MODIFIED SPECTRAL PRP CONJUGATE GRADIENT METHOD FOR SOLVING TENSOR EIGENVALUE COMPLEMENTARITY PROBLEMS

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Abstract. Tensor eigenvalue complementary problems, as a special class of complementary problems, are the generalization of matrix eigenvalue complementarity problems in higher-order. In recent years, tensor eigenvalue complementarity problems have been studied extensively. The research fields of tensor eigenvalue complementarity problems mainly focus on analysis of the theory and algorithms. In this paper, we investigate the solution method for four kinds of tensor eigenvalue complementarity problems with different structures. By utilizing an equivalence relation to unconstrained optimization problems, we propose a modified spectral PRP conjugate gradient method to solve the tensor eigenvalue complementarity problems. Under mild conditions, the global convergence of the given method is also established. Finally, we give related numerical experiments and numerical results compared with inexact Levenberg-Marquardt method, numerical results show the efficiency of the proposed method and also verify our theoretical results.

1. Introduction. Tensors are higher dimensional forms of arrays. Due to the widespread applications in data analysis, biological information, image and signal protecting problem, tensor optimization has been developed rapidly (such as [8, 13, 14, 27, 28, 29, 30, 33, 34, 38, 40, 46]). A real mth order n-dimensional tensor

\[ A = (a_{i_1...i_m}), \quad i_j \in [n], \quad j \in [m] \]

is an multi-array with \( a_{i_1...i_m} \), where \([n] = \{1, 2, \ldots, n\}, \quad [m] = \{1, 2, \ldots, m\} \). Note that \( A \) is an n-order square matrix when \( m = 2 \). The set of all real mth order n-dimensional tensor is denoted as \( T_{m,n} \). \( A \) is called a symmetric tensor if its entries \( a_{i_1...i_m} \) are invariant under any permutation of its indices. A \((d+1)th\) n-dimensional tensor \( A \) is called semi-symmetric tensor if its entries are invariant under any permutation of their backward \( d \) indices, i.e.,

\[ a_{i_0i_1i_2...i_d} = a_{i_0i_1i_2...i_d}, \]

where \( 0 \leq i_0 \leq n, j_1 \cdots j_d \) is any permutation of \( i_1 \cdots i_d, 0 \leq i_1, \cdots, i_d \leq n \). The set of all real mth order n-dimensional symmetric tensor is denoted as \( S_{m,n} \). For a
tensor $A \in T_{m,n}$ and a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $Ax^{m-1}$ and $x^{[m-1]}$ are $n$-dimensional vectors with the $i$th component defined by

$$(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^{n} a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. $Ax^m$ is a value at $x$ of a homogeneous polynomial defined by

$$Ax^m = \sum_{i_1, \ldots, i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} \cdots x_{i_m}.$$

$Ax^{m-2}$ is a matrix in $\mathbb{R}^{n \times n}$ with its $(i, j)$-th component defined by

$$(Ax^{m-2})_{i,j} = \sum_{i_3, \ldots, i_m=1}^{n} a_{i_3 i_4 \ldots i_m} x_{i_3} \cdots x_{i_m}.$$

In the past decade, many scholars have studied tensor and obtained abundant achievements. For instance, various tensors with special structures were given in \([13, 29, 30, 33, 46]\), including copositive tensors, $M$ tensors, $P$-tensors and positive-definite tensors. On the other hand, many kinds of tensor optimization problem have been proposed, such as tensor complementarity problems (TCP) in \([3, 4, 14, 15, 17, 18, 31, 35, 36, 38, 39, 47, 50]\), tensor eigenvalue problems (TEiP) in \([7, 19, 25, 41, 43]\) and tensor eigenvalue complementarity problems (TEiCP) in \([9, 10, 16, 21, 22, 44]\). As an important special case of complementarity problems, tensor eigenvalue complementarity problems have been developing rapidly since the past decades. The research on TEiCP mainly includes theory and algorithms. In the theoretical research work, different mathematical models of TEiCP are proposed and the existence and properties of solution set for TEiCP are studied. For example, in \([21]\), Ling, He and Qi introduced tensor generalized eigenvalue complementarity problem (TGEiCP), i.e. given $A, B \in T_{m,n}$, the TGEiCP($A, B$) is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

$$x \geq 0, \quad (\lambda B - A)x^{m-1} \geq 0, \quad x^T(\lambda Bx^{m-1} - Ax^{m-1}) = 0. \quad (1)$$

The solution $(\lambda, x)$ of TGEiCP($A, B$) \((1)\) is called a Pareto-eigenpair of ($A, B$), where the scale $\lambda$ is called a Pareto-eigenvalue of ($A, B$) and the corresponding vector $x$ is called a Pareto-eigenvector of ($A, B$). The authors gave the existence conditions of solution, the boundary of Pareto-eigenvalue number, the optimal transformation form of TGEiCP ($A, B$)\((1)\) and proposed a projection algorithm for solving it. Inspired by quadratic eigenvalue complementarity problem (QEiCP) proposed in \([32]\), Tensor higher-degree eigenvalue complementarity problems (THDEiCP) beyond the framework of TGEiCP and QEiCP is also proposed in \([22]\). THDEiCP($A, B, C$) is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

$$x \geq 0, \quad (\lambda^m A + \lambda B + C)x^{m-1} \geq 0, \quad x^T(\lambda^m A + \lambda B + C)x^{m-1} = 0, \quad (2)$$

where $A, B, C \in T_{m,n}$. The scalar $\lambda$ and the nonzero vector $x$ satisfying system \((2)\) are respectively called an $m$-degree Pareto-eigenvalue and an associated Pareto-eigenvector of ($A, B, C$). The existence condition of the solution set of problem \((2)\) is given under general condition in \([22]\). Especially, the authors proved that THDEiCP($A, B, C$) \((2)\) can reformulate a weakly coupled homogeneous polynomial optimization problem under the assumption that tensors are symmetric. In \([42]\), Yan and Ling introduced quadratic eigenvalue complementarity problems of tensor
on second-order cone. Based on [42], we consider the following eigenvalue complementarity problem, called tensor quadratic eigenvalue complementarity problem (TQEiCP), which is to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ such that

\[
\begin{cases}
(\lambda^2 A + \lambda B + C)x^{m-1} \geq 0, \\
x \geq 0, \\
x^T(\lambda^2 A + \lambda B + C)x^{m-1} = 0,
\end{cases}
\]

where $A, B, C \in T_{m,n}$. The solution $(\lambda, x)$ of TQEiCP (3) is called a tensor quadratic complementarity eigenpair of $(A, B, C)$, where the scale $\lambda$ is called a tensor quadratic complementarity eigenvalue of $(A, B, C)$ and the vector $x$ is called a tensor quadratic complementarity eigenvector of $(A, B, C)$ corresponding to $\lambda$. TQEiCP($A, B, C$) is called symmetric when $A, B, C$ are symmetric tensors. In [23], the authors proved the equivalent relationship between EiCP-TM with $\lambda > 0$ and an optimization problem and also gave a sufficient and necessary condition of the solution of EiCP-TM under symmetric conditions. Combined with the eigenvalue complementarity problem for tensor and matrix(EiCP-TM) in [23], we consider a new kind of EiCP-TM, defined as finding $\lambda \in \mathbb{R}, x \neq 0$ such that

\[
\begin{cases}
\lambda x - Ax^3 - Bx \geq 0, \\
x \geq 0, \\
x^T(\lambda x - Ax^3 - Bx) = 0, \\
x^T x = 1,
\end{cases}
\]

where $A \in T_{4,n}, B \in \mathbb{R}^{n \times n}$. When $\lambda > 0$, EiCP-TM (4) is the problem studied in [23].

From above statement, we know that many theoretical results of tensor eigenvalue complementarity problems have been obtained. On the other hand, there are also many solution methods proposed for eigenvalue complementarity problems and tensor eigenvalue complementarity problems, such as Lattice projection method (LPM) [1], implementable splitting method [22], Lasserre’s hierarchy of semidefinite relaxations [16], semi-smooth Newton method [10], spectral projected gradient method [44], shifted projected power method [23] and inexact Levenberg-Marquardt method [20]. As far as we know, the nonlinear conjugate gradient methods are not used to solve tensor eigenvalue complementarity problems. The conjugate gradient methods are very efficient for solving large-scale optimization problems because of their simplicity and low storage (one can see [2, 5, 6, 11, 12, 24, 37, 45, 48, 49]. And the PRP conjugate gradient method is widely used to solve the unconstrained optimization problem [5, 37]. So, in this paper, we propose a new modified spectral PRP conjugate gradient method to solve the tensor eigenvalue complementarity problems. The Armijo line search globalization technique [48] is also used in our proposed method.

The structure of this paper is organized as follows. In Section 2, we propose equivalent transformation forms of tensor eigenvalue complementarity problems. Based on equivalent forms, we give a new modified spectral PRP conjugate gradient method with global convergence for solving tensor eigenvalue complementarity problems in Section 3. In Section 4, we give relevant numerical experiments and compare our proposed method with inexact Levenberg-Marquardt method. The numerical results show the effectiveness of our proposed method. Finally, we complete our paper with some conclusions in Section 5.
Throughout the paper, \(A, B, \cdots\), denote tensors, uppercase letters \(A, B, \cdots\), denote matrices, lowercase letters \(x, y, \cdots\), denote vectors, \(\beta, \varepsilon, \cdots\) denote parameters, \(x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\) denotes the inner product of vectors \(x, y\).

2. The equivalent transformation of tensor eigenvalue complementarity problems. We firstly reformulate tensor eigenvalue complementarity problems to unconstrained optimization problems. TGEiCP\((A, B)\) (1) can be rewritten as finding \((\lambda, x, y)\) \(\in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\) such that

\[
y^1 = (\lambda B - A)x^{m-1}, \quad x \geq 0, \quad x^T y^1 = 0, \quad e_n^T x = 1.
\]

THDEiCP\((A, B, C)\) (2) can be rewritten as finding \((\lambda, x, y^2)\) \(\in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\) such that

\[
y^2 = (\lambda^m A + \lambda B + C)x^{m-1}, \quad x \geq 0, \quad x^T y^2 = 0, \quad e_n^T x = 1.
\]

TQEiCP\((A, B, C)\) (3) can be rewritten as finding \((\lambda, x, y^3)\) \(\in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\) such that

\[
y^3 = (\lambda^2 A + \lambda B + C)x^{m-1}, \quad x \geq 0, \quad x^T y^3 = 0, \quad e_n^T x = 1.
\]

EiCP-TM (4) can be rewritten as finding \((\lambda, x, y^4)\) \(\in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n\) such that

\[
y^4 = \lambda x - Ax^3 - Bx \geq 0, \quad x \geq 0, \quad x^T y^4 = 0, \quad e_n^T x = 1,
\]

where \(e_n = (1, 1, \ldots, 1) \in \mathbb{R}^n\). For the convenience of expression, we can express the above four formulas in the following form

\[
\tilde{y} \geq 0, \quad x \geq 0, \quad x^T \tilde{y} = 0, \quad e_n^T x = 1,
\]

(5) where \(\tilde{y} \in \{y^1, y^2, y^3, y^4\}\). For (5), we define a function \(\Phi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}\) by

\[
\Phi(z) = \Phi(x, y, \lambda) = \begin{pmatrix}
\phi(x, y) \\
y^i - y \\
\epsilon_n^T x - 1
\end{pmatrix}, \quad i = 1, 2, 3, 4,
\]

(6) where \(\phi(x, y) = (\varphi(x_1, y_1), \ldots, \varphi(x_n, y_n))^T\) with

\[
\varphi(x_i, y_i) = x_i + y_i - \sqrt{x_i^2 + y_i^2} \quad \forall i \in [n].
\]

It is obvious that \(z = (x, y, \lambda)\) solves (5) if and only if \(\Phi(z) = 0\).

To introduce the framework of a modified spectral PRP conjugate gradient method, we recall some existing results on tensor calculation. For any tensor \(A = (a_{i_1 \ldots i_m}) \in T_{m,n}\) and a vector \(x \in \mathbb{R}^n\), by [26, Lemma 2.1], there is the unique semi-symmetric tensor \(\hat{A} \in T_{m,n}\) such that \(\hat{A}x^{m-1} = Ax^{m-1}\) for all \(x \in \mathbb{R}^n\). Hence, we always assume that \(A \in T_{m,n}\) is semi-symmetric. By [26, Lemma 3.3], the Jacobian of \(Ax^{m-1}\) at \(x\) is given by

\[
(m - 1)Ax^{m-2}.
\]

(7) We next define a merit function for (6), which is defined as

\[
\Psi(z) = \frac{1}{2} \|\Phi(z)\|^2.
\]

Clearly, \(\Psi\) is a continuously differential function for any \(z \in \mathbb{R}^{2n+1}\). From (7), the gradient of \(\Psi\) at \(z\) is given by

\[
\nabla \Psi(z) = H^T \Phi(z),
\]
where
\[ H \in \partial \Phi(z) = \begin{pmatrix} D_a & D_b & 0 \\ \frac{\partial y^i}{\partial x} & -I & \frac{\partial y^i}{\partial \lambda} \\ e^T_n & 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3, 4, \]
and
\[ D_a := \text{diag}\{a_1, \ldots, a_n\}, \quad D_b := \text{diag}\{b_1, \ldots, b_n\}, \]
with
\[ (a_i, b_i) := \begin{cases} \left(1 - \frac{x_i}{\sqrt{x_i^2 + y_i^2}}, 1 - \frac{y_i}{\sqrt{x_i^2 + y_i^2}}\right), & \text{if } (x_i, y_i) \neq (0, 0), \\ (1 - \zeta, 1 - \zeta), & \text{if } (x_i, y_i) = (0, 0), \end{cases} \]
where \((\zeta, \varsigma)\) satisfies \(|(\zeta, \varsigma)| \leq 1\), for any \(i \in [n]\).

3. A new modified spectral PRP conjugate gradient method. In this section, we firstly proposed the new modified spectral PRP conjugate gradient method.

Algorithm 3.1. (Modified spectral PRP Conjugate Gradient Method)

Step 0. Choose an initial point \(z_0 \in \mathbb{R}^n\) and constants \(\delta, \rho \in (0, 1), \varepsilon > 0\). Let \(k := 0\).

Step 1. Compute \(\nabla \Psi(z_k)\). If \(\|\nabla \Psi(z_k)\| \leq \varepsilon\), stop. Otherwise, compute \(d_k\) by
\[ d_k = \begin{cases} -\nabla \Psi(z_k), & \text{if } k = 0, \\ -\theta_k \nabla \Psi(z_k) + \beta_k^{PRP} d_{k-1}, & \text{if } k \geq 1, \end{cases} \]
where
\[ \beta_k^{PRP} = \frac{\nabla \Psi(z_k)^T (\nabla \Psi(z_k) - \nabla \Psi(z_{k-1}))}{\|\nabla \Psi(z_{k-1})\|^2}, \]
\[ \theta_k = \frac{d_k^T (\nabla \Psi(z_k) - \nabla \Psi(z_{k-1}))}{\| \nabla \Psi(z_{k-1}) \|^2} - \frac{d_k^T \nabla \Psi(z_{k-1}) \nabla \Psi(z_k)^T \nabla \Psi(z_{k-1})}{\| \nabla \Psi(z_{k-1}) \|^2 \| \nabla \Psi(z_k) \|^2}. \]

Step 2. Compute \(\alpha_k\) by Armijo line search, that is, \(\alpha_k = \max\{\rho^j, j = 0, 1, 2, \ldots\}\) satisfying
\[ \Psi(z_k + \alpha_k d_k) \leq \Psi(z_k) + \delta \alpha_k \nabla \Psi(z_k)^T d_k. \]

Step 3. Set \(z_{k+1} = z_k + \alpha_k d_k, \quad k := k + 1. \) Go to Step 1.

To ensure the convergence of Algorithm 3.1, we need the following assumptions and lemmas.

Assumption 3.1. The level set \(\Omega = \{z | \Psi(z) \leq \Psi(z_0)\}\) is bounded.

Assumption 3.2. Denote \(\nabla \Psi(z)\) is the gradient of \(\Psi\) and satisfies Lipschitz condition in some neighborhood \(N \subset \Omega\), namely, there exists a positive constant \(L\) such that
\[ \| \nabla \Psi(z_1) - \nabla \Psi(z_2) \| \leq L \| z_1 - z_2 \|, \quad \forall z_1, z_2 \in N. \]

Assumption 3.3. For \(k\) sufficiently large, the following inequalities hold, that is
\[ 0 < \nabla \Psi(z_k)^T \nabla \Psi(z_{k-1}) < 2 \nabla \Psi(z_k)^T \nabla \Psi(z_k). \]

Note that \(\{\Psi(z_k)\}\) is a descending series, so the sequence \(\{z_k\} \subset \Omega\), where \(\{z_k\}\) are generated by Algorithm 3.1. Besides, by the Assumption 3.2, it is clear that there exists a constant \(\eta > 0\) such that
\[ \| \nabla \Psi(z) \| \leq \eta, \forall z \in \Omega. \]
Remark 1. Assumption 3.2 holds naturally for specific tensor eigenvalue complementarity problem. When considering TGEiCP(1), we can get Assumption 3.2.

From the equivalent form of TGEiCP(1) and $\nabla \Psi(z) = H^T \Phi(z)$, we know that there exist $\xi_1, \xi_2, \xi_3, \xi > 0$ such that

$$\|\nabla \Psi(z_1) - \nabla \Psi(z_2)\| = \|H(z_1)^T \Phi(z_1) - H(z_2)^T \Phi(z_2)\|$$

$$\leq \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$+ \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$+ \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$+ \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$\leq \xi_1 \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$+ \xi_2 \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$+ \xi_3 \left\| \frac{D_a^{(1)} \phi(x_1, y_1) - D_a^{(2)} \phi(x_2, y_2)}{x_1} + \lambda_1 (\nabla \Phi(z))\right\|_{x_1} + \frac{\partial((\lambda_1 B - A)x_1^{m-1} - y_1)}{x_2}(\lambda_2 B - A)x_2^{m-1} - y_2 + e_n e_n^T (x_1 - x_2)$$

$$\leq \xi \|z_1 - z_2\|.$$
We also can find \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon > 0 \) such that
\[
\begin{align*}
\|D_b^{(1)} \phi(x_1, y_1) - D_b^{(2)} \phi(x_2, y_2) - (\lambda_1 B - A)x_1^{m-1} + \lambda_2 B - A)x_2^{m-1} + y_1 - y_2 \| \\
&\leq \|D_b^{(1)} \phi(x_1, y_1) - D_b^{(2)} \phi(x_2, y_2)\| \\
&+ \|((\lambda_1 B - A)x_1^{m-1} + (\lambda_2 B - A)x_2^{m-1})\| + \|y_1 - y_2\| \\
&\leq \epsilon_1 \left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\| + \epsilon_2 \left\| \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} \right\| + \epsilon_3 \|y_1 - y_2\| \\
&\leq \epsilon \|z_1 - z_2\|. 
\end{align*}
\]

And there exist \( \zeta_1, \zeta_2, \xi > 0 \) such that
\[
\begin{align*}
\left\| \begin{pmatrix} ((\lambda_1 B - A)x_1^{m-1} - y_1) \partial((\lambda_2 B - A)x_2^{m-1}) \\ -((\lambda_2 B - A)x_2^{m-1} - y_2) \partial((\lambda_1 B - A)x_1^{m-1}) \end{pmatrix} \right\| \\
&\leq \left\| \begin{pmatrix} ((\lambda_1 B - A)x_1^{m-1} - y_1) \partial((\lambda_2 B - A)x_2^{m-1}) \\ -((\lambda_2 B - A)x_2^{m-1} - y_2) \partial((\lambda_1 B - A)x_1^{m-1}) \end{pmatrix} \right\| \\
&+ \left\| \begin{pmatrix} (\lambda_1 B - A)x_1^{m-1} - y_1 \partial((\lambda_2 B - A)x_2^{m-1}) \\ -((\lambda_2 B - A)x_2^{m-1} - y_2) \partial((\lambda_1 B - A)x_1^{m-1}) \end{pmatrix} \right\| \\
&\leq \zeta_1 \left\| \begin{pmatrix} x_1 \\ y_1 \\ \lambda_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ \lambda_2 \end{pmatrix} \right\| + \zeta_2 \left\| \begin{pmatrix} x_1 \\ y_1 \\ \lambda_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ \lambda_2 \end{pmatrix} \right\| \\
&\leq \zeta \|z_1 - z_2\|.
\end{align*}
\]

So, we get
\[
\|\nabla \Phi(z_1) - \nabla \Phi(z_2)\| = \|H(z_1)^T \Phi(z_1) - H(z_2)^T \Phi(z_2)\| \\
\leq L \|z_1 - z_2\|,
\]
where \( L = \epsilon + \xi + \zeta. \)

**Lemma 3.1.** [37] Let \( \{d_k\} \) is computed by Algorithm 3.1, then
\[
\nabla \Phi(z_k)^T d_k = -\|\nabla \Phi(z_k)\|^2
\]
holds for arbitrarily \( k > 0 \).

From Assumption 3.1, 3.2 and Lemma 3.1, we can get the following lemma.

**Lemma 3.2.** Let \( \{d_k\} \) is given by Algorithm 3.1, then
\[
\sum_{k \geq 0} \frac{\|\nabla \Phi(z_k)\|^4}{\|d_k\|^2} < +\infty.
\]

Based on the above assumptions and lemmas, we give the global convergence of Algorithm 3.1 in the following theorem.

**Theorem 3.3.** Let Assumption 3.1-3.3 hold. If \( \{z_k\} \) is generated by Algorithm 3.1, then
\[
\liminf_{k \to \infty} \|\nabla \Phi(z_k)\| = 0.
\]
Proof. If (11) does not hold, there exists a positive constant $\tau > 0$ such that
\[
\| \nabla \Psi(z_k) \| \geq \tau, \forall k \geq 0.
\] (12)

From (8), we get
\[
\| d_k \|^2 = \| -\theta_k \nabla \Psi(z_k) + \beta_k^{PRP} d_{k-1} \|^2
\]
\[
= \theta_k^2 \| \nabla \Psi(z_k) \|^2 - 2 \theta_k \beta_k^{PRP} \nabla \Psi(z_k)^T d_{k-1} + (\beta_k^{PRP})^2 \| d_{k-1} \|^2
\]
\[
= \theta_k^2 \| \nabla \Psi(z_k) \|^2 - 2 \theta_k (d_k + \theta_k \nabla \Psi(z_k))^T \nabla \Psi(z_k) + (\beta_k^{PRP})^2 \| d_{k-1} \|^2
\]
\[
= \theta_k^2 \| \nabla \Psi(z_k) \|^2 - 2 \theta_k d_k^T \nabla \Psi(z_k) - 2 \theta_k \| \nabla \Psi(z_k) \|^2 + (\beta_k^{PRP})^2 \| d_{k-1} \|^2
\]
\[
= -(\theta_k^2 + \theta_k) \| \nabla \Psi(z_k) \|^2 + (\beta_k^{PRP})^2 \| d_{k-1} \|^2
\]
By (9), (12) and Assumption 3.3, we have
\[
\frac{\| d_k \|^2}{\| \nabla \Psi(z_k) \|^4} = \frac{\theta_k^2 + \theta_k}{\| \nabla \Psi(z_k) \|^2} + \frac{(\beta_k^{PRP})^2 \| d_{k-1} \|^2}{\| \nabla \Psi(z_k) \|^4}
\]
\[
= -\frac{(\theta_k + 1)^2}{\| \nabla \Psi(z_k) \|^2} + \frac{1}{\| \nabla \Psi(z_k) \|^2}
\]
\[
+ \frac{\| \nabla \Psi(z_k) \|^2}{\| \nabla \Psi(z_k) \|^4} \left( 1 - \frac{(2 \nabla \Psi(z_k)^T \nabla \Psi(z_k) - \nabla \Psi(z_k)^T \nabla \Psi(z_{k-1})) \nabla \Psi(z_k)^T \nabla \Psi(z_{k-1})}{\| \nabla \Psi(z_k) \|^2} \right)
\]
\[
\leq \frac{1}{\| \nabla \Psi(z_{k-1}) \|^4} + \frac{1}{\| \nabla \Psi(z_k) \|^2}
\]
\[
\leq \sum_{i=0}^{k-1} \frac{1}{\| \nabla \Psi(z_i) \|^2} \leq \frac{k}{\tau^2}.
\]
Then
\[
\sum_{k \geq 1} \frac{\| \nabla \Psi(z_k) \|^4 \| d_k \|^2}{\| \nabla \Psi(z_k) \|^4} \geq \tau^2 \sum_{k \geq 1} \frac{1}{k} = +\infty.
\]
That contradicts (10). This shows (11) holds. We finish the proof of this theorem. \hfill \qed

Remark 2. In Step 2 of Algorithm 3.1, we also can use Wolfe line search [11]
\[\Psi(z_k + \alpha_k d_k) \leq \Psi(z_k) + \rho \alpha_k \nabla \Psi(z_k)^T d_k\]
and
\[\nabla \Psi(z_{k+1})^T d_k \geq \sigma \nabla \Psi(z_k)^T d_k, \rho \in (0, \frac{1}{2}), \sigma \in (\rho, 1)\]
and Armijo-type line search [49]
\[\Psi(z_k + \alpha_k d_k) \leq \Psi(z_k) + \delta_1 \alpha_k \nabla \Psi(z_k)^T d_k - \delta_2 \alpha_k^2 \| d_k \|^2\]
to get the global convergence of Algorithm 3.1.
4. Numerical experiments. In this section, we test four kinds of tensor eigenvalue complementarity problems to show the efficiency of the modified spectral PRP conjugate gradient method (Algorithm 3.1). Compared with the existing references, we give a simple and unified algorithm for solving four kinds of tensor eigenvalue complementarity problems. From the numerical results, we can see that Algorithm 3.1 can detect eigenvalues efficiency with random initial points. The numerical testing is carried out on a Lenovo PC with the code in Matlab version 2013a, the parameters taken as $\varepsilon = 10^{-6}$, $\rho = 0.6$, $\delta = 0.4$ and two examples are taken from [22, 23]. The following tables and figures list the numerical results for the given tensor eigenvalue complementarity problems.

Example 4.1. Consider TGEiCP (1) with $\mathcal{A}, \mathcal{B} \in S_{4,2}$, where $\mathcal{A}, \mathcal{B}$ are given by

$$
\mathcal{A}(::, 1, 1) = \begin{pmatrix} 1.6324 & 1.1880 \\ 1.1880 & 1.5469 \end{pmatrix}, \quad \mathcal{A}(::, 1, 2) = \begin{pmatrix} 1.1880 & 1.5469 \\ 1.5469 & 1.9340 \end{pmatrix}, \\
\mathcal{A}(::, 2, 1) = \begin{pmatrix} 1.1880 & 1.5469 \\ 1.5469 & 1.9340 \end{pmatrix}, \quad \mathcal{A}(::, 2, 2) = \begin{pmatrix} 1.5469 & 1.9340 \\ 1.9340 & 1.0318 \end{pmatrix}, \\
\mathcal{B}(::, 1, 1) = \begin{pmatrix} 0.8147 & 0.5164 \\ 0.5164 & 0.9134 \end{pmatrix}, \quad \mathcal{B}(::, 1, 2) = \begin{pmatrix} 0.5164 & 0.9134 \\ 0.9134 & 0.9595 \end{pmatrix}, \\
\mathcal{B}(::, 2, 1) = \begin{pmatrix} 0.5164 & 0.9134 \\ 0.9134 & 0.9595 \end{pmatrix}, \quad \mathcal{B}(::, 2, 2) = \begin{pmatrix} 0.9134 & 0.9595 \\ 0.9595 & 0.3922 \end{pmatrix}.
$$

We take 30 random initial points $(\lambda_0, x_0)$ uniformly distributed in $(0, 1)^3$ to solve this example by Algorithm 3.1. The numerical results with different initial points are reported in Table 1 and Figure 1. **Eigvalue** denotes the Pareto-eigenvalue, **Eigvector** denotes the corresponding Pareto-eigenvector, **No** denotes number of each solution detected by the method with 30 random initial points. From Table 1, we can see that Algorithm 3.1 can detect three Pareto-eigenvalues, which are 1.9406, 2.0469, and 2.6308 for $(\mathcal{A}, \mathcal{B})$ in Example 4.1. From the numerical results, we can see that Algorithm 3.1 can detect eigenvalues efficiency with random initial points.

| Eigvalue | Eigvector | No |
|----------|-----------|----|
| 1.9406   | $(0.4982, 0.5018)^T$ | 24 |
| 2.0469   | $(0.8817, 0.1183)^T$ | 5  |
| 2.6308   | $(0.0000, 1.0000)^T$ | 1  |

**Figure 1.** Numerical results of Example 4.1 with random initial points.
Example 4.2. Consider THDEiCP($\mathcal{A}, \mathcal{B}, \mathcal{C}$)(2) with $\mathcal{A}, \mathcal{B} \in S_{4,3}, \mathcal{C} = -I$, where $\mathcal{A}, \mathcal{B}$ are given by

\[\mathcal{A}(\cdot, 1, 1) = \begin{pmatrix} 0.6229 & 0.2644 & 0.3567 \\ 0.2644 & 0.0475 & 0.7367 \\ 0.3567 & 0.7367 & 0.1259 \end{pmatrix}, \quad \mathcal{A}(\cdot, 1, 2) = \begin{pmatrix} 0.7563 & 0.5878 & 0.5406 \\ 0.5878 & 0.1379 & 0.0715 \\ 0.5406 & 0.0715 & 0.3725 \end{pmatrix},\]

\[\mathcal{A}(\cdot, 1, 3) = \begin{pmatrix} 0.0657 & 0.4918 & 0.9312 \\ 0.4918 & 0.7788 & 0.9045 \\ 0.9312 & 0.9045 & 0.8711 \end{pmatrix}, \quad \mathcal{A}(\cdot, 2, 2) = \begin{pmatrix} 0.7689 & 0.3941 & 0.6034 \\ 0.3941 & 0.3577 & 0.3465 \\ 0.6034 & 0.3465 & 0.4516 \end{pmatrix},\]

\[\mathcal{A}(\cdot, 2, 3) = \begin{pmatrix} 0.8077 & 0.4910 & 0.2953 \\ 0.4910 & 0.5054 & 0.5556 \\ 0.2953 & 0.5556 & 0.9608 \end{pmatrix}, \quad \mathcal{A}(\cdot, 3, 3) = \begin{pmatrix} 0.7581 & 0.7205 & 0.9044 \\ 0.7205 & 0.0782 & 0.7240 \\ 0.9044 & 0.7240 & 0.3492 \end{pmatrix},\]

\[\mathcal{B}(\cdot, 1, 1) = \begin{pmatrix} 0.6954 & 0.4018 & 0.1406 \\ 0.4018 & 0.9957 & 0.0483 \\ 0.1406 & 0.0483 & 0.9988 \end{pmatrix}, \quad \mathcal{B}(\cdot, 1, 2) = \begin{pmatrix} 0.6730 & 0.5351 & 0.4473 \\ 0.5351 & 0.2853 & 0.3071 \\ 0.4473 & 0.3071 & 0.9665 \end{pmatrix},\]

\[\mathcal{B}(\cdot, 1, 3) = \begin{pmatrix} 0.7585 & 0.6433 & 0.2306 \\ 0.6433 & 0.8966 & 0.3427 \\ 0.2306 & 0.3427 & 0.5390 \end{pmatrix}, \quad \mathcal{B}(\cdot, 2, 2) = \begin{pmatrix} 0.3608 & 0.3941 & 0.5230 \\ 0.3941 & 0.6822 & 0.5516 \\ 0.5230 & 0.5516 & 0.7091 \end{pmatrix},\]

\[\mathcal{B}(\cdot, 2, 3) = \begin{pmatrix} 0.4632 & 0.2043 & 0.2823 \\ 0.2043 & 0.7282 & 0.7400 \\ 0.2823 & 0.7400 & 0.9369 \end{pmatrix}, \quad \mathcal{B}(\cdot, 3, 3) = \begin{pmatrix} 0.8200 & 0.5914 & 0.4983 \\ 0.5914 & 0.0762 & 0.2854 \\ 0.4983 & 0.2854 & 0.1266 \end{pmatrix},\]

We take 10 random initial points ($\lambda_0, x_0$) uniformly distributed in $(0, 1)^4$ for solving this example by Algorithm 3.1 and inexact Levenberg-Marquardt method (ILMM) proposed in [20] respectively. The numerical results are reported in Table 2. Eigvalue denotes the 4-degree Pareto-eigenvalue of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Eigvector denotes the corresponding eigenvector, No. denotes the number of occurrence of each numerical result detected by the method within 10 random initial points, K denotes the number of iterations. From Table 2, we can see that Algorithm 3.1 can detect four eigenvalues 0.3024, 0.2356, 0.2089, 0.0771 and the probability of getting valid results is 90 percent by Algorithm 3.1. Inexact Levenberg-Marquardt method can detect four eigenvalues 0.3024, 0.2356, 0.2089, 0.9807 and its probability of getting valid results is 60 percent.

| Alg.            | Eigvalue       | Eigvector       | No. | K  |
|-----------------|----------------|-----------------|-----|----|
| Algorithm 3.1   | 0.3024         | (0.5479, 1.0000, 0.4521)$^T$ | 3   | 69 |
| Algorithm 3.1   | 0.2356         | (0.5037, 0.4963, 0.0000)$^T$ | 2   | 56 |
| Algorithm 3.1   | 0.2089         | (0.0000, 0.4911, 0.5089)$^T$ | 1   | 41 |
| Algorithm 3.1   | 0.0771         | (0.3328, 0.3372, 0.3301)$^T$ | 3   | 183|
| Algorithm 3.1   | failure        |                 |     |    |
| ILMM            | 0.2089         | (0.0000, 0.4911, 0.5089)$^T$ | 2   | 26 |
| ILMM            | 0.2356         | (0.5037, 0.4963, 0.0000)$^T$ | 2   | 41 |
| ILMM            | 0.3024         | (0.5479, 1.0000, 0.4521)$^T$ | 1   | 45 |
| ILMM            | 0.9807         | (0.0000, 1.0000, 0.0000)$^T$ | 1   | 119|
| ILMM            | failure        |                 |     |    |
Example 4.3. Consider TQEiCP \((A, B, C)\) with \(A, B, C \in S_{0,2}\), where \(C = -A\), and \(A, B\) are given by

\[
\begin{align*}
A(:,:,1,1,1) &= \begin{pmatrix} 0.1518 & 0.4312 \\ 0.4312 & 0.4093 \end{pmatrix}, \\
A(:,:,1,1,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,1,2,1) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,1,2,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,2,1,1) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,1,2) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,2,1) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,2,2) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,1,1,1) &= \begin{pmatrix} 0.1518 & 0.4312 \\ 0.4312 & 0.4093 \end{pmatrix}, \\
A(:,:,1,1,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,1,2,1) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,1,2,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
A(:,:,2,1,1) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,1,2) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,2,1) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
A(:,:,2,2,2) &= \begin{pmatrix} 0.3593 & 0.4671 \\ 0.4671 & 0.2735 \end{pmatrix}, \\
B(:,:,1,1,1) &= \begin{pmatrix} 0.0366 & 0.7056 \\ 0.7056 & 0.5357 \end{pmatrix}, \\
B(:,:,1,1,2) &= \begin{pmatrix} 0.7056 & 0.5357 \\ 0.5357 & 0.5043 \end{pmatrix}, \\
B(:,:,1,2,1) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
B(:,:,1,2,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
B(:,:,2,1,1) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
B(:,:,2,1,2) &= \begin{pmatrix} 0.4321 & 0.4093 \\ 0.4093 & 0.3593 \end{pmatrix}, \\
B(:,:,2,2,1) &= \begin{pmatrix} 0.5357 & 0.5043 \\ 0.5043 & 0.5231 \end{pmatrix}, \\
B(:,:,2,2,2) &= \begin{pmatrix} 0.5043 & 0.5231 \\ 0.5231 & 0.3836 \end{pmatrix}.
\end{align*}
\]
Consider $EiCP-TM(\cdot)$ (Example 4.4), which are 0. We take 20 random initial points ($\lambda_0, x_0$) uniformly distributed in $(0,1)^3$ to solve this example by Algorithm 3.1. The numerical results with different initial points are reported in Table 3. Eigvalue denotes the tensor quadratic eigenvalue, Eigvector denotes the corresponding eigenvector, No denotes number of each solution detected by the method within 20 random initial points. From Table 3, we find Algorithm 3.1 can detect three tensor quadratic complementarity eigenvalues, which are 0.8867, 0.9533, and 0.6 for $(A,B,C)$ in Example 4.3.

Table 3. The numerical results of Example 4.3

| Eigvalue | Eigvector | No |
|----------|-----------|----|
| 0.8867   | (1.0000, 0.0000)$^T$ | 2  |
| 0.9533   | (0.0000, 1.0000)$^T$ | 4  |
| 0.6      | (0.6, 0.4)$^T$    | 14 |

Example 4.4. Consider $EiCP-TM(\cdot)$ (4) with $A \in S_{4,3}$, $B \in \mathbb{R}^{3\times 3}$ defined as

$$
A(\cdot,1,1) = \begin{pmatrix} 0.3402 & 0.4572 & 0.7054 \\ 0.4572 & 0.5875 & 0.4357 \\ 0.7054 & 0.4357 & 0.3185 \end{pmatrix}, \quad A(\cdot,1,2) = \begin{pmatrix} 0.4572 & 0.5875 & 0.4357 \\ 0.5875 & 0.5834 & 0.5211 \\ 0.4357 & 0.5211 & 0.4489 \end{pmatrix},
$$

$$
A(\cdot,1,3) = \begin{pmatrix} 0.7054 & 0.4357 & 0.3185 \\ 0.4357 & 0.5211 & 0.4489 \\ 0.3185 & 0.4489 & 0.3984 \end{pmatrix}, \quad A(\cdot,2,1) = \begin{pmatrix} 0.4572 & 0.5875 & 0.4357 \\ 0.5875 & 0.5834 & 0.5211 \\ 0.4357 & 0.5211 & 0.4489 \end{pmatrix},
$$

$$
A(\cdot,2,2) = \begin{pmatrix} 0.5834 & 0.8038 & 0.4943 \\ 0.5211 & 0.4943 & 0.4346 \end{pmatrix}, \quad A(\cdot,2,3) = \begin{pmatrix} 0.5834 & 0.8038 & 0.4943 \\ 0.5211 & 0.4943 & 0.4346 \end{pmatrix},
$$

$$
A(\cdot,3,1) = \begin{pmatrix} 0.7054 & 0.4357 & 0.3185 \\ 0.4357 & 0.5211 & 0.4489 \\ 0.3185 & 0.4489 & 0.3984 \end{pmatrix}, \quad A(\cdot,3,2) = \begin{pmatrix} 0.5834 & 0.8038 & 0.4943 \\ 0.5211 & 0.4943 & 0.4346 \end{pmatrix},
$$

$$
A(\cdot,3,3) = \begin{pmatrix} 0.3185 & 0.4489 & 0.3984 \\ 0.4489 & 0.4346 & 0.5300 \\ 0.3984 & 0.5300 & 0.7861 \end{pmatrix}, \quad B = \begin{pmatrix} 1.2412 & -0.2937 & -0.3826 \\ -0.2937 & 1.2525 & -0.6042 \\ -0.3826 & -0.6042 & 2.1116 \end{pmatrix}.
$$

We take 10 random initial points ($\lambda_0, x_0$) uniformly distributed in $(0,1)^4$ to solve this example by Algorithm 3.1. The numerical results are reported in Table 4. Eigvalue denotes the eigenvalue of $(A,B)$, Eigvector denotes the corresponding eigenvector, No denotes number of each solution detected by the method within 10 random initial points. From Table 4, we can see that Algorithm 3.1 can detect three eigenvalues, which are 2.9167, 2.9865, and 0.3 for $(A,B)$ in Example 4.4. The experimental results show that the accuracy of Algorithm 3.1 reaching up to 60 percent accuracy.
Table 4. The numerical results of Example 4.4

| Eigvalue     | Eigvector                  | No |
|--------------|----------------------------|----|
| 2.9167       | (0.0232, 0.0000, 0.9967)$^T$ | 3  |
| 2.9865       | (0.0712, 0.0252, 0.9971)$^T$ | 2  |
| 0.3          | (0.1, 0, 0.1)$^T$           | 1  |
| failure      |                            | 4  |

Example 4.5. Consider $EiCP-TM(A, B)(4)$, where $A \in S_{4,10}$ is diagonal tensor with $a_{iiii} = 2, i = \{1, 2, ..., 10\}$ and other elements are 0, $B = b_{ij} \in \mathbb{R}^{10 \times 10}$ with $b_{ij} = 1, i, j = \{1, 2, ..., 10\}$.

We choose random initial points $(\lambda_0, x_0)$ uniformly distributed in $(0, 1)^{11}$ and $(\lambda_0, x_0) = (1, 1, ..., 1) \in \mathbb{R}^{11}$ to solve this example by Algorithm 3.1. We get $(\lambda = 10.0200, x = (0.1000, 0.1000, ..., 0.1000)^T$. The relation between iteration numbers and values of $\Psi(z)$ with different initial points are given in Figure 2. From the numerical results, we can see that Algorithm 3.1 can also solve this example efficiently.

![Figure 2](image-url)

(a) Numerical results of Example 4.5 with random initial points.

(b) Numerical results of Example 4.5 with initial points $(\lambda, x) = (1, 1, ..., 1)^T \in \mathbb{R}^{11}$.

Figure 2. Numerical results of Example 4.5 with different initial points

5. Conclusions. In this paper, we consider four kinds of tensor eigenvalue complementarity problems. By reformulating tensor eigenvalue complementarity problems to unconstrained optimization problem, we give a new modified spectral PRP conjugate gradient method for solving them. The advantages of Algorithm 3.1 for solving
tensor eigenvalue complementarity problems are that the initial points can be chosen randomly and the method is globally convergent under mild conditions. The numerical experiments and comparison with inexact Levenberg-Marquardt method indicate the validity and efficiency of the proposed method.

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