PSEUDOSPIN, SPIN, AND COULOMB DIRAC-SYMMETRIES: 
DOUBLET STRUCTURE AND SUPERSYMMETRIC PATTERNS

A. LEVIATAN

Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
ami@phys.huji.ac.il

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Relativistic symmetries of the Dirac Hamiltonian with a mixture of spherically symmetric Lorentz scalar and vector potentials, are examined from the point of view of supersymmetric quantum mechanics. The cases considered include the Coulomb, pseudospin and spin limits relevant, respectively, to atoms, nuclei and hadrons.

1. Introduction

The Dirac equation serves as the basis for the relativistic description of atoms, nuclei and hadrons. In atoms the relevant potentials felt by the electron are Coulombic vector potentials. A Dirac Hamiltonian with a Coulomb potential exhibits a fine-structure spectrum with characteristic two-fold degeneracy. Relativistic mean fields in nuclei generated by meson exchanges \(^1\), and quark confinement in hadrons \(^2\) necessitate a mixture of Lorentz vector and scalar potentials. Recently symmetries of Dirac Hamiltonians with such Lorentz structure have been shown to be relevant for explaining the observed degeneracies of certain shell-model orbitals in nuclei ("pseudospin doublets") \(^3\), and the absence of quark spin-orbit splitting ("spin doublets") \(^4\), as observed in heavy-light quark mesons. The degenerate doublets associated with the relativistic Coulomb, pseudospin, and spin symmetries are shown in Table 1. In the current contribution we show \(^5\) that the degeneracy patterns and relations between wave functions implied by such relativistic symmetries resemble

| Limit          | Doublet Structure | Dirac labels |
|----------------|--------------------|--------------|
| Coulomb        | \((n, \ell, j = \ell + 1/2)\) \((n - 1, \ell + 1, j' = \ell + 1/2)\) | \(\kappa_1 + \kappa_2 = 0\) |
| Pseudospin     | \((n, \ell, j = \ell + 1/2)\) \((n - 1, \ell + 1, j' = \ell + 3/2)\) | \(\kappa_1 + \kappa_2 = 1\) |
| Spin          | \((n, \ell, j = \ell + 1/2)\) \((n, \ell, j' = \ell - 1/2)\) | \(\kappa_1 + \kappa_2 = -1\) |
the patterns found in supersymmetric quantum mechanics (SUSYQM). The feasibility of such a proposal gains support from the fact that the Dirac Hamiltonian with a vector Coulomb potential is known to be supersymmetric.

2. Dirac Hamiltonian and Supersymmetric Quantum Mechanics

The essential ingredients of SUSYQM\(^7\) are the supersymmetric Hamiltonian\(^7\) and charges\(^7\) which generate the supersymmetry (SUSY) algebra\(^7\) \([\mathcal{H}, Q_\pm] = \{Q_\pm, Q_\mp\} = 0, \{Q_-, Q_+\} = \mathcal{H}\). The partner Hamiltonians \(H_1 = L^1L\) and \(H_2 = LL^1\) satisfy an intertwining relation,

\[
LH_1 = H_2L, \tag{1}
\]

where in one-dimension the transformation operator \(L = \frac{d}{dx} + W(x)\) is a first-order Darboux transformation expressed in terms of a superpotential \(W(x)\). The intertwining relation ensures that if \(\Psi_1\) is an eigenstate of \(H_1\), then also \(\Psi_2 = L\Psi_1\) is an eigenstate of \(H_2\) with the same energy, unless \(L\Psi_1\) vanishes or produces an unphysical state (e.g., non-normalizable). Consequently, as shown in Fig. 1(a), the SUSY partner Hamiltonians \(H_1\) and \(H_2\) are isospectral in the sense that their spectra consist of pair-wise degenerate levels with a possible non-degenerate single state in one sector (when the supersymmetry is exact). The wave functions of the degenerate levels are simply related in terms of \(L\) and \(L^\dagger\). Such characteristic features define a supersymmetric pattern. In what follows we focus the discussion on supersymmetric patterns obtained in selected Dirac Hamiltonians.

The Dirac Hamiltonian, \(H\), for a fermion of mass \(M\) moving in external scalar, \(V_S\), and vector, \(V_V\), potentials is given by \(H = \hat{\alpha} \cdot p + \hat{\beta}(M + V_S) + V_V\), where \(\hat{\alpha}, \hat{\beta}\) are the usual Dirac matrices and we have set the units \(\hbar = c = 1\). When the potentials are spherically symmetric: \(V_S = V_S(r), V_V = V_V(r)\), the operator \(\hat{K} = -\hat{\beta}(\sigma \cdot \ell + 1), (\text{with } \sigma \text{ the Pauli matrices and } \ell = -i r \times \nabla)\), commutes with \(H\) and its non-zero integer eigenvalues \(\kappa = \pm (j + 1/2)\) are used to label the Dirac wave functions \(\Psi_{\kappa, m} = r^{-1}(G\kappa[Y\ell\chi]^{(j)}_m, iF\kappa[Y\ell\chi]^{(j)}_m)\). Here \(G\kappa(r)\) and \(F\kappa(r)\) are the radial wave functions of the upper and lower components respectively, \(Y\ell\) and \(\chi\) are the spherical harmonic and spin function which are coupled to angular momentum \(j\) with projection \(m\). The labels \(\kappa = -(j + 1/2) < 0\) and \(\ell' = \ell + 1\) hold for aligned spin \(j = \ell + 1/2\) \((s_{1/2}, p_{3/2}, \text{etc.})\), while \(\kappa = (j + 1/2) > 0\) and \(\ell' = \ell - 1\) hold for unaligned spin \(j = \ell - 1/2\) \((p_{1/2}, d_{3/2}, \text{etc.})\). Denoting the pair of radial wave functions by

\[
\Phi_\kappa = \begin{pmatrix} G\kappa \\ F\kappa \end{pmatrix}, \tag{2}
\]

the radial Dirac equations can be cast in Hamiltonian form,

\[
H_\kappa \Phi_\kappa = \begin{pmatrix} M + \Delta & -\frac{d}{dr} + \frac{\Sigma}{r} \\ \frac{d}{dr} + \frac{\Sigma}{r} & -(M + \Sigma) \end{pmatrix} \begin{pmatrix} G\kappa \\ F\kappa \end{pmatrix} = E \Phi_\kappa, 
\]

\[
\Delta(r) = V_S + V_V, \quad \Sigma(r) = V_S - V_V. \tag{3}
\]
In analogy to Eq. (1) we now look for radial Dirac Hamiltonians $H_{\kappa_1}$ and $H_{\kappa_2}$ which satisfy an intertwining relation of the form

$$LH_{\kappa_1} = H_{\kappa_2}L.$$  

(4)

Following Ref. [8] we consider a matricial Darboux transformation operator

$$L = A(r) \frac{d}{dr} + B(r),$$

(5)

where $A$ and $B$ are $2 \times 2$ matrices with $r$-dependent entries $A_{ij}(r)$, $B_{ij}(r)$. Relations (4) and (5) should be regarded as a system of equations for the unknown operator $L$ and the so-far unspecified potentials in $H_\kappa$ (3). The transformation
and imposes relations between their respective components.

In the usual application of SUSYQM, one starts from a solvable Hamiltonian \( H_1 \) and uses the intertwining relation to obtain a new solvable Hamiltonian \( H_2 \). In the present case we employ a different strategy, namely, insist that both partner Hamiltonians \( H_N \) and \( H_{\kappa} \) be of the form prescribed in Eq. (3) with the same potentials, and look for solutions of Eq. (4) such that the potentials are independent of \( \kappa \). We find three physically interesting solutions which require \( \kappa_1 + \kappa_2 = 0, 1, -1 \) and lead to the Coulomb, pseudospin, and spin limits respectively.

### 3. The Coulomb limit \( (\kappa_1 + \kappa_2 = 0) \)

The solution of Eq. (4) with \( \kappa_1 + \kappa_2 = 0 \) fix the potentials to be of Coulomb type

\[
V_S = \frac{\alpha_S}{r}, \quad V_V = \frac{\alpha_V}{r},
\]

(omitting constant shifts) with arbitrary strengths, \( \alpha_S, \alpha_V \). In terms of the quantities \( \eta_1 = (\alpha_S M + \alpha_V E)/\lambda, \eta_2 = (\alpha_S E + \alpha_V M)/\lambda, \lambda = \sqrt{M^2 - E^2}, \gamma = \sqrt{\kappa^2 + \alpha_S^2 - \alpha_V^2} \), the quantization condition reads: \( \gamma + \eta_1 = -n_r (n_r = 0, 1, 2, \ldots) \), and leads to the eigenvalues

\[
E_{n_r, \kappa}^{(\pm)} = \frac{-\alpha_S \alpha_V \pm (n_r + \gamma)\sqrt{(n_r + \gamma)^2 - \alpha_S^2 - \alpha_V^2}}{\alpha_V^2 + (n_r + \gamma)^2}.
\]

The \( \kappa \)-dependence enters through the factor \( \gamma \). The spectrum consists of two branches denoted by superscript \( (+) \) and \( (-) \). The eigenfunctions are

\[
\Phi_{n_r, \kappa} = \begin{pmatrix} G_{\kappa} \\ F_{\kappa} \end{pmatrix} = \mathcal{N} \begin{pmatrix} -\sqrt{M + E}[(\kappa + \eta_2)F_1 + n_r F_2] \\ \sqrt{M - E}[(\kappa + \eta_2)F_1 - n_r F_2] \end{pmatrix} \rho^{r} e^{-\rho/2},
\]

\[
\mathcal{N} = \sqrt{\frac{\lambda}{\Gamma(2\gamma + 1)} \left[ \frac{\Gamma(2\gamma + n_r + 1)}{2Mn_r!\eta_2(\kappa + \eta_2)} \right]^{1/2}}.
\]

where \( E = E_{n_r, \kappa}^{(\pm)} \) and \( F_1 = F(-n_r, 2\gamma + 1, \rho), F_2 = F(-n_r, 1, 2\gamma + 1, \rho) \) are confluent hypergeometric functions in the variable \( \rho = 2\lambda r \). The states and energies in each branch are labeled by \( (n_r, \kappa) \). It is also possible to express the results in terms of the principal quantum number \( N \) defined as \( N = n_r + |\kappa|, (N = 1, 2, \ldots) \). For \( n_r \geq 1 \) the eigenvalues in each branch are two-fold degenerate with respect to the sign of \( \kappa \), i.e. \( E_{n_r, \kappa}^{(+)} = E_{n_r, -\kappa}^{(+)} \) and \( E_{n_r, \kappa}^{(-)} = E_{n_r, -\kappa}^{(-)} \). For \( n_r = 0 \) there is only one acceptable state for each \( \kappa \), which has \( \kappa < 0 \) for the \( (+) \) branch and \( \kappa > 0 \) for the \( (-) \) branch. Equivalently, for a fixed principal quantum number \( N \), the allowed values of \( \kappa \) are \( \kappa = \pm 1, \pm 2, \ldots, \pm(N - 1) \), \( -N \) for the \( (+) \) branch and \( \kappa = \pm 1, \pm 2, \ldots, \pm(N - 1), +N \) for the \( (-) \) branch of the spectrum.

Focusing on the set of states with \( \kappa_1 = -\kappa_2 \equiv \kappa \), the levels are separated according to the value of \(|\kappa| = j + 1/2\). For fixed \( \kappa \), \( E^{(+)}_{n_r, \kappa} \) is an increasing function...
of \( n_r \) and, as shown in Fig. 1(b), for each value of \( j \) we have a characteristic supersymmetric pattern. There are two towers of energy levels, one for \(-|\kappa|\) (with \( n_r = 0, 1, 2, \ldots \)) and one for \(+|\kappa|\) (with \( n_r = 1, 2, \ldots \)). The two towers are identical, except that the \( E_{n_r=0,-|\kappa|}^{(+)} \) level at the bottom of the \(-|\kappa|\) tower is non-degenerate. Similar patterns of pair-wise degenerate levels with \( \pm\kappa \) appear also in the \((-)\) branch of the spectrum. However, since for fixed \( \kappa \), \( E_{n_r=0,|\kappa|}^{(-)} \) is a decreasing function of \( n_r \), the non-degenerate \( E_{n_r=0,|\kappa|}^{(-)} \) level is now at the top of the \(+|\kappa|\) tower, resulting in an inverted supersymmetric pattern. The transformation operator is given by

\[
L = a \left( \frac{d}{dr} + \frac{\alpha_S}{\kappa_1} \frac{d}{dr} - \frac{\alpha_V}{\kappa_1} \frac{d}{dr} - \frac{\alpha_S}{\kappa_1} \frac{d}{dr} - \frac{\alpha_V}{\kappa_1} \frac{d}{dr} \right),
\]

where \( \epsilon_{\pm} = \kappa_1 + \alpha_S \alpha_{\pm}/\kappa_1 \) and \( \alpha_{\pm} = (\alpha_S \pm \alpha_V) \). Relation (6) is satisfied with \( C = \frac{\alpha}{\kappa_1} \sqrt{n_r(\gamma - \eta)} \) and \( \kappa_1 + \kappa_2 = 0 \). The operator \( L \) connects degenerate states with \( (n_r \geq 1, \pm \kappa) \), and annihilates the non-degenerate states with \( n_r = 0 \). The condition \( \kappa_1 + \kappa_2 = 0 \) determines the angular momentum couplings in the full Dirac wave functions of the doublet

\[
\Psi_{\kappa_1<0,m} = \frac{1}{r} \left( G_{\kappa|1}[Y_{\ell}\chi|j\rangle_m^j] \right), \quad \Psi_{\kappa_2>0,m} = \frac{1}{r} \left( G_{\kappa_2}[Y_{\ell+1}\chi|j\rangle_m^{j'}] \right)
\]

where \( \kappa_1 = -(\ell + 1) < 0 \), \( j = \ell + 1/2 \) and \( \kappa_2 = +(\ell + 1) > 0 \), \( j' = \ell + 1/2 \).

The explicit solvability and observed degeneracies of the relativistic Coulomb problem are related to the existence of an additional conserved Hermitian operator \( B = -i\hat{K}\gamma_5 \left( H - \hat{\beta} M \right) + \frac{\sigma \cdot r}{r} (\alpha_V M + \alpha_S H) \)

which commutes with the full Dirac scalar and vector Coulomb Hamiltonian, \( H \), but anticommutes with \( \hat{K} \). This operator is a generalization of the Johnson-Lippmann operator \( \hat{G} \) applicable for \( \alpha_S = 0 \).

4. The pseudospin limit \( (\kappa_1 + \kappa_2 = 1) \)

The solution of Eq. (4) with \( \kappa_1 + \kappa_2 = 1 \) requires that the sum of scalar and vector potentials is a constant

\[
\Delta(r) = V_S(r) + V_V(r) = \Delta_0.
\]

In this case the transformation operator is given by

\[
L = b \left( \frac{d}{dr} - \frac{\kappa_1}{r} \frac{d}{dr} \right),
\]

Relation (6) is obeyed with \( \kappa_1 + \kappa_2 = 1 \), \( C = b(M + \Delta_0 - E) \), \( E = E_{\kappa_1} = E_{\kappa_2} \) and, consequently, the radial components satisfy

\[
\frac{dG_{\kappa_1}}{dr} + \frac{\kappa_1}{r} G_{\kappa_1} = \frac{dG_{\kappa_2}}{dr} + \frac{\kappa_2}{r} G_{\kappa_2},
\]

\[
F_{\kappa_1} = F_{\kappa_2}.
\]
The condition $\kappa_1 + \kappa_2 = 1$ determines the form of the full Dirac wave functions of the doublet states

$$
\Psi_{\kappa_1<0,m} = \frac{1}{r} \left( \frac{G_{\kappa_1}}{iF_{\kappa_1}} \right) \left[ Y_{\ell} \chi_{\kappa_1}^{(j)} \right]_{m}^{(j)} \\
\Psi_{\kappa_2>0,m} = \frac{1}{r} \left( \frac{G_{\kappa_2}}{iF_{\kappa_2}} \right) \left[ Y_{\ell+1} \chi_{\kappa_2}^{(j')} \right]_{m}^{(j')} 
$$

where $\kappa_1 = -(\ell + 1) < 0$, $j = \ell + 1/2$ and $\kappa_2 = \ell + 2 > 0$, $j' = \ell + 3/2$.

From Eqs. (15)-(16) we see that the lower components of the two states in the doublet have the same spatial wave function. In particular, they have the same orbital angular momentum $\ell = \ell + 1$, and the same number of nodes, $n$. Eq. (16) and a theorem concerning the nodal structure of Dirac bound states ensure that the corresponding upper components have quantum numbers $(n, \ell, j = \ell + 1/2)$ for $\kappa_1 < 0$ and $(n - 1, \ell + 2, j = \ell + 3/2)$ for $\kappa_2 > 0$. These are precisely the identifying features of pseudospin doublets in nuclei. The latter refer to the empirical observation of quasi-degenerate pairs of normal-parity shell-model orbitals with such non-relativistic quantum numbers. The doublet structure is expressed in terms of the “pseudo” orbital angular momentum, $\tilde{\ell} = \ell + 1$, and “pseudo” spin, $\tilde{s} = 1/2$, which are coupled to $j = \tilde{\ell} \pm \tilde{s}$. Such doublets play a central role in explaining features of nuclei, including superdeformation and identical bands. For potentials with asymptotic behavior as encountered in nuclei, the Dirac eigenstates for which both the upper ($G_{\kappa}$) and lower ($F_{\kappa}$) components have no nodes, can occur only for $\kappa < 0$, and hence do not have a degenerate partner eigenstate (with $\kappa > 0$). These nodeless Dirac states correspond to the shell-model states with $(n = 0, \ell, j = \ell + 1/2)$. For heavy nuclei such states with large $j$, i.e., $0g_{9/2}$, $0h_{11/2}$, $0i_{13/2}$, are the “intruder” abnormal-parity states which, indeed, empirically are found not to be part of a doublet. Altogether, as shown in Fig. 1(c), the ensemble of Dirac states with $\kappa_1 + \kappa_2 = 1$ exhibits a supersymmetric pattern of twin towers with pair-wise degenerate pseudospin doublets sharing a common $\ell$, and an additional non-degenerate nodeless state at the bottom of the $\kappa_1 < 0$ tower. An exception to this rule is the tower with $\kappa_2 = 1$ ($p_{1/2}$ states with $\tilde{\ell} = 0$), which is on its own, because states with $\kappa_1 = 0$ do not exist.

For potentials satisfying the condition of Eq. (13), the Dirac Hamiltonian is invariant under an SU(2) algebra, whose generators are

$$
\hat{S}_\mu = \begin{pmatrix} 0 & \hat{s}_\mu \\ \hat{s}_\mu & 0 \end{pmatrix}.
$$

Here $\hat{s}_\mu = \sigma_\mu/2$ are the usual spin operators, $\hat{s}_\mu = U_\mu \hat{s}_\mu U_\mu$ and $U_\mu = \sigma_\mu/\mu$. This relativistic symmetry has been used to explain the occurrence of pseudospin doublets in nuclei. Eqs. (15)-(16) are derived by exploiting the fact that in the symmetry limit the Dirac eigenfunctions belong to the spinor representation of the pseudospin SU(2) algebra. The relations in Eq. (15) between the radial components of the doublet wave functions, have been tested in numerous realistic mean field calculations in a variety of nuclei, and were found to be obeyed to a good approximation, es-
5. The spin limit ($\kappa_1 + \kappa_2 = -1$)

The solution of Eq. (4) with $\kappa_1 + \kappa_2 = -1$ requires that the difference of the scalar and vector potentials is a constant

$$\Sigma(r) = V_S(r) - V_V(r) = \Sigma_0 .$$

(18)

The transformation operator is given by

$$L = -b \left( \frac{2M + \Sigma_0 + \Delta}{\frac{d}{dr}} \frac{-d}{dr} + \frac{\kappa_1}{r} \right).$$

(19)

It connects the two doublet states as in Eq. (6) with $\kappa_1 + \kappa_2 = -1$, $C = -b(E + M + \Sigma_0)$ and $E = E_{\kappa_1} = E_{\kappa_2}$. The corresponding radial components then satisfy

$$G_{\kappa_1} \frac{dF_{\kappa_1}}{dr} - \frac{\kappa_1}{r} F_{\kappa_1} = \frac{dF_{\kappa_2}}{dr} - \frac{\kappa_2}{r} F_{\kappa_2} .$$

(20)

The condition $\kappa_1 + \kappa_2 = -1$ determines the form of the full Dirac wave functions of the two states in the doublet

$$\Psi_{\kappa_1 < 0, m} = \frac{1}{r} \begin{pmatrix} G_{\kappa_1} [Y_l \chi]_j^m \hfill \\
 iF_{\kappa_1} [Y_{l+1} \chi]_j^m \end{pmatrix}$$

$$\Psi_{\kappa_2 > 0, m} = \begin{pmatrix} G_{\kappa_2} [Y_{l'} \chi]_{j'}^m \hfill \\
 iF_{\kappa_2} [Y_{l'-1} \chi]_{j'}^m \end{pmatrix}$$

(21)

where $\kappa_1 = -(l + 1)$, $j = l + 1/2$ and $\kappa_2 = +l$, $j' = l - 1/2$. Using Eq. (20) we see that the upper components in Eq. (21) share the same spatial wave function, and have quantum numbers $(n, l, j = l + 1/2)$ for $\kappa_1 < 0$ and $(n, l, j = l - 1/2)$ for $\kappa_2 > 0$. As shown in Fig. 1(d), the spectrum consists of towers of states with $\kappa_1 + \kappa_2 = -1$, forming pair-wise degenerate spin doublets. In this case, none of the towers have a single non-degenerate state and hence, the spectrum corresponds to that of a broken SUSY. The tower with $\kappa_1 = -1$ ($s_{1/2}$ states) is on its own, since states with $\kappa_2 = 0$ do not exist.

For potentials satisfying condition (18) the Dirac Hamiltonian is invariant under another SU(2) algebra, whose generators are obtained from Eq. (17) by interchanging $\hat{s}_\mu$ and $\hat{s}_\mu$.

$$\hat{S}_\mu = \begin{pmatrix} \hat{s}_\mu & 0 \\
 0 & \hat{s}_\mu \end{pmatrix} .$$

(22)

Eqs. (20)-(21) follow from the fact that the Dirac eigenfunctions in the spin limit are spinors with respect to this SU(2) algebra. The spin doublets resulting from this relativistic symmetry were argued to be relevant for degeneracies observed in heavy-light quark mesons.
6. Summary

We have examined three symmetry limits of a Dirac Hamiltonian with spherically-symmetric scalar and vector potentials, from a supersymmetric quantum mechanics perspective. In the Coulomb limit the potentials are $1/r$ but their strengths are otherwise arbitrary. In the pseudospin or spin limits there are no restrictions on the $r$-dependence of the potentials but there is a constraint on their sum or difference. These relativistic symmetries lead to degenerate doublets with quantum numbers shown in Table 1, and impose relations between the respective doublet wave functions. The latter relations are precisely the conditions needed for the fulfillment of an intertwining relation, Eq. (4), which is the underlying mechanism of SUSYQM. The resulting supersymmetric patterns exhibit sectors of pair-wise degenerate doublets, with a possible nondegenerate single state in one sector. It is gratifying to note that the indicated supersymmetric patterns are manifested empirically, to a good approximation, in physical dynamical systems. While previous studies have focused on properties of individual doublets in nuclei and hadrons, it is the grouping of several doublets (and intruder levels in nuclei) into larger multiplets, as discussed in the present contribution, which highlights the fingerprints of supersymmetry present in these dynamical systems.

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