BOUNDDEDNESS OF DENOMINATORS OF SPECIAL VALUES OF THE $L$-FUNCTIONS FOR MODULAR FORMS

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Abstract. For a cuspidal Hecke eigenform $F$ for $Sp_n(\mathbb{Z})$ and a Dirichlet character $\chi$ let $L(s, F, \chi, St)$ be the standard $L$-function of $F$ twisted by $\chi$. In [3], Böcherer showed the boundedness of denominators of the algebraic part of $L(m,F,\chi, St)$ at a critical point $m$ when $\chi$ varies. In this paper, we give a refined version of his result. We also prove a similar result for the products of Hecke $L$-functions of primitive forms for $SL_2(\mathbb{Z})$.

1. Introduction

Let $\Gamma^{(n)} = Sp_n(\mathbb{Z})$ be the Siegel modular group of genus $n$. For a cuspidal Hecke eigenform $F$ for $\Gamma^{(n)}$ and a Dirichlet character $\chi$ let $L(s, F, \chi, St)$ be the standard $L$-function of $F$ twisted by $\chi$. In [3], Böcherer showed the boundedness of denominators of the algebraic part of $L(m,F,\chi, St)$ at a critical point $m$ when $\chi$ varies (cf. Remark 2.5). To prove this, Böcherer used congruence of Fourier coefficients of modular forms. In this paper, we give a refined version of the above result without using congruence. We state our main results more precisely. Let $M_k(\Gamma^{(n)})$ be the space of modular forms of weight $k$ for $\Gamma^{(n)}$, and $S_k(\Gamma^{(n)})$ its subspace consisting of cusp forms. We suppose that $k \geq n + 1$. Let $F_1, \ldots, F_e$ be a basis of the space $M_k(\Gamma^{(n)})$ consisting of Hecke eigenforms such that $F_1 = F$. Let $L_{n,k}$ be the composite field of $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_{e-1})$ and $\mathbb{Q}(F_e)$. Let $\mathfrak{E}_F'$ be the ideal of $L_{n,k}$ generated by all $\prod_{i=2}^e (\lambda_F(T_{i-1}) - \lambda_{F_i}(T_{i-1}))$'s $(T_1, \ldots, T_{e-1} \in L'_n)$ and put $\mathfrak{E}_F = \mathfrak{E}_F' \cap \mathbb{Q}(F)$, where $L'_n$ is the Hecke algebra for the Hecke pair $(GSp^+_n(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$. Then, by Theorem 2.2, $\mathfrak{E}_F$ is a non-zero ideal, and therefore $\mathfrak{E}_F$ is a non-zero ideal of $\mathbb{Q}(F)$. Let

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\( \mathcal{I}(l, F, \chi) \) be a certain fractional ideal of \( \mathbb{Q}(F, \chi) \) associated with the value \( L(l, F, \chi, St) \) as defined in Section 2, where \( \mathbb{Q}(F, \chi) \) is the field generated over the Hecke field \( \mathbb{Q}(F) \) of \( F \) by all the values of \( \chi \). Then we prove that we have

\[
\mathcal{I}(m, F, \chi) \subset \langle (C_{n,k} \mathcal{E}_F)^{-1} \rangle_{\mathbb{Q}(F, \chi)[N^{-1}]}
\]

for any positive integer \( m \leq k - n \) and primitive character \( \chi \mod N \) satisfying a certain condition, where \( C_{n,k} \) is a positive integer depending only on \( k \) and \( n \). (For a precise statement, see Theorem 2.3). By this we easily see the following result (cf. Corollary 2.4):

Let \( \mathcal{P}_F \) be the set of prime ideals \( \mathfrak{p} \) of \( \mathbb{Q}(F) \) such that

\[
\text{ord}_\mathfrak{p}(N_{\mathbb{Q}(F, \chi)/\mathbb{Q}(F)}(\mathcal{I}(m, F, \chi))) < 0
\]

for some positive integer \( m \leq k - n \) and primitive character \( \chi \) with conductor not divisible by \( \mathfrak{p} \) satisfying the above condition. Then \( \mathcal{P}_F \) is a finite set. Moreover, there exists a positive integer \( r = r_{n,k} \) depending only on \( n \) and \( k \) such that we have

\[
\text{ord}_\mathfrak{q}(\mathcal{I}(m, F, \chi)) \geq -r[\mathbb{Q}(F, \chi) : \mathbb{Q}(F)]
\]

for any prime ideal \( \mathfrak{q} \) of \( \mathbb{Q}(F, \chi) \) lying above a prime ideal in \( \mathcal{P}_F \) and positive integer \( m \leq k - n \) and primitive character \( \chi \) with conductor not divisible by \( \mathfrak{q} \) satisfying the above condition.

We have also similar results for the products of Hecke \( L \) functions of primitive forms for \( SL_2(\mathbb{Z}) \).

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**Notation** We denote by \( \mathbb{Z}_{>0} \) and \( \mathbb{Z}_{\geq 0} \) the set of positive integers and the set of non-negative integers, respectively.

For a commutative ring \( R \), let \( M_{mn}(R) \) denote the set of \( m \times n \) matrices with entries in \( R \), and especially write \( M_n(R) = M_{nn}(R) \). We often identify an element \( a \) of \( R \) and the matrix \( (a) \) of size 1 whose component is \( a \). If \( m \) or \( n \) is 0, we understand an element of \( M_{mn}(R) \) is the empty matrix and denote it by \( \emptyset \). Let \( GL_n(R) \) be the group consisting of all invertible elements of \( M_n(R) \), and \( \text{Sym}_n(R) \) the set of symmetric matrices of size \( n \) with entries in \( R \). Let \( K \) be a field of characteristic 0, and \( R \) its subring. We say that an element \( A \) of \( \text{Sym}_n(R) \) is non-degenerate if the determinant \( \det A \) of \( A \) is non-zero. For a subset \( S \) of \( \text{Sym}_n(R) \), we denote by \( S^{\text{nd}} \) the subset of \( S \) consisting of non-degenerate matrices. For a subset \( S \) of \( \text{Sym}_n(\mathbb{R}) \) we denote by \( S_{\geq 0} \) (resp. \( S_{>0} \)) the subset of \( S \) consisting of semi-positive definite matrices.
(resp. positive definite) matrices. We say that an element $A = (a_{ij})$ of $\text{Sym}_n(K)$ is half-integral if $a_{ii}$ $(i = 1, \ldots, n)$ and $2a_{ij}$ $(1 \leq i \neq j \leq n)$ belong to $R$. We denote by $\mathcal{H}_n(R)$ the set of half-integral matrices of size $n$ over $R$. We note that $\mathcal{H}_n(R) = \text{Sym}_n(R)$ if $R$ contains the inverse of 2. For an $(m, n)$ matrix $X$ and an $(m, m)$ matrix $A$, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of $X$. Let $G$ be a subgroup of $GL_n(R)$. Then we say that two elements $B$ and $B'$ in $\text{Sym}_n(R)$ are $G$-equivalent if there is an element $g$ of $G$ such that $B' = B[g]$. For two square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

We often write $x \perp Y$ instead of $(x) \perp Y$ if $(x)$ is a matrix of size 1. We denote by $1_m$ the unit matrix of size $m$ and by $O_{m,n}$ the zero matrix of type $(m, n)$. We sometimes abbreviate $O_{m,n}$ as $O$ if there is no fear of confusion.

Let $\mathfrak{b}$ be a subset of $K$. We then denote by $(\mathfrak{b})_R$ the $R$-sub-module of $K$ generated by $\mathfrak{b}$. For a non-zero integer $M$, we put

$$R[M^{-1}] = \{aM^{-s} \mid a \in R, \ s \in \mathbb{Z}_{\geq 0}\}$$

Let $K$ be an algebraic number filed, and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we denote by $\mathcal{O}(\mathfrak{p})$ the localization of $\mathcal{O}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{a}$ be a fractional ideal in $K$. If $\mathfrak{a} = \mathfrak{p}^e\mathfrak{b}$ with a fractional ideal $\mathfrak{b}$ of $K$ such that $\mathcal{O}(\mathfrak{p})\mathfrak{b} = \mathcal{O}(\mathfrak{p})$ we write $\text{ord}_\mathfrak{p}(\mathfrak{a}) = e$. We make the convention that $\text{ord}_\mathfrak{p}(\mathfrak{a}) = \infty$ if $\mathfrak{a} = \{0\}$. We simply write $\text{ord}_\mathfrak{p}(c) = \text{ord}_\mathfrak{p}((c))$ for $c \in K$. For an ideal $\mathfrak{j}$ of $K$, let $\mathfrak{j}^{-1}$ the inverse ideal of $\mathfrak{j}$.

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$.

2. Main result

For a subring $K$ of $\mathbb{R}$ put

$$\text{GSp}_n^+(K) = \{\gamma \in GL_{2n}(K) \mid J_n[\gamma] = \kappa(\gamma)J_n \text{ with some } \kappa(\gamma) > 0\},$$

and

$$\text{Sp}_n(K) = \{\gamma \in \text{GSp}_n^+(K) \mid J_n[\gamma] = J_n\},$$

where $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. In particular, put $\Gamma^{(n)} = \text{Sp}_n(\mathbb{Z})$ as in Introduction. We sometimes write an element $\gamma$ of $\text{GSp}_n^+(K)$ as $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_n(K)$. We define subgroups $\Gamma^{(n)}(N)$ and $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma^{(n)}(N) = \{\gamma \in \Gamma^{(n)} \mid \gamma \equiv 1_{2n} \mod N\},$$

$$\Gamma_0^{(n)}(N) = \{\gamma \in \Gamma^{(n)} \mid \gamma \equiv 1_{2n} \mod N\}.$$
and
\[ I_0^{(n)}(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} | C \equiv O_n \mod N \}. \]

Let \( H_n \) be Siegel’s upper half space of degree \( n \). We write \( \gamma(Z) = (AZ + B)(CZ + D)^{-1} \) and \( j(\gamma, Z) = \det(CZ + D) \) for \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbb{R}) \) and \( Z \in H_n \). We write \( F|_k \gamma(Z) = (\det \gamma)^{k/2} j(\gamma, Z)^{-k} f(\gamma(Z)) \) for \( \gamma \in GSp_n^+(\mathbb{R}) \) and a \( C^\infty \)-function \( f \) on \( H_n \). We simply write \( F|_k \gamma \) for \( F|_k \gamma(Z) \) if there is no confusion. We say that a subgroup \( \Gamma \) of \( \Gamma^{(n)} \) is a congruence subgroup if \( \Gamma \) contains \( \Gamma^{(n)}(N) \) with some \( N \). We also say that a character \( \eta \) of a congruence subgroup \( \Gamma \) is a congruence character if its kernel is a congruence subgroup. For a positive integer \( k \), a congruence subgroup \( \Gamma \) and its congruence character \( \eta \), we denote by \( M_k(\Gamma, \eta) \) (resp. \( M_k^\infty(\Gamma, \eta) \)) the space of holomorphic (resp. \( C^\infty \)) modular forms of weight \( k \) and character \( \eta \) for \( \Gamma \). We denote by \( S_k(\Gamma, \eta) \) the subspace of \( M_k(\Gamma, \eta) \) consisting of cusp forms. If \( \eta \) is the trivial character, we abbreviate \( M_k(\Gamma, \eta) \) and \( S_k(\Gamma, \eta) \) as \( M_k(\Gamma) \) and \( S_k(\Gamma) \), respectively. Let \( dv \) denote the invariant volume element on \( H_n \) defined by
\[ dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq t \leq n} (dx_{jl} \wedge dy_{jl}). \]

Here for \( Z \in H_n \) we write \( Z = (x_{jl}) + \sqrt{-1}(y_{jl}) \) with real matrices \((x_{jl})\) and \((y_{jl})\). For two elements \( F \) and \( G \) of \( M_k^\infty(\Gamma, \eta) \), we define the Petersson scalar product \( \langle F, G \rangle_\Gamma \) of \( F \) and \( G \) by
\[ \langle F, G \rangle_\Gamma = \int_{\Gamma \backslash H_n} F(Z) \overline{G(Z)} \det(\text{Im}(Z))^k dv, \]
provided the integral converges. For \( i = 1, 2 \), let \( \Gamma_i \) be a congruence subgroup with a congruence character \( \eta_i \). Then there exists a congruence subgroup \( \Gamma \) contained in \( \Gamma_1 \cap \Gamma_2 \) and its congruence character \( \eta \) such that \( \eta |_\Gamma = \eta_2 |_\Gamma = \eta \). Then we have \( M_k^\infty(\Gamma, \eta) \supset M_k^\infty(\Gamma_1, \eta_1) \). For elements \( F_1 \) and \( F_2 \) of \( M_k^\infty(\Gamma, \eta_1) \) and \( M_k^\infty(\Gamma_2, \eta_2) \), respectively, the value \( [f^{(n)} : \Gamma]^{-1} \langle F_1, F_2 \rangle_\Gamma \) does not depend on the choice of \( \Gamma \). We denote it by \( \langle F_1, F_2 \rangle \).

Let \( F \) be an element of \( M_k(\Gamma, \eta) \). Then, \( F \) has the following Fourier expansion:
\[ F(Z) = \sum_{A \in \mathcal{H}_n(Z) \geq 0} c_F(A \frac{N}{N}) e(\text{tr}(A Z N)) \]
with some positive integer \( N \), where \( \text{tr} \) denotes the trace of a matrix. For a subset \( S \) of \( \mathbb{C} \), we denote by \( M_k(\Gamma, \eta)(S) \) the set of elements \( F \) of \( M_k(\Gamma, \eta) \) such that \( c_F(A \frac{N}{N}) \in S \) for all \( A \in \mathcal{H}_n(Z) \geq 0 \), and put
$S_k(\Gamma, \eta)(S) = M_k(\Gamma, \eta)(S) \cap S_k(\Gamma, \eta)$. If $R$ is a commutative ring, and $S$ is an $R$ module, then $M_k(\Gamma, \eta)(S)$ and $S_k(\Gamma, \eta)(S)$ are $R$-modules.

For a Dirichlet character $\phi$ modulo $N$, let $\tilde{\phi}$ denote the character of $\Gamma_0^{(n)}(N)$ defined by $\Gamma_0^{(n)}(N) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \phi(\det D)$, and we write $M_k(\Gamma_0^{(n)}(N), \phi)$ for $M_k(\Gamma_0^{(n)}(N), \tilde{\phi})$, and so on.

We denote by $L_n = L_\mathbb{Q}(\text{GSp}_n^+(\mathbb{Q}), \Gamma^{(n)})$ be the Hecke ring over $\mathbb{Q}$ associated with the Hecke pair $(\text{GSp}_n^+(\mathbb{Q}), \Gamma^{(n)})$, and by $L'_n = L_\mathbb{Z}(\text{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$ be the Hecke ring over $\mathbb{Z}$ associated with the Hecke pair $(\text{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}), \Gamma^{(n)})$. For a Hecke eigenform $F$, we denote by $\mathbb{Q}(F)$ the field generated over $\mathbb{Q}$ by the eigenvalues of all Hecke operators $T \in L_n$ with respect to $F$, and call it the Hecke field of $F$. For Dirichlet characters $\chi_1, \ldots, \chi_r$, we denote by $\mathbb{Q}(\chi_1, \ldots, \chi_r)$ the field generated over $\mathbb{Q}$ by all the values of $\chi_1, \ldots, \chi_r$, and by $\mathbb{Q}(F, \chi_1, \ldots, \chi_r)$ the composite field of $\mathbb{Q}(F)$ and $\mathbb{Q}(\chi_1, \ldots, \chi_r)$. For a Hecke eigenform $F$ in $S_k(\Gamma_0^{(n)}(N))$ and a Dirichlet character $\chi$ let $L(s, F, St, \chi)$ be the standard $L$ function of $F$ twisted by $\chi$. For a Dirichlet character $\chi$, we put $\delta_\chi = 0$ or $1$ according as $\chi(-1) = 1$ or $\chi(-1) = -1$. Assume that $\chi$ is primitive, and for any positive integer $m \leq k - n$ such that $m - n \equiv \delta_\chi \text{ mod } 2$ define $\Lambda(m, F, \chi, St)$ as

$$\Lambda(m, F, \chi, St) = \frac{\chi(-1)^n \Gamma(m) \prod_{i=1}^n \Gamma(2k - n - i)L(m, F, St, \chi)}{(F, F) \pi^{-n(n+1)/2+nk+(n+1)m} \sqrt{-1}^{m+n} \tau(\chi)^{n+1}}.$$  

$\tau(\chi)$ is the Gauss sum of $\chi$. For a Dirichlet character $\chi$ let $m_\chi$ be the conductor of $\chi$. The following proposition is essentially due to [4, Appendix, Theorem].

**Proposition 2.1.** Let $F$ be a Hecke eigenform in $S_k(\Gamma^{(n)})(\mathbb{Q}(F))$. Let $m$ be a positive integer not greater than $k - n$ and $\chi$ a primitive character $\chi$ satisfying the following condition:

(C) $m - n \equiv \delta_\chi \text{ mod } 2$, and $m > 1$ if $n > 1$, $n \equiv 1 \text{ mod } 4$ and $\chi^2$ is trivial.

Then $\Lambda(m, F, \chi, St)$ belongs to $\mathbb{Q}(F, \chi)$.

Let $\mathcal{V}$ be a subspace of $M_k(\Gamma^{(n)})$. We say that a multiplicity one holds for $\mathcal{V}$ if any Hecke eigenform in $\mathcal{V}$ is uniquely determined up to constant multiple by its Hecke eigenvalues.

**Theorem 2.2.** Suppose that $k \geq n + 1$. Then a multiplicity one theorem holds for $S_k(\Gamma^{(n)})$.

**Proof.** This is essentially due to Chenevier-Lannes [7, Corollary 8.5.4]. It was proved under a more stronger assumption without using [7,}
Let $F$ be a Hecke eigenform in $S_k(\Gamma^{(n)})$ with $k \geq n + 1$. Then by Theorem 2.2 we have $cF \in S_k(\Gamma^{(n)})(\mathbb{Q}(F))$ with some $c \in \mathbb{C}$. Hence for $A, B \in \mathcal{H}_n(\mathbb{Z})_{>0}$ and an integer $l$ satisfying $(C)$, the value $c_F(A)\overline{c_F(B)}\Lambda(l, F, St, \chi)$ belongs to $\mathbb{Q}(F)$ and does not depend on the choice of $c$. For $A$ and $B$ and an integer $l$ put

$$I_{A,B}(l, F, \chi) = c_F(A)\overline{c_F(B)}\Lambda(l, F, \chi, St).$$

Let $\mathcal{J}(l, F, \chi)$ be the $\mathfrak{O}_{\mathbb{Q}(F)}$-module generated by all $I_{A,B}(l, F, \chi)$’s. Then $\mathcal{J}(l, F, \chi)$ becomes a fractional ideal in $\mathbb{Q}(F, \chi)$. We note that it is uniquely determined by $l$ and the system of eigenvalues of $F$. Let $F_1, \ldots, F_d$ be a basis of $S_k(\Gamma^{(n)})$ consisting of Hecke eigenforms such that $F_1 = F$. Let $K_{n,k}$ be the composite filed $\mathbb{Q}(F_1) \cdots \mathbb{Q}(F_d)$ of $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_d)$. We denote by $\tilde{D}_F$ the ideal of $K_{n,k}$ generated by all $\prod_{i=2}^{d}(\lambda_{F}(T_{i-1}) - \lambda_{F_i}(T_{i-1}))$’s ($T_1, \ldots, T_{d-1} \in \mathfrak{I}'_n$), and put $\tilde{D}_F = D_F \cap \mathbb{Q}(F)$. We make the convention that $\tilde{D}_F = \mathfrak{O}_{K_{n,k}}$ if $d = 1$. Moreover, let $\mathfrak{E}_F$ be the ideal of $\mathbb{Q}(F)$ defined in Section 1. Then our first main result is as follows.

**Theorem 2.3.** Let $F$ be a Hecke eigenform in $S_k(\Gamma^{(n)})$. Then we have

$$\mathcal{J}(m, F, \chi) \subset \langle (\mathfrak{O}^{(n,k)}A_{n,k}\mathfrak{E}_F)^{-1} \rangle_{\mathbb{Q}(F, \chi)[N^{-1}]}$$

for any positive integer $m \leq k - n$ and primitive character $\chi \mod N$ satisfying the condition $(C)$, where $\alpha(n,k)$ is a non-negative integer depending only on $k$ and $n$, and $A_{n,k} = \text{LCM}_{n+1 \leq m \leq k}\{\prod_{i=1}^{n}(2l-2i)(2l-2i+1)\!\}$. In particular if $m \leq k - n - 1$, then

$$\mathcal{J}(m, F, \chi) \subset \langle (\mathfrak{O}^{(n,k)}A_{n,k}\tilde{D}_F)^{-1} \rangle_{\mathbb{Q}(F, \chi)[N^{-1}]}.$$

We will prove the above theorem in Section 5.

**Corollary 2.4.** Let $F$ be a Hecke eigenform in $S_k(\Gamma^{(n)})$. Let $\mathcal{P}_F$ be the set of prime ideals $\mathfrak{p}$ of $\mathbb{Q}(F)$ such that

$$\text{ord}_{\mathfrak{p}}(\mathcal{N}_{\mathbb{Q}(F, \chi)/\mathbb{Q}(F)}(\mathcal{J}(m, F, \chi))) < 0$$

for some positive integer $m \leq k - n$ and primitive character $\chi$ with conductor not divisible by $\mathfrak{p}$ satisfying $(C)$. Then $\mathcal{P}_F$ is a finite set. Moreover, there exists a positive integer $r$ such that we have

$$\text{ord}_{\mathfrak{p}}(\mathcal{J}(m, F, \chi)) \geq -r[\mathbb{Q}(F, \chi) : \mathbb{Q}(F)]$$

Conjecture 8.4.22]. As is written in the postface in that book, this conjecture has been proved [11], and the same proof is available at least even when $k \geq n + 1$. □
for any prime ideal \( q \) of \( \mathbb{Q}(F, \chi) \) lying above a prime ideal in \( \mathcal{P}_F \) and integer \( l \) and primitive character \( \chi \) with conductor not divisible by \( q \) satisfying the condition (C).

Proof. By Theorem 2.3, we have \( p \mid 2^{\alpha(n,k)} A_{n,k} \tilde{E}_F \) if \( p \in \mathcal{P}_F \). This proves the first assertion. Let \( 2^{\alpha(n,k)} A_{n,k} \tilde{E}_F = p_1^{e_1} \cdots p_s^{e_s} \) be the prime factorization of \( 2^{\alpha(n,k)} A_{n,k} \tilde{E}_F \), where \( p_1, \ldots, p_s \) are distinct prime ideals and \( e_1, \ldots, e_s \) are positive integers. We note that for any prime ideal \( p \) of \( \mathbb{Q}(F) \) and prime ideal \( q \) of \( \mathbb{Q}(F, \chi) \) lying above \( p \) we have \( \text{ord}_q(p) \leq [\mathbb{Q}(F, \chi) : \mathbb{Q}(F)] \). Hence \( r = \max\{e_i\}_{1 \leq i \leq s} \) satisfies the required condition in the second assertion. □

Remark 2.5. (1) Let

\[
\Lambda(F, m, \chi) = \frac{\Gamma(m) \prod_{i=1}^{n} \Gamma(2k - n - i)L(m, F, \text{St}, \chi)}{(F, F)_{\pi - n(n+1)/2 + nk+(n+1)m}}.
\]

Then, if \( m \) and \( \chi \) satisfy the condition (C), \( \Lambda(F, m, \chi) \) belongs to \( \mathbb{Q}(F, \chi, \zeta_N) \), where \( \mathbb{Q}(F, \chi, \zeta_N) \) is the field generated over the Hecke field \( \mathbb{Q}(F) \) of \( F \) by all the values of \( \chi \) and the primitive \( N \)-th root \( \zeta_N \) of unity. In [3, Theorem], a similar result has been proved for \( \Lambda(F, m, \chi) \). Our \( L \)-value belongs to \( \mathbb{Q}(F, \chi) \), which is included in \( \mathbb{Q}(F, \chi, \zeta_N) \). Therefore, our result can be regarded as a refinement of Böcherer’s.

(2) Böcherer [3] excluded the case \( m = k - n \). However, we can include this case. We also note that we can get a sharper result if we restrict ourselves to the case \( m < k - n \) as stated in the above theorem.

(3) In [3], the main result was formulated without assuming multiplicity one theorem. However, such a formulation is now unnecessary.

3. Pullback of Siegel Eisenstein series

To prove our main result, first we express a certain modular form as a linear combination of Hecke eigenforms (cf. Theorem 3.7). We have carried out it in [12, Appendix], and here we treat it in a more general setting. We also correct some inaccuracies in [12, Appendix] (cf. Remark 3.8). For a non-negative integer \( m \), put

\[
\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(s - \frac{i - 1}{2}\right).
\]
For a Dirichlet character $\chi$ we denote by $L(s, \chi)$ the Dirichlet $L$-function associated to $\chi$, and put
\[
\mathcal{L}_n(s, \chi) = \Gamma_n(s)\pi^{-ns}L(s, \chi) \prod_{i=1}^{[n/2]} L(2s - 2i, \chi^2) \\ \times \begin{cases} 
\pi^{n/2-s}\Gamma(s - n/2) & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd.}
\end{cases}
\]
Let $n, l$ and $N$ be positive integers. For a Dirichlet character $\phi$ modulo $N$ such that $\phi(-1) = (-1)^l$, we define the Eisenstein series $E_{n,l}^\ast(Z; N, \phi, s)$ by
\[
E_{n,l}^\ast(Z; N, \phi, s) = (\det \text{Im}(Z))^s \mathcal{L}_n(l + 2s, \phi) \\ \times \sum_{\gamma \in T^{(n)}(N) \setminus T^{(n)}(N)} \phi^\ast(\gamma)j(\gamma, Z)^{-l}j(\gamma, Z)^{-2s},
\]
where
\[
T^{(n)}(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} | A \equiv O_n \mod N \}, \\
T^{(n)}(N)_{\infty} = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} | B \equiv O_n \mod N, C = O_n \},
\]
and $\phi^\ast(\gamma) = \phi(\det C)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in T^{(n)}(N)$. Then $E_{n,l}^\ast(Z; N, \phi, s)$ converges absolutely as a function of $s$ if the real part of $s$ is large enough. Moreover, it has a meromorphic continuation to the whole $s$-plane, and it belongs to $M_l^{\infty}(\Gamma_0^{(n)}(N), \phi)$. Moreover it is holomorphic and finite at $s = 0$, which will be denoted by $E_{n,l}^\ast(Z; N, \phi)$. In particular, if $E_{n,l}^\ast(Z; N, \phi)$ belongs to $M_l(\Gamma_0^{(n)}(N), \phi)$, it has the following Fourier expansion:
\[
E_{n,l}^\ast(Z; N, \phi) = \sum_{A \in \mathcal{H}_n(Z) \geq 0} c_{n,l}(A, N, \phi)e(\text{tr}(AZ)).
\]
To see the Fourier coefficient of $E_{n,l}^\ast(Z; N, \phi)$, we define a polynomial attached to local Siegel series. For a prime number $p$ let $\mathbb{Q}_p$ be the field of $p$-adic numbers, and $\mathbb{Z}_p$ the ring of $p$-adic integers. For an element $B \in \mathcal{H}_n(\mathbb{Z}_p)$, we define the Siegel series $b_p(B, s)$ as
\[
b_p(B, s) = \sum_{R \in \text{Sym}_n(\mathbb{Q}_p)/\text{Sym}_n(\mathbb{Z}_p)} e_p(\text{tr}(BR))\nu(R)^{-s},
\]
where $e_p$ is the additive character of $\mathbb{Z}_p$ such that $e_p(m) = e(m)$ for $m \in \mathbb{Z}[p^{-1}]$, and $\nu_p(R) = [R\mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$. We define $\chi_p(a)$ for $a \in \mathbb{Q}_p^\times$
as follows:

\[ \chi_p(a) := \begin{cases} 
+1 & \text{if } Q_p(\sqrt{a}) = Q_p, \\
-1 & \text{if } Q_p(\sqrt{a})/Q_p \text{ is quadratic unramified,} \\
0 & \text{if } Q_p(\sqrt{a})/Q_p \text{ is quadratic ramified.} 
\end{cases} \]

For an element \( B \in \mathcal{H}_n(\mathbb{Z}_p) \) with \( n \) even, we define \( \xi_p(B) \) by

\[ \xi_p(B) := \chi_p((-1)^n/2 \det B). \]

For a nondegenerate half-integral matrix \( B \) of size \( n \) over \( \mathbb{Z}_p \) define a polynomial \( \gamma_p(B,X) \) in \( X \) by

\[ \gamma_p(B,X) := \begin{cases} 
(1 - X) \prod_{i=1}^{n/2} (1 - p^{2i}X^2)(1 - p^{n/2} \xi_p(B)X)^{-1} & \text{if } n \text{ is even,} \\
(1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i}X^2) & \text{if } n \text{ is odd.}
\end{cases} \]

Then it is well known that there exists a unique polynomial \( F_p(B,X) \) in \( X \) over \( \mathbb{Z} \) with constant term 1 such that

\[ b_p(B,s) = \gamma_p(B,p^{-s})F_p(B,p^{-s}) \]

(e.g. \([9]\)). More precisely, we have the following proposition.

**Proposition 3.1.** Let \( B \in \mathcal{H}_m(\mathbb{Z}_p) \). Then there exists a polynomial \( H_p(B,x) \) in \( X \) over \( \mathbb{Z} \) such that

\[ F_p(B,X) = H_p(B,p^{[m+1]/2}X). \]

**Proof.** The assertion follows from \([14]\), Theorem 2. \( \square \)

For \( B \in \mathcal{H}_m(\mathbb{Z}) \) with \( m \) even, let \( B_B \) be the discriminant of \( \mathbb{Q}(\sqrt{(-1)^{m/2} \det B})/\mathbb{Q} \), and \( \chi_B = (\frac{\mathbb{Q}}{4}) \) the Kronecker character corresponding to \( \mathbb{Q}(\sqrt{(-1)^{m/2} \det B})/\mathbb{Q} \). We note that we have \( \chi_B(p) = \xi_p(B) \) for any prime \( p \). We also note that

\[ (-1)^{m/2} \det(2B) = B_B \]

with \( B_B \in \mathbb{Z}_{>0} \). We define a polynomial \( F_p^*(T,X) \) for any \( T \in \mathcal{H}_n(\mathbb{Z}_p) \) which is not-necessarily non-degenerate as follows: For an element \( T \in \mathcal{H}_n(\mathbb{Z}_p) \) of rank \( r \geq 1 \), there exists an element \( \tilde{T} \in \mathcal{H}_r(\mathbb{Z}_p) \) such that \( T \sim_{\mathbb{Z}_p} \tilde{T} \perp O_{n-r} \). We note that \( F_p(\tilde{T},X) \) does not depend on the choice of \( \tilde{T} \). Then we put \( F_p^*(T,X) = F_p(\tilde{T},X) \). For an element \( T \in \mathcal{H}_n(\mathbb{Z}) \) of rank \( r \geq 1 \), there exists an element \( \tilde{T} \in \mathcal{H}_r(\mathbb{Z}) \) such that \( T \sim_{\mathbb{Z}} \tilde{T} \perp O_{n-r} \). Then \( \chi_{\tilde{T}} \) does not depend on the choice of \( \tilde{T} \). We write \( \chi_{T}^* = \chi_{\tilde{T}} \) if \( r \) is even. For a non-negative integer \( m \) and a primitive character \( \phi \) let \( B_{m,\phi} \) be the \( m \)-th generalized Bernoulli number for \( \phi \). In the case \( \phi \) is the principal character, we write \( B_m = B_{m,\phi} \), which is the \( m \)-th Bernoulli number. For a Dirichlet character \( \phi \) we denote by \( \phi_0 \) the primitive character associated with \( \phi \).
Proposition 3.2. Let $n$ and $l$ be positive integers such that $l \geq n + 1$, and $\phi$ a primitive character mod $N$. Then $E_{2n,l}^*(Z; N, \phi)$ is holomorphic and belongs to $M_l(I_0^{(2n)}(N), \phi)$ except the following case:

$l = n + 1 \equiv 2 \mod 4$ and $\phi^2 = 1_N$.

In the case that $E_{2n,l}^*(Z; N, \phi)$ is holomorphic we have the following assertion:

(1) Suppose that $N = 1$ and $\phi$ is the principal character $1$, Then for $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$ of rank $m$, we have

$$c_{2n,l}(B, 1, 1) = (-1)^{l/2+n(n+1)/2}2^{l-1}2^{1/2} \prod_{p | \det(2B)} F_p(B, p^{-m-1})$$

$$\times \left\{ \begin{array}{ll}
\prod_{i=m/2+1}^{n} \zeta(1+2i-2l)L(1+m/2-l, \chi_B^*) & \text{if } m \text{ is even,} \\
\prod_{i=(m+1)/2}^{n} \zeta(1+2i-2l) & \text{if } m \text{ is odd.}
\end{array} \right.$$}

Here we make the convention that $F_p(B, p^{-m-1}) = 1$ and $L(1+m/2-l, \chi_B^*) = \zeta(1-l)$ if $m = 0$.

(2) Suppose that $N > 1$. Then, $c_{2n,l}(B, N, \phi) = 0$ if $B \in \mathcal{H}_{2n}(\mathbb{Z})_{\geq 0}$ is not positive definite. Moreover, for any $B \in \mathcal{H}_{2n}(\mathbb{Z})_{> 0}$ we have

$$c_{2n,l}(B, N, \phi) = (-1)^{n+1}2^{n-1}2^{l-1} \prod_{p | \phi | B} (1-p^{n-l}(\phi \chi_B)_0).$$

Proof. (1) The assertion follows from [11], Theorem 2.3] remarking that

$$L_{2n}(l, 1) = \zeta(1-l) \prod_{i=1}^{n} \zeta(1-2l+2i)(-1)^{(n(n+1)+l)/2}2^{l-1}.$$}

(2) The first assertion follows from [11, Section 5]. Let $B \in \mathcal{H}_{2n}(\mathbb{Z})_{> 0}$. Then,

$$c_{2n,l}(B, N, \phi) = (-1)^{n+1}2^{n-1}2^{l-1} \prod_{p | \phi | B} F_p(B, p^{-m-1}(\phi \chi_B)_0) L(l-n, \phi \chi_B) \prod_{p | \phi | B} (1-p^{n-l}(\phi \chi_B)_0).$$

We have

$$L(l-n, \phi \chi_B) = L(l-n, (\phi \chi_B)_0) \prod_{p | \phi | B} (1-p^{n-l}(\phi \chi_B)_0).$$
\[ \Gamma(l - n)L(l - n, (\phi \chi_B)_0) = (-1)^{(l - \delta(\phi \chi_B)_0)/2} 2^{l - n - 1} |\phi \chi_B|_0 \sqrt{-1}^{-\delta(\phi \chi_B)_0} \times L(1 - l + n, (\phi \chi_B)_0). \]

Moreover, by the functional equation of \( F_p(B, X) \) (cf. [9]), we have
\[ \phi^{2l - 2n - 1}_B \prod_p F_p(B, p^{-l} \phi(p)) = \prod_p F_p(p^{l - 2n - 1} \phi(p), B). \]

Thus the assertion is proved remarking that \( \det(2B) = |\mathfrak{O}_B/\mathfrak{F}_B|. \] \( \square \)

**Corollary 3.3.** Let the notation be as above.

1. Suppose that \( N = 1 \). Then, \( c_{2n, l}(B, 1, 1) \) belongs to \( \langle \prod_{i=1}^{n} ((2l - 2i)(2l - 2i + 1))^{-1} \rangle_Z \) for any \( B \in H_{2n}(\mathbb{Z}) \).

2. Suppose that \( N > 1 \). Then for \( B \in H_{2n}(\mathbb{Z}) \), \( c_{2n,l}(B, N, \phi) \) is an algebraic number. In particular if \( \text{GCD}(\det(2B), N) = 1 \), then \( \tau(\phi)^{-1} \sqrt{-1}^{-l} c_{2n,l}(B, N, \phi) \) belongs to \( \langle (l - n)^{-1} \rangle_{\mathfrak{O}_Q[N^{-1}]} \).

**Proof.** (1) By Proposition 3.1, the product \( \prod_{p|\det(2B)} F_p^*(B, l - m - 1) \) is an integer for any \( m \) and \( B \in H_n(\mathbb{Z}) \) with rank \( m \). By Clausen-von-Staudt theorem, \( \zeta(1 - 2l + 2i) \) belongs to \( \langle ((2l - 2i)(2l - 2i + 1))^{-1} \rangle_Z \). By [2], (5.1), (5.2) and Clausen-von-Staudt theorem, for any positive even integer \( m \) and \( \tilde{B} \in H_m(\mathbb{Z}) \), \( L(1 - l + m/2, \chi_{\tilde{B}}) \) belongs to \( \langle ((2l - m)(2l - m + 1))^{-1} \rangle_Z \). This proves the assertion.

(2) It is well known that \( L(1 - l + n, (\phi \chi_B)_0) \) is algebraic. This proves the first part of the assertion. Suppose that \( \det(2B) \) is coprime to \( N \). Then \( \phi \chi_B \) is a primitive character of conductor \( N|\mathfrak{O}_B| \) and
\[ \tau(\phi \chi_B) = \phi(|\mathfrak{O}_B| \chi_B(N)) \tau(\phi) \tau(\chi_B) = \phi(|\mathfrak{O}_B| \chi_B(N)) \tau(\phi)|\mathfrak{O}_B|^{|1/2} \sqrt{-1}^{-\delta \chi_B}. \]

By [6] or [15], \( N(l - n)L(1 - l + n, \phi \chi_B) \) belongs to \( \mathfrak{O}_{\mathbb{Q}(\phi)} \), and by Proposition 3.1, \( \prod_p F_p(p^{l - 2n - 1} \phi(p), B) \) is an element of \( \mathfrak{O}_{\mathbb{Q}(\phi)} \). Thus the assertion has been proved remarking that \( \sqrt{-1} = \pm \sqrt{-1}^{-\delta \chi_B} \). \( \square \)

Let \( D_{n, \nu}^\phi \) be the differential operator in [4], which maps \( M^\infty_{l_0}(I_{0, 2n}^\nu(N)) \) to \( M^\infty_{l_0+k}(I_{0, 2n}^\nu(N)) \otimes M^\infty_{l_0+k}(I_{0, 2n}^\nu(N)) \). Let \( \chi \) be a primitive character mod \( N \). For a non-negative integer \( \nu \leq k \), we define a function \( \mathfrak{c}_{2n}(Z_1, Z_2, N, \chi) \)
on \( H_n \times H_n \) as
\[
\mathcal{E}_{2n}^{k,\nu}(Z_1, Z_2, N, \chi) = (2\pi \sqrt{-1})^{-\nu} \tau(\chi)^{-n-1} \sqrt{-1}^{-k+\nu} \\
\times D_{n,k-\nu}^0 \left( \sum_{X \in M_n(\mathbb{Z})} \chi(\det X) E^*_2 n,k-\nu(\ast, N, \chi)_{k-\nu} \left( \binom{12n}{O} S(X/N) \right) \right) (Z_1, Z_2)
\]
for \((Z_1, Z_2) \in H_n \times H_n\), where \( S(X/N) = \begin{pmatrix} O_n & X/N \\ tX/N & O_n \end{pmatrix} \). Let \( X \) be a symmetric matrix of size \( 2n \) of variables. Then there exists a polynomial \( P_{n,l}(X) \) in \( X \) such that
\[
D_{n,l}^0 \left( \mathbf{e}(\text{tr}( \begin{pmatrix} A_1 & R/2 \\ tR/2 & A_2 \end{pmatrix} \begin{pmatrix} Z_1 & Z_{12} \\ tZ_{12} & Z_2 \end{pmatrix} )) \right)
= (2\pi \sqrt{-1})^\nu P_{n,l}^\nu \left( \begin{pmatrix} A_1 & R/2 \\ tR/2 & A_2 \end{pmatrix} \right) \mathbf{e}(\text{tr}(A_1 Z_1 + A_2 Z_2))
\]
for \( \begin{pmatrix} A_1 & R/2 \\ tR/2 & A_2 \end{pmatrix} \in H_{2n}(\mathbb{Z}) \geq 0 \) with \( A_1, A_2 \in H_n(\mathbb{Z}) \geq 0 \) and \( \begin{pmatrix} Z_1 & Z_{12} \\ tZ_{12} & Z_2 \end{pmatrix} \in H_{2n} \) with \( Z_1, Z_2 \in H_n \).

**Proposition 3.4.** Under the above notation and the assumption, for a non-negative integer \( l \leq k \) write \( \mathcal{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) \) as
\[
\mathcal{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{A_1, A_2 \in H_n(\mathbb{Z}) \geq 0} c_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)(A_1, A_2) \mathbf{e}(\text{tr}(A_1 Z_1 + A_2 Z_2))
\]
Then we have
\[
c_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)(A_1, A_2)
= \sum_{R \in M_n(\mathbb{Z})} D_{n,l}^{k-l}(\begin{pmatrix} A_1 & R/2 \\ tR/2 & A_2 \end{pmatrix}) c_{2n}^{k,k-l}(\begin{pmatrix} A_1 & R/2 \\ tR/2 & A_2 \end{pmatrix}) \chi(\det R) \tau(\chi)^{-1} \sqrt{-1}^{-l}
\]

**Corollary 3.5.** For any \( A_1, A_2 \in H_n(\mathbb{Z})_0 \), \( c_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)(A_1, A_2) \) belongs to \( \widehat{\mathbb{Q}} \), and in particular if \( \det \left( \begin{pmatrix} 2A_1 & R \\ tR & 2A_2 \end{pmatrix} \right) \) is prime to \( N \), then \( a_{n,l} c_{2n}^{k,k-l}(Z_1, Z_2, N, \chi)(A_1, A_2) \) belongs to \( \mathcal{D}_{\mathbb{Q}(\chi)}[N^{-1}] \), where \( a_{n,l} = \prod_{i=1}^{n} (2l - 2i)(2l - 2i + 1)! \).

Suppose that \( l \leq k \). Then \( \mathcal{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) \) can be expressed as
\[
\mathcal{E}_{2n}^{k,k-l}(Z_1, Z_2, N, \chi) = \sum_{A \in L_n(\mathbb{Z})_0} \mathcal{E}_{2n}^{k,k-l}(Z_1, A, N, \chi) \mathbf{e}(\text{tr}(AZ_2))
\]
with $\mathcal{E}^{k,k-l}_{2n}(Z_1, A, N, \chi)$ a function of $Z_1$. Put

$$\mathcal{G}^{k,k-l}_{2n}(Z_1, A, N, \chi) = \sum_{\gamma \in I_0^{(n)}(N^2) \setminus I^{(n)}} (\mathcal{E}^{k,k-l}_{2n})_{\gamma}(Z_1, A, N, \chi).$$

It is easily seen that $\mathcal{E}^{k,k-l}_{2n}(Z_1, A, N, \chi)$ belongs to $M_k(I_0^{(n)}(N^2))$, and therefore $\mathcal{G}^{k,k-l}_{2n}(Z_1, A, N, \chi)$ belongs to $M_k(\Gamma^{(n)})$. In particular, if $l < k$, then $\mathcal{G}^{k,k-l}_{2n}(Z_1, A, N, \chi)$ belongs to $S_k(\Gamma^{(n)})$.

**Proposition 3.6.** Suppose that $l \leq k$ and let $A \in \mathcal{H}_n(\mathbb{Z})$. Then $a_{n,l}\mathcal{G}^{k,k-l}_{2n}(Z_1, N^2 A, N, \chi)$ belongs to $M_k(\Gamma^{(n)})(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}])$. In particular, if $l < k$, it belongs to $S_k(\Gamma^{(n)})(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}])$.

**Proof.** We have

$$c_{n,l}^{\mathcal{G}^{k,k-l}_{2n}}(Z_1, z_2, N, \chi)(B, N^2 A) = \sum_{R \in M_n(\mathbb{Z})} P_{n,l}^{k-l} \left( \begin{pmatrix} B & R/2 \\ tR/2 & N^2 A \end{pmatrix} \right) C_{n,l} \left( \begin{pmatrix} B & R/2 \\ tR/2 & N^2 A \end{pmatrix} \right) \bar{\chi}(\det R) \tau(\chi)^{-1} \sqrt{-1}^{-l}.$$

We note that det $\begin{pmatrix} 2B \\ tR/2 & 2N^2 A \end{pmatrix}$ is prime to $N$ if and only det $R$ is prime to $N$. Therefore, by Corollary 3.3, $a_{n,l}\mathcal{E}^{k,k-l}_{2n}(Z_1, N^2 A, N, \chi)$ belongs to $M_k(I_0^{(n)}(N^2))(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}])$. By q-expansion principle (cf. [8], [10]), for any $\gamma \in \Gamma^{(n)}$, $a_{n,l}\mathcal{E}^{k,k-l}_{2n}|_{\gamma}(Z_1, N^2 A, N, \chi)$ belongs to $M_k(I^{(n)}(N^2))(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}])$. Hence, $a_{n,l}\mathcal{G}^{k,k-l}_{2n}(Z_1, N^2 A, N, \chi)$ belongs to $M_k(I^{(n)}(N^2))(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}]) \cap M_k(\Gamma^{(n)}) = M_k(\Gamma^{(n)})(\mathcal{O}_{\mathbb{Q}(\chi,\zeta)}[N^{-1}])$. This proves the first of the assertion. The latter is similar. \qed

**Theorem 3.7.** Let $\{F_i\}_{i=1}^d$ be an orthogonal basis of $S_k(\Gamma^{(n)})$ consisting of Hecke eigenforms, and $\{F_i\}_{d+1 \leq i \leq e}$ be a basis of the orthogonal complement $S_k(\Gamma^{(n)})^\perp$ of $S_k(\Gamma^{(n)})$ in $M_k(\Gamma^{(n)})$ with respect to the Petersson product. Then we have

$$\mathcal{G}^{k,k-l}_{2n}(Z, N^2 A, N, \chi) = \sum_{i=1}^d c(n, l) N^{nl} A(l - n, F_i, \chi, \text{St}) c_{F_i}(A) F_i(Z) + \sum_{i=d+1}^e c_i F_i(Z)$$

where $c(n, l) = (-1)^{a(n,l)} 2^{b(n,l)}$ with $a(n,l), b(n,l)$ integers, and $c_i$ is a certain complex number. Moreover we have $c_i = 0$ for any $d+1 \leq i \leq e$ if $l < k$. 

Proof. Put
\[ \mathfrak{G}^{k,k-l}_{2n}(Z_1, Z_2, N, \chi) = \sum_{\gamma \in \Gamma_0(n)(N^2) \setminus \Gamma(n)} \mathfrak{G}^{k,k-l}_{2n}(|k\gamma Z_1, Z_2, N, \chi). \]
Then we have
\[ \mathfrak{G}^{k,k-l}_{2n}(Z_1, Z_2, N, \chi) = \sum_{A \in \mathcal{L}_n(\mathbb{Z}) > 0} \mathfrak{G}^{k,k-l}_{2n}(Z_1, A, N, \chi)e(\text{tr}(AZ_2)) \]
By [H, (3.24)], for any \( \gamma \in \Gamma(n) \) we have
\[ \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(|k\gamma *, -Z_2, N, \chi) \rangle \]
\[ = \langle F_i | k\gamma, \mathfrak{G}^{k,k-l}_{2n}(|k\gamma *, -Z_2, N, \chi) \rangle \]
\[ = \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(*, -Z_2, N, \chi) \rangle \]
\[ = (-1)^{a'(n,l)2^b(n,l)N^{nl}}(\chi)(-1)^{n[n(\Gamma(n) : \Gamma_0(n)(N^2))]^{-1}n(l-k)n-(2n+1)l+n(n+1)/2} \]
\[ \times L(l-n, F_i, \bar{\chi}, St)\Gamma(l-n)\tau(\chi)^{-n-1}\sqrt{1-l} F_i(N^2 Z_2) \]
\[ \times \frac{\Gamma_{2n}(l)\Gamma_n(k-n/2)\Gamma_n(k-(n+1)/2)}{\Gamma_n(l)\Gamma_n(l-n/2)}, \]
with \( a'(n,l), b'(n,l) \in \mathbb{Z} \). We note that we take the normalized Petersson inner product. We also note that
\[ \Gamma_{2n}(l) = \pi^{n/2}\Gamma_n(l)\Gamma_n(l-n/2), \]
and
\[ \Gamma_n(k-n/2)\Gamma_n(k-(n+1)/2) = 2^{\gamma'(n,l)}\pi^{n/2} \prod_{i=1}^{n} \Gamma(2k-n-i) \]
with an integer \( \gamma'(n,l) \). Hence we have
\[ \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(|k\gamma *, -Z_2, N, \chi) \rangle \]
\[ = c(n, l)[\Gamma(n) : \Gamma_0(n)(N^2)]^{-1}N^{nl}\Lambda(l-n, F_i, \bar{\chi}, St)\langle F_i, F_i \rangle F_i(N^2 Z_2), \]
where \( c(n, l) = (-1)^{a(n,l)2^b(n,l)} \) with \( a(n,l), b(n,l) \) integers. On the other hand, we have
\[ \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(*, -Z_2, N, \chi) \rangle = \sum_{A \in \mathcal{L}_n(\mathbb{Z}) > 0} \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(*, A, N, \chi) \rangle e(\text{tr}(AZ_2)). \]
Hence we have
\[ \langle F_i, \mathfrak{G}^{k,k-l}_{2n}(*, A, N, \chi) \rangle = c(n, l)N^{nl}\Lambda(l-n, F_i, \bar{\chi}, St)\langle F_i, F_i \rangle c_{F_i}(N^{-2} A) \]
for any $A$. Now $G_{2n}^{k,k-l}(Z, A, N, \chi)$ can be expressed as

$$G_{2n}^{k,k-l}(Z, A, N, \chi) = \sum_{i=1}^{e} c_i F_i(Z)$$

with $c_i \in \mathbb{C}$. For $1 \leq i \leq d$ we have

$$\langle F_i, G_{2n}^{k,k-l}(*, A, N, \chi) \rangle = c_i \langle F_i, F_i \rangle.$$

Hence we have

$$c_i = c(n, l) N^d A(l - n, F_i, \chi, \text{St}) \langle F_i, F_i \rangle c_{F_i}(N^{-2}A).$$

We note that $\Lambda(l - n, F_i, \chi, \text{St}) = \Lambda(l - n, F_i, \chi, \text{St})$. This proves the assertion. □

**Remark 3.8.** There are errors in [12], Appendix.

1. The factor $\eta^*(\gamma)$ is missing in $E_{n,l}(Z, M, \eta, s)$ on [12], page 125, and it should be defined as

$$E_{n,l}(Z, M, \eta, s) = L(1 - l - 2s, \eta) \prod_{i=1}^{[n/2]} L(1 - 2l - 4s + 2i, \eta^2)$$

$$\times \det(\text{Im}(Z))^s \sum_{\gamma \in \Gamma^{(n)} \setminus \Gamma^{(n)}(M)} j(\gamma, Z)^{-l} \eta^*(\gamma) |j(\gamma, Z)|^{-2s}.$$

Then $E_{n,l}^*(Z, M, \eta, s) = E_{n,l}|W_M(Z, M, \eta, s)$ with $W_M = \begin{pmatrix} O & -1_n \\ M1_n & O \end{pmatrix}$ coincides with the Eisenstein series $E_{n,l}^*(Z, M, \eta, s)$ in the present paper up to elementary factor. However, to quote several results in [4] smoothly, we define $E_{n,l}^*(Z, M, \eta, s)$ as in the present paper. Accordingly we define $G_{2n}^{k,k-l}(Z, A, N, \chi)$ as in our paper. With these changes, Propositions 5.1 and 5.2, and (1) of Theorem 5.3 in [12] should be replaced with Corollary 3.3, Corollary 3.5, and Proposition 3.6, respectively, in the present paper.

2. In [12], we defined $L(m, F, \chi, \text{St})$ as

$$L(m, F, \chi, \text{St}) = \Gamma_C(m) \left( \prod_{i=1}^{n} \Gamma_C(m + k - i) \right) \frac{L(m, F, \chi, \text{St})}{\tau(\chi)^{n+1}(F, F)}.$$

where $\Gamma_C(s) = 2(2\pi)^{-s/2} \Gamma(s)$. However, the factor $\sqrt{-1}^{m+n}$ should be added in the denominator on the right-hand side of the above definition. With this correction, [12], Theorem 2.2 remains valid. Moreover, we
have
$$L(l - n, F, \chi, St) = \frac{\prod_{i=1}^{n} \Gamma_C(l - n + k - i)}{N^{\text{inc}}(n, l) \prod_{i=1}^{n} \Gamma(2k - n - i) \pi^{-n(n+1)/2+nk+(n+1)m}} \Lambda(l - n, F, \chi, St).$$
We note that
$$\prod_{i=1}^{n} \Gamma_C(l - n + k - i)$$
$$N^{\text{inc}}(n, l) \prod_{i=1}^{n} \Gamma(2k - n - i) \pi^{-n(n+1)/2+nk+(n+1)m}$$
is a rational number, and for a prime number $p$ not dividing $N(2k-1)!$, it is $p$-unit. Therefore, (2) of Theorem 5.3 in [12] should be corrected as follows:

Put
$$\widetilde{G}^{k,k-l}_{2n}(Z, N^2 A, N, \chi) = \prod_{i=1}^{n} \Gamma_C(l - n + k - i)$$
$$N^{\text{inc}}(n, l) \prod_{i=1}^{n} \Gamma(2k - n - i) \pi^{-n(n+1)/2+nk+(n+1)m} \times G^{k,k-l}_{2n}(Z, N^2 A, N, \chi).$$
Then $\widetilde{G}^{k,k-l}_{2n}(Z, N^2 A, N, \chi)$ belongs to $G_k(\Gamma^{(n)})(\Omega_{Q(F, \chi, \zeta_N)} \mathfrak{P})$ for any prime ideal $\mathfrak{P}$ of $Q(F, \chi, \zeta_N)$ not dividing $N(2k-1)!$, and we have
$$\widetilde{G}^{k,k-l}_{2n}(Z, N^2 A, N, \chi) = \sum_{i=1}^{d} L(l - n, F_i, \chi, St)c_{F_i}(A)F_i(Z).$$

4. Proof of the main result

Lemma 4.1. Let $r \geq 2$ and let $\{F_1, \ldots, F_r\}$ be Hecke eigenforms $M_k(\Gamma^{(n)}; \lambda_i)$ linearly independent over $\mathbb{C}$, and $G$ an element of $M_k(\Gamma^{(n)})$. Write
$$F_i(Z) = \sum_{A} c_{F_i}(A)e(\text{tr}(AZ))$$
for $i = 1, \ldots, r$ and
$$G(Z) = \sum_{A} c_{G}(A)e(\text{tr}(AZ)).$$
Let $K$ be the composite field of $Q(F_1), \ldots, Q(F_r)$, and $L$ a finite extension of $K$. Let $N$ be a positive integer. Assume that
(1) there exists an element $\alpha \in K$ such that $c_G(A)$ belongs to $\alpha \mathfrak{O}_L[N^{-1}]$ for any $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$

(2) there exist $c_i \in L$ ($i = 1, \ldots, r$) and $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ such that

$$G(Z) = \sum_{i=1}^{r} c_i F_i(Z).$$

Then for any elements $T_1, \ldots, T_{r-1} \in \mathcal{L}'_n$ and $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ we have

$$\prod_{i=1}^{r-1} (\lambda_{F_1}(T_i) - \lambda_{F_{i+1}}(T_i)) c_1 c_{F_1}(A) \in \alpha \mathfrak{O}_L[N^{-1}].$$

Proof. We prove the induction on $r$. The assertion clearly holds for $r = 2$. Let $r \geq 3$ and suppose that the assertion holds for any $r'$ such that $2 \leq r' \leq r - 1$. We have

$$G|_{T_{r-1}}(Z) = \sum_{i=1}^{r} \lambda_{F_i}(T_{r-1}) c_i F_i(Z),$$

and we have

$$G|_{T_{r-1}}(Z) - \lambda_{F_1}(T_{r-1}) G(Z) = \sum_{i=1}^{r-1} (\lambda_{F_i}(T_{r-1}) - \lambda_{F_i}(T_{r-1})) c_i F_i(Z).$$

By Theorem 4.1 and Proposition 4.2 of [10], we have

$$G|_{T_{r-1}}(Z) - \lambda_{T_{r-1}} G(Z) \in \alpha S_k(\Gamma^{(n)})(\mathfrak{O}_L[N^{-1}])$$

Hence, by the induction assumption we prove the assertion.

\[\square\]

**Proof of Theorem 2.3** Let $b(n, l)$ be the integer in Theorem 3.7 and put $\alpha(n, k) = \max_{2 \leq l \leq n - 2} b(n, l)$. Then, $a_{n,k} g_{2n}^{k,l}(Z, N^2 A, N, \chi) \in 2^{\alpha(n,k)} M_k(\Gamma(n))(\mathfrak{O}_{\mathbb{Z}(\chi, \zeta, \bar{\eta})}[N^{-1}])$. Thus, by Theorem 3.7 and Lemma 4.1 for any $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, and $T_1, \ldots, T_e \in \mathcal{L}'_n$, the value

$$\prod_{i=1}^{r-1} (\lambda_{F_i}(T_i) - \lambda_{F_{i+1}}(T_i)) \Lambda(l - n, F, \chi, St) \tilde{c}_F(A)c_F(B)$$

belongs to $(2^{\alpha(n,k)} A_{n,k})^{-1} \mathfrak{O}_{L_{n,k}(\chi, \zeta, \bar{\eta})}[N^{-1}]$, where $e = \dim_{\mathbb{C}} M_k(\Gamma(n))$, and $L_{n,k}$ is the field stated in Section 1. In particular for any $v \in \mathfrak{C}_F$, the value $v \Lambda(l - n, F, \chi, St) \tilde{c}_F(A)c_F(B)$ belongs to $(2^{\alpha(n,k)} A_{n,k})^{-1} \mathfrak{O}_{L_{n,k}(\chi, \zeta, \bar{\eta})}[N^{-1}]$. On the other hand, by Proposition 2.1 the value $\Lambda(l - n, F, \chi, St) \tilde{c}_F(A)c_F(B)$ belongs to $\mathbb{Q}(F, \chi)$, and hence we have

$$v \Lambda(l - n, F, \chi, St) \tilde{c}_F(A)c_F(B) \in (2^{\alpha(n,k)} A_{n,k})^{-1} \mathfrak{O}_F[N^{-1}].$$
This implies that we have
\[ \mathcal{I}(l - n, F, \chi) \subset \langle (2^{\alpha(n,k)}A_{n,k}\bar{E}_F)^{-1} \rangle_{\mathbb{Q}(F,\chi)[N^{-1}]} \cdot \]

**Remark 4.2.** Let the notation be as in Lemma 4.1. Then we have the following.

Let \( p \) be a prime ideal of \( K \). Assume that \( c_1c_{F_1}(A) \) belongs to \( K \) and that \( \text{ord}_p(c_1c_{F_1}(A)) < 0 \) for some \( A \in \mathcal{H}_n(\mathbb{Z}) > 0 \). Then there exists \( i \neq 2 \) such that we have
\[ \lambda_{F_1}(T) \equiv \lambda_{F_i}(T) \mod p \quad \text{for any} \ T \in L_n' \cdot \]

This is a generalization of [10], Lemma 5.1, and it can be proved in the same way. Let \( K_{n,k} \) be the field defined in Section 2. Then, applying the above result to \( L = K_{n,k}(\chi,\zeta_N) \), and using a corrected version of [12], Theorem 5.3 in Remark 3.8 (2), we can remedy the proof of [12], Theorem 3.1.

We also remark that the \( M(2l - 1)! \) in [12], Theorem 3.1 should be \( M(2k - 1)! \).

5. **Boundedness of special values of products of Hecke \( L \)-functions**

For an element \( f(z) = \sum_{m=1}^{\infty} c_f(m)e(mz) \in S_k(SL_2(\mathbb{Z})) \) and a Dirichlet character \( \chi \), we define Hecke’s \( L \) function \( L(s, f, \chi) \) as
\[ L(s, f, \chi) = \sum_{m=1}^{\infty} \frac{c_f(m)}{m^s}. \]

Let \( f \) be a primitive form. Then, for two positive integers \( l_1, l_2 \leq k - 1 \) and Dirichlet characters \( \chi_1, \chi_2 \) such that \( \chi_1(-1)\chi_2(-1) = (-1)^{l_1+l_2+1} \), the value
\[ \frac{\Gamma_C(l_1)\Gamma_C(l_2)L(l_1, f, \chi_1)L(l_2, f, \chi_2)}{\sqrt{-1}^{l_1+l_2+1} \tau((\chi_1\chi_2)0)(f, f)} \]
belongs to \( \mathbb{Q}(f, \chi_1, \chi_2) \) (cf. [17]). We denote this value by \( \mathbf{L}(l_1, l_2; f; \chi_1, \chi_2) \).

In particular, we put
\[ \mathbf{L}(l_1, l_2; f) = \mathbf{L}(l_1, l_2; f; \chi_1, \chi_2) \]
if \( \chi_1 \) and \( \chi_2 \) are the principal characters.

**Theorem 5.1.** Let \( f \) be a primitive form in \( S_k(SL_2(\mathbb{Z})) \). Then we have
\[ \mathbf{L}(l_1, l_2; f; \chi_1, \chi_2) \in \langle (2^{b_k}\zeta(1-k)(k!)^2\bar{D}_f)^{-1} \rangle_{\mathbb{Q}(f;\chi_1,\chi_2)(N_1N_2)^{-1}} \]
with some non-negative integer $b_k$ for any integers $l_1$ and $l_2$ and primitive characters $\chi_1$ and $\chi_2$ of conductors $N_1$ and $N_2$, respectively, satisfying the following conditions:

\[(D1)\] $$(\chi_1 \chi_2)(-1) = (-1)^{l_1+l_2+1}.$$ 
\[(D2)\] $$k - l_1 + 1 \leq l_2 \leq l_1 - 1 \leq k - 2.$$ 
\[(D3)\] Either $l_1 \geq l_2 + 2$, or $l_1 = l_2 + 1$ and $\chi_1$ or $\chi_2$ is non-trivial.

Proof. The proof will proceed by a careful analysis of the proof of [17, Theorem 4] combined with the argument in Theorem 2.3. For a positive integer $\lambda \geq 2$ and a Dirichlet character $\omega \mod N$ such that $\omega(-1) = (-1)^\lambda$ we define the Eisenstein series $G_{\lambda,N}(z,s,\omega)$ ($z \in \mathbb{H}_1$, $s \in \mathbb{C}$) by

\[G_{\lambda,N}(z,s,\omega) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0^{(1)}(N)} \omega(d)(cz+d)^{-\lambda}|cz+d|^{-2s} \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),\]

where $\Gamma_\infty = \{ \pm \left( \begin{array}{cc} 1 & m \\ 0 & 1 \end{array} \right) \mid m \in \mathbb{Z} \}$. It is well known that $G_{\lambda,N}(z,s,\omega)$ is finite at $s = 0$ as a function of $s$, and put

\[G_{\lambda,N}(z,\omega) = G_{\lambda,N}(z,0,\omega).\]

$G_{\lambda,N}(z,\omega)$ is a (holomorphic) modular form of weight $\lambda$ and character $\omega$ for $\Gamma_0^{(1)}(N)$ if $\lambda \geq 3$ or $\omega$ is non-trivial. In the case $\lambda = 2$ and $\omega$ is trivial, $G_{2,N}(z,\omega)$ is a nearly automorphic form of weight 2 for $\Gamma_0^{(1)}(N)$ in the sense of Shimura [18]. We also put

\[\widetilde{G}_{\lambda,N}(z,\omega) = \frac{2\Gamma(\lambda)}{(-2\pi)^{\lambda/2}} L_N(\lambda,\omega) G_{\lambda,N}(z,\omega),\]

where $L_N(s,\omega) = L(s,\omega) \prod_{p|N} (1 - p^{-s}\omega(p))$. Now let $N_i$ be the conductor of $\chi_i$ for $i = 1, 2$. Then, by [16, Theorem 4.7.1] there exists a modular form $g$ of weight $l_1 - l_2 + 1$ and character $\chi_1\chi_2$ for $\Gamma_0^{(1)}(N_1 N_2)$ such that

\[c_g(0) = \begin{cases} 0 & \text{if } \chi_1 \text{ is non-trivial} \\ \frac{-1(N_1 N_2)}{24} & \text{if } l_1 - l_2 = 1 \text{ and both } \chi_1 \text{ and } \chi_2 \text{ are trivial} \\ \frac{-B_{l_1-l_2+1,1} \chi_1 \chi_2}{2(l_1-l_2+1)} & \text{otherwise}, \end{cases}\]

\[c_g(m) = \sum_{0<d|m} \chi_1(m/d) \chi_2(d)d^{l_1-l_2} \quad (m \geq 1),\]

and

\[L(s, g) = L(s, \chi_1)L(s - l_1 + l_2, \chi_2).\]
Since we have \( k \geq l_2, l_1 \), all the Fourier coefficients of \( g \) belong to \((k!)^{-1}\mathcal{D}_{Q(\chi_1, \chi_2)}[(N_1N_2)^{-1}]\). Put \( \lambda = -k + l_1 + l_2 + 1 \). Let \( \delta^{(r)}_\lambda \) be the differential operator in [17], page 788. Then, [17], Lemma 7 we have

\[
g\delta^{(k|-l_1-1)}_{-k+l_1+l_2+1} \tilde{G}_{-k+l_1+l_2+1, N_1N_2}(z, \chi_1\chi_2) = \sum_{\nu=0}^{r} \delta^{(\nu)}_{k-2\nu} h_\nu(z)
\]

with some \( r < k/2 \), and \( h_\nu \in M_{k-2\nu}(\Gamma_0^{(1)}(N_1N_2)) \). By [17], (3.3) and (3.4) and the assumption, all the Fourier coefficients of \( \tilde{G}_{-k+l_1+l_2+1, N_1N_2}(z, \chi_1\chi_2) \) belongs to \((k!)^{-1}\mathcal{D}_{Q(\chi_1, \chi_2)}[(N_1N_2)^{-1}]\) if \(-k + l_1 + l_2 + 1 \geq 3\), or \( \chi_1\chi_2 \) is non-trivial. Moreover, by [17], page 795, \( \tilde{G}_{2, N_1N_2}(z, \chi_1\chi_2) \) is expressed as

\[
\tilde{G}_{2, N_1N_2}(z, \chi_1\chi_2) = \frac{c}{4\pi y} + \sum_{n=0}^{\infty} c_n e(nz),
\]

with \( c, c_n \in 2^{-1}\mathcal{D}_{Q(\chi_1, \chi_2)}[(N_1N_2)^{-1}]\) if \(-k + l_1 + l_2 + 1 = 2\) and \( \chi_1\chi_2 \) is trivial. Hence, by the construction of \( h_0 \), all the Fourier coefficients of \( h_0 \) belong to \((k!)^{-1}\mathcal{D}_{Q(\chi_1, \chi_2)}[(N_1N_2)^{-1}]\). Let \( f_1, \ldots, f_d \) be a basis of \( S_k(SL_2(\mathbb{Z})) \) consisting of primitive forms such that \( f_1 = f \). Then, by [17], Theorem 2, Lemmas 1 and 7], we have

\[
L(l_1, l_2, f_i; \chi_1, \chi_2) \langle f_i, f_i \rangle = d_0[S L_2(\mathbb{Z}) : \Gamma_0^{(1)}(N_1N_2)] \langle f, h_0 \rangle
\]

for any \( i = 1, \ldots, d \), where \( d_0 = (-1)^{a(k, l_1, l_2)} 2^{b(k, l_1, l_2)} \) with some \( a(k, l_1, l_2), b(k, l_1, l_2) \in \mathbb{Z} \). (We note that the Petersson product \( \langle *, * \rangle \) in our paper is \( \frac{n}{2} \) times that in [17].) Define \( h_0(z) \) by

\[
h_0 = d_0 \sum_{\gamma \in \Gamma_0^{(1)}(N_1N_2) \backslash SL_2(\mathbb{Z})} h_0|\gamma(z).
\]

Then, \( h_0 \) belongs to \( M_k(SL_2(\mathbb{Z})) \). We have

\[
\langle f_i, h_0|\gamma \rangle = \langle f_i, h_0 \rangle,
\]

for any \( \gamma \in SL_2(\mathbb{Z}) \), and hence

\[
L(l_1, l_2, f_i; \chi_1, \chi_2) \langle f_i, f_i \rangle = \langle f_i, h_0 \rangle,
\]

and hence we have

\[
h_0(z) = \alpha \tilde{G}_k(z) + \sum_{i=1}^{d} L(l_1, l_2, f_i; \chi_1, \chi_2) f_i(z)
\]

with \( \alpha \in \mathbb{C} \). Put \( b_k = \min\{\min_{l_1, l_2} b(k, l_1, l_2), 0\} \) and \( a_k = 2^{b_k(k!)} \), where \( l_1 \) and \( l_2 \) run over all integers satisfying the conditions (D2) and (D3). By q expansion principle, for any \( \gamma \in SL_2(\mathbb{Z}) \), \( h_0|\gamma \) belongs to \( M_k(\Gamma^{(1)}(N_1N_2))((a_k^{-1})\mathcal{D}_{Q(\chi_1, \chi_2, \zeta_N)}[(N_1N_2)^{-1}]) \). Therefore \( h_0 \) belongs to
Let \( M_k(I^{(1)}(N_1N_2))((a_k^{-1})_{D_{\mathbb{Q}(x_1,x_2,\zeta_N)}}(N_1N_2)\) \( -1)\cap M_k(SL_2(\mathbb{Z})) \). Put \( h = h_0 - \alpha \tilde{G}_k \). Then all the Fourier coefficients of \( h \) belong to \( \langle (2^{k_i}k_i^2\zeta(1 - k))^{-1} \rangle_{D_{\mathbb{Q}(x_1,x_2,\zeta_N)}}(N_1N_2)\). We note that \( L(l_1,l_2; f; \chi_1,\chi_2) \) belongs to \( \mathbb{Q}(f, \chi_1, \chi_2) \). Thus, using Lemma 4.1, we can prove the assertion in the same way as Theorem 2.3. \( \square \)

**Corollary 5.2.** Let \( f \) be a primitive form in \( S_k(SL_2(\mathbb{Z})) \). Let \( Q_f \) be the set of prime ideals \( p \) of \( \mathbb{Q}(f) \) such that

\[
\text{ord}_p(N_{Q(f,\chi_1,\chi_2)}/Q(f)) (L(l_1,l_2; f; \chi_1,\chi_2)) < 0
\]

for some positive integers \( l_1, l_2 \) and primitive characters \( \chi_1, \chi_2 \) with \( p \nmid m_{\chi_1}, m_{\chi_2} \) satisfying the condition (D1), (D2), (D3). Then \( Q_f \) is a finite set. Moreover, there exists a positive integer \( r \) such that we have

\[
\text{ord}_q(L(l_1,l_2; f; \chi_1,\chi_2)) \geq -r[Q(f, \chi_1, \chi_2) : Q(f)]
\]

for any prime ideal \( q \) of \( Q(f, \chi) \) lying above a prime ideal in \( Q_f \) and integer \( l_1, l_2 \) and primitive characters \( \chi_1, \chi_2 \) satisfying the above conditions.

For a prime ideal \( p \) of an algebraic number field, let \( p = p_0 \) be a prime number such that \( (p_0) = \mathbb{Z} \cap p \). Let \( K \) a number field containing \( \mathbb{Q}(f) \). Then there exists a semi-simple Galois representation \( \rho_f = \rho_{f,p} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_p) \) such that \( \rho_f \) is unramified at a prime number \( l \neq p \) and

\[
\det(1_2 - \rho_{f,p}(\text{Frob}_l^{-1}))X = L_l(X,f),
\]

where \( \text{Frob}_l \) is the arithmetic Frobenius at \( l \), and

\[
L_l(X,f) = 1 - c_f(l)X + l^{k-1}X^2.
\]

For a \( p \)-adic representation \( \rho \) let \( \bar{\rho} \) denote the mod \( p \) representation of \( \rho \). To prove our last main result, we provide the following lemma.

**Lemma 5.3.** Let \( p = p_0 \). Let \( k \) be a positive even integer such that \( k < p \). Let \( f \) be a primitive form in \( S_k(SL_2(\mathbb{Z})) \). Let \( a, b \) be integers such that \(-p + 1 < a < b < p - 1\). Suppose that

\[
\bar{\rho}_f^{ss} = \chi^a \oplus \chi^b,
\]

where \( \chi \) is the \( p \)-cyclotomic character. Then \((a,b) = (1-k,0)\).

**Proof.** By [5, Theorem 1.2] and its remark, \( \bar{\rho}_f^{ss} | I_p \) should be

\[
\chi^{1-k} \oplus 1
\]

or

\[
\omega_2^{1-k} \oplus \omega_2^{p(1-k)}
\]

with \( \omega_2 \) the fundamental character of level 2, where \( I_p \) denotes the inertia group of \( p \) in \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). Thus the assertion holds. \( \square \)
Let \( f_1, \ldots, f_d \) be a basis of \( S_k(SL_2(\mathbb{Z})) \) consisting of primitive forms with \( f_1 = f \) and let \( \mathcal{O}_f \) be the ideal of \( \mathbb{Q}(f) \) generated by all \( \prod_{i=2}^{d}(\lambda_f(T(m)) - \lambda_f(T(m)))'s \) \( m \in \mathbb{Z}_{>0} \).

**Theorem 5.4.** Let \( f \) be a primitive form in \( S_k(SL_2(\mathbb{Z})) \). Let \( \chi_1 \) and \( \chi_2 \) be primitive characters of conductors \( N_1 \) and \( N_2 \), respectively, and let \( l_1 \) and \( l_2 \) be positive integers such that \( k - l_1 + 1 \leq l_2 \leq l_1 - 1 \leq k - 2 \). Let \( \mathfrak{p} \) be a prime ideal of \( \mathbb{Q}(f, \chi_1, \chi_2) \) with \( \mathfrak{p} \not| \mathfrak{p} \). Suppose that \( \mathfrak{p} \) divides neither \( \mathcal{O}_{fN_1N_2} \) nor \( \zeta(1-k) \). Then \( L(l_1, l_2; f; \chi_1, \chi_2) \) is \( \mathfrak{p} \)-integral.

**Proof.** The assertion follows from Theorem 5.1 if \( l_1, l_2 \) and \( \chi_1, \chi_2 \) satisfy the conditions \( (D1), (D2), (D3) \). Suppose that \( l_1 = l_2 + 1 \) and \( \chi_1 \) and \( \chi_2 \) are trivial. By Lemma 5.3 there exists a prime number \( q_0 \) such that \( q_0 \) is \( \mathfrak{p} \)-unit and

\[
1 - c_f(q_0)q_0^{-l_2+1} + q_0^{k-2l_2+1} \not\equiv 0 \pmod{\mathfrak{p}}.
\]

As stated in the proof of Theorem 5.1 there exists a modular form \( g \in M_2(\Gamma_0(q_0))\langle \mathcal{O}(\mathfrak{p}) \rangle \) such that

\[
L(s, g) = \zeta(s)\zeta(s-1)(1-q_0^{-s+1}).
\]

We can construct a modular form \( h_0 \in M_k(\Gamma_0(1)(q_0)) \) in the same way as in the proof of Theorem 5.1. Then

\[
(1 - c_f(q_0)q_0^{-l_2+1} + q_0^{k-2l_2+1})L(l_1, l_2; f_i)\langle f_i, f_i \rangle \\
= d_0[SL_2(\mathbb{Z}) : \Gamma_0(1)(q_0)]\langle f_i, h_0 \rangle
\]

with some integer \( d_0 \) prime to \( \mathfrak{p} \) for any \( i = 1, \ldots, d \). Then by using the same argument as above, we can prove that

\[
\text{ord}_{\mathfrak{p}}(L(l_1, l_2; f)(1 - c_f(q_0)q_0^{-l_2+1} + q_0^{k-2l_2+1})) \geq 0.
\]

This proves the assertion. \( \square \)

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