We develop an inequality for the expectation of a product of \( n \) random variables generalizing the recent work of Dedecker and Doukhan (2003) and the earlier results of Rio (1993).

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \((X, Y)\) be a bivariate random vector defined on it. Suppose that \(E(X^2) < \infty\) and \(E(Y^2) < \infty\). Hoeffding proved that

\[
\text{Cov}(X, Y) = \int_{\mathbb{R}^2} \left[ P(X \leq x, Y \leq y) - P(X \leq x)P(Y \leq y) \right] dx \, dy. \tag{1.1}
\]

In [5], Lehmann gave a simple proof of this identity and used it in his study of some concepts of dependence. This identity was generalized to functions \(h(X)\) and \(g(Y)\) with \(E[h^2(X)] < \infty\) and \(E[g^2(Y)] < \infty\) and with finite derivatives \(h'(\cdot)\) and \(g'(\cdot)\) by Newman [6]. Multidimensional versions of these results were proved by Block and Fang [1], Yu [13], and more recently by Prakasa Rao [7]. Related covariance identities for exponential and other distributions are given by Prakasa Rao in [9, 10].

Suppose that \(\mathcal{M}\) is a sub-\(\sigma\)-algebra of \(\mathcal{F}\) and \(Y\) is measurable with respect to \(\mathcal{M}\). Let \(\sigma(X)\) be the sub-\(\sigma\)-algebra generated by the random variable \(X\). Define

\[
\alpha(\mathcal{M}, X) = \sup \left\{ \left| P(A \cap B) - P(A)P(B) \right| : A \in \mathcal{M}, B \in \sigma(X) \right\}. \tag{1.2}
\]

Define

\[
\begin{align*}
Q_X(u) &= \inf \left\{ x : P(|X| > x) \leq u \right\}, \\
G_X(s) &= \inf \left\{ z : \int_0^z Q_X(t) dt \geq s \right\}, \\
H_{X,Y}(s) &= \inf \left\{ t : E(|X| | \{|Y| > t\}) \leq s \right\}.
\end{align*}
\tag{1.3}
\]
Rio [11] proved that
\[
|\text{Cov}(X, Y)| \leq 2 \int_0^{\alpha(\mu, X)/2} Q_Y(u)Q_X(u)du.
\]
(1.4)

Related results are given in [12, page 9]. These results were generalized by Bradley [2] for a strong-mixing process and by Prakasa Rao [8] for \(r\)-th order joint cumulant under \(r\)-th order strong mixing. In a recent work, Dedecker and Doukhan [3] proved that
\[
|E(XY)| \leq \int_0^{\|E(X|\mu]\|_1} H_{X,Y}(t)dt \leq \int_0^{\|E(X|\mu]\|_1} Q_YoG_X(t)dt
\]
(1.5)
and obtained an improved version of the above inequality. If \(X_i, 1 \leq i \leq n\), are positive-valued random variables, it is easy to see that
\[
E(X_1X_2 \cdots X_n) \leq \int_0^1 Q_{X_1}(u)Q_{X_2}(u) \cdots Q_{X_n}(u)du.
\]
(1.6)

For a proof, see [12, Lemma 2.1, page 35].

We now obtain an improved version of the above inequality following the techniques of Dedecker and Doukhan [3] and Block and Fang [1].

2. Main result

Let \(\{X_i, 1 \leq i \leq n\}\) be a sequence of nonnegative random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Then the random variable \(X_i\) can be represented in the form
\[
X_i = \int_0^\infty I_{(x_i, \infty)}(X_i)dx_i,
\]
(2.1)

where
\[
I_{(x_i, \infty)}(X_i) = \begin{cases} 
1 & \text{if } X_i > x_i, \\
0 & \text{if } X_i \leq x_i.
\end{cases}
\]
(2.2)

Hence
\[
E(X_1X_2 \cdots X_n) = E\left[\int_0^\infty I_{(x_i, \infty)}(X_i)dx_i\right]
\]
\[
= \int_{\mathbb{R}^{n-1}} E\left[\int_0^\infty I_{(x_i, \infty)}(X_i)dx_i\right]dx_2 \cdots dx_n
\]
(2.3)
\[
= \int_{\mathbb{R}^{n-1}} E\left[\int_{(x_i, \infty), 2 \leq i \leq n} (X_2, \ldots, X_n)dx_2 \cdots dx_n\right]
\]
by the Fubini’s theorem, where \(\mathbb{R}^{n-1} = \{(x_2, \ldots, x_n) : x_i \geq 0, 2 \leq i \leq n\}\). Observe that
\[
E(X_1I_{[X_i, \infty), 2 \leq i \leq n]} (X_2, \ldots, X_n)) \leq \min \left( E[X_1], E(X_1I_{[X_i, \infty), 2 \leq i \leq n]} (X_2, \ldots, X_n)) \right)
\]
(2.4)
and hence
\[ E(X_1 X_2 \cdots X_n) \leq \int_{\mathbb{R}_+^{n-1}} \left\{ \int_0^{EX_1} \chi_{[E[X_1|X_{[X_{[1}\cdots \alpha \in \mathbb{N}]}(X_2, \ldots, X_n)]>u]}(u)du \right\} dx_2 \cdots dx_n. \] (2.5)

Here \( \chi_A(\cdot) \) denotes the indicator function of the set \( A \). Let
\[ g_{X_1}(x_2, \ldots, x_n) = E[X_1 I_{[X_1>x_i, 2 \leq i \leq n]}(X_2, \ldots, X_n)]. \] (2.6)

Then
\[ E(X_1 X_2 \cdots X_n) \leq \int_{\mathbb{R}_+^{n-1}} \left\{ \int_0^{EX_1} \chi_{[g_{X_1}(x_2, \ldots, x_n)>u]}(u)du \right\} dx_2 \cdots dx_n \]
\[ = \int_0^{EX_1} \left\{ \int_{\{(x_2, \ldots, x_n): g_{X_1}(x_2, \ldots, x_n)>u\}} 1 \right\} dx_2 \cdots dx_n du. \] (2.7)

Let
\[ H_{X_1, X_2, \ldots, X_n}(u) = \lambda[\{x_2, \ldots, x_n): g_{X_1}(x_2, \ldots, x_n)>u\}], \] (2.8)

where \( \lambda \) is the Lebesgue measure on the space \( \mathbb{R}_+^{n-1} \). Hence
\[ E(X_1 X_2 \cdots X_n) \leq \int_0^{EX_1} H_{X_1, X_2, \ldots, X_n}(u)du. \] (2.9)

Observe that
\[ g_{X_1}(x_2, \ldots, x_n) = E[X_1 I_{[X_1>x_i, 2 \leq i \leq n]}(X_2, \ldots, X_n)] \leq \int_0^{E[I_{[X_{[X_{[1}\cdots \alpha \in \mathbb{N}]}(X_2, \ldots, X_n)]}]} Q_{X_1}(u)du \] (2.10)

from the Fréchet’s inequality [4]. Here \( Q_{X_i}(\cdot) \) is the generalized inverse of the function \( T_{X_i}(x) = P(X_i > x) \) as defined earlier. Let
\[ M_{X_i}(y) = \int_0^y Q_{X_i}(t)dt. \] (2.11)

Observe that \( M_{X_i}(\cdot) \) is nondecreasing in \( y \). Let \( G_{X_i}(u) = \inf \{ z : M_{X_i}(z) \geq u \} \) as defined earlier. Let
\[ T_{X_1, \ldots, X_n}(x_2, \ldots, x_n) = P(X_i > x_i, 2 \leq i \leq n). \] (2.12)

Note that
\[ g_{X_1}(x_2, \ldots, x_n) \leq M_{X_1}(E[I_{[X_{[X_{[1]\cdots \alpha \in \mathbb{N}]}(X_2, \ldots, X_n)]}]], \]
\[ g_{X_1}(x_2, \ldots, x_n) > u \implies M_{X_1}(E[I_{[X_{[X_{[1]\cdots \alpha \in \mathbb{N}]}(X_2, \ldots, X_n)]}]) > u \]
\[ \implies E[I_{[X_{[X_{[1]\cdots \alpha \in \mathbb{N}]}(X_2, \ldots, X_n)]}] > G_{X_1}(u) \]
\[ \implies P[X_i > x_i, 2 \leq i \leq n] > G_{X_i}(u). \] (2.13)
Hence the set
\[
\left\{ (x_2, \ldots, x_n) \in \mathbb{R}_+^{n-1} : g_{X_1}(x_2, \ldots, x_n) > u \right\}
\]  
(2.14)
is contained in the set
\[
\left\{ (x_2, \ldots, x_n) \in \mathbb{R}_+^{n-1} : P(X_i > x_i, 2 \leq i \leq n) > G_{X_i}(u) \right\}.
\]  
(2.15)
In particular, it follows that the Lebesgue measure of the former set is less than or equal to that of the latter. Let
\[
Q_{X_2, \ldots, X_n}^\ast(G_{X_1}(u))
\]  
(2.16)
denote the Lebesgue measure of the set (2.15). Then
\[
H_{X_1, X_2, \ldots, X_n}(u) \leq Q_{X_2, \ldots, X_n}^\ast(G_{X_1}(u))
\]  
(2.17)
for all \(0 \leq u \leq 1\). Hence
\[
E(X_1X_2 \cdots X_n) \leq \int_0^{E(X_1)} Q_{X_2, \ldots, X_n}^\ast(G_{X_1}(u)) du.
\]  
(2.18)
We have proved the following inequality.

**Theorem 2.1.** Let \(X_i, 1 \leq i \leq n\), be nonnegative random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Then
\[
E(X_1X_2 \cdots X_n) \leq \int_0^{E(X_1)} H_{X_1, X_2, \ldots, X_n}(u) du \leq \int_0^{E(X_1)} Q_{X_2, \ldots, X_n}^\ast \circ G_{X_1}(u) du,
\]  
(2.19)
where the functions \(H, Q^\ast, \text{ and } G\) are as defined earlier.

**3. Applications**

We now suppose that the random variables \(\{X_i, 1 \leq i \leq n\}\) are arbitrary but with
\[
E \left| X_1X_2 \cdots X_n \right| < \infty.
\]  
(3.1)
Define
\[
g_{X_1}(x_2, \ldots, x_n) = E(\left| X_1 \right| | X_i > x_i, 2 \leq i \leq n)(X_2, \ldots, X_n)),
\]
\[
H_{X_1, X_2, \ldots, X_n}(u) = \lambda[(x_2, \ldots, x_n) : g_{X_1}(x_2, \ldots, x_n) \leq u],
\]  
(3.2)
\[
T_{X_2, \ldots, X_n}(x_2, \ldots, x_n) = P(\left| X_i \right| > x_i, 2 \leq i \leq n),
\]
and define \(M_{X_i}(\cdot), Q_{X_i}(\cdot), Q_{X_2, \ldots, X_n}^\ast, \text{ and } G_{X_i}\) accordingly. The following theorem follows by arguments analogous to those given in Section 2.
Theorem 3.1. Let $X_i, 1 \leq i \leq n$, be arbitrary random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Then

$$E\left(\left|X_1X_2\cdots X_n\right|\right) \leq \int_0^{E(|X_1|)} H_{X_1,X_2,\ldots,X_n}(u)du \leq \int_0^{E(|X_1|)} Q_{X_2,\ldots,X_n}^+oG_{X_1}(u)du,$$  \hspace{1cm} (3.3)

where the functions $H, Q^*, G$ are as defined above.

In particular, for $n = 2$, we have

$$E\left(\left|X_1X_2\right|\right) \leq \int_0^{E(|X_1|)} H_{X_1,X_2}(u)du \leq \int_0^{E(|X_1|)} Q_{X_2}oG_{X_1}(u)du,$$  \hspace{1cm} (3.4)

since $Q_{X_1}^+ = Q_X$ for any univariate random variable $X$. Furthermore,

$$G_{X_1-E(X_1)}(u) \geq G_{X_1}\left(\frac{u}{2}\right), \quad 0 \leq u \leq 1$$ \hspace{1cm} (3.5)

(cf. [3]). Hence

$$E\left(\left|X_1X_2\right|\right) \leq \int_0^{G_{X_1}(E(|X_1|)/2)} Q_{X_2}(u)Q_{X_1}(u)du.$$  \hspace{1cm} (3.6)

Therefore, for any two functions $f_i(\cdot), i = 1,2$, with $f_i(0) = 0$ such that $E|f_1(X_1)f_2(X_2)| < \infty$, we obtain that

$$E\left[\left|f_1(X_1)f_2(X_2)\right|\right] \leq \int_0^{G_{f_2}(E(|f_1(X_1)|)/2)} Q_{f_2}(u)Q_{f_1}(u)du.$$  \hspace{1cm} (3.7)

Applying Theorem 3.1 for the random variables $X_1 - E(X_1), X_2,\ldots,X_n$, we get that

$$E\left[\left|(X_1 - E(X_1))X_2\cdots X_n\right|\right] \leq \int_0^{E(|X_1 - E(X_1)|)} Q_{X_2,\ldots,X_n}^+oG_{X_1-E(X_1)}(u)du.$$  \hspace{1cm} (3.8)

But

$$G_{X_1-E(X_1)}(u) \geq G_{X_1}\left(\frac{u}{2}\right), \quad u \geq 0$$ \hspace{1cm} (3.9)

(cf. [3]). Hence

$$E\left[\left|(X_1 - E(X_1))X_2\cdots X_n\right|\right] \leq \int_0^{G_{X_1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2,\ldots,X_n}^+oG_{X_1}(u)du.$$  \hspace{1cm} (3.10)

Observing that $G_{X_1}(\cdot)$ is the inverse of the function $M_{X_1}(y) = \int_0^y Q_{X_1}(t)dt$, it follows that

$$E\left[\left|(X_1 - E(X_1))X_2\cdots X_n\right|\right] \leq \int_0^{G_{X_1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2,\ldots,X_n}(u)Q_{X_1}(u)du.$$  \hspace{1cm} (3.11)

Hence we have the following result.
Theorem 3.2. Let \( X_i, 1 \leq i \leq n \), be arbitrary random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) with \(E|X_1| < \infty\) and \(E|X_1X_2 \cdots X_n| < \infty\). Then (3.11) holds.

Observe that \( Q_X^* = Q_X \) for any univariate random variable \( X \). Let \( n = 2 \) in Theorem 3.2. Then \( Q_{X_2}^* = Q_{X_1} \), and the above result reduces to

\[
E[ |X_1 - E(X_1)|X_2 | ] \leq \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2}(u)Q_{X_1}(u)du. \tag{3.12}
\]

As a further consequence, we get that

\[
E[ |X_1 - E(X_1)|(X_2 - E(X_2)) | ] \leq \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2 - E(X_2)}(u)Q_{X_1}(u)du. \tag{3.13}
\]

Since

\[
Q_{X_2 - E(X_2)} \leq Q_{X_2} + E|X_2|, \tag{3.14}
\]

we obtain that

\[
E[ |X_1 - E(X_1)|(X_2 - E(X_2)) | ] \\
\leq \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2}(u)Q_{X_1}(u)du + E|X_2| \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_1}(u)du. \tag{3.15}
\]

Let

\[
\alpha(X_1, X_2) = \max \left\{ G_{X_1}^{-1} \left( \frac{E( |X_1 - E(X_1)| )}{2} \right), G_{X_2}^{-1} \left( \frac{E( |X_2 - E(X_2)| )}{2} \right) \right\}. \tag{3.16}
\]

Then it follows that

\[
E[ |X_1 - E(X_1)|(X_2 - E(X_2)) | ] \\
\leq \int_0^{\alpha(X_1, X_2)} Q_{X_1}(u)Q_{X_2}(u)du + \frac{1}{2} \left( E|X_1| \int_0^{\alpha(X_1, X_2)} Q_{X_1}(u)du + E|X_2| \int_0^{\alpha(X_1, X_2)} Q_{X_2}(u)du \right). \tag{3.17}
\]

This inequality is different from the inequality in [12, page 9].

Let \( f_i \) and \( f_j \) be differentiable functions on \( \mathbb{R}_+ \) with \( f_i(0) = 0 \). Let \( X_i, i = 1, 2 \), be non-negative random variables. Suppose that \( E[f_i^2(X_i)] < \infty, i = 1, 2 \). It is easy to see that

\[
f_i(X_i) = \int_0^\infty f_i'(X_i) I_{(X_i, \infty)}(X_i) dx_i. \tag{3.18}
\]

Then

\[
E(f_1(X_1)f_2(X_2)) = E \left[ f_1(X_1) \int_0^\infty f_2'(X_2) I_{(X_2, \infty)}(X_2) dx_2 \right] \\
= \int_{\mathbb{R}_+} E \left[ f_1(X_1) f_2'(X_2) I_{(X_2, \infty)}(X_2) \right] dx_2. \tag{3.19}
\]
by the Fubini’s theorem. Observe that

\[
E(\|f_1(X_1) f_2'(X_2) | I_{[X_2 > x]_2}(X_2)) \\
\leq \min (E[\|f_1(X_1) f_2'(X_2) | I_{[X_2 > x]}(X_2)] , E(\|f_1(X_1) f_2'(X_2) | I_{[X_2 > x]}(X_2)))
\]

(3.20)

and hence

\[
|E(f_1(X_1) f_2(X_2))| \\
\leq \int_{\mathbb{R}^+} \left\{ \int_0^1 E[|f_1(X_1) f_2'(X_2)|] \chi_2(g_1 f_1, f_2(X_2) > u)(u)du \right\} dx_2.
\]

(3.21)

Here \(\chi_A(\cdot)\) denotes the indicator function of the set \(A\). Let

\[
g_{f_1(X_1), f_2'(X_2)}(X_2) = E[\|f_1(X_1) f_2'(X_2) | I_{[X_2 > x]}(X_2)].
\]

(3.22)

Then

\[
|E(f_1(X_1) f_2(X_2))| \leq \int_{\mathbb{R}^+} \left\{ \int_0^1 E[|f_1(X_1) f_2'(X_2)|] \chi_2(g_1 f_1, f_2(X_2) > u)(u)du \right\} dx_2
\]

\[
\leq \int_0^1 \left\{ \int_{\mathbb{R}^+} \chi_{\{x_2 : g_{f_1(X_1), f_2'(X_2)}(X_2) \leq u\}} dx_2 \right\} du.
\]

(3.23)

Let

\[
H_{f_1(X_1), f_2'(X_2)}(u) = \inf \{x_2 : g_{f_1(X_1), f_2'(X_2)}(X_2) \leq u\}.
\]

(3.24)

Then it follows that

\[
|E(f_1(X_1) f_2(X_2))| \leq \int_0^1 H_{f_1(X_1), f_2'(X_2)}(u)du.
\]

(3.25)

An analogous inequality holds by interchanging \(f_1(X_1)\) and \(f_2(X_2)\):

\[
|E(f_1(X_1) f_2'(X_2))| \leq \int_0^1 H_{f'_1(X_1), f_2(X_2)}(u)du.
\]

(3.26)

References

[1] H. W. Block and Z. B. Fang, A multivariate extension of Hoeffding’s lemma, Ann. Probab. 16 (1988), no. 4, 1803–1820.
[2] R. C. Bradley, A covariance inequality under a two-part dependence assumption, Statist. Probab. Lett. 30 (1996), no. 4, 287–293.
[3] J. Dedecker and P. Doukhan, A new covariance inequality and applications, Stochastic Process. Appl. 106 (2003), no. 1, 63–80.
[4] M. Fréchet, Sur la distance de deux lois de probabilité, C. R. Acad. Sci. Paris 244 (1957), 689–692 (French).
[5] E. L. Lehmann, Some concepts of dependence, Ann. Math. Statist. 37 (1966), 1137–1153.
[6] C. M. Newman, Normal fluctuations and the FKG inequalities, Comm. Math. Phys. 74 (1980), no. 2, 119–128.
Rio-type inequality

[7] B. L. S. Prakasa Rao, *Hoeffding identity, multivariance and multicorrelation*, Statistics 32 (1998), no. 1, 13–29.

[8] ———, *Bounds for rth order joint cumulant under rth order strong mixing*, Statist. Probab. Lett. 43 (1999), no. 4, 427–431.

[9] ———, *Covariance identities for exponential and related distributions*, Statist. Probab. Lett. 42 (1999), no. 3, 305–311.

[10] ———, *Some covariance identities, inequalities and their applications: a review*, Proc. Indian Nat. Sci. Acad. Part A 66 (2000), no. 5, 537–543.

[11] E. Rio, *Covariance inequalities for strongly mixing processes*, Ann. Inst. H. Poincaré Probab. Statist. 29 (1993), no. 4, 587–597.

[12] ———, *Théorie Asymptotique des Processus Aléatoires Faiblement Dépendants*, Springer, Paris, 2000.

[13] H. Yu, *A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences*, Probab. Theory Related Fields 95 (1993), no. 3, 357–370.

B. L. S. Prakasa Rao: Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, New Delhi 110 016, India

*E-mail address:* blsp@isid.ac.in