Subchapter 1.1

Generative Adversarial Network: Some Analytical Perspectives

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Ever since its debut, generative adversarial networks (GANs) have attracted tremendous amount of attention. Over the past years, different variations of GANs models have been developed and tailored to different applications in practice. Meanwhile, some issues regarding the performance and training of GANs have been noticed and investigated from various theoretical perspectives. This subchapter will start from an introduction of GANs from an analytical perspective, then move on to the training of GANs via SDE approximations and finally discuss some applications of GANs in computing high dimensional MFGs as well as tackling mathematical finance problems.

1.1.1 Introduction

Generative adversarial networks (GANs), was introduced in 2014 to the machine learning community by (Goodfellow et al., 2014). The key idea behind GANs is to interpret the process of generative modeling as a competing game between two neural networks: a generator $G$ and a discriminator $D$. The generator attempts to fool the discriminator by converting random noise into sample data, while the discriminator tries to identify whether the input sample is fake or true.

Since its introduction, GANs have enjoyed great empirical success, with a wide range of applications especially in image generation and natural language
processing, including high resolution image generation (Denton et al., 2015; Radford et al., 2015), image inpainting (Yeh et al., 2016), image super-resolution (Ledig et al., 2017), visual manipulation (Zhu et al., 2016), text-to-image synthesis (Reed et al., 2016), video generation (Vondrick et al., 2016), semantic segmentation (Luc et al., 2016), and abstract reasoning diagram generation (Kulharia et al., 2017).

Despite the empirical success of GANs, there are well recognized issues in GANs training, such as the vanishing gradient when the discriminator significantly outperforms the generator (Arjovsky and Bottou, 2017), the mode collapse which is believed to be linked with gradient exploding (Salimans et al., 2016), and the challenge of GANs convergence (Barnett, 2018). To improve the performance of GANs training, various approaches have been proposed for amelioration, including different choices of network architectures, loss functions, and regularization. See for instance, a comprehensive survey on these techniques (Wiatrak et al., 2019) and the references therein. Meanwhile, there has been a growing research interest in the theoretical understanding of GANs training. (Berard et al., 2020) proposes a novel visualization method for the GANs training process through the gradient vector field of loss functions. In a deterministic GANs training framework, (Mescheder et al., 2018) demonstrates that regularization improved the convergence performance of GANs. (Conforti et al., 2020) and (Domingo-Enrich et al., 2020) analyze a generic zero-sum minimax game including that of GANs, and connect the mixed Nash equilibrium of the game with the invariant measure of Langevin dynamics.

Recently, GANs have attracted attention in the mathematical finance community, largely due to the clear analogue between simulation of financial time series data and image generation, see for instance (Wiese et al., 2019) and (Wiese et al., 2020). In response to the growing interests of GANs and its computational potential for high-dimensional control problems, stochastic games and backward-stochastic-differential equations, this note provides a gentle introduction of GANs from an analytical perspective, highlights some of the latest development of GANs training in the framework of stochastic differential equations and reviews several representatives GANs applications in asset pricing and simulations of financial time series data.

Throughout this subchapter, the following notations will be adopted, unless otherwise specified.

- The set of $k$ continuously differentiable functions over some domain $\mathcal{X} \subset \mathbb{R}^d$ is denoted by $C^k(\mathcal{X})$ for $k = 0, 1, 2, \ldots$; in particular when $k = 0$, $C^0(\mathcal{X}) = C(\mathcal{X})$ denotes the set of continuous functions.

- Let $p \geq 1$. $L^p_{loc}(\mathbb{R}^d)$ denotes the set of functions $f$ defined on $\mathbb{R}^d$ such that for any compact subset $\mathcal{X}$, $\int_{\mathcal{X}} \|f(x)\|_p^p dx < \infty$.

- Let $J = (J_1, \ldots, J_d)$ be a $d$-tuple multi-index of order $|J| = \sum_{i=1}^d J_i$. For a function $f \in L^1_{loc}(\mathbb{R}^d)$, its $J$th-weak derivative $D^f f \in L^1_{loc}(\mathbb{R}^d)$ is a function
such that for any smooth and compactly supported test function $g$, 

$$
\int_{\mathbb{R}^d} D^J f(x) g(x) dx = (-1)^{|J|} \int_{\mathbb{R}^d} f(x) \nabla^J g(x) dx.
$$

- The Sobolev space $W^{k,p}_{\text{loc}}(\mathbb{R}^d)$ is a set of functions $f$ on $\mathbb{R}^d$ such that for any $d$-tuple multi-index $J$ with $|J| \leq k$, $D^J f \in L^p_{\text{loc}}(\mathbb{R}^d)$.

### 1.1.2 Basics of GANs: an analytical view

**GANs as generative models.** GANs fall into the category of generative models. The procedure of generative modeling is to approximate an unknown probability distribution $P_r$ by constructing a class of suitable parametrized probability distributions $P_\theta$. That is, given a latent space $\mathcal{Z}$ and a sample space $\mathcal{X}$, define a latent variable $Z \in \mathcal{Z}$ with a fixed probability distribution $P_z$ and a family of functions $G_\theta : \mathcal{Z} \rightarrow \mathcal{X}$ parametrized by $\theta$. Then $P_\theta$ is defined as the probability distribution of $G_\theta(Z)$, i.e., $\text{Law}(G_\theta(Z))$.

To distinguish from other generative models, GANs consist of two competing components: a generator $G$ and a discriminator $D$. In particular, the generator $G$ is implemented using a neural network (NN), i.e., function approximators via specific graph structures and network architectures, and it is denoted by $G = G_\theta$ as a parametrized function. Meanwhile, another neural network for the discriminator $D$ assigns a score between 0 to 1 to an input sample, either from the true distribution $P_r$ or the approximated distribution $P_\theta = \text{Law}(G_\theta(Z))$; denote the parametrized $D$ as $D_\omega$. A higher score from the discriminator $D$ would indicate that the sample is more likely to be from the true distribution. GANs are trained by optimizing $G$ and $D$ iteratively until $D$ can no longer distinguish between samples from $P_r$ and those from $P_\theta$.

**GANs as minimax games.** Mathematically, GANs are minimax games as

$$
\min_G \max_D \{ \mathbb{E}_{X \sim P_r}[\log D(X)] + \mathbb{E}_{Z \sim P_z}[\log(1 - D(G(Z)))] \}.
$$

In particular, fixing $G$ and optimizing for $D$ in (1.1.1), the optimal discriminator would be 

$$
D^*_G(x) = \frac{p_r(x)}{p_r(x) + p_\theta(x)},
$$

where $p_r$ and $p_\theta$ are density functions of $P_r$ and $P_\theta$ respectively. Plugging the above $D^*_G$ back to Equation (1.1.1), the following equation holds,

$$
\min_G \left\{ \mathbb{E}_{X \sim P_r} \left[ \log \frac{p_r(X)}{p_r(X) + p_\theta(X)} \right] + \mathbb{E}_{Y \sim P_\theta} \left[ \log \frac{p_\theta(Y)}{p_r(Y) + p_\theta(Y)} \right] \right\} = - \log 4 + 2JS(P_r, P_\theta).
$$
That is to say, training of GANs with Equation (1.1.1) being the objective is equivalent to minimizing Jensen-Shannon (JS) divergence between $P_r$ and $P_\theta$. In other words, through optimization over discriminators, GANs are essentially minimizing proper divergences between the true distribution and the generated distribution over some sample space $\mathcal{X}$.

**GANs and optimal transport.** This view of GANs as an optimization problem with an appropriate divergence function has been instrumental for addressing the instability of GANs training. Variants of GANs with different divergences have been proposed to improve the performance of GANs. For instance, [Nowozin et al., 2016] and [Nock et al., 2017] extend the JS divergence in [Goodfellow et al., 2014] to a broader class of f-divergence. This extension provides the flexibility of choosing various $f$ functions for the loss function in GANs training. [Srivastava et al., 2019] explores scaled Bregman divergence to resolve the issue of support mismatch between $P_r$ and $P_\theta$ in the use of f-divergence and Bregman divergence. This is achieved through introducing a noisy base measure $\mu$ such that $\mu$ is a mixture of $P_r$ and $P_\theta$ convolved with some Gaussian distributions. [Arjovsky et al., 2017] adopts Wasserstein-1 distance that enjoys higher smoothness with respect to the model parameters and consequently leads to a much more stable training of GANs. [Guo et al., 2017] proposes relaxed Wasserstein divergence by generalizing Wasserstein-1 distance with Bregman cost functions to first bypass the restriction on data information geometry in WGAN and achieve faster training. [Salimans et al., 2018] and [Sanjabi et al., 2018] utilize the Sinkhorn loss instead of optimal transport type of loss by interpolating with energy distance and adding entropy regularization. This can significantly reduce the computational burden of optimal transport cost and increase stability of training.

The flexibility of choosing appropriate divergence functions, especially the development of WGANs, leads to the natural connection between GANs and optimal transport problems, established in [Cao et al., 2020a], which identifies sufficient conditions to recast GANs in the framework of optimal transport.

The idea behind this link is intuitive: GANs as generative models are minimax games with the goal to minimize the “error” of the generated sample data against the true sample data; this error is measured under appropriate divergence functions between the true distribution and the generated distribution. Now if this error is viewed as a cost of transporting/fitting the generated distribution into the true distribution, GANs become optimal transport problems.

Indeed, this connection is explicit in the case of WGANs, via the Kantovich duality.

**Theorem 1.1.1.** Suppose that $P_r \in L^1(\mathcal{X})$ and $G \in L^1(P_z)$ where

$$L^1(P_z) = \left\{ f : \mathcal{Z} \to \mathbb{R} : \int_{\mathcal{Z}} |f(z)| P_z(dz) < \infty \right\}.$$

WGAN is an optimal transport problem between $\text{Law}(G(Z))$ and $P_r$. 
As seen in (Cao et al., 2020a), this connection goes beyond the framework of WGANs. Indeed, take any Polish space $\mathcal{X}$ with metric $l$, then $\mathcal{X} \times \mathcal{X}$ is also a Polish space with metric $l'$. Denote $\mathcal{P} (\mathcal{X})$ as the set of all probability distributions over the sample space $\mathcal{X}$, define a generic divergence function

$$W : \mathcal{P} (\mathcal{X}) \times \mathcal{P} (\mathcal{X}) \mapsto \mathbb{R}^+,$$

and take a class of GANs with this divergence $W$. If $W$ can be written as an appropriate optimal cost $W_c$ and if such an optimal transport problem has a duality representation, then GANs model is an transport problem: the discriminator locates the best coupling among $\Pi_G$ under a given $G$, and the generator refines the set of possible couplings $\Pi_G$ to minimize the divergence.

There are earlier studies connecting GANs and optimal transport problems, by different approaches and from different perspectives. For instance, (Salimans et al., 2018) defines a novel divergence called the minibatch energy distance, based on solutions of three associated optimal transport problems. This new divergence is then used to replace the JS divergence for the vanilla GANs. Note that this minibatch energy distance itself is not an optimal transport cost. In (Lei et al., 2019), a geometric interpretation of Wasserstein GANs (WGANs) from the perspective of optimal transport is provided: the latent random variable from the latent space is mapped to the sample space via an optimal mass transport so that the resulted distribution can minimize its Wasserstein distance against the true distribution.

**GANs and MFGs.** In addition to this relation between GANs and optimal transport problems, (Cao et al., 2020a) further associates GANs with mean-field games (MFGs), and design a new algorithm for computing MFGs. This connection between MFGs and GANs can be seen conceptually through the following Table 1.1.1.

| GANs                                      | MFGs                                      |
|-------------------------------------------|-------------------------------------------|
| **Generator G**                           | **Generator G**                           | NN for approximating the map $G : Z \mapsto \mathcal{X}$ |
| **Characterization**                      | Sample data                              | FF equation for consistency |
| of $\mathbb{P}_{r}$                      |                                          |                            |
| **Discriminator D**                       | **Discriminator D**                       | NN for measuring differential residual from the FF equation |
|                                          |                                          | $\mathbb{P}_{\theta}$ and $\mathbb{P}_{r}$ |

Evidently, there is more than one way to establish this connection between MFGs and GANs. Alternatively, one can switch the roles of the generator and
discriminator and view the mean-field term as a generator and the value function as a discriminator.

For certain classes of MFGs, such an interpretation of MFGs as GANs may be explicit. For instance, take the class of periodic MFGs from (Cirant and Nurbekyan, 2018) on flat torus $T^d$ and a finite time horizon $[0,T]$. Such an MFG minimizes the following cost,

$$J_m(t, \alpha) = \mathbb{E} \left[ \int_t^T L(X_\alpha^t, \alpha(X_\alpha^t)) + f(X_\alpha^t, m(t, X_\alpha^t))dt \right], \ t \in [0,T], \ (1.1.2)$$

where $X_\alpha = (X_\alpha^t)_t$ is a $d$-dimensional process with dynamics

$$dX_\alpha^t = \alpha(X_\alpha^t)dt + \sqrt{2\epsilon}dW_t.$$ 

Here $\alpha$ is a control policy, $L$ and $f$ constitute the running cost and $m(t, \cdot)$, for $t \in [0,T]$, denotes the probability density of $X_\alpha^t$ at time $t$.

Now, consider the convex conjugate of the running cost $L$, namely, $H_0(x, p) = \sup_{\alpha \in \mathbb{R}^d} \{ \alpha \cdot p - L(x, \alpha) \}$, and denote $F(x, m) = \int m f(x, z)dz$. Then this class of MFGs can be characterized by the following coupled PDE system as illustrated in (Cirant and Nurbekyan, 2018),

$$\begin{cases}
-\partial_s u - \epsilon \Delta_x u + H_0(x, \nabla_x u) = f(x, m), \\
\partial_s m - \epsilon \Delta_x m - \text{div}(m \nabla_p H_0(x, \nabla u)) = 0, \\
m > 0, m(0, \cdot) = m^0(\cdot), u(T, \cdot) = u^T(\cdot).
\end{cases} \quad (1.1.3)$$

Here the first equation is a Hamilton-Jacobi-Bellman (HJB) equation governing the value function and the second is a Fokker-Planck (FP) equation governing the evolution of the optimally controlled state process, with $m^0$ and $u^T$ the initial functions for $m(t, \cdot)$ and $u(t, \cdot)$, respectively.

Note that this system of equations (1.1.3) is equivalent to the following minimax game

$$\inf_{u \in C^2([0,T] \times T^d)} \sup_{m \in C^2([0,T] \times T^d)} \Phi(m, u), \quad (1.1.4)$$

where

$$\Phi(m, u) = \int_0^T \int_{T^d} \left[ m(-\partial_t u - \epsilon \Delta_x u) + mH_0(x, \nabla_x u) - F(x, m) \right] dx dt$$

$$+ \int_{T^d} \left[ m(T, x)u(T, x) - m^0(x)u(0, x) - m(x, T)u^T(x) \right] dx.$$ 

Therefore, by (1.1.4), the connection between GANs and MFGs is transparent.

Having established the interpretation of MFGs as GANs, the immediate question to ask is whether GANs can be understood as MFGs. (Cao et al., 2020a) further shows that GANs can also be seen as MFGs, under the Pareto Optimality criterion.
Theorem 1.1.2. GANs in (Goodfellow et al., 2014) are MFGs under the Pareto Optimality criterion, assuming that the latent variables $Z$ and true data $X$ are both i.i.d. sampled, respectively, with $E[\log(D(X))]$, $E[\log(1 - D(G(Z)))] < \infty$ for all possible $D$ and $G$.

The above theorem shows that the theoretical framework of GANs in (Goodfellow et al., 2014) can be seen as MFGs under Pareto Optimality criterion, where the generator network is an representative player of infinitely many identical players working in collaboration to defeat the discriminator. In practical training of GANs, however, only finitely many data points, i.e., $N$ latent variables $\{Z_i\}_{i=1}^N$ and $M$ samples from the unknown true distribution $\{X_j\}_{j=1}^M$, are available and therefore GANs in practice can be interpreted as $N$-player cooperative games with players being interchangeable and hence adopting the same strategy. Here, $Z_i \sim \mathbb{P}_z$, $X_j \sim \mathbb{P}_r$ and $\{X_j\}_{j=1}^M \perp \{Z_i\}_{i=1}^N$. The state process for player $i$ is given by the feedforward process within its generator network $G_i$, with the initial layer being $Z_i$ and the final layer being the generated sample $G_i(Z_i)$. Since the players are interchangeable and collaborating, a common generator network $G$ is adopted by all $N$ players to form a symmetric strategy profile $S^G$ for the $N$-player game. These players face with a discriminator $D^{N,M} \in \mathcal{D} = \{D | D : \mathcal{X} \rightarrow [0, 1]\}$ that favors the true samples $X_j$’s. In particular, the collective cost for the $N$ players of choosing a common generator $G$ is given by

$$J^{N,M}(S^G; \{Z_i\}_{i=1}^N, \{X_j\}_{j=1}^M) = \max_{D \in \mathcal{D}} \frac{\sum_{i=1}^N \sum_{j=1}^M \log [D(X_j)] + \log [1 - D(G(Z_i))]}{N \cdot M}$$

and $D^{N,M}$ is given by

$$D^{N,M} = D^{N,M}(\cdot; G, \{Z_i\}_{i=1}^N, \{X_j\}_{j=1}^M) = \operatorname{argmax}_{D \in \mathcal{D}} \frac{\sum_{i=1}^N \sum_{j=1}^M \log [D(X_j)] + \log [1 - D(G(Z_i))]}{N \cdot M}.$$ 

Definition 1.1.3 (Pareto optimality). A strategy profile $S^{G^{N,*}}$ among all possible symmetric strategy profiles is said to be Pareto optimal if for any symmetric strategy profile $S^G$,

$$J^{N,M}(S^{G^{N,*}}; \{Z_i\}_{i=1}^N, \{X_j\}_{j=1}^M) \leq J^{N,M}(S^G; \{Z_i\}_{i=1}^N, \{X_j\}_{j=1}^M).$$

Before characterizing $G^{N,*}$ and $D^{N,M}$, the cost $J^{N,M}$ using the empirical measures $\delta_r^M = \frac{1}{M} \sum_{j=1}^M \delta x_j$ and $\delta_G^N = \frac{1}{N} \sum_{i=1}^N \delta x(Z_i)$ can be rewritten as follows,

$$J^{N,M}(S^G; \{Z_i\}_{i=1}^N, \{X_j\}_{j=1}^M) = \max_{D \in \mathcal{D}} \int_{\mathcal{X}} \log D(x) \delta_r^M(x) + \log(1 - D(x)) \delta_G^N(x) dx.$$ 

Then $D^{N,M}$ and $G^{N,*}$ are naturally characterized by the two empirical distributions.
Proposition 1.1.1. Under a given $G$, a particular $D^{N,M}$ is given by

$$D^{N,M}(x) = \begin{cases} \frac{\delta^M_r(x)}{\delta^M_r(x) + \delta^N_G(x)}, & x \in \{G(Z_1), \ldots, G(Z_N), X_1, \ldots, X_M\}; \\ \frac{1}{2}, & \text{otherwise}; \end{cases}$$

in fact, for $x \not\in \{G(Z_1), \ldots, G(Z_N), X_1, \ldots, X_M\}$, $D^{N,M}(x)$ can take any value in $[0, 1]$.

Theorem 1.1.2. The set of possible $G^{N,*}$s is given by

$$G^{N,*} = \left\{ G \in G : \delta^N_G = \delta^M_r \right\},$$

provided that $G^{N,*} \neq \emptyset$.

The above results show that in practice training of GANs over finitely many samples, the generator can recover the empirical distribution of true samples at best. Moreover, the non-emptiness of $G^{N,*}$ highly depends on the design of $G$ network architecture. This will be discussed in detail in the Section 1.1.3. Theoretically, however, $N$ and $M$ can be taken to infinity, leading the $N$-player cooperative games into MFGs with Pareto optimality criterion as stated in Theorem 1.1.2. Here the mean-field information is given by $G\#P_z = \lim_{N \to \infty} \delta^N_G$ and the convergence of $N$-player games to MFGs is guaranteed by the law of large numbers and the continuous mapping theorem.

1.1.3 GANs Training

In the previous section, it has been pointed out that the success of GANs training depends on the design of network architecture. Apart from choosing a proper network architecture, there have been many practical methods to improve the performance of GANs training. This section is intended to provide mathematical explanation for these practical methods by analyzing GANs training via stochastic differential equation approximation. Before going into detail about GANs training, it is worth revisiting the objective of GANs.

Equilibrium of GANs training. GANs are trained by optimizing $G$ and $D$ iteratively until $D$ can no longer distinguish between true samples and generated samples. Recall that $G_\theta$ denotes the generator parametrized by the neural network with the set of parameters $\theta \in \mathbb{R}^{d_\theta}$, and $D_\omega$ denotes the discriminator parametrized by the other neural network with the set of parameters $\omega \in \mathbb{R}^{d_\omega}$. Under a fixed network architecture, the parametrized version of GANs training is to find

$$v_{GAN}^{GAN} = \min_{\theta} \max_{\omega} L_{GAN}(\theta, \omega),$$

where $L_{GAN}(\theta, \omega) = \mathbb{E}_{X \sim P_r}[\log D_\omega(X)] + \mathbb{E}_{Z \sim P_z}[\log(1 - D_\omega(G_\theta(Z)))]$. (1.1.5)
1.1.3. GANS TRAINING

Remark 1.1.3. From a game theory viewpoint, the objective in (1.1.5), if attained, is in fact the upper value of the two-player zero-sum game of GANs. Meanwhile, the lower value of the game is given by the following maximin problem,

\[ v_{L}^{GAN} = \max_{\omega} \min_{\theta} L_{GAN}(\theta, \omega). \] (1.1.6)

Clearly the following relation holds,

\[ v_{L}^{GAN} \leq v_{U}^{GAN}. \] (1.1.7)

Moreover, if there exists a pair of parameters \((\theta^*, \omega^*)\) such that both (1.1.5) and (1.1.6) are attained, then \((\theta^*, \omega^*)\) is a Nash equilibrium of this two-player zero-sum game. Indeed, if \(L_{GAN}\) is convex in \(\theta\) and concave in \(\omega\), then there is no duality gap hence the equality in (1.1.7) holds by the minimax theorem (see \(\text{Von Neumann, 1959}\) and \(\text{Sion, 1958}\)).

It is worth noting that conditions for such an equality in (1.1.7) is usually not satisfied in many common GANs models, as observed in \(\text{Zhu et al., 2020}\) and analyzed in \(\text{Guo and Mounjid, 2020}\).

GANs training via SGD. As in most deep learning models, stochastic gradient descent (SGD) (or one of its variants) is a standard approach for solving the optimization problem in GANs training. Accordingly, the evolution of parameters of \(\theta\) and \(\omega\) in (1.1.5) by SGD from current step \(t\) to the next step \(t + 1\) is

\[ \omega_{t+1} = \omega_t + \alpha_d \nabla_{\omega} L_{GAN}(\theta_t, \omega_t), \]
\[ \theta_{t+1} = \theta_t - \alpha_g \nabla_{\theta} L_{GAN}(\theta_t, \omega_{t+1}). \] (1.1.8)

Here the \(\alpha_d\) and \(\alpha_g\) denote the step sizes of updating the discriminator and the generator, respectively. Evolution (1.1.8) corresponds to the alternating updating scheme of the algorithm in \(\text{Goodfellow et al., 2014}\) where at each iteration, the discriminator is updated before the generator. One of the main challenges for GANs training is the convergence of such an alternating SGD.

GANs training and SDEs approximation. GANs training is performed on a data set \(\mathcal{D} = \{(z_i, x_j)\}_{1 \leq i \leq N, 1 \leq j \leq M}\), where \(\{z_i\}_{i=1}^{N}\) are sampled from \(\mathbb{P}_z\) and \(\{x_j\}_{j=1}^{M}\) are real image data following the unknown distribution \(\mathbb{P}_r\). The objective of GANs is to solve the following minimax problem

\[ \min_{\theta} \max_{\omega} \Phi(\theta, \omega), \] (1.1.9)

for some cost function \(\Phi\), with \(\Phi\) of a separable form

\[ \Phi(\theta, \omega) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{M} J(D_\omega(x_j), D_\omega(G_\theta(z_i)))}{N \cdot M}. \] (1.1.10)

When the stochastic gradient algorithm (SGA) is performed to solve the minimax problem (1.1.9), the full gradients of \(\Phi\) with respect to \(\theta\) and \(\omega\), denoted
by $g_\theta$ and $g_\omega$, respectively, are estimated over a mini-batch $B$ of batch size $B$, denoted by $g_\theta^B$ and $g_\omega^B$.

Let $\eta_t^\theta > 0$ and $\eta_t^\omega > 0$ be the learning rates at iteration $t = 0, 1, 2, \ldots$, for $\theta$ and $\omega$ respectively, then solving the minimax problem (1.1.9) with SGA and alternating parameter update implies descent of $\theta$ along $g_\theta$ and ascent of $\omega$ along $g_\omega$ at each iteration; within each iteration, the minibatch gradient for $\theta$ and $\omega$ are calculated on different batches. In order to emphasize this difference, $\bar{\theta}$ represents the minibatch for $\theta$ and $B$ for that of $\omega$, with $\bar{\theta}^{i.i.d.} \sim B$. The one-step update can be written as follows.

\[
\begin{align*}
\omega_{t+1} &= \omega_t + \eta_t^\omega g_\omega^B(\theta_t, \omega_t), \\
\theta_{t+1} &= \theta_t - \eta_t^\theta g_\theta^B(\theta_t, \omega_{t+1}).
\end{align*}
\]

(ALT)

Some practical training of GANs uses simultaneous parameter update between the discriminator and the generator, corresponding to a similar yet subtly different form

\[
\begin{align*}
\omega_{t+1} &= \omega_t + \eta_t^\omega g_\omega(\theta_t, \omega_t), \\
\theta_{t+1} &= \theta_t - \eta_t^\theta g_\theta(\theta_t, \omega_t).
\end{align*}
\]

(SML)

For the ease of exposition, the learning rates are assumed to be constant $\eta_t^\theta = \eta_t^\omega = \eta$, with $\eta$ viewed as the time interval between two consecutive parameter updates. In [Guo and Mounjid, 2020], the optimal (variable) learning rate for GANs training is studied under a stochastic control framework.

Let $g_{\theta}^{i,j}$ and $g_{\omega}^{i,j}$ denote $\nabla_\theta J(D_\omega(x_j), D_\omega(G_\theta(z_i)))$ and $\nabla_\omega J(D_\omega(x_j), D_\omega(G_\theta(z_i)))$, respectively, and define the following covariance matrices

\[
\Sigma_\theta(\theta, \omega) = \frac{\sum_i \sum_j [g_{\theta}^{i,j}(\theta, \omega) - g_\theta(\theta, \omega)][g_{\theta}^{i,j}(\theta, \omega) - g_\theta(\theta, \omega)]^T}{N \cdot M},
\]

\[
\Sigma_\omega(\theta, \omega) = \frac{\sum_i \sum_j [g_{\omega}^{i,j}(\theta, \omega) - g_\omega(\theta, \omega)][g_{\omega}^{i,j}(\theta, \omega) - g_\omega(\theta, \omega)]^T}{N \cdot M},
\]

then as the batch size $B$ gets sufficiently large, the classical central limit theorem leads to the following approximation of (ALT),

\[
\begin{align*}
\omega_{t+1} &= \omega_t + \eta g_\omega(\theta_t, \omega_t) \approx \omega_t + \eta g_\omega(\theta_t, \omega_t) + \frac{\eta}{\sqrt{B}} \sum_{i=1}^B (\theta_t, \omega_t) Z_i^1, \\
\theta_{t+1} &= \theta_t - \eta g_\theta(\theta_t, \omega_{t+1}) \approx \theta_t - \eta g_\theta(\theta_t, \omega_{t+1}) + \frac{\eta}{\sqrt{B}} \sum_{i=1}^B (\theta_t, \omega_{t+1}) Z_i^2,
\end{align*}
\]

(1.1.11)

with independent random variables $Z_i^1 \sim N(0, I_{d_\omega})$ and $Z_i^2 \sim N(0, I_{d_\theta})$, $t = 0, 1, 2, \ldots$.

If ignoring the difference between $t$ and $t + 1$, then the approximation could be written in the following form

\[
\begin{align*}
d\begin{pmatrix} \Theta_t \\ W_t \end{pmatrix} &= \begin{pmatrix} -g_\theta(\Theta_t, W_t) \\ g_\omega(\Theta_t, W_t) \end{pmatrix} dt + \sqrt{2 \beta - 1} \begin{pmatrix} \Sigma_\theta(\Theta_t, W_t) & 0 \\ 0 & \Sigma_\omega(\Theta_t, W_t) \end{pmatrix}^{1/2} dW_t, \\
&= \begin{pmatrix} \Sigma_\theta(\Theta_t, W_t) & 0 \\ 0 & \Sigma_\omega(\Theta_t, W_t) \end{pmatrix}^{1/2} dW_t,
\end{align*}
\]

(1.1.12)
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with \( \beta = \frac{2\eta}{\gamma} \) and \( \{W_t\}_{t \geq 0} \) be standard \( (d_\theta + d_\omega) \)-dimensional Brownian motion. This would be the approximation for GANs training of (SML).

Taking the subtle difference between \( t \) and \( t + 1 \) into consideration and thus the interaction between the generator and the discriminator, the approximation for the GANs training process of (ALT) should be

\[
\begin{align*}
\frac{d}{dt}(\Theta_t, W_t) &= \left[ \begin{array}{cc}
-g_\theta(\Theta_t, W_t) & -g_\omega(\Theta_t, W_t) \\
g_\omega(\Theta_t, W_t) & g_\omega(\Theta_t, W_t)
\end{array} \right] dt \\
&+ \frac{\eta}{2} \left[ \begin{array}{cc}
\nabla_\theta g_\theta(\Theta_t, W_t) & -\nabla_\omega g_\theta(\Theta_t, W_t) \\
-\nabla_\theta g_\omega(\Theta_t, W_t) & -\nabla_\omega g_\omega(\Theta_t, W_t)
\end{array} \right] \\
&+ \sqrt{2\beta - 1} \left[ \begin{array}{cc}
0 & \Sigma_\omega(\Theta_t, W_t)^{1/2} \\
\Sigma_\omega(\Theta_t, W_t)^{1/2} & 0
\end{array} \right] dW_t.
\end{align*}
\]

(1.1.13)

Equations (1.1.12) and (1.1.13) can be written in more compact forms

\[
\begin{align*}
\frac{d}{dt}(\Theta_t, W_t) &= b_0(\Theta_t, W_t) dt + \sigma(\Theta_t, W_t) dW_t, \quad \text{(SML-SDE)} \\
\frac{d}{dt}(\Theta_t, W_t) &= b(\Theta_t, W_t) dt + \sigma(\Theta_t, W_t) dW_t. \quad \text{(ALT-SDE)}
\end{align*}
\]

where \( b(\theta, \omega) = b_0(\theta, \omega) + \eta b_1(\theta, \omega) \), with

\[
\begin{align*}
b_0(\theta, \omega) &= \left[ \begin{array}{c}
-g_\theta(\theta, \omega) \\
g_\omega(\theta, \omega)
\end{array} \right], \quad \text{(1.1.14)} \\
b_1(\theta, \omega) &= \frac{1}{2} \begin{pmatrix}
\nabla_\theta g_\theta(\theta, \omega) & -\nabla_\omega g_\theta(\theta, \omega) \\
-\nabla_\theta g_\omega(\theta, \omega) & -\nabla_\omega g_\omega(\theta, \omega)
\end{pmatrix} \\
&= -\frac{1}{2} \nabla b_0(\theta, \omega) b_0(\theta, \omega) - \begin{pmatrix}
\nabla_\omega g_\theta(\theta, \omega) g_\omega(\theta, \omega) \\
0
\end{pmatrix}, \quad \text{(1.1.15)}
\end{align*}
\]

and \( \sigma(\theta, \omega) = \sqrt{2\beta - 1} \left[ \begin{array}{cc}
\Sigma_\omega(\Theta_t, W_t)^{1/2} & 0 \\
0 & \Sigma_\omega(\Theta_t, W_t)^{1/2}
\end{array} \right] \). \quad \text{(1.1.16)}

Note the term \(-\frac{\eta}{2} \left( \nabla_\omega g_\theta(\theta, \omega) g_\omega(\theta, \omega) \right)\) for (ALT-SDE), which highlights the interaction between the generator and the discriminator in GANs training process.

In (Cao and Guo 2020), it is shown that these coupled SDEs are indeed the continuous-time approximations of GANs training processes, with precise error bound analysis, where the approximations are under the notion of weak approximation as in (Li et al. 2019).

**Theorem 1.1.4.** Fix an arbitrary time horizon \( \mathcal{T} > 0 \) and take the learning rate \( \eta \in (0, 1 \wedge \mathcal{T}) \) and the number of iterations \( N = \left\lceil \frac{\mathcal{T}}{\eta} \right\rceil \). Suppose that

1. \( g_{\omega}^{i,j} \) is twice continuously differentiable, and \( g_{\theta}^{i,j} \) and \( g_{\omega}^{i,j} \) are Lipschitz, for any \( i = 1, \ldots, N \) and \( j = 1, \ldots, M \);
1. $\Phi$ is of $C^3(\mathbb{R}^{d_\theta+d_\omega})$, $\Phi \in W^{4,1}_{loc}(\mathbb{R}^{d_\theta+d_\omega})$, and for any multi-index $J = (J_1, \ldots, J_{d_\theta+d_\omega})$ with $|J| = \sum_{i=1}^{d_\theta+d_\omega} J_i \leq 4$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$|D^J \Phi(\theta, \omega)| \leq k_1 \left(1 + \left\|\left(\begin{array}{c} \theta \\ \omega \end{array}\right)\right\|_2^{2k_2}\right)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $\omega \in \mathbb{R}^{d_\omega}$ almost everywhere;

2.c $(\nabla_\theta g_\theta) g_\theta$, $(\nabla_\omega g_\theta) g_\theta$, $(\nabla_\theta g_\omega) g_\theta$ and $(\nabla_\omega g_\omega) g_\omega$ are all Lipschitz.

Then, given any initialization $\theta_0 = \theta$ and $\omega_0 = \omega$, for any test function $f \in C^3(\mathbb{R}^{d_\theta+d_\omega})$ such that for any multi-index $J$ with $|J| \leq 3$ there exist $k_1, k_2 \in \mathbb{N}$ satisfying

$$|\nabla^J f(\theta, \omega)| \leq k_1 \left(1 + \left\|\left(\begin{array}{c} \theta \\ \omega \end{array}\right)\right\|_2^{2k_2}\right),$$

the following weak approximation holds

$$\max_{t=1, \ldots, N} |E f(\theta_t, \omega_t) - E f(\Theta_{t\eta}, \mathcal{W}_{t\eta})| \leq C \eta^2 \quad (1.1.17)$$

for constant $C \geq 0$, where $(\theta_t, \omega_t)$ and $(\Theta_{t\eta}, \mathcal{W}_{t\eta})$ are given by (ALT) and (ALT-SDE), respectively.

**Theorem 1.1.5.** Fix an arbitrary time horizon $T > 0$, take the learning rate $\eta \in (0, 1 \land T)$ and the number of iterations $N = \left\lceil\frac{T}{\eta}\right\rceil$. Suppose

2.a $\Phi(\theta, \omega)$ is continuously differentiable, $\Phi \in W^{3,1}_{loc}(\mathbb{R}^{d_\theta+d_\omega})$ and for any multi-index $J = (J_1, \ldots, J_{d_\theta+d_\omega})$ with $|J| = \sum_{i=1}^{d_\theta+d_\omega} J_i \leq 3$, there exist $k_1, k_2 \in \mathbb{N}$ such that $D^J \Phi$ satisfies

$$|D^J \Phi(\theta, \omega)| \leq k_1 \left(1 + \left\|\left(\begin{array}{c} \theta \\ \omega \end{array}\right)\right\|_2^{2k_2}\right)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $\omega \in \mathbb{R}^{d_\omega}$ almost everywhere;

2.b $g_{\theta}^{i,j}$ and $g_{\omega}^{i,j}$ are Lipschitz for any $i = 1, \ldots, N$ and $j = 1, \ldots, M$.

Then, given any initialization $\theta_0 = \theta$ and $\omega_0 = \omega$, for any test function $f \in C^2(\mathbb{R}^{d_\theta+d_\omega})$ such that for any multi-index $J$ with $|J| \leq 2$ there exist $k_1, k_2 \in \mathbb{N}$ satisfying

$$|\nabla^J f(\theta, \omega)| \leq k_1 \left(1 + \left\|\left(\begin{array}{c} \theta \\ \omega \end{array}\right)\right\|_2^{2k_2}\right),$$

then the following weak approximation holds

$$\max_{t=1, \ldots, N} |E f(\theta_t, \omega_t) - E f(\Theta_{t\eta}, \mathcal{W}_{t\eta})| \leq C \eta \quad (1.1.18)$$

for constant $C \geq 0$, where $(\theta_t, \omega_t)$ and $(\Theta_{t\eta}, \mathcal{W}_{t\eta})$ are given by (SML) and (SML-SDE), respectively.
The above theorems from (Cao and Guo, 2020) make it possible to analyze the convergence of GANs training via the invariant measure of the SDEs.

Convergence of GANs training via invariant measure of SDEs. The invariant measure here in the context of GANs training can be interpreted in the following sense. First of all, the invariant measure $\mu^*$ describes the joint probability distribution of the generator and discriminator parameters $(\Theta^*, W^*)$ in equilibrium. For instance, if the training process converges to the unique minimax point $(\theta^*, \omega^*)$ for $\min_\theta \max_\omega \Phi(\theta, \omega)$, the invariant measure is the Dirac mass at $(\theta^*, \omega^*)$. Having the distribution of $(\theta^*, \omega^*)$, especially the marginal distribution of $\Theta^*$, helps to characterize the probability distribution of the generated samples, $\text{Law}(G_{\Theta^*}(Z))$, and this distribution is in particular useful in the evaluation of GANs performance via metrics such as inception score and Fréchet inception distance. (See (Heusel et al., 2017; Salimans et al., 2016) for more details on these metrics). Besides, from a game perspective, the pair of conditional laws $(\text{Law}(\Theta^*|W^*), \text{Law}(W^*|\Theta^*))$ can be seen as the mixed strategies adopted by the generator and discriminator in equilibrium, respectively.

Theorem 1.1.4. Assume the following conditions hold for (ALT-SDE).

3.a both $b$ and $\sigma$ are bounded and smooth and have bounded derivatives of any order;

3.b there exist some positive real numbers $r$ and $M_0$ such that for any $(\theta^T \omega) \in \mathbb{R}^{d_\theta+d_\omega}$,

$$
(\theta^T \omega) b(\theta, \omega) \leq -r \left\| (\theta^T \omega) \right\|_2^2, \text{ if } \left\| (\theta^T \omega) \right\|_2^2 \geq M_0;
$$

3.c $A$ is uniformly elliptic, i.e., there exists $l > 0$ such that for any $(\theta^T \omega), (\theta'^T \omega') \in \mathbb{R}^{d_\theta+d_\omega}$,

$$
(\theta'^T \omega')^T \sigma(\theta, \omega) \sigma(\theta, \omega)^T (\theta'^T \omega') \geq l \left\| (\theta'^T \omega') \right\|_2^2,
$$

then (ALT-SDE) admits a unique invariant measure $\mu^*$ with an exponential convergence rate.

Similar results hold for the invariant measure of (SML-SDE) with $b$ replaced by $b_0$.

The assumptions 1.a-1.c, 2.a-2.b and 3.a for the regularity conditions of the drift, the volatility, and the derivatives of loss function $\Phi$, are more than mathematical convenience. They are essential constraints on the growth of the loss function with respect to the model parameters, necessary for avoiding the explosive gradient encountered in the training of GANs. Moreover, these conditions put restrictions on the gradients of the objective functions with respect to the parameters. By the chain rule, it requires both careful choices of network structures as well as particular forms of the loss function $\Phi$. 
Dynamics of training loss and FDR. To have a more quantifiable characteristic of the convergence of GANs training, the analysis of the training loss dynamics reveals a fluctuation-dissipation relation (FDR) for the GANs training.

**Theorem 1.1.5.** Assume the existence of an invariant measure $\mu^*$ for $(\text{ALT-SDE})$, then

$$
\mathbb{E}_{\mu^*} \left[ \| \nabla_{\theta} \Phi(\Theta^*, W^*) \|^2_2 - \| \nabla_{\omega} \Phi(\Theta^*, W^*) \|^2_2 \right] = \beta^{-1} \mathbb{E}_{\mu^*} \left[ Tr \left( \Sigma_{\theta}(\Theta^*, W^*) \nabla_{\theta}^2 \Phi(\Theta^*, W^*) + \Sigma_{\omega}(\Theta^*, W^*) \nabla_{\omega}^2 \Phi(\Theta^*, W^*) \right) \right].
$$

The corresponding FDR for the simultaneous update case of $(\text{SML-SDE})$ is

$$
\mathbb{E}_{\mu^*} \left[ \| \nabla_{\theta} \Phi(\Theta^*, W^*) \|^2_2 - \| \nabla_{\omega} \Phi(\Theta^*, W^*) \|^2_2 \right] = \beta^{-1} \mathbb{E}_{\mu^*} \left[ Tr \left( \Sigma_{\theta}(\Theta^*, W^*) \nabla_{\theta}^2 \Phi(\Theta^*, W^*) + \Sigma_{\omega}(\Theta^*, W^*) \nabla_{\omega}^2 \Phi(\Theta^*, W^*) \right) \right].
$$

Note that this FDR relation for GANs training is analogous to that for stochastic gradient descent algorithm on a pure minimization problem in (Yaida, 2019) and (Liu and Theodorou, 2019). This FDR relation in GANs reveals the crucial difference between GANs training of discriminator and generator networks versus training of two independent neural networks. It connects the microscopic fluctuation from the noise of SGA with the macroscopic dissipation phenomena related to the loss function. In particular, the quantity $Tr(\Sigma_{\theta} \nabla_{\theta}^2 \Phi + \Sigma_{\omega} \nabla_{\omega}^2 \Phi)$ links the covariance matrices $\Sigma_{\theta}$ and $\Sigma_{\omega}$ from SGAs with the loss landscape of $\Phi$, and reveals the trade-off of the loss landscape between the generator and the discriminator.

Alternatively, the evolution of the squared norm of the parameters leads to a different type of FDR that will be practically useful for learning rate scheduling.

**Theorem 1.1.6.** Assume the existence of an invariant measure $\mu^*$ for $(\text{SML-SDE})$, then

$$
\mathbb{E}_{\mu^*} \left[ \Theta^*,T \nabla_{\theta} \Phi(\Theta^*, W^*) - W^*,T \nabla_{\omega} \Phi(\Theta^*, W^*) \right] = \beta^{-1} \mathbb{E}_{\mu^*} \left[ Tr(\Sigma_{\theta}(\Theta^*, W^*) + \Sigma_{\omega}(\Theta^*, W^*)) \right].
$$

**Scheduling of learning rate.** Notice that the quantities in $(\text{FDR2})$, including the parameters $(\theta, \omega)$ and first-order derivatives of the loss function $g_{\theta}, g_{\omega}, g_{i,j}^{\theta}$, and $g_{i,j}^{\omega}$, are computationally inexpensive. Therefore, $(\text{FDR2})$ enables customized scheduling of learning rate, instead of predetermined scheduling ones such as Adam or RMSprop optimizer.
For instance, recall that \( g^B_\theta \) and \( g^B_\omega \) are respectively unbiased estimators for \( g_\theta \) and \( g_\omega \), and
\[
\hat{\Sigma}_\theta(\theta, \omega) = \frac{\sum_{k=1}^B [g^I_{k\theta}(\theta, \omega) - g^B_\theta(\theta, \omega)][g^I_{k\theta}(\theta, \omega) - g^B_\theta(\theta, \omega)]^T}{B - 1},
\]
\[
\hat{\Sigma}_\omega(\theta, \omega) = \frac{\sum_{k=1}^B [g^I_{k\omega}(\theta, \omega) - g^B_\omega(\theta, \omega)][g^I_{k\omega}(\theta, \omega) - g^B_\omega(\theta, \omega)]^T}{B - 1},
\]
are respectively unbiased estimators of \( \Sigma_\theta(\theta, \omega) \) and \( \Sigma_\omega(\theta, \omega) \). Now in order to improve GANs training with the simultaneous update, one can introduce two tunable parameters \( \epsilon > 0 \) and \( \delta > 0 \) to have the following scheduling:
\[
\text{if } \left| \frac{\Theta^T g^B_{\theta}(\Theta_t, W_t) - W^T g^B_{\omega}(\Theta_t, W_t)}{\beta^{-1}Tr(\hat{\Sigma}_\theta(\Theta_t, W_t) + \hat{\Sigma}_\omega(\Theta_t, W_t))} - 1 \right| < \epsilon, \text{ then update } \eta \text{ by } (1 - \delta)\eta.
\]

### 1.1.4 Applications of GANs

#### 1.1.4.1 Computing MFGs via GANs

Bases on the conceptual connection between GANs and MFGs, (Cao et al., 2020a) proposes a new computational approach for MFGs, using two neural networks in an adversarial way, summarized in Algorithm 1, in which

- \( u_\theta \) being the NN approximation of the unknown value function \( u \) for the HJB equation,
- \( m_\omega \) being the NN approximation for the unknown mean information function \( m \).

Note that Algorithm 1 can be adapted for broader classes of dynamical systems with variational structures. Such GANs structures are exploited in (Yang et al., 2020) and (Yang and Perdikaris, 2019) to synthesize complex systems governed by physical laws.

To test the performance of Algorithm 1 a class of ergodic MFGs with the following payoff function are considered,
\[
\hat{J}_m(\alpha) = \lim \inf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T L(X^\alpha_t, \alpha(X^\alpha_t)) + f(X^\alpha_t, m(X^\alpha_t))dt \right], \tag{1.1.19}
\]
subject to \( dX^\alpha_t = \alpha(X^\alpha_t)dt + dW_t \), with the cost of control and running cost given by
\[
L(x, \alpha) = \frac{1}{2} |\alpha|^2 + 2\pi^2 \left[ - \sum_{i=1}^d \sin(2\pi x_i) + \sum_{i=1}^d |\cos(2\pi x_i)|^2 \right] - 2 \sum_{i=1}^d \sin(2\pi x_i),
\]
\[
f(x, m) = \ln(m).
Algorithm 1 MFGANs

At \( k = 0 \), initialize \( \theta \) and \( \omega \). Let \( N_\theta \) and \( N_\omega \) be the number of training steps of the inner-loops and \( K \) be that of the outer-loop. Let \( \beta_i > 0, i = 1, 2 \).

for \( k \in \{0, \ldots, K - 1\} \) do

\begin{align*}
\text{Let } m = 0, n = 0, \\
\text{Sample } \{(s_i, x_i)\}_{i=1}^{B_d} \text{ on } [0, T] \times \mathbb{R}^d \text{ according to a predetermined distribution } p_{\text{prior}}, \text{ where } B_d \text{ denotes the number of training samples for updating loss related to FP residual.} \\
\text{Let } \hat{\mathcal{L}}_D(\theta, \omega) = \hat{\mathcal{L}}_{\text{FP}}(\theta, \omega) + \beta_D \hat{\mathcal{L}}_{\text{init}}(\omega), \text{ with} \\
\hat{\mathcal{L}}_{\text{FP}} = \frac{1}{B_d} \left\{ \sum_{i=1}^{B_d} \left[ \partial_s m_\omega(s_i, x_i) + \text{div} \left[ m_\omega(s_i, x_i)b(s_i, x_i, m(s_i, x_i), \alpha^*(s_i, x_i)) \right] \\
- \frac{\sigma^2}{2} \Delta_x m_\omega(s_i, x_i) \right]^2 \right\}, \\
\hat{\mathcal{L}}_{\text{init}} = \frac{\sum_{i=1}^{B_d} [m_\omega(0, x_i) - m^0(x_i)]^2}{B_d},
\end{align*}

where \( m^0 \) is a known density function for the initial distribution of the states and \( \beta_D > 0 \) is the weight for the penalty on the initial condition of \( m \).

for \( m \in \{0, \ldots, N_\omega - 1\} \) do

\begin{align*}
\omega \leftarrow w - \alpha_d \nabla_\omega \hat{\mathcal{L}}_D \text{ with learning rate } \alpha_d. \\
\text{Increase } m.
\end{align*}

end for

Sample \( \{(s_j, x_j)\}_{j=1}^{B_g} \) on \([0, T] \times \mathbb{R}\) according to a predetermined distribution \( p_{\text{prior}} \), where \( B_g \) denotes the number of training samples for updating loss related to HJB residual.

Let \( \hat{\mathcal{L}}_G(\theta, \omega) = \hat{\mathcal{L}}_{\text{HJB}}(\theta, \omega) + \beta_G \hat{\mathcal{L}}_{\text{term}}(\theta), \text{ with}
\begin{align*}
\hat{\mathcal{L}}_{\text{HJB}} &= \frac{1}{B_g} \left\{ \sum_{j=1}^{B_g} \left[ \partial_s u_\theta(s_j, x_j) + \frac{\sigma^2}{2} \Delta_x u_\theta(s_j, x_j) + H_\omega(s_j, x_j, \nabla_x u_\theta(s_j, x_j)) \right]^2 \right\}, \\
\hat{\mathcal{L}}_{\text{term}} &= \frac{\sum_{j=1}^{B_g} u_\theta(T, x_j)^2}{B_g},
\end{align*}

where \( \beta_G > 0 \) is the weight for the penalty on the terminal condition of \( u \).

for \( n \in \{0, \ldots, N_\theta - 1\} \) do

\begin{align*}
\theta \leftarrow \theta - \alpha_g \nabla_\theta \hat{\mathcal{L}}_G \text{ with learning rate } \alpha_g. \\
\text{Increase } n.
\end{align*}

end for

Increase \( k \).

end for

Return \( \theta, \omega \).
In this class of mean-field-games, the associated HJB equation and FP equation are

\[
\begin{align*}
-\epsilon \Delta u + H_0(x, \nabla u) &= f(x, m) + \bar{H}, \\
-\epsilon \Delta m - \text{div} (m \nabla_p H_0(x, \nabla u)) &= 0, \\
\int_{\mathbb{T}^d} u(x) dx &= 0; \ m > 0, \ \int_{\mathbb{T}^d} m(x) dx = 1,
\end{align*}
\]

(1.1.20)

where the convex conjugate \( H_0 \) is given by \( H_0(x, p) = \sup_\alpha \{ \alpha \cdot p - \frac{1}{2} | \alpha |^2 \} - \tilde{f}(x) \). Here, the periodic value function \( u \), the periodic density function \( m \), and the unknown \( \bar{H} \) can be explicitly derived. Indeed, assuming the existence of a smooth solution \((m, u, \bar{H})\), \( m \) in the second equation in (1.1.20) can be written as \( m(x) = \frac{e^{2u(x)}}{\int_{\mathbb{T}^d} e^{2u(x')} dx'} \). Hence the solution to (1.1.20) is given by \( u(x) = \sum_{i=1}^d \sin(2\pi x_i) \) and \( \bar{H} = \ln \left( \int_{\mathbb{T}^d} e^{2\sum_{i=1}^d \sin(2\pi x_i)} dx \right) \). The optimal control policy is also explicitly given by

\[
\alpha^* = \arg \max_\alpha \{ \nabla_x u \cdot \alpha - L(x, \alpha) \}
= \nabla_x u = 2\pi \left( \cos(2\pi x_1) \ldots \cos(2\pi x_d) \right) \in \mathbb{R}^d.
\]

![Figure 1.1.1: One-dimensional test case.](image)

The Algorithm 1 is first tested on a one-dimensional case, with its result highlighted in Figures 1.1.1 and 1.1.2. Figures 1.1.1a and 1.1.1b show the learnt
functions of $u$ and $m$ against the true ones, respectively, and Figure 1.1.1c shows the optimal control, with the accuracy of the learnt functions versus the true ones. The plots of loss in Figures 1.1.2a and 1.1.2b depict the evolution of relative $l_2$ error as the number of outer iterations grows to $K$. Within $10^5$ iterations, the relative $l_2$ error of $u$ oscillates around $3 \times 10^{-2}$, and the relative $l_2$ errors of $m$ decreases below $10^{-3}$. The evolution of the HJB and FP differential residual loss is shown in Figures 1.1.2e and 1.1.2f, respectively. In these figures, the solid line is the average loss among three experiments, with standard deviation captured by

Figure 1.1.2: Losses and errors in the one-dimensional test case.
the shadow around the line. Both differential residuals first rapidly descend to the magnitude of $10^{-2}$ and then the descent slows down accompanied by oscillation.

Algorithm 1 is then applied to a four-dimensional case, with result shown in Figure 1.1.3. Within $2 \times 10^5$ iterations, the relative $l_2$ error of $u$ decreases below $2 \times 10^{-2}$ and that of $m$ decreases to $4 \times 10^{-3}$.

Note the advantage of GANs training when compared with a similar experiment in Test Case 4 in (Carmona and Laurière, 2019) without the adversarial training for two neural networks: algorithms in (Carmona and Laurière, 2019) need significantly larger number of iterations: $10^6$ of iterations versus $2 \times 10^5$ for Algorithm 1 to achieve the same level of accuracy.

A concurrent paper alongside with (Cao et al., 2020a) is the work of (Lin et al., 2020). Using a primal-dual variational formulation associated with the coupled HJB-FP system as in (Cirant and Nurbekyan, 2018), MFGs are recast as GANs in a different way in (Lin et al., 2020), where the density function is seen as the generator and the value function is seen as the discriminator. Based on this alternative interpretation, a GANs-based algorithm named APAC-Net is proposed. Through numerical experiments, this algorithm is shown to be able to solve certain classes of MFGs in dimension up to 100.
There are essentially two different frameworks in which GANs have been adopted in the mathematical finance literature. The first one is to reformulate a constrained control and optimization problem as a minimax problem so that the generator and discriminator networks can be constructed for computational purpose. The second one is to draw the analogy between simulation of financial time series data and image generation such that various statistical and distributional properties can be exploited for performance evaluations. We will review several representative works for each category.

**Asset pricing and minimax problem.**

The work of [Chen et al., 2019](#) is one of the earliest works to identify the minimax structure in a non-linear model for asset pricing. Its primary idea is to exploit the no-arbitrage condition and recast the constrained problem into the minimax framework of GANs. Their objective is to estimate the pricing kernel or stochastic discount factor (SDF) that summarizes the information of the cross-section of returns for different stocks.

Specifically, take the return of asset \( i \in \{1, \ldots, n\} \) at time \( t+1 \) as \( R_{t+1,i} \) and the excess return as \( R_{t+1,i}^e = R_{t+1,i} - R_{t,i} \). Let \( M_{t+1} \) be the SDF satisfying the no-arbitrage condition,

\[
\mathbb{E}_t [M_{t+1} R_{t+1,i}^e] = 0 \iff \mathbb{E}_t [R_{t+1,i}^e] = \left(- \frac{\text{Cov}_t (R_{t+1,i}^e, M_{t+1})}{\text{Var}_t (M_{t+1})}\right) \cdot \frac{\text{Var}_t (M_{t+1})}{\mathbb{E}_T [M_{t+1}]},
\]

where \( \mathbb{E}_t \) stands for expectation conditional on some suitable information by time \( t \). Then assume that

\[
M_{t+1} = 1 - \omega^T R_{t+1}^e = 1 - F_{t+1}, \quad \beta_{t,i} = - \frac{\text{Cov}_t (R_{t+1,i}^e, M_{t+1})}{\text{Var}_t (M_{t+1})},
\]

where \( \omega = (\omega_1, \ldots, \omega_i, \ldots, \omega_n) \) denotes the SDF weights vector which is also the weights vector of the conditional mean-variance efficient portfolio, and \( \beta_{t,i} \) denotes the time-varying exposure to systematic risk for asset \( i \).

In this one-factor model setup, the main quantities to be estimated are the two vectors \( \omega \) and \( \beta_t = (\beta_{t,1}, \ldots, \beta_{t,i}, \ldots, \beta_{t,n}) \). To handle the no-arbitrage constraint, they utilize the unconditional moment conditions: given any \( \sigma \)-algebra generated by some random variable \( Z, \mathcal{F} = \sigma(Z) \),

\[
Y = \mathbb{E}[X|\mathcal{F}] \implies \mathbb{E}[X f(Z)] = \mathbb{E}[Y f(Z)],
\]

for any measurable function \( f \). In particular, let the choice of information be \( \sigma(I_t, I_{t,i}) \), where \( I_t \) represents the macroeconomic conditions at time \( t \) whereas \( I_{t,i} \) denotes information at time \( t \) for the specific stock \( i \), then

\[
\omega_i = \omega_{t,i} = \omega(I_t, I_{t,i}), \quad \beta_{t,i} = \beta(I_t, I_{t,i}).
\]
Consequently, the no-arbitrage condition implies

\[ E \left[ M_{t+1} R_{t+1} e^g(I_t, I_{t,i}) \right] = 0, \]

for any measurable function \( g \); if \( \hat{\omega} \) and \( \hat{\beta} \) correspond to the correct SDF \( \hat{M} \), this is equivalent to

\[ \max_g \frac{1}{N} \sum_{j=1}^{N} \left\| E \left[ \hat{M}_{t+1} R_{t+1,j} g(I_t, I_{t,j}) \right] \right\|^2 = 0. \]

Now, estimating SDF that satisfies the no-arbitrage condition is transformed into the minimax game

\[ \min_{\omega} \max_g \frac{1}{N} \sum_{j=1}^{N} \left\| E \left[ \hat{M}_{t+1} R_{t+1,j} g(I_t, I_{t,j}) \right] \right\|^2, \]

a natural GANs structure.

This proposed GANs model is then compared with an alternative model with the no-arbitrage condition relaxed to a first moment condition given by the one-factor model, \( E[R_{t+1,j}] \propto E[F_{t+1}] \). It is further compared with a second alternative model with both \( \omega \) and \( g \) assumed to be linear. The GANs model is shown to outperform uniformly in terms of Sharp ratio, explained variation, and cross-sectional mean \( R^2 \).

**GANs as financial time series data simulators**

Another application of GANs is to generate financial time series data for both equity and derivatives.

In (Wiese et al., 2019), the main objective is to build a simulator for equity option markets. Instead of dealing with option price directly which is subject to the no-arbitrage constraint, they work with an equivalent and less constrained form called discrete local volatility (DLV).

In this formulation, the time-varying DLV \( \sigma_t \) is seen as a function of strike \( K \) and maturity \( M \). The generator takes the state variable \( S_t = f(\sigma_t, \ldots, \sigma_0) \) as well as some random noise \( Z_{t+1} \) as inputs and set \( X_{t+1} = \log \sigma_{t+1} = g(Z_{t+1}, S_t) \). The discriminator tries to distinguish the true \((X_{t+1}, S_t)\) and the generated \((\tilde{X}_{t+1}, \tilde{S}_t)\). Other calibration techniques such as PCA are also incorporated.

This formulation is compared among different neural network based simulators. The performance evaluation is based on four types of criteria: the distributional metric which is the distance between the empirical probability distribution functions of the generated and historical data, the distributional scores given by skewness and kurtosis scores, the dependency score through the autocorrelation function score for the log-return process and finally the cross-correlation scores for the log-DLV and the DLV log returns. Their numerical results show that the GANs
model outperforms the other benchmark models such as vector autoregressive models, TCN models, and quasi maximum likelihood estimation.

In a closely related work, (Wiese et al., 2020) proposes a special GANs model called the Quant GAN. The main characteristic of Quant GANs is taking temporal convolutional networks (TCNs) as the generator. By choosing appropriate kernel size $K$ and dilation factor $D$, TCNs can carry long-time dependency and avoid abnormal behavior of gradients over time. They show that with Lipschitz constraint on the choices of activation functions and weights, the generated process has as many number of moments as the input latent variable. Finally, they use the inverse Lambert $W$ transform for the real asset log-return processes to copy with the heavy-tail property in the GANs training. In the Lambert $W$ transform, a random variable $X$ with mean $\mu$, variance $\sigma^2$ and cumulative distribution function $F_X$ is transformed into

$$Y = \frac{X - \mu}{\sigma} \exp\left(\frac{\delta(X - \mu)^2}{2\sigma^2}\right) + \mu,$$

with a proper choice of nonnegative parameter $\delta$ so that $Y$ has heavier tail than $X$ if $\delta > 0$. The inverse Lambert $W$ transform is its inverse process.

They propose two different approaches of utilizing TCNs: one is to use the pure TCNs to directly generate time series, the other is to use TCNs to generate drift and volatility process and add another network to represent the noise. They test the simple GARCH model for comparison, with the evaluation of the models based on distributional metrics and dependence scores. In particular, the former include Wasserstein distance and DY metric, i.e., a measurement of the distance between the estimated likelihoods from real and generated data, and the latter include ACF score and the leverage effect score. Their results show that the GANs model with pure TCNs perform the best for the majority of the tests, and that both GANs models dominate the GARCH model.

Other related works include (Takahashi et al., 2019) and (Zhang et al., 2019). The GANs model in (Takahashi et al., 2019) captures statistical properties exhibited in real financial data, such as linear unpredictability, the heavy-tailed price return distribution, volatility clustering, leverage effects, the coarse-fine volatility correlation, and the gain/loss asymmetry. The GANs model in (Zhang et al., 2019) is used to predict stock prices from historical stock data, where long-short-term-memory is adopted as the generator and multi-layer perceptron as the discriminator. In particular, the generator acts as a function characterizing the unknown and possibly complex relation between stock price in the future and historical data.

There are other extensions of GANs models. For instance, in (Cao et al., 2020b), conditional GANs are constructed to simulate quantities that have traditionally been of interest in financial industry. This GANs model enables dynamic data updating for stress tests. Embracing the general idea of adversarial training in GANs, (Cuchiero et al., 2020) proposes a generative adversarial approach for (robust) calibration of local stochastic volatility models; the generation of volatility surfaces follows neural SDEs, where a special deep-hedging-based variance reduction
technique is applied and the adversarial training idea is embedded in evaluating the simulated volatility surfaces: the loss function may come from a family of candidate loss functions to ensure robustness. Recently, a GANs model called COT-GAN is proposed in [Xu et al. 2020] based on causal optimal transport theory. In this work, the temporal causality condition naturally leads to an adversarial framework for GANs and a mixed Sinkhorn distance is proposed to calculate the optimal transport cost with reduced bias. This new framework could be used for generating sequential data including financial time series.

1.1.5 Conclusion and Discussion

This notes covers three major aspects of GANs, essentials of GANs in the optimization and game framework, GANs training via stochastic analysis, and recent applications of GANs in mathematical finance.

Despite its vast popularity and power in data and image generation, GANs face many challenges in implementation and training and remain largely undeveloped in theory. For instance, the well-posedness of GANs as a minimax game has not been fully understood until [Guo and Mounjid 2020] in which the convexity issue is analyzed in details. The connection between mean-field games and GANs via the minimax structure presents GANs’ potential computing power for high dimensional control and optimization problems with variational structures. The next natural test field is forward-backward-stochastic-differential equations, where there is a natural variational structure to retrofit for the minimax game of GANs. Beyond computational power, more explorations are needed to see if convergence and computation complexity results can be obtained, especially given the SDE approximation of GANs training. A small step towards this direction is [Guo and Mounjid 2020], which formulates simple stochastic control problems for learning rate and batch size analysis and shows their impact on error and variance reduction. One also wonders if the empirical success of GANs in data generation can be replicated in the general area of simulation and if robust theoretical analysis can be established.

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