Generating Majorana qubit coherence in Majorana Aharonov-Bohm interferometer

Fei-Lei Xiong,1,* Hon-Lam Lai,1,* and Wei-Min Zhang1,2,†

1Department of Physics and Center for Quantum Information Science, National Cheng Kung University, Tainan 70101, Taiwan
2Physics Division, National Center for Theoretical Sciences, Taipei 10617, Taiwan

We propose an Aharonov-Bohm interferometer consisted of two topological superconducting chains (TSCs) to generate coherence of Majorana qubits, each qubit is made of two Majorana zero modes (MZMs) with the definite fermion parity. We obtain the generalized exact master equation as well as its solution and study the real-time dynamics of the MZM qubit states under various operations. We demonstrate that by tuning the magnetic flux, the decoherence rates can be modified significantly, and dissipationless MZMs can be generated. By applying the bias voltage to the leads, one can manipulate MZM qubit coherence and generate a nearly pure superposition state of Majorana qubit. Moreover, parity flipping between MZM qubits with different fermion parities can be realized by controlling the coupling between the leads and the TSCs through gate voltages.

I. INTRODUCTION

Topological quantum computation has been widely investigated as a promising candidate for realizing fault-tolerant quantum computation due to its robustness against decoherence1,2. The protection against decoherence during the computation process relies on highly-degenerate ground states of the qubit space, which is realized by spatially separated Majorana zero modes (MZMs)1. Theories have predicted that under certain physical conditions, MZMs can exist at the ends of 1D effective spinless p-wave superconductors1, which can be generated by contacting a conventional s-wave superconductor to topological insulators3–6, magnetic atom chains7–12, or semiconductors with strong spin-orbit interaction13–20. After nearly a decade of effort, experimentalists have recently observed some signatures of MZMs in the proximitized nanowire systems21–29.

Because of the non-Abelian exchange property of MZMs, braiding among them correspond to nontrivial unitary transformations, which plays a central role in the scheme of topological quantum computation2. In the literature, there are mainly two kinds of methods to realize the braiding operations, either by changing the physical parameters of the system adiabatically30–34 or by performing experimental measurements systematically35–38. For instance, MZMs can be moved along nanowires by tuning the chemical potential and braiding operations can be performed in T-junction structures30,31. Other proposals include performing braidings by controlling the couplings between different MZMs32, or by fusions of different MZMs in T-junction nanowires34. As for the measurement-based braiding method, effective braiding of two MZMs are done by measuring the joint fermion parity of the MZM pair rather than exchanging their spatial positions. This kind of measurements, as well as the error correction code, can be realized by coupling quantum dots to MZMs35, or by using Aharonov-Bohm interferometers30,37,39–41.

However, in realistic situations, the Majorana qubits are unavoidably coupled to external controlling gates under qubit operations30,42–44. As a consequence, the topological protection against decoherence can be destroyed by, for instance, charge fluctuations of the controlling gates45,46. In this paper, we propose a new scheme other than braiding to manipulate the qubit states of MZMs, where noise effects are taken into account. Our device mainly consists of an Aharonov-Bohm (AB) interferometer, which is constructed by connecting two TSCs with two metal leads (See Fig. 1). The TSCs are tuned to the topological phase through the super-gates so that four MZMs are formed at their ends. Two leads with tunable bias voltage are coupled to the left and right ends of the TSCs, with the coupling strengths being also adjustable through the controlling gates. The magnetic flux threading into the central region of the interferometer can affect the interference pattern of the interferometer.

In this device, unlike the braiding operations, the fermion parity of the MZM states can be intentionally
switched by letting electrons tunnel into and out of the TSCs from the leads. In addition, electrons transport coherently from one end of a TSC to the other through the MZM pairs, the two paths formed by the TSCs interfere with each other. Quantum coherence can therefore be generated and controlled by tuning the lead-TSC couplings through gate voltages or by the applied magnetic flux. Through the exact master equation involving pairing interaction, we shall study the real-time dynamics of this Majorana AB interferometer under various operations. We discover that under the magnetic flux controlling, dissipationless MZM modes can be formed between the leads and the TSCs. Also, intended MZM qubit states with different parities can be prepared by applying a bias to the leads. Finally, parity flipping between the MZM qubits with different fermion parity can be done by controlling the coupling strengths between the leads and the TSCs through gate voltages.

Our paper is organized as follows. In Sec. II, we propose the model of the Majorana AB interferometer, which includes two TSCs contacting with left and right leads. The magnetic flux threading into the central region of the interferometer. We construct the Hamiltonian of the electron tunnelings in this superconducting AB interferometer incorporating the magnetic flux. In Sec. III, we derive the exact master equation for the MZMs localized at the ends of two TSCs. We show that the damping of the MZMs and the couplings induced by the leads are explicitly related to the generalized non-equilibrium Green functions for the MZMs. Furthermore, the density matrix of the four MZMs (two MZM qubits with different parity respectively) can be obtained as the solution to our exact master equation. In Sec. IV, we discuss Majorana qubit state evolution under various kinds of operations. We show that dissipationless MZM modes will be formed by controlling the magnetic flux. Moreover, MZM qubit coherence can be generated and controlled when a bias is applied between the two leads. The MZM qubit state parity can also be flipped by tuning the couplings between the MZMs and the leads. Finally, the conclusions are summarized in Sec. V.

II. THE MAJORANA AB INTERFEROMETER AND ITS MODELING

The Majorana AB interferometer we propose is schematically plotted in Fig. 1. The nanowires labeled α and β are two 1D spinless p-wave superconductor chains, which can be realized by, for example, strong spin-orbit interacting nanowires proximitized by conventional s-wave superconductors. In this paper, we model them as N-site Kitaev chains and in the absence of the magnetic flux, they are described by the Hamiltonians

\[ H_{\alpha(\beta)} = \sum_{j=1}^{N-1} \left[ -\Delta e^{i\phi_{\alpha(\beta),j}} a_{\alpha(\beta),j} a_{\alpha(\beta),j+1} + \Delta e^{i\phi_{\alpha(\beta),j+1}} a_{\alpha(\beta),j} a_{\alpha(\beta),j+1} + \hbar c. \right] 
\]

Here, \( a_{\alpha,j} \) (\( a_{\alpha,j}^\dagger \)) denotes the annihilation (creation) operator of site-j in chain α (similarly for β). The hopping amplitude \( w \) is real-valued and \( \Delta e^{i\phi_{\alpha(\beta),j}} \), with \( \Delta \) and \( \phi_{\alpha(\beta)} \) being real numbers, are the superconducting gap in chain α and β. Two leads, which are labeled L and R and modeled by the free electron gas Hamiltonians

\[ H_{L(R)} = \sum_k \epsilon_{L(R)k} c_{L(R)k}^\dagger c_{L(R)k} , \]

are coupled to the TSCs through the tunneling Hamiltonian

\[ H_I = \sum_k \left( \lambda_{\alpha Lk}(t) c_{Lk}^\dagger a_{\alpha,1} + \lambda_{\beta Lk}(t) c_{Lk}^\dagger a_{\beta,1} + \lambda_{\alpha Rk}(t) c_{Rk} a_{\alpha,N} + \lambda_{\beta Rk}(t) c_{Rk} a_{\beta,N} + h.c. \right) . \]

Here, \( \epsilon_{L(R)k} \) is the single-particle energy of mode-k in lead L(R), with \( c_{L(R)k} \) being the corresponding annihilation and creation operators respectively. Moreover, \( \lambda_{\alpha Lk}(t) \), \( \lambda_{\beta Lk}(t) \), \( \lambda_{\alpha Rk}(t) \), and \( \lambda_{\beta Rk}(t) \) stand for the coupling strengths between the modes in the leads and the ends of the TSCs. They are controlled by tuning the controlling gates in Fig. 1 and can in general be time-dependent. In the following, if not specified, we would omit the time-dependence of the λ’s.

In addition, we apply the magnetic flux threading into the central region of the interferometer \( \Phi = \int A(r) \cdot dr \), where \( A(r) \) stands for the vector potential at position \( r \). The operators \( a_{\alpha,j} \) and \( a_{\beta,j} \) in the Hamiltonians \( H_{\alpha} \) and \( H_{\beta} \) should be converted according to Peierls substitution

\[ a_{\alpha(\beta),j} \rightarrow e^{-i\phi_{\alpha(\beta),j}} a_{\alpha(\beta),j} , \]

and the phase functions satisfy the relation

\[ \phi_{\alpha(\beta),j+1} - \phi_{\alpha(\beta),j} = \frac{e}{\hbar} \int_{\alpha(\beta),j}^{\alpha(\beta),j+1} A(r) \cdot dr . \]

Therefore, the Hamiltonians of the TSCs α and β with magnetic flux threading into the central region of the interferometer can be written as

\[ H_{\alpha(\beta)} = \sum_j \left[ -we^{-i\phi_{\alpha(\beta),j+1}} a_{\alpha(\beta),j} a_{\alpha(\beta),j+1} + \Delta e^{i\phi_{\alpha(\beta),j+1}} e^{-i\phi_{\alpha(\beta),j}} a_{\alpha(\beta),j} a_{\alpha(\beta),j+1} + \hbar c. \right] 
\]

\[ -\sum_j \frac{\mu_{\alpha(\beta)}}{2} [a_{\alpha(\beta),j} a_{\alpha(\beta),j} - \frac{1}{2}] . \]
Apply the substitutions that
\[ \tilde{a}_{\alpha,j} = e^{i\phi_{\alpha,j}}/2 e^{-i\phi_{\alpha,j}} \tilde{a}_{\alpha,j} \],
the Hamiltonians \( H_\alpha \), \( H_\beta \) and \( H_I \) can be expressed in terms of the operators \( \tilde{a}_{\alpha,j} \) and \( \tilde{a}_{\beta,j} \), i.e.,
\[
H_\alpha(\beta) = \sum_j \left[ -w \tilde{a}_{\alpha,j} \tilde{a}_{\alpha,j+1} + \Delta \tilde{a}_{\alpha,j} \tilde{a}_{\alpha,j+1} + h.c. \right],
\]
\[
H_I = \sum_k \left( \lambda_{\alpha Lk} e^{i\phi_{\alpha L}} c_{kL}^{\dagger} \tilde{a}_{\alpha L,1} + \lambda_{\beta Lk} e^{i\phi_{\beta L}} c_{kL}^{\dagger} \tilde{a}_{\beta L,1} \right.
\]
\[
+ \lambda_{\alpha Rk} e^{i\phi_{\alpha R}} c_{kR}^{\dagger} \tilde{a}_{\alpha N} + \lambda_{\beta Rk} e^{i\phi_{\beta R}} c_{kR}^{\dagger} \tilde{a}_{\beta N} + h.c.) \right). (9)
\]
where \( \phi_{\alpha L} = \phi_{\alpha,1} - \phi_{\alpha,1}^{\dagger} \) and \( \phi_{\beta L} = \phi_{\beta,1} + \phi_{\beta,1}^{\dagger} \).

The nanowire chemical potentials \( \mu_\alpha \) and \( \mu_\beta \), the hopping amplitude \( w \), and the pairing parameter \( \Delta \) are tuned so that two MZMs can be generated at the ends of each nanowire, which we denote as \( \gamma_{\alpha L}, \gamma_{\alpha R}, \gamma_{\beta L} \) and \( \gamma_{\beta R} \), respectively. (See the red circles in the ends of two TSCs in Fig. 1). To illustrate this property, we consider an ideal parameter setting that \( \mu_\alpha = \mu_\beta = 0 \) and \( \Delta = w \), then the MZMs possess the explicit form,
\[
\gamma_{\alpha L} = \tilde{a}_{\alpha,1}^{\dagger} + \tilde{a}_{\alpha,1}, \quad \gamma_{\alpha R} = -i \tilde{a}_{\alpha,1}^{\dagger} + i \tilde{a}_{\alpha,1}, \quad \gamma_{\beta L} = \tilde{a}_{\beta,1}^{\dagger} + \tilde{a}_{\beta,1}, \quad \gamma_{\beta R} = -i \tilde{a}_{\beta,1}^{\dagger} + i \tilde{a}_{\beta,1}. \quad (10a)
\]
and the annihilation operators of the zero-energy quasi-particle excitations are
\[
b_{\alpha,0} = \frac{1}{2} (\gamma_{\alpha R} + i \gamma_{\alpha L}), \quad b_{\beta,0} = \frac{1}{2} (\gamma_{\beta R} + i \gamma_{\beta L}). \quad (11)
\]
In this work, we consider the case that the bias between the chains and the leads is much smaller than the superconducting gap and the excitation of quasiparticles in the continuous bands of the TSCs is negligible. As a consequence, in the interaction Hamiltonian (9), the components of the field operators that involved with the non-zero energy bogoliubons in the TSCs can be neglected, Then the interaction Hamiltonian is reduced to
\[
H_I = \frac{1}{2} \sum_k \left( \lambda_{\alpha Lk} e^{i\phi_{\alpha L}} c_{kL}^{\dagger} \gamma_{\alpha L} + \lambda_{\beta Lk} e^{i\phi_{\beta L}} c_{kL}^{\dagger} \gamma_{\beta L} + i \lambda_{\alpha Rk} e^{i\phi_{\alpha R}} c_{kR}^{\dagger} \gamma_{\alpha R} + i \lambda_{\beta Rk} e^{i\phi_{\beta R}} c_{kR}^{\dagger} \gamma_{\beta R} + h.c. \right. \right). (12)
\]
In Eq. (6), the pairing phases at the left and right ends of chain-\( \alpha \) and the left and right ends of chain-\( \beta \) are \(-2\phi_{\alpha L}, -2\phi_{\alpha R}, -2\phi_{\beta L}, \) and \(-2\phi_{\beta R} \), respectively. Following the convention in Feynman’s dealing with the pairing phases in the Josephson junctions, the phases satisfy
\[
\phi_{\alpha L} - \phi_{\beta L} = -\frac{\phi}{2} - \pi \frac{\Phi}{\Phi_0}, \quad (13a)
\]
and
\[
\phi_{\alpha R} - \phi_{\beta R} = -\frac{\phi}{2} + \pi \frac{\Phi}{\Phi_0}, \quad (13b)
\]
where \( \Phi_0 = h/e \) is the flux quantum with \( h \) standing for the Planck constant and \( e \) standing for the elementary charge, \( \phi = \phi_{\alpha} - \phi_{\beta} \) is the initial pairing phase difference between the two TSCs.

Although our modeling of the Majorana AB interferometer is based on modeling the TSCs as Kitaev chains under special physical conditions, it is applicable to more general cases. Generally speaking, when the effective 1D spinless \( p \)-wave superconductors are in the topological phase and the excitation of quasiparticles in the continuous band is negligible, the Hamiltonians (1) characterizing the TSCs can be written as \( H_\alpha = i\epsilon \alpha L \gamma_{\alpha L} \) and \( H_\beta = i\epsilon \beta L \gamma_{\beta L} \gamma_{\beta R} \), where the \( \gamma \)'s are the Majorana operators with wave packets localized near the ends of the TSCs. The energy \( \epsilon_{\alpha L} \sim 0 \) is proportional to the wave-function overlap of the MZMs \( \gamma_{\alpha L} \) and \( \gamma_{\alpha R} \), which is exponentially suppressed by the length of the TSCs. If the TSCs are long enough, the wave packets of the \( \gamma \)'s can be seen as localized and the energy values \( \epsilon_{\alpha L} \) can be treated as zero. The pairing phase differences between the ends of the TSCs can also apply to the relations in Eq. (13). As a consequence, Eq. (12) and Eq. (13) together describe the interaction between the TSCs and the leads, except for the fact that the Majorana operators are no longer in the form of Eq. (10). Thus, the leads and the TSCs together form an Aharonov-Bohm interference ring. Particles exchange and interfere through the MZMs in the TSCs. The dynamics of the system is influenced by the magnetic flux \( \Phi \) and the time-dependent tunneling amplitudes, through which we can generate coherence between the two TSCs and manipulate the MZM qubit states.

III. THE EXACT MASTER EQUATION AND THE DENSITY MATRIX

A. The exact master equation

We treat the MZMs as the principal system, and the two leads as the environment. Suppose that the total system is initially in a product state \( \rho_{\text{tot}}(0) = \rho(0) \otimes \rho_L(0) \otimes \rho_R(0) \), where \( \rho(0) \) is the state of the principal system and \( \rho_L(0) \otimes \rho_R(0) \) is the state of lead L (R). Without loss of generality, we also assume that \( \rho_L(0) \otimes \rho_R(0) \) is the thermal equilibrium state associated to temperature \( T_L \) (\( T_R \)) and chemical potential \( \mu_L \) (\( \mu_R \)). By taking advantage of the path integral approach in the coherent state representation, states of the system can be found to evolve according to the exact master equation
\[
\dot{\rho}(t) = -\frac{1}{i\hbar} [\rho(t), \hat{H}_I(t) + \hat{H}_R(t)] + \sum_{i,j} \frac{\Gamma_{ijL}(t)}{2} \left[ \gamma_{ijL} \rho(t) \gamma_{ijL} - \frac{1}{2} \{\rho(t), \gamma_{ijL} \gamma_{ijL} \} \right] + \sum_{i,j} \frac{\Gamma_{ijR}(t)}{2} \left[ \gamma_{ijR} \rho(t) \gamma_{ijR} - \frac{1}{2} \{\rho(t), \gamma_{ijR} \gamma_{ijR} \} \right]. (14)
\]
In the formula,

$$\tilde{H}_L(t) = \frac{i}{4} \left( [\mathbf{U}_L t^{-1}]_{\alpha \beta} - [\tilde{U}_L t^{-1}]_{\beta \alpha} \right) \gamma_{\alpha L} \gamma_{\beta L}$$  \hspace{1cm} (15)

is the environment-induced renormalized Hamiltonian for the left-side MZMs; \( \Gamma_{Lij}(t) \), \( i, j = \alpha, \beta \) characterize the decoherence rates of the MZMs in the left side and can be explicitly written in terms of the generalized non-equilibrium Green functions \( U_L \) and \( V_L \):

$$\Gamma_{Lij}(t) = [V_L - (\tilde{U}_L t^{-1} V_L + h.c.)]_{ij}.$$  \hspace{1cm} (16)

In Eqs. (15)-(16), \( U_L \) and \( V_L \) are short for the retarded Green’s function \( U_L(t_0, t_0) \) and the correlation function \( V_L(t) \) involving pairing interactions\(^{52}\). \( U_L(t_0, t_0) \) satisfies the integro-differential equation

$$\partial_t U_L(t_0, t_0) + 2 i \int_{t_0}^{t} dt g_L(t, \tau) U_L(\tau, t_0) = 0,$$  \hspace{1cm} (17)

with the initial condition that \( U_L(t_0, t_0) = I \) (\( I \) is a 2 \times 2 identity matrix). The time-nonlocal integral kernel \( g_L(t, \tau) \) is given by

$$g_L(t, \tau) = \frac{i}{2\pi} \int \frac{de}{e^{-i(t-\tau) J_L^e} + e^{i(t-\tau) J_L^h}},$$  \hspace{1cm} (18)

where \( J_L^e \) and \( J_L^h \) are short for the electron spectral density function \( J_L^e(e, t, \tau) \) and the hole spectral density function \( J_L^h(e, t, \tau) \), respectively; and \( J_L^h(e, t, \tau) = J_L^e(e, t, \tau) \). Define that \( J_L^{\alpha \beta}(t) = \frac{1}{2} \sum \delta(\epsilon - \epsilon_{\alpha L}) \lambda_{\alpha L}(t) \lambda_{\beta L}(\tau), \) where \( i \) and \( j \) are either \( \alpha \) or \( \beta \), then the complete expression of \( J_L^e \) is

$$J_L^e = \left( \begin{array}{cc} J_L^{\alpha \alpha} \rho_{\alpha \alpha} & J_L^{\alpha \beta} e^{-i \delta_L / 2} \\ J_L^{\beta \alpha} e^{i \delta_L / 2} & J_L^{\beta \beta} \end{array} \right),$$  \hspace{1cm} (19)

where \( \delta_L = \tilde{\phi}_{\alpha L} - \tilde{\phi}_{\beta L} \). Note that the cross coupling between \( \alpha \) and \( \beta \) is dependent on \( \delta_L \) which has been defined in Eq. (13). \( V_L(t) \) can be written in terms of the retarded Green’s function \( U_L \) that

$$V_L(t) = 2 \int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 U_L(t, \tau_1) \tilde{g}_L(\tau_1, \tau_2) U_L(\tau_2, t),$$  \hspace{1cm} (20)

where the system-environment correlation \( \tilde{g}_L(\tau_1, \tau_2) \) satisfies

$$\tilde{g}_L(\tau_1, \tau_2) = \frac{i}{2\pi} \int_0^\infty \frac{d\epsilon}{e^{-i(\tau_1 - \tau_2) \epsilon} J_L^e + e^{i(\tau_1 - \tau_2) \epsilon} J_L^h}. $$  \hspace{1cm} (21)

Here, \( f^e(\epsilon) = \frac{1}{1 + e^{\epsilon / T_L}} \) and \( f^h(\epsilon) = 1 - f^e(\epsilon) \) are the initial particle number distribution of electrons and holes in lead L respectively. All the relations and conventions are similar for the right-side MZMs, with only the index \( L \) being replaced by \( R \).

As shown in Eq. (14), couplings between the TSCs and the leads induce interactions among the MZMs as well as the dissipation of them. Specifically, the MZMs \( \gamma_{\alpha L} \) and \( \gamma_{\beta L} \) in the left (right) side are coupled to each other through the renormalized Hamiltonian \( \tilde{H}_L \) (\( \tilde{H}_R \)), and dissipate to lead \( L \) (\( R \)) through the dissipation coefficients \( \Gamma_{L(R)i,j} \). All the MZM dynamics can be captured by the Majorana correlation function matrix

$$M(t) = \begin{pmatrix} 0 & \langle i\gamma_{\alpha L} \gamma_{\beta L} \rangle & \langle i\gamma_{\alpha L} \gamma_{\alpha R} \rangle & \langle i\gamma_{\alpha L} \gamma_{\beta R} \rangle \\ -\langle i\gamma_{\alpha L} \gamma_{\beta L} \rangle & 0 & \langle i\gamma_{\alpha R} \gamma_{\beta L} \rangle & \langle i\gamma_{\alpha R} \gamma_{\beta R} \rangle \\ -\langle i\gamma_{\alpha L} \gamma_{\alpha R} \rangle & -\langle i\gamma_{\alpha R} \gamma_{\beta L} \rangle & 0 & \langle i\gamma_{\alpha R} \gamma_{\beta R} \rangle \\ -\langle i\gamma_{\alpha L} \gamma_{\beta R} \rangle & -\langle i\gamma_{\beta L} \gamma_{\beta R} \rangle & -\langle i\gamma_{\alpha R} \gamma_{\beta R} \rangle & 0 \end{pmatrix},$$  \hspace{1cm} (22)

which can be obtained in terms of non-equilibrium Green functions, explicitly,

$$M(t) = U(t_0) \bar{U} \bar{V} U^T - \frac{i}{2} (V - V^T),$$  \hspace{1cm} (23)

where the superscript \( T \) denotes the matrix transpose. Also, the expectation value of the fermion parity\(^{1}\) for the MZM states can be written as

$$\bar{P}(t) = -\langle \gamma_{\alpha L} \gamma_{\alpha R} \gamma_{\beta L} \gamma_{\beta R} \rangle = \bar{P}(t_0) \det(U) + \frac{1}{4} \left( V_{L\alpha \beta} - V_{L\alpha \beta}^* \right) \left( V_{R\alpha \beta} - V_{R\alpha \beta}^* \right).$$  \hspace{1cm} (24)

Note that in Eqs. (22)-(24), we have omitted the time-dependence of the Majorana operators. The Green functions \( U \) and \( V \) can be explicitly expressed as \( U(t, t_0) = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix} \) and \( V(t) = \begin{pmatrix} V_L & 0 \\ 0 & V_R \end{pmatrix} \) respectively.

**B. Exact dynamics of the density matrix**

The MZM density matrix in the AB interferometer can be obtained by solving the master equation. In the following, the basis \( \{ |0\rangle, b^\dagger_{0,\alpha} |0\rangle, b^\dagger_{0,\beta} |0\rangle, b_{0,\alpha} b^\dagger_{0,\beta} |0\rangle \} \) is used, which consists of two Majorana qubit basis with different parities, the even parity qubit basis \( \{ |0\rangle, b^\dagger_{0,\alpha} b^\dagger_{0,\beta} |0\rangle \} \) and the old parity qubit basis \( \{ b^\dagger_{0,\alpha} |0\rangle, b^\dagger_{0,\beta} |0\rangle \} \), where the operator \( b^\dagger_{0,\alpha(\beta)} = \frac{i}{2} (\gamma_{\alpha(\beta)L} - i \gamma_{\alpha(\beta)L}) \) creates a zero-energy Bogoliubov in TSC \( \alpha (\beta) \). This basis corresponds to the zero-energy Bogoliubov occupation in \( \alpha, \beta \) or both TSCs. We consider the case that initially the two nanowires are not correlated, i.e., the system initial state reads

$$\rho(t_0) = \begin{pmatrix} \rho_{00}(t_0) & 0 & 0 & 0 \\ 0 & \rho_{\alpha\alpha}(t_0) & 0 & 0 \\ 0 & 0 & \rho_{\beta\beta}(t_0) & 0 \\ 0 & 0 & 0 & \rho_{dd}(t_0) \end{pmatrix},$$  \hspace{1cm} (25)
where the subscripts 0, \( \alpha, \beta \), and \( d \) correspond to the states \( |0\rangle \), \( b_{0, \alpha}^\dagger |0\rangle \), \( b_{0, \beta}^\dagger |0\rangle \) and \( b_{0, \alpha}^\dagger b_{0, \beta}^\dagger |0\rangle \), respectively. Because there cannot exist coherence between different parity eigenstates of fermions, the density matrix of the two MZM qubits will always possess the form

\[
\rho(t) = \begin{pmatrix}
\rho_{00}(t) & 0 & 0 & \rho_{0d}(t) \\
0 & \rho_{\alpha\alpha}(t) & \rho_{\alpha\beta}(t) & 0 \\
0 & \rho_{\beta\alpha}(t) & \rho_{\beta\beta}(t) & 0 \\
\rho_{d0}(t) & 0 & 0 & \rho_{dd}(t)
\end{pmatrix}.
\]

(26)

At arbitrary time \( t \), the relation between the density matrix elements and the Majorana correlation functions reads

\[
\rho_{00}(t) = \frac{1}{4} \left( 1 + M_{13}(t) + M_{24}(t) + \bar{P}(t) \right),
\]

(27a)

\[
\rho_{\alpha\alpha}(t) = \frac{1}{4} \left( 1 + M_{13}(t) - M_{24}(t) - \bar{P}(t) \right),
\]

(27b)

\[
\rho_{\beta\beta}(t) = \frac{1}{4} \left( 1 - M_{13}(t) + M_{24}(t) - \bar{P}(t) \right),
\]

(27c)

\[
\rho_{dd}(t) = \frac{1}{4} \left( 1 - M_{13}(t) - M_{24}(t) + \bar{P}(t) \right),
\]

(27d)

\[
\rho_{0d}(t) = \frac{1}{4} \left[ -M_{14}(t) + M_{23}(t) + i (M_{12}(t) - M_{34}(t)) \right],
\]

(27e)

\[
\rho_{\alpha\beta}(t) = \frac{1}{4} \left[ -M_{14}(t) - M_{23}(t) - i (M_{12}(t) + M_{34}(t)) \right].
\]

(27f)

where \( M_{ij} \) (\( i, j = 1, 2, 3, 4 \)) is the element of the matrix \( M(t) \). By substituting Eqs. (23) and (24) into Eq. (27), one can obtain the complete solution to the two MZM qubit density matrix at arbitrary time \( t \), which is expressed in terms of the initial condition of the MZM states and the non-equilibrium Green’s functions \( U(t, t_0) \) and \( V(t) \).

Initially, the two MZM qubit density matrix is diagonal and no coherence exists. After the TSC system is coupled to the leads, the off-diagonal matrix elements \( \rho_{0d}(t) \) or \( \rho_{\alpha\beta}(t) \) would, in general, become finite values, i.e., one can generate coherence in each MZM qubit state. Moreover, both the dynamical process and the final state can be manipulated by tuning the magnetic flux and the coupling strengths. In the following section, we shall discuss the cases of various parameter settings. We shall demonstrate that by tuning the magnetic flux \( \Phi \), the bias \( \mu_0 \) and \( \mu_R \), and the coupling strengths \( \lambda \)’s, the MZM qubit states can be modified significantly.

IV. DYNAMICS OF THE MZMS WITH VARIOUS PARAMETER SETTINGS

In this section, we shall study how the coherence dynamics of the MZM qubits are varied under different parameter settings. For clarity, in the following analysis, we set the original pairing phase difference \( \phi \) of the TSCs to be zero (\( \phi = 0 \)) in absence of the magnetic flux [see Eq. (13)], and the initial state of the system as \( \rho(t_0) = |0\rangle\langle 0| \). Firstly, we investigate the general dynamics of two MZM qubits with different parities. For simplicity, we avoid the complicated tunneling effects due to the structure of the leads and simply take the wide-band limit of the spectral density functions: \( \mathbf{J}^0_{L/R}|_{\alpha} = \mathbf{J}^0_{L/R}|_{\beta} = \mathbf{J}^0_{L/R}|_{\beta} = \Gamma_0 \) with \( \Gamma_0 \) standing for a constant. The matrix elements of the retarded Green’s functions are then explicitly given by

\[
[U_{L/R}]_{\alpha\alpha}(t, t_0) = [U_{L/R}]_{\beta\beta}(t, t_0) = \frac{1}{2} \left[ e^{-\gamma_0(t-t_0)} - e^{-\gamma_0(t-t_0)} \right],
\]

(28a)

\[
[U_{L/R}]_{\alpha\beta}(t, t_0) = [U_{L/R}]_{\beta\alpha}(t, t_0) = \frac{1}{2} \left[ e^{-\gamma_0(t-t_0)} - e^{-\gamma_0(t-t_0)} \right],
\]

(28b)

where \( y = \cos [\pi \Phi / \Phi_0] \). Note that \( U(t, t_0) \) and thus the MZM qubit density matrix \( \rho(t) \) show \( 2\Phi_0 \)-periodicity as a function of the magnetic flux \( \Phi \). The diagonal elements of \( U(t, t_0) \) describe the decays of the MZM qubits. From Eq. (28a), one can see that the decay of MZM qubit states consist of two parts with different decay times, namely \( [\Gamma_0(1 + y)]^{-1} \) and \( [\Gamma_0(1 - y)]^{-1} \) respectively. Therefore, if \( y = \pm 1 \), i.e., \( \Phi / \Phi_0 \neq n \) (\( n \) stands for an integer), the MZM qubits will inevitably decay away. Furthermore, it is obvious from Eq. (28a) that for \( y = \pm 1 \), i.e., \( \Phi / \Phi_0 = n \), there exist dissipationless modes and part of the MZM qubit states will not decay. For the parameters considered above, when \( \Phi / \Phi_0 \) is an even integer, the system generates two dissipationless MZM modes reading \( \frac{1}{2}(\gamma_{xL} - \gamma_{xL}) \) and \( \frac{1}{2}(\gamma_{xR} + \gamma_{xR}) \), while for \( \Phi / \Phi_0 \) being an odd integer, the system forms two dissipationless modes reading \( \frac{1}{2}(\gamma_{xL} + \gamma_{xL}) \) and \( \frac{1}{2}(\gamma_{xR} + \gamma_{xR}) \).

On the other hand, the off-diagonal elements of \( U(t, t_0) \) characterize the correlations between MZMs \( \gamma_{xL}(R) \) and \( \gamma_{xL(R)}(R) \), and hence relate to the MZM qubit coherence. One can see from Eq. (28b) that \( [U_{L/R}]_{\alpha\beta} \) increases from zero initially, implying that the correlations between MZMs are building up. If there are no dissipationless modes, these build-up MZM correlations will eventually vanish. When dissipationless MZM modes exist, the MZM correlation functions \( [U_{L/R}]_{\alpha\beta} \) will reach a steady value of 1/2 (for \( \Phi / \Phi_0 \) being odd) or –1/2 (for \( \Phi / \Phi_0 \) being even). To study the MZM qubit dynamics, the evolution of the density matrix elements of the MZM states is shown in Fig. 2. In the case of zero bias, i.e., \( \mu_L = \mu_R = 0 \), the MZM qubit will eventually decay to a maximally mixed state if there is no dissipationless MZM mode. As mentioned above, the qubit coherence (described by the off-diagonal elements of the density matrix) grows from zero initially and fades away as the MZMs decay. Explicitly, \( \text{Re}[\rho_{\alpha\beta}] \) quickly grows from zero to 0.25 within \( t \sim 1/\Gamma_0 \) (see Fig. 2b), then decreases at a decay rate depending on the magnetic flux \( \Phi / \Phi_0 \) (see Eq. (28b)). On the other hand, the dissipationless MZM
mode, which exists when \( \Phi/\Phi_0 = 0, 1 \) or 2, will preserve part of the initial qubit state information and keep the MZM qubits away from a maximally mixed state. Note that two different dissipationless MZM modes are formed at \( \Phi/\Phi_0 = 0 \) and \( \Phi/\Phi_0 = 1 \) (see \( \text{Re}[\rho_{\alpha\beta}] \) in Fig. 2), showing the \( 2\Phi_0 \)-periodicity of the MZM qubit states.

Next, the bias \( \mu_L \) and \( \mu_R \) can be tuned so that two qubit steady states will not become a maximally mixed state. In this case, apart from the mere damping of the MZMs, electrons and holes can be pumped into or out of the two TSCs from the leads. As a result, the even and odd parity qubit states are not equally occupied. Furthermore, MZM qubit coherence with definite parity can also be generated by applying bias (See the curves corresponding to \( \text{Re}[\rho_{\alpha\beta}] \) and \( \text{Im}[\rho_{\alpha\beta}] \) in Fig. 3). If the bias \( \mu_L \) and \( \mu_R \) are large enough, MZM qubits can evolve to a state with almost definite parity and perfect coherence. For instance, in the case of the \( \Phi/\Phi_0 = \frac{3}{2} \), a large anti-symmetric bias (e.g. \( \mu_L = -\mu_R = 10\Gamma_0 \)) leads the system to the almost pure state with odd parity, namely, \( |(\alpha)+i|/\beta\rangle/\sqrt{2} \). While in the case of the \( \Phi/\Phi_0 = \frac{1}{2} \), a large symmetric bias (e.g. \( \mu_L = \mu_R = 10\Gamma_0 \)) leads the system to the almost pure state with even parity, namely, \( |(0)+i|/d\rangle/\sqrt{2} \).

As we demonstrated above, the applied bias leads to the polarization of the MZM state parity. Actually, this parity polarization can also be controlled by tuning the lead-TSC couplings. Specifically, when a bias of \( \mu_L = -\mu_R = 10\Gamma_0 \) is applied, the dominated parity is flipped when the cross-coupling strength \( |J_L|/\alpha\beta/\Gamma_0 \) changes from positive to negative (see Fig. 4). We demonstrate in detail this parity-flip dynamics in Fig. 5, in which the cross-coupling strength of the left-hand side \( |J_L|/\alpha\beta/\Gamma_0 \) is tuned so that it changes from 1 to -1 at different rates. Note that we have fixed the coupling strengths between the TSCs and the right lead. When \( |J_L|/\alpha\beta/\Gamma_0 \) is tuned within a very short time (\( \sim 1/\Gamma_0 \)), the MZMs will relax directly to the even parity state \( |(0)+i|/d\rangle/\sqrt{2} \) (see Fig. 5a). When the changing time of \( |J_L|/\alpha\beta/\Gamma_0 \) becomes a little longer (\( \sim 2/\Gamma_0 \)), as shown in Fig. 5b, the MZMs will relax partially to the odd parity state but then “turn” its relaxation to the even parity state. This is because the coupling changes so fast that the MZM states cannot reach full relaxation. Finally, when the changing time of \( |J_L|/\alpha\beta/\Gamma_0 \) is long enough (\( \sim 5/\Gamma_0 \)), the MZMs relaxes from the initial state \( |0|\rangle/|0\rangle \) to the odd parity state \( |(0)+i|/\beta\rangle/\sqrt{2} \), then relaxes again to the odd parity state, which is a parity flip between two MZM qubit states [see Fig. 5c].

V. CONCLUSION

In this paper, we propose a Majorana Aharonov-Bohm interferometer to control the MZM states. In this device,
magnetic flux, and dissipationless modes can be formed for certain values of the magnetic flux. By setting bias among the leads and the TSCs, MZM qubit states can be drawn away from approaching the maximally mixed state. The fermion parity of the MZM qubit can be polarized and the MZM qubit coherence can also be generated. The parity of the target state can be controlled by setting the bias voltages in a particular configuration, or by tuning the TSC-lead coupling through the controlling gates. If the bias is large enough, the state can evolve to a nearly pure coherent MZM qubit state within the same parity. Moreover, the switch between different parity qubit states can be realized by changing the cross-coupling strength from positive (negative) to negative (positive) at suitable rates.

Acknowledgments

We thank Lian-Ao Wu and Yu-Wei Huang for helpful discussions. This work is supported by the Ministry of Science and Technology of the Taiwan under the Contracts No. MOST-108-2112-M-006-009-MY3.
