol}^x_0 T^a \text{Hol}^x_0.
\] (48)

When we consider instead the field \(G\) that defines our \(BF\) observables,
we obtain:

\[
\frac{\delta}{\delta A^a_{\rho}(x)} \oint G = \int_{x_0}^{x} \text{Hol}^y_{x_0} B^\sigma(y) \text{Hol}^y_{y} \dot{K}^\rho_{\sigma}(x) T^a \text{Hol}^x_{x} dy^\sigma
\]

\[
+ \int_{x}^{x_0} \text{Hol}^x_{x_0} \dot{K}^\rho_{\sigma}(x) T^a \text{Hol}^y_{y} B^\sigma(y) \text{Hol}^x_{x} dy^\sigma
\]

\[
\frac{\delta}{\delta B^a_{\rho}(x)} \oint G = \text{Hol}^x_{x_0} \dot{K}^\rho_{\sigma}(x) T^a \text{Hol}^x_{x}.
\] (49)
The action of performing first the singular deformation of the knot and of its framing and then integrating by parts (see [7]), can be equivalently described by the action of the following operators on the vacuum expectation values:

\[
\frac{\delta}{\delta K^\mu(x)} \to 4\pi \epsilon^{\mu\nu\rho} \sum_a \left( \frac{\delta^2}{\delta \delta B^a_\nu(x(s)) \delta A^a_\rho(x(\tilde{s}))} - \frac{\delta^2}{\delta \delta A^a_\nu(x(s)) \delta B^a_\rho(x(\tilde{s}))} \right)
\]

\[
\frac{\delta}{\delta K^f_\mu(x)} \to 4\pi \epsilon^{\mu\nu\rho} \sum_a \left( \frac{\delta^2}{\delta \delta A^a_\nu(x(s)) \delta B^a_\rho(x(\tilde{s}))} + \frac{\delta^2}{\delta \delta B^a_\nu(x(s)) \delta A^a_\rho(x(\tilde{s}))} \right)
\]

(50)

Here \(x\) is the crossing point of the knot, while \(x(s)\) and \(x(\tilde{s})\) denote the two distinct coordinates of the knot for which we have: \(x(s) = x(\tilde{s}) = x\). The second terms in both the operators (50) correspond to the twist needed in order to restore the standard framing.

It is now apparent that the variation (50) reduces by one the number of \(B-\)fields. Namely when we apply the variation (50) to any term \(w_{2n}(K)\), we obtain a term of order \(10^{2n-1}\).

Moreover by applying the operators (50) to any term \(w_{2n}(K)\), we create one matrix \(T^a\) at the point \(x(s)\) and one matrix \(\tilde{T}^a\) at the point \(x(\tilde{s})\). This will happen both on the knot and on the framing. Exactly as in [7], the Fierz identity generates two contributions. These are:

i) one contribution of order \(2n-1\) relevant to the original knot \(K = K_\). This contribution is zero, due to the results of section 4

\[ ii) \] one contribution of order \(2n-1\) relevant to the 2-component link \(K_0\).

Let us analyze in detail the characteristics of the second contribution \(ii)\). Whenever the two components of the link \(K_0\) are separated by a sphere, then the term \(ii)\) must be zero at any order.

Moreover since the vacuum expectation value of both \(K_+\) and \(K_-\) satisfy separately the denominator surgery formula (section 3), then the series obtained by summing all the contributions \(ii)\) corresponding to the different orders in perturbation theory, must satisfy the same denominator surgery formula. This implies that the series obtained by summing all the contributions \(ii)\) must be proportional by a factor \(f_2(z)\) to the series obtained by summing the connected correlation functions.

\[10]^\text{We recall that the number of } B-\text{fields gives exactly the order of the given term in the perturbative expansion.}\]
There is an arbitrary choice to be made, concerning the term \( f_2(z) \) considered above. Any such choice is equivalent to a choice of the normalization factor for the expectation value (connected correlation functions) of 2-component links.

In order to be consistent with the skein relation (41) (or (42)), we choose:

\[
f_2(z) = z \left\langle W(H; z / (4\pi \sqrt{c_v c_2})) \right\rangle_c \tag{51}
\]

This choice is equivalent to requiring that the \( BF \) expectation value for the Hopf link is exactly given by \( z \).

Thus we have proven, in the framework of perturbation theory, the Alexander-Conway skein relation for the \( BF \) theory.

For any given integer \( s \), we could, in principle, choose a different normalization factor \( f_s(z) \) relevant to the (connected) vacuum expectation values of links with \( s \) components. If we want to preserve the denominator surgery formula, then these normalization factors should satisfy the following equation:

\[
f_s(z) = [f_2(z)]^{s-1},
\]

i.e. only the factors \( f_1(z) \) and \( f_2(z) \) can be assigned independently.

As a conclusion we have the following relation between the Alexander-Conway polynomial for a link \( L \) with \( s \) components and the (connected) vacuum expectation values of the \( BF \) theory:

\[
\Delta_L(z) = f_s(z) \left\langle W_R(L; z / (4\pi \sqrt{c_v c_2})) \right\rangle_c \left\langle W(\bigcirc; z / (4\pi \sqrt{c_v c_2})) \right\rangle_c. \tag{52}
\]

As an example of the previous construction we now perform the explicit computation of \( \delta a_2(K) \equiv a_2(K_+) - a_2(K_-) \).

We shall omit all the terms which give the linking number of \( K \) with \( K_f \), since this has been assumed to be zero. In particular the second halves of the two operators (50) can be omitted.

By taking into account (50), (48) and (49), we obtain \(^{11}\):

\[
\frac{\delta a_2(K)}{\delta K^{\mu}(x)} = \frac{-\epsilon_{\mu\nu\rho}K^{\nu}(x)}{4\pi c_v c_2 N} \left\langle \frac{\delta}{\delta A^{\rho}_{\nu}(\hat{x})} \text{TrHol}_{x_0}^x T^{\nu} \text{Hol}_{x_0}^\nu \int G \right\rangle, \tag{53}
\]

\(^{11}\)In order to avoid a cumbersome notation we write \( x(\tilde{s}) = \hat{x} \) and \( x(s) = x \) (see (50)).
and

\[
\frac{\delta a_2(K)}{\delta K_f^\mu (x)} = -\epsilon_{\mu \nu \rho} \frac{\hat{K}^\nu_f(x)}{4 \pi c_v c_2 N} \frac{\delta}{\delta B^\rho_f(\tilde{x})} \left( \right.
\begin{array}{c}
\text{Tr} \left[ \left( \int_{x_0}^x dy^\sigma \text{Hol}_{x_0}^y B_\sigma(y) \text{Hol}_x^y T^a \text{Hol}_{x_0}^x + 
\right.
\end{array}
\right. 
\]

Then, by taking into account (48), (49) and (46), we can rewrite (53) and (54) as:

\[
\frac{\epsilon_{\mu \nu \rho} \hat{K}^\nu_f(x)}{8 \pi c_v c_2 N} \left( \frac{1}{2} \ln(K_1, f)+\ln(K_1+K_2, f, K_1)+2\ln(K_f+K_2, f, K_2)+\ln(K_1, f, K_1) \right),
\]

Here the products \( \epsilon_{\mu \nu \rho} \epsilon_{\mu' \nu' \rho'} \hat{K}^\nu_f(x) \hat{K}^\rho_f(\tilde{x}) \) and \( \epsilon_{\mu \nu \rho} \epsilon_{\mu' \nu' \rho'} \hat{K}^\nu_f(x) \hat{K}^\rho_f(\tilde{x}) \) have both been normalized to one (as in Ref. [7] and [18]). We now expand all the holonomies retaining only the linear part in \( A \). By taking into account equation (5), we have then to compute terms of the form:

\[
\oint_{\gamma} dx^\sigma \oint_{\gamma'} dy^\rho \left( \text{Tr} A_\sigma(x) B_\rho(y) \right) = 4 \pi \ln(\gamma, \gamma') c_2 N,
\]

for suitable loops \( \gamma \) and \( \gamma' \). Moreover \( N = c_v \) and, by collecting together all the terms in the sum of (53) and (54), we get:

\[
\delta a_2(K) = -\frac{1}{2} \left[ 2\ln(K_{1,f}, K) + \ln(K_f+K_2, f, K_1) + 2\ln(K_f+K_2, f, K_2) + \ln(K_1, f, K_1) \right].
\]
where the knots $K_1$ and $K_2$ (respectively the framings $K_{1,f}$ and $K_{2,f}$) represent the two components of the link $K_0$ (respectively $K_{0,f}$). In force of the bilinearity of the linking number and of equation $\ln(K_f, K) = 0$, we have

$$\ln(K_{1f}, K_1) + \ln(K_{1f}, K_2) + \ln(K_{2f}, K_1) + \ln(K_{2f}, K_2) = 0.$$ 

So (58) can be rewritten in its final form

$$\delta a_2(K) = \ln(K_{2,f}, K_1) = a_1(K_0).$$

We have thus recovered the well-known fact that the first coefficient of the Alexander-Conway polynomial for a two-component link is the linking number of the two components.

### 7 Conclusions

It is now possible to compare the two existing three-dimensional topological field theories, namely the $BF$ theory and the Chern-Simons-Witten theory (CS).

In both cases one can define observables related to knots and links. And in both cases one obtains link-invariants.

The relation between the Chern-Simons vacuum expectation values of the observables and the Jones (or HOMFLY) polynomials can be seen both as a result of Conformal Field Theory (as in the Witten original argument) and (more heuristically) as a consequence of integration by parts techniques (as in [7]. See also the more recent [18]).

The relation between Alexander-Conway polynomials and the $BF$-theory (which is the main result of this paper) relays, for the time being, mainly on the integration by part techniques. But the structure of the perturbative series in the $BF$ case, is considerably simpler than in the CS case. Namely in the $BF$ case we were able to find a close analytical expression for the coefficients of the perturbative expansion.

Actual computations look cumbersome, but it should be possible, with the help of some computer calculation, to compare directly the $BF$-vacuum expectation values of some simple knots, with the corresponding Alexander-Conway polynomial (at least for lower orders). This has been done up to the second order of the perturbative expansion, by referring to the results of [17].
Besides integration by part, other facts support our claim that the $BF$-theory gives directly the Alexander-Conway polynomial. The most significant of these facts is probably the triviality of all the terms of odd order in the perturbative series. Also the introduction of connected correlation functions, allows the recovering, in perturbation theory, of one of the basic property of the Alexander-Conway polynomial, namely the fact that the polynomial is zero for a link that is given by the union of two separated links.

One argument which deserves more work is the comparison between our multiple integrals (normalized by dividing by the corresponding integrals for the unknot) and the knot-invariants of the Kontsevich type $[15]$. One of the main interests of $BF$ theories comes from the fact that these theories are topological field theories that can be defined in both 3 and 4 dimensions.

An observable for 2-knots has already been defined in $[8]$. Techniques completely similar to the one considered in this paper, can be considered also for the 4-dimensional case. This is what we are going to discuss in a forthcoming paper $[16]$, where we plan to connect four-dimensional topological field theories to invariants of 2-knots or, maybe, more generally to invariants of embedded 2-surfaces in 4-manifolds.

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FIGURE CAPTIONS

Fig. 1 A tangle
Fig. 2 Sum of tangles
Fig. 3 Numerator and denominator of a tangle
Fig. 4 $W_R((\mathcal{A} + \mathcal{B})^D; k)W_R(\bigcirc; k)$
Fig. 5 Decomposition of $\mathcal{A} + \mathcal{B}$ and $\bigcirc$
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Fig. 13 Exchange relation for a knot/link
Fig. 14 Exchange relation for a framed knot (Boldface lines denote the knot, light lines denote the framing)
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