Triad Formulations of Canonical Gravity without a fixed reference frame

Joachim Schirmer

Fakultät für Physik der Universität Freiburg
Hermann-Herder-Str. 3, 79104 Freiburg i.Br. / FRG
e-mail: schirm@phyq1.physik.uni-freiburg.de
FR-THEP 95/8 March 1995

Abstract

One can simplify the triad formulations of canonical gravity by abandoning any relation to a fixed coordinate system. That means in case of the ADM formalism that one can determine the momentum by direct derivation of the Lagrange-3-form w.r.t. the time-derivative of the triad-1-forms, thus the momentum is most naturally a 2-form. We apply this concept to the Palatini formulation where we can closely follow Dirac’s concept to find and eliminate the second class constraints. Following the same way for the Ashtekar theory it will turn out to be equivalent to two successive canonical transformations where the first makes explicit use of the spatial dimension being 3 and the second is usually hidden in the use of densities. At the end we can give a simple version of the reality constraints.

1 Technical Preliminaries

Avoiding any coordinate system will eliminate all determinants from the theory, but the associated problems will be contained in the frequently used Hodge operator. Yet the algebra of this operator is simple because our triads are normalized. To have an effective way of handling this operator we first introduce some notations and formulas where we mainly follow the treatment given in [1]. Let \((M, g)\) be a \(m\)-dimensional pseudo-Riemannian manifold. We first define the interior multiplication \(i\) of two forms of different degree.

For \(q \leq p\) let the bilinear mapping \(i : \Omega^q(M) \times \Omega^p(M) \rightarrow \Omega^{p-q}(M); (\mu, \nu) \mapsto i_{\mu,\nu}\) have the following properties:

\[
i_{\mu,\nu} = g^\sharp(\mu, \nu) = \mu_\alpha \nu_\beta g^{\alpha\beta} \quad \text{fr } p = q = 1 \tag{1.1}
\]

\[
i_{\mu}(\nu_1 \wedge \nu_2) = \mu_\alpha \nu_1 \wedge \nu_2 + (-1)^{\beta_1} \nu_1 \wedge \mu_\beta \nu_2 \quad \text{fr } \nu_i \in \Omega^{\alpha_i}(M), \mu \in \Omega^1(M) \tag{1.2}
\]

\[
i_{(\mu_1 \wedge \mu_2)} = i_{\mu_2} \circ i_{\mu_1} \tag{1.3}
\]
These properties define $i$ uniquely. Let $\{a_a\}_{a=1...m}$ be a local basis of the tangential bundle and $\{a^a\}$ be the dual basis. We will use the following abbreviation for the basis of $\Omega^p(M)$:

$$ a^{a_1...a_p} := a^{a_1} \wedge ... \wedge a^{a_p} \quad (1.4) $$

We will not distinguish in the notation between $a_a \in \Gamma_{loc}(TM)$ and $a^a = g_{ab}a^b \in \Omega^1_{loc}(M)$, so usually we will omit the $\flat$-sign. Consequently we do not distinguish between the interior product with a 1-form and the natural injection of the metric dual vector of this 1-form. This identification is possible unless variations or derivations get involved. Then one has to keep in mind if for the definition the metric was used.

In a local cobasis the interior product takes the following form:

$$ i_{\mu \nu} = \frac{1}{q!(p-q)!} \mu^{i_1...i_q} \nu_{i_1...i_qj_{p-q}} a^{j_1...j_{p-q}} \quad (1.5) $$

which is proved in a simple, but tedious calculation. The scalar product of two $p$-forms is now given by:

$$ \langle \mu | \nu \rangle := g^q(\mu, \nu) := i_{\mu \nu} = i_{\nu \mu} \quad \mu, \nu \in \Omega^p(M) \quad (1.6) $$

We notice that the interior product $i_{\mu \nu}$ is the dual mapping of the exterior multiplication $\mu \wedge$ with respect to the scalar product $\langle | \rangle$:

$$ \langle i_{\mu \nu} \omega | \nu \rangle = i_{\nu \mu} i_{\mu \nu} \omega = i_{\mu \nu \omega} = \langle \mu \wedge \nu | \omega \rangle \quad (1.7) $$

We define the Hodge-dual form of a form as interior product with the canonical volume form of the pseudo-Riemannian manifold

$$ * : \Omega^p(M) \longrightarrow \Omega^{m-p}(M) \quad \forall p \quad * \mu := i_{\mu \eta} \quad , \quad (1.8) $$

so in a local basis we have:

$$ * \mu = \frac{1}{p!(m-p)!} \mu^{i_1...i_p} \eta_{i_1...i_pj_{m-p}} a^{j_1...j_{m-p}} \quad (1.9) $$

Using the properties of the interior product it is not difficult to prove the following identities – here $s$ is the signature of the pseudo-metric:

(i) \quad $\langle \star \mu | \star \nu \rangle = (-1)^s \langle \mu | \nu \rangle \quad (1.10)$

(ii) \quad $\star \star \mu = (-1)^{p(m-p)+s} \mu \quad \mu \in \Omega^p(M) \quad (1.11)$

(iii) \quad $\star (\mu \wedge \nu) = i_{\nu} \star \mu \quad \star i_{\nu} \mu = (-1)^{q(m-q)} * \mu \wedge \nu \quad \nu \in \Omega^q(M) \quad (1.12)$

(iv) \quad $\mu \wedge \star \nu = \langle \mu | \nu \rangle \eta = \nu \wedge \star \mu \quad \nu, \mu \in \Omega^p(M) \quad (1.13)$

For an orthonormal co-frame $\{e^i\}_{i=1...m}$ the Hodge operator has the following useful representation:

$$ \star e^{i_1...i_p} = \frac{1}{(m-p)!} e^{i_1...i_p} j_{p+1...j_m} e^{j_{p+1}...j_m} = \frac{1}{(m-p)!} \eta^{i_1j_1}...\eta^{i_pj_p} e_{i_1...j_m} e^{j_{p+1}...j_m} \quad (1.14) $$
For the exterior derivative of these forms one can prove the formula:

\[
d \ast e^{i_1 \ldots i_p} = \frac{1}{(m-p)!} e^{i_1 \ldots i_p} j_{p+1 \ldots j_m} d e^{j_{p+1} \ldots j_m} = d e^l \wedge \ast e^{i_1 \ldots i_p} l
\]  
(1.15)

Let \( \omega^a_b \) be the connection forms of an arbitrary linear connection with respect to an arbitrary basis \( \{a_a\} \). Then the torsion is given by

\[
T^a = D a^a = d a^a + \omega^a_b \wedge a^b ,
\]  
(1.16)

so for a torsionfree connection one has

\[
da^a = -\omega^a_b \wedge a^b \quad \text{and} \quad da^{a_1 \ldots a_p} = -\omega^{a_1}_b \wedge a^{ba_2 \ldots a_p} - \ldots - \omega^{a_p}_b \wedge a^{a_1 \ldots a_{p-1} b}
\]  
(1.17)

(1.18)

The connection forms of a metric linear connection with respect to an arbitrary basis satisfy

\[
g_{ac} \omega^c_b + \omega^c_a g_{cb} = d g_{ab} ,
\]  
(1.19)

so in an orthonormal frame holds

\[
\omega_{ij} + \omega_{ji} = 0.
\]  
(1.20)

Using these equations one finds for an arbitrary connection in an orthonormal basis:

\[
d \ast e^{i_1 \ldots i_p} = d e^l \wedge \ast e^{i_1 \ldots i_p} l = (T^l - \omega^l_k \wedge e^k) \wedge \ast e^{i_1 \ldots i_p} l = T^l \wedge \ast e^{i_1 \ldots i_p} l + \omega^{i_1}_l \wedge \ast e^{i_2 \ldots i_p} + \ldots + \omega^{i_p}_l \wedge \ast e^{i_1 \ldots i_{p-1} l}
\]  
(1.21)

and hence for an arbitrary basis:

\[
d \ast a^{a_1 \ldots a_p} = T^b \wedge \ast a^{a_1 \ldots a_p} b + (\omega^b_{a_1} + d g^b_{a_1}) \wedge \ast a^{i_2 \ldots i_p} + \ldots + (\omega^b_{a_p} + d g^b_{a_p}) \wedge \ast a^{a_1 \ldots a_{p-1} b} - (\omega^a_{a_a} - \frac{1}{2} g^a_{ab} d g_{ab}) \wedge \ast a^{a_1 \ldots a_p}
\]  
(1.22)

For the special case of the Levi-Civit-connection this formula reads:

\[
d \ast a^{a_1 \ldots a_p} = -\omega^{a_1}_b \wedge \ast a^{ba_2 \ldots a_p} - \ldots - \omega^{a_p}_b \wedge \ast a^{a_1 \ldots a_{p-1} b}
\]  
(1.23)

For the convenience of the reader we give the expression of Christoffel symbols of the Levi-Civit-connection in an arbitrary basis:

\[
\omega_{abc} := g_{ad} \omega^d_c (a_b) = \frac{1}{2} \left( g_{ab,c} - g_{bc,a} + g_{ca,b} + C_{abc} - C_{bca} + C_{cab} \right)
\]  
(1.24)

\[C_{abc} = g_{ad} a^d ([a_b, a_c]) = -g_{ad} a^d (a_b, a_c)\]

It is possible to represent the connection form of the Levi-Civit connection by the inner product of an orthonormal basis and its derivatives

\[
\omega_{ij} = \frac{1}{2} \left( i_j d e_i - i_i d e_j - i_{ij} d e_k \cdot e^k \right)
\]  
\[= i_j d e_i - i_i d e_j - \frac{1}{2} i_{ij} (d e_k \wedge e^k)
\]  
\[= (-1)^{m+*} \left[ -e_j \wedge \ast d e_i + e_i \wedge \ast d e_j + \frac{1}{2} i_{ij} \wedge \ast (d e_k \wedge e^k) \right]
\]  
(1.25)
In the following paragraphs one important point is the functional derivative with respect to forms. We will illustrate this method with a well known example. Instead of varying the metric components with respect to a fixed coordinate basis we will vary the orthonormal frame, i.e. we will vary the four 1-forms which are declared to be orthonormal. So varying the metric can be represented by varying with respect to forms. Let us consider the Einstein-Hilbert-action, as usual the variation shall vanish on the boundary of $M$:

$$S[e^\mu] = \frac{1}{2} \int_M R_{\mu\nu} \wedge *e^{\mu\nu}$$

$$\delta S = \frac{1}{2} \int_M [\delta R_{\mu\nu} \wedge *e^{\mu\nu} + R_{\mu\nu} \wedge \delta *e^{\mu\nu}]$$

$$= \frac{1}{2} \int_{\partial M} \delta \omega_{\mu\nu} \wedge *e^{\mu\nu} + \int_M R_{\mu\nu} \wedge \delta e^\rho \wedge *e^{\mu\nu}_{\rho}$$

$$\frac{\delta S}{\delta e^{\rho}} = \frac{1}{2} R_{\mu\nu} \wedge *e^{\mu\nu}_{\rho} = \frac{1}{2} \left[ \langle R_{\mu\nu}|e^{\mu\nu} \rangle e_\rho + \langle R_{\mu\nu}|e^\nu \rangle e^\mu + \langle R_{\mu\nu}|e^\mu \rangle e^\nu \right]$$

$$= - * \left( R_\rho - \frac{1}{2} \Re_\rho \right)$$

(1.27)

Here

$$R_\nu := \iota^\mu R_{\mu\nu}$$

(1.28)

is the Ricci-form, which is symmetric, i.e. $\langle R_\mu|e_\nu \rangle = \langle R_\nu|e_\mu \rangle$ and

$$R := \iota^\mu \Re_\mu = \iota^\mu \iota^\nu R_{\mu\nu}$$

(1.29)

is the Ricci-scalar. It will be useful in the last paragraph to use instead of the orthonormal triads the dual frame of the $m \cdot (m-1)$--forms $*e^\mu$. We will show that one can functionally derive the expression $\int_M e^i \wedge \alpha_i$, with respect to $*e^i$ and obtain a well-defined 1-form. Let $\{\alpha_i\}_{i=1...m}$ be $m \cdot (m-1)$--forms independent of $e$ and define the $m$ 1-forms $\{\beta_i\}_{i=1...m}$ by:

$$*e^i \wedge \beta_j = \alpha_i$$

(1.30)

Then one has

$$\int_M e^i \wedge \alpha_i = \int_M e^i \wedge *e^j \wedge \beta_j = (m-1) \int_M *e^j \wedge \beta_j$$

$$\delta \alpha_i = \delta *e^j \wedge \beta_j + *e^j \wedge \delta \beta_j = 0$$

$$\delta \int_M e^i \wedge \alpha_i = (m-1) \int_M \delta *e^j \wedge \beta_j + \int_M e^i \wedge *e^j \wedge \delta \beta_j$$

$$= (m-1) \int_M \delta *e^j \wedge \beta_j - \int_M e^i \wedge \delta *e^j \wedge \beta_j$$

$$= (m-1) \int_M \delta *e^j \wedge \beta_j - \int_M e^i \wedge \delta e^k \wedge *e^j_{ik} \wedge \beta_j$$

$$= (m-1) \int_M \delta *e^j \wedge \beta_j - (m-2) \int_M \delta e^k \wedge *e^j_k \wedge \beta_j$$
The equation \( *e^j_i \wedge \beta_j = \alpha_i \) is easily solved:

\[
e^k \wedge *e^j_i \wedge \beta_j = -e^k \wedge \beta_k + \eta_{ki} *e^j_i \wedge \beta_j = e^k \wedge \alpha_i
\]

\[
-\langle \beta_k | e^k \rangle + \eta_{ki} \langle \beta_j | e^j \rangle = (-1)^s \langle e_k | *\alpha_i \rangle
\]

\[
\langle \beta_j | e^j \rangle = \frac{(-1)^s}{m-1} \langle e_j | *\alpha^j \rangle
\]

\[
\langle \beta_i | e_j \rangle = (-1)^s \left[ \frac{\eta_{ij}}{m-1} \langle e_k | *\alpha^k \rangle - \langle e_i | *\alpha_j \rangle \right]
\]

\[
\beta_i = (-1)^s \left[ \frac{e_i}{m-1} \langle e_k | *\alpha^k \rangle - \langle e_i | *\alpha_j \rangle e^j \right]
\]

Thus we have proved the formula

\[
\frac{\delta}{\delta *e^i} \int_M e^k \wedge \alpha_k = (-1)^s \left[ \frac{e_i}{m-1} \langle e_k | *\alpha^k \rangle - \langle e_i | *\alpha_j \rangle e^j \right]
\]

which we specialize for the case of \( \text{dim } M = 3, s = 0 \):

\[
\frac{\delta}{\delta *e^i} \int_M e^k \wedge \alpha_k = -e_i^{lm} e^m \alpha_m - \frac{1}{2} e_i \langle \alpha_m | *e^m \rangle
\]

\[ (1.32) \]

2 The ADM formulation in triads

The space-time manifold is assumed to be a parametric set of imbedded spacelike hypersurfaces \( \Sigma_t \), where we call the non-unique parametrization "time"; i.e. there exists a diffeomorphism \( i : R \times \Sigma \rightarrow M \), which can be used for identification. Let \( (x^1, x^2, x^3) \) be a coordinate system for \( \Sigma \), then \( (t, x^1, x^2, x^3) \) is a chart for \( R \times \Sigma \) and \( (\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3) := (t, x^1, x^2, x^3) \circ i^{-1} \) is one for \( M \). We will generally distinguish the analogous quantities on \( M \) and \( \Sigma \) by using a bar for quantities defined on \( M \). So if not defined in another way one obtains the unbarred quantities by pullback with the map \( i_t(x) := i(t, x), \ i_t : \Sigma \rightarrow M \). A pseudo-metric \( \bar{g} \) – signature \((-1,+1,+1,+1)\) – on \( M \) defines a normal \( \bar{e}_0 \) to the subspace of the tangential space which is spanned by \( \{ \frac{\partial}{\partial \bar{t}} \}_{i=1,2,3} \); it is the normal to the \( \{ \bar{t} = \text{const} \} \)-surfaces \( \Sigma_t = i_t \circ \Sigma \) in \( M \). With respect to this normal we decompose the time vector field \( \frac{\partial}{\partial \bar{t}} \):

\[
\frac{\partial}{\partial \bar{t}} = \bar{N} \bar{e}_0 + \bar{N}^\perp
\]

\[ (2.1) \]

At this point it seems that the use of a certain identification \( i \) leads to a gauge fixing of lapse and shift, because it determines at once the time vectorfield and the normal.
But we use this concept only to find the 3+1-decomposed Lagrangian which corresponds to the covariant action principle. When we have obtained it, we will turn our point of view and try to reconstruct the space-time metric from the decomposed data. If we do not know the space-time-metric, the normal on the right-hand-side of the equation will be unknown. By varying lapse and shift, we will vary the normal, and thus it is no surprise that the constraints associated with lapse and shift are equivalent to Einstein’s equations restricted to the normal, i.e. \( \dot{\epsilon}_0 \bar{G} = 0 \). Now let \( \{ \epsilon_{\mu} \}_{\mu=0...3} \) be adapted orthonormal tetrads on \( M \), such that \( \{ \bar{e}_i \}_{i=1,2,3} \) are always parallel to the hypersurfaces \( \Sigma_t \). The metric \( \bar{g} \) will be represented by the dual cotetrads \( \{ \bar{e}^\mu \}_{\mu=0...3} \), \( \bar{g} = \eta_{\mu\nu} \bar{e}^\mu \otimes \bar{e}^\nu \). Using the imbedding \( i_t \) we can pull back the tetrads \( e^\mu = i_t^* \bar{e}^\mu \), \( i_t^* \bar{e}^0 = N i_t^* d\bar{t} = 0 \) We first decompose the Einstein-Hilbert Lagrange form:\(^1\)

\[
\bar{L} = \frac{1}{2} \left( \bar{R}_{\mu\nu} \wedge \ast \bar{e}^{\mu\nu} \right) = \frac{1}{2} \left( 2 \bar{R}_{0i} \wedge \ast \bar{e}^{0i} + \bar{R}_{ij} \wedge \ast \bar{e}^{ij} \right)
\]

\[
= \frac{1}{2} \left[ 2 d\bar{\omega}_{0i} \wedge \ast \bar{e}^{0i} + 2 \bar{\omega}_{0k} \wedge \bar{\omega}^k_i \wedge \ast \bar{e}^{0i} + 3 \bar{R}_{ij} \wedge \ast \bar{e}^{ij} + \bar{\omega}_{i0} \wedge \bar{\omega}^0_j \wedge \ast \bar{e}^{ij} \right]
\]

\[
\overset{(1.23)}{=} \frac{1}{2} \left[ 2 d\left( \bar{\omega}_{0i} \wedge \ast \bar{e}^{0i} \right) - 2 \bar{\omega}_{0i} \wedge \bar{\omega}^0_j \wedge \ast \bar{e}^{ij} + 3 \bar{R}_{ij} \wedge \ast \bar{e}^{ij} + \bar{\omega}_{i0} \wedge \bar{\omega}^0_j \wedge \ast \bar{e}^{ij} \right]
\]

\[
= \frac{1}{2} \left[ 2 d\left( \bar{\omega}_{0i} \wedge \ast \bar{e}^{0i} \right) + \left( 3 \bar{R}_{ij} - \bar{\omega}_{0i} \wedge \bar{\omega}^0_j \right) \wedge \ast \bar{e}^{ij} \right] \quad (2.2)
\]

Neglecting the exact form which turns into a surface integral after integration, we can write the action in the 3+1-decomposed form:

\[
S(\bar{e}) = \int_M \bar{L} = \frac{1}{2} \int_M \left( 3 \bar{R}_{ij} - \bar{\omega}_{0i} \wedge \bar{\omega}^0_j \right) \wedge \ast \bar{e}^{ij}
\]

\[
= \frac{1}{2} \int dt \int_{\Sigma} i_t^* i_{\partial/\partial \bar{t}} \left( 3 \bar{R}_{ij} - \bar{\omega}_{0i} \wedge \bar{\omega}^0_j \right) \wedge \ast \bar{e}^{ij} \quad (2.3)
\]

Here \( i_{\partial/\partial \bar{t}} \) is the natural injection of a vector field in a form. Using \( \bar{e}^0 = \bar{N} d\bar{t} \) and \( i_t^* d\bar{t} = 0 \), this reduces to

\[
S(\bar{e}) = \frac{1}{2} \int_R dt \int_{\Sigma} i_t^* (3 \bar{R}_{ij} - \bar{\omega}_{0i} \wedge \bar{\omega}^0_j) \wedge \ast e^{ij}
\]

\[
= \frac{1}{2} \int_R dt \int_{\Sigma} N (R_{ij} - \omega_{0i} \wedge \omega_{0j}) \wedge \ast e^{ij} \quad , \quad (2.4)
\]

where we have identified \( i_t^* 3 \bar{R}_{ij} \) and \( R_{ij} \), which is defined on \( \Sigma \) as Levi-Civita curvature form to the metric defined by the triads \( \{ e^i \} \). Using the relation \( \omega_{0i} = -i_t K^i \), where \( K \) is the (0,2)-tensor of the extrinsic curvature, one can easily recognize the form given above as the ADM action-integral:

\[
S(e, \dot{e}) = \frac{1}{2} \int dt \int_{\Sigma} N (R_{ij} - \omega_{0i} \wedge \omega_{0j} | e^{ij}) \eta
\]

\[
= \frac{1}{2} \int dt \int_{\Sigma} N (R - K^i K^j_{ij} + K^i_{ij} K^j_{ij}) \eta \quad (2.5)
\]

\(^1\)Greek letters are summed from 0 to 3, Latin letters from 1 to 3
We now define the Lagrange function as follows – the index 3 on the Hodge-operator will be understood for the rest of the paragraph.

\[ L(e^i, \dot{e}^i, N, \vec{N}) = \int_{\Sigma} \frac{N}{2} (R_{ij} - \omega_{0i} \wedge \omega_{0j}) \wedge e^{ij} \]  

(2.6)

To make the dependance of \( \dot{e}^i \) obvious it is necessary to use the relation between the time-derivative of the triads and the extrinsic curvature:

\[ \dot{e}^i = \left. \frac{d}{dt} \right|_{s=1} (i_s^* e^i) = \left. \frac{d}{ds} \right|_{s=0} (i_s^* \Phi_{s/\partial t}^* e^i) = i_t^* L_{\partial/\partial t} \dot{e}^i \]  

(2.7)

because it holds \( i_{t+s} = \Phi_{s/\partial t} \circ i_t \), where \( \Phi_{s/\partial t} \) denotes the flow of the time-vectorfield \( \partial/\partial t \). So we take the Lie-derivative and pull back the result:

\[ L_{\partial/\partial t} \dot{e}^i = L_{\vec{N}} e^i - a^i_j e^j + N \omega^i_0 \]

(2.8)

Here \( a^i_j \) is the rotational parameter which is characteristic for a triad theory. This relation corresponds to the equation

\[ \dot{q}_{ab} = L_{\vec{N}} q_{ab} + 2N K_{ab} \]  

(2.10)

in the usual ADM theory, which could be derived from (2.8). One is tempted to treat the rotational parameter \( a \) like shift and lapse as an additional variable, which turns out to be a gauge parameter, since its time derivative does not appear in the Lagrangian. But this will lead to redundancies: If in (2.10) \( \dot{q}, N \) and \( \vec{N} \) are given, one can determine \( K_{ab} \), and if in (2.8) \( \dot{e}, N \) and \( \vec{N} \) are given, one can determine \( \omega \) and \( a \). So one can not arbitrarily choose \( a \) to determine \( \omega_{0i} \) from \( \dot{e} \). The reason is the symmetry of the extrinsic curvature, which is a consequence of the fact that our original connection on \( M \) was torsionfree.

\[ \vec{T}^0 = 0 \implies i_s^* \vec{T}^0 = i_s^* (de^0 + \vec{\omega}^0_\mu e^\mu) = \omega^0_i \wedge e^i = 0 \]

\[ \implies \langle \omega_{0i} | e_j \rangle = \langle \omega_{0j} | e_i \rangle \]  

(2.11)

We define the symmetric and antisymmetric part of the 1-forms \( \dot{e}^i \) and \( L_{\vec{N}} e^i \) in the following way

\[ \dot{e}^i_{S/A} := \frac{1}{2} (\dot{e}^i \pm \langle \dot{e}^j | e^i \rangle e_j) \]  

(2.12)

\[ L_{\vec{N}} e^i_{S/A} := \frac{1}{2} (L_{\vec{N}} e^i \pm \langle L_{\vec{N}} e^j | e^i \rangle e_j) \]  

(2.13)

and can split the equation (2.8) in these parts:

\[ \dot{e}^i_S = L_{\vec{N}} e^i_S - N \omega^0_i \]

\[ \dot{e}^i_A = L_{\vec{N}} e^i_A - a^i_j e^j \]  

(2.14)

(2.15)
This split will not be possible in the Palatini-case, where the connection is not assumed to be torsionfree and thus the extrinsic curvature is not symmetric. In the Palatini-case the rotational parameter \(a\) will become a variable comparable with lapse \(N\) and shift \(\vec{N}\).

We notice that only \(\dot{e}^i_s\) appears in the Lagrange function. We can now either introduce a fixed basis \(\{a_a\}_{a=1...3}\) and find a variable, such that \(\dot{e}^i_s\) is a proper time derivative – \(\dot{e}^i\) is not the time derivative of \(e^i_s = e^i\). Then we end up with the metric ADM formulation

\[
q\ab = \eta_{ij} e^i (a_a) e^j (a_b) \tag{2.16}
\]

\[
\dot{q}\ab = 2 \langle \dot{e}_i S | e_j \rangle e^i (a_a) e^j (a_b) \tag{2.17}
\]

and of course we can write the Lagrangian in terms of \(q_\ab\) and \(\dot{q}_\ab\). But we can also disregard the fact, that only the symmetric part of \(\dot{e}^i\) appears in the Lagrangian and expect a primary constraint. The momentum form is easily found:

\[
p_i := \frac{\partial L}{\partial \dot{e}^i} = \omega_{0j} \wedge *e^j = \langle \omega_{0j} | e^j \rangle * e_i - \langle \omega_{0j} | e_i \rangle * e^j \tag{2.18}
\]

The momentum is naturally a vector-valued 2-form, generally in a space of dimension \(n\) it is a \(n-1\)-form. This is analogous to the usual ADM formulation, where the momentum-form is a (0,2)-tensorvalued 3-form, where one usually splits the 3-form into a density and \(d^3x\). We define the contracted momentum

\[
p := \langle p_i | e^i \rangle = 2 \langle \omega_{0i} | e^i \rangle \tag{2.19}
\]

and would like to reexpress the extrinsic curvature \(\omega_{0i}\) by a simple calculation as follows

\[
\omega_{0i} = -\langle p_j | e_i \rangle e^j + \frac{1}{2} p e_i, \tag{2.20}
\]

but this would be inconsistent, if \(\langle p_i | e_j \rangle \neq \langle p_j | e_i \rangle\), since we know that the extrinsic curvature is symmetric. Thus it is only possible to obtain \(\omega_{0i}\) from \(p_i\), if \(p_i \wedge e_j = p_j \wedge e_i\). This is a consequence of the nonsolvability of the equation

\[
p_i - \frac{\delta L}{\delta \dot{e}^i} (e_i, \dot{e}^i) = p_i + \frac{1}{N} \left( \dot{e}^k S - L_S e^k \right) \wedge *e_{ik} = 0 \tag{2.21}
\]

which has to be regarded as a constraint equation. Multiplying this constraint by \(e_j\) and taking the antisymmetric part yields the necessary consistency condition

\[
p_i \wedge e_j - p_j \wedge e_i \approx 0 \tag{2.22}
\]

which is an equivalent formulation of the constraint implied by the solvability of (2.21). So this is a primary constraint implied by the symmetry of the extrinsic curvature.
curvature, which is a consequence of the fact that the four-dimensional connection to be constructed from $e^i$ and $\dot{e}^i$ is torsion-free. We can now perform the Legendre-transform.

$$H = \int_{\Sigma} p_i \wedge \dot{e}^i - L$$

$$= \int_{\Sigma} p_i \wedge \dot{e}^i + \int_{\Sigma} p_i \wedge \dot{e}_A - \int_{\Sigma} \frac{N}{2} (R_{ij} - \omega_{0i} \wedge \omega_{0j}) \wedge \ast e^{ij}$$

$$= \int_{\Sigma} p_i \wedge L_{Se}^i - \int_{\Sigma} a^i_j p_i \wedge e^j + \int_{\Sigma} \frac{N}{2} \left[ \langle p_i | \ast e^j \rangle \langle p_j | \ast e^i \rangle - \frac{1}{4} p^2 - R \right] \eta$$

The significance of $a$ is here at first that of a Lagrange multiplier of the rotational constraint. But if one wants to reconstruct the metric space-time one has to relate $a$ to a space-time quantity as given in equation (2.9), in order that the Hamiltonian equation for $\dot{e}$ and the geometric equation (2.8) agree. It seems that the Hamiltonian description does not determine the quantity $b_i := N \bar{\omega}_0(\bar{e}_0)$, but we will show in the following paragraph, that one has $b_i = -\langle dN | e_i \rangle$. Since the time derivatives of lapse $N$ and shift $\vec{N}$ do not appear in the Lagrangian one finds the following primary constraints:

$$p_N = \frac{\delta L}{\delta \dot{N}} \approx 0 \quad p_{\vec{N}} = \frac{\delta L}{\delta \vec{N}} \approx 0$$

which give rise to the following secondary constraints:

$$C_H = \{ H, p_N \} = \eta \left[ \langle p_i | \ast e^j \rangle \langle p_j | \ast e^i \rangle - \frac{1}{2} p^2 - R \right] \approx 0$$

$$C_{Di} = \{ H, p_N \} = -d p_i + p_j \wedge i_d e^j = -D p_i - \omega_k^j (e_i) p_j \wedge e^k \approx 0$$

where in the case of the diffeomorphism constraint a surface term had to be neglected. Using the rotational primary constraint

$$C_{R^j_i} = \frac{1}{2} (p^j \wedge e_i - p_i \wedge e^j)$$

we could redefine the diffeomorphism constraint in a concise way $D p_i \approx 0$, but this will not be possible in the Palatini case, and since we prefer to work with integrated constraints where our diffeomorphism constraint has the simple meaning of the momentum mapping of the action of the diffeomorphism group of $\Sigma$, we stick to the original version. But this is a matter of taste and we will have the choice also in the Ashtekar formulation. We can represent the Hamiltonian as sum of the integrated constraints

$$H(e, p, N, \vec{N}, a) = \int_{\Sigma} N^i C_{Di}(e, p) + \int_{\Sigma} N^j p_j + \int_{\Sigma} a_2^j C_{R^j_i} + \int_{\Sigma} NC_H := H_D(e, p, \vec{N}) + H_R(e, p, a) + H_H(e, p, N)$$

and we will call the integrated versions of the constraints like the constraints itself, diffeomorphism, rotational and Hamiltonian constraint. We can now determine the
Analogously one finds of the Hamiltonian part for the equation of motion.

\[ \dot{e}^i = \frac{\delta H}{\delta p_i} = L_N e^i - a^i_j e^j + N \langle \langle p_j \mid * e^j \rangle \rangle e^i - \frac{1}{2} p e^i \]  
\[ \dot{p}_i = -\frac{\delta H}{\delta e^i} = L_N p_i + a^i_j p_j - N \ast \left( R_i - \frac{1}{2} R e_i \right) + \ast \left( \nabla_i dN - e_i \Delta N \right) - N \left( \langle \langle p_j \mid * e^j \rangle \rangle p_j - \frac{1}{2} p p_i \right) + \frac{N}{2} \left( \langle \langle p_j \mid * e^k \rangle \rangle \langle p_k \mid * e^j \rangle - \frac{1}{2} p^2 \right) * e_i \]  

(2.28)

(2.29)

The Ricci-form \( R_i \) is defined as in \( (1.29) \) and \( \Delta N \) denotes as usual the Laplacian of the lapse function. In the equation for the momentum surface terms had to be neglected in order to derive the first and the last term. We will show the contribution of the lapse function to the equation of motion.

\[ \delta \left[ \langle \langle p_i \mid * e^j \rangle \rangle \langle p_j \mid * e^i \rangle \rangle \eta \right] = \delta \left[ (p_i \wedge e^j) \cdot \ast (p_j \wedge e^i) \right] = \left( \delta p_i \wedge e^j + \delta e^j \wedge p_i \right) \cdot \ast (p_j \wedge e^i) + (p_i \wedge e^j) \cdot \ast \left( p_j \wedge e^i \right) \]

(2.14)

Multiplication by \( \frac{1}{3!} \langle \langle p_i \wedge e^j \rangle \rangle e_{klm} \) leads to

\[ (p_i \wedge e^j) \cdot \ast \left( p_j \wedge e^i \right) = \left( \delta p_j \wedge e^i + \delta e^i \wedge p_j \right) \cdot \ast \left( p_i \wedge e^j \right) - \delta e^k \wedge i_k (p_i \wedge e^j) \cdot \ast \left( p_j \wedge e^i \right) \]

and finally one obtains:

\[ \delta \left[ \langle \langle p_i \mid * e^j \rangle \rangle \langle p_j \mid * e^i \rangle \rangle \eta \right] = \delta p_i \wedge 2e^j \langle \langle p_j \mid * e^i \rangle \rangle + \delta e^i \wedge \left[ 2p_j \langle p_i \mid * e^j \rangle \rangle - * e_i \langle p_j \mid * e^k \rangle \rangle \langle p_k \mid * e^j \rangle \rangle \right] \]

Analogously one finds

\[ \delta (p^2 \eta) = \delta p_i \wedge 2p e^i + \delta e^i (2p p_i - p^2 * e_i) \]

and all momentum terms in the equation of motion are proved. The derivation of the Ricci-scalar term is more difficult:

\[ \delta \left( NR_{ij} \wedge * e^{ij} \right) = N \left[ \delta R_{ij} \wedge * e^{ij} + \delta e^k \wedge \left( R_{ij} \wedge * e^{ij}_k \right) \right] \]

The second term is responsible for the Einstein-force term as shown in \( (1.27) \). It is left to discuss the variation of the curvature-form:

\[ N \delta R_{ij} \wedge * e^{ij} = N d (\delta \omega_{ij} \wedge * e^{ij}) = d (N \delta \omega_{ij} \wedge * e^{ij}) - d N \wedge \delta \omega_{ij} \wedge * e^{ij} \]

modulo exact forms \( = \frac{1}{2} \langle \omega_{ij} | e^j \rangle e^i \wedge * d N \)
We neglect the exact form which, leads to a surface integral, and substitute the variation of the connection form by the variation of the triads:

\[
de^i = -\omega_i^j \wedge e^j
\]

\[
\delta de^i = -\delta \omega_i^j \wedge e^j - \omega_i^j \wedge \delta e^j
\]

\[
i_i \delta de^i = -\langle \delta \omega_i^j | e^i \rangle e^j - i_i \left( \omega_i^j \wedge \delta e^j \right)
\]

Hence we have:

\[
\frac{\mathbf{N}}{2} \delta R_{ij} \wedge *e^{ij} = \langle \delta \omega_{ij} | e^i \rangle e^j \wedge *dN
\]

\[
= -i_i (d\delta e^i + \omega_i^j \wedge \delta e^j) \wedge *dN
\]

\[
= (d\delta e^i \wedge *dN \wedge e_i) + \delta e^i \wedge D \wedge (dN \wedge e_i)
\]

\[
= \delta e^i \wedge * [\nabla_i dN - e_i \Delta N]
\]

(2.30)

Of this derivation we keep in mind the last transformation for later use:

\[
D \wedge (dN \wedge e_i) = *(\nabla_i dN - e_i \Delta N)
\]

(2.31)

Using the equations of motion for each part of the Hamiltonian it is no difficulty to determine the constraint algebra. All surface integrals are consequently neglected.

\[
\{ H_D(\vec{N}), H_D(\vec{M}) \} = H_D([\vec{N}, \vec{M}])
\]

\[
\{ H_D(\vec{N}), H_R(a) \} = H_R(L_N a)
\]

\[
\{ H_D(\vec{N}), H_H(N) \} = H_H(L_N N)
\]

\[
\{ H_R(a_1), H_R(a_2) \} = -H_R([a_1, a_2])
\]

\[
\{ H_R(a), H_H(N) \} = 0
\]

\[
\{ H_H(N), H_H(M) \} = H_D(NdM^2 - M^2dN^2)
\]

\[
= -H_R((dN \wedge dM|e_i^{ij}) + \langle NdM - M^2dN|\omega_i^j \rangle)
\]

(2.32)

We calculate only three of these brackets:

\[
\{ H_D(\vec{N}), H_D(\vec{M}) \} = \int_\Sigma \left[ \frac{\delta H_D(\vec{N})}{\delta e^i} \wedge \frac{\delta H_D(\vec{M})}{\delta p_i} - \frac{\delta H_D(\vec{N})}{\delta p_i} \wedge \frac{\delta H_D(\vec{M})}{\delta e^i} \right]
\]

\[
= \int_\Sigma \left[ -L_{\vec{N}} p^i \wedge L_{\vec{M}} e_i + L_{\vec{N}} e_i \wedge L_{\vec{M}} p_i \right]
\]

\[
= \int_\Sigma p^i \wedge L_{[\vec{N}, \vec{M}]} = H_D([\vec{N}, \vec{M}])
\]

This result reassures that the diffeomorphism constraint \(H_D\) is the momentum mapping of the action of the diffeomorphism group.

\[
\{ H_R(a), H_H(N) \} = \int_\Sigma \left\{ a^i_j p_j \wedge N \left( \langle p_k | * e^i \rangle e^k - \frac{1}{2} p e^i \right) + a^i_j e^j \wedge \right\}
\]
\[
\begin{align*}
&\left[ -N(\langle p_i | * e^j \rangle p_j - \frac{1}{2} p p_i) + \frac{N}{2} (\langle p_i | * e^m \rangle \langle p_m | * e^l \rangle - \frac{1}{2} p^2) * e_i \\
&\quad N * (R_i - \frac{1}{2} R e_i) + * (\nabla_i dN - e_i \Delta N) \right] \\
&= \int_{\Sigma} a^i_j \nabla_i \nabla_j N \eta = 0,
\end{align*}
\]

since \( \nabla_i \nabla_j N \) is symmetric because the connection is torsionfree.

\[
\{H_H(N), H_H(M)\} = \int_{\Sigma} \left[ - * (\nabla_i dN - e_i \Delta N) \wedge M \big( \langle p_j | * e^i \rangle e_j - \frac{1}{2} p e^i \big) \\
+ * (\nabla_i dM - e_i \Delta M) \wedge N \big( \langle p_j | * e^i \rangle e_j - \frac{1}{2} p e^i \big) \right] \\
= \int_{\Sigma} \left( N \nabla_i \nabla_j M - M \nabla_i \nabla_j N \right) p_j \wedge e^i \\
= \int_{\Sigma} \left[ \nabla_i \left( N \nabla^j M - M \nabla^j N \right) - (\nabla_i N \nabla^j M - \nabla_i M \nabla^j N) \right] p_j \wedge e^i \tag{2.34}
= H_D(N dM^2 - M dN^2) \\
- H_R \left( \langle dN \wedge dM | e^i \rangle + \langle N dM - M dN | \omega^i \rangle \right) \tag{2.33}
\]

Here we used the identity

\[
DY^i = (\nabla_j Y^i) e^j = L_{Y^i} e^i + \omega^i_{\ j} (Y) e^j \tag{2.34}
\]

As expected the rotational constraint or more precisely the generated vectorfields on the infinite dimensional manifold of triads and triad-momenta form an ideal. So we can go through the canonical analysis without introducing a fixed coordinate system.

One can also define the Poisson bracket between the 1-form \( e^i \) and the 2-form \( p_j \):

\[
\{e^i(r), p_j(s)\} = \delta^i_j \delta(r, s) \quad r, s \in \Sigma \quad \tag{2.35}
\]

Here \( \delta \) has to be regarded as a \((0,3)\)-tensorfield on \( \Sigma \times \Sigma \), \( \delta(r, s) \) is a 1-form with respect to \( r \) with values in \( \Lambda^2 (T\Sigma) \) or a 2-form with respect to \( s \) with values in \( \Lambda^1 (T\Sigma) \). In a local chart \((U, \{x^a\})_{a=1,2,3} \) \( \delta(r, s) \) has the following form:

\[
\delta(r, s) = \frac{1}{2} \epsilon_{abc} dx^a(s) \wedge dx^b(s) \otimes dx^c(r) \delta(x^1(r) - x^1(s)) \delta(x^2(r) - x^2(s)) \delta(x^3(r) - x^3(s)) \tag{2.36}
\]

Despite the use of the \( \epsilon \)-symbol this is a tensor since in coordinate transformations the product of \( \delta \)-functions transforms with a determinant. Finally we give the representation of the usual ADM momentum \( \pi \), as canonical conjugate to the \((0,2)\)-tensor \( q \) a \((2,0)\)-tensorvalued 3-form in terms of the momentum 2-form and the triads

\[
\pi = \eta^k_{i} e_i \otimes e_j \otimes p_k \wedge e^j = e_i \otimes e_j p^{(i} \wedge e^j), \quad \tag{2.37}
\]
where the symmetrisation is not really necessary, since one can define the metric momentum only on the constraint surface where \( p^i \wedge e^j \) is symmetric. This equation can easily be verified using the well known relation between \( \pi \) and the extrinsic curvature \( K = -\omega_0 \otimes e^i \)

\[
\pi = \left( K - qtr (q^2 K) \right) \otimes \eta,
\]

and equation (2.18).

### 3 The Hilbert-Palatini-action

The Hilbert-Palatini-action was suggested by Palatini [2] mainly in order to avoid second time derivatives of the relevant fields in the action integral. This can also be achieved by neglecting a surface integral

\[
\frac{1}{2} \int_M \bar{R} \eta = \int_M \left[ \frac{1}{2}(d\bar{e}^\mu \wedge \ast d\bar{e}_\mu) - \frac{1}{2}(d\bar{e}_\mu \wedge \bar{e}^\nu) \wedge \ast(d\bar{e}_\nu \wedge \bar{e}^\mu) + \frac{1}{4}(d\bar{e}_\mu \wedge \bar{e}^\mu) \wedge \ast(d\bar{e}_\nu \wedge \bar{e}^\nu) \right]
\]

but then the action is no longer gauge invariant under \( \text{SO}(1,3) \) rotations. Following Palatini’s suggestion one regards the action not only as dependant on the metric, but also on the connection. So the connection is now no longer torsionfree and metric, since if these conditions hold, the connection is uniquely determined by the metric. Unfortunately varying w.r.t. the connection does not yield both necessary conditions, i.e. if one considers a general linear connection the Euler-Lagrange equation for the connection only gives a relation between torsion and non-metricity [3]. In the vacuum case this relation states that the connection is torsionfree, if it is metric, and vice versa. So one usually assumes that one of the conditions for the Levi-Civit connection is satisfied and only varies in the class of either metric or torsionfree connections. Since in case of a non-metric connection the compatibility of the gauge group of the tetrades and the connection form is destroyed – the connection form for a metric connection is \( \text{so}_R(1,3) \)-valued – we prefer to drop the torsion condition here. Since our canonical description shall be equivalent to the covariant formulation, which is of course easier, we first revise the covariant derivation of the equations. We show that assuming that our connection satisfies the condition of metricity

\[
\bar{\omega}_{\mu\nu} + \bar{\omega}_{\nu\mu} = 0
\]

the Euler-Lagrange equation for the connection form yields in the vacuum case that our connection is torsionfree.

\[
S(\bar{e}, \bar{\omega}) = \frac{1}{2} \int_M \bar{R}_{\mu\nu}(\bar{\omega}) \wedge \ast \bar{e}^{\mu\nu}
\]

\[
\delta\bar{\omega} S = \frac{1}{2} \int_M \left[ d\delta \bar{\omega}_{\mu\nu} \wedge \ast \bar{e}^{\mu\nu} + \delta \bar{\omega}_{\mu\nu} \wedge \bar{\omega}^\nu_{\rho} \wedge \ast \bar{e}^{\mu\rho} + \delta \bar{\omega}_{\mu\nu} \wedge \bar{\omega}^\mu_{\rho} \wedge \ast \bar{e}^{\rho\nu} \right]
\]
\[
\frac{1}{2} \int_M \delta \bar{\omega}_{\mu \nu} \wedge \bar{T}^\rho \wedge \ast \bar{e}^{\rho \nu} + d(\delta \bar{\omega}_{\mu \nu} \wedge \ast \bar{e}^{\mu \nu}) \tag{3.3}
\]

Assuming that the variation vanishes on the boundary one obtains
\[
\frac{\delta S}{\delta \bar{\omega}_{\mu \nu}} = \bar{T}^\rho \wedge \ast \bar{e}^{\rho \nu} = 0. \tag{3.4}
\]

This turns out to be a zero torsion condition. We show this abandoning the summation convention for a short while.

\[
\sum_{\rho \neq \mu \nu} \bar{T}^\rho \wedge \ast \bar{e}^{\rho \nu} = \sum_{\rho \neq \mu \nu} \ast \bar{e}_\rho \langle \bar{T}^\rho | \bar{e}^{\rho \nu} \rangle + \ast \bar{e}^{\mu} \sum_{\rho \neq \mu \nu} \langle \bar{T}^\rho | \bar{e}_\rho \rangle = 0
\]

1. \( \langle \bar{T}^\rho | \bar{e}^{\rho \nu} \rangle = 0 \quad \forall \mu, \nu \neq \rho \)

2. \( \sum_{\rho \neq \mu} \langle \bar{T}^\rho | \bar{e}_\rho \rangle = 0 \quad \forall \mu, \nu \)

We write \( (\langle \bar{T}^\rho | \bar{e}^{\rho \nu} \rangle)_{\rho = 1 \ldots m, \rho \neq \nu} =: \vec{x} \) and summarize the second equation for fixed \( \nu \) for all \( \mu \) in the matrix equation
\[
\begin{pmatrix}
0 & 1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1 & 0
\end{pmatrix}
\vec{x} = 0 \tag{3.5}
\]

Since this \((m - 1) \times (m - 1)\)-matrix has determinant \((-1)^m(m - 2)\), \( \vec{x} = 0 \) is the only solution if \( m \neq 2 \) and thus it follows \( \bar{T} = 0 \). Despite this result we keep in mind that the Palatini-theory is another theory than Einstein’s relativity, since spin fields could now couple to torsion [4]. As long as there are no experimental data there is only Hehl’s argument, that a theory where only the translatory part of the Poincaré group couples to geometry seems to be inconsequent. If future experiments really confirm a spin-torsion-coupling it may well be that one has to modify the Einstein-Lagrangian, because it does not give rise to derivative terms for the connection in the field equations. For our canonical description this means that the there is no time-development of the connection, it is fully determined by constraints.

The Hamiltonian description we are seeking for should yield the same result: The connection to be constructed on the four-dimensional manifold \( \mathbb{R} \times \Sigma \sim M \) should prove torsionfree. On \( \Sigma \) one can only observe the pullback of the equation \( \bar{T} = 0 \), so it is useful to realise first the meaning of the pullback of the different components of \( \bar{T} \) on \( \Sigma \). We can differ four equations:

\[
i_t^* \bar{T}^0 = 0 \quad i_t^* \bar{T}^i = 0 \quad i_t^* i_0 \bar{T}^0 = 0 \quad i_t^* i_0 \bar{T}^i = 0
\]

The first equation assures the symmetry of the extrinsic curvature \((\ref{2.11})\). The second implies that the induced connection on \( \Sigma \) is torsionfree:
\[
0 = i_t^* \bar{T}^i = i_t^* (d \bar{e}^i + \bar{\omega}^i_\mu \wedge \bar{e}^{\mu}) = d \bar{e}^i + \bar{\omega}^i_j \wedge \bar{e}^j = 0 \tag{3.6}
\]
The third equation yields an important relation between lapse $N$ and $i^*_t i_0 \omega^0_i$.

$$0 = i^*_t i_0 \bar{T}^0 = i^*_t \left[ i_0 d(\bar{N} d\bar{t}) + \bar{e}^i i_0 \omega^0_i \right] = -\frac{1}{N} dN + e^i i^*_t i_0 \bar{\omega}^0_i$$

Defining $b_i := N i^*_t \bar{\omega}_0i(\bar{e}_0)$ we write this equation as follows:

$$dN + b_i e^i = 0 \quad (3.7)$$

The last equation is the well known relation between the extrinsic curvature, the rotational term and the Lie-derivative along the normal:

$$0 = i^*_t i_0 \bar{T}^i = i^*_t (i_0 d\bar{e}^i + \bar{\omega}^i_\mu (\bar{e}_0) \bar{e}^\mu - \bar{\omega}^i_0) = i^*_t L_{\bar{e}_0} \bar{e}^i + e^j i^*_t \bar{\omega}^j_0(\bar{e}_0) - i^*_t \bar{\omega}^0_i$$

$$N i^*_t L_{\bar{e}_0} \bar{e}^i = -a^i_j e^j - N \omega^0_i \quad (3.8)$$

We will now decompose the Lagrange 4-form in another way than in the last paragraph, since there is a priori no dependance of the connection on the tetrad.

$$\bar{R}_{\mu \nu} \wedge * \bar{e}^{\mu \nu} = 2 \bar{R}_{0i} \wedge * \bar{e}^{0i} + \bar{R}_{ij} \wedge * \bar{e}^{ij}$$

$$= 2d(\bar{\omega}_{0i} \wedge * \bar{e}^{0i}) + 2\bar{\omega}_{0i} \wedge * \bar{e}^{0i} + 2\bar{\omega}_{0j} \wedge * \bar{e}^{0i} + (3\bar{R}_{ij} + \bar{\omega}_{0i} \wedge \bar{\omega}^0_j) \wedge * \bar{e}^{ij}$$

mod. ex. forms

$$= 2\bar{\omega}_{0i} \wedge d\bar{e}^j \wedge * \bar{e}^{0ij} + 2\bar{\omega}_{0j} \wedge * \bar{e}^{0ij} + (3\bar{R}_{ij} + \bar{\omega}_{0i} \wedge \bar{\omega}^0_j) \wedge * \bar{e}^{ij} \quad (3.9)$$

The pullback $i^*_t$ after insertion of the time vector field $\partial/\partial \bar{t}$ will be simple, if we first eliminate $\bar{e}^0$ from the last factor $* \bar{e}^{ij}$. Then the dualised form $* \bar{e}^{ij}$ will contain a factor $*e^0 = \bar{N} d\bar{t}$ in any case and only if the time vector field is inserted there the term will not vanish after the pull back to $\Sigma$. Technically spoken $i^*_t i_0/\partial \bar{t}$ is performed then by changing $*$ to $\wedge^3$ and multiplying with the lapse function.

$$\bar{\mathcal{L}} = \frac{1}{2} \left[ 2\bar{\omega}_{0i} \wedge d\bar{e}^j \wedge i^0 * \bar{e}^{ij} - 2\bar{\omega}_{0j} \wedge \bar{\omega}^i_0 \wedge i^0 * \bar{e}^i - (3\bar{R}_{ij} + \bar{\omega}_{0i} \wedge \bar{\omega}^0_j) \wedge * \bar{e}^{ij} \right]$$

$$= -\bar{\omega}_{0i}(\bar{e}_0) d\bar{e}^j \wedge * \bar{e}^{ij} + \bar{\omega}_{0i} \wedge i_0 d\bar{e}^j \wedge * \bar{e}^{ij} - \bar{\omega}_{0i}(\bar{e}_0) \bar{\omega}^j_0 \wedge * \bar{e}^j - \bar{\omega}^j_0(\bar{e}_0) \bar{\omega}_{0i} \wedge * \bar{e}^{ij} + \frac{1}{2} (3\bar{R}_{ij} + \bar{\omega}_{0i} \wedge \bar{\omega}^0_j) \wedge * \bar{e}^{ij} \quad (3.10)$$

Using the definitions $b_i := N i^*_t \bar{\omega}_{0i}(\bar{e}_0)$ and $a^i_j := N i^*_t \bar{\omega}^j_0(\bar{e}_0)$ we obtain:

$$\mathcal{L} = i^*_t i_0/\partial \bar{t} \bar{\mathcal{L}}$$

$$= \left[ \frac{N}{2} (R_{ij} + \omega_{0i} \wedge \omega_{0j}) - b_i d\bar{e}^j + \omega_{0i} \wedge N i^*_t i_0 d\bar{e}^j \right] \wedge * \bar{e}^{ij} - \left( b_0 \omega^i_j + a^i_j \omega_{0i} \right) \wedge * \bar{e}^j$$

$$= \left[ \frac{N}{2} (R_{ij} + \omega_{0i} \wedge \omega_{0j}) - b_i (d\bar{e}^j + \omega_{0j} \wedge e^k) + a^k_j e_k \wedge \omega_{0i} + \omega_{0i} \wedge N i^*_t i_0 d\bar{e}^j \right] \wedge * \bar{e}^{ij}$$

In order to introduce $\dot{e}^i$ we again have to use the space-time picture:

$$\dot{e}^i = i^*_t L_{\partial/\partial \bar{t}} \bar{e}^i = L_N e^i + N i^*_t L_{\bar{e}_0} \bar{e}^i = L_N e^i + N i^*_t i_0 d\bar{e}^i \quad (3.11)$$
Then we obtain the following Lagrangian:

\[
L(e^i, \dot{e}^i, \omega_0, \omega_{ij}, N, \bar{N}, a^i_j, b_i) = \int \sum \left\{ \dot{e}^i \wedge \omega_{0j} \wedge *e_j^i - \left[ -\frac{N}{2}(R_{ij} + \omega_{0i} \wedge \omega_{0j}) + b_i (de_j + \omega_{jk} \wedge e^k) + a^i_k e_k \wedge \omega_{0j} + L\bar{N}e_i \wedge \omega_{0j} \right] \wedge *e_j^i \right\}
\]  

(3.12)

We notice that the time derivatives of the connection form components \(\omega_{ij}, \omega_{0i}, b_i\) and \(a^i_j\) do not appear in the Lagrangian like those of lapse and shift. This is not surprising, since in the covariant equations \(\frac{\delta S}{\delta \omega_{\mu\nu}} = 0\) did not contain an exterior derivative of the connection form. We can reduce the number of constraints that we get, if we identify at this point

\[
\frac{\delta L}{\delta \dot{e}^i} = p_{e^i} = \omega_{0j} \wedge *e_j^i \quad .
\]

(3.13)

Formally spoken this identification solves the pair of second class constraints

\[
p_{e^i} - \omega_{0j} \wedge *e_j^i \approx 0 \quad p_{\omega_0} \approx 0 \quad .
\]

(3.14)

The spatial part of the connection form \(\bar{\omega}_{0i}\) is then already eliminated as a redundant degree of freedom. Before this identification one would obtain the equation \(L_N e^i + a^i_j - N \omega^i_0 = 0\) as nondynamical Euler-Lagrange-equation for \(\omega_{0i}\). After the identification we obtain this equation in the Hamiltonian picture as equation of motion for \(\dot{e}^i\). The Hamiltonian is now easily obtained \((p_{e^i} \equiv p_i)\):

\[
H(e^i, p_i, \omega_{ij}, N, \bar{N}, a^i_j, b_i) = \int \sum p_i \wedge L\bar{N} e^i - a^i_j e^j \wedge p_i + b_i (de_j + \omega_{jk} \wedge e^k) \wedge *e_j^i + \frac{N}{2} (p_i \left< e^i \right| \left< e^j \right> \left< e^j \right| *e^i) - \frac{1}{2} p^2 - R \right) \eta
\]

(3.15)

Compared to the ADM Hamiltonian there is one additional term. Since \(de^j + \omega^j_k \wedge e^k\) is the torsion of the connection we have an explicit torsion potential. Apart from the two constraints which were already discussed and solved the Lagrangian gives rise to five other primary constraints:

\[
p_N \approx 0 \quad \bar{p}_N \approx 0 \quad p_{aij} \approx 0 \quad p_{b_i} \approx 0 \quad p_{\omega_{ij}} \approx 0
\]

(3.16)

These constraints give rise to the following secondary constraints:

\[
\begin{align*}
C_H &= \{H, p_N\} = \frac{1}{2} (\left< p_i \right| *e^i) \left< p_j \right| *e^j) - \frac{1}{2} p^2 - R \right) \eta \quad \text{Hamiltonian} \\
C_{Di} &= \{H, p_{e^i}\} = -dp_i + p_j \wedge_i de^j \quad \text{diffeomorphism} \\
C_{Rij} &= \{H, p_{aij}\} = -\frac{1}{2} (p^i \wedge e_i - p^j \wedge e^j) \quad \text{rotational} \\
C_{Tij} &= \{H, p_{b_i}\} = (de^j + \omega^j_k \wedge e^k) \wedge *e^i_j = T^j \wedge *e^i_j \quad \text{torsion} \\
C_{Cij} &= \{H, p_{\omega_{ij}}\} = -(dN + b_k e^k) \wedge *e^i_j - NT^k \wedge *e^i_j_k \quad \text{connection constraint}
\end{align*}
\]

(3.17)
The Hamilton function is again a sum of integrated secondary constraints. But only four of the five secondary constraints appear in the Hamiltonian.

\[
H(e^i, p_i, N, \vec{N}, \omega_{ij}, a_{ij}, b_i) = \int_\Sigma N^i C_{Di}^i(e, p) + \int_\Sigma a_{ij} C_{R}^{ij}(e, p) + \int_\Sigma C_H(e, p, \omega) + \int_\Sigma b_i C_T^i(e, \omega) \]

We observe that the rotational, torsion and connection constraint imply the remaining equations which assure that the connection to be constructed on \(IR \times \Sigma\) is torsionfree: Because of the rotational constraint the extrinsic curvature is symmetric. Note that this is a secondary constraint here whereas it is a primary constraint in the ADM description. The other equations are found in the following way:

\[
0 \approx C_C^{ij} \wedge e_j = 2(dN + b_k \wedge e^k) \wedge *e^i + NT^{ij} \wedge *e_j \approx 2(dN + b_k e^k) \wedge *e^i
\]

\[
\Rightarrow \quad dN + b_k e^k \approx 0 \quad (3.18)
\]

because of the torsion constraint and thus both terms in this sum for \(C_C\) vanish separately, hence

\[
T^k \wedge *e^{ij} \approx 0 \quad (3.19)
\]

and this equation implies \(T^k \approx 0\) as shown above (3.4). Note that these arguments are independent of the space dimension. In order to simplify the constraint analysis we substitute torsion and connection constraint by the following equivalents:

\[
C_1^i := de^i + \omega^j \wedge e^j \approx 0 \quad (3.20)
\]

\[
C_2 := dN + b_i de^i \approx 0 \quad (3.21)
\]

We do not expect any tertiary constraints since there is no equivalent on the co-variant level, but it is not obvious that there exists an extension of the Hamiltonian by a sum of integrated primary constraints which does conserve all constraints. To check the absence of tertiary constraint it is necessary to find integration functions \(K_1, K_2, K_3, K_4, \kappa_5\) such that

\[
\{H + \int_\Sigma K_1 p_N + \int_\Sigma K_2^i p_{N^i} + \int_\Sigma K_{3ij} p_{a_{ij}} + \int_\Sigma K_{4i} p_{b_i} + \int_\Sigma \kappa_{5ij} \wedge p_{\omega_{ij}}, C_{H/D/R/1/2}\} \approx 0 \quad (3.22)
\]

We will first analyze the constraints because then it will become obvious how to choose these integration variables. For definition and a survey we summarize once again all constraints we have found. One should note the difference of test integration forms and canonical variables:
Theorem:
The constraints $H_1, \ldots, H_4$ form second class pairs. The other constraints $H_N$, $H_{N}^\perp$, $H_a$, $H_H$, $H_D$, $H_R$ can be substituted by equivalent constraints $H_{N}^\perp$, $H_{N}^\perp$, $H_A$, $H_H$, $H_D$, $H_R$ which are first class. The Poisson bracket relations of those substituted constraints correspond to the relations in the ADM formulation and since they are first class, they also equal the Dirac brackets of the original constraints which one could have calculated directly as well. All surface terms are consequently neglected.

Proof:
The constraints $H_1, \ldots, H_4$ are obviously second class, one obtains for $(\{H_i, H_j\})_{ij=1..4}$ a skewsymmetric matrix of functionals:

$$
(\{H_i, H_j\})_{ij=1..4} = \begin{pmatrix}
0 & 0 & -\int_\Sigma \rho_{ij} \wedge \alpha^i \wedge e^j & 0 \\
0 & 0 & 0 & \int_\Sigma R_i \wedge \beta \wedge e^i \\
-\int_\Sigma \rho_{ij} \wedge \alpha^i \wedge e^j & 0 & 0 & 0 \\
\int_\Sigma R_i \wedge \beta \wedge e^i & \int_\Sigma R_i \wedge \beta \wedge e^i & 0 & 0
\end{pmatrix} =: (S_{ij})_{ij=1..4}
$$

(3.23)

This matrix is invertible and thus the matrix of Poisson brackets of all constraints has at least rank 4 and because the constraints $H_1, \ldots, H_4$ yield this simple symplectic structure we regard these constraints as the fundamental second class pairs. We know that on the surface described by these second class constraints the induced connection is torsionfree, but the extrinsic curvature is not necessarily symmetric. We suppose that all other constraints are first class, since there should not be less first class constraints than in the ADM case. Starting with the primary constraints we can set

$$
H_{N}^\perp := H_{N}^\perp \quad H_A := H_a
$$

(3.24)

since these constraints are first class. $H_N$ does not commute with $H_2$ because of the lapse dependance. Thus we substitute $H_N$ in the following way:

$$
H_N(U; p_N, p_b) := H_N(U; e^i, p_i) - H_4(\langle dU | e_i \rangle; p_b) = \int_\Sigma (U p_N - \langle dU | e_i \rangle p_b)
$$

(3.25)
One can easily check that this constraint commutes also with $H_2$. Regarding the secondary constraints we notice that none of them commutes with $H_1$ and $H_2$ because of their momentum dependance. But the Hamiltonian constraint $H_H$ does not even commute with the primary constraint $H_3$. And indeed calculating for example the Poisson bracket
\[
\{H_H(N), H_H(M)\} = 0
\] (3.26)
we obtain a result which differs from the ADM result, because here the Ricci-scalar-term remains underived since the connection form $\omega$ does not depend on the triads $e$ before restriction to the constraint surface and thus $\frac{\delta H(H(N))}{\delta e^i}$ and $\frac{\delta H(H(N))}{\delta p_i}$ are both linear in $N$. In order to make the Hamiltonian constraint $H_H$ commute with $H_3$ we have to add a term $H_1$ with a certain argument which we now determine:
\[
\tilde{H}_H(X) := H_H(X) - H_1(\alpha_i(X))
\]
\[
\{\tilde{H}_H(X), H_3(\rho_{ij})\} \simeq \left[ -d(X \cdot e^{ij}) - X\omega^i_j \wedge \ast e^k - X\omega^i_k \wedge \ast e^j \right] \wedge \frac{1}{2} \rho_{ij}
\] (3.27)

Thus we obtain $\alpha^i \wedge e^j - \alpha^j \wedge e^i = +dX \wedge \ast e^{ij}$ or equivalently:
\[
\alpha^i \wedge \ast e_{ki} = +dX \wedge e_k
\]
In the same way as we transformed $p_i$ to $\omega_{0i}$ in (2.20) we obtain
\[
\alpha^i = -\langle dX \wedge e_i \rangle e^k + \frac{1}{2} \langle dX \wedge e_j \rangle e^i = (dX \wedge e^i)
\]
and our corrected Hamiltonian constraint $\tilde{H}$ reads:
\[
\tilde{H}(X) := H_H(X) - H_1(\ast(dX \wedge e_i))^4
\] (3.27)
$\tilde{H}$, $H_D$ and $H_R$ do now commute with all primary constraints. Forming the Poisson brackets among themselves we are already back to the ADM algebra if one restricts
\[
3 \simeq \text{denotes: after restriction to the second class surface}
\]
\[
4 \text{The correction term seems to be a pure surface integral}
\]
\[
H_1(\ast(dX \wedge e_i)) = \int_\Sigma \ast(dX \wedge e_i) \wedge (de^i + \omega^i_j \wedge e^j)
\]
\[
= \int_\Sigma D * (dX \wedge e_i) \wedge e^i - \int_\Sigma d \left[ \ast(dX \wedge e_i) \wedge e^i \right]
\]
\[
= \int_\Sigma \ast(\nabla dX - e_i \Delta X) \wedge e^i - \int_\Sigma \ast d \Delta \ldots
\]
\[
= -2 \int_\Sigma \nabla \nabla \nabla X \eta,
\]
but Gauss’s law does not hold for connections with torsion:
\[
\nabla \nabla e^i X \eta \neq L_{(e_i, \nabla)} \eta = d(\nabla e^i X \cdot \ast e^i)
\]
to the second class constraint surface after calculation of the brackets. We will show this for the two most difficult examples:

\[ \{\tilde{H}_H(X), \tilde{H}_H(Y)\} = -\{H_1(\ast(dX \wedge e^i)), H_H(Y)\} - \{H_H(X), H_1(\ast(dY \wedge e_i))\} \]

\[ = -\int \frac{\delta}{\delta e^i} \left[ \ast(dX \wedge e_k) \wedge (d\omega_k^j \wedge e^j) \right] \wedge Y \left( \langle p_j | e^i \rangle e^j - \frac{1}{2}pe^j \right) + (X \leftrightarrow Y) \]

\[ = -\int \left[ \frac{\delta}{\delta e^i} \ast(dX \wedge e_k) \right] \left( d\omega_k^j \wedge e^j \right) \wedge Y \left( \langle p_j | e^i \rangle e^j - \frac{1}{2}pe^j \right) + (X \leftrightarrow Y) \]

\[ \approx \int \left[ \ast(\nabla_i dX - e_i \Delta X) \wedge Y \left( \langle p_j | e^i \rangle e^j - \frac{1}{2}pe^j \right) \right] + (X \leftrightarrow Y) \Rightarrow \]

\[ H_D(XdY^z - YdX^z) - H_R(\langle dX \wedge dY | e^i \rangle + \langle XdY - YdX | \omega^i \rangle) \]  

(3.28)

In order to calculate the bracket \( \{H_D(\tilde{Y}), \tilde{H}_H(X)\} \) we split \( H_H(X) \) in the parts

\[ H_{H\text{kin}}(X; e, p) := \frac{1}{2} \int X \left( \langle p_i | e^j \rangle \langle p_j | e^i \rangle - \frac{1}{2}p^2 \right) \]  

(3.29)

\[ H_{H\text{pot}}(X; e, p, \omega) := -\frac{1}{2} \int X R_{ij}^e(\omega) \wedge *e^{ij} \]  

(3.30)

and obtain:

\[ \{H_D(\tilde{Y}), H_{H\text{kin}}(X)\} = H_{H\text{kin}}(L_{\tilde{Y}}X) \]

\[ \{H_D(\tilde{Y}), H_{H\text{pot}}(X)\} = \frac{1}{2} \int L_{\tilde{Y}}e^j \wedge XR_{jk}(\omega) *e^{jk} \]

\[ \approx \frac{1}{2} \int L_{\tilde{Y}} *e^{jk} \wedge XR_{jk}(\omega) \]

\[ = \frac{1}{2} \int \left[ L_{\tilde{Y}}(X *e^{jk} \wedge R_{jk}) \right. \]

\[ \left. - (L_{\tilde{Y}}X)(R_{jk} \wedge *e^{jk}) - X *e^{jk} \wedge L_{\tilde{Y}}R_{jk}(\omega) \right] \]

mod. bound. terms. \[ \Rightarrow \]

\[ H_{H\text{pot}}(L_{\tilde{Y}}X) - \int L_{\tilde{Y}}e^j \wedge *(\nabla_j dX - e_j \Delta X) \]

\[ \approx H_{H\text{pot}}(L_{\tilde{Y}}X) + \{H_D(\tilde{Y}), H_1(\ast(dX \wedge e^i))\} \]

Thus we have

\[ \{H_D(\tilde{Y}), \tilde{H}(X)\} \simeq H_H(L_{\tilde{Y}}X) \simeq \tilde{H}_H(L_{\tilde{Y}}X) \]  

(3.31)

since the correction term vanishes on the second class constraint surface.

In the last mainly technical step we must change \( \tilde{H}_H, H_D \) and \( H_R \) in such a way that they also commute with \( H_1 \) and \( H_2 \), the secondary second class constraints. It
The Hamiltonian constraint is quadratic in the momentum. We just show this for the case of the rotational constraint:

\[
\{H_{R}(Z_{ij}), H_{1}(\alpha^{i})\} = \int D\alpha^{i} \left[ \int \sum_{j} \nu_{ij} e^{j} \land D\alpha^{i} \right] \approx \int D_{ij} e^{i} \land e^{j} \\
\{H_{R}(Z_{ij}), H_{2}(\beta)\} = \int DZ_{ij} e^{i} \land e^{j}
\]

Thus we substitute \( H_{R} \) in the following way:

\[
H_{R}(Z_{ij}) := H_{R}(Z_{ij}) + K_{R}(Z_{ij}; \omega_{ij}, b_{i}, p_{\omega_{ij}}, p_{b}) := H_{R}(Z_{ij}) - H_{3}(DZ_{ij}) - H_{4}(Z_{ij}b^{j})
\]

We will show now at an example that having proved that there are exactly two pairs of second class constraints one could also have calculated the Dirac brackets of the original constraint functional which vanish after restriction to the constraint surface.

\[
H_{D}(\vec{Y}) := H_{D}(\vec{Y}) + K_{D}(\vec{Y}) \\
K_{D}(\vec{Y}) = K_{D}(\vec{Y}; e, \omega, b, N, p_{\omega}, p_{b})
\]

\[
H_{H}(X) := \tilde{H}(X) + K_{H}(X) \\
K_{H}(X) := -H_{3}\left( \frac{1}{2} \left[ i_{j} \beta_{j} - i_{j} \beta_{i} + e^{k} i_{ij} \beta_{k} \right] \right) - H_{4}\left( b_{j} (X(p_{j}) + e^{j}) - \frac{1}{2} p_{e_{j}} \right)
\]

The Hamiltonian constraint is quadratic in the momentum \( p \), and thus the correction term depends on \( p \). One should convince oneself that despite of the dependance of the integration functions on the canonical variables the Poisson brackets restricted to the second class constraint surface are unchanged. Using Leibniz’s rule for deriving the correction terms the derivation of the integration function yields in any case the constraint functional which vanishes after restriction to the constraint surface.

We will show now at an example that having proved that there are exactly two pairs of second class constraints one could also have calculated the Dirac brackets of the original constraints in order to find the Poisson bracket relations of the corrected first class constraints:

\[
\{H_{H}(X), H_{H}(Y)\}_{D} = \{H_{H}(X), H_{H}(Y)\} - \{H_{H}(X), H_{1}(\cdot)\} S^{ij}(\cdot, \cdot)\{H_{j}(\cdot), H_{H}(Y)\} = -\{H_{H}(X), H_{1}(\alpha)\} S^{13}(\alpha, \rho)\{H_{3}(\rho), H_{H}(Y)\} - \{H_{H}(X), H_{3}(\rho')\} S^{31}(\alpha', \rho')\{H_{1}(\alpha'), H_{H}(Y)\}
\]

where \( S^{ij} \) is the inverse of the matrix of second class constraints. This expression should be independent of the chosen integration forms \( \alpha \) and \( \rho \), and so we choose \( \alpha \) depending on \( Y \) resp. \( \alpha' \) depending on \( X \), such that the product \( S^{13}(\alpha, \rho) \cdot \{H_{3}(\rho), H_{H}(Y)\} \) and \( \{H_{H}(X), H_{3}(\rho')\} S^{31}(\alpha', \rho') \) is 1. The calculation will be equivalent to the determination of the correction term for \( H_{H} \) to \( \tilde{H}_{H} \):

\[
S^{13}(\alpha, \rho) = \left( \int \sum \rho_{ij} \land \alpha^{i} \land e^{j} \right)^{-1}
\]
\{H_3(\rho), H_H(Y)\} = \frac{1}{2} \int \rho_{ij} \wedge dY \wedge \ast e^{ij} = \int \langle \rho_{ij} | e^i \rangle \langle dY | e^j \rangle \eta \\
= \int \rho_{ij} \wedge \ast (dY \wedge e^i) \wedge e^j

So one has to choose \(\alpha^i = \ast (dY \wedge e^i)\) and \(\alpha^{\dot{i}} = \ast (dX \wedge e^{\dot{i}})\) and the calculation proceeds as shown in (2.33):

\{H_H(X), H_H(Y)\}_D = -\{H_H(X), H_1(\ast (dY \wedge e^i))\} + \{H_H(Y), H_1(\ast (dX \wedge e^{\dot{i}}))\} \\
\simeq H_D(XdY^2 - YdX^2) - H_R((dX \wedge dY) e^{\dot{j}}_j + \langle XdY - YdX | \omega^i_j \rangle)

One could as well have chosen \(\rho\) depending on \(X\) and \(\rho'\) depending on \(Y\), such that \(\{H_H(X), H_1(\alpha)\} S^{13}(\alpha, \rho)\) and \(S^{31}(\alpha', \rho')\{H_H(\alpha'), H(Y)\}\) is 1. This calculation is equivalent to the determination of the correction term for \(\bar{H}_H\) to \(H_H\).

Finally we have to show that there is an extension of our Hamiltonian by primary constraints which conserves all constraints. Since our Hamiltonian is a sum of constraints:

\[ H(e, p, N, \vec{N}, \omega, a, b) = H_H(N) + H_1(\ast (b_j e^{\dot{j}i})) + H_D(\vec{N}) + H_R(a) \]

and we know how to correct these constraints in order to get first class constraints we suppose that

\[ \mathcal{H} := \underline{H(e, p, N, \vec{N}, \omega, a, b) + K_H(N) + K_D(\vec{N}) + K_R(a)} + H_N(U) + H_N(\vec{V}) + H_A(W) \]

arbitrary primary first class term

(3.35)

is a first class Hamiltonian. This is actually the most general first class Hamiltonian for our infinite-dimensional problem ([3](1.32)). To prove that the first part is first class, we rewrite it in the following way:

\[ \mathcal{H} = H(e, p, N, \vec{N}, \omega, a, b) + K_H(N) + K_D(\vec{N}) + K_R(a) \]

\[ = H_H(N) + H_1(\ast (b_j e^{\dot{j}i})) + K_H(N) + H_D(\vec{N}) + H_R(a) \]

(3.36)

Forming Poisson brackets with primary constraints yield secondary constraints, forming Poisson brackets with secondary constraints yield secondary constraints, since one obtains the same result as if one had taken the bracket with

\[ H_H(N) + H_D(\vec{N}) + H_R(a) = H_H(N) - H_1(\ast (dN \wedge e^i)) + K_H(N) + H_D(\vec{N}) + H_R(a) \]

apart from a term \(H_1(\cdot)\) which is a result of the different derivations of the integrand functions of the \(H_1\) term and which vanishes on the surface described by the second class constraints.
Thus we have shown that three of the four different equations which represent in 3+1 dimensions that the connection on the fourdimensional manifold is torsionfree are realised by second class constraints, whereas the fourth is obtained by a first class constraint. If one restricts to the second class constraint surface from the beginning the only difference to the ADM formulation is that there is one more secondary constraint and that the part of the phase space where the rotational constraint does not equal zero can be interpreted as a region, where the zero component of the torsion does not vanish. As a by-product we have seen how Dirac’s framework for second class constraints works for a field theory when the canonical variables are forms rather than functions.

4 The Ashtekar formulation

It is well known the main idea of the Ashtekar formulation is the restriction to four dimensions and the use of a complex selfdual connection [6]. We will show that the effect is a canonical transformation in the complexified phase space. In order to obtain a simplification it is necessary to perform a second transformation which is normally done by the use of densities. We are free to decide if we want to regard the selfdual connection as an independent variable or if we prefer to work with the unique selfdual connection given by the selfdual projection of the Levi-Civit-connection. Since we know that the Palatini theory reduces to the ADM theory in the vacuum case – otherwise spin could couple with torsion [4] – we attempt a formulation where the four-dimensional connection is the complex selfdual projection of the Levi-Civit-connection. It is no contradiction that the connection form of the spatial restriction of the connection turns out to be one of the canonical variables, as in the ADM theory it is no contradiction that the momentum consists mainly of one component $\omega_0i$ of the connection. Before describing the canonical formulation we shortly summarize some selfdual notation. Our first step is the transition to the complexified tangential bundle over the real space-time manifold $M$. We consider the complexified frame bundle $L_C(M)$ which carries a certain real structure, because we still regard the manifold as real. We will regard the complexified bundle of tetrades where the gauge group is now $SO_C(1,3)$ as a subbundle of the tangential bundle. As usual there is a unique metric torsionfree connection whose components

$$\bar{\omega}_{\alpha\beta\gamma} = \frac{1}{2} \left[ g_{\alpha\beta,\gamma} - \bar{g}_{\beta\gamma,\alpha} + \bar{g}_{\gamma\alpha,\beta} + \bar{C}_{\alpha\beta\gamma} - \bar{C}_{\beta\gamma\alpha} + \bar{C}_{\gamma\alpha\beta} \right]$$  \hspace{1cm} (4.1)

are real only if they belong to a real basis, for example a coordinate basis.

In the Lie-algebra $so_R(1,3)$ one can define the dualization by

$$A_{\rho\sigma} := \frac{1}{2} \epsilon_{\rho\sigma}^{\mu\nu} A_{\mu\nu}$$  \hspace{1cm} (4.2)
Because of the signature of the metric one finds
\[ \tilde{A} = -A \] (4.3)
so the eigenvalues of the dualisation operator are \( \pm i \) and hence a decomposition in eigenvectors is only possible in the complexified algebra \( so_C(1,3) \). One calls elements of \( so_C(1,3) \) with the property
\[ \tilde{A} = iA \quad \text{selfdual} \]
and
\[ \tilde{A} = -iA \quad \text{antiselfdual}. \]
The subspaces
\[ so_C^+(1,3) := \{ A \in so_C(1,3) | \tilde{A} = iA \} \quad \text{and} \]
\[ so_C^-(1,3) := \{ A \in so_C(1,3) | \tilde{A} = -iA \} \]
form ideals and it holds
\[ so_C(1,3) = so_C^+(1,3) \oplus so_C^-(1,3) \]
For a proof one shows
\[ [\tilde{A},\tilde{B}] = [A,\tilde{B}] = [\tilde{A},B] \] (4.4)
One can introduce the projectors to the selfdual and antiselfdual parts
\[ P^\pm A := \frac{1}{2}(A \mp i\tilde{A}) \quad \quad P^\pm A :=: A^\pm \] (4.5)
and with help of (4.4) one can directly show
\[ P^\pm [A, B] = [P^\pm A, B] = [A, P^\pm B], \] (4.6)
which proves that \( so_C^\pm(1,3) \) form ideals of \( so_C(1,3) \). The connection form with respect to a tetrad base is \( so_C(1,3) \)-valued, and the associated curvature form is a \( so_C(1,3) \)-valued 2-form. Since
\[ \tilde{R}^+ + \tilde{R}^- = \tilde{R} = d\omega + \omega \wedge \omega = \underbrace{d\omega^+ + \omega^+ \wedge \omega^+}_{\text{selfdual}} + \underbrace{d\omega^- + \omega^- \wedge \omega^-}_{\text{anti-selfdual}} + \underbrace{[\omega^+ \wedge \omega^-]}_{0} \] (4.7)
the (anti-) selfdual part of the curvature is the curvature to the (anti-) selfdual part of the connection form. We now return to the action principle. Using the relation
\[ \tilde{R}_{\mu\nu} \wedge \bar{e}^{\nu} = 0 \] (4.8)
we can consider the selfdual Einstein-Hilbert action instead of the usual one.
\[ S = \int_{\Sigma} \tilde{R}_{\mu\nu}^+ \wedge \ast \bar{e}^{\mu} = \frac{1}{2} \int_{\Sigma} (\tilde{R}_{\mu\nu} \wedge \ast \bar{e}^{\mu} - i \tilde{R}_{\rho\sigma} \wedge \bar{e}^{\rho\sigma}) = \frac{1}{2} \int_{\Sigma} \tilde{R}_{\mu\nu} \wedge \ast \bar{e}^{\mu} \] (4.9)
Varying with respect to the tetrades yields (the complexified version of) Einstein’s equations. In order to obtain the Hamiltonian formulation we decompose the Lagrange form \((\epsilon_{0ijk} \equiv \epsilon_{ijk})\):

\[
\mathcal{L} = \tilde{R}^+_{\mu\nu} \wedge \ast \tilde{e}^{\mu\nu}
\]

\[
= 2\tilde{R}^+_{0i} \wedge \ast \tilde{e}^{0i} + \tilde{R}^+_{ij} \wedge \ast \tilde{e}^{ij}
\]

\[
= -i\epsilon_{ijk} \tilde{R}^+_{jk} \wedge \ast \tilde{e}^{0i} + \tilde{R}^+_{ij} \wedge \ast \tilde{e}^{ij}
\]

\[
= +i\epsilon_{ijk} \tilde{N} \tilde{R}^+_{jk} \wedge \ast \tilde{e}^{ij} + \tilde{N} \tilde{R}^+_{ij} \wedge \ast \tilde{e}^{0j}
\]

\[
\text{(4.10)}
\]

Now we use the following equation

\[
\tilde{N} \tilde{N} \tilde{R}^+_{jk} = \tilde{N} \left( d\tilde{\omega}^{+jk} + \tilde{\omega}^{+j}_{\mu} \wedge \tilde{\omega}^{+\mu k} \right) = i\tilde{N}^{e_0} d\tilde{\omega}^{+jk} + di\tilde{N}^{e_0} \tilde{\omega}^{+jk} - \tilde{\omega}^+ D(i\tilde{N}^{e_0} \tilde{\omega}^{+jk})
\]

\[
= L\tilde{N}^{e_0} d\tilde{\omega}^{+jk} - \tilde{\omega}^+ D i\tilde{N}^{e_0} \tilde{\omega}^{+jk},
\]

\[
\text{(4.11)}
\]

where \(L\) denotes the Lie-derivative and obtain

\[
\mathcal{L} = d\tilde{t} \wedge \left[ iL_{\partial/\partial t} \tilde{\omega}^{+jk} \wedge \tilde{e}^{jk} - iL\tilde{N} \tilde{\omega}^{+jk} \wedge \tilde{e}^{jk} - i\tilde{\omega}^+ D i\tilde{N}^{e_0} \tilde{\omega}^{+jk} \wedge \tilde{e}^{jk} + \tilde{N} \tilde{R}^+_{ij} \wedge \ast \tilde{e}^{ij} \right].
\]

\[
\text{(4.12)}
\]

Since

\[
\tilde{R}^+_{ij} = d\tilde{\omega}^{+ij} + \tilde{\omega}^{+ik} \wedge \tilde{\omega}^{+kj} + \tilde{\omega}^{+0i} \wedge \tilde{\omega}^{+0j}
\]

\[
= d\tilde{\omega}^{+ij} + 2\tilde{\omega}^{+ik} \wedge \tilde{\omega}^{+kj}
\]

\[
\text{(4.13)}
\]

one defines a connection form on \(\Sigma\) as follows

\[
A_{ij} := 2i\tilde{t}^* \tilde{\omega}^{+ij} = \omega_{ij} + i\epsilon_{ij}^k \omega_{0k}
\]

\[
\text{(4.14)}
\]

– here \(\omega_{ij}\) and \(\omega_{0i}\) denote the Levi-Civit-connection and the extrinsic curvature on \(\Sigma\) as usual – because for the curvature \(F\) defined by \(A\) holds

\[
F_{ij} = 2i\tilde{t}^* \left( d\tilde{\omega}^{+ij} + 2\tilde{\omega}^{+ik} \wedge \tilde{\omega}^{+kj} \right) = 2i\tilde{t}^* \tilde{R}^+_{ij}.
\]

\[
\text{(4.15)}
\]

Using the definition one obtains

\[
i\tilde{t}^* L_{\partial/\partial t} \tilde{\omega}^{+jk} = \frac{1}{2} \tilde{A}_{jk}
\]

\[
i\tilde{t}^* \tilde{\omega}^+ D i\tilde{N}^{e_0} \tilde{\omega}^{+jk} = \tilde{A} D i\tilde{N}^{e_0} \tilde{\omega}^{+jk} = \frac{1}{2} \tilde{A} D Z^j
\]

\[
Z^j = -Z^k := 2N i\tilde{t}^* i0 \tilde{\omega}^{+jk} = a_{jk} + i\epsilon_{jk}^l b_l.
\]

\[
\text{(4.16)}
\text{(4.17)}
\text{(4.18)}
\]

As usual we pass over to the spatial Lagrange form:

\[
\mathcal{L} = i\tilde{t}^* i\partial/\partial t \mathcal{L}
\]

\[
= \frac{1}{2} \left[ i\tilde{A}_{jk} \wedge \tilde{e}^{jk} - iL_N A_{jk} \wedge \tilde{e}^{jk} - i\tilde{A} D Z^j \wedge \tilde{e}^{jk} + NF_{ij} \wedge \ast \tilde{e}^{jk} \right]
\]

\[
\text{(4.19)}
\]

At this point one could recognize \(\frac{1}{2} i\epsilon_{jk}^l \tilde{A}\) as the canonical momentum of \(-\frac{1}{2} \epsilon_{jk}^l e^jk\). But this would be obvious only in a Palatini-like theory where the connection is not
fixed to be the selfdual projection of the Levi-Civit-connection. On the other hand usually one does not vary w.r.t. $A$ in order to obtain the Levi-Civit-connection. We are more careful and keep in mind that the definition of $A$ implies a dependance on the extrinsic curvature, which is closely linked to the time derivative of the triads. Thus the connection form $A$ itself depends on $\dot{e}$, so every term of the sum in the Lagrange form contributes to the calculation of the momentum. To determine the momentum we write the Lagrangian in the following way

$$L(e, \dot{e}, N, \vec{N}) = \frac{d}{dt} \int \sum i \frac{i}{2} A_{jk} \wedge e^k - \frac{i}{2} \int \sum L_N(A_{jk} \wedge e^k) - \frac{i}{2} \int \sum d(Z_{jk} e^j)$$

$$+ \int \left[ i \dot{e}^j \wedge A_{jk} \wedge e^k - i L_N e^j \wedge A_{jk} \wedge e^k + i Z_{jk} A e^j \wedge e^k + \frac{N}{2} F_{ij} \wedge * e^ij \right]$$

(4.20)

We ignore the boundary terms and notice that one of the terms vanishes in the ADM like Ashtekar theory:

$$Z_{jk} A e^j \wedge e^k = Z_{jk} \omega e^j \wedge e^k + i Z_{jk} \epsilon^{mj} \omega_{0me} e^l \wedge e^k$$

$$= Z_{jk} \omega e^0 \wedge * e^m \wedge e^k$$

$$= Z_{jk} (\omega^0 e^j) - \eta^{jk} (\omega^m e^m) \eta = 0$$

(4.21)

Considering this result we are not surprised since an explicit dependance on $Z_{ij}$ would mean an explicit dependance on the rotational parameter $a_{ij}$ and $b_i$ defined in (2.13) and (3.7), which does not appear in the ADM Lagrangian. But there is a dependance on $\dot{e}_A^i$ (2.13) contrary to the ADM theory in the first term:

$$i \dot{e}^j \wedge A_{jk} \wedge e^k = i \dot{e}_A^j \wedge A_{jk} \wedge e^k + i \dot{e}^j S \wedge A_{jk} \wedge e^k$$

$$= i \dot{e}_A^j \wedge \omega_{jk} \wedge e^k + i \dot{e}^j S \wedge A_{jk} (e^l, \dot{e}_S) \wedge e^k$$

$$= -i \dot{e}_A^j \wedge \omega_{jk} \wedge e^k + i \dot{e}_S^j \wedge A_{jk} (e^l, \dot{e}_S) \wedge e^k$$

and

$$A_{jk} = \omega_{jk} (e) + i \dot{e}_j \frac{1}{N} (L_N e^j - \dot{e}^i S)$$

(4.22)

(4.23)

So the Lagrangian depends linearly on $\dot{e}_A$, but is quadratic in $\dot{e}_S$. This means a second class constraint for $\dot{e}_A$, so that one can disregard this degree of freedom as one knows from the ADM theory. We can guess at this point that the canonical momentum which we derive now has an additional contribution $-ide^j$ and hence $\langle p_i | * e_j \rangle$ is no longer symmetric. We now derive the momentum-2-form.

$$L = \int \sum i \dot{e}^j \wedge A_{jk} \wedge e^k - i L_N e^j \wedge A_{jk} \wedge e^k + \frac{N}{2} F_{ij} \wedge * e^ij$$

(4.24)

$$\delta_{\dot{e}^i} \int \sum i \dot{e}^j \wedge A_{jk} \wedge e^k = i \int \delta \dot{e}^j \wedge A_{ik} \wedge e^k + i \int \dot{e}^j \wedge \delta e_i A_{jk} (e, \dot{e}) \wedge e^k$$

$$i \int \alpha^j \wedge \delta e_i A_{jk} (e, \dot{e}) \wedge e^k = -\frac{N}{2} \int \delta e_i (\dot{e}_S^j \wedge \alpha^j \wedge * e_{ij})$$

$$= -\frac{N}{2} \int \delta e_i (\dot{e}_S \wedge \alpha^j S \wedge * e_{ij})$$

$$= -\frac{N}{2} \int \delta \dot{e}_i \wedge \alpha^j S \wedge * e_{ij}$$
where \( \alpha^i \in \Omega^1(\Sigma) \) and \( \alpha^i S = \frac{1}{2}(\alpha^i + (\alpha^j | e^i) e_i) \) is defined as usual. Hence we obtain

\[
\frac{\delta}{\delta e^i} \int_{\Sigma} i e^j \wedge A_{jk} \wedge e^k = iA_{ij} \wedge e^j - \frac{1}{N} e^j S \wedge *e_{ij}
\]

\[
\frac{\delta}{\delta e^i} \int_{\Sigma} i L_N e^j \wedge A_{jk} \wedge e^k = \frac{1}{N} L_N e^j S \wedge *e_{ij}
\]

\[
\frac{\delta}{\delta e^i} \int_{\Sigma} \frac{N}{2} F_{jk} \wedge *e^{jk} = \frac{\delta}{\delta e^i} \int_{\Sigma} \frac{N}{2} (i^* R_{jk} + i e_{jk} i^* R_{0lj}) \wedge *e^{jk}
\]

\[
= \frac{\delta}{\delta e^i} \int_{\Sigma} \left[ \frac{N}{2} (R_{jk} + \omega_{0j} \wedge \omega_{0k}) \wedge *e^{jk} + i Ni^* \left( R_{0l} \wedge e^l \right) \right]
\]

\[
= -\omega_{0j} \wedge *e^j
\]

\[
p_i = \frac{\delta L}{\delta e^i} = iA_{jk} \wedge e^k - \frac{1}{N} (\dot{e}^j S - L_N \dot{e}^j S + N \omega_0^j) \wedge *e_{ij} = iA_{jk} \wedge e^k \quad . \quad (4.25)
\]

We compare this with the momentum 2-form of the ADM theory

\[
p_i = iA_{ij} \wedge e^j = i\omega_{ij} \wedge e^j - e^k \omega_{0j} \wedge e^j = -ide_i + p_i^{ADM} \quad (4.26)
\]

and notice that the new momentum does not transform homogeneous under rotations. As in the ADM theory the equation (4.25) can only be solved on a submanifold of the phasespace on which the following condition holds:

\[
0 = iA_{ij} \wedge e^j - p_i = -ide_i + \omega_{0l} \wedge *e^l - p_i
\]

\[
\Rightarrow \quad (p_i + ide_i) \wedge e_j - (p_j + ide_j) \wedge e_i = p_i \wedge e_j - p_j \wedge e_i + id(e_{ij}) = 0 \quad (4.27)
\]

Our Hamiltonian is only determined up to this constraint, so we obtain:

\[
H(\epsilon^i, p_i, N, \tilde{N}, Z) = \int_{\Sigma} \epsilon^i \wedge p_i - L - \int_{\Sigma} Z_{ij}(p^i \wedge \epsilon^j + ide^j \wedge \epsilon^i)
\]

\[
= \int_{\Sigma} \left[ p_i \wedge L_N \epsilon^i - Z_{ij}(p^i + ide^i) \wedge \epsilon^j - \frac{N}{2} F_{ij}(p) \wedge *e^{ij} \right] \quad (4.28)
\]

We call the arbitrary integration function of the rotational constraint \( Z \), because if we derive the equations of motion and want to compare the Hamiltonian equation with the geometrical equation obtained by \( i^* L_{Q/\bar{Q}} \) we have again to identify the integration parameter with a space-time quantity by equation (4.18). We will see in a moment that this Hamiltonian is not suitable for further consideration, since the substitution of the curvature form \( F_{ij} \) by momentum terms ends up in a lengthy expression. The substitution of \( A \) by \( p \) on the constraint manifold yields

\[
A_{ij} = -i(\langle p_k | e_{ij} \rangle e^k - \frac{1}{2} p * e_{ij}) \quad p := \langle p_i | * e^i \rangle \quad . \quad (4.29)
\]

For convenience we define also

\[
A_i := \frac{i}{2} \epsilon^{ijk} A_{jk} \quad A_{jk} = -i \epsilon_{ij}^k A_k \quad (4.30)
\]
and analogously \( F_i \) and \( Z_i \). Then the relation between \( A_i \) and \( p_i \) reads
\[
A_i = \langle p_k | * e_i \rangle e^k - \frac{1}{2} p e_i \quad \iff \quad p_i = A_k \wedge * e^k_i
\]
(4.31)
and finally we obtain for the last term of the Hamiltonian
\[
- \frac{N}{2} F_{ij} \wedge * e^{ij} = i N F_k \wedge e^k \quad \text{modulo exact forms}
\]
\[
= \frac{N}{2} \left( \langle p_i | * e^j \rangle \langle p_j | * e^i \rangle - \frac{1}{2} p^2 + 2i \langle p_i | * e_j \rangle \langle de^j | * e^i \rangle - ip \langle de_i | * e^i \rangle \right) \eta \\
- i (dN | e_i) p_j \wedge * e^{ij}.
\]
(4.32)
This is quite an awkward expression for deriving the Hamiltonian equations, but having in mind that
\[
N \frac{1}{2} R \eta = N \left[ -d(e^i \wedge de_i) - \frac{1}{2}(de_i \wedge e^i) \wedge *(de_j \wedge e^j) + \frac{1}{4}(de_i \wedge e^i) \wedge *(de_j \wedge e^j) \right]
\]
(4.33)
one can easily see that one could have obtained this Hamiltonian by a canonical transformation \((e^i, p_i^{ADM}) \mapsto (e^i, p_i^{ADM} - ide_i) =: (e^i, p_i)\) of the ADM Hamiltonian (2.23). So it is not surprising that the Hamiltonian equations one could derive from (1.28) are really equivalent to the ADM equations of motion, but they contain even more terms than in the ADM case. One notices that the canonical transformation is only possible for spatial dimension 3, since otherwise the 2-form \( de_i \) can not be added to the \( n - 1 \)-form \( p_i \). The gauge group \( SO(3) \) acts on the configuration space which is invariant under the canonical transformation. But wheras the action of the group to the momentum can in the ADM case be obtained by the obvious lift to the phase space, one has to transform also the group action to the new phase space in order to obtain the correct action of the group to the new momentum
\[
e^i \mapsto S^i_j e^j \quad p_i^{ADM} \mapsto S^{-1} i p_j^{ADM} = S_i^j p_j^{ADM}
\]
\[
p_i \mapsto S_i^j p_j - id S_i^j \wedge e_j \quad S \in \Sigma \times SO_c(3)
\]
(4.34)
In order to simplify the expression one considers again the Lagrange form and notices:
\[
L = \int_{\Sigma} \left[ - \frac{i}{2} A_{jk} \wedge (e^j k^k) + \frac{i}{2} A_{jk} \wedge L \wedge e^j k^k + \frac{N}{2} F_{ij} \wedge * e^j \right]
\]
(4.35)
One would like to perform another canonical transformation which turns the connection form \( A \) itself into the momentum. We now define and would like to determine the canonical conjugate \( q^i \in \Omega^2(\Sigma) \) such that
\[
\left\{ \int_{\Sigma} q^i \wedge \alpha_i, \int_{\Sigma} A_j \wedge \beta^j \right\}
\]
\[
= \int_{\Sigma} \left[ \left( \frac{\delta}{\delta e^k} \int_{\Sigma} q^i \wedge \alpha_i \right) \wedge \left( \frac{\delta}{\delta p_k} \int_{\Sigma} A_j \wedge \beta^j \right) - \left( \frac{\delta}{\delta p_k} \int_{\Sigma} q^i \wedge \alpha_i \right) \wedge \left( \frac{\delta}{\delta e^k} \int_{\Sigma} A_j \wedge \beta^j \right) \right]
\]
\[
= \int_{\Sigma} \alpha_i \wedge \beta^i
\]
where $\alpha_i \in \Omega^1(\Sigma)$ and $\beta^j \in \Omega^2(\Sigma)$. Using equation (4.31) one can calculate the derivatives of the second term.

$$\frac{\delta}{\delta p_k} \int \Sigma A_j \wedge \beta^j = e_j \langle \beta^j | * e_k \rangle - \frac{1}{2} e^k \langle \beta_j | * e^j \rangle$$

$$\frac{\delta}{\delta e^k} \int \Sigma A_j \wedge \beta^j = \beta^j \langle p_k | * e_j \rangle - \langle \beta^j | * e^j \rangle i_k p_i \wedge e_j - \frac{1}{2} \beta_k p + \frac{1}{2} i_k p_j \wedge e^j \langle \beta_i | * e^i \rangle$$

One can now guess $\frac{\delta}{\delta p_i} \int \Sigma q^i \wedge \alpha_i = 0$, because otherwise one can hardly get rid of the momentum dependence. Consequently one supposes for the canonical coordinate $q^i = c * e^i$ and obtains

$$\frac{\delta}{\delta e^k} \int \Sigma q^i \wedge \alpha_i = c e^i k \wedge \alpha_i = c * e^i k \wedge \alpha_i \quad \text{and}$$

$$c * e^i k \wedge \alpha_i \wedge \left( e_j \langle \beta^j | * e^k \rangle - \frac{1}{2} e^k \langle \beta_j | * e^j \rangle \right) = -c \alpha_i \wedge \beta^i$$

so our new configuration variable is

$$q^i = - * e^i. \quad (4.36)$$

Introducing coordinates $\{x^a\}_{a=1,2,3}$ for a moment we can link these variables with the densities used in Ashtekar’s formulation. Since in integrals one has expressions like

$$\int \Sigma q^i \wedge \alpha_i = \frac{1}{2} \int \Sigma dx^a \wedge dx^b \wedge dx^c q_{ab} \alpha_{ic} = \frac{1}{2} \int \Sigma d^3 x e^{abc} q_{ab} \alpha_{ic}$$

we consider

$$\frac{1}{2} e^{abc} q_{ab} = -\frac{1}{4} \eta^{il} e^{abc} \epsilon_{jkl} e^{jk}_{ab} = -\frac{1}{4} \eta^{il} e^{abc} \epsilon_{jkl} e^{jk} c_{ab} = -\eta^{il} e^{i c} \det e_a = \frac{E^c_i}{\det e^a} = E^c_i \quad (4.37)$$

where $e^i_a = e^i(\partial/\partial x^a)$, $e^i_a = dx^a(\epsilon_i)$ and thus $e^i_a e^a_j = \eta^i_j$. So the densities $E^c_i$ are just the coordinate expressions of the 2-forms $q^i$ – up to a sign and an $\epsilon$-symbol usually hidden in the volume $d^3 x$. The densities $E^a_i$ determine directly the coordinates of the dual triads

$$e^a_i = \frac{E^a_i}{\det^{1/2} E^a_i} \quad (4.38)$$

and these of course determine the triads itself

$$e^i_a = \frac{1}{2 \det e^a_i} e^{ijk} \epsilon_{abc} e^b_j e^c_k = \frac{1}{2 \det^{1/2} E^a_i} e^{ijk} \epsilon_{abc} E^b_j E^c_k \quad (4.39)$$

what insures that the metric can be reconstructed from the new variables.

Substituting $(e^i_i, p_i)$ in the Hamiltonian by $(q^i, A_i)$ one obtains

$$H(q^i, A_i) = \int \Sigma A_k \wedge * e^k i + L_N e^i - Z_{ij} (A_k \wedge * e^k i + i d e^i) \wedge e^j - \frac{N}{2} e^{ij} F_{ij} \wedge e^k$$

$$= \int \Sigma A_k \wedge L_N q^k + Z_k D q^k - i N F_k(A) \wedge * q^k \quad (4.40)$$
where
\[ Dq^k = dq^k + A^k_j \wedge q^j = dq^k - i\epsilon^k_{ji} A^i \wedge q^j \]
\[ D\vec{q} = d\vec{q} + i\vec{A} \wedge \vec{q} \]  
and
\[ F_k = dA_k + i\frac{1}{2} \epsilon_k^{ij} A_i \wedge A_j \]
\[ \vec{F} = d\vec{A} + i\frac{1}{2} \vec{A} \wedge \vec{A} \]

(4.41)

(4.42)

Using the vector notation we obtain what we will call the Ashtekar Hamiltonian:
\[ H(\vec{q}, \vec{A}) = \int_{\Sigma} \left[ \vec{A} \wedge L_{\vec{A}} \vec{q} - \vec{Z} \wedge D\vec{q} - iN \vec{F} \wedge *\vec{q} \right] \]

(4.43)

One easily recognizes the three terms as diffeomorphism, rotational and Hamiltonian part. The derivation of the equations of motion is simple up to a single point: We have to derive a term \[ \int_{\Sigma} \alpha_i \wedge *q^i \] with respect to the 2-form \( q^i \). Reexpressed by the triads the problem reads:
\[ \frac{\delta}{\delta *e^i} \int_{\Sigma} e^m \wedge \alpha_m \quad \alpha_i \in \Omega^2(\Sigma) \]

Since there are as many independent dual triads as triads itself for any space dimension and one can reexpress the metric in terms of coordinates of the dual forms \( *e^i \) as well as in terms of the triads the problem has a simple solution which we described in (1.32). Then we find the following equations of motion:
\[ \dot{\vec{q}} = \frac{\delta H}{\delta \vec{A}} = L_{\vec{N}} \vec{q} - i\vec{Z} \times \vec{q} - iD(N *\vec{q}) \]
\[ \dot{\vec{A}} = -\frac{\delta H}{\delta \vec{q}} = L_{\vec{N}} \vec{A} + D\vec{Z} - N\vec{Ric} \vec{F} - \frac{N}{4} \vec{F} \wedge *\vec{q} \]

(4.44)

(4.45)

Here \( i_j F^j_i \) is the Ricci-form and \( F = \langle F_{ij}|e^i \rangle \) is the Ricci-scalar of \( F \). These equations of motion are the complex extensions of the ADM equations as can be checked using the following relations
\[ A_i = \frac{1}{2} \epsilon_i^{jk} A_{jk} = \frac{1}{2} \epsilon_i^{jk} \omega_{jk} - \omega_{0i} \]
\[ Z_i = \frac{1}{2} \epsilon_i^{jk} Z_{jk} = \frac{1}{2} \epsilon_i^{jk} a_{jk} - b_i = \frac{1}{2} \epsilon_i^{jk} a_{jk} + (dN|e_i) \]

(4.46)

(4.47)

In order to check the equations one needs the equation of motion for \( \omega_{0i} \) which can either be obtained from the ADM equations of motion or by use of the equation of motion for the extrinsic curvature \( K_{ab} \), where the indices are w.r.t. a fixed (coordinate) basis \( \{ dx^a \}_{a=1,2,3} \):
\[ \dot{K}_{ab} = L_{\vec{N}} K_{ab} + 2N K_{ac} K^c_b - NK K_{ab} \]
\[ -NRic R_{ab} + \nabla_a \nabla_b N + \frac{N}{4} (R - K_{cd} K^{cd} + K^2) h_{ab} \]

(4.48)
\[ h_{ab} = \eta_{ij} e^i (\partial_a) e^j (\partial_b) \]
\[ \omega_{0i} = -K_{ab} dx^b e^a_i \]
\[ e^a_i = dx^a (e_i) \]

Using

\[ \dot{e}^a_i = dx^a (\dot{e}_i) = -dx^a (e_j) \dot{e}^j (e_i) \]
\[ \dot{e}^j = L_N \dot{e}^j - a^j_\ell e^\ell - N \omega^j_0 \]

one finally obtains

\[ \dot{\omega}^0_i = - (K_{ab} e^a_i) \, dx^b \]
\[ = L_N \omega^0_i - D(\langle dN | e_i \rangle + a^\ell_i \omega^0_\ell) + \]
\[ N (\text{Ric}_{ij} - K^\ell_i K^j_\ell + K_{ij}) e^\ell - \frac{N}{4} (R - K_{kl} K^{kl} + K^2) \eta_{ij} e^j. \] (4.48)

It is nearly obvious that this equation is the real part of equation (4.45), if one regards all differential geometric objects as real, i.e. \( e^i, \vec{N}, N, a^i_j, \omega^0_i \). Regarding the equations of motion and having in mind the relation \( \langle dN | e_i \rangle + b_i = 0 \) and (4.47) one can easily understand why in the equation of motion for \( \dot{q} \) there appears a derivative term of the lapse contrary to the ADM theory and why there is no derivative term for the lapse in the equation for \( A \). If one decomposed \( Z \) into real and imaginary part another derivative term for the lapse would appear. We notice a difference to the usual Ashtekar formulation. There is no need to densitize our lapse function. If we had done so, the last term of equation (4.45) containing the Ricci-Scalar of \( F \) would vanish. This is not dramatic since we will see in a moment that the Hamiltonian constraint in the Ashtekar formulation is just \( F = 0 \) and since our constraint algebra will prove first class we will never leave the constraint surface. The Ricci-scalar-term corresponds to the last term in equation (4.48) which we recognize as the the Einstein-tensor \( G(n, n) \) or the Hamiltonian constraint of the ADM formulation here expressed by the extrinsic curvature instead of the momentum. But since one does usually not omit the terms with \( *e_i \) in the ADM equation (2.29) whose sum vanishes on the constraint surface and in order to perform a correct constraint analysis, we keep the Ricci-scalar term here. Finally we note that of course \( A \) does not transform homogeneous under rotations as well and that the rotational part in the equation of motion for \( A \) could have been derived easily from the canonical transformation of the \( SO(3) \)-action, using equations (4.31) and (4.34).

Finally we will perform the full constraint analysis. We have already encountered the first primary constraint, the rotational constraint. The secondary constraints associated with lapse and shift read – as usual we disregard all boundary terms:

\[ C_{Di} = \{ H, p_{N_i} \} = \frac{\delta}{\delta N_i} \int_{\Sigma} A_j \wedge L_\vec{N} q^j \]
\[ = A_j \wedge i_i dq^j + dA_j \wedge i_i q^j \]
\[ = i_i A_j Dq^j + F_j \wedge i_i q^j \] (4.49)
\[ C_H = \{ H, p_N \} = -i \vec{F} \wedge \vec{q} = -\frac{1}{2} F \eta \] (4.50)
Using the identity $i_q^j = -\epsilon_{ki}^j * q^k$ and the relation:
\[
(F \wedge \ast q)_i = \epsilon_{ijk}(F^j | e^k)\eta = -\frac{i}{4} \epsilon_{ijm} \epsilon_{krs}(F_{lm}| e^{rs})\eta
\]
\[
 = i(RicF \wedge \ast q)_i
\]  
we can write the diffeomorphism constraint as follows
\[
C_{Di} = i_i A_j Dq^j - i(RicF \wedge \ast q)_i
\]  
and the integrated version then reads
\[
H_D(\vec{N}) = \int_{\Sigma} \vec{A} \wedge L_N \vec{q} = \int_{\Sigma} \left[ \vec{A}(\vec{N}) D\vec{q} - i\vec{N}(RicF \wedge \ast \vec{q}) \right]
\]
We reformulate also the rotational constraint in our variables
\[
C^i_R = Dq^i
\]  
and notice that the differential form of the diffeomorphism constraint can be simplified using the rotational constraint can be simplified using the rotational constraint
\[
\tilde{C}_{Di} = -i_i F \wedge \vec{q} = -i(RicF \wedge \ast \vec{q})_i
\]  
Usually this version of the diffeomorphism constraint is used in the Ashtekar formulation, its integrated form read
\[
\tilde{H}_D(\vec{N}) = -\int_{\Sigma} i_i \vec{F} \wedge \vec{q} = -i \int_{\Sigma} \vec{N}(RicF \wedge \ast \vec{q}),
\]  
but we prefer $H_D = \int_{\Sigma} \vec{A} \wedge L_N \vec{q}$, because it is just the momentum mapping of action of the diffeomorphism group on the phase space. Thus it is not even necessary to calculate the Poisson-bracket between two diffeomorphism constraints in this form. That the differential part of the diffeomorphism constraint has a rotational part is not a special feature of the Ashtekar formalism as we have seen in (2.25). We can write the Hamiltonian as a sum of integrated constraints, neglecting all boundary terms:
\[
H = \int_{\Sigma} N^i C_{Di} + \int_{\Sigma} Z_i C^i_R + \int_{\Sigma} N C_H
\]  
We determine again the constraint algebra, proving that the constraints are first class:
\[
\{H_D(\vec{N}), H_D(\vec{M})\} = H_D(\{\vec{N}, \vec{M}\})
\]
\[
\{H_D(\vec{N}), H_R(\vec{Z})\} = H_R(\{L_N \vec{Z}\})
\]
\[
\{H_D(\vec{N}), H_H(N)\} = H_H(L_N N)
\]
\[
\{H_R(\vec{Z}_1), H_R(\vec{Z}_2)\} = -i H_R(\vec{Z}_1 \times \vec{Z}_2)
\]
\[
\{H_R(\vec{Z}), H_H(N)\} = 0
\]
\[
\{H_H(N), H_H(M)\} = H_D(NdM^t - MdN^t) - H_R(\{\vec{A} | NdM -MdN\})
\]
The first and the third equation are proved as in the previous sections.

\[
\{H_D(\vec{\mathcal{N}}), H_R(\vec{\mathcal{Z}})\} = \int_\Sigma \left[ iL_{\vec{\mathcal{N}}} \vec{A} \wedge (\vec{\mathcal{Z}} \times \vec{q}) + L_{\vec{\mathcal{N}}} \vec{q} \wedge D\vec{Z} \right]
\]

\[
= \int_\Sigma \left[ iL_{\vec{\mathcal{N}}} \vec{\mathcal{Z}}(\vec{A} \times \vec{q}) - L_{\vec{\mathcal{N}}} \vec{\mathcal{Z}} \wedge dq \right]
\]

\[
= + \int_\Sigma L_{\vec{\mathcal{N}}} \vec{\mathcal{Z}} \wedge D\vec{q} = H_R(L_{\vec{\mathcal{N}}} \vec{\mathcal{Z}})
\]

\[
\{H_R(\vec{\mathcal{Z}}_1), H_R(\vec{\mathcal{Z}}_2)\} = \int_\Sigma \left[ iD\vec{\mathcal{Z}}_1 \wedge (\vec{\mathcal{Z}}_2 \times \vec{q}) - iD\vec{\mathcal{Z}}_2 \wedge (\vec{\mathcal{Z}}_1 \times \vec{q}) \right]
\]

\[
= -i \int_\Sigma (\vec{\mathcal{Z}}_1 \times \vec{\mathcal{Z}}_2) \wedge D\vec{q} = -iH_R(\vec{\mathcal{Z}}_1 \times \vec{\mathcal{Z}}_2)
\]

\[
\{H_R(Z), H_H(N)\} = \int_\Sigma \left[ iD\vec{\mathcal{Z}} \wedge D(N \ast \vec{q}) + iN(\vec{\mathcal{Z}} \times \vec{q})(\text{Ric}F + \frac{1}{4}F \ast \vec{q}) \right]
\]

\[
= \left[ (\vec{F} \times \vec{\mathcal{Z}}) \wedge \ast \vec{q} - i(\text{Ric}F \wedge \vec{\mathcal{Z}}) \right]_{4.51} 0
\]

\[
\{H_H(N), H_H(M)\} = \int_\Sigma \left[ -iN(\text{Ric}F + \frac{1}{4}F \ast \vec{q}) \wedge D(M \ast \vec{q}) \right.
\]

\[
+ iM(\text{Ric}F + \frac{1}{4}F \ast \vec{q}) \wedge D(N \ast \vec{q}) \left. \right]
\]

\[
= \int_\Sigma i(NdM - M\text{d}N) \wedge \vec{\mathcal{Z}} \cdot \vec{\mathcal{Z}}
\]

\[
= \int_\Sigma i\text{Ric}F \wedge \ast \vec{q} \cdot (NdM - M\text{d}N) \ast \vec{q}
\]

\[
H_D(NdM^\sharp - M\text{d}N^\sharp) = \int_\Sigma \langle \vec{A} | NdM - M\text{d}N \rangle \wedge D\vec{q}
\]

\[
H_D(NdM^\sharp - M\text{d}N^\sharp) - H_R(\langle \vec{A} | NdM - M\text{d}N \rangle)
\]

So these variables permit a simple derivation of the constraint algebra. At this point we have managed to give an ”ADM like” Ashtekar formalism where only the metric was regarded as dynamical variable whereas the connection was fixed as the Levi-Civita-connection or its selfdual projection. In a Palatini-like Ashtekar theory, where also the connection is regarded as a variable, the solution of the constraints should lead to equation (1.46) by which the arbitrary connection form is linked to the Levi-Civita-connection of the triads and the extrinsic curvature. Our analysis also gives a simple interpretation of the reality constraints. We know that the Ashtekar Hamiltonian is just a canonical transformation of the complexified ADM Hamiltonian, so we consider first the complexified ADM theory. Since the differential equation is real for a real rotational term it is clear that a real initial condition has a real time development, so a pair of real triads and momentum forms \((e^i, p^{ADM}_i)\) as initial condition will produce a real development of the metric, even in the case that the rotational parameter is not real. Let us now suppose that the spatial metric represented by triads \(\eta_{ij}e^i \otimes e^j\) is real for a certain time. Then there is a \(SO_C(3)\)-valued function \(S\) on \(\mathbb{R} \times \Sigma\) such that \(\tilde{e}^i := S^i_j e^j\) are real triads, having real components with respect to a coordinate
basis. One can easily see that \((\tilde{e}^i, \tilde{\eta}_i^{ADM})\), \(\tilde{p}_i^{ADM} := S_i^j p_j^{ADM}\) satisfy the equations of motion (2.28, 2.29) for another rotational term. Considering the split equation for the triads (2.14) one notices that provided lapse and shift are real the extrinsic curvature \(\tilde{\omega}_0 i\) is real and consequently the momentum form \(\tilde{p}_i^{ADM}\) is real. Thus the most general initial condition which leads to a real metric is a pair of real forms \((e^i, p_i^{ADM})\) modulo of course a possible complex rotation. One could write the condition in the form

\[
\eta_{ij} e^i \otimes e^j = h = \text{real} \quad (4.59)
\]

\[
\eta^{k(i} e_i \otimes e_j \otimes p_k^{ADM} \wedge e^{j)} = \pi = \text{real}, \quad (4.60)
\]

but one could as well simplify the second condition by

\[
p_i \otimes e^i = \text{real} \quad (4.61)
\]

and replace the first one by

\[
\eta^{ij} p_i^{ADM} \otimes p_j^{ADM} = \text{real} \quad (4.62)
\]

Now we have only to transform this condition from the complexified ADM theory to the Ashtekar variables. Formally one can restate the reality conditions as \((q^i, A_i - \frac{1}{2} \epsilon^{ijk} \omega_{jk}(q) = -\omega_0 i\) is real modulo a complex rotation or

\[
\eta_{ij} q^i \otimes q^j = \text{real} \quad (4.63)
\]

\[
q^i \otimes \omega_0 (A, q) = \text{real}, \quad (4.64)
\]

but it is quite difficult to obtain \(\omega_{ij}\) from \(q^i\), so the second condition is difficult to check given the coordinates of \(q^i\) and \(A_i\). The first condition is obviously equivalent to the condition that the spatial metric \(h\) is real, so one can reexpress equation (4.60) in terms of Ashtekar variables using

\[
p_i^{ADM} = i\eta_{ij} De^j = -i\eta_{ij} D * q^j, \quad (4.65)
\]

which is a simple consequence of equations (4.25) and (4.26). If one introduces coordinates and uses the expressions (4.38) and (4.39) for the coordinates of triads and dual triads in terms of the densities one finds

\[
\pi^{ab}_{123} = \epsilon^{cde} (e_i \otimes e_j \otimes p^{(i} \wedge e^{j)})(dx^a, dx^b, \partial_c, \partial_d, \partial_e)
\]

\[
= \frac{-i}{\det E^a_i} (\vec{E}^j \times D_f \vec{E}^{(a)} \cdot \vec{E}^{(b)}) = \text{real}
\]

and this is up to the factor \(-1/\det E^a_i\), which is clearly real, the coordinate expression for the second reality constraint, which is usually derived by requiring that the Hamiltonian flow leaves the metric real [7]. So the reality constraints given in the Ashtekar theory just state that the reconstructed ADM metric and momentum is real.
Our analysis has shown that it might be useful to perform calculations in canonical gravity as far as possible without reference to a fixed reference frame. We have seen in which way the coordinate free version of Ashtekar’s formulation is related to the complexified ADM theory and how the rotational parameters of the complexified theory are linked to the real theory. One might ask if triads have a real physical significance or if they are only useful tools for deriving equations for the metric. In Minkowski space one usually derives the energy momentum tensor by considering the variation of the Lagrangian as a consequence of a translation in space-time which could be written as a derivation with respect to the 1-form $dx^\mu$. The obvious generalisation for a curved spacetime is the variation w.r.t. the four orthogonal 1-forms $e^\mu$ what really yields the energy-momentum-tensor. This could be seen as an argument that the tetrades really are physically significant.

Acknowledgment: I would like to thank P. Glößner for help and motivation.

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