Variational principle and energy–momentum tensor for relativistic Electrodynamics of point charges

K. Lechner$^1$ and P.A. Marchetti$^2$

Dipartimento di Fisica, Università degli Studi di Padova,
and
Istituto Nazionale di Fisica Nucleare, Sezione di Padova,
Via F. Marzolo, 8, 35131 Padova, Italia

Abstract

We give a new representation as tempered distribution for the energy–momentum tensor of a system of charged point–particles, which is free from divergent self–interactions, manifestly Lorentz–invariant and symmetric, and conserved. We present a covariant action for this system, that gives rise to the known Lorentz–Dirac equations for the particles and entails, via Noether theorem, this energy–momentum tensor. Our action is obtained from the standard action for classical Electrodynamics, by means of a new Lorentz–invariant regularization procedure, followed by a renormalization. The method introduced here extends naturally to charged $p$–branes and arbitrary dimensions.

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$^1$kurt.lechner@pd.infn.it
$^2$pieralberto.marchetti@pd.infn.it
1 Introduction and Summary

The dynamics of a classical charged point–particle interacting with an electromagnetic field is described by the Maxwell and Lorentz equations,

\[ \partial_\mu F^{\mu\nu} = j^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \]  
\[ \frac{dp^\mu}{ds} = e F^{\mu\nu}(y(s)) u_\nu(s). \]  

(1.1)  
(1.2)

We parametrize the worldline of the particle \( y^\mu(s) \) by the proper time \( s \), and define four–velocity and four–acceleration respectively by,

\[ u^\mu = \frac{dy^\mu}{ds}, \quad w^\mu = \frac{dw^\mu}{ds}. \]

The current is,

\[ j^\mu(x) = e \int u^\mu(s) \delta^4(x - y(s)) \, ds. \]  

(1.3)

Maxwell’s equations admit the general solution,

\[ F^{\mu\nu} = F^{\mu\nu}_{\text{LW}} + F^{\mu\nu}_{\text{in}}, \]

where \( F^{\mu\nu}_{\text{in}} \) is a free external field, and \( F^{\mu\nu}_{\text{LW}} \) amounts to the retarded Lienard–Wiechert (LW) field.

When substituting the total field strength into the Lorentz equation one faces the problem that the LW–field is infinite on the particle’s trajectory, and the quantity,

\[ F^{\mu\nu}_{\text{LW}}(y(s)), \]

(1.4)

diverges. This infinite self–interaction is, of course, a consequence of the point–like nature of the charged particle.

There are several methods of dealing with this divergence, all of them leading to the same conclusions: the quantity (1.4) has a divergent part, which leads to an infinite classical renormalization of the particle’s mass, and a finite part that amounts to,

\[ F^{\mu\nu}_{\text{LW}}(y(s)) \bigg|_{\text{finite}} = -\frac{e}{6\pi} \left( u^\mu \frac{dw^\nu}{ds} - w^\nu \frac{dw^\mu}{ds} \right). \]  

(1.5)

Substituting this into the Lorentz–equation one obtains the relativistic Lorentz–Dirac equation,

\[ \frac{dp^\mu}{ds} = \frac{e^2}{6\pi} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right) + e F^{\mu\nu}_{\text{in}}(y) u_\nu, \]  

(1.6)
which has to be considered as the equation of motion for the charged particle, replacing
the original ill–defined Lorentz–equation (1.2). Here, and in the following, we indicate
the square of a four–vector with $w^2 \equiv w^\mu w_\mu$. Strictly speaking this equation can not
be derived from the fundamental equations (1.1) and (1.2) of classical Electrodynamics
(ED), but has to be postulated.

In its relativistic form equation (1.6) has been derived first by Dirac [1] in 1938, in an
attempt to describe the motion of a classical relativistic charged point–particle, taking
into account its self–interaction. In this seminal paper the equation was derived using in
an essential way the principle of energy–momentum conservation. This conservation law
represents still the basic motivation for the equation itself.

Despite of the fact that (1.6) is a well–defined equation and ensures energy–momentum
conservation, it exhibits several unsatisfactory features, both physically and mathemat-
ically. The key physical drawbacks are that the equation is of the third order in the
time–derivative, and that it admits a class of solutions, so called runaway solutions – see
e.g. [2] – which lead to exponentially growing four–velocities, even in absence of external
forces. When these solutions are eliminated imposing suitable boundary conditions at in-
finity [3, 4], it turns out that the motions allowed by these conditions show up an acausal
behavior, in the sense of a preacceleration: the acceleration is felt before the external
force begins to act. Actually, this preacceleration occurs on a time scale of order,

$$\tau = \frac{e^2}{6\pi m} \sim 10^{-23} s,$$

which is of a factor 1/137 shorter than the time scale $\tau_q = \hbar/m$, where, due to the
Heisenberg uncertainty principle, quantum effects become relevant and the classical theory
has to be abandoned. Hence, physically one can interpret this acausal behavior as an
inconsistency of classical ED of point–particles, which is cured by the quantum formulation
of the theory, see e.g. [5].

Accepting (1.6) as the correct equation of motion for the particle, three questions

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3One can, actually, maintain a second order Lorentz–equation which is free from singularities, if one
introduces a finite cutoff of physical origin on the charge distribution of the would–be point–particle.
This is the philosophy pursued mainly in [6, 7] and references therein. The main unsatisfactory aspects
of this approach are its lack of universality, and the problems related with the implementation of exact
energy–momentum conservation. In the present paper we follow Dirac’s original proposal and insist on
point–like charges, and hence on equation (1.6).
arise automatically, to which we will give new answers in this paper. The first question regards local energy–momentum conservation, in terms of a well–defined conserved energy–momentum tensor. The second regards the existence of an action from which (1.6) can be derived, and the third question is if the energy–momentum tensor can be derived from this action via Noether’s theorem.

Regarding the first question we mention that, although relying on conservation criteria, in his paper [1] Dirac does not construct an energy–momentum tensor, nor does he give a recipe to compute the four–momentum enclosed in a volume $V$ containing the particle. The main obstruction to such a construction arises from the particular form of the naive energy–momentum tensor of the electromagnetic field,

$$
\tau_{\mu\nu}^{em} = F_{\mu\rho}^{LW} F_{\nu \rho}^{LW} + \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma}^{LW} F_{\rho\sigma},
$$

(1.7)

where for simplicity we have set the external field to zero. If we indicate with $R$ the distance from the particle at a certain instant, the fields $F_{\mu\nu}^{LW}$ exhibit near the particle the standard Coulomb–like integrable $1/R^2$ singularities, and hence $\tau_{\mu\nu}^{em}$ exhibits the non–integrable $1/R^4$ singularities. As a consequence the momentum integrals $P^\mu_V = \int_V d^3x \tau_{0\mu}^{em}$ diverge. In a mathematical language the problem can be stated as follows: while the components of $F_{\mu\nu}^{LW}$ are tempered distributions, i.e. elements of $S' \equiv S'(R^4)$, the components of $\tau_{\mu\nu}^{em}$, being products of the formers, are not.

To obtain a well–defined energy–momentum tensor one has to replace $\tau_{\mu\nu}^{em}$ with a tensor $T_{\mu\nu}^{em}$ whose components are elements of $S'$ such that, a) $T_{\mu\nu}^{em}$ coincides with $\tau_{\mu\nu}^{em}$ in the complement of the particle’s worldline, and b) such that the resulting total energy–momentum tensor is conserved, if (1.6) holds. Such a tensor admits then automatically finite four–momentum integrals. This program has been accomplished only rather recently by Rowe in [8], relying partially on [9]. His energy–momentum tensor is defined, rather implicitly, as the distributional derivative of the sum of other elements of $S'$, and entails the additional drawback that it is not manifestly symmetric and traceless. On the other hand it can be shown that the resulting tensor is uniquely determined by the requirements a) and b).

In this paper we present an alternative and simple representation of this tensor, relying on a new Lorentz–invariant regularization procedure, followed by a renormalization,
which leads to a manifestly symmetric, traceless and invariant tensor. In particular, our procedure follows the physical intuition that the renormalized tensor should differ from (1.7) only on the particle’s trajectory: correspondingly in our renormalization scheme we subtract only (divergent) terms which are supported on the world–line, i.e. terms that are proportional to delta–functions supported on the trajectory.

An alternative approach to four–momentum conservation, based on a different regularization procedure, has been proposed in [10]. This approach leads to finite four–momentum integrals, but it does not allow to construct an energy–momentum tensor. Moreover, it lacks manifest Lorentz invariance.

The second question opened by (1.6) is if this equation can be derived from an action principle. It is immediately seen that the resulting equation can not be derived from a canonical action. Indeed, such an action would be necessarily of the type,

\[ I = -\int (m + e A_{\text{in}}^\mu u_\mu + e^2 L_{\text{self}}) ds, \]

where \( L_{\text{self}} \) is an invariant function of the kinematical variables \( y^\mu, u^\mu, w^\mu \) etc, giving rise to the self–interactions at the r.h.s. of (1.6). But for dimensional reasons the unique term with the right dimensions would be the invariant \( L_{\text{self}} \propto u_\mu w^\mu \), and this is zero.

Since there exists no canonical action leading to (1.6), a variational principle giving rise to this equation bears necessarily some unconventional features. The first proposal for an action was made in [11], relying on the previous attempt of [1], and it requires the introduction of two gauge fields: a source–free field and a field generated by the particle via retarded and advanced potentials. As additional – and more serious – drawback this action contains non–local Coulomb interactions between the particles. The more recent attempt in [10] provides an action involving a single gauge field, but employs a regularization which is not manifestly Lorentz–invariant.

In this paper we present an action which is obtained from the standard action for ED, with a single gauge field, in a very simple and natural manner: we introduce a manifestly Lorentz–invariant regularization in the standard action, and renormalize it upon subtracting an infinite mass term. The so obtained action gives rise to (1.6) in the following way. First one varies the action with respect to the gauge field, obtaining a regularized version of the Maxwell equation. Then one varies the action with respect to
the coordinates of the particle and substitutes in the resulting equation the solution of
the regularized Maxwell equation. The limit of this equation, when the regulator goes to
zero, is (1.6).

The third question regards the derivation of the energy–momentum tensor through
Noether’s theorem, from a covariant action. As far as we know this problem has been
addressed only in [12], using a formalism introduced in [13]. This paper is based on a
manifestly Poincarè–invariant, but rather unconventional distribution–valued Lagrangian
density. Its most unusual characteristics are that the minimal interaction term $A^\mu j_\mu$ is
absent, and that the self–interaction of the particle is represented by a total derivative.
The most unsatisfactory feature of the resulting action is, however, that it does not give
rise to the equations of motion, i.e. (1.1) and (1.6). Eventually the role of the Lagrangian
of [12] is restricted to give rise, through translation invariance and Noether’s theorem, to
the conserved energy–momentum tensor given in [8], upon imposing the Lorentz–Dirac
equation (1.6) “by hand”.

The Lagrangian which we propose in the present paper, being in particular manifestly
Poincarè invariant, allows to apply Noether’s theorem in a canonical way, and the resulting
energy–momentum tensor arises precisely in the new form mentioned above.

The formalism used in this paper relies on the power of full–fledged distribution theory.
We remember, indeed, that in the case of point particles the energy–momentum tensor as
well as the Lagrangian density are necessarily distribution valued. The other key ingre-
dient is a manifestly Lorentz–invariant regularization procedure. Within this approach,
in summary, we can set up the entire lagrangian formalism for the ED of point parti-
cles as specified above: construction of a conserved and well–defined energy–momentum
tensor, construction of an action which gives to the correct equations of motion, and
implementation of the Noether–theorem, based on this action.

The method developed in this paper is based on the invariant regularization of the
Green function for the D’Alambertian in eq. (2.6), amounting to the replacement $x^2 \rightarrow
x^2 - \varepsilon^2$, that permits a clear and simple distinction between finite and divergent terms in
the energy–momentum tensor. Since the Green function in arbitrary dimensions depends
always on $x^2$, see [14], our method is very promising for the construction of finite and
conserved energy–momentum tensors for charged systems, for which this tensor has never
been constructed before, [15]. Examples include point particles in a curved background
[16], systems of dyons [17], point particles in higher dimensions [18], and extended ob-
jects in arbitrary dimensions [19]. The knowledge of such a tensor allows a systematic
quantitative analysis of radiation effects in these systems.

The plan of the paper is the following. In section two we introduce the regularization
used in the paper. In section three we present and illustrate our main results, i.e. the
new expression of the renormalized energy–momentum tensor together with its main
properties, the renormalized Lagrangian density giving rise to the equations of motion,
and the Noether–theorem. Section four is devoted to the proof of the properties of the
renormalized energy–momentum tensor, and to a comparison of this tensor with the one
proposed by Rowe [8]. The variational results are proved in section five. Some technical
details are relegated to appendices.

For simplicity we present our results in detail in the case of a single point particle
and for a vanishing external field $F_{\mu\nu}$. The corresponding results for the general case are
summarized in section (3.3).

## 2 Regularization

In Lorentz–gauge the Maxwell equations amount to,

$$\Box A^\mu = j^\mu, \quad \partial_\mu A^\mu = 0,$$

(2.1)

where for a point–particle with charge $e$ and worldline $y^\mu(s)$, the current is given in (1.3).
The solution of these equations can be obtained in terms of the retarded Green function
for the D’Alambertian, supported on the forward light cone,

$$G(x) = \frac{1}{2\pi} H(x^0) \delta(x^2) = \frac{1}{4\pi |\vec{x}|} \delta(x^0 - |\vec{x}|), \quad \Box G(x) = \delta^4(x),$$

(2.2)

where $x^\mu = (x^0, \vec{x})$ are the space–time coordinates, and $H$ denotes the Heaviside step
function. Omitting the external potential $A^\mu_{in}$ one gets as solution the retarded Lienard–
Wiechert potential,

$$A_{LW}^\mu = G \ast j^\mu = \frac{e}{4\pi (x_\nu - y_\nu(s)) u^\nu(s)} \bigg|_{s=s(x)},$$

(2.3)
where $*$ denotes convolution, and the scalar function $s(x)$ is uniquely fixed by the retarded time condition,
\[(x - y(s))^2 = 0, \quad x^0 > y^0(s),\] (2.4)
consequence of the $\delta$–function appearing in (2.2).

Along the worldline of the particle, i.e. for $x^\mu = y^\mu(\lambda)$ for some $\lambda$, the LW–potential $A_{LW}^\mu(y(\lambda))$ diverges. In the present notation this follows from $s(y(\lambda)) = \lambda$. As a consequence also the LW field–strength,
\[F_{LW}^{\mu\nu} = \partial^\mu A_{LW}^\nu - \partial^\nu A_{LW}^\mu,\] (2.5)
diverges along the particle worldline, meaning that a charged particle feels an infinite self–interaction.

To smooth these singularities, while maintaining the structure of the Maxwell equations unaltered, we propose to regularize the retarded Green function in a Lorentz–invariant way. Choosing an UV–regulator $\varepsilon > 0$, with the dimension of length, we introduce a regularized Green function [20],
\[G_\varepsilon(x) = \frac{1}{2\pi} H(x^0)\delta(x^2 - \varepsilon^2) \equiv \frac{1}{2\pi} \delta_+(x^2 - \varepsilon^2) = \frac{1}{4\pi \sqrt{|\mathbf{x}|^2 + \varepsilon^2}} \delta \left( x^0 - \sqrt{|\mathbf{x}|^2 + \varepsilon^2} \right),\] (2.6)
which is still an element of $\mathcal{S}'$. The support of $G_\varepsilon$ is given by the positive–time sheet of the hyperboloid,
\[x^2 = \varepsilon^2.\]
This regularization preserves therefore also causality.

We can now define a regularized LW potential,
\[A_\varepsilon^\mu \equiv G_\varepsilon * j^\mu = \frac{e}{4\pi} \frac{u^\mu(s)}{(x_\nu - y_\nu(s))u^\nu(s)} \mid_{s = s_\varepsilon(x)},\] (2.7)
which differs from (2.3) only through the fact that $s(x)$ is replaced by $s_\varepsilon(x)$, solution of the regularized retarded–time equation,
\[(x - y(s))^2 = \varepsilon^2, \quad x^0 > y^0(s).\] (2.8)
The potential $A_\varepsilon^\mu$ is a regular field, belonging to $C^\infty \equiv C^\infty(\mathbb{R}^4)$, and in particular it is regular on the support of $j^\mu$, i.e. on the worldline of the particle. Also the regularized
field strength,
\[ F_{\varepsilon}^{\mu\nu} \equiv \partial^\mu A_{\varepsilon}^\nu - \partial^\nu A_{\varepsilon}^\mu, \tag{2.9} \]
is an element of \( C^\infty \), and hence everywhere regular. We illustrate these properties giving the explicit formulae for a static particle, at rest in \( \vec{x} = 0 \),
\[
A_0^\varepsilon = \frac{e}{4\pi} \frac{1}{(|\vec{x}|^2 + \varepsilon^2)^{1/2}}, \quad \vec{A}_0 = 0, \quad \vec{E}_0 = \frac{e}{4\pi} \frac{\vec{x}}{(|\vec{x}|^2 + \varepsilon^2)^{3/2}}, \quad \vec{B}_0 = 0, \tag{2.10}
\]
which are all regular in \( \vec{x} = 0 \).

One of the advantages of this regularization is that it preserves the Lorentz–gauge. Indeed, due to the properties of the convolution and thanks to the fact that the current is conserved, the definition (2.7) gives directly,
\[ \partial_\mu A_\mu^\varepsilon = 0. \]
We can then define also a regularized and conserved current through,
\[
j_\nu^\varepsilon = \partial_\mu F_{\varepsilon}^{\mu\nu} = \square A_\nu^\varepsilon, \quad \partial_\nu j_\nu^\varepsilon = 0, \tag{2.11}\]
that belongs to \( C^\infty \), too.

Since the derivative is a continuous operation in \( S' \), it is clear that we have the following limits,
\[
S' - \lim_{\varepsilon \to 0} A_\mu^\varepsilon = A_{LW}^\mu, \tag{2.12}
\]
\[
S' - \lim_{\varepsilon \to 0} F_{\varepsilon}^{\mu\nu} = F_{LW}^{\mu\nu}, \tag{2.13}
\]
\[
S' - \lim_{\varepsilon \to 0} j_\mu^\varepsilon = j_\mu, \tag{2.14}
\]
where \( S' - \lim \) denotes the limit in the (weak) topology of \( S' \). Moreover, since by construction \( G_\varepsilon \) is Lorentz invariant, \( A_\mu^\varepsilon, F_{\varepsilon}^{\mu\nu} \), as well as \( j_\mu^\varepsilon \) are Lorentz covariant tensor fields.

We end this section writing more explicit expressions for \( F_{\varepsilon}^{\mu\nu} \) and \( j_\mu^\varepsilon \), since these will be used in the following. For this purpose we define the vector fields,
\[
R^\mu(x) \equiv x^\mu - y^\mu, \quad \Delta^\mu(x) \equiv (uR) w^\mu - (wR) u^\mu, \tag{2.15}
\]
\[\text{We remember that this means that these limits hold on every test function } \varphi \in S \equiv S(\mathbb{R}^4), \]
\[\lim_{\varepsilon \to 0} A_\mu^\varepsilon(\varphi) = A_{LW}^\mu(\varphi), \quad \text{etc.}\]
where the kinematical quantities \( y, u \) and \( w \) in these expressions are all evaluated at the proper time \( s_\varepsilon(x) \), determined from (2.8). In the following this functional dependence in the regularized quantities will always be understood. For the scalar product of two vectors we use the notation \( (ab) = a_\mu b^\mu \). In particular we have, see (2.8),

\[
R_\mu R^\mu = (R^0)^2 - |\vec{R}|^2 = \varepsilon^2. \tag{2.16}
\]

The expression for the regularized LW field strength can be derived in the same way as the known expression for the standard LW field, (2.5). To calculate it from (2.7) one needs the derivative of the function \( s_\varepsilon(x) \), that is obtained differentiating (2.16),

\[
\frac{\partial s_\varepsilon}{\partial x^\mu} = \frac{R_\mu}{(uR)}. \tag{2.17}
\]

The rest of the calculations is a bit lengthy but straightforward. With the above conventions the results read,

\[
A^\mu_\varepsilon = \frac{e}{4\pi (uR)} u^\mu, \tag{2.18}
\]

\[
F^{\mu\nu}_\varepsilon = \frac{e}{4\pi (uR)^3} \left( R_\mu u^\nu - R^\nu u^\mu + R^\mu \Delta^\nu - R^\nu \Delta^\mu \right) , \tag{2.19}
\]

where we defined,

\[
b^\mu = \frac{dw^\mu}{ds}. \tag{2.20}
\]

The formula (2.18) coincides, actually, with the known expression of the LW field strength \( F^{\mu\nu}_{LW} \) – see for example [4] – the unique difference being that the kinematical quantities are evaluated at the retarded proper time \( s_\varepsilon(x) \), rather then at \( s(x) \).

In the following we will use an asymptotic condition on the worldlines \( y^\mu(s) \) of the particle. We suppose that the particle’s acceleration is zero before a certain instant \( s = \bar{s} \),

\[
w^\mu(s) = 0, \quad \text{for } s < \bar{s}. \tag{2.21}
\]

Since we have in any case \( \lim_{|\vec{x}| \to \infty} s_\varepsilon(x^0, \vec{x}) = -\infty \), this asymptotic condition implies that at fixed time the acceleration \( w^\mu(s_\varepsilon(x)) \) in (2.18) vanishes for sufficiently large \( |\vec{x}| \).

\(^5\) Actually, the results we obtain in this paper are valid also if the acceleration vanishes sufficiently fast for \( s \to -\infty \).
same holds for the corresponding acceleration $w^\mu(s(x))$ in the unregularized field $F_{W_L}^{\mu\nu}$.

This implies that the regularized and unregularized LW–fields have the same fixed–time Coulomb–like asymptotic behavior for large $|\vec{x}|$,

$$F_{\vec{e}}^{\mu\nu} \sim \frac{1}{|\vec{x}|^2}, \quad F_{\text{LW}}^{\mu\nu} \sim \frac{1}{|\vec{x}|^2}. \quad (2.22)$$

These behaviors, that are valid also if the acceleration vanishes sufficiently fast in the remote past, will ensure in particular that the total four–momentum of the electromagnetic field is finite.

## 3 Main results

### 3.1 The energy–momentum tensor

We begin with the construction of a consistent energy–momentum tensor, accepting as equation of motion (1.6).

As explained in the introduction the naive energy–momentum tensor $\tau_{\text{em}}^{\mu\nu}$ in (1.7) is not an element of $S'$—although $F_{\text{LW}}^{\mu\nu}$ is—due its $1/|\vec{R}|^4$ behavior near the worldline. Therefore its derivatives do not even make sense, and the question what the quantity $\partial_\mu \tau_{\text{em}}^{\mu\nu}$ amounts to, is meaningless.

To construct an energy–momentum tensor that is an element of $S'$ we start from the regularized LW field (2.18), and define the regularized electromagnetic energy–momentum tensor as,

$$T_{\vec{e}}^{\mu\nu} = F_{\vec{e}}^{\mu\rho} F_{\vec{e} \rho}^{\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\vec{e}}^{\rho\sigma} F_{\vec{e} \rho \sigma}. \quad (3.1)$$

This tensor belongs to $C^\infty$ and shares with $\tau_{\text{em}}^{\mu\nu}$ the asymptotic behavior for large $|\vec{x}|$ at fixed time, implied by (2.22),

$$T_{\vec{e}}^{\mu\nu} \sim \frac{1}{|\vec{x}|^4}, \quad \tau_{\text{em}}^{\mu\nu} \sim \frac{1}{|\vec{x}|^4}. \quad (3.2)$$

---

6One can evaluate the quantity $|\vec{R}(t, \vec{x})| = |\vec{x} - \vec{y}(s(x))|$ at a point near the worldline, $\vec{x} \approx \vec{y}(t)$, solving (2.4). One obtains,

$$|\vec{R}(t, \vec{x})| = \left( \frac{v \cos \alpha + \sqrt{1 - v^2 \sin^2 \alpha}}{1 - v^2} \right) |\vec{x} - \vec{y}(t)| + o \left( |\vec{x} - \vec{y}(t)|^2 \right),$$

where $\vec{v}$ is the velocity at $t$, and $\alpha$ is the angle between $\vec{v}$ and $\vec{x} - \vec{y}(t)$. Therefore, near the worldline $|\vec{R}|$ represents indeed the spatial distance from the particle, a part from a never vanishing constant.
Moreover, away from the worldline, i.e. for $\vec{x} \neq \vec{y}(t)$, we have the pointwise limit,

$$\lim_{\epsilon \to 0} T_{\epsilon \nu}^\mu(x) = \tau_{\epsilon \nu}^\mu(x).$$

(3.3)

But this limit does not exist in the topology of $S'$, due to the reemerging singularities of $\tau_{\epsilon \nu}^\mu$ along the worldline. Before taking the $S'$-limit one must isolate, and subtract, these singular terms.

In the present framework the form of these singular terms – and this is one more advantage of our regularization – is extremely simple. Since the singularities appear only along the wordline, the counterterm must be covariant, symmetric, traceless and supported along the wordline. These requirements fix it essentially uniquely, a part from an overall constant. We propose as renormalized energy–momentum tensor for the electromagnetic field generated by a point–particle, the expression,

$$T_{\epsilon \nu}^\mu \equiv S' - \lim_{\epsilon \to 0} \left[ T_{\epsilon \nu}^\mu - \frac{e^2}{32 \epsilon} \int \left( u^\mu u^\nu - \frac{1}{4} \eta^\mu\nu \right) \delta^4(x - y(s)) \, ds \right] = S' - \lim_{\epsilon \to 0} \tilde{T}_{\epsilon \nu}^\mu.$$

(3.4)

The first counterterm, proportional to $u^\mu u^\nu$, can be viewed as an electromagnetic mass renormalization and is well known, see [21]. The second counterterm, proportional to $\eta^\mu\nu$, to our knowledge has never been noticed before in the literature, probably because it can not be interpreted as a mass term and amounts, therefore, to a type of singularity that is not present in the Lorentz–equation

7. This term is, however, necessarily present because the divergent part of a traceless tensor must be traceless.

The properties of the tensor (3.4) are indeed summarized by the following

**Theorem I**

1) The limit in (3.4) exists and hence $T_{\epsilon \nu}^\mu \in S'$.

2) $T_{\epsilon \nu}^\mu$ is a Lorentz–covariant, symmetric and traceless tensor field.

3) $T_{\epsilon \nu}^\mu(x) = \tau_{\epsilon \nu}^\mu(x)$ for $x$ in the complement of the particle’s wordline.

4) The four–divergence of $T_{\epsilon \nu}^\mu$ amounts to,

$$\partial^\mu T_{\epsilon \nu}^\mu = -\frac{e^2}{6\pi} \int \frac{dw^\nu}{ds} \left( u^\mu + w^\mu \right) \delta^4(x - y(s)) \, ds.$$  

(3.5)

7The technical reason for the omission of this term stems from the fact that in the literature the momentum conservation equation is derived integrating the formal (divergent) energy–momentum tensor $\tau_{\epsilon \nu}^\mu$ over a restricted class of spacelike surfaces, that are orthogonal to the four velocity $u^\mu$. In this way one looses a divergent term proportional to $\frac{1}{\epsilon} \int (u^\mu u^\nu - \eta^\mu\nu) \delta^4(x - y(s)) \, ds$. 

11
for an arbitrary smooth worldline \( y^\mu(s) \).

5) If one defines the energy momentum tensor of the particle in a standard way as,

\[
T^\mu_\nu^p = m \int u^\mu u^\nu \delta^4(x - y(s))ds,
\]

then the total energy–momentum tensor of the ED of a point–particle,

\[
T^\mu_\nu = T^\mu_\nu^\text{em} + T^\mu_\nu^p,
\]

is conserved, \( \partial_\mu T^\mu_\nu = 0 \), if the particle satisfies the Lorentz–Dirac equation (1.6), with \( F^\mu_\nu = 0 \).

The crucial points to prove are the properties 1) and 4), i.e. the existence of the limit (3.4) and the evaluation of \( \partial_\mu T^\mu_\nu^\text{em} \); we relegate these proofs to section 4.

The limit in (3.4) holds, actually, also in a stronger sense, i.e. on a set of test functions that is larger than \( \mathcal{S} \). Indeed, since for a fixed time \( x^0 = t \), for \( \varepsilon \to 0 \) the tensor \( T^\mu_\nu_\varepsilon(t, \vec{x}) \) develops only a singularity in three–space at the point \( \vec{x} = \vec{y}(t) \), and due to the asymptotic behavior (3.2), the limit (3.4) exists also on “pseudo–test functions” of the form,

\[
\varphi(x) = \delta(x^0 - t) \varphi(\vec{x}),
\]

where \( \varphi(\vec{x}) \) is bounded in \( \mathbb{R}^3 \), and of class \( C^\infty \) in a neighborhood of \( \vec{y}(t) \). For example, \( \varphi(\vec{x}) \) can be a constant, or a characteristic function on a three–volume \( V \). We will take advantage from this fact when considering momentum integrals.

Property 2) holds by construction, and property 3) follows from the fact that the counterterm in (3.4) is supported entirely on the worldline. This property has its physical origin in the fact that the form of \( \tau^\mu_\nu_\text{em} \) off the worldline is regular, and its phenomenological consequences away from the particle, like all classical radiation phenomena, are experimentally very well tested. We were therefore only allowed to change the form of \( \tau^\mu_\nu_\text{em} \) on the worldline. Property 5) follows from property 4), using (1.6).

3.1.1 Momentum integrals

The main purpose of this subsection is to prove, using the above theorem, that the total four–momentum of the electromagnetic field generated by a point–particle at a generic
instant $t$ is given by formula (3.16), as shown first in [8]. In the course of the proof we will illustrate how the new representation (3.4) can be applied in practice.

Formula (3.4) entails indeed an operative definition for the four–momentum of the electromagnetic field, $P^\mu(t)$, contained at the instant $x^0 = t$ in a volume $V$. We define this momentum applying $T^\mu_{em0}$ to the pseudo–test function,

$$\varphi(x) = \delta(x^0 - t)\chi_V(\vec{x}),$$

where $\chi_V(\vec{x})$ is the characteristic function on $V$. By definition we have,

$$P^\mu(t) \equiv T^\mu_{em0}(\varphi) = \lim_{\varepsilon \to 0} \tilde{T}^\mu_{\varepsilon0}(\varphi) = \lim_{\varepsilon \to 0} \int_V \tilde{T}^\mu_{\varepsilon0}(t, \vec{x}) d^3x.$$  \hspace{1cm} (3.10)

This four–momentum is well–defined whenever $\vec{y}(t) \notin \partial V$.

If at the time $t$ the particle is outside $V$, the counterterm in (3.4) does not contribute, and thanks to (3.3) the above definition reduces to,

$$P^\mu(t) = \int_V \tau^\mu_{em0}(t, \vec{x}) d^3x,$$

coinciding with the standard expression.

We illustrate the operative definition (3.10), computing the four–momentum $P^\mu(t)$ for a particle in uniform motion, the result being in particular useful for the derivation of (3.16). Due to Lorentz–invariance it is sufficient to consider a static particle in the origin. From (3.10) we see that we have two equivalent ways to evaluate $P^\mu(t)$: we can evaluate $T^\mu_{em0}(\varphi)$ on a generic pseudo–test function $\varphi$, and then set $\varphi(x) = \delta(x^0 - t)\chi_V(\vec{x})$, or we can evaluate the integral $\int_V \tilde{T}^\mu_{\varepsilon0}(t, \vec{x}) d^3x$, and then take the limit $\varepsilon \to 0$. For a static particle we are able to carry out the first, more ambitious, procedure.

Consider first the energy density. Inserting (2.10) in (3.1) one obtains the regularized energy density,

$$T^0\varepsilon_\varepsilon = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \frac{r^2}{(r^2 + \varepsilon^2)^3}, \quad r \equiv |\vec{x}|,$$

and the 00 component of (3.4) reduces to,

$$T^0_{em0} \equiv S' - \lim_{\varepsilon \to 0} \left( \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \frac{r^2}{(r^2 + \varepsilon^2)^3} - \frac{3e^2}{128\varepsilon} \delta^3(\vec{x}) \right).$$

13
Applying to a generic pseudo–test function $\varphi$ we get,

$$T^{00}_{em}(\varphi) = \lim_{\varepsilon \to 0} \left[ \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{r^2 \varphi(x^0, \vec{x})}{(r^2 + \varepsilon^2)^3} d^4x - \frac{3e^2}{128\varepsilon} \int \varphi(x^0, \vec{0}) dx^0 \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{r^2 (\varphi(x^0, \vec{x}) - \varphi(x^0, \vec{0}))}{(r^2 + \varepsilon^2)^3} d^4x$$

$$= \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{\varphi(x^0, \vec{x}) - \varphi(x^0, \vec{0})}{r^4} d^4x,$$

(3.11)

where we used,

$$\frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{r^2}{(r^2 + \varepsilon^2)^3} d^3x = \frac{3e^2}{128\varepsilon}.$$

The integral in (3.11) is conditionally convergent, meaning that one has first to integrate over angles, and then over $|\vec{x}|$ and $x^0$. Inserting (3.9) we get for the energy contained in $V$ the $t$–independent result,

$$P^0_V = T^{00}_{em}(\varphi) = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{\chi_V(\vec{x}) - \chi_V(0)}{r^4} d^3x.$$

If $V$ does not contain the origin, i.e. the particle, one has $\chi_V(0) = 0$ and this formula reduces to the standard electrostatic energy. If, on the other hand, $V$ is a sphere with radius $\rho$ centered in the origin one has $\chi_V(0) = 1$, and the energy in the sphere amounts to $P^0_V = -\frac{e^2}{8\pi\rho}$. In particular the total energy in whole space is zero.

The spatial momentum $P^i_V(t)$ for a static particle vanishes trivially, since the $0i$ components of both terms in (3.4) are zero, even before taking $\varepsilon \to 0$. Thanks to Lorentz–invariance this means that the total four–momentum of the electromagnetic field generated by a charged particle in uniform motion is zero,

$$P^\mu(t) = 0.$$

(3.12)

We consider now the total four–momentum for a particle in arbitrary motion,

$$P^\mu(t) = \lim_{\varepsilon \to 0} \int \tilde{T}^{\mu0}_\varepsilon (t, \vec{x}) d^3x,$$

where, we remember, the convergence of the integral for large $|\vec{x}|$ is ensured by the asymptotic behavior (3.2). Since for $s < \bar{s}$ the particle is in uniform motion, we have the additional information that,

$$P^\mu(t) = 0, \quad \text{for } t < \bar{t},$$

(3.13)
thanks to (3.12), where $\bar{t} = y^0(\bar{s})$.

To derive an explicit expression for $P^\mu(t)$ we use property 4) of the Theorem I. Define the vector field,

$$K^\mu_\varepsilon = \partial_\nu \bar{T}^{\nu\mu}_\varepsilon. \quad (3.14)$$

Since the derivative is a continuous operation, (3.4) and (3.5) imply,

$$S' - \lim_{\varepsilon \to 0} K^\mu_\varepsilon = -\frac{e^2}{6\pi} \int \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right) \delta^4(x - y(s)) ds, \quad (3.15)$$

where, again, this limit holds also on the pseudo–test functions (3.8). Integrating (3.14) over whole three–space we have,

$$\partial_0 \int \bar{T}^{0\mu}_\varepsilon(t, \vec{x}) d^3x + \int \partial_i \bar{T}^{i\mu}_\varepsilon(t, \vec{x}) d^3x = \int K^\mu_\varepsilon(t, \vec{x}) d^3x.$$

Using the three–dimensional Gauss theorem and (3.2), the second term on the l.h.s. is zero. Taking then the limit $\varepsilon \to 0$ and using (3.15) one obtains,

$$\frac{dP^\mu(s)}{ds} = -\frac{e^2}{6\pi} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right),$$

where we used the variable $s$ instead of $t = y^0(\bar{s})$. Integrating this expression from $-\infty$ to $s$, and using the asymptotic relations (2.21) and (3.13), we get for the total four–momentum of the electromagnetic field,

$$P^\mu(s) = -\frac{e^2}{6\pi} \left( w^\mu(s) + \int_{-\infty}^{s} w^2(\lambda) u^\mu(\lambda) d\lambda \right), \quad (3.16)$$

reproducing the result of [8].

In appendix C we will actually prove that the more implicit expression for $T^{\mu\nu}_{em}$ given by Rowe in [8], defines the same distribution as (3.4).

The formula (3.16) is clearly in agreement with total four–momentum conservation, $P^\mu(s) + p^\mu(s) = \text{constant}$, see (1.6).

3.2 Covariant variational principle

The purpose of Theorem I was the construction of a well–defined energy–momentum tensor. We deduce now the equations of motion and the so constructed energy–momentum tensor – via Noether theorem – from an action principle.
We propose the following regularized Lagrangian density \( \mathcal{L}_\epsilon \equiv \mathcal{L}_\epsilon(A, y) \),

\[
\mathcal{L}_\epsilon = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu_{\epsilon} A_\mu - m_\epsilon \int \delta^4(x - y(s)) \, ds, \tag{3.17}
\]

that is an element of \( S' \) if \( A^\mu \) is sufficiently regular. This Lagrangian differs from the standard Lagrangian for ED by two conceptually simple ingredients. First, the current \( j^\mu \) has been replaced by its regularized counterpart \( j^\mu_{\epsilon} \), see (2.11) and (2.19). Second, we introduced a diverging counterterm for the mass. One can indeed interpret,

\[
m_\epsilon \equiv m - \frac{3\pi^2}{8\epsilon} \left( \frac{e}{4\pi} \right)^2, \tag{3.18}
\]
as the bare mass of the regularized theory, whereas \( m \) is the physical renormalized mass.

We define the regularized action associated to this Lagrangian in a standard way,

\[
I_\epsilon[A, y] = \int \mathcal{L}_\epsilon \, d^4x = -\int \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu_{\epsilon} A_\mu \right) d^4x - m_\epsilon \int ds. \tag{3.19}
\]

Notice that this action is gauge invariant, because the current \( j^\mu_{\epsilon} \) is identically conserved. Moreover, it gives rise to the equations of motion (1.1) and (1.6) according to the following

**Theorem II**

1) The equations of motion for \( A^\mu \) obtained from \( I_\epsilon \) are the regularized Maxwell equations,

\[
\partial_\mu F^{\mu\nu} = j^\nu_{\epsilon}. \tag{3.20}
\]

These equations entail as solution the regularized LW potential \( A^\mu_{\epsilon} \) in (2.17).

2) Consider the equations of motion for \( y^\mu(s) \) derived from \( I_\epsilon \),

\[
L^\mu_{\epsilon}[A, y](s) \equiv \frac{\delta I_\epsilon[A, y]}{\delta y_\mu(s)} = 0.
\]

If one substitutes for \( A^\mu \) the solution \( A^\mu_{\epsilon} \) of (3.20), one has the point–wise limit,

\[
\lim_{\epsilon \to 0} L^\mu_{\epsilon}[A_\epsilon, y](s) = \frac{dp^\mu}{ds} - \frac{e^2}{6\pi} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right), \tag{3.21}
\]
corresponding to the Lorentz–Dirac equation.

Property 1) is obvious. The statement 2) means that one has first to derive the equation of motion for \( y^\mu \) from \( I_\epsilon \), then one must substitute in this equation the regularized LW–field potential \( A^\mu_{\epsilon} \), and eventually perform the limit \( \epsilon \to 0 \). This is *not* equivalent to
substituting \( A_\varepsilon^\mu \) in (3.19) and then deriving the equation of motion for \( y^\mu \) and taking the limit \( \varepsilon \to 0 \). This second procedure would give rise to the equation of motion \( \frac{dp^\mu}{ds} = 0 \), see Theorem III. The reason for the failure of this procedure is that there exists no canonical action giving rise to (1.6), as explained in the introduction.

Since we are not interested in the explicit form of \( L^\mu_\varepsilon \), to prove property 2) it is sufficient to vary \( I_\varepsilon \) w.r.t. \( y^\mu \), set \( A^\mu = A^\mu_\varepsilon \) and then take the limit \( \varepsilon \to 0 \). The equation (3.21) is therefore equivalent to the relation,
\[
\lim_{\varepsilon \to 0} \left. \delta I_\varepsilon[A, y] \right|_{A=A_\varepsilon} = \lim_{\varepsilon \to 0} \left( -\delta \left( m_\varepsilon \int ds \right) - \int A_{\varepsilon \mu} \delta j_\varepsilon^\mu d^4x \right)
\[
= \left( \frac{dp^\mu}{ds} - \frac{e^2}{6\pi} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right) \right) \delta y_\mu ds,
\]
(3.22)
where \( \delta \) means variation w.r.t. \( y^\mu \). The proof of this relation will be given in section 5.1.

The regularity properties of the regularized Lagrangian itself are expressed by

**Theorem III**

*On the solution \( A_\varepsilon^\mu \) the regularized Lagrangian converges for \( \varepsilon \to 0 \),
\[
\mathcal{S}' - \lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon(A_\varepsilon, y) \equiv \mathcal{L} \in \mathcal{S}',
\]
where, for a generic \( \varphi \in \mathcal{S} \),
\[
\mathcal{L}(\varphi) = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int \frac{\varphi(x) - \varphi(y(s(x)))}{(uR)^4} d^4x - m \int \varphi(y(s)) ds.
\]
(3.23)
In this expression the regulator has been everywhere removed.*

The first integral in (3.23) is conditionally convergent, as (3.11). The regularized lagrangian admits therefore a finite limit on the solutions of (3.20). Notice in particular that the divergent contribution of the bare mass in (3.19) canceled out. Along the solutions \( A_\varepsilon^\mu \), also the action \( I_\varepsilon \) itself admits a finite limit. It is obtained evaluating the limit Lagrangian (3.23) on the pseudo–test function \( \varphi(x) = 1 \),
\[
\lim_{\varepsilon \to 0} I_\varepsilon[A_\varepsilon, y] = -m \int ds.
\]
The proof of this theorem is given in section 5.2.

As last result we state the Noether theorem.
Theorem IV

The invariance of the distribution \( \mathcal{L}_\varepsilon \) under space–time translations, i.e. the identity,

\[
\delta_t \mathcal{L}_\varepsilon \equiv \delta A^\mu \frac{\delta \mathcal{L}_\varepsilon}{\delta A^\mu} - \delta y^\mu \frac{\delta \mathcal{L}_\varepsilon}{\delta y^\mu} - a^\mu \partial_\mu \mathcal{L}_\varepsilon = 0, \tag{3.24}
\]

with \( \delta y^\mu = a^\mu \), \( \delta A^\mu = a^\nu \partial_\nu A^\mu \), if evaluated on the solutions of the equations of motion (3.20) and (1.6), implies for \( \varepsilon \to 0 \) the conservation law \( \partial_\mu T^{\mu\nu} = 0 \), where \( T^{\mu\nu} \) is defined in (3.4), (3.6) and (3.7). More precisely, on the solution of the equations of motion one has the relation,

\[
\mathcal{S}' - \lim_{\varepsilon \to 0} \delta_t \mathcal{L}_\varepsilon = a_\nu \partial_\mu T^{\mu\nu}. \tag{3.25}
\]

The significance of this theorem is clear. We want only specify the following conceptual point. Since \( \mathcal{L}_\varepsilon \) is an element of \( \mathcal{S}' \) the variation in (3.24) is defined in the distributional sense. This means that, by definition, we have,

\[
(\delta_t \mathcal{L}_\varepsilon)(\varphi) = \int \mathcal{L}_\varepsilon(x + a)\varphi(x + a) \, d^4x - \int \mathcal{L}_\varepsilon(x)\varphi(x) \, d^4x = a^\mu \int \mathcal{L}_\varepsilon \partial_\mu \varphi \, d^4x + \hat{\delta} \int \mathcal{L}_\varepsilon \varphi \, d^4x, \tag{3.26}
\]

where \( \hat{\delta} \) indicates the variation of the integral w.r.t. \( A \), minus the variation of the integral w.r.t. \( y \). This is due to the fact that the explicit dependence on \( x \) of \( \mathcal{L}_\varepsilon \) is through the difference \( x^\mu - y^\mu(s) \) \(^8\). The Proof of Theorem IV will be given in section 5.3.

3.3 The general case

The results presented so far generalize easily to a generic set of charged point particles with masses \( m_r \) and charges \( e_r \), and to the presence of an external field \( F^{\mu\nu}_{in} \). In this case the total regularized electromagnetic field is given by,

\[
F^{\mu\nu}_\varepsilon = F^{\mu\nu}_{in} + \sum_r F^{\mu\nu}_{(r)\varepsilon}, \tag{3.27}
\]

\(^8\)In the present case the definition (3.26) needs only to be applied to the term multiplying \( m_\varepsilon \) in (3.17) – because all other terms are regular distributions – in which case (3.26) gives, translated back to symbolic notation,

\[
\delta_t \left( \int \delta^4(x - y(s)) \, ds \right) = -a^\mu \partial_\mu \left( \int \delta^4(x - y(s)) \, ds \right) - \int \delta y^\mu \frac{\partial}{\partial y^\mu(s)} \delta^4(x - y(s)) \, ds,
\]

that vanishes identically. In practice we can then always use directly (3.24).
where $F_{(r)}^{\mu\nu}$ is the regularized LW field produced by the $r$–th particle. The electromagnetic energy–momentum tensor, generalizing naturally (3.4), becomes,

$$T_{em}^{\mu\nu} \equiv S' - \lim_{\varepsilon \to 0} \left[ T_{\varepsilon}^{\mu\nu} - \frac{1}{32 \varepsilon} \sum_r e_r^2 \int \left( u_r^\mu u_r^\nu - \frac{1}{4} \eta^\mu\nu \right) \delta^4(x - y_r(s_r)) ds_r \right],$$

(3.28)

with $T_{\varepsilon}^{\mu\nu}$ given by (3.1) and (3.27). There are no divergent counterterms due to the new interactions, because the off–diagonal terms in $T_{\varepsilon}^{\mu\nu}$ are well defined distributions, with only $1/R^2$ singularities near the worldlines, and because $F_{(s)}^{\mu\nu}$ in is supposed to be regular.

The four–divergence of this tensor is,

$$\partial_\mu T_{em}^{\mu\nu} = - \sum_r e_r \int \left[ \frac{e_r^2}{6\pi} \left( \frac{du_r^\mu}{ds_r} + u_r^2 u_r^\nu \right) + \left( F_{in}^{\mu\nu}(y_r) + \sum_{s \neq r} F_{(s)}^{\mu\nu}(y_r) \right) u_r^\mu \right] \delta^4(x - y_r) ds_r,$$

where $F_{(s)}^{\mu\nu}$ is the unregularized LW field of the $s$–th particle. This relation follows directly from (3.5), since the off–diagonal terms in $T_{\varepsilon}^{\mu\nu}$ are free from overlapping divergences, and on them one can simply apply the Leibnitz rule to evaluate the four–divergence. The total energy–momentum tensor $T_{em}^{\mu\nu} + T_p^{\mu\nu}$, with $T_p^{\mu\nu} = \sum_r m_r \int u_r^\mu u_r^\nu \delta^4(x - y_r(s_r)) ds_r$, is then conserved if the particles satisfy the Lorentz–Dirac equations,

$$\frac{dp_r^\mu}{ds} = e_r \frac{e_r^2}{6\pi} \left( \frac{du_r^\mu}{ds} + u_r^2 u_r^\mu \right) + e_r \left( F_{in}^{\mu\nu}(y_r) + \sum_{s \neq r} F_{(s)}^{\mu\nu}(y_r) \right) u_r^\nu. \quad (3.29)$$

Eventually also the regularized Lagrangian density admits the natural generalization,

$$L_{\varepsilon} = -\frac{1}{4} F_\mu F^{\mu\nu} - j_\varepsilon^\mu A_\mu - \sum_r \left( m_r - \frac{3 e_r^2}{128 \varepsilon} \right) \int \delta^4(x - y_r(s_r)) ds_r,$$

where $j_\varepsilon^\mu$ is the sum of the regularized currents of the particles. This Lagrangian gives rise to the Lorentz–Dirac equations (3.29), and entails via Noether theorem the conserved energy–momentum tensor $T_{em}^{\mu\nu} + T_p^{\mu\nu}$. Along the equations of motion $L_{\varepsilon}$ admits again a finite limit for $\varepsilon \to 0$ in the topology of $S'$.

### 4 Energy–momentum tensor

This section is devoted mainly to the proof of Theorem I. The ingredients needed are the same as those needed in the proofs of the other theorems. For this reason we present here the necessary technical tools in some detail, while we will omit them in the proofs for the other theorems.
4.1 Existence of the tensor $T_{\varepsilon}^{\mu\nu}$

We start with the proof of the existence of the limit (3.4), meaning that we have to isolate in $T_{\varepsilon}^{\mu\nu}$ the terms that diverge for $\varepsilon \to 0$, if applied to test functions $\varphi \in S$. Inserting (2.18) in (3.1) we can rewrite the regularized energy–momentum tensor as sum of three terms, characterized respectively by the inverse powers $1/R^4$, $1/R^3$ and $1/R^2$,

$$T_{\varepsilon}^{\mu\nu} = \sum_{i=1}^{3} T_{\varepsilon i}^{\mu\nu},$$

where we count the dimensionful regulator $\varepsilon$ as a power of $R$, since we have $R^2 = \varepsilon^2$, see (2.16). The explicit expressions are,

$$T_{\varepsilon 1}^{\mu\nu} = \left(\frac{e}{4\pi}\right)^2 \frac{1}{(uR)^6} \left( -R^{\mu}\ R^{\nu} + 2(uR)^{\mu\nu} - \varepsilon^2 u^{\mu} u^{\nu} \right) - \text{tr},$$

$$T_{\varepsilon 2}^{\mu\nu} = \left(\frac{e}{4\pi}\right)^2 \frac{1}{(uR)^6} \left( 2(wR)^{\mu\nu} + 2(uR)^{\mu\Delta^{\nu}} - 2\varepsilon^2 u^{\mu\Delta^{\nu}} \right) - \text{tr},$$

$$T_{\varepsilon 3}^{\mu\nu} = \left(\frac{e}{4\pi}\right)^2 \frac{1}{(uR)^6} \left( -\Delta^2 R^{\mu}\ R^{\nu} - \varepsilon^2 \Delta^{\mu}\Delta^{\nu} \right) - \text{tr},$$

where $-\text{tr}$ means the subtraction of the trace of the tensor, and we defined symmetrization by $a^{\mu\nu} = \frac{1}{2}(a^{\mu\nu} + a^{\nu\mu})$.

We will now show that the divergent parts of these tensors, for $\varepsilon \to 0$ in the topology of $S'$, amount to,

$$T_{\varepsilon 1}^{\mu\nu}|_{\text{div}} = \left(\frac{e}{4\pi}\right)^2 \int \left( \frac{\pi^2}{2\varepsilon^2} u^{\mu} u^{\nu} - \frac{1}{4} \eta^{\mu\nu} \right) \ln \varepsilon u^{\mu\nu} \delta^4(x - y(s)) ds,$$

$$T_{\varepsilon 2}^{\mu\nu}|_{\text{div}} = -\left(\frac{e}{4\pi}\right)^2 \frac{16\pi}{3} \ln \varepsilon \int u^{\mu\nu} \delta^4(x - y(s)) ds,$$

$$T_{\varepsilon 3}^{\mu\nu}|_{\text{div}} = 0.$$

Summing up one sees that the logarithmic divergences cancel, and that the polar divergence is canceled by the counterterm in (3.4).

To derive (4.5)–(4.7) one must apply $T_{\varepsilon}^{\mu\nu}$ to a test function $\varphi(x)$, and analyze the limit $\varepsilon \to 0$. In doing this we encounter the technical difficulty that the kinematical variables $y$, $u$ and $w$ appearing in $T_{\varepsilon}^{\mu\nu}$ depend on $x$ in a rather complicated way, since they are evaluated at the proper time $s_\varepsilon(x)$. To disentangle this implicit dependence we use the following identity for a generic function $f \in C^\infty(R)$,

$$f(s_\varepsilon(x)) = \int \delta(s - s_\varepsilon(x)) f(s) ds = \int 2\delta_+([x - y(s)]^2 - \varepsilon^2) u^{\mu}(s) (x - y(s))_{\mu} f(s) ds.$$
When we apply such a function to a test function it is then also convenient to perform in
the resulting integral the shift,

\[ x^\mu \rightarrow x^\mu + y^\mu(s). \]

In the following we make a systematic use of this strategy, beginning with the proof
of (4.5). Applying (4.2) to a test function \( \varphi \) one obtains then,

\[ T_{\varepsilon 1}^{\mu\nu}(\varphi) = A_{\varepsilon}^{\mu\nu}(\varphi) + B_{\varepsilon}^{\mu\nu}(\varphi), \quad (4.9) \]

where, setting momentarily \( e/4\pi = 1 \),

\[
A_{\varepsilon}^{\mu\nu}(\varphi) = \int ds \int d^4x \frac{2\delta+(x^2 - \varepsilon^2)(-x^\mu x^\nu + 2(ux)u(\mu x^\nu) - \varepsilon^2 u^\mu u^\nu - \text{tr})}{(ux)^5} \varphi(y(s)),
\]

\[
B_{\varepsilon}^{\mu\nu}(\varphi) = \int ds \int d^4x \frac{2\delta+(x^2 - \varepsilon^2)(-x^\mu x^\nu + 2(ux)u(\mu x^\nu) - \varepsilon^2 u^\mu u^\nu - \text{tr})}{(ux)^5} \cdot (\varphi(y(s) + x) - \varphi(y(s))). \quad (4.10)
\]

All kinematical quantities are now evaluated in the integration variable \( s \). We have divided
\( T_{\varepsilon 1}^{\mu\nu}(\varphi) \) in two terms, adding and subtracting the term with \( \varphi(y(s)) \). This separation is
convenient because we will see that \( A_{\varepsilon}^{\mu\nu}(\varphi) \) gives rise to the polar divergence in (4.6),
while \( B_{\varepsilon}^{\mu\nu}(\varphi) \) gives rise to the logarithmic divergence.

Actually, \( A_{\varepsilon}^{\mu\nu}(\varphi) \) can be evaluated exactly, it suffices to know the invariant integral,
see appendix A,

\[
\int d^4x \frac{2\delta+(x^2 - \varepsilon^2)(-x^\mu x^\nu + 2(ux)u(\mu x^\nu) - \varepsilon^2 u^\mu u^\nu - \text{tr})}{(ux)^5} x^\mu x^\nu = \frac{\pi^2}{4 \varepsilon} (5u^\mu u^\nu - \eta^{\mu\nu}). \quad (4.11)
\]

This gives,

\[ A_{\varepsilon}^{\mu\nu}(\varphi) = \frac{\pi^2}{2\varepsilon} \int \left( u^\mu u^\nu - \frac{1}{4} \eta^{\mu\nu} \right) \varphi(y(s)) \, ds, \]

amounting to the polar divergence in (4.5).

Consider now \( B_{\varepsilon}^{\mu\nu} \). From a dimensional analysis one sees that \( B_{\varepsilon}^{\mu\nu}(\varphi) \) diverges loga-
rimatically as \( \varepsilon \to 0 \), i.e. \( \ln \varepsilon \). To extract this divergence it is then sufficient to evaluate
the limit,

\[ \lim_{\varepsilon \to 0} \varepsilon \frac{d}{d\varepsilon} B_{\varepsilon}^{\mu\nu}(\varphi). \]

To compute \( \varepsilon \frac{d}{d\varepsilon} B_{\varepsilon}^{\mu\nu}(\varphi) \) it is convenient to rescale first in (4.10) the integration variable
\( x \rightarrow \varepsilon x \), then apply \( \varepsilon \frac{d}{d\varepsilon} \), and then rescale back \( x \rightarrow x/\varepsilon \). Taking then the limit \( \varepsilon \to 0 \)}
the term $\varepsilon^2 u^\mu u^\nu$ drops out and one gets 9,
\[
\lim_{\varepsilon \to 0} \varepsilon \frac{d}{d\varepsilon} B_\varepsilon^{\mu\nu}(\varphi) = \int ds \int d^4 x \frac{2\delta_+(x^2)}{(ux)^5} \left( -x^\mu x^\nu + 2(ux)u^{(\mu}x^{\nu)} - tr \right) \cdot
\]
\[\cdot [\varphi(y(s)) - \varphi(y(s) + x) + x^\alpha \partial_\alpha \varphi(y(s) + x)]
\]
\[= -\frac{8\pi}{3} \int u^\mu u^\nu \frac{d\varphi(y(s))}{ds} ds
\]
\[= \frac{8\pi}{3} \int (u^\mu w^\nu + u^\nu w^\mu) \varphi(y(s)) ds.
\]
This is equivalent to,
\[
B_\varepsilon^{\mu\nu}(\varphi) = \frac{8\pi}{3} \ln \varepsilon \int (u^\mu w^\nu + u^\nu w^\mu) \varphi(y(s)) ds + o(1),
\]
where $o(1)$ means terms that are regular for $\varepsilon \to 0$. This gives rise to the logarithmic divergence in (4.5).

The proof of (4.6) proceeds similarly,
\[
T_\varepsilon^{\mu\nu}(\varphi) = \int ds \int d^4 x \frac{2\delta_+(x^2)}{(ux)^5} \left( 2(wx)x^\mu x^\nu + 2(ux)x^{(\mu}x^{\nu)} - 2\varepsilon^2 u^{(\mu}x^{\nu)} - tr \right) \varphi(y(s) + x),
\]
where here $\Delta^\mu \equiv (ux)^{\mu} - (wx)^{\mu}$. This is again logarithmically divergent and one obtains, operating as above 10,
\[
\lim_{\varepsilon \to 0} \varepsilon \frac{d}{d\varepsilon} T_\varepsilon^{\mu\nu}(\varphi) = \int ds \int d^4 x \frac{2\delta_+(x^2)}{(ux)^5} \left( 2(wx)x^\mu x^\nu + 2(ux)x^{(\mu}x^{\nu)} - tr \right) x^\alpha \partial_\alpha \varphi(y(s) + x)
\]
\[= -\frac{8\pi}{3} \int (u^\mu w^\nu + u^\nu w^\mu) \varphi(y(s)) ds.
\]
This proves (4.6).

The tensor $T_\varepsilon^{\mu\nu}$ admits trivially a finite limit for $\varepsilon \to 0$, since only $1/|\vec{R}|^2$ powers are present. This concludes the proof of property 1) of the theorem.

---

9The resulting expression can be evaluated explicitly noting the integral, with $y \equiv y(s)$,
\[
\int d^4 x \frac{2\delta_+(x^2)}{(ux)^5} x^\mu x^\nu \left[ \varphi(y) - \varphi(y + x) + x^\alpha \partial_\alpha \varphi(y + x) \right] = -\int d\Omega \frac{m^\mu m^\nu}{(um)^3} m^\alpha \partial_\alpha \varphi(y) =
\]
\[
\frac{4\pi}{3} \left( u^\mu \partial^\nu \varphi(y) + u^\nu \partial^\mu \varphi(y) + \eta^{\mu\nu}u^\alpha \partial_\alpha \varphi(y) - 6 u^\mu u^\nu \partial^\rho \varphi(y) \right),
\]
where $\int d\Omega$ denotes the integral over the three–dimensional solid angle, and $m^\mu = (1, \frac{\hat{x}}{|x|})$.

10This time one needs the integral, with $y = y(s)$,
\[
\int d^4 x \frac{2\delta_+(x^2)}{(ux)^5} x^\mu x^\nu x^\alpha \partial_\alpha \varphi(y + x) = -\int d\Omega \frac{m^\mu m^\nu m^\rho}{(um)^3} \varphi(y) =
\]
\[
\frac{4\pi}{3} \left( \eta^{\mu\nu}u^\rho + \eta^{\rho\nu}u^\mu + \eta^{\rho\mu}u^\nu - 6 u^\mu u^\nu u^\rho \right) \varphi(y),
\]
where $\int d\Omega$ is the integral over angles and $m^\mu = (1, \frac{\hat{x}}{|x|})$. 

22
4.2 Conservation of the energy–momentum tensor

The conservation of the total energy–momentum tensor is reduced to the proof of property 4), i.e. to the evaluation of \( \partial_\mu T^{\mu\nu}_{em} \), see (3.5). Since the derivative is a continuous operation in \( S' \), once we have established the existence of the limit (3.4), the operator \( \partial_\mu \) can be interchanged with the limit, and we get,

\[
\partial_\mu T^{\mu\nu}_{em} = S' - \lim_{\varepsilon \to 0} \left( \partial_\mu T^{\mu\nu}_\varepsilon - \frac{\varepsilon^2}{32} \int \left( w^\nu - \frac{1}{4} \partial^\nu \right) \delta^4(x - y(s)) \, ds \right).
\]

Since also the regularized LW–field satisfies the Bianchi identity,

\[
\varepsilon_{\mu\nu\rho\sigma} \partial_\nu F^{\rho\sigma} = 0,
\]

one deduces the standard result,

\[
\partial_\mu T^{\mu\nu}_\varepsilon = -F^{\nu\mu}_\varepsilon j_\varepsilon, \tag{4.12}
\]

and hence,

\[
\partial_\mu T^{\mu\nu}_{em} = S' - \lim_{\varepsilon \to 0} \left( -F^{\nu\mu}_\varepsilon j_\varepsilon - \frac{\varepsilon^2}{32} \int \left( w^\nu - \frac{1}{4} \partial^\nu \right) \delta^4(x - y(s)) \, ds \right). \tag{4.13}
\]

The calculation is therefore reduced to the evaluation of \( F^{\nu\mu}_\varepsilon j_\varepsilon \) for \( \varepsilon \to 0 \), in the topology of \( S' \).

A somewhat delicate point in the evaluation of this limit is the following. The formula for the current (2.19) carries a factor of \( \varepsilon^2 \) in front, due to the fact that in the limit \( \varepsilon \to 0 \) the current \( j_\varepsilon^\mu(x) \) goes to zero for \( x^\mu \) in the complement of the wordline. Actually, it can be seen that when evaluating the limit (2.14), there is only one term in (2.19) that has a non–vanishing limit, more precisely,

\[
S' - \lim_{\varepsilon \to 0} \left( \varepsilon^2 \frac{e}{4\pi} \frac{3u^\mu}{(uR)^5} \right) = j^\mu,
\]

while all other terms converge to zero. These “vanishing” additional terms are however needed in the regularized current to ensure its conservation, and they contribute in the product \( F^{\nu\mu}_\varepsilon j_\varepsilon \) for \( \varepsilon \to 0 \), since \( F^{\nu\mu}_\varepsilon \) has polar terms. Multiplying out (2.18) and (2.19) we obtain indeed,

\[
F^{\nu\mu}_\varepsilon j_\varepsilon = \left( \frac{e}{4\pi} \right)^2 \varepsilon^2 \left[ \left( \frac{2w^2}{(uR)^6} - \frac{(bR)}{(uR)^7} \right) R^\nu + 3 \frac{(wR) - 1}{(uR)^6} w^\nu \right.
\]

\[- 3 \frac{(wR) - 1}{(uR)^7} \left( w^\nu + \left( (wR) - 1 \right) \frac{R^\nu}{(uR)} \right) \right] + o(\varepsilon), \tag{4.14}
\]
where \( b^\mu = dw^\mu /ds \). \( o(\varepsilon) \) denotes terms that vanish for \( \varepsilon \to 0 \) in \( S' \). In the present case these are terms of the form \( \varepsilon^2 /R^4 \), that converge to zero by power counting. We have now to perform the limit of this expression for \( \varepsilon \to 0 \) in \( S' \). This analysis can be performed using the techniques developed in section 4.1, with the difference that now one has to keep also terms that are finite for \( \varepsilon \to 0 \). The computation is deferred to appendix B where we show that,

\[
F_{\varepsilon \mu} j_{\varepsilon \mu} = \left( \frac{e}{4\pi} \right)^2 \int \left[ \frac{8\pi}{3} \left( \frac{dw^\nu}{ds} + w^2 u^\nu \right) - \frac{\pi^2}{2\varepsilon} \left( w^\nu - \frac{1}{4} \partial^\nu \right) \right] \delta^4(x - y(s)) ds + o(\varepsilon), \quad (4.15)
\]

where \( o(\varepsilon) \) denotes again terms that vanish for \( \varepsilon \to 0 \) in \( S' \).

Inserting this in (4.13) one sees that the polar contributions cancel, as they must, and one obtains (3.5).

### 4.3 Comparison with Rowe’s energy–momentum tensor

The electromagnetic energy–momentum tensor introduced by Rowe [8] is written as the sum of three terms elements of \( S' \),

\[
\Theta_{\varepsilon \mu} = \sum_{i=1}^{3} \Theta_{i \mu}^\nu, \tag{4.16}
\]

where,

\[
\Theta_{1 \mu}^\nu = \partial_\alpha K_{1 \mu \nu}^\alpha + \frac{e^2}{16\pi} \int u^\mu \partial^\nu \delta^4(x - y(s)) ds, \quad (4.17)
\]

\[
\Theta_{2 \mu}^\nu = \partial_\alpha K_{2 \mu \nu}^\alpha - \frac{e^2}{6\pi} \int u^\mu u^\nu \delta^4(x - y(s)) ds, \quad (4.18)
\]

\[
\Theta_{3 \mu}^\nu = - \left( \frac{e}{4\pi} \right)^2 \frac{\Delta^2}{(uR)^6} R^\mu R^\nu. \quad (4.19)
\]

The tensors \( K_{i \alpha \mu}^\nu \in S' \) are antisymmetric in \( \alpha \) and \( \mu \), and are given by,

\[
K_{1 \alpha \mu}^\nu = \frac{1}{4} \left( \frac{e}{4\pi} \right)^2 \left( \partial^\mu \left( R^\alpha R^\nu \right) (uR)^4 \right) - \partial^\alpha \left( \frac{R^\mu R^\nu}{(uR)^4} \right), \quad (4.20)
\]

\[
K_{2 \alpha \mu}^\nu = \left( \frac{e}{4\pi} \right)^2 \frac{1}{(uR)^5} (R^\mu \Delta^\alpha - R^\alpha \Delta^\mu) R^\nu. \quad (4.21)
\]

In these expressions no regularization is needed. Notice, indeed, that \( K_{2 \alpha \mu}^\nu \) contains only \( 1/R^2 \) singularities and that \( K_{1 \alpha \mu}^\nu \) contains only derivatives of \( 1/R^2 \) singularities, so that both tensors are elements of \( S' \). This implies that also \( \Theta_{1 \mu}^\nu \) and \( \Theta_{2 \mu}^\nu \) are elements of
$S'$, because the derivative of a distribution is again a distribution. The terms with the $\delta$–functions in (4.17), (4.18) are added to ensure the symmetry of each $\Theta^{\mu\nu}_i$ separately – a property that is rather hidden in Rowe’s formulation, and needs to be proven a posteriori. A more serious practical drawback of the expressions above is that the derivatives appearing are distributional derivatives, and they can not be evaluated applying simply the Leibnitz rule, due to the singularities present on the worldline.

Away from the worldline, i.e. for $x^\mu \neq y^\mu(\lambda)$, one can evaluate the derivatives above using the Leibnitz rule, and it is then straightforward to check that in the complement of the worldline $\Theta^{\mu\nu}_{em}$ coincides with the sum of (4.2)–(4.4), with $\varepsilon = 0$, and hence with our $T^{\mu\nu}_{em}$ in (3.4).

The comparison of $\Theta^{\mu\nu}_{em}$ with $T^{\mu\nu}_{em}$ as elements of $S'$ can be performed applying to the tensors $K^{\alpha\mu\nu}_i$ our $S'$–regularization, $K^{\alpha\mu\nu}_i \rightarrow K^{\alpha\mu\nu}_{\varepsilon i}$, where the regularized tensors are given formally again by (4.20), (4.21), but with the replacement $s(x) \rightarrow s\varepsilon(x)$. On the regularized tensors one can now use the Leibnitz rule to evaluate $\partial_\alpha K^{\alpha\mu\nu}_{\varepsilon i}$, and perform eventually the $S'$–limit for $\varepsilon \rightarrow 0$. Following this procedure in appendix C we prove indeed that,

$$\Theta^{\mu\nu}_{em} = T^{\mu\nu}_{em},$$

as distributions. Here we note only that a comparison of (4.19) with (4.4) yields immediately, since there are only $1/R^2$ singularities present,

$$S' \lim_{\varepsilon \rightarrow 0} T^{\mu\nu}_{\varepsilon 3} = \Theta^{\mu\nu}_3. \quad (4.22)$$

5 Proof of variational results

5.1 Derivation of the Dirac–Lorentz equation

In this subsection we proof property 2) of Theorem II, that is equivalent to the relation (3.22). Instead of computing the variation of $I_\varepsilon$ under a generic variation of $y^\mu(s)$, we compute the variation of the Lagrange density $L_\varepsilon$. The limit of $\delta I_\varepsilon$ for $\varepsilon \rightarrow 0$ can then be obtained integrating $\delta L_\varepsilon$ between two hypersurfaces, and sending $\varepsilon$ to zero. Another reason for proceeding in this way is that the explicit form of $\delta L_\varepsilon$ will also be used in the proof of Theorem IV.
In this subsection with “\(\delta\)” we will always mean a generic smooth variation w.r.t. \(y^\mu\).

Starting from (3.17) we have then,

\[
\delta \mathcal{L}_\varepsilon = -A_{\varepsilon \mu} \delta j^\mu_\varepsilon - m_\varepsilon \delta \left( \int \delta^4(x - y(s)) \, ds \right),
\]

(5.1)

where, according to what we need in (3.22), we have replaced – after variation – \(A_\mu \rightarrow A_{\varepsilon \mu}\).

While the variation of the second term is trivial, to vary the first term it is more convenient to write,

\[
- A_{\varepsilon \mu} \delta j^\mu_\varepsilon = - \delta \left( A_{\varepsilon \mu} j^\mu_\varepsilon \right) + \delta A_{\varepsilon \mu} j^\mu_\varepsilon,
\]

(5.2)

since \(\delta A_{\varepsilon \mu}\) is simpler then \(\delta j^\mu_\varepsilon\). The variation of the first term in this expression can be obtained from the \(S'\)–expansion,

\[
\left. j^\mu_\varepsilon A^\nu_\varepsilon = \left( \frac{e}{4\pi} \right)^2 \int \left( \frac{3\pi^2}{4\varepsilon} u^\mu u^\nu - 4\pi u^\mu w^\nu + o(\varepsilon) \right) \delta^4(x - y(s)) \, ds \right.
\]

(5.3)

upon contracting \(\mu\) and \(\nu\). This expansion can be derived using the method illustrated in appendix B, see (7.7).

The evaluation of the second term in (5.2) is more cumbersome, and the calculation is deferred to appendix D. The result is,

\[
\delta A_{\varepsilon \mu} j^\mu_\varepsilon = \left( \frac{e}{4\pi} \right)^2 \int \left[ \left( - \frac{3\pi^2}{8\varepsilon} (w \, \delta y) - \frac{8\pi}{3} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right) \delta y_\mu \right) \delta^4(x - y(s)) \, ds \\
+ \partial_\mu \int \left( 4\pi (w \, \delta y) u^\mu - \frac{\pi^2}{8\varepsilon} (\delta y^\mu - (u \, \delta y) u^\mu) \right) \delta^4(x - y(s)) \, ds \right] + o(\varepsilon).
\]

(5.4)

Using this in (5.2) one obtains eventually,

\[
\delta \mathcal{L}_\varepsilon = \int \left( mw^\mu - \frac{e^2}{6\pi} \left( \frac{dw^\mu}{ds} + w^2 u^\mu \right) \right) \delta y_\mu \delta^4(x - y(s)) \, ds \\
+ \partial_\mu \int \left( \frac{e^2}{4\pi} (w \, \delta y) u^\mu + \left( m + \frac{e^2}{64\varepsilon} \right) (\delta y^\mu - (u \, \delta y) u^\mu) \right) \delta^4(x - y(s)) \, ds + o(\varepsilon),
\]

(5.5)

which is our final result, holding for generic variations of \(y\). The first line is finite as \(\varepsilon \rightarrow 0\), and the second line, which corresponds to a four–divergence, contains a polar term in \(\varepsilon\).

To derive the equations of motion we compute \(\delta I_\varepsilon\) integrating \(\delta \mathcal{L}_\varepsilon\) between two hypersurfaces, and impose that on them \(\delta y^\mu(s)\) vanishes. The four–divergence drops then out – together with the polar term – and the first line (5.5) gives (3.22). This concludes the proof of Theorem II.
5.2 Proof of Theorem III

The evaluation of the regularized Lagrangian on the solution \( A^\mu = A_\varepsilon^\mu \) requires to evaluate, using (2.18) and (7.6),

\[
-\frac{1}{4} F_{\varepsilon\mu\nu} F_{\varepsilon}^{\mu\nu} = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left( \frac{1}{(uR)^4} - \frac{\varepsilon^2}{(uR)^6} + o(\varepsilon) \right) = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left( \frac{1}{(uR)^4} - \frac{\pi^2}{4\varepsilon} \int \delta^4(x - y(s)) \, ds + o(\varepsilon) \right).
\]

This gives, using (5.3),

\[
\mathcal{L}_\varepsilon(A_\varepsilon, y) = -\frac{1}{4} F_{\varepsilon\mu\nu} F_{\varepsilon}^{\mu\nu} - A_\varepsilon^\mu j_\varepsilon^\mu - m_\varepsilon \int \delta^4(x - y(s)) \, ds
= \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \left( \frac{1}{(uR)^4} - \frac{\pi^2}{\varepsilon} \int \delta^4(x - y(s)) \, ds \right) - m \int \delta^4(x - y(s)) \, ds + o(\varepsilon).
\]

Applying to a test function one gets,

\[
\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon(A_\varepsilon, y)(\varphi) = \lim_{\varepsilon \to 0} \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int ds \int \frac{2\delta_+(x^2 - \varepsilon^2)}{(ux)^3} [\varphi(x + y(s)) - \varphi(y(s))] \, d^4x
- m \int \varphi(y(s)) \, ds,
\]

where we used,

\[
\int d^4x \frac{2\delta_+(x^2 - \varepsilon^2)}{(ux)^3} = \frac{\pi^2}{\varepsilon}.
\]

The integral above converges now for \( \varepsilon \to 0 \), and we can set \( \varepsilon = 0 \) obtaining the conditionally convergent integral,

\[
\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon(A_\varepsilon, y)(\varphi) = \frac{1}{2} \left( \frac{e}{4\pi} \right)^2 \int ds \int \frac{2\delta_+(x^2)}{(ux)^3} [\varphi(x + y(s)) - \varphi(y(s))] \, d^4x - m \int \varphi(y(s)) \, ds.
\]

Shifting \( x \to x - y(s) \) and integrating out the \( \delta \)-function one obtains (3.23).

5.3 Proof of the Noether Theorem IV

The translation invariance of the regularized Lagrangian is expressed by the identity (3.24), \( \delta_t \mathcal{L}_\varepsilon = 0 \), where \( \mathcal{L}_\varepsilon \) is given in (3.17), and \( \delta y^\mu = a^\mu, \delta A^\mu = a^\nu \partial_\nu A^\mu \).

After standard steps one obtains,

\[
\delta_t \mathcal{L}_\varepsilon = (\partial_\mu F^{\mu\nu} - j_\varepsilon^\nu) \delta A_\nu - \partial_\mu (\delta A_\nu F^{\mu\nu}) + A_\mu \delta y j_\varepsilon^\mu + \delta_y \left( m_\varepsilon \int \delta^4(x - y(s)) \, ds \right) - a^\mu \partial_\mu \mathcal{L}_\varepsilon,
\]
where $\delta_y$ means variation w.r.t. $y^\mu$. We evaluate this expression now on the solutions of the regularized Maxwell equation, $A^\mu = A_\varepsilon^\mu$, and on those of the Lorentz–Dirac equation. 

The first term goes then to zero, and for the combination,

$$A_\varepsilon^\mu \delta_y j_\varepsilon^\mu + \delta_y \left( m_\varepsilon \int \delta^4(x - y(s)) \, ds \right),$$

we can use the general expansion (5.1), (5.5), with $\delta y^\mu = a^\mu$, with opposite sign. Thanks to the Lorentz–Dirac equation, the expression (5.5) reduces to a four–divergence and hence $\delta t L_\varepsilon$ becomes a four–divergence, too. Using again (5.3) to evaluate $A_{\varepsilon \nu} j_\varepsilon^\nu$ appearing in $\partial_\mu L_\varepsilon$, after simple algebra one obtains,

$$\delta t L_\varepsilon = a_\nu \partial_\mu \hat{T}^\mu_\nu + o(\varepsilon) = 0,$$

(5.6)

where,

$$\hat{T}^\mu_\nu \equiv F_\varepsilon^{\rho \mu} \partial_\nu A_\varepsilon^\rho + \frac{1}{4} \eta^{\mu \nu} F_\varepsilon^{\rho \sigma} F_\varepsilon_{\rho \sigma} + m \int u^\mu u^\nu \delta^4(x - y(s)) \, ds + \left( \frac{e}{4\pi} \right)^2 \int \left( \frac{\pi^2}{4\varepsilon} u^\mu u^\nu - 4\pi u^\mu u^\nu + \frac{\pi^2}{8\varepsilon} \eta^{\mu \nu} \right) \delta^4(x - y(s)) + o(\varepsilon).$$

(5.7)

Even if it is not obvious from the derivation – see however the footnote below – it can be seen that the limit,

$$S' - \lim_{\varepsilon \to 0} \hat{T}^\mu_\nu \equiv \hat{T}^\mu_\nu,$$

(5.8)

exists. (5.6) implies then that the energy–momentum tensor $\hat{T}^\mu_\nu$ is conserved. It is immediately seen, however, that this tensor does not coincide with the one constructed in Theorem I, see (3.7), nor is it symmetric in $\mu$ and $\nu$. On the other hand this was to be expected, since the Noether theorem leads in general to the canonical energy–momentum tensor, not to the symmetric one.

From the general theory we know however that, if the Lagrangian density is not only translation invariant but also Lorentz invariant, there exists always a symmetrization procedure for the canonical tensor. In the case of classical Electrodynamics this procedure requires to add the divergenceless term $-\partial_\rho (F_\varepsilon^{\rho \mu} A_\varepsilon^\nu)$. In the case at hand this suggests then to add the divergenceless term $^{11}$, see (5.3),

$$-\partial_\rho (F_\varepsilon^{\rho \mu} A_\varepsilon^\nu) = -F_\varepsilon^{\rho \mu} \partial_\rho A_\varepsilon^\nu - j_\varepsilon^\mu A_\varepsilon^\nu$$

$^{11}$The product $F_\varepsilon^{\rho \mu} A_\varepsilon^\nu$ contains terms that behave as $1/R^3$, if $\varepsilon = 0$. This means that a priori its $S'$–limit for $\varepsilon \to 0$ contains logarithmic divergences, $\sim \ln \varepsilon$. However, by inspection i.e. applying the
\[
F_{\varepsilon}^{\mu\nu} \partial_\rho A_\rho^\nu + \left( \frac{e}{4\pi} \right)^2 \int \left( -\frac{3\pi^2}{4\varepsilon} u^\mu u^\nu + 4\pi u^\mu w^\nu + o(\varepsilon) \right) \delta^4(x - y(s)) \, ds.
\]

Adding this to (5.7) one sees that eventually one gets a symmetric tensor, being indeed,

\[
\hat{T}_\varepsilon^{\mu\nu} - \partial_\rho (F_{\varepsilon}^{\rho\mu} A_\rho^\nu) = T^{\mu\nu} + o(\varepsilon),
\]

(5.9)

where \( T^{\mu\nu} \) is defined in (3.7). This concludes the proof of the Noether theorem.

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**6 Appendix A: Invariant integrals**

In the proofs throughout the paper one encounters the invariant integrals,

\[
I_\mu = \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^N} x_\mu,
\]

\[
I_{\mu\nu} = \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^N} x_\mu x_\nu,
\]

\[
I_{\mu\nu\rho} = \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^N} x_\mu x_\nu x_\rho,
\]

where \( \delta_+(x^2 - 1) \equiv H(x^0)\delta(x^2 - 1) \). These integrals can be evaluated considering \( u^\mu \) as an unconstrained variable, not satisfying \( u^2 = 1 \), and writing, for example,

\[
I_{\mu\nu} = \frac{1}{(N - 1)(N - 2)} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial u^\nu} \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^{N-2}}.
\]

This reduces these integrals to the calculation of the “generating function”,

\[
\int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^N} = \frac{\pi^{3/2}}{(u^2)^{N/2}} \frac{\Gamma \left( \frac{N}{2} - 1 \right)}{\Gamma \left( \frac{N+1}{2} \right)}.
\]

product to a test–function and taking \( \varepsilon \to 0 \) one checks that these divergences cancel out, and the limit exists. This means that also,

\[
S' - \lim_{\varepsilon \to 0} \partial_\rho (F_{\varepsilon}^{\rho\mu} A_\rho^\nu),
\]

exists. This fact, together with (5.9), provides an indirect proof for the existence of the limit (5.8).

29
After taking the derivatives w.r.t. \( u^\mu \) one sets again \( u^2 = 1 \). For example,

\[
\int d^4 x \frac{2\delta_+(x^2 - 1)}{(ux)^3} = \frac{\pi^2}{(u^2)^{3/2}},
\]

and hence,

\[
\int d^4 x \frac{2\delta_+(x^2 - 1)}{(ux)^3} x_\mu x_\nu = \frac{1}{3} \cdot 4 \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial u^\nu} \int d^4 x \frac{2\delta_+(x^2 - 1)}{(ux)^3} = \frac{\pi^2}{4(u^2)^{7/2}} \left( 5u^\mu u^\nu - u^2 \eta^\mu\nu \right).
\]

Setting \( u^2 = 1 \) this gives (4.11).

### 7 Appendix B: proof of (4.15)

Before taking the limit of \( \varepsilon \to 0 \) it is convenient to rewrite (4.14) in the form,

\[
F^\nu_\mu J_{\varepsilon \mu} = \left( \frac{e}{4\pi} \right)^2 \varepsilon^2 \left( \frac{2w^2}{(uR)^6} - \frac{(bR)}{(uR)^7} \right) R^\nu - \frac{3w^\nu}{(uR)^6} + \frac{1}{2} \left( (wR) - 1 \right)^2 \partial^\nu \frac{1}{(uR)^6} + o(\varepsilon).
\]

(7.1)

We have used,

\[
\partial^\nu (uR) = u^\nu + ((wR) - 1) \frac{R^\nu}{(uR)},
\]

and we have omitted the term,

\[
3\varepsilon^2 \frac{(wR)w^\nu}{(uR)^6},
\]

because it goes to zero for \( \varepsilon \to 0 \) in \( S' \), for kinematical reasons. More precisely, after applying it to a test function and integrating over \( d^4 x \), for invariance reasons the factor \( R^\mu \) in the numerator gets replaced by \( u^\mu \), and \( (uw) = 0 \). This will become clear below.

Regarding the last term in (7.1) it is convenient to note the identity,

\[
\frac{\varepsilon^2}{2} \left( (wR) - 1 \right)^2 \partial^\nu \frac{1}{(uR)^6} = \frac{\varepsilon^2}{2} \partial^\nu \left[ \left( (wR) - 1 \right)^2 \frac{1}{(uR)^6} \right] - \varepsilon^2 \frac{((wR) - 1)^2}{(uR)^6} \partial^\nu (wR)
\]

\[
= \frac{\varepsilon^2}{2} \partial^\nu \frac{1}{(uR)^6} + \varepsilon^2 \frac{(bR)}{(uR)^7} R^\nu + \varepsilon^2 \frac{w^\nu}{(uR)^6} + o(\varepsilon),
\]

(7.2)

where the \( o(\varepsilon) \) terms are in the same sense as above. Inserting this in (7.1) one sees that the \( (bR) \)-term cancels getting,

\[
F^\nu_\mu J_{\varepsilon \mu} = \left( \frac{e}{4\pi} \right)^2 \varepsilon^2 \left( \frac{2w^2}{(uR)^6} R^\nu - \frac{2w^\nu}{(uR)^6} + \frac{1}{2} \partial^\nu \frac{1}{(uR)^6} \right) + o(\varepsilon).
\]

(7.3)
The behaviour of the three terms present, for $\varepsilon \to 0$ in $S'$, are as follows,

\begin{align}
\frac{2\varepsilon^2 w^2}{(uR)^6} R^\nu &= \frac{8\pi}{3} \int w^2 u^\nu \delta^4(x - y(s)) ds + o(\varepsilon), \quad (7.4) \\
- \frac{2\varepsilon^2 w^\nu}{(uR)^6} &= \int \left( \frac{8\pi}{3} \frac{dw^\nu}{ds} - \frac{\pi^2}{2\varepsilon} w^\nu \right) \delta^4(x - y(s)) ds + o(\varepsilon), \quad (7.5) \\
\frac{\varepsilon^2}{2(uR)^6} &= \frac{\pi^2}{8\varepsilon} \int \delta^4(x - y(s)) ds + o(\varepsilon). \quad (7.6)
\end{align}

For the proofs of these relations, that require to apply the l.h.s. to a test function and to analyze the limit $\varepsilon \to 0$, one can use the technicalities developed in section 4.1. We report explicitly the proof of (7.5),

\begin{align}
\left( - \frac{2\varepsilon^2 w^\nu}{(uR)^6} \right) (\varphi) &= -2\varepsilon^2 \int ds \int d^4 x \frac{2\delta_+ (x^2 - \varepsilon^2)}{(u_x)^5} w^\nu \varphi(y(s) + x) \\
&= -2 \int ds \int d^4 x \frac{2\delta_+ (x^2 - 1)}{(u_x)^5} w^\nu \varphi(y(s) + \varepsilon x) \\
&= -2 \int ds \int d^4 x \frac{2\delta_+ (x^2 - 1)}{(u_x)^5} w^\nu \left( \frac{1}{\varepsilon} \varphi(y(s)) + x^\alpha \partial_\alpha \varphi(y(s)) + o(\varepsilon) \right) \\
&= -2 \int ds \left( \frac{\pi^2}{4\varepsilon} w^\nu \varphi(y(s)) + \frac{4\pi}{3} w^\nu u^\alpha \partial_\alpha \varphi(y(s)) + o(\varepsilon) \right) \\
&= \int ds \left( - \frac{\pi^2}{2\varepsilon} w^\nu + \frac{8\pi}{3\varepsilon} \frac{dw^\nu}{ds} + o(\varepsilon) \right) \varphi(y(s)), \quad (7.7)
\end{align}

where we used the invariant integrals of appendix A.

Inserting (7.4)–(7.6) in (7.3), one obtains (4.15).

### 8 Appendix C: Rowe’s energy–momentum tensor

Before comparing $T^\mu_\nu_{em}$ of (3.4) with $\Theta^\mu_\nu_{em}$ in (4.16), we cast the former in a slightly different form writing, see (4.2)–(4.4),

\[ T^\mu_\nu_{em} = S' - \lim_{\varepsilon \to 0} (D^\mu_\nu_{\varepsilon 1} + D^\mu_\nu_{\varepsilon 2} + T^\mu_\nu_{\varepsilon 3}), \quad (8.1) \]

where,

\begin{align}
D^\mu_\nu_{\varepsilon 1} &= T^\mu_\nu_{\varepsilon 1} - 2 \left( \frac{e}{4\pi} \right)^2 \frac{(wR)}{(uR)^6} (R^\mu R^\nu - \text{tr}) - \frac{e^2}{32\varepsilon} \int \left( u^\mu u^\nu - \frac{1}{4} \eta^\mu_\nu \right) \delta^4(x - y(s)) ds, \\
D^\mu_\nu_{\varepsilon 2} &= T^\mu_\nu_{\varepsilon 2} + 2 \left( \frac{e}{4\pi} \right)^2 \frac{(wR)}{(uR)^6} (R^\mu R^\nu - \text{tr}),
\end{align}
The advantage of this form of presenting $T_{\epsilon m}^{\mu \nu}$ is that the tensors $D_{\epsilon 1}^{\mu \nu}$ converge separately for $\epsilon \to 0$ in the topology of $\mathcal{S}'$, as does $T_{\epsilon 3}^{\mu \nu}$. We have added and subtracted the same term from $T_{\epsilon 1}^{\mu \nu}$ and $T_{\epsilon 2}^{\mu \nu}$, to eliminate the logarithmic divergence from both. This can be seen from (4.5)–(4.7), noting that, 

\[
\frac{(w R)}{(u R)^6} R^\mu R^\nu \bigg|_{\text{div}} = \frac{8 \pi}{3} \ln \epsilon \int u(\mu w^\nu) \delta^4(x - y(s)) \, ds.
\]

We evaluate now the four–divergencies appearing in (4.17), (4.18), using the regularized versions $K_{\epsilon 1}^{\alpha \mu \nu}$ for (4.20), (4.21), obtained replacing $s(x) \to s_\epsilon(x)$. Then one can apply Leibnitz’ rule and, taking into account that now $R^2 = \epsilon^2 \neq 0$, after a straightforward computation one obtains,

\[
\partial_\alpha K_{\epsilon 1}^{\alpha \mu \nu} = D_{\epsilon 1}^{\mu \nu} + \frac{\epsilon^2}{32 \pi} \int \left( w R \right) \left( u^\mu u^\nu - \frac{1}{4} \eta^{\mu \nu} \right) \delta^4(x - y(s)) \, ds \tag{8.2}
\]

\[
+ \left( \frac{\epsilon}{4 \pi} \right)^2 \frac{\epsilon^2}{4} \left[ 15 \left( \frac{w R}{(u R)^3} \right) - \frac{1}{(u R)^6} u^\mu R^\nu + \frac{7 w^\mu u^\nu}{(u R)^6} - \frac{3 w^\mu R^\nu}{(u R)^6} + \frac{2 - 6(w R)}{(u R)^6} \eta^{\mu \nu} \right], \tag{8.3}
\]

\[
\partial_\alpha K_{\epsilon 2}^{\alpha \mu \nu} = D_{\epsilon 2}^{\mu \nu} + \left( \frac{\epsilon}{4 \pi} \right)^2 \frac{\epsilon^2}{2} \left[ 5 \left( \frac{w R}{(u R)^3} \right) - \frac{1}{(u R)^6} \Delta^\mu R^\nu + \frac{w^\mu R^\nu}{(u R)^6} + \frac{1}{(u R)^6} \left( (b R) u^\mu - (u R) b^\mu \right) R^\nu \right.
\]

\[
+ \frac{1}{(u R)^6} \left( 2 \Delta^\mu u^\nu + \Delta^\nu u^\mu \right) + \frac{3(w R)}{2(u R)^6} \eta^{\mu \nu} \right] \tag{8.4}
\]

In $\partial_\alpha K_{\epsilon 1}^{\alpha \mu \nu}$ we have put in evidence the polar part, proportional to $u^\mu u^\nu - \frac{1}{4} \eta^{\mu \nu}$, and in $\partial_\alpha K_{\epsilon 1}^{\alpha \mu \nu}$ we have set $b^\mu = dw^\mu / ds$.

We can now consider the $\mathcal{S}'$–limit for $\epsilon \to 0$ of these relations, applying them to a test–function and sending $\epsilon \to 0$. The calculations, that are straightforward but a bit lengthy, can be performed with the techniques illustrated in appendix B, see (7.7). Since the limits of $\partial_\alpha K_{\epsilon 1}^{\alpha \mu \nu}$ for $\epsilon \to 0$ exist by construction, the r.h.s. of the above relations must also admit finite limits. It can indeed be seen that the square bracket in (8.3) exhibits a divergent contribution that cancels exactly the polar term in (8.2), while the square bracket in (8.4) admits a finite limit by power counting. The results are,

\[
\partial_\alpha K_{\epsilon 1}^{\alpha \mu \nu} = \left( \mathcal{S}' - \lim_{\epsilon \to 0} D_{\epsilon 1}^{\mu \nu} \right) + \frac{\epsilon^2}{2 \pi} \int \left( \frac{1}{6} \left( w^\mu w^\nu + u^\mu w^\nu \right) - \frac{1}{16} u^\mu \partial^\nu \right) \delta^4(x - y(s)) \, ds,
\]

\[
\partial_\alpha K_{\epsilon 2}^{\alpha \mu \nu} = \left( \mathcal{S}' - \lim_{\epsilon \to 0} D_{\epsilon 2}^{\mu \nu} \right) - \frac{\epsilon^2}{6 \pi} \int u^\nu w^\mu \delta^4(x - y(s)) \, ds.
\]

Inserting these expressions in (4.17), (4.18) one gets the symmetric and traceless tensors,

\[
\Theta_1^{\mu \nu} = \left( \mathcal{S}' - \lim_{\epsilon \to 0} D_{\epsilon 1}^{\mu \nu} \right) + \frac{\epsilon^2}{6 \pi} \int (w^\mu w^\nu + u^\nu w^\mu) \delta^4(x - y(s)) \, ds, \tag{8.5}
\]
\[ \Theta_{\mu\nu} = \left( S' - \lim_{\varepsilon \to 0} D_{\varepsilon}^{\mu\nu} \right) - \frac{e^2}{6\pi} \int (u^\mu w^\nu + u^\nu w^\mu) \delta^4(x - y(s)) \, ds. \] (8.6)

These formulae, together with (4.22) and (8.1), imply then \( \Theta_{\nu\mu} = T_{\nu\mu} \), q.e.d.

**9 Appendix D: Proof of (5.4)**

To evaluation of \( \delta A_{\varepsilon \mu} j_{\varepsilon}^\mu \) requires first to evaluate \( \delta A_{\varepsilon \mu} \) under a generic smooth variation of \( y^\mu \). This evaluation is complicated by the fact that \( A_{\varepsilon}^\mu \) – see formula (2.7) – depends on \( y^\mu \) explicitly, but also implicitly through the (regularized) retarded proper time function \( s_{\varepsilon}(x) \), defined in (2.8). To determine \( \delta A_{\varepsilon \mu} \) it is convenient to use the following shortcut.

We compute \( \delta A_{\varepsilon}^\mu \) applied to a test function \( \varphi \in \mathcal{S} \), inserting a \( \delta \)-function as in (4.8), and then shifting \( x \to x + y(s) \),

\[
\int \delta A_{\varepsilon}^\mu \varphi \, d^4x = \frac{e}{4\pi} \delta \int ds \int d^4x \, 2\delta_+ (x^2 - \varepsilon^2) w^{\mu}(s) \varphi(x + y(s)) = \frac{e}{4\pi} \delta \int d^4x \left[ \frac{1}{(uR)} \frac{d\delta y^\mu}{ds} - \partial_\nu \left( \frac{w^{\mu} \delta y^\nu}{(uR)} \right) \right] \varphi,
\]

where in the square bracket all kinematical quantities are evaluated at \( s = s_{\varepsilon}(x) \). This gives,

\[
\delta A_{\varepsilon}^\mu = \frac{e}{4\pi} \left( \frac{1}{(uR)} \frac{d\delta y^\mu}{ds} - \partial_\nu \left( \frac{w^{\mu} \delta y^\nu}{(uR)} \right) \right) = \delta y^\mu - (u \delta y) u^\mu, \quad (u \ell) = 0, \quad \ell^\mu = \frac{d\ell^\mu}{ds},
\]

where

\[
\ell^\mu \equiv \delta y^\mu - (u \delta y) u^\mu, \quad (u \ell) = 0, \quad \ell^\mu \equiv \frac{d\ell^\mu}{ds}.
\]

The appearance of the combination \( \ell^\mu \) is due to the invariance of \( A_{\varepsilon}^\mu \) under reparametrization of the worldline, i.e. \( \delta A_{\varepsilon}^\mu = 0 \) if \( \delta y^\mu = \delta \lambda u^\mu \).

Multiplying out (9.1) and (2.19) one obtains then,

\[
\delta A_{\varepsilon \mu} j_{\varepsilon}^\mu = \left( \frac{e}{4\pi} \right)^2 \varepsilon^2 \left[ \frac{(bR) - (uR)(ub)}{(uR)^7} (R\ell) + 3(1 - (wR)) \right] \cdot \left( \frac{(u\ell) + (\Delta \ell')(uR) - (R\ell')(1 - (wR))}{(uR)^7} - \frac{(1 - (wR))^2 + (uR)^2w^2}{(uR)^8} (R\ell) \right) + o(\varepsilon),
\]

(9.2)

\[ \text{with} \quad \delta = \left( \frac{e}{4\pi} \right)^2 \varepsilon^2 \left[ \frac{(bR) - (uR)(ub)}{(uR)^7} (R\ell) + 3(1 - (wR)) \right] \cdot \left( \frac{(u\ell) + (\Delta \ell')(uR) - (R\ell')(1 - (wR))}{(uR)^7} - \frac{(1 - (wR))^2 + (uR)^2w^2}{(uR)^8} (R\ell) \right) + o(\varepsilon), \]

(9.2)
where the terms $o(\varepsilon)$ correspond to expressions of the type $\varepsilon^2/R^4$, that converge to 0 in the topology of $S'$, by power counting. To evaluate the limit of this expression under $\varepsilon \to 0$ in $S'$, one proceeds precisely as in (7.7). In this way the evaluation of the limit of all terms above is reduced to the determination of the invariant integrals of appendix A.

Let's consider e.g. in the expression above the most singular term, corresponding to $\varepsilon^2(R\ell)/(uR)^8$, that by power counting would give rise to a double pole $1/\varepsilon^2$ in that,

$$
\left( \varepsilon^2 \frac{(R\ell)}{(uR)^8} \right) (\varphi) = \varepsilon^2 \int ds \int d^4x \frac{2\delta_+(x^2 - \varepsilon^2)}{(ux)^7} (x\ell)\varphi(y(s) + x)
$$

$$
= \frac{1}{\varepsilon^2} \int ds \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^7} (x\ell)\varphi(y(s) + \varepsilon x)
$$

$$
= \int ds \int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^7} (x\ell) \left[ \frac{1}{\varepsilon^2} \varphi(y(s)) + \frac{1}{\varepsilon} x^\alpha \partial_\alpha \varphi(y(s)) + \frac{1}{2} x^\alpha x^\beta \partial_\alpha \partial_\beta \varphi(y(s)) + o(\varepsilon) \right].
$$

We compute the invariant integrals according to appendix A,

$$
\int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^7} x^\mu = \frac{8\pi}{15} u^\mu,
$$

$$
\int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^7} x^\mu x^\alpha = \frac{\pi^2}{24} (7u^\mu u^\alpha - \eta^\mu\alpha),
$$

$$
\int d^4x \frac{2\delta_+(x^2 - 1)}{(ux)^7} x^\mu x^\alpha x^\beta = \frac{4\pi}{15} (8u^\mu u^\alpha u^\beta - \eta^\mu\alpha u^\beta - \eta^\alpha\beta u^\mu - \eta^\beta\mu u^\alpha).
$$

Since $(u\ell) = 0$ we see that the double pole cancels, and the remaining two terms give a $1/\varepsilon$ contribution and a finite part. After integration by parts one obtains,

$$
\left( \varepsilon^2 \frac{(R\ell)}{(uR)^8} \right) (\varphi) = \int \left( \frac{4\pi}{15} \ell^\mu - \frac{\pi^2}{24\varepsilon} \ell^\mu \right) \partial_\mu \varphi(y(s)) ds + o(\varepsilon),
$$

i.e. in $S'$,

$$
\left( \varepsilon^2 \frac{(R\ell)}{(uR)^8} \right) = -\partial_\mu \int \left( \frac{4\pi}{15} \ell^\mu - \frac{\pi^2}{24\varepsilon} \ell^\mu \right) \delta^4(x - y(s)) ds + o(\varepsilon).
$$

All other terms in (9.2) by power counting give simple pole contributions and finite parts, too, and a straightforward but a bit lengthy calculation gives (5.4).

References

[1] P.A.M. Dirac, Proc. Roy. Soc. Lond. A 167 148 (1938).
[2] W. Thirring, “A course in Mathematical Physics 2. Classical Field Theory” Springer–Verlag, New York, Wien 1979.

[3] R. Haag, Z. Naturforsch. 10a 752 (1955).

[4] F. Rohrlich, “Classical Charged Particles: Foundation of Their Theory”, Addison–Wesley, Reading, MA, 1965.

[5] E.J. Moniz and D.H. Sharp, Phys. Rev. D 10 1133 (1974).

[6] H. Spohn, “Dynamics of Charged Particles and Their Radiation Field”, Cambridge University Press 2004.

[7] A.D. Yaghjian, “Relativistic Dynamics of a Charged Sphere”, Lecture Notes in Physics, Springer Verlag, Berlin (1992).

[8] E.G.P. Rowe, Phys. Rev. D 18 3639 (1978).

[9] C. Teitelboim, Phys. Rev. D 1 1572 (1970); C. G. van Weert, Phys. Rev. D 9 339 (1974); D. Villarroel, Ann. Phys. (N.Y.) 89 241 (1974).

[10] M. Marino, Ann. Phys. 301 85 (2002).

[11] F. Rohrlich, Phys. Rev. Lett. 12 375 (1964).

[12] E.G.P. Rowe, Phys. Rev. Lett. 53 1779 (1984).

[13] D. Zwanziger, Phys. Rev. D 19 3614 (1979).

[14] R. Courant and D. Hilbert, “Methods of Mathematical Physics”, Vol. II, Wiley, New York (1962).

[15] K. Lechner, in preparation.

[16] B. DeWitt and R. Brehme, Ann. Phys. 9 220 (1960).

[17] F. Rohrlich, Phys. Rev. 150 1104 (1966).

[18] P. Kazinski, L. Lyakhovich and A. Sharapov, Phys. Rev. D 66 025017 (2002).
[19] P. Kazinski and A. Sharapov, Theor. Math. Phys. 143 798 (2005).

[20] F.G. Friedlander, “The Wave Equation on a Curved Space-time”, Cambridge University Press, 1975.

[21] C. Teitelboim, Phys. Rev. D 3 297 (1971).