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Directed transport in a spatially periodic harmonic potential under periodic nonbiased forcing

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Transport of a particle in a spatially periodic harmonic potential under the influence of a slowly time-dependent unbiased periodic external force is studied. The equations of motion are the same as in the problem of a slowly forced nonlinear pendulum. Using methods of the adiabatic perturbation theory we show that for a periodic external force of a general kind the system demonstrates directed (ratchet) transport in the chaotic domain on very long time intervals and obtain a formula for the average velocity of this transport. Two cases are studied: The case of the external force of small amplitude, and the case of the external force with amplitude of order one. The obtained formulas can also be used in case of a nonharmonic periodic potential.

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I. INTRODUCTION

In recent years, studies of transport phenomena in nonlinear systems have been attracting a growing interest. In particular, a large and constantly growing number of papers are devoted to dynamics in systems which allow for directed (on average) motion under unbiased external forces and are referred to as ratchet systems. Their intensive study was motivated by problems of motion of Brownian particles in spatially periodic potentials, unidirectional transport of molecular motors in biological systems, and recognition of “ratchet effects” in quantum physics (see Ref. [1] and references therein). Generally speaking, ratchet phenomena occur due to lack of symmetry in the spatially periodic potential and/or the external forcing. It is interesting, however, to understand microscopic mechanisms leading to these phenomena. A possible approach is to neglect dissipation and noise terms and arrive at a Hamiltonian system with deterministic forcing. Thus, one can make use of results obtained and methods developed in the theory of Hamiltonian chaos. Many papers studying chaotic transport in such Hamiltonian ratchets appeared in the past years (see, e.g., [2–8]). In particular, in [6] the ratchet current is estimated in the case when there are stability islands in the chaotic domain in the phase space of the system. The borders of such islands are “sticky” [9] and this stickiness together with desymmetrization of the islands are responsible for occurrence of the ratchet transport.

We consider the problem of motion of a particle in a periodic harmonic potential $U(q) = \frac{1}{2} \omega_0^2 q^2$, where $q$ is the coordinate and $\omega_0 = \text{const}$, under the influence of unbiased time periodic external forcing. The equations are the same as in the paradigmatic model of a nonlinear pendulum under the action of external torque with zero time average. We study the case when the external forcing is a time-periodic function of large period of order $\epsilon^{-1}$, $0 < \epsilon \ll 1$, and use results and methods of the adiabatic perturbation theory. If $\epsilon$ is small enough, there are no stability islands in the domain of chaotic dynamics (see [10]). Thus, the mechanism of ratchet transport in this system differs from one suggested in [6].

The main objective of this paper is to find a formula for the average velocity $V_q = \langle \dot{q} \rangle$ of a particle in the chaotic domain on very large time intervals. Chaos in the system is a result of multiple separatrix crossings due to slow variation of the external forcing. At each crossing the adiabatic invariant (“action” of the system) undergoes a quasirandom jump (see [11,12]). We show that these jumps result in effective mixing and uniform distribution of the action along a trajectory in the chaotic domain. On the other hand, direction and value of velocity depends on the immediate value of the action. Thus, to find the average velocity of transport on time intervals of order or larger than the mixing time, we find formulas for displacement in $q$ at a given value of the action and then integrate them over the interval of values of the action corresponding to the chaotic domain. We demonstrate that for an external force of general kind (i.e., with zero time average but lowered time symmetry, cf. [5]), there is directed transport in the system and obtain an analytic formula for the average velocity $V_q$ of this transport.

In Sec. II, we obtain the main equations in the case of external force of small amplitude and describe the diffusion of adiabatic invariant due to multiple separatrix crossings. This diffusion makes dynamics chaotic in a large domain of width of order 1. In Sec. III, we derive the formula for the average velocity of transport on very long time intervals (of order of typical diffusion time and larger) and check it numerically. In Sec. IV, we consider the case when the external forcing is not small, of order 1. Width of the chaotic domain in this situation is large, of order $\epsilon^{-1}$. In this case chaos arises as a result of scattering of the adiabatic invariant on the resonance. Formula for the average velocity of transport in this case turns out to be much simpler than in the case of external forcing of small amplitude.

II. MAIN EQUATIONS: DIFFUSION OF THE ADIABATIC INVARIANT

We start with a basic equation describing a nonlinear pendulum under the influence of an external force $\tilde{f}(t)$,
To make the following presentation more clear, we choose
fined up to an additive constant, which we are free to choose.

\[ \dot{q} + \omega_0^2 \sin q = f(t) \]  

We assume that \( f(t) \) is small in amplitude periodic function
of time of large period: \( f(t) = \epsilon f(\tau) = \epsilon f(\tau + 2\pi) \), where \( \tau \equiv t \). Moreover, we assume that \( f(t) \) is a function with zero

time average. This system is Hamiltonian, with time-
dependent Hamiltonian function

\[ H = \frac{\bar{p}^2}{2} - \omega_0^2 \cos q - \epsilon f(\tau) q. \]  

We make a canonical transformation of variables
\((p,q) \rightarrow (\bar{p}, \bar{q})\) using generating function
\( W = [\bar{p} - F(\tau)] \bar{q} \), where \( F(\tau) = -f(\tau) d\tau \). Thus, \( F(\tau) \) is a periodic function defined
up to an additive constant, which we are free to choose.
To make the following presentation more clear, we choose
this constant in such a way that the minimal value of \( F \) is
\( F_{\min} > 4\omega_0 / \pi \). Note that \( \bar{q} = q \). After this transformation, Hamiltonian of the system acquires the form (bars over \( q \) are omitted)

\[ H = \frac{[\bar{p} - F(\tau)]^2}{2} - \omega_0^2 \cos q. \]  

Phase portrait of the system at a frozen value of \( \tau \) (we call it
the unperturbed system) is shown in Fig. 1. The separatrix
divides the phase space into the domains of direct rotations
(above the upper branch of the separatrix), oscillations (be-
tween the separatrix branches), and reverse rotations (below
the lower branch of the separatrix). Introduce the “action” \( I \)
associated with a phase trajectory of the unperturbed system
on this portrait. In the domains of rotation, \( I \) equals an area
between the trajectory, the lines \( q = -\pi, q = \pi \), and the axis
\( \bar{p} = 0 \), divided by 2\( \pi \); in the domain of oscillations, this is an
area surrounded by the trajectory divided by 2\( \pi \). It is known that \( I \) is an adiabatic invariant of (3): Far from the separatrix
its value is preserved along a phase trajectory with the accur-
acy of order \( \epsilon \) on long time intervals (see, e.g., [13]).

Location of the separatrix on the \((q, \bar{p})\) plane depends on
the value of \( F(\tau) \). As \( \tau \) slowly varies, the separatrix slowly
moves up and down, and phase points cross the separatrix and
switch its regime of motion from direct rotations to reverse rotations and vice versa. Recall known results on varia-
tion of the adiabatic invariant when a phase point crosses the
separatrix. The area surrounded by the separatrix is constant,
and hence, capture into the domain of oscillations is impos-
ible in the first approximation (in the exact system, only a
small measure of initial conditions correspond to phase tra-
jectories that spend significant time in this domain; thus their
influence on the transport is small). To be definite, consider
the situation when the separatrix on the phase portrait slowly
moves down. Thus, phase points cross the separatrix and
change the regime of motion from reverse rotation to direct rotation.
Let the action before the separatrix crossing at a
distance of order 1 from the separatrix be \( I = I_0 \) and let the action after the crossing (also at a distance of order 1 from the separatrix) be \( I = I_0 + 1 \). In the first approximation, we have
\( I_s = I_0 + 8\omega_0 / \pi \), i.e., the action increases by the value of the
area enclosed by the separatrix divided by 2\( \pi \) (see, e.g., [14,15]). We shall call this change in the action a “geometric jump.” If the separatrix contour slowly moves up, and a
phase point goes from the regime of direct rotation to the
regime of reverse rotation, the corresponding value of the
action decreases by the same value \( 8\omega_0 / \pi \). Thus, in this
approximation, the picture of motion looks as follows. While
a phase point is in the domain of reverse rotation, the value
of \( I \) along its trajectory stays constant, \( I = I_0 \). After transition
to the domain of direct rotation, this value changes by the value
of the geometric jump. The transition itself in this approx-
imation occurs instantaneously. After the next separatrix
crossing, the adiabatic invariant changes again by the value
of the geometric jump, with the opposite sign, and returns to
its initial value \( I_0 \). We call this approximation adiabatic.

In the next approximation, the value of action at the sepa-
ratix crossing undergoes a small additional jump. Consider
for definiteness the case when the separatrix contour on the
phase portrait moves down, and \( I_0 \) and \( I_s \) are measured when
it is in its uppermost and lowermost positions, accordingly.
Results of [11,12] imply the following formula for the jump
in the adiabatic invariant:

\[ 2\pi (I_s - I_0) = 16\omega_0 + 2\alpha (1 - \xi) \epsilon \Theta \ln(\epsilon \Theta) \]

\[ + \alpha \epsilon \Theta \ln \left( \frac{2\pi (1 - \xi)}{\Gamma^2(\xi)} \right) + 2b \epsilon \Theta (1 - \xi), \]

where \( a = \omega_0^{-1}, b = \omega_0^{-1} \ln(32\omega_0^2), \Theta = 2\pi F'(\tau_s) \). Here \( F' \) is the \( \tau \) derivative of \( F \), \( \tau_s \) is the value of \( \tau \) at the separatrix crossing found in
the adiabatic approximation, \( \Gamma(\cdot | \cdot) \) is the gamma function. Value \( \xi \) is a so-called pseudophase of the
separatrix crossing; it strongly depends on the initial condi-
tions and can be considered as a random variable uniformly
distributed on interval \((0,1)\) (see, e.g., [12]). Thus, value of
the jump in the adiabatic invariant at the separatrix crossings
has a quasirandom component of order \( \epsilon \ln \epsilon \).

Accumulation of small quasirandom jumps due to mul-
tiple separatrix crossings produces diffusion of adiabatic in-
variant (see, e.g., [12]). On a period of \( F(\tau) \) (after two sepa-
ratix crossings) the action changes by a value of order \( \epsilon \ln \epsilon \). Hence, after \( N \sim \epsilon^{-2} (\ln \epsilon)^{-2} \) separatrix crossings the
adiabatic invariant varies by a value of order one. As a result,
in time of order \( t_{ad} \sim \epsilon^{-1} (\ln \epsilon)^{-2} \) the value of adiabatic in-
variant is distributed in all the range of values corresponding
to the domain where phase points cross the separatrix on the phase plane; its distribution is close to the uniform one. We have checked this fact numerically for a sample function $F(\tau)$ at various parameter values. Poincaré sections and distribution histograms of $I$ in all the cases look similar; see an example in Fig. 2.

III. AVERAGE VELOCITY OF THE TRANSPORT

Our aim is to find a formula for average velocity $V_q$ along a phase trajectory on time intervals of order $t_{\text{diff}}$ or larger. We first only take into consideration the geometric jumps, and afterwards, to obtain the final result, we take into account the mixing due to small quasirandom jumps. To simplify the consideration, assume that function $F(\tau)$ has one local minimum $F_{\text{min}}$ and one local maximum $F_{\text{max}}$ on the interval $(0, 2\pi)$. The main results are valid without this assumption.

Introduce $\tilde{I}$, defined in the domains of rotation as follows: it equals the area bordered by the trajectory, the line $\tilde{p}=F(\tau)$, and the lines $q=\pi$, $q=-\pi$, divided by $2\pi$. Thus, $\tilde{I}=|F(\tau)-I|$. Frequency of motion in the domains of rotation is $\omega(\tilde{I})$, where $\omega(I)$ is $F(\tau)>1$ or $F(\tau)<-1$. We do not need an explicit expression for function $\omega(\tilde{I})$. From Hamiltonian (3) we find $\dot{q}=\tilde{p}-F(\tau)$. Consider a phase trajectory of the system frozen at $\tau=\pi$ in a domain of rotation. Let the value of $q$ averaged over a period $T$ of rotation equals $\bar{q}(\tilde{I})=\int_0^T q\,dt/T$. Then the value of $\Delta q$ averaged over a period $T$ of rotation equals $\Delta q(\tilde{I})=\int_0^T \omega[I(\tau)-\bar{I}]\,d\tau.$

Now consider a long phase trajectory in the case of slowly varying $\tau$. Let on the interval $(\tau_1, \tau_2)$ a phase point of (3) be below the separatrix contour. In the adiabatic approximation, the value $I_0$ of the adiabatic invariant along its trajectory is preserved on this interval. Hence, at $\tau \in (\tau_1, \tau_2)$ we have

$$2\pi F(\tau) - 2\pi I_0 \approx 8\omega_0,$$

and the equality here takes place at $\tau=\tau_1$ and $\tau=\tau_2$. In the process of motion on this time interval, $q$ changes by a value

$$\Delta q(I_0) = -\frac{1}{e} \int_{\tau_1}^{\tau_2} \omega[F(\tau) - I_0] \, d\tau.$$  \hspace{1cm} (5)

On the interval $(\tau_2, \tau_1+2\pi)$ the phase trajectory is above the separatrix contour, and the value of the adiabatic invariant equals $\tilde{I}_0=I_0+8\omega_0/\pi$ due to the geometric jump. On this interval we have

$$2\pi F(\tau) - 2\pi I_0 \approx 8\omega_0.$$  \hspace{1cm} (6)

In the process of motion on this time interval, $q$ changes by a value

$$\Delta q(I_0) = \frac{1}{e} \int_{\tau_1}^{\tau_2} \omega[F(\tau) - I_0] \, d\tau.$$  \hspace{1cm} (7)

Total displacement in $q$ on the interval $(\tau_1, \tau_1+2\pi)$ equals $\Delta q(I_0)=\Delta q(I_0)+\Delta q(I_0)$, and the average velocity on this interval is $e\Delta q/I_0/(2\pi)$.

Consider now the motion on a long enough time period $\Delta t=I_{\text{diff}}$. Due to the diffusion in the adiabatic invariant described above, in this time period values of $I_0$, defined as the value of $I$ when the phase point is below the separatrix contour, cover the interval $(I_{\text{min}}, I_{\text{max}}=8\omega_0/\pi)$. Here $I_{\text{min}}=I_{\text{min}}-4\omega_0/\pi$ and $I_{\text{max}}=I_{\text{max}}+4\omega_0/\pi$. Assuming that the distribution of $I$ on this interval is uniform, to find the average velocity, we integrate $e\Delta q/I_0/(2\pi)$ over this interval. Integrating (5) over $I_0$ and changing the order of integration we find

$$\int_{I_{\text{min}}}^{I_{\text{max}}+8\omega_0/\pi} \Delta q(I_0) \, dI_0 = -\frac{1}{e} \int_0^{2\pi} d\tau \int_{I_{\text{min}}}^{I(\tau)\cdot I_{\text{min}}} \omega[F(\tau) - I_0] \, dI_0$$

$$= -\frac{1}{e} \int_0^{2\pi} d\tau \int_{4\omega_0/\pi}^{F(\tau)\cdot I_{\text{min}}} \omega(\eta) \, d\eta.$$  \hspace{1cm} (8)

Now we take into account the equality $\omega(I)=\partial H_0(\tilde{I})/\partial \tilde{I}$ [recall that $H_0(\tilde{I})$ is the Hamiltonian of a nonlinear pendulum as a function of its action variable] and obtain
find the expression for the average velocity
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \omega(\eta) d\eta = -\int_{0}^{2\pi} \{ H_0[F(\tau) - I_{\text{min}}] - H_0^0 \} d\tau, \]
where \( H_0^0 \) is the value of \( H_0 \) on the separatrix. Similarly, integrating (7) we obtain
\[ \int_{I_{\text{min}}}^{I_{\text{max}} - \delta_{0}/\pi^2} \Delta q_d H_0 = \frac{1}{\epsilon} \int_{0}^{2\pi} \{ H_0[I_{\text{max}} - F(\tau)] - H_0^0 \} d\tau. \]
Adding (8) and (9) and dividing by \( 2\pi(F_{\text{max}} - F_{\text{min}})/\epsilon \) we find the expression for the average velocity \( V_q \) of transport on long time intervals,
\[ V_q = \frac{1}{2\pi(F_{\text{max}} - F_{\text{min}})} \times \int_{0}^{2\pi} \{ H_0[I_{\text{max}} - F(\tau)] - H_0^0[I_{\text{max}} - F(\tau)] \} d\tau. \]
In (10), \( H_0^0/I \) can be found as the inverse function to \( \tilde{I}(h) \), which defines action as a function of energy in domains of rotation of a nonlinear pendulum. For the latter function, the following formula holds (see, e.g., [16]):
\[ \tilde{I}(h) = \frac{4}{\pi} \omega_0 \kappa \mathcal{E}(1/\kappa), \quad \kappa \geq 1, \]
where \( \kappa^2 = (1+h/\omega_0^2)/2 \), \( \mathcal{E}(\cdot) \) is the complete elliptic integral of the second kind. If function \( F(\tau) \) has several local extrema on the interval \( (0, 2\pi) \), \( F_{\text{min}} \) and \( F_{\text{max}} \) in (10) are the smallest and largest values of \( F \), respectively.

It can be seen from (10) that \( V_q \) is not zero for function \( F(\tau) \) of general type, and hence there is the directed transport in the system. We checked this formula numerically for a sample function \( F(\tau) = A[1 + 2 \exp(-\alpha \sin(\tau)^2)] \), \( \alpha > 0 \) at various values of parameters \( \epsilon \) and \( \alpha \). Typical plots of \( q \) against time \( t \) are shown in Fig. 3.

The results of numerical checks of formula (10) are represented in Table I. To find numerical values of \( V_q \) presented in the table, we integrated the system with Hamiltonian (3) on a long time interval \( \Delta t = 2\pi \times 10^6/\epsilon \) with a constant time step of \( \pi/100 \) (fifth-order symplectic scheme [17]). Use of a symplectic scheme for long time simulations of Hamiltonian systems is necessary in order to ensure that creeping numerical error do not end up washing off the invariant tori bounding the chaotic domain. The table demonstrates satisfactory agreement between the formula and the numerics.

Finally, we note that formula (10) can be used also in the case of arbitrary (nonharmonic) spatially periodic time independent potential in place of the term \( -\omega_0^2 \cos q \) in (2) and (3). Of course, in this case function \( H_0 \) is different from the Hamiltonian of the nonlinear pendulum, but it always can be found, at least numerically.

IV. CASE OF NOT SMALL EXTERNAL FORCING

In this section we study the case when the external forcing is not small. In this case, amplitude of function \( \tilde{f}(t) \) in (1) is a value of order one, \( \tilde{f}(t) = f(\tau) = f(\tau + 2\pi) \). The equations of motion are
\[ \dot{q} = p, \quad \dot{p} = -\omega_0^2 \sin q + f(\tau), \quad \dot{\tau} = \epsilon. \]
One can see from the second equation, that magnitude of momentum \( p \) can reach values of order \( \epsilon^{-1} \). Making the canonical transformation with generating function \( W = [\tilde{p} - \epsilon^{-1} F(\epsilon t)]q \), where \( F(\tau) \) is defined in Sec. II, we obtain the Hamiltonian
\[ H = \frac{[\tilde{p} - \epsilon^{-1} F(\tau)]^2}{2} - \omega_0^2 \cos q. \]
Introduce \( \tilde{p} = \epsilon \bar{p} \) and rescaled time \( \tilde{t} = \epsilon^{-1} t \). We denote the derivative with respect to \( \tilde{t} \) with prime and thus obtain

\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
\text{\( \epsilon \)} & \text{\( \alpha=4 \)} & \text{\( \alpha=8 \)} & \text{\( \alpha=16 \)} \\
\hline
\text{\( 0.1 \)} & 4.721 & 6.756 & 8.363 \\
\text{\( 0.05 \)} & 4.446 & 6.681 & 8.076 \\
\text{\( 0.01 \)} & 4.298 & 6.211 & 7.442 \\
\text{\( 0.005 \)} & 4.598 & 6.702 & 8.202 \\
\text{\( V_{q_{\text{theor}}} \)} & 4.393 & 6.679 & 8.110 \\
\hline
\end{tabular}
\caption{Numerically found values of \( V_q \) corresponding to various values of parameters \( \epsilon, \alpha \) (four upper rows, \( A=10, \omega_0=1 \)) and theoretical values \( V_{q_{\text{theor}}} \) obtained according to (10) (the bottom row).}
\end{table}
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FIG. 4. (Color online) (a) Poincaré section at \( \tau = 0 \mod 2\pi \) of a long phase trajectory \((5 \times 10^4 \text{ dots})\) of system (13). All the points are mapped onto the interval \( q \in (-\pi, \pi) \). \( F(\tau) = A[1 + 2 \exp[-\alpha(\sin \tau)^2]] \) with \( A = 0.5, \alpha = 4, \epsilon = 0.01, \omega_0 = 1 \). (b) Histogram of \( \bar{p} \) along the same phase trajectory.

This is a system in a typical form for application of the averaging method. We average over fast variable \( q \) and obtain the averaged system

\[
q' = \bar{p} - F(\tau), \quad \bar{p}' = -\epsilon^2 \omega_0^2 \sin q, \quad \tau' = \epsilon^2. \tag{14}
\]

The averaged system describes the dynamics adequately everywhere in the phase space except for a small neighborhood of the resonance at \( \bar{p} = F(\tau) = 0 \), where the “fast” variable \( q \) is not fast. When a phase trajectory of the averaged system crosses the resonance, value of the adiabatic invariant (in this case it coincides with \( \bar{p} \)) undergoes a quasirandom jump of typical order \( \sqrt{\epsilon^2} = \epsilon \) (see, e.g., [13,18]). Thus, the situation in this case is similar to one studied in Sec. II, but the typical value of a jump is of order \( \epsilon \), and, accordingly, the typical diffusion time is \( t_{\text{diff}} \sim \epsilon^{-3} \). Phase trajectories of the averaged system that cross the resonance correspond to values of \( \bar{p} \) belonging to the interval \((F_{\text{min}}, F_{\text{max}})\). Therefore, the chaotic domain of the exact system is, in the main approximation, a strip \( F_{\text{min}} \leq \bar{p} \leq F_{\text{max}} \). Captures into the resonance followed by escapes from the resonance (see [13,18]) are also possible in this system. However, probability of capture is small, of order \( \epsilon \), and hence impact of these phenomena on the transport is small.

In Fig. 4 we represent a sample of Poincaré section of a long phase trajectory of (13) and the corresponding histogram of \( \bar{p} \) for this trajectory. The plots show that the distribution of \( \bar{p} \) is close to the uniform one.

As in Sec. III, one can find value \( \Delta q \) on one period of perturbation, then average over the range of adiabatic invariant corresponding to the chaotic domain, and find the average velocity of transport in this case. Thus we find

\[
\Delta q = \int_0^{2\pi/\epsilon} \frac{d\tau}{\epsilon} \int_0^{2\pi/\epsilon} [\bar{p} - F(\tau)] d\tau = \int_0^{2\pi/\epsilon} \frac{d\tau}{\epsilon} \int_0^{2\pi/\epsilon} [\bar{p} - F(\tau)] d\tau.
\]

To find \( V_q \), we must integrate this expression over \( \bar{p} \) from \( F_{\text{min}} \) to \( F_{\text{max}} \) (i.e., over the chaotic domain) and divide the result by \( (F_{\text{max}} - F_{\text{min}}) \) and by the length of the period of the external forcing \( 2\pi/\epsilon \). Thus we obtain

\[
V_q = \frac{\epsilon}{2\pi(F_{\text{max}} - F_{\text{min}})} \int_0^{F_{\text{max}} - F_{\text{min}}} \Delta q d\bar{p}. \tag{17}
\]

Substituting \( \Delta q \) from (16) and integrating, one straightforwardly obtains

\[
V_q = \frac{1}{4\pi \epsilon} \int_0^{2\pi} [F_{\text{max}} + F_{\text{min}} - 2F(\tau)] d\tau. \tag{18}
\]

Another possibility to find \( V_q \) is to use the already obtained formula (10). This way leads to the same result. When using (10), one should keep in mind that in the considered case \( H_0 \) in this formula is the Hamiltonian of the averaged system, i.e., \( H_0 = F/2, I_{\text{max}} = F_{\text{max}}, I_{\text{min}} = F_{\text{min}}, \) and that according to (13) we should set \( \epsilon^{-1}F \) instead of \( F \) everywhere in the formula. Thus, (10) is much simplified, and we again arrive at formula (18). Factor \( \epsilon^{-1} \) is due to the fact that a phase point spends significant time moving at large velocities corresponding to \( p \sim \epsilon^{-1} \).

Note that formula (18) can be rewritten in a more elegant form as

\[
V_q = \frac{1}{\epsilon} \left( \frac{F_{\text{max}} + F_{\text{min}}}{2} - \langle F(\tau) \rangle \right), \tag{19}
\]

where the angle brackets denote time average. The results of numerical checks of the formula are represented in Table II. As in Sec. IV, we integrated the system with Hamiltonian (13) on a long time interval \( \Delta t = 2\pi \times 10^4/\epsilon \).

Remarkably, formula (19) is valid for arbitrary smooth periodic potential (not necessarily harmonic) in place of the term \( -\omega_0^2 \cos q \) in (2) and (3). The potential may also depend periodically on time with the same period as that of the external force.

V. SUMMARY

To summarize, we have considered the phenomenon of the directed transport in a spatially periodic harmonic poten-
time intervals. Direction and average velocity of the transport in the chaotic domain are independent of initial conditions and determined by properties of the external force. We studied two different cases: The case of small amplitude of the external force and the case, when this amplitude is a value of order one. We have obtained an approximate formula for average velocity of the transport and checked it numerically. The final formulas (10) and (19) are valid for any smooth periodic potential (not necessarily the harmonic one).

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**TABLE II.** Numerically found values of $\varepsilon V_q$ corresponding to various values of parameters $\varepsilon$, $\alpha$ in system (13) for $F(t)=\Delta\{1 + 2 \exp(-\alpha t^2)\}$ (four upper rows, $\Delta=0.5$, $\alpha_0=1$). In the bottom row theoretical values $\varepsilon V_q^{\text{theor}}$ obtained according to (19) are shown.

| $\varepsilon$ | $\alpha=1$ | $\alpha=2$ | $\alpha=4$ |
|--------------|------------|------------|------------|
| $=0.1$       | 0.046      | 0.128      | 0.253      |
| $=0.05$      | 0.046      | 0.112      | 0.225      |
| $=0.01$      | 0.0353     | 0.1050     | 0.2044     |
| $=0.005$     | 0.0369     | 0.1081     | 0.1916     |
| $\varepsilon^{\text{theor}}$ | 0.0389 | 0.1018 | 0.2006 |

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