Generalized Hirota Equations
and Representation Theory
I. The case of $SL(2)$ and $SL_q(2)$

A. Gerasimov, S. Khoroshkin\textsuperscript{1}, D. Lebedev\textsuperscript{2}

ITEP, Moscow, 117 259, Russia

A. Mironov\textsuperscript{3}

Theory Department, P. N. Lebedev Physics Institute, Leninsky prospect, 53,
Moscow, 117924, Russia

and

Niels Bohr Institute, Blegdamsvej, 17, DK-2100 Copenhagen 0, Denmark

A. Morozov\textsuperscript{4}

ITEP, Moscow, 117 259, Russia

\textsuperscript{1}E-mail address: khor@s43.msk.su
\textsuperscript{2}E-mail address: lebedev@vxdesy.desy.de
\textsuperscript{3}E-mail address: mironov@fian.free.net, mironov@nbivax.nbi.dk
\textsuperscript{4}E-mail address: morozov@vxdesy.desy.de
ABSTRACT

This paper begins investigation of the concept of “generalized $\tau$-function”, defined as a generating function of all the matrix elements of a group element $g \in G$ in a given highest-weight representation of a universal enveloping algebra $\mathcal{G}$. In the generic situation, the time-variables correspond to the elements of maximal nilpotent subalgebras rather than Cartanian elements. Moreover, in the case of quantum groups such $\tau$-“functions” are not c-numbers but take their values in non-commutative algebras (of functions on the quantum group $G$). Despite all these differences from the particular case of conventional $\tau$-functions of integrable (KP and Toda lattice) hierarchies (which arise when $G$ is a Kac-Moody (1-loop) algebra of level $k = 1$), these generic $\tau$-functions also satisfy bilinear Hirota-like equations, which can be deduced from manipulations with intertwining operators. The most important applications of the formalism should be to $k > 1$ Kac-Moody and multi-loop algebras, but this paper contains only illustrative calculations for the simplest case of ordinary (0-loop) algebra $SL(2)$ and its quantum counterpart $SL_q(2)$, as well as for the system of fundamental representations of $SL(n)$. 
1 Introduction

The theory of $\tau$-functions is now attracting increasing attention, because of their appearance in the role of non-perturbative partition functions of quantum field theories. So far it was shown (see [1],[2],[3] and references therein) that $\tau$-functions of conventional (perhaps, multicomponent) Toda-lattice hierarchy and its reductions (like KP, KdV, MKdV etc) play this role for certain 0-dimensional theories: $c < 1$ matrix models. However, it seems clear that this application should be much more general, though it is also clear that this requires revision of the very concept of $\tau$-function. The key to such generalization is provided by the group-theoretical interpretation of conventional $\tau$-functions [4],[5]. In this paper we shall follow the suggestion of [6],[3] and introduce a “$\tau$-function” for any Verma module $V$ of any algebra $G$ as a generating function for all the matrix elements $\langle k|g|n \rangle_V$:

$$
\tau_V\{t, \bar{t}|g\} = \left<V\left| \prod_{\alpha>0} e_q(t_\alpha T_\alpha) \right| g \prod_{\alpha>0} e_q(\bar{t}_\alpha T_{-\alpha})\right|0\left|V\right> = \\
= \sum_{k_\alpha \geq 0} \prod_{\alpha>0} \frac{k_\alpha!}{[k_\alpha]!} \prod_{n_\alpha \geq 0} \frac{n_\alpha!}{[n_\alpha]!} \langle k_\alpha|g|n_\alpha \rangle_V.
$$

(1)

Here $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]! = [1][2] \ldots [n]$, $e_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]}$. In the case of Lie algebras $q$-exponents are substituted by the ordinary ones. $T_{\pm \alpha}$ are generators of positive/negative maximal nilpotent subalgebras $N(G)$ and $\bar{N}(G)$ of $G$ with suitably chosen ordering of positive roots $\alpha$, and $t_\alpha$, $\bar{t}_\alpha = t_{-\alpha}$ are the associated “time-variables”. Vacuum state is annihilated by all the positive generators: $T_\alpha|0\rangle_V = 0$ for all $\alpha > 0$. Verma module $V = \{ | n_\alpha \rangle_V = \prod_{\alpha>0} T_{-\alpha}^{n_\alpha}|0\rangle_V \}$ is formed by the action of all the generators $T_{-\alpha}$ for all negative roots $-\alpha$ from maximal nilpotent subalgebra $\bar{N}(G)$. Except for special circumstances all the $\alpha \in N(G)$ are involved and since not all the $T_{-\alpha}$’s are commuting, the so defined $\tau$-function has nothing to do with Hamiltonian integrability (see [3] for detailed description of the specifics of $k = 1$ Kac-Moody

---

1 It can seem from [1] that $N(G)$ and $\bar{N}(G)$ are already fully spanned by “the time-related” elements of $G$ and thus the would-be “Grassmannian element” $g \in G$ can be restricted to belong to Cartan subgroup of $G$ only. This argument, however, is misleading in the quantum group case, when matrix elements belong to a non-commutative “coordinate ring” $A(G)$, while Cartanian $g$’s span no more than its commutative sub-ring. Our construction below does not require $g$ to be Cartanian. See section 4 for more details.
algebras in this context). However, this appears to be the only property of conventional \(\tau\)-functions which is not preserved by our general definition. In particular, the goal of this paper is to explain that bilinear (Hirota) equations are valid for our very general \(\tau\)-function (\[4\]). In this framework they become a very general feature of exact partition functions, essentially very close to bilinear completeness condition.

In [5] it was proposed to associate Hirota equations with Casimir operators in the tensor products of irreducible representations. However, this construction is not very straightforward in general situation (for any representation of any algebra). Also, when associated with non-quadratic Casimirs, it provides equations, which are polylinear rather than bilinear in \(\tau\) and are not very convenient to deal with. Instead of the Casimir-induced construction, we present here a very general and manifest construction, using intertwining operators, which can be also taken as an immediate generalization of the free-fermion approach of \[4\] and of projective functors of \[7\], and makes the whole story about bilinear equations very transparent and simple. The role of the fermion is played by arbitrary vertex operator taken as an intertwining operator into tensor product (see, for example, \[7\] for classical algebras and \[8\] for quantum ones). In this paper we actually describe the general scheme and present explicit calculations only for the simplest case of \(SL(2)\) and \(SL_q(2)\). We emphasize, however, that despite all the specifics of \(rank = 1\) algebras, the construction is absolutely general and does not rely upon any of these specific properties. Let us repeat that the key for applicability of the same construction for \(rank > 1\) case is inclusion of times for all the positive roots \(\alpha\) in the definition of \(\tau\). Sometimes (e.g. for fundamental representations of \(SL(n)\)) the system of bilinear equations can be reduced to the one, involving only smaller number of time-variables (as large as the rank of \(G\)). This brings us back to the field of Hamiltonian integrable systems.

Another aspect of the relation between our construction and conventional \(\tau\)-function is that our set of bilinear equations for \(SL(n)\) algebras can be splitted into that, peculiar for the Kac-Moody \(\widehat{SL(n)}\) case (i.e. arising in the theory of Toda equations) and a set of additional constraints (playing essentially the same role as Virasoro and \(W\)-constraints in matrix models), which specifies the particular finite-parametric solution of the form (\[4\]).
Section 2 describes the generic procedure to deduce bilinear relations. Explicit examples of these equations for \( G = SL_q(2) \) are considered in Section 3. Section 4 demonstrates that, despite the possible subtlety of the notion of “group element” for quantum groups, the set of solutions to bilinear equation for \( SL_q(2) \) is as big for \( q \neq 1 \) as it is for \( q = 1 \). Section 5 describes application of the general procedure to collection of all fundamental representations of \( G = SL(N) \). The corresponding equations are very close to those of the standard Toda-lattice hierarchy. It also contains some comments on quantum case.

2 From intertwining operators to bilinear equations

We suggest the following construction in terms of intertwining operators as a general source of bilinear equations for the \( \tau \)-function (1). One can easily recognize the standard free-fermion derivation of Hirota equations for KP/Toda \( \tau \)-functions as a particular example (for level \( k = 1 \) Kac-Moody algebras \( \hat{G}_{k=1} \), \( V \) is a fundamental representation, \( W \) is the simplest fundamental representation corresponding to the very left root of the Dynkin diagram). Construction below involves a lot of arbitrariness. In order to make the consideration more transparent, we formulate our construction explicitly for finite-dimensional Lie algebras and their \( q \)-counterparts. In the case of \( (q) \)-Affine algebras this actually coincides with that used in [8, 9].

Bilinear equations which we are going to derive are relating \( \tau \)-functions (1) for four different Verma modules \( V, \hat{V}, V', \hat{V}' \). Given \( V, V' \), every allowed choice of \( \hat{V}, \hat{V}' \) provides a separate set of bilinear identities. Of course, not all of these sets are actually independent and can be parametrized by source modules \( V \) and \( V' \) and by a weight of finite-dimensional representation. Also different choices of positive root systems and their ordering in (1) provides equations in somewhat different forms. A more invariant description of the minimal set of bilinear equations for given \( G \) would be clearly interesting to find.

1. Our starting point is embedding of Verma module \( \hat{V} \) into the tensor product \( V \otimes W \),
where $W$ is some irreducible finite-dimensional representation of $\mathcal{G}$. Once $V$ and $W$ are specified, there is only finite number of choices for $\hat{V}$.

Now we define right vertex operator of the $W$-type as homomorphism of $\mathcal{G}$-modules:

$$E_R : \hat{V} \longrightarrow V \otimes W.$$  \hfill (2)

This intertwining operator can be explicitly continued to the whole representation once this is constructed for its vacuum (highest-weight) state:

$$\hat{V} = \left\{ |n_\alpha\rangle_{\hat{V}} = \prod_{\alpha > 0} (\Delta(T_{-\alpha})^{n_\alpha} |0\rangle_{\hat{V}}) \right\},$$  \hfill (3)

where comultiplication $\Delta$ provides the action of $\mathcal{G}$ on the tensor product of representations\(^2\), and

$$|0\rangle_{\hat{V}} = \left( \sum_{\{p_\alpha,i_\alpha\}} A\{p_\alpha,i_\alpha\} \left( \prod_{\alpha > 0} (T_{-\alpha})^{p_\alpha} \otimes (T_{-\alpha})^{i_\alpha} \right) \right) |0\rangle_V \otimes |0\rangle_W.$$  \hfill (4)

For finite-dimensional $W$’s, this provides every $|n_\alpha\rangle_{\hat{V}}$ in a form of finite sums of states $|m_\alpha\rangle_V$ with coefficients, taking values in elements of $W$.

2. The next step is to take another triple, defining left vertex operator\(^4\):

$$\hat{E}_L : \hat{V} \longrightarrow W' \otimes V',$$  \hfill (5)

such that the product $W \otimes W'$ contains unit representation of $\mathcal{G}$. The projection to this unit representation

$$\pi : W \otimes W' \longrightarrow I$$  \hfill (6)

is explicitly provided by multiplication of any element of $W \otimes W'$ by

$$\pi = W\langle 0 | \otimes W'\langle 0 | \left( \sum_{\{i_\alpha,i'_\alpha\}} \pi\{i_\alpha,i'_\alpha\} \left( \prod_{\alpha > 0} (T_{+\alpha})^{-i_\alpha} \otimes (T_{+\alpha})^{-i'_\alpha} \right) \right).$$  \hfill (7)

\(^2\)In the case of Affine algebra, one should use evaluation representation – zero charge representation induced from finite-dimensional one – see the definition of vertex operator in [8].

\(^3\)For Lie algebra it is just $\Delta(T) = T \otimes I + I \otimes T$.

\(^4\)Note the change of ordering at the r.h.s., this is slightly different from $V' \otimes W'$ in the case of quantum groups.
Using this projection, if it is not occasionally orthogonal to the image of $E \otimes E'$, one can build a new intertwining operator

$$\Gamma : \hat{V} \otimes \hat{V}' \xrightarrow{E \otimes E'} V \otimes W \otimes W' \otimes V' \xrightarrow{I \otimes I} V \otimes V',$$

which possesses the property

$$\Gamma(g \otimes g) = (g \otimes g)\Gamma$$

for any group element $g$ such that

$$\Delta(g) = g \otimes g.$$

3. It now remains to take a matrix element of (9) between four states,

$$\langle k' | V \langle k | (g \otimes g)\Gamma | n \rangle_{\hat{V}} | n' \rangle_{\hat{V}'}, = \langle k' | V \langle k | (g \otimes g)\Gamma | n \rangle_{\hat{V}} | n' \rangle_{\hat{V}'},$$

and rewrite this identity in terms of generating functions (10).

For illustrative purposes we give now an explicit example of this calculation for the case of $SL_q(2)$. Formulas for $SL(2)$ arise in the limit $q = 1$.

3. Explicit equations for $SL_q(2)$

3.1 Bilinear identities

To begin with, fix the notations. We consider generators $T_+, T_-$ and $T_0$ of $U_q(SL(2))$ with commutation relations

$$q^{T_0}T_\pm q^{-T_0} = q^{\pm1}T_{\pm},$$

$$[T_+, T_-] = \frac{q^{2T_0} - q^{-2T_0}}{q - q^{-1}},$$

and comultiplication
\[ \Delta(T_\pm) = q^{T_0} \otimes T_\pm + T_\pm \otimes q^{-T_0}, \]
\[ \Delta(q^{T_0}) = q^{T_0} \otimes q^{T_0}. \]

Verma module \( V_\lambda \) with highest weight \( \lambda \) (not obligatory half-integer), consists of the elements

\[ |n\rangle_\lambda \equiv T^n_\lambda |0\rangle_\lambda, \quad n \geq 0, \quad \text{such that} \]
\[ T_-^n |n\rangle_\lambda = |n+1\rangle_\lambda, \]
\[ T_0^n |n\rangle_\lambda = (\lambda-n) |n\rangle_\lambda, \]
\[ T_+^n |n\rangle_\lambda \equiv b_n(\lambda) |n-1\rangle_\lambda, \]

\[ b_n(\lambda) = [n][2\lambda + 1 - n], \quad [x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}, \]
\[ ||n||^2_\lambda \equiv \lambda \langle n | n \rangle_\lambda = \frac{[n]! \Gamma_q(2\lambda + 1)}{\Gamma_q(2\lambda + 1 - n)} = \frac{[2\lambda]![n]!}{[2\lambda - n]!}. \]

Now,
\[ (\Delta(T_-))^n = q^{nT_0} \otimes T_-^n + [n] T_- q^{(n-1)T_0} \otimes T_-^{n-1} q^{-T_0} + \ldots + \]
\[ + [n] T_-^{n-1} q^{T_0} \otimes T_- q^{-(n-1)T_0} + T_-^n \otimes q^{-nT_0}. \]

Let us manifestly derive equations (11) taking for \( W \) an irreducible spin-\( \frac{1}{2} \) representation of \( U_q(SL(2)) \). Then \( \hat{V} = V_{\lambda \pm \frac{1}{2}}, V = V_\lambda \) and the highest weights of \( \hat{V} \) in \( W \otimes V \) or \( V \otimes W \) are:

\[ |0\rangle_{\lambda, \pm \frac{1}{2}} = |+\rangle |0\rangle_\lambda, \quad |+\rangle \equiv |0\rangle_{\frac{1}{2}}, \quad \text{or} \quad |0\rangle_\lambda |+\rangle; \]
\[ |0\rangle_{\lambda, -\frac{1}{2}} = |+\rangle |1\rangle_\lambda - q^{(\lambda+\frac{1}{2})}[2\lambda]|-\rangle |0\rangle_\lambda, \quad |\rangle \equiv |1\rangle_{\frac{1}{2}}, \quad \text{or} \quad (q \rightarrow q^{-1}) |1\rangle_\lambda |+\rangle - q^{-(\lambda+\frac{1}{2})}[2\lambda]|0\rangle_\lambda |\rangle. \]

Entire Verma module is generated by the action of \( \Delta(T_-) \):

\[ \text{Hereafter we omit the symbol of tensor product from the notations of the states } |+\rangle \otimes |0\rangle_\lambda \text{ etc.} \]
\[ |n\rangle_{\lambda+\frac{1}{2}} = (\Delta(T_-))^{\frac{n}{2}} |0\rangle_{\lambda+\frac{1}{2}} \rightarrow q^{n/2} \left( |+\rangle |n\rangle_{\lambda} + q^{-(\lambda+\frac{1}{2})} |n\rangle_{\lambda} |n-1\rangle_{\lambda} \right), \quad (19) \]

\[ \text{or} \quad q^{-n/2} \left( |n\rangle_{\lambda}|+\rangle + q^{(\lambda+\frac{1}{2})} |n-1\rangle_{\lambda} \langle n-1| \right); \]

\[ |n\rangle_{\lambda-\frac{1}{2}} = (\Delta(T_-))^{\frac{n}{2}} |0\rangle_{\lambda-\frac{1}{2}} \rightarrow q^{n/2} \left( |+\rangle |n+1\rangle_{\lambda} + q^{(\lambda+\frac{1}{2})} |n-2\lambda\rangle_{\lambda} - |n\rangle_{\lambda} \right), \quad (20) \]

\[ \text{or} \quad q^{-n/2} \left( |n+1\rangle_{\lambda}|+\rangle + q^{(\lambda+\frac{1}{2})} |n-2\lambda\rangle_{\lambda} \langle n-1| \right); \]

Step 2 to be made in accordance with our general procedure is to project the tensor product of two different \( W \)'s onto singlet state \( S = |+\rangle \langle -| - q |\rangle |+\rangle \langle -| \).

\[ (A|+\rangle + B|-\rangle) \otimes (|+\rangle C + |-\rangle D) \rightarrow AD - qBC. \quad (21) \]

With our choice of \( W \) we can now consider two different cases:

(A) both \( \hat{V} = V_{\lambda-\frac{1}{2}} \) and \( \hat{V}' = V_{\lambda'-\frac{1}{2}} \), or

(B) \( \hat{V} = V_{\lambda-\frac{1}{2}} \) and \( \hat{V}' = V_{\lambda+\frac{1}{2}} \):

**Case A:**

\[ |n\rangle_{\lambda-\frac{1}{2}} |n'\rangle_{\lambda'-\frac{1}{2}} \rightarrow q^{n'-n-1} \left( |n' - 2\lambda'|q^{\lambda'} |n+1\rangle_{\lambda'} |n'\rangle_{\lambda'} - |n - 2\lambda|q^{-\lambda} |n\rangle_{\lambda} |n+1\rangle_{\lambda} \right). \quad (22) \]

**Case B:**

\[ |n\rangle_{\lambda+\frac{1}{2}} |n'\rangle_{\lambda'-\frac{1}{2}} \rightarrow q^{n'-n+1} \left( |n' - 2\lambda'|q^{\lambda'} |n\rangle_{\lambda} |n'\rangle_{\lambda'} - |n|q^{\lambda+1} |n-1\rangle_{\lambda} |n'\rangle_{\lambda} + 1\rangle_{\lambda} \right). \quad (23) \]

---

6This state is a singlet of \( U_q(SL(2)) \). In the case of \( U_q(GL(2)) \) one should account for the \( U(1) \) non-invariance of \( S \). This is the origin of the factor \( \text{det}_q g \) at the r.h.s. of the final equations (20), (34) and (36).
Now we proceed to the step 3. Consider any “group element”, i.e. an element $g$ from some extension of $U_q(G)$, which possesses the property:

$$\Delta(g) = g \otimes g,$$  \hspace{1cm} (24)

and take matrix elements of the formula (9):

$$\lambda^\prime \langle k' | \lambda | k \rangle (g \otimes g) \Gamma = \Gamma g \otimes g | n \rangle \langle n' \rangle \hat{\lambda}^\prime.$$  \hspace{1cm} (25)

The action of operator $\Gamma$ can be represented as:

$$\Gamma | n \rangle \langle n' \rangle \hat{\lambda} = \sum_{l,l'} \langle l | \lambda \rangle \langle l' | \lambda \rangle \Gamma(l,l' | n,n'),$$  \hspace{1cm} (26)

and in these terms (25) turns into:

$$\sum_{m,m'} \Gamma(k,k' | m,m') \frac{||k||^2 ||k'||^2}{||m||^2 ||m'||^2} \langle m | g \rangle \langle m' | g \rangle | n \rangle \langle n' \rangle \hat{\lambda} = \sum_{l,l'} \langle k | g | l \rangle \lambda \langle k' | g | l' \rangle \lambda \Gamma(l,l' | n,n').$$  \hspace{1cm} (27)

In order to rewrite this as a difference equation, we use our definition of $\tau$-function:

$$\tau_{\lambda} (t, \bar{t} | g) = \langle \lambda | e^{it\lambda} g e^{i\bar{t}\lambda} | \lambda \rangle = \sum_{m,n} \langle m | g | n \rangle \lambda \frac{t^m \bar{t}^n}{[m]! [n]!}.$$  \hspace{1cm} (28)

Then, one can write down the generating formula for the equation (27), using the manifest form (22)-(23) of matrix elements $\Gamma(l,l' | n,n')$:

**Case A:**

$$\sqrt{M_+ M_-} \left( q^{\lambda^\prime} D_i^{(0)} D_i^{(2\lambda^\prime)} - q^{-\lambda} \bar{t} D_i^{(2\lambda)} D_i^{(0)} \right) \tau_{\lambda}(t, \bar{t} | g) \tau_{\lambda}(t', \bar{t}' | g) =$$

$$= [2\lambda][2\lambda'] (\det g) \sqrt{M_+ M_-} \left( q^{-(\lambda^\prime+\frac{1}{2})t'} - q^{(\lambda^\prime+\frac{1}{2})t} \right) \tau_{\lambda - \frac{1}{2}}(t, \bar{t} | g) \tau_{\lambda + \frac{1}{2}}(t', \bar{t}' | g).$$  \hspace{1cm} (29)

Here $D_i^{(\alpha)} \equiv \frac{q^{-\alpha} M_i - q^\alpha M_i}{(q - q^{-1}) t}$ and $M^\pm$ are multiplicative shift operators, $M_\pm f(t) = f(q^{\pm 1} t)$.

**Case B:**
\[
\sqrt{M_t M_\bar{t}} \left( q^{\lambda} t^2 D^{(2\lambda)}_b - q^{(\lambda+1)} \bar{t} D^{(0)}(0) \right) \tau_\lambda(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) = \\
\frac{[2\lambda']}{[2\lambda + 1]} \sqrt{M_t M_\bar{t}} \left( q^{\lambda} t^2 D^{(2\lambda+1)}_b - q^{\lambda} t' D^{(0)}(0) \right) \tau_{\lambda + \frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda - \frac{1}{2}}(t', \bar{t}'|g).
\]

Let us note that the derivation of these equations can be presented in the form more close to the original framework of [4]. Namely, we can realize operator \( \Gamma \) in component form as \( E_R^1 \otimes E_L^2 - q E_R^2 E_L^1 \), where \( E_i \)'s are components of the vertex operator\(^7\) (given by fixing different vectors from \( W \)). Then the equation (9) can be rewritten

\[
\hat{\mathcal{V}} \langle 0| e_q(t T_+ + \bar{t} T_-) E_R^1 g e_q(\bar{t} T_-)|0\rangle_V \cdot \hat{\mathcal{V}} \langle 0| e_q(t T_+ + \bar{t} T_-)|0\rangle_V' - \\
- q \hat{\mathcal{V}} \langle 0| e_q(t T_+ + \bar{t} T_-) E_L^2 g e_q(\bar{t} T_-)|0\rangle_V \cdot \hat{\mathcal{V}} \langle 0| e_q(t T_+ + \bar{t} T_-)|0\rangle_V' = \\
= \hat{\mathcal{V}} \langle 0| e_q(t T_+) g E_R^1 e_q(\bar{t} T_-)|0\rangle_V \cdot \hat{\mathcal{V}} \langle 0| e_q(t T_+) g E_L^2 e_q(\bar{t} T_-)|0\rangle_V - \\
- q \hat{\mathcal{V}} \langle 0| e_q(t T_+) g E_R^2 e_q(\bar{t} T_-)|0\rangle_V \cdot \hat{\mathcal{V}} \langle 0| e_q(t T_+) g E_L^1 e_q(\bar{t} T_-)|0\rangle_V'.
\]

We can easily obtain commutation relations of \( E_i \)'s with generators of algebra (see, for example, [8]) as well as their action on vacuum states. Then, it is immediately to commute \( E_i \)'s with \( q \)-exponentials in the expression (31) and to present the result of commuting by the action of difference operators, in complete analogy with the approach of [4]. Certainly, it reproduces the results (29) and (30).

### 3.2 Solutions of bilinear identities

Now let us consider manifest solutions of bilinear identities of the previous subsection. We start with the case of equation (29) and the simplest “group element” \( g = I \), which definitely satisfies (24). Then,

\[
\tau_\lambda = [1 + t\bar{t}]^{2\lambda} \equiv \sum_{i \geq 0} \frac{\Gamma_q(2\lambda + 1)}{\Gamma_q(2\lambda + 1 - i) \cdot i!} \frac{(t\bar{t})^i}{i}.
\]

\(^7\)These components \( E_i \)'s certainly correspond to fermions \( \psi_i \) in [4].
does indeed satisfy \( \text{(29)} \), since

\[
D_t^{(0)}[1 + t\bar{t}]^{2\lambda} = [2\lambda][1 + t\bar{t}]^{2\lambda - 1},
\]

\[
tD_t^{(2\lambda)}[1 + t\bar{t}]^{2\lambda} = -[2\lambda][1 + t\bar{t}]^{2\lambda - 1}.
\]

It is instructive now to write down the classical identity, which can be derived for \( G = SL(2) \) case and is trivially obtained from \( \text{(29)} \) in the limit \( q = 1 \):

\[
\left(2\lambda \frac{\partial}{\partial t'} - 2\lambda' \frac{\partial}{\partial t} + (t' - \bar{t}) \frac{\partial^2}{\partial t \partial t'} \right) \tau_\lambda(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) =
\]

\[
= 4\lambda \lambda'(\det g) (t' - t) \tau_{\lambda - \frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda' - \frac{1}{2}}(t', \bar{t}'|g).
\]

For \( g = I \) it has an evident solution \( \tau_\lambda = (1 + t\bar{t})^{2\lambda} \).

Of course the same \( \text{(32)} \) satisfies the second equation \( \text{(30)} \), which in the classical limit \( q = 1 \) looks like

\[
\left[(t' - \bar{t}) \frac{\partial}{\partial t'} - 2\lambda \right] \tau_\lambda(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) =
\]

\[
= \frac{2\lambda'}{2\lambda + 1} \left[(t - t') \frac{\partial}{\partial t} - 2\lambda - 1 \right] \tau_{\lambda + \frac{1}{2}}(t, \bar{t}|g) \tau_{\lambda' - \frac{1}{2}}(t', \bar{t}'|g).
\]

Thus we see that the same \( \tau \)-function satisfies differently looking equations, arising in the cases \( (A) \) and \( (B) \). Taking other representations to play the role of \( W \), one can derive a lot of other bilinear identities. Say, if \( W \) is the spin-1 representation \( W \) one obtains (in the classical case) the following identity:

\[
\left[ \frac{1}{2\lambda(2\lambda - 1)} \frac{\partial^2}{\partial t^2} + \frac{1}{2\lambda'(2\lambda' - 1)} \frac{\partial^2}{\partial t'^2} - \frac{1}{2\lambda \lambda'} \frac{\partial^2}{\partial t \partial t'} \right.
\]

\[
- \frac{1}{2\lambda \lambda'(2\lambda' - 1)} (t - \bar{t}') \frac{\partial^2}{\partial t \partial t'} + \frac{1}{2\lambda \lambda'(2\lambda - 1)} (t - \bar{t}) \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial t'} + \frac{1}{4\lambda \lambda'(2\lambda' - 1)(2\lambda - 1)} (t - \bar{t}')^2 \frac{\partial^2}{\partial t'^2} \frac{\partial}{\partial t'} \right]
\]

\[
\tau_\lambda(t, \bar{t}|g) \tau_{\lambda'}(t', \bar{t}'|g) =
\]

\[
= (t - t')^2 (\det g)^2 \tau_{\lambda - 1}(t, \bar{t}|g) \tau_{\lambda' - 1}(t', \bar{t}'|g).
\]

---

\footnote{We use systematically notations \( \partial \) for usual derivatives and \( D \) – for \( q \)-difference ones.}
For $SL(2)$ all new identities are, however, corollaries of the first ones.

Now let us note that the general solution of equation (34) is in fact a 3-parametrical one:

$$
\tau_\lambda(t, \bar{t}|g) = (a + bt + c\bar{t} + d\bar{t}\bar{\tau})^{2\lambda},
$$

which is just explicit answer for (28), where $a$, $b$, $c$ and $d$ are elements of the $2 \times 2$ matrix, parametrizing the group element $g$ of $GL(2)$. We shall see in the next section that for $q \neq 1$ the set of solutions is still 3-parametric, if considering in appropriate setting.

However, before we proceed to this issue, it deserves mentioning, what is the relation of our equations to the Liouville one, which is also sometimes associated with $SL(2)$ - but in a slightly different sense. First of all, as all the Hirota-like equations, (34) can be rewritten as a (system of) ordinary differential equations, when expanded in powers of $\epsilon = \frac{1}{2}(t - t')$ and $\bar{\epsilon} = \frac{1}{2}(\bar{t} - \bar{t}')$. For example, for $\lambda = \lambda'$ we obtain from (34):

1. Coefficient in front of $\epsilon$:

$$
\partial \tau_\lambda \bar{\partial} \tau_\lambda - \tau_\lambda \partial \bar{\partial} \tau_\lambda = 2\lambda \tau^{2}_{\lambda - \frac{1}{2}};
$$

2. Coefficient in front of $\bar{\epsilon}$:

$$
2\lambda \tau^{2}_{\lambda} = (2\lambda - 1)(\bar{\partial} \tau_\lambda)^{2};
$$

... (38)

If $\lambda = \frac{1}{2}$, the first one of these is just Liouville equation:

$$
\partial \tau_{\frac{1}{2}} \bar{\partial} \tau_{\frac{1}{2}} - \tau_{\frac{1}{2}} \partial \bar{\partial} \tau_{\frac{1}{2}} = \tau^{2}_{0} = 1,
$$

or

$$
\partial \bar{\partial} \phi = 2e^\phi, \quad \tau_{\frac{1}{2}} = e^{-\phi/2},
$$

while the second one,

$$
\bar{\partial}^{2} \tau_{\frac{1}{2}} = 0,
$$

is very restrictive constraint. Its role is to reduce the huge set of solutions to Liouville equation,

$$
\tau_{\frac{1}{2}}(t, \bar{t}|g) = (1 + A(t)B(i)) \left[ \frac{\partial A \partial B}{\partial t \partial \bar{t}} \right]^{-\frac{1}{2}},
$$

(42)
parametrized by two arbitrary functions $A(t)$ and $B(\bar{t})$, to the 3-parametric family (33). In this sense it plays the same role as the string equation (or Virasoro and W-constraints) in matrix models. In the language of infinite-dimensional Grassmannian there are infinitely many ways to embed $SL(2)$ group into $GL(\infty)$ - and all the cases correspond to solutions (12) (with some $A(t)$ and $B(\bar{T})$) to the $SL(2)$ reduced Toda-lattice hierarchy (i.e. Liouville equation), - but the constraint (11) specifies very concrete embedding: that in the left upper corner of $GL(\infty)$ matrix (associated with linear functions $A(t)$ and $B(\bar{t})$).

4 $\tau$-function for any $g \in SL_q(2)$

4.1 General solution of $SL_q(2)$ bilinear equations

In the previous section we derived Hirota-like equations for the $SL_q(2)$ $\tau$-functions. Formally these equations were derived for any “group element” $g$, but so far we have explicitly checked only that they are indeed satisfied by a $\tau$-function at $g = I$, $\tau_\lambda(t, \bar{t}|g = I) = [1 + t\bar{t}]^{2\lambda}$. It is of course easy to check this as well for any $g$ from Cartan subgroup. However, restricting $g$ to Cartan subgroup, we allow consideration of only (mutually) commuting matrix elements $\langle k|g|n \rangle$ in (1). The purpose of this section is to show explicitly that there is no need in such restriction: the same equations are satisfied for arbitrary choice of matrix elements from the non-commutative coordinate ring $A(G)$.

This result, when compared with the derivation of Hirota equations, implies that the set of the “group elements”, i.e. solutions to $\Delta(g) = g \otimes g$, should be larger than Cartan subgroup. This is indeed true, if we consider $g$ as belonging to $U_q(G|A(G))$, which is defined as the universal enveloping algebra (i.e. the set of formal series in generators $T_\pm$ and $q^{\pm T_0}$) with coefficients in $A(G)$. Unfortunately we are not aware of any explicit construction of this type and present just an illustrative example in section 4.2.

\footnote{To our knowledge, the idea to study non-commutative $\tau$-functions was first proposed in [1].}
Let us begin with the case of $\lambda = \frac{1}{2}$. Then

$$\tau_{\frac{1}{2}}(t, \bar{t}|g) = \langle +|g|+ \rangle + \bar{t}\langle +|g|+ \rangle + t\langle -|g|+ \rangle + t\bar{t}\langle -|g|+ \rangle =$$

$$= a + b\bar{t} + ct + dt\bar{t},$$

where $a, b, c, d$ are elements of the matrix

$$\mathcal{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(44)

with the commutation relations dictated by $\mathcal{T}T\mathcal{R} = \mathcal{R}\mathcal{T}\mathcal{T}$ equation [11]

$$ab = qba,$$

$$ac = qca,$$

$$bd = qdb,$$

$$cd = qdc,$$

$$bc = cb,$$

$$ad - da = (q - q^{-1})bc.$$  

If $b$ or $c$ or both are non-vanishing, $\tau_{\frac{1}{2}}(t, \bar{t}|g)$ with different values of time-variables $t, \bar{t}$ do not commute. Still such $\tau_{\frac{1}{2}}(t, \bar{t}|g)$ does satisfy the same bilinear identity (29), moreover, for this to be true it is essential that commutation relations (45) are exactly what they are. Indeed, the l.h.s. of the equation (29) (using (33) and (45)) is equal to

$$- q^{\frac{1}{2}} \sqrt{M_{\bar{t}}} (b + dt) \sqrt{M_{\bar{t}}} (a + ct') + q^{-\frac{1}{2}} \sqrt{M_{\bar{t}}} (a + ct) \sqrt{M_{\bar{t}}} (b + dt') =$$

$$= (q^{-\frac{1}{2}}ab - q^{\frac{1}{2}}ba) + (q^{-\frac{1}{2}}cd - q^{\frac{1}{2}}dc)tt' + (q^{-\frac{1}{2}}cd - q^{\frac{1}{2}}da)t + (q^{-\frac{1}{2}}ad - q^{\frac{1}{2}}bc)t' =$$

$$= (q^{-\frac{1}{2}}tt' - q^{\frac{1}{2}}t)\det qg,$$

which coincides with the r.h.s. of the equation (29).

To perform the similar check for any half-integer-spin representation, let us note that the
corresponding $\tau$-function can be easily written in terms of $\tau_{1/2}$. Indeed,

$$
|n\rangle_{1} = (q^{T_{0}} \otimes T_{-} + T_{-} \otimes q^{-T_{0}})^{n}|0\rangle_{1/2} = q^{-n/2} (|n\rangle_{1/2} \otimes |0\rangle_{1/2} + [n]q^{\lambda}|n-1\rangle_{1/2} \otimes |1\rangle_{1/2})
$$

(47)

$$
\lambda \langle n| = \lambda_{1/2}\langle 0| \otimes |\lambda\rangle_{1/2} = (q^{T_{0}} \otimes T_{+} + T_{+} \otimes q^{-T_{0}})^{n} = q^{-n/2} (\lambda_{1/2}\langle n| \otimes |\lambda\rangle_{1/2} + [n]q^{\lambda}\lambda_{1/2}\langle n-1| \otimes |1\rangle_{1/2})
$$

Thus

$$
\lambda\langle k|g|n\rangle_{\lambda} = q^{\frac{-n}{2}\lambda} \left[ \lambda_{1/2}\langle k|g|n\rangle_{1/2} \langle +|g|+ \rangle + q^{\lambda}[n] \lambda_{1/2}\langle k|g|n-1\rangle_{1/2} \langle +|g|\rangle + q^{\lambda}[k] \lambda_{1/2}\langle k-1|g|n\rangle_{1/2} \langle -|g|+ \rangle + q^{2\lambda}[k][n] \lambda_{1/2}\langle k-1|g|n-1\rangle_{1/2} \langle -|g|\rangle \right]
$$

(48)

or, in terms of generating ($\tau$-)functions:

$$
\tau_{\lambda}(t, \bar{t}|g) = \sqrt{M_{\bar{t}}/M_{t}} \left( \tau_{\lambda-1/2}(t, \bar{t}|g) \left( a + q^{\lambda}\bar{t}b + q^{\lambda}tc + q^{2\lambda}\bar{t}d \right) \right).
$$

(49)

Applying this procedure recursively we get:

$$
\tau_{\lambda}(t, \bar{t}|g) = \tau_{\lambda-1/2}(q^{-\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g)\tau_{\lambda}(q^{\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g) = 
$$

(50)

if $\lambda \in \mathbb{Z}/2$, then

$$
\tau_{\lambda}(q^{\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g)\tau_{\lambda}(q^{\frac{3}{2}}t, q^{\frac{3}{2}}\bar{t}|g) \ldots \tau_{\lambda}(q^{\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g),
$$

i.e. for half-integer $\lambda$ $\tau_{\lambda}$ is a polynomial of degree $2\lambda$ in $a, b, c, d$.

For example,

$$
\tau_{1}(t, \bar{t}|g) = \tau_{\frac{1}{2}}(q^{-\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g)\tau_{\frac{1}{2}}(q^{\frac{1}{2}}t, q^{\frac{1}{2}}\bar{t}|g) = 
$$

(51)

$$
= (a + q^{\frac{1}{2}}\bar{t}b + q^{\frac{1}{2}}tc + q^{-1}\bar{t}d)(a + q^{\frac{1}{2}}\bar{t}b + q^{\frac{1}{2}}tc + qt\bar{d}) = 
$$

$$
= a^{2} + (q^{\frac{1}{2}}ab + q^{-\frac{1}{2}}ba)\bar{t} + (q^{\frac{1}{2}}ac + q^{-\frac{1}{2}}ca)t + b^{2}\bar{t}^{2} + 
$$

$$
+ (qad + bc + q^{-1}da)\bar{t} + c^{2}t^{2} + (q^{\frac{1}{2}}bd + q^{-\frac{1}{2}}db)t\bar{t}^{2} + 
$$

$$
+ (q^{\frac{1}{2}}cd + q^{-\frac{1}{2}}dc)t^{2}\bar{t} + d^{2}t^{2}\bar{t}^{2}
$$

Using the relations like

$$
q^{\frac{1}{2}}ab + q^{-\frac{1}{2}}ba = [2]q^{\frac{1}{2}}ba = [2]q^{-\frac{1}{2}}ab \text{ etc.}
$$

(52)
one gets for this case
\[
\tau_1(t, \bar{t}|g) = a^2 + [2]q^{-\frac{1}{2}}ab\bar{t} + [2]q^{-\frac{1}{2}}act + b^2\bar{t}^2 + ([2]qbc + [2]da)t\bar{t} + c^2t^2 +
\]
\[+ [2]q^{-\frac{1}{2}}db\bar{t}^2 + [2]q^{-\frac{1}{2}}dct^2\bar{t} + d^2t^2\bar{t}^2.\]

With this explicit expression, one can trivially make the calculations similar to (refcheck) in order to check manifestly equation (29) for \(\lambda = 1, \lambda' = \frac{1}{2} \) \(, 1\) and equation (30) for \(\lambda = \lambda' = \frac{1}{2} \).

Thus, we showed explicitly (for the case of \(SL_q(2)\)) that the quantum bilinear identities have as many solutions as the classical ones, provided the \(\tau\)-function is allowed to take values in non-commutative ring \(A(G)\).

### 4.2 Comment on the notion of “group element”

To avoid possible confusion we present here an illustrative example of how the elements \(g\) from the universal enveloping algebra could look like, in order to guarantee that the “group coproduct” \(\Delta(g) = g \otimes g\) commutes with any intertwining operator. In its turn the intertwining operators are defined as commuting with the “algebra coproduct” \(\Delta(T) \neq T \otimes I + I \otimes T\).

Construction of such elements \(g\) becomes possible if \(g\) is allowed to take values in \(U_q(G|A(G))\): the universal enveloping algebra with coefficients in the non-commutative “coordinate ring” \(A(G)\) (i.e. made out of \(a, b, c, d\)-like objects). Unfortunately we know such \(g\) only in a form which strongly depends on particular representation: more universal formulas still remain to be found.

As everywhere in this paper, we restrict our example to the simplest case: this time to a product of two spin-\(\frac{1}{2}\) representations of \(U_q(SL(2))\), still this will be enough to illustrate the most obscure points of the story.

So the claim is that the “group element” in a spin-\(\frac{1}{2}\) representation is equal to
\[
g = \alpha q^{T_0} + \beta q^{-T_0} + bT_+ + cT_-,
\]
with
\[
\alpha = \frac{aq^{\frac{1}{2}} - dq^{-\frac{1}{2}}}{q - q^{-1}}, \quad \beta = \frac{-aq^{-\frac{1}{2}} + dq^{\frac{1}{2}}}{q - q^{-1}}.
\]
Then
\[
\Delta(g) = g \otimes g = \alpha^2 q^{T_0} \otimes q^{T_0} + \beta^2 q^{-T_0} \otimes q^{-T_0} + \\
+ \alpha \beta q^{T_0} \otimes q^{-T_0} + \beta \alpha q^{-T_0} \otimes q^{T_0} + bc(T_+ \otimes T_- + T_- \otimes T_+) + \\
+ (\alpha q^{T_0} + \beta q^{-T_0}) \otimes bT_+ + bT_+ \otimes (\alpha q^{T_0} + \beta q^{-T_0}) + \\
+ (\alpha q^{T_0} + \beta q^{-T_0}) \otimes cT_- + cT_- \otimes (\alpha q^{T_0} + \beta q^{-T_0}).
\]  
(56)

The r.h.s. does not look like any combination of \( \Delta(T_\pm) = q^{T_0} \otimes T_\pm + T_\pm \otimes q^{-T_0} \) and \( \Delta(q^{nT_0}) = q^{nT_0} \otimes q^{nT_0} \) and it can be a source of confusion. In fact, when acting on the product of two spin-\( \frac{1}{2} \) representations, this expression is just the same as

\[
\left( (aq - d) \Delta(q^{T_0}) + (a - qd) \Delta(q^{-T_0}) - (q + 1)(a - d) \Delta(I) \right) \frac{(a - d)}{(q - 1)(q - q^{-1})} + \\
+ (ad - qbc) \Delta(I) + \frac{bc}{2} \left( q \Delta(T_+) \Delta(T_-) + q^{-1} \Delta(T_-) \Delta(T_+) \right) + \\
+ \frac{q^{-\frac{1}{2}}(a - d)b \Delta(q^{T_0}) - q^\frac{3}{2}b(a - d) \Delta(q^{-T_0})}{q - q^{-1}} \Delta(T_+) + \\
+ \frac{q^\frac{1}{2}(a - d)c \Delta(q^{T_0}) - q^{-\frac{3}{2}}c(a - d) \Delta(q^{-T_0})}{q - q^{-1}} \Delta(T_-)
\]  
(57)

Once this identity is established, it is clear that \( g \) satisfies all the necessary requirements. For it to be true, the actual commutation relations between \( a, b, c, d \) are of course important.

The simplest way to check the identity is just to see explicitly how both expressions act on the four states \(|+\rangle \otimes |+\rangle, \ |+\rangle \otimes |-\rangle, \ |-\rangle \otimes |+\rangle, \ |-\rangle \otimes |-\rangle \), and see that the action is the same. A more instructive way is to observe that, in a given representation, \( q^{nT_0} \) with different \( n \) are, in fact, linearly dependent operators. In particular, in the spin-\( \frac{1}{2} \) representation

\[
q^{T_0} = -q^{-T_0} + s, \\
q^{2T_0} = sq^{T_0} - 1, \\
q^{-2T_0} = -sq^{T_0} + s^2 - 1
\]  
(58)

with \( s = q^{\frac{1}{2}} + q^{-\frac{1}{2}} \) (for spin-\( j \) representation there are exactly \( 2j \) linearly independent operators \( q^{nT_0} \) with integer \( n \)). Using these relations together with

\[
q^{T_0}T_+ = q^{\frac{1}{2}}T_+,
\]  
(59)

17
it is easy to prove the equivalence of two expressions in a formal way.

When higher-spin representations are considered, explicit expressions for group elements are different (they are (homogeneous) polynomials of degree $2j$ in $a, b, c, d$ and in $T_{\pm}$). We are not yet aware of explicit universal formula (for all representations at once, i.e. containing projectors, expressed through $T$’s) even for the $U_q(SL(2))$ case, and it is an interesting open problem to find it for all other groups.

5 Fundamental representations

5.1 The set of fundamental representations of $SL(n)$

This is another important example of our general construction, which is the most close one to conventional case of integrable hierarchies. The reason for this is that, in variance with generic Verma modules for group $G \neq SL(2)$, the fundamental representations are generated by subset of *mutually commuting* operators, not by entire set of generators from maximal nilpotent subalgebra. We describe the basic construction for $G = SL(n)$, since in this case the (finite) Grassmannian construction is the most similar to the conventional infinite-dimensional ($G = \hat{U}(1)$) situation.

There are as many as $r \equiv \text{rank } G = n - 1$ fundamental representations of $SL(n)$. They are the most conveniently described in the language of Young tableaux, namely, in the following way. Let us begin with the simplest fundamental representation $F$ - the $n$-plet, consisting of the states

$$\psi_i = T_{-}^{i-1}|0\rangle_F, \quad i = 1, \ldots, n. \quad (60)$$

Here the distinguished generator $T_-$ is essentially a sum of those for all the $r$ *simple* roots of $G$: $T_- = \sum_{i=1}^{r} T_{-\alpha_i}$. Then all the other fundamental representations $F^{(k)}$ are defined as skew powers of $F = F^{(1)}$:

$$F^{(k)} = \left\{ \Psi_{i_1 \ldots i_k}^{(k)} \sim \psi_{i_1} \ldots \psi_{i_k} \right\} \quad (61)$$
From this description it is clear that $0 \leq k \leq n$, moreover $F^{(0)}$ and $F^{(n)}$ are respectively the singlet and dual singlet representations. $F^{(k)}$ is essentially generated by the operators

$$R_k(T^i_i) \equiv T^i_i \otimes I \otimes \ldots \otimes I + I \otimes T^i_i \otimes \ldots \otimes I + I \otimes I \otimes \ldots \otimes T^i_i.$$  \hspace{1cm} (62)

These operators commute with each other. It is clear that for given $k$ exactly $k$ of them (with $i = 1, \ldots, k$) are independent (note that $R_k(T^i_i) \neq ((R_k(T^j_j))^\dagger)$).

The intertwining operators which are of interest for us are

$$I_{(k)} : F^{(k+1)} \longrightarrow F^{(k)} \otimes F,$$

$$I'_{(k)} : F^{(k-1)} \longrightarrow F^* \otimes F^{(k)} , \text{ and}$$

$$\Gamma_{k|k'} : F^{(k+1)} \otimes F^{(k'-1)} \longrightarrow F^{(k)} \otimes F^{(k')}.$$  \hspace{1cm} (63)

Here

$$F^* = F^{(r)} = \{ \psi^i \sim e^{\psi_{i_1, \ldots, i_r}} \psi_{i_1, \ldots, i_r} \},$$

$$I_{(k)} : \Psi_{i_1, \ldots, j_k+1}^{(k+1)} = \Psi_{i_1, \ldots, i_k}^{(k)} \psi_{i_k+1},$$

$$I'_{(k)} : \Psi_{i_1, \ldots, i_k}^{(k-1)} = \Psi_{i_1, \ldots, i_{k-1}}^{(k)} \psi_i;$$

and $\Gamma_{k|k'}$ is constructed with the help of embedding $I \longrightarrow F \otimes F^*$, induced by the pairing $\psi_i \psi^i$: the basis in linear space $F^{(k+1)} \otimes F^{(k'-1)}$, induced by $\Gamma_{k|k'}$ from that in $F^{(k)} \otimes F^{(k')}$ is:

$$\Psi_{i_1, \ldots, i_k}^{(k)} \Psi_{i'_{k+1}, \ldots, i'_{k'-1}}^{(k')}.$$  \hspace{1cm} (65)

Operation $\Gamma$ can be now rewritten in terms of matrix elements

$$g^{(k)} \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_k \end{array} \right) \equiv \langle \Psi_{i_1, \ldots, i_k} | g | \Psi_{j_1, \ldots, j_k} \rangle = \det_{1 \leq a, b \leq k} \tilde{g}^{i_a}_{j_b} \hspace{1cm} \text{(66)}$$

as follows:

$$g^{(k)} \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_k \end{array} \right) g^{(k') \dagger} \left( \begin{array}{c} i'_1 \ldots i'_k \\ j'_1 \ldots j'_{k'-1} \end{array} \right) =$$

$$= g^{(k+1)} \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_k \end{array} \right) g^{(k'-1)} \left( \begin{array}{c} i'_1 \ldots i'_{k'-1} \\ j'_1 \ldots j'_{k'-1} \end{array} \right).$$  \hspace{1cm} (67)

This is the explicit expression for eq.(9) in the case of fundamental representations, and it is certainly identically true for any $g^{(k)}$ of the form (66).

\[\text{To see this directly it is enough to rewrite the l.h.s. of (67) as:}\]

$$g^{(k)} \left( \begin{array}{c} i_1 \ldots i_k \\ j_1 \ldots j_k \end{array} \right) g^{(k') \dagger} \left( \begin{array}{c} i'_1 \ldots i'_{k'-1} \\ j'_1 \ldots j'_{k'-1} \end{array} \right)$$

\[\text{of the form (66).}\]
Let us note that one can easily construct from the minors (66) local coordinates in the Grassmannian [12]. Then, these coordinates satisfy a set of (bilinear) Plucker relations [12], which are nothing but defining equations of the Grassmannian consisting of all \( k \)-dimensional vector subspaces of an \( n \)-dimensional vector space. Parametrizing determinants (66) by time variables (see (74)), one get a set of bilinear differential equations on the generating function of these Plucker coordinates, which is just \( \tau \)-function [13].

Now let us introduce time-variables and rewrite (67) in terms of \( \tau \)-functions. We shall denote time variables through \( s_i, \bar{s}_i, i = 1, \ldots, r \) in order to emphasize their difference from generic \( t_\alpha, \bar{t}_\alpha \) labeled by all the positive roots \( \alpha \) of \( G \). Note that in order to have a closed system of equations we need to introduce all the \( r \) times \( s_i \) for all \( F^{(k)} \) (though \( \tau^{(k)} \) actually depends only on \( k \) independent combinations of these).

Since the highest weight of representation \( F^{(k)} \) is identified as

\[
|0\rangle_{F^{(k)}} = |\Psi_{1...k}^{(k)}\rangle,
\]

we have:

\[
\tau^{(k)}(s, \bar{s} \mid g) = \langle \Psi_{1...k}^{(k)} | \exp \left( \sum_i s_i R_k(T^i_+) \right) g \exp \left( \sum_i \bar{s}_i R_k(T^i_-) \right) | \Psi_{1...k}^{(k)} \rangle. \quad (69)
\]

Now,

\[
\exp \left( \sum_i s_i R_k(T^i) \right) = \exp \left( R_k \left( \sum_i s_i T^i \right) \right) = \left( \exp \left( \sum_i s_i T^i \right) \right)^\otimes_k = \left( \sum_j P_j(s) T^j \right)^\otimes_k, \quad (70)
\]

where we used the definition of the Schur polynomials

\[
\exp \left( \sum_i s_i z^i \right) = \sum_j P_j(s) z^j, \quad (71)
\]

their essential property being:

\[
\partial_{s_i} P_j(s) = (\partial_{s_i})^i P_j(s) = P_{j-i}(s). \quad (72)
\]

(expansion of the determinant \( g^{(k)} \) in the first column) and now the first two factors can be composed into \( g^{(k+1)} \) (expansion of the determinant \( g^{(k+1)} \) in the first row), thus giving the r.h.s. of (67).
Because of (70), we can rewrite the r.h.s. of (69) as
\[
\tau^{(k)}(s, \bar{s} \mid g) =
\]
\[
= \sum_{i_1, \ldots, i_k, j_1, \ldots, j_k} P_{i_1}(s) \cdots P_{i_k}(s) \langle \Psi^{(k)}_{1+i_1,2+i_2,\ldots,k+i_k} \mid g \mid \Psi^{(k)}_{1+j_1,2+j_2,\ldots,k+j_k} \rangle P_{j_1}(\bar{s}) \cdots P_{j_k}(\bar{s}) =
\]
\[
= \det_{1 \leq \alpha, \beta \leq k} H^\alpha_\beta(s, \bar{s}),
\]
where
\[
H^\alpha_\beta(s, \bar{s}) = \sum_{i,j} P_{i-\alpha}(s) g^i_j P_{j-\beta}(\bar{s}).
\]
This formula can be considered as including infinitely many times \(s_i\) and \(\bar{s}_i\), and it is only due to the finiteness of matrix \(g^i_j \in SL(n)\) that \(H\)-matrix is additionally constrained
\[
\left( \frac{\partial}{\partial s_1} \right)^n H^\alpha_\beta = 0,
\]
\[
\ldots
\]
\[
\frac{\partial}{\partial s_i} H^\alpha_\beta = 0, \text{ for } i \geq n.
\]
The characteristic property of \(H^\alpha_\beta\) is that it satisfies the following “shift” relations (see (72)):
\[
\frac{\partial}{\partial s_i} H^\alpha_\beta = H^\alpha_{\beta+1}, \quad \frac{\partial}{\partial \bar{s}_i} H^\alpha_\beta = H^\alpha_{\beta+1}.
\]
Expressions (73), (74) and (76) are, of course, familiar from the theory of KP and Toda hierarchies (see [2, 3] and references therein).

Coming back to bilinear relation (67), it can be easily rewritten in terms of \(H\)-matrix: it is enough to convolute them with Schur polynomials. For the sake of convenience let us denote
\[
H^{(\alpha_1, \ldots, \alpha_k)}_{(\beta_1, \ldots, \beta_k)} = \det_{1 \leq a, b \leq k} H^{\alpha_a}_{\beta_b}. \text{ In accordance with this notation } \tau^{(k)} = H^{(1, \ldots, k)}_{(1, \ldots, k)}, \text{ while bilinear equation turns into:}
\]
\[
H^{(\alpha_1, \ldots, \alpha_k)}_{(\beta_1, \ldots, \beta_k)} H^{(\alpha'_1, \ldots, \alpha'_{k-1})_{(\beta'_1, \ldots, \beta'_{k-1})}}{\beta_{k+1]}_{\beta_{k+1}}} = H^{(\alpha_1, \ldots, \alpha_k[\alpha'_{k-1})_{(\beta_1, \ldots, \beta_{k+1}]}_{\beta_{k+1}}} H^{(\alpha'_1, \ldots, \alpha_{k-1})}. \quad (77)
\]
Just like original (67) these are just matrix identities, valid for any \(H^\alpha_\beta\). However, after the switch from \(g\) to \(H\) we, first, essentially represented the equations in \(n\)-independent form and, second, opened the possibility to rewrite them in terms of time-derivatives.
For example, in the simplest case of

\[ \alpha_i = i, \quad i = 1, \ldots, k'; \]
\[ \beta_i = i, \quad i = 1, \ldots, k + 1; \]
\[ \alpha'_i = i, \quad i = 1, \ldots, k - 1, \quad \alpha'_k = k + 1; \]
\[ \beta'_i = i, \quad i = 1, \ldots, k - 1 \]

we get:

\[ H \left( \begin{array}{c} \cdots k \\ \cdots k \end{array} \right) H \left( \begin{array}{c} k + 1, 1 \ldots k - 1 \\ k + 1, 1 \ldots k - 1 \end{array} \right) - \]
\[ - H \left( \begin{array}{c} \cdots k - 1, k \\ \cdots k - 1, k + 1 \end{array} \right) H \left( \begin{array}{c} k + 1, 1 \ldots k - 1 \\ k, 1 \ldots, k - 1 \end{array} \right) = \]
\[ = H \left( \begin{array}{c} \cdots k + 1 \\ \cdots k + 1 \end{array} \right) H \left( \begin{array}{c} \cdots k - 1 \\ \cdots k - 1 \end{array} \right) \]

(all other terms arising in the process of symmetrization vanish). This in turn can be represented through \( \tau \)-functions:

\[ \partial_1 \bar{\partial}_1 \tau^{(k)} \cdot \tau^{(k)} - \bar{\partial}_1 \tau^{(k)} \partial_1 \tau^{(k)} = \tau^{(k+1)} \tau^{(k-1)}. \]

This is the usual lowest Toda-lattice equation. For finite \( n \) the set of solutions is labeled by \( g \in SL(n) \) as a result of the additional constraints (75).

We can now use the chance to illustrate the ambiguity of definition of \( \tau \)-function, or, to put it differently, that in the choice of time-variables. Eq.(80) is actually a corollary of two statements: the basic identity (74) and the particular definition (1), which in this case implies (74) with \( P \)'s being ordinary Schur polynomials (71). At least, in this simple situation (of fundamental representations of \( SL(n) \)) one could define \( \tau \)-function not by eq.(1), but just by eq.(73), with

\[ H_{\alpha}^{\alpha}(s, \bar{s}) \longrightarrow \mathcal{H}_{\beta}^{\alpha}(s, \bar{s}) = \sum \mathcal{P}_{i-\alpha}(s) g_i^j \mathcal{P}_{j-\beta}(\bar{s}) \]

with any set of independent functions (not even polynomials) \( \mathcal{P}_\alpha \). Such

\[ \tau_P^{(k)} = \det_{1 \leq \alpha, \beta \leq k} \mathcal{H}_{\beta}^{\alpha} \]

22
still remains a generating function for all matrix elements of $G = SL(n)$ in representation $F^{(k)}$. This freedom should be kept in mind when dealing with “generalized $\tau$-functions”. As a simple example, one can take $P_{\alpha}(s)$ to be $q$-Schur polynomials,

$$\prod_i e_q(s_i z^i) = \sum_j P_j^{(q)}(s) z^j, \text{ or}$$

$$\prod_i e_q^q(s_i z^i) = \sum_j P_j^{(q)}(s) z^j,$$

which satisfy (hereafter we denote $D \equiv D^{(0)}$)

$$D \cdot P_j^{(q)}(s) = (D_{s_1})^i P_j^{(q)}(s) = P_j^{(q)}(s_{j-i}).$$

Then instead of (76) we would have:

$$D \cdot H_{\alpha \beta} = H_{\alpha \beta} + i \cdot \delta_{\alpha \beta},$$

and

$$\tau^{(k)}(s, \overline{s} | g) = \det_{1 \leq \alpha, \beta \leq k} D_{s_1}^{(k-1)} D_{s_1}^{(k-1)} H_1(s, \overline{s}).$$

So defined $\tau$-function satisfies difference rather than differential equations [14, 15]:

$$\tau^{(k)} \cdot D_{s_1} \tau^{(k)} - D_{s_1} \tau^{(k)} \cdot \tau^{(k-1)} \cdot M^+ M^+ \tau^{(k+1)} = \tau^{(k)} \cdot M^+ \tau^{(k+1)},$$

\ldots .

We emphasize, however, that, in a sense, this is just a redefinition of the $SL(n)$, not $SL_q(n)$ $\tau$-function, as generating function of matrix elements. In particular, this $\tau$-function is a $c$-rather than $q$-number function. Still it would have something to do with $SL_q(n)$ group, but as a function of times, i.e. rather in spirit of the connection of $q$-hypergeometric functions to quantum groups (see, for example, [16]).

5.2 Approach to $SL_q(n)$

In order to extend this reasoning to the case of $SL_q(n)$ with $q \neq 1$ we need to go into some more details about the group structure.
The main thing we shall need is the notion of \( q \)-antisymmetrization, to be defined as a sum over all perturbations,
\[
([1, \ldots, k]_q) = \sum_{P} (-q)^{\text{deg } P} (P(1), \ldots, P(k)),
\]
where
\[
\text{deg } P = \# \text{ of inversions in } P.
\]

The first place to use this notion is the definition of \( q \)-determinant:
\[
\det_q A \sim A_{[1]}^{[1]} \cdots A_{[n]}^{[n]} = \sum_{P, P'} (-q)^{\text{deg } P + \text{deg } P'} \prod_{a} A_{P(a)}^{P'(a)},
\]

Note that this is not necessarily the same as \( A_{[1]}^{[1]} \cdots A_{[n]}^{[n]} \), for example for \( n = 2 \) gives
\[
\frac{1}{2} (A_1^1 A_2^2 - q A_2^1 A_1^2 - q A_1^2 A_2^1 + q^2 A_2^2 A_1^1),
\]
while single \( q \)-antisymmetrization would give just \( A_1^1 A_2^2 - q A_2^1 A_1^1 \). Moreover, \( A_{[1]}^{[1]} A_{[2]}^{[2]} = A_1^1 A_2^2 - q A_2^1 A_1^1 \) does not need to vanish, and even \( A_{[1]}^{[1]} A_{[1]}^{[1]} = (1 - q)(A_1^1)^2 \neq 0 \).

The “normal” properties of \( q \)-antisymmetrization are restored only when \( A \) is considered as being an element of \( GL_q(n) \). This means that its elements take values in the non-commutative ring \( A(GL(n)) \) and the following commutation relations - essentially the same as (15) - are imposed:
\[
\forall i, i_1 < j_2, A_{[i_1]}^{[i]} A_{[j_1]}^{[j]} = 0, \quad (ab = qba, cd = qdc)
\]
\[
\forall i_1 < i_2, \forall j A_{[i_1]}^{[j]} A_{[i_2]}^{[j]} = 0, \quad (ac = qca, bd = qdb)
\]
\[
\forall i_1 \neq i_2, j_1 \neq j_2 A_{[i_1]}^{[j_2]} A_{[j_1]}^{[i_2]} = A_{[j_1]}^{[j_2]} A_{[i_1]}^{[i_2]}, \quad (bc = cb)
\]
\[
\forall i_1 < i_2, j_1 < j_2 A_{[i_1]}^{[j_2]} A_{[j_1]}^{[i_2]} = (q - q^{-1}) A_{[j_2]}^{[i_1]} A_{[i_1]}^{[j_2]}, \quad (ad - da) = (q - q^{-1})bc).
\]
For \( A \in GL_q(n) \),
\[
\det_q A = A_{[1]}^{[1]} \cdots A_{[n]}^{[n]} = A_{[1]}^{[1]} \cdots A_{[n]}^{[n]},
\]
and \( A_{[1]}^{[1]} \cdots A_{[k]}^{[k]} = 0 \), if any two of the upper indices coincide (but only provided the lower ones are all \( \neq \) different: it is still true that \( A_{[1]}^{[1]} A_{[1]}^{[1]} = (1 - q)(A_1^1)^2 \neq 0 \) even for \( A \in GL_q(n) \)).

24
The notion of $q$-antisymmetrization is important for us because the $k$-th fundamental representation $F^{(k)}$ of $SL_q(n)$ is the $q$-skew power of $F = F^{(1)}$: for $q \neq 1$ we have instead of (61):

$$F^{(k)} = \left\{ \Psi_{i_1...i_k}^{(k)} \sim \psi_{[i_1\ldots i_k]} q, \quad i_1 < \ldots < i_k \right\}$$

(93)

Now it is necessary to request explicitly that all $i_a$ are different.

All the formulas (63)–(65) for intertwining operators remain exactly the same with antisymmetrization replaced by $q$-antisymmetrization (and obvious definition of $q$-$\epsilon$-symbol).

Instead of (66) we have:

$$g^{(k)} \left( i_1\ldots i_k \atop j_1\ldots j_k \right) \sim \det_q g_{j_a i_a}, \quad i_1 < \ldots i_k, \text{ or } j_1 < \ldots j_k,$$

(94)

and (67) turns into:

$$g^{(k)} \left( i_1\ldots i_k \atop j_1\ldots j_k \right) g^{(k')} \left( i'_1\ldots i'_{k'} \atop j_{k+1}\ldots j'_{k'-1} \right) = g^{(k+1)} \left( i_1\ldots i_k \atop j_1\ldots j_{k+1} \right) g^{(k'-1)} \left( i'_1\ldots i'_{k'-1} \atop j'_1\ldots j'_{k'-1} \right),$$

$$i_1 < \ldots < i_k, \quad \text{or} \quad j_1 < \ldots j_k, \quad i'_1 < \ldots < i'_{k'}, \quad \text{or} \quad j_{k+1} < j'_1 < \ldots < j'_{k'-1},$$

(95)

$$i_1 < \ldots < i_k < i'_{k'}, \quad \text{or} \quad j_1 < \ldots < j_{k+1}, \quad i'_1 < \ldots < i'_{k'-1}, \quad \text{or} \quad j'_1 < \ldots < j'_{k'-1}.$$

(96)

Just as (67) this is nothing but an identity for the matrices from $GL_q(n).$ A feature which is essentially new as compared to (67) is explicit appearance of restrictions on indices $i, j, i', j'$, which makes the translation to the language of generating functions a little more sophisticated.

---

11 It is crucially important here that $g$ indeed belongs to some $GL_q(n)$, i.e. its elements have the proper commutation relations. This is of course implied by the derivation of (12), where $g$ is supposed to be a “group element”. Be it not the case, we would need to understand (14) in the sense of (10) (in particular, to write and instead of or in (11)) and then we would run in a contradiction with (11). To make it more transparent, let us take $k = k' = 1$, then (15) becomes:

$$g^{(1)}_{[j_1\ldots j_2]} = g^{(2)} \left( i \atop j_1 \right) \left( i' \atop j_2 \right)$$

(97)

and the l.h.s. is equal to $g_{j_1}^i g_{j_2}^{i'} - q g_{j_2}^i g_{j_1}^{i'}$, while the r.h.s. would be interpreted as $g_{j_1}^i g_{j_2}^{i'} - q g_{j_2}^i g_{j_1}^{i'} - q g_{j_1}^i g_{j_2}^{i'} + q^2 g_{j_2}^i g_{j_1}^{i'}$. 

25
Let us note that, similar to the classical case, one can construct from the quantum minors \[94\] local coordinates on the quantum flag space \[17\]. As before, in the quantum case there is a set of (quantum) bilinear Plucker relations. Unfortunately, the problems arise when parametrizing Plucker coordinates by time variables.

Just to give an impression of what the result can be, when somehow expressed in terms of \(H\), let us restrict ourselves to the case of \(G = SL_q(2)\) and introduce:

\[
H^1_1 = \tau_F = a + b\bar{t} + ct + d\bar{t}.
\]

If

\[
H^1_2 = D_t H^1_1 = b + dt,
\]

\[
H^2_1 = D_t H^1_1 = c + d\bar{t},
\]

\[
H^2_2 = D_t D_t H^1_1 = d,
\]

we see that \(H_0^a\) is actually not lying in \(GL_q(2)\) (for example, \(H^1_2 H^1_1 \neq H^2_1 H^1_2\)), i.e. a matrix consisting of the \(\tau_F\) and its derivatives, despite these are all elements of \(A(G)\), does not longer belong to \(G_q\). Thus, it is not reasonable to consider \(\text{det}_q H\) (or the definition of \(H\) should be somehow modified). Instead the appropriate formula for the case of \(SL_q(2)\) looks like

\[
\tau_F^{(2)} = \text{det}_q g = H^1_1 H^2_2 - qH^1_2 M^{-} H^2_1 = \tau_F D_t D_t \tau_F - q D_t \tau_F M^{-} D_t \tau_F.
\]

We shall not go into more discussion of transition from \(g\)-identity to \(H\)-identities, because it involves some art in the work with appropriate time-variables, and is not yet brought to a reasonably simple form. Instead we present a few more formulas, which can be illuminating for some readers.

### 5.3 Comments on the quantum case

The first thing we wish to give some more details on is the statement \[13\].

The Lie algebra \(SL(n)\) is generated by operators \(T_{\pm\alpha}\) and Cartan operators \(H_\beta\), such that \([H_\beta, T_{\pm\alpha}] = \pm \frac{1}{2}(\alpha \beta)T_{\pm\alpha}\). All elements of all representations are eigenfunctions of \(H_\beta\),
\[ H_\beta |\lambda\rangle = \frac{1}{2} (\beta \lambda) |\lambda\rangle. \] The highest weight of representation \( F^{(k)} \) is \( \mu_k \). Vectors \( \mu_k \)'s are “dual” to the simple roots \( \alpha_i, i = 1, \ldots, r: (\mu_i \alpha_j) = \delta_{ij}, \) and \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_i \mu_i \).

Representation \( F^{(1)} \) consists of the states
\[
\psi_i = T_{-(i-1)} \cdots T_{-2} T_{-1} \psi_1, \quad i = 1, \ldots, n.
\] (101)

Moreover
\[
T_{-i} \psi_j = \delta_{ij} \psi_{i+1}
\] (102)
(thus, for \( T_- = \sum_{i=1}^r T_{-i} \) \( T_-^i \psi_j = \psi_{j+i} \) and (60) follows), and
\[
\lambda(\psi_i) = \mu_1 - \alpha_1 - \cdots - \alpha_i.
\] (103)

Here \( T_{\pm i} \equiv T_{\pm \alpha_i} \) are generators, associated with the simple roots. Let us denote the corresponding basis in Cartan algebra \( H_i = H_{\alpha_i} \), and \( H_{\pm} |\lambda\rangle = \frac{1}{2} (\alpha_i \lambda) |\lambda\rangle = \lambda_i |\lambda\rangle \). Then
\[
\lambda_i^{(j)} \equiv \lambda_{i}(\psi_j) = \frac{1}{2} (\delta_{ij} - \delta_{i,j-1}).
\] (104)

This formula, together with (101) and (103) implies that \( ||\psi_i||^2 = 1 \), and, since comultiplication formula in the classical case is just \( \Delta(T) = T \otimes I + I \otimes T \), it is obvious that \( \psi[1 \ldots \psi_k] \) are all highest weight vectors (i.e are annihilated by all \( \Delta_k (T_{+i}) \) and, thus by all the \( \Delta_k (T_{+\alpha}) \)).

Quantum universal enveloping algebra \( U_q(SL(n)) \) is generated by \( T_{\pm i} \) and \( q^{\pm H_i} \) with basic commutation relations (\( a_{ij} \) is the Cartan matrix of \( SL(n) \))
\[
q^{H_i} T_{\pm j} q^{-H_i} = q^{\pm a_{ij}} T_{\pm j},
\]
\[
[T_{+i}, T_{-j}] = \delta_{ij} \frac{q^{2H_i} - q^{-2H_i}}{q - q^{-1}}
\] (105)
and comultiplication law
\[
\Delta(T_{\pm i}) = q^{H_i} \otimes T_{\pm i} + T_{\pm i} \otimes q^{-H_i},
\]
\[
\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}.
\] (106)

Comultiplication formulas for \( T_{\pm \alpha} \) in the case of non-simple roots are corollaries of these and look more sophisticated. For example, for the “height 2” \( \alpha \), such that \( T_{\pm \alpha} = \pm [T_{\pm \alpha}, T_{\pm \alpha+1}], \)
we have
\[
\Delta(T-\alpha) = -[\Delta(T\alpha), \Delta(T_{\alpha+1})] = q^{H\alpha} \otimes T\alpha + T\alpha \otimes q^{-H\alpha} + 
(q^{1/2} - q^{-1/2}) \left[ (T-\alpha \otimes T_{\alpha+1})(q^{H_{\alpha+1}} \otimes q^{-H_i}) - (T_{\alpha+1} \otimes T_{\alpha})(q^{H_i} \otimes q^{-H_{\alpha+1}}) \right].
\] (107)

Given multiplication formulas, one can easily check that indeed (93) is true. For example, for \( F^{(2)} \):
\[
\Delta(T+i)(\psi_1 \psi_2 - q \psi_2 \psi_1) = \delta_{i,1}(q^{\lambda^{(1)}_i} \psi_1 \psi_1 - q^{1-\lambda^{(1)}_i} \psi_1 \psi_1) = 0,
\] (108)
because \( \lambda^{(1)}_1 = \frac{1}{2} \). Thus \( \psi^{(2)} \equiv \psi_{[1][2]} \) is indeed the highest weight vector. Similarly \( (i < j) \):
\[
\Delta(T-i)\psi^{(2)}_{ij} = \Delta(T-i)(\psi_i \psi_j - q \psi_j \psi_i) = 
\delta_{ij}q^{-\lambda^{(j)}_i} (\psi_{i+1} \psi_j - q^{1+2\lambda^{(j)}_i} \psi_j \psi_{i+1}) + \delta_{ij}q^{\lambda^{(i)}_j} (\psi_i \psi_{j+1} - q^{1-2\lambda^{(i)}_j} \psi_{j+1} \psi_i).
\] (109)
According to (104) for \( i < j \) \( \lambda^{(i)}_j = 0 \) in all cases, while \( \lambda^{(j)}_i \neq 0 \) only if \( j = i + 1 \), and \( \lambda^{(i+1)}_i = -\frac{1}{2} \). Thus,
\[
\Delta(T-i)\psi^{(2)}_{ij} = \delta_{ij}\psi^{(2)}_{i,j+1} + \delta_{ij}\psi^{(2)}_{i+1,j}(1 - \delta_{i,j+1}).
\] (110)
The rules for the action of all \( \Delta(T-\alpha) \) follow from this. It is easy to describe explicitly the action of all \( \Delta(T+i) \) and also to do the same for all other representations \( F^{(k)} \).

Our second comment concerns the relevance of \( q \)-exponents in eq. (1) in the quantum group case. Of course, they are primarily needed in order to obtain the Hirota equations in the nice form of difference equations. This, however, does not fix the choice completely. Indeed, there are various ways to define \( q \)-numbers and thus \( q \)-derivatives and \( q \)-special functions. These ways are in correspondence with the choices of comultiplication in quantum algebra. Through this text we are using “symmetric” coproduct (13), thus \( q \)-numbers need to be defined in symmetric way as well. For given definition of \( q \)-numbers there is still an ambiguity in the choice of \( q \)-exponent. In (1) and in section 3 we used the simplest definition \( e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \), satisfying \( D_x e_q(x) = e_q(x) \).
\[
\text{12One can also keep in mind the possibility to consider } \alpha \text{-dependent } q \text{ in (1)} \text{ (see for example } [13]). \text{ The reasonable choices of } q_\alpha \text{ could be } q_\alpha = q^{\alpha^2/2} \text{ or } q_\alpha = q^{||\alpha||}, \text{ where } ||\alpha|| \text{ is the “height” of the root } \alpha, ||\alpha|| = |\alpha| = \alpha_+. \]
However, it can appear more reasonable in some situations to use more sophisticated choices, for example,

$$E_q(x) = \sum_n q^{n(n-1)/2} \frac{x^n}{[n]!}, \text{ satisfying } D_x E_q(x) = M^+ E_q(x) = E_q(qx). \quad (112)$$

The advantage of such $q$-exponent is that it satisfies the “summation rule”

$$E_q(x) E_q(y) = E_q(x + y), \text{ for } xy = q^2 y x. \quad (113)$$

With this definition we have, for example, the following “decoupling” property:

$$E_q(\bar{t}_i \Delta_k(T_{-i})) = E_q(\bar{t}_i T_{-i} \otimes q^{-H_i} \otimes \ldots \otimes q^{-H_i}) \times E_q(\bar{t}_i q^{H_i} \otimes T_{-i} \otimes \ldots \otimes q^{-H_i}) \times$$

$$\times \ldots \times E_q(\bar{t}_i q^{H_i} \otimes q^{H_i} \otimes \ldots T_{-i}) \quad (114)$$

for each $i = 1, \ldots r$ (but not for the sum over $i$).

## 6 Conclusion

To summarize, we suggest to begin investigation of the concept of $\tau$-function in the very general framework, defining it as a generating function for all the matrix elements of a group in a given representation. In physical language this notion is of course very close to that of the “non-perturbative partition function”, which contains information about all the amplitudes in the theory. We argued that the bilinear Hirota-like identities, mixing various representations, are a very general feature of such generating functions, as are the analogues of Virasoro constraints and “string equation”. This opens the road to a very general group-theoretical interpretation of non-perturbative partition functions in quantum mechanics, field and string theories.

As a byproduct of this general formalism we derived an example of Hirota equations for quantum groups, which incorporate properly the non-commutative ($q$-number) nature of the corresponding $\tau$-functions. This $q$-number $\tau$-function can be further transformed into a $c$-number one in a particular representation of coordinate ring $A(G)$. Such procedure is
of course reminiscent of the general ideology of the second quantized string theory (when the partition function of the first-quantized model is further substituted into the functional integral over the space of theories).

The most important example which remains to be discussed in order to demonstrate all the new features of “generalized \(\tau\)-functions” is the case of non-fundamental representations of \(G = SL(3)\), where for the first time the existence of “non-Cartanian” time-variables will be essential. As to the fundamental representations of \(G = SL(n)\), there exists a closed subsystem of Hirota equations for them, which does not require more that \(n-1\) time-variables, and all of these can be associated with commuting generators. This example is described in some detail in section 5, though its quantum analogue is to be worked out in more details. In particular, the interpretation within the terms of quantum flag space (in spirit of \([17]\)) is to be obtained in the proposed framework.

Acknowledgements

We are indebted to S. Kharchev and A. Zabrodin for numerous discussions. A.Mironov is grateful to the Niels Bohr Institute, and especially Jan Ambjorn for the kind hospitality and support. A.Morozov acknowledges hospitality and support of the Volterra Center at Brandeis University. The work of A.Gerasimov and A. Mironov is partially supported by grant 93-02-14365 of the Russian Foundation of Fundamental Research.

References

[1] P.Ginsparg and G.Moore, Lectures on 2D gravity and 2D string theory (Tasi 1992), preprint YCTP-P23-92, LA-UR-92-3479, hepth/9304011

P.Di Francesco, P.Ginsparg and J.Zinn-Justin, 2d gravity and random matrices, preprint LA-UR-93-1722, SPhT/93-061, hepth/9306153
[2] A.Marshakov, _Int.J.Mod.Phys._, A8 (1993) 3831
A.Mironov, _2d gravity and matrix models. I. 2d gravity_, Preprint UBC/S-95/93, FIAN/TD-24/93 (August, 1993), hepth/9312212

[3] A.Morozov, _Soviet Physics Uspekhi_, (January 1994), hepth/9303139

[4] E.Date, M.Jimbo, M.Kashiwara and T.Miwa, _Transformation groups for soliton equations_, RIMS Symp. ”Non-linear integrable systems - classical theory and quantum theory” (World Scientific, Singapore, 1983)

[5] V.Kac, _Infinite-dimensional Lie algebras_, Cambridge University press, Cambridge, 1985, chapter 14
V.Kac and M.Wakimoto, _Exceptional hierarchies of soliton equations_, Proceedings of Symposia in Pure Mathematics, 49 (1989) 191

[6] A.Gerasimov, S.Khoroshkin and D.Lebedev, _q-deformations of \( \tau \)-functions: a toy model_, talk presented at the P.Cartier seminar in Ecole Normale, Paris, 1993; (preprint ITEP, 1993)

[7] I.Bernstein and S.Gelfand, _Compositio Math._, 41 (1980) 245

[8] I.Frenkel and N.Reshetikhin, _Comm.Math.Phys._, 146 (1992) 1

[9] B.Davies, O.Foda, M.Jimbo, T.Miwa and A.Nakayashiki, _Comm.Math.Phys._, 151 (1993) 89

[10] J.-L.Gervais and J.Schnittger, _The many faces of the quantum Liouville exponentials_, Preprint LPTENS-93/30 (August, 1993), hepth/9308134

[11] N.Yu.Reshetikhin, L.A.Takhtajan and L.D.Faddeev, _Quantization of Lie groups and Lie algebras_, _Algebra and Analysis_, 1 (1989) 178

[12] P.Griffits and F.Adams, _Topics in algebraic geometry and analytic geometry_, _Mathematical Notes_, 13, Princeton University Press, 1974
[13] M.Sato, *RIMS Kokyuroku*, **439** (1981) 30
Y.Ohta, J.Satsuma, D.Takahashi and T.Tokihiro, *Progr.Theor.Phys.Suppl.*, **94** (1988) 210

[14] K.Kajiwara and J.Satsuma, *J.Phys.Soc.Jpn.*, **60** (1991) 3986
K.Kajiwara, Y.Ohta and J.Satsuma, *q-discrete Toda molecule equation*, [solv-int/9304001](http://arxiv.org/abs/solv-int/9304001)

[15] A.Mironov, A.Morozov and L.Vinet, *On a c-number quantum τ-function*, preprint FIAN/TD-22/93, ITEP M-8/93, CRM-1934, hepth/9312213

[16] R.Floreanini and L.Vinet, *Lett.Math.Phys.*, **27** (1993) 179

[17] H.Awata, M.Noumi and S.Odake, *Heisenberg realization for U_q(SL(n)) on the flag manifold*, preprint YITP/K-1016, (May 1993), hepth/9306010