Anomalous Hydrodynamics of Two-Dimensional Vortex Fluid

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Turbulent flows of incompressible liquid in two dimensions are comprised of dense systems of vortices. Such system of vortices can be treated as a fluid and itself could be described in terms of hydrodynamics. We develop the hydrodynamics of the vortex fluid. This hydrodynamics captures characteristics of fluid flows averaged over fast circulations in the inter-vortex space. The hydrodynamics of the vortex fluid features the anomalous stress absent in Euler’s hydrodynamics. The anomalous stress yields a number of interesting effects. Some of them are: a deflection of stream lines, a correction to the Bernoulli law, accumulation of vortices in regions with high curvature in the curved space. The origin of the anomalous stresses is a divergence of inter-vortex interactions at the micro-scale which manifest at the macro-scale. We obtain the hydrodynamics of the vortex fluid from the Kirchhoff equations for dynamics of point-like vortices.

1. Introduction and main results. Turbulent flows of two dimensional incompressible fluids consist of numerous vortices found essentially at all scales. In many physical realizations larger vortices can be considered as a collective motion of smaller minimal eddies which can not be fragmented further. Fluids with lower bound of vortex circulations could be both quantum and classical. Examples of quantum fluids include superfluid Helium[1,3], superconductors, cold atomic systems (BEC)[4] and electronic liquids in the Fractional Quantum Hall regime[6], models of classical turbulence with quantized vortices, sometimes referred to as models of the quantum turbulence (see e.g. [2]). The lower bound of vortex circulation $2\pi \gamma$ in quantum fluids is defined by the Planck constant. In classical fluids the lower bound for vortex circulation can occur in turbulent flows in the regime of the inverse cascade [8]. There it is determined by the injection scale.

In the regime of interest for this work the minimal eddies are spatially separated by a typical scale $\ell$ which is larger than the size of vortex cores but much smaller than the system size. There is no dissipation on these scales. In such regime the fluid motion can be separated into the fast motion with typical velocity $\gamma / \ell$ describing the rotation of the fluid around vortex cores and the slow motion of vortices themselves. Description of large structures in fluid dynamics requires averaging over fast motions. The averaging essentially eliminates fast motion reducing the fluid dynamics to the hydrodynamics of the vortex matter. Thus, on time scales larger than $\ell^2 / \gamma$ minimal eddies should be treated as a (secondary) dissipation-free fluid.

The idea to treat vortices as a macroscopical system goes back to seminal works of Onsager [1]. Onsager suggested to describe statistical properties of turbulent flows in 2D incompressible fluid by Gibbsian equilibrium statistical mechanics of a gas of point-like vortices [1]. In this paper we address the hydrodynamics of the vortex fluid [8]. Such hydrodynamics is a suitable platform to study the large scale and long time characteristics of flows.

The hydrodynamics of the vortex fluid can be obtained as a result of averaging over fast motion in parcels centered around vortices. The average velocity of each parcel is approximately equal to the velocity of the vortex itself. Therefore, positions of vortices, or Lagrangian coordinates of the vortex fluid are the “slow variables” describing the motion of the fluid. The goal of this paper is to derive the Euler description of the vortex fluid.

We focus on the hydrodynamics of the chiral flow where circulations of all vortices are of the same sign. Chiral flows occur in rotating fluids, and also, emerge as a spontaneous formation of like-circulation stable vortex clusters predicted Onsager [1], and commonly observed in 2D fluids. In this paper we assume that a chiral cluster has already been formed and focus on the vortex dynamics within the cluster.

Our main result is that the hydrodynamics of the chiral vortex flow is anomalous. Anomalous terms in hydrodynamics are conservative pseudo-tensor parts in the stress tensors [10]. They are invariant under translations and proper rotations but change sign under an orientation reversing coordinates transformation. In two dimensions the anomalous term is linear in velocity. In Cartesian coordinates it reads

$$\tau_{xy} = \tau_{yx} = \eta (\nabla_x v_x - \nabla_y v_y),$$
$$\tau_{xx} = -\tau_{yy} = -\eta (\nabla_x v_y + \nabla_y v_x).$$

where $v = (v_x, v_y)$ is the (Eulerian) velocity of the vortex flow and $\eta$ is the anomalous kinetic coefficient.

The anomalous stress is accompanied by the anom-
lous addition to the pressure of the vortex fluid

\[ p = p + 2\eta|\omega| + 2\eta^2 \frac{\Delta \sqrt{|\omega|}}{\sqrt{|\omega|}}, \]  

where \( p \) is the pressure and \( \omega \) is the vorticity of the fluid. We show that, the anomalous kinetic coefficient \( \eta \) is the quarter of the minimal vortex strength

\[ \eta = \frac{\gamma}{4}. \]  

Euler’s incompressible inviscid fluid does not exert a force acting on a shear flow. The vortex fluid does. The anomalous force exerted by the flow on the line element of the vortex fluid acts normally to the vortex shear flow. It produces neither work nor dissipation.

As a result of anomalous force the stream lines of the vortex flow are no longer aligned to the stream lines of the liquid flow. This misalignment causes corrections to the Bernoulli law.

Anomalous forces are revealed in the presence of boundaries and on curved manifolds. In this paper we briefly discuss the latter. On a curved manifold anomalous forces cause vortices to flow toward regions of positive curvature at the expense of the deficit of vortices in regions of negative curvature. This effect gives, yet another definition of the anomalous kinetic coefficient: a response of the vorticity of the flow \( \omega \) to a variation of the (scalar) curvature \( R \)

\[ \eta = \frac{\delta \omega}{\delta R}. \]  

In the paper we discuss few consequences of the anomalous term.

Through the paper we use the complex coordinates \( z = x + iy, \partial = \frac{1}{2}(\partial_x - i\partial_y) \), and the complex vector components \( u = u_x - iu_y \). We use subscripts \( a, b \) to denote the Cartesian components of vectors and subscripts \( i, j, k \) for particle labels. We also use the complex components for symmetric tensors \( \tau = \tau_{xx} - \tau_{yy} - 2i\tau_{xy}, \tau_{zz} = \tau_{xx} + \tau_{yy} \). In complex notations the components of the anomalous forces are applicable to a broader set of fluids which are approximately incompressible at distances away from vortices. The Gross-Pitaevsky equation in the regime where the vortex core is smaller than the separation between vortices is an example \[20\].

A convenient framework to develop the hydrodynamics of the vortex fluid is the Onsager’s description of flows by a system of a large but finite number of point-like vortices. It is based on Kirchhoff \[19\] equations which we now recall (see e.g., \[20\]).

2. Kirchhoff Equations as Lagrangian specification of the vortex flow. In two dimensions the curl of the Euler equation

\[ D_t \mathbf{u} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0 \]  

for an incompressible fluid with a constant density yields the single (pseudo) scalar equation for the vorticity \( \omega \equiv \nabla \times \mathbf{u} = \partial_x u_y - \partial_y u_x = 2i\partial \bar{u} \) (sometimes called Helmholtz equation)

\[ D_t \omega = 0. \]  

In this form the Euler equation has a simple geometrical meaning: the material derivative \( D_t \equiv (\partial_t + \mathbf{u} \cdot \nabla) \) of the vorticity vanishes. Vorticity is transported along the divergence-free velocity field \( \mathbf{u} \).

Helmholtz (1867), and later Kirchhoff (1883) showed that there is a class of solutions of the vorticity equation \[20\] which consists of a finite number of point-like vortices. In this solution the complex velocity of the fluid \( u \) is given by a rational function

\[ u(z, t) = -i \sum_{i=1}^{N} \frac{\gamma_i}{z - z_i(t)}. \]  

The number of vortices (poles) \( N \) and the circulations (residues) \( 2\pi \gamma_i \) do not change in time, while the moving positions of vortices \( z_i(t) \) obey the Kirchhoff equations:

\[ \ddot{z}_i = -i \sum_{j, j \neq i}^{N} \frac{\gamma_j}{(z_i(t) - z_j(t)).} \]  

Kirchhoff equations replace the non-linear PDE \[7\] by a dynamical system. The equations describe chaotic motions of a finite number of vortices if \( N > 3 \). If \( N \) is large equations can approximate virtually any flow \[21\].

We will consider a chiral flow, where all vortices have the counterclockwise circulation \( \omega > 0 \) and focus on the simplest case where all circulations are equal \( \gamma_i = \gamma > 0 \).

Under these specifications the velocity of the fluid, the stream function \( \mathbf{u} = (\partial_y \psi, -\partial_x \psi) \), or \( u = 2i\partial \bar{u} \), and Kirchhoff equations read

\[ u = -\sum_{j=1}^{N} \frac{i \gamma}{|z - z_j|}, \quad \psi = -\gamma \sum_{j=1}^{N} \log |z - z_j|, \]  

\[ \dot{z}_i = v_i, \quad v_i = -\sum_{j, j \neq i}^{N} \frac{i \gamma}{z_i(t) - z_j(t)}. \]  

We want to study this vortex system in the limit of a large number of vortices (\( N \rightarrow \infty \)) with a given mean density \( \bar{\rho} \). Such fluid on the average performs a solid rotation. The frequency of the solid rotation is approximately \( \Omega \approx \pi \gamma \bar{\rho} \). The mean density introduces the length scale \( \ell \approx \bar{\rho}^{-1/2} \) which measures the average distance between vortices. Before we proceed to the large \( N \) limit we describe some basic facts about the Kirchhoff equations.
3. Hamiltonian structure. Kirchhoff equations are Hamiltonian. Holomorphic $z_i$ and anti-holomorphic $\bar{z}_i$ coordinates of vortices are canonical coordinates with Poisson’s brackets
\[
\{z_i, \bar{z}_j\} = i(\pi\gamma)^{-1}\delta_{ij}. \tag{12}
\]
In the case of the chiral flow the Hamiltonian is
\[
H = -2\pi\gamma^2 \sum_{i<j} \log |z_i - z_j|. \tag{13}
\]
In the next sections we compare this Hamiltonian with the energy of an ideal fluid
\[
H = \frac{1}{2} \int u^2 d^2 r. \tag{14}
\]
They are different. Kirchhoff’s Hamiltonian captures only the part of the energy of the fluid which is transported by vortices.

Velocity of vortices and the Kirchhoff equations (11) follow from the canonical structure (12) and the Hamiltonian (13).

\[
v_i \equiv \{H, z_i\} = \frac{i}{\pi\gamma} \frac{\partial H}{\partial z_i}. \tag{15}
\]

In Cartesian coordinates $v_i = \frac{1}{2\pi} (\partial_{yi} H, -\partial_{xi} H)$. The velocity is the skew gradient of the Hamiltonian.

More generally the evolution of a field $O(r_1, \ldots, r_N)$ which depends on vortex positions is given by
\[
\dot{O} = \{H, O\} = \sum_i (v_i \cdot \nabla_{r_i}) O. \tag{16}
\]

Naturally, the vortex coordinates appear as Lagrangian coordinates of the vortex fluid.

4. Eulerian specification of the vortex flow. Kirchhoff equations provide the Lagrangian specification of the vortex flow. We want to proceed to the Eulerian specification. We consider flows where the microscopic density of vortices can be treated as a coarse-grained smooth density function, i.e.
\[
\sum_i \delta(r - r_i) \rightarrow \rho(r) = (2\pi\gamma)^{-1} \omega(r). \tag{17}
\]

Then the Eulerian field $O(r)$ corresponding to the symmetric function of vortex coordinates $O(r_1, \ldots, r_N)$ can be obtained as a result of coarse-graining as
\[
\sum_i \delta(r - r_i) O(r_i) \rightarrow O(r) \rho(r). \tag{18}
\]

In particular, the Eulerian velocity $v(r)$ is defined through the vortex flux
\[
\sum_i \delta(r - r_i) v_i \rightarrow \rho(r) v(r), \tag{19}
\]
where the velocity $v_i$ is given by (11).

Once we defined the vortex flux $\rho v$, we can find the coarse-grained energy (15). We use (15) to write $(\pi\gamma) \rho v = i \sum_i \delta(r - r_i) \partial_i H$ and use the formula $\sum_i \delta(r - r_i) \nabla_{r_i} = \rho \nabla_{\rho} \frac{\delta}{\delta \rho}$ which converts between symmetric function of Lagrangian coordinates $r_i$ and functionals of Eulerian density $\rho$. We obtain the relation
\[
v = i \frac{\delta H}{\delta \rho}. \tag{20}
\]

This formula shows that the vortex fluid is incompressible as the original liquid itself and that the stream function of the vortex flow defined as $v = -2i \partial \psi$ is given by $\psi = -(2\pi\gamma)^{-1} \delta H/\delta \rho = -\delta H/\delta \omega$. In the rest of the paper we use the density of vortices $\rho$ and the vorticity $\omega$ interchangeably.

5. Velocity of the vortex flow and deflections of the vortex stream lines. We start by computing the velocity of the vortex fluid. We use the $\dot{\delta}$ formula $\dot{\delta} \left( \frac{1}{\pi} \right) = \pi \delta(r)$, write the vortex flux as $\rho v = \frac{\pi}{\pi} \left( \sum_i \frac{1}{\pi^2} \right)$, substitute (14), use the identity
\[
2 \sum_{i \neq j} \frac{1}{z_i - z_j} = \left( \sum_i \frac{1}{z_i} \right)^2 + \dot{\partial} \sum_i \frac{1}{z_i} \tag{21}
\]
and obtain the important relation between the velocities of these two flows
\[
v = u - 2\eta i \partial \log \omega = u + \frac{\eta}{\omega} \Delta u , \tag{22}
\]
where $\eta$ is given by (19). We comment that in this work both $\eta$ and $\omega$ are set positive, but the product $\eta \omega$ is positive regardless of the convention.

The last term in (22) is the source of anomalous terms in the hydrodynamics of the vortex fluid. The anomalous terms reflect the discreteness of vortices. Formally, the source of the anomalous term is the careful treatment of $i \neq j$ term in sums over vortices positions in (10,12) when passing to the Euler specification. This is seen from the identity (21). The anomalous terms represent the excluded volume occupied by a vortex. The following simple argument helps to determine the coefficient in front of the anomalous term. The velocity of the fluid close to the vortex core diverges as $\gamma/|z|$. This pole singularity is cancelled by the anomalous term. As a result the velocity of the vortex fluid (or the coarse-grained velocity of the fluid) is a smooth function in contrast to the fluid velocity. The identity (21) illustrates this effect: while each of two terms in the r.h.s. of (21) has double poles, the expression in the l.h.s. possesses only single poles. In the rest of the paper we trace the consequences of the anomalous term.

As an illustration of the anomalous effect let us use (22) to calculate the anomalous correction to the “orbital moment” of the vortex fluid $\int d^2 r \rho (r \times (v - u)) = \eta \int d^2 r (r \cdot \nabla \rho) = 2\eta \int d^2 r \rho$. We see that the correction is equal to
2\eta per vortex. This calculation is a continuous version of the elementary sum rule \( \sum (r_i \times u_i) = 2\eta N(N - 1) \) for Kirchhoff vortices. It isolates the anomalous kinetic coefficient \( \eta \).

The linearized relation of \( \omega \) is \( \psi = (1 + \frac{1}{4}(k^2\Delta))u \), where \( \ell = (2\pi \rho)^{-1/2} \). Consider, for example a shear flow. If the flow is oscillatory, say \( u_i \sim (\cos ky, 0) \) the vortex flow lags the fluid \( v_i = (1 - \frac{1}{4}(k\ell)^2)u_i \). However, if the velocity of the flow falls or rises, say as \( u_i \sim (e^{-ky}, 0) \), then the vortex flow \( v_i = (1 + \frac{1}{4}(k\ell)^2)u_i \) leads the fluid.

Eq. (22) shows that the vortex flow is incompressible as the fluid, but the stream lines of both fluids deflect
\[
\Psi = \psi - \eta \log |\Delta \psi|.
\]
If we treat the vortex flow as a coarse-grained flow we conclude that the coarse graining causes a deflection of stream lines.

Yet another effect is a difference between the vorticities of two flows
\[
\nabla \times \mathbf{v} = \omega + \eta \Delta \log \omega.
\]
We observe that the relation between vorticities is non-linear. This causes interesting effects.

If, for example, a flow is a large vortex patch performing a solid rotation \( v_i = \Omega_i \times r \) (the angular velocity is a vector directed normal to the plane) the relation (24) inside the patch becomes the non-linear Liouville-like equation
\[
2\Omega = \omega + \eta \Delta \log \omega.
\]
Solution of this equation depends on the flow outside the patch and may not correspond to a solid rotation of the fluid \( \omega = 2\Omega \). For example, if the patch is surrounded by the irrotational flow, the equation (25) determines a peculiar structure of the interface. The interface features oscillations with a period of order \( \ell \) propagating well inside the patch. The oscillations are due to the anomalous term and reflect the discrete nature of vortices.

6. The Hamiltonian of the vortex flow. Once we know the vortex velocity, the relation (20) helps to compute the Hamiltonian
\[
\mathcal{H} = H - \eta \int \omega \log |\omega| d^2r.
\]
The anomalous term in (20) (the difference between \( \mathcal{H} \) and \( H \)) is the energy of vortices at rest.

Using Eq. (22) one can express Eq. (26) in terms of velocity of the vortex flow and the vortex density
\[
\mathcal{H} = \frac{1}{2} \int \left[ v^2 - \eta^2 (\nabla \log \rho)^2 \right] d^2r.
\]
The density of vortices is the only independent field in this flow. The Poisson algebra of the density field can be obtained directly from the Kirchhoff canonical brackets (12) and the definition (17). They yield the familiar brackets for the vorticity in incompressible fluid (8)
\[
\{\omega(r), \omega(r')\} = \frac{1}{2} \left[ \nabla_{r'} \times \nabla_r \right] (\omega(r) + \omega(r')) \delta(r - r').
\]

7. Momentum flux tensor and pressure. We may read the momentum flux tensor from the Euler equation (9) written in the form of the conservation law
\[
\dot{u} + \mathbf{v} \cdot \nabla u = 0.
\]
Here \( \Pi \) and \( \Pi_{zz} \) are holomorphic component and the trace of the momentum flux tensor. Let us recover the Euler equation from the Kirchhoff equations by computing the evolution of velocity \( \dot{u} = \{\mathcal{H}, u\} \) with the help of Kirchhoff’s Hamiltonian structure. We obtain
\[
\Pi = -2i\gamma \sum_i \frac{\bar{v}_i}{z - z_i}, \quad \Pi_{zz} = 2\gamma \text{Im} \sum_i \frac{\bar{v}_i}{z - z_i}.
\]
We observe the relation between the vortex fluid and the traceless part of the momentum flux tensor
\[
(2\pi\gamma)\rho v = \omega v = i\delta \Pi,
\]
and find it with the help the expression (22)
\[
\Pi = u^2 - 4i\gamma \partial^2 u.
\]
The last term is the anomaly. The stress exerted by the vortex fluid differs from the stress \( u^2 \) exerted by the fluid itself by the anomalous stress as in (5). The anomalous stress is a pseudo-tensor. It represents a real force that could be measured.

The origin of the anomalous term is the absence of vortex self-interaction in the sum (13). The coarse-grained stress is a smooth function in contrast to the fluid stress \( u^2 \). One can see this from the following simple argument. The velocity of the fluid close to vortex core diverges as \( \gamma/\ell \). The stress of the fluid \( u^2 \) diverges even stronger as \( \gamma^2/\ell^2 \). The second order pole singularity is cancelled by the anomalous term.

The momentum flux tensor (20) can be written in Cartesian components as
\[
\Pi_{ab} = u_a u_b + \delta_{ab} (p + \eta \omega) + \tau_{ab}[u]
\]
and differs from the fluid momentum flux tensor \( u_a u_b + \delta_{ab} p \) by the anomalous tensor \( \eta \omega \delta_{ab} + \tau_{ab}[u] \). The second term here is given by (11) with \( v \) replaced by \( u \). The anomalous correction represents forces exerted by vortices on the fluid.

The anomalous momentum flux tensor is divergence-free \( \nabla_b (\eta \omega \delta_{ab} + \tau_{ab}) = 0 \), and, therefore, does not contribute to the Euler equation of the fluid. Later we see that it contributes to the Euler equation (50) of the vortex fluid.

The anomalous momentum flux tensor (22) has physical implications. Consider for example the Bernoulli law. It states that \( 2p + |u|^2 \) stays constant along stream lines \( \psi = \text{const} \) of a stationary flow of the fluid. Instead, the trace of the momentum flux (52)
\[
\Pi_{zz} = 2p + |u|^2 + 2\eta \omega = \text{const}
\]
does not change along stream lines $\Psi = \text{const}$ of the vortex flow. These lines are different (see (23)).

To conclude this paragraph we mention that the general relation between the momentum flux tensor and the energy remain the same as in the Euler fluid

$$\delta \Pi = 2 \rho \partial \delta \mathcal{H} \delta \rho .$$

(34)

8. Euler equation for the vortex flow. In this paragraph we obtain the analog of the Euler equation for the vortex flow. We treat vortices as elementary constituents of the vortex fluid which density is the vorticity $\rho = (2\pi \gamma)^{-1} \omega$ and velocity $v$ defined by (17,19). Local expressions (22,24) warrant that the Euler equation for the vortex fluid is also local.

The vortex flow is incompressible $\nabla \cdot v = 0$. Let us introduce the material derivative of the vortex flow $D_t = \partial_t + v \cdot \nabla$. Then the Helmholtz equation (7) states that the material derivative of the vortex density vanishes

$$D_t \rho = 0 .$$

(35)

The calculation of the material derivative of the vortex velocity is straightforward albeit tedious. It follows from the relation between the vortex velocity and the fluid velocity (22) and the evolution of the fluid velocity (9). The result is

$$D_t v_a + \nabla_a p + \rho^{-1} \nabla_b [\rho \tau_{ab}] = 0 ,$$

(36)

where $\tau_{ab}$ is the anomalous stress given by (11). The pressure $p$ can be found from the condition of incompressibility of the vortex fluid. It is given by (2) and also has an anomalous correction if compared to the pressure of the fluid $p$.

The anomalous stress exerted by the vortex fluid must be compared with the Coriolis force $-2 \Omega \times v$, where $\Omega$ is the angular velocity of the solid rotation. The shift of velocity in (22) can be seen as a local transformation of coordinates adjusted to a shear flow. Then the anomalous momentum flux tensor in (35) and the anomalous corrections to pressure (2) are the analogs of the Coriolis force and the centrifugal force, respectively.

9. Vortex flow on a curved surface: accumulation of vortices at patches of positive curvature. Anomalous terms are revealed on a curved surface. Here we describe only one effect: accumulation/deficit of vortices in regions of space with positive/negative curvature. A comment is in order. A generalization of the vortex dynamics to the curved space requires the information of the energy of each vortex at rest in the curved background. This energy is additive and was ignored in the flat background. In the curved space it depends on the metric, not necessarily in a universal manner, which causes an additional drift of a single vortex in the presence of curvature gradients. We continue to omit this effect emphasizing the interaction between vortices.

Let us examine how the relation between the stream functions of the fluid and of the vortex fluid (28) changes when the Euclidean metric is deformed into the Riemannian metric. Under this deformation the density of vortices is transformed as the inverse form of volume $\omega \to \omega \sqrt{g}$. Then the difference between stream functions transforms as $\Psi - \psi \to \Psi - \psi - \eta \log \sqrt{g}$. The anomalous transformation changes the relation between vorticities of two fluids (24). The latter becomes

$$2 \Omega = \nabla \times v = \omega + \eta (\Delta_g \log |\omega| - R) ,$$

(37)

where $\Delta_g$ is the Laplace-Beltrami operator on a space with metric $g$ and curvature $R$ and $\Omega$ is an angular velocity of the local rotation of the vortex fluid.

In the leading order of the gradients $\delta \omega$ yields the transformation of the density of vortices at a given vorticity of the vortex flow $\Omega$

$$\delta \omega = \eta \left[ \delta R - \frac{1}{4} (\ell^2 \Delta_g) \delta R + \ldots \right] .$$

(38)

In particular a conical singularity with a deficit angle $\theta$ accumulates an additional $2\eta\theta$ vortices due to the anomalous forces.

This result gives an alternative definition of the anomalous kinetic coefficient $\eta$. It could be seen as a response of the vorticity to a variation of the curvature $\eta = \delta \omega / \delta R$.

The formula establishes a global relation between the total vorticity of the fluid and the vorticity of the vortex fluid. We obtain it by integrating (37) over the volume and using the Gauss-Bonnet formula. We have

$$\int (\omega - \nabla \times v) dV = 4 \pi \eta \chi .$$

(39)

We mention one particular application of this formula: if the vortex patch with the total vorticity $2 \pi \gamma N$ performs a solid rotation with a frequency $\Omega$, the patch occupies the area which is less than $\pi \gamma N / \Omega$ by the anomalous amount $2 \pi \eta \chi / \Omega$. This result illustrates the effect of the anomaly: a local regularization at a micro-scale yields effects on a macro-scale and eventually affects the global relations.

The most interesting effects of the anomalous terms are seen at the boundary. We will discuss the boundary problem elsewhere.

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