ON THE CONFORMAL EIN INVARIANTS

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ABSTRACT. For a compact Riemannian n-manifold \((M, g)\) of positive scalar curvature, the capital Ein invariant of \(g\) is defined to be the infimum over \(M\) of the quotient of the scalar curvature by the maximal eigenvalue of the Ricci curvature. This is a re-scale invariant and belongs to the interval \((0, n]\). For a positive conformal class \([g]\), we define the conformal invariant \(\text{Ein}([g]) := \sup\{\text{Ein}(g) : g \in [g]\}\). In this paper, we prove vanishing theorems for Betti numbers and for the higher homotopy groups of \(M\) under optimal lower bounds on \(\text{Ein}([g])\) assuming that \(g\) is locally conformally flat. We establish an inequality relating our invariant to Schoen-Yau conformal invariant \(d(M, [g])\) from which we deduce a classification result for locally conformally flat manifolds with higher \(\text{Ein}([g])\).

We show that the class of locally conformally flat manifolds with \(\text{Ein}([g]) > k\) is stable under the operation of connected sums for \(0 < k < n - 1\). For a general positive conformal class, we prove in dimension 4 an inequality relating \(\text{Ein}([g])\) to the first and second Yamabe invariants. Similar results are proved in this paper for an analogous conformal invariant, namely the small ein invariant.

Keywords: Ein and ein conformal invariants, positive curvature, locally conformally flat, vanishing theorems.

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References

1. Introduction

Throughout this paper, $(M, g)$ denotes a closed connected Riemannian manifold of dimension $n$. We denote by $\text{Ric}$ and $\text{Scal}$ the Ricci and scalar curvatures of $(M, g)$. Recall that the $k$-th modified Einstein tensor on $(M, g)$ is defined by

\[ \text{Ein}_k(g) := \text{Scal} - k \text{Ric}, \]

Where $k$ is a constant. We bring to the attention of the reader that this modification was used first by Rastall in 1972 in his modified theory of Gravity \[19\] and studied by Bourguignon in \[2\]. We are interested in the positivity properties of these modified tensors. We first remark that for a compact Riemannian $n$-manifold $(M, g)$ with positive scalar curvature and for $0 < k < n$, the tensor $\text{Ein}_k(g)$ is positive definite if and only if at each point of $M$ one has

\[ k < \frac{\text{Scal}}{\rho_{\text{max}}}, \]

Where $\rho_{\text{max}}$ denotes the maximal eigenvalue of the Ricci curvature.

1.1. The capital $\text{Ein}$ invariant. A straightforward consequence of the above characterisation of the positivity of the tensors $\text{Ein}_k(g)$ is the following descent positivity property

\[ \text{For } 0 < k < l < n, \text{ } \text{Ein}_l > 0 \Rightarrow \text{Ein}_k > 0. \]

We therefore define the metric invariant

\[ \text{Ein}(g) := \sup\{k \in (0, n) : \text{Ein}_k(g) > 0\}. \]

We set $\text{Ein}(g) = 0$ if the scalar curvature of $g$ is not positive. An immediate consequence of \[2\] is that for a metric $g$ of positive scalar curvature one has

\[ \text{Ein}(g) = \inf_M \frac{\text{Scal}(g)}{\rho_{\text{max}}(g)}. \]

The above formula shows in particular that the metric invariant $\text{Ein}(g)$ is re-scale invariant, That is for any positive real number $t$, one has

\[ \text{Ein}(tg) = \text{Ein}(g). \]

Recall for later reference that \[12\]

\[ \text{Ein}(g) > n - 1 \Rightarrow \text{Ric} > 0. \]

\[ \text{Ein}(g) = n - 1 \Rightarrow \text{Ric} \geq 0. \]
Conversely, \( \text{Ric} > 0 \) implies only that \( \text{Ein} > 1 \) and this is optimal as shown by the example of Berger metrics on \( S^{2n+1} \). The Ein invariant defines a \textit{pre-order} on the set of Riemannian metrics on \( M \):

\[
(8) \quad g_1 \preceq g_2 \text{ if } \text{Ein}(g_1) \leq \text{Ein}(g_2).
\]

This lead us naturally to the study of maximal metrics with respect to the above pre-order.

We define the smooth invariant \( \text{Ein}(M) \) to be

\[
(9) \quad \text{Ein}(M) = \sup\{\text{Ein}(g) : g \in \mathcal{M}\},
\]

where \( \mathcal{M} \) denotes the space of all Riemannian metrics on \( M \).

1.2. \textbf{The small ein invariant.} The tensors \( \text{Ein}_k(g) \) for negative \( k \) enjoy the following descent positivity property

\[
(10) \quad \text{For } l < k < 0, \quad \text{Ein}_l > 0 \Rightarrow \text{Ein}_k > 0 \Rightarrow \text{Scal} > 0.
\]

We then naturally define \[12\] the metric invariant \( \text{ein}(g) \) to be

\[
(11) \quad \text{ein}(g) := \inf\{k < 0 : \text{Ein}_k(g) > 0\}
\]

We set \( \text{ein}(g) = 0 \) if the scalar curvature of \( g \) is not positive and \( \text{ein}(g) = -\infty \) in case the corresponding set of \( k \)'s is unbounded below.

We remark that if the metric \( g \) has nonnegative Ricci curvature and positive scalar curvature then clearly \( \text{ein}(g) = -\infty \). Otherwise, \( \text{ein}(g) \) can be alternatively defined by

\[
(12) \quad \text{ein}(g) = \sup_{x \in M} \left\{ \frac{\text{Scal}(x)}{\rho_{\text{min}}(x)} : \rho_{\text{min}}(x) < 0 \right\},
\]

where \( \rho_{\text{min}}(x) \) denotes the minimum eigenvalue of Ricci curvature of \( g \) at \( x \in M \).

\textbf{Remark.} We bring to the attention of the reader that Guan and Wang [6], proved that in the case \( \rho_{\text{min}} \) is everywhere negative on \( M \), the existence of a unique conformal metric \( \bar{g} \) whose minimum of Ricci curvature is constant over \( M \) and equal to \(-1\). In particular, if the scalar curvature of \( \bar{g} \) is positive then \( \text{ein}(\bar{g}) \) is nothing but the negative of the infinimum over \( M \) of the scalar curvature of \( \bar{g} \).

A second smooth invariant \( \text{ein}(M) \) of \( M \) is defined as follows

\[
(13) \quad \text{ein}(M) = \inf\{\text{ein}(g) : g \in \mathcal{M}\}.
\]

The above invariants were studied in [12]. In this paper, we restrict our study of the above constants once restricted to a given conformal class. We start by a definition

\textbf{Definition 1.1.} Let \( [g] \) denotes the conformal class of the metric \( g \), we define the following two constants of \( [g] \)

\[
(14) \quad \text{Ein}([g]) = \sup\{\text{Ein}(g) : g \in [g]\} \quad \text{and} \quad \text{ein}([g]) = \inf\{\text{ein}(g) : g \in [g]\}.
\]
We remark first that if the conformal class of \( g \) contains an Einstein metric with positive scalar curvature then \( \text{Ein}(g) = n \). Similarly, if the conformal class of \( g \) contains a metric with non-negative Ricci curvature and with positive scalar curvature then \( \text{ein}(g) = -\infty \).

**Definition 1.2.**
- A metric \( g \) is said to be Ein(\( M \))-maximal if \( \text{Ein}(g) = \text{Ein}(M) \).
- A metric \( g \) is said to be Ein([\( g \)])-maximal if \( \text{Ein}(g) = \text{Ein}(M) \).

2. **Statement of the main results**

The first result is a vanishing theorem

**Theorem A.** Let \((M, g)\) be a compact locally conformally flat \( n \)-manifold and \( p \) an integer such that \( 1 \leq p \leq \frac{n}{2} \).

1. If \( \text{Ein}([g]) > \frac{(n-1)(n-2p)}{n-p-1} \) then the Betti numbers \( b_k \) of \( M \) vanish for \( p \leq k \leq n-p \).
2. If \( p > 1 \) and \( \text{ein}([g]) < \frac{(n-1)(2p-n)}{p-1} \) then the Betti numbers \( b_k \) of \( M \) vanish for \( p \leq k \leq n-p \).

As a consequence we are able to compute the Ein([\( g \)]) and ein([\( g \)]) constants for the product of two space forms of opposite signs as follows

**Corollary A.** Let \( n, d \) be positive integers such that \( d < \frac{n-2}{2} \). Let \((S^{n-d-1}, g_0)\) be the standard sphere of curvature \(+1\) and let \((M^{d+1}, g_1)\) be a compact space form of curvature \(-1\). We denote by \( g \) the Riemannian product of \( g_0 \) and \( g_1 \) on \( S^{n-d-1} \times M^{d+1} \). Then

\[
\text{Ein}([g]) = \text{Ein}(g) = (n-1)\frac{2d-n+2}{d-n+2} \quad \text{and} \quad \text{ein}([g]) = \text{ein}(g) = -(n-1)\frac{n-2d-2}{d}.
\]

In other words the metric \( g \) is Ein([\( g \)])-Maximal and ein([\( g \)])-Minimal.

The next theorem provides a converse result to the previous corollary and the limit case in the above theorem

**Theorem A'.** Let \((M, g)\) be a compact locally conformally flat \( n \)-manifold with \( n > 2 \), and \( p \) an integer such that \( 1 \leq p < \frac{n}{2} \).

1. If \( \text{Ein}(g) = \frac{(n-1)(n-2p)}{n-p-1} \) and the Betti number \( b_p \neq 0 \) then \( M \) is covered by the standard product \( S^{n-p} \times H^p \) of the \((n-p)\)-sphere and the hyperbolic space of dimension \( p \).
2. If \( p > 1 \), \( \text{ein}(g) = \frac{(n-1)(2p-n)}{p-1} \) and the Betti number \( b_p \neq 0 \) then \( M \) is covered by the standard product \( S^{n-p} \times H^p \).
3. If \( p = 1 \), \( \text{ein}(g) = -\infty \) and the Betti number \( b_1 \neq 0 \) then \( M \) is covered by the standard product \( S^{n-1} \times \mathbb{R} \).

Let now \((M, g)\) be a smooth compact connected locally conformally flat Riemannian \( n \)-manifold with positive scalar curvature of dimension \( n \geq 4 \), we denote by \( \tilde{M} \) its universal cover. Schoen and Yau [20], see also [10] [11] [17], proved then that the developing map \( \Phi : \tilde{M} \to S^n \) is a conformal embedding, \( \pi_1(M) \) is isomorphic to a discrete subgroup \( \Gamma \) of Conf(\( M \)), \( \Phi(\tilde{M}) \) is a domain \( \Omega \) in \( S^n \).
and coincides with the complement $\Omega(\Gamma)$ of the limit set $\Lambda(\Gamma)$ of the action of $\pi_1(M) \approx \Gamma$ on the sphere $S^n$. In other words, $(M, g)$ is conformally equivalent to the Kleinian manifold $\Omega/\Gamma$.

Let $\delta := \delta(\Gamma)$ denotes the critical exponent of the Kleinian group $\Gamma$, see [20, 17, 10, 11]. It turns out that $\delta$ depends only on the conformal class of $g$, precisely $\delta$ coincides with the Schoen-Yau conformal invariant $\delta(M, [g])$ of the conformal class $[g]$ of $g$, see Theorem 4 in [11]. If in addition the Kleinian group $\Gamma$ is not elementary then $\delta$ coincides with the Hausdorff dimension of the limit set $\Lambda(\Gamma)$.

Shoen-Yau proved in [20] that for the above manifold one has $\delta(M, [g]) \leq \frac{n-2}{2}$. In the next theorem we generalize this result as follows

**Theorem B.** Let $(M, g)$ be a compact connected locally conformally flat Riemannian manifold of dimension $n \geq 4$ such that $\text{Ein}([g]) > 0$. Then

$$d \leq (n - 2) \frac{n - 1 - \text{Ein}([g])}{2(n - 1) - \text{Ein}([g])}.$$  

Where $d$ denotes the Schoen-Yau conformal invariant $\delta(M, [g])$ of the conformal class $[g]$. Equivalently, we have

$$\text{Ein}([g]) \geq (n - 1) \frac{2d - n + 2}{d - n + 2}.$$  

Recall that the condition $\text{Ein}([g]) > 0$ is equivalent to the existence of a conformal metric in $[g]$ of positive scalar curvature.

The above inequality is optimal as shown by the standard metric on the product $S^{n-d-1} \times \mathbb{H}^{d+1}$.

For more details see Example 3.1

**Corollary B.** Let $(M, g)$ be a compact connected locally conformally flat Riemannian manifold of dimension $n \geq 4$. Let $p$ be an integer such that $2 \leq p \leq n - 1$. Then one has

$$\text{Ein}([g]) > \max\left\{0, \frac{(n - 1)(2p - n)}{p - 1}\right\} \implies \pi_2(M) = \pi_3(M) = ... = \pi_p(M) = 0.$$  

The next Corollary provides a classification of locally conformally flat Riemannian manifold of dimension $n \geq 4$ and of higher Ein

**Corollary B’.** Let $(M, g)$ be a compact connected locally conformally flat Riemannian manifold of dimension $n \geq 4$. If $\text{Ein}([g]) > \frac{(n-1)(n-4)}{n-3}$ then there is a finite covering of $M$ which is either diffeomorphic to $S^n$ or to connected sums of copies of $S^1 \times S^{n-1}$.

The above inequality is optimal as shown by the example of the standard product $S^{n-2} \times \mathbb{H}^2$ whose conformal Ein equals $\frac{(n-1)(n-4)}{n-3}$.

**Remark.** It is remarkable that the lower bound $\frac{(n-1)(n-4)}{n-3}$ for $\text{Ein}([g])$ in Corollary B’ is also the optimal lower bound required for $\text{Ein}(g)$ in order for it to imply positive isotropic curvature (PIC). Precisely, it is not difficult to check that for a locally conformally flat Riemannian manifold of dimension $n \geq 4$ one has

$$\text{Ein}(g) > \frac{(n-1)(n-4)}{n-3} \implies \text{PIC}.$$
The same conclusion does not hold for \( \text{Ein}[g] \leq \frac{(n-1)(n-4)}{n-3} \) as shown by the example of the standard product \( S^{n-2} \times \mathbb{H}^2 \).

The next Theorem C shows that within the class of conformally flat manifolds, the positivity of modified Einstein tensors is preserved under connected sums

**Theorem C.** (1) For \( k \in (-\infty, 2) \), the connected sum of two conformally flat manifolds each one of positive modified Einstein tensor \( \text{Ein}_k \) and with dimension \( \geq 3 \) admits a conformally flat metric of positive \( \text{Ein}_k \) tensor.

(2) For \( k \geq 2 \), the connected sum of two conformally flat manifolds each one of positive \( \text{Ein}_k \) tensor and with dimension \( > k+1 \) admits a conformally flat metric of positive \( \text{Ein}_k \) tensor.

As an application of the above theorem one has,

**Corollary C.** Let \( n \geq 3 \) and \( p \geq 1 \). For each real numbers \( k_1 \in [0, n-1), k_2 \in (-\infty, 0] \), the connected sum of \( p \) copies of \( S^1 \times S^{n-1} \) admits conformally flat metrics \( g_1 \) and \( g_2 \) such that \( \text{Ein}(g_1) > k_1 \) and \( \text{ein}(g_2) < k_2 \).

**Remark.** For \( p > 1 \), the connected sum of \( p \) copies of \( S^1 \times S^{n-1} \) can't admit a conformally flat metric \( g \) with \( \text{Ein}(g) = n - 1 \) as this will imply that \( g \) has non-negative Ricci curvature which is impossible, see [15]. It would be interesting to decide whether there exists on the previous connected sum a conformally flat metric \( g \) with \( \text{Ein}(g) = n - 1 \).

Next, we discuss the case of non conformally flat classes in four dimensions. Let \( (M, g) \) be a compact Riemannian 4-manifold and let \( A \) denotes its Schouten tensor, we denote as usual by \( \sigma_1(A) \) and \( \sigma_2(A) \) respectively the trace of \( A \) and second elementary symmetric function in the eigenvalues of \( A \). We recall two important conformal invariants of \([g]\). The first one is the celebrated Yamabe invariant \( Y[g] \) and is defined by

\[
Y[g] = \inf_{\tilde{g} \in [g]} \frac{1}{(\text{Vol}(\tilde{g}))^{1/2}} \int_M \text{Scal}(\tilde{g}) \mu_{\tilde{g}}.
\]

The second one is \( \int_M \sigma_2(A) \mu_g \), that is the integral over \( M \) of \( \sigma_2(A) \).

The positivity of \( \sigma_1(A) \) and \( \sigma_2(A) \) together imply simultaneously the positivity of the Einstein tensor and the positivity of the Ricci tensor [3]. Consequently they imply \( \text{Ein}([g]) > 2 \) and \( \text{ein}([g]) = -\infty \). The previous simple algebraic property was generalized in [3] and [8] to conformally invariant properties of the conformal class of \( g \), see the main Theorem of [8] and Theorem A in [3]. As a consequence of their results we prove the following

**Theorem D.** Let \( (M, g) \) be a compact 4-dimensional Riemannian manifold with positive Yamabe invariant \( Y[g] \).

(1) If \( \int \sigma_2(A) \mu_g > 0 \) then \( \text{Ein}([g]) > 2 \) and \( \text{ein}([g]) = -\infty \).

(2) If \( \int \sigma_2(A) \mu_g = 0 \) then \( \text{Ein}([g]) \geq 2 \) and \( \text{ein}([g]) = -\infty \).
(3) If $\int \sigma_2(A) \mu_g < 0$ then
\[
\text{Ein}(\{g\}) \geq \frac{4Y[g]}{Y[g] + \sqrt{Y[g]^2 - 96 \int \sigma_2(A) \mu_g}} \quad \text{and} \quad \text{ein}(\{g\}) \leq \frac{4Y[g]}{Y[g] - \sqrt{Y[g]^2 - 96 \int \sigma_2(A) \mu_g}}.
\]

For a Riemannian manifold $(M, g)$ of dimension 4, recall that the Paneitz operator $P_g$ is a fourth order generalization of the usual Laplacian $\Delta$ defined by
\[
P_g(\phi) = \Delta^2 \phi + \frac{2}{3} \delta (\text{Ein}_3(g)) d\phi.
\]
Where $\text{Ein}_3(g) = \text{Scal}_g - 3\text{Ric}$. An important future of $P_g$ is that it is conformally invariant. Precisely, if $\bar{g} = e^{-2u} g$ then
\[
P_{\bar{g}} = e^{4u} P_g.
\]
In particular, the positivity of the Paneitz operator is a conformally invariant property and its kernel is conformally invariant as well. The following theorem is a slightly weaker form of a Theorem due to Eastwood and Singer, see Theorem 5.5 in [5], and also it is due to Gursky and Viacklovsky, see Proposition 6.1 in [8]. It provides a sufficient condition for the non-negativity of Paneitz operator.

**Theorem D’.** If a compact Riemannian 4-manifold $(M, g)$ has the property $\text{Ein}(\{g\}) > 1$ then its Paneitz operator $P_g$ is non negative. Furthermore, the kernel of $P_g$ consists only of constant functions.

Associated with the Paneitz operator is the $Q$-curvature defined by
\[
Q_g = -\frac{1}{12} \Delta \text{Scal}_g + 2\sigma_2(A_g).
\]
One consequence of the previous theorem is

**Corollary D’.** If a compact Riemannian 4-manifold $(M, g)$ has the property $\text{Ein}(\{g\}) > 1$ then it admits a conformal metric with constant $Q$-curvature.

3. $\text{Ein}(\{g\})$ and $\text{ein}(\{g\})$ Constants of a Locally Conformally Flat Class

3.1. **Proof of Theorem A and Corollary A.** We first prove Theorem A as follows

**Proof.** The hypothesis of the theorem implies the existence of a metric $g$ in the conformal class $\{g\}$ such that $\text{Ein}_k(g) > 0$ where $k = \frac{(n-1)(n-2p)}{n-p-1}$ in part (1), and $k = \frac{(n-1)(2p-n)}{p-1}$ in part (2). To prove the theorem we use the descent positivity properties of the modified Einstein tensors and the classical Weitzenböck formula for differential forms
\[
\Delta = \nabla^* \nabla + W,
\]
where $\Delta$ is the Laplacian of differential forms, $\nabla$ is the Levi-Civita connexion and $W$ is the Weitzenböck curvature term. We shall prove that under the theorem hypotheses the Weitzenböck
curvature term is positive and the theorem follows from the previous formula. The curvature term \( W \) once operating on \( p \)-forms takes the following form \[14\]

\[
W_p = \frac{g^{p-2}}{(p-2)!} \left( \frac{g \text{Ric}}{p-1} - 2R \right).
\]

Where \( R \) is the Riemann tensor and all the products are exterior products of double forms. Since the manifold is locally conformally flat then the Weyl tensor vanishes and we have \( R = gA \) where

\[
A = \frac{1}{n-2} \left( \text{Ric} - \frac{\text{Scal}}{2(n-1)} g \right),
\]

is the Schouten tensor. Consequently, we can see that

\[
W_p = \frac{g^{p-2}}{(p-2)!} \left( \frac{g \text{Ric}}{p-1} - 2gA \right) = \frac{g^{p-1}}{(p-1)!} (\text{Ric} - (2p-2)A)
\]

\[
= \frac{g^{p-1}}{(p-1)!} \left( (1 - \frac{2p-2}{n-2}) \text{Ric} + \frac{(p-1)\text{Scal}}{(n-1)(n-2)} g \right)
\]

\[
= \frac{p-1}{(n-1)(n-2)} \frac{g^{p-1}}{(p-1)!} \left( \text{Scal} g - \frac{(n-1)(2p-n)}{p-1} \text{Ric} \right)
\]

\[
= \frac{p-1}{(n-1)(n-2)} \frac{g^{p-1}}{(p-1)!} (\text{Ein}_k)
\]

Note that the last term \( \frac{g^{p-1}}{(p-1)!} (\text{Ein}_k) \) is positive if and only if the sum of the lowest \( p \) eigenvalues of \( \text{Ein}_k \) is positive. In particular, \( \frac{g^{p-1}}{(p-1)!} (\text{Ein}_k) \) is positive if \( \text{Ein}_k \) is positive. This proves the part (2) of the theorem. To prove the first part, it suffices to notice that the curvature term \( W \) once operating on \((n-p)\)-forms takes the form

\[
W_{n-p} = \frac{n-p-1}{(n-1)(n-2)} \frac{g^{n-p-1}}{(n-p-1)!} \left( \text{Scal} g - \frac{(n-1)(n-2p)}{n-p-1} \text{Ric} \right).
\]

Part (2) follows then using Poincaré duality for \( p \neq 1 \). If \( p = 1 \), the condition \( \text{Ein}([g]) > n-1 \) implies the the existence of a metric \( g \) in the conformal class \([g]\) of positive Ricci curvature \[12\]. This completes the proof of Theorem A.

Next, we prove Corollary A.

\[\textbf{Proof.}\] A straightforward computation shows that for the standard product metric \( g \) one has \( \text{Ein}(g) = (n-1) \frac{2d-n+2}{d-n+2} \) and \( \text{ein}(g) = -(n-1) \frac{2d-2}{d-n+2} \). Therefore \( \text{Ein}([g]) \geq (n-1) \frac{2d-n+2}{d-n+2} \) and \( \text{ein}([g]) \leq -(n-1) \frac{2d-2}{d-n+2} \).

Next we use the notations of the theorem, let \( p = n-d-1 \) then \( 2p-n = n-2d-2 > 0 \) and \( k_1 = \frac{(n-1)(2p-n)}{p-1} = (n-1) \frac{2d-n+2}{d-n+2} \), and \( k_2 = \frac{(n-1)(n-2p)}{n-p-1} = -(n-1) \frac{2d-2}{d-n+2} \).

The metric \( g \) being conformally flat then any metric in the conformal class of \( g \) is conformally flat as well. Now since \( b_{(d+1)}(S^{n-d-1} \times M^{d+1}) = b_{(d+1)}(M^{d+1}) \neq 0 \) then by Theorem A no conformally flat metric on \( S^{n-d-1} \times M^{d+1} \) can have \( \text{Ein}([g]) > k_1 \) or \( \text{ein}([g]) < k_2 \). This completes the proof. \[\square\]
3.2. Proof of Theorem A′.

Proof. We prove the capital Ein part of the theorem. The small ein part is completely similar and left to the interested reader.

We proceed as in [15], see also [18] and [13]. We have two possibilities. If \((M, g)\) is locally reducible, then it is either flat or covered by the standard product \(S^k \times H^q\) [15]. Since Ein\((S^k \times H^q)\) = \((n-1)(n-2)\) when \(n = k + q\). With our hypothesis we must have \(k = n - p\) and \(q = p\).

If \((M, g)\) is locally irreducible, we distinguish two cases:

If the holonomy group is not \(SO(n)\) or \(U(n/2)\), then it follows from Berger’s classification of holonomy groups that \((M, g)\) is Einstein, see for instance [1]. Since Ein \(> 0\) then it must be equal to \(n\) which is not our case.

The second case is when the holonomy group is \(SO(n)\) or \(U(n/2)\). The hypothesis Ein\((g)\) = \((n-1)(n-2)\) implies that the Weitzenböck curvature operator of order \(p\) is nonegative, so the Weitzenböck formula shows that \(M\) has a parallel \(p\)-form, hence invariant under the action of the holonomy group. Therefore, at each point of the manifold, we have a \(p\)-form invariant under the action of \(SO(n)\) or \(U(n/2)\). The first possibility can’t occur as \(\wedge^p \mathbb{R}^n\) has no invariant subspaces of dimension 1 under the action of \(SO(n)\). In the second situation, \((M, g)\) must be Kahlerian and conformally flat and consequently flat if \(n > 4\), see 2.68 in [1]. Remains to prove the theorem in the case \(n = 4\). In this case \(p = 1\) and Ein\((g)\) = 3 = \(n - 1\), consequently the Ricci curvature of \((M, g)\) is nonnegative and the theorem follows from the classification of conformally flat manifolds with nonnegative Ricci curvature [18]. \(\square\)

3.3. Nayatani metric and the Ein([g]) invariant: Proof of Theorem B and Corollaries B, B′.

3.3.1. Proof of theorem B.

Proof. It follows from the discussion preceding the statement of Theorem B that \(M = \Omega/\Gamma\) is a Kleinian manifold. We suppose that \(\delta(\Gamma) > 0\). Nayatani [17] constructed a canonical conformally flat metric \(\bar{g} \in [g]\) on \(M = \Omega/\Gamma\) whose Ricci curvature is given by

\[
\text{Ric} = -(n - 2)(\delta + 1)A + (n - 2 - \delta)(\text{tr}_g A)\bar{g},
\]

where \(A\) is a non-negative tensor whose trace \(\text{tr}_g A\) is strictly positive as we supposed the scalar curvature is positive, see [17].

Consequently, one has

\[
\text{Ein}_k(\bar{g}) = \left((n - 1)(n - 2 - 2\delta) - k(n - 2 - \delta)\right)(\text{tr}_g A)\bar{g} + k(n - 2)(\delta + 1)A.
\]

This is clearly positive if \(k < (n - 1)\frac{2\delta - n + 2}{\delta - n + 2}\).

In case \(\delta = 0\), then \((M, g)\) admits a conformal metric \(\bar{g}\) with which \(M\) is locally isometric to \(S^{n-1} \times \mathbb{R}\) [17]. Therefore, in this case one has \(\text{Ein}([g]) \geq \text{Ein}(\bar{g}) = n - 1\). Finally, If \(\delta = -1\), that is \(\pi_1(M)\) is finite, one has \(\text{Ein}([g]) = n\).
The following example shows the optimality of the inequality of Theorem B.

**Example 3.1.** Let $S^d$ be a round $d$-sphere in the $n$-sphere $S^n$ with $1 \leq d \leq n - 3$. Let $\Omega = S^n \setminus S^d$ and $G = \text{Conf}(\Omega) := \{ f \in \text{Conf}(S^n) : f(S^d) = S^d \}$. It turns out that there exists a $G$-invariant conformally flat metric, say $g_0$, on $\Omega$ that makes it isometric to the standard product $S^{n-d-1} \times H^{d+1}$, see [17].

Let now $\Gamma \subset G$ be a Kleinian group with limit set $\Lambda(\Gamma) = S^d$. The manifold $M = \Omega / \Gamma$ is conformally flat and here $\delta = \delta(\Gamma)$ coincides with the Hausdorff dimension $d$ of the limit set $S^d$.

The so obtained metric on $M$ coincides with Nayatani metric up to a positive constant factor. Corollary 2 shows that

$$\text{Ein}(g_0) = \text{Ein}(g_0) = (n - 1) \frac{2d - n + 2}{d - n + 2}.$$ 

### 3.3.2. Proof of Corollary B.

**Proof.** Let $(M, g)$ be compact and locally conformally flat $n$-manifold, and let $p$ be an integer such that $2 \leq p \leq n - 1$. Denote by $d$ the Schoen-Yau conformal invariant of $(M, [g])$.

We distinguish two cases. If $n \geq 2p$, we have $\text{Ein}[g] > 0$ and therefore $[g]$ contains a locally conformally flat metric of positive scalar curvature. The corollary follows then from Schoen-Yau theorem, namely Theorem 4.6(ii) in [20]. In the case where $n < 2p$, one has $\text{Ein}[g] > (n - 1)(2p - n)$ and this implies that

$$(n - 2) \frac{n - 1 - \text{Ein}(g)}{2(n - 1) - \text{Ein}(g)} < \min\{n - p - 1, \frac{(n - 2)^2}{n}\}.$$ 

To prove the last inequality, let $k = \min\{n - p - 1, \frac{(n - 2)^2}{n}\}$. The inequality $(n - 2) \frac{n - 1 - \text{Ein}(g)}{2(n - 1) - \text{Ein}(g)} < k$ is equivalent to $\text{Ein}[g] > (n - 1) \frac{n - 2k}{n - k - 2}$. Letting $k = n - p - 1$, the inequality reads $\text{Ein}[g] > (n - 1) \frac{2p - n}{p - 1}$. For $k = \frac{(n - 2)^2}{n}$, the inequality reads $\text{Ein}[g] > (n - 1)n - 4$. The corollary follows then directly from our Theorem B and Theorem 4.6(i) in [20]. In fact, the later asserts that

$$d < \min\{n - p - 1, \frac{(n - 2)^2}{n}\} \implies \pi_2(M) = \ldots = \pi_p(M) = 0.$$ 

This completes the proof of the corollary. \[\square\]

Analogous results for the small ein invariant holds. We prove for instance the following one

**Proposition 3.1.** Let $(M, g)$ be a compact connected locally conformally flat Riemannian manifold of positive scalar curvature of dimension $n \geq 4$. Then

$$\text{ein}([g]) \leq \frac{2d - n + 2}{d}.$$ 


Proof. We proceed as in the proof of Theorem B using Nayatani metric \( \bar{g} \). Recall that
\[
\text{Ein}_k(\bar{g}) = ((n-1)(n-2-2\delta) - k(n-2-\delta)) (\text{tr}_{\bar{g}} \mathcal{A}) \bar{g} + k(n-2)(\delta + 1) \mathcal{A}.
\]
The tensor \( \mathcal{A} \) is symmetric and non negative, denote by \( \mathcal{A}_{\text{max}} \) its maximum eigenvalue. This is a positive function as the trace of \( \mathcal{A} \) is positive. Then it is easy to see that for \( k < 0 \), the tensor \( \text{Ein}_k(\bar{g}) \) is positive if
\[
((n-1)(n-2-2\delta) - k(n-2-\delta)) \mathcal{A}_{\text{max}} + k(n-2)(\delta + 1) \mathcal{A}_{\text{max}} > 0.
\]
Consequently, \( \text{Ein}_k(\bar{g}) \) for all \( k < 0 \) satisfying \( k > \frac{2n-n+2}{2\delta} \). This completes the proof of the proposition.

We remark that the inequality in Proposition B is not optimal as one can check it for the case of the standard product \( S^{n-d-1} \times H^{d+1} \). The author believes that one can improve this inequality to optimality using a deeper look at the tensor \( \mathcal{A} \).

3.3.3. Proof of Corollary B’.

Proof. First we remark that
\[
\text{Ein}([g]) > (n-1) \frac{n-4}{n-3} \iff (n-2) \frac{n-1 - \text{Ein}([g])}{2(n-1) - \text{Ein}([g])} < 1.
\]
Consequently, under our assumption, Theorem B implies that \( d(M, [g]) < 1 \). The corollary follows then directly from Theorem 6.1 of Izeki [10].

3.4. Connected sums of conformally flat manifolds of positive \( \text{Ein}_k \) tensor: Proof of Theorem C and Corollary C.

3.4.1. Proof of Theorem C.

Proof. This theorem follows from a general theorem due to Hoelzel, see Theorem 6.1 in [9]. We shall use the same notations as in [9]. Let \( C_B(\mathbb{R}^n) \) denote the vector space of algebraic curvature operators \( \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n \) satisfying the first Bianchi identity and endowed with the canonical inner product. Let
\[
C_{\text{Ein}_k > 0} := \{ R \in C_B(\mathbb{R}^n) : \text{Ein}_k(R) > 0 \},
\]
where for a unit vector \( u \), \( \text{Ein}_k(R)(u) = \text{Scal}(R)u - k\text{Ric}(u) \). Here \( \text{Ric} \) and \( \text{Scal}(R) \) denote respectively as usual the first Ricci contraction and the full contraction of \( R \). The subset \( C_{\text{Ein}_k > 0} \) is clearly open, convex, star shaped with respect to the origin and it is an \( O(n) \)-invariant cone. Furthermore, it is easy to check that \( \text{Ein}_k(S^{n-1} \times \mathbb{R}) > 0 \) for \( n-1 > k \). The theorem follows then from Theorem 6.1 in [9].
3.4.2. Proof of Corollary C.

Proof. Recall that the standard metric on \( S^1 \times S^{n-1} \) is locally conformally flat and has its Ein and ein respectively equal to \( n-1 \) and \( -\infty \). The corollary follows then from Theorem C. \( \square \)

4. Ein([g]) and ein([g]) invariants for general conformal classes in four dimensions: Proof of Theorems D and D’

4.1. Proof of Theorem D.

Proof. The first part is a direct consequence of Corollary B to Theorem A in \([3]\). In fact, the condition \( \frac{1}{2} \text{Scal}_g - \text{Ric} > 0 \) is equivalent to \( \text{Ein}_2(g) > 0 \) and therefore implies \( \text{Ein}(g) > 2 \). The remaining two parts are a consequence of the main Theorem in \([8]\) which asserts that under the assumption

\[
4 \int_M \sigma_2(A) \mu_g + \frac{\alpha(\alpha + 1)}{6} (Y[g])^2 > 0,
\]

for an arbitrary positive constant \( \alpha \), the existence of a conformal metric \( \bar{g} \in [g] \) with

\[
(22) \quad \text{Ein}_{\frac{2}{\alpha}}(\bar{g}) > 0 \quad \text{and} \quad \text{Ein}_{\frac{2}{\alpha + 1}}(\bar{g}) > 0.
\]

So if \( \int_M \sigma_2(A) = 0 \) and since \( \alpha > 0 \) is arbitrary we immediately get the conclusions \( \text{Ein}([g]) \geq 2 \) and \( \text{ein}([g]) = -\infty \). Finally, if \( \int_M \sigma_2(A) < 0 \) denote by \( c \) the following positive constant

\[
c = \frac{-24 \int_M \sigma_2(A) \mu_g}{(Y[g])^2}.
\]

The aforementioned theorem guarantees under the condition \( \alpha > \frac{1}{2} \left( \sqrt{4c + 1} - 1 \right) \) the existence of a conformal metric \( \bar{g} \in [g] \) that has the property \([22]\) that is \( \text{ein}([g]) < -\frac{2}{\alpha} \) and \( \text{Ein}([g]) > \frac{2}{\alpha + 1} \). In other words,

\[
\alpha > \frac{1}{2} \left( \sqrt{4c + 1} - 1 \right) \implies \alpha > \frac{-2}{\text{ein}([g])}
\]

and

\[
\alpha > \frac{1}{2} \left( \sqrt{4c + 1} - 1 \right) \implies \alpha > \frac{2}{\text{Ein}([g])} - 1.
\]

From which we conclude that

\[
\text{ein}([g]) \leq \frac{-4}{\sqrt{4c + 1} - 1} \quad \text{and} \quad \text{Ein}([g]) \geq \frac{4}{\sqrt{4c + 1} + 1}.
\]

This completes the proof. \( \square \)

4.2. The positivity of Paneitz operator and the condition \( \text{Ein}(g) > 1 \).
4.2.1. Proof of Theorem D'.

Proof. Note that the condition \( \text{Ein}([g]) > 1 \) implies by definition the existence of a metric \( g_1 \) in the conformal class \([g]\) such that \( \text{Ein}_k(g_1) > 0 \) for some \( k > 1 \) and the theorem follows from Theorem 5.5 in \cite{5}. Also, since \( \text{Ein}_k(g_1) > 0 \implies \text{Ein}_1(g_1) > 0 \) the theorem follows from Proposition 6.1 in \cite{8}. For the sake of completeness we provide a proof of this theorem

Let \( \lambda \) be an eigenvalue of \( P_g \) and \( u \) a corresponding eigenfunction. Using the fact that the Laplacian operator is self-adjoint we get

\[
\int < \lambda u, u > \mu_g = \int < Pu, u > \mu_g = \lambda \int u^2 \mu_g \\
= \int \left\{ < \Delta^2 u, u > + \left( \frac{2}{3} \text{Scal} \ g - 2 \text{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g \\
= \int \left\{ (\Delta u)^2 + \left( \frac{2}{3} \text{Scal} \ g - 2 \text{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g \\
= \int \left\{ \frac{-1}{3} (\Delta u)^2 + \frac{4}{3} (\Delta u)^2 + \left( \frac{2}{3} \text{Scal} \ g - 2 \text{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g.
\]

(23)

The Bochner formula shows that

\[
\frac{4}{3} \int (\Delta u)^2 \mu_g = \frac{4}{3} \int \left\{ |\text{Hess} \ u|^2 + \text{Ric}(\nabla u, \nabla u) \right\} \mu_g.
\]

(24)

Hence, we deduce that

\[
\lambda \int u^2 \mu_g = \int \left\{ \frac{-1}{3} (\Delta u)^2 + \frac{4}{3} |\text{Hess} \ u|^2 + \left( \frac{2}{3} \text{Scal} \ g - \frac{2}{3} \text{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g \\
\geq \frac{2}{3} \int \left\{ \text{Ein}_1(\nabla u, \nabla u) \right\} \mu_g.
\]

(25)

Where in the last step we used Newton-Maclaurin’s identity as follows

\[
(\sigma_1(\text{Hess} \ u))^2 \geq \frac{8}{3} \sigma_2(\text{Hess} \ u) = \frac{4}{3} \left( (\sigma_1(\text{Hess} \ u))^2 - |\text{Hess} \ u|^2 \right).
\]

The first part of the result follows directly. If \( \lambda = 0 \) then clearly \( \nabla u = 0 \) and then the function \( u \) is constant. \( \square \)

4.2.2. Proof of Corollary D'.

Proof. First note that the previous proposition guarantees that the kernel of \( P_g \) consists of constant functions.

Next, since the condition \( \text{Ein}([g]) > 1 \) implies the positivity of the Yamabe constant of the conformal class \([g]\), Theorem B of Gursky \cite{7} shows that \( \int Q_g \mu_g \leq 8\pi^2 \) with equality if and only if the manifold is conformally equivalent to the sphere. In the case \( \int Q_g \mu_g < 8\pi^2 \), a theorem of Djadli-Malchiodi \cite{4} guarantees the existence of a metric with constant \( Q \)-curvature. \( \square \)
Remarks. (1) M. Lai proved in [16] an equivalent version the above theorem and corollary under the condition of 3-positive Ricci curvature. It is easy to show that in four dimensions, a metric $g$ has 3-positive Ricci curvature if and only if $\text{Ein}(g) > 0$.

(2) For higher dimensions $n \geq 4$, let $k = \frac{2n(n-1)}{3n-4}$. It is proved in [12] that $\text{Ein}_k > 0 \implies \Gamma_2(A) > 0$, that is positive scalar curvature and positive $\sigma_2$ curvature, in particular it implies the positivity of the integral of the $Q$-curvature.

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