Cross-over to quasi-condensation: mean-field theories and beyond

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We analyze the cross-over of a homogeneous one-dimensional Bose gas from the ideal gas into the dense quasi-condensate phase. We review a number of mean-field theories, perturbative or self-consistent, and provide accurate evaluations of equation of state, density fluctuations, and correlation functions. A smooth crossover is reproduced by classical-field simulations based on the stochastic Gross-Pitaevskii equation, and the Yang-Yang solution to the one-dimensional Bose gas.

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The achievement of Bose-Einstein condensation in ultracold, dilute atomic vapours has opened a wide research field at the crossroads of quantum optics and condensed-matter physics [1, 2]. It was clear from the beginning that the scenarios of the ideal Bose gas, well-known from thermodynamics, are not sufficient because of interactions between the atoms. In contrast to liquid helium where interactions are strong [3, 4], ultracold vapours can be modelled nearly from first principles, the s-wave scattering length being the main relevant coupling constant. In lower spatial dimensions, interactions qualitatively change the phase diagram and lead to the emergence of a paired vortex phase (Kosterlitz–Thouless transition, 2D) or a quasi-condensate (1D). The phase boundaries are fuzzy, however, and the cross-over region, as it has been called, poses a challenge to conventional pictures. Indeed, thermal and quantum fluctuations that are prominent anyway in lower dimensions have to be modelled in the presence of interactions. As the density is lowered through the cross-over region, density fluctuations become comparable to phase fluctuations, so that a Luttinger liquid approach [5] breaks down. It is then questionable whether one can operate a clean splitting into a 'quasi-condensate' and a 'thermal cloud' familiar from spontaneous symmetry breaking. This may explain why the cross-over is so difficult to describe with mean-field theories that build on the Bogoliubov prescription.

In this paper, we provide a critical assessment of mean-field theories for the description of the cross-over in a homogeneous one-dimensional Bose gas between the ideal gas and the quasi-condensate. These approaches have the common feature that the many-body system is broken down to relevant collective observables that are treated as 'hydrodynamic fields', examples being the (total) density or a c-number valued condensate field. The hydrodynamic fields parametrise an approximate form of the many-body Hamiltonian which is simple enough to be diagonalised in a quasi-particle basis with a well-defined dispersion relation. This permits to compute different mean values and correlation functions. Throughout the paper, we exclude the case of strong interactions which leads to fermionisation (impenetrable bosons, Tonks-Girardeau regime).

There are a number of variants for mean-field theories: some are based on perturbative expansions (weak interactions, weak density fluctuations) whose validity becomes doubtful in the cross-over region, others are constructed in a 'self-consistent' way and may suggest a comprehensive treatment of both regimes. We give an overview on several approaches and work out in detail the equation of state, density fluctuations, and correlation functions. There are numerous approaches that have been implemented to improve on the simple mean-field theories. For a unified-notation review, the readers are referred to Refs. [2, 6]. The "G1" variant of the Hartree-Fock-Bogoliubov approximation attempts to fix the issue of a gapless dispersion relation by carefully observing features a successful theory might have – indeed it had some success in modelling experiments [7, 8]. Much has been written about a gapless spectrum in relation to the Goldstone and Hugenholtz-Pines theorems [9, 10]. We find here that its impact is marginal with respect to the performance of a mean-field theory in the cross-over of the one-dimensional Bose gas.
lar, however (see Figs. 8, 9). They also fail both in providing a smooth description as the chemical potential crosses zero, and predict a critical point (discontinuity in the equation of state). This artefact has been noted before for mean-field theories in three dimensions, see Refs. 1–18. A different fate arises when the self-consistent and gapless theory of Yukalov and Yukalova (17, 19) is extrapolated to a one-dimensional system. The integrals giving the non-condensate density and other quantities diverge in the infrared, similar to the simpler Bogoliubov theory. In Ref. (17), this is claimed to be removed with dimensional regularization, effectively subtracting the divergent piece, although the resulting ‘density’ becomes negative. An infrared regularization has also been operated in the modified Popov theory, but following a different argument: the infrared-divergent pieces were identified as spurious phase fluctuations and eliminated. The resulting expressions are discussed here.

Another important development was the construction of an expansion for large particle numbers, but in a number-conserving way, following arguments laid out in Ref. 20 and extended in Refs. 21–24. We mention that for our system of interest, the homogeneous Bose gas in the thermodynamic limit, the predictions of mean-field theory are qualitatively quite similar, whether it is formulated in a number-conserving way or in the grand-canonical ensemble with symmetry breaking. We have checked this with the example of extended Bogoliubov theory developed by Mora and Castin (25): one key technique, the projector orthogonal to the condensate mode, is irrelevant for a homogeneous system where the elementary excitations naturally appear at finite momentum. The expansions behind these approaches, for example in the fraction of non-condensed particles, are bound to break down in the cross-over because there is no condensate in the dilute phase. In the case of extended Bogoliubov theory, it is the assumption of weak density fluctuations that fails.

It turns out that none of the mean-field theories analyzed here describes the cross-over of the Bose gas from dilute to dense in a satisfactory way: some theories are simply restricted ‘by construction’ to either side of the phase boundary. Other theories give wrong predictions for one side, or suggest a critical point, e.g., a discontinuity in the equation of state. Fortunately, it is possible to follow the cross-over with the help of complex-field simulations (stochastic Gross-Pitaevskii equation, sGP, for a review, see 6, 24). Proposals of this technique date back to Stoof’s group (27, 28), Davis and the Burnett group (29), and Gardiner’s group (30, 31). See also related classical field work by Goral and the Rzążewski group (32). The sGP has been applied to one-dimensional Bose correlations by one of us (13). With a suitably chosen cutoff, its predictions are in excellent agreement with experiments in one (33) and two dimensions (34). We find that these simulations successfully achieve a reasonable modelling of the entire cross-over. Another ‘benchmark’ is provided by the exact solution of the Lieb-Liniger model at finite temperature by Yang and Yang (35, 36). This approach has been used to cross-check perturbative calculations of density fluctuations in the dilute phase by Kheruntsyan and the Shlyapnikov group (37, 38).

To conclude with a comparison to experiments, it should be noted that many setups require modelling beyond the one-dimensional regime, mainly because the transverse confinement is not strong enough. As the ratio between trap frequencies is changed, one observes a “dimensional cross-over” from a true three-dimensional condensate to a one-dimensional quasi-condensate with large phase fluctuations (39). Following relatively early anisotropic expansion experiments (40, 41), theoretical work on this has been performed by Al Khawaja et al. (42) and Gerbier (43). Experimental work by the Bouchoule group (44–47) demonstrated, for example, the breakdown of Hartree-Fock mean field theory by analyzing the density fluctuations, and mapped out the dimensional cross-over (for a review, see Ref. (48)). Setups deeply in the one-dimensional regime have been reported in Refs. 49, 50 where Yang-Yang thermodynamics could be checked. The failure of mean field theories becomes manifest experimentally in the boundary regions of a trapped system (45). For the comparison with theory, the local density approximation (LDA) is often applied. We check the accuracy of this approximation using sGP simulations for both a homogeneous system and a trapped one.

Structure of the paper: the problem setting and a few salient parameters are outlined in Sec. IV. We discuss mean-field theories that do not operate a splitting of the Bose gas in components (Sec. III): the ideal gas and Hartree-Fock theory are covered I. The Bogoliubov approximation in Sec. IIIA introduces the condensate concept, although it suffers from serious infrared divergencies in low dimensions. An extended version that applies to a quasi-condensate gas whose density fluctuations are weak has been developed by Mora and Castin (25). Sec. IIIB So-called self-consistent theories are covered in Sec. IV beginning with the modified Popov theory, Sec. IV A. This is based on suitably regularised expressions for the non-quasi-condensate component. We also illustrate in this section the many-body effects that renormalise the interatomic scattering properties. The last mean-field theory is a variant of Hartree-Fock-Bogoliubov developed by Walser, Sec. IV B.
TABLE I. Hydrodynamic fields. The colons denote normal ordering of the field operators.

| (quasi)condensate | $\phi = \langle \psi \rangle$ | $n_q = |\phi|^2$ |
|-------------------|-----------------|-------------|
| mean density      | $\bar{n} = \langle n \rangle$ | $\langle \psi^\dagger \psi \rangle$ |
| field correlations| $G_1(z - z') = \langle \psi^\dagger(z) \psi(z') \rangle$ | $C(z - z') = \langle n(z)n(z') \rangle - \bar{n}^2$ |
| density correlations | $G_2(z - z') = \langle n(z)n(z') \rangle$ | $ar{n}' = \bar{n} - n_q = G_1(0) - n_q$ |
| thermal density   | $\mu = \langle \psi^\dagger \psi \rangle - \bar{\mu}^2$ | $m' = \langle \psi^\dagger \psi \rangle - \bar{\mu}^2$ |

The results are discussed in Sec. and compared to stochastic simulations with the Gross-Pitaevskii equation. The Appendices summarize more technical material related to high- and low-temperature approximations and to numerical aspects.

I. PROBLEM SETTING

We consider a gas of $N$ bosonic particles of mass $M$, strongly confined into a one-dimensional trap of length $L$, and in thermal equilibrium at temperature $T$. Throughout we work in the thermodynamic limit of a large system with density $\bar{n} = N/L$. This density is controlled by the chemical potential $\mu$, and the interaction energy per particle is given by $g n$ with a positive constant $g$. In the language of second quantization, the Hamiltonian $H$ is

$$H = \int d\bar{z} \left[ \frac{\hbar^2}{2M} \frac{d^2}{d\bar{z}^2} + \frac{g}{2} \psi^\dagger \psi^2 - \mu \psi^\dagger \psi \right]$$

where the field satisfies the bosonic commutation relations $[\psi(z), \psi^\dagger(z')] = \delta(z - z')$. A list of relevant observables is given in Table II. Note that for the homogeneous system we consider in this paper, local averages like the mean density $\bar{n}$ are spatially constant, while correlation functions depend only on the distance $z - z'$ between the observation points.

The characteristic scales for the cross-over can be motivated as follows. For negative chemical potentials, the density is low, and the ideal gas is a good approximation. The statistics of the complex field operator $\psi$ is then Gaussian, and from its fourth moment, one finds that density fluctuations are significant: $\langle n^2 \rangle \approx 2\bar{n}^2$. The cross-over is reached from below when the interaction energy in Eq. (1) becomes relevant, i.e., for $\mu \sim -g\bar{n}$. When the density is estimated with the degenerate ideal gas formula [first term of Eq. (8) below], we get $\mu \sim -\mu_x$ with a characteristic energy scale

$$\mu_x = \left( \frac{GM^{1/2}k_BT}{\hbar} \right)^{2/3}$$

We shall see below that $\mu_x$ gives the typical width of the cross-over region around $\mu = 0$. Repulsive interactions stabilize the gas so that also positive chemical potentials become accessible. On this dense side of the cross-over, density fluctuations get weaker: $\langle n^2 \rangle \approx \bar{n}^2$. The phase still fluctuates strongly enough to prevent the formation of long-range order, leading to the quasi-condensate concept.

TABLE II. Two sets of typical parameters used in simulations (stochastic Gross-Pitaevskii equation).

|                | Na-23     | Rb-87     |
|----------------|-----------|-----------|
| scattering length | 51.97 $a_0$ | 95.41 $a_0$ |
| transv. confinement | 1.46 kHz | 4 kHz |
| interaction $g$ | 0.39 nK $\mu$m | 1.938 nK $\mu$m |
| temperature $T$ | 7 nK | 50 nK |
| thermal wavelength $\lambda$ | 1.74 $\mu$m | 0.33 $\mu$m |
| $\hbar^2k_BT/Mg^2$ | $10^3$ | 74 |
| cross-over chem. pot. $\mu_x$ | 0.70 nK (14.6 Hz) | 12 nK (250 Hz) |
| cross-over density $n_x$ | $1.6 \mu$m$^{-1}$ | $6.1 \mu$m$^{-1}$ |
| healing length $\xi$ | 2.75 $\mu$m | 0.34 $\mu$m |

Typical numbers are listed in Table II for two different atoms [51, 52]. We use the standard formula $g = 2\hbar\omega_0 a_s$ for the one-dimensional interaction constant, assuming that the transverse confinement gives the highest energy scale. We define the thermal wavelength as $\lambda = \hbar(Mk_BT)^{-1/2}$ and the healing length for a given density $n$ as $\xi = \hbar(4Mgn)^{-1/2}$ [see Table IV]. In the cross-over, the two length scales are comparable, while the density is still high enough to be far from the Tonks-Girardeau limit. This can be expressed in terms of the Lieb-Liniger parameter $1/(2n_x\xi)^2 \ll 1$ [35].

An illustration of the relevance of the energy scale $\mu_x$ [Eq. (2)] is provided by Fig. I where we show the equation of state and the normalized density fluctuations for two ‘benchmark theories’: the first is based on numerical simulations of the stochastic Gross-Pitaevskii equation, the second is the $ab\ initio$ solution of the finite-temperature Lieb-Liniger model (Yang-Yang thermodynamics) [35, 36]. We parametrize the data by the dimensionless inverse temperature $\beta\mu_x = \mu_x/k_BT$. Since this scales as $\sim T^{-1/3}$, the temperature range covers nearly two orders of magnitude, but the data ‘collapse’ into a quite narrow band. The scale $\mu_x$ obviously captures the width of the cross-over zone vs. $\mu$ with excellent accuracy. The density fluctuations [right panel] show a slightly larger scatter, but this is due in part to a dependence on the numerical parameters like spatial grid spacing. For the appropriate choice of simulation parameters which give agreement with experimental data, see Refs. [35, 46, 49].
II. ONE-COMPONENT THEORIES

A. Ideal gas

The simplest example is the ideal gas ($g = 0$) where the Hamiltonian is bilinear and diagonal in the plane wave basis (dispersion relation $\epsilon(k) = \hbar^2 k^2 / 2M$, $-\infty < k < +\infty$)

$$H = \int \! dk \ (\epsilon(k) - \mu) \ a_\dagger(k) a(k),$$

with annihilation and creation operators $\{ a(k), a_\dagger(k') \} = \delta(k - k')$. In thermal equilibrium, we have $\{ a(k) \} = 0$ and recover Bose-Einstein statistics ($1/\beta = k_B T$)

$$\langle a_\dagger(k) a(k') \rangle = N(\epsilon(k) - \mu) \delta(k - k')$$

$$N(\epsilon(k) - \mu) = \frac{1}{\exp[\beta(\epsilon(k) - \mu)] - 1}$$

The field correlation function is denoted $G_1(z - z') = \langle \psi_\dagger(z) \psi(z') \rangle$ and given by

$$G_1(x) = \int \! \frac{dk}{2\pi} N(\epsilon(k) - \mu) \exp(ikx)$$

The “Boltzmann approximation” $N(\epsilon - \mu) \approx e^{-\beta(\epsilon - \mu)}$ applies in the regime $\epsilon - \mu \gg k_B T$ and gives a Gaussian correlation function

$$G_1(x) \approx e^{\beta \mu / \sqrt{2\pi} \lambda} e^{-x^2/(2\lambda^2)}$$

with a correlation length set by the thermal wavelength $\lambda$. If large distances are of interest or the chemical potential approaches the critical value $\mu = 0$, a different approximation is required. The Rayleigh-Jeans approximation, $N(\epsilon - \mu) = k_B T / (\epsilon - \mu)$, captures the contribution of small $k$-modes with high degeneracy and yields an exponential shape

$$G_1(x) \approx \frac{x}{\lambda} e^{-|x|/\ell}$$

with a much larger correlation length $\ell = \hbar(-2M\mu)^{-1/2}$ [see Table I].

They are relatively important and involve the positive coefficients $a_1 = -\zeta(\frac{1}{2}) / \sqrt{2\pi} \approx 0.5826$ and $a_2 = -\zeta(\frac{1}{2}) / (\sqrt{2\pi}) \approx 0.0830$ and the regularized zeta function.

The density correlation function [Table II] is computed from the Wick theorem because the Hamiltonian $H$ [Eq. 3] generates Gaussian statistics. This results in

$$C(x) = \bar{n} \delta(x) + |G_1(x)|^2$$

where the first term represents ‘shot noise’ (it arises from putting the field operators into normal order). The second term is called ‘bunching’ and increases the density fluctuations to the level $\langle n^2 \rangle = 2\bar{n}^2$. 

FIG. 1. Illustration of the universal features of the dilute-to-dense cross-over for different atoms and temperatures, when scaled to cross-over units. Symbols (sGP): stochastic Gross-Pitaevskii equation using the parameters of Table III, courtesy of Stuart Cockburn for the Rb data. Solid line (YY): numerical evaluation of the Yang-Yang solution to the finite-temperature Lieb-Liniger model [35, 36], courtesy of Karen Kheruntsyan. The temperature parameter for the YY data is $\delta = 0.0585$, $\beta \mu = 0.238$, $\beta \mu = 0.1$, and $\beta \mu = 0.0585$. (left) Equation of state. (right) Density fluctuations, expressed by the “Mandel parameter” $\Delta n^2 / \bar{n}$. (The name is chosen by analogy to super-Poissonian photon number distributions in laser theory.)
TABLE III. Formulas for mean-field theories.

The dispersion relations are given in the thermodynamic sense: the energy $\varepsilon$ appears in the Bose-Einstein factor $N(\varepsilon)$. Non-condensate density, anomalous density. Equation of state. The lower limits of the chemical potential are taken from Table 7.2 of Ref.\[53\], $a_1 = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.583$. This is based on a low-energy (high-temperature) expansion; higher-order corrections are $O(\beta \mu_x)$ or $O(\beta \mu_x)^{3/2}$.

### Dispersion relation

| Theory               | Formula                                                                 |
|----------------------|-------------------------------------------------------------------------|
| Ideal gas            | $\epsilon - \mu \equiv \hbar^2 k^2/2M - \mu$                         |
| Hartree-Fock         | $\epsilon + 2g n' - \mu$                                              |
| Bogoliubov           | $E \equiv \sqrt{\epsilon(2g n_c + \epsilon)}$                        |
| Mora-Castin          | $E$ (with $g n_c \rightarrow \mu$)                                    |
| modified Popov       | $E$ (with $n_c \rightarrow n_0$)                                      |
| Walser               | $\sqrt{(\epsilon - 2g n'/(2g n_c + \epsilon)}$                       |

### Non-condensate density

| Theory               | Formula                                                                 |
|----------------------|-------------------------------------------------------------------------|
| Ideal gas            | $N(\epsilon - \mu)$                                                    |
| Hartree-Fock         | $N(\epsilon + 2g n' - \mu)$                                            |
| Bogoliubov           | $\frac{\epsilon + gn_c N(E) + \epsilon + gn_c - E}{2E}$ (IR divergent) |
| Mora-Castin          | $\frac{\epsilon}{E} N(E) + \frac{\epsilon - E}{2E}$ (non-positive)   |
| modified Popov       | $\frac{\epsilon}{E} N(E) + \frac{\epsilon - E}{2E} + \frac{g n_0}{2(\epsilon + \mu)}$ |
| Walser               | $\frac{\epsilon + g(n_c - m')}{E} N(E) + \frac{\epsilon + g(n_c - m') - E}{2E}$ |

### Anomalous density

| Theory               | Formula                                                                 |
|----------------------|-------------------------------------------------------------------------|
| Bogoliubov           | $\frac{-g n_c}{E}(N(E) + \frac{1}{2})$                                  |
| Walser               | $\frac{-g(n_c + m')}{E}(N(E) + \frac{1}{2})$                           |

### Equation of state

| Theory               | Formula                                                                 |
|----------------------|-------------------------------------------------------------------------|
| Ideal gas            | $\bar{n} = n_{id}(\mu)$                                                 |
| Hartree-Fock         | $\bar{n} = n_{id}(\mu - 2g\bar{n})$                                    |
| Bogoliubov           | $\mu = gn_c$                                                           |
| Mora-Castin          | $\mu = gn_c + gn'$                                                     |
| modified Popov       | $\mu = gn_0 + 2g n'$                                                   |
| Walser               | $\mu = gn_c + 2g n' + gn'$                                             |

#### B. Interacting gas: Hartree-Fock

Hartree-Fock theory is probably the oldest mean-field theory; it is treating the interactions in a Bose gas in terms of an additional potential (the ‘mean field’). The Hamiltonian is approximated in the plane-wave basis by

$$ H \approx \int \frac{dk}{2\pi} (\epsilon(k) + 2g\bar{n} - \mu) a(k)^\dagger a(k) $$

(10)

This shift of the chemical potential gives the same equation of motion as the full interaction Hamiltonian when correlation functions are factorized in a Gaussian approximation [2]. As long as $2g\bar{n} > \mu$, Bose-Einstein statistics can be applied as for the ideal gas, and we get the following implicit equation for the (mean) density

$$ \bar{n} = \int \frac{dk}{2\pi} \frac{1}{\exp[\beta(\epsilon(k) + 2g\bar{n} - \mu)] - 1} $$

(11)

To work out this formula, we use an ideal-gas chemical potential $\mu_i < 0$ as parameter and plot $\bar{n}_{id}(\mu_i)$ vs. $\mu = \mu_i + 2g\bar{n}_{id}(\mu_i)$. The approximation shown in Eq. (8) can also be used here; it is fairly accurate, as long as $\beta \mu_x \lesssim 0.1$ (see dotted lines in Fig. 2 (left)).
TABLE IV. Correlation lengths

|                         |         |
|-------------------------|---------|
| ideal gas \((\mu < 0)\) | \(\ell = \hbar(2M\mu)^{-1/2} \approx n\lambda^2\) |
| phase diffusion length   | \(\xi = n_\ell\lambda^2\) |
| healing length           | \(\xi = \hbar(4M\mu)^{-1/2}\) |
| extended Bogoliubov \((\mu > 0)\) | \(\ell_e = 2n_\ell\lambda^2\) |
| phase correlation length | \(\xi_e = \hbar(4Mgn_c)^{-1/2}\) |
| density correlation length | \(\xi_m = \hbar(4Mgn_c)^{-1/2}\) |
| Hartree-Fock-Bogoliubov | \(\xi_c = \hbar(4Mgn_c)^{-1/2}\) |
| field and density correlation lengths | \(\xi_m = \hbar(4Mgn_c)^{-1/2}\) |

* Over the distance \(\ell_q\), the phase quadrature \(Y\) [Eq. (23)] has diffused such that its variance is comparable to the condensate density \(n_c = \mu/g\).

In the leading order, we get the explicit expression

\[
\mu \approx 2gn - \frac{k_BT}{2(n\lambda)^2} = 2g\left(n - \frac{n_\ell^2}{4n^2}\right) \tag{12}
\]

Right at the cross-over \(\mu = 0\), we have \(n = 2^{-2/3}n_x \approx 0.63n_x\) where the cross-over density scale \(n_x = \mu_x/g\) is defined by Eq. (2). In the dense case \((\mu \gg \mu_x)\), note again the collapse of the data in cross-over units over a wide range of temperatures. The equation of state \(\mu \approx 2gn\), however, is off by 50% compared to Bogoliubov theory [Eq. (14) below, see Fig. 2(left)]. This will be improved by a more advanced mean-field theory.

The correlation function \(G_1(x)\) of Hartree-Fock theory is formally given by the same integral (5) as for the ideal gas, but evaluated at the self-consistent chemical potential, as shown in Fig. 2. The density correlations are given by Eq. (9) because of Gaussian statistics. They therefore show the same bunching as the ideal Bose gas. At large distances, they feature an exponential decay on a length scale \(\ell \approx n\lambda^2\) [see Eqs. (7) (8)] that is much larger than the thermal wavelength. This behaviour does not capture the strong differences between phase and density fluctuations that characterise the dense phase (5). An improved version of Hartree-Fock including the many-body renormalization of particle interactions is briefly discussed in Sec. IV A 3.

TABLE V. Bogoliubov quasi-particles \((\mu = g|\phi|^2 > 0)\).

|                         |         |
|-------------------------|---------|
| free particle energy    | \(\epsilon(k) = \hbar^2k^2/2M\) |
| dispersion relation     | \(E(k) = \sqrt{\alpha(k)(2\mu + \epsilon(k))}\) |
| Bogoliubov amplitudes   | \(u(k) = e^{i\varphi} \cosh(\frac{1}{2}k\alpha(k))\) |
| (condensate phase \(\varphi\)) | \(v(k) = -e^{i\varphi} \sinh(\frac{1}{2}k\alpha(k))\) |

\[
\cosh\alpha(k) = \frac{\epsilon(k) + \mu}{E(k)} \\
\sinh\alpha(k) = \frac{\epsilon(k) - \mu}{E(k)}
\]

III. EXPANSION AROUND A (QUASI) CONDENSATE

A. Bogoliubov theory

This mean field approach is very successful in three dimensions and implements the concept of spontaneous symmetry breaking in the dense phase. Although it is not directly applicable in lower dimensions, it provides an introduction to the key concepts. The basic idea is the so-called Bogoliubov shift where the field operator is split into a c-number valued field (the ‘condensate’) and fluctuations, \(\psi \rightarrow \phi + \hat{\psi}\). The Hamiltonian is expanded up to second order in the fluctuations, and the condensate is determined by the requirement that the terms linear in \(\psi\) vanish. This gives the Gross-Pitaevskii equation

\[
-\frac{\hbar^2}{2M} \frac{d^2\phi}{dz^2} + g|\phi(z)|^2\phi = \mu\phi \tag{13}
\]

For reasons of thermodynamic stability, one chooses the condensate with the largest possible density—which is spatially constant in a homogeneous system. We get the equation of state

\[
\mu = g|\phi|^2 = gn_c \tag{14}
\]

that leaves the phase of \(\phi\) undetermined. The conventional choice of real and positive \(\phi\) can be interpreted as a spontaneous breaking of the U(1)-symmetry of the field Hamiltonian (1). The self-interaction of the condensate contributes an energy density \(\epsilon_c = -\frac{1}{2}gn_c^2\) to the Hamiltonian, corresponding to a pressure \(p = gn_c^2/2\). These parameters allow for acoustic elementary excitations with a speed of sound \(c\) at long wavelengths set by \(Mc^2 = \partial p/\partial n_c = gn_c\).

1. Quasi-particle spectrum

The part of the Hamiltonian that is of second order in \(\hat{\psi}\) is diagonalized with the help of the Bogoliubov transformation

\[
\hat{\psi}(z) = \int \frac{dk}{\sqrt{2\pi}} \left[ b(k)u(k) e^{ikz} + b^\dagger(k)v(k) e^{-ikz}\right] \tag{15}
\]
FIG. 2. Comparison of one-component mean-field theories. 

(left) equation of state $\bar{n}(\mu)$. Dashed: ideal gas, solid: Hartree-Fock theory. Dashed gray curve: classical approximation (first term of Eq. (8)). Dotted lines, superimposed: low-energy approximations based on Eq. (8). Straight solid line: pure condensate Hartree-Fock asymptote $\mu = 2g\bar{n}$. Chemical potential and density scaled to cross-over units [Eq. (3)]. Arrows: values of $\mu$ selected for right panel.

(right) correlation function $G_1(x)$ for three different densities (marked by red arrows on the left), Hartree-Fock theory. Temperature such that $\beta \mu_x = 0.1$. Dotted curves: low-energy approximation based on Eq. (7); their characteristic length (decay to zero) is $\ell \approx \bar{n} L^2$. The same results would have been obtained for an ideal gas at the chemical potentials (from top to bottom) $\mu \approx -0.285, -0.547, -1.03 \mu_x$ [see Table III].

where the Bogoliubov amplitudes $u$ and $v$ are given in Table V. They are constrained by $|u|^2 - |v|^2 = 1$ to make the operators $b$ and $b^\dagger$ bosonic (commutation relation $[b(k), b^\dagger(k')] = \delta(k - k')$, as shown in Eq. (3)). We note that both $u$ and $v$ are proportional to the phase factor $e^{i\phi}$ involving the condensate phase. The operator $b(k)$ annihilates a quasi-particle with energy $E(k)$ given by the Bogoliubov dispersion relation [Table V] where the acoustic branch involves the speed of sound $c = (gn_c/M)^{1/2}$ consistent with the hydrodynamic argument mentioned above. In this long wavelength limit, the Bogoliubov amplitudes become comparable and large, $u \sim -v \gg 1$.

Finally, the field Hamiltonian is truncated at second order in $\psi$, taking the following form,

$$H \approx \epsilon_0 L + \int dk \, E(k) b^\dagger(k) b(k)$$  \hspace{1cm} (16)

The zero-point energy density $\epsilon_0$ arises by putting the Bogoliubov operators $b, b^\dagger$ into normal order. It is given by the integral

$$\epsilon_0 - \epsilon_c = -\frac{\int \! dk \, E(k) |v(k)|^2}{2\pi} = -\frac{\int \! dk \, \mu^2/2}{2\pi \, E(k) + \epsilon(k) + \mu}$$  \hspace{1cm} (17)

which converges and can be computed analytically (Appendix A, 3)

$$\epsilon_0 = \frac{\mu n_c}{2} - \frac{\mu}{3\pi \xi}$$  \hspace{1cm} (18)

The Bogoliubov correction is small and scales with the Lieb-Liniger parameter $1/(n_c \xi)$ where $\xi$ is the healing length of Table IV. More analytical results at zero temperature in one and higher dimensions can be found in Refs. [54, 55].

The key problem of Bogoliubov theory in one dimension is the infrared divergence of the non-condensate density $n'$. The latter is defined as

$$n' = \langle \psi^\dagger \psi \rangle - \langle \phi \rangle^2 = \langle \psi^\dagger \psi \rangle$$  \hspace{1cm} (19)

In thermal equilibrium with respect to the approximate Hamiltonian [16], the modes corresponding to the $b(k)$’s have an occupation $N(E(k))$ [see Eq. (3)]. Using the expansion [15] of the field operator, the thermal density is given by the integral

$$n' = \int \frac{dk}{2\pi} n'(k)$$  \hspace{1cm} (20)

$$n'(k) = N(E(k)) |u(k)|^2 + [N(E(k)) + 1] |v(k)|^2$$  \hspace{1cm} (21)

The zero temperature limit $N(E) \rightarrow 0$ gives the so-called depletion density that arises by the scattering of virtual particles out of the condensate. Its integral is divergent in the infrared (IR) because the Bogoliubov amplitude scales $|v(k)|^2 \sim 1/k$ at long wavelengths. The temperature-dependent part shows an even stronger divergence $\sim T^2/k^2$ so that Bogoliubov theory is only useful as a conceptual framework. Here is an explicit expression for the non-condensate distribution in $k$-space that will re-surface later [see also Table III]

$$n'(k) = \frac{\mu + \epsilon(k)}{2E(k)} \coth \frac{\beta E(k)}{2} - \frac{1}{2}$$  \hspace{1cm} (22)

For completeness, we mention that in (symmetry-broken) Bogoliubov theory, also the so-called anomalous average $m = \int \frac{dk}{2\pi}$$

\[\frac{\mu + \epsilon(k)}{2E(k)} \coth \frac{\beta E(k)}{2} - \frac{1}{2}\]
\[ \langle \psi | \psi \rangle = \phi^2 + m' \text{ is nonzero. In } k\text{-space, it involves the product of the two Bogoliubov amplitudes,} \]
\[ m'(k) = [2N(E(k)) + 1] u(k) v(k) \tag{23} \]
\[ = -\frac{\mu}{2E(k)} \coth \frac{\beta E(k)}{2} \tag{24} \]
but its integral is also IR-divergent. Note that we fixed the condensate phase to \( \varphi = 0 \) in the second line.

2. **Density and phase diffusion**

Finite results within Bogoliubov theory can be produced by considering spatial increments, similar to Brownian motion. Consider the difference \( \Delta \psi(z, z') = \psi(z) - \psi(z') \) and its real and imaginary parts, assuming real and positive \( \phi \).

The average vanishes, \( \langle \Delta \psi \rangle = \langle X + iY \rangle = 0 \), and for the (co)variances, we find \( \langle X Y + Y X \rangle = 0 \) and the integral representations
\[ \langle X^2 \rangle = \int \frac{dk}{2\pi} \frac{1 - \cos kx}{k^2} \left\{ \frac{e}{2E(k)} \coth \frac{\beta E(k)}{2} - \frac{1}{2} \right\} \tag{25} \]
\[ \langle Y^2 \rangle = \int \frac{dk}{2\pi} \frac{1 - \cos kx}{k^2} \left\{ \frac{e + 2\mu}{2E(k)} \coth \frac{\beta E(k)}{2} - \frac{1}{2} \right\} \]
where \( x = z - z' \) and the arguments of \( e(k) \) and \( E(k) \) have been dropped for brevity. The colons denote normal ordering with respect to the field operators \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) (not with respect to the quasi-particle operators \( b(k), b^\dagger(k) \)). By virtue of the identity
\[ \int \frac{dk}{2\pi} \frac{1 - \cos kx}{k^2} = \frac{|x|}{2} \tag{26} \]
an infrared \( (k \to 0) \) divergence of the integrand translates into a ‘diffusive spreading’ of the quantum field \( \Delta \psi \) as the distance \( x \) between two positions increases. This behaviour appears only in the phase quadrature (imaginary part \( Y \)) which asymptotes to \( \langle Y^2 \rangle \sim D_Y |x - x'| \). The ‘diffusion constant’ is given by the simple and universal expression \( D_Y = M k_B T / \hbar^2 = \lambda^{-2} \). This is illustrated in Fig.4(left) where the dashed line is calculated from the low-energy approximation
\[ \int \frac{dk}{2\pi} \frac{1 - \cos kx}{k^2} = \frac{2 \mu}{\beta E(k)^2} = D_Y \left\{ |x| - \xi (1 - e^{-|x|/\xi}) \right\} \tag{27} \]
with \( \xi \) the healing length [Table IV].

The density quadrature (real part \( X \)) does not diffuse freely; its variance reaches a finite limit given by the integral in Eq. (25) with the cosine dropped. In Appendix A1 we find in the low-energy limit the result
\[ |x| \gg \xi \gg \lambda : \quad \langle X^2 \rangle \approx \frac{\xi}{\lambda^2} - \frac{a_1}{\lambda} - \frac{a_2}{4} \frac{\lambda^2}{\xi^2} \tag{28} \]
Note the close analogy of the sub-leading terms with the ideal gas expansion [8] where the same positive coefficients \( a_1, a_2 \) appear. As shown in Fig.3(right), this agrees well with the (numerically computed) variance \( \langle X^2 \rangle \) at large distances. Note the negative values at low temperature (‘below shot noise’); by analogy to quadrature fluctuations in quantum optics [56, 57], this can be interpreted as the squeezing of the density quadrature due to the nonlinear interaction with the condensate. At zero temperature, the squeezing reaches the level
\[ T = 0, \ |x| \gg \xi : \quad \langle X^2 \rangle \approx -\frac{1}{2\pi \xi^2}, \tag{29} \]
as an elementary integration shows [see Eq. (A10)].

3. **Density correlations**

We finally quote the density correlations \( C(z - z') \) [see Table I]. Due to the Bogoliubov shift, the density operator takes the form \( \hat{n}(z) = |\phi|^2 + \phi^* \hat{\psi}(z) + \hat{\psi}^\dagger(z) \phi + \psi^\dagger(z) \psi(z) \), and we get additional contributions compared to the ideal gas [Eq. (25)]. The second- and fourth-order correlations of the fluctuation \( \psi \) are worked out with the Wick theorem. The result can be written in the form
\[ C(z - z') = \delta(z - z') + 2 \text{Re} \left\{ |\phi|^2 \langle \hat{\psi}(z) \hat{\psi}(z') \rangle + \phi^* \langle \hat{\psi}^\dagger(z) \hat{\psi}(z') \rangle \right\} + \langle \hat{\psi}^\dagger(z) \hat{\psi}(z') \rangle^2 + |\langle \hat{\psi}(z) \hat{\psi}(z') \rangle|^2 \tag{30} \]
This formula is only partially meaningful because the last two terms are both infrared-divergent. The curly bracket can be combined into a regular integral
\[ 2 \text{Re} \left\{ |\phi|^2 \langle \hat{\psi}^\dagger(z) \hat{\psi}(z') \rangle + \phi^* \langle \hat{\psi}^\dagger(z) \hat{\psi}(z') \rangle \right\} \]
\[ = 2 |\phi|^2 \int \frac{dk}{2\pi} \frac{1 - \cos kx}{k^2} \left\{ \frac{e}{2E(k)} \coth \frac{\beta E(k)}{2} - \frac{1}{2} \right\} \tag{31} \]
This is essentially the same as the ‘density quadrature’ \( \langle X^2 \rangle \) [Eq. (25)]. Fig.3(right) with a flip in orientation can thus be interpreted as a plot of the density correlation function. Note the density correlation length \( \xi \) that emerges from the typical \( k\)-scale \( \epsilon(k) \sim \mu \) of the integrand.

The divergences of Bogoliubov theory have been addressed, of course, by the other mean-field theories we analyze now.

B. **Extended Bogoliubov theory**

Mora and Castin [25] have based this theory on an alternative expansion in the dense regime \( \mu > 0 \), using the assump-
The mode expansions sate density δEonian expanded to second order in the fluctuations can be diag-
of zero particles per lattice cell to be negligible. The Hamilton-
by working on a discrete lattice and assuming the probability
ature as in Bogoliubov theory [Table ually conjugate phase and density fluctuation operators
where ˆ ψ(ξ) = µ/g appears in lieu of the condensate density n c. With this proviso, the Bogoliubov amplitudes u(k), ψ(ξ) for the fluctuation operators have the same struc-
tion as in Bogoliubov theory [Table [IV]]. The mode expansions of the fluctuation operators are (k-arguments suppressed for simplicity)

\[ \hat{\theta}(z) = \hat{\varphi} + \frac{n_q^{-1/2}}{2i} \int \frac{dk}{\sqrt{2\pi}} \{ (u - v)b e^{ikz} - h.c. \} \]

\[ \delta n(z) = n_q^{1/2} \int \frac{dk}{\sqrt{2\pi}} \{ (u + v)b e^{ikz} + h.c. \} \]

where \( \hat{\varphi} \) is the operator for the quasi-condensate phase (spatially constant). We have taken the thermodynamic limit where \( z \) is continuous and the momentum conjugate to \( \hat{\varphi} \) can be neglected.

2. Equation of state

The average non-quasi-condensate density vanishes when computed with respect to the second-order Hamiltonian (subscript 2), \( \langle \delta n \rangle_2 = 0 \). Third-order terms in the expansion are needed to describe the non-condensate density and are taken into account in perturbation theory. The resulting equation of state involves the same integrand as the density quadrature \( X \) in Eq. (23):

\[ \mu = g\bar{n} + gn' \]

\[ n' = \int \frac{dk}{2\pi} \{ (u + v)^2 N(E) + v(u + v) \} \]

\[ = \int \frac{dk}{2\pi} \{ \frac{\epsilon}{2E} \coth \frac{\beta E}{2} - \frac{1}{2} \} \]  

This formula can be used to compute the mean density \( \bar{n} = \bar{n}(\mu) \). Its structure is the same as in modified Popov theory [Eq. (26)], and we therefore used the notation \( n' \). Mora-Castin theory does not pretend, however, that \( n' \) can be interpreted as a non-quasi-condensate density. Indeed, the integral (34) becomes negative at low temperatures. At zero temperature, we get [by the same calculation as in Eq. (29)]

\[ T = 0 : \bar{n} = \frac{\mu}{g} + \frac{1}{2\pi \xi} \]

with the healing length \( \xi = h(4M\mu)^{-1/2} \). Since the first term is the quasi-condensate density \( n_q \), the second one can be interpreted as the depletion density. In the opposite limit of high temperatures (low energies), we can use Eq. (28) to get

\[ \bar{n} \approx \frac{\mu}{g} - \left\{ \frac{\xi}{\lambda^2} - \frac{a_1}{\lambda} - \frac{a_2}{4 \xi^2} \right\} \]
which is in excellent agreement with the data plotted in Fig.\[4\] One notes that the density exceeds the linear approximation $n \approx \mu/g$ (light gray) in the dense phase, this is due to the curly bracket in Eq.\[36\] becoming negative. We use cross-over units in this plot [see around Eq.\(2\)] and emphasize that despite the factor 125 in temperature, the scatter of the data is relatively small, also among the mean-field theories. Mora-Castin theory fails to predict a positive total density in the cross-over region: for $\mu \lesssim 2^{-2/3} \mu_x \approx 0.630 \mu_x$ [see Table \[III\]]. This could have been expected because in this range, density fluctuations become so large that the expansion around a ‘quiet’ quasi-condensate breaks down. The size of the density fluctuations can be appreciated from the correlation functions in Figs\[4\] (right) and\[5\] (bottom).

3. Correlation functions

The field correlation function is found as follows [Eq.(146) of Ref.\[25\]]

$$G_1(x) = \bar{n} \exp \left[ -\frac{1}{8\bar{n}} \left( \Delta \hat{\theta}(x)^2 \right)_2 - \frac{1}{8\bar{n}_q} \left( \Delta \hat{n}(x)^2 \right)_2 \right]$$

where the difference operators $\Delta \hat{A}(x) = \hat{A}(x) - \hat{A}(0)$ are similar to the $\Delta \psi$ operator introduced around Eqs.\[25\]. The normal-order prescription $\ldots \ldots$ is with respect to the fluctuation operators $\hat{\psi}$. This expression includes in a perturbative way contributions to the Hamiltonian that are of third order in the fluctuations. The exponent in Eq.\[37\] has the convergent integral representation [Eq.(184) of Ref.\[25\]]

$$\log \frac{G_1(x)}{\bar{n}} = -\frac{1}{\bar{n}} \int \frac{dk}{2\pi} (1 - \cos kx) \left\{ \frac{\mu + \epsilon}{2E} \coth \frac{\beta E}{2} - \frac{1}{2} \right\}$$

(38)

Mora and Castin [25] have recognized the integrand as the non-condensate spectrum of Bogoliubov theory [Eq.\[22\]]. The infrared divergence of the latter therefore yields an exponential decay at large distance $x$: $G_1(x) \sim \exp(-|x|/\ell_\theta)$ [using Eq.\[27\]]. The (phase) correlation length $\ell_\theta = 2\bar{n}\lambda^2$ is twice as large as for the ideal Bose gas (parameter $\ell$ in Eq.\[7\]). A comparison to other mean-field theories is provided in Fig\[5\] (top, center).

At zero temperature, the integral [38] diverges only logarithmically. As explained in Appendix \[A3\] one gets for large $x$ (here, $\gamma \approx 0.577$)

$$T = 0 : \quad \log \frac{G_1(x)}{\bar{n}} \approx -\log(2|x|/\xi) + \gamma - 2 \frac{4\pi \bar{n}_\xi}{\bar{n}_\xi}.$$

(39)

The exponent of this power law has been given earlier by Refs.\[5, 11\], but even the prefactor agrees with Ref.\[55\] in the regime $n_\xi \gg 1$.

For later comparison with the modified Popov theory [Sec.\[IV\]A], we also quote the formula for phase diffusion. Keeping only terms up to second order, one gets indeed the phase quadrature $\langle \hat{Y}^2(x) \rangle$ of Bogoliubov theory [Eq.\[25\]]

$$\langle \Delta \hat{\theta}(x)^2 \rangle_2 = \frac{1}{n\bar{q}} \int \frac{dk}{2\pi} (1 - \cos kx) \left\{ \frac{\epsilon + 2\mu}{2E} \coth \frac{\beta E}{2} - \frac{1}{2} \right\}$$

(40)

This term is at the origin of phase diffusion $\langle \Delta \hat{\theta}(x)^2 \rangle_2 \approx |x|(n_\bar{q}\lambda^2)$ in the exponent of $G_1(x)$.

The density correlations are obtained directly from the expansion \[32\] of the fluctuation operator $\hat{\delta n}(z)$ [Eq.(121) of Ref.\[25\]]

$$C(z - z') = \bar{n} \delta(z - z') + \langle \hat{\delta n}(z) \hat{\delta n}(z') \rangle_2 
\approx \bar{n} \delta(z - z') + 2n'\langle x \rangle$$

(41)

Here, $n'(x)$ is given by Eq.\[34\] with an additional $\cos kx$ under the integral. Note that we recover the same expression as the regular part of Eq.\[51\] in Bogoliubov theory. The density correlation length is therefore of the order of the healing length $\xi$, much shorter than the characteristic phase correlation length $\ell_\theta$ [see after Eq.\[38\]]. A low-energy approximation to Eq.\[41\] can be found by keeping only the classical part $\coth(\beta E/2) \approx 2/(\beta E)$ of the integrand, leading to

$$x \gg \lambda : \quad C(x) \approx \frac{2\bar{n}_\xi}{\lambda^2} e^{-|x|/\xi}$$

(42)

See Fig\[5\] (bottom) for a comparison. For $\mu \lesssim \mu_x$, density fluctuations are clearly too large for the expansion behind Mora-Castin theory to be valid. The squeezing of the density quadrature manifests itself by the non-monotonous behaviour of the pair correlation function $G_2(x)$ as $x$ increases from zero. At zero temperature, the density shows some ‘anti-bunching’

$$T = 0 : \quad G_2(0) = \bar{n}^2 - \frac{\bar{n}}{\pi \xi} < \bar{n}^2,$$

(43)

but this small reduction is of course far from the ‘correlation hole’ of Fermi liquids or the Tonks-Girardeau gas [2, 59].

IV. SELF-CONSISTENT THEORIES

These theories construct a simplified form for the Hamiltonian involving hydrodynamic fields. These are fixed at a later stage by equating them to thermodynamic averages computed
with this approximate Hamiltonian (‘self-consistency’). The simplest example of such a theory is Hartree-Fock [Sec. II B] that works with a single field, the density \( \bar{n} \). More elaborate methods also include a (quasi)condensate or, for example, the anomalous average, and aim at describing the Bose gas also at higher densities. We discuss here two examples in detail.

A. Modified Popov theory

This mean-field theory is based on the idea that low-energy fluctuations actually destroy the long-range order, and there is no condensate in the ordinary sense (long-range order à la Penrose-Onsager [1]). For details of the theory and similar approaches, we refer to Refs. [11–13, 58]. Note that in dimensions 2 and 3, the (‘bare’) interaction constant \( g \) gets renormalized into an energy- (and momentum-) dependent T-matrix [12, 60]. This effect is usually neglected in one-dimensional systems. We provide a brief discussion in Sec. IV A 3. The theory is applied differently on the two sides of the cross-over: on the dilute side, the Hartree-Fock approximation is applied [Sec. II B], while the dense case is outlined now.

1. Equation of state

As the density increases beyond \( \sim n_\sigma \), the density \( \bar{n} = n_q + n' \) of the system is split into the quasi-condensate \( n_q \) and the thermal part \( n' \). The former determines the speed of sound in the (gapless) dispersion relation

\[
E(k) = [2gn_q(k) + \epsilon^2(k)]^{1/2}
\]

The thermal density is given by the convergent integral

\[
n' = \int \frac{dk}{2\pi} \left\{ \frac{\epsilon}{2E} \coth \frac{\beta E}{2} - \frac{1}{2} + \frac{gn_q}{2(\epsilon + \mu)} \right\}
\]

where the first two terms have the same structure as Eq. (34). The last term has been introduced as a counterterm to regularize the zero-temperature (depletion) density. It has the merit of making Eq. (35) positive at all values of \( \mu \) so that an interpretation as the density of the non-quasi-condensate is applicable. The equation of state is written

\[
\mu = g\bar{n} + gn'
\]

It looks formally like Eq. (33) of extended Bogoliubov theory, although the interpretation of the non-condensate density \( n' \) is different. Eq. (46) is an implicit equation for the chemical potential, since \( \mu \) also appears in \( n' \). See Appendix B for details on the numerical procedure.

The zero-temperature analysis can be done similar to Eq. (35), and \( n' \) then describes the quasi-condensate depletion:

\[
T = 0 : \\ n' \approx \frac{\pi/\sqrt{8} - 1}{2\pi\xi}
\]

where the approximation \( \xi \approx \xi_q \) was used [61]. While the scaling with the healing length is the same, the prefactor differs from Eq. (35) due to the counterterm in Eq. (35). At high
can be used to derive the approximation
\[ n' \approx \frac{\xi_q}{\lambda^2} - \frac{a_1}{\xi} + \frac{\xi}{4\sqrt{2}\xi_q^2} + \frac{a_2\lambda}{4\xi_q^4} \]  
(48)

where \( \xi_q = \hbar(4Mn_q)^{-1/2} \) is the healing length of the quasi-condensate density \( n_q \).

In Fig. 5(left), the equation of state (thick dashed) is compared to extended Bogoliubov theory (thin solid). In the dense phase, the difference is small, the self-consistent theory predicts a slightly lower density. In the cross-over region \( \mu \approx 1.89\mu_x \), a ‘critical point’ is reached [see Table III]: below this value, the implicit equation of state has no solution. There is a finite gap to the density given by Hartree-Fock theory (lower lines), which is the appropriate mean-field description on the dilute side [12].

2. Correlation functions

The first-order correlation function can be found, e.g., in Eq.(8) of Ref.[13]
\[ G_1(x) = \tilde{n} \exp\left[-\frac{i}{2}(\Delta \theta^2(x))_{mP}\right], \]
(49)
it involves phase fluctuations given by (subscript for ‘modified Popov’)
\[ \langle \Delta \theta^2(x) \rangle_{mP} = \frac{1}{n_q} \int \frac{dk}{2\pi} (1 - \cos kx) \left\{ \frac{gn_q}{E} \coth \frac{\beta E}{2} - \frac{gn_q}{\epsilon + \mu} \right\} \]
(50)
The first term in curly brackets is proportional to the anomalous average of Bogoliubov theory [24], the second one is the same counterterm as in the thermal density \( n' \) [Eq.(45)] and makes the integral converge in the UV. The IR singularity of the integrand is the same as in Mora-Castin theory [40], so that at large distances, a similar phase diffusion is found:
\[ \langle \Delta \theta^2(x) \rangle \approx |x|/\ell_\theta \] with \( \ell_\theta = n_q\lambda^2 \). The phase coherence length hence grows linearly with the quasi-condensate density. The plots in Fig. 5(top, center) illustrate that the difference \( n_q < \tilde{n} \) makes the predicted phase coherence somewhat smaller than in extended Bogoliubov theory (thin solid). Hartree-Fock theory is even less coherent, as shown in the top panel.

At zero temperature, the phase fluctuations are subdiffusive and increase logarithmically (\(|x| \gg \xi \approx \xi_q\))
\[ T = 0: \quad \langle \Delta \theta^2(x) \rangle_{mP} \approx -\frac{\log(2|x|/\xi) + \gamma - \pi/\sqrt{2}}{2\pi n\xi} , \]
(51)
The power law that this implies for $G_1(x)$ [Eq. (49)] has the same exponent as Eq. (39), but a slightly different prefactor.

Finally, to come to density correlations, we note that Eq. (39) is also valid in the presence of a quasi-condensate as long as one assumes that the fluctuations obey Gaussian statistics. We generalize slightly the expressions of Refs. [11–13] to cover the case $x \neq z'$: as explained around Eq. (37) in Ref. [12], the anomalous averages are removed from Eq. (30), and one gets [Fig. 5(bottom)]

$$G_2(x) = \bar{n}^2 + 2n_qn'(x) + [n'(x)]^2$$

(52)

Here, the function $n'(x)$ is given by Eq. (45) with an additional factor $\cos kx$ inserted under the integral. As noted in Ref. [13], the reduction of density fluctuations, relative to the ideal gas, provides an alternative interpretation of the quasi-condensate density: $G_2(0) = 2\bar{n}^2 - n_q^2$. On the other hand, since $n_q \leq \bar{n}$ by construction, one always has $G_2(0) \geq \bar{n}^2$, and there is no possibility for anti-bunching in modified Popov theory [see Fig. 4(right)].

The density correlations can be approximated quite accurately (dotted lines in Fig. 5(bottom)) by using

$$x \gg \lambda : \quad n'(x) \approx \frac{\xi_q e^{-|x|/\xi_q}}{\lambda^2} + \frac{\xi e^{-|x|/\sqrt{2}\xi}}{\sqrt{32\xi^2}}$$

(53)

The first term results in a formula similar to Eq. (42), but involving the quasi-condensate healing length $\xi_q$. The second is small at low energies and arises from the counter term.

### 3. Renormalised interactions

The scattering between two atoms in a dense gas occurs in a ‘background field’ formed by the other atoms. This leads to an energy- and density-dependent change in the matrix elements of the interaction potential [62]. For completeness, we discuss here the formulas given in Ref. [12], adapted to our notation.

As a first example, consider two atoms in the condensate that collide at zero temperature. The bare interaction constant $g$ is replaced by the two-body T-matrix element [Eq. (7)] of Ref. [12]

$$\frac{1}{T_{2B}(-2\mu)} = \frac{1}{g} + \int \frac{dk}{2\pi} \frac{1}{2(\epsilon + \mu)}$$

(54)

where the denominator involves the pair’s kinetic energy and the change in the condensate energy as two atoms are removed. This integral evaluates to [see Eq. (A14)]

$$T_{2B}(-2\mu) = \frac{g}{1 + g/(\sqrt{32\mu}^2)}$$

(55)

and illustrates that the interactions renormalize to zero as $\mu \to 0$. The magnitude of this effect is small in practical one-dimensional systems because the denominator involves the small Lieb-Liniger parameter $g/(2\mu\xi) \sim (n,\xi)^{-1/2}$.

Our second example are the thermal corrections to the scattering matrix. Consider for simplicity the dilute phase and the many-body effects in Hartree-Fock theory. The average density is worked out as in Eq. (11), with the mean-field shift of the chemical potential replaced by the Hartree-Fock self-energy, $2g\bar{n} \to \Sigma$. According to Eq. (29) of Ref. [12], the renormalised T-matrix is

$$\frac{1}{T_{MB}(-\Sigma)} = \frac{1}{g} + \int \frac{dk}{2\pi} \coth \frac{1}{2(\epsilon + \Sigma - \mu)}$$

(56)

where Eq. (54) has been used. The equations are closed by the self-consistency relation $\Sigma = 2nT_{MB}(-\Sigma)$, Eq. (28) of Ref. [12].

A numerical solution is shown in Fig 6 and illustrates that a critical point appears at $\mu \sim \mu_c$ (the precise value depends on $\beta\mu_c$), where the many-body interactions renormalize to zero and the density diverges. The low-energy approximation for Eq. (56) is [from the techniques of Appendix A2]

$$\frac{1}{T_{MB}(-\Sigma)} \approx \frac{1}{g} + \frac{k_B T}{\sqrt{\Sigma/\mu - 1}} \frac{1}{\sqrt{\Sigma/2} + \sqrt{\Sigma/2 + \mu}} + \frac{2a_2}{k_B T}$$

(57)

This gives, in conjunction with Eq. (8) for the density, a relatively accurate picture (dotted lines in Fig. 6). Note the strong ($\approx 50\%$) reduction of interactions already for $\mu = 0$. We find in particular that the self-energy approaches $\Sigma \to \mu$ at the critical point.

### B. Hartree-Fock-Bogoliubov

The Hamiltonian is approximated in this mean-field theory by the quadratic expression

$$H \approx \epsilon cL + \int dz \left\{ \frac{\hbar^2}{2M} \frac{d\hat{\psi}^\dagger}{dz} \frac{d\hat{\psi}}{dz} + (2g\bar{n} - \mu) \hat{\psi}^\dagger \hat{\psi} + \frac{g}{2} \left( m(\hat{\psi}^2 + m(\hat{\psi})^2) \right) \right\}$$

(58)

where $\hat{\psi}$ is again the fluctuation operator around a condensate field $\phi$. The first term is the condensate energy, the first piece under the integral formally identical to Hartree-Fock theory [Eq. (eq:HF-Hamiltonian)], the total density being split into $\bar{n} = |\phi|^2 + n'$. The last terms involve the anomalous average $m = \phi^2 + m'$ that already appeared in Bogoliubov theory [Eq. (24)]. This Hamiltonian is complemented by the
solve the modified system of extended Bogoliubov theory (see Ref. [17] for a discussion in three dimensions).

A derivation of these equations has been discussed by Griffin et al. [9] who also uses the name ‘Hartree–Fock–Bogoliubov theory’. Keeping the anomalous average in full goes back to Girardeau and Arnowitt (see Ref. [17] for a discussion in three dimensions).

In a homogeneous system with real \( \phi \), one finds the equation of state

\[
\mu = g(\bar{n} + n') + gm' = g|\phi|^2 + g(2n' + m')
\]  

(60)

The anomalous average \( m' < 0 \) reduces the chemical potential relative to extended Bogoliubov and to modified Popov theory [Eqs. (33)–(61)]. This has also been interpreted as a many-body-induced reduction of the particle interactions [2, 62].

The expansion of the operator \( \hat{\psi} \) is the same as in Bogoliubov theory [15], but the amplitudes \( u = u(k), v = v(k) \) solve the modified system

\[
\begin{pmatrix}
\epsilon + 2g\bar{n} - \mu \\
gm \\
\epsilon + 2g\bar{n} - \mu
\end{pmatrix}
\begin{pmatrix}
u \\
g \bar{n}v' \\
u'
\end{pmatrix}
= \begin{pmatrix}
E u \\
\beta E v \\
-E v'
\end{pmatrix}
\]  

(61)

One gets the dispersion relation (using Eqs. (60)):

\[
E = ((\epsilon + 2g\bar{n} - \mu)^2 - g^2|m'|^2)^{1/2}
\]  

(62)

\[
E = \sqrt{(\epsilon - 2gm')(\epsilon + 2g|\phi|^2)}
\]  

(63)

The dispersion relation has the particular feature that it shows a finite gap, \( E(k \to 0) = 2g|\phi|\sqrt{-m'} \). Walser argues, in particular in Ref. [16], that the gap is not in contradiction with the existence of a Goldstone mode due to the U(1)-symmetry of the original theory. The Bogoliubov modes found here are a ‘convenient quasi-particle basis’ to describe the thermodynamic equilibrium state of the Bose gas. The finite gap is essential here to regularize the theory in the infrared. For the linear response of a perturbation to the gas, a different calculation is performed that leads, indeed, to a gapless spectrum of collective excitations. The key difference is that the perturbation also affects the condensate phase which is treated as a dynamical variable, rather than fixed to a symmetry-broken value in Ref. [19].

We note that in the self-consistent HFB theory of Yukalov and Yukalova [17, 19], a gapless dispersion relation for quasi-particles is constructed by introducing a second chemical potential (for the non-condensate particles). Most of this analysis focuses on three dimensions, however. We do not discuss this variant further here because when formulas are extrapolated to the one-dimensional setting, one finds infrared-divergent expressions similar to Bogoliubov theory (Sec. III A). The dimensional regularization suggested in Ref. [19] leads to a negative non-condensate density, similar to extended Bogoliubov theory (Sec. III B).

1. Mean-field densities

The parameters \( n' \) and \( m' \) are determined by consistency from the moments of the fluctuation operator \( \hat{\psi} \) in the gaussian ensemble defined by Eq. (58), for example \( \langle \hat{\psi}^\dagger \hat{\psi} \rangle = n' \). This yields again Eq. (21), as in Bogoliubov theory, but since the expressions for the amplitudes \( u, v \) are different, the resulting integral is regular

\[
n' = \int \frac{dk}{2\pi} \left\{ \frac{\epsilon + g(n_c - m')}{2E} \coth \frac{\beta E}{2} - \frac{1}{2} \right\}
\]  

(64)

Similarly, for the anomalous average \( \langle \hat{\psi}^\dagger \hat{\psi} \rangle = m' \), one finds

\[
m' = - \int \frac{dk}{2\pi} \frac{g(n_c + m')}{2E} \coth \frac{\beta E}{2}
\]  

(65)

This is an implicit equation since \( m' \) also appears in the mode energies \( E \) [Eq. (63)]. The \( T = 0 \) limit has been evaluated in Ref. [63] in terms of elliptic integrals. It has been shown that the behaviour of the condensate depletion is qualitatively similar to Eqs. (35)–(47), except for a logarithmic correction \( \sim \log(\bar{n}\xi)/\xi \) [Eq. (43) of Ref. [63]].

In the opposite limit of high temperatures (low energies, the techniques sketched in Appendix A2 yield (correcting one
2. Correlation functions

The correlation function of the field operator is

$$G_1(x) = n_c + n'(x)$$

(68)

where $n'(x)$ is given by Eq. (64) with an additional factor $\cos k|x|$ under the integral. Due to the gapped dispersion relation, this is regular in the infrared. It shows long-range order at the level of the condensate, $G_1(x \to \infty) = n_c$ [Fig. 5(center)]. Quite different from the previous theories, this version of Hartree-Fock-Bogoliubov theory thus predicts a true condensate (even at one dimension). The non-condensate contribution has, in the leading order, the low-energy (and large-distance) approximation [dotted curve in the Figure]

$$n'(x) \approx \frac{\xi_c}{2\lambda^2} \left( \sqrt{\frac{n_c}{m'}} + 1 \right)$$

(69)

where $\xi_m \sim (-m')^{-1/2}$ may be called the ‘anomalous healing length’ [see Table IV].

For density fluctuations in Walser’s mean field theory, we may use Eq. (60) because it is based on a Gaussian equilibrium ensemble. In distinction to conventional Bogoliubov theory, all integrals are convergent, and we get for the pair correlation function

$$G_2(x) = \bar{n}^2 + 2n_c [n'(x) + m'(x)] + [n'(x)]^2 + [m'(x)]^2$$

(70)

for $m'(x)$, insert $\cos k|x|$ under the integral (65). The analogue of the large-distance approximation (69) is

$$m'(x) \approx -\frac{\xi_c}{2\lambda^2} \left( \sqrt{\frac{n_c}{m'}} e^{-|x|/\xi_m} + e^{-|x|/\xi_c} \right)$$

(71)

When this is inserted into the density correlation function (70), it compares quite well with the numerical calculations, see Fig. 5(bottom).

V. DISCUSSION

A. Complex field simulations and parameters

We use the physical parameters collected in Table IV left column. They correspond to a dimensionless inverse temperature $\beta \mu_x \approx 0.1$. In the following plots, we add for comparison the results of classical field simulations (stochastic Gross-Pitaevskii equation (2)) which were performed for a trapped system with these parameters and the shallow trap frequency $1.4 \text{ Hz} \approx 0.1 \mu_x/h$. A real-space grid with spacing $\Delta \approx
0.5 $\lambda$ was used, slightly larger than the ‘canonical’ choice in extended Bogoliubov theory, $\Delta z = \xi \beta \mu$ [Eq.(176) of Ref. [23]]. If we estimate with Hartree-Fock theory the density of atoms at energies above the cutoff $E_{\text{max}} \sim \hbar^2/(M \Delta z^2)$ (not captured by the simulations), we find a negligible contribution for $\mu \sim 0$. We plot the data against the local chemical potential $\mu - V(z)$, assuming the local-density approximation (LDA) is valid. A discussion of this assumption is provided in Appendix C.

**B. Equation of state**

In Fig. 8 we compare the equations of state for all mean-field theories discussed so far. Coming from negative $\mu$, Hartree-Fock allows to enter smoothly the cross-over region. It fails by 50% in the dense phase, however. One might switch to extended Bogoliubov theory (xB) at $\mu \sim \mu_x$ where the two equations of state cross, but this prescription is lacking a more detailed justification. The self-consistent modified Popov theory involves a finite jump in the density when one switches from HF to its ‘end point’ (inset of Fig. 8). A similar jump appears when the HFB theory proposed by Walser is taken. In the dense phase, the three mean-field theories converge fairly well, but the xB density is systematically higher [see also Fig. 4 (left)]. Note that the stochastic simulation is able to describe the density smoothly throughout the cross-over. Its only deficiency appears in the dilute phase where it joins the classical (low-energy or Rayleigh-Jeans) approximation instead of the full (Bose-Einstein) prediction of the ideal gas.

**C. Density fluctuations**

A survey of the predictions for density fluctuations is given in Fig. 9. We plot the normalized pair correlation function $g_2 = G_2(0)/\bar{n}^2$. The ideal gas and Hartree-Fock theory give a value of 2 typical for a complex Gaussian (or chaotic) field (Wick theorem). In modified Popov theory, this jumps at a ‘critical chemical potential’ $\mu \sim 1.5 \mu_x$ down to a value $g_2 \sim 1.6$, and decreases for $\mu > 1.5 \mu_x$ steeply to the ‘pure condensate’ value $g_2 = 1$. From its critical point on (which is slightly shifted), Walser’s HFB theory behaves similarly. In the extended Bogoliubov theory, the density fluctuations diverge as $\mu \to 0$, and one clearly leaves its region of validity. The stochastic simulation behaves again smoothly and shows that the suppression of density fluctuations is already significant on the dilute side of the cross-over ($\mu < 0$).

**D. Conclusion**

We have analyzed the cross-over of a weakly interacting, homogeneous Bose gas in one dimension in the thermodynamic limit. Interactions (repulsive) stabilize the dilute phase, as the chemical potential increases above zero, but at finite temperature, phase fluctuations persist in the dense phase and preclude any long-range order (quasi-condensate). Using a suitable thermodynamic scaling of the relevant variables, the cross-over can be mapped to a relatively narrow...
range of reduced variables, e.g., $-\mu_x \lesssim \mu \lesssim 3\mu_x$ where $\mu_x \sim (gT)^{2/3}$. We have worked through a portfolio of mean field theories to describe the cross-over. Hartree-Fock theory performs better than the ideal gas model, but fails to capture the equation of state and the reduction of density fluctuations in the quasi-condensed phase. This does not seem to improve when the many-body renormalization of atomic interactions is taken into account. The extended Bogoliubov theory of Mora and Castin breaks down when the cross-over is approached from the dense side, because density fluctuations become too strong. Self-consistent theories (modified Popov of Stoof et al., Hartree-Fock-Bogoliubov of Walser, Holland et al.) predict a critical point in the equation of state because infrared divergences at low (quasi)condensate density enforce a minimal value for the chemical potential. The failure appears for both gapped and gapless quasi-particle spectra. The issue of constructing a number-conserving theory (fixed particle number, canonical ensemble) is of minor importance for the homogeneneous system we were focusing on. It can be checked explicitly from Ref. [25] that the specific features (projection of quasi-particle modes perpendicular to the condensate, condensate phase operator) become irrelevant in the thermodynamic limit.

We could gauge this state of affairs by comparison to two successful models for the cross-over. One is provided by the exact solution of the (Lieb-Liniger) Yang-Yang equations, which gives an easy access to low moments of the density [37, 46, 49]. The second method builds on complex-field simulations (stochastic Gross-Pitaevskii equation) that capture the low-lying modes of the quasi-condensate which can be described classically. With a suitable choice of numerical cutoff, these simulations are essentially unique. Their smooth density profiles through the cross-over region have already compared favorably with experiments. We may expect that the distribution functions (counting statistics) that can be extracted from them (see, e.g., Ref. [64]) may help curing the deficiencies of mean field theories. We have reasons to believe that the failures of mean-field are related to the break-down of the Gaussian approximation to the probability distribution of the quantum field. (For a discussion of beyond-Gaussian correlations in c-field methods, see Ref. [65].) This conclusion is based on the comparison with a classical field theory which will be reported elsewhere [66].

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Appendix A: Low-energy expansions

1. Bose function

The Bose function is also known as polylogarithm \( \text{Li}_\nu(e^x) \):

\[
g_\nu(x) = \sum_{n=1}^{\infty} \frac{e^{nx}}{n^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt \frac{t^{\nu-1}}{e^t-x-1} \quad (A1)
\]

The sum converges only for \( x < 0 \) or a fugacity \( e^x < 1 \). Of interest here is the case \( \nu = 1/2 \) and the ‘high-temperature expansion’ approaching the critical point from below \[67\]

\[
x > 0 : \quad g_{1/2}(x) \approx \sqrt{\frac{\pi}{-x}} + \zeta\left(\frac{1}{2}\right) + \zeta\left(-\frac{1}{2}\right)x + \mathcal{O}(x^2)
\]

(A2)

with coefficients given by the (analytically continued) Zeta function. The first term can be found by expanding the exponential under the integral \((A1)\). Subtracting this convergent integral and expanding the integrand for small \( x \), we observe that the lowest terms provide convergent integrals. They yield the following integral representations for the \( \zeta \)-coefficients

\[
\int_0^\infty \frac{dq}{\pi} \left\{ \frac{1}{e^{q^2/2} - 1} - \frac{2}{q^2} \right\} = \frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} = -a_1 \quad (A3)
\]

\[
\int_0^\infty \frac{dq}{\pi} \left\{ \frac{1}{4 \sinh^2(q^2/4)} - \frac{4}{q^4} \right\} = \frac{\zeta\left(-\frac{1}{2}\right)}{\sqrt{2\pi}} = -a_2 \quad (A4)
\]

where the suggestive substitution \( t = q^2/4 \) was made. The coefficients are approximately \( a_1 \approx 0.5826 \), \( a_2 \approx 0.0830 \).

2. High temperature expansion

As an illustration of the technique, we consider the integral that appears in the non-condensate density \[35\] [see also...
\[ I_1(\beta) = \int \frac{dk}{2\pi} \left\{ \frac{(k^2/2) \coth(\frac{1}{2} \beta E(k))}{2E(k)} - \frac{1}{2} \right\} \] (A5)

\[ \approx \frac{1}{2\beta} - \frac{a_1}{\sqrt{\beta}} - a_2 \sqrt{\beta} \quad (\beta \to 0) \] (A6)

To simplify the notation in this Appendix, we use units where the Bogoliubov dispersion relation is \( E(k) = |k| \sqrt{1 + k^2/4} \).

The dimensionless inverse temperature is \( \beta = M c^2 / k_B T \) with the speed of sound \( c \).

The integrand is even \( k \), and we restrict to \( 0 \leq k < \infty \). Convergence in the infrared is secured by the 'coherence factor' \( k^2/(4E(k)) \) in front of the hyperbolic cotangent. The classical (high-temperature) limit of the latter integrates to the first term in Eq. (A6):

\[ \int_0^\infty \frac{dk}{\pi} \frac{k^2/2}{\beta E^2(k)} = \frac{1}{2\beta} \] (A7)

The next order arises when this classical limit is subtracted from the integrand, and the high-energy approximation \( E(k) \approx k^2/2 \) is applied:

\[ \int_0^\infty \frac{dk}{\pi} \frac{1}{\alpha \beta k^2/2 - 1} - \frac{1}{\beta k^2/2} = -\frac{a_1}{\sqrt{\beta}} \] (A8)

using the substitution \( q = \sqrt{\beta} k \) and the identity (A3). Note that this also includes the last term \(-1/2 (\text{\textquoteleft vacuum subtraction\textquoteright})\) from Eq. (A5).

When the terms in Eqs. (A7) (A8) are subtracted from the integrand, we get an expression that is still integrable both at low and high momentum. We perform again the substitution \( q = \sqrt{\beta} k \) and expand (at fixed \( q \)) for small \( \beta \). The dispersion relation, for example, becomes \( \beta E(k) = \frac{1}{2} q^2 (q^2 + 4\beta)^{1/2} \approx \frac{1}{2} q^2 + \beta + \mathcal{O}(\beta^2/q^2) \). The resulting integral scales like \( \beta^{1/2} \) and, in the leading order, involves the integrand

\[ -\sqrt{\beta} \left\{ \frac{1}{4q^2 (q^2/4)} - \frac{8}{q^4} + \frac{\coth(q^2/4)}{q^2/4} \right\} \] (A9)

The second form makes the subtraction quite transparent that regularize the integrand as \( q \to 0 \). The first and one half of the second term yield \( a_2 \sqrt{\beta} \) from Eq. (A4). The remainder is integrated by parts to make the derivative of the coth appear, taking care of the cancelling poles. We again find the integral of Eq. (A4), but with a different prefactor: \(-2a_2 \sqrt{\beta} \). The sum gives the last term in Eq. (A6).

The next order in this expansion would be \( \mathcal{O}(\beta^{3/2}) \). In modified Popov theory, the last term in the non-condensate density (A5) is integrated elementarily. Since it is temperature-independent, it 'slips' between the \( a_1 \) and \( a_2 \) terms in Eq. (A6).

3. Zero-temperature expansion

The non-condensate density involves two integrals. The first one is

\[ I_a = \int \frac{dk}{2\pi} \frac{\epsilon - E}{2E} \] (A10)

By adopting the units explained after Eq. (A5), a dimensional factor \( 1/(2\xi) \) is pulled out. Here, \( \xi = h(4M\mu)^{-1/2} \) for extended Bogoliubov theory and \( \xi \to \xi_q \) for modified Popov. Make the substitution \( k = 2 \sinh t \) and get

\[ \epsilon = 2 \sinh^2 t, \quad E = 2 \sinh |t| \cosh t \]

\[ I_a = \frac{1}{2\pi \xi} \int_0^\infty dt (\sinh t - \cosh t) = -\frac{1}{2\pi \xi} \] (A11)

The second piece is the term:

\[ I_b = \int \frac{dk}{2\pi} \frac{gn_q}{(\epsilon + \mu)} \] (A12)

which reduces in our units with \( k = \sqrt{2} q \) to

\[ I_b = \frac{gn_q}{2\pi \sqrt{2} \mu \xi} \int_0^\infty dq \frac{\pi \xi}{q^4 + 1} = \frac{1}{2\pi \xi} \frac{\pi \xi}{1 - \frac{1}{2\pi \xi} \frac{\pi \xi}{\xi_q} \frac{\pi \xi}{q}} \] (A13)

The zero-point energy density of Bogoliubov theory, Eq. (17), is integrated similarly. After the substitution \( k = \xi^{-1} \sinh t \),

\[ \epsilon_0 - \epsilon_c = -\frac{\mu}{2\pi \xi} \int_0^\infty dt e^{-2t} \cosh t = -\frac{\mu}{3\pi \xi} \] (A14)

which is Eq. (18).

Some correlation functions involve the integral

\[ C_1(x) = \int \frac{dk}{2\pi} (1 - \cos kx) \frac{\mu}{E(k)} \] (A15)

Due to the \( 1/k \) singularity at \( k = 0 \), it is logarithmically divergent as \( x \to \infty \). We are interested in its asymptotic behaviour. Recall the definition of the cosine integral (here, \( \gamma \approx 0.577 \) is the Euler-Mascheroni constant)

\[ \text{Ci}(x) = \int_0^x dq \frac{\cos q - 1}{q} + \log x + \gamma \] (A16)

and its asymptotic form \( [68] \)

\[ \text{Ci}(x) \approx \frac{\sin x}{x} + \mathcal{O}(1/x^2) \quad (x \to \infty) \] (A17)

Take some \( k_* < \infty \) and subtract in the interval \( 0 \leq k \leq k_* \) the leading term \( 1/E(k) \approx 1/k \)

\[ \int_0^{k_*} \frac{1 - \cos kx}{E(k)} = \log(k_* x) + \gamma - \text{Ci}(k_* x) \] (A18)
\[-\frac{1}{4} \int_0^{k_\star} \frac{dk}{\sqrt{1 + k^2/4(1 + \sqrt{1 + k^2/4})}} \quad \frac{k(1 - \cos kx)}{\sqrt{1 + k^2/4(1 + \sqrt{1 + k^2/4})}}\]

For large \(x\), the cosine integral \(\text{Ci}(k, x)\) vanishes [Eq. (A17)], and by the Riemann-Lebesgue lemma, the \(\cos kx\) can be dropped from the integrand which is regular. The remaining integral is elementary

\[-\frac{1}{4} \int_0^{k_\star} \frac{dk}{\sqrt{1 + k^2/4(1 + \sqrt{1 + k^2/4})}} = -\log \frac{1 + \sqrt{1 + k^2/4}}{2} \quad (A19)\]

Consider now the limit \(k_\star \to \infty\). On the lhs of Eq. (A18), the integrand scales \(\sim 1/k^2\) and falls off sufficiently fast so that one gets the definite integral over \(k = 0 \ldots \infty\). On the rhs, the integrated term \(\text{A19}\) becomes \(-\log(k_\star/4)\) so that the logarithms partially compensate. Re-instating the dimensional prefactor, we finally get

\[C_1(x) = \frac{\log(2x/\xi) + \gamma}{2\pi \xi} \quad (A20)\]

To check this numerically, we keep \(k_\star\) finite and improve the UV-convergence of the integral over \(k_\star \leq k < \infty\) by adding and subtracting \(1/(k^2/2 + 1)\) under the integral. The added term can be integrated explicitly.

To get the full expression for the correlation function, we recall that the integral (38) also contains the zero-temperature density (depletion). In the limit of large \(x\), by the Riemann-Lebesgue lemma, this piece integrates to Eq. (25) in the leading order. Combining with Eq. (A20), we get the result (39).

**Appendix B: Details on numerics**

We solve implicit equations either with a bisection or an iterative scheme, depending on the convergence rate and a priori knowledge about the interval where the solution will be found. In some cases, an interpolation based on parametrically calculated datasets is used. Critical points are determined by minimising the chemical potential as a function of the relevant parameters (e.g., the condensate density, see Fig 7).

**Appendix C: Validity of the local-density approximation**

In Fig 10, we compare results obtained with the stochastic Gross-Pitaevskii equation for trapped systems with different trap frequencies. The red (lower) curve is computed for a homogeneous gas (i.e., a sufficiently large box with periodic boundary conditions). The black (upper) curves are based on the local-density approximation and correspond to increasing axial trapping frequency from left to right. Very good agreement is found on the two asymptotes, but deviations are visible in the cross-over and grow as the trap potential gets steeper. This is consistent with the observation that for the strongest confinement, the inhomogeneity of the potential is significant on the scale of the cross-over: across a displacement of one healing length \(\xi_x\), it changes by a few \(\mu_x\).

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