RESOLUTION OF THE $k$-DIRAC OPERATOR

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ABSTRACT. This is the second part in a series of two papers. The $k$-Dirac complex is a complex of differential operators which are naturally associated to a particular $|2|\text{-graded parabolic geometry.}$ In this paper we will consider the $k$-Dirac complex over the homogeneous space of the parabolic geometry and as a first result, we will prove that the $k$-Dirac complex is formally exact (in the sense of formal power series). Then we will show that the $k$-Dirac complex descends from an affine subset of the homogeneous space to a complex of linear and constant coefficient differential operators and that the first operator in the descended complex is the $k$-Dirac operator studied in Clifford analysis. The main result of this paper is that the descended complex is locally exact and thus it forms a resolution of the $k$-Dirac operator.

1. INTRODUCTION

Let $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ be the standard basis of $\mathbb{R}^{2n}$, $B$ be the standard inner product, $\mathbb{S}$ be the complex space of spinors (see [14, Section 6]) of the complexified Clifford algebra of $(\mathbb{R}^{2n}, B)$ and $U := M(2n, k, \mathbb{R})$ be the vector space of matrices of size $2n \times k$ with real coefficients. We will use matrix coefficients $x_{\alpha i}$, $\alpha = 1, \ldots, 2n$, $i = 1, \ldots, k$ as coordinates on $U$ and we denote by $\partial_{x_{\alpha i}}$ the coordinate vector fields. Let $C^\infty(U, \mathbb{V})$ be the space of smooth functions on $U$ with values in a vector space $\mathbb{V}$. The differential operator

$$D_0 : C^\infty(U, \mathbb{S}) \to C^\infty(U, \mathbb{C}^k \otimes \mathbb{C} \mathbb{S}),$$

is known (see [11] and [21]) as the $k$-Dirac operator. Here $\varepsilon_\alpha$ is the usual action of $\varepsilon_\alpha$ on $\mathbb{S}$ and we view $\mathbb{C}^k \otimes \mathbb{C} \mathbb{S}$ as the vector space of $k$-tuples of spinors. The operator generalizes the $k$-Cauchy-Riemann operator just as the Dirac operator can be viewed as a generalization of the Cauchy-Riemann operator. We assume $k \geq 2$ throughout the article.

The $k$-Dirac operator is an overdetermined, linear and first order differential operator with constant coefficients and so it is natural to look for a resolution of this operator, i.e. to look for a sequence of differential operators which starts with the $k$-Dirac operator and which is locally exact.

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(loosely speaking the image of each operator in the sequence coincides with the kernel of the next operator on a sufficiently small neighborhood of any point \( x \in U \)). In other words, the associated sequence of sheaves is exact. As a potential application, there is an open problem of characterizing the domains of monogenicity, i.e. an open set \( U \) is a domain of monogenicity if at each boundary point there is a null solution of (1.1) which is defined on \( U \) which cannot by continued beyond the boundary point by a null solution. Recall [15, Section 4] that the Dolbeault resolution together with some \( L^2 \) estimates are crucial in a proof of the statement that any pseudoconvex domain is a domain of holomorphy.

If \( n \geq k \), then we will show (see Theorem 3.6) that such a resolution is obtained by descending the \( k \)-Dirac complex. The \( k \)-Dirac complex (see [23] and [25]) is a complex of linear differential operators which are naturally associated to a certain parabolic geometry of type \((G, P)\) where \( G := \text{Spin}(k, 2n + k) \) and \( P \) is a parabolic subgroup associated to a \([2]\)-grading on the Lie algebra \( \mathfrak{g} = \mathfrak{so}(k, 2n + k) \) of \( G \). We will consider here the \( k \)-Dirac complex only over the homogeneous model \( G/P \).

From the point of view of the representation theory, the \( k \)-Dirac complex belongs to so called singular central (or infinitesimal) character and so this complex does not come from the BGG machinery introduced in [9]. It is explained in [23] that the \( k \)-Dirac complex can be constructed using the machinery of the Penrose transform which means that the \( k \)-Dirac complex is the direct image of a relative BGG complex. Even though this might seem a bit clumsy when one is interested only in proving the existence of the complex, the advantage of this approach is that one can show (see [23, Theorem 7.14]) that the \( k \)-Dirac complex is formally exact in the sense of [27], i.e. it induces a long exact sequence of infinite weighted jets at any fixed point.

The Penrose transform (see [3]) uses powerful tools of sheaf theory in the realm of complex manifolds, in particular the Bott-Borel-Weil theorem is crucial. This brings in one technical point, namely in [23] we construct the \( k \)-Dirac complex over the homogeneous space \( G/C/P/C \) of a complex parabolic geometry of type \((G^C, P^C)\) where \( G^C := \text{Spin}(2m, C), \) \( m = k + n \) and \( P^C \) is a parabolic subgroup that is associated to the ”complexified” \([2]\)-grading on \( \mathfrak{g}_C := \mathfrak{g} \otimes C \). However, we will observe in Section 2.6 that a linear and \( G^C \)-invariant operator on \( G^C/P^C \) induces a linear and \( G \)-invariant differential operator on \( G/P \) and thus, the complex from [23] induces the \( k \)-Dirac complex on \( G/P \). This passage is also in Introduction of [2] where it is explained that the \( k \)-Cauchy-Fueter operator (see for example [4] and [12]) can be viewed as the first operator in one of the quaternionic complexes which lives on the Grassmannian of complex 2-planes in \( \mathbb{C}^{2k+2} \).

Let us also mention that the assumption \( n \geq k \) is not needed in this article but only in the first part of the series [23]. However, it is plausible (see [16]) that the machinery of the Penrose transform works also in the case \( n < k \). Unfortunately, due to the representation theory, the Penrose transform does not work for the \( k \)-Dirac operator in dimension \( 2n + 1 \), even though (see

\(^1\)This condition is known as the stable range (see [11] and [21]).
there are \( k \)-Dirac complexes also in odd dimensions but it is not clear whether these complexes are formally exact.

Viewing a differential operator as the direct image of another differential operator is not very useful when one is interested in local formulas. The differential operators in the \( k \)-Dirac complex are of first and second order. While the first order operators can be handled rather easily (see [26]), dealing with second order operators is more difficult. Local formulas for the second order operators in the \( k \)-Dirac complex were given in [25].

As we mentioned above, the resolution of (1.1) is obtained by descending the \( k \)-Dirac complex. The descending of differential operators which are natural to parabolic structures was developed in the recent series of papers [6], [7] and [8] with preliminary paper [5] in full generality for parabolic contact structures. The parabolic geometry of type \((G, P)\) is contact if, and only if \( k = 2 \). Thus, for \( k > 2 \) we have to consider a higher dimensional analogue of the construction. Nevertheless, as we will work only on an open, dense and affine subset of \( G/P \) which is known as the "big cell", the descending procedure will turn out to be rather easy and we will not need a general theory.

Finally, let us summarize the main results of this article. We will fix a "canonical" trivialization of the canonical \( P \)-principal bundle over the big cell so that we can view sections of associated vector bundles as vector valued functions and operators in the \( k \)-Dirac complex as differential operators with polynomial coefficients. In this picture, it is standard to talk about formal power series and we will show in Theorem 3.5 that the formal exactness of the \( k \)-Dirac complex implies the exactness of the complex with formal power series at any fixed point. In Theorem 3.6 we will prove that the descended complex is a resolution of the \( k \)-Dirac operator.

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2. The parabolic geometry of type \((G, P)\)

We will define in Section 2.1 the \(|2|\)-grading on \( g \) which determines the parabolic subgroup \( P \) from Introduction. We will recall in Section 2.2 some well known theory of parabolic geometries, namely we will need some basic properties of invariant differential operators over the homogeneous space \( G/P \) and the filtration of the tangent bundle associated to any \(|2|\)-graded parabolic geometry. In Section 2.3 we will move to the open, dense and
affine subset of $G/P$ which is also called the big cell. We will identify the big cell with a Lie group and fix a trivialization of the $P$-principal bundle over the big cell which leads to several simplifications. In Section 2.4 we will recall some well known theory of invariant differential operators over the big cell and recall the notion of weighted jets. In Section 2.5 we will explain that a linear and invariant operator on the big cell descends to a linear and constant coefficient operator on the affine set $U = M(2n, k, \mathbb{R})$. In Section 2.6 we will consider the complex parabolic geometry of type $(G^C, P^C)$ from Introduction.

A comprehensive introduction into the theory of parabolic geometries is [10]. The concept of weighted jets was originally introduced by Tohru Morimoto, see for example [17] or [18]. As we will work only over the big cell, we will not give here the definition of weighted jets over a general filtered manifold as over the big cell (which we view as a Lie group) we can define the weighted jets via left invariant vector fields.

2.1. Lie algebra $\mathfrak{g}$. Let $\{e_1, \ldots, e_k, \varepsilon_1, \ldots, \varepsilon_{2n}, e_1, \ldots, e_k\}$ be the standard basis of $\mathbb{R}^{2m}$ where $m = n + k$ and $\hbar$ be the symmetric bilinear form such that

$$h(e_i, e_j) = \delta_{ij}, \quad h(e_i, \varepsilon_{\alpha}) = h(e_i, e_{\alpha}) = 0, \quad h(\varepsilon_{\alpha}, \varepsilon_{\beta}) = -\delta_{\alpha\beta}$$

where $i, j = 1, \ldots, k$; $\alpha, \beta = 1, \ldots, 2n$ and $\delta$ is the Kronecker delta. Notice that the associated quadratic form is non-degenerate with signature $(k, 2n + k)$. We will sometimes write $\mathbb{R}^{k, 2n+k}$ to indicate that we consider on $\mathbb{R}^{2m}$ the bilinear form $h$. The associated Lie algebra $\mathfrak{g} := so(h) \cong so(k, 2n + k)$ is

$$\begin{align*}
\mathfrak{g} &:= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
\mathfrak{g}_{-2} &:= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_{-1} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_0 := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}, \\
\mathfrak{g}_1 &:= \left\{ \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_2 := \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}
\end{align*}$$

It is straightforward to verify that

(I) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ where $i, j \in \mathbb{Z}$ and we agree that $\mathfrak{g}_i = \{0\}$ if $|i| > 2$ and

(II) $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as a Lie algebra.

Hence, the direct sum decomposition is a $|2|$-grading (see [10], Definition 3.1.2]). The filtration associated to the grading is $\mathfrak{g} = \mathfrak{g}_{-2} \supset \mathfrak{g}_{-1} \supset \cdots \supset \mathfrak{g}_{-1} \supset \{0\}$ where we put $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$. From the property (I) follows that $\mathfrak{g}_-, \mathfrak{g}_0$ and $\mathfrak{g}^i$ where $i \geq 0$ are subalgebras of $\mathfrak{g}$ and that $\mathfrak{g}_-$ and $\mathfrak{g}^i$, $i \geq 1$ are nilpotent. We call $\mathfrak{p} := \mathfrak{g}_0$ the parabolic subalgebra associated to the grading and put $\mathfrak{p}_+ := \mathfrak{g}_1$. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is the Levi-decomposition of $\mathfrak{p}$ which means that $\mathfrak{g}_0$ is a maximal reductive subalgebra of $\mathfrak{p}$ called the reductive Levi factor and that $\mathfrak{p}_+$ is the nilradical of $\mathfrak{p}$ (see [10], Section 2.1.8)].
The Killing form induces (see [10] Proposition 3.1.2) isomorphisms $g_i \cong g_i^*$ of $g_0$-modules where $^*$ denotes the dual module and $i = -2, \ldots, 2$. By the same Proposition, there is a grading element $E \in g$ which is uniquely determined by $g_i = \{X \in g : [E, X] = iX\}$, $i = -2, -1, \ldots, 2$. The grading element associated to (2.2) corresponds to the block matrix from (2.1) where $A = 1_h$ and all the other matrices are zero.

The algebra $g_0$ is isomorphic to $gl(k, \mathbb{R}) \oplus so(2n)$. Assume that $E$ and $F$ are $gl(k, \mathbb{R})$ and $so(n)$-modules, respectively. Then $E$ and $F$ are also $g_0$-modules by letting the other factor act trivially. We denote by $E \ltimes F$ the exterior tensor product of $E$ and $F$. Notice that the subspaces $[e_1, \ldots, e_k]$ and $[e_1, \ldots, e_{2n}]$ are $g_0$-invariant and that they are isomorphic to the $g_0$-modules $\mathbb{R}^k$ and $\mathbb{R}^{2n}$, respectively. Now it is easy to see that there are isomorphisms of $g_0$-modules:

$$\tag{2.3} g_{-2} \cong \Lambda^2 \mathbb{R}^k, \quad g_{-1} \cong \mathbb{R}^k \ltimes \mathbb{R}^{2n}, \quad g_1 \cong \mathbb{R}^k \ltimes \mathbb{R}^{2n} \quad \text{and} \quad g_2 \cong \Lambda^2 \mathbb{R}^k$$

where we use the $g_0$-invariant inner product $h|_{\mathbb{R}^{2n}}$. The standard basis of $\mathbb{R}^{2n}$ gives the bases

$$\tag{2.4} \{e^r \wedge e^s : 1 \leq r < s \leq k\} \quad \text{and} \quad \{e^i \otimes e_\alpha : i = 1, \ldots, k, \alpha = 1, \ldots, 2n\}$$

of $g_{-2}$ and $g_{-1}$, respectively, and the dual bases

$$\tag{2.5} \{e_r \wedge e_s : 1 \leq r < s \leq k\} \quad \text{and} \quad \{e_i \otimes e_\alpha : i = 1, \ldots, k, \alpha = 1, \ldots, n\}$$

of $g^2$ and $g^1$, respectively.

Finally, let us consider the Lie bracket $\Lambda^2 g_- \to g_-$. This map is homogeneous of degree zero and thus, we can look at each homogeneous part separately. It is clear that the map is non-zero only in the homogeneity -2. Moreover, by the Jacobi identity, the map is $g_0$-equivariant. If we use the isomorphisms from (2.3), then it is easy to see that there is (up to constant) a unique $g_0$-equivariant map which is the composition

$$\tag{2.6} \Lambda^2 g_- \cong \Lambda^2 (\mathbb{R}^{k} \ltimes \mathbb{R}^{2n}) \to \Lambda^2 \mathbb{R}^k \ltimes S^2 \mathbb{R}^{2n} \to \Lambda^2 \mathbb{R}^k \cong g_{-2}$$

where the first map is the obvious projection and in the second map we take the trace with respect to $h|_{\mathbb{R}^{2n}}$. Notice that the map (2.6) is non-degenerate.

### 2.2. Lie groups, homogeneous space and invariant differential operators.

Let $G := \text{Spin}(k, 2n + k)$ and $\text{Ad} : G \to \text{GL}(g)$ be the adjoint representation. We call

$$\tag{2.7} \text{P} := \{g \in G : \text{Ad}(g)(g^i) \subset g^i; \; i = -2, \ldots, 2\}$$

the parabolic subgroup associated to the 2-grading and

$$\tag{2.8} \text{G}_0 := \{g \in G : \text{Ad}(g)(g_i) \subset g_i; \; i = -2, \ldots, 2\}$$

the Levi subgroup of $\text{P}$. Then $\text{P}$ is a closed subgroup of $G$ with Lie algebra $\frak{p}$ and $\text{G}_0$ is a closed subgroup of $\text{P}$ with Lie algebra $\frak{g}_0$. It can be shown (see [22]) that $\text{G}_0$ is isomorphic to $\text{GL}(k, \mathbb{R}) \times \text{Spin}(2n)$. We put $\text{P}_+ := \exp(\frak{p}_+)$. By [10] Theorem 3.1.3, the map $\exp : \frak{p}_+ \to \text{P}_+$ is a diffeomorphism. $\text{P}_+$ is a normal subgroup of $\text{P}$ and $\text{P} = \text{G}_0 \ltimes \text{P}_+$. Hence, there is a surjective Lie group homomorphism $\text{P} \to \text{G}_0$ and thus, any $\frak{g}_0$-module is also a $\frak{p}$-module with trivial action of $\frak{p}_+$. It is well known that any irreducible $\frak{p}$-module arises in this way.
Let us recall that the homogeneous space $G/P$ of the Cartan geometry of type $(G, P)$ carries a natural distribution of co-dimension $\binom{k}{2}$. Let $p : G \to G/P$, $g \mapsto g.P$ be the canonical projection and $\ell_g : G \to G$ and $r^g : G \to G$ be the multiplication by $g \in G$ on the left and on the right, respectively. As $\ell_g$ is a diffeomorphism, the map $T\ell_{g^{-1}} : T_g G \to T_e G = \mathfrak{g}$ is a linear isomorphism and the 1-form $\omega \in \Omega^1(G, \mathfrak{g})$ defined by $\omega_g(X) = T\ell_{g^{-1}}(X)$, $X \in T_g G$ is the Maurer-Cartan form on $G$. We put $T^{-1}_g G := \omega_g^{-1}(\mathfrak{g}^{-1})$ so that $T^{-1}_g G := \bigcup_{g \in G} T^{-1}_g G$ is a distribution on $G$. As $\omega$ is $P$-equivariant in the sense that for every $g \in G$ : $(r^g)^* \omega = \text{Ad}(g^{-1}) \circ \omega$ and $\mathfrak{g}^{-1}$ is a $P$-invariant subspace, it follows that $T^{-1}_g G$ projects under $Tp$ to a well-defined distribution $H$ of co-dimension $\binom{k}{2}$.

Assume now that $\mathcal{V}$ is an irreducible $P$-module. Then the space $\Gamma(\mathcal{V})$ of smooth sections of the associated vector bundle $V := G \times_P \mathcal{V}$ is by [11] Proposition 1.2.7] isomorphic to the space $C^\infty(G, \mathcal{V})^P$ of smooth $\mathcal{V}$-valued $P$-equivariant functions on $G$. More generally, if $U$ is an open subset of the homogeneous space and $\mathcal{V}|_U$ is the restriction of $\mathcal{V}$ to $U$, then $\Gamma(\mathcal{V}|_U) \cong C^\infty(p^{-1}(U), \mathcal{V})^P$. The group $G$ has a canonical action on $C^\infty(G, \mathcal{V})$ given by $g.f = f \circ \ell_{g^{-1}}$, where $f \in C^\infty(G, \mathcal{V})$ is a smooth $\mathcal{V}$-valued function on $G$. As the multiplication on the right commutes with the multiplication on the left, this action descends to an action on $C^\infty(G, \mathcal{V})^P$. Using the isomorphism above, this induces a $G$-action on $\Gamma(\mathcal{V})$.

Assume that $\mathcal{W}$ is another irreducible $P$-module and that

$$D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$$

is a linear differential operator of order $r$ where $W := G \times_P \mathcal{W}$. We call $D$ a $G$-invariant operator (or simply invariant) if it intertwines the action of $G$ on $\Gamma(\mathcal{V})$ and $\Gamma(W)$. Let $J^r \mathcal{V}$ be the vector space of $r$-jets of germs of sections of $\mathcal{V}$ at the origin $eP \in G/P$ where $e \in G$ is the identity element. Then $J^r \mathcal{V}$ is a $P$-module and $D$ determines a $P$-equivariant map

$$J^r \mathcal{V} \to \mathcal{W}, \quad j^r f \mapsto Df(eP)$$

where $j^r f$ is the $r$-th jet of a germ $f$ of a smooth section of $\mathcal{V}$ at $eP$. It is well known (see [10] Section 1.4.10]) that this assignment is a bijection, i.e. a $P$-equivariant homomorphism $J^r \mathcal{V} \to \mathcal{W}$ determines a unique linear and $G$-invariant differential operator of order at most $r$. The order of the induced operator is $< r$ if the homomorphism (2.10) factorizes through the canonical projection $J^r \mathcal{V} \to J^{r-1} \mathcal{V}$.

2.3. Affine subset of the homogeneous space. The group $G$ has a canonical action on the Grassmannian variety of totally isotropic $k$-dimensional subspaces in $\mathbb{R}^{k,2n+k}$. It is straightforward to verify that $P$ is the stabilizer of the totally isotropic subspace $[e_1, \ldots, e_k]$. Hence, we can view $G/P$ as the isotropic Grassmannian and we will do that without further comment. We will now consider an affine subset of the homogeneous space. We will use the following convention. If $A = (a_{ij}) \in M(2m, k, \mathbb{R})$ has maximal rank $k$, then we denote by $[a_{ij}]$ the $k$-dimensional subspace that is spanned by the columns of the matrix $A$. 
The restriction of $p \circ \exp$ to $\mathfrak{g}_-$ induces a map

$$\mathfrak{g}_- \to G/P; \quad \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ Y & -X^T & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1_k \\ X \\ Y - \frac{1}{2}X^T X \end{pmatrix}. \tag{2.11}$$

The map is injective and a moment of thought shows that it is a diffeomorphism onto its image which is an open, dense and affine subset of $G/P$ which is known as the "big cell", see [10, Example 5.1.12]. Let us write $X = (x_{\alpha i})$, $Y = (y_{rs})$ so that we can use $x_{\alpha i}$, $\alpha = 1, \ldots, 2n$, $i = 1, \ldots, k$ and $y_{rs}$, $1 \leq r < s \leq k$ as coordinates on the big cell. It will be convenient to view the big cell as a Lie group with Lie algebra $\mathfrak{g}_-$. 

Put $G_- := \exp(\mathfrak{g}_-)$. Then $G_-$ is a closed subgroup of $G$ with Lie algebra $\mathfrak{g}_-$, the exponential map is a diffeomorphism $\mathfrak{g}_- \to G_-$ and thus, the composition $G_- \to G \xrightarrow{\ell} G/P$, where $\ell$ is the inclusion, induces a diffeomorphism between $G_-$ and the big cell. Hence, we can view $G_-$ as the big cell and the inclusion $\iota$ as a section of $p^{-1}(G_-) \to G_-$. This section determines a trivialization $\Phi : G_- \times P \to p^{-1}(G_-)$ of the $P$-principal bundle over $G_-$. As $p^{-1}(G_-) \subset G$, the map $\Phi$ is simply given by multiplication on $G$.

Let $V$ be a $P$-module and $V$ be the associated vector bundle as in Section 2.2. Recall that $\Gamma(V|_{G_-}) \cong \mathcal{C}^\infty(p^{-1}(G_-), V)^P$. As any $P$-equivariant function on $p^{-1}(G_-)$ is determined by its restriction to $G_-$, it follows that the map

$$\mathcal{C}^\infty(p^{-1}(G_-), V)^P \to \mathcal{C}^\infty(G_-, V), \quad f \mapsto f \circ \iota$$

is an isomorphism of vector spaces. We see that the restriction of the differential operator (2.9) to $G_-$ induces (in the trivialization $\iota$) a linear differential operator

$$\mathcal{C}^\infty(G_-, V) \to \mathcal{C}^\infty(G_-, W). \tag{2.12}$$

We will denote this operator also by $D$ as there is no risk of confusion. If the operator $D$ is invariant, then it is straightforward to verify that (2.12) is a $G_-$-invariant differential operator with respect to the canonical $G_-$-action on both spaces (that is induced by multiplication on left). Let us now recall well known classification of linear and $G_-$-invariant differential operators.

2.4. Invariant differential operators and weighted jets. Let us first recall the definition of the universal enveloping algebra associated to $\mathfrak{g}_-$. In order to simplify notation, we write $gr(X) = i$ if, and only if $X \in \mathfrak{g}_i$ and $i \in \{-1, -2\}$. Recall Section 2.1 that the Lie bracket on $\mathfrak{g}_-$ is compatible with the grading in the sense that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ where $i, j = -1, -2$ and we agree that $\mathfrak{g}_{-2} = \{0\}$ whenever $\ell < -2$.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g}_-)$ associated to $\mathfrak{g}_-$ is the quotient of the tensor algebra $T(\mathfrak{g}_-)$ of $\mathfrak{g}_-$ by the both sided ideal $I$ that is generated by the elements of the form: $X \otimes Y - Y \otimes X - [X, Y]$, $X, Y \in \mathfrak{g}_-$. The grading on $\mathfrak{g}_-$ induces a grading $T(\mathfrak{g}_-) = \bigoplus_{r \geq 0} T_r(\mathfrak{g}_-)$ on the tensor algebra where $T_r(\mathfrak{g}_-)$ is the linear span of all elements $X_{i_1} \otimes \cdots \otimes X_{i_\ell}$ where $\sum_{j=1}^\ell gr(X_{i_j}) = -r$. Notice that the grading is compatible with multiplication. As the Lie bracket on $\mathfrak{g}_-$ is compatible with the grading, it is easy to see that $I = \bigoplus_{r \geq 0} (T_r(\mathfrak{g}_-) \cap I)$. Thus the grading on the tensor algebra induces a
the tensor product

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invariant differential operators

polynomial coefficients. More generally, the vector space of linear and left

follows that

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(2.13) that any

\[ D \]

\[ \in \mathcal{U}(g_-) \]

differential operator with

\[ (2.13) \]

polynomial coefficients. More generally, the vector space of linear and left

invariant differential operators

\[ C^\infty(G_-) \]

as \( \mathcal{U}(g_-) \) without further comment.

A straightforward computation shows that

\[ L_{e^i \otimes e^\alpha} = \partial_{x_{\alpha i}} - \sum_{j=1}^k \frac{1}{2} x_{\alpha j} \partial_{y_{\alpha j}} \quad \text{and} \quad L_{e^i \wedge e^\alpha} = \partial_{y_{\alpha i}} \]

where we use the bases of \( g_-1 \) and \( g_-2 \) given in (2.4) and the convention

\[ \partial_{y_{\alpha i}} = -\partial_{y_{\alpha i}}. \]

Check that the bracket of vector fields is compatible with the Lie bracket on \( g_- \) given in (2.5). From the definition, it immediately follows that \( H|_{g_-} \) is (as a \( C^\infty(G_-) \)-module) generated by \( \{ L_X : X \in g_-1 \} \).

It follows from (2.13) that any \( D \in \mathcal{U}(g_-) \) is a differential operator with

\[ (2.13) \]

\[ D \]

\[ \in \mathcal{U}(g_-) \]

differential operator acting

on \( g_- \) whose value at the identity element \( e \) is \( X \). In the sequel, we will view the algebra of linear differential operators acting

\[ C^\infty(G_-) \]

as \( \mathcal{U}(g_-) \) without further comment.

The algebra of polynomials on the vector space \( g_- \) is naturally isomorphic to the symmetric algebra \( S(p_+) \). The grading \( p_+ = g_1 \oplus g_2 \) induces a grading

\[ S(p_+) = \bigoplus_{r \geq 0} \mathcal{G}^r p_+ \]

where \( \mathcal{G}^r p_+ = \bigoplus_{\ell=0}^{\lfloor \frac{r}{2} \rfloor} S^\ell g_2 \otimes S^{r-2\ell} g_1 \)

and \( \lfloor \rfloor \) denotes the integer part. The grading on \( S(p_+) \) is obviously compatible with multiplication and so \( S(p_+) \) is a graded algebra. As \( g_- \) is diffeomorphic to the big cell \( G_- \), we can view \( S(p_+) \) as polynomials on \( G_- \).

Then there is a natural pairing

\[ (2.15) \]

\[ \mathcal{U}(g_-) \otimes S(p_+) \to \mathbb{R}, \quad (D, f) \mapsto Df(e) \]

which yields a duality \( \mathcal{U}_r(g_-)^* \cong \mathcal{G}^r p_+ \).

Finally, let us recall the concept of weighted jets. Let \( \mathcal{E}(\mathcal{V}) \) be the vector space of germs of smooth sections of the associated bundle \( V \) at \( e \) from Section 2.2. If \( f \in \mathcal{E}(\mathcal{V}) \), then \( f := f \circ \iota \) is a germ of a smooth \( \mathcal{V} \)-valued function at \( e \in G_- \). We write \( \overline{f}_r \sim_r g \) if \( g \in \mathcal{E}(\mathcal{V}) \) and \( Df(e) = Dg(e) \) for each \( D \in \mathcal{U}_r(g_-), \ell \leq r \). It is well known that the equivalence class \( \sim_r \) does not depend on the choice of the trivialization. We denote by \( \mathfrak{J}^r \mathcal{V} \) the quotient of \( \mathcal{E}(\mathcal{V}) \) by the subspace \( \{ f \in \mathcal{E}(\mathcal{V}) : f \sim_{r+1} 0 \} \) and by \( \mathfrak{J}^r f \) the class of \( f \) in \( \mathfrak{J}^r \mathcal{V} \). We call \( \mathfrak{J}^r \mathcal{V} \) the space of weighted \( r \)-jets at \( e \) associated to \( \mathcal{V} \) and \( \mathfrak{J}^r f \) the weighted \( r \)-jet of \( f \) at \( e \).

As the subspace \( \{ f \in \mathcal{E}(\mathcal{V}) : f \sim_{r+1} 0 \} \) is \( P \)-invariant, it follows that the standard action of \( P \) on \( \mathcal{E}(\mathcal{V}) \) descends to an action on \( \mathfrak{J}^r \mathcal{V} \). As \( f \sim_r 0 \) whenever \( f \sim_{r+1} 0 \), it follows that there is a canonical surjective map \( \mathfrak{J}^r \mathcal{V} \to \mathfrak{J}^{r-1} \mathcal{V} \) and we denote its kernel by \( \mathfrak{g} \mathfrak{J}^r \mathcal{V} \). It is well known (see [17].
that there are linear isomorphisms

\begin{equation}
\text{gr}^r \mathbb{V} \cong \mathcal{U}_r(g_-)^* \otimes \mathbb{V} \cong \mathcal{S}^r p_+ \otimes \mathbb{V}.
\end{equation}

From the definitions, it easily follows that for each \( r \geq 0 \) there is a well-defined linear map \( \mathcal{J}^r \mathbb{V} \to \mathcal{J}^r \mathbb{V} \) which is surjective. Here \( \mathcal{J}^r \mathbb{V} \) denotes the vector space of ordinary (unweighted) \( r \)-jets.

**Proposition 2.1.** Suppose that \( \mathbb{V} \) and \( \mathbb{W} \) are irreducible \( G_0 \)-modules on which the grading element \( E \) acts with eigenvalues \( \lambda \) and \( \mu \), respectively and that there is a nonzero, linear and \( G \)-invariant differential operator \( \mathcal{D} \) \((2.1.7)\).

Then \( r := \lambda - \mu \) is a non-negative integer and:

(I) the induced \( P \)-equivariant homomorphism \( \mathcal{D} \) descends to a \( P \)-equivariant map

\begin{equation}
\phi : \mathcal{J}^r \mathbb{V} \to \mathbb{W}
\end{equation}

which does not factorize through the canonical projection \( \mathcal{J}^r \mathbb{V} \to \mathcal{J}^{r-1} \mathbb{V} \) and

(II) the invariant differential operator \( \mathcal{D} \) induced by \( \mathcal{D} \) belongs to \( \mathcal{U}_r(g_-) \).

We call the integer \( r \) the **weighted order** of the differential operator \( \mathcal{D} \).

**Proof.** By Section 2.2, the differential operator \( \mathcal{D} \) is determined by a \( P \)-equivariant homomorphism \( \mathcal{D} \) which is completely determined by a \( P \)-equivariant map as in \( \mathcal{D} \). The map is invariantly defined by \( \phi(f) = Df(eP) \) where \( f \in \mathcal{E}(\mathbb{V}) \). Without loss of generality we may assume that \( \phi \) does not factorize through the canonical projection \( \mathcal{J}^r \mathbb{V} \to \mathcal{J}^{r-1} \mathbb{V} \) which is equivalent to the fact that \( \phi \) is non-zero on \( \text{gr}^r \mathbb{V} \). Hence, \( \phi \) induces an isomorphism between \( \mathbb{W} \) and an irreducible \( P \)-invariant subspace of \( \text{gr}^r \mathbb{V} \). By the assumptions, the grading element acts diagonalizably on \( \text{gr}^r \mathbb{V} \) with finitely many eigenvalues which are of the form \( \lambda, \lambda + 1, \lambda + 2, \ldots, \lambda + r \) where \( \text{gr}^r \mathbb{V} \) is the eigenspace of \( \lambda + r \). Recall the linear isomorphism \( \text{gr}^r \mathbb{V} \cong \mathbb{S}^r \otimes \mathbb{V} \) from \( \mathcal{D} \). From this we see that \( r \) is a non-negative integer and also the claim in (I).

From the definition of \( \phi \) follows that the value of \( Df \) at the origin \( eP \) depends only on \( j^r f \). This implies that \( D \), viewed as the \( G_- \)-invariant operator \( \mathcal{D} \), belongs to \( \mathbb{U}_r(g_-) \). Since \( Df(e) = 0 \) whenever \( j^{r-1}_e f = 0 \), it follows that \( D \in \mathbb{U}_r(g_-) \). The proof is complete. \( \Box \)

As we have seen in the proof of Proposition 2.1, if \( D \) is determined by the map \( \mathcal{D} \), then \( Df(eP) \) where \( f \in \mathcal{E}(\mathbb{V}) \) depends only \( j^r f \). More generally, one can show (see [17] or [20]) that \( j^s Df \) depends only \( j^{r+s} f \) and thus, the operator \( D \) induces for each \( s \geq 0 \) a linear map

\begin{equation}
\text{gr}^{r+s} D : \text{gr}^{r+s} \mathbb{V} \to \text{gr}^s \mathbb{W}, \quad j^{r+s} f \to j^s Df
\end{equation}
which is called the $s$-th prolongation of $D$. From the definition of the map (2.18), it follows that there is a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g}^{r+s}\mathfrak{p}_+ \otimes \mathbb{V} & \stackrel{D}{\longrightarrow} & \mathfrak{g}^s\mathfrak{p}_+ \otimes \mathbb{W} \\
\mathfrak{g}^{r+s}\mathbb{V} & \longrightarrow & \mathfrak{g}^{r+s}\mathbb{W} \\
\end{array}
\]

where the vertical arrows are the isomorphisms from (2.16) and in the first row we view $\mathfrak{g}^{r+s}\mathfrak{p}_+ \otimes \mathbb{V}$ and $\mathfrak{g}^s\mathfrak{p}_+ \otimes \mathbb{W}$ as the vector spaces of $\mathbb{V}$ and $\mathbb{W}$-valued polynomials on $G_-$, respectively, and $D$ as the corresponding differential operator on $G_-$.

2.5. **Descending procedure.** Let $G_{-2} := \exp(\mathfrak{g}_{-2})$. Then it is easy to see that $G_{-2}$ is a closed, abelian and normal subgroup of $G_-$ with Lie algebra $\mathfrak{g}_{-2}$ and that $\exp : \mathfrak{g}_{-2} \rightarrow G_{-2}$ is a diffeomorphism. We will be interested in the space $G_{-2} \setminus G_-$ of right cosets which we for brevity denote by $U$. It is straightforward to verify that the composition

\[
M(2n, k, \mathbb{R}) \cong \mathfrak{g}_{-1} \xrightarrow{\exp} G_- \xrightarrow{q} G_{-2}/G_- = U,
\]

where $q$ is the canonical projection, is a diffeomorphism. For brevity, we denote the origin $G_{-2} e$ of $U$ by $x_0$.

Let $J^r_{x_0}(U, \mathbb{V})$ be the vector space of usual $r$-jets of germs of smooth $\mathbb{V}$-valued functions at the origin $x_0$. We denote by $\text{gr}^r\mathbb{V}$ the kernel of the canonical projection $J^r_{x_0}(U, \mathbb{V}) \rightarrow J^{r-1}_{x_0}(U, \mathbb{V})$. The projection $q$ induces a linear map

\[
q^* : C^\infty(U, \mathbb{V}) \rightarrow C^\infty(G_-, \mathbb{V}), \quad f \mapsto f \circ q
\]

which is obviously injective. This map descends for each $r \geq 0$ to injective linear maps

\[
J^r_{x_0}(U, \mathbb{V}) \hookrightarrow J^r\mathbb{V} \quad \text{and} \quad \text{gr}^r\mathbb{V} \hookrightarrow \text{gr}^r\mathbb{V}
\]

which we also denote by $q^*$. Notice that in this way, $\text{gr}^r\mathbb{V}$ gets identified with the subspace $S^r\mathfrak{g}_{1} \otimes \mathbb{V} \subset \mathfrak{g}^r \otimes \mathbb{V} \cong \text{gr}^r\mathbb{V}$.

**Proposition 2.2.** Suppose that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces of dimensions $s$ and $t$, respectively. Assume that

\[
D : C^\infty(G_-, \mathbb{V}) \rightarrow C^\infty(G_-, \mathbb{W}).
\]

is a linear and $G_-$-invariant operator differential operator from $\mathcal{U}_r(\mathfrak{g}_-)$.

Then there is a unique linear differential operator

\[
\underline{D} : C^\infty(U, \mathbb{V}) \rightarrow C^\infty(U, \mathbb{W})
\]

such that the following diagram

\[
\begin{array}{ccc}
C^\infty(G_-, \mathbb{V}) & \stackrel{D}{\longrightarrow} & C^\infty(G_-, \mathbb{W}) \\
q^* \downarrow & & \downarrow q^* \\
C^\infty(U, \mathbb{V}) & \stackrel{\underline{D}}{\longrightarrow} & C^\infty(U, \mathbb{W})
\end{array}
\]

commutes. Moreover, $\underline{D}$ is a homogeneous constant coefficient operator of order $r$. 

Proof. By the definition, the invariance of $D$ means that $(Df) \circ \ell_{g^{-1}} = D(f \circ \ell_{g^{-1}})$ for every $g \in G_-$ where $\ell_{g^{-1}} : G_- \to G_-$ is multiplication by $g^{-1}$ on left. Assume now that $f \in \mathcal{C}^\infty(G_-, \mathcal{V})$ belongs to the image of $q^*$. It is elementary to see that this is equivalent to $f \circ \ell_{g^{-1}} = f$ for every $g \in G_{-2}$. Thus we see that there is a linear operator (2.23) as claimed. It remains to show that $D$ is a homogeneous differential operator of order at most $r$ (which may be possibly zero). From (2.13), it easily follows that for each $x \in C^\infty(U, \mathcal{V}) : L_{e^t \otimes x_n}(q^* h) = q^* \partial_{x_n} h$ and that $L_{e^t \otimes e^s}(q^* h) = 0$. Hence, the last claim follows from the definition of $\mathcal{U}_e(g_-)$ given in Section 2.4.

With the notation from Proposition 2.2, notice that there is for each $s \geq 0$ a commutative diagram of linear maps

\[
\begin{array}{ccc}
gr^{r+s} \mathcal{V} & \xrightarrow{\text{gr}^{r+s} D} & \text{gr}^s \mathcal{W} \\
q^* \downarrow & & \downarrow q^* \\
gr^{r+s} \mathcal{V} & \xrightarrow{\text{gr}^{r+s} D} & \text{gr}^s \mathcal{W}
\end{array}
\]

where $\text{gr}^{r+s} D$ is the standard linear map $j^{r+s} f \mapsto j^r D f$ that is associated to any linear differential operator of order $r$ and $j^r f$ is the usual $r$-jet of $f$ at $x_0$.

2.6. Complex parabolic geometry. We will now consider the complex analogue of the parabolic geometry of type $(G, P)$ from Sections 2.1 and 2.2. We will proceed in this Section rather quickly as most constructions are similar to the real case and we refer to [23] for a more thorough introduction if necessary.

We have $\mathbb{C}^{2m} = \mathbb{R}^{2m} \otimes \mathbb{C}$ and we denote by $h^C$ the $\mathbb{C}$-bilinear extension of $h$ to $\mathbb{C}^{2m}$. The complexification of $\mathfrak{g}$ is $\mathfrak{so}(2m, \mathbb{C}) \cong \mathfrak{g}^C := \mathfrak{g} \otimes \mathbb{C}$ and the direct sum decomposition $\mathfrak{g}^C = \mathfrak{g}_{-2}^C \oplus \mathfrak{g}_{-1}^C \oplus \cdots \oplus \mathfrak{g}_{2}^C$, where $\mathfrak{g}_{-i}^C := \mathfrak{g} \otimes \mathbb{C}$, is a $[2]$-grading on $\mathfrak{g}^C$. The group $G^C := \text{Spin}(2m, \mathbb{C})$ is a connected and simply connected complex Lie group with Lie algebra $\mathfrak{g}^C$. There is an embedding $\zeta : G \to G^C$ such that $T_e \zeta : T_e G \to T_e G^C$ is the canonical inclusion $\mathfrak{g} = \mathfrak{g} \otimes \mathbb{R} \to \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^C$. We denote by $P^C$ the parabolic subgroup associated via (2.7) to the $[2]$-grading and by $G_0^C$ the Levi factor which is defined as in (2.8). The group $G_0^C$ is isomorphic to $\text{GL}(k, \mathbb{C}) \times \text{Spin}(2n, \mathbb{C})$. It is obvious that $\zeta$ restricts to embeddings $P \hookrightarrow P^C$ and $G_0 \hookrightarrow G_0^C$ which also shows that $\zeta$ descends to an embedding $\zeta : G/P \to G^C/P^C$.

The exponential image $G_0^C$ of $\mathfrak{g}_-^C := \mathfrak{g}_- \otimes \mathbb{C}$ is a closed subgroup of $G^C$ with embedding $\iota^C : G_0^C \to G^C$. The composition $G_0^C \xrightarrow{\iota^C} G^C \xrightarrow{\iota^\mathbb{C}} G^C/P^C$ is a biholomorphism onto its image which identifies $G_0^C$ with an open, dense and affine subset of $G^C/P^C$. It is clear that $\zeta$ restricts to an embedding $G_- \hookrightarrow G_0^C$ and that the pair $G_- \subset G_0^C$ looks as the pair $\mathbb{R}^\ell \subset \mathbb{C}^\ell$ where we for a moment put $\ell := 2nk + \binom{k}{2}$. Hence, $G_0^C$ is a complexification of the real analytic manifold $G_-$. Since $G/P$ and $G^C/P^C$ are homogeneous spaces, also $G^C/P^C$ is a complexification of the real analytic manifold $G/P$ as in [1, Definition 6.6].
Assume now that $V$ is a complex irreducible $P^C$-module. Then $V$ is also a complex irreducible $P$-module by restriction. Let $\mathcal{O}(V)$ be the space of germs of holomorphic sections of $V^C := G^C \times_{P^C} V$ at $eP^C$ and $\mathcal{A}^\omega(V)$ be the space of germs of real analytic sections of $V := G \times_P V$ at $eP$. If we view sections as equivariant functions, then it is easy to see that the pullback of $f \in \mathcal{O}(V)$ along the inclusion $\zeta$ is an element of $\mathcal{A}^\omega(V)$. Hence, there is a well-defined map
\begin{equation}
\zeta^* : \mathcal{O}(V) \to \mathcal{A}^\omega(V).
\end{equation}
As $G^C/P^C$ is a complexification of the real analytic manifold $G/P$, the following fact is well known.

**Lemma 2.3.** The map (2.26) is an isomorphism of infinite dimensional vector spaces.

Recall Section 2.4 for the definition of $\mathfrak{j}^r V$, $r \geq 0$ and that there is a canonical projection $\mathcal{A}^\omega(V) \to \mathfrak{j}^r V$. This induces via the isomorphism (2.20) a surjective linear map $\mathcal{O}(V) \to \mathfrak{j}^r V$. Hence, we can view $\mathfrak{j}^r V$ as a quotient of $\mathcal{O}(V)$ and $\mathfrak{gr}^r V$ as a subspace of this quotient space. Also we can define the $r$-th weighted jet for any germ $f \in \mathcal{O}(V)$. This shows that the vector spaces $\mathfrak{j}^r V$ and $\mathfrak{gr}^r V$ are naturally $P^C$-modules.

Assume now that $W$ is a complex irreducible representation of $P^C$ and that there is a $P^C$-equivariant homomorphism (2.17). Then the homomorphism is also $P^C$-equivariant and it induces a $G^C$-invariant differential operator just as in the real case. Conversely, any such linear and $G^C$-invariant differential operator induces a $P^C$-equivariant homomorphism as in (2.17). As this observation will be needed later on, let us formulate it as a lemma.

**Lemma 2.4.** Assume that $V$ and $W$ are complex and irreducible representations of $P^C$ and that there is a $P^C$-equivariant homomorphism as in (2.17). Then the homomorphism induces a linear and $G^C$-invariant operator
\begin{equation}
G^C \times_{P^C} V \to G^C \times_{P^C} W
\end{equation}
of order at most $r$ where we for brevity write bundles instead of the associated sheaves of germs of holomorphic sections.

Conversely, a linear and $G^C$-invariant differential operator (2.27) induces a $P^C$-equivariant map (2.17) for some integer $r \geq 0$.

### 3. $k$-Dirac Complex and the Resolution of the $k$-Dirac Operator

In the current Section we will apply the theory that we built in Section 2. In Section 3.1 we will set some notation. We will show (see Theorem 3.3) that there is a complex (the $k$-Dirac complex from Introduction) of linear and $G$-invariant differential operators on $G/P$ and that this complex is formally exact (see Proposition 3.4). This uses the results already established in [23] and the discussion from Section 2.6. The $k$-Dirac complex induces a complex of linear and invariant differential operators with polynomial coefficients on $G_-$ and we will show in Section 3.3 that this complex is exact in the sense of formal power series at any given point (see Theorem 3.5). By Proposition 2.2, the complex on $G_-$ induces a complex of linear and constant coefficient differential operators on the set $U := G_{-2} \setminus G_-$. We will prove in Section
that the first operator in the descended complex is the $k$-Dirac operator from Introduction and that the complex is locally exact (see Theorem 3.6).

3.1. Young diagrams. We will use the following notation. We put

$$N_{n++}^k := \{(a_1, \ldots, a_k) : a_i \in \mathbb{Z}, n \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0\}$$

and call $a \in N_{n++}^k$ a partition of the number $|a| := a_1 + \cdots + a_k$. To the partition $a$ we associate a diagram that consists of left justified $a_i$ boxes in the $i$-th row and call it the Young diagram associated to $a$. We denote by $S_k$ the subset of $N_{n++}^k$ consisting of those partitions whose Young diagram is symmetric with respect to the main diagonal. Then we define $q(a)$ and $d(a)$ as the number of boxes in the Young diagram that are “above” and “on” the main diagonal, respectively. We put $r(a) := d(a) + q(a)$ and $S_j^k := \{a \in S_k : r(a) = j\}$. If $a' = (a'_1, \ldots, a'_k) \in N_{n++}^k$, then we write $a < a'$ if $a \neq a'$ and $a_i \leq a'_i$ for each $i = 1, \ldots, k$.

Example 3.1. (1) The Young diagram associated to $a = (4, 2, 0) \in N_{n++}^{3,4}$ is $\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}$. Then $d(a) = 2$, $q(a) = 3$ and $a \not\in S_k$ as there are four boxes in the first row but only two boxes in the first column.

(2) The Young diagram associated to $a' = (3, 2, 1, 0) \in N_{n++}^{4,6}$ is $\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}$. Hence, $d(a) = q(a) = 2$ and we see that $a' \in S_k$.

Example 3.2. (1) If $k = 2$, then

$$S^2 = \{\cdot < \begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} < \begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} \}$$

and

$$S^2_0 := \{\cdot\}, \ S^2_1 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^2_2 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^2_3 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}$$

where we write also the ordering on $S^2$ defined above.

(2) For $k = 3$ we have

$$S^3 = \left\{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} < \begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} < \begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} < \begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array} \right\}$$

and

$$S^3_0 := \{\cdot\}, \ S^3_1 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^3_2 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^3_3 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^3_4 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^3_5 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}, \ S^3_6 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}. \ S^3_7 := \{\begin{array}{c}
\hline
& & & \\
& & & \\
\hline
\hline
\end{array}\}.$$}

See [13] that there is a recursive pattern that relates $S^{k+1}$ to $S^k$.

3.2. The k-Dirac complex. For $a = (a_1, \ldots, a_k) \in S^k$ we denote by $\mathcal{W}_a$ an irreducible $\text{GL}(k, \mathbb{C})$-module with lowest weight

$$-\left(\frac{1 - 2n}{2} - a_k, \ldots, \frac{1 - 2n}{2} - a_1\right).$$

Let $S$ be the complex space of spinors as in Introduction. Recall that $S = \mathcal{S}_+ \oplus \mathcal{S}_-$ where $\mathcal{S}_+$, $\mathcal{S}_-$ are two non-isomorphic complex spinor modules of
Spin$(2n, \mathbb{C})$ as in [23]. Then $\mathcal{V}_a := \mathcal{W}_a \otimes \mathbb{S}$ is an irreducible complex $G^C_0$-module and thus also an irreducible complex representation of $P^C$ and $P$.

We put

\begin{equation}
(3.1) \quad \mathcal{V}_j := \bigoplus_{a \in S^k_j} \mathcal{V}_a, \quad \mathcal{V}_a := G \times_P \mathcal{V}_a \quad \text{and} \quad \mathcal{V}_j = G \times_P \mathcal{V}_j.
\end{equation}

For $a \in S^k_j$, we denote by $s_a$ the $a$-th component of $s \in \Gamma(\mathcal{V}_j)$ so that $s = (s_a)_{a \in S^k_j}$.

**Proposition 3.3.** Assume that $n \geq k \geq 2$, $j \geq 0$, $a \in S^k_j$, $a' \in S^k_{j+1}$ and that $a < a'$. Put $r := |a'| - |a|$.

(a) Then $1 \leq r \leq 2$ and there is an invariant and $\mathbb{C}$-linear differential operator

\begin{equation}
(3.2) \quad D^a_{a'} : \Gamma(\mathcal{V}_a) \to \Gamma(\mathcal{V}_{a'})
\end{equation}

of the weighted order $r$.

(b) There is a complex of differential operators

\begin{equation}
(3.3) \quad \Gamma(\mathcal{V}_0) \xrightarrow{D_0} \Gamma(\mathcal{V}_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{j-1}} \Gamma(\mathcal{V}_j) \xrightarrow{D_j} \Gamma(\mathcal{V}_{j+1}) \xrightarrow{D_{j+1}} \cdots
\end{equation}

where

\begin{equation}
(3.4) \quad (D_j s)_{a'} := \sum_{a \in S^k_j, a < a'} D^a_{a'} s_a, \quad a' \in S^k_{j+1}, \quad s \in \Gamma(\mathcal{V}_j).
\end{equation}

**Proof.** (a) The bounds on $r$ follow easily from the definitions. If there is a nonzero, linear and $G$-invariant differential operator (3.2), then from the action of the grading element on $\mathcal{V}_a$ and $\mathcal{V}_{a'}$ given above and Proposition 2.1, it follows that it is a differential operator of the weighted order $r$. Hence, it is enough to show that there is a $P$-equivariant map $J \ell : \mathcal{V}_a \to \mathcal{V}_{a'}$ for some integer $\ell \geq 0$. But this follows from Lemmata 2.4, [23, 6.1 and 7.8].

(b) This easily follows from [23, Theorem 6.2] and the isomorphism (2.26).

We call the complex of invariant differential operators from Proposition 3.3 the $k$-Dirac complex. Let $a, a', r$ and $D^a_{a'}$ be as in Proposition 3.3. Recall (2.18) that the $s$-th prolongation of $D^a_{a'}$ is a linear map

\begin{equation}
(3.5) \quad \gr D^a_{a'} : \gr^{r+s} \mathcal{V}_{a'} \to \gr^s \mathcal{V}_a.
\end{equation}

This map is homogeneous of degree $-r$. It will be convenient to shift the gradings so that the map (3.5) is homogeneous of degree $-1$. For this we notice that $q(a') = q(a) + r - 1$ and thus, if we put

\begin{equation}
(3.6) \quad \gr^i \mathcal{V}_a[\uparrow] := \gr^{i-q(a)} \mathcal{V}_a
\end{equation}

the map (3.5) becomes

\begin{equation}
(3.7) \quad \gr D^a_{a'} : \gr^i \mathcal{V}_a[\uparrow] \to \gr^{i-1} \mathcal{V}_{a'}[\uparrow]
\end{equation}

\[\text{Notice that we denoted this module in [23] by } \mathcal{V}_{\mu_a}. \text{ More precisely } \mathcal{V}_{\mu_a} \text{ is a summand of } \mathcal{V}_a \text{ given by choosing } S_+ \text{ or } S_- \text{ in } S.\]
where \( i = r + s + q(a) \). If we put
\[
(3.8) \quad \text{gr}^i V_j[\uparrow] = \bigoplus_{a \in S^k_j} \text{gr}^i V_a[\uparrow],
\]
then the differential operator \( D_j \) induces for each \( i \geq 0 \) a linear map
\[
(3.9) \quad \text{gr} D_j \colon \text{gr}^i V_j[\uparrow] \to \text{gr}^{i-1} V_{j+1}[\uparrow].
\]

Let \( V \) be a \( P \)-module. Then by (2.21), we can view for the vector space \( \text{gr}^r V \) as a linear subspace of \( \text{gr}^r V \). We put for each \( a \in S^k \)
\[
(3.10) \quad \text{gr}^i V_a[\uparrow] := \text{gr}^{i-q(a)} V_a
\]
and
\[
(3.11) \quad \text{gr}^i V_j[\uparrow] = \bigoplus_{a \in S^k_j} \text{gr}^i V_a[\uparrow]
\]
so that \( \text{gr}^r V_a[\uparrow] \subset \text{gr}^r V_a[\uparrow] \) and \( \text{gr}^r V_j[\uparrow] \subset \text{gr}^r V_j[\uparrow] \). Then we have the following proposition.

**Proposition 3.4.** The \( k \)-Dirac complex induces for each \( i + j \geq 0 \) a long exact sequence
\[
(3.12) \quad \text{gr}^{i+j} V_0[\uparrow] \xrightarrow{\text{gr} D_0} \text{gr}^{i+j-1} V_1[\uparrow] \to \ldots \to \text{gr}^i V_j[\uparrow] \xrightarrow{\text{gr} D_j} \text{gr}^{i-1} V_{j+1}[\uparrow] \to \ldots
\]
This sequence contains a long exact subsequence
\[
(3.13) \quad \text{gr}^{i+j} V_0[\uparrow] \to \text{gr}^{i+j-1} V_1[\uparrow] \to \ldots \to \text{gr}^i V_j[\uparrow] \to \text{gr}^{i-1} V_{j+1}[\uparrow] \to \ldots
\]
**Proof.** This follows from [23, Theorem 7.14] and the discussion from Section 2.6. \( \square \)

### 3.3. The exactness of the \( k \)-Dirac complex in the sense of formal power series

By the discussion that is given at the end of Section 2.3, the \( k \)-Dirac complex induces a complex
\[
(3.14) \quad C^\infty(G_-, V_0) \xrightarrow{D_0} C^\infty(G_-, V_1) \to \ldots \to C^\infty(G_-, V_j) \xrightarrow{D_j} C^\infty(G_-, V_{j+1}) \to \ldots
\]
of linear and \( G_- \)-invariant differential operators with polynomial coefficients.

We will now show that this complex is exact with formal power series at any point \( x \in G_- \). Since each operator in the complex is invariant, it is certainly enough to show this at the origin \( e \in G_{-2} \). Here we will exploit the fact that with respect to the shifted grading introduced above each operator \( D_j \) is homogeneous of degree -1.

Recall (2.14) that the space of polynomials \( S(p_+) \) over the affine set \( G_- \) has a natural grading \( S(p_+) = \bigoplus_{r \geq 0} \mathcal{S}^r p_+ \). Thus, \( \mathcal{S}^r p_+ \otimes V \) is the vector space of functions \( G_- \to V \) whose components are (with respect to a fixed basis of \( V \)) polynomials from \( \mathcal{S}^r p_+ \). On the other hand by (2.13), we can view \( \mathcal{S}^r p_+ \otimes V \) for any irreducible \( P \)-module as the vector space \( \text{gr}^r V \) and we will do that without further comment. Hence, there is a canonical isomorphism \( \mathcal{S}^r p_+ \otimes V \to \text{gr}^r V \) which appears in the commutative diagram (2.19) as the left vertical arrow.
Let $\tilde{\Phi}(V_j)$ be the vector space of formal power series centered at $e \in G_{-2}$ with values in the vector space $V_j$. So $\Psi \in \tilde{\Phi}(V_j)$ can be written in a unique way as a formal sum $\sum_{i \geq 0} \Psi_i$ where $\Psi_i \in \text{gr}^r V_j[\uparrow]$. We obviously have $D_j \Psi = \sum_{i \geq 0} D_j \Psi_i$.

**Theorem 3.5.** The sequence

\[
\tilde{\Phi}(V_0) \xrightarrow{D_0} \tilde{\Phi}(V_1) \rightarrow \cdots \rightarrow \tilde{\Phi}(V_j) \xrightarrow{D_j} \tilde{\Phi}(V_{j+1}) \rightarrow \cdots
\]

is exact.

**Proof.** Let $\Psi = \sum_{i \geq 0} \Psi_i$ be as above. Then $\Psi_i$ is a $V_j$-valued polynomial function on $G_{-2}$ which (as explained above) we view as an element of $\text{gr}^r V_j[\uparrow]$. Now $D_j \Psi_i$ is a $V_{j+1}$-valued polynomial on $G_{-2}$ which by the commutativity of the diagram (2.19) and (3.9) belongs to $\text{gr}^{r-1} V_{j+1}$. Now we can easily complete the proof. We see that $D_j \Psi = 0$ if, and only if $D_j \Psi_i = 0$ for each $i \geq 0$. Now recall Proposition 3.4 that the complex (3.12) is exact for each $i + j \geq 0$. This immediately implies that also the complex (3.15) is exact. $\square$

Let us now recall a formula for the first operator $D_0$. We have $V_0 \cong S$ and $V_1 \cong \mathbb{C}^k \otimes S$ as vector spaces and we view $V_1$ as the vector space of $k$-tuples of spinors as in (1.1). It is shown in [25] that

\[
D_0 f = \sum_{\alpha=1}^{2n} (\varepsilon_{\alpha} L_{e_{\alpha} \otimes e_\alpha} f, \ldots, \varepsilon_{\alpha} L_{e_{\alpha+1} \otimes e_\alpha} f)
\]

where $f \in C^\infty(G_{-2}, S)$ and $\varepsilon_{\alpha}, \in \text{End}(S)$ is the usual action of $\varepsilon_{\alpha}$ on $S$.

**3.4. The resolution of the $k$-Dirac operator.** By Proposition 2.2 the complex (3.14) descends to a complex

\[
C^\infty(U, V_0) \xrightarrow{D_0} C^\infty(U, V_1) \xrightarrow{D_1} \cdots \rightarrow C^\infty(U, V_j) \xrightarrow{D_j} C^\infty(U, V_{j+1}) \rightarrow \cdots
\]

of linear and constant coefficient differential operators on $M(2n, k, \mathbb{R}) \cong U = G_{-2} \setminus G_{-2}$. From the proof of Proposition 2.2 and (3.16), it immediately follows that the first operator $D_0$ is the $k$-Dirac operator (1.1). It remains to show that the sequence is locally exact.

We will use the following notation. We denote by $x_0 := G_{-2}e$ the origin of $U$ and by $\Phi(V_j)$ the vector space of formal power series centered at the point $x_0$ with values in the vector space $V_j$. An element $\Psi \in \Phi(V_j)$ can be written in a unique way as an infinite formal sum $\sum_{i \geq 0} \Psi_i$ where $\Psi_i \in \text{gr}^r V_j[\uparrow]$. Here we use the fact that for any vector space $V$ the space $\text{gr}^r V$ is naturally isomorphic to the vector space of functions $U \rightarrow V$ whose components are homogeneous polynomials of degree $r$.

**Theorem 3.6.** Assume that $n \geq k \geq 2$. Then the complex (3.17) is a locally exact sequence of linear and constant coefficient differential operators which starts with the $k$-Dirac operator.
Proof. By Theorem a. from [19], it is enough to show that the complex (3.17) is exact with formal power series at the origin $x_0$, i.e. we need to show that the induced complex

(3.18) \[ \Phi(V_0) \xrightarrow{D_0} \Phi(V_1) \rightarrow \ldots \rightarrow \Phi(V_j) \xrightarrow{D_j} \Phi(V_{j+1}) \rightarrow \ldots \]

is exact. Let $\Psi = \sum_{i \geq 0} \Psi_i$ be as above. Arguing as in the proof of Theorem 3.5 and using the commutativity of (2.25), one concludes that $D_j \Psi_i \in \text{gr}^{i-1}V_j$. Since $D_j \Psi = \sum_{i \geq 0} D_j \Psi_i$, we see that $D_j \Psi = 0$ if, and only if $D_j \Psi_i = 0$ for each $i \geq 0$. The exactness of (3.18) then follows from the fact that the complex (3.13) is exact for each $i+j \geq 0$. \[ \square \]

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