Nambu brackets for the electromagnetic field

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A Nambu formulation for the electromagnetic field in the case of stationary charge density and vanishing charge current density is proposed.

I. INTRODUCTION

In Hamiltonian mechanics the governing equations are obtained from the Poisson brackets formalism. Poisson brackets formalism gives a prescription to find time derivative of any function $F$ using the Hamiltonian function $H$ and a bilinear antisymmetric function constructed using the derivatives with respect to the pairs of phase space coordinates $(p_k, q_k)$, $k = 1, ..., N$ known as the Poisson bracket (PB) $\{.,\}_{p_k,q_k}$ i.e.:

$$\frac{dF}{dt} = \sum_k [F, H]_{p_k,q_k}$$

Nambu mechanics \cite{1} is a novel generalization of Hamiltonian mechanics where the prescription to find the time derivative of any function $F$ uses two conserved quantities $I_1$, $I_2$ and a trilinear completely antisymmetric combination of the derivatives with respect to the extended phase space triplets of coordinates $(x_k, y_k, z_k)$, $k = 1, ..., N$ known as the Nambu bracket (NB):

$$\frac{dF}{dt} = \sum_k [F, I_1, I_2]_{x_k,y_k,z_k}$$

Nambu brackets conserve two or more quantities by construction, so for a physical system with a finite number of conserved quantities a Nambu description has less degeneracies than a Poisson description. In this sense we say that Nambu brackets are less degenerate than the Poisson bracket. Névir and Blender \cite{2} established a Nambu field approach to ideal fluid mechanics, and the above-mentioned less degeneracy property allowed Salmon \cite{3,4} to make a first application of this theory. He established that the discretization of the Nambu brackets leads to numerical schemes that conserve energy and vorticity related quantities. He showed that the Arakawa Jacobian can be derived systematically from the antisymmetry of the Nambu representation.

In a recent paper the authors \cite{7} proposed a classification of Nambu brackets where Nambu brackets of first kind (NB I) are those NBs or sums of NBs which involve the same conserved quantities \cite{1}. The sums of NBs that involve different conserved or, alternatively, constitutive quantities are called the Nambu brackets of second kind (NB II) \cite{1}. Like in \cite{5} we call the constitutive quantities those ones which form the basis of Nambu formalism but are not necessarily the constants of motion.

Our main result is an extension of Nambu formalism onto the electromagnetic field in the case of constant electric charge density and vanishing electric current density by using a NB I that involves two conserved quantities neither of them being the energy.

II. ELECTROMAGNETISM

Electromagnetic theory is one of the most fundamental physical theories with applications in almost all areas of physics, ranging from atomic interactions to astrophysical phenomena. The governing equations of electromagnetism are the famous Maxwell equations, which in Lorentz-Heaviside units read:

$$\nabla \cdot E = \rho$$
$$\nabla \cdot B = 0$$
$$\frac{\partial B}{\partial t} = -\nabla \times E$$
$$\frac{\partial E}{\partial t} = \nabla \times B - J$$
and the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J$$

Here $E$ is the electric field, $B$ is the magnetic flux density field, $\rho$ the electric charge density and $J$ is the electric current density. This is a set of linear partial differential equations, but applying a Fourier transform we can obtain a system of formally non-linear ordinary differential equations, where the wavenumber vector $k$ becomes a
quantities are Supercasimirs. Like in \[5\] we say that this kind of quantities are still conserved due to equations (8)-(12) and because of that they are degeneracies of electromagnetic invariant: 

\[ I = \frac{1}{2} \int d^3k (|\vec{E}|^2 + |\vec{B}|^2) \]

Nevertheless the latter quantities are still conserved due to equations (8)-(12) and because of that they are degeneracies of the Nambu bracket. Nevertheless we think that both deductions are close, in the formation we think that both deductions are close, in the sense that both allow us to give a mechanical interpretation of Maxwell equations in empty space. Of course, this is only a mathematical curiosity.

Using this construction we are extending Nambu’s original proposal to describe mechanics within a formalism where energy doesn’t have a superior hierarchy over other conserved quantities (casimirs). We have shown that this is possible for electromagnetism, and in a way that energy takes a merely supercasimir status. It is an interesting feature of this construction that we can construct the Nambu bracket when we apply Fourier transform to the original Maxwell equations and their conservation laws i.e., by changing from linear partial differential equations to non-linear ordinary differential equations. It is also interesting to notice that using the generalized Jacobi identity for Nambu mechanics \[6\], we can find a novel non-trivial invariant \( G \) for this system using as basis the invariants \( I_1, I_2 \) and \( \mathcal{H} \):

\[ G = \int d^3k (\vec{k} \cdot (\vec{E} \times \vec{B})) \]  

III. CONCLUDING REMARKS

We have constructed a Nambu formalism for the Maxwell equations in the case of stationary electric charge density and vanishing electric current. This formalism was found by transforming Maxwell’s linear partial differential equations to Fourier space where they are non-linear ordinary differential equations. Also this formalism conserves energy as a supercasimir and allows us to find a non-trivial invariant by use of the generalized Jacobi identity. We want to notice that by doing this in electromagnetism we have also done it for gravitomagnetism since the latter equations are isomorphic to the Maxwell equations. So, the obtained result has also implication for the general relativity in the limit case of weak gravitational fields \[3\].

A remarkably fact is the existence of a deduction of Maxwell equations in empty space from pure mechanical considerations. This deduction is due to MacCullagh in 1839 (see \[9\]) by means of a hypothetical quasi-elastic body responsive to rotations relative to absolute space. Since Nambu formalism is relative to absolute space and we made the construction by means of a Fourier transformation we think that both deductions are close, in the sense that both allow us to give a mechanical interpretation of Maxwell equations in empty space. Of course, this is only a mathematical curiosity.

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