On Tractability of Ulam’s Metric in Higher Dimensions and Dually Related Hierarchies

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Abstract
Ulam’s metric defines the minimal number of arbitrary extractions and insertion of permutation elements and to get the second permutation. The remaining elements constitutes the longest common subsequence of permutations. In this paper we extend Ulam’s metric $n$-dimensions. One dimension object is defined as a pair of permutations while $n$-dimension object is defined as a pair of $n$-tuples of permutations. For the purpose of encoding we explore $n$-tuple of permutations in order to define duality of mutually related hierarchies. Our very first motivation comes from Murata, Fujiyoshi, Nakatake, and Kajitani paper, in which pairs of permutations are used as a representation of topological relation between an rectangles of a given size that can be placed within the area of minimal size. It is applicable to Very Large Scale Integration VLSI design. Our results concern hardness, approximability, and parameterized complexity within these hierarchies respectively.

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1 Introduction
Permutation $\sigma \in S_n$ is a sequence $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ representing an arrangement of elements of the set $[n]$. Permutations are of great practical importance, as they model solutions to many real-life problems, e.g., in the fields of scheduling [20] and routing [21]. Taking into account significance and applications of permutations, a vast body of research tackled them from different angles, sides and points of view.

One specific research direction has been raised by Stanisław Ulam questions [23]: what is the minimum number of moves to go from some permutation $\sigma_s \in S_n$ to some other permutation $\sigma_t \in S_n$, where a move consists of changing a position of some element in a permutation, and more importantly, what is asymptotic distribution of this number?

The answer for the first question is named the Ulam’s metric $U(\sigma_s, \sigma_t)$. Computationally, it is equivalent to $LCS(\sigma_s, \sigma_t)$ - the Longest Common Subsequence of $\sigma_s$ and $\sigma_t$, and Fredman [10] showed that it can be computed using $n \log n - n \log \log n + O(n)$ comparisons in the worst case, and no algorithm has better worst-case performance. Later, Hunt and Szymanski [13] improved this bound to $O(n \log \log n)$ by exploiting RAM model of computations and a fixed universe of sequence elements.

The latter Ulam’s question drew researcher’s attention over the past 50 years, providing many amazing results and connections between various areas of mathematics, physics and statistics, exhibited and reviewed in [22], [21], and foremost the final answer of Baik, Deift and Johansson [3]: the expected value of the length of $LCS(\sigma_s, \sigma_t)$ for $\sigma_s$ and $\sigma_t$ drawn uniformly at random from $S_n$ equals $2\sqrt{n} - 1.77108n^{1/6} + o(n^{1/6})$.

As permutations often express solutions to many combinatorial optimization problems, the Ulam’s metric is often interpreted as the length of the shortest path connecting (any) two
given solutions in a space spanned by $S_n$. Motivated by surprising discoveries that followed original Ulam’s questions [7], [22], in this paper we rephrase and state the same question, but in higher dimensions: what is the length of the shortest path between two solutions in $S^k_n$,

$$S^k_n = S_n \times \ldots \times S_n, \quad \text{k times}$$

the $k$-dimensional space of permutations?

The question is not of theoretical interest only, as 2-tuples of permutations describe non-overlapping packings of rectangles on a plane in Sequence Pair (SP) representation [18]. The SP has many successful industrial applications in the context of physical layout synthesis of VLSI circuits, where it models a placement of transistors, leaf-cells and macro-blocks on a silicon die [1]. On the other hand, SP can be a solution space for multiprocessor scheduling problems with various definitions of cost functions and constraints [14], [15], [17].

On the other hand, existence and properties of paths between solutions in a solution space are especially important for metaheuristic algorithms [5], performing effective exploration of the solution space based on elimination and explorative properties - incremental moves and neighbourhood structures are essential in the context of guiding the search process. Examples of such methods are hybrid-metaheuristics [4], combining complementary strengths of various techniques to collaboratively tackle hard optimization problems, and hyper-heuristic approaches [6], providing a robust upper-level framework to tune and drive underlying heuristics to adapt to features of solved problems. The efficiency of such methods is based on the connectivity and diameter properties of the solution space - the assumption that one can provide a sequence of moves to every (starting) solution such that every other solution can be reached (especially an optimal one), and the maximal length of such a sequence, respectively. On the other hand, algorithms behind the Ulam’s metric can be immediately applied as crossover operators in evolutionary and path-relinking metaheuristics [5], i.e., given a shortest path between two solutions, a result of their crossover could be either a midpoint, the best, or even all solutions on that path as an offspring.

### 2 Preliminaries

Let a subsequence $s$ of $\sigma \in S_n$ be a sequence $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_m))$ where $1 \leq i_1 < \ldots < i_m \leq n$; denote by $\{s\}$ a set of elements of $s$, i.e., $\{s\} = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_m)\} \subseteq [n]$ and by $l(s) = m$ its length. A set common subsequences of two permutations we denote by

$$CS(\sigma_s, \sigma_t) = \{ \sigma \in [n]^m : m \in [n] \text{ and } \forall i, j \in [m] \land i < j \Rightarrow (\sigma_s^{-1}(\sigma(i)) < \sigma_s^{-1}(\sigma(j)) \land \sigma_t^{-1}(\sigma(i)) < \sigma_t^{-1}(\sigma(j))) \}.$$  

Let $\Gamma \in S^k_n$ be a $k$-tuple of permutations of $[n]$, i.e., $\Gamma = (\sigma^1, \ldots, \sigma^k), \quad \sigma^i \in S_n, \ i \in [k]$. Say an insert move of some $v \in [n]$ in $\Gamma$ consist of moving the element $v$ to some other position in each of $\Gamma$ permutations. Define an insert neighbourhood of $\Gamma, \mathcal{N}(\Gamma)$, as a set of permutation tuples obtained by performing single insert move of any element in $\Gamma$.

Let $f_k : S^k_n \times S^k_n \times 2^n \rightarrow [n]$ be defined in the following way:

$$f_k(\Gamma_s, \Gamma_t, C) = \begin{cases} |C| & \text{if } \forall i \in [k] \ \exists \sigma \in CS(\sigma_s^i, \sigma_t^i) \ \{\sigma\} = C \\ 0 & \text{otherwise} \end{cases}$$

$$U_k(\Gamma_s, \Gamma_t) = \{ [n] \setminus C : C \in LFS(\Gamma_s, \Gamma_t) \}, \quad (1)$$

where $LFS(\Gamma_s, \Gamma_t) = \arg\max_{C \subseteq [n]} f_k(\Gamma_s, \Gamma_t, C)$.
Note that $LFS(\Gamma_s, \Gamma_t)$, as well as $U_k(\Gamma_s, \Gamma_t)$, is a class of equisized sets.

Define a path between $\Gamma_s$ and $\Gamma_t$ in $S_n^k$ as a sequence $\Gamma_0 = \Gamma_s, \Gamma_1, \ldots, \Gamma_m = \Gamma_t$ s.t. $\Gamma_{i+1} \in N(\Gamma_i)$, $i = 0, \ldots, m-1$; let $m$ be the length of that path. The Ulam’s metric in $S_n^k$, is the length of the shortest path between $\Gamma_s$ and $\Gamma_t$, i.e., the minimal number of insert moves transforming $\Gamma_s$ into $\Gamma_t$, where $\Gamma_s, \Gamma_t \in S_n^k$. Following [2] it is not hard to show that the Ulam’s metric is the size of sets in $U_k(\Gamma_s, \Gamma_t)$.

A sequence pair $(\sigma^1, \sigma^2) \in S_n^2$ is a pair of permutations over $[n]$. In the literature, according to Ulam’s original motivation, $U_1$ is often interpreted as sorting of a hand of bridge cards, by sequentially picking cards labelled by numbers which are not in the longest common sequence and putting them back in desired position. It is shown in Figure 1, using sequence pairs representation, the $U_2$ can be analogously interpreted geometrically as a sequence of rectangle packings, in which in each step a single rectangle is taken from a packing and re-inserted in desired place. To more precise explanation how sequence pairs corresponds to rectangle placement see [13].

For two given finite sequences $a$ and $b$, its join is denoted by $a \cdot b$. By $(i)$, we denote a sequence that consists of a single element $i$. For a given pair $(\sigma^1, \sigma^2)$, a subset $B \subseteq [n]$ induces a common subsequence if there is a sequence $(b_1, \ldots, b_l) \in CS(\sigma^1, \sigma^2)$ such that the $\{b_1, \ldots, b_l\} = B$. For given $B$, we also say, that a sequence $(b_1, \ldots, b_l)$ is a $B$-induced subsequence of permutation $\sigma$ if $(b_1, \ldots, b_l)$ is a subsequence of $\sigma$ and $\{b_1, \ldots, b_l\} = B$.

Probably the most convenient form of defining of an NPO problem $A$ is its presentation as a fortuple $(\mathcal{I}(A), fso, cost_A, type)$, where:

- $\mathcal{I}(A)$ is a set of valid instances of $A$.
- $fso$ is a function such that for given an instance $x \in \mathcal{I}(A)$, $fso(x)$ is a set of feasible solutions for $x$. Moreover the question if $y \in fso(x)$ is verifiable in polynomial time with respect to the size of $x$.
- Given an $x \in \mathcal{I}(A)$ and a feasible solution $y$ of $x$, $cost_A(x,y)$ is a positive integer measure of $y$. Additionally $cost_A(x,y)$ is computable in polynomial time.
- $type \in \{\min, \max\}$.

A decision version of $k$-dimensional Longest Fixed Subset Problem (kLFS) is defined as follows: Given a pair of $k$-tuple of permutations $\Gamma = (\Gamma_s, \Gamma_t)$, where $\Gamma_s = (\sigma^1_s, \ldots, \sigma^k_s)$ and $\Gamma_t = (\sigma^1_t, \ldots, \sigma^k_t)$, determine a maximal $l$ and $B \subseteq [n]$ such that $|B| = l$ and $B$ is a feasible solution of $\Gamma$. The cost function $cost_{kLFS}$ satisfies $cost_{kLFS}(\Gamma, B) = |B|$. The $k$-dimensional Largest Fixed Subset Problem (kLFS) is answering the question if for given nonnegative integer $l$ and $(\Gamma_s, \Gamma_t) = ((\sigma^1_s, \ldots, \sigma^k_s), (\sigma^1_t, \ldots, \sigma^k_t))$ there exists $B \subseteq [n]$, such that $|B| = l$ and $B$ is a feasible solution of $\Gamma$.

By $kU$ we denote a dual problem to $kLFS$. The $kU$ is defined over the same inputs as $kLFS$ and if for a given input, a set $B$ is a feasible solution to the $kLFS$ then a set $V \setminus B$ is a feasible solution to the $kU$. The $kU$ is a classic problem and is a simple model for the study of many other problems in the $\mathcal{NP}$-complete class.
a feasible solution to the kU problem. The kU problem consists in find the smallest feasible solution. The cost_{kU} satisfies cost_{kU}(Γ, B) = |B|. The decision version kUD is defined analogously to the kLFSD problem.

In the MAXCLIQUE problem we are given undirected graph G = (V, E) and the task is to find a clique of maximum size. The decision version of the MAXCLIQUE problem – the CLIQUE problem consists in checking for a given m if the graph consists a clique of size m. For the MINVERTEXCOVER problem the task is to find a subset B ⊆ V of minimal cardinality such that, for every (u, v) ∈ E, u ∈ B or v ∈ B.

Solutions of the MAXCLIQUE and the MINVERTEXCOVER problem are related to each other in the following way: subset B of V is a minimum vertex cover if and only if if V \ B is a maximum clique in G. The VERTEXCOVER problem denote decision version of the MINVERTEXCOVER problem.

Let A be a maximization problem. We use Opt_A(x) to refer to the cost of optimal solution for x ∈ I(A). An algorithm A, for A, computes objective function value A(x) for feasible solution of an instance x. Define F(A) = \frac{A(x)}{Opt_A(x)} Function r: \mathbb{Z}^+ \rightarrow \mathbb{R}^+ is an approximation factor of A if for any n, F(A, x) ≤ r(n) for all instances of the size n. We say that algorithm A is r(n)-approximation algorithm. If A is a minimization problem then F(A, x) = \frac{Opt_A(x)}{A(x)}.

Let cost_A(x, y) denote the cost of y that is feasible solution of x ∈ I(A). A reduction of an optimization problem A to an optimization problem B is a pair of polynomially computable functions (f, g) that satisfy two conditions: (1) x' = f(x) ∈ I(B) for any x ∈ I(A), (2) y = g(x, y') is feasible solution of x, for any y' which is feasible solution of x'. A reduction (f, g) of an optimization problem A to an optimization problem B is an S-reduction if (1) Opt_A(x) = Opt_B(x') for any x ∈ I(A), where x' = f(x), and (2) cost_A(x, g(x, y')) = cost_B(x', y'), for any x ∈ I(A) and y' which is feasible solution of x' [8]. For abbreviation, we write A ≤_S B to denote that there exists S-reduction from A to B.

A decision problem A is parametrized problem if it is extended by function κ_A that assigns nonnegative integer to each instance x of A. An algorithm which decides if x ∈ A in time f(κ_A(x)p(|x|)) is called fpt-algorithm, where f is a computable function and p is a polynomial. An R is an fpt-reduction of A to B if for any x ∈ I(A), R(x) is an instance of B, moreover (1) x ∈ A if R(x) ∈ B: (2) R is computable by an fpt-algorithm with respect to κ_A; (3) There is computable function h such that \kappa_B(R(x)) ≤ h(\kappa_A(x)) for any x ∈ I(A).

3 On Tractability of kLFS and kU

3.1 S-reductions and some conclusions

Lemma 1. For any k ∈ [n] there exist S-reductions from kLFS to MAXCLIQUE and from kU to MINVERTEXCOVER.

Proof. For a given (Γ_s, Γ_t) = \langle (σ^s_1, \ldots, σ^s_k), (σ^t_1, \ldots, σ^t_k) \rangle we define a graph G = (V = [n], E), where (i, j) ∈ E if and only if for all r ∈ [k] there are α_s, β_s, α_t, β_t ∈ [k] : α_s < β_s, α_t < β_t and either σ^s_r(α_s) = i, σ^s_r(β_s) = j, σ^t_r(α_t) = i, σ^t_r(β_t) = j or σ^s_r(α_s) = j, σ^s_r(β_s) = i, σ^t_r(α_t) = j, σ^t_r(β_t) = i. This defines the function f transforming any given instance of kLFS instance of MAXCLIQUE.

First, note that any clique C of G, C-induces a common sequences for all pairs (σ^s_i, σ^t_i). Suppose to the contrary, that C does not induce a common sequence for some (σ^s_i, σ^t_i). The clique C has at least two elements. Let σ_s, σ_t be the C-induced subsequences of σ^s_i and σ^t_i, respectively. If σ_s, σ_t are two different sequences over the same set of elements
differs then there is the least index $d$ such that $\alpha_s(d) \neq \alpha_t(d)$. Therefore the element $\alpha_s^{-1}(\alpha_t(d)) > \alpha_t^{-1}(\alpha_s(d))$ and $\alpha_t^{-1}(\alpha_t(d)) < \alpha_s^{-1}(\alpha_s(d))$. The reduction has been defined in such a way that there is no edge $(\alpha_s(d), \alpha_t(d)) \in E$. This is contradiction - $C$ cannot form a clique in $G$. Therefore $g((\Gamma_s, \Gamma_t), C) = C$. So far it has been proven that there is reduction from kLFS to MAXCLIQUE. Note that $cost_{k\text{MaxClique}}(f((\Gamma_s, \Gamma_t))) = G, C) = |C|$ for any clique $C$ of $G$. Additionally, $cost_{k\text{LFS}}((\Gamma_s, \Gamma_t), g((\Gamma_s, \Gamma_t), C)) = C$ can be defined as $|C|$ for any $C$ which is a clique. Hence, the second condition of the S-reduction definition may also be satisfied.

Finally, note that maximum size clique $C$ of the graph $G$, represented as the set of vertex numbers, induces a maximum solution of the kLFS instance. To prove this assume to the contrary that there exists $D \subseteq V$ such that are $D$-induced subsequences for all pairs $\langle \sigma_s^t, \sigma_t^s \rangle$ and $|C| < |D|$ for all cliques $C \subseteq V$. Therefore there are different $i, j \in D$ such that $(i, j) \notin E$. By definition of the reduction there exists $r$, such that either $\sigma_s^r(\alpha_s) = i, \sigma_t^r(\beta_s) = j, \sigma_t^r(\beta_t) = i$ or $\sigma_s^r(\alpha_s) = j, \sigma_t^r(\beta_s) = i, \sigma_t^r(\beta_t) = j$, for some $\alpha_s < \beta_s, \alpha_t < \beta_t$. This contradicts the fact that $i, j$ are elements of a $D$-induced common sequence.

Now we see that the function $g : S^k_n \times S^k_n \times 2^V \to 2^V$ satisfying $g((\Gamma_s, \Gamma_t), C) = C$. Also satisfies $Opt_{k\text{MaxClique}}(G = f((\Gamma_s, \Gamma_t))) = Opt_{k\text{LFS}}((\Gamma_s, \Gamma_t))$ for any $(\Gamma_s, \Gamma_t)$. Therefore, the first condition of the S-reduction definition is also satisfied. Thus, the $(f, g)$ is an S-reduction from kLFS to MAXCLIQUE.

A set $B \subseteq [n]$ is a feasible solution of the kU problem for instance $(\Gamma_s, \Gamma_t)$ if and only if $[n] \setminus B$ is a feasible solution of the kLFS problem for the same instance $(\Gamma_s, \Gamma_t)$. Similarly if $B \subseteq V = [n]$ is a vertex cover of $G$ then $V \setminus B$ is a clique for $G$. Moreover, the minimal vertex cover $B$ for $G$ corresponds to $V \setminus B$ which is a maximum size clique of $G$. Similarly, a maximal longest fixed subset $B$ of $(\Gamma_s, \Gamma_t)$ corresponds to $[n] \setminus B$ which is a minimal solution of the kU problem. The S-reduction $(f', g')$ from $U \to \text{MINVERTEXCOVER}$ is defined by equations $f' = f$ and $g'((\Gamma_s, \Gamma_t), B) = [n] \setminus g((\Gamma_s, \Gamma_t), V \setminus B)$. The equality of the costs can be proven using the same reasoning as before.

An S-reduction is stronger than L-reduction, AP-reduction and PTAS-reduction [5], that are often used for proving the membership to the APX class. Furthermore, if $A \leq_S B$ and $B$ is approximable with some factor, then $A$ is approximable with the same factor. Therefore, by Lemma 1 and the fact that MINVERTEXCOVER there exists 2-approximation algorithm [24], we obtain that there is 2-approximation algorithm for the kU problem.

For verification if an $Y$ is feasible solution for an $x \in I(kLFS)$ one can iteratively check if consecutive elements of $\sigma_s^t$ are in $Y$. These iteration puts consecutive elements to the new sequence $\tau_s^t$ if only they are in $Y$. The same can be done for $\sigma_t^s$, obtaining $\tau_s^t$ for all $i \in [k]$. At the end we check if $\sigma_s^t = \tau_s^t$ for each $i \in [k]$. This can be implemented within $O(kn \log(|Y|))$. Then feasible solutions are verifiable in polynomial time. Since $cost_{kLFS}(Y) = |Y|$, cost is obviously computable in polynomial time.

As a conclusion we obtain:

\textbf{Theorem 2.} The kLFS problem is in NPO for any $k \in [n]$. For any $k \in [n]$ there exists 2-approximation algorithm for kU.

\textbf{Lemma 3.} There exists an S-reduction from the MAXCLIQUE problem to the nLFS problem and from the MINVERTEXCOVER problem to nU problem.

\textbf{Proof.} Let $G = (V, E)$ be undirected graph, where $V = [n]$. We define $\Gamma = (\Gamma_s, \Gamma_t)$, where $\Gamma_s = (\sigma_1^s, \ldots, \sigma_n^s)$ and $\Gamma_t = (\sigma_1^t, \ldots, \sigma_n^t)$. Firstly define permutation $\sigma_s^t$. Let $V^{-i} = V \setminus \{i\}$. Split $V^{-i}$ into two disjoint sets $P^i = \{j \in V^{-i} | j \in E\}$ and $N^i = \{j \in V^{-i} | \neg j \in E\}$. 

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A sequence \( a \) is non repeating if there is no element in \( a \) occurring twice. Let \( a_i \) and \( b_i \) be an increasing non repeating sequences of all elements from \( N^1 \) and \( P^n \), respectively. Define \( \sigma^*_i \) as \((i) \cdot a_i \cdot (i) \cdot b_i \). The second permutation \( \sigma^*_i \) is defined as \( a_i \cdot (i) \cdot b_i \).

**Remark 4.** There is a clique \( K \) in the graph \( G \) if and only if \( K \) is feasible solution of \((\Gamma_s, \Gamma_t)\).

**Remark 5.** If the size of maximal clique in graph \( G \) equals \( m \), then \((\Gamma_s, \Gamma_t)\) has no feasible solution of length greater than \( m \).

The reduction is defined in the following way: \( f(G) = \Gamma, g(G, K) = K \). By Remark 4, for given a feasible solution \( K \) of the \( nLFS \) problem for instance \( \Gamma \), \( g(G, K) \) is feasible solution of the \( MAXCLIQUE \) problem for instance \( G \). Then \( g \) is correctly defined. By Remarks 4 and 5, \( Opt_{MAXCLIQUE}(G) = Opt_{nLFS}(\Gamma) \). For \( MAXCLIQUE \) as well as \( nLFS \) cost of feasible solution is the size of a set which constitutes the solution. Hence \( cost_{nLFS}(\Gamma, K) = cost_{MAXCLIQUE}(G, g(G, K)) = |K| \).

Like in the proof of Lemma 2, presented \( S \)-reduction is also a reduction from \( MINVERTEXCOVER \) to \( nU \). This is because the graph \( G \) which is an instance of the \( MINVERTEXCOVER \) problem is also an instance of the \( MAXCLIQUE \) problem. A maximum solution \( B \) of \( \Gamma \) which is an instance of the \( nLFS \) problem corresponds to \( [n] \setminus B \) which is a minimum solution of the same \( \Gamma \) which is an instance of the \( nU \) this time.

### 3.2 NP-harness and nonapproximability

**Theorem 6.** The 2LFS and 2UD are NP-complete.

**Proof.** By the existence of an \( S \)-reduction (Lemma 2) from \( kLFS \) to \( MAXCLIQUE \) and from \( kU \) to \( MINVERTEXCOVER \), we get that for any \( k \in [n] \), the \( kLFS \) and the \( kUD \) are reducible to the \( CLIQUE \) problem and the \( kUD \) problem, respectively, by polynomial time reductions. Hence \( kLFS \) and \( kUD \) are in \( NP \) and in particular 2LFS and 2UD are in \( NP \).

We let remains to show that 2LFS is in \( NP \)-hard. Let \( \varphi = c_1 \lor \ldots \lor c_m \) be a boolean formula in \( 3CNF \) form, where \( 3CNF \) means that each clause \( c_i \) has three literals. Let \( x_1, \ldots, x_n \) be all variables that occur in \( \varphi \). Assume that variable \( x_i \) occurs a positive \( \alpha \) times in \( \varphi \) as a positive literal and \( \beta \) times as negative one. We construct polynomial time reduction by encoding a satisfiability of \( \varphi \) as 2LFS. We introduce a new set of symbols \( x_1^{(1)} \), \ldots, \( x_i^{(\alpha)} \), \( \neg x_1^{(1)} \), \ldots, \( \neg x_i^{(\beta)} \), where \( x_i^{(j)} \) stands for the \( j \)-th positive literal of \( x_i \) and symbol \( \neg x_i^{(j)} \) stands for the \( j \)-th negative literal of \( x_i \). Let \( C_i = \{x_1^{(1)}, \ldots, x_i^{(\alpha)}, \neg x_1^{(1)}, \ldots, \neg x_i^{(\beta)}\} \) and \( C = \bigcup_{i=1}^n C_i \). Let \( \kappa \) be the bijection from \( C \) onto \([3m]\). Create \( \Gamma(\varphi) = (\Gamma_s, \Gamma_t) = \langle (\sigma_1^*, \sigma_2^*), (\sigma_1^*, \sigma_2^*) \rangle \) which is an instance of 2LFS having solution \( m \) if and only if \( \varphi \) is satisfiable.

The definition of \( \Gamma(\varphi) = (\Gamma_s, \Gamma_t) \) in the presented reduction should preclude the case that if \( Z \) induces common subsequences for \( \langle \sigma^*_1, \sigma^*_1 \rangle \) and \( \langle \sigma^*_2, \sigma^*_2 \rangle \) then both \( \kappa(x_i^{(r)}) \) and \( \kappa(\neg x_i^{(p)}) \) appear in \( Z \) for any \( i \in [n] \) and \( r, p \). It can be realized in the following way: Permutations \( \sigma^*_1 \) and \( \sigma^*_1 \) consist of blocks of substrings \( A_i = (\kappa(x_i^{(1)}), \ldots, \kappa(x_i^{(\alpha)}) \rangle \) and \( B_i = (\kappa(\neg x_i^{(1)}), \ldots, \kappa(\neg x_i^{(\beta)}) \rangle \). We define \( \sigma^*_1 = A_1B_1A_2B_2A_nB_n \) and \( \sigma^*_1 = B_1A_1B_2A_2 \cdots B_nA_n \).

**Remark 7.** If a set \( Z \) induces a common subsequence of \( \sigma^*_1 \) then at most one of \( \kappa(x_i^{(r)}) \) and \( \kappa(\neg x_i^{(p)}) \) is in \( Z \) because \( (1) \kappa(x_i^{(r)}) \) appears before \( \kappa(\neg x_i^{(p)}) \) in \( \sigma^*_1 \), \( (2) \kappa(x_i^{(r)}) \) appears after \( \kappa(\neg x_i^{(p)}) \) in \( \sigma^*_1 \) for any \( i, r, p \).
Let $c_j = x^{(a)}_{j_1} \lor x^{(b)}_{j_2} \lor x^{(c)}_{j_3}$ be the $j$-th clause in $\varphi$, where $x^{(a)}_{j_1}, x^{(b)}_{j_2}, x^{(c)}_{j_3}$ are literals. Define a permutation $\sigma_2^j$ as concatenation of blocks $E_1 E_2 \cdots E_m$ where $E_j = (\kappa(x^{(a)}_{j_1}), \kappa(x^{(b)}_{j_2}), \kappa(x^{(c)}_{j_3}))$ and $\sigma_1^j = E_1^R E_2^R \cdots E_m^R$, where $E_j^R = (\kappa(x^{(c)}_{j_3}), \kappa(x^{(b)}_{j_2}), \kappa(x^{(a)}_{j_1})).$

- **Remark 8.** If a set $Z$ induces a common subsequence of $\sigma_2^j$ and $\sigma_1^j$ then $|Z| \leq m$. If $|Z| \geq m + 1$, there would exist literals $x, y$ belonging to the same clause such that $\kappa(x), \kappa(y) \in Z$. Literals $\kappa(x), \kappa(y)$ that occur in $\sigma_2^j$, appear in reverted order in $\sigma_1^j$. Hence $Z$ does not induce a common subsequence for both $\sigma_2^j$ and $\sigma_1^j$.

Any choice of literals, one from every clause forms the sequence, which is witness of satisfiability of $\varphi$ if the set of chosen literals is consistent.

- **Remark 9.** Assume that $\varphi$ is satisfiable and the set $\{y_1, \ldots, y_m\}$ is a witness of its satisfiability. Then the sequence $(\kappa(y_1), \kappa(y_2), \ldots, \kappa(y_m))$ is a common subsequence of $\sigma_2^j$ and $\sigma_1^j$.

Note that if $y_1, \ldots, y_m$ is a witness of satisfiability then $\{\kappa(y_1), \kappa(y_2), \ldots, \kappa(y_m)\}$ indicates a common subsequence of $\sigma_1^j$ and $\sigma_1^j$. Indeed, by consistency of $\{y_1, \ldots, y_m\}$ there is no $i, j, k$ such that $\kappa(y_i)$ occurs in $A_k$ and $\kappa(y_j)$ occurs in $B_k$. Hence, $\kappa(y_i)$ and $\kappa(y_j)$ occur in the same order in both $\sigma_1^j$ and $\sigma_1^j$. Therefore, by Remark 9, $\{\kappa(y_1), \kappa(y_2), \ldots, \kappa(y_m)\}$ induces common subsequences for pairs $(\sigma_1^j, \sigma_1^j)$ and $(\sigma_2^j, \sigma_2^j)$ of length $m$. By Remarks 7 and 8, the lengths of induced subsequences are not greater than $m$. Thus, the induced subsequences are of the maximal length.

By the above reduction $\varphi$ is satisfiable if and only if the solution of $(\Gamma_s, \Gamma_t)$, as the instance of the the 2UD problem is of the size not greater than $2m$. Hence 2UD is NP-hard.

For a given boolean formula $\varphi$ in 3CNF, the MAX-3SAT problem consists in finding a maximal possible number of clauses that can be satisfied in $\varphi$. Note that the reduction that has been presented in the previous proof is an S-reduction from the MAX-3SAT problem to the 2LFS problem. In the reduction we use the 3CNF with exactly three literals in every clause. It has been proved in [12] that MAX-3SAT is inapproximable with factor better than $\frac{7}{4}$. Since our reduction is an S-reduction also 2LFS cannot have a better approximation. Inapproximability result can easily be generalized to the $k$LFS for $k \leq 2$, because it is enough to repeat construction with $\sigma_1^j = \sigma_1^j$ for $j > 2$.

Analyzing the same reduction and using the same inapproximability result [12], it can be proven that there is no approximation for the 2U problem with an approximation factor better than $\frac{40}{39}$ unless $P = NP$.

- **Theorem 10.** There is no polynomial time approximation algorithm for the 2U problem with approximation factor better than $1 + \frac{1}{39}$ unless $P = NP$.

**Proof.** Assume that there is a polynomial time approximation algorithm $A$ for the 2U problem with the factor $1 + \frac{1}{39} - \epsilon$ for some $\frac{1}{39} > \epsilon > 0$. Let $A(\Gamma(\varphi))$ be a solution for the instance $(\Gamma_s, \Gamma_t)$ of the 2U problem which was created in the proof of Theorem 8. Denote by $|A(\Gamma(\varphi))|$ the size of the set returned by $A$ on the input $\Gamma(\varphi)$. Let $2m + k$ be the optimal solution of the 2U for an instance $\Gamma(\varphi) = \Gamma(\varphi)$, where $m$ is the number of clauses in $\varphi$ which was encoded in $\Gamma(\varphi)$ in the proof of Theorem 8. $2m + k$ is the size of the smallest feasible solution for $\Gamma(\varphi)$ if and only if $m - k$ is the maximal number of clauses that can be satisfied in $\varphi$. Let $\text{opt}_s = m - k$. We have that

$$|A(\Gamma(\varphi))| \leq (2m + k)\left(1 + \frac{1}{39} - \epsilon\right)$$

(2)
Let $A'(\Gamma(\varphi))$ be an algorithm that executes $A$ on the input $\Gamma(\varphi)$ and returns $[3m] \setminus Z$, whereas $A$ returns $Z$ on the input $\Gamma(\varphi)$. $[3m] \setminus Z$ is a feasible solution for instance $\Gamma(\varphi)$ of the problem 2LFS as well as the $\kappa^{-1}(Z)$ is the satisfiability witness for at least $|[3m] \setminus Z|$ clauses of $\varphi$. Since $A(\Gamma(\varphi)) = Z$ and by inequality \(2\) we have:

$$|A'(\Gamma(\varphi))| = |[3m] \setminus Z| \geq (m - k) - \left(\frac{1}{40} - \epsilon\right)(2m + k)$$ (3)

$$\left(\frac{1}{8} - \frac{2}{40}\right)(m - k) \geq \frac{1}{8}(m - k) + \frac{3}{40}k$$ (4)

If $\varphi$ is a formula in the 3CNF form, then there exists an assignment that satisfies at least half of the $\varphi$ clauses. We use this fact to obtain inequalities \(4\). Indeed, $m - k$ is the maximal number of clauses of $\varphi$ which can be satisfied, then $m - k \geq \frac{1}{2}m$ and $k$ denotes the number of clauses satisfied in the optimal assignment, then $1/2m \geq k$. Therefore

$$\left(\frac{1}{8} - \frac{2}{40}\right)(m - k) \geq \frac{1}{8}(m - k) + \frac{3}{40}k$$ (5)

It is easy to note that

$$\epsilon(2m + k) \geq \epsilon(m - k)$$ (6)

Adding inequalities \(5\) and \(6\) by sides we obtain:

$$(m - k) - \frac{1}{40}(2m + k) + \epsilon(2m + k) \geq \frac{7}{8}(m - k) + \epsilon(m - k)$$

$$\left(\frac{40}{1} - \epsilon\right)(2m + k) \geq \left(\frac{7}{8} + \epsilon\right)(m - k) = \left(\frac{7}{8} + \epsilon\right)\text{opt}_\varphi$$ (7)

The left side of inequality \(7\) is equal to the right side of inequality \(3\). Thus

$$|A'(\Gamma(\varphi))| = |[3m] \setminus Z| \geq \left(\frac{7}{8} + \epsilon\right)\text{opt}_\varphi$$

We have shown that $A'$ is an approximation algorithm for MAX-3SAT with an approximation factor $\frac{7}{8} + \epsilon$ for some $\frac{1}{40} > \epsilon > 0$. By the result from \(12\) it is possible only under assumption that $P = NP$. Hence our assumption about the existence of an approximation algorithm with a factor $1 + \frac{1}{40}$ for the 2U problem can be true only under the assumption that $P = NP$.

By the Lemma \(3\) MAXCLIQUE $\leq_S$ nLFS. Therefore there exists an $S$-reduction $(f,g)$ between that problems. We can ask about existence of the solution of the size $l$ for $x' \in \mathcal{I}(nLFS)$. Let $A_{LFS}$ be an algorithm solving that question for any $x \in \mathcal{I}(nMAXCLIQUE)$. One can build an algorithm $A_{MAXC}$ for the MAXCLIQUE problem. We do this in the following steps: Count in polynomial time $x' = f(x)$. Ask if there is a solution for instance $x'$ of the size $l$. Return answer 'yes' if and only if the given answer is 'yes'. Such an
algorithm is correct by definition of S-reduction: existence of a feasible solution \( y' \) for \( x' \) implies existence existence of feasible solution \( y = g(x, y') \) with the same size \( l \). This is by condition \( \text{cost}_{\text{MAXCLIQUE}}(x, g(x, y')) = \text{cost}_{\text{nLFS}}(x', y') \). Therefore, \( \text{nLFS} \) is \( \text{NP} \)-hard. Moreover, by Håstad and Zuckerman results \([11], [25]\), which establishes that approximation of \( \text{MAXCLIQUE} \) within factor \( n^{1-\epsilon} \) is \( \text{NP} \)-hard, we obtain the following nonapproximability theorem:

\[ \text{Theorem 11.} \] The approximation of the \( \text{nLFS} \) problem within \( n^{1-\epsilon} \) is \( \text{NP} \)-hard, for any \( \epsilon > 0 \).

Since there is an \( S \)-reduction from \( \text{MinVertexCover} \) to \( \text{nU} \), the lower bound for the approximation factor of \( \text{MinVertexCover} \) is also the lower bound for the approximation factor of \( \text{nU} \). As a conclusion from Khot and Regev result \([15]\) we obtain:

\[ \text{Theorem 12.} \] There exists no polynomial time \( (2-\epsilon) \)-approximation algorithm for the \( \text{nU} \) problem unless The Unique Game Conjecture is not true.

Let \( G = (V, E) \) be an undirected graph. The product of graph \( G \), denoted by \( G^2 = (V^2, E^2) \) is a graph such that \( V^2 = V \times V \) and \( E^2 = \{ (u, u'), (v, v') \mid \text{either } u = v \lor (u, v') \in E \lor (u', v) \in E \} \). For permutations in \( S_n \), in the proof of the next theorem, we develop approach analogous to application of the next lemma in nonapproximality of the \( \text{MAXCLIQUE} \) problem with an approximation factor better than \( \sqrt{n} \).

\[ \text{Lemma 13.} \] Graph \( G \) has a clique of size \( k \) if and only if graph \( G^2 \) has a clique of size \( k^2 \) \([19]\), chapter 13.

\[ \text{Theorem 14.} \] For any constant \( c \in \mathbb{N} \), it is \( \text{NP} \)-hard to approximate the \( n^{1/c} \)-LFS problem within \( n^{1-\epsilon} \), for any \( \epsilon > 0 \).

Proof. Assume that \( n = \nu^c \). Let \( \Gamma = ((\sigma_1, \ldots, \sigma_{\nu})^c, (\sigma_1^c, \ldots, \sigma_{\nu}^c)) \), where \( \sigma_1, \ldots, \sigma_{\nu} \in S_\nu \). For \( c > 1 \) and \( i \in [\nu] \), let \( f_i : [\nu^{c-1}] \rightarrow [\nu^c] \) satisfy equation \( f_i(k) = (i-1)\nu^{c-1} + k \). Let \( \lambda = (a_1, \ldots, a_{\nu}) \in S_{\nu^{c-1}} \). If \( \nu^{c-1} \) is the domain of \( f \) and the function is total over its domain, we write \( f(\lambda) \) to denote a tuple \( (f(a_1), f(a_2), \ldots, f(a_{\nu})) \).

We will denote by \( \bigcirc \) the generalized concatenation of sequences. If \( a = (i_1, i_2, \ldots, i_n) \in S_\alpha \) is a sequence of indexes, \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are finite sequences, then \( \bigcirc_{i \in \alpha} \lambda_i \) denotes a concatenation \( \lambda_{i_1} \cdot \lambda_{i_2} \cdot \ldots \cdot \lambda_{i_n} \).

We define \( \Lambda = T^c = ((\lambda_1, \ldots, \lambda_{\nu^c}), (\lambda_1^c, \ldots, \lambda_{\nu^c}^c)) \) recursively using \( \Gamma \) and \( T^{c-1} = ((\tau_1, \ldots, \tau_{\nu^{c-1}}), (\tau_1^c, \ldots, \tau_{\nu^{c-1}}^c)) \), where \( \tau_1, \tau_1 \in S_{\nu^{c-1}} \). For \( c = 1 \), \( T^c = \Gamma \). The \( \lambda_i, \lambda_i^c \in S_{\nu^c} \) are defined by the following equations:

\[ \lambda_i = \bigcirc_{k \rightarrow \sigma_k} f_k(\tau_k), \quad \lambda_i^c = \bigcirc_{k \rightarrow \sigma_k^c} f_k(\tau_k^c). \]

\[ \text{Remark 15.} \] Assume that \( \Gamma \) has a feasible solution of size \( m \) and \( T^{c-1} \) has a solution of size \( m^{c-1} \). Then \( \Lambda \) has a solution of size \( m^c \).

Define auxiliary functions \( \text{block}^c \) and \( \text{inblock}^c \) both of type \( [n = \nu^c] \rightarrow [\nu] \):

\[ \text{block}^c(s) = ((s - 1) \div \nu^c) + 1, \quad \text{inblock}^c(s) = ((s - 1) \mod \nu^c) + 1. \]

For a given \( J \subseteq [\nu^c] \) let \( W^c(J) := \{ l \mid \exists r \in J \land l = \text{block}^{c-1}(r) \} \), \( H^c(l, J) := \{ \text{inblock}^{c-1}(r) \mid r \in J \land l = \text{block}^{c-1}(r) \} \), \( \alpha^c(J) = \max \{|H^c(l, J)| \mid l \in [\nu^c] \} \), \( \beta^c(J) := \max \{|l \mid |H^c(l, J)| = \alpha^c(J)\} \).
Remark 16. Assume that \(c > 1\) and \(\Lambda\) has a solution \(J\) of size \(m^c\). Then \(T^{c-1}\) has a solution of the size \(m^{c-1}\) and of the form \(H^c(\beta^c(J), J)\), or \(\Gamma\) has solution of size \(m\) and of the form \(W^c(J)\).

Now we will show, that if there exists a polynomial algorithm solving the \(n^{1/c}\) LFS problem, with an approximation factor within \(O(n^{1-\epsilon})\) for some \(0 < \epsilon < 1\), then there exists a polynomial time algorithm that approximates \(\nu LFS\) with factor \(\nu^{1-\gamma}\) for some \(0 < \gamma < 1\). Instances of the \(\nu LFS\) problem consist of permutations from \(S_n\). Assume the existence of a polynomial time algorithm \(\mathcal{C}\) that approximates \(n^{1/c}\) LFS with a factor \(n^{1-\epsilon}\) and let the polynomial be denoted by \(p\). Approximation algorithm \(\mathcal{B}\) for \(\nu LFS\) works as follows:

1. Create \(\Lambda = ((\lambda_1^1, \ldots, \lambda_c^1), (\lambda_1^2, \ldots, \lambda_c^2))\) as it was defined previously.
2. Run algorithm \(\mathcal{C}\). Assume that it has returned the set \(J \subseteq [n]\).
3. Execute recursive procedure \(\text{EXTRACTLFS}(\Lambda, c, J)\).

One can prove, by induction on \(c\), that for input \(\Lambda = T^c\) and its solution \(J\) the procedure \(\text{EXTRACTLFS}(\Lambda, c, J)\) returns a solution of \(\Gamma\) of size not less than \(|J|^{1/c}\). Induction step can be done with the following assumption.

Assumption 1. Let \(J\) be a solution of \(\nu LFS\) for instance \(T^{c-1}\). Then \(\text{EXTRACTLFS}(T^{c-1}, c-1, J)\) returns a solution of \(\Gamma\) of size not less than \(|J|^{1/(c-1)}\).

In the inductive proof we use Remark 16 and the fact that \(T^1 = \Gamma\) (for the first step of induction).

Corollary 17. If \(\Lambda\) has a solution of size \(m^c\), then \(\Gamma\) has a solution of size \(m\).

We conclude that, only one among the inequalities, either \(|W^c(J)| < |J|^{1/c}\) or \(|H^c(\beta^c(J), J)| < |J|^{(c-1)/c}\) can be true. Indeed, if both inequalities hold there are fewer than \(|J|^{1/c}\) intervals of the form \([\nu^{c-1} + 1, (i+1)\nu^{c-1}]\) that intersect with \(J\) and each interval which contains some element of \(J\) also contains less than \(|J|^{(c-1)/c}\) elements. This means that the interval \([1, \nu^c]\) contains fewer than \(|J|\) elements. This is contradiction. Thus, by Remark 16 the solution returned by \(\text{EXTRACTLFS}(\Lambda, c, J)\) is not smaller than \(|J|^{1/c}\).

The time complexity of the presented reduction depends on the time complexity of the \(T^c\) construction, the complexity of \(\text{EXTRACTLFS}(T^c, c, J)\) and \(\mathcal{C}\). The complexity of the transformation \(T^{c-1}\) into \(T^c\) can be estimated by \(O(\nu^2 \cdot \nu^{c-1})\). The cost of building \(f_k(\tau)\) costs \(O(\nu^{c-1})\) because it depends on the length of \(\tau\) element. This cost is multiplied by \(O(\nu)\) in the process of the concatenation of \(f_k(\tau)\) components and the whole outcome is created \(\nu\) times during \(\lambda\) components construction. Since the whole construction starts from \(\Gamma = T^1\) the overall cost is \(\sum_{i=2}^c O(n^{i+1})\). This is \(O(n^{c+1})\) assuming that \(c\) is a constant. The complexity
3.3 On parametrized complexity

An S-reduction \((f, g)\) preserves the costs values between reduced instance \(x\) of an \(A\) problem and outcome instances \(x'\) of a \(B\) problem. In the case of a minimization problem, the decision version of the problem \(B\) is formulated as the question if \(\text{cost}_R(x', y') \leq l\) for a given \(l \in [n]\), on instance \(x'\) of \(B\), \(y'\) which is a feasible solution of instance \(x'\). Analogously the decision version of the problem \(A\) is the question if \(\text{cost}_A(x, y) \leq l\), for instance \(x\) of \(A\). Assuming that \(l\) is parameter, by the equality of costs, the \(f\) function is also the reduction \(R\). It is easy to note that \(x \in A\) iff \(x' \in B\). The \(R\) is an fpt-algorithm because the \(f\) is polynomially computable. The function \(h\) can be assumed to be identity and \(\kappa_A(x) = \text{cost}_A(x, g(x, y'))\), \(\kappa_B(x) = \text{cost}_B(f(x), y')\), where \(y'\) is feasible solution of the
problem $B$ for an instance $f(x)$. The above shows that if $(f,g)$ is $S$-reductions, then $f$ is an fpt-reduction. Similar argument can be applied for maximization problems.

Therefore, $kLFSD \leq_{FPT} MAXCLIQUE$ and $kUD \leq_{FPT} VERTEXCOVER$. Since MAXCLIQUE is in $W[1]$, VERTEXCOVER is in $FPT$ and both parametrized classes $W[1]$ and $FPT$ are closed under fpt-reductions [9] we obtain:

$\blacktriangleright$ Theorem 19. The $kLFSD$ problem is in $W[1]$ class for any $k \in [n]$. The $kUD$ problem belongs to the $FPT$ class for any $k \in [n]$.

$\blacktriangleright$ Theorem 20. For any constant $c$, the $n^{1/c}LFSD$ problem is $W[1]$-hard.

Proof. (Sketch). Previously we have noticed that if a $(f,g)$ is an $S$-reduction then $f$ is an fpt-reduction. In the proof of Theorem 3 we show the $S$-reduction from CLIQUE to $nLFSD$. Hence there is fpt-reduction from CLIQUE parametrized by the size of required clique to $nLFSD$ parametrized by the required size of set which induces common sequences for all pairs of permutations. Since the CLIQUE is $W[1]$-hard the parametrized $nLFSD$ is $W[1]$-hard under fpt-reductions.

In the proof of Theorem 14 we defined polynomial time reduction which for a given instance $\Gamma$ of $nLFSD$ returns an instance $\Lambda$ of $n^{1/c}LFSD$. Moreover the $\Lambda$ has solution of the size $m^c$ iff the $\Gamma$ has solution of the size $m$. It can be shown by induction over the $c$ that presented reduction satisfies conditions (1) and (2) of the fpt-reduction definition. The condition (3) is satisfied by function $h(y) = y^c$, for any positive integer constant $c$. $\blacktriangleright$

4 Initial Motivation and Conclusions

In order to obtain novelty solution for hypercubes packing we believe that there exists some other metaheuristics which allow to find solution in $n$-dimensional space. This is our motivation to extend the Ulam’s metric to $n$-dimension. We believe that there exists some metaheuristics which use finding a minimal distance between $n$-tuples of permutations providing that this way is computationally effective. For this reason we rely our study on finding a minimal distance between $n$-tuples of permutations. Just because NP-hardness of the $kU$ problem, we realise that finding an effective solution seems almost impossible. Still we hope that these metaheuristics can rely on approximation algorithms. The hope constitutes our motivatin to persue the study in which we proved that there exist some approximation algorithms for $kU$ problem that have the best approximation factors ranged from to $\frac{41}{40}$ and 2. However we need to stress that our findings can not be found as the best ones to resolve the problem because we not obtained the strict factors for each $k$.

We show that hierarchies studied in this paper, namely $kU$ and $kLF$, refer to each other just in a way like $MAXCLIQUE$ and $MINVERTEXCOVER$ do. Additionally, we shed light on the question how our hierarchies can be placed in the map of parameterized classes of computational complexity.

References

1 I.L. Markov J. Hu A.B. Kahng, J. Lienig. VLSI Physical Design: From Graph Partitioning to Timing Closure. Springer, New York, 2011.
2 David Aldous and Persi Diaconis. Longest increasing subsequences: From patience sorting to the baik-deift-johansson theorem. Bull. Amer. Math. Soc. 36:413–432, 1999.
3 Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. J. Amer. Math. Soc., 4:1119—-1178, 1999. doi:https://doi.org/10.1090/S0894-0347-99-00307-0
C. Blum, M. Aguilera, A. Roli, and M. Sampels. Hybrid metaheuristics: an emerging approach to optimization. *Studies in Computational Intelligence*, 114, 2008.

C. Blum and A. Roli. Metaheuristics in combinatorial optimization: Overview and conceptual comparison. *ACM Computing Surveys*, 35(3):268–308, 2003.

E. Burke, M. Gendreau, M. Hyde, G. Kendall, G. Ochoa, E. Ozcan, and R. Qu. Hyper-heuristics: a survey of the state of the art. *Journal of Operations Research Society*, 64:1695–1724, 2013.

Ivan Corwin. Commentary on “longest increasing subsequences: from patience sorting to the baik-deift-johansson theorem” by david aldous and persi diaconis. *Bull. Amer. Math. Soc.*, 55:363–374, 2018. [doi:https://doi.org/10.1090/bull/1623](https://doi.org/10.1090/bull/1623).

Pierluigi Crescenzi. A short guide to approximation preserving reductions. In *Proceedings of the Twelfth Annual IEEE Conference on Computational Complexity*, Ulm, Germany, June 24–27, 1997, pages 262–273. IEEE Computer Society, 1997. [doi:10.1109/CCC.1997.612321](https://doi.org/10.1109/CCC.1997.612321).

Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. [doi:10.1007/3-540-29953-X](https://doi.org/10.1007/3-540-29953-X).

Michael L. Fredman. On computing the length of longest increasing subsequences. *Discrete Mathematics*, 11:29–35, 1975.

Johan Håstad. Clique is hard to approximate within n^{1-epsilon}. In *37th Annual Symposium on Foundations of Computer Science, FOCS ’96*, Burlington, Vermont, USA, 13-16 October, 1996, pages 627–630. IEEE Computer Society, 1996. [doi:10.1109/SFCS.1996.548522](https://doi.org/10.1109/SFCS.1996.548522).

Johan Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):789–859, 2001.

S. Imahori, M. Yagiura, and T. Ibaraki. Improved local search algorithms for the rectangle packing problem with general spatial costs. *European Journal of Operational Research*, 167:48–67, 2005.

Andrzej Kozik. Handling precedence constraints in scheduling problems by the sequence pair representation. *J. Comb. Optim.*, 33(2):445–472, 2017. [doi:10.1007/s10878-015-9973-8](https://doi.org/10.1007/s10878-015-9973-8).

Andrzej Kozik. Scheduling under the network of temporo-spatial proximity relationships. *Computers and Operations Research*, 84:106–115, 2017. [doi:https://doi.org/10.1016/j.cor.2017.03.011](https://doi.org/10.1016/j.cor.2017.03.011).

H. Murata, K. Fujiyoshi, S. Nakatake, and Y. Kajitani. VLSI module placement based on rectangle-packing by the sequence pair. *IEEE Trans. on CAD of ICs.*, 15:1518–1524, 1996.

Christos M. Papadimitriou. *Computational complexity*. Addison-Wesley, Reading, Massachusetts, 1994.

Michael L. Pinedo. *Scheduling. Theory, Algorithms, and Systems*. Springer-Verlag, New York, 2012.

D. Vigo.P.Toth. *The vehicle routing problem*. Society for Industrial and Applied Mathematics, Philadelphia, 2002.

Dan Romik. *The Surprising Mathematics of Longest Increasing Subsequences*. Cambridge University Press, New York, 2015.

S.M. Ulam. Some ideas and prospects in biomathematics. *Ann. Rev. Biophys. Bioeng.*, 1:277–292, 1972.

Vijay V. Vazirani. *Approximation algorithms*. Springer, 2001.

David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(1):103–128, 2007. [doi:10.4086/toc.2007.v003a006](https://doi.org/10.4086/toc.2007.v003a006).
Appendix

5.1 Proof of Remark 4

Proof. We consider two cases (1) if \( i \in K \) and (2) if \( i \notin K \). A conditional split of \( K \) is defined to be a pair of disjoint sets \( K^+ \) and \( K^- \) that satisfy the following conditions:

1. \[
K^+_i = \begin{cases} 
K \cap P^i & \text{if } i \notin K \\
(K \cap P^i) \cup \{i\} & \text{if } i \in K 
\end{cases}
\]

2. \( K^-_i = K \cap N^i \)

We will denote by \( \sigma_i \) the non repeating sequence of all elements from \( K^-_i \) in the increasing order, by \( \beta_i \) the non repeating sequence of all elements from \( K^+_i \) in the increasing order. It is easily seen that depending on if \( i \in K \) or \( i \notin K \) the join sequence \( \alpha_i \cdot \beta_i \) has either \(|K| - 1\) or \(|K|\) elements.

Notice that in case \( i \in K \) the sequence \( \alpha_i \) is empty and \( \beta_i \) consists of elements from \( K \setminus \{i\} \). In this case \( \sigma^+_i \) contains subsequence \( (i) \cdot \beta_i \). In permutation \( \sigma^+_i \) element \( i \) appears as the first element of the sequence. Since \( \beta_i \) contains elements from \( K^+_i \) and \( K^+_i \), by definition, is a subset of \( P^i \) and both \( \beta_i \) have the same order, then \( \beta_i \) is subsequence of \( \beta_i \).

Permutation \( \sigma^-_i \) contains subsequence \( (i) \cdot \beta_i \) in this case. Thus \( \sigma^-_i \) also contains subsequence \( (i) \cdot \beta_i \) and \( (i) \cdot \beta_i \) is subsequence of \( \sigma^-_i \) and \( \sigma^-_i \). The common subsequence \( (i) \cdot \beta_i \) is \( K \)-induced subsequence for both \( \sigma^+_i \) and \( \sigma^-_i \).

Consider the case \( i \notin K \). In this case \( \alpha_i \) does not have to be empty sequence, unlike it was previously. Since \( K^-_i \subseteq N^i \) and \( K^+_i \subseteq P^i \), the sequence \( \alpha_i \) is subsequence of \( a_i \) and \( \beta_i \) is subsequence of \( b_i \). In permutation \( \sigma^+_i \), element \( i \) precede \( a_i \cdot b_i \), \( \alpha_i \cdot \beta_i \) is subsequence of \( a_i \cdot b_i \). Hence \( \alpha_i \cdot \beta_i \) is subsequence of \( (i) \cdot a_i \cdot b_i = \sigma^+_i \). By definition \( \sigma^+_i = a_i \cdot (i) \cdot b_i \). Hence \( \alpha_i \cdot \beta_i \) is subsequence of \( \sigma^+_i \). Thus \( \alpha_i \cdot \beta_i \) is \( K \)-induced subsequence for \( \sigma^+_i \) and \( \sigma^-_i \).

Now assume that a set \( K \subseteq [n] \) induces common subsequences for all \( (\sigma^+_i, \sigma^-_i) \), where \( i \in [n] \).

Let \( \gamma_i \) be a subsequence of \( \sigma^+_i = (i) \cdot a_i \cdot b_i \) and \( \sigma^-_i = a_i \cdot (i) \cdot b_i \). Let \( i \in K \). It is easy to see that \( i \) is the first element of \( \gamma_i \), hence \( \gamma_i = (i) \cdot \gamma'_i \), where \( \gamma'_i \) is the subsequence of \( b_i \). By definition of \( b_i \) all its elements are in \( P_i \). This means that \( iEx \) for all \( x \in \{\gamma'_i\} \). Moreover \( \{\gamma'_i\} = K \setminus \{i\} \). Now we can make a conclusion that for all different \( i, x \in K \) \( iEx \). This means that \( K \) is a clique in \( G \).

5.2 Proof of Remark 5

Proof. Suppose that some \( B \subseteq V \), which satisfies \(|B| > m\), induces subsequences for all \( (\sigma^+_i, \sigma^-_i) \). Subgraph of \( G \) induced by \( B \) cannot be a clique. Therefore, there are \( \alpha, \beta \in B \), such that \( \alphaEx\beta \). In permutation \( \sigma^+_\alpha \) element \( \alpha \) precedes \( \beta \), by definition of \( \sigma^+_\alpha \), \( \alpha \) precedes all other members of \( V^\alpha \). In permutation \( \sigma^+_\alpha \) elements from \( a_\alpha \) precedes \( \alpha \). By definitions of \( U^\alpha \) and \( a_\alpha \) we have \( \beta \) occurs in \( a_\alpha \). Thus \( \alpha \) and \( \beta \) appear in \( \sigma^+_\alpha \) and \( \sigma^+_\alpha \) in reverse order. Therefore, there is no sequence induced by \( B \) which is common subsequence of \( \sigma^+_\alpha \) and \( \sigma^+_\alpha \). This is contradiction.

5.3 Proof of Remark 15

Proof. Let \( B^{\nu-1} \subseteq [\nu^{\nu-1}] \) be a set, which induces common subsequence for all \( (\tau^+_\nu, \tau^-_\nu) \). Let \( B \subseteq [\nu] \) be a set, which induces common subsequence for all \( (\sigma^+_\nu, \sigma^-_\nu) \). Any pair of different
elements $i, j \in B$ appears in the same order in $\sigma^s_i$ and $\sigma^s_j$ for all $x \in [\nu]$. This means that $i$ occurs in the earlier position than $j$ in $\sigma^s_i$ if and only if $i$ occurs in the earlier position than $j$ in $\sigma^s_j$. Similar remark holds for all pairs $(\tau^s_i, \tau^s_j) \in S^{\tau^s}_{\sigma^c-1}$ and $k, l \in [\nu^{c-1}]$.

We claim that $\mathcal{B}^c = \{(j-1)\nu^{c-1} + k \mid j \in B, k \in B^{-\nu^c}\}$ is solution of $\Lambda$. On the contrary, suppose that $\mathcal{B}^c$ do not induces common subsequence of all pairs $(\lambda^s_{\nu}, \lambda^s_l)$. Thus we may find $y$ such that $\mathcal{B}^c$-induced sequences $\rho$ and $\gamma$, for $\lambda^s_{\nu}$ and $\lambda^s_l$, and $\rho \neq \gamma$, respectively. Then there are elements $(i - 1)\nu^{c-1} + l$ and $(j - 1)\nu^{c-1} + k$ that cause first difference in $\rho$ and $\gamma$. Thus either $i \neq j$ or $i = j$. For the first case, by definition of $\lambda^s_{\nu}$ and $\lambda^s_l$, without loss of generality, one can assume that in the sequence $\bigcirc_{r \rightarrow \sigma^s_{i}} f_r(\tau^c_{i})$, element $(i - 1)\nu^{c-1} + l$ appears before element $(j - 1)\nu^{c-1} + k$ and in the $\bigcirc_{r \rightarrow \sigma^s_{j}} f_r(\tau^c_{j})$ they are in the opposite order. This holds for any $l, k \in [\nu]$ because all elements from $f_1([\nu^{c-1}])$ appear before elements from $f_j([\nu^{c-1}])$ if $i$ appears before $j$ in $\sigma^s_i$. Elements $i, j$ are in the opposite order in $\sigma^s_{i}$. By definition of $\mathcal{B}^c$, $i, j \in B$ and we assumed at the beginning that $\mathcal{B}$ induces common subsequence in all $(\sigma^s_{i}, \sigma^s_{j})$, in particular in $(\sigma^s_{i}, \sigma^s_{j})$. This is contradiction.

In the case $i = j$, element $(i - 1)\nu^{c-1} + l$ appears before $(i - 1)\nu^{c-1} + k$ in $f_1(\tau^c_{i})$ and in the $f_1(\tau^c_{j})$ they are in the opposite order. This means that $l, k$ are in different orders in $\tau^c_{i}$ and $\tau^c_{j}$. By definition of $\mathcal{B}^c$, $k, l \in B^{-\nu^c}$, but it is assumed $B^{-\nu^c}$ induces common subsequence for all $(\tau^c_{i}, \tau^c_{j})$, in particular in $(\tau^c_{i}, \tau^c_{j})$. This is contradiction. ▶

5.4 Proof of Remark 16

Proof. Assume that there is $J \subseteq [\nu]$, of the size $m^c$ that is solution of $\Lambda$.

Let $R(l) = \{s \in J \mid \text{ inblock}^c = \{s\} \in H^c(l, J)\}$. First, notice that $J = \bigcup_{l \in W^c(J)} R(l)$ and $R(l)$ are pairwise disjoint sets. It is easy to see if $|W^c(J)| < m$, then there exists $l$ such that $|R(l)| \geq m^c$. At least one of two cases holds: $W^c(J)$ is a set which is feasible solution of $\Gamma$ or $H^c(l, J)$ is feasible solution of $T^{c-1}$. In particular if $l$ be chosen to make $H^c(l, J)$ of maximal size then $l$ be equal $\beta^c(J)$.

For any $p, r \in W^c(J)$ there are $\pi = (p-1)\nu^{c-1} + k, \rho = (r-1)\nu^{c-1} + l \in J$. By definition of $\lambda^s_{\nu}, \lambda^s_l$, if $\pi$ and $\rho$ appear in the same order in $\lambda^s_{\nu}, \lambda^s_l$ then $p$ and $\rho$ are in the same order in $\sigma^s_{i}$ and $\sigma^s_{j}$. Thus, indeed $W^c(J)$ is a solution of $\Gamma$.

For any $a, b \in H^c(l, J)$, there are $\pi, \rho \in J$ such that $a = \text{ inblock}^c(\pi), b = \text{ inblock}^c(\rho)$, $l = \text{ block}^c(\pi)$ and $l = \text{ block}^c(\rho)$. By definition of $\text{ block}^c$ and $\text{ inblock}^c$, $\pi = (l - 1)\nu^{c-1} + a$ and $\rho = (l - 1)\nu^{c-1} + b$.

For all $i \in [\nu]$, numbers $\pi$ and $\rho$ occur in the same order in $\lambda^s_{\nu}$ and $\lambda^s_l$. Sequences $f_1(\tau^c_{i})$ and $f_1(\tau^c_{j})$ are substrings of $\lambda^s_{\nu}$ and $\lambda^s_l$, respectively. Moreover $\pi$ and $\rho$ occur in $f_1(\tau^c_{i})$ and $f_1(\tau^c_{j})$ in the same order. It means that, for each $i \in [\nu]$, $a$ and $b$ appear in the same order in $\tau^c_{i}$ and $\tau^c_{j}$ (by definition of $f_1$). Thus, if $a, b \in H^c(l, J)$, then $a$ and $b$ appear in the same order in $\tau^c_{i}$ and $\tau^c_{j}$. Therefore, for all $l \in [\nu]$, $H^c(l, J)$ is feasible solution of $T^{c-1}$. In case $W^c(J) < m, R(\beta^c(J)) \geq m^c$, this implies that $H^c(\beta^c(J), J) > m^c$. ▶