TWO NEW DIFFERENT KINDS OF CONVEX DOMINATED FUNCTIONS AND INEQUALITIES VIA HERMITE-HADAMARD TYPE

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ABSTRACT. In this paper, we establish two new convex dominated function and then we obtain new Hadamard type inequalities related to this definitions.

1. INTRODUCTION

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and let \( a, b \in I \), with \( a < b \). The following inequality
\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]
is known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if \( f \) is concave.

In [1], Godunova and Levin introduced the following class of functions.

**Definition 1.** A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is said to belong to the class of \( Q(I) \) if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in (0, 1) \) satisfies the inequality;
\[
 f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}
\]
They also noted that all nonnegative monotonic and nonnegative convex functions belong to this class and also proved the following motivating result:

If \( f \in Q(I) \) and \( x, y, z \in I \), then
\[
 f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \geq 0.
\]

In [2], Dragomir et al. defined the following new class of functions.

**Definition 2.** A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is \( P \) function or that \( f \) belongs to the class of \( P(I) \), if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in [0, 1] \) satisfies the following inequality;
\[
 f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).
\]

In [2], Dragomir et al. proved the following inequalities of Hadamard type for class of \( Q(I) \) – functions and \( P \) – functions.
Theorem 1. Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then the following inequalities hold:

$$f \left(\frac{a + b}{2}\right) \leq \frac{4}{b - a} \int_a^b f(x)dx$$

and

$$\frac{1}{b - a} \int_a^b p(x) f(x)dx \leq \frac{f(a) + f(b)}{2}$$

where $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}$, $x \in [a, b]$.

Theorem 2. Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then the following inequality holds:

$$f \left(\frac{a + b}{2}\right) \leq 2 \int_a^b f(x)dx \leq 2[f(a) + f(b)].$$

In [3] and [4], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a $g$-convex-dominated function. In [5], Hwang et al. established some inequalities of Fejér type for $g$-convex-dominated functions. Finally, in [6] Kavurmacı et al. introduced several new different kinds of convex-dominated functions and then gave H-H type inequalities for this classes of functions.

The main purpose of this paper is to introduce two new convex-dominated function and then present new H-H type inequalities related to these definitions.

2. $(g, Q(I))$-Convex Dominated Functions

Definition 3. Let a nonnegative function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ belong to the class of $Q(I)$. The real function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called $(g, Q(I))$-convex dominated on $I$ if the following condition is satisfied:

$$(2.1) \quad \left| \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda} - f(\lambda x + (1 - \lambda) y) \right| \leq \frac{g(x)}{\lambda} + \frac{g(y)}{1 - \lambda} - g(\lambda x + (1 - \lambda) y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

The next simple characterisation of $(g, Q(I))$-convex dominated functions holds.

Lemma 1. Let a nonnegative function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ belong to the class of $Q(I)$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a real function. The following statements are equivalent:

1. $f$ is $(g, Q(I))$-convex dominated on $I$.
2. The mappings $g - f$ and $g + f$ are $(g, Q(I))$-convex on $I$.
3. There exist two $(g, Q(I))$-convex mappings $l, k$ defined on $I$ such that

$$f = \frac{1}{2} (l - k) \quad \text{and} \quad g = \frac{1}{2} (l + k).$$
Theorem 3. \[Q \in \{a, b\}
Theorem 3. By Definition 1 with \(\lambda = \frac{1}{2}\), we have that

\[
g (\lambda x + (1 - \lambda) y) - g (x) \leq f(x) \cdot \frac{1}{\lambda} - f(y)
\]

\[
\leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda} - f (\lambda x + (1 - \lambda) y)
\]

for all \(x, y \in I\) and \(\lambda \in (0, 1)\). The two inequalities may be rearranged as

\[
(g + f) (\lambda x + (1 - \lambda) y) \leq \frac{(g + f)(x)}{\lambda} + \frac{(g + f)(y)}{1 - \lambda}
\]

and

\[
(g - f) (\lambda x + (1 - \lambda) y) \leq \frac{(g - f)(x)}{\lambda} + \frac{(g - f)(y)}{1 - \lambda}
\]

which are equivalent to the \( (g, Q(I))\) -convexity of \(g + f\) and \(g - f\), respectively.

2\(\Leftrightarrow\)3 Let us define the mappings \(f, g\) as \(f = \frac{1}{2} (l - k)\) and \(g = \frac{1}{2} (l + k)\). Then if we sum and subtract \(f, g\), respectively, we have \(g + f = l\) and \(g - f = k\). By the condition 2 of Lemma 1, the mappings \(g - f\) and \(g + f\) are \((g, Q(I))\) -convex on \(I\), so \(l, k\) are \((g, Q(I))\) -convex mappings on \(I\) too.

Theorem 3. Let a nonnegative function \(g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) belong to the class of \(Q(I)\) and the real function \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is \((g, Q(I))\) -convex dominated on \(I\). If \(a, b \in I\) with \(a < b\) and \(f, g \in L_1[a, b]\), then one has the inequalities:

\[
\left| \frac{4}{b - a} \int_a^b f(x) \, dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{4}{b - a} \int_a^b g(x) \, dx - g \left( \frac{a + b}{2} \right)
\]

and

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b p(x) \, dx \right| \leq \frac{g(a) + g(b)}{2} - \frac{1}{b - a} \int_a^b p(x) g(x) \, dx
\]

for all \(x, y \in I\) and \(p(x)\) as in Theorem 1.

Proof. By Definition 1 with \(\lambda = \frac{1}{2}\), \(x = ta + (1-t)b\), \(y = (1-t)a + tb\) and \(t \in [0, 1]\), as the mapping \(f\) is \((g, Q(I))\) -convex dominated function, we have that

\[
\left| 2 \left[ f (ta + (1-t)b) + f ((1-t)a + tb) \right] - f \left( \frac{a + b}{2} \right) \right|
\]

\[
\leq 2 \left[ g (ta + (1-t)b) + g ((1-t)a + tb) \right] - g \left( \frac{a + b}{2} \right).
\]

Integrating the above inequality over \(t\) on \([0, 1]\), the first inequality is proved.

Since \(f\) is \((g, Q(I))\) -convex dominated using Definition 1 with \(x = a\), \(y = b\) and \(t \in [0, 1]\), we can write

\[
\left| (1-t) f(a) + tf(b) - t (1-t) f (ta + (1-t)b) \right|
\]

\[
\leq (1-t) g(a) + tg(b) - t (1-t) g (ta + (1-t)b)
\]

and

\[
\left| tf(a) + (1-t) f(b) - t (1-t) f ((1-t)a + tb) \right|
\]

\[
\leq tg(a) + (1-t) g(b) - t (1-t) g ((1-t)a + tb).
\]
Then, adding above inequalities we have
\[
[[f(a) + f(b)] - t (1 - t) [f(\lambda a + (1 - \lambda) b)] - f(\lambda x + (1 - \lambda) y)]
\leq [g(a) + g(b)] - t (1 - t) [g(\lambda a + (1 - \lambda) b)] - g(\lambda x + (1 - \lambda) y)
\]
Integrating the resulting inequality over \( t \) on \([0, 1]\), we get the second inequality. The proof is completed. \( \square \)

3. \((g, P(I))\)-convex dominated functions

**Definition 4.** Let a nonnegative function \( g : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) belong to the class of \( P(I) \). The real function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is called \((g, P(I))\)-convex dominated on \( I \) if the following condition is satisfied:

\[
(3.1) \quad T \leq [g(x) + g(y)] - g(\lambda x + (1 - \lambda) y)
\]
for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

The next simple characterisation of \((g, P(I))\)-convex dominated functions holds.

**Lemma 2.** Let a nonnegative function \( g : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) belong to the class of \( P(I) \) and \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a real function. The following statements are equivalent:

1. \( f \) is \((g, P(I))\)-convex dominated on \( I \).
2. The mappings \( g - f \) and \( g + f \) are \((g, P(I))\)-convex on \( I \).
3. There exist two \((g, P(I))\)-convex mappings \( l, k \) defined on \( I \) such that

\[
f = \frac{1}{2} (l - k) \quad \text{and} \quad g = \frac{1}{2} (l + k)
\]

**Proof.** \( 1 \Leftrightarrow 2 \) The condition \((3.1)\) is equivalent to

\[
(g(\lambda x + (1 - \lambda) y) - g(x) + g(y))
\leq [f(x) + f(y)] - f(\lambda x + (1 - \lambda) y)
\leq [g(x) + g(y)] - g(\lambda x + (1 - \lambda) y)
\]
for all \( x, y \in I \) and \( \lambda \in [0, 1] \). The two inequalities may be rearranged as

\[
(g + f)(\lambda x + (1 - \lambda) y) \leq (g + f)(x) + (g + f)(y)
\]
and

\[
(g - f)(\lambda x + (1 - \lambda) y) \leq (g - f)(x) + (g - f)(y)
\]
which are equivalent to the \((g, P(I))\)-convexity of \( g + f \) and \( g - f \), respectively.

\( 2 \Leftrightarrow 3 \) Let we define the mappings \( f, g \) as \( f = \frac{1}{2} (l - k) \) and \( g = \frac{1}{2} (l + k) \). Then if we sum and subtract \( f, g \) respectively, we have \( g + f = l \) and \( g - f = k \). By the condition 2 of Lemma 2, the mappings \( g - f \) and \( g + f \) are \((g, P(I))\)-convex on \( I \), so \( l, k \) are \((g, P(I))\)-convex mappings on \( I \) too. \( \square \)

**Theorem 4.** Let a nonnegative function \( g : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) belong to the class of \( P(I) \). The real function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is \((g, P(I))\)-convex dominated on \( I \). If \( a, b \in I \) with \( a < b \) and \( f, g \in L_1[a, b] \), then one has the inequalities:

\[
\left| \frac{2}{b - a} \int_a^b f(x) \, dx - f\left(\frac{a + b}{2}\right) \right| \leq \frac{2}{b - a} \int_a^b g(x) \, dx - g\left(\frac{a + b}{2}\right)
\]
and
\[ |f(a) + f(b)| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq |g(a) + g(b)| - \frac{1}{b-a} \int_a^b g(x) \, dx \]

for all \( x, y \in I \).

**Proof.** By Definition 4 with \( \lambda = \frac{1}{2}, x = ta + (1-t)b, y = (1-t)a + tb \) and \( t \in [0,1] \), as the mapping \( f \) is \((g, P(I))\)−convex dominated function, we have
\[
|f(ta + (1-t)b) + f((1-t)a + tb)| - f\left(\frac{a+b}{2}\right) \leq |g(ta + (1-t)b) + g((1-t)a + tb)| - g\left(\frac{a+b}{2}\right).
\]

Integrating the above inequality over \( t \) on \([0,1]\), the first inequality is proved.

Since \( f \) is \((g, P(I))\)−convex dominated using Definition 4 with \( x = a, y = b \) and \( t \in [0,1] \), we can write
\[
||f(a) + f(b)| - f(ta + (1-t)b)| 
\leq |g(a) + g(b)| - g(ta + (1-t)b).
\]

Integrating the above resulting inequality over \( t \) on \([0,1]\) we get the second inequality. The proof is completed. \( \square \)

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