We present the field equations governing the equilibrium of rapidly rotating neutron stars in scalar-tensor theories of gravity, as well as representative numerical solutions. The conditions for the presence of a nontrivial scalar field and the deviations from the general relativistic solutions are studied. Two examples of scalar-tensor theories are examined – one case that is equivalent to the Brans-Dicke theory and a second case, that is perturbatively equivalent to Einstein’s General Relativity in the weak field regime, but can differ significantly for strong fields. Our numerical results show that rapidly rotating neutron star models with a nontrivial scalar field exist in both cases and that the effect of the scalar field is stronger for rapid rotation. If we consider values of the coupling parameters in accordance with current observations, only the second example of scalar-tensor theories has significant influence on the neutron star structure. We show that scalarized, rapidly rotating neutron stars exist for a larger range of the parameters than in the static case, since a nontrivial scalar field is present even for values of the coupling constant $\beta > -4.35$, and that these solutions are energetically more favorable than the general relativistic ones. In addition, the deviations of the rapidly rotating scalar-tensor neutron stars from the general-relativistic solutions is significantly larger than in the static case.

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I. INTRODUCTION

Einstein’s General Relativity theory (GR) has been tested in various astrophysical scenarios and its agreement with observations is remarkable. But, many generalizations of GR exist, which are also compatible with current observational uncertainties. Moreover, there exist phenomena, such as dark matter and the accelerated expansion of the universe, which do not fit well in the standard GR framework and alternative theories of gravity are often employed to explain them. Alternative theories of gravity also originate from theories trying to unify all the interactions such as Kaluza-Klein theories, higher dimensional theories of gravity, etc. (see [1–4] and references therein).

The mediator of the gravitational interaction in GR is the metric tensor $g_{\mu\nu}$. One of the most natural generalizations of GR is to include an additional mediator – a scalar field $\Phi$. In fact, there exists a whole class of scalar-tensor theories of gravity (STT), which originate from the works of Jordan, Fierz and Brans & Dicke [5–9]. In STT, there is no direct interaction between the sources of gravity and the scalar field, which means that the weak equivalence principle is satisfied. To test
these alternative theories, it is interesting to explore solutions representing relativistic stars in STT (in the region of parameter space that is compatible with current observations) and investigate their deviation from corresponding solutions in GR.

Because of their high densities and compactness, neutron stars are an ideal laboratory for testing gravity in the strong field regime. In contrast, most current astrophysical constraints on gravity theories are derived from observations in weak field environments. Specifically, observations of neutron stars may test strong-field deviations from GR, for STT that are indistinguishable from GR in the weak field regime [10, 11]. In the static spherically symmetric case, it is already known that there exists nonuniqueness of solutions representing neutron stars for a particular class of STT and a certain range of parameters: in addition to the general-relativistic solutions with a trivial scalar field, solutions with a nontrivial scalar field exist, which are energetically more favorable than their GR counterpart [4, 10–14] and their stability was studied in [15, 16]. Under this spontaneous scalarization, neutron star models can have significantly different properties than in GR. Similarly, black hole solutions in STT with a nontrivial scalar field were found and their dynamics was studied in [17–24].

Astrophysical implications, such as the effect of a nontrivial scalar field on gravitational wave emission and the redshift of spectral lines in X-rays and γ-rays have been considered in [25–27]. Neutron star mergers in scalar-tensor theories of gravity were studied in [28] and the collapse of a spherical neutron star to a black hole was examined in [29]. The transition of a neutron star with a zero scalar field to a scalarized state was considered in [30] and the scalar gravitational waves from Oppenheimer-Snyder collapse were examined in [31]. All these studies showed that the scalar field has distinct properties, some of which could potentially be used as observational probes. However, so far all studies were limited to either static, or slowly-rotating models (to first order in the angular velocity) [11, 12]. In the latter case, rotational corrections to the mass, radius and scalar field, were not obtained, because they are of higher order with respect to the rotational frequency. The extension of these studies to rapid rotation is important from an astrophysical point of view, because accreting neutron stars in Low-Mass-X-Ray-Binaries (LMXBs) are known to rotate with frequencies up to 700 Hz [32], while progenitors of magnetars are theorized to have been born rapidly rotating [33, 34]. In addition, in binary neutron star mergers a quasi-stable merger remnant with rapid differential rotation is now considered as the generic outcome [35–37].

Here, we present the first study of rapidly rotating neutron stars in scalar-tensor theories of gravity. The required field equations are derived and they are solved numerically by using a modification of the rns code [38]. Our aim is to check for the existence of solutions with a nontrivial scalar field for two different examples of STT and for different values of the parameters, and to determine the deviations from GR. For STT, current observations set tight constraints on the possible values of the scalar field coupling constants [3, 11, 39, 40]. We find that the range of parameters for which scalarized rapidly rotating neutron star models exist is considerably larger than in the nonrotating case. The strength of the scalar field also increases when rapid rotation is considered and several stellar properties, such as mass, radius and angular momentum change significantly.

The paper is structured as follows. In Sec. III we give the basic theoretical background and present the field equations. A brief overview of the numerical method, the changes induced by the presence of the scalar field and the various tests we apply in order to check our code are given
in Sec. III. In Sec. IV we present the numerical solutions describing rapidly rotating neutron stars with a nontrivial scalar field. Two examples of STT are considered: one, which is equivalent to the Brans-Dicke theory, and a second one, which is perturbatively equivalent to GR in the weak field regime, but can differ significantly for strong fields. We end the paper with a summary and discussion.

II. BASIC EQUATIONS

The general form of the gravitational action of the scalar-tensor theories in the physical Jordan frame is given by \[1–4\]

\[
S = \frac{1}{16\pi G_\ast} \int d^4x \sqrt{-\tilde{g}} \left[ F(\Phi)\tilde{R} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2U(\Phi) \right] + S_m \left[ \Psi_m; \tilde{g}_{\mu\nu} \right].
\]

Here, \(G_\ast\) is the bare gravitational constant, and \(\tilde{R}\) is the Ricci scalar curvature with respect to the space-time metric \(\tilde{g}_{\mu\nu}\). The dynamics of the scalar field \(\Phi\) is governed by the functions \(F(\Phi), Z(\Phi)\) and \(U(\Phi)\). In order for the gravitons to carry positive energy we must have \(F(\Phi) > 0\). The nonnegativity of the scalar field kinetic energy requires that \(2F(\Phi)Z(\Phi) + 3(dF(\Phi)/d\Phi)^2 \geq 0\). The matter fields are collectively denoted by \(\Psi_m\) and their action \(S_m\) depends on \(\Psi_m\) and the space-time metric \(\tilde{g}_{\mu\nu}\). Here we consider the phenomenological case when the matter action does not involve the scalar field in order for the weak equivalence principle to be satisfied. \(^2\)

From a mathematical point of view it is convenient to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by \[1–4\]

\[
g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu}.
\]

Introducing the scalar field \(\varphi\) via the equation

\[
\left( \frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left( \frac{d\ln(F(\Phi))}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)}
\]

and defining

\[
A(\varphi) = F^{-1/2}(\Phi), \quad 2V(\varphi) = U(\Phi)F^{-2}(\Phi),
\]

the Einstein frame version of the action \([1]\) takes the form

\(^1\) In fact, the three functions governing the dynamics of the scalar field \(\Phi\) can be reduced to two functions by a simple redefinition of the scalar field. For example, a widely used parametrization is the Brans-Dicke one given by \(F(\Phi) = \Phi\) and \(Z(\Phi) = \frac{\omega(\Phi)}{\Phi}\). In this parametrization the functions \(F(\Phi)\) and \(Z(\Phi)\) are reduced to one function, namely \(\omega(\Phi)\).

\(^2\) In general we can consider the more general case when there are direct interactions between the matter fields. However we leave this for future study.
\[ S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 4V(\varphi)\right) + S_m[\Psi_m; A^2(\varphi)g_{\mu\nu}], \] (5)

where \( R \) is the Ricci scalar curvature with respect to the Einstein metric \( g_{\mu\nu} \). By varying this action with respect to the Einstein frame metric \( g_{\mu\nu} \) and the scalar field \( \varphi \), we find the field equations in the Einstein frame

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_* T_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \partial_\alpha \varphi \partial_\beta \varphi - 2V(\varphi)g_{\mu\nu} , \]

\[ \nabla^\mu \nabla_\mu \varphi = -4\pi G_* k(\varphi)T + \frac{dV(\varphi)}{d\varphi} \] (6)

where

\[ k(\varphi) = \frac{d \ln (A(\varphi))}{d\varphi}. \] (7)

The Einstein frame energy-momentum tensor \( T_{\mu\nu} \) is related to the Jordan frame one \( \tilde{T}_{\mu\nu} \) via

\[ T_{\mu\nu} = A^2(\varphi) \tilde{T}_{\mu\nu}. \]

The relations between the quantities in both frames are explicitly given by the following equations:

\[ \varepsilon = A^4(\varphi) \tilde{\varepsilon}, \]
\[ p = A^4(\varphi) \tilde{\rho}, \]
\[ u_\mu = A^{-1}(\varphi) \tilde{u}_\mu. \] (10)

The contracted Bianchi identities give the following conservation law for the Einstein frame energy-momentum tensor

\[ \nabla_\mu T^\mu_{\nu} = k(\varphi) \partial_\nu \varphi. \]

In accordance with the main purpose of the present paper we consider stationary and axisymmetric spacetimes. In mathematical terms this means that the spacetimes admit one asymptotically timelike at infinity Killing vector field \( \xi \) and another axial spacelike Killing vector field \( \eta \) with periodic orbits and commuting with \( \xi \) such that their flows leave invariant not only the metric (i.e. \( \mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\eta g_{\mu\nu}=0 \)) but also the scalar field and the matter fields, i.e.
\[ \mathcal{L}_\xi \varphi = \mathcal{L}_\eta \varphi = 0, \]  
\[ \mathcal{L}_\xi T_{\mu\nu} = \mathcal{L}_\eta T_{\mu\nu} = 0, \]  
(11)

with \( \mathcal{L} \) being the Lie derivative. It is rather natural from a physical point of view to require the set of fixed points under the flow of \( \eta \) to be nonempty. In other words we require \( \eta \) to have a nontrivial axis of symmetry where \( \eta \) vanishes.

We should note that in the Jordan frame we also have \( \mathcal{L}_\xi \tilde{g}_{\mu\nu} = \mathcal{L}_\eta \tilde{g}_{\mu\nu} = 0 \) and \( \mathcal{L}_\xi \tilde{T}_{\mu\nu} = \mathcal{L}_\eta \tilde{T}_{\mu\nu} = 0 \) since the scalar field \( \varphi \) is invariant under the Killing flows.

We will impose one more condition, namely the stationary and axisymmetric spacetime to be circular. In other words, the 2-dimensional surfaces orthogonal to the Killing fields \( \xi \) and \( \eta \) to be integrable. According to the Frobenius theorem this holds if and only if the Frobenius conditions are satisfied, namely \[ \text{[41, 42]} \]

\[ \eta_{[\mu} \xi_{\nu]} \nabla_{\alpha} \xi_{\beta]} = 0, \]  
(13)

\[ \xi_{[\mu} \eta_{\nu]} \nabla_{\alpha} \eta_{\beta]} = 0. \]  
(14)

Using the properties of the Killing fields and the field equations, the Frobenius conditions can be written in the form

\[ \xi^\mu T_{\mu[\nu} \xi_{\alpha]} \eta_{\beta]} = 0, \]  
(15)

\[ \eta^\mu T_{\mu[\nu} \xi_{\alpha]} \eta_{\beta]} = 0. \]  
(16)

Written in this form, the Frobenius conditions themselves impose restrictions on the energy-momentum tensor. Specifically, for the case of a perfect fluid we obtain (taking into account that \( \xi^\mu u_\nu \neq 0 \) and \( \varepsilon + p \neq 0 \))

\[ u_{[\nu} \xi_{\alpha]} \eta_{\beta]} = 0, \]  
(17)

which means that at every spacetime point the 4-velocity \( u^\mu \) lies in the plane spanned by \( \xi \) and \( \eta \), i.e. \( u^\mu = u^\xi \xi^\mu + u^\eta \eta^\mu \). This also holds in the physical Jordan frame, i.e. \( \tilde{u}^\mu = \tilde{u}^\xi \xi^\mu + \tilde{u}^\eta \eta^\mu \) since \( \mathcal{L}_\xi \eta \varphi = 0 \).

Using the fact that \( u \) is a linear combination of \( \xi \) and \( \eta \) as well as that \( \xi \) and \( \eta \) commute, we obtain from (11) the natural consequence

\[ \mathcal{L}_\xi \eta \varepsilon = \mathcal{L}_\xi \eta p = \mathcal{L}_\xi \eta \tilde{\varphi} = \mathcal{L}_\xi \eta \tilde{p} = 0. \]  
(18)

The circularity condition allows us to simplify considerably the spacetime metric. The circular spacetimes admit a foliation by integrable 2-surfaces orthogonal to the Killing fields \( \xi \) and \( \eta \) and the spacetime is (locally) a product manifold \( M = M_\parallel \times M_\perp \). Here \( M_\parallel \) is a 2-dimensional Lorentz manifold to which the Killing fields \( \xi \) and \( \eta \) are tangential and \( M_\perp \) is a 2-dimensional Riemannian manifold orthogonal to \( \xi \) and \( \eta \). The spacetime metric is then given by the sum \( g = g_\parallel + g_\perp \) where
$g_\parallel$ is a Lorentz metric on $M_\parallel$ and $g_\perp$ is a Riemannian metric on $M_\perp$. In a coordinate presentation we have

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\parallel ab} dx^a dx^b + g_{\perp ij} dx^i dx^j,$$

(19)

where both metrics depend only on the coordinates $x^k$ on $M_\perp$, namely $g_{\parallel ab} = g_{\parallel ab}(x^k)$, $g_{\perp ij} = g_{\perp ij}(x^k)$.

Now we can choose convenient coordinates on $M_\perp$ and $M_\parallel$ which automatically fix convenient spacetime coordinates. Since $(M_\perp, g_{\perp ij})$ is a two-dimensional Riemannian manifold we can always choose the coordinates $(r, \theta)$ so that the metric has the conformal form

$$g_{\perp ij} dx^i dx^j = e^{2\alpha} (dr^2 + r^2 d\theta^2)$$

(20)

with $\alpha = \alpha(r, \theta)$. Concerning the 2-dimensional Lorentz manifold $(M_\parallel, g_{\parallel ab})$, the natural choice of the coordinates on it are the coordinates adapted to the Killing fields, i.e. the coordinates $t$ and $\phi$ for which $\zeta = \partial_t$ and $\eta = \partial_\phi$. Denoting by $\omega$ the minus scalar product of the Killing fields normalized by the norm of $\eta$, i.e. $\omega = -g(\zeta, \eta)/g(\eta, \eta)$, the Lorentz metric can be written in the form

$$g_{\parallel ab} dx^a dx^b = -A^2 dt^2 + B^2 r^2 \sin^2 \theta (d\phi - \omega dt)^2$$

(21)

with $\omega = \omega(r, \theta)$, $A = A(r, \theta)$ and $B(r, \theta)$. It turns out that it is more convenient in dealing with the field equations to use the functions $\sigma$ and $\gamma$ instead of $A$ and $B$, defined by $A^2 = e^{\sigma+\gamma}$ and $B^2 = e^{\gamma-\sigma}$. Summarizing, we have chosen coordinate $t, \phi, r, \theta$ in which the spacetime metric takes the form

$$ds^2 = -e^{-\sigma} dt^2 + e^{\gamma-\sigma} r^2 \sin^2 \theta (d\phi - \omega dt)^2 + e^{2\alpha} (dr^2 + r^2 d\theta^2)$$

(22)

with all metric functions depending only on $r$ and $\theta$.

Having the explicit form of the metric, we proceed to write the dimensionally reduced field equations. For this purpose it is convenient to use the proper velocity $v$ of the fluid given by

$$v = (\Omega - \omega) r \sin \theta e^{-\sigma},$$

(23)

where $\Omega$ is the fluid angular velocity defined by $\Omega = \frac{\partial \phi}{\partial t}$. $\Omega$ and $v$ are the same in both Einstein and Jordan frames. Indeed, for the angular velocity $\tilde{\Omega}$ in the Jordan frame we have $\tilde{\Omega} = \frac{\partial \phi}{\partial t} = \frac{\mathcal{A}(\phi) \partial \phi}{\mathcal{A}(\phi) \partial t} = \frac{\partial \phi}{\partial \tilde{t}} = \Omega$. In the same way the factor $\mathcal{A}(\phi)$ cancels out in the definition of the proper velocity $\tilde{v}$ in the Jordan frame which gives that $\tilde{v} = v$.

The fluid four velocity in the Einstein frame then is

$$u^\mu = \frac{e^{-(\sigma+\gamma)/2}}{\sqrt{1 - v^2}} [1, 0, 0, \Omega].$$

(24)
From a numerical point of view it is more convenient to use the following angular coordinate \( \mu = \cos \theta \) instead of \( \theta \). With the help of this coordinate, the dimensionally reduced Einstein equations for the metric functions \( \gamma, \sigma \) and \( \omega \) are the following:

\[
\left( \Delta + \frac{1}{r} \partial_r - \frac{\mu}{r^2} \partial_\mu \right) \left( \gamma e^{\gamma/2} \right) = e^{\gamma/2} \left\{ (16 \pi p - 4 V(\varphi)) e^{2\alpha} + \frac{\gamma}{2} \left[ (16 \pi p - 4 V(\varphi)) e^{2\alpha} - \frac{1}{2} (\partial_r \gamma)^2 - \frac{1}{2} \frac{\mu^2}{r^2} (\partial_\mu \gamma)^2 \right] \right\}, \tag{25}
\]

\[
\Delta (\sigma e^{\gamma/2}) = e^{\gamma/2} \left\{ 8 \pi (\epsilon + p) e^{2\alpha} \frac{1 + v^2}{1 - v^2} + r^2 (1 - \mu^2) e^{-2\sigma} \left[ (\partial_r \omega)^2 + \frac{1}{r^2} (\partial_\mu \omega)^2 \right] + \frac{1}{r} \partial_r \gamma - \frac{\mu}{r^2} \partial_\mu \gamma \right\}.
\]

\[
\Delta (\omega e^{\gamma/2 - \sigma}) = e^{\gamma/2 - \sigma} \left\{ -16 \pi \frac{(\epsilon + p)(\Omega - \omega)}{1 - v^2} e^{2\alpha} + \omega \left[ -\frac{1}{r} \partial_r \left( \frac{1}{2} \partial_\gamma + 2 \varphi \right) + \frac{\mu}{r^2} \partial_\mu \left( \frac{1}{2} \partial_\gamma + 2 \varphi \right) - \frac{1}{4} (\partial_r \gamma)^2 - \frac{1}{4} \frac{\mu^2}{r^2} (\partial_\mu \gamma)^2 + \right.ight.
\]

\[
\left. + (\partial_r \sigma)^2 + \frac{1}{r^2} (\partial_\mu \sigma)^2 - r^2 (1 - \mu^2) e^{-2\sigma} \left[ (\partial_r \gamma)^2 + \frac{1}{r^2} (\partial_\mu \omega)^2 \right] \right \}.
\]

Here, the differential operator \( \Delta \) is defined by

\[
\Delta = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1 - \mu^2}{r^2} \partial_\mu^2 - \frac{2 \mu}{r^2} \partial_\mu.
\]

For the metric function \( \alpha \) we obtain two first order partial differential equations. However, in the numerical method in order to determine \( \alpha \) we need only one of them, namely the following

\[
\partial_\mu \alpha = - \frac{\partial_\mu \gamma + \partial_\mu \sigma}{2} - \left\{ (1 - \mu^2) (1 + r \partial_r \gamma)^2 + [-\mu + (1 - \mu^2) \partial_\mu \gamma]^2 \right\}^{-1} \times \tag{29}
\]

\[
\left\{ \frac{1}{2} \left[ r \partial_r (r \partial_r \gamma) + r^2 (\partial_r \gamma)^2 - (1 - \mu^2) (\partial_\mu \gamma)^2 - \partial_\mu [ (1 - \mu^2) \partial_\mu \gamma] + \mu \partial_\mu \gamma \right] \times [-\mu + (1 - \mu^2) \partial_\mu \gamma] +
\right.
\]

\[
\left. + \frac{1}{4} [-\mu + (1 - \mu^2) \partial_\mu \gamma] \times \left[ r^2 (\partial_r \gamma + \partial_r \sigma)^2 - (1 - \mu^2) (\partial_\mu \gamma + \partial_\mu \sigma)^2 + 4 r^2 (\partial_\varphi)^2 - 4 (1 - \mu^2) (\partial_\mu \varphi)^2 \right] +
\right.
\]

\[
\left. + \mu \partial_r \gamma [1 + r \partial_r \gamma] - (1 - \mu^2) r (1 + r \partial_r \gamma) \left[ \partial_\mu \partial_r \gamma + \partial_\mu \sigma \partial_r \gamma + \frac{1}{2} (\partial_\mu \gamma + \partial_\mu \sigma) (\partial_r \gamma + \partial_r \sigma) + 2 \partial_\mu \varphi \partial_r \varphi \right] +
\right.
\]

\[
\left. + \frac{1}{4} (1 - \mu^2) e^{-2\sigma} \left[ -[-\mu + (1 - \mu^2) \partial_\mu \gamma] [r^4 (\partial_r \omega)^2 - r^2 (1 - \mu^2) (\partial_\mu \omega)^2] +
\right.
\]

\[
\left. + 2 (1 - \mu^2) r^3 \partial_\mu \omega \partial_r \omega (1 + r \partial_r \gamma) \right\}.
\]
The above equations have to be supplemented with the field equation for the scalar field and
the equation for hydrostatic equilibrium

$$\Delta \varphi = -\partial_r \gamma \partial_r \varphi - \frac{1}{r^2} \partial_{\mu} \gamma \partial_\mu \varphi + \left[ 4\pi k(\varphi)(\varepsilon - 3p) + \frac{dV(\varphi)}{d\varphi} \right] \epsilon^2, \quad (30)$$

$$\frac{\partial_i \tilde{p}}{\tilde{\varepsilon} + \tilde{p}} - \left[ \partial_i (\ln \upsilon^i) - \upsilon^i \upsilon_\varphi \partial_i \Omega - k(\varphi) \partial_i \varphi \right] = 0. \quad (31)$$

Here we use the Jordan (physical) frame pressure $\tilde{p}$ and energy density $\tilde{\varepsilon}$ which are related to
the Einstein frame ones via (10). On the other hand, the Einstein frame four velocity is utilized
because the equations take a simpler form in this case.

To close the system describing the equilibrium configurations of rapidly rotating neutron stars
in the scalar-tensor theories we should specify the equations of state for the matter. In the present
paper we use a polytropic equation of state (EoS):

$$\tilde{\varepsilon} = K \tilde{\rho}^\Gamma \frac{\Gamma - 1}{\Gamma - 1} + \tilde{\rho}^2, \quad \tilde{p} = K \tilde{\rho}^\Gamma \frac{\Gamma - 1}{\Gamma - 1},$$

where $\tilde{\rho}$ is the rest mass density in the Jordan frame, $N$ is the polytropic index and $K$ is the polytropic constant.

The global physical characteristics of the rapidly rotating neutron stars in which we are inter-
ested, are the mass $M$, the angular momentum $J$, the total baryon rest mass $M_0$ and the circum-
ferential radius of the star $R_c$ in the physical Jordan frame. In contrast with general relativity,
the definition of mass in scalar-tensor theories of gravity is in general quite subtle because scalar-
tensor theories violate in general the strong equivalence principle. This results in the appearance
of different possible masses as a measure of the total energy of the star [43–47]. It turns out that
only the so-called “tensor” mass has natural energy-like properties; for example the tensor mass
is positive definite, it decreases monotonically by the emission of gravitational waves and it is
well defined even for dynamical spacetimes [43–45]. Moreover, it is the tensor mass that leads to
a physically acceptable picture in the theory of the stars – the tensor mass peaks at the same point
as the particle number, which is a crucial property for the stability of the static stars [46, 47]. There-
fore, it is the tensor mass that should be taken as the physical mass$^3$. By definition the tensor mass is
just the Arnowitt-Deser-Misner (ADM) mass in the Einstein frame. As a direct consequence of
this definition and using the Komar integral we find the following expression for the tensor mass:

$$M = \int_{\text{Star}} \left[ (\varepsilon + 3p) + 2(\varepsilon + p) \frac{\Omega}{\Omega - \omega} \frac{v^2}{1 - v^2} - \frac{1}{2\pi} V(\varphi) \right] \sqrt{-g} d^3x. \quad (33)$$

$^3$ In some specific scalar-tensor theories it is possible for the ADM masses defined in both frames to coincide. This
is the case for example with the second scalar-tensor theory considered in the present paper. Because of the special
dependence $A(\varphi) = \exp(1/2\beta \varphi^2)$ and the condition $\lim_{\varphi \to \infty} \varphi = 0$, it is not difficult to show that both frame metrics
have the same leading asymptotic which results in the coincidence of the masses.
The angular momentum $J$ is defined as usual and it is the same in both frames. The Komar integral gives the following equation for the angular momentum expressed in terms of the Einstein frame quantities:

$$J = \int_{\text{Star}} (\epsilon + p) \frac{v^2}{1 - v^2} \frac{1}{\Omega - \omega} \sqrt{-g} d^3x. \quad (34)$$

Concerning the total baryon rest mass, the equation in the physical Jordan frame is the usual one

$$M_0 = \int_{\text{Star}} \tilde{\rho} \tilde{u}^\mu d\Sigma_\mu = \int_{\text{Star}} \tilde{\rho} \tilde{u}^t \sqrt{-\tilde{g}} d^3x, \quad (35)$$

where $\tilde{\rho}$ is the rest mass density in the Jordan frame. The above equation, expressed in terms of the four-velocity and metric in the Einstein frame, takes the form

$$M_0 = \int_{\text{Star}} A^3(\varphi) \tilde{\rho} u^t \sqrt{-\tilde{g}} d^3x. \quad (36)$$

The circumferential radius $\tilde{R}_e$ of the star in the physical Jordan frame is defined to be the normalized by $2\pi$ circumference of the circle in the equatorial plane where the pressure vanishes, i.e.

$$\tilde{R}_e = A(\varphi) r e^{(\gamma - \sigma)/2} |_{r=r_e, \theta=\pi/2}, \quad (37)$$

where $r_e$ is the coordinate equatorial radius of the star given by $\tilde{p}(r_e, \theta = \pi/2) = 0$.

The Kepler angular velocity $\Omega_K$ of a neutron star is defined as the angular velocity of a free particle in circular orbit in the $\theta = \pi/2$ plane. The mass-shedding or Kepler limit along a sequence of rotating stellar models is reached when the angular velocity of the neutron star fluid at the equator is equal to the Kepler angular velocity. In terms of Einstein frame quantities it is given by

$$\Omega_K = \left( \omega + \frac{\omega'}{8 + \gamma' - \sigma' + 2k(\varphi)\varphi'} + \sqrt{\left[ \frac{\omega'}{8 + \gamma' - \sigma' + 2k(\varphi)\varphi'} \right]^2 + \frac{\epsilon^2u(\sigma' + \gamma' + 2k(\varphi)\varphi')}{r_e^2(8 + \gamma' - \sigma' + 2k(\varphi)\varphi')} \right) \bigg|_{r=r_e, \theta=\pi/2} \quad (38)$$

where $'(\cdot)$ stands for $\partial_r$. But as we pointed out, $\Omega$ has the same values in both Einstein and Jordan frames so this equation gives us also the physical or Jordan frame Kepler angular velocity.

The procedure of introducing dimensionless variables is the same as in [48]. We set $c = G = 1$ and as $K^{N/2}$ has units of length it can be used as a fundamental length scale of the system.

### III. NUMERICAL METHOD

We follow the numerical method of H. Komatsu, Y. Eriguchi, and I. Hachisu [49, 50] with the modifications introduced in [48]. As a base we use the $\text{rns}$ code [38] where the additional terms
and equations coming from the scalar field are implemented. Below we will describe very briefly some of the main points of the method.

First we have to note that for convenience, instead of the radial coordinate $r$ we use a compactified one given by

$$r \equiv r_e \left( \frac{s}{1 - s} \right),$$

(39)

where the integration domain $r \in [0, \infty)$ is mapped to $s \in [0, 1)$ and the equatorial radius of the star is always at $s = 0.5$.

The essence of the method is that the differential equations (25)–(27) are transformed into an integral form using the Green functions which enable us to handle the boundary conditions in a simple manner. The boundary conditions come from the requirements for regularity of the functions at the center of the star and the rotational axes, and asymptotic flatness at infinity. For each differential equation a specific Green function is chosen that fulfills the corresponding boundary conditions as it is explained in detail in [49]. The introduction of a scalar field alters only the source terms in the integral representation of the metric potential equations (25)–(27). But a new second order differential equation for the scalar field appears, that is eq. (30), and we have to derive its integral representation. One can easily conclude that the appropriate Green function in this case is the same as for the metric potential $\sigma$ and after taking into account the equatorial and axial symmetry we obtain

$$\phi = - \sum_{n=0}^{\infty} P_{2n}(\mu) \left[ \left( \frac{1 - s}{s} \right)^{2n+1} \int_0^s ds' \frac{s'^{2n}}{(1 - s')^{2n+1}} \int_0^1 d\mu' P_{2n}(\mu') \tilde{S}_\phi(s', \mu') + \right.
$$

$$+ \left. \left( \frac{s}{1 - s} \right)^{2n} \int_0^1 ds' \frac{(1 - s')^{2n-1}}{s'^{2n+1}} \int_0^1 d\mu' P_{2n}(\mu') \tilde{S}_\phi(s', \mu') \right],$$

(40)

where $P_n(\mu)$ are the associated Legendre polynomials. The effective source $\tilde{S}_\phi(s, \mu)$ is defined as

$$\tilde{S}_\phi(s, \mu) = -s^2(1 - s)^2 \partial_s \gamma \partial_s \phi - (1 - \mu^2) \partial_\mu \gamma \partial_\mu \phi +
$$

$$+ \left. \frac{r^2 s^2}{(1 - s)^2 e^{2\alpha}} \left[ 4 \pi k(\phi)(\varepsilon - 3p) + \frac{dV(\phi)}{d\phi} \right] \right].$$

The asymptotic behavior at infinity ($\phi \sim O(1/r)$ for large $r$) is secured by the choice of the Green function.

The only differential equation that is not transformed into integral form is the first order differential equation for the metric function $\alpha$ – Equation (29). The flatness condition at the rotation axis requires that

$$\alpha = \frac{\gamma - \sigma}{2} \quad \text{at} \quad \mu = \pm 1,$$

(42)

and we can integrate equation (29) with the above condition. The asymptotic flatness of $\alpha$ at infinity is fulfilled automatically, because all other metric potentials, as well as the scalar field, satisfy the correct boundary conditions at infinity.
The last equation we have to address in detail is the equation for hydrostationary equilibrium given by (31). Taking into account the standard integrability condition related to the rotational law \[51, 52\]

\[ u^t u_\phi = F[\Omega], \quad (43) \]

we can transform equation (31) into:

\[ H - \ln u^t + \ln A(\phi) + \int_{\Omega_c}^{\Omega} F[\Omega] d\Omega = \text{const}, \quad (44) \]

where \( \Omega_c \) is the angular velocity at the center and \( H \) is the specific enthalpy which is defined up to an additive constant as

\[ H = \int_0^\rho \frac{d\bar{\rho}}{\bar{\varepsilon} + \bar{\rho}}. \quad (45) \]

In the present paper we assume uniform rotation and that is why we can set \( F[\Omega] = 0 \). The differentially rotating case will be considered elsewhere.

The numerical procedure of solving the differential equations is explained in detail in \[48, 49, 53\]. An important point is that the ratio of the polar to the equatorial radius \( r_p/r_e \) is used as an input parameter instead of the angular velocity of the star. This choice is more suitable, for example, when nonuniqueness of the solutions is present in the case of rapid rotation, or in the case when the matter has toroidal topology instead of spheroidal (\( r_e \) is the outer radius of the torus then). The second input parameter is the energy density at the center of the star. The procedure for finding solutions is the standard one – we have to supply first an initial approximation for the metric functions, the energy density and the scalar field, and after substituting them in equations (25)–(27), (29)–(31), we obtain the new updated values. A solution is obtained when the difference between the values of selected properties (such as the radius of the star) at two consecutive iterations is small enough.

After implementing the required changes, we checked the modified \( rns \) code successfully against several limiting cases:

- **The GR limit.** When we set the potential \( V(\phi) \) and the coupling function \( k(\phi) \) to zero, the code converges toward the general relativistic solutions with trivial scalar field.

- **The nonrotating limit.** To be able to make a more profound verification of our code, we created a new one dimensional code that solves the ordinary differential equations describing nonrotating neutron stars in STT. For this purpose it is most convenient to use the equations given in \[4, 10, 11\]. Those equations are much simpler and it is easier to converge to a solution with nontrivial scalar field as ordinary differential equations are solved with only one shooting parameter – the central value of the scalar field. The comparison between the two codes showed an excellent agreement. Our results also agree well with the results for static neutron stars in STT presented in \[10, 25\].

- **The slow rotation approximation (of order \( \Omega \)).** The rotating scalar-tensor neutron stars in slow rotation approximation were studied in \[11, 12\]. In this approximation the rotational corrections to the mass, radius, scalar field, etc. cannot be calculated, as they are of higher
order with respect to the angular velocity of the star, but we can compute for example the 
$g_{t\phi}$ component of the metric and the angular momentum. To make a systematic comparison,
we chose to implement the slow rotation modifications given in [11] in our one dimensional
code. The results produced by rns are in a very good agreement with those produced with
the slow rotation code.

- **The mass and the angular momentum.** There are two independent ways to determine
the mass and the angular momentum of the star that are valid also in the rapidly rotating
case. First, one can use the integral Equations (33) and (34). Alternatively, $M$ and $J$
can be determined from the asymptotic form of the metric components at infinity
$g_{tt}|_{r\to\infty} \approx -(1 - 2M/r)$ and $g_{t\phi}|_{r\to\infty} \approx -2J \sin^2 \theta/r$. The two values of $M$
and $J$ coincide within roughly 0.3% even in the rapidly rotating case.

### IV. RESULTS

In our studies we set the scalar field potential $V(\phi)$ to zero. We will consider two representative
scalar-tensor theories given by the following functions $^4 A(\phi)$:

- $\ln A(\phi) = k_0 \phi$,
- $\ln A(\phi) = \frac{1}{2} \beta \phi^2$,

with zero background value for the scalar field, i.e.

$$\lim_{r\to\infty} \phi = 0. \quad (46)$$

The first case is equivalent to the Brans-Dicke scalar-tensor theory. In the second case, the STT
is perturbatively equivalent to GR in the weak field regime. For strong fields, though, interesting
new effects can appear (such as a bifurcation due to nonuniqueness of solutions) that were
already observed for static neutron stars [10,11] and black holes [17,18]. Moreover, typically it is
energetically more favorable for the compact object to possess a nontrivial scalar field. Also, the
field equations are invariant under the change of sign of the scalar field, $\phi \to -\phi$, for the second
class of STT. Therefore, two nontrivial branches of solutions exist, which differ only in the sign
of the scalar field, but otherwise have identical global properties, such as mass, radius, angular
momentum, etc.

For large values of the parameters $k_0$ and $\beta$, the STT neutron stars can differ significantly from
the general relativistic solutions, but the experiments set constraints on these parameters, which
currently are roughly $k_0 < 4 \times 10^{-3}$ and $\beta > -4.8$ [3,40]. On their own, such small values of $k_0$
would lead to a very weak scalar field, with a negligible effect on the neutron star structure, but
the constraints on $\beta$ allow for the development of a nontrivial scalar field (spontaneous scalariza-
tion [10]) which can alter the neutron star properties significantly.

We will consider a polytropic equation of state given by Equation (32) with $N = 0.7463$ and
$K = 1186$, which is chosen to match equation of state II in [54] ($K$ is given in dimensionless units

$^4 k_0$ is usually denoted by $\alpha$ in most previous publications, but we use a different notation, as in our case $\alpha$ is reserved for one of the metric potentials.
$c = G = M_c = 1)$. It is widely used in the literature on static neutron stars in STT \cite{10, 16, 25} and it is a convenient model for comparisons.

Let us also briefly discuss how we converge to a nontrivial solution: In the case when \( \ln A(\varphi) = k_0 \varphi \), a unique solution of the field equations exists for a given central energy density and axis ratio, and the numerical scheme converges to it, starting from a static GR solution (i.e. with a trivial scalar field) of the same central energy density. But, in the case with \( \ln A(\varphi) = \frac{1}{2} \beta \varphi^2 \), three different equilibrium solutions can exist for a certain range of parameters: one with a trivial scalar field (equivalent to the GR solution) and two nontrivial solutions with positive and negative signs of the scalar field, but with identical global characteristics given by \((33)\)–\((38)\). To converge to the nontrivial solutions, it suffices to use a static GR solution with a nontrivial scalar field as an initial guess (this solution is found with a new TOV solver in \texttt{rns}), solving the equations given in \cite{10, 11}). In cases where for a given central energy density no nontrivial solutions exist in the static limit, but a nontrivial scalar field is expected in the rapidly rotating case, we use the trivial (GR) static solution together with a (chosen) constant scalar field as an initial guess. With careful choices for the initial scalar field, the numerical method in \texttt{rns} then also converges to the nontrivial solutions, when they exist.

A. Scalar-tensor theories with \( \ln A(\varphi) = k_0 \varphi \)

This example of STT is actually equivalent to the Brans-Dicke theory. Here, only one neutron star solution of the field equations exist for a given central energy density and axis ratio and neutron star models have a nontrivial scalar field for any nonzero value of \( k_0 \). Because of the observational constraint \( k_0 < 4 \times 10^{-3} \), we will consider this case very briefly.

Two sequences of solutions were calculated for the case \( k_0 = 4 \times 10^{-3} \) – the static one and a sequence of models rotating at the mass-shedding limit with angular velocity \((38)\). We find that even though a nontrivial scalar field is present and its value increases for rapid rotation, it is very weak and the total mass, radius and angular momentum of the stars are the same as the corresponding quantities in the GR limit within the numerical error.

A contour plot of the scalar field is shown in Fig. \ref{fig1}. The left panel displays the scalar field of a static neutron star, while the right panel shows the scalar field of a star rotating at the mass-shedding limit. The Cartesian coordinates \((x, z)\) are obtained by the usual transform of spherical polar coordinates and are normalized by the equatorial coordinate radius \( r_e \). The surface of the star is shown as a thick dashed line. The two models have the same central energy density \( \tilde{\varepsilon}_c = 1.25 \times 10^{15} \text{g/cm}^3 \) while their mass is \( M = 1.77 M_\odot \) and \( M = 2.20 M_\odot \), correspondingly. In the rapidly rotating case, the isosurfaces of the energy density and the scalar field no longer coincide – the flattening of the scalar field isosurfaces due to the rotation is weaker than for the fluid variables, which is an expected effect. In the right panel, a well known effect could also be observed – the surface of the star forms a cusp at the equator, when reaching the mass-shedding limit.

B. Scalar-tensor theories with \( \ln A(\varphi) = \frac{1}{2} \beta \varphi^2 \)

This example of STT is of particular interest, because, it is indistinguishable from GR in the weak field regime but it can differ significantly when strong fields are considered. In the latter
FIG. 1: Contour plot of the neutron star scalar field $\varphi$ in the nonrotating case (left panel) and for a model rotating at the mass-shedding limit (right panel); see text for details. The thick, dashed line represents the neutron star surface.

case, all solutions of the GR field equations are also solutions of the STT field equations with a trivial scalar field, but additional solutions can also exist. For example, for certain values of the parameter $\beta$ and in a certain range of central densities nonuniqueness of the neutron star solutions is present – in addition to the solutions describing neutron stars with a zero scalar field (trivial solutions), neutron star solutions with a nonzero scalar field exist (nontrivial solutions), which are also energetically more favorable (spontaneous scalarization).

FIG. 2: The mass as a function of the central energy density (left panel) and of the radius (right panel) for static sequences of neutron stars (solid lines) and sequences of stars rotating at the mass-shedding limit (dotted lines). The trivial solutions coincide with the GR limit ($\beta = 0$). For $\beta = -4.2$ nontrivial solutions do not exist in the nonrotating case.

We will consider three different values of the parameter $\beta$: $-4.8$, $-4.5$ and $-4.2$. The value $\beta = -4.8$ is the lower limit set by the binary pulsar experiments. On the other hand, according to [16] spontaneous scalarization should only occur for $\beta < -4.35$ in the nonrotating case. As an intermediate value, we choose $\beta = -4.5$. Because rapid rotation allows for spontaneous scalar-
ization for larger values of $\beta$ than in the nonrotating case (as we demonstrate below), we also use the value of $\beta = -4.2$.

To investigate the effect of rapid rotation, we compare solutions at the mass-shedding limit to nonrotating solutions, for the chosen values of $\beta$. Fig. 2 shows the mass as a function of the central energy density (left panel) and as a function of the circumferential equatorial radius. The black lines represent the solutions with zero scalar field, and the colored lines correspond to different nonzero values of $\beta$. Solid lines are nonrotating models, while models at the mass-shedding limit are shown as dotted lines. Equilibrium models with intermediate rotation rates exist in-between the two limits. We observe that the branching of the neutron star solutions with a nontrivial scalar field from the GR solutions occurs for a larger range of central energy densities for rapidly rotating stars than for nonrotating ones. In addition, we find that it occurs for larger values of $\beta$ (such as -4.2), for which there are no nontrivial solutions in the nonrotating case.

Fig. 3 shows the central value of the scalar field as a function of the central energy density (left panel) and the angular velocity of the star as a function of the normalized angular momentum (right panel). At a certain critical central energy density $\varepsilon_{\text{crit}}^\text{min}$, the nontrivial solutions branch out from the trivial ones. As the central energy density is increased, the scalar field first increases, reaches a maximum and then decreases again, until the nontrivial branch of solutions merges with the trivial one at $\varepsilon_{\text{crit}}^\text{max}$. Thus, nontrivial solutions exist only for densities between $\varepsilon_{\text{crit}}^\text{min}$ and $\varepsilon_{\text{crit}}^\text{max}$ and these two values depend on the parameter $\beta$ and on the rotation rate. Although we only focus on a representative EoS here, this range of central energy densities is also EoS dependent.

The difference in mass, radius and angular momentum between the trivial and the nontrivial solutions increases with the rotation rate. For example, the nontrivial models with $\beta = -4.5$ have

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5 The figure shows only the negative scalar field solutions – as we mentioned earlier, a second branch of solutions exists with positive scalar field and with the same global characteristics, such as mass, radius, angular momentum, etc.
almost the same equilibrium properties as the trivial ones in the static limit, but at rapid rotation the strength of the scalar field increases significantly and then the mass, radius and angular momentum also differ considerably. Moreover, for $\beta = -4.2$ there are no nontrivial solutions in the static limit, but such solutions exist for fast rotation. This means that the maximum value of $\beta$ for which scalarization is possible increases considerably for high rotational rates.

Fig. 4 shows the moment of inertia, $I = J/\Omega$, as a function of the angular velocity, for sequences with a fixed value of the central energy density. The moment of inertia can reach higher values for scalarized models, compared to the trivial solution, at all rotation rates (even at slow rotation) for $\beta = -4.8$ and $\beta = -4.5$. In the case of $\beta = -4.2$, nontrivial solutions do not exist in the nonrotating limit, but above a certain rotational frequency the nontrivial branch of solutions appears.

From the above figures it is evident that the strongest effect of the scalar field for rotating models is on the angular momentum and the moment of inertia of the star, which can differ up to a factor of two, compared to the GR case, for the lowest possible value of $\beta = -4.8$. Because the moment of inertia is affected even at slow rotation, it could become useful as a sensitive astrophysical probe. Other observables, such as the frequencies of emitted gravitational waves (when stars are dynamically perturbed) should also be affected by the presence of a nontrivial scalar field.

The distribution of the scalar field inside the star is qualitatively very similar to the one shown in Fig. 1 for $\ln A(\phi) = k_0 \phi$, that is, the rotation causes flattening of the isosurfaces of the scalar field, which is weaker than the flattening of the matter, so that the scalar field isosurfaces do not coincide with the density isosurfaces. Instead, the density isosurfaces coincide with the isosurfaces of an effective potential, that can be defined through (31), requiring a barotropic equilibrium.

![FIG. 4: Moment of inertia as a function of the angular velocity, for different values of $\beta$. All the models have the same central energy density $\tilde{\varepsilon}_c = 1.5 \times 10^{15} \text{ g/cm}^3$.](image)

In the static case, it was shown that for a given baryon mass the scalarized neutron stars have
a lower total mass compared to the trivial solution (when both solutions exist) and therefore they are energetically more favorable [10]. In Fig. 5 we plot the relative energy $1 - M_0/M$, versus the rest mass for a given, representative value of the angular momentum $c J / (G M_c^2) = 1.38$. We find that all nontrivial solutions have a lower relative energy and are thus energetically more favorable than the corresponding trivial solutions. This result holds also for any other value of the angular momentum in the parameter space where both trivial and nontrivial solutions are possible. In Fig. 5 each line folds over at a cusp, which represents a turning point along the fixed-$J$ sequence, where secular instability to collapse sets in. To make this clearer, we show a magnification of the cusp area in the right panel of Fig. 5.

![Graphs showing relative energy vs rest mass](image)

**FIG. 5**: The relative energy $1 - M_0/M$ as a function of the baryon rest mass for a fixed value of the angular momentum $c J / (G M_c^2) = 1.38$. In the right panel a magnification of the turning point area is shown.

As we pointed out in Section IV B, nontrivial solutions can exist in the nonrotating case only if $\beta < -4.35$ (for equation of state II [16]). Determining a corresponding limit in the rapidly rotating case requires some care in choosing appropriate initial conditions for the numerical iteration, when considering central densities for which only the trivial solution exists in the nonrotating limit. We find that along the sequence of models rotating at the mass-shedding limit, nontrivial solutions can exist for approximately $\beta < -3.9$, which is a significantly less stringent limit than in the nonrotating case. A detailed study of the critical $\beta$ at different rotation rates, as well as for different equations of state, will be presented in a forthcoming publication.

V. SUMMARY AND DISCUSSION

For the first time, we present the field equations governing rapidly rotating neutron stars in scalar-tensor theories of gravity. We solve these equations numerically, by extending the rns code [38], implementing the required modifications coming from the scalar field. The accuracy of our numerical solutions was checked by a comparison with an independent code in the slow-rotation approximation.

For these first, representative numerical solutions, we employ a specific polytropic equation of state, that is widely used in the literature on nonrotating neutron stars in STT. Furthermore, we
assume rigid rotation and a vanishing potential $V(\phi)$. We consider two examples of STT – when $\ln A(\phi) = k_0 \phi$ and when $\ln A(\phi) = \frac{1}{2} \beta \phi^2$, where $k_0$ and $\beta$ are constants. In the first case, which is equivalent to the Brans-Dicke theory, all of the solutions possess a nontrivial scalar field, but $k_0$ is limited to low values by observations, which leads to a very weak scalar field and models that differ only marginally from their GR counterparts. Our results show that the strength of the scalar field could increase for rapid rotation, but still the total mass, radius and angular momentum of the STT models are almost indistinguishable from the GR case, even for stars rotating at the mass-shedding limit.

The second example is more interesting, because it is perturbatively equivalent to GR, but significant deviations from GR are present when strong fields are considered. For example, it was shown that in the static case, for certain values of $\beta$, nonuniqueness of the neutron star solutions is present – in addition to the neutron stars with a trivial scalar field, additional nontrivial solutions appear, which are energetically more favorable [10]. We find that such nontrivial solutions are present also in the rapidly rotating case for a larger range of central energy densities. The critical value of $\beta$ for which scalarized neutron stars exist also changes from $\beta < -4.35$ in the nonrotating case to approximately $\beta < -3.9$ for models rotating at the mass-shedding limit. Furthermore, we show that the nontrivial solutions are energetically more favorable than the corresponding trivial ones.

For a given value of $\beta$, the changes (when compared to GR) in the equilibrium properties of scalarized models, such as mass, radius and angular momentum, increase with rotation, which is due to an increased strength of the scalar field. For example, for the limiting value of $\beta$ set by observations ($\beta = -4.8$), the angular momentum of neutron stars with a nontrivial scalar field rotating at the mass-shedding limit can be up to two times larger than for the corresponding models with a zero scalar field. It is interesting that the effect of scalarization on the moment of inertia is significant even in the slow-rotation limit. Such large changes in the neutron star properties may lead to interesting observational effects.

Even though normal pulsars rotate at most at a small fraction of the mass-shedding limit, it is conceivable that at least magnetars are born rapidly rotating, in which case our results would apply. The subsequent spin-down could cause a transition from a scalarized solution to a trivial solution, as the nonrotating limit or the axisymmetric instability limit are approached. Furthermore, the recycling of old neutron stars through accretion takes the stars through the region where the scalarized solutions are energetically more favorable. Such scenarios are worth exploring for possible observational signatures. In forthcoming publications, we are planning a more detailed study of models with different equations of state and rotation laws, as well as a study of the astrophysical implications of the scenarios mentioned briefly above.

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