Scale-free nonlinear conservative cascades
and their stationary spectra

Dmitri O. Pushkin

Department of Theoretical and Applied Mechanics,
University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Hassan Aref

Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0217, USA

(Dated: March 26, 2004)

Abstract

We show that a variety of complex processes can be viewed from the unified standpoint of scale-free nonlinear conservative cascades. Examples include certain turbulence models, percolation, cluster coagulation (aggregation) and fragmentation, ‘coarse-grained’ forest fire model of self-organized criticality, and scale-free network growth. We classify such cascades by the values of three indices, and show how power-law steady spectra may arise. The power-law exponent is proven to depend only on the values of the three indices by a simple algebraic formula.

PACS numbers:
The cascade idea is central to our understanding of an amazingly broad scope of natural phenomena. One of the earliest mathematical cascade models, the Galton-Watson process [1], was suggested in 1874 as a solution to the "problem of the extinction of families." This kind of branching cascades was later used for theoretical treatment of chemical and nuclear chain reactions [2], cosmic radiation [3], and statistics of polymers [4] among other physical and biological problems.

Cluster fragmentation is another common type of cascades [5]. Typical examples of fragmentation are rock fracture, breakup of liquid droplets [6], and polymer degradation [7]. Fragmentation preserves the total mass of clusters. Both branching processes and fragmentation are linear cascades, as the agents are assumed to not interfere with each other.

Studies of turbulence have provided a wealth of examples of nonlinear cascades. Among them are the Richardson cascade of energy in the inertial range that underlies the now classical approach of Kolmogorov [8], the Kraichnan reverse enstrophy cascade for two-dimensional turbulence [9], and various random cascade models suggested as theoretical explanations of the unsolved problem of turbulence intermittency [10].

More recently the cascade idea has been frequently evoked in connection with a variety of complex processes. A self-organized branching cascade process was put forth as a mean-field theory for avalanches [11]. A cascade model was suggested as a ‘coarse-grained’ approximation to the forest fire model, displaying self-organized criticality [12]. The classic result for the asteroidal size distribution [13] has been re-considered from the perspective of collisional fragmentation cascades [14]. Mesoscopic rainfall [15], cascades of reconnecting magnetic loops in solar flares [16], and cascades of data networks [17] are further examples from this vast list.

We remark, however, that other important complex systems and processes – e.g. multiplicative random processes, percolation [18], cluster coagulation [19], Scheidegger model of river networks, directed Abelian sandpile model [20], and scale-free random networks [21] – can be advantageously considered as nonlinear cascades.

Nonlinear cascades are characterized by strong interaction between the agents. Unlike linear cascades, they are are poorly understood. Their studies have usually relied on system-specific features and assumptions and, as a result, the general picture has remained unknown.

The purpose of the present paper is to study a general class of scale-free nonlinear conser-
TABLE I: Some scale-free conservative cascades with stationary power-law spectra, \( n(s) \propto s^{-\tau} \), discovered in various contexts. For all of them \( \tau = 1 + (m + \alpha)/h \).

| Cascade type                              | \( h \) | \( m \) | \( \alpha \) | \( \tau \) | Notes                        |
|-------------------------------------------|--------|--------|-------------|--------|------------------------------|
| Turbulent energy                          | 3/2    | 0      | 1           | 5/3    | Kolmogorov (1941) [8]        |
| Enstrophy                                 | 3/2    | 2      | 1           | 3      | Kraichnan (1967) [9]         |
| Passive tracer                            | 1      | 0      | 0           | 1      | Batchelor (1959) [22]        |
| Percolation                               | 2      | 1      | 2           | 5/2    |                              |
| Diffusion limited cluster aggregation     | 2      | 1      | 0           | 3/2    |                              |
| Cluster coagulation in a shear flow        | 2      | 1      | 1           | 2      | Hunt (1982) [23]             |
| Scale-free cluster coagulation            | 2      | 1      | \( \alpha \) \((3 + \alpha)/2\) | Pushkin and Aref (2002) [24] |
| Cascade model of forest fires              | 2      | 1      | 1           | 2      | Turcotte (1999) [12]         |
| Directed Abelian sandpile model           | 3      | 1      | 0           | 4/3    | Dhar and Ramaswami (1989) [20] |
| 1D-diffusion limited cluster aggregation  | 3      | 1      | 0           | 4/3    | Takayasu (1989) [25]         |
| Asteroid collisional fragmentation        | 2      | 1      | 2/3         | 11/6   | Dohnanyi (1969) [13]         |
| Scale-free collisional fragmentation      | 2      | 1      | \( \alpha \) \((3 + \alpha)/2\) | Tanaka et. al. (1996) [14]  |
| ‘Random scission’ fragmentation            | 1      | 1      | 1           | 3      | Ben-Naim and Krapivsky (2000) [26] |
| Scale-free random networks                 | 1      | 1      | 1           | 3      | Barabasi and Albert (1999) [21] |

Conservative cascades, and to show that seemingly different systems reveal common features when considered from this unified viewpoint. In particular, we derive a nonlinear cascade equation, and demonstrate how it can give rise to stationary power-law spectra. We show that the power-law exponent depends only on three indices characterizing a particular cascade: the ‘conservation law’ index \( m \), the scale homogeneity index \( \alpha \), and the nonlinearity index \( h \). This result is illustrated by Table 1.

Let us set the stage. We will think of cascades as statistical collective systems of interacting agents of different type. Examples of agents are turbulent eddies, coalescing rain droplets, avalanches, connected network clusters and polymer chains, genetic family trees, etc. As a result of an interaction of two or more agents, ‘new’ agents form, whereas ‘old’ agents annihilate. For instance, the result of hydrodynamic interactions between turbulent eddies leading to exchange of their energies, is viewed as annihilation of ‘old’ eddies and...
creation of ‘new’ eddies having altered energies in their place. Any such interaction can be written formally as

$$A_{s_1} + A_{s_2} + \ldots \rightarrow A'_{s_1} + A'_{s_2} + \ldots; \quad \text{or} \quad \sum_{s_i} \nu_{s_i} A_{s_i} = 0. \quad (1)$$

Here $s$ labels various agent types, and can be discrete or continuous. In the latter case summation has to be substituted by integration. Here we will assume $s$ to be a real number, and refer to it as ‘size’. $\nu_{s_i}$’s are taken positive for ‘new’ agents and negative for ‘old’ ones. Although this notation is routinely used for chemical reactions, it is quite unusual in other fields, e.g. turbulence.

We assume that agents of the same type are indistinguishable in statistical sense, and that a broad distribution of agent types emerges as a result of interactions.

A cascade state is specified by the agent distribution, also called spectrum, $n(t, s)$. Common examples of agent distributions are the turbulent energy spectrum $E(t, k)$ in homogeneous isotropic turbulence, where the scalar wavenumber $k$ plays the role of $s$; the avalanche size distribution $n(t, s)$; and the distribution of masses of clusters formed due to cluster coagulation or/and fragmentation.

The primary goal of a cascade theory is to predict the spectrum $n$. We derive a novel formal nonlinear cascade equation, which describes temporal evolution of $n$, from the master equation \[27\]:

$$\dot{P}_t[N(s)] = \sum_{N_1(s)} w[N(s), N_1(s)]P_t[N_1(s)] - w[N_1(s), N(s)]P_t[N(s)]. \quad (2)$$

Here $P_t[N(s)]$ is the time-dependent probability of the microstate $N$, square brackets emphasize functional dependence; $w[N, N_1]$ is the probability of transition from the microstate $N_1$ to the microstate $N$ per unit time; summation is carried over all microstates compatible with the interactions \[11\]. Then the macroscopic spectrum $n(t, s)$ can be obtained as an average over all microstates:

$$n(t, s) = \langle N(s) \rangle \equiv \sum_N N(s)P_t[N]. \quad (3)$$

After multiplying \[2\] by $N(s)$, summing it over all microstates $N$, regrouping the terms, and introducing $\Delta N(s) \equiv N(s) - N_1(s)$, we obtain

$$\dot{n}(t, s) = \sum_{\Delta N} \Delta N(s) \sum_N w[N + \Delta N, N]P_t[N].$$
When an interaction is of the type (1), $\Delta N(s)$ has the form:

$$\Delta N(s) = \nu(s, \{s_i\}) \equiv \sum_{s_i} \nu_{s_i} \delta(s - s_i). \quad (4)$$


Clearly, $\nu(s, \{s_i\})$ is the number of agents of size $s$ created due to an ‘elementary’ interaction of agents of sizes $\{s_i\}$. Now we can define the interaction probability per unit time:

$$W(\{s_i\}, [N(s)]) = w[N(s) + \nu(s, \{s_i\}), N(s)], \quad (5)$$

and its average:

$$\mathcal{T}(t, \{s_i\}) \equiv \langle W(\{s_i\}, [N]) \rangle. \quad (6)$$

The physical sense of $\mathcal{T}$ is interaction strength between the agents of sizes $\{s_i\}$. Thus, we arrive at the *nonlinear cascade equation*:

$$\dot{n}(t, s) = \langle \nu(s, \{s_i\}) \rangle, \quad (7)$$

where the following notation has been introduced: for any function $f(\{s_i\})$,

$$\langle \langle f(\{s_i\}) \rangle \rangle(t) \equiv \sum_{\{s_i\}} f(\{s_i\}) \mathcal{T}(t, \{s_i\}). \quad (8)$$

The summation in (8) is carried over all sets $\{s_i\}$. At this point equation (7) is not closed, as the interaction strength $\mathcal{T}$ depends on the microdistribution $N$. The necessary closure can be provided for particular types of interactions by mean field-type approximations or renormalization procedures among other methods [28]. However, we will show that for conservative scale-free cascades important conclusions about asymptotic spectra can be found by-passing the closure problem.

Let us illustrate the above ideas with two examples. For cluster coagulation [19]:

$$A_{s_i} + A_{s_j} \to A_{s_{i+j}},$$

$$\nu(s_i, s_j, s_{i+j}) = \delta_{s_i+s_j} - \delta_{s_i} - \delta_{s_j},$$

$$W(t; s_i, s_j, s_{i+j}; [N]) = K(s_i, s_j) N_{s_i} N_{s_j}, \text{ for any } i, j.$$

Here $K(i, j)$ is the probability that two cluster of sizes $s_i$ and $s_j$ aggregate per unit time. $K$ is commonly called coagulation kernel. In the mean-field approximation:

$$\mathcal{T} = K(s_i, s_j) n_{s_i} n_{s_j},$$

5
and (7) becomes the celebrated Smoluchowski coagulation equation:

\[
\dot{n}_s(t) = \frac{1}{2} \sum_{s_i=1}^{s-1} K(s_i, s - s_i) n_{s_i}(t) n_{s-s_i}(t) - n_s(t) \sum_{s_i=1}^{\infty} K(s, s_i) n_{s_i}(t).
\] (9)

As another example consider two-dimensional isotropic turbulence. In the inviscid limit the energy balance equation for a turbulent flow in a cyclic box of side \(D\) reads [9]:

\[
\partial_t E(k) = T(k), \quad T(k) = \frac{1}{2} \int_0^\infty dp \, dq \, T(k, p, q),
\]

where

\[
T(k, p, q) = 2\pi k \text{Im}\{(D/2\pi)^4(2\pi/|\sin(p, q)|) \times (k_m \delta_{ij} + k_j \delta_{im}) \langle u_i'(k)u_j(p)u_m(q) \rangle\},
\]

\[
(k = p + q, \ k = |k|, \ p = |p|, \ q = |q|).
\]

This problem can be easily written in the form (10):

\[
A_k + \nu_p A_p + \nu_q A_q = 0,
\]

\[
\nu(s; k, p, q) = \delta(s - k) + \nu_p \delta(s - p) + \nu_q \delta(s - q),
\]

\[
\nu_p = T(p, k, q)/T(k, p, q), \quad \nu_q = T(q, p, k)/T(k, p, q),
\]

\[
T(k, p, q) = T(k, p, q).
\]

The condition that a cascade is scale-free has several consequences: first, the reaction type defined by (11) is independent of the physical scale of agent sizes. More exactly, any two sets of agent sizes, \(\{s_i\}\) and \(\{s'_i\}\), such that \(s'_i = \lambda s_i\) for some positive \(\lambda\), allow reactions of the same type, i.e. \(\nu_{s_i} = \nu_{s'_i}\).

Next, interactions should have no characteristic size. In mathematical terms: if \(s'_i = \lambda s_i, \ N' = \lambda^{-1} N\), then

\[
\langle W(\{s'_i\}, [N']) \rangle = \lambda^\alpha \langle W(\{s_i\}, [N]) \rangle.
\] (11)

For instance, for homogeneous isotropic turbulence \(\alpha = 1\) and this value can be traced to the gradient operator in the Navier-Stokes equations. For coagulation, \(\alpha\) is the homogeneity degree of the coagulation kernel.

Finally, the interactions must be scalable in agent density. Thus, if \(s'_i = s_i, \ N' = \lambda N\),

\[
\langle W(\{s'_i\}, [N']) \rangle = \lambda^h \langle W(\{s_i\}, [N]) \rangle.
\] (12)
For example, for homogeneous isotropic turbulence \( h = 3/2 \), because the interactions depend on the third order velocity correlation function, and velocity scales as square root of energy. For coagulation, \( h = 2 \) due to binary collisions.

Let the conservation law read:

\[
\sum_{s_j} \nu(s_j, \{s_i\}) s_j^m = 0, \text{ for any set } \{s_i\}. \tag{13}
\]

For instance, the energy conservation law for turbulence yields \( m = 0 \), while enstrophy conservation for 2D turbulence results in \( m = 2 \). Mass conservation in coagulation and fragmentation cascades yields \( m = 1 \).

The nonlinear cascade equation can be used for treating open systems, e.g. a constantly stirred turbulent flow, a coagulating system with a source of the smallest clusters, or a driven self-organizing system. Let us assume (without limitation of generality) a direct cascade, i.e. \( s > s_0 \), where \( s_0 \) is the smallest size in the system. Then, we supplement equation (7) with the ‘boundary condition’:

\[
n(t, s_0) = n_0 \text{ for all } t, \tag{14}
\]

and look for stationary solutions. We are particularly interested in the limit \( s \gg s_0 \). For scale-free conservative cascades this problem can be easily solved in a general way. Because the conserved quantity flux, \( E \), is well-defined in the limit \( s/s_0 \to \infty \), one can expect that solutions of (7), (14) depend on \( E \) and ‘forget’ the microscale \( s_0 \). These expectations are corroborated below.

It follows from (7) and the definition of \( E \) that:

\[
E = \sum_{\{s_i\}} \langle W(\{s_i\}, [N]) \rangle \sum_{s_i} \nu_{s_i} s_i^m \theta(S - s_i), \text{ for any } s_0 < S < \infty. \tag{15}
\]

Here \( \theta(x) \) is the step function. A lot of instances of scale-free conservative cascades are known to evolve steady power-law spectra. Now we can understand how they arise. Indeed, it follows from the definition of spectrum (3), the properties (11) and (12), and the arbitrariness of \( S \) in (15), that

\[
n(s, s_0, E) = E^{1/h} \lambda^\tau n(\lambda s, \lambda s_0, 1), \text{ for any positive } \lambda,
\]

\[
\tau = 1 + \frac{m + \alpha}{h}. \tag{16}
\]
Hence $n$ has the form:

$$n(s, s_0, E) = E^{1/h} s^{-\tau} f(s_0/s). \quad (17)$$

For wide classes of interactions the right hand side of (15) has a finite limit as $s_0/S \to 0$. These classes depend on asymptotic properties of interactions. They have been determined for cluster coagulation [24] and weak wave turbulence ('locality conditions') [29]. In this limit the scale $s_0$ drops out of the equations, and the function $f$ has a finite limit:

$$\lim_{x \to 0} f(x) = C, \quad \text{and thus,} \quad n(s) \approx C E^{1/h} s^{-\tau}. \quad (18)$$

In turbulence the constant $C$ was first introduced by Kolmogorov [5], and is often denoted as $C_2$. It is clearly interaction-dependent and, therefore, non-universal. For cluster coagulation an expression for $C$ was found in [24].

At this point a remark is appropriate: the choice of size variable, which one faces looking for a model for a cascade process, is quite arbitrary. For example, one could characterize cluster coagulation by the cluster diameter distribution, rather than the cluster mass distribution. Let the new size variable $\hat{s} = s^q$. Then $\hat{n}(\hat{s}) d\hat{s} = n(s) ds$, and $(\tau - 1) \to q(\tau - 1)$. Clearly, (16) must obey this transformation rule. It does so, indeed, as under this transform \( \{m, \alpha, h\} \to \{qm, q\alpha, h\} \).

The self-similar distribution (18) is our central result. Table 1 demonstrates that for many complex processes, which can be viewed as scale-free conservative cascades, the exponents of power-law spectra can be obtained in nearly automatic fashion. More of examples of self-similar spectra can doubtlessly be found scattered throughout the scientific literature in various fields. Because values of the indices for a particular cascade are often apparent due to their clear physical meaning, our result has considerable heuristic powers.

The semblance between phenomena in the fields of turbulence, cluster coagulation (aggregation) and fragmentation, self-organized criticality, critical phenomena, and, more recently, econophysics – has long been perceived, e.g. [30]. In certain instances it has led to new quantitative results: e.g. the self-similar spectra due to cluster coagulation have been derived for three coagulation mechanisms – Brownian, laminar shear, and sedimentation in gravity – using reasoning patterned on the Kolmogorov 1941 theory of turbulence [23]. Our approach justifies and generalizes such results.

It is notable that most of well-studied cascade processes – branching processes, fragmentation, (strange/anomalous) diffusions, Lévy flights, multiplicative processes, and cascade
models of turbulence intermittency – are linear cascades with $h = 1$. Although just a particular class of scale-free cascades, their investigation has led to rich developments in our understanding and modeling of complex phenomena. We expect, therefore, that investigation of nonlinear cascades is essential to further progress in this field.

[1] H. W. Watson, J. Antropol. Inst. Great Britain and Ireland 4, 138 (1874).
[2] N. Semenov, *Chemical kinetics and chain reactions*, Oxford U. Press (1935); D. Hawkins and S. Ulam, *Theory of multiplicative processes, I*, Los Alamos Scientific Laboratory, LADC-265, (1944); C. J. Everett and S. Ulam, Proc. Nat. Acad. Sci. 34, 403 (1948).
[3] H. J. Bhabha and W. Heitler, Proc. Royal Soc. London A 159, 432 (1937); J. F. Carlson and J. R. Oppenheimer, Phys. Rev. 51, 220 (1937).
[4] M. Gordon, Proc. Roy. Soc. (London) A268, 240 (1962).
[5] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 31, 99 (1941). A. F. Filippov, Theory Probab. Appl. 6, 275 (1961).
[6] R. Shinnar, J. Fluid Mech. 10, 259 (1961).
[7] R. M. Ziff and E. D. McGrady, J. Phys. A: Math. Gen. 18, 3027 (1985).
[8] A. N. Kolmogorov 1941, Dokl. Acad. Nauk SSSR 31, 538 (1941).
[9] R. H. Kraichnan, Phys. Fluids 10 (7), 1417 (1967).
[10] B. B. Mandelbrot, J. Fluid Mech. 62, 331 (1974); U. Frisch, *Turbulence*, Cambridge U. Press, Cambridge (1995).
[11] S. Zapperi, K. B. Lauritsen, and H. E. Stanley, Phys. Rev. Lett. 75, 4071 (1995).
[12] D. L. Turcotte, Rep. Prog. Phys. 62, 1377 (1999).
[13] J. S. Dohmnyi, J. Geophys. Res. 74, 2531 (1969).
[14] D. R. Williams and G. W. Wetherill, Icarus 107 (1), 117 (1994); H. Tanaka, S. Inaba, and K. Nakazawa, Icarus 123, 450 (1996).
[15] V. K. Gupta and E. L. Waymire, J. Appl. Meteorol. 32, 251 (1993).
[16] D. Hughes, M. Paczuski, R. O. Dendy, P. Helander, and K. G. McClements, Phys. Rev. Lett. 90 (13), 131101 (2003).
[17] A. Feldman, A. C. Gilbert, and W. Willinger, Comp. Commun. Rev. 28 (4), 42 (1998).
[18] D. Stauffer, *Introduction to percolation theory*, Taylor and Francis, London (1985).
[19] D. O. Pushkin and H. Aref, *Cluster coagulation* in *Encyclopedia of Nonlinear Science*, ed. A. Scott, Fitzroy Dearborn (2004).

[20] D. Dhar, cond-mat/9909009 (1999); D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).

[21] A.-L. Barabasi and R. Albert, Science **286**, 509512, 1999.

[22] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).

[23] J. R. Hunt, J. Fluid Mech. **122**, 169 (1982).

[24] D. O. Pushkin and H. Aref, Phys. Fluids **14**(2), 694, (2002).

[25] H. Takayasu, Phys. Rev. Lett. 63, 2563 (1989).

[26] E. Ben-Naim and P. L. Krapivsky, Phys. Lett. A **275**, 48 (2000).

[27] N. G. van Kampen, *Stochastic processes in Physics and Chemistry*, North-Holland, Amsterdam (1981).

[28] N. Goldenfeld, *Lectures on phase transitions and the renormalization group*, Addison-Wesley, Reading, Massachusetts (1992).

[29] V. E. Zakharov, V. S. L’vov, and G. Falkovich, *Kolmogorov spectra of turbulence*, Springer-Verlag (1992).

[30] G. Eyink and N. Goldenfeld, Phys. Rev. E **50**(6), 4679 (1994); R. N. Mantegna and H. E. Stanley, Nature **376**(6535), 46 (1995); P. Bak, K. Chen, J. A. Scheinkman, and M. Woodford, Ricerche Economiche **47**, 3 (1993); D. Sornette, *Why Stock-markets crash: critical events in complex financial systems*, Princeton U. Press, Princeton (2003).