ON LANDAU-GINZBURG SYSTEMS AND $D^b(X)$ OF PROJECTIVE BUNDLES

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Abstract. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$ be a Fano projective bundle over $\mathbb{P}^s$ and denote by $Crit(X) \subset (\mathbb{C}^*)^n$ the solution scheme of the Landau-Ginzburg system of equations of $X$. We describe a map $E : Crit(X) \to Pic(X)$ whose image $E = \{E(z) | z \in Crit(X)\}$ is the full strongly exceptional collection described by Costa and Miró-Roig in [15]. We further show that $Hom(E(z), E(w))$ for $z, w \in Crit(X)$ can be described in terms of a monodromy group acting on $Crit(X)$.

1. Introduction and Summary of Main Results

Let $X$ be a smooth algebraic manifold and let $D^b(X)$ be the bounded derived category of coherent sheaves on $X$, see [22, 42]. A fundamental question in the study of $D^b(X)$ is the question of existence of exceptional collections $E = \{E_1, ..., E_N\} \subset D^b(X)$. Such collections satisfy the property that the adjoint functors

$$RHom_X(T, -) : D^b(X) \to D^b(A_L) ; \quad - \otimes_{A_L} T : D^b(A_L) \to D^b(X)$$

are equivalences of categories where $T := \bigoplus_{i=1}^N E_i$ and $A_L = End(T)$ is the corresponding endomorphism ring. The first example of such a collection is

$$E = \{\mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(s)\} \subset Pic(\mathbb{P}^s)$$

found by Beilinson in [7]. When $X$ is a toric manifold one further asks the more refined question of wether $D^b(X)$ admits an exceptional collection whose elements are line bundles $E \subset Pic(X)$, rather than general elements of $D^b(X)$?

Let $X$ be a $s$-dimensional toric Fano manifold given by a Fano polytope $\Delta$ and and let $\Delta^\circ$ be the polar polytope of $\Delta$. Let $f_X = \sum_{n \in \Delta^\circ \cap \mathbb{Z}^s} z^n \in \mathbb{C}[z_1^+, ..., z_s^+]$ be the Landau-Ginzburg potential associated to $X$, see [3, 21, 36]. Recall that the Landau-Ginzburg system of equations is given by

$$z_i \frac{\partial}{\partial z_i} f_X(z_1, ..., z_s) = 0 \quad \text{for} \quad i = 1, ..., s$$

and denote by $Crit(X) \subset (\mathbb{C}^*)^s$ the corresponding solution scheme. Consider the following example:
In this work we consider the next simplest case \[ \rho \in \mathbb{E} \] and the solution scheme \( \text{Crit} \in \mathbb{E}^s \) is given by \( z_k = (e^{\frac{2\pi ki}{s+1}}, \ldots, e^{\frac{2\pi ki}{s+1}}) \) for \( k = 0, \ldots, s \).

In particular, in the case of projective space, one has the map \( E : \text{Crit}(\mathbb{P}^s) \to \text{Pic}(\mathbb{P}^s) \) given by \( z_k \mapsto \mathcal{O}(k) \), associating elements of the Beilinson exceptional collection to elements of the solution scheme \( \text{Crit}(\mathbb{P}^s) \). In [26] we asked, motivated by the Dubrovin-Bayer-Manin conjecture, whether it is possible to similarly introduce exceptional maps \( E : \text{Crit}(X) \to \text{Pic}(X) \) for more general classes of toric Fano manifolds \( X \).

In this work we consider the next simplest case \( \rho(X) := rk(\text{Pic}(X)) = 2 \) which, according to Kleinschmidt’s classification theorem [29], consists of projective bundles of the form

\[
X = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i) \right) \quad \text{with} \quad \sum_{i=1}^{r} a_i \leq s, a_i \leq a_{i+1}
\]

The Picard group is expressed in this case by \( \text{Pic}(X) = \pi^*H \cdot \mathbb{Z} \oplus \xi \mathbb{Z} \) where \( \pi^*H \) is the pull-back of the positive generator \( H \in \text{Pic}(\mathbb{P}^s) \) via \( \pi : X \to \mathbb{P}^s \) and \( \xi \) is the tautological line bundle of \( X \). On the “derived category side”, it follows from a result of Costa and Miró-Roig in [15] that the collection \( \mathcal{E}_X = \{ E_{kl} \}_{k=0, l=0}^{s, r} \subset \text{Pic}(X) \) where

\[
E_{kl} := k \cdot \pi^*H + l \cdot \xi \in \text{Pic}(X) \quad \text{for} \quad 0 \leq k \leq s, 0 \leq l \leq r
\]

is a full strongly exceptional collection. On the other hand, on the ”Landau-Ginzburg side”, the Landau-Ginzburg potential is given by

\[
f(z, w) = \sum_{i=1}^{s} z_i + \sum_{i=1}^{r} w_i + \frac{w_1^{a_1} \cdots w_r^{a_r}}{z_1 \cdots z_s} + \frac{1}{w_1 \cdots w_r}
\]

In the trivial case, when \( X = \mathbb{P}^s \times \mathbb{P}^r \) is a product, that is \( a_1 = \ldots = a_r = 0 \), one can readily verify that the solution scheme is given by

\[
\text{Crit}(\mathbb{P}^s \times \mathbb{P}^r) = \{ (z_k, w_l) | z_k \in \text{Crit}(\mathbb{P}^s), w_l \in \text{Crit}(\mathbb{P}^r) \} \subset (\mathbb{C}^*)^{s+r}
\]

Hence set \( E(z_k, w_l) := E_{kl} \in \text{Pic}(\mathbb{P}^s \times \mathbb{P}^r) \). However, in general, the solution scheme \( \text{Crit}(X) \) is not given in terms of roots of unity. In order to overcome this and define the map \( E \) in general we note that the Landau-Ginzburg potential \( f_X \) is an element of the space

\[
L(\Delta^0) := \left\{ \sum_{n \in \Delta^0 \cap \mathbb{Z}^d} e^{u_n z^n} | u_n \in \mathbb{C} \right\} \subset \mathbb{C}[z^\pm]
\]
Theorem A: \(\lim_{t \to -\infty} (\Theta(Crit(X; f_i)))\) = \(\left\{ \left( e^{2\pi i \left( \frac{1}{(k+1)(l+1)} + \frac{1}{s+t} \right)} , e^{2\pi i t} \right) \right\}_{k=0, l=0}^{s, r} \subset T^2\).

In particular, the collection of roots of unity of (a) enables us to generalize the definition of the exceptional map \(E : Crit(X) \to Pic(X)\) to any Fano projective bundle.

As mentioned, when studying exceptional collections, one is interested in the endomorphism algebra \(A_{E} = End \left( \oplus_{i=1}^{N} E_{i} \right) \simeq \oplus_{i=1,j=1}^{N} Hom(E_i, E_j)\). A choice of basis for the \(Hom\)-groups expresses the algebra \(A_{E}\) as the path algebra of a quiver with relations whose vertex set is \(E\), see [18, 28]. In our case there is a natural choice of such bases and we denote the resulting quiver by \(\tilde{Q}(a)\). In the Landau-Ginzburg setting, we introduced in [26], the monodromy action

\[ M : \pi_1(L(\Delta^0) \setminus R_X, f_X) \to Aut(Crit(X)) \]

where \(R_X \subset L(\Delta^0)\) is the hypersurface of all elements such that \(Crit(X; f)\) is non-reduced. The main feature of the exceptional map \(E\) is that the quiver \(\tilde{Q}(a)\) and, in particular, the structure of \(Hom(E(z_i), E(z_j))\), could further be related to the geometry of the monodromy action \(M\).

For any \((n, m) \in (\mathbb{Z}^+)^{s+1} \times (\mathbb{Z}^+)^{r+1}\) and \(t \in \mathbb{R}\) consider the loop \(\gamma_{(n,m)}^t : [0, 1) \to L(\Delta^0)\) given by

\[ \gamma_{(n,m)}^t(\theta) := \sum_{i=1}^{s} e^{2\pi i m_{0i} \theta} z_i + \sum_{i=1}^{r} e^{2\pi i m_{0i} \theta} w_i + e^{t} e^{2\pi i m_{0i} \theta} \prod_{j=1}^{r} w_j \frac{1}{\prod_{i=1}^{s} z_i} + e^{2\pi i m_{0i} \theta} \prod_{i=1}^{r} w_i \]

Let \(\eta_t : [0, 1] \to L(\Delta^0)\) be the segment connecting \(f_X\) to \(f_t\). Each loop \(\gamma_{(n,m)}^t\) gives rise to a monodromy element:

\[ \Gamma_{(n,m)} := \lim_{t \to -\infty} [\eta_t^{-1} \circ \gamma_{(n,m)}^t \circ \eta_t] \in \pi_1(L(\Delta^0 \setminus R_X, f_X) ) \]
We use the exceptional map $E$ to express the solution scheme as 

$$\text{Crit}(X) = \{(k, l)\}_{k, l=0}^{s, r} \simeq \mathbb{Z}/(s+1) \oplus \mathbb{Z}/(r+1)\mathbb{Z}$$

where $(k, l)$ is the solution such that $E((k, l)) = E_{kl}$. For $(n, m) \in (\mathbb{Z}^+)^{s+1} \times (\mathbb{Z}^+)^{r+1}$ set 

$$|(n, m)|_1 := \sum_{i=0}^s n_i - \sum_{i=0}^r a_i m_i ; \quad |(n, m)|_2 := \sum_{i=0}^r m_i$$

and consider the rectangle 

$$D^+(k, l) = \left\{ (n, m) \left| -k < |(n, m)|_1 \leq s - k, \quad 0 < |(n, m)|_2 \leq r - l \right\} \subset (\mathbb{Z}^+)^{s+1} \times (\mathbb{Z}^+)^{r+1}$$

We define the following spaces via the monodromy action 

$$\text{Hom}_{\text{mon}}((k_1, l_1), (k_2, l_2)) := \bigoplus_{(n, m) \in M((k_1, l_1), (k_2, l_2))} \mathbb{C} \Gamma_{(n, m)} \quad \text{for} \quad (k_1, l_1), (k_2, l_2) \in \text{Crit}(X)$$

where 

$$M((k_1, l_1), (k_2, l_2)) = \{(n, m) | M(\Gamma_{(n, m)})(k_1, l_1) = (k_2, l_2) \text{ and } (n, m) \in D^+(k, l)\}$$

We show the following property of the map $E$:

**Theorem B** (M-aligned property): For any two solutions $(k, l), (k', l') \in \text{Crit}(X)$ the following holds 

$$\text{Hom}(E_{k_1 l_1}, E_{k_2 l_2}) \simeq \text{Hom}_{\text{mon}}((k_1, l_1), (k_2, l_2))$$

The rest of the work is organized as follows: In section 2 we recall relevant facts on projective Fano bundles and their derived categories of coherent sheaves. In section 3 we study variations of the Landau-Ginzburg system, prove Theorem A and define the exceptional map. In section 4 we prove Theorem B and describe the quiver monodromy correspondence. In section 5 we discuss concluding remarks and relations to further topics of mirror symmetry.

### 2. Relevant Facts on Toric Fano Manifolds

Let $N \simeq \mathbb{Z}^n$ be a lattice and let $M = N^\vee = \text{Hom}(N, \mathbb{Z})$ be the dual lattice. Denote by $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$ the corresponding vector space. Let $\Delta \subset M_\mathbb{R}$ be an integral polytope and let 

$$\Delta^\circ = \{ n \mid (m, n) \geq -1 \text{ for every } m \in \Delta \} \subset N_\mathbb{R}$$

be the *polar* polytope of $\Delta$. The polytope $\Delta \subset M_\mathbb{R}$ is said to be *reflexive* if $0 \in \Delta$ and $\Delta^\circ \subset N_\mathbb{R}$ is integral. A reflexive polytope $\Delta$ is said to be *Fano* if every facet of $\Delta^\circ$ is the convex hull of a basis of $M$. 


To an integral polytope $\Delta \subset M_\mathbb{R}$ associate the space

$$L(\Delta) = \bigoplus_{m \in \Delta \cap M} \mathbb{C} m$$

of Laurent polynomials whose Newton polytope is $\Delta$. Denote by $i_\Delta : (\mathbb{C}^*)^n \to \mathbb{P}(L(\Delta)^\vee)$ the embedding given by $z \mapsto [z^m \ | \ m \in \Delta \cap M]$. The toric variety $X_\Delta \subset \mathbb{P}(L(\Delta)^\vee)$ corresponding to the polytope $\Delta \subset M_\mathbb{R}$ is defined to be the compactification of the image $i_\Delta((\mathbb{C}^*)^n) \subset \mathbb{P}(L(\Delta)^\vee)$. A toric variety $X_\Delta$ is said to be Fano if its anticanonical class $-K_{X}$ is Cartier and ample. In [4] Batyrev shows that $X_\Delta$ is a Fano variety if $\Delta$ is reflexive and, in this case, the embedding $i_\Delta$ is the anti-canonical embedding. The Fano variety $X_\Delta$ is smooth if and only if $\Delta^\circ$ is a Fano polytope.

Denote by $\Delta(k)$ the set of $k$-dimensional faces of $\Delta$ and denote by $V_X(F) \subset X$ the orbit closure of the orbit corresponding to the facet $F \in \Delta(k)$ in $X$, see [20, 35]. In particular, consider the group of toric divisors

$$Div_T(X) := \bigoplus_{F \in \Delta(n-1)} \mathbb{Z} \cdot V_X(F)$$

Assuming $X$ is a smooth the group $Pic(X)$ is described in terms of the short exact sequence

$$0 \to M \to Div_T(X) \to Pic(X) \to 0$$

where the map on the left hand side is given by $m \to \sum_F \langle m, n_F \rangle \cdot V_X(F)$ where $n_F \in \mathbb{N}_\mathbb{R}$ is the unit normal to the hyperplane spanned by the facet $F \in \Delta(n-1)$. In particular, note that

$$\rho(X) = \text{rank}(Pic(X)) = |\Delta(n-1)| - n$$

Moreover, when $\Delta$ is reflexive one has $\Delta^\circ(0) = \{n_F \mid F \in \Delta(n-1)\} \subset N_\mathbb{R}$. We thus sometimes denote $V_X(n_F)$ for the $T$-invariant divisor $V_X(F)$. We denote by $Div_T^+(X)$ the semi-group of all toric divisors $\sum_F m_F \cdot V_X(F)$ with $0 \leq m_F$ for any $F \in \Delta(n-1)$.

Let $X$ be a smooth projective variety and let $\mathcal{D}^b(X)$ be the derived category of bounded complexes of coherent sheaves of $O_X$-modules, see [22, 42]. For a finite dimensional algebra $A$ denote by $\mathcal{D}^b(A)$ the derived category of bounded complexes of finite dimensional right modules over $A$. Given an object $T \in \mathcal{D}^b(X)$ denote by $A_T = Hom(T,T)$ the corresponding endomorphism algebra.

**Definition 2.1:** An object $T \in \mathcal{D}^b(X)$ is called a tilting object if the corresponding adjoint functors

$$RHom_X(T,-) : \mathcal{D}^b(X) \to \mathcal{D}^b(A_T) \ ; \ - \otimes^L_{A_T} T : \mathcal{D}^b(A_T) \to \mathcal{D}^b(X)$$

are equivalences of categories. A locally free tilting object is called a tilting bundle.
An object $E \in D^b(X)$ is said to be exceptional if $\text{Hom}(E, E) = \mathbb{C}$ and $\text{Ext}^i(E, E) = 0$ for $0 < i$. We have:

**Definition 2.2:** An ordered collection $\mathcal{E} = \{E_1, ..., E_N\} \subset D^b(X)$ is said to be an exceptional collection if each $E_j$ is exceptional and $\text{Ext}^i(E_k, E_j) = 0$ for $j < k$ and $0 \leq i$. An exceptional collection is said to be strongly exceptional if also $\text{Ext}^i(E_j, E_k) = 0$ for $j \leq k$ and $0 < i$. A strongly exceptional collection is called full if its elements generate $D^b(X)$ as a triangulated category.

The importance of full strongly exceptional collections in tilting theory is due to the following properties, see [8, 28]:

- If $\mathcal{E}$ is a full strongly exceptional collection then $T = \bigoplus_{i=1}^N E_i$ is a tilting object.

- If $T = \bigoplus_{i=1}^N E_i$ is a tilting object and $\mathcal{E} \subset \text{Pic}(X)$ then $\mathcal{E}$ can be ordered as a full strongly exceptional collection of line bundles.

By a result of Kleinschmidt’s [29] the class of toric manifolds with $rk(\text{Pic}(X)) = 2$ consists of the projective bundles

$$X_a = \mathbb{P} \left( O_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r O_{\mathbb{P}^s}(a_i) \right) \quad \text{with} \quad 0 \leq a_1 \leq ... \leq a_r$$

see also [17]. Set $a_0 = 0$. Consider the lattice $N = \mathbb{Z}^{s+r}$ and let $v_1, ..., v_s$ be the standard basis elements of $\mathbb{Z}^s$ and $e_1, ..., e_r$ be the standard basis elements of $\mathbb{Z}^r$. Set $v_0 = -\sum_{i=1}^s u_i + \sum_{i=1}^r a_i e_i$ and $e_0 = -\sum_{i=1}^r e_i$. Let $\Delta_a^0 \subset N_\mathbb{R}$ be the polytope whose vertex set is given by

$$\Delta_a^0(0) = \{v_0, ..., v_s, e_0, ..., e_r\}$$

It is straightforward to verify that $\Delta_a$, the polar of $\Delta_a^0$, is a Fano polytope if and only if $\sum_{i=1}^r a_i \leq s$. In particular, in this case $X_a \simeq X_{\Delta_a}$, see [17]. One has

$$\text{Pic}(X_a) = \xi \cdot \mathbb{Z} \oplus \pi^*H \mathbb{Z}$$

where $\xi$ is the class of the tautological bundle and $\pi^*H$ is the pullback of the generator $H$ of $\text{Pic}(\mathbb{P}^s) \simeq H \cdot \mathbb{Z}$ under the projection $\pi : X_a \to \mathbb{P}^s$. Note that the following holds

$$[V_X(v_i)] = \pi^*H \quad ; \quad [V_X(e_0)] = \xi \quad ; \quad [V_X(e_j)] = \xi - a_i \cdot \pi^*H$$

for $0 \leq i \leq s$ and $1 \leq j \leq r$. 
It follows from results of Costa and Miró-Roig in [15] that the collection of line bundles $\mathcal{E} = \{E_{kl}\}_{k=0,l=0}^{s,r} \subset \text{Pic}(X)$ where

$$E_{kl} := k \cdot \pi^*H + l \cdot \xi$$

for $0 \leq k \leq s$, $l \leq r$

is a full strongly exceptional collection. In the next section we describe how the solution scheme $\text{Crit}(X) \subset (\mathbb{C}^*)^{r+s}$ can be associated with similar invariants by considering asymptotic variations of the Landau-Ginzburg system of equations of $X_a$.

3. Variations of the LG-system and roots of unity

Let $X$ be a $n$-dimensional toric Fano manifold given by a Fano polytope $\Delta \subset M_{\mathbb{R}}$ and let $\Delta^o \subset N_{\mathbb{R}}$ be the corresponding polar polytope. Set

$$L(\Delta^o) := \left\{ \sum_{n \in \Delta^o \cap \mathbb{Z}^n} u_n z^n | u_n \in \mathbb{C}^* \right\} \subset \mathbb{C}[z_1^\pm, ..., z_n^\pm]$$

We refer to

$$z_i \frac{\partial}{\partial z_i} f_u(z_1, ..., z_n) = 0 \quad \text{for} \quad i = 1, ..., n$$

as the LG-system of equations associated to an element $f_u(z) = \sum_{n \in \Delta^o \cap \mathbb{Z}^n} u_n z^n$ and denote by $\text{Crit}(X; f_u) \subset (\mathbb{C}^*)^n$ the corresponding solution scheme. We refer to the element $f_X(z) = \sum_{n \in \Delta^o \cap \mathbb{Z}^n} z^n$ as the LG-potential of $X$. In particular for the projective bundle $X_a$ the Landau-Ginzburg potential is given by

$$f(z, w) = 1 + \sum_{i=1}^s z_i + \sum_{i=1}^r w_i + \frac{w_1^{a_1} \cdot ... \cdot w_r^{a_r}}{z_1 \cdot ... \cdot z_s} + \frac{1}{w_1 \cdot ... \cdot w_r} \in L(\Delta^o_a)$$

We consider the 1-parametric family of Laurent polynomials

$$f_u(z, w) := 1 + \sum_{i=1}^s z_i + \sum_{i=1}^r w_i + e^u \cdot \frac{w_1^{a_1} \cdot ... \cdot w_r^{a_r}}{z_1 \cdot ... \cdot z_s} + \frac{1}{w_1 \cdot ... \cdot w_r} \in L(\Delta^o_a)$$

for $u \in \mathbb{C}$. Let $\text{Arg} : (\mathbb{C}^*)^n \to \mathbb{T}^n$ be the argument map given by

$$(r_1 e^{2\pi i \theta_1}, ..., r_n e^{2\pi i \theta_n}) \mapsto (\theta_1, ... \theta_n)$$

In general, the image $A(V) := \text{Arg}(V) \subset \mathbb{T}^n$ of an algebraic subvariety $V \subset (\mathbb{C}^*)^n$ under the argument map is known as the co-amoeba of $V$, see [37]. For $1 \leq i \leq s$ and $1 \leq j \leq r$ consider the following sub-varieties of $(\mathbb{C}^*)^{s+r}$:

$$V^u_i = \left\{ z_i - e^u \prod_{l=1}^r w_i^{a_l} / z_i = 0 \right\} \quad ; \quad W^u_j = \left\{ w_i + a_i e^u \prod_{l=1}^r w_i^{a_l} - 1 / w_i = 0 \right\}$$

Clearly, by definition $\text{Crit}(X; f_u) = (\bigcap_{i=1}^s V^u_i) \cap (\bigcap_{i=1}^r W^u_i)$. For the co-amoeba one has

$$A(\text{Crit}(X; f_u)) \subset \left( \bigcap_{i=1}^s A(V^u_i) \right) \cap \left( \bigcap_{i=1}^r A(W^u_i) \right) \subset \mathbb{T}^{s+r}$$
Let \((\theta_1, ..., \theta_s, \delta_1, ..., \delta_r)\) be coordinates on \(T^{s+r}\). We have, via straight-forward computation:

**Lemma 3.1:** For \(1 \leq i \leq s\) and \(1 \leq j \leq r\):

1. \(\lim_{t \to -\infty} A(V^t_i) = \left\{ \theta + \sum_{i=1}^s \theta_i - \sum_{j=1}^r a_j \delta_j = 0 \right\} \subset T^{s+r}\)

2. \(\lim_{t \to -\infty} A(W^t_j) = \left\{ \delta_j + \sum_{j=1}^r \delta_j = 0 \right\} \subset T^{s+r}\)

Let \(\Theta : (\mathbb{C}^*)^{s+r} \to (\mathbb{C}^*)^2\) be the map given by

\[(z_1, ..., z_s, w_1, ..., w_r) \mapsto Arg\left( \frac{\prod_{i=1}^s w_{a_i}}{\prod_{i=1}^s z_i} \cdot \frac{1}{\prod_{i=1}^r w_i} \right)\]

We have:

**Proposition 3.2:**

\[
\lim_{t \to -\infty} (\Theta(Crit(X; f_i))) = \left\{ \left( \frac{l \sum_{i=1}^r a_i}{(s+1)(r+1)} + \frac{k}{s+1} \cdot \frac{l}{r+1} \right) \right\}^{s,r}_{k=0, l=0} \subset T^2
\]

**Proof:** Set \(A_i = \lim_{t \to -\infty} A(V^t_i)\) and \(B_j = \lim_{t \to -\infty} A(W^t_j)\) for \(1 \leq i \leq s\) and \(1 \leq j \leq r\). If \((\theta, \delta) \in \bigcap_{j=1}^r B_j\) then \(\delta = \delta_1 = ... = \delta_r\) and \((r+1)\delta = 0\) in \(T\). If

\[
(\theta, \delta) \in \left( \bigcap_{i=1}^s A_i \right) \cap \left( \bigcap_{j=1}^r B_j \right)
\]

then \(\theta = \theta_1 = ... = \theta_s\) and \(\delta = \frac{l}{r+1}\) for some \(0 \leq l \leq r\). As \((s+1)\theta - \sum_{j=1}^r a_j \delta\) we get \(\theta = \sum_{j=1}^r \frac{a_j}{(s+1)(r+1)} + \frac{k}{s+1}\) for \(1 \leq k \leq s\). As there are exactly \((r+1)(s+1)\) such elements \((\theta, \delta)\) we get \(\lim_{t \to -\infty} A(Crit(X; f_i)) = (\bigcap_{i=1}^s A_i) \cap (\bigcap_{j=1}^r B_j)\). □

Each solution \((z, w) \in Crit(X)\) extends to a unique smooth curve \((z(t), w(t)) \subset (\mathbb{C}^*)^{s+r}\) for \(t \leq 0\) satisfying \((z^t, w^t) \in Crit(X; f_i)\). Set \(\rho_n := \frac{2\pi i}{(n+1)}\) and \(\theta_{n,m}(a) := \frac{2\pi i \sum a_k}{(n+1)(m+1)}\) for \(n, m \in \mathbb{Z}\). By Proposition 3.2 we have

\[
\lim_{t \to -\infty} \Theta(z(t), w(t)) = (l \cdot \theta_{r,s}(a) + k \cdot \rho_s, l \cdot \rho_r)
\]

For some \(0 \leq k \leq s\) and \(0 \leq l \leq r\). In particular, set \(k(z, w) := k\) and \(l(z, w) := l\). We are now in position to define:

**Definition 3.3** (Exceptional map): Let \(E : Crit(X) \to Pic(X)\) be the map given by

\[
E(z, w) := k(z, w) \cdot \pi^*H + l(z, w) \cdot \xi
\]
for \((z, w) \in Crit(X)\).

Let us note the following remark:

**Remark 3.4** (Geometric viewpoint): Consider the Riemann surface

\[
C_s(a) := \left\{ \left( \frac{w_1^{a_1} \cdots w_r^{a_r}}{z_1 \cdots z_s}, \frac{1}{w_1 \cdots w_r}, u \right) \mid (z, w) \in Crit(X; f_u) \right\} \subset (\mathbb{C}^*)^3
\]

Denote by \(\pi : C_s(a) \to \mathbb{C}^*\) the projection on the third factor which expresses \(C_s(a)\) as an algebraic fibration over \(\mathbb{C}^*\) of rank \(N = \chi(X)\). Denote by \(C_s(a; u) = \pi^{-1}(u)\) for \(u \in \mathbb{C}^*\). A graphic illustration of \(C_s(a)\) together with the curves \(C(a; t)\) for \(0 \leq t \leq \infty\) for the Hirzebruch surface \(X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))\) is as follows:

An amusing analogy can be drawn between the resulting dynamics and the cue game of "pool". Indeed, consider the Riemann surface \(C_s(a)\) as a "pool table", the cusps of the surface as the "pockets", and the set \(C_s(a; 0) \simeq Crit(X)\) as an initial set of "balls". In this analogy the dynamics of \(C_s(a; t)\) describes the path in which the balls approach the various "pockets" of the table as \(t \to \pm \infty\).

Set \(\Theta_{\pm}(X) = \lim_{t \to \pm \infty} \Theta(Crit(X; f_t)) \subset \mathbb{T}^2\). Note that defintion 3.3 of the exceptional map utilized only the sets \(\Theta_{-}(X)\). It is interesting to ask whether \(\Theta_{+}(X)\) can also be interpreted in terms of the exceptional map \(E\). Consider the following example:

**Example 3.5** (The Hirzebruch surface): Let \(X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))\) be the Hirzebruch surface. Recall that

\[
X = \{ ([z_0 : z_1 : z_2], [\lambda_0 : \lambda_1]) | \lambda_0 z_0 + \lambda_1 z_1 = 0 \} \subset \mathbb{P}^2 \times \mathbb{P}^1
\]

Denote by \(p : X \to \mathbb{P}^2\) and \(\pi : X \to \mathbb{P}^1\) the projection to the first and second factor, respectively. Note that \(p\) expresses \(X\) as the blow up of \(\mathbb{P}^2\) at the point \([0 : 0 : 1]\) \(\in \mathbb{P}^2\) and \(\pi\) is the fibration map. The group \(Pic(X)\) is described, in turn, in the following two ways

\[
Pic(X) \simeq p^*H_{\mathbb{P}^2} \cdot \mathbb{Z} \oplus E \cdot \mathbb{Z} \simeq \pi^*H_{\mathbb{P}^1} \cdot \mathbb{Z} \oplus \xi \cdot \mathbb{Z}
\]
Where \( E \) is class of the the line bundle whose first Chern class \( c_1(E) \in H^2(X; \mathbb{Z}) \) is the Poincare dual of the exceptional divisor and \( \xi \) is the class of the tautological bundle of \( \pi \). The exceptional collection is expressed in these bases by

\[
\mathcal{E}_X = \{0, p^*H_{p^2} - E, 2p^*H_{p^2} - E, p^*H_{p^2} \} = \{0, \pi^*H_{\pi^1}, \pi^*H_{\pi^1} + \xi, \xi \}
\]

Let us note that we have \( p_* \{0, p^*H_{p^2}, 2p^*H_{p^2} - E \} = \{0, H_{p^2}, 2H_{p^2} \} = \mathcal{E}_{p^2} \), while we think of the additional element \( p^*H - E \) as "added by the blow up".

On the other hand, direct computation gives \( \Theta_+(X) = \mu(3) \cup \mu(1) \) where \( \mu(n) = \{ e^{\frac{2\pi i}{n}} | k = 0, ..., n - 1 \} \subset \mathbb{T} \) is the set of \( n \)-roots of unity for \( n \in \mathbb{N} \) (see illustration in Remark 3.4). For \( (k, l) \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) let \( \gamma_{kl}(t) = (z_{kl}(t), w_{kl}(t)) \in (\mathbb{C}^*)^{s+r} \) be the smooth curve defined by the condition \( (z_{kl}(t), w_{kl}(t)) \in \text{Crit}(X; f_t) \) for \( t \in \mathbb{R} \) and \( E(z_{kl}(0), w_{kl}(0)) = E_{kl} \). Define the map \( I^+: \text{Crit}(X) \to \Theta_+(X) \) by

\[
I^+(z_{kl}, w_{kl}) := \lim_{t \to -\infty}(\Theta(z_{kl}(t), w_{kl}(t)))
\]

By direct computation

\[
I^+((z_{00}, w_{00})) = \rho_3^0 ; \quad I^+((z_{01}, w_{01})) = \rho_3^1 ; \quad I^+((z_{11}, w_{11})) = \rho_3^2 ; \quad I^+((z_{10}, w_{10})) = 1
\]

where \( \rho = e^{\frac{2\pi i}{3}} \in \mu(3) \). Similarly, define the map \( I^- : \text{Crit}(X) \to \Theta_-(X) \), on the other hand, taking \( t \to -\infty \) in the limit. Note that this is the way we defined the exceptional map \( E \) in the first place. We thus view the map \( I : \Theta_-(X) \to \Theta_+(X) \) given by \( I = I^+ \circ (I^-)^{-1} \) as a "geometric interpolation" between the bundle description of \( \mathcal{E}_X \) and the blow up description of \( \mathcal{E}_X \).

## 4. Monodromies and the Endomorphism Ring

Given a full strongly exceptional collection \( \mathcal{E} = \{E_i\}_{i=1}^N \subset \text{Pic}(X) \) one is interested in the structure of its endomorphism algebra

\[
A_{\mathcal{E}} = \text{End} \left( \bigoplus_{i=1}^N E_i \right) = \bigoplus_{i,j=0}^N \text{Hom}(E_i, E_j) = \bigoplus_{i,j=0}^N H^0(X; E_j \otimes E_i^{-1})
\]

Our aim in this section is to show how this algebra is naturally reflected in the monodromy group action of the Landau-Ginzburg system, in our case. Note that, in our case

\[
\text{Div}_T(X) = \left( \bigoplus_{i=1}^s \mathbb{Z} \cdot V_X(v_i) \right) \bigoplus \left( \bigoplus_{i=0}^r \mathbb{Z} \cdot V_X(e_i) \right)
\]

First, we have:
Proposition 4.1 Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i))$ be a projective Fano bundle and let $L_{kl} = k \cdot \pi^* H + l \cdot \xi \in \text{Pic}(X)$ be any element. Then

$$H^0(X; L_{kl}) \simeq \left\{ \sum_{i=0}^{s} n_i V_X(v_i) + \sum_{i=0}^{r} m_i V_X(e_i) \middle| m \mid = l \text{ and } |n| = k + \sum_{i=0}^{r} m_i a_i \right\} \subset \text{Div}_T^+(X)$$

Recall that a quiver with relations $\tilde{Q} = (Q, R)$ is a directed graph $Q$ with a two sided ideal $R$ in the path algebra $\mathbb{C}Q$ of $Q$, see [18]. In particular, a quiver with relations $\tilde{Q}$ determines the associative algebra $A_{\tilde{Q}} = \mathbb{C}Q/R$, called the path algebra of $\tilde{Q}$. In general, a collection of elements $C \subset \mathcal{D}^b(X)$ and a basis $B \subset A_C := \text{End}(\bigoplus_{E \in C} E)$ determine a quiver with relations $\tilde{Q}(C, B)$ whose vertex set is $C$ such that $A_C \simeq A_{\tilde{Q}(C, B)}$, see [28]. By Proposition 4.1 the algebra $A_C$ comes with the basis \{ $V(v_0), ..., V(v_s), V(e_0), ..., V(e_r)$ \}. We denote the resulting quiver by $Q_s(a_0, ..., a_r)$. For example, the quiver $Q_3(0, 1, 2)$ for $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2))$ is the following:

\[
\begin{array}{cccc}
E_{00} & \rightarrow & E_{10} & \rightarrow & E_{20} & \rightarrow & E_{30} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{01} & \rightarrow & E_{11} & \rightarrow & E_{21} & \rightarrow & E_{31} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{02} & \rightarrow & E_{12} & \rightarrow & E_{22} & \rightarrow & E_{32}
\end{array}
\]

On the other hand, on the Landau-Ginzburg side, let $R_X \subset L(\Delta^\circ)$ be the hypersurface of all $f \in L(\Delta^\circ)$ such that $\text{Crit}(X; f)$ is non-reduced. Whenever $\text{Crit}(X)$ is reduced, one obtains, via standard analytic continuation, a monodromy map of the following form

$$M : \pi_1(L(X) \setminus R_X, f_X) \rightarrow \text{Aut}(\text{Crit}(X))$$

For a divisor $D = \sum_{i=0}^{s} n_i V_X(v_i) + \sum_{i=0}^{r} m_i V_X(e_i) \in \text{Div}_T(X)$ and $u \in \mathbb{C}$ consider the loop

$$\gamma^u_D(\theta) := \sum_{i=1}^{s} e^{2\pi in_i \theta} z_i + \sum_{i=1}^{r} e^{2\pi im_i \theta} w_i + e^u \cdot e^{2\pi in_0 \theta} \prod_{i=1}^{r} w_i^{a_i} \prod_{i=1}^{s} z_i + e^{2\pi im_0 \theta} \prod_{i=1}^{r} w_i$$
For $\theta \in [0, 1)$. To a loop $\gamma^t : [0, 1] \to L(\Delta^s)$ with base point $\gamma(0) = \gamma(1) = f_t$ we associate the loop
\[
\tilde{\gamma}^t(\theta) := \begin{cases} 
(1 - 3\theta)f_X + 3\theta f_t & \theta \in [0, \frac{1}{3}) \\
\gamma^t(3\theta - 1) & \theta \in \left[\frac{1}{3}, \frac{2}{3}\right] \\
(3\theta - 1)f_t + (3\theta - 2)f_X & \theta \in (\frac{2}{3}, 1]
\end{cases}
\]
Define $\Gamma_D := \lim_{t \to -\infty} [\tilde{\gamma}^t_D] \in \pi_1(L(\Delta^s) \setminus R_X, f_X)$ and set $\tilde{M}_D := M(\Gamma_D) \in \text{Aut}(\text{Crit}(X))$. Express the solution scheme as
\[
\text{Crit}(X) = \{(z_{kl}, w_{kl})\}_{k=0,l=0}^{s+r} \simeq \mathbb{Z}/(r + 1)\mathbb{Z} \oplus \mathbb{Z}/(s + 1)\mathbb{Z}
\]
where $E((z_{kl}, w_{kl})) = E_{kl}$. We have:

**Theorem 4.2** For $(k, l) \in \mathbb{Z}/(s + 1)\mathbb{Z} \oplus \mathbb{Z}/(r + 1)\mathbb{Z} \simeq \text{Crit}(X)$ the monodromy action satisfies:

(a) $\tilde{M}_{V(v_j)}(k, l) = (k + 1, l)$ for $j = 0, \ldots, s$.

(b) $\tilde{M}_{V(v_j)}(k, l) = (k - a_j, l + 1)$ for $j = 0, \ldots, r$.

**Proof:** For a divisor $D \in \text{Div}_T(X)$ and $\theta \in [0, 1)$ Set
\[
V^{u,\theta}_{D,i} := \left\{ e^{2\pi i n_0 \theta} z_i - e^u e^{2\pi in_0 \theta} \prod_{i=1}^{r} \frac{w_n^i}{z_i} = 0 \right\}; \quad W^{u,\theta}_{D,i} := \left\{ e^{2\pi in_j \theta} w_j + a_i e^u e^{2\pi in_0 \theta} \prod_{i=1}^{r} \frac{w_n^i}{z_i} - e^{2\pi in_0 \theta} = 0 \right\}
\]
where $1 \leq i \leq s$, $1 \leq j \leq r$ and $u \in \mathbb{C}$. Let $(\theta_1, \ldots, \theta_s, \delta_1, \ldots, \delta_r)$ be coordinates on $T^{s+r}$. It is clear that:

- $A^{\theta}_{D,j} := \lim_{t \to -\infty} A(V^{t,\theta}_{D,j}) = \left\{ \theta_i + \sum_{i=1}^{s} \theta_i - \sum_{j=1}^{r} a_j \delta_j + (n_i - n_0)\theta = 0 \right\} \subset T^{s+r}$

- $B^{\theta}_{D,j} := \lim_{t \to -\infty} A(W^{t,\theta}_{D,j}) = \left\{ \delta_j + \sum_{j=1}^{r} \delta_j + (m_j - m_0)\theta = 0 \right\} \subset T^{s+r}$

For $D = V(v_0)$ we have $(\theta, \delta) \in \bigcap_{j=1}^{r} B^{t,\theta}_{D,j}$ then $\delta := \delta_1 = \ldots = \delta_r$ and $(r + 1)\delta = 0$ hence $\delta = \frac{l}{r+1}$ for some $0 \leq l \leq r$. Assume further that $(\theta, \delta) \in \bigcap_{i=1}^{s} A^{t,\theta}_{D,i} \cap \bigcap_{j=1}^{r} B^{t,\theta}_{D,j}$ then $\tilde{\delta} = \delta_1 = \ldots = \delta_s$ and $(s + 1)\tilde{\delta} - \sum_{j=1}^{r} \frac{a_j l}{r+1} - \theta = 0$. Hence, $\tilde{\delta} = \frac{k}{s+1} + \frac{l \sum_{j=1}^{r} a_j}{(s+1)(r+1)} + \frac{\theta}{s+1}$ for some $0 \leq k \leq s$.

For $D = V(v_1)$ if $(\theta, \delta) \in \bigcap_{i=1}^{s} A^{t,\theta}_{D,i} \cap \bigcap_{j=1}^{r} B^{t,\theta}_{D,j}$ then $\tilde{\delta} = \theta_1 = \ldots = \theta_s$ and $\theta_i = \tilde{\theta} - \delta$ and again $(s + 1)\tilde{\delta} - \sum_{j=1}^{r} \frac{a_j l}{r+1} - \theta = 0$. Hence, $\tilde{\delta} = \frac{k}{s+1} + \frac{l \sum_{j=1}^{r} a_j}{(s+1)(r+1)} + \frac{\theta}{s+1}$ for some $0 \leq k \leq s$.

For $D = V(e_0)$ we have $(\theta, \delta) \in \bigcap_{j=1}^{r} B^{t,\theta}_{D,j}$ then $\delta := \delta_1 = \ldots = \delta_r$ and $(r + 1)\delta = \theta$ hence $\delta = \frac{\theta l}{r+1}$ for some $0 \leq l \leq r$. Assume further that $(\theta, \delta) \in \bigcap_{i=1}^{s} A^{t,\theta}_{D,i} \cap \bigcap_{j=1}^{r} B^{t,\theta}_{D,j}$ then $\tilde{\delta} = \theta_1 = \ldots = \theta_s$ and $\theta_i = \tilde{\theta} - \delta$ and again $(s + 1)\tilde{\delta} - \sum_{j=1}^{r} \frac{a_j l}{r+1} - \theta = 0$. Hence, $\tilde{\delta} = \frac{k}{s+1} + \frac{l \sum_{j=1}^{r} a_j}{(s+1)(r+1)} + \frac{\theta}{s+1}$ for some $0 \leq k \leq s$. 

\[ \tilde{\theta} = \theta_1 = \ldots = \theta_s \text{ and } (s + 1) \tilde{\theta} - \sum_{j=1}^{r} a_j \frac{(l+\theta)}{r+1} = 0. \] Hence, \[ \tilde{\theta} = \frac{k}{s+1} + \frac{(l+\theta) \sum_{j=1}^{r} a_j}{(s+1)(r+1)} \] for some \( 0 \leq k \leq s. \)

For \( D = V(e_j) \) we have \((\theta, \delta) \in \bigcap_{j=1}^{r} B^e_{D,j}\) then \( \delta := \delta_1 = \ldots = \delta_j = \ldots = \delta_r \) and \( \delta_j = \delta - \theta \) hence \((r + 1) \delta = \theta \) and \( \delta = \frac{l+\theta}{r+1} \) for some \( 0 \leq l \leq r. \) Assume \((\theta, \delta) \in (\bigcap_{i=1}^{s} A^t_{D,i}) \cap (\bigcap_{j=1}^{r} B^e_{D,j}) \) then \( \tilde{\theta} = \theta_1 = \ldots = \theta_s \) and \((s + 1) \tilde{\theta} - \sum_{j=1}^{r} a_j \frac{(l+\theta)}{r+1} + a_j \theta = 0. \) Hence, \( \tilde{\theta} = -\frac{k-a_j \theta}{s+1} + \frac{(l+\theta) \sum_{j=1}^{r} a_j}{(s+1)(r+1)} \) for some \( 0 \leq k \leq s. \ \Box \)

For instance, consider the following example:

**Example** (monodromies for \( X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)) \)): The following diagram outlines the corresponding monodromies on \( \mathbb{T}^2: \)

Blue lines describe the monodromy action of \( v_0, v_1, v_2, v_3 \) (which are, in practice, all linear in the horizontal direction), black lines describe the action of \( e_0 \) while red and green lines describe the action of \( e_1, e_2 \) respectively.

For a divisor \( D \in \text{Div}_T(X) \) set

\[ |D|_1 := \sum_{i=0}^{s} n_i - \sum_{i=0}^{r} a_i m_i \quad ; \quad |D|_2 = \sum_{i=0}^{r} m_i \]

Set

\[ \text{Div}^+(k, l) := \{ \text{Div}0 < k + |D|_1 \leq s \text{ and } 0 < l + |D|_2 \leq r \} \subset \text{Div}^+_T(X) \]
For two solutions \((k_1, l_1), (k_2, l_2) \in \mathbb{Z}/(s + 1)\mathbb{Z} \oplus \mathbb{Z}/(r + 1)\mathbb{Z}\) we define

\[
\text{Hom}_{\text{mon}}((k_1, l_1), (k_2, l_2)) := \bigoplus_{D \in M((k_1, l_1), (k_2, l_2))} \tilde{M}_D \cdot \mathbb{Z}
\]

where

\[
M((k_1, l_1), (k_2, l_2)) := \left\{ D \mid \tilde{M}_D (k_1, l_1) = (k_2, l_2) \text{ and } D \in \text{Div}^+(k_1, l_1) \right\}
\]

We have:

**Corollary 4.3 (M-Aligned property):** For any two solutions \((k_1, l_1), (k_2, l_2) \in \text{Crit}(X)\) the following holds

\[
\text{Hom}(E_{k_1 l_1}, E_{k_2 l_2}) \simeq \text{Hom}_{\text{mon}}((k_1, l_1), (k_2, l_2))
\]

Furthermore, the composition map

\[
\text{Hom}(E_{k_1 l_1}, E_{k_2 l_2}) \otimes \text{Hom}(E_{k_2 l_2}, E_{k_3 l_3}) \to \text{Hom}(E_{k_1 l_1}, E_{k_3 l_3})
\]

is induced by the map

\[
\text{Mon}((k_1, l_1), (k_2, l_2)) \times \text{Mon}((k_2, l_2), (k_3, l_3)) \to \text{Mon}((k_1, l_1), (k_3, l_3))
\]

given by \((D_1, D_2) \mapsto D_1 + D_2\).

5. **Discussion and Concluding Remarks**

We would like to conclude with the following remarks and questions:

(a) **Monodromies and Lagrangian submanifolds:** A leading source of interest for the study of the structure of \(\mathcal{D}^b(X)\), in recent years, has been their role in the famous homological mirror symmetry conjecture due to Kontsevich, see [28]. For a toric Fano manifold \(X\) denote by \(X^\circ\) the toric variety given by \(\Delta^\circ\), the polar polytope of \(\Delta\). It is generally accepted, that in this setting, the analog of the HMS-conjecture relates the structure of \(\mathcal{D}^b(X)\) to the structure of \(\text{Fuk}(\tilde{Y}^\circ)\), where \(\tilde{Y}^\circ\) is a disingularization of a hyperplane section \(Y^\circ\) of \(X^\circ\), see [2, 32, 40]. It is thus natural to pose the following question:

**Question:** Is it possible to naturally associate a Lagrangian submanifold \(L(z) \subset \tilde{Y}^\circ\) to a solution \(z \in \text{Crit}(X)\) with the property

\[
\text{HF}(L(z), L(w)) \simeq \text{Hom}_{\text{mon}}(z, w) \quad \text{for } z, w \in \text{Crit}(X)
\]

where \(\text{HF}\) stands for Lagrangian Floer homology?
(b) **Further toric Fano manifolds:** The Landau-Ginzburg potential of a toric Fano manifold $X$ could always be written in the form $f_X(z) := \sum_{i=1}^{n} z_i + \sum_{j=1}^{\rho(X)} z^{n_j}$ where $\rho(X) = \text{rk}(\text{Pic}(X))$, by taking an automorphism of the polytope $\Delta$. Consider the map $\Theta : (\mathbb{C}^*)^n \to \mathbb{T}^\rho$ given by

$$(z_1, ..., z_n) \mapsto \text{Arg}(z_1^{n_1}, ..., z_1^{n_n})$$

For an element $f_u(z) := \sum_{i=1}^{n} z_i^n + \sum_{j=1}^{\rho} e^{u_j} z^{n_j} \in L(\Delta^o)$ and $i = 1, ..., n$ define the hypersurfaces

$$V_i(u_1, ..., u_n) = \left\{ z_i \frac{\partial}{\partial z_i} f_u = 0 \right\} \subset (\mathbb{C}^*)^n$$

It is interesting to ask to which extent the study of the properties of the "co-tropical LG-system of equations"

$$\bigcap_{i=1}^{n} \Theta(V_i(u_1, ..., u_n)) \subset \mathbb{T}^\rho$$

for $|u| \to \infty$ could be further related to exceptional collections $\mathcal{E}_X \subset \text{Pic}(X)$ and their quivers for other, more general, examples of toric Fano manifolds.

Let us note that the zero set $V(f) = \{ f = 0 \} \subset (\mathbb{C}^*)^n$ of an element $f \in L(\Delta^o)$ is an affine Calabi-Yau hypersurface. In [4] Batyrev introduced $\mathcal{M}(\Delta^o)$ the toric moduli of such affine Calabi-Yau hyper-surfaces which is a $\rho(X)$-dimensional singular toric variety obtained as the quotient of $L(\Delta^o)$ by appropriate equivalence relations. In [4] Batyrev further shows that $\text{PH}^{n-1}(V(f)) \simeq \text{Jac}(f)$, where $\text{Jac}(f)$ is the function ring of the solution scheme $\text{Crit}(X; f) \subset (\mathbb{C}^*)^n$.

In this sense our approach could be viewed as a suggesting that in the toric Fano case homological data about the structure of $\mathcal{D}^b(X)$, could, in fact, be extracted from the local behavior around the boundary of the B-model moduli, which in our case is $\mathcal{M}(\Delta^o)$, rather than the Fukaya category appearing in the general homological mirror symmetry conjecture, whose structure is typically much harder to analyze.

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