Reverse engineering of a non-lossy adiabatic Hamiltonian for non-Hermitian systems

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We generalize the quantum adiabatic theorem to the non-Hermitian system and build a strict adiabaticity condition to make the adiabatic evolution non lossy when taking into account the effect of adiabatic phase. According to the strict adiabaticity condition, the non-adiabatic couplings and the effect of the imaginary part of adiabatic phase should be eliminated as much as possible. Also the non-Hermitian Hamiltonian reverse engineering method is proposed for adiabatically driving an artificial quantum state. Concrete two-level system is adopted to show the usefulness of the reverse engineering method. We obtain the desired target state by adjusting extra rotating magnetic fields at a predefined time. Furthermore, the numerical simulation shows that certain noise and dissipation in the systems are no longer undesirable, but play a positive role in the scheme. Therefore, the scheme is quite useful for quantum information processing in some dissipative systems.

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I. INTRODUCTION

As an essential task in many areas of quantum information science ranging from quantum information processing \cite{1, 2} and coherent manipulation of quantum systems \cite{3, 4} to high-precision measurements \cite{5, 6}, the quantum-state engineering (QSE) \cite{7, 8}, has attracted much attention, which promotes the development of experimental technique and theoretical scheme. The quantum adiabatic theorem (QAT), an important way of realizing QSE, has been widely studied and the basic properties of the QAT are being scrutinized both theoretically and experimentally \cite{9, 10}. The basic idea of QAT can be summarized as follows: if the control parameters in the time-dependent Hamiltonian change slowly, the system will follow closely along an eigenstate trajectory up to a adiabatic phase factor when it is initially in one of the eigenstates. Thereinto, the adiabatic phase is a complicated factor which can be divide into dynamical phase and geometrical phase \cite{11}. Interestingly, in the Hermitian adiabatic Hamiltonians scenario, one can focus on the dynamics of the eigenstate, and neglect the complicated phase factor since it can be discarded as the common pure phase. However, in practice the quantum system inevitably interacts with the surrounding environment, e.g. the non-Hermitian (NH) systems \cite{12, 13}. In this case, the complicated adiabatic phase factor could not be simply discarded as the common pure phase any more, since it generally is not a pure (real) phase factor. Then, the ideal robustness and the intended dynamics may be spoiled by the accumulation of the imaginary part of the adiabatic phase due to noise and undesired interactions. Thus, it is very worthwhile to look for the novel methods which are robustness, and figure out the strict adiabaticity condition for NH Hamiltonians when taking into account the effect of adiabatic phase.

In fact, several authors have paid attention to the study of adiabaticity in NH systems \cite{14, 15}. For example, Miniatura et al. have set a rough estimate of an adiabaticity condition by analogy with the Hermitian counterpart and recognized the importance of the nonadiabatic transition \cite{16}. Subsequently, Sun has devoted to the generalization of the high-order adiabatic approximation method for the NH quantum systems by using perturbation theory and integration by parts, and obtained an adiabaticity condition similar to the Hermitian one with the damping factor and the oscillating factor \cite{17}. Recently, Dridi et al. have established a generalization of the Davis-Dykhnpe-Pechukas formula by the complex time method and showed a general adiabatic approximation for lossy two-state models \cite{18, 19}. More recently, Ibáñez and Muga have generalized the concept of population for NH systems to characterize adiabaticity and worked out an approximate adiabaticity criterion \cite{20}. Indeed, the adiabaticity of a given NH system has been discussed well with those excellent methods \cite{21, 22}. However, in principle above methods didn’t give a clear quantitative analysis about the dynamics of the bare state in the eigenstate and the adiabatic phase. In some cases, we may observe a false adiabaticity due to the problematic or obscure population concept \cite{23, 24}. In addition, they will also be limited severely by the presence of the strong dissipation effects in some applications \cite{25, 26}. Above problems make designing perfect scheme to reach the intended dynamics for the NH systems very challenging.

In this paper, we will introduce a novel method to solve the problems shown above. Different from the previous schemes \cite{27, 28, 29}, which are proposed to explore the adiabatic approximation condition for a given NH system, we are dedicated to setting a strict adiabaticity condition to make the adiabatic evolution non lossy when taking into account the effect of adiabatic phase, and exploring the Hamiltonian which will exactly satisfy the strict condition via the reverse engineering method. The scheme has following advantages: (1) We take the effect of adiabatic phase into consideration, and make the adiabatic process be a strict one without lossy in NH system. (2) By using the reverse engineering method, we can design the Hamiltonian to realize the intended dynamics according to the demand. (3) The noise or certain dissipation in the systems can do positive job in the scheme. We can obtain the desired target state by adjusting extra rotating magnetic fields at a predefined time even in the dissipative system. Therefore, the scheme makes it possible to realize the QSE for some dissipative systems.

The rest of this paper is arranged as follows. In Sec. II we briefly review some important properties of the NH Hamiltonians, and build a strict adiabaticity condition which contains two parts: an auxiliary adiabaticity condition with respect to adiabatic phase and a general adiabatic condition given via the Feshbach P-Q partitioning technique \cite{22, 23}. In Sec. III we explicitly discuss how to engineer the NH Hamiltonian which could exactly satisfy the strict adiabaticity condition. Then, we consider a concrete two-level system example to show the usefulness of our reverse engineering method. Both experimental feasibility and population engineering are discussed step by step in Sec. IV. Finally we give a summary in Sec. V.
II. BASIC THEORIES

A. NH Hamiltonians: basic formulas

For NH systems, the usual approximations and criteria are not necessarily valid, so the results which are applicable for Hermitian systems have to be reconsidered and modified. We first briefly recall some important properties of the NH Hamiltonians $\hat{H}_0$ [24, 25]. Consider an arbitrary time-dependent NH Hamiltonian $\hat{H}(t)$ with $N$ nondegenerate instantaneous eigenstates $\{|\phi_n(t)\rangle\}$, $n=1,2,...,N$. It satisfies following eigenvalue equation

$$H(t)|\phi_n(t)\rangle = E_n(t)|\phi_n(t)\rangle. \tag{1}$$

As the adjoint operator of $\hat{H}(t)$, $\hat{H}(t)^\dagger$, will also satisfy following eigenvalue equation

$$\hat{H}(t)^\dagger |\phi_n(t)\rangle = E_n^*(t)|\phi_n(t)\rangle, \tag{2}$$

where $\{|\phi_n(t)\rangle\}$ are the instantaneous eigenstates of $\hat{H}(t)^\dagger$ and also the biorthogonal partners of $\{|\phi_n(t)\rangle\}$, the asterisk means complex conjugate. The biorthogonal partners are normalized to satisfy the biorthogonality relation

$$\langle \phi_n(t)|\phi_m(t)\rangle = \delta_{nm}, \tag{3}$$

and the closure relation

$$\sum_n \langle \phi_n(t)|\phi_n(t)\rangle = \sum_n |\phi_n(t)\rangle\langle \phi_n(t)| = 1. \tag{4}$$

With above properties, the Hamiltonian and its adjoint can be rewritten as

$$\hat{H}(t) = \sum_n |\phi_n(t)\rangle E_n(t)|\phi_n(t)\rangle, \tag{5}$$

$$\hat{H}(t)^\dagger = \sum_n |\phi_n(t)\rangle E_n^*(t)|\phi_n(t)\rangle.$$

B. The auxiliary adiabaticity condition for the NH systems with respect to adiabatic phase

According to the adiabatic theorem, a state with initial condition $|\phi(0)\rangle = |\phi_0(0)\rangle$ will evolve adiabatically if its dynamics is well approximated by $|\phi(t)\rangle \approx e^{i\hat{H}_0(t)}|\phi_0(t)\rangle$. Furthermore, if $|\phi(t)\rangle$ is the instantaneous state of the system Hamiltonian $\hat{H}(t)$ and $|\phi(t)\rangle$ satisfies the Schrödinger equation $\dot{\phi}(t) = i\hat{H}(t)|\phi(t)\rangle$, we can obtain the adiabatic phase

$$\beta_n(t) = \int_0^t [-E_n(t') + i\langle \phi_n(t')|\phi_n(t')\rangle] dt'. \tag{7}$$

However, this ansatz of the adiabaticity for the NH system is not strict. The imaginary part of the adiabatic phase will induce the decay of system and cause confusion about the validity of the adiabaticity. Consequently, it’s necessary to forcibly eliminate $\text{Im}[\beta_n(t)]$ to keep the adiabatic scheme working well, that is, we should insure

$$-\text{Im}[E_n(t)] + \text{Re}[i\langle \phi_n(t)|\phi_n(t)\rangle] = 0. \tag{8}$$

Then, the adiabatic phase can be safely discarded as a common pure phase when we investigate the dynamics of the target state, even in the NH Systems. Notice that, Eq. (3) is the auxiliary adiabaticity condition which allows one to make the adiabatic evolution non lossy with respect to adiabatic phase, which is the primary result to be used in following work.

C. The general adiabatic condition for the NH systems

In general, a state at time $t$ can be expressed as

$$|\Psi(t)\rangle = \sum_n \Psi_n(t)|\phi_n(t)\rangle, \tag{9}$$

where the phase factor $\beta_n(t)$ satisfies Eq. (7) for arbitrary $n$ and $\Psi_n(t)$ is considered as a complex function. It’s obvious that $\Psi_n(t)$ is the key coefficient associated with the dynamics of $|\phi_n(t)\rangle$. Therefore, an exact dynamical equation for $\Psi_n(t)$ is highly desirable. Assuming $|\Psi(t)\rangle$ satisfies the Schrödinger equation, we can obtain the following equations,

$$i\dot{\Psi}_n(t) = -i \sum_{m\neq n} \langle \phi_n(t)|\phi_m(t)\rangle e^{i(\beta_m(t) - \beta_n(t))} \Psi_m(t)$$

$$= \sum_{m\neq n} H'_{nm} |\Psi_m(t)\rangle / |\Psi_n(t)\rangle, \tag{10}$$

$$i|\Psi'_n(t)\rangle = \sum_{m\neq n} H'_{mn} |\Psi_m(t)\rangle / |\Psi_n(t)\rangle = H'(t)|\Psi(t)\rangle \tag{11}$$

where $H'_{mn} = -i \langle \phi_n(t)|\phi_m(t)\rangle e^{i(\beta_m(t) - \beta_n(t))}$. According to adiabatic theorem, $|\phi_n(t)\rangle$ will evolve adiabatically if the term on the left-hand side of Eq. (10) approaches to zero. Moreover, it is interesting to find that the form of Eq. (10) is similar to the form of the artificial Schrödinger equation Eq. (11) for the vector $|\Psi'_n(t)\rangle = [\Psi_1(t), \Psi_2(t), \Psi_3(t),...\Psi_n(t)]^T$ (the superscript $T$ denotes the transpose operator) with the rotating representation Hamiltonian $H'(t)$. So we can deal with $|\Psi_n(t)\rangle$ with the help of Eq. (11). In fact, $H'(t)$ describes the coupling transitions between the instantaneous eigenstates $\{|\phi_n(t)\rangle\}$, the so-called non-adiabatic couplings.

We should stress that in this paper we don’t intend to research fully adiabatic dynamics (for all modes). The problem we address is that the adiabatic dynamics of one target component (for one mode). Without loss of generality, the target component can be denoted as $\Psi_1(t)$, corresponding to the target eigenstate $|\phi_1(t)\rangle$ of $H(t)$. In order to obtain a better understanding of the adiabatic dynamics of $\Psi_1(t)$, the Feshbach P-Q partitioning technique [34, 35] is introduced. According to the P-Q partitioning technique, the state $|\Psi'_n(t)\rangle$ and the rotating representation Hamiltonian $H'(t)$ in the Schrödinger equation Eq. (11) can be always partitioned into the following form,

$$|\Psi'_n(t)\rangle = \begin{bmatrix} P(t) \\ Q(t) \end{bmatrix}, \quad H'(t) = \begin{pmatrix} 0 & R \\ W & D \end{pmatrix}, \tag{12}$$

where $P$ associated with the target state is equal to $\Psi_1(t)$, while $Q$ associated with $(N-1)$-dimensional vector denotes the rest of the state spaces. The vector $R = [R_2, R_3, ..., R_{N_1}]$ with $R_{n} = -i \langle \phi_1(t)|\phi_m(t)\rangle e^{i(\beta_m(t) - \beta_1(t))}$, while, the vector $W = [W_2 W_3 ... W_{N_1}]$ with $W_{n} = -i \langle \phi_1(t)|\phi_m(t)\rangle e^{i(\beta_m(t) - \beta_1(t))}$.

The $(N-1)\times(N-1)$-matrix $D \equiv \sum_{m\neq n} D_{mn} |\Psi_m(t)\rangle\langle \Psi_n(t)|$, where $D_{mn} = -i \langle \phi_n(t)|\phi_m(t)\rangle e^{i(\beta_m(t) - \beta_n(t))}$. Substituting Eq. (12) into Eq. (11), we obtain the following equations

$$i\dot{P} = RQ, \quad i\dot{Q} = WP + DQ. \tag{13}$$

The formal solution of Eq. (13) can be written as

$$i\dot{P} = -iR(t) \int_0^t G(t,s)W(s)P(s)ds + R(t)G(t,0)Q(0), \tag{14}$$

where $G(t,s) = T_{W} \{ \exp[-i \int_s^t D(s')ds'] \}$ is the time-ordered evolution operator. Under the condition $P(0) = 1$ and $Q(0) = 0$, we have the exact dynamical equation for the P part

$$\dot{P} = -R(t) \int_0^t G(t,s)W(s)P(s)ds = -\int_0^t g(t,s)P(s)ds, \tag{15}$$
where $g(t, s) = R(t)G(t, s)W(s)$ is an effective propagator which plays a very important role in the analysis of adiabaticity. Notice that, the general adiabatic approximation condition is $\int g(t, s)P(s)ds = 0$, that is, the propagator $g(t, s) = 0$ or $g(t, s)$ is factored by a rapid oscillating function $\Delta(t)$, which is also the primary result to be used in following work.

For an effective two-level system, the associated rotating representation Hamiltonian $H'(t)$ reads

$$H'(t) = -i \left( \begin{array}{cc} \phi_1(t) \phi_2(t) & e^{-i\Delta \beta(t)} \\ e^{i\Delta \beta(t)} & \phi_3(t) \phi_4(t) \end{array} \right),$$

(16)

where $\Delta \beta(t) = \beta_2(t) - \beta_1(t)$. When the effective two-level system is initially in the eigenstate $|\phi_1(t)\rangle$, the propagator $g(t, s)$ reads

$$g(t, s) = -\langle \phi_2(t)|\phi_2(s)\rangle \langle \phi_3(s)|e^{i\int_{t'}^{s'}(\beta_2(s') - \beta_1(s'))ds'}.$$ 

(17)

Notice that, Eqs. (16) and (17) are also the primary results to be used in following work.

III. THE NH HAMILTONIAN REVERSE ENGINEERING METHOD AND APPLICATIONS

A. The NH Hamiltonian reverse engineering method

In this section, we will start with an engineering method about how to engineer the yet unknown NH Hamiltonian which could exactly satisfy the strict adiabaticity condition. From the special properties of the NH Hamiltonian [see Eq. (3)], one can conclude that the design process can be divided into two steps: designing the eigenvectors and modifying the eigenvalues. Here we should make some remarks on the eigenvectors designs. (1) The goal of our scheme is driving the eigenvectors of an initial Hamiltonian into those of a final Hamiltonian, so the designed eigenvectors must connect the initial state with the target state. (2) Our scheme is working in the NH Hamiltonians scenario, the eigenvectors must satisfy the biorthogonality relation and the closure relation. (3) The eigenvectors must evolve adiabatically, that is, they should satisfy the general adiabatic condition which has been discussed in Sec. III C. Once the eigenvectors designs are completed, we can reconsider and modify the eigenvalues resorting to normalization ambiguities in the eigenvectors of NH Hamiltonians. More specifically, we should consider the auxiliary adiabaticity condition with respect to adiabatic phase for the new eigenvector in this step.

Before the elaborating on manipulating a two-level system to the target state, we will give a simple restriction on eigenvectors to satisfy the biorthogonality relation and the closure relation from the view of mathematics. Without loss of generality, for a $n$ dimensions system, we assume the eigenstates $\{ |\phi_n(t)\rangle \}$ of $H(t)$ read

$$\langle \phi_1(t) | = A_{11}(t)|1\rangle + A_{21}(t)|2\rangle + \cdots + A_{n1}(t)|n\rangle,$$

$$\langle \phi_2(t) | = A_{12}(t)|1\rangle + A_{22}(t)|2\rangle + \cdots + A_{n2}(t)|n\rangle,$$

where $|1\rangle (l=1, 2, 3 \cdots n)$ is the bare state for the system and $A_{jk}(t) (j, k = 1, 2, 3 \cdots n)$ is a desirable function associated with the bare state $|j\rangle$ in $|\phi_j(t)\rangle$. In a similar manner, the biorthogonal states of $\{ |\phi_n(t)\rangle \}$ are expressed as

$$\langle \phi_1(t) | = \hat{A}_{11}(t)|1\rangle + \hat{A}_{12}(t)|2\rangle + \cdots + \hat{A}_{n1}(t)|n\rangle,$$

$$\langle \phi_2(t) | = \hat{A}_{12}(t)|1\rangle + \hat{A}_{22}(t)|2\rangle + \cdots + \hat{A}_{n2}(t)|n\rangle,$$

where $|l\rangle (l=1, 2, 3 \cdots n)$ also is the bare state for the system and $\hat{A}_{jk}(t) (j, k = 1, 2, 3 \cdots n)$ is a desirable function associated with the state $|k\rangle$ in $|\phi_k(t)\rangle$. Let’s introduce two matrix constructed by $A_{jk}(t)$ and $\hat{A}_{jk}(t)$, respectively,

$$A^T(t) = \begin{pmatrix} A_{11}(t) & A_{21}(t) & \cdots & A_{n1}(t) \\
A_{12}(t) & A_{22}(t) & \cdots & A_{n2}(t) \\
\vdots & \vdots & \ddots & \vdots \\
A_{1n}(t) & A_{2n}(t) & \cdots & A_{nn}(t) \end{pmatrix},$$

(20)

therefore, mathematically, the determinant of $A^T(t)$ and $A(t)$ should never be zero for the reverse engineered biorthogonal partners.

B. Engineering quantum states by the reverse engineering method

As an example, we now demonstrate how to engineer quantum state of a single qubit by means of the reverse engineering method. For the sake of simplicity, we assume the eigenstates $\{ |\phi_n(t)\rangle \}$ of $H(t)$ read

$$\langle \phi_1(t) | = A_{11}(t)|1\rangle + A_{21}(t)|2\rangle,$$

$$\langle \phi_2(t) | = A_{12}(t)|1\rangle + A_{22}(t)|2\rangle.$$ 

(22)

The choice of coefficients $A_{11}(t)$ and $A_{21}(t)$ is various, we can choose the interested state as the target state $|\phi_1(t)\rangle$. Without loss of generality, by setting $A_{11}(t) = -\lambda(t)\sin\alpha(t)$, $A_{21}(t) = \cos\alpha(t)$, and $A_{22}(t) = \lambda(t)\cos\alpha(t)$, we can obtain

$$A(t) = \begin{pmatrix} -\lambda(t)\sin\alpha(t) & \lambda(t)\cos\alpha(t) \\
\lambda(t)\cos\alpha(t) & \sin\alpha(t) \end{pmatrix},$$

(23)

where $\lambda(t)$ and $\alpha(t)$ are time-dependent complex functions. Obviously, $A(t)$ will be an invertible matrix if $\lambda(t)\neq0$ is established all the time. Then, we can obtain the accurate solution of $A'(t)$

$$A'(t) = \frac{1}{\lambda(t)} \begin{pmatrix} \sin\alpha(t) & \cos\alpha(t) \\
\cos\alpha(t) & \sin\alpha(t) \end{pmatrix}.$$ 

(24)

Now, we start to consider the general adiabatic condition for the designed system and calculate following matrix elements,

$$\langle \phi_1(t)|\phi_1(t)\rangle = \frac{\lambda(t)}{\lambda(t)} \sin^2\alpha(t),$$

$$\langle \phi_2(t)|\phi_2(t)\rangle = \frac{\lambda(t)}{\lambda(t)} \cos^2\alpha(t),$$

$$\langle \phi_2(s)|\phi_1(s)\rangle = -\alpha \frac{\lambda(s)}{\lambda(s)} \sin \alpha(s) \cos \alpha(s),$$

$$\langle \phi_1(t)|\phi_2(t)\rangle = \alpha - \frac{\lambda(t)}{\lambda(t)} \sin \alpha(t) \cos \alpha(t).$$

(25)

Then, $|\phi_1(t)\rangle$ will adiabatically evolve if the propagator $g(t, s) = 0$ or $g(t, s)$ is factored by a rapid oscillating function. Mathematically, the simplest choice is setting $\langle \phi_1(t)|\phi_2(t)\rangle = 0$ (we also can set $\langle \phi_2(s)|\phi_1(s)\rangle = 0$), and $\lambda(t)$ can be solved as

$$\lambda(t) = \tan\alpha(t),$$

(26)
where \(\alpha(t) \neq \pi r / 2, \pi r Z\). Here, we should note that \(g(t, s)\) will also be factored by a rapid oscillating function if \(\lambda(t)\) is a constant and \(\approx 0\). In fact, this kind of setting was examined in detail in Ref. \(^{31}\) by Ibáñez and Muga. However, the weakness of this kind of setting is quite obvious, the target state \(|\phi(t)\rangle\) could not be engineered to reach an arbitrary target state in a short time as \(\approx 0\). For the sake of generality and giving more choices for the realization of QSE, \(\lambda(t)\) will be chosen as Eq. (22) in the paper. Up till now, we have successfully completed the eigenvectors designs and obtained following unnormalized eigenvectors

\[
|\phi_1(t)\rangle = -\sin^2(\alpha(t)|1\rangle + \cos(\alpha(t)|2\rangle), \\
|\phi_2(t)\rangle = \sin(\alpha(t)|1\rangle + \cos(\alpha(t)|2\rangle). \tag{27}
\]

According to Eq. (4), the system Hamiltonian takes the form

\[
H(t) = \left( E_1(t) + \Delta_E(t) \cos^2(\alpha(t)) - \Delta_E(t) \sin^2(\alpha(t)) \right) \Delta_E(t) \cos^2(\alpha(t)) \tag{28}
\]

where \(\Delta_E(t) \equiv E_2(t) - E_1(t)\) is eigenvalue difference of the system, and it can not equal zero due to the nondegeneracy.

Until now, the eigenvalues of NH Hamiltonians are still undetermined, although the eigenvectors designs are completed. We should reconsider and modify the normalization eigenvectors residing to the eigenvalues ambiguity in the eigenvectors of NH Hamiltonians. One can find that following states are also the eigenvectors of Eq. (28) with the same eigenvalues,

\[
|\tilde{\phi}_1(t)\rangle = f_1(t)|\phi_1(t)\rangle, \\
|\tilde{\phi}_2(t)\rangle = f_2(t)|\phi_2(t)\rangle, \tag{29}
\]

where \(f_1(t)\) and \(f_2(t)\) can be arbitrary non-zero functions. Then, the biorthogonal partners of \(|\phi_1(t)\rangle, |\phi_2(t)\rangle\) read

\[
|\tilde{\phi}_1(t)| = \frac{1}{f_1(t)}|\phi_1(t)|, \\
|\tilde{\phi}_2(t)| = \frac{1}{f_2(t)}|\phi_2(t)|. \tag{30}
\]

By calculating, we can find the propagator \(g(t, s)\) is also factored by a rapid oscillating function for the new eigenvector \(|\tilde{\phi}_1(t)\rangle\). That is, \(|\tilde{\phi}_1(t)\rangle\) will continue to evolve adiabatically in current system without additional Hamiltonians, even though \(f_1(t)\) is an arbitrary non-zero function. Substituting \(|\tilde{\phi}_1(t)\rangle\) into Eq. (6), we obtain

\[
|\tilde{\phi}(t)\rangle = e^{i\tilde{\phi}_1(t)}|\tilde{\phi}_1(t)\rangle = e^{i\tilde{\phi}_1(t)}f_1(t)|\phi_1(t)\rangle, \tag{31}
\]

where the adiabatic phase for the new eigenread vectors

\[
\beta_1(t) = \int_0^t \left[ E_1(t') + i\tilde{\phi}_1(t')|\phi_1(t')\rangle\right] dt' = \int_0^t \left[ E_0(t') + i\tilde{\phi}_1(t')|\phi_1(t')\rangle + i\alpha(t)|1\rangle\right] dt'. \tag{32}
\]

As a consequence, the normalization ambiguities in the eigenvectors only generates a constant multiplication factor \(f_1(t)\), and the target state \(|\phi(t)\rangle\) always evolves adiabatically in current system. Furthermore, when the auxiliary adiabatic condition with respect to the adiabatic phase [see Eq. (5)] is taken into account,

\[
\text{Im}[E_1(t)] = \text{Re}[\tilde{\phi}_1(t)|\phi_1(t)\rangle] = \text{Re}[\sin(\alpha(t)|1\rangle], \tag{33}
\]

the target state won’t suffer strong exponential variations which is remarkable for quantum information processing.

We can find that Eq. (25) can be expressed in terms of the Pauli matrices as

\[
H(t) = \frac{\Delta_E(t)}{2} \sigma_x - \frac{\delta(t)}{2} \sigma_y + \frac{\delta(t)}{2} \sigma_z + E_0(t) \mathbb{1}, \tag{34}
\]

where \(\delta(t) = \Delta_E(t) \cos(2\alpha(t))\) and \(E_0(t) = E_1(t) + \Delta_E(t)/2\) are the time-independent variables, and \(\mathbb{1}\) denotes the unit matrix. In fact, the real part of \(E_0(t)\) can be ignored by applying appropriate energy shift, which doesn’t play a negative role in the investigation of population of system. The system can be mapped onto the Hamiltonian

\[
H(t) = \frac{1}{2} \left( \Delta_E(t) \sigma_x - \delta(t) \sigma_y + \delta(t) \sigma_z + \text{Im}[E_0(t)] \mathbb{1} \right). \tag{35}
\]

It can be easily found there are only two variables, \(\Delta_E(t)\) and \(\alpha(t)\), in Eq. (25). Thus, the crucial NH Hamiltonian engineering can be cast into the \(\Delta_E(t)\) design and the \(\alpha(t)\) design. Theoretically speaking, besides the consistency condition \(\alpha(t) \neq 2n/\pi, \Delta_E(t) \neq 0\) and \(\sin(\alpha(t)) \approx 1\) (it should be noted that the initial state could make connection with the target state \(|\phi(t)\rangle\) by setting \(\sin(\alpha(t)) \approx 1\) according to Eq. (27)), there is almost no limit on the choices of \(\Delta_E(t)\) and \(\alpha(t)\) for engineering the system to reach an arbitrary target state at a predefined time. However, the choices of \(\alpha(t)\) and \(\Delta_E(t)\) will affect evolution speed for the target state and the feasibility in the practical realization. Especially, when the term \(\text{Im}[E_0(t)]\) in in Eq. (35) does not equal zero, the practical realization of this Hamiltonian is significantly challenged in experiments. We shall explore in the following subsection to find an appropriate physical model that can incorporate the resulting Hamiltonian.

### IV. EXPERIMENTAL FEASIBILITY AND NUMERICAL EXAMPLES

For the purpose of convenience, we consider a simple case of Eq. (35),

\[
\text{Im}[E_0(t)] = \text{Im}[E_1(t)] + \frac{\Delta_E(t)}{2} \approx 0. \tag{36}
\]

The Hamiltonian of system reduces

\[
H(t) = \frac{1}{2} \left( \Delta_E(t) \sigma_x - \delta(t) \sigma_y + \delta(t) \sigma_z \right). \tag{37}
\]

In general, there is no simple “real” field interaction leading to Eq. (37), since the off-diagonal terms of the resulting Hamiltonian are different. For example, we assume a semiclassical description of the interaction between a “real” magnetic field \(B(t)\) and a rotating spin qubit, where \(B(t) = B_x(t)e_x + B_y(t)e_y + B_z(t)e_z\), \(B_x(t) = M_0 \sin(\omega t)/2(\pi)\) is the Bohr magneton, and \(B_x(t)\) is real variable. Then, the Hamiltonian of this system reads

\[
H(t) = \begin{pmatrix} B_x(t) & B_z(t) + B_y(t) \\ B_z(t) + B_y(t) & -B_x(t) \end{pmatrix}, \tag{38}
\]

we can find that the off-diagonal terms are complex conjugate of each other which does meet the requirements. However, we may obtain the resulting Hamiltonian if the magnetic field \(B(t)\) is the complex signal field rather than the real signal field, for example,

\[
\begin{align*}
B_x(t) &\to A_x(t)e^{i\Theta_x(t)} = \text{Re}[\Delta_E(t)] + \text{Im}[\Delta_E(t)], \\
B_y(t) &\to A_y(t)e^{i\Theta_y(t)} = \text{Im}[\delta(t)] - i\text{Re}[\delta(t)], \\
B_z(t) &\to A_z(t)e^{i\Theta_z(t)} = \text{Re}[\delta(t)] + i\text{Im}[\delta(t)], \tag{39}
\end{align*}
\]

where \(A_x\) is the amplitude and \(\Theta_x\) is the phase. In fact, a similar complex signal field has been discussed in detail in Refs. \(^{52}\) and \(^{54}\) and references therein. Additionally, the phase \(\Theta_x\) can also be considered as the dissipation factor which is introduced by the noise (e.g., the dephasing effects due to the collisions or phase fluctuations of the magnetic fields or when the rotating-wave approximation fails for the strong magnetic fields \(^{53}\)). Therefore, the resulting Hamiltonian Eq. (37) is accessible experimentally with the complex signal field or the real signal field under some dissipation effects.

Now, let’s focus on how to design \(\Delta_E(t)\) and \(\alpha(t)\) from an experimental viewpoint. At first, we can write \(\alpha(t)\) in polar form

\[
\alpha(t) = \rho(t) \exp(i\theta(t)), \tag{40}
\]

where \(\rho(t)\) and \(\theta(t)\) are time-dependent real variables. It is useful to rewrite \(\Delta_E(t)\), taking into account Eq. (35) and Eq. (39), as

\[
\Delta_E(t) = \text{Re}[\Delta_E(t)] - i\sin[2\rho(t)\cos(\theta(t))] \cosh[2\rho(t)\sin(\theta(t))]. \tag{41}
\]
where the real part of $\Delta_E(t)$ is a undetermined parameter and the selection of $\text{Re}[\Delta_E(t)]$ seems quite arbitrary mathematically. However, $\Delta_E(t)$ is physically associated with the eigenvalue difference of the system [see Eq. (25)]. Thus, we should guarantee the modulus of $\text{Re}[\Delta_E(t)]$ is relatively large, otherwise, the system will undergo transitions between $|\varphi_1(t)\rangle$ and $|\varphi_2(t)\rangle$ constantly. Furthermore, $\Delta_E(t)$ is also associated with the magnetic field, we should consider the experimental technology for the magnetic field engineering. Once $\text{Re}[\Delta_E(t)] > 0$, $\rho(t)$, and $\theta(t)$ are fixed, which means the magnetic fields $B$ is fixed. However, it should be emphasized that an arbitrary choice of $\rho(t)$ and $\theta(t)$ will typically lead to singularities on the magnetic field. We will detailedly discuss this problem in following physical model.

In the above derivation, we have considered a simple case of Eq. (24) that is $\text{Re}[\Delta_E(t)] = 0$. Now, we will discuss the experimental feasibility for the physical model when $\text{Im}[\Delta_E(t)] > 0$. For convenient discussion, we assume $\text{Im}[\delta(t)]=\Gamma(t)$, where $\Gamma(t)$ is a time-dependent real coefficient. In this case, the Eq. (35) can be written as

$$H(t) = \frac{1}{2}|\Delta_E(t)|\sigma_x - i\delta(t)\sigma_y + \text{Re}[\delta(t)]\sigma_z$$

$$+ \left(i \frac{\text{Im}[\Delta_E(t)] + \frac{\Gamma(t)}{2}}{0} \frac{\text{Im}[\Delta_E(t)] - \frac{\Gamma(t)}{2}}{0} \right). \quad (42)$$

Note that the difference of the order of magnitude between $\text{Im}[\Delta_E(t)]$ and $\Gamma(t)$ is little, otherwise, the problem seems to be equivalent to above simple example. More specially, setting $\text{Im}[\Delta_E(t)] + \Gamma(t)/2 = 0,$ (43)

we will find the resulting Hamiltonian in Eq. (42) can be accessible in the following physically setting: a spin qubit or atom passes through a region of rapidly varying magnetic field $B=|\Delta_E(t)|\sigma_x - i\delta(t)\sigma_y + \text{Re}[\delta(t)]\sigma_z/2M_0$, and the spin qubit or atom suffers a radiation process with the dissipation rate $\Gamma(t)$ (e.g. the spontaneous decay; in some cases, $\Gamma(t)$ can be controlled as an effective decay rate by other interacting systems, e.g. Ref. [122]). This is remarkable, since the noise and certain dissipation in the systems are no longer undesirable, but play an integral part in our scheme.

From an experimental view point, we should consider the $\Delta_E(t)$ design and the $\alpha(t)$ design for current physical model. Similar to above derivation, $\alpha(t)$ is still in polar form. Substituting Eqs. (33) and (40) into Eq. (35), we will find $\Delta(t)$ satisfies following equation

$$-\text{Re}[\Delta_E(t)]\Omega_1 = \text{Im}[\Delta_E(t)](1 + \Omega_2) + \Omega_3,$$  

$$\Omega_1 = \text{sin}[2\rho(t) \cos \theta(t)] \sinh[-2\rho(t) \sin \theta(t)],$$

$$\Omega_2 = \cos[2\rho(t) \cos \theta(t)] \cosh[2\rho(t) \sin \theta(t)],$$

$$\Omega_3 = \text{sin}[2\rho(t) \cos \theta(t)] \cosh[2\rho(t) \sin \theta(t)]. \quad (45)$$

Furthermore, $\Gamma(t)$ can be simplified as $\Gamma(t)= -\text{Im}[\Delta_E(t)] - \Omega_3$. Apparently, once $\text{Im}[\Delta_E(t)]$ is specified, the magnetic field $B$ and $\Gamma(t)$ are straightforwardly calculated with Eqs. (44) and (45). On the other hand, the form of $\text{Im}[\Delta_E(t)]$ can be derived with the inversion strategy, if the form of dissipation rate $\Gamma(t)$ is fixed. This is remarkable, since we can choose appropriate extra magnetic fields to adiabatically drive an artificial quantum state for certain dissipative quantum system. Up to now, we have in principle constructed the magnetic fields according to the $\alpha(t)$ design and specified dissipation rate $\Gamma(t)$. However, the $\alpha(t)$ designs are problematic, as an arbitrary choice of $\rho(t)$ and $\theta(t)$ will typically lead to singularities on the right-hand side of Eq. (45) (for instance, $\Omega_1$ (i=1, 2, 3) will jump abruptly when $2\rho(t) \cos \theta(t)=\pi\sigma$ or $2\rho(t) \sin \theta(t)=\pi$). In general, $\Omega_1$ will also introduce singularities in magnetic fields, then, we could not construct the finite and smooth magnetic fields. Thus, we should design $\rho(t)$ and $\theta(t)$ to avoid the singularities. It is advisable to fix $\rho(t)$ or $\theta(t)$ first, then design the other one to avoid the singularities. A simple example is

$$\rho(t) = \frac{\pi}{2} - \sigma - \xi \sin \mu t, \ \theta(t) = \zeta + \sin \nu t, \quad (46)$$

where $\mu$ and $\nu$ are constant frequencies related to the concrete phase engineering, and $\sigma$ is an extremely small constant to keep the consistency condition. By choosing appropriate parameters (such as $\xi=0.4\pi$, $\zeta=0.08\pi$, and $\mu=0.5\pi$), we can construct the finite and smooth magnetic fields. For an intuitive grasp of the change of magnetic fields with different parameters, we display the time evolution of magnetic fields with different parameters in the dissipation system, we plot the time evolution of magnetic fields with different parameters in the dissipation system, we plot the time evolution of magnetic fields with different parameters in the dissipation system, we plot the time evolution of magnetic fields with different parameters in the dissipation system.
FIG. 2: Time evolution of the relative populations for the states $|1\rangle$ and $|2\rangle$ with different magnetic field parameters. The parameters are the same as shown in the caption of Fig. 1 (a) $\Gamma(t)=100\Omega$, $\mu=\nu=0.5\Omega$; (b) $\Gamma(t)=100\Omega$, $\mu=\nu=0.4\Omega$ ($\xi=0.4\pi$ and $\zeta=0.08\pi$); (c) $\Gamma(t)=100\Omega \exp[-(t-t_0)^2/T^2]$, $T=\sqrt{2}/\Omega$; (d) $\Gamma(t)=100\Omega \exp[-(t-t_0)^2/T^2]$, $T=\sqrt{0.01}/\Omega$ ($t_0=\pi/\Omega$, $\mu=\nu=0.5\Omega$, $\xi=0.4\pi$, and $\zeta=0.08\pi$).

Now we start to study the population engineering of the bare state in the target state. However, as shown in Eq. (29), the target state seems to be no natural normalization. For an intuitive grasp of the change of the population engineering of the bare basic, we will use the relative population $P^t_i$ ($i=1,2$) to study the effects of different magnetic fields on the population engineering, where the relative population is defined as $P^t_i=P_i/(P_1+P_2)$, and $P_i$ is the population for the bare state $|i\rangle$. We consider a realistic case of an extremely small population in the bare state $|2\rangle$ for the initial state

$$|\phi(0)\rangle \approx |\phi_2(0)\rangle = \sqrt{1-o^2}|1\rangle + o|2\rangle,$$

(48)

where $o$ is an extremely small constant. In Fig. 2, we plot the time evolution of the relative populations for the bare states $|1\rangle$ and $|2\rangle$ with the same parameters as shown in the caption of Fig. 1. We can find that the relative populations $P^t_1$ and $P^t_2$ almost have the same evolving tendency in Figs. 2(a) and 2(c), and a perfect full relative population inversion is compiled when $\Omega t=\pi$. It should be noted that the time for a full relative population inversion is about 50 ns which is short, if $\Omega=2\pi \times 10$ KHz. Figure 2(b) also clearly shows a full relative population inversion when $\Omega t=1.3\pi$. However, the time evolution of the relative populations in Fig. 2(d) are complicated, which are quite different from others. The reason for this result is that the choice of the parameters in Fig. 2(d) is problematic or false, more particularly, the $T$ is too short and the consistency condition is invalid in this case.

To judge the validity of our scheme for adiabatically driving, we should compare the real population engineering with the ideal population engineering [see Eq. (24)]. The ideal population engineering with different $\alpha(t)$ are given in Fig. 3. As shown in Eq. (24), the ideal population engineering only depends on the $\alpha(t)$ design. In other words, the population engineering will be identical for the same $\alpha(t)$ design independent of other parameters. Thus, if our scheme is valid, Figs. 2(a), 2(c) and 3(a) (Figs. 2(b) and 3(b)) should be identical, since the $\alpha(t)$ designs for them are identical. Obviously, the results are consistent with our deduction, hence our scheme can work well even under noise if the parameters are chosen appropriately. In addition, we can get more interested target states with different $\alpha(t)$ designs.

FIG. 3: The ideal population engineering with different $\alpha(t)$. $\alpha(t)$ is based on Eqs. (10) and (16): (a) $\mu=\nu=0.5\Omega$, $\xi=0.4\pi$ and $\zeta=0.08\pi$; (b) $\mu=\nu=0.4\Omega$, $\xi=0.4\pi$ and $\zeta=0.08\pi$.

V. DISCUSSION AND CONCLUSION

We have generalized the quantum adiabatic theorem to the NH system and provided a strict adiabaticity condition to make the adiabatic evolution non-lossy. The strict adiabaticity condition can be regarded as a non-trivial generalization of adiabaticity conditions for the Hermitian Hamiltonians presented by Jing et al. [37]. According to the strict adiabatic condition, one should eliminate the non-adiabatic couplings and the effect of the imaginary part of adiabatic phase as much as possible. The NH Hamiltonian reverse engineering method has been proposed to adiabatically drive an artificial quantum state. A concrete two-level system example was discussed to show the usefulness of the reverse engineering method in the paper, and numerical simulation showed that our scheme can work well even under noise if the parameters are chosen appropriately. Furthermore, we can obtain the desired target state by adjusting extra rotating magnetic fields at a predefined time. Specifically, the noise or certain dissipation in the systems are no longer undesirable, but can play a positive role in our scheme. Therefore, our scheme is powerful and reliable for the quantum information processing.

The present work bears some common elements with the quantum control in open quantum systems, including the idea of using dissipation as a resource (e.g. dissipative quantum dynamics (DQD) [33] and the NH shortcuts to adiabaticity schemes [32, 33]), so it is worth stressing the similarities and differences. In fact, the basic idea of DQD can be summarized as follows: the interaction between the system and the environment is modulated to make the target state become the stationary state of the system. Therefore, some specific dissipative factors are no longer undesirable, but can be regarded as important resources. For the NH shortcuts to adiabaticity schemes, the dissipative factors are also introduced to the system to cancel somehow the non-adiabatic losses. In this way, one can improve dramatically the fidelity of the adiabatic passage. However, a common problem which one may encounter via DQD or the NH shortcuts to adiabaticity is how to use the specific dissipative factors or employ the appropriate interaction between the system and the environment. Furthermore, those methods may also be limited severely for some applications (the non-adiabatic dynamics processes), since the starting point of them generally is to improve a given (adiabatic) dynamics process.

Among the differences with the present works [32, 28, 29, 11, 50], the most prominent point is: using the reverse engineering method, we can easily obtain the Hamiltonian to realize the intended dynamics without loss, which allows one to design the Hamiltonian according to the demand. The main task we should consider is
how to physically realize the resulting NH Hamiltonian. Sometimes, the resulting NH Hamiltonian may be hard to be realized (a common potential problem of the NH shortcuts to adiabaticity). However, we should note that the difficulty to realize the NH Hamiltonian may be solved by enlarging the system with the aid of Naimark extensions \cite{51}. Furthermore, in a sense, all the resulting NH Hamiltonian (even the problematic NH Hamiltonian) may help us with a deeper understanding on the problem: which dissipative factors are the specific dissipative factors that can be used as a resource to realize QSE, and promote the development of quantum information science in NH system frames.

Furthermore, any quantum system whose Hamiltonian is possible to be simplified into the form in Eq. \eqref{35} (the basic for the simplified Hamiltonian can be arbitrary dressed states, as long as, the dressed states satisfy the biorthogonality relation and closure relation), the scheme can be implemented straightforward. This might lead to a useful step toward realizing fast and noise-resistant quantum information processing for multi-qubit systems in current technology. The applications or extensions of this work may be in fields, such as n-dimensional systems \cite{52, 53} (for instance, the three-dimensional systems for the stimulated Raman adiabatic passage), superadiabatic treatments \cite{54, 55}, and non-adiabatic evolution of NH quantum systems \cite{56}.

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