Horizons of coalescing black holes on the Eguchi–Hanson space

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Abstract
Using the numerical method, we study the dynamics of coalescing black holes on the Eguchi–Hanson base space. The effects of a difference in spacetime topology on the black hole dynamics is discussed. We analyze the appearance and disappearance process of marginal surfaces. In our calculation, the area of a coverall black hole horizon at the creation time in the coalescing black hole solutions on the Eguchi–Hanson space is larger than that in the five-dimensional Kastor–Traschen solutions. This fact suggests that the black hole production on the Eguchi–Hanson space is easier than that on the flat space.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the framework of the braneworld scenario, higher dimensional black holes are expected to be produced in a future linear collider [1–6]. By observing physical phenomena associated with the black holes we might obtain evidence for the existence of extra dimensions. Such black holes, which evaporate by the Hawking radiation, are also expected to play crucial roles in the yet unaccomplished theoretical development to reconcile gravitational interactions with quantum description of nature.

So far, many authors have focused mainly on asymptotically flat and stationary higher dimensional black holes since they would be idealized models if such black holes are small enough for us to neglect the tension of a brane or the size of extra dimensions. It has been clarified that such asymptotically flat higher dimensional black hole solutions have a richer structure than the four-dimensional one [7–10]. However, there is no reason to restrict the
asymptotic structures of higher dimensional spacetimes to the flat spacetime. Then, we do not have to restrict ourselves to black hole solutions with asymptotic flatness. In fact, higher dimensional black holes would admit a variety of asymptotic structures. For example, the black hole solutions in Kaluza–Klein theory admit the structure of a twisted $S^1$ fiber bundle over four-dimensional Minkowski spacetime [11–13] or a direct product of $S^1$ and four-dimensional Minkowski spacetime [14].

Recently, the coalescing black hole solutions on the Eguchi–Hanson space (CBEH) are constructed in the five-dimensional Einstein–Maxwell theory with a positive cosmological constant [15]. These solutions are asymptotic locally de Sitter spacetime; the topology of the radial coordinate $r = \text{const}$ surfaces is not a sphere $S^3$ but the lens space. In this paper, the behavior of black holes at the early time and the late time are mainly discussed. The reason for this restriction is that it is easy to analyze the structure of solutions in such a region, which one can regard as that of the five-dimensional Reissner–Nordström–de Sitter solution (RNdS). As a result, it is clarified that the solutions describe the physical situation such that two black holes with the topology of $S^3$ coalesce and change into a single black hole with the topology of the lens space $L(2; 1) = S^3/\mathbb{Z}_2$.

Another solution of Einstein–Maxwell theory with a positive cosmological constant in arbitrary dimensions had already been found by London [16]. These solutions, which are the generalization of the Kastor–Traschen solution [17] to higher dimensions, describe the dynamical situation such that the arbitrary number of multi black holes with a spherical topology coalesce into a single black hole with a spherical topology in asymptotically de Sitter spacetime. The two black hole case of the five-dimensional Kastor–Traschen solutions (5DKT) describes that the two black holes with $S^3$ coalesce into a single black hole with $S^3$.

The purpose of this paper is to investigate the global structure of the CBEH and the 5DKT by the numerical approach and to clarify the effects on coalescence of black holes brought about by the difference in the asymptotic structure between both solutions. Following the numerical method in [18–20], where they discussed how marginal surfaces evolve with time in the four-dimensional Kastor–Traschen solutions, we numerically investigate the existence and the time evolution of marginal surfaces. Especially, we focus on the appearance and disappearance process of marginal surfaces. We also discuss the time evolution of these areas.

The rest of the paper is organized as follows. In section 2, we review the CBEH and the 5DKT. We show the method to search for marginal surfaces in section 3. Then, the time sequences of marginal surfaces and these areas is shown in section 4. Section 5 is devoted to the summary and discussion.

2. Brief review

2.1. Five-dimensional Kastor–Traschen solutions

First, let us consider the 5DKT [16], namely, the black hole solutions on a flat base space. Especially, we concentrate on the solution with two black holes whose masses are $m_1$ and $m_2$ at early time:

$$\text{d}s^2 = a^2\left[-H_{\text{KT}}^2 \text{d}r^2 + H_{\text{KT}}(\text{d}r^2 + r^2 \text{d}\Omega_3^2)\right], \quad (1)$$

where $H_{\text{KT}}$ is given by

$$H_{\text{KT}} = \lambda r + \frac{m_1}{|r - r_1|^2} + \frac{m_2}{|r - r_2|^2}, \quad (2)$$

with the position vector on the four-dimensional Euclid space $r$. $r_1$ and $r_2$ are the positions of point sources. We can set $r_1 = (0, 0, 0, 1)$ and $r_2 = (0, 0, 0, -1)$ without a loss of generality.
2.1.1. Early time. Let us focus on the neighborhood of $r = r_i$ ($i = 1, 2$). In the new coordinate $\tilde{r} = |r - r_i|$, we can write the metric (1) as

$$ds^2 \simeq a^2 \left[ -\left( \lambda \tau + \frac{m_1}{r^2} \right)^{-2} d\tau^2 + \left( \lambda \tau + \frac{m_2}{r^2} \right) \left\{ d\tilde{r}^2 + \tilde{r}^2 d\Omega_3^2 \right\} \right],$$

where $d\Omega_3^2$ is the metric of a unit 3-sphere. This is identical to the metric of the RN\(\Lambda\)S with mass parameter $m_i$ except for the conformal factor $a^2$ which does not contribute to the horizon condition $\theta_{out} = 0$, where $\theta_{out}$ is the outgoing null expansion on the $\tau = \text{const}$ and $\tilde{r} = \text{const}$ surface.

For this metric, let us introduce a variable $x := \lambda \tau \tilde{r}^2$, and then horizons occur at $x$ satisfying

$$\lambda^2 (x + m_i)^3 - 4x^2 = 0.$$  

(4)

For $m_i < 16/(27\lambda^2)$, there are three horizons, i.e., the inner and outer black hole horizons and the de Sitter horizon, which correspond to the three real roots $x_m[m_i] < x_{BH}[m_i] < x_{AS}[m_i]$, respectively.

If $m_i < 16/(27\lambda^2)(i = 1, 2)$, the horizon radius $\tilde{r}^2_{BH} := x_{BH}[m_i]/(\lambda \tau)$ satisfies $\tilde{r}_{BH} \ll |r_i| = 1$ at an early time $\tau \ll 0$. This fact means that we can find an approximately spherical and sufficiently small black hole horizon around $r = r_i$. Hence, an outer trapped region always exists around $r = r_i$.

2.1.2. Late time. Next, we study the asymptotic behavior of the metric for large $r := |r|$, where we assume that $r$ is much larger than the coordinate distance $|r_1 - r_2| = 2$ between the two masses $m_1$ and $m_2$. Then, the metric takes the following form:

$$ds^2 \simeq a^2 \left[ -\left( \lambda \tau + \frac{m_1 + m_2}{r^2} \right)^{-2} d\tau^2 + \left( \lambda \tau + \frac{m_1 + m_2}{r^2} \right) \left\{ d\tilde{r}^2 + \tilde{r}^2 d\Omega_3^2 \right\} \right].$$

(5)

This metric resembles that of the RN\(\Lambda\)S with mass equal to $m_1 + m_2$. If we assume $m_1 + m_2 < 16/(27\lambda^2)$, the horizon radius $\tilde{r}^2_{BH} := x_{BH}[m_1 + m_2]/(\lambda \tau)$ satisfies $\tilde{r}_{BH} \gg |r_1 - r_2| = 2$ at late time $\tau \to -0$. Then the approximate form of the metric (5) is valid around $r = r_{BH}$. Hence, an approximately spherical black hole horizon can be found around $r = r_{BH}$ in the metric (1).

2.2. Black holes on the Eguchi–Hanson base space

Second, we give a brief review on the CBEH [15] whose metric is given by

$$ds^2 = a^2 \left[ -H_{EH}^2 d\tau^2 + \frac{1}{6} H_{EH} \left\{ V^{-1} dR^2 + V^{-1} R^2 d\Omega_3^2 + V (d\psi + \omega_\phi d\phi)^2 \right\} \right],$$

(6)

where

$$H_{EH} = \lambda \tau + \frac{2m_1}{|R - R_1|} + \frac{2m_2}{|R - R_2|},$$

(7)

$$V^{-1} = \frac{1}{|R - R_1|} + \frac{1}{|R - R_2|},$$

(8)

$$\omega_\phi = \frac{z - 1}{|R - R_1|} + \frac{z + 1}{|R - R_2|},$$

(9)

$$d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

(10)

and $\mathbf{R} = (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta)$ is the position vector on the three-dimensional Euclid space and positions of point sources $\mathbf{R}_1$ and $\mathbf{R}_2$ are set to be $(0, 0, 1)$ and $(0, 0, -1)$. The
range of angular coordinates is defined by $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. This metric is given by equation (9) in [15], rewriting as $R \rightarrow aR$, $\tau \rightarrow a\tau$, $\lambda \rightarrow \lambda/a$, $m_1 \rightarrow \lambda a^2 m_1$ and $m_2 \rightarrow a^2 m_2$. This is a solution of the five-dimensional Einstein equation with a positive cosmological constant and the Maxwell equation with a gauge potential 1-form is given by

$$A = \pm \frac{\sqrt{3}}{2} aH_{\text{BH}}^1 \, \text{d}r.$$  (11)

In order to focus on the coalescence of two black holes, we only consider the contracting phase $\lambda < 0$. Though $\tau$ runs the range $(-\infty, \infty)$, in this paper we only investigate the region $-\infty < \tau \leq 0$.

2.2.1. Early time. First, let us focus on the neighborhood of $R = R_i$ ($i = 1, 2$). In terms of the new coordinate $\bar{r}^2 := |R - R_i|^2/2$, the metric can be written in the form

$$\text{d}s^2 \simeq a^2 \left[ -\left( \lambda \tau + \frac{m_1}{\bar{r}^2} \right)^{-2} \, \text{d}\tau^2 + \left( \lambda \tau + \frac{m_1}{\bar{r}^2} \right) \left\{ \text{d}\bar{r}^2 + \frac{\bar{r}^2}{4} \text{d} \Omega_5^2 + \frac{\bar{r}^2}{4} \left( \frac{1}{2} \text{d}\psi + \cos \theta \, \text{d}\phi \right)^2 \right\} \right].$$  (12)

This is equivalent to the metric of the RNdS which has the mass equal to $m_1$ written in the cosmological coordinate. Hence like the 5DKT, we can conclude that a nearly spherical and small black hole horizon can be found around $R = R_i$ in the metric (1) at the early time, and sufficiently small spheres with the topology of $S^3$ centered at $R = R_i$ are always outer trapped.

2.2.2. Late time. Next, we study the asymptotic behavior of the metric (6) in the region where $R$ is much larger than the coordinate distance $|R_i - R| = 2$. Here, let us introduce a new coordinate $\hat{r}^2 := |R - R_i|/2$, and then the metric takes the following form:

$$\text{d}s^2 \simeq a^2 \left[ -\left( \lambda \tau + \frac{m}{\hat{r}^2} \right)^{-2} \, \text{d}\tau^2 + \left( \lambda \tau + \frac{m}{\hat{r}^2} \right) \left\{ \text{d}\hat{r}^2 + \frac{\hat{r}^2}{4} \text{d} \Omega_5^2 + \frac{\hat{r}^2}{4} \left( \frac{1}{2} \text{d}\psi + \cos \theta \, \text{d}\phi \right)^2 \right\} \right],$$  (13)

where $m = 2(m_1 + m_2)$. This resembles the metric of the RNdS solution with mass equal to $m$, and if we assume $m < 16/(27\lambda^2)$, a nearly spherical black hole horizon can be found in the metric (6) with $\hat{r} = r_{\text{BH}}[m]$ at late time $\tau \rightarrow -0$.

However, we note that the metric form of (13) differs from that of the RNdS solution in the following point; each $\hat{r} = \text{const}$ surface is topologically the lens space $L(2; 1) = S^3/\mathbb{Z}_2$, while it is diffeomorphic to $S^3$ in the RNdS solution. We can regard $S^3$ and the lens space $L(2; 1) = S^3/\mathbb{Z}_2$ as examples of Hopf bundles, i.e., $S^1$ bundle over $S^2$. The difference between these metrics appears in equations (12) and (13): $\text{d}\psi$ in the metric (12) is replaced by $\text{d}\psi/2$ in the metric (13). Therefore, at late time, the topology of the trapped surface is the lens space $L(2; 1) = S^3/\mathbb{Z}_2$ in the metric (6).

2.3. Comparison

The above results suggest that both solutions describe the coalescence of black holes. (In fact, using the numerical techniques, Nakao et al showed that the four-dimensional Kastor–Traschen solutions describe such a physical process [20].) Between both solutions, there exists an essential difference, namely, in the 5DKT, two black holes with the topology of $S^3$ coalesce into a single black hole with the topology of $S^3$, while in the CBEH, two black holes
with the topology of $S^3$ coalesce and change into a single black hole with the topology of $L(2; 1) = S^3/\mathbb{Z}_2$. In the next section, we investigate how two black holes coalesce in the two solutions by pursuing the time evolution of marginal surfaces.

3. Method to search for marginal surfaces

Here, we seek marginal surfaces on $\tau = \text{const}$ surfaces, which are defined as surfaces of co-dimension 2 such that the outgoing orthogonal null geodesics have zero convergence $\theta_{\text{out}}$ on the surfaces. The metrics (1) and (16) are decomposed into the form

$$g_{ab} = -n_an_b + h_{ab},$$  \hspace{1cm} (14)

where $n^a := H_{\text{EHKKT}}a^{-1}(\partial/\partial \tau)^a$ and $h_{ab}$ denote the timelike unit vector normal to the $\tau = \text{const}$ surfaces and the induced metric on the surfaces, respectively.

In our numerical computation, we use the coordinate system $(\tau, z, \rho, \phi, \psi)$. The metric (1) is written as

$$ds^2 = a^2 \left[ -H_{\text{KKT}} d\tau^2 + H_{\text{EH}} \left( dz^2 + d\rho^2 + \rho^2 (d\phi^2 + \sin^2 \phi d\psi^2) \right) \right],$$  \hspace{1cm} (15)

in this coordinate system, and the metric (6) is written as

$$ds^2 = a^2 \left[ -H_{\text{EH}}^2 d\tau^2 + \frac{1}{H_{\text{KKT}}} H_{\text{KKT}} \left( dz^2 + d\rho^2 + \rho^2 (d\phi^2 + \sin^2 \phi d\psi^2) \right) + V(d\psi + \omega \phi d\phi) \right].$$  \hspace{1cm} (16)

Let $s^a$ be the spacelike unit vector normal to such marginal surfaces on the $\tau = \text{const}$ surfaces, and consider the marginal surfaces as $(z, \rho, \phi, \psi) = (z(v), \rho(v), \phi, \psi)$, i.e., they are parametrized by $v, \phi$ and $\psi$ on the $\tau = \text{const}$ surfaces. Then, the metric $h_{ab}$ can be written in the following form:

$$h_{ab} = s_as_b + \delta_{ij}(e^i)_a(e^j)_b,$$  \hspace{1cm} (17)

where $\delta_{ij} = \text{diag}(1, 1, 1)$ and $(e^i)_a (i = 1, 2, 3)$ are triplet bases on the marginal surface. In the case of the CBEH, we can set $s^a$ and $(e^i)_a$ in the forms

$$(e_1)^a = \frac{2\sqrt{2V}}{a \sqrt{H_{\text{EH}}(\rho^2 + z^2)}} \left( \frac{\partial}{\partial z} \right)^a + \rho \left( \frac{\partial}{\partial \rho} \right)^a,$$  \hspace{1cm} (18)

$$(e_2)^a = \frac{2\sqrt{2V}}{a \sqrt{H_{\text{EH}}} (\rho^2 + \omega_\phi^2 V^2)} \left( \frac{\partial}{\partial \phi} \right)^a,$$  \hspace{1cm} (19)

$$(e_3)^a = \frac{2\sqrt{2V}}{a \rho \sqrt{H_{\text{EH}}} V} \left( -\omega_\phi V^2 \left( \frac{\partial}{\partial \phi} \right)^a + \sqrt{\rho^2 + \omega_\phi^2 V^2} \left( \frac{\partial}{\partial \psi} \right)^a \right),$$  \hspace{1cm} (20)

$$s^a = \pm \frac{2\sqrt{2V}}{a \sqrt{H_{\text{EH}}} (\rho^2 + z^2)} \left( \rho \left( \frac{\partial}{\partial z} \right)^a - z \left( \frac{\partial}{\partial \rho} \right)^a \right),$$  \hspace{1cm} (21)

where the sign $\pm$ of $s^a$ should be chosen so that $s^a$ directs outward. On the other hand, in the case of the 5DKT, we can set these vectors in the forms

$$(e_1)^a = \frac{1}{a \sqrt{H_{\text{KKT}}} (\rho^2 + z^2)} \left( \frac{\partial}{\partial z} \right)^a + \rho \left( \frac{\partial}{\partial \rho} \right)^a,$$  \hspace{1cm} (22)

$$(e_2)^a = \frac{1}{a \rho \sqrt{H_{\text{KKT}}} } \left( \frac{\partial}{\partial \phi} \right)^a,$$  \hspace{1cm} (23)
\[(e_3)^a = \frac{1}{a \rho \sqrt{H_{KT} \sin \phi}} \left( \frac{\partial}{\partial \psi} \right)^a, \tag{24} \]
\[s^a = \pm \frac{1}{a \sqrt{H_{KT} (\rho^2 + z^2)}} \left( \rho \left( \frac{\partial}{\partial z} \right)^a - z \left( \frac{\partial}{\partial \rho} \right)^a \right). \tag{25} \]

The expansion \(\theta_{\text{out}}\) of the null congruence which is normal to the marginal surface is given by
\[\theta_{\text{out}} = (h^{ab} - s^a s^b) \nabla_b (n_a + s_a) = - \sum_{i=1,2,3} s_a (e_i)^b D_b (e_i)^a + s^a s^b k_{ab} - \text{tr} k, \tag{26} \]
where \(k\) is the trace of the extrinsic curvature \(k_{ab}\) of the \(\tau = \text{const}\) surface. By the definition of a marginal surface, the expansion vanishes \(\theta_{\text{out}} = 0\) on the surface. On the other hand, the expansion of the ingoing null congruence which is normal to the marginal surface is given by
\[\theta_{\text{in}} = (h^{ab} - s^a s^b) \nabla_b (n_a - s_a) = \sum_{i=1,2,3} s_a (e_i)^b D_b (e_i)^a + s^a s^b k_{ab} - \text{tr} k. \tag{27} \]

Since \(\theta_{\text{out}}\) vanishes on the marginal surface by its definition, we have
\[\theta_{\text{in}} = \theta_{\text{in}} + \theta_{\text{out}} = 2 s^a s^b k_{ab} - 2 \text{tr} k = \frac{3 \lambda}{a} < 0. \tag{28} \]

Equations (18)–(21) or equations (22)–(25) with equation (26) give a second-order ordinary differential equation
\[\theta_{\text{out}} (\ddot{z}, \dot{\rho}, \dot{z}, \dot{\rho}) = 0 \tag{29} \]
for marginal surfaces on the \(\rho-z\) plane. We find smooth closed curves on the \(\rho-z\) plane, \(\rho = \rho(v), z = z(v)\), satisfying equation (29). It should be noted that equation (29) does not depend on the parameter \(a\). By using the freedom in the choice of the parameter \(v\), following Cadez [18, 20], we fix \(v\) by
\[\dot{z}^2 + \dot{\rho}^2 = \left( \frac{8 V}{H_{EH}} \right)^2, \tag{30} \]
in the case of the CBEH, and
\[\dot{z}^2 + \dot{\rho}^2 = H_{KT}^{-2}, \tag{31} \]
in the case of the 5DKT.

Using these parametrizations (30) and (31) of \(v\) and imposing the equations on the boundary conditions \(\dot{z} = 0\) at the \(z\)-axis, we can numerically search for marginal surfaces.

4. Time evolution of horizons

4.1. Marginal surfaces

We would like to pursue how two black holes evolve with time and coalesce for both solutions. We restrict the range of the mass parameters to
\[m_1 + m_2 < \frac{8}{27 \lambda^2}, \tag{32} \]
so that a black hole horizon exists after the coalescence. In this paper, we consider the case where two black holes at early time have equal masses and set them to be \( m_1 = m_2 = 1/(8\lambda^2) \) and \( \lambda = -1/(2\sqrt{2}) \) for both solutions. Under this assumption, since there is a reflection symmetry \( z \rightarrow -z \), it is sufficient to consider only the region of \( z \geq 0 \). In general, several marginal surfaces exist on each timeslice. We label each marginal surface which corresponds to a black hole horizon and a de Sitter horizon at the early time as BHE and dSE, respectively, and label each marginal surface which corresponds to a black hole horizon and a de Sitter horizon at the late time as BHL and dSL, respectively. Some of marginal surfaces appear or disappear in pairs with another marginal surface. We label the marginal surfaces other than black hole horizons and de Sitter horizons as MS\(_i\) (\( i = 1, 2, \ldots \)). To avoid confusion, we do not depict marginal surfaces which are not related to the appearance and disappearance of BHE, dSE, BHL and dSL.

Figures 1–3 show the time sequence of marginal surfaces in the 5DKT. Before \( \tau = -250 \), there are two black hole horizons BHE and two de Sitter horizons dSE enclosing each BHE. In addition, there is a marginal surface MS\(_1\) surrounding the two black hole horizons. After the lapse of time, at a time within the period \(-160 < \tau < -140\), another de Sitter horizon dSL appears in pairs with another marginal surface MS\(_2\). After a brief interval, each dSE disappears in pairs with MS\(_1\), and MS\(_2\) pinches off at a time in \(-100 < \tau < -80\). Finally, at a time in \(-10 < \tau < -5\), a new black hole horizon BHL appears in pairs with a new marginal surface MS\(_3\), and then it asymptotically approaches to the black hole horizon of the RNdS with the mass parameter \( m_1 + m_2 \).

On the other hand, figures 4–6 show the time sequence of marginal surfaces in the CBEH. Before \( \tau = -250 \), there exist two BH\(_E\) and two dS\(_E\). At a time within the period \(-230 < \tau < -220\), dS\(_E\) appears in pairs with MS\(_1\). After a brief interval, dS\(_E\) disappears in pairs with MS\(_1\) at a time in \(-140 < \tau < -120\). Finally, BH\(_L\) appears in pairs with MS\(_2\) at a time in \(-10 < \tau < -5\), and BH\(_L\) approaches a black hole horizon of the RNdS with the mass \( 2m_1 + 2m_2 \) whose horizon topology is the lens space \( L(2; 1) = S^3/\mathbb{Z}_2 \).

We can see that two solutions differ in the number of marginal surfaces which are related to the appearance and disappearance of dS\(_E\) and dS\(_L\). In each solution, the situation does not essentially depend on the choice of the parameters \( m_1, m_2 \) and \( \lambda \). Hence, this result suggests that this difference does not come from the difference in the choice of the parameters but in the asymptotic structures.
Figure 2. Time evolution of marginal surfaces in the 5DKT in \(-250 < \tau < -100\). Each left frame is the scaled-up figure of the region near BH_E. At \(\tau = -250\), BH_E, dS_E and MS_1 exist. dS_L and MS_2 appear at same time during \(-160 < \tau < -140\).

4.2. Areas of horizons

First, for later convenience, we introduce the areas of horizons in the RNdS with the horizon topology of \(L(n; 1) = S^3/\mathbb{Z}_n\) given by

\[
A_n(r_{BH}[m']) = \frac{2\pi^2 a d^3 r_{BH}[m']}{n} \left( \lambda \tau + \frac{m'}{r_{BH}[m']} \right)^{3/2},
\]

\[
A_n(r_{dS}[m']) = \frac{2\pi^2 a^3 r_{dS}[m']}{n} \left( \lambda \tau + \frac{m'}{r_{dS}[m']} \right)^{3/2},
\]

(33)  (34)
Figure 3. Time evolution of marginal surfaces in the 5DKT in $-80 < \tau < -1$. The left frame at $\tau = -80$ is the scaled-up figure of the region near BH$_E$. dS$_{E}$ and MS$_1$ disappear at the same time and MS$_2$ pinches off during $-100 < \tau < -80$. Finally, MS$_3$ and BH$_L$ appear at the same time during $-2 < \tau < -1$. At $\tau = -1$, dS$_L$ exists in the outside of the frame.

where $m'$ is the mass parameter in the metric form written by

$$ds^2 = a^2 \left[ - \left( \frac{\lambda \tau + m'}{r^2} \right)^2 d\tau^2 + \left( \frac{\lambda \tau + m'}{r^2} \right) \left\{ d\tau^2 + \frac{r^2}{4} d\Omega^2_{S^2} + \frac{r^2}{4} \left( \frac{1}{n} \, d\psi + \cos \theta \, d\phi \right)^2 \right\} \right].$$

(35)
Figure 4. Time evolution of marginal surfaces in the CBEH when BH_E, dS_E, BH_L, dS_L and two marginal surfaces MS_1, MS_2 exist. The vertical axis denotes the values of \( \tau \). A pair of marginal surfaces connected by a dashed line appears or disappears at one time.

In the previous work [15], we pointed out that after two black holes with the horizon topology of \( S^3 \) coalesce, the area of the eventual single black hole in the CBEH is larger than that in the 5DKT, where we assume that each black hole in the CBEH has the same mass and area as that in the 5DKT. The difference is essentially due to the asymptotic structure. While the 5DKT is asymptotically de Sitter and each surface enclosing two black holes has the topological structure of \( S^3 \), the topological structure of those in the CBEH is \( L(2; 1) = S^3 / \mathbb{Z}_2 \). In this sense, the CBEH is not asymptotically de Sitter but asymptotically locally de Sitter. Namely, the horizon radius of a black hole in the spacetime whose spatial infinity has the lens space \( L(2; 1) = S^3 / \mathbb{Z}_2 \) becomes larger than that of the spacetime which has asymptotically Euclidean timeslices even if they have the same mass.

Using the results in the previous work, the ratios of areas of the black hole horizon and de Sitter horizon at the early time in the CBEH to those in the 5DKT become

\[
\frac{A_{EH}}{A_{KT}} = \frac{A_i(r_{BH}[m_1])}{A_i(r_{BH}[m_1])} = 1, \quad \frac{A_{dS}}{A_{dS}} = \frac{A_i(r_{dS}[m_1])}{A_i(r_{dS}[m_1])} = 1, \quad \text{(36)}
\]

where EH and KT denote the quantities associated with the CBEH and the 5DKT, respectively.

On the other hand, those at the late time become

\[
\frac{A_{EH}}{A_{BH}} = \frac{A_2(r_{BH}[2(m_1 + m_2)])}{A_1(r_{BH}[m_1 + m_2])} = C_1, \quad \frac{A_{dS}}{A_{dS}} = \frac{A_2(r_{dS}[2(m_1 + m_2)])}{A_1(r_{dS}[m_1 + m_2])} = C_2, \quad \text{(37)}
\]

where \( C_1 \) and \( C_2 \) are some constants determined by the values of \( \lambda, m_1 \) and \( m_2 \). In our setting, we find \( C_1 = 0.332 \ldots \) and \( C_2 = 2.350 \ldots \). Here, it should be noted that the area of the black hole horizon in the CBEH is larger than that in the 5DKT, but reversely the area of the de Sitter horizon in the CBEH is smaller than that in the 5DKT.

Here, we numerically study how the areas of black hole horizons evolve with time in the 5DKT and the CBEH. The area of each marginal surface on \( \tau = \text{const} \) surfaces is computed as

\[
A_{EH} = \frac{\pi^2 a^3}{2} \int_{v_1}^{v_2} \rho \sqrt{\frac{H_{EH}^3(z^2 + \hat{\rho}^2)}{V}} \mathrm{d}v, \quad A_{KT} = 4\pi a^3 \int_{v_1}^{v_2} \rho^2 \sqrt{H_{KT}^3(z^2 + \hat{\rho}^2)} \mathrm{d}v, \quad \text{(38)}
\]

where \( v_1 \) and \( v_2 \) satisfy \( z(v_1) = z(v_2) = 0 \).

Figures 7 and 8 show the evolution of the areas of de Sitter horizons dS_E and dS_L. Figures 9 and 10 show the time evolution of the areas of black hole horizons BH_E and
Figure 5. Time evolution of marginal surfaces in the CBEH in $-250 < \tau < -140$. Each left frame is the scaled-up figure of the region near BH$_E$. At $\tau = -250$, BH$_E$ and dS$_E$ exist. MS$_I$ and dS$_I$ appear at same time during $-250 < \tau < -220$.

BH$_L$. The time evolution of the areas of BH$_E$ and BH$_L$ is shown in figure 11. In fact, from these figures we can confirm that these values of areas asymptotically approach to the values computed from equations (36) and (37). From figure 11, we see that the area of BH$_L$ in the CBEH at the appearance is larger than that in the 5DKT. It suggests that the impact parameter at the appearance of black holes in a spacetime with asymptotically locally Euclidean timeslices may be larger than that in a spacetime with asymptotically Euclidean timeslices.
Figure 6. Time evolution of marginal surfaces in the CBEH in $-120 < \tau < -5$. Each left frame is the scaled-up figure of the region near BH$_E$, dS$_G$ and MS$_1$ disappear at the same time during $-140 < \tau < -120$. Finally, MS$_2$ and BH$_E$ appear at the same time during $-10 < \tau < -5$. At $\tau = -5$, dS$_E$ exists in the outside of the frame.

5. Summary and discussion

We have studied the evolution of marginal surfaces in the CBEH and the 5DKT. We have numerically searched for marginal surfaces in each timeslice and calculated the areas of the horizons. Each marginal surface corresponding to the black hole or de Sitter horizon at the early or the late time appears or disappears in pairs with another marginal surface. We have shown the time evolution of marginal surfaces in figures 1–6.
Figure 7. Time evolution of the area of $dS_E$. The vertical axis denotes the area normalized by $A_1(r_{dS_m^1})$.

Figure 8. Time evolution of the area of $dS_L$. The vertical axis denotes the area normalized by $A_1(r_{dS_{m_1+m_2}})$.

The area at the appearance of the black hole enclosing both pre-existent black holes in the CBEH is larger than that in the 5DKT. This suggests that the black hole production on the Eguchi–Hanson base space will be easier than that on the flat base space. It comes from the difference in the asymptotic structure between these solutions. The results of this paper give us the suggestion that the black hole dynamics may be notably affected by the topological structure of the extra dimensions. In the context of TeV gravity scenarios, the topology of the bulk space might be nontrivial. Hence if our living higher dimensional world admits the asymptotic structure of the lens-space topology, the black hole production rate in the linear collider might give us some information or the restriction to the model about the asymptotic structure of the extra dimensions.

Although throughout this paper, we focus on the time evolution of marginal surfaces on a certain timeslice, finally we also comment on the event horizon. In fact, we have searched for the event horizon numerically by tracing null geodesics from the sufficiently future region to the past region [21]. $BH_E$ and $BH_L$ are almost identical to the cross-sections of the event
Figure 9. Time evolution of the area of BHs. The vertical axis denotes the area normalized by $A_1(r_{BH_1})$.

Figure 10. Time evolution of the area of BHs. The vertical axis denotes the area normalized by $A_1(r_{BH_{m1+m2}})$.

horizon with the timeslice $\tau = \text{const}$ surfaces at the sufficiently early and late time, respectively, because the spacetime asymptotically becomes stationary in the sufficiently future and past regions.

From a general viewpoint, Siino discussed the topology change of an event horizon in the four-dimensional spacetime [22, 23] which is asymptotically stationary far in the future and showed that the nontrivial topology changes are caused by the set of endpoints of the event horizon, the so-called crease set of the event horizon, where the event horizon is indifferentiable. Therefore, we expect that the difference between the topological structures of the crease sets will play an essential role in causing the difference in topology change in both solutions. In higher dimensional spacetimes, the structure of such crease sets is more complex than that in four dimensions since the topology of an event horizon far in the future is not determined uniquely since the event horizons in higher dimensional stationary spacetimes.
can admit various topologies [24, 25] in contrast to four-dimensional ones, which is restricted only to $S^3$ [26]. The detailed analysis about the event horizon will be discussed in near future.

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