Abstract

In this paper, we describe the category of bi-equivariant vector bundles on a bi-equivariant smooth (partial) compactification of a connected reductive algebraic group with normal crossing boundary divisors. Our result is a generalization of the description of the category of equivariant vector bundles on toric varieties established by A.A. Klyachko [Math. USSR. Izvestiya 35 No.2 (1990)]. As an application, we prove splitting of equivariant vector bundles of low rank on the wonderful compactification of an adjoint simple group in the sense of C. De Concini and C. Procesi [Lecture Notes in Math. 996 (1983)]. Moreover, we present an answer to a problem raised by B. Kostant in the case of complex groups.

Keywords: reductive group, the wonderful compactification, vector bundle, category, splitting.

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Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic zero. Then we have a $G \times G$-action on $G$ defined as $(g_1, g_2).x = g_1.x.g_2^{-1}$. Let $X$ be a $G \times G$-equivariant smooth partial compactification of $G$ with normally crossing boundary divisors. (Here, a nonsingular toric variety (cf. Oda [23]) and the wonderful compactification of an adjoint semisimple group in the sense of De Concini-Procesi [11] satisfy the condition.) By the general theory (Theorem 1.2), such a partial compactification is described by a fan $\Sigma$.

A $(G \times G)$-equivariant vector bundle is a vector bundle $E$ on $X$ with an action of $G \times G$ which is linear on each fiber and makes the following diagram commutative:

\[
\begin{array}{ccc}
V(E) & \xrightarrow{g^*} & V(E) \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & X
\end{array}
\]

Here, $V(E)$ denotes the total space of $\mathcal{E}$.

Let $EV(X)$ be the category of $G \times G$-equivariant vector bundles and morphisms of $G \times G$-equivariant coherent $O_X$-modules. (Our actual setting is to handle a finite cover $\tilde{G}$ of $G$ instead of $G$. Here we restrict the situation for simplicity.) Our goal (cf. Theorem 4) is to describe this category explicitly via linear algebra data.

Let $e$ be the point of $X$ corresponding to the identity element of $G$. Choose a maximal torus $T$ of $G$. An equivariant vector bundle $\mathcal{E}$ is completely determined by the following data:

- $\mathcal{E} \otimes k(e)$: the fiber at $e$;
- A family of subspaces

\[
F^\tau(n, \mathcal{E}) := \{ v \in \mathcal{E} \otimes k(e); \exists \lim_{t \to \infty} t^n(1 \times \tau(t))v \in V(\mathcal{E}) \} \subseteq \mathcal{E} \otimes k(e)
\]
for every one-parameter subgroup $\tau : \mathbb{G}_m \to T$ and every $n \in \mathbb{Z}$. In other words, the asymptotic behavior of elements of $\mathcal{E} \otimes k(e)$ with respect to all one-parameter subgroups.

Thus, the problem is to find a sufficient condition for the existence of an equivariant vector bundle which corresponds to a given asymptotic datum.

Klyachko \cite{Kl} described the category of equivariant vector bundles on toric varieties by using the above asymptotic data $(\mathcal{E} \otimes k(e), \{F^\tau (n, \mathcal{E})\}_\tau)$. His description is based on the fact that there are no “local” obstruction for the existence of an equivariant vector bundle with a given asymptotic behavior. (For another description in this case, see Kaneyama \cite{Kan1, Kan2}.)

In this paper, we extend such a description to the compactification of an arbitrary reductive algebraic group by introducing a new constraint which we call the transversality condition.

Let $\Sigma(1)$ be the set of one-skeletons of the fan $\Sigma$ which is identified with a set of one-parameter subgroups of $T$. We put $P\tau := \{g \in G; \exists \lim_{t \to 0} \tau(t) g \tau(t)^{-1} \in G\}$. Put $\mathfrak{g} := \text{Lie}G$. For each root $\alpha$, let us denote the $\alpha$-root space of $\mathfrak{g}$ by $\mathfrak{g}_\alpha$.

**Definition A.** We define the category $\mathcal{E}(X)^+$ by the following:

- **(Objects)** Pairs $(V, \{F^\tau (\bullet)\}_{\tau \in \Sigma(1)})$ consisting of a $G$-module $V$ and exhausting $\mathbb{Z}$-indexed decreasing $P\tau$-filtrations $F^\tau (\bullet)$ of $V$ such that:
  - For each $\sigma \in \Sigma$, there exists a basis $B^\sigma$ of $V$ which spans every $F^\tau (n)$ when $\tau \in \sigma(1)$.
- **(Morphisms)** Let $(V_1, \{F^\tau_1 (\bullet)\}_{\tau \in \Sigma(1)})$, $(V_2, \{F^\tau_2 (\bullet)\}_{\tau \in \Sigma(1)}) \in \text{Ob} \mathcal{E}(X)^+$. Then, we define

\[
\text{Hom}_{\mathcal{E}(X)^+} (\{V_1, \{F^\tau_1 (\bullet)\}_{\tau \in \Sigma(1)}) , (V_2, \{F^\tau_2 (\bullet)\}_{\tau \in \Sigma(1)}) \} := \{f \in \text{Hom}_G (V_1, V_2) ; f (F^\tau_1 (n)) \subset F^\tau_2 (n) \text{ for every } n \in \mathbb{Z} \text{ and every } \tau \in \Sigma(1)\}.
\]

We also define a full-subcategory $\mathcal{E}(X)$ of $\mathcal{E}(X)^+$ by the following:

- $(V, \{F^\tau (\bullet)\}_{\tau \in \Sigma(1)}) \in \text{Ob} \mathcal{E}(X)^+$ is in $\text{Ob} \mathcal{E}(X)$ if, and only if, $F^\tau (\bullet)$ satisfies the following ($\tau$-) transversality condition for each $\tau \in \Sigma(1)$.

**(Transversality Condition)** For every $n \in \mathbb{Z}$ and each root $\alpha$ of $\mathfrak{g}$ such that $\langle \tau, \alpha \rangle < 0$, we have $\mathfrak{g}_\alpha F^\tau (n) \subset F^\tau (n + \langle \tau, \alpha \rangle)$.

**Examples B.**

1. If $X = G$, then $\mathcal{E}(X) = \mathcal{E}(X)^+ = \text{Rep}G$.

2. **(Case $G = PGL_2$ and $X = \mathbb{P}^3$)** Fix a triangular decomposition $\mathfrak{sl}_2 = \mathfrak{n}^- \oplus \mathfrak{b}$. Then, an object of $\mathcal{E}(X)^+$ is a pair consisting of a $PGL_2$-module $V$ and its $\mathfrak{b}$-stable decreasing filtration $F(\bullet)$. Transversality condition says $\mathfrak{n}^- F(n) \subset F(n - 1)$ for every $n$.

**Proposition C.** (cf. Proposition 2.35 + 3.10). The map

\[
\Xi : \text{Ob}EV(X) \ni \mathcal{E} \mapsto (\mathcal{E} \otimes k(e), \{F^\tau (n, \mathcal{E})\}_{\tau \in \Sigma(1)}) \in \text{Ob} \mathcal{E}(X)^+
\]

is injective. Moreover, it gives a faithful covariant functor $EV(X) \to \mathcal{E}(X)^+$ which we denote by the same letter $\Xi$.

**Remark D.** Let $D_\tau$ be the $G \times G$-equivariant divisor corresponding to $\tau \in \Sigma(1)$. Then, tensoring $O_X(nD_\tau)$ at the LHS corresponds to shifting the degree of $F^\tau$ by $-n$ at the RHS.

The idea in the proof of Proposition C is to measure the asymptotic behavior of an equivariant vector bundle by bounding it with a certain family of “standard” vector bundles instead of using the asymptotic behavior of elements on the identity fiber.

Next, our key result is as follows:
Proposition E (cf. Proposition 3.12). The image of the map

\[ \Xi : \text{Ob}EV(X) \to \text{Ob}\mathcal{C}(X)^+ \]

is contained in \( \text{Ob}\mathcal{C}(X) \).

In the above, there are two choices of families of “standard” vector bundles. It turns out that each of them yields different constraints about filtrations. Proposition E follows from the comparison of these constraints by means of the theory of Tannakian categories (cf. [13]).

Now we state our main result.

Theorem F (cf. Theorem 4.1). We have a category equivalence

\[ \Xi : \text{EV}(X) \overset{\simeq}{\to} \mathcal{C}(X). \]

For the proof of Theorem F we construct the inverse functor \( \Phi \) of \( \Xi \). The construction of \( \Phi \) depends on the fact that the boundary behavior of equivariant vector bundles is completely controlled by their restriction to certain toric varieties. Thus, we devote ourselves to check that the reconstruction process prescribed by \( \Phi \) is compatible with the action of the “unipotent part”.

If \( G \) is commutative, the left and the right multiplications are essentially the same. In this case, the transversality condition is a void condition. However, we cannot identify the left and the right multiplications if \( G \) is non-commutative. To establish Theorem F in this setting, we need to control the \( G \times G \)-action via a single group \( G \), which is the stabilizer of the \( G \times G \)-action on \( G \) at the identity element of \( G \). This is why the transversality condition appears. Moreover, it imposes a strong restriction on the existence of equivariant vector bundles. A typical example of this kind of phenomena is the following.

Example G. In the same setting as in Examples 2), simple objects are classified as follows: Let \( V_n \) be an irreducible \( PGL_2 \)-module of dimension \( 2n+1 \). Then, every irreducible equivariant vector bundle of rank \( 2n+1 \) has \( V_n \) as its identity fiber. From this fact, one can prove that irreducible equivariant vector bundles are classified by Dynkin quivers of type \( A_{2n+1} \) up to line bundle twist. In particular, there are only finitely many (namely \( 2^{2n} \)) possibilities.

For more details, see Example 3. In general, a simple equivariant vector bundle may have nontrivial equivariant deformations even in the case of the wonderful compactification.

Kostant has raised the question of the existence of a canonical extension of an equivariant vector bundle on a symmetric space to its wonderful compactification in order to deduce representation theoretic data from asymptotic expansions of matrix coefficients (see §5.2). Our description gives the following answer to his problem in the case of complex groups.

Theorem H (Theorem 5.8). Let \( G \) be a semisimple adjoint group, and let \( X \) be its wonderful compactification. For every \( G \)-module \( V \), there exists a unique \( G \times G \)-equivariant vector bundle \( \mathcal{E}_V \) on \( X \) which satisfies the following properties:

1. \( \mathcal{E}_V \otimes k(e) \cong V \) as a \( G \)-module;

2. For every \( v \in \mathcal{E}_V \otimes k(e) \) and every one-parameter subgroup \( \tau : \mathbb{G}_m \to G \), there exist a limit value \( \lim_{t \to 0} (\tau(t) \times \tau(t)^{-1})v \) in \( V(\mathcal{E}_V) \);

3. Every \( G \times G \)-equivariant vector bundle \( \mathcal{E} \) with the above two properties can be \( G \times G \)-equivariantly embedded into \( \mathcal{E}_V \).
Though the above Kostant problem is known by some experts, there is no existing literature. Thus, we also present the whole picture of his problem in §5.2. (The author learned about this problem from Prof. Brion, Prof. Kostant, and Prof. Uzawa. He wants to express gratitude to them.)

As a bonus of our description, we have the following result.

**Theorem I (= Corollary 5.5).** Let $G$ be an adjoint simple group and let $X$ be its wonderful compactification. Then, every $G \times G$-equivariant vector bundle of rank less than or equal to $r = \text{rk } G$ splits into a direct sum of line bundles.

This kind of material is treated in §5.1. For a classical (simple) group, the wonderful compactification is obtained by successive blowing-ups of a partial flag variety of an overgroup of $G$ (cf. Brion [8]). On projective spaces or Grassmannians, we have a splitting criterion of vector bundles in terms of the vanishing of intermediate cohomologies (see Horrocks [14], Ottaviani [25], or Arrondo and Graña [1]). Thus, if we have an analogous result of [18] 1.2.1 (a vector bundle is equivariant if its infinitesimal deformation is zero), we may find a splitting criterion of vector bundles using our result. There are plenty of vector bundles with nontrivial infinitesimal deformations, but a direct sum of line bundles on the wonderful compactification has no nontrivial infinitesimal deformation (cf. Tchoudjem [31], [32], [33] and K [17]). However, there exists a line bundle with non-vanishing intermediate cohomologies in this case. Hence, a naive reformulation of the splitting criterion of vector bundles in terms of cohomology vanishing has a counter-example in this case.

The organization of this paper is as follows. In §1 we fix our notation and introduce objects which we concern. In particular, we assume the notation and terminology introduced in §1 in the whole paper unless stated otherwise. In §2 we present some fundamental results which is needed to formulate our main theorem. More precisely, in §2.1 we devote ourselves to the preparation of the working ground. In §2.2 we construct a functor $\Xi$ in its full generality (Proposition 2.35). In §3 we develop a technique to discuss the boundary behavior of equivariant vector bundles and prove that the “image” of $\Xi$ is contained in $\mathcal{C}(X)$ (Proposition 3.12). In §4 we state and prove Theorem 4.1, which is our main result. Finally, we deal with some consequences of Theorem 4.1 in §5.

This paper is the main body of the author’s Doctoral Dissertation at University of Tokyo.

# 1 Notation and Terminology

## 1.1 Notation on algebraic groups

The general reference for the material in this subsection is Springer’s book [28].

Let $G$ be a connected reductive group of rank $r$ over an algebraically closed field $k$ of characteristic zero. There exists a connected finite cover $\tilde{G}$ of $G$ which is isomorphic to the direct product of a torus $\tilde{T}_0$ and a simply connected semisimple algebraic group $\tilde{G}_s$. We put $\tilde{Z}(G) := \ker[\tilde{G} \to G]$. For a group $H$, we denote its center by $Z(H)$. We have $\tilde{Z}(G) \subset Z(\tilde{G})$.

We put $G_{ad} := G/Z(G)$, $T_{ad} := T/Z(G)$, and $\tilde{T}_s := \tilde{G}_s \cap \tilde{T}$.

Let $B$ and $B^-$ be (mutually opposite) Borel subgroups of $G$, with a unique common maximal torus $T$; let $N_G(T)$ be its normalizer, and put $W = N_G(T)/T$ the corresponding Weyl group. Let $U$ and $U^-$ be the unipotent radicals of $B$ and $B^-$, respectively. We denote by $\tilde{B}$, $\tilde{T}$... the preimages of $B$, $T$... in $\tilde{G}$. 

For a torus $S$, we denote the weight lattice $\text{Hom}(S, \mathbb{G}_m)$ of $S$ by $X^*(S)$ and the coweight lattice $\text{Hom}(\mathbb{G}_m, S)$ of $S$ by $X_*(S)$. We put $X^*(S)_\mathbb{R} := X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. We regard $X^*(T)$ as a subset of $X^*(\tilde{T})$. We have a natural $\mathbb{Z}$-bilinear pairing

$$\langle , \rangle : X_*(T) \times X^*(\tilde{T}) \to \mathbb{Q}.$$ 

Let $\Delta \subset X^*(T)$ be the root system of $(G, T)$ and let $\Delta^+$ be its subset of positive roots defined by $B$. Let $g, b, \ldots$ be the Lie algebras of $G, B, T \ldots$ For each $\alpha \in \Delta$, we fix a root vector $e_\alpha \in g$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta^+$ be the set of simple roots. We denote by $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset X_*(\tilde{T}_s)$ the set of simple coroots, which we consider as a subset of $X_*(\tilde{T})$ via natural inclusion $X_*(\tilde{T}_s) \subset X_*(\tilde{T})$. The set of fundamental coweights $\{\omega_1^\vee, \omega_2^\vee, \ldots, \omega_\ell^\vee\}$ is defined as the set of $\mathbb{Z}$-linear forms on $X^*(T)$ such that $\omega_i^\vee(\alpha_j) = \delta_{i,j}$ ($1 \leq i, j \leq \ell$). Every $\omega_i^\vee$ defines an element of $X_*(T_{\text{ad}})$, which we also denote by $\omega_i^\vee$.

Let $\tilde{Z}(G)^\vee$ be the character group of $\tilde{Z}(G)$ and let $h$ be its order. Since $\tilde{Z}(G)$ is contained in $Z(G)$, we have natural surjection $X^*(\tilde{T}) \to \tilde{Z}(G)^\vee$. For each $\lambda \in X^*(\tilde{T})$, we denote its image in $\tilde{Z}(G)^\vee$ by $\tilde{\lambda}$. $\lambda \in X^*(\tilde{T})$ is called a dominant weight if, and only if, $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ for every $1 \leq i \leq \ell$. For a dominant weight $\lambda$, we denote by $W_\lambda$ the irreducible rational representation of $G$ with highest weight $\lambda$. Let $v_{w_0\lambda}$ be a lowest weight vector of $V_\lambda$, where $w_0$ is the longest element of $W$.

For every $\tau \in X_*(T)$, we define the following three Lie subalgebras of $g$:

$$l^\tau := t \oplus \bigoplus_{\alpha \in \Delta: (\tau, \alpha) = 0} ke_\alpha, u^\tau := \bigoplus_{\alpha \in \Delta: (\tau, \alpha) > 0} ke_\alpha, \text{ and } p^\tau := l^\tau \oplus u^\tau.$$ 

We also put $u^- := u^\tau$. We denote by $L^\tau, U^\tau$ the corresponding subgroups of $G$. $P^\tau$ is a parabolic subgroup of $G$. $P^\tau = L^\tau U^\tau$ is the Levi decomposition of $P^\tau$ such that $T \subset L^\tau$.

For a group $H$, we denote the diagonal embedding $H \hookrightarrow H \times H$ by $\Delta^d$. Also, we denote by $V^H$ the space of $H$-fixed vectors of $V$ for a $H$-module $V$.

### 1.2 Notation on partial compactifications

Here we assume that readers are familiar with the standard material in the theory of toric varieties (cf. Oda [23] Chapter 1). The contents in this subsection are found in the papers by De Concini-Procesi [12], Uzawa [34], and Knop’s survey [19].

For every fan $\Sigma$ in $X_*(T)_\mathbb{R}$, we denote by $\Sigma(n)$ the set of $n$-dimensional cones of $\Sigma$. By abuse of notation, we denote the fan consisting of a cone $\sigma$ of $\Sigma$ and its faces by the same letter $\sigma$. Moreover, we denote the integral generator of a one-dimensional cone $\tau$ by the same letter. (Thus, $\tau$ defines a one-parameter subgroup of $T$ when $\Sigma$ is a fan of $X_*(T)_\mathbb{R}$.)

We define a fan $\Sigma_0$ of $X_*(T_{\text{ad}})_\mathbb{R}$ by

$$\Sigma_0 := \left\{ \Sigma \in S_{\mathbb{R}}: \ominus \omega_i^\vee; S \subset \{1, 2, \ldots, \ell\} \right\}.$$ 

$\Sigma_0$ is the fan consisting of the dominant Weyl co-chamber and its faces.

Let $X_0$ be the wonderful compactification of $(G_{\text{ad}} \times G_{\text{ad}})/\Delta^d(G_{\text{ad}})$ in the sense of De Concini-Procesi [11]. We consider $X_0$ as a $G \times G$-equivariant compactification of $G_{\text{ad}}$ via the quotient map $G \times G \to G_{\text{ad}} \times G_{\text{ad}}$.

**Definition 1.1 (Regular embeddings).** A regular embedding $X$ of $G$ is a $G \times G$-equivariant smooth partial compactification of $X$ with the following conditions:

1. $X_0$ is the wonderful compactification of $(G_{\text{ad}} \times G_{\text{ad}})/\Delta^d(G_{\text{ad}})$.
2. $X_0$ has a natural $G_{\text{ad}} \times G_{\text{ad}}$-action.
3. $X_0$ is obtained by taking the quotient of $X_0$ by the $G_{\text{ad}} \times G_{\text{ad}}$-action.
4. $X$ is a $G \times G$-equivariant compactification of $X_0$.

1. $X \setminus G$ is a union of normal crossing divisors $D_1, \ldots, D_p$;
2. Each $D_i$ is smooth and is the closure of a single $G \times G$-orbit;
3. Every $G$-orbit closure in $X$ is a certain intersection of $D_1, \ldots, D_p$;
4. For each $x \in X$, the total space of the normal bundle of $(G \times G)x$ in $X$ contains a dense $G \times G$-orbit.

**Theorem 1.2 (Uzawa [34] 3.5. See also [12]).** Let $\Sigma$ be a fan of $X_*(T)_R$ such that the following two conditions hold:

1. The natural quotient map $q : X_*(T) \to X_*(T_{ad})$ yields a morphism $\Sigma \to \Sigma_0$ of fans;
2. The toric variety $T(\Sigma)$ corresponding to $\Sigma$ is nonsingular.

Then, there exists a unique regular embedding $X(\Sigma)$ of $G$ such that:

a. We have an embedding $T(\Sigma) \subset X(\Sigma)$;

b. Each $G \times G$-orbit in $X(\Sigma)$ intersects with a unique $T$-orbit in $T(\Sigma)$;

c. There exists a dominant $G \times G$-equivariant morphism $\pi : X(\Sigma) \to X_0$.

Its converse is also true. We denote by $O_\sigma$ the closure of a $G \times G$-orbit $O_\sigma$ corresponding to $\sigma \in \Sigma$. If $\tau \in \Sigma(1)$, we also denote $O_\tau$ by $D_\tau$. (We regard it as a prime divisor.)

Theorem 1.2 is more or less a consequence of the results in the references we list above and known by experts. However, since the author does not know an appropriate reference, we provide a proof. In the proof, we need the following modification of a theorem of Strickland, Theorem 1.3 which is obtained by pulling back the original version $(X(\Sigma) = X_0$ case) via $\pi$.

**Theorem 1.3 (Local structure theorem [30] cf. [6] 1.1 or [9]).** Under the same settings as in Theorem 1.2, the map

$$U^- \times T(\Sigma) \times U \longrightarrow X(\Sigma), \quad (g, x, h) \mapsto (g, h)x,$$

is an open embedding.

**Proof of Theorem 1.2.** We write the images of $B$ and $B^-$ in $G_{ad}$ by $B_{ad}$ and $B_{ad}^-$, respectively. Then $B_{ad}B_{ad}^- \subset G_{ad}$ is an open dense subset. Hence, $G_{ad} \cong G_{ad} \times G_{ad}/\Delta^d(G_{ad})$ and its compactification is a spherical variety. For a reductive group $K$ and its (mutually opposite) Borel subgroups $B_K$ and $B_K^-$ with a common torus $T_K$, we define $\Lambda_K$ as follows:

$$\Lambda_K := \left(k\left(\left(\left(K \times K\right)/\Delta^d(K)\right)^{B_K \times B_K^-}\right) - \{0\}\right)/k^*.$$

Here superscript $(B_K \times B_K^-)$ means the eigenpart with respect to the $B_K \times B_K^-$-action. Then, we have $\text{Hom}(\Lambda_K, \mathbb{Z}) \cong X_*(T_K)$. As is described in Brion [6] (2.2 Remarques i)), the wonderful compactification corresponds to the colored fan $(\Sigma_0, \emptyset)$ (= usual fan in $X_*(T_{ad})$). Then, by Knop [19] 3.3 and 4.1, we have a $G \times G$-equivariant (partial) compactification $X'(\Sigma)$ of $(G \times G)/\Delta^d(G)$ with a $G \times G$-equivariant dominant map

$$X'(\Sigma) \to X_0$$

for each fan $\Sigma$ satisfying 1. Here each cone of $\Sigma$ corresponds to a unique $G \times G$-orbit of $X'(\Sigma)$ by [19] 3.2 and 3.3. Since $\Lambda_G \cong \Lambda_T$, the closure of $T \subset X'(\Sigma)$ contains $T(\Sigma)$ by [19] 4.1. Since the smoothness is an open condition, Theorem 1.3 and the equality $(G \times G).T(\Sigma) = X'(\Sigma)$ asserts that $X'(\Sigma)$ is smooth if, and only if, $T(\Sigma)$ is smooth. Therefore, setting $X(\Sigma) := X'(\Sigma)$ completes the proof of Theorem 1.2.

\[\square\]
From now on, we always assume the assumptions of Theorem 1.2. For simplicity, we may write $X$ instead of $X (\Sigma)$.

Let $t_\sigma$ be the inclusion $\overline{O}_\sigma \to X (\Sigma)$ corresponding to $\sigma \in \Sigma$. We define $u^+_{\sigma} := \sum_{\tau \in \sigma(1)} u_+^\tau \subset g$. Similarly, we write $\mathfrak{t}^\sigma$ for $\cap_{\tau \in \sigma(1)} \mathfrak{t}^\tau$. Let $U_{\tau}^\sigma, U_{\sigma}^\tau$, and $L^\sigma$ be the algebraic subgroups of $G$ corresponding to $u^+_{\sigma}, u^+_{\sigma}$, and $\mathfrak{t}^\sigma$, respectively. Denote $L^\sigma U_{\tau}^\sigma$ by $P^\sigma$ (this is a parabolic subgroup of $G$).

Since $X (\Sigma)$ dominates $X_0$, the same arguments as in [11] §5.2 assert the existence of a $G \times G$-equivariant fibration $\pi_\sigma : \overline{O}_\sigma \to G/P^\sigma \times G/P^\sigma$. We denote the point of $X (\Sigma)$ corresponding to the identity element of $G$ by $e$. We have $T (\Sigma) \subset T.e \subset X$. Hence, $x_\sigma := \lim_{t \to \infty} (1 \times \prod_{\tau \in \sigma(1)} ^\tau(t)) e$ exists in $T (\Sigma)$ for each $\sigma \in \Sigma$. Let $G^\tau_m$ be the image of $G_m$ via $(1 \times \tau^{-1}) : G_m \to T \times T$ for each $\tau \in \Sigma(1)$. We denote by $G^\sigma$ the stabilizer of the $G \times G$-action at $x_\sigma$. Then we have

$$G^\sigma = \Delta^d (L^\sigma) \prod_{\tau \in \sigma(1)}^\tau G_{\sigma_m} (U_{\tau}^\sigma \times U_{\sigma}^\tau) = \Delta^d (L^\sigma) \prod_{\tau \in \sigma(1)}^\tau \text{Im} \tau \times 1 (U_{\tau}^\sigma \times U_{\sigma}^\tau).$$

This is another consequence of the fact that $X (\Sigma)$ dominates $X_0$.

For simplicity, we denote $\otimes_{\mathcal{O}_X}$ by $\otimes_X$, or even by $\otimes$ when there is no risk of confusion.

### 1.3 The category $\mathcal{C} (\Sigma)$

In order to formulate our main theorem, we introduce some notation and a category $\mathcal{C} (\Sigma)_c$, which contains the category $\mathcal{C} (X) = \mathcal{C} (X (\Sigma))$ in the introduction as a full subcategory. This enhancement is necessary in order to handle all line bundles on $X$ in the main theorem (cf. Theorem 2.7).

We denote the universal enveloping algebra of a Lie algebra $\mathfrak{a}$ by $U (\mathfrak{a})$. For each $\tau \in \Sigma(1)$ and every $n \in \mathbb{Z}$, we define

$$U (g)^\tau_n := \{ X \in U (g) ; \tau(t)X\tau(t)^{-1} = t^n X \text{ for every } t \in G_m(k) \cong k^\times \}.$$

We have $U (g) = \oplus_{n \in \mathbb{Z}} U (g)^\tau_n$. For each $\tau$-stable subalgebra $\mathfrak{f}$ of $g$, we put $U (f)^\tau_n := U (f) \cap U (g)^\tau_n$. Since the $\tau$-action on $g$ is semi-simple, we have $U (f) = \oplus_{n \in \mathbb{Z}} U (f)^\tau_n$. Every Lie algebra defined in [11] satisfies this property.

**Remark 1.4.** We have $U (\mathfrak{t}^\tau) = U (p^\tau)^\tau_0$ and $U (u^+_{\tau})^\tau_n = U (u^-_{\tau})^\tau_n = 0$ for all $n > 0$.

Now we introduce our linear algebra data, the category $\mathcal{C} (\Sigma)$. The most remarkable property of $\mathcal{C} (\Sigma)$ is the $(\tau)$-transversality condition.

**Definition 1.5.** Let $\tau \in \Sigma(1)$. Let $V$ be a $\tilde{G} \times \tilde{Z} (G)$-module. A $\tau$-standard filtration $F (\bullet)$ is a decreasing filtration of $V$ indexed by $\mathbb{Z}$ such that:

1. For each $n \in \mathbb{Z}$, $F (n)$ is a $\tilde{P}^\tau \times \tilde{Z} (G)$-module via restriction $\tilde{P}^\tau \subset \tilde{G}$;
2. $F (-n) = V$ and $F (n) = \{0\}$ for $n >> 0$.

A $\tau$-standard filtration $F (\bullet)$ is called an $\tau$-transversal filtration if, and only if, the following condition holds:

- **(Transversality condition)** For every $n, m \in \mathbb{Z}$, we have
  $$U (u^-_{\tau})^\tau_m F (n) \subset F (n + m).$$
Definition 1.6. Let $V$ be a finite dimensional vector space. A family of linear subspaces \( \{U_\lambda\}_{\lambda \in \Lambda} \) forms a distributive lattice if, and only if, there exists a basis $B$ of $V$ such that $B \cap U_\lambda$ is a basis of $U_\lambda$ for every $\lambda \in \Lambda$.

For a category $Z$, we denote the class of objects of $Z$ by $\text{Ob} Z$. Also, for every $\mathcal{X}, \mathcal{Y} \in \text{Ob} Z$, we denote the set of morphisms of $\mathcal{X}$ to $\mathcal{Y}$ by $\text{Hom}_Z (\mathcal{X}, \mathcal{Y})$.

Definition 1.7 (Category $\mathcal{C}(\Sigma)_c$). Let $\mathcal{C}(\Sigma)_c$ and $\mathcal{C}(\Sigma)_c^l$ be categories defined as follows:

(Objects) We define $\text{Ob} \mathcal{C}(\Sigma)_c$ as pairs \( (V, \{F^\tau (\bullet)\}_{\tau \in \Sigma(1)}) \) such that the following three conditions hold:

1. $V$ is a $\tilde{G} \times \tilde{Z} (G)$-module;
2. For each $\tau \in \Sigma(1)$, $F^\tau (\bullet)$ is an $\tau$-standard filtration of $V$;
3. For each $\sigma \in \Sigma$, a family of subspaces $\{F^\tau (n)\}_{\tau \sigma(1), n \in \mathbb{Z}}$ of $V$ forms a distributive lattice.

Let $\mathcal{C}(\Sigma)_c$ be the full-subcategory of $\mathcal{C}(\Sigma)_c^l$ obtained by replacing condition 2. by the following stronger condition:

4. For each $\tau \in \Sigma(1)$, $F^\tau (\bullet)$ is an $\tau$-transversal filtration of $V$.

(Morphisms) For \( (V_1, \{F^\tau_1 (\bullet)\}_{\tau \in \Sigma(1)}), (V_2, \{F^\tau_2 (\bullet)\}_{\tau \in \Sigma(1)}) \in \text{Ob} \mathcal{C}(\Sigma)_c^l \), we define

\[
\text{Hom}_{\mathcal{C}(\Sigma)_c^l} \left( (V_1, \{F^\tau_1 (\bullet)\}_{\tau \in \Sigma(1)}), (V_2, \{F^\tau_2 (\bullet)\}_{\tau \in \Sigma(1)}) \right) := \{ f \in \text{Hom}_{\tilde{G} \times \tilde{Z}(G)} (V_1, V_2) : f (F^\tau_1 (n)) \subseteq F^\tau_2 (n) \text{ for every } n \in \mathbb{Z} \text{ and every } \tau \in \Sigma(1) \}.
\]

For simplicity, we may write $(V, \{F^\tau (\bullet)\}_{\tau \in \Sigma(1)})$ instead of $(V, \{F^\tau (\bullet)\}_{\tau \in \Sigma(1)})$.

Remark 1.8. Replacing $\tilde{G} \times \tilde{Z}(G)$-modules in Definition 1.7 by $G \times 1$-modules, we obtain the two fullsubcategories $\mathcal{C}(X)^+ \subseteq \mathcal{C}(\Sigma)_c^l$ and $\mathcal{C}(X) \subseteq \mathcal{C}(\Sigma)_c$ defined in the introduction.

Definition 1.9 (Category $\mathcal{C}(\Sigma)$). We define subcategories $\mathcal{C}(\Sigma) \subseteq \mathcal{C}(\Sigma)_c$ and $\mathcal{C}(\Sigma)^l \subseteq \mathcal{C}(\Sigma)_c^l$ as follows:

(Objects) $\text{Ob} \mathcal{C}(\Sigma) := \text{Ob} \mathcal{C}(\Sigma)_c$ and $\text{Ob} \mathcal{C}(\Sigma)^l := \text{Ob} \mathcal{C}(\Sigma)_c^l$.

(Morphisms) For $(V_1, \{F^\tau_1 (\bullet)\}), (V_2, \{F^\tau_2 (\bullet)\}) \in \text{Ob} \mathcal{C}(\Sigma)^l$, we define

\[
\text{Hom}_{\mathcal{C}(\Sigma)^l} \left( (V_1, \{F^\tau_1 (\bullet)\}), (V_2, \{F^\tau_2 (\bullet)\}) \right) := \{ f \in \text{Hom}_{\tilde{G} \times \tilde{Z}(G)} (V_1, V_2) : f \text{ satisfies the conditions (L) and (R), where} \}
\]

\[
\begin{align*}
(L) & \quad f (F^\tau_1 (n)) = f (V_1) \cap F^\tau_2 (n) \text{ for every } n \in \mathbb{Z} \text{ and every } \tau \in \Sigma(1); \\
(R) & \quad \{ f (V_1), \{F^\tau_2 (n)\}_{\tau \sigma(1), n \in \mathbb{Z}} \} \text{ forms a distributive lattice for every } \sigma \in \Sigma.
\end{align*}
\]

We regard $\mathcal{C}(\Sigma)$ as a fullsubcategory of $\mathcal{C}(\Sigma)^l$.

1.4 The category $\text{EV}(\Sigma)$ and the map $\Xi$

For a $G$-variety $Y$, we denote the category of $G$-equivariant coherent sheaves on $Y$ by $\text{Coh}^G X(Y)$.

Definition 1.10 (Category of equivariant bundles). We define two subcategories $\text{EV}(\Sigma)$ and $\text{EV}(\Sigma)_c$ of $\text{Coh}^G \tilde{G} \times \tilde{G} X(\Sigma)$ as follows:

- (Objects) We define $\text{Ob} \text{EV}(\Sigma)$ and $\text{Ob} \text{EV}(\Sigma)_c$ as

\[
\{ \tilde{G} \times \tilde{G} \text{-equivariant vector bundles on } X(\Sigma) \}.
\]
• (Morphisms) We define the morphisms of $EV(\Sigma)_c$ as the morphisms of $\tilde{G} \times \tilde{G}$-equivariant coherent $\mathcal{O}_{X(\Sigma)}$-modules. We define the morphisms of $EV(\Sigma)$ as morphisms in $EV(\Sigma)_c$ such that both their kernel and cokernel exist in $EV(\Sigma)$. We call a morphism of $EV(\Sigma)$ a $(\tilde{G} \times \tilde{G}$-equivariant) vector bundle morphism.

Remark 1.11. The category $EV(\Sigma)_c$ introduced above contains the category $EV(X) = EV(X(\Sigma))$ in the introduction as a fullsubcategory.

We denote the total space of a $\tilde{G} \times \tilde{G}$-equivariant vector bundle $\mathcal{E}$ by $V(\mathcal{E})$.

Definition 1.12. For each $\tilde{G} \times G$-equivariant vector bundle $\mathcal{E}$, we define a pair $\Xi(\mathcal{E})$ as follows.

$$\Xi(\mathcal{E}) := (B(\mathcal{E}), \{F^\tau(n)\}_{\tau \in \Sigma(1), n \in \mathbb{Z}})$$

Here $B(\mathcal{E})$ and $F^\tau(n)$ are vector spaces such that:

- $B(\mathcal{E}) := \mathcal{E} \otimes_X k(e)$ is the identity fiber of $\mathcal{E}$;
- $F^\tau(n) := \{v \in B(\mathcal{E}) : \exists \lim_{t \to \infty} t^n (1 \times \tau(t)) v \in V(\mathcal{E})\}$ for every $\tau \in \Sigma(1)$ and every $n \in \mathbb{Z}$.

In particular, $F^\tau(\bullet)$ is a decreasing filtration of $V$ for each $\tau \in \Sigma(1)$.

A refinement of $\Xi$ gives an equivalence of $EV(\Sigma)_c$ to $\mathcal{E}(\Sigma)_c$ in 

2 Foundational results

2.1 Preliminaries

2.1.1 Isotypical decompositions

Let $X = X(\Sigma)$ be the $G \times G$-equivariant partial compactification of $G$ associated with $\Sigma$. Then, the natural projection $\tilde{G} \times \tilde{G} \to G \times G$ induces a $\tilde{G} \times \tilde{G}$-action on $X(\Sigma)$. Thus, by the definition of $\tilde{Z}(G)$, the $1 \times \tilde{Z}(G)$-action on $X(\Sigma)$ is trivial. Let $\mathcal{E}$ be a $1 \times \tilde{Z}(G)$-equivariant vector bundle on $X(\Sigma)$ (e.g. a $\tilde{G} \times \tilde{G}$-equivariant vector bundle). Then, $1 \times \tilde{Z}(G)$ operates on $\mathcal{E} \otimes_X k(x)$ for every $x \in X(\Sigma)$. For each $\chi \in \tilde{Z}(G)^\vee$, we define a sheaf of abelian groups $\mathcal{E}_\chi$ on $X(\Sigma)$ as follows:

$$\mathcal{E}_\chi(U) := \text{Hom}_{1 \times \tilde{Z}(G)}(k \boxtimes \chi, \mathcal{E}(U)) \quad \text{for every Zariski open subset } U \subset X(\Sigma).$$

Then, $\mathcal{E}_\chi$ is an $\mathcal{O}_X$-submodule of $\mathcal{E}$. The following lemmas easily follow from standard representation theory.

Lemma 2.1. Let $\mathcal{E}$ and $\mathcal{F}$ be $1 \times \tilde{Z}(G)$-equivariant vector bundles. Choose $\chi, \xi \in \tilde{Z}(G)^\vee$. If $\chi \neq \xi$, then we have $\text{Hom}_{(\mathcal{O}_X, 1 \times \tilde{Z}(G))}((\mathcal{E}_\chi, \mathcal{F}_\xi)) = 0$. □

Lemma 2.2. Every $\tilde{G} \times \tilde{G}$-equivariant vector bundle $\mathcal{E}$ admits the following isotypical decomposition:

$$\mathcal{E} \cong \oplus_{\chi \in \tilde{Z}(G)^\vee} \mathcal{E}_\chi.$$

Moreover, each $\mathcal{E}_\chi$ is a $\tilde{G} \times \tilde{G}$-equivariant vector bundle. □

Definition 2.3 (Isotypical components of the category $EV(\Sigma)_c$). For each $\chi \in \tilde{Z}(G)^\vee$, we define two categories $EV(\Sigma, \chi)_c$ and $EV(\Sigma, \chi)$ as follows:

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• (Objects) We define $\text{Ob}EV(\Sigma, \chi)_c$ and $\text{Ob}EV(\Sigma, \chi)$ by

$$\{E \in \text{Ob}EV(\Sigma)_c : \mathcal{E}_\chi \cong \mathcal{E}\}.$$  

• (Morphisms) We regard $EV(\Sigma, \chi)_c$ and $EV(\Sigma, \chi)$ as full-subcategories of $EV(\Sigma)_c$ and $EV(\Sigma)$, respectively.

Notice that an object of $EV(\Sigma, 1)$ can be identified with a $\tilde{G} \times G$-equivariant vector bundle on $X$.

**Corollary 2.4 (Isotypical decomposition of $EV(\Sigma)_c$).** We have a direct sum decomposition

$$EV(\Sigma)_c \cong \bigoplus_{\chi \in \tilde{Z}(G)^\vee} EV(\Sigma, \chi)_c$$

as categories.

**Proof.** This is a direct consequence of Lemma 2.1 and Lemma 2.2. □

**Definition 2.5 (Isotypical components of the category $C(\Sigma)_c$).** For each $\chi \in \tilde{Z}(G)^\vee$, we define three categories $C(\Sigma, \chi)_c$, $C(\Sigma, \chi)_l$, and $C(\Sigma, \chi)_l^{\text{tr}}$ as follows:

• (Objects) We define $\text{Ob}C(\Sigma, \chi)_c$ and $\text{Ob}C(\Sigma, \chi)$ by

$$\{(V, \{F^\tau(\bullet)\}) \in \text{Ob}C(\Sigma)_c : \text{Hom}_{1 \times \tilde{Z}(G)}(k \otimes \chi, V) \cong V\}.$$  

We also define $\text{Ob}C(\Sigma, \chi)_l^{\text{tr}}$ by

$$\{(V, \{F^\tau(\bullet)\}) \in \text{Ob}C(\Sigma)_c : \text{Hom}_{1 \times \tilde{Z}(G)}(k \otimes \chi, V) \cong V\}.$$  

• (Morphisms) We consider $C(\Sigma, \chi)_c$, $C(\Sigma, \chi)_l$, and $C(\Sigma, \chi)_l^{\text{tr}}$ as full-subcategories of $C(\Sigma)_c$, $C(\Sigma)_c$, and $C(\Sigma)_c$, respectively.

We have an analogous decomposition to that of Corollary 2.4 for $C(\Sigma)_c$.

**Corollary 2.6 (Isotypical decomposition of $C(\Sigma)_c$).** We have a direct sum decomposition

$$C(\Sigma)_c \cong \bigoplus_{\chi \in \tilde{Z}(G)^\vee} C(\Sigma, \chi)_c$$

as categories.

**Proof.** The assertion follows directly from the complete reducibility of a $\tilde{Z}(G)$-module. □

### 2.1.2 Some lemmas about equivariant structures

For equivariant structures on line bundles, we have the following celebrated result of Steinberg.

**Theorem 2.7 (Steinberg [29] and GIT [22] 1.4-1.6).** Assume that a connected, simply connected semisimple linear algebraic group $K$ acts on a normal projective algebraic variety $Y$. Then, every line bundle on $Y$ admits a unique $K$-linearization. □
Until Lemma 2.8, we assume that $X$ is complete. In vector bundle case, a naive extension of Theorem 2.7 is false. For an arbitrary irreducible $\tilde{G}$-module $V$ of dimension $\geq 2$, $V \times X$ has at least three $\tilde{G} \times \tilde{G}$-equivariant vector bundle structures. One is given by

$$(\tilde{G} \times \tilde{G}) \times (V \times X) \ni (g_1, g_2, v, x) \mapsto (v, (g_1, g_2).x) \in V \times X,$$

that is $\tilde{G} \times \tilde{G}$-equivariantly isomorphic to a direct sum $\mathcal{O}_X^{\oplus \dim V}$ of trivial line bundles. The others are given by

$$(\tilde{G} \times \tilde{G}) \times (V \times X) \ni (g_1, g_2, v, x) \mapsto (g_1.v, (g_1, g_2).x) \in V \times X,$$ \hspace{1cm} (2.1.1)

$$(\tilde{G} \times \tilde{G}) \times (V \times X) \ni (g_1, g_2, v, x) \mapsto (g_2.v, (g_1, g_2).x) \in V \times X. \hspace{1cm} (2.1.2)$$

We denote the $\tilde{G} \times \tilde{G}$-equivariant vector bundles \ref{2.1.1} and \ref{2.1.2} by $V \otimes \mathcal{O}_X$ and $\mathcal{O}_X \otimes V$, respectively. Moreover, we also write $V \otimes \mathcal{L}$ and $\mathcal{L} \otimes V$ for their twist by a $\tilde{G} \times \tilde{G}$-equivariant line bundle $\mathcal{L}$.

Hence, to obtain a proper analogue of Steinberg’s theorem for vector bundles, we must impose some auxiliary condition.

**Lemma 2.8.** Let $\mathcal{E} \hookrightarrow \mathcal{F}$ be an inclusion of vector bundles on $X$ as coherent $\mathcal{O}_X$-modules. Assume that both $\mathcal{E}$ and $\mathcal{F}$ have $\tilde{G} \times \tilde{G}$-equivariant structures such that $\mathcal{E}|_{\tilde{G} \times \tilde{G}} \cong \mathcal{F}|_{\tilde{G} \times \tilde{G}}$ as $\tilde{G} \times \tilde{G}$-equivariant $\mathcal{O}_{\tilde{G}}$-modules. Then, the $\tilde{G} \times \tilde{G}$-equivariant structure of $\mathcal{E}$ is given by the restriction of that of $\mathcal{F}$. □

**Corollary 2.9.** Let $\mathcal{E}$ be a vector bundle on $X$. If we fix a compatible $(\tilde{G} \times \tilde{G}, k[\Gamma(G)])$-module structure on $\mathcal{E}(G) (= \Gamma(G, \mathcal{E}))$, then $\tilde{G} \times \tilde{G}$-equivariant vector bundle structure on $\mathcal{E}$ is unique if it exists. □

Let $\{0\}$ be the fan consisting of a unique cone $\{0\} \subset X*(T)_R$.

**Lemma 2.10.** We have an equivalence of categories $\text{Ind} : \mathcal{C}(\{0\}) \cong \mathcal{E}(\{0\})$. Its inverse functor is given by the restriction to the fiber at $e$.

**Proof.** We have $X(\{0\}) = G$. In particular, $X(\{0\})$ is a homogeneous space under $\tilde{G} \times \tilde{G}$-action. Its isotropy group at $e$ is isomorphic to $\tilde{G} \times \tilde{Z}(G)$ by definition, the set of one-dimensional cones of $\{0\}$ is an empty set. Hence, we have $\mathcal{C}(\{0\}) \cong \text{Rep}\tilde{G} \times \tilde{Z}(G)$. Thus, the desired equivalence is standard (cf. Chriss and Ginzburg [10], 5.2.16). □

**Lemma 2.11.** Assume that $X = G$ (i.e. $\Sigma = \{0\}$). Let $\text{Ind}$ be as in Lemma 2.10. Then, we have

$$\text{Ind}(V_\lambda \otimes k) \cong V_\lambda \otimes \mathcal{O}_G \cong \text{Ind}(k \boxtimes \tilde{\lambda}^{-1}) \otimes V_\lambda$$

for every $\lambda \in X^*(T)$. Here $\tilde{\lambda}$ is the image of $\lambda$ in $\tilde{Z}(G)^\vee$ as in 1.3.

**Proof.** As a $\tilde{G} \times \tilde{Z}(G)$-module, we have $V_\lambda \otimes \mathcal{O}_X \boxtimes X k(e) \cong V_\lambda \otimes k$. Hence, we have the first isomorphism. Next, we prove $\text{Ind}(V_\lambda \otimes k) \cong \text{Ind}(k \boxtimes \tilde{\lambda}^{-1}) \otimes V_\lambda$ to complete the proof. $k \boxtimes \tilde{\lambda}^{-1}$ is trivial as a $\tilde{G} \times 1(\subset \tilde{G} \times \tilde{Z}(G))$-module. Thus, we have a $\tilde{G} \times 1$-module isomorphism $\text{Ind}(k \boxtimes \tilde{\lambda}^{-1}) \otimes V_\lambda \otimes X k(e) \cong V_\lambda$. Here the action of $1 \times \tilde{Z}(G) \subset \tilde{G} \times \tilde{Z}(G)$ is the right action. This action is trivial since $\tilde{Z}(G)$ acts by $-\tilde{\lambda} + \tilde{\lambda} = 0$. As a result, we have $\text{Ind}(V_\lambda \otimes k) \cong \text{Ind}(k \boxtimes \tilde{\lambda}^{-1}) \otimes V_\lambda$. □

\[\text{12}\]
2.1.3 Equivariant divisors

In this section, we introduce and explain the notion of equivariant divisors. First, we present their definition. Recall that $O_\sigma$ is the closure of the $G \times G$-orbit of $X$ corresponding to $\sigma \in \Sigma$.

**Definition 2.12 (Equivariant Divisors).** An equivariant divisor is a $\mathbb{Z}$-linear formal sum of $D_\tau$ for every $\tau \in \Sigma(1)$. For two equivariant divisors $D_1 = \sum n_\tau^1 D_\tau$ and $D_2 = \sum n_\tau^2 D_\tau$, we say $D_1 \geq D_2$ if, and only if, we have $n_\tau^1 \geq n_\tau^2$ for every $\tau \in \Sigma(1)$. Moreover, $D_1$ is said to be sufficiently large if, and only if, $n_\tau^1 \gg 0$ for every $\tau \in \Sigma(1)$.

Before we exploit some properties of equivariant divisors, we need a result.

**Lemma 2.13.** There exists a fan $\Sigma_+$ such that $X(\Sigma_+)$ is a complete regular embedding of $G$ which (equivariantly) contains $X(\Sigma)$.

**Proof.** By the theory of spherical embeddings, we have a $G \times G$-equivariant complete embedding $Y$ of $G$ which contains $X(\Sigma)$ (cf. [19]). Then, we successively blow-up along the singular locus (which is $G \times G$-stable) to obtain a regular embedding. Now Theorem 1.2 gives the result. \qed

We fix one $\Sigma_+$ of Lemma 2.13 hereafter.

**Theorem 2.14 (See [6] 2.2, [4] 2.4, and [18] 2.1-2.2).** We have the following:

1. For each $\sigma \in \Sigma(r)$, the restriction map

$$\iota_\sigma^* : \bigoplus_{\tau \in \sigma(1)} \mathbb{Z}D_\tau \rightarrow \text{Pic}^{1 \times \tilde{G}}O_\sigma$$

is an injection.

2. We have the following short exact sequence.

$$0 \rightarrow \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z}D_\tau \rightarrow \text{Pic}^{1 \times \tilde{G}}X(\Sigma) \xrightarrow{\kappa} \tilde{Z}(G)^{\vee} \rightarrow 0$$

3. For each equivariant divisor $D = \sum_{\tau \in \Sigma(1)} n_\tau D_\tau$, $G_m^\tau$ acts on $O_X(D) \otimes_X k(x_\tau)$ by weight $-n_\tau$. ($G_m^\tau$ is the image of $G_m$ under $1 \times \tau^{-1}$. See [1,2])

Here $\text{Pic}^{1 \times \tilde{G}}$ is the $1 \times \tilde{G}$-equivariant Picard group. Moreover, the image of $\kappa$ determines the right $1 \times \tilde{Z}(G)(\subset \tilde{G} \times \tilde{Z}(G))$-module structure of the identity fiber.

**Remark 2.15.** The origin of Theorem 2.14 is somewhat complicated. If $G$ is an adjoint semisimple group and $X$ is complete, the above formulation is essentially due to Bifet [4]. For another extreme of our scope, namely toric varieties, the description of the equivariant Picard group is a corollary of Klyachko’s theorem [18]. In the meantime, Brion [6] established a general description of Picard groups using $B \times B^-$-orbits. Hence, Theorem 2.14 is essentially a corollary of his result (but not a direct consequence). Anyway, since the author could not find a proper reference to this form of the theorem, we provide a proof.

For the proof of Theorem 2.14 we need some preparation.

The following is a modification of Strickland’s Theorem [30] 2.4 to our setting. It is easily deduced from [30] 2.4 by twisting a character of $T_0 \times T_0$. 

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Theorem 2.16 (Strickland’s theorem cf. [30] 2.4). Let \( p_0 \) be the unique \( B \times B^- \)-fixed point in \( X_0 \). For each \( \lambda \in X^*(\tilde{T}) \), there exists a \( \tilde{G} \times \tilde{G} \)-equivariant line bundle \( L_\lambda \) on \( X \) such that \( L_\lambda \otimes \mathcal{O}_{X_0} k(p_0) \cong \lambda^{-1} \otimes \kappa \) as \( \tilde{T} \times \tilde{T} \)-modules. Moreover, \( L_\lambda \) is unique up to isomorphism.

Recall the morphism \( \pi : X(\Sigma) \rightarrow X_0 \) introduced in Theorem 1.2.

Corollary 2.17. For each \( \sigma \in \Sigma(r) \), the map

\[ j_\sigma : \text{Pic}^{1 \times \tilde{G}} X(\Sigma) \ni L \mapsto L \otimes_X k(x_\sigma) \in X^*(\tilde{T}) \cong \text{Rep}(1 \times \tilde{T}) \]

is a surjection.

Proof. \( O_\sigma \) is a closed \( G \times G \)-orbit of \( X(\Sigma) \) with dimension \( \dim G - r \). Thus, \( \pi(O_\sigma) \) is the closed \( G \times G \)-orbit of \( X_0 \). As a result, \( \pi(x_\sigma) \) is the unique \( B \times B^- \)-fixed point of \( X_0 \). Hence, for each \( \lambda \in X^*(\tilde{T}) \), we have \( j_\sigma(\pi^* L_\lambda) = \lambda \).

Next, we review the key proposition of [13].

Theorem 2.18 (Klyachko [18] 2.1.1). For each \( \sigma \in \Sigma \), we have the following:

1. Every \( 1 \times T \)-equivariant vector bundle \( E \) on \( T(\sigma) \) is uniquely written as \( E \times T(\sigma) \) by a \( T \)-module \( E \). (Here \( T \) acts on both \( E \) and \( T(\sigma) \).)
2. Let \( E, F \) be \( T \)-modules. Then, two \( 1 \times T \)-equivariant vector bundles \( E \times T(\sigma) \) and \( F \times T(\sigma) \) are isomorphic if, and only if, \( E \cong F \) as \( T \)-modules.
3. Let \( E \) be a \( T \)-module. Let \( E = \bigoplus_{\lambda \in X^*(\tilde{T})} E^\lambda \) be its \( T \)-isotypical decomposition. Choose \( v \in E^\lambda \subset E \otimes k(\epsilon) \) and \( \tau \in \Sigma(1) \). Then,

\[ \lim_{t \rightarrow \infty} t^n (1 \times \tau(t))v \]

exists in \( E \times T(\sigma) \) if, and only if, \( \langle \tau, \lambda \rangle \leq -n \).

Proof of Theorem 2.14 First, we prove 1). We restrict our attention to \( T(\Sigma) \). Let \( \{\tau_1, \tau_2, \ldots, \tau_r\} \) be the \( \mathbb{Z} \)-basis of \( X_*(T) \) which spans \( \sigma \). Let \( \{\mu_1, \mu_2, \ldots, \mu_r\} \) be its dual basis (i.e. \( \langle \tau_i, \mu_j \rangle = \delta_{i,j} \)). By Theorem 1.3 \( T(\sigma) \cap D_\tau \) is a \( 1 \times T \)-stable divisor if \( \tau \in \sigma(1) \). Moreover, by the definition of the coordinate ring of toric varieties [23] 1.2, we have \( \mathcal{O}_X(D_{\tau_1}) \otimes_X k(x_\sigma) = \mu_i \) as \( \tilde{T} \)-module for each \( 1 \leq i \leq r \). As a result, the image of \( D_{\tau_1} \) under the composition map \( \bigoplus_{i=1}^r \mathbb{Z} D_{\tau_1} \rightarrow \text{Pic}^{1 \times \tilde{G}} X(\Sigma) \rightarrow X^*(\tilde{T}) \) is \( \mu_i(\epsilon X^*(\tilde{T})) \) for each \( i \). In particular, 1) follows. Combined with Theorem 2.18 1), we have 3) if \( \Sigma = \Sigma^+ \). Thus, 3) follows from Lemma 2.13 by restriction.

Now we prove 2). If we have \( \sum_{\tau \in \Sigma(1)} a_\tau D_\tau \sim 0 \) in \( \text{Pic}^{1 \times \tilde{G}} X(\Sigma^+) \), then there exist \( f \in k[\tilde{G}]^\times \) such that \( \sum_{\tau \in \Sigma(1)} a_\tau D_\tau = \text{div}(f) \). By a theorem of Rosenlicht, we have \( k[\tilde{G}]^\times = k^X X^*(\tilde{G}) \). Hence, 1) yields \( a_\tau = 0 \) for every \( \tau \in \Sigma(1) \). Therefore, we have a short exact sequence

\[ 0 \rightarrow \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z} D_\tau \rightarrow \text{Pic}^{1 \times \tilde{G}} X(\Sigma^+) \rightarrow \text{Pic}^{1 \times \tilde{G}} G \rightarrow 0. \]

We have \( \text{Pic}^{1 \times \tilde{G}} G \cong \tilde{Z} G^\vee \). By restricting the above short exact sequence to \( X(\Sigma) \), we obtain 2) and the assertion about \( \kappa \).

By Theorem 2.14 the pullback of the line bundle \( L_\lambda \) defined in Theorem 2.16 corresponds to an element \( D^\lambda \) of \( \text{Pic}^{1 \times \tilde{G}} X(\Sigma) \). We denote the \( \tilde{G} \times \tilde{G} \)-equivariant line bundle \( \pi^* L_\lambda \) by \( L_\lambda \) (when we want to stress characters) or \( \mathcal{O}_X(D^\lambda) \) (when we want to stress divisors).

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Therefore, we have the result if $\Sigma = \Sigma + D$ the above arguments. As a result, we conclude general case follows.

Hence,

$$\oplus \text{Pic}$$

surjection. By Theorem 2.14 2) (applied to the case $\Sigma = \Sigma + L$ Lemma 2.21.

2.2.1 Bounding an equivariant bundle by standard ones

In this subsection, we rewrite the definition of the functor $\Xi$ in order to handle subtle behaviors at the boundaries. Main ingredients in this subsection are the introduction of another

$\Xi$.

2.2 Redefinition of the functor $\Xi$

In this subsection, we rewrite the definition of the functor $\Xi$ in order to handle subtle behaviors at the boundaries. Main ingredients in this subsection are the introduction of another functor $\Xi'$ (Definition 2.20 and Definition 2.28), the proof of $\Xi = \Xi'$ (Proposition 2.30), and a description of $\Xi$ (Proposition 2.35).

2.2.1 Bounding an equivariant bundle by standard ones

Lemma 2.21. Let $E$ be a $\tilde{G} \times \hat{G}$-equivariant vector bundle. Assume that we have $V \boxtimes k \subset \Gamma (X, E)$ (as $\tilde{G} \times \hat{G}$-modules) for some $\hat{G}$-module $V$. Then we have a unique $\hat{G} \times \tilde{G}$-equivariant inclusion $V \otimes \mathcal{O}_X \hookrightarrow E$ such that its image is generated by $V \boxtimes k$.

Proof. There exists a $\tilde{G}$-module $Z$ such that we have $Z \boxtimes k \cong E_1 \otimes_X k(e)$ as $\tilde{G} \times \tilde{Z}(G)$-modules (cf. Lemma 2.10). Then, we have $Z \otimes \mathcal{O}_G \cong E_1 | G$ by Lemma 2.11. By looking at $1 \times \tilde{Z}(G)$-action, we have $\Gamma (G, E)^{1 \times \tilde{G}} = \Gamma (G, E_1)^{1 \times \tilde{G}}$. (Here superscript $1 \times \tilde{G}$ means the fixed part as
in \(\{1,1\}\) Hence, we have \(\Gamma (G, \mathcal{E})^{1 \times \tilde{G}} \cong Z \boxtimes k\) as \(\tilde{G} \times G\)-modules. As a consequence, we have an induced inclusion \(V \boxtimes k \subset Z \boxtimes k\) as \(\tilde{G} \times G\)-submodules of \(\Gamma (X, \mathcal{E})\). Thus, the induced \(\mathcal{O}_X\)-module morphism \(\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{E}\) is a \(\tilde{G} \times G\)-equivariant inclusion when restricted to \(G\) from Lemma \(2.11\) and Lemma \(2.10\). Hence, Lemma \(2.8\) yields the result.

For a \(\tilde{G} \times G\)-equivariant vector bundle \(\mathcal{E}\) and an equivariant divisor \(D\), we often write \(\mathcal{E}(D)\) instead of \(\mathcal{E} \otimes_X \mathcal{O}_X(D)\).

**Corollary 2.22.** Let \(\mathcal{E}\) be a \(\tilde{G} \times G\)-equivariant vector bundle. Let \(W\) be a \(\tilde{G}\)-module. Assume that we have \(V \boxtimes k = \Gamma (G, \mathcal{E})^{1 \times \tilde{G}}\) for some \(G\)-module \(V\). Then, for a sufficiently large equivariant divisor \(D\), every \(\tilde{G} \times G\)-equivariant morphism \(W \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{E}\) factors through \(V \otimes \mathcal{O}_X(-D)\). In particular, we have a natural isomorphism

\[
\text{Hom}_{\mathcal{E}V(\Sigma,1)_c}(W \otimes \mathcal{O}_X (-D), \mathcal{E}) \cong \text{Hom}_{\tilde{G}}(W, V).
\]

**Proof.** Let us choose a (sufficiently large) equivariant divisor \(D_0\) such that \(V \boxtimes k \subset \Gamma (X, \mathcal{E}(D_0))\) as a \(\tilde{G} \times G\)-submodule. We have \(V \boxtimes k \subset \Gamma (X, \mathcal{E}(D))\) for all \(D \geq D_0\). Thus, we have \(V \boxtimes k = \Gamma (X, \mathcal{E}(D))^{1 \times \tilde{G}}\). By Lemma \(2.21\) we have a \(\tilde{G} \times G\)-equivariant inclusion \(V \otimes \mathcal{O}_X \subset \mathcal{E}(D)\). A \(\tilde{G} \times G\)-equivariant morphism \(W \otimes \mathcal{O}_X \rightarrow \mathcal{E}(D)\) defines a \(\tilde{G} \times G\)-module morphism between their sections \(\Gamma (X, W \otimes \mathcal{O}_X) = W \boxtimes k \rightarrow V \boxtimes k = \Gamma (X, \mathcal{E}(D))^{1 \times \tilde{G}}\). Since \(W \otimes \mathcal{O}_X\) is generated by its global sections, every \(\tilde{G} \times G\)-equivariant morphism \(W \otimes \mathcal{O}_X \rightarrow \mathcal{E}(D)\) factors through \(V \otimes \mathcal{O}_X\). Therefore, twisting by \(\mathcal{O}_X (-D)\) completes the proof.

From now on, we sometimes deal with \(\tilde{G} \times G\)-equivariant vector bundles (= objects in \(\mathcal{E}V(\Sigma,1)\)). This strange restriction comes from two rather technical reasons. On one hand, the representation categories of \(\tilde{G}\) and \(G\) are essentially different. Hence, we cannot forget the \(\tilde{G}\)-action completely. On the other hand, we use equivariant divisors which correspond only to \(G \times G\)-equivariant line bundles.

For each \(\tilde{G} \times G\)-equivariant vector bundle \(\mathcal{E}\) and an equivariant divisor \(D\), we denote the sheaf \(B(\mathcal{E}) \otimes_X \mathcal{O}_X(D)\) by \(B(\mathcal{E})^D\).

**Corollary 2.23.** For every \(\mathcal{E} \in \text{Ob} \mathcal{E}V(\Sigma,1)_c\), there exists a sufficiently large equivariant divisor \(D_0\) such that:

- \(B(\mathcal{E})^{-D} \subset \mathcal{E} \subset B(\mathcal{E})^D\) are inclusions of \(\tilde{G} \times G\)-equivariant coherent sheaves for every \(D \geq D_0\);

- The composition map \(B(\mathcal{E})^{-D} \subset B(\mathcal{E})^D\) is the tensor product of the identity map of \(B(\mathcal{E})\) and a unique \(\tilde{G} \times G\)-equivariant inclusion \(\mathcal{O}_X (-D) \subset \mathcal{O}_X(D)\).

Moreover, such a series of inclusions is unique up to automorphism of \(B(\mathcal{E})\) as a \(\tilde{G}\)-module.

**Proof.** We have \(B(\mathcal{E}) \boxtimes k \subset B(\mathcal{E}) \otimes k[G] \cong \Gamma (G, \mathcal{E})\) as \(\tilde{G} \times G\)-modules by the algebraic Peter-Weyl theorem and Lemma \(2.11\). Hence, there exists a sufficiently large equivariant divisor \(D_0\) such that we have \((B(\mathcal{E}) \boxtimes k) \subset \Gamma (X, \mathcal{E}(D))\) for every \(D \geq D_0\). Thus, we have a \(\tilde{G} \times G\)-equivariant embedding \(B(\mathcal{E})^{-D} \subset \mathcal{E}\) by Lemma \(2.21\). Moreover, such an embedding is uniquely determined up to an automorphism of \(B(\mathcal{E})\) as a \(\tilde{G}\)-module. Next, we apply the same argument to \(\mathcal{E}^\vee\). Then, enlarge \(D_0\) if necessary, we obtain \(B(\mathcal{E})^* \otimes \mathcal{O}_X(-D) \subset \mathcal{E}^\vee\) for every \(D \geq D_0\). Hence, we have a \(\tilde{G} \times G\)-equivariant inclusion \(\mathcal{E} \subset B(\mathcal{E})^D\) since the both vector bundles have the same rank. By Corollary \(2.22\) we can use an automorphism of \(B(\mathcal{E})\) (as \(\tilde{G}\)-modules) to obtain the desired series of inclusions. 

\[\square\]
Lemma 2.24 (Main observation of this subsection). Consider two objects $\mathcal{E}$ and $\mathcal{F}$ of $EV(\Sigma, 1)_c$ such that $B(\mathcal{E}) \cong B(\mathcal{F})$ as $\hat{G}$-modules. Then, we can take the intersection $\mathcal{E} \cap_{D_0} \mathcal{F}$ of $\mathcal{E}$ and $\mathcal{F}$ as $\hat{G} \times \hat{G}$-equivariant coherent subsheaves of $B(\mathcal{E})^{D_0}$ for a sufficiently large equivariant divisor $D_0$. Moreover, we have the following commutative diagram of $\hat{G} \times \hat{G}$-equivariant coherent sheaves:

$$
\begin{array}{c}
B(\mathcal{E})^{D_0} \\ \cup \\
\mathcal{E} \cap_{D_0} \mathcal{F} \\ \cong \\
\mathcal{E} \cap D \mathcal{F}
\end{array}
$$

for every $D \geq D_0$. \hspace{1cm} (2.2.1)

Proof. We have an embedding $\mathcal{E} \hookrightarrow B(\mathcal{E})^{D_0}$ as $\hat{G} \times \hat{G}$-equivariant coherent sheaves by Corollary 2.23. By the isomorphism $B(\mathcal{E}) \cong B(\mathcal{F})$, we also have an embedding $\mathcal{F} \hookrightarrow B(\mathcal{E})^{D_0}$ as $\hat{G} \times \hat{G}$-equivariant coherent sheaves. Hence, we can take their intersection. Here, $\mathcal{E} \cap_{D_0} \mathcal{F}$ is a $\hat{G} \times \hat{G}$-equivariant coherent subsheaf of $B(\mathcal{E})^{D_0}$ because the $\hat{G} \times \hat{G}$-equivariant structure of $B(\mathcal{E})^{D_0}$ preserves both $\mathcal{E}$ and $\mathcal{F}$ by Lemma 2.28. By Corollary 2.22 every $\hat{G} \times \hat{G}$-equivariant embedding $B(\mathcal{E})^* \otimes O_X(-D) \hookrightarrow \mathcal{E}^\vee$ (resp. $\mathcal{F}^\vee$) factors through $B(\mathcal{E})^* \otimes O_X(-D_0) \hookrightarrow \mathcal{E}^\vee$ (resp. $\mathcal{F}^\vee$). By taking their dual, we obtain the second assertion. \hfill \square

Remark 2.25. From now on, we freely use the notion of the intersection of two $\hat{G} \times \hat{G}$-equivariant vector bundles in the sense of Lemma 2.24. Notice that such an intersection is defined if, and only if, we specify an isomorphism between identity fibers (as $\hat{G} \times \hat{Z}(G)$-modules) and a sufficiently large equivariant divisor.

2.2.2 Redefinition of $\Xi$

Now we redefine the map $\Xi$ introduced in §1.3. We first introduce another map $\Xi'$ and prove $\Xi = \Xi'$ at Proposition 2.30. At the same time, we extend the definition of the map $\Xi$ to the whole of $ObEV(\Sigma)_c$.

Definition 2.26 ($\Xi'$ for $EV(\Sigma, 1)_c$). For each $\mathcal{E} \in ObEV(\Sigma, 1)_c$, we define a pair $\Xi'(\mathcal{E}) := (B(\mathcal{E}), \{ F^n(\bullet, \mathcal{E}) \}_{\tau \in \Sigma(1)})$ by the following rules:

- $B(\mathcal{E}) := \mathcal{E} \otimes_X k(e)$ as a $\hat{G} \times \hat{Z}(G)$-module. (By assumption, the $1 \times \hat{Z}(G)$-action is trivial.)

- For each $\tau \in \Sigma(1)$, let $\phi_\tau : B(\mathcal{E}) \otimes O_X \to B(\mathcal{E}) \otimes k(x_\tau)$ be the residual map at $x_\tau$. Then we define $\quad F^n(\tau, \mathcal{E}) := \phi_\tau(\mathcal{E}(-nD_\tau) \cap_D B(\mathcal{E}) \otimes O_X)$ for every $n \in \mathbb{Z}$ and a sufficiently large equivariant divisor $D$.

We call this map $\Xi'$. Here the existence (and the stability) of intersections with respect to $D$ is guaranteed by Lemma 2.24. By fixing an isomorphism $k(e) \cong k(x_\tau)$, we have an inclusion $\quad F^n(\tau, \mathcal{E}) \subset B(\mathcal{E})$ for every $\tau \in \Sigma(1)$ and every $n \in \mathbb{Z}$.

Remark 2.27. The isomorphism $k(e) \cong k(x_\tau)$ in Definition 2.26 looks like quite artificial. However, since we deal with only vector spaces, we cannot detect the diagonal $\mathbb{G}_m$-action for each vector space. If $X$ is complete, we can use the space of the global sections of $B(\mathcal{E}) \otimes O_X$ to canonicalize $k(e) \cong k(x_\tau)$. 

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For a rational number $x$, $\lfloor x \rfloor$ denotes the maximal integer which does not exceed $x$.

**Definition 2.28 ($\Xi'$ for $EV(\Sigma,c)$).** Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ be a set of complete representative of $X^*(T) \to \tilde{Z}(G)^\vee$. For each $1 \leq i \leq h$, we have a $\tilde{G} \times \tilde{G}$-equivariant line bundle $\mathcal{L}_{\lambda_i}$ defined before Lemma 2.19. By Lemma 2.2, every $\tilde{G} \times \tilde{G}$-equivariant vector bundle $\mathcal{E}$ is uniquely written as $\mathcal{E} \cong \bigoplus_{1 \leq i \leq h} \mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}$ for some $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_h \in \text{Ob}EV(\Sigma,1)$. Using the above decomposition, we define a pair $\Xi'_\Lambda(\mathcal{E})$ as the direct sum

$$\Xi'_\Lambda(\mathcal{E}) = \left( \bigoplus_{i=1}^h B(\mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}), \bigoplus_{i=1}^h \mathcal{F}^{\tau}(\bullet, \mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}) \right)_{\tau \in \Sigma(1)}$$

of pairs $(B(\mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}), \mathcal{F}^{\tau}(\bullet, \mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}))_{\tau \in \Sigma(1)}$ determined by the following rules:

- $B(\mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}) := B(\mathcal{E}_i) \otimes (k \boxtimes \lambda_i)$ as a $\tilde{G} \times \tilde{G}$-equivariant vector bundle $\mathcal{E}$.
- For each $\tau \in \Sigma(1)$, let $\phi^\tau_\tau : B(\mathcal{E}_i) \otimes \mathcal{O}_X \to B(\mathcal{E}_i) \otimes k(x_\tau)$ be the residual map at $x_\tau$.

Then, we define

$$\mathcal{F}^{\tau}(n + [\tau(\lambda_i)], \mathcal{E}_i \otimes \mathcal{L}_{\lambda_i}) := \phi^\tau_\tau(\mathcal{E}_i(-nD_\tau)) \otimes B(\mathcal{E}_i) \otimes \mathcal{O}_X \otimes (k \boxtimes \lambda_i)$$

for every $n \in \mathbb{Z}$ and a sufficiently large equivariant divisor $D$.

We have an inclusion

$$\mathcal{F}^{\tau}(n, \mathcal{E}) = \bigoplus_{i=1}^h \mathcal{F}^{\tau}(n, \mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}) \subset \bigoplus_{i=1}^h B(\mathcal{E}_i \otimes_{\mathcal{O}} \mathcal{L}_{\lambda_i}) = B(\mathcal{E})$$

for every $\tau \in \Sigma(1)$ and every $n \in \mathbb{Z}$.

**Lemma 2.29.** $\Xi'_\Lambda$ is independent of the choice of $\Lambda$.

**Proof.** Let $\Lambda_1 = \{\lambda'_1, \lambda'_2, \ldots\}$ and $\Lambda_2 = \{\lambda_1, \lambda_2, \ldots\}$ be two choices of $\Lambda$. By changing the numbers of the elements of $\Lambda_1$, we can assume $\lambda_i - \lambda'_i \in X^*(T)$ for every $1 \leq i \leq h$. We put $\Lambda' := \{\lambda_1, \lambda_2, \ldots, \lambda_i, \lambda'_{i+1}, \ldots\}$. Then, it suffices to check the assertion $\Xi'_\Lambda = \Xi'_{\Lambda'}$ for an arbitrary $0 \leq i \leq h$. For each $\lambda \in X^*(T)$, we have $\langle \tau, \lambda \rangle \in \mathbb{Z}$ for every $\tau \in \Sigma(1)$. Hence, $D_{\lambda'_i} - D_{\lambda_i} = \sum_{\tau \in \Sigma(1)} \langle \tau, \lambda'_i - \lambda_i \rangle D_\tau$ is an equivariant divisor by Lemma 2.19. As a consequence, we have

$$\mathcal{F}^{\tau}(n + [\langle \tau, \lambda_i \rangle], \mathcal{E}(D_{\lambda_i}))$$

$$= \mathcal{F}^{\tau}(n + [\langle \tau, \lambda_i \rangle] + \langle \tau, \lambda'_i - \lambda_i \rangle, \mathcal{E}(D_{\lambda_i} + \langle \tau, \lambda'_i - \lambda_i \rangle D_\tau))$$

$$= \mathcal{F}^{\tau}(n + [\langle \tau, \lambda'_i \rangle] + \langle \tau, \lambda'_i - \lambda_i \rangle, \mathcal{E}(D_{\lambda_i} + D_{\lambda'_i - \lambda_i}))$$

$$= \mathcal{F}^{\tau}(n + [\langle \tau, \lambda'_i \rangle], \mathcal{E}(D_{\lambda'_i}))$$

for every $\tau \in \Sigma(1)$ by the construction of Definition 2.28. Therefore, Definition 2.28 does not depend on the choice of $\lambda_i$. \hfill \Box

Now we come to one of the main result of this subsection.

**Proposition 2.30.** For each $\mathcal{E} \in \text{Ob}EV(\Sigma,1)_c$, we have $\Xi'(\mathcal{E}) \cong \Xi(\mathcal{E})$ as vector spaces equipped with families of filtrations.
Let $\tau \in \Sigma(1)$. The definition of $F^\tau(\bullet, \mathcal{E})$ makes sense if we restrict ourselves to $T(\tau)$. Thus, we restrict our attention to $T(\tau)$. $\mathcal{E}|_{T(\tau)}$ is isomorphic to $E \times T(\tau)$ for some $T$-module $E$ by Theorem 2.18. Let $E = \bigoplus_{\lambda \in X^*(T)} E^\lambda$ be the isotypical decomposition of $E$.

Let $E_0$ be a trivial $G^*_m$-representation such that $\dim E = \dim E_0$. By Theorem 2.18, we have $F^\tau(n, \mathcal{E}) = \bigoplus_{(\tau, \lambda) + n \leq 0} E^\lambda$. We write $T(\tau) = \text{Spec} R$, where $R$ is the $(1 \times T)$-algebra.

Let $\mathcal{E}$ be the primary definition of $\Xi$ hereafter because it prolongs the original $\Xi$ in (1.4) and is independent of the choice of $\Lambda$ by Lemma 2.29.

By comparing term by term, we conclude $F^\tau(\bullet, \mathcal{E})$ makes sense if we restrict ourselves to $T(\tau)$. Thus, we restrict our attention to $T(\tau)$. $\mathcal{E}|_{T(\tau)}$ is isomorphic to $E \times T(\tau)$ for some $T$-module $E$ by Theorem 2.18. Let $E = \bigoplus_{\lambda \in X^*(T)} E^\lambda$ be the isotypical decomposition of $E$.

Let $E_0$ be a trivial $G^*_m$-representation such that $\dim E = \dim E_0$. By Theorem 2.18, we have $F^\tau(n, \mathcal{E}) = \bigoplus_{(\tau, \lambda) + n \leq 0} E^\lambda$. We write $T(\tau) = \text{Spec} R$, where $R$ is the $(1 \times T)$-algebra.

Let $\mathcal{E}$ be the primary definition of $\Xi$ hereafter because it prolongs the original $\Xi$ in (1.4) and is independent of the choice of $\Lambda$ by Lemma 2.29.

2.2.3 Making $\Xi$ into a functor

In the definition of $\Xi$ in (1.4) we only need the closure of $(1 \times T)e$ (inside $X(\Sigma)$) to define $F^\tau$. As a consequence, our definition of $\Xi$ coincides with the composition of the restriction to $T(\Sigma)$ and Klyachko’s functor (1.8) 0.1. Therefore, we obtain the following from Klyachko’s description (1.8) 2.3 by using Definition 2.28 and Proposition 2.30.

Corollary 2.31. Let $\mathcal{E} \in \text{Ob} E(V(\Sigma))$. For each $\tau \in \Sigma(1)$, $F^\tau(\bullet, \mathcal{E})$ is a decreasing filtration of $B(\mathcal{E})$. Moreover, we have $F^\tau(-n, \mathcal{E}) = B(\mathcal{E})$ and $F^\tau(n, \mathcal{E}) = 0$ for $n > 0$. 

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Corollary 2.32. Let $E \in \mathbb{Ob}EV(\Sigma)_c$. For each $\sigma \in \Sigma$, $\{F^\tau(n,E)\}_{\tau \in \sigma(1), n \in \mathbb{Z}}$ forms a distributive lattice.

Lemma 2.33. Let $E \in \mathbb{Ob}EV(\Sigma)_c$. For each $\tau \in \Sigma(1)$ and $n \in \mathbb{Z}$, $F^\tau(n,E)$ is a $\tilde{P}^\tau \times \tilde{Z}(G)$-submodule of $B(E)$.

Proof. By Definition 2.22, it suffices to prove the assertion for $\mathbb{Ob}EV(\Sigma)_c$. By Lemma 2.21 $E(-nH) \cap_D B(E) \otimes O_X$ is a $G \times G$-equivariant $O_X$-module for a sufficiently large equivariant divisor $H$. Thus, its fiber at $x_\tau$ admits an action of the stabilizer at $x_\tau$. Hence, $F^\tau(n,E)$ is a $G^\tau$-submodule of $B(E)$. By the definition of $\triangle^d$, we have $\tilde{P}^\tau = \tilde{L}U^\tau_\tau \hookrightarrow \triangle^d \tilde{L}^\tau(U^\tau_\tau \times 1) \subset \tilde{G}^\tau$. By Lemma 2.11 the $\tilde{P}^\tau = \tilde{L}U^\tau_\tau$-action on $B(E) \cong B(E) \otimes O_X \otimes_X k(x_\tau)$ coincides with the action of stabilizer $\triangle^d\tilde{P}^\tau$ on $B(E) \cong B(E) \otimes O_X \otimes_X k(x_\tau)$ (notice that we have fixed an isomorphism $k(e) \cong k(x_\tau)$ in Definition 2.26).

Lemma 2.34. Let $E_1, E_2 \in \mathbb{Ob}EV(\Sigma)_c$ and let $f : E_1 \rightarrow E_2$ be a morphism of $EV(\Sigma)_c$. Then, we have a $\tilde{P}^\tau \times \tilde{Z}(G)$-module morphism $f' : F^\tau(n,E_1) \rightarrow F^\tau(n,E_2)$ such that the morphism of their ambient spaces $f' : B(E_1) \rightarrow B(E_2)$ is obtained by the specialization of $f$ at $e$ for every $\tau \in \Sigma(1)$ and every $n \in \mathbb{Z}$.

Proof. By Definition 2.28 and Corollary 2.31 it suffices to prove the assertion for $\mathbb{Ob}EV(\Sigma, 1)_c$. We have a natural morphism $f' : B(E_1) \rightarrow B(E_2)$ induced by $f$. We have $B(E_i) = \Gamma(G, (\Sigma)_i)^{1\times \tilde{G}}$ for $i = 1, 2$. Let $D$ be a sufficiently large divisor. By Lemma 2.21 and Corollary 2.22 we have a $\tilde{G} \times \tilde{G}$-equivariant inclusion $\Gamma(G, (\Sigma)_1)^{1\times \tilde{G}} \otimes O_X (-D) \subset E_1$ such that $f$ induces a morphism $\Gamma(G, (\Sigma)_1)^{1\times \tilde{G}} \otimes O_X (-D) \rightarrow \Gamma(G, (\Sigma)_2)^{1\times \tilde{G}} \otimes O_X (-D) \subset E_2$. Enlarging $D$ if necessary, we have the following commutative diagram of $\tilde{G} \times \tilde{G}$-equivariant coherent sheaves by Corollary 2.20.

Here we can take the intersection of $E_1(-nD)$ and $B(E_1) \otimes O_X$ in $B(E_1)^D$, $f'^\otimes_{id}$ termwise to define a morphism

$$E_1(-nD) \cap_D B(E_1) \otimes O_X \rightarrow E_2(-nD) \cap_D B(E_2) \otimes O_X.$$

Applying $\phi_\tau$ (in Definition 2.22) to both sides, we obtain a $\tilde{G}^\tau$-module morphism $F^\tau(n,E_1) \rightarrow F^\tau(n,E_2)$ because all sheaves and morphisms in the above diagram are $\tilde{G} \times \tilde{G}$-equivariant. Hence, we obtain the result from the same argument as in the proof of Lemma 2.33.

Proposition 2.35. $\Xi$ defines a faithful covariant functor $\Xi : EV(\Sigma)_c \rightarrow \mathbb{Ob}(\Sigma)_c$.

Proof. By Corollary 2.31 Corollary 2.32 and Lemma 2.33 the range of the map $\Xi$ is contained in $\mathbb{Ob}(\Sigma)_c$. By Lemma 2.34 we have a morphism (of sets)

$$\Xi : \text{Hom}_{EV(\Sigma)_c}(E, F) \rightarrow \text{Hom}_{\mathbb{Ob}(\Sigma)_c}(\Xi(E), \Xi(F)).$$
for every $\mathcal{E}, \mathcal{F} \in \text{Ob} \mathcal{E}V(\Sigma)_c$. By the similar argument as in the proof of Lemma 2.34 we can
deduce that $\Xi(f) \circ \Xi(g) = \Xi(f \circ g)$ for two morphisms $f, g$ in $\mathcal{E}V(\Sigma)_c$. Thus, $\Xi$
becomes a covariant functor from $\mathcal{E}V(\Sigma)_c$ to $\mathcal{E}(\Sigma)_c$. For each nonzero morphism of $\mathcal{E}V(\Sigma)_c$, its
restriction to $G$ is nontrivial since a locally free $\mathcal{O}_X$-module cannot contain torsion submodule. Then, it induces a nontrivial morphism between the identity fibers by Lemma 2.10. Hence, $\Xi$
must be faithful.

3 Actions and toric slice

3.1 Toric decomposition

Here we develop a method to study of $\hat{G} \times G$-equivariant vector bundles via the study of
equivariant vector bundles of a certain one-dimensional toric variety.

3.1.1 Reduction to $\mathbb{A}^1$

First, we fix an arbitrary one-dimensional cone $\tau$ of $\Sigma$.

We define $\mathbb{A}_\tau := \mathbb{G}^r_{m, \tau} \subset T\left(\Sigma \setminus \{x_{-\tau}\}\right)$. We have $\mathbb{A}_\tau = \mathbb{G}^r_{m, \tau} \cup \{x_{\tau}\}$ (as sets) and $\mathbb{A}_\tau \cong \mathbb{A}^1$ (as $\mathbb{G}^r_{m, \tau}$-toric varieties). By Theorem 1.3, we have an open embedding $B^-.\mathbb{A}_\tau.B \hookrightarrow X(\Sigma)$. Hence, we have the following composition of faithful functors

$$\text{Res}^\tau : \text{Coh} \hat{G} \times G X(\Sigma) \rightarrow \text{Coh} \hat{B}^-.B (B^-.\mathbb{A}_\tau.B) \rightarrow \text{Coh} \mathbb{G}^r_{m, \mathbb{A}_\tau},$$

where the second functor is the composition of the restriction (cf. [10] 5.2.16) and restriction
of the group. Since all of the previous constructions use faithful functors, we can restrict the
bounding sheaves of Corollary 2.23 to $\mathbb{A}_\tau$ by Res$^\tau$. In particular, the definition of $F^\tau(\bullet, \mathcal{E})$
(see Definition 1.12) factors through Res$^\tau$. Let $t$ be a local uniformizer of $x_{\tau}$ on $\mathbb{A}_\tau$ which is
an eigenfunction with respect to $\mathbb{G}^r_{m, \mathbb{A}_\tau}$.

**Lemma 3.1.** For each $\mathcal{E} \in \text{Ob} \mathcal{E}V(\Sigma, 1)_c$ and a sufficiently large integer $N$, we have the
following inclusions of $(\Delta^d(\hat{L}^\tau)\mathbb{G}^r_{m, t})$-modules:

$$B(\mathcal{E}) \otimes t^N k[t] \subset \Gamma(\mathbb{A}_\tau, \mathcal{E}|_{\mathbb{A}_\tau}) \subset B(\mathcal{E}) \otimes t^{-N} k[t] = \Gamma(\mathbb{A}_\tau, B(\mathcal{E})^{ND\tau}|_{\mathbb{A}_\tau}).$$

Here $\Delta^d(\hat{L}^\tau)$ acts on $B(\mathcal{E})$ as a subgroup of $\hat{G}$ via $\hat{L}^\tau \subset \hat{G}$ and acts on $t$ trivially. $\mathbb{G}^r_{m, \mathbb{A}_\tau}$ acts on $B(\mathcal{E})$ trivially and acts on $t$ by weight one.

**Proof.** By Corollary 2.23 all we have to do is to check the compatibility with respect to the
group actions. Since $\Delta^d(\hat{L}^\tau)$ fixes $\mathbb{A}_\tau$ pointwise, it acts on $t$ trivially (cf. [1.2]). By the
definition of $B(\mathcal{E}) \otimes \mathcal{O}_X$ (2.12), $\Delta^d(\hat{L}^\tau)$ acts on $B(\mathcal{E})$ as $\hat{L}^\tau \subset \hat{G}$ and $\mathbb{G}^r_{m, \mathbb{A}_\tau}$ acts on $B(\mathcal{E})$
trivially. Since $x_{\tau}$ is a limit of $t.e$ with $t \rightarrow 0$ in $\mathbb{G}^r_{m, \mathbb{A}_\tau}$, $\mathbb{G}^r_{m, \mathbb{A}_\tau}$ acts on $t$ by degree one.

**Corollary 3.2.** Let $\mathcal{E} \in \text{Ob} \mathcal{E}V(\Sigma, 1)_c$. Let $N$ be an integer. Then, we have

$$\Gamma(\mathbb{A}_\tau, (B(\mathcal{E})^{ND\tau} \cap D \mathcal{E})|_{\mathbb{A}_\tau}) = \bigoplus_{n=-\infty}^{N} F^\tau(n, \mathcal{E}) \otimes k t^{-n}.$$

as $\Delta^d(\hat{L}^\tau)\mathbb{G}^r_{m, \mathbb{A}_\tau}$-submodules of $B(\mathcal{E}) \otimes t^{-N} k[t]$ for a sufficiently large equivariant divisor $D$. In
particular, the RHS is the $\mathbb{G}^r_{m, \mathbb{A}_\tau}$-isotypical decomposition.
Proof. By Lemma 3.1, \( \mathbb{G}_m^\tau \) acts on \( B(\mathcal{E}) \otimes kt^n \subset B(\mathcal{E}) \otimes t^{-N} k[t] = \Gamma(\mathcal{A}_\tau, \mathcal{B}(\mathcal{E})^{ND_\tau}|_{\mathcal{A}_\tau}) \) by degree \( n \). Thus, we have the \( \mathbb{G}_m^\tau \)-isotypical decomposition
\[
\Gamma(\mathcal{A}_\tau, \mathcal{B}(\mathcal{E})^{ND_\tau}) \cong \bigoplus_{n=-\infty}^{N} B(\mathcal{E}) \otimes kt^{-n}.
\]

By Corollary 2.23, we have
\[
\Gamma(A_\tau, \mathcal{B}(\mathcal{E})|_{\mathcal{A}_\tau}) \subset \Gamma(\mathcal{A}_\tau, \mathcal{B}(\mathcal{E})^{MD_\tau}|_{\mathcal{A}_\tau})
\]
as a compatible \((\mathbb{G}_m^\tau, k[t])\)-submodule for \( M > 0 \). Hence, we have the \( \mathbb{G}_m^\tau \)-isotypical decomposition
\[
\Gamma(\mathcal{A}_\tau, \mathcal{B}(\mathcal{E})|_{\mathcal{A}_\tau}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{F}^\tau(n, \mathcal{E}) \otimes kt^{-n}.
\]
Since the latter is \( k[t] \)-free, \( \mathcal{F}^\tau(\bullet, \mathcal{E}) \) is a decreasing filtration of \( B(\mathcal{E}) \). If \( \tau \neq \xi \in \Sigma(1) \), then twisting by \( D_\xi \) has no effect on coherent sheaves on \( \mathcal{A}_\tau \). Moreover, twisting \( D_\tau \) is equivalent to multiplying \( t^{-1} \). Thus, we have
\[
\mathcal{F}^\tau(n, \mathcal{E}) = \phi(\mathcal{E}(-nD_\tau) \cap_D \mathcal{B}(\mathcal{E}))
\]
\[
= \Gamma(\mathcal{A}_\tau, \mathcal{E}(-nD_\tau) \cap_D \mathcal{B}(\mathcal{E}) \otimes kt^0) = \mathcal{F}^\tau(n, \mathcal{E})
\]
as vector subspaces of \( B(\mathcal{E}) \) by Definition 2.26 (Here we put \( D|_{\mathcal{A}_\tau} = MD_\tau|_{\mathcal{A}_\tau} \)). Therefore, we have
\[
\Gamma(\mathcal{A}_\tau, (\mathcal{B}(\mathcal{E})^{ND_\tau} \cap_D \mathcal{E})|_{\mathcal{A}_\tau}) = \bigoplus_{n=-\infty}^{N} \mathcal{F}^\tau(n, \mathcal{E}) \otimes kt^{-n} \subset B(\mathcal{E}) \otimes t^{-N} k[t]
\]
as compatible \((\mathbb{G}_m^\tau, k[t])\)-modules. By Lemma 2.33, each inclusion \( \mathcal{F}^\tau(n, \mathcal{E}) \hookrightarrow B(\mathcal{E}) \) is a \( \Delta^d(\tilde{L}^\tau) \)-module inclusion. \( \square \)

For a \( \tilde{G} \)-module \( V \), we define the formal loop space \( V^\tau_{[t]} \) of \( V \) with respect to \( \tau \in \Sigma(1) \) as follows:
\[
V^\tau_{[t]} := \lim_{N} \Gamma(\mathcal{A}_\tau, V \otimes \mathcal{O}_X(ND_\tau)|_{\mathcal{A}_\tau}) = V \otimes \lim_{N} t^{-N} k[t] \cong V \otimes k[t, t^{-1}]
\]
(3.1.1)

Here \( t, t^{-1} \), and \( V \otimes k \) generates \( V^\tau_{[t]} \).

**Definition 3.3.** We define the category \( \mathfrak{B}(\Sigma)^\tau \) as follows:

- **(Objects)** Triples \((R, V, \iota)\) such that:
  - \( R \) is a compatible \((\Delta^d(\tilde{L}^\tau) \mathbb{G}_m^\tau, k[t])\)-module;
  - \( V \) is a \( \tilde{G} \)-module;
  - \( \iota : R \hookrightarrow V^\tau_{[t]} \) is an injective morphism of compatible \((\Delta^d(\tilde{L}^\tau) \mathbb{G}_m^\tau, k[t])\)-modules.

  For simplicity, we may also denote it by \((R \subset V^\tau_{[t]}\)).

- **(Morphisms)** We define the morphism by the following commutative diagram of compatible \((\Delta^d(\tilde{L}^\tau) \mathbb{G}_m^\tau, k[t])\)-modules:
\[
\begin{array}{ccc}
(R_1)_{[t]} & \hookrightarrow & (V_1)_{[t]} \in \text{Ob} \mathfrak{B}(\Sigma)^\tau \\
\downarrow & & \downarrow \\
(R_2)_{[t]} & \hookrightarrow & (V_2)_{[t]} \in \text{Ob} \mathfrak{B}(\Sigma)^\tau
\end{array}
\]

Here \((V_1)_{[t]} \rightarrow (V_2)_{[t]}\) is the morphism induced by a \( \tilde{G} \)-module map \( V_1 \rightarrow V_2 \).
For each $E \in \text{Ob} EV (\Sigma, 1)_c$, we define an object $\rho^\tau_{\infty} (E) \in \text{Ob} \mathcal{B} (\Sigma)^\tau$ as follows:

$$\rho^\tau_{\infty} (E) := \left( \Gamma (\mathbb{A}_\tau, E|_{\mathbb{A}_\tau}) = \lim_{\leftarrow} N \Gamma (\mathbb{A}_\tau, (E \cap_D \mathcal{B}(E)^{ND},)|_{\mathbb{A}_\tau}) \subset B (E)^\tau \right)$$

Here $D$ is a sufficiently large equivariant divisor which depends on $E$ (cf. Lemma 2.21).

**Remark 3.4.** It is important NOT to forget the ambient space $B (E)^{\tau[t]}$.

**Corollary 3.5.** For each $E \in \text{Ob} EV (\Sigma, 1)_c$, we have

$$\rho^\tau_{\infty} (E) \sim \left( \bigoplus_{n \in \mathbb{Z}} F^\tau (n, E) \otimes k^t \right) \subset B (E)^{\tau[t]}$$

as $\Delta^d (\tilde{L}^\tau) \mathbb{G}_m^\tau$-modules. In particular, the above direct sum decomposition is the $\mathbb{G}_m^\tau$-isotypical decomposition.

**Proof.** The assertion follows from the definition of $\rho^\tau_{\infty}$ and Corollary 3.2. □

**Corollary 3.6.** $\rho^\tau_{\infty}$ defines a functor $EV (\Sigma, 1)_c \to \mathcal{B} (\Sigma)^\tau$.

**Proof.** This follows immediately from Proposition 2.35 and Corollary 3.5. □

### 3.1.2 Basic computation

We work under the same settings as in §3.1.1. In particular, $\tau \in \Sigma (1)$. Here, we compute $\rho^\tau_{\infty}$ for two kinds of $\tilde{G} \times G$-equivariant vector bundles. The description of the first series of $\tilde{G} \times G$-equivariant vector bundles is a direct consequence of the definition.

**Corollary 3.7.** For a $\tilde{G}$-module $V$, we have $\rho^\tau_{\infty} (V \otimes \mathcal{O}_X) = \left( V \otimes k[t] \subset V^{\tau[t]} \right)$.

**Proof.** It is a direct consequence of Corollary 3.5. □

To describe our second series of $\tilde{G} \times G$-equivariant vector bundles, we need a preparation.

**Definition 3.8.** Let $\lambda$ be a dominant weight. Let $v^*_{w_0 \lambda}$ be a highest weight vector of $V_{w_0 \lambda}$ (cf. §1.1). We define

$$V^\tau_{\lambda} (n) := \{ v \in V_{\lambda}; \tau(t)(v \otimes v^*_{w_0 \lambda}) = t^n v \otimes v^*_{w_0 \lambda} \text{ for all } t \in \mathbb{G}_m (k) \subset k \}$$

for every $n \in \mathbb{Z}$. We have a direct sum decomposition $V_{\lambda} = \bigoplus_{n \geq 0} V^\tau_{\lambda} (n)$. Moreover, we define

$$F^{\tau}_{\max} (m, V_{\lambda}) := \bigoplus_{n \geq m} V^\tau_{\lambda} (n) \text{ for all } m \in \mathbb{Z}.$$

We call this filtration the maximal filtration of the $\tilde{G} \times \tilde{Z}(G)$-module $V_{\lambda} \otimes k$.

Let $V$ be a finite dimensional rational $\tilde{G}$-module. We denote its ($\tilde{G}$-) irreducible decomposition by $\bigoplus_{\mu} V_{\mu}^{\otimes \mu}$. Then, we define

$$F^{\tau}_{\max} (m, V) := \bigoplus_{\mu} F^{\tau}_{\max} (m, V_{\mu})^{\otimes \mu} \text{ for all } m \in \mathbb{Z}.$$

We call this filtration the maximal filtration of the $\tilde{G} \times \tilde{Z}(G)$-module $V \otimes k$. It is independent of the choice of an irreducible decomposition of $V$. 23
Taking account the weight decomposition of $V$ and Definition \ref{def:weight_decomposition} we see that $F_{\max}^\tau (\bullet , V)$ is an $\tau$-transversal filtration.

Now we can state and prove the description of our second series of $G \times G$-equivariant vector bundles.

**Proposition 3.9.** We have $\Xi (L_{-w,\lambda} \otimes V_{\lambda}) = (V_{\lambda}, \{F_{\max}^\tau (\bullet , V_{\lambda})\}_{\tau \in \Sigma(1)})$ for every dominant weight $\lambda$. (Here $L_{\lambda}$ is defined before Lemma \ref{lem:weight_decomposition}.)

**Proof.** We have $L_{-w,\lambda} \in \text{Ob}EV (\Sigma, -w_0 \lambda)$ by Theorem \ref{thm:weight_decomposition} and the definition. Thus, we have $L_{-w,\lambda} \otimes V_{\lambda} \in \text{Ob}EV (\Sigma, 1)$, we compute the boundary behavior as in \ref{thm:boundary_behavior} via Proposition \ref{prop:boundary_behavior} for each direction. We choose an arbitrary $\sigma \in \Sigma(1)$. By the remarks at the beginning of \ref{cor:boundary_behavior}, we restrict ourselves to $(L_{-w,\lambda} \otimes V_{\lambda})|_{A_{\sigma}}$. By Theorem \ref{thm:boundary_behavior} 1), we have

$$(L_{-w,\lambda} \otimes V_{\lambda})|_{A_{\sigma}} \cong A_{\sigma} \times ((w_0 \lambda)^{-1} \otimes V_{\lambda})$$

as $G_m$-equivariant vector bundles on $A_{\sigma}$. Therefore, we have

$$F_{\sigma}^\tau (m,A_{\sigma} \times ((w_0 \lambda)^{-1} \otimes V_{\lambda})) = \bigoplus_{n \geq m} V_{\lambda}^\tau (n) = F_{\max}^\tau (m, V_{\lambda})$$

for every $m \in \mathbb{Z}$ by Theorem \ref{thm:boundary_behavior} 3). \hfill $\Box$

### 3.1.3 Injectivity of $\Xi$

The rest of this subsection is devoted to the proof of Proposition \ref{prop:injectivity}.

**Proposition 3.10.** $\Xi$ induces an injective map $\text{Ob}EV (\Sigma)_c \to \text{Ob}\mathcal{C} (\Sigma)_p$.

First, we prove a simple special case.

**Lemma 3.11.** Let $\sigma \in \Sigma(1)$. Let $\mathcal{E}_1, \mathcal{E}_2 \in \text{Ob}EV (\sigma, 1)_c$ be such that $\Xi (\mathcal{E}_1) \cong \Xi (\mathcal{E}_2)$. For a sufficiently large equivariant divisor $D$, we fix inclusions $B (\mathcal{E}_1)^{-D} \subset \mathcal{E}_1, \mathcal{E}_2 \subset B (\mathcal{E}_1)^{D}$ which gives rise to an isomorphism $\Xi (\mathcal{E}_1) \cong \Xi (\mathcal{E}_2)$ (see Corollary \ref{cor:boundary_behavior}). Then, we have $\mathcal{E}_1 = \mathcal{E}_2$ as a $G \times G$-equivariant coherent subsheaves of $B (\mathcal{E}_1)^{D}$.

**Proof.** We prove by induction on $p$. We have $\mathcal{E}_1 \cap_{H} B (\mathcal{E}_1)^{p D_{\tau}} = \mathcal{E}_2 \cap_{H} B (\mathcal{E}_1)^{p D_{\tau}}$ for every sufficiently small integer $p$. By assumption, we have $\mathcal{O}_{\tau} = \mathcal{O}_{\tau}$. Hence, every $G \times G$-equivariant $O_{X(\tau)}$-module annihilated by $O_{X(\tau)} (-D_{\tau})$ is a $\tilde{G} \times G$-equivariant vector bundle on $O_{\tau}$. In particular, it is determined by its fiber at $x_{\tau}$. Suppose

$$\mathcal{E}_1 \cap_{D} B (\mathcal{E}_1)^{p D_{\tau}} = \mathcal{E}_2 \cap_{D} B (\mathcal{E}_1)^{p D_{\tau}}$$

and $F_{\sigma} (p + 1, \mathcal{E}_1)$ as a subspace of $B (\mathcal{E}_1)$ for some integer $p$. Let $\mathcal{Q}_{p+1} (\mathcal{E}_1)$ be a $\tilde{G} \times G$-equivariant $O_{\mathcal{O}_{\tau}}$-submodule of $B (\mathcal{E}_1) \otimes O_{\mathcal{O}_{\tau}}$ such that $\mathcal{Q}_{p+1} (\mathcal{E}_1) \otimes O_{X} (\mathcal{O}_{\tau} (p + 1)) \cong F_{\sigma} (p + 1, \mathcal{E}_1)$ as $\tilde{G} \times G$-modules. We put $\mathcal{Q}_{p+1} (\mathcal{E}_1) := \mathcal{Q}_{p+1} (\mathcal{E}_1) \otimes O_{X} ((p + 1) D_{\tau})$. Then, we have the following commutative diagrams

\[
\begin{align*}
0 & \rightarrow B (\mathcal{E}_1)^{p D_{\tau}} \rightarrow B (\mathcal{E}_1)^{(p+1) D_{\tau}} \quad \overset{\varphi^p}{\rightarrow} B (\mathcal{E}_1)^{(p+1) D_{\tau}}|_{D_{\tau}} \rightarrow 0 \quad (\text{exact}) \\
0 & \rightarrow \mathcal{E}_i \cap_{D} B (\mathcal{E}_1)^{p D_{\tau}} \rightarrow \mathcal{E}_i \cap_{D} B (\mathcal{E}_1)^{(p+1) D_{\tau}} \rightarrow \mathcal{Q}_{p+1} (\mathcal{E}_1) \rightarrow 0 \quad (\text{exact})
\end{align*}
\]

for $i = 1, 2$.  

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Claim 1. In the diagram \( \text{(3.12)} \), the extension \( E_i \cap_D B(E_1)^{(p+1)D_\tau} \) of \( Q_{p+1}(E_1) \) by \( E_i \cap_D B(E_1)^{pD_\tau} \) (as \( \tilde{G} \times G \)-equivariant coherent \( \mathcal{O}_X \)-modules) is uniquely determined by the inclusions \( s^p \) and \( u \).

**Proof of Claim.** The preimage of \( Q_{p+1}(E_1) \) under \( \phi_i^p \) is uniquely determined in \( B(E_1)^{(p+1)D_\tau} \). By standard homological algebra (cf. Rotman [20] Theorem 7.19 and Ex. 7.20-7.22), the ambiguity of extension is given by a morphism

\[
h: Q_{p+1}(E_1) \to B(E_1)^{pD_\tau}/(E_i \cap_D B(E_1)^{pD_\tau})
\]
as \( \tilde{G} \times G \)-equivariant coherent \( \mathcal{O}_X \)-modules. \( B(E_1)^{pD_\tau}/(E_i \cap_D B(E_1)^{pD_\tau}) \) is a successive extension of elements in \( \{ B(E_1)^{pD_\tau}/(\phi_i^{q-1})^{-1}(Q_q(E_1)) : q \leq p \} \) as \( \tilde{G} \times G \)-equivariant coherent \( \mathcal{O}_X \)-modules by construction. For each \( q \leq p \), \( B(E_1)^{qD_\tau}/(\phi_i^{q-1})^{-1}(Q_q(E_1)) \) is an \( \mathcal{O}_D \)-module. Moreover, \( G^*_\mathfrak{m} \) acts on its fiber at \( x_\tau \) by weight \( -q \) (cf. Theorem [21.43]). Similarly, \( G^*_\mathfrak{m} \) acts on \( Q(E_1,p+1) \otimes_k k(x_\tau) \) by weight \( -(p+1) \). Therefore, we have

\[
\text{Hom}_{\tilde{G} \times G, \mathcal{O}_\mathfrak{O}_\tau} \left( Q_{p+1}(E_1), B(E_1)^{qD_\tau}/(\phi_i^{q-1})^{-1}(Q_q(E_1)) \right) = 0.
\]

Every \( \mathcal{O}_X \)-morphism from \( Q_{p+1}(E_1) \) descends to a \( \mathcal{O}_\mathfrak{O}_\tau \)-morphism. Hence, we have \( h = 0 \) by induction on \( q \).

By the above claim, we have \( E_i \cap_D B(E_1)^{(p+1)D_\tau} = E_2 \cap_D B(E_1)^{(p+1)D_\tau} \). Thus, the induction on \( p \) proceeds.

**Proof of Proposition 3.10.** Let \( E_1, E_2 \in \text{ObEV}(\Sigma)_c \) such that \( \Xi(E_1) \cong \Xi(E_2) \). By Corollary 2.21 and Corollary 2.20, we need only to prove the case \( E_1, E_2 \in \text{ObEV}(\Sigma,1)_c \). By Lemma 3.11, we have \( E_1|\mathfrak{X}(\tau) = E_2|\mathfrak{X}(\tau) \) as a \( \tilde{G} \times \tilde{G} \)-equivariant coherent subsheaf of \( B(E_1)^D|\mathfrak{X}(\tau) \) for a sufficiently large equivariant divisor \( D \). Collecting these isomorphisms (as coherent \( \mathcal{O}_X \)-submodules of \( B(E_1)^D|\mathfrak{X}(\tau) \) for all \( \tau \in \Sigma(1) \)), we have \( E_1 = E_2 \) on \( \cup_{\tau \in \Xi(1)} X(\tau) \). Thus, we have \( E_1 = E_2 \) as vector bundles because a vector bundle on a normal variety is uniquely determined by its restriction to a complement of a codimension two locus. Hence, we have \( E_1 = E_2 \) as \( \tilde{G} \times \tilde{G} \)-equivariant vector subbundles of \( B(E_1)^D \) by Corollary 2.9.

### 3.2 Proof of Transversality

This subsection is devoted to prove the following proposition.

**Proposition 3.12.** For each \( E \in \text{ObEV}(\Sigma)_c \), we have \( \Xi(E) \in \text{ObC}(\Sigma)_c \).

Since the transversality condition (Definition 1.5) is closed under direct sums, we can reduce the problem to \( \text{ObEV}(\Sigma,1)_c \) by Corollary 2.3 and Corollary 2.20.

We fix an arbitrary \( \tau \in \Sigma(1) \).

#### 3.2.1 Left actions

**Definition 3.13.** For each \( \tilde{G} \)-module \( V \), we introduce a left \( u^*_+ \)-action \( L_\tau \) on \( V^*_\tau \) as follows:

\[
L_\tau(X): V^*_\tau \ni \sum_{n \in \mathbb{Z}} v_n \otimes t^n \mapsto \sum_{n \in \mathbb{Z}} (Xv_n) \otimes t^n \in V^*_\tau
\]
for every $X \in u_+^\tau$. We extend it to a left $g$-action $\tilde{L}_\tau$ on $V^\tau_{[t]}$ as follows:

$$
\tilde{L}_\tau(X) : V^\tau_{[t]} \ni \sum_{n \in \mathbb{Z}} v_n \otimes t^n \mapsto \sum_{n \in \mathbb{Z}} (Xv_n) \otimes t^n \in V^\tau_{[t]}
$$

for every $X \in g$.

**Lemma 3.14.** For every $\mathcal{E} \in \text{ObEV}(\Sigma, 1)_c$, the $G_m^\tau$-isotypical decomposition

$$
\rho^\tau_\infty(\mathcal{E}) = \left( \bigoplus_{n \in \mathbb{Z}} F^\tau_n (\mathcal{E}) \otimes kt^n \right) \subset B(\mathcal{E})^\tau_{[t]}
$$

identifies the differential of the unipotent radical of the $F^\tau$-action on $F^\tau(\bullet)$ (coming from Proposition 2.23) with the induced action of the left $u^\tau_+$-action. In particular, $\rho^\tau_\infty(\mathcal{E})$ is preserved by the left $u^\tau_+$-action.

**Proof.** Since the construction of $F^\tau(\bullet)$ factors through $A_\tau$, this is a rephrasement of Lemma 2.11 and Corollary 3.5. \qed

### 3.2.2 Right action

Here we introduce a $u^-_+$-action on $B(\bullet)^\tau_{[t]}$ which is compatible with the natural $u^-_+$-action on the fiber of $\mathcal{O}_X \otimes B(\bullet)$ at $x_\tau$. What we do here is only to switch from left to right in the definition of $L_\tau$. However, this is little complicated because our description heavily relies on Definition 2.26, where we choose $B(\bullet) \otimes \mathcal{O}_X$ as the standard $\tilde{G} \times G$-equivariant vector bundle (its alternative was $\mathcal{O}_X \otimes B(\bullet)$).

We define $\tau G_m$ to be the image of $(\tau \times 1) : G_m \to T \times T$. Let $V$ be a $\tilde{G}$-module. Then, $\tau G_m$ acts on a $\tilde{G} \times G$-equivariant vector bundle $V \otimes \mathcal{O}_X$. We have $\tau G_m \subset \Delta^d(\tilde{L}^\tau)G_m^\tau$. In particular, $\tau G_m$ acts on $V^\tau_{[t]}$. Fix a set of representatives $\{\lambda_1, \lambda_2, \ldots, \lambda_h\}$ of $\tilde{Z}(G)^\vee$ in $X^*(\tilde{T})$. Then, we define $R(\mathcal{E})$ for each $\mathcal{E} \in \text{ObEV}(\Sigma, 1)_c$ as follows:

$$
R(\mathcal{E}) := \bigoplus_{i=1}^h \bigoplus_{\lambda - \lambda_i \in X^*(T)} L_{-\lambda_i} \otimes V^\otimes_{\lambda_i}^\tau.
$$

Here $\bigoplus_{\lambda \in X^*(T)} V^\otimes_{\lambda}^\tau$ is the $\tilde{G}$-isotypical decomposition of $B(\mathcal{E})$. For an equivariant divisor $D$, let $R(\mathcal{E})^D$ denote $R(\mathcal{E}) \otimes \mathcal{O}_X(D)$. By Lemma 2.11, we have $B(\mathcal{E}) \otimes \mathcal{O}_G \cong R(\mathcal{E})|_G$ as $\tilde{G} \times \tilde{Z}(G)$-modules. Then, by using Corollary 2.23 repeatedly, we have

$$
\cdots \subset B(\mathcal{E})^{-D} \subset R(\mathcal{E}) \subset B(\mathcal{E})^D \subset R(\mathcal{E})^{2D} \subset \cdots
$$

for a sufficiently large equivariant divisor $H$. As a result, we have the following isomorphism of $(\Delta^d(\tilde{L}^\tau)G_m^\tau, k[t, t^{-1}])$-modules.

$$
B(\mathcal{E})^\tau_{[t]} = \lim_{\rightarrow N} \Gamma(\mathbb{A}_\tau, B(\mathcal{E})^{ND_\tau}|_{\mathbb{A}_\tau}) \cong \lim_{\rightarrow N} \Gamma(\mathbb{A}_\tau, R(\mathcal{E})^{ND_\tau}|_{\mathbb{A}_\tau})
$$

**Lemma 3.15.** For every $\mathcal{E} \in \text{ObEV}(\Sigma, 1)_c$, there exist a section $s : R(\mathcal{E})|_{x_\tau} \hookrightarrow B(\mathcal{E})^\tau_{[t]}$ which yields the following $\tau G_m^\tau$-isotypical decomposition:

$$
B(\mathcal{E})^\tau_{[t]} = \bigoplus_{n \in \mathbb{Z}} t^n s(R(\mathcal{E})|_{x_\tau}).
$$
Proof. We have the residual map $\psi^R_\tau(\lambda) : (L_{-w_0\lambda} \otimes V_\lambda)|_{A_\tau} \to (L_{-w_0\lambda} \otimes V_\lambda)|_{x_\tau}$. By Proposition 3.9 we have an inclusion of $\tau\tilde{G}_m$-modules

$$(L_{-w_0\lambda} \otimes V_\lambda)|_{x_\tau} \overset{s_\lambda}{\rightarrow} \bigoplus_{n=0}^{\langle \tau, -w_0\lambda \rangle} V^\tau_\lambda(n) \otimes kt^{-n} \subset \bigoplus_{n=0}^{\langle \tau, -w_0\lambda \rangle} \bigoplus_{m \geq -n} V^\tau_\lambda(n) \otimes kt^m \subset (V_\lambda)^{\tau}$$

induced from a $\tau\tilde{G}_m$-module section of $\psi^R_\tau(\lambda)$. Here the image of $s_\lambda$ is contained in a $\tau\tilde{G}_m$-isotypical component since $(L_{-\lambda_i} \otimes V_\lambda)|_{x_\tau}$ is of the form $\lambda_i^{\otimes \dim V_\lambda}$ regarded as a representation of $\tau\tilde{G}_m$. Moreover, the image of $s_\lambda$ generates $(V_\lambda)^{\tau}$ as a $k[t, t^{-1}]$-module.

Let $\phi^R_\tau : \mathcal{R}(E) \to \mathcal{R}(E)|_{x_\tau}$ be the residual map. It factors through $\psi^R_\tau : \mathcal{R}(E)|_{A_\tau} \to \mathcal{R}(E)|_{x_\tau}$. We define a $\tau\tilde{G}_m$-module inclusion $\mathcal{R}(E)|_{x_\tau} \overset{s}{\rightarrow} V^{\tau}_0$ as

$$s := \sum_{i=1}^{h} t^{\langle \tau, w_0\lambda_i - \lambda_i \rangle}, s_\lambda = \sum_{i=1}^{h} s_i.$$  

Since we have $\langle \tau, w_0\lambda - \lambda_i \rangle \in \mathbb{Z}$ for every $\tau \in \Sigma(1)$, the above construction is well-defined. $s$ is a sum of the inclusions of the form $s_\lambda$ with $t$-twist. Hence, the image of $s$ generates $V^\tau_0$ as a $k[t, t^{-1}]$-module. By the above construction, the image of $s_i$ is contained in a $\tau\tilde{G}_m$-isotypical component. Therefore, the image of $s_i$ form a $\tau\tilde{G}_m$-isotypical component since $\tilde{Z}(G) \times 1$ acts on the various $\tilde{L}_i$ by distinct characters.

Since $\mathcal{R}(E)|_{x_\tau}$ has a natural $u_-^\tau$-action induced from $(1 \times U_0^\tau) \subset \tilde{G}_\tau$, we can introduce a $u_-^\tau$-action on the image of $s$.

**Definition 3.16.** We define a right $u_-^\tau$-action $R_\tau$ on $B(E)_0^{\tau}$ by

$$R_\tau(X) : B(E)_0^{\tau} \ni \oplus_{n \in \mathbb{Z}} t^n.v_n \mapsto \oplus_{n \in \mathbb{Z}} t^n.(Xv_n) \in B(E)_0^{\tau}$$

for every $X \in u_-^\tau$.

**Corollary 3.17.** $\rho^\tau_\infty(E)$ is stable under the right $u_-^\tau$-action for every $E \in \text{ObEV}(\Sigma, 1)_c$.

**Proof.** By using $\mathcal{R}(E)^{-D} \subset \mathcal{E} \subset \mathcal{R}(E)^D$ instead of $B(E)^{-D} \subset \mathcal{E} \subset B(E)^D$ in Definition 2.26 we define

$$R^F \equiv \phi^R_\tau (E(-nD_\tau) \cap D \mathcal{R}(E)) \subset \mathcal{R}(E)|_{x_\tau}$$

for every $n \in \mathbb{Z}$. This is a $\tilde{P}^{-\tau}$-submodule of $B(E)$ by a similar argument as in the proof of Lemma 2.33 via the same isomorphism $k(e) \cong k(x_\tau)$ as in Definition 2.26. Then, we have

$$\Gamma(A_\tau, \mathcal{E}|_{A_\tau}) = \bigoplus_{m \in \mathbb{Z}} F^\tau(n, \mathcal{E}) \otimes kt^{-n} = \bigoplus_{m \in \mathbb{Z}} t^{-n}.R^F(n, \mathcal{E}).$$

as compatible $(k[t], \tau\tilde{G}\Delta^d(\tilde{L}^\tau))$-submodules of $B(E)_0^{\tau}$. Here the latter direct sum gives the $\tau\tilde{G}$-isotypical decomposition. Hence, the result follows from the definition of $\rho^\tau_\infty(E)$. 

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3.2.3 Compatibility

We work under the same assumptions as in §3.2.1. The main technical result of this subsection is the following.

**Theorem 3.18.** Let $\alpha$ be a root of $\mathfrak{g}$ such that $e_\alpha \in u_-^\tau$ (i.e. $\langle \tau, \alpha \rangle < 0$). Then, the right $u_-^\tau$-action $R_\tau(e_\alpha)$ of $e_\alpha$ is written as

$$R_\tau(e_\alpha) = c\bar{L}_\tau(e_\alpha) t^{-\langle \tau, \alpha \rangle},$$

where $\bar{L}_\tau$ is the left $\mathfrak{g}$-action and $c$ is a non-zero constant.

We first prove our main result (Proposition 3.12) by assuming Theorem 3.18.

**Final step of the proof of Proposition 3.12.** By Corollary 3.5, we have

$$\rho_\tau^\infty(E) \sim (\bigoplus_{n \in \mathbb{Z}} F^\tau(n, E) \otimes kt^{-n}) \subset B(E)[t].$$

Let $\alpha$ be a root such that $e_\alpha \in u_-^\tau$. By Corollary 3.17, $R_\tau(e_\alpha)$ must preserve $\rho_\tau^\infty(E)$ in $B(E)[t]$. By Theorem 3.18, it suffices to check whether the composition of the $\bar{L}_\tau(e_\alpha)$-action and multiplying $t^{-\langle \tau, \alpha \rangle}$ preserves $\rho_\tau^\infty(E)$ or not. Then, to preserve $\rho_\tau^\infty(E)$, we must have

$$(1 \otimes t^{-\langle \tau, \alpha \rangle}).e_\alpha F^\tau(n, E) \otimes kt^{-n} \subset F^\tau(n + \langle \tau, \alpha \rangle, E) \otimes kt^{-n - \langle \tau, \alpha \rangle}.$$ 

Hence, we have

$$U(u_\tau^\tau)^m F^\tau(n, E) \subset F^\tau(n + m, E)$$

This is the $\tau$-transversality condition. Hence, we have proved the result for $\tau$. Since $\tau$ is an arbitrary one-dimensional cone, we obtain the result.

Before the proof of Theorem 3.18 we need some preparations. We define two full-subcategories $\mathfrak{L}$ and $\mathfrak{R}$ of $EV(\Sigma, 1)_c$ as follows:

- $\text{Ob} \mathfrak{L} := \{V \otimes O_X; V \in \text{Rep}\tilde{G}\};$
- $\text{Ob} \mathfrak{R} := \{\oplus_{\lambda \in X^*(\tilde{T})} L_{\lambda} \otimes V_{\lambda}^{\otimes n}; n \in \mathbb{Z}_{\geq 0}\}.$

We put $\mathcal{V}^{\mathfrak{R}}_\lambda := L_{\lambda} \otimes V_\lambda$ for each $\lambda \in X^*(\tilde{T})$. $\mathfrak{L}$ is a rigid tensor category (cf. Deligne-Milne [13]) by the usual tensor product over $O_X$.

**Lemma 3.19.** There exists a tensor structure $\otimes'$ on $\mathfrak{R}$ which makes $\mathfrak{R}$ a rigid tensor category which is equivalent to $\text{Rep}\tilde{G}$.

**Proof.** Since $\mathfrak{R}$ is semisimple as an abelian category, it suffices to construct tensor products for each pair of irreducible objects. For each dominant weights $\lambda, \gamma$, we write an irreducible decomposition of $V_\lambda \otimes V_\gamma$ by

$$V_\lambda \otimes V_\gamma = \bigoplus_{\epsilon : \text{dominant}} V_{\epsilon}^{\otimes m_{\lambda, \gamma}}.$$ 

Again by semisimplicity, we can twist the above direct sum decomposition isotypical componentwise. We have

$$\mathcal{V}^{\mathfrak{R}}_\lambda \otimes \mathcal{V}^{\mathfrak{R}}_\gamma = L_{\omega_0(\lambda + \gamma)} \otimes \bigoplus_{\epsilon : \text{dominant}} V_{\epsilon}^{\otimes m_{\lambda, \gamma}}.$$ 

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We twist the component \( L^\nu_{w_0(\lambda+\gamma)} \otimes V^{\oplus m_{\lambda,\gamma}}_\epsilon \) by \( L^\nu_{w_0(\epsilon - \lambda - \gamma)} \). Notice that we have \( (\tau, w_0(\epsilon - \lambda - \gamma)) \in \mathbb{Z} \) for every \( \tau \in \Sigma(1) \). In particular, \( D_{-w_0(\epsilon - \lambda - \gamma)} \) is an equivariant divisor. Hence, defining the twisted tensor product \( \otimes' \) by

\[
\nu^R_\lambda \otimes' \nu^R_\gamma := \bigoplus_{\epsilon : \text{dominant}} (\nu^{R}_{\epsilon})^{\oplus m_{\lambda,\gamma}}
\]
yields the result.

Hence, we obtain two different rigid tensor categories \( \mathcal{L} \) and \( \mathfrak{R} \) which are equivalent to \( \text{Rep}\tilde{G} \). We denote their images under \( \rho^\infty_\tau \) by \( \tilde{\mathcal{L}}^\tau \) and \( \hat{\mathfrak{R}}^\tau \), respectively. Then, we have the following:

\[
\rho^\infty_\tau (\mathcal{E} \otimes_X \mathcal{F}) \cong \left( (\Gamma(\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau}) \otimes k[t]) \Gamma(\mathcal{A}_\tau, \mathcal{F})|_{\mathcal{A}_\tau} \right) \subset (B(\mathcal{E}) \otimes_k B(\mathcal{F}))|_{[t]}
\]

for every \( \mathcal{E}, \mathcal{F} \in \text{Ob}EV(\Sigma, 1)_c \), and

\[
\rho^\infty_\tau (\nu^R_\lambda \otimes' \nu^R_\gamma) \cong \bigoplus_{\epsilon : \text{dominant}} t^{(\tau,w_0(\epsilon - \lambda - \gamma))} \rho^\infty_\tau \left( L^\nu_{w_0(\lambda+\gamma)} \otimes V^{\oplus m_{\lambda,\gamma}}_\epsilon \right).
\]

By the above formulas, we equip \( \tilde{\mathcal{L}}^\tau \) and \( \hat{\mathfrak{R}}^\tau \) with tensor products \( \otimes_0 \) and \( \otimes'_0 \) that arise from \( \otimes_X \) and \( \otimes' \), respectively. We also define a tensor product \( \otimes_0 \) of \( \mathfrak{B}(\Sigma)^\tau \) by

\[
(A_1 \subset (V_1)^t_{[t]}) \otimes_0 (A_2 \subset (V_2)^t_{[t]}) := \left( (A_1 \otimes_k[t] A_2) \subset (V_1 \otimes V_2)^t_{[t]} \right)
\]

for each \( (A_1 \subset (V_1)^t), (A_2 \subset (V_2)^t) \in \text{Ob} \mathfrak{B}(\Sigma)^\tau \). This tensor product coincides with the tensor product of \( EV(\Sigma, 1)_c \) via \( \rho^\infty_0 \). Therefore, it is an extension of \( \otimes_0 \) in \( \tilde{\mathcal{L}} \).

**Proof of Theorem 3.18** Exponential \( \exp R_\tau (e_\alpha) \) of \( R_\tau (e_\alpha) \) acts on any object of \( \hat{\mathfrak{R}}^\tau \) and commutes with the tensor product \( \otimes_0 \) of two elements of \( \hat{\mathfrak{R}} \) (in \( \mathfrak{B}(\Sigma)^\tau \)). By construction, the right \( u_- \)-action commutes with \( t \)-twists. Hence, \( \exp R_\tau (e_\alpha) \)-action commutes with the tensor product \( \otimes'_0 \) of \( \hat{\mathcal{L}} \). We define a twisted right \( u_-^\circ \)-action on formal loop spaces by the following:

\[
R(e_\alpha) t^{(\tau,\alpha)} : V^\tau_{[t]} \ni v \mapsto R(e_\alpha) v \mapsto \left( R(e_\alpha) t^{(\tau,\alpha)} \right) v \in V^\tau_{[t]}
\]

for each \( V \in \text{Rep}\tilde{G} \). Here, the weight of this endomorphism of \( V^\tau_{[t]} \) with respect to \( G^\tau_m \)-action is zero. Thus, \( R(e_\alpha) t^{(\tau,\alpha)} \) preserves \( \rho^\infty_\tau (V \otimes \mathcal{O}_X) \). Hence, it operates on objects of \( \tilde{\mathcal{L}} \) compatibly with \( \otimes_0 \). Moreover, its image under the residual map \( \psi_\tau : V \otimes \mathcal{O}_X \to V \otimes k(x_\tau) \) is non-zero if \( R_\tau (e_\alpha) \) operates \( V^\tau_{[t]} \) nontrivially. \( \psi_\tau \) commutes with the decomposition rule of the tensor category \( \text{Rep}\tilde{G} \). Hence, \( \psi_\tau \) is a fiber functor of \( \tilde{\mathcal{L}} \). Here \( \exp (s R(e_\alpha) t^{(\tau,\alpha)}) \) commutes with \( \psi_\tau \) for every \( s \in k \). Thus, \( \exp (s R(e_\alpha) t^{(\tau,\alpha)}) \) defines some element of \( \tilde{G} \) (as an automorphism of \( V \otimes k(x_\tau) \) for every \( V \in \text{Rep}\tilde{G} \)) by [13] II Prop. 2.8. Hence, its derivative \( R(e_\alpha) t^{(\tau,\alpha)} \) defines some element of \( \hat{g} \). The weight of \( R(e_\alpha) t^{(\tau,\alpha)} \) is \( \alpha \) by \( \Delta^d(\hat{T}) \)-action. Moreover, every root space of a reductive Lie algebra is one-dimensional. Therefore, \( R(e_\alpha) t^{(\tau,\alpha)} \) must operate as a nonzero constant multiple of \( L_\tau (e_\alpha) \).

**Remark 3.20** The term "tensor category" in [13] (and this paper) is the same as "symmetric tensor category" in the standard notation after the invention of quantum groups (cf. Bakalov and Kirillov [2]).
4 Main theorem

Now we can state our main result as follows:

**Theorem 4.1.** We have an equivalence of categories

$$
\Xi : EV(\Sigma)_c \cong C(\Sigma)_c.
$$

Moreover, \( \Xi \) induces a category equivalence \( EV(\Sigma) \cong C(\Sigma) \).

**Example 4.2 (Tangent bundles).** Let \( G \) be a semisimple adjoint group. Let \( X = X(\Sigma_0) \) be its wonderful compactification. Then, the set of one-dimensional cones of \( \Sigma_0 \) is represented by the set of fundamental co-weights \( \omega^\vee_1, \ldots, \omega^\vee_r \) (cf. \$1.2). In this setting, we have

$$
\Xi(TX) \cong (g, \{ F^{\omega^\vee_i} (\bullet) \}),
$$

where

$$
F^{\omega^\vee_i} (n) = \begin{cases}
\bigoplus_{(\omega^\vee_i, \alpha) \geq n} ke_\alpha & (n \geq 2) \\
kh_i \bigoplus \bigoplus_{(\omega^\vee_i, \alpha) \geq n} ke_\alpha & (n = 1) \\
g & (n \leq 0)
\end{cases}
$$

Here \( e_\alpha \) is a root vector of \( g \) and we put \( H_i := [e_\alpha, e_{-\alpha}] \in t \). If we replace \( F^{\omega^\vee_i} (1) \) by \( \bigoplus_{(\omega^\vee_i, \alpha) \geq 1} ke_\alpha \), then the corresponding vector bundle is the “logarithmic” tangent bundle of \( X \) along the reduced union of boundary divisors. (cf. Bien-Brión [3].)

**Example 4.3 (Case \( G = PGL_2 \)).** In this case, the only nontrivial (partial) compactification of \( PGL_2 \) is the projectification \( \mathbb{P}(M_2) \) of the set of two by two matrices \( M_2 \), which is the wonderful compactification. Here, we have \( \tilde{G} = SL_2 \). Moreover, \( SL_2 \times SL_2 \)-action on \( \mathbb{P}(M_2) \) factors through the following action:

$$
SL_2 \times SL_2 \times M_2 \ni (g_1, g_2, A) \mapsto g_1Ag_2^{-1} \in M_2.
$$

We have a unique boundary divisor \( D \) of \( \mathbb{P}(M_2) \) described as the zeros of the determinant.

Let \( \alpha \) be a unique positive root of \( PGL_2 \) and let \( \varpi \) be the unique fundamental coweight of \( PGL_2 \).

For a \( SL_2 \)-module \( V \), we have the \( \tilde{T} \)-isotypical decomposition \( V = \bigoplus_{\lambda \in X^*(\tilde{T})} V^\lambda \). In this case, we define the character \( chV \) of \( V \) as \( chV := \sum_{\lambda \in X^*(\tilde{T})} (\dim V^\lambda) e^\lambda \).

Let \( E \) be a \( SL_2 \times SL_2 \)-equivariant vector bundle on \( X \) such that \( E \otimes_X k(e) \cong sl_2 \boxtimes k \) as \( SL_2 \times \mathbb{Z}/2\mathbb{Z} \)-modules. By Theorem 4.1, we can classify such \( SL_2 \times SL_2 \)-equivariant vector bundles in \( \text{Ob} C(\Sigma_0) \). We have \( \Xi(E) \cong (sl_2 \boxtimes k, F^{\varpi} (\bullet)) \). Then, we have the following classification by the transversality condition of Definition 1.5.

Every object in \( C(\Sigma_0) \) with the form \( (sl_2 \boxtimes k, F^{\varpi} (\bullet)) \) is given by one of the following four
objects up to degree shift:

\[
\begin{align*}
\text{ch} F^\omega (n, \mathfrak{sl}_2 \otimes O_X) &= \begin{cases} 
0 & (n \geq 1) \\
\epsilon^\alpha + \epsilon^0 + \epsilon^{-\alpha} & (n \leq 0)
\end{cases} \\
\text{ch} F^\omega (n, O_X \otimes \mathfrak{sl}_2) &= \begin{cases} 
0 & (n \geq 2) \\
\epsilon^\alpha & (n = 1) \\
\epsilon^\alpha + \epsilon^0 & (n = 0) \\
\epsilon^\alpha + \epsilon^0 + \epsilon^{-\alpha} & (n \leq -1)
\end{cases} \\
\text{ch} F^\omega (n, TX) &= \begin{cases} 
0 & (n \geq 2) \\
\epsilon^\alpha + \epsilon^0 & (n = 1) \\
\epsilon^\alpha + \epsilon^0 + \epsilon^{-\alpha} & (n \leq 0)
\end{cases} \\
\text{ch} F^\omega (n, T^*X) &= \begin{cases} 
0 & (n \geq 1) \\
\epsilon^\alpha & (n = 0) \\
\epsilon^\alpha + \epsilon^0 + \epsilon^{-\alpha} & (n \leq -1)
\end{cases}
\end{align*}
\]

We say a $SL_2 \times SL_2$-equivariant vector bundle $E$ on $\mathbb{P}(M_2)$ irreducible, if there exists no proper $SL_2 \times SL_2$-equivariant vector subbundle in $EV(\Sigma_0)$. Applying the inverse functor of $\Xi$, we obtain:

Every irreducible rank three $SL_2 \times SL_2$-equivariant vector bundle on $\mathbb{P}^3$ is isomorphic to $\mathfrak{sl}_2 \otimes O_X$, $O_X \otimes \mathfrak{sl}_2$, $TX$, or $T^*X$ up to line bundle twist.

Similarly, we can prove that the number of isomorphism classes of irreducible $SL_2 \times SL_2$-equivariant vector bundles of rank $n$ on $\mathbb{P}^3$ is just $2^n - 1$ up to line bundle twist.

Remark 4.4 (Tensor structures). All categories in Theorem 4.1 have natural tensor products. In fact, $\Xi$ is a tensor functor when we restrict to $EV(\Sigma, 1)$ or $EV(\Sigma, 1)$. However, due to our presentation of the category $\mathcal{C}$, $\Xi$ does not preserve natural tensor products in general.

We postpone the proof of our main theorem until §4.2 since we need the inverse functor $\Phi : \mathcal{C}(\Sigma) \rightarrow EV(\Sigma)$ in order to prove Theorem 4.1.

4.1 Construction of the inverse functor

In this subsection, we construct an inverse functor $\Phi$ of $\Xi$. To achieve this, we construct some $\tilde{G} \times G$-equivariant vector bundle from a given object $(V, \{F^\tau(\bullet)\})$ of $\mathcal{C}(\Sigma, 1)_c$.

4.1.1 The functor $\Phi$

Here we devote ourselves to construct a functor $\Phi$ by assuming the following Proposition 4.5. This Proposition is proved in §4.1.2.

Proposition 4.5. Let $A := (V, \{F^\tau(\bullet)\}) \in \text{Ob} \mathcal{C}(\Sigma, 1)_c$ and choose an arbitrary $\tau \in \Sigma(1)$. Assume that the following condition holds:

\[\diamond \ F^\xi (0) = V \text{ and } F^\xi (1) = \{0\} \text{ for every } \xi \in \Sigma(1) \backslash \{\tau\}.\]

Then, there exist a $\tilde{G} \times G$-equivariant vector bundle $\Phi^\tau(A)$ with the following properties:

a. $\Xi(\Phi^\tau(A)) = (V, \{F^\tau(\bullet)\});$

b. There exist $\tilde{G} \times G$-equivariant inclusions $V \otimes O_X (-ND_\tau) \hookrightarrow \Phi^\tau(A) \hookrightarrow V \otimes O_X (ND_\tau)$ for a sufficiently large integer $N$. 

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Moreover, for each $\mathcal{B} := (V_1, \{F^\tau_1(\bullet)\}) \in \text{ObC}(\Sigma, 1)_c$ which satisfies $\diamond$, we have a natural inclusion

$$\text{Hom}_{\mathcal{C}(\Sigma)_c}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Hom}_{\text{EV}(\Sigma)_c}(\Phi^\tau(\mathcal{A}), \Phi^\tau(\mathcal{B})).$$

For a $\tilde{G}$-module $V$, we define a filtration $F_{null}(\bullet, V)$ of $V$ indexed by $\mathbb{Z}$ as follows:

$$F_{null}(n, V) = 0 \text{ if } n \geq 1 \text{ and } V \text{ if } n \leq 0.$$ This is clearly a $\tau$-transversal filtration for every $\tau \in \Sigma(1)$.

For every $\mathcal{A} = (V, \{F^\tau(\bullet)\}) \in \text{ObC}(\Sigma, 1)_c$ and every $\tau \in \Sigma(1)$, we define

$$\mathcal{A}_\tau := (V, \{F_{null}(\bullet, V)\}_{\eta \in \Sigma(1) \setminus \{\tau\}} \cup \{F^\tau(\bullet)\}) \in \text{ObC}(\Sigma, 1)_c.$$ $\mathcal{A}_\tau$ satisfies the assumption of Proposition 4.5. We put $D_0 := \sum_{\tau \in \Sigma(1)} D_\tau$ and $D_\eta := \sum_{\tau \in \Sigma(1) \setminus \eta} D_\tau$ for each $\eta \in \Sigma(1)$. For each $\mathcal{A} \in \text{ObC}(\Sigma, 1)_c$, we define

$$\Phi_1(\mathcal{A}) := \bigcap_{\tau \in \Sigma(1)} \Phi^\tau(\mathcal{A}_\tau) \otimes_X \mathcal{O}_X(MD_\tau^c) \subset V \otimes \mathcal{O}_X(MD_0)$$

Here we assume that $M$ is a sufficiently large integer and drop the sufficiently large equivariant divisor $D$ needed to define the intersection (Lemma 3.3) by letting $D = MD_0$. We use this convention throughout this subsection. It is clear that $\Phi_1(\mathcal{A})$ does not depend on the choice of $M$. We have $\Phi_1(\mathcal{A})|_{X(\tau)} \cong \Phi^\tau(\mathcal{A}_\tau)|_{X(\tau)}$ for each $\tau \in \Sigma(1)$.

**Lemma 4.6.** For every $(V, \{F^\tau(\bullet)\}) \in \text{ObC}(\Sigma, 1)_c$, $\Phi_1((V, \{F^\tau(\bullet)\}))$ is a $\tilde{G} \times G$-equivariant vector bundle.

**Proof.** What we need to show is that $\Phi_1((V, \{F^\tau(\bullet)\}))$ is a vector bundle. We restrict our construction from $X(\Sigma)$ to $T(\Sigma)$ to obtain the corresponding object in Klyachko's category. We fix an inclusion $\Phi_1((V, \{F^\tau(\bullet)\})) \otimes_X \mathcal{O}_X(MD_\tau^c) \subset V \otimes \mathcal{O}_X(MD_0)$. It induces the corresponding inclusion between the identity fibers. For each $\sigma \in \Sigma$, the family of filtrations $F_{null}(\bullet, V)$ and $\{F^\tau(\bullet)\}_{\tau \in \Sigma(1)}$ forms a distributive lattice. Hence, they satisfies the condition of Klyachko's category [1]. As a result,

$$\bigcap_{\tau \in \Sigma(1)} (\Phi^\tau((V, \{F^\tau(\bullet)\})) \otimes_X \mathcal{O}_X(MD_\tau^c)|_{T(\Sigma)}) \subset V \otimes \mathcal{O}_X(MD_0)|_{T(\Sigma)}$$

is a vector bundle for a sufficiently large integer $M$. Since $\Phi_1((V, \{F^\tau(\bullet)\}))$ is a $\tilde{G} \times G$-equivariant coherent sheaf, the local structure theorem (Theorem 1.3) asserts that its restriction to the open subset

$$T(\Sigma) \times \mathbb{A}^{2 \dim U} \cong U^{-1} \cdot T(\Sigma).U \subset X(\Sigma)$$

is still a vector bundle. Further, this open subset meets all $(\tilde{G} \times G)$-orbits, which yields the result.

It follows that the map $\Phi_1: \text{ObC}(\Sigma, 1)_c \rightarrow \text{ObEV}(\Sigma, 1)_c$ satisfies $\Xi \circ \Phi_1 = \text{id}$ on $\text{ObC}(\Sigma, 1)_c$.

**Lemma 4.7.** For every $\mathcal{A}, \mathcal{B} \in \text{ObC}(\Sigma, 1)_c$, we have a natural inclusion

$$\text{Hom}_{\mathcal{C}(\Sigma)_c}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Hom}_{\text{EV}(\Sigma)_c}(\Phi_1(\mathcal{A}), \Phi_1(\mathcal{B})).$$
Proof. Put \( \mathcal{A} := (V, \{ F^\tau (\bullet) \} ) \) and \( \mathcal{B} := (V_1, \{ F_1^\tau (\bullet) \} ) \). For each morphism \( f : (V, \{ F^\tau (\bullet) \} ) \rightarrow (V_1, \{ F_1^\tau (\bullet) \} ) \) in \( \mathcal{C}(\Sigma, 1)_c \), we have a base morphism \( V \rightarrow V_1 \). Hence, we have a corresponding morphism \( f_\tau : (V, \{ F^\tau (\bullet) \} )_\tau \rightarrow (V_1, \{ F_1^\tau (\bullet) \} )_\tau \) for every \( \tau \in \Sigma(1) \). By Proposition 4.5, we have a corresponding morphism \( f'_\tau : \Phi^\tau ((V, \{ F^\tau (\bullet) \} )_\tau ) \rightarrow \Phi^\tau ((V_1, \{ F_1^\tau (\bullet) \} )_\tau ) \). Then, by taking the intersection of the both sides compatibly with \( f \otimes \text{id} : V \otimes O_X(MD_0) \rightarrow V_1 \otimes O_X(MD_0) \), we have a corresponding morphism
\[
\text{id} : \bigcap_{\tau \in \Sigma(1)} (\Phi^\tau ((V_1, \{ F_1^\tau (\bullet) \} ))_\tau ) \otimes O_X(MD_0) = \Phi_1 ((V_1, \{ F_1^\tau (\bullet) \} )).
\]
If \( f \) is nontrivial, then its base morphism \( V \rightarrow V_1 \) is also nontrivial. Hence, \( f'' \) induces nontrivial morphism.

For each \( \lambda \in X^*(\tilde{T}) \), we have a category equivalence \( \lambda : \mathcal{C}(\Sigma, 1)_c \rightarrow \mathcal{C}(\Sigma, \bar{\lambda})_c \) induced from
\[
\text{Ob} \mathcal{C}(\Sigma, 1)_c \ni (V \otimes k, \{ F^\tau (\bullet) \}) \mapsto (V \otimes \bar{\lambda}, \{ F^\tau (\bullet - [\tau, \lambda]) \}) \in \text{Ob} \mathcal{C}(\Sigma, \bar{\lambda})_c.
\]
For \( \lambda \in X^*(T) \), \( \lambda \) yields a category auto-equivalence which corresponds to \( \otimes X O_X(D^\lambda) \) via \( \Xi \) (from Lemma 2.19 and Definition 2.28). Hence, we have the following commutative diagram.
\[
\begin{array}{ccc}
\mathcal{C}(\Sigma, 1)_c & \xrightarrow{\lambda} & \mathcal{C}(\Sigma, 1)_c \\
\Phi_1 \downarrow & & \downarrow \Phi_1 \\
EV(\Sigma, 1)_c & \xrightarrow{\otimes X O_X(D^\lambda)} & EV(\Sigma, 1)_c
\end{array}
\]
for every \( \lambda \in X^*(T) \).

Similarly, we have a category auto-equivalence \( \lambda \) on \( \mathcal{C}(\Sigma)_c \) for every \( \lambda \in X^*(\tilde{T}) \) as the direct sum of the composition of the inverse of \( \mu \) and \( \lambda + \mu \) in
\[
\mathcal{C}(\Sigma, \bar{\mu})_c \xrightarrow{\lambda} \mathcal{C}(\Sigma, 1)_c \xrightarrow{\lambda + \mu} \mathcal{C}(\Sigma, \bar{\lambda} + \bar{\mu})_c \text{ for } \mu \in X^*(\tilde{T}).
\]
Notice that the above construction is independent of the representative \( \mu \) of \( \bar{\mu} \). By the category auto-equivalence \( \lambda \) on \( \mathcal{C}(\Sigma)_c \), define the category equivalence \( \Phi_\lambda \) as
\[
\Phi_\lambda := (\otimes X \mathcal{L}_\lambda) \circ \Phi_1 \circ (\lambda)^{-1} : \mathcal{C}(\Sigma, \bar{\lambda})_c \rightarrow EV(\Sigma, \bar{\lambda})_c.
\]
Now, we extend the category equivalence from \( \mathcal{C}(\Sigma, 1)_c \rightarrow EV(\Sigma, 1)_c \) to \( \mathcal{C}(\Sigma)_c \rightarrow EV(\Sigma)_c \) by setting \( \Phi := \otimes \chi \in Z(G)_c \Phi_\lambda \).

Summarizing the above, we have the following result.

**Proposition 4.8.** \( \Phi : \mathcal{C}(\Sigma)_c \rightarrow EV(\Sigma)_c \) is a faithful covariant functor. Moreover, \( \Xi \circ \Phi \) is the identity on \( \text{Ob} \mathcal{C}(\Sigma)_c \).

### 4.1.2 Simple case

Here we prove Proposition 4.5. We fix \( \tau \in \Sigma(1) \) which is the same as in the statement of Proposition 4.5. First, we recall some notation and prove a preliminary result.

By the description of 4.2, we have \( O_\tau \cong (G \times G)/G^\tau \). Moreover, the base point \( 1 \times 1 \text{ mod } G^\tau \) corresponds to \( x_\tau \). We denote the Lie algebra of \( G^\tau \) by \( g^\tau \).
Lemma 4.9. For every $\mathcal{E} \in \text{ObEV} (\Sigma, 1)_c$, combining $\Delta^d(\tilde{L}_m^\tau)$-action, the left $u_+^\tau$-action, and the right $u_-^\tau$-action, we can introduce a $\mathfrak{g}^\tau$-module structure on

$$\Gamma (\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau}) / \tau \Gamma (\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau}) (\cong \mathcal{E} \otimes_X k(\tau_x) \text{ as } \mathcal{O}_{X, \tau_x}\text{-modules})$$

which coincides with the natural $\mathfrak{g}^\tau$-module structure on $\mathcal{E} \otimes_X k(\tau_x)$.

Proof. We have $\mathfrak{g}^\tau = \text{Lie}(\Delta^d(\tilde{L}_m^\tau)) + (u_+^\tau + u_-^\tau) \subset \mathfrak{g} \oplus \mathfrak{g}$ (see 1.2). By construction, $\Gamma (\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau})$ inherits a natural $\Delta^d(\tilde{L}_m^\tau)$-action from $\mathcal{E}$. Hence, the only nontrivial part is about the left $u_+^\tau$-action and the right $u_-^\tau$-action. We show that the left $u_+^\tau$-action and the right $u_-^\tau$-action defined in 3.5 indeed satisfies the above property. For each $\alpha, \beta \in \Delta$ such that $e_\alpha \in u_+^\tau$ and $e_\beta \in u_-^\tau$, we have $[L_\tau(e_\alpha), R_\tau(e_\beta)] = [L_\tau(e_\alpha), \tilde{L}_\tau(e_\beta)]t^{-(\tau, \alpha + \beta)}$. $\rho^\infty_{\infty} (\mathcal{E})$ is preserved by both the left $u_+^\tau$-action and the right $u_-^\tau$-action.

We have $\langle \tau, \alpha + \beta \rangle > \langle \tau, \beta \rangle$. As a result, the left $u_+^\tau$-action and the right $u_-^\tau$-action commute on $\Gamma (\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau}) / \tau \Gamma (\mathcal{A}_\tau, \mathcal{E}|_{\mathcal{A}_\tau})$. Therefore, we can restrict ourselves to $\Delta^d(\tilde{L}_m^\tau)(U_+^\tau \times 1)$. In this case, the assertion is a consequence of Lemma 2.8 and the isotypical decomposition of Corollary 3.5. \hfill \Box

Proof of Proposition 4.9. Let $N$ be an integer such that $F^\tau (N - 1) = \{0\}$ and $F^\tau (-N) = V$. By induction on $n$, we construct a $\tilde{G} \times G$-equivariant vector bundle $\Phi^\tau_n (\mathcal{A})$ with the following properties (which we call $(\#)_n$):

1. $V \otimes \mathcal{O}_X (-ND_\tau) \hookrightarrow \Phi^\tau_n (\mathcal{A}) \hookrightarrow V \otimes \mathcal{O}_X (nD_\tau)$ as $\tilde{G} \times G$-equivariant coherent sheaves;
2. $F^\tau (m, \Phi^\tau_n (\mathcal{A})) = F^\tau (m + N - n)$ for $m > -N$;
3. $F^\tau (m, \Phi^\tau_n (\mathcal{A})) = V$ for $m \leq -N$.

Here we put $\Phi^\tau_n (\mathcal{A}) := \Phi^\tau_n (\mathcal{A})$. Since $(\#)_N$ is the same as the conditions of Proposition 4.5, $\Phi^\tau_n (\mathcal{A})$ has the desired property if such an induction proceeds. We have $F^\tau (m, \Phi^\tau_n (\mathcal{A})) = F^\tau (m + 2N) = 0$ for $m > -N$. Thus, $\Phi^\tau_n (\mathcal{A}) := V \otimes \mathcal{O}_X (-ND_\tau)$ satisfies the property $(\#)_N$.

Then, assuming $(\#)_n$, we construct $\Phi^\tau_{n+1} (\mathcal{A})$ with the property $(\#)_{n+1}$.

We denote $\text{coker} [\Phi^\tau_n (\mathcal{A}) \rightarrow \Phi^\tau_n (\mathcal{A}) \otimes_X \mathcal{O}_X (D_\tau)]$ by $Q^\tau_{n+1} (\mathcal{A})$. Consider the following short exact sequence of $\tilde{G} \times G$-equivariant coherent sheaves:

$$0 \rightarrow \Phi^\tau_n (\mathcal{A}) \rightarrow \Phi^\tau_n (\mathcal{A}) \otimes_X \mathcal{O}_X (D_\tau) \rightarrow Q^\tau_{n+1} (\mathcal{A}) \rightarrow 0 \quad (4.1.1)$$

Since $\mathcal{O}_X (-D_\tau)$ annihilates $Q^\tau_{n+1} (\mathcal{A})$, we can regard $Q^\tau_{n+1} (\mathcal{A})$ as a $\tilde{G} \times G$-equivariant vector bundle on $D_\tau$. To proceed the proof, we need a lemma.

Lemma 4.10. We have a natural inclusion of $\mathfrak{g}^\tau$-modules

$$\frac{V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F^\tau (m + N - n) \otimes kt^{-m-1}}{V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F^\tau (m + N - n) \otimes kt^{-m-1}} \hookrightarrow Q^\tau_{n+1} (\mathcal{A}) \otimes_X k(\tau_x).$$

Proof. By Corollary 3.5, we have

$$\frac{V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F^\tau (m + N - n) \otimes kt^{-m-1}}{V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F^\tau (m + N - n) \otimes kt^{-m-1}} \cong Q^\tau_{n+1} (\mathcal{A}) \otimes_X k(\tau_x).$$

Since each term is $\left( \Delta^d(\tilde{L}_m^\tau), L_\tau, R_\tau, k[t] \right)$-stable by the induced action from $V^\tau_0$, we have the result by Lemma 4.9. \hfill \Box
We denote by \( S^0_{n+1}(A) \) the \( \tilde{G} \times G \)-equivariant vector bundle on \( O_\tau \) whose fiber at \( x_\tau \) is isomorphic to the LHS of the Lemma 4.10 as a \( g^\tau \)-module. Hence, we have an inclusion of \( G \times G \)-equivariant vector bundles \( S^0_{n+1}(A) \subset Q_{n+1}(A)|_{O_\tau} \). We define a \( G \times G \)-equivariant coherent subsheaf \( S_{n+1}(A) \) of \( Q_{n+1}(A) \) as follows:

\[
S_{n+1}(A)(U) := \{ s \in \Gamma(U, Q_{n+1}(A)); s|_{O_\tau} \in \Gamma(U \cap O_\tau, S^0_{n+1}(A)) \}
\]

for every Zariski open set \( U \subset D_\tau \).

The preimage \( '\Phi^\tau_{n+1}(A) \) of \( S_{n+1}(A) \) by \( \Phi^\tau_n(A) \otimes_X O_X(D_\tau) \rightarrow Q_{n+1}(A) \) is a \( \tilde{G} \times G \)-equivariant \( O_X \)-module by the assumption (on \( \Phi^\tau_n(A) \)) and the construction (of \( Q_n(A) \) and \( Q_{n+1}(A) \)).

Since we have \( '\Phi^\tau_{n+1}(A) \subset '\Phi^\tau_n(A) \otimes_X O_X(D_\tau) \), it follows that \( '\Phi^\tau_{n+1}(A) \) satisfies the condition 1 of (\( \mathbb{z} \))\(_{n+1} \).

We show that \( '\Phi^\tau_{n+1}(A) \) is a vector bundle in order to check (\( \mathbb{z} \))\(_{n+1} \). We have a composition

\[
\text{Res} : \text{coh}^\tilde{G} \times G X(\Sigma) \rightarrow \text{coh}^\tilde{G} \times G \mathcal{B} U^{-T}(\Sigma). U \stackrel{\cong}{\rightarrow} \text{coh}^\tilde{T} \times T T(\Sigma)
\]

of restriction functors. Here the second equivalence is a consequence of the local structure theorem (Theorem 4.3). This also implies that a \( \tilde{G} \times G \)-equivariant coherent sheaf on \( X(\Sigma) \) is a vector bundle if, and only if, its restriction to \( T(\Sigma) \) is a vector bundle. Hence, what has to be proved is:

**Lemma 4.11.** For each \( \sigma \in \Sigma \) such that \( \tau \in \sigma(1) \), \( '\Phi^\tau_{n+1}(A)|_{T(\sigma)} \) is a vector bundle.

*Proof.* \( S_{n+1}(A)|_{T(\sigma)} \) is isomorphic to the sheaf obtained by replacing \( U \) by \( U \cap T(\sigma) \) and taking tensor products with \( O_{T(\sigma)} \) at the RHS of (4.1.2). Since \( T(\sigma) \) is smooth, we can write \( T(\sigma) := \text{Spec} \mathcal{R}, \mathbb{W} = k[t, t_2, \ldots, t_m, t^\pm_{m+1}, \ldots, t^\pm_1] \). Here we assume that \( D_\tau \cap T(\sigma) \) is defined by \( t = 0 \) and each of \( t, t_2, \ldots, t_m \) spans a extremal ray of the dual cone \( \sigma^\vee \). By Theorem 2.18 1), \( '\Phi^\tau_n(A)|_{T(\sigma)} \) is written as \( V_0 \otimes \mathcal{R} \) by some \( 1 \times T \)-module \( V_0 \). We have

\[
(\Phi^\tau_n(A) \otimes_X O_X(D_\tau)|_{T(\sigma)}) \otimes_{O_{T(\sigma)}} k(x_\tau) \cong Q_{n+1}(A) \otimes_X k(x_\tau).
\]

Hence, we have

\[
Q_{n+1}(A)|_{T(\sigma)} \cong V_0 \otimes t^{-1}k[t_2, \ldots, t_m, t^\pm_{m+1}, \ldots, t^\pm_1]
\]

as a \( \mathcal{R}/t\mathcal{R} \)-module. For a vector subspace \( V_1 := S^0_{n+1}(A) \otimes_X k(x_\tau) \) of \( V_0 \), we have

\[
S^0_{n+1}|_{T(\sigma)} \cong V_0 \otimes t^{-1}k[t_2, \ldots, t_m, t^\pm_{m+1}, \ldots, t^\pm_1] \cap V_1 \otimes t^{-1}k[t^\pm_2, \ldots, t^\pm_1] = V_1 \otimes t^{-1} \mathcal{R}/\mathcal{R}
\]

Thus, we have

\[
'\Phi^\tau_{n+1}(A)|_{T(\sigma)} \cong V_0 \otimes \mathcal{R} + V_1 \otimes t^{-1} \mathcal{R} \subset V_0 \otimes t^{-1} \mathcal{R}.
\]

Since this module is \( \mathcal{R} \)-free, we obtain the result. \( \square \)

**Continuation of the proof of Proposition 4.5**

The definition of \( \rho \) (\( '\Phi^\tau_{n+1}(A) \)) makes sense since \( '\Phi^\tau_{n+1}(A) \) is a \( \tilde{G} \times G \)-equivariant vector bundle. We have

\[
\rho(\Phi^\tau_{n+1}(A)) = \left( V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F^\tau (m + N - n) \otimes k t^{-m-1} \subset V^\tau_{[\ell]} \right)
\]

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by the comparison with the corresponding extension over \(T(\tau)\) via \(\text{Res.}\).

In other words, \(\Phi_{n+1}^r(A)\) satisfies 2) and 3) of \((z)_{n+1}\). Hence, putting \(\Phi_{n+1}^r(A) := \Phi_{n+1}^r(A)\) proceeds the induction on \(n\).

Now we construct a morphism \(\Phi^r(f) : \Phi^r(A) \to \Phi^r(B)\) from \(f : A \to B\). Enlarge \(N\) if necessary to assume \(F_1^r(N) = \{0\}\) and \(F_1^r(-N) = V_1\). We construct a morphism \(f_n^r \colon \Phi_n^r(A) \to \Phi_n^r(B)\) by induction on \(n\). For \(n = -N\), we put \(f_{-N}^r := f \otimes \text{id} : V \otimes \mathcal{O}_X(-ND_\tau) \to V_1 \otimes \mathcal{O}_X(-ND_\tau)\). Assume that we have \(f_n^r\). For each morphism \(f : A \to B\), we have the following associated commutative diagram of compatible \(\left(\triangle^d(\tilde{L}^r)\mathcal{G}^r_m, L_\tau, R_\tau, k[t]\right)\)-modules for all \(n \in \mathbb{Z}\).

\[
\begin{array}{c}
V \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F_1^r (m + N - n) \otimes kt^{-m-1} & \hookrightarrow & V_1^r[t] \\
\downarrow & & \downarrow \\
V_1 \otimes t^N k[t] + \oplus_{m \in \mathbb{Z}} F_1^r (m + N - n) \otimes kt^{-m-1} & \hookrightarrow & (V_1)^r[t]
\end{array}
\]

Hence, we have the following commutative diagram arising from the \(\mathfrak{g}^r\)-module homomorphism of Lemma \[4.9\]

\[
\begin{array}{c}
\mathcal{S}_{n+1}(A) & \hookrightarrow & \mathcal{Q}_{n+1}(A) & \leftarrow & \Phi_{n+1}(A) \otimes \mathcal{O}_X(D_\tau) \\
\downarrow & & \downarrow & & \downarrow f_n^r \otimes \text{id} \\
\mathcal{S}_{n+1}(B) & \hookrightarrow & \mathcal{Q}_{n+1}(B) & \leftarrow & \Phi_{n}(B) \otimes \mathcal{O}_X(D_\tau)
\end{array}
\]

Hence, taking preimages yield a morphism \(f_{n+1}^r : \Phi_{n+1}^r(A) \to \Phi_{n+1}^r(B)\) of \(\tilde{G} \times G\)-equivariant \(\mathcal{O}_X\)-modules. Therefore, the induction on \(n\) proceeds. Thus, setting \(\Phi^r(f) := f_N^r\) yields a morphism \(\Phi_{\text{hom}}^r : \text{Hom}_{\mathcal{E}(\Sigma, 1)_c}(A, B) \to \text{Hom}_{\mathcal{E}V(\Sigma, 1)_c}(\Phi^r(A), \Phi^r(B))\). Since \(f : V \to V_1\) yields a morphism between the identity fibers of \(\Phi^r(A)\) and \(\Phi^r(B)\), the injectivity of \(\Phi_{\text{hom}}^r\) is clear.

### 4.2 Proof of Theorem 4.1

By Proposition \[3.10\] and Proposition \[4.8\], both \(\Xi\) and \(\Phi\) define one-to-one correspondences. By Proposition \[2.35\] and Proposition \[4.8\], both \(\Xi\) and \(\Phi\) are faithful. Hence, we have the following category equivalence:

\[\Xi : EV(\Sigma)_c \xrightarrow{\cong} C(\Sigma)_c.\]

Next, we want to check what happens if we restrict our attention to \(EV(\Sigma)\).

**Lemma 4.12.** Let \(\mathcal{E}, \mathcal{F} \in \text{Ob}EV(\Sigma, 1)_c\) and let \(f : \mathcal{E} \to \mathcal{F}\) be a morphism in \(EV(\Sigma, 1)\). Then, for each \(\tau \in \Sigma(1)\), we have

\[
\Xi(f) \left( F^r(n, \mathcal{E}) \right) = \Xi(f) \left( B(\mathcal{E}) \right) \cap F^r(n, \mathcal{F}) \quad \text{for all } n \in \mathbb{Z}
\]

and

\[
\{ \Xi(f)(B(\mathcal{E})) \cap F^r(n, \mathcal{F}) \}_{n \in \mathbb{Z}} \text{ forms a distributive lattice.}
\]

**Proof.** Consider the restriction to \(A_\tau\) as in §3.1.1. Since a \(\mathcal{G}_m^r\)-equivariant vector bundle on \(A_\tau\) splits, we have

\[
\mathcal{F}|_{A_\tau} \cong \text{Im} f|_{A_\tau} \oplus \text{coker} f|_{A_\tau}
\]

as \(\mathcal{G}_m^r\)-equivariant vector bundles. Hence, we have a splitting

\[
F(n, \mathcal{F}) = (F(n, \mathcal{F}) \cap \text{Im} \Xi(f)) \oplus (F(n, \mathcal{F}) \cap \text{coker} \Xi(f))
\]

for every \(n \in \mathbb{Z}\). \(\square\)
Let \( \mathcal{E}, \mathcal{F} \in \text{Ob}EV(\Sigma) \) (to indicate \( \text{Ob}EV(\Sigma) \)) and let \( f : \mathcal{E} \to \mathcal{F} \) be a morphism of \( EV(\Sigma) \). Then \( \Xi(f) \) satisfies the condition (L) of Definition 1.7 by Lemma 2.12. By definition, \( \text{coker} f \) is a \( G \times \tilde{G} \)-equivariant vector bundle. Thus, \( \Xi(f) \) also satisfies the condition (R) of Definition 1.7 by Lemma 2.14. Hence, \( \Xi(f) \) is a morphism of \( \mathcal{C}(\Sigma) \).

We prove its converse. Suppose \( (V_1, \{ F_1^\tau(\bullet) \}), (V_2, \{ F_2^\tau(\bullet) \}) \in \text{Ob}\mathcal{C}(\Sigma, 1) \) and let \( f' : (V_1, \{ F_1^\tau(\bullet) \}) \to (V_2, \{ F_2^\tau(\bullet) \}) \) be a morphism in \( \mathcal{C}(\Sigma, 1) \). Then, we have a morphism \( \Phi(f') \) of \( EV(\Sigma, 1)_c \). By the definition of morphisms of \( \mathcal{C}(\Sigma, 1) \), we have

\[
\text{Coker} f' := (\text{coker} f', \{ F_2^\tau(\bullet) / f'(F_1^\tau(\bullet)) \}) \in \text{Ob}\mathcal{C}(\Sigma, 1).
\]

Our construction of \( \Phi \) coincides with that of Klyachko’s when restricted to \( T(\Sigma) \). Therefore, the condition (R) of Definition 1.7 yields

\[
\Phi((V_1, \{ F_1^\tau(\bullet) \}))|_{T(\sigma)} \cong \Phi((f'(V_1), \{ f'(F_1^\tau(\bullet)) \}))|_{T(\sigma)} \oplus \text{coker}(\Phi(f'))|_{T(\sigma)}
\]

for each \( \sigma \in \Sigma \). Thus, both \( \text{coker} \Phi(f')|_{T(\Sigma)} \) and \( \text{Im} \Phi(f')|_{T(\Sigma)} \) are vector bundles. Hence, \( (G \times \tilde{G})T(\Sigma) = X(\Sigma) \) asserts that both \( \text{coker} \Phi(f') \) and \( \text{Im} \Phi(f') \) are vector bundles. By the similar argument, \( \ker \Phi(f') \) is also a vector bundle. Therefore, \( \Phi(f') \) is a morphism of \( EV(\Sigma) \).

By the arguments before Proposition 1.8 we obtain the category equivalence

\[
\Xi : EV(\Sigma) \to \mathcal{C}(\Sigma)
\]

from the equivalence of the ambient categories.

## 5 Consequences of the main theorem

### 5.1 Comparison with Klyachko’s category

In this subsection, we compare our category \( \mathcal{C}(\Sigma) \) to that of Klyachko’s.

We have a natural \( \tilde{T} \times \tilde{T} \)-equivariant embedding \( T(\Sigma) \hookrightarrow X(\Sigma) \). We denote the category of \( \tilde{T} \times \tilde{T} \)-equivariant vector bundles on \( T(\Sigma) \) by \( EV(\Sigma)_T \) (to distinguish it from \( EV(\Sigma) \), the category of \( G \times \tilde{G} \)-equivariant vector bundles on \( X(\Sigma) \)). Then, pullback defines a functor

\[
\text{rest}^T_{\Sigma} : EV(\Sigma) \to EV(\Sigma)_T.
\]

Let \( \mathcal{C}(\Sigma)_T \) be the category \( \mathcal{C}(\Sigma) \) in 1.13 which is obtained by replacing \( G \) with \( T \). We define a functor restricting the \( \tilde{G} \)-action to the \( \tilde{T} \)-action by

\[
\text{rest}^T_{\Sigma} : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)_T.
\]

Then, we have \( \Xi \circ \text{rest}^T_{\Sigma} = \text{rest}^T_{\Sigma} \circ \Xi \) since the construction of \( \Xi \) in 1.14 is in terms of \( T \)-orbits of \( e \). Define a full-subcategory \( \mathcal{C}(\Sigma)_T \) of \( \mathcal{C}(\Sigma)_T \) as follows:

\[
\text{Ob}\mathcal{C}(\Sigma)_T := \{ (V, \{ F^T(\bullet) \}) \in \text{Ob}\mathcal{C}(\Sigma)_T : V \text{ is trivial as a } \tilde{T}_s\text{-module} \}.
\]

Here we regard \( \tilde{T} \cap \tilde{G}_s \) as a subgroup of \( \tilde{T} \times \tilde{Z}(T) \).

In this case, we can naturally regard \( V \) as a \( \tilde{G}_s \)-module with trivial \( \tilde{G}_s \)-action. Since the transversality condition of Definition 1.9 is a void condition for a direct sum of trivial \( \tilde{G}_s \)-modules, we have the following section of \( \text{rest}^T_{\Sigma} \).

\[
\text{sect}^T_{\Sigma} : \text{Ob}\mathcal{C}(\Sigma)_T \ni (V, \{ F^T(\bullet) \}) \mapsto (V, \{ F^T(\bullet) \}) \in \text{Ob}\mathcal{C}(\Sigma).
\]
sect\Sigma naturally gives rise to a functor which we denote by the same letter. Its inverse functor is \text{rest}_\Sigma^\prime.

We review the affine local description of Klyachko. Notice that the left \( T \)-action and (the inverse of) the right \( T \)-action on \( T(\Sigma) \) are the same since \( T \) is commutative. As a result, a \( T \times T \)-equivariant vector bundle on \( T(\Sigma) \) is a vector bundle on \( T(\Sigma) \) with two commutative \( T \)-equivariant structures.

**Corollary 5.1 (cf. Klyachko [18, 6.1.5 2]).** For each \( \sigma \in \Sigma(r) \), every \( T \times T \)-equivariant vector bundle on \( T(\sigma) \) splits into a direct sum of \( T \times T \)-equivariant line bundles. \( \square \)

Now we state an analogous statement of Corollary 5.1 in the wonderful setting, which the author does not know any non-equivariant counterpart.

**Theorem 5.2.** Let \( G \) be an adjoint semisimple group and let \( X = X_0 = X(\Sigma_0) \) be its wonderful compactification as defined in [11]. Then, a \( \tilde{G} \times \tilde{G} \)-equivariant vector bundle \( E \) on \( X \) splits into a direct sum of \( \tilde{G} \times \tilde{G} \)-equivariant line bundles if, and only if, \( E \otimes_X k(e) \) is trivial as a \( \tilde{G} \)-module.

**Proof.** We have \( (V, \{ F^\tau(\bullet) \}_{\tau \in \Sigma_0(1)}) := \Xi(E) \in \text{Ob}_C(\Sigma_0)^T_\tilde{G} \). By assumption, we have \( \tilde{G}_s \cong \tilde{G} \).

Hence, \( V \) is a direct sum of trivial \( \tilde{G} \times 1 \)-modules for each \( (V, \{ F^\tau(\bullet) \}_{\tau \in \Sigma_0(1)}) \in \text{Ob}_C(\Sigma_0)^T_\tilde{G} \).

Let \( V = \bigoplus_{\chi \in \tilde{Z}(G) \setminus \Sigma} V_{\chi} \) be the isotypical decomposition of \( V \) with respect to the \( 1 \times \tilde{Z}(G) \)-action. By Corollary 2.6, we have

\[
(V, \{ F^\tau(\bullet) \}_{\tau \in \Sigma_0(1)}) = \bigoplus_{\chi \in \tilde{Z}(G) \setminus \Sigma} \left( V_{\chi}, \{ F^\tau(\bullet) \}_{\tau \in \Sigma_0(1)} \right).
\]

\( \Sigma_0 \) consists of a unique \( r \)-dimensional cone \( \sigma \) and its faces. For each \( \chi \in \tilde{Z}(G) \setminus \Sigma \), we have a basis \( B^\sigma_{\chi} \) of \( V_{\chi} \) such that \( F^\tau(n)_{\chi} \) is spanned by a subset of \( B^\sigma_{\chi} \) for every \( \tau \in \sigma(1) \) and every \( n \in \mathbb{Z} \). Thus, \( (V, \{ F^\tau(\bullet) \}_{\tau \in \Sigma_0(1)}) \in \text{Ob}_C(\Sigma_0)^T_\tilde{G} \) splits into a direct sum of one-dimensional objects. Sending by \( \Phi \circ \text{sect}_{\Sigma_0} \), we obtain the result. \( \square \)

We formulate corollaries of the above result by using the following well-known result.

**Lemma 5.3 (Minimal dimension of modules cf. [35]).** Let \( G \) be an adjoint simple group. Then, the minimal dimension \( d_G \) of a nontrivial representation of a simply connected cover \( \tilde{G} \) of \( G \) is the following: 1) \( d_{A_r} = r + 1 \), 2) \( d_{B_r} = 2r + 1 \), 3) \( d_{C_r} = 2r \), 4) \( d_{D_r} = 2r \), 5) \( d_{E_6} = 27 \), 6) \( d_{E_7} = 56 \), 7) \( d_{E_8} = 248 \), 8) \( d_{F_4} = 26 \), 9) \( d_{G_2} = 7 \).

Together with Theorem 5.2, we obtain the following

**Corollary 5.4.** Let \( G \) be an adjoint simple group and let \( X = X_0 \) be its wonderful compactification. Then, every \( \tilde{G} \times \tilde{G} \)-equivariant vector bundle of rank less than \( d_G \) splits into a direct sum of line bundles.

Since \( d_G \) is always greater than the rank \( r = \text{rk}G \) of a simple group \( G \), we have the following.

**Corollary 5.5.** Let \( G \) be an adjoint simple group and let \( X = X_0 \) be its wonderful compactification. Then, every \( \tilde{G} \times \tilde{G} \)-equivariant vector bundle of rank less than or equals to \( \text{rk}G \) splits into a direct sum of line bundles.

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5.2 Kostant’s Problem

The first part of this subsection is independent of the other parts of this paper. A general reference for the material in this subsection is the book of Borel-Wallach [5].

Kostant [20] raised the question of existence of a canonical extension of an equivariant vector bundle on a (complexified) symmetric space to its wonderful compactification. His question is connected to the existence of a vector bundle corresponding to the asymptotic behavior of the minimal K-type of a given unitary representation. More precisely, he seeks an algebraic framework to handle the celebrated Casselman theorem (cf. [5] Chapter X Theorem 2.4) about the $n_0$-coinvariants of an unitary representation using the boundary behavior of equivariant vector bundles. In our setting, his general conjecture is as follows.

Conjecture 1 (Kostant’s Problem). Let $G_0$ be a real reductive linear Lie group, and $K_0$ its maximal compact subgroup. We fix an Iwasawa decomposition $G_0 = K_0 A_0 N_0$ of $G_0$. Here $A_0$ is an abelian group which normalizes $N_0$. We define $G$, $K$, ... to be the complexification of $G_0$, $K_0$, ... respectively. Let $e \in G/K$ be the point corresponding to $[K] \in G/K$. Then, for each irreducible $K$-module $V$ which appears as the minimal $K_0$-type of an unitary representation of $G_0$, we have a $G$-equivariant vector bundle $\mathcal{E}_V$ on the wonderful compactification $Y$ of $G/K$ with the following properties:

1. There exists an isomorphism $\mathcal{E}_V|_{G/K} \cong G \times_K V$ of $G$-equivariant vector bundles;
2. For every $v \in V \cong \mathcal{E}_V \otimes_{\mathcal{O}_Y} k(e)$, and every one-parameter subgroup $\tau : \mathbb{G}_m \to A$, the limit value $\lim_{t \to \infty} \tau(t)v$ exists in the total space $V(\mathcal{E}_V)$ of $\mathcal{E}_V$;
3. For every $G$-equivariant vector bundle $\mathcal{E}$ on $Y$ with the above two properties, we have a $G$-equivariant embedding $\mathcal{E} \hookrightarrow \mathcal{E}_V$.

Remark 5.6. The wonderful compactification of a symmetric space is an algebraic analogue of the Oshima compactification [24] in the real-analytic setting. Sato [27] described the analogous complex-analytic compactification in the case of adjoint semisimple groups, which coincides with the wonderful compactification in the sense of De Concini-Procesi [11].

In the rest of this subsection, we assume that $X = X_0 = X(\Sigma_0)$ is the wonderful compactification of an adjoint semisimple group over $\mathbb{C}$ and use the notation and terminology introduced in [11] and [12].

We define the canonical extension of a $G \times G$-equivariant vector bundle corresponding to $G$-module $V$ on $G \cong (G \times G)/\Delta^d(G)$ as follows. By Lemma 2.10, $V \otimes \mathcal{O}_G$ is the only $G \times G$-equivariant vector bundle on $G$ whose identity fiber is isomorphic to $V$ (as a $G$-module). Let $V = \bigoplus_{\lambda \in X^+} V^\lambda$ be the $T$-isotypical decomposition of $V$.

Then, we define the canonical filtration of $V$ with respect to $\varpi \in \Sigma_0(1)$ as follows:

$$F_{\text{can}}^\varpi(n, V) := \bigoplus_{\langle \varpi, \lambda \rangle \geq 2n} V^\lambda \subset V \text{ for each } n \in \mathbb{Z}.$$}

We regard $V$ as a $\hat{G} \times \hat{Z}(G)$-module via natural surjection.

Lemma 5.7. For each $G$-module $V$, $\left( V, \{ F_{\text{can}}^\varpi(\bullet, V) \}_{\varpi \in \Sigma_0(1)} \right)$ is an object of $\mathcal{E}(\Sigma)$.

Proof. Since $F_{\text{can}}^\varpi(n, V)$ is a direct sum of $T$-isotypical components for every $\varpi \in \Sigma_0(1)$, $\{ F^\varpi(n) \}_{\varpi \in \Sigma_0(1), n \in \mathbb{Z}}$ clearly forms a distributive lattice. Moreover, it is a $p^\varpi$-stable decreasing filtration since $p^\varpi V^\lambda \subset \bigoplus_{\langle \varpi, \mu \rangle \geq 0} V^{\lambda+\mu}$. Similarly, we have

$$e_\alpha : \bigoplus_{\langle \varpi, \lambda \rangle \geq 2n} V^\lambda \to \bigoplus_{\langle \varpi, \lambda \rangle \geq 2n} V^{\lambda+\alpha} \subset \bigoplus_{\langle \varpi, \lambda \rangle \geq 2n+\langle \varpi, \alpha \rangle} V^\lambda \subset \bigoplus_{\langle \varpi, \lambda \rangle \geq 2n+2\langle \varpi, \alpha \rangle} V^\lambda$$

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for every negative root $\alpha$. Hence, $F_{\text{can}}^\omega(\bullet, V)$ is a $\varpi$-transversal filtration.

Now we define the canonical extension $\mathcal{E}_V$ of a $G$-module $V$ by

$$\mathcal{E}_V := \Phi \left( (V, \{F_{\text{can}}^\omega(\bullet, V)\}_{\varpi \in \Sigma_0(1)}) \right).$$

This is a $G \times G$-equivariant vector bundle on $X$ which is isomorphic to $V \otimes \mathcal{O}_G$ when restricted to $G$. Since $G \times G$ is the complexification of a complex group $G$ regarded as a real group, $G(=(G \times G)/\Delta^d(G))$ is a symmetric space. Here the group is $G \times G$ and its (complexified) Cartan involution $\vartheta$ swaps its first and second factor. Iwasawa’s abelian subgroup is defined as the exponential of the maximal abelian subalgebra in the set of semisimple elements of $-1$-eigenpart of the induced $\vartheta$-action on $g \oplus g$. Thus, in our case, it is the anti-diagonal embedding of the torus $T$ into $G \times G$ defined by $t \mapsto (t, t^{-1})$. Let $a : T \rightarrow X$ be the action coming from $T \times G \ni (t, g) \mapsto t.g.t \in G$. By the above arguments, this is the (complexified) action of Iwasawa’s abelian subgroup.

Now we present an answer to Kostant’s problem in this case.

**Theorem 5.8.** For every $G$-module $V$, $\mathcal{E}_V$ satisfies the following properties:

1. $\mathcal{E}_V \otimes_X \mathbb{C}(e) \cong V \otimes \mathbb{C}$ as a $\hat{G} \times Z(G)$-module;
2. For every $v \in \mathcal{E}_V \otimes_X \mathbb{C}(e)$, there exist a limit value $\lim_{t \rightarrow 0} a(t)v$ in $V(\mathcal{E}_V)$, the total space of $\mathcal{E}_V$;
3. Let $\mathcal{E}$ be another $G \times G$-equivariant vector bundle with the above two properties. Then, we have a $G \times G$-equivariant inclusion $\mathcal{E} \hookrightarrow \mathcal{E}_V$. In other word, $\mathcal{E}_V$ is maximal among the $G \times G$-equivariant vector bundles with the above two properties.

**Proof.** 1) follows from the definition of $\mathcal{E}_V$. Since the $a$-action preserves $T(\Sigma_0)$, we restrict our attention to $T(\Sigma_0)$. Let $\varpi \in \Sigma_0(1)$. $G_m$ acts on $O_X(D_{\varpi}) \otimes_X k(x_{\varpi})$ via $a \circ \varpi$-action by degree $-2$. For each weight $\lambda$ of $V$, $V^\lambda \otimes O_X|T(\Sigma_0)$ is a $T \times T$-equivariant vector subbundle of $V \otimes O_X|T(\Sigma_0)$. $G_m$ acts on $V^\lambda \otimes O_X|T(\Sigma_0) \otimes_{O_T(\Sigma_0)} k(x_{\varpi})$ via $a \circ \varpi$-action by degree $\langle \varpi, \lambda \rangle$. Hence, the condition of the convergence with respect to the $\varpi$-direction ($s \rightarrow 0$ in $s \in A^1 \supset G_m$) is given as

$$V^\lambda \cap F^\varpi \left( \left\lfloor \frac{\langle \varpi, \lambda \rangle}{2} \right\rfloor + 1 \right) = \{0\}$$

by Corollary 5.5 and Theorem 2.15. Hence, if the filtration $F^\varpi(\bullet)$ of $V$ satisfies

$$V^\lambda \subset F^\varpi \left( \left\lfloor \frac{\langle \varpi, \lambda \rangle}{2} \right\rfloor \right)$$

and (5.2.1) for every $\lambda \in X^*(T)$, then the corresponding vector bundle $\mathcal{E}|_{T(\varpi)}$ is a maximal $T \times T$-equivariant vector bundle on $T(\varpi)$ such that:

a. $\mathcal{E}|_{T(\varpi)}$ is contained in $V \otimes O_X(D)|_{T(\varpi)}$ for a sufficiently large equivariant divisor $D$ (see Corollary 2.23);

b. Limit value $\lim_{t \rightarrow 0} (\varpi \times (\varpi)^{-1})(t) v$ exists in $V(\mathcal{E}|_{T(\varpi)})$ for every $v \in \mathcal{E}|_{T(\varpi)} \otimes_{O_T(\varpi)} k(e)$.

We can check $F^\varpi(n)$ and $F^\varpi(\xi)$ $T$-isotypical componentwise because $F^\varpi(n)$ is $T$-stable for each $n \in \mathbb{Z}$. $F^\varpi_{\text{can}}(\bullet, V)$ satisfies (5.2.1) and (5.2.2) for every fundamental coweight $\xi$. Therefore, $\mathcal{E}_V|_{T(\varpi)}$ is maximal as a $T$-equivariant vector subbundle of $V \otimes O_X(D)|_{T(\varpi)}$ which satisfies
the conditions a) and b). Thus, every element of $E_{|T(\Sigma_0)} \otimes \sigma_{T(\Sigma_0)} k(e)$ converges when $t \in T$ approaches to zero in an arbitrary way. This yields 2). For every $G \times G$-equivariant vector bundle $E$ such that $E_{|T(\Sigma)}$ satisfies the conditions a) and b), we have a $T \times T$-equivariant embedding $E_{|T(\Sigma)} \subseteq E_{V_{|T(\Sigma)}}$. By Theorem 4.1 we can recover $E$ from the condition a) and a family of filtrations $\{F^\xi (\bullet , E)\}_{\xi \in \Sigma_0(1)}$ as vector subspaces of $V$. As a result, we have a $G \times G$-equivariant embedding $E \hookrightarrow E_{V}$. □

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