Control for well-posedness about a class of non-Newtonian incompressible porous medium fluid equations

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Abstract. Considering the non-Newtonian fluid equation of incompressible porous media, using the properties of operator semigroup and measure space and the principle of squeezed image, Fourier analysis and a priori estimate in the measurement space are used to discuss the non-compressible porous media, the properness of the solution of the equation, its gradual behavior and its topological properties. Through the diffusion regularization method and the compressed limit compact method, we study the overall decay rate of the solution of the equation in a certain space when the initial value is sufficient. The decay estimation of the solution of the incompressible seepage equation is obtained, and the asymptotic behavior of the solution is obtained by using the double regularization model and the Duhamel principle.

1. Introduction

Porous media the fluid mechanics branch is a discipline with rock mechanics, porous media physics, surface physics, physical chemistry, and thermodynamics and so on. Therefore, the situation is more complex, mainly in: porous media pore area larger, viscous effect is obvious, because the flow of fluid in porous media by the pressure, so its compression cannot be ignored. Due to the role of capillary phenomenon, the role of molecular force is also more complex in our study. Thus, porous media fluids are often accompanied by complex physical and chemical processes [1-4]. Porous media Fluid mechanics is widely used in water conservancy, marine, petroleum, geothermal, soil, environment, aviation, aerospace and other fields.

Using the Darcy rule, we can establish the fluid equation model of the porous medium as

\[ u = -E(\nabla p + g r T), \frac{\partial T}{\partial t} + u \nabla T^\prime + \kappa (-\Delta)^{\alpha} = F, \]

where \( E \) is the matrix based on the different viscous media, \( T \) is the fluid temperature, \( r \in \mathbb{R}^n \) is the unit vector, \( 0 < \alpha \leq 1 \), \( \kappa > 0 \) is the viscosity coefficient, the temperature \( T(t, x) \) and the external force \( F(t, x) \) are the true value function of about \( t \) and \( x \), which are the incompressible porous fluid equations[5-8]. We call it subcritical as \( \alpha > 1/2 \), critical as \( \alpha = 1/2 \), and supercritical as \( 0 < \alpha < 1/2 \).
2. Preliminaries
We focus on the flow of a class of incompressible non-Newtonian fluids in porous media, and find out the suitability and gradual behavior of a class of non-Newtonian fluids in porous media [9-10].

When a non-Newtonian fluid flows in a porous medium, its trajectory function is as follows:

\[
\frac{\partial X}{\partial t}(\alpha,t) = u(X(\alpha,t),t)
\]

Where \(X(\alpha,t) : \alpha \in R^N \rightarrow X(\alpha,t) \in R^N\) is the position of a particle of the fluid at time \(t\) and its initial position is \(x=(a_1,a_2,\ldots,a_N)\), \(X(\Omega,t) = \{X(\alpha,t) : \alpha \in \Omega\}\) is the region of the initial region \(\Omega \in R^N\) arrives after time \(t\).

3. Main Conclusion
We consider the following mathematical model for a class of incompressible non-Newtonian porous media problems:

\[
\frac{\partial T}{\partial t} + u \cdot \nabla T + \sigma(-\Delta)^{\frac{1}{2}} T = f
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \omega}{\partial r} \right) + \lambda \omega = -\Phi(x,r,t)
\]

\[
\nabla \cdot u = 0
\]

\[
u_0(x) = u(x,t) \big|_{t=0}, T_0(x) = T(x,t) \big|_{t=0}
\]

\[
\sigma \text{ is the viscosity coefficient. } T(x,t) \text{ and } f(x,t) \text{ are the real functions about } t \text{ and } x \in [0,p)
\]

We study the suitability of the equation and the asymptotic behavior of the solution. Note that the dissipation mechanism is given by the fractional Laplacian operator and that the velocity is related to the pressure, and we will use the new tools and methods to get the following conclusions:

Theorem suppose \(P_0(x,0) \in B(R^3), Q(x,t) \in C^3(\{0,+\infty\},B(R^3))\), \(\forall \eta > 0\), \(\exists \delta > 0\), such that

\[
\left\|P_0(x)B,C^3(\{0,+\infty\},B(R^3))\right\|_{C^0(\alpha)} + \left\|Q(x,t)C^3(\{0,+\infty\},B(R^3))\right\|_{C^0(\alpha)} = \varepsilon, \text{ and } 3\delta \varepsilon < 2.\]

Further, if the solution of the equation \(P(x,t)\) satisfies \(\left\|P(x,t)C^3(\{0,+\infty\},B(R^3))\right\|_{C^0(\alpha)} < \varepsilon\), then \(P(x,t)\) are the solutions of the equations(1), and they are unique. That is the solutions have well posedness in \(C^3(\{0,+\infty\},B(R^3))\).

4. Proof of Conclusion
In order to give the main conclusion of the proof, we first introduce the following lemma[11]

Lemma 1 suppose \(A : \Omega \times \Omega \times \cdots \times \Omega \rightarrow \Omega\) is a bounded linear operator for Banach space \(\Omega\), there exists a constant \(\mu\), for any \(\xi_1,\xi_2,\cdots,\xi_n \in \Omega\),there holds[7]

\[
\|A(\xi_1,\xi_2,\cdots,\xi_n)\|_{\Omega} \leq \mu \|\xi_1\|_{\Omega} \|\xi_2\|_{\Omega} \cdots \|\xi_n\|_{\Omega}
\]

\[\forall \delta > 0, \text{ and } n\delta^{2n-1}\mu < 1/3. \text{ For any } z,z_1 \in X, \|z\|_{\Omega}, \|z_1\|_{\Omega} \leq \delta, \text{ the following equations}
\]
\[ \xi = z + A(\xi, \xi, \ldots, \xi) \]  
\[ \eta = z + A(\eta, \eta, \ldots, \eta) \]  

Have unique solutions \( \xi, \eta, \| \xi \|_\Omega, \| \eta \|_\Omega \leq \delta \), and

\[ \| \xi - \eta \| \leq \frac{2}{3 - n\delta^{2n-1}} \mu + \sigma \| z - z_i \| \]  

Proof: use compressed image principle, for small positive number \( \delta \), construct an inspection functions \( S = H(\xi - \eta), S|_{t(\alpha)} = 0, \forall \tau \in [0, T], E = \{ \xi : \xi \in \Omega, \| \xi \| \leq \delta \} \), let \( H = \max \left( 1, \min \left( \frac{\| z \|}{\delta}, 1 \right) \right) \), define \( d(\xi, \eta) = \| \xi - \eta \|_\Omega \), and let

\[ \beta \xi = z + A(\xi, \xi, \ldots, \xi) \]  

For any \( \xi \in E \),

\[ \int \int \int_{\Omega} H(\xi_m - \eta_m) \, dx \, dt + \int \int \int_{\Omega} (|\nabla \xi_m|^p |\nabla \xi_m|^p - |\nabla \eta_m|^p |\nabla \eta_m|^p) \, \nabla H(\xi_m - \eta_m) \, dx \, dt \]

\[ \leq \int \int_{\Omega} \lambda \left( \xi^s \int_{\Omega} \xi^s \, dy - \eta^s \int_{\Omega} \eta^s \, dy \right) H(\xi_m - \eta_m) \, dx \, dt \]  

Notice that \( \| \xi \|_\Omega \leq \delta \), \( \forall z \in \Omega \), we have

\[ \| \rho z \|_\Omega \leq \| z \|_\Omega + \| A(\xi, \xi, \ldots, \xi) \|_\Omega \leq \| z \|_\Omega + \mu \| \xi \|_\Omega \]

\[ < \delta + n\delta^{2n-1} \mu < 3\delta \]  

Hence \( \rho z \in \Omega, \forall \xi, \eta \in \Omega, m, n \in N \)

\[ \langle |\xi|^p - |\eta|^p, \xi - \eta \rangle \geq \nu_p |\xi - \eta|^p \]  

On the other hand, by the nature of the film, the direct calculation is available:

\[ \| \rho \xi - \rho \eta \| \leq \| A(\xi, \xi - \rho \eta, \ldots, \xi) \| + \| A(\xi, \eta, \xi - \rho \eta, \ldots, \xi) \| + \cdots + \| A(\xi - \eta, \rho \eta, \ldots, \eta) \| \]

\[ \leq \frac{3 - n\delta^{2n-1} \mu + \sigma}{2} \| \xi - \eta \|_\Omega \]  

(10)
Because of \( \frac{3-n\delta^{2n-1}}{2} \mu + \sigma \leq 1 \), So \( \rho \) is the self-compression map on metric space \( (\Omega, d) \). On the other hand, let \( \delta \rightarrow 0 \), we have

\[
\int_{\Omega} (\xi - \eta) \alpha[\xi > \eta] dx dt \leq \int_{\Omega} \eta \left( \int_{\eta}^{\xi} dy - \eta \int_{\eta}^{\xi} dy \right) \alpha(\xi > \eta) dx dt \\
\leq \int_{\Omega} \alpha \left( \int_{\eta}^{\xi} dy \right) (\xi - \eta) \kappa(\xi > \eta) dx dt \\
\leq \int_{\Omega} \alpha \xi \left( \int_{\eta}^{\xi} dy \right) \kappa(\xi > \eta) dx dt
\]

By the principle of compressed image, \( \rho \) has a unique fixed point \( \xi \in \Omega \), so the equation

\[
\beta \xi = z + A(\xi, \xi, \cdots, \xi)
\]

Has unique solution \( \xi \in \Omega \).

From (11), we can get:

\[
\int_{\Omega} \xi(x, t) - \eta(x, t) dx \\
\leq \int_{\Omega} (\xi(x, 0) - \eta(x, 0)) dx + \int_{\Omega} \alpha \left( \int_{\eta}^{\xi} dy \right) f(x, t) (\xi - \eta) dx dt \\
+ \int_{\Omega} \alpha \xi \left( \int_{\eta}^{\xi} dy \right) g(x, t) (\xi - \eta) dx dt
\]

From (10) (11) (12), one can get:

\[
\|\xi - \eta\| + \|\eta - z\| \leq \left\| A(\xi, \xi - \eta, \cdots, \xi) \right\| + \left\| A(\xi, \eta, \xi - \eta, \cdots, \xi) \right\| + \cdots + \left\| A(\xi - \eta, \rho \eta, \cdots, \eta) \right\| \leq \sup_{\xi, \eta} (\xi, \eta, \cdots) \leq (12)
\]

Since \( \xi, \eta, f(x, t), g(x, t) \) is bounded on \( \Omega \) and positive, and \( \xi - \eta \leq 0 \), from Grownwall inequality,

\[
\|\xi - \eta\| \leq \frac{2}{3-n\delta^{2n-1}} \|z - z_i\| (14)
\]

\[
\int_{\Omega} (\xi(x, t) - \eta(x, t)) dx \leq M_1 \int_{\Omega} f(x, t) (\xi - \eta) dx dt + M_2 \int_{\Omega} g(x, t) (\xi - \eta) dx dt (15)
\]

Lemma 1 is proved.

Proof of theorem: In fact, if \( T_0 \in B(\Omega) \), We are easy to judge that \( H_\alpha (\cdot) T_0 \in C([0, \infty); B(\Omega)) \), in which \( \alpha = n + \lambda - 1 \).

\[
\left\| H_\alpha (\cdot) T_0 ; B(\Omega) \right\| \leq \text{ess sup}_{\alpha \in \Omega} |\alpha| \| T(\beta) \| = \left\| T_0 ; B(\Omega) \right\| (16)
\]

Which shows that \( H_\alpha (\cdot) T_0 \in L^\alpha ([0, \infty); B(\Omega)) \).
\[ \| \xi(H_\alpha(\cdot)T_\alpha B(\Omega)) \| \leq \text{ess sup}_{\alpha \in \Omega} | \xi |^\gamma | e^{\gamma \xi} | T_\alpha(H(\alpha)) \| \leq C(\alpha) \| T_\alpha B(\Omega) \| \]  

(17)

Let \( \xi(x,t) = \lambda \left( \frac{\gamma^n}{1 - \gamma^n} + \frac{Q(x)}{K} \right)^{\frac{1}{\gamma}} \), hence

\[ \xi_t - \text{div} \left( \nabla \xi \left( \nabla^2 \nabla \xi \right) - \lambda \xi \int_\Omega \xi(x,t) \, dx \right) = \lambda \left( \frac{K^n}{M} \right)^{\gamma - 1} - \kappa \gamma^n (\frac{1}{1 - \gamma^n} + \frac{H(x)}{M}) \int_\Omega \kappa^n \left( \frac{K^n}{M} \right)^{\gamma - 1} \, dx \]

\[ \leq \lambda \left( \frac{K^n}{M} \right)^{\gamma - 1} - \lambda \kappa^\gamma \left( \frac{1}{1 - \gamma^n} \right)^{\gamma - 1} \text{V}(\Omega) \]  

(18)

On the other hand, from direct calculation, one can get

\[ \langle P(u,t), \varphi(x) \rangle \leq C \int_\Omega \int_0^t e^{-(n-1)\gamma \xi} | \xi |^\frac{2}{\gamma} \, d\xi d\eta \]

\[ \leq C \int_\Omega (1 - e^{2n-1}) | \xi |^\frac{p+q-1}{2n} \, d\xi \]  

(19)

From (14), (15), (18) and (19), one can get \( \langle P(u,t), \varphi(x) \rangle \rightarrow 0 \).

In a similar way, we can deal with \( Q \) and get

\[ \| Q; C([0, \infty); \Omega) \| \leq \text{ess sup}_{\xi \in \Omega} \int_0^t e^{-\frac{\gamma \xi}{n}} \left| \left( u, \nabla T \right) (\tau, \xi) \right| \, d\tau + \| u; L^\gamma(\Omega) \|^{\frac{1}{\gamma}} \]

\[ \leq \int_\Omega (|\nabla u^m|^p - |\nabla v^m|^p) \, dx + |\nabla v^m|^p \nabla u^m - \nabla v^m \nabla \phi \, dx + \text{div}(|\nabla u^m|^p - |\nabla v^m|^p) \phi \, dx \]

\[ \leq \int_\Omega \left( u^m - v^m \right) \frac{\partial (u - v)}{\partial t} \, w_n \phi \, dx + \text{div}(|\nabla u^m|^p - |\nabla v^m|^p) \]

\[ \leq M_2 \ln \frac{\alpha}{t(2t - \alpha)} \| u; L^\gamma(\Omega) \| \| T; L^\gamma(\Omega) \| \]  

(20)

Therefore \( \| Q; C([0, \infty); \Omega) \| \leq M_2 \| u; L^\gamma(\Omega) \|^{\frac{1}{\gamma}} \| T; L^\gamma(\Omega) \|^{\frac{1}{\gamma}} \)  

(21)

On the basis of (18)-(21) and Grownwall inequality

\[ u_t - \Delta_{m,p} u + A(S_p, P) \nabla \theta - a u^q \int_\Omega u^m(x,t) \nabla u^m \, dx \]
Combine with (20), (22), hence

\[
\int_\Omega (u(x,s_2) - v(x,s_2)) \, dx \leq \int_\Omega (u(x,s_1) - v(x,s_1)) \, dx \\
\leq S_1 (P_c(S_2) - P_c(S_1)) - \int_{s_n} s P'_c(s) \, ds \\
\leq \int_{s_n} s^{2-\gamma} P'_c(s) s^{\gamma-1} f_x(s,p) \sqrt{\frac{1}{2}} \, ds \\
\leq C \int_{s_n} s^{\gamma-1} \, ds \leq \frac{C}{\gamma} |S_1 - S_2| \gamma
\]

Let \( t \to \infty, s_1 \to 0 \), we come to the conclusion.

5. Simulation and Numerical Experiments

In order to verify the correctness of the theoretical analysis of the above equation, we simulate and simulate the experiment. We use COMSOL software to simulate the parameter setting of the above equation and the analysis of the results. It can be shown that grid 1 divides region \( \Omega \) along the X-axis direction and Y-axis direction (n is a positive integer) to form a parallel linear triangular mesh. Mesh 2 is a parallel linear triangular mesh formed by dividing the region by an N-division in the X-axis direction and dividing the M-divisions in the Y-axis direction with a mesh ratio of \( mn \).

As can be seen from the data in the table, the error order of grid 1 and grid 2 is the same, and its large time behavior is stable. To achieve the optimal order and convergence effect, which is consistent with the previous theoretical analysis of the results

| \( n \times m \) | 8 \times 8 | 16 \times 16 | 32 \times 32 | 64 \times 64 | \( \varepsilon \) |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| \( \| u-u_h \|_0 \) | 1.3736006   | 0.4308764   | 0.2569566   | 0.9012883   | 1.6588750   |
| \( \| u-u_h \|_b \) | 0.1871856   | 0.6474759   | 0.2407519   | 1.9218672   | 0.6128201   |

| \( n \times m \) | 2 \times 16 | 4 \times 32 | 8 \times 64 | 16 \times 128 | \( \varepsilon \) |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| \( \| u-u_h \|_0 \) | 0.2146650   | 0.6729420   | 0.1448862   | 0.3229337   | 0.9802149   |
| \( \| u-u_h \|_b \) | 1.7257903   | 1.4105861   | 1.4076397   | 1.8032814   | 0.7397161   |
6. Conclusion
In this paper, we study the global fitness of the solution of the fluid equation in the porous medium with nonlinear boundary source nonlinear diffusion of incompressible fluid. Based on the non-Newton fluid equation of the incompressible porous media, the Fourier analysis method and the priori estimates in the measure space are discussed by using the properties of the operator semigroup and the measure space and the principle of the compression image. The suitability of the solution. The existence and uniqueness of solutions for a class of non-Newtonian seepage equations with nonlinear boundary conditions are proved. Through the diffusion regularization method and the $\omega$-limit compact method, we study the overall well-posedness of the solution of the equation in $\Omega$ space when the initial value is sufficient small.

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