Geometrical Underpinning of Finite Dimensional Hilbert space

M. Revzen

Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel

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Finite geometry is employed to underpin operators in finite, d, dimensional Hilbert space. The central role of mutual unbiased bases (MUB) states projectors is exhibited. Interrelation among operators in Hilbert space, revealed through their (finite) dual affine plane geometry (DAPG) underpinning is studied. Transcription to (finite) affine plane geometry (APG) is given and utilized for their interpretation.

I. INTRODUCTION

Several recent studies [6, 15–18] consider the relation between finite geometry and finite dimensional Hilbert space. Such relation, aside from its intrinsic interest, reveals interrelation among physical operators that may be hidden otherwise. The present work emphasizes a particular branch of finite geometry: dual affine plane geometry (DAPG) [6, 21, 23, 24]. The physical operators that play the central role in this work are the mutual unbiased bases (MUB) state’s projectors [3, 4, 9, 12]. The association of MUB projectors with affine plane geometry (APG) was treated in depth in [6] who emphasized the relevance of the trace in these studies. In the present work we are concerned with underpinning of the physical operators with both geometries and indicate the inter transitions between DAPG and APG. Our approach allows the definition of mappings of Hilbert space operators onto the lines and points of the geometries in addition to exhibiting the geometry based interrelation among operators.

The finite dimensional characteristics of both affine plane geometry (APG) and its dual (DAPG) are listed in section II. We describe therein the interrelations among points and lines required for a realization of the geometries. Mutual unbiased bases (MUB) whose state’s projectors are central to our study are defined and discussed briefly in the next section, section III. In section IV we present our underpinning scheme of DAPG for Hilbert space operators. The section contains some geometrically based interrelations among operators. In section V we study further the so called line operators i.e. operators underpinned with DAPG lines. We give here the equation governing the DAPG lines and derive some explicit properties of of the operators. APG is considered in section VI where we discuss the DAPG and APG underpinned operators. In section VII we briefly indicate the geometrically assisted mappings of operators in Hilbert space onto lines and points of the geometry. The last section, section VIII, contained brief summary and discussions.

II. FINITE GEOMETRY AND HILBERT SPACE OPERATORS

We now briefly review the essential features of finite geometry required for our study [6, 20, 21, 23, 24]. A finite plane geometry is a system possessing a finite number of points and lines. There are two kinds of finite plane geometry: affine and projective. We shall confine ourselves to affine plane geometry (APG) which is defined as follows. An APG is a non empty set whose elements are called points. These are grouped in subsets called lines subject to:

1. Given any two distinct points there is exactly one line containing both.
2. Given a line L and a point S not in L (S ⊄ L), there exists exactly one line L’ containing S such that L ∩ L’ = ∅. This is the parallel postulate.
3. There are 3 points that are not collinear.

It can be shown [23, 24] that for d = p^n (a power of prime) APG can be constructed (our study here is for d=p) and the following properties are, necessarily, built in:

a. The number of points is d^2; S_α, α = 1, 2, ...d^2 and the number of lines is d(d+1); L_j, j = 1, 2,...d(d + 1).
b. A pair of lines may have at most one point in common: L_j ∩ L_k = λ; λ = 0, 1 for j ≠ k.
c. Each line is made of d points and each point is common to d+1 lines: L_j = ∪α S_α, S_α = ∩j L_j.
d. If a line L_j is parallel to the distinct lines L_k and L_i then L_k ∥ L_i. The d^2 points are grouped in sets of d parallel lines. There are d+1 such groupings.
e. Each line in a set of parallel lines intersect each line of any other set: L_j ∩ L_k = 1; L_j ∥ L_k.

The above items will be referred to by APG (x), with x=a,b,c,d or e.

The existence of APG implies [20, 23, 24] the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we shall study extensively this, DAPG, we list the corresponding properties for it. We shall refer
a. The number of lines is \( d^2 \), \( L_j, j = 1, 2, \ldots d^2 \). The number of points is \( d(d+1) \), \( S_\alpha, \alpha = 1, 2, \ldots d(d+1) \).

b. A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.

c. Each point is common to \( d \) lines. Each line contain \( d+1 \) points.

d. The \( d(d+1) \) points may be grouped in sets, \( R_\alpha \), of \( d \) points each no two of a set share a line. Such a set is designated by \( \alpha' \in \{\alpha \cup M_\alpha\} \), \( \alpha' = 1, 2, \ldots d(d+1) \). \( (M_\alpha \) contain all the points not connected to \( \alpha \) - they are not connected among themselves.) i.e. such a set contain \( d \) disjoined (among themselves) points. There are \( d+1 \) such sets: 

\[
\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha=1}^{d} R_\alpha; \tag{1}
\]

\[
R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}; \tag{2}
\]

\[
R_\alpha \cap R_{\alpha'} = \emptyset, \alpha \neq \alpha'. \tag{3}
\]

e. Each point of a set of disjoint points is connected to every other point not in its set.

### III. FINITE DIMENSIONAL MUTUAL UNBIASED BASES, MUB, BRIEF REVIEW

In a finite, \( d \)-dimensional, Hilbert space two complete, orthonormal vectorial bases, \( B_1, B_2 \), are said to be MUB if and only if \((B_1 \neq B_2)\) \[3–6, 9, 11–13, 16, 19, 26, 28, 30\]

\[
\forall |u\rangle, |v\rangle \in B_1, B_2 \text{ resp.}, \quad |\langle u|v\rangle| = 1/\sqrt{d}. \tag{4}
\]

The physical meaning of this is that knowledge that a system is in a particular state in one basis implies complete ignorance of its state in the other basis.

Ivanovic \[10\] proved that there are at most \( d+1 \) MUB, pairwise, in a \( d \)-dimensional Hilbert space and gave an explicit formulae for the \( d+1 \) bases in the case of \( d=p \) (prime number). Wootters and Fields \[4\] constructed such \( d+1 \) bases for \( d=p^m \) with \( m \) an integer. Variety of methods for construction of the \( d+1 \) bases for \( d=p^m \) are now available \[11, 13, 19\]. Our present study is confined to \( d\neq 2 \).

We now give explicitly the MUB states in conjunction with the algebraically complete operators \[1, 9\] set: \( \hat{Z}, \hat{X} \). Thus we label the \( d \) distinct states spanning the Hilbert space, termed the computational basis, by \( |n\rangle, \quad n = 0, 1, \ldots d-1; |n+d\rangle = |n\rangle \)

\[
\hat{Z}|n\rangle = \omega^n|n\rangle; \quad \hat{X}|n\rangle = |n+1\rangle, \quad \omega = e^{i2\pi/d}. \tag{5}
\]

The \( d \) states in each of the \( d+1 \) MUB bases \[11\] are the states of computational basis (CB) and

\[
|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{0}^{d-1} \omega^{\frac{d}{2}n(n-1)-nm}|n\rangle; \quad b, m = 0, 1, \ldots d-1. \tag{6}
\]

Here the \( d \) sets labeled by \( b \) are the bases and the \( m \) labels the states within a basis. We have \[11\]

\[
\hat{X}\hat{Z}^b|m; b\rangle = \omega^m|m; b\rangle. \tag{7}
\]

For later reference we shall refer to the computational basis (CB) by \( b=-1 \). Thus the above gives \( d+1 \) bases, \( b=-1,0,1,\ldots d-1 \) with the total number of states \( d(d+1) \) grouped in \( d+1 \) sets each of \( d \) states. We have of course,

\[
\langle m; b|m'; b'\rangle = \delta_{m,m'}; \quad |\langle m; b|m'; b'\rangle| = \frac{1}{\sqrt{d}}, \quad b \neq b'. \tag{8}
\]

We remark at this junction that the eigen values of the CB might be considered finite dimensional modulated position values ("q") and the eigenvalues of shifting operator, \( X \), modulated momentum ("p"). This completes our discussion of MUB.
IV. DAPG UNDERPINNING OF D-DIMENSIONAL HILBERT SPACE

We first list some direct consequences of DAPG. DAPG(c) allows the definition;

\[ S_\alpha = \frac{1}{d} \sum_{j \in \alpha} L_j. \]  

(9)

This implies,

\[ \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{\alpha' \in \alpha \cup M_\alpha} d L_j, \]  

(10)

leading via DAPG(d) to

\[ \sum_{\alpha}^{d+1} \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{d(d+1)}{d} \sum_{\alpha}^{d} S_{\alpha} = \frac{d+1}{d+1} \sum_{\alpha}^{d} S_{\alpha}. \]  

(11)

We thus have,

\[ \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{\alpha' \in \alpha \cup M_\alpha} d^2 L_j = \frac{1}{d+1} \sum_{\alpha}^{d} S_{\alpha}. \]  

(12)

Now the underpinning of Hilbert space operators with DAPG will be undertaken. We consider d=p, a prime. For d=p we may construct d+1 MUB [3, 6, 10, 11]. Points will be associated with MUB state projectors. To this end we recall that we designate the MUB states by \(|m, b⟩\). with \(b = 0, 1, 2...d − 1\) labels the eigenfunction of, resp. \(XZ^b\). \(m\) labels the state within a basis. We designate the computational basis, CB, by \(b=-1\). The projection operator defined by,

\[ \hat{A}_\alpha \equiv |m, b⟩⟨b, m|; \ \alpha = \{b, m\}; \ \alpha = -1, 0, 1, 2...d − 1; \ m = 0, 1, 2...d − 1. \]  

(13)

The point label, \(\alpha = (m, b)\) is now associated with the projection operator, \(A_\alpha\). We now consider a realization, possible for d=p, a prime, of a d dimensional DAPG, as points marked on a rectangular whose horizontal width (x-axis) is made of d+1 columns of points. Each column is labeled by \(b\), and its vertical height (y axis) is made of \(d\) points each marked with \(m\). The total number of points is \(d(d+1)\) - there are \(d\) points in each of the \(d+1\) columns. We associate the \(d\) points \(m = 0, 1, 2...d − 1\) in each set, labeled by \(b\), \((\alpha ∼ (m, b))\) to the disjointed points of DAPG(d), \(R_\alpha\) viz. for fixed \(b \ \alpha' \in \alpha \cup M_\alpha\) form a column. The columns are arranged according to their basis label, \(b\). The first being \(b=-1\), \(\alpha_{-1} = (m, -1); m = 0, 1, ...d − 1\), pertains to the computational basis (CB). Lines are now made of \(d+1\) points, each of different \(b\). A point \(S_{\alpha}\) underpins a Hilbert space state projector, \(A_\alpha\), i.e. \(A_{\alpha}^2 = A_\alpha\), and \(trA_\alpha = 1\). We designate the line operator underpinned with \(L_j\), by \(P_j\). Thus the above relations now hold with \(S_{\alpha} \leftrightarrow A_\alpha\): \(L_j \leftrightarrow P_j\).

Now DAPG(c) (and Eq.13, 12) implies that \(A_\alpha\): \(\alpha = 0, 1, 2...d − 1; \ \alpha \in \alpha' \cup M_\alpha\) forms an orthonormal basis for the d-dimensional Hilbert space:

\[ \sum_{m}^{d} |m, b⟩⟨b, m| = \sum_{\alpha' \in \alpha \cup M_\alpha}^{d(d+1)} \hat{A}_{\alpha'} = \hat{1} \]  

(14)

e.g. for d=3 the underpinning’s schematics is,

\[
\begin{pmatrix}
  m\b & -1 & 0 & 1 & 2 \\
  0 & A_{(0,-1)} & A_{(0,0)} & A_{(0,1)} & A_{(0,2)} \\
  1 & A_{(1,-1)} & A_{(1,0)} & A_{(1,1)} & A_{(1,2)} \\
  2 & A_{(2,-1)} & A_{(2,0)} & A_{(2,1)} & A_{(2,2)}
\end{pmatrix}
\]
Eq. (9) implies,

\[ A_\alpha = \frac{1}{d} \sum_{j \in \alpha} P_j. \]  

(15)

Evaluating

\[ \sum_{\alpha \in j} A_\alpha = \frac{1}{d} \sum_{\alpha \in j} \sum_{j' \in \alpha} P_j = \frac{1}{d} \left( \sum_{j' \neq j} P_{j'} + (d + 1)P_j \right) = I + P_j. \]  

(16)

i.e.,

\[ P_j = \sum_{\alpha \in j} A_\alpha - I. \]  

(17)

Eq. (8) implies,

\[ \text{tr} A_\alpha A_\alpha' = \begin{cases} 1; & \alpha = \alpha' \\ 0; & \alpha \neq \alpha'; \alpha \in \alpha' \cup M_{\alpha'} \\ \frac{1}{d}; & \alpha \neq \alpha'; \alpha \ni \alpha' \cup M_{\alpha'}. \end{cases} \]  

(18)

Hence, using Eq. (9), (8),

\[ \text{tr} A_\alpha P_j = \begin{cases} \sum_{\alpha' \neq \alpha} \text{tr} A_\alpha A_{\alpha'} = 1; & \alpha \in j \\ \sum_{\alpha' \neq \alpha} \text{tr} A_\alpha A_{\alpha'} - A_\alpha = 0; & \alpha \ni j. \end{cases} \]  

(19)

Trivially

\[ \text{tr} P_j = \sum_{\alpha \in j} \text{tr} A_\alpha - 1 = 1. \]  

(20)

\[ \text{tr} P_j P_{j'} = \sum_{\alpha' \in j'} \text{tr} P_j A_{\alpha'} - 1 = \begin{cases} \frac{d}{d} j = j' \\ 0; \ j \neq j', \end{cases} \]  

(21)

i.e.

\[ \text{tr} P_j P_{j'} = d \delta_{j,j'}. \]  

(22)

An alternative view of the Lambda function is gained via

\[ \text{tr} A_\alpha P_j = \frac{1}{d} \sum_{j} P_j P_j = \begin{cases} \frac{1}{d} (\text{tr} P_j^2 + \text{tr} \sum_{j' \ni \alpha} P_j P_j) = 1; & j \in \alpha \\ \sum_{j' \neq j} P_{j'} P_j = 0; & j \ni \alpha. \end{cases} \]  

(23)

Note that the case of \( j \ni \alpha \) implies \( j \in M_{\alpha} \).

These are summarized by

\[ \text{tr} A_\alpha P_j = \begin{cases} 1; & \alpha \in j, \\ 0; & \alpha \ni j, \end{cases} \]  

(24)

and

\[ \text{tr} A_\alpha P_j = \begin{cases} 1; & j \in \alpha \\ 0; & j \ni \alpha. \end{cases} \]  

(25)
V. GEOMETRIC UNDERPINNING OF MUB QUANTUM OPERATORS: THE LINE OPERATOR

We now consider a particular realization of DAPG of dimensionality \( d = p \neq 2 \) which is the basis of our present study. We arrange the aggregate the \( d(d+1) \) points, \( \alpha \), in a \( d \times (d+1) \) matrix like rectangular array of \( d \) rows and \( d+1 \) columns. Each column is made of a set of \( d \) points \( R_\alpha = \bigcup_{\alpha' \in \mathbb{M}_d} S_{\alpha'} \); DAPG(d). We label the columns by \( b = -1,0,1,2,\ldots,d-1 \) and the rows by \( m=0,1,2,\ldots,d-1 \). (Note that the first column label of \( -1 \) is for convenience and does not designate negative value of a number.) Thus \( \alpha = m(b) \) designate a point by its row, \( m \), and its column, \( b \); when \( b \) is allowed to vary - it designate the point’s row position in every column. We label the left most column by \( b = -1 \) and with increasing values of \( b \), the basis label, as we move to the right. Thus the right most column is \( b = d-1 \). We now assert that the \( d+1 \) points, \( m_j(b), b = 0,1,2,\ldots,d-1 \), and \( m_j(-1) \), that form the line \( j \) which contain the two (specific) points \( m(-1) \) and \( m(0) \) is given by (we forfeit the subscript \( j \) - it is implicit),

\[
m(b) = \frac{b}{2}(c-1) + m(0), \text{ mod}[d] \quad b \neq -1, \\
m(-1) = c/2.
\]  

(26)

The rationale for this particular form will be clarified below. Thus a line \( j \) is parameterized fully by \( j = (m(-1), m(0)) \). (Note: since \( b \) takes on the values \(-1,0\) in our line labeling a more economic label for \( j \) is \( \alpha \), not designate negative value of a number.) Thus \( \alpha \) study. We arrange the aggregate the \( d(d+1) \) points, \( m(b), m(0) \), can have \( d \) values the number of lines is \( d^2 \); the number of points in a line is evidently \( d+1 \): a point for each \( b \). DAPG(a).

2. The linearity of the equation precludes having two points with a common value of \( b \) on the same line, DAPG(d). Now consider two points on a given line, \( m(b_1), m(b_2); b_1 \neq b_2 \). We have from Eq.(26), \((b \neq -1, b_1 \neq b_2)\)

\[
m(b_1) = \frac{b_1}{2}(c-1) + m(0), \text{ mod}[d] \\
m(b_2) = \frac{b_2}{2}(c-1) + m(0), \text{ mod}[d].
\]

(27)

These two equation determine uniquely \((for d=p, prime)\) \( m(-1) \) and \( m(0) \). DAPG(b).

For fixed point, \( m(b), c \leftrightarrow m(0) \) i.e the number of free parameters is \( d \) (the number of points on a fixed column). Thus each point is common to \( d \) lines. That the line contain \( d+1 \) is obvious. DAPG(c).

3. As is argued in 2 above no line contain two points in the same column (i.e. with equal \( b \)). Thus the \( d \) points, \( \alpha \), in a column form a set \( R_\alpha = \bigcup_{\alpha' \in \mathbb{M}_d} S_{\alpha'} \), with trivially \( R_\alpha \cap R_{\alpha'} = \emptyset \), \( \alpha \neq \alpha' \), and \( \bigcup_{\alpha=1}^{d} S_{\alpha} = \bigcup_{\alpha=1}^{d} R_\alpha \). DAPG(d).

4. Consider two arbitrary points \( \textit{not} \) in the same set, \( R_\alpha \) defined above: \( m(b_1), m(b_2) \) \( (b_1 \neq b_2) \). The argument of 2 above states that, \( \textit{for} d=p \), there is a unique solution for the two parameters that specify the line containing these points. DAPG(e).

We illustrate the above for \( d=3 \), where we explicitly specify the points contained in the line \( j = (m(-1) = (1,-1), m(0) = (2,0)) \)

\[
\begin{pmatrix}
0 & -1 & 0 & 1 & 2 \\
1 & (1,-1) & \cdot & (1,1) & \cdot \\
2 & \cdot & (2,0) & \cdot & \cdot
\end{pmatrix}
\]

For example the point \( m(1) \) is gotten from

\[
m(1) = \frac{1}{2}(2-1) + 2 = 1 \text{ mod}[3] \rightarrow m(1) = (1,1).
\]

Similar calculation gives the other point: \( m(2) = (0,2) \). i.e. the line \( j = (1,2) \) contains the points \((1,-1),(2,0),(1,1)\) and \((0,2)\).

The geometrical line, \( L_j, j = (1,2) \) given above upon being transcribed to its operator formula is via Eq.(17),

\[
P_{j=(1,2)} = A_{(1,-1)} + A_{(2,0)} + A_{(1,1)} + A_{(0,2)} - \hat{1}.
\]  

(28)
Evaluating the point operators, \( \hat{A}_\alpha \),

\[
A_{(1,-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
A_{(2,0)} = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix},
A_{(1,1)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix},
A_{(0,2)} = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix},
\]

(29)

and evaluating the sum, Eq. (28), gives

\[
P_{j:(m(-1)=1,m(0)=2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}.
\]

(30)

This operator obeys \( P_{j:(-1,2)} = \hat{I} \). We shall now show that this is quite general, viz \( P_{j} = \hat{I}, \forall j \).

Returning to Eqs. (13,6), these equations imply that, the projection operators \( A_{\alpha} \), in the CB representation are given by,

\[
(A_{\alpha=m,b})_{n,n'} = \begin{cases} \omega^{s/2}, & s = (n-n') \cdot \left( \frac{b}{2} |n+n'-1| - m \right), \\ \delta_{n,n'} \delta_{c/2,n}, & b \neq -1. \end{cases}
\]

(31)

Consider two distinct columns, \( b,b' \) (\( b,b' \neq -1 \)) and given the matrix elements \( n,n' \) (\( n \neq n' \)) of a projector \( (A_{\alpha=m,b})_{n,n'} \), compare it with \( (A_{\alpha'=m',b'})_{n,n'} \). If \( s \) (Eq. (31)) is \( s' \) i.e. \( \frac{b}{2} (n-n') - m \neq \frac{b'}{2} (n+n'-1) - m' \) pick another projector in the same column, \( b' \) (i.e vary \( m' \)). Since \( m' = 0,1,2...d-1 \) there is one (and only one) \( (A_{\alpha'=m',b'})_{n,n'} \) such that \( (A_{\alpha=m,b})_{n,n'} = (A_{\alpha'=m',b'})_{n,n'} \). Now consider another matrix element \( (A_{\alpha})_{\bar{n},\bar{n}} \). We have trivially that \( (A_{\alpha})_{\bar{n},\bar{n}} = (A_{\alpha'})_{\bar{n},\bar{n}} \) iff \( \bar{n} + \bar{n}' = n + n' \). i.e. all matrix elements (\( n,n' \)) with \( n+n'=c \) (constant) are such that \( (A_{\alpha})_{n,n'} = (A_{\alpha'})_{n,n'} \). These elements are situated along a line perpendicular to the diagonal of the matrices. We refer to this perpendicular as FV (foliated vector), it is parameterized by \( c \).

We now assert that all other (non diagonal) matrix elements are unequal. i.e. for \( b \neq b' \), \( (A_{\alpha=m,b})_{n,n'} \neq (A_{\alpha'=m',b'})_{n,n'}, \forall n,n' \in \text{FV} \). Proof: Let two elements \( n,n' \) and \( 1,1 \) with \( n \neq n' ; l \neq l' \) in the two matrices be equal. Thus \( c=n+n', c'=l+l' \):

\[
\frac{b}{2} (c-1) - m = \frac{b'}{2} (c-1) - m', \quad \text{and}
\]

\[
\frac{b}{2} (c'-1) - m = \frac{b'}{2} (c'-1) - m',
\]

(32)

These imply \( c=c' \), QED. Now consider \( s=0 \). Then all the matrix elements along FV are 1/d. We have then that for \( (A_{\alpha})_{n,n'} = (A_{\alpha'})_{n,n'} = \omega^s/d, s \neq 0, \) d-1 matrix elements along FV are all distinct. The diagonal is common to all.

We have thus a prescription for d projectors, \( A_{(m,b)} \), one for each \( b \), \( b \neq -1 \), all having equal matrix elements along FV labelled by \( c \). We supplement these with the projector \( A_{(c/2,-1)} = |c/2 \rangle \langle c/2 | \) to have the d+1 "points" constituting a line \( j \). (\( c/2 \) being a state in the CB.) Thus our line is formed as follows: It emerges from \( A_{(c/2,-1)} \) continues to \( A_{(m(0),0)} \) in the b=0 column. Then it continues to the points \( A_{(m(b),b} \) in succession: \( b=1,2...d-1 \) with \( m(b) \) determined by

\[
\frac{b}{2} (c-1) - m(b) = \frac{b+1}{2} (c-1) - m(b+1).
\]

Thus the two parameters, \( c=2m(-1) \) and \( m(1) \), determine the line i.e. \( j=(m(-1),m(1)) \). The general formula for the line, Eq. (26) now acquires a meaning in terms of the point operators, \( A_{(m,b),b} \). The discussion of the properties of the line thus defined confirm that these lines form a realization of DAPG lines. The analysis above indicate that the line operator, \( P_j \), may be labelled by two indices \( P_{j:(m(-1),m(0))} \). We now list some important consequences of this. We have shown that the matrix elements along a FV direction are the same for all the point operators \( A_{\alpha \in j} \). Indeed that is how we defined our lines. On the other hand we argued that the matrix elements not along the FV are all distinct. Hence summing up d such terms residing on a fixed line \( P_j \) (excluding the \( b=-1 \) and the diagonal term) sums up for each matrix element \( n,n' \) the d distinct roots of unity for matrix elements not on FV, hence for all \( c \),

\[
\sum_{\alpha \in j, \alpha \neq \alpha_{-1}} (\hat{A}_\alpha - \hat{I})_{n,n'} = 0; \ n,n' \ni n+n'=c; \ \alpha_{-1} = |c/2 \rangle \langle c/2 |.
\]

(33)
Thus \((\hat{P}_j)_{n,n'} = (\sum_{\alpha\in j} \hat{A}_\alpha - \hat{I})_{n,n'} \neq 0\) only along FV, and is 1 along the diagonal at \(c/2=m(-1)\). The sum over \(\alpha \in j\) of the matrix elements on a FV, which are equal for all \(\hat{A}_{\alpha\in j,\neq-1}\), simply cancel the \(1/d\). This is illustrated in Eqs.(point1), (pj).

Quite generally,

\[
(P_{j=m(-1),m(0)})_{n,n'} = \begin{cases} \omega^{-(n-n')m(0)} \delta_{((n-n')2m(-1))} & (34) \\ 0 & \text{otherwise}. \end{cases}
\]

Thus,

\[
(\hat{P}_j^2)_{j=m(-1),m(0)} = \delta_{n,n'}. \ i.e. \ \hat{P}_j^2 = \hat{I} \forall j. \ (35)
\]

In appendix A we show that \(\hat{P}_j^2 = \hat{I} \forall j\) implies the operator relation,

\[
\sum_{\alpha\neq\alpha'} A_{\alpha} \hat{A}_{\alpha'} = \sum_{\alpha\in j} \hat{A}_\alpha. 
\]

VI. AFFINE PLANE GEOMETRY (APG)

We now recast our DAPG underpinning into an APG one. This is achieved by interchanging lines with points. For notational convention we refer to \(m(-1)\) and \(m(0)\) of the DAPG (cf. Eq.) by \(\xi\) and \(\eta\) respectively in their APG image. Thus a line in DAPG \(j=\{m(-1),m(0)\}\) is a point \((\xi,\eta)\) in its APG image, \(\xi,\eta = 0,1,...d-1\). We now construct the following realization of a \(d\) (=prime) dimensional APG: consider \(d\cdot d\) points arranged in a square array of \(d\) columns and \(d\) rows. Each point is an image of DAPG line and is specified by \((\xi,\eta)\) in its APG image, and is \(1\) along the diagonal at \(c/2=m(-1)\). The sum over \(\xi,\eta\) of the matrix elements on a FV, which are equal for all \(\hat{A}_{\alpha\in j,\neq-1}\), simply cancel the \(1/d\). This is illustrated in Eqs.(point1), (pj).

Thus,

\[
(\hat{P}_j^2)_{j=m(-1),m(0)} = \delta_{n,n'}. \ i.e. \ \hat{P}_j^2 = \hat{I} \forall j. \ (35)
\]

In appendix A we show that \(\hat{P}_j^2 = \hat{I} \forall j\) implies the operator relation,

\[
\sum_{\alpha\neq\alpha'} A_{\alpha} \hat{A}_{\alpha'} = \sum_{\alpha\in j} \hat{A}_\alpha. 
\]

The consistency requirement is thus fulfilled: Consider APG line (e.g. \(\eta = r\xi + s\ mod[d]\)) for their lines DAPG images to share a point we must have, Eq.(,), simply cancel the \(1/d\). This is illustrated in Eqs.(point1), (pj).

Thus,

\[
(\hat{P}_j^2)_{j=m(-1),m(0)} = \delta_{n,n'}. \ i.e. \ \hat{P}_j^2 = \hat{I} \forall j. \ (35)
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Thus,

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(\hat{P}_j^2)_{j=m(-1),m(0)} = \delta_{n,n'}. \ i.e. \ \hat{P}_j^2 = \hat{I} \forall j. \ (35)
\]

In appendix A we show that \(\hat{P}_j^2 = \hat{I} \forall j\) implies the operator relation,
Thus the interrelation among the operators, $A_\alpha$ and $P_j$ are identical whether given within DAPG or APG. e.g. Eq.():

$$A_\alpha = \frac{1}{d} \sum_{j \in \alpha} P_j$$

is, within, DAPG gives the point operator $A_\alpha$ in terms of the d line operators $P_j$ - while within APG this very same equation gives the very same Hilbert space operator, $A_\alpha$, now a line operator in terms of the d point operators $P_j$.

As an example let us consider for $d=3$ the APG line $\eta = \xi + 1$. The APG points on this line are $(0,1),(1,2)$ and $(2,0)$ reflecting DAPG line operators, cf Eq.(34),

$$P(0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}; \quad P(1,2) = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}; \quad P(2,0) = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}. \quad (37)$$

Now, Eq.(??) relates these to the DAPG point (i.e. the MUB projector) $|0,2\rangle\langle 2,0|=A_{(0,2)}$

$$\frac{1}{3} \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} \right) = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (38)$$

where the last matrix is $A_{(0,2)}$, cf. Eq.(29).

This completes our discussion of geometrical underpinning of finite dimensional Hilbert space concomitant with operators interrelationships. We now turn to its possible use in the mappings of Hilbert space operators onto the phase space like points and lines of the geometry.

**VII. MAPPING ONTO PHASE SPACE**

We now define a mapping of Hilbert space operators, e.g. an arbitrary operator, $B$, onto the phase space - like lines of DAPG. The mapping is defined by [29],

$$B \Rightarrow V(j; B) \equiv tr BP_j, \quad (39)$$

Here $P_j$ is a line operator within DAPG. (Alternatively we could have cast the mappings within APG as is clear from the discussion in the previous section.) The density operator may be expressed in terms of $V(j; \rho)$:

$$\rho = \frac{1}{d^2} \sum_{j} (tr \rho P_j) P_j. \quad (40)$$

It can be shown that $V(j; \rho)$ plays the role of quasi distribution [29] in the phase space like lines of DAPG. Thus for example the expectation value of an arbitrary operator $B$ we have,

$$tr \rho B = \frac{1}{d} \sum_{j} V(j; \rho)V(j; B). \quad (41)$$

The quasi distribution may be reconstructed from the expectation values of the point operator $A_\alpha$ i.e. MUB state projector’s expectation value (obtained, e.g. by measurements),

$$tr \rho A_\alpha = \frac{1}{d} \sum_{j} V(j; \rho)V(j; A_\alpha) = \frac{1}{d} \sum_{j} V(j; \rho) \Lambda_{\alpha,j}. \quad (42)$$

Thence

$$\frac{1}{d} \sum_{\alpha \in \alpha' \cup M_{\alpha'}} \sum_{j} V(j; \rho) \Lambda_{\alpha,j} = V(j; \rho). \quad (43)$$
VIII. SUMMARY AND CONCLUDING REMARKS

It is of interest that, if we associate the CB states with the position variable, q, of the continuous problem and its Fourier transform state, viz b=0 (cf. Eq.(3)), with the momentum, p, we have that the line of the finite dimension problem is parameterized with "initial" values of "q" and "p" i.e. m(-1) and m(0).

Finite geometry stipulates interrelations among lines and points. The stipulations for the (finite) dual affine plane geometry (DAPG) was shown to conveniently accommodate association of geometric lines and points with projectors of states of mutual unbiased bases (MUB). The latter act in a (finite dimensional, d) Hilbert space. This underpinning of Hilbert space operator with DAPG reveal some novel inter operators relations. Noteworthy among these are Hilbert space operators, $P_j$, $j = 1, 2, ..., d^2$, which are underpinned with DAPG lines, $L_j$ that abide by $P_j^2 = I$ $\forall j$, and are mutually orthogonal, $trP_jP_{j'} = d\delta_{j,j'}$. These allow their utilization for general mapping of Hilbert space operators onto the phase space like lines and points of DAPG in close analogy with the mappings within the continuum of Hilbert space operators onto phase space via the well known Wigner function [8, 25]. If we associate the computational basis (CB) states with the position variable, q, of the continuous problem, and its Fourier transform state (i.e. states of the basis that is diagonal for translation operator, X (cf. Eq.(3)) with the momentum, p, we have that the line of the finite dimension problem is parameterized with these phase space like variables. We present a transcription from DAPG to (finite) affine plane geometry (APG) underpinnings. Within the latter such labelling is natural and, further more, the line operators here include points that are aligned in a straight line. This interpret the APG operator underpinning that is given in [5] as due to the association of MUB state projectors with points within DAPG.

Appendix: Fluctuation Distillation Formula

Given, Eq(34), \( \hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - I \) and, Eq(35), \( \hat{P}_j^2 = I \), implies

\[
(\sum_{\alpha \in j} \hat{A}_\alpha - I)(\sum_{\alpha' \in j} \hat{A}_{\alpha'} - I) = I.
\]

Thus,

\[
\sum_{\alpha, \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = 2 \sum_{\alpha \in j} \hat{A}_\alpha.
\]

Recalling that, Eq(13), $A^2_\alpha = A_\alpha$ allows

\[
\sum_{\alpha \neq \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = \sum_{\alpha \in j} \hat{A}_\alpha.
\]

QED

[1] J. Schwinger, Proc. Nat. Acad. Sci. USA 46, 560 (1960).
[2] W. Schleich, Quantum Optics in Phase Space, Wiley-Vch (2001).
[3] W. K. Wootters, Ann. Phys.(N.Y.) 176, 1 (1987).
[4] W. K. Wootters and B. D. Fields, Ann. Phys. (N.Y.) 191, 363 (1989).
[5] K. S. Gibbons, M. J. Hoffman and W. K. Wootters, Phys. Rev. A 70, 062101 (2004).
[6] W. K. Wootters, Found. of Phys. 36, 112 (2006).
[7] F. C. Khanna, P. A. Mello and M. Revzen, submitted for publication (2011).
[8] D. Ellinas and A. J. Bracken, Phys. Rev. A, 78 052106 (2008).
[9] A. Kalev, M. Revzen and F. C. Khanna, Phys. Rev. A 80, 022112 (2009).
[10] I. D. Ivanovic, J. Phys. A, 14, 3241 (1981).
[11] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury and F. Vatan, Algorithmica 34, 512 (2002).
[12] I. Bengtsson, AIP Conf. Proc. 750, 63-69 (2005); quant-ph/0406174
[13] A. Vourdas, Rep. Math. Phys. 40, 367 (1997), Rep. Prog. Phys. 67, 267 (2004).
[14] A. B. Klimov, L. L. Sanchez-Soto and H. de Guise, J. Phys. A: Math. Gen. 38, 2747 (2005).
[15] M. Saniga, M. Planat and H. Rosu, J. Opt. B: Quantum Semiclassic Opt. 6 L19 (2004).
[16] M. Planat and H. C. Rosu, Europ. Phys. J. 36, 133 (2005).
[17] M. Planat, H. C. Rosu and S. Perrine, Foundations of Physics 36, 1662 (2006).
[18] M. Combescure, quant-ph/0605090 (2006).
[19] A. B. Klimov, C. Munos and J. L. Romero, quant-ph/0605113 (2005).
[20] M. Grassl, Electronic Notes in Discrete Mathematics 20, 151 (2005).
[21] T. Bar-On, Jour. Math. Phys. 50, 072106 (2009).
[22] M. Revzen quant-phys/0912.5433 (2009).
[23] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error Correcting Codes, North Holland, Amsterdam, (1977).
[24] S. A. Shirakova, Russ. Math. Surv. 23 47 (1968).
[25] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).
[26] M. Revzen, P.A. Mello, A. Mann and L.M. Johansen, Phys. Rev. A 71, 022103 (2005)
[27] Amir Kalev, Ady Mann, Pier A. Mello and Michael Revzen, Phys. Rev A 79, 014104 (2009).
[28] M. Revzen, Phys. Rev. A 81, 012113 (2010).
[29] M. Revzen, arXiv:quant-ph/1111.2846.
[30] A. Peres, Quantum Theory: Concepts and Methods (Kluwer, Dordrect, 1995).
Geometrical Underpinning of Finite Dimensional Hilbert Space

M. Revzen
Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel
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Finite geometry is employed to underpin operators in finite, d, dimensional Hilbert space. The central role of mutual unbiased bases (MUB) states projectors is exhibited. Interrelation among operators in Hilbert space, revealed through their (finite) dual affine plane geometry (DAPG) underpinning is studied. Transcription to (finite) affine plane geometry (APG) is given and utilized for their interpretation.

I. INTRODUCTION

Several recent studies [6,15–18] consider the relation between finite geometry and finite dimensional Hilbert space. Such relation, aside from its intrinsic interest, reveals interrelation among physical operators that may be hidden otherwise. The present work emphasizes a particular branch of finite geometry: dual affine plane geometry (DAPG) [6, 21, 23, 24]. The physical operators that play the central role in this work are the mutual unbiased bases (MUB) state’s projectors [3, 4, 9, 12]. The association of MUB projectors with affine plane geometry (APG) was treated in depth in [6] who emphasized the relevance of the trace in these studies. In the present work we are concerned with underpinning of the physical operators with both geometries and indicate the inter transitions between DAPG and APG. Our approach allows the definition of mappings of Hilbert space operators onto the lines and points of the geometries in addition to exhibiting the geometry based interrelation among operators.

The finite dimensional characteristics of both affine plane geometry (APG) and its dual (DAPG) are listed in section II. We describe therein the interrelations among points and lines required for a realization of the geometries. Mutual unbiased bases (MUB) whose state’s projectors are central to our study are defined and discussed briefly in the next section, section III. In section IV we present our underpinning scheme of DAPG for Hilbert space operators. The section contains some geometrically based interrelations among operators. In section V we study further the so called line operators i.e. operators underpinned with DAPG lines. We give here the equation governing the DAPG lines and derive some explicit properties of of the operators. APG is considered in section VI where we discuss the DAPG and APG underpinned operators. In section VII we briefly indicate the geometrically assisted mappings of operators in Hilbert space onto lines and points of the geometry. The last section, section VIII, contained brief summary and discussions.

II. FINITE GEOMETRY AND HILBERT SPACE OPERATORS

We now briefly review the essential features of finite geometry required for our study [6, 20, 21, 23, 24]. A finite plane geometry is a system possessing a finite number of points and lines. There are two kinds of finite plane geometry: affine and projective. We shall confine ourselves to affine plane geometry (APG) which is defined as follows. An APG is a non empty set whose elements are called points. These are grouped in subsets called lines subject to:
1. Given any two distinct points there is exactly one line containing both.
2. Given a line L and a point S not in L (S ∋ L), there exists exactly one line L’ containing S such that L ∩ L’ = ∅. This is the parallel postulate.
3. There are 3 points that are not collinear.
It can be shown [23, 24] that for d = p^n (a power of prime) APG can be constructed (our study here is for d=p) and the following properties are, necessarily, built in:
a. The number of points is d^2; S_α, α = 1, 2,...d^2 and the number of lines is d(d+1); L_j, j = 1, 2,...d(d + 1).
b. A pair of lines may have at most one point in common: L_j ∩ L_k = λ; λ = 0, 1 forj ≠ k.
c. Each line is made of d points and each point is common to d+1 lines: L_j = ∪_{α=1}^{d} S_α, S_α = ∩_{j=1}^{d+1} L_j.
d. If a line L_j is parallel to the distinct lines L_k and L_l then L_k ∥ L_l. The d^2 points are grouped in sets of d parallel lines. There are d+1 such groupings.
e. Each line in a set of parallel lines intersect each line of any other set: L_j ∩ L_k = 1; L_j ∥ L_k.

The above items will be referred to by APG (x), with x=a,b,c,d or e.

The existence of APG implies [20, 23, 24] the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we shall study extensively this, DAPG, we list the corresponding properties for it. We shall refer
to these by DAPG(y):

a. The number of lines is \( d^2, \ L_j, \ j = 1, 2, ..., d^2 \). The number of points is \( d(d+1), \ S_\alpha, \ \alpha = 1, 2, ..., d(d+1) \).

b. A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.

c. Each point is common to \( d \) lines. Each line contain \( d+1 \) points.

d. The \( d(d+1) \) points may be grouped in sets, \( R_\alpha \), of \( d \) points each no two of a set share a line. Such a set is designated by \( \alpha' \in \{ \alpha \cup M_\alpha \} \), \( \alpha' = 1, 2, ..., d \) (\( M_\alpha \) contain all the points not connected to \( \alpha \) - they are not connected among themselves.) i.e. such a set contain \( d \) disjoined (among themselves) points. There are \( d+1 \) such sets:

\[
\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha'=1}^{d} R_{\alpha'};
\]

\[
R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'};
\]

\[
R_\alpha \cap R_{\alpha'} = \emptyset, \ \alpha \neq \alpha'.
\]

e. Each point of a set of disjoint points is connected to every other point not in its set.

### III. FINITE DIMENSIONAL MUTUAL UNBIASED BASES, MUB, BRIEF REVIEW

In a finite, \( d \)-dimensional, Hilbert space two complete, orthonormal vectorial bases, \( \mathcal{B}_1, \mathcal{B}_2 \), are said to be MUB if and only if \((\mathcal{B}_1 \neq \mathcal{B}_2) \) \[3-6, 9, 11-13, 16, 19, 26, 28, 30\]

\[
\forall |u\rangle, \ |v\rangle \in \mathcal{B}_1, \mathcal{B}_2 \text{ resp., } |\langle u|v\rangle| = 1/\sqrt{d}.
\]

The physical meaning of this is that knowledge that a system is in a particular state in one basis implies complete ignorance of its state in the other basis.

Ivanovic \[10\] proved that there are at most \( d+1 \) MUB, pairwise, in a \( d \)-dimensional Hilbert space and gave an explicit formulae for the \( d+1 \) bases in the case of \( d=p \) (prime number). Wootters and Fields \[4\] constructed such \( d+1 \) bases for \( d=p^m \) with \( m \) an integer. Variety of methods for construction of the \( d+1 \) bases for \( d=p^m \) are now available \[11, 13, 19\]. Our present study is confined to \( d=p \neq 2 \).

We now give explicitly the MUB states in conjunction with the algebraically complete operators \[1, 9\] set: \( \hat{Z}, \hat{X} \).

Thus we label the \( d \) distinct states spanning the Hilbert space, termed the computational basis, by \( |n\rangle, \ n = 0, 1, ..., d-1; |n+d\rangle = |n\rangle \)

\[
\hat{Z}|n\rangle = \omega^n|n\rangle; \ \hat{X}|n\rangle = |n+1\rangle, \ \omega = e^{i2\pi/d}.
\]

The \( d \) states in each of the \( d+1 \) MUB bases \[3,11\] are the states of computational basis (CB) and

\[
|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{0}^{d-1} \omega^{\frac{1}{2}(n-1)-nm}|n\rangle; \ b, m = 0, 1, ..., d-1.
\]

Here the \( d \) sets labeled by \( b \) are the bases and the \( m \) labels the states within a basis. We have \[11\]

\[
\hat{X}\hat{Z}^b|m; b\rangle = \omega^m|m; b\rangle.
\]

For later reference we shall refer to the computational basis (CB) by \( b=-1 \). Thus the above gives \( d+1 \) bases, \( b=-1, 0, 1, ..., d-1 \) with the total number of states \( d(d+1) \) grouped in \( d+1 \) sets each of \( d \) states. We have of course,

\[
\langle m; b|m'; b'\rangle = \delta_{m,m'}; \ |\langle m; b|m'; b'\rangle| = \frac{1}{\sqrt{d}}, \ b \neq b'.
\]

We remark at this junction that the eigen values of the CB might be considered finite dimensional modulated position values ("q") and the eigenvalues of shifting operator, \( X \), modulated momentum ("p").

This completes our discussion of MUB.
IV. DAPG UNDERPINNING OF D-DIMENSIONAL HILBERT SPACE

We first list some direct consequences of DAPG. DAPG(c) allows the definition:

\[ S_\alpha = \frac{1}{d} \sum_{j \in \alpha} L_j. \]  

(9)

This implies,

\[ \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{j} d^2 L_j, \]

leading via DAPG(d) to

\[ \sum_{\alpha} \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \sum_{\alpha} S_\alpha = \frac{d+1}{d} \sum_{j} d d^2 L_j. \]

(11)

We thus have,

\[ \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{j} d^2 L_j = \frac{1}{d+1} \sum_{\alpha} S_\alpha. \]

(12)

Now the underpinning of Hilbert space operators with DAPG will be undertaken. We consider \( d=p \), a prime. For \( d=p \) we may construct \( d+1 \) MUB \[ 3, 6, 10, 11 \]. Points will be associated with MUB state projectors. To this end we recall that we designate the MUB states by \( |m, b\rangle \), with \( b = 0, 1, 2, \ldots d-1 \) labels the eigenfunction of, resp. \( XZ^b \). \( m \) labels the state within a basis. We designate the computational basis, CB, by \( b=-1 \). The projection operator defined by,

\[ \hat{A}_\alpha \equiv |m, b\rangle \langle b, m|; \quad \alpha = \{b, m\}; \quad b = -1, 0, 1, 2, \ldots d-1; \quad m = 0, 1, 2, \ldots d-1. \]

(13)

The point label, \( \alpha = (m, b) \) is now associated with the projection operator, \( A_\alpha \). We now consider a realization, possible for \( d=p \), a prime , of a d dimensional DAPG, as points marked on a rectangular whose horizontal width (x-axis) is made of \( d+1 \) columns of points. Each column is labeled by \( b \), and its vertical height (y axis) is made of \( d \) points each marked with \( m \). The total number of points is \( d(d+1) \) - there are \( d \) points in each of the \( d+1 \) columns. We associate the \( d \) points \( m = 0, 1, 2, \ldots d-1 \) in each set, labeled by \( b \) (\( \alpha \sim (m, b) \)) to the disjointed points of DAPG(d), \( R_\alpha \) viz. for fixed \( b \) \( \alpha' \in \alpha \cup M_\alpha \) form a column. The columns are arranged according to their basis label, \( b \). The first being \( b=-1, \alpha_1 = (m, -1); m = 1, \ldots d-1 \), pertains to the computational basis (CB). Lines are now made of \( d+1 \) points, each of different \( b \). A point \( S_\alpha \) underpins a Hilbert space state projector, \( A_\alpha \), i.e. \( A_\alpha^2 = A_\alpha \), and \( trA_\alpha = 1 \). We designate the line operator underpinned with \( L_j \), by \( P_j \). Thus the above relations now hold with \( S_\alpha \leftrightarrow A_\alpha; \quad L_j \leftrightarrow P_j \).

E.g. for \( d=3 \) the underpinning’s schematics is,

\[
\begin{pmatrix}
m \backslash b & -1 & 0 & 1 & 2 \\
0 & A_{(0,-1)} & A_{(0,0)} & A_{(0,1)} & A_{(0,2)} \\
1 & A_{(1,-1)} & A_{(1,0)} & A_{(1,1)} & A_{(1,2)} \\
2 & A_{(2,-1)} & A_{(2,0)} & A_{(2,1)} & A_{(2,2)} \\
\end{pmatrix}
\]

DAPG(c) (and Eq.(13), (12)) implies that \( A_\alpha; \quad \alpha = 0, 1, 2, \ldots d-1; \quad \alpha \in \alpha' \cup M_\alpha \), forms an orthonormal basis for the \( d \)-dimensional Hilbert space:

\[
\sum_{m}^d |m, b\rangle \langle b, m| = \sum_{\alpha' \in \alpha \cup M_\alpha} \hat{A}_{\alpha'} = \hat{I}
\]

\[
\sum_{\alpha}^{d(d+1)} \hat{A}_\alpha = (d+1) \hat{I}.
\]

(14)
Eq. (9) implies,

$$ A_\alpha = \frac{1}{d} \sum_{j \in \alpha} P_j. \quad (15) $$

Evaluating

$$ \sum_{\alpha \in j} A_\alpha = \frac{1}{d} \sum_{\alpha \in j} \sum_{j' \in \alpha} P_j = \frac{1}{d} \left[ \sum_{j' \neq j} P_{j'} + (d+1)P_j \right] = I + P_j, \quad (16) $$

i.e.,

$$ P_j = \sum_{\alpha \in j} A_\alpha - I. \quad (17) $$

Eq. (8) implies,

$$ trA_\alpha A_{\alpha'} = \begin{cases} 1; & \alpha = \alpha' \\ 0; & \alpha \neq \alpha'; \alpha \in \alpha' \cup M_{\alpha'} \\ \frac{1}{d}; & \alpha \neq \alpha'; \alpha \ni \alpha' \cup M_{\alpha'}. \end{cases} \quad (18) $$

Hence, using Eq. (9), (8),

$$ trA_\alpha P_j = \begin{cases} \sum_{\alpha' \ni \alpha} trA_\alpha A_{\alpha'} = 1; & \alpha \in j. \\ \sum_{\alpha' \neq \alpha} trA_\alpha A_{\alpha'} - A_\alpha = 0; & \alpha \ni j. \end{cases} \quad (19) $$

Trivially

$$ trP_j = \sum_{\alpha \in j} trA_\alpha - 1 = 1. \quad (20) $$

$$ trP_j P_{j'} = \sum_{\alpha' \in j'} trP_j A_{\alpha'} - 1 = \begin{cases} d & j = j' \\ 0 & j \neq j'. \end{cases} \quad (21) $$

i.e.

$$ trP_j P_{j'} = d \delta_{j,j'}. \quad (22) $$

An alternative view of $trA_\alpha P_j$ is gained via

$$ trA_\alpha P_j = \frac{1}{d} \sum_{j' \in \alpha} P_{j'} P_j = \begin{cases} \frac{1}{d} (trP_j^2 + tr \sum_{j' \ni \alpha} P_{j'} P_j) = 1; & j \in \alpha \\ \sum_{j' \neq j} P_{j'} P_j = 0; & j \ni \alpha. \end{cases} \quad (23) $$

Note that the case of $j \ni \alpha$ implies $j \in M_\alpha$. These are summarized by

$$ trA_\alpha P_j \equiv A_{\alpha,j} = \begin{cases} 1; & \alpha \in j, \\ 0; & \alpha \ni j, \quad = \begin{cases} 1; & j \in \alpha \\ 0; & j \ni \alpha. \end{cases} \quad (24) $$
V. GEOMETRIC UNDERPINNING OF MUB QUANTUM OPERATORS: THE LINE OPERATOR

We now consider a particular realization of DAPG of dimensionality $d = p, \neq 2$ which is the basis of our present study. We arrange the aggregate the $(d(d+1))$ points, $m$, in a $d \times (d+1)$ matrix like rectangular array of $d$ rows and $d+1$ columns. Each column is made of a set of $d$ points $R_{m} = \bigcup_{\alpha} R_{\alpha}$; DAPG(d). We label the columns by $b=-1,0,1,2,...,d-1$ and the rows by $m=0,1,2...d-1.$ (Note that the first column label of -1 is for convenience and does not designate negative value of a number.) Thus $m=b$ designate a point by its row, $m$, and its column, $b$; when $b$ is allowed to vary - it designate the point’s row position in every column. We label the left most column by $b=-1$ and with increasing values of $b$, the basis label, as we move to the right. Thus the right most column is $b=d-1$. We now assert that the $d+1$ points, $m_{b}(b), b=0,1,2,...d-1$, and $m_{b}(-1)$, that form the line $j$ which contain the two (specific) points $m(-1)$ and $m(0)$ is given by (we forfeit the subscript $j$ - it is implicit),

$$m(b) = \frac{b}{2} (c - 1) + m(0), \mod[d] \ b \neq -1,$$

$$m(-1) = c/2. \tag{25}$$

The rationale for this particular form will be clarified below. Thus a line $j$ is parameterized fully by $j = (m(-1), m(0))$. (Note: since $b$ takes on the values -1 and 0 in our line labeling a more economic label for $j$ is $j = (m-1, m_0)$ i.e. the $m$ values for $b=-1$ and 0. We shall use either when no confusion should arise.) We now prove that the set $j = 1, 2, 3...d^2$ lines covered by Eq.(25) with the points as defined above form a DAPG.

1. Since each of the parameters, $m(-1)$ and $m(0)$, can have $d$ values the number of lines is $d^2$; the number of points in a line is evidently $d+1$: a point for each $b$. DAPG(a).
2. The linearity of the equation precludes having two points with a common value of $b$ on the same line, DAPG(b). Now consider two points on a given line, $m(b_1), m(b_2)$; $b_1 \neq b_2$. We have from Eq.(25), $(b \neq -1, b_1 \neq b_2)$

$$m(b_1) = \frac{b_1}{2} (c - 1) + m(0), \mod[d]$$

$$m(b_2) = \frac{b_2}{2} (c - 1) + m(0), \mod[d]. \tag{26}$$

These two equations determine uniquely (for $d=p$, prime) $m(-1)$ and $m(0)$. DAPG(b).

For fixed point, $m(b)$, $c \leftrightarrow m(0)$ i.e the number of free parameters is $d$ (the number of points on a fixed column). Thus each point is common to $d$ lines. That the line contain $d+1$ is obvious. DAPG(c).

3. As is argued in 2 above no line contain two points in the same column (i.e. with equal $b$). Thus the $d$ points, $\alpha$, in a column form a set $R_{\alpha} = \bigcup_{\alpha} R_{\alpha}$, with trivially $R_{\alpha} \cap R_{\alpha'} = \emptyset$, $\alpha \neq \alpha'$, and $\bigcup_{\alpha=1}^{d} R_{\alpha} = \bigcup_{\alpha=1}^{d} S_{\alpha}$. DAPG(d).

4. Consider two arbitrary points not in the same set, $R_{\alpha}$ defined above: $m(b_1), m(b_2)$ $(b_1 \neq b_2)$. The argument of 2 above states that, for $d=p$, there is a unique solution for the two parameters that specify the line containing these points. DAPG(c).

We illustrate the above for $d=3$, where we explicitly specify the points contained in the line $j = (m(-1) = (1, -1), m(0) = (2, 0))$

$$m\backslash b \quad -1 \quad 0 \quad 1 \quad 2 \quad$$

$$0 \quad \cdot \quad \cdot \quad (0, 2)$$

$$1 \quad (1, -1) \quad (1, 1) \quad \cdot$$

$$2 \quad (2, 0) \quad \cdot \quad \cdot$$

For example the point $m(1)$ is gotten from

$$m(1) = \frac{1}{2}(2 - 1) + 2 = 1 \mod[3] \rightarrow m(1) = (1, 1).$$

Similar calculation gives the other point: $m(2)=(0,2)$. i.e. the line $j=(1,2)$ contains the points $(1,-1),(2,0),(1,1)$ and $(0,2)$.

The geometrical line, $L_j$, $j = (1, 2)$ given above upon being transcribed to its operator formula is via Eq.(17),

$$P_{j=1,2} = A_{(1,-1)} + A_{(2,0)} + A_{(1,1)} + A_{(0,2)} - I. \tag{27}$$
Evaluating the point operators, \( \hat{A}_\alpha \),

\[
A_{(1,-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{(2,0)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad A_{(1,1)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad A_{(0,2)} = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix},
\]

(28)

and evaluating the sum, Eq. (27), gives

\[
P_{\hat{\alpha};(m(-1)=1,m(0)=2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}.
\]

(29)

This operator obeys \( P_{\hat{\alpha};(m(-1)=1,m(0)=2)}^2 = \hat{I} \). We shall now show that this is quite general, viz \( P_{\hat{\alpha};j}^2 = \hat{I}, \forall j \).

Returning to Eqs. (27), these equations imply that, the projection operators \( A_\alpha \), in the CB representation are given by,

\[
(A_{\alpha=m,b})_{n,n'} = \begin{cases} \frac{\omega^s}{\delta_{n,n'}\delta_2}, & s = (n-n')(b|n+n'-1) - m, \quad b \neq -1, \\ \delta_{n,n'}\delta_2, & b = -1. \end{cases}
\]

(30)

Consider two distinct columns, \( b,b' \neq -1 \) and given the matrix elements \( n,n' \) \( (n \neq n') \) of a projector \( (A_{\alpha=m,b})_{n,n'} \), compare it with \( (A_{\alpha'=m',b'})_{n,n'} \). If \( s \) \( (\text{Eq.}(30)) \) is \( \neq s' \) i.e. \( \frac{b}{2}(n+n') - 1 \neq m \neq \frac{b'}{2}(n+n') - 1 \).\( m' \) pick another projector in the same column, \( b' \) (i.e vary \( m' \)). Since \( m' = 0,1,2,..d-1 \) there is one (and only one) \( (A_{\alpha'=m',b'})_{n,n'} \) such that \( (A_{\alpha=m,b})_{n,n'} = (A_{\alpha'=m',b'})_{n,n'} \). Now consider another matrix element \( (A_{\alpha})_{n,n'} \). We have trivially that \( (A_{\alpha})_{n,n'} = (A_{\alpha'})_{n,n'} \) iff \( \tilde{n} + \tilde{n}' = n + n' \). i.e. all matrix elements \( (n,n') \) with \( n+n' = c \) (constant) are such that \( (A_{\alpha})_{n,n'} = (A_{\alpha'})_{n,n'} \). These elements are situated along a line perpendicular to the diagonal of the matrices. We refer to this perpendicular as FV (foliated vector), it is parameterized by \( c \).

We now assert that all other (non diagonal) matrix elements are unequal. i.e. for \( b \neq b' \), \( (A_{\alpha=m,b})_{n,n'} \neq (A_{\alpha'=m',b'})_{n,n'} \), \( \forall \ n,n' \ni \text{FV} \). Proof: Let two elements \( n,n' \) and \( 1,1' \) with \( n \neq n' \); \( 1 \neq 1' \) in the two matrices be equal. Thus \( \langle c=n+n', c'=1+1' \rangle:

\[
\frac{b}{2}(c-1) - m = \frac{b'}{2}(c-1) - m', \quad \text{and}
\]

\[
\frac{b}{2}(c'-1) - m = \frac{b'}{2}(c'-1) - m',
\]

(31)

These imply \( c=c' \), QED. Now consider \( s=0 \). Then all the matrix elements along FV are \( 1/d \). We have then that for \( (A_{\alpha})_{n,n'} = (A_{\alpha'})_{n,n'} = \omega^s/d, \ s \neq 0, \text{d-1} \text{matrix elements along FV are all distinct. The diagonal is common to all. We have thus a prescription for d projectors, } A_{(m,b)}, \text{one for each } b, (b \neq -1), \text{all having equal matrix elements along FV labelled by } c. \text{ We supplement these with the projector } A_{(c/2,-1)} = |c/2⟩⟨c/2| \text{ to have the d+1 "points" constituting a line } j. (c/2) \text{ being a state in the CB.) Thus our line is formed as follows: It emerges from } A_{(c/2,-1)} \text{ continues to } A_{(m(0),0)} \text{ in the b=0 column. Then it continues to the points } A_{(m,b),b} \text{ in succession: } b=1,2,...d-1 \text{ with } m(b) \text{ determined by}

\[
\frac{b}{2}(c-1) - m(b) = \frac{b+1}{2}(c-1) - m(b+1).
\]

Thus the two parameters, \( c=2m(-1) \) and \( m(1) \), determine the line i.e. \( j=(m(-1),m(1)) \). The general formula for the line, Eq. (25) now acquires a meaning in terms of the point operators, \( A_{(m,b),b} \). The discussion of the properties of the line thus defined confirm that these lines form a realization of DAPG lines. The analysis above indicate that the line operator, \( P_j \), may be labelled by two indices \( P_j=(m(-1),m(0)) \). We now list some important consequences of this. We have shown that the matrix elements along a FV direction are the same for all the point operators \( A_{\alpha \in j} \). Indeed that is how we defined our lines. On the other hand we argued that the matrix elements not along the FV are all distinct. Thence summing up d such terms residing on a fixed line \( P_j \) (excluding the } b=-1 \text{ and the diagonal term) sums up for each matrix element \( n,n' \) the d distinct roots of unity for matrix elements not on FV, hence for all c,

\[
\sum_{\alpha \in j, \alpha \geq \alpha-1} \hat{A}_\alpha - \hat{f}_{n,n'} = 0; \ n, n' \ni n+n' = c; \ \alpha-1 = |c/2⟩⟨c/2|.
\]

(32)
Thus \((\hat{P}_2)_{n,n'} = (\sum_{\alpha \in j} \hat{A}_\alpha - \hat{I})_{n,n'} \neq 0\) only along \(FV\), and is 1 along the diagonal at \(c/2=m(-1)\). The sum over \(\alpha \in j\) of the matrix elements on a \(FV\), which are equal for all \(\hat{A}_{\alpha \in j,\neq -1}\), simply cancel the \(1/d\). This is illustrated in Eqs.(point1),(pj). Quite generally,

\[
(P_{j=m(-1),m(0)})_{n,n'} = \begin{cases} 
\omega^{-(n-n')m(0)}\delta_{((n+n'),2m(-1))} \\
0 \quad \text{otherwise.}
\end{cases}
\]  

Thus,

\[
(\hat{P}^2_{j=m(-1),m(0)})_{n,n'} = \delta_{n,n'}. \quad \text{i.e. } \hat{P}^2_{j} = \hat{I} \forall j.
\]  

In appendix A we show that \(\hat{P}^2_{j} = \hat{I} \forall j\) implies the operator relation,

\[
\sum_{\alpha \neq \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = \sum_{\alpha \in j} \hat{A}_\alpha.
\]

\[\text{VI. AFFINE PLANE GEOMETRY (APG)}\]

We now recast our DAPG underpinning into an APG one. This is achieved by interchanging lines with points. For notational convenience we refer to \(m(-1)\) and \(m(0)\) of the DAPG (cf. Eq.()) by \(\xi\) and \(\eta\) respectively in their APG image. Thus a line in DAPG \(j=(m(-1),m(0))\) is a point \((\xi, \eta)\) in its APG image, \(\xi, \eta = 0, 1, \ldots, d-1\). We now construct the following realization of a \(d\) (=prime) dimensional APG: consider \(d \cdot d\) points arranged in a square array of \(d\) columns and \(d\) rows. Each point is an image of DAPG line and is specified by \(s, \eta\)  with \(s=0,1,\ldots,d-1\) and \(\eta=0,1,\ldots,d-1\). The \(d\) lines "emerging" from the point \(s'=0,1,\ldots,d-1\) divide the \(d\) points residing on a APG line share a point. The proof is as follows:

1. The \(d\) points \(\xi = 0, 1, \ldots, d-1\) on the line \(\xi = s'\) trivially share the point \(m(-1)=s'\). They are the images of the d lines "emerging" from the point \(s'\) in the column \(b=-1\) of the DAPG realization.

2. Consider two arbitrary APG points \((\eta', \xi')\) and \((\eta'', \xi'')\) that lie on the line \(\eta = r\xi + s \mod[d]\) For their lines DAPG images to share a point we must have, Eq.(), \(b/2(2\xi' - 1) + r\xi' + s\) equals \(b/2(2\xi'' - 1) + r\xi'' + s \mod[d]\), i.e. \(b+r=0, \mod[d]\). Thus their common point is \(m(b) = -r/2(2\xi' - 1) + r\xi' + s\), \(m(b) = -r/2(2\xi'' - 1) + r\xi'' + s\) i.e. in the column \(b=-r\mod[d]\), row \(m=s+r/2\mod[d]\). This is true for all the \(d\) points on that line. QED.
Thus the interrelation among the operators, $A_\alpha$ and $P_j$ are identical whether given within DAPG or APG. e.g. Eq.(24)

$$A_\alpha = \frac{1}{d} \sum_{j \in \alpha} P_j$$

is, within, DAPG gives the point operator $A_\alpha$ in terms of the d line operators $P_j$ - while within APG this very same equation gives the very same Hilbert space operator, $A_\alpha$, now a line operator in terms of the d point operators $P_j$.

As an example let us consider for $d=3$ the APG line $\eta = \xi + 1$. The APG points on this line are $(0,1),(1,2)$ and $(2,0)$ reflecting DAPG line operators, cf Eq.(33),

$$P_{(0,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega & 0 \end{pmatrix}; \quad P_{(1,2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix}; \quad P_{(2,0)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix}. \quad (36)$$

Now, Eq.(27) relates these to the DAPG point (i.e. the MUB projector) $|0,2\rangle\langle 2,0| = A_{(0,2)}$

$$\frac{1}{3} \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega & 0 \end{pmatrix} \right] + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix} \right] = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (37)$$

where the last matrix is $A_{(0,2)}$, cf. Eq.(28).

This completes our discussion of geometrical underpinning of finite dimensional Hilbert space concomitant with operators interrelationships. We now turn to its possible use in the mappings of Hilbert space operators onto the phase space like points and lines of the geometry.

VII. MAPPING ONTO PHASE SPACE

We now define a mapping of Hilbert space operators, e.g. an arbitrary operator, $B$, onto the phase space - like lines of DAPG. The mapping is defined by $29$,

$$B \Rightarrow V(j;B) \equiv trBP_j. \quad (38)$$

Here $P_j$ is a line operator within DAPG. (Alternatively we could have cast the mappings within APG as is clear from the discussion in the previous section.) The density operator may be expressed in terms of $V(j;\rho)$:

$$\rho = \frac{1}{d} \sum_{j} (tr\rho P_j)P_j. \quad (39)$$

It can be shown that $V(j;\rho)$ plays the role of quasi distribution $29$ in the phase space like lines of DAPG. Thus for example the expectation value of an arbitrary operator $B$ we have,

$$tr\rho B = \frac{1}{d} \sum_{j} V(j;\rho)V(j;B). \quad (40)$$

The quasi distribution may be reconstructed from the expectation values of the point operator $A_\alpha$ i.e. MUB state projector’s expectation value (obtained, e.g. by measurements),

$$tr\rho A_\alpha = \frac{1}{d} \sum_{j} V(j;\rho)V(j;A_\alpha) = \frac{1}{d} \sum_{j} V(j;\rho)\Lambda_{\alpha,j}. \quad (41)$$

Thence

$$\frac{1}{d} \sum_{\alpha \in \alpha' \cup M_\alpha'} \sum_{j} V(j;\rho)\Lambda_{\alpha,j} = V(j;\rho). \quad (42)$$
VIII. SUMMARY AND CONCLUDING REMARKS

It is of interest that, if we associate the CB states with the position variable, q, of the continuous problem and its Fourier transform state, viz b=0 (cf. Eq. (8)), with the momentum, p, we have that the line of the finite dimension problem is parameterized with "initial" values of "q" and "p" i.e. m(-1) and m(0).

Finite geometry stipulates interrelations among lines and points. The stipulations for the (finite) dual affine plane geometry (DAPG) was shown to conveniently accommodate association of geometric lines and points with projectors of states of mutual unbiased bases (MUB). The latter act in a (finite dimensional, d) Hilbert space. This underpinning of Hilbert space operator with DAPG reveal some novel inter operators relations. Noteworthy among these are Hilbert space operators, \( P_j \), j = 1, 2, ..., d, which are underpinned with DAPG lines, \( L_j \) that abide by \( P_j^2 = \hat{I} \) \( \forall j \), and are mutually orthogonal, \( \text{tr} P_j P_{j'} = d \delta_{jj'} \). These allow their utilization for general mapping of Hilbert space operators onto the phase space like lines and points of DAPG in close analogy with the mappings within the continuum of Hilbert space operators onto phase space via the well known Wigner function [8, 25]. If we associate the computational basis (CB) states with the position variable, q, of the continuous problem, and its Fourier transform state (i.e. states of the basis that is diagonal for translation operator, X (cf. Eq. (8)) with the momentum, p, we have that the line of the finite dimension problem is parameterized with these phase space like variables. We present a transcription from DAPG to (finite) affine plane geometry (APG) underpinnings. Within the latter such labelling is natural and, further more, the line operators here include points that are aligned in a straight line. This interpret the APG operator underpinning that is given in [5] as due to the association of MUB state projectors with points within DAPG.

Appendix: Fluctuation Distillation Formula

Given, Eq(34), \( \hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I} \) and, Eq(34), \( \hat{P}_j^2 = \hat{I} \), implies

\[
\left( \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I} \right) \left( \sum_{\alpha' \in j} \hat{A}_\alpha' - \hat{I} \right) = \hat{I}.
\]

Thus,

\[
\sum_{\alpha, \alpha' \in j} \hat{A}_\alpha \hat{A}_\alpha' = 2 \sum_{\alpha \in j} \hat{A}_\alpha.
\]

Recalling that, Eq(13), \( A_\alpha^2 = A_\alpha \) allows

\[
\sum_{\alpha \neq \alpha' \in j} \hat{A}_\alpha \hat{A}_\alpha' = \sum_{\alpha \in j} \hat{A}_\alpha.
\]

QED

[1] J. Schwinger, Proc. Nat. Acad. Sci. USA 46, 560 (1960).
[2] W. Schleich, Quantum Optics in Phase Space, Wiley-Vch (2001).
[3] W. K. Wootters, Ann. Phys. (N.Y.) 176, 1 (1987).
[4] W. K. Wootters and B. D. Fields, Ann. Phys. (N.Y.) 191, 363 (1989).
[5] K. S. Gibbons, M. J. Hoffman and W. K. Wootters, Phys. Rev. A 70, 062101 (2004).
[6] W. K. Wootters, Found. of Phys. 36, 112 (2006).
[7] F. C. Khanna, P. A. Mello and M. Revzen, submitted for publication (2011).
[8] D. Ellinas and A. J. Bracken, Phys. Rev. A, 78 052106 (2008).
[9] A. Kalev, M. Revzen and F. C. Khanna, Phys. Rev. A 80, 022112 (2009).
[10] I. D. Ivanovic, J. Phys. A, 14, 3241 (1981).
[11] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury and F. Vatan, Algorithmica 34, 512 (2002).
[12] I. Bengtsson, AIP Conf. Proc. 750, 63-69 (2005); quant-ph/0406174
[13] A. Vourdas, Rep. Math. Phys. 40, 367 (1997), Rep. Prog. Phys. 67, 267 (2004).
[14] A. B. Klimov, L. L. Sanchez-Soto and H. de Guise, J. Phys. A: Math. Gen. 38, 2747 (2005).
[15] M. Saniga, M. Planat and H. Rosu, J. Opt. B: Quantum Semiclassic Opt. 6 L19 (2004).
[16] M. Planat and H. C. Rosu, Europ. Phys. J. 36, 133 (2005).
[17] M. Planat, H. C. Rosu and S. Perrine, Foundations of Physics 36, 1662 (2006).
[18] M. Combescure, quant-ph/0605090 (2006).
[19] A. B. Klimov, C. Munos and J. L. Romero, quant-ph/0605113 (2005).
[20] M. Grassl, Electronic Notes in Discrete Mathematics 20, 151 (2005).
[21] T. Bar-On, Jour. Math. Phys. 50, 072106 (2009).
[22] M. Revzen quant-phys/0912.5433 (2009).
[23] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error Correcting Codes, North Holland, Amsterdam, (1977).
[24] S. A. Shirakova, Russ. Math. Surv. 23 47 (1968).
[25] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).
[26] M. Revzen, P.A. Mello, A. Mann and L.M. Johansen, Phys. Rev. A 71, 022103 (2005)
[27] Amir Kalev, Ady Mann, Pier A. Mello and Michael Revzen, Phys. Rev A 79, 014104 (2009).
[28] M. Revzen, Phys. Rev. A 81, 012113 (2010).
[29] M. Revzen, arXiv:quant-ph/1111.2846.
[30] A. Peres, Quantum Theory: Concepts and Methods (Kluwer, Dordrect, 1995).