THEORETICAL AND NUMERICAL ANALYSIS OF THE RAPID POINTWISE STABILIZATION OF COUPLED STRING-BEAM SYSTEMS

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Abstract. We consider a pointwise stabilization problem for a coupled wave and plate equations. We prove under rather general assumptions, that such systems can stabilized so as to have arbitrarily high decay rates and are exactly controllable. We propose a numerical approximation of the model and we study numerically the construction of the feedback law leading to exponential decay with arbitrarily large rate.

1. Introduction

Let $\xi, \eta \in (0, \pi)$ are given points, fix four real numbers $A, B, C, D$ and consider the coupled string-beam system, more precisely we have the following partial differential equations with pointwise dissipation:

$$
\begin{align*}
&y_{1,t} - y_{1,xx} + Ay_1 + Cy_2 = v_1(t)\delta_\xi & \text{in } \mathbb{R} \times (0, \pi), \\
&y_{2,t} + y_{2,xxxx} + By_1 + Dy_2 = v_2(t)\delta_\eta & \text{in } \mathbb{R} \times (0, \pi), \\
&y_1(t,0) = y_1(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\
&y_2(t,0) = y_2(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\
&y_{2,xx}(t,0) = y_{2,xx}(t,\pi) = 0 & \text{for } t \in \mathbb{R}, \\
&y_1(0,x) = y_{10}(x) \text{ and } y_1(t,0,x) = y_{11}(x) & \text{for } x \in (0, \pi), \\
&y_2(0,x) = y_{20}(x) \text{ and } y_2(t,0,x) = y_{21}(x) & \text{for } x \in (0, \pi)
\end{align*}
$$

where $v_1(t), v_2(t)$ are the control functions in $L^2_{\text{loc}}(\mathbb{R})$, $\delta_\xi$ and $\delta_\eta$ denotes the Dirac mass at some given points $\xi$ and $\eta \in (0, \pi)$.

The coupled structural model has been of great interest in recent year; for details about the physical motivation for the model see [19], [16] and the references therein. Mathematical analysis of coupled partial differential equations is detailed in [15], [20]. The question of controllability and stabilization for such models has been widely treated in a series of relevant works [1], [2], [3], [4], [13], [19], [18], [22]. Many works were devoted to the construction of explicit feedback laws and to the proof of exponential decay by different methods; see, e.g., [20], [13], [15]. It is known that this type of feedback does not yield arbitrarily large decay rates. It was pointed out earlier by Haraux and Jaffard [2], [11], [10] that the observability and controllability properties depend heavily on the location of the observation of control point. For the stabilization another difficulty appears because the suitable function spaces, as we will show, are not Sobolev spaces.

In this paper we apply another approach for the stabilization of the coupled string-beam system (1.1), which similar in sprit to the HUM. This method, developed by Komornik [15] is as general as the former one; however, it provides stronger results with simpler proofs and it’s the first time that we apply this method to prove the exponential stability of the coupled sting-beam systems with pointwise control. The main result of this paper is to introduce functions spaces depending on the arithmetical properties of the stabilization point and to give an estimate on
exponential decay that is valid for regular initial data, as a result we will construct pointwise feedbacks leading to arbitrarily large prescribed decay rates. Numerical tests and explicit construction of the feedback are presented.

The method used is based on a regularity results combined with an observability inequality for the corresponding undamped problem. See [15].

The paper is organised as follows. The statement and the proof of the main results are given in the sections 2 and 3 respectively. The last section is devoted to the numerical approximation of the coupled string-beam system and the explicit construction of the feedback law.

2. Statement of the main result

In order to formulate our result, we assume that $\xi/\pi$ and $\eta/\pi$ are irrational, so that $\sin k\xi$ and $\sin k\eta$, don’t vanish for any $k = 1, 2, \ldots$, we denote by $Z$ the linear hull of the functions $w_k(x) := \sqrt{2/\pi} \sin kx$, $k = 1, 2, \ldots$, and we denote by $D^{\alpha}_\xi$ and $(D^{\beta}_\eta)'$ for every $\alpha \in \mathbb{R}$ and $\beta \in \{\xi, \eta\}$ the Hilbert spaces obtained by completing $Z$ with respect to norms given by the following formulae:

$$\left\| \sum a_k w_k \right\|^2_{D^{\alpha}_\xi} := \sum k^{2\alpha} \sin^2(k\beta) |a_k|^2,$$

$$\left\| \sum a_k w_k \right\|^2_{(D^{\beta}_\eta)'} := \sum k^{-2\alpha} \sin^{-2}(k\beta) |a_k|^2.$$

If we identify $L^2(0, \pi)$ with its dual and take into account that

$$\| \sum a_k w_k \|^2_{L^2(0, \pi)} := \sum |a_k|^2,$$

then $(D^{\alpha}_\xi)'$ is the dual space of $D^{\alpha}_\xi$.

Fix $\xi, \eta \in (0, \pi)$ such that $\xi/\pi$ and $\eta/\pi$ are irrational and introduce the Hilbert space

$$\mathcal{H}_{\xi, \eta} := (D^{0}_\xi \times D^{-1}_\xi)' \times (D^{0}_\eta \times D^{-2}_\eta)'.'$$

The problem (1.1) is well posed in the Hilbert space $\mathcal{H}_{\xi, \eta}$ in the following sense:

**Proposition 2.1.** for any given initial and final data

$$(y_{10}, y_{11}, y_{20}, y_{21}) \in \mathcal{H}_{\xi, \eta}$$

and

$$v_1, v_2 \in L^2(0, T; \mathcal{H}_{\xi, \eta})$$

The system has a unique weak solution satisfying

$$(y_1, y_{1,t}, y_2, y_{2,t}) \in C([0, T]; \mathcal{H}_{\xi, \eta}),$$

and the linear mapping

$$(y_{10}, y_{11}, y_{20}, y_{21}, v_1, v_2) \mapsto (y_1, y_{1,t}, y_2, y_{2,t})$$

is continuous with these topologies.

We shall study the controllability of the system.

**Definition.** Fix $\xi, \eta \in (0, \pi)$ such that $\xi/\pi$ and $\eta/\pi$ are irrational. the system (1.1) is exactly controllable if for any given initial and final data

$$(y_{10}, y_{11}, y_{20}, y_{21}) \in \mathcal{H}_{\xi, \eta}$$

and

$$(z_{10}, z_{11}, z_{20}, z_{21}) \in \mathcal{H}_{\xi, \eta}$$

there exist control functions

$$v_1, v_2 \in L^2(0, T; \mathcal{H}_{\xi, \eta})$$
such that the corresponding solution of (1.1) satisfies the final condition
\[ (y_1, y_{1t}, y_2, y_{2t})(T) = (z_{10}, z_{11}, z_{20}, z_{21}). \]

**Theorem 2.2.** If \( T > 2\pi \), then the system (1.1) is exactly controllable for almost all choices of \((A, B, C, D) \in \mathbb{R}^4\).

Finally, we are looking for stabilizing feedback laws of the form
\[
(v_1, v_2)(t) := ((P_1 y_{1t} + Q_1 y_1)(t, \xi), (P_2 y_{2t} + Q_2 y_2)(t, \eta)),
\]
leading to arbitrarily high decay rates.

**Theorem 2.3.** Fix \( \xi \) and \( \eta \in (0, \pi) \) such that \( \xi/\pi \) and \( \eta/\pi \) are irrational. For almost all choices of \((A, B, C, D) \in \mathbb{R}^4\) and for every positive number \( \omega \) there exist two linear operators
\[
(P_1, Q_1, P_2, Q_2) : \mathcal{H}_{\xi, \eta} \to D_1^{-1} \times D_2^{-2},
\]
and a positive constant \( M \) such that the problem (1.1) is well posed in \( \mathcal{H}_{\xi, \eta} \) and its solutions satisfy the inequality
\[
\| (y_1, y_{1t}, y_2, y_{2t}) \|_{\mathcal{H}_{\xi, \eta}} \leq M e^{-\omega t} \| (y_{10}, y_{11}, y_{20}, y_{21}) \|_{\mathcal{H}_{\xi, \eta}}
\]
for all \((y_{10}, y_{11}, y_{20}, y_{21}) \in \mathcal{H}_{\xi, \eta} \) and \( t \geq 0 \).

**Remark.** It follows from some results of Komornik and Loreti that the system (1.1) can not be exactly controllable for some exceptional choices of the parameters \( A, B, C, D \); see [10] and [17] for explicit counter examples concerning an equivalent observability problem.

The proofs are based on the study of the dual problem
\[
\begin{align*}
\frac{d}{dt} u_{1t} - u_{1xx} + Au_1 + Cu_2 &= 0 & \text{in } \mathbb{R} \times (0, \pi), \\
\frac{d}{dt} u_{2t} + u_{2xxx} + Bu_1 + Du_2 &= 0 & \text{in } \mathbb{R} \times (0, \pi), \\
u_1(t, 0) = u_1(t, \pi) &= 0 & \text{for } t \in \mathbb{R}, \\
u_2(t, 0) = u_2(t, \pi) &= 0 & \text{for } t \in \mathbb{R}, \\
u_{2xx}(t, 0) = u_{2xx}(t, \pi) &= 0 & \text{for } t \in \mathbb{R}, \\
u_1(0, x) = u_{10}(x) \text{ and } u_{1t}(0, x) = u_{11}(x) & \text{ for } x \in (0, \pi), \\
u_2(0, x) = u_{20}(x) \text{ and } u_{2t}(0, x) = u_{21}(x) & \text{ for } x \in (0, \pi), \\
u(t) = u_1(t, \xi) + u_2(t, \xi) & \text{ for } t \in \mathbb{R}.
\end{align*}
\]

We prove that under some conditions this dual problem is observable. Feedbacks of this type are important for the engineering applications: as we will show in this paper, on various numerical aspect of these feedbacks, and we can see the works of Bourquin et al. [4] on physical experiences.

3. **Proof of the main results**

We consider the abstract observability problem (2.3), if the initial data are given by the formula
\[
\begin{align*}
u_{10}(x) &= \sum_{k=1}^{\infty} a_k \sin kx, & \nu_{11}(x) &= \sum_{k=1}^{\infty} b_k \sin kx, \\
u_{20}(x) &= \sum_{k=1}^{\infty} \alpha_k \sin kx, & \nu_{21}(x) &= \sum_{k=1}^{\infty} \beta_k \sin kx.
\end{align*}
\]
with only finitely many non vanishing coefficients \( a_k \) \( b_k \), \( \alpha_k \) \( \beta_k \), then a simple computation shows that

\[
    u_1(t, x) = \sum_{k=1}^{\infty} (c_k e^{ikt} + c_{-k} e^{-ikt}) \sin kx
\]

and

\[
    u_2(t, x) = \sum_{k=1}^{\infty} (d_k e^{ik^2t} + d_{-k} e^{-ik^2t}) \sin kx
\]

with

\[
    c_k = \frac{1}{2} (a_k - i \frac{b_k}{k}), \quad c_{-k} = \frac{1}{2} (a_k + i \frac{b_k}{k}),
\]

and

\[
    d_k = \frac{1}{2} (\alpha_k - i \frac{\beta_k}{k^2}), \quad d_{-k} = \frac{1}{2} (\alpha_k + i \frac{\beta_k}{k^2}).
\]

If \( T > 2\pi \), then using Parseval’s equality and a result of Haraux [10] it follows that

\[
    \int_0^T (|u_1(t, \xi)|^2 + |u_2(t, \eta)|^2) dt = \sum_{k=1}^{\infty} (|a_k|^2 + k^{-4} |b_k|^2) \sin^2 k\xi + (|\alpha_k|^2 + k^{-4} |\beta_k|^2) \sin^2 k\eta.
\]

It can be rewritten in the form

\[
    \int_0^T |u_1(t, \xi)|^2 + |u_2(t, \eta)|^2 dt \leq \|u_{10}\|^2_{D_\xi^0} + \|u_{20}\|^2_{D_\eta^0} + \|u_{11}\|_{(D_\xi^{-1})} + \|u_{21}\|_{D_\eta^{-2}}.
\]

We rewrite (2.3) as a first-order system

\[
    (3.1) \quad U' = A^* U, \quad U(0) = U_0, \quad \psi = B^* U
\]

by setting

\[
    U := (u_1, u_2, u_{11}, u_{21}),
    \quad U_0 := (u_{10}, u_{20}, u_{11}, u_{21}),
    \quad A^*(u_1, u_2, v_1, v_2) := (v_1, v_2, \Delta u_1 - A u_1 - C u_2, -\Delta^2 u_2 - B u_1 - D u_2)
\]

and

\[
    B^*(u_1, u_2, v_1, v_2) := (u_1(\xi), u_2(\eta)).
\]

We introduce the dual space of the Hilbert spaces \( H_{\xi, \eta} \) denoted by \( H'_{\xi, \eta} := D_\xi^0 \times D_\eta^0 \times D_\xi^{-1} \times D_\eta^{-2}, \mathcal{G} := \mathbb{R}^2 \) and we define the domain of definition of the linear operators \( A^* \) and \( B^* \) by

\[
    D(A^*) = D(B^*) = D_\xi^1 \times D_\eta^2 \times D_\xi^0 \times D_\eta^0.
\]

**Proposition 3.1.** The system (2.3) verify the following four assumptions:

- **(H1)**: The operator \( A^* \) generates a strongly continuous group of automorphisms \( e^{tA^*} \) in \( H'_{\xi, \eta} \).
- **(H2)**: \( D(A^*) \subset D(B^*) \), and there exists a constant \( c \) such that \( \|B^* U_0\|_{\mathcal{G}} \leq c\|A^* U_0\|_{H'_{\xi, \eta}} \) for all \( U_0 \in D(A^*) \).
Proposition 3.2. Also [5] for a proof of this proposition.

We rewrite (1.1) as a linear evolution problem

\( H \)

We can see [15] for the necessity of the assumptions \((H1)-(H3)\). Next we define the solution of (1.1) by transposition. Fix \( X \) has a unique weak solution \( X \) for all \( T > 0 \) proved in [15]. Fix two numbers \( M \) with some constant \( c \) arbitrary. Multiply the equation (3.3) by the solution \( u = (v_1, v_2) \in L^2(0; \mathbb{R}^2) \) arbitrary. Multiply the equation (3.3) by the solution \( U \) of the equation in (3.2). Integrating by part formally between 0 and \( T \in \mathbb{R} \), we easily obtain the identity

\[ (3.4) \]

\[ < X(T), U(T) >_{H\ell, \eta} = < X_0, U_0 >_{H\ell, \eta} + \int_0^T < u(s), B^s U >_{\mathbb{R}^2} \ ds. \]

Hence we define a solution of (3.3) as a continuous function \( X : \mathbb{R} \rightarrow H\ell, \eta \) satisfying the identity (3.4) for all \( U_0 \in H\ell, \eta \) and for all \( T \in \mathbb{R} \). This definition is justified by the following lemma.

Lemma 3.3. Assume \((H1)-(H4)\). For any given \( X_0 \in H\ell, \eta \) and \( u \in L^2(0;\mathbb{R}^2) \), the problem (3.3) has a unique solution. Moreover, we have the estimates.

\[ (3.5) \]

\[ \| X \|_{L^\infty(0; T; H\ell, \eta)} \leq M_T(\| X_0 \|_{H\ell, \eta} + \| u \|_{L^2(0; \mathbb{R}^2)}) \]

with some constant \( M_T \) which does not depend on the particular choice of \( X_0 \) and for all \( T > 0 \).

In order to prove the stabilization estimate, we need to recall a general result proved in [15]. Fix two numbers \( T > |I'|, \omega > 0 \), set \( T_\omega = T + (2\omega)^{-1} \), define

\[ \epsilon_\omega(s) = \begin{cases} e^{-2\omega s} & \text{if } 0 \leq s \leq T, \\ 2\omega e^{-2\omega T} (T_\omega - s) & \text{if } T \leq s \leq T_\omega, \end{cases} \]

and set

\[ \langle \Lambda_\omega U_0, \tilde{U}_0 \rangle_{\Lambda \ell, \gamma} := \int_0^{T_\omega} \epsilon_\omega(s)(B e^{-A} U_0, B e^{-A} \tilde{U}_0)_{\gamma} \ ds. \]

Then \( \lambda_\omega \) is a self-adjoint, positive definite isomorphism \( \lambda_\omega \in L(H, H') \). Let us denote by \( J : \gamma \rightarrow \gamma' \) the canonical Riesz anti-isomorphism.

The following result is a special case of a theorem obtained in [15].
Theorem 3.4. Assume (H1)-(H4) and fix $\omega > 0$ arbitrarily. Then the problem
\begin{equation}
(3.6)

v' = (-A - JB^*\Lambda^-_\omega)v, \quad v(0) = v_0,
\end{equation}
is well-posed in $\mathcal{H}$. Furthermore, there exists a constant $M$ such that the solutions of (3.6) satisfy the estimates
\begin{equation}
(3.7)

\|v(t)\|_{\mathcal{H}} \leq M\|v_0\|_{\mathcal{H}} e^{-\omega t}
\end{equation}
for all $v_0 \in \mathcal{H}$ and for all $t \geq 0$.

In other words, this theorem asserts that the feedback law

$$W = -JB^*\Lambda^-_\omega v$$

uniformly stabilizes the control problem
\begin{equation}
(3.8)

v' = -Av + BW, \quad v(0) = v_0
\end{equation}

with a decay rate at least equal to $\omega$.

The well-posedness means here that (3.6) has a unique solution $v \in C(\mathbb{R}; \mathcal{H})$ for every $v_0 \in \mathcal{H}$.

3.1. Proof of Theorem 2.3. Since hypothesis (H1) – (H4) are all satisfied, we may apply Theorem 3.4. In order to write down explicitly the stabilization result, we multiply the equation (1.1) by $u$ and we integrate by parts as follows (we use all conditions in (1.1) and (2.3)). This shows that if we write (2.3) in the form (3.3), then its dual (3.2) corresponds to (1.1). Furthermore, writing the operator

$$\Lambda^{-1}_\omega : (D^2_\xi)' \times (D^2_\eta)' \rightarrow (D^0_\xi)' \times (D^0_\eta)' \rightarrow D^{-1}_\xi \times D^0_\xi \times D^{-2}_\eta \times D^0_\eta$$

in the matrix form

$$\Lambda^{-1}_\omega = \begin{pmatrix}
-P_1 & Q_1 & \Lambda_{13} & \Lambda_{14} \\
\Lambda_{21} & -P_2 & Q_2 & \\
\Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\
\Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44}
\end{pmatrix},$$

we have
\begin{equation}
(3.9)

(v_1(t), v_2(t)) = -((P_1y_1 + Q_1y_1)(t, \xi), (P_2y_2 + Q_2y_2)(t, \eta)).
\end{equation}

4. Numerical Approximation

To perform a numerical computation we use a Faedo-Galerkin method. This allows us to approach numerically the operator $\Lambda_\omega$ using the family of the functions $w_k(x) := \sqrt{2/\pi} \sin kx$. An approximate solution $y^N = (y^N_1, y^N_2)$ of the coupled system (1.1) is a solution of the problem:

\begin{equation}
(4.1)

\begin{cases}
y^N_{1,tt} - y^N_{1,xx} + Ay^N_1 + Cy^N_2 = v^N_1(t)\delta_\xi & \text{in } (0, T) \times (0, \pi), \\
y^N_{2,tt} + y^N_{2,xxxx} + By^N_1 + Dy^N_2 = v^N_2(t)\delta_\eta & \text{in } (0, T) \times (0, \pi), \\
y^N_1(t, 0) = y^N_1(t, \pi) = 0 & \text{for } t \in (0, T), \\
y^N_2(t, 0) = y^N_2(t, \pi) = 0 & \text{for } t \in (0, T), \\
y^N_{1,xx}(t, 0) = y^N_{1,xx}(t, \pi) = 0 & \text{for } t \in (0, T), \\
y^N_1(0, x) = y^N_1(0, x) \text{ and } y^N_1(t, x) = y^N_1(t, x) & \text{for } x \in (0, \pi), \\
y^N_2(0, x) = y^N_2(0, x) \text{ and } y^N_2(t, x) = y^N_2(t, x) & \text{for } x \in (0, \pi)
\end{cases}
\end{equation}
In order to compute the state feedback law \( v^N(t) = (v_1^N(t), v_2^N(t)) = \mathcal{F}(y^N(t), \partial_t y^N(t)) \), an adjoint state is introduced: let \( u^N(s, x) = (u_1^N(s, x), u_2^N(s, x)) \) be the solution of the coupled adjoint system:

\[
\begin{cases}
  u_{1ss}^N - u_{1xx}^N + Au_1^N + Cu_2^N = 0 & \text{in } (0, S) \times (0, \pi), \\
  u_{2ss}^N + u_{2xxx}^N + Bu_1^N + Du_2^N = 0 & \text{in } (0, S) \times (0, \pi), \\
  u_1^N(s, 0) = u_1^N(s, \pi) = 0 & \text{for } s \in (0, S), \\
  u_2^N(s, 0) = u_2^N(s, \pi) = 0 & \text{for } s \in (0, S), \\
  u_1^N(0, x) = u_1^N(0, x) = u_1^N(x) & \text{for } x \in (0, \pi), \\
  u_2^N(0, x) = u_2^N(0, x) = u_2^N(x) & \text{for } x \in (0, \pi). 
\end{cases}
\]  

(4.2)

Where \( s \) denotes a fictitious time, and \( S \) a fictitious time horizon. The solution \( U^N(s, x) = (u_1^N(s, x), u_2^N(s, x)) \) of (4.2) depends linearly on initial conditions \( U_0^N := (u_{10}^N, u_{20}^N, u_{11}^N, u_{21}^N) \). Hence one can define the approximate bilinear controllability gramian for any solution \( \tilde{U}^N = (\tilde{u}_1^N, \tilde{u}_2^N) \) of (4.2) with initial conditions \( \tilde{U}_0^N := (\tilde{u}_{10}^N, \tilde{u}_{20}^N, \tilde{u}_{11}^N, \tilde{u}_{21}^N) \) and parameter \( \omega \) as:

\[
a_{\omega, S}(U_0^N, \tilde{U}_0^N) := \int_0^S e^{-2\omega s} (u_1^N(s, \xi)u_1^N(s, \xi) + u_2^N(s, \eta)\tilde{u}_2^N(s, \eta)) \, ds.
\]

(4.3)

As pointed out in [6] we consider, from numerical point of vue, the function \( e^{-2\omega s} \) initially introduced in [14] and which gives similar results as the general theory [15]. For any \( Z = \{z_{10}, z_{20}, z_{11}, z_{21}\} \) let \( U_0^N = \{u_{10}^N, u_{20}^N, u_{11}^N, u_{21}^N\} \) be the unique solution of the variational equation:

\[
a_{\omega, S}(U_0^N, Z) = \int_0^\pi [(z_{11}u_{10}^N - z_{10}u_{11}^N) + (z_{21}u_{20}^N - z_{20}u_{21}^N)] \, dx.
\]

(4.4)

Let know define the operator \( L_2 \) by \( \{u_{10}^N, u_{20}^N, u_{11}^N, u_{21}^N\} = L_2(\{z_{10}, z_{20}, z_{11}, z_{21}\}) \) and let also define: \( P_1 : \mathbb{R}^4 \rightarrow \mathbb{R} \) and \( P_2 : \mathbb{R}^4 \rightarrow \mathbb{R} \) respectively as the projection on the first and second component, i.e. \( P_1(\{a, b, c, d\}) = a \) and \( P_2(\{a, b, c, d\}) = b \). The feedback law \( v^N(t) = (v_1^N(t), v_2^N(t)) \) is then given by:

\[
v_1^N(t) = -P_1L_2(y_1^N, y_2^N, (y_1^N)', (y_2^N)')(x = \xi)
\]

(4.5)

and

\[
v_2^N(t) = -P_2L_2(y_1^N, y_2^N, (y_1^N)', (y_2^N)')(x = \eta)
\]

(4.6)

To compute the feedback \( v_N(t) \) we use the expansion of initial conditions:

\[
u_1^N(x) = \sum_{k=1}^N \alpha_k^0 \sin kx, \quad u_1^N(x) = \sum_{k=1}^N \alpha_k^1 \sin kx
\]

and

\[
u_2^N(x) = \sum_{k=1}^N \beta_k^0 \sin kx, \quad u_2^N(x) = \sum_{k=1}^N \beta_k^1 \sin kx
\]

an approximate solution of problem (2.3) is then given by \( u_N(t, x) = \sum_{k=1}^N \alpha_k(t) \sin kx \) and \( u_2^N(t, x) = \sum_{k=1}^N \beta_k(t) \sin kx \). With:

\[
\alpha_k(t) = \frac{1}{2}(\alpha_k^0 - i\alpha_k^1) e^{ikt} + \frac{1}{2}(\alpha_k^0 + i\alpha_k^1) e^{-ikt}
\]

and

\[
\beta_k(t) = \frac{1}{2}(\beta_k^0 - i\beta_k^1) e^{ikt} + \frac{1}{2}(\beta_k^0 + i\beta_k^1) e^{-ikt}
\]
by setting \( m_i = \sin(i\xi) \) and \( n_i = \sin(i\eta) \) the second hand side of (4.3) is given by:

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^{S} e^{-2su} (\alpha_i(s) \tilde{\alpha}_j(s) m_i m_j + \beta_i(s) \tilde{\beta}_j(s) n_i n_j) \, ds
\]

Let us know define \( \{\phi_0^N(t), \psi_0^N(t), \phi_1^N(t), \psi_1^N(t)\} \) as a solution at each time \( t \geq 0 \), of:

\[
\{\phi_0^N(t), \psi_0^N(t), \phi_1^N(t), \psi_1^N(t)\} = L_N \{y_1^N(t), y_2^N(t), (y_1^N(t))', (y_2^N(t))'\}
\]

We can then write the operator \( L_N \) in a matrix form by setting:

\[
K^{\omega,4}_{AN} \{\phi_0^N(t), \psi_0^N(t), \phi_1^N(t), \psi_1^N(t)\} = \{(y_1^N(t))', (y_2^N(t))', -y_1^N(t), -y_2^N(t)\}
\]

Where the matrix \( K^{\omega,4}_{AN} \) is defined by relation (4.3),(4.7). Let us know define the inverse \( K^{-1}_{\omega,AN} \) by a block matrix:

\[
K^{-1}_{\omega,AN} = (K^{\alpha\beta}), \alpha, \beta = 1,4.
\]

the control law is then given by:

\[
v^N(t) = (v_1^N(t), v_2^N(t)) = \left( \sum_{k=1}^{N} (\phi_0^N)_km_k, \sum_{k=1}^{N} (\psi_0^N)_kn_k \right)
\]

Where \( \phi_0^N(t) = K^{11}\alpha'(t) - K^{12}\alpha(t) \) and \( \psi_0^N(t) = K^{23}\beta'(t) - K^{24}\beta(t) \)

We compute the solution of the coupled system for \( N = 7 \) and value of parameters \( A = 3, B = 2, C = 1 \) and \( D = \frac{1}{2} \). The damping points are chosen as \( \xi = \frac{\sqrt{2}}{3} \) and \( \eta = \frac{\sqrt{2}}{4} \). We show the efficiency of the feedback for two values of parameter \( \omega \) in figure 1 (\( \omega = 1 \)) and figure 2 (\( \omega = 10 \)).
In figures 3 and 4 we compare the energy of the coupled system in the space $D_0^0 \times D_1^1 \times D_0^0 \times D_2^2$ (solid line) to the classical energy in the natural space $L^2(0, \pi) \times H_0^1(0, \pi) \times L^2(0, \pi) \times (H^2(0, \pi) \cap H_0^1(0, \pi))$ (dashed dot line) for two values of the parameter $\omega = 1$ and $\omega = 10$. The computations show that we have also uniform decay of the energy in the natural space.

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