A Zero-Sum Deterministic Impulse Controls Game in Infinite Horizon with a New HJBI-QVI

Brahim El Asri · Hafid Lalioui · Sehail Mazid

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Abstract
In the present paper, we study a two-player, zero-sum, deterministic differential game with both players adopting impulse controls in infinite-time horizon, under rather weak assumptions on the cost functions. We prove by means of the dynamic programming principle that the lower and upper value functions are continuous and viscosity solutions to the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) quasi-variational inequality (QVI). We define a new HJBI-QVI for which, under a proportional property assumption on the maximizing player cost, the value functions are the unique viscosity solution. We then prove that the lower and upper value functions coincide.

Keywords Deterministic differential game · Impulse control · Infinite horizon · Dynamic programming principle · Viscosity solution · Quasi-variational inequality

Mathematics Subject Classification 49K35 · 49L25 · 49N70 · 90C39 · 93C20

1 Introduction
Differential games are concerned with the problem that multiple players make decisions, according to their own advantages and trade-off with other peers, in the context of dynamic systems. The theory of two-player (or, two-person), zero-sum, differential
games was initiated by Isaacs [26] and Pontryagin et al. [9] at the beginning of 60’s, in the early 80’s the theory of viscosity solutions was pioneered by the seminal papers of Crandall and Lions [12] and Crandall et al. [13, 14]. The notion of strategies and the rigorous definitions of lower and upper value functions are due to Elliott and Kalton [23, 24], Evans and Souganidis [25] began to study differential games by means of the viscosity theory, proving that the two value functions are the unique viscosity solution to the corresponding Hamilton–Jacobi–Bellman–Isaacs (HJBI) partial differential equations (PDEs) for finite-time horizon problem, Bardi and Capuzzo-Dolcetta [3] described the implementation of the viscosity solutions approach to a number of significant model problems in optimal deterministic control and differential games. The deterministic differential games and impulse control problems, apart from the mathematical interest in its own right, enjoy a wide range of applications in various fields of engineering, such as medicine, biology, economics and finance, see for more information Bensoussan and Lions [6] and [26]. The deterministic impulse control problems in finite-time horizon were studied by many authors, Yong [35] considered impulse control game problems where one player takes continuous controls whereas the other uses impulse control, El Farouq et al. [21] treated a mini-max problem driven by two controls, one is continuous and another impulsive. For the infinite-time horizon case, as considered in the present paper, we cite the works by Barles [4], Dharmatti and Shaiju [15, 17], and Dharmatti and Ramaswamy [16] (see also [28–30]).

In this paper, we consider the system’s state $y_x(\cdot)$ of a two-player, zero-sum, deterministic differential game involving two impulse controls, one for each player, in infinite-time horizon described by the solution of the following dynamical equation:

$$y_x(t) := x + \int_0^t b(y_x(s))ds + \sum_{m \geq 1} \xi_m \mathbb{I}_{\{\tau_m, +\infty\}}(t) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}$$

$$+ \sum_{k \geq 1} \eta_k \mathbb{I}_{\{\rho_k, +\infty\}}(t),$$

(1)

where $m, k \in \mathbb{N}^*$ and $y_x(t)$ denotes the state variable of the system at time $t$, $\mathbb{R}^n$—valued, with initial state $y_x(0) =: x$. The state $y_x(\cdot)$ is driven by two impulse controls, $u$ control of player-$\xi$ defined by a double sequence $u := (\tau_m, \xi_m)_{m \geq 1}$ and $v$ control of player-$\eta$ defined by a double sequence $v := (\rho_k, \eta_k)_{k \geq 1}$, the actions $\xi_m$ and $\eta_k$ take values in two convex cones $U$ and $V$ subsets of $\mathbb{R}^n$, respectively. The infinite product $\prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}$ signifies that when the two players act together on the system at the same time, we take into account only the action of player-$\eta$. The gain (resp. cost) functional $J$ for maximizing player-$\xi$ (resp. minimizing player-$\eta$) is defined by

$$J(x; u, v) := \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t)dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}$$

$$+ \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k),$$

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where $c$ and $\chi$ are the impulse cost functions for player-$\xi$ and player-$\eta$, respectively, the function $f$ represents the running gain and the discount factor $\lambda$ is a fixed positive real, these three functions are $\mathbb{R}$-valued. We note that the cost of a player is the gain for the other (zero-sum), meaning that when a player performs an action he/she has to pay a positive cost, resulting in a gain for the other player.

Regarding the existing literature, our paper considers under more general assumptions a new class of two-player, zero-sum, deterministic differential games where each players adopts impulse control with impulses against strategies. Such a game, which involves two impulsive controls on a trajectory, has applications in various fields such as mathematical finance, cybersecurity, marketing and operations research (see e.g [3, 5–8, 10, 32]). Our closest related works are the stochastic games involving impulse controls considered in Azimzadeh [1], Cosso [11] and El Asri and Mazid [19]. Authors in [11, 19] studied a finite-time horizon stochastic impulse controls problem based on the dynamic programming principle (DPP) approach and viscosity solutions theory, they proved that the differential game related to the considered problem admits a value, however in both works the authors impose a stronger constraint that involves both cost functions, which is given by

$$\exists h : [0, T] \rightarrow (0, +\infty) \text{ such that } c(t, \xi_1 + \eta + \xi_2) \leq c(t, \xi_1) - \chi(t, \eta) + c(t, \xi_2) - h(t),$$

where $c$ and $\chi$ are the impulse cost functions from $U$ and $V$, respectively, $\xi_1, \xi_2 \in U$ and $\eta \in V$, as a consequence they had to require $V \subset U \subset \mathbb{R}^n$. The terminology of a quasi-variational inequality (QVI) was introduced in [6] to deal with impulse control problems. The definition of lower and upper value functions for a differential game as defined in [23–25] leads to the characterization of the values of the game as the unique viscosity solution to a corresponding QVI. Moreover the relationship between the theory of two-player, zero-sum, deterministic differential games and viscosity solutions was first shown in [25], Barron et al. [2] and Souganidis [33, 34].

Our aim in this work is to investigate the two-player, zero-sum, deterministic impulse controls differential game problem in infinite-time horizon given by the dynamical equation (1) and related to the gain/cost functional $J$. In particular, we describe the problem by a classic HJBI-QVI, which we replace by a new HJBI-QVI in order to characterize, in viscosity solution sense under rather weak assumptions on the cost functions and by means of the DPP, the value function of the differential game studied as the unique viscosity solution to the defined new HJBI-QVI. In this work we only adopt, in addition to the classical assumptions of the impulse control problems, a proportional property assumption on the maximizing player’s cost function $c$ which is given by

$$\forall k > 0, \forall \xi \in U \text{ we have } c(k\xi) \leq kc(\xi),$$

with $c(0) = 0$, note that assumption (3) is of great interest in the literature, as an application we cite the work developed in recent years in the field of biology, see Mailleret and Grognard [31]. For our differential game, the associated QVI is given
by the following double-obstacle HJBI equation:

\[
\max \left\{ \min \left[ \lambda v(x) - Dv(x).b(x) - f(x), v(x) - \mathcal{H}_{sup}^c v(x) \right] \right\} = 0,
\]

where the Hamiltonian involves only the first order partial derivatives, \( Dv(.) \) denotes the gradient of the function \( v : \mathbb{R}^n \rightarrow \mathbb{R} \), and the first (resp. second) obstacle is defined through the use of the minimum (resp. maximum) cost operator \( \mathcal{H}_{inf}^X \) (resp. \( \mathcal{H}_{sup}^c \)), where

\[
\mathcal{H}_{inf}^X v(x) := \inf_{\eta \in V} [v(x + \eta) + \chi(\eta)] \quad \text{(resp. \( \mathcal{H}_{sup}^c v(x) := \sup_{\xi \in U} [v(x + \xi) - c(\xi)] \)).
\]

Using the DPP, we prove the existence of the value functions for our differential game as viscosity solutions to the HJBI-QVI (4), but the uniqueness of the viscosity solution for this QVI is not guarantee under standing assumptions on the players’ costs, which means that the value function cannot enjoy anymore the property being the unique viscosity solution to the classic HJBI-QVI (4). Furthermore, we define a new HJBI-QVI, where the term of impulsion \( v(.) - \mathcal{H}_{sup}^c v(.) \) is replaced by the differential term \( \mathcal{F}_{inf}^c(Dv(.)) \) defined, for all \( x \in \mathbb{R}^n \), by means of the operator \( \mathcal{F}_{inf}^c \) as follows:

\[
\mathcal{F}_{inf}^c(Dv(x)) := \inf_{\xi \in U : \|\xi\|=1} \left[ -Dv(x).\xi + c(\xi) \right].
\]

Therefore, under assumption (3) and classical assumptions of the impulse control problems, we show the existence and the uniqueness results in the viscosity solution sense for the defined new HJBI-QVI. Indeed, for the existence result, we give an equivalence in the viscosity super-solution sense between the classic HJBI-QVI (4) and the new HJBI-QVI, then, for the uniqueness, we establish a comparison theorem.

It should be pointed out, as explained in the above, that we are concerned with a two impulse controls differential game in a general setting framework, thing that extends earlier works existing in the literature involving one impulse control only (see e.g. [4, 20, 21, 35, 36]). We mention that as alternative to the classic HJBI-QVI, which may require additional assumption similar to (2) on both players’ costs to obtain comparison and uniqueness results \( (\exists \alpha \in (0, +\infty) : c(\xi_1 + \eta + \xi_2) \leq c(\xi_1) - \chi(\eta) + c(\xi_2) - \alpha) \), our paper suggests a new version of such a QVI. Our contribution consists then in relating solutions of the two QVIs using the DPP and the viscosity solution approaches. We also mention that we can use the same approach of changing the original QVI of the game problem into the equivalent QVI with the differential term when dealing with games involving hybrid controls (continuous, switching and impulse) [15–17], stochastic differential games with impulse and switching strategies, and the HJBI equation arising from stochastic games with impulses. Indeed, the non-local cost operator \( \mathcal{H}_{sup}^c \) will remain unchanged. Hence, we extend the range of potential applications of our study.

The remainder of the paper is organized as follows: in Sect. 2, we present the impulse controls differential game studied, we give its related definitions and assumptions and
we introduce our new HJBI-QVI. In Sect. 3, we prove some classic results on the lower and upper value functions, we first show that both satisfy the DPP property, then we prove that they are continuous in $\mathbb{R}^n$. Section 4 is devoted, on one hand, to the viscosity characterization of the classic HJBI-QVI (4) by deducing that both value functions are its viscosity solutions, and, on the other hand, to prove that the new HJBI-QVI has the same bounded continuous viscosity super-solutions as the classic HJBI-QVI (4), then to deduce the viscosity characterization of the new HJBI-QVI. Further, in Sect. 5, we look more carefully to the new HJBI-QVI by proving that the value functions of our infinite-time horizon two-player, zero-sum, deterministic, impulse controls differential game are his unique viscosity solution. Finally, we deduce that the lower and upper value functions coincide, thus the game admits a value.

2 Assumptions and Setting of the Game Problem

2.1 Assumptions

Throughout this paper, we let $n$, the dimension of the state, be a fixed positive integer, the time variable $T$ be in $[0, +\infty]$, $k, m \in \mathbb{N}^*$ and we let the discount factor $\lambda$ be a fixed positive real.

Let us assume $H1$:

$[H_{b,f}]$ The functions $b : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ are bounded and Lipschitz continuous with constant $C_b$ and $C_f$ respectively.

$[H_{c,\chi}]$ The cost functions $c : U \to \mathbb{R}^+$ and $\chi : V \to \mathbb{R}^+$ are from two convex cones of $\mathbb{R}^n$, $U$ and $V$, respectively, into $\mathbb{R}^+$, non negative, continuous and satisfy the following:

$$c(0) = 0, c(\xi) \neq 0 \text{ for } \xi \neq 0 \text{ and } \inf_{\eta \in V} \chi(\eta) > 0.$$ 

Also for all $\xi_1, \xi_2 \in U$ and $\eta_1, \eta_2 \in V$, we let the impulse cost functions be such that

$$\begin{cases} c(\xi_1 + \xi_2) \leq c(\xi_1) + c(\xi_2); \\ \chi(\eta_1 + \eta_2) < \chi(\eta_1) + \chi(\eta_2). \end{cases}$$

Moreover, we assume $H2$ that encompass the proportional impulse costs for maximizing player-$\xi$, that is the function $c$ satisfies the inequality

$$\forall k > 0, \forall \xi \in U \text{ we have } c(k\xi) \leq kc(\xi).$$

Remark 1 Regarding assumption $H1$, the assumption on function $b$ implies that there exists a unique global solution $y_\lambda(.)$ to the above dynamical equation (1), while the assumption on $f$ and $[H_{c,\chi}]$ provide the classical framework for the study of the impulse control problems. Assumption $H2$, which is of great interest in the literature,
leads to the existence and the uniqueness of the viscosity solutions for the new HJBI-QVI defined hereafter in equation (7).

For the rest of the paper we denote by $|.|$ and $\|.|$ the Euclidean vector norm in $\mathbb{R}$ and $\mathbb{R}^n$, respectively, and for a bounded and continuous function $F$ from $\mathbb{R}^n$ to $\mathbb{R}$ (resp. $\mathbb{R}^n$) we define

$$\|F\|_\infty := \sup_{x \in \mathbb{R}^n} |F(x)| \quad \text{(resp.} \|F\|_\infty := \sup_{x \in \mathbb{R}^n} \|F(x)\|).$$

\[\square\]

### 2.2 Impulse Controls Game Problem

Here we shall be interested in the two-player, zero-sum, deterministic, impulse controls differential game described in the introduction set by the dynamical equation (1) and related to the gain/cost functional $J$. The time horizon (the interval in which time varies) is infinite. The state of the system $y_x(t)$ at the instant $t$ lies in $\mathbb{R}^n$, with initial value $y_x(0) =: x$. The mapping $t \to y_x(t)$ describes the evolution of the system’s state provided, for any $t \in [0, +\infty[$, by the deterministic model

$$y_x(t) := x + \int_0^t b(y_x(s)) ds + \sum_{m \geq 1} \xi_m \mathbb{I}_{[\tau_m, +\infty)}(t) \prod_{k \geq 1} \mathbb{I}_{[\tau_m \neq \rho_k]}(t) + \sum_{k \geq 1} \eta_k \mathbb{I}_{[\rho_k, +\infty)}(t).$$

The impulse-time sequences $\{\tau_m\}_{m \geq 1}$ and $\{\rho_k\}_{k \geq 1}$ are two non-decreasing sequences of $[0, +\infty]$ which satisfy $\tau_m, \rho_k \to +\infty$ when $m, k \to +\infty$, and the impulse-value sequences $\{\xi_m\}_{m \geq 1}$ and $\{\eta_k\}_{k \geq 1}$ are two sequences of elements of $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, respectively. The state $y_x(.)$ of the system is driven by two impulse controls, $(\tau_m, \xi_m)_{m \geq 1}$ control of player-$\xi$ and $(\rho_k, \eta_k)_{k \geq 1}$ control of player-$\eta$. The infinite product $\prod_{k \geq 1} \mathbb{I}_{[\tau_m \neq \rho_k]}$ signifies that when the two players act together on the system at the same time, only the action of minimizing player-$\eta$ is taking into account. We call $\mathcal{U}$ (resp. $\mathcal{V}$) the space of all impulse controls $u$ (resp. $v$) for maximizing player-$\xi$ (resp. minimizing player-$\eta$) and we denote $u := (\tau_m, \xi_m)_{m \geq 1}$ (resp. $v := (\rho_k, \eta_k)_{k \geq 1}$).

For any initial state $x$, the impulse controls $u$ and $v$ generate a trajectory $y_x(.)$ solution of the equation (1). We are then given a gain (resp. cost) functional $J(x; u, v)$ for player-$\xi$ (resp. player-$\eta$), which represents the criterion to maximize (resp. minimize) by applying the control $u$ (resp. $v$), by

$$J(x; u, v) := \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} \mathbb{I}_{[\tau_m \neq \rho_k]}$$

$$+ \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k),$$

where $c$ and $\chi$ are the impulse cost functions (jump costs) from $U$ and $V$, respectively, $\mathbb{R}^+\text{-valued and satisfy the assumption } [H_{c, \chi}]$, the running gain $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the assumption $[H_{b, f}]$ and the fixed positive real $\lambda > 0$, represents the discount factor.

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Typically, in a two-player game, the player who moves first would not choose a fixed action. Instead, he/she prefers to employ a strategy which can give different responses to different future actions the other player will take. Hence, besides the admissible controls, we follow [23, 24] to define the notion of non-anticipative strategies for our game problem as follows:

**Definition 1 (Non-Anticipative Strategy)** The non-anticipative strategy set \( \mathcal{A} \) for player-\( \xi \) is the collection of all non-anticipative maps \( \alpha \) from \( V \) to \( U \), i.e., for any \( T > 0 \) and for any \( v_1 \) and \( v_2 \) in \( V \), if \( v_1 \equiv v_2 \) on \( [0, T] \), then \( \alpha (v_1) \equiv \alpha (v_2) \) on \( [0, T] \).

Similarly, the non-anticipative strategy set \( \mathcal{B} \) for player-\( \eta \) is the collection of all non-anticipative maps \( \beta \) from \( U \) to \( V \), i.e., for any \( T > 0 \) and for any \( u_1 \) and \( u_2 \) in \( U \), if \( u_1 \equiv u_2 \) on \( [0, T] \), then \( \beta (u_1) \equiv \beta (u_2) \) on \( [0, T] \).

In the game, player-\( \xi \) aims to maximize the gain functional \( J \) and contrarily player-\( \eta \) aims to minimize his cost, also given by \( J \). We may now give the definition of the lower and upper value functions for our differential game.

**Definition 2 (Value Functions)** We define the lower value function \( V^-(\cdot) \) and the upper value function \( V^+ (\cdot) \) of the two-player, zero-sum, deterministic, impulse controls differential game by

\[
\begin{align*}
V^-(x) &:= \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} J(x; u, \beta(u)); \\
V^+(x) &:= \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} J(x; \alpha(v), v).
\end{align*}
\]

If \( V^-(x) = V^+(x) \) we say that the game, with initial state \( x \) at time \( t = 0 \), admits a value and we denote by \( V(x) := V^-(x) = V^+(x) \) the value function of the game.

### 2.3 New HJBI Quasi-Variational Inequality

For the impulse controls differential game studied in the present paper, the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) quasi-variational inequality (QVI) turns out to be the same for the two value functions, because of the two players can not act simultaneously on the system, and it is given by the double-obstacle equation

\[
\max \left\{ \min \left[ \lambda v(x) - Dv(x) . b(x) - f(x), v(x) - \mathcal{H}^c_{\text{sup}} v(x) \right] , v(x) - \mathcal{H}^\chi_{\text{inf}} v(x) \right\} = 0 ,
\]

where \( \mathcal{H}^\chi_{\text{inf}} \) and \( \mathcal{H}^c_{\text{sup}} \) are, respectively, the non-local minimum and maximum cost operators defined as follows:

\[
\begin{align*}
\mathcal{H}^\chi_{\text{inf}} v(x) &:= \inf_{\eta \in \mathcal{V}} \left[ v(x + \eta) + \chi(\eta) \right] ; \\
\mathcal{H}^c_{\text{sup}} v(x) &:= \sup_{\xi \in \mathcal{U}} \left[ v(x + \xi) - c(\xi) \right].
\end{align*}
\]
and $Dv(.)$ denotes the gradient of the function $v : \mathbb{R}^n \to \mathbb{R}$.

This paper defines the new HJBI-QVI (7), where the term of impulsions $v(.) - H_{sup}^c v(.)$ is replaced by the differential term $F_{inf}^c(Dv(.))$ through the use of the operator $F_{inf}^c$, as follows:

$$\max \left\{ \min \left[ \lambda v(x) - Dv(x).b(x) - f(x), F_{inf}^c(Dv(x)) \right] \right\} = 0,$$

(7)

where the operator $F_{inf}^c$ is defined by

$$F_{inf}^c(Dv(x)) := \inf_{\xi \in U, \parallel \xi \parallel = 1} \left[ -Dv(x).\xi + c(\xi) \right].$$

(8)

Note that a differential term as in (8) was introduced in Barles [5] and used, to deal with the particular case of null infimum jump costs in the infinite-time horizon impulse control problem, in El Farouq [22].

**Remark 2** The main objectives of this paper are:

(i) To focus on the existence of the solution in viscosity sense for both quasi-variational inequalities (6) and (7);

(ii) To show that the new HJBI-QVI (7) admits the lower and upper value functions as the unique solution of viscosity.

For the rest of the paper we call QVI (6) the classic HJBI-QVI, we call QVI (7) the new HJBI-QVI and we adopt the following definition of the viscosity solution:

**Definition 3** (*Viscosity Solution*) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. $V$ is called:

(i) A viscosity sub-solution of the classic HJBI-QVI (resp. new HJBI-QVI) if for any $\bar{x} \in \mathbb{R}^n$ and any function $\phi \in C^1(\mathbb{R}^n)$ such that $V(\bar{x}) = \phi(\bar{x})$ and $\bar{x}$ is a local maximum point of $V - \phi$, we have

$$\max \left\{ \min \left[ \lambda V(\bar{x}) - D\phi(\bar{x}).b(\bar{x}) - f(\bar{x}), V(\bar{x}) - H_{inf}^c V(\bar{x}) \right] \right\} \leq 0,$$

resp. $\max \left\{ \min \left[ \lambda V(\bar{x}) - D\phi(\bar{x}).b(\bar{x}) - f(\bar{x}), F_{inf}^c(D\phi(\bar{x})) \right] \right\} \leq 0$;

(ii) A viscosity super-solution of the classic HJBI-QVI (resp. new HJBI-QVI) if for any $\bar{x} \in \mathbb{R}^n$ and any function $\phi \in C^1(\mathbb{R}^n)$ such that $V(\bar{x}) = \phi(\bar{x})$ and $\bar{x}$ is a local minimum point of $V - \phi$, we have

$$\max \left\{ \min \left[ \lambda V(x) - D\phi(x).b(x) - f(x), V(x) - H_{sup}^c V(x) \right] \right\} \leq 0,$$
\[
V(x) - \mathcal{H}^X_{inf}V(x) \geq 0
\]
(resp. \[
\max \left\{ \min \left[ \lambda V(x) - D\phi(x) \cdot b(x) - f(x), \mathcal{F}^c_{inf}(D\phi(x)) \right] \right\},
\]
\[
V(x) - \mathcal{H}^X_{inf}V(x) \geq 0
\];

(iii) A viscosity solution of the classic HJBI-QVI (resp. new HJBI-QVI) if it is both a viscosity sub-solution and super-solution of the classic HJBI-QVI (resp. new HJBI-QVI).

\[\square\]

2.4 Preliminary Results

Letting \( y_x(.) \) and \( y_{x'}(.) \) be two trajectories generated by \( u \in U \) and \( v := \beta(u) \in V \) from \( x \) and \( x' \), respectively, where \( \beta \in B \). We then have the following characterization of the trajectories, for which the proof follows from Gronwall’s Lemma and can be found in Lions [27]:

**Lemma 1** Under assumption \( H1 \), for any \( x, x' \in \mathbb{R}^n \) and any \( t \geq 0 \), we have

\[
\| y_x(t) - y_{x'}(t) \| \leq \exp(C_b t)\| x - x' \|.
\]

\[\square\]

We show hereafter that the lower and upper value functions are bounded in \( \mathbb{R}^n \).

**Proposition 1** Under assumption \( H1 \), the lower and upper value functions are bounded in \( \mathbb{R}^n \).

**Proof** We make the proof only for the lower value function \( V^- \), the other case being analogous. By the definition of \( V^- \), for all \( x \in \mathbb{R}^n \) and all non-anticipative strategy \( \beta \in B \) we have

\[
V^-(x) \leq \sup_{u \in U} J(x; u, \beta(u)).
\]

Then, considering the set of non-anticipative strategies \( \beta(u) := (\rho_k, \eta_k)_{k \geq 1} \) for player-\( \eta \) where there is no impulse-time, i.e., \( \rho_1 = +\infty \), we get

\[
V^-(x) \leq \sup_{u \in U} \left[ \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \right].
\]

Next, for all \( \varepsilon > 0 \), there exists a strategy \( u^\varepsilon := (\tau_{m}^{\varepsilon}, \xi_{m}^{\varepsilon}) \in U \) such that

\[
V^-(x) \leq \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_{m}^{\varepsilon}) \exp(-\lambda \tau_{m}^{\varepsilon}) + \varepsilon.
\]
Since $c$ is a non negative function and $f$ is bounded, we then get the existence of a constant $C > 0$ such that

$$V^-(x) \leq C.$$ 

Similarly, for the set of strategies $u \in \mathcal{U}$ for player-$\xi$ for which there is no impulse-time, i.e., $\tau_1 = +\infty$, we have

$$V^-(x) \geq \inf_{\beta \in \mathcal{B}} \left[ \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t) dt + \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k) \right].$$ 

Let $\varepsilon > 0$, then there exists a strategy $\beta^\varepsilon(u) := (\rho_k^\varepsilon, \eta_k^\varepsilon) \in \mathcal{V}$ where $\beta^\varepsilon \in \mathcal{B}$, for which we have

$$V^-(x) + \varepsilon \geq \int_0^{+\infty} f(y_x(t)) \exp(-\lambda t) dt + \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon).$$ 

Since $\chi$ is a non negative function and $f$ is bounded, we deduce the existence of a constant $C_1 > 0$ such that

$$V^-(x) \geq -C_1.$$ 

Hence we obtain the thesis.

\[ \square \]

### 3 Dynamic Programming and Continuity of the Value Functions

We now present, for the infinite-time horizon, two-player, zero-sum, deterministic, impulse controls differential game studied, the dynamic programming principle (DPP) in Theorem 1 bellow. The DPP, as one of the principle and most commonly used approaches in solving optimal control problems, meaning that an optimal control viewed from today will remain optimal when viewed from tomorrow and stands for a basic property in dealing with our problem. We begin with the following Lemma 2 for which the proof is similar to [11]:

**Lemma 2** Under assumption $H1$, we have

$$V^-(x) = \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} J(x; u, \beta(u));$$

$$V^+(x) = \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} J(x; \alpha(v), v),$$

where $\mathcal{U}$ and $\mathcal{V}$ contain all the impulse controls in $\mathcal{U}$ and $\mathcal{V}$, respectively, which have no impulses at time 0. Similarly, $\mathcal{A}$ and $\mathcal{B}$ contain all the non-anticipative strategies with values in $\mathcal{U}$ and $\mathcal{V}$, respectively.\[ \square \]
\textbf{Theorem 1} (Dynamic Programming Principle) Under assumption $H1$, given $x \in \mathbb{R}^n$ and $T \geq 0$, we have the dynamic programming principle:

\begin{align}
V^-(x) = \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} \left[ \int_0^T f(y_x(t)) \exp(-\lambda t) dt \right. \\
&- \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \mathbb{I}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \\
&\left. + \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k) \mathbb{I}_{\{\rho_k \leq T\}} + V^-(y_x(T)) \exp(-\lambda T) \right].
\end{align}

and

\begin{align}
V^+(x) = \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} \left[ \int_0^T f(y_x(t)) \exp(-\lambda t) dt \right. \\
&- \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \mathbb{I}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \\
&\left. + \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k) \mathbb{I}_{\{\rho_k \leq T\}} + V^+(y_x(T)) \exp(-\lambda T) \right].
\end{align}

\textbf{Proof} We give the proof only for the lower value function $V^-$, similarly for $V^+$. We first let $\varepsilon > 0$, $u \in \mathcal{U}$ and assume, for some $x \in \mathbb{R}^n$ and some $T \geq 0$, that $V^-(x) < W_T(x)$, where $W_T(x)$ is the right-hand side of (9). We let the difference be $W_T(x) - V^-(x) = 2\varepsilon$ and we choose $\beta^\varepsilon$ a non-anticipative strategy that approximates $V^-(x)$ up to $\varepsilon$, and denote $\beta^\varepsilon(u) := (\rho_k^\varepsilon, \eta_k^\varepsilon)_{k \geq 1}$ the jumps it produces. We then have

\begin{align}
W_T(x) - \varepsilon \geq \sup_{u \in \mathcal{U}} \left[ \int_0^T f(y_x(t)) \exp(-\lambda t) dt \right. \\
&- \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \mathbb{I}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k^\varepsilon\}} \\
&\left. + \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \mathbb{I}_{\{\rho_k^\varepsilon \leq T\}} + J(y_x(T); u, \beta^\varepsilon(u)) \exp(-\lambda T) \right].
\end{align}

The above inequality can be rewritten, for $\mathcal{U}_0$ and $\mathcal{U}_T$ the restrictions of $\mathcal{U}$ to $[0, T] \times U$ and $[T, +\infty] \times U$, respectively, as follows:

\begin{align}
W_T(x) - \varepsilon \geq \sup_{u \in \mathcal{U}_0} \left[ \int_0^T f(y_x(t)) \exp(-\lambda t) dt \right. \\
&- \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \mathbb{I}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k^\varepsilon\}} \\
&\left. + \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \mathbb{I}_{\{\rho_k^\varepsilon \leq T\}} + \sup_{u \in \mathcal{U}_T} J(y_x(T); u, \beta^\varepsilon(u)) \exp(-\lambda T) \right].
\end{align}
Observe that once \( y_x(T) \) is known, the knowledge of \( u \) over \([0, T] \times U\) is useless to evaluate \( \sup_{u \in \mathcal{U}_T} J \). Therefore, since \( \lambda > 0 \), the restriction of \( \mathcal{U} \) to \( \mathcal{U}_T \) satisfies

\[
V^-(y_x(T)) \exp(-\lambda T) \leq \sup_{u \in \mathcal{U}_T} J(y_x(T); u, \beta^x(u)) \exp(-\lambda T),
\]

replacing in inequality (10) leads to a contradiction. Finally, for all \( x \in \mathbb{R}^n \) and all \( T > 0 \), we deduce

\[
V^-(x) \geq W_T(x).
\]

Now let us assume to the contrary, for some \( x \in \mathbb{R}^n \) and some \( T > 0 \), that \( V^-(x) > W_T(x) \) and let the difference be \( V^-(x) - W_T(x) = 3\varepsilon \) for \( \varepsilon > 0 \). We denote by \( \beta^x_1 \in \mathcal{B} \) the non-anticipative strategy that approximates \( W_T(x) \) up to \( \varepsilon \), where \( \beta^x_1(u) := (\rho_k^x, \eta_k^x)_{k \geq 1} \) are the jumps it produces for \( u := (\tau_m, \xi_m)_{m \geq 1} \in \mathcal{U} \). We then get

\[
V^-(x) - 2\varepsilon \geq \sup_{u \in \mathcal{U}_0} \int_0^T f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} [\tau_m \neq \rho_k^x] \\
+ \sum_{k \geq 1} \chi(\eta_k^x) \exp(-\lambda \rho_k^x) \prod_{\rho_k^x \leq T} \left[ V^-(y_x(T)) \exp(-\lambda T) \right],
\]

therefore, for any \( u \in \mathcal{U}_0 \) the restriction of \( \mathcal{U} \) to \([0, T] \), we have

\[
V^-(x) - 2\varepsilon \geq \int_0^T f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} [\tau_m \neq \rho_k^x] \\
+ \sum_{k \geq 1} \chi(\eta_k^x) \exp(-\lambda \rho_k^x) \prod_{\rho_k^x \leq T} \left[ V^-(y_x(T)) \exp(-\lambda T) \right].
\]

Furthermore, we choose a non-anticipative strategy \( \beta^x_2 \in \overline{\mathcal{B}} \) of the game over \([T, +\infty) \) that approximates \( V^-(y_x(T)) \) again up to \( \varepsilon \). The concatenation \( \beta^x \) of \( \beta^x_1 \) and \( \beta^x_2 \) is a non-anticipative strategy of the game over \([0, +\infty) \), then we deduce for the non-anticipative strategy \( \beta^x \in \mathcal{B} \) and all control \( u \in \mathcal{U} \) the following:

\[
V^-(x) - \varepsilon \geq J(x; u, \beta^x(u)),
\]

a contradiction. Finally, for all \( x \in \mathbb{R}^n \) and all \( T > 0 \), we deduce

\[
V^-(x) \leq W_T(x).
\]

The proof is now complete. \( \square \)

We next use the DPP to show that both lower value function and upper value function are continuous in \( \mathbb{R}^n \).
Theorem 2  Under assumption $H1$, the lower and upper value functions are continuous in $\mathbb{R}^n$.

Proof  We make the proof only for the lower value function $V^-$, the other case being analogous. We first show that $V^-$ is upper semi-continuous. For any $x, x' \in \mathbb{R}^n$ and $T \geq 0$, according to the DPP for the lower value function, we have

$$V^-(x) = \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} \left[ \int_0^T f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} + \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m = \rho_k\}} \right]$$

and

$$V^-(x') = \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} \left[ \int_0^T f(y_{x'}(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} + \sum_{k \geq 1} \chi(\eta_k) \exp(-\lambda \rho_k) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m = \rho_k\}} + V^-(y_{x'}(T)) \exp(-\lambda T) \right].$$

Now fix an arbitrary $\varepsilon > 0$ and pick, for $\beta^\varepsilon \in \mathcal{B}$, a strategy $\beta^\varepsilon(u) := (\rho_k^\varepsilon, \eta_k^\varepsilon)_{k \geq 1}$ which satisfies the following:

$$V^-(x') + \varepsilon \geq \sup_{u \in \mathcal{U}} \left[ \int_0^T f(y_{x'}(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k^\varepsilon\}} + \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m = \rho_k^\varepsilon\}} + V^-(y_{x'}(T)) \exp(-\lambda T) \right].$$

Further, we pick a control $u^\varepsilon := (\tau_m^\varepsilon, \xi_m^\varepsilon)_{m \geq 1} \in \mathcal{U}$ which satisfies the following:

$$V^-(x) - \varepsilon \leq \int_0^T f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m^\varepsilon) \exp(-\lambda \tau_m^\varepsilon) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}} + \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon = \rho_k^\varepsilon\}} + V^-(y_x(T)) \exp(-\lambda T).$$
Since the inequality (11) holds for any control $u$, then $u^\varepsilon$ satisfies

$$V^-(x') + \varepsilon \geq \int_{0}^{T} f(y_x(t)) \exp(-\lambda t) dt - \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m^\varepsilon) \mathbb{1}_{\{\tau_m^\varepsilon \leq T\}} \prod_{k \geq 1} \mathbb{1}_{\{\tau_k^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \mathbb{1}_{\{\rho_k^\varepsilon \leq T\}} + V^-(y_{x'}(T)) \exp(-\lambda T).$$

It follows from the two last inequalities that

$$V^-(x) - V^-(x') \leq \int_{0}^{T} \left| f(y_x(t)) - f(y_{x'}(t)) \right| \exp(-\lambda t) dt$$

$$- \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m^\varepsilon) \mathbb{1}_{\{\tau_m^\varepsilon \leq T\}} \prod_{k \geq 1} \mathbb{1}_{\{\tau_k^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \mathbb{1}_{\{\rho_k^\varepsilon \leq T\}} + V^-(y_x(T)) \exp(-\lambda T)$$

$$+ \sum_{m \geq 1} c(\xi_m) \exp(-\lambda \tau_m^\varepsilon) \mathbb{1}_{\{\tau_m^\varepsilon \leq T\}} \prod_{k \geq 1} \mathbb{1}_{\{\tau_k^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$- \sum_{k \geq 1} \chi(\eta_k^\varepsilon) \exp(-\lambda \rho_k^\varepsilon) \mathbb{1}_{\{\rho_k^\varepsilon \leq T\}} - V^-(y_{x'}(T)) \exp(-\lambda T) + 2\varepsilon.$$

Thus, we get

$$V^-(x) - V^-(x') \leq \int_{0}^{T} \left| f(y_x(t)) - f(y_{x'}(t)) \right| \exp(-\lambda t) dt$$

$$+ \left| V^-(y_x(T)) - V^-(y_{x'}(T)) \right| \exp(-\lambda T) + 2\varepsilon.$$

By Lemma 1, the Lipschitz continuity of $f$ and the boundedness of $V^-$, we deduce that there exists a constant $C > 0$ such that

$$V^-(x) - V^-(x') \leq C_f \|x - x'\| \int_{0}^{T} \exp((C_b - \lambda)t) dt + 2C \exp(-\lambda T) + 2\varepsilon. \quad (12)$$

Therefore, if $\lambda \neq C_b$, we obtain

$$V^-(x) - V^-(x') \leq \frac{C_f}{C_b - \lambda} \|x - x'\| \left[ \exp((C_b - \lambda)T) - 1 \right]$$

$$+ 2C \exp(-\lambda T) + 2\varepsilon. \quad (13)$$

Now we choose $T$ such that $\exp(-C_b T) = \|x - x'\|^{1/2}$ with $\|x - x'\| < 1$. Hence, in the right-hand side of (13), the first term goes to 0 when $x \to x'$, i.e., $T \to +\infty$, indeed, it is equal to

$$\frac{C_f}{C_b - \lambda} \|x - x'\|^{1/2} \left( \exp(-\lambda T) - \|x - x'\|^{1/2} \right),$$
while the second term goes to 0 where $T \to +\infty$. We then deduce, by letting $x \to x'$ and $\varepsilon \to 0$, the upper semi-continuity of the lower value function

$$\limsup_{x \to x'} V^-(x) \leq V^-(x').$$

In the case where $\lambda = C_b$, it suffice to let some $\hat{\lambda} < \lambda = C_b$, so we go back to (12) and we proceed, since $\exp((C_b - \lambda)T) < \exp((C_b - \hat{\lambda})T)$ and $\exp(-\lambda T) < \exp(-\hat{\lambda} T)$, as above with the case $\hat{\lambda} \neq C_b$, we then conclude by letting $x \to x'$ and $\varepsilon \to 0$.

Analogously we get the lower semi-continuity

$$\liminf_{x \to x'} V^-(x) \geq V^-(x').$$

Then the lower value function is continuous in $\mathbb{R}^n$. \hfill $\Box$

We now focus on the viscosity characterization of the lower and upper value functions through the classic and new HJBI-QVIs.

## 4 Viscosity Characterization

The aim of the present Section is to show that the lower and upper value functions are viscosity solutions to both classic HJBI-QVI and new HJBI-QVI. To do so we first give, in Lemma 3, some properties of the value functions, hence we get the aim for the classic HJBI-QVI. Furthermore, in Theorem 4, we show equivalence in the viscosity super-solution sense between the classic HJBI-QVI and the new HJBI-QVI, then we conclude. We begin with the following technical lemma:

**Lemma 3** Under assumption $H1$, the lower value function satisfies, for all $x \in \mathbb{R}^n$, the following properties:

1. $V^-(x) \leq \mathcal{H}^c_{\sup} V^-(x)$;
2. $V^-(x) \leq \mathcal{H}^\varepsilon_{\inf} V^-(x)$;
3. Let $x \in \mathbb{R}^n$ be such that $V^-(x) < \mathcal{H}^\varepsilon_{\inf} V^-(x)$, then $V^-(x) = \mathcal{H}^c_{\sup} V^-(x)$.

The same results hold true for the upper value function $V^+$.

**Proof** We give the proof for the lower value function $V^-$, similarly for $V^+$. The proof of (i) follows immediately from the assumption on the cost function $c$,

$$V^-(x) = V^-(x + 0) - c(0) \leq \sup_{\xi \in \mathcal{U}^-} [V^-(x + \xi) - c(\xi)] = \mathcal{H}^c_{\sup} V^-(x).$$

To prove (ii), we first let $x \in \mathbb{R}^n$ and consider, for player-$\eta$, the strategy $\beta(u) := (\rho_k, \eta_k)_{k \geq 1} \in \mathcal{V}$ where $\beta \in \mathcal{B}$. Next, choose $\beta' \in \mathcal{B}$ such that $\beta'(u) := (0, \eta; \rho_2, \eta_2; \rho_3, \eta_3; \ldots)$, we then obtain

$$V^-(x) \leq \sup_{u \in \mathcal{U}} J(x; u, \beta'(u)) \leq \sup_{u \in \mathcal{U}} J(x + \eta; u, \beta(u)) + \chi(\eta),$$
from which we deduce the following inequality:

\[ V^-(x) \leq \inf_{\eta \in V} \left[ V^-(x + \eta) + \chi(\eta) \right]. \]

Now, in order to deal with (iii), we assume for any \( x \in \mathbb{R}^n \) that \( V^-(x) < \mathcal{H}_{\inf}^X V^-(x) \). From the DPP for \( V^- \), by taking \( T = 0 \), we get

\[
V^-(x) = \inf_{\rho_1 \in [0, +\infty)} \sup_{\tau_1 \in [0, +\infty]}, \xi \in U \left[ -c(\xi) \mathbb{1}_{\tau_1 = 0} \mathbb{1}_{\rho_1 = +\infty} + \chi(\eta) \mathbb{1}_{\rho_1 = 0} 
+ V^-(x + \xi \mathbb{1}_{\tau_1 = 0}) \mathbb{1}_{\rho_1 = +\infty} + \eta \mathbb{1}_{\rho_1 = 0} \right],
\]

therefore

\[
V^-(x) = \inf_{\rho_1 \in [0, +\infty)} \left[ \inf_{\eta \in V} \left[ \chi(\eta) + V^-(x + \eta) \right] \mathbb{1}_{\rho_1 = 0} 
+ \sup_{\tau_1 \in [0, +\infty]}, \xi \in U \left[ -c(\xi) \mathbb{1}_{\tau_1 = 0} + V^-(x + \xi \mathbb{1}_{\tau_1 = 0}) \right] \mathbb{1}_{\rho_1 = +\infty} \right].
\]

Since \( V^-(x) < \mathcal{H}_{\inf}^X V^-(x) \), we get

\[
V^-(x) = \sup_{\tau_1 \in [0, +\infty], \xi \in U^-} \left[ -c(\xi) \mathbb{1}_{\tau_1 = 0} + V^-(x + \xi \mathbb{1}_{\tau_1 = 0}) \right].
\]

Therefore

\[
V^-(x) \geq \sup_{\xi \in U^-} \left[ V^-(x + \xi) - c(\xi) \right].
\]

Combining with property (i) we deduce

\[
V^-(x) = \mathcal{H}_{\sup}^C V^-(x).
\]

\[ \square \]

4.1 Viscosity Characterization of the Classic HJBI-QVI

Now we are ready to show the relation between our zero-sum deterministic impulse controls differential game and the classic HJBI-QVI (6), indeed, we prove the following theorem:

**Theorem 3** Under assumption \( H1 \), the lower and upper value functions are viscosity solutions to the classic Hamilton-Jacobi-Bellman-Isaacs quasi-variational inequality.
Proof: The proof is based on the DPP and it is inspired from [4], we give the proof only for $V^-$, the other case being analogous. We first prove the sub-solution property. Suppose that $V^- - \phi$ achieves its local minimum at $\overline{x}$ in $B_{\delta}(\overline{x})$, where $B_{\delta}(\overline{x})$ is the open ball of center $\overline{x}$ and radius $\delta > 0$, with $V^- (\overline{x}) = \phi(\overline{x})$, where $\phi$ is a function in $C^1(\mathbb{R}^n)$. From Lemma 3 we always have $V^- (\overline{x}) - \mathcal{H}_{inf}^{\delta} V^- (\overline{x}) \leq 0$ and $V^- (\overline{x}) - \mathcal{H}_{sup}^{\delta} V^- (\overline{x}) \leq 0$, then the sub-solution property follows directly. Now, in order to prove the super-solution property, we suppose that $V^- - \phi$ achieves its local minimum at $\overline{x}$ in $B_{\delta}(\overline{x})$, with $V^- (\overline{x}) = \phi(\overline{x})$, where $\phi$ is a function in $C^1(\mathbb{R}^n)$. Then if $V^- (\overline{x}) - \mathcal{H}_{inf}^{\delta} V^- (\overline{x}) = 0$ there is nothing to prove. Otherwise, we suppose that $V^- (\overline{x}) - \mathcal{H}_{inf}^{\delta} V^- (\overline{x}) < -\varepsilon_1 < 0$, for $\varepsilon_1 > 0$. Then, without loss of generality, we can assume that

$$V^- (\overline{x}) - \mathcal{H}_{inf}^{\delta} V^- (\overline{x}) < -\varepsilon_1 < 0, \text{ on } B_{\delta}(\overline{x}). \quad (14)$$

Next, let $T > 0$, $0 < \varepsilon < \varepsilon_1 \exp(-\lambda T)$ and consider $\tau_1 = +\infty$, i.e., no impulse for player-$\xi$. Furthermore, for $\rho^2_{\overline{x}} > 0$, we pick a strategy $v^\varepsilon := (\rho^k, \eta^k)_{k \geq 1} \in \mathcal{V}$ for player-$\eta$. Next, define the exit time

$$t' := \inf \{ t \geq 0 : y_{\overline{x}}(t) \notin B_{\delta}(\overline{x}) \},$$

and let $T \leq t'$ sufficiently small, which, due to DPP for $V^-$, satisfies the following:

$$V^- (\overline{x}) \geq \int_0^{T \wedge \rho^1} f(y_{\overline{x}}(t)) \exp(-\lambda t) dt + \chi(\eta^1) \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]}$$

$$+ V^- (y_{\overline{x}}(T \wedge \rho^1)) \exp(-\lambda (T \wedge \rho^1)) - \varepsilon$$

$$\geq \int_0^{T \wedge \rho^1} f(y_{\overline{x}}(t)) \exp(-\lambda t) dt + \exp(-\lambda \rho^1) \chi(\eta^1) \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]}$$

$$+ V^- (y_{\overline{x}}(T)) \exp(-\lambda T) \mathbb{I}_{[\rho^1_{\overline{x}} > T]} - \varepsilon$$

$$\geq \int_0^{T \wedge \rho^1} f(y_{\overline{x}}(t)) \exp(-\lambda t) dt + \mathcal{H}_{inf}^{\delta} V^- (y_{\overline{x}}(\rho^1_{\overline{x}})) \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]}$$

$$+ V^- (y_{\overline{x}}(T)) \exp(-\lambda T) \mathbb{I}_{[\rho^1_{\overline{x}} > T]} - \varepsilon.$$

It then follows from assumption (14) that

$$V^- (\overline{x}) \geq \int_0^{T \wedge \rho^1} f(y_{\overline{x}}(t)) \exp(-\lambda t) dt + V^- (y_{\overline{x}}(\rho^1_{\overline{x}})) \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]}$$

$$+ V^- (y_{\overline{x}}(T)) \exp(-\lambda T) \mathbb{I}_{[\rho^1_{\overline{x}} > T]} + \varepsilon_1 \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]} - \varepsilon.$$

Since $\varepsilon_1 \exp(-\lambda \rho^1) \mathbb{I}_{[\rho^1_{\overline{x}} \leq T]} - \varepsilon > 0$, then without loss of generality, we only need to consider a strategy $v^\varepsilon \in \mathcal{V}$ for which $T < \rho^1_{\overline{x}}$, then we get

$$V^- (\overline{x}) + \varepsilon \geq \int_0^T f(y_{\overline{x}}(t)) \exp(-\lambda t) dt + V^- (y_{\overline{x}}(T)) \exp(-\lambda T).$$
As $T \to 0$, we have

$$\|y_x(T) - x\| \to 0,$$

from which we deduce

$$V^-(y_x(T)) \geq \phi(y_x(T)) + [V^-(x) - \phi(x)],$$

it follows, when $\varepsilon$ goes to 0, that

$$1 - \exp(-\lambda T) V^-(x) \geq 1 \frac{T}{T} \int_0^T f(y_x(t)) \exp(-\lambda t) dt + \frac{\phi(y_x(T)) - \phi(x)}{T} \exp(-\lambda T).$$

We use the fact that

$$\phi(y_x(T)) - \phi(x) = \int_0^T b(y_x(s)).D\phi(y_x(s)) ds,$$

then we let $T \to 0$ to get

$$\lambda V^-(x) - D\phi(x).b(x) - f(x) \geq 0.$$

Finally, since Lemma 3 implies $V^-(x) - \mathcal{H}^C_{sup} V^-(x) = 0$ when assuming (14), we get the super-solution property

$$\max\left\{ \min\left[ \lambda V^-(x) - D\phi(x).b(x) - f(x), V^-(x) - \mathcal{H}^C_{sup} V^-(x) \right], V^-(x) - \mathcal{H}^\lambda_{inf} V^-(x) \right\} \geq 0.$$

4.2 Viscosity Characterization of the New HJBI-QVI

This Section is devoted to proving that both lower and upper value functions of our impulse controls differential game are viscosity solutions to the new HJBI-QVI (7). We start by proving the following theorem:

**Theorem 4** Under assumptions $H1$ and $H2$, a bounded and continuous function $v$ is a viscosity super-solution to the classic HJBI-QVI if and only if it is a viscosity super-solution to the new HJBI-QVI.

**Proof** Assume first that $v$ is a bounded and continuous viscosity super-solution to the new HJBI-QVI. Then, for a function $\phi \in C^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ such that $v(x) = \phi(x)$ and $x$ is a local minimum point of $v - \phi$, we have

$$\max\left\{ \min\left[ \lambda v(x) - D\phi(x).b(x) - f(x), \mathcal{F}^c_{inf} D\phi(x) \right], v(x) - \mathcal{H}^\lambda_{inf} v(x) \right\} \geq 0.$$

$\square$
If \( v(x) - \mathcal{H}_{inf}^c v(x) \geq 0 \) then we are done with the first implication. Else we assume
\[
\min \left[ \lambda v(x) - D\phi(x) \cdot b(x) - f(x), \mathcal{F}_{inf}^c D\phi(x) \right] \geq 0.
\]
It follows that \( \mathcal{F}_{inf}^c D\phi(x) \geq 0 \). Let us first assume that \( v \in C^1(\mathbb{R}^n) \), we then have for all \( \xi \in U \) with \( \|\xi\| = 1 \), for all \( x \in \mathbb{R}^n \),
\[
-Dv(x).\xi \geq -c(\xi).
\]
That is, for all \( \xi \in U \), for all \( x \in \mathbb{R}^n \),
\[
-Dv(x).\xi \geq -c(\xi),
\]
and, when \( \xi = 0 \), we also get \( -Dv(x).\xi \geq -c(\xi). \) Since
\[
v(x) - v(x + \xi) = \int_0^1 \frac{d}{ds} \left( v(x + s\xi) \right) ds = \int_0^1 -Dv(x + s\xi).\xi ds \geq -c(\xi).
\]
We then obtain for all \( \xi \in U \),
\[
v(x) - \left[ v(x + \xi) - c(\xi) \right] \geq 0,
\]
which means that
\[
v(x) - \mathcal{H}_{sup}^c v(x) \geq 0.
\]
Finally, \( v \) is a viscosity super-solution to the classic HJBI-QVI when \( v \in C^1(\mathbb{R}^n) \).

We obtain the same result even \( v \) is not in \( C^1(\mathbb{R}^n) \). It suffices to make the same regularization \( v_\varepsilon \in C^\infty(\mathbb{R}^n) \) for \( v \), as in the proof in [22, Theorem 3.2] (see also [27]), which converges uniformly toward \( v \) in \( \mathbb{R}^n \). Let \( \theta \) be a positive function in \( C^\infty(\mathbb{R}^n) \) with \( \text{supp} \, \theta(x) \subset B_1(0) \), where \( B_1(0) \) is the open ball of center 0 and radius 1, and \( \int_{\mathbb{R}^n} \theta(x) dx = 1 \). We then define, for \( \varepsilon > 0 \) and \( x \in \mathbb{R}^n \), the function \( \theta_\varepsilon \) by the following:
\[
\theta_\varepsilon(x) := \frac{1}{\varepsilon^n} \theta \left( \frac{x}{\varepsilon} \right).
\]
Further, we define in \( \mathbb{R}^n \) the regularization
\[
v_\varepsilon(x) := \int_{\mathbb{R}^n} v(y) \theta_\varepsilon(x - y) dy.
\]
The function $v_{\varepsilon}$ is bounded, belongs to $C^\infty(\mathbb{R}^n)$ and satisfies

$$\sup_{x \in \mathbb{R}^n} |v_{\varepsilon}(x) - v(x)| \leq \sup_{\|x-y\| \leq \varepsilon} |v(x) - v(y)|,$$

then it converges to $v$ as $\varepsilon$ goes to 0.

In addition, for all $x \in \mathbb{R}^n$ and since $v$ is a viscosity super-solution to the new HJBI-QVI, the regularization $v_{\varepsilon}$ satisfies $F_{\inf}^c D v_{\varepsilon}(x) \geq \delta(\varepsilon)$, where $\delta(\varepsilon)$ goes to 0 with $\varepsilon$.

Therefore, by the same computation as in above, we get for all $\xi \in U$,

$$v_{\varepsilon}(x) - \left[ v_{\varepsilon}(x + \xi) - c(\xi) \right] \geq \delta(\varepsilon),$$

that is, when $\varepsilon$ goes to 0,

$$v(x) - \left[ v(x + \xi) - c(\xi) \right] \geq 0.$$

Hence we get the desired result

$$v(x) - H_{\sup}^c v(x) \geq 0.$$

Assume now that $v$ is a bounded and continuous viscosity super-solution to the classic HJBI-QVI. And let $x \in \mathbb{R}^n$ be a global minimum point of $v - \phi$, where $\phi$ is a function in $C^1(\mathbb{R}^n)$ and $v(x) = \phi(x)$. We then have

$$\max \left\{ \min \left[ \lambda v(x) - D\phi(x).b(x) - f(x), v(x) - H_{\sup}^c v(x) \right], v(x) - H_{\inf}^c v(x) \right\} \geq 0.$$

If $v(x) - H_{\inf}^c v(x) \geq 0$ then we are done. Else we assume

$$\min \left[ \lambda v(x) - D\phi(x).b(x) - f(x), v(x) - H_{\sup}^c v(x) \right] \geq 0.$$

Which gives

$$v(x) - \sup_{\xi \in U} \left[ v(x + \xi) - c(\xi) \right] \geq 0,$$

thus, for all $\xi \in U$, we get

$$v(x) - v(x + \xi) + c(\xi) \geq 0.$$

Since for all $\xi \in U$,

$$\phi(x) - \phi(x + \xi) + c(\xi) \geq v(x) - v(x + \xi) + c(\xi),$$
then we have

$$\phi(x) - \phi(x + \xi) + c(\xi) \geq 0.$$  

We can then deduce, under assumption $H_2$, for all $k > 0$ the following:

$$\frac{\phi(x) - \phi(x + k\xi)}{k} \geq -\frac{c(k\xi)}{k} \geq -c(\xi).$$

Hence, by letting $k \to 0$, we get for all $\xi \in U$,

$$-D\phi(x)\cdot\xi + c(\xi) \geq 0.$$  

Therefore, we obtain

$$\mathcal{F}'_{\inf} D\phi(x) \geq 0.$$  

Finally, $v$ is a viscosity super-solution to the new HJBI-QVI. \hfill \Box

**Theorem 5** Under assumptions $H_1$ and $H_2$, the lower and upper value functions are viscosity solutions to the new Hamilton-Jacobi-Bellman-Isaacs quasi-variational inequality.

**Proof** We give the proof for the lower value function $V^-$, similarly for $V^+$. We first prove the viscosity sub-solution property. We let $\phi$ be a function in $C^1(\mathbb{R}^n)$ and $\bar{x} \in \mathbb{R}^n$ such that $V^-(\bar{x}) = \phi(\bar{x})$ and $\bar{x}$ is a local maximum of $V^--\phi$. Since we have proved in Lemma 3 that $V^-(\bar{x}) - \mathcal{H}_{\inf} V^-(\bar{x}) \leq 0$, then if $\mathcal{F}'_{\inf} D\phi(\bar{x}) \leq 0$ there is nothing to prove. Otherwise, for all $x \in \mathbb{R}^n$ we assume that $\mathcal{F}'_{\inf} D\phi(x) > 0$. We then get, for $\xi \neq 0$,

$$\inf_{\xi \in U, \|\xi\| = 1} \left[-D\phi(x)\cdot\xi + c(\xi)\right] > 0.$$  

Then for all $\xi \in U\setminus\{0\}$, for all $x \in \mathbb{R}^n$,

$$-D\phi(x)\cdot\xi > -c(\xi),$$

because when $\xi \in U\setminus\{0\}$ with $\|\xi\| \neq 1$ we get from assumption $H_2$ that

$$-D\phi(x)\cdot\frac{\xi}{\|\xi\|} > -c\left(\frac{\xi}{\|\xi\|}\right) \geq -\frac{c(\xi)}{\|\xi\|}.$$  

Since

$$\phi(x) - \phi(x + \xi) = \int_{0}^{1} -\frac{d}{ds}(\phi(x + s\xi))ds = \int_{0}^{1} -D\phi(x + s\xi)\cdot\xi ds > -c(\xi),$$

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we get
\[ \phi(\bar{x}) - \phi(\bar{x} + \xi) + c(\xi) > 0, \]

furthermore
\[ \phi(\bar{x}) - \phi(\bar{x} + \xi) + c(\xi) \leq V^-(\bar{x}) - V^-(\bar{x} + \xi) + c(\xi), \]
hence
\[ V^-(\bar{x}) - [V^-(\bar{x} + \xi) - c(\xi)] > 0. \]

Thus, for \( \xi \neq 0 \), whenever \( F_{\inf}^c D\phi(\bar{x}) > 0 \) we have \( V^-(\bar{x}) - \mathcal{H}_{\sup}^c V^-(\bar{x}) > 0 \). Next, an analogous computation to the proof of the viscosity super-solution sense for the classic HJBI-QVI, Theorem 3, leads to the following viscosity sub-solution property:
\[ \max \left\{ \min \left[ \lambda V^-(\bar{x}) - D\phi(\bar{x}).b(\bar{x}) - f(\bar{x}), F_{\inf}^c D\phi(\bar{x}) \right], V^-(\bar{x}) - \mathcal{H}_{\inf}^X V^-(\bar{x}) \right\} \leq 0. \]

For the proof of the viscosity super-solution property, since we have proved in Theorem 3 that \( V^- \) is a viscosity solution to the classic HJBI-QVI, then, due to the result in Theorem 4, the lower value function \( V^- \) is a viscosity super-solution to the new HJBI-QVI. \( \square \)

5 Uniqueness of the Viscosity Solution of the New HJBI-QVI

We prove in the present Section, via a comparison theorem, that the new HJBI-QVI (7) has a unique bounded and continuous solution in viscosity sense. As a consequence, the lower and upper value functions of the differential game coincide, since they are both viscosity solutions to the new HJBI-QVI. Hence the differential game has a value. We first give the following classical lemma which also appears in [35]:

Lemma 4 Let \( v \) be a bounded uniformly continuous function and \( x_0 \in \mathbb{R}^n \) such that
\[ v(x_0) \geq \mathcal{H}_{\inf}^X v(x_0). \]
Then there exists an element \( y \) in \( \mathbb{R}^n \) such that
\[ \exists \delta > 0, \forall x \in \overline{B}_\delta(y) : v(x) < \mathcal{H}_{\inf}^X v(x), \]
where \( \overline{B}_\delta(y) \) is the closed ball of center \( y \) and radius \( \delta \).

Proof Fix \( \epsilon > 0 \) and let \( x_0 \in \mathbb{R}^n \) such that \( v(x_0) \geq \mathcal{H}_{\inf}^X v(x_0) \). Then there exists \( \eta_0 \in V \) such that
\[ v(x_0) \geq v(x_0 + \eta_0) + \chi(\eta_0) - \epsilon, \]
then we get for all $\eta \in V$,

$$
    v(x_0 + \eta_0 + \eta) + \chi(\eta) - v(x_0 + \eta_0 + \eta_0) \geq v(x_0 + \eta_0 + \eta) + \chi(\eta_0) - v(x_0) - \varepsilon
$$

$$
    \geq \chi(\eta_0) + \chi(\eta) - \chi(\eta_0 + \eta) - \varepsilon.
$$

Thus, from assumption H1 and by letting $\varepsilon \to 0$, we deduce

$$
    v(x_0 + \eta_0) < \mathcal{H}^{X}_{\inf} v(x_0 + \eta_0).
$$

Now we take $y = x_0 + \eta_0$, then, since for all $\eta \in V$ we have

$$
    v(y) < v(y + \eta) + \chi(\eta) - \varepsilon,
$$

we obtain, by uniform continuity of $v$ where $C_v$ is the modulus of continuity, for all $x \in \mathbb{R}^n$,

$$
    v(x) < v(x + \eta) + \chi(\eta) + C_v(\|x - y\|) - \varepsilon.
$$

Hence there exists $\delta > 0$ such that for all $x \in B_\delta(y)$ we have

$$
    v(x) - \mathcal{H}^{X}_{\inf} v(x) < 0.
$$

We next prove the following useful lemma:

**Lemma 5** Let $v: \mathbb{R}^n \to \mathbb{R}$ be a bounded and continuous viscosity super-solution to the new HJBI-QVI. If $v(x) - \mathcal{H}^{X}_{\inf} v(x) < 0$, then, for any $\mu$, $\alpha$ and $K$ such that

$$
    0 < \mu < 1, \quad K > \|f\|_\infty / \lambda, \quad \text{and}
$$

$$
    0 < \alpha < (1 - \mu) \min \left( \inf_{\xi \in U, \|\xi\| = 1} c(\xi), \left( \lambda K - \|f\|_\infty \right) / \|b\|_\infty \right),
$$

the function $v^*(x) := \mu v(x) + \alpha \sqrt{\|x\|^2 + 1} + K (1 - \mu)$ is a strict viscosity super-solution to the new HJBI-QVI.

**Proof** Let $v$ be a bounded continuous viscosity super-solution to the new HJBI-QVI, $\phi^* \in C^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be a local minimum point of $v^* - \phi^*$ such that $v(x) - \mathcal{H}^{X}_{\inf} v(x) < 0$.

Then, for all $x \in B_\delta(x)$, where $B_\delta(x)$ is the open ball of center $x$ and radius $\delta > 0$, we have

$$
    \mu v(x) + \alpha \sqrt{\|x\|^2 + 1} - \phi^*(x) \leq \mu v(x) + \alpha \sqrt{\|x\|^2 + 1} + K (1 - \mu),
$$

then

$$
    v(x) - \frac{\phi^*(x) - \alpha \sqrt{\|x\|^2 + 1}}{\mu} \leq v(x) - \frac{\phi^*(x) - \alpha \sqrt{\|x\|^2 + 1}}{\mu},
$$

$$
    \square$$
this inequality means that $x$ is a local minimum point of $v - \phi$, where
\[
\phi(x) = \left( \phi^*(x) - \alpha \sqrt{\|x\|^2 + 1} \right) / \mu.
\]
Then, since $v(x) - H^X_{inf} v(x) < 0$ and $v$ is viscosity super-solution to the new HJBI-QVI, we have the following:
\[
\min \left[ \lambda v(x) - D\phi(x).b(x) - f(x), F^c_{inf} D\phi(x) \right] \geq 0,
\]
thus
\[
\lambda v(x) - D\phi(x).b(x) - f(x) \geq 0, \quad \text{and} \quad F^c_{inf} D\phi(x) \geq 0.
\]
On one hand, we get
\[
\lambda v(x) - \frac{1}{\mu} D\phi^*(x).b(x) + \frac{\alpha x}{\mu \sqrt{\|x\|^2 + 1}} b(x) - f(x) \geq 0,
\]
then
\[
\lambda \mu v(x) - D\phi^*(x).b(x) - \mu f(x) \geq -\alpha \|b\|_\infty,
\]
thus
\[
\lambda v^*(x) - D\phi^*(x).b(x) - f(x) \geq \lambda K (1 - \mu) - (1 - \mu) \|f\|_\infty - \alpha \|b\|_\infty.
\]
Finally, since $\alpha < (1 - \mu) (\lambda K - \|f\|_\infty) / \|b\|_\infty$, we obtain
\[
\lambda v^*(x) - D\phi^*(x).b(x) - f(x) > 0. \tag{15}
\]
On the other hand, since $F^c_{inf} D\phi(x) \geq 0$, we get
\[
F^c_{inf} \left[ \frac{D\phi^*(x)}{\mu} - \frac{\alpha x}{\mu \sqrt{\|x\|^2 + 1}} \right] \geq 0,
\]
then
\[
\inf_{\xi \in U, \|\xi\|=1} \left[ -D\phi^*(x).\xi + \frac{\alpha x}{\sqrt{\|x\|^2 + 1}} .\xi + \mu c(\xi) \right] \geq 0,
\]
it follows that
\[
\inf_{\xi \in U, \|\xi\|=1} \left[ -D\phi^*(x).\xi + c(\xi) + (\mu - 1)c(\xi) \right] \geq -\alpha,
\]
from which we deduce, since \( \mu - 1 < 0 \), the following:

\[
\inf_{\xi \in U, \|\xi\| = 1} \left[ -D\phi^*(x) . \xi + c(\xi) + (\mu - 1) \inf_{\xi \in U, \|\xi\| = 1} c(\xi) \right] \\
\geq \inf_{\xi \in U, \|\xi\| = 1} \left[ -D\phi^*(x) . \xi + c(\xi) + (\mu - 1) c(\xi) \right] \\
\geq -\alpha,
\]

therefore

\[
(\mu - 1) \inf_{\xi \in U, \|\xi\| = 1} c(\xi) + \inf_{\xi \in U, \|\xi\| = 1} \left[ -D\phi^*(x) . \xi + c(\xi) \right] \geq -\alpha.
\]

Finally, from assumption \( H1 \) and since \( \alpha < (1 - \mu) \inf_{\xi \in U, \|\xi\| = 1} c(\xi) \), we obtain

\[
F^c_{\inf} D\phi^*(x) > 0. \tag{16}
\]

Then the two strict inequalities (15) and (16) imply that

\[
\min \left[ \lambda v^*(x) - D\phi^*(x) . b(x) - f(x), F^c_{\inf} D\phi^*(x) \right] > 0.
\]

Thus, we get that \( v^* \) is a strict viscosity super-solution to the new HJBI-QVI. \( \Box \)

We are now in a position to prove the comparison theorem which is an essential result to conclude.

**Theorem 6** (Comparison Theorem) Under assumption \( H1 \), if \( u \) is a bounded and continuous viscosity sub-solution to the new HJBI-QVI and \( v \) is a bounded and continuous viscosity super-solution to the new HJBI-QVI, then we have

\[
\forall x \in \mathbb{R}^n : u(x) \leq v(x).
\]

**Proof** Let \( u \) and \( v \) be a bounded and continuous viscosity sub-solution and super-solution, respectively, to the new HJBI-QVI. Our aim is to show, by contradiction, that \( u \leq v \).

We denote by \( M = \sup_{x \in \mathbb{R}^n} (u(x) - v^*(x)) \) the maximal value of \( u - v^* \), where \( v^* \) is defined as in Lemma 5. We let \( R = (\|u\|_{\infty} + \|v\|_{\infty})/\alpha \), then we have for all \( x \in \mathbb{R}^n \) such that \( \|\xi\| \geq R \),

\[
u(x) \leq \|u\|_{\infty} + (1 - \mu) \|v\|_{\infty} \leq v^*(x),
\]

that is \( u(x) \leq v^*(x) \) for all \( x \in \mathbb{R}^n \setminus \overline{B}_R(0) \) where \( \overline{B}_R(0) \) is the closed ball in \( \mathbb{R}^n \) of radius \( R \) centered at 0. Let us now assume that there exists \( \hat{x} \in B_R(0) \), the open ball, such that

\[
M = u(\hat{x}) - v^*(\hat{x}) > 0,
\]
if it is not the case, i.e., $M \leq 0$, then the proof follows immediately by letting $\mu \to 1$ and $\alpha \to 0$. The proof will now be divided into three steps:

**Step 1.** We can find $x \in B_R(0)$ and $\delta > 0$ such that

$$\sup_{x \in \overline{B}_\delta(x)} (u(x) - v^*(x)) \geq u(\overline{x}) - v^*(\overline{x}) > 0,$$

and for all $x \in \overline{B}_\delta(\overline{x})$,

$$v(x) < \mathcal{H}_{\inf}^X v(x),$$

where $\overline{B}_\delta(\overline{x})$ is the closed ball of center $\overline{x}$ and radius $\delta$.

In fact, if $v(\hat{x}) < \mathcal{H}_{\inf}^X v(\hat{x})$, then considering the continuity of $u$, $v$ and $\mathcal{H}_{\inf}^X$ we obtain the result by taking $\overline{x} = \hat{x}$.

Otherwise, we let $v(\hat{x}) \geq \mathcal{H}_{\inf}^X v(\hat{x})$, then, for some $\eta' \in V$, the result in Lemma 4 gives

$$v(\hat{x} + \eta') < \mathcal{H}_{\inf}^X v(\hat{x} + \eta'),$$

we then take $\overline{x} = \hat{x} + \eta'$ to deduce

$$\exists \delta > 0, \ \forall x \in \overline{B}_\delta(\overline{x}) : v(x) < \mathcal{H}_{\inf}^X v(x). \quad (17)$$

Furthermore, when $v^*(\hat{x}) \geq \mathcal{H}_{\inf}^X v^*(\hat{x})$, we fix $\varepsilon > 0$, then for $\alpha \in (0, 1)$ there exists $\eta' \in V$ such that

$$v^*(\hat{x}) \geq v^*(\hat{x} + \eta') + \chi(\eta') - \alpha \varepsilon, \quad (18)$$

which gives

$$u(\hat{x} + \eta') - v^*(\hat{x} + \eta') \geq u(\hat{x} + \eta') + \chi(\eta') - v^*(\hat{x}) - \alpha \varepsilon$$

$$\geq u(\hat{x}) - v^*(\hat{x}) - \alpha \varepsilon,$$

thus, for $\overline{x} = \hat{x} + \eta'$, we get

$$u(\overline{x}) - v^*(\overline{x}) \geq M - \alpha \varepsilon, \quad (19)$$

in the case where $v^*(\hat{x}) < \mathcal{H}_{\inf}^X v^*(\hat{x})$ it suffice to choose $\eta' \in V$ for which (18) holds, then we proceed analogously to get (19).

Therefore, by taking $\alpha$ sufficiently small, we get

$$\sup_{x \in \overline{B}_\delta(\overline{x})} (u(x) - v^*(x)) \geq M > 0.$$
As a consequence we consider
\[ M = \sup_{x \in \overline{B}_\delta(x)} (u(x) - v^*(x)). \]

**Step 2.** Let \( \varepsilon \) be a positive real number, \((x, y) \in \overline{B}_\delta(x) \times \overline{B}_\delta(x)\) and consider \( \psi_\varepsilon \) the test function defined as follows:

\[ \psi_\varepsilon : \overline{B}_\delta(x) \times \overline{B}_\delta(x) \to \mathbb{R} \]

\[ (x, y) \to \psi_\varepsilon(x, y) := u(x) - v^*(y) - \frac{\|x - y\|^2}{\varepsilon^2}. \]

Let \((x_m, y_m)\) be the maximal point of \( \psi_\varepsilon \) and denote \( M_{\psi_\varepsilon} = \max_{x \in \overline{B}_\delta(x) \times \overline{B}_\delta(x)} \psi_\varepsilon(x, y) = \psi_\varepsilon(x_m, y_m) \). \( M_{\psi_\varepsilon} \) exists, since on the one hand, \( \psi_\varepsilon \) is a bounded and continuous function on a bounded set, and on the other hand, it is negative in a neighborhood of the boundary \( \|x\| = \delta \) or \( \|y\| = \delta \), and, by hypothesis, positive for some \( x = y \). So, the search for the maximum can then be restricted to a compact set \( B_{\delta - \gamma}(x) \times B_{\delta - \gamma}(x) \).

Therefore
\[ u(x_m) - v^*(y_m) - \frac{\|x_m - y_m\|^2}{\varepsilon^2} \geq u(x) - v^*(y) - \frac{\|x - y\|^2}{\varepsilon^2}. \tag{20} \]

- Firstly, for \( x = x_m \), we get for all \( y \in \overline{B}_\delta(x) \),

\[ u(x_m) - v^*(y_m) - \frac{\|x_m - y_m\|^2}{\varepsilon^2} \geq u(x_m) - v^*(y) - \frac{\|x_m - y\|^2}{\varepsilon^2}, \]

then \( y_m \) is a local minimal point of \( y \to (v^* - \phi_{v^*})(y) \) with

\[ \phi_{v^*}(y) = u(x_m) - \frac{\|x_m - y\|^2}{\varepsilon^2}. \]

From the inequality (17) and since \( y_m \in \overline{B}_\delta(x) \) we get

\[ v(y_m) < \mathcal{H}_{\inf}^x v(y_m), \]

in addition \( v \) is a viscosity super-solution to the new HJBI-QVI, then by applying the result in Lemma 5, we find the following:

\[ \min \left[ \lambda v^*(y_m) - D_y \phi_{v^*}(y_m) \cdot b(y_m) - f(y_m), \mathcal{F}_{\inf}^c D_y \phi_{v^*}(y_m) \right] > 0, \]

thus, we obtain
\[ \lambda v^*(y_m) - D_y \phi_{v^*}(y_m) \cdot b(y_m) - f(y_m) > 0, \text{ and } \mathcal{F}_{\inf}^c D_y \phi_{v^*}(y_m) > 0. \tag{21} \]
Secondly, for \( y = y_m \), we get for all \( x \in B_\delta(x) \),

\[
u(x_m) - v^*(y_m) - \frac{\|x_m - y_m\|^2}{\varepsilon^2} \geq u(x) - v^*(y_m) - \frac{\|x - y_m\|^2}{\varepsilon^2},
\]

then \( x_m \) is a local maximal point of \( x \to (u - \phi_u)(x) \) with

\[
\phi_u(x) = v^*(y_m) + \frac{\|x - y_m\|^2}{\varepsilon^2}.
\]

Since \( u \) is a viscosity sub-solution to the new HJBI-QVI, we get

\[
u(x_m) - \mathcal{H}_{\inf}^u u(x_m) \leq 0,
\]

and

\[
\lambda(u(x_m) - D_x \phi_u(x_m).b(x_m) - f(x_m) \leq 0, \quad \text{or} \quad \mathcal{F}_{\inf}^{\mathcal{C}} D_x \phi_u(x_m) \leq 0. \tag{22}
\]

From (21) and (22), since \( \mathcal{F}_{\inf}^{\mathcal{C}} D_y \phi_u^*(y_m) = \mathcal{F}_{\inf}^{\mathcal{C}} D_x \phi_u(x_m) \), we get

\[
\lambda v^*(y_m) - D_x \phi_u^*(y_m).b(y_m) - f(y_m) > 0,
\]

and

\[
\lambda(u(x_m) - D_x \phi_u(x_m).b(x_m) - f(x_m) \leq 0.
\]

It follows that

\[
\lambda \left( u(x_m) - v^*(y_m) \right) + \left[ D_y \phi_u^*(y_m).b(y_m) - D_x \phi_u(x_m).b(x_m) \right] + \left( f(y_m) - f(x_m) \right) \leq 0.
\]

Then

\[
\lambda \left( u(x_m) - v^*(y_m) \right) + 2 \frac{\|x_m - y_m\|^2}{\varepsilon^2} \left( b(y_m) - b(x_m) \right) + \left( f(y_m) - f(x_m) \right) \leq 0.
\]

Since \( f \) and \( b \) are Lipschitz with constants \( C_f \) and \( C_b \), respectively, we deduce

\[
\lambda \left( u(x_m) - v^*(y_m) \right) \leq 2C_b \frac{\|x_m - y_m\|^2}{\varepsilon^2} + C_f \|x_m - y_m\|. \tag{23}
\]

**Step 3.** Proving now that \( \forall \beta > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0 : \frac{\|x_m - y_m\|^2}{\varepsilon^2} \leq \beta \), and showing the contradiction.

By taking \( x = y \) in (20) we get, for all \( x \in B_\delta(x) \), \( u(x) - v^*(x) \leq M_{\psi \varepsilon} \) then \( 0 < M_{\psi \varepsilon} \leq M_{\psi \varepsilon} \).
Let $r^2 = \|u\|_\infty + \|v^*\|_\infty$ then $0 < M^\psi_\varepsilon \leq r^2 - \frac{\|x_m - y_m\|^2}{\varepsilon^2}$, it follows that $\|x_m - y_m\| \leq \varepsilon r$.

Since $u$ is upper semi-continuous, we have

$$\forall \beta > 0, \exists \varepsilon_0 > 0, \forall \varepsilon \leq \varepsilon_0 : u(x_m) \leq u(y_m) + \beta,$$

then

$$u(x_m) - v^*(y_m) \leq u(y_m) - v^*(y_m) + \beta \leq M + \beta.$$

Using $M \leq M^\psi_\varepsilon$ we get

$$\frac{\|x_m - y_m\|^2}{\varepsilon^2} \leq \beta, \text{ and } M \leq u(x_m) - v^*(y_m),$$

then from (23) we deduce

$$\lambda (u(x_m) - v^*(y_m)) \leq 2C_b \beta + C_f \sqrt{\beta}.$$

By sending $\beta$ to 0, we get $u(x_m) - v^*(y_m) \leq 0$, from which yields for all $x \in \overline{B}_\delta(x)$,

$$u(x) - v^*(x) \leq u(x_m) - v^*(y_m) \leq 0,$$

thus we get the contradiction $M \leq 0$.

Hence, by letting $\mu \to 1$ and $\alpha \to 0$, we deduce for all $x \in \mathbb{R}^n$ the desired comparison

$$u(x) \leq v(x).$$

\[\Box\]

**Theorem 7** Under assumption $H1$, the new HJBI-QVI has a unique bounded and continuous viscosity solution.

**Proof** Assume that $v_1$ and $v_2$ are two viscosity solutions to the new HJBI-QVI. We first use $v_1$ as a bounded and continuous viscosity sub-solution and $v_2$ as a bounded and continuous viscosity super-solution and we recall the comparison theorem. Then we change the role of $v_1$ and $v_2$ to get $v_1(x) = v_2(x)$ for all $x \in \mathbb{R}^n$. \[\Box\]

Finally, we give the main result of the paper.

**Corollary 1** Under assumptions $H1$ and $H2$, the lower and upper value functions coincide and the value function $V := V^- = V^+$ of the infinite-time horizon, two-player, zero-sum, deterministic, impulse controls differential game is the unique viscosity solution to the new HJBI-QVI. \[\Box\]
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Declarations

Conflict of interest  The authors have not disclosed any competing interests.

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