Minimax Signal Detection in Ill-Posed Inverse Problems

Yuri I. Ingster*
Department of Mathematics II,
Saint Petersburg State Electrotechnical University,
Russia

Theofanis Sapatinas
Department of Mathematics and Statistics,
University of Cyprus,
Cyprus

and

Irina A. Suslina
Department of Mathematics,
Saint Petersburg State University of Information Technologies, Mechanics and Optics,
Russia

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Abstract

Ill-posed inverse problems arise in various scientific fields. We consider the signal detection problem for mildly, severely and extremely ill-posed inverse problems with $l^q$-ellipsoids (bodies) for Sobolev, analytic and generalized analytic classes of functions under the Gaussian white noise model. We restrict our attention to the range $q \in (0, 2]$, consisting of the “standard” case $q = 2$ and the “sparse” case $q \in (0, 2)$, that has received considerable attention in the nonparametric estimation literature over the last decade, with the “sparse” case $q \in (0, 2)$ considered mostly in well-posed problems. We study both rate and sharp asymptotics for the error probabilities in the minimax setup. By construction, however, the derived tests are, often, non-adaptive. In such cases, minimax rate-optimal adaptive tests of rather simple structure are also constructed.

The above formulation and the results obtained are based on the singular value decomposition of the operator involved. We also consider the minimax signal detection problem for mildly ill-posed inverse problems with Besov classes (bodies) of functions $B_{q,t}^{\alpha}$, $\alpha > 0$, $q \in (0, \infty)$, $t \in (0, \infty)$, that arises through the wavelet-vaguelette decomposition of a special class of homogeneous operators (such as integration or fractional integration). Restricting again our attention to the “sparse” case $q \in (0, 2)$, both rate and sharp asymptotics for the error probabilities in the minimax setup are studied. Minimax rate-optimal non-adaptive and adaptive tests are also constructed.

Keywords: Analytic functions, Ill-posed inverse problems, Besov Spaces, Minimax testing, Singular value decomposition, Sobolev spaces, Wavelet-Vaguelette decomposition

Mathematics Subject Classification (2000): Primary 62G10, 62G20; Secondary 62C20

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1 Introduction

We consider the detection problem in linear operator equations from noisy data under the assumption that the singular values of the operator decrease polynomial, exponential or power-exponential fast and that the underlying solution is also polynomial, exponential or power-exponential smooth in the Fourier domain. More precisely, consider the Gaussian white noise model
\[ dY_\varepsilon(t) = Af(t)dt + \varepsilon dW(t), \quad t \in D, \] (1.1)
where \( A \) is a known linear bounded operator acting on a Hilbert space \( \mathcal{H} \) with values on \( L^2(D), D \subset \mathbb{R}, \) \( W \) is a standard Wiener process on \( D, \varepsilon > 0 \) is a small parameter (the noise level) and \( f \in \mathcal{H} \) is the unknown response function (that one needs to detect or estimate). In what follows, we assume that \( \mathcal{H} \subset L^2(D) \).

Although the Gaussian white noise model (1.1) is continuous and real data are typically discretely sampled, versions of it have been extensively studied in the nonparametric literature and are considered as idealized models that provide, subject to some limitations, approximations to many, sampled-data, nonparametric models (see, e.g., [4], [10], [21]). It may also seem initially rather remote. One may, however, be helped by the observation that what it really means is the following: for any function \( f, \psi \),
\[
\langle f, \psi \rangle = \int_D f(t)\psi(t) dt,
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product. Equality, \( A \) being injective (meaning that \( A \) has a trivial nullspace). In most cases of interest, \( A^* \) is the adjoint of \( A \) and assume that \( A^*A \) is a compact operator on \( \mathcal{H} \), which is equivalent to \( A \) being compact on \( \mathcal{H} \). Denote with \( \mathbb{N} \) the set of natural numbers, i.e., \( \mathbb{N} = \{1, 2, \ldots\} \). Then, an application of the spectral theorem for self-adjoint compact operators on Hilbert spaces ensures the existence of a (complete) orthonormal basis system of eigenfunctions \( \{\varphi_k\}_{k \in \mathbb{N}} \) of \( A^*A \) with corresponding positive eigenvalues \( \{\rho_k\}_{k \in \mathbb{N}} \) (since \( A \) is injective, which is equivalent to \( A^*A \) being positive). More precisely, we have the following representation of \( A^*A \)
\[
A^*Af = \sum_{k \in \mathbb{N}} \rho_k \langle f, \varphi_k \rangle \varphi_k = \sum_{k \in \mathbb{N}} b_k^2 \theta_k \varphi_k,
\] (1.2)
where \( b_k = \sqrt{\rho_k} > 0, k \in \mathbb{N}, b_1 > b_2 > \ldots > 0, \) and \( \{\theta_k\}_{k \in \mathbb{N}} \) are the so-called Fourier coefficients of \( f \) with respect to \( \{\varphi_k\}_{k \in \mathbb{N}} \). The expansion (1.2) is called the singular value decomposition (SVD) of \( A \) with singular values \( \{b_k\}_{k \in \mathbb{N}} \) with respect to \( \{\varphi_k\}_{k \in \mathbb{N}} \) and \( \theta_k = \langle f, \varphi_k \rangle, k \in \mathbb{N} \) (see, e.g., [21], Chapter VI, pp. 203-204). Note also that, by Parseval’s equality, \( \theta = \{\theta_k\}_{k \in \mathbb{N}} \in l^2, \) where \( l^2 = \{\theta : \sum_{k \in \mathbb{N}} \theta_k^2 < \infty\} \).

Clearly, \( \|A\varphi_k\| = b_k, k \in \mathbb{N} \). Let now \( \psi_k \) (the normalized image of \( \varphi_k \)) be determined by the relation \( \psi_k = (A\varphi_k)/\|A\varphi_k\| = b_k^{-1}A\varphi_k, k \in \mathbb{N} \). Then, the basis system \( \{\psi_k\}_{k \in \mathbb{N}} \) is orthonormal since, for \( k, l \in \mathbb{N} \), one has
\[
\|\psi_k\|^2 = b_k^{-2} \langle A\varphi_k, A\varphi_k \rangle = b_k^{-2} \langle A^*A\varphi_k, \varphi_k \rangle = b_k^{-2}b_k^2 \|\varphi_k\|^2 = 1, \quad \langle \psi_k, \psi_l \rangle = 0, \quad k \neq l,
\]
and, furthermore, \( A^*\psi_k = b_k^{-1}A^*A\varphi_k = b_k \varphi_k \). Denote by \( y_k = \langle dY_\varepsilon, \psi_k \rangle = \int_D \psi_k(t)dY_\varepsilon(t) \) and \( \xi_k = \langle dW, \psi_k \rangle = \int_D \psi_k(t)dW(t), k \in \mathbb{N} \), the Fourier coefficients of \( dY_\varepsilon \) and \( dW \), respectively, with respect to \( \{\psi_k\}_{k \in \mathbb{N}} \). We then have the following expansion
\[
y_k = \langle Af, \psi_k \rangle + \varepsilon \xi_k = \langle Af, b_k^{-1}A\varphi_k \rangle + \varepsilon \xi_k = b_k \theta_k + \varepsilon \xi_k, k \in \mathbb{N},
\]
where $\xi_k, k \in \mathbb{N}$, are independent and identically distributed (iid) standard Gaussian (i.e., $\mathcal{N}(0,1)$) random variables.

Thus, the Gaussian white noise model \((1.1)\) generates an equivalent discrete observational model in the sequence space, called the Gaussian sequence model,

$$y_k = b_k \theta_k + \varepsilon \xi_k, \quad k \in \mathbb{N},$$

where $\varepsilon > 0$ is a small parameter, $\{b_k\}_{k \in \mathbb{N}}$ and $\{\theta_k\}_{k \in \mathbb{N}}$ are, respectively, the (positive) singular values of (the injective compact operator) $A$ and the Fourier coefficients of $f$, with respect to $\{\varphi_k\}_{k \in \mathbb{N}}$, and $\xi_k \sim \mathcal{N}(0,1), k \in \mathbb{N}$.

Therefore, the problem of detection or estimation of the unknown response function $f$ based on the observation of a trajectory $\{Y_\varepsilon = Y_\varepsilon(t)\}, t \in D$, generating from the Gaussian white noise model \((1.1)\), corresponds to the problem of detection or estimation of the unknown sequence $\theta = \{\theta_k\}_{k \in \mathbb{N}}$ based on observations $y = \{y_k\}_{k \in \mathbb{N}}$ generating from the Gaussian sequence model \((1.3)\). Since the goal is to detect or estimate $\theta = \{\theta_k\}_{k \in \mathbb{N}}$ and not $b \theta = \{b_k \theta_k\}_{k \in \mathbb{N}}$ in the Gaussian sequence model \((1.3)\), one has to remove $b = \{b_k\}_{k \in \mathbb{N}}$, which is equivalent to inverting the operator $A$ in the Gaussian white noise model \((1.1)\). Hence, the effect of the ill-posedness of the inverse problem is clearly seen in the decay of $b_k$ as $k \to \infty$. As $k \to \infty$, $b_k \theta_k$ usually gets weaker and is then more difficult to detect or estimate $\theta = \{\theta_k\}_{k \in \mathbb{N}}$.

Observe now that Gaussian sequence model \((1.3)\) can be rewritten in the (equivalent) form

$$x_k = \theta_k + \varepsilon \sigma_k \xi_k, \quad k \in \mathbb{N},$$

where $x_k = y_k/b_k$ and $\sigma_k = b_k^{-1} > 0, k \in \mathbb{N}$. In this situation, the difficulty of ill-posedness, and hence any asymptotic results, is measured by the rates (type of growth) of $\sigma_k$ as $k \to \infty$. For polynomial rates, i.e., $\sigma_k \asymp k^\beta, \beta > 0, as k \to \infty$, the inverse problem is called mildly (or softly) ill-posed, for exponential rates, i.e., $\sigma_k \asymp \exp(\beta k), \beta > 0, as k \to \infty$, is called severely ill-posed, and for the case where $\sigma_{k+1}/\sigma_k \to \infty$ as $k \to \infty$, is called extremely ill-posed. Note that an extremely ill-posed inverse problem includes power-exponential rates, i.e., $\sigma_k \asymp \exp(\beta k^\gamma), \beta > 0, \gamma > 1, as k \to \infty$ (in what follows, $\asymp$ is the symbol of equivalence in order of growth, i.e., the relation $c_n \asymp d_n$ means that there exists constants $0 < C_1 \leq C_2 < \infty$ and $n_0$ large enough such that $C_1 \leq c_n/d_n \leq C_2$ for $n \geq n_0$. We say that $c_n(\kappa) \asymp d_n(\kappa)$ uniformly over $\kappa \in \mathcal{K}$, if the similar inequalities hold true for all $\kappa \in \mathcal{K}$ with constants $0 < C_1 \leq C_2 < \infty$ and $n_0$ which do not depend on $\kappa$. The relation $c_n \sim d_n$ means that for any $\delta \in (0,1)$ there exists $n_0$ large enough such that $1 - \delta \leq c_n/d_n \leq 1 + \delta$ for $n \geq n_0$. The uniform version of the relation $c_n(\kappa) \sim d_n(\kappa), \kappa \in \mathcal{K}$, is defined similarly.)

The study of ill-posed inverse problems (in the presence of additive random noise) was initiated in 1960-ies (see \([2], [26]\)) and has been in the focus of recent statistical literature, mostly in the context of estimation of the unknown response function $f$ based on observations from the Gaussian white noise model \((1.1)\). Several methods of estimation were proposed such as Tikhonov-Phillips type regularization techniques, recursive estimation procedures in Hilbert spaces and projection (or Galerkin) methods. For a survey on these and other recent results, we refer to, e.g., \([5], [6]\) and \([12]\). The SVD of $A$, described above, is a natural way of projection for ill-posed inverse problems, leading to the equivalence between the Gaussian white noise model \((1.1)\) and the Gaussian sequence models \((1.3)-(1.4)\).

Note, however, that the Gaussian sequence models \((1.3)-(1.4)\) are not confined to the above situation only and they appear in many other situations. For example, they also describe the estimation of a signal from direct observations with correlated data, see \([12]\). Moreover, the theoretical results presented in subsequent sections are actually derived for the Gaussian...
sequence model (1.3) (or, equivalently, (1.4)) and are, therefore, independent of the way one can explore to arrive at this model. (We point out here that an analogous representation for certain homogeneous operators based on the wavelet-vaguelette decomposition (WVD), that simultaneously quasi-diagonalize both the operator and the a-priori information of the unknown response function, exists, see [9]. This representation will be discussed in Section 3 in order to justify minimax detection for mildly ill-posed problems with Besov classes (bodies) of functions $B^q_{q,t}$, $\alpha > 0$, $q \in (0,2)$, $t \in (0,\infty)$.)

An important element of the Gaussian sequence models (1.3)-(1.4) is the prior information about the sequence $\theta = \{\theta_k\}_{k \in \mathbb{N}}$. Successful detection or estimation of the sequence $\theta = \{\theta_k\}_{k \in \mathbb{N}}$ is possible only if its elements $\theta_k$, $k \in \mathbb{N}$, tend to zero sufficiently fast as $k$ tends to infinity, meaning that the underlying response function $f$ in the Gaussian white noise model (1.1) is sufficiently smooth. A standard assumption on the smoothness of $f$ is to suppose that the sequence $\theta = \{\theta_k\}_{k \in \mathbb{N}}$ belongs to an $l^q$-ellipsoid (body), $0 < q < \infty$, in $l^2$, of semi-axes $L/a_k$, $k \in \mathbb{N}$, i.e.,

$$\tilde{\Theta} = \tilde{\Theta}_q(a, L) = \{\theta \in l^2 : \sum_{k \in \mathbb{N}} |a_k \theta_k|^q \leq L^q\} , \quad (1.5)$$

where $a = \{a_k\}_{k \in \mathbb{N}}$, $a_k \geq 0$, $a_k \to \infty$ as $k \to \infty$, and $L > 0$. (Note that the requirement $a_k > 0$, $a_k \to \infty$ as $k \to \infty$, ensures that $\tilde{\Theta}_q(a, L)$ is a compact subset of $l^2$.) The sequence $a = \{a_k\}_{k \in \mathbb{N}}$ characterizes the “shape” of the ellipsoid while the parameter $L$ characterizes its “size”. This means that for large values of $k$, the elements $\theta_k$, $k \in \mathbb{N}$, will decrease in $k$ and, hence, will be small for large $k$. (Certainly, the set $\tilde{\Theta}_q(a, L)$ is a ball of the radius $L$ in $l^2$ with respect to the norm $|\theta|_{a,q} = \left(\sum_{k \in \mathbb{N}} |a_k \theta_k|^q\right)^{1/q}$.)

Over the last decade, the range $q \in (0,2]$ has attracted considerable attention in the nonparametric estimation literature, with the case $q \in (0,2)$ considered mostly in well-posed problems. This range contains the “standard” case $q = 2$ and the “sparse” case $q \in (0,2)$. In what follows, we also restrict our attention to these cases, i.e., we consider minimax signal detection in ill-posed problems with $l^q$-ellipsoids for the range $q \in (0,2]$.

The functional sets of the form (1.5) that are often used in various ill-posed inverse problems are the Sobolev classes of functions (see [28]) and the classes of analytic functions (see [15]). The Sobolev classes of functions are of the form

$$W_q(\alpha, L) = \{f = \sum_{k \in \mathbb{N}} \theta_k \varphi_k : \theta \in \tilde{\Theta}_q(\alpha, L)\},$$

where $\tilde{\Theta}_q(\alpha, L) = \tilde{\Theta}_q(a, L)$, $a = \{a_k\}_{k \in \mathbb{N}}$, $a_1 = 1$ and, for $k = 2, 3, \ldots$ and for some $\alpha > 0$,

$$a_k = \begin{cases} (k - 1)^{\alpha}, & \text{if } k \text{ is odd,} \\ k^{\alpha}, & \text{if } k \text{ is even}, \end{cases} \quad \text{i.e., } a_k \sim k^{\alpha}.$$

The class of analytic functions is of the form

$$A_q(\alpha, L) = \{f = \sum_{k \in \mathbb{N}} \theta_k \varphi_k : \theta \in \tilde{\Theta}_q(\alpha, L)\},$$

where $\tilde{\Theta}_q(\alpha, L) = \tilde{\Theta}_q(a, L)$, $a = \{a_k\}_{k \in \mathbb{N}}$, $a_k = \exp(\alpha k)$, $k \in \mathbb{N}$, for some $\alpha > 0$. We also consider the class of generalized analytic functions defined as

$$G_q(\alpha, L) = \{f = \sum_{k \in \mathbb{N}} \theta_k \varphi_k : \theta \in \tilde{\Theta}_q(\alpha, L)\}.$$
where \( \Theta_q(a, L) = \tilde{\Theta}_q(a, L) \), \( a = \{a_k\}_{k \in \mathbb{N}} \), \( \lim \inf_{k \to \infty} a_{k+1}/a_k \in (1, \infty) \). This class includes the cases where \( a_k = \exp(\alpha k^\tau) \), \( k \in \mathbb{N} \), for some \( \alpha > 0 \) and \( \tau \geq 1 \) (the case \( \tau = 1 \) corresponds to the class of analytic functions). The Sobolev class of functions has been associated, e.g., with the estimation of derivatives of smooth functions while the classes of analytic and generalized analytic functions has been used in, e.g., the estimation of the initial or boundary conditions in partial differential equations (see, e.g., [5], [6], [7]).

Despite the growing number of works for the estimation problem in ill-posed inverse problems under the Gaussian white noise model (1.1) (see, e.g., [3], [5], [6], [7], [13], [14], [22], [23]), very little work exists for the corresponding detection problem, see, Section 4.3.3 of [17] and [11] (although their results are obtained from models that are neither formulated nor immediately seen as particular ill-posed inverse problems). It is our aim to present a general framework for the minimax detection study of the aforementioned ill-posed inverse problems. In particular, both rate and sharp asymptotics for the error probabilities in the minimax setup are studied in detail and minimax tests are constructed. By construction, however, the derived tests are, often, non-adaptive. In such case, minimax rate-optimal adaptive tests of rather simple structure for the various ill-posed inverse problems under study are also constructed. (We pinpoint that non-asymptotic minimax rates of testing for some of the ill-posed inverse problems under consideration are recently studied in [20].)

Before proceeding to the theoretical results, we briefly mention some illustrative examples arising in various scientific fields that lead to the Gaussian sequence model (1.3) (or the equivalent model (1.4)). The various models presented below are scattered throughout the literature, see, e.g., [5], [6], [7], [13], [14] and [20].

- **differentiation** \( (f \in L^2([0, 1]), \text{periodic on } [0, 1], \{\varphi_k\}_{k \in \mathbb{N}} \text{ being the complex trigonometric system on } [0, 1]) \). The goal is to detect or estimate the \( m \)-th derivative \( f(t) = g^{(m)}(t) \) (for some \( m \in \mathbb{N} \)), based on the observation of a trajectory \( \{Y_\varepsilon = Y_\varepsilon(t)\}, t \in [0, 1] \), obeying the Gaussian white noise model (1.1) with \( D = [0, 1], \mathcal{H} = \{f : f \in L^2([0, 1]), \int_0^1 f(t)dt = 0\}, \mathcal{H} \subset L^2([0, 1]) \) and \( Af(t) = Ag^{(m)}(t) = g(t) \). This problem corresponds to a mildly ill-posed inverse problem since \( b_k \to 0 \) (or, equivalently, \( \sigma_k \to \infty \)) polynomially (with \( \beta = m \)) fast as \( k \to \infty \).

- **the Dirichlet problem of the Laplacian on the unit circle** \( (f \in L^2([0, 2\pi]), \text{periodic on } [0, 2\pi], \{\varphi_k\}_{k \in \mathbb{N}} \text{ being the trigonometric system on } [0, 2\pi]) \). The goal is to detect or estimate the boundary condition \( f \) based on the observation of a trajectory \( \{Y_\varepsilon = Y_\varepsilon(\varphi)\}, \varphi \in [0, 2\pi] \), obeying the Gaussian white noise model (1.1) with \( \varphi \) in place of \( t \), \( D = [0, 2\pi], \mathcal{H} = L^2([0, 2\pi]) \) and \( Af(t) = u(r_0, \varphi) \), where \( u(r, \varphi), r \in [0, 1], \varphi \in [0, 2\pi] \), is the solution of the Dirichlet problem of the Laplacian on the unit circle in polar coordinates with boundary condition \( u(1, \varphi) = f(\varphi) \). This problem corresponds to a severely ill-posed inverse problem since \( b_k \to 0 \) (or, equivalently, \( \sigma_k \to \infty \)) exponentially fast as \( k \to \infty \).

- **the heat conductivity equation** \( (f \in L^2([0, 1]), \text{periodic on } [0, 1], \{\varphi_k\}_{k \in \mathbb{N}} \text{ being the complex trigonometric system on } [0, 1]) \). The goal is to detect or estimate the initial condition \( f \) based on the observation of a trajectory \( \{Y_\varepsilon = Y_\varepsilon(x)\}, x \in [0, 1] \), obeying the Gaussian white noise model (1.1) with \( x \) in place of \( t \), \( D = [0, 1], \mathcal{H} = L^2([0, 1]) \) and \( Af(t) = u(T, x), \text{where } u(t, x), t > 0, x \in [0, 1], \text{is the solution of the heat conductivity equation with periodic boundary conditions and initial condition } u(0, x) = f(x) \). This problem corresponds to an extremely ill-posed inverse problem since \( b_k \to 0 \) (or, equivalently, \( \sigma_k \to \infty \)) power-exponentially fast (with \( \gamma = 2 \)) as \( k \to \infty \).
• deconvolution \((f \in L^2([0, 1]), \text{ periodic on } [0, 1], \{\varphi_k\}_{k \in \mathbb{N}} \text{ being the complex trigonometric system on } [0, 1])\). The goal is to detect or estimate the response function \(f\) based on the observation of a trajectory \(\{Y_\varepsilon = Y_\varepsilon(t)\}_{t \in [0, 1]}\), obeying the Gaussian white noise model (1.1) with \(D = [0, 1], \mathcal{H} = L^2([0, 1])\) and \(Af(t) = (g * f)(t) = \int_0^1 f(u)g(t - u)du, \) i.e., \(A\) is the convolution operator on \(L^2([0, 1])\), where the unknown kernel (or blurring function) \(g \in L^2([0, 1])\) is also periodic on \([0, 1]\). This problem corresponds to a mildly, severely or extremely ill-posed inverse problem, depending on the decay of \(b_k = |\nu_k|\) to zero as \(k \to \infty\), where \(\nu_k, k \in \mathbb{N}\), are the Fourier coefficients of \(g\).

The above formulation and the various ill-posed inverse problems considered are based on the singular value decomposition of \(A\), under the Gaussian white noise model (1.1). However, it is well known that although offers a kind of diagonalization, SVD has its limitations rooted in the fact that the resulting eigenfunctions derive from the operator under study, not from the signal to be recovered. To this end, under the Gaussian white noise model (1.1), we also consider the signal detection problem for mildly ill-posed inverse problems with Besov classes (bodies) of functions, \(\mathcal{B}^\alpha_{q,t}, \alpha > 0, q \in (0, \infty), t \in (0, \infty)\). This problem arises through the WVD of a special class of homogeneous operators (such as integration or fractional integration), and simultaneously quasi-diagonalizes both the operator \(A\) and the a-priori regularity of the function \(f\) to be detected. Restricting again our attention to the “sparse” case \(q \in (0, 2)\), both rate and sharp asymptotics for the error probabilities in the minimax setup are studied. Minimax rate-optimal non-adaptive and adaptive tests are also constructed.

The rest of the paper is organized as follows. The general statement of minimax signal detection in ill-posed inverse problems is given in Section 2. To help the readers, a short description of the main results and a comparison with similar results obtained in the corresponding estimation problems are presented in Section 3. The general methods for the study of minimax signal detection in ill-posed inverse problems with \(l^2\)-ellipsoids are given in Section 4.1. In Sections 4.2–4.7, we provide a complete treatment to the minimax signal detection problem for mildly, severely and extremely ill-posed inverse problems with \(l^q\)-ellipsoids, \(q \in (0, 2]\), for Sobolev, analytic and generalized analytic classes of functions under the Gaussian sequence model (1.3). We study both rate and sharp asymptotics for the error probabilities in the minimax setup. By construction, the derived tests are, often, non-adaptive. In Section 5 for the ill-posed inverse problems under consideration, we also construct minimax rate-optimal adaptive tests of rather simple structure. We point out that, sharp, rate and rate-adaptive optimality results for the case of mildly ill-posed inverse problems with (i) \(l^q\)-ellipsoids, \(q \in (0, 2]\), for Sobolev classes of functions, and (ii) Besov classes of functions, \(\mathcal{B}^\alpha_{q,t}, \alpha > 0, t \in (0, \infty), q \in (0, 2)\), are of different nature, more complicated, and do not follow directly from the general framework considered above. However, as we reveal in Sections 4.7 and 6, these results can be obtained from a hitherto unknown link with the sequence models considered in this paper and results obtained in other contexts and presented in Chapters 4, 6 and 7 of [17]. Since these results are scattered in the cited reference, and not not immediately seen as ill-posed inverse problems, for completeness and an immediate access, we formulate and present them in Sections 4.7 and 6. The proofs are postponed until the Appendix.
2 Signal detection in the Gaussian sequence model: the minimax framework

Consider the Gaussian sequence model (1.3). In order to avoid having a trivial minimax hypothesis testing problem (i.e., trivial power), one usually needs to remove a neighborhood around the functional parameter under the null hypothesis and to add some additional constraints, that are typically expressed in the form of some regularity conditions, such as constraints on the derivatives, of the unknown functional parameter of interest (see, e.g., [17], Sections 1.3-1.4).

In view of the above observation, the main object of our study is the hypothesis testing problem

$$H_0 : \theta = 0, \quad \text{versus} \quad H_1 : \sum_{k \in \mathbb{N}} |a_k \theta_k|^q \leq 1, \quad \sum_{k \in \mathbb{N}} \theta_k^2 \geq r_\varepsilon^2,$$  \hspace{1cm} (2.1)

where $\theta = \{\theta_k\}_{k \in \mathbb{N}} \in l^2$, $a = \{a_k\}_{k \in \mathbb{N}}$, $a_k \geq 0$, $a_k \to \infty$ as $k \to \infty$, $r_\varepsilon > 0$, $r_\varepsilon \to 0$, is a given family, and $q \in (0,2]$. It means that the set under the alternative corresponds to an $l^q$-ellipsoid of semi-axes $1/a_k$, $k \in \mathbb{N}$, with an $l^2$-ball of radius $r_\varepsilon$ removed. (For simplicity in the calculations of the main results in subsequent sections, we focus attention on ellipsoids of the form (1.5) with “size” $L = 1$.)

Consider now the sequence $\eta = \{\eta_k\}_{k \in \mathbb{N}}$ with elements $\eta_k = b_k \theta_k = \theta_k/\sigma_k$, $k \in \mathbb{N}$. Recall that, in the ill-posed inverse problems under consideration, $\sigma_k = 1/b_k \to \infty$ or $b_k \to 0$, as $k \to \infty$. Hence, $\eta \in l^2$, and the Gaussian sequence model (1.3) is of the form

$$y_k = \eta_k + \varepsilon \xi_k, \quad k \in \mathbb{N}. \quad (2.2)$$

The hypothesis testing problem (2.1) can also be written in the following equivalent form

$$H_0 : \eta = 0, \quad \text{versus} \quad H_1 : \eta \in \Theta_q(r_\varepsilon), \quad (2.3)$$

where the set under the alternative, i.e., $\Theta_q(r_\varepsilon)$, is determined by the constraints

$$\Theta_q = \{\eta \in l^2 : \sum_{k \in \mathbb{N}} |a_k \sigma_k \eta_k|^q \leq 1\}, \quad \Theta_q(r_\varepsilon) = \{\eta \in \Theta_q : \sum_{k \in \mathbb{N}} \sigma_k^2 \eta_k^2 \geq r_\varepsilon^2\}, \quad (2.4)$$

i.e., the set under the alternative corresponds to an $l^q$-ellipsoid of semi-axes $1/(a_k \sigma_k)$, $k \in \mathbb{N}$, with an $l^2$-ellipsoid of semi-axes $r_\varepsilon/\sigma_k$, $k \in \mathbb{N}$, removed.

We are therefore interesting in the minimax efficiency of the hypothesis testing problem (2.3)-(2.4) for a given family of sets $\Theta_\varepsilon = \Theta_q(r_\varepsilon) \subset l^2$. It is characterized by asymptotics, as $\varepsilon \to 0$, of the minimax error probabilities in the problem at hand. Namely, for a (randomized) test $\psi$ (i.e., a measurable function of the observation $y = \{y_k\}_{k \in \mathbb{N}}$ taking values in $[0,1]$), the null hypothesis is rejected with probability $\psi(y)$ and is accepted with probability $1 - \psi(y)$. Let $P_{\varepsilon, \eta}$ be the probability measure for the Gaussian sequence model (2.2) and denote by $E_{\varepsilon, \eta}$ the expectation over this probability measure. Let $\alpha_\varepsilon(\psi) = E_{\varepsilon, 0}\psi$ be its type I error probability, and let $\beta_\varepsilon(\Theta_\varepsilon, \psi) = \sup_{\eta \in \Theta_\varepsilon} E_{\varepsilon, \eta}(1 - \psi)$ be its maximal type II error probability. We consider two criteria of asymptotic optimality:

1. The first one corresponds to the classical Neyman-Pearson criterion. For $\alpha \in (0,1)$, we set

$$\beta_\varepsilon(\Theta_\varepsilon, \alpha) = \inf_{\psi : \alpha_\varepsilon(\psi) \leq \alpha} \beta_\varepsilon(\Theta_\varepsilon, \psi).$$

We call a family of tests $\psi_{\varepsilon, \alpha}$ \textit{asymptotically minimax} if

$$\alpha_\varepsilon(\psi_{\varepsilon, \alpha}) \leq \alpha + o(1), \quad \beta_\varepsilon(\Theta_\varepsilon, \psi_{\varepsilon, \alpha}) = \beta_\varepsilon(\Theta_\varepsilon, \alpha) + o(1),$$


where \( o(1) \) is a family tending to zero; here, and in what follows, unless otherwise stated, all limits are taken as \( \varepsilon \to 0 \).

(2) The second one corresponds to the total error probabilities. Let \( \gamma_{\varepsilon}(\Theta_{\varepsilon}, \psi) \) be the sum of the type I and the maximal type II error probabilities, and let \( \gamma_{\varepsilon}(\Theta_{\varepsilon}) \) be the minimax total error probability, i.e.,

\[
\gamma_{\varepsilon}(\Theta_{\varepsilon}) = \inf_{\psi} \gamma_{\varepsilon}(\Theta_{\varepsilon}, \psi),
\]

where the infimum is taken over all possible tests. We call a family of tests \( \psi_{\varepsilon} \) \textit{asymptotically minimax} if

\[
\gamma_{\varepsilon}(\Theta_{\varepsilon}, \psi_{\varepsilon}) = \gamma_{\varepsilon}(\Theta_{\varepsilon}) + o(1).
\]

It is known that (see, e.g., [17], Chapter 2) that often the sharp asymptotics are of Gaussian type, i.e.,

In other words, it means that, for small \( \varepsilon \), one can detect all sequences \( \eta \in \Theta_{q}(r_{\varepsilon}) \) if the ratio \( r_{\varepsilon}/r_{\varepsilon}^{*} \) is large, whereas if this ratio is small then it is impossible to distinguish between the null and the alternative hypothesis, with small minimax total error probability. Hence, the rate optimality problem corresponds to finding the separation rates \( r_{\varepsilon}^{*} \) and to constructing asymptotically minimax consistent families of tests.

For the set of the form (2.4), we use the notation \( \gamma_{\varepsilon}(\Theta_{q}(r_{\varepsilon})) = \gamma_{\varepsilon}(r_{\varepsilon}) \) and \( \beta_{\varepsilon}(\Theta_{q}(r_{\varepsilon}), \alpha) = \beta_{\varepsilon}(r_{\varepsilon}, \alpha) \), and we are interesting in the minimal decreasing rates for the sequence \( r_{\varepsilon} \) such that \( \gamma_{\varepsilon}(r_{\varepsilon}) \to 0 \). Namely, we say that the positive sequence \( r_{\varepsilon}^{*} \to 0 \) is a \textit{separation rate}, if

\[
\gamma_{\varepsilon}(r_{\varepsilon}) \to 1, \quad \text{and} \quad \beta_{\varepsilon}(r_{\varepsilon}, \alpha) \to 1 - \alpha \quad \text{for any} \ \alpha \in (0, 1), \quad \text{as} \quad r_{\varepsilon}/r_{\varepsilon}^{*} \to 0,
\]

and

\[
\gamma_{\varepsilon}(r_{\varepsilon}) \to 0, \quad \text{and} \quad \beta_{\varepsilon}(r_{\varepsilon}, \alpha) \to 0 \quad \text{for any} \ \alpha \in (0, 1), \quad \text{as} \quad r_{\varepsilon}/r_{\varepsilon}^{*} \to \infty.
\]

In other words, it means that, for small \( \varepsilon \), one can detect all sequences \( \eta \in \Theta_{q}(r_{\varepsilon}) \) if the ratio \( r_{\varepsilon}/r_{\varepsilon}^{*} \) is large, whereas if this ratio is small then it is impossible to distinguish between the null and the alternative hypothesis, with small minimax total error probability. Hence, the rate optimality problem corresponds to finding the separation rates \( r_{\varepsilon}^{*} \) and to constructing asymptotically minimax consistent families of tests.

On the other hand, the sharp optimality problem corresponds to the study of the asymptotics of the quantities \( \beta_{\varepsilon}(\Theta_{\varepsilon}, \alpha) \), \( \gamma_{\varepsilon}(\Theta_{\varepsilon}) \) (up to vanishing terms) and to the construction of asymptotically minimax families of tests \( \psi_{\varepsilon,a} \) and \( \psi_{\varepsilon} \), respectively. We shall see (see Section 4.1) that often the sharp asymptotics are of Gaussian type, i.e.,

\[
\beta_{\varepsilon}(r_{\varepsilon}, \alpha) = \Phi(H^{(\alpha)} - u_{\varepsilon}) + o(1), \quad \gamma_{\varepsilon}(r_{\varepsilon}) = 2\Phi(-u_{\varepsilon}/2) + o(1),
\]

where \( \Phi \) is the standard Gaussian distribution function, \( H^{(\alpha)} \) is its \((1-\alpha)\)-quantile, i.e., \( \Phi(H^{(\alpha)}) = 1-\alpha \). The quantity \( u_{\varepsilon} = u_{\varepsilon}(r_{\varepsilon}) \) is the value of the specific extreme problem (4.1) on the sequence space \( L_{2} \), and the extreme sequence of this problem determines the structure of the asymptotically minimax families of tests \( \psi_{\varepsilon,a} \) and \( \psi_{\varepsilon} \). Moreover, we shall see that if \( u_{\varepsilon}(r_{\varepsilon}) \to \infty \), then \( \gamma_{\varepsilon}(r_{\varepsilon}) \to 0 \), \( \beta_{\varepsilon}(r_{\varepsilon}, \alpha) \to 0 \), and if \( u_{\varepsilon}(r_{\varepsilon}) \to 0 \), then \( \gamma_{\varepsilon}(r_{\varepsilon}) \to 1 \), \( \beta_{\varepsilon}(r_{\varepsilon}, \alpha) \to 1 - \alpha \) for any \( \alpha \in (0, 1) \), i.e., the family \( u_{\varepsilon}(r_{\varepsilon}) \) characterizes \textit{distinguishability} in the testing problem. The separation rates \( r_{\varepsilon}^{*} \) are usually determined by the relation \( u_{\varepsilon}(r_{\varepsilon}^{*}) \asymp 1 \) (see, e.g., [10], [17]). Hence, sharp and rate optimality problems correspond to the study of the extreme problem (4.1) and of the asymptotics of the family \( u_{\varepsilon}(r_{\varepsilon}) \).

Hereafter, the relations \( A_{\varepsilon} \sim B_{\varepsilon} \) and \( A_{\varepsilon} \asymp B_{\varepsilon} \) as \( \varepsilon \to 0 \) are defined similarly to those mentioned in Section 4. Let also \( A_{\{\varepsilon\}} \) be the indicator function of a set \( A \) and let \( (a)_{+} = \max\{0, a\} \).
3 Minimax signal detection in ill-posed inverse problems: a short description of the main results

Sharp and rate optimality results for the specific ill-posed inverse problems under consideration are discussed in detail in Section 4. In this section, we give a short description of these results for mildly and severely ill-posed inverse problems with $l^q$-ellipsoids, $q \in (0, 2]$, for Sobolev and analytic classes of functions.

First, consider the “standard” case $q = 2$. The asymptotics of the quality of testing $u_\varepsilon(r_\varepsilon)$ as $r_\varepsilon \to 0$ is presented in the following table:

| Detection Problem | Sobolev classes | analytic classes |
|-------------------|-----------------|-----------------|
| mildly ill-posed   | $c_1 \varepsilon^{-2r_\varepsilon^{(4\alpha+1\beta+1)/2}}$ | $c_2 \varepsilon^{-2r_\varepsilon^2(\log r_\varepsilon^{-1})-2\beta-1/2}$ |
| severely ill-posed | $\varepsilon^{-2r_\varepsilon^2e^{-2\beta r_\varepsilon^{-1/\alpha}}}$ | $\varepsilon^{-2r_\varepsilon^{2(\alpha+\beta)/\alpha}}$ |

Here, $c_1 = c_1(\alpha, \beta) > 0$, $c_2 = c_2(\alpha, \beta) > 0$ are some constants. We have the sharp asymptotics of the form (2.8) for mildly ill-posed inverse problems with Sobolev and analytic classes of functions, with either $a_k \sim A k^\alpha$ or $a_k \sim \exp(\alpha k)$, and $\sigma_k \sim B k^\beta$, $k \in \mathbb{N}$. Furthermore, the separation rates $r_\varepsilon^*$ as $\varepsilon \to 0$ are presented in the following table:

| Detection Problem | Sobolev classes | analytic classes |
|-------------------|-----------------|-----------------|
| mildly ill-posed   | $\varepsilon^{4\alpha/(4\alpha+4\beta+1)}$ | $\varepsilon^{(\log \varepsilon^{-1})^{\alpha+\beta}/(\alpha+\beta)}$ |
| severely ill-posed | $((\log \varepsilon^{-1})/\beta)^{-\alpha}$ | $\varepsilon^{\alpha/(\alpha+\beta)}$ |

Note that the separation rates are sharp in severely ill-posed inverse problems with Sobolev class of functions, with $a_k \sim k^\alpha$ and $b_k \sim \exp(-\beta k)$, $k \in \mathbb{N}$. It means that

$$
\gamma_\varepsilon(r_\varepsilon) \to 0 \quad \text{as} \quad \lim r_\varepsilon/r_\varepsilon^* > 1, \quad \gamma_\varepsilon(r_\varepsilon) \to 1 \quad \text{as} \quad \lim r_\varepsilon/r_\varepsilon^* < 1.
$$

(Similar non-asymptotic minimax rates are recently given in [20].) We do not present the results for extremely ill-posed problems with the class of generalized analytic functions in this short description because the asymptotics are more complicated in this case. Roughly speaking, these asymptotics are determined by a piecewise linear function $u_\varepsilon^{lin} = u_\varepsilon^{lin}(r_\varepsilon)$ and, principally, seem to be of a new type, see Section 4.6 and remarks therein for details. (Moreover, in subsequent sections, we consider this case only for $q = 2$.)

Consider now the “sparse” case $q \in (0, 2)$. Then, the results noted above still hold true for severely ill-posed problems with Sobolev and analytic classes of functions. For mildly ill-posed problems with the class of analytic functions, we also get the same separation rates $r_\varepsilon^*$. However, the situation for mildly ill-posed problems with Sobolev classes of functions is more delicate. More precisely, let $\alpha > 0$, $\beta > 0$ and set $\lambda = (\alpha + \beta)/2 - \beta/q$. If $\lambda > 0$, then the sharp asymptotics are of the Gaussian type (2.8) with

$$
u_\varepsilon = c_3 \varepsilon^{-(2\alpha+1/q-1/2)/(\alpha+\beta(1-2/q))} r_\varepsilon^{2(\alpha+\beta+1/q)/(\alpha+\beta(1-2/q))}
$$

for some constant $c_3 = c_3(\alpha, \beta, q) > 0$, while the separation rates $r_\varepsilon^*$ are of the form

$$r_\varepsilon^* = \varepsilon^{2(\alpha+1/q-1/2)/(2(\alpha+\beta)+1/q)}.
$$

On the other hand, if $\lambda \leq 0$, then the sharp asymptotics are of the following degenerate type

$$
\beta_\varepsilon(\alpha) = (1 - \alpha)\Phi(-D_\varepsilon) + o(1), \quad \gamma_\varepsilon = \Phi(-D_\varepsilon) + o(1),
$$
where
\[ D_\varepsilon = n_\varepsilon^{-\beta} r_\varepsilon / \varepsilon - \sqrt{2 \log(n_\varepsilon)}, \quad n_\varepsilon = r_\varepsilon^{-1/\alpha}, \]
while the separation rates \( r_\varepsilon^* \) are of the form
\[ r_\varepsilon^* = \Lambda \varepsilon^{\alpha/(\alpha + \beta)} (\log(\varepsilon^{-1}))^{\alpha/(2(\alpha + \beta)}, \quad \Lambda = (2/(\alpha + \beta))^{\alpha/(2(\alpha + \beta)).} \]

We now compare the above mentioned minimax rates of testing with the corresponding minimax rates of estimation. The minimax estimation problem for the Gaussian white noise model (1.1) (or, equivalently, for the Gaussian sequence model (1.3)) was studied very intensively in statistical ill-posed inverse problems, see, e.g., [5] (and references there in), [6], [7], [13] and [14]. The main object of the study is the minimax quadratic risk, defined by
\[ R_\varepsilon^2(\mathcal{F}) = \inf \sup_{f \in \mathcal{F}} E_{\varepsilon, f} \| \hat{f} - f \|^2, \]
where the infimum is taken over all possible estimators \( \hat{f} \) of \( f \) based on observations from the Gaussian white noise model (1.1).

For the main types of the ill-posed inverse problems and classes of functions under consideration with \( l^2 \)-ellipsoids, \( q = 2 \), the rates of \( R_\varepsilon(\mathcal{F}) \) as \( \varepsilon \to 0 \) are presented in the following table (see, e.g., [5]):

| Estimation Problem   | Sobolev classes | analytic classes |
|----------------------|-----------------|-----------------|
| mildly ill-posed      | \( \varepsilon^{\alpha/(2(\alpha + \beta) + 1)} \) | \( \varepsilon^{(\log \varepsilon^{-1})^{\alpha/(2(\alpha + \beta)}} \) |
| severely ill-posed    | \( (\log \varepsilon^{-1})^{-\alpha} \) | \( \varepsilon^{\alpha/(\alpha + \beta)} \) |

For mildly ill-posed problems with \( l^q \)-ellipsoids, \( q \in (0, 2) \), for Sobolev classes of functions, one has for \( \lambda > 0 \),
\[ R_\varepsilon \approx \varepsilon^{(\alpha - 1/2 + 1/q)/(\alpha + \beta + 1/q)}, \]
while, for \( \lambda < 0 \), the rates of \( R_\varepsilon \) in the estimation problem coincide with the separation rates \( r_\varepsilon^* \) in the corresponding detection problem (see, e.g., [17], Section 2.8, and references therein.)

Observe that the minimax rates of testing are faster than the corresponding minimax rates of estimation (as it is common in nonparametric inference, see, e.g., [17], Sections 2.10 and 3.5.1), except for the cases of mildly ill-posed problems with \( l^q \)-ellipsoids for Sobolev classes of functions with \( q \in (0, 2) \) and \( \lambda > 0 \) and the cases of severely ill-posed inverse problems with \( l^2 \)-ellipsoids for Sobolev and analytic classes of functions. (To the best of our knowledge, we are not aware of any minimax estimation results with the case of severely ill-posed inverse problems with \( l^q \)-ellipsoids, \( q \in (0, 2) \), for Sobolev and analytic classes of functions.)

Returning to the signal detection problem, note that, except for the case of mildly ill-posed inverse problems with \( l^q \)-ellipsoids, \( q \in (0, 2] \), for analytic classes of functions and for the case of mildly ill-posed problems with \( l^q \)-ellipsoids, \( q \in (0, 2) \) (and \( \lambda \leq 0 \)), for Sobolev classes of functions, the aforementioned separation rates \( r_\varepsilon^* \) hold true for the known parameters \( (\alpha, \beta, q) \) associated with the classes of functions and the ill-posed inverse problems under consideration. The rate-optimal tests depend on the parameters \( (\alpha, \beta, q) \) for these cases as well. In practice, the parameters \( \alpha \) and \( q \) associated with the considered functional classes are typically unknown and, very often, the statistician is not confident about the value of the parameter \( \beta \) that characterizes the spectrum of the operator \( A^*A \). For unknown parameters \( (\alpha, \beta, q) \in \Sigma \subset \mathbb{R}_+^2 \times (0, 2] := (0, \infty) \times (0, \infty) \times (0, 2] \), we have the so-called adaptive problems: in order to distinguish between the null hypothesis and the ‘combined’ alternative, which corresponds to a wide enough compact set \( \Sigma \subset \mathbb{R}_+^2 \times (0, 2] \), it does not suffice to just require
\( u_\varepsilon = u_\varepsilon(r_\varepsilon(\alpha, \beta, q), \alpha, \beta, q) \to \infty \) for all \((\alpha, \beta, q) \in \Sigma\); instead, one needs that it should tend to \( \infty \) faster than some family \( u_\varepsilon^{ad} \to \infty \), which is a 'payment' for adaptation, see [25].

Adaptive rate optimality results for the specific ill-posed inverse problems under consideration are discussed in detail in Section 5. Below, we give a short description of these results for mildly and severely ill-posed inverse problems with \( l^q \)-ellipsoids, \( q \in (0, 2] \), for the Sobolev and analytic classes of functions.

We shall see that \( u_\varepsilon^{ad} \approx 1 \) for mildly ill-posed inverse problems with \( l^q \)-ellipsoids, \( q \in (0, 2] \) for the class of analytic functions and for the Sobolev class with \( \lambda \leq 0 \), while for mildly ill-posed inverse problems with \( l^q \)-ellipsoids for the Sobolev class of functions with \( \lambda > 0 \) one has

\[
 u_\varepsilon^{ad} = \sqrt{\log \log \varepsilon^{-1}}.
\]

On the other hand, for severely ill-posed inverse problems with \( l^q \)-ellipsoids for the Sobolev and analytic classes of functions (as well as for extremely ill-posed inverse problems with \( l^2 \)-ellipsoids for the class of generalized analytic functions), we shall see that

\[
 u_\varepsilon^{ad} = \log \log \varepsilon^{-1}.
\]

These yield the adaptive separation rates \( r_\varepsilon^{ad} \) presented in the following table (here \( q = 2 \) for mildly ill-posed problems with ellipsoids for the Sobolev classes):

| Detection Problem | Sobolev classes | analytic classes |
|-------------------|-----------------|-----------------|
| mildly ill-posed   | \((\tilde{\varepsilon}_1)^{4/(\alpha+4\beta+1)}\) \(\varepsilon(\log \varepsilon^{-1})^{\beta+1/4}\) | \((\tilde{\varepsilon}_2)^{\alpha/(\alpha+\beta)}\) |
| severely ill-posed | \(((\log \varepsilon^{-1})/\beta)^{-\alpha}\) \((\tilde{\varepsilon}_2)^{\alpha/(\alpha+\beta)}\) |

where

\[
 \tilde{\varepsilon}_1 = \varepsilon^{4/\log \log \varepsilon^{-1}}, \quad \tilde{\varepsilon}_2 = \varepsilon^{\sqrt{\log \log \varepsilon^{-1}}}
\]

On the other hand, for the “sparse” case \( q \in (0, 2) \) and \( \lambda > 0 \), for mildly ill-posed problems with \( l^q \)-ellipsoids for Sobolev classes of functions, the adaptive separation rates \( r_\varepsilon^{ad} \) are of the form

\[
 r_\varepsilon^{ad} = \tilde{\varepsilon}_2^{(2\alpha+1/q-1/2))//(2(\alpha+\beta)+1/q)}.
\]

We have the same adaptive rates for mildly ill-posed problems with Besov classes for \( q \in (0, 2) \), \( \lambda > 0 \).

We conclude this short description by saying that, as we shall show in Section 5, the rate-optimal adaptive tests are of rather simple structure for all problems under consideration (except for mildly ill-posed problems with \( l^q \)-ellipsoids, \( q \in (0, 2) \), for Sobolev classes of functions, and Besov classes): they are based on combinations of tests based on a grid of centered and normalized statistics of \( \chi^2 \)-type and on simple thresholding. For mildly ill-posed problems with \( l^q \)-ellipsoids, \( q \in (0, 2) \), for Sobolev classes of functions, the rate-optimal adaptive tests are more complicated (see Remark 6.4 in Section 6).

4 Minimax signal detection in ill-posed inverse problems: rate and sharp asymptotics

4.1 A general result for \( l^q \)-ellipsoids: the “standard” case \( q = 2 \)

Consider the Gaussian sequence model (2.2). We are interested in the hypothesis testing problem (2.3) with the set under the alternative \( \Theta_\varepsilon = \Theta_2(r_\varepsilon) \) given by (2.4) with \( q = 2 \).
Consider now the extreme problem

$$w_k^2 = u_k^2(r_\varepsilon) = \frac{1}{2\varepsilon^4} \inf_{\eta \in \Theta(r_\varepsilon)} \sum_{k \in \mathbb{N}} \eta_k^4.$$  \hfill (4.1)

Suppose that $\Theta(r_\varepsilon) \neq \emptyset$ and $u_\varepsilon > 0$, and let there exist an extreme sequence $\{\tilde{\eta}_k\}_{k \in \mathbb{N}}$ in the extreme problem (4.1). (Observe the uniqueness of a nonnegative extreme sequence $\{\tilde{\eta}_k\}$, $k \in \mathbb{N}$, because, by passing to the sequence $\{z_k\}_{k \in \mathbb{N}}$ with elements $z_k = \tilde{\eta}_k^2$, $k \in \mathbb{N}$, we obtain the minimization problem of a strictly convex function under linear constraints.) Denote

$$w_k = \frac{\tilde{\eta}_k^2}{\sqrt{2} \sum_{k \in \mathbb{N}} \tilde{\eta}_k^4}, \quad k \in \mathbb{N}, \quad w_0 = \sup_{k \in \mathbb{N}} w_k,$$  \hfill (4.2)

and consider the following families of test statistics and tests

$$t_\varepsilon = \sum_{k \in \mathbb{N}} w_k \left( \left( y_k / \varepsilon \right)^2 - 1 \right), \quad \psi_{\varepsilon,H} = \mathbb{I}_{\{t_\varepsilon > H\}}.$$  \hfill (4.3)

(Note that the values of $\tilde{\eta}_k$, $w_k$, $k \in \mathbb{N}$, and $w_0$ depend on $\varepsilon$, i.e., $\tilde{\eta}_k = \tilde{\eta}_{k,\varepsilon}$, $w_k = w_{k,\varepsilon}$, $k \in \mathbb{N}$, and $w_0 = w_{0,\varepsilon}$.)

The key tool for the study of the above mentioned hypothesis testing problem is the following general theorem.

**Theorem 4.1** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with $q = 2$. Let $u_\varepsilon$ be determined by the extreme problem (4.1), let the coefficients $w_k$, $k \in \mathbb{N}$, and $w_0$ be as in (4.2), and consider the family tests $\psi_{\varepsilon,H}$ given by (4.3). Then

(1) (a) If $u_\varepsilon \to 0$, then $\beta_\varepsilon(r_\varepsilon, \alpha) \to 1 - \alpha$ for any $\alpha \in (0, 1)$ and $\gamma_\varepsilon(r_\varepsilon) \to 1$, i.e., minimax testing is impossible. If $u_\varepsilon = O(1)$, then $\lim \inf \beta_\varepsilon(r_\varepsilon, \alpha) > 0$ for any $\alpha \in (0, 1)$ and $\lim \inf \gamma_\varepsilon(r_\varepsilon) > 0$, i.e., minimax consistent testing is impossible.

(b) If $u_\varepsilon \asymp 1$ and $w_0 = o(1)$, then the family of tests $\psi_{\varepsilon,H}$ of the form (4.3) with $H = H^{(\alpha)}$ and $H = u_\varepsilon/2$ are asymptotically minimax, i.e.,

$$\alpha_\varepsilon(\psi_{\varepsilon,H^{(\alpha)}}) \leq \alpha + o(1),$$

$$\beta_\varepsilon(\Theta(r_\varepsilon), \psi_{\varepsilon,H^{(\alpha)}}) = \beta_\varepsilon(r_\varepsilon, \alpha) + o(1),$$

$$\gamma_\varepsilon(\Theta(r_\varepsilon), \psi_{\varepsilon,u_\varepsilon/2}) = \gamma_\varepsilon(r_\varepsilon) + o(1),$$

and the sharp asymptotics (2.8) hold true, i.e.,

$$\beta_\varepsilon(r_\varepsilon, \alpha) = \Phi(H^{(\alpha)} - u_\varepsilon) + o(1),$$

$$\gamma_\varepsilon(r_\varepsilon) = 2\Phi(-u_\varepsilon/2) + o(1).$$

(2) If $u_\varepsilon \to \infty$, then the family of tests $\psi_{\varepsilon,H}$ of the form (4.3) with $H = T_\varepsilon$ are asymptotically minimax consistent for any $c \in (0, 1)$ and a family $T_\varepsilon \sim cu_\varepsilon$, i.e., $\gamma_\varepsilon(\Theta(r_\varepsilon), \psi_{\varepsilon,T_\varepsilon}) \to 0$.

The proof is given in the Appendix, Section 7.1.

Theorem 4.1 shows that the asymptotics of the quality of testing is determined by the asymptotics of values $u_\varepsilon$ of the extreme problem (4.1). In order to make use of it, one needs to study the extreme problem (4.1). This problem is studied by using Lagrange techniques...
multipliers. Then, the extreme sequence in the above mentioned extreme problem is of the form
\[ \eta_k^2 = z_0^2 \sigma_k^2 (1 - A a_k^2)_+, \quad k \in \mathbb{N}, \quad (4.4) \]
where the quantities \( z_0 = z_{0, \varepsilon} \) and \( A = A_\varepsilon \) are determined by the equations
\[
\begin{aligned}
\sum_{k \in \mathbb{N}} \sigma_k^2 \eta_k^2 &= r_\varepsilon^2, \\
\sum_{k \in \mathbb{N}} a_k^2 \sigma_k^2 \eta_k^2 &= 1.
\end{aligned}
\quad (4.5)
\]
The equations (4.5) are immediately rewritten in the form
\[
\begin{aligned}
r_\varepsilon^2 &= z_0^2 J_1, \\
1 &= z_0^2 A^{-1} J_2,
\end{aligned}
\quad (4.6)
\]
and, hence, the extreme problem (4.1) takes the form
\[ u_\varepsilon^2 = \varepsilon^{-4} z_0^2 J_0/2, \quad (4.7) \]
where
\[
\begin{aligned}
J_1 &= \sum_{k \in \mathbb{N}} \sigma_k^4 (1 - A a_k^2)_+, \\
J_2 &= A \sum_{k \in \mathbb{N}} a_k^2 \sigma_k^4 (1 - A a_k^2)_+, \\
J_0 &= J_1 - J_2 = \sum_{k \in \mathbb{N}} \sigma_k^4 (1 - A a_k^2)_+^2.
\end{aligned}
\]
It is also convenient to rewrite (4.6) and (4.7) in the form
\[ r_\varepsilon^2 = A \frac{J_1}{J_2}, \quad u_\varepsilon^2 = \left( \frac{r_\varepsilon}{\varepsilon} \right)^4 \frac{J_0}{2 J_1^2}. \quad (4.8) \]
Note that, the first equation in (4.8) is used to calculate \( A \) which determines the efficient dimension \( m \) in the specific ill-posed inverse problems considered below: if \( a_k \) is an increasing sequence (it is assumed further), the efficient dimension is a quantity \( m = m_\varepsilon \in [1, \infty) \) such that \( A a_k^2 \leq 1 < A a_k^2 + 1 \).

**Remark 4.1** Since, in the ill-posed problems under consideration, \( \sigma_k \to \infty \) as \( k \to \infty \), it is immediate that \( \sum_{k \in \mathbb{N}} \sigma_k^4 = \infty \). Under this condition, and the fact that \( a_k \to \infty \) as \( k \to \infty \), one can see that, for \( r_\varepsilon \) small enough, the equations in (4.6) have a unique solution (see Proposition 7.2 in the Appendix, Section 7.8).

**Remark 4.2** Let \( u_\varepsilon = u_\varepsilon (r_\varepsilon) \) be the value of the extreme problem (4.1) with sequences \( a = \{a_k\}_{k \in \mathbb{N}} \) and \( \sigma = \{\sigma_k\}_{k \in \mathbb{N}} \) associated with the set under the alternative \( \Theta_\varepsilon = \Theta(r_\varepsilon) \) given by (2.4), and let \( \tilde{u}_\varepsilon = \tilde{u}_\varepsilon (r_\varepsilon) \) be the corresponding value of the extreme problem similar to (4.1) with sequences \( \tilde{a} = C a = \{Ca_k\}_{k \in \mathbb{N}} \) and \( \tilde{\sigma} = D \sigma = \{D \sigma_k\}_{k \in \mathbb{N}} \) in (2.4), for some positive constants \( C \) and \( D \). Then, it is easily seen that the relation \( \tilde{u}_\varepsilon (r_\varepsilon) = (CD)^{-2} u_\varepsilon (Cr_\varepsilon) \) holds true.
4.2 Application to mildly ill-posed inverse problems with the Sobolev class of functions

Consider first the “standard” case \( q = 2 \). Then, the following statement is true.

**Theorem 4.2** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with \( q = 2 \). Let \( a_k = k^α \) and \( σ_k = k^β \), \( k ∈ \mathbb{N} \), \( α > 0 \), \( β > 0 \). Then

(a) The sharp asymptotics (2.3) hold with the value \( u_ε \) of the extreme problem (4.7) determined by (7.7).

(b) The asymptotically minimax family of tests \( ψ_{ε,H} \) are determined by the family of test statistics \( t_ε \) given by (4.3) with coefficients \( w_k \), \( k ∈ \mathbb{N} \) and \( w_0 \) as in (4.2), and with extreme sequence \( \{m_k\}_{k∈\mathbb{N}} \) given by (7.6) with \( m \) determined by (7.7).

(c) The separation rates are of the form

\[
r_ε^* = ε^{4α/(4α+4β+1)}.
\]  

(4.9)

The proof is given in the Appendix, Section 7.2.

**Remark 4.3** It follows from the evaluations of the functions \( J_0 \), \( J_1 \) and \( J_2 \) that their asymptotics are determined by the tails of the sequences \( a_k = k^α \) and \( σ_k = k^β \), \( k ∈ \mathbb{N} \), \( α > 0 \), \( β > 0 \). For this reason, in view of Remark 4.2, we get the sharp asymptotics (7.8) for the sequences \( a_k ∼ k^α \) and \( σ ∼ k^β \), \( k ∈ \mathbb{N} \), \( α > 0 \), \( β > 0 \), and similar rate asymptotics for the sequences \( a_k ≍ k^α \), \( σ ≍ k^β \), \( k ∈ \mathbb{N} \), \( α > 0 \), \( β > 0 \). In both cases, the separation rates are still of the form (4.9).

A family of asymptotically minimax consistent tests of simple structure

If \( u_ε → ∞ \), then one can construct a family of asymptotically minimax consistent tests of simpler structure than (4.3). Indeed, observe that, by (7.7), one has

\[
r_ε a_ε m ∼ c_1^α, \quad c_1 = c_1(α, β) > 1, \quad u_ε ∼ \frac{r_ε^2}{ε^2σ_m^2}, \quad m = [m] ∈ \mathbb{N}
\]  

(4.10)

where \([m]\) is the integral part of \( m \). Hence, for an integer-valued family \( m = m_ε → ∞ \), one has

\[
a_{m+1} r_ε ≥ B + o(1), \quad B > 1, \quad u_ε ∼ \frac{r_ε^2}{ε^2σ_m^2}.
\]  

(4.11)

For each \( m ∈ \mathbb{N} \), consider the following families of test statistics and tests

\[
t_{ε,m} = \frac{1}{√2m} \sum_{k=1}^{m} (y_k/ε)^2 - 1, \quad ψ_{ε,H} = Π_{\{t_{ε,m}>H\}}.
\]  

(4.12)

Then, the following statement is true.

**Theorem 4.3** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with \( q = 2 \). Let \( a_k = k^α \) and \( σ_k = k^β \), \( k ∈ \mathbb{N} \), \( α > 0 \), \( β > 0 \). Let the value \( u_ε \) of the extreme problem (4.7) be determined by (7.7), and assume that \( u_ε → ∞ \). Then, the family of tests \( ψ_{ε,H} \), given by (4.12) with \( m = m_ε \) satisfying (4.11) and \( H = H_ε → ∞ \), is asymptotically minimax consistent, i.e., \( α_ε(ψ_{ε,H_ε}, Θ_ε) → 0 \) as \( H_ε < (c + o(1))u_ε \).

The proof is given in the Appendix, Section 7.3.

Unlike the “standard” case \( q = 2 \), the “sparse” case \( q ∈ (0,2) \) is not directly linked with Theorem 4.1 and will be considered separately in Section 4.7.
4.3 Application to severely ill-posed inverse problems with the class of analytic functions

The following statement is true.

**Theorem 4.4** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with \( q \in (0, 2) \). Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = \exp(3k) \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). Then

(a) The asymptotically minimax consistent family of tests \( \psi_{\varepsilon, H} \) are determined by the family of test statistics \( t_{\varepsilon} \) given by (4.3) with coefficients \( w_k \), \( k \in \mathbb{N} \), as in (4.2), and with extreme sequence \( \{\tilde{\eta}_k\}_{k \in \mathbb{N}} \) given by (7.14) with \( m \) determined by (7.15).

(b) The separation rates are of the form

\[
 r^*_\varepsilon = \varepsilon^{\alpha/(\alpha + \beta)}. \tag{4.13}
\]

The proof is given in the Appendix, Section 7.4.

**Remark 4.4** We do not consider sharp asymptotics in this case, since the assumption \( w_0 = o(1) \) does not hold for \( \beta > 0 \) in the “standard” case \( q = 2 \). Indeed,

\[
 w_0 = \frac{\max_{1 \leq k \leq m} \tilde{\eta}^2_k}{\sqrt{2} \sum_{k=1}^m \tilde{\eta}^2_k} \times \frac{\varepsilon^2 \exp(2\beta m)}{\varepsilon^2 \exp(2\beta m)} \times 1 = 1.
\]

**Remark 4.5** Similar to Remark 4.3, the asymptotics (7.16) hold true for the sequences \( a_k \asymp \exp(\alpha k) \) and \( \sigma_k \asymp \exp(\beta k) \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). In this case, the separation rates are still of the form (4.13). Remark 4.4 still applies to these cases too.

4.4 Application to severely ill-posed inverse problems with the Sobolev class of functions

The following statement is true.

**Theorem 4.5** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4). Let \( a_k = k^\alpha \) and \( \sigma_k = \exp(3k) \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). Then

(a) The asymptotically minimax consistent family of tests \( \psi_{\varepsilon, H} \) are determined by the family of test statistics \( t_{\varepsilon} \) given by (4.3) with coefficients \( w_k \), \( k \in \mathbb{N} \), as in (4.2), and with extreme sequence \( \{\tilde{\eta}_k\}_{k \in \mathbb{N}} \) given by (7.17) with \( m \) determined by (7.18).

(b) The separation rates are of the form

\[
 r^*_\varepsilon = \left( \frac{\log(\varepsilon^{-1})}{\beta} \right)^{-\alpha}. \tag{4.14}
\]

The proof is given in the Appendix, Section 7.5.

**Remark 4.6** It is worth mentioned that a stronger result is possible in this case. In view of (7.18), the relation (4.14) determines sharp separation rates \( r^*_\varepsilon \) in the following sense.

(a) if \( \lim inf (r_\varepsilon/r^*_\varepsilon) \geq 1 \), then \( u_\varepsilon \to \infty \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \to 0 \).

(b) If \( \lim sup (r_\varepsilon/r^*_\varepsilon) < 1 \), then \( u_\varepsilon \to 0 \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \to 1 \), and the minimax testing is impossible.

Moreover the relation

\[
 (r^*_\varepsilon)^{-1/\alpha} = \left( \frac{\log(\varepsilon^{-1}) - \alpha \log \log(\varepsilon^{-1})}{\beta} \right) + O(1),
\]

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determines the sharper separation rates \( r_\varepsilon^* \) in the following sense.

(c) If \( \lim \inf \left( \frac{r_\varepsilon^{-1/\alpha}}{r_\varepsilon^*} \right) = -\infty \), then \( u_\varepsilon \to \infty \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \to 0 \).

(d) If \( \lim \sup \left( \frac{r_\varepsilon^{-1/\alpha}}{r_\varepsilon^*} \right) = +\infty \), then \( u_\varepsilon \to 0 \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \to 1 \) and the testing is asymptotically impossible.

**Remark 4.7** We do not consider sharp asymptotics in this case, since the assumption \( w_0 = o(1) \) does not hold for \( \beta > 0 \) in the “standard” case \( q = 2 \). Indeed,

\[
\max_{1 \leq k \leq m} \frac{\eta_k^2}{\tilde{\eta}_k^4} \geq B_1 z_0^2 \exp(2\beta m)/m, \quad B_1 > 0; \quad \sqrt{2} \sum_{1 \leq k \leq m} \tilde{\eta}_k^4 \asymp \varepsilon^2 u_\varepsilon \asymp z_0^2 \exp(2\beta m)/m,
\]

and, therefore,

\[
w_0 = \frac{\max_{1 \leq k \leq m} \tilde{\eta}_k^2}{\sqrt{2} \sum_{1 \leq k \leq m} \tilde{\eta}_k^4} \geq B \frac{z_0^2 \exp(2\beta m)}{z_0^2 \exp(2\beta m)} \asymp 1 \neq 0, \quad B > 0.
\]

**Remark 4.8** Similar to Remark 4.3, the asymptotics (7.19) and the sharp separation rates (4.14) mentioned in Remark 4.6 hold true for the sequences \( \sigma_k \asymp \exp(\beta k) \), \( k \in \mathbb{N} \), \( \beta > 0 \). The dependence on the sequence \( \{a_k\}_{k \in \mathbb{N}} \) is, however, more delicate. One can actually show that the separation rates (4.14) mentioned in Remark 4.6 are still of the same form for \( a_k \sim k^\alpha \), \( k \in \mathbb{N} \), \( \alpha > 0 \). Remark 4.7 still applies to these cases too.

### 4.5 Application to mildly ill-posed inverse problems with the class of analytic functions

Here, we consider the “standard” case \( q = 2 \). (The “sparse” case \( q \in (0, 2) \) will be discussed in Remark 4.11.) The following statement is true.

**Theorem 4.6** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4), with \( q = 2 \). Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = k^\beta \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). Then

(a) The sharp asymptotics (2.8) hold with the the value \( u_\varepsilon \) of the extreme problem (4.1) determined by (7.27).

(b) The asymptotically minimax family of tests are determined by the test statistics \( t_\varepsilon \) given by (4.3) with coefficients \( w_k \), \( k \in \mathbb{N} \) and \( w_0 \) as in (4.2), and with extreme sequence \( \{\tilde{\eta}_k\}_{k \in \mathbb{N}} \) given by (7.20) with \( m \) determined by (7.20).

(c) The separation rates are of the form

\[
r_\varepsilon^* = \varepsilon(\log \varepsilon^{-1})^{\beta + 1/4}.
\]

The proof is given in the Appendix, Section 7.6

**Remark 4.9** It is also easy to see that, uniformly over \( (\alpha, \beta) \in \Sigma \), for any compact set \( \Sigma \subset \mathbb{R}^2_+ \), the efficient dimension \( m = m_\varepsilon(\alpha, \beta) \) satisfies

\[
m_\varepsilon(\alpha, \beta) \sim \frac{2 \log(\varepsilon^{-1}) - \log(u_\varepsilon)}{2\alpha} \asymp \log(\varepsilon^{-1}) \quad \text{as} \quad \log(u_\varepsilon) = o(\log(\varepsilon^{-1})).
\]

**Remark 4.10** Similar to Remark 4.3, the asymptotics (7.26) hold true for the sequences \( a_k \sim \exp(\alpha k) \) and \( \sigma_k \sim k^\beta \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). Similar rate asymptotics hold true for the sequences \( a_k \asymp \exp(\alpha k) \) and \( \sigma_k \asymp k^\beta \), \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). In both cases, the separation rates are still of the form (4.15).
Remark 4.11 The rate asymptotics of Theorem 4.6 hold true uniformly in $q \in [\delta, 2]$, for any $\delta \in (0, 2)$. Indeed, in view of the embedding
\[ \Theta_q \subset \Theta_2, \quad \Theta_q(r_\varepsilon) \subset \Theta_2(r_\varepsilon), \]
for the sets defined by (2.14), it suffices to establish the lower bounds. Let $\Theta_q^\alpha(L)$ be the set that is determined by the constraints
\[ \Theta_q^\alpha(L) = \{ \eta \in l^2 : \sum_{k \in \mathbb{N}} |\exp(\alpha k)k^\beta \eta_k|^q \leq L^q \}. \]
(Note that the set $\Theta_q^\alpha(1)$ just corresponds to the set under the alternative considered above.)

For any $\alpha_1 = \alpha + \delta$, $\delta > 0$, the following embedding is true:
\[ \Theta_2^\alpha(cL) \subset \Theta_q^\alpha(L), \quad c = c(q, \delta) = (\exp(2q\delta/(2-q)) - 1)^{(2-q)/2q}, \quad c(q, \delta) \to \exp(\delta) \text{ as } q \to 2. \]

Using Hölder’s inequality, the embedding (4.18) follows easily on noting that
\[ \sum_{k \in \mathbb{N}} |\exp(\alpha k)k^\beta \eta_k|^q = \sum_{k \in \mathbb{N}} |\exp(\alpha k)k^\beta \eta_k|^q \exp(-q\delta) \leq \left( \sum_{k \in \mathbb{N}} (\exp(\alpha k)k^\beta \eta_k)^2 \right)^{2/q} \left( \sum_{k \in \mathbb{N}} \exp(-2kq\delta/(2-q)) \right)^{1-q/2} = \left( c^{-2} \sum_{k \in \mathbb{N}} (\exp(\alpha k)k^\beta \eta_k)^2 \right)^{2/q}. \]

Since the separation rates from Theorem 4.6 do not depend on $\alpha$ and $c$, in view of Remark 4.10 the rate asymptotics of Theorem 4.6 hold true uniformly in $q \in [\delta, 2]$, for any $\delta \in (0, 2)$.

A family of asymptotically minimax adaptive consistent tests of simple structure

A family of asymptotically minimax consistent tests of simple structure, that is also adaptive, in the sense that it does not depend on the unknown parameters $\alpha$, $\beta$ and $q \in (0, 2]$, is constructed as follows. Let a compact set $\Sigma = \{ (\alpha, \beta) \} \subset \mathbb{R}_+^2$ be given. Denote by $\Theta_{\varepsilon, \alpha, \beta}(r)$ the set under the alternative given by (2.4) with $r = r_\varepsilon(\alpha, \beta)$ and $q = 2$. Let $u_{\varepsilon, \alpha, \beta}(r)$ be the value of the extreme problem (4.1) for the set $\Theta_{\varepsilon} = \Theta_{\varepsilon, \alpha, \beta}(r)$. Observe that, for $a_k = \exp(\alpha k)$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, and for $\varepsilon$ small enough, in view of (4.16),
\[ c \log(\varepsilon^{-1}) \leq m_\varepsilon(\alpha, \beta) \leq C \log(\varepsilon^{-1}), \]

as
\[ \sup_{(\alpha, \beta) \in \Sigma} |\log(u_{\varepsilon, \alpha, \beta}(r_\varepsilon(\alpha, \beta)))| = o(\log(\varepsilon^{-1})), \]

where the constant $c$ and $C$ satisfy
\[ 0 < \max_{(\alpha, \beta) \in \Sigma} \alpha^{-1} < C, \quad 0 < c < \min_{(\alpha, \beta) \in \Sigma} \alpha^{-1}. \]

Set
\[ u_\varepsilon(\Sigma) = \inf_{(\alpha, \beta) \in \Sigma} u_{\varepsilon, \alpha, \beta}(r_\varepsilon(\alpha, \beta)). \]

Then, the following statement is true.
Theorem 4.7 Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = k^\beta \), \( k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Consider the Gaussian sequence model \((2.2)\) and the hypothesis testing problem \((2.3)\) where \( \Theta_{x,\alpha,\beta}(r) \) denotes the set under the alternative given by \((2.4)\) with \( r = r_{x}(\alpha, \beta) \) and \( q = 2 \), and let \( r_{x}(\alpha, \beta) \) be taken in such way that \( u_{x}(\Sigma) \) in \((4.19)\) satisfies \( u_{x}(\Sigma) \to \infty \). Then, the family of tests \( \psi_{x,H} \) given by \((4.12)\) with \( \tilde{m} = C \log(\epsilon^{-1}) + O(1) \in \mathbb{N} \), \( \tilde{m} \geq C \log(\epsilon^{-1}) \) is adaptive and asymptotically minimax consistent, i.e., \( \alpha_{x}(\psi_{x,H}) \to 0 \) as \( H \to \infty \) and one can take \( H = H_{x} \to \infty \) such that \( \beta_{x}(\psi_{x,H}, \Theta_{x,\alpha,\beta}(r_{x}(\alpha, \beta))) \to 0 \), uniformly over \((\alpha, \beta) \in \Sigma\). In view of the embedding \((4.17)\), these hold true uniformly in \( q \in (0, 2) \).

The proof is given in the Appendix, Section 4.7.

Remark 4.12 As stated in Theorem 4.7, the family of tests \( \psi_{x,H} \) given by \((4.12)\) with \( \tilde{m} = C \log(\epsilon^{-1}) + O(1) \in \mathbb{N} \) is adaptive with respect to the unknown parameters \( \alpha, \beta \) and \( q \in (0, 2) \). Furthermore, there is no price to pay for this adaptation, meaning that both non-adaptive and adaptive procedures for the considered ill-posed problem share, asymptotically, the same separation rates. However, this is usually an exception to the rule and in most cases there is a price to pay for the adaptation which is usually appear in the form of an extra log-log factor in the separation rates. The study of adaptivity in the remaining considered ill-posed problems and the construction of appropriate rate optimal families of tests is the theme of Section 4.4.

4.6 Application to extremely ill-posed inverse problems with the class of generalized analytic functions

We consider the case \( q = 2 \) only. Assume that \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{\sigma_k\}_{k \in \mathbb{N}} \) are increasing sequences such that
\[
\lim_{k \to \infty} \frac{\sigma_{k+1}}{\sigma_k} \to \infty, \quad \liminf_{k \to \infty} \frac{a_{k+1}}{a_k} = c, \quad c \in (1, \infty]. \tag{4.20}
\]
In order to describe the asymptotics of the value \( u_{x} = u_{x}(r) \) of the extreme problem \((4.1)\), we introduce the following functions.

Let \( m \in \mathbb{N}, m 
\Delta_{m} = [1/a_{m}, 1/a_{m-1}] \), and for \( r < 1/a_{1} \) take \( m = m(r) \geq 2 \) such that \( r \in \Delta_{m}^{s} \). Consider now the piecewise quadratic (in \( r^{2} \)) function defined by
\[
(u_{x}^{s}(r))^{2} = \frac{1}{2\epsilon^{4}(a_{m}^{2} - a_{m-1}^{2})^{2}} \left( \frac{(a_{m}^{2}r^{2} - 1)^{2}}{\sigma_{m}^{4}} + \frac{(1-a_{m-1}^{2}r^{2})^{2}}{\sigma_{m}^{4}} \right), \tag{4.21}
\]
and the piecewise linear (in \( r^{2} \)) function defined by
\[
u_{x}^{lin}(r) = \frac{1}{\epsilon^{2}(a_{m}^{2} - a_{m-1}^{2})} \left( \frac{a_{m}^{2}r^{2} - 1}{\sigma_{m}^{4}} + \frac{(1-a_{m-1}^{2}r^{2})}{\sigma_{m}^{4}} \right) \tag{4.22}
\]
Then, the following statement is true.

Theorem 4.8 Let \( u_{x} = u_{x}(r) \) be the value of the extreme problem \((4.1)\). Let \((u_{x}^{s}(r))^{2}\) be the piecewise quadratic (in \( r^{2} \)) function defined by \((4.21)\) and let \( u_{x}^{lin}(r) \) be the piecewise linear (in \( r^{2} \)) function defined by \((4.22)\), where \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{\sigma_k\}_{k \in \mathbb{N}} \) be increasing sequences satisfying \((4.20)\). Then
\[
(a) \text{ (sharp asymptotics of } u_{x}) \text{ The families } u_{x}(r) \text{ and } u_{x}^{s}(r) \text{ are related by }
\]
\[
u_{x}(r_{x}) \sim u_{x}^{s}(r_{x}) \quad \text{as } r_{x} \to 0. \tag{4.23}
\]
 sharp in the follows sense. Let $r$.

Therefore, in order to obtain $u_{\varepsilon}^\text{lin}(r_{\varepsilon}) \to 0$, one needs to take $r_{\varepsilon}$ such that $r_{\varepsilon}/r_{\varepsilon}^* \to 0$, and if $a_{m+1}/a_{m} \to c$, $1 < c < \infty$, then, for $u_{\varepsilon}^\text{lin}(r_{\varepsilon}) \to 0$, one needs to take $r_{\varepsilon} < r_{\varepsilon}^*/c$. Thus, the conditions for distinguishability and non-distinguishability could be non-symmetric in these problems.

Remark 4.13 It is easy to see that the relation (4.24) is true uniformly over all sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{\sigma_k\}_{k \in \mathbb{N}}$ such that $\sigma_{k+1}/\sigma_k \geq B_k$, $B_k \to \infty$, and $a_{k+1}/a_k > c$, as $k \geq k_0$, $k_0 \geq 1$.

Remark 4.14 We do not consider sharp asymptotics in this case, since the assumption $w_0 = o(1)$ does not hold under assumption (4.20). Indeed, using (7.31)-(7.32), it can be easily seen that

$$\sum_{k=1}^{m} \tilde{n}_k^4 \sim \tilde{n}_{m-1}^4 + \tilde{n}_m^4,$$

and, therefore,

$$w_0 = \frac{\max_{1 \leq k \leq m} \tilde{n}_k^2}{\sqrt{2 \sum_{k=1}^{m} \tilde{n}_k^2}} \sim \frac{\max(\tilde{n}_{m-1}^2, \tilde{n}_m^2)}{\sqrt{2(\tilde{n}_{m-1}^4 + \tilde{n}_m^4)}} \geq 1/2 \not\to 0.$$

Remark 4.15 The relation $u_{\varepsilon}^\text{lin}(r_{\varepsilon}^*) \propto 1$ determines the separation rates $r_{\varepsilon}^*$ that are rather sharp in the follows sense. Let $r_{\varepsilon}^* = a_m^{-1}$ for some $m \in \mathbb{N}$, $m \to \infty$, and let $r^2 = (1+b)(r_{\varepsilon}^*)^2 \in (a_m^{-2}, a_{m-1}^{-2})$, $b > 0$. Then, one has

$$u_{\varepsilon}^\text{lin}(r) = u_{\varepsilon}^\text{lin}(r_{\varepsilon}^*)(1 + k_mb),$$

where, as $m \to \infty$,

$$k_m = \frac{\sigma_m^2}{\sigma_{m-1}^2} \frac{1 - (\sigma_{m-1}a_{m-1}/\sigma_m a_m)^2}{1 - (a_{m-1}/a_m)^2} \sim \frac{\sigma_m^2}{\sigma_{m-1}^2(1 - (a_{m-1}/a_m)^2)} \times \frac{\sigma_m^2}{\sigma_{m-1}^2} \to \infty.$$

Therefore, in order to obtain $u_{\varepsilon}^\text{lin}(r_{\varepsilon}) \to \infty$, it suffices to take $r_{\varepsilon} = r_{\varepsilon}^*(1 + \delta)$ for any $\delta > 0$. On the other hand, let $r^2 = (1-b)(r_{\varepsilon}^*)^2 \in (a_{m+1}^{-2}, a_m^{-2})$, $b \in (0, 1)$. Then, similarly, one has

$$u_{\varepsilon}^\text{lin}(r) = u_{\varepsilon}^\text{lin}(r_{\varepsilon}^*)(1 - l_mb),$$

where, as $m \to \infty$,

$$l_m = \frac{1 - (\sigma_m a_m / \sigma_{m+1} a_{m+1})^2}{1 - (a_m/a_{m+1})^2} \sim \frac{1}{1 - (a_m/a_{m+1})^2} \times 1.$$
Remark 4.16 Let us consider the example

$$a_k = \exp(ak^\tau), \quad \alpha > 0, \quad \tau \geq 1, \quad \sigma_k = \exp(\beta k^\gamma), \quad \beta > 0, \quad \gamma > 1.$$ 

For the moment, let us forget that $m \in \mathbb{N}$ and define $m = m(r) \in \mathbb{R}_+$ by the equality $r = a_m^{-1}$, i.e., $m(r) = (\log(r^{-1})/\alpha)^{1/\tau}$. Set also

$$\hat{u}_\varepsilon(r) = (\varepsilon m(r) \sigma_{m(r)})^{-2} = (r/\varepsilon)^2 \exp(-2\beta(\log(r^{-1})/\alpha)^{\gamma/\tau}).$$

Observe that $\hat{u}_\varepsilon(r) = u_\varepsilon^{\text{lin}}(r)$ as $r = a_m^{-1}$, $m \in \mathbb{N}$. On the other hand, one can check that the function $\hat{u}_\varepsilon(r)$ is a convex function in $r^2$ for $r > 0$ small enough. Therefore, $\hat{u}_\varepsilon(r) < u_\varepsilon^{\text{lin}}(r)$ as $r \neq a_m^{-1}$ for any $m \in \mathbb{N}$, and the condition $\hat{u}_\varepsilon(r) \rightarrow \infty$ implies $u_\varepsilon^{\text{lin}}(r) \rightarrow \infty$. However, it is possible that $u_\varepsilon^{\text{lin}}(r) \rightarrow \infty$ when $\hat{u}_\varepsilon(r) = O(1)$, in general. For instance, let $\tau = \gamma$. Then

$$\hat{u}_\varepsilon(r) = \varepsilon^{-2}r^{2+2\beta/\alpha}.$$

If $r_\varepsilon = a_m^{-1}$ and $u_\varepsilon^{\text{lin}}(r_\varepsilon) \propto 1$, then it was noted in Remark 4.15 that $u_\varepsilon^{\text{lin}}(r_\varepsilon(1 + \delta)) \propto 1$ for any $\delta > 0$, but $\hat{u}_\varepsilon(r_\varepsilon(1 + \delta)) \propto 1$. The same holds for $\gamma < \tau$.

The relation $\hat{u}_\varepsilon(r_\varepsilon) \propto 1$ determines the family $\tilde{r}_\varepsilon$. Note that if $r_\varepsilon/\tilde{r}_\varepsilon \rightarrow \infty$, then $\hat{u}_\varepsilon(r_\varepsilon) \rightarrow \infty$, and since $u_\varepsilon^{\text{lin}}(r) \geq \hat{u}_\varepsilon(r)$, this yields $u_\varepsilon^{\text{lin}}(r) \rightarrow \infty$ and $\gamma_\varepsilon(r_\varepsilon) \rightarrow 0$ by Theorem 4.8. However, this family is not a family of separation rates, at least if $\gamma \leq \tau$, because the condition $r_\varepsilon/\tilde{r}_\varepsilon \rightarrow 0$ does not guaranty that $\gamma_\varepsilon(r_\varepsilon) \rightarrow 1$.

More precisely, there exists a sequence $\varepsilon_m \rightarrow 0$ and $\tilde{r}_m = o(\tilde{r}_m)$ such that $\gamma_{\varepsilon_m}(\tilde{r}_m) \rightarrow 0$. In fact, observe that if $\gamma \leq \tau$, then the function $\hat{u}_\varepsilon(r)$ satisfies (uniformly over $\varepsilon > 0$ since $\varepsilon^2$ is a factor in $u_\varepsilon(r)$):

$$\hat{u}_\varepsilon(Br) \propto \hat{u}_\varepsilon(r) \quad \text{iff} \quad B \gg 1, \quad r \rightarrow 0.$$ 

Take a sequence $m \rightarrow \infty$ and put $r_m = a_m^{-1}$, $(r_m^{(1)})^2 = r_m^2(1 + \delta_m)$, $r_m^2 = r_m^2(1 + \delta)$, where $\delta_m \rightarrow 0$, $\delta_m \sigma_m^2/\sigma_{m-1}^2 \rightarrow \infty$, $\delta > 0$. Observe that similarly to evaluations in Remark 4.15, one has, uniformly over $\varepsilon > 0$,

$$\hat{u}_\varepsilon(\tilde{r}_m) \propto \hat{u}_\varepsilon(r_m^{(1)}) \propto \hat{u}_\varepsilon(r_m) = u_\varepsilon^{\text{lin}}(r_m) \ll u_\varepsilon^{\text{lin}}(r_m^{(1)}) \ll u_\varepsilon^{\text{lin}}(\tilde{r}_m), \quad m \rightarrow \infty.$$ 

Take now $\tilde{r}_m$ and $\varepsilon_m$ such that

$$\hat{u}_{\varepsilon_m}(\tilde{r}_m) \propto u_{\varepsilon_m}^{\text{lin}}(r_m^{(1)}) \propto 1.$$ 

This implies $\hat{u}_{\varepsilon_m}(\tilde{r}_m) \gg \hat{u}_{\varepsilon_m}(r_m)$ and $\tilde{r}_m \gg r_m \gg \tilde{r}_m$. By construction, we see that the sequence $\tilde{r}_m$ satisfies $\hat{u}_{\varepsilon_m}(\tilde{r}_m) \propto 1$ and $\tilde{r}_m = o(\tilde{r}_m)$, but $u_{\varepsilon_m}^{\text{lin}}(\tilde{r}_m) \rightarrow \infty$, which yields $\gamma_{\varepsilon_m}(\tilde{r}_m) \rightarrow 0$.

A family of asymptotically minimax consistent tests of simple structure

The family of tests given by (4.3) are determined by the sequence $\{w_k\}_{k \in \mathbb{N}}$ given by (4.2) and are rather complicate. Furthermore, as revealed in Remark 4.14, the condition $w_0 = o(1)$ does not hold under assumption (4.20) and, hence, these families of tests are rate optimal only. We describe below another rate optimal family of tests that is of simpler structure.

This procedure is determined by a family $m = m(r_\varepsilon)$ such that $r_\varepsilon \in \Delta^*_m = [1/a_m, 1/a_{m-1}]$, $m \in \mathbb{N}$, $m \geq 2$. Take $\alpha \in (0, 1)$ small enough and consider the collection $T_{m,k}$, $1 \leq k \leq m$, such that

$$T_{m,m} = T_{m,m-1} = \Phi^{-1}((1 - \alpha/6)), \quad T_{m,k} = \Phi^{-1}((1 - c\alpha/(m - k - 1)^2)), \quad (4.25)$$
where \( c \) is taking in such way that \( \sum_{k=1}^{m} \Phi(-T_{m,k}) = \alpha/2 \). (Note that this yields \( \sum_{k=1}^{m} k^{-2} = 1/(6c) \).

Consider now the following families of events and tests

\[
\mathcal{Y}_{\varepsilon,\alpha} = \{ y : |y_k| < T_{m,k} \varepsilon, \ k = 1, 2, \ldots, m \}, \quad \psi_{\varepsilon,\alpha} = \mathbb{I}_{\mathcal{Y}_{\varepsilon,\alpha}},
\]

where, hereafter, \( \mathcal{C} \) denotes the complement of an arbitrary set \( C \in \mathbb{R} \).

Then, the following statement is true.

**Theorem 4.9** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with \( q = 2 \). Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) be increasing sequences satisfying (4.20). Then, the family of tests given by (4.26) with the collection \( T_{m,k} \), \( 1 \leq k \leq m \) described by (4.25) is asymptotically minimax consistent, i.e., \( \alpha_{\varepsilon}(\psi_{\varepsilon,\alpha}) \leq \alpha \) and one can take \( u_{\varepsilon}^{\text{lin}} = u_{\varepsilon}^{\text{lin}}(r_{\varepsilon}) \to \infty \) such that \( \beta_{\varepsilon}(\psi_{\varepsilon,\alpha}, \Theta(r_{\varepsilon})) \to 0 \).

The proof is given in the Appendix, Section 7.9.

**Remark 4.17** It is evident that the statement \( \alpha_{\varepsilon}(\psi_{\varepsilon,\alpha}) \leq \alpha \) in Theorem 4.9 holds uniformly for each \( \varepsilon \) small enough such that \( r_{\varepsilon}a_2 \leq 1 \). From the proof of Theorem 4.9 it is also evident that this statement does not depend on the assumption (4.20).

### 4.7 Mildly ill-posed inverse problems with \( l^q \)-ellipsoids for Sobolev classes of functions: the “sparse” case \( q \in (0, 2) \)

Here, we consider mildly ill-posed inverse problems with \( l^q \)-ellipsoids for the Sobolev classes of functions, for the “sparse” case \( q \in (0, 2) \). As pointed out previously, unlike the “standard” case \( q = 2 \), the sharp and rate optimality results for the “sparse” case \( q \in (0, 2) \) are of different nature and are not directly linked with Theorem 4.11 but can be obtained from a hitherto unknown link with results obtained in another context and presented in Sections 4.4.2-4.4.3 of [17]. Since these results are scattered in the cited reference, and not not immediately seen as ill-posed inverse problems, for completeness and an immediate access to these results, we formulate and present them below.

Consider the extreme problem

\[
u_{\varepsilon}^2 = 2 \inf_{i \in \mathbb{N}} h_i^2 \sinh^2(z_i^2/2),
\]

where the infimum is taken over sequences \( (h_i, z_i), \ h_i \in [0, 1], \ z_i \geq 0, \ i \in \mathbb{N} \), such that

\[
\sum_{i \in \mathbb{N}} i^{2/(\alpha+\beta)}h_i z_i^2 \geq (\bar{r}_{\varepsilon}/\varepsilon^2), \quad \sum_{i \in \mathbb{N}} i^{1/(\alpha+\beta)}h_i z_i^{1/(\alpha+\beta)} \leq (1/\varepsilon^q),
\]

where \( \bar{r}_{\varepsilon} = r_{\varepsilon}(1 - \delta_{\varepsilon}), \ \delta_{\varepsilon} > 0, \ \delta_{\varepsilon} \to 0, \ \delta_{\varepsilon} \log(\varepsilon^{-1}) \to \infty \).

Set \( \lambda = (\alpha+\beta)/2 - \beta/q \). If \( \lambda > 0 \), then, there exist extreme sequences \( h_{i,\varepsilon} \in (0, 1], \ z_{i,\varepsilon} > 0, \) in the problem (4.27)-(4.28), and we have the asymptotics of the form

\[
u_{\varepsilon}^2 \sim c_0 n h_0^2;
\]

where the quantities \( n = n_{\varepsilon} \) and \( h_0 = h_{0,\varepsilon} \) are determined by the relations

\[
c_1 n^{\beta+1/2} h_0^{1/2} \sim r_{\varepsilon}/\varepsilon \quad c_2 n^{\alpha+\beta+1/2} h_0^{1/2} \sim 1/\varepsilon,
\]

where the quantities \( n = n_{\varepsilon} \) and \( h_0 = h_{0,\varepsilon} \) are determined by the relations

\[
c_1 n^{\beta+1/2} h_0^{1/2} \sim r_{\varepsilon}/\varepsilon \quad c_2 n^{\alpha+\beta+1/2} h_0^{1/2} \sim 1/\varepsilon,
\]

\[
(4.30)
\]
for some constants $c_l = c_l(\alpha, \beta, q) > 0$, $l = 0, 1, 2$, which, in turn, imply
\[
u = c_3 \epsilon^{-(2\alpha+1/q-1/2)/(\alpha+\beta(1-2/q))} r_\epsilon^{(2(\alpha+\beta)+1/q)/(\alpha+\beta(1-2/q))}
\]

for some constant $c_3 = c_3(\alpha, \beta, q) > 0$. (The quantity $n = n_\epsilon \to \infty$ plays the role of the “efficient dimension” in the problem.)

Set
\[
Q_{\epsilon,i} = \sqrt{2(\log i + \log \log i + 2 \log \log (\epsilon^{-1}))}
\]
and consider the events
\[
\mathcal{Y}_\epsilon = \{y = \{y_i\}_{i \in \mathbb{N}} : (\sup_{i \in \mathbb{N}} |y_i|/|\epsilon Q_{\epsilon,i}|) > 1\}
\]
and the following families of test statistics
\[
l_\epsilon(y) = u_\epsilon^{-1} \sum_{i \in \mathbb{N}} h_{\epsilon,i} \xi(y_i/\epsilon, z_{\epsilon,i}), \quad \xi(t, z) = e^{z^2/2} \cosh(t z) - 1,
\]
and tests
\[
\psi_{\epsilon,H} = \mathbb{I}(l_\epsilon(y) > H) \cap \mathcal{Y}_\epsilon, \quad \psi_{\epsilon} = \mathbb{I}_{\mathcal{Y}_\epsilon}, \quad \psi_{\epsilon,\alpha} = \alpha + (1 - \alpha) \mathbb{I}_{\mathcal{Y}_\epsilon}.
\]

Then, the following statement is true.

**Theorem 4.10** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) with $q \in (0, 2)$. Let $a_k = k^\alpha$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, and set $\lambda = (\alpha + \beta)/2 - \beta/q$. Then

(a) If $\lambda > 0$, then the sharp asymptotics are of the Gaussian type (2.8) with $H = H(\alpha)$ and $H = u_\epsilon/2$ are asymptotically minimax, i.e.,
\[
\alpha_{\epsilon}(\psi_{\epsilon,H(\alpha)}) \leq \alpha + o(1), \quad \beta_{\epsilon}(\Theta(r_\epsilon), \psi_{\epsilon,H(\alpha)}) = \beta_{\epsilon}(r_\epsilon, \alpha) + o(1),
\]
\[
\gamma_{\epsilon}(\Theta(r_\epsilon), \psi_{\epsilon,\alpha}) = \gamma_{\epsilon}(r_\epsilon) + o(1).
\]

(b) If $\lambda \leq 0$, then the sharp asymptotics are of the following degenerate type
\[
\beta_{\epsilon}(r_\epsilon, \alpha) = (1 - \alpha) \Phi(-D_\epsilon) + o(1), \quad \gamma_{\epsilon}(r_\epsilon) = \Phi(-D_\epsilon) + o(1),
\]
where
\[
D_\epsilon = n_\epsilon^{-\beta} r_\epsilon/\epsilon - \sqrt{2 \log(n_\epsilon)}, \quad n_\epsilon = r_\epsilon^{-1/\alpha}.
\]
The tests $\psi_{\epsilon}^D$ and (the randomized) tests $\psi_{\epsilon,\alpha}^D$ of the form (4.34) are asymptotically minimax, i.e.,
\[
\alpha_{\epsilon}(\psi_{\epsilon,\alpha}^D) = \alpha + o(1), \quad \beta_{\epsilon}(\Theta(r_\epsilon), \psi_{\epsilon,\alpha}^D) = \beta_{\epsilon}(r_\epsilon, \alpha) + o(1),
\]
\[
\gamma_{\epsilon}(\Theta(r_\epsilon), \psi_{\epsilon,\alpha}^D) = \gamma_{\epsilon}(r_\epsilon) + o(1).
\]
(c) If $\lambda > 0$, then the separation rates are of the form
\[
r_\epsilon^* = \epsilon^{(2\alpha+1/q-1/2)/(2(\alpha+\beta)+1/q)}.
\]
(d) If $\lambda \leq 0$, then the sharp separation rates are of the form
\[
r_\epsilon^* = \Lambda \epsilon^{\alpha/(\alpha+\beta)} (\log(\epsilon^{-1}))^{\alpha/2}(\alpha+\beta), \quad \Lambda = (2/(\alpha + \beta))^{\alpha/2(\alpha+\beta)}.
\]
We are interested in how large $u_0$. We say that the family $u \in \Theta$ constructs asymptotically minimax adaptive consistent families of tests $\gamma$. We are aiming at finding conditions for either $\gamma \in \Theta$ in Theorem 4.5 and 6.1 in [17], and noting (from their proofs) that the events (thresholding rule) (4.155) in [17] can be replaced by the events (thresholding rule) (4.32). Its proof is omitted.

Remark 4.18 Similarly to Remark 4.3, we get the sharp asymptotics (4.31) for the sequences $a_k \sim k^\alpha$ and $\sigma_k \sim k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, and similar rate asymptotics for the sequences $a_k \asymp k^\alpha$, $\sigma \asymp k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$. In both cases, the separation rates are still of the form given in Theorem 4.10.

5 Minimax signal detection in ill-posed inverse problems: adaptivity and rate optimality

The families of tests described in Section 4.1 (except those described in Theorem 4.7) depend on a parameter $\kappa \in \Sigma \subset \mathbb{R}^+_n \times (0, 2]$, $n \geq 2$, associated with the sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{\sigma_k\}_{k \in \mathbb{N}}$, and $q \in (0, 2]$, that are involved in the ill-posed inverse problems under consideration, that is usually unknown in practice. For example, if $a_k = \exp(\alpha k^\gamma)$ and $\sigma_k = \exp(\beta k^\gamma)$, $k \in \mathbb{N}$, $\alpha > 0$, $\tau \geq 1$, $\beta > 0$, $\gamma > 1$, and if $q \in (0, 2]$, then $\kappa \in \Sigma$.

Therefore, it is of paramount importance to construct families of tests that do not depend on the unknown parameter $\kappa$ and, at the same time, provide the best possible asymptotical minimax efficiency. These families of tests are called adaptive (to the parameter $\kappa$), and the formal setting is as follows.

5.1 Adaptive distinguishability and adaptive separation rates

Let a set $\Sigma = \{\kappa\}$ and a family $r_\ve(k)$, $\kappa \in \Sigma$, be given, where $\ve > 0$ is small. Let the set $\Theta_\ve(k, r_\ve(k))$ be determined by the constraints (2.4) with $a_k = a_k(\kappa)$, $\sigma_k = \sigma_k(\kappa)$, $k \in \mathbb{N}$, $q = q(\kappa)$, and $r_\ve = r_\ve(\kappa)$, and set

$$\Theta_\ve(\Sigma) = \bigcup_{\kappa \in \Sigma} \Theta_\ve(\kappa, r_\ve(\kappa)).$$

We are interested in the following hypothesis testing problem

$$H_0 : \eta = 0, \quad \text{versus} \quad H_1 : \eta \in \Theta_\ve(\Sigma).$$

We are aiming at finding conditions for either $\gamma_\ve(\Theta_\ve(\Sigma)) \rightarrow 1$ or $\gamma_\ve(\Theta_\ve(\Sigma)) \rightarrow 0$, and to constructing asymptotically minimax adaptive consistent families of tests $\psi_\ve^{ad}$ such that $\gamma_\ve(\Theta_\ve(\Sigma), \psi_\ve^{ad}) \rightarrow 0$ as $\gamma_\ve(\Theta_\ve(\Sigma)) \rightarrow 0$.

Let $u_\ve(\kappa) = u_\ve(\kappa, r_\ve(\kappa))$ be the value of the extreme problem (4.4) for the set $\Theta_\ve = \Theta_\ve(\kappa, r_\ve(\kappa))$. Set

$$u_\ve(\Sigma) = \inf_{\kappa \in \Sigma} u_\ve(\kappa).$$

We are interesting in how large $u_\ve(\Sigma)$ should be in order to provide the relation $\gamma_\ve(\Theta(\Sigma)) \rightarrow 0$. We say that the family $u_\ve^{ad} = u_\ve^{ad}(\Sigma) \rightarrow \infty$ characterizes adaptive distinguishability if there exist constants $0 < d = d(\Sigma) \leq D = D(\Sigma) < \infty$ such that

$$\gamma_\ve(\Theta_\ve(\Sigma)) \rightarrow 1 \quad \text{as} \quad \limsup_{\kappa \in \Sigma} u_\ve(\kappa)/u_\ve^{ad} < d,$$

$$\gamma_\ve(\Theta_\ve(\Sigma)) \rightarrow 0 \quad \text{as} \quad \liminf_{\kappa \in \Sigma} u_\ve(\kappa)/u_\ve^{ad} > D.$$
We call a family $r^{ad}_\varepsilon(\kappa)$, $\kappa \in \Sigma$, such that $u^{ad}_\varepsilon \asymp u_\varepsilon(\kappa, r^{ad}_\varepsilon(\kappa))$, the family of adaptive separation rates.

Note that the relation $\gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \to 0$ is possible if $u_\varepsilon(\Sigma) \to \infty$. It was shown in Theorem 4.7 that this relation suffices for the construction of minimax adaptive consistent families of tests for mildly ill-posed inverse problems with the class of analytic functions. However this implication does not hold in the remaining ill-posed inverse problems under consideration. In these cases, adaptive distinguishability conditions and adaptive separation rates are sought, and they are the goal of the subsequent sections. In contrast to Theorem 4.7 there is price to pay for the adaptation. We show that $u^{ad}_\varepsilon = \sqrt{\log \log \varepsilon^{-1}}$ for the mildly ill-posed inverse problems with the Sobolev class of functions and $u^{ad}_\varepsilon = \log \log \varepsilon^{-1}$ for other problems under consideration (except the case mildly ill-posed inverse problems with the class of analytic functions). These yield a loss in the separation rates in terms of an extra $\sqrt{\log \log \varepsilon^{-1}}$ factor for severely ill-posed problems with the Sobolev class, and in terms of an extra $\sqrt{\log \log \varepsilon^{-1}}$ factor for severely problems with analytic classes of functions. A similar loss in the separation rates for a well-posed signal detection problem was first observed in [25].

As we shall show below, the derived families of tests are of simple structure. In particular, for the mildly ill-posed inverse problems with the Sobolev class of functions, these are of the form

$$
\psi^{ad}_\varepsilon = \mathbb{I}_{\{\sup_k |y_k| > H_k\}}, \quad m_k = 2^k, \quad H_k = \sqrt{C \log(k)}, \quad k \geq L, \quad C > 2,
$$

where $L = L_\varepsilon$, $L_\varepsilon \to \infty$, is an integer-valued family and $t_{\varepsilon,m}$ are centered and normalized version of $\chi^2$-statistics that correspond to the first $m$ observations $(y_1, y_2, \ldots, y_m)$ in (1.3).

For the severely ill-posed inverse problems with the Sobolev class of functions or the class of analytic functions, the derived families of tests are of the form

$$
\psi^{ad}_\varepsilon = \mathbb{I}_{\{\sup_k |y_k| > H_k\}}, \quad H_k = \sqrt{2\log(k)}, \quad k < L, \quad H_k = \sqrt{C \log(k)}, \quad k \geq L, \quad C > 2,
$$

where $L = L_\varepsilon$, $L_\varepsilon \to \infty$, is an integer-valued family.

Finally, for the severely ill-posed inverse problems with the generalized analytic class of functions, the derived tests are of the form

$$
\psi^{ad}_\varepsilon = \mathbb{I}_{\{\sup_k |y_k| > T_{\varepsilon,k}\}}, \quad T_{\varepsilon,k} = \max \left( T_\varepsilon, \sqrt{2(\log(k) + \log \log(k))} \right)
$$

for a family $T_\varepsilon \to \infty$.

### 5.1.1 Mildly ill-posed inverse problems with the Sobolev class of functions

Consider first the “standard” case $q = 2$. Let $a_k = k^\alpha$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$. Set $\kappa = (\alpha, \beta)$ and let $\Sigma$ be a compact subset of $\mathbb{R}^2_+$. We show that, under a weak assumption on the set $\Sigma$,

$$
u^{ad}_\varepsilon = \sqrt{\log \log(\varepsilon^{-1})}.
$$

This corresponds to the adaptive separation rates

$$
r^{ad}_\varepsilon(\kappa) = (\varepsilon \sqrt{\log \log(\varepsilon^{-1})})^{4\alpha/(4\alpha + 4\beta + 1)}, \quad (5.1)
$$

The rate optimal adaptive family of tests is of the following structure. Take a collection $m_k = 2^k$, $k \in \mathbb{N}$, $k \geq L = L_\varepsilon$ with the integer-valued family $L_\varepsilon \to \infty$, $L_\varepsilon = o(\log(\varepsilon^{-1}))$, and take a family of test statistics $t_{\varepsilon,m_k}$ of the form (4.12). Consider the following families of thresholds and tests

$$
H_k = \sqrt{C \log(k)}, \quad \mathcal{Y}_\varepsilon = \{ y : t_{\varepsilon,m_k} \leq H_k, \quad \forall k \geq L_\varepsilon \}, \quad \psi_\varepsilon = \mathbb{I}_{\mathcal{Y}_\varepsilon}, \quad C > 2. \quad (5.2)
$$
Denote also
\[ \phi(\kappa) = \frac{4}{4\alpha + 4\beta + 1}, \quad \phi(\Sigma) = \{\phi(\kappa) : \kappa \in \Sigma\} \subset (0, \infty). \quad (5.3) \]

Then, the following statement is true.

**Theorem 5.1** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) for \( q = 2 \). Let \( a_k = k^\alpha \) and \( \sigma_k = k^\beta, k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Then

(a) (lower bounds) Let the set \( \phi(\Sigma) \) given by (5.3) contains an interval \([a, b]\), \( 0 < a < b < \infty \). Then, there exists constant \( d > 0 \) such that if \( \limsup_{\kappa \in \Sigma} u_\varepsilon(\kappa)/\sqrt{\log \log(\varepsilon^{-1})} \leq d \), then \( \gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \to 1 \).

(b) (upper bounds) For the family of tests \( \psi_\varepsilon \) given by (5.2), \( \alpha(\psi_\varepsilon) = o(1) \) and there exists constant \( D = D(\Sigma) > 0 \) such that if \( \liminf_{\kappa \in \Sigma} u_\varepsilon(\kappa)/\sqrt{\log \log(\varepsilon^{-1})} > D \), then \( \beta_\varepsilon(\psi_\varepsilon, \Theta_\varepsilon(\Sigma)) = o(1) \).

(c) (adaptive separation rates) The adaptive separation rates \( r_\varepsilon^{ad}(\kappa), \kappa \in \Sigma \), are given by (5.1).

The proof is given in the Appendix, Section 7.10.

**Remark 5.1** In view of Remark 4.3 similar rate optimality results and the same adaptive separation rates hold for the sequences \( a_k \sim k^\alpha \) and \( \sigma_k \sim k^\beta, k \in \mathbb{N}, \alpha > 0, \beta > 0 \), and the sequences \( a_k \asymp k^\alpha \) and \( \sigma_k \asymp k^\beta, k \in \mathbb{N}, \alpha > 0, \beta > 0 \).

As in the case of rate and sharp asymptotics, unlike the “standard” case \( q = 2 \), the “sparse” case \( q \in (0, 2) \) is not directly linked with Theorem 4.1 and, hence, adaptive separation rates for the “sparse” case \( q \in (0, 2) \) will be considered separately in Section 6. (In fact, adaptivity results for the (more general) Besov class of functions will be presented therein.)

**5.1.2 Severely ill-posed inverse problems with the class of analytic functions**

Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = \exp(\beta k), k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Set \( \kappa = (\alpha, \beta, q) \) and let \( \Sigma \) be a compact subset of \( \mathbb{R}^2 \times (0, 2] \). We show that, under a weak assumption on the set \( \Sigma \),

\[ u_\varepsilon^{ad} = \log \log(\varepsilon^{-1}). \]

This corresponds to the adaptive separation rates

\[ r_\varepsilon^{ad}(\kappa) = (\varepsilon \sqrt{\log \log(\varepsilon^{-1})})^{\alpha/(\alpha + \beta)}. \quad (5.4) \]

The rate optimal adaptive family of tests is of the following structure. Take an integer-valued \( L = L_\varepsilon \) with \( L_\varepsilon \to \infty \), \( L_\varepsilon = o(\log \log(\varepsilon^{-1})) \). Consider the following families of thresholds and tests

\[ H_k = \begin{cases} \sqrt{2 \log(L)}, & k < L, \\ \sqrt{C \log(k)}, & k \geq L, \end{cases} \quad \mathcal{Y}_\varepsilon = \{y : |y_k| \leq \varepsilon H_k\}, \quad \psi_\varepsilon = \mathbb{I}_{\mathcal{Y}_\varepsilon}, \quad C > 2. \quad (5.5) \]

Denote also
\[ \phi(\kappa) = \frac{1}{2(\alpha + \beta)}, \quad \phi(\Sigma) = \{\phi(\kappa) : \kappa \in \Sigma\} \subset (0, \infty). \quad (5.6) \]
Theorem 5.2 Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4). Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = \exp(\beta k) \), \( k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Then

(a) (lower bounds) Let the set \( \phi(\Sigma) \) given by (5.6) contains an interval \([a, b]\), \(0 < a < b < \infty\). Then, there exists constant \( d > 0 \) such that if \( \limsup_{k \in \Sigma} u_\varepsilon(k) / \log \log(\varepsilon^{-1}) \leq d \), then \( \gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \rightarrow 1 \).

(b) (upper bounds) For the family of tests \( \psi_\varepsilon \) given by (5.5), \( \alpha(\psi_\varepsilon) = o(1) \) and there exists constant \( D = D(\Sigma) > 0 \) such that if \( \liminf_{k \in \Sigma} u_\varepsilon(k) / \log \log(\varepsilon^{-1}) > D \), then \( \beta_\varepsilon(\Theta_\varepsilon(\Sigma), \psi_\varepsilon) = o(1) \).

(c) (adaptive separation rates) The adaptive separation rates \( r_\varepsilon^{ad}(\kappa), \kappa \in \Sigma \), are given by (5.4).

The proof is given in the Appendix, Section 7.11.

Remark 5.2 In view of Remark 4.5, similar rate optimality results and the adaptive separation rates (5.4) hold for the sequences \( a_k \sim \exp(\alpha k) \) and \( \sigma \sim \exp(\beta k) \), \( k \in \mathbb{N}, \alpha > 0, \beta > 0 \).

5.1.3 Severely ill-posed inverse problems with the Sobolev class of functions

Let \( a_k = k^\alpha \) and \( \sigma_k = \exp(\beta k) \), \( k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Set \( \kappa = (\alpha, \beta, q) \) and let \( \Sigma \) be a compact subset of \( \mathbb{R}_+^2 \times (0, 2] \). We show that, under a weak assumption on the set \( \Sigma \),

\[ u_\varepsilon^{ad} = \log \log(\varepsilon^{-1}) \]

This corresponds to the adaptive separation rates

\[ r_\varepsilon^{ad}(\kappa) = \left( \frac{2 \log(\varepsilon^{-1}) - 2 \alpha \log \log(\varepsilon^{-1}) - \log \log \log(\varepsilon^{-1})}{2 \beta} \right)^{-\alpha} \sim \left( \frac{\log(\varepsilon^{-1})}{\beta} \right)^{-\alpha}. \]  

(5.7)

The rate optimal adaptive family of tests is of the following structure. Take an integer-valued \( L = L_\varepsilon \) with \( L_\varepsilon \rightarrow \infty \), \( L_\varepsilon = o(\log \log(\varepsilon^{-1})) \), and consider the families of thresholds and tests given by (5.5).

Theorem 5.3 Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4). Let \( a_k = k^\alpha \) and \( \sigma_k = \exp(\beta k) \), \( k \in \mathbb{N}, \alpha > 0, \beta > 0 \). Then

(a) (lower bounds) Let the set \( \Sigma \) contains an interval of \((a, b) : \beta \in [1/2b, 1/2a], 0 < a < b < \infty, \) and a fixed \( \alpha > 0 \). Then, there exists constant \( d > 0 \) such that if \( \limsup_{k \in \Sigma} u_\varepsilon(k) / \log \log(\varepsilon^{-1}) \leq d \), then \( \gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \rightarrow 1 \).

(b) (upper bounds) For the family of tests \( \psi_\varepsilon \) given by (5.5), \( \alpha(\psi_\varepsilon) = o(1) \) and there exists constant \( D = D(\Sigma) > 0 \) such that if \( \liminf_{k \in \Sigma} u_\varepsilon(k) / \log \log(\varepsilon^{-1}) > D \), then \( \beta_\varepsilon(\psi_\varepsilon, \Theta_\varepsilon(\Sigma)) = o(1) \).

(c) (adaptive separation rates) The adaptive separation rates \( r_\varepsilon^{ad}(\kappa), \kappa \in \Sigma \), are given by (5.4).

The proof is given in the Appendix, Section 7.12.

Remark 5.3 It is worth mentioned that a stronger result is possible in this case. In view of (5.18), the relation (5.7) determines sharp adaptive separation rates \( r_\varepsilon^{ad}(\kappa), \kappa \in \Sigma \), in the following sense.

(a) If \( \liminf (r_\varepsilon(\kappa) / r_\varepsilon^{ad}(\kappa)) > 1 \), then \( u_\varepsilon \rightarrow \infty \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \rightarrow 0 \).

(b) If \( \limsup (r_\varepsilon(\kappa) / r_\varepsilon^{ad}(\kappa)) < 1 \), then \( u_\varepsilon \rightarrow 0 \), i.e., \( \gamma_\varepsilon(r_\varepsilon) \rightarrow 1 \), and the minimax testing is impossible.
Remark 5.4 In view of Remark 4.8, similar rate optimality results and the adaptive separation rates (5.7) (as well as the sharp adaptive separation rates mention in Remark 5.3) hold for the sequences $a_k \sim k^\alpha$ and $\sigma \asymp \exp(\beta k)$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$.

5.1.4 Extremely ill-posed inverse problems with the class of generalized analytic functions

We consider the case $q = 2$ only. By the results of Section 4.6, in order to obtain distinguishability conditions, we can replace $u_\varepsilon(\kappa)$, $\kappa \in \Sigma$, by $u_{\varepsilon,\text{lin}}(\kappa) = u_{\varepsilon,\text{lin}}(\kappa, r_\varepsilon(\kappa))$, determined by (4.22), for $a_k = a_k(\kappa)$ and $\sigma_k = \sigma_k(\kappa)$, $k \in \mathbb{N}$. Set

$$u_{\varepsilon,\text{lin}}(\Sigma) = \inf_{\kappa \in \Sigma} u_{\varepsilon,\text{lin}}(\kappa).$$

We are interested in how large $u_{\varepsilon,\text{lin}}(\Sigma)$ should be in order to provide the relation $\gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \to 0$.

Assume below the uniform version of (4.20): let $\sigma_k(\kappa)$ and $a_k(\kappa)$, $k \in \mathbb{N}$, be increasing sequences such that, for all $\kappa \in \Sigma$ and some constants $0 < b < B$,

$$b \leq a_1(\kappa) \leq B, \quad b \leq \sigma_1(\kappa) \leq B,$$ (5.8)

and, for some increasing sequence $\tau_k > 1$, $\tau_k \to \infty$ and some $c_0 > 1$ for all $\kappa \in \Sigma$ and $k \in \mathbb{N}$,

$$\sigma_{k+1}(\kappa)/\sigma_k(\kappa) \geq \tau_k, \quad a_{k+1}(\kappa)/a_k(\kappa) \geq c_0.$$ (5.9)

Similarly to $u_{\varepsilon,\text{ad}}$, one can consider a family $u_{\varepsilon,\text{lin,ad}}$ which characterizes adaptive distinguishability. We show that, under some assumption on the set $\Sigma$, one has

$$u_{\varepsilon,\text{lin,ad}} = \log \log(\varepsilon^{-1}).$$

For $\kappa \in \Sigma$ and for $A > 0$ large enough, let an integer $m = m(A, \kappa)$ be defined by the relations

$$a_{m-1}(\kappa)\sigma_{m-1}(\kappa) \leq A < a_m(\kappa)\sigma_m(\kappa).$$ (5.10)

Under (5.8) and (5.9), one has

$$a_{m-1}\sigma_{m-1} \geq b^2 c_0^{m-2} \prod_{k=1}^{m-2} \tau_k,$$

which yields

$$\sup_{\kappa \in \Sigma} m(A, \kappa) = o(\log(A)) \quad \text{as} \quad A \to \infty.$$ (5.11)

Set

$$\mathcal{M}(A, \Sigma) = \{m(A, \kappa) : \kappa \in \Sigma\}, \quad M(A, \Sigma) = \#(\mathcal{M}(A, \Sigma)).$$

Since $M(A, \Sigma) \leq \max_{m \in \mathcal{M}(A, \Sigma)} m$, one has, by (5.11),

$$M(A, \Sigma) = o(\log(A)), \quad \text{as} \quad A \to \infty.$$ (5.12)

Let $m = m(A, \kappa)$ be defined by (5.10) and set $L(A, \Sigma) := \sup_{\kappa \in \Sigma} \log(m(A, \kappa))$. By (5.11) we have, as $A \to \infty$,

$$\lim \sup L(A, \Sigma) / \log \log(A) \leq 1.$$ (5.12)
For the lower bounds we suppose one can find quantities $b > 0$, $C \geq 1$ such that
\[
\liminf_{A \to \infty} \frac{\log(M(A, \Sigma))}{\log(A)} = b, \quad \sup_{\kappa \in \Sigma} u_{\varepsilon}^{\text{lin}}(\kappa, r_{\varepsilon}(\kappa)) \leq C u_{\varepsilon}^{\text{lin}}(\Sigma).
\] (5.13)
(The first relation in (5.13) is fulfilled for the example mentioned in Remark 4.16 at least if the set $\Sigma = \{(\alpha, \tau, \beta, \gamma)\}$ contains an interior point.)

The rate optimal adaptive family of tests is of the following structure. Take a family $T_{\varepsilon} \to \infty$ such that $T_{\varepsilon} = o(\sqrt{\log \log(\varepsilon^{-1})})$, and take a family of sequences $T_{\varepsilon, k}$ of the form
\[
T_{\varepsilon, k} = \max \left( T_{\varepsilon}, \sqrt{2(\log(k) + \log \log(k))} \right).
\]
This family satisfies
\[
\sum_{k \in \mathbb{N}} \Phi(-T_{\varepsilon, k}) \asymp T_{\varepsilon}^{-3} + \sum_{k > 2 \exp(T_{\varepsilon}^2/2)} e^{-T_{\varepsilon, k}^2/2} T_{\varepsilon, k}^{-3} + \sum_{k > 2 \exp(T_{\varepsilon}^2/2)} \frac{1}{k(\log(k))^{3/2}} = o(1).
\]
Consider the following families of events and of tests
\[
\mathcal{Y}_{\varepsilon} = \{ y : |y_k| \leq \varepsilon T_{\varepsilon, k}, \quad \forall \ k \in \mathbb{N} \}, \quad \psi_{\varepsilon} = \mathbb{I}_{\mathcal{Y}_{\varepsilon}}.
\]
(5.14)

Then, the following statement is true.

**Theorem 5.4** Consider the Gaussian sequence model (2.3) and the hypothesis testing problem (2.3) with the set under the alternative given by (2.4) for $q = 2$. Let $\{a_k\}_{k \in \mathbb{N}}$ and $\{\sigma_k\}_{k \in \mathbb{N}}$ be increasing sequences satisfying (5.8) and (5.9). Then

(a) (lower bounds) Assume (5.13). Then, there exists a constant $d > 0$ such that if $\limsup u_{\varepsilon}^{\text{lin}}(\Sigma)/\log \log(\varepsilon^{-1}) \leq d$, then $\gamma_{\varepsilon}(\Theta_{\varepsilon}(\Sigma)) \to 1$.

(b) (upper bounds) Assume (5.12). For the family of tests $\psi_{\varepsilon}$ given by (5.14), there exists a constant $D > 0$ such that if $u_{\varepsilon}^{\text{lin}}(\Sigma) > D \log \log(\varepsilon^{-1})$, then $\gamma_{\varepsilon}(\Theta_{\varepsilon}(\Sigma), \psi_{\varepsilon}) = o(1)$.

(c) (adaptive separation rates) The adaptive separation rates $r_{\varepsilon, a}(\kappa)$, $\kappa \in \Sigma$, are determined by the relation $u_{\varepsilon, a}^{\text{lin}} \asymp u_{\varepsilon}^{\text{lin}}(\kappa, r_{\varepsilon, a}(\kappa))$.

The proof is given in the Appendix, Section 7.13.

6 Mildly ill-posed inverse problems with the Besov class of functions: the “sparse” case $q \in (0, 2)$

The various ill-posed inverse problems considered in Sections 1-5, under the Gaussian white noise model (1.1), are based on the SVD of the involved operator $A$. However, it is well known that although offers a kind of diagonalisation, SVD has its limitations rooted in the fact that the resulting eigenfunctions derive from the operator $A$, not from the function $f$ to be recovered. Unfortunately, it is in general impossible, although it would be very pleasant, to have a system of eigenfunctions which simultaneously diagonalizes both $A$ and the a-priori regularity of $f$. On the other hand, [9] showed that there exists a WVD of a special class of homogeneous operators (such as integration or fractional integration), that simultaneously quasi-diagonalises both the operator $A$ under study and the a-priori regularity of the function $f$ to be recovered (see, also, [1]).

In view of the above discussion, under the Gaussian white noise model (1.1), we also consider the signal detection problem for mildly ill-posed inverse problems with Besov classes.
(bodies) of functions, \( B_{q,t}^{\alpha} \), \( \alpha > 0 \), \( q \in (0, \infty) \), \( t \in (0, \infty) \), that arises naturally through WVD when \( A \) is assumed to belong to the special class of homogeneous operators mentioned above. Restricting again our attention to the “sparse” case \( q \in (0, 2) \), both rate and sharp asymptotics for the error probabilities in the minimax setup are studied. Both minimax rate-optimal non-adaptive and adaptive tests are also constructed.

6.1 WVD, Besov spaces and the Gaussian sequence model

For a special class of homogeneous operators \( A \) acting on Hilbert spaces, \cite{9} showed that an WVD exists that involves three set of functions, an orthonormal (regular) wavelet basis, and two near-orthogonal sets, \( \{u_\nu\}_{\nu \in \Lambda} \) and \( \{v_\nu\}_{\nu \in \Lambda} \), that satisfy certain biorthogonality and near-orthogonality relations (see (10)-(12) in \cite{9}), and the following quasi-singular relations:

\[
A\psi_\nu = \kappa_j v_\nu, \quad A^* u_\nu = \kappa_j \psi_\nu, \quad \nu \in \Lambda,
\]

where the set \( \Lambda = \{\nu : \nu = (i,j), j \in \mathbb{N}, i = 1, 2, \ldots, 2^j \} \) be of dyadic structure, \( (j \text{ and } i \text{ correspond to be the resolution and spatial indices, respectively}), \kappa_j \) be the the quasi-singular values (depending on \( j \) but not on \( i \)).

Consider the Gaussian white noise model (1.1) and assume that \( f \) belongs to the Besov ball

\[
B_{q,t}^{\alpha}(L) = \{\theta \in l^2 : |\theta|^\alpha_{q,t} < L \}, \quad \alpha > 0, \quad q \in (0, \infty), \quad t \in (0, \infty), \quad L > 0,
\]

where the wavelet coefficients \( \theta = \{\theta_\lambda\}_{\lambda \in \Lambda}, \theta_\lambda = \langle f, \psi_\lambda \rangle, \lambda \in \Lambda, \) satisfy

\[
|\theta|^\alpha_{q,t} = \left( \sum_{j \in \mathbb{N}} 2^{\alpha t j} \left( \sum_{i=1}^{2^j} |\theta_{ij}|^q \right)^{t/q} \right)^{1/t}, \quad \alpha > 0, \quad q > 0, \quad t > 0,
\]

with obvious modifications for the cases \( q = \infty \) and/or \( t = \infty \). Under a regular enough (i.e., a number of vanishing moments and continuous derivatives) wavelet basis \( \{\psi_\lambda\}_{\lambda \in \Lambda} \) in \( L_2(0,1) \), the Besov norm \( |\cdot|_{q,t}^\alpha \) in the sequence space is equivalent to the Besov norm \( \|\cdot\|_{q,t}^\alpha \) in the function space (i.e., in \( L_2(0,1) \)), with the smoothness parameter \( \tilde{\alpha} = \alpha - 1/2 + 1/q > 0 \), \( q \in [1, \infty] \), \( t \in [1, \infty] \).

Note also that the Besov norm \( |\cdot|_{q,q}^\alpha \) (i.e., for \( t = q \)) is equivalent to the Sobolev norm \( |\cdot|_{a,q} \), and that there are not direct relationships between the Besov norm \( \|\cdot\|_{q,t}^\alpha \) and the Sobolev norm \( \|\cdot\|_{\tilde{\alpha},q} \) in \( L_2(0,1) \); however the following relation exists: for some constants \( c_1 > 0 \) and \( c_2 > 0 \), not depending on \( f \),

\[
c_1 \|f\|_{q,\infty}^{\tilde{\alpha}} \leq \|f\|_{\tilde{\alpha},q} \leq c_2 \|f\|_{q,1}^{\tilde{\alpha}}.
\]

(See, e.g., \cite{17}, Section 2.7, and references therein).

Assume now that, in the Gaussian white noise model (1.1), \( A \) belongs to the special class of homogeneous operators mentioned above. Then, according to (35)-(36) in \cite{9}, the following Gaussian sequence model emerges

\[
y_\nu = \theta_\nu + \varepsilon \sigma_j \xi_\nu, \quad \nu \in \Lambda,
\]

where \( y_\nu \) are the empirical WVD coefficients, \( \sigma_j \) are the reciprocal of the quasi-singular values \( \kappa_j \), and \( \xi_\nu \) are iid \( \mathcal{N}(0,1) \) random variables.
6.2 Rate and sharp optimality results

Let now $a_\nu = 2^{\alpha \nu}$, $\alpha > 0$, and take $\sigma_\nu = 2^{\beta j}$, $\beta > 0$, in (6.1). The latter sequence (of variances) arise as the reciprocal of the quasi-singular values of certain homogeneous operators using the WVD (see Theorem 3 in [9]). (On replacing the indices $(i,j)$, $j \in \mathbb{N}$, $i = 1, 2, \ldots, 2^j$, by $k = 2^j + i$, then we get $\sigma_k \sim k^{\beta j}$, $\beta > 0$, $k \in \mathbb{N}$, justifying the term mildly ill-posed inverse problem used above.)

In view of the above, the alternative set (2.4) is now determined by

$$\Theta_q(r_\varepsilon) = \{ \eta \in l^2 : |\eta|_{\alpha + \beta,q,t} \leq 1, \ |\eta|_{\beta,2,2} \geq r_\varepsilon \}.$$  (6.2)

Consider the extreme problem

$$u_\varepsilon^2 = \inf_{j \in \mathbb{N}} \sum_{j \in \mathbb{N}} 2^{j+1} h_j^2 \sinh^2(z_j^2/2),$$  (6.3)

where the infimum is taken over sequences $(h_j, z_j)$, $h_j \in [0, 1]$, $z_j \geq 0$, $j \in \mathbb{N}$, such that

$$\sum_{j \in \mathbb{N}} 2^{j(2\beta+1)} h_j^2 z_j^2 \geq (\tilde{r}_\varepsilon^2/\varepsilon^2), \quad \sum_{j \in \mathbb{N}} 2^{j(\alpha+\beta)+t/q} h_j^{1/q} z_j^t \leq (1/\varepsilon^t),$$  (6.4)

where $\tilde{r}_\varepsilon = r_\varepsilon(1 - \delta_\varepsilon)$, $\delta_\varepsilon \geq 0$ with $\delta_\varepsilon \to 0$.

Set $\lambda = (\alpha + \beta)/2 - \beta/q$. If $\lambda > 0$ and $q \leq t$, then, there exist extreme sequences $h_{j,\varepsilon} \in (0, 1]$, $z_{j,\varepsilon} > 0$ in the problem (6.3)-(6.4) and we have the asymptotics

$$u_\varepsilon^2 \sim c_0 2^{j_0} h_{0}^2,$$  (6.5)

where the quantities $h_0$ and $j_0$ are determined by the relations

$$c_1 2^{j_0(\beta+1/2)} h_{0}^{1/2} \sim r_\varepsilon/\varepsilon, \quad c_2 2^{j_0(\alpha+\beta+1/q)} h_{0}^{1/q} \sim 1/\varepsilon,$$  (6.6)

for some positive, 1-periodic functions (in $j_0$) $c_l = c_l(\alpha, \beta, q, t, j_0) > 0$, $l = 1, 2$, which, in turn, imply the asymptotics of the form (4.31) for some positive, 1-periodic function (in $j_0$) $c_3 = c_3(\alpha, \beta, q, t, j_0)$.

Set

$$Q_{\varepsilon,j} = \sqrt{2(j \log(2) + \log j + 2 \log \log(\varepsilon^{-1}))}$$

and consider the events

$$\mathcal{Y}_\varepsilon = \{ y_\nu = \{ y_{ij} \} : \sup_{j \in \mathbb{N}} \max_{1 \leq i \leq 2^j} |y_{ij}|/\varepsilon Q_{\varepsilon,j} > 1 \},$$  (6.7)

the family of test statistics

$$l_\varepsilon(y) = u_\varepsilon^{-1} \sum_{j \in \mathbb{N}} h_{j,\varepsilon} l_{j}(y_j, z_{\varepsilon,j}), \quad l_j(y_j, z) = \sum_{i=1}^{2^j} \xi(y_{ij}/\varepsilon, z), \quad \xi(t, z) = e^{z^2/2} \cosh(tz) - 1,$$  (6.8)

and the family of tests of the form (4.34) with events and test statistics defined, respectively, by (6.7) and (6.8).

Then, the following statement is true.
Theorem 6.1 Consider the Gaussian sequence model \( (2.2) \) and the hypothesis testing problem \( (2.3) \) with the set under the alternative given by \( (6.3) \) with \( t > 0 \) and \( q \in (0, 2) \). Let \( a_\nu = 2^{\nu j} \) and \( \sigma_\nu = 2^{\nu j} \), \( j \in \mathbb{N} \), \( \alpha > 0 \) and \( \beta > 0 \), and set \( \lambda = (\alpha + \beta)/2 - \beta/q \). Then

(a) If \( \lambda > 0 \) and \( q \leq t \), then the tests are of the Gaussian type \( (2.8) \) with \( u_\epsilon \) from \( (6.3) \). The tests \( \psi^{G}_{\alpha,\beta,q,t} \) of the form \( (4.34) \) with \( H = H^{(\alpha)} \) and \( H = u_\epsilon/2 \) are asymptotically minimax, i.e.,

\[
\alpha_\epsilon(\psi^{G}_{\alpha,\beta,q,t}) \leq \alpha + o(1),
\beta_\epsilon(\Theta(r_\epsilon), \psi^{G}_{\alpha,\beta,q,t}) = \beta_\epsilon(r_\epsilon, \alpha) + o(1),
\gamma_\epsilon(\Theta(r_\epsilon), \psi^{G}_{\alpha,\beta,q,t}) = \gamma_\epsilon(r_\epsilon) + o(1).
\]

(b) If \( \lambda \leq 0 \) and \( t \leq q \) or if \( \lambda < 0 \) and \( t > q \), then, there exist constants \( c_1 = c_1(\alpha, \beta, q, t) \), \( c_1 \geq 1 \geq c_2 > 0 \), such that

\[
(1 - \alpha)\Phi(-D_{\epsilon,c_1}) + o(1) \leq \beta_\epsilon(r_\epsilon, \alpha) \leq (1 - \alpha)\Phi(-D_{\epsilon,c_2}) + o(1),
\Phi(-D_{\epsilon,c_1}) + o(1) \leq \gamma_\epsilon(r_\epsilon) \leq \Phi(-D_{\epsilon,c_2}) + o(1),
\]

where

\[
D_{\epsilon,c} = c n^{-\beta} r_\epsilon / \epsilon - \sqrt{2\log(n_\epsilon)}, \quad n_\epsilon = r_\epsilon^{-\alpha}.\]

The tests \( \psi^{D}_{\epsilon,\alpha} \) and \( \psi^{D}_{\epsilon,\alpha} \) of the form \( (4.34) \) determine the upper bounds, i.e.,

\[
\alpha_\epsilon(\psi^{D}_{\epsilon,\alpha}) \leq \alpha + o(1),
\beta_\epsilon(\Theta(r_\epsilon), \psi^{D}_{\epsilon,\alpha}) \leq (1 - \alpha)\Phi(-D_{\epsilon,c_2}) + o(1),
\gamma_\epsilon(\Theta(r_\epsilon), \psi^{D}_{\epsilon,\alpha}) \leq \Phi(-D_{\epsilon,c_2}) + o(1).
\]

(c) If \( \lambda > 0 \), then the separation rates are of the form (this holds for \( q > t \) as well)

\[
r^{*}_\epsilon = \epsilon^{(2\alpha + 1/q - 1)/2(\alpha + \gamma)/\alpha + 1/q}.
\]

(d) If \( \lambda \leq 0 \) and \( t \leq q \) or if \( \lambda < 0 \) and \( q < t \), then the separation rates are of the form

\[
r^{*}_\epsilon = \epsilon^{\alpha/(\alpha + \beta)}(\log(\epsilon^{-1}))^{\alpha/2(\alpha + \beta)}.
\]

Moreover, there exist constants \( \Lambda_1 \geq \Lambda_2 > 0 \) such that

\[
\text{if} \quad \liminf r_\epsilon/n^{*}_\epsilon > \Lambda_1 \quad \text{then} \quad \beta_\epsilon(r_\epsilon, \alpha) \to 0, \quad \gamma_\epsilon(r_\epsilon) \to 0,
\]

and

\[
\text{if} \quad \limsup r_\epsilon/n^{*}_\epsilon < \Lambda_2 \quad \text{then} \quad \beta_\epsilon(r_\epsilon, \alpha) \to 1 - a, \quad \gamma_\epsilon(r_\epsilon) \to 1.
\]

The above theorem can be obtained by using the general minimax hypothesis testing framework considered in Section 2 and Theorems 4.6 and 6.2 in [17]. Its proof is omitted.

Remark 6.1 Similarly to Remark 4.3 we get the sharp asymptotics \( (6.3) \) for the sequences \( a_\nu \sim 2^{\nu j} \) and \( \sigma_\nu \sim 2^{\nu j} \), \( j \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \), and similar rate asymptotics for the sequences \( a_\nu \sim 2^{\nu j} \) and \( \sigma_\nu \sim 2^{\nu j} \), \( j \in \mathbb{N} \), \( \alpha > 0 \), \( \beta > 0 \). In both cases, the separation rates are still of the form given in Theorem 6.1.

Remark 6.2 Observe that the tests \( \psi^{D}_{\epsilon,\alpha} \) and \( \psi^{D}_{\epsilon,\alpha} \) of the form \( (4.34) \), with events and test statistics defined, respectively, by \( (6.7) \) and \( (6.8) \), do not depend on \( \alpha, \beta \) and \( q \), that determine the particular ill-posed inverse problem and the class of functions, i.e., these tests are adaptive (with respect to \( \alpha, \beta \) and \( q \)) in the region \( \lambda < 0 \).
6.3 Rate-adaptive optimality results

As pointed out previously, unlike the “standard” case \( q = 2 \), similar to the sharp and rate optimality results, adaptivity results for the “sparse” case \( q \in (0, 2) \) are of different nature and are not directly linked with Theorem [14] but can be obtained from a hitherto unknown link with results obtained in another context and presented in Sections 7.1.3 and 7.4.1 of [17]. For completeness and an immediate access to these results, we formulate and present them below.

Let \( a_{\nu} = 2^{\alpha_{\nu}}, \sigma_{\nu} = 2^{\beta_{\nu}}, j \in \mathbb{N}, \alpha > 0, \beta > 0, \kappa = (\alpha, \beta, q), q \in (0, 2), \tau = (\kappa, t) \in \Sigma \), where \( \Sigma \) is a compact subset of \( \mathbb{R}_+^2 \times (0, 2) \times \mathbb{R}_+ \). Set \( \lambda = \lambda(\kappa) = (\alpha + \beta)/2 - \beta/q \) and recall from Remark [5.2] that if \( \lambda < 0 \), then the tests \( \psi_{\varepsilon}^D, \psi_{\varepsilon,\alpha} \) provide distinguishability and do not depend in \( \tau \), i.e., these tests are adaptive and we have no losses in the separation rates.

Assume now \( \inf_{\tau \in \Sigma} \lambda(\kappa) > 0 \). Then, as in Section [5.1.1] one can show that

\[
u_{\varepsilon}^{ad} = \sqrt{\log \log(\varepsilon^{-1})}\]

which, in turn, by results of Section [6.5] yields the following adaptive separation rates (depending on \( \kappa \) only)

\[
u_{\varepsilon}^{ad}(\kappa) = \varepsilon^{(2\alpha+1/q-1/2)/(2(\alpha+\beta)+1/q)}, \quad \bar{\varepsilon} = \varepsilon^{4 \log \log(\varepsilon^{-1})}. \tag{6.9}\]

We now construct a family of tests that are rate-optimal adaptive. First, let

\[
\psi_{\varepsilon,0} = \mathbb{I}_{Y_{\varepsilon,0}}, \quad Y_{\varepsilon,0} = \{y_{\nu} = \{y_{ij}\} : \sup_{j \in \mathbb{N}} \max_{1 \leq i \leq 2^j} \{|y_{ij}|/(\varepsilon T_{\varepsilon,j}) > 1\}, \tag{6.10}\]

where

\[
T_{\varepsilon,j} = \begin{cases} \sqrt{2C J_{\varepsilon,0}}, & \text{if } j \leq J_{\varepsilon,0}, \\ 2(C j + \log j), & \text{if } j > J_{\varepsilon,0}, \end{cases}
\]

Next, for \( j \geq J_{\varepsilon,0} \), consider the test statistics

\[
l_j = 2^{-(j+1)/2} \sum_{i=1}^{2^j} (y_{ij}^2/\varepsilon^2 - 1), \]

and set

\[
\psi_{\varepsilon,1} = \mathbb{I}_{Y_{\varepsilon,1}}, \quad Y_{\varepsilon,1} = \{y_{\nu} = \{y_{ij}\} : \sup_{J_{\varepsilon,0} \leq j} l_j/T_j > 1\}, \quad T_j = 2\sqrt{\log j}. \tag{6.11}\]

Lastly, fix a small enough value \( c \in (0, (\log_2 2)/4) \) and take

\[
K = K(j) = (\log j)/2, \quad K(c, j) = K + cj. \tag{6.12}\]

For all resolution levels \( j \) such that

\[
J_{\varepsilon,0} \leq j \leq J_{\varepsilon,1} \propto (\log \log \varepsilon^{-1}) \log \varepsilon^{-1},
\]

let us consider the collection of test statistics

\[
l_{j,k} = \sum_{i=1}^{2^j} l_{ij,k} = (2^{j+1} \sinh^2(z_{j,k}^2/2))^{-1/2} \sum_{i=1}^{2^j} \xi(y_{ij}/\varepsilon, z_{j,k}), \quad 1 \leq k \leq K(c, j), \]

where

\[
z_{j,k} = \begin{cases} e^{k-1}/\sqrt{j}, & 1 \leq k \leq K, \\ \sqrt{k - K}, & K < k \leq K(c, j). \end{cases}
\]
Set
\[ \psi_{\varepsilon,j,k} = \mathbb{I}_{j,k > t_j}, \quad t_j = \sqrt{5 \log j}, \]
and consider the family of tests
\[ \psi_{\varepsilon} = \max\{\psi_{\varepsilon,0}, \psi_{\varepsilon,1}, \max_{j \geq 0} \max_{1 \leq k \leq \kappa(c,j)} \psi_{\varepsilon,j,k}\}. \quad (6.13) \]

Then, the following statement is true.

**Theorem 6.2** Consider the Gaussian sequence model (2.2) and the hypothesis testing problem (2.3) with the set under the alternative given by (6.2) with \( q \in (0, 2) \). Let \( a_\nu = 2^{\alpha j} \) and \( \sigma_\nu = 2^{\beta j}, \; j \in \mathbb{N}, \; \alpha > 0, \; \beta > 0 \) and \( t > 0 \). Set \( \lambda = \lambda(\kappa) = (\alpha + \beta)/2 - \beta/q \) and assume that \( \inf_{\tau \in \Sigma} \lambda(\kappa) > 0 \)

(a) (lower bounds) Set \( \phi(\tau) = \alpha + \beta + 1/2q \) and assume there exists set \( \Sigma_0 \subset \Sigma \) such that the set \( \phi(\Sigma_0) = \{\phi(\tau) : \tau \in \Sigma_0\} \subset \mathbb{R}_+ \) contains an interval \( (a, b) \), \( 0 < a < b \). Then, there exists a constant \( c = c(\Sigma) > 0 \) such that if \( \lim \sup \sup_{\tau \in \Sigma_0} u_{\tau}^2(\tau, \rho_\varepsilon(\tau))/\log \log \varepsilon^{-1} < c \), then \( \beta_\varepsilon(\alpha, \Theta_\varepsilon(\Sigma)) \to (1 - \alpha) \) for all \( \alpha \in (0, 1) \) and \( \gamma_\varepsilon(\Theta_\varepsilon(\Xi)) \to 1 \).

(b) (upper bounds) For the family of tests \( \psi_{\varepsilon} \) given by (6.13), there exists a constant \( C = C(\Sigma) > 0 \) such that if \( \lim \inf \sup_{\tau \in \Sigma} u_{\tau}^2(\Sigma)/\log \log \varepsilon^{-1} > C \), then, one has \( \alpha_\varepsilon(\psi_{\varepsilon}) = o(1) \) and \( \beta_\varepsilon(\psi_{\varepsilon}, \Theta_\varepsilon(\Sigma)) = o(1) \). These relations imply \( \beta_\varepsilon(\alpha, \Theta_\varepsilon(\Sigma)) = o(1) \) for all \( \alpha \in (0, 1) \) and \( \gamma_\varepsilon(\Theta_\varepsilon(\Xi)) = o(1) \).

The above theorem can be obtained by using the general minimax hypothesis testing framework considered in Section 2, the adaptive setup considered in Section 5, Theorem 7.2 in [17] and the family of tests in Section 7.4.1 [17], and noting that although the assumption in Theorem 6.2 (a) is slightly wider than the one in Theorem 7.2 in [17] it suffices for the lower bounds in the case under consideration. Its proof is omitted.

Using Theorem 6.2 the following relations, with adaptive critical radii given by (6.9), are easily obtained.

**Corollary 6.1** Under the formulation in Theorem 6.2, then

(a) (lower bounds) There exists constant \( d = d(\Sigma) > 0 \) such that if \( \lim \sup \sup_{\tau \in \Sigma} r_{\varepsilon}(\tau)/r_{\varepsilon,ad}(\kappa) < d \), then \( \beta_\varepsilon(\alpha, \Theta_\varepsilon(\Sigma)) \to 1 - \alpha \) and \( \gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \to 1 \).

(b) (upper bounds) There exists constant \( D = D(\Sigma) > 0 \) such that if \( \lim \inf \sup_{\tau \in \Sigma} r_{\varepsilon}(\tau)/r_{\varepsilon,ad}(\kappa) > D \), then \( \beta_\varepsilon(\alpha, \Theta_\varepsilon(\Sigma)) \to 0 \) and \( \gamma_\varepsilon(\Theta_\varepsilon(\Sigma)) \to 0 \).

(c) (adaptive separation rates) The adaptive separation rates \( r_{\varepsilon,ad}(\kappa), \kappa \in \Sigma, \) are given by (6.7).

**Remark 6.3** In view of Remark 6.1 similar rate optimality results and the same adaptive separation rates hold for the sequences \( a_\nu \sim 2^{\alpha j} \) and \( \sigma_\nu \sim 2^{\beta j}, \; j \in \mathbb{N}, \; \alpha > 0, \; \beta > 0 \), and the sequences \( a_\nu \sim 2^{\alpha j} \) and \( \sigma_\nu \sim 2^{\beta j}, \; j \in \mathbb{N}, \; \alpha > 0, \; \beta > 0 \).

**Remark 6.4** As pointed out in Section 6.1 for \( q = t \), the Besov norm \( | \cdot |_{a,q}^\alpha \) is equivalent to the Sobolev norm \( | \cdot |_{a,q} \). Furthermore, the adaptive rates in Theorem 6.2 do not depend on \( t \) associated with the Besov norm. Hence, the same rate-adaptive optimality results hold true for Sobolev classes of functions with \( H^q \)-ellipsoids, \( q \in (0, 2) \).
7 Appendix: Proofs

7.1 Proof of Theorem 4.1

To prove the theorem, we utilize techniques and results presented in Chapters 3 and 4 of [17] for minimax hypothesis testing in infinite dimensional settings. In particular, in order to get the lower bounds, we replace the minimax problem by a Bayesian one. Let \( \pi = \pi_\varepsilon \) be a prior (probability measure) on the sequence space such that \( \pi(\Theta(r)) = 1 \). Let \( P_{\varepsilon, \pi} = E_{\varepsilon}(P_{\varepsilon, \eta}) \) be the mixture over \( \pi \) and \( L_{\varepsilon, \pi} = dP_{\varepsilon, \pi}/dP_{\varepsilon, 0} = E_{\pi}dP_{\varepsilon, \eta}/dP_{\varepsilon, 0} \) be the likelihood ratio. Denote by \( \beta(0, P_0, \alpha), \gamma(0, P_0) \) the minimal type I error probability for a given level \( \alpha \) and the minimal total error probability, respectively, for testing the simple null hypothesis \( H_0 : P = P_0 \) against the simple alternative hypothesis \( H_1 : P = P_1 \), for the measure \( P \) of the observations. It is well known (see, e.g., [17], Section 2.4.2) that, for any \( \alpha \),

\[
\beta_\varepsilon(\Theta(r), \alpha) \geq \beta(0, P_{\pi}, \alpha), \quad \gamma_\varepsilon(\Theta(r)) \geq \gamma(0, P_{\pi, \eta}) = 1 - \frac{1}{2}\|P_{\varepsilon, 0} - P_{\varepsilon, \pi}\|_1, \quad (7.1)
\]

where \( \|P_{\varepsilon, 0} - P_{\varepsilon, \pi}\|_1 = E_{\varepsilon, 0}|L_{\varepsilon, \pi} - 1| \) is the variation distance. Therefore, as \( \varepsilon \to 0 \), in order for \( \gamma_\varepsilon(\Theta(r)) \to 1 \), it suffices \( \|P_{\varepsilon, 0} - P_{\varepsilon, \pi}\|_1 \to 0 \). Since \( \|P_{\varepsilon, 0} - P_{\varepsilon, \pi}\|_1^2 \leq \|P_{\varepsilon, 0} - P_{\varepsilon, \pi}\|_2^2 = E_{\varepsilon, 0}(L_{\varepsilon, \pi})^2 - 1 \), it suffices \( E_{\varepsilon, 0}(L_{\varepsilon, \pi})^2 \to 1 \). By (2.5), this leads to \( \beta_\varepsilon(\Theta(r)), \alpha \to 1 - \alpha \). Also the relation \( E_{\varepsilon, 0}(L_{\varepsilon, \pi})^2 = O(1) \) implies \( \liminf \gamma_\varepsilon(r) > 0 \) and \( \beta_\varepsilon(r, \alpha) > 1 - \alpha \) for any \( \alpha \in (0, 1) \), see [17], Proposition 2.12.

To proceed, we need a definition. A set \( V \) is called sign-symmetric (or orthosymmetric) if \( v = \{v_i\}_{i \in \mathbb{N}} \in V \), then \( \tilde{v} = \{\pm v_i\}_{i \in \mathbb{N}} \in V \) for all changes of signs of the coordinates. Observe now that \( \Theta(r) \) is a sign-symmetric set. For \( \eta \in \Theta(r), \) let us consider the product prior \( \pi_\varepsilon = \prod_{k \in \mathbb{N}} \pi_{\varepsilon, k} \), where \( \pi_{\varepsilon, k} = \frac{1}{2}(\delta_{\eta_k} + \delta_{-\eta_k}) \), \( k \in \mathbb{N} \), are symmetric two-points priors. Note that, \( \pi_\varepsilon(\Theta(r)) = 1 \) by the sign-symmetric condition. Let \( P_{\varepsilon, \eta_k} \) be the measure for \( y_k = \eta_k + \varepsilon k, \ \xi_k \sim N(0, 1), k \in \mathbb{N} \), and

\[
L_{\varepsilon, \eta_k}^{(k)} = dP_{\varepsilon, \eta_k}/dP_{\varepsilon, 0} = \exp((-\eta_k^2/2\varepsilon^2 + \eta_k y_k/\varepsilon^2)), \\
L_{\varepsilon, \pi_k}^{(k)} = E_{\pi_k}(L_{\varepsilon, \eta_k}^{(k)}) = e^{-\eta_k^2/2\varepsilon^2} \cosh(y_k \eta_k/\varepsilon^2).
\]

Simple calculations give \( E_{\varepsilon, 0}(L_{\varepsilon, \eta_k}^{(k)})^2 = \cosh(\eta_k^2/\varepsilon^2), k \in \mathbb{N} \). Therefore, using the inequality \( \cosh(\eta) \leq \exp(t^2/2) \) (which follows from Taylor’s expansions), we have

\[
E_{\varepsilon, 0}(L_{\varepsilon, \pi}^2) = E_{\varepsilon, 0} \prod_{k \in \mathbb{N}} E_{\pi_k}(L_{\varepsilon, \eta_k}^{(k)})^2 = E_{\varepsilon, 0} \prod_{k \in \mathbb{N}} E_{\pi_k}(L_{\varepsilon, \eta_k}^{(k)} \prod_{k \in \mathbb{N}} \cosh(\eta_k^2/\varepsilon^2) \leq \exp(-4 \sum_{k \in \mathbb{N}} \eta_k^4/2) \to 1,
\]

if \( u_\varepsilon^2(\eta) = \frac{1}{2}e^{-4 \sum_{k \in \mathbb{N}} \eta_k^4} \to 0 \), and in order to get the best \( \eta \in \Theta(r) \) we use the extreme problem (4.1). Also if \( u_\varepsilon^2(\eta) = O(1) \), then we get \( E_{\varepsilon, 0}(L_{\varepsilon, \pi}) = O(1) \). This completes part (1)(a) of the theorem.

In order to get the sharp lower bounds, take \( \pi \) that corresponds to the extreme sequence \( \tilde{\eta}_\varepsilon \) of the problem (4.1). In order to get the relation

\[
\beta_\varepsilon(P_{\varepsilon, 0}, P_{\varepsilon, \pi}, \alpha) \geq \Phi(H^{(\alpha)} - u_\varepsilon) + o(1), \quad \gamma_\varepsilon(P_{\varepsilon, 0}, P_{\varepsilon, \pi}) \geq 2\Phi(-u_\varepsilon/2) + o(1),
\]

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it suffices to show that
\[ \log(L_{ε,π}) = -u_{ε}^2/2 + u_{ε}ξ_{ε} + δ_{ε}, \] (7.2)
where \( u_{ε} = u_{ε}(N_{ε}) = \mathcal{O}(1) \) and \( ξ_{ε} \rightarrow ξ \sim \mathcal{N}(0, 1) \), \( δ_{ε} \rightarrow 0 \) in \( P_{ε,0} \)-probability (see [17], Section 4.3.1). Setting \( x_{ε,k} = y_{k}/ε, \ v_{ε,k} = \bar{y}_{k}/ε, \ k \in \mathbb{N} \), we note that \( \sum_{k \in \mathbb{N}} v_{ε,k}^4 = 2u_{ε}^2. \) We have
\[ \log(L_{ε,π}) = \sum_{k \in \mathbb{N}} (-v_{ε,k}^2/2 + \log(\cosh(x_{ε,k}v_{ε,k}))). \]

Using the inequality \( |\log(\cosh(t)) - t^2/2 + t^4/12| \leq Bt^6, \ t \in \mathbb{R} \), for some \( B > 0 \), we have [7.2] with \( δ_{ε} = δ_{ε,1} + δ_{ε,2} \), where
\[ ξ_{ε} = \frac{1}{2u_{ε}} \sum_{k \in \mathbb{N}} v_{ε,k}^2(2x_{ε,k} - 1), \quad δ_{ε,1} = \frac{1}{12} \sum_{k \in \mathbb{N}} v_{ε,k}^4(3 - x_{ε,k}^4), \quad |δ_{ε,2}| \leq δ_{ε,3} = B \sum_{k \in \mathbb{N}} v_{ε,k}^6x_{ε,k}^6. \]

Since \( x_{ε,k} \sim \mathcal{N}(0, 1), \ k \in \mathbb{N} \), under \( P_{ε,0} \), the relations \( δ_{ε,1} \rightarrow 0, \ δ_{ε,2} \rightarrow 0 \) follow from \( E_{ε,0}δ_{ε,1} = 0 \) and, for some constants \( B_l > 0, \ l = 1, 2, 3, 4, \)
\[ \text{Var}_{ε,0}δ_{ε,1} = B_1 \sum_{k \in \mathbb{N}} v_{ε,k}^8 \leq B_2u_{ε}^2w_0^2 = o(1), \quad \text{E}_{ε,0}δ_{ε,3} = B_3 \sum_{k \in \mathbb{N}} v_{ε,k}^6 \leq B_4u_{ε}^2w_0 = o(1), \]

since \( δ_{ε,3} \geq 0. \) Also, \( E_{ε,0}ξ_{ε} = 0, \text{Var}_{ε,0}ξ_{ε} = 1 \) and the asymptotic \( \mathcal{N}(0, 1) \) normality of \( ξ_{ε} \) under \( P_{ε,0} \) follows from Lyapunov condition: if \( z_{ε,k} = v_{ε,k}^2(x_{ε,k}^2 - 1) \) then
\[ \sum_{k \in \mathbb{N}} E_{ε,0}z_{ε,k}^2/(\sum_{k} E_{ε,0}z_{ε,k}^2)^2 \leq Bw_0^2 \rightarrow 0 \]
for some \( B > 0. \) This completes the lower bounds of part (1)(b) of the theorem.

In order to obtain the upper bounds let us calculate the expectations and variances of the statistics \( t_{ε} \) of the form [4.3] for a sequence \( w_k = \tilde{y}_k/u_{ε}, \ w_k \geq 0, \ k \in \mathbb{N}, \sum_{k \in \mathbb{N}} w_k^2 = 1/2; \ w_0 = \sup_{k \in \mathbb{N}} w_k \in (0, 2^{-1/2}), \) where \( \tilde{y} \) and \( u_{ε}^2 \) are the extreme sequence and the extreme value in [5.1]. We have
\[ E_{ε,0}t_{ε} = 0, \quad \text{Var}_{ε,0}t_{ε} = 1, \quad E_{ε,0}t_{ε} = ε^{-2} \sum_{k \in \mathbb{N}} w_k \tilde{y}_k^2 : = h_{ε}(η), \] (7.3)
\[ \text{Var}_{ε,0}t_{ε} = 1 + 4ε^{-2} \sum_{k \in \mathbb{N}} w_k^2 \tilde{y}_k^2, \quad 1 \leq \text{Var}_{ε,0}t_{ε} \leq 1 + 4h_{ε}(η)w_0. \] (7.4)

The key point is the following lemma.

**Lemma 7.1**
\[ \inf_{η \in \Theta(r_{ε})} h_{ε}(η) = u_{ε}. \]

**Proof.** Denote \( τ_k = \eta_k^2, \ \tilde{τ}_k = \tilde{y}_k^2, \ k \in \mathbb{N}, \ w = \{w_k\}_{k \in \mathbb{N}} \) and consider the set \( \Upsilon = \{τ = \{τ_k\}_{k \in \mathbb{N}} : η = \{η_k\}_{k \in \mathbb{N}} \in \Theta(r_{ε})\}. \) Observe that \( \Upsilon \) is a convex set (i.e., \( \Theta(r_{ε}) \) is a quadratically convex set) in the sequence space \( l_2 \) and \( h_{ε}(η) = ε^{-2}(τ, w) = ε^{-2}(τ, \tilde{τ})/(\sqrt{2}\|\tilde{τ}\|), \) where \((·,·)\)
and \(\|·\|\) stand, respectively, for the scalar product and the norm in \( l_2 \). We have to check that \( G := \inf_{τ \in \Upsilon} ||τ||^2, \)
where
\[ \bar{τ} \in \Upsilon, \quad ||\bar{τ}|| = \inf_{τ \in \Upsilon} ||τ||. \] (7.5)

First observe that \( G \leq ||\tilde{τ}||^2 \) since \( \tilde{τ} \in \Upsilon, \) and it suffices to check that \( G \geq ||\tilde{τ}||^2 \). Suppose there exists \( ζ_0 \in \Upsilon \) such that \( (ζ_0, \bar{τ}) < ||\tilde{τ}||^2, \) which is equivalent to \( r := ((ζ_0 - \bar{τ}), \bar{τ}) < 0. \)
Consider the interval \( \tau(t) = \tilde{\tau} + t(\tau_0 - \tilde{\tau}) \), \( \tau(t) \in \Upsilon \) for all \( t \in [0,1] \), by convexity of \( \Upsilon \). We have, for \( t \in (0,1) \) small enough,
\[
\|\tau(t)\|^2 = \|\tilde{\tau}\|^2 + 2t\tau + t^2\|\tau_0 - \tilde{\tau}\|^2 < \|\tilde{\tau}\|^2.
\]
This contradicts to (7.5). The lemma now follows. □

Return to the proof of the upper bounds. Let \( u_\varepsilon \to \infty \). Then, applying the Chebyshev inequality and (7.3) we have, for \( H \to \infty \),
\[
\alpha_\varepsilon(\psi_{\varepsilon,H}) = P_{\varepsilon,0}(t_\varepsilon > H) \leq \frac{\text{Var}_{\varepsilon,0}t_\varepsilon}{H^2} \to 0.
\]

For the alternative hypothesis, applying the Chebyshev inequality once again, (7.4) and Lemma 7.1, we have, for \( H = cu_\varepsilon \), \( c \in (0,1) \) and uniformly over \( \eta \in \Theta(r_\varepsilon) \),
\[
\beta_\varepsilon(\psi_{\varepsilon,H,\eta}) = P_{\varepsilon,\eta}(t_\varepsilon \leq H) = P_{\varepsilon,\eta}(h_\varepsilon(\eta) - t_\varepsilon \geq h_\varepsilon(\eta) - H) \leq \frac{\text{Var}_{\varepsilon,\eta}t_\varepsilon}{(h_\varepsilon(\eta) - H)^2} \leq \frac{1 + 4w_0h_\varepsilon(\eta)}{(1 - c)h_\varepsilon(\eta))^2} \to 0.
\]
This completes part (2) of the theorem.

Let \( u_\varepsilon \sim 1 \) and \( w_0 = o(1) \). Observe that \( t_\varepsilon = \xi_\varepsilon \) where \( \xi_\varepsilon \) is the statistic from the proof of the lower bounds, and it was shown that \( t_\varepsilon \) is asymptotically standard Gaussian under \( P_{\varepsilon,0} \). This yields \( \alpha_\varepsilon(\psi_{\varepsilon,H}) = \Phi(-H) + o(1) \). In order to evaluate type II error probability, let us divide the set \( \Theta(r_\varepsilon) \) into two sets \( \Theta_{\varepsilon,1} = \{ \eta \in \Theta(r_\varepsilon) : h_\varepsilon(\eta) < h_\varepsilon \} \) and \( \Theta_{\varepsilon,2} = \{ \eta \in \Theta(r_\varepsilon) : h_\varepsilon(\eta) > h_\varepsilon \} \), where \( h_\varepsilon \to \infty \), \( h_\varepsilon w_0 \to 0 \). Similarly to evaluation above, we get \( \sup_{\eta \in \Theta_{\varepsilon,2}} \beta_\varepsilon(\psi_{\varepsilon,H,\eta}) \to 0 \) for any \( H = O(1) \). Let \( \eta \in \Theta_{\varepsilon,1} \). By (7.4), we have \( \text{Var}_{\varepsilon,\eta}t_\varepsilon = 1 + o(1) \). Observe that the statistics \( t_\varepsilon = (t_\varepsilon - h_\varepsilon(\eta))/\sqrt{\text{Var}_{\varepsilon,\eta}t_\varepsilon} \) are asymptotically standard Gaussian under \( P_{\varepsilon,\eta} \). This follows from Lyapunov’s condition, since \( t_\varepsilon - h_\varepsilon(\eta) = \sum_{k \leq KL} \hat{t}_{\varepsilon,k} \), where \( \hat{t}_{\varepsilon,k} \) are independent and \( P_{\varepsilon,\eta} \)-distributed as \( w_k(\xi_k^2 - 1 + 2v_k\xi_k) \), where \( v_k = \eta_k/\varepsilon \), \( \xi_k \sim \mathcal{N}(0,1) \), \( k \in \mathbb{N} \). Therefore, one has, for some constant \( B > 0 \), uniformly over \( \eta \in \Theta_{\varepsilon,1} \),
\[
E_{\varepsilon,\eta}\hat{t}_{\varepsilon,k}^4 \leq B(w_k^4 + v_k^4), \quad \sum_{k \in \mathbb{N}} (w_k^4 + v_k^4) \leq w_0^2 \sum_{k \in \mathbb{N}} w_k^2 + w_0^2 \left( \sum_{k \in \mathbb{N}} w_k v_k^2 \right) \leq w_0^2/2 + w^2 h_\varepsilon \to 0.
\]
It also follows from the asymptotic normality of \( \hat{t}_\varepsilon \) that, uniformly over \( \eta \in \Theta_{\varepsilon,1} \),
\[
\beta_\varepsilon(\psi_{\varepsilon,H,\eta}) = P_{\varepsilon,\eta}(t_\varepsilon \leq H) = P_{\varepsilon,\eta}(t_\varepsilon \leq (H - h_\varepsilon(\eta))/\sqrt{\text{Var}_{\varepsilon,\eta}t_\varepsilon} = \Phi(H - h_\varepsilon(\eta)) + o(1).
\]
By Lemma 7.1 and evaluation over \( \eta \in \Theta_{\varepsilon,2} \) above, we get
\[
\beta(\Theta(r_\varepsilon),\psi_{\varepsilon,H}) = \Phi(H - \inf_{\eta \in \Theta(r_\varepsilon)} h_\varepsilon(\eta)) + o(1) = \Phi(H - u_\varepsilon) + o(1).
\]
Taking \( H = H^{(\alpha)} \) and \( H = u_\varepsilon/2 \), it completes the upper bounds for part (1) (b) of the theorem. The theorem now follows.

### 7.2 Proof of Theorem 4.2

Set \( A = m^{-\alpha} \) in (4.4); the quantity \( m = m_\varepsilon \) determines the efficient dimension in the problem. Then, the extreme sequence (4.4) in the extreme problem (4.1) takes the form
\[
\hat{q}_k^2 = z_0^2 k^{2\beta} (1 - (k/m)^2\alpha)^2, \quad 1 \leq k \leq m,
\]
(7.6)
while the equations for \( z_0 = z_{0,\varepsilon}, \ m = m_\varepsilon \) and \( u_\varepsilon \) take the form (4.7), (4.8), where

\[
\begin{align*}
J_1 & = \frac{1}{m} \sum_{1 \leq k \leq m} \left( \frac{k}{m} \right)^{4\beta} \left( 1 - \left( \frac{k}{m} \right)^{2\alpha} \right), \\
J_2 & = \frac{1}{m} \sum_{1 \leq k \leq m} \left( \frac{k}{m} \right)^{2\alpha} \left( 1 - \left( \frac{k}{m} \right)^{2\alpha} \right), \\
J_0 & = J_1 - J_2 = \frac{1}{m} \sum_{1 \leq k \leq m} \left( \frac{k}{m} \right)^{4\beta} \left( 1 - \left( \frac{k}{m} \right)^{2\alpha} \right)^2.
\end{align*}
\]

We consider the situation \( m \to \infty \) and \( \varepsilon \to 0 \) as \( \varepsilon \to 0 \). Let us now find the asymptotics of the sums \( J_1, J_2 \) and \( J_0 \) as \( m \to \infty \). Replacing the sums \( J_1, J_2 \) and \( J_0 \) by integrals, after some calculations, we have, as \( m \to \infty \),

\[
\begin{align*}
J_1 & \sim \frac{2\alpha}{(4\beta + 1)(4\beta + 2\alpha + 1)} := d_1, \\
J_2 & \sim \frac{2\alpha}{(4\alpha + 4\beta + 1)(4\beta + 2\alpha + 1)} := d_2, \\
J_0 & \sim \frac{8\alpha^2}{(4\beta + 1)(4\alpha + 4\beta + 1)(4\beta + 2\alpha + 1)} := d_0.
\end{align*}
\]

These yield

\[
r_\varepsilon \sim c_1 \varepsilon^{-\alpha} m^{-\alpha}, \quad u_\varepsilon^2 \sim c_2 \varepsilon^{-4} r_\varepsilon^{-4\alpha+4\beta+1})/\alpha, \quad \text{(7.7)}
\]

where \( c_1 = (d_1/d_2)^{1/(2\alpha)}, \ c_2 = (d_2/d_1)^{1/(4\beta+1)}/(2d_1^2) \). Hence, the value \( u_\varepsilon \) of the extreme problem (4.11) and the efficient dimensions \( m = m_\varepsilon \) satisfy

\[
u_\varepsilon \sim c_2 \varepsilon^{-2} (c_1/m)^{(4\alpha+4\beta+1)/2}, \quad m \sim c_1 c_2 \varepsilon^{-4} u_\varepsilon^{-2} / c_2^{-1/(4\alpha+4\beta+1)} \text{.} \quad \text{(7.8)}
\]

Observe also that, for the extreme sequence determined by (7.6), one has

\[
w_0 = \frac{\max_{1 \leq k \leq m} \eta_k^2}{\sqrt{2} \sum_{k=1}^m \eta_k^4} \leq \frac{B z_0^2 m^{2\beta}}{\varepsilon^2 m^{2\beta+1/2}} \times m^{-1/2} \to 0, \quad B > 0.
\]

The theorem now follows on applying Theorem 4.1.

### 7.3 Proof of Theorem 4.3

To prove the theorem, we need the following proposition. (Note that its validity is true for a wider class of sequences \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{\sigma_k\}_{k \in \mathbb{N}} \), which cover all ill-posed inverse problems of interest.)

**Proposition 7.1** Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{\sigma_k\}_{k \in \mathbb{N}} \) be positive increasing sequences. Assume that there exists \( B > 1, a > 0, \ m = m_\varepsilon \in \mathbb{N}, \ v_0 > 0 \) such that, as \( 0 < \varepsilon < \varepsilon_0 \),

\[
r_\varepsilon a_{m+1} \geq B, \quad u_\varepsilon \leq \frac{a \varepsilon^2}{\varepsilon^2 \sqrt{m \sigma_m^2}}. \quad \text{(7.9)}
\]

Set

\[
h_{m}(\eta) = \frac{1}{\varepsilon^2 \sqrt{2m}} \sum_{k=1}^{m} \eta_k^2, \quad h_{m} = \inf_{\eta \in \Theta_\varepsilon} h_{m}(\eta). \quad \text{(7.10)}
\]

Then, there exists \( b = b(B, a) > 0 \) such that \( h_{m} \geq bu_\varepsilon \) as \( 0 < \varepsilon < \varepsilon_0 \).
Proof. By definition of $\Theta_\varepsilon$, since the sequences $\{a_k\}_{k \in \mathbb{N}}$ and $\{\sigma_k\}_{k \in \mathbb{N}}$ increase as $k \to \infty$, and by (7.9), we have, for $\eta \in \Theta(\varepsilon)$,

$$
\sum_{k=1}^m \eta_k^2 \geq \frac{1}{\sigma_m^2} \sum_{k=1}^m \sigma_k^2 \eta_k^2 \geq \frac{1}{\sigma_m^2} \left( r_\varepsilon^2 - \sum_{k=m+1}^{\infty} \sigma_k^2 \eta_k^2 \right) \geq \frac{1}{\sigma_m^2} \left( r_\varepsilon^2 - \frac{1}{a_m^2} \sum_{k=m+1}^{\infty} a_k^2 \sigma_k^2 \eta_k^2 \right)
$$

$$
\geq \frac{1}{\sigma_m^2} \left( r_\varepsilon^2 - \frac{1}{a_m^2} \right) = \frac{b_1 r_\varepsilon^2}{\sigma_m^2}, \quad b_1 = 1 - B^{-2} > 0. \quad (7.11)
$$

Therefore, we have

$$
h_\tilde{m} \geq \frac{b_1 r_\varepsilon^2}{\varepsilon^2 \sqrt{2m} \sigma_m^2} \geq b u_\varepsilon, \quad b = b_1/(\sqrt{2a}). \quad (7.12)
$$

The proposition now follows.

We are now ready to prove Theorem 4.3. By the asymptotic normality of $t_\tilde{m}$ under $P_{0,\varepsilon}$ as $\tilde{m} \to \infty$ (see [17], Lemma 3.1), we have $\alpha(\psi_{0,H}) = \Phi(-H) + o(1) \to 0$ as $H = H_{\varepsilon} \to \infty$.

In order to evaluate type II error probability for the test $\psi_{H,H}$, take $h_{\tilde{m}}(\eta)$ and $h_{\tilde{m}}$ as in (7.10). By the asymptotic normality of $t_\tilde{m} - h_{\tilde{m}}(\eta)$ under $P_{H,\varepsilon}$ as $\tilde{m} \to \infty$ (see [17], Lemma 3.1), we have

$$
\beta_\varepsilon(\psi_{H,H}, \eta) \leq \Phi(H - h_\tilde{m}(\eta)) + o(1), \quad \beta_\varepsilon(\psi_{H,H}, \Theta_\varepsilon) \leq \Phi(H - h_{\tilde{m}}) + o(1). \quad (7.13)
$$

Proposition 7.1 implies that $\beta_\varepsilon(\psi_{H,H}, \Theta_\varepsilon) \to 0$ as $H_{\varepsilon} \leq (c + o(1)) u_\varepsilon \to \infty$, $c \in (0, b)$. The theorem now follows.

### 7.4 Proof of Theorem 4.4

We first consider the “standard” case $q = 2$. Let the efficient dimension $m = m_\varepsilon$ be determined by $A = \exp(-2\alpha m)$ in (4.4). Then, the extreme sequence (4.4) in the extreme problem (4.1) takes the form

$$
\eta_k^2 = z_0^2 \exp(2\beta k)(1 - \exp(2\alpha(k - m))), \quad 1 \leq k \leq m, \quad (7.14)
$$

while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and $u_\varepsilon$ take the form (4.7), (4.8) where

$$
J_1 = \sum_{1 \leq k \leq m} \exp(4\beta k)(1 - \exp(2\alpha(k - m))),
$$

$$
J_2 = \exp(-2\alpha m) \sum_{1 \leq k \leq m} \exp((2\alpha + 4\beta)k)(1 - \exp(2\alpha(k - m))),
$$

$$
J_0 = J_1 - J_2 = \sum_{1 \leq k \leq m} \exp(4\beta k)(1 - \exp(2\alpha(k - m)))^2.
$$

We consider the situation $m \to \infty$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us now find the asymptotics of the sums $J_1$, $J_2$ and $J_0$ as $m \to \infty$. After some calculations, we have, as $m \to \infty$,

$$
J_1 \asymp J_2 \asymp J_0 \asymp \exp(4\beta m)
$$

and, hence, using (4.8), we get the relations

$$
r_\varepsilon \asymp \exp(-\alpha m), \quad u_\varepsilon^2 \asymp \varepsilon^{-4} r_\varepsilon^4 (\alpha + \beta)/\alpha. \quad (7.15)
$$
Hence, the value $u_\varepsilon$ of the extreme problem (4.11) and the efficient dimensions $m = m_\varepsilon$ satisfy
\[ u_\varepsilon \asymp \varepsilon^{-2} \exp(-2(\alpha + \beta)m), \quad m = \frac{2 \log(\varepsilon^{-1}) - \log(u_\varepsilon)}{2(\alpha + \beta)} + O(1). \] (7.16)

Hence, the “standard” case $q = 2$ for the theorem follows on applying Theorem 4.4.

Consider now the “sparse” case $q \in (0, 2)$. The embedding (4.17) yields $\gamma(\psi, \Theta_q(r_\varepsilon)) \leq \gamma(\psi, \Theta_2(r_\varepsilon))$. Therefore, it suffices to establish the lower bounds. Take $m = \max\{k : r_\varepsilon \exp(ak) \leq 1\}$, and consider the vector $\eta_m$ that contains only one non-zero coordinate, the value $z_n = r_\varepsilon \exp(-\beta n)$ at position $m$. One can easily check that $\eta_m \in \Theta_q(r_\varepsilon)$ for any $q \in (0, 2)$. Therefore, one cannot distinguish between $H_0$ and $H_1$ if $z_\alpha = o(\varepsilon)$, which is equivalent to $r_\varepsilon = o(r_\varepsilon^*)$, where $r_\varepsilon^*$ is obtained by combining $u_\varepsilon \asymp 1$ and (7.15). In view of the above and the results for the “standard” case $q = 2$, the “sparse” case $q = 2$ for the theorem also follows. Hence, the theorem follows.

7.5 Proof of Theorem 4.5

We first consider the “standard” case $q = 2$. Let the efficient dimension $m = m_\varepsilon$ be determined by $A = m^{-2\alpha}$ in (4.4). Then, the extreme sequence (4.4) in the extreme problem (4.1) takes the form
\[ z_{\alpha k}^2 = z_k^2 \exp(2\beta k) \left(1 - \left(\frac{k}{m}\right)^{2\alpha}\right)_+, \quad 1 \leq k \leq m, \] (7.17)
while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and $u_\varepsilon$ take the form (4.7), (4.8) where
\[ J_1 = \sum_{1 \leq k \leq m} \exp(4k\beta) \left(1 - \left(\frac{k}{m}\right)^{2\alpha}\right), \]
\[ J_2 = \sum_{1 \leq k \leq m} \exp(4k\beta) \left(\frac{k}{m}\right)^{2\alpha} \left(1 - \left(\frac{k}{m}\right)^{2\alpha}\right), \]
\[ J_0 = J_1 - J_2 = \sum_{1 \leq k \leq m} \exp(4k\beta) \left(1 - \left(\frac{k}{m}\right)^{2\alpha}\right)^2. \]

We consider the situation $m \to \infty$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us now find the asymptotics of the sums $J_1$, $J_2$ and $J_0$ as $m \to \infty$. Take $\delta > 0$, $\delta \to 0$, such that $m\delta \to \infty$, $m\delta \gg \log(m)$, $m\delta^2 \to 0$ as $m \to \infty$. Set $k = m - l$. Then,
\[ J_1 = \sum_{1 \leq k \leq m, \ l = m - k} e^{4\beta(m-l)} \left(1 - \left(1 - \frac{l}{m}\right)^{2\alpha}\right) = e^{4m\beta} \sum_{m(1-\delta) \leq k \leq m, \ l = m - k} e^{-4l\beta} \left(\frac{2l\beta}{m} + O(l^2/m^2)\right) + e^{4m\beta} \sum_{1 \leq k < m(1-\delta), \ l = m - k} e^{-4l\beta} \left(1 - \left(1 - \frac{l}{m}\right)^{2\alpha}\right) \]
\[ := \exp(4m\beta)(A + B). \]

For the term $A$, we have
\[ A = \frac{2\alpha}{m} \left(\sum_{m(1-\delta) \leq k \leq m, \ l = m - k} le^{-4l\beta} + m^{-1}O\left(\sum_{m(1-\delta) \leq k \leq m, \ l = m - k} l^2e^{-4l\beta}\right)\right) = \frac{2\alpha}{m} \left(A_1 + O(m^{-1})\right). \]
Set \( t = \exp(-4\beta) \). Then, the sum \( A_1 \) can be rewritten in the form

\[
A_1 = A_1(m, \beta) = \exp(-4\beta) \left( \sum_{m(1-\delta) \leq k \leq m} t^k \right)' = 1,
\]

where \( ()' \) denotes differentiation with respect to \( t \). Therefore,

\[
A = \frac{2\alpha A_1(m, \beta)}{m} + O(m^{-2}) \asymp m^{-1}.
\]

For the term \( B \), we have

\[
B = \sum_{1 \leq k < m(1-\delta), \ l = m-k} \exp(-4l\beta) \left( 1 - \left( 1 - \frac{l}{m} \right)^{2\alpha} \right) = O(\exp(-4\beta m\delta)) = o(1/m^k), \ k \in \mathbb{N}.
\]

Hence, combining the two terms, we get the asymptotics

\[
J_1 = \frac{2\alpha \exp(4\beta m) A_1(m, \beta)}{m} (1 + O(m^{-1})) \asymp \frac{\exp(4\beta m)}{m}.
\]

For the asymptotics of \( J_2 \), let us first rewrite it in the form

\[
J_2 = -J_1 + \sum_{1 \leq k \leq m} \exp(4\beta k) \left( 1 - \left( \frac{k}{m} \right)^{4\alpha} \right) = -J_1(\alpha, \beta) + J_1(2\alpha, \beta).
\]

The asymptotics of \( J_1(2\alpha, \beta) \) are studied similar to ones of \( J_1 = J_1(\alpha, \beta) \), and we get

\[
J_2 = 2\alpha m^{-1} e^{4\beta m} A_1(m, \beta)(1 + O(m^{-1})) \sim J_1.
\]

For the asymptotics of \( J_0 \), we use the second order Taylor's formula to get

\[
\left( 1 - \frac{l}{m} \right)^{2\alpha} = 1 - \frac{2\alpha l}{m} + \frac{\alpha(2\alpha - 1)l^2}{m^2} + O\left( \frac{l^3}{m^3} \right).
\]

Repeating the considerations that we used for \( J_1 \), \( l = m - k \) we have for

\[
J_0 = \exp(4m\beta) \left( \frac{4\alpha^2}{m^2} \sum_{m(1-\delta) \leq k \leq m} \exp(-4l\beta) \left( l^2 + O\left( \frac{l^3}{m} \right) \right) \right) + O(\exp(4\beta m(1 - \delta)))
\]

\[
= \exp(4m\beta) \left( \frac{4\alpha^2}{m^2} \sum_{m(1-\delta) \leq k \leq m} \exp(-4l\beta) l^2 + O(m^{-3}) \right) + o(\exp(4\beta m) m^{-3})
\]

\[
= \exp(4m\beta) \left( \frac{4\alpha^2}{m^2} A_2 + O(m^{-3}) \right) + o(\exp(4m\beta) m^{-3}).
\]

Taking derivatives as in the calculation of \( A_1 \), we get \( A_2 = A_2(m, \beta) \asymp 1 \), which implies

\[
J_0 \asymp \frac{\exp(4\beta m)}{m^2}.
\]

Thus, using (4.8), we obtain the following asymptotics

\[
J_1 \sim J_2 \asymp \frac{\exp(4\beta m)}{m}, \quad J_0 \asymp \frac{\exp(4\beta m)}{m^2}, \quad r^2 = -2\alpha \left( \frac{J_0}{J_1} \right) = m^{-2\alpha} \left( 1 + \frac{J_0}{J_1} \right) = m^{-2\alpha} \left( 1 + \frac{B}{m} \right), \quad B \asymp 1,
\]

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and, hence, we get the relations
\[ r_{\varepsilon}^{-1/\alpha} = m + O(1), \quad u_{\varepsilon}^2 \asymp \varepsilon^{-4} r_{\varepsilon}^4 \exp(-4\beta r_{\varepsilon}^{-1/\alpha}). \] (7.18)

Hence, the value \( u_\varepsilon \) of the extreme problem \((4.1)\) and the efficient dimensions \( m = m_\varepsilon \) satisfy
\[ u_\varepsilon^2 \sim \varepsilon^{-4} m^{-4\alpha} \exp(-4m\beta A_2/(2A_1^2)), \quad m \sim \frac{2\log(\varepsilon^{-1}) - 2\alpha \log \log(\varepsilon^{-1}) - (\log(u_\varepsilon))}{2\beta} + D, \]
(7.19)
where \( D \asymp 1 \) hold true uniformly over \((\alpha, \beta) \in \Sigma\) for any compact set \( \Sigma \subset (0, \infty) \times (0, \infty)\). Hence, the “standard” case \( q = 2 \) for the theorem follows on applying Theorem 4.1.

Consider now the “sparse” case \( q \in (0, 2)\). In view of the embedding \((4.17)\), it suffices to establish the lower bounds. Take \( m = \max\{k : r_k \varepsilon^\alpha \leq 1\} \), \( m = r^{-1/\alpha} + O(1) \), and consider the vector \( \eta_m \) that contains only one non-zero coordinate, the value \( z_n = r_\varepsilon \exp(-\beta n) \) at position \( m \). One can easily check that \( \eta_m \in \Theta_q(r_\varepsilon) \) for any \( q > 0 \). Therefore, one cannot distinguish between \( H_0 \) and \( H_1 \) if \( z_n = o(\varepsilon) \), which is equivalent to
\[ r_\varepsilon/\varepsilon = o(\exp(-\beta r_\varepsilon^{-1/\alpha} + O(1))). \]

However, this is equivalent to \( u_\varepsilon \to 0 \), where \( u_\varepsilon \) is determined by \((7.18)\). In view of the above and the results for the “standard” case \( q = 2 \), the “sparse” case \( q = 2 \) for the theorem also follows. Hence, the theorem follows.

### 7.6 Proof of Theorem 4.6

Let the efficient dimension \( m = m_\varepsilon \) be determined by \( A = \exp(-2m\alpha) \) in \((4.4)\). Then, the extreme sequence \((4.4)\) in the extreme problem \((4.1)\) takes the form
\[ \eta_{\varepsilon}^2 = z_0^2 \varepsilon^\beta (1 - \exp(2\alpha(k - m)))_+, \quad 1 \leq k \leq m, \] (7.20)
while the equations for \( z_0 = z_{0,\varepsilon}, \ m = m_\varepsilon \) and \( u_\varepsilon \) take the form \((4.7), (4.8)\), where
\[
\begin{align*}
J_1 &= \sum_{1 \leq k \leq m} k^{4\beta} (1 - \exp(-2\alpha(m - k))), \\
J_2 &= \sum_{1 \leq k \leq m} k^{4\beta} \exp(-2\alpha(m - k))(1 - \exp(-2\alpha(m - k))), \\
J_0 &= J_1 - J_2 = \sum_{1 \leq k \leq m} k^{4\beta} (1 - \exp(-2\alpha(m - k)))^2.
\end{align*}
\]

We consider the situation \( m \to \infty \) and \( r_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Let us now find the asymptotics of the sums \( J_1, J_2 \) and \( J_0 \) as \( m \to \infty \). We have
\[
J_1 = \sum_{1 \leq k \leq m} k^{4\beta} - \sum_{1 \leq k \leq m} k^{4\beta} \exp(-2\alpha(m - k)) = A - B, \] (7.21)
where
\[
A = \sum_{1 \leq k \leq m} k^{4\beta} = m^{4\beta + 1} \sum_{1 \leq k \leq m} \left( \frac{k}{m} \right)^{4\beta} \frac{1}{m} \sim m^{4\beta + 1} \frac{1}{4\beta + 1}.
\]
Let us now evaluate the term \( B \) in the sum \((7.21)\). Let \( k = m - l \). Then,
\[
B = \sum_{1 \leq k \leq m} k^{4\beta} \exp(-2\alpha(m - k)) = m^{4\beta} \sum_{1 \leq k \leq m, \ l=m-k} (1 - l/m)^{4\beta} \exp(-2\alpha l) = m^{4\beta} B_1.
\]
Let $\alpha > 0$. Using the Taylor’s formula, and since the series $\sum_{l=1}^{\infty} l^k \exp(-2\alpha l)$, $k = 1, 2$, converges, we get

$$B_1 = \sum_{1 \leq k \leq m, \ l = m - k} \left( 1 - \frac{4\beta l}{m} + O \left( \frac{l^2}{m^2} \right) \right) \exp(-2\alpha l) \times 1.$$  

Therefore, combining the terms $A, B$ and $B_1$, we get

$$J_1 \sim \frac{m^{4\beta + 1}}{4\beta + 1}.$$  

(7.22)

Similarly, for $J_2$, letting $k = m - l$, we have

$$J_2 = m^{4\beta} \sum_{1 \leq k \leq m, \ l = m - k} \left( 1 - \frac{l}{m} \right)^4 \exp(-2\alpha l)(1 - \exp(-2\alpha l)) \propto m^{4\beta}.$$  

(7.23)

By (7.22) and (7.23), we have

$$J_0 = J_1 - J_2 \sim J_1 \sim c_1 m^{4\beta + 1}, \ J_2 \propto m^{4\beta},$$  

(7.24)

where $c_1 = 1/(4\beta + 1)$. Hence using (4.8), we get the relations

$$r_\varepsilon \propto m^{1/2} \exp(-\alpha m), \ u_\varepsilon^2 \sim d_1 (r_\varepsilon/\varepsilon)^4 m^{-(4\beta + 1)} \sim d_2 (r_\varepsilon/\varepsilon)^4 (\log r_\varepsilon^{-1})^{-(4\beta + 1)},$$  

(7.25)

where $d_1 = 1/(2c_1)$ and $d_2 = d_1 \alpha^{4\beta + 1}$. Hence, the value $u_\varepsilon$ of the extreme problem (4.1) and the efficient dimensions $m = m_\varepsilon$ satisfy

$$u_\varepsilon^2 \propto \varepsilon^{-4} \exp(-4\alpha m) m^{-(4\beta - 1)}, \ m \sim \log(r_\varepsilon^{-1/\alpha}).$$  

(7.26)

Observe also that, for the extreme sequence determined by (7.20), one has

$$w_0 = \frac{\max_{1 \leq k \leq m} \eta_k^2}{\sqrt{2 \sum_{k=1}^{m} \eta_k^2}} \leq \frac{B \varepsilon^2 m^{2\beta}}{\varepsilon^2 \eta_0^{2\beta + 1/2}} \propto m^{-1/2} \rightarrow 0.$$  

Hence, the theorem follows on applying Theorem 4.1.

### 7.7 Proof of Theorem 4.7

In view of the embedding (4.17), we only need to consider the case $q = 2$. It was shown in the proof of Theorem 4.3 that $\alpha(\psi_{\varepsilon,H}) \rightarrow 0$ as $H \rightarrow \infty$. In order to evaluate the type II error probability, it suffices to consider only the case where $u_\varepsilon(r_\varepsilon(\alpha, \beta)) = o(\log(\varepsilon^{-1}))$, uniformly over $(\alpha, \beta) \in \Sigma$. Similar to the proof of Theorem 4.3, we have the relation (7.13), and it suffices to evaluate the quantity

$$h_\varepsilon(\alpha, \beta) = \inf_{\eta \in \Theta_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha, \beta))} \frac{1}{\varepsilon^2 \sqrt{2m}} \sum_{k=1}^{m} \eta_k^2.$$  

Since $D m_\varepsilon(\alpha, \beta) \geq \tilde{m} \geq m_\varepsilon(\alpha, \beta)$ and $D = C(1 + o(1))/c > 0$, we have

$$h_\varepsilon(\alpha, \beta) \geq d h_\varepsilon^*(\alpha, \beta), \quad d = D^{-1/2}(1 + o(1)),$$  

where $h_\varepsilon^*(\alpha, \beta) = \inf_{\eta \in \Theta_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha, \beta))} \frac{1}{\varepsilon^2 \sqrt{2m}} \sum_{k=1}^{m} \eta_k^2$.

By (7.22) and (7.23), we have

$$J_0 = J_1 - J_2 \sim J_1 \sim c_1 m^{4\beta + 1}, \ J_2 \propto m^{4\beta},$$  

(7.24)

where $c_1 = 1/(4\beta + 1)$. Hence using (4.8), we get the relations

$$r_\varepsilon \propto m^{1/2} \exp(-\alpha m), \ u_\varepsilon^2 \sim d_1 (r_\varepsilon/\varepsilon)^4 m^{-(4\beta + 1)} \sim d_2 (r_\varepsilon/\varepsilon)^4 (\log r_\varepsilon^{-1})^{-(4\beta + 1)},$$  

(7.25)

where $d_1 = 1/(2c_1)$ and $d_2 = d_1 \alpha^{4\beta + 1}$. Hence, the value $u_\varepsilon$ of the extreme problem (4.1) and the efficient dimensions $m = m_\varepsilon$ satisfy

$$u_\varepsilon^2 \propto \varepsilon^{-4} \exp(-4\alpha m) m^{-(4\beta - 1)}, \ m \sim \log(r_\varepsilon^{-1/\alpha}).$$  

(7.26)

Observe also that, for the extreme sequence determined by (7.20), one has

$$w_0 = \frac{\max_{1 \leq k \leq m} \eta_k^2}{\sqrt{2 \sum_{k=1}^{m} \eta_k^2}} \leq \frac{B \varepsilon^2 m^{2\beta}}{\varepsilon^2 \eta_0^{2\beta + 1/2}} \propto m^{-1/2} \rightarrow 0.$$  

Hence, the theorem follows on applying Theorem 4.1.
\[ h^*_\varepsilon(\alpha, \beta) = \frac{1}{\varepsilon^2 \sqrt{2\tilde{m}(\alpha, \beta)}} \inf_{\eta \in \mathcal{O}_{\varepsilon, \alpha, \beta}(r_{\varepsilon}(\alpha, \beta))} \sum_{k=1}^{\tilde{m}(\alpha, \beta)} \eta_k^2, \]

with \( \tilde{m}(\alpha, \beta) = [m_\varepsilon(\alpha, \beta)] \), where \([a]\) is the integral part of \(a\). By (7.25), the assumptions of Proposition 7.1 are fulfilled uniformly over \((\alpha, \beta) \in \Sigma\). In particular one can take \(a > 0\) such that (7.9) holds true for all \((\alpha, \beta) \in \Sigma\), as \(\varepsilon\) is small enough. Applying now Proposition 7.1, we have

\[ h^*_\varepsilon(\alpha, \beta) \geq bu_{\varepsilon, \alpha, \beta}(r_{\varepsilon}(\alpha, \beta)) \geq bu_{\varepsilon}(\Sigma). \]

Therefore, we get

\[ h_{\varepsilon}(\alpha, \beta) \geq b_1 u_{\varepsilon}(\Sigma) \rightarrow \infty, \]

\[ b_1 = bd. \]

By (7.13), this implies that it suffices to take

\[ H_{\varepsilon} \rightarrow \infty, \quad H_{\varepsilon} < b_2 u_{\varepsilon}(\Sigma) \]

with any \(b_2 \in (0, b_1)\). The theorem now follows.

7.8 **Proof of Theorem 4.8**

Before we prove the theorem we need the following result.

Recall, that the extreme sequence (4.4) in the extreme problem (4.1) is of the form

\[ \eta_k^2 = z_0^2 \sigma_k^2 (1 - A \alpha_k^2), \quad k \in \mathbb{N}, \quad (7.27) \]

where the quantities \(z_0 = z_{0, \varepsilon}\) and \(A = A_{\varepsilon}\) are determined by the equations

\[ \begin{aligned} &\sum_{k \in \mathbb{N}} a_k^2 \sigma_k^2 \eta_k^2 = 1, \\ &\sum_{k \in \mathbb{N}} \sigma_k^2 \eta_k^2 = r_{\varepsilon}^2. \end{aligned} \]

and, thus, the value of the extreme problem (4.1) takes the form

\[ u_{\varepsilon}^2 = \frac{1}{2\varepsilon^4} \sum_{k \in \mathbb{N}} \eta_k^4. \]

Consider now the following “truncated” version of the above system of equations

\[ \begin{aligned} &\sum_{k=1}^{m} a_k^2 \sigma_k^2 \eta_k^2 = 1, \\ &\sum_{k=1}^{m} \sigma_k^2 \eta_k^2 = r_{\varepsilon}^2, \quad u_{\varepsilon}^2 = \frac{1}{2\varepsilon^4} \sum_{k=1}^{m} \eta_k^4. \end{aligned} \quad (7.28) \]

In order to solve the equations (7.27)-(7.28), let us define a function \(r(A)\), \(A \in (0, a_2^{-2})\) as follows. Take \(m = m(A) \in \mathbb{N}\), \(m \geq 2\), such that \(a_{m+1}^{-2} \leq A \leq a_m^{-2}\) and set

\[ r(A) = \left( \frac{\sum_{k=1}^{m} \sigma_k^4 (1 - A a_k^2)}{\sum_{k=1}^{m} \sigma_k^2 a_k^4 (1 - A a_k^2)} \right)^{1/2}, \quad A \in (0, a_2^{-2}). \quad (7.29) \]

Then, for \(r_{\varepsilon}\) small enough, the quantity \(A = A_{\varepsilon}\) in (7.27) is determined by the equation

\[ r_{\varepsilon} = r(A_{\varepsilon}). \quad (7.30) \]

Note first that \(r(A)\) is a positive continuous functions in \(A \in (0, a_2^{-2})\). The following proposition ensures the existence of a unique solution in (7.30). (Note that its validity does not depend on the assumption (4.20).)

**Proposition 7.2** The function \(r(A)\) defined in (7.29) is strictly increasing in \(A \in (0, a_2^{-2})\).
Proof. Let \( a_{m+1}^2 \leq A < a_m^2 \), \( m \geq 2 \). Introduce a probability measure \( P = \{ p_i \}_{i \in I} \) on the set \( I = \{ 1, 2, \ldots, m \} \) such that \( p_i = \sigma_i^4 / \sum_{k=1}^{m} \sigma_k^4 \), \( i \in I \). Set

\[
H_m(A) = \left( \sum_{k=1}^{m} \sigma_k^4 a_k^2 (1 - A a_k^2) \right)^2, \quad m \geq 2.
\]

We consider \( a = \{ a_i \}_{i \in I} \) as random variable on the set \( I \). Then, we have

\[
(r^2(A))'_A = \frac{(\sum_{k=1}^{m} \sigma_k^4 (\sum_{k=1}^{m} \sigma_k^4 a_k^2) - (\sum_{k=1}^{m} \sigma_k^4 a_k^2)^2)}{H_m(A)}
\]

\[
= \frac{(\sum_{k=1}^{m} a_k^2 p_k - (\sum_{k=1}^{m} \sigma_k^4 p_k^2) (\sum_{k=1}^{m} \sigma_k^4)^2)}{H_m(A)}
\]

\[
= \frac{(E_P(a^4) - (E_P(a^2))^2) (\sum_{k=1}^{m} \sigma_k^4)^2}{H_m(A)} = \frac{\text{Var}_P(a^2) (\sum_{k=1}^{m} \sigma_k^4)^2}{H_m(A)} > 0,
\]

where \((\cdot)'_A\) denotes differentiation with respect to \( A \). The proposition now follows. \(\square\)

We are now ready to prove part (a) of the theorem. Let \( A = A_\varepsilon \) be the solution of (7.30). It then follows from (4.20) that

\[
\begin{align*}
\sum_{k=1}^{m-2} a_k^2 \tilde{h}_k^2 &= \tau_1 a_{m-1}^2 \tilde{h}_{m-1}^2 = \sum_{k=1}^{m-2} \tilde{h}_k^2 = \tau_2 \tilde{h}_{m-1}^2, \\
\sum_{k=1}^{m-2} \sigma_k^2 \tilde{h}_k^2 &= \tau_2 \sigma_{m-1}^2 \tilde{h}_{m-1}^2, \\
\sum_{k=1}^{m-2} \sigma_k^4 \tilde{h}_k^2 &= \tau_0 \sigma_{m-1}^4 \tilde{h}_{m-1}^2,
\end{align*}
\]

(7.31)

where \( \tau_i = \tau_{m,i}(A) \), \( i = 0, 1, 2 \), are such that

\[
\tau_1 \sim \frac{\sigma_{m-2}^4 a_{m-2}^2 (1 - A a_{m-2}^2)}{\sigma_{m-1}^4 a_{m-1}^2 (1 - A a_{m-1}^2)} = o(1), \quad \tau_2 \sim \frac{\sigma_{m-2}^4 (1 - A a_{m-2}^2)}{\sigma_{m-1}^4 (1 - A a_{m-1}^2)} = o(1), \quad \tau_0 = o(1).
\]

(7.32)

Therefore, we can rewrite the equations (7.28) in the form

\[
\begin{align*}
\theta_1 a_{m-1}^2 \sigma_{m-1}^2 \tilde{h}_{m-1}^2 + \sigma_{m-1}^2 \sigma_{m-1}^2 \tilde{h}_{m-1}^2 &= 1, \\
2 \tilde{h}_{m-1}^2 &= \epsilon^{-4} (\theta_0 \tilde{h}_{m-1}^4 + \tilde{h}_{m}^4)/2, \\
\sigma_{m-2}^2 \tilde{h}_{m-1}^2 &= \epsilon^{-4} (\theta_0 \tilde{h}_{m-1}^4 + \tilde{h}_{m}^4)/2,
\end{align*}
\]

(7.33)

with \( \theta_i = \theta_{m,i}(A) = 1 + \tau_{m,i}(A) \sim 1 \), \( i = 0, 1, 2 \). Setting \( z_1 = \tilde{h}_{m-1}^2 \), \( z_2 = \tilde{h}_{m}^2 \) we find \( z = (z_1, z_2) \) from (7.33):

\[
z_1 = \frac{a_{m-1}^2 \tilde{h}_{m-1}^2}{(\theta_0 a_{m-1}^2 - \theta_1 a_{m-1}^2 \sigma_{m-2}^2 \sigma_{m-2}^2)} - 1, \quad z_2 = \frac{\theta_2 - a_{m-1}^2 r_{m-1}^2 \theta_1}{(\theta_0 a_{m-1}^2 - \theta_1 a_{m-1}^2 \sigma_{m-2}^2 \sigma_{m-2}^2)}; \quad u_{m-1}^4 \sim \frac{\|z\|^2}{2 \epsilon^4}.
\]

We have \( \tilde{h}_{m}^4 = 0 \) (this corresponds to \( A = a_m^{-2} \)) as \( r_{m-1} = r_{m-1}^2 := a_{m-1}^2 \theta_{m-1}^2 \), where by (7.32)

\[
\theta_{m-1} = \theta_{m,2}(a_m^2)/\theta_{m,1}(a_m^2) > 1, \quad \theta_{m,1} \sim 1.
\]

The conditions \( z_1 > 0 \), \( z_2 \geq 0 \) correspond to

\[
a_{m-1}^2 \leq r_{m-1}^2 \leq a_{m-1}^2 \theta_{m-1}.
\]

(7.34)

By (7.32) and the definition of \( r_{m} \) we have, as \( m \to \infty \),

\[
r_{m}^2 = \frac{1}{a_m^2} \left( 1 + \frac{\sigma_{m-1}^4}{\sigma_{m}^4} \cdot \frac{(1 - a_{m-1}^2/a_m^2)(1 - a_{m-1}^2/a_{m+1}^2)}{(1 - a_{m-1}^2/a_m^2)} \right), \quad \frac{1}{a_{m}^2} > \frac{1}{a_{m-1}^2}.
\]

Recalling the monotonicity of \( r(A) \), we see that if \( a_{m+1}^2 \leq A \leq a_m^2 \), then \( r_{m} = r_{A_\varepsilon} \in \Delta_{m} = [r_m, r_{m-1}] = [a_m^{-1}(1 + o(1)), a_{m-1}^{-1}(1 + o(1))] \), where \( r_i > a_i^{-1}, i = m - 1, m \).
Let $u_{\varepsilon,\min} = \min_{r_\varepsilon \in \Delta_m} u_{\varepsilon}(r_\varepsilon)$. Thus, we get

$$u_{\varepsilon}(r_\varepsilon) \geq u_{\varepsilon,\min} \times (\varepsilon^2 a_m^2 \sigma_m^2)^{-1} \quad \text{as} \quad r_\varepsilon \in \Delta_m.$$  \hspace{1cm} (7.35)

Let us now consider the interval $\Delta_m^* = [r_{m,1}, r_{m-1,1}]$, $r_{l,1} = 1/a_l$. For $r_\varepsilon \in \Delta_m$, we set $\bar{z} = (\bar{z}_1, \bar{z}_2)$, $z^* = (z^*_1, z^*_2)$,

$$\bar{z}_1 = \frac{a_m^2 r_\varepsilon^2 - 1}{(\theta_2 a_m^2 - \theta_1 a_m^2 - 1) a_m^2}, \quad \bar{z}_2 = \frac{1 - a_m^2 - r_\varepsilon^2}{(\theta_2 a_m^2 - \theta_1 a_m^2) \sigma_m^2}, \quad \bar{u}_\varepsilon = \bar{u}_\varepsilon(r_\varepsilon) = \frac{||\bar{z}||}{\sqrt{2\varepsilon^2}};$$

$$z^*_1 = \frac{a_m^2 r_\varepsilon^2 - 1}{(a_m^2 - a_m^2) a_m^2}, \quad z^*_2 = \frac{1 - a_m^2 - r_\varepsilon^2}{(a_m^2 - a_m^2) \sigma_m^2}, \quad u_{\varepsilon}^* = u_{\varepsilon}^*(r_\varepsilon) = \frac{||z^*||}{\sqrt{2\varepsilon^2}}.$$  \hspace{1cm} (7.36)

Note that, for some $B > 0$,

$$|u_{\varepsilon}^*(r_2) - u_{\varepsilon}^*(r_1)| \leq B(r_2^2 - r_1^2)/\varepsilon^2 \sigma_m^2, \quad \text{as} \quad r_{m,1} \leq r_1 < r_2 \leq r_{m-1,1},$$  \hspace{1cm} (7.37)

and it is easily seen that

$$\bar{u}_\varepsilon(r_\varepsilon) \sim u_{\varepsilon}^*(r_\varepsilon) \quad \text{as} \quad r_\varepsilon \to 0; \quad u_{\varepsilon}^*(r) \geq u_{\varepsilon}(r) \quad \forall \quad r > 0.$$  \hspace{1cm} (7.38)

Also, for $\delta = \tilde{z} - z$ and for $r_\varepsilon \in \Delta_m \cap \Delta_m^* = [r_m, r_{m-1,1}]$, we have

$$||\delta|| = O((a_m^2 - r_\varepsilon^2 + 1)/a_m^2 \sigma_m^2) = \varepsilon^2 o(u_{\varepsilon,\min}).$$  \hspace{1cm} (7.39)

These yields, as $r_\varepsilon \in [r_{m,1}, r_{m-1,1}]$,

$$u_{\varepsilon}(r_\varepsilon) \sim u_{\varepsilon}^*(r_\varepsilon) = \frac{1}{\sqrt{2\varepsilon^2}} \left( \frac{(a_m^2 - r_\varepsilon^2 - 1)^2}{a_m^2 - a_m^2} + \frac{(1 - a_m^2 - r_\varepsilon^2)^2}{\sigma_m^2} \right)^{1/2}. \hspace{1cm} (7.40)$$

Let $r_\varepsilon \in [r_{m-1,1}, r_{m-1}] \subset \Delta_m^*$. Observe that

$$0 \leq u_{\varepsilon}^*(r_\varepsilon) - u_{\varepsilon}(r_\varepsilon) \leq u_{\varepsilon}^*(r_{m-1,1}) - u_{\varepsilon}(r_{m-1,1}) + |u_{\varepsilon}^*(r_\varepsilon) - u_{\varepsilon}^*(r_{m-1,1})| = \xi_1 + \xi_2,$$

where $\xi_1 = u_{\varepsilon}^*(r_{m-1,1}) - u_{\varepsilon}(r_{m-1,1})$, $\xi_2 = |u_{\varepsilon}^*(r_\varepsilon) - u_{\varepsilon}^*(r_{m-1,1})|$. By (7.41),

$$\xi_1 = o(u_{\varepsilon}(r_{m-1,1})) = o(u_{\varepsilon}(r_\varepsilon)).$$

Applying (7.40) for the interval $\Delta_m^*$ we get

$$\xi_2 \leq B \frac{r_{m-1,1}^2 - r_{m-1,1}^2}{\varepsilon^2 \sigma_m^2}.$$  \hspace{1cm} (7.42)

Since $r_{m-1,1}^2 - r_{m-1,1}^2 = (\theta_{m-1} - 1)/a_{m-1}^2$, using (7.43), we have $\theta_{m-1} - 1 = O(\sigma_m^2/a_{m-1}^4)$. By (7.44),

$$\xi_2 = O \left( \frac{\sigma_m^2}{\varepsilon^2 a_m^2 \sigma_m^4} \right) = o(u_{\varepsilon}(r_\varepsilon)),$$

as $r_\varepsilon \in \Delta_m^*$. This completes part (a) of the theorem.

We now prove part (b) of the theorem. For $r_\varepsilon \in \Delta_m^*$, $m \in \mathbb{N}$, $m \geq 2$, consider the piecewise linear (in $r^2$) function $u_{\varepsilon,\min}^*(r)$ defined in (7.45). We then have, at the break points,

$$u_{\varepsilon,\min}^*(1/a_m) = \frac{1}{\varepsilon^2 a_m^2 \sigma_m^2}, \quad u_{\varepsilon,\min}^*(1/a_{m-1}) = \frac{1}{\varepsilon^2 a_m^2 \sigma_m^2}. \hspace{1cm} (7.46)$$

Using the standard inequalities $(x + y)/\sqrt{2} \leq \sqrt{x^2 + y^2} \leq x + y$, $x \geq 0$, $y \geq 0$, we get, for $r > 0$ small enough,

$$u_{\varepsilon,\min}^*(r)/2 \leq u_{\varepsilon,\min}^*(r) \leq u_{\varepsilon,\min}^*(r)/\sqrt{2}. \hspace{1cm} (7.47)$$

Part (b) of the theorem follows by (7.48) and part (a) of the theorem.

Parts (c) and (d) of the theorem follow immediately by combining Theorem 4.1 and part (b) of the theorem. The theorem now follows.
7.9 Proof of Theorem 4.9

For type I error probability, we have

\[ \alpha(\psi_{\varepsilon, \alpha}) = P_{\varepsilon, 0}(Y_{\varepsilon, \alpha}) \leq \sum_{k=1}^{m} P_{\varepsilon, 0}(|y_k| \geq T_{m,k}\varepsilon) \leq 2 \sum_{k=1}^{m} \Phi(-T_{m,k}) = \alpha. \]

In order to evaluate type II error probability, observe that

\[ \beta(\psi_{\varepsilon, \alpha}, \eta) = P_{\varepsilon, \eta}(Y_{\varepsilon, \alpha}) \leq \min_{1 \leq k \leq m} P_{\varepsilon, \eta}(|y_k| < T_{m,k}\varepsilon) \leq \Phi(\min_{1 \leq k \leq m} (T_{m,k} - \varepsilon^{-1}\eta_k)), \]

and it suffices to check that

\[ \inf_{\eta \in \Theta(r_\varepsilon)} \left( \max_{1 \leq k \leq m} (\varepsilon^{-1}\eta_k - T_{m,k}) \right) \rightarrow \infty \text{ as } u_{\varepsilon}^{lin}(r_\varepsilon) \rightarrow \infty, \ r_\varepsilon \in \Delta^*_m. \quad (7.40) \]

The following proposition is useful to our goal.

**Proposition 7.3** Let assume \((4.20)\) holds true. Let \(r_\varepsilon \in \Delta^*_m\), consider the collection \(H_{m,k}, \ 1 \leq k \leq m\) satisfying \(0 < H_{m,k} \leq B_1(m - k + 1)^{B_2}\) for some \(B_l > 0, \ l = 1, 2\) if \(1 \leq k \leq m - 2\) and \(H_{m,m} = H_{m,m-1} = 1\). Then

\[ \inf_{\eta \in \Theta(r_\varepsilon)} \max_{1 \leq k \leq m} \varepsilon^{-2}H_{m,k}^{-1}\eta_k^2 \geq u_{\varepsilon}^{lin}(r_\varepsilon)(1/(2\sqrt{2}) + o(1)). \]

**Proof.** Let \(\eta \in \Theta(r_\varepsilon)\), take

\[ r_\varepsilon^2 = \frac{1 - t}{a_m^2} + \frac{t}{a_{m-1}^2}, \quad t \in [0, 1], \quad (7.41) \]

and suppose that

\[ \max_{1 \leq k \leq m - 2} \varepsilon^{-2}H_{m,k}^{-1}\eta_k^2 \leq u_{\varepsilon}^{lin}(r_\varepsilon). \]

On noting that \(u_{\varepsilon}^{lin}\), in view of \((7.41)\), takes the form

\[ u_{\varepsilon}^{lin}(r) = \frac{1 - t}{\varepsilon^2\sigma_m^2a_m^2} + \frac{t}{\varepsilon^2\sigma_{m-1}^2a_{m-1}^2}, \quad t \in [0, 1], \]

we then get

\[ \sum_{k=1}^{m-2} \sigma_k^2 \eta_k^2 \leq \varepsilon^2 u_{\varepsilon}^{lin}(r_\varepsilon) \sum_{k=1}^{m-2} \sigma_k^2 H_{m,k} \times \sigma_{m-2}^2 \left( \frac{1 - t}{\sigma_m^2a_m^2} + \frac{t}{\sigma_{m-1}^2a_{m-1}^2} \right) =: \delta = o(r_\varepsilon^2). \]

Set \(\tilde{\eta} = (0, \ldots, 0, \eta_{m-1}, \eta_m, \ldots)\). It follows from the estimation above that \(\tilde{\eta} \in \Theta(\tilde{r}_\varepsilon)\), \(\tilde{r}_\varepsilon^2 = r_\varepsilon^2 - b\delta\) for some \(b > 0\), and

\[ u_{\varepsilon}(\tilde{r}_\varepsilon) \geq u_{\varepsilon}^{lin}(\tilde{r}_\varepsilon)(1/2 + o(1)) \geq (1/2 + o(1)) \left( u_{\varepsilon}^{lin}(r_\varepsilon) - \frac{B\delta}{\sigma_{m-1}^2}\varepsilon_2 \right) \sim \frac{1}{2} \left( 1 - \frac{B\sigma_{m-2}^2}{\sigma_{m-1}^2} \right) u_{\varepsilon}^{lin}(r_\varepsilon) \sim u_{\varepsilon}^{lin}(r_\varepsilon)/2 \geq u_{\varepsilon}(r_\varepsilon)(1/\sqrt{2} + o(1)). \]

This implies

\[ \varepsilon^{-4} \sum_{k \in \mathbb{N}} \eta_k^4 \varepsilon^{-4} \sum_{k=m-1}^{\infty} \eta_k^4 \geq 2u_{\varepsilon}(\tilde{r}_\varepsilon) \geq u_{\varepsilon}^2(\varepsilon_3^2)(1 + o(1)). \]
Since $\sum_{k\in N} a_k^2 \sigma_k^2 \leq 1$, we have $\eta_k \leq (a_k\sigma_k)^{-1}$ and

$$
\sum_{k=m+1}^{\infty} \eta_k^4 \leq \sum_{k=m+1}^{\infty} (a_k\sigma_k)^{-4} \leq (a_{m+1}\sigma_{m+1})^{-4} \sum_{k=m+1}^{\infty} \frac{(a_{m+1}\sigma_{m+1})^4}{(a_k\sigma_k)^4} \sim (a_{m+1}\sigma_{m+1})^{-4} = o(\varepsilon^4 u_\varepsilon^2(r_\varepsilon)).
$$

Thus, for $m$ large enough,

$$
\varepsilon^{-4} \max(\eta_{m-1}^4, \eta_m^4) \geq \varepsilon^{-4}(\eta_{m-1}^4 + \eta_m^4)/2 \geq u_\varepsilon^2(r_\varepsilon)(1/2 + o(1)),
$$

which yields $\varepsilon^{-2} \max(\eta_{m-1}^2, \eta_m^2) \geq u_\varepsilon(r_\varepsilon)(1/\sqrt{2} + o(1)) \geq u_l^{\text{lin}}(r_\varepsilon)(1/2\sqrt{2} + o(1))$. The proposition now follows. □

We are now ready to complete the proof of the theorem. Note that $T_{m,k} \geq \Phi^{-1}(1 - c\alpha)$ are bounded away from 0. The collection $H_{m,k} = (T_{m,k}/T_{m,m})^2$ satisfies the assumption of Proposition 7.3 since

$$
T_{m,k} = \Phi^{-1} \left( 1 - \frac{c\alpha}{(m - k - 1)^2} \right) \sim \sqrt{2 \log \left( \frac{(m - k - 1)^2}{c\alpha} \right)} \quad \text{as} \quad m - k \to \infty.
$$

Applying now Proposition 7.3 to this collection we get that there exists $k$, $1 \leq k \leq m$ such that

$$
\varepsilon^{-1} \eta_k \geq 8^{-1/4}(T_{m,k}/T_{m,m})\sqrt{u_l^{\text{lin}}(r_\varepsilon)}(1 + o(1)),
$$

which yield

$$
\max_{1 \leq k \leq m} (\varepsilon^{-1} \eta_k - T_{m,k}) \to \infty \quad \text{as} \quad u_l^{\text{lin}}(r_\varepsilon) \to \infty.
$$

This implies (7.40). The theorem now follows.

### 7.10 Proof of Theorem 5.1

We first obtain the lower bounds. Take a collection $\kappa_l$ such that

$$
\phi(\kappa_l) = a + l\delta_\varepsilon, \quad 1 \leq l \leq L = L_\varepsilon, \quad \phi(\kappa_{L+1}) = b, \quad \delta = \delta_\varepsilon = \frac{(b - a)}{L} \sim \frac{\log(3)}{\log(\varepsilon^{-1})}.
$$

Assume, without loss of generality, that $u_\varepsilon(\kappa_l) \sim \sqrt{\log \log(\varepsilon^{-1})}$ uniformly in $l = 1, 2, \ldots, L$. Observe that $\log(L) \sim \log \log(\varepsilon^{-1})$. Set

$$
\mu_l \sim \left( \varepsilon \left( \log \log(\varepsilon^{-1}) \right)^{1/4} \right)^{-\phi(\kappa_{l})}.
$$

By construction, we have

$$
\mu_l - \mu_{l-1} \sim \mu_{l-1} \left( \exp \left( \delta \log \left( \varepsilon^{-1} \left( \log \log(\varepsilon^{-1}) \right)^{-1/4} \right) \right) - 1 \right)
\quad = \quad \mu_{l-1}(3(1 + o(1))) - 1 \sim 2\mu_{l-1}.
$$

Set

$$
\Delta_l = \{ k \in N : \mu_l - 1 < k \leq \mu_l \}, \quad M_l = \#(\Delta_l)2 \sim \mu_{l-1}.
$$

Take a collection $z_l > 0$ such that

$$
z_l^2 M_l \sigma_m^2(\kappa_l) \sigma_m^2(\kappa_l) = 1, \quad 1 \leq l \leq L.
$$
By \((7.7), (7.8)\), the relation \((7.44)\) implies that, as the quantity \(d\) in Theorem 5.1 (a) is small enough (this corresponds to \(r_\varepsilon(\kappa)\) small enough), one has

\[
 z_l^2 M_l \sigma_{m_{l-1}}^2(\kappa_l) \geq r_\varepsilon^2(\kappa_l), \quad 1 \leq l \leq L.
\]  

(7.45)

Set \(u_l^2 = M_l z_l^2/(2\varepsilon^4)\). Observe that the relations \((7.44), (7.42), (7.43)\) imply

\[
u_l^2 \sim 3 \log \log(\varepsilon^{-1})/4 \sim u_l^2(\kappa_l),
\]

(7.46)

Therefore the relations \(z_l^2 = 2\varepsilon^4 u_l^2/M_l\), \((7.42), (7.46)\) and \((7.43)\) imply

\[z_l = o(\varepsilon).
\]

(7.47)

Consider the priors

\[\pi_l = \prod_{k \in \Delta_l} \left( \delta_{z_l e_k} + \delta_{-z_l e_k} \right)/2, \quad \pi = \frac{1}{L} \sum_{l=1}^L \pi_l,
\]

where \(\{e_k\}_{k \in \mathbb{N}}\) is the standard basis in \(l^2\) and \(\delta_\eta\) is the Dirac mass at the point \(\eta \in l^2\). The relations \((7.44), (7.45)\) imply, for \(d = d(\Sigma)\) small enough, \(\pi_l(\Theta(\kappa_l(r_\varepsilon(\kappa_l)))) = 1, \pi(\Theta(\Sigma)) = 1\). Let \(P_{\pi_l} = E_{\varepsilon_l} P_{\varepsilon, \eta}\) and \(P_\pi = E_{\varepsilon_l} P_{\varepsilon, \eta}\) be the mixtures over the priors. It suffices to check that

\[E_{\varepsilon, 0} \left( (dP_\pi/dP_{\varepsilon, 0} - 1)^2 \right) = o(1).
\]

(7.48)

Using evaluations similar to \([17]\) (see formulae (3.64)–(3.69)), we have

\[
E_{\varepsilon, 0} \left( (dP_\pi/dP_{\varepsilon, 0} - 1)^2 \right) = \frac{1}{L^2} \sum_{l=1}^L \left( E_{\varepsilon, 0} (dP_{\pi_l}/dP_{\varepsilon, 0})^2 - 1 \right) \leq \frac{1}{L^2} \sum_{l=1}^L \left( e^{\tilde{u}_l^2} - 1 \right),
\]

where \(\tilde{u}_l^2 = 2M_l \sinh^2(z_l^2/(2\varepsilon^2)) \sim u_l^2\) by \((7.47)\). By \((7.46)\) one has

\[\max_l u_l^2 \log(L) \sim 3/4 < 1.
\]

(7.49)

This yields \((7.48)\) and completes part (a) of the theorem.

We now obtain the upper bounds. Recall that we have, in Theorem 5.1 (b), \(L_\varepsilon = o(\log(\varepsilon^{-1})), L_\varepsilon \to \infty\). It follows from the exponential inequality for \(\chi^2\)-statistics that

\[\log(P_{\varepsilon, 0}(t_m > H)) \leq -H^2/2(1 + o(1)) \quad \text{as} \quad H \to \infty,
\]

(7.50)

see, e.g., (5.22) in \([18]\). This implies that, for the type I error probability,

\[\alpha(\psi_\varepsilon) \leq \sum_{l=L_\varepsilon}^\infty P_{\varepsilon, 0}(t_m > H_l) \leq \sum_{l=L_\varepsilon}^\infty l^{-C/2 + o(1)} \to 0 \quad \text{as} \quad L_\varepsilon \to \infty.
\]

Let us evaluate the type II error probability. It suffices to consider the case \(u_\varepsilon = D \sqrt{\log \log(\varepsilon^{-1})}\) with \(D\) larger enough. Observe that (see \((7.13)\))

\[\beta_\varepsilon(\psi_\varepsilon, \eta) \leq \min_{l \geq L_\varepsilon} P_{\varepsilon, \eta}(t_m \leq H_l) = \min_{l \geq L_\varepsilon} \Phi(H_l - h_m(\eta)) + o(1),
\]

(7.51)
where \( h_m(\eta) \) is determined by (7.10). Therefore uniformly over \( \kappa \in \Sigma \),

\[
\beta_\varepsilon(\psi_\varepsilon, \Theta_\varepsilon, \kappa) \leq \Phi(\sqrt{C \log L} - \max_{l \geq L_\varepsilon} h_m(\kappa)) + o(1), \quad h_m(\kappa) = \inf_{\eta \in \Theta_\varepsilon, \kappa} h_m(\eta).
\]

For \( \kappa \in \Sigma \), let us set \( m_{\varepsilon}(\kappa) = (\varepsilon^{-4} \log \log(\varepsilon^{-1}))^{1/(4\alpha + 4\beta + 1)} \) and take \( l \) such that \( m_{\varepsilon}(\kappa) < m_l \), i.e.,

\[
m_l = cm_{\varepsilon}(\kappa), \quad c \in (1, 2], \quad l \sim \frac{4 \log(\varepsilon^{-1})}{(4\alpha + 4\beta + 1) \log(2)} > L_\varepsilon.
\]

It follows from (7.7), (4.11) that, for \( D = D_{\text{max}}(\Sigma) \) larger enough, \( r_\varepsilon(\kappa) a_{m_l + 1}(\kappa) \geq B + o(1) \), with \( B = B(\Sigma) > 1 \) that could be taken common for all \( \kappa \in \Sigma \). It follows from (4.11) that the assumptions of Proposition 7.1 are fulfilled for \( m_\varepsilon = m_l \) with some \( a(\Sigma) = \sup_{\kappa \in \Sigma} a(\kappa) > 0 \), uniformly over \( \kappa \in \Sigma \). Applying Proposition 7.1 one can take \( b = b(\Sigma) \) such that, uniformly over \( \kappa \in \Sigma \), \( h_{m_\varepsilon}(\kappa) \geq bu_\varepsilon(\kappa) \). Thus, it suffices take \( D(\Sigma) > \max(D_{\text{max}}(\Sigma), C/b(\Sigma)) \). This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (7.7). The theorem now follows.

### 7.11 Proof of Theorem 5.2

We first obtain the lower bounds. Take a collection \( \kappa_l \) such that \( \phi(\kappa_l) = a_\varepsilon + l\delta_\varepsilon \), \( 1 \leq l \leq L = L_\varepsilon \), \( \phi(\kappa_L) = b_\varepsilon \), where \( a < a_\varepsilon < b_\varepsilon < b \), \( a_\varepsilon = a + o(1) \), \( b_\varepsilon = b + o(1) \) and take \( L \) such that

\[
\delta_\varepsilon = \frac{b_\varepsilon - a_\varepsilon}{L} < \frac{2}{2 \log(\varepsilon^{-1}) - \log \log \log(\varepsilon^{-1})},
\]

\[
m_l = [\phi(\kappa_l)(2 \log(\varepsilon^{-1}) - \log \log \log(\varepsilon^{-1}) - \log(c))] \in \mathbb{N}, \quad c < \exp(-1/2),
\]

where \([a]\) is the integral part of \( a \). By construction, \( m_l - m_{l-1} \sim 2 \).

Applying (7.15), we see that, if \( u_\varepsilon(\kappa) < d \log \log(\varepsilon^{-1}) \) for all \( \kappa \in \Sigma \) and some \( d > 0 \), then

\[
u_\varepsilon(\kappa) = \varepsilon^{-2} \frac{2(\alpha(\kappa) + \beta(\kappa))/\alpha(\kappa)}{\log(\varepsilon^{-1})} < d_1 \log \log(\varepsilon^{-1})
\]

for all \( \kappa \in \Sigma \) and some \( d_1 \). Observe that, for any \( c > 0 \) from the definition of \( m_l \) above, one can take \( d \) small enough (this corresponds to \( r_\varepsilon(\kappa) \) small enough) such that \( d_1 \leq c \). This yields

\[
\exp(-\alpha m_l) \geq r_\varepsilon(\kappa_l).
\]

(7.51)

For \( \kappa_l \in \Sigma \), let us take \( z_l = \eta_l e_m_l \), where \( \eta_l = \exp(-(\alpha l + \beta_l) m_l) \) and \( \{e_l\}_{l \in \mathbb{N}} \) is the standard basis in \( l^2 \). By (7.51) this yields \( \eta_l \in \Theta_\varepsilon(\kappa_l, r_\varepsilon(\kappa_l)) \) for any \( q = q_l > 0 \). Let us consider the prior

\[
\pi = \frac{1}{L} \sum_{l=1}^{L} \delta_{z_l}
\]

and the mixture \( P_\pi \) over \( \pi \). Since \( \pi(\Theta_\varepsilon(\Sigma)) = 1 \), it suffices to verify that (see [17], Section 2.5.2, Propositions 2.11, 2.12)

\[
E_{\varepsilon, 0}(dP_\pi/dP_{\varepsilon, 0} - 1)^2 \to 0.
\]

(7.52)

One has

\[
E_{\varepsilon, 0}(dP_\pi/dP_{\varepsilon, 0} - 1)^2 = \frac{1}{L^2} \sum_{l=1}^{L} E_{\varepsilon, 0}(dP_{\varepsilon, z_l}/dP_{\varepsilon, 0} - 1)^2 = \frac{1}{L^2} \sum_{l=1}^{L} (e_{\eta_l}^2/\varepsilon^2 - 1).
\]

(7.53)
The relation (7.52) holds true as $L \propto \log(\varepsilon^{-1})$ and for $c$ small enough

$$\max_{1 \leq i \leq L} \eta_i^2 / \varepsilon^2 \leq c \log \log(\varepsilon^{-1}) \sup_{n \in \Sigma} \exp(2(\alpha_l + \beta_l)) = c_1 \log \log(\varepsilon^{-1}), \ c_1 < 1. \quad (7.54)$$

Thus (7.51) holds true under the assumption of the theorem for $d$ small enough. This completes part (a) of the theorem.

In order to obtain the upper bounds, we need the following (general) proposition and its corollary.

**Proposition 7.4** Let $b = \{b_i\}_{i \in \mathbb{N}}$ and $c = \{c_i\}_{i \in \mathbb{N}}$ be positive sequences, $b = \{b_i\}_{i \in \mathbb{N}}$ be an increasing sequence, $b_i \to \infty$ and $c_i b_i \to \infty$ as $i \to \infty$. Let also $r > 0$ be a small enough quantity and let $X = \{x | x = \{x_i\}_{i \in \mathbb{N}}\}$ be a set of sequences $x = \{x_i\}_{i \in \mathbb{N}}$ that are determined by the constraints

$$\sum_{i \in \mathbb{N}} b_i c_i x_i \leq 1, \quad \sum_{i \in \mathbb{N}} c_i x_i \geq r, \quad x_i \geq 0 \quad \forall \ i \in \mathbb{N}.$$

Consider the extreme problem

$$w = w(r) = \inf_{x \in X} \phi(x), \quad \phi(x) = \sup_{i \in \mathbb{N}} x_i.$$

Then, the extreme sequences $x^* = \{x^*_i\}_{i \in \mathbb{N}}$ such that $\phi(x^*) = w$ is of the form:

$$x^*_i = w, \quad i = 1, 2, \ldots, m - 1, \quad x^*_m = w_0, \quad x_i = 0 \quad \text{as} \quad i > m,$$

where the quantities $w$ and $w_0$, $0 \leq w_0 \leq w$, are of the form

$$w = \frac{r b_m - 1}{\sum_{i=1}^{m-1} c_i (b_m - b_i)}, \quad w_0 = \frac{\sum_{i=1}^{m-1} c_i (1 - r b_i)}{c_m \sum_{i=1}^{m-1} c_i (b_m - b_i)},$$

and the integer $m$ is determined by the inequalities

$$B_m \leq r \leq B_{m-1}, \quad B_k = \frac{\sum_{i=1}^{k} c_i}{\sum_{i=1}^{k} b_i c_i}, \quad k = 1, 2, \ldots, m. \quad (7.55)$$

One further obtains the inequalities

$$C_m \leq w \leq C_{m-1}, \quad C_k = \frac{1}{\sum_{i=1}^{k} b_i c_i}, \quad k = 1, 2, \ldots, m. \quad (7.56)$$

**Proof.** In order to find a minimum of a convex function defined on a convex set $X$, we use the methods of sub-differentials (see [27]). Consider $X$ and $\phi$ as in the statement of the proposition, and let $x \in X$. Then, the structure of $X$ implies that $\lim_{i \to \infty} x_i = 0$ and there exists $i \in \mathbb{N}$ such that $x_i > 0$.

Let us consider the sets $I(x)$ consisting of the indices $i \in \mathbb{N}$ such that $x_i = \sup_{i \in \mathbb{N}} x_i$. Then $I(x) \neq \emptyset$, $x \in X$, and for $i \in I(x)$ we have $x_i > 0$. The sub-differential of the convex function $\phi(x) = \sup_i x_i$ consists of sequences $d = \{d_i\}_{i \in \mathbb{N}}$ such that $d_i \geq 0$, $i \in \mathbb{N}$, $d_i = 0$ for $i \notin I(x)$, and $\sum_{i \in \mathbb{N}} d_i = 1$ (see Lemma 1 in Section 1.4.1 of [27]). We get the following relations for the extreme sequence $x^*$:

$$d_i = \lambda c_i - \mu c_i b_i + \varepsilon_i, \quad i \in \mathbb{N},$$

where $\lambda, \mu, \varepsilon_i$ are determined by the constraints

$$\sum_{i=1}^{m-1} b_i c_i x_i \leq 1, \quad \sum_{i=1}^{m-1} c_i x_i \geq r, \quad x_i \geq 0 \quad \forall \ i \in \mathbb{N}.$$
where \( \lambda \geq 0, \mu \geq 0 \) and \( d_i, \varepsilon_i, i \in \mathbb{N} \), are non-negative quantities such that: if \( \lambda > 0 \), then \( \sum_{i \in \mathbb{N}} c_i x_i^r = r \); if \( \mu > 0 \), then \( \sum_{i \in \mathbb{N}} b_i c_i x_i^r = 1 \); if \( i \notin I(x^*) \), then \( d_i = 0 \) and \( x_i^e \varepsilon_i = 0, i \in \mathbb{N} \), \( \sum_{i \in \mathbb{N}} d_i = 1 \). These relations are possible if \( \lambda > 0, \mu > 0 \) only, and it can be rewritten in the form

\[
d_i = \lambda c_i (1 - b_i/B) + \varepsilon_i, \quad i \in \mathbb{N}, \quad B > 0.
\]

Since \( b_i > 0 \) increases in \( i \in \mathbb{N} \), and \( b_i \to \infty \), as \( i \to \infty \), then \( d_i > 0, \varepsilon_i = 0, i \in \mathbb{N} \), \( x_i^r = \sup_{i \in \mathbb{N}} x_i := w > 0 \) as \( i \leq m - 1 \), where \( m = m(B) = \max \{ i : b_i \leq B \} \) and \( x_i^r = 0 \) as \( i > m \). The quantities \( B \) and \( x_m^r := w_0 \) are taken such that \( b_m = B, d_m = \varepsilon_m \geq 0 \),

\[
w \sum_{i=1}^{m-1} b_i c_i + w_0 b_m c_m = 1, \quad w \sum_{i=1}^{m-1} c_i + w_0 c_m = r, \quad 0 \leq w_0 \leq w.
\]

The proposition now follows. \( \square \)

**Corollary 7.1** Let \( a_k = \exp(\alpha k) \) and \( \sigma_k = \exp(\beta k) \), \( k \in \mathbb{N} \), \( \alpha > 0 \) and \( \beta > 0 \). Let \( r_\varepsilon > 0, r_\varepsilon \to 0 \). Set \( m = -(\log r_\varepsilon)/\alpha + O(1) \). Then, for \( m_1 = m + c \) and \( c > 0 \) large enough, one has

\[
\inf_{\eta \in \Theta(r_\varepsilon)} \max_{l \leq m_1} \eta_i^2 \exp(-2m(\alpha + \beta)) \asymp \varepsilon^2 u_\varepsilon.
\]

**Proof.** We apply Proposition \( \square \) to \( i = k \in \mathbb{N} \), \( b_i = a_i^2, c_i = \sigma_i^2, x_i = \eta_i^2 \), \( X = \Theta(r_\varepsilon) \) and \( r = r_\varepsilon^2 \). It then follows from (7.55), (7.56) that

\[
\inf_{\eta \in \Theta(r_\varepsilon)} \sup_{i \in \mathbb{N}} \eta_i^2 \exp(-2m(\alpha + \beta)), \quad m = -\frac{\log r_\varepsilon}{\alpha} + O(1).
\]

Therefore and by (7.5) we have \( \exp(-2m(\alpha + \beta)) \asymp r_\varepsilon^{2(\alpha + \beta)/\beta} \asymp \varepsilon^2 u_\varepsilon \). It suffices now to check that we can replace \( \sup_{i \in \mathbb{N}} \) by \( \max_{l \leq m_1} \) for \( m_1 = m + c \) and \( c > 0 \) large enough. This follows immediately from the inequalities \( a_i^2 \sigma_i^2 \eta_i^2 \leq 1, i \in \mathbb{N} \). This completes the proof of the corollary. \( \square \)

We are now ready to obtain the upper bounds. One has

\[
\alpha(\psi_\varepsilon) \leq \sum_{l=1}^{\infty} P_{\varepsilon,0}(|y_l|/\varepsilon > H_l) = 2 \sum_{l=1}^{\infty} \Phi(-H_l) \asymp \frac{1}{\sqrt{\log(L)}} + \sum_{l=L}^{\infty} \frac{1}{l^{C/2} \sqrt{\log(l)}} \to 0.
\]

Let us now evaluate the type II error probability. In view of the embedding (4.17), it suffices to consider the case \( q = 2 \). We have

\[
\beta_\varepsilon(\psi_\varepsilon, \eta) \leq \min_{l \geq L} P_{\varepsilon, \eta}(|y_l|/\varepsilon \leq H_l) \leq \min_{l \geq L} \Phi(H_l - |\eta_l|/\varepsilon).
\]

It suffices to verify that, uniformly over \( \kappa \in \Sigma \),

\[
\inf_{\eta \in \Theta(\kappa, r_\varepsilon(\kappa))} \max_l (\eta_l^2 / \varepsilon^2 - H_l^2) \to \infty.
\] (7.57)

We apply Corollary 7.1 Since

\[
m = \frac{2 \log(\varepsilon^{-1}) - \log(u_\varepsilon) + O(1)}{2(\alpha + \beta)} = O(\log(\varepsilon^{-1}))
\]

and, as \( L < l \leq m_1 = m + c, c = O(1) \),

\[
H_l^2 = C \log(l) \leq C \log(m_1) \leq C \log(\varepsilon^{-1}) + O(1),
\]

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it follows from Corollary 7.1 that
\[
\inf_{\eta \in \Theta_{i}(\kappa)} \max_{l} (\frac{\eta^2}{\varepsilon^2} - H_l^2) \geq \inf_{\eta \in \Theta_{i}(\kappa)} \max_{l} (\frac{\eta^2}{\varepsilon^2} - H_l^2) \geq b_u - C \log \log(\varepsilon^{-1}) \rightarrow \infty,
\]
as \lim \inf \frac{u_{\varepsilon}}{\log \log(\varepsilon^{-1})} > D, for D large enough. This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (7.15). The theorem now follows.

### 7.12 Proof of Theorem 5.3

We first obtain the lower bounds. Set \( H = (\varepsilon^2 \log^{2\alpha}(\varepsilon^{-1}) \log \log(\varepsilon^{-1}))^{-1} \). Take a collection \( \kappa_l = (\alpha, \beta_l) \in \Sigma \) such that
\[
\frac{1}{\beta_l} = 2a_{\varepsilon} + \frac{2l}{\log(H)}, \quad 1 \leq l \leq L = L_{\varepsilon}, \quad \frac{1}{\beta_L} = 2b_{\varepsilon},
\]
where \( L \asymp \log(H) \sim 2 \log(\varepsilon^{-1}) \), and \( a < a_{\varepsilon} < b_{\varepsilon} < b \), \( a_{\varepsilon} = a + o(1), \ b_{\varepsilon} = b + o(1) \) are taken in such way that \( m_l = \log(H a^{-2\alpha})/2\beta_l \in \mathbb{N} \). By construction, we have \( m_l = m_{l-1} + 1 \) and
\[
m_l^{-2\alpha} \exp(-2\beta_l m_l) \sim (a_{\varepsilon})^{2\alpha} \varepsilon^2 \log(\varepsilon^{-1}) \leq 2^{-2\alpha} \varepsilon^2 \log(\varepsilon^{-1})(1 + o(1)). \tag{7.58}
\]

Assume, without loss of generality, that \( u_{\varepsilon}(\kappa_l) \asymp \log \log(\varepsilon^{-1}) \), uniformly in \( l = 1, 2, \ldots, L \). Taking into account (7.18) and (7.19), we can assume that, for \( d \) small enough (this corresponds to \( r_{\varepsilon}(\kappa_l) \) small enough),
\[
m_l^{-\alpha} \geq r_{\varepsilon}(\kappa_l). \tag{7.59}
\]
For \( l = 1, 2, \ldots, L \), let us take \( \eta_l = z_l \epsilon_{m_l} \), where \( z_l = m_l^{-\alpha} \exp(-\beta_l m_l) \) and \( \{ \epsilon_l \}_{l \in \mathbb{N}} \) be the standard basis in \( \ell^2 \). By (7.59), this yields \( \eta_l \in \Theta_{\varepsilon}(\kappa_l) \) for any \( q = q_l > 0 \). The following steps are along the lines of the proof of part (a) of Theorem 5.2. We consider the prior
\[
\pi = \frac{1}{L} \sum_{l=1}^{L} \delta_{\eta_l}
\]
and the mixture \( P_{\pi} \) over the prior \( \pi \). Since \( \pi(\Theta(\Sigma)) = 1 \), it suffices to verify (7.52). By (7.53), this relation holds true as
\[
\lim \sup \frac{\max_{1 \leq l \leq L} z_l^2/\varepsilon^2}{\log(L)} < 1. \tag{7.60}
\]
By construction, we have \( \log(L) \sim \log \log(\varepsilon^{-1}) \), and by (7.58), \( z_l^2/\varepsilon^2 \leq 2^{-2\alpha} \log \log(\varepsilon^{-1})(1 + o(1)) \). This implies (7.60). This completes part (a) of the theorem.

In order to obtain the upper bounds, we need the following corollary.

**Corollary 7.2** Let \( a_k = k^{\alpha} \) and \( \sigma_k = \exp(\beta k), \ k \in \mathbb{N}, \ \alpha > 0, \ \beta > 0 \). Let \( r_{\varepsilon} > 0, \ r_{\varepsilon} \rightarrow 0 \). Set \( m = r_{\varepsilon}^{-\alpha} + O(1) \). Then, for \( m_1 = m + c \) and \( c > 0 \) large enough, one has
\[
\inf_{\eta \in \Theta(\varepsilon)} \max_{1 \leq l \leq m_1} \eta_l^2 \times m^{-2\alpha} \exp(-2m\beta) \times \varepsilon^2 u_{\varepsilon}, \quad \text{as} \quad m_1 > m.
\]

**Proof.** The first rate relation follows from Proposition 7.4 and is similar to the proof of Corollary 7.1, the second one follows from (7.19). This completes the proof of the corollary. \( \square \)

We now obtain the upper bounds. In view of the embedding (4.17), it suffices to consider the case \( q = 2 \). We work along the lines of the proof of part (b) of Theorem 5.2 and apply Corollary 7.2, (7.18), and (7.19). This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (7.18). The theorem now follows.
7.13 Proof of Theorem 5.4

We first obtain the lower bounds. By making \( r_\varepsilon(\kappa) \) larger, we can assume, without loss of generality, that \( C = 1 \), i.e., for all \( \kappa \in \Sigma \),

\[
u^{lin}_\varepsilon(\kappa, r_\varepsilon(\kappa)) = \sup_{\kappa \in \Sigma} \nu^{lin}_\varepsilon(\kappa, r_\varepsilon(\kappa)) = \nu^{lin}_\varepsilon(\Sigma),
\]

and, some \( d > 0 \), \( \nu^{lin}_\varepsilon(\Sigma)/\log(\varepsilon^{-1}) = d \). Taking \( A_\varepsilon = (\varepsilon \sqrt{\nu^{lin}_\varepsilon(\Sigma)})^{-1} \), find a collection \( \kappa_l, 1 \leq l \leq M = M_\varepsilon \times M(A_\varepsilon, \Sigma) \) such that, for \( m(A_\varepsilon, \kappa_l) = m_l \), one has

\[|m_l - m_k| > 1, \forall k, l = 1, ..., M, \ k \neq l; \ r_\varepsilon(\kappa_l) \in \Delta^*_m.\]

Observe that \( \log \log(A_\varepsilon) \sim \log \log(\varepsilon^{-1}) \) and that, by (5.13),

\[\log(M(A_\varepsilon, \Sigma)) \sim \log(M), \ \lim \inf \log(M)/\log(\varepsilon^{-1}) = b > 0.\]

For each \( l = 1, 2, ..., M \), take \( t_l \in [0, 1] \) such that

\[r_\varepsilon^2(\kappa_l) = \frac{1 - t_l}{a_{m_l}^2(\kappa_l)} + \frac{t_l}{a_{m_l-1}^2(\kappa_l)}.\]

Let us now consider a collections of vectors \( \eta^l = (0, 0, ..., 0, \eta^l_{m_l-1}, \eta^l_{m_l}, 0, 0, ...) \) with

\[\eta^l_{m_l-1} = \frac{\sqrt{t_l}}{a_{m_l-1}(\kappa_l)\sigma_{m_l-1}(\kappa_l)}, \ \eta^l_{m_l} = \frac{\sqrt{1 - t_l}}{a_{m_l}(\kappa_l)\sigma_{m_l}(\kappa_l)}.\]

One can easily check that \( \eta^l \in \Theta_{\kappa_l}(r_\varepsilon(\kappa_l)) \) and

\[\varepsilon^{-2}\|\eta^l\|_2^2 = u^{lin}_\varepsilon(r_\varepsilon(\kappa_l)) = u^{lin}_\varepsilon(\Sigma), \ (\eta^l, \eta^k) = 0, \forall k, l = 1, 2, ..., M, \ k \neq l. \quad (7.61)\]

We now work along similar lines of the proof of part (a) of Theorem 5.2. We consider the prior

\[\pi = \frac{1}{M} \sum_{l=1}^{M} \delta_{\eta^l}\]

and the mixture \( P_\pi \) over \( \pi \). Since \( \pi(\Theta_\varepsilon(\Sigma)) = 1 \), it suffices to verify (7.52). Similarly to (7.53), one has, by (7.61),

\[E_{\varepsilon,0}(dP_\pi/dP_{\varepsilon,0} - 1)^2 = M^{-2} \sum_{l=1}^{M} (\exp(\|\eta^l\|_2^2/\varepsilon^2) - 1) = M^{-1} \exp(u^{lin}_\varepsilon(\Sigma)).\]

Therefore, the relation (7.52) holds true as

\[\lim \sup \frac{u^{lin}_\varepsilon(\Sigma)}{\log(M)} < 1.\]

By (7.13), it suffices to take \( d \in (0, b) \). This completes part (a) of the theorem.

We now obtain the upper bounds. First, by (5.14), we have

\[\alpha(\psi_\varepsilon) \leq \sum_{k=1}^{\infty} P_{\varepsilon,0}(|y_k|/\varepsilon \geq T_{\varepsilon,k}) = 2 \sum_{k=1}^{\infty} \Phi(-T_{\varepsilon,k}) = o(1).\]
Next, let $\eta \in \Theta_{\epsilon, \kappa}(r_\epsilon(\kappa))$. We have

$$\beta_\epsilon(\psi_\epsilon, \eta) \leq \inf_{k \in \mathbb{N}} P_{\epsilon, \eta}(|y_k|/\epsilon < T_{\epsilon, k}) \leq \inf_{k \in \mathbb{N}} \Phi(T_{\epsilon, k} - \epsilon^{-1}|y_k|),$$

and it suffices to check that, uniformly over $\kappa \in \Sigma$ and $\eta \in \Theta_{\epsilon, \kappa}(r_\epsilon(\kappa))$,

$$\sup_{k \in \mathbb{N}} (\epsilon^{-1}|\eta_k| - T_{\epsilon, k}) \to \infty. \quad (7.62)$$

Let $m = m(A_{\epsilon, \kappa}, \kappa)$ where $A_{\epsilon, \kappa} = (\epsilon \sqrt{u_{\epsilon, \kappa}(\kappa, r_\epsilon(\kappa))})^{-1}$. We have $r_\epsilon(\kappa) \in \Delta_m^*$. Since the sequence $T_{\epsilon, k}^2 \sim 2\log(k)$ increases in $k$, the relation (7.62) follows from

$$\liminf \frac{\max_{1 \leq k \leq m} \epsilon^{-2}\eta_k^2}{\max(T_{\epsilon, k}^2, 2\log(m))} > 2. \quad (7.63)$$

Applying Proposition [7.3] to the collection $H_{m, k} = 1, \ k = 1, 2, \ldots, m$, we have

$$\max_{1 \leq k \leq m} \epsilon^{-2}\eta_k^2 \geq u_{\epsilon, \infty}(\kappa, r_\epsilon(\kappa))(1/(2\sqrt{2}) + o(1)) \geq u_{\epsilon, \infty}(\Sigma)(1/(2\sqrt{2}) + o(1)).$$

Also, since $m(A, \kappa)$ increases in $A$, and $A_{\epsilon, \kappa} \leq bA_{\epsilon}$ where $A_{\epsilon} = (\epsilon \sqrt{\log \log(\epsilon^{-1})})^{-1}$, we have

$$2\log(m) \leq 2L(bA_{\epsilon}, \Sigma) \leq 2B \log\log(\epsilon^{-1})(1 + o(1)), \quad T_{\epsilon}^2 \leq \log\log(\epsilon^{-1}).$$

Therefore, the relation (7.63) holds true as $u_{\epsilon, \infty}(\Sigma) > D \log \log(\epsilon^{-1})$, for $D > 2\sqrt{2} \max(2B, 1)$. This completes part (b) of the theorem.

Part (c) of the theorem follows immediately from parts (a) and (b) of the theorem and the definition $r_\epsilon^{ad}(\kappa)$. The theorem now follows.

References

[1] Abramovich, F. and Silverman, B.W. (1998). Wavelet decomposition approaches to statistical inverse problems. *Biometrika*, 85, 115–129.

[2] Bakushinskii, A.B. (1969). On the construction of regularizing algorithms under random noise. *Soviet Math. Doklady*, 189, 231–233.

[3] Belitser, E.N. and Levit, B.Y. (1995). On minimax filtering on ellipsoids. *Math. Methods Statist.*, 4, 259–273.

[4] Brown, L.D. and Low, M.G. (1996). A constraint risk inequality with applications to nonparametric functional estimation. *Ann. Statist.*, 24, 2524–2535.

[5] Cavalier, L. (2008). Nonparametric statistical inverse problems. *Inverse Problems*, 24, 034004 (19pp).

[6] Cavalier, L. and Tsybakov, A.B. (2002). Sharp adaptation for inverse problems with random noise. *Probab. Theory Relat. Fields*, 123, 323–354.

[7] Cavalier, L., Golubev, G.K., Lepskii, O. and Tsybakov, A.B. (2004). Block thresholding and sharp adaptive estimation in severely ill-posed inverse problems. *Theory Probab. Appl.*, 48, 426–446.
[8] Deans, S.R. (1983). *The Radon Transform and Some of its Applications*. New York: John Wiley and Sons.

[9] Donoho, D.L. (1995). Nonlinear solution of linear inverse problems by wavelet-vaguelette decomposition. *Appl. Comput. Harmon. Anal.*, 2, 101–126.

[10] Efromovich, S. and Samarov, A. (1996). Asymptotic equivalence of nonparametric regression and white noise models has its limits. *Statist. Probab. Lett.*, 28, 143–145.

[11] Ermakov, M.S. (1990). Minimax detection of a signal in a Gaussian white noise. *Theory Probab. Appl.*, 35, 667–679.

[12] Goldenshluger, A. and Pereverzev, S.V. (2000). Adaptive estimation of linear functionals in Hilbert scales from indirect white noise observations. *Probab. Theory Related Fields*, 118, 169–186.

[13] Golubev, G.K. and Khasminskii, R.Z. (1999). Statistical approach to some inverse boundary problems for partial differential equations. *Problems Inform. Transm.*, 35, 51–66.

[14] Golubev, G.K. and Khasminskii, R.Z. (2000). Statistical approach to Cauchy problem for Laplace equation In *State of the Art in Probability and Statistics*, A Festschrift for W.R. van Zwet, (Eds. de Gunst, M., Klaassen, C. and van der Vaart, A.), IMS Lecture Notes Monograph Series, Ser. 36, pp. 419–433, Beachwood, OH: Institute of Mathematical Statistics.

[15] Ibragimov, I.A. and Khasminskii, R.Z. (1985). On nonparametric estimation of the value of a linear functional in Gaussian white noise. *Theory Probab. Appl.*, 29, 18–32.

[16] Ingster, Yu.I. (1993). Asymptotically minimax hypothesis testing for nonparametric alternatives. I, II, III. *Math. Methods Statist.*, 2, 85–114, 171–189, 249–268.

[17] Ingster, Yu.I. and I. A. Suslina, I.A. (2003). *Nonparametric Goodness-of-Fit Testing under Gaussian Model*. Lectures Notes in Statistics, Vol. 169, New York: Springer-Verlag.

[18] Ingster, Yu.I. and I. A. Suslina, I.A.(2004). Nonparametric hypothesis testing for small type I errors. I *Math. Methods Statist.*, 13, 409–459.

[19] Johnstone, I.M. (1999). Wavelet shrinkage for correlated data and inverse problems: adaptivity results. *Statistica Sinica*, 9, 51–83

[20] Laurent, B., Loubes, J.M. and Marteau, C. (2009). Non asymptotic minimax rates of testing in signal detection with heterogeneous variances. *Preprint*. arXiv:0912.2423v1 [math.ST], 12 December 2009.

[21] Nussbaum, M. (1985). Spline smoothing in regression models and asymptotic efficiency in $L_2$. *Ann. Statist.*, 13, 984–997.

[22] Nussbaum, M. (1999). Minimax risk: Pinsker bound. In *Encyclopedia of Statistical Sciences*, (Ed. Kotz, S.), Update Vol. 3, pp. 451–460, New York: John Wiley & Sons.

[23] Pinsker, M.S. (1980). Optimal filtering of square integrable signals in Gaussian white noise. *Problems Info. Trans.*, 16, 120–133.
[24] Reed, M. and Simon, B. (1980). *Methods of Modern Mathematical Physics I: Functional Analysis.* Revised and Enlarged Edition, San Diego: Academic Press.

[25] Spokoiny, V.G. (1996). Adaptive hypothesis testing using wavelets. *Ann. Statist.*, 24, 2477–2498.

[26] Sudakov, V.N. and Khalfin, L.A. (1964). Statistical approach to ill-posed problems in mathematical physics. *Soviet Math. Doklady.*, 157, 1094–1096.

[27] Tikhomirov, V.M. (1976). *Some Problems of Approximation Theory.* Moscow: Moscow State University Press.

[28] Wahba, G. (1990). *Spline Models for Observational Data.* SIAM: Philadelphia.