\textbf{Abstract.} Structure of the quotient modules in $H^2(\Gamma^2)$ is very complicated. A good understanding of some special examples will shed light on the general picture. This paper studies the so-call $N_\varphi$-type quotient modules, namely, quotient modules of the form $H^2(\Gamma^2) \ominus [z - \varphi]$, where $\varphi(w)$ is a function in the classical Hardy space $H^2(\Gamma)$ and $[z - \varphi]$ is the submodule generated by $z - \varphi(w)$. This type of quotient modules serve as good examples in many studies. A notable feature of the $N_\varphi$-type quotient module is its close connections with some classical single variable operator theories.

1. Introduction

Let $H^2(\Gamma^2)$ be the Hardy space on the two dimensional torus $\Gamma^2$. We denote by $z$ and $w$ the coordinate functions. Shift operators $T_z$ and $T_w$ on $H^2(\Gamma^2)$ are defined by $T_z f = zf$ and $T_w f = wf$ for $f \in H^2(\Gamma^2)$. Clearly, both $T_z$ and $T_w$ have infinite multiplicity. A closed subspace $M$ of $H^2(\Gamma^2)$ is called a submodule (over the algebra $H^\infty(D^2)$), if it is invariant under multiplications by functions in $H^\infty(D^2)$, where $D$ stands for the unit disk. Equivalently, $M$ is a submodule if it is invariant for both $T_z$ and $T_w$. The quotient space $N := H^2(\Gamma^2) \ominus M$ is called a quotient module. Clearly $T_z^* N \subset N$ and $T_w^* N \subset N$. And for this reason $N$ is also said to be backward shift invariant. In the study here, it is necessary to distinguish the classical Hardy space in the variable $z$ and that in the variable $w$, for which we denote by $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$, respectively. $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ are thus different subspaces in $H^2(\Gamma^2)$. We will simply write $H^2(\Gamma)$ when there is no need to tell the difference. In $H^2(\Gamma)$, it is well known as the Beurling theorem that if $M \subset H^2(\Gamma)$ is invariant for $T_z$, then $M = qH^2(\Gamma)$ for an inner function $q(z)$. The structure of submodules in $H^2(\Gamma^2)$ is much more complex, and there is a great amount of works on this subject in recent years. A good reference of this work can be found in [3]. One natural approach to the problem is to find and study some relatively simple submodules, and hope that the study will generate concepts and general techniques that will lead to a better understanding of the general picture. This in fact has become an interesting and encouraging work.

In this paper, we look at submodules of the form $[z - \varphi(w)]$, where $\varphi$ is a function in $H^2(\Gamma_w)$ with $\varphi \neq 0$ and $[z - \varphi(w)]$ is the closure of $(z - \varphi)H^\infty(\Gamma^2)$ in $H^2(\Gamma^2)$. For simplicity we denote $[z - \varphi(w)]$ by $M_\varphi$. One good way of studying $M_\varphi$ is through the so-called two variable Jordan block $(S_z, S_w)$ defined on the quotient module

$$N_\varphi := H^2(\Gamma^2) \ominus M_\varphi.$$
For every quotient module $N$, the two variable Jordan block $(S_z, S_w)$ is the compression of the pair $(T_z, T_w)$ to $N$, or more precisely,

$$S_z f = P_N z f, \quad S_w f = P_N w f, \quad f \in N,$$

where $P_N : H^2(\Gamma^2) \to N$ is the orthogonal projection. This paper studies interconnections between the quotient module $N_\varphi$, the two variable Jordan block $(S_z, S_w)$ and the function $\varphi$. Some related work has been done in [14, 22, 23]. By [14], $N_\varphi \neq \{0\}$ if and only if $\varphi(D) \cap D \neq \emptyset$. For convenience, we let

$$\Omega_\varphi = \{ w \in D : |\varphi(w)| < 1 \},$$

and assume throughout the paper that $N_\varphi \neq \{0\}$, i.e., $\varphi(D) \cap D \neq \emptyset$. The paper is organized as follows.

Section 1 is introduction.

Section 2 introduces some useful tools and states a few related known results.

Section 3 studies the spectral properties of the operators $S_z$ and $S_w$. It is interesting to see how these properties depend on the function $\varphi$.

A notable phenomenon in many cases is the compactness of the defect operators $I - S_z S_z^*$ and $I - S_w S_z^*$. Section 4 aims to study how the compactness is related to the properties of $\varphi$.

The quotient module $N_\varphi$ has very rich structure. Indeed, when $\varphi$ is inner, $N_\varphi$ can be identified with the tensor product of two well-known classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L^2_a(D)$. Section 5 makes a detailed study of this case.

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2. Preliminaries

For every $\lambda \in D$, we define a left evaluation operator $L(\lambda)$ from $H^2(\Gamma^2)$ to $H^2(\Gamma_w)$ and a right evaluation operator $R(\lambda)$ from $H^2(\Gamma^2)$ to $H^2(\Gamma_z)$ by

$$L(\lambda) f(w) = f(\lambda, w), \quad R(\lambda) f(z) = f(z, \lambda), \quad f \in H^2(\Gamma^2).$$

Clearly, $L(\lambda)$ and $R(\lambda)$ are operator-valued analytic functions over $D$. Restrictions of $L(\lambda)$ and $R(\lambda)$ to quotient spaces $N$, $M \ominus zM$ and $M \ominus wM$ play key roles in the study here. The following lemma is from [4].

Lemma 2.1. The restriction of $R(\lambda)$ to $M \ominus wM$ is equivalent to the characteristic operator function for $S_w$.

The following spectral relations are thus clear. Details can be found in [4] and [18].
(a) \( \lambda \in \sigma(S_w) \) if and only if \( R(\lambda) : M \ominus wM \to H^2(\Gamma_z) \) is not invertible,
(b) \( \dim \ker (S_w - \lambda I) = \dim \ker (R(\lambda)|_{M \ominus wM}) \).
(c) \( S_w - \lambda I \) has a closed range if and only if \( R(\lambda)(M \ominus wM) \) is closed,
(d) \( S_w - \lambda I \) is Fredholm if and only if \( R(\lambda)|_{M \ominus wM} \) is Fredholm, and in this case
\[
\text{ind} (S_w - \lambda I) = \text{ind} (R(\lambda)|_{M \ominus wM}).
\]
Restrictions \( T^*_z|_{M \ominus wM} \) and \( T^*|_{M \ominus wM} \) are also important here, and for simplicity they are denoted by \( D_z \) and \( D_w \), respectively. Clearly,
\[
D_z f(z, w) = \frac{f(z, w) - f(0, w)}{z}, \quad D_w f(z, w) = \frac{f(z, w) - f(z, 0)}{w}.
\]
And it is not hard to check that the ranges of \( D_z \) and \( D_w \) are subspaces of \( N \). The following lemma (cf. [22]) gives a description of the defect operators for \( S_z \), and it will be used often.

**Lemma 2.2.** On a quotient module \( N \),

(i) \( S_z^* S_z + D_z D_z^* = I \);
(ii) \( S_z S_z^* + (L(0)|_N)^* L(0)|_N = I \).

A parallel version of Lemma 2.2 for \( S_w \) will also be used.

The operator \( D_z \) is a useful tool in this study. We first note that
\[
D_z^* f = P_M zf, \quad f \in N.
\]
So if \( D_z^* f = 0 \), then \( zf \in N \). Clearly \( zf \in \ker L(0)|_N \). Conversely, if \( h \) is in \( \ker L(0)|_N \), then we can write \( h = zh_0 \). One checks easily that \( h_0 \in \ker D_z^* \). This observation shows that
\[
\ker D_z^* = \ker L(0)|_N.
\]
So on \( N_\varphi \), since \( L(0) \) is injective (cf. [14]), \( D_z^* \) has trivial kernel, i.e., the range \( R(D_z) \) is dense in \( N_\varphi \). The following theorem describes \( \bar{R}(D_z) \) in detail.

**Theorem 2.3.** Let \( N \) be a quotient module of \( H^2(\Gamma^2) \) and \( M = H^2(\Gamma^2) \ominus N \). Suppose that \( R(D_z) \) is dense in \( N \). Let \( f \in N \). Then \( f \in R(D_z) \) if and only if there exists a positive constant \( C_f \) depending on \( f \) such that \( \|\langle S_z^* h, f \rangle\| \leq C_f \|L(0)h\| \) for every \( h \in N \).

**Proof.** Suppose that \( f \in R(D_z) \). Let \( g \in M \ominus zM \) with \( T_z^* g = f \). We have \( g = zf + L(0)g \). Then for \( h \in N \),
\[
|\langle S_z^* h, f \rangle| = |\langle h, zf \rangle| = |\langle h, g - L(0)g \rangle| = |\langle h, L(0)g \rangle| = |\langle L(0)h, L(0)g \rangle| \leq \|L(0)g\| \|L(0)h\|.
\]
To prove the converse, suppose that there exists a positive constant \( C_f \) satisfying
\[
|\langle S_z^* h, f \rangle| \leq C_f \|L(0)h\|
\]
for every \( h \in N \). Since \( L(0) \) on \( N \) is one to one, we have a map \( \Lambda \) defined by
\[
\Lambda : L(0)N \supset u(w) \to L(0)^{-1} u \to \langle S_z^* L(0)^{-1} u, f \rangle \in \mathbb{C}.
\]
Note that \( L_0^{-1}u \in N \). Obviously, \( \Lambda \) is linear and
\[
|\Lambda u| = |\langle S_z^*L(0)^{-1}u, f \rangle| \leq C_f \|L(0)L(0)^{-1}u\| = C_f \|u\|.
\]
Hence by the Hahn-Banach theorem, \( \Lambda \) is extendable to a bounded linear functional on \( H^2(\Gamma_w) \) and there exists \( v(w) \in H^2(\Gamma_w) \) satisfying \( \langle u, v \rangle = \Lambda u \) for every \( u \in L(0)N \). We have
\[
\langle u, v \rangle = \langle S_z^*L(0)^{-1}u, f \rangle = \langle L(0)^{-1}u, zf \rangle.
\]
Since \( v(w) \in H^2(\Gamma_w) \), \( \langle u, v \rangle = \langle L(0)^{-1}u, v \rangle \). Therefore
\[
\langle L(0)^{-1}u, zf - v \rangle = 0
\]
for every \( u \in L(0)N \). Since \( L_0^{-1}(L(0)N) = N \), we get \( zf - v \perp N \). Hence \( zf - v \in M \). Since \( v(w) \in H^2(\Gamma_w) \), we have \( T_z^*(zf - v) = f \in N \). This implies that \( zf - v \in M \Theta zM \). Thus we get \( f \in R(D_z) \).

In the case of \( N_\varphi \), [14] provides a very useful description of the functions in the space. Let \( \varphi(w) \in H^2(\Gamma_w) \). For \( f(w) \in H^2(\Gamma_w) \), we formally define a function
\[
(T_\varphi^*f)(w) = \sum_{n=0}^{\infty} a_n w^n,
\]
where
\[
a_n = \int_0^{2\pi} \overline{\varphi(e^{i\theta})} f(e^{-i\theta}) e^{-in\theta} d\theta / 2\pi = \langle f(w), \varphi(w)w^n \rangle.
\]
Generally, \( T_\varphi^*f \) may not be in \( H^2(\Gamma_w) \). When \( T_\varphi^*f \in H^2(\Gamma_w) \), we can define \( T_\varphi^*f = T_\varphi^*(T_\varphi^*f) \). Inductively if \( T_\varphi^{*n}f \in H^2(\Gamma_w) \), we can define \( T_\varphi^{*(n+1)}f = T_\varphi^*(T_\varphi^{*n}f) \). For convenience, we let
\[
A_\varphi f(z, w) = \sum_{n=0}^{\infty} z^n T_\varphi^{*n}f(w)
\]
be an operator defined at every \( f \in H^2(\Gamma_w) \) for which \( A_\varphi f \in H^2(\Gamma_\varphi^2) \). Then it is shown in [14] that \( L(0) \) is one-to-one on \( N_\varphi \) and
\[
N_\varphi = \{ A_\varphi f : f \in H^2(\Gamma_w), \sum_{n=0}^{\infty} \|T_\varphi^{*n}f\|^2 < \infty \}.
\]
It is easy to see that \( L(0)A_\varphi f = f \). Moreover by [14, Corollary 2.8], \( L(0)N_\varphi \) is dense in \( H^2(\Gamma_w) \).

The following two lemmas are needed for the study of \( \sigma(S_z) \).

**Lemma 2.4.** Let \( \varphi(w), g(w) \in H^2(\Gamma_w) \) and \( \psi(w) \in H^\infty(\Gamma_w) \). Then \( T_\varphi^*T_\psi^*g = T_\varphi^*T_\psi^*g \). Moreover if \( T_\varphi^*g \in H^2(\Gamma_w) \), then \( T_\varphi^*T_\psi^*g = T_\varphi^*T_\psi^*g \).

**Proof.** Let \( n \geq 0 \). Then by the definitions above,
\[
\langle T_\varphi^*T_\psi^*g, z^n \rangle = \langle g, \psi z^n \rangle = \langle T_\psi^*g, z^n \rangle.
\]
Thus $T_\varphi^*T_\psi^*g = T_\varphi^*g$. Suppose that $T_\varphi^*g \in H^2(\Gamma_w)$. We have $\overline{\varphi g - T_\varphi^*g} \in zH^1$. Hence

$$\langle T_\psi^*T_\varphi^*g, z^n \rangle = \langle T_\varphi^*g, \psi z^n \rangle = \int_0^{2\pi} \overline{\varphi(e^{i\theta})}g(e^{i\theta})\overline{\psi(e^{i\theta})}e^{-in\theta}d\theta/2\pi = \langle g, \psi \varphi z^n \rangle.$$

Thus we get our assertion. \qed

Let $w_0 \in \Omega_\varphi$. The following lemma follows easily from the calculation

$$T_\varphi^* \frac{1}{1 - \overline{w_0}w} = \frac{\varphi(w_0)}{1 - \overline{w_0}w}.$$

**Lemma 2.5.** For $w_0 \in \Omega_\varphi$, we have

$$\frac{1}{(1 - \varphi(w_0)z)(1 - \overline{w_0}w)} \in \mathbb{N}_\varphi.$$

### 3. The Spectra of $S_z$ and $S_w$

The spectra of $S_z$ and $S_w$ on $\mathbb{N}_\varphi$ is evidently dependent on $\varphi$. This section aims to figure out how they are exactly related. Lemma 2.1 and the description in (2.1) are helpful to this end.

**Proposition 3.1.** $\varphi(D) \cap \overline{D} \subset \sigma(S_z) \subset \varphi(D) \cap \overline{D}$.

**Proof.** Let $w_0 \in \varphi(D) \cap D$. Then

$$S_z^* \left( \frac{1}{(1 - \varphi(w_0)z)(1 - \overline{w_0}w)} \right) = \sum_{n=1}^{\infty} \left( \frac{\varphi(w_0)}{1 - \overline{w_0}w} \right)^n \varphi(w_0)^n \frac{1}{(1 - \varphi(w_0)z)(1 - \overline{w_0}w)}.$$

By Lemma 2.5, $\varphi(w_0)$ is a point spectrum of $S_z^*$. Thus we get $\overline{\varphi(D) \cap \overline{D}} \subset \sigma(S_z)$.

Let $\lambda \notin \varphi(D)$. Then $1/(\varphi(w) - \lambda) \in H^\infty(\Gamma_w)$. Let $F \in \mathbb{N}_\varphi$. We have

$$S_{1/(\varphi-\lambda)}^* F = S_{1/(\varphi-\lambda)}^* \sum_{n=0}^{\infty} (T_\varphi^{*n} L(0)F) z^n = \sum_{n=0}^{\infty} (T_\varphi^{*n} T_{1/(\varphi-\lambda)} L(0)F) z^n \text{ by Lemma 2.4.}$$
Hence
\[ S_{1/((\varphi-\lambda))} S_{z-\lambda}^* F = \sum_{n=0}^{\infty} (T_{n} S_{1/((\varphi-\lambda))} L(0) S_{z-\lambda}) z^n \]
\[ = \sum_{n=0}^{\infty} (T_{n} S_{1/((\varphi-\lambda))} T_{n-\lambda} L(0) F) z^n \]
\[ = \sum_{n=0}^{\infty} (T_{n} L(0) F) z^n \text{ by Lemma 2.4} \]
\[ = F. \]

Also we have
\[ S_{z-\lambda}^* S_{1/((\varphi-\lambda))} F = \sum_{n=1}^{\infty} (T_{n} T_{1/((\varphi-\lambda))} L(0) F) z^{n-1} - \sum_{n=0}^{\infty} (T_{n} T_{1/((\varphi-\lambda))} L(0) F) z^n \]
\[ = \sum_{n=1}^{\infty} (T_{n} T_{1/((\varphi-\lambda))} L(0) F) z^{n-1} - \sum_{n=0}^{\infty} (T_{n} T_{1/((\varphi-\lambda))} L(0) F) z^n \]
\[ = \sum_{n=0}^{\infty} (T_{n} T_{1/((\varphi-\lambda))} L(0) F) z^n \]
\[ = F. \]

Thus \((S_{z-\lambda})^{-1} = S_{1/((\varphi-\lambda))}\) and hence \(\lambda \notin \sigma(S_z)\).

Since \(\|S_z\| \leq 1\), we have our assertion. \(\square\)

For a submodule \(M\) in \(H^2(\Gamma^2)\), the quotient space \(M \ominus zM\) is a wandering subspace for the multiplication by \(z\) and we have
\[ M = \sum_{n=0}^{\infty} \oplus z^n (M \ominus zM). \]

For a fixed \(\lambda \in D\) and every \(f \in M\), we write \(f = \sum_{j=0}^{\infty} z^j f_j\) for some unique sequence \(\{f_j\}\) in \(M \ominus zM\). So
\[ f = \sum_{j=0}^{\infty} \lambda^j f_j + \sum_{j=0}^{\infty} (z^j - \lambda^j) f_j, \]
which means that \(f = h_1 + (z - \lambda)h_2\) for some \(h_1 \in M \ominus zM\) and \(h_2 \in M\). If \(h_1 + (z - \lambda)h_2 = 0\), then \(h_1 + zh_2 = \lambda h_2\), and hence \(|\lambda|^2 \|h_2\|^2 = \|h_1\|^2 + \|h_2\|^2\), which is possible only if \(h_1 = h_2 = 0\). This observation shows that \(M\) can be expressed as the direct sum
\[ M = (M \ominus zM) + (z - \lambda)M, \]
(3.1)

We now look at the spectral properties of \(S_w\).

**Proposition 3.2.** On \(N_\varphi\),

(i) \(\Omega_\varphi \subset \sigma(S_w)\).

(ii) \(S_w - \alpha I\) is Fredholm for every \(\alpha \in \Omega_\varphi\) and \(\text{ind} (S_w - \alpha I) = -1\).
Proof. We use Lemma 2.1 to this end.

(i) It is sufficient to show \( \Omega_\varphi \subset \sigma(S_w) \). If \( \alpha \in \Omega_\varphi \), then for any function \((z - \varphi)h(z, w)\) in \( M_\varphi \otimes wM_\varphi \), \((z - \varphi(\alpha))h(z, \alpha)\) vanishes at \( \varphi(\alpha) \), and therefore \( R(\alpha)(M_\varphi \otimes wM_\varphi) \subset (z - \varphi(\alpha))H^2(\Gamma_z) \neq H^2(\Gamma_z) \). By Lemma 2.1, \( \alpha \in \sigma(S_w) \).

(ii) It is equivalent to show that \( R(\alpha)|_{M_\varphi \otimes wM_\varphi} \) is Fredholm with index \(-1\). We first show that \( R(\alpha) \) is injective on \( M_\varphi \otimes wM_\varphi \) for every \( \alpha \in \Omega_\varphi \). Let \((z - \varphi)h(z, w)\) be in \( M_\varphi \). Then there is a sequence of polynomials \( \{p_n(z, w)\} \) such that \((z - \varphi)p_n\) converges to \((z - \varphi)h\) in the norm of \( H^2(\Gamma^2) \). Since \( R(\alpha) \) is a bounded operator, \((z - \varphi(\alpha))p_n(z, \alpha)\) converges to \((z - \varphi(\alpha))h(z, \alpha)\), which, by the fact \( |\varphi(\alpha)| < 1 \), implies that \( p_n(z, \alpha) \) converges to \( h(z, \alpha) \) in \( H^2(\Gamma_z) \). Since for every \( f \in H^2(\Gamma_z) \), we have \( \|\varphi f\| = \|\varphi\|\|f\| \) and hence

\[
\|(z - \varphi)f\| \leq \|zf\| + \|\varphi f\| = (1 + \|\varphi\|)\|f\| < \infty,
\]

\((z - \varphi)p_n(z, \alpha)\) converges to \((z - \varphi)h(z, \alpha)\) in \( M_\varphi \). It follows that

\[
\lim_{n \to \infty} \frac{(z - \varphi)p_n - p_n(\cdot, \alpha)}{w - \alpha} = (z - \varphi)\frac{h - h(\cdot, \alpha)}{w - \alpha},
\]

which concludes that \((z - \varphi)\frac{h - h(\cdot, \alpha)}{w - \alpha} \in M_\varphi \). If \((z - \varphi)h(z, w)\) is in \( M_\varphi \otimes wM_\varphi \) such that \((z - \varphi(\alpha))h(z, \alpha) = 0\), then \( h(z, \alpha) = 0 \), and it follows from the observation above that

\[
(z - \varphi)h = (w - \alpha)(z - \varphi)\frac{h}{w - \alpha} \in (w - \alpha)M_\varphi,
\]

and hence by (3.1) \((z - \varphi)h(z, w) = 0\) which concludes that \( R(\alpha) \) is injective on \( M_\varphi \otimes wM_\varphi \).

In the proof of (i), we showed that \( R(\alpha)(M_\varphi \otimes wM_\varphi) \subset (z - \varphi(\alpha))H^2(\Gamma_z) \). On the other hand, for every \( g \in H^2(\Gamma_z) \), \((z - \varphi)g\) is in \( M_\varphi \) by (3.2), and by (3.1)

\[
(z - \varphi(\alpha))g \in R(\alpha)(M_\varphi) = R(\alpha)(M_\varphi \otimes wM_\varphi).
\]

This shows that

\[
R(\alpha)(M_\varphi \otimes wM_\varphi) = (z - \varphi(\alpha))H^2(\Gamma_z),
\]

i.e., \( R(\alpha)|_{M_\varphi \otimes wM_\varphi} \) has a closed range with codimension 1, and this completes the proof in view of Lemma 2.1.

\[\square\]

**Corollary 3.3.** If \( \varphi \) is bounded with \( \|\varphi\|_\infty \leq 1 \), then \( \sigma(S_w) = \bar{D} \) and \( \sigma_e(S_w) = \Gamma \).

*Proof.** By Proposition 3.2 and the fact that \( S_w \) is a contraction, \( \sigma(S_w) = \bar{D} \) and \( \sigma_e(S_w) \subset \Gamma \). Since \( \text{ind}(S_w) = -1 \), \( \sigma_e(S_w) \) is a closed curve, and therefore \( \sigma_e(S_w) = \Gamma \).

\[\square\]

We will mention another somewhat deeper consequence of Proposition 3.2 near the end of this section. Here we continue to study the Fredholmness of \( S_\varphi \). Unfortunately, the techniques used for Proposition 3.2(ii) can not be applied directly to the case here and a technical difficulty seems hard to overcome. So instead we use (3.1) in the case here. We begin with some simple observations.

**Lemma 3.4.** Let \( \varphi(w) = b(w)h(w) \) be the inner-outer factorization of \( \varphi(w) \). Then \( ker S_\varphi^* = H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w) \).
Proof. Since the functions in $H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)$ depend only on $w$, the inclusion

$$H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w) \subset \ker S_z^*$$

is easy to check.

If $f$ is a function in $N_\varphi$ such that $S_z^*f = 0$, then $\bar{zf}$ is orthogonal to $H^2(\Gamma^2)$ which means $f$ is independent of the variable $z$. Since for every non-negative integer $j$

$$0 = \langle (z - \varphi)w^j, f \rangle = \langle -\varphi w^j, f \rangle,$$

$f$ is in $H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)$. \qed

**Theorem 3.5.** Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of $\varphi$ and

$$\alpha = \inf_{w \in D} |h(w)|.$$

Then $S_z^*$ has a closed range if and only if $\alpha \neq 0$, and in this case $S_z^*N_\varphi = N_\varphi$.

Proof. Write $K_b = H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)$. By Lemma 3.4, $\ker S_z^* = K_b$.

Suppose that $\alpha > 0$. Then $h(w)^{-1} \in H^\infty(\Gamma_w)$ and $\|T_{h^{-1}w}^*\| = \|h^{-1}\|_\infty = \alpha^{-1}$. Let $F \in N_\varphi \ominus K_b$. We can write $(L(0)F)(w) = b(w)f(w)$. Then by (2.1)

$$\|F\|^2 = \left\| \sum_{n=0}^{\infty} z^n T_{\varphi}^* b f \right\|^2$$

$$= \sum_{n=0}^{\infty} \|T_{\varphi}^* b f\|^2$$

$$\geq \|f\|^2 + \|T_{\varphi}^* b f\|^2$$

$$= \|f\|^2 + \|T_h^* f\|^2$$

$$= \|f\|^2 + \alpha^2 \|T_{h^{-1}}^* f\|^2$$

$$\geq \|f\|^2 + \alpha^2 \|f\|^2$$

by Lemma 2.4

$$= (1 + \alpha^2) \|L(0)F\|^2.$$

Since by Lemma 2.2 $\|S_z^* F\|^2 + \|L(0)F\|^2 = \|F\|^2$,

$$\|S_z^* F\|^2 = \|F\|^2 - \|L(0)F\|^2 \geq \left(1 - \frac{1}{1 + \alpha^2}\right) \|F\|^2 = \frac{\alpha^2}{1 + \alpha^2} \|F\|^2.$$

This implies that $S_z^*$ is bounded below on $N_\varphi \ominus K_b$, and hence $S_z^*$ has a closed range.

Suppose that $\alpha = 0$. Let $\{w_k\}_k$ be a sequence in $D$ satisfying $|h(w_k)| < 1$ and $h(w_k) \to 0$ as $k \to \infty$. Let

$$F_k(z, w) = \frac{b(w)}{1 - w_k w} + \sum_{n=1}^{\infty} z^n b(w_k) \frac{(n-1)h(w)w}{1 - w_k w}.$$

Then

$$\|F_k\|^2 \geq \left\| \frac{1}{1 - w_k w} \right\|^2.$$
Using the fact that \( T_g(1/(1 - \bar{w}_k w)) = g(w_k)(1/(1 - \bar{w}_k w)) \) for every \( g \in H^2(\Gamma_w) \), we have
\[
F_k(z, w) = \sum_{n=0}^{\infty} z^n T_{\varphi}^n b(w) \frac{1}{1 - \bar{w}_k w} \in N_{\varphi} \ominus K_b,
\]
and therefore
\[
S_z^* F_k = \sum_{n=0}^{\infty} z^n b(w_k)^n \frac{h(w_k)(n+1)}{1 - \bar{w}_k w},
\]
and
\[
\|S_z^* F_k\| \leq \left\| \frac{1}{1 - \bar{w}_k w} \right\|^2 \frac{|h(w_k)|^2}{1 - |h(w_k)|^2}.
\]
It follows
\[
\|S_z^* F_k\| \leq \frac{|h(w_k)|^2}{1 - |h(w_k)|^2} \|F_k\|^2.
\]
This implies that \( S_z^* \) is not bounded below on \( N_{\varphi} \ominus K_b \). Since \( S_z^* \) is one-to-one on \( N_{\varphi} \ominus K_b \), \( S_z^* (N_{\varphi} \ominus K_b) \) is not a closed subspace. Since \( S_z^* (N_{\varphi}) = S_z^* (N_{\varphi} \ominus K_b) \), \( S_z^* \) does not have a closed range.

Next we shall prove that \( S_z^* N_{\varphi} = N_{\varphi} \) when \( \alpha > 0 \). Let \( g(w) \in L(0)N_{\varphi} \). We have
\[
\sum_{n=0}^{\infty} \|T_{\varphi}^n T_{h^{-1}}^* bg\|^2 = \|T_{h^{-1}}^* bg\|^2 + \sum_{n=1}^{\infty} \|T_{\varphi}^{n-1} g\|^2 \\
\leq \|h^{-1}\|_{\infty}^2 \|g\|^2 + \|L(0)^{-1} g\|^2 \\
< \infty.
\]
Hence \( T_{h^{-1}}^* bg \in L(0)N_{\varphi} \), and
\[
S_z^* L_0^{-1} T_{h^{-1}}^* bg = \sum_{n=1}^{\infty} z^{n-1} T_{\varphi}^n T_{h^{-1}}^* bg \\
= \sum_{n=1}^{\infty} z^{n-1} T_{\varphi}^{n-1} g \\
= L_0^{-1} g.
\]
This implies that \( S_z^* N_{\varphi} = N_{\varphi} \).

**Corollary 3.6.** With notations as in Theorem 3.5, the following conditions are equivalent.

(i) \( \alpha \neq 0 \).
(ii) \( S_z^* \) has a closed range.
(iii) \( S_z^* N_{\varphi} = N_{\varphi} \).
(iv) \( T_{\varphi}^* L(0)N_{\varphi} = L(0)N_{\varphi} \).

Theorem 3.5 in particular shows that \( S_z \) is injective when \( \alpha > 0 \). This is in fact a general phenomenon on \( N_{\varphi} \). The following fact (cf. [5, p.85]) is need to this end.
Lemma 3.7. Let \( h(w) \) be an outer function on \( \Gamma_w \). Then there is a sequence of outer functions \( \{h_k\}_k \) in \( H^\infty(\Gamma_w) \) such that \( \|h_k h\|_\infty \leq 1 \) and \( h_k h \to 1 \) a.e. on \( \Gamma_w \) as \( k \to \infty \).

Theorem 3.8. \( S_z \) is injective on \( N_\varphi \).

Proof. We show that \( S_z^* \) has a dense range. Let \( \varphi(w) = b(w)h(w) \) be the inner-outer factorization of \( \varphi \). By Lemma 3.7, there is a sequence \( \{h_k\}_k \) in \( H^\infty(\Gamma_w) \) such that

\[
\|h_k h\|_\infty \leq 1 \quad \text{and} \quad h_k h \to 1 \quad \text{a.e. on } \Gamma_w \quad \text{as } k \to \infty.
\]

Let \( g(w) \in L(0)N_\varphi \). By Lemma 2.4, we have

\[
\sum_{n=0}^{\infty} \|T_{\varphi}^n T_{h_k}^* b g\|^2 = \|T_{h_k}^* b g\|^2 + \sum_{n=1}^{\infty} \|T_{h_k}^* T_{\varphi}^{n-1} g\|^2 \leq \|h_k\|^2 \|g\|^2 + \sum_{n=1}^{\infty} \|T_{\varphi}^{n-1} g\|^2 \quad \text{by (3.3)}
\]

\[
= \|h_k\|^2 \|g\|^2 + \|L(0)^{-1} g\|^2 < \infty.
\]

Hence \( T_{h_k}^* b g \in L(0)N_\varphi \), and we have

\[
\|S_z^* L(0)^{-1} T_{h_k}^* b g - L(0)^{-1} g\|^2 = \sum_{n=0}^{\infty} \|T_{\varphi}^{n+1} T_{h_k}^* b g - T_{\varphi}^n g\|^2 = \sum_{n=0}^{\infty} \|T_{h_k}^* T_{\varphi}^n g\|^2 \leq \sum_{n=0}^{\infty} \|(h_k h - 1)T_{\varphi}^n g\|^2 = \int_0^{2\pi} |(hh_k)(e^{i\theta}) - 1|^2 \sum_{n=0}^{\infty} |(T_{\varphi}^n g)(e^{i\theta})|^2 \frac{d\theta}{2\pi}.
\]

Since \( g \in L(0)N_\varphi \),

\[
\sum_{n=0}^{\infty} |T_{\varphi}^n g|^2 \in L^1(\Gamma_w).
\]

Hence by (3.3) and the Lebesgue dominated convergence theorem,

\[
\|S_z^* L(0)^{-1} T_{h_k}^* b g - L(0)^{-1} g\|^2 \to 0 \quad \text{as } k \to \infty.
\]

This implies that \( S_z^* \) has a dense range. \( \square \)
Corollary 3.9. Let \( \varphi(w) = b(w)h(w) \) be the inner-outer factorization of \( \varphi(w) \). Then the following are equivalent.

(i) \( S_z \) is Fredholm.
(ii) \( b(w) \) is a finite Blaschke product and \( h^{-1}(w) \in H^\infty(\Gamma_w) \).

In this case, \( -\text{ind}(S_z) \) is the number of zeros of \( b(w) \) in \( D \) counting multiplicities.

Proof. We let \( \alpha = \inf_{w \in D} |h(w)| \). \( S_z \) is Fredholm if and only if \( S_z^* \) is Fredholm, and by Lemma 3.4 and Theorem 3.5 this is equivalent to \( b \) being a finite Blaschke product and \( \alpha > 0 \). Clearly, \( \alpha > 0 \) if and only if \( h^{-1}(w) \in H^\infty(\Gamma_w) \).

\( \Box \)

A quotient module \( N \) is said to be essentially reductive if both \( S_z \) and \( S_w \) are essentially normal, i.e., \( [S_z^*, S_z] \) and \( [S_w^*, S_w] \) are both compact. Essential reductivity is an important concept and has been studied recently in various contexts. In the context here, it will be interesting to see what type of \( \varphi \) makes \( N_\varphi \) essentially reductive. Proposition 3.2 has a couple of consequences to this end. A general study will be made in a different paper.

Corollary 3.10. For every \( \varphi \in H^2(\Gamma_w) \), \( [S_z^*, S_w] \) is Hilbert-Schmidt on \( N_\varphi \).

Proof. We let \( R_z \) and \( R_w \) denote the multiplications by \( z \) and \( w \) on the submodule \( M_\varphi \), respectively. It then follows from Proposition 3.2 and Theorem 2.3 in [21] that \([R_z^*, R_z][R_w^*, R_w] \) is Hilbert-Schmidt, and the corollary thus follows from Theorem 2.6 in [21].

In the case \( \varphi \) is in the disk algebra \( A(D) \), there is a sequence of polynomials \( p_n \to \varphi \) in \( A(D) \), and hence \( [S_z^*, p_n(S_w)] \to [S_z^*, \varphi(S_w)] \) in operator norm. Since \( S_z = \varphi(S_w) \) on \( N_\varphi \), we easily obtain the following corollary.

Corollary 3.11. If \( \varphi \in A(D) \), then \( S_z \) is essentially normal.

Question 1. For what \( \varphi \in H^2(\Gamma_w) \) is \( S_w \) essentially normal on \( N_\varphi \)?

In the case \( \varphi \) is inner, this question can be settled by direct calculations. We will do it in Section 5.

4. Compactness of \( L(0)|_N \) and \( D_z \)

In view of Lemma 2.2, the compactness of \( L(0)|_N \) or \( D_z \) will give us much information about the operator \( S_z \). So to determine whether \( L(0)|_N \) or \( D_z \) is compact for a certain quotient module \( N \) is of great interests. In the case of \( N_\varphi \), the compactness is undoubtly dependent on the properties of \( \varphi \). This section aims to unveil the connection.

We first look at the compactness of \( L(0)|_{N_\varphi} \). For each fixed \( \zeta \in D \), we denote by \( Z_\varphi(\zeta) \) the number of zeros of \( \zeta - \varphi(w) \) in \( D \) counting multiplicities. This integer-valued function has an important role to play in this study. As a matter of fact, in [22, Theorem 5.2.2], the second author showed that if \( L(0) \) on \( N_\varphi \) is compact, then \( Z_\varphi(\zeta) \) is a finite constant on \( D \). The following study describes the function \( \varphi \) for which this is the case.

Lemma 4.1. Let \( \varphi(w) = b(w)h(w) \) be the inner-outer factorization of \( \varphi \). Then \( Z_\varphi(\zeta) \) is a finite constant on \( D \) if and only if \( b \) is a finite Blaschke product and \( |h(w)| \geq 1 \) for every \( w \in D \).
Proof. It is easy to see that that $b$ is a finite Blaschke product and $|h(w)| \geq 1$ for every $w \in D$ if and only if
\[
\liminf_{|w| \to 1} |\varphi(w)| \geq 1.
\]
Suppose that $c = Z_\varphi(\zeta)$ for every $\zeta \in D$. To prove the necessity by contradiction, we assume that there exists a sequence $\{w_n\}_n$ in $D$ such that $\sup |\varphi(w_n)| < 1$ and $|w_n| \to 1$. We may assume that $\varphi(w_n) \to \zeta_0 \in D$. Then there exists $r_0, 0 < r_0 < 1$, such that the number of zeros of $\zeta_0 - \varphi(w)$ in $r_0D$ equals to $\alpha$. By the Hurwitz theorem, for a large positive integer $n_0$, the number of zeros of $\varphi(w_{n_0}) - \varphi(w)$ in $r_0D$ equals to $\alpha$. Further, we may assume that $w_{n_0} \notin r_0D$. Hence the number of zeros of $\varphi(w_{n_0}) - \varphi(w)$ in $D$ is greater than $c$ which contradicts the fact that $Z_\varphi(\zeta)$ is a constant.

The sufficiency is an easy consequence of Rouché’s theorem in Complex Analysis. In fact, if $b(w)$ is a finite Blaschke product and $h(w)$ is an outer function with $|h(w)| \geq 1$ on $D$, then by Rouché’s theorem, for each $\zeta \in D$ the number of zeros of $\zeta - \varphi(w)$ in $D$ coincides with the number of zeros of $b(w)$ in $D$. So $Z_\varphi(\zeta)$ is a finite constant. \hfill \Box

**Theorem 4.2.** Let $\varphi(w) = b(w)h(w)$ be the inner-outer factorization of $\varphi$. Then the following conditions are equivalent.

(i) $L(0)$ on $N_\varphi$ is compact.
(ii) $b$ is a finite Blaschke product and $|h(w)| \geq 1$ for every $w \in D$.

Proof. (i) $\Rightarrow$ (ii) If $L(0)$ on $N_\varphi$ is compact, then by Theorem 5.2.2 in [22] $Z_\varphi(\zeta)$ is a finite constant, and (ii) thus follows from Lemma 4.1.

(ii) $\Rightarrow$ (i) For any positive integer $m$, we have
\[
H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w) = \bigoplus_{j=0}^{m-1} b^j(w)(H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)).
\]
Since $b$ is a finite Blaschke product, $\dim(H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)) < \infty$ and $H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)$ is contained in the disk algebra $A(D)$. One easily sees that
\[
T_\varphi b^j(w)(H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)) \subset b^{j-1}(w)(H^2(\Gamma_w) \ominus b(w)H^2(\Gamma_w)),
\]
so that
\[
H^2(\Gamma_w) \ominus b^m(w)H^2(\Gamma_w) \subset L(0)N_\varphi.
\]
Then
\[
L(0)N_\varphi = (H^2(\Gamma_w) \ominus b^mH^2(\Gamma_w)) \ominus (b^mH^2(\Gamma_w) \cap L(0)N_\varphi)
\]
and hence
\[
N_\varphi = L(0)^{-1}(H^2(\Gamma_w) \ominus b^mH^2(\Gamma_w)) + L(0)^{-1}(b^mH^2(\Gamma_w) \cap L(0)N_\varphi),
\]
which is in fact a direct sum because $L(0)|_{N_\varphi}$ is injective. For simplicity we write this decomposition as
\[
N_\varphi = N_{1,m} + N_{2,m}.
\]
Since \( \dim(N_{1,m}) < \infty \), to prove that \( L(0) \) on \( N_{\varphi} \) is compact it is sufficient to prove that 

\[
\lim_{m \to \infty} \|L(0)|_{N_{2,m}}\| = 0, \text{ i.e.,}
\]

\[
\sup_{b^m g \in L(0)N_{\varphi}} \frac{\|b^m g\|^2}{\|L(0)^{-1}b^m g\|^2} \to 0 \quad \text{as } m \to \infty.
\]

Let \( b^m g \in L(0)N_{\varphi} \) and \( 0 \leq n \leq m \). By Lemma 2.4, 

\[
T_h^* b^{m-1} g = T_{\varphi} b^m g \in H^2(\Gamma_w),
\]

so that

\[
T_h^{*2} b^{m-2} g = T_h^* T_h^* b^{m-1} g = T_h^* T_h^* b^{m-1} g = T_{\varphi}^{*2} b^m g \in H^2(\Gamma_w).
\]

Repeating this, we have

\[
(4.1) \quad T_h^{*n} b^m g = T_{\varphi}^{*n} b^m g \in H^2(\Gamma_w).
\]

Using the fact that \( L(0)A_{\varphi} f = f \), i.e.,

\[
L^{-1}(0) f = \sum_{j=0}^{\infty} z^j T_{\varphi}^{*j} f,
\]

and that \( \|h^{-1}\|_{\infty} \leq 1 \), we calculate that

\[
\sup_{b^m g \in L(0)N_{\varphi}} \frac{\|b^m g\|^2}{\|L(0)^{-1}b^m g\|^2} = \sup_{b^m g \in L(0)N_{\varphi}} \frac{\|g\|^2}{\sum_{j=0}^{\infty} \|T_{\varphi}^{*j} b^m g\|^2}
\]

\[
\leq \sup_{b^m g \in L(0)N_{\varphi}} \frac{\|g\|^2}{\sum_{j=0}^{m} \|T_{\varphi}^{*j} b^m g\|^2} = \sup_{b^m g \in L(0)N_{\varphi}} \frac{\|g\|^2}{\sum_{j=0}^{m} \|T_{\varphi}^{*j} b^{m-j} g\|^2} \quad \text{by (4.1)}
\]

\[
\leq \sup_{b^m g \in L(0)N_{\varphi}} \frac{\|g\|^2}{\sum_{j=0}^{m} \|T_{\varphi}^{*j} b^{m-j} g\|^2} \quad \text{by Lemma 2.4}
\]

\[
= \frac{1}{m+1}.
\]

So it follows that \( \lim_{m \to \infty} \|L(0)|_{N_{2,m}}\| = 0 \) and this completes the proof.

\[ \Box \]

**Corollary 4.3.** If \( L(0) \) and \( R(0) \) are both compact on \( N_{\varphi} \) then \( \varphi \) is a finite Blaschke product.

**Proof.** If \( R(0) \) is compact on \( N_{\varphi} \), then by the parallel statement of Theorem 5.2.2 in [22] for \( R(0) \), the number of zeros of \( z - \varphi(\lambda) \) in \( D \) is a constant with respect to \( \lambda \in D \). Since \( N_{\varphi} \) is non-trivial, this constant is equal to 1. So \( \|\varphi\|_{\infty} \leq 1 \), and it follows that \( \|h\|_{\infty} \leq 1 \). If \( L(0) \) is also compact on \( N_{\varphi} \), then by Theorem 4.2 \( h \) is a constant of modulus 1, hence \( \varphi \) is a finite Blaschke product.

\[ \Box \]

In fact the converse of Corollary 4.3 is also true and we will see it in Section 5.
Next we study the compactness of $D_z$. In fact, the compactness of $D_z$ and that of $L(0)|_{N_{\varphi}}$ are closely related.

**Theorem 4.4.** If $\varphi$ is bounded, then $L(0)|_{N_{\varphi}}$ is compact if and only if $D_z$ is compact.

**Proof.** The fact that the compactness of $L(0)|_{N_{\varphi}}$ implies the compactness of $D_z$ follows from Theorem 3.7 and [22, Theorem 5.3.1].

To show that the compactness of $D_z$ implies that of $L(0)|_{N_{\varphi}}$, we first check that $S_z$ is Fredholm in this case. If $D_z$ is compact, then by Lemma 2.2 $S_z^*S_z$ is Fredholm, and hence $S_z^*$ has closed range. Moreover, it follows from Theorem 3.8 that $S_z^*$ is in fact onto. So it remains to show that $S_z^*$ has a finite dimensional kernel. If we let $\varphi = bh$ be the inner-outer factorization of $\varphi$, then by Lemma 3.4 we need to show that $H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$ is a finite dimensional subspace in $N_{\varphi}$, or equivalently, $b$ is a Blaschke product. For every $f \in H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$ and integers $i, j \geq 0$, one checks that

$$\langle D_z^* f, (z - \varphi)z^i w^j \rangle = \langle zf, (z - \varphi)z^i w^j \rangle = \langle f, z^i w^j \rangle.$$ 

So $D_z^* f$ is orthogonal to $(z - \varphi)z^i w^j$ when $i \geq 1$. Therefore,

$$\|D_z^* f\| = \|P_{M_\varphi}zf\| \geq \sup_{\|(z - \varphi)p\| \leq 1} |\langle zf, (z - \varphi)p \rangle|,$$

$p$ are polynomials in $H^2(\Gamma_w)$

$$= \sup_{\|(z - \varphi)p\| \leq 1} |\langle f, p \rangle|.$$ 

Since

$$\|(z - \varphi)p\|^2 = \|p\|^2 + \|\varphi p\|^2 \leq \|p\|^2 (1 + \|\varphi\|_{\infty}^2),$$

we have

$$\|D_z^* f\| \geq \sup_{\|p\| \leq (1 + \|\varphi\|_{\infty}^2)^{-1/2}} |\langle f, p \rangle| = (1 + \|\varphi\|_{\infty}^2)^{-1/2}\|f\|,$$

which means $D_z^*$ is bounded below by a positive constant on $H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$. Since $D_z$ is compact, $H^2(\Gamma_w) \ominus bH^2(\Gamma_w)$ is finite dimensional, and this concludes that $S_z$ is Fredholm.

Now we show that $L(0)|_{N_{\varphi}}$ is compact. For this matter, we recall the equality (cf. Proposition 5.1.1 in [22])

$$S_z D_z + (L(0)|_N)^*(L(0)|_{M \ominus zM}) = 0.$$

Since $D_z$ is compact, $(L(0)|_N)^*(L(0)|_{M \ominus zM})$ is compact. Since we have shown that $S_z$ is Fredholm in this case, $L(0)|_{M \ominus zM}$ is Fredholm by Lemma 2.1, and therefore $L(0)|_{N_{\varphi}}$ is compact.

The following example gives a simple illustration for the compactness of $L(0)|_{N_{\varphi}}$.

**Example 1.** We consider a function $\varphi(w) = aw$, where $a \in \mathbb{C}$ and $a \neq 0$. Let

$$R_j = \sqrt{1 + |a|^2 + \cdots + |a|^{2j}}$$

and

$$e_j = \frac{w^j + (\overline{a}z)w^{j-1} + \cdots + (\overline{a}z)^j}{R_j}.$$
Then it is not difficult to check that \( \{e_j\}_j \) is an orthonormal basis of \( N_\varphi \), and one verifies that
\[
\|L(0)e_j\|^2 = \left\| \frac{w\bar{z}}{R_j} \right\|^2 = R_j^{-2}.
\]
So if \(|a| < 1\), then \( \|L(0)e_j\|^2 \geq 1 - |a|^2 \) and hence \( L(0) \) on \( N \) is not compact. If \(|a| \geq 1\), then \( \lim_{j \to \infty} \|L(0)e_j\| = 0 \) which shows that \( L(0) \) on \( N \) is compact.

It is clear by Corollary 3.11 that \( S_z \) is essentially normal in this case. It is easy to give a direct calculation of \( [S_z^*, S_z] \). In fact,
\[
S_z e_j = \frac{aR_j}{R_{j+1}} e_{j+1}, \quad S_z^* e_j = \frac{\bar{a}R_{j-1}}{R_j} e_{j-1}.
\]
so
\[
(S_z^* S_z - S_z S_z^*) e_j = |a|^2 \left( \frac{R_j^2}{R_{j+1}^2} - \frac{R_{j-1}^2}{R_j^2} \right) e_j
= \left( \frac{|a|^2 + \cdots + |a|^{2(j+1)}}{1 + |a|^2 + \cdots + |a|^{2j}} - \frac{|a|^2 + \cdots + |a|^{2j}}{1 + |a|^2 + \cdots + |a|^{2j}} \right) e_j
:= c_j e_j.
\]
It is clear that \( c_j \to 0 \) as \( j \to \infty \). One also observes that \( S_z \) on \( N_{aw} \) is hypernormal.

By [14], we know that \( \|S_z\| = \|\varphi\|_\infty \) if \( \|\varphi\|_\infty \leq 1 \), and \( \|S_z\| = 1 \) for other cases. In the last part of this section, we calculate the norm and the essential norm of \( L(0)|N_\varphi \) and \( S_z \).

First we recall that the essential norm \( \|A\|_e \) is the norm of \( A \) in the Calkin algebra.

Since \( \|S_z^* F\|^2 + \|L(0)F\|^2 = \|F\|^2 \) for every \( F \in N_\varphi \), we have
\[
\|S_z^*\|^2 = \sup_{F \in N_\varphi, \|F\| = 1} \|S_z^* F\|^2 = 1 - \inf_{F \in N_\varphi, \|F\| = 1} \|L(0)F\|^2,
\]
(4.2)
\[
\inf_{F \in N_\varphi, \|F\| = 1} \|S_z^* F\|^2 = 1 - \sup_{F \in N_\varphi, \|F\| = 1} \|L(0)F\|^2 = 1 - \|L(0)\|^2.
\]

Hence
\[
\inf_{F \in N_\varphi, \|F\| = 1} \|L(0)F\| = \begin{cases} \sqrt{1 - \|\varphi\|_\infty^2}, & \text{if } \|\varphi\|_\infty \leq 1 \\ 0, & \text{other cases.} \end{cases}
\]

**Proposition 4.5.** Let \( \alpha = \inf_{w \in D} |\varphi(w)| \). Then \( \alpha < 1 \) and \( \|L(0)|N_\varphi\| = \sqrt{1 - \alpha^2} \).

**Proof.** By [14, Corollary 2.7], \( \varphi(D) \cap D \neq \emptyset \). Hence \( \alpha < 1 \). Let
\[
F = \frac{2}{(1 - \varphi(w_0)\bar{z})(1 - \bar{w}_0 w)}
\]
Let \( w_0 \in \Omega_\varphi \). Then by Lemma 2.5, \( F \in N_\varphi \) and
\[
\frac{\|L(0)F\|^2}{\|F\|^2} = 1 - |\varphi(w_0)|^2.
\]
This implies $1 - |\varphi(w_0)|^2 \leq \|L(0)\|^2$. Thus we get

$$\sqrt{1 - \alpha^2} \leq \|L(0)\| \leq 1.$$ \hfill (4.3)

If $\alpha = 0$, then $\|L(0)\| = 1$.

Suppose that $\alpha > 0$. Then $(1/\varphi)(w) \in H^\infty(\Gamma_w)$, and by Lemma 2.4 we have $T^*_1/\varphi^nT^*_n = I$ on $L(0)N_\varphi$ for every $n \geq 1$. Let $h \in L(0)N_\varphi$. We have

$$\|h\| = \|T^*_1/\varphi^nT^*_n h\| \leq \|T^*_1/\varphi^n\| \|T^*_n h\| = \|1/\varphi\|_\infty \|T^*_n h\| = \|T^*_n h\|/\alpha^n.$$  

Then $\alpha^n\|h\| \leq \|T^*_n h\|$ for every $h \in L(0)N_\varphi$ and $n$. Hence

$$\|h\|^2 \frac{1}{1 - \alpha^2} \leq \sum_{n=0}^{\infty} \|T^*_n h\|^2 = \|L_0^{-1} h\|^2$$

for every $h \in L(0)N_\varphi$, and $\|L(0) F\|^2 \leq (1 - \alpha^2) \|F\|$ for every $F \in N_\varphi$. Therefore $\|L(0)\| \leq \sqrt{1 - \alpha^2}$. By (4.3), $\|L(0)\| = \sqrt{1 - \alpha^2}$. \hfill $\Box$

A combination of (4.2), Propositions 3.1 and Proposition 4.5 leads to the following

**Corollary 4.6.** Let $\alpha = \inf_{w \in D} |\varphi(w)|$. Then $S^*_z$ is invertible if and only if $\alpha > 0$. In this case, 

$$\|S^*_z\|^{-1} = \inf_{F \in N_\varphi, \|F\| = 1} \|S^*_z F\| = \alpha.$$  

**Theorem 4.7.** Let $\varphi(w) \in H^\infty(\Gamma_w)$ with $N_\varphi \neq \{0\}$. Let $\varphi(w) = b(w) h(w)$ be the outer-inner factorization of $\varphi$. Suppose that $L(0)$ on $N_\varphi$ is not compact. Let $\gamma = \liminf_{|w|\to 1} |\varphi(w)|$.

Then $\gamma < 1$ and $\|L(0)\|_e = \sqrt{1 - \gamma^2}$. Moreover $\|L(0)\|_e \neq \|L(0)\|$ if and only if $b(w)$ is a non-constant finite Blaschke product and $1/h(w) \in H^\infty(\Gamma_w)$.

**Proof.** By Theorem 4.2, $\gamma < 1$. Take a sequence $\{w_j\}_j$ in $D$ such that $|\varphi(w_j)| \to \gamma$ and $|w_j| \to 1$ as $j \to \infty$. We have

$$\|L(0)k_{w_j}\| = \sqrt{1 - |w_j|^2} \sqrt{1 - |\varphi(w_j)|^2} \|\frac{1}{1 - \varphi(w_j)}\|$$

$$= \sqrt{1 - |\varphi(w_j)|^2} \to \sqrt{1 - \gamma^2}.$$  

Let $K$ be a compact operator from $N_\varphi$ to $H^2_\infty(\Gamma_w)$. Since $k_{w_j} \to 0$ weakly in $N_\varphi$, $\|(L(0) + K)k_{w_j}\| \to \sqrt{1 - \gamma^2}$. Hence $\|L(0)\|_e \geq \sqrt{1 - \gamma^2}$.

Suppose that $\gamma = 0$. Then $1 \leq \|L(0)\|_e \leq \|L(0)\| \leq 1$. In this case, either $b$ is not a finite Blaschke product or $1/h \notin H^\infty(\Gamma_w)$. 

Suppose that $0 < \gamma < 1$. Then $b$ is a finite Blaschke product. By Proposition 4.5, $\|L(0)\| = \sqrt{1 - \alpha^2}$, where $\alpha = \inf_{w \in D} |\varphi(w)|$. We note that $\alpha \leq \gamma$. If $\alpha = \gamma$, then we have $\|L(0)\| = \|L(0)\|_e$. In this case, $b$ is a constant function and $1/h \in H^\infty(\Gamma_w)$.

If $\alpha < \gamma$, then $b$ is a non-constant finite Blaschke product and $1/h \in H^\infty(\Gamma_w)$. This implies that $\alpha = 0$ and $\|L(0)\| = 1$. In this case we shall prove that $\|L(0)\|_e = \sqrt{1 - \gamma^2}$.

We note that $\|1/h\| \leq 1/\gamma$. The idea of the proof is the same as that of Theorem 4.2. We have

$$\sup_{b^m \in L(0)N_\varphi} \frac{\|b^m\|}{\|L^{-1}(0)b^m\|_e^2} \leq \sup_{b^m \in L(0)N_\varphi} \frac{\|b^m\|}{\sum_{n=0}^m \|T_h b^m - n g\|^2} = \sup_{b^m \in L(0)N_\varphi} \frac{\|b^m\|}{\sum_{n=0}^m \gamma^n \|T_h b^m - n g\|^2} \leq \frac{1}{\sum_{n=0}^m \gamma^n}.$$ 

Hence $\|L(0)\|_e \leq \sqrt{1 - \gamma^2}$, so that we obtain $\|L(0)\|_e = \sqrt{1 - \gamma^2}$.

\begin{theorem}
Let $\varphi(w) \in H^2(\Gamma_w)$ satisfying and $\varphi \neq 0$. Then $\|S_z\|_e = \|S_z\|$.
\end{theorem}

\begin{proof}
First, suppose that $0 < \|\varphi\|_\infty < 1$. Let $K$ be a compact operator on $N_\varphi$. Let $\{w_j\}_j$ be a sequence in $D$ such that $|\varphi(w_j)| \to \|\varphi\|_\infty$ as $j \to \infty$. Then $K w_j \to 0$ as $j \to \infty$. One easily sees that $\|S_z^* w_j\| = |\varphi(w_j)|$, so that $\|S_z^* w_j\| \to \|\varphi\|_\infty$ as $j \to \infty$. Hence $\|S_z^* + K\| \geq \|\varphi\|_\infty$.

By [14, Proposition 3.5], $\|S_z^*\| = \|\varphi\|_\infty$, so that $\|S_z\|_e = \|S_z^*\|_e = \|\varphi\|_\infty = \|S_z\|$. 

Next, suppose that $1 \leq \|\varphi\|_\infty \leq \infty$. By [14, Proposition 3.5], $\|S_z\|_e = 1$. Suppose that $\liminf_{|w| \to 1} |\varphi(w)| \geq 1$. By Theorem 4.2, $L(0)$ is compact on $N_\varphi$. Since $S_z S_z^* = I - L^*(0)L(0)$, $\|S_z S_z^*\|_e = 1, so that \|S_z\|_e = 1.

Suppose that $\liminf_{|w| \to 1} |\varphi(w)| < 1$. Then there exists a sequence $\{\alpha_j\}_j \subset D$ satisfying the following conditions; $|\alpha_j| \to 1$ as $j \to \infty$, and for each $j$ there exists a sequence $\{w_{j,l}\}_l$ in $\Omega_\varphi$ such that $|w_{j,l}| \to 1$ and $\varphi(w_{j,l}) \to \alpha_j$ as $l \to \infty$. Let $K$ be a compact operator on $N_\varphi$. Then $\|(S_z^* + K) w_{j,l}\| \to |\alpha_j|$ as $l \to \infty$. Since $|\alpha_j| \to 1$, we have $\|S_z^* + K\| \geq 1$. Hence $\|S_z\|_e = \|S_z^*\|_e = 1 = \|S_z\|$.
\end{proof}

5. The Case When $\varphi$ is Inner

This section gives a detailed study for the case when $\varphi$ is inner. On the one hand, the fact that $\varphi$ is inner makes this case very computable, and, as a consequence, many of the earlier results have a clean illustration in this case. On the other hand, the case has a close connection with the two classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L^2_\varphi(D)$. This fact suggests that the space $N_\varphi$ indeed has very rich structure.

Some preparations are needed to start the discussion. With every inner function $\theta(w)$ in the Hardy space $H^2(\Gamma)$ over the unit circle $\Gamma$, there is an associated contraction $S(\theta)$ on $H^2(\Gamma) \ominus \theta H^2(\Gamma)$ defined by $S(\theta)f = P_{\theta}wf$, $f \in H^2(\Gamma) \ominus \theta H^2(\Gamma)$.
where $P_\theta$ is the projection from $H^2(\Gamma)$ onto $H^2(\Gamma) \ominus \theta H^2(\Gamma)$. The operator $S(\theta)$ is the classical Jordan block, and its properties have been very well studied (cf. [1, 18]). We will state some of the related facts later in the section. Here, we display an orthonormal basis for $N_\varphi$.

**Lemma 5.1.** Let $\varphi(w)$ be a one variable non-constant inner function. Let $\{\lambda_k(w)\}_{k=0}^m$ be an orthonormal basis of $H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)$, and

$$e_j = \frac{w^j + w^{j-1}z + \cdots + z^j}{\sqrt{j+1}}$$

for each integer $j \geq 0$. Then $\{\lambda_k(w)e_j(z,\varphi(w)); k = 0, 1, 2, \ldots, m, j = 1, 2, \ldots\}$ is an orthonormal basis for $N_\varphi$.

**Proof.** First of all, we have the facts that

$$N_\varphi = \left\{ A_\varphi f : f \in H^2(\Gamma_w), \sum_{n=0}^\infty \|T_{\varphi^n}^* f\|^2 < \infty \right\},$$

and

$$H^2(\Gamma_w) = \sum_{j=0}^\infty \oplus \varphi^j(w)(H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)).$$

Write

$$E_{k,j} = \lambda_k(w)e_j(z,\varphi(w)).$$

Then if $(k, j) \neq (s, t)$ and $j \leq t$,

$$\langle E_{k,j}, E_{s,t} \rangle = \frac{1}{\sqrt{j+1}\sqrt{t+1}} \sum_{l=0}^j \sum_{i=0}^t \langle \lambda_k(w)\varphi^{j-l}(w)z^l, \lambda_s(w)\varphi^{t-i}(w)z^i \rangle$$

$$= \frac{(j+1)\langle \lambda_k(w), \varphi^{j-1}(w)l_s(w) \rangle}{\sqrt{j+1}\sqrt{t+1}}$$

$$= 0,$$

and $\|E_{k,j}\| = 1$ for every $k, j$. Let $f(w) \in H^2(\Gamma_w)$ and write

$$f(w) = \sum_{j=0}^\infty \oplus \left( \sum_{k=1}^m a_{k,j}\lambda_k(w) \right)\varphi^j(w), \quad \sum_{j=0}^\infty \sum_{k=0}^m |a_{k,j}|^2 < \infty.$$ 

Then

$$\sum_{n=0}^\infty \|T_{\varphi^n}^* f(w)\|^2 = \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^m |a_{k,j}|^2 = \sum_{j=0}^\infty (j+1) \sum_{k=0}^m |a_{k,j}|^2.$$ 

Hence

$$\sum_{n=0}^\infty z^n T_{\varphi^n}^* f(w) \in N_\varphi \iff \sum_{j=0}^\infty (j+1) \sum_{k=0}^m |a_{k,j}|^2 < \infty.$$
In this case, we have
\[
\sum_{n=0}^{\infty} z^n \mathcal{T}_{\varphi^n} f(w) = \sum_{j=0}^{m} \left( \sum_{k=0}^{m} a_{k,j} \lambda_k(w) \right) (\varphi^j(w) + \varphi^{j-1}(w) z + \cdots + z^j) = \sum_{j=0}^{\infty} \sum_{k=0}^{m} \sqrt{j+1} a_{k,j} E_{k,j}.
\]
This shows that \( \{E_{k,j}\}_{k,j} \) is an orthonormal basis of \( N_\varphi = H^2(\Gamma^2) \ominus M_\varphi \).

The operators \( L(0)|_{N_\varphi} \), \( R(0)|_{N_\varphi} \) and \( D_z \) are easy to calculate in this case. In fact, one checks that
\[
L(0) E_{k,j} = \frac{\lambda_k(w) \varphi^j(w)}{\sqrt{j+1}},
\]
and
\[
R(0) E_{k,j} = \frac{\lambda_k(0)(\varphi(0)^j + \varphi(0)^{j-1} z + \cdots + z^j)}{\sqrt{j+1}}.
\]
So \( L(0)|_{N_\varphi} \) and \( R(0)|_{N_\varphi} \) are both compact if \( m < \infty \), that is, \( \varphi(w) \) is a finite Blaschke product. We summarize this observation and Corollary 4.3 in the following corollary.

Corollary 5.2. For \( \varphi \in H^2(\Gamma_w) \), \( L(0) \) and \( R(0) \) are both compact on \( N_\varphi \) if and only if \( \varphi \) is a finite Blaschke product.

The operator \( D_z \) is also easy to calculate in this case. One first verifies that
\[
X_{k,j} := \frac{\lambda_k(w)}{\sqrt{j+2}} (ze_j(z, \varphi(w)) - \sqrt{j+1} \varphi^{j+1}(w)), \quad 0 \leq k \leq m, \quad 0 \leq j < \infty,
\]
is an orthonormal basis for \( M_\varphi \ominus z M_\varphi \). Then
\[
D_z X_{k,j} = \frac{\lambda_k(w)e_j(z, \varphi(w))}{\sqrt{j+2}} = \frac{1}{\sqrt{j+2}} E_{k,j}
\]
which is also compact if \( \varphi(w) \) is a finite Blaschke product.

Two other observations are also worth mentioning. First one calculates that
\[
\langle z E_{k,j}, E_{s,t} \rangle = \frac{1}{\sqrt{j+1} \sqrt{t+1}} \sum_{l=0}^{j} \sum_{i=0}^{t} \langle z \lambda_l(w) \varphi^{j-l}(w) z^l, \lambda_s(w) \varphi^{t-i}(w) z^i \rangle = \frac{1}{\sqrt{j+1} \sqrt{t+1}} \sum_{l=0}^{j} \sum_{i=0}^{t} \langle \lambda_l(w), \lambda_s(w) \varphi^{t+l-i-j}(w) z^{i-l-1} \rangle.
\]
Hence
\[
\langle z E_{k,j}, E_{s,t} \rangle = 0 \iff t = j + 1 \text{ and } k = s,
\]
and
\[
S_z E_{k,j} = \langle S_z E_{k,j}, E_{k,j+1} \rangle E_{k,j+1}
\]
\[
= \frac{1}{\sqrt{j+1}\sqrt{j+2}} \sum_{l=0}^{j} \langle \lambda_k(w), \lambda_l(w) \rangle E_{k,j+1}
\]
\[
= \frac{\sqrt{j+1}}{\sqrt{j+2}} E_{k,j+1}.
\]

This calculation reminds us of the Bergman shift $B$ on the Bergman space $L_a^2(D)$ with the orthonormal basis \{\sqrt{j+1}\zeta_j\}. In fact, if we define the operator $U : N_\varphi \rightarrow (H^2(\Gamma) \ominus \varphi H^2(\Gamma)) \otimes L_a^2(D)$ by
\[
U(E_{k,j}) = \lambda_k(w)\sqrt{j+1}\zeta_j,
\]
then $U$ is clearly a unitary operator, and one checks that
\[
US_z = (I \otimes B)U.
\]

So from this viewpoint $N_\varphi$ can be identified as $(H^2(\Gamma) \ominus \varphi H^2(\Gamma)) \otimes L_a^2(D)$. As both $H^2(\Gamma) \ominus \varphi H^2$ and $L_a^2(D)$ are classical subjects, this observation indicates that the space $N_\varphi$ indeed has very rich structure.

The other observation is about the range $R(D_z)$. Let $F \in N_\varphi$. Then by Theorem 2.3,
\[
F \in T_z^*(M_\varphi \ominus zM_\varphi) \iff \sup_{G \in N_\varphi, \|G\| = 1} \left| \langle S_z^* G, F \rangle \right| < \infty.
\]

Write
\[
F = \sum_{k=0}^{m} \sum_{j=0}^{\infty} a_{k,j} E_{k,j}, \quad \sum_{k=0}^{m} \sum_{j=0}^{\infty} |a_{k,j}|^2 < \infty,
\]
\[
G = \sum_{k=0}^{m} \sum_{j=0}^{\infty} b_{k,j} E_{k,j}, \quad \sum_{k=0}^{m} \sum_{j=0}^{\infty} |b_{k,j}|^2 = 1.
\]

Then
\[
\left| \langle S_z^* G, F \rangle \right| = \left| \left\langle \sum_{k=0}^{m} \sum_{j=0}^{\infty} b_{k,j} E_{k,j}, \sum_{k=0}^{m} \sum_{j=0}^{\infty} a_{k,j} S_z E_{k,j} \right\rangle \right|
\]
\[
= \frac{\left| \sum_{k=0}^{m} \sum_{j=0}^{\infty} b_{k,j} \lambda_k(w) \varphi_j(w) \right|}{\left\| \sum_{k=0}^{m} \sum_{j=0}^{\infty} b_{k,j} \lambda_k(w) \varphi_j(w) \right\|}
\]
\[
= \frac{\sqrt{\sum_{k=0}^{m} \sum_{j=0}^{\infty} |b_{k,j}|^2}}{\sqrt{\sum_{k=0}^{m} \sum_{j=0}^{\infty} |b_{k,j}|^2}}
\]
\[
= \frac{\sqrt{\sum_{k=0}^{m} \sum_{j=0}^{\infty} \sqrt{j+1}|b_{k,j}|}}{\sqrt{\sum_{k=0}^{m} \sum_{j=0}^{\infty} \sqrt{j+1}|b_{k,j}|}}.
\]
and
\[
\sup_{G \in N_\varphi, \|G\| = 1} \frac{|\langle S_\varphi^* G, F \rangle|}{\|L(0) G\|} = \sqrt{\sum_{k=0}^{m} \sum_{j=0}^{\infty} (j + 1)|a_{k,j}|^2}.
\]

Write \( c_{k,j} = \sqrt{j + 1} a_{k,j} \), then we have \( F \in D_z(M_\varphi \ominus z M_\varphi) \) if and only if
\[
F = \sum_{k=0}^{m} \sum_{j=0}^{\infty} c_{k,j} E_{k,j} = \sum_{k=0}^{m} \sum_{j=0}^{\infty} |c_{k,j}|^2 < \infty.
\]

So
\[
U(R(D_z)) = \left( H^2(\Gamma) \ominus \varphi H^2(\Gamma) \right) \otimes H^2(\Gamma).
\]

It follows directly from (5.1) that \( S_z \) on \( N_\varphi \) is essentially normal if and only if \( \varphi \) is a finite Blaschke product. Now we take a look at the essential normality of \( S_w \). Some facts about the space \( H^2(\Gamma) \ominus \varphi H^2(\Gamma) \) need to be mentioned here. We recall that the Jordan block \( S(\varphi) \) is defined by
\[
S(\varphi) g = P_\varphi w g, \quad g \in H^2(\Gamma) \ominus \varphi H^2(\Gamma),
\]
where \( P_\varphi \) is the orthogonal projection from \( H^2(\Gamma) \) onto \( H^2(\Gamma) \ominus \varphi H^2(\Gamma) \). The two functions \( P_\varphi 1 \) and \( P_\varphi \bar{w} \varphi \) play important roles here, and we let the operator \( T_0 \) on \( H^2(\Gamma) \ominus \varphi H^2(\Gamma) \) be defined by
\[
T_0 g = \langle g, P_\varphi \bar{w} \varphi \rangle P_\varphi 1.
\]
One verifies
\[
T_0^* T_0 g = \|P_\varphi 1\|^2 \langle g, P_\varphi \bar{w} \varphi \rangle P_\varphi \bar{w} \varphi, \quad T_0 T_0^* g = \|P_\varphi \bar{w} \varphi\|^2 \langle g, P_\varphi 1 \rangle P_\varphi 1,
\]
and
\[
(5.2) \quad I - S(\varphi)^* S(\varphi) = \|P_\varphi 1\|^{-2} T_0^* T_0, \quad I - S(\varphi) S(\varphi)^* = \|P_\varphi \bar{w} \varphi\|^2 T_0 T_0^*.
\]

For every \( g \in H^2(\Gamma) \ominus \varphi H^2(\Gamma) \), we decompose \( w g \) as
\[
w g = S(\varphi) g + (I - P_\varphi) w g.
\]
Using the facts that \( (I - P_\varphi) w g = \langle w g, \varphi \rangle, \quad P_\varphi 1 = 1 - \bar{\varphi}(0) \varphi \) and \( S_\varphi = S_z \), where \( S_\varphi g = P_{N_\varphi} \varphi g \), we have
\[
S_w g e_j = \sum_{m,n} \langle w g e_j, E_{m,n} \rangle E_{m,n}
\]
\[
= \sum_{m,n} \left( \langle S(\varphi) g e_j + \langle w g, \varphi \rangle \frac{\varphi P_\varphi 1}{1 - \bar{\varphi}(0) \varphi} e_j, E_{m,n} \rangle E_{m,n} \right)
\]
\[
= \langle S(\varphi) g e_j + \langle w g, \varphi \rangle \sum_{m,n} \frac{\varphi P_\varphi 1}{1 - \varphi(0) \varphi} e_j, E_{m,n} \rangle E_{m,n}
\]
\[
= \langle S(\varphi) g e_j + \langle g, \bar{w} \varphi \rangle (1 - \bar{\varphi}(0) S_z)^{-1} S_z (P_\varphi 1 \cdot e_j) \rangle.
\]

So
\[
(5.3) \quad U S_w U^* = S(\varphi) \otimes I + T_0 \otimes (1 - \bar{\varphi}(0) B)^{-1} B.
\]

For further discussion, we assume \( \varphi \) is not a singular inner function, i.e., \( \varphi \) has a zero in \( D \). We first look at the case when \( \varphi(0) = 0 \). In this case (5.3) reduces to the cleaner expression
\[
(5.4) \quad U S_w = (S(\varphi) \otimes I + T_0 \otimes B) U.
\]
Using (5.4) and the fact \( S^*(\varphi)T_0 = 0 \), one easily verifies that

\[
US_w^*S_w^*U^* = S(\varphi)^*S(\varphi) \otimes I + T_0^*T_0 \otimes B^*B,
\]

and

\[
US_w^*S_w^*U^* = S(\varphi)^*S(\varphi) \otimes I + T_0^*T_0 \otimes BB^*.
\]

Then by (5.2)

\[
U[S_w^*, S_w]U^* = (I - S(\varphi)S(\varphi)^*) \otimes I + (I - S(\varphi)^*S(\varphi)) \otimes I
\]

\[
+ T_0^*T_0 \otimes B^*B - T_0^*T_0 \otimes BB^*
\]

\[
= T_0^*T_0 \otimes (I - BB^*) - T_0^*T_0 \otimes (I - B^*B).
\]

Since \( T_0 \) is of rank 1 and it is well-known that \( I - BB^* \) and \( I - BB^* \) are Hilbert-Schmidt, (5.5) implies that \([S_w^*, S_w]\) is Hilbert-Schmidt. The Hilbert-Schmidt norm of \([S_w^*, S_w]\) can be readily calculated in this case. First of all, \( P_{\mathcal{N}_\varphi}1 = 1 \) and \( P_{\mathcal{N}_\varphi}w\varphi = w\varphi \). Let \( \lambda_k, k = 0, 1, 2, \ldots \), be an orthonormal basis of \( H^2(\Gamma) \otimes \varphi H^2(\Gamma) \) and \( \lambda_0 = 1 \). Then by (5.5),

\[
[S_w^*, S_w] \lambda_k e_j = \frac{(T_0^*T_0^*\lambda_k)e_j}{j+1} - \frac{(T_0^*T_0^*\lambda_k)e_j}{j+2} = \frac{\lambda_k(0)e_j}{j+1} - \frac{\langle \lambda_k, w\varphi \rangle w\varphi e_j}{j+2},
\]

and one calculates that

\[
\sum_k \| [S_w^*, S_w] \lambda_k e_j \|^2 = \frac{1}{(j+1)^2} + \frac{1}{(j+2)^2} - \frac{2|\varphi'(0)|^2}{(j+1)(j+2)},
\]

from which it follows that

\[
\| [S_w^*, S_w] \|^2_{HS} = \frac{\pi^2}{3} - 1 - 2|\varphi'(0)|^2.
\]

In the case \( \varphi(0) \neq 0 \), we need an additional general fact. For \( \alpha \in D \), we let \( x_\alpha(w) = \frac{\alpha - w}{1 - \bar{\alpha}w} \). So if we let operator \( U_\alpha \) be defined by

\[
U_\alpha(f)(z, w) := \frac{1 - |\alpha|^2}{1 - \bar{\alpha}w} f(z, x_\alpha(w)), \quad f \in H^2(D^2),
\]

then it is well-known that \( U_\alpha \) is a unitary. We let \( M' = U_\alpha([z - \varphi]) = [z - \varphi(x_\alpha)] \) and \( N' = H^2(D^2) \ominus M' \). The two variable Jordan block on \( N' \) is denoted by \((S'_w, S'_w)\). Then by [25],

\[
U_\alpha S_w U_\alpha^* = S'_w, \quad U_\alpha S_w U_\alpha^* = x_\alpha(S'_w).
\]

Since \( x_\alpha(x_\alpha(w)) = w \), we also have

\[
U_\alpha x_\alpha(S_w) U_\alpha^* = S'_w.
\]

So if \( \varphi(0) \neq 0 \), we pick any zero of \( \varphi \), say \( \alpha \). Since \( \varphi(x_\alpha(0)) = \varphi(\alpha) = 0 \), \([S_w^*, S_w]\) is Hilbert-Schmidt by the above calculations, and it then follows that \([S_w^*, S_w]\) is Hilbert-Schmidt (cf. [20, Lemma 1.3]). So in conclusion, when \( \varphi \) is not singular \([S_w^*, S_w]\) is Hilbert-Schmidt on \( \mathcal{N}_\varphi \).

These calculations on \( S_z \) and \( S_w \) prove the following theorem.
Theorem 5.3. Let \( \varphi \) be an one variable inner function. Then \( N_\varphi \) is essentially reductive if and only if \( \varphi \) is a finite Blaschke product.

On \( N_\varphi \), the commutator \([S^*_z, S_w]\) can also be easily calculated. One sees that

\[
US_z^*S_wU = (I \otimes B^*) \left( S(\varphi) \otimes I + T_0 \otimes (1 - \overline{\varphi(0)} B)^{-1} B \right)
\]

and

\[
US_wS_z^*U = \left( S(\varphi) \otimes I + T_0 \otimes (1 - \overline{\varphi(0)} B)^{-1} B \right) (I \otimes B^*)
\]

So

\[
[S^*_z, S_w] = T_0 \otimes [B^*, (1 - \overline{\varphi(0)} B)^{-1} B].
\]

It was shown in [26] that

\[
(5.6) \quad \text{tr} \ [f^*(B), g(B)] = \int_D f'(w)\overline{g'(w)}dA,
\]

where \( f \) and \( g \) are analytic functions on \( D \) that are continuous on \( \bar{D} \) and the derivatives \( f' \) and \( g' \) are in \( L^2_a(D) \). Using (5.6), one easily verifies that \([B^*, (1 - \overline{\varphi(0)} B)^{-1} B]\) is trace class with \( \text{tr} \ [B^*, (1 - \overline{\varphi(0)} B)^{-1} B] = 1 \). Therefore, \([S^*_z, S_w]\) is trace class with

\[
\text{tr} \ [S^*_z, S_w] = \text{tr} \ T_0 \cdot \text{tr} \ [B^*, (1 - \overline{\varphi(0)} B)^{-1} B]
\]

\[
= \text{tr} \ T_0
\]

\[
= \varphi'(0).
\]

Example 2. As we have remarked before that \( S_z \) on \( N_w \) is equivalent to the Bergman shift \( B \) and \( S_z = S_w \) in this case, and moreover \( \varphi' = 1 \). So from the calculations above

\[
\text{tr} \ [B^*, B] = 1, \quad \text{and} \quad \| [B^*, B] \|_{H.S.}^2 = \frac{\pi^2}{3} - 3.
\]

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