CURTIS-TITS GROUPS GENERALIZING KAC-MOODY GROUPS OF TYPE $A_n$

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Abstract. In [12] we define a Curtis-Tits group as a certain generalization of a Kac-Moody group. We distinguish between orientable and non-orientable Curtis-Tits groups and identify all orientable Curtis-Tits groups as Kac-Moody groups associated to twin-buildings.

In the present paper we construct all orientable and non-orientable Curtis-Tits groups with diagram $\tilde{A}_n$ over a field $k$. The resulting groups are quite interesting in their own right. The orientable ones are related to Drinfel’d’s construction of vector bundles over a non-commutative projective line and to the classical groups over cyclic algebras. The non-orientable ones are related to q-CCR algebras in physics and have symplectic, orthogonal and unitary groups as quotients.

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1. Introduction

In [40] J. Tits defines a group of Kac-Moody type to be a group with a twin-root datum, which implies that they are symmetry groups of Moufang twin-buildings.

In [3] P. Abramenko and B. Mühlherr generalize a celebrated theorem of Curtis and Tits on groups with finite BN-pair [14, 38] to groups of Kac-Moody type. This theorem states that a Kac-Moody group $G$ is the universal completion of an amalgam of rank two (Levi) subgroups, as they are arranged inside $G$ itself. Similar results on Curtis-Tits-Phan type amalgams have been obtained in [4, 5, 6, 10, 11, 24, 25, 26, 27]. For an overview of that subject see [23].

In [12] we take this result as a starting point and define a Curtis-Tits amalgam over a given diagram to be an amalgam of groups such that the sub-amalgam corresponding to a two-vertex sub-diagram is the amalgam of Levi subgroups of some rank-2 group of Lie type. There is no a priori reference to an ambient group, nor to the existence of an associated (twin-) building. Indeed, there is no a priori guarantee that the amalgam will not collapse.

We then classify those Curtis-Tits amalgams whose diagram is simply-laced and has no circuits of length at most 3 and whose groups are $k$-groups for some field $k$ of order at least 4 (for similar results in special cases see [17, 22]). Such amalgams are described by the diagram as follows. For each node the amalgam contains a rank-1 group which is a copy of $\text{SL}_2(k)$, for each non-edge the rank-2 group is the direct product of the node groups, and for each edge the rank-2 group is a copy of $\text{SL}_3(k)$ such that the node groups form a
“standard pair”. The condition on the diagram and field ensure that if the amalgam has a non-trivial completion $G$, then it contains a torus that is generated by certain uniquely determined tori in the groups of the amalgam. We obtain the following result.

**Classification Theorem** Let $\Gamma$ be a simply laced Dynkin diagram with no triangles and $k$ a field with at least 4 elements. There is a natural bijection between isomorphism classes of Curtis-Tits amalgams over the field $k$ on the graph $\Gamma$ and elements of the set $\{\Phi: \pi(\Gamma, i_0) \to \langle \tau \rangle \times \text{Aut}(k) | \Phi \text{ is a group homomorphism} \}$, where $\tau$ has order 2.

Here, $\pi(\Gamma, i_0)$ denotes the (first) fundamental group of the graph $\Gamma$ with base point $i_0$. The group $\text{Aut}(k) \times \langle \tau \rangle$ should be viewed as a subgroup of the stabilizer in $\text{Aut}(\text{SL}_2(k))$ of a fixed torus in $\text{SL}_2(k)$; $\tau$ denotes the transpose-inverse map w.r.t. that torus.

We call amalgams corresponding to homomorphisms $\Phi$ whose image lies inside $\text{Aut}(k)$ “orientable”; others are called “non-orientable”. It is not at all immediate that all non-orientable amalgams arising from the Classification Theorem are non-collapsing, i.e. that their universal completion is non-trivial. We shall call a non-trivial group a Curtis-Tits group if it is the universal completion of a Curtis-Tits amalgam. It is shown that orientable Curtis-Tits amalgams are precisely those arising from the Curtis-Tits theorem applied to a group of Kac-Moody type. Thus, groups of Kac-Moody type are orientable Curtis-Tits groups.

### 1.1. main results

The purpose of the present paper is to construct orientable and non-orientable Curtis-Tits groups with diagram $\tilde{A}_{n-1}$ and to study their properties.

The paper is structured as follows. In Section 2 we introduce the relevant notions about amalgams and in Section 3 we describe the automorphisms needed to define all possible Curtis-Tits amalgams of type $\Gamma = \tilde{A}_{n-1}$. For the reader’s convenience we have included this definition here.

**Definition 1.1.** Given an element $\delta \in \text{Aut}(k) \times \langle \tau \rangle \leq \text{Aut}(\text{SL}_2(k))$ there is a Curtis-Tits amalgam $A^{\delta}$ of type $\tilde{A}_{n-1}$, which we now construct. We index the nodes of $\Gamma$ cyclically with elements from $I = \{1, 2, \ldots, n\}$ (modulo $n$). For each $i \in I$ we let $G_i = \text{SL}_2(k)$ and $A^\delta = \{G_i, G_{ij} | i, j \in I\}$ with connecting maps $\psi = \{\psi_{i,j} | i, j \in I\}$, where

- (SCT1) for any vertex $i$, the group $G_i = \text{SL}_2(k)$ and for each pair $i, j \in I$,
  
  $$G_{i,j} \cong \left\{ \begin{array}{ll} \text{SL}_2(k) & \text{if } \{i, j\} = \{i, i+1\} \\ G_i \times G_j & \text{if } \{i, j\} \neq \{i, i+1\} \end{array} \right.;$$

- (SCT2) For $i = 1, 2, \ldots, n - 1$ we have
  
  $$\psi_{i,i+1}: G_i \to G_{i,i+1} \quad \psi_{i+1,i}: G_{i+1} \to G_{i,i+1}$$
  
  $$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$
and we have

\( \psi_{n,1} : G_n \to G_{n,1} \quad \psi_{1,n} : G_1 \to G_{1,n} \)

\[ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A^\delta \end{pmatrix} , \]

whereas for all other pairs \((i,j)\), \( \psi_{i,j} \) is the natural inclusion of \( G_i \) in \( G_i \times G_j \).

For \( \tilde{A}_{n-1} \) type diagrams, the Classification Theorem reduces to Corollary 3.3 which asserts that, indeed, every Curtis-Tits amalgam is isomorphic to \( A^\delta \) for some \( \delta \in \text{Aut}(k) \times \langle \tau \rangle \). We have chosen our setup such that \( A^\text{id} \) is the amalgam resulting from applying the Curtis-Tits theorem to the split Kac-Moody group \( \text{SL}_n(k[t, t^{-1}]) \) of type \( \tilde{A}_{n-1} \). For more details see the proof of Corollary 3.3.

Theorem 1 then shows that all Curtis-Tits amalgams with diagram \( \tilde{A}_{n-1} \) corresponding to elements in the torsion subgroup of \( \text{Aut}(k) \times \langle \tau \rangle \) have non-trivial universal completion. More precisely, we prove the following.

**Theorem 1.** Let \( n \geq 4 \) and \( |k| \geq 4 \). For each element \( \delta \in \text{Aut}(k) \times \langle \tau \rangle \) of finite order \( s \), there exists a non-trivial completion \( G^\delta \) of \( A^\delta \) inside a split Kac-Moody group of type \( \tilde{A}_{nm-1} \); \( m = s \) if \( \delta \in \text{Aut}(k) \) and \( m = 2s \) otherwise. Moreover, if \( \delta \in \text{Aut}(k) \) or \( \delta = \tau \), then \( G^\delta \) is the universal completion of \( A^\delta \).

The first part of Theorem 1 follows from the classification of amalgams in Corollary 3.3 and the construction of their completions in Section 4. Since the proof of the last statement in Theorem 1 is more involved we distribute it over Theorems 2 and 3 below.

In Section 5 we exhibit a non-trivial completion for orientable Curtis-Tits groups and prove the following.

**Theorem 2.** Let \( n \geq 4 \) and \( |k| \geq 4 \). The universal completion of \( A^\delta \) is \( \text{SL}_n(R) \).

Here \( R = k\{t, t^{-1}\} \) is the ring of skew Laurent polynomials with coefficients in the field \( k \) such that for \( x \in k \) we have \( t^{-1}xt = x^\delta \) (recall that \( \delta \in \text{Aut}(k) \)). We use a suitable definition of a determinant to identify \( \text{SL}_n(R) \).

Then in Section 6 we do the same for the non-orientable amalgam \( A^\tau \) and prove the following.

**Theorem 3.** Let \( n \geq 4 \) and let \( k \) be a field of size at least 5. The universal completion \( G^\tau \) of \( A^\tau \) is the group of symmetries in \( \text{SL}_{2n}(k[t, t^{-1}]) \) of the \( \sigma \)-sesquilinear form \( \beta \).

Here \( k[t, t^{-1}] \) denotes the ring of Laurent polynomials with coefficients in the field \( k \) and commuting indeterminate \( t \); \( \sigma \) is the automorphism fixing \( k \) pointwise and interchanging \( t \) and \( t^{-1} \) and \( \beta \) is an asymmetric \( \sigma \)-sesquilinear form.

Since the completions in Theorems 2 and 3 are a special case of those constructed in Section 4, we offer the following natural conjecture.

**Conjecture 1.** For each element \( \delta \in \text{Aut}(k) \times \langle \tau \rangle \), the universal completion of the amalgam \( A^\delta \) is a non-trivial subgroup \( G^\delta \) of the automorphism group of a combinatorial building of type \( \tilde{A} \).
In Section 7 we explore connections between Curtis-Tits groups of type $\tilde{A}_{n-1}$ and other areas of mathematics and mathematical physics. In particular we see that these groups are quite interesting in their own right. The orientable ones are related to Drinfel’d’s construction of vector bundles over a non-commutative projective line and to classical groups over cyclic algebras. The non-orientable ones are related to q-CCR algebras in physics and have symplectic, orthogonal and unitary groups as quotients.

Finally, we note that some of these groups have been studied in a different context, namely that of abstract involutions of Kac-Moody groups [20]. In that paper, also connectedness, but not simple-connectedness, of geometries such as those defined in Section 6 is proved.

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2. CT-GROUPS

In this section we introduce the notion of a Curtis-Tits group over a commutative field and define their category. Throughout the paper $k$ will be a commutative field.

**Definition 2.1.** Let $V$ be a vector space of dimension 3 over $k$. We call $(S_1, S_2)$ a standard pair for $S = SL(V)$ if there are decompositions $V = U_i \oplus V_i$, $i = 1, 2$, with $\dim(U_i) = 1$ and $\dim(V_i) = 2$ such that $U_1 \subseteq V_2$ and $U_2 \subseteq V_1$ and $S_i$ centralizes $U_i$ and preserves $V_i$.

One also calls $S_1$ a standard complement of $S_2$ and vice-versa. We set $D_1 = N_{S_1}(S_2)$ and $D_2 = N_{S_2}(S_1)$. A simple calculation shows that $D_i$ is a maximal torus in $S_i$, for $i = 1, 2$. In general if $G \cong SL_3(k)$, then $(G_1, G_2)$ is a standard pair for $G$ if there is an isomorphism $\psi: G \to S$ such that $\psi(G_i) = S_i$ for $i = 1, 2$.

**Definition 2.2.** A simply laced Dynkin diagram over the set $I$ is a simple graph $\Gamma = (I, E)$. That is, $\Gamma$ has vertex set $I$, and an edge set $E$ that contains no loops or double edges.

**Definition 2.3.** An amalgam over a set $I$ is a collection $\mathcal{A} = \{G_i, G_{i,j} \mid i, j \in I\}$ of groups, together with a collection $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$ of monomorphisms $\varphi_{i,j}: G_i \hookrightarrow G_{i,j}$, called inclusion maps. A completion of $\mathcal{A}$ is a group $\hat{G}$ together with a collection $\phi = \{\phi_i, \phi_{i,j} \mid i, j \in I\}$ of homomorphisms $\phi_i: G_i \to \hat{G}$ and $\phi_{i,j}: G_{i,j} \to \hat{G}$, whose images generated $\hat{G}$, such that for any $i, j$ we have $\phi_{i,j} \circ \varphi_{i,j} = \phi_i$. For simplicity we denote by $\hat{G}_i = \varphi_{i,j}(G_i) \leq G_{i,j}$. The amalgam $\mathcal{A}$ is non-collapsing if it has a non-trivial completion. A completion $(\hat{G}, \hat{\phi})$ is called universal if for any completion $(G, \phi)$ there is a unique surjective group homomorphism $\pi: \hat{G} \to G$ such that $\phi = \pi \circ \hat{\phi}$.

**Definition 2.4.** Let $\Gamma = (I, E)$ be a simply laced Dynkin diagram. A Curtis-Tits amalgam over $\Gamma$ is a non-collapsing amalgam $\mathcal{A}(\Gamma) = \{G_i, G_{i,j} \mid i, j \in I\}$, with connecting maps $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$, such that
(CT1) for any vertex $i$, the group $G_i = \text{SL}_2(k)$ and for each pair $i, j \in I$, 
\[ G_{i,j} \cong \begin{cases} \text{SL}(V_{i,j}) & \text{if } \{i, j\} \in E \\ G_i \times G_j & \text{if } \{i, j\} \notin E \end{cases}, \]
where $V_{i,j}$ is a 3-dimensional vector space over $k$;
(CT2) if $\{i, j\} \in E$ then $(\bar{G}_i, \bar{G}_j)$ is a standard pair in $G_{i,j}$.

From now on $\Gamma = (I, E)$ will be a Dynkin diagram without circuits of length $\leq 3$ and without isolated nodes. Also, $A = A(\Gamma) = \{G_i, G_{i,j} \mid i, j \in I\}$ will be a non-collapsing Curtis-Tits amalgam over $\Gamma$ with connecting maps $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$.

It is proved in [12] that if the Dynkin diagram is has no circuits of length at most 3, then the following is well-defined.

**Definition 2.5.** For $i, j \in I$, we let $\bar{D}_i = N_{\bar{G}_j}(\bar{G}_i) \cap \bar{G}_i$, where $\{i, j\} \in E$. Note that this defines $\bar{D}_i$ for all $i$ since $\Gamma$ has no isolated nodes. We also denote $D_i = \varphi_{i,j}^{-1}(\bar{D}_i)$.

We then have the following.

**Lemma 2.6.** [12, Section 2] If $\{i, j\} \in E$, then $\bar{D}_i$ and $\bar{D}_j$ are contained in a unique common maximal torus $D_{i,j}$ of $G_{i,j}$.

**Definition 2.7.** Note that a torus in $\text{SL}_2(k)$ uniquely determines a pair of opposite root groups $X_+$ and $X_-$. We now choose one root group $X_i$ normalized by the torus $D_i$ of $G_i$ for each $i$. An orientable Curtis-Tits (OCT) amalgam (respectively orientable Curtis-Tits (OCT) group) is a non-collapsing Curtis-Tits amalgam that admits a system $\{X_i \mid i \in I\}$ of root groups as above such that for any $i, j \in I$, the groups $\varphi_{i,j}(X_i)$ and $\varphi_{j,i}(X_j)$ are contained in a common Borel subgroup $B_{i,j}$ of $G_{i,j}$.

2.1. **morphisms.** In this subsection, for $k = 1, 2$, let $\Gamma^k = (I^k, E^k)$ be a Dynkin diagram. Now, for $k = 1, 2$, let $A^k = \{G_{i,j}^k \mid i, j \in I^k\}$ be a Curtis-Tits amalgam whose Dynkin diagram $\Gamma^k$ has no circuits of length $\leq 3$.

**Definition 2.8.** A homomorphism between the amalgams $A_1(\Gamma)$ and $A_2(\Gamma)$ is a collection $\phi = \{\phi_i, \phi_{i,j} \mid i, j \in I^1\}$ of group homomorphisms $\phi_i : G_i^1 \to G_i^2$ and $\phi_{i,j} : G_{i,j}^1 \to G_{i,j}^2$ such that
\[ \phi_{i,j} \circ \varphi_{i,j}^1 = \varphi_{i,j}^2 \circ \phi_i. \]
We call $\phi$ an isomorphism of amalgams if $\phi_i$ and $\phi_{i,j}$ are bijective for all $i, j \in I$, and $\phi^{-1} = \{\phi_i^{-1}, \phi_{i,j}^{-1} \mid i, j \in I^1\}$ is a homomorphism of amalgams.
3. CLASSIFICATION OF CURTIS-TITS AMALGAMS OF TYPE $\tilde{A}_{n-1}$

3.1. the role of $\text{Aut}(k) \times \langle \tau \rangle$. In this subsection we describe all amalgams of type $\tilde{A}_{n-1}$ using the Classification Theorem.

To this end we first discuss certain automorphisms of the Curtis-Tits amalgam with diagram $A_2$. Let $W$ be a (left) vector space of dimension $n$ over $k$. Let $G = \text{SL}(W)$ act on $W$ as the matrix group $\text{SL}_n(k)$ with respect to some fixed basis $E = \{e_i \mid i = 1, 2, \ldots, n\}$. Let $\tau \in \text{Aut}(\text{SL}_n(k))$ be the automorphism given by
\[
A \mapsto A^{-1}
\]
where $A$ denotes the transpose of $A$.

Let $\Phi = \{(i, j) \mid 1 \leq i \neq j \leq n\}$. For any $(i, j) \in \Phi$ and $\lambda \in k$, we define the root group
\[
X_{i,j} = \{X_{i,j}(\lambda) \mid \lambda \in k\}, \text{ where } X_{i,j}(\lambda) \text{ acts as }
\]
\[
e_j \mapsto e_j + \lambda e_i \quad \text{and} \quad e_k \mapsto e_k \quad \text{for all } k \neq j.
\]
Let $\Phi_+ = \{(i, j) \in \Phi \mid i < j\}$ and $\Phi_- = \{(i, j) \in \Phi \mid j < i\}$. We call $X_{i,j}$ positive if $(i, j) \in \Phi_+$ and negative otherwise. Let $H$ be the torus of diagonal matrices in $\text{SL}_n(k)$ and for $\varepsilon \in \{+, -\}$, let $X_\varepsilon = \langle X_{i,j} \mid (i, j) \in \Phi_\varepsilon \rangle$ and $B_\varepsilon = H \ltimes X_\varepsilon$. The following lemma describes the action of $\tau$ on these root groups.

**Lemma 3.1.** $X_{i,j}^\tau = X_{j,i}$ for all $(i, j) \in \Phi$ and $B_\varepsilon^\tau = B_{-\varepsilon}$, for $\varepsilon \in \{+, -\}$.

Let $\Gamma L_n(k)$ be the group of all semilinear automorphisms of the vector space $W$ and let $\text{PGL}_n(k) = \Gamma L_n(k)/Z(\Gamma L_n(k))$. Then $\Gamma L_n(k) \cong \text{GL}_n(k) \rtimes \text{Aut}(k)$, where we view $\alpha \in \text{Aut}(k)$ as an element of $\Gamma L_n(k)$ by setting $((a_{i,j})_{i,j=1}^n)^\alpha = (a_{i,j}^\alpha)_{i,j=1}^n$. The automorphism group of $\text{SL}_n(k)$ can be expressed using $\text{PGL}_n(k)$ and $\tau$ as follows [33].

**Lemma 3.2.**
\[
\text{Aut}(\text{SL}_n(k)) = \begin{cases} 
\text{PGL}_n(k) & \text{if } n = 2; \\
\text{PGL}_n(k) \rtimes \langle \tau \rangle & \text{if } n \geq 3.
\end{cases}
\]

**Corollary 3.3.** When $n \geq 4$ and $|k| \geq 4$, every Curtis-Tits amalgam with diagram $\tilde{A}_{n-1}$ is isomorphic to $\mathcal{A}^\delta$ for some unique $\delta \in \text{Aut}(k) \times \langle \tau \rangle$.

**Proof.** Let $\Gamma = (I, E)$, where $I = \{1, \ldots, n\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$. Select $\gamma = (1, 2, \ldots, n, 1)$ as a generator for the fundamental group $\pi(\Gamma, 1)$ of $\Gamma$ with base point 1. Let $\mathcal{A}_0$ be the amalgam with diagram $\tilde{A}_{n-1}$ arising from the Curtis-Tits theorem applied to the Kac-Moody group $\text{SL}_n(k[t, t^{-1}])$. Following [12] all other Curtis-Tits amalgams with diagram $\tilde{A}_{n-1}$ are of type $\mathcal{A}_0$ and they are all classified up to isomorphism by a homomorphism $\Phi \colon \pi(\Gamma, 1) \to \text{Aut}(k) \times \langle \tau \rangle$. Such a homomorphism is completely determined by $\Phi(\gamma)$. Thus, the Curtis-Tits amalgams with diagram $\tilde{A}_{n-1}$ are classified by an element $\delta \in \text{Aut}(k) \times \langle \tau \rangle$. Here $\delta = \text{id}$ corresponds to the amalgam $\mathcal{A}_0$ itself and in general $\delta$ corresponds to the amalgam $\mathcal{A}_0$ in which all vertex-edge inclusions are those of $\mathcal{A}_0$ except
that the inclusion map $G_1 \hookrightarrow G_{n,1}$ is given by $\delta$ (see the proof of Theorem 1 of loc. cit. and the section on twists of Kac-Moody groups).

\[\square\]

**Remark 3.4.** The conditions $n \geq 4$ and $|k| \geq 4$ are required in the proof of the Classification Theorem. If the amalgam has a non-trivial completion, these conditions guarantee the existence of a torus generated by certain tori of the groups in the amalgam. By Theorem 1 in fact all Curtis-Tits amalgams described in Corollary 3.3 have a non-trivial completion.

If one considers those Curtis-Tits amalgams of type $\tilde{A}_2$ or of type $\tilde{A}_{n-1}$ over a field $k$ of at most 3 elements, such that if the amalgam has a non-trivial completion, such a torus exists, then also those amalgams are in bijection with elements of $\text{Aut}(k) \times \langle \tau \rangle$. However, in the absence of such a guarantee, the methods developed in [13] show that there are 8 pairwise non-isomorphic amalgams, but the only non-collapsing two are those corresponding to the elements of $\text{Aut}(k) \times \langle \tau \rangle = \{\text{id}, \tau\}$.

Our next goal is to construct universal completions of each one of the amalgams $\mathcal{A}^\delta$.

### 4. Mixed Completions

Let $\delta \in \text{Aut}(k) \times \langle \tau \rangle$ and let $\mathcal{A}^\delta$ be as in Definition 1.1. We shall now construct a completion of $\mathcal{A}^\delta$ assuming that $\delta = \alpha \tau$ for some $\alpha \in \text{Aut}(k)$ of finite order $s > 1$. The cases where $\delta \in \text{Aut}(k)$ or $\delta = \tau$ will be dealt with in Section 5 and 6 respectively. In those sections we shall not only construct the amalgam and a completion, but we shall also prove that those completions are universal.

Let $A = k[t, t^{-1}]$ be the ring of Laurent polynomials in the commuting variable $t$ with coefficients in the field $k$ and let $\sigma \in \text{Aut}(A)$ be the automorphism that fixes $k$ element wise and interchanges $t$ and $t^{-1}$.

We will now construct an amalgam $L^\delta$ inside $\text{SL}_{2_{sn}}(A)$ that is isomorphic to the amalgam $\mathcal{A}^\delta$ from Corollary 3.3. Consider the following matrices:

\[
F = \begin{pmatrix} 0 & I_{2_{sn-1}} \\ I_{2_{sn}} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} tI_{sn} & 0_{sn} \\ 0_{sn} & J_{sn} \end{pmatrix}.
\]

Let $H = \langle \alpha, \tau \rangle \leq \text{Aut}(\text{SL}_2(k))$. We now define an injective homomorphism $\Phi: H \hookrightarrow \text{Aut}(\text{SL}_{2_{sn}}(A))$ by letting

\[
\Phi(\alpha): X \mapsto F^{-n}X^\alpha F^n \\
\Phi(\tau): X \mapsto F^{-sn}J^{-1t}X^{-\sigma}JF^{sn}
\]

Also define the map $i: \text{SL}_2(k) \rightarrow \text{SL}_{2_{sn}}(A)$ by

\[
A \mapsto \begin{pmatrix} A & \cdot \\ \cdot & I_{2_{sn-2}} \end{pmatrix}.
\]

Next, for $k = 1, \ldots, n + 1$, define $\phi_k: \text{SL}_2(k) \rightarrow \text{SL}_{2_{sn}}(A)$ by setting

\[
\phi_1(A) = \Pi_{\eta \in H} \Phi(\eta)(i(A)) \\
\phi_k(A) = F^{-1} \phi_{k-1}(A) F
\]
We now let $L_i$ be the image of $SL_2(k)$ under $\phi_i$ and $L_{i,j}$ be the subgroup of $SL_{sn}(A)$ generated by $L_i$ and $L_j$. Note that as groups $L_1 = L_{n+1}$.

**Definition 4.1.** For each $i,j \in \{1, 2, \ldots, n\}$, let $\varphi_{i,j} : L_i \hookrightarrow L_{i,j}$ be the natural inclusion map. Then we define the following amalgam:

$$L_\delta = \{ L_i, L_{i,j}, \varphi_{i,j} \mid i,j \in I \}.$$  

Now let  

$$G_\delta = \langle L_i, L_{i,j} \mid i,j \in \{1, \ldots, n\} \rangle \leq SL_{sn}(A)$$  

**Proposition 4.2.** The group $G_\delta$ is a completion for the amalgam $L_\delta$, which is isomorphic to $A_\delta$.

**Proof.** That $L_\delta$ is contained in, and generates $G_\delta$ follows by definition of $\phi_k$. We claim that the collection $\phi = \{ \phi_i, \phi_{i,j} \mid i,j \in I \}$ is the required isomorphism between $A_\delta$ and $L_\delta$. This is completely straightforward since we have $\phi_{n+1} \circ \phi_{i}^{-1} = \alpha^{s-1} \tau = \delta^{-1}$.

We conjecture that it is in fact the universal completion.

5. ORIENTABLE CURTIS-TITS GROUPS

Let $k[T, T^{-1}]$ be the ring of Laurent polynomials over the field $k$ and let $\delta \in Aut(k)$.

**Theorem 5.1.** If $\delta$ has order $s$ then the universal completion $G_\delta$ of $A_\delta$ is a simply connected Kac-Moody group of type $\tilde{A}_{n-1}$. It is a subgroup of finite index $\aleph$ inside $SL_{sn}(k[T, T^{-1}])$. Moreover if the norm $k \to k_\delta$ is surjective then $\aleph = ns[(k_\delta)^* : ((k_\delta)^*)^{sn}]$.

5.1. **linear groups over twisted laurent polynomials.** Let $k$ be a commutative field and $\delta \in Aut(k)$. The ring of twisted Laurent polynomials is the non-commutative ring 

$$R = k\{t, t^{-1}\}$$

where $t^{-1}xt = x^\delta$ for all $x \in k$. For some given $n \geq 1$, let $I = \{1, 2, \ldots, n\}$ and let $M$ be an $n$-dimensional free left $R$-module with ordered basis $E = \{e_1, \ldots, e_n\}$. The group of all $R$-linear invertible transformations of $M$ is denoted $GL_R(M)$. Representation of transformations as matrices w.r.t. the basis $E$ acting from the left yields the usual identification:

$$End_R(M) \to M_n(R)$$

$$g \mapsto (g_{i,j})_{i,j \in I}, \quad \text{where, for all } j \in I, ge_j = \sum_i g_{i,j}e_i$$

Note that since $R$ is in general not commutative, for $a, b, c \in End_R(M)$ with $ab = c$, we have

$$c_{ik} = \sum_{j \in I} b_{jk}a_{i,j}.$$  

At the very end of [40] it is claimed that a realization of the Kac-Moody group $G_\delta$ can be obtained as a subgroup of index $n$ inside $PGL_n(k\{t, t^{-1}\})$. We shall now proceed to give an explicit description of this realization.
Consider the following collection \( \mathcal{L}^\delta = \{L_i, L_{i,j} \mid i, j = 1, \ldots, n\} \) of subgroups of \( \text{SL}_n(k(t, t^{-1})) \). For \( i = 1, 2, \ldots, n-1 \), let

\[
L_i = \left\{ \begin{pmatrix} I_i & A \\ 0 & I_{n-i-2} \end{pmatrix} \mid A \in \text{SL}_2(k) \right\}
\]

and

\[
L_n = \left\{ \begin{pmatrix} d^{i-1} & ct^{-1} \\ tb & \alpha \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(k) \right\}
\]

Moreover, for each \( i, j \in I \) we let

\[
L_{i,j} = \langle L_i, L_j \rangle.
\]

Finally we let the inclusion maps \( \varphi_{i,j} \) be given by natural inclusion of subgroups of \( \text{GL}_R(M) \).

**Proposition 5.2.** We have an isomorphism of CT amalgams \( \mathcal{L}^\delta \cong \mathcal{A}^\delta \).

**Proof.** Consider the following matrix:

\[
F = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}.
\]

We now define the automorphism \( \Phi \) of \( \text{PGL}_n(k(t, t^{-1})) \) given by \( X \mapsto F^{-1}XF \). We first note that we have isomorphisms \( \phi_i \colon \text{SL}_2(k) \to L_i \). For \( i = 1, 2, \ldots, n-1 \) we take

\[
\phi_i \colon A \mapsto \begin{pmatrix} I_i & A \\ 0 & I_{n-i-2} \end{pmatrix}.
\]

Moreover, we define

\[
\phi_n \colon \text{SL}_2(k) \to L_n,
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^{i-1} & ct^{-1} \\ tb & \alpha \end{pmatrix}.
\]

One verifies that, for \( i = 1, 2, \ldots, n \) we have \( \phi_i = \Phi^i \circ \phi_0 \). In particular \( \phi_n \) is an isomorphism. We now turn to the rank 2 groups. For distinct \( i, j \in \{1, 2, \ldots, n\} \), let \( \phi_{i,j} \) be the canonical isomorphism between \( G_{i,j} = \langle G_i, G_j \rangle \) and \( L_{i,j} = \langle L_i, L_j \rangle \) induced by \( \phi_i \) and \( \phi_j \). Note that this implies that \( \phi_{i,j+1} = \Phi^{i-1} \circ \phi_{1,2} \).

We claim that the collection \( \phi = \{\phi_i, \phi_{i,j} \mid i, j \in I\} \) is the required isomorphism between \( \mathcal{A}^\delta \) and \( \mathcal{L}^\delta \). This is completely straightforward except for the maps \( \phi_1, \phi_{n,1} \). Note that

\[
\phi_{n,1} \colon \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} tet^{-1} & tft^{-1} \\ th^{-1} & tit^{-1} \\ tb & tc \end{pmatrix} \begin{pmatrix} d^{t-1} \\ g^{t-1} \\ a \end{pmatrix}.
\]

Thus we have

\[
\phi_{i,j} \circ \psi_{i,j} = \phi_{i,j} \circ \phi_i,
\]
for all $i, j \in I$. \hfill \square

5.2. a presentation over the ring $k[T, T^{-1}]$. In the case when the order of the automorphism $\delta \in \text{Aut}(k)$ is finite we give another interpretation of the group $G^\delta$. To do so, let $s = |\delta|$ and consider the rings

\[
R = k\{t, t^{-1}\} \\
A = k[T, T^{-1}]
\]

where $T = t^s$. Note that $T$ commutes with $k$ so that $A$ is the usual ring of Laurent polynomials in $T$ over $k$.

Now let $M$ be the free left $R$-module of dimension $n$ with basis $e_1, \ldots, e_n$. Then $M$ is also a free $A$-module of dimension $sn$ with basis $B = \{b_{ni+j} = t^ie_j \mid i = 0, \ldots, s-1$ and $j = 1, \ldots, n\}$, ordered lexicographically (that is, $t^ie_j < t^ke_l$ whenever $i < k$ and $j, l$ are arbitrary, or $i = k$ and $j < l$) so that $b_u < b_v$, whenever $u < v$. Using the basis $B$ we have an embedding

\[\rho: \text{End}_R(M) \hookrightarrow \text{End}_A(M) \cong M_{sn}(A).\]

Scalar multiplication on $M$ by the element $t \in R$ is a $\delta^{-1}$-semi $A$-linear transformation on $M$ and so we can interpret this as an element from $G_{sn}(A)$, acting on the basis $B$ as $x^n$, where $x$ is given by

\[x = \left( \frac{T}{T_{n-1}} \right).\]

From now on we shall denote this transformation by $\theta$. Now an $A$-linear map $g$ represents an $R$-linear transformation precisely if it satisfies $\theta g = g\theta$. We phrase this in a lemma.

**Lemma 5.3.** $\text{End}_R(M)^\rho = C_{M_{sn}(A)}(\theta)$.

In matrix notation this means that $x^n g^\delta^{-1} = g x^n$. More explicitly, if we represent $g$ with respect to $B$ as a block-matrix $g = (g_{i,j})_{i,j=1}^n$, where $g_{i,j} \in M_n(A)$, then the condition $\theta g = g\theta$ is equivalent to choosing $g_{i,j}$ randomly, and setting

\[
\begin{align*}
g_{i+1,j+1} &= g_{i,j}^{\delta^{-1}} & 1 \leq i, j \leq s - 1 \\
g_{i+1,1} &= g_{i,s-1}^{\delta^{-1}} T^{-1} & i = 1, \ldots, s - 1.
\end{align*}
\]

**Definition 5.4.** For any $g \in \text{End}_R(M)$, let $\det_R(g) = \det_A(g^\rho)$, where the latter denotes the determinant in the matrix ring $M_{sn}(A)$.

**Lemma 5.5.** We have $\text{GL}_R(M) = \{g \in \text{End}_R(M) \mid \det_R(g) \in A^*\}$.

**Proof.** Let $g \in \text{End}_R(M)$. Clearly if $g \in \text{GL}_R(M)$, then $g^\rho$ is invertible in $M_{sn}(A)$ so that $\det_R(g) \in A^*$, the ring of units of $A$. Conversely, suppose that $\det_R(g) \in A^*$, and let $g^{-1}$ be its inverse in $M_{sn}(A)$. Since $g \in C_{M_{sn}(A)}(\theta)$, so is $g^{-1}$ and the result follows from Lemma 5.3. \hfill \square

**Lemma 5.6.** Consider the map $\det_R: \text{End}_R(M) \to A$ and assume that the norm $N_\delta: k \to k^\delta$ is surjective. Then, we have the following.
• The image of GL$_R$($M$) under det$_R$ is equal to \( \{ \lambda T^l \mid \lambda \in k^\delta, l \in \mathbb{Z} \} \).

• The image of $Z_n(R) = Z(GL_R(M))$ under det$_R$ is equal to \( \{ \lambda^{sn}T^{lsn} \mid \lambda \in k^\delta, l \in \mathbb{Z} \} \).

Proof. The relation $x^ng^\delta = gnx^n$ implies that \( \det(g^\delta) = \det(g) \), that is, \( \det(g) \in k[T, T^{-1}]^* = \{ aT^l \mid a \in k^\delta, l \in \mathbb{Z} \} \). This shows \( \subseteq \). Moreover, note that the element \( x \in GL_R(M)^\rho \) has determinant \( T \) and the diagonal matrix corresponding to the transformation \( e_1 \rightarrow ae_1 \) with \( a \in k \) and \( e_i \rightarrow e_i \) for all \( i \geq 2 \) has determinant \( N_\delta(a) \). This shows the inclusion \( \supseteq \) and we have proved part (a).

(b) As in commutative algebra it is clear that any element of $Z_n(R)$ must be of the form \( z \text{id} \), for some \( z \in R \). Moreover, since such an element must commute with all other scalar matrices, \( z \) must belong to \( Z(R)^* = (A^\delta)^* = \{ aT^l \mid a \in (k^\delta)^*, l \in \mathbb{Z} \} \). The image of \( z \text{id} \) under \( \rho \) is a matrix of the form \( zI_{sn} \) and therefore has determinant \( z^{sn} \). \( \square \)

From now on we shall make the following assumption:

(S) The norm \( N_\delta: k \rightarrow k^\delta \) is surjective.

Corollary 5.7. The index [PGL$_n(R)$: PSL$_n(R)$] = \( sn[k^\delta : (k^\delta)^sn] \).

Proof. We have [PGL$_n(R)$: PSL$_n(R)$] = [GL$_R(M)$: SL$_n(R)$ \( \cdot \) Z$_n(R)$] = [(A$^\delta$)$^*$: ((A$^\delta$)$^*$)$^{sn}$], so the result follows from Lemma 5.6. \( \square \)

5.3. proof of theorem 5.1. Let \( \Delta = (\Delta_+, \Delta_, O_\varphi) \) be the affine twin-building of type \( M = \tilde{A}_{ns-1} \) over the set \( I = \{ 0, 1, \ldots, ns - 1 \} \) afforded by \( V = M \otimes_A k(T) \). Consider the standard twin-apartment \( \Sigma = (\Sigma_+, \Sigma_-) \) corresponding to the A-basis \( \mathcal{B} = \{ b_{ni+j} = t^ie_j \mid i = 0, \ldots, s - 1; j = 1, \ldots, n \} \). For \( \varepsilon = +, - \), let \( v_\varepsilon \) be the discrete valuation on \( k \) such that \( v_\varepsilon(T^\varepsilon) = 1 \) (where \( T^\varepsilon \) is short for \( T^{1} \)) and let \( O_\varepsilon \subseteq k \) be its valuation ring. In this section we shall also need the flag complex \( \Theta_\varepsilon \) of \( \Delta_\varepsilon \). We describe this here following [2]. Let \( \varepsilon \in \{ +, - \} \). The objects of \( \Delta_\varepsilon \) are the \( T \)-invariant chains of $A$-lattices of rank \( ns \) in \( V \) of the form:

\[ \cdots T^\varepsilon \Lambda \subseteq \Lambda \subseteq T^{-\varepsilon} \Lambda \subseteq \cdots \]

We denote this chain \([\Lambda]\). The collection of objects of \( \Delta_\varepsilon \) is denoted \( O_\varepsilon \). We have a type function $\text{typ}_\varepsilon: O_\varepsilon \rightarrow I$ given as follows. The type of the standard object \([\Lambda_{\varepsilon,l}]\) is \( l \in I \), where

\[ \Lambda_{\varepsilon,l} = \langle Tb_0, \ldots, T_{b_{l-1}}, b_l, \ldots, b_{ns} \rangle_{O_\varepsilon}, \]

and in general $\text{typ}_\varepsilon([g\Lambda_{\varepsilon,l}]) = \varepsilon v_\varepsilon(\det(g)) \mod (ns)$, for any \( g \in GL_{ns}(k(T)) \). Call objects \( x, y \in O_\varepsilon \) incident, written \( x \star y \), if \( x \cup y \) is a \( T \)-invariant chain. A flag \( F \) is a subset \( F \subseteq O_\varepsilon \) such that \( x, y \in F \) implies \( x \star y \). Let \( \mathcal{F}_\varepsilon \) be the collection of all flags of \( \Delta_\varepsilon \) ordered by \( \subseteq \). We have a type function $\text{typ}_\varepsilon: \mathcal{F}_\varepsilon \rightarrow I$, given by $\text{typ}_\varepsilon(F) = \{ \text{typ}_\varepsilon(x) \mid x \in F \}$. Then, $\Theta_\varepsilon = (\mathcal{F}_\varepsilon, \subseteq, \text{typ}_\varepsilon, I)$ is the flag-complex of \( \Delta_\varepsilon \) in the sense of [28].

Call objects \( x_+ \in O_+ \) and \( x_- \in O_- \) opposite if there is some \( A \)-basis \( \{ a_0, \ldots, a_{sn} \} \) for \( M \) such that \( x_\varepsilon = \langle a_0, \ldots, a_{sn} \rangle_{O_\varepsilon} \). Call flags \( F_+ \subseteq O_+ \) and \( F_- \subseteq O_- \) opposite if for each \( x_+ \in F_+ \) there is an \( x_- \in F_- \) opposite to it. The resulting relation is called \( O_\varepsilon \) and
\( \Theta = (\Theta_+, \Theta_-, \text{Op}_\Theta) \) is called a twin flag complex. Since \( I \) is finite \( \Theta_\varepsilon \) is a combinatorial building and \( \Theta \) is a combinatorial twin-building in the sense of [28].

Note that \( c_\varepsilon = \{[\Lambda_{\varepsilon,i}] \mid i \in I \} \) is a chamber of \( \Theta_\varepsilon \) and \((c_+, c_-)\) is a pair of opposite chambers of \( \Theta \). In the notation of [2], we have \( c_\varepsilon = c_\varepsilon(\mathcal{B}) \).

Let \( H \) be a group of not-necessarily type-preserving automorphisms of \( \Delta \). Then each \( h \in H \) induces a permutation of the node set \( I \) of \( M \) denoted \( \text{typ}(h) \) and we set \( \text{typ}(H) = \{ \text{typ}(h) \mid h \in H \} \). Clearly \( \text{typ}(h) \) preserves adjacency and non-adjacency in \( M \) and so \( \text{typ}(H) \) is a group of automorphisms of \( M \). Now \( H \) is a deck-transformation group in the sense of [28] provided

\[ \text{i typ}(H) \text{ is a deck transformation group of } M, \text{ that is, no non-trivial element of } H \text{ sends a node } i \text{ to a node at distance } \leq 2 \text{ from } i, \]

\[ \text{ii typ}(h) = \text{id}_M \text{ implies } h = \text{id}_\Delta. \]

We let \( \tilde{I} \) be the set of \( \text{typ}(H) \)-orbits in \( I \) and let \( \gamma : I \to \tilde{I} \) be the corresponding covering map. Let \( \tilde{M} \) be the diagram on \( \tilde{I} \) such that \( \gamma(i) \) and \( \gamma(j) \) are adjacent in \( \tilde{M} \) if and only if \( i \) and \( j \) are adjacent in \( M \).

**Lemma 5.8.** Let \( \varepsilon = +, - \). Then, \( \theta \) acts as a type-permuting automorphism on the combinatorial twin-building \( \Theta \). Moreover, \( \theta \) fixes the pair of opposite chambers \((c_+, c_-)\) and \( \langle \theta \rangle \) is a deck-transformation group of \( \Theta \). Hence, \( \langle \theta \rangle \) satisfies the conditions of Theorem B in M"{u}hlherr’s paper [28].

**Proof.** That \( \theta \) acts as an automorphism of \( \Theta \) follows from the fact that it sends free \( \mathcal{O}_\varepsilon \) lattices to free \( \mathcal{O}_\varepsilon \) lattices while preserving their rank and inclusion among such lattices. Since it is \( \delta^{-1} \)-semilinear over \( k(T) \), it preserves the \( A \)-module \( M \), thereby preserving the opposition relation \( \text{Op}_\Theta \).

That \( \theta \) preserves \( c_\varepsilon \) is an easy exercise. In fact \( \theta \) permutes the objects of \( c_\varepsilon \) by sending the object of type \( i \) to the object of type \( i + n \) modulo \( ns \). Since \( \theta \) is an automorphism of \( \Theta \) the graph automorphism \( \text{typ}(\theta) \) permutes the types by sending type \( i \) to type \( i + n \) modulo \( ns \).

We now have to check that \( \langle \theta \rangle \) satisfies conditions (i) and (ii) above. (i) follows since \( n \geq 2 \). (ii) follows since \( \theta \) has order \( s \) on \( \Theta \). \( \square \)

Thus, we have \( \gamma(i) = \{i, i + n, \ldots, i + (s - 1)n\} \), so we can represent \( \tilde{I} = \{0, 1, 2, \ldots, n - 1\} \) and \( \tilde{M} = \tilde{A}_{n-1} \). Naturally \( \theta \) induces a graph automorphism on \( W \). Since there can be no confusion, we shall also denote this map by \( \theta \). The group \( \tilde{W} = \{w \in W \mid w^\theta = w\} \) is a Coxeter group with diagram \( \tilde{A}_{n-1} \).

For \( \varepsilon = +, - \), let \( \tilde{\Delta}_\varepsilon = \{d \in \Delta_\varepsilon \mid \theta(d) = d\} \), and let \( \text{Op}_\varepsilon \) be the opposition relation induced on \( \tilde{\Delta}_+ \times \tilde{\Delta}_- \cup \tilde{\Delta}_- \times \tilde{\Delta}_+ \) by the opposition relation of \( \Delta \). Let \( \tilde{\delta}_s \) be the \( \tilde{W} \)-valued distance function induced by \( \tilde{\delta}_s \). Let \( \tilde{\Delta} = (\tilde{\Delta}_+ \times \tilde{\Delta}_-, \tilde{\delta}_s) \).

**Proposition 5.9.** \( \tilde{\Delta} \) is a twin-building with diagram \( \tilde{A}_{n-1} \).
Proof. By Lemma 5.8 Θ and \( G = \langle \theta \rangle \) satisfy the conditions of Theorem B of [28]. That theorem claims that \( \Theta \) contains a combinatorial twin-building \( \tilde{K} = (\tilde{K}_+, \tilde{K}_-, \tilde{O}_p) \). We shall now describe its construction.

For \( \varepsilon = +, - \), let \( \tilde{F}_\varepsilon \) be the collection of \( F_\varepsilon \in F_\varepsilon \) fixed by \( \theta \) and let \( \tilde{C}_\varepsilon \) be the collection of \( I \)-flags in \( \tilde{F}_\varepsilon \) at finite distance from \( c_\varepsilon \). Here \( c = (c_+, c_-) \) is the pair of opposite chambers fixed by \( \theta \) by assumption. Since \( I \) is finite in our case, any two chambers in \( F \) are at finite distance and so \( \tilde{C}_\varepsilon = \{ D \in F_\varepsilon \mid \theta(D) = D \} \). Note here that since \( \Theta \) is a combinatorial building itself, we could rephrase this as \( \tilde{C}_\varepsilon = \{ d \in \Delta_\varepsilon \mid \theta(d) = d \} \). Now \( \tilde{K} \) is defined by setting \( \tilde{K}_\varepsilon = (\tilde{S}_\varepsilon, \xi, \text{typ}_\varepsilon, \tilde{I}) \) and letting \( \tilde{O}_p \) be the restriction of \( O_p \) to \( \tilde{S}_+ \times \tilde{S}_- \cup \tilde{S}_- \times \tilde{S}_+ \); here \( \tilde{S}_\varepsilon = \{ F \in \tilde{F}_\varepsilon \mid F \subseteq D \) for some \( D \in \tilde{C}_\varepsilon \} \). Since \( \tilde{I} \) is finite, \( \tilde{C}_\varepsilon \) is the chamber system of \( \tilde{K}_\varepsilon \). Thus it suffices to show that \( \tilde{K}_\varepsilon \) equipped with the \( \tilde{W} \)-valued distance function and \( \delta_\varepsilon \) is the codistance determined by \( \tilde{O}_p \).

Lemma 5.10. The group \( C_G(\theta) \) is an automorphism group of the twin-building \( \tilde{\Delta} \) that is transitive on pairs of opposite chambers.

Proof. Identify \( \Delta \) with the twin-building obtained from the Birkhoff decomposition associated to the twin-BN pair \( (B_+, B_-, N) \) for \( G \). Here \( B_+ \) and \( B_- \) are the stabilizers of the fundamental chambers \( c_+ \) and \( c_- \) and \( N \) is the stabilizer of the twin-apartment \( \Sigma \) on \( c_+ \) and \( c_- \). Thus \( \Delta_\varepsilon \) is identified with \( G/B_\varepsilon \) via \( g \in G \mapsto gB_\varepsilon \). Moreover, setting \( W = N/(N \cap B_+) = N/(N \cap B_-) \), the co-distance function is given by \( \delta_\varepsilon(gB_+, hB_-) = w \in W \), when \( B_+g^{-1}hB_- \) is the unique double coset \( B_+wB_- \) in \( B_+WB_- \).

Since \( \theta \) fixes \( c_+ \) and \( c_- \), it preserves \( B_+ \) and \( B_- \), so that the action of \( \theta \) on \( \Delta \) is given entirely by its action on \( G \). It readily follows that the subgroup \( C_G(\theta) \) of \( G \) preserves \( \tilde{\Delta} \) as well as \( \tilde{O}_p \) and hence is an automorphism group of \( \tilde{\Delta} \).

It suffices to show that \( C_G(\theta) \) is transitive on pairs of opposite chambers in \( \tilde{\Delta} \). Consider a pair of opposite chambers \( (d_+, d_-) \in \tilde{\Delta}_+ \times \tilde{\Delta}_- \). Then, since \( G \) is transitive on pairs of opposite chambers of \( \Delta \), there is a \( g \in G \) so that \( (d_+, d_-) \) corresponds to \( (gB_+, gB_-) \). Note that this implies that \( g^\theta = gh \) for some \( h \in B_+ \cap B_- = H \). We claim that there is some \( g' \in C_G(\theta) \) with \( (g'B_+, g'B_-) = (gB_+, gB_-) \). Thus it suffices to show that there is some \( k \in H \) with \( (gk)^\theta = ghk^\theta = gk \), that is so that

\[
(2) \quad hkk^\theta = k.
\]

Note also that since \( g^\theta = ghh^\theta \cdots h^{\theta-1} = g \) we have

\[
(3) \quad hh^\theta \cdots h^{\theta-1} = \text{id}_{sn}
\]

Writing \( h = \text{diag}\{h_1, \ldots, h_s\} \) and \( k = \text{diag}\{k_1, \ldots, k_s\} \) as block-diagonal matrices with respect to the basis \( B \) so that \( h_i, k_i \in M_n(A) \) we find that (2) can be solved provided that \( h_s^\theta h_{s-1}^\theta \cdots h_1^\theta = \text{id}_n \), which is precisely equivalent to (3).
Therefore the fixed complex $\tilde{\Delta}$ consists of those chambers $gB_+, gB_-$, where $g \in C_G(\theta)$. Thus, the result follows.

\[ \square \]

Proof. (of Theorem 2)

By Proposition 5.9 $\tilde{\Delta}$ is a twin-building with diagram $\tilde{A}_{n-1}$. In particular, $\tilde{\Delta}$ satisfies condition (co) of [29]. By Lemma 5.10, $C_G(\theta)$ is an automorphism group of $\tilde{\Delta}$ that is transitive on pairs of opposite chambers. Then, by the twin-building version of the Curtis-Tits theorem [3] the automorphism group $C_G(\theta)$ of $\tilde{\Delta}$ is the universal completion of the amalgam $B = \{B_i, B_{ij} \mid i, j \in I\}$ of Levi-components of rank 2 and 3. Now consider the amalgam $\rho(L^\theta)$. One verifies easily that, for each $i, j \in I$, $\text{SL}_2(k) \cong \rho(L_i) \leq B_i$ and $\text{SL}_3(k) \cong \rho(L_{ij}) \leq B_{ij}$, when $\{i, j\}$ is an edge of the diagram (In fact we have $B_i = \rho(L_i)D$ and $B_{ij} = \rho(L_{ij})D$, where $D$ is the maximal split torus stabilizing $c_+$ and $c_-$. Using the more precise version of the Curtis-Tits theorem [37] for the group $\langle B_{ij} \mid i \in I - \{k\} \rangle$ one shows that the universal completion of the amalgam $B$ equals the universal completion of $\rho(L^\theta)$. Alternatively one could obtain this result along the lines of the proof of Theorem 3.

The result follows from Proposition 5.2 and the fact that $C_G(\theta) = \text{SL}_n(R)$ by Lemmas 5.3 and 5.5.

\[ \square \]

6. THE NON-ORIENTABLE CURTIS-TITS GROUP $G^\tau$

In this section $A = k[t, t^{-1}]$ denotes the ring of commuting Laurent polynomials with coefficients in the field $k$. As before let $V$ be a $k(t)$-vector space of dimension $2n$, where $n \geq 4$, with (ordered) basis $H = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Let $M$ be the free $A$-lattice spanned by this basis. Furthermore we let $G = \text{SL}(M)$.

In this subsection we introduce a sesquilinear form $\beta$ on $V$ and an involution $\theta$ of $G$ such that the fixed group $G^\theta$ is precisely the group of symmetries of $\beta$ in $G$. In Subsection 6.1 we will prove that $G^\theta$ is flag-transitive on a geometry $\Delta^\theta$. In Subsection 6.2 we prove that the geometry $\Delta^\theta$ is connected and simply connected which by Tits’ Lemma implies that the group $G^\theta$ is the universal completion of the amalgam of maximal parabolics. We then observe that the amalgam of parabolics of rank 2 and 3 in $G^\theta$ with respect to this action on $\Delta^\theta$ is isomorphic to $A^\tau$. Moreover, we note that the maximal parabolics are all linear groups over $k$. Theorem 3 will then follow by applying the Curtis-Tits theorem for linear groups applied to the maximal parabolics.

We shall sometimes identify elements of $V$ as (column) vectors with respect to the basis $H$. With this identification $G \cong \text{SL}_{2n}(A)$ acts on $V$ as a group of matrices with respect to the basis $H$ by left multiplication.

Let $\sigma$ be the involutory automorphism of $k(t)$ that fixes $k$ pointwise and interchanges $t$ and $t^{-1}$. Define a $\sigma$-sesquilinear form $\beta$ on $V$, that is,

$$\beta(\lambda u, \mu v) = \lambda \beta(u, v)\mu^\sigma$$

for all $u, v \in V$ and $\lambda, \mu \in k(t)$. 


such that the Gram matrix of $H$ with respect to $\beta$ equals

$$B = \begin{pmatrix} 0_n & I_n \\ tI_n & 0_n \end{pmatrix}.$$

**Definition 6.1.** The *right adjoint* of a transformation $g \in \Gamma L(V)$, is the transformation $g^* \in \Gamma L(V)$ such that $\beta(gu, g^*v) = \beta(u, v)$ for all $u, v \in V$.

We define the automorphism $\theta : G \mapsto G$ by

$$g \mapsto tB^{-1}t^{-1}g \sigma t$$

Here $tB$ denotes the transpose of $B$.

**Lemma 6.2.** The automorphism $\theta$ is an involution of $G$ and for each $g \in G$ we have $g^\theta = g^\ast$.

**Proof.** For $u, v \in V$ and $g \in G$ we have $\beta(gu, g^\theta v) = t(gu)B(g^\theta v)^\sigma = t^u g t^g B^\sigma v^\sigma = t^u tB^\sigma v^\sigma = \beta(u, v)$ and since this holds for all $u, v \in V$ and $\beta$ is non-degenerate, we are done. That $\theta$ is an involution now follows. □

Let

$$SU(V, \beta) := \{g \in SL_{2n}(A) | \forall x, y \in V, \beta(gx, gy) = \beta(x, y)\}.$$

**Corollary 6.3.** The unitary group $SU(V, \beta)$ is the fixed group $G^\theta = \{g \in G | g^\theta = g\}$.

We will now construct an amalgam $\mathcal{L}^r$ inside $SL_{2n}(A)$ that is isomorphic to the amalgam $\mathcal{A}^r$ from Corollary 3.3. Consider the following matrix:

$$F = \begin{pmatrix} 0 & I_{2n-1} \\ 1 & 0 \end{pmatrix}.$$

We now define the automorphism $\Phi$ of $SL_{2n}(A)$ given by $X \mapsto F^{-1}XF$. Also define the map $i : SL_2(k) \rightarrow SL_{2n}(A)$ by

$$A \mapsto \begin{pmatrix} A & I_{2n-2} \end{pmatrix}.$$

Next, for $k = 1, \ldots, n + 1$, let $\phi_k : SL_2(k) \rightarrow SL_{2n}(A)$ by

$$\phi_k(A) = \Phi^{k-1}(i(A)) \cdot \theta(\Phi^{k-1}(i(A)))$$

and let $L_k$ be the image of $\phi_k$. Note that for each $k = 1, \ldots n - 1$ we have

$$L_k = \left\{ \begin{pmatrix} I_{k-1} & A \\ A & I_{n-k-1} \\ I_{n-k-1} & I_{n-k-1} \end{pmatrix} | A \in SL_2(k) \right\}$$
and
\[
L_n = \left\{ \begin{pmatrix} a & I_{n-2} \\ -ct^{-1} & a \\ -bt & c & d & I_{n-2} \\ \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(k) \right\}.
\]

Let \( I = \{1, 2, \ldots, n\} \). We shall denote the diagonal torus in the group \( L_i \) by \( D_i \) for each \( i \in I \). For distinct \( i, j \in I \), let \( \phi_{i,j} \) be the canonical isomorphism between \( G_{i,j} = \langle G_i, G_j \rangle \) and \( L_{i,j} = \langle L_i, L_j \rangle_G \) induced by \( \phi_i \) and \( \phi_j \). It follows that \( L_{i,j} \cong \text{SL}_3(k) \) if \( i-j \equiv \pm 1 \) (mod \( n \)) and \( G_{i,j} \cong L_i \times L_j \) otherwise.

**Definition 6.4.** For each \( i, j \in \{1, 2, \ldots, n\} \), let \( \varphi_{i,j} : L_i \hookrightarrow L_{i,j} \) be the natural inclusion map. Then we define the following amalgam:
\[
\mathcal{L}^\tau = \{ L_i, L_{i,j}, \varphi_{i,j} \mid i, j \in I \}.
\]

**Proposition 6.5.** The amalgam \( \mathcal{L}^\tau \) is contained in \( G^\theta \) and is isomorphic to \( \mathcal{A}^\tau \).

**Proof.** That \( \mathcal{L}^\tau \) is contained in \( G^\theta \) follows by definition of \( \phi_k \) and the fact that \( \theta \) has order 2. We claim that the collection \( \phi = \{ \phi_{i,j} \mid i, j \in I \} \) is the required isomorphism between \( \mathcal{A}^\tau \) and \( \mathcal{L}^\tau \). This is completely straightforward since we have \( \phi_1 \circ \phi_{n+1}^{-1} = \tau \).

We finish this subsection with some observations on the form \( \beta \) and the action of \( G \) on \( V \).

**Lemma 6.6.** \( \beta \) is a non-degenerate trace-valued \((\sigma, t)\)-sesquilinear form, that is for all \( u, v \in V \) we have \( \beta(v, u) = t\beta(u, v)^{\sigma} \) and there exists \( x \in k(t) \) such that \( \beta(u, u) = x + x^{\sigma} t \).

**Proof.** The first claim follows easily from the fact that \( t^i B^\sigma = B \). To prove the second claim, let \( u = \sum_{i=1}^n \lambda_i e_i + \mu_i f_i \) and let \( u' = \sum_{i=1}^n \lambda_i' e_i + \mu_i' f_i \). Then
\[
\beta(u, u') = \sum_{i=1}^n \lambda_i \mu_i^{\sigma} + \mu_i \lambda_i^{\sigma} t = t\beta(u', u)^{\sigma}
\]

In particular, setting \( u = u' \) we get \( x = \sum_{i=1}^n \mu_i \lambda_i^{\sigma} \).

**Definition 6.7.** Given a \( k(t) \)-basis \( \{a_1, \ldots, a_{2n}\} \) for \( V \), the right dual basis for \( V \) with respect to \( \beta \) is the unique basis \( \{a_1^*, \ldots, a_{2n}^*\} \) such that \( \beta(a_i, a_j^*) = \delta_{ij} \) (note the order within \( \beta \)).

**Lemma 6.8.** Let \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) be a basis for \( V \) with Gram matrix \( B \). Then its right-dual basis is \( \{b_1, \ldots, b_n, ta_1, \ldots, ta_n\} \).

The following is an easy generalization of Lemma 6.2.

**Lemma 6.9.** If \( g \in \text{GL}(V) \) is represented by a matrix \( (g_{ij}) \) with respect to \( \{a_1, \ldots, a_{2n}\} \), then \( g^* \) is represented by \( t(g_{ij}^*)^{-1} \) with respect to \( \{a_1^*, \ldots, a_{2n}^*\} \).
Corollary 6.10. The right dual of an $A$-basis for $M$ is an $A$-basis for $M$.

Proof. This follows from Lemma 6.8 and 6.9 by observing that $GL(M)$ is transitive on such bases and invariant under the map $(g_{ij}) \mapsto (g'_{ij})^{-1}$. \hfill \Box

6.1. the geometry $\Delta^\theta$ for $G^\theta$. We now describe a geometry $\Delta^\theta$ on which $G^\theta$ acts flag-transitively so that $L^\sigma$ is the amalgam of parabolic subgroups of rank 2 and 3 in $G^\theta$. Let $\Delta$ be the twin-building for the group $G = SL_{2n}(A)$ with twinning determined by $M$ (for a construction see Subsection 5.3). Let $(W,S)$ be the Coxeter system with diagram $\Gamma$ of type $\tilde{A}_{2n-1}$. Call $S = \{s_i \mid i = 0, \ldots, 2n - 1\}$.

We define a relaxed incidence relation on $\Delta_\varepsilon$ as follows. We say that $d_\varepsilon$ and $e_\varepsilon$ are $(i, \theta(i))$-adjacent if and only if $d_\varepsilon$ and $e_\varepsilon$ are in a common $\{i, \theta(i)\}$-residue. In this case we write

$$d_\varepsilon \approx_i e_\varepsilon,$$

where we let $i \in \{1, \ldots, n\}$. Note that the residues in this chamber system are $J$-residues of $\Delta_\varepsilon$ where $J^\theta = J$. In Subsection 6.2 we shall see that the resulting chamber system $(\Delta_\varepsilon, \approx)$ is simply connected.

Definition 6.11. For each $O_\varepsilon$-lattice $\Lambda_\varepsilon$ we let

$$\Lambda^\theta_\varepsilon = \{v \in V \mid \beta(u,v) \in O_\varepsilon \text{ for all } u \in \Lambda_\varepsilon\}.$$

Lemma 6.12. (a) If $\{a_1, \ldots, a_{2n}\}$ is a basis for $V$ with right dual $\{a^*_1, \ldots, a^*_{2n}\}$ with respect to $\beta$, then $\Lambda^\theta_\varepsilon(a_1, \ldots, a_{2n}) = \Lambda_{-\varepsilon}(a^*_1, \ldots, a^*_{2n})$.

(b) For all $i, j$ we have $(t^j a^*_i)^* = t^j a^*_i$ so $\Lambda^0_\varepsilon(t^j a_1, \ldots, t^{2n} a_{2n}) = \Lambda_{-\varepsilon}(t^j a^*_1, \ldots, t^{2n} a^*_{2n})$.

(c) $\theta$ reverses inclusion of lattices.

Proof. Parts (a) and (b) are straightforward consequences of the fact that $\beta$ is $\sigma$-sesquilinear. Part (c) follows from Definition 6.11. \hfill \Box

The standard ordered $t$-hyperbolic basis for $M$ is $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ whose Gram matrix is given by $B$. The standard chamber in $\Delta_\varepsilon$ is $c_{\varepsilon}(e_1, \ldots, e_n, f_1, \ldots, f_n)$.

Proposition 6.13. The map $\theta$ induces isomorphisms $\theta : \Delta_\varepsilon \rightarrow \Delta_{-\varepsilon}$ where $typ(\theta) : I \rightarrow I$ is the graph isomorphism defined by $i \rightarrow i - n \mod (2n)$. Moreover, $\theta$ interchanges the standard chambers $c_+$ and $c_-$. \hfill \Box

Proof. By Lemma 6.12 (a) and (c) $\theta$ sends admissible chains of $O_\varepsilon$-lattices to admissible chains of $O_{-\varepsilon}$-lattices. In particular, it interchanges $\Delta_\varepsilon$-objects with $\Delta_{-\varepsilon}$-objects while preserving incidence. Thus $\theta$ induces the required isomorphisms. We now analyze how types are permuted by $\theta$.

Let $X_{i,\varepsilon}$ be the object of type $i$ on $c_\varepsilon$. We show that $X^\theta_{i,\varepsilon} = X_{n+i,\varepsilon}$. This follows immediately from Lemmas 6.12 and 6.8. In particular $c_+$ and $c_-$ are interchanged.

We now consider an arbitrary lattice $\Lambda_\varepsilon = \langle a_1, \ldots, a_{2n}\rangle_{O_\varepsilon}$, where $\{a_1, \ldots, a_{2n}\}$ is some $k(t)$-basis for $M$ (note that this is always possible as the Kac Moody group $SL_{2n}(A)$ acts flag transitively on the twin building).
Let \( g \) be the transformation sending \( e_i \) to \( a_i \) and \( f_i \) to \( a_{n+i} \) for \( i = 1, 2, \ldots, n \). Since \( g \in \text{GL}(M) \), it follows that \( \det g = at^l \in A^* \) for some \( a \in k, l \in \mathbb{Z} \). Now \( s^{-1}g^{-\sigma} = g^\theta s^{-1} \) is the transformation sending \( e_1, \ldots, f_n \) to \( a_1^\ast, \ldots, a_n^\ast \). Taking determinants we see that the type of \( \Lambda_\lambda^\theta \) is \( e_v(\det(g)^{-\sigma}t^n) = e_v(\det(g)) - n \mod (2n) \). \( \square \)

**Definition 6.14.** Let

\[
\Delta^\theta = \{ (d_+, d_-^\theta) \mid d_+ \text{ opp } d_-^\theta \}
\]

Adjacency is given by \( \approx \). Residues of \( \Delta^\theta \) are the intersections of residues of \( (\Delta, \approx) \) with \( \Delta^\sigma \).

**Lemma 6.15.** \( (d_+, d_-) \in \Delta^\theta \) if and only if there is an \( A \)-basis \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) for \( M \) whose Gram matrix is \( B \) and such that \( d_\varepsilon = c_\varepsilon(a_1, \ldots, a_n, b_1, \ldots, b_n) \).

**Proof.** As in the proof of Proposition 6.13, one verifies that any such basis gives rise to a pair of chambers in \( \Delta^\theta \). Conversely, let \( (d_+, d_-) \in \Delta^\theta \). That means that \( d_- = d_+^\theta \). Let \( \Sigma = \Sigma(d_+, d_-) \) be the twin-apartment containing \( d_+ \) and \( d_- \). Then \( \Sigma^\theta = \Sigma \). Let \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) be an \( A \)-basis for \( M \) such that \( \Sigma = \Sigma\{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) and \( d_\varepsilon = c_\varepsilon(a_1, \ldots, a_n, b_1, \ldots, b_n) \), where \( X_0 = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle \cap \varepsilon \) has type 0. Let \( \{a_1^\ast, \ldots, a_n^\ast, b_1^\ast, \ldots, b_n^\ast\} \) be the right dual basis with respect to \( \beta \). Then,

\[
\Sigma = \Sigma\{a_1^\ast, \ldots, a_n^\ast, b_1^\ast, \ldots, b_n^\ast\} \\
d_\varepsilon = c_\varepsilon(a_1^\ast, \ldots, a_n^\ast, b_1^\ast, \ldots, b_n^\ast)
\]

Note that by Corollary 6.10 and Lemma 6.12 both bases are \( A \)-bases for \( M \). Note that the type of the lattice \( \langle a_1^\ast, \ldots, a_n^\ast, b_1^\ast, \ldots, b_n^\ast \rangle \cap \varepsilon = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle_{\Delta^\theta} \cap \varepsilon \) is \( n \). Now consider the \( k(t) \)-linear map

\[
\phi: V \rightarrow V \\
b_i \mapsto a_i^\ast \\
ta_i \mapsto b_i^\ast
\]

for all \( i = 1, 2, \ldots, n \). It is easy to check that \( \phi \) is a type-preserving automorphism of \( \Delta_\varepsilon \) such that \( d_\varepsilon^\phi = d_\varepsilon \) since it is a \( k(t) \)-linear map that sends the object of type \( i \) on \( d_\varepsilon \) to the object of type \( i \) on \( d_\varepsilon \). This implies that \( \phi \in H = N \cap B_+ \cap B_- \) and it follows (see e.g. [1]) that

\[
b_i = \lambda_i a_i^\ast \\
ta_i = \mu_i b_i^\ast
\]

where \( \lambda_i, \mu_i \in k^* \) and in fact since \( (a_i^\ast)^* = ta_i \) we have \( \mu_i = \lambda_i^{-1} \). Without modifying the chambers \( d_\varepsilon \), we may scale so that \( \lambda_i = 1 \) for all \( i \), that is

\[
b_i = a_i^\ast \\
ta_i = b_i^\ast
\]

so the Gram matrix of \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) is \( B \). \( \square \)

**Theorem 6.16.** The group \( G^\theta \) acts flag-transitively on \( \Delta^\theta \).
Proof. Let \((d_+,d_-) \in \Delta^\theta\). By Lemma 6.15 there exists an \(A\)-basis \(\{a_1,\ldots,a_n,b_1,\ldots,b_n\}\) for \(M\) whose Gram matrix is \(B\). The \(A\)-linear map
\[
\phi: V \to V
\]
\[e_i \mapsto a_i\]
\[f_i \mapsto b_i\]
for all \(i = 1,2,\ldots,n\) belongs to \(G^\theta\) and sends \((c_+,c_-)\) to \((d_+,d_-)\).

We now discuss the parabolic subgroups of the action of \(G\) on \(\Delta^\theta\).

Lemma 6.17. Let \(J \subseteq I\) be \(\theta\)-invariant and let \((R,R^\theta)\) be a pair of residues of \(\Delta^\theta\) of type \(J\). Then,
(a) we can write \(J = K \uplus K^\theta\) such that \(M_K\) and \(M_{K^\theta}\) are in separate connected components of \(M_J\).
(b) \(R = S \times T^\theta\) and \(R^\theta = S^\theta \times T\), where \(S\) and \(T\) are of type \(K\).
(c) Assume \(R\) lies on the standard chamber \(c_+\) so that all parabolic groups are standard. \(\text{Stab}_{G^\theta}(R) = L^\theta_J\), where \(L_J\) is the Levi subgroup of the standard parabolic subgroup of type \(J\) in \(G\) stabilizing \(R\).
(d) We have \(L_J = L_K \oplus L_{K^\theta}\) and \(L^\theta_J = \{gg^\theta \mid g \in L_K\}\).

Proof. (a) Since \(J \neq I\) there is some type \(i\) is missing from \(J\) and hence so is \(\theta(i)\). Thus \(M_{I - \{i,\theta(i)\}}\) has two connected components interchanged by \(\theta\).
(b) This is general building theory.
(c) The simultaneous stabilizer of \(R\) and its opposite \(R^\theta\) in \(G\) is the Levi component \(L_R\).
Now \(G^\theta \cap L_R = L^\theta_R\).
(d) From (b) we have \(L_R = L_S \oplus L_{T^\theta}\) and since \(T^\theta\) and \(S^\theta\) are opposite, \(L_{T^\theta} = L_{S^\theta}\). The second part follows immediately.

Remark 6.18. Note that the amalgam of minimal parabolic groups in \(G^\theta\) for the action on \(\Delta^\theta\) is close to \(L^\tau\), but not equal. Indeed, each parabolic contains a large portion of the maximal torus that stabilizes the standard pair of opposite chambers.

6.2. simple connectedness. In this subsection we will prove that the chamber system \((\Delta^\theta,\approx)\) is connected and simply-connected. In order to do so we shall in fact prove a stronger result, namely that \((\Delta^\theta,\sim)\) is connected and simply connected. Namely,

Lemma 6.19. Suppose that \(X\) is a sub set of \(\Delta_+\) such that \((X,\sim)\) is connected and simply connected. Then \((X,\approx)\) is also connected and simply connected.

Proof. Note that each rank \(r < n\) residue of \((\Delta_+,\sim)\) is included in a residue of rank \(\leq r\) of \((\Delta_+\approx)\). Since connectedness is a statement about rank 1 residues and simple connectedness is a statement about rank 2 residues, we are done.
We will use the techniques developed in [15] to show that \((\Delta^{\theta}, \sim)\) is simply connected. In the terminology of loc. cit. a collection \(\{C_m\}_{m \in \mathbb{N}}\) of subsets of a chamber system \(D\) over \(I\) is a filtration if the following are satisfied:

- **F1** For any \(m \in \mathbb{N}\) \(C_m \subseteq C_{m+1}\),
- **F2** \(\bigcup_{m \in \mathbb{N}} C_m = D\),
- **F3** For any \(m \in \mathbb{N}_{>0}\), if \(C_{m-1} \neq \emptyset\), there exists an \(i \in I\) such that for any \(c \in C_m\), there is a \(d \in C_{m-1}\) that is \(i\)-adjacent to \(c\).

It is called a residual filtration if the intersections of \(C\) with any given residue is a filtration of that residue.

For any \(c \in \Delta\), let \(|c| = \min\{\lambda \mid c \in C_\lambda\}\). For a subset \(X \subseteq D\) we accordingly define \(|X| = \min\{|c| \mid c \in X\}\) and \(\text{aff}(X) = \{c \in X \mid |c| = |X|\}\). We shall make use of the following result from loc. cit..

**Theorem 6.20.** Suppose \(C\) is a residual filtration on \(D\) such that for any rank 2 residue \(R\), \(\text{aff}(R)\) is connected and any rank 3 residue \(R\), \(\text{aff}(R)\) is simply connected, then the following are equivalent.

(a) \(D\) is simply connected,
(b) \(C_n\) is simply connected for all \(n \in \mathbb{N}\).

We now let \(D\) be the chamber system \(\Delta^+\), with adjacency relations \(\approx\) \((i \in I)\). We then define a residual filtration \(C\) on \(\Delta^+\) with the property that \(C_0 \cong \Delta^\theta\). We shall first prove that \(D\) is simply connected. In order to obtain simple connectedness of \(\Delta^\theta\) it will suffice to show that \(C\) satisfies the conditions of the theorem.

### 6.3. the filtration \(C\).

In order to define the filtration \(C\) we first let

\[ W^\theta = \{w \in W \mid \exists d_\varepsilon \in \Delta_\varepsilon: w = \delta_\varepsilon(d_\varepsilon, d_\varepsilon^\theta)\}. \]

We also fix an injective map \(|\cdot|: W^\theta \to \mathbb{N}\) such that whenever \(l(w) > l(w')\), we have \(|w| > |w'|\) and for any \(m \in \mathbb{N}\). We then define a filtration on \(\Delta^+\) using \(|\cdot|\) as follows: Let

\[ C_m = \{c_+ \in \Delta^+ \mid |\delta_\varepsilon(c_+, c_+^\theta)| \leq m\}. \]

In the remainder of this section we prove that \(C\) is a residual filtration. First however, we will need some technical lemmas about \(W^\theta\). Let

\[ W(\theta) = \{u \in W \mid u^\theta = u^{-1}\}. \]

These elements are called twisted involutions in [35] and [20]. Some of the results below have somewhat weaker forms in the most general case of a quasi-twist. See [20] for details on both twisted involutions and of the corresponding geometries.

We now characterize \(W(\theta)\) as follows:

**Lemma 6.21.**

\[ W(\theta) = \{w(w^{-1})^\theta \mid w \in W\}. \]

More precisely, given any \(u \in W(\theta)\) there exists a word \(w \in W\) such that \(w(w^{-1})^\theta\) is a reduced expression for \(u\).
Proof. It is obvious that we have $\supseteq$. We now proceed to prove the reverse inclusion. Let $u \in W^\theta$. We prove that $u$ can be written as a reduced expression of the form $w(w^{-1})^\theta$ by induction on $l = l(u)$. If $l = 0$, then $1 = u = 1 \cdot (1^{-1})^\theta$. Now let $l \geq 1$ and write $u = s_i \cdots s_i$. By assumption we can also write $u = s_{\theta(i_1)} \cdots s_{\theta(i_1)}$. Consider $u' = s_i u s_{\theta(i_1)}$. Note that $u' \in W(\theta)$. We note the following: $l(s_i u s_{\theta(i_1)}) < l(u)$ and so writing $s_i u = s_i s_{\theta(i_1)} \cdots s_{\theta(i_1)}$ it follows from the exchange property that there is some $j$ such that $s_i u = s_{\theta(i_1)} \cdots s_{\theta(i_1)}$. There are two cases:

1. $j > 1$
2. $j = 1$

In case (i) it follows that $l(s_i u s_{\theta(i_1)}) = l(u) - 2$. By induction we have a word $w'$ of length $(l(u) - 2)/2$ such that $u = s_i w'(w'^{-1})^\theta s_{\theta(i_1)}$, and since this expression has length $l(u)$ it is reduced and we are done.

In case (ii) it follows that $s_i u s_{\theta(i_1)} = u$. This means that $u$ can also be written in the form $u = s_i s_{\theta(i_1)} \cdots s_i s_{\theta(i_1)}$. Repeating this process we either decrease the length as in case (i), or $u$ has the property that it can be written such that any of the $s_i$ come first. By Theorem 2.16 of [30] this means that if $J = \{i_1, \ldots, i_i, \theta(i_1), \ldots, \theta(i_1)\}$, then $W_J$ is finite and $u$ is the longest word in $W_J$. In particular $J \neq I$, then since $\text{typ}(\theta)$ acts on $\widetilde{A}_{2n-1}$ by interchanging opposite nodes, there is a subset $K \subseteq J$ such that $J$ is the disjoint union of $K$ and $K^\theta$. As a consequence, $u = w_K(w_K)^\theta$. 

The following lemma characterizes $W^\theta$.

Lemma 6.22. $W^\theta = W(\theta)$.

Proof. Let $c_\varepsilon \in \Delta_\varepsilon$. Then $u = \delta_\varepsilon(c_\varepsilon, c_\varepsilon^\theta)$ satisfies $u^\theta = u^{-1}$. Therefore the inclusion $\subseteq$ follows by definition. Conversely, consider a chamber $c_\varepsilon$ such that $c_\varepsilon$ opp $c_\varepsilon^\theta$. Then the apartment $\Sigma(c_\varepsilon, c_\varepsilon^\theta)$ is preserved by $\theta$ and identifying it with the Coxeter group we see that $\theta$ acts on $\Sigma$ as it acts on $W$. Let $u \in W(\theta)$. Then, by Lemma 6.21 it is of the form $w(w^{-1})^\theta$ for some $w \in W$. Let $d_\varepsilon$ be the chamber such that $\delta_\varepsilon(c_\varepsilon, d_\varepsilon) = w$, then $\delta_\varepsilon(d_\varepsilon, d_\varepsilon^\theta) = w(w^{-1})^\theta = u$ as desired.

In the sequel we shall use the following notation for projections. Given a residue $R$ of $\Delta_\varepsilon$, we denote projection from $\Delta_\varepsilon$ onto $R$ by $\text{proj}_R$ and denote (co-) projection from $\Delta_{-\varepsilon}$ onto $R$ by $\text{proj}^*_R$.

Lemma 6.23. Suppose that $c_\varepsilon \in \Delta_\varepsilon$ satisfies $\delta_\varepsilon(c_\varepsilon, c_\varepsilon^\theta) = w$, let $i \in I$ and suppose that $\pi$ is the $\sim_i$-panel on $c_\varepsilon$. Then,

(a) If $l(s_i w) > l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ except one satisfy $\delta_\varepsilon(d_\varepsilon, d_\varepsilon^\theta) = w$.

The remaining chamber $\tilde{c}_\varepsilon$ satisfies $\delta_\varepsilon(\tilde{c}_\varepsilon, (\tilde{c}_\varepsilon)^\theta) = s_i w s_{\theta(i)}$.

(b) If $l(s_i w) < l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ satisfy $\delta_\varepsilon(d_\varepsilon, d_\varepsilon^\theta) = s_i w s_{\theta(i)}$.

In particular, if $w = 1$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ except one satisfy $\delta_\varepsilon(d_\varepsilon, d_\varepsilon^\theta) = 1$. 

Proof. (a) In this case, by the twin-building axioms, there is a unique chamber, called \( \hat{c} = \text{proj}_\pi^*(c^\theta) \) such that \( \delta_\ast(\hat{c}, c^\theta) = s_i w \). Let \( d_\varepsilon \) be any other chamber in \( \pi \). Then, again by the twin-building axioms we have \( \delta_\ast(d_\varepsilon, c^\theta) = w \). By applying \( \theta \) we see that \( \delta_\ast(d^\theta_\varepsilon, c^\theta) = w^\theta = w^{-1} \). It follows that for any other chamber \( d' \in \pi \) we either have \( \delta_\ast(d^\theta_\varepsilon, d') = w^\theta s_{\theta(i)} \) or \( w^\theta \). Note here that \( l(w^\theta s_{\theta(i)}) = l(w^\theta) + 1 \). However, \( \delta_\ast(d_\varepsilon, d^\theta_\varepsilon) \in W^\theta \), where all lengths are even. Since \( w^\theta \in W^\theta \), \( w^\theta s_{\theta(i)} \notin W^\theta \) and so we must have \( \delta_\ast(d_\varepsilon, d^\theta_\varepsilon) = w \). By the same token, the distance \( \delta_\ast(\hat{c}, c^\theta) = s_i w s_{\theta(i)} \).

(b) In this case, by the twin-building axioms, every chamber \( d_\varepsilon \in \pi \setminus \{ c_\varepsilon \} \) satisfies \( \delta_\ast(d_\varepsilon, c^\theta) = s_i w \), since now \( c_\varepsilon = \text{proj}_\pi^*(c^\theta) \), which is unique. Applying \( \theta \) we see that \( \delta_\ast(d^\theta_\varepsilon, c^\theta) = s_{\theta(i)} w^\theta \). It follows that for any other chamber \( d^\theta \in \pi \) we either have \( \delta_\ast(d^\theta_\varepsilon, d^\theta) = s_{\theta(i)} w^\theta s_i \) or \( s_{\theta(i)} w^\theta \). However, since \( w^\theta \in W^\theta \), by looking at the lengths, \( w^\theta s_i \notin W^\theta \), and so we must have \( \delta_\ast(d^\theta_\varepsilon, d_\varepsilon) = s_{\theta(i)} w^\theta s_i \), and we are done. \( \square \)

Lemma 6.24. \( \theta \) does not commute with any reflection.

Proof. Let \( r \) be any reflection such that \( r^\theta = r \). Then in fact \( r \in W^\theta \). However, all elements of \( W^\theta \) have even length and \( r \) being a conjugate of a fundamental reflection does not.

Lemma 6.25. For \( u \in W^\theta \) and \( i \in I \), we have \( l(s_i u s_{\theta(i)}) = l(u) + 2 \).

Proof. By Lemma 6.21 \( u \) has a reduced expression of the form \( w w^{-\theta} \). First note that by Lemma 6.24 we cannot have \( s_i u s_{\theta(i)} = u \) because that would imply that the reflection \( w^{-1} s_i w \) is fixed by \( \theta \). There are two cases to consider, namely,

(a) \( l(s_i u) > l(u) \),

(b) \( l(s_i u) < l(u) \).

In case (a) note that \( l(s_i u) = l(u s_{\theta(i)}) > l(u) \), so that by Proposition 4.1(b) of [15] we have \( l(s_i u s_{\theta(i)}) = l(u) + 2 \) or \( s_i u s_{\theta(i)} = u \). The latter is impossible by the preceding argument.

In case (b) consider \( u' = s_i u \) and assume that \( l(s_i u s_{\theta(i)}) = l(u) \). We now have \( l(u' s_{\theta(i)}) = l(s_i u s_{\theta(i)}) = l(u) > l(u') \) and \( l(s_i u') = l(u) > l(s_i u) = l(u') \). Applying the aforementioned Proposition again, we find that either \( l(s_i u' s_{\theta(i)}) = l(u') + 2 \) or \( s_i u' s_{\theta(i)} = u' \). In the first case we find that \( l(s_i u) = l(u s_{\theta(i)}) - 2 \), which contradicts the equality \( l(s_i u) = l(s_i u^\theta) = l(s_{\theta(i)} u^{-1}) = l(u s_{\theta(i)}) \). The second case is ruled out as in (a). \( \square \)

We define the following subset of a given residue \( R \):

\[ A_\theta(R) = \{ c \in R \mid l(\delta_\ast(c, c^\theta)) \text{ is minimal among all such distances} \} \]

Lemma 6.26. Let \( R \) be a \( J \)-residue. Let \( c \in A_\theta(R) \) and let \( w = \delta_\ast(c, c^\theta) \). Then, \( d \in A_\theta(R) \) if and only if \( w = \delta_\ast(d, d^\theta) \). Moreover, \( w \) is determined by the fact that for any \( j \in J \) we have \( l(s_j w) = l(w) + 1 \).

Proof. First note that by Lemma 6.23, \( \{ \delta_\ast(x, x^\theta) \mid x \in R \} = \{ u w u^\theta \mid u \in W_J \} \). Moreover, the coset \( W_J u W_{\theta(j)} \) has a minimal element \( m \) that is characterized by the fact that \( l(s_j m) = l(m) + 1 \) and \( l(m s_{\theta(j)}) = l(m) + 1 \) for all \( j \in J \). We claim that \( w \) has that property as well. Namely, let \( j \in J \) have the property that \( l(w s_{\theta(j)}) = l(s_j w) < l(w) \). Then, by
Lemma 6.23 (b) any element $d$ in the $j$-panel on $c$ has the property that $\delta_s(d, d^\theta) = s_jws_{\theta(j)}$ and by Lemma 6.25 this must have length $l(w) - 2$, a contradiction to the fact that $c \in A_\theta(R)$. Thus, $w$ satisfies the conditions on $m$ and it follows that $w = m$. \hfill \Box

Proposition 6.27. Let $c \in R$ and let $w = \delta_s(c, c^\theta)$. The following are equivalent:

i. $c \in A_\theta(R)$,

ii. $w = w_R$, the unique element of minimal length in $W_JwW_{\theta(J)}$,

iii. $c \in C_k$, where $k = \min\{l \mid C_l \cap R \neq \emptyset\}$.

In particular, we have $A_\theta(R) = \text{aff}(R)$.

Proof. By Lemma 6.26 (i) and (ii) are equivalent. Since $\cdot$ is strictly increasing, also (ii) and (iii) are equivalent. \hfill \Box

Proposition 6.28. $\mathcal{C}$ is a residual filtration.

Proof. Part (F1) and (F2) are immediate. Now let $R$ be a $J$-residue, suppose that $R \cap C_{n-1} \neq \emptyset$ and let $c \in R \cap C_n - C_{n-1}$. Let $w = \delta_s(c, c^\theta)$. By Proposition 6.27, $c \not\in A_\theta(R)$ and so, by Lemma 6.26, there exists a $j \in J$ with $l(s_jw) < l(w)$. Therefore by Lemma 6.25, any $j$-neighbor $d$ of of $c$ has $l(\delta(d, d^\theta)) = l(w) - 2$ and therefore belongs to $C_{n-1}$. \hfill \Box

Proposition 6.28 allows us to apply Theorem 6.20 and, by Proposition 6.27, in order to show simple connectedness of $\Delta^\theta$, it suffices to show that $\text{aff}(R) = A_\theta(R)$ is connected when $R$ has rank 2 and is simply connected when $R$ has rank 3. We shall first obtain some general properties of $A_\theta(R)$ and then verify the connectedness properties using concrete models of $A_\theta(R)$.

Proposition 6.29. (See Corollary 7.4 of [11]) For $\varepsilon = \pm$, let $S_\varepsilon \subsetneq \subset R_\varepsilon$ be residues of $\Delta_\varepsilon$ such that $S_\varepsilon = \text{proj}_{R_\varepsilon}(R_{-\varepsilon})$ and let $x_\varepsilon \in R_\varepsilon$ be an arbitrary chamber and assume in addition that $R_{-\varepsilon} = R^\theta_\varepsilon$ and $x_{-\varepsilon} = x^\theta_\varepsilon$, for $\varepsilon = \pm$. Then, $x_\varepsilon \in A_\theta(R_\varepsilon)$ if and only if

i. $x_\varepsilon$ belongs to a residue opposite to $S_\varepsilon$ in $R_\varepsilon$ whose type is also opposite to the type of $S_\varepsilon$ in $R_\varepsilon$ and

ii. $\text{proj}_{S_\varepsilon}(x_\varepsilon) \in A_\theta(S_\varepsilon)$.

Proof. This is exactly the same as the proof in loc. cit. noting that it suffices for $\theta$ to be an isomorphism between $\Delta_+$ and $\Delta_-$ that preserves lengths of codistances. \hfill \Box

Lemma 6.30. With the notation of Proposition 6.29, $\text{proj}_{S_\varepsilon}^*$, $\text{proj}_{S_{-\varepsilon}}^*$ define adjacency preserving bijections between $S_{-\varepsilon}$ and $S_\varepsilon$ and $(\text{proj}_{S_\varepsilon}^*)^{-1} = \text{proj}_{S_{-\varepsilon}}^*$. Let $l = \max\{l(\delta_s(c_\varepsilon, d_{-\varepsilon})) \mid c_\varepsilon \in S_\varepsilon, d_{-\varepsilon} \in S_{-\varepsilon}\}$. Then, $d_{-\varepsilon} = \text{proj}_{S_{-\varepsilon}}^*(c_\varepsilon)$ if and only if $l(\delta_s(c_\varepsilon, d_{-\varepsilon})) = l$.

Proof. This is the twin-building version of the main result of [16]. \hfill \Box

In view of Proposition 6.29, in order to study $A_\theta(R)$ entirely inside $R$ we need to know what $A_\theta(S)$ looks like if $\text{proj}_{S}^* \circ \theta$ is a bijection on $S$. From now on we shall write $\theta_S = \text{proj}_{S}^* \circ \theta$. 

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Corollary 6.31. In the notation of Proposition 6.29, \( \theta_{S_{\varepsilon}} \) has order 2.

Proof. Pick any \( c \in S_{\varepsilon} \). Then \( l(\delta_{x}(c^{\theta}, \text{proj}_{S_{\varepsilon}}^{\pi}(c)^{\theta})) = l(\delta_{x}(c, \text{proj}_{S_{\varepsilon}}^{\pi}(c))) \). Therefore, by Lemma 6.30, \( \text{proj}_{S_{\varepsilon}}^{\pi}(c^{\theta}) = (\text{proj}_{S_{\varepsilon}}^{\pi}(c))^{\theta} \). The claim of the lemma follows.

Lemma 6.32. Let \( R \) be a residue of type \( M_{J} \cong A_{m} \) for some \( m \) and assume that \( \text{proj}_{R}^{\theta} \) defines a bijection between \( R \) and \( R^{\theta} \). Then, \( \theta_{R} \) is a type preserving automorphism of \( R \).

Proof. Note first that both \( \theta \) and \( \text{proj}_{R}^{\theta} \) define a bijection between the type set of \( R \) and the type set of \( \theta(R) \). Both maps can either be equal or differ by opposition. We now prove that they cannot differ by opposition.

Let \( x \in A_{\theta}(R) \) and consider an arbitrary twin-apartment \( \Sigma \) on \( x \) and \( x^{\theta} \). Note that \( \text{proj}_{R}^{\theta}(x) \in \Sigma \) and \( \text{proj}_{R}^{\theta}(x^{\theta}) \in \Sigma \). Moreover, since \( x \in A_{\theta}(R) \), the chambers \( \text{proj}_{R}^{\theta}(x) \) and \( x^{\theta} \) are opposite in \( R^{\theta} \cap \Sigma \).

Let \( y = \text{proj}_{\pi}^{\pi}(x^{\theta}) \), where \( \pi \) is the \( j \)-panel on \( x \) in \( R \). Then \( y \in \Sigma \cap R \) and \( l(\delta_{x}(y^{\theta}, y^{\theta})) = l(\delta_{x}(x, x^{\theta}))+2 \) by Lemma 6.23. More precisely, that lemma says that \( y^{\theta} = \text{proj}_{S_{\varepsilon}}^{\pi}(y) \). In particular \( y^{\theta} \in \Sigma \).

Note that \( l(\delta_{x}(x, \text{proj}_{R}^{\theta}(x))) = l(\delta_{x}(y, \text{proj}_{R}^{\theta}(y))) \), but \( l(\delta_{x}(x, x^{\theta})) \neq l(\delta_{x}(y, y^{\theta})) \). Therefore, by definition of projection \( \delta_{-\varepsilon}(\text{proj}_{R}^{\theta}(y), y^{\theta}) \neq \delta_{-\varepsilon}(\text{proj}_{R}^{\theta}(x), x^{\theta}) = w_{\theta}(j) \). Therefore if \( \text{proj}_{R}^{\theta}(y) \) and \( \text{proj}_{R}^{\theta}(x) \) are \( j' \) adjacent, then \( j' \) and \( \theta(j) \) are not opposite.

Proposition 6.33. Assume the terminology of Proposition 6.29. Then, we have the following.

i \( \theta_{S_{\varepsilon}} \) cannot preserve a panel.

ii \( S_{\varepsilon} \) cannot be of type \( A_{1} \);

iii \( S_{\varepsilon} \) cannot be of type \( A_{2} \);

iv if \( S_{\varepsilon} \) has type \( A_{1} \times A_{1} \), then either \( A_{\theta}(S_{\varepsilon}) = S_{\varepsilon} \) or \( \theta_{S_{\varepsilon}} \) interchanges the types;

Proof. Suppose \( \pi \) is an \( i \)-panel that is preserved by \( \theta_{S_{\varepsilon}} \). Thus the bijection \( \text{proj}_{S_{\varepsilon}}^{\pi} : S_{\varepsilon}^{\theta} \to S_{\varepsilon} \) restricts to a bijection between \( \pi^{\theta} \) and \( \pi \). Note that this bijection is \( \text{proj}_{S_{\varepsilon}}^{\pi} \).

However, by Lemma 6.25 we see that there is a chamber \( c_{\varepsilon} \in \pi \) and a \( w \in W^{\theta} \) with the property that \( \delta_{x}(c_{\varepsilon}, c_{\varepsilon}^{\theta}) = s_{\varepsilon}w_{\theta}(i) \) and \( \delta_{x}(d_{\varepsilon}, d_{\varepsilon}^{\theta}) = w \), for all \( d_{\varepsilon} \in \pi - \{c_{\varepsilon}\} \) and \( l(s_{\varepsilon}w_{\theta}(i)) = l(w) + 2 \). From the twin-building axioms it now follows that \( c_{\varepsilon} = \text{proj}_{S_{\varepsilon}}^{\pi}(d_{\varepsilon}^{\theta}) \) for all \( d_{\varepsilon} \in \pi \). Thus, \( \text{proj}_{S_{\varepsilon}}^{\pi} \) is not bijective on \( \pi^{\theta} \), hence neither is \( \text{proj}_{S_{\varepsilon}}^{\pi} \) on \( S_{\varepsilon}^{\theta} \), a contradiction.

Part (ii) follows immediately from (i). To see (iii) note that in this case \( S_{\varepsilon} \) is a projective plane and any automorphism of order 2 necessarily has a fixed point. This fixed point is a panel that is preserved by \( \text{proj}_{S_{\varepsilon}}^{\pi} \), contradicting (i).

(iv) Suppose \( S_{\varepsilon} \) has type \( A_{1} \times A_{1} \). Then, by (i) \( \theta_{S_{\varepsilon}} \) cannot preserve a panel. Therefore if it fixes type, then, \( \theta_{S_{\varepsilon}} \) has no fixed points so that \( A_{\theta}(S_{\varepsilon}) = S_{\varepsilon} \).

□

Lemma 6.34. Assume the terminology of Proposition 6.29 and set \( R = R_{\varepsilon} \) and \( S = S_{\varepsilon} \) for some \( \varepsilon = \pm \). Suppose that \( R \neq S \) and \( S = A_{\theta}(S) \). If \( R \) has rank 2, then \( A_{\theta}(R) \) is connected and if \( R \) has rank 3, then \( A_{\theta}(R) \) is connected and simply connected.

Proof. By Proposition 6.29, \( A_{\theta}(R) \) is the geometry opposite \( S \). Connectedness is proved in [9, Theorem 2.1], [8, Theorem 3.12] [1, Proposition 7]. Now let \( R \) have rank 3. If the
diagram of \( R \) is disconnected, \( A_\theta(R) \) is the product of connected residues of rank \( \leq 2 \), hence it is simply connected. Finally suppose \( R \) has type \( A_3 \). If \( S \) is a chamber then we are done by \([1]\). In view of Proposition 6.33 this leaves the case where \( S \) has type \( A_1 \times A_1 \). Now \( A_\theta(R) \) is the geometry of all points, lines and planes of a projective 3-space that are opposite a fixed line \( l \). That is the points and planes are those not incident to \( l \) and the lines are those not intersecting \( l \). Consider any closed gallery \( \gamma \) in \( A_\theta(R) \). It corresponds to a path of points and lines that all belong to \( A_\theta(R) \). One easily verifies the following: Any two points are on some plane. Hence the collinearity graph \( \Xi \) on the point set of \( A_\theta(R) \) has diameter 2. Any triangle in \( \Xi \) lies on a plane. Given any line \( m \) and two points \( p_1 \) and \( p_2 \) off that line, there is a point \( q \) on \( m \) that is collinear to \( p_1 \) and \( p_2 \) since lines have at least three points. It follows that quadrangles and pentagons in \( \Xi \) can be decomposed into triangles. Since triangles are geometric, that is, there is some object incident to all points and lines of that triangle, \( \gamma \) is null-homotopic.  

\[ \square \]

**Lemma 6.35.** If \( R \) has rank 2, then \( A_\theta(R) \) is connected.

**Proof.** There are two cases: \( R \) has type \( A_2 \) or \( A_1 \times A_1 \). If \( R \) has type \( A_2 \), then by Proposition 6.33, \( S \) is a chamber and so by Lemma 6.34 we are done. Now let \( R \) have type \( A_1 \times A_1 \), then \( S \) is a chamber, in which case we are done again, or it is \( R \). By Proposition 6.33, either \( A_\theta(R) = R \), which is connected, or \( \theta_R \) switches types and \( A_\theta(R) \) is a complete bipartite graph with a perfect matching removed. This is connected since panels have at least three elements.  

\[ \square \]

**Lemma 6.36.** Assume the notation of Proposition 6.29. Suppose that \( R \cong R_1 \times R_2 \) and \( S \cong S_1 \times S_2 \), where typ\((S_i) \subseteq \text{typ}(R_i)\) for \( i = 1, 2 \). Suppose moreover, that \( \theta_S \) preserves the type sets \( I_i \) of the residue \( S_i \) (not necessarily point-wise). Then,

\[
i \ \theta_R = \theta_{R_1} \times \theta_{R_2},
\]

\[
ii \ \ A_\theta(R) \cong A_\theta(R_1) \times A_\theta(R_2).
\]

**Proof.** For \( i = 1, 2 \), let \( J_i = \text{typ}(R_i) \) and let \( I_i = \text{typ}(S_i) \). (i) Note that if, for \( i = 1, 2 \), \( R_i \) is a residue of type \( j_i \) in \( R \) then \( R_i \cap R_2 = \{ c \} \) for some chamber \( c \) and, for any \( x \in R_i \), \( \text{proj}_{R_i}(x) = c \). By assumption on \( S \) the same is true for residues \( S_i \) of type \( I_i \). Note further that the same applies to the residues \( R^\theta \) and \( S^\theta \). Recall now that the isomorphism \( R \cong R_1 \times R_2 \) is given by \( x \mapsto (x_1, x_2) \), where \( x_i = \text{proj}_{R_i}(x) \). Thus in order to prove (i) it suffices to show that

\[ \text{proj}_{R_i} \circ \text{proj}_{R} \circ \theta = \text{proj}_{R_i} \circ \theta \circ \text{proj}_{R_i}. \]

However, note that in fact

\[ \theta_R = \text{proj}_{R} \circ \theta = \text{proj}_{S} \circ \theta, \]

By Lemma 7.3 of \([11]\) we have \( \text{proj}_{S} = \text{proj}_{S} \circ \text{proj}_{S} \circ \theta \) so that

\[ \theta_R = \text{proj}_{S} \circ \theta = \text{proj}_{S} \circ \text{proj}_{S} \circ \theta, \]
and the same holds for $R_i$ and $S_i$. Since $\theta$ is an isomorphism we also have $	ext{proj}_{S'} \circ \theta = \theta \circ \text{proj}_S$, so that
\[
\theta_{R_i} = \text{proj}_{S_i} \circ \text{proj}_{S'} \circ \theta = \text{proj}_{S_i} \circ \theta \circ \text{proj}_{S_i}, \text{ for } i = 1, 2.
\]
Note at this point that $\text{proj}_{S}(x) = \text{proj}_{S}((x_1, x_2)) = (\text{proj}_{S_1} \circ \text{proj}_{R_1}(x)), \text{proj}_{S_2} \circ \text{proj}_{R_2}(x))$. In other words: $\text{proj}_{S} = \text{proj}_{S_1} \times \text{proj}_{S_2} = (\text{proj}_{S_1} \circ \text{proj}_{R_1}, \text{proj}_{S_2} \circ \text{proj}_{R_2})$. Thus in order to prove (i) it suffices to show that
\[
\text{proj}_{S_i} \circ \text{proj}^* \circ \theta \circ \text{proj}_{S} = \text{proj}_{S_i} \circ \theta \circ \text{proj}_{S_i}, \text{ for } i = 1, 2.
\]
This is equivalent to showing that on $S$ we have
\[
\text{proj}_{S_i} \circ \text{proj}^* \circ \theta = \text{proj}_{S_i} \circ \theta \circ \text{proj}_{S_i}, \text{ for } i = 1, 2.
\]
To see this, first pick some $x \in S$ and note that if $x$ lies on the $I_2$-residue $S''_2$, then $x, \text{proj}_{S_1}(x) \in S''_2$, thus $\theta(x), \theta \circ \text{proj}_{S_1}(x) \in S''_2$. But since $\theta_S$ is type-preserving, we have $\text{proj}^*_{S} \circ \theta(x), \text{proj}^*_{S} \circ \theta \circ \text{proj}_{S_1}(x) \in \text{proj}^*_{S}(S''_2) = S''_1$, and $S''_1$ is again of type $I_2$. Therefore, the projection on $S_1$ of these two chambers is the same, namely $S_1 \cap S''_1$. Namely, $\text{proj}_{S_1} \circ \text{proj}^*_{S} \circ \theta(x) = \text{proj}_{S_1} \circ \text{proj}^*_{S} \circ \theta \circ \text{proj}_{S_1}(x) = S_1 \cap S''_1$. Noting that $\text{proj}_{S_1} \circ \text{proj}^*_{S}(y) = \text{proj}_{S_1}^*\circ \delta_{i}(x, \theta_R(x))$ for any $y \in S''$, we have $\text{proj}_{S_1} \circ \text{proj}^*_{S} \circ \theta(x) = (\text{proj}_{S_1} \circ \text{proj}^*_{S}) \circ \theta \circ \text{proj}_{S_1}(x) = \text{proj}_{S_1} \circ \theta \circ \text{proj}_{S_1}(x)$, that is, $\text{proj}_{S_i} \circ \theta \circ \text{proj}_{S_1}(x)$, which proves the claim.

(ii) Let $x = (x_1, x_2) \in R_1 \times R_2$, and suppose $R \subseteq \Delta_i$. Then, by (i),
\[
\delta_{i}(x, x^\theta) = \delta((x_1, x_2), \theta_{R_1}(x_1, x_2)) = \delta((x_1, x_2), \theta_{R_1}(x_1, x_2)) = \delta_1((x_1, x_2), \theta_{R_1}(x_1, x_2)) = \delta_1((x_1, x_2), \theta_{R_1}(x_1, x_2)) = \delta_2(x_1, \theta_{R_1}(x_1)) \cdot \delta_2(x_2, \theta_{R_1}(x_2)).
\]
Since $A_{\theta}(R_1) \times A_{\theta}(R_2) \subseteq R_1 \times R_2$, we see that $\delta(x, \theta_{R_1}(x))$ is maximal if and only if $\delta(x_i, \theta_{R_i}(x_i))$ is maximal for $i = 1, 2$. Thus $A_{\theta}(R) \cong A_{\theta}(R_1) \times A_{\theta}(R_2)$. \hfill $\Box$

**Theorem 6.37.** Suppose that $|k| \geq 5$. If $R$ has rank 3, then $A_{\theta}(R)$ is connected and simply 2-connected.

**Proof.** The residue $R$ has one of three possible types: $A_3$, $A_2 \times A_1$, or $A_1 \times A_1 \times A_1$. In view of Lemma 6.34 we will ignore the cases where $S = A_{\theta}(S)$ is a proper residue of $R$.

Since $S$ is a residue, but not a chamber, a panel, or a residue of type $A_2$, and $S \neq A_{\theta}(S)$, it follows from Proposition 6.33 that either $R = S$ or $S$ has type $A_1 \times A_1$ and $\theta_S$ switches types on $S$. The theorem will now follow from Lemmas 6.38 and 6.39. \hfill $\Box$

**Lemma 6.38.** If $R$ has disconnected diagram of rank 3, then $A_{\theta}(R)$ is connected and simply connected.

**Proof.** We show that in all cases Lemma 6.36 applies. If $R$ has type $A_1 \times A_1 \times A_1$, then let $\theta$ act on the types of $R$. It either fixes all types or it has two orbits $I_1$ and $I_2$, where we may assume $|I_2| = 2$. Moreover, if $S$ has type $A_1 \times A_1$, then we can write $S \cong S_1 \times S_2$, where $S_1 = \{e\} \subseteq R_1$, $S_2 = R_2$ and $R_i$ has type $I_i$, for $i = 1, 2$. If $R = S$, then we can take $S_i = R_i$, where $R_i$ as above. One verifies that Lemma 6.36 applies.
We now turn to the case, where $R$ has type $A_2 \times A_1$. Let $J_i$ be the underlying type set of type $A_i$. Since $\theta$ is an adjacency preserving permutation of $R$ of order 2, it must preserve the type sets $J_1$ and $J_2$. In particular if $S$ has type $A_1 \times A_1$, $\theta_S$ must be type preserving. Take $R_i$ to be a residue on $c \in R$ of type $J_i$. Let $S_1 = R_1$ and let $S_2 = S \cap R_2$. Now again Lemma 6.36 applies.

By Lemma 6.36, $A_\theta(R) \cong A_\theta(R_1) \times A_\theta(R_2)$. By Lemma 6.35, $A_\theta(R_i)$ is connected, hence $A_\theta(R)$ is connected and simply connected. □

Lemma 6.39. If $R$ is of type $A_3$ and $|k| \geq 5$ then the geometry $A_\theta(R)$ is connected and simply connected.

Proof. Case 1: $S = R$. By Lemma 6.32, $\theta_R$ is given by an involutory semilinear map $\phi$ on a 4-dimensional vector space. Since $S = R$, we also know that $\phi$ has no fixed points. We now define the objects of the geometry $A_\theta(R)$. All points and all planes of PG$(V)$ belong to $A_\theta(R)$. The only lines in the geometry are those 2-dimensional spaces of $V$ that are not fixed by $\phi$. These will be called good lines. Points will be denoted by lowercase letters, good lines will be denoted by uppercase letters and planes will be denoted by greek letters.

We now describe incidence. We shall use containment relations only for containment in PG$(V)$, not to be confused with incidence in $A_\theta(R)$. Any point contained in a good line will be incident to it and any plane containing a good line will be incident to it. A point $p$ will be incident to a plane $\pi$ if and only if $p \subseteq \pi$ and $p \not\subseteq \pi^\phi$.

We now gather some basic properties of $A_\theta(R)$. Any plane $\pi$ is incident to any point $p$ that is not contained in the only bad line $\pi \cap \pi^\phi$ of $\pi$. It follows that any two points incident to a plane will be collinear. and any point $p$ is incident to all planes $\pi$ so that $p \subseteq \pi$ but $\pi$ does not contain the only bad line $\langle p, p^\phi \rangle$ containing $p$. If a line $L$ is incident to a plane $\pi$, then all but one point incident to $L$ is incident to $\pi$.

Connectivity is quite immediate since any two points $p_1, p_2$ that are not collinear will be collinear to any other point not in the unique bad line $\langle p_1, p_2 \rangle$ on $p_1$ (and $p_2$).

In order to prove simple connectivity we first reduce any path to a path in the collinearity graph. Indeed any path $p_1\pi_2$ will be homotopically equivalent to the path $p_1Lp_2$ where $L = \langle p_1, p_2 \rangle$. Any path $p\pi L$ will be homotopically equivalent to the path $pL'p\pi L$ where $p'$ is a point on $L'$ that is also incident to $\pi$ and $L' = \langle p, p' \rangle$. Note that since $p'$ is incident to $\pi$, $L'$ is a good line. Finally a path $L_1\pi L_2$ is homotopically equivalent to the path $L_1p_1L'p_2L_2$ where $p_i$ are points on $L_i$ that are incident to $\pi$ and $L' = \langle p_1, p_2 \rangle$.

Therefore, to show simple connectedness we can restrict to paths in the collinearity graph. Note also the fact that if $p$ is a point and $L$ is a good line not incident to $p$ then $p$ will be collinear to all but at most one point on $L$ (namely the intersection of the unique bad line on $p$ and $L'$ if this intersection exists). This enables the decomposition of any path in the collinearity graph to triangles. Indeed, the diameter of the collinearity graph is two and so any path can be decomposed into triangles, quadrangles and pentagons. Moreover, if $p_1, p_2, p_3, p_4$ is a quadrangle then, since $|k| \geq 4$, the line $\langle p_2, p_3 \rangle$ will admit a point collinear to both $p_1$ and $p_4$ decomposing the quadrangle into triangles. Similarly, if $p_1, p_2, p_3, p_4, p_5$ is a pentagon, then there will be a point on the good line $\langle p_3, p_4 \rangle$ that is
collinear to \( p_1 \). Thus, the pentagon decomposes into quadrangles. Therefore it suffices to decompose triangles into geometric triangles.

Assume that \( p_1, p_2, p_3 \) is a triangle. The plane \( \pi = \langle p_1, p_2, p_3 \rangle \) is incident to all three (good) lines in the triangle and so, either the triangle is geometric and then we are done, or one of the points is not incident to \( \pi \). Let us assume that \( p_1 \) is not incident to \( \pi \).

Consider a plane \( \pi' \) that contains the line \( \langle p_2, p_3 \rangle \) and so that \( p_2 \) and \( p_3 \) are incident to \( \pi' \). This is certainly possible since \(|k| \geq 4\) and one only need to stay clear of the planes \( \langle p_2, p_3, p_3^\phi \rangle \) and \( \langle p_2, p_3, p_2^\phi \rangle \). Note that by choice of \( \pi' \), any line \( L \) with \( p_i \subseteq L \subseteq \pi' \) is good.

Let now for each \( i = 2, 3 \)

\[
L_i = \{ L \subseteq \pi' \mid L \not\subseteq \langle p_2, p_3 \rangle, p_i \subseteq L, \text{ and } p_i \text{ is incident to the plane } \langle p_1, p_i, L \rangle \}.
\]

We have \( L_i = |k| - 1 \), the only lines of \( \pi' \) on \( p_i \) not in \( L_i \) are \( \langle p_2, p_3 \rangle \) and \( \langle p_1, p_i, p_i^\phi \rangle \cap \pi' \). Note that if \( L \in L_i \) then \( L \) only admits one point not incident to \( \pi' \). Pick lines distinct lines \( L_{i,j} \in L_i \) with \( j = 1, 2, 3 \). Of the 9 intersection points at most 6 are not incident to one of the three planes that they define. Pick any one of the remaining 3 points and use it as the point \( p \) above.

Case 2: \( S \) of type \( A_1 \times A_1 \). The geometry is rather similar to the previous one. There is a line \( L \) so that \( S \) is the residue corresponding to \( L \) and the map \( \theta_S \) induces a pairing between points of \( L \) and planes on \( L \). The geometry \( A_\theta(R) \) is described as follows. The points of the geometry are all the points of \( V \) not in \( L \), the lines of the geometry are all the lines of \( V \) not intersecting \( L \) and the planes are all planes of \( V \) not containing \( L \).

We now describe incidence. Any line included in a plane is incident to it and any point included in a line is incident to it. A point \( p \) is incident to a plane \( \pi \) if and only if the plane \( \pi' = \langle p, L \rangle \) is not paired to the point \( p' = L \cap \pi \).

We now gather a few useful properties of this geometry. Note a number of similarities with the previous geometry. Any plane \( \pi \) is incident to all the points \( p \subseteq \pi \) so that \( p \) is not contained in the bad line \( \pi' \cap \pi \) where \( \pi' \) is the plane paired to the point \( \pi \cap L \). Similarly any point \( p \) is incident to any plane \( \pi \) if \( p \subseteq \pi \) and \( \langle p, p' \rangle \not\subseteq \pi \) where \( p' \) is the point paired to the plane \( \langle p, L \rangle \). If \( p \) is a point and \( L \) is a good line not incident to \( p \) then \( p \) will be collinear to all but one point on \( L \); namely the non-collinear point on \( L \) is the intersection of \( L \) with the bad plane \( \langle p, L \rangle \).

Any two points \( p_1, p_2 \) that are not collinear have the property that \( \langle p_1, p_2 \rangle \) intersects \( L \) and so any point not in \( \langle p_1, p_2, L \rangle \) will be collinear to both \( p_1 \) and \( p_2 \). In particular, the geometry \( A_\theta(R) \) is connected and the diameter of the collinearity graph is 2.

The reduction to the collinearity graph is a little more involved because not every two points on a good plane will be collinear. However any two non-collinear points incident to a good plane \( \pi \) are collinear to any other point incident to \( \pi \) since \( L \) intersects \( \pi \) in exactly one point.

The previous remark immediately shows that a path of type \( p_1 \pi p_2 \) can be replaced by a path \( p_1, L_1, p', L_2, p_2 \), where all elements are incident to \( \pi \). Suppose we have a path of type \( p \pi L \). Since \( \pi \) is incident to all but one point on the line \( L \) and \( p \) is collinear to all but one point on the line \( L \), we can replace this path by one of type \( p_1 L_1 p_2 L \), where all objects are
incident to \( \pi \). Suppose we have a path of type \( L_1 \pi L_2 \). This reduces to the previous case since all but one point of \( L_1 \) are incident to \( \pi \).

As before, given any line \( L \) and two points \( p_1 \) and \( p_2 \) not on \( L \), there are only two points on \( L \) that are not collinear to at least one of \( p_1 \) and \( p_2 \). The proof that all paths in the collinearity graph decompose into triangles is identical. Therefore it suffices to show that any triangle decomposes into geometric triangles.

Finally we need to modify the argument above to decompose triangles. The only difference is once more the fact that not every two points incident to a good plane are collinear. As a consequence the sets \( L_i \) only have \(|k| - 2\) lines because one needs to exclude the space \( \langle p_i, \pi' \cap L \rangle \). Moreover, each line of \( L_i \) has three forbidden points. Namely, in addition to the two as in the previous case, it has one point that is not collinear to \( p_1 \) since it lies on the plane \( \langle p_1, L \rangle \). If \(|k| \geq 5\), then we can choose four lines from \( L_2 \) and \( L_3 \) and see that out of the 16 intersection points at most 9 are bad. Pick any one of those remaining points \( p \) and notice that it is collinear to all of the \( p_i \) and \( pp_i \) is a geometric triangle for all \( i \neq j \).

This decomposes the initial triangle into geometric triangles. \( \square \)

For \( \theta \)-invariant \( J \subset I \) let \( B_J \) be the stabilizer in \( G^\theta \) of \( (R_J, R_J^\theta) \subseteq \Delta^\theta \), where \( R_J \) is the \( J \)-residue of \( \Delta_+ \) on \( c_+ \). Let \( B = \{ B_J \mid J \subset I, J^\theta = J \} \) and note that this is the amalgam of maximal parabolic subgroups of \( G^\theta \) for the action on \( \Delta^\theta \). For \( k \in \{1, \ldots, n\} \) write \( B_k = B_{I-\{k, \theta(k)\}} \).

**Lemma 6.40.** The group \( G^\theta \) is the universal completion of the amalgam \( B \).

**Proof.** By Lemma 6.35 and Theorem 6.37 the residual filtration \( C \) satisfies the conditions of Theorem 6.20. It follows that \( (\Delta^\theta, \sim) \) is connected and simply connected and by Lemma 6.19, so is \( (\Delta^\theta, \approx) \). Therefore, by Tits’ Lemma [39, Corollaire 1], \( G^\theta \) is the universal completion of the amalgam \( B \). \( \square \)

In order to prove Theorem 3 we shall have to prove that the universal completion of \( L^\tau \) equals the universal completion of \( B \). To this end we set up some notation.

Recall that \( L^\tau = \{ L_i, L_{ij} \mid i, j \in \{1, 2, \ldots, n\} \} \) and that \( D_i \) denotes the diagonal torus of \( L_i \leq G^\theta \). For \( \theta \)-invariant \( J \subset I \) let \( A_J = \langle L_i, L_{ij} \mid i, j \in J \cap \{1, \ldots, n\} \rangle_{G^\theta} \). For \( i, j \in \{1, 2, \ldots, n\} \) we shall write \( A_i = A_{\{i, \theta(i)\}}, A_i = A_{I-\{i, \theta(i)\}}, \) and \( A_{ij} = A_{\{i, j, \theta(i), \theta(j)\}} \). For any \( k \in \{1, \ldots, n\} \), let

\[
A_k = \{ A_i, A_{ij} \mid i, j \in \{1, \ldots, n\} - \{k\} \}
\]

**Lemma 6.41.** For each \( k \in \{1, \ldots, n\} \) the universal completion of \( A_k \) is \( A_k \cong \text{SL}_n(k) \).

**Proof.** This follows from the Curtis-Tits theorem. \( \square \)

Let \( \tilde{G} \) be the universal completion of \( L^\tau \).
Lemma 6.42. There exists a unique surjective homomorphism $\Phi: \tilde{G} \to G^\theta$.

Proof. By construction, $G^\theta$ contains a copy of $L^\tau$. Moreover, for each $k \in \{1, \ldots, n\}$ the subgroup $\langle A_k, D_k \rangle \leq G^\theta$ equals the maximal parabolic group $B_k$. Hence by Lemma 6.40 the amalgam $L^\tau$ generates $G^\tau$. The existence of $\Phi$ follows from the universality of $\tilde{G}$.

Lemma 6.43. For each $k \in \{1, \ldots, n\}$ there exists a subgroup $\tilde{A}_k \leq \tilde{G}$ such that $\Phi: \tilde{A}_k \to A_k$ is an isomorphism.

Proof. Let $\tilde{A}_k$ be the subgroup of $\tilde{G}$ generated by the amalgam $A_k \subseteq \tilde{G}$. By Lemma 6.41 there is a unique surjective homomorphism $A_k \to \tilde{A}_k$. The composition of this homomorphism with $\Phi: \tilde{A}_k \to A_k$ is the identity on $A_k$ so by universality property of $A_k$ this is the identity map.

Proof. (of Theorem 3) By Lemma 6.42 we have a surjective homomorphism $\Phi: \tilde{G} \to G^\theta$. We shall now prove that it has an inverse by showing that $\tilde{G}$ contains a copy of $B$. The result then follows from Lemma 6.40.

Let $D = \langle D_i \mid i \in \{1, \ldots, n\} \rangle \leq G^\theta$. Note that $D \cong D_1 \times \cdots \times D_n$ and that $B_J = A_J D$ (product of subgroups). In fact $B_J$ is a quotient of $A_J \times D$, with respect to conjugation in $G^\theta$.

We shall first reconstruct a copy, called $\tilde{D}$, of $D$ inside $\tilde{G}$. We adopt the tilde convention, so that for any element $x$ or subgroup $H$ from $A$, $\tilde{x}$ and $\tilde{H}$ denote their images in $\tilde{G}$. Notice that $D$ is not a subgroup of the amalgam $A$, and consequently we cannot use this convention to define $\tilde{D}$. Therefore we define it indirectly as follows. Let $\tilde{D}$ be equal to the product of all the subgroups $\tilde{D}_i$ (notice that $D_i \subseteq A_i \in A$ and hence $\tilde{D}_i$ is defined). We claim that there exists an isomorphism $D \to \tilde{D}$ extending the isomorphisms $D_i \to \tilde{D}_i$. Indeed $D_i$ and $D_j$ are both contained in $A_{i,j}$ and they commute element-wise. Therefore $\tilde{D}_i$ and $\tilde{D}_j$ also commute element-wise. This proves that there exists a surjective homomorphism $\phi: \Pi_{i=1}^n \tilde{D}_i \to \tilde{D}$. However, we also have a canonical isomorphism of abstract groups $\gamma: \Pi_{i=1}^n D_i \to \Pi_{i=1}^n \tilde{D}_i$. Here the former group is $D$. Now the composition $\phi \circ \gamma$ is a surjective homomorphism, which restricts to an isomorphism $\tilde{D}_i \to \tilde{D}_i$ for each $i$. The restriction $\Phi: \tilde{D} \to D$ is surjective and has $\phi \circ \gamma$ as its inverse. This proves our claim.

In a similar spirit, for $\theta$-invariant $J \subseteq I$, define $\tilde{B}_J$ to be the product of the subgroups $\tilde{A}_J$ with $\tilde{D}$. For this definition to make sense, we must show that every $\tilde{D}_i$ normalizes $\tilde{A}_J$. Note that $A_J$ is generated by the subgroups $A_j$ with $j \in J \cap \{1, \ldots, n\}$. Inside $A_{i,j}$ we see that $D_i$ normalizes $A_i$ and $A_j$. Hence $\tilde{D}_i$ normalizes every $\tilde{A}_J$, implying that $\tilde{D}_i$ normalizes $\tilde{A}_J$. Thus the subgroups $\tilde{B}_J$ are well defined.

We claim that with respect to the natural homomorphism $\Phi: \tilde{G} \to G^\theta$, $\tilde{B}_J$ maps isomorphically onto the group $B_J$. This map is clearly surjective. Since $\tilde{D}$ normalizes $\tilde{B}_J$, there is a surjective homomorphism $A_J \times \tilde{D} \to \tilde{B}_J$, whose kernel is $K = \{(a, a^{-1}) \in \tilde{A}_J \times \tilde{D} \mid a \in \tilde{A}_J \cap \tilde{D}\}$. Similarly, there is a surjective homomorphism $A_J \times D \to B_J$. 


whose kernel is \( K = \{(a, a^{-1}) \in A_J \times D \mid a \in A_J \cap D\} \). The natural homomorphism \( \Phi \) now induces a surjective homomorphism \((\tilde{A}_J \times \tilde{D})/K \rightarrow (A_J \times D)/K\). Now note that \( A_J \cap D = \langle D_i \mid i \in J \cap \{1, \ldots, n\}\rangle \). Moreover, \( \tilde{A}_J \cap \tilde{D} = \langle \tilde{D}_i \mid i \in J \cap \{1, \ldots, n\}\rangle \). This shows that the natural homomorphism is in fact injective. \( \square \)

7. APPLICATIONS AND CONNECTIONS

7.1. the orientable curtis-tits groups \( SL_n(A) \). Consider the ring \( R = k\{t, t^{-1}\} \) of skew Laurent polynomials. More precisely if \( x \in k \), then \( t^{-1}xt = x^\delta \) for some fixed automorphism \( \delta \) of \( k \).

In Section 5 we constructed a group \( G \leq GL_n(R) \) that is a completion of the Curtis-Tits amalgam \( A^\delta \) corresponding to the automorphism \( \delta \) (see Definition 1.1). Moreover, for \( \delta \) of finite order we showed that \( G \) can be regarded as \( SL_n(R) \), for a suitable definition of a determinant \( \det_R \), and that it is the universal completion of \( A^\delta \) and hence is a Kac-Moody group.

Let \( \Delta = (\langle \Delta_+, \delta_+ \rangle, \langle \Delta_-, \delta_- \rangle, \delta_e) \) be the twin-building associated to the Kac-Moody group \( G \). Then, the pairs of maximal residues from \( \Delta_+ \) and \( \Delta_- \) that are opposite for the twinning correspond to vector bundles over the non-commutative projective line \( \mathbb{P}^1(\delta) \) in the sense of Drinfel’d. More precisely, let \( k\{t\}, k\{t^{-1}\} \leq k\{t, t^{-1}\} \) be the corresponding skew polynomial rings and fix \( M \) a free \( k\{t, t^{-1}\} \)-module of rank \( r \). Following [21] and [36] one can define a rank \( r \) vector bundle over the non-commutative projective line \( \mathbb{P}^1(\delta) \) as a collection \((M_+, M_-, \phi_+, \phi_-)\) where \( M_\varepsilon \) is a free \( r \)-dimensional module over \( k\{t^\varepsilon\} \) and \( \phi_\varepsilon : M_\varepsilon \otimes k\{t, t^{-1}\} \rightarrow M \) is an isomorphism of \( k\{t, t^{-1}\} \)-modules. By analogy to the commutative case (see [31, 32] for example) one can describe the building structure in terms of these vector bundles. We intend to explore these relations to number theory in a future paper.

To give a different perspective on these groups we note that the skew Laurent polynomials are closely related to cyclic algebras as defined by Dickson. More precisely let \( k' \leq k \) be a cyclic field extension, of degree \( n \), and let \( \delta \) be the generator of its Galois group. Given any \( a \in k' \), define the \( k' \)-algebra \( (k/k', \delta, a) \) to be generated by the elements of \( k \), viewed as an extension of \( k' \), together with some element \( u \) subject to the following relations:

\[
u^n = a, xu = ux^\delta \quad \text{for} \quad x \in k.
\]

These algebras are central simple algebras. The celebrated Brauer-Hasse-Noether theorem states that every central division algebra over a number field \( k' \) is isomorphic to \((k/k', \delta, a)\) for some \( k, a, \delta \).

For each \( a \in k' \) one constructs the map \( \epsilon_a : k\{t\} \rightarrow (k/k', \delta, a) \) via \( t \mapsto u \). This induces a map \( \epsilon_a : SL_n(R) \rightarrow SL_n((k/k', \delta, a)) \), realizing the linear groups over cyclic algebras as completions of the Curtis-Tits amalgams.

7.2. the non-orientable groups \( G^\tau \). Let \( V \) be a free \( A \)-module of rank \( 2n \) with basis \( \{e_i, f_i \mid i = 1, \ldots, n\} \). In this case \( k\{t, t^{-1}\} \) denotes the ring of commutative Laurent polynomials in the variable \( t \) over a field \( k \). The group \( G^\tau \) is the isometry group of the unique non-symmetric \( \sigma \)-sesquilinear form \( \beta \) on \( V \) with the property that \( \beta(e_i, e_j) = \beta(f_i, f_j) = \)
0, \beta(e_i, f_j) = t \delta_{ij} and \beta(f_i, e_j) = \delta_{ij} where \sigma: k[t, t^{-1}] \to k[t, t^{-1}] is the identity on k and interchanges t and t^{-1}. More precisely

\[ G^\tau := \{ g \in \text{SL}_2n(k[t, t^{-1}]) | \forall x, y \in V, \beta(gx, gy) = \beta(x, y) \} \]

In Section 6 we proved that \( G^\tau \) is the Curtis-Tits group corresponding to the element \( \tau \) from Theorem 1.

It turns out that the group \( G^\tau \) has some very interesting natural quotients and that its action on certain Clifford-like algebras are related to phenomena in quantum physics.

Let \( \bar{k} \) denote the algebraic closure of \( k \). For any \( a \in \bar{k}^* \) consider the specialization map \( \epsilon_a: k[t, t^{-1}] \to \bar{k} \) given by \( \epsilon_a(f) = f(a) \). The map induces a homomorphism \( \epsilon_a: \text{SL}_2n(k[t, t^{-1}]) \to \text{SL}_2n(k(a)) \). In some instances the map commutes with the automorphism \( \sigma \) and so one can define a map \( \epsilon_a: G^\tau \to \text{SL}_2n(\bar{k}) \).

The most important specialization maps are those given by evaluating \( t \) at \( a = \pm 1 \) or \( a = \zeta \), a \((q^n + 1)-\)st root of 1 where \( q \) is a power of the characteristic.

Consider first \( a = -1 \). In this case the automorphism \( \sigma \) becomes trivial. Note that for \( g \in G^\tau \) we have \( \epsilon_{-1}(g) \in \text{Sp}_{2n}(k) \). In this case, the image of the group \( G^\tau \) is the group generated by the Curtis-Tits amalgam \( \mathcal{A}^\tau \) inside \( \text{Sp}_{2n}(k) \). Preliminary studies suggest that we have equality. Similarly, if \( a = 1 \), the automorphism \( \sigma \) is trivial and the map \( \epsilon_1 \) takes \( G^\tau \) into \( \text{SO}_{2n}^+(k) \). Preliminary results suggest that in fact the image of this map is \( \Omega_{2n}^+(k) \).

Finally assume that \( k = \mathbb{F}_q \) and \( a \in \mathbb{F}_q \) is a primitive \((q + 1)-\)st root of 1. The \( \mathbb{F}_q \)-linear map \( \mathbb{F}_q(a) \to \mathbb{F}_q(a) \) induced by \( \sigma \) sends \( a \) to \( a^{-1} \). Thus, \( \sigma \) coincides with the Frobenius automorphism of the field \( \mathbb{F}_q(a) = \mathbb{F}_{q^2} \). It is easy to verify that a change of coordinates \( e_i' = e_i \) and \( f_i' = a f_i \) where \( b^2 = a \) standardizes the Gram matrix of \( \beta \circ (\epsilon_a \times \epsilon_a) \) to a hermitian one, thus identifying the image of \( \epsilon_a \) with a subgroup of a conjugate of the unitary group \( \text{SU}_{2n}(q) \). Again, preliminary results suggest that in fact the image of this map is isomorphic to \( \text{SU}_{2n}(q) \). This easily generalizes to the case where \( a \) is a \((q^n + 1)-\)st root of unity and indeed to other cases where \( a \) is Galois-conjugate to \( a^{-1} \).

An intriguing connection comes from mathematical physics, where the form \( \beta \) has been considered in the context of \( q \)-CCR algebras (see for example [19, 7]). The related infinite dimensional Clifford algebra is a higher GK-dimensional version of Manin’s quantum plane. This algebra is related to both the Clifford algebra of the orthogonal groups and the Heisenberg algebra for the symplectic groups in a similar fashion.

These applications will be discussed in more detail in an upcoming paper.

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