Quasi-Normal Modes of a Schwarzschild White Hole

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We investigate perturbations of the Schwarzschild geometry using a linearization of the Einstein vacuum equations within a Bondi-Sachs, or null cone, formalism. We develop a numerical method to calculate the quasi-normal modes, and present results for the case \( \ell = 2 \). The values obtained are different to those of a Schwarzschild black hole, and we interpret them as quasi-normal modes of a Schwarzschild white hole.

I. INTRODUCTION

The theory of a linear perturbation of a black hole was developed some time ago [1, 2, 3, 4]; see also the textbook [5] and the review [6]. The essential idea is that the vacuum Einstein equations are linearized about the Schwarzschild (or Kerr) geometry described by the usual \((t, r, \theta, \phi)\) coordinates. Then a standard separation of variables ansatz is applied, with metric quantities behaving as an unknown function of \( r \times Y_{\ell m}(\theta, \phi) \exp(i\nu t) \) (actually, the angular dependence is somewhat more complicated, and the technical details can be found in the literature). There results an ordinary differential equation in \( r \), and the quasi-normal modes are obtained by finding the special values of \( \nu \) for which solutions exist that satisfy appropriate boundary conditions in the neighbourhood of the event horizon, and of infinity.

Quasi-normal mode theory has become a cornerstone of modern general relativity theory. They have been seen in numerical relativity simulations of binary black hole coalescence. And, while not yet actually observed, it is strongly expected that they will be measured by the LIGO collaboration, and certainly by LISA, yielding precise information about the parameters describing a black hole from some coalescence event.

In the usual approach to linear perturbations of a black hole, the linearization is performed using standard Schwarzschild (or Kerr) coordinates \((t, r, \theta, \phi)\). It is also possible to perform the linearization using Bondi-Sachs coordinates, which is a coordinate system based on outgoing null cones. This has been done in previous work, in order to obtain analytic solutions of the linearized Einstein equations for the purpose of testing numerical relativity codes. As with the usual approach, one ends up with a second order ordinary differential equation involving \( \ell \) and \( \nu \) as parameters, Eq. (5). However, when the quasi-normal modes were calculated for this equation, it was found that they are not the standard ones. Different physical problems are considered in the two cases, as illustrated in the Penrose diagram of Schwarzschild spacetime (Fig. 1). \( K \) is a typical hypersurface used in finding the quasi-normal modes of a black hole, and the direction of wave propagation at the boundaries of \( K \) is shown by arrows. On the other hand, \( N \) is a typical hypersurface used in finding the quasi-normal modes of Eq. (5). From the direction of wave propagation on \( N \), the resulting quasi-normal modes can be interpreted as being those of a white hole.

![Penrose diagram](image)

**FIG. 1:** Penrose diagram illustrating the differences, in terms of location and boundary conditions, between the hypersurfaces \( K \) and \( N \).

The plan of this paper is as follows. Section II summarizes previous work on the Bondi-Sachs metric and linearized solutions within that framework. Section III describes our approach to calculating the quasi-normal modes, and Sec. IV presents the results. We end with a Conclusion, Sec. V.

II. BACKGROUND MATERIAL

The Bondi-Sachs formalism uses coordinates \( x^i = (u, r, x^A) \) based upon a family of outgoing null hypersurfaces. We label the hypersurfaces by \( u = \text{constant} \), null rays by \( x^A \) \((A = 2, 3)\), and the surface area coordinate by \( r \). In this coordinate system the Bondi-Sachs metric [5, 8, 9] takes the form

\[
ds^2 = - \left[ e^{2\beta} \left( 1 + \frac{W}{r} \right) - r^2 h_{AB} U^A U^B \right] du^2 - 2 e^{2\beta} du dr - 2r^2 h_{AB} U^B dx^A + r^2 h_{AB} dx^A dx^B , \tag{1}
\]

where \( h^{AB} h_{BC} = \delta_D^A \) and \( \det(h_{AB}) = \det(q_{AB}) \), with \( q_{AB} \) being a unit sphere metric. We represent \( q_{AB} \) by means of a complex dyad \( q_A \). For example, in the case that the angular coordinates are spherical polar \((\theta, \phi)\), the dyad takes the form

\[
q_A = (1, i \sin \theta). \tag{2}
\]
For an arbitrary Bondi-Sachs metric, \( h_{AB} \) can be represented by its dyad component
\[
J = h_{AB} q^A q^B / 2,
\]
(3)
We also introduce the spin-weighted field \( U = U^A q_A \), as well as the (complex differential) \( \eth \) operators \( \eth \) and \( \bar{\eth} \). In Schwarzschild space-time, \( W = -2M, \beta = 0, U^A = 0 \) and \( J = 0 \).

We use \( Z_{\ell m} \), rather than \( Y_{\ell m} \), as spherical harmonic basis functions, where \( \ell \) and \( m \) are integers.

We assume the following ansatz, representing a small perturbation of the Schwarzschild geometry
\[
\beta = \Re(\beta_0(r)e^{i\nu u})Z_{\ell m}, \quad U = \Re(U_0(r)e^{i\nu u})\eth Z_{\ell m},
\]
\[
J = \Re(J_0(r)e^{i\nu u})\bar{\eth}^2 Z_{\ell m},
\]
\[
W = -2M + \Re(w_0(r)e^{i\nu u})Z_{\ell m}.
\]
(4)
Using the above ansatz, Ref. [11] constructed the resulting linearized Einstein vacuum equations. As expected, the angular and time dependence factored out, and a system of ordinary differential equations (in \( r \)) was obtained. As discussed in Ref. [11], the system can be manipulated to give
\[
x^3(1 - 2xM) \frac{d^2 J_2}{dx^2} + 2\frac{dz}{dx}(2x^2 + i\nu x - 7x^3 M)
-2(x(\ell^2 + \ell - 2)/2 + 8Mx^2 + i\nu)J_2 = 0
\]
(5)
where \( J_2(x) = d^2 J_0(dx^2) \) and \( x = 1/r \). (Actually, Ref. [11] gave Eq. (5) only in the case \( \ell = 2 \), and here we give the formula for general \( \ell \).

### III. Problem Specification

We note that Eq. (5) has singularities at \( x = 0 \) and \( x = 0.5M \). The problem is to find values of \( \nu \) for which there exists a solution to Eq. (5) that is regular everywhere in the interval \([0, 0.5M]\); these values of \( \nu \) are the quasi-normal modes. This is the same situation that is faced when finding the quasi-normal modes of a black hole. The first solution to this problem was obtained by using series solutions around the singular points, and a numerical solution of an ordinary differential equation within the interior of the interval [4]. Subsequently, it was shown [12] how the theory of 3-term recurrence relations [13] for the series solution about \( x = 0.5M \) could be used to determine the quasi-normal modes.

It is straightforward to write Eq. (5) with the origin transferred to \( x = 0.5M \), and then to evaluate the recurrence relation satisfied by a regular solution (see Eqs. (24) and (25) below). We find a 4-term, rather than a 3-term, recurrence relation. While it may be that the quasi-normal modes could be found by an approach similar to that of [12], this is not a practical option since there does not seem to be available a well-developed mathematical theory of 4-term recurrence relations.

Instead, we proceed along the lines used in [4]. We construct the asymptotic series about the essential singularity at \( x = 0 \), and use it to find a solution to within a specified tolerance at a point \( x_0 > 0 \). We then use this solution as initial data for a numerical solution of Eq. (5) in the range \((x_0, x_c)\) where \( x_c < 0.5M \); actually, as in [4], we do not integrate Eq. (5) directly but first convert it to first-order Ricatti form. Finally, we construct the regular series solution about \( x = 0.5M \) and use it to find a solution at \( x = x_c \). Then a value of \( \nu \) is a quasi-normal mode if the difference at \( x = x_c \) between the regular series solution and the numerical solution, vanishes.

### A. Asymptotic series solution about the essential singularity at \( x = 0 \)

Since the singularity is essential, the resulting series solution has radius of convergence zero, although it is asymptotic. We use [14] to determine rigorous bounds on the error of approximating the solution by its first \( n \) terms. Note that a series solution has radius of convergence zero, although it is a singular series solution about \( \ell \).

Following [14], we let the solution to Eq. (5) be
\[
J_2(z) = L_\nu(z) + \epsilon_\nu(z)
\]
(12)
where \( L_n(z) = \exp(\lambda_1 z)z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s} \), \( a_{s,1} \) and define the residual \( R_n(z) \) by

\[
\frac{d^2 L_n(z)}{dz^2} + f(z) \frac{dL_n(z)}{dz} + g(z)L_n(z) = \frac{R_n(z)}{z},
\]

with

\[
|R_n(z)| \leq \frac{B_n}{z^{n+1}}
\]

in some region \(|z| > b\) and where \( B_n \) is calculable. Ref. \[14\] obtains a bound on \( \epsilon_n(z) \) provided the quantity \( C(n, b, \nu) \) defined immediately below satisfies \( C < 1 \), where

\[
C(n, b, \nu) = \frac{\beta \sqrt[p]{\pi} \Gamma \left( \frac{1}{2}(n+1) + 1 \right)}{|2i\nu|\Gamma \left( \frac{1}{2}(n+1) + \frac{1}{2} \right)(n+1)}
\]

where \( \beta \) is bounded by

\[
\beta \leq |4i\nu| + \left| \frac{1 + i\nu}{b - 2} \right| + \left| \frac{32}{b(b - 2)} \right|
\]

\[
+ |2i\nu| \left| \frac{3 - 4i\nu}{b - 2} \right|
\]

Given \( \nu \) and \( b \), we use numerics to determine conditions on \( n \) such that \( C < 0.99 \) and then we bound \( \epsilon_n(z) \) by

\[
|\epsilon_n(z)| \leq \frac{2B_n}{\beta(1 - C(n, b, \nu)|z|^{n+1}}.
\]

We also need to bound the error \( \epsilon'_n(x) \) in using a finite series to estimate \( \frac{dJ_n(x)}{dx} \). Noting that

\[
\frac{dJ_2(x)}{dx} = -z^2 \frac{dJ_2(z)}{dz},
\]

the bound on the error is

\[
|\epsilon'_n(x)| \leq \frac{2|i\nu|B_n}{\beta(1 - C(n, b, \nu)|z|^{n+1}}.
\]

### 1. Numerical implementation

We have written Matlab code that takes as input \( \nu \) and \( b \), and then finds \( \beta \) and the lowest value of \( n \) such that \( C < 0.99 \). Then the code finds the maximum of the absolute values of \( \epsilon_n(b) \) and \( \epsilon'_n(x = 1/b) \). A bisection method program takes \( \nu \) as input and refines \( b \) until the absolute value of the maximum error is in the range \((0.5, 1) \times \) machine precision (about \( 2 \times 10^{-16} \)). The code returns the values of \( 1/b \) and \( L_n(b)/L'_n(x = 1/b) \).

### B. Numerical integration of Eq. (5)

The first step is to transform Eq. (5) into first-order Ricatti form. Defining a new dependent variable \( v(x) \) by

\[
J_2(x) \rightarrow v(x) = \frac{1}{J_2(x)} \frac{dJ_2(x)}{dx},
\]

we obtain

\[
x^3(1 - 2x) \left( \frac{dv}{dx} + v^2 \right) + 2x(2x + i\nu - 7x^2)v - 2(2x + 8x^2 + i\nu) = 0.
\]

The numerical integration of Eq. (22) near the singularity at \( x = 0 \) can be tricky, because we need the result to be as accurate as possible. We found that a fourth order Runge-Kutta scheme (ode45 in Matlab) performed better than the stiff schemes, provided stringent tolerance conditions were used (specifically, RelTol = 10\(^{-12}\), AbsTol = 10\(^{-12}\), MaxStep = 2 \times 10^{-6}). Under these conditions each integration to \( x_c \) (=0.25) takes of order 100s.

### C. Series solution about the regular singularity at \( x = 0.5M \)

We first make the transformation

\[
x \rightarrow s = 1 - 2x
\]

to Eq. (5) and obtain

\[
s^3(1 - s)^3 \frac{d^2 J_2(s)}{ds^2} - (1 - s)(4i\nu - 3 + 10s - 7s^2) \frac{dJ_2(s)}{ds} - 4(i\nu + 3 - 5s + 2s^2)J_2(s) = 0.
\]

This equation has a series solution \( \sum_0^\infty a_n s^n \) that satisfies the recurrence relation

\[
a_0 = 1, a_1 = 4 + 3i\nu, a_2 = \frac{15(4 + 3i\nu)}{2(1 - i\nu)(3 - 4i\nu)}, a_n = a_{n-1} \frac{4 + 3n^2 - 2n}{n(4i\nu - n - 2)} + a_{n-2} \frac{4 + 3n^2 + 2n}{n(4i\nu - n - 2)} + a_{n-3} \frac{1 - n)(1 + n)}{n(4i\nu - n - 2)}.
\]

The radius of convergence of the above series is \( s < 1 \), and, given \( \nu \), the numerical evaluation of the coefficients, and then of the series, is straightforward. Using \( x_c = 0.25 \) means that we need to evaluate the series at \( s = 0.5 \). We terminate summation of the series at the first term smaller than \( 10^{-18} \) (typically, about 60 terms), and thus expect the result to be accurate to within machine precision (about \( 2 \times 10^{-16} \)).
IV. RESULTS

We have written a Matlab program that, given a value of \( \nu \), first uses the asymptotic series to find the value \( v_0 \) of \( v(x) \) (as defined in Eq. (21)) at \( x = x_0 = 1/b \), and then integrates numerically Eq. (22) between \( x_0 \) and \( x_c = 0.25 \), obtaining a complex number \( v_+ = v(x_c) \); and secondly uses the regular series about \( x = 0.5 \) to find \( v_- = v(x_c) \). Defining

\[
g_\nu = v_+ - v_-, \tag{26}
\]

the quasi-normal modes are those values of \( \nu \) such that \( g_\nu \) is indistinguishable from zero.

We calculated \( g_\nu \) for values of \( \nu \) in the range \( \nu = a + ib \), \( 0.1 \leq a \leq 1.07 \), \( 0.05 \leq b \leq 0.89 \), in increments of 0.03. The results are shown in the contour plot in Fig. 2. The green line is the zero contour of \( \Re(g_\nu) \), and the red line is the zero contour of \( \Im(g_\nu) \), and the blue line is the boundary of a region where the computation is probably unreliable (because the computed curve oscillates, indicating that a smaller step-length is required). Clearly, the quasi-normal modes lie at the intersection of a red and a green line, and from the plot we can read off an estimate for the lowest mode, \( \nu = 0.9 + 0.63i \). We then applied a secant method, obtaining a final estimate for the lowest quasi-normal mode at

\[
\nu = 0.883 + 0.614i. \tag{27}
\]

In this case, \( x_0 = 0.036493228795438 \), \( v_0 = 0.03683521818950 + 0.000637428772012i \), \( \beta = 0.988517790240599 \), and 62 terms were used in the asymptotic series. The contour plot indicates another quasi-normal mode at about \( \nu = 1.06 + 0.63i \), but we did not investigate further.

In Fig 2, we have used the value in Eq. (27), and vary the numerical methods so as to determine the accuracy with which \( g_\nu \) has been determined. In Fig 3 the integration between \( x_0 \) and \( x_i \) is carried out with different values of MaxStep, \( 2 \times 10^{-6}, 10^{-6} \) and \( 5 \times 10^{-7} \), and also an error of an amount \((1+i) \times 10^{-15}\) is introduced into the value of \( v_0 \) at \( x_0 \) in the case \( \text{MaxStep} = 2 \times 10^{-6} \). Also, numerical integration of Eq. (22) as well as a series solution is used in the range \((x_c, 0.5)\). The various curves lie on top of each other and are visually indistinguishable. Taking all these options into account, the maximum value noted for \( g_\nu \) was \((6.02 + 5.87i) \times 10^{-3}\). Using intermediate results from the secant root-finding process to estimate

\[
\frac{\partial \nu}{\partial g_\nu} = 3.95 + 0.69i, \tag{28}
\]

it follows that the possible error in Eq. (27) is

\[
\left| (3.95 + 0.69i) \times (6.02 + 5.87i) \times 10^{-4}\right| = 0.003, \tag{29}
\]

so that Eq. (27) should be amended to read

\[
\nu = 0.883 + 0.614i + 0.003k \tag{30}
\]

where \( k \) is a complex number satisfying \(|k| \leq 1\).

![Fig. 2: Contour plot in the complex plane of \( \nu \) showing the contours where \( \Re(g_\nu) = 0 \) (red) and \( \Im(g_\nu) = 0 \) (green).](image)

![Fig. 3: The real (solid line) and imaginary (dotted line) parts of \( v(x) \) in the quasi-normal mode case \( \nu = 0.883 + 0.614i \).](image)

The lowest quasi-normal mode of a Schwarzschild black hole is at \( \nu = 0.37367 + 0.08896i \). We have used this value in our code, and obtained Fig 4, from which it is clear that this value of \( \nu \) is not a quasi-normal mode of Eq. (5).

V. CONCLUSION

Using a linearization of the vacuum Einstein equations about the Schwarzschild geometry, within a Bondi-Sachs framework, we have constructed a numerical procedure to calculate the quasi-normal modes. The value of the lowest mode in the case \( \ell = 2 \) is not a quasi-normal mode.
mode of a Schwarzschild black hole, and further the lowest quasi-normal mode of a Schwarzschild black hole is not a quasi-normal mode of Eq. (5). As discussed in the Introduction, this apparent discrepancy can be avoided by interpreting the quasi-normal modes of Eq. (5) as being those of a white hole rather than those of a black hole.

The results obtained depend crucially on the validity of Eq. (5), and thus it is important to discuss the extent to which this has been verified. Eq. (5) was derived in [11], and there has been no subsequent, independent, derivation. Nevertheless, Eq. (5) has been subject to some consistency checks, since ref. [11] confirmed that solutions obtained also satisfy the remaining Einstein equations (the constraint equations). Further, in the case $M = 0$, solutions based on Eq. (5) have been used as analytic solutions for the testing of numerical relativity codes based on the Bondi-Sachs metric, and the expected order of convergence was observed [13, 10].

The evidence for the existence of black holes is now very strong. However, the question about the existence of white holes is much more problematic, since such objects cannot form from regular initial data, but instead must have been created as part of the creation of the universe. The present work provides a possible observational signature of a white hole, since it is in principle possible for a gravitational wave detector to extract the parameters of a quasi-normal mode from a gravitational wave signal.

Acknowledgments

NTB and ASK would like to thank the National Research Foundation of South Africa for financial support. NTB thanks Max-Planck-Institut für Gravitationsphysik, for hospitality. We thank E. Rosinger for discussion and for drawing our attention to ref. [14].

[1] T. Regge and J. Wheeler, Phys. Rev. 108, 1063 (1957).
[2] F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).
[3] C. V. Vishveshwara, Phys. Rev. D 1, 2870 (1970).
[4] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London A344, 441 (1975).
[5] S. Chandrasekhar, The Mathematical Theory of Black Holes (Oxford University Press, Oxford, England, 1983).
[6] K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativ. 2, 2 (1999), http://www.livingreviews.org/lrr-1999-2.
[7] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London A269, 21 (1962).
[8] R. Sachs, Proc. Roy. Soc. London A270, 103 (1962).
[9] N. T. Bishop, R. Gómez, L. Lehner, M. Maharaj, and J. Winicour, Phys. Rev. D 56, 6298 (1997), gr-qc/9708065.
[10] R. Gómez, L. Lehner, P. Papadopoulos, and J. Winicour, Class. Quantum Grav. 14, 977 (1997), gr-qc/9702002.
[11] N. T. Bishop, Class. Quantum Grav. 22, 2393 (2005).
[12] E. Leaver, Proc. R. Soc. London, Ser. A 402, 285 (1985).
[13] W. Gautschi, SIAM Review 9, 24 (1967).
[14] F. Olver, Asymptotics and special functions (Academic Press, New York, 1974).
[15] M. C. Babiuc, N. T. Bishop, B. Sziulágyi, and J. Winicour, Phys. Rev. D79, 084001 (2009), gr-qc/0808.0861.
[16] C. Reisswig, N. T. Bishop, C. W. Lai, J. Thornburg, and B. Sziulágyi, Class. Quantum Grav. 24, S327 (2007).