Abstract. The purpose of this article is to give another proof on the existence of a diffusion on a junction, which has been already done in [4], but with time dependent coefficients. Such a process can be seen as a couple \((x, i)\) with \(x\) a one dimensional continuous diffusion whose coefficients depends on the edge \(i\) where it is located. We then provide an Itô’s formula for this process. Finally, we give an estimate of the local time of the process at the junction point.

1. Introduction

Originally introduced by Freidlin and Sheu in [4] and Freidlin and Wetzell in [5], stochastic diffusions in graphs have attracted a lot of attention in the last 20 years. In [5] the authors studied an elliptic linear operator on a junction, with Kirchoff condition, and the process generated by its semi group. More precisely, given a junction \(J = \bigcup_{i=1}^{I} J_i\), with \(J_i = \{0, \{i\} \times (0, +\infty)\}, i = 1, \ldots, I\), and \((\sigma_i, b_i)\) regular functions from \(\mathbb{R}_+\) to \(\mathbb{R}\), the authors have defined the elliptic operator \(L\) on the set of twice continuous differentiable function at the junction \(C^2(J)\), by

\[
L : \begin{cases}
C^2(J) \to C(J) \\
f = f_i(x) \mapsto \left( x \to \frac{1}{2} \sigma_i^2(x) \partial_{x,x} f_i(x) + b_i(x) \partial_x f_i(x), \ (x, i) \in J_i \right)
\end{cases}
\]

with domain

\[
D(L) := \left\{ f \in C^2(J), \sum_{i=1}^{I} \alpha_i \partial_x f(0, i) = 0 \right\},
\]

where \(\alpha_1, \ldots, \alpha_I\) are non-negative constants such that \(\alpha_1 + \cdots + \alpha_I = 1\). The authors of [5] have proved that there exists a continuous Markov process \(X = (x, i)\) defined on \(J\) generated by \(L\). Then [4] showed that there exists a one dimensional Wiener process \(W\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), adapted to the natural filtration of \(X = (x, i)\),
such that the process $x$ satisfies on each time interval $[0, T]$ the stochastic differential equation:

$$dx(t) = \sigma_i(t)(x(t))dW(t) + b_i(t)(x(t))dt + dl(t), \quad 0 \leq t \leq T,$$

where $l$ is a nondecreasing process starting from zero and satisfying:

$$\mathbb{P}\left(\int_0^T 1_{\{x(s) > 0\}}dl(s)_{0 \leq t \leq T} = 0\right) = 1.$$

Moreover, [4] gives the following Itô’s formula:

$$df(X(t)) = L\left(f_i(t)(x(t))\right)dt + \partial_x f_i(t)(x(t))\sigma_i(t)(x(t))dW(t) + \sum_{i=1}^l \alpha_i \partial_x f_i(0)dl(t),$$

for any sufficiently smooth test function $f$.

The aim of this note is to provide a different method for the construction of the diffusion on the junction $J$, which allows to deal with time dependent borel coefficients $(b_i, \sigma_i)$ for the stochastic differential equation (1).

Our main idea is to build the process as the limit of the càdlàg process $X^\delta$ (with parameter $\delta > 0$) which jumps at position $\delta$ on the edge $J_i$ with probability $\alpha_i$ each time it reaches the junction 0. We prove that the process $X^\delta$ converges weakly, as $\delta$ tends to 0, to a continuous diffusion process $X$ valued in $J$ and with dynamics similar to (1). The additional randomness due to the process $l$, prevent us from completely describing the behavior of the component $x$ with its dynamics (1). We therefore use the previous convergence to establish an Itô’s formula for the process $X$, which completely characterizes it. We then focus on the process $l$ involved in (1). This process can be seen as the local time of $x$ at the vertex. We therefore provide an estimate of this process as the limit of the quadratic variation of $x$ over the times spent at the neighborhood of 0.

With this construction, we will be able in a future work to formulate a stochastic control problem (in the weak sens) on a junction, with control at the junction point. They will be two types of control. The first ones will appear in the coefficients $(b_i, \sigma_i)$, and are classical from a mathematical point of view for a problem of control. The second ones are the terms $\alpha_i$ appearing in (2), in front of the term of reflection $l$, that are called control at the junction point. More precisely, the controls $\alpha_i$ will be progressively measurable process

\[^1\text{Acronym for right-continuous and left-limited}\]
valued in the set of the probability measures of moving to another edge. Hence the link with the theory of non linear parabolic PDE with Neumann boundary condition at the junction, see for instance [7] will be done.

The paper is organized as follows. In Section 2 we introduce the mathematical material that will be used throughout this work, and we state our central Theorem 2.3. Section 3 deals with the construction of the non-homogeneous diffusion on the junction. Finally, Section 4 is dedicated to Itô’s formula, and local time estimates on the vertex.

2. Main results

In this section, we state our main result Theorem 2.3, namely the existence of a weak solution of the following stochastic differential equation at the junction

\[ dy(t) = b_j(t, y(t))dt + \sigma_j(t, y(t))dB(t) + dl(t), \quad 0 \leq t \leq T, \]

where \( l \) is a nondecreasing process starting from zero and satisfying:

\[ \int_0^t 1_{\{y(s) > 0\}} dl(s)_{0 \leq t \leq T} = 0, \]

such that we have the following Itô’s formula:

\[ \left( f_j(t, y(t)) - f_j(0, y_0) = \int_0^t L(f)(s, Y(s))ds \right. \]

\[ + \int_0^t \partial_y f_j(s, y(s))\sigma_j(s, y(s))dB(s) + \sum_{i=1}^I \alpha_i \int_0^t \partial_y f_i(s, 0)dl(s) \left. \right)_{0 \leq t \leq T}, \]

where the Dynkin operator \( L(f) \) is defined by

\[ L(f)(t, (y, j)) = \partial_t f_j(t, y) + \partial_y f_j(t, y)b_j(t, y) + \frac{1}{2} \partial^2_{y,y} f_j(t, y)\sigma_j^2(t, y), \]

for any sufficiently smooth test function \( f \).

2.1. Notations and preliminary results. Let \( I \in \mathbb{N}^* \), be the number of edges. We recall that a junction \( J \) consists in a vertex and a finite number \( I \) of edges. More precisely:

\[ J = \left\{ X = (x, i), \quad x \in \mathbb{R}_+ \text{ and } i \in \{1, \ldots, I\} \right\}, \]
where all the points \((0, i), i = 1, \ldots, I\), are identified to the vertex denoted by 0. We can then write

\[ J = \bigcup_{i=1}^{I} J_i, \]

with \(J_i := \mathbb{R}_+ \times \{i\}\) and \(J_i \cap J_j = \{0\}\) for \(i \neq j\).

We introduce the following Polish spaces, that will be useful in the sequel for the construction of the process in Section 3:

(i) \((\mathcal{J}, d^\mathcal{J})\), where \(d^\mathcal{J}\) is defined by

\[
\forall ((x, i), (y, j)) \in \mathcal{J}^2, \quad d^\mathcal{J}((x, i), (y, j)) = \begin{cases} |x - y| & \text{if } i = j, \\ x + y & \text{if } i \neq j. \end{cases}
\]

(ii) \((\mathcal{U}, d^\mathcal{U})\), the set of sequences valued in \(\{1, \ldots, I\}\), where \(d^\mathcal{U}\) is defined by

\[
\forall (u, z) \in \mathcal{U}^2, \quad d^\mathcal{U}(u, z) = \sum_{n=0}^{+\infty} \frac{|u_n - z_n|}{2^n}.
\]

(iii) \((\mathbb{N}, d^\mathbb{N})\), where \(d^\mathbb{N}\) is defined by

\[
\forall (u, p) \in \mathbb{N}, \quad d^\mathbb{N}(u, p) = |u - p|.
\]

As explained in introduction, the limit process is valued in the space of continuous functions from \([0, T]\) to \(\mathcal{C}^\mathcal{J}(\mathbb{R}_+, \mathcal{J})\), which we denote by \(\mathcal{C}^\mathcal{J}([0, T])\). This space is endowed with the following uniform norm \(d^\mathcal{J}_{(0,T)}\), defined by

\[
\forall \left((x(\cdot), i(\cdot)), (y(\cdot), j(\cdot))\right) \in \mathcal{C}^\mathcal{J}([0, T])^2, \quad d^\mathcal{J}_{(0,T)}(X, Y) = \sup_{0 \leq s \leq T} d^\mathcal{J}\left((x(s), i(s)), (y(s), j(s))\right),
\]

and then is Polish.

We recall that for \((E, d^E)\) a Polish space \(E\), endowed with its metric \(d^E\), the Polish space \((\mathcal{D}^E([0, T]), d^E_{sko})\), consists of càdlàg maps defined on \([0, T]\), valued in \(E\) endowed with the Skohokhod’s metric \(d^E_{sko}\) defined by

\[
d^E_{sko}(z, u) = \inf_{\lambda \in \Lambda} \left\{ \sup_{t \neq s} \ln \left(\frac{\lambda(t) - \lambda(s)}{t - s}\right) \right\} \vee \sup_{0 \leq t \leq T} d^E\left(z(\lambda(t)), u(t)\right),
\]

where \(\Lambda\) is a compact set of \(\mathbb{R}_+\) and \(\lambda(t)\) is the càdlàg version of \(\lambda\).
where $\Lambda$ denotes the set of continuous increasing homeomorphism from $[0, T]$ to itself, and $|.|_{(0,T)}$ the standard uniform norm of $C([0, T])$ (see, for instance, [1]).

**Remark 2.1.** We recall that $d^E_{sko}$ is equivalent to the following metric defined on $D^E([0, T])$

\[
\forall (z(\cdot), u(\cdot)) \in D^E([0, T]),
\]

\[
d^E_{sko}(z, u) = \inf_{\lambda \in \Lambda} \left\{ |\lambda - Id|_{(0,T)} \vee \sup_{0 \leq t \leq T} d^E \left( z(\lambda(t)), u(t) \right) \right\},
\]

see for instance Theorem 12.1 in [1].

We get therefore, that $(D^J([0, T]), d^J_{sko}), (D^U([0, T]), d^U_{sko}), (D^N([0, T]), d^N_{sko})$ are Polish spaces.

As it has been said in introduction, for $\delta > 0$, the approximate càdlàg process $X^\delta$ jumps at position $\delta$ on the edge $J_i$ with probability $\alpha_i$ each time it reaches the junction. Therefore it is valued in the space $D^J_{\delta}([0, T])$ defined by

\[
D^J_{\delta}([0, T]) := \left\{ (x(\cdot), i(\cdot)) \in D^J([0, T]), \text{ if } d^J((x(s), i(s)), (x(s^-), i(s^-)) \neq 0, \right\}
\]

then $x(s^-) = 0$ and $x(s) = \delta \}$,

namely all the càdlàg maps whose jumps are of size $\delta$ and can only occur at the junction point. This set inherits of the structure of $D^J([0, T])$.

**Proposition 2.2.** $(D^J_{\delta}([0, T]), d^J_{sko})$ is closed in $(D^J([0, T]), d^J_{sko})$, and then a Polish space.

**Proof.** Let $X_n(\cdot) = (x_n(\cdot), i_n(\cdot))$ be a sequence in $D^J_{\delta}([0, T])$ which converges to $X(\cdot) = (x(\cdot), i(\cdot)) \in D^J([0, T])$, and let us show that $X(\cdot) \in D^J_{\delta}([0, T])$. There exists a sequence of continuous increasing homeomorphism $\lambda_n : [0, T] \rightarrow [0, T]$, such that :

\[
\sup_{t \neq s} \left| \ln \left( \frac{\lambda_n(t) - \lambda_n(s)}{t - s} \right) \right| \vee d^J_{(0,T)}(X_n \circ \lambda_n, X) \xrightarrow{n \rightarrow +\infty} 0.
\]

Using Remark 2.1, we get that

\[
|\lambda_n - Id|_{(0,T)} \vee d^J_{(0,T)}(X_n \circ \lambda_n, X) \xrightarrow{n \rightarrow +\infty} 0.
\]
Let $t \in [0, T]$, we have

$$x(t) - x(t-) = x(t) - x_n(\lambda_n(t)) + x_n(\lambda_n(t)) - x_n(\lambda_n(t-)) + x_n(\lambda_n(t-)) - x(t-). \quad (4)$$

From (4), we get

$$|x(t) - x_n(\lambda_n(t))| + |x_n(\lambda_n(t-)) - x(t-)| \xrightarrow{n \to +\infty} 0.$$ 

Knowing that the sequence $X_n \circ \lambda_n(\cdot)$ is valued too in $D_\delta^J([0, T])$, we have therefore using (4)

$$x(t) - x(t-) = x(t) - x_n(\lambda_n(t)) + x_n(\lambda_n(t-)) - x(t-),$$

or, $x(t) - x(t-) = x(t) - x_n(\lambda_n(t)) + \delta + x_n(\lambda_n(t-)) - x(t-),$

and sending $n$ to $+\infty$ we obtain finally

$$x(t) - x(t-) \in \{0, \delta\}.$$ 

We now prove that a discontinuity of $x(\cdot)$ only can occurs for $x(\cdot-) = 0$. Fix $t \in [0, T]$ such that $x(t) - x(t-) = \delta$. Let $\varepsilon \leq \frac{\delta}{4}$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $|x_n \circ \lambda_n - x|_{(0, T)} \leq \varepsilon$. We therefore get using (4)

$$x_n(\lambda_n(t)) - x_n(\lambda_n(t-)) \geq \delta - 2\varepsilon,$$

and hence

$$x_n(\lambda_n(t)) - x_n(\lambda_n(t-)) \geq \frac{\delta}{2},$$

and then necessarily, for all $n \geq n_0$, $x_n(\lambda_n(t)) = \delta$ and $x_n(\lambda_n(t-)) = 0$. Hence $x(t) = \delta$, $x(t-) = 0$, that completes the proof. \qed

2.2. Assumption and main results. We state in this subsection the central Theorem 2.3 of this note. For this, let $T > 0$, we set

$$\mathcal{J}_T = [0, T] \times \mathcal{J}.$$
Let us introduce the following data

\[
\begin{cases}
(\Omega, \mathcal{A}, \mathbb{P}) \text{ a probability space,} \\
W \text{ a standard one dimensional Brownian motion defined on } (\Omega, \mathcal{A}, \mathbb{P}), \\
Z = (Z_n)_{n \geq 0} \text{ a random variable valued in } \mathcal{U}, \text{ defined on } (\Omega, \mathcal{A}, \mathbb{P}), \\
(\alpha_i) \in (0, 1)^I, \\
x_0 = y_0 \in (0, +\infty),
\end{cases}
\]

and for each \(i \in \{1 \ldots I\}\)

\[
\begin{cases}
\sigma_i \in L_\infty((0, T) \times (0, +\infty), \mathbb{R}) \\
b_i \in L_\infty((0, T) \times (0, +\infty), \mathbb{R})
\end{cases}
\]

We suppose furthermore that the data satisfy the following assumptions:

**Assumption (H)**

(i) \(\exists c > 0, \forall i \in \{1 \ldots I\}, \forall (t, x) \in [0, T] \times [0, +\infty), \sigma_i(t, x) \geq c,\)

(ii) \(\exists (|b|, |\sigma|) \in (0, +\infty)^2, \forall i \in \{1 \ldots I\},\)

\[
\sup_{(t, x) \in [0, T] \times [0, +\infty)} |b_i(t, x)| + \sup_{t \in [0, T], (x, y) \in [0, +\infty) \times \mathbb{R}^2, x \neq y} \frac{|b_i(t, x) - b_i(t, y)|}{|x - y|} \leq |b|,
\]

(iii) \(\sup_{(t, x) \in [0, T] \times [0, +\infty)} |\sigma_i(t, x)| + \sup_{t \in [0, T], (x, y) \in [0, +\infty) \times \mathbb{R}^2, x \neq y} \frac{|\sigma_i(t, x) - \sigma_i(t, y)|}{|x - y|} \leq |\sigma|,
\]

(iv) \(Z = (Z_n)_{n \geq 0}\) are independent and identically distributed (i.i.d.), with \(\mathbb{P}(Z_n = i) = \alpha_i.\)

**Theorem 2.3.** Assume assumption (H). There exists a term

\[
\left( E, \mathcal{E}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mu, (Y(t) = (y(t), j(t), l(t)))_{0 \leq t \leq T}, (B(t))_{0 \leq t \leq T}, (\alpha(t))_{0 \leq t \leq T} \right),
\]

where:

(i) \(E, \mathcal{E}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mu\) is a filtered probability space which satisfies the usual hypotheses,

(ii) \((B(t))_{0 \leq t \leq T}\) is a standard one dimensionnal brownian motion \((\mathcal{G}_t)_{0 \leq t \leq T}\) adapted,

(iii) \((\alpha(t))_{0 \leq t \leq T}\) is a \((\mathcal{G}_t)_{0 \leq t \leq T}\) adapted process valued in the set of probability measures
on $\{1, \ldots, I\}$, such that
\[ \forall t \in [0, T], \quad \alpha(t)(i) = \alpha_i, \quad \mu \text{ a.s,} \]

(that we call a constant control at the junction point),
(iv) $(Y(t) = (y(t), j(t)))_{0 \leq t \leq T}$ is a $(G_t)_{0 \leq t \leq T}$ adapted process valued in the set continuous maps $C^J([0, T])$ at the junction,
(v) $(l(t))_{0 \leq t \leq T}$ is a $(G_t)_{0 \leq t \leq T}$ adapted process valued in the set of continuous nondecreasing real maps of $[0, T]$ satisfying
\[ \begin{cases} 
  l(0) = 0 \\
  \int_0^t 1_{\{y(s) > 0\}} dl(s) = 0, \quad 0 \leq t \leq T 
\end{cases} \mu \text{ a.s.} \]

(vi) we have furthermore the following Itô’s formula
\[
\left( f(t, Y(t)) - f(0, Y(0)) = \int_0^t L(f)(s, Y(s)) ds \right. \]
\[ + \int_0^t \partial_y f_j(s)(y(s)) \sigma_j(s)(s, y(s)) dB(s) + \sum_{i=1}^I \alpha_i \int_0^t \partial_y f_i(s, 0) dl(s) \bigg)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \quad (5) \]

where the Dynkin operator $L(f)$ is defined by
\[
L(f)(t, (y, j)) = \partial_t f_j(t, y) + \partial_y f_j(t, y) b_j(t, y) + \frac{1}{2} \partial_{y,y}^2 f_j(t, y) \sigma_j^2(t, y),
\]
for any $(t, (y, j)) \in [0, T] \times J$, and the function test $f$ belongs to the following functionnal space
\[
C^{1,2}(J_T) := \left\{ f : J_T \rightarrow \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x), \quad \forall i \in \{1 \ldots I\}, \quad f_i \in C^{1,2}_b([0, T] \times [0, +\infty)), \right. \]
\[ \forall (i, j) \in \{1 \ldots I\}^2, \quad i \neq j, \quad \forall t \in [0, T], \quad f_i(t, 0) = f_j(t, 0) \} \).

3. Construction of the process on the junction

In this section, we fix a terminal time $T > 0$, and we build a diffusion process $(X(t) = (x(t), i(t)))_{0 \leq t \leq T}$ valued in $C^J([0, T])$. The continuous process $(X(t) = (x(t), i(t)))_{0 \leq t \leq T}$ is approximated by a classical càdlàg process $X^\delta$ with parameter $\delta > 0$ which jumps at position $\delta$ on the edge $J_i$ with probability $\alpha_i$ each time it reaches the junction.

To this end we consider a family of processes $(X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T}$, depending on a
parameter $\delta > 0$, such that $(X^\delta(t))_{0 \leq t \leq T}$ is valued in $D^F_\delta([0,T])$. We prove the tightness of this sequence and get a continuous limit satisfying a stochastic differential equation on $J$.

### 3.1. Diffusion with jumps at the vertex.

We consider the sequence of diffusion processes $(x_n^\delta)_{n \geq 0}$ and stopping times $(\tau_n^\delta)_{n \geq 0}$ defined recursively by

$$
\begin{align*}
\forall t \geq 0, \quad x_0^\delta(t) &= x_0 + \int_0^t b_{\delta\nu}(x_0^\delta(s)) ds + \int_0^t \sigma_{\delta\nu}(s, x_0^\delta(s)) dW(s), \quad 0 \leq t \leq T, \\
\tau_0^\delta &= 0,
\end{align*}
$$

and, for $n \geq 0$,

$$
\begin{align*}
\forall t \geq 0, \quad x_{n+1}^\delta(t) &= \delta + \int_0^t 1_{[\tau_{n+1},+\infty)}(s) b_{\delta\nu}(s, x_{n+1}^\delta(s)) ds + \\
&+ \int_0^t 1_{[\tau_{n+1},+\infty)}(s) \sigma_{\delta\nu}(s, x_{n+1}^\delta(s)) dW(s), \quad 0 \leq t \leq T, \\
\tau_{n+1}^\delta &= \inf \{ t > \tau_n^\delta : x_n^\delta(t) = 0 \},
\end{align*}
$$

We now define the process $X^\delta(t) = (x^\delta(t), i^\delta(t))_{0 \leq t \leq T}$ valued in $D^F_\delta([0,T])$ by

$$
\begin{align*}
\forall t \geq 0, \quad x^\delta(t) &= \sum_{n \geq 0} x_n^\delta(t) 1_{[\tau_n^\delta, \tau_{n+1}^\delta)}(t), \\
i^\delta(t) &= \sum_{n \geq 0} Z_n 1_{[\tau_n^\delta, \tau_{n+1}^\delta)}(t),
\end{align*}
$$

We notice that the process $(x^\delta(t))_{0 \leq t \leq T}$ is valued in $(0, +\infty)$ and satisfies the following equation, for all $t \in [0, T]$

$$
\begin{align*}
x^\delta(t) &= x_0 + \int_0^{\tau_1^\delta \wedge t} b_{\delta\nu}(s, x^\delta(s)) ds + \int_0^{\tau_1^\delta \wedge t} \sigma_{\delta\nu}(s, x^\delta(s)) dW(s) + \\
&+ \sum_{n \geq 1} 1_{[\tau_n^\delta, \tau_{n+1}^\delta)} \left( \delta + \int_{\tau_n^\delta}^t b_{\delta\nu}(s, x^\delta(s)) ds + \int_{\tau_n^\delta}^t \sigma_{\delta\nu}(s, x^\delta(s)) dW(s) \right), \quad P \text{ a.s.}
\end{align*}
$$

From the ellipticity assumption ($\mathcal{H}$) (i), we underline that each stopping time $(\tau_n^\delta)_{n \geq 0}$ is almost surely finite. We denote by $(\mathcal{F}_t^\delta)_{0 \leq t \leq T}$ the right continuous and complete filtration
generated by \( \left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T} \). The dynamics can be rewritten as

\[
\begin{align*}
X^\delta(0) &= (x_0, Z_0) \\
\frac{dx^\delta(t)}{dt} &= b_i(t,x^\delta(t))dt + \sigma_i(t,x^\delta(t))dW(t) + \delta d\left( \sum_{0 \leq s \leq t} 1_{\{x^\delta(s-) = 0\}} \right) \quad \text{P a.s.} \quad (8)
\end{align*}
\]

\[
i^\delta(t) = Z \int_{0 \leq s \leq t} 1_{\{x^\delta(s-) = 0\}} dW(t)
\]

In the next subsection we prove that the sequence \( \left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T} \) is C-tight, namely up to a subsequence \( \left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T} \) converges weakly to a random variable \( \left( X(t) = x(t), i(t) \right)_{0 \leq t \leq T} \), with continuous paths. In other words the limit \( \left( X(t) = x(t), i(t) \right)_{0 \leq t \leq T} \) is valued in \( C^J([0,T]) \).

3.2. **Weak convergence.** For \( Y \in D^J([0,T]) \), we denote by \( \omega_T(Y,. \) the modulus of continuity of \( Y \) on \( [0,T] \) defined by

\[
\omega_T(Y, \theta) = \sup \left\{ d^J(Y(s),Y(u)), (u,s) \in [t,t+\theta], 0 \leq t \leq t + \theta \leq T \right\}, \quad (9)
\]

for all \( \theta \in [0,T] \).

**Proposition 3.1.** There exists a constant \( C \) independent of \( \delta \), depending only on

\[
\left( T, \|b\|, \|\sigma\| \right),
\]

such that

\[
\mathbb{E}_T \left[ |x^\delta|_0^2 \right] \leq C \left( 1 + x_0^2 + \delta^2 \right), \quad (10)
\]

\[
\forall \theta \in (0,T], \quad \mathbb{E}_T \left[ \omega_T(X^\delta, \theta)^2 \right] \leq C \left( \delta^2 + \theta \ln \left( \frac{2T}{\theta} \right) \right). \quad (11)
\]

**Proof.** Let \( t \in [0,T] \). We use the stopping times \( (\tau^\delta_n) \) defined in (4) and we introduce the following random times

\[
\tau^\delta_t := \inf \left\{ \tau^\delta_n, \tau^\delta_n \geq t \right\} \quad \text{and} \quad \tau^\delta := \sup \left\{ \tau^\delta_n, \tau^\delta_n \leq t \right\}.
\]
Using (6) and (8), we have
\[
|x^\delta(t)|^2 \leq 3 \left( |x^\delta(\tau^\delta)|^2 + \left| \int_{\tau^\delta}^t b_i^\delta(s) \, ds \right|^2 + \left| \int_{\tau^\delta}^t \sigma_i^\delta(s, x^\delta(s)) \, dW(s) \right|^2 \right) \leq \\
3 \left( |x_0|^2 + \delta^2 + T^2 |b|^2 + 2 \sup_{t \in [0, T]} \left| \int_{0}^{t} \sigma_i^\delta(s)(x^\delta(s)) \, dW(s) \right|^2 \right), \quad \mathbb{P} \text{ a.s.}
\]

Using assumption (H) and Doob-Meyer inequality, we get that there exists a constant $C$ independent of $\delta$, depending only on $(T, |b|, |\sigma|)$ such that
\[
\mathbb{E}^\mathbb{P} \left[ |x^\delta|_{(0, T)}^2 \right] \leq C \left( 1 + x_0^2 + \delta^2 \right).
\]

Let $\theta \in (0, T]$, and $(t, s) \in [0, T]^2$, such that $s \leq t$ and $|t - s| \leq \theta$. We define the process
\[
(V^\delta(t))_{0 \leq t \leq T}
\]
by
\[
V^\delta(t) = \int_{0}^{t} b_i^\delta(s, x^\delta(s)) \, ds + \int_{0}^{t} \sigma_i^\delta(s, x^\delta(s)) \, dW(s), \quad \mathbb{P} \text{ a.s.}
\]

We have:

\[
\begin{align*}
\text{if } \sum_{s \leq u \leq t} 1_{\{x^\delta(s-) = 0\}} = 0, \text{ then } & d^J(X^\delta(s), X^\delta(t)) \leq \omega_T(V^\delta, \theta), \\
\text{if } \sum_{s \leq u \leq t} 1_{\{x^\delta(s-) = 0\}} \geq 1, \text{ then } & d^J(X^\delta(s), X^\delta(t)) \leq \\
d^J(X^\delta(s), X^\delta(T^\delta - )) + d^J(X^\delta(T^\delta - ), X^\delta(T^\delta + )) + d^J(X^\delta(T^\delta + ), X^\delta(t)) \leq & \delta + 2\omega_T(V^\delta, \theta), \quad \mathbb{P} \text{ a.s.}
\end{align*}
\]

Using Theorem 1 of [3], and assumption (H), we get that there exists a constant $C$ independent of $\delta$, depending only on $(T, |b|, |\sigma|)$, such that
\[
\mathbb{E}^\mathbb{P} \left[ \omega_T(V^\delta, \theta)^2 \right] \leq C \theta \ln \left( \frac{2T}{\theta} \right).
\]

Therefore
\[
\mathbb{E}^\mathbb{P} \left[ \omega_T(X^\delta, \theta)^2 \right] \leq C \left( \delta^2 + \theta \ln \left( \frac{2T}{\theta} \right) \right).
\]

This completes the proof. \qed
Now we have the necessary tools to prove the C-tightness of the sequence \( (X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T} \).

**Theorem 3.2.** The sequence \( (X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T} \) is C-tight.

**Proof.** Let \( \delta > 0, \theta > 0, \eta > 0 \) and \( h > 0 \). Using Proposition 3.1 and Markov inequality, there exists a constant \( C \), independent of \( \delta \) and depending only on \( (T, |b|, |\sigma|) \), such that

\[
P\left( |x^\delta|_{(0,T)} > h \right) \leq C \frac{\sqrt{1 + x_0^2 + \delta^2}}{h},
\]

\[
P\left( \omega_T(X^\delta, \theta) \geq \eta \right) \leq C \frac{\sqrt{\delta^2 + \theta \ln \left( \frac{2T}{\theta} \right)}}{\eta}.
\]

Therefore we get

\[
\lim_{h \to +\infty} \limsup_{\delta \to 0} P \left( |x^\delta|_{(0,T)} > h \right) = 0;
\]

\[
\lim_{\eta \to 0^+} \limsup_{\theta \to 0} \limsup_{\delta \to 0} P \left( \omega_T(X^\delta, \theta) \geq \eta \right) = 0.
\]

Theorem 3.21 and Proposition 3.26 of [6] then imply the tightness of \( (X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T} \) as \( \delta \) tends to 0.

From Prokhorov’s Theorem, there is a subsequence, denoted in the same way, such that

\[
\mathbb{P}_{X^\delta} \xrightarrow{\delta \to 0} \text{law} \mu,
\]

where \( \mu \) is a probability measure defined on \( C^J([0,T]) \).

**3.3. Identification of the limit law.** The aim of this subsection is to identify the law \( \mu \).

To this end, we transform the weak convergence of \( X^\delta \) into a strong one using Skorokhod’s representation Theorem. Using Tikhonov’s Theorem, we know that \( \mathcal{U} \) is compact, therefore it is easy to get that the random variable \( (X^\delta, W, Z) \) is precompact for the weak topology in the space \( C^J([0,T]) \times C([0,T]) \times U \). There exists a probability space \( (E, \mathcal{E}, \mu) \) and random variables \( (Y^\delta = (y^\delta, j^\delta), B^\delta, A^\delta) \) and \( (Y = (y, j), B, A) \), taking their values
in the following Polish spaces
\[
\left( \mathcal{D}_\delta^J([0,T]) \times \mathcal{C}([0,T]) \times \mathcal{U}, \quad d_{sko}^J, \cdot |_{(0,T)} + d^{\mathcal{I}} \right),
\]
and respectively
\[
\left( \mathcal{C}^J([0,T]) \times \mathcal{C}([0,T]) \times \mathcal{U}, d^J_{(0,T)} + \cdot |_{(0,T)} + d^{\mathcal{I}} \right),
\]
such that
\[
\forall \delta > 0, \quad (Y^\delta, B^\delta, A^\delta) \overset{law}{=} (X^\delta, W, Z)
\]
\[
(Y, B, A) \overset{law}{=} (X, W, Z)
\]
and
\[
d_{sko}^J(Y^\delta, Y) + |B^\delta - B|_{(0,T)} + d^U(A^\delta, A) \xrightarrow{\mu-a.s.} 0.
\] (12)

To make easier the reading, in the rest of these notes, we denote by \( \left(N^\delta(t)\right)_{0 \leq t \leq T} \) the following process valued in \( \mathbb{N} \), whose paths are defined by
\[
N^\delta(t) = \sum_{0 \leq s \leq t} 1_{\{y^\delta(s-)=0\}}, \quad \mu \text{ a.s.}
\] (13)

**Proposition 3.3.** For any \( \delta > 0 \), the process \( \left(y^\delta(t), j^\delta(t)\right)_{0 \leq t \leq T} \) satisfies
\[
\left( \begin{array}{l}
y^\delta(t) = y_0 + \int_0^t b^\delta_j(s, y^\delta(s))ds + \int_0^t \sigma^\delta_j(s, y^\delta(s))dB^\delta(s) + \delta N^\delta(t) \\
j^\delta(t) = A^\delta_{N^\delta(t)}
\end{array} \right)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \] (14)

**Proof.** Let \( i \in \{1 \ldots I\} \), and \( \gamma > 0 \). We define \( (b^\gamma_i, \sigma^\gamma_i) \) by
\[
\sigma^\gamma_i(t, x) = \sigma_i(t, x)1_{x > \gamma} + \frac{\sigma_i(t, \gamma)}{\gamma} x 1_{0 \leq x \leq \gamma}, \quad \text{if} \ (t, x) \in [0, T] \times [0, +\infty),
\]
\[
b^\gamma_i(t, x) = b_i(t, x)1_{x > \gamma} + \frac{b_i(t, \gamma)}{\gamma} x 1_{0 \leq x \leq \gamma}, \quad \text{if} \ (t, x) \in [0, T] \times [0, +\infty).
\]
Therefore for all \( i \in \{1 \ldots I\} \), \( (b^\gamma_i, \sigma^\gamma_i) \in L_\infty((0, T) \times (0, +\infty), \mathbb{R}) \) and satisfies the following continuity condition at the junction point
\[
\forall t \in [0, T], \quad \forall i, j \in \{1 \ldots I\}, \quad i \neq j, \quad \sigma^\gamma_i(t, 0) = \sigma^\gamma_j(t, 0) = 0, \quad b^\gamma_i(t, 0) = b^\gamma_j(t, 0) = 0.
\]
Moreover, using assumption \((\mathcal{H})\), it is to get that
\[
\exists (|b^\gamma|, |\sigma^\gamma|) \in (0, +\infty)^2, \quad \forall i \in \{1 \ldots J\},
\]
\[
\sup_{(t,x) \in [0,T] \times [0, +\infty)} |b_i^\gamma(t, x)| + \sup_{t \in [0,T], (x, y) \in [0, +\infty), x \neq y} \frac{|b_i^\gamma(t, x) - b_i^\gamma(t, y)|}{|x - y|} \leq |b^\gamma|,
\]
\[
\sup_{(t,x) \in [0,T] \times [0, +\infty)} |\sigma_i^\gamma(t, x)| + \sup_{t \in [0,T], (x, y) \in [0, +\infty), x \neq y} \frac{|\sigma_i^\gamma(t, x) - \sigma_i^\gamma(t, y)|}{|x - y|} \leq |\sigma^\gamma|. \tag{15}
\]

**Step 1.** For \(n \geq 0\), we start by proving that the following maps
\[
\rho := \begin{cases}
D_\delta^J([0, T]) \to D^R([0, T]) \\
(x(\cdot), i(\cdot)) \mapsto \left( \sum_{0 \leq s \leq t} 1_{\{x(s) - x(s-) = \delta\}} \right)_{0 \leq t \leq T},
\end{cases}
\]
\[
\kappa := \begin{cases}
\mathcal{U} \times D_\delta^J([0, T]) \to D^U([0, T]) \\
(Z, (x(\cdot), i(\cdot))) \mapsto \left( Z \sum_{0 \leq s \leq t} 1_{\{x(s) - x(s-) = \delta\}} \right)_{0 \leq t \leq T},
\end{cases}
\]
\[
\psi_{n, \gamma} := \begin{cases}
\mathcal{C}([0, T]) \times D_\delta^J([0, T]) \to D^R([0, T]) \\
(w(\cdot), (x(\cdot), i(\cdot))) \mapsto \left( \sum_{j=0}^{2^n - 1} \sigma_j^\gamma i^{\frac{j}{2^n}}(\frac{j}{2^n}, x(\frac{j}{2^n}))(w(j+1) - w(j)) \right)_{0 \leq t \leq T},
\end{cases}
\]
\[
\phi_{n, \gamma} := \begin{cases}
D_\delta^J([0, T]) \to D^R([0, T]) \\
(x(\cdot), i(\cdot)) \mapsto \left( x(0) + \frac{1}{n} \sum_{j=0}^{n-1} b_j^\gamma i^{\frac{j}{2^n}}(\frac{j}{2^n}, x(\frac{j}{2^n})) \right)_{0 \leq t \leq T},
\end{cases}
\]
are continuous.

To this end let \(Z^k, w^k(\cdot), (x^k(\cdot), i^k(\cdot))\) converge to \((Z, w(\cdot), (x(\cdot), i(\cdot)))\) in \(\mathcal{U} \times \mathcal{C}[0, T] \times D_\delta^J([0, T])\).

Using Remark 2.1, we get that for all \(\varepsilon > 0\), there exists \(k_\varepsilon \in \mathbb{N}\), and a sequence \(\lambda^k\) valued in \(\Lambda\) (the set of continuous increasing homeomorphism from \([0, T]\) to itself), such that for all \(k \geq k_\varepsilon\)
\[
\sum_{n \geq 0} \left| \frac{Z_n^k - Z_n}{2^n} \right| + |w^k - w|_{(0, T)} + |\lambda^k - Id|_{(0, T)} + d_{(0, T)}^J((x^k \circ \lambda^k(\cdot), i^k \circ \lambda^k(\cdot), (x(\cdot), i(\cdot))) \leq \varepsilon. \tag{16}
\]
Substep 1.1 \( \rho \) and \( \kappa \) are continuous:

Let \( \varepsilon < \frac{\delta}{4} \wedge \frac{1}{2} \). We have

\[
\frac{\delta}{2} \geq x^k(\lambda^k(s)) - x^k(\lambda^k(s-)) - (x(s) - x(s-)) \geq -\frac{\delta}{2}.
\]

(17)

for each \( s \in [0, T] \).

Since, from Proposition 2.2 \( D^\gamma_\delta([0, T]) \) is closed, \( (x(\cdot), i(\cdot)) \in D^\gamma_\delta([0, T]) \). Using (17) and the assumption \( \varepsilon < \delta/4 \), we then have

\[
\left\{ s \in [0, t], \ x(s) - x(s-) = \delta \right\} = \left\{ s \in [0, t], \ x^k(\lambda^k(s)) - x^k(\lambda^k(s-)) = \delta \right\}
\]

which implies that

\[
\sum_{0 \leq s \leq \lambda^k(t)} 1_{\{x^k(s)-x^k(s-)=\delta\}} = \sum_{0 \leq s \leq t} 1_{\{x(s)-x(s-)=\delta\}},
\]

(18)

and

\[
|\lambda^k - Id|_{(0, T)} \vee \sup_{0 \leq t \leq T} d^N\left( \sum_{0 \leq s \leq \lambda^k(t)} 1_{\{x^k(s)-x^k(s-)=\delta\}}, \sum_{0 \leq s \leq t} 1_{\{x(s)-x(s-)=\delta\}} \right) \leq \varepsilon,
\]

namely \( \rho \) is continuous.

On the other hand we get using (16) and (18), that for all \( t \in [0, T] \)

\[
d^U\left( \kappa\left(Z^k, (x^k(\cdot), i^k(\cdot))\right)(\lambda^k(t)), \kappa\left(Z, (x(\cdot), i(\cdot))\right)(t) \right) = \frac{\left| Z^k_n \left\{ \sum_{0 \leq s \leq \lambda^k(t)} 1_{\{x^k(s)-x^k(s-)=\delta\}} = n \right\} - Z_n 1_{\{\sum_{0 \leq s \leq t} 1_{\{x(s)-x(s-)=\delta\}} = n\} \right|}{2^n} \leq \varepsilon.
\]

Therefore

\[
|\lambda^k - Id|_{(0, T)} \vee \sup_{0 \leq t \leq T} d^U\left( \kappa\left(Z^k, (x^k(\cdot), i^k(\cdot))\right)(\lambda^k(t)), \kappa\left(Z, (x(\cdot), i(\cdot))\right)(t) \right) \leq \varepsilon,
\]

and \( \kappa \) is continuous.

Substep 1.2 Let us show now that \( \psi_{n, \gamma} \) and \( \phi_{n, \gamma} \) are continuous.
Let \( j \in \{0, \ldots 2^n - 1\} \), and \( t \in [0, T] \), we write
\[
\begin{align*}
& \left| \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{i^{\lambda_k(t)}}{2^n}, \frac{x^k(i^{\lambda_k(t)})}{2^n} \right) - \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, \frac{x(j/2^n)}{2^n} \right) \right| \\
& \quad + \left| \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, 0 \right) - \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, 0 \right) \right| + \left| \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, 0 \right) - \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, 0 \right) \right|.
\end{align*}
\]

Since
\[
\forall t \in [0, T], \ \forall i, j \in \{1 \ldots I\}, \ i \neq j, \ \sigma^\gamma_i(t, 0) = \sigma^\gamma_j(t, 0) = 0,
\]
we get from assumption (15) and (16)
\[
\left| \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{i^{\lambda_k(t)}}{2^n}, \frac{x^k(i^{\lambda_k(t)})}{2^n} \right) - \sigma^\gamma_{i^{\frac{j}{2^n}}} \left( \frac{j}{2^n}, \frac{x(j/2^n)}{2^n} \right) \right| \leq |\sigma^\gamma| \varepsilon.
\]

Therefore, simple computations allows us to write that there exists a constant \( C > 0 \), independent of \( k \), depending only on the data \((n, |\sigma^\gamma|, |w|_{(0,T)})\) such that
\[
|\lambda^k - Id|_{(0,T)} \vee \sup_{0 \leq t \leq T} \left( \psi_{n,\gamma}(w^k, (x^k, i^k)) \circ \lambda^k(t), \psi_{n,\gamma}(w(t), (x(t), i(t)) \right) \leq C(\varepsilon + \sup_{|t-s| \leq \varepsilon} |w(t) - w(s)|).
\]

We conclude then that \( \psi_{n,\gamma} \) is continuous.

With the same arguments used for the continuity of \( \psi_{n,\gamma} \), we can show then that \( \phi_{n,\gamma} \) is continuous.

**Step 2.**

We have now from the results of **Step 1** the mains tools to prove Proposition 3.3.

First of all, it is easy to get since \( y^\delta(\cdot) \) and \( x^\delta(\cdot) \) have the same law (recall that \( x_0 = y_0 \in (0, +\infty) \))
\[
\mu(y^\delta(0) = y_0 = x_0) = \mathbb{P}(x^\delta(0) = x_0) = 1.
\]

Let \((g, h) \in C_b \left( \mathcal{D}([0, T]), \mathbb{R} \right) \times C_b \left( \mathcal{D}^d([0, T]), \mathbb{R} \right). \) Since \( \rho, \kappa, \psi_{n,\gamma}, \phi_{n,\gamma} \) are continuous, we have
\[ \mathbb{E}^\mu \left[ \int_{0}^{T} \sigma_{i(s)}^\gamma(s, x^\delta(s)) dW(s) - \int_{0}^{T} \sigma_{i(s)}^{\gamma\delta}(s, x^\delta(s)) dW(s) \right] = \mathbb{E}^\mu \left[ \int_{0}^{T} \sigma_{j(s)}^{\gamma\delta}(s, y^\delta(s)) dB^\delta(s) - \int_{0}^{T} \sigma_{j(s)}^{\gamma\delta}(s, y^\delta(s)) dB^\delta(s) \right] = 0. \]
Therefore up to a subsequence $\gamma_p$,

$$
\left| \int_0^T \sigma_i^{\delta}(s,x^\delta(s))dW(s) - \int_0^T \sigma_{\gamma_p}^{\delta}(s,x^\delta(s))dW(s) \right|_{(0,T)} \xrightarrow{P-a.s.,\gamma_p \to 0} 0,
$$

and

$$
\left| \int_0^T \sigma_j^{\delta}(s,y^\delta(s))dB^\delta(s) - \int_0^T \sigma_{\gamma_p}^{\delta}(s,y^\delta(s))dB^\delta(s) \right|_{(0,T)} \xrightarrow{\mu-a.s.,\gamma_p \to 0} 0.
$$

Using the same arguments, we can show that up to another subsequence $(\gamma_{p'},n_{k'})$

$$
\lim_{\gamma_{p'} \to 0} \limsup_{n_{k'} \to +\infty} \left| \int_0^T b_i^{\delta}(s,x^\delta(s))ds - \phi_{n_{k'},\gamma_{p'}}(X^\delta)(\cdot) \right|_{(0,T)} \xrightarrow{P-a.s.} 0,
$$

$$
\lim_{\gamma_{p'} \to 0} \limsup_{n_{k'} \to +\infty} \left| \int_0^T b_j^{\delta}(s,y^\delta(s))ds - \phi_{n_{k'},\gamma_{p'}}(Y^\delta)(\cdot) \right|_{(0,T)} \xrightarrow{\mu-a.s.} 0.
$$

On the other hand using that

$$
\sum_{0 \leq s \leq \cdot} 1_{\{x(s) - x(-) = \delta\}} = \sum_{0 \leq s \leq \cdot} 1_{\{x(-) = 0\}}, \quad P \text{ a.s.,}
$$

$\forall s \in [0, T], \quad \mu\left((y^\delta(s) - y^\delta(-) = \delta, y^\delta(-) > 0\right) = \mathbb{P}\left((x^\delta(s) - x^\delta(-) = \delta, x^\delta(-) > 0\right) = 0,$

and applying Lebesgue’s Theorem, we get from (19) that for any $g \in C_b\left(\mathcal{D}([0,T]), \mathbb{R}\right)$

$$
\lim_{\gamma_{p'} \to 0} \limsup_{n_{k'} \to +\infty} \mathbb{E}^p \left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(X^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(W,X^\delta)(\cdot) + \delta\rho(X^\delta)(\cdot)\right)\right) \right]
$$

$$
= \mathbb{E}^p \left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(X^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(W,X^\delta)(\cdot) + \delta\rho(X^\delta)(\cdot)\right)\right) \right],
$$

and

$$
\lim_{\gamma_{p'} \to 0} \limsup_{n_{k'} \to +\infty} \mathbb{E}\left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(Y^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(B^\delta,Y^\delta)(\cdot) + \delta\rho(Y^\delta)(\cdot)\right)\right) \right]
$$

$$
= \mathbb{E}\left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(Y^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(B^\delta,Y^\delta)(\cdot) + \delta\rho(Y^\delta)(\cdot)\right)\right) \right].
$$

Therefore for any $g \in C_b\left(\mathcal{D}([0,T]), \mathbb{R}\right)$

$$
\mathbb{E}^p \left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(X^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(W,X^\delta)(\cdot) + \delta\rho(X^\delta)(\cdot)\right)\right) \right]
$$

$$
= \mathbb{E}\left[ g\left(\left(\phi_{n_{k'},\gamma_{p'}}(Y^\delta)(\cdot) + \psi_{n_{k'},\gamma_{p'}}(B^\delta,Y^\delta)(\cdot) + \delta\rho(Y^\delta)(\cdot)\right)\right) \right].
$$
Considering a sequence \( g_n \) of \( C_b(\mathcal{D}([0,T]),\mathbb{R}) \), uniformly bounded, converging to \( x \to 1_{\{x=0\}} \) in the pointwise sense, we get that
\[
\mu\left( \begin{array}{c}
y^\delta(t) = y_0 - \int_0^t b_{j^\delta(s)}(s, y^\delta(s)) \, ds - \int_0^t \sigma_{j^\delta(s)}(s, y^\delta(s)) \, dB^\delta(s) + \\
\delta\left( \sum_{0 \leq s \leq t} 1_{\{y^\delta(s-) = 0\}} \right)
\end{array} \right)_{0 \leq t \leq T} = 1.
\]
Using the same arguments and the continuity of \( \kappa \) we also infer that
\[
\mu\left( \begin{array}{c}
j^\delta(t) = A^\delta \sum_{0 \leq s \leq t} 1_{\{y^\delta(s-) = 0\}}
\end{array} \right)_{0 \leq t \leq T} = 1,
\]
that completes the proof of Proposition 3.3.

\[\square\]

Remark 3.4. As a consequence of \((14)\) and Proposition 3.1, it is easy to get that there exists a constant \( C \) independent of \( \delta \), depending only on
\[(T, |b|, |\sigma|, y_0),\]
such that
\[
\mathbb{E}\mu\left[ |y^\delta(\cdot)|^2_{(0,T)} \right] \leq C \left( 1 + \delta^2 \right), \quad \mathbb{E}\mu\left[ |\delta N^\delta(\cdot)|^2_{(0,T)} \right] \leq C \left( 1 + \delta^2 \right).
\]

For simplicity, we denote by \((Y^\delta = (y^\delta, j^\delta), B^\delta, A^\delta)\) the subsequence converging almost surely to \((Y = (y, j), B, A)\). Using Proposition 3.3 we know that the process \(Y^\delta = (y^\delta, j^\delta)\) satisfies
\[
\begin{cases}
y^\delta(\cdot) = y_0 + \int_0^{\gamma^\delta_{n+1}} b_{j^\delta(s)}(s, y^\delta(s)) \, ds + \int_0^{\gamma^\delta_{n+1}} \sigma_{j^\delta(s)}(s, y^\delta(s)) \, dB^\delta(s) \\
+ \sum_{n \geq 1} 1_{\{\gamma^\delta_n \leq s < \gamma^\delta_{n+1}\}} \left( \delta + \int_{\gamma^\delta_n}^{\gamma^\delta_{n+1}} b_{j^\delta(s)}(s, y^\delta(s)) \, ds + \int_{\gamma^\delta_n}^{\gamma^\delta_{n+1}} \sigma_{j^\delta(s)}(s, y^\delta(s)) \, dB^\delta(s) \right), \mu \text{ a.s. (21)}
\end{cases}
\]
\[
j^\delta(\cdot) = \sum_{n \geq 0} A^\delta_n 1_{\{\gamma^\delta_n \leq s < \gamma^\delta_{n+1}\}}(\cdot)
\]
where we have defined recursively the following stopping times
\[
\gamma^\delta_0 = 0, \quad \gamma^\delta_{n+1} = \inf\left\{ T \geq t > \gamma^\delta_n : y^\delta(t) = 0 \right\}, \mu \text{ a.s. (22)}
\]
We denote by \((G^\delta_t)_{0 \leq t \leq T}\) the right continuous and complete filtration generated by \(Y^\delta(t) = (y^\delta(t), j^\delta(t))\). We also denote by \((G_t)_{0 \leq t \leq T}\) the right continuous and complete filtration generated by the process \(Y(t) = (y(t), j(t))\). We now state some properties of the process \(Y(t) = (y(t), j(t))\).

**Proposition 3.5.** The process \(Y^\delta(t) = (y^\delta(t), j^\delta(t))\) converges uniformly to \(Y(t) = (y(t), j(t))\), namely:

\[
\mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta, Y) \xrightarrow{\mu-a.s.} 0, \quad (23)
\]

and therefore

\[
|y^\delta - y|_{(0,T)} \xrightarrow{\mu-a.s.} 0. \quad (24)
\]

**Proof.** From \([12]\) and Remark 2.1 there exists a (random) change of time \(\lambda^\delta\) such that

\[
|\lambda^\delta - Id|_{(0,T)} \vee \mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta \circ \lambda^\delta, Y) \xrightarrow{\mu-a.s.} 0.
\]

We then have

\[
\mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta, Y) \leq \mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta \circ \lambda^\delta \circ (\lambda^\delta)^{-1}, Y \circ (\lambda^\delta)^{-1}) + \mathcal{d}_{(0,T)}^\mathcal{J}(Y \circ (\lambda^\delta)^{-1}, Y)
\]

\[
\leq \mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta \circ \lambda^\delta, Y) + \mathcal{d}_{(0,T)}^\mathcal{J}(Y \circ (\lambda^\delta)^{-1}, Y), \quad \mu \text{ a.s.}
\]

From Theorem 3.2, \(Y\) is continuous and hence uniformly continuous on \([0, T]\). Therefore we get

\[
\mathcal{d}_{(0,T)}^\mathcal{J}(Y^\delta, Y) \xrightarrow{\mu-a.s.} 0,
\]

and consequently

\[
|y^\delta - y|_{(0,T)} \xrightarrow{\mu-a.s.} 0.
\]

\[\square\]

Next we will study the behavior of the jump part of \(y^\delta(\cdot)\). For this we state first the following Lemma.
Lemma 3.6. Fix $M > 0$. There exists a constant $C > 0$, independent of $\delta$, depending only on the data
\[ (T, M, |b|, |\sigma|, y_0), \]
introduced in assumption $(H)$, such that
\[ \mathbb{E}^\mu\left[ \exp(My^\delta(T)) \right] \leq C \exp(M\delta). \] (25)

Proof. We define the following map $\phi$ by
\[ \phi := \begin{cases} 
[0, +\infty) \to \mathbb{R} \\
x \mapsto \exp(Mx) - Mx - 1 
\end{cases}. \]

Let $k \geq 0$, we introduce the following stopping times
\[ \gamma^\delta_{\leq T} := \sup \left\{ \gamma_{n+1}^\delta : \gamma_n^\delta \leq T \right\}, \quad \theta^\delta_k := \inf \left\{ s \in [\gamma^\delta_{\leq T}, T] : y^\delta(s) \geq k \right\}, \]
where the sequence $(\gamma^\delta_{n+1})_{n \geq 0}$ is defined in (22).

Hence, applying Itô's formula in the following interval $(\gamma^\delta_{\leq T}, T \wedge \theta^\delta_k)$, using (21) and Remark 3.4, we get
\[ \mathbb{E}^\mu\left[ \exp(My^\delta(T \wedge \theta^\delta_k)) \right] = \left( \exp(M\delta) - M\delta \right) 1_{\{\gamma^\delta_{\leq T} \geq \gamma^\delta_1\}} + \left( \exp(My_0) - My_0 \right) 1_{\{\gamma^\delta_{\leq T} = 0\}} \]
\[ + \mathbb{E}^\mu\left[ My^\delta(T \wedge \theta^\delta_k) \right] + \mathbb{E}^\mu\left[ \int_{\gamma^\delta_{\leq T}}^{T \wedge \theta^\delta_k} \left( \frac{1}{2} \sigma^2_{j^\delta(u)}(u, y^\delta(u)) \partial_x \phi(y^\delta(u)) + b_{j^\delta(u)}(u, y^\delta(u)) \partial_x \phi(y^\delta(u)) \right) du \right] \leq C \exp(M\delta) \left( 1 + \mathbb{E}^\mu\left[ \int_{\gamma^\delta_{\leq T}}^{T \wedge \theta^\delta_k} \exp(My^\delta(u)) du \right] \right), \]
where $C$ is a constant depending only on
\[ (T, M, |b|, |\sigma|, y_0). \]

Hence sending $k \to +\infty$, we get using monotone convergence's Theorem, and Fubini’s Theorem
\[ \mathbb{E}^\mu\left[ \exp(My^\delta(T)) \right] \leq C \exp(M\delta) \left( 1 + \mathbb{E}^\mu\left[ \int_{\gamma^\delta_{\leq T}}^{T} \exp(My^\delta(u)) du \right] \right) \]
\[ \leq C \exp(M\delta) \left( 1 + \int_{\gamma^\delta_{\leq T}}^{T} \mathbb{E}^\mu\left[ \exp(My^\delta(u)) du \right] \right) \leq C \exp(M\delta) \left( 1 + \int_{0}^{T} \mathbb{E}^\mu\left[ \exp(My^\delta(u)) du \right] \right). \]
We conclude finally using Gronwall’s Lemma to the following measurable map

\[ \rho : [0, T] \rightarrow \mathbb{R} \\
\text{s} \mapsto \mathbb{E}^{\mu} \left[ \exp(M y^\delta(s)) \right]. \]

□

We now study the behavior of the jump part of \( y^\delta(\cdot) \). To this end, we estimate the time spent by \( y(\cdot) \) at the junction point, and we state the following proposition that will be useful in the sequel.

**Proposition 3.7.** We have

\[ \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \mathbb{E}^{\mu} \left[ \int_0^T 1_{\{y^\delta(s) \leq \varepsilon\}} ds \right] = 0, \] (26)

and therefore

\[ \mu \left( \int_0^T 1_{\{y(s) = 0\}} ds = 0 \right) = 1. \]

**Proof.** Let \( \varepsilon > \delta \), and \( \beta^\varepsilon \in \mathcal{C}([0, +\infty), \mathbb{R}_+) \) satisfying

\[ \forall x \geq 2\varepsilon, \, \beta^\varepsilon(x) = 0, \, \forall x \geq 0, \, 1_{\{x \leq \varepsilon\}} \leq \beta^\varepsilon(x) \leq 1. \] (27)

Let \( u^\varepsilon \in \mathcal{C}^2([0, +\infty)) \) be the unique solution of the following ordinary second order differential equation

\[
\begin{cases}
\partial_{xx} u^\varepsilon(x) - M \partial_x u^\varepsilon(x) = 2\beta^\varepsilon(x)/c^2, & \text{if } x \in (0, +\infty), \\
\partial_x u^\varepsilon(0) = 0,
\end{cases}
\] (28)

where \( c \) is the constant of ellipticity defined in assumption \((\mathcal{H})\), and \( M \) is given by

\[ M = \frac{|b|}{2c^2}. \]

For each \( i \in \{1 \ldots I\} \), the solution is given by

\[ u^\varepsilon(x) = \int_0^x \exp \left( M z \right) \int_0^z \frac{2\beta_i^\varepsilon(u)}{c^2} \exp(-Mu)dudz. \]
By the assumption on $\beta_\varepsilon$, and assumption $(\mathcal{H})$, we get
\[ \forall x \geq 0, \ 0 \leq \partial_x u^\varepsilon(x) \leq 4\varepsilon/c^2 \exp(Mx), \ 0 \leq u^\varepsilon(x) \leq \frac{4\varepsilon}{Mc^2}(\exp(Mx) - 1). \quad (29) \]

Recall the definition of the stopping times $(22)$
\[ \gamma_0^\delta = 0, \ \gamma_{n+1}^\delta = \inf \{ T \geq t > \gamma_n^\delta : y^\delta(t) = 0 \}, \ \mu \text{ a.s.} \]

We write then
\[
\begin{align*}
&\quad u^\varepsilon(y^\delta(T)) - u^\varepsilon(y_0) = \\
&\quad \left( \sum_{n \geq 0} u^\varepsilon(y^\delta(\gamma_{n+1}^\delta)) - u^\varepsilon(y^\delta(\gamma_n^\delta +)) \right) \\
&\quad + \left( \sum_{n \geq 0} u^\varepsilon(y^\delta(\gamma_n^\delta +)) - u(y^\delta(\gamma_n^\delta -)) \right) \ \mu \text{ a.s.} \quad (30)
\end{align*}
\]

For $(30)$, we apply Itô’s formula and use $(28)$ on each interval $(\gamma_n^\delta, \gamma_{n+1}^\delta)$ to get (after an argument of localization with stopping times)
\[
\begin{align*}
&\quad \mathbb{E}^\mu \left[ \sum_{n \geq 0} u^\varepsilon(y^\delta(\gamma_{n+1}^\delta)) - u^\varepsilon(y^\delta(\gamma_n^\delta +)) \right] \\
&= \mathbb{E}^\mu \left[ \int_0^T \left( \frac{1}{2} \sigma^2_{y^\delta(s)}(s, y^\delta(s)) \partial_x x u^\varepsilon(y^\delta(s)) + b_{y^\delta(s)}(s, y^\delta(s)) \partial_x x u^\varepsilon(y^\delta(s)) \right) ds \right] \\
&= \mathbb{E}^\mu \left[ \int_0^T \frac{1}{2} \sigma^2_{y^\delta(s)}(s, y^\delta(s)) \left( \partial_x x u^\varepsilon(y^\delta(s)) + \frac{b_{y^\delta(s)}(s, y^\delta(s))}{2} \sigma^2_{y^\delta(s)}(s, y^\delta(s)) \partial_x x u^\varepsilon(y^\delta(s)) \right) ds \right] \\
&\geq \mathbb{E}^\mu \left[ \int_0^T \frac{1}{2} c^2 \left( \partial_x x u^\varepsilon(y^\delta(s)) - M \partial_x x u^\varepsilon(y^\delta(s)) \right) ds \right] \geq \mathbb{E}^\mu \left[ \int_0^T \beta^\varepsilon(y^\delta(s)) ds \right].
\end{align*}
\]

We now study the term $(31)$. We get using $(29)$ the following upper bound
\[
\mathbb{E}^\mu \left[ |(31)| \right] \leq \frac{4\varepsilon}{Mc^2} \delta \exp(M\delta) \mathbb{E}^\mu \left[ N^\delta(T) \right].
\]

Using Remark $3.4$, it is easy to get that there exists a constant $K > 0$, independent of $\delta$ such that
\[
\mathbb{E}^\mu \left[ \delta N^\delta(T) \right] \leq K(1 + \delta^2)^{1/2}.
\]
Hence
\[
\mathbb{E}^\mu \left[ \int_0^T \beta(y^\delta(s))ds \right] \leq \mathbb{E}^\mu \left[ u^\varepsilon(y^\delta(T)) - u^\varepsilon(y_0) \right] + \frac{4\varepsilon}{M^2} \exp(M\delta) \mathbb{E}^\mu \left[ \delta N^\delta(T) \right] \\
\leq \frac{4\varepsilon}{M^2} \mathbb{E}^\mu \left[ \exp(My^\delta(T)) -1 \right] + \frac{4\varepsilon}{M^2} \exp(M\delta)K(1 + \delta^2)^{1/2}.
\]

We get then using Lemma 3.6
\[
\lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ \int_0^T \beta^\varepsilon(y^\delta(s))ds \right] = 0.
\]

Finally, using that \(x \to 1_{\{0 \leq x < \varepsilon\}}\) is lower semicontinuous, \(x \to 1_{\{0 \leq x \leq \varepsilon\}} \leq \beta^\varepsilon(x)\), Proposition 3.5, and Fatou Lemma we get
\[
\mathbb{E}^\mu \left[ \int_0^T 1_{\{y(s) = 0\}}ds \right] \leq \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ \int_0^T 1_{\{y^\delta(s) \leq \varepsilon\}}ds \right] = 0.
\]

Therefore
\[
\mu \left( \int_0^T 1_{\{y(s) = 0\}}ds = 0 \right) = 1,
\]
that completes the proof.

**Proposition 3.8.** The following map

\[
\psi := \begin{cases} 
\mathcal{D}^{\mathbb{R}^+}([0, T]) \times \mathcal{D}^{\mathbb{R}^+}_c([0, T]) \to \mathbb{R}_+ , \\
(y, l) \mapsto \int_0^T 1_{\{y(s) > 0\}}dl(s) 
\end{cases}
\]

(where \(\mathcal{D}^{\mathbb{R}^+}_c([0, T])\) is the set of càdlàg nondecreasing maps valued in \(\mathbb{R}_+\), is lower semicontinuous at any \((y, l) \in \mathcal{C}([0, T], \mathbb{R}_+) \times \left( \mathcal{C}([0, T], \mathbb{R}_+) \cap \mathcal{D}^{\mathbb{R}^+}_c([0, T]) \right)\).

**Proof.** Let \(\left( y_k(\cdot) \right)\) be a sequence in \(\mathcal{D}^{\mathbb{R}^+}([0, T])\) converging to \(y(\cdot) \in \mathcal{C}([0, T], \mathbb{R}_+)\), and \(\left( l_k(\cdot) \right) \) in \(\mathcal{D}^{\mathbb{R}^+}_c([0, T])\) converging to \(l(\cdot) \in \left( \mathcal{C}([0, T], \mathbb{R}_+) \cap \mathcal{D}^{\mathbb{R}^+}_c([0, T]) \right)\).

Using Proposition 3.5 we get that \(\left( y_k(\cdot) \right)\) converges uniformly to \(y(\cdot)\), and with the same arguments \(\left( l_k(\cdot) \right)\) converges uniformly to \(l(\cdot)\). Let \(p \geq 0\) and \(\phi_p \in \mathcal{C}([0, +\infty))\) a continuous sequence uniformly bounded, converging from below to \(x \to 1_{\{x > 0\}}\) in the
pointwise sense, as \( p \to +\infty \). We write then
\[
\left| \int_0^T \phi_p(y_k(u))dl_k(u) - \int_0^T \phi_p(y(u))dl(u) \right| \leq \int_0^T \left| \phi_p(y_k(u)) - \phi_p(y(u)) \right|dl_k(u)
\]
\[
+ \left| \int_0^T \phi_p(y(u))dl_k(u) - \int_0^T \phi_p(y(u))dl(u) \right| \leq \left| \phi_p(y_k(\cdot)) - \phi_p(y(\cdot)) \right|_{(0,T)} l_k(T)
\]
\[
+ \left| \int_0^T \phi_p(y(u))dl_k(u) - \int_0^T \phi_p(y(u))dl(u) \right|.
\]
Hence we get using that \( l_k(T) \) is uniformly bounded in \( k \), and that \( l_k \) converges weakly to \( l \)
\[
\forall p \geq 0, \quad \lim_{k \to +\infty} \int_0^T \phi_p(y_k(u))dl_k(u) = \int_0^T \phi_p(y(u))dl(u).
\]
Finally writing
\[
\int_0^T 1_{\{y_k(u) > 0\}} dl_k(u) \geq \int_0^T \phi_p(y_k(u))dl_k(u),
\]
we get
\[
\forall p \geq 0, \quad \liminf_{k \to +\infty} \int_0^T 1_{\{y_k(u) > 0\}} dl_k(u) \geq \int_0^T \phi_p(y(u))dl(u),
\]
and hence using Lebesgue’s Theorem
\[
\liminf_{k \to +\infty} \int_0^T 1_{\{y_k(u) > 0\}} dl_k(u) \geq \limsup_{p \to +\infty} \int_0^T \phi_p(y(u))dl(u) = \int_0^T 1_{\{y(u) > 0\}} dl(u).
\]
That completes the proof. \( \square \)

We are now ready to identify the SDE satisfied by the limit process \( (y(t))_{0 \leq t \leq T} \).

**Theorem 3.9.** There exists a nondecreasing process \( (l(t))_{0 \leq t \leq T} \) such that \( (y(t))_{0 \leq t \leq T} \) satisfies the stochastic differential equation
\[
\begin{cases}
  y(0) = y_0 \\
  dy(t) = b_{j(t)}(t, y(t))dt + \sigma_{j(t)}(t, y(t))dB(t) + dl(t), \quad 0 \leq t \leq T
\end{cases}
\]
\( \mu \text{ a.s.} \) \ (32)
with
\[
\begin{cases}
  l(0) = 0 \\
  \int_0^t 1_{\{y(s) > 0\}}dl(s) = 0, & 0 \leq t \leq T
\end{cases}
\]

\[\mu \text{ a.s.}\]

Moreover the process \(N^\delta(t)\) converges uniformly to \(l(t)\), namely
\[
\left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0,T)} \xrightarrow{\delta \to 0} 0, \quad \mu \text{ a.s.}
\]

**Proof. Step 1:** We start to prove that up to a subsequence again denoted \(\delta\)
\[
\lim_{\delta \to 0} \left| y_0 + \int_0^\cdot b_j(s,y_j(s))ds + \int_0^\cdot \sigma_j(s,y_j(s))dB(s) \right. \\
- \left. \left( y_0 + \int_0^\cdot b_j(s,y(s))ds + \int_0^\cdot \sigma_j(s,y(s))dB(s) \right) \right|_{(0,T)} = 0.
\]

For all \(\gamma > 0\), we introduce the map \((t,(x,i)) \mapsto \sigma_i^\gamma(t,x)\) defined by
\[
\forall (t,(x,i)) \in [0,T] \times \mathcal{J}, \quad \sigma_i^\gamma(t,x) = \sigma_i(t,x)1_{x > \gamma} + \frac{\sigma_i(t,\gamma)}{\gamma}x1_{0 \leq x \leq \gamma}
\]
for \((x,i) \in \mathcal{J}\).

For \(n \geq 0\), we define the functions \(\psi_{n,\gamma}\) from \(\mathcal{C}[0,T] \times \mathcal{D}^\mathcal{J}([0,T])\) to \(\mathcal{D}([0,T])\) by
\[
\psi_{n,\gamma}(w(\cdot),(x(\cdot),i(\cdot))) = \left( \sum_{j=0}^{2^n-1} \sigma_i^\gamma(x(T/2^n),x(T/2^n))(w(\cdot)+\xi_{j}t) - w(\cdot)\right)_{0 \leq t \leq T},
\]
for all \((w(\cdot),(x(\cdot),i(\cdot))) \in \mathcal{C}[0,T] \times \mathcal{D}^\mathcal{J}([0,T])\). We write then
\[
\left| \int_0^\cdot \sigma_j^\delta(s,y_j^\delta(s))dB^\delta(s) - \int_0^\cdot \sigma_j(s,y(s))dB(s) \right|_{(0,T)} \leq \quad (34)
\]
\[
\left| \int_0^\cdot \sigma_j^\delta(s,y_j^\delta(s))dB^\delta(s) - \int_0^\cdot \sigma_j^\delta(s,y_j^\delta(s))dB^\delta(s) \right|_{(0,T)} + \quad (35)
\]
\[
\left| \int_0^\cdot \sigma_j^\gamma(s,y_j^\delta(s))dB^\delta(s) - \psi_{n,\gamma}(B^\delta,(y^\delta,i^\delta))\right|_{(0,T)} + \quad (36)
\]
\[
\left| \psi_{n,\gamma}(B^\delta,(y^\delta,i^\delta))\right|_{(0,T)} + \quad (37)
\]
\[
\left| \psi_{n,\gamma}(B,(y,i))\right|_{(0,T)} + \quad (38)
\]
\[
\left| \int_0^\cdot \sigma_j^\gamma(s,y(s))dB(s) - \int_0^\cdot \sigma_j(s,y(s))dB(s) \right|_{(0,T)} . \quad (39)
\]
Now we prove that \(31\) tends to 0 as \(\delta\) goes to 0 in \(L^2(E)\). Using Burkholder-Davis-Gundy inequality and assumption \((H)\), there exists a constant \(C > 0\), independent of \(\delta, n, \gamma\), such that for \((35)\) and \((39)\)

\[
\mathbb{E}^\mu \left[ \left| \int_0^T \sigma^\delta_j(s, y^\delta(s)) dB^\delta(s) - \int_0^T \sigma^\gamma_j(s, y^\delta(s)) dB^\delta(s) \right|^2_{(0,T)} \right] \leq C \mathbb{E}^\mu \left[ \int_0^T 1_{\{y^\delta(s) \leq \gamma\}} ds \right],
\]

\[
\mathbb{E}^\mu \left[ \left| \int_0^T \sigma^\gamma_j(s, y(s)) dB(s) - \int_0^T \sigma_j(s, y(s)) dB(s) \right|^2_{(0,T)} \right] \leq C \mathbb{E}^\mu \left[ \int_0^T 1_{\{y(s) \leq \gamma\}} ds \right].
\]

Using then Proposition 3.5 and Proposition 3.7, we get

\[
\lim_{\gamma \to 0} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ (35) \right] \leq \lim_{\gamma \to 0} \mathbb{E}^\mu \left[ \int_0^T 1_{\{y(s) \leq \gamma\}} ds \right] = \mathbb{E}^\mu \left[ \int_0^T 1_{\{y(s) = 0\}} ds \right] = 0,
\]

and with the same arguments \(\lim_{\gamma \to 0} \mathbb{E}^\mu \left[ (39) \right] = 0\). For \((36)\), we get that there exists a constant \(C(\gamma)\) independent of \(\delta, n\), depending only on \(\gamma\) such that

\[
\mathbb{E}^\mu \left[ \left| \int_0^T \sigma^\gamma_j(s, y^\delta(s)) dB^\delta(s) - \psi_n,\gamma(B^\delta, (y^\delta, i^\delta)) (\cdot) \right|^2_{(0,T)} \right] \leq C(\gamma) \mathbb{E}^\mu \left[ \sup \left\{ \left| \sigma^\gamma_j(s, y^\delta(s)) - \sigma^\gamma_j(t, y^\delta(t)) \right|, \ |t - s| \leq \frac{1}{2^k}, \ (t, s) \in [0, T] \right\} \right].
\]

Using Proposition 3.5, the continuity of \(\sigma^\gamma_i\) at the junction point, and the uniform Lipschitz continuity in its second variable (assumption \((H)\)), we deduce then by Lebesgue’s Theorem

\[
\limsup_{\delta \to 0} \mathbb{E}^\mu \left[ \left| \int_0^T \sigma^\gamma_j(s, y^\delta(s)) dB^\delta(s) - \psi_n,\gamma(B^\delta, (y^\delta, i^\delta)) (\cdot) \right|^2_{(0,T)} \right] \leq C(\gamma) \mathbb{E}^\mu \left[ \sup \left\{ \left| \sigma^\gamma_j(s, y(s)) - \sigma^\gamma_j(t, y(t)) \right|^2, \ |t - s| \leq \frac{1}{2^k}, \ (t, s) \in [0, T] \right\} \right].
\]

Finally using the continuity of the paths of the process \(Y(\cdot)\) and once again the continuity of \(\sigma^\gamma\) we get

\[
\lim_{n \to +\infty} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ \left| \int_0^T \sigma^\gamma_j(s, y^\delta(s)) dB^\delta(s) - \psi_n,\gamma(B^\delta, (y^\delta, i^\delta)) (\cdot) \right|^2_{(0,T)} \right] = 0.
\]
Similarly we have \( \lim_{n \to +\infty} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ (37) \right] = 0 \). Finally for (37) we get that there exists a constant \( C(\gamma) \) independent of \((n, \delta)\) such that

\[
\left| \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) - \psi_{n,\gamma}(B, (y, i))(\cdot) \right|_{(0,T)} \leq 2^{n+1} C(\gamma) \left( |B^\delta - B|_{(0,T)} + |\sigma^\gamma_j(\cdot, y^\delta(\cdot)) - \sigma^\gamma_j(\cdot, y(\cdot))|_{(0,T)} \right).
\]

Using once more Proposition 3.3 and assumption \((H)\), and the uniform Lipschitz continuity of \(\sigma^\gamma\) in its second variable, we get by Lebesgue’s Theorem that \( \lim_{\delta \to 0} \mathbb{E}^\mu \left[ (37) \right] = 0 \). Therefore we have

\[
\lim_{\delta \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t b^\delta_{j(s)}(s, y^\delta(s)) dB^\delta(s) - \int_0^t \sigma_{j(s)}(s, y(s)) dB(s) \right|^2_{(0,T)} \right] \leq \lim_{\gamma \to 0} \limsup_{n \to +\infty} \mathbb{E}^\mu \left[ (31) + (34) + (36) + (37) + (38) + (39) \right] = 0.
\]

With the same arguments, we can show that

\[
\lim_{\delta \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t b^\delta_{j(s)}(s, y^\delta(s)) ds - \int_0^t b_{j(s)}(s, y(s)) ds \right|_{(0,T)} \right] = 0.
\]

Choosing again a subsequence \(\delta\), to get an almost convergence sense, we have \(\mu\) almost surely

\[
\lim_{\delta \to 0} \left| y_0 + \int_0^t b^\delta_{j(s)}(s, y^\delta(s)) ds + \int_0^t \sigma^\delta_{j(s)}(s, y^\delta(s)) dB^\delta(s) - (y_0 + \int_0^t b_{j(s)}(s, y(s)) ds + \int_0^t \sigma_{j(s)}(s, y(s)) dB(s)) \right|_{(0,T)} = 0.
\]

We recall that from Proposition 3.3 we have

\[
\mu\left( \left( y^\delta(t) = y_0 + \int_0^t b^\delta_{j(s)}(s, y^\delta(s)) ds + \int_0^t \sigma^\delta_{j(s)}(s, y^\delta(s)) dB^\delta(s) + \delta \left( \sum_{0 \leq s \leq t} 1_{(y^\delta(s) = 0)} \right) \right)_{0 \leq t \leq T} \right) = 1.
\]

The continuity of both processes \(\left( y(t) \right)_{0 \leq t \leq T}\) and \(\left( \int_0^t b_{j(s)}(s, y(s)) ds + \int_0^t \sigma_{j(s)}(s, y(s)) dB(s) \right)_{0 \leq t \leq T}\), implies that \(\left( \delta \left( \sum_{0 \leq s \leq t} 1_{(y^\delta(s) = 0)} \right) \right)_{0 \leq t \leq T}\) tends almost surely uniformly to a continuous process \(\left( l(t) \right)_{0 \leq t \leq T}\). Since \(\left( \delta \left( \sum_{0 \leq s \leq t} 1_{(y^\delta(s) = 0)} \right) \right)_{0 \leq t \leq T}\) has nondecreasing trajectories, it implies that \(\left( l(t) \right)_{0 \leq t \leq T}\) is a nondecreasing process.
Step 2: We now prove that

\[
\mu\left( \begin{array}{l}
l(0) = 0 \\
\left( \int_0^t 1_{\{y(s) > 0\}} dl(s) \right)_{0 \leq t \leq T} = 0 \\
\end{array} \right) = 1,
\]

namely \( (l(t))_{0 \leq t \leq T} \) can increases only when the process \( (Y(t) = (y(t), j(t)))_{0 \leq t \leq T} \) reaches the junction point 0. The fact that \( l(0) = 0 \), \( \mu \) almost surely is obvious. Since the paths of \( y^\delta(\cdot) \) and \( \delta N^\delta(\cdot) \) converge uniformly to the continuous processes \( y(\cdot) \) and \( l(\cdot) \), we have using Proposition 3.8 that

\[
\liminf_{\delta \to 0} \int_0^T 1_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) \geq \int_0^T 1_{\{y(s) > 0\}} dl(s), \quad \mu \text{ a.s.}
\]

Hence, using the lower semi continuity of the real map \( x \mapsto 1_{\{x > 0\}} \), we obtain

\[
\liminf_{\delta \to 0} \left\{ \int_0^T 1_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right\} \geq \left\{ \int_0^T 1_{\{y(s) > 0\}} dl(s) > 0 \right\}, \quad \mu \text{ a.s.}
\]

Therefore, using Fatou’s Lemma we have

\[
\liminf_{\delta \to 0} \mu\left( \int_0^T 1_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) \geq \mu\left( \liminf_{\delta \to 0} \int_0^T 1_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) \\
\geq \mu\left( \int_0^T 1_{\{y(s) > 0\}} dl(s) > 0 \right),
\]

and using that \( \mu\left( \int_0^T 1_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) = 0 \), we get finally

\[
\mu\left( \left( \int_0^t 1_{\{y(s) > 0\}} dl(s) \right)_{0 \leq t \leq T} = 0 \right) = 1.
\]

We conclude then that \( (y(t))_{0 \leq t \leq T} \) satisfies the differential stochastic equation (32), with the required conditions on \( (l(t))_{0 \leq t \leq T} \). \( \square \)

4. Itô’s Formula and Local Time Estimate at the Junction

4.1. Proof of Theorem 2.3 and Itô’s formula. The stochastic differential equation (32) satisfied by the process \( (y(t))_{0 \leq t \leq T} \), does not completely characterize the process, since the randomness due to \( (j(t))_{0 \leq t \leq T} \) is hidden in the process \( (l(t))_{0 \leq t \leq T} \). In order
to better characterize the behavior of the process \( Y(t) = (y(t), j(t)) \) \( \forall t \leq T \), we test its dynamic against regular maps \( f \) defined on the junction. This is the aim of the Itô’s formula; (iv) of Theorem 2.3. The points (i) to (v) of Theorem 2.3 have been already seen previously and proved in Theorem 2.3 and 3.2.

Proof. Let \( f \in C_b^{1,2}(J_T) \). Suppose first that \( f \) does not depend on time:

\[
\forall \left( t, (y, j) \right) \in J_T, \quad f_j(t, y) = f_j(y).
\]

We first show that

\[
\left( f(Y(t)) - f(Y(0)) \right) = \int_0^t L(f)(s, (y(s), j(s)))ds + \int_0^t \partial_y f_j(s)(y(s))\sigma_{j(s)}(s, y(s))dB(s) + \sum_{i=1}^l \alpha_i \partial_y f_i(0)l(t) \right)_{0 \leq t \leq T} \mu \text{ a.s.}
\]

We recall the definition of the stopping times \( \gamma^\delta \)

\[
\gamma^\delta_0 = 0, \quad \gamma^\delta_{n+1} = \inf \left\{ t > \gamma^\delta_n : y^\delta(t) = 0 \right\}, \mu \text{ a.s.}
\]

Let \( t \in [0, T] \), we have then the following decomposition

\[
f_j^\delta(t)(y^\delta(t)) - f_j^\delta(0)(y_0) = \sum_{n \geq 0} f_j^\delta(t \wedge \gamma^\delta_{n+1}-) \left( y^\delta(t \wedge \gamma^\delta_{n+1}+) - f_j^\delta(t \wedge \gamma^\delta_{n+1}) \right) + \sum_{n \geq 0} f_j^\delta(t \wedge \gamma^\delta_{n+1}+) \left( y^\delta(t \wedge \gamma^\delta_{n+1}) - f_j^\delta(t \wedge \gamma^\delta_{n+1}-) \right), \mu \text{ a.s.}
\]

For \( \text{(40)} \), we apply Itô’s formula on each interval \( \left( t \wedge \gamma^\delta_n+, t \wedge \gamma^\delta_{n+1}- \right) \), to get

\[
\sum_{n \geq 0} f_j^\delta(t \wedge \gamma^\delta_{n+1}-) \left( y^\delta(t \wedge \gamma^\delta_{n+1}-) \right) - f_j^\delta(t \wedge \gamma^\delta_{n+1}) \left( y^\delta(t \wedge \gamma^\delta_{n+1}) \right) = \int_0^t L(f)(s, (y^\delta(s), \tilde{j}^\delta(s)))ds + \int_0^t \sigma_{j^\delta(s)}(s, y^\delta(s))\partial_y f_j^\delta(s)(y^\delta(s))dB^\delta(s), \mu \text{ a.s.}
\]
Using the same arguments as in the proof of Theorem 3.9 it is easy to show that up to a subsequence $\delta$

$$\lim_{\delta \to 0} \left| \int_0^T L(f)(s, (y^\delta(s), j^\delta(s))) ds + \int_0^T \sigma_{j^\delta(s)}(s, y^\delta(s)) \partial_x f_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) \right|_{(0,T)} = 0, \mu \text{ a.s.}$$

Using the continuity of $f$ at the junction point, namely $\forall (i,j) \in \{1 \ldots I\}^2, f_i(0) = f_j(0) = f(0)$, we get for (41) (where we recall that $N^\delta(t) = \sum_{0 \leq s \leq t} 1_{\{y^\delta(s-\omega) = 0\}}, \mu \text{ a.s.})$

$$\sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_n^\delta+)}(y^\delta(t \wedge \gamma_n^\delta+)) - f_{j^\delta(t \wedge \gamma_n^\delta-)}(y^\delta(t \wedge \gamma_n^\delta-)) = \left( \sum_{n=0}^{N^\delta(t)} f_{j^\delta(\gamma_n^\delta)}(\delta) - f_{j^\delta(\gamma_n^\delta)}(0) \right) 1_{\{N^\delta(t) > 0\}}$$

$$= \left( \sum_{n=0}^{N^\delta(t)} \delta \partial_y f_{j^\delta(\gamma_n^\delta)}(0) + \delta \bar{\varepsilon}_{j^\delta(\gamma_n^\delta)}(\delta) \right) 1_{\{N^\delta(t) > 0\}}, \mu \text{ a.s.}$$

where for each $i \in \{1 \ldots I\}$, $\lim_{\delta \to 0} \varepsilon_i(\delta) = 0$.

From Remark 3.4 we know that there exists a constant $C > 0$ independent of $\delta$, such that

$$\mathbb{E}^\mu \left[ |\delta N^\delta(\cdot)|_{(0,T)} \right] \leq C(1 + \delta^2)^{1/2}, \quad (42)$$

hence

$$\mathbb{E}^\mu \left[ \left| \sum_{n=0}^{N^\delta(\cdot)} \delta \bar{\varepsilon}_{j^\delta(\gamma_n^\delta)}(\delta) \right|_{(0,T)} \right] \leq C(1 + \delta^2)^{1/2} \max_{i \in \{1 \ldots I\}} |\varepsilon_i(\delta)|,$$

which means that, up to a subsequence $\delta$, we have

$$\lim_{\delta \to 0} \left| \sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_n^\delta+)}(y^\delta(t \wedge \gamma_n^\delta+)) - f_{j^\delta(t \wedge \gamma_n^\delta-)}(y^\delta(t \wedge \gamma_n^\delta-)) \right. \left. - \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \delta \partial_y f_i(0) 1_{\{j^\delta(\gamma_n^\delta) = i\}} \right) 1_{\{N^\delta(\cdot) > 0\}} \right|_{(0,T)} = 0, \mu \text{ a.s.}$$
Recall that from assumption \((\mathcal{H})\), \((\sum_{i=1}^{I} \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}})_{n \geq 0}\) are i.i.d. Using the law of large numbers, there exists \(A \in \mathcal{E}\) with \(\mu(A) = 1\), such that for all \(\omega \in A\)

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n} \sum_{i=1}^{I} \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}} = \sum_{i=1}^{I} \alpha_i \partial_x f_i(0),
\]
since for all \(\delta\), for all \(i \in \{1 \ldots I\}\), \(\mathbb{P}^{\mu} [\mathbf{1}_{\{j^i(\gamma_n^i) = i\}}] = \alpha_i\).

On the other hand we have

\[
\mathcal{A} = \bigcap_{\eta \in \mathbb{Q}_+^*} \bigcup_{K \in \mathbb{N}^*} \mathcal{A}_{\eta,K},
\]

where

\[
\mathcal{A}_{\eta,K} = \left\{ \omega \in \mathcal{A}, \ \forall n \geq K, \ \left| \frac{1}{n} \sum_{k=0}^{n} \sum_{i=1}^{I} \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}} - \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) \right| \leq \eta \right\}.
\]

Given \(\eta \in \mathbb{Q}_+^*\) and \(K \in \mathbb{N}^*\), we have

\[
\left| \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \delta \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}} \right) \mathbf{1}_{\{N^\delta(\cdot) > 0\}} - l(\cdot) \left( \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) \right) \right|_{(0,T)} \leq \ (43)
\]

\[
\left| \frac{l(\cdot) - \delta N^\delta(\cdot)}{N^\delta(\cdot)} \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}} \right) \mathbf{1}_{\{N^\delta(\cdot) > 0\}} \right|_{(0,T)} + \ (44)
\]

\[
\left| l(\cdot) \right|_{(0,T)} \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \partial_y f_i(0) \mathbf{1}_{\{j^i(\gamma_n^i) = i\}} - \left( \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) \right) \right) \times \ (45)
\]

\[
\left( \mathbf{1}_{\{0 < N^\delta(\cdot) \leq K\}} + \mathbf{1}_{\{N^\delta(\cdot) > K\}} \left( \mathbf{1}_{\{(A_{\eta,K}) + \mathbf{1}_{\{A_{\eta,K}\}}\}} \right) \right) \right|_{(0,T)}, \ \mu \text{ a.s.}
\]

For \((44)\), we have the following upper bound

\[
\text{(44)} \leq \max_{i \in \{1 \ldots I\}} \left\{ \left| \partial_x f_i(0) \right| \right\} \delta N^\delta(\cdot) - l(\cdot) \right|_{(0,T)}.
\]

Recall that from \((33)\) in Theorem \((33)\), we know that

\[
\left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0,T)} \xrightarrow{\delta \to 0} 0, \ \mu \text{ a.s.,}
\]

hence using Lebesgue’s Theorem

\[
\lim_{\delta \to 0} \mathbb{E}^{\mu} \left[ \text{(44)} \right] = 0.
\]
We now consider the term \(I_{15}\), we have
\[
\forall t \in [0, T], \quad 0 \leq l(t)1_{\{0 < N^\delta(t) \leq K\}} = (l(t) - \delta N^\delta(t) + \delta N^\delta(t))1_{\{0 < N^\delta(t) \leq K\}} \\
\leq \left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0, T)} + \delta K, \quad \mu \text{ a.s.,}
\]
and then using Lebesgue’s Theorem and once again \(33\), we have
\[
\lim_{\delta \to 0} \mathbb{E}^\mu \left[ \left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0, T)} + \delta K \right] = 0.
\]
Finally we have the following upper bounds
\[
\mathbb{E}^\mu \left[ \left| l(\cdot) \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \partial_y f_i(0) 1_{\{y^{\delta}(\cdot) = i\}} - \left( \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) \right) \right) 1_{\{0 < N^\delta(\cdot) \leq K\}} \right]_{(0, T)} \\
\leq \eta \mathbb{E}^\mu \left[ \left| l(\cdot) \right|_{(0, T)} \right],
\]
\[
\mathbb{E}^\mu \left[ \left| l(\cdot) \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^{I} \partial_y f_i(0) 1_{\{y^{\delta}(\cdot) = i\}} - \left( \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) \right) \right) 1_{\{N^\delta(\cdot) > K\}} 1_{\{A_0, K\}} \right]_{(0, T)} \\
\leq C \mathbb{E}^\mu \left[ \left| l(\cdot) \right|_{(0, T)} 1_{\{A_{0, K}\}} \right].
\]
Therefore
\[
\lim_{\eta \to 0} \lim \sup_{K \to +\infty} \limsup_{\delta \to 0} \mathbb{E}^\mu \left[ I_{13} \right] = 0.
\]
We finally get, that, up to a subsequence \(\delta\)
\[
\lim_{\delta \to 0} \left| f_{j^\delta}(y(\cdot)) - f_{j^\delta(0)}(y^\delta(0)) - \left( \int_0^t L(f)(s, (y(s), j(s))) ds + \int_0^t \partial_y f_j(s)(y(s)) \sigma_j(s, y(s)) dB(s) + \sum_{i=1}^{I} \alpha_i \partial_y f_i(0) l(\cdot) \right) \right|_{(0, T)} = 0, \quad \mu \text{ a.s.}
\]
This completes the proof of \textbf{Step 1}.

\textbf{Step 2.} We generalize the result obtained in \textbf{Step 1}, for \(f \in C^{1,2}_b(J_T)\). To this end, fix
for \( t \in [0, T] \) and set \( (t^n_k := kt/n)_{0 \leq k \leq n} \). We have

\[
f_j(t) \left( t, y(t) \right) - f_j(0) \left( 0, y(0) \right) = \sum_{k=1}^{n} f_j(t^n_k) \left( t^n_k, y(t^n_k) \right) - f_j(t^n_{k-1}) \left( t^n_{k-1}, y(t^n_{k-1}) \right)
\]

\[
+ \sum_{k=1}^{n} f_j(t^n_k) \left( t^n_{k-1}, y(t^n_{k-1}) \right) - f_j(t^n_{k-1}) \left( t^n_{k-1}, y(t^n_{k-1}) \right).
\]

(46)

Using the continuity of \( f \) at the junction point, the interior regularity of the \((\partial_t f_i)_{i \in \{1, \ldots, l\}}\), the continuity of the paths of \( y(\cdot) \) and that \( \int_0^T 1_{\{y(s) = 0\}} ds = 0 \) \( \mu \) a.s, we get

\[
\left| \sum_{k=1}^{n} f_j(t^n_k) \left( t^n_k, y(t^n_k) \right) - f_j(t^n_{k-1}) \left( t^n_{k-1}, y(t^n_{k-1}) \right) - \int_0^T \partial_t f_j(s) \left( s, y(s) \right) ds \right|_{(0, T)} \xrightarrow{n \to +\infty} 0, \ \mu \text{ a.s.}
\]

Using the result obtained in Step 1, we get for (47)

\[
\sum_{k=1}^{n} f_j(t^n_k) \left( t^n_k, y(t^n_k) \right) - f_j(t^n_{k-1}) \left( t^n_{k-1}, y(t^n_{k-1}) \right) =
\]

\[
\sum_{k=1}^{n} \int_{t^n_{k-1}}^{t^n_k} \left( \partial_y f_j(s) \left( t^n_{k-1}, y(s) \right) b_j(s, y(s)) + \frac{1}{2} \partial^2_{yy} f_j(s) \left( t^n_{k-1}, y(s) \right) \sigma^2_j(s, y(s)) \right) ds +
\]

\[
\int_{t^n_{k-1}}^{t^n_k} \partial_y f_j(s) \left( t^n_{k-1}, y(s) \right) \sigma_j(s, y(s)) dB(s) + \left( l(t^n_k) - l(t^n_{k-1}) \right) \left( \sum_{i=1}^{l} \alpha_i \partial_y f_i(t^n_{k-1}, 0) \right).
\]

Finally, using the regularity of \( f \), and the fact that \( \int_0^T 1_{\{y(s) = 0\}} ds = 0, \ \mu \) a.s (Proposition 3.7), it is easy to check that up to a subsequence \( n_p \)

\[
\left| \sum_{k=1}^{n_p} f_j(t^n_{kp}) \left( t^n_{kp}, y(t^n_{kp}) \right) - f_j(t^n_{kp-1}) \left( t^n_{kp-1}, y(t^n_{kp-1}) \right) \right|
\]

\[
\left( \int_0^T \left( \partial_y f_j(s) \left( s, y(s) \right) b_j(s, y(s)) + \frac{1}{2} \partial^2_{yy} f_j(s) \left( s, y(s) \right) \sigma^2_j(s, y(s)) \right) ds +
\]

\[
\int_0^T \partial_y f_j(s) \left( s, y(s) \right) \sigma_j(s, y(s)) dB(s) + \left( l(t^n_k) - l(t^n_{k-1}) \right) \left( \sum_{i=1}^{l} \alpha_i \partial_y f_i(t^n_{k-1}, 0) \right) \right|_{(0, T)} \xrightarrow{n \to +\infty} 0, \ \mu \text{ a.s.}
\]

that completes the proof. \( \square \)

4.2. \textbf{Local time estimate at the junction.} We complete this Section by giving a local time estimate of the process \( Y(t) = \left( (y(t), j(t)) \right)_{0 \leq t \leq T} \) at the junction point. This estimate is reminiscent of the local time for the reflected Brownian motion.
Theorem 4.1. We have:

$$\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \sum_{j=1}^{I} \int_0^l \sigma_j^2(s,0) 1_{0 \leq y(s) \leq \varepsilon, j(s) = j} \, ds \right| - l(\cdot) \right|_{(0,T)}^2 = 0,$$

and, more generally, for any subset $\mathcal{I} \subset \{1, \ldots, I\}$:

$$\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \sum_{k \in \mathcal{I}} \sum_{j=1}^{I} \int_0^l \sigma_j^2(s,0) 1_{0 \leq y(s) \leq \varepsilon, j(s) = j} \, ds \right| - l(\cdot) \right|_{(0,T)}^2 = 0.$$

Proof. Let $\varepsilon > 0$. We define the following sequence $\phi^\varepsilon$ on $\mathcal{J}$ by

$$\forall (y,j) \in \mathcal{J}, \quad \phi^\varepsilon((y,j)) = \phi^\varepsilon(y) = \begin{cases} \frac{y^2}{2\varepsilon} & \text{if } 0 \leq y \leq \varepsilon, \\ y - \frac{\varepsilon}{2} & \text{if } y \geq \varepsilon, \end{cases} \text{ and } y \in J_j.$$

Step 1 : We start by showing that

$$\left( \phi^\varepsilon(Y(\cdot)) - \phi^\varepsilon(Y(0)) \right) = \int_0^l \left( \partial_y \phi^\varepsilon(y(s)) b_j(s,y(s)) + \frac{1}{2} \partial_{yy} \phi^\varepsilon(y(s)) \sigma_j(y(s))^2 \right) ds + \int_0^l \partial_y \phi^\varepsilon(y(s)) \sigma_j(s,y(s)) dB(s) \right)_{0 \leq t \leq T}, \mu \text{ a.s.} (48)$$

Let $\eta > 0$, and $\phi^{\varepsilon, \eta} \in C^2(\mathcal{J})$, the sequence of smooth functions satisfying the following conditions $(T)$:

$$\begin{cases} \phi^{\varepsilon, \eta} = \phi^\varepsilon \text{ on } [0, \varepsilon/2] \cup [2\varepsilon, +\infty), \\
\partial_{yy} \phi^{\varepsilon, \eta} - \partial_{yy} \phi^\varepsilon \xrightarrow{n \to 0} 0,
\end{cases} \quad \begin{cases} |\partial_y \phi^{\varepsilon, \eta} - \partial_y \phi^\varepsilon|_{[0, +\infty)} \xrightarrow{n \to 0} 0, \\
|\phi^{\varepsilon, \eta} - \phi^\varepsilon|_{[0, +\infty)} \xrightarrow{n \to 0} 0. \end{cases}$$

Let $a > \varepsilon > 0$. We introduce the following stopping time

$$\gamma_a := \inf\{t \in [0,T], \ y(t) = a\}, \ \mu \text{ a.s.}$$

Applying Itô’s formula of Theorem 2.3 to $\phi^{\varepsilon, \eta} \in C^2(\mathcal{J})$, we have

$$\left( \phi^{\varepsilon, \eta}(Y(\cdot \land \gamma_a)) - \phi^{\varepsilon, \eta}(Y(0)) \right) = \int_0^{\land \gamma_a} \left( \partial_y \phi^{\varepsilon, \eta}(y(s)) b_j(s,y(s)) + \frac{1}{2} \partial_{yy} \phi^{\varepsilon, \eta}(y(s)) \sigma_j(s,y(s))^2 \right) ds + \int_0^{\land \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_j(s,y(s)) dB(s) \right)_{0 \leq t \leq T}, \mu \text{ a.s.} (49)$$
We are going to send \( \eta \to 0 \) in [19]. We start by showing that, up to a sub sequence \( \eta \),
\[
\lim_{\eta \to 0} \left| \int_0^{\gamma a} \partial_y \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s)) dB(s) - \int_0^{\gamma a} \partial_y \phi^\varepsilon(y(s)) \sigma_j(s, y(s)) dB(s) \right|_{(0,T)} = 0, \ \mu \text{ a.s.}
\]
Using assumption (H) and Burkholder-Davis-Gundy inequality, we get that there exists a constant \( C > 0 \), independent of \( a, \varepsilon \), and \( \eta \), such that
\[
\mathbb{E}^\mu \left[ \left| \int_0^{\gamma a} \partial_y \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s)) dB(s) - \int_0^{\gamma a} \partial_y \phi^\varepsilon(y(s)) \sigma_j(s, y(s)) dB(s) \right|^2_{(0,T)} \right] \leq C \mathbb{E}^\mu \left[ \int_0^T \left( \partial_y \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s)) - \partial_y \phi^\varepsilon(y(s)) \sigma_j(s, y(s)) \right)^2 ds \right] \leq C \left( \left| \partial_y \phi^\varepsilon,\eta(\cdot) - \partial_y \phi^\varepsilon(\cdot) \right|^2_{(0,a)} \right).
\]
Hence using conditions (\( T \)), we get that up to a sub sequence \( \eta \)
\[
\lim_{\eta \to 0} \left| \int_0^{\gamma a} \partial_y \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s)) dB(s) - \int_0^{\gamma a} \partial_y \phi^\varepsilon(y(s)) \sigma_j(s, y(s)) dB(s) \right|_{(0,T)} = 0, \ \mu \text{ a.s.}
\]
With the same arguments, we get that up to a sub sequence ( denoted in the same way by \( \eta \))
\[
\lim_{\eta \to 0} \left| \int_0^{\gamma a} \partial_y \phi^\varepsilon,\eta(y(s)) b_j(s, y(s)) ds - \int_0^{\gamma a} \partial_y \phi^\varepsilon(y(s)) b_j(s, y(s)) ds \right|_{(0,T)} = 0, \ \mu \text{ a.s.}
\]
Finally, as the stochastic differential equation [32] has uniform Lipschitz (with respect to the second variable) and bounded coefficients, the process \( \left( y(t) \right)_{0 \leq t \leq T} \) has a continuous density on \((0, +\infty)\) (see Proposition 1.1.2 and Theorem 1.8.3 of [2]). Let us denote by \( C \) its bound on \([\varepsilon/2, a]\). We have then using conditions assumption (H)
\[
\mathbb{E}^\mu \left[ \left| \int_0^{\gamma a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s))^2 ds - \int_0^{\gamma a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_j(s, y(s))^2 ds \right|_{(0,T)} \right] \leq \mathbb{E}^\mu \left[ \int_0^T \left| \frac{1}{2} \sigma_j(s, y(s))^2 (\partial_{y,y} \phi^\varepsilon(y(s)) - \phi^\varepsilon,\eta(y(s))) 1_{\{\varepsilon/2 \leq y(s) \leq a\}} \right| ds \right] \leq \frac{1}{2} \left| \int_0^T \sigma^2 \left\| \partial_{y,y} \phi^\varepsilon,\eta \right\|_{L^1(\varepsilon/2, a)} \right| ds.
\]
Then, by condition (\( T \)), we have
\[
\lim_{\eta \to 0} \mathbb{E}^\mu \left[ \left| \int_0^{\gamma a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon,\eta(y(s)) \sigma_j(s, y(s))^2 ds - \int_0^{\gamma a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_j(s, y(s))^2 ds \right|_{(0,T)} \right] = 0.
\]
This proves that, for any $a > \varepsilon$,

$$
(\phi^{\varepsilon}(Y(\cdot \wedge \gamma_a)) - \phi^{\varepsilon}(Y(0)) = \int_0^{\gamma_a} \left( \partial_y \phi^{\varepsilon}(y(s))b_j(s, y(s)) + \frac{1}{2} \partial_{yy} \phi^{\varepsilon}(y(s))\sigma_j(s, y(s))^2 \right) ds + \int_0^{\gamma_a} \partial_y \phi^{\varepsilon}(y(s))\sigma_j(s, y(s))dB(s) \right)_{0 \leq t \leq T}, \ \mu \text{ a.s.}
$$

Finally, since the process $y(\cdot)$ has continuous paths, we have

$$
\lim_{a \to +\infty} \gamma_a = +\infty, \ \mu \text{ a.s.}
$$

Hence sending $a$ to $+\infty$, we obtained (58), and that completes the proof of Step 1.

**Step 2**: we prove the main result of this Proposition.

Using the result obtained in Step 1 and Theorem 3.9, we get using the expression of the derivatives of $\phi^{\varepsilon}$:

$$
\mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \int_0^\infty \sigma^2_j(s, y(s))1_{\{0 \leq y(s) \leq \varepsilon\}} ds - \int_0^\infty \phi^{\varepsilon}(Y(\cdot)) - \phi^{\varepsilon}(Y(0)) - (y(\cdot) - y_0)^2 \right|_{(0,T)}^2 \right] \leq 5 \mathbb{E}^\mu \left[ \left| \phi^{\varepsilon}(Y(\cdot)) - \phi^{\varepsilon}(Y(0)) - (y(\cdot) - y_0)^2 \right|_{(0,T)}^2 \right] + \mathbb{E}^\mu \left[ \left| \int_0^\infty \sigma_j(s, y(s))1_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|_{(0,T)}^2 \right] + \mathbb{E}^\mu \left[ \left| \int_0^\infty b_j(s, y(s))1_{\{0 \leq y(s) \leq \varepsilon\}} ds \right|_{(0,T)}^2 \right].
$$

For (50), recalling that, for all $Y = (y, j) \in \mathcal{J}$, $|\phi^{\varepsilon}(y) - y| \leq 2\varepsilon$, we get by Lebesgue’s Theorem

$$
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \phi^{\varepsilon}(Y(\cdot)) - \phi^{\varepsilon}(Y(0)) - (y(\cdot) - y_0)^2 \right|_{(0,T)}^2 \right] = 0.
$$

On the other hand for (51), using assumption (H) and Burkholder-Davis-Gundy inequality, there exists a constant $C > 0$ independent of $\varepsilon$, such that

$$
\mathbb{E}^\mu \left[ \left| \int_0^\infty \sigma_j(s, y(s))1_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|_{(0,T)}^2 \right] \leq C \mathbb{E}^\mu \left[ \int_0^T 1_{\{0 \leq y(s) \leq \varepsilon\}} ds \right].
$$
and therefore using Proposition \(3.7\), we get
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t \sigma_j(s,y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|^2_{(0,T)} \right] = 0.
\]

Similarly, we have for \(52\)
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t b_j(s,y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right|^2 \right] = 0.
\]

Finally, for \(53\) and \(54\), it is easy to check using assumption \((\mathcal{H})\) that
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t \sigma_j(s,y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} dB(s) - \int_0^t \sigma_j(s,y(s)) dB(s) \right|^2_{(0,T)} \right] = 0,
\]
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \int_0^t b_j(s,y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} ds - \int_0^t b_j(s,y(s)) ds \right|^2_{(0,T)} \right] = 0.
\]

We get finally
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \int_0^t \sigma^2_j(s,y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds - l(\cdot) \right|^2_{(0,T)} \right] = 0.
\]

To conclude, we remark that for all \(t \in [0,T]\)
\[
\frac{1}{2\varepsilon} \int_0^t \sigma^2_j(s,y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds = \sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^t \sigma^2_j(s,0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s) = j\}} ds + 
\sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^t \left( \sigma^2_j(s,y(s)) - \sigma^2_j(s,0) \right) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s) = j\}} ds , \ \mu \text{ a.s.}
\]

Using the uniform Lipschitz continuity of all the \((\sigma_i)_{i \in \{1 \ldots I\}}\) at each edge \(J_i\), with respect to the second variable (assumption \((\mathcal{H})\)), we get that there exists a constant \(C > 0\), independent of \(\varepsilon\) such that for all \(j \in \{1 \ldots I\}\), for all \(s \in [0,T]\)
\[
\left| \sigma^2_j(s,y(s)) - \sigma^2_j(s,0) \right| \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} \leq C\varepsilon \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}}, \ \mu \text{ a.s.,}
\]
and then using Proposition \(3.7\)
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left| \sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^t \left( \sigma^2_j(s,y(s)) - \sigma^2_j(s,0) \right) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s) = j\}} ds \right|_{(0,T)} \right] \leq
\]
\[
C \lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right] = 0.
\]
We get then the required result, namely
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left( \frac{1}{2\varepsilon} \sum_{j=1}^I \int_0^T \sigma_j^2(s,0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s) = j\}} ds \right)^2 \right] - l(\cdot)^2_{(0,T)} = 0.
\]

We notice that the second approximation, which is for any \( \mathcal{I} \subset \{1, \ldots I\} \)
\[
\lim_{\varepsilon \to 0} \mathbb{E}^\mu \left[ \left( \frac{1}{2\varepsilon} \sum_{k \in \mathcal{I}} \sum_{j \in \mathcal{I}} \int_0^T \sigma_j^2(s,0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s) = j\}} ds \right)^2 \right] - l(\cdot)^2_{(0,T)} = 0,
\]
can be proved with the same arguments above considering the same map \( \phi^\varepsilon \), but vanishing on each edge whose indexes belong to \( \{1, \ldots I\} \setminus \mathcal{I} \).

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