Disordered $d$-wave superconductors with chiral symmetry

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A two-dimensional lattice model for $d$-wave superconductor with chiral symmetry is studied. The field theory at the band center is shown to be in the universality class of $U(2n)/O(2n)$ and $U(2n)$ nonlinear sigma model for the system with broken and unbroken time-reversal symmetry, respectively. Vanishing of the beta function implies extended states at the band center. Density of state vanishes as a cubic function of the energy at the band center for the former case, while linear for the latter.

Symmetries play a crucial role in random critical phenomena. The rotational and time-reversal invariances specify the well-known universality classes whose underlying symmetries are orthogonal, symplectic, and unitary [7]. The variety of universality classes, however, has turned out to be richer. Above all, Altland and Zirnbauer [8] have recently discussed universality classes for quasi-particles in disordered spin-singlet BCS Hamiltonian, denoted in their article by $C$ and $CI$ for the case with broken and unbroken time-reversal symmetry, respectively. These universality classes are actually involved with vortex in $s$-wave superconductors [9], quasi-particle transport in $d$-wave superconductors [10], and the spin quantum Hall (QH) transitions [11]. Especially, dirty $d$-wave superconductors have attracted a lot of interest, because they can be described by Dirac Fermions [8] due to gapless quasi-particle spectrum. Random Dirac Fermions actually give nontrivial critical points [11]. The criticality of the density of state (DOS) of disordered $d$-wave superconductors is, for example, still controversial [11,12].

Another possibility is the class for the two-sublattice model [11], where Hamiltonian has a special symmetry which will be specified momentarily. This symmetry will be referred to as chiral symmetry [11]. One example in two-dimension is the random flux model [13] and another is the random hopping fermions with $\pi$-flux [14], both of which have an isolated delocalized states at the band center. Various universality classes, including those mentioned above, are well summarized by Zirnbauer [11].

In this paper, we study a system with chiral symmetry in $d$-wave superconductors. The basic idea is that the pure $d$-wave lattice system is a kind of two-sublattice model. It is, therefore, natural to include randomness keeping the same symmetry of the pure Hamiltonian. The model we study has a $SU(2)$ symmetry as well as the chiral symmetry. It is also interesting to study a system with such enhanced symmetries. We show that the field theory of the model with broken and unbroken time-reversal symmetry is in the universality class of $U(2n)/O(2n)$ and $U(2n)$ nonlinear sigma model (NLSM), respectively. It is interesting to compare the classes to those of two-sublattice models without $SU(2)$ symmetry, which are in the class of $Sp(n)/U(2n)$ and $Sp(n)$ NLSM for the case with broken and unbroken time-reversal symmetry, respectively. We conclude that regardless of the time-reversal symmetry the present model has extended states at the band-center, which are associated with diffusive spin transport [15]. We discuss the behavior of the quasi-particle density of state (DOS), which should vanish at the band center as $E^3$ and $E$ in broken and unbroken time-reversal cases, respectively.

Let us start from the lattice Hamiltonian of a $d$-wave superconductor defined on the square lattice in two-dimension [12],

$$H = \sum_{\langle i,j \rangle} \left( t_{ij} \sum_{\sigma} c_{i\sigma}^\dagger c_{j\sigma} + \Delta_{ij} c_{i\uparrow}^\dagger c_{j\downarrow} + \Delta_{ij} c_{i\downarrow}^\dagger c_{j\uparrow} \right).$$

(Hermiticity and $SU(2)$ symmetry (spin-rotation) require $t_{ij} = t_{ji}^*$ and $\Delta_{ij} = \Delta_{ji}$, respectively. In the absence of randomness, the pure $d$-wave Hamiltonian is given by $t_{j,j+\hat{x}} = t_{j,j+\hat{y}} = -t_0$, $\Delta_{j,j+\hat{x}} = -\Delta_{j,j+\hat{y}} = \Delta_0$, and others are zero, where $\hat{x} = (1,0)$ and $\hat{y} = (0,1)$. It is easy to see that the pure Hamiltonian change the sign under the transformation $c_{ij\sigma} \to (-)^{j_x+j_y} c_{ij\sigma}$, which is the chiral symmetry as mentioned-above. Therefore, the pure system has the $SU(2)$, time-reversal, and chiral symmetry. By introducing disorder which breaks some symmetries explicitly but keeps the others, the model shows various kinds of universality classes. Those studied so far are as follows: If we keep only $SU(2)$ symmetry, the (replicated) model has $Sp(n)$ symmetry, which is spontaneously broken to $U(2n)$. If we keep in addition the time-reversal invariance, the model has an enhanced symmetry $Sp(n) \times Sp(n)$, which is broken to $Sp(n)$ [15]. In this paper, the symbol $\langle i,j \rangle$ in Eq. (1) is restricted to the nearest neighbor pairs, which keeps the chiral symmetry, and universality classes of the $d$-wave superconductors unknown so far are investigated.

The pure Hamiltonian has four nodes, where gapless quasi-particle excitations exist [16]. To study the low-energy properties of the system governed by them, let us take the continuum limit near the nodes [16].

$$c_{j\uparrow}/a \sim i^{j_x+j_y} \chi_{11}(x) - i^{-j_x-j_y} \chi_{12}(x) + i^{-j_x+j_y} \chi_{12}(x) - i^{j_x-j_y} \chi_{12}(x),$$
where $a$ is a lattice constant, $x = aj$, and indices $\sigma, \alpha$, and $i$ of the field $\chi_{\sigma\alpha i}$ are associated with spin, particle-hole, and nodes, respectively. Namely, the field $\chi$ at each $x$ lives in the space $V = C^2 \otimes C^2 \otimes C^2$. The pure Hamiltonian is then $H_p = \int d^2x \chi^\dagger \chi$, where

$$H_p = 1_2 \otimes \begin{pmatrix} -v_0 \gamma_0 \partial_0 & (x \leftrightarrow y) \end{pmatrix},$$

and the coordinates have been transformed as $x, y \rightarrow x \pm y \sqrt{2}$. Here the explicit matrix in Eq. (3) denotes the node space, and gamma matrices denotes the particle-hole space, defined by $\gamma_1 = r \sigma_3$ and $\gamma_2 = r^{-1} \sigma_1$ with $r = \sqrt{v_F/v_\Delta}$, and $v = \sqrt{v_Fv_\Delta}$, where $v_F = 2\sqrt{2}\Delta a$ and $v_\Delta = 2\sqrt{2}\Delta a$.

The symmetries of the lattice model are translated into the continuum model as

$$CCH^{-1} = -H^t,$$

$$\mathcal{P}H \mathcal{P}^{-1} = -H,$$

$$\mathcal{T}HT^{-1} = -H,$$

which describe, respectively, the SU(2), chiral, and time-reversal symmetry. The total Hamiltonian density is given by $H = H_p + \mathcal{H}_d$, where $\mathcal{H}_d$ is the Higgs potential satisfying Eq. (4). In what follows, we study the model with broken and unbroken time-reversal symmetry at the same time. For the former case, the third condition should be omitted. It is easy to write down explicit disorder potentials using these conditions, though it is not necessary in the following calculation.

To study the DOS and the conductance of the spin transport, let us define the Green functions,

$$G(x) = tr_V x (ie-H)^{-1} |x|,$$

$$K(x, x') = tr_V |\langle x | (ie-H)^{-1} |x' \rangle|^2,$$

where $tr_V$ is the trace in the $V$ space. Although it may be easy to introduce the generating functional of these Green functions, we have to prepare some notations. Firstly, the generating functional expressed by path-integrals over Fermi fields is defined in a standard way by introducing replica. $\chi_i \rightarrow \chi_{\sigma i}$ and $\tilde{\chi}_i \rightarrow \tilde{\chi}_{\sigma i}$, where $i$ and $\alpha$ are indices denoting $V$ and replica space $W_R = C^\alpha$, respectively. Lagrangian density is then $\mathcal{L} = -tr_{W_R} \tilde{\chi} (ie-H) \chi$, where $tr_{W_R}$ is the trace in the replica space. Moreover, we introduce an auxiliary space to reflect the symmetries in the $V$ space to an auxiliary field introduced later [See Eq. (2)], chiral, and time-reversal symmetry, respectively, and in this extended space, Fermi fields are denoted by $\tilde{\psi}_{\sigma i}$ and $\psi_{\sigma i}$, which are subject to

\begin{align*}
\tilde{\psi} &= \gamma x^{\dagger} C^{-1}, & \tilde{\psi} &= \mathcal{C} \tilde{\psi} \gamma^{-1}, \\
\psi &= i \tau \tilde{\psi} \mathcal{P}^{-1}, & \psi &= i \mathcal{P} \psi \tau^{-1}, \\
\tilde{\psi} &= i \tau \tilde{\psi} \mathcal{T}^{-1}, & \tilde{\psi} &= i \mathcal{T} \tilde{\psi} \tau^{-1}.
\end{align*}

Matrices $\gamma, \tau$, and $\pi$ are defined in the $W$ space, given by $\gamma = 1_{n} \otimes \sigma_3 \otimes 1_2 \otimes 1_2$, $\tau = 1_{n} \otimes 1_2 \otimes \sigma_3 \otimes 1_2$, and $\pi = 1_{n} \otimes 1_2 \otimes 1_2 \otimes \sigma_3$. The identity $tr_W (\omega \tilde{\psi} \psi - \tilde{\psi} \mathcal{H}_d \psi)$ leads to the generating functional $Z = \int D\tilde{\psi}D\psi e^{-S}$ with

$$S = -\int d^2x tr_W \left( \omega \tilde{\psi} \psi - \tilde{\psi} \mathcal{H}_d \psi + J \tilde{\psi} \psi \right).$$

Green functions are expressed as

$$G(x) = dW_A \lim_{n \to 0} \left( \langle \tilde{\psi} \psi \rangle_{12}(x) \right),$$

$$K(x, x') = -d_W \lim_{n \to 0} \left( \langle \tilde{\psi} \psi \rangle_{12}(x') \langle \tilde{\psi} \psi \rangle_{21}(x) \right),$$

where $1 = \sigma \otimes \otimes \otimes \otimes \otimes$ and $2 = \beta \otimes \otimes \otimes \otimes$, and $d_W$ is the dimension of the auxiliary space, given by 4 and 8 for broken and unbroken time-reversal case, respectively. Assume that disorder potentials obey the Gaussian distribution $\int \mathcal{D}d^2x \exp(-\frac{1}{2} \int d^2x tr_{V} \mathcal{H}_d^{2})$. Then it is easy to average over them, as usual. Here, some comments may be useful: Firstly, disorder potentials are integrated out by using $-\frac{1}{2g} tr_{V} \mathcal{H}_d^{2} + tr_{V} \mathcal{H}_d \tilde{\psi} = -\frac{1}{2g} (tr_{V} \mathcal{H}_d - g \psi \tilde{\psi})^2 + \frac{1}{2g} tr_{V} (\psi \tilde{\psi})^2$. It turns out that the integration over $\mathcal{H}_d$ is automatic because $\psi \tilde{\psi}$ satisfy the same symmetries as those of $\mathcal{H}_d$ due to Eq. (8).

Secondly, $tr_V (\psi \tilde{\psi})^2 = -tr_W (\psi \tilde{\psi})^2$, which is converted into Yukawa-type interactions by introducing an auxiliary matrix field defined in the $W$ space into the action in the form $\frac{1}{2g} tr_W (Q + \psi \tilde{\psi} - \omega)^2$. Integrating out the Fermi fields, we end up with an effective action

$$S = -\frac{1}{2g} \int d^2x tr_W (Q^2 - 2Q \omega)^2 - \frac{1}{dW_A} \ln \mathrm{Det}_V \omega \left( 1 \otimes Q - \mathcal{H}_p \otimes 1 \right).$$

Here we have set $J = 0$ for simplicity and the anti-Hermitian auxiliary field $Q = -Q^\dagger$ is subject to

$$Q = -\gamma Q^\dagger \gamma^{-1}, \quad Q = -\tau Q \tau^{-1}, \quad Q = -\pi Q \pi^{-1}.$$
The effective action has $U(2n)$ and $U(2n) \times U(2n)$ symmetry for the model with broken and unbroken time-reversal symmetry, respectively. To be concrete, let us consider the former case. For the action function to be invariant under the group action $Q \rightarrow gQg^{-1}$, $g$ should satisfy $g = g\gamma g^{-1}$ and $\gamma g = g^{-1}$. They are explicitly given by

$$g = V \left( \begin{array}{c} u \\ i \end{array} \right) V^\dagger, \quad Q = V \left( \begin{array}{c} -q^\dagger \\ g \end{array} \right) V^\dagger,$$

where matrices in Eq. (12) denote the chiral space of $W$, and $u$ and $u^\dagger$ unitary matrix in the replica and SU(2) space of $W$, while $g$ is a complex matrix with a condition $q^\dagger = q$. $V$ is a matrix defined by $V = V_1V_2$ with $V_1 = 1_n \otimes \mu \otimes \mu$ and $V_2 = 1_n \otimes 1_2 \otimes \frac{i}{2}(1_2 + \sigma_3) + 1_n \otimes \sigma_2 \otimes \frac{i}{2}(1_2 - 3\sigma_3)$, where $\mu = \frac{1}{\sqrt{2}}(\sigma_3 + \sigma_2)$. It is, therefore, easy to show that the saddle-point solution is invariant under the group action if $u^\dagger u = 1$. Thus, the $U(2n)$ symmetry is spontaneously broken to $O(2n)$, and the saddle-point manifold is $U(2n)/O(2n)$.

Local quantum fluctuation around the saddle-point yields Goldstone mode, which governs the low energy properties of the present system. To derive an effective action of this mode integrating out the massive mode, let us decompose the field $Q$ into transverse and longitudinal modes $Q(x) = T(x)(Q_0 + L(x))T^\dagger(x)$, where

$$T = \begin{pmatrix} U \\ 0 \end{pmatrix} T^\dagger, \quad L = \begin{pmatrix} -P \\ 0 \end{pmatrix} T^\dagger.$$

Here $U(x) \in U(2n)/O(2n)$ and $P(x)$ is a real matrix with $P^\dagger = P$. Next transformation properties of $U(x)$ and $P(x)$ fields should be examined. The action of global $u \in U(2n)$ induces $uU(x) = U'(x)h(u, U(x))$, where $U'(x) \in U(2n)$ and $h(u, U) \in O(2n)$. It should be noted that $h(u, U)$ is a nonlinear function of $U(x)$ dependent on $x$. Therefore, we have the following transformation laws of $U$ and $P$, $U \rightarrow uU^{-1}$ and $P \rightarrow hP^{-1}$.

Keeping this in mind, we proceed to calculate the action of the Goldstone mode. The point is the derivative expansion of the ln Det term in Eq. (13).

$$\ln \text{Det}_{V \otimes W}(1 \otimes Q - \mathcal{H}_p \otimes 1) \approx \text{Tr}_{V \otimes W} \Delta_F G - \frac{1}{2} \text{Tr}_{V \otimes W} (\Delta_F G)^2,$$

where $\Delta_F$ is the free propagator defined by $\Delta_F^{-1} = 1 \otimes Q_0 - \mathcal{H}_p \otimes 1$ and $G$ is

$$G = (1 \otimes T^\dagger)(\mathcal{H}_p \otimes 1)(1 \otimes T) - \mathcal{H}_p \otimes 1 = \left( \begin{array}{c} v_{\gamma\mu} t^{T^\dagger} \partial_{\nu} T \\ (x \leftrightarrow y) \end{array} \right).$$

The explicit matrix in the above denotes the node space in $V$. It is stressed that the effective action should be independent of the gauge associated with the local $O(2n)$ transformation. To see this, let us write down the field $T^\dagger \partial_{\mu} T$ in $W$ space,

$$T^\dagger \partial_{\mu} T = V \left( \begin{array}{c} V_\mu + A_\mu \\ V_\mu - A_\mu \end{array} \right) V^\dagger,$$

where $A_\mu, V_\mu = \frac{1}{4}(U^\dagger \partial_{\mu} U + \partial_{\mu} U^\dagger)$. The transformation laws of these fields are $A_\mu \rightarrow hA_\mu h^{-1}$ and $V_\mu \rightarrow hV_\mu h^{-1} + h\partial_{\mu} h^{-1}$, which tells that $V_\mu$ is a gauge field associated with the hidden local $O(2n)$ symmetry. In the leading order of the derivative expansion, therefore, only the gauge covariant $A_\mu$ field appears in the action.

By using these, manifestly gauge-invariant action of the Goldstone mode can be obtained. The principal term is calculated as $\frac{1}{2ac^2} \text{Tr}_{V \otimes W} (\Delta_F G)^2 = -\frac{2c}{6} \text{Tr}_{W_{RS}} A_\mu^2$. After gauge-fixing $U^1 = U$, we have

$$S = \int d^2x t \text{Tr}_{W_{RS}} \left[ \frac{1}{2b} \partial_{\mu} U^2 \partial_{\nu} U - \frac{\mu c}{g} (U^2 + U^{-2}) \right].$$

Here $W_{RS}$ denotes a part of the space $W$ restricted to the replica and SU(2) spaces omitting the chiral space, and $\frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$. This coupling constant is just the same as that derived by Senthil at al. So far we have obtained an effective action for broken time-reversal system. It is easy to follow similar calculations for the unbroken time-reversal system. The results are the same as Eq. (17) including the coupling constant, but with $U(x) \in U(2n)$ and a breaking term $\frac{2c}{6} \text{Tr}_{W_{RS}} R(U^2 + U^{-2})$.

In the process of the renormalization, it is shown that the operator $\frac{2c}{6a} \text{Tr}_{W_{RS}} A_\mu = \frac{1}{2a^2} \text{Tr}_{W_{RS}} U^2 \partial_{\mu} U^{-2}$ is needed. In the leading order we have taken above, the coupling constant is $\frac{1}{2} = 0$. However, higher loop expansion of the ln Det term actually gives a finite coupling constant. After the replica limit $n \rightarrow 0$, the renormalization group equations at one-loop order are given by

$$\frac{db}{dl} = 0, \quad \frac{dc}{dl} = -\alpha_{1e}^2, \quad \frac{d\zeta}{dl} = \frac{1}{2} \left( \beta c + \frac{b^2}{c} \right),$$

where $(\alpha_{1e}, \beta_1) = (\frac{1}{4}, \frac{1}{2})$ and $(\frac{1}{4}, 0)$ for $U(2n)/O(2n)$ and $U(2n)$ models, respectively. The spin conductance is related only with the coupling constant $b$ as $\frac{1}{\gamma} = 2\alpha_{1e}^2$, which is obtained by the bare diffusion constant calculated from Eq. (8). Therefore, vanishing of the beta function for $b$ implies delocalized state at zero energy, which is due to the chiral symmetry. On the other hand, the behavior of the DOS depends on the coupling constant $c$. Since $\epsilon$ has the same dimension of the energy, we can compute, according to Gade, a rough estimate of the DOS for finite $E$ as $\rho(E) \sim \frac{1}{2} c e^{-4c(\epsilon/b)^{-1}} \ln(E)$. This yields an enhancement of DOS near the zero energies. However, the result from perturbative calculations
needs alternative consideration for DOS at the zero energy.

![Graph](image)

**FIG. 1.** Schematic illustration of the density of state as a function of energy. Dashed-line denotes DOS of pure system, while dashed-dotted-line (full-line) denotes DOS for broken (unbroken) time-reversal symmetry. The bump means the enhancement obtained by the renormalization group analysis.

Since delocalization occurs at the zero energy, the localization length is quite long, i.e., the order of the system size $L$ near the zero energy. Accordingly, in the energy scale much smaller than the Thouless energy $\frac{1}{L^2}$, where $D$ is a spin diffusion constant, quasi-particles are diffusing all around the system keeping the symmetry of the Hamiltonian, and hence, the spatial dependence of the system is smeared out and we can describe the system by a random matrix theory. Taking only zero mode of $Q$ into account, we actually have from Eq. (1) an effective action of a random matrix theory with the symmetries $\rho$ but defined in $V = C^N \otimes C^2 \otimes C^2 \otimes C^2$, where $N = L^2$. The Hamiltonian $H$ is now not a field but a quantum mechanical one subject to Eq. (1). To calculate the DOS near the zero energy of such a random matrix ensemble, it may be convenient to rotate the basis by a orthogonal transformation and to switch into more convenient basis. We can choose $\mathcal{C} = \mathcal{I}_N \otimes \mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_2$, $\mathcal{P} = \mathcal{I}_N \otimes \mathcal{I}_2 \otimes \mathcal{I}_2 \otimes \mathcal{I}_3$. Let us omit the SU(2) space in the above, because it is irrelevant. A little thought tells that the Hamiltonian describes the tangent space of $\text{Sp}(N, N)/\text{Sp}(N) \times \text{Sp}(N)$ and $U(N, N)/U(N) \times U(N)$ for the case with broken and unbroken time-reversal symmetry, respectively, and accordingly, belongs to CII (chiral GSE) and $\text{AIII (chiral GUE)}$ in Zirnbauer’s classification [3]. We can now diagonalize the Hamiltonian as $H = U \text{diag}(\theta, -\theta) U^\dagger$, where $U$ is a unitary matrix. The Jacobian of the change of variables from $H_{ij}$ to $U_{ij}$ and $\theta_i$ is $J = \text{const.} \Pi_i \theta_i^\alpha \Pi_{i < j} (\theta_i^2 - \theta_j^2)^{\beta_2}$, where $(\alpha_2, \beta_2) = (3, 4)$ and $(1, 2)$ for broken and unbroken time-reversal case, respectively. $\beta_2$ describes the level repulsion of eigenvalues each other, while $\alpha_2$ describes the repulsion of each pair $\theta$ and $-\theta$ and determines the DOS for small $\theta$ [4]. Based on this argument, we expect that $\text{DOS vanishes at zero energy as } E^3$ and $E$, respectively, for the system with broken and unbroken time-reversal symmetry. Interposing the results, we expect the behavior of DOS as Fig. 1.

It is quite interesting to test the present conjectures by numerical calculations. Especially a cubic dependence of DOS on the energy has not been known so far in disordered $d$-wave superconductors.

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