We show that a standard calculation of the splitting function for the nonsinglet structure function $h_1$ does not lead to the expected result \cite{1}. The calculation is compared to the corresponding derivation of the splitting function for the nonsinglet polarized structure function $g_1$. We analyze possible explanations for the unexpected result and discuss its implications.

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I. INTRODUCTION

In the Drell-Yan process a quark-antiquark pair is annihilated to a virtual photon, which produces lepton pairs. The corresponding cross section contains a contribution from quarks of different chirality: a quark originating from one nucleon may return to this nucleon in the squared scattering amplitude with its chirality flipped. This is possible because the intermediate photon does not remember the distribution of quark chirality on the scattering nucleon. This gives rise to the leading twist chiral odd distribution function $h_1$, which first appeared in the discussion of the transverse polarized Drell-Yan process \cite{2,3}. One should emphasize that such chiral odd quark distribution functions do not appear in totally inclusive deep inelastic scattering because the quark chirality is conserved at each vertex of the scattering process.

The quark helicity distribution $g_1$ has a partonic interpretation in the chiral basis. This is not the case for $h_1$ which gets a partonic interpretation only by switching to the transversity basis \cite{4}. Looking on transversely polarized nucleons $h_1$ is interpreted as the probabilistic asymmetry of finding quarks in eigenstates of the transverse Pauli-Lubanski vector with eigenvalue $\pm 1/2$. One should be aware, that $g_1$ loses its partonic interpretation in the transversity basis.

There are several attempts to measure the transversity distribution $h_1$, see e.g. \cite{5,6,6}. For general reasons the transversity distribution has to obey restrictions as Soffer’s inequality \cite{13} for twist-2 distribution functions \cite{20}. The anomalous dimension of $h_1$ and the corresponding splitting function can be found in literature \cite{12,14}. This result is found by direct evaluation of the splitting function. Nevertheless the behavior at Bjorken-$x$ equal to 1 remains unfixed in this procedure. In the case of the helicity distribution $g_1$ the behavior for $x=1$ is fixed by a corresponding sum rule. For $h_1$ such a sum rule is lacking. So the full $h_1$ splitting function was determined by a general argument, which demands the splitting function of $h_1$ to behave exactly as the splitting function of $g_1$ for $x \uparrow 1$. To our knowledge, no explicit calculation (without additional arguments) of the full splitting function of the transversity distribution exists until now.

This paper is organized as follows. First, the forward scattering amplitudes corresponding to the helicity and transversity operator are defined. We derive the leading order anomalous dimension of $g_1$ and $h_1$ using a dispersion relation and show that the behavior of the splitting function for $h_1$ does not match the expected result found in literature in the limit $x \uparrow 1$. We discuss possible explanations and implications of our result.

II. DEFINITIONS

A. Helicity operator

The forward scattering amplitude

$$T_{\mu\nu}(p, q, s) = \frac{i}{2} \int d^4 y \, e^{ipy} \times$$

$$\langle p, s | T [j_{\mu5}(y) j_{\nu}(0) + j_{\nu}(y) j_{\mu5}(0)] | p, s \rangle$$

(1)

contributes to the nonsinglet polarized structure function $g_1(x, Q^2) := g^T_1(x, Q^2) - g^A_1(x, Q^2)$, where $j_{\nu}(y) = \bar{\psi}(y) \gamma_\mu \psi(y)$ is a vector current and $j_{\mu5}(y) = \bar{\psi}(y) \gamma_\mu \gamma_5 \psi(y)$ denotes an axial-vector current. $g_1(x, Q^2)$ depends on all possible Lorentz invariants $Q^2 = -q^2$ and Bjorken $x$ with $x = Q^2/2p \cdot q$. Here $p$ and $q$ are the four-momenta of the hadron and virtual photon, respectively. The latter defines the momentum scale of the scattering process. $T$ denotes the time-ordered product and summation over flavor indices is assumed. In order to get the part relevant for $g_1$ one has to project the forward scattering amplitude on the correct Lorentz structure:

$$g_1 \frac{i\varepsilon_{\mu\nu\lambda\rho} q^\lambda s^\rho}{p \cdot q} = \frac{1}{\pi} \text{Abs} \, T_{\mu\nu} \left| \frac{i\varepsilon_{\mu\nu\lambda\rho} q^\lambda s^\rho}{p \cdot q} \right|$$

(2)

Abs denotes the absorptive part, i.e. the discontinuity across a cut \cite{15}

If higher twist operators are neglected, the moments of the polarized structure function can be identified with the twist-2 operator matrix elements $A_{g_1, n}$ in the framework of the operator product expansion:
\[ M_{g_1,n} = \int_0^1 dx \, x^n g_1(x) = C_{g_1,n} A_{g_1,n}. \]

Note, that we dropped the \( Q^2 \)- and the renormalization scale dependence, which are not important for our purpose. \( C_n \) is the Wilson-coefficient which depends on the renormalization scale and the strong coupling. It contains the full perturbation series of leading twist. Consequently, the structure function transfers such that this contribution to the cross section is not suppressed by powers of \( Q^2 \) for large momentum transfers such that this contribution to the cross section is of leading twist. Consequently, the structure function \( h_1 \) of Eq. (2) is determined by the divergent part of the forward scattering amplitudes. By Wick expansion of the current products we find 32 Feynman graphs contributing to \( h_1 \), which can be identified with the transversity distribution in QCD. Nevertheless, quark chirality may be flipped in Drell-Yan processes, in which quarks of different chirality are connected to one virtual photon. This process is not suppressed by powers of \( Q^2 \) for large momentum transfers such that this contribution to the cross section is of leading twist.

As for the polarized structure function, we get the contributions of the forward scattering amplitudes to the structure function \( h_1 \) by projection on the corresponding Lorentz structure:

\[ h_1(x)s_{\perp \mu} = \frac{1}{\pi} \text{Abs} T_{\mu} \bigg|_{s_{\perp \mu}}. \]

The corresponding moments are defined as in (3)

\[ M_{\perp,n} = \int_0^1 dx \, x^n [h_1(x) + (-)^n h_1(-x)] = C_{\perp,n} A_{\perp,n}. \]

The last term again corresponds to the leading twist contribution in the operator product expansion with the Wilson coefficient \( C_{\perp,n} \) and the chiral odd operator matrix element \( A_{\perp,n} \).

### III. DERIVATION OF SPLITTING FUNCTIONS

#### A. Wick expansion

The splitting functions for the operators in Eqs. (1) and (4) are determined by the divergent part of the forward scattering amplitudes. By Wick expansion of the current products we find 32 Feynman graphs contributing to the splitting functions. These diagrams can be divided into two classes which transform into each other by substituting the gluon momentum \( k \rightarrow -k \). Since the gluon loop integrals are symmetric it is sufficient to consider 16 graphs. The forward scattering amplitude \( T \) is real on the real axis, i.e. \( T^* = T \). The remaining 16 graphs split up into 8 diagrams and their complex conjugates which contain the same loop integral and a Lorentz structure in inverse order. The 8 relevant diagrams left consist of 4 diagrams (given below) with different loop integrals and the corresponding exchange graphs. The latter are obtained by substituting the momentum of the virtual photon \( q \rightarrow -q \) and interchanging the external vertices simultaneously. The Feynman diagrams are depicted in Fig. III A.

\[
\begin{align*}
\gamma_\mu &\to q - q + i\eta \\
\gamma_\mu &\to q - q + i\eta
\end{align*}
\]

**FIG. 1.** The Feynman graphs contributing to the forward scattering amplitude with \( \Gamma_\nu = 1, \gamma_\nu \), respectively: The upper graphs represent the selfenergy (left) and the box graph (right) and the lower ones depict the vertex corrections.

The amplitude associated to the selfenergy reads

\[
\mathcal{M}_S = -C_F \overline{\psi}(p) \Gamma_\nu \frac{1}{\not{p} + \not{q} + i\eta} \gamma_\rho \gamma_\beta \gamma_\alpha \times \\
\frac{1}{\not{p} + \not{q} + i\eta} \gamma_\mu \gamma_5 u(p) I_S^\beta,
\]

where \( I_S^\beta \) is given in the Appendix. In analogy we get for the vertex corrections

\[
\mathcal{M}_{V1} = C_F \overline{\psi}(p) \gamma_\rho \gamma_\alpha \Gamma_\nu \frac{1}{\not{p} + \not{q} + i\eta} \times \\
\gamma_\mu \gamma_5 u(p) I_V^\alpha\beta,
\]

and
\[ \mathcal{M}_{V2} = -C_F \bar{\pi}(p) \Gamma_\nu \frac{1}{\not{p} + \not{q} + i\eta} \gamma_{\rho} \gamma_\sigma \gamma_{\mu} \gamma_{\nu} \gamma_5 \gamma_\rho u(p) \gamma^\nu \mathcal{I}_{V}^{\alpha \beta}, \]

and finally for the boxgraph

\[ \mathcal{M}_B = C_F \bar{\pi}(p) \gamma_\rho \gamma_\sigma \Gamma_\nu \gamma_{\rho} \gamma_{\nu} \gamma_5 \gamma_\lambda \gamma_\lambda u(p) \mathcal{I}_{B}^{\alpha \beta \lambda}. \]

The integrals \( \mathcal{I}_{V}^{\alpha \beta} \) and \( \mathcal{I}_{B}^{\alpha \beta \lambda} \) can be found in the Appendix too. The exchange graphs are not shown. \( \Gamma_\nu \) stands for \( \gamma_\nu \) or for identity accounting for the helicity or transversity operator, respectively. We have to isolate the divergent parts contributing to the Lorentz structures which correspond to \( g_1 \) or \( h_1 \).

**B. Dispersion relation**

In order to calculate the contributions of the above Feynman diagrams to the splitting functions we have to consider the following expression

\[ \frac{1}{\not{p} + \not{q} + i\eta} = \frac{\not{p} + \not{q} - i\eta}{Q^2 (\omega - 1) + \eta^2} \]

with

\[ (p + q)^2 = -Q^2 (1 - \omega) \quad \text{and} \quad \omega = 1/x. \]

The limit \( \eta \downarrow 0 \) cannot be taken easily at this stage of calculation. For example the Dirac identity

\[ \lim_{\eta \downarrow 0} \frac{1}{\chi \pm i\eta} = \mathcal{P} \left( \frac{1}{\chi} \right) \mp i\pi \delta(\chi) \]

is not sufficient to deduce the resulting distributions. Although the imaginary part of the r.h.s. of Eq. (11) generates some \( \delta \)-type distribution

\[ \lim_{\eta \downarrow 0} \frac{-i\eta}{Q^2 (\omega - 1) + \eta^2} = -i\pi \delta \left( \sqrt{Q^2 (\omega - 1)} \right) \]

\[ = -2i\pi \sqrt{Q^2 (\omega - 1)} \delta(Q^2 (\omega - 1)) \]

this distribution does not contribute to the moments in Eqs. (3) and (4) owing to the measure \( dx \).

We deduce the singularity structure of the splitting functions using the well-known dispersion relation (see e.g. [13])

\[ T(x) = \sum_{n=0}^{\infty} \left( \frac{1}{x} \right)^{n+1} \frac{1}{\pi} \int_{-1}^{1} dy y^n \operatorname{Abs} T(y) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{x} \right)^{n+1} \frac{1}{\pi} \int_{0}^{1} dy y^n \times \]

\[ \frac{1}{\pi} \left( \operatorname{Abs} T(y) + (-1)^n \operatorname{Abs} T(-y) \right). \]

\( T \) denotes the forward scattering amplitude accounting for \( g_1 \) and \( h_1 \), i.e. \( T_{\mu\nu} \) and \( T_\mu \), respectively. These amplitudes are analytic functions of \( Q^2 \) and \( x \). They possess a pole at \( x = 1 \) and a cut on the real axis for \( x < 1 \). Through the absorptive part \( \operatorname{Abs} T \) we isolate the discontinuities across the cut. The dispersion relation Eq. (15) connects the behavior of \( T \) in the regular (but unphysical) region \( x > 1 \) with the physical region \( x < 1 \), i.e. the coefficients of the Laurent expansion for large \( x \) correspond to the moments of the structure functions. Having expanded the amplitude \( T \), the full information about the singularity structure is contained in the corresponding coefficients. In this representation the limit \( \eta \downarrow 0 \) can be taken for each coefficient separately. In this way the anomalous dimensions are calculated. By taking the inverse Mellin transformation we find the corresponding splitting functions.

In the case of \( g_1 \) crossing symmetry holds, so that Eq. (15) gives rise to even moments only. Odd moments cancel identically and there is no way of getting information on the missing moments. Nevertheless, in order to calculate the splitting function we will have to evaluate the inverse Mellin transformation of the anomalous dimension, which, for this reason should be known for all moments. As it will become evident later, our main interest lies on the singularity structure for \( x \uparrow 1 \), in particular we are interested in contributions like \( \delta(1 - x) \) to the forward scattering amplitude \( T \). The \( \delta \)-distribution is an element of the Schwartz-Sobolev space \( \mathcal{S}_{-1} ([0, 1]) \), cf. [17]. Its domain of definition is the dual space \( \mathcal{S}'_1 ([0, 1]) \), i.e. \( \langle \delta_y f \rangle = f(\chi) \) exists for all functions \( f \) that are element of \( \mathcal{S}_1 ([0, 1]) \subset L_2 ([0, 1]) \). This is also the necessary requirement for the definition of the plus-function

\[ \langle (1 - x)^{-1} f \rangle = \int_{0}^{1} dx \frac{f(x) - f(1)}{1 - x} \]

which will be of importance later. The singular behaviour of the splitting function at \( x \uparrow 1 \) corresponds to high moments. But for large \( n \) the weight functions \( x^n \) form a Cauchy series with respect to the norm of the Schwartz-Sobolev space \( \mathcal{S}_1 ([0, 1]) \)

\[ ||x^{n+1} - x^n||_{\mathcal{E}_1}^2 = O \left( \frac{1}{n} \right). \]

The convergence is even stronger \( (O(1/n^3)) \) for the \( L_2 \)-norm, but, e.g., it does not persist for the \( \mathcal{S}_2 \)-norm. This Schwartz-Sobolev space \( \mathcal{S}_2 ([0, 1]) \) is the domain of definition of the derivative of the \( \delta \)-distribution \( \langle \delta'_y f \rangle = -f'(\chi) \). If we now assume that the singularity structure of the splitting function does not contain such distributions, but only the Dirac-\( \delta \) or the plus-function, we may restrict ourselves to the (lowest) Schwartz-Sobolev space \( \mathcal{S}_1 ([0, 1]) \). Then the inequality of Schwarz

\[ \langle g \rangle \leq ||g||_{\mathcal{E}_{-1}} ||f||_{\mathcal{E}_1}, \]
implies that
\[
\int_0^1 dx \, x^{n+1} \text{Abs } T - \int_0^1 dx \, x^n \text{Abs } T \rightarrow 0 \quad (19)
\]
holds for \( n \uparrow \infty \). Note that the contribution of \( \delta(1-x) \) to the moments is a constant. From Eq. (19) the behavior of the anomalous dimension for \( n \uparrow \infty \) can be determined by the even moments only. The singular part \( T_{\text{sing}} \) of \( T \) corresponds to high moments, so that it is already fixed in this way (as long as it can be defined within \( \mathcal{G}_{-1} \)).

Now let us analyze the regular part \( T_{\text{reg}} \) of the forward scattering amplitude \( T = T_{\text{reg}} + T_{\text{sing}} \). Since \( T \) is assumed to be an analytic function, we suppose that its moments possess an analytic continuation into the complex plane, i.e. we presume that
\[
M(s) = \int_0^1 dx \, x^s \text{Abs } T_{\text{reg}} \quad (20)
\]
is an analytic function for \( s > 1 \). This assumption is very reasonable but not necessary for our main result, which mainly concerns the singular part. After a conformal mapping of the complex plane onto the unit sphere \( \mathbb{S}_2 \) the known even moments describe a convergent series of the (supposed to be) analytic function \( M(s) \). Since we have subtracted the singular part \( T_{\text{sing}} \) which is completely determined by the behaviour of the high moments – the remaining regular part approaches zero in this limit
\[
\int_0^1 dx \, x^n \text{Abs } T_{\text{reg}} \rightarrow 0. \quad (21)
\]
Now the theorem of identity for analytic functions can be applied. (If two analytic functions coincide at a convergent series, then they coincide everywhere.) It follows that the result for even moments remains correct for odd moments as well. This means that we get all moments by calculating the even moments only (under the assumptions made).

We require the same assumptions to hold for \( h_1 \) and proceed in the same way.

C. Results for \( g_1 \)

The forward Compton scattering amplitude corresponding to the polarized structure function \( g_1 \) reads after expansion in \( \omega = 1/x \):
\[
T_{\mu \nu} \bigg|_{i \varepsilon \mu \lambda \rho s^\lambda s^\rho} = 2i \varepsilon_{\mu \lambda \rho} q^\lambda s^\rho \sum_{n=0}^\infty C_{g_1,n} A_{g_1,n} \omega^{n+1}.
\]
(22)

We have computed all one-gluon-exchange diagrams mentioned above with the vertex \( \Gamma_\mu = \gamma_\mu \). In the framework of dimensional regularization with \( d = 4 - \varepsilon \) the divergent part appears as coefficient of \( 2/\varepsilon \). In order to analyze the calculation we present the results for all graphs separately. After projection onto the appropriate Lorentz structure
\[
\alpha_{\mu \nu} = \frac{2 i \varepsilon_{\mu \lambda \rho} q^\lambda s^\rho}{p \cdot q} \quad (23)
\]
we get using the results from the Appendix
\[
M_{g_1,\mu \nu} = - C_F \alpha_{\mu \nu} \frac{\omega^{n+1}}{\varepsilon} \sum_{n=0}^\infty \omega^n. \quad (24)
\]
Note that this geometrical series can be summed up to yield \( \omega/(1-\omega) \) in the unphysical regime \( \omega < 1 \). For the physically interesting absorptive part a \( \delta(1-x) \)-distribution is generated. The results for the vertex corrections and the box graph read
\[
M_{V_1,\mu \nu} = C_F \alpha_{\mu \nu} \frac{\omega^{n+1}}{\varepsilon} \sum_{n=0}^\infty \frac{2}{1 + \frac{n}{k+1}} \quad (25)
\]
\[
M_{V_2,\mu \nu} = C_F \alpha_{\mu \nu} \frac{\omega^{n+1}}{\varepsilon} \sum_{n=0}^\infty \frac{2}{1 + \frac{n}{k+1}} \quad (26)
\]
\[
M_{B,\mu \nu} = - C_F \alpha_{\mu \nu} \frac{\omega^{n+1}}{\varepsilon} \sum_{n=0}^\infty \frac{2}{(1+n)(2+n)} \quad (27)
\]
The gluon propagator was taken in Feynman-gauge so that it is reduced to the \( g_{\mu \nu} \)-term. Nevertheless, we checked that the sum of the contributions arising from the \( k_{\mu} k_{\nu} \)-term vanishes as required by gauge invariance. Taking these results together we find
\[
M_{\mu \nu} = C_F \alpha_{\mu \nu} \frac{\omega^{n+1}}{\varepsilon} \times \left( \frac{4}{1+n} - \frac{2}{(1+n)(2+n)} + 4S_n - 3 \right) \quad (28)
\]
where
\[
S_n = \sum_{k=1}^n \frac{1}{k} \quad (29)
\]
The leading order anomalous dimension becomes
\[
\gamma_{g_1,n} = \frac{C_F}{(4\pi)^2} \left( \frac{4}{1+n} - \frac{2}{(1+n)(2+n)} + 4S_n - 3 \right) \quad (30)
\]
and the inverse Mellin transformation generates
\[ P_{g_1}(x) = \frac{C_F}{(4\pi)^2} \left( 4 - 2(1-x) - \frac{4}{(1-x)_+} - 3\delta(1-x) \right) \]  

(31)

for the splitting function. This result is well known in literature [13]. Usually, the distribution \( 3\delta(1-x) \) is fixed by the sum rule

\[ \int_0^1 dx P_{g_1}(x) = 0 \]  

(32)

and is not calculated directly (as we did here by using the dispersion relation Eq. (15)). Due to conservation of helicity the integral over the quark distribution does not change with \( Q^2 \), i.e. \( \gamma_{g_1,0} = 0 \). As transversity is not conserved a corresponding sum rule for the transversity distribution is not known.

D. Results for \( h_1 \)

In the same way as for the polarized structure function we expand Eq. (4) in \( \omega \):

\[ T_{\mu\nu}^{s_{1\mu}} = s_{1\mu} \sum_{n=0}^{\infty} C_{\perp n} A_{\perp n} \omega^{n+1}. \]  

(33)

In this case we find for the one gluon exchange graphs (again using the results from the Appendix) after projection on \( s_{1\mu} \)

\[ M_{S_{\mu}} = -C_F s_{1\mu} 2 \sum_{n=0}^{\infty} \omega^{n+1}, \]  

(34)

\[ M_{V_{1\mu}} = C_F s_{1\mu} 2 \sum_{n=0}^{\infty} \omega^{n+1} \left( 4 + \sum_{k=1}^{n} \frac{2}{k+1} \right), \]  

(35)

\[ M_{V_{2\mu}} = C_F s_{1\mu} 2 \sum_{n=0}^{\infty} \omega^{n+1} \left( 1 + \sum_{k=1}^{n} \frac{2}{k+1} \right), \]  

(36)

\[ M_{B_{\mu}} = 0. \]  

(37)

The boxgraph does not lead to any divergent contribution to the Lorentz structure \( s_{1\mu} \).

Adding up all graphs we get

\[ M_{\mu} = C_F s_{1\mu} 2 \sum_{n=0}^{\infty} \omega^{n+1} \left( \frac{4}{1+n} + 4S_n \right). \]  

(38)

This leads to the anomalous dimension

\[ \gamma_{\perp,\mu} = \frac{C_F}{(4\pi)^2} \left( \frac{4}{1+n} + 4S_n \right) \]  

(39)

and after an inverse Mellin transformation to the splitting function:

\[ P_{\perp}(x) = \frac{C_F}{(4\pi)^2} \left( 4 - \frac{4}{(1-x)_+} - 3\delta(1-x) \right). \]  

(40)

This is the splitting function corresponding to the operator defined in Eq. (4). Additionally, according to the method of Ioffe and Khodjamirian, this result may be identified with the splitting function of the (twist-2) transversity distribution as defined in Eq. (13) in the Appendix. However, our result Eq. (40) differs from the one found in literature for the transversity distribution [3]

\[ P_{\perp}^{\text{lit}}(x) = \frac{C_F}{(4\pi)^2} \left( 4 - \frac{4}{(1-x)_+} - 3\delta(1-x) \right) \]  

(41)

at \( x = 1 \).

IV. DISCUSSION

The splitting functions Eqs. (31) and (40) for \( g_1 \) and \( h_1 \), respectively, were calculated using the same method. In the case of the operator corresponding to the longitudinally polarized structure function the expected result was found. Nevertheless, the analogous calculation for the transversity operator leads to a result which differs from the one in literature. We want to elucidate possible reasons for this discrepancy.

A. Sum rules

As emphasized in Sec. III B a direct calculation of the splitting function without using the dispersion relation Eq. (15) does not fix the behavior at \( x = 1 \). This is done by the sum rule Eq. (32) in the case of the helicity asymmetry operator. We showed above, that the dispersion relation leads to exactly the result expected from the sum rule without any further argument.

One may observe that the splitting functions for the helicity asymmetry and the transversity operator Eqs. (31) and (40) differ by an additional term proportional to \( 1-x \) which vanishes for \( x = 1 \) and an additional \( \delta(1-x) \). This distribution enters \( P_{g_1}(x) \) with the same weight as the missing term found in literature Eq. (41) for \( P_{\perp} \). This is not accidental. As for the transversity operator no sum rule corresponding to Eq. (32) is known, the behavior for \( x = 1 \) of the transversity splitting function was fixed with the general argument [14,19], that the probability of a transversity flip should vanish for \( x = 1 \). It follows from this (reasonable) requirement that the splitting functions \( P_{\perp} \) and \( P_{g_1} \) should behave identically at \( x = 1 \). We want to emphasize that by using this argument, the divergent structure at \( x = 1 \) is not calculated explicitly but fixed by hand.
B. Comparing both calculations

The Feynman graphs contributing to the splitting functions differ solely by the vertex $\Gamma_{\nu}$ which is $\gamma_{\nu}$ in the case of the helicity asymmetry operator and unity for the transversity operator. For this reason the divergent part of one gluon exchange contributions are equal for the self energy $\mathcal{M}_S$ as well as for the vertex correction $\mathcal{M}_{V_2}$ which do not involve the differing vertex into the loop integral. Comparing the explicitly calculated results Eqs. (24) and (26) with Eqs. (34) and (36), respectively, this statement is confirmed.

Looking at the results for the boxgraph Eqs. (27) and (29) we observe a difference. The boxgraph is not divergent for the transversity operator due to an extra factor $4-d$ appearing by projection on the corresponding Lorentz structure. After an inverse Mellin transformation the contribution Eq. (27) to the anomalous dimension is translated into the $1-x$ term in Eq. (31). This term vanishes for $x = 1$ and for this reason is not a good candidate to explain the difference at $x = 1$.

In this way only the vertex correction $\mathcal{M}_{V_1}$ with a gluon exchanged at the $\gamma_{\nu}$ or the unity vertex, respectively, may be responsible for the difference at $x = 1$. Comparing Eqs. (25) and (35) the result differ by an additional constant contribution to the anomalous dimension. An inverse Mellin transformation of a constant leads exactly to the additional term $-3\delta (1-x)$ for the polarized splitting function $P_{Sg}$. It is plausible that two graphs with one gluon exchanged at different vertices do not lead to the same result. Anyhow, we will have to look for a conceptual problem if we want to save the general statement of vanishing transversity flip probability at $x = 1$.

C. Assumptions

We based the identification of the calculated anomalous dimension with the one of the transversity distribution on the argument of Ioffe and Khodjamirian [4]. Here it was shown, that the operator used in the forward scattering amplitude Eq. (3) indeed has a relation to the definition of the transversity distribution Eq. (13). Their argument remains correct on the leading twist level, which is appropriate to our purpose. The identification may be affected by anomalies. But these appear first in second order in the strong coupling. This implies that our first order calculation remains unaffected from possible anomalies.

In addition the method of Ioffe and Khodjamirian is based on light cone dominance. One may contest, that this assumption is not appropriate to describe the splitting function at $x = 1$. Here a potentially unbounded number of zero-momentum gluons appears, which are not well described by free quark operators. Nevertheless, the identification of the structure function (defined in Eq. (3)) with the transversity distribution (defined in Eq. (13)) on the leading twist level and in leading order of perturbation theory is unaffected by the above consideration, so that the leading order anomalous dimension should be reproduced correctly. Consequently, if the method of Ioffe and Khodjamirian is applicable to the problem under consideration, we will have to look for other reasons for the discrepancy.

Our calculation of the anomalous dimension for $g_1$ and $h_1$ rely both on the assumption that $\text{Abs} T$ in Eq. (3) is element of the Schwartz-Sobolev space $\mathcal{S}_{-1}$. The result Eq. (3) for $g_1$ is in accordance with literature which is not the case for $h_1$, so that the above assumption may be questioned for the transversity operator. If $\text{Abs} T_{\mu}$ is not element of the Schwartz-Sobolev space $\mathcal{S}_{-1}$, the irregular part contains other contributions than a $\delta$-distribution. For example derivatives of $\delta$ could appear. As the difference of the results Eqs. (31) and (33) is element of $\mathcal{S}_{-1}$, it cannot be explained in this way. This means that the reason for the discrepancy is not related to our assumptions.

The assumption that the regular part of the forward scattering amplitude is an analytic function, is completely independent of the problem at $x = 1$. Analyticity is necessary to ensure that our calculation of the terms with $x < 1$ reproduces the correct regular part of the splitting function.

D. Implications

It follows from the analysis of the assumptions that our result for the transversity splitting function can be interpreted in two alternative ways: First, the correspondance of Eqs. (3) and (13) shown by Ioffe and Khodjamirian contains a conceptual difficulty at $x = 1$. Alternatively, our result implies a non-vanishing probability of transversity flip by emitting infrared gluons from quarks. This is really hard to believe. Especially, the number of infrared gluon emissions is arbitrary large so that the transversity becomes completely statistically distributed. On the other hand the regularization of the infrared regime was done by hand, exactly in order to avoid this surprising interpretation. It was not deduced from first principles of QCD. Our calculation shows that (under the assumptions we made) the vanishing transversity flip probability for zero-momentum gluons does not follow analytically.

APPENDIX

A. Distribution functions

The polarized quark distribution is defined in leading twist by
Again $x = -q^2/(2p \cdot q)$ is the Bjorken variable. $n^\nu$ is a light cone vector with $n^2 = 0$ of dimension $(mass)^{-1}$, $p$ is the proton four-momentum with $p^2 = m^2$ and $p \cdot n = 1$, and $s$ is the proton spin four-vector with $s^2 = -1$. In deep inelastic electron proton scattering the distribution $g_1$ describes the asymmetry of nucleons polarized parallel or antiparallel to the longitudinally polarized lepton.

The definition of $h_1$ in terms of an operator matrix element reads [2,3]:

$$\int \frac{d\lambda}{4\pi} e^{ix\lambda} \langle p, s | \bar{\psi}(0) \gamma_\mu \gamma_5 \psi(\lambda n) | p, s \rangle = g_1(x, q^2) \rho_{\mu} s \cdot n.$$  

(42)

is obviously infrared divergent only, which leads to a maximum divergence of $1/\varepsilon$. In the case of the transversal structure function $h_1$ the boxgraph is supplied with a prefactor of $4 - d$ resulting from the according Lorentz structure. Therefore it yields no contributions to the anomalous dimension. On the other hand the boxgraph leads to a term $\sim (1 - x)$ (see Eq. (27)) for the polarized structure function $g_1$. However, since we are mainly interested in the behavior of the splitting functions in the limit $x \uparrow 1$, the boxgraph is not relevant for our purposes.

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