We apply a new general method of anholonomic frames with associated nonlinear connection structure to construct new classes of exact solutions of Einstein–Dirac equations in five dimensional (5D) gravity. Such solutions are parametrized by off–diagonal metrics in coordinate (holonomic) bases, or, equivalently, by diagonal metrics given with respect to some anholonomic frames (pentads, or funfbein, satisfying corresponding constraint relations). We consider two possibilities of generalization of the Taub NUT metric in order to obtain vacuum solutions of 5D Einstein equations with effective renormalization of constants (by higher dimension anholonomic gravitational interactions) having distinguished anisotropies on an angular parameter or on extra dimension coordinate. The constructions are extended to solutions describing self–consistent propagations of 3D Dirac wave packets in 5D anisotropic Taub NUT spacetimes. We show that such anisotropic configurations of spinor matter can induce gravitational 3D solitons being solutions of Kadomtsev–Petviashvili or of sine–Gordon equations.

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I. INTRODUCTION

Recently one has proposed a new method of construction of exact solutions of the Einstein equations on (pseudo) Riemannian spaces of three, four and extra dimensions (in brief, 3D, 4D,...), by applying the formalism of anholonomic moving frames [1]. There were constructed static solutions for black holes / tori, soliton–dilaton systems and wormhole / flux tube configurations and for anisotropic generalizations of the Taub NUT metric [2]; all such solutions being, in general, with generic local anisotropy. The method was elaborated following the geometry of anholonomic frame (super) bundles and associated nonlinear connections (in brief, N–connection) [3,4] which has a number of applications in generalized Finsler and Lagrange geometry, anholonomic spinor geometry, (super) gravity and strings with anisotropic (anholonomic) frame structures.

In this paper we restrict our considerations for the 5D Einstein gravity. In this case the N–connection coefficients are defined by some particular parametrizations of funfbein, or pentadic, coefficients defining a frame structure on (pseudo) Riemannian spacetime and describing a gravitational and matter field dynamics with mixed holonomic (unconstrained) and anholonomic (constrained) variables. We emphasize that the Einstein gravity theory in arbitrary dimensions can be equivalently formulated with respect to both holonomic (coordinate) and anholonomic frames. In the anholonomic cases the rules of partial and covariant derivation are modified by some pentad transforms. The point is to find such values of the anholonomic frame (and associated N–connection) coefficients when the metric is diagonalized and the Einstein equations are written in a simplified form admitting exact solutions.

The class of new exact solutions of vacuum Einstein equations describing anisotropic Taub NUT like spacetimes [2] is defined by off–diagonal metrics if they are given with respect to usual coordinate bases. Such metrics can be anholonomically transformed into diagonal ones with coefficients being very similar to the coefficients of the isotropic Taub NUT solution but having additional dependencies on the 5th coordinate and angular parameters.

We shall use the term locally anisotropic (spacetime) space (in brief, anisotropic space) for a (pseudo) Riemannian space provided with an anholonomic frame structure induced by a procedure of anholonomic diagonalization of a off–diagonal metric.
The Hawking’s \textsuperscript{[3]} suggestion that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang–Mills instanton holds true on anisotropic spaces but in this case both the metric and instanton have some anisotropically renormalized parameters being of higher dimension gravitational vacuum polarization origin. The anisotropic Euclidean Taub-NUT metric also satisfies the vacuum Einstein’s equations with zero cosmological constant when the spherical symmetry is deformed, for instance, into ellipsoidal or even toroidal configuration. Such anisotropic Taub-NUT metrics can be used for generation of deformations of the space part of the line element defining an anisotropic modification of the Kaluza-Klein monopole solutions proposed by Gross and Perry \textsuperscript{[6]} and Sorkin \textsuperscript{[7]}

In the long-distance limit, neglecting radiation, the relative motion of two such anisotropic monopoles can be also described by geodesic motions, like in Ref. \textsuperscript{[3]}, but these motions are some anholonomic ones with associated nonlinear connection structure and effective torsion induced by the anholonomy of the systems of reference used for modelling anisotropies. The torsion and N–connection corrections vanish if the geometrical objects are transferred with respect to holonomic (coordinate) frames.

From the mathematical point of view, the new anholonomic geometry of anisotropic Taub-NUT spaces is also very interesting. In the locally isotropic Taub-NUT geometry there are four Killing-Yano tensors \textsuperscript{[9]}. Three of them form a complex structure realizing the quaternionic algebra and the Taub-NUT manifold is hyper-Kähler. In addition to such three vector-like Killing-Yano tensors, there is a scalar one which exists by virtue of the metric being of class $D$, according to Petrov’s classification. Anisotropic deformations of metrics to off–diagonal components introduce substantial changes in the geometrical picture. Nevertheless, working with respect to anholonomic frames with associated nonlinear connection structure the basic properties and relations, even being anisotropically modified, are preserved and transformed to similar ones for deformed symmetries \textsuperscript{[2]}

The Schrödinger quantum modes in the Euclidean Taub-NUT geometry were analyzed using algebraic and analytical methods \textsuperscript{[14,15]}. The Dirac equation was studied in such locally isotropic curved backgrounds \textsuperscript{[11]}. One of the aims of this paper is to prove that this approach can be developed as to include into consideration anisotropic Taub-NUT backgrounds in the context of the standard relativistic gauge-invariant theory \textsuperscript{[10,11]} of the Dirac field.

The purpose of the present work is to develop a general SO$(4,1)$ gauge-invariant theory of the Dirac fermions \textsuperscript{[12]} which can be considered for locally anisotropic spaces, for instance, in the external field of the Kaluza-Klein monopole \textsuperscript{[11]} which is anisotropically deformed.

Our goal is also to point out new features of the Einstein theory in higher dimension spacetime when the locally anisotropic properties, induced by anholonomic constraints and extra dimension gravity, are emphasized. We shall analyze such effects by constructing new classes of exact solutions of the Einstein–Dirac equations defining 3D soliton–spinor configurations propagating self–consistently in an anisotropic 5D Taub NUT spacetime.

We note that in this paper the 5D spacetime is modeled as a direct time extension of a 4D Riemannian space provided with a corresponding spinor structure, i. e. our spinor constructions are not defined by some Clifford algebra associated to a 5D bilinear form but, for simplicity, they are considered to be extended from a spinor geometry defined for a 4D Riemannian space.

We start in Section II with an introduction in the Einstein–Dirac theory formulated with respect to anholonomic frames with associated nonlinear connection structure. We write down the Einstein and Dirac equations for some classes of metric ansatz which can be diagonalized via anholonomic transforms to a locally anisotropic basis. In Section III we construct exact solutions for 3D Dirac wave packets propagating in anisotropic backgrounds. In Section IV we outline two classes of 5D locally anisotropic solutions of vacuum Einstein equations generalizing the well known Taub NUT metric for the cases of anisotropic (angular and/or extra dimension) polarizations of constants and metric coefficients. Section V is devoted to such solutions of the 5D Einstein – Dirac equations which are constructed as generalizations of Taub NUT anisotropic vacuum metrics to configurations with Dirac spinor energy–momentum source. In Section VI we prove that the Dirac spinor field in such anisotropic spacetimes can also induce 3D dimensional solitons which can treated as an anisotropic, in general, non–trivially typological, soliton–Dirac wave packet configuration propagating self–consistently in anisotropically deformed Taub NUT spacetime. Finally, in Section VII we conclude the work.

II. EINSTEIN–DIRAC EQUATIONS WITH ANHOLONOMIC VARIABLES

In this Section we introduce an ansatz for pseudo Riemannian off–diagonal metrics and consider the anholonomic transforms diagonalizing such metrics. The system of field Einstein equations with the spinor matter energy–momentum tensor and of Dirac equations are formulated on 5D pseudo–Riemannian spacetimes constructed as a trivial extension by the time variable of a 4D Riemannian space (an anisotropic deformation of the Taub NUT instanton \textsuperscript{[3]}).
A. Ansatz for metrics

We consider a 5D pseudo–Riemannian spacetime of signature (+, −, −, −, −), with local coordinates

\[ u^\alpha = (x^i, y^a) = (x^0 = t, x^1 = r, x^2 = \theta, y^3 = s, y^4 = p), \]

– or more compactly \( u = (x, y) \) – where the Greek indices are conventionally split into two subsets \( x^i \) and \( y^a \) labeled respectively by Latin indices of type \( i, j, k, \ldots \) and connections) into sets of mixed holonomic–anholonomic variables (coordinates) provided respectively with ‘holonomic’ indices of type \( a, b, c, \ldots \) and, its dual basis, \( \tilde{a}, \tilde{b}, \tilde{c}, \ldots \). The 5D (pseudo) Riemannian metric

\[ ds^2 = g_{\alpha\beta} du^\alpha du^\beta \]

is given by a metric ansatz parametrized in the form

\[ g_{\alpha\beta} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & g_{1,2} & 0 & 0 & 0 \\
0 & 0 & h_{3,4} & 0 & 0 \\
0 & 0 & 0 & h_{1,4} & 0 \\
0 & 0 & 0 & 0 & h_{2,4}
\end{bmatrix}, \]

(2)

where the coefficients are some functions of type \( g_{1,2} = g_{1,2}(x^1, x^2), h_{3,4} = h_{3,4}(x^1, x^2, s), w_{1,2} = w_{1,2}(x^1, x^2, s), n_{1,2} = n_{1,2}(x^1, x^2, s). \)

Both the inverse matrix (metric) as well the metric (2) is off–diagonal with respect to the coordinate basis

\[ \partial_\alpha \equiv \frac{\partial}{\partial u^\alpha} = (\partial_t = \frac{\partial}{\partial x^i}, \partial_\theta = \frac{\partial}{\partial y^a}) \]

(4)

and, its dual basis,

\[ d^\alpha \equiv du^\alpha = (d^t = dx^i, d^\theta = dy^a). \]

(5)

The metric (1) with coefficients (2) can be equivalently rewritten in the diagonal form

\[ \delta s^2 = dt^2 + g_{1,2}(x) (dx^1)^2 + g_{2,2}(x) (dx^2)^2 + h_{3,4}(x, s) (dy^3)^2 + h_{1,4}(x, s) (dy^4)^2, \]

(6)

if instead the coordinate bases (1) and (3) we introduce the anholonomic frames (anisotropic bases)

\[ \delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_t = \partial_t - N_{b}^{\alpha}(u) \partial_b, \delta_\theta = \frac{\partial}{\partial y^a}) \]

(7)

and

\[ \delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_{k}^{a}(u) dx^k) \]

(8)

where the \( N \)–coefficients are parametrized

\[ N_{0}^{i} = 0, N_{1,2}^{j} = w_{1,2} \text{ and } N_{1,2}^{4} = n_{1,2} \]

and define the associated nonlinear connection (N–connection) structure, see details in Refs [3,1,2,4]. The anisotropic frames (3) and (4) are anholonomic because, in general, they satisfy some anholonomy relations,

\[ \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W_{\alpha\beta}^\gamma \delta_\gamma, \]

with nontrivial anholonomy coefficients

\[ W_{ij}^{k} = 0, W_{aj}^{k} = 0, W_{ia}^{k} = 0, W_{ab}^{k} = 0, W_{ab}^{a} = 0, \]

(9)

\[ W_{ij}^{a} = -\Omega_{ij}^{a}, W_{bj}^{a} = -\partial_b N_{j}^{a}, W_{ia}^{a} = \partial_a N_{i}^{a}, \]

where

\[ \Omega_{ij}^{a} = \delta_j N_{i}^{a} - \delta_i N_{j}^{a} \]

is the N–connection curvature. Conventionally, the N–coefficients decompose the spacetime variables (tensors, spinors and connections) into sets of mixed holonomic–anholonomic variables (coordinates) provided respectively with 'holonomic' indices of type \( i, j, k, \ldots \) and with 'anholonomic' indices of type \( a, b, c, \ldots \). Tensors, metrics and linear connections with coefficients defined with respect to anisotropic frames (3) and (4) are distinguished (d) by N–coefficients into holonomic and anholonomic subsets and called, in brief, d–tensors, d–metrics and d–connections.
B. Einstein equations with anholonomic variables

The metric (1) with coefficients (2) (equivalently, the d–metric (6)) is assumed to solve the 5D Einstein equations

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \]  

where \( \kappa \) and \( \Upsilon_{\alpha\beta} \) are respectively the coupling constant and the energy–momentum tensor.

The nontrivial components of the Ricci tensor for the metric (1) with coefficients (2) (equivalently, the d–metric (6)) are

\[ R_1^1 = R_2^2 = - \frac{1}{2g_1 g_2} \left[ g_1^{**} - \frac{g_1' g_2^*}{2g_1} - \frac{(g_2^*)^2}{2g_2} \right], \]

\[ R_3^3 = R_4^4 = - \frac{\beta}{2h_3 h_4}, \]

\[ R_{31} = -u_1 \frac{\beta}{2h_4} - \frac{\alpha_1}{2h_4}, \]

\[ R_{32} = -u_2 \frac{\beta}{2h_4} - \frac{\alpha_2}{2h_4}, \]

\[ R_{41} = \frac{h_4}{2h_3} \left[ n_{1**} + \gamma n_{1}' \right], \]

\[ R_{42} = -\frac{h_4}{2h_3} \left[ n_{2**} + \gamma n_{2}' \right], \]

where, for simplicity, the partial derivatives are denoted \( h^* = \partial h/\partial x \), \( f' = \partial f/\partial x^2 \) and \( f^* = \partial f/\partial s \).

The scalar curvature is computed

\[ R = 2 \left( R_1^1 + R_3^3 \right). \]

In result of the obtained equalities for some Ricci and Einstein tensor components, we conclude that for the metric ansatz (3) the Einstein equations with matter sources are compatible if the coefficients of the energy–momentum d–tensor give with respect to anholonomic bases (7) and (8) satisfy the conditions

\[ \Upsilon_0^0 = \Upsilon_1^1 + \Upsilon_3^3, \quad \Upsilon_1^1 = \Upsilon_2^2 = \Upsilon_1, \quad \Upsilon_3^3 = \Upsilon_4^4 = \Upsilon_3, \]

and could be written in the form

\[ R_1^1 = -\kappa \Upsilon_3, \]

\[ R_3^3 = -\kappa \Upsilon_1, \]

\[ R_{3i} = \kappa \Upsilon_{3i}, \]

\[ R_{4i} = \kappa \Upsilon_{4i}, \]

where \( i = 1, 2 \) and the left parts are given by the components of the Ricci tensor (11)–(14).

The Einstein equations (10), equivalently (16)–(19), reduce to this system of second order partial derivation equations:

\[ g_2^{**} - \frac{g_1' g_2^*}{2g_1} - \frac{(g_2^*)^2}{2g_2} \]

\[ g_1'' - \frac{g_1' g_2'}{2g_1} - \frac{(g_1')^2}{2g_1} = -2g_1 g_2 \Upsilon_3, \]

\[ h_4'^* - \frac{(h_4^*)^2}{2h_4} - \frac{h_1^* h_3^*}{2h_3} = -2h_3 h_4 \Upsilon_1, \]

\[ \beta w_i + \alpha_i = -2h_4 \kappa \Upsilon_{3i}, \]

\[ n_i'^* + \gamma n_i^* = -2h_4 \kappa \Upsilon_{4i}, \]
where

\[\alpha_1 = h_4^\bullet - \frac{h_1^\bullet}{2} \left( \frac{h_2^\bullet}{h_3} + \frac{h_4^\bullet}{h_4} \right),\]  
(24)

\[\alpha_2 = h_4^{\prime\bullet} - \frac{h_2^\bullet}{2} \left( \frac{h_2^\bullet}{h_3} + \frac{h_4^\bullet}{h_4} \right),\]  
(25)

\[\beta = h_4^{\prime\prime\bullet} - \frac{(h_2^\bullet)^2}{2h_4} \left( \frac{h_2^\prime h_3^\bullet}{2h_3} \right),\]  
(26)

\[\gamma = \frac{3h_4^\bullet}{2h_4} - \frac{h_3^\bullet}{h_3},\]  
(27)

and the partial derivatives are denoted, for instance,

\[g_{i}\partial^i = \partial g_i^2/\partial x^1 = \partial g_2/\partial r,\]
\[g_{\prime}^{\prime} = \partial g_1/\partial x^2 = \partial g_1/\theta,\]
\[h_3^\prime = \partial h_3/\partial s = \partial h_3/\partial \phi \quad \text{(or } \partial h_3/\partial y^4, \text{ for } s = y^4).\]

C. Dirac equations in anisotropic spacetimes

The problem of definition of spinors in locally anisotropic spaces and in spaces with higher order anisotropy was solved in Refs. [4]. In this paper we consider locally anisotropic Dirac spinors given with respect to anholonomic frames with associated N–connection structure on a 5D (pseudo) Riemannian space \(V^{(1,2,2)}\) constructed by a direct time extension of a 4D Riemannian space with two holonomic and two anholonomic variables.

Having an anisotropic d–metric

\[g_{\alpha\beta} = (g_{ij}(u), h_{ab}(u)) = (1, g_i^\bullet(u), h_a(u)),\]
\[\hat{i} = 1, 2; \hat{i} = 0, 1, 2; a = 3, 4,\]

defined with respect to an anholonomic basis (7) we can easily define the funfbein (pentad) fields

\[f_{\mu} = f_{\mu}^\nu \delta_{\mu} = \{f_1^\nu, f_2^\nu, f_3^\nu, f_4^\nu \},\]  
(28)

\[f^\nu_{\mu} = f^\nu_{\mu} \delta^\nu = \{f_1^\nu, f_2^\nu, f_3^\nu, f_4^\nu \},\]

satisfying the conditions

\[g_{\alpha\beta} = f_{\alpha}^\mu f_{\beta}^\nu \delta_{\mu} \delta_{\nu}, \quad h_{ab} = f_a^\nu f_b^\nu \delta_{\mu} \delta_{\nu},\]
\[g_{ij} = diag[1, -1, -1] \quad \text{and} \quad h_{ab} = diag[-1, -1].\]

For a diagonal d-metric of type (6) we have

\[f_1^\nu = \sqrt{|g_1^\bullet|} \delta_{\mu}^\nu \quad \text{and} \quad f_4^\nu = \sqrt{|h_4^\bullet|} \delta_{\mu}^\nu,\]

where \(\delta_{\mu}^\nu\) and \(\delta_{\nu}^\mu\) are Kronecker’s symbols.

We can also introduce the corresponding funfbein which are related with the off–diagonal metric ansatz (2) for \(g_{\alpha\beta},\)

\[e_{\mu} = e_{\mu}^\nu \partial_{\nu} \quad \text{and} \quad e_{\mu}^\nu = e_{\mu}^\nu \partial_{\nu},\]  
(29)

satisfying the conditions

\[g_{\alpha\beta} = e_{\alpha}^\nu e_{\beta}^\nu \quad \text{for} \quad g_{\mu\nu} = diag[1, -1, -1, -1, -1],\]
\[e_{\alpha}^\mu = \delta_{\alpha}^\mu \quad \text{and} \quad e_{\alpha}^\mu = \delta_{\mu}^\alpha.\]

The Dirac spinor fields on locally anisotropic deformations of Taub NUT spaces,
\[ \Psi (u) = [\Psi^\Gamma (u)] = [\psi^\dagger (u), \chi^\dagger (u)], \]

where \( \tilde{I} = 0, 1 \), are defined with respect to the 4D Euclidean tangent subspace belonging the tangent space to \( V^{(1,2,2)} \). The \( 4 \times 4 \) dimensional gamma matrices \( \gamma^{\alpha \beta} = [\gamma^1_\alpha, \gamma^2_\alpha, \gamma^3_\alpha, \gamma^4_\alpha] \) are defined as to satisfy the relation

\[ \left\{ \gamma^{\alpha \gamma} \gamma^{\beta \gamma} \right\} = 2g^{\alpha \beta}, \tag{30} \]

where \( \left\{ \gamma^{\alpha \gamma} \gamma^{\beta \gamma} \right\} \) is a symmetric commutator, \( g^{\alpha \beta} = (-1, -1, -1, -1) \), which generates a Clifford algebra distinguished on two holonomic and two anholonomic directions (hereafter the primed indices will run values on the Euclidean and/or Riemannian, 4D component of the 5D pseudo–Riemannian spacetime). In order to extend the \( (30) \) relations for unprimed indices \( \alpha, \beta \ldots \) we conventionally complete the set of primed gamma matrices with a matrix \( \gamma^{\alpha 4} \)

i.e. write \( \gamma^{\alpha} = [\gamma^{\alpha 1}, \gamma^{\alpha 2}, \gamma^{\alpha 3}, \gamma^{\alpha 4}] \) when

\[ \left\{ \gamma^{\alpha \gamma} \gamma^{\beta \gamma} \right\} = 2g^{\alpha \beta}. \]

The coefficients of gamma matrices can be computed with respect to coordinate bases \( (\hat{\theta}) \) or with respect to anholonomic bases \( (\hat{\theta}) \) by using respectively the funfbein coefficients \( (28) \) and \( (29) \),

\[ \gamma^{\alpha} (u) = e_{\alpha}^a (u) \gamma^a \text{ and } \tilde{\gamma}^{\beta} (u) = f_{\beta}^a (u) \gamma^a, \]

were by \( \gamma^{\alpha} (u) \) we denote the curved spacetime gamma matrices and by \( \tilde{\gamma}^{\beta} (u) \) we denote the gamma matrices adapted to the N–connection structure.

The covariant derivation of Dirac spinor field \( \Psi (u), \nabla_\alpha \Psi \), can be defines with respect to a pentad decomposition of the off–diagonal metric \( (\hat{\theta}) \).

\[ \nabla_\alpha \Psi = \left[ \partial_\alpha + \frac{1}{4} C_{\alpha \beta \gamma} (u) e_{\beta}^a (u) \gamma^{\beta \gamma} \right] \Psi, \tag{31} \]

where the coefficients

\[ C_{\alpha \beta \gamma} (u) = \left( D_{\gamma} e_{\alpha}^a \right) e_{\beta a} e_{\gamma}\]

are called the rotation Ricci coefficients; the covariant derivative \( D_{\gamma} \) is defined by the usual Christoffel symbols for the off–diagonal metric.

We can also define an equivalent covariant derivation of the Dirac spinor field, \( \tilde{\nabla}_\alpha \Psi \), by using pentad decompositions of the diagonalized d–metric \( (\hat{\theta}) \).

\[ \tilde{\nabla}_\alpha \Psi = \left[ \delta_\alpha + \frac{1}{4} C_{\alpha \beta \gamma}^{(\hat{\theta})} (u) f^{\beta} (u) \gamma^{\beta \gamma} \right] \Psi, \tag{32} \]

where there are introduced N–elongated partial derivatives and the coefficients

\[ C_{\alpha \beta \gamma}^{(\hat{\theta})} (u) = \left( D^{[\hat{\theta}]_{\gamma}} f_{\alpha}^a \right) f_{\beta a} f_{\gamma}\]

are transformed into rotation Ricci d–coefficients which together with the d–covariant derivative \( D^{[\hat{\theta}]}_{\gamma} \) are defined by anholonomic pentads and anholonomic transforms of the Christoffel symbols.

For diagonal d–metrics the funfbein coefficients can be taken in their turn in diagonal form and the corresponding gamma matrix \( \tilde{\gamma}^{\alpha} (u) \) for anisotropic curved spaces are proportional to the usual gamma matrix in flat spaces \( \gamma^\alpha \).

The Dirac equations for locally anisotropic spacetimes are written in the simplest form with respect to anholonomic frames,

\[ (i \tilde{\gamma}^{\alpha} (u) \nabla^\alpha - \mu) \Psi = 0, \tag{33} \]

where \( \mu \) is the mass constant of the Dirac field. The Dirac equations are the Euler equations for the Lagrangian

\[ \mathcal{L}^{(1/2)} = \sqrt{|g|} \left\{ [\Psi^+ (u) \tilde{\gamma}^{\alpha} (u) \nabla^\alpha \Psi (u) - (\nabla^\alpha \Psi^+ (u)) \tilde{\gamma}^{\alpha} (u) \Psi (u)] - \mu \Psi^+ (u) \Psi (u) \right\}, \tag{34} \]
where by $\Psi^+ (u)$ we denote the complex conjugation and transposition of the column $\Psi (u)$.

Varying $L^{(1/2)}$ on d–metric [34] we obtain the symmetric energy–momentum d–tensor

$$\Upsilon_{\alpha\beta} (u) = \frac{i}{4} [\Psi^+ (u) \bar{\gamma}_\alpha (u) \nabla^\beta \Psi (u)$$
$$+ \Psi^+ (u) \bar{\gamma}_\beta (u) \nabla^\alpha \Psi (u)$$
$$- (\nabla^\alpha \Psi^+ (u)) \bar{\gamma}_\beta (u) \Psi (u)$$
$$- (\nabla^\beta \Psi^+ (u)) \bar{\gamma}_\alpha (u) \Psi (u)].$$

We choose such spinor field configurations in curved spacetime as to be satisfied the conditions (15).

One can introduce similar formulas to (33)–(35) for spacetimes provided with off-diagonal metrics with respect to holonomic frames by changing of operators $\bar{\gamma}_\alpha (u) \rightarrow \gamma_\alpha (u)$ and $\nabla_\beta \rightarrow \nabla_\beta$.

III. ANISOTROPIC TAUB NUT – DIRAC SPINOR SOLUTIONS

By straightforward calculations we can verify that because the conditions

$$D [\delta \gamma^\alpha] \alpha \beta (u) = 0$$

are satisfied the Ricci rotation coefficients vanishes,

$$C_{\alpha\beta\gamma} (u) = 0$$

and the anisotropic Dirac equations (33) transform into

$$(i \bar{\gamma} \alpha (u) \delta_\alpha - \mu) \Psi = 0.$$

Further simplifications are possible for Dirac fields depending only on coordinates $(t, x^1 = r, x^2 = \theta)$, i. e. $\Psi = \Psi(x^k)$ when the equation (36) transforms into

$$(i \gamma^0 \partial_t + i \gamma^1 \frac{1}{\sqrt{|g_1|}} \partial_1 + i \gamma^2 \frac{1}{\sqrt{|g_2|}} \partial_2 - \mu) \Psi = 0.$$

The equation (38) simplifies substantially in $\zeta$–coordinates

$$(t, \zeta^1 = \zeta^1 (r, \theta), \zeta^2 = \zeta^2 (r, \theta)),$$

defined as to be satisfied the conditions

$$\frac{\partial}{\partial \zeta^1} = \frac{1}{\sqrt{|g_1|}} \partial_1$$
$$\frac{\partial}{\partial \zeta^2} = \frac{1}{\sqrt{|g_2|}} \partial_2$$

and we get

$$(-i \gamma^0 \frac{\partial}{\partial t} + i \gamma^1 \frac{\partial}{\partial \zeta^1} + i \gamma^2 \frac{\partial}{\partial \zeta^2} - \mu) \Psi (t, \zeta^1, \zeta^2) = 0.$$

The equation (38) describes the wave function of a Dirac particle of mass $\mu$ propagating in a three dimensional Minkowski flat plane which is imbedded as an anisotropic distribution into a 5D pseudo–Riemannian spacetime.

The solution $\Psi = \Psi(t, \zeta^1, \zeta^2)$ of (38) can be written

$$\Psi = \Psi(t, \zeta^1, \zeta^2)$$

with the condition that $k_0$ is identified with the positive energy and $\varphi^0 (k)$ and $\chi_0 (k)$ are constant bispinors. To satisfy the Klein–Gordon equation we must have

$$k^2 = k_0^2 - k_1^2 - k_2^2 = \mu^2.$$
The Dirac equations implies

\[(\sigma^i k_i - \mu)\varphi^0(k) \text{ and } (\sigma^i k_i + \mu)\chi^0(k),\]

where \(\sigma^i(i = 0, 1, 2)\) are Pauli matrices corresponding to a realization of gamma matrices as to a form of splitting to usual Pauli equations for the bispinors \(\varphi^0(k)\) and \(\chi^0(k)\).

In the rest frame for the horizontal plane parametrized by coordinates \(\zeta = \{t, \zeta^1, \zeta^2\}\) there are four independent solutions of the Dirac equations,

\[\varphi^0_{(1)}(\mu, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^0_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\]

\[\chi^0_{(1)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi^0_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.\]

In order to satisfy the conditions (15) for compatibility of the equations (20)–(23) we must consider wave packets of type (for simplicity, we can use only superpositions of positive energy solutions)

\[\Psi^{(+)}(\zeta) = \int \frac{d^3p}{2\pi^3} \frac{\mu}{\sqrt{\mu^2 + (k^2)^2}} \times \sum_{[\alpha]=1,2,3} b(p, [\alpha])\varphi^{[\alpha]}(k) \exp [-ik_i\zeta^i]\]

when the coefficients \(b(p, [\alpha])\) define a current (the group velocity)

\[J^2 = \sum_{[\alpha]=1,2,3} \int \frac{d^3p}{2\pi^3} \frac{\mu}{\sqrt{\mu^2 + (k^2)^2}} |b(p, [\alpha])|^2 \frac{p^2}{\sqrt{\mu^2 + (k^2)^2}} \equiv \langle p^2 \rangle >\]

with \(|p^2| \sim \mu\) and the energy–momentum d–tensor has the next nonrivial coefficients

\[\Upsilon^0_0 = 2T(\zeta^1, \zeta^2) = k_0 \Psi^+ \gamma_0 \Psi, \quad \Upsilon^1_1 = -k_1 \Psi^+ \gamma_1 \Psi, \quad \Upsilon^2_2 = -k_2 \Psi^+ \gamma_2 \Psi\]

where the holonomic coordinates can be reexpressed \(\zeta^i = \zeta^i(x^i)\). We must take two or more waves in the packet and choose such coefficients \(b(p, [\alpha])\), satisfying corresponding algebraic equations, as to have in (40) the equalities

\[\Upsilon^1_1 = \Upsilon^2_2 = \Upsilon(\zeta^1, \zeta^2) = \Upsilon(x^1, x^2),\]

required by the conditions (35).

**IV. TAUB NUT SOLUTIONS WITH GENERIC LOCAL ANISOTROPY**

The Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time coordinate in a trivial way. There are anisotropic variants of such solutions when anisotropies are modelled by effective polarizations of the induced magnetic field. The aim of this Section is to analyze such Taub–NUT solutions for both cases of locally isotropic and locally anisotropic configurations.

**A. A conformal transform of the Taub NUT metric**

We consider the Taub NUT solutions and introduce a conformal transformation and a such redefinition of variables which will be useful for further generalizations to anisotropic vacuum solutions.
1. The Taub NUT solution

This locally isotropic solution of the 5D vacuum Einstein equations is expressed by the line element

\[ ds^{2}_{(5D)} = dt^2 + ds^{2}_{(4D)}; \]

\[ ds^{2}_{(4D)} = -V^{-1}(dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2) - V(dx^4 + A_i dx^i)^2 \]

where

\[ V^{-1} = 1 + \frac{m_0}{r}, m_0 = \text{const.} \]

The functions \( A_i \) are static ones associated to the electromagnetic potential,

\[ A_r = 0, A_{\theta} = 0, A_{\phi} = 4m_0 (1 - \cos \theta) \]

resulting into "pure" magnetic field

\[ \vec{B} = \text{rot} \vec{A} = m_0 \frac{\vec{r}}{r^3}. \]

of a Euclidean instanton; \( \vec{r} \) is the spherical coordinate’s unity vector. The spacetime defined by (42) has the global symmetry of the group \( G_s = SO(3) \otimes U_4(1) \otimes T_\mu(1) \) since the line element is invariant under the global rotations of the Cartesian space coordinates and \( y^4 \) and \( t \) translations of the Abelian groups \( U_4(1) \) and \( T_\mu(1) \) respectively. We note that the \( U_4(1) \) symmetry eliminates the so called NUT singularity if \( y^4 \) has the period \( 4\pi m_0 \).

2. Conformally transformed Taub NUT metrics

With the aim to construct anisotropic generalizations it is more convenient to introduce a new 5th coordinate,

\[ y^4 \to \varsigma = y^4 - \int \mu^{-1}(\theta, \phi) d\xi(\theta, \phi), \]

with the property that

\[ d\varsigma + 4m_0(1 - \cos \theta) d\theta = dy^4 + 4m_0(1 - \cos \theta) d\phi, \]

which holds for

\[ d\xi = \mu(\theta, \phi) d(\varsigma - y^4) = \frac{\partial \xi}{\partial \theta} d\theta + \frac{\partial \xi}{\partial \phi} d\phi, \]

when

\[ \frac{\partial \xi}{\partial \theta} = 4m_0(1 - \cos \theta) \mu, \]

\[ \frac{\partial \xi}{\partial \phi} = -4m_0(1 - \cos \theta) \mu, \]

and, for instance,

\[ \mu = (1 - \cos \theta)^{-2} \exp[\theta - \phi]. \]

The changing of coordinate (44) describe a reorientation of the 5th coordinate in a such way as we could have only one nonvanishing component of the electromagnetic potential

\[ A_{\theta} = 4m_0 (1 - \cos \theta). \]

The next step is to perform a conformal transform,
and to consider the 5D metric

\[ ds^2_{(4D)} \rightarrow ds^2_{(4D)} = V ds^2_{(4D)} \]

and to consider the 5D metric

\[
\begin{align*}
    ds^2_{(5D)} &= dt^2 + ds^2_{(4D)}; \\
    ds^2_{(4D)} &= -(dr^2 + r^2 d\theta^2) - r^2 \sin^2 \theta d\varphi^2 \\
    &\quad - V^2 (d\zeta + A_\theta d\theta)^2,
\end{align*}
\]

(not being an exact solution of the Einstein equations) which will transform into some exact solutions after corresponding anholonomic transforms.

Here, we emphasize that we chose the variant of transformation of a locally isotropic non–Einsteinian metrics into an anisotropic one solving the vacuum Einstein equations in order to illustrate a more simple procedure of construction of 5D vacuum metrics with generic local anisotropy. As a matter of principle we could remove vacuum isotropic anisotropic one solving the vacuum Einstein equations in order to illustrate a more simple procedure of construction Ref. [2]) which will be extended to configurations with spinor matter field source.

B. Anisotropic Taub NUT solutions with magnetic polarization

We outline two classes of exact solutions of 5D vacuum Einstein equations with generic anisotropies (see details in Ref. [2]) which will be extended to configurations with spinor matter field source.

1. Solutions with angular polarization

The ansatz for a d–metric (1), with a distinguished anisotropic dependence on the angular coordinate \( \varphi \), when \( s = \varphi \), is taken in the form

\[
\begin{align*}
    \delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\
    \delta s^2_{(4D)} &= - (dr^2 + r^2 d\theta^2) - r^2 \sin^2 \theta d\varphi^2 \\
    &\quad - V^2 (d\zeta + n_2(\theta, \varphi) d\theta),
\end{align*}
\]

where the values \( \eta^2_{(\varphi)}(\theta, \varphi) \) (we use non–negative values \( \eta^2_{(\varphi)} \) not changing the signature of metrics) and \( n_2(\theta, \varphi) \) must be found as to satisfy the vacuum Einstein equations in the form (20) – (23). We can verify that the data

\[
\begin{align*}
    x^0 &= t, x^1 = r, x^2 = \theta, x^3 = s = \varphi, x^4 = \zeta, \\
    g_0 &= 1, g_1 = -1, g_2 = -r^2, g_3 = -r^2 \sin^2 \theta, \\
    h_4 &= V^2(\varphi) \eta^2_{(\varphi)}, \eta^2_{(\varphi)} = [1 + \varpi(r, \theta) \varphi]^2, w_i = 0; \\
    n_{0, 1} &= 0; n_2 = n_{2[0]}(r, \theta) + n_{2[1]}(r, \theta)/[1 + \varpi(r, \theta) \varphi]^2.
\end{align*}
\]

give an exact solution. If we impose the condition to obtain in the locally isotropic limit just the metric (45), we have to choose the arbitrary functions from the general solution of (23) as to have

\[ \eta^2_{(\varphi)} = [1 + \varpi(r, \theta) \varphi]^2 \rightarrow 1 \text{ for } \varpi(r, \theta) \varphi \rightarrow 0. \]

For simplicity, we can analyze only angular anisotropies with \( \varpi = \varpi(\theta) \), when

\[ \eta^2_{(\varphi)} = \eta^2_{(\varphi)}(\theta, \varphi) = [1 + \varpi(\theta) \varphi]^2. \]

In the locally isotropic limit of the solution for \( n_2(r, \theta, \varphi) \), when \( \varpi \varphi \rightarrow 0 \), we could obtain the particular magnetic configuration contained in the metric (15) if we impose the condition that

\[ n_{2[0]}(r, \theta) + n_{2[1]}(r, \theta) = A_\theta = 4m_0 (1 - \cos \theta), \]
which defines only one function from two unknown values \( n_{2[0]}(r, \theta) \) and \( n_{2[1]}(r, \theta) \). This could have a corresponding physical motivation. From the usual Kaluza–Klein procedure we induce the 4D gravitational field (metric) and 4D electromagnetic field (potentials \( A_i \)), which satisfy the Maxwell equations in 4D pseudo–Riemannian spacetime. For the case of spherical, locally isotropic, symmetries the Maxwell equations can be written for vacuum magnetic fields without any polarizations. When we introduce into consideration anholonomic constraints and locally anisotropic gravitational configurations the effective magnetic field could be effectively renormalized by higher dimension gravitational field. This effect, for some classes of anisotropies, can be modeled by considering that the constant \( m_0 \) is polarized,

\[
m_0 \to m(r, \theta, \varphi) = m_0 \eta_m(r, \theta, \varphi)
\]

for the electromagnetic potential and resulting magnetic field. For "pure" angular anisotropies we write that

\[
n_2(\theta, \varphi) = n_{2[0]}(\theta) + n_{2[1]}(\theta) / (1 + \varpi(\theta) \varphi)^2
\]

\[
= 4m_0 \eta_m(\theta, \varphi) (1 - \cos \theta),
\]

for

\[
\eta^2_n(\theta, \varphi) = \eta^2_{n[0]}(\theta) + \eta^2_{n[1]}(\theta) / (1 + \varpi(\theta) \varphi)^2.
\]

This could result in a constant angular renormalization even \( \varpi(\theta) \varphi \to 0 \).

2. Solutions with extra–dimension induced polarization

Another class of solutions is constructed if we consider a d–metric of the type (6), when \( s = \varsigma \), with anisotropic dependence on the 5th coordinate \( \varsigma \),

\[
\delta s^2 = dt^2 - \delta s^2_{(4D)},
\]

\[
\delta s^2_{(4D)} = -(dr^2 + r^2 d\theta^2) - r^2 \sin^2 \theta d\varphi^2 - \nu(r) \eta^2_0(\theta, \varsigma) d\varsigma^2,
\]

\[
\delta \varsigma = d\varsigma + w_3(\theta, \varsigma) d\theta,
\]

where, for simplicity, we omit possible anisotropies on variable \( r \), i. e. we state that \( \eta_{(\varsigma)} \) and \( w_2 \) are not functions of \( r \).

The data for a such solution are

\[
x^0 = t, x^1 = r, x^2 = \theta, y^3 = s = \varsigma, y^4 = \varphi,
\]

\[
g_0 = 1, g_1 = -1, g_2 = -r^2, g_4 = -r^2 \sin^2 \theta,
\]

\[
h_1 = \nu(r) \eta^2_0, \eta^2_0 = \eta^2_0(\theta, \varsigma) \eta^2_0; n_{0,1} = 0;
\]

\[
w_{0,1,2} = 4m_0 \eta_m(\theta, \varsigma) (1 - \cos \theta), n_{0,2} = 0,
\]

\[
n_{1,2} = n_{1,2[0]}(r, \theta) + n_{1,2[1]}(r, \theta) \int \eta^{-3}_n(r, \theta, \varsigma) d\varsigma,
\]

where the function \( \eta_{(\varsigma)} = \eta_{(\varsigma)}(r, \theta, \varsigma) \) is an arbitrary one as follow for the case \( h^4_4 = 0 \), for angular polarizations we state, for simplicity, that \( \eta_{(\varsigma)} \) does not depend on \( r \), i. e. \( \eta_{(\varsigma)} = \eta_{(\varsigma)}(\theta, \varsigma) \). We chose the coefficient

\[
w_4 = 4m_0 \eta_m(\theta, \varsigma) (1 - \cos \theta)
\]

as to have compatibility with the locally isotropic limit when \( w_2 \simeq A_\theta \) with a "polarization" effect modeled by \( \eta_m(\theta, \varsigma) \), which could have a constant component \( \eta_m \simeq \eta_{m[0]} = \text{const} \) for small anisotropies. In the simplest cases we can fix the conditions \( n_{1,2[0,1]}(r, \theta) = 0 \). All functions \( \eta^2_0, \eta_m \) and \( n_{1,2[0,1]} \) can be treated as some possible induced higher dimensional polarizations.
V. ANISOTROPIC TAUB NUT–DIRAC FIELDS

In this Section we construct two new classes of solutions of the 5D Einstein–Dirac fields in a manner as to extend the locally anisotropic Taub NUT metrics defined by data (46) and (47) as to be solutions of the Einstein equations (20)–(23) with a nonvanishing diagonal energy momentum d–tensor

\[ r, \phi_n \] of the Einstein equations becomes linear (21)

\[ \{2\} \]

\[ r, \phi_n \] with a nonvanishing diagonal energy momentum d–tensor the locally anisotropic Taub NUT metrics defined by data (46) and (47) as to be solutions of the Einstein equations

\[ \text{for a Dirac wave packet satisfying the conditions (21) and (22).} \]

A. Dirac fields and angular polarizations

In order to generate from the data (46) a new solution with Dirac spinor matter field we consider instead of a linear dependence of polarization, \( \eta(\varphi) \sim [1 + \varphi (r, \theta)] \), an arbitrary function \( \eta(\varphi) (r, \theta, \varphi) \) for which \( h_4 = V^2(r) \eta^2(\varphi) (r, \theta, \varphi) \) is an exact solution of the equation (21) with \( \Upsilon = \Upsilon (r, \theta) \). With respect to the variable \( \eta^2(\varphi) (r, \theta, \varphi) \) this component of the Einstein equations becomes linear

\[ \eta^{**} + r^2 \sin^2 \theta \Upsilon \eta(\varphi) = 0 \]  \( (48) \)

which is a second order linear differential equation on variable \( \varphi \) with parametric dependencies of the coefficient \( r^2 \sin^2 \theta \Upsilon \) on coordinates \( (r, \theta) \). The solution of equation (48) is to be found following the method outlined in Ref. [17].

\[ \eta(\varphi) = C_1 (r, \theta) \cosh[\varphi r \sin \theta \sqrt{\Upsilon (r, \theta)}] + C_2 (r, \theta), \]

\[ \Upsilon (r, \theta) < 0; \]  \( (49) \)

\[ = C_1 (r, \theta) + C_2 (r, \theta) \varphi, \Upsilon (r, \theta) = 0; \]  \( (50) \)

\[ = C_1 (r, \theta) \cos[\varphi r \sin \theta \sqrt{\Upsilon (r, \theta)}] + C_2 (r, \theta), \]

\[ \Upsilon (r, \theta) > 0, \]  \( (51) \)

where \( C_{1,2} (r, \theta) \) are some functions to be defined from some boundary conditions. The first solution (49), for negative densities of energy should be excluded as unphysical, the second solution (50) is just that from (46) for the vacuum case. A new interesting physical situation is described by the solution (51) when we obtain a Taub NUT anisotropic metric with periodic anisotropic dependencies on the angle \( \varphi \) where the periodicity could vary on coordinates \( (r, \theta) \) as it is defined by the energy density \( \Upsilon (r, \theta) \). For simplicity, we can consider a package of spinor waves with constant value of \( \Upsilon = \Upsilon_0 \) and fix some boundary and coordinate conditions when \( C_{1,2} = C_{1,2}[0] \) are constant. This type of anisotropic Taub NUT solutions are described by a d–metric coefficient

\[ h_4 = V^2(r) \cos^2[\varphi r \sin \theta \sqrt{\Upsilon_0} + C_2[0]]. \]  \( (52) \)

Putting this value into the formulas (24), (23) and (22) for coefficients in equations (22) we can express \( \alpha_{1,2} = \alpha_{1,2}[h_3, h_4, \Upsilon_0] \) and \( \beta = \beta[h_3, h_4, \Upsilon_0] \) (we omit these rather simple but cumbersome formulas) and in consequence we can define the values \( w_{1,2} \) by solving linear algebraic equations:

\[ w_{1,2} (r, \theta, \varphi) = \alpha_{1,2} / \beta (r, \theta, \varphi). \]

Having defined the values (52) it is a simple task of two integrations on \( \varphi \) in order to define

\[ n_2 = n_2[0] (r, \theta) \left[ \ln \frac{1 + \cos \kappa}{1 - \cos \kappa} + \frac{1}{1 - \cos \kappa} + \frac{1}{1 - \sin \kappa} \right] \]

\[ + n_2[1] (r, \theta), \]  \( (53) \)

were

\[ \kappa = \varphi r \sin \theta \sqrt{\Upsilon_0} + C_2[0]. \]

\[ n_{2[0,1]} (r, \theta) \] are some arbitrary functions to be defined by boundary conditions. We put \( n_{0,1} = 0 \) to obtain in the vacuum limit the solution (46).
Finally, we can summarize the data defining an exact solution for an anisotropic (on angle $\varphi$) Dirac wave packet – Taub NUT configuration:

$$x^0 = t, x^1 = r, x^2 = \theta, y^3 = s = \varphi, y^4 = \varsigma,$$

$$g_0 = 1, g_1 = -1, g_2 = -r^2, h_3 = -r^2 \sin^2 \theta,$$

$$h_4 = V^2(r) \eta_{(r)}^2 \eta_{(\varphi)}, \eta_{(\varphi)} = C_1(r, \theta) \cos \kappa(r, \theta, \varphi),$$

$$w_1 = 0, n_{1,0} = 0, n_{2} = n_2(r, \theta, \kappa(r, \theta, \varphi)) \text{ see (53)},$$

$$\Psi = \Psi^{(+)}(\zeta, x^1, x^2) \text{ see (39)},$$

$$\Upsilon = \Upsilon^{(+)}(\zeta, x^1, x^2) \text{ see (40)}.$$ 

This solution will be extended to additional soliton anisotropic configurations in the next Section.

### B. Dirac fields and extra dimension polarizations

Now we consider a generalization of the data (47) for generation of a new solution, with generic local anisotropy on extra dimension 5th coordinate, of the Einstein – Dirac equations. Following the equation (22) we conclude that there are no nonvacuum solutions of the Einstein equations (with $\Upsilon \neq 0$) if

$$h_4^* = 0$$

which impose the condition $\Upsilon = 0$ for $h_3, h_4 \neq 0$. So, we have to consider that the $d$-metric component $h_4 = -r^2 \sin^2 \theta$ from the data (47) is generalized to a function $h_4(r, \theta, \varsigma)$ satisfying a second order nonlinear differential equation on variable $\varsigma$ with coefficients depending parametrically on coordinates $(r, \theta)$.

The equation (i. e. (22)) can be linearized (see Ref. [17]) if we introduce a new variable $h_4 = h_2$, 

$$h^{**} - \frac{h_3^*}{2h_3} h^* + h_3 \Upsilon h = 0,$$

which, in its turn, can be transformed:

a) to a Riccati form if we introduce a new variable $v$, for which $h = v^*/v$,

$$v^* + v^2 - \frac{h_3^*}{2h_3} v + h_3 \Upsilon = 0; \quad (55)$$

b) to the so-called normal form [17],

$$\lambda^{**} + I \lambda = 0, \quad (56)$$

obtained by a redefinition of variables like

$$\lambda = h \exp \left[ -\frac{1}{4} \int \frac{h_3^*}{h_3} d\varsigma \right] = h h_3^{-1/4},$$

where

$$I = h_3 \Upsilon - \frac{1}{16} \frac{h_3^*}{h_3} + \frac{1}{4} \left( \frac{h_3^*}{h_3} \right)^*.$$

We can construct explicit series and/or numeric solutions (for instance, by using Mathematica or Maple programs) of both type of equations (55) and normal (56) for some stated boundary conditions and type of polarization of the coefficient $h_3(r, \theta, \varsigma) = V^2(r) \eta_{(r)}^2 \eta_{(\varphi)}(r, \theta, \varsigma)$ and, in consequence, to construct different classes of solutions for $h_4(r, \theta, \varsigma)$. In order to have compatibility with the data (47) we must take $h_4$ in the form

$$h_4(r, \theta, \varsigma) = -r^2 \sin^2 \theta + h_{4(c)}(r, \theta, \varsigma),$$

where $h_{4(c)}(r, \theta, \varsigma)$ vanishes for $\Upsilon \to 0$.

Having defined a value of $h_4(r, \theta, \varsigma)$ we can compute the coefficients (24), (25) and (26) and find from the equations (22)

$$w_{1,2}(r, \theta, \varsigma) = \alpha_{1,2}(r, \theta, \varsigma) / \beta(r, \theta, \varsigma).$$
From the equations (23), after two integrations on variable $\varsigma$ one obtains the values of $n_{1,2}(r, \theta, \varsigma)$. Two integrations of equations (23) define

$$n_{i}(r, \theta, \varsigma) = n_{i[0]}(r, \theta) \int_{0}^{\varsigma} dz \int_{0}^{z} ds P(r, \theta, s) + n_{i[1]}(r, \theta),$$

where

$$P = \frac{1}{2} \left( \frac{h_{3}^{*}}{h_{3}} - \frac{3}{h_{4}} \right)$$

and the functions $n_{i[0]}(r, \theta)$ and $n_{i[1]}(r, \theta)$ on $(r, \theta)$ have to be defined by solving the Cauchy problem. The boundary conditions of both type of coefficients $w_{1,2}$ and $n_{1,2}$ should be expressed in some forms transforming into corresponding values for the data (47) if the source $\Upsilon \rightarrow 0$. We omit explicit formulas for exact Einstein–Dirac solutions with $\varsigma$–polarizations because their forms depend very strongly on the type of polarizations and vacuum solutions.

VI. ANHOLONOMIC SOLITON–TAUB NUT–DIRAC FIELDS

In the next subsections we analyze two explicit examples when the spinor field induces two dimensional, depending on three variables, solitonic anisotropies.

A. Kadomtsev–Petviashvili type solitons

By straightforward verification we conclude that the d–metric component $h_{4}(r, \theta, s)$ could be a solution of Kadomtsev–Petviashvili (KdP) equation (18) (the first methods of integration of 2+1 dimensional soliton equations where developed by Dryuma [19] and Zakharov and Shabat [20])

$$h_{4}^{**} + \epsilon \left( h_{4} + 6 h_{4} h_{4}' + h_{4}'' \right)' = 0, \epsilon = \pm 1,$$

if the component $h_{3}(r, \theta, s)$ satisfies the Bernoulli equations (17)

$$h_{3}^{*} + Y(r, \theta, s) (h_{3})^{2} + F_{\epsilon}(r, \theta, s) h_{3} = 0,$$

where, for $h_{3} \neq 0$,

$$Y(r, \theta, s) = \kappa \Upsilon \frac{h_{4}}{h_{4}}$$

and

$$F_{\epsilon}(r, \theta, s) = \frac{h_{4}^{*}}{h_{4}} + \frac{2 \epsilon}{h_{4}} \left( h_{4} + 6 h_{4} h_{4}' + h_{4}'' \right)'.$$

The three dimensional integral variety of (58) is defined by formulas

$$h_{3}^{-1}(r, \theta, s) = h_{3}^{-1}(x, r, \theta) E_{\epsilon}(x, \theta, s) \times \frac{Y(r, \theta, s)}{F_{\epsilon}(r, \theta, s)} ds,$$

where

$$E_{\epsilon}(r, \theta, s) = \exp \int F_{\epsilon}(r, \theta, s) ds$$

and $h_{3}(r, \theta)$ is a nonvanishing function.

In the vacuum case $Y(r, \theta, s) = 0$ and we can write the integral variety of (58)

$$h_{3}^{(vac)}(r, \theta, s) = h_{3}^{(vac)}(x, r, \theta) \exp \left[ - \int F_{\epsilon}(r, \theta, s) ds \right].$$

We conclude that a solution of KdP equation (58) could be generated by a non–perturbative component $h_{4}(r, \theta, s)$ of a diagonal h–metric if the second component $h_{3}(r, \theta, s)$ is a solution of Bernoulli equations (58) with coefficients determined both by $h_{4}$ and its partial derivatives and by the $\Upsilon_{1}$ component of the energy–momentum d–tensor (see (11)). The parameters (coefficients) of (2+1) dimensional KdP solitons are induced by gravity and spinor constants and spinor waves coefficients defining locally anisotropic interactions of packets of Dirac’s spinor waves.
B. (2+1) sine–Gordon type solitons

In a similar manner we can prove that solutions \( h_4(r, \theta, s) \) of (2+1) sine–Gordon equation (see, for instance, \([21–23]\))

\[
h_4^{**} + \kappa E(r, \theta) \frac{h_3}{h_4^2} (h_3)^2 + F(r, \theta, s) h_3 = 0, h_4^* \neq 0,
\]

also induce solutions for \( h_3(r, \theta, s) \) following from the Bernoulli equation

\[
F(r, \theta, s) = \frac{h_4^*}{h_4} + 2 \frac{h_4'' - \bar{h}_4 - \sin(h_4)}{h_4'^2}.
\]

The general solutions (with energy–momentum sources and in vacuum cases) are constructed by a corresponding redefinition of coefficients in the formulas from the previous subsection. We note that we can consider both type of anisotropic solitonic polarizations, depending on angular variable \( \varphi \) or on extra dimension coordinate \( \varsigma \). Such classes of solutions of the Einstein–Dirac equations describe three dimensional spinor wave packets induced and moving self–consistently on solitonic gravitational locally anisotropic configurations. In a similar manner, we can consider Dirac wave packets generating and propagating on locally anisotropic black hole (with rotation ellipsoid horizons), black tori, anisotropic disk and two or three dimensional black hole anisotropic gravitational structures \([1]\). Finally, we note that such gravitational solitons are induced by Dirac field matter sources and are different from those soliton solutions of vacuum Einstein equations originally considered by Belinski and Zakharov \([24]\).

VII. CONCLUSIONS

We have argued that the anholonomic frame method can be applied for construction on new classes of Einstein–Dirac equations in five dimensional (5D) spacetimes. Subject to a form of metric ansatz with dependencies of coefficients on two holonomic and one anholonomic variables we obtained a very simplified form of field equations which admit exact solutions. We have identified two classes of solutions describing Taub NUT like metrics with anisotropic dependencies on angular parameter or on the fifth coordinate. We have shown that both classes of anisotropic vacuum solutions can be generalized to matter sources with the energy–momentum tensor defined by some wave packets of Dirac fields. Although the Dirac equation is a quantum one, in the quasi–classical approximation we can consider such spinor fields as some spinor waves propagating in a three dimensional Minkowski plane which is imbedded in a self–consistent manner in a Taub–NUT anisotropic spacetime. At the classical level it should be emphasized that the results of this paper are very general in nature, depending in a crucial way only on the locally Lorentzian nature of 5D spacetime and on the supposition that this spacetime is constructed as a trivial time extension of 4D spacetimes. We have proved that the new classes of solutions admit generalizations to nontrivial topological configurations of 3D dimensional solitons (induced by anisotropic spinor matter) defined as solutions Kadomtsev–Petviashvili or sine–Gordon equations.

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