Ray tracing in FLRW flat space-times

Giovanni Acquaviva\textsuperscript{(a)}\textsuperscript{∗}, Luca Bonetti\textsuperscript{(a)}\textsuperscript{†}, Guido Cognola\textsuperscript{(a)}\textsuperscript{‡}, and Sergio Zerbini\textsuperscript{(a)}\textsuperscript{§}

\textsuperscript{(a)} Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare - Gruppo Collegato di Trento
Via Sommarive 14, 38123 Povo, Italia

Abstract

In this work we take moves from the debate triggered by Melia \textit{et al.} in \cite{9} and followed by opposite comments by Lewis and Oirschot in \cite{10,11}. The point in question regards the role of the Hubble horizon as a limit for observability in a cosmological setting. We propose to tackle the issue in a broader way by relating it to the causal character of the Hubble surface and to the tracing of null trajectories, focusing on both three-fluids and generalized Chaplygin gas models. The results should make clear that for quite reasonable and physically motivated models, light rays reaching a comoving observer at $R(t_0) = 0$ have never traveled a distance greater than the proper radius of the horizon until $t_0$.

1 Introduction

Relativistic theories of gravity on flat FLRW space-times have become important in modern cosmology after the discovery of the current cosmic acceleration, the rising of the dark energy issue and the confirmation of inflationary models. Among the several descriptions of the current accelerated expansion of the universe, the simplest one considers the introduction of a small positive cosmological constant in the framework of General Relativity, so that one is dealing with a perfect fluid whose equation of state parameter $\omega = -1$. This fluid model is able to describe the current cosmic acceleration. Also other forms of fluid (phantom, quintessence, inhomogeneous fluids, etc.) satisfying suitable equation of state are not excluded, since the observed small value of cosmological constant leads to several conceptual problems – the debate on vacuum energy and the coincidence problem, among others. For this reason, several different approaches to the dark energy issue have been proposed. Among them, modified theories of gravity \cite{1}–\cite{5} represent an interesting extension of Einstein’s theory. Unfortunately, a large class of these modified models admit future singularities, the worst being the so-called Big Rip singularity \cite{6} (for a general discussion, see for example \cite{7}).

\textsuperscript{∗} gioacqua@gmail.com
\textsuperscript{†} bonetti659@gmail.com
\textsuperscript{‡} cognola@science.unitn.it
\textsuperscript{§} zerbini@science.unitn.it
With these models in mind, we revisit in a deep and analytic way the analysis – proposed first in [8] and recently reproposed in the context of the debate [9, 10, 11] – of light trajectories in FLRW models and the role of the Hubble horizon as an observational limit for comoving observers, hopefully elucidating some points.

We restrict our analysis to a flat FLRW model, which is also a spherically symmetric dynamical space-time admitting a dynamical horizon. For the sake of completeness, we briefly review the general formalism [12, 13, 14, 15, 16] that will be useful in the following.

Recall that any spherically symmetric dynamical space-time has a metric which can locally be expressed in the form

\[
\gamma_{ij}(x^i)dx^idx^j + R^2(x)d\Omega^2, \quad i,j = 0,1, \quad x = \{x^i\} \equiv \{x^0, x^1\}, \quad (1.1)
\]

where the two-dimensional metric

\[
d\gamma^2 = \gamma_{ij}(x)dx^idx^j \quad (1.2)
\]

is referred to as the “normal” metric, \(\{x^i\}\) being the coordinates of the corresponding two-dimensional “normal” space and \(R(x)\) the areal radius, which is a scalar quantity in the normal space. Finally \(d\Omega^2\) is the metric of a two-dimensional sphere \(S^2\). Associated with the areal radius, there exists a spherical surface \(S(x) = 4\pi R^2(x)\). It will be useful to define also the expansions related to the horizon surface, that is the rate of change of the area transverse to bundles of null rays orthogonal to the horizon. In spherical symmetry and double null coordinates, the two expansions are given by

\[
\theta_{\pm} = \frac{\partial_{\pm} S}{S} = \frac{2}{R} \partial_{\pm} R, \quad (1.3)
\]

where \(R\) is the areal radius. A marginal surface is defined by \(\theta_+ = 0\), and it is future if \(\theta_- < 0\) and past if \(\theta_- > 0\). Moreover the sign of \(\partial_- \theta_+\) discerns whether the horizon is \textit{inner} (positive) or \textit{outer} (negative).

To make use of a covariant formulation, one may introduce the normal space scalar quantity proportional to \(\theta_+ \theta_-\), namely

\[
\Phi(x) = \gamma^{ij}(x)\partial_i R(x)\partial_j R(x), \quad (1.3)
\]

The surface \(S\) is said trapped if the related scalar \(\Phi(x) < 0\), untrapped if \(\Phi(x) > 0\), and marginal if \(\Phi(x) = 0\). The dynamical trapping horizon according to Hayward is a surface foliated by the marginal surfaces, namely is the solution of equation

\[
\Phi(x)\big|_{x = x_H} = 0, \quad (1.4)
\]

provided that \(\partial_\theta \Phi|_{x = x_H} \neq 0\). The trapping horizon is a quite natural generalization of the event horizon, which, in the dynamical setting, is “teleological” in its definition [17].

For the sake of clarity, we give two examples. The first one is the static metric describing Schwarzschild space-time

\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2d\Omega^2. \quad (1.5)
\]
In this gauge the coordinates are \( x = (t, r) \), the areal radius concides with \( r \) and the normal metric
\[
d\gamma^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)}. \tag{1.6}
\]
The horizon is a static one and, in this case, concides with the event horizon: it is given by equation (1.4)
\[
\Phi|_H = 1 - \frac{2M}{r_H} = 0, \tag{1.7}
\]
namely one gets the usual Schwarzschild radius \( r_H = 2M \). We also stress the fact that the formalism is covariant. For example in the Painleve's system of coordinates \((v, r)\), with a different time coordinate \(v\), the normal metric is static but not diagonal, namely
\[
d\gamma^2 = - \left(1 - \frac{2M}{r}\right) dv^2 - 2\sqrt{\frac{2M}{r}} dv dr. \tag{1.8}
\]
Again, the horizon is located at \( r_H = 2M \), but now the normal metric evaluated on the horizon is regular and null.

The second example is the one we are mainly interested in. Let us consider the flat FLRW space-time, the metric usually being written in the form
\[
ds^2 = -dt^2 + a^2(t) \left(dr^2 + r^2 d\Omega^2\right). \tag{1.9}
\]
The coordinates are \( x = (t, r) \), the areal radius is \( R = a(t) r \) and the normal metric simply reads
\[
d\gamma^2 = -dt^2 + a^2(t) dr^2. \tag{1.10}
\]
Thus,
\[
\Phi|_H = \left[-(\partial_t R)^2 + \frac{1}{a^2(t)}(\partial_r R)^2\right] = - a^2 r^2 + 1 \bigg|_H = 0, \quad \dot{a} = \frac{d a}{d t}, \tag{1.11}
\]
namely the trapping horizon is located at \( r_H = 1/\dot{a} \), and in terms of areal (or proper) radius reads
\[
R_H = a(t) r_H = \frac{1}{H(t)}, \tag{1.12}
\]
where the Hubble parameter \( H(t) \) is defined by
\[
H(t) = \frac{\dot{a}}{a} = \frac{d \ln a}{d t}. \tag{1.13}
\]
The quantity \( R_H \) is known as the Hubble sphere, but we may also refer to it as the Hubble dynamical horizon in the Hayward terminology. In this true dynamical case, the normal metric evaluated on the dynamical horizon reads
\[
d\gamma^2_H = -dt^2 + a(t)^2 (dr_H)^2 \frac{\dot{H}^2 + 2\dot{H} H^2}{H^4} dt^2 = \left[ \left(\frac{d R_H}{dt}\right)^2 - 2 \frac{d R_H}{dt} \right] dt^2. \tag{1.14}
\]
The sign of the line element, related to the value of the quantity \( \dot{R}_H \), determines the causal character of the horizon surface, which is timelike, spacelike or null according to whether \( d\gamma^2_H \) is negative,
positive or vanishing respectively. For example, in the de Sitter case, $R_H = 1/H_0$, and thus the corresponding horizon is null, as for a generic static black hole. As already stressed by other authors, when $d\gamma_H^2 \neq 0$, photons may cross the dynamical horizon several times, but this is not surprising and this property depends on the cosmological model (see the examples presented in Sec. \[9\], and in \[11\]). The horizon surface is always space-like for decreasing $R_H$. This case will be discussed in detail in Section 4 of the paper.

The paper is organized as follows. In Sec. 2 we review some past and future singularity scenarios in a flat FLRW universe. In Sec. 4 we analyze the causal character of cosmological horizons: this in turn introduces the topic of Sec. 3 where the ray tracing of null trajectories is discussed. Conclusions are given in Sec. 5.

2 From $\Lambda$CDM to Big Rip solutions

Here we review the conditions under which cosmological past and future singular solutions like Big Bang, Little Rip and Big Rip may be present. We recall the form of flat FLRW space-time in the spherical coordinates $(t, r, \theta, \psi)$

$$ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right) = d\gamma^2 + a^2(t)r^2 d\Omega^2. \quad (2.1)$$

For our discussion it is convenient to introduce the coordinate defined by the proper radius $R = r a(t)$. Thus, one has

$$ds^2 = -(1 - H^2 R^2) dt^2 - 2 R H dR dt + dR^2 + R^2 d\Omega^2. \quad (2.2)$$

This expression suggests the introduction as evolution parameter of the quantity $y = \ln a(t)$, largely used in inflationary and dark energy models (for example, see the recent paper \[18\]). As a result, the cosmic time may be expressed as

$$t(y) = \int \frac{dy}{H(y)} \quad (2.3)$$

and the normal metric, the only relevant for our discussion, in the new coordinates $(y, R)$ reads

$$d\gamma^2 = - \left( \frac{1}{H^2} - R^2 \right) dy^2 - 2 R dy dR + dR^2. \quad (2.4)$$

It is easy to check that the trapping horizon is again given by the Hubble horizon $R_H = 1/H$, and we may rewrite

$$t(y) = \int dy R_H(y). \quad (2.5)$$

We must supply this “new kinematic” FLRW framework with the dynamics of gravity.

One may assume a generalization of the Friedmann equation and matter energy conservation together with a suitable equation of state, namely

$$H^2 = \frac{\chi}{3} F(\rho), \quad (2.6)$$

$$\frac{d\rho}{dy} + 3(p + \rho) = 0, \quad p = p(\rho). \quad (2.7)$$

Here $\chi = 8\pi G$. In general $F(\rho)$ has to be non-negative. For further generalizations of Friedmann equation see \[19\] and references therein.
2.1 Standard equation of state

Recall that in general relativity $F(\rho) = \sum_i \rho_i$ is linear in the density species. With $\omega_i$ constant quantities one has the simple equations of state

$$ p_i = \omega_i \rho_i, \quad (2.8) $$

and assuming matter conservation for every species one gets

$$ \frac{d\rho_i}{dy} = -3(1 + \omega_i)\rho_i, \quad (2.9) $$

which can be solved to give

$$ \rho_i = c_i e^{-3(1+\omega_i)y}. \quad (2.10) $$

We thus have

$$ H^2 = \frac{\chi}{3} \sum_i c_i e^{-3(1+\omega_i)y}. \quad (2.11) $$

The associated Hubble horizon is

$$ R_H(y) = \sqrt{\frac{3}{\chi}} \frac{e^{3y/2}}{(\sum_i c_i e^{-3\omega_i y})^{1/2}}. \quad (2.12) $$

As a consequence, the solution of Friedmann equation may be expressed as

$$ t(y) = t(y_0) + \sqrt{\frac{3}{\chi}} \int_{y_0}^y dx \frac{e^{3x/2}}{(\sum_i c_i e^{-3\omega_i x})^{1/2}}. \quad (2.13) $$

One-fluid model. The simplest example is the one-fluid model with equation of state $p = \omega \rho$. In such a case one has

$$ t(y) = t(y_0) + \sqrt{\frac{3}{c\chi}} \int_{y_0}^y dx e^{3(1+\omega)x/2}. \quad (2.14) $$

If $1 + \omega > 0$, then we may choose $y_0 = -\infty$ with $t(-\infty) = 0$ (the Big Bang) and so

$$ t(y) = \sqrt{\frac{3}{c\chi}} \frac{2}{3(1 + \omega)} e^{3(1+\omega)y/2} = \frac{2}{3(1 + \omega)} a^{3(1+\omega)}, \quad (2.15) $$

which, after inversion, gives the usual flat FLRW solution as a function of the time $t$. In the special case $\omega = -1$ there is no Big Bang and from (2.14) one trivially gets the de Sitter solution $t \simeq y = \ln a$. 

5
Three-fluids model. Now we consider an interesting phenomenological generalization of previous case describing (dark) matter, radiation and dark energy (cosmological constant or phantom matter). The total energy density and equations of state read

$$\rho_T = \rho_m + \rho_r + \rho_f, \quad p_m = 0, \quad p_r = \frac{1}{3} \rho_r, \quad p_f = \omega_f \rho. \quad (2.16)$$

For phantom matter, we make the choice $1 + \omega_f = -\delta < 0$. The energy-matter conservation gives

$$\rho_m = c_0 e^{-3y}, \quad \rho_r = c_r e^{-y}, \quad \rho_f = c_f e^{-(1+\omega_f)y}. \quad (2.17)$$

From the Friedmann equation we have

$$R_H(y) = \sqrt{\frac{3}{\chi}} \left( \frac{e^{3y/2}}{c_0 + c_r e^{-y} + c_f e^{(3+3\delta)y}} \right)^{1/2} \quad (2.18)$$

and hence

$$\sqrt{\frac{\chi}{3}} t(y) = \int_{-\infty}^{y} \frac{e^{3x/2} dx}{(c_0 + c_r e^{-x} + c_f e^{(3+3\delta)x})^{1/2}} \quad (2.19)$$

where $y_0 = -\infty$, and $t(-\infty) = 0$ because here the integrand is summable. This is the initial Big Bang singularity of this model.

On the other hand, the behaviour for large $y$ characterizes the future singularities. In fact, for $\delta = 0$ (the so called ΛCDM model), $t \to \infty$ as soon as $y \to \infty$.

In the case of phantom component, $\delta > 0$ and small, the integral converges for $y \to \infty$. As a result a singularity is present for $y$ and $a$ at a future finite time given by

$$\sqrt{\frac{\chi}{3}} t_s = \int_{-\infty}^{\infty} \frac{e^{3/2x} dx}{(c_0 + c_r e^{-x} + c_f e^{(3+3\delta)x})^{1/2}} \quad (2.20)$$

This is the well known Big Rip singularity associated with the presence of a phantom fluid [6]. In a two-fluids model, namely putting $c_r = 0$, one has $c_f = 1 - c_0$, and $H_0^2 = \frac{\chi}{3}$ (the Hubble parameter is a constant). The integral can be computed and reads

$$t_s = \frac{1}{\sqrt{\pi H_0}} \frac{\Gamma \left( \frac{\delta}{2(1+\delta)} \right)}{(1+\delta)c_0^{\delta}} \frac{1}{\Gamma \left( \frac{\delta}{2} \right)} \left( c_0 c_f \right)^{-1/2} \quad (2.21)$$

For small value of $\delta$ one easily gets

$$t_s \simeq \frac{1}{H_0} \left[ c_0 (1 - c_0) \right]^{-1/2} \frac{\Gamma \left( \frac{\delta}{2} \right)}{\Gamma \left( \frac{\delta}{2(1+\delta)} \right)} \quad (2.22)$$

Thus, the smaller is $\delta$, and in the future the finite singularity will be located with respect to $\frac{1}{H_0}$ (roughly the age of our universe).

Coming back to (2.19), its inversion would give the FLRW solution for the ΛCDM model. As is well known, in general the inversion of this equation is a difficult task and numerical analysis is required. In the next Section we shall see that the inversion will not be strictly necessary.
However in a two-fluids model (matter or radiation, plus cosmological constant $\delta = 0$), the inversion is possible since one has (here $\omega$ is either 0 or 1/3)

$$\sqrt{\frac{X}{3}} t(y) = \int_y^{\infty} \frac{e^{3x/2}}{(ce^{-3\omega x} + cf e^{3x})^{1/2}} dx = \frac{2}{3(1 + \omega)\sqrt{cf}} \sinh^{-1} \left( e^{3(1+\omega)y/2} \right), \quad (2.23)$$

and this gives the well known result

$$a(t) = \left( \frac{c}{cf} \right)^{\frac{1}{3}(1+\omega)} \sinh \left( \sqrt{\frac{X}{3}} \frac{c}{cf} \frac{(1 + \omega)}{2} t \right) \left( \frac{3}{2} \right)^{\frac{1}{4}(1+\omega)}. \quad (2.24)$$

### 2.2 Modified equation of state

Another possibility that has been investigated by several authors is to keep the Friedmann equation with matter conservation but to modify the equation of state, for example considering

$$p = \omega \rho - A \rho^{-\gamma}, \quad A > 0. \quad (2.25)$$

As a result one has

$$-3y = \int \frac{\rho^{\gamma}}{(1 + \omega)\rho^{\gamma+1} - A} d\rho. \quad (2.26)$$

Let us consider first a generalized model for dark energy, where $\omega = -1$ (see, for example, [20]). Considering $\gamma = -b - \frac{1}{2}$, one has

$$y = \frac{2}{3A(1 - 2b)} \rho^{1/2-b}, \quad \rho = (C_b y)^{\frac{2}{1-2b}}, \quad (2.27)$$

where $C_b = 3A(1 - 2b)/2$. For $b = 0$, one has the so-called Little Rip behaviour $R_H = \sqrt{\frac{3}{X}} \frac{3A}{2y}$ (see [21]), there is no Big Bang and it is possible to show that

$$H(t) = H_0 e^{B(0)t}, \quad (2.28)$$

with $B$ constant. If $b \neq 0$, the solution may be written in the form

$$H(t) = H_0 \left( 1 - 2bB(b)(t - t_0) \right)^{-\frac{1}{2b}}. \quad (2.29)$$

If $b < 0$ one has a Little Rip singularity, but if $b > 0$ one has a Big Rip singularity [20].

If $\omega + 1 > 0$, then one is dealing with a Chaplygin gas and its generalizations [22, 23]. In this case, one obtains

$$\rho = e^{-3y} \left( \frac{1 + Ae^{3\alpha y}}{(1 + \omega)} \right)^{\frac{1}{\alpha}}, \quad \alpha = (1 + \omega)(1/2 - b) > 0. \quad (2.30)$$

The Hubble horizon is $R_H = \frac{1}{H}$, and the time reads

$$\sqrt{\frac{X}{3}} t(y) = \int_y^{-\infty} dx e^{3x/2x} \left( \frac{1 + Ae^{3\alpha x}}{(1 + \omega)} \right)^{-\frac{1}{2\alpha}}. \quad (2.31)$$

There is a Big Bang singularity, but no future singularity because the above integral diverges as $y \to \infty$. 

---

7
3 Ray tracing in FLRW space-times

We here describe the null trajectories followed by light rays in flat FLRW. This analysis allows to visualize the range of possible trajectories followed by massless bodies in a given space-time. In particular it is possible to determine i) whether or not a comoving observer sitting in the origin at time $t_0$ will receive ingoing light rays and ii) the maximum proper radius reached by these light rays before $t_0$. In the following, we reformulate the analysis presented in [8] and recently in [9, 10], with the main aim to present analytic results.

From equation (2.4) we find for radial ingoing photon geodesics

$$\frac{dR_\gamma}{dy} = R_\gamma - \frac{1}{H} \equiv R_\gamma - R_H.$$ (3.1)

The general solution of eq. (3.1) is given by

$$R_\gamma(y) = e^y \left( C - \int_{-\infty}^{y} e^{-x} R_H(x) dx \right).$$ (3.2)

Here we have assumed the existence of a Big Bang initial singularity $y_0 \to -\infty$. Providing the model through the specification of $R_H$ and appropriate initial conditions, one can trace ingoing light rays.

First we discuss the Hubble horizon behavior. In the standard one-fluid model, one has

$$H^2 = \frac{\chi}{3} \rho = H_0^2 e^{-3(1+\omega)y}, \quad H_0^2 = \frac{\chi c^2}{3}.$$ (3.3)

Thus, the Hubble horizon is always expanding according to

$$R_H = \frac{1}{H_0} e^{3(1+\omega)y/2}.$$ (3.4)

For the generalized Chaplygin case we have an increasing but asymptotically constant function in $y$

$$R_H(y) = \sqrt{\frac{3(1+\omega)^{1/\alpha}}{\chi}} \frac{e^{3y/2}}{(1 + A e^{3\alpha y})^{1/\alpha}}.$$ (3.5)

Here $R_H(-\infty) = 0$, which is the Big-Bang singularity. For $y \to \infty$, $R_H(y)$ reaches its maximum given by

$$R_H^{\text{max}} = \sqrt{\frac{3(1+\omega)^{1/\alpha}}{\chi}}.$$ (3.6)

A similar behaviour is present for the three-fluids model in the case $\delta = 0$, see (2.18), and the maximum for $y \to \infty$ now reads

$$R_H^{\text{max}} = \sqrt{\frac{3}{\chi c_f}}.$$ (3.7)
In the case of phantom field ($\delta > 0$), since for $y \to \infty$ one has $R_H(y) \to 0$, it follows that there exists a local maximum at finite $y = y^*$, given by the solution of the transcendental equation

$$3c_0 + 4c_r e^{-y^*} = 3\delta c_f e^{(3+3\delta)y^*}.$$ 

(3.8)

At the Big Bang $y \to -\infty$ one has

$$\left.\frac{dR_H}{dy}\right|_{y=-\infty} = 0, \quad \left.\frac{e^{-y}dR_H}{dy}\right|_{y=-\infty} = 0.$$ 

(3.9)

With regard to photon tracing, in the standard one fluid model 

$$R_\gamma(y) = e^y \left( C - \frac{2}{(1+3\omega)H_0} e^{(3\omega+1)y/2} \right).$$ 

(3.10)

The photon trajectory, chosen an arbitrary $C$, always reaches the origin again, namely $R_\gamma(y_1) = 0$ at

$$H_0C = e^{(3\omega+1)y_1/2}$$

(3.11)

In presence of dark energy, the situation changes. In fact, in the three-fluids and generalized Chaplygin models, having Big Bang singularities, one has

$$\left.\frac{dR_\gamma}{dy}\right|_{y=-\infty} = 0, \quad \left.\frac{e^{-y}dR_\gamma}{dy}\right|_{y=-\infty} = C.$$ 

(3.12)

which gives a physical meaning to the integration constant $C$. Furthermore, in these cases, the crucial fact is the existence of the finite integral

$$C^* = \int_{-\infty}^{\infty} e^{-x} R_H(x) \, dx < \infty.$$ 

(3.13)

The corresponding constant in the one-fluid model is obviously divergent. For the two fluids model one has

$$C^*(\delta) = \frac{1}{\sqrt{\pi}H_0} \frac{\Gamma \left( \frac{2+3\delta}{6(1+\delta)} \right)}{6(1+\delta)} \frac{\Gamma \left( \frac{1}{6(1+\delta)} \right)}{\Gamma \left( \frac{1}{2(1+\delta)} \right)} \left( \frac{c_0}{c_f} \right)^{\frac{1}{6(1+\delta)}},$$ 

(3.14)

while for the Chaplygin gas

$$C^*(\alpha) = R_H(\alpha)|_M \frac{\Gamma \left( \frac{1}{6\alpha} \right)}{6\alpha \Gamma \left( \frac{1}{2\alpha} \right)} \frac{A^{-1/6\alpha}}{-\Gamma \left( \frac{1}{3\alpha} \right)}.$$ 

(3.15)

As a consequence, one can distinguish between three cases.

The first one is the most interesting from the physical point of view and it is realized when $C < C^*$. In this case, for $y \to \infty$ one has $R_\gamma(y) \to -\infty$. Of course, only positive values of $R_\gamma$ are physically relevant, thus there exists $y_1$ such that

$$C = \int_{-\infty}^{y_1} e^{-x} R_H(x) \, dx, \quad R_\gamma(y_1) = 0,$$ 

(3.16)
namely these photons emitted at the Big Bang may be observed at the origin after a finite “time” $y_1$ and their trajectories are hence given by

$$R_\gamma(y) = e^y \int_{y_1}^{y} e^{-x} R_H(x) \, dx$$  \hfill (3.17)

For this class of trajectories there exists an extremal $\frac{dR_\gamma}{dy} = 0$ at $y_M$, which defines the horizon crossing

$$R_\gamma|_M = R_H|_M.$$  \hfill (3.18)

Making use of photon trajectory equation one has on the extremal

$$\left. \frac{d^2R_\gamma}{d^2y} \right|_M = -\left. \frac{dR_H}{dy} \right|_M.$$  \hfill (3.19)

In order for the light ray to eventually reach the origin, the moment of the horizon crossing should correspond to a maximum of the trajectory. From the last equation, this means that $dR_H/dy > 0$ at $y_M$, i.e. the horizon’s proper radius has to be an increasing function in a neighborhood of $y_M$. In both the Chaplygin gas and the $\delta = 0$ three-fluids models, the horizon radius is always an increasing function of $y$. On the other hand, for $\delta > 0$ and smaller it has to be $y_M < y^*$ ($y^*$ being the time corresponding to the maximum value of horizon radius $R_H$) because in this range $R_H$ is increasing. In general, for this class of photon trajectories one has the trivial but important property (see [9])

$$R_\gamma(y_M) = R_H(y_M) < R_H(y^*).$$  \hfill (3.20)

This property supports the claim put forward graphically in Ref. [9] and gives an important global geometric characterization of the Hayward trapping horizon $R_H = 1/H$.

In the other two cases ($C > C^*$ and $C = C^*$) $R_\gamma$ is never vanishing. Furthermore, when $y \to \infty$, for $C > C^*$ it follows $R_\gamma(y) \to \infty$, while for $C = C^*$ one has $R_\gamma(y) \to 0$.

For the $\delta = 0$ and the Chaplygin gas models there are no extremal points. In fact, if there were an extremal, due to equation (3.19) this should be a local maximum, and that would contradict $R_\gamma(\infty) = \infty$.

In the phantom case there exists a first extremal (and we have seen that this is a local maximum), but there exists also a second extremal, which has to be a local minimum in order to be compatible with $R_\gamma \to \infty$. In any case, this class of photon trajectories cannot ever be observed at the origin.

4 \hspace{1em} Hubble horizon and its causal character

The causal characterization of the Hubble horizon in different models can be useful in order to better clarify the behavior of light trajectories – a topic that we have addressed in previous sections. In a flat FLRW model the horizon is a spherically symmetric surface located at $r_H \dot{a}(t) = 1$. Evaluating the normal metric on the horizon one may rewrite

$$d\gamma^2_H = \frac{dR_H}{dy} \left( \frac{dR_H}{dy} - 2R_H \right) \, dy^2.$$  \hfill (4.1)

Recall that the sign of the line element determines the causal character of the horizon surface, in particular for $d\gamma^2_H < 0$ ($> 0$) the horizon will be timelike (spacelike).
Standard cosmologies. We can promptly recall a couple of known examples. In the de Sitter case $H(t) = H_0$ and constant, so that $d\gamma_H^2$ vanishes identically: hence its null character.

In the one-fluid model, one has

$$H^2 = \frac{\chi}{3} = H_0^2 e^{-3(1+\omega)y}, \quad H_0^2 = \frac{\chi c}{3}. \quad (4.2)$$

Thus

$$R_H(y) = \frac{1}{H_0} \frac{c(1+\omega)}{2} y, \quad (4.3)$$

and the line element

$$d\gamma^2_H = R_H(y) \frac{9}{4} R^2 (1 + \omega) \left( -\frac{1}{3} + \omega \right) dy^2,$$

so that the horizon is timelike for $-1 < \omega < 1/3$. The values $\omega = -1$ (cosmological constant) and $\omega = 1/3$ (radiation-dominated cosmologies) give the horizon a null character. On the other hand, models with $\omega > 1/3$ (including stiff matter) contain a spacelike horizon.

Big Bang and/or Big Rip. Here we consider the model containing Big Bang as well as Big Rip solutions that has been presented in the previous section: the three-fluids model given in eq. (2.16).

We recall that

$$R_H(y) = \sqrt{\frac{\chi}{3} e^{3y/2}} , \quad D = c_0 + c_r e^{-y} + c_f e^{(3+3\delta)y}. \quad (4.4)$$

Thus

$$\frac{dR_H}{dy} = \frac{R_H}{2D} N, \quad N = \left( 3D - \frac{dD}{dy} \right), \quad (4.5)$$

and one has

$$d\gamma^2_H = -\frac{R^2_H(y)}{4D^2} \left( c_0 + (4 + 3\delta)c_f e^{(3+3\delta)y} \right) N dy^2. \quad (4.6)$$

As a consequence the causal nature of the horizon depends on $N$. For $\omega_f = -1$ or $\delta = 0$ (the standard $\Lambda$CDM model) it turns out that

$$N = 3c_0 + 4c_r e^{-y} > 0. \quad (4.7)$$

Thus, in this case the Hubble horizon is timelike, approaching a null character (i.e. $d\gamma^2_H \to 0$) for $y \to \infty$. The same fact holds true for the Chaplygin gas model. This is consistent with the results of the previous section. In the phantom scenario, one has

$$N = 3c_0 + 4c_r e^{-y} - 3\delta c_f e^{(3+3\delta)y}. \quad (4.8)$$

A direct calculation, making use of equation (3.8) leads to

$$N = -\frac{3c_0}{e^{(3+3\delta)y}} \left( e^{(3+3\delta)y} - e^{(3+3\delta)y^*} \right) - 4c_r e^{-y} \frac{e^{(4+3\delta)y} - e^{(4+3\delta)y^*}}{e^{(4+3\delta)y^*}}. \quad (4.9)$$

Thus, if $y < y^*$ the Hubble horizon is timelike, if $y = y^*$ it is null and for $y > y^*$ it is spacelike. These three ranges correspond to the ranges in which the horizon’s radius is increasing, instantaneously stationary and decreasing, respectively. Again, given that the light rays can cross the horizon toward the origin only if the horizon itself is timelike and increasing, the range in which this can occur is $y < y^*$, the same result of the previous section.
**Time dependent dark energy model.** Eventually, it may be of some interest to discuss the example presented in [11]. While it is not a physically motivated scenario, it is nevertheless an example that shows the dependence on the Hubble horizon of the dynamics. The model is defined by the usual Friedmann equations and equations of state for ordinary matter and the “dark energy” component with a suitable time dependent barotropic factor:

\[
H^2 = \frac{\chi}{3}(\rho_M + \rho_E), \quad p_M = 0, \quad p_E = \omega(t)\rho_E, \quad \omega(t) = -1 + \delta + g(t),
\]  
(4.10)

where \(\delta > 0\) but small (strictly dark energy contribution) and \(g(t) > 0\) (a non-dark part). Assuming energy conservation for \(\rho_M\) and \(\rho_E\),

\[
\dot{H} = -\frac{\chi}{2}\left(\rho_M + (1 + \omega(t))\rho_E\right).
\]  
(4.11)

As a result, the evolution of the Hubble horizon is given by

\[
\dot{R}_H = \frac{3}{2}\frac{\rho_M + (-\delta + g(t))\rho_E}{\rho_M + \rho_E},
\]  
(4.12)

The model is not exactly solvable due to the time dependence of \(\omega(t)\). Also the use of the evolution parameter \(y\), as in our approach, does not simplify the computation, and it seems difficult to reproduce analytically the results of Ref. [11]. However, as reported in the cited paper, one may distinguish between two regimes: the first one valid for small values of \(t\), when the dark energy component can be neglected, and the second one valid for large values of \(t\), when the dark energy component dominates.

In the first regime one obtains \(R_H \simeq \frac{3}{2}t\), while in the second regime one has

\[
R_H \simeq \frac{3}{2}\int_{t_1}^t (-\delta + g(t'))dt',
\]  
(4.13)

valid for \(t_1\) sufficiently large. If the positive function \(g(t)\) is locally summable then \(R_H\) may eventually diverge at infinity only. Moreover, if \(g(t_2) = \delta\) for \(t_2 \in (t_1, \infty)\), then \(R_H(t_2)\) is a local minimum for \(R_H\) and in such a case the behavior of \(R_H\) in the whole range \(t \in (0, \infty)\) is the one considered in [11]. In that paper the function \(g(t)\) has been chosen in such a way that for very large \(t\) the dark energy contribution becomes negligible and the universe falls down in the standard expanding phase with non a negative barotropic parameter. Thus there exist photon trajectories which cross the dynamical horizon and arrive at the origin. As we already said in the Introduction this is not a surprising behavior, because the causal nature of Hubble surface may change according to Eq. (1.14) or (4.1).

5 Conclusions

We start our concluding remarks by discussing one of the perhaps confusing concepts that has risen in the debate, that is the present size of the horizon. It should be clear that if one takes into account limits of observation for an observer sitting in the origin at time \(t_0\), the size of the horizon at that time, \(R_H(t_0)\), plays no role: the information about what happens at \(R_H(t_0)\) will reach the origin at a time \(t > t_0\) (if the horizon allows it, and this depends on the future behavior of the model).
In this paper we focused on the analysis of particular physically motivated models presenting a Big Bang singularity and different future behavior, a Big Rip ($\delta > 0$ three-fluid model) or no singularity ($\delta = 0$ and generalized Chaplygin gas models): in any case we restricted the analysis to expanding universes, clearly the most interesting in view of the present behavior of our own Universe. With these models in mind, we showed that the present-day observational horizon (identified by the maximum proper radius attained by the ingoing light paths reaching the origin now) cannot be larger than the maximum proper radius attained by the Hubble horizon, which is what eq. (3.20) expresses. One has to keep in mind that light rays are able to cross the horizon toward the origin only if the horizon is increasing (and timelike). With regard to this issue, we have confirmed the results presented in [11]. Furthermore, we have provided a sufficient condition for incoming light rays to reach the origin through the general condition $C < C^*$ [see Eq. (3.13) and discussions], which actually applies to every expanding model with a Big Bang initial singularity, including also the old one-fluid standard model, where $C^* = \infty$, and for which the condition is always satisfied.

References

[1] S. Nojiri and S. D. Odintsov, eConfC 0602061, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007)] [hep-th/0601213].
[2] S. Nojiri and S. D. Odintsov, Phys. Rept. 505 59, (2011), [arXiv:1011.0544 [gr-qc]].
[3] S. Capozziello and M. Francaviglia, Gen. Rel. Grav. 40, 357 (2008) [arXiv:0706.1146 [astro-ph]].
[4] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]].
[5] R. Myrzakulov, L. Sebastiani and S. Zerbini, “Some aspects of generalized modified gravity models,” [arXiv:1302.4646 [gr-qc]], to appear in IJMPD (2013).
[6] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. 91, 071301 (2003) [astro-ph/0302506].
[7] S. 'i. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D 71, 063004 (2005) [hep-th/0501025].
[8] G. F. R. Ellis and T. Rothman, American J. Phys. 61, 883 (1993). [arXiv:hep-th/9505061 [hep-th]].
[9] F. Melia, JCAP 1209, 029 (2012) [arXiv:1206.6192 [astro-ph.CO]]; O. Bikwa, F. Melia and A. Shevchuk, MNRAS, 421 3356 (2012), [arXiv:1112.4774 [astro-ph.CO]].
[10] G. F. Lewis and P. van Oirschot, MNRAS Letters, 423 26 (2012), [arXiv:1203.0032 [astro-ph.CO]].
[11] G. F. Lewis, MNRAS Letters, 431 25 (2013), [arXiv:1203.0032 [astro-ph.CO]].
[12] H. Kodama, Prog. Theor. Phys. 63, 1217 (1980).
[13] S.A. Hayward Phys. Rev. D 53, 1938 49 (1996).
[14] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini and S. Zerbini, Class. Quant. Grav. 26, 062001 (2009).

[15] R. Di Criscienzo, S. A. Hayward, M. Nadalini, L. Vanzo and S. Zerbini, Class. Quant. Grav. 27, 015006 (2010).

[16] L. Vanzo, G. Acquaviva, R. Di Criscienzo, “Tunnelling Methods and Hawking’s radiation: achievements and prospects,” Class. Quant. Grav. 28, 183001 (2011). [arXiv:1106.4153 [gr-qc]].

[17] A. Ashtekar and B. Krishnan, Living Rev. Rel. 7, 10 (2004) [gr-qc/0407042].

[18] H. Wei, L.F. Wang and X.J. Guo, Phys. Rev. D 86, 083003 (2012).

[19] G. Cognola, R. Myrzakulov, L. Sebastiani and S. Zerbini, arXiv:1304.1878 [gr-qc].

[20] I. Brevik, R. Myrzakulov, S. Nojiri and S. D. Odintsov, Phys. Rev. D 86, 063007 (2012)

[21] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D 84, 063003 (2011) [arXiv:1106.4996 [astro-ph.CO]].

[22] A. Y. .Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B 511, 265 (2001) [gr-qc/0103004].

[23] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D 66, 043507 (2002) [gr-qc/0202064].