Once more on the $\theta$-vacua in $2+1$ dimensional QED and $3+1$ dimensional gluodynamics.

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Abstract

Two different but tightly connected problems, $U(1)$ and strong CP violation problems, are discussed in two different models which exhibit both asymptotic freedom and confinement. One of them is the 3d Polyakov’s model of compact QED and the other is 4d gluodynamics. It is shown that although both these models possess the long range interactions of the topological charges, only in the former case physics does not depend on $\theta$; while the latter exhibits an explicit $\theta$-dependence.

The crucial difference is due to the observation, that the pseudoparticles of 4d gluodynamics possess an additional quantum number, apart of the topological charge $Q$. 

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1. Introduction.

In the present letter I investigate some problems related to the \( \theta \) vacua in the \( SU(2) \) gauge theories.

As is known one can add to the standard Lagrangian the so called \( \theta \) term:

\[
\Delta L = \theta Q
\]

where \( Q \) is the topological charge which can be written for the 4d gluodynamics and for 2+1 QED correspondingly as follows:

\[
Q = \frac{1}{32\pi^2} \int d^4 x G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} = \frac{1}{32\pi^2} \int d^4 x \partial_\mu K_\mu, \quad a = 1, 2, 3, \quad \mu, \nu = 1, 2, 3, 4.
\]  

\[
Q = \frac{1}{8\pi} \int d^3 x \epsilon_{ijk} \partial_i (G^a_{jk} \phi^a) = \frac{1}{4\pi} \int B_k dS_k, \quad i, j, k = 1, 2, 3.
\]

Here \( G^a_{\mu\nu} \) is the field strength tensor and \( \phi^a \) is the scalar field which is present in the definition of the instanton solution in the 2+1 theory (a monopole, in the terminology of the 3+1 theories) \[1, 2\]. In the unitary gauge \( \phi^a \) becomes constant \( \phi^a \rightarrow (0, 0, 1) \) at large distances and the combination \( \epsilon_{ijk} G^a_{jk} \phi^a \) reduces to \( B_i \). This is just the reason why this theory is called the 2+1 QED \[3\].

As is known, the \( \theta \) term preserves renormalizability of the theory but is \( P \) and \( T \) odd. As it can be seen from (2,3) the \( \theta \) term is a full divergence, therefore it is equivalent to certain boundary conditions.

The standard line of reasoning in this case is as follows. Because the boundary conditions can influence to the physics far from the boundary in the ordered phase only, and because confinement occurs in a disordered phase, one can think that the all \( \theta \) vacua are equivalent. If so, the nonvanishing surface integral (2,3) due to a separate topological solution could be an artifact of the dilute gas approximation which ignores completely the most profound features of the considering models - confinement. Indeed the surface integral does not vanish since the topological field falls off too slowly at large distances due to the presence of massless particles in the original Lagrangian, which is not present, however, in the physical spectrum. Therefore one might hope that the confinement effects would make the surface integral vanishes. Such a viewpoint based on the analysis 2+1 QED \[4\] (see also the recent paper \[5\] on this topic), was strongly advocated by Polyakov.
Indeed, Vergeles has shown \[4\] that as soon as quasiparticles interaction becomes strong enough, the $\theta$ dependence disappear from the effective long distance Lagrangian and physics becomes $\theta$ independent. We will shortly reproduce this result in the next section, and emphasize generality of this result and its independence on the space dimensions.

At the same time, as is known \[6\],\[7\],\[8\] the $\theta$-dependence of physics is linked to the $U(1)$ problem. If we believe that the resolution of the $U(1)$ problem appears within the framework of the papers \[8\],\[7\], we must assume that the correlator

$$K = i \int d^4x \langle 0 | T \left( \frac{1}{32\pi^2} G^a_{\mu\nu}(x) \bar{G}^a_{\mu\nu}(x) \right) \frac{1}{32\pi^2} G^a_{\mu\nu}(0) \right) 0 \rangle \quad (4)$$

is nonzero. It means that the vacuum expectation value (vev) of the topological density

$$\langle | \frac{1}{32\pi^2} G^a_{\mu\nu} \bar{G}^a_{\mu\nu} | \rangle = \frac{1}{2} K \theta, \quad \theta \ll 1. \quad (5)$$

is nonzero too. But as was shown in ref.\[8\] the nonzero vev $\langle G \bar{G} \rangle \neq 0$ does imply the CP-violation in physical transition and leads, for example, to the mixing of the heavy quarkonium levels with $J^P = 0^+$ and $J^P = 0^-$ in terms of $\langle G \bar{G} \rangle$. Only dispersion relations are used to translate $\langle G \bar{G} \rangle \neq 0$ into a proof of CP-violation in physical effects.

So, if we believe that $U(1)$ problem is solved in the framework of Witten-Veneziano approach \[8\],\[7\] we automatically get $K \neq 0 \quad (4)$ and therefore, the nontrivial $\theta$ dependence.

With introduction of light quarks (u,d,s) the value of correlator $K(\mathbb{I})$ changes \[8\] but $\theta$ is still experimentally observable quantity. Moreover, a few strict results (computation of $\eta \rightarrow \pi \pi$ decay \[8\] and electrical dipole moment \[8\]) were obtained using the soft meson technique.

I will not consider in this paper the theory of light quarks and confine myself by consideration of pure gluodynamics. I will discuss the resolution of the aforementioned apparent paradox (the coexistence of the strong quasiparticle interaction with the nontrivial $\theta$ dependence) within framework of the dynamical toron approach which was discussed early in context of different field theories, see

\[2\]It is obviously that $K \sim m_q^4$ irrespectively to the number of light flavours \[8\].
ref.\[10\] and references therein. I would like to recall that in all known cases, the toron calculations give, at least, selfconsistent results. Thus, one may expect that apparent puzzle should be solved in an automatic way.

We discuss the statistical ensemble of quasiparticles which, presumably \[11\], describes the grand partition function of the 4d YM theory and which possesses long distances strong interaction. It will be shown that this ensemble describes the system with nontrivial \(\theta\)- dependence irrespectively to the strength of quasiparticle interaction . Crucial point, in compare with analogous calculation of ref.\[4\] in Polyakov’s model, is a very nontrivial algebraic structure of the quasiparticle interaction.

Let us now recall some basic facts about the toron’s role in two and four dimensional physics. Besides that we give some arguments why quasiparticles with fractional topological charge should be considered in those cases and what physical effects arise due to fluctuations with fractional \(Q\).

The self dual ”toron” solution with fractional topological charge was first considered by ’t Hooft\[12\] (but in quite different context). It is defined in a box of size \(L_{\mu}\), smeared over this box and exists when the sizes \(L_{\mu}\) satisfy certain relations. The calculation of gluino condensate \(<\lambda^2>\) in the supersymmetric YM theory (SYM) based on the ’t Hooft solution was carried out in ref.\[13\]. However by many reasons (in particular, difficulties with introduction of fields in the fundamental representation and consideration of the ensemble interacting quasiparticles) the ’t Hooft solution \[12\] and corresponding calculation \[13, 14\] can be considered as only illustrative example with fractional topological charge.

Nevertheless, I believe that solutions with a fractional \(Q\) may play an important role in theory, but these solutions should be formulated in another way (see\[10\] and references therein). The corresponding pseudoparticle can be understood as a point defect when the regularization parameter (which is present in the definition of the solution) goes to zero. We keep the term ’toron’ introduced in ref.\[12\]. By this we emphasize the fact that the new solution also minimizes the action and carries the topological charge \(Q = 1/2\), so it possesses all the characterestics ascribed to the standard toron\[12\].

All calculations, based on the solution \[10]\ demonstrate its very nontrivial role for different field
theories. Most glaringly these effects appear in supersymmetric variants of a theory. In particular, in the supersymmetric $CP^{N-1}$-theories, the torons (point defects) can ensure a nonvanishing value for the $\langle \bar{\psi} \psi \rangle \sim \exp(2i\pi k/N + i\theta/N)$ with right $\theta$-dependence. Such behavior is in agreement with the value of the Witten index which equals $N$\cite{15} and in agreement with the large $N$-expansion \cite{16}. In analogous way, the chiral condensates can be obtained for 4d theories: supersymmetric YM (SYM), \cite{13}, supersymmetric QCD (SQCD). In these cases a lot of various results are known from independent consideration (such as the dependence of condensates on parameters $m, g$; the Konishi anomaly equation and so on...). \cite{15}. Toron approach is in agreement with these general results. The same approach can be used for physically interesting theory of QCD with $N_f = N_c$. In this case an analogous calculation of $\langle \bar{\psi} \psi \rangle$ does possible because of cancellation of nonzero modes, like in supersymmetric theories. For this theory the contribution of the toron configurations to the chiral condensate has been calculated and is equal to: $\langle \bar{\psi} \psi \rangle = -\pi^2 \exp(5/12) 2^4 4^3 \Lambda^3$ \cite{10} (see also E. Cohen \cite{17}). As is well known in any consistent mechanism for chiral breaking a lot of problems, such as: the $U(1)$-problem, the number of discrete vacuum states, the $\theta$-puzzle, low energy theorems and so on, must be solved in an automatic way. We have checked that all these properties \cite{10} are consistent with the toron calculation.

To give some insight about the algebraic structure of the quasiparticle interaction, I would like to cite some results for $2dCP^{N-1}$ model which is known \cite{13} possesses nontrivial $\theta$-dependence. It turns out that the toron gas contribution to grand partition function in $2dCP^{N-1}$ model reduces to the classical Coulomb system(CCS) \cite{11}:

$$Z = \sum_{k=0}^{\infty} \frac{\lambda^{k_1 + k_2}}{(k_1)!(k_2)!} \sum_{\mu_\alpha, q_\alpha} \prod_{i=1}^{k_1 + k_2} d^2 x_i \exp(-\epsilon_{int.}),$$

$$\epsilon_{int.} = -4 \sum_{i > j} q_i \bar{\mu}_i \ln(x_i - x_j)^2 q_j \bar{\mu}_j + 2 \ln L^2 (\sum_i q_i \bar{\mu}_i)^2,$$

$$\lambda = c \frac{M_0}{f(M_0)} \exp(-\frac{\pi}{f(M_0)}),$$

where $N$ different kinds of torons classified by the weight $\bar{\mu}_\alpha$ of fundamental representation of the $SU(N)$ group and $q_i$ is the sign of the topological charge. Besides that, in formula (6) the value $f(M_0)$ is the bare coupling constant and $M_0$ is ultraviolet regularization, so that in eq.(6) there
appears the renormalization invariant combination $\lambda$. As can be seen from (6) the configurations only satisfying the neutrality requirement
\[ \sum_i q_i \vec{\mu}_i = 0 \] (7)
are essential in thermodynamic limit $L \to \infty$. However we will consider the system in the box with size $L$ and so we keep this term for the future analysis.

Using the correspondence between CCS and the Toda field theory
\[ Z_\theta = \int D\vec{\phi} \exp(-\int d^2x L_{\text{eff}}), \]
\[ L_{\text{eff}} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \sum_{\tilde{\mu}} \lambda^2 \exp(i4\sqrt{\pi} \tilde{\mu}_\alpha \vec{\phi} + i\theta/N) - \sum_{\tilde{\mu}} \lambda^2 \exp(-i4\sqrt{\pi} \tilde{\mu}_\alpha \vec{\phi} - i\theta/N). \]
the expectations of different values (the vacuum energy, the topological density, the Wilson line ...) was calculated. All results (confinement, right dependence on $\theta/N$ and so on) are precisely what one obtains from the large $N$ expansion. In this effective field theory, $\vec{\phi}$ represents the $N-1$ component scalar potential, the sum over $\tilde{\mu}_\alpha$ in (8) is over the $N$ weights of the fundamental representation of $SU(N)$ group. Note, that the first interaction term is related to torons and the second one to antitorons. Besides that, since we wish to discuss the $\theta$ dependence, we also include a term proportional to the topological charge density $\frac{\theta}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}$ to the starting lagrangian (formula, analogous to eqs.(1, 2, 3)), and corresponding track from this to the effective lagrangian(8).

The most important result from ref.[11] is the nontrivial dependence on $\theta$ of the topological density and susceptibility, the values which are relevant for the solution of the $U(1)$ problem:
\[ < \frac{\epsilon_{\mu\nu} F_{\mu\nu}}{4\pi} >_\theta \sim \sin(\frac{\theta}{N}). \]
\[ \int d^2x < \frac{\epsilon_{\mu\nu} F_{\mu\nu}}{4\pi}(x), \frac{\epsilon_{\mu\nu} F_{\mu\nu}}{4\pi}(0) >_\theta \sim \frac{1}{N} \cos(\frac{\theta}{N}). \]
Here $\frac{F_{\mu\nu}\epsilon_{\mu\nu}}{4\pi}$ is the topological density in 2d$CP^{N-1}$ model. The vacuum characteristics listed above are analogous to eqs.[4][3] in 4d YM theory.

The reason to have the nontrivial $\theta$-dependence (444) as well as the strong quasiparticle interaction $\sim \ln(x_i-x_j)^2$ (3) in this $2dCP^{N-1}$ model is the presence of the nontrivial algebraic structure $\sim q_i \tilde{\mu}_i q_j \tilde{\mu}_j$. 
in the expression for \( \epsilon_{\text{int}} \) \(^3\). Just this fact was crucial in the analysis of the 2d\( CP^{N-1} \) model. As we observed early, \(^\text{11}\) the same structure for quasiparticle interaction take place in the 4d YM theory in the striking contrast with 2+1 Polyakov’s model \(^\text{3}\), where the interaction energy proportional to the topological (magnetic) charges \(^\text{3}\) of quasiparticles only:

\[
\epsilon_{\text{int}} \sim \frac{q_i q_j}{|x_i - x_j|}. \tag{11}
\]

Just this difference leads, as will be shown below, to the existence of the nontrivial \( \theta \) dependence in 4d YM theory in spite of the fact of the strong quasiparticle interaction \( \sim \ln(x_i - x_j)^2 \).

2. Comparative analysis of the 2+1d QED and 3+1d gluodynamics.

We start from the short review of the results of ref.\(^\text{4}\) to demonstrate the independence of \( \theta \) the partition function in Polyakov’s model \(^\text{3}\) . To do this let us consider the contribution to the partition function of the configurations with fixed numbers of \( n_1 \) monopoles and \( n_2 \) antimonopoles. As has been shown by Polyakov \(^\text{3}\) the corresponding contribution reduces to the following expression:

\[
Z_{\theta}^{NQ} \equiv \exp(iQ\theta) \exp(-F_{NQ}) = \exp(iQ\theta) \frac{1}{n_1!n_2!} \prod_{i=1}^{N} d^3x_i \lambda^N \exp\left(-\sum_{i \neq j}^{N} \frac{q_i q_j}{|x_i - x_j|}\right) \tag{12}
\]

\[Q = n_1 - n_2, \quad N = n_1 + n_2, \quad q_i = \pm 1, \quad i = 1, N\]

Here, \( \lambda \) is fugacity of the obtained Coulomb gas and it is determined by the ultraviolet behaviour of the theory.

If the Coulomb interaction could be ignored, then \(^\text{12}\) yields for free energy \( F_\theta \) the \( \theta \)-dependent expression. This is a well known result obtaining in the framework of dilute noninteracting gas approximation.

In our case the partition function with a fixed topological charge

\[
Z_{\theta}^Q \equiv \exp(iQ\theta) \exp(-F_Q) = \sum_{N} Z_{\theta}^{NQ} \rightarrow \int dN Z_{\theta}^{NQ} \tag{13}
\]

can be obtained by the steepest descent method with respect \( N \). With \( L \), the linear size of the system, going to infinity, we get the following expression for the noninteracting case:

\[
\exp(-F_Q) = \int dN \exp\left[N \ln(L^3 \lambda) - \frac{N + Q}{2} \ln \left( \frac{N + Q}{2} \right) - \frac{N - Q}{2} \ln \left( \frac{N - Q}{2} \right) - 1 \right] \sim \tag{14}
\]

\[7\]
\[ \exp(N_0) \exp\left(-\frac{Q^2}{N_0}\right), \]

where we have used formula \( \ln(k!) \sim k \ln k - k \) and \( N_0 \) is to be determined from the equation:

\[
\ln(L^3 \lambda) = \frac{1}{2} \ln\left(\frac{N_0^2 - Q^2}{4}\right), \quad \frac{Q}{N} \ll 1, \quad N_0 \simeq 2L^3 \lambda. \tag{15}
\]

Now the partition function for the dilute noninteracting gas is evaluated in the following way:

\[
Z_\theta \equiv \exp(-F_\theta) \sim \exp(2L^3 \lambda) \int dQ \exp(i\theta Q) \exp\left(-\frac{Q^2}{4\lambda L^3}\right) \sim \exp[2\lambda L^3 (1 - \frac{\theta^2}{2})]\tag{16}
\]

\[
F_\theta \simeq -2\lambda L^3 (1 - \frac{\theta^2}{2}), \quad \theta \ll 1
\]

As it can be seen from (16) the essential terms are those which have the charges

\[
|Q| \sim \lambda L^3 \theta, \quad \frac{|Q|}{N} \sim \theta \tag{17}
\]

and our approximation \( Q/N \rightarrow 0 \) can be justified in the limit \( \theta \rightarrow 0 \). So, as expected, the free energy \( F_\theta \) does depend on \( \theta \) explicitly.

Now we return to the Polyakov’s model. It is known that the Coulomb interaction in plasma cannot be ignored. Hence, formula (16) is not true. To give some estimation in this case let us place our system to the box with size \( L \). In this case, the excessive charge deposits on the wall of the box and the free energy is the same as for neutral Debye plasma plus the Coulomb energy [4]:

\[
\exp(-F_Q) \sim \int dN \exp[N \ ln(L^3 \lambda) - \frac{N + Q}{2} (\ln \frac{N + Q}{2} - 1) - \frac{N - Q}{2} (\ln \frac{N - Q}{2} - 1) - \frac{Q^2}{L}] \tag{18}
\]

Now the partition function with a fixed charge can be obtained by the same way as before (14). The estimates of \( F_\theta \) is now less trivial:

\[
Z_\theta \equiv \exp(-F_\theta) \sim \exp(2L^3 \lambda) \int dQ \exp(i\theta Q) \exp\left(-\frac{Q^2}{4\lambda L^3}\right) \exp\left(-\frac{Q^2}{L}\right) \tag{19}
\]

Here the new factor \( \exp\left(-\frac{Q^2}{L}\right) \) is due to the Coulomb interaction, and now the essential configurations have charges:

\[
|Q| \sim L\theta, \quad \frac{Q}{N} \sim \frac{\theta}{\lambda L^2} \rightarrow 0 \tag{20}
\]
and therefore:

$$F_\theta \sim -2\lambda L^3 + L\theta^2 = -2\lambda L^3 (1 - \frac{\theta^2}{2\lambda L^2}).$$

(21)

Thus, at $L \to \infty$ the free energy $F_\theta$ in the Polyakov’s model does not depend on $\theta$ and coincides with the corresponding expression at $\theta = 0$. Let me emphasize that this result is due to the strong interaction of quasiparticles.

At first sight this derivation looks rather general. Moreover, one can suspect that as soon as we have a strong enough interaction and corresponding trace of it in the formula analogous to (19), we will obtain the expression for the free energy, like (21), which does not exhibit any $\theta$ dependence at $L \to \infty$. This is indeed the case for the simplest algebraic structure $\sim q_i q_j$ for the quasiparticle interaction. Moreover, the arguments, given above are in perfect agreement with intuitive picture (discussed in Introduction) connecting $\theta$ independence of the effective lagrangian and confinement.

Let us now proceed to the analysis of 4d gluodynamics. In this case, like in $2d \mathbb{C}P^1$ model, a toron classified by two numbers: the sign of the topological charge $q_i$ and the isotopic projection $I_i$:

$$|q_i, I_i >$$

(22)

The details are in the original paper [11], and I would like only mention here the definition of the toron isotopic projection.

Let us introduce along with papers [13] an additional object into the theory, so called measuring operator $C$ such that

$$D_\mu C(x) = 0$$

(23)

and consider the integral

$$I = \int tr(C F_{\mu\nu}) d\sigma_{\mu\nu}$$

(24)

where the toron lies in a plane $\sigma_{\mu\nu}$. In this case the different choice of $C$ obeing eq.(23) yeilds different isospin directions. It is useful to keep in mind some analogy with monopole’s classification. In this case the role of $C$ plays a Higgs field [20], which far from the core satisfies the same equation (23). Let
us note, that an each quasiparticle is classified by $I_3$ proection and so has a nontrivial transformation properties under the $SU(2)$ gauge group. However due to the neutrality condition in the infinite volume limit (8), the vacuum is the singlet state under the gauge transformation. Moreover, the answer does not depend on the choice of the $z$ axis (the axis of quantization) when the sum over all possible isospins for all quasiparticles will be done.

With this in mind let us consider in more details the grand partition function for the 4d YM theory [11] which looks like (6) with some trivial changes:

$$\lambda_{CPN-1} \rightarrow \lambda_{YM} \sim \left(\frac{M_0^{11/3}}{g^4(M_0)}\exp\left(-\frac{4\pi^2}{g^2(M_0)}\right)\right)^{\frac{2}{\pi}}$$

$$d^2x \rightarrow d^4x, \quad -4\ln(x_i - x_j)^2 \rightarrow -\frac{2}{3}\ln(x_i - x_j)^2, \quad -4\ln L \rightarrow -\frac{2}{3}\ln L.$$  \hspace{1cm} (25)

The last replacements in (25) is the direct consequence of renorminvariance of the theory [11]. Now let us examine the contribution to the partition function of the terms with fixed numbers of $n_1(m_1)$ torons (antitorons) with isospin $up$ and $n_2(m_2)$ torons (antitorons) with isospin $down$. In this case the integration over $N$ can be done as before by the steepest descent method with respect $N$ (let us note that the term related with interaction does not depend on $N$) and so the formula analogous to eq.(19) for the free energy $F_\theta$ in 4d gluodynamics looks as follows:

$$Z_\theta \equiv \exp(-F_\theta) \sim \exp(2L^4\lambda) \int dQ \exp\left(\frac{\theta}{2} Q\right) \exp\left(-\frac{Q^2}{4AL^4}\right) \int d(Q_1 - Q_2) \exp\left(-\frac{2}{3}(Q_1 - Q_2)^2\ln L\right)$$

$$Q \equiv (Q_1 + Q_2), \quad n_1 + n_2 = 1/2(N + Q_1 + Q_2), \quad m_1 + m_2 = 1/2(N - Q_1 - Q_2)$$

(26)

Here the $\frac{1}{2}Q_1 = \frac{1}{2}(n_1 - m_1)$ is the topological charge carried by the quasiparticles with isospins $up$ and $\frac{1}{2}Q_2 = \frac{1}{2}(n_2 - m_2)$ is the same for quasiparticles with isospins $down$. While obtaining (26) we took into account the form of the last term in eq.(8) (with corresponding replacement (25) for transition from $CP^{N-1}$ model to 4d gluodynamics):

$$\left(\sum_i \mu_i q_i\right)^2 \ln L \rightarrow (n_1 - m_1 - n_2 + m_2)^2 \ln L = (Q_1 - Q_2)^2 \ln L.$$ \hspace{1cm} (27)

The trace of this interaction is the appearance of the last term in eq.(26) analogous to the corresponding contribution $\sim \exp(-\frac{Q^2}{4})$ in eq.(19) for 2+1 QED. Besides that, we have the factor $(i\frac{\theta}{2})$ in eq.(26)
instead of \((i\theta)\) in eqs.\((16,19)\): it is the direct consequence of the fractional value for the toron topological charge equals one half.

Now we see the crucial difference between the expressions for free energy in the 4d gluodynamics \((26)\) and 2+1d QED \((19)\). If the topological quasiparticles were classified by topological charge only (it would corresponds \(Q_2 = 0\) in the formula \((26)\)) we would obtain the \(\theta\) independent expression for free energy, just as it has happened in 2+1d QED, see eq.\((21)\).

But fortunately, we have less trivial expression for 4d gluodynamics:

The last term does not depend on combination \(Q_1 + Q_2\) and is factorized out. The nontrivial on \(\theta\) integral over \(d(Q_1 + Q_2)\) is reduced to the noninteraction gas case \((16)\) and does depend on \(\theta\) explicitly at \(L \to \infty\):

\[
F_\theta \simeq -2\lambda L^4(1 - \frac{1}{2}(\frac{\theta}{2})^2), \quad \theta \ll 1
\]

(28)

Let me emphasize that this result has been obtained due to the relevant new quantum number classifying the quasiparticle. Besides that I interpret this result as the expansion of

\[
F_\theta \sim -|\cos(\frac{\theta}{2})|
\]

(29)

obtained in \(\text{[11]}\) by quite different method. As discussed in that paper the reason for such \(\theta\) dependence is existence of the two vacuum solutions (for \(SU(2)\) group) minimizing vacuum energy. Although each of the separate solutions has a \(\theta\) period of \(4\pi\), the overall minimum has a \(\theta\) period of \(2\pi\) because the solutions jump from one value to another at \(\theta = \pi\). Very important is that two solutions have been prepared for such jumps from the very beginning. Indeed, as soon as we allowed one half topological charge, the number of the classical vacuum states is increased by the same factor two in compare with a standard classification, counting only integer winding numbers \(|n\rangle\).

Of course, vacuum transitions eliminate this degeneracy. However the trace of enlargement number of the classical vacuum states does not dissapear. Vacuum states now classified by two numbers: \(0 \leq \theta < 2\pi\) and \(k = 0,1\). Let me repeat that origin for this is our main assumption that fractional
charge is admitted and therefore the number of classical vacuum states is multiplied by a factor two.

I have to note that the same situation takes place in the supersymmetric YM theory, but in this case the vacuum states are still degenerate after vacuum transitions. The number $k$ in this case just numerates different vacua at the same $\theta$. However, the gauge classification for the winding vacua before perturbation for both models (supersymmetric and nonsupersymmetric one) is the same.

We close this discussion by a remark that the analogous $\theta/N$ dependence was discovered in gluodynamics at large $N$ \cite{6}, \cite{7}, \cite{21}. In these papers was argued that the vacuum energy at large $N$ appears in the form $E \sim E(\theta/N)$. Such a function can be periodic in $\theta$ with period $2\pi$ only if there are many vacuum states for given values of $\theta$. Indeed, gluodynamics can be understood as a QCD with very large quark’s mass. In this limit effective lagrangian can be founded \cite{21} and it turns out that the number of vacua is of order $N$ at $N \to \infty$. This fact actually is coded in the effective lagrangian containing the multi-branched logarithm $\log \det(U)$. In the Veneziano approach \cite{7} the same fact can be seen from the formula for multiple derivation of the topological density $Q$ with respect to $\theta$ at $\theta = 0$.

$$\frac{\partial^{2n-1}}{\partial \theta^{2n-1}} <Q(x)> \sim \left(\frac{1}{N}\right)^{2n-1}, n = 1, 2... \quad (30)$$

So, eq. (29) definitely does not contradict to results for large $N$ expansion.

It is instructive to understand this result in terms of the effective field description. Just like in $2dCP^{N-1}$ model, the grand partition function for toron gas in 4d gluodynamics can be re-expressed in terms of effective field theory very similar to eq. (3). The difference with $2dCP^{N-1}$ model only in the kinetic term, which has the standard form $\sim (\partial_{\mu} \phi)^2$ in $2dCP^{N-1}$ model and looks more complicated in 4d gluodynamics $\sim (\Box \phi)^2$ \cite{11}. This difference, however, does not influence on the arguments given bellow. Only the interaction part of the effective lagrangian, depending on $\theta$, is relevant for our analysis.

For Polyakov’s model the effective lagrangian has the same Sine-Gordon form \cite{8}, but with a crucial difference. Namely, the grand partition function \cite{12} for 2+1d QED can be re-expressed in
terms of a field theory with lagrangian

\[ L_{\text{eff}} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \lambda \cos(\phi + \theta) \]  

(31)

but without any additional summing over \( \mu_\alpha \). After shift \( \phi \to \phi - \theta \), the \( \theta \) dependence in the \( L_{\text{eff}} \) disappear in according with our formula (21).\(^3\)

In contrast with it the situation in the \( 2dCP^{N-1} \) model and 4d gluodynamics is quite different because in these cases the \( \theta \) parameter can not be removed from effective lagrangian (8) in agreement with formula (28).

3. Final remarks.

The main point of this Letter is that the dynamical resolution of the \( U(1) \) problem in gluodynamics looks very naturally in terms of ensemble of the quasiparticles with the fractional topological charges (torons), which have a nontrivial long range interactions. This approach is in a perfect agreement with Witten-Veneziano solution of the \( U(1) \) problem in framework of large N-expansion [6], [7]. Besides that this approach demonstrates a self-consistency of the different calculation in a various field theories [10], where the results are known beforehand.

We have discovered that the algebraic structure of the quasiparticle interaction is just what is needed for the solution of \( U(1) \) problem irrespectively to the strength of interaction. This result in gluodynamics is in a striking contrast with that for the 2+1 QED, where strong quasiparticle interaction eliminates the \( \theta \) dependence in the theory.

The other point which I would like to mention here is as follows. Although we have discussed the dynamical solution of the \( U(1) \) problem within the toron framework, the obtaining results have a more general origin. Indeed, we have shown that the nontrivial \( \theta \) dependence does appear in the effective lagrangian \(^4\) in spite of the fact of strong quasiparticle interaction, provided the quasiparticles are

\(^3\)Let us note, that in 4d gluodynamics this summing reproduces in the effective Lagrangian the well known \( Z_N \) Weyl symmetry of the original YM theory.

\(^4\)This is an essential requirement for the solution of the \( U(1) \) problem
classified by the new quantum number, the weight of representation of the group.

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References

[1] A.M.Polyakov, JETP.Lett. 20, 194 (1988).
[2] G’t Hooft, Nucl.Phys. B79, 276 (1974).
[3] A.Polyakov, Phys.Lett. B59, 82 (1975); Nucl.Phys.B120, 428 (1977).
[4] S.Vergeles, Nucl.Phys. B152, 330 (1979).
[5] S.Samuel, Preprint CCNY-HEP-91/5, May 1991.
[6] E.Witten, Nucl.Phys. B156, 269 (1979).
[7] G.Veneziano, Nucl.Phys. B159, 213 (1979).
[8] M.Shifman, A.Vainshtein and V.Zakharov, Nucl.Phys. B166, 493 (1980).
[9] R.Crewther et al. Phys.Lett. B88, 123 (1979).
[10] A.R.Zhitnitsky, Nucl.Phys. B340, 56 (1990).
[11] A.R.Zhitnitsky, Nucl.Phys. B374, 183 (1992).
[12] G’t Hooft, Commun.Math.Phys. 81 (1981) 267.
[13] E. Cohen and C. Gomez, Phys.Rev.Lett. 52 (1984) 237.
[14] C. Gomez, Phys.Lett.B 141 (1984),366.

[15] E.Witten, Nucl.Phys.B202,253 (1982).

[16] A.D’Adda,P.Di Vecchia and M.Luscher, Nucl.Phys.B14,63 (1978).

   E.Cremer and J.Scherk, Phys.Lett.B74,341 (1978).

   H.Eichenherr, Nucl.Phys.B146,215 (1978).

   E.Witten, Nucl.Phys.B149,285 (1979).

[17], E.Cohen, 1984, communication from referee.

[18] D.Amati et al., Phys.Rep.162,169(1988).

[19] H.G.Loos,Ann. Phys. (N.Y.)36 (1966),486.

   Z.F.Ezawa and H.C. Tze,Nucl.Phys. 100 (1975),1.

[20] F.Englert and P. Windey,Phys.Rev. D14(1976),2728;

   P.Goddard, J.Nuyts and D.Olive, Nucl.Phys. B125 (1977),1.

[21] E.Witten,Ann.Phys.(N.Y.)128,(1980),363.