The effect of Dirac phase on acoustic vortex in media with screw dislocation

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Abstract

We study acoustic vortex in media with screw dislocation using the Katanaev-Volovich theory of defects. It is shown that the screw dislocation affects the beam’s orbital angular momentum and changes the acoustic vortex strength. This change is a manifestation of topological Dirac phase and is robust against fluctuations in the system.

Keywords: Acoustic vortex, Dirac phase, Screw dislocation, Thermal noise

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I. INTRODUCTION

The linearized equations of elasticity are analogous with Maxwell equations [1]. This analogy enables us to suggest new phenomena for elastic waves by knowing their optical counterparts. In particular, vortex phenomena in optics can be mapped onto acoustic vortex. An optical vortex beam focuses on rings rather than points and has helical wavefront, Fig. 1, [2]. The difference between this kind of beam and a plane wave is just an overall phase factor, $e^{il\varphi}$. The angle $\varphi$ is the polar angle in cylindrical coordinates for a beam with axis parallel to $z$ and $l$ is the optical vortex strength or the angular momentum that is carried by the helical beam [3, 4]. When the helical beam interacts with a microscopic particle, the orbital angular momentum can be transferred to the particle and make it spin around the beam axis. A wide range of applications have been recently found for this orbital angular momentum transfer. For example we can mention, particle trapping [5] in optical tweezers to manipulate micrometer-sized particles [6] and remote control of particles [7]. Optical vortex also has application in information encoding [8].

Acoustic vortex as the classical counterpart of optical vortex has been studied [9–12] and generated [13–15], recently. Since acoustic vortices can transfer orbital angular momentum to particles [15, 16], like their optical counterparts, they can be applied to particle trapping in acoustic tweezers and remote controlling, too. In addition acoustic vortices, potentially, can be used in sonar experiments [13]. Although similar properties to optical vortex is expected for acoustic vortex, there are limited studies in this area.

In this letter, acoustic vortex in media with screw dislocation is studied using the Katanaev-Volovich theory of defects. The motivation for studying defects is that they usually exist in crystalline solids and have strong effect on their physical properties [17–25].
In the presence of defects, we are confronting with complicated boundary conditions. This difficulty persuades physicists to introduce new approaches such as Katanaev-Volovich theory of defects in solids [24–30]. Katanaev-Volovich theory is a geometrical approach based on the isomorphism existing between the theory of defects in solids and three-dimensional gravity. In this formalism, elastic deformation which is introduced in the medium by defects is replaced by a non-Euclidean metric. According to this theory, at distances much larger than the lattice spacing where the continuum limit is valid, the solid can be described by a Riemann-Cartan manifold. Dislocations and disclinations of the medium are respectively associated with torsion and curvature of the manifold. We will show that the screw dislocation changes the acoustic vortex strength. This change is due to Dirac phase and is robust against fluctuations. Dirac phase belongs in the category of non-integrable phase factors that appear in many different areas of Physics [31–33]. Dirac showed that when a particle transports in an external electromagnetic field, its wave function acquires a phase term in addition to usual dynamic phase factor [34, 35].

The letter is organized as follow. In section II, we review the screw dislocation in Katanaev-Volovich formalism. Section III is devoted to the Dirac phase of acoustic waves in media with screw dislocation. The effect of noise on Dirac phase is discussed in section IV. Finally the conclusion is presented in section V.

II. SCREW DISLOCATION IN KATANAEV-VOLOVICH FORMALISM

Consider a point in an undeformed medium with coordinates $x^i$ with Euclidean metric $\delta_{ij}$. If the deformation due to the defect is described by a displacement vector $U^i(x)$, the point will have coordinates $y^i = x^i + U^i(x)$. According to the Katanaev-Volovich approach the effect of the elastic deformation $U^i$ is considered by introducing metric $g_{ij}$ which can be expressed in terms of the initial metric $\delta_{ij}$ as [27–29]

$$g_{ij} := \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \delta_{kl}, \quad i, j = 1, 2, 3.$$ 

One of the defects, which we are interested in, is a screw dislocation. In a screw dislocation the Burgers vector is parallel to the dislocation line. This kind of defect, which corresponds to a singular torsion along the defect line, is described by the following metric [26, 36]

$$ds^2 = g_{ij}dx^idx^j = (dz + \beta d\phi)^2 + d\rho^2 + \rho^2 d\phi^2,$$
where the parameter $\beta$ is related to the Burgers vector, $b$, by $\beta = \frac{b}{2\pi}$ and the screw dislocation line is oriented along the $z$-axis of the cylindrical coordinates ($\rho, \varphi, z$). To derive the above metric, we have used the displacement vector $U = (0, 0, \beta \varphi)$ associated with a screw dislocation in cylindrical coordinates. The metric tensor $g_{ij}$ is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^2 + \rho^2 & \beta \\ 0 & \beta & 1 \end{pmatrix}$$

and carries no curvature.

The torsion two-form associated with this defect is defined as $T^a = T_{ij}^a dx^i \wedge dx^j$, ($a \equiv \{\rho, \varphi, z\}$), that the only non-vanishing component is given by [25]

$$T^z = 2\pi \beta \delta^2(\rho) d\rho \wedge d\varphi,$$

where $\delta^2(\rho)$ is the two-dimensional delta function in flat space and reveals the singularity in torsion. Also the torsion in tensor notation can be written as

$$T^a_{ij} = \partial_i e^a_j - \partial_j e^a_i,$$

(2)

where $e^a_i$ are triad components. Comparison of equation (2) with the field strength $F_{ij} = \partial_i A_j - \partial_j A_i$ in the electromagnetism and the singular value of the torsion field, indicates a similarity between this case and the Aharanov-Bohm effect [37], where the Burgers vector plays the role of the magnetic flux.

### III. DIRAC PHASE OF ACOUSTIC WAVES

The dynamic of displacement vector field $U(x, t)$ in an elastic medium without defect is governed by (see e.g. [38])

$$\partial_t^2 U^i = \frac{\mu}{\rho} \nabla^2 U^i + \frac{(\lambda + \mu)}{\rho} \partial^i \partial_j U^j,$$

(3)

where $\lambda$ and $\mu$ are the Lame coefficients and $\rho$ is the density of the medium. According to the Katanaev-Volovich approach, media with defects can be treated by nontrivial metric, $g_{ij}$. Therefore, the covariant generalization of equation (3) gives the displacement vector dynamics in media with defects

$$\partial_t^2 U^i = \frac{\mu}{\rho} \widetilde{\nabla}^2 U^i + \frac{(\lambda + \mu)}{\rho} \widetilde{\nabla}^i \widetilde{\nabla}_j U^j.$$

(4)
The displacement vector can be decomposed covariantly into transversal, \(U^T_i\), and longitudinal, \(U^L_i\), parts \([28]\),

\[ U^i = U^T_i + U^L_i, \]

which satisfy the following relations

\[ \tilde{\nabla}_i U^T_i = 0, \]
\[ \tilde{\nabla}_i U^L_j - \tilde{\nabla}_j U^L_i = 0. \]

As far as we know from vector analysis, it is always possible to express a vector as the sum of the curl of a vector and the gradient of a scalar. So equation (4) decomposes into two independent equations for transverse and longitudinal parts of the displacement vector field,

\[ \frac{1}{v_T^2} \partial^2 U^T_i - \tilde{\nabla}^2 U^T_i = 0, \quad \frac{1}{v_L^2} \partial^2 U^L_i - \tilde{\nabla}^2 U^L_i = 0, \]

where

\[ v_T^2 = \frac{\mu}{\rho}, \quad v_L^2 = \frac{(\lambda + 2\mu)}{\rho}, \]

are the speeds of transverse and longitudinal parts in the medium. So by decomposition of (4) every longitudinal or transverse component of the displacement vector, \(U\), satisfies a separate scalar wave equation in the curved space. \(\tilde{\nabla}^2\) in (5) is the Laplace-Beltrami operator which is given by \(\tilde{\nabla}^2 = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j)\), where \(g\) is the determinant of the metric tensor \(g_{ij}\) and \((g^{ij})^{-1}\) is its inverse. This decomposition enables us to study each mode separately. Here we are interested in the longitudinal mode but similar results can be deduced for the transverse mode, too.

Using the metric tensor (1) for screw dislocation, the longitudinal part of the elastic wave in (5) takes the form

\[ \left\{ \frac{1}{\rho} \partial_\rho (\rho \partial_\rho) + \frac{1}{\rho^2} (\partial_\varphi - \beta \partial_z)^2 + \partial^2_z \right\} U^L_i(\rho, \varphi, z, t) = \frac{1}{v_L^2} \partial^2_t U^L_i(\rho, \varphi, z, t). \]

Considering a monochromatic paraxial wave as

\[ U^L_i(\rho, \varphi, z, t) = e^{-i\omega t} e^{ikz} u^L_i(\rho, \varphi), \]

yields to the following equation for longitudinal elastic wave

\[ \left\{ \frac{1}{\rho} \partial_\rho (\rho \partial_\rho) + \frac{1}{\rho^2} (\partial_\varphi - i\beta k)^2 - k^2 \right\} u^L_i(\rho, \varphi) = -\frac{\omega^2}{v_L^2} u^L_i(\rho, \varphi). \]
Equation (6) implies that $\partial_\varphi \rightarrow \partial_\varphi - ik\beta$ with respect to the defect free case ($\beta = 0$) in which the Laplacian operator is given in a flat space. In the other words, the $z$-component of angular momentum has changed according to $L_z \rightarrow L_z - k\beta$. The angular momentum of the acoustic wave along its axis is modified by the presence of the defect that is due to the torque exerted by the strain field of the dislocation. Introducing a momentum operator as $P = -i\nabla$ converts (6) into a time-independent Schrödinger-like equation with a gauge potential

$$\left(P - A\right)^2 u_{Li}(\rho, \varphi) = \frac{\omega^2}{v_L^2} u_{Li}(\rho, \varphi),$$

where the corresponding vector gauge potential is

$$A = \frac{k\beta}{\rho} \hat{e}_\varphi.$$  (8)

Since this gauge is curl free, $\nabla \times A = 0$, the perfect analogy is seen between acoustic waves in media with screw dislocation and the Aharanov-Bohm effect. Note that, the torsion field is invariant under gauge transformations of the potential,

$$A \rightarrow A + \nabla \Lambda.$$  

According to this correspondence, Dirac phase factor method [34, 39] can be used here (See the appendix). Thus, the solution of the wave equation (7) has the following property

$$u_{Li}(\rho, \varphi) = \exp \left\{ i \int_C A \cdot d\mathbf{r} \right\} u_{Li}^0(\rho, \varphi),$$

where $u_{Li}^0(\rho, \varphi)$ is the solution of the defect free case and $C$ is the beam trajectory (See Fig. 1). Substituting (8) into (9) yields

$$u_{Li}(\rho, \varphi) = e^{i \int_0^\varphi k\beta d\varphi} u_{Li}^0(\rho, \varphi).$$

This means that $u_{Li}(\rho, \varphi)$ differs from $u_{Li}^0(\rho, \varphi)$ just in a phase factor $e^{i\gamma}$ that

$$\gamma = \int_0^\varphi k\beta d\varphi,$$

is called Dirac phase. In other words, the coupling of torsion with angular momentum leads to an additional phase factor in the solution when the screw dislocation is present.

Hitherto we found that the difference between the solutions to the acoustic wave equation in the presence and in the absence of defects is just manifested in a phase factor, (10).
we only need to find the solution for the defect free case. In this case, \( \beta = 0 \), the solution of the wave equation (6) can be easily found as

\[
u^L_i(\rho, \varphi) = R(\rho)e^{il\varphi},
\quad (11)
\]

where \( R(\rho) \) is the radial solution of the Helmholtz equation and \( e^{il\varphi} \) represents acoustic vortex carrying the angular momentum \( l \), acoustic vortex strength, along the paraxial axis. According to equations (10) and (11) the solution of the wave equation in the presence of screw dislocation is shown as

\[
u^L_i(\rho, \varphi) = R(\rho)e^{i(l+\beta k)\varphi}.
\]

Therefore, the screw dislocation results in the change of the acoustic vortex strength from \( l \) to \( l + \beta k \). This change, due to Dirac phase, is proportional to the magnitude of Burgers vector or in other words the flux of torsion.

IV. THE EFFECT OF NOISE ON DIRAC PHASE

The presence of noise is an inevitable subject in physical systems, such as the ubiquitous thermal fluctuation. The effect of noise on the Dirac phase can be treated similar to the problem of electrons in media with screw dislocations [40]. The same procedure is used to show that considering a white noise leads to the average zero for the Dirac phase. This kind of noise coincides with uncorrelated nature of thermal noise. Indeed, the variance of the Dirac phase diminishes with time as

\[
\langle \Delta \gamma^2(T) \rangle \propto \frac{1}{T}
\]

where \( T \) is the period of the beam’s rotation on its circular trajectory (Fig. 1). Therefore, \( \gamma \) coincides with its noiseless value in the limit \( T \to \infty \). As a result, in spite of the dynamic phase, the Dirac phase of elastic waves is robust against fluctuations in the system.

V. CONCLUSION

In this letter we studied the effect of the screw dislocation on an acoustic wave using the Katanaev-Volovich theory of defects. This theory which is a geometrical approach uses
the isomorphism between the theory of solids and three-dimensional gravity to suppress the
technical complications due to adding defects to the system. It was shown that the screw
dislocation changes the acoustic vortex strength by coupling of torsion with orbital angular
momentum. This change is a manifestation of the topological Dirac phase and is robust
against fluctuations. For a white noise, coincides with the nature of thermal noise, the effect
of fluctuations on the Dirac phase diminishes as $\frac{1}{T}$ where $T$ is the period of beam’s rotation.

Appendix: Dirac phase factor method

Consider two Schrödinger-like wave equations with two different gauge fields $A_1, A_2 = A_1 + \nabla \Lambda$

$$\left(\mathbf{P} - A_1\right)^2 u_1^{Li}(\rho, \varphi) = \frac{\omega^2}{v_L^2} u_1^{Li}(\rho, \varphi), \quad (12)$$

and

$$\left(\mathbf{P} - A_1 - \nabla \Lambda\right)^2 u_2^{Li}(\rho, \varphi) = \frac{\omega^2}{v_L^2} u_2^{Li}(\rho, \varphi). \quad (13)$$

These wave equations are equal due to the gauge invariance, so we are going to find the
relation between $u_1^{Li}$ and $u_2^{Li}$ to make this equality possible. We claim that

$$u_2^{Li} = e^{i\Lambda(x)} u_1^{Li}, \quad (14)$$

and put it in (13) which yields to

$$\left(\mathbf{P} - A_1 - \nabla \Lambda\right) e^{i\Lambda(x)} u_1^{Li}(\rho, \varphi) = \frac{\omega^2}{v_L^2} e^{i\Lambda(x)} u_1^{Li}(\rho, \varphi).$$

Rewriting the last equation as follow

$$e^{-i\Lambda(x)}(\mathbf{P} - A_1 - \nabla \Lambda) e^{i\Lambda(x)}(\mathbf{P} - A_1 - \nabla \Lambda) e^{i\Lambda(x)} u_1^{Li}(\rho, \varphi) = \frac{\omega^2}{v_L^2} u_1^{Li}(\rho, \varphi).$$

and comparing with (12) leads to

$$e^{-i\Lambda(x)}(\mathbf{P} - A_1 - \nabla \Lambda(x)) e^{i\Lambda(x)} = \mathbf{P} - A_1.$$

Putting $\mathbf{P} = -i\nabla$ will complete the proof. If we consider (12) for a defect free case, $A_1 = 0$,
and (13) for a case with defect, $A_2 = A = \nabla \Lambda(x)$, then according to (14)

$$u_2^{Li} = \exp \left\{ i \int_C A \cdot d\mathbf{r} \right\} u_1^{Li}.$$ 

Therefore the two solutions are related by a phase factor which is a function of the gauge
field.

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