WHAT ABOUT $A,B$ IF $AB-BA$ AND $A$ COMMUTE.

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Abstract. Let $A,B$ be complex $n \times n$ complex matrices such that $AB-BA$ and $A$ commute. We show that, if $n=2$ then $A,B$ are simultaneously triangularizable and if $n \geq 3$ then there exists such a couple $A,B$ such that the pair $(A,B)$ has not property L of Motzkin-Taussky and such that $B$ and $C$ are not simultaneously triangularizable.

Notations. i) If $U$ is a square matrix, then $\sigma(U)$ denotes the spectrum of $U$.

ii) Let $A,B$ be complex $n \times n$ complex matrices. If there exists an invertible matrix $P$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular then we say that $A$ and $B$ are ST.

iii) Denote by $I_n$ and $0_n$ the identity matrix and the zero matrix of dimension $n$.

Definition. (See [3]). A pair $(A,B)$ of complex $n \times n$ matrices is said to have property L if for a special ordering of the eigenvalues $(\lambda_i)_{1 \leq n},(\mu_i)_{1 \leq n}$ of $A,B$, the eigenvalues of $xA+yB$ are $(x\lambda_i+y\mu_i)_{1 \leq n}$, for all values of the complex numbers $x,y$.

Remark. If $A,B$ are ST then $(A,B)$ has property L, but the converse is false (see [3]).

We deal with the couples $(A,B)$ such that $AB-BA$ and $A$ commute. If $(A,B)$ is such a couple, then for every complex numbers $\lambda, \mu$, $(A+\lambda I_n,B+\mu I_n)$ is another one. Then we may assume that $A$ and $B$ are invertible matrices, or on the contrary, that they are singular. In the sequel, we put $C=AB-BA$.

Several well-known results are gathered in the following Proposition.

Proposition 1. Let $A,B$ be complex $n \times n$ matrices. We assume that $C$ and $A$ commute. Then $C$ is a nilpotent matrix and the pair $(B,C)$ has property L of Motzkin-Taussky. Moreover, if $A,B$ are invertible, then $A^{-1}B^{-1}C,B^{-1}A^{-1}C$ and $B^{-1}C$ are nilpotent matrices.

Proof. $C$ is nilpotent by virtue of [1]. According to [2], one has, for every real $t$, $e^{tA}Be^{-tA}=B+tC$ and therefore $\sigma(B+tC)=\sigma(B)$. Reasoning by a continuity argument, we can conclude that the pair $(B,C)$ has property L.

Now we assume that $A,B$ are invertible. One has $A^{-1}CB^{-1}=CA^{-1}B^{-1}=ABA^{-1}B^{-1}-I_n$. In [3] Theorem 2, it is shown that $ABA^{-1}B^{-1}-I_n$ is a nilpotent matrix. Since $\sigma(A^{-1}B^{-1}C)=\sigma(CA^{-1}B^{-1})=\{0\}$ and $\sigma(B^{-1}A^{-1}C)=\sigma(A^{-1}CB^{-1})=\{0\}$, we conclude that $A^{-1}B^{-1}C$ and $B^{-1}A^{-1}C$ are also nilpotent matrices. Finally the fact that $CB^{-1}$ is nilpotent (or equivalently $B^{-1}C$ is nilpotent) is also proven in [3] (see the proof of theorem 1).

There are strong relations on the one hand between $A$ and $C$ and on the other hand between $B$ and $C$. We may wonder whether $A$ and $B$ are simultaneously triangularizable or, at least, the pair $(A,B)$ has property L. We have a positive answer in the following case.
Definition. A complex matrix $A$ is said to be non-derogatory if for all $\lambda \in \sigma(A)$, the number of Jordan blocks of $A$ associated with $\lambda$ is 1.

Proposition 2. If $A$ is a non-derogatory matrix and if $AC = CA$, then $A$ and $B$ are ST.

Proof. Necessarily, $C$ is a polynomial in $A$. According to [5, Theorem 1], $A$ and $B$ are ST. □

Remark. i) Note that the set of derogatory matrices is included in the set NS of non-separable matrices (they have at least one multiple eigenvalue). NS is an algebraic variety in $M_n(C)$ of codimension 1 and therefore is a null set in the sense of Lebesgue measure (see [6] for an outline of the proof).

ii) If we fix the matrix $A$, then the equation $A(AB - BA) = (AB - BA)A$ is linear in the unknown $B$. More precisely $B \in \ker(\phi)$ where $\phi : X \to A^2 X + X A^2 - 2 X A X$. Therefore $\phi = A^2 \otimes I + I \otimes (A^T)^2 - 2 A \otimes A^T = \psi^2$ where $\psi = A \otimes I - I_n \otimes A^T$. Thus, if $\sigma(A) = (\lambda_i)_i$, then $\sigma(\psi) = (\lambda_i - \lambda_j)_i,j$ and $\sigma(\phi) = ((\lambda_i - \lambda_j)^2)_i,j$. We deduce that the expression $i(A) = \dim(\ker(\psi^2)) - \dim(\ker(\psi))$ is linked to the existence of $B$ such that $AB - BA$ and $A$ commute and such that $A, B$ are not ST.

Now we prove our main result.

Proposition 3. i) If $n = 2$ and $CA = AC$, then $A$ and $B$ are ST.

ii) If $n \geq 3$ then there exists a couple $A, B$ such that $AB - BA$ and $A$ commute, satisfying

- the pair $(A, B)$ has not property L.
- $B$ and $C$ are not ST.

Proof. i) According to a previous remark, we may assume that $A$ is derogatory, that is a scalar matrix, and we conclude immediately.

ii) It is sufficient to find such a counterexample $(A_0, B_0)$ when $n = 3$. Indeed, if $n > 3$, then consider the couple $(A_0 \oplus 0_{n-3}, B_0 \oplus 0_{n-3})$.

Now we suppose $n = 3$ and $A_0$ is chosen as a derogatory matrix, for instance

$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here $\psi$ is nilpotent, $\dim(\ker(\psi)) = 5$, $\dim(\ker(\psi^2)) = 8$ and we search associated matrices $B$. Amongst numerous solutions, we choose this one

$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

- $(A_0, B)$ has not property L because $\sigma(A_0) = \{0\}$ and for every couple of complex numbers $(t, x)$, $\chi_{t A_0 + B}(x) = x^3 - t$.
- We observe that $\text{Trace}(B^2 C^2) = -1$, that implies that $B$ and $C$ are not ST. □

To show that two complex matrices are ST, the McCoy Theorem (see [8]) contains no finite verification procedure. The following test admits a finite one (see [7, Theorem 6]).

Theorem 1. Two $n \times n$ complex matrices $A$ and $B$ are ST if and only if for every $k \in \{1, n^2 - 1\}$, each matrix in the form $U_1 \cdots U_k(AB - BA)$ (where, for every $i$, $U_i$ is $A$ or $B$) has a zero trace.

Proposition 4. If $n \geq 4$, then there exist derogatory matrices $A_1$ such that $A_1$ and each associated matrix $B$ are ST.
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Proof. We take $n = 4$ and $A_1 = \text{diag}(J_2, J_2)$ where $J_2$ is the Jordan nilpotent block of dimension 2. Here $\text{dim}(\ker(\psi)) = 8$, $\text{dim}(\ker(\psi^2)) = 12$ and $i(A_1) = \frac{1}{4} < i(A_0)$.

The associated matrices $B$ are in the form $B = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix}$ where each $*$ represents an arbitrary complex entry. Using Theorem 1, we verify (with Maple software) that the 65534 considered matrices have a zero trace. Thus $A_1$ and $B$ are $ST$. □

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