Quantum histories without contrary inferences

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Abstract

In the consistent histories formulation of quantum theory it was shown that it is possible to retrodict contrary properties. We show that this problem do not appear in our formalism of generalized contexts for quantum histories.
I. INTRODUCTION.

In the consistent histories formulation of quantum theory \cite{2} \cite{3} \cite{4}, the probabilistic predictions and retrodictions depend on the choice of a consistent set. It was shown that this freedom allows the formalism to retrodict two contrary properties \cite{1}. This is not a problem for the defenders of the theory, because each retrodiction is obtained in a different consistent sets of histories, i.e. in different descriptions of the physical system not to be considered simultaneously \cite{5} \cite{13}. However, this fact is considered by some authors as a serious failure of the theory of consistent histories \cite{1} \cite{6} \cite{14}.

We are going to analyze this problem with our formalism of generalized contexts \cite{7} \cite{8}, developed to deal with expressions involving properties at different times. The formalism is an alternative to the theory of consistent histories, which has proved to be useful for the time dependent description of the logic of quantum measurements \cite{9}, the decay processes \cite{10} and the double slit experiment with and without measurement instruments \cite{8}. More recently \cite{11} we have discussed the relation of our formalism with the theory of consistent histories.

In section II we show that there is no possibility for contrary inferences in ordinary quantum mechanics. In section III we discuss the retrodiction of contrary properties in the theory of consistent histories. In section IV we show that there are no retrodiction of contrary properties in our formalism of generalized contexts. The main conclusions are given in section V.

II. CONTRARY PROPERTIES IN AN ORDINARY QUANTUM CONTEXT.

In quantum mechanics, a property $p$ is represented by a projector $\Pi_p$ in the Hilbert space $\mathcal{H}$, or alternatively by the corresponding Hilbert subspace $V_p = \Pi_p \mathcal{H}$. By definition \cite{1}, two quantum properties $p$ and $q$ are said to be contrary if they satisfy the order relation $p \leq q$, which can also be expressed in terms of the inclusion of the corresponding Hilbert subspaces in the form

$$\Pi_p \mathcal{H} \subseteq (I - \Pi_q) \mathcal{H}.$$  \hspace{1cm} (1)

The inclusion of subspaces is equivalent to the following relation between the correspond-
ing projectors (see \cite{12}, section 1.3)

\[
\Pi_p(I - \Pi_q) = (I - \Pi_q)\Pi_p = \Pi_p,
\]

from which we easily deduce that \(\Pi_p\Pi_q = \Pi_q\Pi_p = 0\), that means the projectors \(\Pi_p\) and \(\Pi_q\) are orthogonal.

As they also commute, \(p\) and \(q\) are compatible properties. The projectors \(\Pi_p\), \(\Pi_q\) and \(\Pi_{p\lor q} = I - \Pi_p - \Pi_q\) form a projective decomposition of the Hilbert space, i.e. they are orthogonal and their sum is the identity operator. Therefore, the properties \(p\), \(q\) and \(\overline{p \lor q}\) can be considered the atomic properties generating a context of quantum properties with well defined probabilities \cite{8}.

For any state of the system represented by a state operator \(\rho\), the probability of any property \(p'\) in the context is obtained with the Born rule, i.e. \(Pr_\rho(p') = \text{Tr}(\rho \Pi_{p'})\). For the atomic properties \(p\), \(q\) and \(\overline{p \lor q}\) we obtain

\[
Pr_\rho(p) + Pr_\rho(q) + Pr_\rho(p \lor q) = 1
\]  

Equation (2)

From this equation we easily deduce that if \(Pr_\rho(p) = 1\) then \(Pr_\rho(q) = 0\) and if \(Pr_\rho(q) = 1\), \(Pr_\rho(p) = 0\).

We conclude that in ordinary quantum mechanics it is impossible for any state \(\rho\) that two contrary properties \(p\) and \(q\) have probability equal to one. These results are the stochastic version of contrary proposition in ordinary logic. They can be interpreted by saying that whenever the property \(p\) (\(q\)) is true, the property \(q\) (\(p\)) is false. By the way, this result also justify to have given the name contrary to quantum properties \(p\) and \(q\) satisfying equation (1).

More generally, it is easy to see that if \(p\) and \(q\) are contrary properties, it is not possible to have a state \(\rho\) and another property \(r\) for which

\[
Pr_\rho(p|r) = 1, \quad Pr_\rho(q|r) = 1.
\]

Equation (3)

Taking into account that \(p\), \(q\) and \(r\) should be represented by commuting projectors, so that the conditional probabilities be well defined, we would have

\[
Pr_\rho(p|r) = \frac{\text{Tr}(\rho \Pi_p \Pi_r)}{\text{Tr}(\rho \Pi_r)} = \text{Tr}(\rho^* \Pi_p) = Pr_\rho(p), \quad Pr_\rho(q|r) = \text{Tr}(\rho^* \Pi_q) = Pr_\rho(q),
\]

where \(\rho^* \equiv \frac{\Pi_r \rho \Pi_r}{\text{Tr}(\rho \Pi_r)}\). Taking into account equation (2) with \(\rho = \rho^*\) we conclude that there are no state \(\rho\) and property \(r\) for which equations (3) can be both valid.
III. CONTRARY PROPERTIES IN THE THEORY OF CONSISTENT HISTORIES.

In the theory of consistent histories $n$ different contexts of properties at each time $t_j$ ($j = 1, \ldots, n$), satisfying a state dependent consistency condition, can be used to define a family of consistent histories, i.e. a set of $n$ times sequences of properties with well defined probabilities \[2\] \[3\] \[4\]. According to the theory, each possible family of consistent histories is an equally valid description of the quantum system. In general it is not possible to include two different families in a single larger one. Different families of this kind are complementary descriptions of the system, which the theory excludes to be considered simultaneously.

A discussion on the logical aspects of the theory was opened by Adrian Kent \[1\], who first pointed out that it is possible the retrodiction of contrary properties in different families of consistent histories, i.e.

\[
\Pr_{\rho_{t_0}}(p, t_1| r, t_2) = 1, \quad \Pr_{\rho_{t_0}}(q, t_1| r, t_2) = 1, \quad (4)
\]

where $\rho_{t_0}$ is the state of the system at time $t_0$, $p$ and $q$ are contrary properties at time $t_1 > t_0$ and $r$ is a property at time $t_2 > t_1$ (see references \[1\] and \[5\] for explicit expressions of $p$, $q$, $r$ and $\rho_{t_0}$).

The first equation above is valid for the consistent family that includes $p$ and $\overline{p}$ at time $t_1$ together with $r$ and $\overline{r}$ at time $t_2$. It gives the retrodiction of property $p$ at time $t_1$ conditional to property $r$ at time $t_2$. The second equation is valid for the consistent family including $q$ and $\overline{q}$ at time $t_1$ together with $r$ and $\overline{r}$ at time $t_2$, and it gives the retrodiction of property $q$ at time $t_1$ conditional to property $r$ at time $t_2$.

From the point of view of the theory of consistent histories equations \[4\] cannot be interpreted as the retrodiction of two contrary properties, because they are valid in two different and complementary descriptions, which cannot be included in a single consistent family \[13\] \[5\]. However, some authors have considered the results given in equations \[4\] as a serious objection for the internal consistency of the theory of consistent histories \[1\] \[14\] \[6\].
IV. CONTRARY PROPERTIES IN THE FORMALISM OF GENERALIZED CONTEXTS.

In this section contrary quantum properties will be considered from the point of view of our formalism of generalized contexts. We start with a brief description of the formalism, which was presented in full details in our previous papers [7] [8].

Quantum mechanics do not give a meaning to the joint probability distribution of observables whose operators do not commute. It can only deal with a set of properties belonging to a context.

A context of properties $C_i$ at time $t_i$ is obtained starting from a set of atomic properties $p_{ki}^i$ ($k_i \in \sigma_i$) represented by projectors $\Pi_{k_i}^i$ corresponding to a projective decomposition of the Hilbert space $\mathcal{H}$, i.e. verifying

$$\sum_{k_i \in \sigma_i} \Pi_{k_i}^i = I, \quad \Pi_{k_i}^i \Pi_{k'_i}^i = \delta_{k_i k'_i} \Pi_{k_i}^i.$$  

Any property $p$ of the context $C_i$ is represented by a sum of the projectors of the projective decomposition,

$$\Pi_p = \sum_{k_i \in \sigma_p} \Pi_{k_i}^i, \quad \sigma_p \subset \sigma_i.$$  

(5)

Any property $p$ of the context $C_i$ is represented by a sum of the projectors of the projective decomposition,

$$\sum_{k_i \in \sigma_p} \Pi_{k_i}^i \Pi_{k_i}^i = I, \quad \Pi_{k_i}^i \Pi_{k'_i}^i = \delta_{k_i k'_i} \Pi_{k_i}^i.$$  

(6)

The context $C_i$ is an orthocomplemented distributive lattice, with the complement $\overline{p}$ of a property $p$ defined by $\Pi_{\overline{p}} \equiv I - \Pi_p$ and the order relation $p \leq p'$ defined by $\Pi_p \mathcal{H} \subseteq \Pi_{p'} \mathcal{H}$.

A well defined probability (i.e. additive, non negative and normalized) is defined by the Born rule $\text{Pr}_{t_i}(p) \equiv \text{Tr}(\rho_{t_i} \Pi_p)$ on the context $C_i$. In Heisenberg representation, the probability of a property $p$ at time $t_i$ can be written in terms of the state at a reference time $t_0$, i.e.

$$\text{Pr}_{t_i}(p) = \text{Tr}(\rho_{t_0} \Pi_p), \quad \Pi_{p,0} \equiv U(t_0, t_i) \Pi_p U(t_i, t_0), \quad U(t_i, t_0) = e^{-i\frac{\pi}{\hbar} H(t_i - t_0)}.$$  

Taking into account equations (6) and (7), the Heisenberg representation of the property $p$ of the context $C_i$ at time $t_i$ is given by

$$\Pi_{p,0} = \sum_{k_i \in \sigma_p} \Pi_{k_i}^i,$$  

(8)

where the projectors $\Pi_{p,0}^{k_i} = U(t_0, t_i) \Pi_{k_i}^i U(t_i, t_0)$ represent the time translation of the atomic properties $p_{ki}^i$ from time $t_i$ to the time $t_0$. The projectors $\Pi_{p,0}^{k_i}$ also satisfy equations (5).

The Heisenberg representation of the context $C_i$ at time $t_i$ suggest a generalization of quantum mechanics for including the joint probability of properties belonging to different contexts $C_1, ..., C_i, ..., C_n$ corresponding to $n$ different times $t_1 < ... < t_i < ... < t_n$. 

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By extending what is a common assumption in ordinary quantum mechanics, we proposed to give a meaning to the joint probability of properties at different times if they correspond to commuting projectors in Heisenberg representation. This will be the case if the atomic properties generating each of the $n$ contexts are represented by projectors satisfying

$$[\Pi_{k,0}^{i}, \Pi_{j,0}^{j}] = 0, \quad i, j = 1, \ldots, n, \quad k_i \in \sigma_i, \quad k_j \in \sigma_j.$$ 

If these projectors commute the projectors $\Pi_{0}^{k} \equiv \Pi_{1,0}^{k_{1}} \ldots \Pi_{i,0}^{k_{i}} \ldots \Pi_{n,0}^{k_{n}}$, with $k = (k_1, \ldots, k_n)$ and $k_i \in \sigma_i$, form a projective decomposition of the Hilbert space $\mathcal{H}$, as they satisfy

$$\sum_{k} \Pi_{0}^{k} = I, \quad \Pi_{0}^{k} \Pi_{0}^{k'} = \delta_{k'k} \Pi_{0}^{k}, \quad k, k' \in \sigma_1 \times \ldots \times \sigma_n.$$ 

In our formalism we postulate that an expression of the form “property $p_{k_{1}}^{1}$ at time $t_{1}$ and \ldots and $p_{k_{n}}^{n}$ at time $t_{n}$” is an atomic generalized property $p^{k}$ with the Heisenberg representation given by the projector $\Pi_{0}^{k}$. A generalized context is defined by all the generalized properties $p$ having a Heisenberg representation given by an arbitrary sum of the projectors $\Pi_{0}^{k}$, i.e.

$$\Pi_{p} = \sum_{k \in \sigma_{p}} \Pi_{0}^{k},$$

where $\sigma_{p}$ is a subset of $\sigma_1 \times \ldots \times \sigma_n$. The generalized context is an orthocomplemented distributive lattice, with the complement $\overline{p}$ of $p$ defined by $\Pi_{\overline{p}} = I - \Pi_{p}$, and the order relation $p \leq p'$ defined by the inclusion of the corresponding Hilbert subspaces ($\Pi_{p} \mathcal{H} \subseteq \Pi_{p'} \mathcal{H}$).

An extension of the Born rule provides a definition of an additive, non negative and normalized probability on the generalized context, given by

$$\text{Pr}(p) \equiv \text{Tr}(\rho_{t_{0}} \Pi_{p}). \quad (9)$$

We are now going to analyze the retrodiction of contrary properties in the formalism of generalized contexts. We consider a state $\rho_{t_{0}}$ at time $t_{0}$, two contrary properties $p$ and $q$ at time $t_{1} > t_{0}$ and another property $r$ at time $t_{2} > t_{1}$, and we search for the possibility to obtain for both conditional probabilities the results $\text{Pr}_{\rho_{t_{0}}}(p, t_{1}| r, t_{2}) = 1$ and $\text{Pr}_{\rho_{t_{0}}}(q, t_{1}| r, t_{2}) = 1$.

The projectors $\Pi_{p}$ and $\Pi_{p,0} = U(t_{0}, t_{1}) \Pi_{p} U(t_{1}, t_{0})$ are respectively Schrödinger and Heisenberg representations of the property $p$ at time $t_{1}$. Analogously, $\Pi_{q}$ and $\Pi_{q,0} = U(t_{0}, t_{1}) \Pi_{q} U(t_{1}, t_{0})$ are representations of the property $q$ at time $t_{1}$. Moreover, $\Pi_{r}$ and $\Pi_{r,0} = U(t_{0}, t_{2}) \Pi_{r} U(t_{2}, t_{0})$ are representations of the property $r$ at time $t_{2}$. 

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The conditional probabilities are meaningful in our formalism if the following compatibility conditions are satisfied

$$[\Pi_{p,0}, \Pi_{r,0}] = 0, \quad [\Pi_{q,0}, \Pi_{r,0}] = 0, \quad (10)$$

while the contrary properties $p$ and $q$ are represented by orthogonal projectors, and therefore

$$[\Pi_{p,0}, \Pi_{q,0}] = 0. \quad (11)$$

The commutation relations given in equations (10) and (11) are the compatibility conditions required to consider a two times generalized context including the contrary properties $p$ and $q$ at time $t_1$ and property $r$ at time $t_2$, in which both conditional probabilities $\Pr_{\rho_{t_0}}(p, t_1 | r, t_2)$ and $\Pr_{\rho_{t_0}}(q, t_1 | r, t_2)$ are meaningful.

In our formalism, the required retrodictions would have the explicit forms

$$\Pr_{\rho_{t_0}}(p, t_1 | r, t_2) = \frac{\text{Tr}(\rho_{t_0} \Pi_{p,0} \Pi_{r,0})}{\text{Tr}(\rho_{t_0} \Pi_{r,0})} = 1, \quad \Pr_{\rho_{t_0}}(q, t_1 | r, t_2) = \frac{\text{Tr}(\rho_{t_0} \Pi_{q,0} \Pi_{r,0})}{\text{Tr}(\rho_{t_0} \Pi_{r,0})} = 1.$$

Taking into account the commutation relations given in equations (10), the previous equations are equivalent to

$$\text{Tr}(\rho_{t_0}^* \Pi_{p,0}) = 1, \quad \text{Tr}(\rho_{t_0}^* \Pi_{q,0}) = 1, \quad \rho_{t_0}^* = \frac{\Pi_{r,0} \rho_{t_0} \Pi_{r,0}}{\text{Tr}(\Pi_{r,0} \rho_{t_0} \Pi_{r,0})}. \quad (12)$$

As $\Pi_{p,0}$ and $\Pi_{q,0}$ represent contrary properties at the same time $t_0$, we can follow the arguments given at the end of section II to show that there is no $\rho_{t_0}^*$ for which both equations given in equations (12) can be valid. Therefore we conclude that the problem of retrodiction of contrary properties do not arise in our formalism of generalized contexts for quantum histories.

V. CONCLUSIONS.

In ordinary quantum mechanics, contrary properties are represented by orthogonal subspaces of the Hilbert space associated with the physical system. In section II, we proved that given two contrary properties $p$ and $q$, there is no state $\rho$ and property $r$ for which the probability of $p$ conditional to $r$ and the probability of $q$ conditional to $r$ can be both equal to one. Therefore, there is no possibility of contrary inferences in ordinary quantum mechanics. This result corresponds to a state and properties considered at a single time.
As we discussed in section III, this is not the case for the theory of consistent histories, where a state at time $t_0$, two contrary properties $p$ and $q$ at time $t_1 > t_0$ and another property $r$ at time $t_2 > t_1$ can be found in such a way that the probability of $p$ conditional to $r$ and the probability of $q$ conditional to $r$ are both equal to one. Although these conditional probabilities are defined in different sets of consistent histories [5] [13], some authors have considered this fact as a serious problem for the logical consistency of the theory [1] [6] [14].

The main purpose of this paper was to analyze the problem of contrary inferences in the framework of our formalism of generalized contexts. In this formalism, as it was explained in section IV, ordinary contexts of properties at different times can be used to obtain a valid set of quantum histories if they satisfy a compatibility condition. This condition is given by the commutation of the projectors corresponding to the time translation of the properties to a single common time. These compatibility conditions are state independent, an important difference with respect to the state dependent consistency conditions of the theory of consistent histories. Each quantum history has a Heisenberg representation given by a projection operator, and each valid set of quantum histories is generated by a projective decomposition of the Hilbert space. As a consequence, a generalized context of quantum histories has the logical structure of a distributive orthocomplemented lattice of subspaces of the Hilbert space, i.e. the same logical structure of the quantum properties of an ordinary context. It is because of this logical structure that in our formalism there is no place for the retrodiction of contrary properties.

Recently we have analyzed the relations of our formalism with the theory of consistent histories [11]. Our formalism was also successful in describing the time dependent logic of quantum measurements [9], the quantum decay process [10] and the double slit experiment with and without measurement instruments [8]. The results of this paper encourages us to continue our future research considering more applications of the formalism of generalized contexts.

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