Pion Properties in the $1/N_c$-corrected NJL model

M. Oertel, M. Buballa and J. Wambach
Institut für Kernphysik, TU Darmstadt,
Schloßgartenstr. 9, 64289 Darmstadt, Germany

Abstract

We investigate the effect of mesonic fluctuations on the pion propagator in the Nambu–Jona-Lasinio (NJL) model by explicitly taking into account $1/N_c$-corrections. Because of the non-renormalizability of the model we have to regularize the meson loops with an independent cutoff parameter $\Lambda_M$. Whereas for moderate values of $\Lambda_M$ the pion properties change only quantitatively we encounter strong instabilities for larger values of $\Lambda_M$.

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1 Introduction

It is generally believed that chiral symmetry, which is an approximate symmetry of quantum chromodynamics, is spontaneously broken in vacuum. Together with the linear sigma model the Nambu–Jona-Lasinio (NJL) model plays the role of a prototype for this mechanism. The original papers were published long time before QCD was developed. By the observation that massless pions emerge as a consequence of the spontaneous symmetry breaking they had direct impact on the formulation of the Goldstone theorem. Today the NJL-model, which was originally a model of interacting nucleons, is reinterpreted as a schematic quark model. Being much simpler than QCD, it is an attractive tool for studying consequences of chiral symmetry breaking as well as its restoration at large temperatures or densities, although its applicability to real processes is limited by several shortcomings of the model, in particular the lack of confinement and non-renormalizability.

*e-mail:micaela.oertel@physik.tu-darmstadt.de
So far, most NJL model calculations have been performed in leading order in $1/N_c$, the inverse number of colors. Some authors have also considered corrections in next-to-leading order in $1/N_c$, e.g. to discuss $\pi - \pi$ scattering [3] or meson loop corrections to the pion electromagnetic form factor [4]. With the appropriate choice of model parameters chiral symmetry is spontaneously broken in leading order, resulting in the appearance of massless pions and a massive sigma meson. If care is taken in the choice of correction terms [5, 6] the pion emerges massless also in next-to-leading order. However, in a recent paper Kleinert and Van den Bossche argue that for the physical value, $N_c = 3$, the spontaneous breakdown of chiral symmetry does not occur, as a consequence of strong fluctuations, which come about as higher-order corrections in the $1/N_c$-expansion [9]. At present this paper is subject to many controversial discussions.

In the present article we want to study the effect of mesonic fluctuations on the pion propagator by calculating $1/N_c$-corrections explicitly. However, instead of trying to give a definite answer to the question whether or not chiral symmetry is spontaneously broken, we take the point of view that there is no unique solution in a non-renormalizable model. At each order one encounters new divergences, which are in principle related to additional parameters (e.g. counter terms). In the present case it is rather natural to introduce a new cutoff parameter $\Lambda_M$ to regularize the meson loops [7, 8]. By varying this cutoff from zero to larger values we can smoothly turn on the mesonic fluctuations and study their effects.

Although we cannot directly investigate the question of a possible restoration of chiral symmetry within our scheme, one might find hints for an instability. This is a similar situation as in many-body systems, where e.g. complex energy eigenvalues in the excitation spectrum indicate an instability of the ground state [10]. In fact, as we will discuss in Sec. 3, at large values of $\Lambda_M$ the pion propagator receives “unphysical” poles, i.e. poles at complex or space-like momenta. We also find negative wave function renormalization constants corresponding to residues with the wrong sign. The main point we want to make in this letter is that the existence of such signals depends on the choice of the meson cutoff $\Lambda_M$. Obviously, for small values of $\Lambda_M$ the $1/N_c$-corrected results are close to the leading order ones and no instability shows up. Ultimately, of course, the cutoff should be fixed by a fit to observables, for instance the decay width of vector mesons [11].

2 The Model

In this section we give a brief outline of our scheme for describing mesons within the NJL model in next-to-leading order in $1/N_c$. To a large extent our model is based on Ref. [7]. More details will be presented elsewhere [11].

We consider the standard NJL Lagrangian for two flavors and three colors with scalar-isoscalar and pseudoscalar-isovector interaction:

$$\mathcal{L} = \bar{\psi}(i\slashed{D} - m_0)\psi + g \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\bar{\slashed{D}}\psi)^2 \right].$$

(1)

Here $g$ is a dimensionful coupling constant of order $1/N_c$. In the limit $m_0 = 0$ the Lagrangian is symmetric under $SU(2)_L \times SU(2)_R$ transformations.
In leading order in $1/N_c$ mesons are described by iterated quark-antiquark loops,

$$D^{qq}_{M}(q^2) = \frac{-2g}{1 - 2g \Pi^{qq}_{M}(q^2)}.$$  (2)

This is illustrated in the lower part of Fig. 1. For simplicity we will call $D^{qq}_{M}$ a “propagator”, although strictly speaking it should be interpreted as $g^{2}_{Mqq} \tilde{D}^{qq}_{M}$, where $\tilde{D}^{qq}_{M}$ is the renormalized meson propagator and $g_{Mqq}$ is a meson quark coupling constant. The quark-antiquark polarization diagrams are given by

$$\Pi^{qq}_{M}(q^2) = -i N_f N_c \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ \Gamma_{M} \frac{1}{p + q - m + i\varepsilon} \Gamma_{M} \frac{1}{p - m + i\varepsilon} \right],$$  (3)

with $\Gamma_{M} = i\gamma_5$ for the pion and $\Gamma_{M} = 1$ for the sigma channel. In Eq. (3) $m$ denotes the constituent quark mass which is a solution of the gap equation

$$m = m_0 + 2ig(N_fN_c)\int \frac{d^4p}{(2\pi)^4} \text{tr} \frac{1}{p - m + i\varepsilon}.$$  (4)

It is now straightforward to show that for $m_0 = 0$ but $m \neq 0$ (“Nambu-Goldstone mode”) the right hand side of Eq. (3) has a pole at $q^2 = 0$ in the pion channel and at $q^2 = 4m^2$ in the sigma channel, corresponding to $m_{\pi} = 0$ and $m_{\sigma} = 2m$. If $m_0/m$ is small but not exactly zero the pion mass becomes non-zero and is approximately given by the Gell-Mann Oakes Renner (GOR) relation [12]:

$$m^2 \approx f^2 \langle \bar{\psi}\psi \rangle.$$  (5)

The $1/N_c$-correction terms to the inverse meson propagators, are shown in the upper part of Fig. 1. These diagrams contain mesonic fluctuations calculated in leading order.
in $1/N_c$, as described above. When iterated together with $\Pi_M^{q\bar{q}}$ the new terms lead to an improved meson propagator,

$$D_M(q^2) = \frac{-2g}{1 - 2g(\Pi_M^{q\bar{q}}(q^2) + \delta\Pi_M(q^2))},$$

(6)

where $\delta\Pi(q^2)$ denotes the sum of the correction terms. It can be shown analytically that the pion constructed in this way is again massless in the chiral limit $[7, 11]$. Note that we do not use the improved meson propagators $D_M(q^2)$ for evaluating the correction terms $\delta\Pi(q^2)$. Such a selfconsistent procedure spoils the $1/N_c$ counting scheme and leads to inconsistencies with chiral symmetry.

A non-renormalizable model is incomplete without defining how to regularize divergent loops. To leading order in $1/N_c$ one has to deal with quadratically divergent integrals, the $q\bar{q}$ polarization diagrams (Eq. (3)) and the gap equation (Eq. (4)). These divergences can be regularized by standard methods, such as the Pauli-Villars scheme or a sharp cutoff [4]. In next-to-leading order the situation is more involved. As an example we consider diagram (a) of Fig. 1. This diagram contains two intermediate $q\bar{q}$-mesons, which have to be calculated in a first step. We are then left with three loops: two quark triangles and an integration over the four-momentum of the intermediate $q\bar{q}$-mesons. It is quite natural to regularize the quark triangles in the same way as the $q\bar{q}$ polarization diagrams which enter into the intermediate mesons. In fact, this is necessary in order to preserve chiral symmetry. However, there is no stringent reason, why the remaining meson loop should also be regularized in this way. We therefore follow Refs. [7] and [8] and introduce a meson cutoff as an independent parameter.

For computational convenience we work in the rest frame of the meson (i.e. the external three-momentum is zero) and regularize the internal meson loops by a 3-dimensional sharp cutoff $\Lambda_M$ in momentum space. On the other hand, since the internal mesons can of course not be chosen to be at rest, we should use a covariant scheme for their computation. As in Ref. [7] we employ the Pauli-Villars scheme to regularize the quark loops. Because of the quadratic divergences we need two regulators. The regulator masses are chosen in the standard way as $M_1 = \sqrt{m^2 + \Lambda_q^2}$ and $M_2 = \sqrt{m^2 + 2\Lambda_q^2}$, with a free parameter $\Lambda_q$. This “mixed” scheme with a covariant regularization of the quark loops and a non-covariant regularization of the meson loops was chosen for entirely practical reasons. However, we expect qualitatively similar results for other schemes, e.g. if we take a covariant sharp cutoff for both, quark and meson loops.

In this context we should also comment on Ref. [13] where $1/N_c$-corrections to the quark condensate are studied within a NJL-type model with a separable non-local interaction. This interaction generates a form factor at the quark-vertices and can be chosen such that the 3-momentum of each quark is limited to absolute values less than a certain cutoff parameter. In this way also the 3-momentum of the meson loops is automatically restricted without introducing an additional cutoff. Note that this does not contradict the statement that additional regulators are necessary if a non-renormalizable model is extended to higher loop orders: Because of the non-local interaction the model of Ref. [13] is not identical to the NJL model but a modification which contains no longer any divergences. Although this is a quite appealing feature, the model has the disadvantage of being manifestly non-covariant (the form factors depend on the 3-momenta of the quarks).
Figure 2: Squared pion mass as a function of the current quark mass $m_0$ for different meson loop cutoffs: $\Lambda_M = 0$ MeV (solid), 500 MeV (dashed), 900 MeV (dashed-dotted) and 1300 MeV (dotted). The calculated points are explicitly marked.

There are similar models with 4-momentum dependent form factors (in Euclidean space) \[14, 15\] but these models have wrong analytical properties, which might be very disturbing in the context of this article. Therefore - and because of the numerical effort - we decided to restrict our investigations on the standard NJL model with the regularization scheme described above.

3 Numerical Results

We are now ready to study the influence of $1/N_c$-corrections on the NJL pion propagator and related quantities. We begin with the leading order, which corresponds to a meson cutoff $\Lambda_M = 0$. With $\Lambda_q = 800$ MeV, $g\Lambda_q^2 = 2.9$ and $m_0 = 6.1$ MeV we obtain a reasonable fit for the pion mass, the pion decay constant and the quark condensate: $m^{(0)}_\pi = 140$ MeV, $f^{(0)}_\pi = 93.5$ MeV and $\langle \bar{\psi}\psi \rangle^{(0)} = -2 (241.1 \text{ MeV})^3$. Here and in the following the superscript $(0)$ is used to denote quantities which are calculated in leading order in $1/N_c$. The above parameters correspond to a relatively small constituent quark mass of 260 MeV.

Now we turn on the mesonic fluctuations by taking a non-zero meson cutoff $\Lambda_M$. Fig. 3 displays the squared pion mass as a function of the current quark mass $m_0$ for different values of $\Lambda_M$. Obviously all points which correspond to the same meson cutoff lie almost on a straight line through the point $(m_0 = 0, m_{\pi}^2 = 0)$. The latter was calculated analytically whereas all other points are numerical results. This demonstrates the consistency of our scheme with chiral symmetry and the stability of the numerics.

Fig. 3(a) displays the behavior of $m_{\pi}^2$, $f_{\pi}^2$ and the quark condensate with increasing
Figure 3: (a) The ratios $m_{\pi}^2/m_{\pi}^{2(0)}$ (solid), $f_{\pi}^2/f_{\pi}^{2(0)}$ (dashed), $\langle \bar{\psi} \psi \rangle/\langle \bar{\psi} \psi \rangle^{(0)}$ (dashed-dotted) and the combination $-m_0\langle \bar{\psi} \psi \rangle/m_{\pi}^2 f_{\pi}^2$ (dotted) as a function of the meson loop cutoff $\Lambda_M$. (b) Inverse pion propagator, $D^{-1}_{\pi}$, multiplied by $-2g$ as a function of the 4-momentum squared for various meson cutoffs: $\Lambda_M = 0$ (solid), 900 MeV (dashed), 1300 MeV (dashed-dotted) and 1500 MeV (dotted).

All other parameters are kept constant at the values given above. As one can see the mesonic fluctuations lead to a reduction of $f_{\pi}$ while $m_{\pi}$ is increased. There are two $1/N_c$-correction terms to the quark condensate, which have been discussed, e.g., in Ref. [13]. We find that the absolute value of the quark condensate decreases at smaller values of $\Lambda_M$ but goes up again for $\Lambda_M \gtrsim 900$ MeV.

The leading-order quantities $m_{\pi}^2(0)$, $f_{\pi}^2(0)$ and $\langle \bar{\psi} \psi \rangle^{(0)}$, are in almost perfect agreement with the GOR relation, Eq. (5). From the (almost) linear dependence of $m_{\pi}^2$ on $m_0$, as shown in Fig. 2, one might expect that this is also the case for $\Lambda_M > 0$. If one carefully expands both sides of Eq. (5) up to next-to-leading order in $1/N_c$, one can show analytically that our model is consistent with the GOR relation [14]. However, for the quantities $m_{\pi}^2$ and $f_{\pi}^2$ as they follow from the $1/N_c$-corrected inverse pion propagator, the l.h.s. of Eq. (5) also receives higher-order terms (i.e. terms beyond next-to-leading order) in $1/N_c$, which are not present on the r.h.s. and therefore violate the GOR relation. This can be seen in Fig. 3(a), where the ratio of the right hand side and the left hand side of Eq. (5) is displayed by the dotted line. For $\Lambda_M \leq 900$ MeV the relation holds within 30%. However, when the meson cutoff is further increased the deviation grows rapidly. This indicates that higher-order corrections in $1/N_c$ become important and our perturbative scheme should no longer be trusted in this regime.

The behavior of the pion mass becomes more clear from Fig. 3(b) where the inverse pion propagator $D^{-1}_{\pi}(q^2)$ is plotted as a function of $q^2$ for different values of $\Lambda_M$. In the chiral limit all lines in this plot would go through $-2gD^{-1}_{\pi} = 0$ at $q^2 = 0$. For $m_0 > 0$
the $q^2 = 0$-value of the leading-order result (solid line) is shifted up to $m_0/m$, the ratio of current and constituent quark mass, and the curve crosses the zero-line at a positive value of $q^2$, corresponding to a finite pion mass. For $\Lambda_M \lesssim 900$ MeV the main effect of the $1/N_c$-corrections is a further increase of the $q^2=0$-value, while the slope does not change significantly (dashed line). This leads to a moderate increase of the pion mass. If we continue to increase $\Lambda_M$, the $q^2=0$-value of $-2gD_\pi^{-1}$ moves down again. However, at the same time the function becomes more flat and this causes the accelerated increase of the pion mass one sees in Fig. 3(a). For $\Lambda_M \gtrsim 1250$ MeV the curve even turns around at larger $q^2$ and crosses the zero line a second time (dashed-dotted line). At $\Lambda_M \approx 1350$ MeV the two poles of $D_\pi$ merge and upon further increasing $\Lambda_M$ disappear from the positive real $q^2$-axis (dotted line).

Obviously the pion becomes unstable if the mesonic fluctuations are too strong. In fact, already the second pole of the propagator in the regime 1250 MeV $\lesssim \Lambda_M \lesssim 1350$ MeV is unphysical because the residue has the wrong sign. This would correspond to an imaginary pion-quark coupling constant and to a negative value for $f_\pi^2$. We found that this behavior is mainly due to diagram (b) in Fig. 1. This diagram has a peak in the $q\bar{q}$ continuum which has an imaginary part with the “wrong” sign. When $\Lambda_M$ exceeds a certain value, this contribution becomes larger than the sum of all other contributions, such that the imaginary part of the inverse pion propagator, $\text{Im}D_\pi^{-1}$, gets the “wrong” sign. Via dispersion relations this peak can be related to the turn-around of $\text{Re}D_\pi^{-1}$ below the $q\bar{q}$ threshold which is responsible for the second pole. Because of the unphysical features of the second pole we used the first one to determine the pion mass for $\Lambda_M = 1300$ MeV in Fig. 2.

It is quite reasonable that the instabilities of the pion propagator indicate an instability of the underlying ground state with a spontaneously broken chiral symmetry against mesonic fluctuations. On the other hand it is clear that our model is not applicable to study the process of chiral symmetry restoration itself. Since $1/N_c$-corrections have been built in only perturbatively, the range of validity of the model is restricted to the regime where these corrections are small. Therefore the rise of $|\langle \bar{\psi}\psi \rangle|$ for $\Lambda_M \gtrsim 900$ MeV should not be taken too seriously.

The results of our calculations can be summarized by noting that the structure of the pion propagator stays relatively stable for $\Lambda_M \lesssim \Lambda_q$ although sizable changes in $m_\pi$, $f_\pi$ and the quark condensate are found. For larger values of $\Lambda_M$ instabilities occur, which might be related to instabilities of the underlying ground state.

Unfortunately, for numerical reasons we cannot perform calculations in the exact chiral limit, $m_0 = 0$, where the internal pion propagator $D_\pi^{q\bar{q}}$ has a pole for $q^2 = 0$. However, as mentioned above, one can show analytically for the improved pion propagator that $D_\pi^{-1}(0) = 0$ in this case, independent of $\Lambda_M$. Furthermore, it is quite reasonable to expect that the slope of $D_\pi^{-1}$ will behave similarly to what we found for $m_0 = 6.1$ MeV. This implies that above a certain value of $\Lambda_M$ the residue at $q^2 = 0$ gets again the “wrong” sign, indicating an instability.

In Fig. 3 we did not change the parameters which were determined in leading order by fitting $f_\pi^{(0)}$, $m_\pi^{(0)}$ and $\langle \bar{\psi}\psi \rangle^{(0)}$. Of course, if one wants to apply the model to describe physical processes a refit of these observables should be performed including the $1/N_c$-corrections. This was partially done in Refs. [7] and [8] (without taking into account the
full momentum dependence of the quark triangles (see Fig. (a)) and the other quark loops), where for each choice of $\Lambda_M$ the new $f_\pi$ was fitted to the empirical value. Since the $1/N_c$-corrections to $f_\pi$ are negative this means that the leading-order term $f_\pi^{(0)}$ has to be larger than the empirical value. As a consequence, the onset of the instabilities is shifted to much larger values of $\Lambda_M$. In order to estimate the effect, we recall that without refit the second pion pole shows up at $\Lambda_M \approx 1250$ MeV. At this point we find $f_\pi = 35$ MeV and $m_\pi = 216$ MeV. A simple way to get the correct pion decay constant $f_\pi^{\text{exp}} = 93$ MeV is to rescale the parameters such that all quantities of dimension energy are multiplied by a factor $f_\pi^{\text{exp}}/f_\pi$, which is about 2.7 for the above example. Hence, the point at which the second pion pole emerges corresponds to a rescaled cutoff $\Lambda_M \approx 3300$ MeV. Of course, the pion mass is rescaled by the same factor and is now much too large ($\sim 570$ MeV). This can be easily cured by choosing a smaller current quark mass, which does not affect $f_\pi$ very much. Making use of the (almost) linear dependence of $m_\pi^2$ on $m_0$ we obtain $m_0 \approx 1$ MeV. However, with these parameters we would strongly overestimate the quark condensate. Performing a more careful parameter study, it turns out that a simultaneous fit of $f_\pi$ and $\langle \bar{\psi} \psi \rangle$ is only possible for $\Lambda_M \lesssim 700$ MeV. Therefore the region where the instabilities show up in the pion propagator seems to be far away from a realistic parameter set. Ultimately, one should of course try to determine also $\Lambda_M$ itself, e.g. by fitting the decay width of vector mesons.

4 Conclusions

We have calculated the pion propagator within an NJL model which was extended to explicitly include meson loops in a $1/N_c$-expansion. As already pointed out in Refs. 9 and 8 this requires the introduction of a new cutoff parameter, reflecting the non-renormalizability of the NJL model. Similar to these authors an independent meson loop cutoff $\Lambda_M$ has been introduced in addition to the cutoff $\Lambda_q$ which was used to regularize the quark loops. Since the importance of the $1/N_c$-correction terms depends strongly on $\Lambda_M$, the question of how much the results are altered and in particular whether the spontaneous breakdown of chiral symmetry is spoiled by mesonic fluctuations 9, cannot be uniquely answered. Whereas for large values of $\Lambda_M$ instabilities show up in the pion propagator, this is not the case at lower values of $\Lambda_M$. First estimates seem to indicate that the region of instabilities is quite far away from a realistic parameter set, leaving enough room for further applications of the model to physical processes.

The instabilities in the pion propagator might be a hint for instabilities of the underlying ground state against mesonic fluctuations. However, we should clearly state, that this question cannot be assessed within the formalism we have presented here. Similarly the stable pion propagator we find at lower values of $\Lambda_M$ does not prove that the ground state is stable. (Of course it should be stable if $\Lambda_M$ is sufficiently small.) Here further careful studies are necessary including a comparison of different regularization schemes.
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