Wu-Yang Monopoles
and
Non-Abelian Seiberg-Witten Equations

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Abstract

Some exact solutions of the SU(2) Seiberg-Witten equations in Minkowski spacetime are given.
1. Introduction

The asymptotic freedom of non-Abelian gauge theories makes them unsuitable for doing reliable low energy calculations because in that limit the theory enters into its strong coupling regime. Seiberg and Witten [1] used duality arguments within the framework of N=2 supersymmetric Yang-Mills theory to bring an answer to this long standing problem. As a remarkable byproduct Witten [2] has shown that the Donaldson invariants of 4-manifolds can be determined by essentially counting the solutions of a set of massless magnetic monopole equations of the dual Abelian gauge theory [3],[4]. It was noted that the Seiberg-Witten equations do not admit any square integrable solutions. On the other hand Freund [5] has shown that the pair \((A, \psi)\) consisting of a Dirac monopole potential together with a simple ansatz describing a positive chirality spinor field in Minkowski spacetime satisfies the Seiberg-Witten equations analytically continued to a Lorentzian spacetime. In fact the ansatz \((A, \psi)\) was constructed by Gürsey [6] long before the Seiberg-Witten theory. The simplicity of the Freund-Gürsey solution encourages the exploration of other solutions. A non-Abelian generalization of Seiberg-Witten equations have been recently considered both from the physical [7] and the mathematical [8] points of view. The most natural SU(2) extension of Seiberg-Witten equations in Minkowski spacetime admits a Wu-Yang magnetic pole solution and it is our purpose here to demonstrate this solution explicitly.

2. SU(2) Seiberg-Witten Equations

SU(2) monopole equations consists of the pair of equations

\[ \rho^+(F) = k \Psi^\dagger \gamma \wedge \gamma \Psi \]  
\[ *\gamma \wedge D_A \Psi = 0 \]  

where \(A = (A^1, A^2, A^3)\) is a su(2) Lie algebra valued Yang-Mills potential 1-form. We may also write \(A = A^a T_a\) where \(\{T_a\}\) are the anti-hermitian generators of su(2) satisfying \([T_a, T_b] = \epsilon_{abc} T_c\). The corresponding Yang-Mills field (curvature) 2-form

\[ F = dA + e A \wedge A. \]  

\(\rho^+\) projects the self-dual part of \(F\). We introduced a gauge constant \(e\) for
later convenience. Then the local gauge transformations are given by
\[ A \rightarrow UAU^{-1} + \frac{1}{e} UdU^{-1}. \] (4)

\( \Psi = (\psi^1, \psi^2, \psi^3) \) is a su(2) Lie algebra valued 2-spinor of positive chirality. The exterior covariant derivative of \( \Psi \) is defined as
\[ D_A \Psi = d\Psi + e[A, \Psi]. \] (5)

The constant \( k \) in (1) can always be absorbed by a scaling of \( \Psi \).

We fix the orientation of space-time by letting \( *1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \).

\( * \) denotes the Hodge map related to the metric
\[ g = -e^0 \otimes e^0 + \vec{e} \otimes \vec{e} \] (6)

where \( \{ e^a = (e^0, \vec{e}) \} \) is an orthonormal coframe. We used in (1) and (2) the following \( Cl(3, 1) \) Clifford algebra basis
\[ \Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma^* & 0 \end{pmatrix} \] (7)

where \( \gamma = \gamma_a e^a \) and \( \gamma_a = (I, \vec{\sigma}) \). \( I \) is the \( 2 \times 2 \) identity matrix and \( \vec{\sigma} \) are the standard Pauli matrices.

3. Static, rotationally symmetric solutions

We will be considering static solutions so that it is convenient to work with either local Cartesian coordinates \( (t, x, y, z) \) or spherical polar coordinates \( (t, r, \theta, \phi) \). These are related through the coordinate transformations
\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \]

We assume the gauge potentials are independent of \( t \) and Coulomb gauge is chosen so that \( A_0 = 0 \). We take \( e = \frac{1}{2} \) and adjust \( k \) in a suitable way. With these assumptions, the SU(2) Seiberg-Witten equations (1) and (2) reduce to the following coupled 3-dimensional equations:

\[ *\sigma \wedge (d\psi^1 - \frac{1}{2} A^2 \psi^3 + \frac{1}{2} A^3 \psi^2) = 0 \] (8)
\[ *\sigma \wedge (d\psi^2 - \frac{1}{2} A^3 \psi^1 + \frac{1}{2} A^1 \psi^3) = 0 \] (9)
\[ *\sigma \wedge (d\psi^3 - \frac{1}{2} A^1 \psi^2 + \frac{1}{2} A^2 \psi^1) = 0 \] (10)
\begin{align}
F^1 &= \frac{1}{4}(\psi^2 \Sigma \psi^3 - \psi^3 \Sigma \psi^2) \\
F^2 &= \frac{1}{4}(\psi^3 \Sigma \psi^1 - \psi^1 \Sigma \psi^3) \\
F^3 &= \frac{1}{4}(\psi^1 \Sigma \psi^2 - \psi^2 \Sigma \psi^1)
\end{align}

where $\Sigma = \sigma \wedge \sigma$.

i) Abelian solution:

We let $\psi^1 = \frac{\sqrt{2}}{2r}(\xi + \eta)$, $\psi^2 = \frac{i\sqrt{2}}{2r}(-\xi + \eta)$ and $\psi^3 = 0$, where

\begin{align*}
\xi &= \left(\begin{array}{c}
\cos \frac{\theta}{2} e^{-i\phi} \\
\sin \frac{\theta}{2}
\end{array}\right) = \frac{1}{\sqrt{2r(r-z)}} \left(\begin{array}{c}
x - iy \\
r - z
\end{array}\right) \\
\eta &= \left(\begin{array}{c}
\sin \frac{\theta}{2} e^{i\phi} \\
-\cos \frac{\theta}{2}
\end{array}\right) = \frac{1}{\sqrt{2r(r-z)}} \left(\begin{array}{c}
r - z \\
-(x + iy)
\end{array}\right)
\end{align*}

Then $A^1 = 0$, $A^2 = 0$, $A^3 = -(1 + \cos \theta)d\phi$. This solution describes a magnetic monopole of strength 1, whereas the Freund-Gürsey solution describes a monopole of strength $\frac{1}{2}$. We note a Dirac string singularity along the +ve z-axis [9].

ii) Non-Abelian solution:

We start with Wu-Yang monopole potentials written in terms of Cartesian coordinates [10]

\begin{align*}
A^1 &= \frac{1}{r^2}(zdy - ydz) \\
A^2 &= \frac{1}{r^2}(xdz - zdx) \\
A^3 &= \frac{1}{r^2}(ydx - xdy).
\end{align*}

It should be noted that these are free of any line singularity. A full solution is obtained for

$\psi^1 = \frac{\sqrt{3}}{2r}(a\xi + a^*\eta)$
\[ \psi^2 = \frac{\sqrt{3}}{2r}(b\xi + b^*\eta) \]
\[ \psi^3 = \frac{\sqrt{3}}{2r}(c\xi + c^*\eta) \]  \hspace{1cm} (15)

where
\[ a = \frac{(r - z)}{2r} - \frac{(x + iy)^2}{2r(r - z)}, \quad b = i\frac{(r - z)}{2r} + i\frac{(x + iy)^2}{2r(r - z)}, \quad c = \frac{(x + iy)}{r}. \]

4. Concluding Comments

In fact the non-Abelian solution (14) and (15) above may be related to the Abelian solution by a singular gauge transformation followed by a scaling of the spinor field. The required gauge transformation is given in 2 × 2 matrix notation for convenience as [11]

\[ U = e^{-i\sigma^3 x/2} e^{i\sigma^2 x/4} e^{i\sigma^3 \frac{x}{4}}. \]  \hspace{1cm} (16)

Then the transformation rules for the components of Ψ are found from
\[
\begin{pmatrix}
\psi^3 \\
\psi^1 + i\psi^2 \\
-\psi^3
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
\psi^3 \\
\psi^1 + i\psi^2 \\
-\psi^3
\end{pmatrix}U^{-1}
\]

by substituting \( U \) from above.

The Abelian solution can be obtained in any gauge group; for example \( SU(N) \). In this case the Lie algebra can be split into the Cartan subalgebra \( \mathfrak{t} \) and the roots \( \alpha \), and we can set all the gauge potentials in the root directions to zero while setting all the monopole fields in the Cartan directions to zero. The Dirac equation for the monopoles becomes

\[ \ast \sigma \wedge (d\psi^\alpha + \alpha(A)\psi^\alpha) = 0. \]

This is solved by simply taking the Freund-Gürsey solution for the monopoles and taking \( \alpha(A) = A^0 \) where \( A^0 \) is the Abelian Dirac potential. For \( SU(2) \) this implies that \( A = 2A^0 \) which is why the magnetic charge of our solution is 1 rather than \( \frac{1}{2} \). To complete the analysis, all that is left is to work out the gauge field strength 2-forms from the remaining equations. This can be done consistently after an appropriate scaling of the spinors. Once the Abelian
solution to the $SU(N)$ Seiberg-Witten equations is found, it may then be
gauge rotated to any non-Abelian solution.

It is to be expected that the moduli space of non-Abelian Seiberg-Witten
monopole equations has a structure richer than that of ordinary Donaldson
theory. Accordingly we have been able to show that $SU(2)$ Seiberg-Witten
equations admit a simple Wu-Yang magnetic pole solution. We think this
provides one other reason why $SU(2)$ Seiberg-Witten equations should be
investigated more closely.

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References

[1] N. Seiberg, E. Witten, Nucl. Phys. B426 (1994) 19
    (Err. ibid. B430 (1994) 485)

[2] E. Witten, Math. Res. Lett. 1 (1995) 1

[3] R. Flume, L. O’Raifeartaigh, I. Sachs, Brief resume of Seiberg-Witten theory
    [hep-th/9611118]

[4] S. Akbulut, Lectures on Seiberg-Witten invariants in Proceedings of
    4th Gökova Geometry-Topology Conference p.95 Edited by S. Akbulut,
    T. Önder, R. J. Stern (TUBITAK, Ankara, 1996)

[5] P. G. O. Freund, J. Math. Phys. 36 (1995) 2673

[6] F. Gürsey in Gauge Theories and Modern Field Theory Edited
    by R. Arnowitt, P. Nath (MIT Press, 1976)

[7] J. M. F. Labastida, M. Mariño, Nucl. Phys. B448 (1995) 373

[8] C. Okonek, A. Teleman, Comm. Math. Phys. 180 (1996) 36

[9] P. A. M. Dirac, Proc. Roy. Soc. A133 (1931) 60

[10] T. T. Wu, C. N. Yang, in The Properties of Matter Under Unusual
    Conditions Edited by H. Mark, S. Fernbach (Wiley, 1969)

[11] J. Arafune, P. G. O. Freund, C. J. Goebel, J. Math. Phys. 16 (1975) 433