Multilinear Class-Specific Discriminant Analysis

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Abstract—There has been a great effort to transfer linear discriminant techniques that operate on vector data to high-order data, generally referred to as Multilinear Discriminant Analysis (MDA) techniques. Many existing works focus on maximizing the inter-class variances to intra-class variances defined on tensor data representations. However, there has not been any attempt to employ class-specific discrimination criteria for the tensor data. In this paper, we propose a multilinear subspace learning technique suitable for applications requiring class-specific tensor models. The method maximizes the discrimination of each individual class in the feature space while retaining the spatial structure of the input. We evaluate the efficiency of the proposed method on two problems, i.e. facial image analysis and stock price prediction based on limit order book data.

I. INTRODUCTION

Over the past two decades, several subspace learning techniques have been proposed for computer vision and pattern recognition problems. The aim of subspace learning is to find a set of bases that optimizes a given objective function (or criterion) enhancing properties of interest in the learnt subspace. The obtained projection can be subsequently used either as a means of preprocessing or as a classifier. One of the methods used as a preprocessing step is Principal Component Analysis (PCA) ([1]), which finds a projection maximizing the data dispersion. While PCA retains most spectral information, it is an unsupervised method that does not utilize labeling information to increase class discrimination. Linear Discriminant Analysis (LDA) ([2]) is one of the most widely used discriminant learning techniques due to its success in many applications ([3], [4], [5]). In LDA, each class is assumed to follow a unimodal distribution and is represented by the corresponding class mean vector. LDA optimizes the ratio between inter-class and intra-class scatters. Several extensions have been proposed in order to relax these two assumptions ([6], [7], [8]). An important characteristic of LDA is that the maximal dimensionality of the learnt subspace is limited by the number of classes C forming the problem at hand. For problems where the objective is to discriminate one class from all other alternatives, i.e. for binary problems like face verification, this might not be an optimal choice for class discrimination.

To tackle the latter limitation of LDA, class-specific discriminant analysis techniques were proposed ([9], [10], [11], [12]). In the class-specific setting, a unimodal distribution is still assumed for the class of interest (hereafter noted as positive class), and the objective is to determine class-specific projections discriminating the samples forming the positive class from the rest samples (forming the negative class) in the subspace. By defining suitable out-of-class and in-class scatter matrices, the maximal subspace dimensionality is limited by the cardinality of the positive class, leading to better class discrimination and classification ([9], [11], [13]). Various extensions have been proposed to utilize class-specific formulation. For example, ([13]) proposed a solution to optimize both the class representation and the projection; in addition, approximate and incremental learning solutions were proposed in ([12], [14]).

While being able to overcome the limitation in subspace dimensionality of LDA, there is yet a limitation in the existing Class-Specific Discriminant Analysis (CSDA) methods. These methods are defined on vector data. Since many types of data are naturally represented by (high-order) matrices, generally referred as tensors, exploiting vector-based learning approaches might lead to the loss of spatial information being available on the data. For example, a grayscale image is naturally represented as a matrix (i.e. second order tensor), a color image is represented as a third order tensor and a multi-dimensional time series is represented as a third order tensor. Vectorizing such high-order tensors results to high-dimensional vectors, leading to high computational costs and the small sample size problem ([15]). In order to address such issues, generalizations of many linear subspace learning methods to multilinear ones have been proposed, including MPCA ([16]) and CMDA ([17]) as the multilinear extensions of PCA, GTDA ([18]) and DATER ([19]) as the multilinear extensions of LDA.

With the potential advantage of using tensorial data representations in (binary) verification problems, in this work, we propose to extend the class-specific discrimination criterion for tensor-based learning and formulate the Multilinear Class-Specific Discriminant Analysis (MCSDA) method. Moreover, we provide a time complexity analysis for the proposed method and compare it with its vector counterparts. We conducted experiments in two problems involving data naturally represented in a tensor form, i.e. facial image analysis and stock price prediction based on limit order book data. Experimental results show that the proposed MCSDA is able to outperform related tensor-based and vector-based methods and to compare favourably with recent methods.

The rest of the paper is organized as follows. Section 2 introduces the notations used throughout the paper, as
well as related prior works. In section 3, we formulate the proposed MCSDA method and provide our analysis on its time complexity. Section 4 presents our experimental analysis, and conclusions are drawn in Section 5.

II. Notations and Prior Work

We start by introducing the notations used throughout the paper and related definitions from multilinear algebra. In addition, previous works in discriminant analysis utilizing multi-class and class-specific criteria are briefly reviewed.

A. Multilinear Algebra Concepts

In this paper, we denote scalar values by either low-case or upper-case characters (x, y, X, Y . . .), vectors by low-case bold-face characters (x, y, . . .), matrices by upper-case bold-face characters (A, B, . . .) and tensors by calligraphic capital characters (X, Y, . . .). A tensor is a multilinear matrix with K modes, and is defined as \( X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_K} \), where \( I_k \) denotes the dimension in mode-\( k \). The entry in the \( i_k \)th index in mode-\( k \) for \( k = 1, \ldots, N \) is denoted as \( X_{i_1, i_2, \ldots, i_K} \).

Definition 1 (Mode-\( k \) Fiber and Mode-\( k \) Unfolding): The mode-\( k \) fiber of a tensor \( X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_K} \) is a vector of \( I_k \)-dimensional, given by fixing every index but \( i_k \). The mode-\( k \) unfolding of \( X \), also known as mode-\( k \) matricization, transforms the tensor \( X \) to matrix \( X_{(k)} \), which is formed by arranging the mode-\( k \) fibers as columns. The shape of \( X_{(k)} \) is \( \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_{k-1} \times I_{k+1} \times \cdots \times I_K} \) with \( I_k = \prod_{i=1,i \neq k}^{K} I_i \).

Definition 2 (Mode-\( k \) Product): The mode-\( k \) product between a tensor \( X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_K} \) and a matrix \( W \in \mathbb{R}^{I_2 \times I_K} \) is another tensor of size \( I_1 \times \cdots \times J_{k-1} \times J_k \times \cdots \times I_K \) and denoted by \( X \times_k W \). The element of \( X \times_k W \) is defined as \( (X \times_k W)_{i_1, i_2, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_K} = \sum_{i_{k-1}}^{I_{k-1}} \sum_{i_{k+1}}^{I_{k+1}} \cdots \sum_{i_{I_K}}^{I_{I_K}} X_{i_1, i_2, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_K} W_{j_k, i_{k+1}} \).

With the definition of mode-\( k \) product and mode-\( k \) unfolding, the following equation holds

\[
(X \times_k W)_{(k)} = WX_{(k)}
\]

For convenience, we denote \( X \times_1 W_1 \times_2 \cdots \times_K W_K \) by \( X^{\prod_{k=1}^{K} \times_k W_k} \).

B. Linear Discriminant Analysis

Let us denote by \( X = [x_1, \ldots, x_N] \in \mathbb{R}^{D \times N} \) a set of \( N \) D-dimensional vectors, each of which has an associated class label \( l_j \) (\( j = 1, \ldots, N \)) belonging to the label set \( \{c_i \mid i = 1, \ldots, C \} \). \( n_i \) is the number of samples in class \( c_i \). Let \( x_{i,j} \) denote the \( j \)th sample of class \( c_i \). The mean vector of class \( c_i \) is calculated as \( m_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{i,j} \). The mean vector of the entire set is \( m = \frac{1}{N} \sum_{i=1}^{C} \sum_{j=1}^{n_i} x_{i,j} = \frac{1}{N} \sum_{i=1}^{C} n_i m_i \).

Linear Discriminant Analysis (LDA) seeks an othonormal projection matrix \( W \in \mathbb{R}^{D \times d} \) that maps each sample \( x_i \) to a lower \( d \)-dimensional feature space (\( d \leq D \)) in which samples from different classes are highly discriminated. \( W \) is obtained by maximizing the ratio between the inter-class and intra-class variances in the feature space (21), i.e.

\[
J(W) = \frac{\sum_{i=1}^{C} n_i \|W^T m_i - W^T \bar{m}\|^2_F}{\sum_{i=1}^{C} n_i \|W^T x_{i,j} - W^T \bar{m}\|^2_F} = \frac{\text{tr}(W^T S_b W)}{\text{tr}(W^T S_w W)}
\]

where \( S_b = \sum_{i=1}^{C} n_i (m_i - \bar{m})(m_i - \bar{m})^T \) denotes the between-class scatter matrix and \( S_w = \sum_{i=1}^{C} \sum_{j=1}^{n_i} (x_{i,j} - m_i)(x_{i,j} - m_i)^T \) denotes the within-class scatter matrix. By maximizing \( J(W) \) in (2), the dispersion between the data and the corresponding class mean is minimized while the dispersion between each class mean and the total mean is maximized in the projected subspace. The columns of \( W \) are formed by the eigenvectors corresponding to the \( d \leq C - 1 \) largest eigenvalues of \( S_w^{-1} S_b \).

C. Class-Specific Discriminant Analysis

While LDA seeks to project all data samples to a common subspace where the data samples between classes are expected to be highly discriminated, class-specific discriminant analysis (CSDA) learns a subspace discriminating the class of interest from everything else. For a C-class classification problem, application of CSDA leads to the determination of \( C \) different discriminant subspaces \( \mathbb{R}^{d_i}, d_i < D, i = 1, \ldots, C \) in an One-versus-Rest manner, where \( d_i \) is the dimensionality of the ith subspace that discriminates samples of class \( c_i \) from the rest.

Let us denote \( p, n \) the positive and negative labels, respectively. The optimal mapping \( W \) is obtained by maximizing the following criterion

\[
J(W) = \frac{D_O}{D_I}
\]

where \( D_O = \sum_{j,j \neq p} \|W^T x_j - W^T m_p\|^2_F \) is the out-of-class distance and \( D_I = \sum_{j,j \neq p} \|W^T x_j - W^T m_p\|^2_F \) is the in-class distance, respectively. That is the positive class is assumed to be unimodal and the optimal projection matrix \( W \) maps the positive class vectors as close as possible to the positive class mean \( m_p \), while keeping the negative samples far away from \( m_p \) in the subspace. \( J(W) \) in (3) can be expressed as

\[
J(W) = \frac{\text{tr}(W^T S_O W)}{\text{tr}(W^T S_I W)}
\]

with

\[
S_O = \sum_{j,j \neq p} (x_j - m_p)(x_j - m_p)^T, S_I = \sum_{j,j \neq p} (x_j - m_p)(x_j - m_p)^T
\]

denoting the out-of-class and in-class scatter matrices, respectively. The solution of (4) is obtained by the eigenvectors corresponding to the \( d_i \) largest eigenvalues of \( S_I^{-1} S_O \).

The optimal dimensionality \( d_i \) may vary for each class. For classes that are already highly discriminated from the others, fewer dimensions may be needed as compared to classes that are densely mixed with other classes. Since the rank of \( S_I \) is at most \( n_p - 1 \), \( n_p \) is the number of samples from positive
class), the dimensionality of the learnt subspace can be at most \( \min(n_p - 1, D) \).

### D. Multilinear Discriminant Analysis

Several works have extended multi-class discriminant analysis criterion in order to utilize the natural tensor representation of the input data (21, 22, 19, 18, 17). We denote the set of \( N \) tensor samples as \( \{X_1, X_2, \ldots, X_N\} \), each with an associated class label \( l_j \) \( (j = 1, \ldots, N) \) belonging to the label set \( \{c_i \ | \ i = 1, \ldots, C\} \). The mean tensor of class \( c_i \) is calculated as \( M_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j} \) and the total mean tensor is \( M = \frac{1}{C} \sum_{i=1}^{C} \sum_{j=1}^{n_i} n_i M_i \). MDA seeks a set of projection matrices \( W_k \in \mathbb{R}^{I_1 \times \cdots \times I_K} \) that map \( X_{i,j} \) to \( Y_{i,j} \in \mathbb{R}^{I_1' \times \cdots \times I_K'} \) with the subspace projection defined as

\[
Y_{i,j} = X_{i,j} \prod_{k=1}^{K} \times_k W_k^T
\]  

Similar to LDA, the set of optimal projection matrices are obtained by maximizing ratio between the inter-class and intra-class distances, measured in the tensor subspace \( \mathbb{R}^{I_1' \times \cdots \times I_K'} \):

\[
J(W_1, \ldots, W_K) = \frac{D_b}{D_w}
\]

where

\[
D_b = \sum_{i=1}^{C} \sum_{j=1}^{n_i} \left\| X_{i,j} \prod_{k=1}^{K} \times_k W_k^T - M \prod_{k=1}^{K} \times_k W_k^T \right\|_F^2
\]

\[
D_w = \sum_{i=1}^{C} \sum_{j=1}^{n_i} \left\| X_{i,j} \prod_{k=1}^{K} \times_k W_k^T - M_i \prod_{k=1}^{K} \times_k W_k^T \right\|_F^2
\]

are, respectively, the between-class and within-class distances.

An iterative approach is usually employed to solve the optimization problem in (7). For example (17) proposed CMDA algorithm that assumes orthogonal constraints on each projection matrix \( W_k \) \( k = 1, k = 1, \ldots, K \) and optimizes (7) by iteratively solving the following trace ratio problem for each mode-\( k \):

\[
J(W_k) = \frac{\text{tr}(W_k^T S^b_k W_k)}{\text{tr}(W_k^T S^w_k W_k)}
\]

where

\[
S^b_k = \sum_{i=1}^{C} \sum_{j=1}^{n_i} \left[ (X_{i,j} - M) \prod_{q \neq k}^{K} \times_q W_q^T \right] \left[ (X_{i,j} - M) \prod_{q \neq k}^{K} \times_q W_q^T \right]^T\]

\[
S^w_k = \sum_{i=1}^{C} \sum_{j=1}^{n_i} \left[ (X_{i,j} - M) \prod_{q \neq k}^{K} \times_q W_q^T \right] \left[ (X_{i,j} - M) \prod_{q \neq k}^{K} \times_q W_q^T \right]^T
\]

are the between-class and within-class scatter matrices in mode-\( k \).

CMDA first initializes \( W_k, k = 1, \ldots, K \) with all ones. At each iteration, the algorithm sequentially updates \( W_k \) by maximizing (10) while keeping the rest projection matrices fixed.

### III. Multilinear Class-Specific Discriminant Analysis

In this section, we formulate the proposed multilinear version of CSA, called Multilinear Class-Specific Discriminant Analysis (MCSDA). MCSDA finds a set of projection matrices that map \( I_1 \times \cdots \times I_K \)-dimensional tensor space to a smaller tensor as defined in (9). The objective function of MDA is to find a tensor subspace in which the distances of the negative samples from the positive mean tensor are maximized and the distances of the positive samples from it are minimized.

Let us denote by \( M_p = \frac{1}{n_p} \sum_{j,l_j=p} X_j \) the mean tensor of the positive class. The out-of-class and in-class distances are defined as follows

\[
D_O = \sum_{j,l_j \neq p} \left\| X_j \prod_{k=1}^{K} \times_k W_k^T - M_p \prod_{k=1}^{K} \times_k W_k^T \right\|_F^2
\]

\[
D_I = \sum_{j,l_j = p} \left\| X_j \prod_{k=1}^{K} \times_k W_k^T - M_p \prod_{k=1}^{K} \times_k W_k^T \right\|_F^2
\]

The MCSDA criterion is then expressed as

\[
J(W_1, \ldots, W_K) = \frac{D_O}{D_I}
\]

As in case of MDA, the objective in (14) exposes a dependency between each \( W_k \). We therefore optimize (14) by applying an iterative optimization process. In order to optimize for each \( W_k, D_O \) and \( D_I \) need to be expressed as functions of \( W_k \). This can be done for \( D_O \) by utilizing the relation in (11), i.e.

\[
D_O^k = \sum_{j,l_j \neq p} \left\| W_k^T \left[ X_j \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)} - W_k^T \left[ M_p \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)} \right\|_F^2
\]

\[
= \text{tr} \left( W_k^T \sum_{j,l_j \neq p} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)}^T \right)
\]

where \( D_O^k \) denotes \( D_O \) after unfolding the projected tensor in mode-\( k \). Let us denote \( S_O^k \) the out-of-class scatter matrix in mode-\( k \), which is defined as

\[
S_O^k = \sum_{j,l_j \neq p} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)}^T
\]

Then \( D_O^k \) in (15) is expressed as \( D_O^k = \text{tr} \left( W_k^T S_O^k W_k \right) \). In a similar manner, the in-class distance calculated in mode-\( k \) is expressed as \( D_I^k = \text{tr} \left( W_k^T S_I^k W_k \right) \) with

\[
S_I^k = \sum_{j,l_j = p} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)} \left[ (X_j - M_p) \prod_{q \neq k}^{K} \times_q W_q^T \right]_{(k)}^T
\]

Finally, the class-specific criterion with respect to \( W_k \)

\[
J(W_k) = \frac{\text{tr}(W_k^T S_O^k W_k)}{\text{tr}(W_k^T S_I^k W_k)}
\]

MCSDA starts by initializing \( W_k \) with ones. At each iteration, it updates \( W_k \) by maximizing (18), while keeping the rest projection matrices fixed. A detailed description of the
Algorithm 1 MCSDA

Input: Training tensor $X_{ij} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$ and respective labels $c_i$; Subspace dimensionality $I'_1 \times \cdots \times I'_K$; maximum iteration $\tau$ and threshold $\epsilon$.

1: Initialize $W_k(0), k = 1, \ldots, K$ with $1$
2: for $t \leftarrow 1$ to $\tau$ do
3: for $k \leftarrow 1$ to $K$ do
4: Calculate $S_O$ according to (16) using $W_k(t-1)$
5: Calculate $S_T$ according to (17) using $W_k(t-1)$
6: Update $W_k(t)$ by solving (18)
7: end for
8: if $\sum_{k=1}^{K} \| W_k(t)W_k^T(t-1) - I \|_F \leq \epsilon$ then
9: Terminate
10: end if
11: end for

Output: Projection matrices $W_k, k = 1, \ldots, K$

MCSDA optimization process is presented in Algorithm 1.

A. Complexity Discussion

It is clear that the number of parameters of the tensor version is much lower compared to the vector version. Suppose the dimensionality of each tensor sample is $\mathbb{R}^{I_1 \times \cdots \times I_K}$, which corresponds to a vectorized sample in $\mathbb{R}^{\sum I_k}$. Given the tensor subspace is $\mathbb{R}^{I'_1 \times \cdots \times I'_K}$, the number of parameters for MCSDA is equal to $\sum_{k=1}^{K} I_k I'_k$. The corresponding CSA model projects each vectorized sample from $\mathbb{R}^{\sum I_k}$ to $\mathbb{R}^{\sum I'_k}$, requiring $\sum_{k=1}^{K} I_k I'_k$ parameters. In order better understand the difference between the two cases, let us consider the following example. For an image of size $30 \times 30$ pixels projected to a scalar, the tensor model learns $30 \times 30 \times 30$ parameters, while the vector model learns $30 \times 30 = 900$ parameters.

Regarding time complexity, let us denote $I = \prod_{k=1}^{K} I_k$ the total number of elements in input space and $I' = \prod_{k=1}^{K} I'_k$ the total number of elements in the learnt subspace. The solution of CSA involves the following steps:

- Calculation of $S_O$ and $S_I$ defined in (5) having time complexity of $O(NI^2)$.
- Calculation of $S_I^{-1}S_O$ requires matrix inversion of $S_I$ and matrix multiplication between $S_I^{-1}$ and $S_O$, having time complexity of $O(2I^3)$.
- Eigenvalue decomposition of $S_I^{-1}S_O$ having time complexity of $O(\frac{2}{3}I^3)$.

Thus the total time complexity of CSA is

$$O(NI^2 + \frac{13}{2}I^3)$$

(19)

MCSDA employs an iterative process parameterized by the terminating threshold $\epsilon$ and the number of maximum iteration $\tau$. At each iteration, MCSDA requires the following computation steps:

- Calculation of $S_T^k$ and $S_O$ requires the projection of $X_j$ to $\mathbb{R}^{I'_1 \times \cdots \times I'_K}$ having time complexity of $O(NI'_kI)$.
- Calculation of $(S_T^k)^{-1}S_O$ requires matrix inversion of $S_T^k$ and matrix multiplication between $(S_T^k)^{-1}$ and $S_O$, having time complexity of $O(2I'_k^3)$.
- Eigenvalue decomposition of $(S_T^k)^{-1}S_O$ having time complexity of $O(\frac{2}{3}I'_k^3)$.

Hence, the computational cost to update $W_k$ of MCSDA is $O(NI'_kI + \frac{13}{2}I'_k^3)$. Let $\tau$ be the number of maximum iteration, the maximum cost of computation of MCSDA is

$$O(\tau NI \sum_{k=1}^{K} I'_k + \frac{13}{2} \tau \sum_{k=1}^{K} I'_k^3)$$

(20)

Due to the iterative nature of MCSDA, it is not straightforward to compare the time complexity of MCSDA with that of CSA. Our experiments showed that with the maximum number of iteration set to $\tau = 20$, MCSDA already achieves good performance. In addition, for frequently encountered data, the number of tensor modes $K$ ranging from 2 to 4. For example, grayscale images, EEG multichannel data or time-series financial data has $K = 2$ while RGB images has $K = 3$ or video data has $K = 4$. Comparing the first two terms of (19) and (20) and noting the fact that the dimensions of the projected space are usually much smaller than the input, it is easy to see that $NI^2 = NI \prod_{k=1}^{K} I_k > NI \sum_{k=1}^{K} I_k$. Comparing the second term of (19) and (20), it is also clear that $\frac{13}{2}I^3 = \frac{13}{2} \prod_{k=1}^{K} I'_k^3 > \frac{13}{2} \tau \sum_{k=1}^{K} I'_k^3$.

To conclude, the solution of the vector model is more costly in terms of computation as compared to the tensor model. Moreover, the vector approach with $O(I^3)$ becomes impractical when $I$ scales to the order of thousands or more, which is usually the case. In contrast, the tensor approach with $O(I'_k^3)$ is scalable with high-dimensional input.

IV. EXPERIMENTS

In this section, we provide experiments conducted in order to evaluate the effectiveness of the proposed MCSDA and compare it with related discriminant analysis methods, namely vector-based Class-Specific Discriminant Analysis (CSDA) and Multilinear Discriminant Analysis (MDA) (17).

It should be noted that the class-specific methods model $C$ classes as $C$ binary problems, we therefore conducted the experiments in which $C$ one-vs-rest MDA classifiers are learned. We performed the benchmark in three publicly available datasets coming from two application domains: face verification and stock price prediction based on limit order book data. Detailed description of the datasets and the corresponding experimental settings are provided in the following subsections.

Since all the competing methods are subspace methods, after learning the optimal projection matrices, one can train any classifier on the data representations in the discriminant subspace to boost the performance. For example, the distance between training sample and each class mean vectors can be used as the training data for SVM classifier. In the test phase, a test sample is projected to the discriminant subspace and distances between test sample and each class mean are used.
as feature vector fed to the learnt SVM classifier, similar to \cite{13}. Since the goal of this paper is to directly compare the discrimination ability of MCSDA, compared to that of CSDA and MDA, we do not train any other classifier in the discriminant space. In the test phase, the similarity score is calculated as the inverse of the distance between the test sample and the positive mean in the discriminant space. The similarity scores are used to evaluate the performance of each algorithm, based on different metrics as described next.

A. Facial Image Datasets

Since tensor is a natural representation of image data, we employ two facial image datasets, namely ORL and CMU PIE, with different sizes to compare the performance of the tensor-based and vector-based methods. The ORL dataset \cite{23} consists of 400 facial images depicting 40 persons (10 images each). The images were taken at different times with different conditions in terms of lighting, facial expressions (smile/neutral) and facial details (open/closed eyes, with/without glasses). All of the images were captured in frontal position with a tolerance of rotation and tilting up to 20 degrees. The CMU PIE dataset \cite{24} consists of 64 individuals with 41,368 facial images in total. The images were captured with 13 different camera positions and 21 flashes under different pose, illumination and expression. All images in 5 near frontal positions (C05, C07, C09, C27, C29) of 8 individuals (55, 57, 58, 59, 61, 66, 67, 68, 69) were used in our experiments. Moreover, all images used in our experiments are in grayscale format.

Using the above datasets, we formulate multiple face verification problems. That is, a class-specific model is learned for a person of interest, either using class-specific or the multi-class (in this case binary) criterion. During the test phase, image a test image is presented and the model decides whether the image depicts the person of interest or not \cite{9, 12, 14}. We measure the performance of each model by calculating the Average Precision (AP) metric. This process is applied multiple times (equal to the number of persons in each dataset) and the performance of each approach is obtained by calculating the mean Average Precision (mAP) over all sub-experiments. We applied multiple experiments based on five different train/test split sizes, where $k$ percent of the data is randomly selected for training and the rest for testing with $k \in \{0.1, 0.2, 0.25, 0.35, 0.5\}$. For each value of $k$, 5 experiments were repeated and the average result is reported.

Regarding the preprocessing step, all facial images were cropped and resized to $40 \times 30$ pixels. For tensor-based approaches, we keep the projected dimension of both mode-$1$ and mode-$2$ equal, ranging from 2 to 20. The maximum number of iterations is set to $\tau = 20$ and the terminating threshold $\epsilon = 1e^{-5}$. To ensure stability, we regularized $S_{w}^{c}$ in MDA, $S_{l}^{c}$ in MCSDA and $S_{l}$ in CSDA by adding a scaled version of the identity matrix (using a value of $\lambda = 0.01$). We also investigated the case when additional information is available by generating HOG images \cite{23} from the original images and concatenating the original image and its HOG image to form a 3-mode tensors of size $40 \times 30 \times 2$. The results from the enriched version are denoted by CSDA-H, MCSDA-H and MDA-H.

B. Limit Order Book Dataset

In addition to image data, (multi-dimensional) time-series, like limit order book (LOB) data, also have a natural representation as tensors of two modes. In our experiments, a recently introduced LOB dataset, called FI-2010 \cite{26}, was used. FI-2010 collects order information of 5 Finnish stocks in 10 consecutive days, resulting in more than 4 millions of order events. For every 10 consecutive order events a 144-dimensional feature vector is extracted and a corresponding label is defined indicating the prospective change (increase, decrease or stationary) of the mid-price after the next 10 order events. For the vector models, each sample is of size 144 dimension, representing information from 10 most recent order events. In order to take into account more information in the recent past, our tensor models exploits a tensor sample of size $144 \times 10$, representing information from 100 most recent order events.

We followed the standard day-based anchored cross-validation sets provided by the database with 9 folds in total. For the tensor-based models, we varied the projected dimension of the first mode from 5 to 60 with a step of 5 and the second mode from 1 to 8. The values of $\tau$ and $\epsilon$ were the same as those used in the face verification experiments. Since FI-2010 is an unbalanced dataset with the mid-price remaining stationary most of the time, we cross-validated based on average $f1$ score per class and also report the corresponding accuracy, average precision per class, average recall per class. Since our experimental protocol is the same with that used in \cite{27} for the Bag-of-Words (BoF) and Neural Bag-of-Words (N-BoF) models, we directly report their results. In addition, we report the baseline results from the database \cite{24} using Single Layer Feed-forward Network (SLFN) and Ridge Regression (RR).

C. Results

The results from 2 facial datasets are presented in Table \ref{Table1} and Table \ref{Table2}. Moreover, the last column of Tables 1 and 2 shows the relative computation time ($\ell$) of each method measured on the same machine (normalized with respect to the computation time of the proposed MCSDA method). Comparing the vector model and the tensor model utilizing class-specific criterion, it is clear that CSDA slightly outperforms the proposed MCSDA. However, as can be seen, the computational time (normalized with respect to the training time of MCSDA) of CSDA is higher (by one or two orders of ten). The computational efficiency of the proposed MCSDA over CSDA becomes more significant when the dimension of the input scales up. While the number of elements in the input doubles, computation time of MCSDA-H scales favourably while CSDA-H requires approximately 7 times more computation compared to CSDA. The result justifies our analysis in the time complexity discussion section above.
Comparing the two tensor-based approaches, the proposed MCSDA outperforms MDA in most of the configurations of \( k \), while their computational times are similar. Regarding the exploitation of enriched information, we can observe that all competing methods achieved some improvements. The benefit of additional information is marginal when the training data is small but clearly visible when 50% of the data is used for training for the tensor-based methods. In contrast, the benefit of additional information for the vector-based model is very small.

The results for stock prediction using LOB data are presented in Table III. While the performance of MCSDA was not better than its vector counterpart in the above face verification experiments, MCSDA outperforms all competing methods in the stock prediction problem, including the more complex neural network-based bag-of-words model N-BoF (21).

The difference in the relative performance between the vector-based and tensor-based variants in the two different application domains can be explained by looking into the optimal dimensionality of the subspaces obtained for both CSDA and MCSDA. In the two image verification problems, the optimal dimensionality of the subspace obtained for MCSDA is equal to \( 7 \times 7 = 49 \) dimensions for both ORL and CMU PIE datasets. For CSDA, the optimal subspace dimensionality is equal to \( 27 \times 27 = 729 \) dimensions for ORL and \( 25 \times 25 = 625 \) dimensions for CMU PIE. This result shows that in the CSDA case, the number of parameters is much higher compared to its tensor counterpart. In facial images, several visual cues are usually necessary to perform the verification. Since the vector approach estimates many more parameters, more visual cues can be captured, which leads to better performance, compared to MCSDA. However, this comes with a much higher computational cost.

In the stock prediction problem, the difference between the number of estimated parameters for MCSDA and CSDA is small. Particularly, over 10 folds the average number of parameters estimated for MCSDA is approximately equal to 6300, while for CSDA is slightly over and equal to 6000. Since multilinear class-specific projection (MCSDA) can perform the projection along temporal mode (mode-2) too, MCSDA can potentially capture important temporal cues required to predict future movements in stock price. The experiment in stock price prediction problem shows the potential of multilinear techniques in general, and MCSDA in particular, in exploiting the multilinear structure of the time-series data.

### V. Conclusions

In this paper, we proposed a tensor subspace learning method that utilizes the intrinsic tensor representation of the data together with the class-specific discrimination criterion. We provided a theoretical discussion of the time complexity of the proposed method, compared with its vector counterpart. Experimental results show that the MCSDA is computationally efficient and scalable with performance close to its vector counterpart in face verification problems, while outperformed other competing methods in a stock price prediction problem based on Limit Order Book data.

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### Table I

| Dataset       | k = 0.1 | k = 0.2 | k = 0.25 | k = 0.35 | k = 0.5 |
|---------------|---------|---------|---------|---------|---------|
| CSDA          | 76.81   | 87.08   | 91.42   | 93.72   | 97.81   |
| MDA           | 63.99   | 75.21   | 79.26   | 81.41   | 88.41   |
| MCSDA         | 72.20   | 84.74   | 87.65   | 92.05   | 95.90   |

### Table II

| Dataset       | k = 0.1 | k = 0.2 | k = 0.25 | k = 0.35 | k = 0.5 |
|---------------|---------|---------|---------|---------|---------|
| CSDA          | 76.81   | 89.49   | 93.09   | 93.91   | 95.76   |
| MDA           | 79.77   | 88.29   | 89.67   | 90.07   | 91.20   |
| MCSDA         | 79.85   | 89.06   | 90.36   | 91.45   | 92.57   |

### Table III

| Method   | Accuracy  | Precision | Recall  | F1       |
|----------|-----------|-----------|---------|----------|
| RR       | 49.00 ± 2.66 | 43.30 ± 9.9 | 45.54 ± 5.2 | 42.52 ± 1.22 |
| SLFN     | 63.22 ± 7.04 | 62.66 ± 8.83 | 41.28 ± 4.94 | 38.24 ± 6.19 |
| CSDA     | 79.15 ± 4.89 | 40.26 ± 0.66 | 42.89 ± 1.87 | 40.29 ± 0.69 |
| MDA      | 83.92 ± 3.23 | 45.08 ± 1.37 | 45.92 ± 1.64 | 45.33 ± 1.33 |
| MCSDA    | 83.66 ± 3.69 | 46.11 ± 1.29 | 48.00 ± 1.63 | 46.72 ± 1.29 |
| N-BoF    | 67.59 ± 7.34 | 39.26 ± 0.96 | 31.44 ± 2.53 | 36.25 ± 2.86 |
| BoF      | 62.70 ± 6.73 | 42.28 ± 0.87 | 41.41 ± 3.68 | 41.63 ± 1.90 |
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