LOWER BOUNDS FOR DILATION, WIRELENGTH, AND EDGE CONGESTION OF EMBEDDING GRAPHS INTO HYPERCUBES

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Abstract. Interconnection networks provide an effective mechanism for exchanging data between processors in a parallel computing system. One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance. Thus it becomes the first choice of topological structure of parallel processing and computing systems. In this paper, lower bounds for the dilation, wirelength, and edge congestion of an embedding of a graph into a hypercube are proved. Two of these bounds are expressed in terms of the bisection width. Applying these results, the dilation and wirelength of embedding of certain complete multipartite graphs, folded hypercubes, wheels, and specific Cartesian products are computed.

1. Introduction

A suitable interconnection network is an important part for the design of a multicomputer or multiprocessor system. Such a network is usually modeled by a symmetric graph, where the nodes represent the processing elements and the edges represent the communication channels. Desirable properties of an interconnection network include symmetry, embedding capabilities, relatively small degree, small diameter, scalability, robustness, and efficient routing [35]. One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [36].

The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machines based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel, and Connection Machines have been implemented commercially [10]. Hypercubes are very popular models for parallel computation because of their symmetry and relatively small number of interprocessor connections. The hypercube embedding problem is the problem of mapping a communication graph into a hypercube multiprocessor. Hypercubes are known to simulate other structures such as grids and binary trees [8, 26].

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Graph embedding is an important technique that maps a guest graph into a host graph, usually an interconnection network \[6\]. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the \textit{dilation}. The dilation of an embedding is defined as the maximum distance between a pair of vertices of the host graph that are images of adjacent vertices of the guest graph. It is a measure for the communication time needed when simulating one network on another \[23\]. A closely related and important cost criterion is the \textit{wirelength of an embedding} which is the sum of the dilations in the host graph of edges in the guest graph. The wirelength of a graph embedding arises from VLSI design, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on \[21, 31, 41\].

Another important cost criterion is the \textit{edge congestion}. The edge congestion of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge of the host graph. An embedding with a large edge congestion faces many problems, such as long communication delay, circuit switching, and the existence of different types of uncontrolled noise. In data networking, network edge congestion occurs when a link or node is carrying so much data that its quality of service deteriorates. Typical effects include packet loss or the blocking of new connections. Therefore, a minimum edge congestion is a most desirable feature in network embedding \[11, 28\].

Graph embeddings have been well-studied for a number of networks \[2, 3, 8, 11, 12, 13, 14, 15, 22, 27, 33, 26, 29, 34, 41\]. In this paper, we study the dilation, wirelength, and congestion of embeddings of graphs into hypercubes and proceed as follows. In the next section concepts needed are formally introduced. In Section 3 we consider the dilation of embedding into \(Q_n\) and in the main result characterize the graphs \(G\) with \(\text{dil}(G, Q_n) \leq n - 1\): they are precisely the graphs which contain a perfect anti-matching. Then, in Section 4 we prove a general lower bound on the edge congestion and a lower bound on the wirelength of embeddings into hypercubes. Both bounds are expressed in terms of the bisection width. In the remainder of the paper these results are applied to particular graphs. The results obtained in Sections 5 to 8 are summarized in Table 1 where \(K_{2^{n-p},...,2^{n-p}}\) denotes the complete \(2^p\)-partite graph with parts of size \(2^{n-p}\).

**Table 1. Summary of Sections 5 to 8**

| Embedding \(f : G \rightarrow Q_n\) | Dilation | Wirelength |
|-----------------------------------|----------|------------|
| \(G = K_{2^{n-p},...,2^{n-p}}\)   | \(n\) for \(p = n\) \(n - 1\) for \(n - \log_2(n + 1)\) \(< p \leq n - 1\) (Corollary 1) | \(n2^{2n-p-2}(2^p - 1)\) (Theorem 4) |
| \(G = FQ_n\)                     | 2 \[27\] | \(n2^n\) (Theorem 5) |
| \(G = W_{2^n}\)                 | \(n\) (Theorem 1) | \((n + 2)2^{n-1}\) (Theorem 6) |
| \(G = K_{2^{n/2}} \square K_{2^{n/2}}\) \((n\ even)\) | \(\leq n/2\) | \(n2^{3n/2-2}\) (Theorem 8) |
2. Preliminaries

In this section we give basic definitions and preliminaries related to embedding problems. Let $G$ and $H$ be finite graphs. Then an embedding of $G$ into $H$ is a pair $(f, P_f)$, where $f : V(G) \rightarrow V(H)$ is a one-to-one mapping and $P_f$ is a mapping from $E(G)$ to the set of paths in $H$ such that each edge $e = uv \in E(G)$ is assigned a path $P_f(e)$ in $H$ between $f(u)$ and $f(v)$. By abuse of language we will also refer to an embedding $(f, P_f)$ simply by $f$.

The length of $P_f(e)$ is the dilation $dil_f(e)$ of the edge $e$ (with respect to the embedding $(f, P_f)$). The maximum dilation over all edges of $G$ is called the dilation of the embedding. The dilation of embedding $G$ into $H$ is the minimum dilation taken over all embeddings of $G$ into $H$ and denoted by $\text{dil}(G, H)$, cf. [3]. In other words,

$$\text{dil}(G, H) = \min_f \max_{e \in E(G)} \text{dil}_f(e),$$

where the minimum is taken over all injections $f : V(G) \rightarrow V(H)$. It is not necessary to specify $P_f$ when we are interested in the dilation of embedding $G$ into $H$, because we can take $P_f(uv)$ to be any shortest path between $f(u)$ and $f(v)$. The diameter of $H$, denoted by $\text{diam}(H)$, is the maximum distance between any two vertices of $H$, and this provides a trivial upper bound for $\text{dil}(G, H)$.

The wirelength of an embedding $f = (f, P_f)$ of $G$ into $H$ is

$$WL_f(G, H) = \sum_{e \in E(G)} \text{dil}_f(e_G),$$

and the wirelength of embedding $G$ into $H$ is

$$WL(G, H) = \min_f WL_f(G, H),$$

see [26]. We also refer to [2, 3, 30] for the search of embeddings of $G$ into $H$ that attain the value $WL(G, H)$.

Let $f = (f, P_f)$ be an embedding of $G$ into $H$. For $e = uv \in E(H)$, let $EC_f(e)$ denote the number of edges $e' \in E(G)$ such that $e$ is in the path $P_f(e')$ between $f(u)$ and $f(v)$ in $H$. Then the edge congestion of $f$ is

$$EC_f(G, H) = \max_{e \in E(H)} EC_f(e),$$

and the edge congestion of embedding $G$ into $H$ is

$$EC(G, H) = \min_f EC_f(G, H),$$

where the minimum is taken over all embeddings $f = (f, P_f)$ of $G$ into $H$. Note that

$$WL_f(G, H) \sum_{e \in E(H)} EC_f(e_H).$$

The above problems can be specialized to several distance problems. For example, if the host graph is a path, then the dilation of embedding graph into a path is the bandwidth problem. Further distance problems are listed in Table 2 where $P_n$, $C_n$ and $K_n$, are the path, the cycle, and the complete graph of order $n$, respectively, and $K_{a,b}$ is the complete bipartite graphs on $a + b$ vertices.

Let $n \geq 1$. The $n$-cube ($n$-dimensional hypercube) has vertex set $\{0, 1\}^n$, and vertices $x, y \in V(Q_n)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit. Equivalently, the vertices of $Q_n$ can be identified with integers $0, 1, \ldots, 2^n - 1$, where vertices $i$ and $j$ are adjacent if and only if $i - j = \pm 2^p$ for some integer $p \geq 0$.

For disjoint subsets $A, B \subseteq V(G)$ we will use the notation $E_G(A, B) = \{uv \in E(G) \mid u \in A, v \in B\}$ and $E_G(A) = \{uv \mid u, v \in A\}$. If $x$ is a vertex of a graph $G$, then $N_i(x)$ is the set of vertices in $G$ at distance $i$ from $x$. The maximum degree of $G$ is denoted by $\Delta(G)$ and its order by $n(G)$. 


Thus, there exists at least one vertex $G$ of $\text{Proof}$. If Lemma 1. could be applied elsewhere. From $\text{diam}(Q_n) = n$ we obtain immediately the upper bound $\text{dil}(G, Q_n) \leq n$ for every graph $G$. To characterize graphs $G$ of order $2^n$ with $\text{dil}(G, Q_n) = n$, we need the following concept. If $G$ is a graph, then the set of pairs of vertices $X = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ forms an anti-matching, if $x_1y_1 \notin E(G), \ldots, x_ky_k \notin E(G)$. That is, $X$ is an anti-matching if $x_1y_1, \ldots, x_ky_k$ form a matching in the complement of $G$. In addition, we say that $X$ is a perfect anti-matching if $|X| = n(G)$. Now we have the following characterization.

**Theorem 1.** Let $G$ be a graph of order $2^n$. Then $\text{dil}(G, Q_n) \leq n - 1$ if and only if $G$ contains a perfect anti-matching.

**Proof.** Let $\text{dil}(G, Q_n) \leq n - 1$ and let $(f, P_f)$ be a corresponding embedding of $G$ into $Q_n$. Consider the partition $X' = \{(u, \pi) \mid u \in V(Q_n)\}$ of $V(Q_n)$ into $2^{n-1}$ pairs. Then, since $\text{dil}_f(e) \leq n - 1$ holds for any edge $e$ of $G$, we must have that $f^{-1}(u)$ is not adjacent to $f^{-1}(\pi)$. It follows that $X = \{(f^{-1}(u), f^{-1}(\pi)) \mid u \in V(Q_n)\}$ is a perfect anti-matching.

Conversely, let $X$ be a perfect anti-matching of $G$. We now define an embedding $(f, P_f)$ of $G$ into $Q_n$ by mapping each pair $(x, y)$ from $X$ into an antipodal pair of vertices of $Q_n$, that is, $(f(x), f(y)) = (v, \pi)$ for some $v \in V(Q_n)$. Since each vertex of $Q_n$ has a unique antipodal vertex, it follows that $\text{dil}_f(e) \leq n - 1$ for each edge $e$ of $G$ and hence $\text{dil}(G, Q_n) \leq n - 1$. \hfill $\square$

To determine a large class of graphs $G$ with $\text{dil}(G, Q_n) = n - 1$ we state the following lemma that could be applied elsewhere.

**Lemma 1.** If $G$ is a graph of order $2^n$, then

$$\text{dil}(G, Q_n) \geq \max \left\{ k : \sum_{i=1}^{k-1} \binom{n}{i} < \Delta(G) \right\}.$$

**Proof.** Let $k$ be a positive integer satisfying $\binom{n}{0} + \cdots + \binom{n}{k-1} < \Delta(G)$, and let $(f, P_f)$ be an embedding of $G$ into $Q_n$. Let $u$ be a vertex of $G$ with $\text{deg}(u) = \Delta(G)$ and let $f(u) = v \in V(Q_n)$. Setting $X = N_1(v) \cup \cdots \cup N_{k-1}(v)$, the assumption on $k$ implies that

$$|X| = |N_1(v)| + \cdots + |N_{k-1}(v)| = \sum_{i=1}^{k-1} \binom{n}{i} < \Delta(G).$$

Thus, there exists at least one vertex $w$ of $G$ adjacent to $u$ such that $f(w) \in V(Q_n) \setminus X$. It follows that the length of $P_f(uw)$ is at least $k$ and consequently $\text{dil}(G, Q_n) \geq k$. \hfill $\square$

### Table 2. Various embedding problems

| Embedding $f : G \rightarrow H$ | Dilation | Wirelength | Congestion |
|--------------------------------|----------|------------|------------|
| $H = P_n$                      | Bandwidth [17] | MinLA [1] | Cutwidth [16] |
| $H = C_n$                      | Cyclic bandwidth [24] | Cyclic wirelength [4] | Cyclic cutwidth [7] |
| $G = K_n$                      | $\text{diam}(H)$ [40] | Wiener index of $H$ [39] | Forwarding index of $H$ [18] |
| $G = K_{a,b}$                  | Makespan [37] | Average load | Maximum load |

**3. Graphs with large dilation of embedding into hypercubes**

The dilation, the wirelength, and the edge congestion problem are different in the sense that an embedding that gives the minimum dilation need not give the minimum congestion (wirelength) and vice-versa. From $\text{diam}(Q_n) = n$ we obtain immediately the upper bound $\text{dil}(G, Q_n) \leq n$ for every graph $G$. To characterize graphs $G$ of order $2^n$ with $\text{dil}(G, Q_n) = n$, we need the following concept. If $G$ is a graph, then the set of pairs of vertices $X = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ forms an anti-matching, if $x_1y_1 \notin E(G), \ldots, x_ky_k \notin E(G)$. That is, $X$ is an anti-matching if $x_1y_1, \ldots, x_ky_k$ form a matching in the complement of $G$. In addition, we say that $X$ is a perfect anti-matching if $|X| = n(G)$. Now we have the following characterization.

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Conversely, let $X$ be a perfect anti-matching of $G$. We now define an embedding $(f, P_f)$ of $G$ into $Q_n$ by mapping each pair $(x, y)$ from $X$ into an antipodal pair of vertices of $Q_n$, that is, $(f(x), f(y)) = (v, \pi)$ for some $v \in V(Q_n)$. Since each vertex of $Q_n$ has a unique antipodal vertex, it follows that $\text{dil}_f(e) \leq n - 1$ for each edge $e$ of $G$ and hence $\text{dil}(G, Q_n) \leq n - 1$. \hfill $\square$

To determine a large class of graphs $G$ with $\text{dil}(G, Q_n) = n - 1$ we state the following lemma that could be applied elsewhere.

**Lemma 1.** If $G$ is a graph of order $2^n$, then

$$\text{dil}(G, Q_n) \geq \max \left\{ k : \sum_{i=1}^{k-1} \binom{n}{i} < \Delta(G) \right\}.$$
Combining Lemma 1 with Theorem 1 we obtain the following corollary.

**Corollary 1.** If \( G \) is a graph of order \( 2^n \) with \( \Delta(G) \geq 2^n - n - 1 \) that contains a perfect anti-matching, then \( \text{dil}(G, Q_n) = n - 1 \).

**Proof.** By Theorem 1, \( \text{dil}(G, Q_n) \leq n - 1 \) and the reverse inequality follows from Lemma 1 and
\[
\sum_{i=1}^{n-2} \left( \begin{array}{c} n \\ i \\ \end{array} \right) = 2^n - \left( \begin{array}{c} n \\ 0 \\ \end{array} \right) - \left( \begin{array}{c} n \\ n - 1 \\ \end{array} \right) - \left( \begin{array}{c} n \\ n \\ \end{array} \right) = 2^n - n - 2 < \Delta(G).
\]

For illustration, consider \( G = K_{4,4,4,4} \). Then \( \Delta(G) = 12 \geq 2^4 - 4 - 1 \). Since \( G \) contains a perfect anti-matching, \( \text{dil}(G, Q_4) = 3 \). In Figure 1 a corresponding embedding is shown.

**Figure 1.** \( \text{dil}(K_{4,4,4,4}, Q_4) = 3 \)

4. TWO LOWER BOUNDS IN TERMS OF THE BISECTION WIDTH

In this section we give a general lower bound on the edge congestion and on the wirelength of embedding into hypercubes. Both bound involve the bisection width that is defined as follows. The bisection width \( BW(G) \) of a graph \( G \) is the minimum number of edges necessary in an edge cut so that the sizes of the two sides of the cut differ by at most one.

The first lower bound is established by the following averaging argument.

**Theorem 2.** If \( G \) and \( H \) are graphs of the same order, then
\[
\text{EC}(G, H) \geq \frac{BW(G)}{BW(H)}.
\]

**Proof.** Set \( n = n(G) (= n(H)) \). Let \( A \cup B \) be a partition of \( V(H) \) such that \( |A| = \lfloor n/2 \rfloor \), \( |B| = \lceil n/2 \rceil \), and \( |E_H(A, B)| = BW(H) \), and let \( f : G \rightarrow H \) be an embedding with \( \text{EC}_f(G, H) = \text{EC}(G, H) \). Then
\[
BW(G) \leq |E_G(f^{-1}(A), f^{-1}(B))| \leq \sum_{e \in E_H(A, B)} \text{EC}_f(e). \tag{2}
\]

where the first inequality follows from the definition of the bisection width, and the second one from the fact that for every edge \( uv \in E_G(f^{-1}(A), f^{-1}(B)) \) the path \( P_f(uv) \) has to use at least one edge from \( E_H(A, B) \). Finally, we conclude
\[
\frac{BW(G)}{BW(H)} = \frac{BW(G)}{|E_H(A, B)|} \leq \frac{1}{|E_H(A, B)|} \sum_{e \in E_H(A, B)} \text{EC}_f(e) \leq \max_{e \in E(H)} \text{EC}_f(e) = \text{EC}_f(G, H) = \text{EC}(G, H). \quad \Box
\]
Corollary 2. If $G$ is a graph of order $2^n$, then
\[ EC(G, Q_n) \geq \frac{BW(G)}{2^{n-1}}. \]

Proof. The result follows from Theorem 2 and the fact that $BW(Q_n) = 2^{n-1}$, see [38]. □

The lower bound for the wirelength is the following.

Theorem 3. If $G$ is a graph of order $2^n$, then $WL(G, Q_n) \geq n \cdot BW(G)$.

Proof. Let $E(Q_n) = S_1 \cup \cdots \cup S_n$ be the partition of $E(Q_n)$ where $S_i$ consists of the edges whose incident vertices differ in the $i$th coordinate. Then $Q_n \setminus S_i$ consists of two vertex-disjoint $(n-1)$-cubes. Moreover, $|S_i| = 2^{n-1} = BW(Q_n)$, and we can now estimate as follows, where the minimum is taken over all embeddings $f : G \to Q_n$:
\[ WL(G, Q_n) = \min_f \sum_{e \in E(Q_n)} EC_f(e) = \min_f \sum_{i=1}^n \sum_{e \in S_i} EC_f(e) \geq \min_f \sum_{i=1}^n BW(G) = n \cdot BW(G). \]

The first equality above holds by the second equality in [1], and the second equality follows because the sets $S_i$ form a partition of $E(Q_n)$. □

5. Embeddings of Complete Multipartite Graphs

In this section we consider the complete $p$-partite graph $G = K_{n_1, \ldots, n_p}$ of order $2^n$. Recall that $G$ contains $p$ independent sets containing $n_i$, $i \in [p]$, vertices, and all possible edges between vertices from different parts.

From Theorem [1] it follows that $\text{dil}(G, Q_n) = n$ if at least one of the $p_i$ is odd and $\text{dil}(G, Q_n) \leq n-1$ otherwise (that is, if all the $p_i$ are even). We will next determine the wirelengths of embedding certain complete $p$-partite graphs into $n$-cubes, and start by determining the bisection widths of balanced complete multipartite graphs.

Lemma 2. If $G$ is the complete $t$-partite graph $K_{2r, \ldots, 2r}$, with $2tr$ vertices, $t \geq 2$, $r \geq 1$, then
\[ BW(G) = r^2t(t-1). \]

Proof. Let $V(G) = A \cup B$ be a partition with $|A| = |B| = tr$ such that $A$ and $B$ each contains half of the vertices in every maximal independent set. Then
\[ BW(G) \leq |E_G(A, B)| = |E(G)| - |E_G(A)| - |E_G(B)| = 4r^2 \left( \frac{t}{2} - 2r^2 \left( \frac{t}{2} \right) \right) = r^2t(t-1). \]

For the reverse inequality, let $V(G) = A \cup B$ be a partition with $|A| = |B| = tr$ and $|E_G(A, B)| = BW(G)$. For $i = 1, \ldots, t$, let $n_i$ be the number of vertices of $A$ in the $i$-th independent set. Then we apply the Cauchy-Schwarz inequality to obtain the bound
\[ |E_G(A)| = \sum_{1 \leq i < j \leq t} n_i n_j = \frac{1}{2} \left( \sum_{i=1}^t n_i^2 \right) - \frac{1}{2} \left( \sum_{i=1}^t n_i^2 \right) \leq \frac{r^2t(t-1)}{2}. \]

For the same reason, $|E_G(B)| \leq r^2t(t-1)/2$, and consequently,
\[ BW(G) = |E_G(A, B)| = |E(G)| - (|E_G(A)| + |E_G(B)|) \geq 4r^2 \left( \frac{t}{2} \right) - r^2t(t-1) = r^2t(t-1). \] □

We now want to determine the wirelength for embedding the complete $2p$-partite graph $G = K_{2^n-p, \ldots, 2^n-p}$, $1 \leq p < n$ into $Q_n$. 

Theorem 4. If $G$ is the complete $2^p$-partite graph $K_{2n-p,\ldots,2n-p}$, where $1 \leq p < n$, then

$$WL(G, Q_n) = n \cdot 2^{2n-p-2} (2^p - 1).$$

Proof. Combining Lemma 2 for $t = 2^p$ and $r = 2^{n-p-1}$ with Theorem 3 we obtain

$$WL(G, Q_n) \geq n \cdot BW(G) = n \cdot 2^{2n-p-2} (2^p - 1).$$

In order to prove the reverse inequality we construct an embedding $f : G \to Q_n$ in the following way. Let $V(G) = V_1 \cup \cdots \cup V_{2^p}$ be the partition of $V(G)$ into $2^p$ independent sets of size $2^{n-p}$. Label the vertices of $V(G)$ with the numbers $0, 1, 2, \ldots, 2^n - 1$, in such a way that

$$V_i = \{ j2^{p+1} + i - 1 : j = 0, 1, 2, \ldots, 2^{n-p-1} - 1 \} \cup \{ j2^{p+1} - i : j = 1, 2, 3, \ldots, 2^{n-p-1} \}.$$

Recall that $V(Q_n) = \{0, 1\}^n$ and define $f : V(G) \to V(Q_n)$ by mapping $x \in V(G)$ to its binary representation. That is, $x = x_1 \cdot 2^{n-1} + x_2 \cdot 2^{n-2} + \cdots + x_n \cdot 2^0$ is mapped to $f(x) = x_1 x_2 \ldots x_n$. We will now show that for this embedding, $WL_f(G, Q_n) = n \cdot 2^{2n-p-2} (2^p - 1)$, which concludes the proof. For $x, y \in \{0, 1\}^n$, let $d(x, y)$ denote their Hamming distance, that is, $d(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$, which is equal to the distance between $x$ and $y$ in $Q_n$. For every $i \in \{1, 2, \ldots, 2^p\}$,

$$\sum_{\{x:y\} \in (V_i)} d(f(x), f(y)) = n2^{2(n-1-p)},$$

and consequently,

$$WL_f(G, Q_n) = \sum_{\{x:y\} \in (V(Q_n))^2} d(x, y) - \sum_{i=1}^{2^p} \sum_{\{x:y\} \in (V_i)} d(f(x), f(y)) = n2^{2n-2} - 2^p n2^{2(n-1-p)} = n2^{2n-2-p}(2^p - 1).$$

6. Embeddings of folded hypercubes

The $n$-dimensional folded hypercube $FQ_n$ is the graph obtained from $Q_n$ by adding, for every vertex $x = x_1 \ldots x_n$, the edge between $x$ and its antipodal vertex $\overline{x} = \overline{x_1} \ldots \overline{x_n}$, where $\overline{x_i} = 1 - x_i$, cf. [41]. Folded hypercubes are of wide current interest in reliability and fault tolerance of interconnection networks, see [9, 21, 25, 42]. In this section we add to the known results on folded hypercubes their wirelength.

Theorem 5. For all $n \geq 2$, $WL(FQ_n, Q_n) = n2^n$.

Proof. In [32] it was proved that $BW(FQ_n) = 2^n$, and with Theorem 3 this implies $WL(FQ_n, Q_n) \geq n2^n$. To prove the reverse inequality, let $id : FQ_n \to Q_n$ be the identity embedding, that is, $id(v) = v$, for every $v \in V(FQ_n)$ (and where the second $v$ is considered as a vertex of $Q_n$). For this embedding, the $n2^{n-1}$ edges of $FQ_n$ that are also the edges of $Q_n$ contribute $n2^{n-1}$ to the wirelength. and each of the $2^{n-1}$ diagonal edges contributes $n$ to the wirelength. We conclude

$$WL_{id}(FQ_n, Q_n) = n2^{n-1} + 2^{n-1}n = n \cdot 2^n.$$

7. Embeddings of wheels

Let $n \geq 4$. The wheel $W_n$ of order $n$ is the graph obtained from a cycle $C_{n-1}$ by adding a new vertex $x$ and joining $x$ to all the vertices of the cycle. Vertex $x$ is called the hub of the wheel and the edges incident with $x$ are spokes of the wheel. Since the hub vertex is adjacent to all other vertices, the wheel does not contain an anti-matching, and by Theorem 1, $dil(W_{2n}, Q_n) = n$. We next determine the wirelength of embedding $W_{2n}$ into $Q_n$. 
Theorem 6. If \( n \geq 2 \), then \( WL(W_{2^n}, Q_n) = (n + 2)2^{n-1} \).

Proof. For every embedding the spokes contribute

\[
\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}
\]

to the wirelength. Since \( Q_n \) is bipartite, the \( 2^n - 1 \) cycle edges contribute at least \( 2^n \). This gives a lower bound of \( n2^{n-1} + 2^n = (n + 2)2^{n-1} \), and using a Gray code this bound can be achieved. \( \square \)

8. Embeddings of some Cartesian products

Recall that the Cartesian product \( G \square H \) of graphs \( G \) and \( H \) is the graph with the vertex set \( V(G) \times V(H) \), vertices \((g, h)\) and \((g', h')\) being adjacent if either \( g = g' \) and \( hh' \in E(H) \), or \( h = h' \) and \( gg' \in E(G) \). For more information on this product see the book [21]. We will use the following result on the edge-isoperimetric problem for Cartesian products of complete graphs (see [5] for a survey of related results).

Theorem 7. ([19]) Let \( G = K_{p_1} \square K_{p_2} \square \cdots \square K_{p_t} \) with \( p_1 \leq p_2 \leq \cdots \leq p_t \), and let \( m \) be an integer with \( 1 \leq m \leq p_1p_2 \cdots p_t \). If \( H \) is a subgraph of \( G \) with \( |V(H)| = m \), then \( |E(H)| \leq |E(H^*)| \), where \( H^* \) is the subgraph induced by the first \( m \) vertices in lexicographic order, where we take

\[
V(G) = \{(a_1, a_2, \ldots, a_t) : 1 \leq a_i \leq p_i \text{ for } i = 1, 2, \ldots, t\},
\]

and the lexicographic order is defined by \((a_1, a_2, \ldots, a_t) < (b_1, b_2, \ldots, b_t)\) if there exists an index \( i \) with \( a_i < b_i \) and \( a_j = b_j \) for all \( j < i \).

As a corollary, we obtain the bisection widths for products of complete graphs.

Corollary 3. If \( G = K_{2p_1} \square K_{p_2} \square K_{p_3} \square \cdots \square K_{p_t} \) with \( 2p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_t \), then \( BW(G) = p_1^2p_2 \cdots p_t \).

Proof. We have \( |E(G)| = p_1p_2 \cdots p_t (2p_1 + p_2 + p_3 + \cdots + p_t - t) \), and for any \( A \subseteq V(G) \) with \( |A| = p_1p_2 \cdots p_t \), Theorem 7 implies

\[
|E_G(A)| \leq \frac{1}{2} p_1p_2 \cdots p_t (p_1 + p_2 + \cdots + p_t - t)
\]

with equality if \( A \) is the set of vertices \((a_1, \ldots, a_t)\) with \( a_1 \leq p_1 \) or the set of vertices \((a_1, \ldots, a_t)\) with \( a_1 > p_1 \). This implies that for every partition \( V(G) = A \cup B \) with \( |A| = |B| = p_1p_2 \cdots p_t \),

\[
|E_G(A, B)| = |E(G)| - |E_G(A)| - |E_G(B)|
\]

\[
\geq p_1p_2 \cdots p_t (2p_1 + p_2 + p_3 + \cdots + p_t - t) - p_1p_2 \cdots p_t (p_1 + p_2 + \cdots + p_t - t) = p_1^2p_2 \cdots p_t
\]

with equality if \( A \) is the set of vertices \((a_1, \ldots, a_t)\) with \( a_1 \leq p_1 \). \( \square \)

Theorem 8. Let \( n \) be a positive even integer, and let \( G = K_{2^{n/2}} \square K_{2^{n/2}} \). Then

\[
WL(G, Q_n) = n2^{3n/2 - 2}.
\]

Proof. From Corollary 3 with \( t = 2, p_1 = 2^{n/2 - 1} \) and \( p_2 = 2^{n/2} \), we obtain \( BW(G) = 2^{3n/2 - 2} \), and together with Theorem 8 this implies the lower bound. To prove the upper bound we embed \( G \) into \( Q_n \), by mapping a vertex \((a, b) \in V(G) = \{0, 1, \ldots, 2^{n/2} - 1\} \times \{0, 1, \ldots, 2^{n/2} - 1\}\) with

\[
a = \sum_{i=1}^{n/2} a_i 2^{n/2 - i},
\]

\[
b = \sum_{i=1}^{n/2} b_i 2^{n/2 - i}
\]
to the binary string \( a_1a_2\ldots a_n/b_1b_2\ldots b_n/2 \in \{0,1\}^n = V(Q_n) \). For \( i \in \{1,\ldots,n\} \), let \( V(Q_n) = A_i \cup B_i \) be the partition with \( A_i = \{ x \in V(Q_n) : x_i = 0 \} \) and \( B_i = \{ x \in V(Q_n) : x_i = 1 \} \). Then

\[
E(Q_n) = \bigcup_{i=1}^{n} E_{Q_n}(A_i, B_i)
\]

and since each of the sets \( f^{-1}(A_i) \) and \( f^{-1}(B_i) \) induces a subgraph of \( G \) isomorphic to \( K_{2^{n/2-1}} \), we obtain

\[
WL_f(G, Q_n) = \sum_{i=1}^{n} \sum_{e \in E_{Q_n}(A_i, B_i)} EC_f(e) = \sum_{i=1}^{n} |E_{Q_n}(f^{-1}(A_i), f^{-1}(B_i))|
= \sum_{i=1}^{n} BW(G) = n \cdot BW(G) = n2^{3n/2-2}.
\]

### Concluding Remarks

In this paper, we have obtained lower bounds for dilation, wirelength, and edge congestion of an embedding of graphs into \( n \)-cubes. In particular, we found lower bounds for congestion and wirelength in terms the bisection width of the guest graph. This technique allows the computation of the exact dilation and wirelength for a range of networks. An open problem remains to determine the edge congestion of these networks. It also opens up the study of embedding parameters which remains unsolved for several good candidates of guest architectures.

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