On mixture representations for the generalized Linnik distribution and their applications in limit theorems

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Abstract: We present new mixture representations for the generalized Linnik distribution in terms of normal, Laplace and generalized Mittag-Leffler laws. In particular, we prove that the generalized Linnik distribution is a normal scale mixture with the generalized Mittag-Leffler mixing distribution. Based on these representations, we prove some limit theorems for a wide class of statistics constructed from samples with random sized including, e. g., random sums of independent random variables with finite variances, in which the generalized Linnik distribution plays the role of the limit law. Thus we demonstrate that the scheme of geometric (or, in general, negative binomial) summation is far not the only asymptotic setting (even for sums of independent random variables) in which the generalized Linnik law appears as the limit distribution.

Key words: generalized Linnik distribution; Mittag-Leffler distribution; exponential distribution; Weibull distribution; Laplace distribution; strictly stable distribution; gamma distribution, generalized gamma distribution, random sum; central limit theorem; normal scale mixture; folded normal distribution; sample with random size

1 Introduction

In this paper we continue the research we started in \cite{17, 18}. We study the interrelationship between the (generalized) Linnik and (generalized) Mittag-Leffler distributions. In \cite{17} we showed that along with the traditional and well-known representation of the Linnik distribution as the scale mixture of a strictly stable law with exponential mixing distribution, there exists another representation of the Linnik law as the normal scale mixture with the Mittag-Leffler mixing distribution. The former representation makes it possible to treat the Linnik law as the limit distribution for geometric random sums of independent identically distributed random variables (r.v.’s) in which summands have infinite variances. The latter normal scale mixture representation opens the way to treating the Linnik distribution as the limit distribution in the central limit theorem for random sums of independent random variables in which summands have finite variances. Moreover, being scale mixtures of normal laws, the Linnik distributions can serve as the one-dimensional distributions of a special subordinated Wiener process often used as models of the evolution of stock prices and financial indexes. Strange as it may seem, the results concerning the possibility of representation of the Linnik distribution as a scale mixture of normals were never explicitly presented in the literature in full detail before the paper \cite{17} saw the light, although the property of the Linnik distribution to be a normal scale

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mixture is something almost obvious. Perhaps, the paper [24] was the closest to this conclusion and exposed the representability of the Linnik law as a scale mixture of Laplace distributions with the mixing distribution written out explicitly.

Here we consider the generalized Linnik distribution and prove new mixture representations for it in terms of normal, Laplace, exponential and stable laws and establish the relationship between the mixing distributions in these representations. In particular, we prove that the generalized Linnik distribution is a normal scale mixture with the generalized Mittag-Leffler mixing distribution. Based on these representations, we prove some limit theorems for a wide class of rather simple statistics constructed from samples with random sizes including, e.g., random sums of independent random variables with finite variances and maximum random sums, in which the generalized Linnik distribution plays the role of the limit law. Thus we demonstrate that the scheme of geometric (or, in general, negative binomial) summation is far not the only asymptotic setting (even for sums of independent r.v.’s) in which the generalized Linnik law appears as the limit distribution.

The presented material substantially relies on the results of [17] and [30]. The paper is organized as follows. Section 2 presents the definitions and basic properties of the Linnik, generalized Linnik, and Mittag-Leffler distributions. Section 3 contains basic definitions and auxiliary results. In Section 4 we prove the representability of the generalized Linnik distribution as the normal scale mixture of normal laws with the generalized Mittag-Leffler mixing distribution. We show that if the ‘generalizing’ parameter $\nu$ does not exceed 1, then the generalized Linnik distribution is a scale mixture of ‘ordinary’ Linnik distributions with the same characteristic parameter $\alpha$. We prove the representation of the generalized Linnik distribution as a scale mixture of the Laplace laws with the mixing distribution explicitly determined as that of the randomly scaled ratio of two independent r.v.’s with the same strictly stable distribution concentrated on the nonnegative halfline. Here we also discuss some properties of the generalized Mittag-Leffler distribution. In Sections 5 and 6 we prove and discuss some criteria (that is, necessary and sufficient conditions) for the convergence of the distributions of rather simple statistics constructed from samples with random sizes including, e.g., random sums of independent r.v.’s with finite variances to the generalized Linnik law.

2 The Linnik and Mittag-Leffler distributions

2.1 The Linnik distributions

In 1953 Yu. V. Linnik [31] introduced the class of symmetric probability distributions defined by the characteristic functions

$$f^L_\alpha(t) = \frac{1}{1 + |t|^\alpha}, \quad t \in \mathbb{R},$$

where $\alpha \in (0, 2]$. Later the distributions of this class were called Linnik distributions [25] or $\alpha$-Laplace distributions [35]. In this paper we will keep to the first term that has become conventional. With $\alpha = 2$, the Linnik distribution turns into the Laplace distribution corresponding to the density

$$f^\Lambda(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}.$$  \hspace{1cm} (2)

A r.v. with Laplace density (2) and its distribution function will be denoted $\Lambda$ and $F^\Lambda(x)$, respectively. A random variable with the Linnik distribution with parameter $\alpha$ will be denoted $L_\alpha$. Its distribution function will be denoted $F^L_\alpha$. As this is so, from (1) and (2) it follows that $F^L_\alpha(x) \equiv F^\Lambda(x), \quad x \in \mathbb{R}$.

The Linnik distributions possess many interesting analytic properties such as unimodality [28] and infinite divisibility [6], existence of an infinite peak of the density for $\alpha \leq 1$ [6], etc.
In [22, 23] a detailed investigation of analytic and asymptotic properties of the density of the Linnik distribution was carried out. However, perhaps, most often Linnik distributions are recalled as examples of geometric stable distributions.

This means that if $X_1, X_2, \ldots$ are independent r.v.'s whose distributions belong to the domain of attraction of an $\alpha$-strictly stable symmetric law and $N_{B_1,p}$ is the r.v. independent of $X_1, X_2, \ldots$ and having the geometric distribution

$$P(N_{B_1,p} = n) = p(1 - p)^{n-1}, \quad n = 1, 2, \ldots, \quad p \in (0, 1),$$

then for each $p \in (0, 1)$ there exists a constant $a_p > 0$ such that

$$a_p(X_1 + \ldots + X_{N_{B_1,p}}, p) \Rightarrow L_\alpha$$

as $p \to 0$, see, e. g., [5] or [14] (the symbol $\Rightarrow$ hereinafter denotes convergence in distribution).

In [34], Pakes showed that the probability distributions known as generalized Linnik distributions which have characteristic functions

$$\phi_{\alpha,\nu,\theta}(t) = 1 + e^{-\theta \text{sgn}(t)|t|^\alpha} e^{-\nu|t|}, \quad t \in \mathbb{R}, \quad |\theta| \leq \min\left\{\frac{\pi \alpha}{2}, \frac{\pi - \pi \alpha}{2}\right\}, \quad \nu > 0, \quad (3)$$

play an important role in some characterization problems of mathematical statistics. The class of probability distributions with density functions corresponding to ch.f. (3) have found some interesting properties and applications, see [1, 2, 6, 10, 24–26, 29] and related papers. In particular, they are good candidates to model financial data which exhibits high kurtosis and heavy tails [33]. Here we concentrate our attention at the symmetric case $\theta = 0$.

### 2.2 The Mittag-Leffler distributions

The Mittag-Leffler probability distribution is the distribution of a nonnegative r.v. $M_\delta$ whose Laplace transform is

$$\psi_\delta(s) \equiv \mathbb{E}e^{-sM_\delta} = \frac{1}{1 + \lambda s^\delta}, \quad s \geq 0, \quad (4)$$

where $\lambda > 0$, $0 < \delta \leq 1$. For simplicity, in what follows we will consider the standard scale case and assume that $\lambda = 1$.

The origin of the term Mittag-Leffler distribution is due to the fact that the probability density of $M_\delta$ corresponding to Laplace transform (4) has the form

$$f_\delta^M(x) = \frac{1}{x^{1-\delta}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{\delta n}}{\Gamma(\delta n + 1)} = -\frac{d}{dx} E_\delta(-x^\delta), \quad x \geq 0, \quad (5)$$

where $E_\delta(z)$ is the Mittag-Leffler function with index $\delta$ that is defined as the power series

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)}, \quad \delta > 0, \quad z \in \mathbb{Z}.$$

Here $\Gamma(s)$ is Euler’s gamma-function,

$$\Gamma(s) = \int_0^{\infty} z^{s-1} e^{-z} dz, \quad s > 0.$$}

The distribution function (d.f.) corresponding to density (5) will be denoted $F_\delta^M(x)$.

With $\delta = 1$, the Mittag-Leffler distribution turns into the standard exponential distribution, that is, $F_1^M(x) = [1 - e^{-x}]1(x \geq 0)$, $x \in \mathbb{R}$ (here and in what follows the symbol $1(C)$ denotes the indicator function of a set $C$). But with $\delta < 1$ the Mittag-Leffler distribution density has...
the heavy power-type tail: from the well-known asymptotic properties of the Mittag-Leffler function it can be deduced that if $0 < \delta < 1$, then

$$ f^M_\delta(x) \sim \frac{\sin(\delta \pi) \Gamma(\delta + 1)}{\pi x^{\delta+1}} $$

as $x \to \infty$, see, e. g., [12].

It is well-known that the Mittag-Leffler distribution is geometrically stable. The history of the Mittag-Leffler distribution is discussed in [17].

The Mittag-Leffler distributions are of serious theoretical interest in the problems related to thinned (or rarefied) homogeneous flows of events such as renewal processes or anomalous diffusion or relaxation phenomena, see [9, 38] and the references therein.

Let $\nu > 0$, $\delta \in (0, 1]$. By analogy to (3), the distribution of a nonnegative r.v. $M_{\delta, \nu}$ defined by the Laplace–Stieltjes transform

$$ \psi_{\delta, \nu}(s) \equiv Ee^{-sM_{\delta, \nu}} = \frac{1}{(1 + s^\delta)^\nu}, \quad s \geq 0, $$

will be called the generalized Mittag-Leffler distribution, see [11, 32] and the references therein.

### 3 Basic notation and auxiliary results

Most results presented below actually concern special mixture representations for probability distributions. However, without any loss of generality, for the sake of visuality and compactness of formulations and proofs we will represent the results in terms of the corresponding r.v.’s assuming that all the r.v.’s mentioned in what follows are defined on the same probability space $(\Omega, \mathcal{A}, P)$.

The r.v. with the standard normal d.f. $\Phi(x)$ will be denoted $X$,

$$ P(X < x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2}dz, \quad x \in \mathbb{R}. $$

Let $\Psi(x)$, $x \in \mathbb{R}$, be the d.f. of the maximum of the standard Wiener process on the unit interval, $\Psi(x) = 2\Phi(\max(0, x) - 1, x \in \mathbb{R}$. It is easy to see that $\Psi(x) = P(|X| < x)$. Therefore, sometimes $\Psi(x)$ is said to determine the half-normal or folded normal distribution.

A r.v. having the gamma distribution with shape parameter $r > 0$ and scale parameter $\lambda > 0$ will be denoted $G_{r, \lambda}$,

$$ P(G_{r, \lambda} < x) = \int_0^x g(z; r, \lambda)dz, \quad \text{with} \ g(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1}e^{-\lambda x}, \ x \geq 0, $$

where $\Gamma(r)$ is Euler’s gamma-function, $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx, r > 0$.

In this notation, obviously, $G_{1,1}$ is a r.v. with the standard exponential distribution: $P(G_{1,1} < x) = [1 - e^{-x}] \mathbf{1}(x \geq 0)$ (here and in what follows $\mathbf{1}(A)$ is the indicator function of a set $A$).

The gamma distribution is a particular representative of the class of generalized gamma distributions (GG distributions), which were first described in [36] as a special family of lifetime distributions containing both gamma and Weibull distributions.

A **generalized gamma (GG) distribution** is the absolutely continuous distribution defined by the density

$$ f(x; r, \alpha, \lambda) = \frac{\alpha |\lambda|^r x^{\alpha r-1}e^{-\lambda x^\alpha}}{\Gamma(r)}, \quad x \geq 0, $$

with $\alpha \in \mathbb{R}$, $\lambda > 0$, $r > 0$. 

A r.v. with the density $g(x; r, \alpha, \lambda)$ will be denoted by $G_{r, \alpha, \lambda}$.

It is easy to see that

$$
G_{r, \alpha, \mu} \xrightarrow{d} G_{r, \mu} \xrightarrow{d} \mu^{-1/\alpha} G_{r, 1} \xrightarrow{d} \mu^{-1/\alpha} G_{r, \alpha, 1}.
$$

For a r.v. with the Weibull distribution, a particular case of GG distributions corresponding to the density $g(x; 1, \alpha, 1)$ and the d.f. $\left[1 - e^{-x^\alpha}\right]1(x \geq 0)$ with $\alpha > 0$, we will use a special notation $W_\alpha$. Thus, $G_{1, 1} \xrightarrow{d} W_1$. It is easy to see that $W_{1/\alpha} \xrightarrow{d} W_\alpha$.

The d.f. and the density of the strictly stable distribution with the characteristic exponent $\alpha$ and shape parameter $\theta$ defined by the characteristic function

$$
p_{\alpha, \theta}(t) = \exp \left\{-|t|^\alpha \exp\left\{-\frac{1}{2}i\theta \alpha \text{sgn} t\right\}\right\}, \quad t \in \mathbb{R},
$$

with $0 < \alpha \leq 2$, $|\theta| \leq \min\{1, \frac{2}{\alpha} - 1\}$, will be denoted by $P_{\alpha, \theta}(x)$ and $p_{\alpha, \theta}(x)$, respectively (see, e.g., [39]). Any r.v. with the d.f. $P_{\alpha, \theta}(x)$ will be denoted $S_{\alpha, \theta}$. For definiteness, $S_1, 1 \xrightarrow{d} 1$ (throughout the paper the symbol $\xrightarrow{d}$ will denote the coincidence of distributions).

From (6) it follows that the characteristic function of a symmetric ($\theta = 0$) strictly stable distribution has the form

$$
p_{\alpha, 0}(t) = e^{-|t|^\alpha}, \quad t \in \mathbb{R}.
$$

(7)

From (7) it is easy to see that $S_{2, 0} \xrightarrow{d} \sqrt{2}X$.

**Lemma 1.** Let $\alpha \in (0, 2], \alpha' \in (0, 1]$. Then

$$
S_{\alpha, \alpha', 0} \xrightarrow{d} S_{\alpha, 0} S_{\alpha', 1}^{1/\alpha}
$$

where the r.v.'s on the right-hand side are independent.

**Proof.** See, e.g., Theorem 3.3.1 in [39].

**Corollary 1.** A symmetric strictly stable distribution with the characteristic exponent $\alpha$ is a scale mixture of normal laws in which the mixing distribution is the one-sided strictly stable law ($\theta = 1$) with the characteristic exponent $\alpha/2$:

$$
S_{\alpha, 0} \xrightarrow{d} X \sqrt{2S_{\alpha/2, 1}}
$$

(8)

with the r.v.'s on the right-hand side being independent.

In terms of d.f.s the statement of Corollary 1 can be written as

$$
P_{\alpha, 0}(x) = \int_{0}^{\infty} \Phi\left(\frac{x}{\sqrt{2z}}\right) dP_{\alpha/2, 1}(z), \quad x \in \mathbb{R}.
$$

Let $\gamma > 0$. The distribution of the r.v. $W_\gamma$:

$$
P(W_\gamma < x) = \left[1 - e^{-x^\gamma}\right]1(x \geq 0),
$$

is called the Weibull distribution with shape parameter $\gamma$. It is obvious that $W_1$ is the r.v. with the standard exponential distribution: $P(W_1 < x) = \left[1 - e^{-x}\right]1(x \geq 0)$.

It is easy to see that if $\gamma > 0$ and $\gamma' > 0$, then $P(W_\gamma^{1/\gamma} \geq x) = P(W_{\gamma'} \geq x^{\gamma'}) = e^{-x^{\gamma'}} = P(W_{\gamma \gamma'} \geq x), x \geq 0$, that is, for any $\gamma > 0$ and $\gamma' > 0$

$$
W_{\gamma \gamma'} \xrightarrow{d} W_{\gamma'}^{1/\gamma}.
$$

(9)

In the paper [7] it was shown that any gamma distribution with shape parameter no greater than one is mixed exponential. For convenience, we formulate this result as the following lemma.
Lemma 2 [7]. The density of a gamma distribution \( g(x; r, \mu) \) with \( 0 < r < 1 \) can be represented as
\[
g(x; r, \mu) = \int_0^\infty ze^{-zx}p(z; r, \mu)dz,
\]
where
\[
p(z; r, \mu) = \frac{\mu^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{1(z \geq \mu)}{(z-\mu)^r}
\]
(10)
Moreover, a gamma distribution with shape parameter \( r > 1 \) cannot be represented as a mixed exponential distribution.

Lemma 3 [19]. For \( r \in (0,1) \) let \( G_{r,1} \) and \( G_{1-r,1} \) be independent gamma-distributed r.v.’s. Let \( \mu > 0 \). Then the density \( p(z; r, \mu) \) defined by (10) corresponds to the r.v.
\[
Z_{r,\mu} = \frac{\mu(G_{r,1} + G_{1-r,1})}{G_{r,1}} \xrightarrow{d} \mu Z_{r,1} = \mu(1 + \frac{1}{r}Q_{1-r,r}),
\]
where \( Q_{1-r,r} \) is the r.v. with the Snedecor–Fisher distribution defined by the probability density
\[
q(x; 1-r, r) = \frac{(1-r)^{1-r}r^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{1}{x^r[r+(1-r)x]}, \quad x \geq 0.
\]

Remark 1. It is easily seen that the sum \( G_{r,1} + G_{1-r,1} \) has the standard exponential distribution: \( G_{r,1} + G_{1-r,1} \xrightarrow{d} W_1 \). However, the numerator and denominator of the expression defining the r.v. \( Z_{r,\mu} \) are not independent.

The statements of Lemmas 2 and 3 mean that if \( r \in (0,1) \), then
\[
G_{r,\mu} \xrightarrow{d} W_1 \cdot Z_{r,\mu}^{-1}
\]
(11)
where the r.v.’s on the right-hand side are independent.

The following statement has already become folklore. Without any claims for priority, its proof was given in [17] as an exercise.

Lemma 4. For any \( \delta \in (0,1] \), the Mittag-Leffler distribution with parameter \( \delta \) is a scale mixture of a one-sided stable distribution with the Weibull mixing distribution with parameter \( \delta/2 \), that is,
\[
M_{\delta} \xrightarrow{d} S_{\delta,1}W_{\delta} \xrightarrow{d} S_{\delta,1}\sqrt{W_{\delta/2}},
\]
where the r.v.’s on the right-hand side are independent.

Let \( \rho \in (0,1) \). In [26] it was demonstrated that the function
\[
f^K_{\rho}(x) = \frac{\sin(\pi \rho)}{\pi \rho[x^2 + 2x \cos(\pi \rho) + 1]}, \quad x \in (0, \infty),
\]
(12)
is a probability density on \((0, \infty)\). Let \( K_{\rho} \) be a random variable with density (12).

Lemma 5 [26]. Let \( 0 < \delta < \delta' \leq 1 \) and \( \rho = \delta/\delta' < 1 \). Then
\[
M_{\delta} \xrightarrow{d} M_{\delta'} K_{\rho}^{1/\delta}
\]
where the r.v.’s on the right-hand side are independent.

In [17] it was shown that for any \( \delta \in (0,1) \)
\[
K_{\delta}^{1/\delta} \xrightarrow{d} \frac{S_{\delta,1}}{S_{\delta,1}'},
\]
where \( S_{\delta,1} \) and the r.v.’s on the right-hand side are independent. So, with \( \delta' = 1 \), from Lemma 5 we obtain

**Corollary 2** [17, 26]. Let \( 0 < \delta < 1 \). Then the Mittag-Leffler distribution with parameter \( \delta \) is mixed exponential:

\[
M_{\delta} \overset{d}{=} W_1 \cdot K_{\delta}^{1/\delta} \overset{d}{=} W_1 \cdot \frac{S_{\delta,1}}{S_{\delta,1}'}
\]

where the involved r.v.’s are independent.

Let \( 0 < \alpha < \alpha' \leq 2 \). In [24] it was shown that the function

\[
f_{Q,\alpha,\alpha'}^{d}(x) = \frac{\alpha' \sin(\pi \alpha/\alpha')}{\pi [1 + x^{2\alpha} + 2x^\alpha \cos(\pi \alpha/\alpha')]} \cdot x^{\alpha-1}, \quad x > 0,
\]

is a probability density on \((0, \infty)\). Let \( Q_{\alpha,\alpha'} \) be a r.v. whose probability density is \( f_{Q,\alpha,\alpha'}^{d}(x) \).

**Lemma 6** [24]. Let \( 0 < \alpha < \alpha' \leq 2 \). Then

\[
L_{\alpha} \overset{d}{=} L_{\alpha'} Q_{\alpha,\alpha'},
\]

where the r.v.’s on the right-hand side are independent.

With \( \alpha' = 2 \) we have

**Corollary 3** [24]. Let \( 0 < \alpha < 2 \). Then the Linnik distribution with parameter \( \alpha \) is a scale mixture of Laplace distributions corresponding to density (5):

\[
L_{\alpha} \overset{d}{=} \Lambda Q_{\alpha,2}
\]

where the r.v.’s on the right-hand side are independent.

Now we recall some representations of the Linnik distribution as a normal or Laplace scale mixtures from [17]. In all the products of r.v.’s mentioned below the multipliers are assumed independent.

In [6] the following statement was proved. Here its formulation is extended with the account of (9).

**Lemma 7** [6]. For any \( \alpha \in (0, 2] \), the Linnik distribution with parameter \( \alpha \) is a scale mixture of a symmetric stable distribution, that is,

\[
L_{\alpha} \overset{d}{=} S_{\alpha,0} \cdot W_1^{1/\alpha},
\]

where the r.v.’s on the right-hand side are independent.

**Lemma 8** [17]. Let \( \alpha \in (0, 2] \), \( \alpha' \in (0, 1] \). Then

\[
L_{\alpha \alpha'} \overset{d}{=} S_{\alpha,0} \cdot M_{\alpha'}^{1/\alpha}.
\]

As far as we know, although the property of the Linnik distribution to be a normal scale mixture is something almost obvious by virtue of Lemmas 8 and 1, only in the paper [17] the mixing distribution was recognized as the Mittag-Leffler law, as is seen from the following statement.

**Corollary 4** [17]. For each \( \alpha \in (0, 2] \), the Linnik distribution with parameter \( \alpha \) is the scale mixture of zero-mean normal laws with mixing Mittag-Leffler distribution with twice less parameter \( \alpha/2 \):

\[
L_{\alpha} \overset{d}{=} X \sqrt{2M_{\alpha/2}^{2}},
\]

where the r.v.’s on the right-hand side are independent.
Lemma 9 [17]. For each \( \alpha \in (0, 2] \), the Linnik distribution with parameter \( \alpha \) is the scale mixture of the Laplace laws corresponding to density (6) with mixing distribution being that of the ratio of two independent r.v.’s having one and the same one-sided strictly stable distribution with characteristic exponent \( \alpha/2 \):

\[
L_\alpha \overset{d}{=} A \sqrt{\frac{S_{\alpha/2,1}}{S_{\alpha/2,1}'}}
\]

where the r.v.’s on the right-hand side are independent.

In [17] it was shown that if \( S_{\alpha,1} \) and \( S_{\alpha,1}' \) be two independent r.v.’s having one and the same one-sided strictly stable distribution with characteristic exponent \( \alpha \in (0, 1) \), then \( S_{\alpha,1}/S_{\alpha,1}' \overset{d}{=} K_{\alpha}^{1/\alpha} \overset{d}{=} Q_{2\alpha,2}^2 \), that is, the probability density \( p_\alpha(x) \) of the ratio \( S_{\alpha,1}/S_{\alpha,1}' \) has the form

\[
p_\alpha(x) = f_{\alpha,1}^Q(x) = \frac{\sin(\pi \alpha) x^{\alpha-1}}{\pi [1 + x^{2\alpha} + 2x^\alpha \cos(\pi \alpha)]}, \quad x > 0.
\]

Now consider the folded normal mixture representation for the Mittag-Leffler distribution.

Lemma 10 [17]. For \( \delta \in (0, 1] \) the Mittag-Leffler distribution with parameter \( \delta \) is a scale mixture of half-normal laws:

\[
M_\delta \overset{d}{=} |X| \cdot \sqrt{2W_1 \left( \frac{S_{\delta,1}}{S_{\delta,1}'} \right)^2}.
\]

4 New mixture representations for the generalized Linnik distribution

In this section we present some results containing new mixture representations for the generalized Linnik distribution. These results generalize and improve some results of [34] and [30]. We begin from the following well-known result due to Devroye and Pakes.

Lemma 11 [6, 34]. Let \( \alpha \in (0, 2], \nu > 0 \). Then

\[
L_{\alpha,\nu} \overset{d}{=} S_{\alpha,0} \cdot G_{\nu,1}^{1/\alpha} \overset{d}{=} S_{\alpha,0} \cdot G_{\nu,\alpha,1}.
\]

From Corollary 1 (see relation (8)) it follows that for \( \nu > 0 \) and \( \alpha \in (0, 2] \)

\[
L_{\alpha,\nu} \overset{d}{=} X \cdot \sqrt{2S_{\alpha/2,1}} \cdot G_{\nu,1}^\alpha \overset{d}{=} X \cdot \sqrt{2S_{\alpha/2,1}} \cdot G_{\nu,\alpha/2,1}.
\]

(14)

that is, the generalized Linnik distribution is a normal scale mixture.

Notice that the Laplace–Stieltjes transform of the mixing distribution in (14) has the form

\[
\psi(s; \nu, \alpha/2) = \mathbb{E}e^{-sS_{\alpha/2,1}}G_{\nu,\alpha/2,1} = \mathbb{E}\mathbb{E}[e^{-sS_{\alpha/2,1}}G_{\nu,\alpha/2,1} | G_{\nu,\alpha/2,1}] =
\]

\[
= \int_0^\infty \mathbb{E}e^{-sS_{\alpha/2,1}}g(x; \nu, \alpha/2, 1) dx = \frac{\alpha}{2\Gamma(\nu)} \int_0^\infty e^{-x^{\alpha/2}/(1+s^{\alpha/2})} x^{\alpha/2-1} dx =
\]

\[
= \frac{\alpha}{2\Gamma(\nu)(1 + s^{\alpha/2})^\nu} \int_0^\infty e^{-x^{\alpha/2}/(1+s^{\alpha/2})} x^{\alpha/2-1} dx = \frac{1}{(1 + s^{\alpha/2})^\nu}, \quad s \geq 0,
\]

(15)

corresponding to the generalized Mittag-Leffler distribution with parameters \( \alpha/2 \) and \( \nu \), that is, \( M_{\alpha/2,\nu} \overset{d}{=} S_{\alpha/2,1}G_{\nu,\alpha/2,1} \) (see [11, 32]). So, by analogy with Corollary 4 we obtain the following result.
Theorem 1. If \( \alpha \in (0, 2] \) and \( \nu > 0 \), then

\[
L_{\alpha, \nu} \overset{d}{=} X \cdot \sqrt{2M_{\alpha/2, \nu}},
\]

where the r.v.'s on the right-hand side are independent. In other words, the generalized Linnik distribution is a normal scale mixture with the generalized Mittag-Leffler mixing distribution.

Let \( \alpha \in (0, 2], \alpha' \in (0, 1) \) and \( \nu > 0 \). Using Lemmas 1 and 11 we obtain the following chain of relations:

\[
L_{\alpha \alpha', \nu} \overset{d}{=} S_{\alpha \alpha', 0} \cdot G_{\nu,1}^{1/\alpha} \overset{d}{=} S_{\alpha, 0} \cdot S_{\alpha', 1}^{1/\alpha} \cdot G_{\nu,1}^{1/\alpha} \overset{d}{=} L_{\alpha, \nu} \cdot S_{\alpha', 1}^{1/\alpha}.
\]

Hence, the following statement holds representing the generalized Linnik distribution as a scale mixture of the generalized Linnik distributions with greater characteristic parameter.

Theorem 2. Let \( \alpha \in (0, 2], \alpha' \in (0, 1) \) and \( \nu > 0 \). Then

\[
L_{\alpha \alpha', \nu} \overset{d}{=} L_{\alpha, \nu} \cdot S_{\alpha', 1}^{1/\alpha}.
\]

Now let \( \nu \in (0, 1] \). From representation (11) and Lemma 7 we obtain the chain

\[
L_{\alpha, \nu} \overset{d}{=} S_{\alpha, 0} \cdot G_{\nu,1}^{1/\alpha} \overset{d}{=} S_{\alpha, 0} \cdot W_{1/\alpha} \cdot Z_{\nu,1}^{-1/\alpha} \overset{d}{=} L_{\alpha} \cdot Z_{\nu,1}^{-1/\alpha}
\]

yielding the following statement relating generalized and ‘ordinary’ Linnik distributions.

Theorem 3. If \( \nu \in (0, 1] \) and \( \alpha \in (0, 2] \), then

\[
L_{\alpha, \nu} \overset{d}{=} L_{\alpha} \cdot Z_{\nu,1}^{-1/\alpha}.
\]

In other words, with \( \nu \in (0, 1] \) and \( \alpha \in (0, 2] \), the generalized Linnik distribution is a scale mixture of ordinary Linnik distributions.

From (16) and Lemma 9 we obtain the following representation of the generalized Linnik distribution as a scale mixture of Laplace distributions.

\[
L_{\alpha, \nu} \overset{d}{=} \Lambda \cdot Z_{\nu,1}^{-1/\alpha} \cdot \sqrt{\frac{S_{\alpha/2,1}}{S_{\alpha/2,1}^2}}.
\]

Furthermore, from Corollary 4 (see relation (13)) it follows that, if \( \nu \in (0, 1) \) and \( \alpha \in (0, 2] \), then

\[
L_{\alpha, \nu} \overset{d}{=} X \cdot Z_{\nu,1}^{-1/\alpha} \cdot \sqrt{2M_{\alpha/2}}.
\]

Since normal scale mixtures are identifiable [37], from representation (17) and Theorem 1 we obtain the following result representing the generalized Mittag-Leffler distribution as a scale mixture of ‘ordinary’ Mittag-Leffler distributions.

Theorem 4. Let \( \nu \in (0, 1] \) and \( \delta \in (0, 1] \). Then

\[
M_{\delta, \nu} \overset{d}{=} Z_{\nu,1}^{-1/\delta} \cdot M_{\delta},
\]

where the r.v.'s on the right-hand side are independent.

Let \( \delta \in (0, 1] \). From relations (15) with \( \alpha/2 \) replaced by \( \delta \) we see that

\[
M_{\delta, \nu} \overset{d}{=} S_{\delta, 1} \cdot G_{\nu, \delta/2,1} \overset{d}{=} S_{\delta, 1} \cdot G_{\nu,1}^{1/\delta}
\]

Now we have all tools required for proving an analog of Theorem 2 for generalized Mittag-Leffler distributions.
Let $\alpha \in (0, 2]$, $\alpha' \in (0, 1)$ and $\nu > 0$. From Theorem 1 we have
\[ L_{\alpha\alpha', \nu} \overset{d}{=} X \cdot \sqrt{2M_{\alpha'/2, \nu}} \]
and
\[ L_{\alpha, \nu} \overset{d}{=} X \cdot \sqrt{2M_{\alpha/2, \nu}}. \]
From Theorem 2 we have
\[ X \cdot \sqrt{2M_{\alpha'/2, \nu}} \overset{d}{=} L_{\alpha\alpha', \nu} \overset{d}{=} L_{\alpha, \nu} \cdot S_{\alpha', 1}^{1/\alpha} \overset{d}{=} X \cdot \sqrt{2M_{\alpha/2, \nu}} \cdot S_{\alpha, 1}^{1/\alpha}. \]
Hence, by virtue of identifiability of normal scale mixtures we have
\[ M_{\alpha\alpha', \nu} \overset{d}{=} M_{\alpha/2, \nu} \cdot S_{\alpha', 1}^{2/\alpha} \]
Therefore, re-denoting $\alpha/2 = \delta$, $\alpha' = \delta'$ we obtain the following result
\[ \text{Theorem 5. Let } \delta \in (0, 1], \delta' \in (0, 1) \text{ and } \nu > 0. \text{ Then} \]
\[ M_{\delta\delta', \nu} \overset{d}{=} M_{\delta, \nu} \cdot S_{\delta', 1}^{1/\delta}. \]
That is, any generalized Mittag-Leffler distribution is a scale mixture of generalized Mittag-Leffler distributions with greater parameter.

5 Convergence of the distributions of random sums to the generalized Linnik distribution

In applied probability it is a convention that a model distribution can be regarded as well-justified or adequate, if it is an asymptotic approximation, that is, if there exists a rather simple limit setting (say, schemes of maximum or summation of random variables) and the corresponding limit theorem in which the model under consideration manifests itself as a limit distribution. The existence of such limit setting can provide a better understanding of real mechanisms that generate observed statistical regularities.

As we have already mentioned, the Linnik distributions are geometrically stable. Geometrically stable distributions are only possible limits for the distributions of geometric random sums of independent identically distributed r.v.’s. As this is so, the distributions of the summands belong to the domain of attraction of the strictly stable law with some characteristic exponent $\alpha \in (0, 2]$ and hence, for $0 < \alpha < 2$ have infinite moments of orders greater or equal to $\alpha$. As concerns the case $\alpha = 2$, where the variance is finite, within the framework of the scheme of geometric summation in this case the only possible limit law is the Laplace distribution [13].

As we will demonstrate below, the generalized Linnik distributions can be limiting for negative binomial sums of independent identically distributed r.v.’s. Negative binomial random sums turn out to be important and adequate models of total precipitation volume during wet (rainy) periods in meteorology [20, 21]. However, in this case the summands also must have distributions from the domain of attraction of a strictly stable law with some characteristic exponent $\alpha \in (0, 2]$ and hence, with $\alpha \in (0, 2)$, have infinite variances. If $\alpha = 2$, then the only possible limit distribution for negative binomial random sums is the so-called variance gamma distribution which is well known in financial mathematics [8].

However, when the (generalized) Linnik distributions are used as models of statistical regularities observed in real practice and an additive structure model is used of type of a (stopped) random walk for the observed process, the researcher cannot avoid thinking over the following question: which of the two combinations of conditions can be encountered more often,
- the distribution of the number of summands (the number of jumps of a random walk) is negative binomial (asymptotically gamma), but the distributions of summands (jumps) have so heavy tails that, at least, their variances are infinite, or

- the second moments (variances) of the summands (jumps) are finite, but the number of summands exposes an irregular behavior so that its very large values are possible?

Since, as a rule, when real processes are modeled, there are no serious reasons to reject the assumption that the variances of jumps are finite, the second combination at least deserves a thorough analysis.

As it was demonstrated in the preceding section, the (generalized) Linnik distributions even with $\alpha < 2$ can be represented as normal scale mixtures. This means that they can be limit distributions in analogs of the central limit theorem for random sums of independent r.v.’s with finite variances. Such analogs with ‘ordinary’ Linnik limit distributions were presented in [17]. Here we will extend these results to generalized Linnik distributions. It will be demonstrated that the scheme of negative binomial summation is far not the only asymptotic setting (even for sums of independent r.v.’s!) in which the generalized Linnik law appears as a limit distribution.

We will begin with the limit theorem for negative binomial random sums in which the generalized Linnik law is the limit distribution. For this purpose we will use the following auxiliary result. Consider a sequence of r.v.’s $W_1, W_2, \ldots$ Let $N_1, N_2, \ldots$ be natural-valued r.v.’s such that for every $n \in \mathbb{N}$ the r.v. $N_n$ is independent of the sequence $W_1, W_2, \ldots$ In the following statement the convergence is meant as $n \to \infty$. The symbol $\Rightarrow$ will denote convergence in distribution. Recall that a random sequence $N_1, N_2, \ldots$ is said to infinitely increase in probability ($N_n \overset{P}{\to} \infty$), if $P(N_n \leq m) \to 0$ for any $m \in (0, \infty)$.

**Lemma 12** [15, 16]. Assume that there exist an infinitely increasing (convergent to zero) sequence of positive numbers $\{b_n\}_{n \geq 1}$ and a r.v. $W$ such that

$$b_n^{-1}W_n \Rightarrow W. \quad (18)$$

If there exist an infinitely increasing (convergent to zero) sequence of positive numbers $\{d_n\}_{n \geq 1}$ and a r.v. $N$ such that

$$d_n^{-1}b_{N_n} \Rightarrow N, \quad (19)$$

then

$$d_n^{-1}W_{N_n} \Rightarrow W \cdot N, \quad (20)$$

where the r.v.’s on the right-hand side of (20) are independent. If, in addition, $N_n \overset{P}{\to} \infty$ in probability and the family of scale mixtures of the d.f. of the r.v. $W$ is identifiable, then condition (19) is not only sufficient for (20), but is necessary as well.

Let $X_1, X_2, \ldots$ be independent identically distributed random variables such that their common distribution belongs to the domain of attraction of a strictly stable law with the characteristic exponent $\alpha \in (0, 2)$. This means that there exists a $c \in (0, \infty)$ such that

$$\frac{1}{cn^{1/\alpha}} \sum_{i=1}^{n} X_i \Rightarrow S_{\alpha, 0} \quad (21)$$

as $n \to \infty$. For simplicity, without any restriction of generality, we will assume that $c = 1$.

Consider a r.v. $\mathcal{N}_{\nu, p}$ having the negative binomial distribution with parameters $r > 0$ and $p \in (0, 1)$:

$$P(\mathcal{N}_{\nu, p} = k) = \frac{\Gamma(\nu + k + 1)}{(k - 1)!\Gamma(\nu)} \cdot p^\nu (1 - p)^{k - 1}, \quad k = 1, 2, \ldots, \quad (22)$$

In this case $E\mathcal{N}_{\nu, p} = \nu / p$. 

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Lemma 13. If \( p \to 0 \), then

\[ pN_{B, p} \Rightarrow G_{\nu, 1}. \]

The proof is a simple exercise on characteristic functions.

Let \( n \in \mathbb{N} \). Assume that \( p = 1/n \) and denote \( N_n = NB_{\nu, 1/n} \). Also assume that for each \( n \geq 1 \) the random variable \( N_n \) is independent of \( X_1, X_2, \ldots \) Let \( W_n = X_1 + \ldots + X_n \). Put \( b_n = d_n = n^{1/\alpha} \). Then from Lemma 13 it follows that

\[ b_n/d_n = (N_n/n)^{1/\alpha} \Rightarrow G^{1/\alpha}_{\nu, 1} \]

as \( n \to \infty \). We will treat this relation as condition (19) in Lemma 12. As condition (18) we will treat relation (21) (with \( c = 1 \)). As a result, from Lemmas 11 and 13 and we obtain the following statement establishing the convergence of the distributions of negative binomial random sums to the generalized Linnik distribution.

Theorem 6. Let \( X_1, X_2, \ldots \) be independent identically distributed r.v.’s such that their common distribution belongs to the domain of attraction of a strictly stable law with the characteristic exponent \( \alpha \in (0, 2] \). Let \( N_n \) be a random variable having the negative binomial distribution with parameters \( \nu > 0 \) and \( p = 1/n \). Then

\[ \frac{1}{n^{1/\alpha}} \sum_{i=1}^{N_n} X_i \Rightarrow L_{\alpha, \nu} \]

as \( n \to \infty \).

Remark 2. If \( \alpha = 2 \), then the limit r.v. has the form

\[ L_{2, \nu} \overset{d}{=} X \cdot \sqrt{2G_{\nu, 1}}. \]

The distribution of this r.v. is a normal scale mixture with respect to the gamma distribution. This is the well-known variance gamma distribution, a popular heavy-tailed model in financial mathematics.

As we have already noted, the representation for the generalized Linnik distribution as a scale mixture of normals obtained above opens the way for the construction in this section of a random-sum central limit theorem with the generalized Linnik distribution as the limit law. Moreover, in this “if and only if” version of the random-sum central limit theorem the generalized Mittag-Leffler distribution must be the limit law for the normalized number of summands.

Consider independent not necessarily identically distributed random variables \( X_1, X_2, \ldots \) with \( \mathbb{E}X_i = 0 \) and \( 0 < \sigma_i^2 = \text{Var}X_i < \infty \), \( i \geq 1 \). For \( n \in \mathbb{N} \) denote

\[ S_n^* = X_1 + \ldots + X_n, \quad B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2. \]

Assume that the r.v.’s \( X_1, X_2, \ldots \) satisfy the Lindeberg condition: for any \( \tau > 0 \)

\[ \lim_{n \to \infty} \frac{1}{B_n^2} \sum_{i=1}^{n} \int_{|x| \geq \tau B_n} x^2 d\mathbb{P}(X_i < x) = 0. \] (23)

It is well known that under these assumptions

\[ \mathbb{P}(S_n^* < B_n x) \Rightarrow \Phi(x) \]

(this is the classical Lindeberg central limit theorem).

Let \( N_1, N_2, \ldots \) be a sequence of integer-valued nonnegative r.v.’s defined on the same probability space so that for each \( n \in \mathbb{N} \) the r.v. \( N_n \) is independent of the sequence \( X_1, X_2, \ldots \)
Denote $S_{N_n}^* = X_1 + \ldots + X_{N_n}$. For definiteness, in what follows we assume that $\sum_{j=1}^{0} = 0$. Everywhere in what follows the convergence will be meant as $n \to \infty$.

Let $\{d_n\}_{n \geq 1}$ be an infinitely increasing sequence of positive numbers.

The proof of the main result of this section is based on the following version of the random-sum central limit theorem.

**Lemma 14** [15]. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (23) hold. Moreover, let $N_n \xrightarrow{p} \infty$. A d.f. $F(x)$ such that

$$P\left(\frac{S_{N_n}^*}{d_n} < x\right) = F(x)$$

exists if and only if there exists a d.f. $H(x)$ satisfying the conditions

$$H(0) = 0, \quad F(x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{y}}\right) dH(y), \quad x \in \mathbb{R},$$

and $P(B_{N_n}^2 < x d_n^2) \implies H(x)$.

**Proof.** This statement is a particular case of a result proved in [15], also see Theorem 3.3.2 in [8].

The following theorem gives a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of random sums of independent identically distributed r.v.’s with finite variances to the generalized Linnik distribution.

**Theorem 7.** Let $\alpha \in (0, 2], \nu > 0$. Assume that the r.v.’s $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (23) hold. Moreover, let $N_n \xrightarrow{p} \infty$. Then the distributions of the normalized random sums $S_{N_n}^*$ converge to the generalized Linnik law with parameters $\alpha$ and $\nu$, that is,

$$\frac{S_{N_n}^*}{d_n} \implies L_{\alpha, \nu}$$

with some $d_n > 0$, $d_n \to \infty$, if and only if

$$\frac{B_{N_n}^2}{d_n^2} \implies 2M_{\alpha/2, \nu}.$$

**Proof.** This statement is a direct consequence of Theorem 1 and Lemma 14 with $H(x) = P(2M_{\alpha/2, \nu} < x)$.

Note that if the r.v.’s $X_1, X_2, \ldots$ are identically distributed, then $\sigma_i = \sigma$, $i \in \mathbb{N}$, and the Lindeberg condition holds automatically. In this case it is reasonable to take $d_n = \sigma \sqrt{n}$. Hence, from Theorem 5 in this case it follows that for the convergence

$$\frac{S_{N_n}^*}{\sigma \sqrt{n}} \implies L_{\alpha, \nu}$$

to hold it is necessary and sufficient that

$$\frac{N_n}{n} \implies 2M_{\alpha/2, \nu}.$$

One more remark is that with $\alpha = 2$ Theorem 5 involves the case of convergence to the Laplace distribution.
6 Convergence of the distributions of statistics constructed from samples with random sizes to the generalized Linnik distribution

In classical problems of mathematical statistics, the size of the available sample, i.e., the number of available observations, is traditionally assumed to be deterministic. In the asymptotic settings it plays the role of infinitely increasing known parameter. At the same time, in practice very often the data to be analyzed is collected or registered during a certain period of time and the flow of informative events each of which brings a next observation forms a random point process. Therefore, the number of available observations is unknown till the end of the process of their registration and also must be treated as a (random) observation. In this case the number of available observations as well as the observations themselves are unknown beforehand and should be treated as random to avoid underestimation of risks or error probabilities.

Therefore it is quite reasonable to study the asymptotic behavior of general statistics constructed from samples with random sizes for the purpose of construction of suitable and reasonable asymptotic approximations. As this is so, to obtain non-trivial asymptotic distributions in limit theorems of probability theory and mathematical statistics, an appropriate centering and normalization of r.v.’s and vectors under consideration must be used. It should be especially noted that to obtain reasonable approximation to the distribution of the basic statistics, both centering and normalizing values should be non-random. Otherwise the approximate distribution becomes random itself and, for example, the problem of evaluation of quantiles or significance levels becomes senseless.

In asymptotic settings, statistics constructed from samples with random sizes are special cases of random sequences with random indices. The randomness of indices usually leads to that the limit distributions for the corresponding random sequences are heavy-tailed even in the situations where the distributions of non-randomly indexed random sequences are asymptotically normal see, e.g., [3, 4, 8].

Consider a problem setting that is traditional for mathematical statistics. Let r.v.’s \( N_1, N_2, \ldots, X_1, X_2, \ldots \) be defined on one and the same probability space \((\Omega, \mathcal{A}, P)\). Assume that for each \( n \geq 1 \) the r.v. \( N_n \) takes only natural values and is independent of the sequence \( X_1, X_2, \ldots \). Let \( T_n = T_n(X_1, \ldots, X_n) \) be a statistic, that is, a measurable function of \( X_1, \ldots, X_n \). For every \( n \geq 1 \) define the random variable \( T_{N_n} \) as

\[
T_{N_n}(\omega) = T_{N_n}(\omega) \left( X_1(\omega), \ldots, X_{N_n}(\omega) \right)
\]

for each \( \omega \in \Omega \). As usual, the symbol \( \Rightarrow \) denotes convergence in distribution.

A statistic \( T_n \) is said to be asymptotically normal, if there exist \( \sigma > 0 \) and \( \theta \in \mathbb{R} \) such that

\[
P \left( \sigma \sqrt{n} \left( T_n - \theta \right) < x \right) \Rightarrow \Phi(x).
\]  

(24)

**Lemma 15 [16].** Assume that \( N_n \rightarrow \infty \) in probability. Let the statistic \( T_n \) be asymptotically normal in the sense of (24). A distribution function \( F(x) \) such that

\[
P \left( \sigma \sqrt{n} \left( T_{N_n} - \theta \right) < x \right) \Rightarrow F(x),
\]

exists if and only if there exists a d.f. \( H(x) \) satisfying the conditions

\[
H(0) = 0, \quad F(x) = \int_0^\infty \Phi(x\sqrt{y})dH(y), \quad x \in \mathbb{R}, \quad P(N_n < nx) \Rightarrow H(x).
\]
The following theorem gives a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of statistics, which are suggested to be asymptotically normal in the traditional sense but are constructed from samples with random sizes, to the generalized Linnik distribution.

**Theorem 8.** Let $\alpha \in (0, 2], \nu > 0$. Assume that the r.v.'s $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and, moreover, let $N_n \xrightarrow{P} \infty$. Let the statistic $T_n$ be asymptotically normal in the sense of (24). Then the distribution of the statistic $T_{N_n}$ constructed from samples with random sizes $N_n$ converges to the generalized Linnik law, that is,

$$\sigma \sqrt{n}(T_{N_n} - \theta) \Rightarrow L_{\alpha, \nu},$$

if and only if

$$\frac{N_n}{n} \Rightarrow \frac{1}{2} M^{-1}_{\alpha/2, \nu}.$$

**Proof.** This statement is a direct consequence of Theorem 1 and Lemma 15 with $H(x) = P(M^{-1}_{\alpha/2, \nu} < 2x)$.

**Acknowledgement** The research is partially supported by the Russian Foundation for Basic Research, project 17-07-00717.

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