Tikekar superdense stars in electric fields

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Abstract

We present exact solutions to the Einstein-Maxwell system of equations with a specified form of the electric field intensity by assuming that the hypersurface \( \{t = \text{constant}\} \) are spheroidal. The solution of the Einstein-Maxwell system is reduced to a recurrence relation with variable rational coefficients which can be solved in general using mathematical induction. New classes of solutions of linearly independent functions are obtained by restricting the spheroidal parameter \( K \) and the electric field intensity parameter \( \alpha \). Consequently it is possible to find exact solutions in terms of elementary functions, namely polynomials and algebraic functions. Our result contains models found previously including the superdense Tikekar neutron star model [R. Tikekar, J. Math. Phys. 31, 2454 (1990)] when \( K = -7 \) and \( \alpha = 0 \). Our class of charged spheroidal models generalise the uncharged isotropic Maharaj and Leach solutions [S. D. Maharaj and P. G. L. Leach, J. Math. Phys. 37, 430 (1996)]. In particular, we find an explicit relationship directly relating the spheroidal parameter \( K \) to the electromagnetic field.

1 Introduction

Exact solutions of the Einstein-Maxwell field equations are of crucial importance in relativistic astrophysics. These solutions may be utilised to model a charged relativistic star as they are matchable to the Reissner-Nordstrom exterior at the boundary. A recent review of Einstein-Maxwell solutions is given by Ivanov1. It is interesting to

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observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided. In this situation the gravitational attraction is counterbalanced by the repulsive Coulombian force in addition to the pressure gradient. Einstein-Maxwell solutions are also important in studies involving the cosmic censorship hypothesis and the formation of naked singularities. The presence of charge affects values for redshifts, luminosities and maximum mass for stars. Consequently the Einstein-Maxwell system, for a charged star, has attracted considerable attention in various investigations.

In an attempt to generate exact solutions in some cases Vaidya and Tikekar\textsuperscript{2} proposed that the geometry of the spacelike hypersurfaces generated by \{\(t = \text{constant}\)\} are that of the 3-spheroid. This spheroidal condition provides a clear geometrical interpretation which is not the case in many other exact solutions. Knutsen\textsuperscript{3} was the first to consider the pressure gradients of stars with spheroidal geometry and showed that they were negative. Also note that spheroidal geometries exhibit the important physical feature of being stable with respect to radial pulsations\textsuperscript{4}. Tikekar\textsuperscript{5} comprehensively studied a particular spheroidal geometry and showed that it could be applied to superdence neutron stars with densities in the range of \(10^{14}\) g cm\(^{-3}\). Maharaj and Leach\textsuperscript{6} found all spheroidal solutions, for uncharged stars, that could be expressed in terms of elementary functions. Mukherjee \textit{et al}\textsuperscript{7} showed that it was possible to express the general solution in terms of Gegenbauer functions, an alternate form of the general solution was found by Gupta and Jasim\textsuperscript{8}. These uncharged solutions can be extended to models in the presence of electromagnetic field. Spheroidal models in the presence of an electric field have been extensively studied by Sharma \textit{et al}\textsuperscript{9}, Patel and Koppar\textsuperscript{10}, Patel \textit{et al}\textsuperscript{11}, Tikekar and Singh\textsuperscript{12}, and Gupta and Kumar\textsuperscript{13}. These investigations have been motivated on the grounds that by restricting the geometry of the hypersurfaces \{\(t = \text{constant}\)\} to be spheroidal produces neutral and charged stars which are consistent with observations for dense astronomical objects. Models with spheroidal geometry can be directly related to particular physical situations: the maximum mass is in agreement with values for cold compact objects\textsuperscript{14}, values for densities are consistent with strange matter\textsuperscript{15}, the equation of state is consistent with a compact X-ray binary pulsar Her X-1\textsuperscript{16}, relevance to equation of state for stars compared of quark-diquark mixtures in equilibrium\textsuperscript{17}, and uniform charged dust in equilibrium\textsuperscript{18}. Spheroidals geometries are relevant in core-envelope steller models, core consisting of isotropic fluid and envelope with anisotropic fluid, as shown by Thomas \textit{et al}\textsuperscript{19} and Tikekar and Thomas\textsuperscript{20}.

It is clear that stars with spheroidal geometry have a number of different physical applications and therefore require deeper investigation. In this paper our objective is to generate a class of charged spheroidal solutions corresponding to a physically reasonable form for the electric field intensity. Our intention is to obtain simple forms
for the solution that highlights the role of the spheroidal parameter. In section 2 we express the Einstein-Maxwell field equations for the static spherically symmetric line element. The condition of pressure isotropy becomes a second order linear differential equation which facilitates the integration process. We assume a solution in a series form which yields recurrence relations, which we manage to solve from first principles in section 3. It is then possible to exhibit exact solutions to the Einstein-Maxwell system. In section 4 we present polynomials as first solutions and product of polynomials and algebraic functions as second linearly independent solutions. In addition we express the general solutions in terms of elementary functions. We demonstrate that solutions found previously are special cases of our general treatment. Finally in section 5 we briefly discuss the physical viability of our solutions. We emphasise that our simple approach of utilising the method of Frobenius for series yields a rich family of Einstein-Maxwell solutions in terms of elementary functions. This approach was utilised by Thirukkanesh and Maharaj\textsuperscript{21} to produce electromagnetic solutions to the Einstein-Maxwell system that contains the Durgapal and Bannerji neutron star model\textsuperscript{22}.

2 The isotropic model

On physical grounds it is necessary for the gravitational field to be static and spherically symmetric to describe the internal structure of a charged relativistic sphere. Therefore we assume that the interior of a spherically symmetric star is described by the line element

\[ ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

(1)

in Schwarzschild coordinates \((t, r, \theta, \phi)\) where \(\nu(r)\) and \(\lambda(r)\) are arbitrary functions. For charged perfect fluids the Einstein-Maxwell system of field equations can be written in the form

\[ \frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\lambda'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2 \]

(2a)

\[ -\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p - \frac{1}{2}E^2 \]

(2b)

\[ e^{-2\lambda} \left( \nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r} \right) = p + \frac{1}{2}E^2 \]

(2c)

\[ \sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)' \]

(2d)

for the line element (1). The field equations (2) are the same as those in Thirukkanesh and Maharaj\textsuperscript{21}; we are utilising units in which the coupling constant and the speed of light are unity. The energy density \(\rho\) and the pressure \(p\) are measured relative to the comoving fluid 4-velocity \(u^a = e^{-\nu}\delta_0^a\) and primes denote differentiation with respect to
the radial coordinate \( r \). In the system (2), the quantities \( E \) and \( \sigma \) are the electric field intensity and the proper charge density respectively.

To integrate the system (2) it is necessary to choose two of the variables \( \nu, \lambda, \rho, p \) or \( E \). In our approach we specify \( \lambda \) and \( E \). In the integration procedure we make the choice

\[
e^{2\lambda(r)} = \frac{1 - K r^2 / R^2}{1 - r^2 / R^2}
\]

where \( K \) is an arbitrary constant. The form (3) for the gravitational potential \( \lambda \) restricts the geometry of the 3-dimensional hypersurfaces \( \{ t = \text{constant} \} \) to be spheroidal. When \( K = 0 \) the hypersurfaces \( \{ t = \text{constant} \} \) become spherical. For the choice \( K = 0 \) familiar spacetimes are regainable for particulars forms of the metric function \( e^{2\nu} \), e.g. the choice \( \nu = 0 \) gives the metric of Einstein’s universe. On eliminating \( p \) from (2b) and (2c), for the particular form (3), we obtain

\[
(1 - Kr^2 / R^2)^2 E^2 = (1 - Kr^2 / R^2)(1 - r^2 / R^2) \left( \nu'' + \nu'^2 - \frac{\nu'}{r} \right) - (1 - K)(r / R^2) \left( \nu' + \frac{1}{r} \right) + \frac{1}{R^2} \left( 1 - Kr^2 / R^2 \right)
\]

which is the condition of pressure isotropy with a nonzero electromagnetic field.

It is convenient at this point to introduce the transformation

\[
\psi(x) = e^{\nu(r)}, \quad x^2 = 1 - \frac{r^2}{R^2}
\]

Then the condition of pressure isotropy (4) becomes

\[
(1 - K + Kx^2)\ddot{\psi} - Kx\dot{\psi} + \left( \frac{(1 - K + Kx^2)^2 R^2 E^2}{x^2 - 1} + K(K - 1) \right) \psi = 0
\]

in terms of new variables \( \psi \) and \( x \); dots denote differentiation with respect to \( x \). The Einstein-Maxwell system (2) implies

\[
\rho = \frac{1 - K (3 - K + Kx^2)}{R^2 (1 - K + Kx^2)^2} - \frac{1}{2} E^2 \quad (7a)
\]

\[
p = \frac{1}{R^2 (1 - K + Kx^2)} \left( -2x \frac{\dot{\psi}}{\psi} + K - 1 \right) + \frac{1}{2} E^2 \quad (7b)
\]

\[
\sigma^2 = \frac{[2xE - (1 - x^2)E]^2}{R^2 (1 - x^2)(1 - K + Kx^2)} \quad (7c)
\]

in terms of the variable \( x \). Thus \( \rho, p \), and \( \sigma \) are defined in terms of \( E \) in (7).

The solution of the Einstein-Maxwell system depends on the integrability of (6). Clearly (6) is integrable once \( E \) is specified. A variety of choices for \( E \) is possible;
however only a few are physically reasonable. We need to choose $E$ such that closed form solutions are possible. We make the choice

$$E^2 = \frac{\alpha K(x^2 - 1)}{R^2(1 - K + Kx^2)^2} \tag{8}$$

where $\alpha$ is constant. A similar form of $E$ was also used by Sharma et al$^9$ and Tikekar and Singh$^{12}$. The electric field intensity $E$ in (8) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameters $\alpha$ and $K$. Thus this choice for $E$ is physically reasonable and useful in the study of the gravitational behaviour of charged stars. On substituting (8) into (6) we obtain

$$(1 - K + Kx^2)\frac{d^2\psi}{dx^2} - Kx\frac{d\psi}{dx} + K(\alpha + K - 1)\psi = 0 \tag{9}$$

This is a second order differential equation which is linear in $\psi$. We expect that our investigation of (9) will produce viable models of charged stars since the special case $\alpha = 0$ yields models consistent with neutron stars.

3 Series solution

It is possible to express the solution of (9) in terms of special functions namely the Gegenbauer functions as demonstrated by Sharma et al$^9$. However that form of the solution is not particularly useful because of the analytic complexity of the special functions involved. In addition the role of parameters of physical interest, such as the spheroidal parameter $K$, is lost or obscured in the representation as Gegenbauer functions. The representation of the solutions in a simple form is necessary for a detailed physical analysis. Consequently we attempt to obtain a general solution to the differential equation (9) in a series form using the method of Frobenius. Later we will indicate that it is possible to extract solutions in terms of polynomials and algebraic functions for particular parameter values.

As the point $x = 0$ is a regular point of (9), there exist two linearly independent solutions of the form of a power series with centre $x = 0$. Thus we assume

$$\psi(x) = \sum_{i=0}^{\infty} a_i x^i \tag{10}$$

where the constants $a_i$ are the coefficients of the series. For a legitimate solution we need to determine the coefficients $a_i$ explicitly. On substituting (10) into (9) we obtain
after simplification

\[(1 - K)2.1a_0 + K(\alpha + K - 1)a_0 + [(1 - K)3.2a_3 + K(\alpha + K - 2)a_1]x\]

\[+ \sum_{i=2}^{\infty}((1 - K)(i + 1)(i + 2)a_{i+2} + K[\alpha + K - 1 + i(i - 2)]a_i)x^i = 0\] \hspace{1cm} (11)

in increasing powers of \(x\). For equation (11) to be valid for all \(x\) in the interval of convergence we require

\[(1 - K)2.1a_2 + K(\alpha + K - 1)a_0 = 0\] \hspace{1cm} (12a)

\[(1 - K)3.2a_3 + K(\alpha + K - 2)a_1 = 0\] \hspace{1cm} (12b)

\[(1 - K)(i + 1)(i + 2)a_{i+2} + K[\alpha + K - 1 + i(i - 2)]a_i = 0, \ i \geq 2\] \hspace{1cm} (12c)

Equation (12c) is the the linear recurrence relation governing the structure of the solution.

The recurrence relation (12c) consists of variable, rational coefficients. It does not fall in the known class of difference equations and has to be solved from first principles. It is possible to solve (12c) using the principle of mathematical induction. We first consider the even coefficients \(a_0, a_2, a_4, \ldots\). Equation (12a) implies

\[a_{2.1} = \left(\frac{K}{K - 1}\right) \frac{1}{(2.1)!} \prod_{q=1}^{1} [\alpha + K - 1 + (2q - 2)(2q - 4)]a_0\] \hspace{1cm} (13)

where we have utilised the conventional symbol \(\prod\) to denote multiplication for the first term. We now assume the pattern

\[a_{2p} = \left(\frac{K}{K - 1}\right)^p \frac{1}{(2p)!} \prod_{q=1}^{p} [\alpha + K - 1 + (2q - 2)(2q - 4)]a_0\] \hspace{1cm} (14)

for the coefficient \(a_{2p}\) which is the inductive step. We now establish that this is true for the next coefficient \(a_{2(p+1)}\). Replacing \(i\) with \(2p\) in (12c) we obtain

\[a_{2(p+1)} = \left(\frac{K}{K - 1}\right) \left(\frac{\alpha + K - 1 + 2p(2p - 2)}{(2p + 2)(2p + 1)}\right) a_{2p}\]

\[= \left(\frac{K}{K - 1}\right) \left(\frac{\alpha + K - 1 + 2p(2p - 2)}{(2p + 2)(2p + 1)}\right) \times\]

\[\left(\frac{K}{K - 1}\right)^p \frac{1}{(2p)!} \prod_{q=1}^{p} [\alpha + K - 1 + (2q - 2)(2q - 4)]a_0\]

\[= \left(\frac{K}{K - 1}\right)^{p+1} \frac{1}{[2(p+1)!]} \prod_{q=1}^{p+1} [\alpha + K - 1 + (2q - 2)(2q - 4)]a_0\] \hspace{1cm} (15)
where we have used (14). Hence by mathematical induction all the even coefficients 
a_{2i}
can be written in terms of the coefficient \(a_0\). These coefficients generate a pattern

\[
a_{2i} = \left(\frac{K}{K - 1}\right)^i \frac{1}{(2i)!} \prod_{q=1}^{i} [\alpha + K - 1 + (2q - 2)(2q - 4)] a_0
\]

(16)

for the even coefficients \(a_0, a_2, a_4, \ldots\)

We can obtain a similar formula for the odd coefficients \(a_1, a_3, a_5, \ldots\). From (12b) we have

\[
a_{2,1+1} = \left(\frac{K}{K - 1}\right)^1 \frac{1}{(2 + 1)!} \prod_{q=1}^{p} [\alpha + K - 1 + (2q - 1)(2q - 3)] a_1
\]

(17)

for the first term. We now assume

\[
a_{2p+1} = \left(\frac{K}{K - 1}\right)^p \frac{1}{(2p + 1)!} \prod_{q=1}^{p} [\alpha + K - 1 + (2q - 1)(2q - 3)] a_1
\]

(18)

for the coefficient \(a_{2p+1}\). We then establish that this is true for the next coefficient \(a_{2(p+1)+1}\). Replacing \(i\) with \((2p + 1)\) in (12c) we obtain

\[
a_{2(p+1)+1} = \left(\frac{K}{K - 1}\right)^{p+1} \frac{1}{[2(p + 1) + 1]} \prod_{q=1}^{p+1} [\alpha + K - 1 + (2q - 1)(2q - 3)] a_1
\]

(19)

on utilising (18). Hence by using mathematical induction all the odd coefficients \(a_{2i+1}\) can be written in terms of the coefficient \(a_1\). These coefficients generate a pattern which is clearly of the form

\[
a_{2i+1} = \left(\frac{K}{K - 1}\right)^i \frac{1}{(2i + 1)!} \prod_{q=1}^{i} [\alpha + K - 1 + (2q - 1)(2q - 3)] a_1
\]

(20)

for the odd coefficients \(a_1, a_3, a_5, \ldots\)

The coefficients \(a_{2i}\) are generated from (16). The coefficients \(a_{2i+1}\) are generated from (20). Hence the difference equation (12c) has been solved and all nonzero coefficients are expressible in terms of the leading coefficients \(a_0\) and \(a_1\). From (10), (16)
and (20) we establish that

\[
\psi(x) = a_0 \left( 1 + \sum_{i=1}^{\infty} \left( \frac{K}{K-1} \right)^i \frac{1}{(2i)!} \prod_{q=1}^{i} \left( \alpha + K - 1 + (2q-2)(2q-4) \right) x^{2i} \right) + a_1 \left( x + \sum_{i=1}^{\infty} \left( \frac{K}{K-1} \right)^i \frac{1}{(2i+1)!} \prod_{q=1}^{i} \left( \alpha + K - 1 + (2q-1)(2q-3) \right) x^{2i+1} \right)
\]

(21)

where \(a_0\) and \(a_1\) are arbitrary constants. Clearly (21) is of the form

\[
\psi(x) = a_0 \psi_1(x) + a_1 \psi_2(x)
\]

(22)

where

\[
\psi_1(x) = \left( 1 + \sum_{i=1}^{\infty} \left( \frac{K}{K-1} \right)^i \frac{1}{(2i)!} \prod_{q=1}^{i} \left( \alpha + K - 1 + (2q-2)(2q-4) \right) x^{2i} \right)
\]

(23a)

\[
\psi_2(x) = \left( x + \sum_{i=1}^{\infty} \left( \frac{K}{K-1} \right)^i \frac{1}{(2i+1)!} \prod_{q=1}^{i} \left( \alpha + K - 1 + (2q-1)(2q-3) \right) x^{2i+1} \right)
\]

(23b)

are linearly independent solutions of (9). Thus we have found the general series solution to the differential equation (9) for the choice of the electromagnetic field \(E\) given in (8). The solution (21) is expressed in terms of a series with real arguments unlike the complex arguments given by software packages. The series (23a) and (23b) converge if there exist a nonnegative value for the radius of convergence. Note that the radius of convergence of the series is not less than the distance from the centre \((x = 0)\) to the nearest root of the leading coefficient of the differential equation (9). Clearly this is possible for a wide range of values for \(K\).

4 Solutions with elementary functions

It is interesting to observe that the series in (23) terminates for restricted values of the parameters \(\alpha\) and \(K\). This will happen when \(\alpha + K\) takes on specific integer values. Utilising this feature it is possible to generate solutions in terms of elementary functions by determining the specific restriction on \(\alpha\) and \(K\) for a terminating series. Solutions in terms of polynomials and algebraic functions can be found. We use the recurrence relation (12c), rather than the series (23), to find the elementary solutions as this is simpler.
4.1 Polynomial solutions

We first consider polynomials of even degree. It is convenient to set

\[ i = 2(j - 1) \]  \hspace{1cm} (24a)  \\
\[ K + \alpha = 2 - (2n - 1)^2 \]  \hspace{1cm} (24b)  

where \( n > 1 \) is fixed integer in (12c). This leads to

\[ a_{2j} = -\gamma \frac{(n + j - 2)(n - j + 1)}{2j(2j - 1)} a_{2j-2} \]  \hspace{1cm} (25)  

where we have set \( \gamma = 4 - \frac{4}{4n(n-1)+\alpha} \). We note that (25) implies \( a_{2(n+1)} = 0 \). Consequently the remaining coefficients \( a_{2(n+2)}, a_{2(n+3)}, \ldots \) vanish. Equation (25) may be solved to yield

\[ a_{2j} = (-\gamma)^j \frac{(n + j - 2)!}{(n - j)!(2j)!} j^{2j} \]  \hspace{1cm} (26)  

where we have set \( a_0 = \frac{1}{n(n-1)} \). With the help of (26) we can express the polynomial in even powers of \( x \) in the form

\[ f_1(x) = \sum_{j=0}^{n} (-\gamma)^j \frac{(n + j - 2)!}{(n - j)!(2j)!} x^{2j} \]  \hspace{1cm} (27)  

for \( K + \alpha = 2 - (2n - 1)^2 \).

We now consider polynomials of odd degree. For this case we let

\[ i = 2(j - 1) + 1 \]  \hspace{1cm} (28a)  \\
\[ K + \alpha = 2(1 - 2n^2) \]  \hspace{1cm} (28b)  

where \( n > 0 \) is fixed integer in (12c). We obtain

\[ a_{2j+1} = -\mu \frac{(n + j - 1)(n - j + 1)}{2j(2j + 1)} a_{2j-1} \]  \hspace{1cm} (29)  

where we have set \( \mu = 4 - \frac{4}{(4n^2-1)+\alpha} \). We observe that (29) implies \( a_{2(n+1)+1} = 0 \). Consequently the remaining coefficients \( a_{2(n+2)+1}, a_{2(n+3)+1}, \ldots \) vanish. Equation (29) can be solved to yield

\[ a_{2j+1} = (-\mu)^j \frac{(n + j - 1)!}{(n - j)!(2j + 1)!} j^{2j+1} \]  \hspace{1cm} (30)  

where we have set \( a_1 = \frac{1}{n} \). With the assistance of (30) we can express the polynomial in odd powers of \( x \) as

\[ g_1(x) = \sum_{j=0}^{n} (-\mu)^j \frac{(n + j - 1)!}{(n - j)!(2j + 1)!} x^{2j+1} \]  \hspace{1cm} (31)  

for \( K + \alpha = 2(1 - 2n^2) \).

The polynomial solutions (27) and (31) comprise the first solution of (9) for appropriate values of \( K + \alpha \).
4.2 Algebraic solutions

We take the second solution of (9) to be of the form

\[ \psi(x) = u(x)(1 - K + Kx^2)^{3/2} \]  

(32)

when \( u(x) \) is an arbitrary polynomial. Particular solutions found in the past are special cases of this general form; the factor \((1 - K + Kx^2)^{3/2}\) helps to simplify the integration process. This motivates the algebraic form for \( \psi \) as a generic solution to the differential equation (9). On substituting \( \psi \) in (9) we obtain after simplification

\[ (1 - K + Kx^2) \frac{d^2u}{dx^2} + 5Kx \frac{du}{dx} + K(\alpha + K + 2)u = 0 \]  

(33)

which is a linear differential equation for \( u(x) \).

As in §4.1 we can find two classes of polynomial solutions for \( u(x) \), in even powers of \( x \) and in odd powers of \( x \), for certain values of \( K + \alpha \). As the point \( x = 0 \) is a regular point of (33), there exists two linearly independent solutions of the form of power series with centre \( x = 0 \). Therefore we can write

\[ u(x) = \sum_{i=0}^{\infty} b_i x^i \]  

(34)

where \( b_i \) are the coefficients of the series. Substituting (34) in (33) we obtain

\[ (1 - K + Kx^2) \frac{d^2u}{dx^2} + 5Kx \frac{du}{dx} + K[\alpha + K + 2 + i(i + 4)]b_i x^i = 0 \]  

(35)

For equation (35) to hold true for all \( x \) we require that

\[ (1 - K)2.1b_2 + K(\alpha + K + 2)b_0 + [(1 - K)3.2b_3 + K(\alpha + K + 7)b_1]x \]

\[ + \sum_{i=2}^{\infty} \{(1 - K)(i + 2)(i + 1)b_{i+2} + K[\alpha + K + 2 + i(i + 4)]b_i\} x^i = 0 \]  

(36a)

\[ (1 - K)3.2b_3 + K(\alpha + K + 7)b_1 = 0 \]  

(36c)

which governs the coefficients.

We first consider even powers of \( x \). Replacing \( i \) with \( 2(j - 1) \) and assuming \( K + \alpha = 2(1 - 2n^2) \) in (36c), where \( n > 0 \) is fixed integer, we obtain

\[ b_{2j} = -\mu \frac{(n + j)(n - j)}{2j(2j - 1)} b_{2j-2} \]  

(37)

where we have set \( \mu = 4 - \frac{4}{4n^2 - 1 + \alpha} \). From (37) we have that \( b_{2n} = 0 \) and subsequent coefficients \( b_{2(n+1)}, b_{2(n+2)}, \ldots \) vanish. Then (37) has the solution

\[ b_{2j} = (-\mu)^j \frac{(n + j)!}{(n - j - 1)!(2j)!}, \quad 0 \leq j \leq n - 1 \]  

(38)
where we have set \( b_0 = n \). On using (34) and (38) the polynomial in even powers of \( x \) leads to the expression

\[
g_2(x) = (1 - K + Kx^2)^{3/2} \sum_{j=0}^{n-1} (-\mu)^j \frac{(n+j)!}{(n-j-1)!(2j)!} x^{2j} \tag{39}
\]

for \( K + \alpha = 2(1 - 2n^2) \).

We now consider odd powers of \( x \). Replacing \( i \) with \( 2(j - 1) + 1 \) and assuming \( K + \alpha = 2 - (2n - 1)^2 \) in (36c), where \( n > 1 \) is fixed integer, we obtain

\[
b_{2j+1} = -\gamma \frac{(n+j)(n-j-1)}{2j(2j+1)} b_{2j-1} \tag{40}
\]

where we have set \( \gamma = 4 - \frac{4}{4n(n-1) + \alpha} \). From (40) we have that \( b_{2(n-1)+1} = 0 \) and subsequent coefficients \( b_{2n+1}, b_{2(n+1)+1}, \ldots \) vanish. Then equation (40) has the solution

\[
b_{2j+1} = (-\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!}, \quad 0 \leq j \leq n - 2 \tag{41}
\]

where we have set \( b_1 = n(n-1) \). On using (34) and (41) the polynomial in odd powers of \( x \) leads to the result

\[
f_2(x) = (1 - K + Kx^2)^{3/2} \sum_{j=0}^{n-2} (-\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!} x^{2j+1} \tag{42}
\]

for \( K + \alpha = 2 - (2n - 1)^2 \).

The algebraic solutions (39) and (42) comprise the second solution of (9) for appropriate values of \( K + \alpha \). The solutions (39) and (42) are expressed as products of algebraic functions and polynomials, and they are clearly linearly independent from (31) and (27), respectively.

### 4.3 Elementary functions

We have obtained two classes of polynomial solutions (27) and (31) in §4.1 to the differential equation (9). Also we have found two classes of algebraic solutions (39) and (42) in §4.2. By collecting these results we can express the general solution to (9) in two categories. The first category of solution for \( \psi(x) = f(x) \) is given by

\[
f(x) = A f_1(x) + B f_2(x)
\]

\[
= A \sum_{j=0}^{n} (-\gamma)^j \frac{(n+j-2)!}{(n-j)!(2j)!} x^{2j} + B(1 - K + Kx^2)^{3/2} \sum_{j=0}^{n-2} (-\gamma)^j \frac{(n+j)!}{(n-j-2)!(2j+1)!} x^{2j+1} \tag{43}
\]
for the values
\[
\gamma = 4 - \frac{4}{4n(n-1)+\alpha} \quad (44a)
\]
\[
K + \alpha = 2 - (2n-1)^2. \quad (44b)
\]

The second category of solution for \( \psi(x) = g(x) \) has the form
\[
g(x) = Ag_1(x) + Bg_2(x)
\]
\[
= A \sum_{j=0}^{n} (-\mu)^j \frac{(n+j-1)!}{(n-j)!(2j+1)!} x^{2j+1}
\]
\[
+ B(1-K+Kx^2)^{3/2} \sum_{j=0}^{n-1} (-\mu)^j \frac{(n+j)!}{(n-j-1)!(2j)!} x^{2j} \quad (45)
\]
for the values
\[
\mu = 4 - \frac{4}{4n^2-1+\alpha} \quad (46a)
\]
\[
K + \alpha = 2(1-2n^2) \quad (46b)
\]
where \( A \) and \( B \) are arbitrary constants.

It is remarkable that these solutions are expressed completely as combinations of polynomials and algebraic functions. From our general class of solutions (43) and (45) it is possible to generate particular solutions found previously. Consider one example with \( \alpha = 0 \) and \( K = -7(n = 2) \). Then \( \gamma = \frac{7}{2} \) and it is easy to verify that equation (43) becomes
\[
\psi = A \left( 1 - \frac{7}{2} x^2 + \frac{49}{24} x^4 \right) + Bx \left( 1 - \frac{7}{8} x^2 \right)^{3/2} . \quad (47)
\]
Thus we have regained the Tikekar\footnote{5} solution for a superdense neutron star from our general solutions. Many other particular solutions found in the literature are also contained in our general solutions, eg. the model of Patel and Koppar\footnote{10}. The solutions (43) and (45) reduce to the Maharaj and Leach\footnote{6} model when \( \alpha = 0 \). Our solutions are applicable to a charged superdense star with spheroidal geometry. When \( \alpha = 0 \) we obtain uncharged relativistic stars which model ultradense barotropic matter.

5 Physical conditions

In the general solution of Sharma \textit{et al.}\footnote{9} it is not possible to isolate the spheroidal parameter \( K \) as that solution is given in terms of special functions. Our solutions are in terms of simple elementary functions which facilitate a study of the physical features,
in particular the role of $K$. The exact solutions (43) and (45) make it possible to analyse the role of the spheroidal parameter $K$ and it’s connection to the electromagnetic field. In particular it is possible to make the following comment about the special role that the spheroidal parameter $K$ has in charged solutions. The form of the solution for the uncharged relativistic star is similar to (43) and (45); however the models are different because the coefficients of the polynomials (namely $\gamma$ and $\mu$) differ by the parameter $\alpha$. If the parameter $\alpha \geq 0$ then we observe that

$$K(\alpha \neq 0) < K(\alpha = 0)$$

Hence the presence of charge directly affects the spheroidal geometry through the parameter $K$. The geometry of the hypersurfaces $\{t = \text{constant}\}$ in the spacetime manifold is related to the electromagnetic field via the relationships $K = 2 - (2n-1)^2 - \alpha$ and $K = 2(1-2n^2) - \alpha$. Such explicit relationships connecting the spacetime geometry to the energy momentum (or electromagnetic field) are rare in exact solutions. The presence of charge $\alpha$ decreases the value of the spheroidal parameter $K$ in our solutions.

We make a few brief comments about the physics of the models found in this paper. If $0 < K < 1(\alpha < 0)$ then $\rho$ remains positive in the region

$$(1 - x^2) < \frac{3(1 - K)}{K(1 - K - \alpha/2)} \Rightarrow r^2 < \frac{3R^2(1 - K)}{K(1 - K - \alpha/2)}$$

which restricts the size of the configuration. When $K < 0(\alpha > 0)$ there is no restriction on $\rho$. Hence $\rho$ is positive in the interior of the star. It is clear from (7a) and (8) that $d\rho/dr < 0$ for $K < 0(\alpha > 0)$. Consequently the energy density decreases from the centre to the boundary. For the pressure to be vanish at the boundary $r = a$ we require that

$$\left(\frac{2}{R^2\sqrt{1 - a^2/R^2}}\left[\frac{\dot{\psi}}{\psi}\right]_{r=a} - \frac{1}{a^2}\right) \frac{1 - a^2/R^2}{1 - Ka^2/R^2} + 1 + \frac{\alpha Ka^2/R^2}{2R^2(1 - Ka^2/R^2)^2} = 0$$

(49)

where $\psi$ is given by (43) or (45). This will constrain the values of the constants $A$ and $B$. The solution of the Einstein-Maxwell equations for $r > a$ is given by the Reissner-Nordstrom metric as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(50)

where $m$ and $q$ are the total mass and charge of the star respectively. To match the line element (11) with the Reissner-Nordstrom metric (50) across the boundary at $r = a$ we require the continuity of the gravitational potentials and of the radial electric field at $r = a$. Continuity of the gravitational potentials yields the relationships between
the constants $A, B, K, a$ and $R$ as

\[
\left(1 - \frac{2m}{a} + \frac{q^2(a)}{a^2}\right) = \left[A\psi_1(a) + B\psi_2(a)\right]^2 \tag{51a}
\]

\[
\left(1 - \frac{2m}{a} + \frac{q^2(a)}{a^2}\right)^{-1} = \frac{1 - Ka^2/R^2}{1 - a^2/R^2} \tag{51b}
\]

The continuity of electric field yields the form

\[
q^2(a) = -\frac{\alpha Ka^6/R^4}{(1 - Ka^2/R^2)^2} \tag{52}
\]

for the charge at the boundary. This shows that continuity of the metric functions across the boundary $r = a$ is easily achieved. The matching conditions at $r = a$ may place restrictions on the metric coefficients $\nu$ and its first derivative for uncharged matter; and the pressure may be nonzero if there is a surface layer of charge. However there are sufficient free parameters to satisfy the necessary conditions that arises for a particular spheroidal model. It is interesting to note that our solutions may be interpreted as models for relativistic anisotropic stars where the parameter $\alpha$ plays a role of the anisotropy factor. Isotropic and uncharged stars can be regained when $\alpha = 0$. Chaisi and Maharaj\textsuperscript{23}, Dev and Gleiser\textsuperscript{24,25} and Maharaj and Chaisi\textsuperscript{26} provide some recent treatments involving the physics of anisotropic matter.

**Acknowledgements**

KK thanks the National Research Foundation and the University of KwaZulu-Natal for financial support, and also extends his appreciation to the South Eastern University of Sri Lanka for granting study leave. SDM acknowledges that this work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation. We are grateful to the referee for a careful reading of the manuscript.

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