Review

Quantifying entanglement resources

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Received 19 February 2014, revised 24 April 2014
Accepted for publication 25 April 2014
Published 8 October 2014

Abstract

We present an overview of the quantitative theory of single-copy entanglement in finite-dimensional quantum systems. In particular we emphasize the point of view that different entanglement measures quantify different types of resources, which leads to a natural interdependence of entanglement classification and quantification. Apart from the theoretical basis, we outline various methods for obtaining quantitative results on arbitrary mixed states.

This article is part of a special issue of \textit{Journal of Physics A: Mathematical and Theoretical} devoted to ‘50 years of Bell’s theorem’.

Keywords: entanglement measures, entanglement classification, quantum Information

(Some figures may appear in colour only in the online journal)

1. Preamble

The quantitative description of entanglement started with Bell’s inequalities 50 years ago [1]. The next milestones from the theory point of view were Werner’s seminal work on the precise mathematical characterization of entanglement for mixed quantum states [2], and the advent of multipartite entanglement [3, 4]. Since then, thousands of studies on quantum information have appeared for which entanglement is a central concept. Therefore it may seem surprising that to date there is still no comprehensive quantitative theory of entanglement. Even worse, if a non-specialist wants a quick overview of how to quantitatively characterize entanglement he will encounter a puzzling diversity of entanglement classifications and entanglement measures, based on different concepts and methods without obvious connections between them.
However, the situation is much better than it might seem at first glance, at least for the simplest subtopic of entanglement theory, which comprises the entanglement resources contained in a few systems with a finite number of levels. There has been enormous progress during the past decade, such that many features of the conceptual framework have already become apparent. It is realistic to expect a reasonably complete theory of the basic concepts within the next few years.

In this review, we describe the state of affairs in this subfield of quantitative entanglement theory. Rather than covering each and every detail, our aim is to outline the well-established and commonly used concepts for single-copy entanglement, and to illuminate the logical connections between them. We put particular emphasis on the resource character of entanglement. That is, entanglement is a necessary prerequisite to carry out a certain procedure or protocol. The more entanglement there is (or the higher its quality is) the more likely it is that the protocol will be successful. Different entanglement measures quantify distinct resources, and different resources require their specific entanglement measures. The relation between measures, resources and the structure of the quantum-mechanical state space is a central issue of this research. It is needless to mention that at present we are able to mathematically distinguish many more types of entanglement than we know protocols, for which they might serve as a resource.

More technically, we explain the basic concepts of the quantitative theory for single-copy entanglement and then sketch the structure of entanglement between two finite-dimensional systems, as well as common entanglement measures and methods to evaluate (or estimate) them for arbitrary states. Further, we discuss multipartite entanglement, in particular that of three and more qubits, and the polynomial measures that distinguish different types of genuine multipartite entanglement.

Due to this narrow focus, there are many topics that cannot be discussed or even be mentioned. In particular, this article does not review

- asymptotic entanglement measures and protocols, which we consider very briefly, however, essentially only to mark down the field of single-copy entanglement. Asymptotic measures and resources are discussed in detail in the review by Plenio and Virmani [5], as well as in the review by the Horodecki family [6],
- the relation between entanglement and nonlocality. All relevant aspects of this topic are extensively discussed by Brunner et al in [7]. There is also a shorter overview of the main concepts by Werner and Wolf [8],
- the entanglement of infinitely many degrees of freedom (continuous variables). There are up-to-date overviews of the relevant concepts by Adesso and Illuminati [9], and by Braunstein and van Loock [10],
- entanglement for indistinguishable particles and in many-body systems. There are two recent reviews of these subjects by Tichy et al [11] and by Amico et al [12],
- entanglement for relativistic particles. These concepts were reviewed, e.g., by Alsing and Fuentes [13] as well as by Peres and Terno [14].

Apart from the sources mentioned so far there are excellent texts on quantum information and entanglement, e.g., A Peres’ textbook [15], J Preskill’s lecture notes on quantum computation [16], the textbook by MA Nielsen and IL Chuang [17], lecture notes by MM Wolf [18], and the textbook by I Bengtsson and K Zyczkowski [19].
2. Entanglement as a resource

2.1. The LOCC paradigm

At the very basis of entanglement theory lies the paradigm of Local Operations and Classical Communication (LOCC) formulated by Bennett et al [20]. Under this paradigm, a state is distributed among different parties who can perform arbitrary local operations (including measurements and operations involving additional local systems, so-called ancillas), and in addition can communicate with each other over a normal classical channel; however, they are not able to exchange quantum systems. Under those restrictions, not all transformations of states are possible. In particular, it is not possible to create arbitrary states from scratch this way. This gives rise to the basic definition: a state is separable if it can be created using only local operations and classical communications, and it is called entangled otherwise [21, 22].

The fact that entangled states cannot be generated locally makes them a resource [23]. This resource is used in different tasks in quantum computation [24], quantum communication [25] and quantum cryptography [26]. Those tasks are usually written in the form of a protocol, which uses the entanglement as resource. While quantum computation protocols are not normally written in this form, but instead in the language of quantum gates, any quantum computation can also be done using measurement-based quantum computation [27], also known as one-way quantum computation. In this form, it uses an LOCC protocol as well (where the local operations are projective measurements) that consumes an entangled state, typically a cluster state, as a resource. However, the question of whether entanglement is a necessary resource for quantum computation is still open.

The resource character of entanglement immediately leads to two questions. The first one is obvious: how much of that resource do we have? This is the question of entanglement quantification. Answering this question leads to so-called entanglement monotones [28] or entanglement measures [20, 29, 30], which are the topic of this review. However, as we will see, entanglement monotones are not only useful to answer this question, but also for answering the second one.

The second question arises most naturally in situations where there are more than two parties, so-called multipartite states, but turns out to be relevant even in bipartite states (just two parties): how many different types of this resource do we have? This is the question of entanglement classification.

The same resource' in the previous paragraph means that we can use those states as resources for the same tasks, although possibly with lower efficiency. Obviously if we can convert the state $|\psi\rangle$ into the state $|\phi\rangle$ using LOCC, then for any task that works starting with the state $|\phi\rangle$, also works starting with the state $|\psi\rangle$: just convert $|\psi\rangle$ to $|\phi\rangle$ and then run the corresponding protocol on $|\phi\rangle$. Also note that this conversion need not be successful every time; as long as it is with non-vanishing probability, we can still use it for the protocol, just with lower efficiency: only in those cases where the conversion succeeds, so does the protocol.

Those operations which can be performed using LOCC but may fail are known as Stochastic Local Operations and Classical Communication (SLOCC) [28, 31–34]. In particular, states that can be transformed into one another are called SLOCC equivalent. SLOCC equivalent states therefore contain the same resource. The equivalence classes under SLOCC equivalence are also called SLOCC classes or entanglement classes. They are obviously invariant under invertible SLOCC transformations.

Note that for a specific task, it is possible that states from different SLOCC classes can be used to perform it. The most obvious example is when conversion by SLOCC is only possible
one way, where the given protocol works on the destination state. Therefore if one is interested only in specific tasks, it makes sense to apply some coarse graining to the SLOCC classes, which only distinguishes the entanglement properties of interest. To distinguish the classes of such a classification from the SLOCC classes, we speak of families in that case. Of course such a family is also invariant under invertible SLOCC transformations, since any SLOCC class is contained in it either completely or not at all.

2.2. Single copy versus asymptotic entanglement properties

There is one point which we glossed over in the previous section: Namely the question of what comprises a specific task. Here, two different approaches exist. The first is to take a single state at a time, and operate on that. This approach is therefore called single copy. When looking at entanglement this way, the relevant questions are whether one can convert a single copy of state $|\psi\rangle$ into a single copy of state $|\phi\rangle$, and the probability of doing so. A typical protocol adhering to this restriction is superdense coding [35]: it uses a single shared Bell state to allow the transmission of two classical qubits. To transfer two additional qubits, an additional shared Bell pair can be used. However, there is no operation involving both Bell pairs. Another such protocol is quantum teleportation [36] where each shared Bell pair is used individually to teleport one qubit, and nothing produced in that transmission (not even the classical information) is used for transmission of further qubits.

The other approach, which is more in line with classical information theory, is to take an unlimited number of copies of a state as resource, and allow local transformations between those copies. In this case, the relevant quantities are asymptotic quantities (for example, the number of generated systems in state $|\phi\rangle$ per initial system in state $|\psi\rangle$, in the limit where the number of initial systems goes to infinity). Also, it allows that the final state is only approximated, as long as the error can be made arbitrarily small. Typical protocols using this approach are quantum compression [37, 38] and entanglement distillation protocols [39].

It is obvious that any single-copy protocol is also an asymptotic protocol, therefore any transformation that is possible in the single-copy is also possible asymptotically. However the reverse is not true. For example, for three qubits, there are two inequivalent types of three-qubit entanglement [33], called GHZ-type and $W$-type entanglement, named after the prominent states contained in those classes, namely the Greenberger–Horne–Zeilinger (GHZ) state $|\text{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ resp. the $W$ state $|\text{W}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. In single-copy protocols, neither of those states can be transformed into each other using SLOCC. However, asymptotic protocols are able to transform GHZ states into $W$ states and vice versa [40–42].

This relation means that any entanglement classification based on asymptotic protocols is also a coarse-grained single-copy classification. Moreover it means that any entanglement measure suitable for asymptotic protocols is also suitable for single-copy protocols.

In this review, we adopt the single-copy viewpoint.

2.3. Mathematical description of SLOCC operations and SLOCC equivalence

The basis of the mathematical description of SLOCC operations is the concept of an instrument [43, 44]. An instrument can be thought of as a set of quantum channels $T_i$ with associated probability $p_i$ that the corresponding channel is selected, and a classical output giving the selected output. Since classical communication is allowed, there is no need to explicitly distinguish between the knowledge on different sites.
Every quantum channel has a Kraus decomposition \([45, 46]\), that is, any quantum channel \(T\) can be written as
\[
T(\rho) = \sum_k G_k \rho G_k^\dagger \quad \text{with} \quad \sum_k G_k^\dagger G_k = 1
\] (1)
where \(I\) is the identity operator. The operators \(G_k\) are called the Kraus operators of the channel. Now it is easy to see that \(T_k(\rho) = G_k \rho G_k^\dagger / \text{tr}(G_k \rho G_k^\dagger)\) is also a (very special) channel. Thus the channel can be modelled as applying the special instrument giving the ‘one-operator channel’ \(T_k\) with probability \(p_k\) and subsequently discarding the result of the instrument. This in turn means that any instrument can be considered such a fine-grained instrument, followed by discarding part of the classical information, which essentially causes a mixture of the corresponding output states. Note that the one-operator channels always map pure states onto pure states.

Since we are interested in local operations, we also need local instruments. A local instrument only acts on a single subsystem, therefore its Kraus decomposition consists of the tensor product of a valid Kraus operator acting on that system and identities acting on all the other systems.

An SLOCC protocol is built of a sequence of local instrument applications, where the information gained from previous instruments can be used to select further instruments. Unlike an LOCC protocol, an SLOCC protocol may fail, that is, it suffices if one of the possible outcomes is the desired one. The obtained classical information therefore does not only define which step to do next, but also whether the protocol succeeded or failed. In the end, this means that a state can be reached from another state by SLOCC iff there is a local channel with postselection from one state to the other.

As described above, a single term of the Kraus decomposition defines, after normalization, a channel by itself. However, for SLOCC operations, those terms, while employing only local operators, are in general not themselves describing local channels, that is, it is in general not possible to implement that channel locally without postselection. Only if the transformation can be done by LOCC (that is, with certainty), is the channel local. Such a channel that can be implemented with SLOCC through postselection has been termed a local filtering operation \([34, 47, 48]\).

Note that local unitary transformations are also a specific type of local channel.

As we have seen above, discarding information is equivalent to the mixing of the corresponding output states \([49]\). However, it has to be stressed that for local operations, not all mixtures can be produced that way, since it is necessary to be able to produce each of the states with a sequence of local channels from the original state. However, note that mixing with a separable state is always possible because those can, by definition, be created from scratch with SLOCC.

When looking at channels converting pure states to pure states, it is easy to see that they have only one Kraus operator. Since concatenation of local channels means forming products of the channels’ Kraus operators, this implies a simple rule to decide whether one pure state can be transformed into another \([33]\): the state \(|\psi\rangle\) can be transformed into the state \(|\phi\rangle\) if there is an operator \(G = G_1 \otimes \ldots \otimes G_n\) (where the tensor product factors are for the different subsystems) so that \(G |\psi\rangle = |\phi\rangle\).

Obviously this also leads to a simple criterion of SLOCC equivalence of pure states: the operator \(G\) has to be invertible \([33]\). In other words:

Two pure states \(|\psi\rangle\) and \(|\phi\rangle\) living in the multipartite Hilbert space \(H_1 \otimes \ldots \otimes H_n\) are SLOCC equivalent iff there exists an operator \(G \in \text{GL}(H_1) \otimes \ldots \otimes \text{GL}(H_n)\) so that \(G |\psi\rangle = |\phi\rangle\). We refer to such an operation as a local GL operation.
Obviously a local GL operation will map product states to product states; therefore no non-product state can be produced from a product state. On the other hand, obviously a pure product state can be created locally from an arbitrary state by doing a complete projective measurement followed by a unitary operation depending on the outcome.

The total set of separable states is therefore mathematically characterized as the mixture of product states [2].

3. Properties of entanglement measures

This section is about what conditions entanglement measures fulfil. The first subsection (3.1) gives the properties every function needs to have in order to be called an entanglement measure or an entanglement monotone, while the following sections describe the additional restrictions which one may require. Note that here, the terms ‘entanglement measure’ and ‘entanglement monotone’ are usually used synonymously for anything fulfilling the two properties described in section 3.1.

3.1. Main properties

Given that the defining property of entanglement is that it cannot be produced by SLOCC, at first it seems obvious that an entanglement measure should be strictly non-increasing under SLOCC. Such measures indeed do exist; one example is the Schmidt rank of bipartite states [50, 51] (see also section 5.2.1). However while those measures can be quite useful for classification of entanglement, they do not adequately quantify their resource character. This is because, by definition, they are constant on every SLOCC class. However not all states in a given SLOCC class are equally good resources. For example, all states of the form \(|\alpha\rangle \langle \beta| + |00\rangle \langle 11| \) where \(\alpha \neq 0\) and \(\beta \neq 0\) are SLOCC equivalent but, for example with superdense coding, only in the case \(\alpha = \beta\) can two classical bits be correctly transmitted with probability 1 (up to experimental errors).

For that reason, one only demands that the entanglement measure does not increase on average under SLOCC [28, 52]. Not increasing on average means that if the protocol transforms the initial state \(\rho\) into the final state \(\rho_n\) with probability \(p_n\), then the measure \(\mu\) must fulfil the inequality

\[
\sum_n p_n \mu(\rho_n) \leq \mu(\rho)
\]

Since such measures are necessarily constant on the set of separable states, usually the trivial additional constraint

\[
\rho \text{ separable } \rightarrow \mu(\rho) = 0
\]

is added.

Note that those properties imply that the set of states on which the measure vanishes is invariant under SLOCC operations. That is, every entanglement measure already implies a rough classification of states into two classes: those where the entanglement measure vanishes (which includes all separable states, but also may include certain entangled states), and those where it does not.

Sometimes authors only impose the even weaker condition that the measure has to decrease under (deterministic) LOCC.
3.2. Convexity and the convex roof

Another common requirement for entanglement measures is convexity [28]:

\[ p_1 + p_2 = 1 \implies \mu(p_1 \rho_1 + p_2 \rho_2) \leq p_1 \mu(\rho_1) + p_2 \mu(\rho_2). \] (4)

Physically it means that mixing two states should never increase entanglement. It seems intuitive because mixing certainly looks like a local operation (and is indeed classified as local operation by Vidal). Measures fulfilling conditions (2), (3) and (4) were called entanglement monotones by Vidal [28].

Certainly whenever there is an SLOCC protocol to generate the state \( \rho_1 \), and another SLOCC protocol to generate the state \( \rho_2 \), there also exists an SLOCC protocol to generate any mixture of both states, by randomly executing one of the two protocols (with appropriately chosen probabilities), and then discarding the information on which the protocols have been run. However, that is an SLOCC operation from the original state to the mixture, not a mixing operation by itself. Also, if both states are available at the same time, then of course it is easy to mix them by randomly selecting one of them, discarding the other, and discarding the information on which one was chosen. However in that case, the mixing starts from the product state \( \rho_1 \otimes \rho_2 \), not from \( \rho_1 \) or \( \rho_2 \). Therefore convexity is not a strictly necessary condition for entanglement measures. Indeed, there are entanglement measures which are not convex, like the logarithmic negativity [49]. Here, the term ‘entanglement monotone’ is generally used for all measures fulfilling conditions (2) and (3).

It is easy to see from (2) and (3) that if \( \mu_1(\rho) \) and \( \mu_2(\rho) \) are entanglement monotones, then the minimum of both, \( \mu(\rho) = \min \{ \mu_1(\rho), \mu_2(\rho) \} \), is also an entanglement monotone. However, in general the minimum of two convex functions is not itself a convex function. For example, in the multipartite case, to be discussed later, a bipartite entanglement measure may be applied to different bipartitions. Then the minimum of that measure will also be an entanglement measure, but in general it will not be convex, even if the bipartite measure is.

Nonetheless, convexity is often taken as an additional requirement due to the interpretation of mixing as a loss of information, which of course should not increase entanglement. A general way to construct convex entanglement monotones is the convex roof extension [53]. The convex roof extension takes a measure \( \mu \) that is defined only on the pure states and extends it to the mixed states as

\[ \mu(\rho) = \min_{\text{decomposition}} \sum_i p_i \mu(\psi_i) \] (5)

where the minimum goes over all decompositions of \( \rho \), that is, over all sets \( \{(p_i, \psi_i)\} \) so that \( \sum_i p_i = 1 \) and \( \sum_i p_i |\psi_i\rangle \langle \psi_i| = \rho \). Note that this construction can be used to produce new measures \( \tilde{\mu} \) for measures \( \mu \) which are also defined on mixed states, by defining for pure states \( \tilde{\mu}(\psi) = \mu(\psi) \) and convex roof-extending \( \tilde{\mu} \). For example, the negativity (see section 5.2.5) is defined for all bipartite mixed states, but its convex roof gives a different measure known as the convex roof extended negativity (CREN) (see section 5.2.6).

An important property of the convex roof is that it is the largest convex function which agrees with the original function on the pure states [53, 54].

The convex roof extension has the advantage that it automatically constructs a convex entanglement monotone on the mixed states from an entanglement monotone defined only on the pure states. However it has the disadvantage that it is in general hard to compute.
3.3. Homogeneity and SL invariance

The close connection between SLOCC operations and local GL transformations noted in section 2.3 suggests adding related conditions on the measures. Invariance under local GL operations is obviously too strong, since it would only allow measures which are invariant under SLOCC transformations. However by splitting the GL transformations into SL transformations, that is invertible transformations of determinant 1, and a single prefactor, two very useful conditions can be imposed.

The first condition is homogeneity. It means that for any state \( \rho \) and any positive number \( \lambda \), the measure \( \mu \) has the property

\[
\mu(\lambda \rho) = \lambda^\alpha \mu(\rho)
\]

with some exponent \( \alpha \), called the degree of the homogeneity. The same condition can also be written down for pure states \( |\psi\rangle \), but one has to be careful: the density matrix for the state \( |\psi\rangle \) is the projector \( |\psi\rangle \langle \psi| \), which is quadratic in \( \psi \). Therefore any measure which is homogeneous of degree \( n \) in the state vector is homogeneous of degree \( n/2 \) in the density matrix. Therefore it is always important which quantity the degree refers to.

Note that for convex roof extended measures, homogeneity on the pure states automatically implies homogeneity on the mixed states as well.

The second condition is invariance under transformations \( S = S_1 \otimes \ldots \otimes S_n \in \text{SL}(d_1, \mathbb{C}) \otimes \ldots \otimes \text{SL}(d_n, \mathbb{C}) \) where \( S_j \) acts on the \( j \)th subsystem and \( \det S_j = 1 \). We refer to those transformations as local SL transformations (or, for short, LSL transformations). That is, if \( S \) is a local SL transformation, then

\[
\mu(S \rho) = \mu(\rho).
\]

For convex roof extended measures, invariance under local SL transformations of the measure on pure states generally does not imply invariance on the mixed states; however if the measure is of homogeneous degree 1 in the density matrix (degree 2 in the state vector), the local SL invariance is carried over to the mixed states as well [55].

Verstraete et al [56] have shown that for any local SL invariant, a convex roof extended measure of homogeneous degree 1 is automatically an entanglement monotone. Indeed, for systems of qubits, any homogeneous measure in the state vector which is local SL invariant on pure states gives an entanglement monotone if and only if the homogeneity degree in the state vector is nonnegative and not larger than 4 [57].

Another reason why these two conditions are very useful is that methods exist which systematically build measures fulfilling them, based on local SL invariant polynomials in the state coefficients [56, 58, 59], see section 6.4.3.

3.4. Dimension-independence and additivity

There are other common requirements for entanglement measures, which are related to their use for asymptotic protocols. The first one is that the measure must not depend on the Hilbert space dimension: that is, the very same measure can be applied to systems of arbitrary Hilbert space dimension, and will give the same result for the same state embedded in a larger Hilbert space. This is important for asymptotic protocols because they do not work only on the state \( \rho \), but on the state \( \rho^\otimes N \) for the limit of large \( N \). Therefore to make any statements regarding such measures, they have to be applicable for arbitrary \( N \). Consequently, all asymptotic measures are dimension-independent, but also I-concurrence (if defined with a dimension-independent prefactor), as well as the negativity and related measures.
The second requirement, which rests on the first, is additivity. This is the requirement that
\[ \mu(\rho \otimes \sigma) = \mu(\rho) + \mu(\sigma) \]  
(8)
where, in this case, the tensor product is not between different subsystems, but between different states shared by the parties. However, this equation can be hard to prove. Generally, a less strict inequality can be proved, the subadditivity \[ \mu(\rho \otimes \sigma) \leq \mu(\rho) + \mu(\sigma). \]  
(9)
One quantity which is known to be subadditive is the entanglement of formation.

4. Connections between entanglement measures and other important concepts

4.1. Normal form
An important concept in entanglement is Verstraete’s normal form \[ [56]. \]  
The normal form of a state \( \rho \) is a (generally not normalized) state in the closure of the local SL orbit of \( \rho \) whose reduced density matrices are all multiples of the unit matrix. That is, there exists a set of local SL matrices \( S(t) \) parameterized by \( t \) such that
\[ \rho_{\text{NF}} = \lim_{t \to \infty} S(t) \rho S(t)^\dagger, \quad \text{tr}_{BC}.\rho_{\text{NF}} = \lambda \mathbb{1}_A, \quad \text{tr}_{AC}.\rho_{\text{NF}} = \lambda \mathbb{1}_B, \quad \ldots \]  
(10)
where \( \mathbb{1}_X \) is the unit operator for system \( X \).

In most cases, the normal form of a state is LSL-equivalent to the original state, but in some cases the limit is explicitly needed. This is especially the case for states where the normal form is zero.

There exists an explicit and efficient iterative algorithm to calculate the normal form (or, in the case that it is only reached asymptotically, a close approximation for it), which is also given in \[ [56]. \]

All local SL invariant monotones reach their maximum value on a pure state in normal form. This means that states whose normal form is zero cannot have their entanglement quantified by any local SL invariant measure.

For pure states, the reverse is also true: if the normal form is non-zero, the state is measured by at least one LSL-invariant entanglement monotone. For mixed states this is not true, as can be seen by the fact that the completely mixed state is already in normal form, but, as a separable state, obviously cannot be measured by any entanglement measure.

Since the normal form is obtained using local SL operations, it can be used to calculate/estimate the value of homogeneous LSL-invariant measures, as long as the value/an estimate of the normalized state is known for the corresponding normal form: if the monotone \( \mu \) is homogeneous of degree \( \alpha \) in the density matrix, then
\[ \mu(\rho) = \left( \text{tr} \rho_{\text{NF}} \right) ^\alpha \mu \left( \frac{\rho_{\text{NF}}}{\text{tr} \rho_{\text{NF}}} \right). \]  
(11)

4.2. Entanglement witnesses
An entanglement witness \[ [61–63] \] is an observable which has a nonnegative expectation value on all separable states, but a negative expectation value on at least some entangled states. A state is said to be detected by a witness if it has a negative expectation value. While any detected state is, by construction, entangled, the reverse is not true: for any entanglement
witness, there exist entangled states it does not detect. However for every entangled state, there exists an entanglement witness which detects it [62]. An extensive overview on entanglement detection by witnesses was given by Gühne and Tóth [64].

An entanglement witness is called optimal if there is no entanglement witness which detects a proper superset of the states detected by it. It is also possible to define class-specific entanglement witnesses, like Schmidt number witnesses [65] or witnesses for GHZ-type entanglement [66].

Another important concept is optimality relative to a subset of states [63, 67, 68], where only a subset of states is considered for detecting optimality (of course the witness must not detect any unentangled states, including those outside that subset—otherwise it would not be an entanglement witness).

Entanglement witnesses are ultimately a geometric concept, since they split the space of bounded operators (and especially the set of positive operators of trace 1, that is, the density matrices) into two half-spaces, one positive and one negative. The hyperplane of zero expectation value is a supporting plane of the set of unentangled states iff the witness is optimal.

Entanglement detection through witnesses is particularly useful for the assessment of experimental entanglement generation (cf. e.g., recent experiments such as [69–71]). In such experiments the full density matrix (or parts of it) is determined. The question of whether experimental data of this kind are compatible with the presence of entanglement was raised early on, e.g., in [72, 73].

A general theory of quantifying entanglement by means of entanglement witnesses was developed by Brandao and coworkers [74, 75] and by Gühne et al. [76, 77]. For example, given two arbitrary positive numbers $m$ and $n$, the function

$$E_{n,m} = \max \left\{ 0, - \min_{W \in \mathcal{M}_{n,m}} \text{tr} W \rho \right\}$$

is an entanglement monotone, where $\mathcal{M}_{n,m}$ is the set of all entanglement witnesses $W$ fulfilling $-n1 \leq W \leq m1$ [74].

As is discussed in section 7.5 also the converse is true: entanglement measures (e.g., polynomial invariants) may be used to derive well-known entanglement witnesses [78].

5. Bipartite entanglement

5.1. Schmidt decomposition and SLOCC classes

We consider a quantum system consisting of two subsystems $A$ and $B$ with dimensions $\dim H_A = d$ and $\dim H_B = d'$, so that the Hilbert space of the composite system is $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$. We will also call $\mathcal{H}$ a $d \times d'$-dimensional system. Henceforth we assume $d \leq d'$. With orthonormal bases $\{|a\rangle\}, \{|b\rangle\}$ a pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{a=1}^d \sum_{b=1}^{d'} \psi_{ab} |a\rangle \otimes |b\rangle = \sum_{a,b} \psi_{ab} |ab\rangle.$$ (13)

Pure bipartite states have the important property that there are always orthonormal bases $\{|j_A\rangle\}, \{|j_B\rangle\}$ such that [16, 17]
\[ |\psi\rangle = \sum_{j=1}^{r(\psi)} \sqrt{\lambda_j} \left| jj \right> \quad \text{(14)} \]

with real positive numbers \(\lambda_j\), the so-called Schmidt coefficients. The Schmidt rank \(r(\psi) \leq d\) corresponds to the rank of the reduced density matrices of the subsystems \(r(\psi) = \text{rank}(\text{tr}_B |\psi\rangle\langle\psi|) = \text{rank}(\text{tr}_A |\psi\rangle\langle\psi|)\). With this we note that a pure bipartite state is separable if its Schmidt rank equals 1, otherwise it is entangled. In particular, we may define the maximally entangled state in \(d\) dimensions

\[ \sum_{j=1}^d \frac{1}{\sqrt{d}} \left| jj \right> \quad \text{(15)} \]

Evidently the rank of the reduced state \(\rho_A = \text{tr}_B |\psi\rangle\langle\psi|\) does not change under arbitrary invertible operations \(A \in \text{GL}(d, \mathbb{C})\) (and analogously for subsystem \(B\)), so that \(r(\psi)\) is an entanglement monotone [50, 51]. Hence, there are \(d\) different SLOCC classes for \(\psi \in \mathcal{H}_A \otimes \mathcal{H}_B\), each of which is characterized by its Schmidt rank.

A mixed state \(\rho\) of the composite system is represented by a positive definite bounded Hermitian operator acting on the vectors \(\psi \in \mathcal{H}\), i.e., \(\rho \in \mathcal{B}(\mathcal{H})\). It has a decomposition into pure states \((\{p_j, \psi_j\})\)

\[ \rho = \sum_{j=1}^\ell p_j \pi_{\psi_j} \quad \text{with} \quad \pi_{\psi_j} = \left| \psi_j \right> \left< \psi_j \right| \quad \text{(16)} \]

where \(\ell \geq \text{rank}(\rho)\) is called the length of the decomposition. The weights \(p_j > 0\) obey \(\sum_j p_j = 1\) and \(\text{tr} \pi_{\psi_j} = 1\) if not stated otherwise. That is, a mixed state can be regarded as a convex combination of pure states. Note that there are infinitely many ways to decompose a state [79, 80]: Given \((\{p_j, \psi_j\})\) and a unitary matrix \(U\) with at least \(\ell\) columns we find another decomposition \((\{q_k, \phi_k\})\) where

\[ \left| \psi_k \right> = \frac{1}{\sqrt{q_k}} \left| \phi_k \right>, \quad q_k = \left< \phi_k \left| \phi_k \right> \right. \quad \text{and} \quad \left. \left| \phi_k \right> = \sum_{j=1}^\ell U^{(j)} \sqrt{p_j} \left| \psi_j \right> \right. \]

This ambiguity is at the origin of many difficulties in entanglement theory.

As to the entanglement classes of bipartite states, we generalize the Schmidt rank to mixed states in a spirit similar to the convex roof: The Schmidt number [81] is the smallest possible maximal Schmidt rank occurring in any pure-state decomposition of \(\rho\)

\[ r(\rho) = \min_{\{\rho, \psi_j\}} \max_j r(\psi_j) \quad \text{(17)} \]

As opposed to the convex roof, the maximum Schmidt rank gets minimized, not the average. Also \(r(\rho)\) is an entanglement monotone [81], see also [82]. In particular, we say \(\rho\) is separable if \(r(\rho) = 1\), that is, if it can be decomposed into pure product states [2]

\[ \rho \text{ separable} \iff \rho = \sum_{j=1}^\ell p_j \pi_{a_j} \otimes \pi_{b_j} \quad \text{(18)} \]

with \(a_j \in \mathcal{H}_A\) and \(b_j \in \mathcal{H}_B\).

The states of a given Schmidt number \(k\) form a compact convex set \(S_k\) which on their part build a hierarchy \(S_1 \subset S_2 \subset \ldots \subset S_k\) [65]. This hierarchy describes an SLOCC classification for the states of a bipartite system. However, it is not the only one. Another example is the
classification of bipartite states with respect to the sign of the partial transpose (i.e., whether or not the partial transpose has negative eigenvalues). Note that this alternative classification has little in common with the one based on Schmidt numbers: The class $S_1$ belongs entirely to the PPT class and, moreover, there is the conjecture that the class $S_d$ does not contain PPT-entangled states [65].

These considerations provide a clear illustration of the fact that there is no such thing like ‘the’ SLOCC classification of a given system. Basically any SLOCC-invariant criterion (or a combination of several criteria) induces an SLOCC classification. The question is whether or not the criterion is appropriate to characterize a certain resource.

The concept of Schmidt decomposition can also be applied to mixed states. In that case, except for $d = d' = 2$, it is not possible to transform the state into a state-independent basis. However with SL transformations it is possible to transform it into the form

$$
\rho = \frac{1}{d^d} \left( 1_d \otimes 1_d + \sum_k \xi_k J_k^A \otimes J_k^B \right)
$$

(19)

where the $J_k^X$ are traceless [83]. Note that (19) is a special representation of the normal form (see section 4.1).

5.2. The most important bipartite entanglement measures

In this section, we look at the most important bipartite entanglement measures for single copies.

5.2.1. The Schmidt number. The Schmidt number (17) is an entanglement monotone which is strictly nonincreasing under SLOCC. As explained in section 3.1, this implies that it does not quantify a resource, but it allows classification of the entanglement. Indeed, for pure states this classification is complete, that is, two pure states are SLOCC equivalent iff they have the same Schmidt rank. For mixed states, there exist entanglement criteria like bound entanglement which are not covered by the Schmidt number.

Note that, strictly speaking, one has to subtract 1 from the Schmidt number in order to get an entanglement monotone, since otherwise it does not fulfil condition (3), section 3.1.

Sometimes, especially when studying asymptotic properties, the logarithm of the Schmidt rank is used. This is because the Schmidt rank of tensor products is the product of the Schmidt ranks, and therefore the logarithm of the Schmidt rank is additive.

A generalization of the Schmidt number to multipartite states (as the minimum number of product components) was studied in [84].

5.2.2. The $k$-concurrences. Gour [85] introduced a hierarchy of entanglement monotones for $d \times d$ systems, which measure Schmidt rank specific entanglement, called $k$-concurrence (where $2 \leq k \leq d$). The $k$-concurrence is defined for pure states as

$$
C_k(\psi) = N_k^{(d)} \left( \sum_{i_1 < i_2 < \ldots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \right)^{1/k}, \quad N_k^{(d)} = d \binom{d}{k}^{-1/k}
$$

(20)

where $N_k^{(d)}$ is a normalization factor chosen in a way that $C_k(\psi) = 1$ for the maximally entangled state, where $\lambda_1 = \ldots = \lambda_d = 1/d$. For mixed states, it is defined by convex roof extension.
The $k$-concurrence is nonzero exactly if the Schmidt number of the state is at least $k$. Unless $k = d$, it is not invariant under local SL transformations, however it is homogeneous of degree 2 in the state coefficients resp. of degree 1 in the density matrix.

The $k$-concurrences are ordered: if $k > k'$, then $C_k(\rho) < C_{k'}(\rho)$ for all $|\rho\rangle$. This was proven by Gour for pure states, but is easily extended to mixed states using the properties of the convex roof.

For pure states, the $k$-concurrences together completely determine the Schmidt coefficients of the state.

Note that the choice of the normalization factors $N^{(d)}_k$ by Gour means that the $k$-concurrence of a given state depends on the dimension of the Hilbert space, even if the support of the state is a true subspace. It is of course possible to choose normalization factors $N_k$ which are independent of $d$, which then makes the $k$-concurrences independent of the Hilbert space dimension, and also gives a natural way to apply them to bipartite systems with different dimensions $d$ and $d'$ for both systems.

The next two measures are special cases of the $k$-concurrence.

### 5.2.3. I-concurrence

The $I$-concurrence was defined by Rungta et al. [86, 87] using the concept of a ‘universal inverter’ defined by $S_d(\rho) = (\text{tr}(\rho)1_d - \rho)$ (they allowed an arbitrary normalization factor, which they later set to 1). They then defined the pure state $I$-concurrence as

$$C_I(\psi) = \sqrt{\langle \psi | S_d \otimes S_d (|\psi\rangle\langle \psi|) |\psi\rangle}$$

resulting in the square root of the linear entropy of the reduced density matrix

$$C_I(\psi) = \sqrt{\frac{1}{2} \left( \left( \text{tr} \rho_A^2 \right)^2 - \text{tr} \left( \rho_A^2 \right) \right)}.$$  

For mixed states the $I$-concurrence is defined by convex roof extension.

This is actually the 2-concurrence of section 5.2.2, except for the different normalization factor: equation (20) leads to a prefactor $d/(d - 1)$ instead of 2 under the square root for $k = 2$.

Usually the $I$-concurrence is referred to just as the concurrence, without further qualification. The $I$-concurrence is nonzero iff the state is entangled.

### 5.2.4. G-concurrence

The $G$-concurrence is the $k$-concurrence for $k = d$. It is the geometric mean of the Schmidt coefficients times a constant factor (d in Gour’s normalization) [85],

$$C_G(\psi) = d \left( \lambda_1 \cdots \lambda_d \right)^{1/d}.$$  

Alternatively it can be defined by the determinant of the reduced density matrix:

$$C_G(\psi) = d \left( \det \rho_{1n} \right)^{1/d}.$$  

It is nonzero exactly for states of maximal Schmidt rank. Unlike all other $k$-concurrences it is invariant under local SL operations. Indeed, it is the only SL-invariant convex-roof extended bipartite monotone.

Although for each dimension the $G$-concurrence is a $k$-concurrence, it is not dimension-independent even with a dimension-independent normalization factor, because it is a different $k$-concurrence for different dimensions. Note that for $d \neq d'$, there exists no SL-invariant measure at all.
5.2.5. Negativity. The negativity is defined both for pure and mixed states as [88–91]
\[\mathcal{N} = \frac{1}{2} (\| \rho^{T_A} \|_1 - 1)\]  
(25)
where \(\| \Delta \|_1\) denotes the trace norm, \(\| A \|_1 = \text{tr} \sqrt{A^* A}\), and \(\rho^{T_A}\) the partial transpose, \(\rho \otimes \sigma = A^T \otimes B\). In particular, the negativity is not a convex-roof extended measure.

Another way to describe the negativity is that it is the absolute value of the sum of the negative eigenvalues of \(\rho^{T_A}\). Note that this means that the negativity is zero iff the partial transpose \(\rho^{T_A}\) is positive, that is, has no negative eigenvalues. In that case, the state \(\rho\) is also called a positive partial transpose (PPT) state. The set of PPT states is invariant under SLOCC. The fact that a non-positive partial transpose implies entanglement was actually discovered earlier by Peres [92] and is also known as Peres condition.

The negativity can be zero even for entangled states. Such states are called PPT-entangled states. All PPT-entangled states are bound entangled, that is, their entanglement cannot be distilled [94]. Whether the reverse is also true is still an open question.

5.2.6. Convex-roof extended negativity (CREN). The convex-roof extended negativity is defined as the convex-roof extension (5) of the negativity (25) on pure states [95]:
\[\mathcal{N}^{\text{CREN}}(\rho) = \min_{\text{decompositions}} \sum_i \rho_i \mathcal{N}(\psi_i).\]  
(26)
Given that the negativity is convex, it is always a lower bound to the CREN:
\[\mathcal{N}(\rho) \leq \mathcal{N}^{\text{CREN}}(\rho).\]  
(27)
The CREN is nonzero iff the state is entangled.

5.2.7. Logarithmic negativity. The logarithmic negativity is defined as [49]
\[\ln(\rho) = \log_2 \| \rho^{T_A} \|_1\]  
(28)
with the same definitions as in section 5.2.5.

Like negativity, the logarithmic negativity vanishes exactly on the PPT states. It has been linked to the cost of entanglement under PPT-preserving operations [96], an asymptotic measure based on a slightly different set of operations than SLOCC.

The logarithmic negativity is not convex.

5.2.8. The geometric measure of entanglement and other distance-based measures. The geometric measure of entanglement for pure states is defined as [97, 98]
\[E_G(\psi) = 1 - \max_{|\phi_i \rangle \otimes |\phi_2 \rangle} \left| \left( \langle \phi_1 | \otimes \langle \phi_2 | \right) \psi \right|^2.\]  
(29)
For mixed states, it is usually defined by convex-roof extension. Note that it can be made homogeneous if we replace the number 1 on the right-hand side of equation (29) by \(\langle \psi | \psi \rangle\).

The geometric measure is non-zero iff the state is entangled.

Another distance-based measure is the robustness [31]. The robustness is
\[R(\rho) = \min_{\sigma \text{ separable}} R(\rho \| \sigma)\]  
(30)
where the quantity
\[
R(\rho \| \sigma) = \min \{ s \geq 0 : (\rho + s\sigma)/(1 + s) \text{ is separable} \}
\]
is the robustness relative to the separable state \(\sigma\). The geometrical meaning of \(R(\rho \| \sigma)\) is that of the ‘mixing line’ from \(\rho\) to \(\sigma\), the fraction \(1/(1 + R(\rho \| \sigma))\) consists of separable states.

5.2.9. Entanglement of formation and other entropy-based measures. In this section, we briefly review some measures based on the von Neumann entropy. Those measures are generally connected with the asymptotic viewpoint, therefore we will not consider them in detail, but only list them for completeness.

Historically, the first well-established entanglement measure is the entanglement of formation (EoF) [20]. Its definition is motivated by the asymptotic protocol viewpoint, however it can be calculated directly on the single state.

The EoF is defined for pure states as the von-Neumann entropy of the reduced density matrix
\[
E_F(\psi) = S(\rho_A)
\]
where \(S(\rho) = \text{tr}(-\rho \log \rho)\) is the von Neumann entropy and \(\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)\) is the reduced density matrix of system \(A\). Note that \(S(\rho_A) = S(\rho_B)\), therefore the choice of subsystem does not matter. For pure states, the entanglement of formation is also called entanglement entropy.

For mixed states, the EoF is defined by a convex roof extension. It is nonzero iff the state is entangled.

Further, the EoF is subadditive [60, 99] but not additive [100].

It is conjectured to equal the entanglement cost which is defined as the number of Bell states per copy needed to create asymptotically many copies of the state and is given by the explicitly asymptotic expression [101, 102]
\[
E_C(\rho) = \lim_{n \to \infty} \frac{E_F(\rho^\otimes n)}{n}.
\]

Another measure that is explicitly defined asymptotically is the distillable entanglement [20]. It is defined as the asymptotic number of Bell states which can be extracted per copy of the given state. For pure states, EoF and distillable entanglement agree [103].

The relative entropy of entanglement, introduced by Vedral et al [30, 52] is defined as
\[
E_R(\rho) = \min_{\rho \text{ separable}} \text{tr}\left(\rho \ln \frac{\rho}{\sigma}\right)
\]

Another entropic measure is the squashed entanglement [104]. It is defined as
\[
E_{sq}(\rho) = \inf \left\{ \frac{1}{2} I(A; B|E) : \rho_{AB} = \text{tr}_E \rho_{ABE} \right\}
\]
where the infimum goes over all such extensions of \(\rho_{AB}\) with unbounded dimension of \(E\), and
\[
I(A; B|E) = S(AE) + S(BE) - S(ABE) - S(E)
\]
is the quantum conditional mutual information of \(\rho_{ABE}\).

The squashed entanglement is convex, additive on tensor products, upper bounded by the EoF, and lower bounded by distillable entanglement.
5.3. Two-qubit entanglement

The simplest system that can be entangled consists of two qubits. It is currently the only system whose entanglement properties have been completely characterised both for pure and mixed states.

Two-qubit systems generally have quite unique properties [105]. For pure states, there are just two SLOCC classes, that is, there exists only one type of entanglement (the other SLOCC class contains the separable states). Consequently, for pure states there is effectively only one entanglement monotone, that is, different entanglement monotones, when restricted to pure states, can be written as function of each other. Indeed, as it turns out, even for mixed states this generally holds, even in cases where it is not obvious.

Every two-qubit state is SLOCC-equivalent to a Bell-diagonal state [34], that is, a mixture of the four Bell states $\psi_+ = (|00\rangle + |11\rangle)/\sqrt{2}$, $\psi_- = (|01\rangle + |10\rangle)/\sqrt{2}$ which also is its normal form.

5.3.1. Wootters’ analytical solution for the concurrence. For two qubits we have $d = 2$, therefore the whole family of $k$-concurrence consists of only one concurrence, the 2-concurrence, which in this case is both the $I$-concurrence (and thus is non-zero exactly if the state is entangled) and the $G$-concurrence (and thus is invariant under local SL operations).

Indeed, the concurrence was originally defined for two-qubit systems [20, 106, 107], and the concurrences of sections 5.2.2 to 5.2.4 are generalizations of that.

For pure states $\psi = |\psi_+\rangle = |00\rangle + |11\rangle$ the concurrence is [107, 108]

$$C(\psi) = 2 \left| \psi_{00} \psi_{11} - \psi_{01} \psi_{10} \right|. \quad (37)$$

Alternatively, it can be written as an expectation value of an antilinear operator [107]:

$$C(\psi) = \left| \left\langle \psi | \sigma_y \otimes \sigma_y | \psi^* \right\rangle \right|. \quad (38)$$

with the Pauli matrix $\sigma_y = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$.

For mixed states, the concurrence is defined by the convex-roof extension. It is one of the rare cases where a method to calculate the convex roof without optimization is known [107]. To calculate it, one needs the eigenvalues $\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ of the matrix

$$R = \rho (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \quad (39)$$

where $\rho^*$ denotes the complex-conjugated density matrix, that is, the matrix where in the computational basis all matrix elements are the complex conjugate of the corresponding matrix element in $\rho$. Then the mixed-state concurrence is

$$C(\rho) = \max \left\{ 0, \sqrt{\eta_1} - \sqrt{\eta_2} - \sqrt{\eta_3} - \sqrt{\eta_4} \right\}. \quad (40)$$

A special case is the mixture of a Bell state with an orthogonal separable state, like $\sqrt{p} \rho^* \| \psi^+ \rangle + \sqrt{1-p} |01\rangle$. For such states, Abouraddy et al [109] found that the concurrence simply equals the weight $p$ of the Bell state.

5.3.2. Other measures related to the concurrence. Since, for two qubits, the pure state negativity equals the concurrence [110], the CREN also agrees with the concurrence.

For two qubits, the geometric measure of entanglement can be calculated from the concurrence as [111]
The EoF can be calculated from the concurrence as [106, 107]

$$E_G(\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - C(\rho)^2} \right).$$

(41)

with $$H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$$.

Remarkably, these relations hold not only for pure, but also for mixed states. This is because for qubits, concurrence, geometric entanglement and entanglement of formation have optimal decompositions in which all states have equal measure.

Another relation that has been shown in [78] relates the concurrence to the fully entangled fraction introduced in [20] and the standard projection witness

$$\mathcal{W}_{2\text{ qubits}} = \frac{1}{2} - |\phi^+\rangle \langle \phi^+|.$$  

(43)

Applying the witness $$\mathcal{W}_{2\text{ qubits}}$$ and optimizing the state over local SL operations gives, up to a sign, the concurrence for entangled states:

$$C(\rho) = \max_{S = S_1 \otimes S_2} \left[ \frac{1}{2} \left( \rho S \rho \mathcal{S}^\dagger \right) \right].$$

(44)

where $$S_{1/2} \in SL(2, \mathbb{C})$$. But the fully entangled fraction is [20]$$F(\rho) = \max_{U \in SU(2)} \left| \langle \phi^+| U \rho U^\dagger |\phi^+\rangle \right|^2.$$  

(45)

where $$|\phi^+\rangle = |U_1 \otimes U_2 \rangle$$ is optimized over the maximally entangled states, and $$U_{1/2} \in SU(2)$$. This differs from the first term in (44) only in the more restricted group. Therefore the concurrence is basically the local-SL optimized fully entangled fraction.

### 5.3.3. Two-qubit monotonies that do not depend on the concurrence.

The above discussion seems to imply that the concurrence is the only entanglement monotone for two-qubit mixed states. However, this is not the case.

The mixed state negativity, while strictly positive for entangled states, does not agree with the concurrence [110].

Verstraete et al [112] identified another monotone based on the Lorentz singular values of the density matrix, namely

$$M(\rho) = \max \{0, -s_0 + s_1 + s_2\}.$$  

(46)

where $$s_0 \geq s_1 \geq s_2 \geq |s_3|$$ are the (local SL invariant) Lorentz singular values, that is, the state has the normal form $$s_0 |1 \rangle \langle 1| + s_1 |\sigma_1 \otimes \sigma_1 + \sigma_2 \sigma_2 \otimes \sigma_2 + \sigma_2 \sigma_2 \rangle \langle \sigma_2 \sigma_2|$$, where $$\sigma_x, \sigma_y$$ and $$\sigma_z$$ are the Pauli matrices.

Liang et al [113] derived a complete set of monotonies for Bell-diagonal entangled states. If $$p_1 > p_2 > p_3 > p_4$$ are the mixing coefficients of the Bell states, the entanglement monotonies are given by them as

$$E_1(\rho_{\text{Bell}}) = p_1,$$

$$E_2(\rho_{\text{Bell}}) = \frac{1 - 2p_2}{p_1 + p_4},$$

(47)

(48)
\[ E_3(\rho_{\text{Bell}}) = \frac{1 - 2p_2 - 2p_3}{p_4}. \] (49)

Obviously, \( E_2 \) is only defined if \( p_3 > 0 \) and \( E_3 \) only if \( p_4 > 0 \). An entangled Bell diagonal state \( \rho \) is convertible to another Bell entangled state \( \rho' \) if and only if all \( E_i(\rho) \geq E_i(\rho') \).

As such, the functions do not fulfill the conditions on monotones written above. The first obvious point is that they are only valid on entangled states and do not vanish on separable states. However, owing to the simple structure of Bell-diagonal states, this is easy to fix: the border of separable states is at \( p_1 = \frac{1}{2} \), at which all three values of \( E_i \) are constant:

\[ E_1\left(p_1 = \frac{1}{2}\right) = \frac{1}{2}, \quad E_2\left(p_1 = \frac{1}{2}\right) = E_3\left(p_1 = \frac{1}{2}\right) = 2. \]

Therefore, \( \tilde{E}_i(\rho_{\text{Bell}}) := \max \{0, E_i(\rho_{\text{Bell}}) - E_i(p_1 = \frac{1}{2})\} \) satisfies that condition. Note that \( 2\tilde{E}_1 \) turns out to be the concurrence.

The second point is that they are only defined on Bell-diagonal states. Given that each SL orbit corresponds to a specific Bell state, one obvious way to do it is to replace \( 1 \) with \( + + + \) and then extend them on the SL orbit according to their degree of homogeneity. If this is combined with the previous change, \( E_1 \), being homogeneous of degree \( 1 \) in the density matrix, gives exactly the concurrence, and \( E_2 \) and \( E_3 \), being of degree \( 0 \), give entanglement measures that are constant on the complete orbit and strictly non-increasing under SLOCC. The latter are therefore no resource measures.

A final problem is that \( E_2 \) and \( E_3 \) diverge when \( p_3 \) resp. \( p_4 \) tend to zero (that is, if the rank of the density matrix goes below \( 2 \) resp. \( 3 \)). But given that they are not resource measures anyway, this seems a minor point (and can easily be fixed by applying an appropriate monotonic function).

### 5.4. Partial transpose, concurrence, and negativity

Historically, one of the first criteria, as to check whether or not a generic mixed state is separable, was the partial-transpose criterion \([61, 92]\), that is, whether or not the partial transpose (cf section 5.2.5) of a given state is positive. The definition of the negativity measures is built on the partial transpose. On the other hand, there are concurrence-based entanglement quantifiers which seem to bear little relation to the partial transpose. In the following we sketch the connection between them, as well as with other interesting concepts.

Consider a pure product state \( \psi \in \mathcal{H}_A \otimes \mathcal{H}_B \) (with \( \dim \mathcal{H}_A = d \), \( \dim \mathcal{H}_B = d' \))

\[ |\psi\rangle = |a\rangle \otimes |b\rangle = \sum_{jk} \psi_{jk} |j\rangle |k\rangle = \sum_{jk} a_j b_k |j\rangle |k\rangle \] (50)

where \( jk \) can be read as a joint (two-digit) index on the wavefunction \( \psi \). The elements of the density matrix are

\[ \rho = |\psi\rangle \langle \psi| = \sum_{jkm} a_j b_k a_j^* b_m^* |j\rangle \langle k| \rightarrow \rho_{jk,lm} = a_j b_k a_j^* b_m^*. \] (51)

In a matrix representation, the off-diagonal element \( \rho_{jk,lm} \) is located in the same row (or column) as the diagonal element \( \rho_{lm,lm} \) (or \( \rho_{lm,lm} \), respectively). At that position, the partial transpose \( \rho_B^T \) has the element
Consequently, a product state $\psi$ obeys the condition
\[
\left| \left( \rho_{jk}^{T_a} \right)_{jk,lm} \right|^2 - \left( \rho_{jk}^{T_b} \right)_{jk,jk} \left( \rho_{jm}^{T_a} \right)_{lm,lm} = \left| \psi_{jm} \psi_{lk} \right|^2 - \left| \psi_{jk} \psi_{km} \right|^2 = 0.
\]
Violation of the condition (53) for any pair of levels $\{j, l\}$ of party $A$ and $\{k, m\}$ in $B$, respectively, means that $\psi$ cannot be a product state. Correspondingly, we may define
\[
C(\psi)^2 = \sum_{jk,lm} \left| \psi_{jm} \psi_{lk} - \psi_{jk} \psi_{km} \right|^2
\]
as a measure of the total violation of the product-state condition for $\psi$ on the bipartition $H_A \otimes H_B$. It turns out [108] that $C(\psi)$ is invariant under local unitaries in $H_A, H_B$ and that it is an alternative definition for the $I$-concurrence (22)
\[
C(\psi) = \sqrt{\sum_{jk,lm} \left| \psi_{jm} \psi_{lk} - \psi_{jk} \psi_{km} \right|^2} = \sqrt{2 \left[ \left( \text{tr} \rho_A \right)^2 - \text{tr} \rho_A^2 \right]}
\]
where $\rho_A = \text{tr}_B \langle \psi | \psi \rangle$.

One may also note that $C(\psi)$ corresponds to the Euclidean length of a vector with components $C_{jk,lm}$, the so-called concurrence vector [114–117]. On the other hand, (54) may be viewed as a 2-norm on the concurrence vector. Accordingly one might expect that the corresponding 1-norm
\[
\tilde{\mathcal{N}}(\psi) = \sum_{jk,lm} \left| \psi_{jm} \psi_{lk} - \psi_{jk} \psi_{km} \right|
\]
is an entanglement measure as well [118]. However, this expression is not invariant under local unitaries. Nonetheless, it is interesting that the minimum of $\tilde{\mathcal{N}}(\psi)$ is obtained for the Schmidt decomposition of $\psi$ and that, with the restriction $j < l, k < m$, it is equal to the negativity $\mathcal{N}(\psi)$. This implies that for pure states
\[
2\mathcal{N}(\psi) \geq C(\psi) = \sqrt{4 \sum_{j<l, k<m} \left| \psi_{jk} \psi_{lm} - \psi_{jm} \psi_{lk} \right|^2}
\]
\[
\geq 2 \sqrt{\frac{2}{d(d-1)}} \mathcal{N}(\psi).
\]
For the second inequality we have used the fact that the quadratic always exceeds the arithmetic mean.

5.5. Bounds on the Schmidt number and the concurrence of mixed states

It is a common feature of most entanglement measures that they are easy to compute on pure states, but—due to their definitions via optimization procedures—it is hard to evaluate or even just estimate them on mixed states. The negativity measures are a notable exception to this rule. Therefore it is important to find reliable lower bounds for the entanglement measures and the Schmidt number to characterize the entanglement resources of a given mixed state. A
special case of the Schmidt-number estimation is the separability problem, that is, the question of whether or not the Schmidt number of a state is at least two [81].

The negativity of the maximally entangled state of Schmidt rank \( k \) is \( \mathcal{N}(\Psi_k) = k(k-1)/(2k) = \frac{1}{2}(k-1) \). Therefore, for a state \( \rho = \sum_j p_j \pi_{\psi_j} \) (with definition (16)) of Schmidt number \( k \) (i.e., \( r(\psi_j) \leq k \)) we have [119]

\[
\mathcal{N}(\rho) \leq \mathcal{N}^{\text{CREN}}(\rho) \leq \sum_j p_j \mathcal{N}(\psi_j) \leq \frac{k-1}{2}
\]

so that

\[
r(\rho) \geq 2\mathcal{N}(\rho) + 1, \tag{58}
\]

i.e., the negativity provides a lower bound for the Schmidt number (for alternative bounds, see [120]). Analogously we obtain \( C(\Psi_k) = \sqrt{2(k-1)/k} \) and hence

\[
r(\rho) \geq \frac{2}{2 - C(\rho)^2}, \tag{59}
\]

(with the normalization in (55)), so non-zero concurrence implies entanglement. Note that for mixed states there is no relation analogous to (56), since the negativity vanishes for PPT-entangled states while the concurrence does not. However, (57) leads to

\[
C(\rho) \geq \frac{2}{\sqrt{d(d-1)}} \mathcal{N}^{\text{CREN}}(\rho) \geq \frac{2}{\sqrt{d(d-1)}} \mathcal{N}(\rho). \tag{60}
\]

Note that for two-qubit states [121] two times the CREN equals the concurrence, so the resulting inequality for mixed states seems to invert the pure-state inequality (56). In the general case it is not easy to estimate \( C(\rho) \) and there is a vast literature on lower bounds of the concurrence (e.g., [87, 122–131]). Here we focus on a powerful analytical method that is both simple and easily generalizable to the multipartite case. It is based on ideas by Gühne and Seevinck [132] and Huber et al [133–136].

Consider a subset \( M \) of \( \eta \) level pairs \( \{j, lm\} \) (\( j < l, k < m \)) of the sum for the concurrence in (55). By using the inequality \( \sqrt{a_1^2 + \ldots + a_n^2} \geq (a_1 + \ldots + a_n)/\sqrt{n} \) for real nonnegative numbers \( a_1, \ldots, a_n \) as well as the triangle inequalities one finds

\[
C(\psi) \geq \frac{2}{\sqrt{\eta}} \sum_{jk \in \mathcal{M}} \left| \psi_{jk} \psi_{jm} \psi_{jm} \psi_{ik} \right|
\]

\[
\geq \frac{2}{\sqrt{\eta}} \sum_{jk \in \mathcal{M}} \left( \left| \psi_{jk} \psi_{jm} \right| - \sqrt{\left| \psi_{jm} \right|^2 \left| \psi_{ik} \right|^2} \right)
\]

\[
\geq \frac{2}{\sqrt{\eta}} \sum_{jk \in \mathcal{M}} \left( \left| \rho_{jk,lm} \right| - \sqrt{\rho_{jm,lm} \rho_{ik,ik}} \right). \tag{61}
\]

In the last line of (61) we have used \( \rho = \pi_{\psi} \). Due to the convexity of the concurrence and of the functions on the right-hand side of (61) this relation can directly be extended to mixed states \( \rho = \sum_j p_j \pi_{\psi_j} \)

\[
C(\rho) \geq \frac{2}{\sqrt{\eta}} \sum_{jk \in \mathcal{M}} \left( \left| \rho_{jk,lm} \right| - \sqrt{\rho_{jm,lm} \rho_{ik,ik}} \right). \tag{62}
\]

This relation may be regarded as a set (for different level subsets \( \mathcal{M} \)) of witness inequalities for bipartite entanglement in terms of density matrix elements, as for separable states the
right-hand side of (62) cannot be positive. At the same time, the matrix element difference provides a lower bound for the concurrence of the mixed state $\rho$. Note that there is also a simple method to optimize this bound: local unitary operations on the parties $A$ and $B$ do not change the concurrence, however, they may change the value of the right-hand side in (62). That is, the estimate can be improved by maximizing the right-hand side over local unitaries.

5.6. Axisymmetric states

5.6.1. Definition. As is the case in many other areas of physics, exact solutions of nontrivial problems provide a testbed for the concepts of the theory, the models for the observed phenomena and for approximate methods. In entanglement theory, this role is played by exact solutions which are sometimes possible for special states of high symmetry and/or problems of reduced rank [137–139]. An early example is the Werner states for two qudits that are invariant under $U \otimes U$ operations where $U$ denotes an arbitrary one-qudit unitary transformation. From symmetry one concludes that $d \times d$ Werner states can be represented by a linear combination of the projectors onto the symmetric and antisymmetric subspaces. For two qubits (where the antisymmetric subspace has only one dimension), the Werner states are locally equivalent to

$$\rho^W(p) = p |\Psi_z\rangle \langle \Psi_z| + \frac{1-p}{4} 1_d.$$  \hfill (63)

The states $\rho^W(p)$ are particularly interesting because they are mixtures of a maximally entangled state with the completely mixed state. The latter is often used as a model of white noise and serves to characterize the robustness of the entanglement. In fact, in [31] the minimal noise admixture $\tilde{p}$ at which the mixed state $\frac{\tilde{p}}{4^2} 1_d + (1 - \tilde{p}) \pi_{\psi}$ becomes separable was termed the random robustness of $\psi$ (note, however, that this quantity is not an entanglement monotone [140]).

If one attempts to obtain a state analogous to (63) in higher dimensions from symmetry considerations, the appropriate requirement is invariance under $U \otimes U^*$ and the resulting one-parameter family of states

$$\rho^{iso}(p) = p |\Psi_0\rangle \langle \Psi_0| + \frac{1-p}{d^2} 1_{d^2}$$  \hfill (64)

is called the isotropic states [137]. Several interesting exact results were obtained for isotropic states, such as the separability criterion [31, 141], the entanglement of formation [142] and the convex roofs of both $I$-concurrence and the square of the $I$-concurrence [87]. An arbitrary $d \times d$ state $\rho$ can be projected onto the isotropic states by a so-called twirling operation

$$P_{iso}(\rho) = \int dU \left( U \otimes U^* \right) \rho \left( U \otimes U^* \right)$$  \hfill (65)

where $dU$ is the Haar measure. Note that $P_{iso}$ combines local operations and mixing, that is, it is an LOCC operation and therefore the entanglement cannot increase in the mapping $P_{iso}: \rho \rightarrow \rho^{iso}$. Although (65) involves a continuous average the twirling operation can be represented by a finite sum (this is a consequence of the Krein–Milman and Caratheodory theorems [143] for the finite-dimensional compact group of $(U \otimes U^*)$ transformations).

By modifying the symmetry requirements it is possible to generate and study other families of symmetric states. The rotationally invariant states provide another interesting example, i.e., they do not change under rotations $R \in SO(3)$ [144]. Here we consider in some detail a different option. Recently it was noticed that the symmetry requirement for isotropic
states can be relaxed so that one obtains a two-parameter family for all finite-dimensional $d \times d$ systems, the axisymmetric states $\Psi_d$. We expect that these states will be instrumental in the further development of methods for bipartite mixed states. Axisymmetric states have the same symmetries as the maximally entangled state $\Psi_d$, that is,

(i) exchange of two qudits,

(ii) simultaneous permutations of the basis states for both parties e.g., $|1\rangle_A \leftrightarrow |2\rangle_A$ and $|1\rangle_B \leftrightarrow |2\rangle_B$,

(iii) simultaneous phase rotations

\[
V(\varphi_1, \ldots, \varphi_{d-1}) = e^{i \sum_j \varphi_j \theta_j} \otimes e^{-i \sum_i \varphi_i \theta_i}
\]

where $\varrho_j$ ($j = 1, \ldots, d-1$) are the diagonal generators of the group SU(d). The resulting states $\rho^{\text{axi}}$ for a $d \times d$-dimensional system have the diagonal matrix elements

\[
\rho^{\text{axi}}_{ji, ij} = \frac{1}{d^2} + a, \quad \rho^{\text{axi}}_{jk, ik} = \frac{1}{d^2} - \frac{a}{d-1} \quad (j \neq k)
\]

($j, k = 1, \ldots, d$) and off-diagonal entries

\[
\rho^{\text{axi}}_{ji, km} = \begin{cases} b & \text{for } l = j, m = k \\ 0 & \text{otherwise.} \end{cases}
\]

Each $\rho^{\text{axi}}$ is a mixture of three states, therefore the family can be represented by a triangle in a plane, see figure 1. The only pure state of this family is $|\Psi_d\rangle \langle \Psi_d|$ in the right upper corner of the triangle. We choose the coordinates $x$ and $y$ such that the Euclidean metric coincides with the Hilbert–Schmidt metric (where we define $D_{\text{HS}}(A, B) \equiv \text{tr} [A - B][A - B]^\dagger$). Then, the coordinates are in the ranges

\[
-\frac{1}{d \sqrt{d-1}} \leq y \leq \frac{\sqrt{d-1}}{d}
\]

**Figure 1.** The family of $d \times d$ axisymmetric states $\rho^{\text{axi}}$ (here for $d = 4$). It is characterized by two real parameters (for all $d$). The only pure state in the family is $\Psi_d$ in the right upper corner. The completely mixed state $\frac{1}{d^2}1_{d \times d}$ is located at the origin. One can clearly identify the hierarchy of convex sets $S_k$ of increasing Schmidt number $k$ ($1 \leq k \leq d$).

Each $\rho^{\text{axi}}$ is a mixture of three states, therefore the family can be represented by a triangle in a plane, see figure 1. The only pure state of this family is $|\Psi_d\rangle \langle \Psi_d|$ in the right upper corner of the triangle. We choose the coordinates $x$ and $y$ such that the Euclidean metric coincides with the Hilbert–Schmidt metric (where we define $D_{\text{HS}}(A, B) \equiv \text{tr} [A - B][A - B]^\dagger$). Then, the coordinates are in the ranges

\[
-\frac{1}{d \sqrt{d-1}} \leq y \leq \frac{\sqrt{d-1}}{d}
\]
\[ -\frac{1}{\sqrt{d(d-1)}} \leq x \leq \sqrt{\frac{d-1}{d}}. \]  

(68)

hence \( a = y\sqrt{d-1/d} \) and \( b = x/\sqrt{d(d-1)} \). The fully mixed state \( \frac{1}{d} I \) lies at the origin. Thus the isotropic states form a subset of the axisymmetric family: they are located on the straight line connecting the right upper triangle corner to the origin.

Clearly, there is a twirling operation \( \mathcal{P}_{\text{axi}}: \rho \to \rho^{\text{axi}} \) analogous to (65) also for axisymmetric states

\[ \mathcal{P}_{\text{axi}}(\rho) = \int dVV\rho V^\dagger \]  

(69)

where now the integral denotes an average over both discrete and continuous symmetry operations (i)–(iii).

5.6.2. Entanglement properties of axisymmetric states. The SLOCC classes of axisymmetric states in the Schmidt-number classification can be determined by using optimal Schmidt-number witnesses [65]. The result is illustrated in figure 1. The borders of the classes \( S_k \) are straight lines parallel to the left lower border of the triangle and divide the right lower border into \( d \) equal parts. Moreover, there is always a part of the Schmidt number 2 close to the left upper corner.

Now we study the entanglement resources by calculating both the negativity and the concurrence for \( \rho^{\text{axi}}(x, y) \) for \( x \geq 0 \). The negativity is straightforwardly obtained as

\[ \mathcal{N}\left(\rho^{\text{axi}}(x, y)\right) = \max\left\{ 0, \frac{1}{2} \left[ \sqrt{d(d-1)}x + \sqrt{d-1}y - \frac{d-1}{d} \right] \right\}. \]  

(70)

For the concurrence we use the lower bound (60) including all off-diagonal matrix elements of \( \rho^{\text{axi}}(x, y) \). The result is

\[ C\left(\rho^{\text{axi}}(x, y)\right) \geq \max\left\{ 0, \frac{2}{\sqrt{d(d-1)}} \left[ \sqrt{d(d-1)}x + \sqrt{d-1}y - \frac{d-1}{d} \right] \right\}. \]  

(71)

We note that both measures depend linearly on \( x \) and \( y \), so their graphs are planes. They both vanish on the line \( y = \sqrt{d-1/d} - x\sqrt{d} \), that is, the border of \( S_1 \). Moreover, they both assume their exact maximum for \( \Psi_k \). Since, on the other hand, the measures are convex, they both represent the largest possible convex functions with the corresponding exact behavior for \( S_1 \) and for \( \Psi_\alpha \). Hence, the right-hand sides of (70) and (71) give the exact convex-roof extended negativity and the exact concurrence for the axisymmetric states, and they coincide (up to a normalization factor). We mention that the axisymmetric states share this property with the rotationally invariant states discussed by Manne and Caves [144].

It is remarkable that both \( \mathcal{N}(\rho^{\text{axi}}) \) and \( C(\rho^{\text{axi}}) \) are constant along the borders between the SLOCC classes. This means that for the axisymmetric states \( \mathcal{N} \) and \( C \) are class-specific entanglement measures, i.e., they measure the Schmidt number \( k \leq d \).
\[
 r(\rho_{\text{axi}}) = k \iff \begin{cases}
 k - 2 \leq 2\mathcal{N}(\rho_{\text{axi}}) \leq k - 1 \\
 k - 2 \leq \frac{d(d - 1)}{2} C(\rho_{\text{axi}}) \leq k - 1.
\end{cases}
\] (72)

Note that axisymmetric states have positive partial transpose iff they are separable.

We conclude this section with two remarks. First, the fact that the exact concurrence (or more precisely, the 2-concurrence, see section 5.2.2) has the shape of a plane for the axisymmetric states and gives rise to the conjecture that each \(k\)-concurrence can be determined exactly in this case and is represented by a plane, with its zero line coinciding with the border between the classes \(S_{k-1}\) and \(S_k\). While for \(d = 3\) this has indeed been proven [145], for \(d > 3\) it is an open question.

The second remark regards the observation that the fidelity \(F\) of an arbitrary \(d \times d\) state with the maximally entangled state \(\Psi_d\) remains unchanged when \(\rho\) is projected onto the axisymmetric states

\[
 F = \langle \Psi_d | \rho | \Psi_d \rangle = \langle \Psi_d | \rho_{\text{axi}}(\rho) | \Psi_d \rangle.\] (73)

On the other hand, for axisymmetric states the entanglement measures \(C\) and \(\mathcal{N}\) depend only on the fidelity with \(\Psi_d\). Consequently, we obtain an alternative method to estimate these measures for arbitrary states: project \(\rho\) to the axisymmetric states (which does not increase the amount of entanglement) and use the entanglement measure of the image as a lower bound. Clearly, this bound can be improved by unitary optimization before the projection:

\[
 C(\rho) \geq C(\rho_{\text{opt}}) = \max_{U_A \otimes U_B} C\left(\mathbb{P}_{\text{axi}}\left(\left[ U_A \otimes U_B \right] \rho \left[ U_A \otimes U_B \right] \right)\right).\] (74)

As long as the optimization is only over local unitaries, this method is equivalent to the estimate (62) taking into account the off-diagonal matrix elements \(\rho_{ij,kl}\) (\(i < k\)). In fact, (62) can be rewritten for this choice of \(\mathcal{M}\) as

\[
 C(\rho) \geq \sqrt{\frac{2d}{d-1}} \min\left( \rho \left[ |\Psi_d\rangle\langle\Psi_d| - \frac{1}{d} |\psi_d\rangle\langle\psi_d| \right] \right)\] (75)

which just links the optimal witness for Schmidt number 2 by Sanpera et al [65] with a lower bound for the concurrence.

6. Quantification of multipartite entanglement

6.1. \(k\)-separability and genuine entanglement of multipartite states

Multipartite quantum systems contain more than two individual subsystems. Their pure states are vectors in the Hilbert space \(\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N}\) with dim \(\mathcal{H}_{A_i} = d_i\). In analogy with bipartite states, the pure state \(|\psi\rangle\) of an \(N\)-partite system is called fully separable (often also just separable) if it is a product of states of the individual systems

\[
 |\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_N\rangle.\] (76)

If this is not the case then \(\psi\) contains entanglement. Then, if it can be written as a product of \(k\) factors (\(1 < k < N\))
it is called $k$-separable. If a state is not a product (77) for any $k > 1$, it is truly $N$-partite entangled. It is now common to say it is genuinely entangled.

Alternatively, a state is said to be $k$-party entangled if none of the factors in

$$|\psi\rangle = \bigotimes_{j=1}^m |\phi_j\rangle$$

contain genuine entanglement of more than $k$ parties [64]. For mixed states we have analogous definitions for the corresponding convex combinations. In particular, a mixed $N$-party state is fully separable if it can be written as

$$\rho = \sum_{j=1}^N p_j \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \cdots \otimes \rho_k^{(N)}$$

where $\rho_k^{(j)}$ is a state of the $j$th system only, and it is genuinely entangled if it cannot be written as a convex combination of biseparable (2-separable) states.

The subdivision into $k$-separable states does not change under SLOCC, so it induces an entanglement classification. While this classification is somewhat obvious, states of more than two parties are far more subtle: as Dür et al noticed [33], for three qubits there are two inequivalent, genuinely entangled types of state: the Greenberger–Horne–Zeilinger (GHZ) state and the $W$ state. For four or more qubits there are infinitely many inequivalent classes of genuinely entangled states. It is far more difficult to answer how these states and their entanglement can be classified. Whenever we refer to the classification of multipartite entanglement, we have in mind the classification of genuinely entangled states, rather than the one according to the separability classes.

As for the quantitative characterization of multipartite entanglement, to date the most relevant questions concern the quantification of genuine entanglement, and entanglement in bipartitions of multipartite states. For the important three-qubit case, this covers all separability classes. Therefore, we restrict our discussion to measures of bipartite and genuine entanglement. Furthermore, as the vast majority of studies has been carried out for qubit systems, we focus on measures for multi-qubit entanglement and mention higher-dimensional systems where it is appropriate.

6.2. Measures for bipartite entanglement in multipartite states

6.2.1. Concurrence, 1-tangle, and negativity. Consider a pure $N$-party state $\psi = \bigotimes_{j=1}^N |\psi\rangle$ and a single-party bipartition $A_j|A_1, \ldots, A_j-1, A_{j+1}, \ldots A_N$. The local reduced state of the party $A_j$ is

$$\rho_{A_j} = \text{tr}_{A_1,\ldots,A_{j-1},A_{j+1},\ldots,A_N} |\psi\rangle\langle\psi|$$

that is, we take the trace over the degrees of freedom of all the other parties (analogously for mixed multipartite states $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. In principle, any bipartite entanglement measure from section 5.2 that is defined on the local state can be used to describe the entanglement in such bipartitions of multipartite states. The appropriate choice depends on the resource that is to be described.
Often, the concurrence (section 5.2.3) is considered
\[
C_{A_1 A_{j+1} \ldots A_N}(\psi) \equiv C_{A_i}(\psi) = \sqrt{2 \left( 1 - \text{tr} \rho_{A_i}^2 \right)}.
\] (81)
If the \( j \)th party is a qubit, one often defines the 1-tangle [146]
\[
\tau_{A_j | A_1 \ldots A_{j-1} A_{j+1} \ldots A_N}(\psi) \equiv \tau_{A_j}(\psi) = \left( C_{A_j}(\psi) \right)^2.
\] (82)
We mention that if all parties are qubits, the average of this quantity over all parties was termed global entanglement and considered in [147, 148]
\[
Q(\psi) = \frac{1}{N} \sum_{j=1}^{N} \tau_{A_j},
\] (83)
Note that, essentially, there is no difference between the definitions (81) and (82) as long as the multipartite state is pure. However, it is relevant for mixed states since, in general, (convex roof[\( \sqrt{T_{A_{j}}^{2}} \)] \( \neq \) convex roof[\( \tau_{A_{j}} \)].
Alternatively, one may use, e.g., the negativity to quantify the entanglement in \( \rho_{A_i} \)
\[
\mathcal{N}_{A_i}(\psi) = \frac{1}{2} \left( \| \rho^{T_{A_i}} \|_1 - 1 \right)
\] (84)
which is particularly convenient for mixed multipartite states just because this quantity is easily calculated.
We mention that the measures discussed by Emary [149] amount to an application of the k-concurrence to multipartite states.

6.2.2. Concurrence of two qubits in a multipartite state. Another method of analyzing bipartite entanglement in a multipartite state \( \psi \) is to consider the bipartition \( A_1 A_{j} | A_1 \ldots A_{j-1} A_{j+1} \ldots A_N \) and the corresponding two-party reduced density matrix
\[
\rho_{A_1 A_{j}} = \text{tr}_{A_{j+1} \ldots A_N} |\psi \rangle \langle \psi|.
\] (85)
This is particularly interesting for multi-qubit states, because then the Wootters concurrence \( C(\rho_{A_{j},A_{i}}) \equiv C_{A_{i},A_{j}}(\psi) \equiv C_{A_{i},A_{j}} \) can be evaluated exactly using the method of section 5.3.1 (for an important application to condensed matter theory see, e.g., [150, 151]).
With the quantities defined so far a profound law of quantum correlations can be stated: it is the monogamy of bipartite qubit entanglement which was conjectured by Coffman et al [146] and proved by Osborne and Verstraete [152]
\[
\tau_{A_i} \geq C_{A_{i},A_{j}}^2 + C_{A_{i},A_{s}}^2 + \ldots + C_{A_{i},A_{N}}^2,
\] (86)
where \( \rho_{A_i} \) and \( \rho_{A_i A_j} \) are the reduced one-qubit and two-qubit states, respectively, of a pure \( N \)-qubit state \( \psi \). Note that on the right-hand side in (86) it does not matter whether the convex roof is taken for \( C \) or \( C^0 \). This is because for two qubits it is always possible to find an optimal pure-state decomposition with the same concurrence for all its elements.
We also mention a variation of (86) for the negativities of the bipartitions [153]
\[
\mathcal{N}_{A_i}^2 \geq \mathcal{N}_{A_{i}A_{j}}^2 + \mathcal{N}_{A_{i}A_{s}}^2 + \ldots + \mathcal{N}_{A_{i}A_{N}}^2
\] (87)
which follows immediately taking into account (57), (60) in section 5.4. Other interesting monogamy inequalities related to the Osborne–Verstraete relation (86) were found for the concurrences of states with higher-dimensional local systems [154] and for the entanglement of formation [155, 156].

6.2.3. Concurrence of assistance. Instead of asking, as in the previous section, about the minimum entanglement contained in a two-qubit reduced state $\rho_{AA_{jk}}$, one might wonder what the possible maximum concurrence compatible with this state is. Thus one can define concurrence of assistance [157]

$$C_{AA_{jk}}^\sharp(\psi) = \max_{\text{decompositions}} \sum_j p_j C(\psi_j)$$ (88)

where the optimization is over the decompositions of $\rho_{AA_{jk}} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. In the multipartite case, this concept is also termed localizable entanglement [158, 159]. Operationally, $C_{AA_{jk}}^\sharp$ corresponds to the maximum average single-copy entanglement the two parties $A_j$ and $A_k$ can achieve through local operations and classical communication by the other parties in the multipartite state. According to the Hughston–Jozsa–Wootters theorem [80], $C_{AA_{jk}}^\sharp$ depends only on the reduced two-qubit state. The calculation of $C_{AA_{jk}}^\sharp$ is simple: one follows the procedure of calculating the eigenvalues $r_j$ of the $R$ matrix in (39) for $\rho_{AA_{jk}}$ and obtains

$$C_{AA_{jk}}^\sharp(\psi) = \frac{4}{\sqrt{\sum_j r_j}}$$ (89)

The concept of the concurrence of assistance can also be generalized to higher-dimensional systems by replacing Wootters’ concurrence with the $G$ concurrence [160].

6.3. Measures for genuine multipartite entanglement

If one simply wants to make sure that a multipartite state is genuinely entangled it suffices to check that it is not separable on any bipartition (or, conversely, that it has a minimum amount of entanglement in each bipartition). Consequently any bipartite measure can be used for this purpose, however, clearly one would choose those measures which can easily be calculated or estimated. The obvious choices—the concurrence and the negativity—lead to the methods that have been most successful in recent years.

6.3.1. Concurrence of genuine multipartite entanglement. The concept of minimum entanglement on all bipartitions in a multipartite state was first applied by Pope and Milburn [161] invoking the von Neumann entropy. Subsequently, Scott [162] and Love et al [163] studied a generalization to the global entanglement (83) by considering the set of averaged linear entropies for general bipartite splits. We mention that, for very large numbers of parties (and therefore bipartitions), one can also study the statistics of purities over bipartitions [164, 165]. However, since purities are not entanglement monotones, the relation of the results to entanglement properties of the states is not clear.

A technically simpler quantity is the concurrence of genuine multipartite entanglement, for short GME concurrence [134]. Consider all possible bipartitions $\gamma = \{P_j | Q_j \}$ of a pure multipartite state $\psi$. Then
The extension of GME concurrence to mixed states is via the convex-roof extension (5), section 3.2. Note the difference of this definition with Akhtarshenas’ work [116] where all linear entropies are added, so that the corresponding concurrence is nonzero as soon as a single bipartition has entanglement. By considering such sums for all possibilities of $k$ partitions, this idea can be used to quantify the $k$-nonseparability of a pure state [166]. Definition (90) has the advantage that lower bounds can readily be found by a straightforward extension of the lower bound of section 5.5 to the multipartite setting [134–136, 167]. To this end, one selects a subset $\mathcal{M}$ of $\eta$ multilevel index pairs $\{ M, M' \}$ as in section 5.5, only that now these index pairs may originate from any bipartition $\gamma_j$ of $\mathcal{H}$, not just a single one. That is, if we consider a particular bipartition $\gamma_j = \{ A_j, B_j \}$, the indices $M, M'$ may be decomposed in parts $M \rightarrow K_j L_j$, $M' \rightarrow K'_j L'_j$ where $K_j, K'_j \in A_j$ and $L_j, L'_j \in B_j$. Then the bound reads

$$C_{GME}(\rho) \geq \frac{2}{\sqrt{\eta}} \sum_{M = K_j L_j, M' = K'_j L'_j} \left( |\rho_{K_j L_j K'_j L'_j}| - \sum_{\gamma_j} \sqrt{|\rho_{K_j L_j K'_j L'_j}|} \right).$$

(91)

The prime in the last sum means that all terms do not necessarily have to be summed up. This is because by summing over all choices $\{ M, M' \}$ and corresponding bipartitions $\gamma_j$, repetitions of diagonal elements may occur. It suffices to take into account the maximum number of repetitions that occur for a single bipartition. In practice, selecting the subset $\mathcal{M}$ in order to produce a good bound may have a subtle solution, so it is difficult to establish a general rule.

The simplest example of this method is a biseparability criterion and GME concurrence bound based on the three-qubit GHZ state. For the off-diagonal index pair we choose the $\{ 000, 111 \}$, so $\eta = 1$. For three qubits (A, B, C) there are three bipartitions $\gamma_1 = \{ ABC \}$, $\gamma_2 = \{ BAC \}$ and $\gamma_3 = \{ CAB \}$. The corresponding permuted index combinations $\{ K_j L'_j, K'_j L_j \}$ are $\{ 100, 011 \}$, $\{ 010, 101 \}$, $\{ 001, 110 \}$ which all occur once. Hence we find

$$C_{GME}(\rho) \geq 2( |\rho_{000,111}| - \sqrt{|\rho_{100,100}|^2 + |\rho_{011,111}|^2} - \sqrt{|\rho_{001,010}|^2 + |\rho_{010,010}|^2} - \sqrt{|\rho_{001,010}|^2 + |\rho_{010,010}|^2} ) .$$

(92)

This entanglement detection criterion was first derived by Gühne and Seevinck [132] and generalized by Huber et al [133]. The method is rather powerful and enables entanglement detection and quantification in numerous situations that were previously inaccessible. An immediate conclusion in [132] was, for example, that GHZ-diagonal states contain entanglement as soon as the fidelity of one GHZ state exceeds $\frac{1}{3}$. The exact GME concurrence of GHZ-diagonal states was found in [168]. It is worth noting that this approach can be extended to $k$-separability, as well as combined with permutation invariance to produce lower bounds on entanglement for the permutation-invariant part of a state [169] that apply to arbitrary multipartite states, very much in the spirit of section 7.4.

It is interesting to reflect upon the structure of (92) which lower bounds the concurrence by a difference between off-diagonal and diagonal elements of the density matrix. In fact, this is the essential idea behind various detection and classification schemes of entanglement:
given a particular entanglement class, the modulus of an off-diagonal element of a density matrix\(^4\) in that class cannot exceed a certain value that depends on related diagonal elements (here: determined via the partial transpose).

### 6.3.2. Genuine multipartite negativity

As an alternative to GME concurrence one may consider the genuine multipartite negativity (GMN) which can be defined directly on mixed states [171, 172]

\[
N_{\text{GME}}(\rho) = \min_{\rho = \sum \rho_j \rho_j} \mu(\rho_j),
\]

\[
\mu(\rho) = \min_{\gamma_i} \left( \left| \rho^{T_{\gamma_i}} \right| - 1 \right)
\]

where \(\rho^{T_{\gamma_i}}\) is the partial transpose with respect to party \(A_j\) in the bipartition \(\gamma_j\) of the multipartite state \(\rho\), so that \(\mu(\rho)\) is the bipartite negativity, minimized over all bipartitions of \(\rho\). The first equation defines \(N_{\text{GME}}\) as the convex hull of \(\mu(\rho)\) and guarantees its convexity. Rather than the border of the biseparable states, this quantity detects whether or not a state is a mixture of PPT states. The advantage, however, is that it can be implemented favorably in a semidefinite program [173]. Via this method, various interesting bounds for white-noise tolerance could be found, i.e., the maximum weight \(1 - p^*\) for which an entangled \(N\)-qubit state \(\psi_{\text{ent}}\) remains genuinely entangled in a mixture

\[
p^* \left| \psi_{\text{ent}} \right\rangle \left\langle \psi_{\text{ent}} \right| + \frac{1 - p}{2^N} I^{2^N}.
\]

For example, it could be established that for \(N\)-qubit linear cluster states [174] the white-noise tolerance tends to 1 for large \(N\) [171].

### 6.4. Measures based on polynomial invariants

In research on entanglement classification and quantification, invariant polynomial functions were investigated early on [29, 50, 175–183]. The initial focus was on invariants under local unitaries. This view changed with the investigation of SLOCC [28, 31–34] and in particular at the point when Coffman et al discovered the residual tangle [146], which redirected the attention towards the special linear group [34, 112, 184–186]. Local unitary invariance is characterized by many more parameters than SL invariance, therefore it appears to describe more subtle resources. In particular, it is linked to the deterministic interconvertibility of states, see, e.g., [50, 187, 188], but also, for example, to topological properties of multipartite states [189–191]. Here we exclusively consider invariance properties with respect to the special linear group, and their relation with entanglement.

#### 6.4.1. Three-tangle

It is a salient feature of multipartite entanglement that there are locally inequivalent classes of genuinely entangled states [33]. For example, for any qubit number \(N\) the corresponding GHZ state

\[
\left| \text{GHZ}_N \right\rangle = \frac{1}{\sqrt{2}} (|00\ldots0\rangle + |11\ldots1\rangle)
\]

\(^4\) An approach for entanglement detection based on the maximization of (generalized) off-diagonal matrix elements was considered in [170].
cannot be transformed by local invertible operations into the $W$ state

$$\left| W_N \right\rangle = \frac{1}{\sqrt{N}} (|0\ldots001\rangle + |0\ldots010\rangle + \ldots + |1\ldots000\rangle)$$

(96)

where all basis states are understood to contain $N$ entries. Similarly, the $N$-qubit cluster state is locally inequivalent to any $N$-qubit GHZ state (or $W$ state) [192, 193]. By applying the entanglement measures in the preceding sections it would be difficult, if not impossible, to distinguish such inequivalent classes. However, there is an elegant way out of this problem. It was noticed by Coffman et al that a polynomial function of the coefficients in a quantum state may help to distinguish the GHZ from the $W$ state for three qubits [149]. They termed it the residual tangle of the three-qubit state $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\tau_{\text{res}}(\psi) = 4 \left| d_1 - 2d_2 + 4d_3 \right|.$$ (97)

For reasons explained in section 3.3 (see also [55, 194]), we reserve the name ‘three-tangle’ for the square root of $\tau_{\text{res}}$

$$\tau_3 = \sqrt{\tau_{\text{res}}}.$$ (98)

Dür et al proved that both $\tau_3$ and $\tau_{\text{res}}$ are entanglement monotones. Intriguingly, it turned out only afterwards [184, 185] that these quantities—just as Wootters’ concurrence (37)—are invariant under SL(2, $\mathbb{C}$) transformations on each qubit. Further it was noted [184, 185, 195] that both the concurrence and the three-tangle are related to Cayley’s hyperdeterminant [196].

It is easily checked that

$$\tau_3(\text{GHZ}_3) = 1, \quad \tau_3(W_3) = 0$$

(99)

and one can conclude that it is impossible to convert a single copy of the $W$ state with nonvanishing probability into a GHZ state by means of invertible local operations. For three qubits the three-tangle (just as the concurrence for two qubits) is the only independent LSL-invariant polynomial. Correspondingly, all pure three qubit states $\psi$ with $\tau_3(\psi) \neq 0$ are locally equivalent to the GHZ state. We have

$$0 \leq \tau_3(\psi) \leq 1$$

(100)

which suggests calling the GHZ state maximally entangled.

We may use the three-tangle to quickly illustrate the peculiarities of multipartite entanglement. In section 5.3 it was mentioned that for two qubits the concurrence for the superposition of a Bell state and an orthogonal product state equals the weight of the Bell state [109]. If we consider an analogous superposition for three qubits

5 The term ‘three-tangle’ was first used, to our knowledge, by Dür et al [33]. It adapts to the nomenclature by Coffman et al [146] to call a quantity ‘tangle’ if it is of degree 4 in the state coefficients. Therefore the degree-2 quantity in (98) should not be called ‘three-tangle’, in principle. However, since the name has become so popular during the past decade we continue using it.
we find [197]

\[ \tau_{\text{res}}(p, \varphi) = \left| p^2 - \frac{8\sqrt{6}}{9} \sqrt{p(1-p)^3} e^{3i\varphi} \right| \]  

(102)

which vanishes, for example, for \( \varphi = 0 \) and \( p = \frac{4\sqrt{2}}{3 + 4\sqrt{2}} \).

That is, while for two qubits the effect of the orthogonal unentangled states is merely to proportionally reduce the weight of the maximally entangled state, for three qubits the action of superposing a W state is more complex—it may be more harmful with respect to GHZ-type entanglement (e.g., for \( \varphi = 0 \), \( p \geq p_0 \)) as well as less harmful (for \( \varphi = \pi \) and all \( p \)).

### 6.4.2. Four-qubit invariants

While for two and three qubits there is only a single independent invariant polynomial, for four qubits there are infinitely many. The invariant polynomials form an algebra (actually a ring), so there is a set of generating polynomials. In algebraic geometry, the Hilbert series is a standard tool to find degrees and dimensions for polynomial spaces. Its application becomes increasingly difficult for larger systems, however, in the four-qubit case a complete set of generating polynomials is known and was first described by Luque and Thibon [198]. It consists of one degree-2 polynomial, three degree-4 polynomials (among which only two are algebraically independent), and one degree-6 polynomial. Note that, in order to obtain an entanglement monotone with nice properties also for the convex-roof extension (cf section 3.3), the appropriate power of the polynomials’ modulus has to be taken, so that the result is of homogeneous degree 2 in the state coefficients.

We consider four-qubit states \( \psi \) in a four-qubit Hilbert space \( \mathcal{H}_{ABCD} = (\mathbb{C}^2)^{\otimes 4} \). The degree-2 polynomial is the straightforward generalization of Wootters’ concurrence (38)

\[ H(\psi) = \left| \left\{ \psi^\dagger \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \right\} \right| \].  

(104)

With this definition \( H \) has a (physically irrelevant) prefactor of 2 compared to the definitions in [198] and [199]. We note that this quantity, in a strict sense, cannot be a measure of genuine entanglement as it yields 1 on a tensor product of two two-qubit Bell states [200]. Therefore this kind of measure needs to be complemented with a measure of genuine multipartite entanglement. On the other hand, the convex roof of \( H \) can be evaluated exactly by using the procedure from Wootters [107] and Uhlmann [201].

The degree-4 invariants are denoted \( L, M, \) and \( N \) and can be written in terms of the pure-state coefficients \( |\psi\rangle = \sum_{i_4} \psi_{i_4} |i_4\rangle \)

\[ L(\psi) = \begin{pmatrix} \psi_{0000} & \psi_{0100} & \psi_{1000} & \psi_{1100} \\ \psi_{0001} & \psi_{0101} & \psi_{1001} & \psi_{1101} \\ \psi_{0010} & \psi_{0110} & \psi_{1010} & \psi_{1110} \\ \psi_{0011} & \psi_{0111} & \psi_{1011} & \psi_{1111} \end{pmatrix} \]  

(105)

and \( M, N \) analogous to the second and the third, or the second and the fourth qubit exchanged, respectively. These invariants are not independent, since

\[ L + M + N = 0. \]  

(106)
In [57] it was shown that the squared moduli of $L, M, N$ are nothing but the determinants of the two-qubit reduced density matrices of the four-qubit state $\psi$:

$$|L|^2 = \det\left(\text{tr}_{CD} \pi_\psi\right) \quad \text{(107a)}$$

$$|M|^2 = \det\left(\text{tr}_{BD} \pi_\psi\right) \quad \text{(107b)}$$

$$|N|^2 = \det\left(\text{tr}_{BC} \pi_\psi\right). \quad \text{(107c)}$$

One sees that also $L, M, N$ may be nonzero on separable states (for example, on a tensor product of Bell states). Note also that, according to (107a)–(107c) the quartic invariants may be regarded as (powers of) $G$-concurrences on $4 \times 4$ bipartite systems, that is, they are SL (4, C) invariants (cf also [149]).

To get a complete set of generators, one independent degree-6 invariant is required. A possible choice is the degree-6 filter $\tilde{\mathcal{F}}_{1}^{(4)}$ [58] that is discussed in section 6.4.4. The sextic polynomial $\tilde{\mathcal{F}}_{1}^{(4)}$ is symmetric under qubit permutations and defines another permutation-symmetric polynomial $W$ via

$$\tilde{\mathcal{F}}_{1}^{(4)} = 32W - H^3. \quad \text{(108)}$$

In [198] the degree-6 polynomial $D_{xt}$ was used instead, which belongs to a family of three invariants that obey

$$W = D_{xy} + D_{xz} + D_{xt} \quad \text{(109)}$$

and

$$\frac{1}{2}H(N - M) = 3D_{xy} - W, \quad \text{(110a)}$$

$$\frac{1}{2}H(L - N) = 3D_{xz} - W, \quad \text{(110b)}$$

$$\frac{1}{2}H(M - L) = 3D_{xt} - W. \quad \text{(110c)}$$

There are a few more four-qubit polynomials of particular interest. These include the degree-8 filter $\tilde{\mathcal{F}}_{2}^{(4)}$, and the degree-12 filter $\tilde{\mathcal{F}}_{3}^{(4)}$ [58, 199, 202] whose precise definitions are also given in section 6.4.4. The peculiarity of the filters $\tilde{\mathcal{F}}_{1}^{(4)}, \tilde{\mathcal{F}}_{2}^{(4)}, \tilde{\mathcal{F}}_{3}^{(4)}$ is that they vanish in any biseparable state of the Hilbert space $\mathcal{H}_{ABCD}$ and generate an ideal of polynomial invariants [199]. Finally, we also mention the degree-24 four-qubit hyperdeterminant $\text{Det}$ whose relation to four-qubit entanglement was discussed in [195] (we quote the result from [199]):

$$2^{12}3^{3} \text{Det} = -2\tilde{\mathcal{F}}_{1}^{(4)}A + \left(128\Sigma - H^4\right)B - \left(256\Pi + \frac{1}{8}H^6\right)^2 \quad \text{(111)}$$

where

$$\Sigma = L^2 + M^2 + N^2 \quad \text{(112a)}$$

$$\Pi = (L - M)(M - N)(N - L) \quad \text{(112b)}$$
\[
A = \frac{5}{512} H^9 + \frac{5}{16} WH^6 - \frac{9}{2} \Sigma H^5 + 2 \left(5W^2 - 24\Pi\right)H^3 - 240W\Sigma H^2 + 768\Sigma^2 H + 192W \left(3W^2 + 8\Pi\right) \tag{112c}
\]

\[
B = \frac{1}{256} H^8 - \frac{17}{2} \Sigma H^4 - 96\Pi H^2 + 256\Sigma^2. \tag{112d}
\]

It is worthwhile noting that the hyperdeterminant is maximized by the four-qubit state \(|X_4\rangle\) in (157c), cf. [203].

6.4.3. Invariants of degree 2 and 4. It is evident from the discussion of the four-qubit polynomials that even for small systems with \(N \lesssim 10\) qubits it becomes exceedingly complicated to work with invariants of a higher degree. Therefore, one would hope to extract much of the relevant physical information from the invariants of a lower degree, that is, degrees 2 and 4 (there are no nontrivial invariants of odd-integer degree [185]). Therefore it is useful to present a set of standard rules on how to construct such polynomials [57, 58, 202]. In the following we assume that the pure state \(\psi\) is always an element of the correct Hilbert space corresponding to the quantity under consideration.

First we recall that the two-qubit concurrence (38) can be written as

\[
C(\psi) = \left| \left\langle \psi \left| \sigma_1 \otimes \sigma_1 \right| \psi \right\rangle \right| = \left| \left\langle \psi^* \left| \sigma_1 \otimes \sigma_1 \right| \psi \right\rangle \right| \equiv \left| H^{(2)}(\psi) \right|. \tag{113}
\]

where we introduce a shorthand notation for the expectation value

\[
H^{(2)}(\psi) = \left| \left\langle \psi^* \left| \sigma_1 \otimes \sigma_1 \right| \psi \right\rangle \right| \equiv \left\langle \sigma_2 \sigma_2 \right\rangle. \tag{114}
\]

Here \(\psi^*\) is the state \(\psi\) with complex conjugate entries. Moreover, we enumerate the Pauli operators as \(\sigma_1 \equiv \sigma_x, \sigma_2 \equiv \sigma_y, \sigma_3 \equiv \sigma_z\) and \(\sigma_0 \equiv \mathbb{I}_2\). Straightforward algebra shows that the invariant for the residual tangle (97) can be expressed as (where we use the definition \(\tau_{\text{res}} \equiv \text{tr}

\[
B_1^{(3)}(\psi) = -\left| \left\langle \psi \left| \sigma_0 \otimes \sigma_2 \otimes \sigma_2 \right| \psi^* \right\rangle \left\langle \psi \left| \sigma_0 \otimes \sigma_2 \otimes \sigma_2 \right| \psi \right\rangle + \right.
\]

\[
+ \left| \left\langle \psi \left| \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \right| \psi^* \right\rangle \left\langle \psi \left| \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \right| \psi \right\rangle + \right.
\]

\[
+ \left| \left\langle \psi \left| \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \right| \psi^* \right\rangle \left\langle \psi \left| \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \right| \psi \right\rangle \right| \tag{115}
\]

and, if we use the shorthand notation of (114)

\[
B_1^{(3)}(\psi) = \sum_{\mu=0,1,3} (-1)^{\mu+1} \left| \left\langle \sigma_\mu \sigma_2 \sigma_2 \right\rangle \left\langle \sigma_\mu \sigma_2 \sigma_2 \right\rangle \right| \equiv \left| \langle \sigma_\mu \sigma_2 \sigma_2 \rangle \langle \sigma_\mu \sigma_2 \sigma_2 \rangle \right| \equiv \left| \left\langle \sigma_\mu \sigma_2 \sigma_2 \right\rangle \left\langle \sigma_\mu \sigma_2 \sigma_2 \right\rangle \right| \tag{116}
\]

where, in the last line, we use the summation convention with a ‘metric’ \([-1, 1, 0, 1]\). Why does this work? This is because, for \(S \in \text{SL}(2, \mathbb{C})\)

\[
S\sigma_2S^T = \sigma_2, \quad (S \otimes S)\sigma_\mu \otimes \sigma_\mu (S \otimes S)^T = \sigma_\mu \otimes \sigma_\mu
\]

that is, the expression in (115), (116) is local SL invariant on each qubit by construction. Note that one cannot simply use \(\langle \sigma_2 \sigma_2 \rangle\) because combining a real symmetric operator \(A\) with an
odd number of $\sigma_2$ operators, the antilinear expectation value always vanishes
\[
\left< \psi^* \left| \sigma_2^{\otimes (2n+1)} \otimes A \right| \psi \right> = -\left< \psi^* \left| \sigma_2^{\otimes (2n+1)} \otimes A \right| \psi^* \right>
= +\left< \psi^* \left| \sigma_2^{\otimes (2n+1)} \otimes A \right| \psi^* \right> = 0. \tag{118}
\]

In particular we have for pure single-qubit states $\phi$
\[
\left< \phi^* \sigma_2 \left| \phi \right> = 0. \quad \left( \left< \phi \right| \otimes \left< \phi \right> \left( \sigma_2^* \otimes \sigma_2^* \right) \left| \phi \right> \otimes \left| \phi \right> \right) = 0. \tag{119}
\]

Hence, if a single qubit is separable in the three-qubit state $\psi$ in (115) the expression must vanish—just as is the case for the three-tangle. The two operators in (119) were called combs and, accordingly, the technique to systematically build polynomial invariants for qubit states the invariant-comb method [58, 199, 202]. In [199] it was also shown that any invariant that can be constructed with a so-called $\Omega$-process (a standard method, in classical invariant theory owed to Cayley [204], to systematically obtain invariants) can also be generated by means of the comb approach.

Now we can write down the four-qubit invariants: for the degree-2 polynomial $H \equiv H^{(4)}$ we find
\[
H^{(4)}(\psi) = \left< \sigma_2 \sigma_2 \sigma_2 \sigma_2 \right> \tag{120}
\]
As degree-4 invariants we obtain
\[
B_{1,2}^{(4)} = \left< \sigma_2 \sigma_2 \sigma_2 \sigma_2 \right> \left< \sigma^* \sigma^* \sigma^* \sigma^* \right> \tag{121a}
B_{1,3}^{(4)} = \left< \sigma_2 \sigma_2 \sigma_2 \sigma_2 \right> \left< \sigma^* \sigma^* \sigma_2 \sigma_2 \right> \tag{121b}
B_{1,4}^{(4)} = \left< \sigma_2 \sigma_2 \sigma_2 \sigma_2 \right> \left< \sigma^* \sigma_2 \sigma_2 \sigma^* \right>. \tag{121c}
\]

Note that these polynomials are not permutation invariant. It turns out that [199]
\[
L = \frac{1}{48} \left( B_{1,3}^{(4)} - B_{1,4}^{(4)} \right) \tag{122a}
M = \frac{1}{48} \left( B_{1,4}^{(4)} - B_{1,2}^{(4)} \right) \tag{122b}
N = \frac{1}{48} \left( B_{1,2}^{(4)} - B_{1,3}^{(4)} \right). \tag{122c}
\]
We also mention the useful identity [199]
\[
\left( H^{(4)} \right)^2 = \frac{1}{3} \left( B_{1,2}^{(4)} + B_{1,3}^{(4)} + B_{1,4}^{(4)} \right). \tag{123}
\]

Obviously this scheme of generating degree-4 invariants can be extended to any qubit number.
• Even qubit number \( n = 2k \): there is always one degree-2 invariant

\[
\hat{H}^{(2k)}(\psi) = \left\langle \sigma_2 \right\rangle \otimes^{2k}
\]  

and \( \frac{1}{2} n (n - 1) \) degree-4 polynomials of the type

\[
B_{a,b}^{(2k)}(\psi) = \left\langle \sigma_2 \ldots \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \right\rangle \left\langle \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \right\rangle
\]  

where the contractions are located at positions \( a \) and \( b \) in the brackets, respectively.

• For an odd qubit number, \( n = 2k + 1 \), there is no degree-2 invariant and \( n \) degree-4 invariants of the type

\[
B_{a}^{(2k + 1)}(\psi) = \left\langle \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \right\rangle \left\langle \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \right\rangle
\]

with the contraction at position \( a \) in the bracket. We note the invariants \( B_{a}^{(2k + 1)}(\psi) \) are evidently not invariant under qubit permutations. Therefore one can also introduce, in addition to those, an explicitly permutation-invariant polynomial

\[
B_{a}^{(2k + 1)}(\psi) = \sum_{a=1}^{2k+1} B_{a}^{(2k + 1)}(\psi).
\]

In [57] it was shown that \( B_{a}^{(2k + 1)}(\psi) \) (for odd \( n \)) and the square of \( \hat{H}^{(2k)}(\psi) \) (for even \( n \)) coincide with the degree-4 invariants of Wong and Christensen [200]. Clearly, for both even and odd \( n \), there are also degree-4 polynomials with more than two contractions.

6.4.4. Invariants of higher degree. With the shorthand notation from the previous section we can write the precise definitions of the invariants \( F_{j}^{(4)} \), \( j = 1, 2, 3 \), from section 6.4.2. We start with \( F_{1}^{(4)} \) and recall that a shortcoming of the invariants \( B_{j}^{(4)}(\psi) \) was that they give non-zero values for biseparable states. Consider therefore the definition

\[
F_{1}^{(4)} = \left\langle \sigma_2, \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma^2 \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma_2, \sigma_2, \sigma^2 \right\rangle.
\]  

First, one notes that separability of any single qubit is excluded for nonvanishing \( F_{1}^{(4)} \). This is a very general fact and not only relevant for qubit invariants [56]. As for the two-qubit bipartitions, the expression (128) is constructed in such a way that for each choice of a two-qubit partition, there is at least one expectation value, for which a single \( \sigma_2 \) is paired with a real symmetric operator, so that it vanishes if this two-qubit bipartition is separable.

This is quite remarkable: we see that the invariant polynomials can be built in such a way that separability on any bipartition is excluded. The price to pay for this is increasing complexity and higher degree of the polynomial.

In this spirit, the other filter invariants \( F_{2}^{(4)} \) (degree 8) and \( F_{3}^{(4)} \) (degree 12) can also be defined:

\[
F_{2}^{(4)} = \left\langle \sigma_2, \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma^2 \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma_2, \sigma_2, \sigma^2 \right\rangle
\]

\[
F_{3}^{(4)} = \frac{1}{2} \left\langle \sigma_2, \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma^2 \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma_2, \sigma_2, \sigma^2 \right\rangle
\]

\[
\times \left\langle \sigma_2, \sigma_2, \sigma_2 \right\rangle \left\langle \sigma^2 \sigma_2, \sigma_2, \sigma_2 \right\rangle.
\]

In [199] many other examples of higher-degree invariants for five qubits can also be found.
We mention that it is possible to rewrite the antilinear expectation values of the comb approach in terms of directly measurable quantities which, in general, leads to SL invariants of higher degree [205]. However, the approach becomes really cumbersome even for degree-4 invariants such as the three-tangle (cf also [206]). Therefore, we quote only the simplest possibility, the concurrence for pure two-qubit states

\[
|C(\psi)|^2 = \frac{1}{4} M_{\mu \nu} M_{\kappa \lambda} \langle \psi | \sigma_\mu \otimes \sigma_\kappa | \psi \rangle \langle \psi | \sigma_\nu \otimes \sigma_\lambda | \psi \rangle.
\]  

(130)

Curiously, \(M_{\mu \nu}\) is given by the full Minkowski metric \(\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}\). Since this quantity consists of the correlation functions of local observables only, it is—in principle—directly accessible experimentally. The problem here is that an experiment always deals with mixed states \(\rho\). The correlation functions in (130) are then given by \(\text{tr} (\rho \sigma_\mu \otimes \sigma_\kappa)\), however, it is not clear whether these correlation functions are related to the convex roof of the concurrence in a more direct way than via the fidelity estimate 5.3.2. Note that the issue of directly measuring entanglement measures is subtle and requires careful consideration, see, e.g., [207].

To summarize this section on polynomial invariants, we have discussed in some detail one method for the systematic generation of local SL-invariant polynomials for multi-qubit states, known as the invariant comb approach [58, 199, 202]. Various other authors have proposed the generation and investigated the properties of such invariants with respect to entanglement, based on mathematical or physical motivation (e.g., [208, 209, 211–214, 262]. Moreover, we have not touched upon higher-dimensional local systems at all, although there exist results in the literature in this direction (cf, for example, [205, 215, 216, 263]). In particular, Gour and Wallach [59] have developed a method to generate invariants systematically for higher-dimensional systems.

One can conclude that, from the physics point of view, the relevant polynomials required for the complete characterization of multi-party entanglement in finite-dimensional systems are probably known. As it stands at the moment, there is no agreement on which of them play a primary role in entanglement quantification, and what their precise meaning could be.

6.4.5. Other local SL invariants. In section 5.3.3 we have already described the Lorentz singular values of a two-qubit density matrix, which are LSL invariant, but cannot obviously be expressed in terms of polynomials. Here we briefly discuss yet another type of LSL invariant that can be defined for any \(N\)-qubit density matrix.

The density matrix \(\rho\) of a \(d\)-dimensional system can be written in terms of the \(d^2 - 1\) generators \(T_j\) of the SU\((d)\) algebra

\[
\rho = \frac{1}{d} I_d + \frac{1}{2} \sum_{j=1}^{d^2-1} x_j T_j
\]  

(131)

which we refer to as Bloch representation of the state\(^6\) [29, 175, 217]. Here, the real entries \(x_j\) are defined as

\[
x_j = \text{tr} (\rho T_j), \quad \text{tr} T_j = 0, \quad \text{tr} T_j T_k = 2 \delta_{jk}.
\]  

(132)

Hence for a single-qubit density matrix \(\rho\) we have (with the Pauli matrices \(\{T_j\} = \{\sigma_1, \sigma_2, \sigma_3\}\) and \(\sigma_0 \equiv I_2\))

\(^6\) For this quantity, other names like coherence vector [217] or correlation tensor [218] are also used.
Now, \( x = (x_0, x_1, x_2, x_3) \) may be regarded as a four-vector in \( \mathbb{R}^4 \). For its Minkowskian length one finds
\[
\| x \| = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \det \rho.
\] (134)

However, we know that \( \det \rho \) is an invariant under SL (2, \( \mathbb{C} \)) operations. Hence, the application of SLOCC to \( \rho \) can be identified with Lorentz transformations on the four-vector \( x \). Local unitary operations on \( \rho \) correspond to rotations of \( x \) in \( \mathbb{R}^3 \). The four-vectors \( x \) for physical states \( \rho \) are located within the ‘forward light cone’, with the pure states on the surface.

This idea can be generalized to \( N \) qubits [219]. Here we illustrate only the two-qubit case for which, in the Bloch representation,
\[
\rho = \frac{1}{4} \sum_{j,k=0}^3 x_{jk} \sigma_j \otimes \sigma_k.
\] (135)

The Minkowskian length for the tensor \( x \) in (135) is
\[
\| x \|^2 = (x_{00})^2 - \sum_{j=1}^3 \left[ (x_{j0})^2 + (x_{0j})^2 \right] + \sum_{j,k=1}^3 (x_{jk})^2
\] (136)

which is invariant under Lorentz transformations on each index—or, in other words, it is invariant under local SL transformations to the two-qubit state \( \rho \) (cf also (130)). Note that the analog Euclidean length of \( x \) is
\[
\text{tr} \, \rho^2 = \frac{1}{4} \sum_{j,k=0}^3 x_{jk}^2
\] (137)

that is, the purity of \( \rho \).

The Minkowskian length discussed in this section is an LSL invariant defined directly on the density matrix (as opposed to the convex-roof construction, e.g., for the concurrence). Although this length is homogeneous of degree 1 in the density matrix, it is not automatically an entanglement monotone. For that purpose, one needs additional conditions. For example, it would be sufficient if one could make sure that this quantity is also a convex function on the state space.

### 7. Three qubits

The simplest multipartite quantum system consists of three qubits. This system is already sufficiently complex that no complete analytical solution is known, but on the other hand simple enough to avoid most of the difficulties connected with multipartite entanglement. Moreover, for the important entanglement measures methods to calculate good bounds are known.

#### 7.1. Pure three-qubit states

One peculiar property of multipartite entanglement can already be seen for the three-qubit case: unlike in the two-qubit case, not all pure states can be generated from a single
maximally entangled state. There are two inequivalent SLOCC classes for genuinely multipartite entangled classes [33]: the GHZ class with the representative state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

(138)

and the W class with the representative state

$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

(139)

While conversion between those two classes is not possible in either direction, as we will see there are still good reasons to consider the GHZ state the unique (up to local unitary operations) maximally entangled state.

Any pure three-qubit state is unitarily equivalent to a state of the form [220]

$$\psi = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle$$

(140)

where \(\sum_{k=0}^{4} \lambda_k^2 = 1\), all \(\lambda_k \geq 0\) and \(0 \leq \varphi \leq \pi\). In this form, several entanglement measures take on a particularly simple form:

- The three-tangle (98) is
  \(\tau_3 = 2\lambda_0\lambda_4\).

- The two-qubit concurrences (40) after tracing out one qubit are
  \[C_{AB} = 2\lambda_0\lambda_3,\]
  \[C_{AC} = 2\lambda_0\lambda_2,\]
  \[C_{BC} = 2|\lambda_1\lambda_4 e^{i\varphi} - \lambda_2\lambda_3|\]

(141a-d)

- The I-concurrence for splitting the first qubit from the rest is
  \[C_{AB|C} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}.\]

(141e)

Note that the quantities \(J_1\) to \(J_4\) of [220] are
\[J_1 = \frac{1}{4}C_{BC}^2, J_2 = \frac{1}{4}C_{AC}^2, J_3 = \frac{1}{4}C_{AB}^2, J_4 = \frac{1}{4}\tau_3^2.\]

Using these quantities, the SLOCC classification of pure three-qubit states is straightforward (following [220] with the criteria, but ignoring their subclasses which are not SLOCC classes; see [33] for a detailed description of the SLOCC classes):

- The GHZ class consists of all states with \(\tau_3 > 0\).
- The W class consists of states with \(\tau_3 = 0\), \(C_{AB} > 0\), \(C_{AC} > 0\) and \(C_{BC} > 0\).
- The three biseparable classes \(A - BC\), \(B - AC\) and \(C - AB\) all have the corresponding concurrence \(C_{BC}\), \(C_{AC}\) or \(C_{AB}\) nonzero, and the others zero.
- For the completely separable states, all measures vanish.

From equations (141a) to (141e), one can also immediately obtain the Coffman–Kundu–Wootters monogamy relation [149]

$$C_{AB|C}^2 = C_{AB}^2 + C_{AC}^2 + \tau_3^2.$$  

(142)

from which they originally derived the residual tangle \(\tau_{res} = \tau_3^2\).  

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There are other quantities which are invariant under local unitary transformations, but not under local SL transformations on any qubit, like the Kempe invariant [182, 221]. However, they do not describe SLOCC properties of the state [222].

Note that the two-qubit concurrences $C_{AB}$, $C_{AC}$ and $C_{BC}$ are not three-qubit monotones, as can be easily seen by the fact that the GHZ state (all three two-qubit concurrences equal 0) can be converted via SLOCC to the tensor product of a Bell state (concurrence 1) with a pure one-qubit state, thus increasing the concurrence on the corresponding two qubits.

7.2. Hierarchy of mixed states

The entanglement classification of pure states can be extended to mixed states by considering the classes of the pure states in the decomposition. An obvious definition seems to be: a state is of type X if it can be decomposed into states of type X, that is, the set of states of type X is the convex hull of the set of pure states of type X. However the sets defined in this way have some striking properties which suggest a different definition: it turns out that with the above definition, all states, except for a set of W-type states (which is of measure zero) are GHZ-type states, and similarly for biseparable and completely separable states (whose sets of zero measure of course contain all the pure states of that class). That is, there exists a hierarchy of entanglement types, which leads to a slightly different definition of mixed state classes, as was defined by Acín et al [66]:

- The set of separable states is, of course, the convex hull of the pure separable (i.e. product) states.
- The set of biseparable states is the convex hull of the set of the pure biseparable and separable states.
- The set of W-type states is the convex hull of the pure W-type, biseparable and separable states.
- The set of GHZ-type states is the convex hull of the pure GHZ-type, W-type, biseparable and separable states, that is, all states.
It is, however, often more useful to define the classes in an exclusive manner, that is, to call a state X-type entangled only if it is not in one of the lower classes (e.g. a state is W-type entangled if it can be decomposed into states of a maximally W type, but not into states of a maximally biseparable type). This exclusive classification is what we use below.

Note that the biseparable states have a substructure [223, 224]: there are three classes of state which are biseparable on a specific decomposition, that is, they can be decomposed into pure states which are all biseparable on that decomposition. The total set of biseparable states is the convex hull of the union of those three sets.

Since pure states that are biseparable on each bipartition are fully separable, one would guess that the same is also true for mixed states; however this is not the case. There exist mixed states which are biseparable on each bipartition, yet are not separable [225]. Those states are PPT entangled.

This hierarchy of entanglement is schematically depicted in figure 2.

The mixed state entanglement classes can be distinguished using entanglement measures: a state is GHZ-type entangled iff the three-tangle (98) does not vanish. A state is W-type entangled iff the GME concurrence (90) does not vanish, but the three-tangle does. For biseparability, an appropriate measure is the convex roof of the square root of the global entanglement (83) (the square root is to get homogeneous degree 2 in the state vector). The biseparable states are exactly those for which this measure does not vanish, but the GME concurrence does.

Separability on a specific bipartition can be checked using the concurrence (81) for that bipartition. Note that vanishing of all three concurrences is not sufficient for separability.

7.3. Exact treatment of GHZ-symmetric states

As mentioned in section 5.6, mixed states of low rank and/or high symmetry may help considerably to elucidate the structure and properties of the state space with respect to entanglement. Also for three-qubit states, important progress could be achieved by finding the exact solutions for such specific problems. The first example of this kind was solved by Lohmayer et al [197] who considered rank-2 mixtures

$$\rho(p) = p |\text{GHZ}\rangle\langle \text{GHZ}| + (1-p)|W\rangle\langle W|. \quad (143)$$

They found the exact residual tangle $\tau_{\text{res}}$ as well as the concurrence and the 1-tangle for this family. The spirit of the method behind this calculation (termed the convex characteristic curve) was applied before (e.g., in [20] and [142]) and is outlined in Osterloh et al [226].

Subsequently, Eltschka et al extended this result to rank-2 mixtures of generalized GHZ and W states, that is, $|\text{gGHZ}\rangle = a |000\rangle + b |111\rangle$, $|\text{gW}\rangle = c |001\rangle + d |010\rangle + e |100\rangle$ [227]. Jung et al provided exact solutions of $\tau_{\text{res}}$ for the rank-3 problem [228]

$$\rho(p, q) = p |\text{GHZ}\rangle\langle \text{GHZ}| + q |W\rangle\langle W| + (1-p-q) |\overline{W}\rangle\langle \overline{W}| \quad (144)$$

as well as for a family of rank-4 states [229].

$$|\overline{W}\rangle = \frac{1}{\sqrt{3}} (|011\rangle + |101\rangle + |110\rangle) \quad (145)$$

is the bit-flipped W state. Further, He et al considered $\tau_{\text{res}}$ for certain states with up to rank 8 [230]. Finally, in [231], the three-tangle (i.e., $\tau_3 = \sqrt{\tau_{\text{res}}}$) for the states (143) and (144) was found.

In this section we take a closer look at another set of three-qubit states for which the relevant entanglement properties have been derived exactly. To our knowledge it is the only
completely, qualitatively and quantitatively solved set which covers all main entanglement classes except PPT entanglement.

Invariance under symmetries has proved to be a valuable tool in studying entanglement since the seminal work on bipartite states by Werner [2], and on tripartite states by Eggeling and Werner [232]. Since the GHZ state is the maximally entangled three-qubit state, it is desirable for it to be contained in the invariant set. To achieve this, it is necessary to use all or a subset of the symmetries of that state, which are [233]:

(i) qubit permutations,
(ii) simultaneous three-qubit flips (i.e., application of $\sigma_x \otimes \sigma_x \otimes \sigma_z$),
(iii) qubit rotations about the $z$ axis of the form

$$U \left( \phi_1, \phi_2 \right) = e^{i\phi_1 \sigma_z} \otimes e^{i\phi_2 \sigma_z} \otimes e^{-i\left(\phi_1 + \phi_2\right) \sigma_z}.$$ (146)

Here, $\sigma_x$ and $\sigma_z$ are Pauli operators. Invariance under the operations (i)–(iii) is called GHZ symmetry and states that are invariant under that symmetry are called GHZ-symmetric states. Except for the qubit permutations, all those operations are local and therefore do not change the entanglement type. Also, the only effect of the qubit permutations on the entanglement is that they permute the different subclasses of bipartite entanglement, which implies that the biseparability properties of the GHZ-symmetric states are the same on all bipartitions.

There are two GHZ-symmetric pure states, the standard GHZ-state $|\text{GHZ}\rangle \equiv |\text{GHZ}_+\rangle$ (138) and the sign-flipped GHZ state $|\text{GHZ}_-\rangle = (|000\rangle - |111\rangle)/\sqrt{2}$. The complete set of GHZ-symmetric states consists of mixtures of those two states and the mixed state $\rho = \sum_{klm=00}^{11}|klm\rangle \langle klm|$. The colours have the same meaning as in figure 2: the blue kite in the middle are the separable states, the two green triangles surrounding it are the biseparable states. The roughly triangular yellow areas are the W-type entangled states, and are bounded by the red, curved GHZ-W line. Above the GHZ-W line there lie the GHZ-type entangled states.

![Figure 3. Entanglement classes of the GHZ-symmetric states.](image_url)

A GHZ-symmetric state $\rho^S$ is fully specified by two independent real parameters. A possible choice is

$$x \left( \rho^S \right) = \frac{1}{2} \left[ \langle \text{GHZ}_+ | \rho^S | \text{GHZ}_+ \rangle - \langle \text{GHZ}_- | \rho^S | \text{GHZ}_- \rangle \right]$$ (147a)

$$y \left( \rho^S \right) = \frac{1}{\sqrt{3}} \left[ \langle \text{GHZ}_+ | \rho^S | \text{GHZ}_+ \rangle + \langle \text{GHZ}_- | \rho^S | \text{GHZ}_- \rangle - \frac{1}{2} \right]$$ (147b)

such that the Euclidean metric in the ($x,y$) plane coincides with the Hilbert–Schmidt metric on the density matrices. The completely mixed state is located at the origin. The three corners of
For the GHZ-symmetric states, the entanglement classes have been determined exactly [233].

The separable states live in the kite with corners given by the four points $P_r, (0, 0), P_m = (0, \sqrt{3}/4)$ and $(-\frac{1}{2}, 0)$. The biseparable states are those states which are not separable in the kite with corners $P_r, (\frac{1}{2}, 1/(4\sqrt{3}))$, $P_m$ and $(-\frac{1}{2}, 1/(4\sqrt{3}))$.

For the GHZ-symmetric states, the entanglement classes have been determined exactly [233]. It turns out that all of the permutation symmetric SLOCC classes except PPT entangled states are present. The separable states live in the kite with corners given by the four points $P_r, (\frac{1}{2}, 0), P_m = (0, \sqrt{3}/4)$ and $(-\frac{1}{2}, 0)$. The biseparable states are those states which are not separable in the kite with corners $P_r, (\frac{1}{2}, 1/(4\sqrt{3}))$, $P_m$ and $(-\frac{1}{2}, 1/(4\sqrt{3}))$.

The $W$-type states are the remaining states in the shape between $P_r$ and the ‘GHZ-$W$ line’ given in a parametrized form by

$$x^w(v) = \frac{v^3 + 8v^3}{8(4 - v^2)}$$

$$y^w(v) = \frac{\sqrt{3} 4 - v^2 - v^4}{4 - v^2}.$$  \quad (148)

All remaining states are of GHZ-type.

Not only is the classification of GHZ-symmetric states known, but also the most important entanglement measures.

As could already be guessed from the GHZ-$W$ line, the most complicated is the three-tangle. It is calculated as follows [234].
Given a GHZ-symmetric three-qubit state \( \rho^{S} \) with coordinates \((x, y)\), one first determines the straight line that connects the GHZ state at \((1/2, \sqrt{3}/4)\) with the point \((x, y)\). This line intersects the GHZ-W-line at the point \((x^{W}, y^{W})\). Then the three-tangle \( \tau_{3}(\rho^{S}(x, y)) \) is given by (cf figure 4)

\[
\tau_{3}(x, y) = \begin{cases} 
0 & \text{for } x < x^{W}, y < y^{W} \\
\frac{x - x^{W}}{\frac{1}{2} - x^{W}} & \text{otherwise.}
\end{cases}
\tag{149}
\]

Since the GHZ-symmetric states are a subset of the GHZ-diagonal states, the GME concurrence calculated in [168] can be applied. Rewritten in \((x,y)\) coordinates, it is given by

\[
C_{\text{GME}}(\rho^{S}(x, y)) = 2 \max \left\{ 0, x + \frac{1}{2} \sqrt{3} y - \frac{3}{8} \right\}.
\tag{150}
\]

The permutation symmetry implies that the negativities are equal on all three bipartitions. The negativity for any bipartition is [68]

\[
\mathcal{N}(\rho^{S}(x, y)) = \max \left\{ 0, \frac{1}{8} - \frac{1}{2\sqrt{3}} y - |t| \right\},
\tag{151}
\]

see figure 4.

From this result, the concurrences for the three bipartite splits (which are, again, equal) can be easily determined from (60), which in this case, due to the smaller dimension \(d = 2\), is simply \( C_{A|BC}(\rho) \geq 2N_{A|BC}(\rho) \). On the other hand, for the GHZ states clearly \( C_{A|BC}(\rho) = 2N_{A|BC}(\rho) \), and certainly for the separable states the concurrence vanishes. Now by direct decomposition, it is clear that the concurrence cannot be larger than the linear interpolation between values of the separable states and the GHZ state. But that linear interpolation agrees with the lower bound given by \(2N_{A|BC}\), which therefore equals the concurrence.

### 7.4. Arbitrary three-qubit mixed states

There have been various attempts to obtain numerical estimates for the three-tangle (or the residual tangle) in arbitrary mixed states, e.g., [235–237], but also analytical approaches, e.g., [194, 205, 238].

The GHZ-symmetric states are useful in deriving a lower bound (in principle, analytical) of the three-tangle for mixed states [78, 239]. The procedure is as follows.

(i) Calculate the normal form with the algorithm in [56], remember the trace of the normal form and renormalize. If the normal form vanishes, the three-tangle does, too, and the procedure finishes.

(ii) Optimize the normal form obtained in step 1 over local unitary transformations in order to minimize the entanglement loss in the next step.

(iii) Project onto the set of GHZ-symmetric states by the twirling operation

\[
R_{\text{GHZ}}(\rho) = \int_{U: U^{\dagger}[\text{GHZ}]=\text{[GHZ]}} U \rho U^{\dagger} \, dU,
\tag{152}
\]

calculate the three-tangle of the projected state, and multiply with the trace from step 1 to obtain the lower bound.
The twirling operation can be described by the two simple equations
\[
x(\rho) = \frac{1}{2} (\rho_{000,111} + \rho_{111,000}) \quad (153a)
\]
\[
y(\rho) = \frac{1}{\sqrt{3}} (\rho_{000,000} + \rho_{111,111} - \frac{1}{4}) \quad (153b)
\]

Note that the same procedure can be used for any measure that is known on the GHZ-symmetric states, with the exception that the first step has to be omitted because it only works for measures that are invariant under local SL transformation, which for three qubits holds only for the three-tangle (note that for mixed states, even the residual tangle is not invariant under local SL operations).

Rodrigues et al. [237] have developed an algorithm to obtain an upper bound based on what they call the ‘best zero-E decomposition’, a generalisation of the best separable approximation [242] and best W approximation [66]. It is based on decomposing a mixed state into a pure state with nonzero measure and a mixed state with zero measure, and optimizing this decomposition.

For the GME-concurrence, a method to calculate the lower bound is described in section 6.3.1.

7.5. Optimal witnesses for three qubits

The knowledge of the exact properties of GHZ-symmetric states also allows one to explicitly derive optimal witnesses for different types of entanglement [68]. Due to the convex shape of the border between GHZ-type and W-type states, there exists a complete continuous family of optimal witnesses for GHZ-type entanglement, corresponding to the different points of the border. In particular, the well-known projection witness [66]
\[
\mathcal{W}_{\text{proj}} = \frac{3}{4} I - |\text{GHZ}\rangle \langle \text{GHZ}| \quad (154)
\]
turns out not to be an optimal witness, but can be improved to [68]
\[
\mathcal{W}_{\text{opt}} = \frac{3}{4} I - |\text{GHZ}\rangle \langle \text{GHZ}| - \frac{3}{7} |\text{GHZ}_-\rangle \langle \text{GHZ}_-|. \quad (155)
\]

Those witnesses can also be used as estimators for the actual three-tangle [78, 239]. For the witness (155) one obtains the analytical lower bound [239]
\[
\tau_{3,\text{approx}} = \max \left( 0, \left[ \pm \frac{9}{7} (\rho_{000,111} + \rho_{111,000}) + \frac{20}{7} (\rho_{000,000} + \rho_{111,111} - 3) \right] \right). \quad (156)
\]

\footnote{We mention that optimization over arbitrary local operations was used also to improve the detection of entangled states by witnesses, cf. [240, 241].}
8. Towards the general case

8.1. Maximal entanglement in multipartite systems

Among the many distinctive features of multipartite entanglement, it is worth highlighting the ambiguity of the concept of maximal entanglement. It was discussed, e.g., in [33, 163, 243–247]. Recall that for bipartite systems the unique (up to local unitaries) maximally entangled state is $\Psi_d$, cf (15). This is because any pure $d \times d$ bipartite state can be produced from $\Psi_d$ by means of SLOCC (including non-invertible operations). The situation is different for multipartite systems (with more than three parties): there are, e.g., many inequivalent types of genuine entanglement that cannot be converted into one another (see section 8.2).

The obvious requirement for maximal entanglement, in particular if one advocates the point of view of characterizing multipartite entanglement by local SL-invariant polynomials, is to require maximal mixedness of each local party (i.e., the state needs to be stochastic). This is exactly the condition to select the state in each orbit that maximizes all non-vanishing entanglement measures based on local SL-invariant polynomials [56]. This condition can, but need not, be complemented by additional requirements. Let us first check what we obtain for four qubits. There are three states that obviously fulfill that condition

$$|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$  \hspace{1cm} (157a)

$$|\text{Cl}_4\rangle = \frac{1}{2}(|0000\rangle + |0111\rangle + |1011\rangle + |1100\rangle)$$  \hspace{1cm} (157b)

$$|X_4\rangle = \frac{1}{\sqrt{6}}(|1111\rangle + |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$$  \hspace{1cm} (157c)

the GHZ state, the cluster state, and yet another state $X_4$ [58, 163, 194, 203]. These are the three ‘irreducibly balanced states’ for four qubits [245]. Clearly, they are not the only stochastic four-qubit states, however, we can immediately see several interesting facts. These states have the following Schmidt ranks across their two-qubit bipartitions: 2 (GHZ), 2 or 4 (cluster), 3 ($X_4$). This means they cannot be locally equivalent. Further, we see that all the reduced states are maximally mixed on their span. Hence, requiring this as an additional property [163, 194, 245] would yield precisely the states (157a)–(157c) (up to local unitaries).

One might think that, as an additional condition, one could require that all $k$-qubit reduced states after tracing out more than half of the parties (i.e., $N/2 \leq k \leq N - 1$) be maximally mixed. However, this is in general not possible for $N$-qubit states [243, 246].

8.2. Classifications of four-qubit states

Dür et al [33] pointed out that for multipartite systems with four or more qubits there are infinitely many SLOCC classes. That is, there are one or more continuous labels to specify one equivalence class. On the other hand, it is not generally true that a protocol can be run only with the states from exactly one SLOCC class. To illustrate this, assume a protocol that works with the four-qubit $W$ state

8 If one defines ‘maximal entanglement’ in the sense that all other states can be generated from them by means of LOCC (rather than SLOCC), then already for three qubits there are infinitely many maximally entangled states [187, 248].
It is conceivable that the protocol still works well if a small component $|1111\rangle$ is added to the state

$$|\phi_{\epsilon}\rangle = |W\rangle = \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle).$$  \hspace{1cm} (158)$$

However, the states are SLOCC inequivalent. Despite this, a quantifier of the resource for that protocol should measure both $\phi_0$ and $\phi_{\epsilon}$ with a nonzero value.

This gives rise to the idea that some coarse graining needs to be applied to the set of SLOCC classes, that is, SLOCC-inequivalent states get bunched together in families according to some (SLOCC-invariant) criterion (cf also section 2.1). Usually, this coarse-grained set of families is referred to as SLOCC classification. It also appears evident that a reasonable criterion for bunching classes will depend on the resource under consideration. Consequently, there is no absolutely preferable SLOCC classification. Rather, for a given resource one classification may be more adequate than another.

The first classification for four qubits was presented by Verstraete et al [249] and later elaborated on by Chiterental and Djokovic [250]. This approach essentially classifies the normal forms of four-qubit states. Further classifications (both for four qubits and also larger systems) were worked out by Miyake and Verstraete [251], Mandilara et al [252], Lamata et al [253, 254], Bastin et al [255, 256], Borsten et al [257], Li et al [258], Viehmann et al [231], Sharma and Sharma [213], and recently by Huber and de Vicente [259], Walter et al [260], and by Gour and Wallach [59].

Here we want to highlight and compare only two of the above results: the strikingly elegant classification for four-qubit symmetric states [255] and a corresponding polynomial classification [231]. There are various points that motivate this choice. First, the symmetric subspace of four qubits has far fewer parameters than the complete space, however, it displays many of the essential features that are added when moving on from the three to the four-qubit case: there is a single continuous label for most of the SLOCC classes, it contains distinct maximally entangled states, and there are two independent polynomial invariants that do not vanish on the symmetric space. Further it appears feasible to build a hierarchy also for the mixed states: one defines the hierarchy for pure states and then extends it, in the spirit of section 7.2 to the mixed states. Conversely, this could also provide a clear recipe for how to determine the family of a given mixed state. Note that it is an important practical requirement for an SLOCC classification that it be possible to determine the family of a given mixed state.

### Table 1. Comparison of the polynomial characterization and the SLOCC classification of symmetric four-qubit states [255]. The representatives are given in the basis of the symmetric four-qubit Dicke states $D^{(k)}_4$. For the continuous parameter in the $X$ family we have $\lambda^2 \neq 2/3$ and $h(\lambda) = 2 + \lambda^2$, $f(\lambda) = -8 + 4\lambda^2 - (102\lambda^4 + 5\lambda^6)/9$.  

| $\mathcal{D}_{(n)}$ | representative | $H$ | $F^{(4)}$ | type |
|--------------------|----------------|-----|-----------|------|
| $\mathcal{D}_4$   | $D^{(0)}_4$    | 0   | 0         | separable |
| $\mathcal{D}_{3,1}$ | $D^{(1)}_4$    | 0   | 0         | $W$     |
| $\mathcal{D}_{2,2}$ | $D^{(2)}_4$    | 1   | $-5/9$   | $D^{(2)}_4$ |
| $\mathcal{D}_{2,1,1}$ | $D^{(0)}_4 + D^{(2)}_4$ | 1   | $-5/9$   | $D^{(2)}_4$ |
| $\mathcal{D}_{1,1,1}$ | $|0000\rangle + |1111\rangle + iD^{(2)}_4$ | $h(\lambda)$ | $f(\lambda)$ | $X_4$ |

$|\phi_0\rangle = |W\rangle = \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$.  \hspace{1cm} (158)$$
So, altogether, with the symmetric four-qubit case we find ourselves on solid ground without debatable assumptions.

According to [255, 261] every symmetric $N$-qubit state can be written

$$\left| \psi_S \right\rangle = \nu \sum_{1 \leq j \neq \ldots \neq j_N \leq N} \left| \epsilon_{j_1} \ldots \epsilon_{j_N} \right\rangle$$

where $\nu$ is a normalization factor and $\epsilon_j$ denote one-qubit directions. Symmetric $N$-qubit states can be expanded into $N$-qubit Dicke states with $k$ excitations [262]

$$\left| D_N^{(k)} \right\rangle = \left( \frac{N!}{k!} \right) \frac{1}{k} \sum_{k} \mathcal{P}_k \left( \left| 1_1 1_2 \ldots 1_k 0_{k+1} \ldots 0_N \right\rangle \right)$$

where the sum is over the permutations $\mathcal{P}_k$ of the $N$ entries. It turns out that two symmetric states $\left| \psi_S \right\rangle = \nu \sum \left| \epsilon_{j_1} \ldots \epsilon_{j_N} \right\rangle$ and $\left| \psi_S' \right\rangle = \nu' \sum \left| \epsilon_{j_1}' \ldots \epsilon_{j_N}' \right\rangle$ belong to the same SLOCC class if and only if there exists a single invertible local operation that converts each state $\epsilon_j$ into $\epsilon_j'$. Thus, SLOCC equivalence of two states $\psi_S, \psi'_S$ can be excluded already if they have different degeneracy configuration $\mathcal{D}_{(n_j)}$, that is, if the multiplicities $n_j$ of the distinct directions $\epsilon_j$ in $\psi_S$ and $\psi'_S$ do not coincide.

For four qubits there are only five possible degeneracy configurations and, hence, five SLOCC families which consist of four single orbits and one continuous family, cf table 1. Note that here the arrangement of all SLOCC classes into five families occurs naturally because the degeneracy configuration is invariant under SLOCC.

On the other hand, we may check what the invariant polynomials of section 6.4.2 yield for the symmetric four-qubit states [231]. That is, since in table 1 all SLOCC classes are listed, we just add the values of the polynomials for each representative to the table. For symmetric four-qubit states, the polynomials of degree 4 are not independent $B_{1,4}^{(4)} = H^2$, therefore we need to consider only $H$ (degree 2) and the sextic invariant $F_1^{(4)}$. As expected, all polynomials vanish on the separable states and the $W$ states. For the orbits $\mathcal{D}_{2,2}$ (stable) and $\mathcal{D}_{2,1,1}$ (semistable) only $H \neq 0$ while for the family $\mathcal{D}_{2,1,1}$ both $H$ and $F_1^{(4)}$ do not vanish. We see that the resources of the GHZ state and the Dicke state $D_4^{(4)}$—according to the polynomial measures $H$ and $F_1^{(4)}$—are not strictly distinct. However, we may equally well find polynomials that measure only the GHZ state

$$\mu_{\text{GHZ}} = F_1^{(4)} + \frac{5}{9} H^3$$

or only the Dicke state

$$\mu_{\text{Dicke}} = F_1^{(4)} + H^3 \equiv 32 W$$

(for the last identity, cf (108)).

Hence, the polynomial classification admits a certain freedom for precise characterization of the resource. Note, however, that the local SL-invariant polynomials alone are not sufficient for a complete classification, in particular if one wants to include the mixed states. While for distinguishing the separable states $\mathcal{D}_4$ from the genuinely entangled families, the GME concurrence is appropriate, there is considerable fine structure in the state space (even $\mathcal{D}_{2,1,1}$ from $\mathcal{D}_{2,2}$ are not distinguished by the invariant polynomials). For this purpose, the covariants [208, 224, 263–265] possibly provide a solution. But this remains to be worked out in the future.
8.3. Monogamy

An interesting fundamental question is whether there are strict monogamy relations similar to (142) for systems of four and more parties. The Osborne–Verstraete monogamy inequality (86) suggests that there are generalizations in terms of quantities derived from local SL-invariant quantities for the subsystems. For example, for pure four-qubit states \( \psi \), and denoting the parties \( A, B, C, D \), one might guess for the difference between 1-tangle and squared concurrences for party \( A \)

\[
R_A = \tau_A - C_{AB}^2 - C_{AC}^2 - C_{AD}^2
\]

\[
= \tau_{3,ABC}^2 + \tau_{3,ABD}^2 + \tau_{3,ACD}^2 + \tau_{4}^2.
\] (164)

Indeed one finds states where a monogamy equality in this spirit holds [266]. For example, for a three-qubit state \( \phi^{(3)} \), we have the monogamy (142). If we now generate a four-qubit state \( \phi^{(4)} \) from \( \phi^{(3)} \) according to the rule (‘telescoping’)

\[
\left| \phi^{(3)} \right> = \sum_{jkl} \phi_{jkl}^{(3)} \left| jkl \right> \rightarrow \left| \phi^{(4)} \right> = \sum_{jkl} \phi_{jkl}^{(3)} \left| jkl \right>,
\] (165)

one finds that the three-tangle of \( \phi^{(3)} \) translates into a four-tangle that, not so surprisingly, equals one of the degree-4 invariants \( B_j^{(4)} \) from section 6.4.2, cf [266]. However, this relation is not valid for generic four-qubit states, and counterexamples to the assumption (164) can be constructed.

On the other hand, there is indeed a relation that holds for all four-qubit states, which is not of the form (164). To this end, define the global entanglement

\[
\tau_1(\psi) = \frac{1}{4} \left( \tau_A(\psi) + \tau_B(\psi) + \tau_C(\psi) + \tau_D(\psi) \right)
\] (166)

and the average pure-state linear entropy (in analogy with (81))

\[
\tau_2(\psi) = \frac{1}{3} \left( C_{AB}^2(\psi) + C_{AC}^2(\psi) + C_{AD}^2(\psi) \right).
\] (167)

Then one has [246]

\[
\tau_2(\psi) = \frac{1}{3} \left( 4\tau_1(\psi) - \left| H(\psi) \right|^2 \right)
\] (168)

with the degree-2 local SL invariant \( H \) from (104).

Hence, there are strong indications that general monogamy relations do exist, however, the precise rules and conditions remain an open question at this point.

9. Conclusion

We have reviewed the topic of quantifying single-copy entanglement resources of a few finite-dimensional parties. We have witnessed enormous progress in this field in recent years. While a decade ago, essentially the case of two qubits could be considered solved, to date the three-qubit problem appears tractable to a large extent. Also many aspects of \( d \times d \) bipartite systems have been understood at a quantitative level. In addition, considerable insight into more complex systems has been gained, in particular, regarding the case of four qubits. The general mathematical framework with tools from convex optimization and algebraic geometry has been identified and applied successfully. The application of the SLOCC paradigm and its
mathematical model, the representation of SLOCC by local invertible operations, i.e., the group $SL(d, \mathbb{C})$ were instrumental in these developments.

Let us conclude by specifying some of the open challenges for the near future. At the moment, there is still insufficient understanding of the relation between the entanglement measures (the resource characterization) and the corresponding entanglement families. Compared to the three-qubit case, the big step forward was to realize that GHZ entanglement is a resource that is not contained in all genuinely entangled three-qubit states, and that it is measured by the three-tangle. For four qubits we know—in principle—all possible measures and at least some of the interesting families, but we do not know how to precisely characterize the relation between them.

We believe that this problem can be solved only by gaining more insight into the structure of the space of mixed states, because then one possibly can decide which type of characterization is relevant, and which is not. This means, in turn, that better tools for the evaluation of convex-roof extended entanglement measures are required, possibly also more exact solutions for relevant examples of mixed states that contain distinct types of multipartite entanglement resources. Once a more thorough understanding of the entangled states is obtained, this needs to be complemented by an analogous characterization/classification of the states in the nullcone of the local $SL$-invariant polynomials. Finally, we consider it essential to achieve a much better understanding of Verstraete’s normal form and its relation to any kind of entanglement.

Acknowledgments

It is a pleasure to thank T Bastin, D Braun, R Lohmayer, A Osterloh, and O Viehmann for our fruitful collaboration and countless stimulating discussions. In particular, we would like to thank A Uhlmann for sharing his intuition and deep physical insight with us.

We gratefully acknowledge illuminating conversations with D Bruß, J-L Chen, Z-H Chen, J Eisert, O Gühne, P Horodecki, M Huber, P Hyllus, M Johansson, M Kleinnmann, B Kraus, L Lamata, P J Love, Z-H Ma, T Moroder, D-K Park, M B Plenio, N Schuch, C Schwemmer, E Solano, S Szalay, G Tóth, J de Vicente, M Wallquist, M M Wolf, R Zeier, and Z Zimboras.

This work was funded by the German Research Foundation within SPP 1386 (C.E.), by Basque Government grant IT-472-10 and MINECO grant FIS2012-36673-C03-01 (J.S.). Moreover, the authors thank J Fabian and K Richter for their invaluable support.

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