ČEBYŠĚV SUBSPACES OF JBW*-TRIPLES

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Abstract. We describe the one-dimensional Čebyšěv subspaces of a JBW*-triple $M$, by showing that for a non-zero element $x$ in $M$, $Cx$ is a Čebyšěv subspace of $M$ if, and only if, $x$ is a Brown-Pedersen quasi-invertible element in $M$. We establish a complete description of all Čebyšěv JBW*-subtriples of $M$, establishing that a non-zero JBW*-subtriple $N$ of $M$ if a Čebyšěv subspace if and only if one of the following statements holds:

$(a)$ $N$ and $M$ have rank one or two;
$(b)$ $N = Cc$, where $c$ is a complete tripotent in $M$;
$(c)$ $N$ has rank greater or equal than 3 and $N = M$.

1. Introduction

Let $V$ be a subspace of a Banach space $X$. The subspace $V$ is called a Čebyšěv (Chebyshev) subspace of $X$ if and only if for each $x \in X$ there exists a unique point $x_0 \in V$ such that $\text{dist}(x, V) = ||x - x_0||$.

Let $K$ be a compact Hausdorff space. A classical theorem due to A. Haar establishes that an $n$-dimensional subspace $V$ of the space $C(K)$, of all continuous complex-valued functions on $K$, is a Čebyšěv subspace of $C(K)$ if, and only if, any non-zero $f \in V$ admits at most $n - 1$ zeros (cf. [17] and the monograph [32, p. 215]). Having in mind the Riesz representation theorem, and the characterization of the extreme points of the closed unit ball in the dual space of $C(K)$, we can easily see that, in the above conditions, $V$ is an $n$-dimensional Čebyšěv subspace of $C(K)$ if, and only if, there exists no set $\{\delta_{t_1}, \ldots, \delta_{t_n}\}$ of $n$-mutually orthogonal, pure states such that $V \subset \bigcap_{i=1}^{n} \ker(\delta_{t_i})$. This result implies that any non-zero $f$...
in $C(K)$ spans a Čebyšev subspace of the latter space if, and only if, $f$ is invertible in the algebra $C(K)$.

Later on, J.G. Stampfli proved in [33, Theorem 2], that the scalar multiples of the unit element in a von Neumann algebra $M$ is a Čebyšev subspace of $M$. In [24], D.A. Legg, B.E. Scranton, and J.D. Ward characterize the semi-Čebyšev and finite dimensional Čebyšev subspaces of $K(H)$, the algebra of compact operators on an infinite-dimensional Hilbert space $H$. They conclude that, for a separable Hilbert space $H$, there exist Čebyšev subspaces of every finite dimension in $K(H)$ [24, Theorem 3], when $H$ is not separable $K(H)$ has no finite-dimensional Čebyšev subspaces [24, Corollary 2].

A.G. Robertson continued with the study on Čebyšev subspaces of von Neumann algebras in [28], where he established the following results:

**Theorem 1.** ([28, Theorem 6]) Let $x$ be a non-zero element in a von Neumann algebra $M$. Then, the one dimensional subspace $Cx$ is a Čebyšev subspace of $M$ if and only if there is a projection $p$ in the center of $M$ such that $px$ is left invertible in $pM$ and $(1-p)x$ is right invertible in $(1-p)M$.

**Theorem 2.** ([28, Theorem 6]) Let $N$ be finite dimensional $*$-subalgebra of an infinite dimensional von Neumann algebra $M$. Suppose $N$ has dimension $>1$. Then $N$ is not a Čebyšev subspace of $M$.

A.G. Robertson and D. Yost prove in [29, Corollary 1.4] that in an infinite dimensional C*-algebra $A$ admits a finite dimensional $*$-subalgebra $B$ which is also a Čebyšev in $A$ if and only if $A$ is unital and $B = \mathbb{C}1$.

The results proved by Robertson and Yost were complement by G.K. Pedersen, who shows that if $A$ is a C*-algebra without unit and $B$ is a Čebyšev C*-subalgebra of $A$, then $A = B$ (compare [27, Theorem 4]).

The previous results of Robertson [28] and Pedersen [27, Theorem 2] also prove the following equivalent reformulation of Theorem 1: for each non-zero element $x$ in a von Neumann algebra $M$, the following statements are equivalent:

(a) $Cx$ is a Čebyšev subspace of $M$;
(b) $x$ is Brown-Pedersen quasi-invertible in $M$;
(c) For each pure state (i.e. for each extreme point of the positive part of the closed unit ball of $M^*$) $\varphi \in M^*$, and for each unitary $u \in M$, we have $\varphi(x^*x) + \varphi(uxx^*u) > 0$.

Then, the one dimensional subspace $Cx$ is a Čebyšev subspace of $M$ if and only if there is a projection $p$ in the center of $M$ such that $px$ is left invertible in $pM$ and $(1-p)x$ is right invertible in $(1-p)M$.

A renewed interest on Čebyšev subspaces of C*-algebras has led M. Namboodiri, S. Pramod, and A. Vijayarajan to revisit and generalize the previous contributions of Robertson, Yost and Pedersen in [26].
On the other hand, $C^*$-algebras can be regarded as elements in a strictly wider class of complex Banach spaces called JB$^*$-triples (see §2 for the detailed definitions). Many geometric properties studied in the setting of $C^*$-algebras have been also explored in the bigger class of JB$^*$-triples. However Čebyšëv subspaces and the theory of best approximations remains unexplored in the class of JB$^*$-triples. In this note we present the first results about Čebyšëv subspaces and Čebyšëv subtriples in Jordan structures.

In Section 2 we prove that for a non-zero element $x$ in a JB$^*$-triple $M$, $Cx$ is a Čebyšëv subspace of $M$ if, and only if, $x$ is a Brown-Pedersen quasi-invertible element in $M$ (see Theorem 6). This result generalizes the result established by Robertson in Theorem 1 (cf. [28]), but it also add a new perspective from an independent argument.

In Section 3 we establish a precise description of the JB$^*$-subtriples of a JB$^*$-triple $M$ which are Čebyšëv subspaces in $M$. We should remark that in the setting of von Neumann algebras and $C^*$-algebras, the scarcity of non-trivial Čebyšëv $^*$-subalgebras is endorsed with the following results: If an infinite dimensional von Neumann algebra, $M$, contains a finite dimensional von Neumann subalgebra $N$ which is a Čebyšëv subspace in $M$, then $N$ must be one dimensional (compare Theorem 2 or [28, Theorem 6]). Furthermore, an infinite dimensional $C^*$-algebra $A$ admits a finite dimensional $^*$-subalgebra $B$ which is also a Čebyšëv in $A$ if and only if $A$ is unital and $B = \mathbb{C}1$ (cf. [29, Corollary 1.4]). If $A$ is a $C^*$-algebra without unit and $B$ is a Čebyšëv $C^*$-subalgebra of $A$, then $A = B$ (compare [27, Theorem 4]). The first main difference in the setting of JB$^*$-triples is the existence of Čebyšëv JB$^*$-subtriples with arbitrary dimensions; complex Hilbert spaces and spin factors give a complete list of examples (compare Remark 7 and comments before it).

In our main result about Čebyšëv JB$^*$-subtriples (cf. Theorem 13), we establish the following criterium: Let $N$ be a non-zero Čebyšëv JB$^*$-subtriple of a JB$^*$-triple $M$. Then exactly one of the following statements holds:

(a) $N$ and $M$ are rank one JB$^*$-triples. Moreover, $M$ is a complex Hilbert space and $N$ is a closed subspace of arbitrary dimension;
(b) $N = \mathbb{C}e$, where $e$ is a complete tripotent in $M$;
(c) $N$ and $M$ have rank two, but $N$ may have arbitrary dimension;
(d) $N$ has rank greater or equal than 3 and $N = M$.

It should be remarked at this point that the techniques applied by Robertson, Yost [28, 29] and Pedersen [27] in the setting of von Neumann algebras do not make any sense in the wider setting of JB$^*$-triples. The techniques developed in this paper are completely independent and provide new arguments to understand the Čebyšëv JB$^*$-subtriples of a von Neumann algebra (Corollary 14).
2. One-dimensional Čebyšev subspaces and subtriples of JBW*-triples

A complex Jordan triple system is a complex linear space $E$ equipped with a triple product which is bilinear and symmetric in the external variables and conjugate linear in the middle one and satisfies the Jordan identity:

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$.

A JB*-triple is a complex Jordan triple system $E$ which is a Banach space satisfying the additional “geometric” axioms:

(a) For each $x \in E$, the operator $L(x, x)$ is hermitian with non-negative spectrum;

(b) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

Every $C^*$-algebra is a JB*-triple with respect to the triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^* + cb^*).$$

Every JB*-algebra (i.e. a complex Jordan Banach *-algebra satisfying

$$\|U_a(a^*)\| = \|a\|^3,$$

for every element $a$, where $U_a(x) := 2(a \circ x) - a^2 \circ x$, cf. [18, §3.8]) is a JB*-triple under the triple product defined by

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

The space $B(H, K)$ of all bounded linear operators between complex Hilbert spaces, although rarely is a $C^*$-algebra, is a JB*-triple with the product defined in (2.2). In particular, every complex Hilbert space is a JB*-triple.

Other examples of JB*-triples are given by the so-called Cartan factors. A Cartan factor of type 1 is a JB*-triple which coincides with the Banach space $B(H, K)$ of bounded linear operators between two complex Hilbert spaces, $H$ and $K$, where the triple product is defined by (2.2). Cartan factors of types 2 and 3 are JB*-triples which can be identified the subtriples of $B(H)$ defined by $II^C = \{x \in B(H) : x = -jx^*j\}$ and $III^C = \{x \in B(H) : x = jx^*j\}$, respectively, where $j$ is a conjugation on $H$. A Cartan factor of type 4 or $IV$ is a spin factor, that is, a complex Hilbert space provided with a conjugation $x \mapsto \bar{x}$, where the triple product and the norm are defined by

$$\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

and $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - \langle x/\bar{x} \rangle^2}$, respectively. The Cartan factors of types 5 and 6 consist of finite dimensional spaces of matrices over the eight dimensional complex Cayley division algebra $\mathbb{O}$; the type $VI$ is the space of all hermitian 3x3 matrices over $\mathbb{O}$, while the type $V$ is the subtriple of 1x2 matrices with entries in $\mathbb{O}$ (compare [25], [16], and [12, §2.5]).
A JB*-triple $W$ is called a JBW*-triple if it has a predual $W_*$. It is known that a JBW*-triple admits a unique isometric predual and its triple product is separately $\sigma(W, W_*)$-continuous (see [3]). The second dual $E^{**}$ of a JB*-triple $E$ is a JBW*-triple with respect to a triple product which extends the triple product of $E$ (cf. [13]).

For more detail of the properties of JB*-triples and JBW*-triples the reader is referred to the monographs [12] and [11].

Given an element $a$ in a JB*-triple $E$, the symbol $Q(a)$ will denote the conjugate linear operator on $E$ defined by $Q(a)(x) = \{a, x, a\}$.

An element $e \in E$ is called a *tripotent* when $\{e, e, e\} = e$. Each tripotent $e \in E$ induces a decomposition of $E$, called the *Peirce decomposition*, in the form $E = E_2(e) \oplus E_1(e) \oplus E_0(e)$, where $E_i(e)$ is the $\frac{i}{2}$ eigenspace of the operator $L(e, e)$, $i = 0, 1, 2$. This decomposition satisfies the following *Peirce rules*:

$$ \{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0 $$

and

$$ \{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i+j+k}(e), $$

when $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. The projection $P_k(e)$ of $E$ onto $E_k(e)$ is called the *Peirce k-projection*. It is known that Peirce projections are contractive (cf. [15, Corollary 1.2]) and satisfy:

$$ P_2(e) = Q(e)^2, \quad P_1(e) = 2(L(e, e) - Q(e)^2), $$

and

$$ P_0(e) = Id_E - 2L(e, e) + Q(e)^2. $$

The separate weak*-continuity of the triple product of a JBW*-triple $M$ implies that Peirce projections associated with a tripotent $e$ in $M$ are weak*-continuous.

It is known that the Peirce-2 subspace $E_2(e)$ is a JB*-algebra with unit $e$, Jordan product $x \circ_e y := \{x, e, y\}$ and involution $x^{*e} := \{e, x, e\}$, respectively. Since surjective linear isometries and triple isomorphisms on a JB*-triple coincide (cf. [22, Proposition 5.5]), the triple product in $E_2(e)$ is uniquely given by

$$ \{x, y, z\} = (x \circ_e y^{*e}) \circ_e z + (z \circ_e y^{*e}) \circ_e x - (x \circ_e z) \circ_e y^{*e}, $$

$x, y, z \in E_2(e)$.

We shall make use of the following property: given a tripotent $e \in E$ and an element $\lambda$ in the unit sphere of $C$, the mapping:

$$ S_\lambda(e) : E \rightarrow E, \quad S_\lambda(e) = \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e), $$

is a surjective linear isometry on $E$ and a triple isomorphism (compare [15, Lemma 1.1]).

A tripotent $e \in E$ is said to be *unitary* if the operator $L(e, e)$ coincides with the identity map $I_E$ on $E$; that is, $E_2(e) = E$. We shall say that $e$ is
complete or maximal when \( E_0(e) = E \). When \( E_2(e) = P_2(e)(E) = \mathbb{C}e \neq \{0\} \), we say that \( e \) is minimal.

The complete tripotents of a JB*-triple \( E \) coincide with the real and complex extreme points of its closed unit ball \( E_1 \) (cf. [5, Lemma 4.1] and [23, Proposition 3.5] or [12, Theorem 3.2.3]). Consequently, the Krein-Milman theorem assures that every JBW*-triple admits an abundant set of complete tripotents [12, Corollary 3.2.4].

When \( a \) is an element in a JBW*-triple \( M \), the sequence \( (a^{\frac{1}{n}}) \) converges in the weak*-topology of \( M \) to a tripotent, denoted by \( r(a) \), called the range tripotent of \( a \). The tripotent \( r(a) \) is the smallest tripotent \( e \in M \) satisfying that \( a \) is positive in the JBW*-algebra \( M_2(e) \) (see [14, page 322]).

Let \( a \) be an element in a JB*-triple \( E \). It is known that the JB*-subtriple \( E_a \) generated by \( a \), identifies with some \( C_0(L) \) where \( \|a\| \in L \subseteq [0, \|a\|] \) with \( L \cup \{0\} \) compact (cf. [22, 1.15]). Moreover, there exists a triple isomorphism \( \Psi : E_a \rightarrow C_0(L) \) such that \( \Psi(a)(t) = t \). Clearly, the range tripotent \( r(a) \) can be identified with the characteristic function \( x_{(0,|a|]} \in C_0(L)^* \) (see [7, beginning of §2]).

We recall that an element \( x \) in a Jordan algebra \( J \) with unit \( e \) is called invertible if there exists an element \( y \) such that \( x \circ y = e = x \circ x^2 \circ y = x \). The element \( y \) is called the inverse of \( x \), and is denoted by \( x^{-1} \). Inverse of any element \( x \) in a Jordan algebra \( J \) is unique whenever it exists. The set of all invertible elements in \( J \) is denoted by \( J^{-1} \).

An element element \( a \) in a JB*-triple \( E \) is called von Neumann regular if and only if there exists \( b \in E \) such that

\[
Q(a)(b) = a, \quad Q(b)(a) = b, \quad \text{and} \quad [Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0.
\]

When \( a \) is von Neumann regular, the (unique) element \( b \in E \) satisfying the above conditions is called the generalized inverse of \( a \), and is denoted by \( a^\dagger \). It is known that an element \( a \in E \) is von Neumann regular if, and only if, \( Q(a) \) has norm-closed image if, and only if, the range tripotent \( r(a) \) of \( a \) lies in \( E \) and \( a \) is positive and invertible element of the JB*-algebra \( E_2(r(a)) \) (compare [10]). Furthermore, when \( a \) is von Neumann regular, \( Q(a)Q(a^\dagger) = Q(a^\dagger)Q(a) = P_2(r(a)) \) and \( L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a)) \) [10, page 192].

Given a pair of elements \( a, b \) in a JB*-triple \( E \), the Bergmann operator associated to \( a \) and \( b \) is the mapping \( B(a, b) : E \rightarrow L(E) \) defined by \( B(a, b) = \text{Id}_E - 2L(a, b) + Q(a)Q(b) \) (cf. [12, page 22]).

An element \( a \) in a JB*-triple \( E \) is said to be Brown-Pedersen quasi-invertible (BP-quasi-invertible for short) when it is von Neumann regular with generalized inverse \( b \) such that the Bergman operator \( B(a, b) \) vanishes; in such a case, \( b \) is called the BP-quasi inverse of \( a \). The set of BP-quasi invertible elements in \( E \) is denoted by \( E_1q^{-1} \) [34]. It is established in [34] that
an element \( a \in E \) is BP-quasi-invertible if, and only if, one of the following equivalent statements holds:

(i) \( a \) is von Neumann regular, and its range tripotent \( r(a) \) is an extreme point of the closed unit ball \( E_1 \) of \( E \);

(ii) There exists a complete tripotent \( e \in E \) such that \( r(a) = e \) and \( a \) is positive and invertible in the JB*-algebras \( E_2(e) \).

We recall that two elements \( a, b \) in a JB*-triple, \( E \), are said to be orthogonal (written \( a \perp b \)) if \( L(a, b) = 0 \). Lemma 1 in [8] shows that \( a \perp b \) if and only if one of the following nine statements holds:

\[
\{a, a, b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b); \quad E^*_2(r(a)) \perp E^*_2(r(b)); \quad r(a) \in E^*_0(r(b)); \quad a \in E^*_0(r(b)); \quad b \in E^*_0(r(a)); \quad E_a \perp E_b \quad \{b, b, a\} = 0.
\]

(2.5)

Let \( e \) be a tripotent in a JB*-triple \( E \). Lemma 1.3(a) in [15] shows that

\[
\|x_2 + x_0\| = \max\{\|x_2\|, \|x_0\|\},
\]

for every \( x_2 \in E_2(e) \) and every \( x_0 \in E_0(e) \). Combining this result with the equivalences in (2.5) we see that

\[
\|a + b\| = \max\{\|a\|, \|b\|\},
\]

whenever \( a \) and \( b \) are orthogonal elements in a JB*-triple.

Given a subset \( M \subseteq E \), we write \( M^\perp_E \) (or simply \( M^\perp \)) for the (orthogonal) annihilator of \( M \) defined by \( M^\perp_E = \{y \in E : y \perp x, \forall x \in M\} \). If \( e \in E \) is a tripotent, then \( e^\perp = E_0(e) \), and \( a^\perp = (E^*0(r(a)))^\cap E \), for every \( a \in E \) (cf. [9, Lemma 3.2]).

**Lemma 3.** Let \( V \) be a non-zero Čebyšëv subspace of a JBW*-triple \( M \). Then \( V \cap M^{-1}_q \neq \emptyset \), where \( M^{-1}_q \) denotes the set of BP-quasi invertible elements of \( M \).

**Proof.** Arguing by contradiction, we suppose that \( V \cap M^{-1}_q = \emptyset \).

Let us take \( x \in V \) with \( \|x\| = 1 \). By assumptions, \( x \notin M^{-1}_q \). Under these conditions, the range complete tripotent of \( x \), \( r(x) \) is not complete in \( M \) or \( x \) is not invertible in the JBW*-algebra \( M_2(r(x)) \). By [19, Lemma 3.12], there exists a complete tripotent \( e \) in \( M \) such that \( r(x) \leq e \).

We shall identify the JB*-subtriple, \( M_x \), of \( M \) generated by \( x \) with some \( C_0(L) \) where \( 1 = \|x\| \in L \subseteq [0, 1]\) with \( L \cup \{0\} \) compact (cf. [22, 1.15]). We further know that there exists a triple isomorphism \( \Psi : M_x \rightarrow C_0(L) \) such that \( \Psi(x)(t) = t \), and the range tripotent \( r(x) \) identifies with the
characteristic function \( \chi_{[0,\|x\|]\setminus L} \in C_0(L)^{**} \) (see page 2). It is clear that, under this identification,

\[
\|r(x) - \lambda x\| = 1 - |\lambda| \inf \{|x(t)| : t \in L\} \leq 1,
\]

for every \(|\lambda| \leq 1\) in \(\mathbb{C}\). When \(e = r(x)\), the element \(x\) is not invertible in the JBW*-algebra \(M_2(r(x))\), and hence \(\|e - x\| = \|r(x) - x\| = 1\). When \(e \geq r(x)\), we have \(\|e - r(x)\| = 1\). Thus, applying \(e - r(x) \perp r(x)\) and (2.6), we further known that

\[
\|e - \lambda x\| = \|e - r(x) + r(x) - \lambda x\| = \max \{\|e - r(x)\|, \|r(x) - \lambda x\|\} = 1.
\]

We observe that, since \(e\) is a complete tripotent, \(e \in M_q^{-1}\), and hence \(e \notin V\). Since \(V\) is a Čebyšëv subspace, there exists a unique best approximation, \(c_\nu(e) \in V\), of \(e\) in \(V\) satisfying \(\text{dist}(e, V) = \|e - c_\nu(e)\| > 0\).

If \(\text{dist}(e, V) = \|e - c_\nu(e)\| \geq 1\), we would have \(1 = \|e\| \geq \text{dist}(e, V) = 1\), and

\[
1 = \|e - c_\nu(e)\| = \text{dist}(e, V) = \|e - \lambda x\|,
\]

for every \(|\lambda| \leq 1\), contradicting the uniqueness of the best approximation of \(e\) in \(V\). We can therefore assume that \(\text{dist}(e, V) < 1\). Consequently, there exits \(y \in V\) with \(|e - y| < 1\). Corollary 2.4. in [20] implies that \(y \in M_q^{-1} \cap V\), which is impossible. \(\square\)

Let \(e\) be a tripotent in a JB*-triple \(E\). Let us recall that \(e\) is a tripotent in the JBW*-triple \(E^{**}\), and that Peirce projections associated with \(e\) on \(E^{**}\) are weak*-continuous. Goldstine’s theorem assures that \(E\) is weak*-dense in \(E^{**}\), and hence, \(E_k^{**}(e)\) coincides with the weak*-closure of \(E_k(e)\) in \(E^{**}\), for every \(k = 0, 1, 2\). In particular, \(e\) is complete in \(E^{**}\) whenever \(e\) is a complete tripotent in \(E\). Moreover, since the orthogonal complement of a tripotent \(e\) in a JB*-triple \(F\) coincides with \(F_0(e)\), we have:

**Lemma 4.** Let \(e\) be a complete tripotent in a JB*-triple \(E\). Then \(\{e\}^{\perp}_{E^{**}} = \{0\}\), that is, \(e\) is not orthogonal to any non-zero element in \(E^{**}\). \(\square\)

The following technical result is part of the folklore in the theory of best approximation (see [28, Lemma 3] or [32, Theorem 2.1]).

**Lemma 5.** ([28, Lemma 3]). Let \(x\) be an element in complex a Banach space \(X\) such that \(\mathbb{C}x\) is not a Čebyšëv subspace of \(X\). Then there exists an extreme point \(\phi\) of the closed unit ball of \(X^*\), a vector \(y \in X\) and a scalar \(\lambda \in \mathbb{C}\setminus \{0\}\) such that

(a) \(\phi(x) = 0\);
(b) \(\phi(y) = \|y\| = \|y - \lambda x\|\). \(\square\)

We can characterize now the one dimensional Čebyšëv subspaces of a JBW*-triple.

**Theorem 6.** Let \(x\) be a non-zero element in a JBW*-triple \(M\). The following statements are equivalent:

(a) \(\mathbb{C}x\) is a Čebyšëv subspace of \(M\);
(b) $x$ is a Brown-Pedersen quasi-invertible element in $M$.

Proof. The implication $(a) \Rightarrow (b)$ follows from Lemmas 3.

$(b) \Rightarrow (a)$ Suppose $x$ is BP-quasi invertible in $M$. We note that the support tripotent, $r(x)$, of $x$ is complete in $M$, and hence a complete tripotent in $M^{**}$ (cf. Lemma 4 and comments before it).

Suppose that $Cx$ is not a Čebyšev subspace of $M$. By Lemma 5 there exists an extreme point $\phi$ of the closed unit ball of $M^*$, $\lambda \in \mathbb{C}\{0\}$, and $y \in M$ such that $\phi(x) = 0$ and $\phi(y) = \|y\| = \|y - \lambda x\|$.

The support tripotent $v = s(\phi)$ of $\phi$ in $M^{**}$ is a (non-zero) minimal tripotent in $M^{**}$ satisfying $\phi = P_2(v)^*\phi = \phi P_2(v)$ and $\phi(z)v = P_2(v)(z)$, $\forall z \in M^{**}$ (cf. [15, Proposition 4]). Therefore, $P_2(v)(x) = \phi(x)v = 0$.

We may suppose that $\|y\| = 1$. Since $P_2(v)(y) = \phi(y)v = v$, Lemma 1.6 in [15] implies that $P_1(v)(y) = 0$, which shows that $y = v + P_0(v)y$. We similarly get $P_1(v)(y - \lambda x) = 0$ (we simply observe that $\phi(y - \lambda x) = \|y\| = \|y - \lambda x\| = 1$). Therefore, $P_1(v)(x) = 0$, and $x = P_0(v)x \in (M^{**})_0(v) = ((M^{**})_2(v))^\perp$, implying that $x \perp v$. The equivalent statements in (2.5) prove that $r(x) \perp v$, which contradicts Lemma 4. \qed

The above Theorem 6 generalizes the previously commented results obtained by Robertson [28] (compare Theorem 1). In order to find a triple version of the reformulation established by Pedersen in [27, Theorem 2], stated as statement (c) in page 2, we recall some notation.

For each functional $\varphi$ in the predual of a JBW*-triple $W$, and for each $z$ in $W$ with $\varphi(z) = \|\varphi\|$, and $\|z\| = 1$, the mapping $x \mapsto \|x\|_{\varphi} := (\varphi\{x, x, z\})^{1/2}$ defines a pre-Hilbertian semi-norm on $W$. Moreover, $\varphi\{x, x, w\} = \varphi\{x, x, z\}$ whenever $w \in W$ with $\varphi(w) = \|\varphi\|$ and $\|w\| = 1$ (cf. [1, Proposition 1.2]). It is known that

$$(2.7) \quad \|\varphi(x)\| \leq \|x\|_{\varphi},$$

for every $x \in W$ (see [2, page 258]).

The inequality in (2.7) together with Lemma 5 imply the following property: Let $x$ be a non-zero element in a JBW*-triple $M$ such that $Cx$ is a Čebyšev subspace of $M$. Then for each extreme point $\varphi$ of the closed unit ball of $M^*$ we have $\|x\|_{\varphi} \geq 0$. It would be interesting to know under what additional hypothesis, the condition $\|x\|_{\varphi} \geq 0$, for every extreme point $\varphi$ of the closed unit ball of $M^*$, implies that $x$ is BP-quasi invertible.

3. Čebyšev Subtriples of JBW*-Triples

In this section, we shall determine the JBW*-subtriples of a JBW*-triple $M$ which are Čebyšev subspaces in $M$. Let us recall that in the case of an infinite dimensional von Neumann algebra $M$, if a finite dimensional von Neumann subalgebra $N$ of $M$ is a Čebyšev subspace in $M$ then $N$ must be one dimensional (compare Theorem 2 or [28, Theorem 6]). Furthermore, an infinite dimensional C*-algebra $A$ admits a finite dimensional *-subalgebra
which is also a Čebyšev in $A$ if and only if $A$ is unital and $B = \mathbb{C}1$ (cf. [29, Corollary 1.4]). The scarcity of non-trivial Čebyšev $C^\ast$-subalgebras in general $C^\ast$-algebras can be better understood with the following result due to G.K. Pedersen: If $A$ is a $C^\ast$-algebra without unit and $B$ is a Čebyšev $C^\ast$-subalgebra of $A$, then $A = B$ (compare [27, Theorem 4]).

The first main difference in the setting of JB$^\ast$-triples is the existence of Čebyšev JB$^\ast$-subtriples with arbitrary dimensions. For example, let $E = H$ be a complex Hilbert space regarded as a type 1 Cartan factor with the Hilbert norm and the product
\[
\{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x),
\]
where $\langle ., . \rangle$ denotes the inner product of $H$. The Orthogonal Projection theorem tells that any closed subspace of $H$ is a Čebyšev subspace of $H$ and clearly a JB$^\ast$-subtriple.

The following remark provides an additional example.

**Remark 7.** Let $E$ be a spin factor with triple product and norm given by
\[
\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},
\]
and $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - |\langle x/\bar{x} \rangle|^2}$, respectively, where $x \mapsto \bar{x}$ is a conjugation on $E$, and $\langle . . \rangle$ denotes the inner product of $E$. Let $K$ be a closed subspace of $E$ with $\overline{K} = K$. Clearly, $K$ is a JB$^\ast$-subtriple of $E$. Since $K$ is a closed subspace of the complex Hilbert space $E$, there exists an orthogonal projection $P$ of $E$ onto $K$. Since $E = K \bigoplus H$, where $H = (I - P)(E)$ with $\langle K/H \rangle = 0$. Since $\overline{K} = K$, we also have $\overline{P} = H$.

Given $\eta \in K$ and $\xi \in H$, it is easy to check that
\[
\|\eta + \xi\|^2 = \langle \eta + \xi/\eta + \xi \rangle + \sqrt{\langle \eta + \xi/\eta + \xi \rangle^2 - |\langle \eta + \xi/\bar{\eta} + \bar{\xi} \rangle|^2}
\]
\[
\geq \langle \eta/\eta \rangle + \langle \xi/\xi \rangle + \sqrt{\langle \eta/\eta \rangle^2 - |\langle \eta/\bar{\eta} \rangle|^2} + \langle \xi/\xi \rangle^2 - |\langle \xi/\bar{\xi} \rangle|^2
\]
Moreover, $\|\eta + \xi\|^2 = \|\eta\|^2$ if and only if $\xi = 0$. This shows that $P : E \to E$ is a bi-contractive for the norm $\| . \|$, and for each $x \in E$, $P(x)$ is the unique best approximation of $x$ in $K$. Therefore, $K$ is a Čebyšev JB$^\ast$-subtriple of $E$. We observe that the dimensions of $E$ and $K$ can be arbitrarily big.

The next property of Čebyšev subspaces is probably part of the folklore in the theory of best approximation in normed spaces, but we couldn’t find an exact reference.

**Lemma 8.** Let $V$ be a Čebyšev subspace of a normed space $X$. For each $x \in X$, we denote by $c_V(x)$ the unique element in $V$ satisfying $\|x - c_V(x)\| = \ldots$
dist}(x,V). Let \( P : X \to X \) be a contractive projection such that \( P(V) \subseteq V \). Then
\[
P(c_V(P(x))) = c_V(P(x)),
\]
for every \( x \in X \).

**Proof.** Let \( x \) be an element in \( X \). The condition \( \|P\| \leq 1 \) implies that
\[
\left\| P(x) - P(c_V(P(x))) \right\| \leq \left\| P(x) - c_V(P(x)) \right\| = \text{dist}(P(x), V).
\]
The element \( P(c_V(P(x))) \in P(V) \subseteq V \). Thus, the uniqueness of the best approximation in \( V \) proves that \( P(c_V(P(x))) = c_V(P(x)) \). \( \square \)

**Proposition 9.** Let \( F \) be a Čebyšev JB*-subtriple of a JB*-triple \( E \). Suppose \( e \) is a non-zero tripotent in \( F \). Then \( E_0(e) = F_0(e) \). Consequently, every complete tripotent in \( F \) is complete in \( E \).

**Proof.** Since \( e \) is a tripotent in \( F \) and the latter is a JB*-subtriple of \( E \), \( e \) is a tripotent in \( E \) and \( F_0(e) \subseteq E_0(e) \). Arguing by contradiction, let us assume that there exists \( b \in E_0(e) \setminus F_0(e) = E_0(e) \setminus F \neq \emptyset \). Since \( \text{dist}(b,F) > 0 \) and \( F \) is a Čebyšev subspace, there exists a unique \( c_F(b) \in F \) such that \( \|b - c_F(b)\| = \text{dist}(b,F) \).

Since \( F_0(e)(F) \subseteq F \) and \( F_0(e)(b) = b \), Lemma 8 implies that
\[
P_0(e)(c_F(b)) = c_F(b) \in F_0(e).
\]
Having in mind that \( e \in E_2(e) \perp E_0(e) \ni b - c_F(b) \), we deduce, via \((2.6)\), that
\[
\|b - c_F(b) - \lambda e\| = \max\{\|b - c_F(b)\|, |\lambda|\} = \|b - c_F(b)\| = \text{dist}(b,F),
\]
for every \( |\lambda| \leq \text{dist}(b,F) \). This contradicts the uniqueness of the best approximation, \( c_F(b) \), of \( b \) in \( F \), because \( c_F(b) + \lambda e \in F \) for every \( |\lambda| \leq \text{dist}(b,F) \). \( \square \)

**Proposition 10.** Let \( F \) be a Čebyšev JB*-subtriple of a JB*-triple \( E \). Suppose \( e \) is a tripotent in \( F \) with \( F_0(e) = \{e\}^F \neq 0 \). Then \( E_2(e) = F_2(e) \).

**Proof.** Clearly \( F_2(e) \subseteq E_2(e) \). We have to show that \( E_2(e) \subseteq F_2(e) \). Suppose, on the contrary, that \( E_2(e) \setminus F_2(e) = E_2(e) \setminus F \neq \emptyset \). Pick \( b \in E_2(e) \setminus F \). Since \( F \) is a Čebyšev subspace of \( E \), there exists a unique \( c_F(b) \in F \) satisfying \( \|b - c_F(b)\| = \text{dist}(b,F) > 0 \).

By Lemma 8 applied to \( P = P_2(e) \), \( X = E \) and \( V = F \), we deduce that \( P_0(e)(c_F(b)) = c_F(b) \).

By hypothesis, \( F_0(e) = \{e\}^F \neq 0 \). So, there exists a norm-one element \( z \in F_0(e) \). The conditions \( b, c_F(b) \in F_2(e) \) and \( z \in F_0(e) \) combined with \((2.6)\) give
\[
\|b - c_F(b) - \lambda z\| = \max\{\|b - c_F(b)\|, |\lambda|\} = \|b - c_F(b)\| = \text{dist}(b,F),
\]
for every $|\lambda| \leq \text{dist}(b, F)$, which contradicts the uniqueness of the best approximation of $b$ in $F$ because $c_F(b) - \lambda z \in F$, for every $\lambda$ in the above conditions. \hfill \Box

Let $e$ and $v$ be tripotents in a JB*-triple $E$. We shall say that $v \leq e$, when $e - v$ is a tripotent in $E$ with $e - v \perp v$ (compare the notation in [15]).

We state now a technical result.

**Proposition 11.** Let $e$ and $h$ be non-zero tripotents in a JB*-triple $E$ with $e \in E_1(h)$ and $h \in E_1(e)$. Suppose $e_1$ is another non-zero tripotent in $E$ satisfying $e_1 \leq e$. Let $C$ denote the JB*-subtriple of $M_{2 \times 4}(\mathbb{C})$ generated by $\overline{e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\overline{h} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $\overline{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and let $B$ be the JB*-subtriple of $E$ generated by $e$, $e_1$ and $h$. Then there exists a triple isomorphism $\Phi : B \rightarrow C$ satisfying $\Phi(e) = \overline{e}$, $\Phi(e_1) = \overline{e_1}$, and $\Phi(h) = \overline{h}$. Moreover $C$ coincides with

$$\left\{ \begin{pmatrix} \alpha & 0 & \gamma & 0 \\ 0 & \beta & 0 & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$ 

**Proof.** Since $e_1 \leq e$ (that is, $e - e_1$ is a tripotent and $e_1 \perp e - e_1$), we see that $E_2(e_1) \subset E_2(e)$. Let $e_2 = e - e_1$. Let $h = P_2(e_1)(h) + P_1(e_1)(h) + P_0(e_1)(h)$. By the Peirce rules,

$$E_2(e) \ni P_2(e_1)(h) = \{e_1, e_1, h\} \in E_{2-2+1}(e) = E_1(e),$$

and hence $P_2(e_1)(h) = 0$. Therefore, $h = P_1(e_1)(h) + P_0(e_1)(h)$. Since

$$\frac{1}{2} P_1(e_1)(h) + \frac{1}{2} P_0(e_1)(h) = \frac{1}{2} h = \{e,e,h\} = \{e_1, e_1, h\} + \{e_2, e_2, h\}$$

$$= \frac{1}{2} P_1(e_1)(h) + \{e_2, e_2, P_1(e_1)(h)\} + \{e_2, e_2, P_0(e_1)(h)\},$$

and by Peirce rules $\{e_2, e_2, P_1(e_1)(h)\} \subset E_{0-0+(e_1)}$, we deduce that

$$\{e_2, e_2, P_1(e_1)(h)\} = 0,$$

and $\{e_2, e_2, P_0(e_1)(h)\} = \frac{1}{2} P_0(e_1)(h)$.

That is, $P_1(e_1)(h) = E_0(e_2)$, $P_0(e_1)(h) = E_1(e_2)$. We similarly get

$$P_1(e_1)(h) = P_0(e_2)(h) \in E_0(e_2) \cap E_1(e_1),$$

$$P_0(e_1)(h) = P_1(e_2)(h) \in E_0(e_1) \cap E_1(e_2).$$

By Peirce rules,

$$E_0(e_1) \ni \{P_1(e_1)(h), P_1(e_1)(h), P_0(e_1)(h)\}$$

$$= \{P_0(e_2)(h), P_0(e_2)(h), P_1(e_2)(h)\} \in E_1(e_2).$$

Having in mind that $e_1 \perp e_2$, we can easily see that $E_0(e_1) \cap E_1(e_2) = \{0\}$. Therefore

$$\{P_1(e_1)(h), P_1(e_1)(h), P_0(e_1)(h)\} = \{P_0(e_2)(h), P_0(e_2)(h), P_1(e_2)(h)\} = 0,$$

which proves that $P_1(e_1)(h) \perp P_0(e_1)(h)$. Now, $h$ being a tripotent implies that $P_1(e_1)(h)$ and $P_0(e_1)(h)$ are orthogonal tripotents.
We also observe that
\[ \frac{1}{2} e = \{h, h, e\} = \{P_1(e_1)(h), P_1(e_1)(h), e\} + \{P_0(e_1)(h), P_0(e_1)(h), e\} \]
= (by Peirce rules) \( \{P_1(e_1)(h), P_1(e_1)(h), e\} + \{P_0(e_1)(h), P_0(e_1)(h), e\} \)
proving that \( \{P_1(e_1)(h), P_1(e_1)(h), e\} = \frac{1}{2} e_1 \) and \( \{P_0(e_1)(h), P_0(e_1)(h), e\} = \frac{1}{2} e_2 \).

Let us define \( \tilde{e}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), \( \tilde{h}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), and \( \tilde{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \). It is easy to check that \( B \equiv \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}h_1 \oplus \mathbb{C}h_2 \), and \( C \equiv \mathbb{C} \tilde{e}_1 \oplus \mathbb{C} \tilde{e}_2 \oplus \mathbb{C} \tilde{h}_1 \oplus \mathbb{C} \tilde{h}_2 \).

Finally, defining a linear bijection \( \Phi : B \to C \) satisfying \( \Phi(e_1) = \tilde{e}_1 \), \( \Phi(e_2) = \tilde{e}_2 \), \( \Phi(h_1) = \tilde{h}_1 \), and \( \Phi(h_2) = \tilde{h}_2 \), it is easy to check that \( \Phi \) is a triple isomorphism.

Let \( E \) be a JB*-triple. A subset \( S \subseteq E \) is said to be orthogonal if \( 0 \not\in S \) and \( x \perp y \) for every \( x \neq y \) in \( S \). The minimal cardinal number \( r \) satisfying \( \text{card}(S) \leq r \) for every orthogonal subset \( S \subseteq E \) is called the rank of \( E \) (and will be denoted by \( r(E) \)). Given a tripotent \( e \in E \), the rank of the Peirce-2 subspace \( E_2(e) \) will be called the rank of \( e \).

Theorem 3.1 in [4] combined with Proposition 4.5.(iii) in [6] assure that a JB*-triple is reflexive if and only if it is isomorphic to a Hilbert space if, and only if, it has finite rank.

Suppose \( E \) is a rank-one JB*-triple. The above comments show that \( E \) is reflexive and hence a JBW*-triple. Let \( e \) be a complete tripotent in \( E \). Since the rank of \( e \) is smaller than the rank of \( E \), we deduce that \( e \) is a minimal tripotent in \( E \). Proposition 3.7 in [9] and its proof show that \( E = \{e\}^{\perp\perp} = \{0\}^{\perp} \) is a rank-one Cartan factor of the form \( L(H, \mathbb{C}) \), where \( H \) is a complex Hilbert space or a type 2 Cartan factor \( II_3 \) (it is known that \( II_3 \) is JB*-triple isomorphic to a 3-dimensional complex Hilbert space). This shows the following:

**Lemma 12.** Every JB*-triple of rank one is JB*-isomorphic and hence isometric to a complex Hilbert space regarded as a type 1 Cartan factor. \( \square \)

The theorem describing the Čebyšev JBW*-subtriples of a JBW*-triple can be stated now. We shall show that the examples given in Remark 7 and the comments before it are essentially the unique examples of non-trivial Čebyšev JBW*-subtriples.

**Theorem 13.** Let \( N \) be a non-zero Čebyšev JBW*-subtriple of a JBW*-triple \( M \). Then exactly one of the following statements holds:
(a) \( N \) and \( M \) are rank one JBW*-triples. Moreover, \( M \) is a complex Hilbert space and \( N \) is a closed subspace of arbitrary dimension;
(b) \( N = \mathbb{C}e \), where \( e \) is a complete tripotent in \( M \);
(c) $N$ and $M$ have rank two, but $N$ may have arbitrary dimension;
(d) $N$ has rank greater or equal than 3 and $N = M$.

**Proof.** We can always find a complete tripotent $e$ in $N$ (see the comments in page 6). Proposition 9 implies that $e$ is complete in $M$ (i.e. $M_0(e) = \{0\}$).

We have three possibilities:

(i) $e$ has rank one in $N$;
(ii) $e$ has rank 2 in $N$;
(iii) $e$ has rank greater or equal than 3 in $N$.

(i) Suppose first that $e$ has rank one in $N$. In this case, $e$ is a minimal and complete tripotent in $N$. Therefore, $N$ is a complex Hilbert space regarded as a type 1 Cartan factor (cf. Lemma 12 or Proposition 3.7 in [9]).

We claim that $e$ has rank one in $M$ or $N_1(e) = \{0\}$.

Suppose, on the contrary to our claim, that $e$ is not rank one in $M$ and $N \neq Ce$ (i.e. $N_1(e) \neq \{0\}$). Then, we can find a non-zero tripotent $e_1 \in M$ with $e_1 \not\leq e$. If $N_1(e) \neq \{0\}$ (i.e. $N \neq Ce$), then we can find a non-zero tripotent $h$ in $N_1(e)$ satisfying $\{e, e, h\} = \frac{1}{2}h$ and $\{h, h, e\} = \frac{1}{2}e$ (just consider $N$ as a complex Hilbert space regarded as a type 1 Cartan factor).

Let us observe that $h$ is a complete tripotent in $N$, and hence it is also complete in $M$ (Proposition 9).

Let $C$ denote the JB*-subtriple of $M_{2 \times 4}(\mathbb{C})$ generated by

$$\tilde{e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tilde{e}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \end{pmatrix},$$

and let $B$ be the JB*-subtriple of $M$ generated by $e, e_1$ and $h$. By Proposition 11 there exists a triple isomorphism $\Phi : B \to C$ satisfying $\Phi(e) = \tilde{e}$, $\Phi(e_1) = \tilde{e}_1$, and $\Phi(h) = \tilde{h}$. Moreover $C$ coincides with

$$\left\{ \begin{pmatrix} \alpha & 0 & \gamma & 0 \\
0 & \beta & 0 & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$ 

We define $a_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \end{pmatrix} \in C \equiv B \subseteq M$.

By (2.4) the mappings

$$S_{-1}(e) = P_2(e) - P_1(e) + P_0(e) = P_2(e) - P_1(e)$$

and

$$S_{-1}(h) = P_2(h) - P_1(h) + P_0(h) = P_2(h) - P_1(h)$$

are surjective linear isometries on $M$. Thus, $\|S_{-1}(e)S_{-1}(h)(x)\| = \|x\|$, for every $x \in M$.

The JBW*-triple $N$ decomposes in the form $N = Ce \oplus Ch \oplus K$, where $K$ is the orthogonal complement of $\{e, h\}$ in the Hilbert space $N$. It can be easily checked that

$$S_{-1}(e)S_{-1}(h)(\lambda e + \mu h + k) = -\lambda e - \mu h + k,$$
and
\[ S_{-1}(e)S_{-1}(h)(a_0) = -a_0. \]

Therefore,
\[ \|a_0 - (\lambda e + \mu h)\| \leq \|a_0 - (\lambda e + \mu h + k)\|, \]
for every \( \lambda e + \mu h + k \in N \), showing that \( \text{dist}(a_0, N) = \text{dist}(a_0, Ce \oplus Ch) \).

Considering the identification of \( B \) and \( C \) given above, it is a good exercise to see that
\[
\text{dist}(a_0, N) = \text{dist}(a_0, Ce \oplus Ch) = \text{dist}(\tilde{a}_0, \tilde{C}e \oplus \tilde{C}h)
\]
\[
= \text{dist}\left( \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbb{C} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \sqrt{2}.
\]
However
\[
\|a_0 - h\| = \left\| \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\| = \sqrt{2}
\]
and
\[
\|a_0 - e\| = \left\| \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\| = \sqrt{2},
\]
which contradicts that \( a_0 \) admits a unique best approximation in \( N \). This proves the claim.

If \( e \) has rank one in \( M \), then \( M \) is a rank one JB*-triple and hence a complex Hilbert space regarded as a type 1 Cartan factor and we are in statement (a). If \( N = Ce \) (i.e. \( N_1(e) = \{0\} \)) we are in case (b).

(ii) We assume now that \( e \) has rank 2 in \( N \). Then there exist two non-zero minimal, mutually orthogonal tripotents \( e_1, e_2 \in N \) with \( e = e_1 + e_2 \). Propositions 9 and 10 show that \( M_2(e_j) = N_2(e_j) \), and \( M_0(e_j) = N_0(e_j) \), for every \( j \in \{1, 2\} \). Since \( e \) is complete in \( M \), we also have \( M = M_2(e) \oplus M_1(e) \). We shall prove the desired statement by showing that \( e_1 \) and \( e_2 \) are minimal in \( M \). The statement concerning the dimension of \( N \) follows from the example in Remark 7.

Suppose, contrary to our claim, that, for example \( e_2 \) is not minimal in \( M \). Then we can find two non-zero mutually orthogonal tripotents \( u_1, u_2 \) in \( M_2(e_2) \) with \( e_2 = u_1 + u_2 \). The projections \( P_2(e_1), P_2(u_1), P_2(u_2) : M \to M \) are contractive and have mutually orthogonal images in \( M \). By (2.6),
\[
\| (P_2(e_1) + P_2(u_1) + P_2(u_2))(u_1 - n) \| = \max\{\| P_2(e_1)(u_1 - n) \|, \| P_2(u_1)(u_1 - n) \|, \| P_2(u_2)(u_1 - n) \| \} \leq \| u_1 - n \|,
\]
for every \( n \in N \). It is easy to see that
\[
(P_2(e_1) + P_2(u_1) + P_2(u_2))(u_1) = u_1,
\]
and
\[
(P_2(e_1) + P_2(u_1) + P_2(u_2))(n) = \lambda e_1 + \mu e_2,
\]
for suitable \( \lambda, \mu \in \mathbb{C} \). Therefore
\[
\text{dist} \left( u_1, Ce_1 \oplus Ce_2 \right) = \inf \{\| u_1 - (\lambda e_1 + \mu e_2) \| : \lambda, \mu \in \mathbb{C} \} \leq \text{dist}(u_1, N).
\]
Moreover, by orthogonality, dist \((u_1, C e_2)\).

On the other hand,
\[
\|u_1 - \lambda e_2\| = \|(1 - \lambda)u_1 - \lambda(e_2 - u_1)\| = \max\{|1 - \lambda|, |\lambda|\},
\]
which implies that
\[
\text{dist}(u_1, N) = \text{dist}(u_1, C e_2) = \inf_{\lambda \in \mathbb{C}} \max\{|1 - \lambda|, |\lambda|\} = \frac{1}{2}.
\]

However, the elements for each \(|\mu| < \frac{1}{2}\), the element \(\mu e_1 + \frac{1}{2} e_2 \in N\) and the distance \(\|u_1 - (\mu e_1 + \frac{1}{2} e_2)\| = \frac{1}{2}\), contradicting the uniqueness of the best approximation of \(u_1\) in \(N\).

(iii) Suppose now that \(e\) has rank greater or equal than 3 in \(N\). We shall show that \(M = N\). Under the present assumptions, we can find three non-zero mutually orthogonal tripotents \(e_1, e_2, e_3\) with \(e_1 + e_2 + e_3 = e\). Clearly, \(N_0(e_j + e_k) \neq \{0\}\) for every \(k \neq j\) in \(\{1, 2, 3\}\). Propositions \(9\) and \(10\) assure that \(M_2(e_j + e_k) = N_2(e_j + e_k)\), \(M_0(e_j + e_k) = N_0(e_j + e_k)\), \(M_2(e_j) = N_2(e_j)\), and \(M_0(e_j) = N_0(e_j)\), for every \(k \neq j\) in \(\{1, 2, 3\}\). In the Peirce decomposition
\[
M = M_2(e_1 \oplus M_1(e_1 \oplus M_0(e_1)),
\]
we have \(M_2(e_1) = N_2(e_1)\) and \(M_0(e_1) = N_0(e_1)\). Pick \(x \in M_1(e_1)\). Since \(e_1 \perp e_j\ (j = 2, 3)\) we have \(M_1(e_1) \cap M_2(e_j) = \{0\}\) for every \(j = 2, 3\). Therefore
\[
x = P_1(e_2)(x) + P_0(e_2)(x),
\]
where \(P_0(e_2)(x) \in M_0(e_2) = N_0(e_2) \subseteq N\) and \(P_1(e_2)(x) \in P_1(e_2)(N_1(e_1))\).

Since
\[
\frac{1}{2} P_0(e_2)(x) + \frac{1}{2} P_1(e_2)(x) = \frac{1}{2} x = \{e_1, e_1, x\}
\]
\[
= \{e_1, e_1, P_0(e_2)(x)\} + \{e_1, e_1, P_1(e_2)(x)\},
\]
it follows from Pierce rules that
\[
\frac{1}{2} P_1(e_2)(x) = \{e_1, e_1, P_1(e_2)(x)\},
\]
and hence \(P_1(e_2)(x) \in M_1(e_1) \cap M_1(e_2)\). The condition \(e_1 \perp e_2\) leads us to \(\{e_1 + e_2, e_1 + e_2, P_1(e_2)(x)\} = P_1(e_2)(x)\), which means that
\[
P_1(e_2)(x) \in M_2(e_1 + e_2) = N_2(e_1 + e_2) \subseteq N.
\]
We have therefore shown that \(x = P_1(e_2)(x) + P_0(e_2)(x) \in N\), which implies that \(M_1(e_1) \subseteq N\) and consequently \(M = N\). This concludes the proof. \(\square\)

Let us recall that a \(C^*\)-algebra is reflexive if and only if it if finite dimensional (cf. [30, Proposition 2]). Consequently, a \(C^*\)-algebra has finite rank if and only if it is finite dimensional. In particular, the result established by Robertson in [28, Theorem 6] (see Theorem 2) is a direct consequence of our last corollary:
Corollary 14. Let $M$ be an infinite dimensional von Neumann algebra. Let $N$ be a Čebyšëv $JBW^*$-subtriple of $M$. Then $N = \mathbb{C}e$, where $e$ is a complete (maximal) partial isometry of $M$ or $M = N$. Accordingly, every proper Čebyšëv von Neumann subalgebra of $M$ is of the form $\mathbb{C}1$. □

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