MORI DREAM SURFACES ASSOCIATED WITH CURVES WITH ONE PLACE AT INFINITY

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Abstract. We study a class of rational surfaces (considered in [CPR05]) associated to curves with one place at infinity and explicitly describe generators of the Cox ring and global sections of line bundles on these surfaces. In particular, we show that their Cox rings are finitely generated, i.e. they are Mori dream spaces. We also compute their global Zariski semigroups at infinity (consisting of line bundles which have no base points ‘at infinity’) and global Enriques semigroups (generated by closures of curves in \(\mathbb{C}^2\)). In particular, we show that the global Zariski semigroups at infinity and Enriques semigroups of surfaces corresponding to pencils which are equisingular at infinity are isomorphic, which answers a question of [CPRL02]. We also give an effective algorithm to determine if a (rational) surface admits systems of numerical curvettes (these surfaces were also considered in [CPR05]).

1. Introduction

Pencils of curves on \(\mathbb{P}^2\) form a classical topic in algebraic geometry. A particular class \(\mathcal{V}_1\) of pencils that appeared in numerous works (see e.g. [Moh74], [Sat77], [ABCN96], [Suz99], [ABCN00], [CPRL02], [GM04], [Mon07]) consists of pencils at infinity corresponding to curves with one place at infinity, i.e. pencils \(V := \mathbb{C}(F, H^d)\), where \(H\) is the equation of a straight line \(L\) (the line at infinity), and \(F\) is a polynomial of degree \(d\) such that the curve \(C := \{F = 0\}\) has one place at infinity (i.e. \(C\) intersects \(L\) only at one point \(P\) and the germ of \(C\) at \(P\) is irreducible). [CPR05] studied a more general class \(\mathcal{V} \supseteq \mathcal{V}_1\) of pencils where \(F\) is of the form \(F = F_1 \cdots F_k\), where each \(F_j\), \(1 \leq j \leq k\), defines a curve with one place at infinity. In this article we continue the study of [CPRL02] and [CPR05] on line bundles on the surfaces \(X_V\) resulting from (minimal) resolutions of basepoints of pencils \(V \in \mathcal{V}\) and show that they are Mori dream spaces, i.e. their Cox rings are finitely generated. Finite generation of Cox rings of surfaces have been extensively studied (see e.g. [BP04], [HT04], [TVAV09], [AHL10], [Ott13]), but we would like to point out, as it was noted in [CPRL02], that the surfaces \(X_V\) are unlike ‘typical examples’ of Mori dream spaces in the sense that in general the anti-canonical bundles on these surfaces are far from being nef, so that the Mori theory does not apply. Cox rings of \(X_V\) for a special class of \(V \in \mathcal{V}\) were in fact shown to be finitely generated in [GM05]; however, our method is (different and) explicit (in the sense that we explicitly describe the generators of the Cox ring) and this enables us to tackle the sharper problem of characterizing some interesting classes of line bundles on these surfaces:

The main problem studied in [CPRL02] was to achieve (for surfaces of the form \(X_V\), \(V \in \mathcal{V}_1\)) global analogues of the work of Enriques and Zariski, i.e. to characterize line bundles on \(X_V\) which are generated by global sections (Zariski), and those which have global sections supported on (the strict transform of) curves that properly intersect the line at infinity and the

\[\text{We write } \mathbb{C}(F_1, \ldots, F_k) \text{ to denote the vector space over } \mathbb{C} \text{ spanned by } F_1, \ldots, F_k.\]
exceptional curves (Enriques). We completely solve this problem and give explicit description of the global Enriques and Zariski semigroups. In particular we answer the question asked in [CPRLO2] about these semigroups:

**Question 1.1** ([CPRLO2]). Let $V_i := \langle F_i, H^d \rangle \in \mathcal{V}_1$, $i = 1, 2$. Identify $\mathbb{P}^2 \setminus \{H = 0\}$ with $\mathbb{C}^2$, and assume $V_1$ and $V_2$ are equisingular at infinity, i.e. the curves $C_i$ defined by $F_i$ (of the same degree $d$) have equisingular germs 'at infinity' i.e. at the the points $O_i := C_i \cap \{H = 0\}$, $1 \leq i \leq 2$. Under what conditions are the global Enriques semigroups and/or global Zariski semigroups on $X_{V_i}$ isomorphic?

We show that the answer to Question 1.1 is “always” (Corollary 6.4). In fact our results extend to the larger class $\mathcal{V}$: if $V_1, V_2 \in \mathcal{V}$ are equisingular at infinity, then Corollary 6.4 states that

1. the global Enriques semigroups of $X_{V_1}$ and $X_{V_2}$ are isomorphic.
2. the ‘Zariski semigroups at infinity’ (consisting of divisors which have no base point on $X_{V_1} \setminus \mathbb{C}^2$) of $X_{V_1}$ and $X_{V_2}$ are isomorphic.

Pick $V = \mathbb{C}(F_1^{a_1}, \ldots, F_k^{a_k}, H^d) \in \mathcal{V}$. Then $X_V$ can be considered naturally as a compactification of $\mathbb{C}^2$ (by identifying $\mathbb{C}^2$ with the complement of $\{H = 0\}$ in $\mathbb{P}^2$). Set $f_i := F_i|_{\mathbb{C}^2}$, $1 \leq i \leq k$. As in [CPR05] and [CPRLO2], we show that the divisors $D_{ij}$ corresponding to approximate roots (introduced by Abhyankar and Moh [AM73]) of the $f_i$’s play a crucial role in the structure of line bundles on $X_V$: they essentially generate the Cox ring of $X_V$ (Theorem 4.5 and Remark 4.2) and tropically generates the global Enriques semigroup $P^d(X_V)$:

**Definition 1.2.** Given $S \subseteq \mathbb{Z}^k$, the tropical closure of $S$ is the smallest semigroup $\bar{S} \subseteq \mathbb{Z}^k$ containing $S$ which is also closed under taking (coordinatewise) maximum, i.e. if $\alpha_i := (\alpha_{i1}, \ldots, \alpha_{ik}) \in \bar{S}$, $1 \leq i \leq 2$, then $\max\{\alpha_1, \alpha_2\} := (\max\{\alpha_{11}, \alpha_{21}\}, \ldots, \max\{\alpha_{1k}, \alpha_{2k}\}) \in \bar{S}$. We say that $S$ tropically generates $T \subseteq \mathbb{Z}^k$ iff $T = \bar{S}$.

**Theorem 1.3** (follows from Corollary 5.3 and Remark 4.2). Let $\Gamma_0, \ldots, \Gamma_N$ be the irreducible components of $X_V \setminus \mathbb{C}^2$. Consider the corresponding identification of $\text{Pic} X_V$ with $\mathbb{Z}^{N+1}$. Then $\{D_{ij}\}_{i,j}$ tropically generates $P^d(X_V) \subseteq \mathbb{Z}^{N+1}$.

The global Zariski semigroup at infinity $\bar{P}_\infty(X_V)$ can also be expressed in terms of (products of) the approximate roots of the $f_i$’s. The statements however turn out to be a bit more technical, and we refer the reader to Theorem 5.7 for a precise description of $\bar{P}_\infty(X)$ for surfaces $X$ in a more general class $S_{\text{pol}}$, and to Corollary 6.3 for a version customized for $X_V$ for $V \in \mathcal{V}_1$.

Our results on Cox ring and global Enriques and Zariski semigroups are valid for a class $S_{\text{pol}}$ of surfaces which (strictly) contains all $X_V$, $V \in \mathcal{V}$. The class $S_{\text{pol}}$ appears naturally in the study of compactifications of $\mathbb{C}^2$. Indeed, if $X$ is a compactification of $\mathbb{C}^2$ then to each irreducible curve $\Gamma$ ‘at infinity’ on $X$ (i.e. $\Gamma$ is an irreducible component of $X_V \setminus \mathbb{C}^2$) one can associate a (finite) sequence of elements in $\mathbb{C}[x, x^{-1}, y]$ which describe the order of vanishing of rational functions along $\Gamma$; these are called key forms associated to $C$. The key forms are global variants of key polynomials of valuations introduced by MacLane [Mac36], and they can be used to determine effectively various properties of the associated (divisorial) valuation, see e.g. [Mon13a], [Mon13b], [MN13]. In particular, [Mon13b, Theorem 1.8] uses key forms to characterize a class $C$ of divisorial valuations on $\mathbb{C}(x, y)$ whose skewness (introduced in
can be ‘read from’ the slope of an extremal ray in the cone of curves on an associated compactification of $\mathbb{C}^2$. On the other hand, [CPR05] introduced the class $S_{\text{num}}$ of compactifications of $\mathbb{C}^2$ which admit systems of numerical curvettes. Theorem 3.2 shows that for a compactification $X$ of $\mathbb{C}^2$, $X \in S_{\text{num}}$ iff the divisorial valuation associated to each curve at infinity on $X$ is in $\mathcal{C}$; in particular it gives an effective algorithm to determine if $X \in S_{\text{num}}$.

The class $S_{\text{pol}}$, which is the central topic of this article, is the subset of $S_{\text{num}}$ for which the key forms associated to the curvettes at infinity are polynomials. In particular $S_{\text{pol}}$ contains all $X_V$, $V \in \mathcal{V}$ (Proposition 3.8). A geometric interpretation of the class $S_{\text{pol}}$ comes from the following observation: if $X$ is a compactification of $\mathbb{C}^2$ which dominates $\mathbb{P}^2$, then $X \in S_{\text{pol}}$ iff $X$ admits a system of numerical curvettes which ‘essentially come from affine curves’ (Theorem 3.4).

The organization of this article is follows: Section 2 and the appendix (Section 7) contains some background regarding key forms, semidegrees (which are negatives of divisorial valuations), and associated degree-wise Puiseux series. In Section 3 we introduce the classes $S^+_{\text{pol}} \subseteq S_{\text{pol}} \subseteq S_{\text{num}}$ of surfaces, characterize them in terms of associated ‘systems of curvettes’, and show that each $X_V$ for $V \in \mathcal{V}$ (resp. $V \in \mathcal{V}_1$) is a member of $S_{\text{pol}}$ (resp. $S^+_{\text{pol}}$). Theorem 4.5 in Section 4 gives an explicit description of Cox rings of surfaces in $S_{\text{pol}}$ and in Section 5 we describe global Enriques semigroups and global Zariski semigroups at infinity of surfaces in $S_{\text{pol}}$. Finally, in Section 6 we reformulate the results of Sections 4 and 5 for surfaces of the form $X_V$ for $V \in \mathcal{V}_1$ in terms of the approximate roots of the associated polynomial(s).

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2. Background

2.1. Semidegrees and degree-wise Puiseux series. Let $X$ be a normal complete algebraic surface containing $U := \mathbb{C}^2$, and $\Gamma$ be an irreducible curve at infinity, i.e. an irreducible component of $X \setminus U$. Let $\delta$ be the semidegree on $\mathbb{C}(X)$ determined by $\Gamma$, i.e. $\delta(f)$, where $f \in \mathbb{C}(X)$, is the order of pole of $f$ along $\Gamma$ (in other words, $\delta$ is the negative of the divisorial discrete valuation on $\mathbb{C}(X)$ associated to $\Gamma$). Choose coordinates $(x, y)$ on $U$ such that $\delta(x) > 0$ and let $\tilde{p} := \delta(x)$. Then there exists $\phi(x) \in \mathbb{C}[x^{1/\tilde{p}}, x^{-1/\tilde{p}}]$, and a rational number $r \in \mathbb{Q}$ such that $r < \text{ord}_x(\phi)$ and

$$\delta(f(x, y)) = \tilde{p} \deg_x \left( f(x, y) \big|_{y=\phi(x)+\xi x^r} \right)$$

for all $f \in \mathbb{C}(x, y)$, where $\xi$ is an indeterminate ([Mon11, Theorem 1.2]). We call $\tilde{\phi}(x) := \phi(x) + \xi x^r$ the generic degree-wise Puiseux series associated to $\delta$. The field of degree-wise Puiseux series in $x$ is

$$\mathbb{C}(\langle x \rangle) := \bigcup_{p=1}^{\infty} \mathbb{C}(x^{1/p}) = \left\{ \sum_{j \leq k} a x^{j/p} : k, p \in \mathbb{Z}, \ p \geq 1 \right\},$$

where for each integer $p \geq 1$, $\mathbb{C}(x^{1/p})$ denotes the field of Laurent series in $x^{1/p}$. We refer to Section 7 for some notions, e.g. conjugacy, Puiseux pairs, e.t.c. of degree-wise Puiseux
series. The usual factorization of polynomials in terms of Puiseux series (see e.g. [CA00, Section 1.5]) implies the following

**Theorem 2.1.** Let \( f \in \mathbb{C}[x,y] \). Then there are unique (up to conjugacy) degree-wise Puiseux series \( \phi_1, \ldots, \phi_k \), a unique non-negative integer \( m \) and \( c \in \mathbb{C}^* \) such that

\[
f = cx^m \prod_{i=1}^{k} (y - \psi_{ij}(x))
\]

where \( \psi_{ij} \) is a conjugate of \( \psi_i \).

The following proposition, which is a straightforward corollary of [Mon11, Proposition 4.2], illustrates a relation between degree-wise Puiseux factorization of polynomials and the behaviour at infinity of the curves they define:

**Proposition 2.2.** Let \( X \) be a normal algebraic surface containing \( \mathbb{C}^2 \). Let \( \Gamma_1, \ldots, \Gamma_N \) be the irreducible components of \( X \setminus \mathbb{C}^2 \) and \( \delta_j, 1 \leq j \leq N \), be the semidegree on \( \mathbb{C}(X) \) induced by \( \Gamma_j \). Choose coordinates \( (x,y) \) on \( \mathbb{C}^2 \) such that \( \delta_j(x) > 0 \) for each \( j \), \( 1 \leq j \leq N \). For each \( j \), \( 1 \leq j \leq N \), let \( \phi_j(x,\xi) := \phi_j(x) + \xi x_\infty \) be the generic degree-wise Puiseux series associated to \( \delta_j \). Let \( f \in \mathbb{C}[x,y] \) and \( C \) be the closure in \( X \) of the curve \( \{ f = 0 \} \subseteq \mathbb{C}^2 \).

1. Assume \( C \) is proper and \( C \cap \Gamma_j \cap \Gamma_k = \emptyset \) for all \( j \neq k \). Then the degree-wise Puiseux factorization of \( f \) is of the form

\[
f = c \prod_{i=1}^{N} \prod_{j=1}^{n_i} \prod_{k=1}^{\bar{q}_{ij}} (y - \psi_{ijk}(x))
\]

where \( c \in \mathbb{C}^* \), \( n_i := |C \cap \Gamma_i| \), and \( \psi_{ijk}'s \) are conjugates of some \( \psi_{ij} \) of the form

\[
\phi_i(x) + \xi_{ij} x_\infty + \text{l.d.t.}
\]

where \( \xi_{ij} \in \mathbb{C} \) and l.d.t. stands for ‘terms with lower degree’ in \( x \).

2. Assume in addition that each \( C \) intersects each \( \Gamma_j \) transversally (at points where both \( \Gamma_j \) and \( X \) are nonsingular). Then in (2), for all \( i,j \), \( \bar{q}_{ij} = \bar{p}_i \), and \( \psi_{ij} - \phi_i(x) - \xi_{ij} x_\infty \in \mathbb{C}((x^{-1}/\bar{p}_i)) \).

### 2.2. Key forms.

Let \( X, \Gamma, \delta \) and \( \tilde{\phi}(x,\xi) = \phi(x) + \xi x_\infty \) be as in Section 2.1. One can associate to \( \delta \) a finite sequence of elements in \( \mathbb{C}[x,x^{-1},y] \) called the **key forms** of \( \delta \) (see [Mon13a, Definition 3.17]). The sequence starts with \( f_0 := x, f_1 := y \), and for each \( n \geq 1 \), the \( (n + 1) \)-th key form \( f_{n+1} \) is an element in \( \mathbb{C}[f_0, f_0^{-1}, f_1, f_2, \ldots, f_n] \) whose \( \delta \)-value is smaller than the ‘expected’ value. An algorithm and detailed example for the computation of key forms of \( \delta \) from \( \phi \) and \( \omega \) appears in [Mon13a, Section 3.3].

### Example 2.3.

| \( \phi(x) + \xi x_\infty \) | key forms |
|-----------------------------|----------|
| \( \xi x^{p/q} \)           | \( x, y \) |
| \( cx^{p/q} + \xi x^q \), \( c \in \mathbb{C}^*, q > 0, \) | \( x, y, y^d - cx^p \) |
| \( p, q \) rel. prime integers, \( \omega < \frac{p}{q} \) | \( x, y, y^d - x^p \) |
| \( x^{5/2} + x^{-3/2} + \xi x^{-5/2} \) | \( x, y, y^d - x^3, y^d - x^5 - 2x \) |
| \( x^{5/2} + x^{-1} + x^{-3/2} + \xi x^{-5/2} \) | \( x, y, y^d - x^3, y^d - x^5 - 2x^4 y^2 - x^5 - 2x^4 y - 2x \) |

Theorem 2.4 below sums up the properties of key forms that we need. The first assertion of Theorem 2.4 follows from defining properties of key forms ([Mon13a, Proposition 3.28]), the
second assertion follows from [Mon13a, Proposition 4.2] and the third assertion is a corollary of [Mon13b, Theorem 1.4].

**Theorem 2.4.** Let the key forms of $\delta$ be $f_0, \ldots, f_n$.

1. Let $p$ be the polydromy order (see Definition 7.1) of $\phi(x)$. Then there is $\psi(x) \in \mathbb{C}((x^{-1/p}))$ such that:

$$\psi(x) = \phi(x) + \text{terms with } x\text{-degree } \leq r,$$

2. The following are equivalent:
   - (a) $f_k$ is a polynomial for all $k$, $0 \leq k \leq n$.
   - (b) $f_n$ is a polynomial.
   - (c) There exists a polynomial $f$ with degree-wise Puiseux factorization of the form:

$$f = \prod_{i=1}^{k} \prod_{j} (y - \psi_{i,j}(x)), \quad \text{where for each } i,$$

$$\psi_i = \phi(x) + \xi_i x^r + \text{l.d.t.}, \quad \xi_i \in \mathbb{C}.$$

3. The following are equivalent:
   - (a) $\delta(f) \geq 0$ for all $f \in \mathbb{C}[x,y] \setminus \{0\}$.
   - (b) $\delta(f_n) \geq 0$.
   - (c) $\delta(f) \geq 0$ for some $f \in \mathbb{C}[x,x^{-1},y]$ which satisfies (4).

**Proposition 2.5 ([Mon11, Propositions 4.2 and 4.7]).** Let $\delta$ be as in Theorem 2.4. Assume $\delta$ is not the usual degree in $(x,y)$ coordinates, Then there exists a unique compactification $X'$ of $\mathbb{C}^2$ such that

1. $X'$ is projective and normal.
2. $X'_\infty := X' \setminus \mathbb{C}^2$ has two irreducible components $C'_1, C'_2$.
3. The semidegree on $\mathbb{C}[x,y]$ corresponding to $C'_1$ and $C'_2$ are respectively deg and $\delta$.

All singularities of $X'$ are rational (which implies in particular that all Weil divisors are $\mathbb{Q}$-Cartier). Let $(C'_i, C'_j), 1 \leq i, j \leq 2$, denote the intersection number of $C'_i$ and $C'_j$. Then

1. $(C'_2, C'_2) < 0$,
2. $(C'_1, C'_i) = -q \delta(f_{i+1}), \quad \text{where } q \text{ is a positive rational number},$
3. Let $D'_2 := C'_1 - \frac{(C'_1, C'_1)}{(C'_1, C'_2)} C'_2$. Then $(C'_1, D'_2) = 0$ and $(C'_2, D'_2) = 1$.

### 3. Surfaces admitting systems of numerical (semi-affine) curvettes

**Definition 3.1.** [CPR05] introduced the notion of surfaces admitting systems of numerical curvettes. Here we extend the scope of the definition: Let $X$ be a normal complete algebraic surface containing $U := \mathbb{C}^2$ as a dense open subset, and let $\Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus U$. An effective Weil divisor $\Delta$ on $X$ is called a numerical $\Gamma_i$-curvette, $1 \leq i \leq N$, if $(\Delta, \Gamma_i') > 0$ and $(\Delta, \Gamma_j') = 0$ for all $j \neq i$. We say that $X$ admits a system of numerical curvettes, or equivalently, $X \in S_{\text{num}}$, iff there exists a numerical $\Gamma_i$-curvette for all $i, 1 \leq i \leq N$. 


Theorem 3.2. Let $X$ be as in Definition 3.1. For each $j$, $1 \leq j \leq N$, let $\delta_j$ be the semidegree associated to $\Gamma_j$ and $f_j$ be the last key form of $\delta_j$. Then the following are equivalent:

(a) $X$ admits a system of numerical curvettes.

(b) For each $j$, $1 \leq j \leq N$, either $\delta_j(f_j) \geq 0$ or $f_j$ is a polynomial.

Proof. Choose a system of coordinates $(x, y)$ on $\mathbb{C}^2 = U$ such that no $\delta_j$ is the degree in $(x, y)$-coordinates, and let $X_0 \cong \mathbb{P}^2$ be the usual compactification of $U$ given by $(x, y) \mapsto [x : y : 1]$. Let $\tilde{X}$ be the minimal normal surface containing $U$ which dominates both $X$ and $X_0$ (via morphisms induced by identification of $U$). Then $\tilde{X} \setminus U$ has $N + 1$ irreducible components $\tilde{\Gamma}_0, \ldots, \tilde{\Gamma}_N$, where $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_i$, $1 \leq i \leq N$) is the strict transform of the line at infinity on $X_0$ (resp. $\Gamma_i$). It follows that the semidegree associated to $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_i$, $1 \leq i \leq N$) is the degree in $(x, y)$-coordinates (resp. $\delta_j$). Let $\pi : \tilde{X} \rightarrow X$ be the morphism induced by identification of $U$.

(a) $\Rightarrow$ (b): Assume $X$ admits a system of numerical curvettes. Fix $j$, $1 \leq j \leq N$. Let $\Delta_j := D_j + \sum_{i=1}^{N} a_i \Gamma_i$ be a numerical $\Gamma_j$-curvette on $X$, where $D_j$ is an effective Weil divisor such that $\text{Supp}(D_j) \not\subseteq \Gamma_i$ for any $i$, and $a_i \geq 0$ for all $i$, $1 \leq i \leq N$. Let $\tilde{\Delta}_j := \pi^*(\Delta_j)$. Then $\tilde{\Delta}_j$ is of the form $\tilde{D}_j + \sum_{i=1}^{N} a_i \tilde{\Gamma}_i$, where $a_0 \geq 0$ and $\tilde{D}_j$ is the strict transform of $D_j$. Moreover, $\tilde{\Delta}_j$ is a numerical $\tilde{\Gamma}_j$-curvette on $\tilde{X}$. Let $X_j$ be the surface $X$ constructed in Proposition 2.5 with $\delta = \delta_j$. Let $\pi_j : \tilde{X} \rightarrow X_j$ be the natural birational morphism, and $\Gamma_j'$ be the image of $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_j$) on $X_j$ (note that $\Gamma_0' = C_1$ and $\Gamma_j' = C_2$ in the notation of Proposition 2.5). Set $\Delta_j' := \pi_j^*(\tilde{\Delta}_j) = D_j' + a_0 \Gamma_0' + a_j \Gamma_j'$, where $D_j' := \pi_j^*(\tilde{D}_j)$.

Claim 3.2.1. $\Delta_j' \neq 0$. Moreover, $\Delta_j = \pi_j^*(\Delta_j')$.

Proof. Indeed, the only way $\Delta_j'$ can be zero is if $a_0 = a_j = 0$ and $\tilde{D}_j = 0$. But then we would have $(\tilde{\Delta}_j, \tilde{\Gamma}_0) = 0$. Since the intersection matrix of $\bigcup_{1 \leq i \leq N} \tilde{\Gamma}_i$ is negative definite, this would imply that $\tilde{\Delta}_j = 0$, which is a contradiction. This proves the first assertion. The second assertion follows from the fact that the coefficients of exceptional curves in the pullback of a divisor are uniquely determined by the requirement that the intersection number of the pullback with every exceptional curve is zero.

W.l.o.g. we may assume $\delta_j(f_j) < 0$. Then Proposition 2.5 implies that $(\Gamma_0', \Gamma_0') > 0$. Since $(\tilde{\Delta}_j, \tilde{\Gamma}_0) = 0$, it follows that

\[(D_j', \Gamma_0') = (D_j', \Gamma_0') + a_0(\Gamma_0', \Gamma_0') + a_j(\Gamma_j', \Gamma_0') = 0.
\]

Since $(\Gamma_0', \Gamma_0')$ and $(\Gamma_j', \Gamma_0')$ are positive, identity (5) implies that $a_0 = a_j = (D_j', \Gamma_0') = 0$. Let $f$ be the polynomial in $\mathbb{C}[x, y]$ that defines $D_j' \cap \mathbb{C}^2$. Since $D_j'$ does not intersect $\Gamma_0'$, it follows that $f$ satisfies identity (4) for $\delta = \delta_j$. Theorem 2.4 then implies that $f_j$ is a polynomial, as required to complete the proof that (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Fix $j$, $1 \leq j \leq N$. Let $\pi_j : \tilde{X} \rightarrow X_j$ be as above. We claim that there is a numerical $\Gamma_j'$-curvette $D_j'$ on $X_j$. Indeed, if $\delta_j(f_j) \geq 0$, the existence of $D_j'$ follows from Proposition 2.5. On the other hand, if $f_j$ is a polynomial, then set $D_j'$ to be the closure in $X_j$ of the curve defined by $f_j$ on $\mathbb{C}^2$. Proposition 2.2 then implies that $D_j'$ is a numerical $\Gamma_j'$-curvette, which proves the claim. Now note that $\pi_*(\pi_j^*(D_j'))$ is a numerical $\Gamma_j$-curvette on $\tilde{X}$, which completes the proof of the theorem. \qed
**Definition 3.3.** Let $X$ be a normal complete algebraic surface such that

1. $X$ contains $U := \mathbb{C}^2$ as a dense open subset, and
2. the identity map of $U = \mathbb{C}^2$ extends to a morphism $X \to X_0 := \mathbb{P}^2$.

Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus U$, where $\Gamma_0$ is the strict transform of the line at infinity on $X_0$. An effective Weil divisor $\Delta$ on $X$ is called a **numerical semi-affine $\Gamma_i$-curvette**, $1 \leq i \leq N$, if

1. $\Delta$ is a numerical $\Gamma_i$-curvette,
2. $\text{Supp}(\Delta)$ contains neither $\Gamma_0$ nor $\Gamma_i$, and
3. $\text{Supp}(\Delta_i) \cap U \neq \emptyset$.

We say that $X$ admits a system of numerical semi-affine curvettes if there exists a numerical semi-affine $\Gamma_i$-curvette for all $i$, $1 \leq i \leq N$.

**Theorem 3.4.** Let $X$ be as in Definition 3.3. For each $j$, $1 \leq j \leq N$, let $\delta_j$ be the semidegree associated to $\Gamma_j$ and $f_j$ be the last key form of $\delta_j$. Then the following are equivalent:

(a) $X$ admits a system of numerical semi-affine curvettes which are Cartier divisors.
(b) $X$ admits a system of numerical semi-affine curvettes.
(c) For each $j$, $1 \leq j \leq N$, the last key form of $\delta_j$ is a polynomial.
(d) For each $j$, $1 \leq j \leq N$, all the key forms of $\delta_j$ are polynomials.

**Proof.** (b) $\Rightarrow$ (c): Fix $j$, $1 \leq j \leq N$. Let $X_j$ be as in the proof of Theorem 3.2, $\pi_j : X \to X_j$ be the natural morphism, $\Gamma'_0 := \pi_j(\Gamma_0)$ and $\Gamma'_j := \pi_j(\Gamma_j)$. Let $\Delta_j := D_j + \sum_{i=0}^{N} a_i \Gamma_i$ be a numerical semi-affine $\Gamma_j$-curvette on $X$, where $D_j$ is a non-zero effective Weil divisor such that $\text{Supp}(D_j) \not\supseteq \Gamma_i$ for any $i$, $a_0 = a_j = 0$, and $a_i \geq 0$, $0 \leq i \leq N$. Set $\Delta'_j := \pi_j(\Delta_j) = D'_j$, where $D'_j := \pi_j(D_j)$. It follows that $(D'_j, \Gamma'_0) = (D_j, \Gamma_0) = 0$. Let $f$ be the polynomial in $\mathbb{C}[x, y]$ that defines $D'_j \cap \mathbb{C}^2$. The same arguments as in the paragraph following the proof of Claim 3.2.1 then show that the last key form of $\delta_j$ is a polynomial, as required.

(c) $\Rightarrow$ (a): Fix $j$, $1 \leq j \leq N$. Let $\pi_j : X \to X_j$ be as above. Let $D'_j$ be the closure in $X_j$ of the curve defined by the last key form of $\delta_j$ on $U = \mathbb{C}^2$. Theorem 2.4 implies that $D'_j \cap \Gamma'_0 = \emptyset$, so that $D'_j$ is a numerical semi-affine $\Gamma'_j$-curvette $D'_j$ on $X_j$. Moreover, it follows from Proposition 2.5 that for some $n \geq 1$, $nD'_j$ is a Cartier divisor. Then $\pi_j^*(nD'_j)$ is a Cartier divisor on $X$ which is also a numerical semi-affine $\Gamma_j$-curvette, as required.

The equivalence (d) $\iff$ (c) follows from Theorem 2.4. Since the implication (a) $\Rightarrow$ (b) is obvious, the proof of Theorem 3.4 is complete.

We are now ready to define the central object of this article.

**Definition 3.5.** We denote by $S_{\text{pol}}$ the collection of normal complete algebraic surfaces $X$ which satisfy the following properties:

1. $X$ contains $U := \mathbb{C}^2$ as an open subset,
2. for each irreducible curve $\Gamma \subseteq X \setminus U$, all key forms associated to (the semidegree corresponding to) $\Gamma$ are polynomials (i.e. regular functions on $U$).
3. $S_{\text{pol}}^+$ is the subcollection of $S_{\text{pol}}$ consisting of $X \in S_{\text{pol}}$ which satisfies in addition:

   (3) For each semidegree $\delta$ on $\mathbb{C}[U] = \mathbb{C}[x, y]$ associated to some irreducible curve on $X \setminus U$, $\delta(f) \geq 0$ for all $f \in \mathbb{C}[x, y] \setminus \{0\}$ (by Theorem 2.4 this is equivalent to saying that $\delta(f_{\delta}) \geq 0$, where $f_{\delta}$ is the last key form for $\delta$).
Remark 3.6. If $X$ is a complete rational surface which dominates $\mathbb{P}^2$ via a birational morphism, then it follows from Theorem 3.4 that $X \in S_{\text{pol}}$ iff $X$ admits a system of numerical semi-affine curvettes.

Remark 3.7. It is not hard to see (e.g. follows from Theorem 4.5 below) that every surface in $S_{\text{pol}}$ is in fact projective.

Now we describe an important subclass of $S_{\text{pol}}$. Recall that in the introduction we considered collections $V$ of pencils $V := \mathbb{C}\langle F, H^d \rangle$ where

1. $H$ is the equation of a straight line $L$, and
2. $F = F_1^{a_1} \cdots F_k^{a_k}$, where each $F_j$, $1 \leq j \leq k$, defines a curve with one place at infinity, and $\sum_{j=1}^{k} a_j \deg(F_j) = d$.

For a $V \in V$, recall that $X_V$ denotes the surface resulting from (minimal) resolutions of basepoints of $V$ on $\mathbb{P}^2$. Also recall that $V_1$ is the subset of $V$ consisting of pencils for which $F$ itself defines an irreducible curve with one place at infinity.

Proposition 3.8.

1. $X_V \in S_{\text{pol}}$ for each $V \in V$.
2. If $V \in V_1$, then $X_V \in S_{\text{pol}}^+$.

Proof. Identify $U := \mathbb{C}^2$ with $\mathbb{P}^2 \setminus L$. Denote the irreducible components of $X_V \setminus U$ by $\Gamma_0, \ldots, \Gamma_N$, where $\Gamma_0$ is the strict transform of $L$. For each $i$, $0 \leq i \leq N$, let $\delta_i$ be the associated semidegree on $\mathbb{C}[x,y] := \mathbb{C}[U]$. At first assume $V \in V_1$, i.e. $F$ defines a curve with one place at infinity. In that case dual graph of the irreducible components of $V$ is of the form as in Figure 1.

![Figure 1. Dual graph of curves at infinity on $X_V$ for $V \in V_1$](image)

It follows from a result of Abhyankar and Moh that there are polynomials $f_1, \ldots, f_s, f_{s+1} = F|_{\mathbb{C}^2}$, (where $s$ is the number of ‘vertical segments’ in Figure 1), such that for each $j$, $1 \leq j \leq s + 1$, the closure $C_j$ in $X_V$ of the curve $\{f_j = 0\} \subseteq \mathbb{C}^2$ is a numerical $\Gamma_j$-curvette. Fix $j$, $1 \leq j \leq s + 1$, and define

$[\Gamma_j', \Gamma_j] := \begin{cases} \text{‘L-shaped segment’ between (and including) } \Gamma_j' \text{ and } \Gamma_j & \text{if } 1 \leq j \leq s, \\ \text{‘horizontal segment’ between (and including) } \Gamma_{j+1}' \text{ and } \Gamma_{j+1} & \text{if } j = s + 1. \end{cases}$

It follows that for each $l$ such that $\Gamma_l \in [\Gamma_j', \Gamma_j]$, identity (4) is satisfied with $f = f_j$ and $\delta = \delta_l$, and therefore all the key forms of $\delta_l$ are polynomials (via assertion 2 of Theorem 2.4). Moreover, it is not hard to compute that $\delta_l(f_j) \geq \delta_i(f_j) \geq 0$, which implies that $\delta_l$ is
non-negative on all non-zero polynomials (via assertion 3 of Theorem 2.4). This proves that $X_V \in S_{pol}^+$.}

Now consider the general case that $F = F_1^{a_1} \cdots F_k^{a_k}$ for $k \geq 1$ and positive integers $a_1, \ldots, a_k$. Then for each $i$, $0 \leq i \leq N$, one of the following (mutually exclusive) cases holds (see e.g. [CPR05, Proof of Theorem 2]):

Case 1. $\Gamma_i$ is the strict transforms of some irreducible curve on $X_{V_j} \setminus U$, $1 \leq j \leq k$, where $V_j$ is the pencil generated by $F_j$ and $H^{\deg(F_j)}$.

Case 2. There exists $j$, $1 \leq j \leq k$, such that $-\delta_i$ is a monomial valuation centered at $O_j$, where $O_j$ is the (unique) point at infinity on the closure $C_j$ in $X_{V_j}$ of the curve $F_j|_U = 0$. Moreover,

(a) $O_j$ is the point of intersection of $C_j$ and the ‘last exceptional curve’ $\Gamma_{j,N_j}$ on $X_{V_j}$ (i.e. $\Gamma_{j,N_j}$ is the exceptional divisor of the last blow up performed in the resolution of the pencil $V_j$).

(b) $O_j$ does not belong to any irreducible component of $X_{V_j} \setminus U$ other than $\Gamma_{j,N_j}$.

In Case 1, we have already seen that the key forms corresponding to $\delta_i$ are polynomials. So assume Case 2 holds. But then it can be shown that there exists a numerical $\Gamma_i$-curvettes on $X$ (see [CPR05, Proof of Theorem 2]). Let $\delta_{j,N_j}$ be the semidegree on $\mathbb{C}[x,y]$ associated to $\Gamma_{j,N_j}$. It follows from the construction of $X_{V_j}$ that $f_j := F_j|_U$ is the last key form of $\delta_{j,N_j}$ and $\delta_{j,N_j}(f_j) = 0$. Property 2b then implies that $\delta_i$-value of the last key form of $\delta_i$ is negative. Theorem 3.2 then implies that the last key form of $\delta_i$ is a polynomial. It follows from Theorem 2.4 that all key forms of $\delta_i$ are polynomials, as required to complete the proof of the proposition.

\[\square\]

4. Cox ring of $X \in S_{pol}$

Notation 4.0. Throughout the rest of this article we denote by $X$ a surface from the class $S_{pol}$. Let $\Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus \mathbb{C}^2$, and $\delta_i$, $1 \leq i \leq N$, be the order of pole along $\Gamma_i$. Choose coordinates $(x, y)$ on $\mathbb{C}^2$ such that $\delta_i(x) > 0$ for each $i$. Let $\tilde{\phi}_i(x, \xi) := \phi_i(x) + \xi x^{\gamma_i}$, $1 \leq i \leq N$, be the generic degree-wise Puiseux series of $\delta_i$. Fix $i$, $1 \leq i \leq N$. Let the formal Puiseux pairs (Definition 7.2) of $\phi_i$ be $(q_{i,1}, p_{i,1}), \ldots, (q_{i,l_i+1}, p_{i,l_i+1})$. Recall that the characteristic exponents (Definition 7.2) of $\phi_i$ are $r_{i,j} := \frac{q_{i,j}}{p_{i,j}}$, $1 \leq j \leq l_i + 1$. For each $j$, $0 \leq j \leq l_i$, consider the semidegree $\delta_{ij}$ with generic degree-wise Puiseux series $\tilde{\phi}_{ij}(x, \xi) := [\phi_i(x)]_{r_{i,j+1}} + \xi x^{r_{i,j+1}}$, where $[\phi_i(x)]_{r_{i,j+1}}$ denotes the sum of all terms of $\phi_i$ with degree (in $x$) greater than $r_{i,j+1}$.

Note that $\delta_{ij} = \delta_i$. For each $j$, $0 \leq j \leq l_i$, let $f_{ij}$ be the last key form of $\delta_{ij}$. The following lemma records some properties of $f_{ij}$’s. It is an immediate corollary of Theorem 2.4.

Lemma 4.1. Fix $i, j$, $1 \leq i \leq N$, $0 \leq j \leq l_i$. Then

1. $f_{ij} = 0$ is a curve with one place at infinity.

2. $f_{ij}$ has a degree-wise Puiseux root $\phi_{ij}$ (which is unique up to conjugacy) such that

(a) Puiseux pairs of $\phi_{ij}$ are $(q_{i,1}, p_{i,1}), \ldots, (q_{i,j}, p_{i,j})$, and

(b) $\phi_{ij} = [\phi_i(x)]_{r_{i,j+1}} + \text{terms with degree (in } x\text{) less than or equal to } r_{i,j+1}$.\[\square\]

2Even though [CPR05, Theorem 2] assumes that $\gcd(a_1, \ldots, a_k) = 1$, the proof for existence of systems of numerical curvettes on $X_V$ does not use this assumption.
Remark 4.2. Assume that $X = X_V$ for some $V = \mathbb{C}(F, H^d) \in V$, where $F := F_1^{a_1} \cdots F_k^{a_k}$ such that $f_j := F_j|_{C^2}$ defines a curve with one place at infinity for each $j$, $1 \leq j \leq k$. Let \( \{g_{\nu,j}\}_{\nu} \) be the approximate roots [AM73] of $f_{\nu}$, $1 \leq \nu \leq k$. Then \( \{g_{\nu,j}\}_{\nu} \) satisfies the properties of \( \{f_{ij}\}_{i,j} \) of Lemma 4.1, i.e. for each $i, j$, $1 \leq i \leq N$, $0 \leq j \leq l_i$, there exists $i', j'$ such that properties $1_{(ij)}$ and $2_{(ij)}$ of Lemma 4.1 hold with $f_{ij} := g_{i',j'}$.

Definition 4.3. The Cox ring, or the total coordinate ring, of $X$ is

\[ \mathcal{R}(X) := \bigoplus_{(d_1, \ldots, d_N) \in \mathbb{Z}^N} \Gamma \left( X, \mathcal{O} \left( \sum d_j \Gamma_j \right) \right) \]

For $d := (d_1, \ldots, d_N) \in \mathbb{Z}^N$ and $f \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$, we denote by $(f)_d$ the ‘copy’ of $f$ in the $d$th graded component of $\mathcal{R}(X)$. Let $\vec{e}_j$ be the $j$-th unit vector in $\mathbb{Z}^N$. Moreover, for $f \in \mathcal{C}(x, y)$ we denote by $\vec{d}(f)$ the element $(\delta_1(f), \ldots, \delta_N(f)) \in \mathbb{Z}^N$.

Convention 4.4. For each $d := (d_1, \ldots, d_N) \in \mathbb{Z}^N$ and $f \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$, we identify $f$ with $f|_{C^2} \in \mathcal{C}[x, y]$.

Theorem 4.5. Pick any collection of (not necessarily pairwise distinct) polynomials $f_{ij}$, $1 \leq i \leq N$, $0 \leq j \leq l_i$, which satisfy properties $1_{(ij)}$ and $2_{(ij)}$ of Lemma 4.1. Let $N' := \sum_{i=0}^{N} (l_i+1)$. For all $\gamma := (\gamma_{ij})_{i,j} \in \mathbb{Z}_0^{N'}$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, define $g_{\alpha,\beta,\gamma} := x^\alpha y^\beta \prod_{i,j} f_{ij}^{\gamma_{ij}} \in \mathcal{C}(x, y)$.

1. $(f_{ij})_{\gamma_{ij}}$’s together with $(x)_\delta(x), (y)_\delta(y)$ and $(1)_{\vec{e}_j}$, $1 \leq j \leq N$, generate $\mathcal{R}(X)$ as a $\mathbb{C}$-algebra.
2. Let $\vec{d} := (d_1, \ldots, d_N) \in \mathbb{Z}^N$. Let $\mathcal{E}_{\vec{d}}$ be the collection of all $(\alpha, \beta, \gamma) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{N'}$ such that

\[ \alpha \delta_k(x) + \beta \delta_k(y) + \sum_{i,j} \gamma_{ij} \delta_k(f_{ij}) \leq d_k \]

is satisfied for all $k, 1 \leq k \leq N$. Then $\Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$ is a generated as a vector space over $\mathbb{C}$ by \{ $g_{\alpha,\beta,\gamma} : (\alpha, \beta, \gamma) \in \mathcal{E}_{\vec{d}}$ \}.

3. Let $\vec{d} := (d_1, \ldots, d_N) \in \mathbb{Z}^N$. Define

\[ \mathcal{E}'_{\vec{d}} := \{(a, b) \in \mathbb{Z}_{\geq 0}^2 : b = \beta + \sum_{i,j} \gamma_{ij} \deg_y(f_{ij}) = \deg_y(g_{\alpha,\beta,\gamma}) \}

for some $\beta \in \mathbb{Z}_{\geq 0}$ and $\gamma := (\gamma_{ij})_{i,j} \in \mathbb{Z}_{\geq 0}^{N'}$ such that $(\alpha, \beta, \gamma) \in \mathcal{E}_{\vec{d}}$. Then $\dim \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j)) = |\mathcal{E}_{\vec{d}}|$. For each $(a, b) \in \mathcal{E}'_{\vec{d}}$ pick any $\beta \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \mathbb{Z}_{\geq 0}^{N'}$ such that $(a, \beta, \gamma) \in \mathcal{E}_{\vec{d}}$ and $b = \deg_y(g_{\alpha,\beta,\gamma})$, and define $g_{a,b} := g_{a,\beta,\gamma}$. Then \{ $g_{a,b} : (a, b) \in \mathcal{E}'_{\vec{d}}$ \} is a basis of $\Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$.

Proof. Note that for all $(\alpha, \beta, \gamma) \in \mathcal{E}_{\vec{d}}$, we have

\[ \vec{d}(g_{\alpha,\beta,\gamma}) = \vec{d}(x^\alpha y^\beta \prod_{i,j} f_{ij}^{\gamma_{ij}}) = \alpha \vec{d}(f) + \beta \vec{d}(y) + \sum_{i,j} \vec{d}(f_{ij}) \leq \vec{d} \]

(where “$\leq$” denotes coordinate-wise inequality). In particular,

\[ (\alpha, \beta, \gamma) \in \mathcal{E}_{\vec{d}} \iff g_{\alpha,\beta,\gamma} \in \Gamma \left( X, \mathcal{O}(\sum d_j \Gamma_j) \right). \]
This implies that assertion 1 follows from assertion 2. Moreover, assertion 3 also is a corollary of assertion 2. Therefore it suffices to prove assertion 2.

Pick \( \vec{d} \in \mathbb{Z}^N \) and \( h \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j)) \). We will show by induction on \( \deg_y(h) \) that \( h \) belongs to the vector space generated by \( G_{\vec{d}} := \{ g_{\alpha \beta \gamma} : (\alpha, \beta, \gamma) \in \mathcal{E}_{\vec{d}} \} \). Indeed, if \( \deg_y(h) = 0 \), then it is clear. So assume \( \deg_y(h) \geq 1 \), and consider the degree-wise Puiseux factorization of \( h \):

\[
h = cz^m \prod_{i=1}^M \prod (y - \psi_{ij}(x)).
\]

Fix \( i, 1 \leq i \leq M \). For each \( j, j', 1 \leq j \leq N, 1 \leq j' \leq d_i^* \) (where \( d_i^* \) is the number of conjugates of \( \psi_i \)), define

\[
s_{ijj'} := \deg_x(\psi_{ij}(x) - \tilde{\phi}_j(x, \xi)) = \deg_x(\psi_{ij}(x) - \phi_j(x) - \xi x^{r_j}),
\]

\[
s_i := \min\{s_{ijj'}\}_{j', j}.
\]

Pick \( j_i, j_i \) such that \( s_i = s_{ij_i j_i} \). We may assume w.l.o.g. that \( j_i = 1 \). Let

\[
\psi_i^* := [\psi_{ij_1}(x)]_{s_i} = [\phi_{j_i}(x)]_{s_i}.
\]

For each \( k, 0 \leq k \leq l_{j_i} \), let \( \phi_{j_ik} \) be a degree-wise Puiseux root of \( f_{j_ik} \) (recall that \( f_{j_ik} \)'s satisfy properties 1(iii) and 2(iii) of Lemma 4.1). Since \( s_i \geq r_{j_i} \), it follows that if \( \psi_i^* \neq 0 \), then there is a unique \( k_i, 0 \leq k_i \leq l_{j_i} \), such that

(a) \( \phi_{j_ik} \) has the same Puiseux pairs as \( \psi_i^* \), and
(b) a conjugate of \( \phi_{j_ik} \) is of the form \( \psi_i^* + \) terms with degree less than or equal to \( s_i \).

Define

\[
h_i := \prod_{\psi_{ik} \text{ is a conjugate of } \psi_i} (y - \psi_{ik}(x)) \in \mathbb{C}(\langle x \rangle)[y].
\]

Note that \( h = \prod_{i=1}^M h_i \) and \( \deg_y(h_i) = \) number of conjugates of \( \psi_i \). Let

\[
m_i := \begin{cases} \deg_y(h_i), & \text{if } \psi_i^* = 0, \\ \text{number of conjugates of } \psi_i, & \text{otherwise.} \end{cases}
\]

\[
h_i^* := \begin{cases} y^{m_i}, & \text{if } \psi_i^* = 0, \\ f_{j_ik_i}^{m_i}, & \text{otherwise.} \end{cases}
\]

Note that \( m_i \) is a positive integer and \( h_i^* \in \mathbb{C}[x, y] \). Define

\[
h^* := cz^m \prod_{i=1}^M h_i^* \in \mathbb{C}[x, y].
\]

**Claim 4.5.1.** \( \delta_j(h^*) \leq \delta_j(h) \) for each \( j, 1 \leq j \leq N \).

**Proof.** Fix \( i, j, 1 \leq i \leq M, 1 \leq j \leq N \). It suffices to show that \( \delta_j(h_i^*) \leq \delta_j(h_i) \). As in (1),

\[
\delta_j(h_i) = \delta_j(x) \deg_x \left( h_i(x, y)|_{y=\phi_j(x, \xi)} \right)
\]

\[
= \delta_j(x) \deg_x \left( \prod_{k=1}^{d_i^*} (\tilde{\phi}_j(x, \xi) - \psi_{ik}(x)) \right)
\]

\[
= \delta_j(x) \sum_{k=1}^{d_i^*} s_{ikj}.
\]
where $d_i^* := \deg_y(h_i) = \deg_y(h_i^*)$. If $\psi_i^* = 0$, then $s_{ikj} \geq \deg_x(\tilde{\phi}_j(x, \xi))$ for all $k$, which implies that

$$\delta_j(h_i) \geq \delta_j(x)d_i^* \deg_x(\tilde{\phi}_j(x, \xi)) = d_i^*\delta_j(y) = \delta_j(y^{d_i^*}) = \delta_j(h_i^*),$$

as required to prove the claim. So assume $\psi_i^* \neq 0$. Define

$$s_{ij} := \min\{s_{ij'j}j'\},$$

$$s'_{ij} := \min\{\deg_x(\tilde{\phi}_j(x, \xi) - \phi(x)) : \phi \text{ is a conjugate of } \phi_{j, ki}\}.$$

Claim 4.5.2 and Lemma 7.6 below then imply that $\delta_j(h_i) \geq \delta_j(h_i^*)$, as required to prove Claim 4.5.1.

\[\square\]

**Claim 4.5.2.** $s_{ij} \geq s'_{ij}$.

\[\text{Proof.}\] Indeed, replacing $\phi_j$ by one of its conjugates if necessary, we may assume that $s_{ij} = s_{i1j}$. Similarly, property (b) of $\phi_{j, ki}$ implies that replacing $\phi_{j, ki}$ by one of its conjugates if necessary, we may assume that $\phi_{j, ki}$ is of the form $[\psi_{11}(x)]_{s_i} + \text{terms with degree less than or equal to } s_i$. Since $s_{ij} \geq s_i$, this implies that $\deg_x(\tilde{\phi}_j(x, \xi) - \phi_{j, ki}(x)) \leq s_{ij}$, as required. \[\square\]

Claim 4.5.1 implies that $h^* \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$, so that $h - h^*$ is also an element of $\Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$. Since $\deg_y(h - h^*) < \deg_y(h)$, the inductive hypothesis implies that $h - h^*$ is in the vector space generated by $\mathcal{G}_{d^*}$. Now note that by construction $h^* = e_{\alpha\beta\gamma}$ for some $\alpha, \beta, \gamma$. Claim 4.5.1 and (6) then imply that $g_{\alpha\beta\gamma} \in \mathcal{G}_{d^*}$ and complete the proof of the theorem. \[\square\]

5. **Global Enriques and Zariski Semigroups of Divisors on $X \in S_{pol}$**

Global Enriques and Zariski semigroups of line bundles on surfaces of the form $X_V$, where $V$ is a pencil at infinity, were introduced in \cite{CPR1}. In this section we adapt these notions for surfaces in $S_{pol}$ and then compute these semigroups. We continue with the set up introduced in Notation 4.0.

**Definition 5.1.** Recall that the divisor class group $\text{Cl}(X)$ (of Weil divisors on $X$ modulo linear equivalence) is isomorphic to $\mathbb{Z}^N$ and generated by $\Gamma_1, \ldots, \Gamma_N$. The global Enriques semigroup $P^{st}(X)$ of $X$ consists of all $D \in \text{Cl}(X)$ such that $D$ is linearly equivalent to an effective Weil divisor whose support does not contain any of the $\Gamma_j$’s. The global Zariski semigroup, or the characteristic semigroup of $X$ is the semigroup $\hat{P}(X)$ generated by all divisors in $\text{Cl}(X)$ which are base point free. The Zariski semigroup at infinity $\hat{P}_{\infty}$ is generated by all divisors in $\text{Cl}(X)$ which have no base point ‘at infinity’, i.e. the base locus consists of finitely many points on $\mathbb{C}^2$.

**Remark 5.2.** If $X \in S_{pol}^+$ (e.g. if $X = X_V$ for some $V \in V_1$) then $\hat{P}_{\infty} = \hat{P}$.

For $\bar{d} \in \mathbb{Z}^N$, we denote by $\Gamma_{\bar{d}}$ the divisor $\sum_j d_j \Gamma_j \in \text{Cl}(X)$. Theorem 4.5 gives an immediate characterization of those $\bar{d}$ for which $\Gamma_{\bar{d}} \in P^{st}(X)$.
Corollary 5.3. Let $f_{ij}, 1 \leq i \leq N, 0 \leq j \leq l_i$, be as in Theorem 4.5. For each $\mathbf{d} \in \mathbb{Z}^N$ and each $j, 1 \leq j \leq N$, define:

$$M_{\mathbf{d},j} := \max \{ \delta_j(g_{\alpha \beta \gamma}) : (\alpha, \beta, \gamma) \in \mathcal{E}_\mathbf{d} \}$$

$$= \max \left\{ \alpha \delta_j(x) + \beta \delta_j(y) + \sum_{i,j} \gamma_{ij} \delta_j(f_{ij}) : (\alpha, \beta, \gamma) \in \mathcal{E}_\mathbf{d} \right\}.$$

Then $\Gamma_{\mathbf{d}} \in \mathbb{P}^{st}(X)$ iff $d_j = M_{\mathbf{d},j}$ for all $j, 1 \leq j \leq N$. □

Definition 5.4. Fix $i, 1 \leq j \leq N$. Let $\delta_i := \delta_i/\delta_i(x)$ be the ‘normalized’ version of $\delta_i$. Recall that the formal Puiseux pairs (Definition 7.2) of $\delta_i$ are $(q_{i,1}, p_{i,1}), \ldots, (q_{i,l_i+1}, p_{i,l_i+1})$. Define:

$$p_i := p_{i,1} \cdots p_{i,l_i},$$

$$\tilde{p}_i := \delta_i(x) = p_{i,l_i+1} = p_{i,1} \cdots p_{i,l_i+1}.$$ 

Let $f_{ij}, 1 \leq i \leq N, 0 \leq j \leq l_i$, as in Theorem 4.5. For each $i, j$, let $\phi_{ij}$ be a degree-wise Puiseux root of $f_{ij}$. Fix $i, 1 \leq i \leq N$, and define $\mathcal{N}_i := \{ i' \colon \tilde{\delta}_{i'} \leq \tilde{\delta}_i \}$. Recall (Property 2$_{(ij)}$ implies that $\tilde{f}_{il}$ has a generic degree-wise Puiseux root $\tilde{\phi}_{il}(x)$ such that

$$\phi_{il} = [\phi_i(x)]_{r_i} + \text{terms with degree (in } x \text{) less than or equal to } r_i.$$

For each $i' \in \mathcal{N}_i$, (replacing $\phi_{i'}$ by a conjugate if necessary we may assume that) there exists $c_{i'} \in \mathbb{C}$ such that

$$\phi_{i'l_{i'}} = [\phi_i(x)]_{r_i} + c_{i'} x^{r_i} + \text{l.d.t.} \quad \quad (7)$$

Note that $c_{i'}$ is unique only up to a multiplication by a $p_{i,l_i+1}$-th root of unity. For each $c \in \mathbb{C}$, define

$$\mathcal{N}_{i,c} := \left\{ i' \in \mathcal{N}_i \setminus \{ i \} : c^{p_{i,l_i+1}} = c \right\}.$$ 

Then $\mathcal{N}_i = \bigcup_{c \in \mathbb{C}} \mathcal{N}_{i,c} \cup \{ i \}$. For convenience we record the following lemma which is a direct consequence of Lemma 4.1:

Lemma 5.5. If $c_{ii} \neq 0$, then $p_{i,l_i+1} = 1$. In particular, identity (7) for $i := i'$ uniquely determines $c_{ii}$. □

Remark-Definition 5.6. For each $f \in \mathbb{C}[x,y] \setminus \{ 0 \}$, and each $i, 1 \leq i \leq N$, we write $\text{LC}_i(f)$ for the coefficient of $y^{\delta_i}$ in $f|_{y=\tilde{\phi}_i(x)}$. Note that $\text{LC}_i(f)$ can be factored as $\xi^r h(\xi^{p_{i,l_i+1}})$, where

1. $r$ is the number of degree-wise Puiseux roots $\psi(x)$ of $f$ such that $\text{deg}_x(\phi_i - \psi(x)) < r_i$,
2. $h$ is a polynomial in one variable,
3. $p_{i,l_i+1} \text{deg}(h)$ is the number of degree-wise Puiseux roots $\psi(x)$ of $f$ such that $\text{deg}_x(\phi_i - \psi(x)) = r_i$. 

Theorem 5.7. For $\vec{a} := (a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geq 0}$, define

$$d_i(\vec{a}) := \sum_{j=1}^N a_j \omega_{ij}, \quad 1 \leq i \leq N,$$

where

$$\omega_{ij} := \begin{cases} p_{j,l} t_{j+1} \delta_{i}(f_{jl}) & \text{if } i \notin N_j, \\ p_{j,l} t_{j+1} \tilde{p}_{i} \tilde{\delta}_{j}(f_{jl}) & \text{if } i \in N_j. \end{cases}$$

$\vec{d}_a := (d_1(\vec{a}), \ldots, d_N(\vec{a}))$,

$$m_i(\vec{a}) := \sum_{\nu \in \mathbb{N}_i} \frac{p_{i}}{\tilde{p}_{\nu} a_{i,\nu}}, \quad 1 \leq i \leq N,$$

$$m_{i,c}(\vec{a}) := \sum_{\nu \in \mathbb{N}_{i,c}} \frac{p_{i}}{\tilde{p}_{\nu} a_{i,\nu}}, \quad c \in \mathbb{C}, \quad 1 \leq i \leq N.$$

Let $f_{ij}$’s be as in Theorem 4.5. Then for each $\vec{d} \in \mathbb{Z}^N$, the following are equivalent:

(a) $\mathcal{O}(\sum d_j \Gamma_j)$ has no base point at infinity,

(b) $\vec{d} = \vec{d}_a$ for some $\vec{a} \in \mathbb{Z}^N_{\geq 0}$ such that for each $i, 1 \leq i \leq N$,

\begin{align*}
(8) & \quad d_i(\vec{a}) = \max \left\{ \delta_i \left( g_{a_\beta \gamma} \right) : (\alpha, \beta, \gamma) \in \mathcal{E}_{d_a} \right\}, \\
(9) & \quad m_i(\vec{a}) = \max \left\{ \deg_{c_i} \left( L c_i \left( g_{a_\beta \gamma} \right) \right) : (\alpha, \beta, \gamma) \in \mathcal{E}_{d_a}, \delta_i(g_{a_\beta \gamma}) = d_i(\vec{a}) \right\}, \\
(10) & \quad m_{i,c}(\vec{a}) = \min \left\{ \ord_{c_i - c} \left( L c_i \left( g_{a_\beta \gamma} \right) \right) : (\alpha, \beta, \gamma) \in \mathcal{E}_{d_a}, \delta_i(g_{a_\beta \gamma}) = d_i(\vec{a}) \right\}, \\
(11) & \quad a_i p_{i,l+1} + m_{i,c}(\vec{a}) = \max \left\{ \ord_{c_i - c} \left( L c_i \left( g_{a_\beta \gamma} \right) \right) : (\alpha, \beta, \gamma) \in \mathcal{E}_{d_a}, \delta_i(g_{a_\beta \gamma}) = d_i(\vec{a}) \right\}.
\end{align*}

Proof. (a) $\Rightarrow$ (b): Assume $\mathcal{O}(\sum d_j \Gamma_j)$ has no base point at infinity. Let $C$ be the divisor corresponding to a generic global section of $\mathcal{O}(\sum d_j \Gamma_j)$. By Bertini’s theorem $C$ is transversal to the curve at infinity, and for each $j$,

1. $C$ intersects each $C_j$ (transversally) at $a_j$ distinct points for some $a_j \geq 0$,
2. $C \cap C_i \cap C_j = \emptyset$ for each $i \neq j$,
3. $C$ does not pass through the ‘zero point’ of $C_j$ for any $j$.

(note that $a_1, \ldots, a_N$ are independent of $C$). Pick $h \in \mathbb{C}[x, y]$ which defines $C \cap \mathbb{C}^2$. Then $h$ has a degree-wise Puiseux factorization of the form

$$h = c \prod_{i=1}^{N} a_i \prod_{j=1}^{\tilde{p}_i} \left( y - \psi_{ijk}(x) \right)$$

where $c \in \mathbb{C}^*$, and $\psi_{ijk}$’s are conjugates of some $\psi_{ij}$ of the form $\phi_i(x) + \xi_{ij} x^{r_i} + \text{l.o.t.}$ for $\xi_{ij} \in \mathbb{C}^*$ which are ‘base point free’ in the sense that

(12) for every finite set $S \subseteq \mathbb{C}^*$, if $h$ is generic enough, then $\{\xi_{ij}\}_{j \in S} = \emptyset$. 

A straightforward computation then shows that for each $i, i', j, 1 \leq i, i' \leq N, 1 \leq j \leq a_i$,

$$\delta_{i'} \left( \prod_{k=1}^{\hat{p}_i} (y - \psi_{ijk}(x)) \right) = \omega_{i'i}, \text{ so that}$$

$$\delta_{i'}(h) = \sum_{i=1}^{N} a_i \omega_{i'i} = d_{i'}(\vec{a}).$$

(13)

It follows that $\vec{d} = \vec{d}_{\vec{a}}$. The first assertion of the following claim is a straightforward implication of the ‘base point freeness’ of $\xi_{ij'}$’s, and the second assertion follows from the genericness of $h$.

Claim 5.7.1. Fix $i', 1 \leq i' \leq N$. Let

$$\eta_{i'}(\xi) := \prod_{i \notin \mathcal{N}_{i'} \setminus \{i'\}} \left( \xi_{P_{i',ij'+1}}^{P_{i',ij'+1}} - \xi_{ij'+1}^{P_{ij'+1}} \right) \sigma_{ij}^{P_{ij'+1}}.$$ 

Then

1. $Lc_{i'}(h) = c_{i'} \eta_{i'}(\xi) \prod_{j=1}^{a_{ij'}} \left( \xi_{P_{ij'+1}}^{P_{ij'+1}} - \xi_{ij'+1}^{P_{ij'+1}} \right)$ for some $c' \in \mathbb{C}^\ast$.

2. Let $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j \Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. Then $Lc_{i'}(h') = \eta_{i'}(\xi) \mu_{i'}(\xi)$ for some $\mu_{i'} \in \mathbb{C}[\xi]$.

\[\square\]

Theorem 4.5, identity (13), Claim 5.7.1 and Lemma 5.5 together imply that $\vec{a}$ satisfies (8), (9) and (10).

Now we show that $\vec{a}$ satisfies (11). Recall the procedure from the proof of Theorem 4.5 to write $h$ as a linear combination of the $g_{\alpha \beta \gamma}$. Set $h_0 := h$, and inductively write $h_{k+1} = h_k - h_k^\ast$, where $h_k^\ast$ is computed from $h_k$ in the same way as $h^\ast$ is computed from $h$ in the proof of Theorem 4.5.

Claim 5.7.2. Fix $i', 1 \leq i' \leq N$. Let $k_{i'} \geq 0$ be the smallest integer such that $\delta_{i'}(h_k^\ast) = d_{i'}(h)$. Then for each $k, 0 \leq k \leq k_{i'}$, the degree-wise Puiseux factorization of $h_k$ is of the form

$$h_k = g_k(x, y) \prod_{i \in \mathcal{N}_{i'}} \prod_{j=1}^{a_i} \prod_{k=1}^{\hat{p}_i} (y - \psi_{ijk}(x)),$$

(15)

for some $g_k(x, y) \in \mathbb{C}(\langle x \rangle)[y]$.

Proof. We prove the claim by induction on $k$. It is clearly true if $k = 0$. Now pick $k, 0 \leq k < k_{i'}$, such that (15) holds for $k$. We will show that (15) holds for $k + 1$. Write the degree-wise Puiseux factorization of $h_k$ as

$$h_k = cx^{m_k} \prod_{k'=1}^{K} \prod_{\psi_{k'} \text{ is a conjugate of } \psi_{k'}} (y - \theta_{k'}(x))$$

Recall that $h_k^\ast$ is formed from (15) by replacing each $h_{kk'} := \prod_{l'} (y - \theta_{k'}(x))$ by $h_{kk'}^\ast := \prod_{l'}$, for certain $i_{k'}, j_{k'}, n_{k'}$. Since $k < k_{i'}$, we have $\delta_{i'}(h_k) > \delta_{i'}(h_k^\ast)$. This implies that there exists
$k'$ such that $\delta_{\nu'}(h_{kk'}) > \delta_{\nu'}(h_{kk'}^*)$. It then follows (e.g. due to Lemma 7.6) that

$$s_{\nu'k'} := \min_{\nu'} \{\deg_{x}(\phi_{\nu'}(x) + \xi x^{\nu'} - \theta_{\nu'}(x)) \geq \min_{l} \{\deg_{x}(\phi_{\nu'}(x) + \xi x^{\nu'} - \phi_{\nu'j_{k'}}(x)) \geq r_{\nu'}$$

where $\phi_{\nu'j_{k'}}$'s on the right hand side runs over all conjugates of $\phi_{\nu'j_{k'}}$. Pick $l_{0}'$ (resp. $l_{0}$) for which the minimum is achieved on the left (resp. right) hand side of the preceding inequality. Then

$$\psi_{\nu'k'}(x) = \phi_{\nu'}(x) + bx^{\nu'} + 1.d.t.$$ 

$$\phi_{\nu'j_{k'}}(x) = \phi_{\nu'}(x) + \text{terms with degree less than } s_{\nu'k'}.$$ 

Now fix $i \in \mathcal{N}_{\nu'}$. Then $\phi_{i}$ has a conjugate which agrees with $\phi_{\nu'}$ up to degree $r_{\nu'}$, so that

$$\min_{l} \{\deg_{x}(\phi_{i}(x) + \xi x^{\nu'} - \phi_{\nu'j_{k'}}(x)) \leq \min_{\nu'} \{\deg_{x}(\phi_{\nu'}(x) + \xi x^{\nu'} - \theta_{\nu'}(x)) \}.$$ 

Lemma 7.6 then implies that $\delta_{i}(h_{kk'}) > \delta_{i}(h_{kk'}^*)$, so that $\delta_{i}(h_{k}) > \delta_{i}(h_{k}^*)$. By induction hypothesis the (closure in $X$ of the) curve $h_{k} = 0$ intersects $\Gamma_{i}$ transversally at $a_{i}$ points. It follows that the curve defined by $h_{k+1} = h_{k} - h_{k}^*$ also intersects $\Gamma_{i}$ transversally at each of the $a_{i}$ points. This implies that (15) holds for $k + 1$, as required. \hfill $\square$

Fix $i,j$, $i \in \mathcal{N}_{\nu'} \setminus \{i'\}$, $1 \leq j \leq a_{i}$. Claim 5.7.2 implies that $h_{ij} := \prod_{k=1}^{a_{ij}} (y - \psi_{ijk}(x))$ divides $h_{k_{ij}}$ in $C(\langle x \rangle)[y]$. Recall that $\psi_{ijk}$'s are conjugates of $\psi_{i}$ and $\psi_{ij} = \phi_{i}(x) + \xi x^{\nu'} + 1.d.t..$ The arguments in the proof of Theorem 4.5 show that the factor $h_{ij}^*$ of $h_{k_{ij}}^*$ corresponding to $h_{ij}$ can be defined to be $j_{h_{ij}}^{P',i_{i}+1}$. It follows that $L_{c_{i}}(h_{k_{ij}}^*)$ has a factor of the form

$$\sigma(\xi) := \prod_{i \in \mathcal{N}_{\nu'}} \prod_{j=1}^{a_{i}} L_{c_{i}}(h_{ij}^*) = \prod_{i \in \mathcal{N}_{\nu'}} \prod_{j=1}^{a_{i}} (\xi^{P_{i}^{j}+1} - c_{i,j}^{P_{i}^{j}+1}) = \eta_{\nu'}(\xi)(\xi^{P_{i}^{j}+1} - c_{i,j}^{P_{i}^{j}+1})a_{i}.\nu'$$

Since $\deg(\sigma) = m_{\nu'}(\tilde{a})$, identity (9) implies that $L_{c_{i}}(h_{k_{ij}}^*) = c_{i} \sigma(\xi)$ for some $c_{i} \in C^{*}$, and consequently we get, using Lemma 5.5, that

$$\operatorname{ord}_{\xi-c_{i,j}^{P_{i}^{j}+1}}(L_{c_{i}}(h_{k_{ij}}^*)) = \operatorname{ord}_{\xi-c_{i,j}^{P_{i}^{j}+1}}(\sigma(\xi)) = m_{i,j}^{P_{i}^{j}+1}(\tilde{a}) + a_{i,j}^{P_{i}^{j}+1}.\nu'$$

On the other hand, assertion 2 of Claim 5.7.1, coupled with identity (9) and Lemma 5.5 imply that $\operatorname{ord}_{\xi-c_{i,j}^{P_{i}^{j}+1}}(h_{i}^*) = m_{i,j}^{P_{i}^{j}}(\tilde{a}) + a_{i,j}^{P_{i}^{j}+1}$. for all $h_{i}^* \in \Gamma(X,\mathcal{O}(\sum_{j=1} d_{j}\Gamma_{j}))$ such that $\delta_{\nu'}(h_{i}^*) = d_{i}$. The proves (11), and finishes the proof that (b) $\Rightarrow$ (a).

(b) $\Rightarrow$ (a): Pick $\tilde{a}$ that satisfies (8), (9), (10), (11), and set $\tilde{d} := \tilde{d}_{i}$. At first the following claim:

**Claim 5.7.3.** Fix $i, 1 \leq i \leq N$. Let $\eta_{i}(\xi)$ be as in (14). A generic $h_{i} \in \Gamma(X,\mathcal{O}(\sum_{j=1} d_{j}\Gamma_{j}))$ satisfies $L_{c_{i}}(h_{i}) = c_{i}(\xi) \prod_{j=1}^{a_{i,j}} (\xi^{P_{i,j}^{j}+1} - \xi^{P_{i,j}^{j}+1})$ for some $c_{i} \in C^{*}$ and $\{\xi_{j}\}_{i,j} \subseteq C^{*}$ that satisfies the ‘base point free condition’ (12). In other words, assertion 1 of Claim 5.7.1 is true for $i' := i$.

**Proof.** We prove the claim by induction on $|\mathcal{N}_{i}|$. Indeed, if $|\mathcal{N}_{i}| = 1$, then $\eta_{i}(\xi) = 1$, $m_{i,j_{i}}(\tilde{a}) = 0$, and $m_{i}(\tilde{a}) = p_{i,j_{i}+1}a$. By assumption there exists $h_{1}, h_{2} \in \Gamma(X,\mathcal{O}(\sum_{j=1} d_{j}\Gamma_{j}))$ such that

1. $\delta_{i}(h_{1}) = \delta_{i}(h_{2}) = d_{i}$,
2. $L_{c_{i}}(h_{1}) = c_{i} \xi_{c_{i}}(\tilde{a})$, $c_{i} \in C^{*}$, and
3. $\operatorname{ord}_{\xi-c_{i}}(L_{c_{i}}(h_{2})) = 0$. 


It follows that for a generic $\lambda \in \mathbb{C}^*$, the roots $\xi_{ij}$ of $\text{Lc}_r(h_1 + \lambda h_2)$ satisfies the base point free condition (12). Remark-Definition 5.6 then implies that Claim 5.7.3 holds for $i$.

Now pick $i'$ such that Claim 5.7.3 holds for all $i$ with $|N_i| < |N_{i'}|$.

**Claim 5.7.4.** Let $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j \Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. Then $\text{Lc}_{i'}(h') = \eta_{i'}(\xi)\mu_{i'}(\xi)$ for some $\mu_{i'} \in \mathbb{C}[\xi]$. In other words, assertion 2 of Claim 5.7.1 holds for $i'$.

**Proof.** Pick a generic $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j \Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. It suffices to show that Claim 5.7.4 holds for $h'$. Fix $i \in N_{i'} \setminus \{i'\}$. Property (8) implies that $\delta_i(h) = d_i$. Let $\zeta$ be a primitive $p_{i, i+1}$-th root of unity. The inductive hypothesis implies that $h'$ has $a_{i}p_{i, i+1}$ degree-wise Puiseux roots of the form

$$\psi_{ijk}(x) = \phi_i(x) + c_{i}x^{r_i} + \text{l.d.t.}$$

for $1 \leq k \leq p_{i, i+1}$, $1 \leq j \leq a_i$, and $\xi_{ij}$'s that satisfy the 'base point free condition' (12). For each such root, there are $p_{i}/p_{i'}$ conjugates which are of the form

$$\phi_{i'}(x) + c_{i'}x^{r_{i'}} + \text{l.d.t.}$$

where $c_{i'}$ is as in Definition 5.4. It follows that each $i \in N_{i'} \setminus \{i'\}$ contributes a factor of $\eta_{i'}(\xi) := (\xi^{d_{i'}}\zeta^{i'_{1}+1} - c_{i'}\zeta^{i'_{1}+1})\eta_{i\prime}/\phi_{i'}$ to $\text{Lc}_{i'}(h')$. Consequently $\text{Lc}_{i'}(h')$ has a factor of the form $\prod_{i \in N_{i'} \setminus \{i'\}} \eta_{i'}(\xi) = \eta_{i'}(\xi)$.

**Proof of Claim 5.7.3 continued.** Using property (11) we can pick $(\alpha, \beta, \gamma) \in \mathcal{E}_d$ such that $\delta_{i'}(g_{\alpha \beta \gamma}) = d_{i'}$ and $\text{ord}_{\xi - c_{i'} \gamma} \left( \text{Lc}_{i'}(g_{\alpha \beta \gamma}) \right) = a_{i'}p_{i\prime}t_{i\prime+1} + m_{i', c_{i'}}(\alpha)$. Since $\text{ord}_{\xi - c_{i'} \gamma}(\eta_{i'}(\xi)) = m_{i', c_{i'}}(\alpha)$, applying Claim 5.7.4 to $h := g_{\alpha \beta \gamma}$ and using property (9), we see

$$\deg_{\xi} \left( \text{Lc}_{i'}(g_{\alpha \beta \gamma}) \right) \geq a_{i'}p_{i', i+1} + \deg_{\xi}(\eta_{i'}(\xi)) = \sum_{i \in N_{i'}} \bar{p}_i a_i/p_{i'} = m_{i'}(\alpha) \geq \deg_{\xi} \left( \text{Lc}_{i'}(g_{\alpha \beta \gamma}) \right).$$

It follows that $\deg_{\xi} \left( \text{Lc}_{i'}(g_{\alpha \beta \gamma}) \right) = m_{i'}(\alpha)$, and $\text{Lc}_{i'}(g_{\alpha \beta \gamma}) = c(\xi - c_{i'} \gamma)^{a_{i'}p_{i', i+1} + 1}\eta_{i'}(\xi)$ for some $c \in \mathbb{C}^*$. On the other hand, property (10) and Claim 5.7.4 imply that there is $(\alpha', \beta', \gamma') \in \mathcal{E}_d$ such that $\delta_{i'}(g_{\alpha' \beta' \gamma'}) = d_{i'}$ and $\text{Lc}_{i'}(g_{\alpha' \beta' \gamma'}) = \eta_{i'}(\xi)\mu(\xi)$ for some $\alpha' \in \mathbb{C}^*$ and $\mu(\xi) \in \mathbb{C}[\xi]$ with $\text{ord}_{\xi - c_{i'} \gamma}(\mu(\xi)) = 0$. It follows that for a generic $\lambda \in \mathbb{C}^*$, the roots $\xi_{ij}$ of $\text{Lc}_{i'}(g_{\alpha \beta \gamma} + \lambda g_{\alpha' \beta' \gamma'})$ satisfy the base point free condition (12). Remark-Definition 5.6 then implies Claim 5.7.3 holds for $i'$, and concludes the proof of Claim 5.7.3.

Now pick a generic $h \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$. Claim (5.7.3) implies that $h$ has a factorization of the form

$$h = g(x, y) \prod_{i=1}^{N} \prod_{j=1}^{a_i} \prod_{k=1}^{b_{ij}} (y - \psi_{ijk}(x)),$$

where

1. $g(x, y) \in \mathbb{C}[[x]][y]$,
2. $a_i = \sum_{j=1}^{a_i} b_{ij}$, $1 \leq i \leq N$, and
3. $\psi_{ijk}$'s are conjugates of some $\psi_{ij}$ of the form $\phi_i(x) + \xi_{ij} x^{r_i} + \text{l.o.t.}$ for $\xi_{ij} \in \mathbb{C}^*$ which satisfy the 'base point free' condition (12).
Using the 'base point freeness' property of $\xi_{ij}$'s, it is straightforward to compute that for each $i, i', j, 1 \leq i, i' \leq N, 1 \leq j \leq a_i$

$$\delta_{i'} \left( \prod_{k=1}^{b_{ij}} (y - \psi_{ijk}(x)) \right) = b_{ij} \omega_{i'},$$

so that

$$\delta_{i'}(h/g) = \sum_{i=1}^{N} a_i \omega_{i'} = d_{i'}.$$ 

It follows that $\delta_{i'}(g) = 0$ for all $i', 1 \leq i' \leq N,$ and consequently, $g$ is a constant. But then (16) and the 'base point freeness' property of $\xi_{ij}$'s imply that for generic $h_1, h_2 \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$, the closures $C_i, 1 \leq i \leq 2,$ on $X$ of the curves defined by $h_i$ on $\mathbb{C}^2$ intersect distinct and non-singular points of $X \setminus \mathbb{C}^2$. This proves (a), and completes the proof of Theorem 5.7. □

6. The case of $X_V, V \in \mathcal{V}$

In this section we illustrate Theorem 4.5, Corollary 5.3 and Theorem 5.7 for the case of $X_V$ for $V \in \mathcal{V}$. We start with the case of $\mathcal{V}_1$.

Let $V : = \mathbb{C}(G, H^d) \in \mathcal{V}_1$. Identify $\mathbb{P}^2 \setminus \{ H = 0 \}$ with $\mathbb{C}^2$. Then $g := G|_{\mathbb{C}^2}$ defines a curve $C$ with one place at infinity. Denote the irreducible components of $X_V \setminus \mathbb{C}^2$ by $\Gamma_0, \ldots, \Gamma_N$, where $\Gamma_0$ is the strict transform of the line at infinity $H = 0$. Recall that the dual graph of the curves at infinity are of the form as in Figure 2. Recall that there are polynomials $g_1, \ldots, g_s, g_{s+1} = g$ (the approximate roots of $g$ introduced by Abhyankar and Moh) such that each $g_k$ is a $\Gamma_{ik}$-curvette. Choose coordinates $(x, y)$ on $\mathbb{C}^2$ such that the point at infinity on $C$ is on the closure of the $x$-axis. Then each $g_k, 1 \leq k \leq s + 1$, is a monic polynomial in $y$ (as an element of $\mathbb{C}[x][y]$). Set $g_0 := x$.

**Figure 2.** Dual graph of curves at infinity on $X_V$ for $V \in \mathcal{V}_1$

Fix $k, 0 \leq k \leq N$. Let $l_k, 0 \leq l_k \leq s$, be such that $\Gamma_k \in [\Gamma_{il_k+1}, \Gamma_{il_k+1}]$ (where $[\Gamma_{ij}, \Gamma_{ij}]$ is defined as in the proof of Proposition 3.8). Also, let $\delta_k$ be the semidegree on $\mathbb{C}[x, y]$ associated to $\Gamma_k$.

**Remark 6.1.** The above definition of $l_k$ agrees with the definition in Notation 4.0, i.e. the degree-wise Puiseux series associated to $\delta_k$ has precisely $l_k + 1$ formal Puiseux pairs (Definition 7.2) $(q_{k,1}, p_{k,1}), \ldots, (q_{k,l_k+1}, p_{k, l_k+1})$. Moreover,

1. $p_{k,j} = \gcd(\delta_k(g_0), \ldots, \delta_k(g_{j-1}))$, 
   \[ 1 \leq j \leq l_k + 1. \]

2. We can set $f_{kk} := g_{lk}$ (where the correspondence between $f_{kk}$ and $\delta_k$ is as in Lemma 4.1).
Recall that \( \{g_1, \ldots, g_{s+1}\} \) satisfies the properties of \( \{f_{ij}\}_{i,j} \) of Theorem 4.5 for \( X = X_V \) (Remark 4.2). For each \( k_1, k_2, 1 \leq k_1 \leq k_2 \leq s + 1 \), define

\[
e_{k_1, k_2} := \frac{\deg(g_{k_2})}{\deg(g_{k_1})}.
\]

Let \( B := \{i : 0 \leq i \leq N, \text{ and } \Gamma_i \text{ is on the (only) } \text{‘horizontal segment’ (which has } \Gamma_0 \text{ and } \Gamma_N \text{ on the two ends)} \} \). The following lemma, which follows from more or less straightforward computations, computes \( \mathcal{N}_i, \mathcal{N}_{i,c}, e_i \) and \( L_c \) (from Definition 5.4 and Remark-Definition 5.6), \( 1 \leq i \leq N \).

**Lemma 6.2.** Set \( f_{kl} := g_{kl}, 0 \leq k \leq N \).

1. Fix \( i, k, 0 \leq i, k \leq N \). Then \( k \in \mathcal{N}_i \) iff one of the following conditions holds:
   
   (a) \( k = i \).
   
   (b) \( i \in B \) and the position of \( \Gamma_k \) on the graph of Figure 2 is below and/or to the right of \( \Gamma_i \).
   
   (c) \( \Gamma_i \) and \( \Gamma_k \) are both on the same ‘vertical segment’ of the graph of Figure 2 and \( \Gamma_k \) is below \( \Gamma_i \).

2. (a) If \( k \notin \{j_1, \ldots, j_s\} \), then \( N_{k,c} := N_k \setminus \{k\} \).
   
   (b) If \( k = j_q, 1 \leq q \leq s \), then \( c_{kk} = 0 \) and \( N_k = N_{j_q,0} \cup N_{j_q,c} \cup \{j_q\} \) for some \( c_q^* \neq 0 \). Moreover,
   
   (i) \( N_{j_q,0} \) consists of all \( l \) such that \( l \neq j_q \) and the position of \( \Gamma_l \) on the graph in Figure 2 is on the vertical segment between \( \Gamma_{j_q} \) and \( \Gamma_i \).
   
   (ii) \( N_{j_q,c} = N_{j_q} \setminus \{N_{j_q,0} \cup \{j_q\}\} \).

3. (a) If \( k \notin B \), then \( L_e(g_1^{a_1} \cdots g_{s+1}^{a_{s+1}}) = c(\xi - c_{kk})^{a_{kk} + 1} \) for some \( c \in \mathbb{C}^* \). If in addition \( k \neq i_{k+1} \), then \( c_{kk} = 0 \).
   
   (b) If \( k \in B \setminus \{j_1, j_2, \ldots, j_s\} \), then \( L(e)(g_1^{a_1} \cdots g_{s+1}^{a_{s+1}}) = c(\xi - c_{kk})^\sum_{i=1}^{s+1} a_i e_{i+1} \) for some \( c \in \mathbb{C}^* \).

The following result follows from combining Theorem 4.5, Corollary 5.3 and Theorem 5.7 with the above observations (plus Remark 5.2). At first we recall some notations: let \( \delta_j, 0 \leq j \leq N \), be the \( j \)-th unit vector in \( \mathbb{Z}^{N+1} \). For \( \vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^N \), write \( \Gamma_{\vec{d}} \) for the divisor \( \sum_{j=0}^N d_j \Gamma_j \in \text{Cl}(X_V) \) and if \( f \in \Gamma(X_V, \mathcal{O}(\Gamma_{\vec{d}})) \), we denote by \( (f)_{\vec{d}} \) the ‘copy’ of \( f \) in the \( \vec{d} \)-th graded component of \( \mathcal{R}(X_V) \). Moreover, for \( f \in \mathcal{C}(x, y) \) we denote by \( \delta(f) \) the element \( (\delta_0(f), \ldots, \delta_N(f)) \in \mathbb{Z}^{N+1} \).

**Corollary 6.3.**

1. \( (g_j)_{\vec{d}(g_j)} \)’s together with (1) \( \epsilon_j \), \( 0 \leq j \leq N \), generate \( \mathcal{R}(X_V) \) as a \( \mathbb{C} \)-algebra.

2. Let \( \vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^{N+1} \). Let \( E_{\vec{d}} \) be the collection of all \( \vec{a} := (a_0, \ldots, a_{s+1}) \in \mathbb{Z}^{s+2}_{\geq 0} \) such that \( \sum_{j=0}^{s+1} a_j \delta_k(g_j) \leq d_k \) for all \( k, 0 \leq k \leq N \). Then \( \Gamma(X, \mathcal{O}(\Gamma_{\vec{d}})) \) is a generated as a vector space over \( \mathbb{C} \) by \( \{\prod_{j=0}^{s+1} g_j^{a_j} : \vec{a} \in E_{\vec{d}}\} \).

3. Let \( \vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^{N+1} \). Define \( E'_{\vec{d}} := \{(a_0, \sum_{j=1}^{s+1} a_j \deg_y(g_j)) : (a_0, \ldots, a_{s+1}) \in E_{\vec{d}}\} \subseteq \mathbb{Z}^{s+2}_{\geq 0} \). Then \( \dim \Gamma(X, \mathcal{O}(\Gamma_{\vec{d}})) = |E'_{\vec{d}}| \). For each \( (a, b) \in E'_{\vec{d}} \), pick any \( \vec{a} \in E_{\vec{d}} \)
such that \( a = \alpha_0 \) and \( b = \sum_{j=0}^{s+1} \alpha_j \deg(g_j) \), and define \( g_{a,b} := \prod_{j=0}^{s+1} g_j^{\alpha_j} \). Then \( \{ g_{a,b} : (a,b) \in \mathcal{E}_d \} \) is a basis of \( \Gamma(X, \mathcal{O}(\Gamma_d)) \).

(4) For each \( \vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^{N+1} \), the following are equivalent:
(a) \( \Gamma_d \in \mathcal{P}^{{\mathcal{F}(X_Y)}} \).
(b) \( d_j = \max \left\{ \sum_{k=0}^{s+1} \alpha_k \delta_j(g_k) : (\alpha_0, \ldots, \alpha_{s+1}) \in \mathcal{E}_d \right\} \) for all \( j, 0 \leq j \leq N \).
(c) For each \( j, 0 \leq j \leq N \), there exists \( (\alpha_0, \ldots, \alpha_{s+1}) \in \mathbb{Z}^{s+2} \) such that \( \sum_{k=0}^{s+1} \alpha_k \delta_j(g_k) = d_j \) and \( \sum_{k=0}^{s+1} \alpha_k \delta_j(g_k) \leq d_i \) for all \( i, 0 \leq i \leq N \).

(5) For each \( k, 0 \leq k \leq N \), and each \( \vec{\alpha} := (\alpha_0, \ldots, \alpha_{s+1}) \in \mathbb{Z}^{s+2} \), define

\[
\mu_k(\vec{\alpha}) := \begin{cases} 
\alpha_{i_k+1} & \text{if } k \notin \mathcal{B} \\
\sum_{i \geq i_k+1} \alpha_i \epsilon_{i_k+1,i} & \text{if } k \in \mathcal{B}.
\end{cases}
\nu_k(\vec{\alpha}) := \begin{cases} 
\alpha_{i_k+1} & \text{if } k \notin \mathcal{B} \\
\sum_{i \geq i_k+1} \alpha_i \epsilon_{i_k+1,i} & \text{if } k \in \mathcal{B} \setminus \{j_1, \ldots, j_s\} \\
\alpha_q & \text{if } k = j_q, 1 \leq q \leq s.
\end{cases}
\]

Let \( \vec{a} := (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1} \). Define \( d_k(\vec{a}) := \sum_{i=0}^{N} a_i \omega_{k,i}, 0 \leq k \leq N \), where

\( \omega_{k,i} := \begin{cases} 
p_{i,k} \delta_k(f_{i,k}) & \text{if } i \notin \mathcal{N}_i \\
\prod_{j=1}^{k+1} \frac{p_{i,j}}{p_{k,j}} \delta_l(f_{i,k}) & \text{if } i \in \mathcal{N}_i.
\end{cases} \)

and set \( \vec{d}_a := (d_0(\vec{a}), \ldots, d_N(\vec{a})) \). For each \( k, 0 \leq k \leq N \), define

\[
m_k(\vec{a}) := \sum_{i \in \mathcal{N}_k} \frac{\prod_{j=1}^{k+1} p_{i,j}}{\prod_{j=1}^{k} p_{k,j}} a_i, \\
n_k(\vec{a}) := \sum_{i \in \mathcal{N}_k, \epsilon_{kk}} \frac{\prod_{j=1}^{k+1} p_{i,j}}{\prod_{j=1}^{k} p_{k,j}} a_i.
\]

Then for each \( \vec{d} \in \mathbb{Z}^{N+1} \), the following are equivalent:
(a) \( \Gamma_d \in \mathcal{P}(X_Y) \).
(b) \( \vec{d} = \vec{d}_a \) for some \( \vec{a} \in \mathbb{Z}^{N+1} \) such that for each \( k, 0 \leq k \leq N \),

\[
d_k = \max \left\{ \sum_{j=0}^{s+1} \alpha_j \delta_k(g_j) : \vec{a} \in \mathcal{E}_d \right\},
\]

\[
m_k(\vec{a}) = \max \left\{ \mu_k(\vec{a}) : \vec{a} \in \mathcal{E}_d, \sum_{j=0}^{s+1} \alpha_j \delta_k(g_j) = d_k \right\},
\]

\[
n_k(\vec{a}) = \min \left\{ \nu_k(\vec{a}) : \vec{a} \in \mathcal{E}_d, \sum_{j=0}^{s+1} \alpha_j \delta_k(g_j) = d_k \right\},
\]

\[
ak p_{k+1} + n_k(\vec{a}) = \max \left\{ \nu_k(\vec{a}) : \vec{a} \in \mathcal{E}_d, \sum_{j=0}^{s+1} \alpha_j \delta_k(g_j) = d_k \right\}.
\]

Let \( V_i := \mathbb{C}[G_i, H^0] \in \mathcal{V}, 1 \leq i \leq 2 \). Identify \( \mathbb{P}^2 \setminus \{H = 0\} \) with \( \mathbb{C}^2 \). Assume that the germs at infinity of curves defined by \( G_1 \) and \( G_2 \) are equisingular. Then there is a natural bijection between the sets of irreducible curves at infinity on \( X_{V_i} \)'s which induces an isomorphism \( \rho : \text{Pic}(X_{V_1}) \cong \text{Pic}(X_{V_2}) \).
Corollary 6.4.

1. For each line bundle \( \mathcal{L} \in \text{Pic}(X_{V_1}) \), \( \dim_C(\Gamma(X_{V_1}, \mathcal{L})) = \dim_C(\Gamma(X_{V_2}, \mathcal{O}(\rho(\mathcal{L})))) \).
2. \( \rho \) induces isomorphisms \( P^s(X_{V_1}) \cong P^s(X_{V_2}) \) and \( \bar{P}_\infty(X_{V_1}) \cong \bar{P}_\infty(X_{V_2}) \). In particular, if \( D \) is an effective divisor with support at infinity (i.e. \( \text{Supp}(D) \subseteq X_{V_1} \setminus \mathbb{C}^2 \)), then \( D \in \bar{P}_\infty(X_{V_1}) \) iff \( \rho(D) \in \bar{P}_\infty(X_{V_2}) \).
3. If \( V_1, V_2 \in \mathcal{V}_1 \), then \( \bar{P}(X_{V_1}) \cong \bar{P}(X_{V_2}) \).

Proof. If \( V_1, V_2 \in \mathcal{V}_1 \), then the statements of Corollary 6.4 follow from Corollary 6.3 and Remark 5.2. The general case follows via almost the same arguments. \( \square \)

7. Appendix - Preliminaries on Degree-wise Puiseux Series

Definition 7.1 (Degree-wise Puiseux series). The field of \textit{degree-wise Puiseux series} in \( x \) is

\[
\mathbb{C}((x)) := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j \leq k} a_j x^{j/p} : k, p \in \mathbb{Z}, p \geq 1 \right\},
\]

where for each integer \( p \geq 1 \), \( \mathbb{C}((x^{-1/p})) \) denotes the field of Laurent series in \( x^{-1/p} \). Let \( \phi \) be a degree-wise Puiseux series in \( x \). The \textit{polydromy order} (terminology taken from [CA00]) of \( \phi \) is the smallest positive integer \( p \) such that \( \phi \in \mathbb{C}((x^{-1/p})) \). For any \( r \in \mathbb{Q} \), let us denote by \( [\phi]_{> r} \) (resp. \( [\phi]_{\geq r} \)) sum of all terms of \( \phi \) with order greater than (resp. greater than or equal to) \( r \). Then the \textit{Puiseux pairs of \( \phi \)} are the unique sequence of pairs of relatively prime integers \( (q_1, p_1), \ldots, (q_k, p_k) \) such that the polydromy order of \( \phi \) is \( p_1 \cdots p_k \), and for all \( j, 1 \leq j \leq k, \)

1. \( p_j \geq 2, \)
2. \( [\phi]_{\frac{q_j}{p_1 \cdots p_j}} \in \mathbb{C}((x^{-\frac{1}{p_0 - \frac{1}{p_1 \cdots p_j}}})) \) (where we set \( p_0 := 1 \)), and
3. \( [\phi]_{\frac{q_j}{p_1 \cdots p_j}} \notin \mathbb{C}((x^{-\frac{1}{p_0 - \frac{1}{p_1 \cdots p_j}}})) \).

The exponents \( q_j/(p_1 \cdots p_j) \), \( 1 \leq j \leq k \), are called the \textit{characteristic exponents} of \( \phi \). Let \( \phi = \sum_{q \leq q_0} a_q x^{q/p} \), where \( p \) is the polydromy order of \( \phi \). Then the \textit{conjugates} of \( \phi \) are \( \phi_j := \sum_{q \leq q_0} a_q x^{q/p} \zeta^j, 1 \leq j \leq p \), where \( \zeta \) is a primitive \( p \)-th root of unity.

Definition 7.2. Let \( \delta \) be a semidegree and, as in (1), let \( \hat{\phi}(x, \xi) := \phi(x) + \xi x^r \) be the \textit{generic degree-wise Puiseux series} associated to \( \delta \). Let the Puiseux pairs of \( \hat{\phi} \) be \( (q_1, p_1), \ldots, (q_l, p_l) \). Express \( r \) as \( q_{l+1}/(p_1 \cdots p_{l+1}) \) where \( p_{l+1} \geq 1 \) and \( \gcd(q_{l+1}, p_{l+1}) = 1 \). Then the \textit{formal Puiseux pairs} of \( \hat{\phi} \) are \( (q_1, p_1), \ldots, (q_{l+1}, p_{l+1}) \). Note that

1. \( \delta(x) = p_1 \cdots p_{l+1}, \)
2. it is possible that \( p_{l+1} = 1 \) (as opposed to other \( p_k \)'s, which are always \( \geq 2 \)).

The \textit{formal characteristic exponents} of \( \hat{\phi} \) are \( q_j/(p_1 \cdots p_j), 1 \leq j \leq l + 1 \).

Example 7.3. Let \( (p, q) \) are integers such that \( p > 0 \) and \( \delta \) be the weighted degree on \( \mathbb{C}(x, y) \) corresponding to weights \( p \) for \( x \) and \( q \) for \( y \). Then \( \hat{\phi} = \xi x^{\delta / p} \) (i.e. \( \phi = 0 \)).

The rest of this section is devoted to a proof of Lemma 7.6 below, which is used in the proofs of Theorem 4.5 and Theorem 5.7.
Definition 7.4. Let \( \phi = \sum_j a_j x^{q_j/p} \in \mathbb{C}((x)) \) be a degree-wise Puiseux series with polydromy order \( p \) and \( r \) be a multiple of \( p \). Then for all \( c \in \mathbb{C} \) we define

\[
c \ast_r \phi := \sum_j a_j c^{q_j/(p \cdot p_j)} x^{q_j/p}.
\]

Remark 7.5. Let \( \phi, p \) be as in Definition 7.4. Then

1. Every conjugate of \( \phi \) is of the form \( \zeta^k \ast_p \phi \) where \( 0 \leq k \leq p - 1 \) and \( \zeta \) is a primitive \( p \)-th root of unity.
2. Let \( d \) and \( e \) be positive integers, and \( c \in \mathbb{C} \). Then \( c \ast_{pde} \phi = c^e \ast_{pd} \phi = c^{de} \ast_p \phi \).

Lemma 7.6. Let \( \delta \) be a semidegree with generic degree-wise Puiseux series \( \tilde{\phi}(x, \xi) = \phi(x) + \xi x^r \). Let \( \psi_1, \psi_2 \in \mathbb{C}((x)) \), and \( g_1, g_2 \in \mathbb{C}((x))[y] \) be defined as

\[
g_i = \prod_{\psi_{ij} \text{ is a conjugate of } \psi_i} (y - \psi_{ij}(x)).
\]

Let \( \epsilon_{ij} := \deg_y(\tilde{\phi}(x, \xi) - \psi_{ij}(x)) \), and \( \epsilon_i := \min_j \{\epsilon_{ij}\} \). If \( \epsilon_1 \geq \epsilon_2 \), then

\[
\frac{\delta(g_1)}{\deg_y(g_1)} \geq \frac{\delta(g_2)}{\deg_y(g_2)}.
\]

Proof. W.l.o.g. (replacing \( \psi_i \)'s by some of their conjugates if necessary) we may assume that \( \epsilon_i = \deg_x(\tilde{\phi}(x, \xi) - \psi_i(x)) \) for each \( i \). Then we can write:

\[
\begin{align*}
\phi(x) &= \phi_1(x) + \phi_2(x) + \phi_3(x, \xi) \\
\psi_1(x) &= \phi_1(x) + \tilde{\psi}_1(x) \\
\psi_2(x) &= \phi_1(x) + \phi_2(x) + \tilde{\psi}_2(x),
\end{align*}
\]

where all terms of \( \phi_{j+1} \) and \( \tilde{\psi}_j \) have lower degree in \( x \) than each term of \( \phi_j \) for \( j = 1, 2 \). Let the polydromy orders of \( \phi_1, \phi_1 + \phi_2, \psi_1, \psi_2 \) be respectively \( p_1, p_1 p_2, p_1 q_1 \), and \( p_1 p_2 q_2 \). Let
\( \zeta_1, \zeta_2, \hat{\zeta}_1, \hat{\zeta}_2 \) be primitive roots of unity of orders \( p_1, p_1p_2, p_1q_1, p_1p_2q_2 \) respectively. Then

\[
\delta(g_1) = \delta(x) \sum_{j=0}^{p_1q_1-1} \deg_x \left( \phi(x, \xi) - \hat{\zeta}_1^j *_{p_1q_1} \psi_1(x) \right)
\]

\[
= \delta(x) \sum_{j_1=0}^{q_1-1} \sum_{j=0}^{p_1-1} \deg_x \left( \phi_1(x) + \phi_2(x) + \xi x^r - \hat{\zeta}_1^{jp_1+j} *_{p_1q_1} (\phi_1(x) + \psi_1(x)) \right)
\]

\[
= \delta(x) \sum_{j_1=0}^{q_1-1} \sum_{j=0}^{p_1-1} \deg_x \left( \phi_1(x) + \phi_2(x) + \xi x^r - \hat{\zeta}_1^{jp_1+j} *_{p_1q_1} \psi_1(x) \right)
\]

\[
= \delta(x) \sum_{j_1=0}^{q_1-1} \left( \deg_x \left( \phi_2(x) + \xi x^r - \hat{\zeta}_1^{jp_1} *_{p_1q_1} \psi_1(x) \right) + \sum_{j_1=1}^{p_1-1} \deg_x \left( \phi_1(x) - \hat{\zeta}_1^j *_{p_1} \phi_1(x) \right) \right)
\]

\[
= q_1 \delta(x) \left( \epsilon_1 + \sum_{j=1}^{p_1-1} \deg_x \left( \phi_1(x) - \hat{\zeta}_1^j *_{p_1} \phi_1(x) \right) \right),
\]

\[
\delta(g_2) = \delta(x) \sum_{j=0}^{p_1p_2q_2-1} \deg_x \left( \phi(x, \xi) - \hat{\zeta}_2^j *_{p_1p_2q_2} \psi_2(x) \right)
\]

\[
= \delta(x) \sum_{j_2=0}^{p_2q_2-1} \sum_{j_1=0}^{p_1-1} \deg_x \left( \phi_1(x) + \phi_2(x) + \xi x^r - \hat{\zeta}_1^{jp_1+j} *_{p_1p_2q_2} (\phi_1(x) + \phi_2(x) + \psi_2(x)) \right)
\]

\[
= \delta(x) \sum_{j_2=0}^{p_2q_2-1} \left( \deg_x \left( \phi_2(x) + \xi x^r - \hat{\zeta}_1^{jp_1} *_{p_1p_2q_2} (\phi_2(x) + \psi_2(x)) \right) + \sum_{j_1=1}^{p_1-1} \deg_x \left( \phi_1(x) - \hat{\zeta}_1^j *_{p_1} \phi_1(x) \right) \right)
\]

\[
\leq p_2q_2 \delta(x) \left( \epsilon_1 \sum_{j=1}^{p_1-1} \deg_x \left( \phi_1(x) - \hat{\zeta}_1^j *_{p_1} \phi_1(x) \right) \right).
\]

It follows that \( \delta(g_2)/(p_1p_2q_2) \leq \delta(g_1)/(p_1q_1) \), as required.

\[
\square
\]

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