Persistence and Life Time Distribution in Coarsening Phenomena

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Abstract

We investigate the life time distribution $P(\tau,t)$ in one dimensional and two dimensional coarsening processes modelled by Ising-Glauber dynamics at zero temperature. The life time $\tau$ is defined as the time that elapses between two successive flips in the time interval $(0,t)$ or between the last flip and the observation time $t$. We calculate $P(\tau,t)$ averaged over all the spins in the system and over several initial disorder configurations. We find that asymptotically the life time distribution obeys a scaling ansatz: $P(\tau , t) = t^{-1/\phi(\xi)}$, where $\xi = \tau/t$. The scaling function $\phi(\xi)$ is singular at $\xi = 0$ and 1, mainly due to slow dynamics and persistence. An independent life time model where the life times are sampled from a distribution with power law tail is presented, which predicts analytically the qualitative features of the scaling function. The need for going beyond the independent life time models for predicting the scaling function for the Ising-Glauber systems is indicated.

Coarsening phenomenon [1], is a simple example of a dynamical process which is slow, and which becomes slower as time proceeds. This phenomenon is found in several nonequilibrium systems, \textit{e.g.} phase separation in binary alloys, grain growth, growth of soap bubbles, magnetic bubbles \textit{etc.} A striking feature of the coarsening phenomenon is the dynamical scale invariance. The domain structure at different times are statistically similar to each other but for a rescaling length $L(t)$. The rescaling length can be taken as the typical linear size of a domain and it increases with time as $t^{1/z}$. It was soon realized that the dynamical scaling exponent $z$ is not adequate to describe completely the coarsening dynamics, see below.

An interesting question asked in this context concerns the persistence probability $P_0(t)$, that a local order parameter does not change its sign until the observation time $t$. In other words, $P_0(t)$ refers to the fraction of the total volume that remains unswept by the domain walls until time $t$, in the context of three dimensional coarsening phenomenon. This quantity, exhibits, asymptotically, a power law decay, $P_0(t) \sim t^{-\theta}$, where $\theta$ is called the persistence exponent, and it is independent of $z$, the dynamical scaling exponent.

The persistence phenomenon has a long history. The earliest question on persistence was perhaps asked and answered by W. A. Whitworth in 1878 and later by J. Bertrand in 1887, see Feller [3]. The question relates to two candidates $P$ and $Q$, who poll $p$ votes and $q$ votes respectively, with candidate $P$ winning the election by a margin of $x = p - q$ votes.
Let $n = p + q$ denote the total number of votes polled and there are no invalid votes! The ballot problem consists of finding the probability that throughout the counting process the candidate $P$ leads. In the language of random walks, this persistence probability equals the ratio of the number of walks that start at origin, remain above origin and eventually reach the site $x$ after $n$ steps to the total number of walks that start at origin and reach site $x$ after $n$ steps. A simple application of reflection principle yields the persistence probability as $x/n$, see Feller [2], which equals the fractional excess votes polled by $P$ over $Q$.

The next question on persistence arose in the context of simple random walks on a one dimensional lattice. Let us consider the walks that start at origin and take $2n$ steps. The total number of random walks is thus $2^{2n}$ and we consider that all these walks to be equally probable. Let us collect the fraction of random walks that visit origin for the last time at step $2k$. This fraction is given by,

$$P(2k, 2n) = \frac{(2k)!}{k!k!(n-k)!(n-k)!} \frac{1}{2^{2n}}$$

$P(2k, 2n)$, given above, is called the discrete arc-sine distribution of order $n$. It is $\cup$-shaped; it is maximum at either ends of the support i.e. at $k = 0$ and at $k = n$. It is symmetric. Let us define a scaling variable $\xi_k = k/n$ and express $P(2k, 2n) = n^{-1}\phi(\xi_k)$. The continuous arc-sine distribution,

$$f(\xi) = \frac{1}{\pi} \frac{1}{\xi^{1/2}(1-\xi)^{1/2}}$$

provides a very good fit to the discrete scaling function $\phi(\xi_k)$, especially for large $n$ and $k$ not too close to 0 or $n$. Eq. (2) was first derived by P. Lévy [3] for the Brownian motion, which is a continuum version of the random walk problem considered above. The persistence probability distribution as given by the arc-sine law is a surprising result, since contrary to intuition, it says that in a duration of time $t$, the probability for the last zero crossing is maximum at time zero and at time $t$. In other words, the probability is highest for the Brownian particle to be left or right of the origin persistently. The $\cup$ shaped arc-sine distribution with singularities at either ends of the support is the forerunner to all the surprising results found in the recent times in the context of persistence in Ising and Potts spin systems, coarsening systems etc., including the ones presented in this paper. For a review of the persistence phenomenon see Majumdar [4].

The phenomenon of persistence has been studied extensively both theoretically [4,5] and experimentally [6] in the recent times. Also the notion of persistence of local order has been extended to persistence of global order, of domains, and of patterns [5]. We propose here yet another generalization of the phenomenon of persistence for characterizing the slow dynamics of the coarsening process.

Accordingly, we investigate first the one dimensional zero temperature Ising-Glauber dynamics for the coarsening phenomenon. Consider a one dimensional array of $N$ Ising spins \{$S_i = \pm 1 : i = 1, N$\}. The spin system is prepared in a homogeneous, disordered, high temperature phase. Operationally this means that we assign to each spin a value of $+1$ or $-1$ independently and with equal probability. A spin interacts with its nearest neighbours only and the interaction is ferromagnetic. The Hamiltonian is given by, $H = -J \sum_{(i,j)} S_i S_j$, where the sum runs over nearest neighbour pairs and $J$ measures the strength of ferromagnetic...
interaction. We set $J = 1$, without loss of generality. Periodic boundary condition is imposed. At time $t = 0$, we quench the system to zero temperature. Upon the temperature quench, the system does not order instantaneously or even immediately. Instead, domains of equilibrium broken-symmetry phases form, grow and the domain structure coarsens.

The dynamics is simulated as follows. A spin is selected randomly. It is always flipped if the resulting configuration has lower energy; the spin is never flipped if the energy is raised; the spin is flipped with probability half if the change in energy is zero. A set of $N$ consecutive spin-flip attempts constitutes a Monte Carlo time Step (MCS) which sets the unit of time. Spin-flips occur relatively often in the initial times since there would exist a large number of lattice sites where the spin flips are energetically favourable. As time proceeds, domains of up spins and down spins form, coalesce and grow. Since only the spins in the domain boundaries can flip, the dynamics slows down. The dynamics becomes slower and slower since the number of domains decreases with time.

We say that when a spin flips, it dies and is reborn instantaneously in the flipped orientation. We denote by $\tau$ the time that elapses between two consecutive spin flips, and call it the life time. Thus a single spin can have many lives of possibly different life times. $\tau$ is a random variable and we are interested in its distribution obtained from an ensemble of all the lives of all the spins in the system. Further we average the life time distribution over several initial disorder configurations. Suppose during the time interval $(0, t)$, a spin flips for the last time at say $t_L$ and does not flip until time $t$. Then we take $t - t_L$ as the last life time of the spin. The average life time distribution is obviously dependent on the observation time $t$. We denote this by $P(\tau, t)$. We define a scaling variable $\xi = \tau/t$, and make a scaling ansatz that for $t \to \infty$, $P(\tau, t) \sim t^{-1}\phi(\xi)$. Fig. 1 depicts $tP(\tau, t) \ vs \ \xi$ for several values of $t$ on a semi-log graph. We find that the life time distributions for large $t$, collapse reasonably well establishing the validity of the scaling ansatz. In Fig. 1, we see that the collapse occurs for $t \geq 4000$ MCS. We have also shown the life time distributions for $t = 500$ and $1000$, and they deviate, though not considerably, from the scaling curve. One can see from Fig. 1 that the scaling curve is $\cup$ shaped and exhibits singularities at either ends of its support $(0, 1)$, a feature observed in the arc-sine law arising in the context of Brownian motion.

Consider now the distribution $P(\tau, t)$, in the limit $\tau \to t$. This limit requires that the spin never flips during the time interval $(0, t)$. Let us define $Q(t) = P(\tau = t, t)$. Clearly $Q(t)$ is the same as the persistence probability $P_0(t)$ but for a time dependent normalization $N(t)$, defined as the average number of lives per spin. In other words, $Q(t) = P_0(t)/N(t)$. Fig. 2 depicts $Q(t) \ vs. \ t$ on a log-log graph. The points fall on a straight line, and a linear least square estimate gives the slope as $-\theta_L = -0.87$, implying that $Q(t) \sim t^{-\theta_L}$. Fig. 3 depicts a similar plot for the average number of lives per spin. We find that $N(t) \sim t^{\theta_N}$, with $\theta_N$ estimated as 0.49. The exponent $\theta_N$ lends itself to a simple explanation. It is related to the number of zero crossing of a Brownian particle in a duration of time $t$, which goes [2] as $t^{1/2}$. From these two exponents $\theta_L$ and $\theta_N$, we can now calculate the standard persistence exponent as $\theta = \theta_L - \theta_N = 0.38$, a result consistent with earlier results [3]. The singularities at $\xi = 0$ and at $\xi = 1$, of the scaling curve $\phi(\xi)$, can be understood as follows. In the discussions below, we always take the limit $\tau \to \infty$, $t \to \infty$ such that $\xi = \tau/t$ is finite. Let us first consider the singularity at $\xi = 0$. We notice that only those spins in the domain boundaries can flip. As time progresses, the number of domains decreases as $\tau^{-1/2}$. Also once a spin flips the probability it flips next in time $t$ is related to the first return to origin of a
Brownian particle, and is proportional to $\tau^{-1/2}$. Thus we get, $P(\tau, t) = t^{-1}\phi(\xi) \sim t^{-1/2}\tau^{-1/2}$. Hence in the limit $\xi \to 0$, we have, $\phi(\xi) \sim \xi^{-1/2}$. For understanding the other singularity (at $\xi = 1$), we use the notion of persistence. The limit $\xi \to 1$ implies $\tau \to t$. In this limit, $P(\tau, t) = t^{-1}\phi(\xi) \sim t^{-\theta_L}$. Hence $\phi(\xi) \sim t^{-\theta_L+1}$; also $t$ is of the order of $(1 - \xi)^{-1}$. Thus we find that in the limit $\xi \to 1$, $\phi(\xi) \sim (1 - \xi)^{-\left(1-\theta_L\right)}$.

Next we consider the two dimensional zero temperature Ising-Glauber model for coarsening phenomenon. In Fig. 4 we depict the distribution of the life time in the scaling form. The scaling curve looks qualitatively the same as the one found for the one dimensional case. The data collapse for large observation times $t$ is very clear. In Fig. 5 we depict $Q(t)$ vs. $t$ on a log-log graph. The data points fall on a straight line showing that $Q(t) \sim t^{-\theta_L}$ and the exponent $\theta_L = 0.58$. Fig. 6 depicts the average number of lives per spin, $N(t)$ vs. $t$ on a log-log graph, and we find that $N(t) \sim t^{\theta_N}$. The exponent $\theta_N = 0.36$. Unlike the one dimensional Ising-Glauber dynamics, this exponent does not have a simple analytical explanation. It has to be calculated only numerically, atleast as of now. From the numerical estimates of $\theta_L$ and $\theta_N$, we can calculate the standard persistence exponent as $\theta = \theta_L - \theta_N = 0.22$, a result which matches with an earlier finding [3].

At this stage, one can think of a simple model for the coarsening dynamics, in terms of sampling the life times independently and randomly from a distribution, which we denote by $\rho_1(\tau)$. The persistence exponent appears in this model as a parameter and our interest is to investigate the nature of the $\cup$ shaped life time distribution. In such an approach, we neglect the spin-spin correlations that are present in the Ising model. The hope is that such simple models would help us understand the nature of the persistent events underlying these distributions and help us in investigating persistence in extended nonequilibrium statistical mechanical models. The distribution $\rho_1(\tau)$ is taken as power-law tailed. The reason is clear, if we consider that within the scope of independent life time model, the persistence probability is

$$P_0(t) \sim t^{-\theta_L} = \int_t^\infty \rho_1(\tau) d\tau,$$

which implies that for large $\tau$, we have,

$$\rho_1(\tau) \sim \frac{\theta_L}{\tau^{1+\theta_L}}.$$  

We take the above as the distribution of a single life time. Also $1 \leq \tau \leq \infty$ and $\theta_L < 1$. We set the lower limit of $\tau$ at unity for ensuring normalization. Let $\rho_m(\tau)$ denote the distribution of the sum of $m$ independent realizations of $\tau$. It is easily seen that,

$$\rho_m(\tau) = \theta_L^{-1/m} \rho_1\left(\frac{\tau}{\theta_L^{1/m}}\right).$$  

With this model for the basic life time distribution, we are now ready to derive the scaling function. We define $t_n = \sum_i^n \tau_i$, the sum of $n$ independent realizations of the random variable $\tau$, sampled from the distribution $\rho_1(\tau)$. We define $\xi = \tau/t_n$. Formally we have,

$$\phi(\xi) = \int d\tau \int dt_n \delta(\xi - \tau/t_n) \rho_1(\tau) \rho_{n-1}(t_n - \tau).$$
Noting that $\delta(\xi - \tau/t_n) = t_n\delta(\tau - \xi t_n)$, and carrying out the integration over $\tau$, we get,

$$
\phi(\xi) = \int_\beta^\infty dt_n t_n \rho_1(\xi t_n)(n - 1)^{-1/\theta L} \rho_1 \left( \frac{t_n (1 - \xi)}{(n - 1)^{1/\theta L}} \right)
$$

(7)

where $\beta$ is the lower limit of the integration, obtained suitably, see below. In the above we substitute for $\rho_1(\cdot)$ given by Eq. (4) and carry out the integration. We get,

$$
\phi(\xi) = \frac{(n - 1)\theta L}{2\xi^{1+\theta L}(1 - \xi)^{1+\theta L}} \beta^{-2\theta L}
$$

(8)

We note that the argument of the function $\rho_1(\cdot)$ occurring in Eq. (7) must be greater than unity. This requirement gives rise to the following inequalities,

$$
\xi t_n > 1
$$

(9)

$$
\frac{t_n (1 - \xi)}{(n - 1)^{1/\theta L}} > 1
$$

(10)

From the above we get $\beta$ as,

$$
\beta = \max \left\{ \frac{1}{\xi}, \frac{(n - 1)^{1/\theta L}}{1 - \xi} \right\}
$$

(11)

Finally we get,

$$
\phi(\xi) = \begin{cases} 
\frac{(n-1)\theta L}{2(1-\xi)^{1+\theta L}} \frac{1}{\xi^{1+\theta L}} & \text{for } 0 < \xi < \xi^* \\
\frac{\theta L}{2(n-1)\xi^{1+\theta L}} \frac{1}{(1-\xi)^{1+\theta L}} & \text{for } \xi^* < \xi < 1
\end{cases}
$$

(12)

where,

$$
\xi^* = \frac{1}{1 + (n - 1)^{1/\theta L}}
$$

(13)

First observation we make is that the scaling function is singular at the either ends of the support, a feature observed in the one and two dimensional Ising-Glauber models presented in this paper. We find that the scaling function is explicitly a function of $n$, the number of lives of a spin. The exponents characterizing the singularities at $\xi = 0$ and at $\xi = 1$ are both equal. However, for the Ising-Glauber models, the exponents are different. Hence we may need to go beyond the independent life time models, for making contact with the observations on Ising-Glauber dynamics. A correlated life time model could prove useful in this context. Work in this direction is in progress and will be reported soon [9].

We dedicate this work to the memory of Klaus W. Kehr.
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FIG. 1. The life time distribution function plotted against the scaling variable, for the zero temperature one dimensional Ising-Glauber coarsening.
FIG. 2. The persistence probability $Q(t)$ vs. $t$, for the zero temperature one dimensional Ising-Glauber coarsening. The solid line corresponds to the linear least square fit. The slope is $-0.8744$. 
FIG. 3. The average number of lives per spin, for the zero temperature one dimensional Ising-Glauber coarsening. The solid line is the linear least square fit. The slope is 0.4853
FIG. 4. The life time distribution plotted against the scaling variable, for the zero temperature two dimensional Ising-Glauber coarsening
FIG. 5. The persistence probability $Q(t)$ vs. $t$, for the zero temperature two dimensional Ising-Glauber coarsening. The solid line is the linear least square fit. The slope is $-0.5828$. 
FIG. 6. The average number of lives per spin for the zero temperature two dimensional Ising-Glauber coarsening. The solid line is the linear least square fit. The slope is 0.3656.