Resonance–assisted tunneling in 4D normal–form Hamiltonians

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A significant enhancement of tunneling may be caused by the presence of nonlinear resonances in the corresponding classical phase space. We describe this resonance–assisted tunneling using a 4D normal–form Hamiltonian as it captures all fundamental features. We consider single as well as double resonances and study the morphology of quantum states. Quantitatively we describe tunneling in terms of the weight of these states in classically disconnected regions. By applying perturbative methods we reveal the underlying mechanism and obtain excellent qualitative and quantitative agreement with numerical data. Using a minimal $4 \times 4$ matrix model, an intuitive understanding of the basic features of resonance-assisted tunneling in the normal–form Hamiltonians is obtained.

I. INTRODUCTION

Quantum tunneling connects classically disjoint regions and therefore is one of the most prominent features of quantum mechanics. In particular, the description of tunneling through energy barriers allowed for computing of molecular ground states [1] and explaining radioactive decay [2–3]. However, classical barriers may arise not only from potential barriers but can be generated by the classical dynamics in phase space leading to the concept of dynamical tunneling [4–5]. The paradigmatic example for this is tunneling between dynamically not connected regions of a mixed phase space in which chaotic and regular motion coexists. There, for instance, tunneling between two regular regions, separated by chaotic dynamics, is moderated by chaos–assisted tunneling through the chaotic component of phase space [6–8]. The tunneling between regular and chaotic regions is described by regular–to–chaotic tunneling [9, 10]. Being a fundamental quantum mechanical effect, dynamical tunneling is of relevance in many different fields of physics, e.g. vibrational spectra of molecules [11, 12], systems of ultra cold atoms [13, 14], optical microcavities [15, 21], microwave billiards [22, 24], and explains power–law level repulsion at small energy spacings in systems with a mixed phase space [25].

Independent of the concrete system the presence of classical nonlinear resonances drastically enhances tunneling effects between disconnected regions by the mechanism of resonance–assisted tunneling [20, 28], which has been demonstrated by recent experiments [21, 29]. A lot of effort has been put into its theoretical understanding [9, 12, 20, 23, 30, 49]. Although resonance–assisted tunneling is expected to occur in any number of dimensions, most of the work has concentrated on 2D systems. For these the universal description of the classical dynamics in the vicinity of a nonlinear resonance is given by a pendulum–like 2D normal–form Hamiltonian. Such Hamiltonians arise from secular perturbation theory [20, 28] or normal form theory describing the surrounding of periodic trajectories [30]. Classically, these normal–form Hamiltonians provide an interpolating flow for the dynamics of 2D symplectic maps. The corresponding quantum maps are approximated by the quantized normal–form Hamiltonians. Thus, the universal features of resonance–assisted tunneling can be described in terms of the normal–form Hamiltonian.

Dynamical tunneling and specifically resonance–assisted tunneling has also been studied in higher–dimensional systems [51–55]. Generically in such systems resonances of higher rank arise, which are not present in 2D systems. The double resonance first occurring in a 4D system is the simplest case showing new types of dynamics. The vicinity of a double resonance is captured by an associated 4D normal–form Hamiltonian. The quantum systems corresponding to these classical normal–form Hamiltonians have been extensively studied in the context of the vibrational dynamics of chemical molecules [11, 12, 50, 61]. Especially the intramolecular energy transfer is heavily influenced by classical nonlinear resonances, as they allow for couplings between different vibrational modes [11, 12]. However, a quantitative description based on resonance–assisted tunneling has not been worked out so far.

In this paper we give a qualitative as well as a quantitative description of resonance–assisted tunneling in 4D normal–form systems. To this end, we study the classical dynamics of the simplest normal–form Hamiltonians, which describe the dynamics of either an integrable single coupled resonance or a non–integrable double resonance. Subsequently we study the phase–space localization of the eigenstates of the quantized systems in classically disjoint regions. We find that tunneling in the vicinity of the single resonance shows the same characteristics as 2D systems, i.e. a drastic enhancement of tunneling. Such an enhancement is also seen in the case of the double resonance, but occurs in a much more complex fashion. Additionally, we also observe the suppression of tunneling in this case. Utilizing a perturbative expansion [92] of these states, previously employed for the case of 2D quantum maps in Ref. [9], we are able to describe both enhancement as well as suppression of tunneling. Following and exploiting some of the theoretical methods
successfully applied to 2D systems we provide a first step towards a detailed understanding of resonance-assisted tunneling in higher dimensions. The detailed description thereof in generic 4D quantum maps, however, is out of the scope of this paper and subject to further research.

This paper is organized as follows: In Sec. II a brief sketch of the general approach is given. By means of the instructive example of tunneling in the case of a single coupled resonance we introduce the methods and notation in Sec. IIIA. Sec. IIIB treats the more complex case of a double resonance. For both cases the classical normal form model as well as the associated quantum system is investigated. Quantitatively resonance-assisted tunneling is studied by means of numerical diagonalization and a perturbative description of the underlying mechanism is presented. The key features of resonance-assisted tunneling in the normal–form Hamiltonians can be intuitively understood within a simple 4 $\times$ 4 matrix model presented in Sec. IIIC. Finally, a summary and outlook is given in Sec. IIID.

II. RESONANCE ASSISTED TUNNELING

Classical nonlinear resonances are ubiquitous in Hamiltonian dynamical systems and manifest themselves by means of resonance-assisted tunneling in the corresponding quantum system. In order to explain the quantum features based on classical properties, we utilize a universal description of the classical dynamics in terms of a truncated 4D normal–form Hamiltonian for both single and double resonances \[63, 64\]. The corresponding 4D phase space is separated into dynamically disjoint regions by the resonance channels associated with the nonlinear resonances. The eigenstates of the quantum Hamiltonian predominantly localize on classical quantizing tori, located in one of these regions. However, due to tunneling, there is also non–vanishing probability in the other, classically forbidden, regions. We introduce the weight in these regions as a quantitative measure of resonance-assisted tunneling which shows similar characteristics as known from two dimensional quantum maps \[48\]. A perturbative description of resonance-assisted tunneling is obtained for both single coupled resonances and double resonances. This allows for an intuitive understanding of the enhancement and, in the case of double resonances, also the suppression of tunneling in terms of perturbative paths along a discrete action grid.

A. Single coupled resonance

First the instructive case of the single coupled resonance in a 4D Hamiltonian system is considered, which will turn out to be quite similar to 2D systems with a single dominating resonance. To this end we start with the classical resonant Hamiltonian \[63, 65\],

\[ H(\theta, I) = H_0(I) + V(\theta) \]  

(1)

obtained from normal form analysis and truncation. It is expressed in action-angle coordinates \( I = (I_1, I_2) \) and \( \theta = (\theta_1, \theta_2) \) of \( H_0(I) \). Keeping only the non-resonant, quadratic order, gives

\[ H_0(I) = \frac{1}{2} (I - I_{\text{res}})^T M (I - I_{\text{res}}) \]  

(2)

for some real symmetric matrix \( M \) and where the resonance has been shifted to \( I_{\text{res}} \). In contrast, the lowest order resonant term is given by \( V(\theta) = 2V_\pi \cos(s \theta) \) and the resonance vector \( s = (s_1, s_2) \). By means of a canonical transformation either the resonance vector can be brought to the form \( s = (s_1, 0) \), i.e. an uncoupled resonance, or \( M \) may be transformed to diagonal form. Here, we choose \( M = M_{\text{res}}^2 I \). Since the potential \( V(\theta) \) is 2$\pi$ periodic the phase space of the system is the cylinder $T^2 \times \mathbb{R}^2$. Furthermore, the system is integrable because energy as well as the polyad number \( s_2 I_1 - s_1 I_2 \) are conserved \[66\].

For concrete numerical calculations and visualization we choose \( s = (1, 1) \), \( V_\pi = 0.1 \), \( M_{\text{res}} = 0.8 \) and \( I_{\text{res}} = (1.0, 1.0) \). Instead of restricting the 4D phase space to a 3D energy manifold and introducing a Poincaré section to obtain a 2D representation of the dynamics, we show the 3D hyperplane for a fixed value of \( \theta_2 = 0 \) and visualize the remaining three coordinates \((\theta_1, I_1, I_2)\). This allows for an intuitive representation of the phase-space structures for different energies. In Fig. 1 this hyperplane for \( \theta_2 = 0 \) is shown. Regular dynamics takes place on 2D-tori so that they appear as 1D curves. Furthermore, the 4D phase space is foliated into invariant 3D planes of constant polyad number. In the 3D hyperplane they appear as 2D planes resembling the dynamics of a pendulum. In Fig. 1 orbits in four such planes are shown. The stable and unstable fixed points of these pendulums correspond to elliptic and hyperbolic 1D-tori in the full phase space. Their projection onto the action coordinates is called resonance center line \[67, 68\] and fulfills

\[ s_1 \frac{\partial H_0(I)}{\partial I_1} + s_2 \frac{\partial H_0(I)}{\partial I_2} = 0. \]  

(3)

The families of the elliptic and hyperbolic 1D-tori build up the skeleton of the resonance channel \[69\]. The regular structures form a tilted tube in phase space as shown in Fig. 1. Note, that the geometry of the 4D normal form system, represented in the section with \( \theta_2 = 0 \), is very similar to that of a corresponding 4D symplectic map, represented in a 3D phase space slice \[70\].

The quantum system is obtained from the classical Hamiltonian Eq. (1) by means of Weyl quantization \[71\], which yield the operators \( \hat{\theta} \) and \( \hat{I} \) as well as the Hamiltonian \( \hat{H} = H_0(I) + V(\hat{\theta}) \). The eigenvalue equation

\[ \hat{H} |\psi_m\rangle = E |\psi_m\rangle \]  

(4)

gives the eigenstates and eigenenergies of the system.

The periodicity of \( V(\hat{\theta}) \) implies, analogously to Bloch’s theorem \[72\], the discretization of action space, in terms
\[ \mathbf{I}_m = \hbar (\mathbf{m} + \mathbf{\vartheta}) \quad \text{with} \quad \mathbf{m} \in \mathbb{Z}^2. \]  

Here, \( \hbar \) denotes an effective Planck’s constant, which may be defined as the ratio of Planck’s constant and a typical action of the system and plays the role of a semiclassical parameter, i.e. \( \hbar \to 0 \) corresponds to the semiclassical limit. Furthermore, we fix the Bloch phase \( \mathbf{\vartheta} = (0, 0) \). Evaluation of the Hamiltonian in the action basis Eq. (5) gives the matrix elements

\[ \hat{H}_{m,n} = H_0(\mathbf{I}_n) \delta_{m,n} + V_s(\delta_{m,n+s} + \delta_{m,n-s}). \]  

As we are interested in quantum states in the vicinity of the resonance channel near \( \mathbf{I} = (0, 0) \) we restrict the action grid to a rectangle, outside which the wave functions are supposed to carry negligible probability. Truncation at \( |I_1|, |I_2| \leq 10 \) gives rise to a finite-dimensional Hilbert space.

Note, that \( H_0(\hat{\mathbf{I}}) \) is diagonal in action space and therefore its eigenstates \( |\mathbf{I}_m\rangle \) are labeled by the quantum number \( \mathbf{m} \in \mathbb{Z}^2 \). By means of semiclassical EBK quantization this relates them to the quantizing torus \( \mathbf{I}_m \), Eq. (5), of the classical system given by \( H_0(\mathbf{I}) \). As long as \( V(\mathbf{\vartheta}) \) is a small perturbation of \( H_0(\mathbf{I}) \) this association of quantum numbers remains valid and the eigenstates of interest localize on quantizing tori of the perturbed Hamiltonian with the same action as \( \mathbf{I}_m \). Our choice of \( \mathbf{\vartheta} = (0, 0) \) as well as \( \mathbf{I}_{\text{res}}, M_{\text{res}}, \) and \( V_s \) ensures the existence of an eigenstate \( |\psi_0\rangle \equiv |\psi_{\mathbf{m}}\rangle \) with quantum number \( \mathbf{m} = (0, 0) \) which localizes on the corresponding quantizing torus \( \mathbf{I}_0 = (0, 0) \) close to, but still outside of,
the resonance channel. In Fig. 1, this torus is depicted as a white line. The choice \( m = (0, 0) \) is convenient as it allows to study quantum states associated with the same torus when varying \( \hbar \).

In order to visualize the eigenstates \(|\psi\rangle\) in phase space we use the Husimi representation \[73\]

\[
H_\psi(\theta, I) = \frac{1}{\pi\hbar^2} |\alpha_{\text{coh}}(\theta, I)|^2 |\psi\rangle^2,
\]

defined by the overlap of \(|\psi\rangle\) with a coherent state \(|\alpha_{\text{coh}}(\theta, I)\rangle\) of minimal uncertainty centered at \((\theta, I)\) given in action representation as

\[
\langle I | \alpha_{\text{coh}}(\theta_0, I_0) \rangle = \frac{1}{\sqrt{\pi\hbar}} \exp \left(-\frac{(I - I_0)^2}{2\hbar} + i \frac{\hbar}{\sqrt{\pi}} I_0 \theta_0 \right),
\]

and considered on the torus \[74\].

By fixing \( \theta_2 = 0 \) the Husimi function can be compared with the classical phase space structures. In Fig. 1(a) the eigenstate \(|\psi_0\rangle\) for \( 1/\hbar = 0.3450 \) is shown, localizing on the quantizing torus \( I_0 \). Note that this representation is similar to the 3D Husimi representation on the 3D phase space slice \[70\].

The effect of resonance-assisted tunneling can be measured by the weight of \(|\psi_0\rangle\) on the opposite side of the resonance \( s = (s_1, s_2) \). In order to determine this weight we formally follow Ref. \[48\] and introduce a region \( \Lambda \) in action space given by

\[
\Lambda := \{ (I_1, I_2) \in \mathbb{R}^2 : I_2 \geq \frac{s_1}{s_2} (I_1 - I_{\text{res},1}) + I_{\text{res},2} + \Lambda_1 \}. \tag{9}
\]

The trivial extension of \( \Lambda \) in \( \theta_1 \)-direction is shown as blue shaded region in Fig. 1. The boundary is chosen parallel to the resonance center line. It is located on the opposite side of the resonance channel with respect to the quantizing torus \( I_0 = (0, 0) \). Its distance from the resonance channel is controlled by \( \Lambda_1 \). As \( \Lambda \) is dynamically separated from the quantizing torus \( I_0 \), the weight of the state \(|\psi_0\rangle\) on \( \Lambda \) measures the strength of tunneling over the resonance channel. For the quantum system this translates into the projector \[48\]

\[
\hat{P}_\Lambda |I\rangle = \chi_\Lambda(I) |I\rangle, \tag{10}
\]

where \( \chi_\Lambda \) denotes the characteristic function,

\[
\chi_\Lambda(I) = \begin{cases} 
1 & \text{for } I \in \Lambda \\
0 & \text{for } I \notin \Lambda.
\end{cases} \tag{11}
\]

The weight \( w_0 \) of \(|\psi_0\rangle\) in \( \Lambda \) is determined by

\[
w_0 = \|\hat{P}_\Lambda |\psi_0\rangle\|^2. \tag{12}
\]

Note, that opening the system in \( \Lambda \) leads to decay rates showing qualitatively the same results as the weight \( w_0 \) \[48\].
thonormal basis of action states $|I_m\rangle$, in which $H_0(\mathbf{I})$ is diagonal, and consider $V(\hat{\theta})$ as a perturbation. The only non-vanishing matrix elements of $V(\hat{\theta})$ occur between action states whose quantum numbers differ by $\pm s$

$$\langle I_m | V(\hat{\theta}) | I_{m+s} \rangle = V_s.$$  

(13)

The perturbative expansion of a state localizing predominantly on the quantizing torus $I_m$ is thus given by \[|\psi_{\text{pert}}^m\rangle = |I_m\rangle + \sum_{l \in \mathbb{Z}\setminus\{0\}} A_m^{(l)} |I_{m+l}s\rangle.\] 

Note that only states with the same polyad number $s_2 I_1 - s_1 I_2$ contribute to this perturbative expansion. For fixed $l$ the coefficient $A_m^{(l)}$ can be decomposed into distinct contributions $\lambda_{m}^\Gamma$, that differ in the order of perturbation theory. Each contribution is uniquely associated with a sequence $\Gamma = [t_1, t_2, \ldots, t_k]$ of couplings with $t_i \in \{\pm s\}$ which defines a path on the action grid. The length of the path is given by the number of elements in the sequence and is denoted as $|\Gamma| := k$. It coincides with the order of perturbation theory in which $\Gamma$ contributes.

A path can contribute to $A_m^{(l)}$ only if $|\Gamma| \geq |l|$. Graphically such a path $\Gamma$ connects $I_m$ with $I_{m+l}s$ on the discrete action grid, where intermediate subsequent actions differ by $\pm s$ in their quantum numbers.

Schematically this is illustrated in Fig. 3 where the action grid is indicated by gray points and the region $\Lambda$ is shaded blue. Figure 3(a) shows the path $\Gamma = [s, s]$ of length $|\Gamma| = 2$ connecting $I_0$, marked as green point, with $I_{(2,2)}$. In Fig. 3(b) the path $\Gamma = [s, s, s]$ of length $|\Gamma| = 3$ is shown.

Let $\mathcal{M}_m^\prime$ denote the set of all paths which connect $I_m$ with $I_{m+l}s$ and for which no intermediate action coincides with $I_m$, i.e. paths coming back to their starting point are excluded in the following according to perturbation theory. Each path $\Gamma \in \mathcal{M}_m^\prime$ then contributes with

$$\lambda_{m}^\Gamma = V_s |\Gamma| \prod_{i=1}^{|\Gamma|} \frac{1}{H_0(I_m) - H_0(I_m + \sum_{j=1}^{|\Gamma|} t_j)},$$

which includes the unperturbed energies of the intermediate actions. Finally, the contributions from different paths add up to the coefficients

$$A_m^{(l)} = \sum_{\Gamma \in \mathcal{M}_m^\prime} \lambda_{m}^\Gamma.$$ 

(16)

Inserting the state $|\psi_{\text{pert}}^m\rangle$, Eq. (14) into Eq. (12) we find for the weight in the region $\Lambda$

$$w_m = \sum_{l \in \mathbb{N}} \sum_{I_{m+l}s \in \Lambda} |A_m^{(l)}|^2.$$ 

(17)

From the 2D case it is known that it suffices to take only the unique shortest path $\Gamma_l = [s, s, \ldots, s]$ of length $l$ into account. The sum in Eq. (17) may thus be rewritten as a sum over all positive integers $l$ starting from the smallest integer $l_{\text{min}}$ for which $I_{m+l_{\text{min}}s}$ is a member of $\Lambda$ holds and where we approximate $A_m^{(l)} \approx \lambda_{m}^\Gamma$. In particular, neglecting all but the lowest contributing order $l_{\text{min}}$ of perturbation theory, i.e. including only the shortest path, Eq. (17) reduces to

$$w_m = |\lambda_{m}^{\Gamma_{\text{min}}}|^2.$$ 

(18)

Note that the paths shown in Fig. 3 are exactly the shortest for the two different values of $1/h$, respectively. This lowest order approximation already leads to excellent agreement with the weights $w_0$ obtained from diagonalization of the Hamiltonian, see Fig. 2 where the prediction Eq. (18) is shown as blue crosses comparing very well with the numerical results. This is because higher orders are suppressed by $|V_s|$, i.e. by at least one order of magnitude.

The emergence of resonance–assisted tunneling peaks can be traced back to the energy denominators in Eq. (15). The contribution $\lambda_{m}^\Gamma$ diverges whenever an intermediate state $|I_{m+l}s\rangle$ and $|I_m\rangle$ are resonant, i.e. they are energetically degenerate with respect to $H_0(I)$, which

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig3.png}
\caption{Scheme of perturbation theory for the single coupled resonance. The action grid $I_m$ is depicted as gray points, the quantizing torus $I_0 = (0, 0)$ as green point. The dashed line shows the position of the resonance center line, see Eq. (3). The circle indicates the level set of constant energy $H_0(I_0)$. The blue shaded area shows the region $\Lambda$, see Eq. (9). The red line indicates the shortest path $\Gamma$. In (a) in the non-resonant case for $1/h = 0.1314$ and $|\Gamma| = 2$. In (b) the resonant case for $1/h = 0.1638$ and $|\Gamma| = 3$.}
\end{figure}
gives rise to the peaks. This mechanism is well known from 2D systems. It is further illustrated in Fig. 3(a) where the non-resonant case is shown. There no intermediate state lies on the circular level set of constant energy $H_0(I) = H_0(I_0) = \text{constant}$. In contrast Fig. 3(b) depicts the resonant case where both $I_0$ and $I_{(2,2)}$ have approximately the same unperturbed energy. Note, that in the case of exact degeneracy of $I_0$ with an intermediate state, the perturbative prediction diverges and thus loses its validity. This situation, which in principle could be treated by degenerate perturbation theory, is also ignored in the work on 2D system. Furthermore, quantization jumps occur in the perturbative result as well and coincide with the values of $1/h$ at which the length $l_{\text{min}}$ of the shortest path into the region $\Lambda$ changes by one. Despite being a 4D system due to integrability and the both classically and quantum mechanically conserved polyad number $s_2 I_1 - s_1 I_2$ the results closely resemble what is known from 2D systems.

**B. Double resonance**

Resonances of higher rank are possible in higher-dimensional systems only. The minimal example is the case of double resonances occurring in 4D systems. Using normal-form theory and truncation yields an effective Hamiltonian

$$H(\theta, I) = H_0(I) + 2V_r \cos(r \theta) + 2V_s \cos(s \theta),$$

where $r, s$ gives rise to two different resonance channels parametrized by the corresponding resonance center line, see Eq. (19). At the intersection of both resonance center lines the double resonance condition is fulfilled. In general, Eq. (19) gives rise to non-integrable dynamics, except for specific choices of $r, s$ [64]. In particular, due to the second resonant term the polyad numbers corresponding to either of the two resonance vectors are no longer conserved. However, whenever perturbations are small the system still can be considered as near-integrable.

In the following we choose $r = (r_1, r_2) = (1, 0)$ and $s = (s_1, s_2) = (1, 1)$ as linearly independent resonance vectors with prefactors $V_r = V_s = 0.05$ and $H_0(I)$ as given by Eq. (2) for $M_{\text{res}} = 1$ and $I_{\text{res}} = (1.0, 1.0)$. The phase space is shown in the 3D hyperplane for fixed $\theta_2 = 0$ in Fig. 4. Both resonance vectors give rise to a resonance channel intersecting in a so-called resonance junction at the minimum of $H_0(I)$ at $I_{\text{res}} = (1.0, 1.0)$. At this point the double resonance condition is fulfilled. This gives rise to four equilibria with different types of stability, namely one elliptic-elliptic at $(\theta_1, \theta_2) = (\pi, 0)$, one hyperbolic-hyperbolic at $(0, 0)$ and two elliptic-hyperbolic at $(0, \pi)$ and $(\pi, \pi)$ respectively [65, 70]. Together with their invariant manifolds or attached families of elliptic 1D tori they organize the phase space close to $I_{\text{res}}$. Along the resonance center line of each resonance and sufficiently far away from $I_{\text{res}}$ the phase space locally resembles the phase space of a 2D pendulum and is governed by the associated resonance channel. Following the resonance channels towards the junction at $I_{\text{res}}$ chaotic layers of both resonance channels begin to overlap and form a connected stochastic layer. As usual in non-integrable systems also resonances of higher order occur. These regular resonance structures are not shown, but are indirectly seen by the small holes in the chaotic layers.

From Fig. 4 we conclude, that phase space and action space are each divided into four dynamically separated regions by the resonance channels and the resonance center lines, respectively. Therefore, tunneling across both of the resonance channels is expected.

Accounting for the additional second resonant term in Eq. (19) the matrix elements of the quantized Hamiltonian read

$$\hat{H}_{m,n} = H_0(I_n) \delta_{m,n} + V_a (\delta_{m,n+s} + \delta_{m,n-s}) + V_r (\delta_{m,n+r} + \delta_{m,n-r}).$$

Our choice of parameters guarantees the existence of a quantizing torus $I_0$ for all values of the effective Planck’s constant. This allows for studying tunneling of the eigenstate $|\psi_0\rangle$ in terms of its weight in a properly chosen region $\Lambda$ in action space located on opposite sides of both resonance channels. We choose

$$\Lambda := \{ (I_1, I_2) \in \mathbb{R}^2 : \begin{align*}
I_2 &\geq -\frac{s_1}{s_2} (I_1 - I_{\text{res},1}) + I_{\text{res},2} + \Lambda_1, \\
I_1 &\geq \Lambda_2. \end{align*}$$

For numerical computations we set $\Lambda_1 = 1.7$ and $\Lambda_2 = 2.7$. In Fig. 4 the region $\Lambda$ is shown as light blue region, trivally extended in $\theta_1$-direction. Using $\Lambda$ in the definition of the projector Eq. (10) the weight $w_0$ is again given by Eq. (12).

The weight $w_0$ of $|\psi_0\rangle$ on $\Lambda$ is shown in Fig. 5 as a function of the inverse effective Planck constant $1/h$ by orange bullets. Besides the overall exponential decay of the weight in the region $\Lambda$ various peaks are visible. These peaks occur in a complicated manner and have a strong variation of their widths. Furthermore, between the peaks also a drastic decrease of the weight $w_0$ is observed, e.g. for $1/h \approx 0.146$. Apart from peaks and suppression also plateaus of nearly constant weight over some interval of $1/h$ are present.

In Fig. 4 Husimi representations of $|\psi_0\rangle$ are shown for different values of $1/h$. For the nonresonant case at $1/h = 0.375$, see Fig. 4(a), the state $|\psi_0\rangle$ localizes mainly on the quantizing torus $I_0$ (white line). The Husimi representations for $1/h = 0.1991$ and $1/h = 0.1591$ in Fig. 4(b, c) show resonant eigenstates with a significant weight in dynamically distinct regions. Both cross the resonance junction, however, the morphology of the
FIG. 4. Phase space of the double resonance shown in a 3D hyperplane for $\theta_2 = 0$ fixed. Regular structures are shown as 1D black curves whereas chaotic structures shown as points. The eigenstate $|\psi\rangle$ maximally localizing on the classical quantizing torus $I_0 = (0, 0)$ (white line) is shown in a 3D Husimi phase-space representation (see colorbar) in (a) for $1/h = 0.375$, (b) for $1/h = 0.1991$ and (c) for $1/h = 0.1591$. The region $\Lambda$, Eq. (21), trivially extended in $\theta_1$-direction, is shown in light blue.
into account the coefficients in Eq. (24) only using the shortest paths, is shown as blue crosses. 

FIG. 5. Weight $w_0$ of state $|\psi_0\rangle$, see Eq. (12), as function of the inverse effective Planck constant $1/h$ for the double resonance as shown as orange bullets. The perturbative prediction [27], only using the shortest paths, is shown as blue crosses.

states differs which can be explained in terms of perturbation theory, which is discussed next.

In order to explain the underlying mechanism of resonance-assisted tunneling in the case of a double resonance we use a perturbative approach. As in Sec. II A we start with the unperturbed system, but also consider the second resonant term allowing for two non-vanishing matrix elements

\begin{align}
\langle I_m | V(\theta) | I_{m+r}\rangle &= V_r, \\
\langle I_m | V(\theta) | I_{m+s}\rangle &= V_s,
\end{align}

(22) (23)

for quantum numbers differing by $\pm s$ and $\pm r$. This gives rise to the perturbative expansion [62, 77]

\[ |\psi_{m}^{\text{pert}}\rangle = |I_m\rangle + \sum_{(k,l)\in\mathbb{Z}^2\setminus\{0\}} A_{m}^{(k,l)} |I_{m+kr+is}\rangle \]

(24)

of the state localizing predominantly on the quantizing torus $I_m$. As Eq. (22) and Eq. (23) suggest there are twice the number of possibilities of subsequent couplings in each order of perturbation theory compared to the case of the single resonance. Thus, in the perturbative expansion of $|\psi_{m}^{\text{pert}}\rangle$ an unperturbed state $|I_{m+kr+is}\rangle$ contributes to a coefficient obtained by all paths $\Gamma$ which connect the initial quantizing torus $I_m$ with the final torus $I_{m+kr+is}$ in the discrete action grid, where subsequent actions differ in their quantum numbers by $\pm r$ or $\pm s$, respectively. Let $\mathcal{M}_{m}^{k,l}$ be the set of these paths. In contrast to Sec. II A here every path $\Gamma \in \mathcal{M}_{m}^{k,l}$ corresponds to a sequence $\Gamma = \{t_1, t_2, \ldots, t|\Gamma|\}$ with $t_i \in \{\pm r, \pm s\}$. Again, paths which return to $I_m$ at some point are excluded from $\mathcal{M}_{m}^{k,l}$. Taking all paths into account the coefficients in Eq. (24) read

\[ A_{m}^{(k,l)} = \sum_{\Gamma \in \mathcal{M}_{m}^{k,l}} \chi_{m}^{\Gamma} \]

(25)

and every path contributes with

\[ \lambda_{m}^{\Gamma} = \prod_{i=1}^{|\Gamma|} \frac{V_{t_i}}{H_0(I_m) - H_0(I_{m+\sum_{j=1}^{i} t_j})}, \]

(26)

where we define $V_{t_i} = V_t$ for $t \in \{\pm r, \pm s\}$. Inserting the obtained state $|\psi_{m}^{\text{pert}}\rangle$ of equation Eq. (24) into Eq. (12), and using the definition of the projection operator $P_\Lambda$, see Eq. (10), we find

\[ w_{m} = \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{|\mathcal{M}_{m}^{k,l}|} |A_{m}^{(k,l)}|^2. \]

(27)

In order to apply Eq. (27) we consider only the shortest paths $\Gamma$, which reach $\Lambda$ as they give rise to the lowest order contributions. Let $n$ be the length of these paths. For our choice of the resonance vectors $r$ and $s$ this restricts the relevant endpoints $I_{m+kr+is}$ to those, where $k+l = n$ for positive $k$ and $l$. Given $(k,l) \in \mathbb{N}^2$ the set $\mathcal{M}_{m}^{k,l}$ then contains $\binom{n}{k}$ of these shortest paths. Here, taking only the shortest paths into account, i.e. the lowest order of perturbation theory, gives excellent agreement with the weights obtained from numerical diagonalization. This is seen in Fig. 5 where the perturbative prediction of $w_0$, Eq. (27), shown as blue crosses, is compared with the numerically obtained weights. The perturbative description covers the overall exponential decay as well as the peaks and plateaus. Note that in general considering only the shortest paths may not be sufficient. One such example is the peak near $1/h = 0.5751$. As in the case of the single resonance the peaks arise, whenever an intermediate state along one or more perturbative paths is resonant with $I_m$. Furthermore, quantization jumps occur when the minimal length of paths, which reach $\Lambda$, increases by one.

FIG. 6. Perturbative prediction using Eq. (27) incorporating an increasing number of paths sorted by their absolute value of contribution $|\chi_{m}^{\Gamma}|$. Prediction for one path is shown as crosses, for two paths as circles, and for all paths as squares.
FIG. 7. Scheme of perturbation theory for the double resonance. The action grid $I_m$ is depicted as gray points, the quantizing torus $I_0 = (0, 0)$ as green point. The dashed lines show the position of the two resonance center line, see Eq. (3). The circle indicates the level set of constant energy $H_0(I_0)$. The blue shaded area shows the region $\Lambda$, see Eq. (21). In (a) for $1/h = 0.1991$ five paths with an resonant intermediate state and in (b) for $1/h = 0.1591$ two resonant paths with different intermediate state are shown. In Fig. 4(b, c) both states are shown in a Husimi representation. In (c) for $1/h = 0.1465$ two canceling paths are shown in (d) for $1/h = 0.2244$ six non-resonant paths are shown.

In contrast to the single resonance case there is a larger number of shortest paths which enter the prediction (27). As all these paths have the same length it is a priori not clear which of them will give the dominant contribution and which paths need to be taken into account to achieve the accuracy presented in Fig. 5. To this end we sort all contributing paths by their absolute value $|\lambda_{m}^{\Gamma}|$ in descending order. In Fig. 6 the perturbative result is shown for an increasing number of shortest paths $\Gamma$. Using only the path with the greatest absolute value for a given inverse Planck’s constant $1/h$ is shown as crosses. While the positions of the peaks are already resolved, the plateau-like structures show a mismatch of several orders of magnitude.

In general there is not necessarily one single dominating path, but several paths may give rise to similar contributions. This case is illustrated in Fig. 7(a) for $1/h = 0.1991$ where on the action grid five paths containing the resonant intermediate state $I_{(3,1)}$ fulfilling $H_0(I_0) = H_0(I_{(3,1)})$ are shown. All these paths contribute to $w_0$ within the same order of magnitude. Thus, considering only the most contributing path resolves the
position of the peaks while adding all resonant paths refines the prediction. The sixth most contributing path, however, gives rise to a contribution which is two orders of magnitude smaller since no intermediate state is located on the circular level set of constant energy $H_0(I_0)$. Therefore, excluding these nonresonant paths (if other resonant paths exist) does not affect the result.

In Fig. 7(b) two points of the action grid $I_{(2,0)}$ and $I_{(2,2)}$ are close to the level set of energy $H_0(I_0)$ for $1/\hbar = 0.1591$. Thus both paths with a resonant intermediate state are essential for an accurate prediction of the weight. This also nicely explains the different morphologies of the Husimi-distributions, see Fig. 4(b, c). Whereas in Fig. 4(c) the state $|\psi_0\rangle$ has a higher density in phase space at the position of both resonant intermediate states $I_{(2,0)}$ and $I_{(2,2)}$ in Fig. 7(b) only the intermediate state $I_{(3,1)}$ shows a higher density.

In contrast to the case where several paths with resonant intermediate states constructively interfere also the case of destructive interference of tunneling paths occurs. This is best seen in the perturbative prediction arising from two paths depicted in Fig. 5 as circles. For specific values of $1/\hbar$, e.g. around $1/\hbar = 0.1465$, a drastic decrease of $w_0$ can be observed. This happens if the two paths $\Gamma_1$ and $\Gamma_2$ under consideration fulfill $\lambda_m^{I_1} = -\lambda_m^{I_2}$. Fig. 7(c) illustrates this scenario, where two paths cancel each other. Taking only the two most contributing paths into account thus gives a qualitative description of the observed suppression of tunneling. Such mechanism is also known from 2D maps if several single resonances are present [9, 10]. The basic features, i.e. the overall exponential decay, the peaks, and the suppression of tunneling, are already captured if only two paths are taken into account. Whereas considering all shortest paths compensates the effect of destructive interference of paths as depicted in Fig. 5. Furthermore it emphasizes the necessity to consider all paths for a quantitatively accurate prediction of the plateau-like structures pronounced in the regime of large $1/\hbar$. Schematically the importance for considering all paths is depicted in Fig. 7(d) showing only the first six most contributing nonresonant paths for $1/\hbar = 0.2244$. By considering all paths the weight of plateaus-like structures can be quantitatively predicted.

Note that for arbitrary resonance vectors, higher orders of perturbation theory may be necessary for instance if there are paths with a resonant intermediate state which need an additional step to reach $\Lambda$. In other words, the shortest paths not necessarily lead to the largest contribution. If necessary these longer paths can be computed easily to refine the prediction. In the considered system including only the shortest path in the perturbative expression (27) allows for a successful prediction of the numerically obtained weight $w_0$, see Eq. (12), over several orders of magnitude.

![FIG. 8. Matrix model: (a) Weight $w_0$ as a function of $\hbar$ obtained via Eq. (29) shown as orange line and $w_{0,\text{pert}}$ calculated using Eq. (32) as blue line. (b) Energies $E_i$ (solid lines) of the states $|\psi_i\rangle$ as a function of $\hbar$ in black, red, green and gray for $i = 0, 1, 2, 3$. The vertical dashed lines indicate the values of $\hbar$ for the cases of enhancement and suppression.](image)

**C. $4 \times 4$ matrix model**

The underlying mechanism of the enhancement and suppression of the weight $w_m$ in the situation of a double resonance can be explained within a minimal $4 \times 4$ matrix model. To this end we consider four states $|I_i\rangle$ resembling the unperturbed action states $|I_m\rangle$ corresponding to the quantizing actions $I_0 = I_{(0,0)} = (0, 0), I_1 = I_{(1,0)} = (h, 0), I_2 = I_{(1,1)} = (h, h), I_3 = I_{(2,1)} = (2h, h)$. The unperturbed energies of these states are given by $E_i = H_0(I_i)$, see Eq. (2). We allow for couplings between these states according to Eq. (22) and Eq. (23) The matrix representation then reads

$$\hat{H}_{4 \times 4} = \begin{pmatrix} E_0 & V_r & V_s & 0 \\ V_r & E_1 & 0 & V_s \\ V_s & 0 & E_2 & V_r \\ 0 & V_s & V_r & E_3 \end{pmatrix}. \quad (28)$$

Diagonalization of $\hat{H}_{4 \times 4}$ for a fixed value of $\hbar$ yields the eigenstates $|\psi_i\rangle$ in the basis of action eigenstates. We choose for $H_0$ the parameters $M_{\text{res}} = 1$ and $I_{\text{res}} = (0.35, 0.55)$ and for the resonances $r = (1, 0)$ and $s = (1, 1)$ with corresponding coupling strengths $V_r = 0.0025$ and $V_s = 0.005$, respectively. As the perturbations are small the eigenstates will predominantly resemble one of the action states and are labeled accordingly. Thus,
Resonance-assisted tunneling between \( \mathbf{I}_0 \) and \( \mathbf{I}_3 \) can be quantified by the overlap

\[ w_0 = |\langle \psi_0 | \mathbf{I}_3 \rangle|^2 \tag{29} \]

between the state \( |\psi_0\rangle \) associated with \( \mathbf{I}_0 \) and the state \( |\mathbf{I}_3\rangle \). The states \( |\mathbf{I}_1\rangle \) and \( |\mathbf{I}_2\rangle \) deal as intermediate states. In Fig. 8(a) the weight \( w_0 \) is depicted as a function of \( \hbar \) as (orange) line. It resembles the basic features of resonance-assisted tunneling observed in the full system with a double resonance. In particular, the 4 \times 4 matrix models shows three peaks of enhancement as well as suppression in between the second and third peak.

In the following, we treat the 4 \times 4 matrix model also perturbatively. In order to compute the weight \( w_0 \), Eq. (29), there are only two paths to be considered. They connect \( \mathbf{I}_0 \) with \( \mathbf{I}_3 \) and differ by their intermediate step. Explicitly they are given by \( \Gamma_1 = [r, s] \) and \( \Gamma_2 = [s, r] \) and contribute with

\[
\lambda_1 = \frac{V_r}{E_0 - E_1} \frac{V_s}{E_0 - E_3}, \tag{30}
\]

\[
\lambda_2 = \frac{V_s}{E_0 - E_2} \frac{V_r}{E_0 - E_3}. \tag{31}
\]

The perturbatively computed weight is given by

\[ w_{0, \text{pert}} = |\lambda_1 + \lambda_2|^2 \tag{32} \]

and is shown as (blue line in Fig. 8(a)). They are in perfect agreement with the numerically obtained weights as long as the coupling terms are sufficiently small. The position of the peaks are given by the denominators of Eq. (30) and Eq. (31), whenever there is a degeneracy of the unperturbed state \( |\mathbf{I}_0\rangle \) with either one of the remaining three states. This is exactly the case for \( \hbar = 2I_{\text{res},1} \), \( \hbar = I_{\text{res},1} + I_{\text{res},2} \), and \( \hbar = 4I_{\text{res},1} + 2I_{\text{res},2} \). Instead, destructive interference occurs if \( \lambda_1 = -\lambda_2 \) which holds for \( \hbar = 4I_{\text{res},1} + 2I_{\text{res},2} \). The above values of \( \hbar \) are marked in Fig. 8 as black dashed lines.

Additionally, in Fig. 8(b) the energy levels \( \tilde{E} \) of the eigenstates \( |\psi_i\rangle \) are shown. The black line corresponds to \( \tilde{E}_0 \) and shows crossings with either the energies of the intermediate states (green and red) or the energy of the final state (gray). The positions of these avoided crossings match with the positions of the peaks of enhancement of the weight \( w_0 \). On the other hand in the case of suppression, the energy difference between \( \tilde{E}_0 \) and the energy of the two intermediate states is equal in size, but of opposite sign.

### III. SUMMARY AND OUTLOOK

In this paper we studied resonance-assisted tunneling by exploiting the universal description of the classical dynamics in the vicinity of a nonlinear resonance in terms of normal–form Hamiltonians. In particular we concentrated on single as well as double resonances in 4D normal–form Hamiltonians, where we visualized classical phase space in a suitable hyperplane. Numerical diagonalization of the quantized normal–form Hamiltonians yielded its eigenstates, whose Husimi representation allowed for the comparison with classical phase-space structures. There, we observed enhanced probability on classical tori located on opposite sides of the resonance channels. In particular, for specific values of the effective Planck’s constant, this effect becomes significantly enhanced due to resonance–assisted tunneling. We introduced the weight of an eigenstate localizing on one side of the resonance channel in a disjoint phase-space region as a quantitative measure of tunneling, which gives qualitatively the same behavior as, e.g., phase splittings or tunneling rates studied in 2D system. That is, we found an overall exponential decay as well as prominent peaks, where tunneling is enhanced over several orders of magnitude, and, in the case of the double resonance, situations where tunneling is suppressed. Such suppression is known from 2D systems as well, when multiple single resonances are involved, which typically is observed in the deep semiclassical regime for which also small resonance chains become important. This is different in 4D systems as there typically are double resonances, which dominate resonance–assisted tunneling even in the quantum regime of large \( \hbar \).

In order to predict the weight and understand the mechanism for the peaks and cases of suppression the use of the normal–form Hamiltonians allowed us to perform the perturbative calculation in both the single and the double resonance case. While, due to integrability, tunneling across the single resonance turns out to be effectively 2D the situation in the double resonance is more involved. This was visualized by introducing paths on the discrete action grid. There, for the single resonance just one path needs to be considered while for the double resonance multiple paths must be included to obtain an accurate description. In particular, this allows for destructive interference of different paths leading to the suppression of tunneling for specific values of \( \hbar \). In contrast, the mechanism causing the peaks is the same for both cases and corresponds with the one known from 2D systems. Crucial for enhancement in either case is the energetic degeneracy of action states with respect to the unperturbed Hamiltonian. Note that for different sets of resonance vectors even higher orders of perturbation theory may yield the dominant contribution in comparison with lower orders, i.e. shorter paths. For the concrete systems studied in this paper this was not the case and it was sufficient to take only the lowest order in terms of the shortest paths into account. Furthermore, we presented a minimal 4 \times 4 matrix model, which captures the observed features of resonance–assisted tunneling in 4D and allowed for a simplified explanation of the observed phenomena.

The perturbative description provides a first step towards a detailed understanding of resonance–assisted tunneling for general higher–dimensional systems. For a
universal description of a generic system, e.g. a 4D quantum map, several presently open problems have to be solved: For example, in phase space the relevant resonances for a given regime of $\hbar$ needs to be identified. For these resonances the parameters for the construction of the normal–form Hamiltonian have to be extracted. Furthermore for these local approximations of the resonances a canonical transformation to the phase space of the map needs to be constructed. The subsequent quantization would then allow for a quantitative prediction of the tunneling rates and resonance-assisted tunneling peaks. Beyond that the development of a semiclassical description only based on the classical properties of the nonlinear resonance would be desirable. Another phenomenon, occuring in at least 4D symplectic maps or 6D Hamiltonians is the famous Arnold diffusion. It provides a classical transport mechanism connecting different regions in phase space. Its interplay with tunneling is not clear at present.

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