A new integrable equation valued on a Cayley–Dickson algebra

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Abstract
We introduce a new integrable equation valued on a Cayley–Dickson (C–D) algebra. In the particular case in which the algebra reduces to a complex one the new interacting term in the equation cancels and the equation becomes the known Korteweg–de Vries (KdV) equation. For each C–D algebra the equation has an infinite sequence of local conserved quantities. We obtain a Bäcklund transformation in the sense of Walquist–Estabrook for the equation for any Cayley–Dickson algebra, and relate it to a generalized Gardner equation. From this, the infinite sequence of conserved quantities follows directly, we give the explicit expression for the first few. From the Bäcklund transformation we get the Lax pair and the one-soliton and two-soliton solutions generalizing the known solutions for the quaternion valued KdV equation. From the Gardner equation we obtain the generalized modified KdV equation which also has an infinite sequence of conserved quantities. The new integrable equation is preserved under a subgroup of the automorphisms of the C–D algebra. In the particular case of the algebra of octonions, the equation is invariant under $SU(3)$.

Keywords: integrable systems, conservation laws, partial differential equations, rings and algebras

1. Introduction

Since the work of Miura and Gardner et al [1–5], carried out in the context of the Korteweg–de Vries (KdV) equation, a lot of interesting results about the so-called integrable systems have been obtained. The fascinating theory developed from that time involves, among other features, several methods to solve non-linear partial differential equations which arise in a wide range of mathematical and physical situations. This is the case, for example, of the Hirota
bilinear method [6], which allows one to obtain multisolitonic solutions and of the Bäcklund transformation obtained by Wahlquist and Estabrook [7] with its corresponding generating formula of solutions (including multisolitonic solutions) which acts as a nonlinear superposition principle.

One important aspect of these systems is the existence of infinite conserved quantities generalizing the well known property of totally integrable systems describing a finite number of degrees of freedom. See, for example [8], for a review of these systems. Also the quantum formulation of such systems and its classical limit obtained from the commuting conserved charges is an interesting topic.

A nice idea due to Gardner [1] allows one to obtain the infinite conserved quantities for the KdV equation using simultaneously a parametric auxiliar equation (the Gardner equation) and a parametric transformation (the Gardner transformation), relating solutions between both equations.

In [9] Chen obtained the Wahlquist–Estabrook formulation for the KdV equation starting from the associated Lax equations and using a discrete symmetry of the associated Gardner equation (see also [10]). Satsuma in [11] used the Wahlquist–Estabrook transformation to get the corresponding infinite sequence of conserved quantities.

Following these ideas, it is a natural question to ask which of the possible extensions of the KdV equation still possess the integrability properties of the KdV equation. An integrable Grassmann valued extension of KdV was obtained in [12]. A class of extensions of KdV equation arises by introducing supersymmetry. Several integrable supersymmetric extensions were given in [13–18]. These extensions are a special subset of the set of the coupled extensions. Coupled extensions of the KdV equation form a category by itself, containing among others the ones given in [19–21]. In particular, in [22, 23] we considered a \( \lambda \)-coupled KdV system. The case \( \lambda = 0 \) plays a relevant role in the description of 3D gravity [24].

Interesting extensions of the KdV equation follow by considering the defining field to be valued either in associative algebras [25], see also [26, 27], or in non-associative algebras arising as non-commutative generalizations of Jordan ones [28]. The complex KdV equation is also integrable with interesting blow-up properties [29].

Another extension of KdV can be given by considering a supersymmetric extension and then substituting the defining fields by Clifford valued fields [30, 31] (in order to analize a supersymmetric breaking procedure with nice stability properties for the resulting solitonic solutions) or by operators acting in some general space of functions [32].

In the present work we introduce a new integrable equation with the field valued on a Cayley–Dickson (C–D) algebra. The equation we consider introduces a new interaction term which couples the field to an external source. The term is reminiscent of the interaction of a vector and a spinor field arising in supersymmetric theories. In our case we consider the external field \( \nu \) a constant field valued on the C–D algebra. It could be interpreted as a mean value of the external field. The interaction term, which is zero for the real and complex algebras, depends explicitly on the structure constants of the C–D algebra. The integrable equation is naturally invariant under the automorphisms of the algebra which preserve the interacting terms. For the octonions the equation is invariant under \( SU(3) \). In this sense the new term we introduce is a symmetry breaking term in order to reduce the original global symmetry to a physical relevant one. \( SU(3) \) is relevant in the description of fundamental particles.

We notice that the non-associative algebras considered by Svinolupov and Sokolov [28] do not contain the octonion algebra nor the C–D algebras beyond it. In fact, they consider among others: Lie algebras, Jordan algebras, left-symmetric algebras, L-T algebras. The basic identity they use in their paper is given in equation (0.18) of the above reference. It is
strafight, by considering three elements of the octonionic imaginary basis not belonging to a quaternionic subspace, to verify that (0.18) is not satisfied for the octonion algebra and hence for any Cayley–Dickson algebra beyond it. For example, refer to the Fano plane in section 3, take $e_1, e_3, e_7 = e_1 e_2$ and use definitions in the referred paper.

We prove in this work that the field equation we introduce here is an integrable equation in the sense that it has an infinite sequence of conserved quantities. To do this we will introduce a Bäcklund transformation in the sense of Wahlquist–Estabrook and relate it to a generalized Gardner equation. From it one can directly obtain the infinite sequence of conserved quantities. The product on the transformation is defined in terms of the corresponding algebra. Its existence is a non-trivial result. Besides we obtain the associated Lax pair and obtain from the Bäcklund transformation the one-soliton and two-soliton solutions.

In [33, 34] an operatorial non-commuting associative approach to analyze non-linear evolution equations was introduced. In [33] multisolitonic solutions for the quaternion KdV equation were found. They have interesting properties beyond the one in the scalar KdV equation, showing the relevance of considering such extension. In [33] only two conserved quantities were found. In [34] a Miura transformation was obtained. In the present work we extend the soliton solution to an octonion valued one. We obtain, using explicit properties of the octonion algebra, the one-soliton and two-soliton solutions. They arise directly from the Bäcklund transformation we introduce. Besides we obtain an infinite sequence of conserved quantities for the non-linear evolution equation (2), not only for the octonion valued KdV equation but also for any C–D algebra valued KdV equation. The Miura transformation and a new modified KdV equation valued on any C–D algebra follow from the Gardner equation. It is also an integrable equation.

In connection with physics theories it is important to mention that, in particular, the octonion algebra is directly related to supersymmetric theories. For example octonion truncations of the supermembrane theories are interesting models for describing aspects of the unification of the fundamental forces in nature. Recently higher spin constructions have been related to the KdV equation [35]. See [36, 37] for a direct relation of C–D algebras and dimensionally reduced supersymmetric theories.

In section 2 we define a new equation with values on a general C–D algebra and we analyze its global symmetries. In section 3 we consider the particular case of the octonions, we use an explicit representation of the algebra of the $G_2$ exceptional Lie group of automorphisms of the octonions. In section 4 we introduce a Bäcklund transformation in the sense of Wahlquist–Estabrook for a KdV equation valued on a C–D algebra. In section 5 we relate the Wahlquist–Estabrook construction to a generalized Gardner transformation and Gardner equation, and we give an explicit Lax pair for the equation (2). We also obtain a generalized modified KdV equation. In section 6 we prove the existence of an infinite sequence of conserved quantities. In section 7 we obtain, using the Bäcklund transformation, the one and two solitonic solutions for the particular case of the octonion KdV equation. In section 8 we give the conclusions.

2. An integrable equation valued on the C–D algebra

The C–D algebras are constituted by a sequence of algebras, starting with the reals $\mathbb{R}$, obtained inductively with what is called the Cayley–Dickson process. At every stage of this process, a new algebra with twice the dimension of the previous one is formed by considering pairs of elements in the preceding algebra, with multiplication given by $(p, q)(r, s) = (pr - s'q, sp + qr^*)$ where $(p, q)^* = (p^*, -q)$ is the conjugation map and $a^* = a$ for $a \in \mathbb{R}$. The first four algebras generated by this process are, precisely, the normed division algebras: the reals $\mathbb{R}$, the
complex numbers \( \mathbb{C} \), the quaternions \( \mathbb{H} \), and the octonions \( \mathbb{O} \). The octonion algebra is a non-commutative, non-associative algebra, and alternative normed division algebra. All further algebras in this process have zero divisors and lack the alternative property; the next algebra in the sequence is known as the sedenions \( \mathbb{S} \). Nonetheless, all of the C–D algebras are power associative.

The Cayley–Dickson (C–D) algebra of dimension \( 2^n \) contains a basis \( \{e_0, e_1, \ldots, e_{2^n-1}\} \) with the following relations:

\[
e_0 e_0 = e_0, \quad e_0^* = e_0, \quad e_0 e_1 = e_1 e_0 = e_1, \quad e_i e_j = -e_j e_i, \quad e_i^* = -e_i, \quad e_i e_j = -e_j e_i,
\]

for \( i, j \in \{1, \ldots, 2^n - 1\} \) and \( i \neq j \).

We denote by \( C_{ijk} \) the structure constants defined by \( [e_i, e_j] = \sum_{k=1}^{2^n-1} C_{ijk} e_k \), with \( C_{ijk} \) being totally antisymmetric on the indices \( i, j, k \).

Any element \( x \) of the algebra can be expressed into its real and imaginary parts

\[
x = A_0 e_0 + A e_i = a + \bar{A}
\]

where \( a^* = a \) and \( \bar{A}^* = -\bar{A} \).

The associator of any three elements \( x, y, \) and \( z \) of the algebra is defined by

\[
[x, y, z] = (xy)z - x(yz).
\]

It is zero for associative algebras, as \( \mathbb{R} \), \( \mathbb{C} \), and \( \mathbb{H} \). It is skew symmetric for alternative algebras as the octonion algebra. The C–D algebras beyond the octonions fail to have the alternative property (that is, \( (x^2)y \neq x(xy) \) and \( (xy)y \neq x(y^2) \), in general). However, all of them are power associative algebras.

In what follows we assume \( u = u(x, t) \) a function with domain in \( \mathbb{R} \times \mathbb{R} \) valued on the C–D algebra. If we denote \( e_i, i = 1, \ldots, 2^n - 1 \) the imaginary basis of the C–D algebra, \( u \) can be expressed (for each pair of \( x \) and \( t \) in their domain) as

\[
u(x, t) = b(x, t) + \vec{B}(x, t),
\]

where \( b(x, t) \) is the real part and \( \vec{B} = \sum_{i=1}^{2^n-1} B_i(x, t) e_i \) its imaginary part.

The equation formulated on the C–D algebra is given by

\[
u_t + \nu_{xxx} + \frac{1}{2}(\nu^2)_x + [v, u] = 0,
\]

where the product in \( \nu^2 \) and in \( [\nu, u] \) is the product on the C–D algebra, \( v \) is a constant element on the C–D algebra, which can be interpreted as an external field.

If \( v = a + \bar{A} \) then

\[
[v, u] = (vy - uv) = [a + \bar{A}, b + \vec{B}] = [\bar{A}, \vec{B}] = A_j B_k C_{ijk} e_i,
\]

where \( [e_i, e_j] = \sum_{k=1}^{2^n-1} C_{ijk} e_k \), \( C_{ijk} \) are the structure constants characterizing the C–D algebra. For every C–D algebra \( C_{ijk} \) are completely antisymmetric on the three indices \( i, j, k \).

When \( \vec{B} = \vec{0} \) the equation reduces to the scalar KdV equation. For the real and complex algebras \( [v, u] = 0 \), \( [v, u] \) is always pure imaginary.

In terms of \( b \) and \( \vec{B} \) the equation can be re-expressed as

\[
b_t + b_{xxx} + bb_x - \sum_{i=1}^{2^n-1} B_i B_{ix} = 0,
\]

\[
4
\]
\[(B_i)_{t} + (B_i)_{x x x} + (bB_i)_{x} + \sum_{j,k=1}^{N-1} A_jB_kC_{jki} = 0. \tag{4}\]

In the case \(v = 0\), the space of solutions of the system (3) and (4) for any \(B_i, i = 1, \ldots, N\) is contained in the space of solutions of (2) for every Cayley–Dickson algebra with a number of imaginary generators greater than \(N\). In fact, if at \(t = 0\) \(B_i = 0\) for \(i \in I\) then equation (4) implies that \(B_i = 0\) for all \(t\) and \(i \in I\). That is, the system (3) and (4) with \(B_i, i = 1, \ldots, N\) is a subsystem of (2) for any C–D algebra with more imaginary generators than \(N\), in a way that all solutions of the subsystem are solutions of the C–D system. The subsystem is not invariant under the automorphisms of the C–D algebra. It is mapped under them into another subsystem whose space of solutions is also contained in the space of solutions of the C–D system (2). The space of solutions of the C–D algebra is manifestly invariant under the automorphisms of it. The converse is also valid: the space of solutions of (2) for a given C–D algebra is contained into the space of solutions of (3) and (4) for a given set \(B_i, i = 1, \ldots, N\) with \(N\) greater than the number of imaginary generators of the C–D algebra.

In the case \(v \neq 0\) we do not know which is the relation between the system (3) and (4), for any number of \(B_i, i = 1, \ldots, N\) and antisymmetric constants \(C_{ijk}\), and the system associated to a C–D algebra. The reason is that the interactions term \(A_jb_kC_{jki}\) combine in a non-trivial way the different components \(B_i\). Certainly, if they are not equivalent, the symmetries of the C–D system are going to be lost, so we lose one of the main properties for describing a physical system.

Equation (2) is invariant under Galileo transformations:

\[\begin{align*}
\tilde{x} &= x + ct, \\
\tilde{t} &= t, \\
\tilde{u} &= u + c, \\
\tilde{v} &= v,
\end{align*}\]

where \(c\) is a real constant.

Additionally, equation (2), when \(v \neq 0\), is invariant under the subgroup of automorphisms of the C–D algebra which preserves \(v\).

If under an automorphism

\[u \rightarrow \phi(u)\]

then

\[u_1u_2 \rightarrow \phi(u_1u_2) = \phi(u_1)\phi(u_2)\]

and consequently

\[\left[\phi(u)\right]_t + \left[\phi(u)\right]_{x x x} + \frac{1}{2} \left[\phi(u)^2\right]_x = 0.\]

We explicitly analyze the symmetry under the group \(G_2\), the automorphisms of the octonions in the next section.

In the case of the octonion algebra when \(v = 0\), the equation (4) and hence (3), is invariant under the action on \(B_i, i = 1, \ldots, 7\), by a rotation belonging to \(SO(7)\) and hence under \(G_2\), a subgroup of \(SO(7)\). In the general case, when \(v \neq 0\), the symmetry reduces to the subgroup of \(G_2\) which preserves \(v\). For example, if \(v\) is equal to one of the imaginary generators of the algebra then the subgroup is \(SU(3)\) [38].
3. Symmetries of the equation in the particular case of the octonions

We examine here in more detail, some relevant symmetries for equation (2) in the case of octonions.

According to Cartan’s classification of simple Lie groups, $G_2$ is the smallest exceptional Lie group. It is the group of automorphisms of the octonions. The tangent space to a group of automorphisms is an algebra of derivations. Therefore the Lie algebra $g_2$ of the Lie group $G_2$ is $\text{Der}(\mathbb{O})$. The elements in $\text{Der}(\mathbb{O})$ can be expressed as linear combinations of maps $D_{a,b} : \mathbb{O} \to \mathbb{O}$, for $a, b \in \mathbb{O}$, given by

$$D_{a,b}(x) = \frac{1}{2} \left( [a,x] + [a,[b,x]] + [[a,b],x] \right) = [[a,b],x] - 3[a,b,x]$$

where $[a,b] = ab - ba$ is the commutator and the bracket with three entries is the associator $[a,b,x] = (ab)x - a(bx)$. The associator for the octonionic algebra is totally antisymmetric.

The map satisfies the Leibniz rule

$$D_{a,b}(xy) = D_{a,b}(x)y + xD_{a,b}(y)$$

and the generalized Jacobi identity

$$D_{a,b}(D_{c,d}) = D_{D_{a,b}(c),d} + D_{c,D_{a,b}(d)} + D_{c,d}(D_{a,b}).$$

We denote the pure imaginary basis of the octonionic algebra by $e_i$, $i = 1, \ldots, 7$ and its multiplication rule is represented by the Fano plane in figure 1.

For each index $p \in \{1, \ldots, 7\}$ consider the three pairs of indices $(i, j), (r, s)$ and $(u, v)$ such that $e_p = e_i e_j = e_r e_s = e_u e_v$, then

$$D_{e_i e_j} + D_{e_r e_s} + D_{e_u e_v} = 0 \quad \text{(5)}$$

is identically satisfied.

A basis for $g_2$ can be given by 14 of these maps grouped in seven pairs, each of them constituted by $D_{e_i e_j}$ and $D_{e_r e_s}$ where $e_i e_j = e_r e_s = e_p$ for each $p \in \{1, \ldots, 7\}$.

Let us now consider the transformation of the octonionic KdV equation under an infinitesimal $G_2$ transformation

$$u \to u + \sum_{ij} \lambda^j D_{e_i e_j}(u) \quad \text{(6)}$$

where $\lambda^j = -\lambda^i$ are the infinitesimal parameters of the transformation and the summation is only over the 14 elements of the basis of $g_2$. The parameters $\lambda^j$ are independent of $(x, t)$ since the $G_2$ symmetry is a global one.

The transformation of (2) under (6) is

$$\phi \equiv (\delta u)_t + (\delta u)_{xxx} + \frac{1}{2} (\delta u \cdot u + u \cdot \delta u)_x + [\delta v, u] + [v, \delta u]$$

where $\delta u = \sum_{ij} \lambda^j D_{e_i e_j}(u)$.

Using now the Leibnitz rule for the derivation map we obtain

$$\phi = \lambda^j D_{e_i e_j} \left( u_t + u_{xxx} + \frac{1}{2} (u^2)_x + [v, u] \right) = 0$$

which shows that equation (2) is invariant under the transformation given in (6) if $v = 0$ and under the subgroup $SU(3)$ if $v$ is equal to one of the imaginary generators of the octonion algebra.
4. Bäcklund transformation for the Cayley–Dickson equation

The Wahlquist–Estabrook (WE) transformation [7] for the real KdV equation can be straightforwardly generalized for the complex KdV one. We prove, in what follows, its extension for the C–D algebras. It is a non-trivial result due to the new interacting term and to the fact that the field is valued in general on a non-associative, non-alternative algebra.

We define \( w \) through \( u = wx \). From (2) we obtain

\[
Q(w) \equiv w_t + w_{xxx} + \frac{1}{2}(wx)^2 + [v,w] = C(t)
\]

where \( C(t) \) is a function of \( t \) only. As in the case of the real KdV equation we can redefine

\[
\tilde{w} = w - \int_{-\infty}^{t} C(\tau) d\tau.
\]

We then obtain \( Q(\tilde{w}) = 0 \). In what follows we assume that this redefinition has been performed and consider \( Q(w) = 0 \) in order to simplify the notation.

The Bäcklund transformation for the C–D equation has the expression

\[
w_x + w'_x = \eta - \frac{1}{12} (w - w')^2
\]

(7)

\[
w_t + w'_t = \frac{1}{12} [ (w - w')^2 ]_{xx} - \frac{1}{2} w_x^2 - \frac{1}{2} w'_x^2 - [v,w + w']
\]

(8)

where \( w = w(x,t) \) and \( w' = w'(x,t) \) are valued on the C–D algebra.

We now prove the following proposition.

**Proposition 1.** If \( w \) and \( w' \) are solutions of (7) and (8), satisfying \( \Re(e^{w - w'}) \neq 0 \), then \( u = w_x \) and \( u' = w'_x \) are solutions of the C–D equation.

**Proof of proposition 1.** We consider \( Q(w) + Q(w') \), and obtain

\[
Q(w) + Q(w') = 0.
\]

We now consider the integrability condition for (7) and (8):

\[
(7)_t - (8)_x = 0.
\]
We get,
\[-\frac{1}{12} \left[ (w - w')_x (w - w') + (w - w') (w - w')_x \right] - \frac{1}{12} \left[ (w - w')^2 \right]_{xxx} + \frac{1}{2} \left[ w_{xx} w_x + w_x w_{xx} \right] + \frac{1}{2} \left[ w_x w_s' + w_s' w_x' \right] + \left[ v, w_x + w_x' \right] = 0. \tag{10}\]

Also,
\[-\frac{1}{12} \left[ (w - w')^2 \right]_{xxx} = -\frac{1}{12} (w - w')_{xxx} (w - w') - \frac{1}{4} (w - w')_{xx} (w - w')_x - \frac{1}{12} (w - w') (w - w')_{xxx}. \tag{11}\]

The fourth and fifth terms of (10) combine with the second and third terms of the right hand side member of equation (11) to give
\[
\begin{align*}
\frac{1}{2} \left[ w_{xx} w_x + w_x w_{xx} \right] &+ \frac{1}{2} \left[ w_x w_s' + w_s' w_x' \right] - \frac{1}{4} (w - w')_{xx} (w - w')_x \\
- \frac{1}{4} (w - w')_x (w - w')_{xx} &+ \frac{1}{4} (w_{xx} + w_{xx}') (w_x + w_x') + \frac{1}{4} (w_{xx} + w_{xx}') (w_x + w_x'). \tag{12}\end{align*}
\]

We may now use (7) to obtain
\[
\begin{align*}
\frac{1}{4} (w_{xx} + w_{xx}') (w_x + w_x') + \frac{1}{4} (w_{xx} + w_{xx}') (w_{xx} + w_{xx}') &= -\frac{1}{48} \left[ (w - w')_x (w - w') + (w - w') (w - w')_x \right] (w_x + w_x') \\
- \frac{1}{48} (w_{xx} + w_{xx}') &\left[ (w - w')_x (w - w') + (w - w') (w - w')_x \right]. \tag{13}\end{align*}
\]

Furthermore, using the definition of the associator,
\[
(w_x + w_x') \left[ (w - w')_x (w - w') \right] = \left[ (w_x + w_x') (w_x - w_x') \right] (w - w') - \left[ w_x + w_x', w_x - w_x', w - w' \right].
\]
\[
\left[ (w - w')_x (w - w') \right] (w_x + w_x') = (w - w') \left[ (w_x - w_x') (w_x + w_x') \right] + \left[ w - w', w_x - w_x', w_x + w_x' \right].
\]

Summation of the right hand members of these two equations yields
\[
\begin{align*}
\left[ (w_x)^2 - (w_x')^2 \right] (w - w') + (w - w') \left[ (w_x)^2 - (w_x')^2 \right] &+ \left[ w_x w_{xx} - w_{xx} w_x \right] (w - w') \\
+ (w - w') \left[ w_x w_x' - w_{xx} w_x \right] + \left[ w - w', w_x - w_x', w_x + w_x' \right] - \left[ w_x + w_x', w_x - w_x', w - w' \right].
\end{align*}
\]

In the same way
\[
\begin{align*}
\left[ (w_x - w_x') \left[ (w - w') \right] (w_x + w_x') = (w_x - w_x') \left[ (w - w') (w_x + w_x') \right] + \left[ w_x - w_x', w - w', w_x + w_x' \right],
\end{align*}
\]

\[
\begin{align*}
(w_x + w_x') \left[ (w - w') (w_x - w_x') \right] = (w_x + w_x') \left[ (w - w') (w_x - w_x') \right] (w_x - w_x') - \left[ w_x + w_x', w - w', w_x - w_x' \right].
\end{align*}
\]

Using (7) and the power associative property of the Cayley–Dickson algebras we get
\[(w_x - w'_x) [(w - w') (w_x + w'_x)] = (w_x - w'_x) [(w_x + w'_x) (w - w')] = (w_x - w'_x) (w_x + w'_x)] \]

\[\cdot (w - w') - [w_x - w'_x, w_x + w'_x, w - w'] = \left( (w_x^2 - (w'_x)^2) \right) (w - w') \]

\[+ (w_x w'_x - w'_x w_x) (w - w') - [w_x - w'_x, w_x + w'_x, w - w'], \]

\[\left[ (w_x + w'_x) (w - w') \right] (w_x - w'_x) = \left[ (w - w') (w_x + w'_x) \right] (w_x - w'_x) \]

\[= (w - w') \left( (w_x^2 - (w'_x)^2) \right) + (w - w') (w'_x w_x - w_x w'_x) + [w - w', w_x + w'_x, w_x - w'_x]. \]

Summation of the right hand members of these two equations yields

\[\left( (w_x^2 - (w'_x)^2) \right) (w - w') + (w - w') \left( (w_x^2 - (w'_x)^2) \right) + (w_x w'_x - w'_x w_x) (w - w') + \]

\[+ (w - w') (w'_x w_x - w_x w'_x) - [w_x - w'_x, w_x + w'_x, w - w'] + [w - w', w_x + w'_x, w_x - w'_x]. \]

We then have from (13)

\[\frac{1}{4} (w_{xx} + w'_{xx}) (w_x + w'_x) + \frac{1}{4} (w_x + w'_x) (w_{xx} + w'_{xx}) = - \frac{1}{24} \left( (w_x^2 - (w'_x)^2) \right) (w - w') \]

\[- \frac{1}{24} (w - w') \left( (w_x^2 - (w'_x)^2) \right) - \frac{1}{48} \left[ (w - w', w_x - w'_x, w_x + w'_x) \right] \]

\[- [w_x + w'_x, w_x - w'_x, w - w'] - [w_x - w'_x, w_x + w'_x, w - w'] + [w - w', w_x + w'_x, w_x - w'_x] \]

\[+ [w_x - w'_x, w - w', w_x + w'_x] - [w_x + w'_x, w - w', w_x - w'_x]]. \tag{14} \]

We may now evaluate explicitly the associators, to do so we replace \( w_x + w'_x \) by its expression given from (7), for example

\[[w_x - w'_x, w_x + w'_x, w - w'] = [w_x - w'_x, \eta, w - w'] - \frac{1}{12} \left[ w_x - w'_x, (w - w')^2, w - w' \right]. \]

Since \( \eta \) is real, the first term of the right hand side member is zero. The second term involves \( w - w' = a + \tilde{v} \), where \( a \) is its real part and \( \tilde{v} \) its imaginary one. We have, using the properties of the basis for any C–D algebra,

\[(w - w')^2 = a^2 + 2a\tilde{v} - \|\tilde{v}\|^2, \]

\[\left[ w_x - w'_x, a^2 + 2a\tilde{v} - \|\tilde{v}\|^2, a + \tilde{v} \right] = [w_x - w'_x, 2a\tilde{v}, \tilde{v}] = 2a [w_x - w'_x, \tilde{v}, \tilde{v}], \]

which is zero for any alternative algebra like the octonions but it is not zero for a generic C–D algebra.

However, we notice that using this result and the corresponding ones for the other associators in (14), the associators cancel by pairs: the first with the second, the third with the fifth and the fourth with the sixth.

For the latter term in (10) we have, using (7) and denoting \( w - w' = d + \tilde{D} \) and \( v = a + \tilde{A} \),

\[ [v, w_x + w'_x] = - \frac{1}{12} \left[ v, (w - w')^2 \right] = - \frac{1}{6} d [\tilde{A}, \tilde{D}] \]
also
\[
- \frac{1}{12} (w - w') (v, w - w') - \frac{1}{12} (v, w - w') (w - w') = - \frac{1}{12} (d + \vec{D}) \left( [\vec{A}, \vec{B}] \right) (d + \vec{D}) = - \frac{1}{6} d [\vec{A}, \vec{B}].
\]

We then obtain
\[
[Q(w) - Q(w')] (w - w') + (w - w') [Q(w) - Q(w')] = 0. \tag{15}
\]
If the real part of \( w - w' \) is different from zero: \( \Re (w - w') \neq 0 \), the above equation implies
\[
Q(w) - Q(w') = 0. \tag{16}
\]

In fact, if we denote
\[
w - w' = a + \vec{v},
Q(w) - Q(w') = c + \vec{w},
\]
the decomposition into its real and imaginary parts, then \((a + \vec{v})(c + \vec{w}) + (c + \vec{w})(a + \vec{v}) = 0\) imply that its real part is zero and so is its imaginary part.
That is,
\[
2ac + \vec{v} \vec{w} + \vec{w} \vec{v} = 0,
2a\vec{v} + 2c\vec{v} = 0,
\]
where due to the properties of the basis of the C–D algebra \( \vec{v} \vec{w} + \vec{w} \vec{v} \) is real.
If \( a \neq 0 \), we then get
\[
\vec{w} = - \frac{c}{a} \vec{v},
2ac + \vec{v} \vec{w} + \vec{w} \vec{v} = \frac{2c}{a} (a^2 - \vec{v} \vec{v}) = 0
\]
but \( a^2 - \vec{v} \vec{v} = a^2 + \| \vec{v} \|^2 \neq 0 \), hence we must have \( c = 0 \) and \( \vec{w} = 0 \), that is, equation (16).
Finally, from (9) and (16) we get
\[
Q(w) = Q(w') = 0
\]
and hence \( u = w \), and \( u' = w' \) are both solutions of the C–D KdV equation.

Remark 1. The converse of proposition 1 is not valid. Given \( u \) and \( u' \) solutions of (2) then \( w, w' \) defined through \( u = w \), \( u' = w' \) respectively, do not generally satisfy equations (7) and (8).

5. The Bäcklund transformation, the generalized Gardner equation and the Lax pair

We will assume, as in proposition 1, \( \Re (w - w') \neq 0 \). In the previous section we showed that equations (7) and (8) imply
\[ Q(w) - Q(w') = 0. \]

We have
\[ Q(w) - Q(w') = (w - w') + (w - w')_{xxx} + \frac{1}{2} (w_x')^2 - \frac{1}{2} (w_x')^2 + [v, w - w'] \]

where
\[ (w_x')^2 - (w_x')^2 = \frac{1}{2} (w_x + w_x') (w_x - w_x') + \frac{1}{2} (w_x - w_x') (w_x + w_x'). \]

We may now use (7) to obtain
\[ (w - w')_t + (w - w')_{xxx} + \frac{1}{4} \left( \eta - \frac{1}{12} (w - w')^2 \right) (w_x - w_x') \]
\[ + \frac{1}{4} (w_x - w_x') \left[ \eta - \frac{1}{12} (w - w')^2 \right] + [v, w - w'] = 0. \]

We introduce the field \( r(x, t) \) through the relation \( w - w' = 2\epsilon (r - \frac{3}{\epsilon^2}) \) where \( \epsilon \neq 0 \) is a real parameter. We get
\[ \eta - \frac{1}{12} (w - w')^2 = -\frac{1}{3} \epsilon^2 \left( r^2 - \frac{6r}{\epsilon^2} \right) \]
provided \( \eta = \frac{1}{\epsilon^2} \).

We finally obtain from (17)
\[ r_t + r_{xxx} + \frac{1}{2} \left( rr_x + r_x r \right) - \frac{1}{12} \left( (r^2)_{xx} + r_x (r^2) \right) \epsilon^2 + [v, r] = 0. \]

This equation, where \( r(x, t) \) is valued on the C–D algebra and the product is the one of the algebra, is the generalized Gardner equation. It allows us to obtain an infinite sequence of conserved quantities, as we will show in the next section.

**Proposition 2.** If \( w \) and \( w' \) are solutions of the Bäcklund equations (7) and (8) and \( \text{Re}(w - w') \neq 0 \), then \( r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon} (w - w') \) satisfy the generalized Gardner equation (18) and

\[ u = w_x + r_x + \frac{1}{6} \epsilon^2 r^2 \]
\[ u' = w_x' + r_x - \frac{1}{6} \epsilon^2 r^2 \]
are solutions of the C–D KdV equation.

If \( r(x, t) \) is a solution of the generalized Gardner equation (18), then \( u \) and \( u' \) given by (19) and (20) are solutions of the C–D equation and \( w, w' \) are solutions, by choosing the integration constants in a way that \( r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon} (w - w') \), of the Bäcklund equations (7) and (8).

**Remark 2.** We notice that \( u' \) is obtained from \( u \) by changing \( \epsilon \rightarrow -\epsilon \).

**Proof of proposition 2.** From the previous argument in this section \( r(x, t) \) is a solution of the Gardner equation. In addition, from (7)
\[ w_x + w'_x = 2r - \frac{1}{3} \epsilon^2 r^2, \quad (21) \]

and from the definition of \( r \) in terms of \( w - w' \)
\[ w_x - w'_x = 2\epsilon r_x, \quad (22) \]

From these two equations we obtain (19) and (20). Proposition 1 ensures that \( u, u' \) are solutions of (2).

If \( r(x, t) \) is a solution of the Gardner equation then defining \( w \) and \( w' \) from (19) and (20) we conclude that (21) and (22) are satisfied. By fixing the integration constant obtained from (22) in a way that
\[ r = \frac{3}{\epsilon^2} + \frac{1}{2\epsilon} (w - w'), \]

(17) is then satisfied. Since this is the integrability condition of (7) and (8) we obtain that \( w \) and \( w' \) are solutions of (7) and (8) (we notice that (7) arises directly from (21)). We now apply proposition 1 to show that \( u \) and \( u' \) given by (19) and (20) are solutions of (2).

Lax introduced an approach to construct a large class of nonlinear evolution equations with multisolitonic solutions. Besides, the Bäcklund transformation is a natural approach to obtain multisolitonic solutions starting from trivial ones.

In [9] a general way of obtaining the Bäcklund transformation from the Lax equations was found. Following these ideas one can obtain the Lax pair for the nonlinear equation (2) valued on a C–D algebra from the Bäcklund transformation we have considered. It is given by
\[
\mathcal{L} = [\mathcal{L}, \mathcal{P}],
\]
\[
\mathcal{L} = -\partial_x^2 - \frac{1}{6} u, \\
\mathcal{P} = 4\partial_x^3 + u\partial_x + \frac{1}{2} \partial_x u + 6v,
\]
defined for \( u(x, t) \) and \( v \) valued on a C–D algebra. The eigenvalued problem for the operator \( \mathcal{L} \), however, is well-defined on a Hilbert space constructed for division algebras only. Hence, this Lax pair can in principle be formulated up to the octonion algebra but not beyond it.

From (18) we obtain directly the modified KdV equation. First we multiply by \( \epsilon \), redefine \( \hat{r} = c \epsilon r \) and take the limit \( \epsilon \to \infty \).

We obtain,
\[ \hat{r} + \hat{r}_{xxx} - \frac{1}{12} \left( \left( \hat{r}^2 \right) \hat{r}_x + \hat{r}_x \left( \hat{r}^2 \right) \right) + [v, \hat{r}] = 0, \]
the modified KdV equation defined on a C–D algebra, with \( \hat{r} \) valued on it. The product is understood as the product in the C–D algebra.

6. An infinite sequence of conserved quantities

We first show that \( \int_{-\infty}^{+\infty} \Re [r(x, t)] \, dx \) is a conserved quantity of the generalized Gardner equation (18).
Proposition 3. Let $r(x,t)$ be a solution of the Gardner equation (18) and assume $r(x,t) \in \mathcal{L}(\mathbb{R})$, the Schwartz space of functions on $\mathbb{R}$, then $\int_{-\infty}^{+\infty} \Re \{r(x,t)\} \, dx$ is a conserved quantity.

Proof of proposition 3. Taking the real part of equation (18), we obtain

$$\Re \{r\}, + \Re \{r\}_{xxx} + \frac{1}{2} \Re \{r^2\}_x - \frac{1}{12} \epsilon^2 \Re \{r^2 r_x + r r^2\} = 0.$$ 

We now show that $\Re \{r^2 r_x + r r^2\}$ is a total derivative. In fact, we consider $r = a + \tilde{A}$ where $a$ is the real part of $r$ and $\tilde{A}$ its imaginary part. We get

$$r^2 = a^2 + \tilde{A}^2 + 2a\tilde{A}, \quad \tilde{A} = -\|\tilde{A}\|^2,$$

$$r^2 r_x + r r^2 = \left(a^2 - \|\tilde{A}\|^2 + 2a\tilde{A}\right) a_x + \left(a_x + \tilde{A}_x\right) \left(a^2 - \|\tilde{A}\|^2 + 2a\tilde{A}\right)$$

$$= 2a^2 a_x - 2\|\tilde{A}\|^2 a_x + 2a \left(\tilde{A} a_x + A_x\tilde{A}\right) + 2a^2 \tilde{A}_x + 2aa_x\tilde{A} - 2\|\tilde{A}\|^2 \tilde{A}_x,$$

$$\Re \{r^2 r_x + r r^2\} = \frac{2}{3} \left(\epsilon a^3\right)_x - 2\|\tilde{A}\|^2 a_x - 2a \left(\|\tilde{A}\|^2\right)_x = \left(\frac{2}{3} \epsilon a^3 - 2a\|\tilde{A}\|^2\right)_x.$$

Then

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \Re \{r(x,t)\} \, dx$$

$$= \int_{-\infty}^{+\infty} \left[-(\Re \{r\})_{xxx} - \frac{1}{2} (\Re \{r^2\})_x + \frac{1}{12} \epsilon^2 \left(\frac{2}{3} (\Re \{r\})^3 - 2\Re \{r\} \|\Im \{r\}\|^2\right)_x\right] \, dx = 0.$$

We are now able to construct, following a very well-known approach [1, 12, 13, 32, 39, 40], an infinite sequence of conserved quantities. Assuming a formal expansion of $r$ in powers of $\epsilon$, we can invert (19). We obtain

$$r = u - u\epsilon + \left(u_{xx} + \frac{1}{6} u^2\right) \epsilon^2 - \left(u_{xxx} + \frac{1}{3} u^2\right) \epsilon^3$$

$$+ \left(u_{xxx} + \frac{1}{3} u^2\right) \epsilon^4 + \frac{1}{6} \left(u_{xxxx} + \frac{1}{3} u^3 + u_{xx} u + (u_x)^2\right) \epsilon^4 + \cdots.$$ 

Using proposition 3 we get an infinite sequence of conserved quantities.

The first few of them are

$$H_1 = \int_{-\infty}^{+\infty} \Re \{u\} \, dx,$$

$$H_2 = \int_{-\infty}^{+\infty} \left((\Re \{u\})^2 - \|\Im \{u\}\|^2\right) \, dx,$$

$$H_3 = \int_{-\infty}^{+\infty} \left(\frac{1}{3} (\Re \{u\})^3 - (\Re \{u\})^2 + \|\Im \{u\}\|^2 - \Re \{u\} \|\Im \{u\}\|^2\right) \, dx.$$ 

Also, if $v = 0$, we have that $\int_{-\infty}^{+\infty} \Im \{u\} \, dx$ is a conserved quantity.
In addition to the above conserved quantities, valid for any solution \( r(x, t) \) of the Gardner equation, also valid is the following property for particular solutions of the equation.

Assuming \( v = 0 \) and that there exist solutions of the Gardner equation for which \( ||m(r)||^2 \) is constant. That is, if \( r = a + A \), suppose there are solutions such that \( \bar{A}A + A\bar{A} = 0 \). Then \( \int_{-\infty}^{+\infty} \text{Im}(r)dx \) is conserved. In fact, for these particular solutions we have

\[
(r^2)_{x} + r_{x}(r^2) = (r^3)_{x} - \left(2||m(r)||^2\bar{A}\right)_{x},
\]

where \( r^3 = a^3 - 3a||m(r)||^2 + 3a^2\bar{A} - ||m(r)||^2\bar{A} \) and \( ||m(r)||^2 = -\bar{A}A \).

7. Soliton solutions from the Bäcklund transformation

In this section we obtain the one-soliton and two-soliton solutions of the octonion KdV equation with and without interaction term \([v, -\cdot]\). As we emphasized earlier this is a symmetry breaking term in order to reduce the symmetry group, preserving the KdV equation, to \( SU(3) \) (the group related to the physics of quarks).

As it is well-known, the inverse of an octonion \( \alpha \ (\alpha \in \mathbb{O}) \), is \( \frac{1}{\alpha} \equiv \frac{\bar{\alpha}}{||\alpha||^2} \), where \( \alpha = \alpha_0 + \bar{\alpha} \) and \( \bar{\alpha} = \alpha_0 - \bar{\alpha} \), \( \bar{\alpha} \) is its pure imaginary part which is expressed in terms of the basis elements \( e_1, \ldots, e_7 \) and \( \alpha_0 \) is its real part.

If \( F(x, t) \) is an octonion valued function with values different from zero then

\[
\partial_x \left( \frac{1}{F} \right) = -\frac{1}{2}(\partial_x F) \left( \frac{1}{F^2} \right) \partial_x F,
\]

the associator of the three factors in this expression is zero.

We consider \( f = e^{-\lambda x + \lambda t} \) a wave function as usual in the construction of solitary waves. We then notice that \( w_1 = 6\lambda \left( \frac{1}{\alpha + f} \right) (\alpha - f), \alpha \in \mathbb{O}\setminus\mathbb{R}_- \), is a regular solution of the Bäcklund equation (7). In fact, the left hand member of (7) is

\[
\begin{align*}
w_{1x} &= 6\lambda \left[ \left( \frac{1}{\alpha + f} \right) (\alpha - f) + \left( \frac{1}{\alpha + f} \right) (\alpha - f) \right] = 6\lambda \left( \frac{-f_x}{(\alpha + f)^2} \right) (\alpha - f) + \left( \frac{1}{\alpha + f} \right) (-f_x) \\
&= -6\lambda f_x \left( \frac{1}{(\alpha + f)^2} \right) (\alpha - f + \alpha + f) = \frac{12\lambda^2 af}{(\alpha + f)^2},
\end{align*}
\]

(23)

where no ambiguities arise because only \( \alpha \) and \( \bar{\alpha} \) are involved in the expression, hence any associator is zero.

On the other side, the right hand member of (7) is

\[
\eta - \frac{1}{12}36\lambda^2 \left( \frac{1}{(\alpha + f)} (\alpha - f) \right)^2 = \eta - 3\lambda^2 \left( \frac{1}{(\alpha + f)^2} \right) (\alpha - f)^2,
\]

since \( \frac{1}{\alpha + f} \) and \( (\alpha - f) \) commute and the associator of any three octonions in the expression is zero.

If \( \eta = 3\lambda^2 \) the above expression reduces to

\[
3\lambda^2 \left( \frac{1}{\alpha + f} \right)^2 (\alpha + f)^2 - 3\lambda^2 \left( \frac{1}{(\alpha + f)^2} \right) (\alpha - f)^2 = \frac{12\lambda^2 af}{(\alpha + f)^2},
\]

which is the same as the left hand member of (7).
When $\alpha$ is real and positive the expression $u_1 = w_1 x$ is the one-soliton solution for the scalar KdV equation. The presence of an octonion in the expression introduces non-trivial properties on the multisolitonic solution. In the particular case in which $\alpha$ is a quaternion this solution was found in [33]. In that work it arises from a non-commutative operatorial approach which is valid for the quaternion algebra but not valid for the non-associative octonion algebra. In our case, besides its extension to a more general algebra, it arises as a first step analysis of the Bäcklund equations. The multisoliton solutions in [33] have interesting and non-trivial properties, not present in the scalar case.

It is straightforward to show that $w_2 = -6\lambda \left( \frac{1}{f-\alpha} \right) (f + \alpha)$ is also a solution of (7) (change $\alpha$ to $-\alpha$). If $\alpha \in \mathbb{O}\setminus\mathbb{R}$ then $w_1$ and $w_2$ are regular solutions of (7). In distinction when $\alpha$ is real and different from zero then one of them, $w_1$ or $w_2$, is singular at some value of $x$.

From (7) it follows that $u_1 = w_1 x$ and $u_2 = w_2 x$ are solutions of

$$u_{xx} + \frac{1}{2} u^2 - \lambda^2 u = 0$$

and of the KdV equation

$$u_t + u_{xxx} + \frac{1}{2} (u^2)_x = 0.$$  
(25)

Also if $\alpha = \alpha_0 + \vec{\alpha}$ satisfies $\vec{\alpha} = \gamma \vec{v}$, $\gamma$ real, then $u_1$ and $u_2$ are solutions of equation (2). In that case $[v,u] = 0$ and $v$ reduces the symmetry group of the space of solutions to $SU(3)$.

The two-soliton solution is obtained from (7) and the permutability property of the Bäcklund transformation,

$$w = 12 \frac{(\eta_2 - \eta_1)}{w_2 - w_1},$$

$\eta_1$ and $\eta_2$ are the parameters associated to $u_1$ and $u_2$, respectively.

It can be verified, after several calculations where careful treatment of the associator of some expressions has to be considered, that $u = w_2$ satisfies (25).

It generalizes the two soliton solution of the scalar and quaternion [33] solutions. Moreover, it can be shown that for any octonions $\alpha$ and $\beta$, $u = w$ with

$$w = 12 \left( \eta_\alpha - \eta_\beta \right) \left( \frac{1}{w_\alpha - w_\beta} \right),$$

satisfies (25), where

$$w_\alpha = 6\lambda_\alpha \left( \frac{1}{\alpha + f_\alpha} \right) (f - \alpha),$$

$$f_\alpha = \exp \left( -\lambda_\alpha x + \lambda^3_\alpha t \right), \quad \eta_\alpha = 3\lambda^3_\alpha$$

and the analogous expression for $w_\beta$ in terms of the parameter $\lambda_\beta$ and the octonion $\beta$.

When $\vec{\alpha} = \gamma_1 \vec{v}, \vec{\beta} = \gamma_2 \vec{v}$ for any real $\gamma_1, \gamma_2$ and $\alpha_0, \beta_0, u$ is also a solution of equation (2). In this case we have a six parameter space of solutions: the real parts of the octonions $\alpha_0, \beta_0$, the scaling factors $\gamma_1, \gamma_2$ and the soliton parameters $\lambda_\alpha$ and $\lambda_\beta$. 


8. Conclusions

We introduced a new integrable equation valued on a general Cayley–Dickson algebra. The non-linear equation reduces to the complex KdV equation when the algebra reduces to the complex one. One of the interacting terms, the self interacting one, is formally the same as the one in the KdV equation, however it is now valued on a non-associative and a non-alternative algebra. The other interacting term does not appear in the KdV equation and is similar to the interaction of a vector and a spinor field on a supersymmetric theory. It is a symmetry breaking term. For the octonion algebra the symmetry of the KdV equation reduces to the $SU(3)$ group, which is interesting since it is expected that the physics of the quarks can be modelled in terms of octonions.

We obtained a Bäcklund transformation in the sense of W–E for the non-linear integrable equation. It is a non-trivial transformation since it works for a general C–D algebra with a new interaction term, which is absent in the scalar KdV equation. From the Bäcklund transformation we got the Lax pair associated to the KdV equation valued on the C–D algebra. The extension of the corresponding eigenvalued problem can in principle be well-formulated on a Hilbert space defined on a division algebra, that is up to the octonions but not beyond them.

We also obtained the associated Gardner transformation for the non-linear equation. From it, by inverting the Gardner transformation we show the existence of an infinite sequence of conserved quantities for any C–D algebra. From the generalized Gardner equation we obtained the modified KdV equation on a C–D algebra. It also has an infinite sequence of conserved quantities.

The new non-linear equation we proposed has additional symmetries besides its invariance under Galileo transformations. It is invariant under the automorphisms of the C–D algebra which preserve the interaction source $v$. In particular, for the octonion algebra, it is invariant under the $SU(3)$ group.

The Bäcklund transformation besides its use in proving the existence of an infinite sequence of conserved quantities allows one, in the usual way, to obtain solutions to the non-linear equation starting from the trivial one. From it we got the one-soliton and two-soliton solutions valued on the octonion algebra. They are the generalizations of the quaternion solutions found in [33].

An interesting open problem is the existence of a bi-Hamiltonian structure associated to all the integrable systems we have considered. For the KdV equation the first and second Hamiltonian structures were obtained in [4, 41]. Recently the bi-Hamiltonian structures of KdV type for several integrable systems was obtained in [42].

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