ON WEAK MODEL SETS OF EXTREMAL DENSITY

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Abstract. The theory of regular model sets is highly developed, but does not cover examples such as the visible lattice points, the $k$th power-free integers, or related systems. They belong to the class of weak model sets, where the window may have a boundary of positive measure, or even consists of boundary only. The latter phenomena are related to the topological entropy of the corresponding dynamical system and to various other unusual properties. Under a rather natural extremality assumption on the density of the weak model set, we establish its pure point diffraction nature. We derive an explicit formula that can be seen as the generalisation of the case of regular model sets. Furthermore, the corresponding natural patch frequency measure is shown to be ergodic. Since weak model sets of extremal density are generic for this measure, one obtains that the dynamical spectrum of the hull is pure point as well.

1. Introduction

The theory of regular model sets, which are also known as cut and project sets with sufficiently nice windows, is well established; see [1] and references therein for general background. One cornerstone of this class is the pure pointedness of the diffraction measure [21, 36, 9]. Equivalently, this means that the dynamical spectrum of the uniquely ergodic hull defined by the model set is pure point as well; compare [25, 5, 27]. The regularity of the window is vital to the existing proofs such as that in [36], and also enters the characterisation of regular model sets via dynamical systems [6].

For quite some time, systems such as the visible lattice points or the $k$th power-free integers have been known to be pure point diffractive as well [10]. These point sets can also be described as model sets, but here the windows are no longer regular. In fact, for each of these examples, the window consists of boundary only, which has positive measure, and many other properties of regular model sets are lost, too. In particular, there are many invariant probability measures on the orbit closure (or hull) of the point set under the translation action of the lattice. Yet, as explicit recent progress has shown, the natural cluster (or patch) frequency measure of this hull is ergodic and the visible points are generic for this measure [3]. Consequently, the dynamical spectrum is still pure point, by an application of the general equivalence theorem [5]. Since this example is one out of a large class with similar properties, it is natural to ask for a general approach that includes all of them. Such a class is provided by weak model sets, where one allows more general windows. This name was coined by Moody [31, 32], see also [1 Rem. 7.4], and apparently was first looked at by Schreiber [37].

It is the purpose of this paper to derive some key results for weak model sets. To this end, we begin with the visible lattice points as a motivating example. Then, we start from
the general setting of model sets for a general cut and project scheme \((G, H, \mathcal{L})\), see Eq. \((2)\) below for a definition, but investigate the diffraction properties for the larger class of windows indicated above. It turns out that this is indeed possible under one fairly natural assumption, namely that of maximal or minimal density for a given van Hove averaging sequence in the group \(G\). This assumption guarantees pure point diffractivity (Theorems \(7\) and \(9\)).

In a second step, we analyse the ergodicity of the cluster frequency measure for the very van Hove sequence, which then results in the dynamical properties of the hull we are after. In particular, we establish that weak model sets of extremal density have pure point dynamical spectrum, and calculate the latter. Finally, we apply our results to coprime lattice families, which encompasses the \(k\)-visible lattice points in \(d\)-space as well as other examples of arithmetic origin, such as \(k\)-free or (coprime) \(B\)-free integers \([33, 3, 17, 12, 24]\) and their generalisations to analogous systems in number fields \([14, 3, 11]\). This way, we demonstrate that and how the theory of weak model sets provides a natural framework for a unified treatment of such systems.

In parallel to our approach, Keller and Richard \([23]\) have developed an alternative view on model sets via a systematic exploitation of the torus parametrisation for such systems; compare \([2, 19, 36, 6]\). Their work includes weak model sets and provides an independent way to derive several of our key results. In this sense, the two approaches are complementary and, in conjunction, give a more complete picture of a larger class of model sets than understood previously, both concretely and structurally.

2. Preliminaries and background

Our general reference for background and notation is the recent monograph \([1]\). Here, we basically summarise some key concepts and their extensions in the generality we need them. Let \(G\) be a locally compact Abelian group (LCAG), and denote the space of translation bounded (and generally complex) measures on \(G\) by \(\mathcal{M}_\infty(G)\). Here and below, measures are viewed as linear functionals on the space \(C_c(G)\) of continuous functions with compact support, which is justified by the general Riesz–Markov theorem; see \([13]\) for general background. In this setting, we use \(\mu(g)\) and \(\int_G g \, d\mu\) for an integrable function \(g\) as well as \(\mu(A) = \int_A 1_A \, d\mu\) for a Borel set \(A\) exchangeably. For a measure \(\mu\), we define its twisted version \(\tilde{\mu}\) by \(\tilde{\mu}(g) = \mu(\tilde{g})\) for \(g \in C_c(G)\) as usual, where \(\tilde{g}(x) := g(-x)\).

If \(\mu\) is a finite measure on \(G\), we define its norm as \(\|\mu\| = |\mu|(G)\), where \(|\mu|\) denotes the total variation of \(\mu\). More generally, for \(\nu \in \mathcal{M}_\infty(G)\) and any compact set \(K \subseteq G\), we define

\[
\|\nu\|_K = \sup_{t \in G} |\nu|(t + K).
\]

It is clear that \(\nu \in \mathcal{M}_\infty(G)\) means \(\|\nu\|_K < \infty\) for any compact \(K \subseteq G\).

**Fact 1.** Let \(\mu\) be a finite measure on \(G\), let \(\nu \in \mathcal{M}_\infty(G)\) and \(g \in C_c(G)\). If \(\text{supp}(g) \subseteq K\), with \(K \subseteq G\) compact, one has the estimates

\[
\|\mu * \nu * g\|_\infty \leq \|\mu\| \|\nu\|_K \|g\|_\infty \quad \text{and} \quad |(\mu * \nu)(g)| \leq \|\mu\| \|\nu\|_K \|g\|_\infty.
\]
Proof. Since $\nu * g$ defines a continuous function, $\|\nu * g\|_\infty \leq \|\nu\|_K \|g\|_\infty$ follows from standard arguments. Then, one finds

$$\left| (\mu \ast \nu \ast g)(x) \right| = \left| \int_G (\nu * g)(x - y) \, d\mu(y) \right| \leq \int_G \left| (\nu * g)(x - y) \right| \, d\|\mu\|(y) \leq \int_G \|\nu * g\|_\infty \, d\|\mu\| = \|\nu * g\|_\infty \|\mu\| \leq \|\mu\| \|\nu\|_K \|g\|_\infty,$$

which proves the first claim.

Next, observe that $(\mu \ast \nu)(g) = \int_{G \times G} g(x + y) \, d\mu(x) \, d\nu(y) = (\mu \ast \nu \ast g)(0)$, where $g$ is defined by $g_\cdot(x) := g(-x)$, so that $\|(\mu \ast \nu)(g)\| \leq \|\mu \ast \nu \ast g\|_\infty$, and the second claim follows from the first because $\|g\|_\infty = \|g\|_\infty$. □

Let $H$ be a compactly generated LCAG, hence (up to isomorphism) of the form $\mathbb{R}^d \times \mathbb{Z}^n \times \mathbb{K}$ for some integers $d, n \geq 0$ and some compact Abelian group $\mathbb{K}$; compare [20, Thm. 9.8]. We assume $H$ to be equipped with its Haar measure $\theta = \theta_H$, where we follow the standard convention that this is Lebesgue measure on $\mathbb{R}^d$, counting measure on $\mathbb{Z}^n$ and normalised on compact groups, so $\theta_{\mathbb{K}}(\mathbb{K}) = 1$. The Haar measure on $G$ is denoted by $\theta_G$, where we will use $dt$ instead of $d\theta_G(t)$ for integration over (subsets of) $G$. Also, we will write $\text{vol}(A)$ instead of $\theta_G(A)$ for measurable sets $A \subset G$.

The covariogram function $c_W$ of a relatively compact Borel set $W \subseteq H$ is the real-valued function $c_W$ defined by

$$c_W(x) = \bigl(1_W \ast \mathring{1}_W\bigr)(x),$$

where convolution is defined via $\theta_H$ as usual. Note that the value at 0 is given by

$$c_W(0) = \int_H \bigl|1_W(x)\bigr|^2 \, d\theta_H(x) = \theta_H(W). \tag{1}$$

Fact 2. Let $W$ be a relatively compact Borel set in a compactly generated LCAG $H$. Then, the corresponding covariogram function $c_W$ is bounded and uniformly continuous on $H$.

Proof. Both $1_W$ and $\mathring{1}_W$ are elements of $L^1(H) \cap L^\infty(H)$, whence $1_W \ast \mathring{1}_W$ is well-defined. The convolution of an $L^1$ function with an $L^\infty$ function is uniformly continuous and bounded by standard arguments [35, Thm. 1.1.6]. □

Next, we need a cut and project scheme (CPS) as introduced in [29], coded by a triple $(G, H, \mathcal{L})$; see also [30, 31, 1] for background. Here, we use a LCAG $G$ as direct space, another LCAG $H$ as internal space, and a lattice $\mathcal{L} \subset G \times H$ subject to some further restrictions as follows,

$$\begin{align*}
G \xleftarrow{\pi} G \times H & \xrightarrow{\pi_{\text{int}}} H \quad \cup \quad \cup \quad \cup \text{dense} \\
\pi(\mathcal{L}) \xleftarrow{1^{-1}} \mathcal{L} & \xrightarrow{\pi_{\text{int}}(\mathcal{L})} L^* \quad \parallel \quad \parallel \\
L & \xrightarrow{\ast} L^*
\end{align*}\tag{2}$$
Here, $\pi$ and $\pi_{\text{init}}$ denote the natural projections. Since the lattice is located within $G \times H$ such that its projection into $G$ is $1 - 1$, one inherits a well-defined $*$-map from $L$ into $H$, which will become important later on. Note that, when $G$ is torsion-free, the $*$-map has a unique extension to $\mathbb{Q}L$, which is a particularly useful property for $G = \mathbb{R}^d$. In our exposition below, we will further assume that $G$ is $\sigma$-compact and $H$ is compactly generated.

A projection set (or cut and project set) in the strict sense is any set of the form

$$\mathcal{W}(W) = \{x \in L \mid x^* \in W\}$$

with $W \subseteq H$. Such a set is called a model set, if $W \subset H$ is relatively compact with non-empty interior. When $\emptyset \neq W = \overline{W}^c$ is compact, the window is called proper. When, in addition, $\theta_H(\partial W) = 0$, the model set is called regular. In this situation, a highly developed theory is at hand; see [1] and references therein for background. The known results easily generalise to relatively compact windows, when $\theta_H(W^c) = \theta_H(\overline{W}^c)$, which is often needed for practical examples. Here, we are interested in the significantly more general situation where one only demands $W \subseteq H$ to be a relatively compact set with $\theta_H(\overline{W}) > 0$, without further conditions. The corresponding cut and project set $\mathcal{W}(W)$ is then called a weak model set, and one generally has the chains of inclusions

$$\{\text{regular model sets}\} \subseteq \{\text{model sets}\} \subseteq \{\text{weak model sets}\} \subseteq \{\text{projection sets}\}.$$  

Note that weak model sets can have rather curious properties. In particular, they need neither be Meyer sets nor even Delone sets. A classic example, which we will discuss below again in some detail, is provided by the visible points of a lattice in Euclidean space. Let us also stress that our condition $\theta_H(\overline{W}) > 0$ essentially excludes point sets with vanishing upper density. We will not consider more general situations in this paper.

Remark 1. If $W \subset H$ is relatively compact, there is a compact neighbourhood $K$ of $0 \in H$ such that $W \subseteq K \subseteq H$. If we had started with a general LCAG $H$ in our CPS, we could now reduce the CPS to one with the group $H_0$ generated by $K$ instead of $H$. In this sense, our assumption that $H$ be compactly generated is no restriction.

For our extensions below, we also need the concept of a weighted model set. By this we mean a marked set of the form $\{(x, h_x) \mid x \in \mathcal{W}(W)\}$ where the $h_x$ are real or complex numbers, usually assumed bounded. Of particular relevance is the case that the weights satisfy $h_x = c(x^*)$ with a continuous, real- or complex-valued function $c$ on $H$. Particularly nice properties emerge when $c$ is compactly supported [20]. Moreover, if $c$ is also positive definite (or a linear combination of functions of that class), one obtains a powerful extension of the Poisson summation formula to weighted Dirac combs [1][34].

To formulate it, we need a dual to the CPS of Eq. (2). First, given an LCAG $G$, its dual, $\hat{G}$, is the set of continuous characters $\chi: G \rightarrow S^1$, which is an LCAG again, with multiplication of characters as group operation. For our purposes, it is advantageous to write the group additively, by identifying a character $\chi(h) = \chi_u(\cdot)$ with a pairing $\langle u, \cdot \rangle$, so that $\chi_u \chi_v = \chi_{u+v}$ in analogy to $\chi_u(x) = e^{2\pi iux}$ in the important case $G = \mathbb{R}^d$, where $ux$ is the standard inner product in $\mathbb{R}^d$. Now, using this additive notation, and observing the natural isomorphism
The dual CPS \([30, 31]\) is given by

\[
\widehat{G} \times \widehat{H} \cong \hat{G} \times \hat{H}, \quad \pi \left( \mathcal{L}^0 \right) \xleftarrow{1-1} L^0 \xrightarrow{\pi_{\text{int}}} \pi_{\text{int}}(\mathcal{L}^0)
\]

without further restrictions on \(\hat{G}\) and \(\hat{H}\). Here, to define \(\mathcal{L}^0\), we make use of the fact that \(L\) from the original CPS \((2)\) has the form \(L = \{ (x, x^*) \mid x \in L \}\), which permits us to define

\[
\mathcal{L}^0 := \{ (u, v) \in \hat{G} \times \hat{H} \mid \langle u, x \rangle \langle v, x^* \rangle = 1 \text{ for all } x \in L \}
\]

which is a lattice for the new CPS; compare \([30, \text{Sec. 5}]\) and references therein for more.

The important properties indicated in Eq. \((3)\) are inherited from the original CPS \([30]\), so we have once again a well-defined \(*\)-map, for which we use the same symbol. In particular, \(\mathcal{L}^0\) can also be written as \(\mathcal{L}^0 = \{ (u, u^*) \mid u \in L^0 \}\). Under the isomorphism \(\widehat{G} \times \widehat{H} \cong \hat{G} \times \hat{H}\), one sees that \(\mathcal{L}^0\) becomes the annihilator of \(L \subset G \times H\) in the dual group \(\hat{G} \times \hat{H}\), and also that one has a natural isomorphism

\[
\widehat{\mathcal{L}}^0 \cong \mathbb{T} = (G \times H) / L.
\]

Note that, in the Euclidean setting, \(\mathcal{L}^0\) coincides with the standard dual lattice \(\mathcal{L}^*\) of \(\mathcal{L}\).

In this setting, we have the following important result.

**Theorem 3.** Consider a CPS \((G, H, \mathcal{L})\) according to Eq. \((2)\), and fix some \(c \in C_c(H)\) that is a positive definite function on \(H\). Then, the weighted Dirac comb

\[
\omega_c := \sum_{x \in L} c(x^*) \delta_x
\]

is a translation bounded pure point measure that is Fourier transformable, with

\[
\hat{\omega}_c = \text{dens}(\mathcal{L}) \sum_{u \in \mathcal{L}^0} \hat{c}(-u^*) \delta_u,
\]

where \(L^0 = \pi(\mathcal{L}^0)\) according to the dual CPS of Eq. \((3)\). Here, \(\hat{\omega}_c\) is a translation bounded and positive pure point measure on \(\hat{G}\).

**Sketch of proof.** The result is a consequence of the Poisson summation formula (PSF) together with the uniform distribution of the lifted points in the window; see \([32]\) and references therein. In fact, it is an interesting observation that the validity of the PSF, via Weyl sums, can be used to derive the uniform distribution — the PSF thus appears in a double role \([34]\). The factor \(\text{dens}(\mathcal{L})\) stems from the PSF, compare \([1, \text{Thm. 9.1 and Lemma 9.3}]\) for the Euclidean case.

The general version of the claim as stated here is proved in \([34]\) as well as in \([38, \text{Prop. 12.2}]\). Note that part (iii) of this proposition, which is what we need here, does not use or need the assumption of Fourier transformability of \(\omega_c\).
Given a \(\sigma\)-compact LCAG \(G\), an averaging sequence \(A = (A_n)_{n\in\mathbb{N}}\) consists of relatively compact open sets \(A_n\) with \(A_n \subset A_{n+1}\) for all \(n \in \mathbb{N}\) and \(\bigcup_{n\in\mathbb{N}} A_n = G\). Here, \(\sigma\)-compactness of \(G\) is equivalent to the existence of such an averaging sequence. Now, \(A\) is called van Hove if, for any compact \(K \subset G\),
\[
\lim_{n \to \infty} \frac{\text{vol}(\partial K A_n)}{\text{vol}(A_n)} = 0,
\]
where, for an arbitrary open set \(B \subset G\),
\[
\partial K B := (B + K \setminus B) \cup ((B^c - K) \cap B),
\]
with \(B^c\) the complement of \(B\) in \(G\), is the (closed) \(K\)-boundary of \(B\). The existence of van Hove sequences in \(\sigma\)-compact LCAGs is shown in [36]. Note that each van Hove sequence is also Følner, but not vice versa; compare the discussion in [5] and references therein.

Averaging sequences are needed to define the density of a point set \(\Lambda \subset G\),
\[
\text{dens}(\Lambda) := \lim_{n \to \infty} \frac{\text{card}(\Lambda \cap A_n)}{\text{vol}(A_n)},
\]
provided the limit exists. More generally, if the existence of \(\text{dens}(\Lambda)\) is not clear, one has to work with lower and upper densities according to
\[
\text{dens}(\Lambda) := \liminf_{n \to \infty} \frac{\text{card}(\Lambda \cap A_n)}{\text{vol}(A_n)} \quad \text{and} \quad \overline{\text{dens}}(\Lambda) := \limsup_{n \to \infty} \frac{\text{card}(\Lambda \cap A_n)}{\text{vol}(A_n)},
\]
which always exist, with \(0 \leq \text{dens}(\Lambda) \leq \overline{\text{dens}}(\Lambda) \leq \infty\). When \(\Lambda\) is uniformly discrete, one has \(\text{dens}(\Lambda) < \infty\). Moreover, as a result of \([22, 38]\), one also gets the following estimate.

**Fact 4.** Let \(\Lambda = \bigvee (W)\) be a projection set for the CPS of Eq. (2), with relatively compact window \(W \subset H\). Then, one has
\[
\text{dens}(L) \theta_H(W^\circ) \leq \text{dens}(\Lambda) \leq \overline{\text{dens}}(\Lambda) \leq \text{dens}(L) \theta_H(W),
\]
relative to any fixed van Hove sequence \(A\) in \(G\). When \(\text{dens}(\Lambda) = \overline{\text{dens}}(\Lambda)\), the density of \(\Lambda\) exists, relative to \(A\), and satisfies the corresponding inequality. \(\square\)

In our context, we also need the van Hove property for a meaningful definition of the autocorrelation of a translation bounded measure \(\omega \in \mathcal{M}^\infty(G)\) with respect to \(A\). Consider
\[
\gamma_\omega^{(n)} := \frac{\omega|_{A_n} \ast \omega|_{A_n}}{\text{vol}(A_n)},
\]
which gives a well-defined sequence of positive definite measures on \(G\). As a consequence of the translation boundedness of \(\omega\), this sequence has at least one vague accumulation point, each of which is called an autocorrelation measure of \(\omega\); see [21] or [1] for background. If only one such accumulation point exists, \(\gamma_\omega := \lim_{n \to \infty} \gamma_\omega^{(n)}\) exists and is called the autocorrelation of \(\omega\) relative to \(A\).

By construction, any autocorrelation measure \(\gamma\) is a positive definite measure, and hence Fourier transformable by standard arguments [13]. Its Fourier transform, \(\hat{\gamma}\), is then a translation bounded positive measure on the dual group, \(\hat{G}\), and called the (corresponding) diffraction measure. If the autocorrelation for \(A\) is unique, then so is the diffraction measure, which is thus also referenced via \(A\). It is this measure \(\hat{\gamma}\) that we will explore below, and ultimately use to gain access to the dynamical spectrum of a natural dynamical system defined via \(\omega\).
3. Visible lattice points as guiding example

By definition, the visible points of a lattice in Euclidean space are the lattice points that are visible from the origin. Although all results below hold in much greater generality, we prefer to begin with the visible points

\[ V = \{(x, y) \in \mathbb{Z}^2 \mid \gcd(x, y) = 1\} = \mathbb{Z}^2 \setminus \bigcup_{p \in \mathbb{P}} p\mathbb{Z}^2 \]

of the (unimodular) square lattice \( \mathbb{Z}^2 \subset \mathbb{R}^2 \), where \( \mathbb{P} \) denotes the set of rational primes. A central patch of the set \( V \) is illustrated in Figure 1. We refer the reader to [10, 1, 3] for proofs of the subsequent results, which we repeat here in an informal manner. In Section 6 we shall discuss a substantial extension in the form of coprime sublattice families.

It is well known that \( V \) is non-periodic and has arbitrarily large holes, so it fails to be a Delone set. Nevertheless, its natural density exists and is equal to

\[ \text{dens}(V) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}. \]

The term ‘natural’ refers to the use of centred, nested discs as averaging regions; see [1] and [10] Appendix for a more detailed discussion of this aspect. In other words, we use a van Hove sequence \( \mathcal{A} \) of centred, open discs with increasing radius. Discs can be replaced by other bodies with nice boundaries, but one has to work with tied densities in the sense of [10] in order to deal with the holes in \( V \) properly.

Moreover, relative to \( \mathcal{A} \), one can explicitly compute the natural autocorrelation measure \( \gamma_V \) together with its Fourier transform \( \widehat{\gamma}_V \), the diffraction measure of \( V \). It turns out that the latter is a positive pure point measure which is translation bounded and supported on the points of \( \mathbb{Q}^2 \) with square-free (s.f.) denominator, so

\[ \widehat{\gamma}_V = \sum_{k \in \mathbb{Q}^2 \text{ s.f.}} I(k) \delta_k, \quad \text{where } I(k) = \left(\frac{6}{\pi^2} \prod_{p \mid \text{den}(k)} \frac{1}{1 - p^2}\right)^2. \]

Fig. 2 illustrates the diffraction measure.

To compare this with the dynamical spectrum, let us define the (discrete) hull of \( V \) as

\[ \mathcal{X}_V = \overline{\{t + V \mid t \in \mathbb{Z}^2\}}, \]

with the closure being taken in the (metric) local topology, where two subsets of \( \mathbb{Z}^2 \) are close if they agree on a large ball around the origin. The hull \( \mathcal{X}_V \) is then compact and the translational action of \( \mathbb{Z}^2 \) on the hull is continuous, so \( (\mathcal{X}_V, \mathbb{Z}^2) \) is a topological dynamical system. Since \( V \) contains holes of arbitrary size, the empty set is an element of \( \mathcal{X}_V \). As a result, \( \mathcal{X}_V \) fails to be minimal, and the set \( V \) is non-periodic, but not aperiodic in the terminology of [1]. Also, the hull is rather ‘large’ in terms of the variety of its members, unlike what one is used to from hulls of substitution generated point sets. Astonishingly, one can explicitly characterise the
elements of $X_V$ as the subsets of $\mathbb{Z}^2$ that miss at least one coset modulo the subgroup $p\mathbb{Z}^2$ for any prime $p \in \mathbb{P}$ (for instance, $V$ itself misses by definition the zero coset $0 + p\mathbb{Z}^2$ modulo $p\mathbb{Z}^2$ for all $p \in \mathbb{P}$).

There is a natural Borel probability measure $\nu$ on the hull $X_V$ that originates from the natural patch frequencies of $V$ in space. More precisely, the frequency $\nu(\mathcal{P})$ of a $\rho$-patch $\mathcal{P} = (V-t) \cap B_\rho(0)$ of $V$ at location $t$ (the natural density of all such $t$’s) can again be calculated explicitly and one can then assign this very value to the cylinder set $C_\mathcal{P}$ of elements $A$ of the hull with $A \cap B_\rho(0) = \mathcal{P}$. This can then uniquely be extended to a $\mathbb{Z}^2$-invariant probability measure, also called $\nu$, on the hull $X_V$, and one obtains a measure-theoretic dynamical system $(X_V, \mathbb{Z}^2, \nu)$. The measure $\nu$ gives no weight to the empty set (as a member of $X_V$), and the system becomes aperiodic in the measure-theoretic sense of [1, Def. 11.1].

Here, one is also interested in the dynamical spectrum, that is the spectrum of the corresponding unitary representation $U$ of $\mathbb{Z}^2$ on the Hilbert space $L^2(X_V, \nu)$, with the standard
Figure 2. Diffraction of the visible points of $\mathbb{Z}^2$. A point measure at $k$ with intensity $I(k)$ is shown as a disk centred at $k$ with area proportional to $I(k)$. Shown are the intensities with $I(k)/I(0) \geq 10^{-6}$ and $k \in [0,2]^2$. Its lattice of periods is $\mathbb{Z}^2$.

The inner product

$$\langle f | g \rangle = \int_{X \nu} \overline{f(x)} g(x) \, d\nu(x).$$

The system $(X \nu, \mathbb{Z}^2, \nu)$ has pure point dynamical spectrum if and only if the eigenfunctions span all of $L^2(X \nu, \nu)$. Since $V$ is $\nu$-generic, the individual diffraction measure of $V$ coincides with the diffraction measure of the system $(X \nu, \mathbb{Z}^2, \nu)$ in the sense of [5, 8]. By the general equivalence theorem [5, Thm. 7], the pure point nature of the dynamical spectrum follows. Moreover, the spectrum (in additive notation) is nothing but the set of points in $\mathbb{Q}^2$ with square-free denominator, which form a subgroup of $\mathbb{Q}^2$. 


The set $V$ is a weak model set in our above terminology, originating from the CPS $(G, H, \mathcal{L})$ of Eq. (2) with LCAGs $G = \mathbb{Z}^2$, equipped with its natural discrete topology, and $H := \prod_p \mathbb{Z}^2/p\mathbb{Z}^2$, where $\mathbb{Z}^2/p\mathbb{Z}^2$ is a quotient group of order $p^2$ and $H$ is endowed with the product topology with respect to the discrete topology on its factors. In particular, the internal group is compact. The lattice $\mathcal{L}$ (a discrete and co-compact subgroup of $\mathbb{Z}^2 \times H$) is defined by the diagonal embedding of $\mathbb{Z}^2$, 

$$L = \{(x, \iota(x)) \mid x \in \mathbb{Z}^2\},$$

where $\iota(x) = (x_p)_{p \in \mathbb{P}}$, with $x_p$ the reduction of $x$ mod $p$, is the $\ast$-map in this case. In fact, one even has $\pi(\mathcal{L}) = \mathbb{Z}^2$ and $\pi_{\text{int}}(\mathcal{L}) = H$ here. With

$$W := \prod_{p \in \mathbb{P}} (\mathbb{Z}^2/p\mathbb{Z}^2 \setminus \{0\}) \subset H,$$

one clearly obtains $V = \Lambda(W) \subset \mathbb{Z}^2$. Note that $W$ is compact and satisfies $W = \partial W$, so has no interior. Moreover, $W$ has positive measure $\theta_H(W) = \prod_p (1 - \frac{1}{p}) = 1/\zeta(2)$ with respect to the normalised Haar measure $\theta_H$ on the compact group $H$. Note that $\theta_H(W)$ coincides with the natural density of $V$.

Employing the ergodicity of the frequency measure $\nu$, one can in fact show that the dynamical system $(\mathbb{X}_V, \mathbb{Z}^2, \nu)$ is isomorphic with the Kronecker system $(H, \mathbb{Z}^2, \theta_H)$, with the action of $\mathbb{Z}^2$ being given by addition of $\iota(x)$. Here, we have also used the isomorphism $H \simeq (\mathbb{Z}^2 \times H)/\mathcal{L}$.

The setting of this section can be generalised to many similar systems, all arithmetic in nature; compare [3] and references therein. Let us thus develop a wider scheme for weak model sets that comprises all of them.

4. Diffraction of weak model sets

Let us assume that a CPS $(G, H, \mathcal{L})$ is given, including some normalisation of the Haar measures $\theta_G$ and $\theta_H$ as indicated earlier. Since one often has to deal with several different lattices within $G \times H$, we do not assume $\mathcal{L}$ to be unimodular, so factors $\text{dens}(\mathcal{L})$ will appear in our formulas. For a single lattice $\mathcal{L}$, one could rescale $\theta_H$ to make $\mathcal{L}$ unimodular, which is another convention that is also often used.

**Proposition 5.** Let $(G, H, \mathcal{L})$ be a CPS as in Eq. (2), with $G$ being $\sigma$-compact and $H$ compactly generated, and let $\emptyset \neq W \subseteq H$ be compact. Next, consider the weak model set $\Lambda = \Lambda(W)$ and assume that a van Hove averaging sequence $A$ is given relative to which the density $\text{dens}(\Lambda)$ and the autocorrelation measure $\gamma_A$ are to be defined. Then, the following statements are equivalent.

1. The lower density of $\Lambda$ is maximal, $\text{dens}(\Lambda) = \text{dens}(\mathcal{L})\theta_H(W)$;
2. The density of $\Lambda$ exists and is maximal, $\text{dens}(\Lambda) = \text{dens}(\mathcal{L})\theta_H(W)$;
3. The autocorrelation of $\Lambda$ exists and satisfies $\gamma_A = \text{dens}(\mathcal{L})\omega_{\nu_W}$.

Here, $c_W = 1_W \ast \tilde{1}_W$ is the covariogram function of $W$.

**Proof.** The equivalence of statements (1) and (2) follows from Fact [4]. If $\gamma_A$ exists relative to $A$, we know that also $\text{dens}(\Lambda)$ exists relative to $A$, because $\gamma_A(\{0\}) = \text{dens}(\Lambda)$. Now, if
\[ \gamma_\alpha = \text{dens}(\mathcal{L}) \omega_{cW}, \] one obtains via Eq. (1) that
\[
\frac{\text{dens}(A)}{\text{dens}(\mathcal{L})} = \frac{\gamma_\alpha(\{0\})}{\text{dens}(\mathcal{L})} = \omega_{cW}(\{0\}) = (1_W \ast \tilde{1}_W)(0) = \theta_H(W),
\]
which establishes the implication (3) \(\Rightarrow\) (2).

For the converse direction, assume the existence of \(\text{dens}(A)\) relative to the averaging sequence \(A\). By [20] Thm. 8.13, we know that all LCAGs are normal, so we have Urysohn’s lemma at our disposal, which we now employ for an approximation argument as follows. Here, the compactness of \(W\) implies the existence of compact set \(K_g \subseteq H\) with \(W \subseteq K_g^\circ\) together with a \([0,1]\)-valued continuous function \(g\) with supp \((g) \subseteq K_g\) that is 1 on \(W\). Indeed, employing the regularity of the Haar measure \(\theta_H\), there is even a net of \([0,1]\)-valued functions \(g_\alpha \in C_c(H)\) with \(W \subseteq \text{supp}(g_\alpha) \subseteq K_g\), all with the same \(K_g\), such that \(1_{K_g} \geq g_\alpha \geq 1_W\) holds for all \(\alpha\) together with \(\lim_\alpha \theta_H(g_\alpha) = \theta_H(W)\).

Let us first observe that the choice of \((g_\alpha)\) implies \(\lim_\alpha (g_\alpha \ast \tilde{g_\alpha}) = 1_W \ast \tilde{1}_W\) in \(C_c(H)\), because, employing \(\|f \ast h\|_\infty \leq \|f\|_1 \|h\|_\infty\) with \(\|1_W\|_\infty = \|g_\alpha\|_\infty = 1\) and \(\|h\|_1 = \|h\|_1\), one finds the estimate
\[
\|1_W \ast \tilde{1}_W - g_\alpha \ast \tilde{g_\alpha}\|_\infty \leq \|1_W \ast (\tilde{1}_W - \tilde{g_\alpha})\|_\infty + \|\tilde{g_\alpha} \ast (1_W - g_\alpha)\|_\infty \leq 2\|1_W - g_\alpha\|_1.
\]

Now, consider the weighted Dirac combs \(\omega_{g_\alpha, \tilde{g_\alpha}}\), which are positive and positive definite pure point measures, all supported within the common model set \(\mathcal{L}(K_g - K_g)\), which is thus a Meyer set as well. Since also \(\text{supp}(\omega_{1_W \ast \tilde{1}_W}) \subseteq \mathcal{L}(K_g - K_g)\), it follows that
\[
\omega_{g_\alpha, \tilde{g_\alpha}} \longrightarrow \omega_{1_W \ast \tilde{1}_W} = \omega_{cW}
\]
pointwise at each \(x \in \mathcal{L}(K_g - K_g)\), and hence also in the vague topology.

To simplify the exposition, let us assume for the remainder of this proof that \(\text{dens}(\mathcal{L}) = 1\), which means no restriction as we could rescale \(\theta_H\) accordingly. Let \(\varepsilon > 0\) and choose some \(f \in C_c(G)\), so \(\text{supp}(f) \subseteq K\) for some compact set \(K \subseteq G\). Since the measures \(\omega_{g_\alpha}\) and \(\delta_A\) are equi-translation bounded, there is constant \(C\) such that
\[
\|\omega_{g_\alpha}\|_K < C \quad \text{and} \quad \|\delta_A\|_K < C.
\]

Moreover, there is some index \(M\) such that \(\theta_H(g_\alpha) \leq \theta_H(W) + \frac{\varepsilon}{1 + C\|f\|_\infty}\) holds for all \(\alpha \geq M\). We also know, possibly after adjusting the index \(M\), that
\[
|\omega_{cW}(f) - \omega_{g_\alpha, \tilde{g_\alpha}}(f)| < \varepsilon
\]
holds for all \(\alpha \geq M\). With \(A_n := \Lambda \cap A_n\), we now employ an \(8\varepsilon\)-argument to establish the convergence of the autocorrelation sequence \((\frac{1}{\text{vol}(A_n)} \delta_{A_n} \ast \delta_{A_n})_{n \in \mathbb{N}}\) as follows.

Let \(\alpha \geq M\) be fixed. With the abbreviation \(\omega_{\alpha,n} = \omega_{g_\alpha}|_{A_n}\), we have
\[
\omega_{g_\alpha, \tilde{g_\alpha}} = \lim_{n \to \infty} \frac{\omega_{\alpha,n} \ast \tilde{\omega}_{\alpha,n}}{\text{vol}(A_n)}
\]
as a consequence of the results of [34], see also [26] for an alternative proof. Consequently, there is some integer $N_1$ so that, for all $n \geq N_1$, we have

$$\left| \omega_{\varphi_n} (f) - \frac{(\omega_{\varphi_n \ast \varphi_n} (f))}{\text{vol}(A_n)} \right| < \varepsilon.$$ 

Next, recall that $\omega_{\varphi_n}$ is a norm-almost periodic measure [9, 34], so it is amenable with the mean being given by $\lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \omega_{\varphi_n} (A_n) = \theta_H (g_{\varphi_n})$. There is thus some $N_2$ so that $\theta_H (g_{\varphi_n}) \geq \frac{1}{\text{vol}(A_n)} \omega_{\varphi_n} (A_n) - \frac{2\varepsilon}{1 + C \|f\|_{\infty}}$ holds for all $n > N_2$. Finally, due to our density condition for $A = \lambda (W)$, there exists some $N_3$ such that $\frac{1}{\text{vol}(A_n)} \delta_A (A_n) > \theta_H (W) - \frac{3\varepsilon}{1 + C \|f\|_{\infty}}$ is satisfied for all $n > N_3$.

Now, if $n > \max\{N_1, N_2, N_3\}$, we have

$$\frac{\delta_A (A_n)}{\text{vol}(A_n)} > \theta_H (W) - \frac{3\varepsilon}{1 + C \|f\|_{\infty}} \geq \frac{\omega_{\varphi_n} (A_n)}{\text{vol}(A_n)} - \frac{2\varepsilon}{1 + C \|f\|_{\infty}} \geq \frac{\omega_{\varphi_n} (A_n)}{\text{vol}(A_n)} - \frac{3\varepsilon}{1 + C \|f\|_{\infty}}$$

and hence $\omega_{\varphi_n} (A_n) - \delta_A (A_n) < \frac{3\varepsilon \text{vol}(A_n)}{1 + C \|f\|_{\infty}}$. As $\omega_{\varphi_n} \geq \delta_A$, the finite measure $\frac{1}{\text{vol}(A_n)} (\omega_{\varphi_n} - \delta_A)$ is positive. Consequently,

$$\|\omega_{\varphi_n} - \delta_A\| = (\omega_{\varphi_n} - \delta_A) (G) = (\omega_{\varphi_n} - \delta_A) (A_n) < \frac{3\varepsilon \text{vol}(A_n)}{1 + C \|f\|_{\infty}}.$$ 

Clearly, this also implies $\|\omega_{\varphi_n} - \delta_A\| < \frac{3\varepsilon \text{vol}(A_n)}{1 + C \|f\|_{\infty}}$. Put together, we can now estimate

$$\left| \frac{\omega_{c_W} (f) - \left(\frac{\delta_A \ast \varphi_n}{\text{vol}(A_n)}\right)}{\text{vol}(A_n)} \right| \leq \left| \omega_{c_W} (f) - \omega_{\varphi_n} \varphi_n \right| + \left| \omega_{\varphi_n} \varphi_n - \frac{(\omega_{\varphi_n} \ast \omega_{\varphi_n}) (f)}{\text{vol}(A_n)} \right| \leq \left| \frac{(\omega_{\varphi_n} \ast \omega_{\varphi_n}) (f)}{\text{vol}(A_n)} - \frac{(\delta_A \ast \varphi_n) (f)}{\text{vol}(A_n)} \right| + 2\varepsilon \leq \left| \frac{\omega_{\varphi_n} - \delta_A}{\text{vol}(A_n)} \right| \|\omega_{\varphi_n} - \delta_A\|_{K} \|f\|_{\infty} + \|\delta_A\|_{K} \left| \frac{\omega_{\varphi_n} - \delta_A}{\text{vol}(A_n)} \right| \|f\|_{\infty} + 2\varepsilon \leq \frac{6\varepsilon}{1 + C \|f\|_{\infty}} C \|f\|_{\infty} + 2\varepsilon < 8\varepsilon,$$ 

where the second last line follows from Fact 1 and the last from Eq. (3). This completes our argument.

\[\Box\]

**Remark 2.** Let us note that the estimate in Eq. (5) implies the uniform convergence of the net $(\varphi_n \ast \varphi_n)$ of continuous functions to the limit $c_W$. The latter must then be continuous, too, in line with Fact 2. Since this type of argument can also be used for measurable sets $W$ that are merely relatively compact, one sees the continuity of $c_W$ in an alternative way. \[\Diamond\]
Remark 3. In the setting of Proposition 5, let $\Lambda_1 = \mathcal{A}(W_1)$ and $\Lambda_2 = \mathcal{A}(W_2)$ be two weak model sets with compact windows such that, for the same van Hove sequence $\mathcal{A}$, we have

$$\text{dens}(\Lambda_i) = \text{dens}(\mathcal{L}) \theta_H(W_i), \quad \text{for } i \in \{1, 2\}.$$  

Then, in complete analogy to the proof of Proposition 5, one can show that

$$\lim_{n \to \infty} \frac{\delta_{\Lambda_1 \cap A_n} * \delta_{\Lambda_2 \cap A_n}}{\text{vol}(A_n)} = \text{dens}(\mathcal{L}) \omega_{1_{W_1} * 1_{W_2}}$$

holds for the mixed correlation (or Eberlein convolution) between $\Lambda_1$ and $\Lambda_2$. 

Let us extend our previous result to more general windows, namely to relatively compact sets $W \subset H$ with $\theta_H(W) > 0$. One then finds the following result.

Corollary 6. Consider the setting of Proposition 5 with van Hove averaging sequence $\mathcal{A}$ and $\Lambda = \mathcal{A}(W)$, but only assume $W \subset H$ to be relatively compact. If $\text{dens}(\Lambda) = \text{dens}(\mathcal{L}) \theta_H(W)$, or equivalently $\text{dens}(\Lambda) = \text{dens}(\mathcal{L}) \theta_H(W)$, one also has $\gamma_\Lambda = \text{dens}(\mathcal{L}) \omega_1$, where $\omega_1$ is the covariogram function of $1_W$.

Proof. The two conditions are equivalent by Fact 4; compare the proof of Proposition 5. With respect to $\mathcal{A}$, we find via Eq. (4) that

$$\text{dens}(\mathcal{L}) \theta_H(W) = \text{dens}(\Lambda) \leq \text{dens}(\mathcal{L}(W)) \leq \text{dens}(\mathcal{L}(\overline{W})).$$

This implies $\text{dens}(\mathcal{L}(W)) = \text{dens}(\mathcal{L}(\overline{W}))$ and thus the existence of the limit

$$\text{dens}(\mathcal{L}(W)) = \lim_{n \to \infty} \frac{\text{card}(\mathcal{L}(\overline{W}) \cap A_n)}{\text{vol}(A_n)} = \text{dens}(\mathcal{L}) \theta_H(W).$$

As $\Lambda \subseteq \mathcal{L}(\overline{W})$, the argument also implies that the point set $\mathcal{L}(\overline{W}) \setminus \Lambda$ has zero density. Consequently, due to the inclusion relation, $\Lambda = \mathcal{L}(W)$ and $\mathcal{L}(\overline{W})$ possess the same autocorrelation measure relative to $\mathcal{A}$, so $\gamma_\Lambda = \gamma_{\mathcal{L}(\overline{W})}$. The claim now follows from Proposition 5. \qed

Our derivation so far motivates the following concept.

Definition 1. For a given CPS $(G, H, \mathcal{L})$ with $G$ $\sigma$-compact and $H$ compactly generated, a projection set $\mathcal{A}(W)$ is called a weak model set of maximal density relative to a given van Hove averaging sequence $\mathcal{A}$ if the window $W \subseteq H$ is relatively compact with $\theta_H(W) > 0$, if the density of $\mathcal{L}(W)$ relative to $\mathcal{A}$ exists, and if the density condition $\text{dens}(\mathcal{L}(W)) = \text{dens}(\mathcal{L}) \theta_H(W)$ is satisfied.

Note that, in view of Fact 4, the two conditions on the density can be replaced by the single maximality condition $\text{dens}(\mathcal{L}(W)) = \text{dens}(\mathcal{L}) \theta_H(W)$, which is equivalent. If $W$ has zero measure in $H$, the corresponding projection set would have upper density zero, and is not of interest in our setting. Thus, we formulate our general diffraction result as follows.
Theorem 7. Let \( \Lambda = \Lambda(W) \), with \( W \subseteq H \) compact and \( \theta_H(W) > 0 \), be a weak model set of maximal density for the CPS \((G, H, L)\), in the setting of Proposition 5. Then, the autocorrelation \( \gamma_\Lambda \) is a strongly almost periodic pure point measure. It is Fourier transformable, and \( \hat{\gamma}_\Lambda \) is a translation bounded, positive, pure point measure on \( \hat{\Gamma} \). It is explicitly given by

\[
\hat{\gamma}_\Lambda = \sum_{u \in L^0} |a(u)|^2 \delta_u, \quad \text{with amplitude } a(u) = \frac{\operatorname{dens}(\Lambda)}{\theta_H(W)} \widehat{1}_W(-u^*),
\]

where \( \widehat{1}_W \) is a bounded, continuous function on the dual group \( \hat{H} \) and \( L^0 = \pi(L^0) \subset \hat{G} \) is the corresponding Fourier module in additive notation.

More generally, if \( W \) is relatively compact with \( \theta_H(W) > 0 \), but \( \operatorname{dens}(\Lambda) = \operatorname{dens}(L) \theta_H(W) \) as in Corollary 6, the previous formula holds with \( W \) replaced by \( W \).

Proof. Note that, under our assumptions, we have \( \operatorname{dens}(L) = \operatorname{dens}(\Lambda) \theta_H(W) \) in both cases. If \( W \) is compact, the density assumption gives us \( \gamma_\Lambda = \theta_H(W) \omega_{c_W} \) by an application of Proposition 5. Since \( c_W = 1_W * \widehat{1}_W \) is continuous (by Fact 2), compactly supported (because \( \operatorname{supp}(c_W) \subseteq W - W \)) and positive definite by construction, we can invoke Theorem 3, which proves the claim on \( \hat{\gamma}_\Lambda \).

The extension to a relatively compact window \( W \), under our density assumption, is a consequence of Corollary 6. \( \square \)

Remark 4. A regular model set with proper window \( W \) is automatically a weak model set of maximal density, and thus a special case of Theorem 7. Also, if \( W \) is regular, but fails to be proper, we still get a special case when \( \theta_H(W^c) = \theta_H(W) \) is satisfied. \( \diamond \)

Let us note in passing that the explicit formula for the diffraction measure is nice and systematic, and a direct generalisation of the known formula for regular model sets; compare [1, Thm. 9.4]. In particular, we still have a well-defined meaning of the intensities in terms of an amplitude (or Fourier–Bohr coefficient), which will be useful for calculations with superpositions of Dirac combs. However, the result is also a bit deceptive in the sense that it will be difficult to actually calculate the amplitudes \( a(u) \) in this generality — unless one has an underlying arithmetic structure as in our guiding example.

In fact, the analogy with the properties of regular model sets goes further, via the following result on the amplitudes (or Fourier–Bohr coefficients), which resembles the original result in [21], but cannot be proved by the methods of that paper due to the absence of uniform densities. Recall that, due to our additive notation of the dual groups, the character \( \chi_u \) defined by \( u \in \hat{G} \) is written as \( \chi_u(\cdot) = \langle u, \cdot \rangle \), with \( \langle -u, \cdot \rangle = \langle u, \cdot \rangle \).

Proposition 8. Under the conditions of Theorem 7 the limit

\[
a_u := \lim_{n \to \infty} \frac{1}{\operatorname{vol}(A_n)} \sum_{t \in \Lambda \cap A_n} \langle u, t \rangle = \lim_{n \to \infty} \frac{1}{\operatorname{vol}(A_n)} \int_{A_n} \langle u, t \rangle \, d\delta_\Lambda(t)
\]

exists for each \( u \in \hat{G} \), and one has \( \hat{\gamma}_\Lambda(\{u\}) = |a_u|^2 \).
Moreover, \( a_u = 0 \) if \( u \notin L^0 \) with \( L^0 \) from the dual CPS according to (3), and \( a_u \) agrees with the amplitude \( a(u) \) of Theorem 7 otherwise.

**Proof.** The proof is methodically similar to the one of Proposition 5. Let \( u \in \hat{G} \) be fixed. For each \( \varepsilon > 0 \), there is some index \( M \) and some integer \( N \) such that
\[
0 \leq \omega_{\hat{g}_\alpha}(A_n) - \delta_{\dot{A}}(A_n) < \varepsilon \text{ vol}(A_n)
\]
holds for \( \alpha = M \) and all \( n > N \), and actually also for all \( \alpha > M \), due to the monotonicity of the functions \( g_\alpha \). Consequently, for all \( n > N \) and \( \alpha > M \), we have
\[
\left| \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}(\omega_\alpha - \delta_{\dot{A}})(t) \right| < \varepsilon.
\]

Now, fix some \( \alpha > M \) and observe that \( \omega_{\hat{g}_\alpha} \) is norm-almost periodic, so the mean
\[
\mathbb{M}(\langle u, \cdot \rangle \omega_{\hat{g}_\alpha}) = \lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}\omega_{\hat{g}_\alpha}(t)
\]
extists and satisfies \( \overline{\omega_{\hat{g}_\alpha \ast \hat{g}_\alpha}}(\{u\}) = |\mathbb{M}(\langle u, \cdot \rangle \omega_{\hat{g}_\alpha})|^2 \). Now, if \( u \notin L^0 \), we have \( \overline{\omega_{\hat{g}_\alpha \ast \hat{g}_\alpha}}(\{u\}) = 0 \) as a consequence of [26]. Thus, there is some integer \( N' \) such that
\[
\left| \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}\omega_{\alpha}(t) \right|^2 < \varepsilon^2
\]
holds for all \( n > N' \). Together with Eq. (7), this gives
\[
\left| \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}\delta_{\dot{A}}(t) \right| < 2\varepsilon
\]
for all \( n > N' \), and thus \( a_u = 0 \) in this case.

If \( u \in L^0 \), we have \( \overline{\omega_{\hat{g}_\alpha \ast \hat{g}_\alpha}}(\{u\}) = |\hat{g}_\alpha|^2(-u^*) \) by [26]. Due to the choice of the net \((g_\alpha)\), we know that \( \hat{\omega}_{\hat{g}_\alpha \ast \hat{g}_\alpha}(u) = |\hat{g}_\alpha|^2(-u^*) \) for all \( v \in H \), hence \( |\hat{\omega}_{\hat{g}_\alpha \ast \hat{g}_\alpha}(u) - \hat{g}_\alpha(-u^*)| < \varepsilon \) for some \( \alpha > M \). As \( \mathbb{M}(\langle u, \cdot \rangle \omega_{\hat{g}_\alpha}) = \hat{g}_\alpha(-u^*) \), there is some \( N_1 > N \) such that, for all \( n > N_1 \), one has
\[
|\hat{g}_\alpha(-u^*) - \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}\omega_{\alpha}(t)| < \varepsilon.
\]
Combined with Eq. (7) again, we get
\[
\left| \hat{\omega}_{\hat{g}_\alpha \ast \hat{g}_\alpha}(u) - \frac{1}{\text{vol}(A_n)} \int_{A_n} \langle u, t \rangle \text{d}\omega_{\alpha}(t) \right| < 3\varepsilon
\]
for all \( N > N_1 \). This gives \( a_u = \hat{\omega}_{\hat{g}_\alpha \ast \hat{g}_\alpha}(u) \) as claimed. \( \Box \)

At this point, one should realise that relatively little can be said in general when the maximality condition is not satisfied, as becomes evident from the large class of Toeplitz sequences. They can be realised as model sets with proper windows [7], hence also as weak model sets. The irregular Toeplitz sequences then display a wealth of possible phenomena. However, there is still one other class of weak model sets which behaves nicely, too.
Definition 2. For a given CPS $(G, H, \mathcal{L})$ as in Definition 1 a projection set $\Lambda(W)$ is called a weak model set of minimal density relative to a given van Hove averaging sequence $\mathcal{A}$ if the window $W \subseteq H$ is relatively compact with $\theta_H(W^o) > 0$, if the density of $\Lambda(W)$ relative to $\mathcal{A}$ exists, and if the density condition $\text{dens}(\Lambda(W)) = \text{dens}(\mathcal{L}) \theta_H(W^o)$ is satisfied.

Note first that weak model sets of minimal density, in this setting, are Meyer sets, and thus perhaps less interesting than their counterparts with maximal density. In analogy to before, we could alternatively demand $\text{dens}(\Lambda(W)) = \text{dens}(\mathcal{L}) \theta_H(W^o)$, which then entails the existence of the density via Fact 4. Also, one could extend the setting by only asking for $W^o$ to be relatively compact, but not $W$, which might take us outside the Meyer set class; we will not pursue this further here. Nevertheless, repeating our approximation with weighted Dirac combs $\omega_h$, this time from below via a suitable net $(h_n)$ of compactly supported functions on $H$ such that $0 \leq \omega_h \leq \delta_{\Lambda(W^o)}$ holds, one finds the following analogue of Theorem 7.

Theorem 9. Let $(G, H, \mathcal{L})$ be a CPS as in Theorem 7 and let $W \subseteq H$ be relatively compact with $\theta_H(W^o) > 0$. Consider the weak model set $\Lambda = \Lambda(W)$ and assume that it is of minimal density for a given van Hove averaging sequence $\mathcal{A}$ in $G$, in the sense of Definition 2. Then, the autocorrelation measure of $\Lambda$ resp. $\delta_{\Lambda}$ relative to $\mathcal{A}$ exists and satisfies $\gamma_{\Lambda} = \text{dens}(\mathcal{L}) \omega_{c_{W^o}}$, where $c_{W^o}$ is the covariogram function of $1_{W^o}$. Moreover, $\gamma_{\Lambda}$ is a translation bounded, positive, pure point measure, which is given by the formulas from Theorem 7 with $W$ replaced by $W^o$. Finally, the statement of Proposition 8 holds as well.

Proof. Instead of repeating our previous arguments with an approximation from below, let us employ an alternative argument that emphasises the complementarity. First, under our minimality assumption, the weak model sets $\Lambda = \Lambda(W)$ and $\Lambda_0 := \Lambda(W^o)$ possess the same autocorrelation measure (provided the limit along $\mathcal{A}$ exists, which we still have to show), as $\Lambda(W \setminus W^o)$ is a point set of zero density for $\mathcal{A}$.

Now, let $K \subseteq H$ be a compact set that contains $W$, is proper, and satisfies $\theta_H(\partial K) = 0$, which is clearly possible. Consequently, $\Lambda_1 := \Lambda(K)$ is a regular model set with all the nice properties, in particular relative to $\mathcal{A}$. It is thus also a weak model set of maximal density for $\mathcal{A}$.

Next, consider $A_2 := \Lambda(K \setminus W^o)$, which is another weak model set of maximal density relative to $\mathcal{A}$. As $\delta_{\Lambda_0} = \delta_{\Lambda_1} - \delta_{A_2}$, we can employ Remark 3 to relate the various correlation measures. With $A_{i,n} := A_i \cap A_n$, we get for the approximating autocorrelations of $\delta_{\Lambda_0}$ that

$$\lim_{n \to \infty} \frac{\delta_{A_{0,n}} \ast \delta_{A_{0,n}}}{\text{vol}(A_n)} = \lim_{n \to \infty} \frac{(\delta_{A_{1,n}} - \delta_{A_{2,n}}) \ast (\delta_{A_{1,n}} - \delta_{A_{2,n}})}{\text{vol}(A_n)}$$

$$= \lim_{n \to \infty} \left(\frac{\delta_{A_{1,n}} \ast \delta_{A_{1,n}}}{\text{vol}(A_n)} + \frac{\delta_{A_{2,n}} \ast \delta_{A_{2,n}}}{\text{vol}(A_n)} - \frac{\delta_{A_{2,n}} \ast \delta_{A_{1,n}}}{\text{vol}(A_n)} - \frac{\delta_{A_{1,n}} \ast \delta_{A_{2,n}}}{\text{vol}(A_n)}\right)$$

$$= \text{dens}(\mathcal{L}) \left(\omega_{1_K} + \omega_{1_K \setminus W^o} \ast 1_{K \setminus W^o} - \omega_{1_K \setminus W^o} \ast 1_{K \setminus W^o} - \omega_{1_K} \ast 1_{K \setminus W^o}\right)$$

$$= \text{dens}(\mathcal{L}) \omega_{c_{W^o}}.$$
which also establishes the existence of the limit and thus completes our argument. □

At this stage, in the spirit of Fact 4, one can derive the following sandwich result for an arbitrary autocorrelation of a weak model set.

**Corollary 10.** Let \( \Lambda \) be a weak model set for the CPS \((G, H, \mathcal{L})\) from above, with relatively compact window \( W \). If \( \gamma \) is any autocorrelation of \( \Lambda \), it satisfies the measure inequality

\[
0 \leq \text{dens}(\mathcal{L}) \omega_{c_{W \circ}} \leq \gamma \leq \text{dens}(\mathcal{L}) \omega_{c_{W \circ}},
\]

with \( c_A \) the covariogram function of the measurable set \( A \).

**Proof.** Select nets \((h_\alpha)\) and \((g_\alpha)\) of continuous functions with \( h_\alpha \nearrow 1_{W \circ} \) and \( g_\alpha \searrow 1_{\Xi} \) in analogy to our previous arguments. Also, let \( \mathcal{B} = (B_n)_{n \in \mathbb{N}} \) be a van Hove sequence relative to which the autocorrelation of \( \Lambda \) is \( \gamma \). Then, as \( \omega_{h_\alpha} \) and \( \omega_{g_\alpha} \) are norm-almost periodic measures, their autocorrelation measures with respect to \( \mathcal{B} \) exist and are given by \( \omega_{h_\alpha * \tilde{h}_\alpha} \) and \( \omega_{g_\alpha * \tilde{g}_\alpha} \), respectively. Both are positive and positive definite measures.

Now, we have \( 0 \leq \omega_{h_\alpha} \leq \delta_A \leq \omega_{g_\alpha} \) by construction, which implies

\[
0 \leq \text{dens}(\mathcal{L}) \omega_{h_\alpha * \tilde{h}_\alpha} \leq \gamma \leq \text{dens}(\mathcal{L}) \omega_{g_\alpha * \tilde{g}_\alpha}
\]

by standard arguments. Since \( h_\alpha * \tilde{h}_\alpha \to c_{W \circ} \) and \( g_\alpha * \tilde{g}_\alpha \to c_{\Xi} \), we obtain the claimed inequality by taking the limits of the previous inequality in \( \alpha \). □

The general spectral theory can now be developed further, aiming at a result on the dynamical spectrum of the hull of weak model sets of extremal density. For this, we first need to construct a suitable measure and establish its ergodicity.

### 5. Hull, ergodicity and dynamical spectrum

Let us fix a CPS \((G, H, \mathcal{L})\) with a \( \sigma \)-compact LCAG \( G \), a compactly generated LCAG \( H \) and a lattice \( \mathcal{L} \subseteq G \times H \) as before, and let \( \Lambda = \lambda(W) \) with compact \( W \subseteq H \) be a weak model set of maximal density, relative to a fixed van Hove averaging sequence \( \mathcal{A} = (A_n)_{n \in \mathbb{N}} \). The (geometric) hull of \( \Lambda \) is the orbit closure \( G + \Lambda \) in the local topology; compare [1, Sec. 5.4] for background. Note that our point set \( \Lambda \) is an FLC set, so that the local topology suffices (it is a special case of a Fell topology [5]). The group \( G \) acts continuously on the hull by translations.

In view of our further reasoning, we now represent \( \Lambda \) by its Dirac comb \( \delta_A \), which is a translation bounded, positive pure point measure with support \( \Lambda \). Its hull is

\[
\Xi_A := \{ \delta_t * \delta_A \mid t \in G \},
\]

where the closure is taken in the vague topology. By standard arguments, \( \Xi_A \) is vaguely compact, with a continuous action of \( G \) on it. Clearly, \( \delta_t * \delta_A = \delta_{t+A} \), so that the topological dynamical systems \((\Lambda, G)\) and \((\Xi_A, G)\) are topologically conjugate.
Let now \((g_\alpha)\) be the net of \(C_c(H)\)-functions with \(1_{K_g} \geq g_\alpha \geq 1_W\) from the proof of Proposition 5, and consider the weighted Dirac combs
\[
\omega_{g_\alpha} = \sum_{x \in \lambda(K_g)} g_\alpha(x^*) \delta_x,
\]
where \(K_g \subseteq H\) is the compact set introduced earlier that covers the supports of all \(g_\alpha\). Since each \(\omega_{g_\alpha}\), as well as \(\delta\Lambda\), is supported in the same Meyer set \(\lambda(K_g)\), we have pointwise (and hence norm) convergence \(\lim_\alpha \omega_{g_\alpha} = \delta\Lambda\). Moreover, for each comb \(\omega_{g_\alpha}\), there is a hull \(X_\alpha = \{\delta t \ast \delta_{g_\alpha} | t \in G\}\) that is compact in the vague topology and defines a topological dynamical system \((X_\alpha, G)\). In fact, one has more; see [26, Thm. 3.1] as well as [28].

Fact 11. Each dynamical system \((X_\alpha, G)\) is minimal and admits precisely one \(G\)-invariant probability measure, \(\mu_\alpha\) say, and is thus strictly ergodic. Moreover, the system is topologically conjugate to its maximal equicontinuous factor, wherefore it has pure point diffraction and dynamical spectrum, and the hull possesses a natural structure as a compact Abelian group. □

Clearly, the Dirac comb \(\delta_{\lambda(K_g)}\) is translation bounded, so there is a compact set \(K \subseteq G\) and a constant \(C > 0\) such that \(\|\delta_{\lambda(K_g)}\|_K \leq C\). By construction, we also have \(\Lambda \subseteq \lambda(K_g)\). Consequently, both our Dirac comb \(\delta_\Lambda\) and the measures \(\omega_{g_\alpha}\) are elements of \(\mathbb{Y} := \{\nu \in M^\infty(G) | \|\nu\|_K \leq C\}\), which is a compact subset of \(M^\infty(G)\). In fact, for all \(\alpha\), we have the relation
\[(8) \quad 0 \leq \delta_\Lambda \leq \omega_{g_\alpha} \leq \delta_{\lambda(K_g)} \in \mathbb{Y}\]
as an inequality between pure point measures. Moreover, we also have \(X_\Lambda \subseteq \mathbb{Y}\) as well as \(X_\alpha \subseteq \mathbb{Y}\) for all \(\alpha\). Clearly, the measures \(\nu_\alpha\) have a trivial extension to measures on \(\mathbb{Y}\), still called \(\mu_\alpha\), such that \(\text{supp}(\mu_\alpha) = X_\alpha\). In particular, \(\omega_{g_\alpha}\) is then generic for \(\mu_\alpha\). We can now work within \(\mathbb{Y}\) for approximation purposes. To do so, we need a smoothing operation, which is based on the linear mapping \(\phi: C_c(G) \rightarrow C(\mathbb{Y})\), \(c \mapsto \phi_c\), where
\[\phi_c(\nu) := (\nu \ast c)(0)\]
This is the standard approach to lift continuous functions on \(G\) with compact support to continuous functions on a compact measure space such as \(\mathbb{Y}\). It underlies the fundamental relation between diffraction and dynamical spectra via the Dworkin argument; compare [5, 8] and references therein.

Theorem 12. For each \(c \in C_c(G)\) and each \(\varepsilon > 0\), there exists some bound \(M\) and some integer \(N\) such that
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| \phi_c(\delta_{-t} \ast \omega_{g_\alpha}) - \phi_c(\delta_{-t} \ast \delta_\Lambda) \right| dt < \varepsilon
\]
holds for all \(\alpha \geq M\) and all \(n > N\).

Proof. In view of the linearity of the mapping \(c \mapsto \phi_c\), it suffices to prove the claim for non-negative \(c\) that are not identically zero, where \(\|c\|_1 = \int_G c(t) dt > 0\). The extension to general
c is then a standard $4\varepsilon$-argument via splitting $c$ into its real and imaginary parts and writing a real-valued function as a difference of non-negative functions.

From the proof of Proposition 5 in conjunction with Eq. (8), we know that, given $\varepsilon > 0$, there is an index $M$ and an integer $N$ such that

\begin{equation}
0 \leq \frac{(\omega g\alpha - \delta A)(A_n)}{\text{vol}(A_n)} < \frac{\varepsilon}{2\|c\|_1}
\end{equation}

holds for all $\alpha > M$ and $n > N$.

With the abbreviation $\nu_\alpha := \omega g\alpha - \delta A$, one has $0 \leq \phi_c(\delta_{-t} * \omega g\alpha) - \phi_c(\delta_{-t} * \delta A) = (\nu_\alpha * c)(t)$. If $B = \text{supp}(c)$, it is clear that the two functions $(\nu_\alpha * c)|_{A_n}$ and $\nu_{\alpha,n} * c$ agree on the complement of the compact set $\partial^B A_n$, where $\nu_{\alpha,n} := \nu_\alpha|_{A_n}$. Thus, for each $\alpha$,

$$\lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \int_G ((\nu_{\alpha,n} * c)(t) - (\nu_\alpha * c)|_{A_n}(t)) \, dt = 0$$

as a consequence of the van Hove property of $A$. The limit is even uniform in $\alpha$, because the absolute value of the integral, which effectively only runs over the set $\partial^B A_n$, is bounded by $2\|\delta_{\lambda(K)}\|_B \|c\|_\infty \text{vol}(\partial^B A_n)$ as a result of Eq. (8) in conjunction with Fact 1, the latter applied with $\mu = \delta_0$. So, possibly after adjusting $N$, we know that

\begin{equation}
\left| \frac{1}{\text{vol}(A_n)} \int_G ((\nu_{\alpha,n} * c)(t) - (\nu_\alpha * c)|_{A_n}(t)) \, dt \right| < \frac{\varepsilon}{2}
\end{equation}

holds for all $n > N$ and all $\alpha$.

Now, since also $(\nu_{\alpha,n} * c)(t) \geq 0$, one finds

$$0 \leq \frac{1}{\text{vol}(A_n)} \int_{A_n} \left( \phi_c(\delta_{-t} * \omega g\alpha) - \phi_c(\delta_{-t} * \delta A) \right) \, dt$$

\begin{align*}
&\leq \frac{1}{\text{vol}(A_n)} \int_{A_n} (\nu_{\alpha,n} * c)(t) \, dt + \frac{\varepsilon}{2} \\
&= \frac{1}{\text{vol}(A_n)} \int_G \int_G 1_{A_n}(t) c(t - y) \, dt \, d\nu_{\alpha,n}(y) + \frac{\varepsilon}{2} \\
&= \frac{1}{\text{vol}(A_n)} \int_G \nu_{\alpha,n}(A_n - t') c(t') \, dt' + \frac{\varepsilon}{2} \\
&\leq \frac{\nu_\alpha(A_n)}{\text{vol}(A_n)} \|c\|_1 + \frac{\varepsilon}{2} < \varepsilon
\end{align*}

where the second line follows from the inequality in Eq. (10). Fubini’s theorem gives the ensuing identity, while the next step results from setting $t' = t - y$ and applying Fubini again. Since

$$0 \leq \nu_{\alpha,n}(A_n - t') \leq \nu_{\alpha,n}(G) = \nu_\alpha(A_n),$$

the last estimate is a consequence of Eq. (9). □
Corollary 13. For each $c \in C_c(G)$ and each $\varepsilon > 0$, there exists some bound $M$ and some integer $N$ such that, for any $h \in C_u(G)$,
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| \left( \phi_c(\delta_{-t} \ast \omega_{g_a}) - \phi_c(\delta_{-t} \ast \delta) \right) h(t) \right| dt < \varepsilon \|h\|_\infty
\]
holds for all $\alpha > M$ and all $n > N$.

Proof. One clearly has the estimate
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| \phi_c(\delta_{-t} \ast \omega_{g_a}) - \phi_c(\delta_{-t} \ast \delta) \right| dt \leq \frac{\|h\|_\infty}{\text{vol}(A_n)} \int_{A_n} \left| \phi_c(\delta_{-t} \ast \omega_{g_a}) - \phi_c(\delta_{-t} \ast \delta) \right| dt,
\]
where $\|h\|_\infty < \infty$ by our assumptions. The claim now follows from Lemma 12. \qed

The next step consists in extending the orbit average estimate to a sufficiently large class of functions so that we can later apply the Stone–Weierstrass theorem. For this, we first need the algebra generated by functions of type $\phi_c$. Since $\phi$ is a linear map, we only need to extend Lemma 12 to products of such functions as follows.

Proposition 14. Let $k \in \mathbb{N}$ and choose arbitrary functions $c_1, \ldots, c_k \in C_c(G)$ and $\varepsilon > 0$. Then, there exists some bound $M$ and some integer $N$ such that
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| \prod_{i=1}^{k} \phi_{c_i}(\delta_{-t} \ast \omega_{g_a}) - \prod_{j=1}^{k} \phi_{c_j}(\delta_{-t} \ast \delta) \right| dt < \varepsilon
\]
holds for all $\alpha > M$ and all $n > N$.

Proof. For $k = 1$, the claim is just Lemma 12 and hence true. Assume the claim to hold for some $k \in \mathbb{N}$ and consider the case $k + 1$. For any $\nu \in \mathbb{Y}$ and $c \in C_c(G)$, the function defined by $t \mapsto \phi_c(\delta_{-t} \ast \nu)$ is an element of $C_u(G)$. So, consider
\[
0 \leq C := \max_{1 \leq i \leq k+1} \sup_{\nu \in \mathbb{Y}} |\phi_{c_i}(\nu)| < \infty,
\]
which is an upper bound to the absolute value of $\phi_{c_i}$ along any orbit in $\mathbb{Y}$.

Now, let $g = \prod_{i=1}^{k} \phi_{c_i}$ and $h = \phi_{c_{k+1}}$. Choose $M$ and $N$ such that
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| g(\delta_{-t} \ast \omega_{g_a}) - g(\delta_{-t} \ast \delta) \right| dt < \frac{\varepsilon}{2(C+1)}
\]
and
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| h(\delta_{-t} \ast \omega_{g_a}) - h(\delta_{-t} \ast \delta) \right| dt < \frac{\varepsilon}{2(C+1)^k}
\]
holds for all $\alpha > M$ and $n > N$, which is possible under our assumptions.
Then, we have
\[
\frac{1}{\text{vol}(A_n)} \int_{A_n} \left| g(\delta_{-t} \ast \omega_{g_\alpha}) h(\delta_{-t} \ast \delta_\Lambda) - g(\delta_{-t} \ast \delta_\Lambda) h(\delta_{-t} \ast \delta_\Lambda) \right| dt \\
\leq \frac{1}{\text{vol}(A_n)} \int_{A_n} \left| (g(\delta_{-t} \ast \omega_{g_\alpha}) - g(\delta_{-t} \ast \delta_\Lambda)) h(\delta_{-t} \ast \delta_\Lambda) \right| dt \\
+ \frac{1}{\text{vol}(A_n)} \int_{A_n} \left| (h(\delta_{-t} \ast \omega_{g_\alpha}) - h(\delta_{-t} \ast \delta_\Lambda)) g(\delta_{-t} \ast \delta_\Lambda) \right| dt < \varepsilon
\]
by our previous assumptions. Here, the second term is estimated by an application of Corollary 13 with \(\sup_{t \in G} |g(\delta_{-t} \ast \delta_\Lambda)| \leq C^k\), while the estimate of the first term works as in the proof of the same corollary.

At this point, we may consider the algebra \(A\) of continuous functions on \(\mathbb{Y}\) that is generated by the functions \(\phi_c\) with arbitrary \(c \in C_c(G)\) together with the constant function 1. This algebra is dense in \(C(\mathbb{Y})\) by the Stone–Weierstrass theorem. We thus have a suitable algebra of functions at our disposal to assess equality of probability measures on \(\mathbb{Y}\). Moreover, via Proposition 14, we will be able to assess ergodicity properties as well. In our present context, it would suffice to consider real-valued functions, as all our measures will be positive or signed. Nevertheless, we will discuss the general case of complex-valued functions, as this causes no extra complications.

To continue, observe that we have a net \((\phi_n)\) of invariant probability measures on \(\mathbb{Y}\), which is compact. There is a converging subnet which defines a measure \(\mu\) that is also \(G\)-invariant. Our next step will be to show that this measure is ergodic and that the net itself converges to \(\mu\), so \(\mu\) is unique.

**Proposition 15.** Consider the net \((\mu_n)\) of ergodic \(G\)-invariant probability measures on \(\mathbb{Y}\). Then, for all \(c_1, \ldots, c_k \in C_c(G)\), the net \((\mu_n(\phi_{c_1} \cdot \ldots \cdot \phi_{c_k}))\) is a Cauchy net and hence convergent. Moreover, the limit satisfies
\[
\lim_{\alpha} \mu_\alpha \left( \prod_{i=1}^k \phi_{c_i} \right) = \lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \int_{A_n} \prod_{i=1}^k \phi_{c_i}(\delta_{-t} \ast \omega_{g_\alpha}) dt.
\]

**Proof.** Since \((\mathcal{X}_\alpha, G, \mu_\alpha)\) is uniquely ergodic by Fact 11, we have
\[
\lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \int_{A_n} \prod_{i=1}^k \phi_{c_i}(\delta_{-t} \ast \omega_{g_\alpha}) dt = \mu_\alpha \left( \prod_{i=1}^k \phi_{c_i} \right)
\]
by the stronger version of Birkhoff’s ergodic theorem for the orbit average of a continuous function. This holds for any \(\alpha\).

For a suitable index \(M\) and arbitrary \(\alpha, \alpha' \succ M\), we can now estimate the difference \(|\mu_\alpha(\prod_{i=1}^k \phi_{c_i}) - \mu_{\alpha'}(\prod_{i=1}^k \phi_{c_i})|\) by means of a 4\(\varepsilon\)-argument on the basis of Proposition 14 and Eq. (11). This establishes the Cauchy property by standard arguments.

The second claim is another 3\(\varepsilon\)-argument of a similar kind, again using Eq. (11) and Proposition 14. We leave the details to the reader. \(\Box\)
Theorem 16. The net \((\mu_\alpha)\) of ergodic, \(G\)-invariant probability measures from Proposition 15 converges, and the limit, \(\mu\) say, is an ergodic, \(G\)-invariant probability measure on \(\mathcal{Y}\). Moreover, our weak model set \(\Lambda = \lambda(W)\) of maximal density is generic for \(\mu\).

Proof. Let \(\mu\) be the limit of a fixed subnet of \((\mu_\alpha)\). Since all \(\mu_\alpha\) are \(G\)-invariant probability measures on \(\mathcal{Y}\) and this property is preserved under vague limits, \(\mu\) is a \(G\)-invariant probability measure as well. Via Proposition 15 we know the evaluation of \(\mu\) on all elements of \(\mathcal{A}\), which is dense in \(C(\mathcal{Y})\) and thus determines \(\mu\) completely.

As a consequence, the limit \(\mu'\) of any other convergent subnet of \((\mu_\alpha)\) must agree with \(\mu\) on \(\mathcal{A}\), whence \(\mu = \mu'\), and our original net \((\mu_\alpha)\) is convergent, with limit \(\mu\). Our construction thus determines a unique measure \(\mu\) on \(\mathcal{Y}\). As a vague limit of ergodic measures, it is ergodic as well.

For all \(f \in \mathcal{A}\), we also know from Proposition 15 that

\[
\mu(f) = \lim_{n \to \infty} \frac{1}{\text{vol}(A_n)} \int_{A_n} f(\delta_{\Lambda - t}) \, dt,
\]

whence this also holds for all \(f \in C(\mathcal{Y})\). Consequently, \(\Lambda\) is generic for \(\mu\), which completes our argument. \(\Box\)

Remark 5. Since the measure \(\mu\) constructed above is a regular Borel measure, we can use Eq. (12) to give a nice geometric interpretation for \(\mu\). As our weak model set \(\Lambda\) of maximal density is a point set of finite local complexity, \(\mu\) induces a unique probability measure \(\mu_0\) on the discrete hull \(X_0 := \{\Lambda' \in X_\Lambda \mid 0 \in \Lambda'\}\) via a standard filtration. Now, \(\mu_0\) is specified by its values on the cylinder sets \(Z_{K,\mathcal{P}} = \{\Lambda' \in X_0 \mid \Lambda' \cap K = \mathcal{P}\}\), where \(K \subseteq G\) is compact and \(\mathcal{P}\) a \(K\)-cluster of \(\Lambda\). An inspection of Eq. (12) reveals that the measure of \(Z_{K,\mathcal{P}}\) is nothing but the cluster frequency of \(\mathcal{P}\), defined with respect to the van Hove sequence \(\mathcal{A}\). So, our measure \(\mu\) is the cluster (or patch) frequency measure for \(\Lambda\) relative to \(\mathcal{A}\). \(\diamondsuit\)

Our approach started with an individual weak model set \(\Lambda\) of maximal density, which is then pure point diffractive by Theorem 7. Now, we also have a measure-theoretic dynamical system \((X_\Lambda, G, \mu)\) with an ergodic measure \(\mu\) as constructed above. Relative to the van Hove sequence \(\mathcal{A}\), it is the cluster frequency measure.

Moreover, our weak model set of maximal density is generic for this measure \(\mu\) by Theorem 16 so we know that the individual autocorrelation \(\gamma_\Lambda\) of \(\Lambda\) is also the autocorrelation of the dynamical system, and its Fourier transform, \(\hat{\gamma}_\Lambda\), is the diffraction measure both of \(\Lambda\) and of our dynamical system \(\mu\). Note that the equivalence theorem only needs genericity, but not ergodicity, though the possible statements on the diffraction of a given element of the hull is then even weaker.

By the general equivalence theorem between diffraction and dynamical spectrum in the pure point situation [5, 8], we thus have the following consequence.

Corollary 17. Let \(\Lambda\) be a weak model set of maximal density, relative to a fixed van Hove averaging sequence \(\mathcal{A}\), for a CPS \((G, H, L)\) as above. Then, \(\Lambda\) is pure point diffractive and the dynamical system \((X_\Lambda, G, \mu)\) with the measure \(\mu\) from Theorem 16 has pure point dynamical spectrum. \(\Box\)
Remark 6. If the CPS is irredundant in the sense of [36], see also [6], we can also immediately give the dynamical spectrum, which (in additive notation) is $L^0 = \pi(L^0)$, with the lattice $L^0$ from the dual CPS in Eq. (3).

As in Section 4, general statements seem difficult when the maximality condition for $\text{dens}(A)$ is violated. Examples for the possible complications can once again be taken from the family of Toeplitz sequences, viewed as (weak) model sets with proper windows [7]. Still, repeating our above analysis for weak model sets of minimal density, one obtains the following analogous result.

Corollary 18. Let $\Lambda$ be a weak model set of minimal density, relative to a fixed van Hove averaging sequence $A$, for a CPS $(G, H, L)$ as above. Then, the autocorrelation of $\Lambda$ relative to $A$ exists, and $\Lambda$ is pure point diffractive. Moreover, the dynamical system $(X_\Lambda, G, \mu)$, where $\mu$ is the cluster frequency measure relative to $A$, has pure point dynamical spectrum. The spectrum can be calculated as in Remark 6.

Remark 7. It is important to note that Corollary 18 is a result on the measure-theoretical eigenvalues. It may indeed happen (as in the visible lattice points of Section 3 and their arithmetic generalisations) that the topological point spectrum is trivial. A difference between topological and measure-theoretic spectrum is also well-known and studied in the theory of Toeplitz sequences (see [15] and references therein), which can be described as weak model sets as well [7].

Let us turn our attention to a versatile class of point sets that comprise the arithmetic example of Section 3 as well as its various generalisations.

6. Application to coprime sublattice families

Given a lattice $\Gamma \subset \mathbb{R}^d$, we consider a countable family of proper sublattices $(\Gamma_n)_{n \in \mathbb{N}}$ with the coprimality property that

$$\Gamma_i + \Gamma_j = \Gamma$$

holds for all $i \neq j$. In fact, with $\Gamma_F := \bigcap_{n \in F} \Gamma_n$ for $F \subset \mathbb{N}$ finite and $\Gamma_\emptyset := \Gamma$, we further assume the validity of

$$\Gamma_F + \Gamma_{F'} = \Gamma_{F \cap F'}$$

for all finite $F, F' \subset \mathbb{N}$, which represents some general gcd-law of our lattice family. Finally, we assume the (absolute) convergence condition

$$\sum_{n \in \mathbb{N}} \frac{1}{|\Gamma : \Gamma_n|} < \infty.$$  

We call such a system a coprime sublattice family, which, by definition, is infinite. Similar to the case of the visible lattice points, this setting gives rise to the set

$$V = \Gamma \setminus \bigcup_{n \in \mathbb{N}} \Gamma_n.$$
Let us note in passing that a finite family would simply result in a crystallographic (or fully periodic) point set, hence not to a situation outside the class of regular model sets. This is not of interest to us here.

The coprimality condition clearly implies the Chinese remainder theorem for pairwise coprime sublattices, hence
\[
\frac{\Gamma}{\Gamma_F} = \frac{\Gamma}{\bigcap_{n \in F} \Gamma_n} \simeq \prod_{n \in F} \frac{\Gamma}{\Gamma_n}
\]
for any finite subset \( F \subset \mathbb{N} \). In particular, one has the index formula
\[
(17) \quad \left[ \frac{\Gamma}{\Gamma_F} : \frac{\Gamma}{\Gamma} \right] = \prod_{n \in F} \left[ \frac{\Gamma}{\Gamma_n} : \frac{\Gamma}{\Gamma_n} \right].
\]

In particular, the lattices \( \Gamma_n \) are mutually commensurate, and any finite subset of them still has a common sublattice of finite index in \( \Gamma \). Observing the relation \((\bigcap_{n \in F} \Gamma_n)^* = \bigcap_{n \in F} \Gamma_n^* \) for the dual lattices in this situation, where \( \Gamma_n^* \cap \Gamma_m^* = \Gamma^* \) for \( m \neq n \) as a result of the coprimality condition \((13)\), Eq. \((17)\) is equivalent to
\[
[I_F^* : \Gamma^*] = \left[ \left( \bigcap_{n \in F} \Gamma_n^* \right) : \Gamma^* \right] = \prod_{n \in F} \left[ \Gamma_n^* : \Gamma_n^* \right].
\]

What is more, \( V \) gives rise to a CPS \((G, H, \mathcal{L})\) as in Eq. \((2)\), with \( G = \Gamma \) and the compact group \( H := \prod_{n \in \mathbb{N}} \Gamma/\Gamma_n \), where \( \Gamma/\Gamma_n \) is a quotient group of order \( [\Gamma : \Gamma_n] = |\det(\Gamma_n)|/|\det(\Gamma)| \). The lattice is given by \( \mathcal{L} = \{(x, \iota(x)) \mid x \in \Gamma\} \), where the \( \ast \)-map \( \iota \) is again the natural (diagonal) embedding of \( \Gamma \) in \( H (x \mapsto (x + \Gamma_n)_{n \in \mathbb{N}}) \). Indeed, we can write \( V \) as a cut and project set, \( V = \lambda(W) \), with window
\[
(18) \quad W = \prod_{n \in \mathbb{N}} \left( (\Gamma/\Gamma_n) \setminus \{0 + \Gamma_n\} \right).
\]

With standard arguments, and using the convergence condition \((15)\), it is easy to verify the following result.

**Fact 19.** The window \( W \) of Eq. \((18)\) is a compact subset of \( H \) with empty interior, and has positive measure
\[
\theta_H(W) = \prod_{n \in \mathbb{N}} \left( 1 - \frac{1}{[\Gamma : \Gamma_n]} \right)
\]
with respect to the normalised Haar measure \( \theta_H \) on \( H \). \qed

The following result characterises the cases where \( V \) is of maximal density. Throughout, we let \( A = (A_m)_{m \in \mathbb{N}} \) be a van Hove sequence of centred balls, with \( A_m = B_m(0) \), say.

**Proposition 20.** The weak model set \( V = \lambda(W) \) is of maximal natural density for the CPS \((\Gamma, H, \mathcal{L})\) constructed above if and only if
\[
(19) \quad \lim_{N \to \infty} \text{dens} \left( (\bigcup_{n > N} \Gamma_n) \setminus \bigcup_{n \leq N} \Gamma_n \right) = 0.
\]

**Proof.** For all \( N \geq 1 \), one has \((\bigcup_{n > N} \Gamma_n) \setminus \bigcup_{n \leq N} \Gamma_n = (\bigcup_{n > N} \Gamma_n) \cap (\Gamma \setminus \bigcup_{n \leq N} \Gamma_n)\) and \( V = \Gamma \setminus \bigcup_{n \leq N} \Gamma_n = V_N \setminus R_N \), where
\[
V_N = \Gamma \setminus \bigcup_{n \leq N} \Gamma_n \quad \text{and} \quad R_N = (\bigcup_{n > N} \Gamma_n) \cap V_N.
\]
For fixed $N \in \mathbb{N}$, it is now clear that
\[ \text{dens}(V_N) - \text{dens}(R_N) \leq \text{dens}(V) \leq \text{dens}(V_N) - \text{dens}(R_N). \]

The set $V_N$ is crystallographic with lattice of periods $\bigcap_{n \leq N} \Gamma_n$. Consequently, the natural density of $V_N$ exists, so $\text{dens}(V_N) = \text{dens}(V_N) = \text{dens}(V_N)$. By the inclusion-exclusion formula for sublattice densities, $\text{dens}(V_N)$ can be computed as
\begin{equation}
\text{dens}(V_N) = \frac{1}{|\det(\Gamma)|} \prod_{n \leq N} \left( 1 - \frac{1}{|\Gamma : \Gamma_n|} \right) \prod_{n \in \mathbb{N}} \left( 1 - \frac{1}{|\Gamma : \Gamma_n|} \right).
\end{equation}

Thus, by the convergence from Eq. (20), the density of $V$ exists and is equal to
\begin{equation}
\text{dens}(\mathcal{L} \theta_H(W)) = \frac{1}{|\det(\Gamma)|} \prod_{n \in \mathbb{N}} \left( 1 - \frac{1}{|\Gamma : \Gamma_n|} \right)
\end{equation}
if and only if $\lim_{N \to \infty} \text{dens}(R_N) = 0$. In fact, since $V_N = V \cup R_N$, the density of $V$ exists if and only if the density of $R_N$ exists for all $N \in \mathbb{N}$. $\square$

In particular, $V$ is of maximal density if the lattice family has light tails in the sense of [12], which means that $\lim_{N \to \infty} \text{dens}(\bigcup_{n > N} \Gamma_n) = 0$. This is somewhat reminiscent of the situation of regular versus irregular Toeplitz sequences when described as model sets; see [3]. Note that the complement set $\Gamma \setminus V$ is another weak model set for the same CPS, which has minimal density (in our above terminology) if and only if $V$ has maximal density.

Via our general spectral results from Sections 4 and 5 on weak model sets of maximal density, we now get the following consequence.

**Corollary 21.** Given the maximal density property (19), the point set $V \subset \Gamma$ from Eq. (16) is pure point diffractive, and its hull has pure point dynamical spectrum with respect to the natural cluster frequency measure.

The dynamical spectrum is $\Sigma = +_{n \in \mathbb{N}} \Gamma_n^*$, and the diffraction measure of $\delta_V$ reads
\[ \widehat{\gamma}_V = \sum_{k \in \Sigma} |a(k)|^2 \delta_k, \quad \text{with} \quad a(k) = \text{dens}(V) \prod_{n \in F_k} \frac{1}{1 - [\Gamma : \Gamma_n]}, \]
where $F_k := \bigcap \{ F \subset \mathbb{N} \text{ finite} \mid k \in +_{n \in F} \Gamma_n^* \}$, for each $k \in \Sigma$, is a unique finite subset of $\mathbb{N}$.

**Proof.** The first claim on the pure point nature of the two types of spectra is clear from our above derivation. The calculation of the dynamical spectrum is an elementary consequence of $+_{n \in \mathbb{N}} \Gamma_n^*$ being the spectrum of the crystallographic set $V_N$ and taking the limit. Also the diffraction measure can be calculated this way. Clearly, any point $k \in \Sigma$ is contained in a set of the form $\Gamma_F^* = +_{n \in F} \Gamma_n^*$ for some finite $F \subset \mathbb{N}$. Dualising Eq. (11) gives the relation $\Gamma_F^* \cap \Gamma_F' = \Gamma_{F \cap F'}^*$, which implies the claim on $F_k$.

Since we now know that the diffraction measure is pure point, we know its general form and only need to calculate the amplitude $a(k)$ for a given $k \in \Sigma$. This can be done by a simple inclusion-exclusion argument as follows, which is justified by the norm convergence.
of the sequence of approximating crystallographic systems obtained by suitable truncation. Observe that we have
\[ \delta_V = \sum_{F \subset \mathbb{N}} (-1)^{\text{card}(F)} \delta_{\Gamma_F}. \]
This gives, in the sense of tempered distributions,\n\[ \hat{\delta}_V = \sum_{F \subset \mathbb{N}} (-1)^{\text{card}(F)} \text{dens}(\Gamma_F) \delta_{\Gamma_F}. \] by an application of Poisson’s summation formula; compare \[ \text{Thm. 9.1}. \] From the structure of the lattices \( \Gamma_F \), one now obtains the amplitude as
\[
a(k) = \sum_{F \supseteq F_k} \frac{(-1)^{\text{card}(F)}}{\prod_{m \in F} [\Gamma : \Gamma_m]} \prod_{n \in F_k \setminus F} \left(1 - \frac{1}{[\Gamma : \Gamma_n]}\right) \\
= \left(\text{dens}(\Gamma) \prod_{m \in \mathbb{N} \setminus F_k} \left(1 - \frac{1}{[\Gamma : \Gamma_m]}\right)\right) \prod_{n \in F_k} \frac{-1}{[\Gamma : \Gamma_n]} \left(1 - \frac{1}{[\Gamma : \Gamma_n]}\right)^{-1} \\
= \text{dens}(V) \prod_{n \in F_k} \frac{1}{1 - [\Gamma : \Gamma_n]^{\frac{1}{k}},}
\]
where the sums are over finite subsets of \( \mathbb{N} \) and our previous formula for the density of \( V \) was used in the last step. \( \square \)

The various examples of \( B \)-free systems and their generalisations, which are all covered, can be seen as coprime lattice families with an arithmetic structure. Also, they all fall into the class of weak model sets of maximal density, and are thus special cases of the general theory of weak model sets. In particular, their spectral properties get a nice and general explanation in this way.

Acknowledgements

It is a pleasure to thank Gerhard Keller and Christoph Richard for helpful discussions, and for sharing their approach from reference [23] with us prior to publication. NS was supported by NSERC, under grant 2014-03762. This work was also supported by the German Research Foundation (DFG), within the CRC 701.

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