Relationships between \( \tau \)-functions and Fredholm determinant expressions for gap probabilities in random matrix theory

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Abstract
The gap probabilities at the hard and soft edges of scaled random matrix ensembles with orthogonal symmetry are known in terms of \( \tau \)-functions. Extending recent work relating to the soft edge, it is shown that these \( \tau \)-functions, and their generalizations to contain a generating function parameter, can be expressed as Fredholm determinants. These same Fredholm determinants also occur in exact expressions for gap probabilities in scaled random matrix ensembles with unitary and symplectic symmetry.

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1. Introduction

In the 1950s Wigner introduced random real symmetric matrices to model the highly excited energy levels of heavy nuclei (see [13]). From the experimental data, a natural statistic to calculate empirically is the distribution of the spacing between consecutive levels, normalized so that the spacing is unity. For random real symmetric matrices \( X \) with independent Gaussian entries such that the joint probability density function (pdf) for the elements is proportional to \( e^{-\text{Tr}(X^2)/2} \) (such matrices are said to form the Gaussian orthogonal ensemble (GOE)), Wigner used heuristic reasoning to surmise that the spacing distribution is well approximated by the functional form

\[
p_W^1(s) := \frac{\pi s}{2} e^{-\pi s^2/4}.
\]  

In the limit of infinite matrix size, it was subsequently proved by Gaudin that the exact spacing distribution is given by

\[
p_1(s) = \frac{d^2}{ds^2} \det(I - K_{0,1}^{\text{bulk}}),
\]  

where \( I \) is the identity matrix and \( K_{0,1}^{\text{bulk}} \) is the Fredholm determinant associated with the kernel of the bulk density matrix.
In terms of this solution, introduce the corresponding \( \tau \) determinant of the corresponding eigenvalues (1.2) was computed and shown to differ from the differential operator for the so-called prolate spherical functions, and from the numerical approximation (1.1) by no more than a few per cent. The Fredholm determinant in (1.2) is itself a probabilistic quantity. Thus, let \( E_{1}^{\text{bulk}}(0; (0, s)) \) denote the probability that for the infinite GOE, scaled so that the mean spacing is unity, the interval \((0, s)\) of the spectrum contains no eigenvalues. Then

\[
E_{1}^{\text{bulk}}(0; (0, s)) = \det(\mathbb{1} - K_{(0, s)}^{\text{bulk}, +}).
\]  

(1.4)

In applications of random matrices to the eigenspectrum of quantum Hamiltonians, two other ensembles in addition to the GOE are relevant. These are the Gaussian unitary ensemble (GUE) of complex Hermitian matrices and the Gaussian symplectic ensemble (GSE) of Hermitian matrices with real quaternion elements. For the infinite limit of such ensembles of matrices, scaled so that the mean density is unity, let \( E_{2}^{\text{bulk}}(0; (0, s)) \) and \( E_{4}^{\text{bulk}}(0; (0, s)) \), respectively, denote the probabilities that the interval \((0, s)\) is free of eigenvalues. Then it is known \([4, 5]\) that

\[
E_{2}^{\text{bulk}}(0; (0, s)) = \det(\mathbb{1} - K_{(0, s)}^{\text{bulk}, +}) - \det(\mathbb{1} - K_{(0, s)}^{\text{bulk}, -}),
\]  

(1.5)

while

\[
E_{4}(0; (0, s)) = \frac{1}{4} (\det(\mathbb{1} - K_{(0, 2s)}^{\text{bulk}, +}) + \det(\mathbb{1} - K_{(0, 2s)}^{\text{bulk}, -})),
\]  

(1.6)

where \( K_{j}^{\text{bulk}, -} \) denotes the integral operator \( K_{j}^{\text{bulk}} \) restricted to odd eigenfunctions.

The remarkable structure exhibited by (1.4)–(1.6) can also be seen in certain Painlevé transcendent evaluations of the gap probabilities \([8]\). These expressions are given in terms of the solution of the \( \sigma \) form of the PIII equation

\[
(t \sigma ')^{2} - v_{1} v_{2} (\sigma ')^{2} + \sigma ' (4 \sigma' - 1)(\sigma - t \sigma') - \frac{1}{43} (v_{1} - v_{2})^{2} = 0,
\]  

(1.7)

with

\[
v_{1} = v_{2} = a = \pm \frac{i}{2}
\]

subject to the boundary condition

\[
\sigma(t; a) \sim \frac{t^{1+a}}{2^{2a+2a} \Gamma(1+a) \Gamma(2+a)}.
\]  

(1.8)

In terms of this solution, introduce the corresponding \( \tau \) functions by

\[
\tau_{\text{III}}(s; a) := \exp \left( - \int_{0}^{s} \frac{\sigma(t; a)}{t} \, dt \right).
\]  

(1.9)

Then

\[
E_{1}^{\text{bulk}}(0; (0, 2s)) = \tau_{\text{III}}((\pi s)^{2}; -1/2),
\]  

(1.10)

\[
E_{2}^{\text{bulk}}(0; (0, 2s)) = \tau_{\text{III}}((\pi s)^{2}; -1/2) \tau_{\text{III}}((\pi s)^{2}; 1/2),
\]  

(1.11)

\[
E_{4}^{\text{bulk}}(0; (0, 2s)) = \frac{1}{2} (\tau_{\text{III}}((\pi s)^{2}; -1/2) + \tau_{\text{III}}((\pi s)^{2}; 1/2)).
\]  

(1.12)
Comparison of the results (1.4)–(1.6) with the results (1.10)–(1.12) shows
\[
\det(I - K^{\text{bulk,} \pm}(0, 2s)) = \tau_{\text{III}}((\pi s)^2; \mp 1/2).
\] (1.13)

It is the objective of this paper to give formulae analogous to (1.13) for both the soft and hard edge scalings. In doing so we will be relating known \(\tau\)-function evaluations of these quantities to some recently derived Fredholm determinant formulae in the case of the soft edge and to some new Fredholm determinant formulae in the case of the hard edge. Further, these identities will be generalized to include a generating function type parameter \(\xi\).

2. Soft edge scaling

Soft edge scaling refers to shifting the origin to the neighbourhood of the largest, or smallest, eigenvalue where it is required that the support of the eigenvalue density is unbounded beyond this eigenvalue and then scaling so that the average eigenvalue spacings in this neighbourhood are of order unity.

The soft edge scaling can be made precise in the case of the Gaussian and Laguerre ensembles. For this let us define a random matrix ensemble by its eigenvalue pdf, assumed to be of the functional form
\[
\frac{1}{C_N} \prod_{l=1}^{N} g(x_l) \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta},
\] (2.1)

and denote the corresponding probability that the interval \(J\) is free of eigenvalues by \(E_{\beta}^{\text{soft}}(0; J; g(x); N)\). For the Gaussian ensembles with \(\beta = 1\) or 2, the soft edge scaling is defined by
\[
\lim_{N \to \infty} E_{\beta}^{\text{soft}}(0; (\sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}}, \infty); e^{-x^2/2}; N) = E_{\beta}^{\text{soft}}(0; (s, \infty)),
\] (2.2)

while for \(\beta = 4\) a more natural definition (see the formulae of [1]) is
\[
\lim_{N \to \infty} E_{4}^{\text{soft}}(0; (\sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}}, \infty); e^{-x^2}; N) = E_{4}^{\text{soft}}(0; (s, \infty)).
\] (2.3)

It is expected that for a large class of weights \(g(x)\) in (2.1), the soft edge limit of the gap probabilities exists and is equal to that for the Gaussian ensembles (see [3] for some proofs related to this statement). This can be checked explicitly in the case of the Laguerre ensembles (i.e. the weight \(g(x) = x^a e^{-x}, x > 0\) in (2.1), up to scaling of \(x\)). Thus, for \(\beta = 1\) or 2 we have
\[
\lim_{N \to \infty} E_{\beta}(0; (4N + 2(2N)^{1/3}s, \infty); x^a e^{-\beta x^2/2}; N) = E_{\beta}^{\text{soft}}(0; (s, \infty)),
\] (2.4)

while for \(\beta = 4\)
\[
\lim_{N \to \infty} E_{4}(0; (4N + 2(2N)^{1/3}s, \infty); x^a e^{-x}; 2N) = E_{4}^{\text{soft}}(0; (s, \infty)).
\] (2.5)

A number of exact expressions are known for \(E_{\beta}^{\text{soft}}\). Let us first consider those involving Painlevé transcendent. These can in turn be grouped into two types. The first of these relates to the particular Painlevé II transcendent \(q(s)\), specified as the solution of the Painlevé II equation
\[
q'' = sq + 2q^3 + \alpha
\] (2.6)

with \(\alpha = 0\) and subject to the boundary condition
\[
q(s) \sim Ai(s),
\] (2.7)
where $\text{Ai}(s)$ denotes the Airy function. One has \cite{15,17} (see \cite{7} for a simplified derivation of the latter two)

\begin{align}
E_{\text{soft}}^2(0; (s, \infty)) &= \exp \left( - \int_s^\infty (t-s)q^2(t) \, dt \right), \quad (2.8) \\
E_{\text{soft}}^1(0; (s, \infty)) &= \exp \left( - \frac{1}{2} \int_s^\infty (t-s)q^2(t) \, dt \right) \exp \left( \frac{1}{2} \int_s^\infty q(t) \, dt \right), \quad (2.9) \\
E_{\text{soft}}^4(0; (s, \infty)) &= \frac{1}{2} \exp \left( - \frac{1}{2} \int_s^\infty (t-s)q^2(t) \, dt \right) \\
& \times \left( \exp \left( \frac{1}{2} \int_s^\infty q(t) \, dt \right) + \exp \left( - \frac{1}{2} \int_s^\infty q(t) \, dt \right) \right). \quad (2.10)
\end{align}

The alternative Painlevé expressions relate to the $\sigma$-form of the PII equation:

\begin{equation}
\left( H''_{II} \right)^2 + 4(H'_{II})^3 + 2H'_{II} \left( H'_{II} - H_{II} \right) - \frac{1}{4}(\alpha + \frac{1}{2})^2 = 0. \quad (2.11)
\end{equation}

Introduce the auxiliary Hamiltonian

\begin{equation}
h_{II}(t; \alpha) := H_{II}(t; \alpha) + \frac{t^2}{8} \quad (2.12)
\end{equation}

and the corresponding $\tau$-function

\begin{equation}
\tau_{II}(s; \alpha) = \exp \left( - \int_s^\infty h_{II}(t; \alpha) \, dt \right). \quad (2.13)
\end{equation}

Then from \cite{10} we know that

\begin{align}
E_{\text{soft}}^1(0; (s, \infty)) &= \tau_{II}^+(s; 0), \quad (2.14) \\
E_{\text{soft}}^2(0; (s, \infty)) &= \tau_{II}^-(s; 0) \tau_{II}^+(s; 0), \quad (2.15) \\
E_{\text{soft}}^4(0; (s, \infty)) &= \frac{1}{2}(\tau_{II}^+(s; 0) + \tau_{II}^-(s; 0)), \quad (2.16)
\end{align}

where $\tau_{II}^\pm(s, 0)$ is specified by \eqref{eq:2.13} with $h_{II}(t; 0)$ in \eqref{eq:2.12} subject to the boundary condition $h_{II}(t; 0) \sim \pm \frac{1}{2} \text{Ai}(t)$ as $t \to \infty$.

We now turn our attention to Fredholm determinant expressions for the gap probabilities at the soft edge. The best known is the $\beta = 2$ result \cite{6}

\begin{equation}
E_{\text{soft}}^2(0; (s, \infty)) = \det(\mathbb{I} - K_{(0, \infty)}^\text{soft}), \quad (2.17)
\end{equation}

where $K_{(s, \infty)}^\text{soft}$ is the integral operator on $(s, \infty)$ with kernel

\begin{equation}
K_{(s, \infty)}^\text{soft}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x-y}. \quad (2.18)
\end{equation}

This can be rewritten \cite{15} as

\begin{equation}
E_{\text{soft}}^2(0; (s, \infty)) = \det(\mathbb{I} - \tilde{K}_{(0, \infty)}^\text{soft}), \quad (2.19)
\end{equation}

where $\tilde{K}_{(0, \infty)}^\text{soft}$ is the integral operator on $(0, \infty)$ with kernel

\begin{equation}
\tilde{K}_{(0, \infty)}^\text{soft} = \int_0^\infty \text{Ai}(s + x + t)\text{Ai}(s + y + t) \, dt,
\end{equation}

which in turn implies

\begin{equation}
E_{\text{soft}}^2(0; (s, \infty)) = \det(\mathbb{I} - V_{(0, \infty)}^\text{soft}) \det(\mathbb{I} + V_{(0, \infty)}^\text{soft}), \quad (2.20)
\end{equation}
where \( V_{\text{soft}}^{(0, \infty)} \) is the integral operator on \((0, \infty)\) with kernel
\[
V_{\text{soft}}^{(0, \infty)}(x, u) = \text{Ai}(x + u + s). \tag{2.21}
\]

Recently it has been conjectured by Sasamoto \cite{14}, and subsequently proved by Ferrari and Spohn \cite{11} that
\[
E_1^{\text{soft}}(0; (s, \infty)) = \det(\mathbb{I} - V_{\text{soft}}^{(0, \infty)}), \tag{2.22}
\]
which is the soft edge analogue of the evaluation of \( E_1^{\text{bulk}}(0; (0, s)) \) \cite{14}. Comparing \( (2.20), (2.22) \) with \( (2.15) \), we see immediately that
\[
\tau^{\pm}_s(s; 0) = \det(\mathbb{I} - V_{\text{soft}}^{(0, \infty)}). \tag{2.23}
\]
This is the soft edge analogue of the bulk identity \((1.13)\).

3. Hard edge scaling

The Laguerre ensemble has its origin in positive definite matrices \( X^\dagger X \) where \( X \) is an \( n \times N \) matrix \((n \gg N)\) with real \((\beta = 1)\), complex \((\beta = 2)\) or real quaternion \((\beta = 4)\) entries. Being positive definite the eigenvalue density is strictly zero for \( x < 0 \); for this reason the neighbourhood of \( x = 0 \) is referred to as the hard edge. The hard edge scaling limit takes \( N \to \infty \) while keeping the mean spacing between eigenvalues near \( x = 0 \) of order unity. In relation to the gap probabilities, this can be accomplished by the limits
\[
E^{\text{hard}}_\beta(0; (0, s); a) := \lim_{N \to \infty} E_\beta\left(0; \left(0, \frac{s}{4N}\right); x^a e^{-\beta x^2}; N\right)
\]
for \( \beta = 1, 2 \), while for \( \beta = 4 \)
\[
E^{\text{hard}}_4(0; (0, s); a) := \lim_{N \to \infty} E_4\left(0; \left(0, \frac{s}{4N}\right); x^a e^{-x^2}; \frac{N}{2}\right).
\]

As for the soft edge, there are two classes of Painlevé evaluations of the gap probability at the hard edge. The first involves the solution \( \tilde{q}(t) \) of the nonlinear equation
\[
t(\tilde{q}^2 - 1)(t\tilde{q})' = \tilde{q}(t\tilde{q})^2 + \frac{1}{2}(t - a^2)\tilde{q} + \frac{1}{4}t\tilde{q}^3(\tilde{q}^2 - 2) \tag{3.1}
\]
(a transformed version of the Painlevé V equation) subject to the boundary condition
\[
\tilde{q}(t; a) \sim \frac{1}{2\sqrt{\Gamma(1 + a)}} e^{a/2}. \tag{3.2}
\]
Thus \cite{16}
\[
E^{\text{hard}}_2(0; (0, s); a) = \exp\left(-\frac{1}{4} \int_0^s \left(\log \frac{s}{t}\right) \tilde{q}(t; a) \, dt\right), \tag{3.3}
\]
while \cite{7}
\[
E^{\text{hard}}_1\left(0; (0, s); \frac{a - 1}{2}\right) = \exp\left(-\frac{1}{8} \int_0^s \left(\log \frac{s}{t}\right) \tilde{q}(t; a) \, dt\right) \exp\left(-\frac{1}{4} \int_0^s \frac{\tilde{q}(t; a)}{\sqrt{t}} \, dt\right), \tag{3.4}
\]
\[
E^{\text{hard}}_4(0; (0, s); a + 1) = \frac{1}{2} \exp\left(-\frac{1}{8} \int_0^s \left(\log \frac{s}{t}\right) \tilde{q}(t; a) \, dt\right) \times \left(\exp\left(-\frac{1}{4} \int_0^s \frac{\tilde{q}(t; a)}{\sqrt{t}} \, dt\right) + \exp\left(\frac{1}{4} \int_0^s \frac{\tilde{q}(t; a)}{\sqrt{t}} \, dt\right)\right). \tag{3.5}
\]
For the second class of Painlevé evaluations at the hard edge, we recall the $\sigma$-form of the $PV$ equation
\[
(t\sigma')^2 - (\sigma - t\sigma' + 2(\sigma')^2 + (v_0 + v_1 + v_2 + v_3)\sigma')^2 + 4(v_0 + \sigma')(v_1 + \sigma')(v_2 + \sigma')(v_3 + \sigma') = 0. \tag{3.6}
\]
Set
\[
v_0 = 0, \quad v_1 = v_2 = v_1, \quad v_2 = v_3 - v_1, \quad v_3 = v_4 - v_1 \tag{3.7}
\]
and let
\[
xh^{\pm}(x; a) = \sigma^{\pm}(x; a) - \frac{1}{4} x^2 + \frac{a}{2} x - \frac{a(a - 1)}{4}, \tag{3.8}
\]
where $\sigma^{\pm}(x; a)$ satisfies (3.6) with $t \mapsto 2x$, subject to the boundary condition consistent with
\[
xh^{\pm}(x; a) \sim x \to 0^+ \mp xa + \frac{1}{2} a + 1/\Gamma(a + 1). \tag{3.9}
\]
Further, introduce the $\tau$-function
\[
\tau^{\pm}(s; a) = \exp \int_0^x xh^{\pm}(x; a) \, dx. \tag{3.10}
\]
In terms of this quantity [10]
\[
E_{\text{hard}}^1((0, s); a - 1/2) = \tau^{\pm}(\sqrt{s}; a), \tag{3.11}
\]
\[
E_{\text{hard}}^2((0, s); a) = \tau_\psi^{\pm}(\sqrt{s}; a) \tau_\psi^{\mp}(\sqrt{s}; a), \tag{3.12}
\]
\[
E_{\text{hard}}^4((0, s); a + 1) = \frac{1}{2}(\tau_\psi^{\pm}(\sqrt{s}; a) + \tau_\psi^{\mp}(\sqrt{s}; a)), \tag{3.13}
\]
where the parameters (3.7) are specified by
\[
v_1 = v_3 = -(a - 1)/4, \quad v_2 = v_4 = (a + 1)/4. \]
In relation to Fredholm determinant expressions for the gap probabilities at the soft edge, analogous to (2.17) we have [6]
\[
E_{\text{soft}}^2((0, s); a) = \det(\mathbb{I} - K_{(0, 1)}), \tag{3.14}
\]
where $K_{(0, 1)}$ is the integral operator on $(0, 1)$ with kernel
\[
K_{(0, 1)}(x, y) = J_0(\sqrt{xy})J_0(\sqrt{xy})J_0(\sqrt{xy}) \frac{J_x(\sqrt{xy}) - \sqrt{x} J_0(\sqrt{xy})}{x - y}. \tag{3.15}
\]
This can be rewritten [16] as
\[
E_{\text{soft}}^2((0, s); a) = \det(\mathbb{I} - \tilde{K}_{(0, 1)}), \tag{3.16}
\]
where $\tilde{K}_{(0, 1)}$ is the integral operator on $(0, 1)$ with kernel
\[
\tilde{K}_{(0, 1)}(x, y) = \frac{s}{4} \int_0^1 J_0(\sqrt{sxu}) J_0(\sqrt{syu}) \, du. \tag{3.17}
\]
Because
\[
\tilde{K}_{(0, 1)} = (\psi_{(0, 1)})^2, \tag{3.18}
\]
where $\psi_{(0, 1)}$ is the integral operator on $(0, 1)$ with kernel
\[
\psi_{(0, 1)}(x, y) = \frac{1}{2} J_0(\sqrt{xy}), \tag{3.19}
\]
it follows that
\[
E_2^{\text{hard}}(0, s); a) = \det(\mathbb{I} - V_{(0,1)}^{\text{hard}}) \det(\mathbb{I} + V_{(0,1)}^{\text{hard}}).
\] (3.20)

For \(\beta = 1\), a Fredholm determinant expression analogous to the result (2.22) holds true. This is proved with the help of the three following lemmas, which are modelled on the strategy used in [11] to prove (2.22).

**Lemma 1.** Let \(V = V_{(0,1)}^{\text{hard}}\) and \(\rho(x) = 1/\sqrt{x}\) for \(x > 0\). Let \(\langle f|g\rangle_{(0,1)} = \int_0^1 f(x)g(x)\,dx\) be the scalar product in \(L^2(0, 1)\). Also let \(\delta_1\) denote the delta function at \(1\); that is, \(\langle \delta_1|f\rangle_{(0,1)} = f(1)\). Then,
\[
\left( E_1^{\text{hard}}(0, s); \frac{a-1}{2} \right)^2 = \det(\mathbb{I} - V) \det(\mathbb{I} + V) \langle \delta_1| (\mathbb{I} + V)^{-1}\rho \rangle_{(0,1)}.
\]

**Proof.** We know from [7] that
\[
\left( E_1^{\text{hard}}(0, s); \frac{a-1}{2} \right)^2 = \det(\mathbb{I} - K_{(0,s)}^{\text{hard}} - C \otimes D),
\]
where \(K_{(0,s)}^{\text{hard}}\) and \(C \otimes D\) are integral operators on \((0, s)\) whose kernels are, respectively, \(K_{(x,y)}^{\text{hard}}\) (see equation (3.15)) and \(J_a(\sqrt{x})(1/2\sqrt{y}) f^{\infty} J_a(t)\,dt\). Note that \(f \otimes g\) stands for an integral operator with kernel
\[
(f \otimes g)(x, y) = f(x)g(y).
\] (3.21)

We now make use of \(\sqrt{s} J_a(\sqrt{x}) = 2(V\delta_1)(x)\) and \(\int_0^1 J_a(y)\,dy = 1\) for showing that
\[
(C \otimes Df)(x) = \frac{J_a(\sqrt{x})}{2} \int_0^x \left( 1 - \int_0^y J_a(t)\,dt \right) \frac{f(y)}{\sqrt{y}}\,dy = \frac{\sqrt{s}}{2} J_a(\sqrt{x}) \int_0^1 \left( \frac{1}{\sqrt{t}} - \sqrt{s} \frac{J_a(\sqrt{st})}{\sqrt{t}} \right) f(st)\,dy = (V\delta_1)(x) \int_0^1 (\rho(y) - (V\rho)(y)) f(sy)\,dy.
\]

Then, by recalling equations (3.14)–(3.18), we get
\[
\left( E_1^{\text{hard}}(0, s); \frac{a-1}{2} \right)^2 = \det(\mathbb{I} - V^2 - V\delta_1 \otimes (\mathbb{I} - V)\rho) = \det(\mathbb{I} - V) \det(\mathbb{I} + V) \det(\mathbb{I} - (\mathbb{I} + V)^{-1}\rho \otimes V\delta_1).
\] (3.22)

But \(\mathbb{I} - (\mathbb{I} + V)^{-1}\rho \otimes V\delta_1\) is a degenerate operator of rank 1 (see, e.g. [17, equation (17)]). This means that equation (3.22) can be written as
\[
\left( E_1^{\text{hard}}(0, s); \frac{a-1}{2} \right)^2 = \det(\mathbb{I} - V) \det(\mathbb{I} + V) (1 - \langle \delta_1| (\mathbb{I} + V)^{-1}\rho \rangle_{(0,1)}).
\]

The use of \(\langle \rho|\delta_1 \rangle = 1\) finishes the proof. \(\square\)

**Lemma 2.** Let \(\Delta\) be the operator defined by \((\Delta f)(x) = x\partial_x f(x)\) and let \(\otimes\) be the direct product defined in equation (3.21). Then, for \(V = V_{(0,1)}^{\text{hard}}\),
\[
2s\frac{\partial}{\partial \bar{s}} V = (\mathbb{I} + 2\Delta)V, \quad \Delta V = -V\Delta + V\delta_1 \otimes \delta_1 - V,
\]

and consequently,
\[
s\frac{\partial}{\partial \bar{s}} V = \frac{1}{2}(\mathbb{I} - V^2)^{-1}V(\mathbb{I} + 2\Delta) - (\mathbb{I} - V^2)^{-1}V\delta_1 \otimes \delta_1(\mathbb{I} + V)^{-1}.
\]
Proof. Firstly, the definition of \( V = V_{\text{hard}}^{(0,1)} \) (given in equation (3.19)) and the property \( s \partial_s J_a(\sqrt{sxt}) = x \partial_x J_a(\sqrt{sxt}) \) directly imply that
\[
\frac{d}{ds}(V_f)(x) = s \frac{d}{ds} \left( \sqrt{s} \int_0^1 J_a(\sqrt{sxt}) f(t) \, dt \right) = \frac{1}{2} (V_f)(x) + (\Delta V_f)(x),
\]
which is the desired result. Secondly, by using \( x \partial_x J_a(\sqrt{sxt}) = t \partial_t J_a(\sqrt{sxt}) \) and by integrating by parts, we find
\[
(\Delta V_f)(x) = \sqrt{s} \int_0^1 \sqrt{s} \partial_t (J_a(\sqrt{sxt})) f(t) \, dt = (V\delta_1)(x) \langle \delta_1 | f(0,1) \rangle - (V_f)(x) - (V /\Delta_1 f)(x),
\]
as expected. Finally, by exploiting
\[
2s \frac{d}{ds} (\mathbb{I} + V)^{-1} = \sum_{n \geq 1} (-1)^n s \frac{d}{ds} V^n
\]
we get
\[
2s \frac{d}{ds} (\mathbb{I} + V)^{-1} = \sum_{n \geq 1} (-1)^n s \frac{d}{ds} V^n = \sum_{n \geq 1} (-1)^n \sum_{k=0}^{n-1} V^k \left( 2s \frac{d}{ds} V \right)^{n-k-1}
\]
\[
= -V(\mathbb{I} + V)^{-2} + 2 \sum_{n \geq 1} (-1)^n \sum_{k=0}^{n-1} V^k \Delta V^{n-k}.
\]

But, for any operators \( O \) and \( P \) such that \( OV = -VO - P \), we have [11, lemma 3]
\[
\sum_{n \geq 1} (-1)^n \sum_{k=0}^{n-1} V^k O V^{n-k} = (\mathbb{I} - V^2)^{-1} VO + (\mathbb{I} - V^2)^{-1} P(\mathbb{I} + V)^{-1}.
\]

In our case, \( O = \Delta \) and \( P = -V \delta_1 \otimes \delta_1 + V \). Therefore,
\[
2s \frac{d}{ds} (\mathbb{I} + V)^{-1} = -V(\mathbb{I} + V)^{-2} + 2(\mathbb{I} - V^2)^{-1} V(\mathbb{I} + V)^{-1}
\]
\[
+ 2(\mathbb{I} - V^2)^{-1} V \Delta - (\mathbb{I} - V^2)^{-1} V \delta_1 \otimes \delta_1 (\mathbb{I} + V)^{-1}.
\]

This turns out to be equivalent to the last equation we wanted to prove. \( \square \)

Lemma 3. Let \( M \) be a symmetric, trace class operator in \( L^2(0,1) \). Then,
\[
\text{Tr}[ (\mathbb{I} + 2\Delta) M ] = \langle \delta_1 | M \delta_1 \rangle_{(0,1)}.
\]

Proof. Set \( \{ f_i \} \) and \( \{ \lambda_i \} \), respectively, the orthonormal eigenfunctions and the eigenvalues of \( M \). On the one hand, we have
\[
\langle \delta_1 | M \delta_1 \rangle_{(0,1)} = \sum_i \lambda_i f_i(1)^2.
\]
On the other hand, we have
\[ \text{Tr}[(\mathbb{I} + 2\Delta)M] = \sum_i \langle f_i | (1 + 2\Delta)M f_i \rangle_{(0,1)} = \sum_i \lambda_i (1 + 2\langle f_i | \Delta f_i \rangle_{(0,1)}). \]

But integration by parts gives
\[ \langle f_i | \Delta f_i \rangle_{(0,1)} = \int_0^1 f_i(x) x \frac{\partial}{\partial x} f_i(x) \, dx = f_i(1)^2 - 1 - \int_0^1 f_i(x) x \frac{\partial}{\partial x} f_i(x) \, dx. \]

Consequently, \( \text{Tr}[(\mathbb{I} + 2\Delta)M] = \sum_i \lambda_i f_i(1)^2 \) and the lemma follows. \( \square \)

**Proposition 1.** We have
\[ E_1^{\text{hard}} \left( (0, s), \frac{a - 1}{2} \right) = \det(\mathbb{I} - V_{(0,1)}^{\text{hard}}) \] and consequently
\[ \tau_1^+(\sqrt{s}) = \det(\mathbb{I} - V_{(0,1)}^{\text{hard}}). \]

**Proof.** From lemma 1, we know that the proposition is true if
\[ \det((\mathbb{I} - V)(\mathbb{I} + V)^{-1}) = \langle \rho | (\mathbb{I} + V)^{-1} \delta_1 \rangle_{(0,1)} \] or, equivalently, if
\[ \ln \det((\mathbb{I} - V)(\mathbb{I} + V)^{-1}) = \langle \rho | (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}. \]

But from the fact that \( V \to 0 \) as \( s \to 0 \), we deduce that equation (3.25) holds if and only if
\[ s \frac{\partial}{\partial s} \ln \det((\mathbb{I} - V)(\mathbb{I} + V)^{-1}) = \frac{\partial}{\partial s} \ln \langle \rho | (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}. \]

By virtue of \( s \frac{\partial}{\partial s} \ln \det(M) = \text{Tr}(M^{-1} \delta \partial \delta M) \), the latter equation reads
\[ \text{Tr} \left[ (\mathbb{I} - V^2)^{-1} 2\frac{\partial}{\partial s} V \right] = - \frac{\langle \delta_1 | \rho (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}}{\langle \delta_1 | (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}}. \] (3.26)

Using the cyclicity of the trace and lemma 3, we find that
\[ \text{Tr} \left[ (\mathbb{I} - V^2)^{-1} 2\frac{\partial}{\partial s} V \right] = \text{Tr} \left[ (\mathbb{I} - V^2)^{-1} (\mathbb{I} + 2\Delta) V \right] = \langle \delta_1 | (\mathbb{I} - V^2)^{-1} V \delta_1 \rangle_{(0,1)}. \] (3.27)

Furthermore, lemma 2 and \( (\mathbb{I} + 2\Delta) \rho = 0 \) imply that
\[ \frac{\langle \delta_1 | \rho (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}}{\langle \delta_1 | (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}} = \frac{\langle \delta_1 | (\mathbb{I} - V^2)^{-1} V \delta_1 \otimes \delta_1 (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}}{\langle \delta_1 | (\mathbb{I} + V)^{-1} \rho \rangle_{(0,1)}}, \]
\[ = \langle \delta_1 | (\mathbb{I} - V^2)^{-1} V \delta_1 \rangle_{(0,1)}. \] (3.28)

The comparison of equations (3.27) and (3.28) finally establishes the validity of equation (3.26), and the proposition follows. \( \square \)

By comparing (3.23) with (3.11), and then equating (3.12) and (3.20), we obtain the hard edge analogue of (2.23).

**Corollary 1.** One has
\[ \tau_1^+(\sqrt{s}) = \det(\mathbb{I} - V_{(0,1)}^{\text{hard}}). \] (3.29)

We remark that the evaluation of the hard edge gap probability (3.23) and the identity (3.29) contain the evaluation of the soft edge gap probability (2.22) and the identity (2.23), as a limiting case. This follows from the limit formula (see, e.g. [2]),
\[ E_1^{\text{soft}}(0; (s, \infty)) = \lim_{a \to \infty} E_1^{\text{hard}} \left( 0; (0, a^2 - (2a^2)^{3/2}, a - 1/2 \right). \]
4. Generating function generalization

The probabilistic quantity $E_{\text{bulk}}^2((0, s); \xi)$ is the first member of the sequence
$\{E_{\text{bulk}}^2(n; (0, s))\}_{n=0,1,\ldots}$, where $E_{\text{bulk}}^2(n; (0, s))$ denotes the probability that the interval $(0, s)$ contains exactly $n$ eigenvalues. Introducing the generating function for this sequence by

$$E_{\text{bulk}}^2((0, s); \xi) := \sum_{n=0}^{\infty} (1 - \xi)^n E_{\text{bulk}}^2(n; (0, s)), \quad (4.1)$$

it is well known that \[12\]

$$E_{\text{bulk}}^2((0, s); \xi) = \det(I - \xi K_{\text{bulk}}^{(0,s)}) = \det(I - \xi K_{\text{bulk}}^{(0,s)^+}) \det(I - \xi K_{\text{bulk}}^{(0,s)^-}). \quad (4.2)$$

Thus, to obtain from the Fredholm determinant expressions (1.5) for $E_{\text{bulk}}^2((0, s); \xi)$ expressions for the generating function (4.1), one merely multiplies the kernel by $\xi$.

This immediately raises the question as to whether the formula (1.13) admits a generalization upon multiplying the kernel by $\xi$? The answer is that it does, with the only change being in the initial condition (1.8) satisfied by the transcendent $\sigma(t; a)$ in (1.9). Thus specify $\sigma(t; a)$ as again satisfying (1.7) but now subject to the boundary condition

$$\sigma(t; a; \xi) \sim_{t \to 0^+} \frac{\xi t^{1+a}}{2^{1+2a} \Gamma(1+a) \Gamma(2+a)}.$$ 

Then with

$$\tau_{\text{III}}(s; a; \xi) := \exp\left(-\int_0^t \frac{\sigma(t; a; \xi)}{t} \, dt\right)$$

we have \[16, 8\]

$$\det(I - \xi K_{\text{bulk}}^{(0,2s)^\pm}) = \tau_{\text{III}}((\pi s)^2, \mp 1/2; \xi). \quad (4.3)$$

Now, the gap probabilities at the soft and hard edges can similarly be generalized to generating functions. Thus, in an obvious notation

$$E_{\text{soft}}^2((s, \infty); \xi) = \sum_{n=0}^{\infty} (1 - \xi)^n E_{\text{soft}}^2(n; (s, \infty)),$$

$$E_{\text{hard}}^2((0, s); a; \xi) = \sum_{n=0}^{\infty} (1 - \xi)^n E_{\text{hard}}^2(n; (0, s); a).$$

Analogous to (4.2), it is fundamental in random matrix theory that (2.19) and (3.16) generalize (see, e.g. \[9\]) to give

$$E_{\text{soft}}^2((s, \infty); \xi) = \det(I - \xi \tilde{K}_{(0,\infty)}^{\text{soft}})$$

$$= \det(I - \sqrt{\xi} \tilde{V}_{(0,\infty)}^{\text{soft}}) \det(I + \sqrt{\xi} \tilde{V}_{(0,\infty)}^{\text{soft}}) \quad (4.4)$$

and

$$E_{\text{hard}}^2((0, s); a) = \det(I - \xi \tilde{K}_{(0,1)}^{\text{hard}})$$

$$= \det(I - \sqrt{\xi} \tilde{V}_{(0,1)}^{\text{hard}}) \det(I + \sqrt{\xi} \tilde{V}_{(0,1)}^{\text{hard}}). \quad (4.5)$$

Also, analogous to the situation with $E_{\text{bulk}}^2((0, s); \xi)$ we know from \[10, 15, 16\] that the $\tau$-function formulae in (2.16) and (3.12) for $E_{\text{soft}}^2((s, \infty))$ and $E_{\text{hard}}^2(0; (0, s))$ require only modification to the boundary condition satisfied by the corresponding transcendent to
generalize to \( \tau \)-function formulae for the generating functions. Explicitly, in relation to \( E_2^{\text{soft}} \), in (2.11) and (2.12) again set \( \alpha = 0 \), but now require that \( H_II \) and thus \( h_II \) depend on an auxiliary parameter \( \xi \) by specifying the boundary condition
\[
h_II^\pm(t; 0; \xi) \sim \pm \frac{\sqrt{\xi}}{2} \text{Ai}(t).
\] (4.6)

Then, with
\[
\tau_II^\pm(s; \alpha; \xi) = \exp \left( - \int_{0}^{\xi} h_II^\pm(t; \alpha; \xi) \, dt \right),
\]
we have [15]
\[
E_2^{\text{soft}}((s, \infty); \xi) = \tau_II^+(s; 0; \xi) \tau_II^-(s; 0; \xi),
\] (4.7)
where the superscripts refer to the corresponding sign in (4.6). And generalizing the identity implied by the equality between (2.9) and (2.14) \( \tau_II^+ \) admits the further Painlevé transcendent form [10, 17]
\[
\tau_II^\pm(s; 0; \xi) = \exp \left( - \frac{1}{2} \int_{s}^{\infty} (t - s)q^2(t; \xi) \, dt \right) \exp \left( \mp \frac{1}{2} \int_{s}^{\infty} q(t; \xi) \, dt \right),
\] (4.8)
where \( q(t; \xi) \) satisfies (2.6) with \( \alpha = 0 \) subject to the boundary condition
\[
q(s; \xi) \sim s \to \infty \sqrt{\xi} \text{Ai}(s).
\] (4.9)

At the hard edge again specify \( \tilde{h}_V^\pm \) in terms of \( \sigma^\pm \) by (3.8) but now modify the boundary condition (3.9) by multiplying it by \( \sqrt{\xi} \), thus requiring that
\[
x \tilde{h}_V^\pm(x; a; \xi) \sim x \to 0^+ \pm \sqrt{\xi} x a + 1 / \Gamma(1 + a).
\]

With the corresponding \( \tau \) function specified by
\[
\tau_V^\pm(s; a; \xi) = \exp \int_{0}^{s} \tilde{h}_V^\pm(s; a; \xi) \, dx,
\]
we then have [10]
\[
E_2^{\text{hard}}((0, s); a; \xi) = \tau_V^+(s; a; \xi) \tau_V^-(s; a; \xi).
\] (4.10)
Analogous to (4.8) \( \tau_V^\pm \) admits the further Painlevé transcendent form [7, 10]
\[
\tau_V^\pm(s; a; \xi) = \exp \left( - \frac{1}{8} \int_{0}^{s} \left( \log \frac{s}{t} \right) \tilde{q}^2(t; a; \xi) \, dt \right) \exp \left( \mp \frac{1}{4} \int_{0}^{s} \tilde{q}(t; a; \xi) \sqrt{t} \, dt \right),
\] (4.11)
where \( \tilde{q}(t; a; \xi) \) satisfies (3.1) but now with the boundary condition
\[
\tilde{q}(t; a; \xi) \sim t \to 0^+ \sqrt{\xi} t^{1/2}.
\]

This with \( \xi = 1 \) reduces (in the ‘+’ case) to the equality implied by (3.11) and (3.4).

The general \( \xi \) bulk identity (4.3) leads us to investigate if, as is true at \( \xi = 1 \) according to (2.23) and (2.29), the factors in the Fredholm determinant factorizations (4.4), (4.5) coincide with those in the \( \tau \)-function factorizations (4.7), (4.10). The answer is that they do coincide, but to show this requires some intermediate working. We will detail this working for the soft edge and be content with a sketch in the hard edge as the strategy is very similar.
Lemma 4. With $q(t; \xi)$ as in (4.8)
\[
\exp \left( - \int_s^\infty q(t; \xi) \, dt \right) = 1 - \int_s^\infty \left[ (1 - \xi K^\text{soft})^{-1} A^s \right] (y) B^s(y) \, dy,
\] (4.12)
where $A^s$ is the operator which multiplies by $\sqrt{\xi} \Ai(x)$, while
\[
B^s(y) := 1 - \sqrt{\xi} \int_y^\infty \Ai(x) \, dx.
\] (4.13)

Proof. We closely follow the working in [7], referring to equations therein as required. Introduce the notation
\[
\phi(x) = \sqrt{\xi} \Ai(x), \quad Q(x) = \left[ (1 - \xi K^\text{soft})^{-1} \right] \psi(x)
\]
so that
\[
\int_s^\infty \left[ (1 - \xi K^\text{soft})^{-1} A^s \right] (y) B^s(y) \, dy = \int_s^\infty dy Q(y) \left( 1 - \int_y^\infty \phi(v) \, dv \right) =: u_e.
\] (4.14)
The strategy is to derive coupled differential equations for $u_e$ and
\[
q_e := \int_s^\infty dy \rho(s, y) \left( 1 - \int_y^\infty \phi(v) \, dv \right),
\] (4.15)
where $\rho(s, y)$ denotes the kernel of the integral operator $(1 - \xi K^\text{soft})^{-1}$.

According to the working of [7, equations (3.11)-(3.14)] the sought equations are
\[
\frac{du_e}{ds} = -q(s; \xi) q_e, \quad \frac{dq_e}{ds} = q(s; \xi) (1 - u_e).
\] (4.16) (4.17)
where $q(s; \xi)$ enters via the fact that $Q(y) = q(s; \xi)$. Since $Q(y)$ is smooth while $\rho(s, y)$ is equal to the delta function $\delta(s - y)$ plus a smooth term, we see from (4.14), (4.15) that the equations (4.16), (4.17) must be solved subject to the boundary conditions
\[
u_e \to 0, \quad q_e \to 1 \quad \text{as} \ s \to \infty.
\]

It is simple to verify that the solution subject to these boundary conditions is
\[
u_e(s) = 1 - q_e(s) = 1 - \exp \left( - \int_s^\infty q(x; \xi) \, dx \right),
\]
and (4.12) follows.

□

Lemma 5. One has
\[
1 - \int_s^\infty \left[ (1 - \xi K^\text{soft})^{-1} A^s \right] (y) B^s(y) \, dy = \langle \delta_0 | (1 + \sqrt{\xi} V^\text{soft}(0, \infty))^{-1} \rangle_{(0, \infty)}.
\] (4.18)

Proof. Changing variables $y \mapsto y + s$ and noting from (4.13) that
\[
B^s(y + s) = \left[ 1 - \sqrt{\xi} V^\text{soft}(0, \infty) \right] (1)(y)
\]
shows that the left-hand side of (4.18) is equal to
\[
1 - \langle \delta_0 | \sqrt{\xi} V^\text{soft}(0, \infty) \left[ 1 + \sqrt{\xi} V^\text{soft}(0, \infty) \right]^{-1} \rangle_{(0, \infty)}.
\]
This reduces to the right-hand side upon noting $\langle \delta_0 | 1 \rangle_{(0, \infty)} = 1$.

The sought $\xi$ generalization of (2.23) can now be established.
Proposition 2. One has
\[ \tau^{\pm}_{\Omega}(s; 0; \xi) = \det(\mathbb{I} \mp \sqrt{\xi} V^{\text{soft}}_{(0,\infty)}). \] (4.19)

Proof. The well known fact [16] that
\[ \exp \left(-\frac{1}{4} \int_0^s \left( \log \frac{t}{\xi} \right) q^2(t; a; \xi) \, dt \right) = \det(\mathbb{I} - \xi K^{\text{soft}}_{(0,\infty)}), \] (4.20)

together with (4.8), lemmas 4 and 5 tell us that
\[ (\tau^{\pm}_{\Omega}(s; 0; \xi))^2 = \det(\mathbb{I} - \xi K^{\text{soft}}_{(0,\infty)}) \langle \delta_0 | (\mathbb{I} + \sqrt{\xi} V^{\text{soft}}_{(0,\infty)})^{-1} \rangle. \] (4.21)

Recalling (4.4) we see that (4.19) in the '+' case is equivalent to the identity
\[ \det(\mathbb{I} - \sqrt{\xi} V^{\text{soft}}_{(0,\infty)}) = \det(\mathbb{I} + \sqrt{\xi} V^{\text{soft}}_{(0,\infty)}) \langle \delta_0 | (\mathbb{I} + \sqrt{\xi} V^{\text{soft}}_{(0,\infty)})^{-1} \rangle. \] (4.22)

With \( \xi = 1 \) this is precisely the identity established in [11]. Inspection of the details of the derivation (on which, as already mentioned, our lemmas 1–3 are based) show that the workings remain valid upon multiplying \( V^{\text{soft}}_{(0,\infty)} \) by a scalar, so (4.21) is true, and thus so is (4.19) in the '+' case. The validity of the '−' case now follows from the use of (4.20) and the '+' case in (4.4). \( \square \)

At the hard edge, analogous to the result (4.19) we would like to show that (3.29) admits a \( \xi \)-generalization. The \( \xi \)-generalization of the \( \tau \)-function on the left-hand side is given by (4.10). In relation to that expression we know that [16]
\[ \exp \left(-\frac{1}{2} \int_0^t \frac{q(t; a; \xi)}{\sqrt{t}} \, dt \right) = 1 - \int_0^t \left(\mathbb{I} - \xi K^{\text{hard}}_{(0,\infty)}\right)^{-1} A^b \left(\mathbb{I} + \sqrt{\xi} V^{\text{hard}}_{(0,1)}\right) \] (4.23)

where \( A^b \) is the operator which multiplies by \( \sqrt{\xi} J_a(\sqrt{\xi}) \), while
\[ B^b(y) = \frac{1}{2\sqrt{\xi}} \left(1 - \sqrt{\xi} \int_0^y J_a(t) \, dt \right) \] (cf (4.12)). Proceeding as in the proof of lemma 1 (and using the notation therein) shows that the right-hand side of (4.22) is equal to
\[ \langle \delta_1 | (\mathbb{I} + \sqrt{\xi} V^{\text{hard}}_{(0,1)})^{-1} \rho \rangle_{(0,1)}. \]

With these preliminaries noted, our sought result can be established.

Proposition 3. One has
\[ \tau^{\pm}_{\Omega}(s; a; \xi) = \det(\mathbb{I} \mp \sqrt{\xi} V^{\text{hard}}_{(0,1)}), \] (4.24)

Proof. According to the above results, the '+' case is equivalent to the identity
\[ \det(\mathbb{I} - \sqrt{\xi} V^{\text{hard}}_{(0,1)}) = \det(\mathbb{I} + \sqrt{\xi} V^{\text{hard}}_{(0,1)}) \langle \delta_1 | (\mathbb{I} + \sqrt{\xi} V^{\text{hard}}_{(0,1)})^{-1} \rho \rangle_{(0,1)}, \] (4.25)

which in the case \( \xi = 1 \) is precisely (3.24). The derivation given of the latter identity carries over unchanged with \( V \mapsto \sqrt{\xi} V \), thus verifying (4.25). The '−' case can now be deduced from (4.5). \( \square \)
We conclude by noting a $\xi$-generalization which holds in the bulk but not at the hard or soft edge. Thus, in the bulk, with the generating function for $\{E_1^{\text{bulk}}(n; (0, s))\}_{n=0,1,...}$, specified by

$$E_1^{\text{bulk}, \pm}((0, s); \xi) = \sum_{n=0}^{\infty} (1 - \xi)^n 2E_1^{\text{bulk}}(n; (0, s)) + E_1^{\text{bulk}}(2n \mp 1; (0, s)),$$

the identity (1.4) admits the simple generalization (see e.g. [9])

$$E_1^{\text{bulk}, \pm}((0, s); \xi) = \det(\| + \sqrt{\xi} K_{(0,s)}^{\text{bulk}, \pm}).$$

(4.25)

However, the corresponding $\xi$ generalizations of (2.22) and (3.23) cannot hold true, as the corresponding integral operators are not positive definite but rather have both positive and negative eigenvalues. The Fredholm determinant $\det(\| - \xi V_{\text{soft}}(0, \infty))$ (for example) thus vanishes for some negative $\xi$, in contradiction to the behaviour of $\sum_{n=0}^{\infty} (1 - \xi)^n E_1^{\text{soft}}(n, (s, \infty))$.

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