CONTINUITY OF THE DATA-TO-SOLUTION MAP FOR THE FORQ EQUATION IN BESOV SPACES

JOHN HOLMES¹, FERIDE TIĞLAY², RYAN THOMPSON³

¹DEPARTMENT OF MATHEMATICS AND STATISTICS, WAKE FOREST UNIVERSITY, WINSTON SALEM, NC 27109, USA
²DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA
³DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF NORTH GEORGIA, DAHLONEGA, GA 30597, USA

Abstract. For Besov spaces $B^s_{p,r}(\mathbb{R})$ with $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, $p \in (1, \infty]$ and $r \in [1, \infty)$, it is proved that the data-to-solution map for the FORQ equation is not uniformly continuous from $B^s_{p,r}(\mathbb{R})$ to $C([0,T];B^s_{p,r}(\mathbb{R}))$. The proof of non-uniform dependence is based on approximate solutions and the Littlewood-Paley decomposition.

1. Introduction

We consider the following initial value problem for the integrable partial differential equation, known as the FORQ equation:

$$u_t + uu_x - u^3 + (1 - \partial_x^2)^{-1}(\frac{2}{3}u^3 + uu_x^2) + (1 - \partial_x^2)^{-1}(\frac{1}{3}u_x^3) = 0$$

(1.1)

$$u(x,0) = u_0(x),$$

(1.2)

where $x \in \mathbb{R}$ and $u_0 \in B^s_{p,r}(\mathbb{R}) = B^s_{p,r}$ where $(1 - \partial_x^2)^{-1}f = \mathcal{F}^{-1}(\frac{1}{1+\xi^2}\widehat{f}(\xi))$. The FORQ equation was derived in Fokas [5], Fuchssteiner [9], Olver and Rosenau [18] by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. It was also derived by Qiao [19] who showed an entire integrable hierarchy known as the MCH hierarchy and the existence of a Lax pair. Qiao also showed that FORQ admits solitary solutions including peakons, solitary traveling wave solutions of the form

$$u(x,t) = \sqrt{c}e^{-|x-ct|}.$$

Peakons were discovered by Fornberg and Whitham [7] and then by Camassa and Holm [2] as solutions to shallow water wave equations, and take the general form

$$u(x,t) = ce^{-|x-q(t)|}.$$

The Camassa-Holm (CH) equation

$$(1 - \partial_x^2)^2 u_t = uu_{xxx} + 2u_x u_{xx} - 3uu_x,$$

(1.3)

is an integrable equation closely related to the FORQ equation, which also admits peakons; for more information about peakon solutions, we refer the reader to Holm and Ivanov [14]. Originally derived by Fokas and Fuchssteiner [6] in the context of hereditary symmetries, it was later derived by [2] from the Euler equations through asymptotic expansions. One may note that the CH equation has quadratic rather than cubic nonlinearities, and this plays an important role in the

Date: October 12, 2020.
2010 Mathematics Subject Classification. 35Q35, 35Q80.
analysis of these two equations. Local well-posedness results for the CH equation were discovered
by Li and Olver [17], Constantine and Escher [3], Rodriguez-Blanco [20] and Danchin [4] among
others.

Well-posedness in the sense of Hadamard for the Cauchy problem (1.1)-(1.2) was shown by Himonas
and Matzavinos [13] in Sobolev spaces $H^s$, $s > 5/2$ in both the periodic and non periodic cases.
Well-posedness in Besov spaces $B^{s}_{p,r}$ with $p \in [1, \infty]$, $r \in [1, \infty)$ and $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$ and at the
critical index corresponding to $s = 5/2$, $p = 2$, $r = 1$, by Fu, Gui, Liu and Qu [8] (additionally the
case $r = \infty$ is considered, however, the continuity of the data-to-solution map is established in a
weaker topology). These authors also derive a precise blow-up scenario and a lower bound on the
maximal time of existence for solutions. The methodology used in the well-posedness argument in
Besov spaces is based upon the ideas in [4].

Ill-posedness for the Cauchy problem (1.1) in $H^s$ for any $s < 3/2$ was demonstrated by studying
the interaction of multi-peakon solutions in Himonas and Holliman [10]. This was extended to a
related family of equations in [15]. Well-posedness in the gap, $\frac{3}{2} < s < \frac{5}{2}$, remains an open question
for the FORQ initial value problem, and appears to be challenging due to the $u^3_x$ term.

Continuity of the data-to-solution map is an important part of the well-posedness theory, and is
very delicate for the CH equation and related problems. In fact, Himonas and Kenig [12] and
Himonas, Kenig and Misiolek [11] prove that the data-to-solution map for the CH equation in
Sobolev spaces is not uniformly continuous in the non-periodic and periodic cases respectively.
They use the method of approximate solutions, conservation of the $H^1$ norm, and commutator
estimates. More recently, a similar result has been shown in Besov spaces by Li, Yu and Zhu [16].
This result does not extend cleanly to the FORQ equation. Rather, using some ideas from [16] as
well as some ideas from [13] we are able to deal with the cubic nonlinearities in the FORQ equation
and prove nonuniform dependence of the data-to-solution map in Besov spaces. In particular, our
main result can be stated as follows.

**Theorem 1.1.** Assume $s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}$, with $p \in (1, \infty]$ and $r \in [1, \infty)$. Then the data-to-
solution map for the initial value problem (1.1)-(1.2) is not uniformly continuous from $B^{s}_{p,r}(\mathbb{R})$ to
$C([0, T]; B^{s}_{p,r}(\mathbb{R}))$.

The case $p = 1$ is not covered by our theorem. Similar to the difficulties in proving well-posedness,
our estimates fail in this case due to the $u^3_x$ term in the FORQ equation.

Our paper is organized as follows. In the next section, Section 2, we present some preliminary
results and introduce notation used in Besov space. Section 3 presents the proof of Theorem 1.1 and
introduces some important estimates readers may find useful for related problems. In particular
we introduce two sequences of functions which we will use to construct our approximate solutions.
We next provide some necessary estimates concerning these functions, and then outline our proof
before providing the details.

2. Preliminary result and notation

In this section, we will recall some conclusions on the properties of Littlewood-Paley decomposition,
and Besov spaces; these results may be found in [1]. We begin with the Littlewood-Paley decomposition.
Lemma 2.1. (Littlewood-Paley decomposition). There exists a couple of smooth radial functions \((\chi, \varphi)\) valued in \([0,1]\) such that \(\chi\) is supported in the ball \(B = \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}\) and \(\varphi\) is supported in the ring \(C = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\). Moreover,
\[
\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1
\]
and
\[
\text{supp } \varphi(2^{-q'}) \cap \text{supp } \varphi(2^{-q''}) = \emptyset, \quad \text{if } |q - q'| \geq 2,
\]
\[
\text{supp } \chi(\cdot) \cap \text{supp } \varphi(2^{-q}) = \emptyset, \quad \text{if } |q| \geq 1.
\]
Then for \(u \in S'(\mathbb{R})\) the nonhomogeneous dyadic blocks are defined as follows:
\[
\Delta_q u = 0, \quad \text{if } q \leq -2,
\]
\[
\Delta_{-1} u = \chi(D) u = \mathcal{F}_x^{-1} \chi \mathcal{F} u,
\]
\[
\Delta_q u = \varphi(2^{-q} D) \mathcal{F}_x^{-1} \varphi(2^{-q} \xi) \mathcal{F} u, \quad \text{if } q \geq 0.
\]
Thus \(u = \sum_{q \in \mathbb{Z}} \Delta_q u\) in \(S'(\mathbb{R})\).

Remark. The low frequency cut-off \(S_q\) is defined by
\[
S_q u = \sum_{p=-1}^{q-1} \Delta u = \mathcal{F}_x^{-1} \chi(2^{-q}\xi) \mathcal{F}_x u, \quad \forall q \in \mathbb{N}.
\]
We can see that
\[
\Delta_p \Delta_q u = 0, \quad \text{if } |p - q| \geq 2,
\]
\[
\Delta_q (S_{p-1} u \Delta_p u) = 0, \quad \text{if } |p - q| \geq 5, \quad \forall u, v \in S'(\mathbb{R})
\]
as well as \(\|\Delta_q u\|_{L^p} \leq \|u\|_{L^p}\), and \(\|S_q u\|_{L^p} \leq C\|u\|_{L^p}\), \(\forall p \in [1, \infty]\), with the aid of Young’s Inequality, where \(C\) is a positive constant independent of \(q\).

Using the Littlewood-Paley decomposition, we may now define the Besov space.

Definition 2.1. (Besov Spaces) Let \(s \in \mathbb{R}, p, r \in [1, \infty]\) and \(u \in S'(\mathbb{R})\). Then we define the Besov space of functions as
\[
B^s_{p,r} = B^s_{p,r}(\mathbb{R}) = \{u \in S'(\mathbb{R}) : \|u\|_{B^s_{p,r}} < \infty\},
\]
where
\[
\|u\|_{B^s_{p,r}} = \begin{cases} \left( \sum_{q \geq -1} 2^{sq} \|\Delta_q u\|_{L^p}^r \right)^{1/r} & \text{if } 1 \leq r < \infty \\ \sup_{q \geq -1} 2^{sq} \|\Delta_q u\|_{L^p} & \text{if } r = \infty. \end{cases}
\]
In particular, \(B^\infty_{p,r} = \bigcap_{s \in \mathbb{R}} B^s_{p,r}\).

There are several important, albeit standard results which ensure Besov spaces are amenable to studying partial differential equations.

Lemma 2.2. Let \(s \in \mathbb{R}, 1 \leq p, r, p_j, r_j \leq \infty, j = 1, 2\), then
\begin{enumerate}
\item Topological properties: \(B^s_{p,r}\) is a Banach space which is continuously embedded in \(S'(\mathbb{R})\).
\item Density: \(C^\infty_0\) is dense in \(B^s_{p,r} \iff p, r \in [1, \infty)\).
\item Embedding: \(B^s_{p_1,r_1} \hookrightarrow B^{s-(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2,r_2}\), if \(p_1 \leq p_2\) and \(r_1 \leq r_2\).
\[
B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}, \text{ locally compact if } s_1 < s_2.
\]
\end{enumerate}
(4) Algebraic properties: \( \forall s > 0, B^s_{p,r} \cap L^\infty \) is a Banach algebra. \( B^s_{p,r} \) is a Banach algebra if and only if \( s > \frac{1}{p} \) or \( (s \geq \frac{1}{p} \text{ and } r = 1) \). In particular, \( B^{1/p}_{p,1} \) is continuously embedded in \( B^{1/p}_{p,\infty} \cap L^\infty \) and \( B^{1/p}_{p,\infty} \cap L^\infty \) is a Banach algebra.

(5) 1-D Moser-type estimates: For \( s > 0 \),
\[
\|fg\|_{B^s_{p,r}} \leq C(\|f\|_{B^s_{p,r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^s_{p,r}}).
\]
(6) For \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \),
\[
\|fg\|_{B^s_{p,r}^{-2}} \leq C\|f\|_{B^s_{p,r}^{-2}} \|g\|_{B^s_{p,r}^{-1}}.
\]
(7) Interpolation:
\[
\|f\|_{B^s_{p,r}^{-1+(\theta-1)s}2} \leq \|f\|_{B^s_{p,r}^{\theta}} \|g\|_{B^s_{p,r}^{1-\theta}} \|g\|_{B^s_{p,r}^{-1}}, \quad \forall f \in B^s_{p,r} \cap B^{s_1}_{p,r}, \quad \forall \theta \in [0, 1].
\]

Finally, we shall use the following transport equation estimate extensively.

**Proposition 2.1.** Let \( p, r \in [1, \infty] \) and \( \sigma > 1 + \frac{1}{p} \) or \( \sigma = 1 + \frac{1}{p} \) when \( r = 1 \). Assume \( v_0 \in B^\sigma_{p,r} \), \( g \in L^1([0,T]; B^\sigma_{p,r}) \) and \( \partial_x f \in L^1([0,T]; B^{\sigma-1}_{p,r}) \). Let \( v \in L^\infty([0,T]; B^\sigma_{p,r} \cap C([0,T]; S')) \) solve the following transport equation,
\[
\begin{aligned}
\frac{dv}{dt} + f v_x &= g, \\
v(x,0) &= v_0(x).
\end{aligned}
\]

Then there exists a constant \( C \) which depends upon \( p, r \) and \( \sigma \), such that
\[
\|v(t)\|_{B^\sigma_{p,r}} \leq e^{V(t)} \left( \|v(0)\|_{B^\sigma_{p,r}} + \int_0^t e^{-V(s)} \|g(s)\|_{B^\sigma_{p,r}} ds \right),
\]
where \( V(t) = \int_0^t \|\partial_x f\|_{B^{\sigma-1}_{p,r}} ds \). If \( -\min\{\frac{1}{p}, 1-\frac{1}{p}\} \leq \sigma < 1 + \frac{1}{p} \), the above estimate holds with \( V(t) = \int_0^t \|\partial_x f\|_{B^{\sigma}_{p,\infty} \cap L^\infty} ds \).

### 3. Proof of Theorem [1.1](#)

Our proof of non-uniform dependence relies upon choosing two sequences of initial data, \( u_{0,n}(x) \) and \( v_{0,n}(x) \), which converge to each other as \( n \) tends towards \( \infty \) in \( B^s_{p,r} \), however, the corresponding solutions, denoted \( u_n \) and \( v_n \), remain bounded away from each other for any positive time. We begin by outlining the proof.

We let \( \tilde{\phi}(\xi) \) be a smooth bump function equal to 1 when \( |\xi| \leq 1/4 \) and support on the interval \( |\xi| < 1/2 \), and we choose two constants \( \delta \) and \( \sigma \) which satisfy \( 0 < \delta < \frac{1}{8} \), \( \max\{1+\frac{1}{p}, s-\frac{9}{8}\} < \sigma < s-1 \) and \( 0 < \frac{8\delta}{p} < s - \sigma - 1 \). For integers \( n \geq 10 \), we choose the initial data
\[
\begin{aligned}
u_{0,n}(x) &= 2^{-n} \phi(2^{-n} x) \cos \left( \frac{17}{12} \cdot 2^n x \right), \\
v_{0,n}(x) &= u_{0,n}(x) + 2^{-n} \phi(2^{-n} x),
\end{aligned}
\]
which satisfies the following estimate.
Lemma 3.1. For any $s, s' \in \mathbb{R}$ and $(p, r) \in [1, \infty] \times [1, \infty)$, we have

\[
\|2^{-s\frac{1}{2^n}} \phi(2^{-s\delta_n} x)\|_{B^s_{p', r}} = 2^{s\frac{1}{2^n} - s\frac{1}{2^n}} \|\phi\|_{B^s_{p', r}} \tag{3.2}
\]

\[
\|2^{-ns-s\frac{1}{2^n}} \phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right)\|_{B^s_{p', r}} \leq 2^{n(s'-s)} \|\phi\|_{B^s_{p', r}} \tag{3.3}
\]

\[
\|2^{-ns-s\frac{1}{2^n}} \phi(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right)\|_{B^s_{p', r}} \leq 2^{n(s'-s)} \|\phi\|_{B^s_{p', r}} \tag{3.4}
\]

\[
\lim_{n \to \infty} 2^{-ns-s\frac{1}{2^n}} \phi^3(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right)\|_{B^s_{p', r}} \approx 1. \tag{3.5}
\]

Proof. Using

\[
\mathcal{F} \left( \phi(2^{-s\delta_n} \cdot) \right) (\xi) = 2^{s\delta_n} \tilde{\phi}(2^{s\delta_n} \xi)
\]

we see that $\text{supp}(\mathcal{F}[\phi(2^{-s\delta_n} x)]) \subseteq \{ \xi : |\xi| \in \left[ \frac{2^n}{1}, \frac{2^n}{1} \right] \}$, from which we can see

\[
\Delta \phi(2^{-s\delta_n} x) = \begin{cases} 
\phi(2^{-s\delta_n} x) & \text{if } q = -1, \\
0 & \text{if } q \geq 0.
\end{cases}
\]

Therefore,

\[
\|2^{-s\frac{1}{2^n} - s\frac{1}{2^n}} \phi(2^{-s\delta_n} x)\|_{B^s_{p', r}} = 2^{-s'\frac{1}{2^n} - s\frac{1}{2^n}} \|\phi(2^{-s\delta_n} x)\|_{L^p} = 2^{-s'\frac{1}{2^n} - s\frac{1}{2^n}} \|\phi\|_{L^p},
\]

which completes the first estimate in the lemma. Next, we compute the support of

\[
\mathcal{F} \left[ \phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right) \right] (\xi), \text{ and the support of } \mathcal{F} \left[ \phi(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right) \right] \text{ and show that they both are subsets of } \{ \xi : |\xi| \in \left[ \frac{17}{12} 2^n - \frac{1}{12}, \frac{17}{12} 2^n + \frac{1}{12} \right] \}.\]

Indeed, using the identity $\sin(\theta) = \frac{1}{2} i(e^{-i\theta} - e^{i\theta})$ we have

\[
\mathcal{F}[\phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right)] = \int e^{-ix\xi} \phi(2^{-s\delta_n} x) \sin(2^n x) dx \tag{3.6}
\]

\[
= i \int \left( e^{-ix(\xi + \frac{17}{12} 2^n)} - e^{-i(\xi - \frac{17}{12} 2^n)} \right) \phi(2^{-s\delta_n} x) dx \tag{3.7}
\]

\[
= \frac{i 2^{s\delta_n}}{2} \int \left( e^{-i(2^{s\delta_n} n)(\xi + \frac{17}{12} 2^n)} - e^{-i(2^{s\delta_n} n)(\xi - \frac{17}{12} 2^n)} \right) \phi(x) dx, \tag{3.8}
\]

which clearly equals $\frac{i 2^{s\delta_n}}{2} \tilde{\phi}(2^{s\delta_n} \xi + \frac{17}{12} 2^{(1+s)\delta_n}) - \frac{i 2^{s\delta_n}}{2} \tilde{\phi}(2^{s\delta_n} \xi - \frac{17}{12} 2^{(1+s)\delta_n})$. A similar calculation shows

\[
\mathcal{F}[\phi(x) \cos \left( \frac{17}{12} 2^n x \right)] = \frac{i 2^{s\delta_n}}{2} \tilde{\phi}(2^{s\delta_n} \xi + \frac{17}{12} 2^{(1+s)\delta_n}) + \frac{i 2^{s\delta_n}}{2} \tilde{\phi}(2^{s\delta_n} \xi - \frac{17}{12} 2^{(1+s)\delta_n}),
\]

and therefore it has the same support. From here it is easy to see that

\[
\Delta \phi(2^{-s\delta_n} x) \cos(2^n x) = \begin{cases} 
\phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right) & \text{if } q = n, \\
0 & \text{if } q \neq n.
\end{cases}
\]

Therefore, we may compute

\[
2^{-ns-s\frac{1}{2^n}} \phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right)\|_{B^s_{p', r}} = 2^{-ns-s\frac{1}{2^n} - n} \|\phi(2^{-s\delta_n} x) \cos \left( \frac{17}{12} 2^n x \right)\|_{L^p} \tag{3.9}
\]

\[
\leq 2^{n(s'-s)-s\frac{1}{2^n}} \|\phi(2^{-s\delta_n} x)\|_{L^p} = 2^{n(s'-s)} \|\phi(x)\|_{L^p}, \tag{3.10}
\]

which completes the second and third estimates. We now consider the fourth quantity and notice that a similar calculation shows that

\[
\mathcal{F}[\phi^3(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right)] = \frac{i 2^{s\delta_n}}{2} \phi^3(2^{s\delta_n} \xi + \frac{17}{12} 2^{(1+s)\delta_n}) - \frac{i 2^{s\delta_n}}{2} \phi^3(2^{s\delta_n} \xi - \frac{17}{12} 2^{(1+s)\delta_n}).
\]

Therefore, we can compute the $B^s_{p', r}$ norm by

\[
2^{-ns-s\frac{1}{2^n}} \|\phi^3(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right)\|_{B^s_{p', r}} = 2^{-s\frac{1}{2^n}} \|\phi^3(2^{-s\delta_n} x) \sin \left( \frac{17}{12} 2^n x \right)\|_{L^p}. \tag{3.11}
\]
A similar calculation from the second estimate shows that the last line is bounded from above. A change of variables yields
\[ \|\phi^3(2^{-\delta n}x)\sin(\frac{17}{12}2^n x)\|^p_{L_p} = 2^{\delta n} \int_{-\infty}^{\infty} |\phi^3(x)\sin(\frac{17}{12}2^{n+\delta n} x)|^p dx, \]
and a simple calculation shows the remaining integral is bounded from below by a constant independent of \( n \).

We summarize several important estimates in the following corollary.

**Corollary 3.1.** For any \( s' \in \mathbb{R} \), \( s > 2 + 1/p \) and \((p, r) \in [1, \infty] \times [1, \infty)\), we have
\[
\|u_{0,n}\|_{L^{\infty}} \approx 2^{-n(s+\frac{1}{p})}, \quad \|u_{0,n}\|_{B^{s'}_{p,r}} \lesssim 2^n(s'-s), \quad \|v_{0,n}\|_{B^{s'}_{p,r}} \lesssim 2^n(s'-s) + 2^n(\frac{s}{p} - \frac{s'}{s'} - \frac{1}{p} - \frac{1}{2}),
\]
\[
\|v_{0,n}\|_{L^{\infty}} \approx 2^{-\frac{1}{2}n}, \quad \|\partial_x v_{0,n}\|_{B^{s'}_{p,r}} \lesssim 2^n(\frac{s}{p} - \frac{1}{2} - \delta) + 2^n(s'+1-s), \quad \|\partial_x v_{0,n}\|_{L^{\infty}} \lesssim 2^{-(\frac{1}{2}+\delta)n},
\]
\[
\|\partial^2_{x} v_{0,n}\|_{B^{s'}_{p,r}} \lesssim 2^{-(\frac{1}{2} + 2\delta)n} + 2^n(s'+2-s), \quad \|\partial^2_{x} v_{0,n}\|_{L^{\infty}} \lesssim 2^{-(\frac{1}{2}+2\delta)n} + 2^n(2-s-\delta).
\]

We let \( T < \inf \{T_n\} \), \( T_n \) the minimum of the lifespan of the solutions \( u_n \) and \( v_n \) (the well-posedness argument ensures that this infimum is positive, as long as the initial data is uniformly bounded).

Our proof proceeds by showing the following results.

1. \( \|u_{0,n}\|_{B^{s'}_{p,r}} \approx 1 \), and for all for all \( t \in [0, T] \), \( \lim_{n \to \infty} \|u_{n} - u_{0,n}\|_{B^{s'}_{p,r}} = 0 \)
2. \( \|v_{0,n}\|_{B^{s'}_{p,r}} \approx 1 \), \( \lim_{n \to \infty} \|u_{n} - v_{0,n}\|_{B^{s'}_{p,r}} = 0 \), and for all for all \( t \in [0, T] \), \( \lim_{n \to \infty} \|v_{n} - w_{n}\|_{B^{s'}_{p,r}} = ct^\beta, 1 < \beta \), where the approximate solution, \( w_n(x, t) \), is defined as
\[
w_n = v_{0,n} + t v_{0,n} \partial_x v_{0,n}.
\]

3. We will show that there exists a positive constant \( c \), such that for any small positive time
\[
\liminf_{n \to \infty} \|w_n - u_{0,n}\|_{B^{s'}_{p,r}} \geq ct.
\]

4. By the triangle inequality, this yields that for any small positive \( t \)
\[
\|u_{n} - v_{n}\|_{B^{s'}_{p,r}} \geq \|w_{n} - u_{0,n}\|_{B^{s'}_{p,r}} - \|u_{n} - u_{0,n}\|_{B^{s'}_{p,r}} - \|v_{n} - w_{n}\|_{B^{s'}_{p,r}}.
\]

Combining the previous results, we can conclude that there exists a constant \( c \) such that for any small \( t > 0 \), \( \liminf_{n \to \infty} \|u_{n} - v_{n}\|_{B^{s'}_{p,r}} \geq ct \), which proves the theorem.

**Proof of (1).** By Lemma 3.1 \( \|u_{0,n}\|_{B^{s'}_{p,r}} = C \), where \( C \) is a constant which depends only upon \( \phi \).

The local well-posedness result tells us that for all \( n \), there exists \( T > 0 \) with \( u_n \in C([0, T]; B^{s'}_{p,r}) \).

We re-write the FORQ equation as
\[
u_t + (u^2 - (\partial_x u)^2)\partial_x u + N(u) = 0.
\]

We estimate \( \|u_{n} - u_{0,n}\|_{B^{s'}_{p,r}} \) by setting \( \tilde{u}_n = u_n - u_{0,n} \) and notice that \( \tilde{u}_n \) satisfies the following transport equation
\[
\partial_t \tilde{u}_n + (u_n^2 - (\partial_x u_n)^2)\partial_x \tilde{u}_n = -(u_n^2 - (\partial_x u_n)^2)\partial_x u_{0,n} - N(u_n).
\]

We now apply Proposition 2.1 to conclude that
\[
\|\tilde{u}\|_{B^{s'-1}_{p,r}} \leq e^{CV(t)} \int_0^t e^{-CV(\tau)} \|u_n^2 - (\partial_x u_n)^2\|_{B^{s'-1}_{p,r}} d\tau,
\]
where \( V(t) = \int_0^t \|\partial_x (u_n^2 - (\partial_x u_n)^2)\|_{B^{s'-1}_{p,r}} d\tau \).
Using property (4) of Lemma 2.2 we have
\[ \| \partial_x(u_n^2 - (\partial_x u_n)^2) \|_{B^{s-1}_p} \leq \| (u_n^2 - (\partial_x u_n)^2) \|_{B^{s-1}_p} \leq \| u_n \|_{B^{s-1}_p}^2 + \| \partial_x u_n \|_{B^{s-1}_p}^2 \leq 2 \| u_n \|_{B^{s-1}_p}, \]
(3.17)
so \( V(t) \) is bounded for all \( t \in [0, T] \). Next, we use property (5) of Lemma 2.2 to estimate
\[ \| (u_n^2 - (\partial_x u_n)^2) \partial_x u_0 \|_{B^{s-1}_p} \leq \| u_n^2 - (\partial_x u_n)^2 \|_{B^{s-1}_p} \| \partial_x u_0 \|_{L^\infty} + \| (u_n^2 - (\partial_x u_n)^2) \|_{L^\infty} \| \partial_x u_0 \|_{B^{s-1}_p}. \]
We use property (4) in Lemma 2.2 and the estimate found in Lemma 3.1 and its corollary to find
\[ \| (\partial_x u_n)^2 \|_{L^\infty} \leq \| (\partial_x u_n)^2 \|_{B^{s-2}_p} \leq \| u_0^2 \|_{B^{s-2}_p} \leq 2^{-2n}. \]
Using these estimates, we find
\[ \| (u_n^2 - (\partial_x u_n)^2) \partial_x u_0 \|_{B^{s-1}_p} \leq 2^{n(1-s+\frac{4}{p})} + 2^{-2} \leq 2^{n\beta}, \]
(3.18)
where \( \beta = \max\{1 + \frac{2}{p} - s, -2\} < -1 \). Next we estimate
\[ \| N(u_n) \|_{B^{s-1}_p} \leq \| \frac{2}{3} u_n^3 + u_n (\partial_x u_n)^2 \|_{B^{s-1}_p} + \| \frac{1}{3} (\partial_x u_n)^3 \|_{B^{s-1}_p}. \]
(3.19)
We use property (6) of Lemma 2.2 to estimate the second term on the right hand side, while we use the algebra property (4) to estimate the first term on the right hand side of the above inequality
\[ \| N(u_n) \|_{B^{s-1}_p} \leq \| u_n \|_{B^{s-2}_p} + \| u_n \|_{B^{s-2}_p} \| \partial_x u_n \|_{B^{s-2}_p} + \| \partial_x u_n \|_{B^{s-2}_p}^2 \| \partial_x u_n \|_{B^{s-2}_p}, \]
(3.20)
where \( \bar{s} = \max\{s - 1, 3/2\} \). Using Corollary 3.1 we find
\[ \| N(u_n) \|_{B^{s-1}_p} \leq 2^{-6n} + 2^{-2n} 2^{-n} + 2^{n(2\bar{s} - 2s)} 2^{n(\bar{s} - 1 - s)} \leq 2^{-2n}. \]
(3.21)
Combining estimates (3.18) and (3.21) with inequality (3.16) we find
\[ \| \bar{u}_n \|_{B^{s-1}_p} \leq 2^{n\beta} + 2^{-2n}, \]
(3.22)
From the well-posedness argument, we have the following estimate on the solution
\[ \| u_n \|_{B^{s+1}_p} \leq \| u_0 \|_{B^{s+1}_p} \approx 2^n, \]
(3.23)
which by the triangle inequality, implies \( \| \bar{u}_n \|_{B^{s+1}_p} \leq 2^n \). Now, applying the interpolation inequality, property (7) of Lemma 2.2 we find
\[ \| \bar{u}_n \|_{B^{s-1}_p} \leq \| \bar{u}_n \|_{B^{s-1}_{p,r}}^{1/2} \| \bar{u}_n \|_{B^{s+1}_{p,r}}^{1/2} \leq 2^{\frac{1}{2} \beta n} 2^{\frac{1}{2} n}, \]
(3.24)
which clearly tends towards zero as \( n \) tends towards infinity, and this completes the proof of (1).

**Proof of (2).** It is clear that and \( \lim_{n \to \infty} \| u_{0,n} - v_{0,n} \|_{B^{s-1}_p} = 0 \). We must evaluate \( \lim_{n \to \infty} \| v_n - w_n \|_{B^{s+1}_p} \) where \( w_n = v_0, u_0 - \tau v_0 \partial_x v_0 \). We denote \( w_n = v_n - w_n \) and we find that
\[ \partial_t w_n + (v_n^2 - (\partial_x v_n)^2) \partial_x w_n = -(v_n^2 - (\partial_x v_n)^2) \partial_x w_n - N(v_n) + v_0 \partial_x v_0, \]
and after adding and subtracting \( v_0^2 \partial_x w_n \) this is equivalent to
\[ \partial_t w_n + (v_n^2 - (\partial_x v_n)^2) \partial_x w_n = (\partial_x v_n)^2 \partial_x w_n - w_n (v_n + v_0) \partial_x w_n + v_0 \partial_x v_0 (v_n + v_0) \partial_x w_n + v_0^2 \partial_x (v_0 \partial_x v_0) - N(v_n). \]
(3.26)
We now apply Proposition 2.1 to conclude that
\[ \| \bar{w} \|_{B^{s-1}_p} \leq e^{CV(t)} \int_0^t e^{-CV(r)} \| (\partial_x v_n)^2 \partial_x w_n - N(v_n) - \bar{w}_n (v_n + v_0) \partial_x w_n \|_{B^{s-1}_p} dr, \]
(3.27)
where \( V(t) = \int_0^t \| \partial_x (v_n^2 - (\partial_x v_n)^2) \|_{B_{p,r}^{-1}}^2 \, dt \). As in the proof of (1), \( V(t) \) is bounded and we use Corollary 3.1 to find
\[
\|N(v_n)\|_{B_{p,r}^{-1}} \lesssim \|v_n\|_{B_{p,r}^{-1}}^3 + \|v_n\|_{B_{p,r}^{-1}}^2 \|\partial_x v_n\|_{B_{p,r}^{-1}}^2 + \|\partial_x v_n\|_{B_{p,r}^{-1}}^2 \|\partial_x v_n\|_{B_{p,r}^{-2}}^2 \\
\lesssim 2^n(\frac{4}{3} - \frac{1}{2})^2 + 2^n(\frac{4}{3} - \frac{1}{2}) \left(2^n(\frac{4}{3} - \frac{1}{2}) + 2^n(n - s)\right)^2 + \left(2^n(\frac{4}{3} - \frac{1}{2}) + 2^n(n - s)\right)^3
\] (3.29)
where \( \bar{\sigma} = \max\{\sigma, \frac{2}{3}\} \). Using \( \delta < \frac{1}{8} \), we see that \( \delta/p - 1/2 < -3/8 \) and therefore we obtain
\[
\|N(v_n)\|_{B_{p,r}^{-1}} \lesssim 2^{-\frac{2n}{3}} + 2^{-\frac{2n}{3}} \left(2^{-\frac{2n}{3}} + 2^n(n - s)\right)^2 + 2^{-\frac{2n}{3}} \lesssim 2^{-\frac{2n}{3}}.
\] (3.30)

Next, we estimate \( \| (\partial_x v_n)^2 \partial_x w_n \|_{B_{p,r}^{-1}} = \| (\partial_x v_n)^2 \partial_x (v_0 - t v_n^2 \partial_x v_n) \|_{B_{p,r}^{-1}} \). We break this into two pieces and find
\[
\| (\partial_x v_n)^2 \partial_x (v_0) \|_{B_{p,r}^{-1}} \lesssim \| \partial_x v_n \|_{B_{p,r}^{-1}}^2 + \| \partial_x v_n \|_{B_{p,r}^{-1}} \| \partial_x v_n \|_{L^\infty} + \| \partial_x v_n \|_{B_{p,r}^{-1}} \| \partial_x v_n \|_{B_{p,r}^{-1}} \| \partial_x v_n \|_{B_{p,r}^{-1}}^2 + \| v_n \|_{B_{p,r}^{-1}} \| \partial_x v_n \|_{B_{p,r}^{-1}} \| \partial_x v_n \|_{L^\infty}^2.
\]

Now, using the estimates found in Corollary 3.1 we have
\[
\| (\partial_x v_n)^2 \partial_x (v_0) \|_{B_{p,r}^{-1}} \lesssim 2^{n(\frac{4}{3} - \frac{1}{2})} 2^n(\frac{4}{3} - \frac{1}{2}) - n(\frac{1}{2} + \delta) + \| \partial_x v_n \|_{B_{p,r}^{-1}}^2 \left(2^n(\frac{4}{3} - \frac{1}{2}) 2^n(\frac{4}{3} - \frac{1}{2}) + 2^n(n - s)\right)^2 + 2^n(n - s) 2^n(\frac{1}{2} + \delta).
\] (3.31)

In order to continue, we must use the estimate for \( \| \partial_x v_n \|_{B_{p,r}^{-1}} \) found in the following lemma.

**Lemma 3.2.** For \( 0 < \delta < \frac{1}{8} \), \( \max\{1 + \frac{1}{p}, s - \frac{9}{8}\} < \sigma < s - 1 \) and \( 0 < \frac{8s}{p} < s - \sigma - 1 \), we have
\[
\| \partial_x v_n \|_{B_{p,r}^{-1}} \lesssim \| \partial_x v_0 \|_{B_{p,r}^{-1}} \approx 2^n(\frac{4}{3} - \frac{1}{2} - \delta).
\]

**Proof.** Notice \( \omega = \partial_x v_n \) is a solution to the following transport equation
\[
\partial_t \omega + (v_n^2 - (\partial_x v_n)^2) \partial_x \omega = -\partial_x (v_n^2 - (\partial_x v_n)^2) \omega - \partial_x N(v_n).
\]

We apply Proposition 2.1 to find
\[
\| \omega \|_{B_{p,r}^{-1}} \leq e^{V(t)} \left(\| \omega(0) \|_{B_{p,r}^{-1}}^2 + \int_0^t e^{-V(s)} \| \partial_x (v_n^2 - (\partial_x v_n)^2) \omega + \partial_x N(v_n) \|_{B_{p,r}^{-1}}^2 \, ds\right),
\]
where \( V(t) = \int_0^t \| \partial_x (v_n^2 - (\partial_x v_n)^2) \|_{B_{p,r}^{-1} \cap L^\infty}^2 \, ds \). Clearly \( V(t) \) is bounded independent of \( n \). We use the algebra property to find
\[
\| \partial_x (v_n^2 - (\partial_x v_n)^2) \omega \|_{B_{p,r}^{-1}} \lesssim \| v_n^2 - (\partial_x v_n)^2 \|_{B_{p,r}^{-1}} \| \omega \|_{B_{p,r}^{-1}} \lesssim \| \omega \|_{B_{p,r}^{-1}}^2
\]
\[
\| \partial_x N(v_n) \|_{B_{p,r}^{-1}} \lesssim \| v_n^3 \|_{B_{p,r}^{-1}}^2 + \| v_n (\partial_x v_n)^2 \|_{B_{p,r}^{-1}}^2 + \| (\partial_x v_n)^3 \|_{B_{p,r}^{-2}}^2
\] (3.32)
\[
\lesssim 2^{-\frac{2n}{3}} + \| \omega \|_{B_{p,r}^{-1}}^2.
\] (3.33)

Making these substitutions into the above, we now have
\[
\| \omega \|_{B_{p,r}^{-1}} \leq \| \omega(0) \|_{B_{p,r}^{-1}}^2 + \int_0^t 2^{-\frac{2n}{3}} + \| \omega \|_{B_{p,r}^{-1}}^2 \, ds,
\]
and now applying Grönwall’s inequality yields the result. 
\( \square \)
Applying this lemma, our inequality \([3.31]\) becomes
\[
\left\| (\partial_x v_n)^2 \right\|_{B_{\sigma,r}^p} \lesssim 2^n(\sigma+\frac{1}{4}) 2^{n\left(\frac{4}{p} - \frac{1}{2}\right)} 2^{-n\left(\frac{1}{2} + \delta\right)} + 2^n\left(\frac{4}{p} - \frac{1}{2}\right) 2^n(\sigma+1-s) 2^{-n\left(\frac{1}{2} + \delta\right)} + 2^n(\sigma+1-s) 2^{-n\left(\frac{1}{2} + \delta\right)}
\]
\[
\approx 2^n(\sigma+\frac{1}{4} - \delta) + 2^n(\sigma+\frac{4}{p} - 2\delta) \approx 2^n(\sigma+\frac{1}{4} - \delta).
\]  
(3.34)

Next we estimate \(\left\| (\partial_x v_n)^2 \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p}\), and find
\[
\left\| (\partial_x v_n)^2 \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p} \lesssim \left\| (\partial_x v_n)^2 \right\|_{B_{\sigma,r}^p} \left\| \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p}
\]
\[
+ \left\| (\partial_x v_n)^2 \right\|_{B_{\sigma,r}^p} \left\| \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{L^\infty}.
\]
(3.35)

We use the previous lemma to get
\[
\left\| (\partial_x v_n)^2 \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p} \lesssim 2^n(\sigma+\frac{1}{4} - 2\delta) \left\| \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p}
\]
\[
+ \left\| (\partial_x v_n)^2 \right\|_{B_{\sigma,r}^p} \left\| \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{L^\infty}.
\]
(3.36)

We use \(\|\partial_x (tv_{0,n}^3 \partial_x v_{0,n})\|_{L^\infty} \lesssim t \|v_{0,n}(\partial_x v_{0,n})^2\|_{L^\infty} + t \|v_{0,n}(\partial_x v_{0,n})^2\|_{L^\infty}\) and we use
\[
\|\partial_x (tv_{0,n}^3 \partial_x v_{0,n})\|_{B_{\sigma,r}^p} \lesssim t \|v_{0,n}(\partial_x v_{0,n})^2\|_{B_{\sigma,r}^p} + t \|v_{0,n}\|_{L^\infty} \|\partial_x v_{0,n}\|_{B_{\sigma,r}^p} + t \|v_{0,n}\|_{B_{\sigma,r}^p}, \|\partial_x v_{0,n}\|_{L^\infty}
\]
and apply the estimates found in Corollary 3.1 to conclude
\[
\left\| (\partial_x v_n)^2 \partial_x (tv_{0,n}^3 \partial_x v_{0,n}) \right\|_{B_{\sigma,r}^p} \lesssim t2^n(\sigma+\frac{1}{4} - 2\delta) \left(2^n(2\sigma+\frac{1}{4}) + 2^n(\sigma+1-s)
\right)
\]
\[
+ t2^n(\sigma+1-s) \left(2^n(\sigma+\frac{1}{4}) + 2^n(\sigma+1-s)\right)
\]
\[
\lesssim t2^n(\sigma+\frac{1}{4} - \delta).
\]
(3.37)

Combining estimates \([3.35]\) and \([3.37]\), we obtain
\[
\left\| (\partial_x v_n)^2 \partial_x w_n \right\|_{B_{\sigma,r}^p} \lesssim 2^n(\sigma+\frac{1}{4} - \delta).
\]

In order to estimate the next term in inequality \([3.27]\) we shall need to estimate \(\|\partial_x w_n\|_{B_{\sigma,r}^p}\). We have
\[
\left\| \partial_x w_n \right\|_{B_{\sigma,r}^p} \lesssim \left\| \partial_x v_n \right\|_{B_{\sigma,r}^p} + t \|v_{0,n}\|_{B_{\sigma,r}^p} \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} + t \|v_{0,n}\|_{B_{\sigma,r}^p} \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} + t \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} + t \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} + t \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} + t \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p}
\]
\[
\lesssim 2^{-n\left(\frac{1}{p} + \delta\right)n} + 2^n(\sigma+1-s) + t2^n(\sigma+1-s) \left(2^{-n\left(\frac{1}{p} + \delta\right)n} + 2^n(\sigma+1-s)\right)
\]
\[
+ t2^n(\sigma+1-s) \left(2^{-n\left(\frac{1}{p} + \delta\right)n} + 2^n(\sigma+1-s)\right)
\]
\[
\lesssim 2^n(\sigma+1-s) + 2^n(\sigma+1-s),
\]
(3.38)

and using the condition \(0 < \frac{28}{p} < s - \sigma - 1\) we see that all of the exponents are negative, and therefore \(\|\partial_x w_n\|_{B_{\sigma,r}^p} \lesssim 1\). Using this estimate, we now find
\[
\left\| \partial_x w_n \right\|_{B_{\sigma,r}^p} \lesssim \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p} \left\| \partial_x v_{0,n} \right\|_{B_{\sigma,r}^p} \|w_n\|_{B_{\sigma,r}^p} \lesssim \|w_n\|_{B_{\sigma,r}^p}.
\]
(3.41)

We now estimate \(\|v_{0,n}^2 \partial_x v_{0,n} v_{0,n} \|_{B_{\sigma,r}^p}\). Using the algebra property, and Corollary 3.1, we find
\[
\|v_{0,n}^2 \partial_x v_{0,n} v_{0,n} \|_{B_{\sigma,r}^p} \leq \|v_{0,n}\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p} \|v_{0,n}\|_{B_{\sigma,r}^p}
\]
\[
\lesssim \left(2^n(\sigma+1-s) + 2^n(\sigma+1-s)\right)^3 \left(2^n(\sigma+1-s) + 2^n(\sigma+1-s)\right)
\]
\[
\lesssim 2^n(3\sigma-3s) + 2^n(\sigma+1-s).
\]
(3.42)
We estimate \( \|v_{0,n}^2 \partial_x (v_{0,n}^2 \partial_x v_{0,n})\|_{B^s_{p,r}} \) using properties (4) and (5) of Lemma 2.2 and find
\[
\|v_{0,n}^2 \partial_x (v_{0,n}^2 \partial_x v_{0,n})\|_{B^s_{p,r}} \lesssim \|v_{0,n}^2\|_{B^s_{p,r}}^3 + \|v_{0,n}^4\|_{L^\infty} + \|v_{0,n}^2\|_{B^s_{p,r}} \|v_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}},
\]
and using the estimates found in Corollary 3.1 we have
\[
\|v_{0,n}^2 \partial_x (v_{0,n}^2 \partial_x v_{0,n})\|_{B^s_{p,r}} \lesssim \left(2^n(\sigma-s) + 2^n(\frac{s}{2} - \frac{1}{2})\right)^2 + 2^{-n} \left(2^{-n} + 2^n(\frac{s}{2} - \frac{1}{2})\right) \lesssim 2^n(\sigma-s) + 2^{-2n} \lesssim 2^n(\sigma-s).
\]
Combining the previous estimates, and substituting into inequality (3.27) we find
\[
\|w\|_{B^s_{p,r}} \lesssim \int_0^t \left(2^{n(\sigma+s-\frac{3}{2}-\delta)-s} + \|w_n(\tau)\|_{B^s_{p,r}} + \tau 2^n(\sigma-s)\right) d\tau.
\]
By Grönwall’s inequality, we have
\[
\|w\|_{B^s_{p,r}} \lesssim 2^n(\sigma+s-\frac{3}{2}-\delta-s) + 2^n(\sigma-s)t^2.
\]
A simple calculation also shows that
\[
\|w_n\|_{B^s_{p,r}+2} \lesssim 2^n.
\]
We now apply the interpolation lemma using the last inequality and inequality (3.48) to find
\[
\|w_n\|_{B^s_{p,r}} \lesssim \|w_n\|_{B^s_{p,r,2}}^{1-\theta} \|w_n\|_{B^s_{p,r,2}}^\theta \lesssim \left(2^n(\sigma+s-\frac{3}{2}-\delta-s) + 2^n(\sigma-s)t^2\right)^\theta 2^n(1-\theta),
\]
where \(\theta = 2(2+s-\sigma)^{-1} > 1/2\). Therefore, we find
\[
\|w_n\|_{B^s_{p,r}} \lesssim \left(2^n(\sigma+s+\frac{2}{3} - \delta - 2s) + 2^{2\sigma+s-\frac{3}{2}-\delta-2s}t^2 + 2^n(2\sigma-2s) t^4\right)^{(2+s-\sigma)^{-1}}
\]
\[
= \left(2^n(\frac{2\sigma-2\delta}{3}) + 2^n(\frac{s}{2} - \delta) t^2 + t^4\right)^{(2+s-\sigma)^{-1}}.
\]
Now taking the liminf of both sides, we see
\[
\liminf_{n \to \infty} \|w_n\|_{B^s_{p,r}} \leq 4^{(2+s-\sigma)^{-1}} = t^{2\theta}.
\]
**Proof of (3).** We now show that for small \( t \geq 0 \),
\[
\liminf_{n \to \infty} \|w_n - u_{0,n}\|_{B^s_{p,r}} \geq ct.
\]
We have
\[
\|w_n - u_{0,n}\|_{B^s_{p,r}} \|2^{-\frac{1}{2} n} \phi(2^{-\delta n}) - v_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}},
\]
and by the triangle inequality
\[
\|w_n - u_{0,n}\|_{B^s_{p,r}} \geq t\|v_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}} - \|2^{-\frac{1}{2} n} \phi(2^{-\delta n})\|_{B^s_{p,r}}.
\]
Now we break \(\|v_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}}\) into pieces. Let \(v_{0,n} = u_{0,n} + u_\ell\), where \(u_\ell\) is the low frequency part of the initial data. We have
\[
\|v_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}} \geq \|u_{0,n}^2 \partial_x u_{0,n}\|_{B^s_{p,r}} - \|u_{0,n}^2 \partial_x u_\ell\|_{B^s_{p,r}} - 2\|u_{0,n} u_\ell \partial_x v_{0,n}\|_{B^s_{p,r}} - \|u_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}}.
\]
The last three terms above tend to zero as \( n \to \infty \). Indeed, using property (4) of Lemma 2.2 we have
\[
\|u_{0,n}^2 \partial_x u_\ell\|_{B^s_{p,r}} \lesssim \|u_\ell\|_{L^\infty} \|\partial_x u_\ell\|_{B^s_{p,r}} \lesssim 2^{\frac{2n}{r}} n^{-n} 2^n(\delta - 2 \delta - \delta),
\]
(3.55)
which tends to zero as $n \to \infty$. Similarly, using property (5) of Lemma 2.2 we can bound the third term on the right hand side by

$$
\|u_{0,n}u_t \partial_x v_{0,n}\|_{B^s_{p,r}} \lesssim \|u_{0,n}u_t\|_{L^\infty} \|\partial_x v_{0,n}\|_{B^s_{p,r}} + \|u_{0,n}u_t\|_{L^\infty} \|\partial_x v_{0,n}\|_{L^\infty}
$$

(3.56)

$$
\lesssim 2^{-(s+\frac{1}{2})n^2} + 2^\frac{2}{n} - \frac{1}{2}n \left(2^{-(\frac{1}{2}+\delta)n} + 2^{-(s-1)n}\right)
$$

(3.57)

which again tends to zero as $n \to \infty$. Finally,

$$
\|u_{0,n}^2 \partial_x v_{0,n}\|_{B^s_{p,r}} \lesssim \|u_{0,n}\|_{L^\infty}^2 \|\partial_x v_{0,n}\|_{B^s_{p,r}} + \|u_{0,n}\|_{B^s_{p,r}}^2 \|\partial_x v_{0,n}\|_{L^\infty}
$$

(3.58)

$$
\lesssim 2^{-2n} - 2^{\frac{2}{p}n^2} + 2^{-(\frac{1}{2}+\delta)n}
$$

(3.59)

and again, this tends towards zero as $n$ grows towards infinity. The first term, $\|u_{0,n}^2 \partial_x u_{0,n}\|_{B^s_{p,r}}$, is bounded below by a constant by the last estimate in Lemma 3.1. Combining the above estimates, the proof of (3) is complete.

**Proof of (4).** By the triangle inequality, this yields that for any positive $t$

$$
\|u_t - v_t\|_{B^s_{p,r}} \geq \|u_t - u_{0,n}\|_{B^s_{p,r}} - \|u_t - u_{0,n}\|_{B^s_{p,r}} - \|v_t - w_t\|_{B^s_{p,r}}
$$

(3.60)

Combining the previous results, we can conclude that there exists a constant $c$ such that for any $t \in [0,T]$, $\liminf_{n \to \infty} \|u_t - v_t\|_{B^s_{p,r}} \geq c_1 t - c_2 t^\theta$, which proves the theorem since $2\theta > 1$.

**REFERENCES**

[1] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011.

[2] Roberto Camassa and Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[3] Adrian Constantin and Joachim Escher. Global weak solutions for a shallow water equation. *Indiana Univ. Math. J.*, 47(4):1527–1545, 1998.

[4] Darryl D. Holm and Rossen I. Ivanov. Smooth and peaked solitons of the CH equation. *J. Phys. A*, 43(43):434003, 18, 2010.

[5] John Holmes and Rajan Puri. Non-uniqueness for the ab-family of equations. *Journal of Mathematical Analysis and Applications*, 493(2):124563, 2021.
[16] Jinlu Li, Yanghai Yu, and Weipeng Zhu. Non-uniform dependence on initial data for the Camassa-Holm equation in Besov spaces. *J. Differential Equations*, 269(10):8686–8700, 2020.

[17] Yi A. Li and Peter J. Olver. Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation. *J. Differential Equations*, 162(1):27–63, 2000.

[18] Peter J. Olver and Philip Rosenau. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E (3)*, 53(2):1900–1906, 1996.

[19] Zhijun Qiao. A new integrable equation with cuspons and W/M-shape-peaks solitons. *J. Math. Phys.*, 47(11):112701, 9, 2006.

[20] Guillermo Rodríguez-Blanco. On the Cauchy problem for the Camassa-Holm equation. *Nonlinear Anal.*, 46(3, Ser. A: Theory Methods):309–327, 2001.