Kröger’s type upper bounds for Dirichlet eigenvalues of the fractional Laplacian

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Abstract

The purpose of this paper is to provide an upper bound for the increasing sequence of eigenvalues \( \{ \lambda_{s,i}(\Omega) \} \) to the Dirichlet problem

\[
(-\Delta)^s u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

where \((-\Delta)^s\) is the fractional Laplacian operator defined in the principle value sense, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with \(N \geq 1\). We were able to establish an upper bound of the sum of the eigenvalues. This important result is obtained by a subtle computation of Rayleigh quotient for specific functions. Our method is inspired with Kröger’s one in [22].

Keywords: Dirichlet eigenvalues; Fractional Laplacian.

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1 Introduction and main results

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\) with the integer \(N \geq 1\). The main purpose of this paper is to study the upper bounds of eigenvalues of the Dirichlet problem

\[
\begin{cases}
(-\Delta)^s u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(1.1)

where \((-\Delta)^s\) is the fractional laplacian defined in the following sense (principle value):

\[
(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy
\]

(1.2)

with \(c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(1-s\right)}\) and \(\Gamma\) being the Gamma function, see e.g. [32]. Recall that, for \(s \in (0,1)\), the fractional Laplacian of a function \(u \in C_0^\infty(\mathbb{R}^N)\) can also be defined by:

\[
\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \hat{u}(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.
\]

Here and in the sequel both \(\mathcal{F}\) and \(\hat{\cdot}\) denote the Fourier transform.

During the last years, there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially for the fractional Laplacian. This was motivated by numerous applications, which necessitated a significant progress in the theory of linear and nonlinear partial differential equations, see basic properties [29], regularities [2, 31], Liouville property [3], general nonlocal operator [6], fractional Pohozaev identity [32], singularities [14, 15], uniqueness [16], fractional variational setting [11, 17, 20, 33] and the references therein.
To analyze the fractional Dirichlet eigenvalues, we denote $\mathbb{H}_0^s(\Omega)$ the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < +\infty.
$$

We shall see that $\mathbb{H}_0^s(\Omega)$ is a Hilbert space with inner product

$$
\mathcal{E}_s(u, w) = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy
$$

and the induced norm $\|u\|_s = \sqrt{\mathcal{E}_s(u, u)}$. A function $u \in \mathbb{H}_0^s(\Omega)$ will then be called an eigenfunction of (1.1) corresponding to the eigenvalue $\lambda$ if

$$
\mathcal{E}_s(u, w) = \lambda \int_{\Omega} uw \, dx \quad \text{for all } w \in \mathbb{H}_0^s(\Omega).
$$

Here if necessary, the above inner product is replaced by $\mathcal{E}_s(u, w) = \lambda \int_{\Omega} uw \, dx$ for complex functions $u, w \in \mathbb{H}_0^s(\Omega)$. It is known that problem (1.1) admits a sequence of real eigenvalues

$$
0 < \lambda_{s,1}(\Omega) < \lambda_{s,2}(\Omega) \leq \cdots \leq \lambda_{s,i}(\Omega) \leq \lambda_{s,i+1}(\Omega) \leq \cdots
$$

and corresponding eigenfunctions $\phi_i, i \in \mathbb{N}$ such that the following holds:

(a) $\lambda_{s,i}(\Omega) = \min \{ \mathcal{E}_s(u, u) : u \in \mathbb{H}_0^s(\Omega), \|u\|_{L^2(\Omega)} = 1 \}$, where

$$
\mathbb{H}_1(\Omega) := \mathbb{H}_0^s(\Omega) \quad \text{and} \quad \mathbb{H}_{0,i}(\Omega) := \{ u \in \mathbb{H}_0^s(\Omega) : \int_{\Omega} u \phi_i \, dx = 0 \text{ for } i = 1, \ldots, i-1 \} \quad \text{for } i > 1;
$$

(b) $\{ \phi_i : i \in \mathbb{N} \}$ is an orthonormal basis of $L^2(\Omega)$;

(c) $\phi_1$ is strictly positive in $\Omega$. Moreover, $\lambda_{s,1}(\Omega)$ is simple, i.e., if $u \in \mathbb{H}_0^s(\Omega)$ satisfies (1.1) in weak sense with $\lambda = \lambda_{s,1}(\Omega)$, then $u = t \phi_1$ for some $t \in \mathbb{R}$;

(d) $\lim_{i \to \infty} \lambda_{s,i}(\Omega) = +\infty$.

In the classical setting ($s = 1$), the asymptotic behavior of eigenvalues attracted the attention of mathematicians since 1912. Indeed, in [30], he was able to show that the $k$-th eigenvalue $\mu_k(\Omega)$ of Dirichlet problem with $s = 1$, i.e., the Laplacian, has the asymptotic behavior $\lambda_{1,k}(\Omega) \sim C_N(k|\Omega|)^{\frac{1}{N}}$ as $k \to +\infty$, where $C_N = (2\pi)^{\frac{N}{2}}|B_1|^{-\frac{N}{N+2}}$. Later, Pólya [30] (in 1960) proved that

$$
\lambda_{1,k}(\Omega) \geq C(k/|\Omega|)^{\frac{1}{N}}
$$

(1.3)

holds for $C = C_N$ and any tiling Domain $D$ in $\mathbb{R}^2$, (his proof also works in dimension $N \geq 3$). He also conjectured that (1.3) holds with $C = C_N$ for any bounded domain in $\mathbb{R}^N$. Lieb [25] proved (1.3) with a positive constant $C$ for general bounded domain and Li-Yau [24] improved the constant $C = \frac{N}{N+2}C_N$. With this famous constant, (1.3) is now called Brezis-Lieb-Yau inequality. It has played a crucial role in the study of linear elliptic operators [9,10,12,13,22,25,28]. The upper bounds of Dirichlet eigenvalues are derived by Kröger in [22] by calculating the Rayleigh quotient by using a sequence of functions approaching the characterized function of $\Omega$. We also refer to Yang’s upper bounds of the Dirichlet’s eigenvalues in [8,9] in the following way:

$$
\lambda_{1,k}(\Omega) \leq c(N,k)k^{\frac{2}{N}} \lambda_{1,1}(\Omega) \quad \text{for some } c(N,k) > 0.
$$

For the fractional laplacian $(-\Delta)^s$, the Wely’s estimate was shown in [12] and the lower bounds of the Dirichlet’s eigenvalues were formulated in [19,35] in the following

$$
\lambda_{s,k}(\Omega) \geq \frac{(2\pi)^{2s}N}{N+2s}(|B_1||\Omega|)^{-\frac{2s}{N}} k^{\frac{2s}{N}}.
$$

(1.4)
In particular the lower bounds for Klein-Gordon operators $\sqrt{-\Delta + m^2}$ are obtained in [19]. For the upper bounds of fractional Dirichlet eigenvalues, Yang type inequality has been obtained in [7]:

$$\lambda_{s,k}(\Omega) \leq c(N,k)k^{\frac{2s}{N}}\lambda_{s,1}(\Omega)$$

for some $c(N,k) > 0$ and some $s \in (0,1)$.

However, this type of inequality heavily depends on a very precise estimates of $\lambda_{s,1}(\Omega)$. For a more detailed account about that, the reader can refer to [17].

Despite the importance and the numerous relevant applications of the establishment of an upper bound for the eigenvalues for (1.1), the literature remained silent until very recently the survey [17]. The nonlocal aspect makes this problem very complicated. Additionally, Caffarelli and Silvestre extension does not help in this case. Therefore, all techniques developed to address the bounds of eigenvalues for (1.1) when $s = 1$, do not extend to the fractional setting.

The main objective of this work is to provide an upper bound for the sum of eigenvalues of (1.1). The main result of this paper is:

**Theorem 1.1.** Assume that $\Omega$ is a bounded domain in $\mathbb{R}^N$ such that for some $R > 0$,

$$B_R \subset \Omega \subset B_{2R}$$

and there exists $c_0 > 0$ such that

$$|\Omega_1| \leq c_0 R^{N-1},$$

where $\Omega_1 = \{x \in \Omega : \rho(x) = \text{dist}(x, \partial \Omega) = 1\}$. Let $\{\lambda_{s,i}(\Omega)\}_{i \in \mathbb{N}}$ be the increasing sequence of eigenvalues of problem (1.1). Then there exists $c_1 > 0$ independent of $k$ such that for $k \in \mathbb{N}$

$$\sum_{i=1}^{k} \lambda_{s,i}(\Omega) \leq \frac{(2\pi)^{2s} N}{N + 2s} (|B_1||\Omega|)^{-\frac{2s}{N}} k^{1+\frac{2s}{N}} + c_1 k^{1+\frac{2s}{N}}.$$

Compared with the lower bound (1.1), our upper bound in Theorem 1.1 provides an sharp main term $\frac{(2\pi)^{2s} N}{N + 2s} (|B_1||\Omega|)^{-\frac{2s}{N}} k^{1+\frac{2s}{N}}$. Our proof is inspired by the method of Kr"oger in [22]. The major difficulty is to do estimates for $(-\Delta)^s(w_\sigma(x)e^{iz})$, where

$$w_\sigma(x) = \eta_0(\sigma^{-1} \rho(x)), \quad \forall x \in \mathbb{R}^N.$$ 

Here $\eta_0$ is a $C^2$ increasing function such that

$$\eta_0(t) = 1 \text{ if } t \geq 1, \quad \eta_0(t) = 0 \text{ if } t \leq 0.$$

Indeed, we have the following decomposition:

$$(-\Delta)^s(w_\sigma(x)e^{iz}) = w_\sigma(x)(-\Delta)^s e^{iz} + e^{iz}(-\Delta)^s w_\sigma(x) + L^s_\sigma w_\sigma,$$

where

$$L^s_\sigma w_\sigma = c_{N,s} \int_{\mathbb{R}^N} \frac{(w_\sigma(x) - w_\sigma(\tilde{x}))(e^{iz} - e^{i\tilde{z}})}{|x - \tilde{x}|^{N+2s}} d\tilde{x}.$$ 

The dominating term is $w(x)(-\Delta)^s e^{iz}$. For the latter, we obtain the following identity:

$$(-\Delta)^s e^{iz} = |z|^{2s} e^{iz}, \quad \forall x \in \mathbb{R}^N$$

for any given $z \in \mathbb{R}^N$.

Together with the lower bound of the sum of eigenvalues, we can obtain the limit as following:

**Corollary 1.2.** Under the assumptions of Theorem 1.1 we have that

$$\lim_{k \to +\infty} k^{-1-\frac{2s}{N}} \sum_{i=1}^{k} \lambda_{s,i}(\Omega) = \frac{(2\pi)^{2s} N}{N + 2s} (|B_1||\Omega|)^{-\frac{2s}{N}}.$$ 

Throughout this paper, $e$ denotes the Euler number, $\rho(x) = \text{dist}(x, \partial \Omega)$ for $x \in \mathbb{R}^N$, $B_r(x) \subset \mathbb{R}^N$ is an open ball of radius $r$ centered at $x \in \mathbb{R}^N$, and we put $B_r := B_r(0)$ for $r > 0$. The rest of this paper is organized as following: Section 2 is devoted to the normalization of the constant $d_\sigma(s)$. In Section 3, we provide the proofs of our results.
2 Preliminary

For an integer $m \geq 2$, we denote
\[
E_m(m + 2s) = \int_0^\infty \frac{t^{m-2}}{(1 + t^2)^{(m+2)s}} dt,
\]
then we have that
\[
E_2(2 + 2s) = \int_0^\infty \frac{1}{(1 + t^2)^{1+s}} dt = \frac{1}{2} \int_0^\infty \frac{t^{-\frac{1}{2}}}{(1 + t)^{1+s}} dt = \frac{1}{2} B\left(\frac{1}{2}, s + \frac{1}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma(s + \frac{1}{2}) \Gamma(1 + s),
\]
and for $m \geq 4$
\[
E_m(m + 2s) = \int_0^\infty \frac{1}{(1 + t^2)^{\frac{1}{2}+s}} d(1 + t^2) = \frac{1}{1 + 2s}
\]
and
\[
E_3(3 + 2s) = \frac{1}{2} \int_0^\infty \frac{1}{(1 + t^2)^{\frac{3}{2}+s}} d(1 + t^2) = \frac{1}{1 + 2s}
\]
and for $m \geq 4$
\[
E_m(m + 2s) = m - 3 \quad m - 2 \quad m - 2 \quad m - 2
\]
and for $n \geq 5$
\[
E_m(m + 2s) = E_3(3 + 2s) \cdot \frac{2}{3 + 2s} \cdot \frac{m - 3}{m - 2 + 2s} = \frac{1}{1 + 2s} \cdot \frac{2}{3 + 2s} \cdot \frac{m - 3}{m - 2 + 2s}
\]
We remark that for $n \geq 4$ and $s = \frac{1}{2}$, we have that
\[
E_m(m + 1) = E_2(3) \cdot \frac{1}{3} \cdot \frac{m - 3}{m - 1} = \frac{1}{m - 1}, \quad m \text{ is even}
\]
and
\[
E_m(m + 1) = \frac{1}{2} \cdot \frac{2}{4} \cdot \frac{m - 3}{m - 1} = \frac{1}{m - 1}, \quad m \text{ is odd}.
\]
For $N \geq 2$, denote
\[
b_N(s) = \frac{2}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{N+2s}{2}\right)} E_N(N + 2s)
\]
and $b_N(s) \equiv 1$ for $N = 1$.

Lemma 2.1. Let $b_N(s)$ be defined in (2.3), then for $N \geq 2$
\[
b_N(s) = 1.
\]

Proof. It is obvious for $N = 2, 3$.

When $N \geq 5$ is odd, in view of (2.2), we obtain that
\[
b_N(s) = \frac{2}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{N+2s}{2}\right) - 1} E_{N-2}(N + 2s - 2) \cdot \frac{N - 3}{N + 2s - 2} \cdot \frac{N - 2}{N + 2s - 2}
\]
\[
= \frac{2}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{N+2s}{2} - 1\right)} E_{N-2}(N + 2s - 2)
\]
\[
= \ldots
\]
\[
= \frac{2}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{3+2s}{2}\right)} E_3(3 + 2s)
\]
\[
= 1,
\]
where we used $E_3(3 + 2s) = \frac{1}{1 + 2s}$.

When $N \geq 4$ is even, in view of (2.1), we obtain that

$$b_N(s) = \frac{2}{\Gamma(s + \frac{1}{2})} \frac{N + 2s}{N - 1} \frac{\Gamma(N - 1)}{\Gamma(N - \frac{1}{2})} E_{N-2}(N + 2s - 2) \frac{N - 3}{N + 2s - 2}$$

$$= \frac{2}{\Gamma(s + \frac{1}{2})} \frac{\Gamma(N - 1)}{\Gamma(N - \frac{1}{2})} E_{N-2}(N + 2s - 2)$$

$$= \cdots$$

$$= \frac{2}{\Gamma(s + \frac{1}{2})} \frac{\Gamma(1 + s)}{\Gamma(\frac{1}{2})} E_2(2 + 2s)$$

$$= 1.$$

This completes the proof. \hfill \square

3 Upper bounds

The following lemma plays an important role in our proof of Theorem 1.1.

Lemma 3.1. For fixed $z \in \mathbb{R}^N \setminus \{0\}$, denote

$$v_z(x) = e^{ix \cdot z}, \quad \forall x \in \mathbb{R}^N,$$

then

$$(-\Delta)^s v_z(x) = |z|^{2s} v_z(x), \quad \forall x \in \mathbb{R}^N. \quad (3.1)$$

Proof. Without loss of generality, we only need to calculate (3.1) with $z = te_1$, where $t > 0$ and $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$. For this, we write

$$v_t(x) = v_t(x_1) = e^{itx_1}, \quad x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

Note that for $N \geq 2$

$$(-\Delta)^s v_t(x) = c_{N,s} \text{p.v.} \int_{\mathbb{R}^N} \frac{v_t(x) - v_t(y)}{|x - y|^{N + 2s}} dy$$

$$= c_{N,s} \left( c_{N,s} \int_{\mathbb{R}^N} \frac{v_t(x_1) - v_t(y_1)}{|x_1 - y_1|^{1 + 2s}} dy_1 \int_{\mathbb{R}^{N-1}} \frac{1}{(|y'|^2 + 1)^{N-1}} dy'ight)$$

$$= \frac{2}{\Gamma(1 + 2s)} \frac{\Gamma(N - 1)}{\Gamma(N - \frac{1}{2})} \int_0^\infty \frac{t^{N-2}}{(t^2 + 1)^{N - \frac{1}{2}}} dt (-\Delta)^s_{\mathbb{R}} v_t(x_1)$$

$$= (-\Delta)^s_{\mathbb{R}} v_t(x_1),$$

where we used lemma 2.1

$$\omega_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})},$$

and

$$\frac{c_{N,s}}{c_{1,s}} = \pi^{-\frac{N-1}{2}} \frac{\Gamma(N + 2s)}{\Gamma(1 + 2s)}.$$

Now we claim that

$$(-\Delta)^s_{\mathbb{R}} v_t(x_1) = t^{2s} v_t(x_1), \quad \forall x_1 \in \mathbb{R}. \quad (3.2)$$

Indeed, observe that $-\Delta v_t = t^2 v_t$ in $\mathbb{R}$ and then

$$(|\xi|^2 - t^2) \hat{v}_t = \mathcal{F}(-\Delta v_t - t^2 v_t) = 0,$$
which implies that 
\[ \text{supp}(\hat{v}_t) \subset \{ \pm t \}. \]
Thus, we have that 
\[ \mathcal{F}
\left((-\Delta)^s_{\mathbb{R}} v_t - t^{2s} v_t \right)(\xi_1) = (|\xi_1|^{2s} - t^{2s})\hat{v}_t(\xi_1) = 0 \]
and 
\[ (-\Delta)^s_{\mathbb{R}} v_t = t^{2s} v_t \quad \text{in} \; \mathbb{R}. \]

Now we can conclude that 
\[ (-\Delta)^s v_t(x) = (-\Delta)^s_{\mathbb{R}} v_t(x) = t^{2s} v_t(x), \quad \forall x \in \mathbb{R}^N. \]
This completes the proof. \hfill \Box

Let \( \eta_0 \) be a \( C^2 \) increasing function such that \( \|\eta_0\|_{C^2}, \|\eta_0\|_{C^2} \leq 2 \)
\[ \eta_0(t) = 1 \quad \text{if} \; t \geq 1, \quad \eta_0(t) = 0 \quad \text{if} \; t \leq 0. \]
For \( \sigma > 0 \), denote
\[ w_\sigma(x) = \eta_0(\sigma^{-1}\rho(x)), \quad \forall x \in \mathbb{R}^N. \quad (3.3) \]
Observe that \( w_\sigma \in \mathbb{H}^s(\Omega) \) and 
\[ w_\sigma \to 1 \quad \text{in} \; \Omega \quad \text{as} \; \sigma \to 0^+. \]

**Lemma 3.2.** Let \( B_R \subset \Omega \subset B_{2R} \), then 
\[ |(-\Delta)^s w_\sigma(x)| \leq 2c_{N,s}\omega_{N-1}\sigma^{-2s} \quad \text{for} \; x \in \Omega. \]

**Proof.** For \( x \in \Omega \), we have that 
\[ |2w_\sigma(x) - w_\sigma(x + \zeta) - w_\sigma(x - \zeta)| \leq \min\{2, \|w_\sigma\|_{C^2}\zeta^2\} \]
\[ \leq \min\{2, \sigma^{-2}\|\eta_0\|_{C^2}\zeta^2\} \]
We use an equivalent definition
\[ \frac{2}{c_{N,s}}|(-\Delta)^s w_\sigma(x)| = \left| \int_{\mathbb{R}^N} \frac{2w_\sigma(x) - w_\sigma(x + \zeta) - w_\sigma(x - \zeta)}{\zeta^{N+2s}} d\zeta \right| \]
\[ \leq \int_{\mathbb{R}^N} \min\{2, \sigma^{-2}\|\eta_0\|_{C^2}\zeta^2\} \frac{1}{\zeta^{N+2s}} d\zeta \]
\[ \leq 2\sigma^{-2} \int_{B_2} \frac{\zeta^2}{\zeta^{N+2s}} d\zeta + \int_{\mathbb{R}^N \setminus B_2} \frac{2}{\zeta^{N+2s}} d\zeta \]
\[ \leq 4\omega_{N-1}\sigma^{-2s}, \]
where \( \|\eta_0\|_{C^2} \leq 2 \). This completes the proof. \hfill \Box

**Lemma 3.3.** Let \( B_R \subset \Omega \subset B_{2R} \) and 
\[ \mathcal{L}^s_z w_\sigma(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{(w_\sigma(x) - w_\sigma(\tilde{x}))(e^{i\xi \cdot z} - e^{i\xi \cdot \tilde{x}})}{|x - \tilde{x}|^{N+2s}} d\tilde{x}. \]
Then we have that for \( x \in \Omega \)
(i) for \( s \in (\frac{1}{2}, 1) \),
\[ \frac{1}{c_{N,s}}|\mathcal{L}^s_z w_\sigma(x)| \leq \frac{\omega_{N-1}}{1 - s} \sigma^{-1}|z|^{2s-1} + \frac{\omega_{N-1}}{2s - 1} |z|^{2s-1} + \frac{\omega_{N-1}}{2s} \tilde{R}^{-2s}; \]
(ii) for $s = \frac{1}{2}$,

$$
\frac{1}{c_{N,s}}|L^s w_\sigma(x)| \leq \frac{\omega_{N-1}}{1-s} \sigma^{-1} + \omega_{N-1} (\log |z| + \log(4R)) + \frac{\omega_{N-1}}{2s} R^{-1};
$$

(iii) for $s \in (0, \frac{1}{2})$,

$$
\frac{1}{c_{N,s}}|L^s w_\sigma(x)| \leq \frac{\omega_{N-1}}{1-s} \sigma^{-1} |z|^{2s-1} + \frac{\omega_{N-1}}{1-2s} (4R)^{1-2s} + \frac{\omega_{N-1}}{2s} R^{-2s}.
$$

**Proof.** Note that

$$
|e^{i\hat{\varphi}z} - e^{ixz}| \leq \min\{2, |z||\hat{x} - x|\}
$$

and

$$
|w_\sigma(x) - w_\sigma(\hat{x})| \leq \frac{2}{\sigma} |x - \hat{x}|, \quad |\hat{x}| < 3R.
$$

For $x \in \Omega$ and $|z| > 1$, we have that

$$
\frac{1}{c_{N,s}}|L^s w_\sigma(x)| \leq \int_{\mathbb{R}^N} \frac{|w_\sigma(x) - w_\sigma(\hat{x})| |e^{i\hat{\varphi}z} - e^{ixz}|}{|x - \hat{x}|^{N+2s}} \, d\hat{x}
$$

and

$$
\frac{1}{c_{N,s}}|L^s w_\sigma(x)| \leq \int_{B_{4R}} \frac{2\sigma^{-1}|x - \hat{x}| \min\{2, |z||\hat{x} - x|\}}{|x - \hat{x}|^{N+2s}} \, d\hat{x} + \int_{\mathbb{R}^N \setminus B_{4R}} \frac{2}{|x - \hat{x}|^{N+2s}} \, d\hat{x}
$$

where

$$
2\sigma^{-1}|z| \int_{B_{\frac{1}{2}}(x)} |x - \hat{x}|^{2s-N-2s} \, d\hat{x} \leq \frac{\sigma^{-1} \omega_{N-1}}{1-s} |z|^{2s-1},
$$

and

$$
\int_{\mathbb{R}^N \setminus B_{R}} \frac{2}{|x|^{N+2s}} \, dx \leq \frac{\omega_{N-1}}{2s} R^{-2s}
$$

and

$$
4\sigma^{-1} \int_{B_{4R} \setminus B_{\frac{1}{2}}(x)} |x - \hat{x}|^{1-N-2s} \, d\hat{x} \leq \begin{cases} 
\sigma^{-1} \omega_{N-1} |z|^{2s-1} & \text{if } s \in (\frac{1}{4}, 1), \\
\sigma^{-1} \omega_{N-1} (\log |z| + \log(4R)) & \text{if } s = \frac{1}{2}, \\
\sigma^{-1} \omega_{N-1} (4R)^{1-2s} & \text{if } s \in (0, \frac{1}{2}).
\end{cases}
$$

This completes the proof.

**Proof of Theorem 1.1.** Denote

$$
\Phi_k(x, y) = \sum_{i=1}^{k} \phi_i(x) \phi_i(y)
$$

and

$$
\hat{\Phi}_k(z, y) = (2\pi)^{-\frac{N}{2}} \int_{x \in \mathbb{R}^N} \Phi_k(x, y) e^{ixz} \, dx,
$$

here $\hat{\Phi}_k$ is the Fourier transform with respect to $x$.

Denote

$$
v_\sigma(x, z) = w_\sigma(x) e^{ixz}.
$$
Note that the projection of $v_\sigma$ onto the subspace of $L^2(\Omega)$ spanned by $\phi_i$ can be written in terms of the Fourier transform $\eta_\sigma \Phi_k$ of $w_\sigma$ with respect to the $x$-variable:

$$\int_\Omega v_\sigma(x, z) \Phi_k(x, y) dx = (2\pi)^{N/2} \mathcal{F}_x(w_\sigma \Phi_k)(z, y).$$

Denote

$$v_{\sigma, k}(z, y) = v_\sigma(z, y) - (2\pi)^{N/2} \mathcal{F}_x(w_\sigma \Phi_k)(z, y)$$

and the Rayleigh-Ritz formula shows that

$$\lambda_{s,k+1}(\Omega) \int_\Omega |v_{\sigma, k}(z, y)|^2 dy \leq \int_\Omega \left| v_{\sigma, k}(z, y)(-\Delta)^{\sigma}_y v_{\sigma, k}(z, y) \right| dy$$

for any $z \in \mathbb{R}^N$ and $\sigma > 0$. Thus, we can conclude that

$$\lambda_{s,k+1}(\Omega) \leq \inf_{\sigma > 0} \frac{\int_{B_r} \int_\Omega |v_{\sigma, k}(z, y)(-\Delta)^{\sigma}_y v_{\sigma, k}(z, y)| dy dz}{\int_{B_r} \int_\Omega |v_{\sigma, k}(z, y)|^2 dy dz}.$$

An elementary calculation yields that

$$\int_{B_r} \int_\Omega |v_{\sigma, k}(z, y)|^2 dy dz = \int_{B_r} \int_\Omega |v_\sigma(z, y)|^2 dy dz - (2\pi)^N \int_{B_r} \int_\Omega \sum_{i=1}^k |\mathcal{F}_x(w_\sigma \phi_i)(z)|^2 \phi_i(y)^2 dy dz$$

$$\geq \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y) dy - (2\pi)^N \int_{B_r} \int_\Omega \sum_{i=1}^k |\mathcal{F}_x(w_\sigma \Phi_k)(z, y)(-\Delta)^{\sigma}_y \mathcal{F}_x(w_\sigma \Phi_k)(z, y)| dy dz,$$

where $r \geq \sigma^{-1}$.

On the other hand,

$$\int_{B_r} \int_\Omega |v_{\sigma, k}(z, y)(-\Delta)^{\sigma}_y v_{\sigma, k}(z, y)| dy dz = \int_{B_r} \int_\Omega |v_\sigma(z, y)(-\Delta)^{\sigma}_y v_\sigma(z, y)| dy dz$$

$$- (2\pi)^N \int_{B_r} \int_\Omega \sum_{i=1}^k |\mathcal{F}_x(w_\sigma \Phi_k)(z, y)(-\Delta)^{\sigma}_y \mathcal{F}_x(w_\sigma \Phi_k)(z, y)| dy dz,$$

where

$$\int_{B_r} \int_\Omega \sum_{i=1}^k |\mathcal{F}_x(w_\sigma \Phi_k)(z, y)(-\Delta)^{\sigma}_y \mathcal{F}_x(w_\sigma \Phi_k)(z, y)| dy dz = \int_{B_r} |\mathcal{F}_x(w_\sigma \phi_i)(z)|^2 dz$$

and

$$\int_{B_r} \int_\Omega |v_\sigma(z, y)(-\Delta)^{\sigma}_y v_\sigma(z, y)| dy dz \leq \int_{B_r} \int_\Omega |w_\sigma^2(y)(-\Delta)^{\sigma}_y e^{iyz}| dy dz$$

$$+ \int_{B_r} \int_\Omega |w_\sigma(y)(-\Delta)^{\sigma}_y w_\sigma(y)| dy dz + \int_{B_r} \int_\Omega |L_2 w_\sigma(y)| dy dz$$

$$\leq \frac{\omega_{N-1}}{N+2s} \int_\Omega w_\sigma^2(y) dy + \frac{\omega_{N-1} r^N}{N} \frac{2c_{N,s}}{\sigma^{2s}} \int_\Omega w_\sigma(y) dy$$

$$+ \frac{\omega_{N-1} r^{N+2s-1}}{\sigma(N+2s-1)(1-s)} \frac{1}{|\Omega|} \phi_s(r, R)|\Omega| + \frac{\omega_{N-1} r^N}{2\sigma s N} R^{-2s} |\Omega|$$

$$:= \frac{\omega_{N-1}}{N+2s} \int_\Omega w_\sigma^2(y) dy + \Psi_s(r, \sigma)$$
with
\[
\phi_s(r, R) = \begin{cases} 
\frac{\omega_{N-1} r^{N+2s-1}}{(2s-1)(N+2s-1)} & \text{if } s \in \left(\frac{1}{2}, 1\right), \\
\frac{\omega_{N-1} r^N \log r + \log(4R)}{N} & \text{if } s = \frac{1}{2}, \\
\frac{\omega_{N-1} r^N}{N(1-2s)} (4R)^{1-2s} & \text{if } s \in \left(0, \frac{1}{2}\right).
\end{cases}
\] (3.4)

Observe that Parseval’s identity implies that
\[
\int_{B_r} |F_x(w_\sigma \phi_i)(z)|^2 dz \leq \int_\Omega |(w_\sigma \phi_i)|^2 dx \leq 1.
\]

and if \( \frac{\omega_{N-1} r^N}{N} r^{N+2s} \geq \frac{\omega_{N-1} r^N}{N} f_\Omega w_\sigma^2(y) dy > (2\pi)^N k \), we have that
\[
\lambda_{s,k+1}(\Omega) \leq \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y) dy + \frac{\omega_{N-1} r^{N+2s-1}}{\sigma(2N+2s)} \int_\Omega \frac{\omega_{N-1} r^N}{N} \int_{B_r} |F_x(w_\sigma \phi_i)(z)|^2 dz
\]
\[
\leq \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y) dy + \frac{\omega_{N-1} r^N}{N} \int_{B_r} |F_x(w_\sigma \phi_i)(z)|^2 dz
\]
\[
\leq \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y) dy - (2\pi)^N k.
\] (3.5)

Therefore, choosing \( \sigma = r^{-\frac{s}{2}} \), we have that
\[
\Psi_s(r, \sigma) = \frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma(y) dy + \frac{\omega_{N-1} r^{N+2s-1}}{\sigma(2N+2s)(N+2s-1)(1-s)} |\lambda_i(\Omega)| + \left| \frac{1}{\sigma} \phi_s(r, R) \right| |\Omega|
\]
\[
+ \frac{\omega_{N-1} r^N}{\sigma(2N)} R^{-2s} |\Omega|
\]
\[
\leq 2c_N \omega_{N-1} \frac{r^N}{N} \int_\Omega w_\sigma(y) dy r^{N+s} + \frac{\omega_{N-1} |\Omega|}{N(2s-1)(1-s)} r^{2s} + 2\pi^{-N} \Psi_s(r, \sigma)
\]
\[
+ r^\frac{s}{2} \phi_s(r, R) |\Omega| + \frac{\omega_{N-1} r^{N+\frac{s}{2}}}{2sN} R^{-2s} |\Omega|,
\]

now choosing \( r > 1 \) large such that
\[
\frac{\omega_{N-1} r^N}{N} \int_\Omega w_\sigma^2(y) dy = (2\pi)^N (k + 1),
\]

we derive that
\[
\sum_{i=1}^k \lambda_s,i(\Omega) \leq (2\pi)^{-N} \frac{\omega_{N-1} r^{N+2s}}{N+2s} \int_\Omega w_\sigma^2(y) dy + (2\pi)^{-N} \Psi_s(r, \sigma)
\]
\[
\leq \frac{N}{N+2s} (2\pi)^2 s (\omega_{N-1})^{-\frac{s}{2}} \left( \int_\Omega w_\sigma^2(y) dy \right)^{-\frac{2s}{N}} (k + 1)^{1+\frac{s}{N}} + (2\pi)^{-N} \Psi_s(r, \sigma)
\]
\[
\leq \frac{N}{N+2s} (2\pi)^2 s |B_1|^{-\frac{2s}{N}} |\Omega|^{-\frac{2s}{N}} \left( 1 + \frac{c}{k} \right) (k + 1)^{1+\frac{s}{N}} + (2\pi)^{-N} \Psi_s(r, \sigma),
\]
where \( \sigma \sim k^{-\frac{s}{N}} \), and
\[
\left( \int_\Omega w_\sigma^2(y) dy \right)^{-\frac{2s}{N}} \leq |\Omega|^{-\frac{2s}{N}} \left( 1 + \frac{c}{k} \right)
\]
by the assumption of \( |\Omega| \leq c_0 R^{N-1} \), using (1.5), we have that
\[
\Psi_s(r, \sigma) \leq c_2 \left( k^{1+\frac{s}{N}} + k^{1+\frac{3s^2}{2N^2}} + k^{1+\frac{2s}{N}} + k^{1+\frac{3s}{N}} \max\{2s-1,0\} + \frac{3s}{N} \log k \right)
\]
\[
\leq c_3 k^{1+\frac{s}{N}}.
\]
where \(c_2, c_3 > 0\) could be chosen independently of \(k\).

In conclusion, we have that

\[
\sum_{i=1}^{k} \lambda_{s,i}(\Omega) \leq \frac{N(2\pi)^{2s}}{N + 2s} (|B_1||\Omega|)^{-\frac{s}{N}} k^{1+s\frac{s}{N}} + c_0 k^{1+s\frac{s}{N}}. \tag{3.6}
\]

This completes the proof. \(\Box\)

**Proof of Corollary 1.2.** From [35, Corollary 2.2], using the Berezin-Li-Yau method, a lower bound could be derived as following

\[
\sum_{i=1}^{k} \lambda_{s,i}(\Omega) \geq \frac{N(2\pi)^{2s}}{N + 2s} (|B_1||\Omega|)^{-\frac{s}{N}} k^{1+s\frac{s}{N}}.
\]

which, combining with (3.6), implies

\[
\lim_{k \to +\infty} k^{-1-s\frac{s}{N}} \sum_{i=1}^{k} \lambda_{s,i}(\Omega) = \frac{(2\pi)^{2s} N}{N + 2s} (|B_1||\Omega|)^{-\frac{s}{N}}.
\]

We complete the proof. \(\Box\)

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