Hyperconfluent third-order supersymmetric quantum mechanics

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Abstract

The hyperconfluent third-order supersymmetric quantum mechanics, in which all the factorization energies tend to a common value, is analyzed. It will be shown that the final potential can also be achieved by applying consecutively a confluent second-order and a first-order SUSY transformations, both with the same factorization energy. The technique will be applied to the free particle and the Coulomb potential.

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1. Introduction

Nowadays, supersymmetric quantum mechanics (SUSY QM) has became the standard technique for generating solvable potentials from a given initial one [1–6]. Moreover, it represents a powerful tool for designing Hamiltonians with a fixed prescribed spectrum (see, e.g., [7–9]). In its higher order version, in which the differential intertwining operators are of order greater that 1, it is well known that several seed solutions of the initial stationary Schrödinger equation with an appropriate behavior are in general required for calculating the new potential, the eigenfunctions of the corresponding Hamiltonian, etc [10–18]. If for some reason just one of those seeds is available, one is driven to what could be called as hyperconfluent higher order SUSY QM, in which all the factorization energies which are involved tend to a common value.

In the past, several works dealing with the confluent second-order SUSY QM have been elaborated [19–22]. Up to our knowledge, however, the hyperconfluent third-order SUSY QM has not been addressed explicitly. Of course, several papers involving the third-order SUSY QM have been published, but they are centered mainly in the case when the factorization energies are all different (see e.g. [23–29] and references therein).

In this paper, we aim to fill the gap by studying in detail the hyperconfluent third-order SUSY QM. In order to achieve this, we have arranged the paper as follows. In the next section,
we will briefly review the confluent second-order SUSY QM. In section 3, we will analyze the direct approach to the hyperconfluent third-order SUSY QM, while in section 4 we will address the corresponding iterative method. Section 5 explores the requirements that the seed solution has to obey in order to produce non-singular transformations as well as the eigenfunctions of the SUSY-generated Hamiltonians. In section 6, we will illustrate our general treatment by means of two specific examples, the free particle and the Coulomb potential. Our conclusions will be presented in section 7.

2. Confluent second-order SUSY QM

Let us consider a one-dimensional Schrödinger Hamiltonian

\[ H_0 = -\frac{d^2}{dx^2} + V_0(x). \]  

(1)

The domain of definition or the corresponding system is denoted as \( D = [x_l, x_r] \). Thus, depending on the problem we are dealing with, and the consequent identification of \( x_l \) and \( x_r \), this domain could be the full real line, the positive semi-axis or a finite interval. The eigenfunctions and eigenvalues associated with the discrete part of the spectrum of \( H_0 \), denoted by \( \psi_n(x), E_n, n = 0, 1, \ldots \), satisfy the stationary Schrödinger equation

\[ H_0 \psi_n = -\psi_n'' + V_0 \psi_n = E_n \psi_n, \]  

as well as the boundary conditions

\[ \psi_n(x_l) = \psi_n(x_r) = 0. \]  

(3)

From now on we are going to suppose that all the eigenfunctions and eigenvalues of \( H_0 \) are known. In the general formulation of the second-order SUSY QM one looks for a new Hamiltonian

\[ H_2 = -\frac{d^2}{dx^2} + V_2(x), \]  

(4)

which is intertwined with \( H_0 \) by a second-order operator \( B^+_2 \) in the following way:

\[ H_2 B^+_2 = B^+_2 H_0, \]  

(5)

where

\[ B^+_2 = \frac{d^2}{dx^2} - \eta(x) \frac{d}{dx} + \gamma(x). \]  

(6)

By plugging these expressions in the intertwining relationship (5), decoupling the resulting system of equations and solving it we arrive at [5, 6]

\[ V_2 = V_0 - 2\eta', \]  

(7)

\[ \gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - V_0 + \frac{\epsilon_1 + \epsilon_2}{2}, \]  

(8)

\[ \eta = \{ \ln[W(u_1, u_2)] \}', \]  

(9)

where \( u_1, u_2 \) are two seed solutions of the initial stationary Schrödinger equation associated with the factorization energies \( \epsilon_1, \epsilon_2 \) (in general different):

\[ H_0 u_i = -u_i'' + V_0 u_i = \epsilon_i u_i, \quad i = 1, 2, \]  

(10)
and \( W(u_1, u_2) \equiv u_1u'_2 - u'_1u_2 \) denotes their Wronskian. Note that the seeds \( u_1, u_2 \) could obey or not the boundary conditions of equation (3).

The **confluent second-order SUSY QM** arises now as a limit procedure of the previous formalism when \( \epsilon_1 \to \epsilon_2 \to \epsilon \). Note that, if the potential \( V_2 \) is going to be different from the initial one, then \( u_1 \) and \( u_2 \) cannot be just chosen as two linearly independent solutions of equation (10), since then \( W(u_1, u_2) = \text{constant} \) and therefore \( V_2 = V_0 \). In order to produce non-trivial results, the right choice is to take \( u_1 \) as a standard eigenfunction of \( H_0 \), while \( u_2 \) becomes a generalized eigenfunction of rank 2 of \( H_0 \), both associated with \( \epsilon \), namely

\[
(H_0 - \epsilon)u_1 = 0 \quad \Rightarrow \quad u_1'' = (V_0 - \epsilon)u_1, \tag{11}
\]

\[
(H_0 - \epsilon)u_2 = u_1 \quad \Rightarrow \quad (H_0 - \epsilon)^2u_2 = 0 \quad \Rightarrow \quad u_2'' = (V_0 - \epsilon)u_2 - u_1, \tag{12}
\]
i.e. we are employing a Jordan chain of length 2. Expressing this in matrix language \([7, 8]\), this specific choice of basis \([u_1, u_2]\) means that, in the restriction to the two-dimensional subspace of functions belonging to \( \text{Ker}(B^+ \omega \gamma) \), the initial Hamiltonian \( H_0 \) is represented by a matrix \((H_0)\) having a non-trivial Jordan structure of standard type:

\[
(H_0) = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}. \tag{13}
\]

Note that, if \( u_1 \) is given, then it is possible to determine the general solution \( u_2 \) to the second-order equation (12) and to then calculate explicitly \( W(u_1, u_2) \). An alternative (and shorter) procedure runs as follows. First of all it is straightforward to show that

\[
W'(u_1, u_2) = u_1u''_2 - u_2u''_1 = -u_1^2, \tag{14}
\]

where we have used equations (11) and (12). This implies that

\[
W(u_1, u_2) = w_0 - \int_{x_0}^x u_1^2(y) \, dy \equiv w(x), \tag{15}
\]

where \( x_0 \in (x_l, x_r) \).

Let us note that, in order that the new potential

\[
V_2 = V_0 - 2[\ln(W(u_1, u_2))]'' = V_0 - 2[\ln(w)]'' \tag{16}
\]
does not have additional singularities with respect to \( V_0 \), then \( w(x) \) must not have nodes in \((x_l, x_r)\). This can be achieved by choosing a Schrödinger seed solution \( u_1 \) such that \([20]\)

\[
\lim_{x \to x_l} u_1 = 0, \quad v_- \equiv \int_{x_l}^{x_0} u_1^2(y) \, dy < \infty, \quad \text{or} \tag{17}
\]

\[
\lim_{x \to x_r} u_1 = 0, \quad v_+ \equiv \int_{x_0}^{x_r} u_1^2(y) \, dy < \infty. \tag{18}
\]

With this choice, it turns out that \( w(x) \) becomes nodeless either for \( w_0 \leq -v_- \) in the first case or for \( w_0 \geq v_+ \) in the second one. Moreover, departing from the normalized bound states \( \psi_n(x) \) of \( H_0 \) the normalized ones \( \psi_n^{(2)}(x) \) of \( H_2 \) can be built up in the following way:

\[
\psi_n^{(2)}(x) = \frac{B_n^+ \psi_n(x)}{E_n - \epsilon}. \tag{19}
\]

In addition, there is an eigenfunction of \( H_2 \) associated with \( \epsilon \) which also becomes square integrable (here we are using a notation for this state which is appropriate for the purpose of this paper):

\[
u_1^{(2)}(x) \propto \frac{u_1(x)}{w(x)} \tag{20}
\]
Note that the confluent algorithm has been used to create bound states above the ground state energy of $H_0$ [20]. This possibility of spectral manipulation typically was outside the goals of the standard first-order SUSY QM. Moreover, the use of just one eigenfunction of $H_0$ in the confluent case is advantageous compared with the second-order SUSY QM with $\epsilon_1 \neq \epsilon_2$, which requires the knowledge of two Schrödinger seed solutions.

### 3. Hyperconfluent third-order SUSY QM: direct approach

In turn, let us analyze the hyperconfluent third-order SUSY QM, for which the three factorization energies converge to the same $\epsilon$-value, namely $\epsilon_i \to \epsilon$, $i = 1, 2, 3$. Similarly as for the second-order case of section 2, here we are going to use a Jordan chain of length 3 of generalized eigenfunctions $\{u_1, u_2, u_3\}$ such that $u_1$ and $u_2$ obey equations (11) and (12), while $u_3$ satisfies

$$\begin{align*}
(H_0 - \epsilon)u_3 &= u_2 \\
\Rightarrow (H_0 - \epsilon)^3 u_3 &= 0 \\
\Rightarrow u''_3 &= (V_0 - \epsilon)u_3 - u_2.
\end{align*}$$

Equations (11), (12) and (21) mean that in the three-dimensional subspace of functions belonging to Ker($B_3^\dagger$) this choice of basis implies that the matrix representing $H_0$ has a non-trivial Jordan structure of standard type\(^1\):

$$
(H_0) = \begin{pmatrix}
\epsilon & 1 & 0 \\
0 & \epsilon & 1 \\
0 & 0 & \epsilon
\end{pmatrix}.
$$

Now, the hyperconfluent third-order SUSY partner Hamiltonians $H_0$ and $H_3$ are intertwined by the third-order operator $B_3^\dagger$ in the following way:

$$
H_3 B_3^\dagger = B_3^\dagger H_0,
$$

where $H_0$ is given by equation (1), $H_3$ has the standard Schrödinger form

$$
H_3 = -\frac{d^2}{dx^2} + V_3(x)
$$

and $V_3$ is expressed in terms of the initial potential and the three seeds $u_1$, $u_2$, $u_3$ as follows:

$$
V_3 = V_0 - 2[\ln[W(u_1, u_2, u_3)]]'',
$$

with $W(u_1, u_2, u_3)$ denoting the Wronskian of $u_1$, $u_2$ and $u_3$ (we will give the explicit expression for $B_3^\dagger$ in the next section). A straightforward calculation leads to

$$
W(u_1, u_2, u_3) = \begin{vmatrix}
u_1 & u_2 & u_3 \\
u_1' & u_2' & u_3' \\
u_1'' & u_2'' & u_3''
\end{vmatrix} = u''_1 W(u_2, u_3) - u''_2 W(u_1, u_3) + u''_3 W(u_1, u_2).
$$

Now, by using equations (11), (12) and (21) it turns out that

$$
W(u_1, u_2, u_3) = u''_1 W(u_1, u_3) - u''_3 W(u_1, u_2).
$$

Recall that $W(u_1, u_2) = w(x)$ was calculated in a simple way in the previous section; thus, a similar procedure to obtain $W(u_1, u_3)$ can be followed, leading to

$$
W(u_1, u_3) = w_1 - \int_{x_0}^x u_1(y)u_2(y) dy.
$$

Given $u_1$, and consequently the $w$ of equation (15), it remains just to express $u_2$ in terms of them. Let us first of all note that

$$
w = W(u_1, u_2) = u''_1 \left(\frac{u_2}{u_1}\right)'.
$$

\(^1\) Note that the matrices ($H_0$) of equations (13) and (22) are non-Hermitian despite $H_0$ is Hermitian.
Henceforth
\[ u_2 = u_1 \left[ \beta_1 + \int_{x_0}^{x} \frac{w(y)}{u_1'(y)} dy \right] . \]  
(30)

Thus, a straightforward calculation leads to
\[ \int_{x_0}^{x} u_1(y) u_2(y) dy = w_0 \beta_1 - \beta_1 w(x) - w(x) \int_{x_0}^{x} \frac{w(y)}{u_1(y)} dy + \int_{x_0}^{x} \left[ \frac{w(y)}{u_1(y)} \right]^2 dy. \]  
(31)

By plugging equations (15), (28) and (31) into equation (27) we arrive at
\[ W(u_1, u_2, u_3) = u_1 \left\{ f_0 - \int_{x_0}^{x} \left[ \frac{w(y)}{u_1(y)} \right]^2 dy \right\} \equiv u_1 f, \]  
(32)

with
\[ f(x) = f_0 - \int_{x_0}^{x} \left[ \frac{w(y)}{u_1(y)} \right]^2 dy, \]  
(33)

and \( f_0 = w_1 - w_0 \beta_1 \). Finally, the potential of equation (25) becomes
\[ V_2(x) = V_0(x) - 2[\ln[u_1(x)]]'' - 2[\ln[f(x)]]'' , \]  
(34)

where \( f(x) \) is given by equation (33).

4. Hyperconfluent third-order SUSY QM: iterative approach

Now, we are going to apply two consecutive SUSY transformations departing from the initial Hamiltonian \( H_0 \): a confluent second-order one for generating \( V_2 \) from \( V_0 \), which employs the two generalized eigenfunctions \( u_1 \) and \( u_2 \) associated with \( \epsilon \) satisfying equations (11) and (12) of section 2; then a first-order transformation in order to obtain \( V_3 \) from \( V_2 \), which makes use of the general solution of the stationary Schrödinger equation of \( H_2 \) associated with \( \epsilon \).

As for the confluent second-order transformation, in section 2 we saw that the new potential \( V_2 \) is given by
\[ V_2 = V_0 - 2[\ln[W(u_1, u_2)]]'' = V_0 - 2[\ln(w)]'' , \]  
(35)

where the Wronskian \( W(u_1, u_2) = w(x) \) of the two generalized eigenfunctions \( u_1, u_2 \) of \( H_0 \) associated with \( \epsilon \) is given by equation (15).

Concerning the first-order transformation, it turns out that \( H_2 \) and \( H_3 \) are intertwined by a first-order operator \( A_3^+ \) in the following way:
\[ H_3 A_3^+ = A_3^+ H_2, \]  
(36)

where \( H_2 \) and \( H_3 \) are given by equations (4) and (24), respectively, and
\[ A_3^+ = - \frac{d}{dx} + \ln[u^{(2)}'], = - \frac{d}{dx} + \frac{u^{(2)'}'}{u^{(2)'}} , \]  
(37)

with \( u^{(2)} \) being the general solution of the Schrödinger equation
\[ H_2 u^{(2)} = \epsilon u^{(2)}. \]
$u_2^{(2)} = u_1^{(2)} \int_0^x dy / \left[ u_1^{(2)}(y) \right]^2$. Thus, the solution $u^{(2)}$ we are looking for to implement the first-order transformation takes the form

$$
\begin{align*}
    u^{(2)} &= c_1 u_1^{(2)} + c_2 u_2^{(2)} = -c_2 u_1 \left\{ -c_1 u_1' - \int_{x_0}^x \left[ \frac{w(y)}{u_1(y)} \right]^2 dy \right\}. 
\end{align*}
$$

Hence, the final potential $V_3$ resulting from applying the first-order SUSY transformation to the Hamiltonian $H_2$, when using the seed solution given in equation (38), becomes

$$
V_3 = V_2 - 2[\ln(u^{(2)})]' = V_0 - 2[\ln(u_1)]'' - 2 \left\{ \ln \left( -\frac{c_1}{c_2} - \int_{x_0}^x \left[ \frac{w(y)}{u_1(y)} \right]^2 dy \right) \right\}''.
$$

Note that the two hyperconfluent third-order SUSY partner potentials $V_3(x)$ of $V_0(x)$ given by equations (34) and (39) are exactly the same if it is taken $f_0 = -c_1/c_2$.

We can give, finally, the explicit expression for the third-order operator $B_3^+$ intertwining the initial and final Hamiltonians $H_0$ and $H_1$ (see equation (23)):

$$
B_3^+ = A_3^+ B_2^+,
$$

where $B_2^+$ and $A_3^+$ are given by equations (6) and (37), respectively.

5. Non-singular transformations and bound states of $H_3$

As can be seen from equation (25), in order that the potential $V_3$ has no additional singularities compared with those of $V_0$, the Wronskian $W(u_1, u_2, u_1)$ given in equation (32) should not have nodes inside $D$. This implies that both functions $u_1$ and $f$ in this factorized expression should be free of zeros in this domain, in particular the seed solution $u_1$ which automatically leads to the restriction $\epsilon \leq E_0$, where $E_0$ is the ground state energy of $H_0$. Moreover, for the second factor $f(x)$ of equation (33) to be nodeless, the function $w/u_1$ should vanish either to the left edge $x_l$ of $D$ or to the right one $x_r$. Here, we are going to discuss in detail just the first case; the second one can be addressed in a similar way.

Let us choose first of all a nonphysical Schrödinger seed solution $u_1$ without nodes in $D$, obeying equation (11) for $\epsilon < E_0$. Moreover, it is supposed that $u_1$ also satisfies equation (17). Since $w/u_1$ should vanish for $x \to x_l$, we must have

$$
\lim_{x \to x_l} w = w_0 + v_- = 0,
$$

which implies that $w_0$ has to be taken as

$$
w_0 = -v_- = -\int_{x_l}^{x_0} u_1^2(y) dy.
$$

Therefore,

$$
w(x) = -\int_{x_l}^x u_1^2(y) dy.
$$

With this specific choice of $u_1$ and $w$, for most of the typical quantum mechanical problems it turns out that

$$
\lim_{x \to x_l} \frac{w(x)}{u_1(x)} = 0 \quad \text{and} \quad \lim_{x \to x_l} \frac{w(x)}{|u_1(x)|} = \infty.
$$

Hence, the domain of the parameter $f_0$ such that $f(x)$ is nodeless in $(x_l, x_r)$ is given by

$$
f_0 < -\sigma_- = -\int_{x_l}^{x_0} \frac{w^2(y)}{u_1^2(y)} dy.
$$
Let us note that in this $f_0$-domain the third-order intertwining operator $B_3^+$ of equation (40) transforms the normalized eigenfunctions $\psi_n$ of $H_0$ into normalized eigenfunctions $\psi_n^{(3)}$ of $H_3$ in the following way:

$$\psi_n^{(3)}(x) = \frac{B_3^+ \psi_n}{\sqrt{(E_n - \epsilon)^3}}.$$  

(46)

Moreover, the eigenfunction $\psi_\epsilon^{(3)}$ of $H_3$ associated with the eigenvalue $\epsilon$ (compare equation (38)),

$$\psi_\epsilon^{(3)}(x) \propto \frac{1}{u^{(2)}(x)} \propto \frac{w(x)}{u_1(x) f(x)},$$  

(47)

turns out to be square-integrable in $D$, which implies that the spectrum of $H_3$ becomes

$$\text{Sp}(H_3) = \{ \epsilon \} \cup \text{Sp}(H_0).$$  

(48)

Note that, when $f_0 \to -\sigma_-$, the hyperconfluent third-order transformation remains non-singular but the eigenstate $\psi_\epsilon^{(3)}$ is not longer square-integrable. Thus, in this limit the two Hamiltonians $H_3$ and $H_0$ become isospectral.

On the other hand, for $u_1(x)$ being chosen as the normalized ground state eigenfunction $\psi_0(x)$ of $H_0$ associated with $E_0$, it turns out that the previous equations (41)–(47) remain valid, the only difference is that now

$$\nu_- = \int_{x_l}^{x_u} u_1^2(y) \, dy < 1.$$  

Thus, for $f_0$ satisfying equation (45), it turns out that $\epsilon = E_0 \in \text{Sp}(H_3)$, which implies that

$$\text{Sp}(H_3) = \text{Sp}(H_0),$$  

(49)

i.e. the transformation is again strictly isospectral. However, when $f_0 \to -\sigma_-$ the $\psi_\epsilon^{(3)}$ of equation (47) is no longer normalizable, meaning that in this limit $\epsilon = E_0 \notin \text{Sp}(H_3)$, namely

$$\text{Sp}(H_3) = \text{Sp}(H_0) - \{ E_0 \}.$$  

(50)

In this case, through the hyperconfluent third-order SUSY transformation somehow we ‘delete’ the ground state energy of $H_0$ for generating $H_3$.

6. Examples

Next, let us apply the previous formalism to two physically interesting examples, the free particle and the Coulomb potential.

6.1. Free particle

The general solution of the stationary Schrödinger equation (11) for the free particle with a negative factorization energy $\epsilon = -k^2$, $k > 0$ (for which $V_0(x) = 0$), is given by

$$u_1(x) = A e^{kx} + B e^{-kx}.$$  

(51)

In order to apply our method, let us use a nonphysical seed solution $u_1(x)$ satisfying equation (17) for $x_l = -\infty$, i.e. let us make in equation (51) $B = 0$ and $A = 1$ so that

$$u_1(x) = e^{kx}.$$  

(52)

With this choice, the calculation of equation (43) leads to

$$w(x) = \frac{e^{2kx}}{2k}.$$  

(53)
Moreover, the evaluation of equation (33) with $x_0 = 0$ produces
\[ f(x) = f_0 + \frac{e^{2kx}}{8k^3}. \] (54)
Note that this function does not have nodes for
\[ f_0 < -\sigma_+ = -\frac{1}{8k^3}. \]

Hence, it is convenient to reparametrize this domain in the following way:
\[ f_0 = -\frac{1}{8k^3} \frac{e^{2kx_1}}{8k^3}, \] (55)
where $x_1 \in (-\infty, \infty)$. Thus, it is straightforward to show that
\[ f(x) = \frac{e^{k(x + x_1)}}{4k^3} \cosh[k(x - x_1)]. \] (56)

Finally, by plugging equations (52) and (56) into equation (34), the hyperconfluent third-order SUSY partner potential of the free particle turns out to be
\[ V_3(x) = -2k^2 \text{sech}^2[k(x - x_1)]. \] (57)
This is the well-known Pöschl–Teller potential with one bound state at $\epsilon = E_0 = -k^2$, which has also been derived through first-order SUSY (see e.g. [2], page 30) and confluent second-order SUSY techniques [20].

6.2. Coulomb potential

Working in spherical coordinates, separating the angular ones $\theta, \phi$ and making $\hbar = e = m = 1$, the three-dimensional stationary Schrödinger equation for the Coulomb potential $-e^2/r$ leads to a one-dimensional problem characterized by the effective potential
\[ V_0(r) = -\frac{2}{r} + \frac{\ell(\ell + 1)}{r^2}, \] (58)
where $0 \leq r < \infty$, $\ell = 0, 1, \ldots$. The discrete energy levels $E_n$ of $H_0$, for a fixed value of $\ell$, take the form $E_n = -1/(n + \ell + 1)^2$, $n = 0, 1, 2, \ldots$. In order to apply our method, let us employ here the normalized ground state eigenfunction
\[ u_1(r) = \frac{1}{(\ell + 1)\sqrt{(2\ell + 1)!}} \left( \frac{2r}{\ell + 1} \right)^{\ell+1} e^{-\frac{2r}{\ell+1}}, \] (59)
associated with the eigenvalue $E_0 = -1/(\ell + 1)^2$. Let us start by calculating the $w(r)$ of equation (43) with $r_1 = 0$, which leads to
\[ w(r) = -\gamma(2\ell + 3, \frac{2r}{\ell+1}) = e^{-\frac{2r}{\ell+1}} \sum_{k=0}^{2\ell+2} \frac{1}{k!} \left( \frac{2r}{\ell + 1} \right)^k - 1 \]
\[ = -e^{-\frac{2r}{\ell+1}} \sum_{k=2\ell+3}^{\infty} \frac{1}{k!} \left( \frac{2r}{\ell + 1} \right)^k, \] (60)
$\gamma(a, x)$ being an incomplete Gamma function. Using this result and the expression for $u_1(r)$ of equation (59) it turns out that
\[ \frac{w(r)}{u_1(r)} = -(\ell + 1)\sqrt{(2\ell + 1)!} e^{-\frac{2r}{\ell+1}} \sum_{k=2\ell+3}^{\infty} \frac{1}{k!} \left( \frac{2r}{\ell + 1} \right)^{k-\ell-1}, \] (61)
which vanishes for $r \to 0$, as required. The calculation of the $f(r)$ of equation (33) with $r_0 = 0$ now produces

$$f(r) = f_0 - \frac{(\ell + 1)^3(2\ell + 1)!}{2} \sum_{k=2\ell+3}^{\infty} \sum_{m=2\ell+3}^{\infty} \frac{\gamma(k + m - 2\ell - 1, \frac{2r}{\ell + 1})}{k!m!}$$

$$= f_0 - \frac{\gamma(2\ell + 3, \frac{2r}{\ell + 1})}{2 \Gamma(2\ell + 4)} r^2 F_2 \left( 1, 2; 3, 2\ell + 4; \frac{2r}{\ell + 1} \right)$$

$$+ \frac{(\ell + 1)^2}{4} \sum_{m=0}^{\infty} \frac{\gamma(m + 2\ell + 5, \frac{2r}{\ell + 1})}{(m + 2)(m + 2\ell + 3)!}.$$  (62)

which is nodeless in $(0, \infty)$ for $f_0 \leq 0$. The hyperconfluent third-order SUSY partner of the effective potential (58) finally becomes

$$V_3(r) = -2 + \frac{(\ell + 1)(\ell + 2)}{r^2} + 2 \left[ \frac{w^2(r)}{f(r)u_1^2(r)} \right]' ,$$  (63)

where $u_1(r)$, $w(r)$ and $f(r)$ are given by equations (59), (60) and (62), respectively.

The first two terms of equation (63) correspond to an effective potential different from the initial one (compare equation (58)). This difference is also reflected in the energy levels of a potential composed only of these two terms, which are given by $E_n = -1/(n + \ell + 1)^2$, $n = 1, 2, \ldots$. Thus, it is natural to interpret that the third term of equation (63) is mainly responsible for supporting the ground state energy of $V_3(r)$ at $E_0 = -1/(\ell + 1)^2$.

Let us note that the family of hyperconfluent third-order SUSY partner potentials given by equation (63) is different from the ones which have been derived either by first-order SUSY [30–32, 15, 16] or by second-order SUSY transformations [15, 16, 21] (just compare the centrifugal terms of each family).

In particular, for $\ell = 0$ it turns out that, departing from the Coulomb potential without centrifugal term, $V_0(r) = -2/r$, we arrive at a new one-dimensional potential with a non-trivial centrifugal term given by

$$V_3(r) = -2 + \frac{2}{r^2} + 2 \left[ \frac{w^2(r)}{f(r)u_1^2(r)} \right]' ,$$  (64)

where now

$$u_1(r) = 2r e^{-r}, \quad w(r) = -\frac{1}{2} \gamma(3, 2r) = (2r^2 + 2r + 1)e^{-2r} - 1 ,$$  (65)

$$f(r) = f_0 - \frac{1}{12} \gamma(3, 2r) r^2 F_2 \left( 1, 2; 3, 4; 2r \right) + \frac{1}{4} \sum_{m=0}^{\infty} \frac{\gamma(m + 5, 2r)}{(m + 2)(m + 3)!} .$$  (66)

As an illustration, the isospectral potentials $V_0(r) = -2/r$ and the $V_3(r)$ of equations (64)–(66) for $f_0 = -1/10$ as functions of $r$ are shown in figure 1. The corresponding energy levels of $H_3$ and $H_0$ are given by $E_n = -1/(n + 1)^2$, $n = 0, 1, 2, \ldots$.

Let us note that the well arising in $V_3(r)$ at the neighborhood of $r \approx 1$ is induced by the third term of equation (64), which is also responsible for supporting the ground state energy at $E_0 = -1$. Let us recall that this level was not present in the effective potential of equation (58) with $\ell = 1$. 

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Figure 1. Coulomb potential $V_0(r) = -2/r$ (gray curve) and its hyperconfluent third-order SUSY partner $V_3(r)$ given by equations (64)–(66) for $f_0 = -1/10.$

7. Conclusions

In this paper, we have addressed the hyperconfluent third-order SUSY QM through two different (but equivalent) approaches, namely direct and iterative one. It was found the explicit expression for the Wronskian, which is the most relevant quantity to determine the form of the new potentials, the eigenfunctions of the associated Hamiltonians, etc.

The requirements for the seed solution to produce non-singular SUSY transformations were also explicitly determined. Note that, from considerations taking into account the order of the transformation, through the hyperconfluent third-order SUSY QM one obtains a three-parametric family of potentials (for a fixed factorization energy). However, since we had to impose two requirements on the solutions employed in the iterative approach, it turns out that the non-singular potentials for $r \in (0, \infty)$ belong just to a one-parametric subset of the general three-parametric family which is able to build up.

Our general procedure was illustrated by means of the free particle and the Coulomb potential. In particular, the last case illustrates clearly that the non-singular one-parametric family of potentials derived through the hyperconfluent third-order SUSY QM is different either from the set which can be achieved from a first-order SUSY transformation [30–32, 15, 16] or from the one which can be generated through the confluent second-order SUSY transformation [21].

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References

[1] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267
[2] Junker G 1996 Supersymmetric Methods in Quantum and Statistical Physics (Berlin: Springer)
