ON THE $C^*$-ALGEBRA GENERATED BY THE KOOPMAN REPRESENTATION OF A TOPOLOGICAL FULL GROUP

EDUARDO SCARPARO

Abstract. Let $(X, T, \mu)$ be a Cantor minimal system and $[\![T]\!]$ the associated topological full group. We analyze $C^*_\pi([\![T]\!])$, where $\pi$ is the Koopman representation attached to the action of $[\![T]\!]$ on $(X, \mu)$.

Specifically, we show that $C^*_\pi([\![T]\!]) = C^*_\pi([\![T]\!]')$ and that the kernel of the character $\tau$ on $C^*_\pi([\![T]\!])$ coming from containment of the trivial representation is a hereditary $C^*$-subalgebra of $C(X) \rtimes \mathbb{Z}$. Consequently, $\ker \tau$ is stably isomorphic to $C(X) \rtimes \mathbb{Z}$, and $C^*_\pi([\![T]\!]')$ is not AF.

We also prove that if $G$ is a finitely generated, elementary amenable group and $C^*(G)$ has real rank zero, then $G$ is finite.

1. Introduction

In this work, we study the real rank zero and AF properties for certain classes of group $C^*$-algebras. The motivations are the classical equivalence between amenability of a group and nuclearity of its $C^*$-algebra, and the equivalence between local finiteness of a group and finiteness of its uniform Roe algebra worked out in [21], [11] and [20].

For a compact metric space $X$, both $C(X)$ being AF and having real rank zero are equivalent to total disconnectedness of $X$.

If a group $G$ is countable and locally finite, then $C^*(G)$ is clearly AF (hence it has real rank zero). Conversely, in [11] Theorem 2, Kaniuth proved that if $G$ is a nilpotent group and $C^*(G)$ has real rank zero, then $G$ is locally finite.

In section 2 we show that if $G$ is a finitely generated, elementary amenable group, and $C^*(G)$ has real rank zero, then $G$ is finite (Theorem 2.4). Our proof relies on the fact that infinite, finitely generated, elementary amenable groups virtually map onto $\mathbb{Z}$ [7, Chapter I, Lemma 1].

Let $(X, T, \mu)$ be a Cantor minimal system and $\pi$ the Koopman representation associated to the action of the topological full group $[\![T]\!]$ on $(X, \mu)$.

Notice that $C^*([\![T]\!])$ does not have real rank zero, since $[\![T]\!]$ maps onto $\mathbb{Z}$ (by [19] Theorem 1.1(i)), or [5] Proposition 5.5). On the other hand, by results of Matui, the commutator subgroup $[\![T]\!]'$ is simple [13] and non-locally finite (this follows from much sharper results from [14]). Hence, commutators of topological full groups form a class which is not covered by Theorem 2.4.

Furthermore, it was proven by Juschenko and Monod [8] that $[\![T]\!]$ is amenable.

In Section 3 we prove that $C^*([\![T]\!]')$ is not AF. This is done by showing that $C^*_\pi([\![T]\!]) = C^*_\pi([\![T]\!]')$, and that the kernel of the character $\tau$ on $C^*_\pi([\![T]\!])$ coming...
from containment of the trivial representation is a hereditary $C^*$-subalgebra of $\mathcal{C}(X) \rtimes \mathbb{Z}$. Consequently, $\text{ker } \tau$ is stably isomorphic to $\mathcal{C}(X) \rtimes \mathbb{Z}$, and $C^*_\tau([\mathcal{T}])'$ is not AF and has real rank zero.

In Section 4, we discuss examples coming from odometers.

2. Elementary Amenable Groups and Real Rank Zero

Recall that the class of elementary amenable groups is the smallest class of groups containing all abelian and all finite groups, and closed under taking subgroups, quotients, extensions and inductive limits.

In order to show Theorem 2.4, we will need the fact, due to Hillman ([7, Chapter I, Lemma 1]), that if $G$ is an infinite, finitely generated, elementary amenable group, then there is a finite index subgroup of $G$ which admits a homomorphism onto $\mathbb{Z}$. For completeness, we provide a proof of this result in Lemma 2.2.

Let $B$ denote the class of groups consisting of finite groups and $\mathbb{Z}$. The next lemma is immediate from the proof of [16, Corollary 2.1]. For the convenience of the reader, we give a proof of it by recalling Osin’s argument and notation.

**Lemma 2.1.** The class of elementary amenable groups is the smallest class of groups containing $B$ and closed under taking direct limits with injective connecting maps and extensions by groups from $B$.

**Proof.** Let $\mathcal{E}(B)$ be the smallest class of groups containing $B$ and closed under taking subgroups, quotients, extensions and inductive limits. Clearly, $\mathcal{E}(B)$ is equal to the class of elementary amenable groups.

Let $\mathcal{E}_0(B)$ be the class consisting only of the trivial group. Supposing that $\alpha > 0$ is an ordinal and we have already defined the classes $\mathcal{E}_\beta(B)$ for all ordinals $\beta < \alpha$, put

$$\mathcal{E}_\alpha(B) = \bigcup_{\beta < \alpha} \mathcal{E}_\beta(B)$$

if $\alpha$ is a limit ordinal. If $\alpha$ is a successor ordinal, define $\mathcal{E}_\alpha(B)$ to be the class of groups that can be obtained from groups of the class $\mathcal{E}_{\alpha-1}(B)$ by taking direct limits or by taking an extension of a given group by a group from $B$.

By [16, Lemma 3.1], each $\mathcal{E}_\alpha$ is closed under taking quotients. Hence, in the definition of $\mathcal{E}_\alpha$ for $\alpha$ a successor ordinal, we could have considered only direct limits with injective connecting maps.

Therefore, it follows from [16, Theorem 2.1] that $\mathcal{E}(B)$ is the smallest class of groups containing $B$ and closed under taking direct limits with injective connecting maps and extensions by groups from $B$.

**Lemma 2.2 ([7]).** If $G$ is an infinite, finitely generated, elementary amenable group, then there is a subgroup of finite index of $G$ which admits a homomorphism onto $\mathbb{Z}$.

**Proof.** Let $\mathcal{A}$ be the class of all finite groups, all non-finitely generated groups, and all groups containing a finite index subgroup which maps onto $\mathbb{Z}$.

We claim that $\mathcal{A}$ contains the class of elementary amenable groups. Obviously, $\mathcal{A}$ contains all finite groups, it contains $\mathbb{Z}$, and it is closed under taking inductive limits with injective connecting maps, and extensions by $\mathbb{Z}$.
Let us check that $A$ is also closed under taking extensions by finite groups. Let $H \in A$, $F$ be a finite group, and $G$ a group which fits into the short exact sequence $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$.

If $G$ is infinite and finitely generated, then also $H$ is infinite and finitely generated. Hence, $H$ contains a finite index subgroup $H'$ which maps onto $\mathbb{Z}$. Since $F$ is finite, also $H'$ has finite index in $G$. Therefore, $G \in A$.

By Lemma 2.2, it follows that $A$ contains the elementary amenable groups. □

A $C^*$-algebra $A$ is said to have real rank zero if every hereditary $C^*$-subalgebra of $A$ has an approximate unit of projections (not necessarily increasing). We refer the reader to, for example, [3, Section V.7] for other equivalent definitions of real rank zero.

**Lemma 2.3.** If $A$ is an infinite-dimensional, real rank zero $C^*$-algebra, then it contains a sequence of non-zero, orthogonal projections.

**Proof.** Since $A$ is infinite-dimensional, there is a sequence $(a_n)_{n \in \mathbb{N}} \subset A$ of non-zero, positive elements such that $a_j a_k = 0$ when $j \neq k$ (see, for example, [9, Exercise 4.6.13] or [13]).

For each $n \in \mathbb{N}$, take a non-zero projection $p_n$ in the hereditary (hence real rank zero) $C^*$-subalgebra $A_n A_n^*$. By construction, $p_j p_k = 0$ when $j \neq k$. □

**Theorem 2.4.** If $G$ is a finitely generated, elementary amenable group and $C^*(G)$ has real rank zero, then $G$ is finite.

**Proof.** Suppose $G$ is infinite. By Lemma 2.2 there is a subgroup $H$ of $G$ with finite index $n$, and $\Phi: H \rightarrow \mathbb{Z}$ a surjective homomorphism. Let $\varphi: C^*(H) \rightarrow C^*(\mathbb{Z})$ be the $\ast$-homomorphism induced by $\Phi$, and $\varphi_n: M_n(C^*(H)) \rightarrow M_n(C^*(\mathbb{Z}))$ the inflation of $\varphi$.

There is an injective $\ast$-homomorphism $\psi: C^*(G) \rightarrow M_n(C^*(H))$ such that the image of $\varphi_n \circ \psi$ is infinite-dimensional. For the convenience of the reader, we sketch the construction of $\psi$, which is standard.

Let $x_1, \ldots, x_n \in G$ be such that $x_1 = e$ and $G = \sqcup_{i=1}^n x_i H$. Consider the following unitary defined on canonical basis vectors:

$$U: \bigoplus_{i=1}^n \ell^2(H) \rightarrow \ell^2(G)$$

$$\delta_{i,h} \mapsto \delta_{x_i h}.$$ 

Let $S: B(\ell^2(G)) \rightarrow M_n(B(\ell^2(H)))$ be the isomorphism induced by $U$.

By using the left regular representations $\lambda_G$ and $\lambda_H$, we see $C^*(G)$ as contained in $B(\ell^2(G))$ and analogously for $C^*(H)$.

It is easy to check that $S(\lambda_G(g)) \in M_n(C^*(H))$ for every $g \in G$. Hence, $S(C^*(G)) \subset M_n(C^*(H))$. Furthermore, for $h \in H$, we have that $S(\lambda_G(h))_{1,1} = \lambda_H(h)$. Let $\psi := S|_{C^*(G)}$. Then $\varphi_n(\psi(C^*(G)))$ is infinite-dimensional.

Hence, by Lemma 2.3, $M_n(C^*(\mathbb{Z})) \simeq M_n(C(\mathbb{T}))$ contains a sequence of non-zero, orthogonal projections. Since $T$ is connected, we get a contradiction. Hence, $G$ is finite. □

**Remark 2.5.** Recall that a $C^*$-algebra $A$ is said to have property (SP) if every non-zero hereditary $C^*$-subalgebra of $A$ contains a non-zero projection. Furthermore,
A is said to have residual property (SP) if every quotient of A has property (SP) (see [17] Section 7 for more details about these properties).

In the proof of Theorem 2.4, the only aspects of real rank zero that were used are that it implies property (SP) and that having real rank zero is closed under taking quotients. In particular, Theorem 2.4 remains true if one replaces “real rank zero” by “residual property (SP)”.

3. Koopman representation of a topological full group

Given a unitary representation \( \pi \) of a group \( G \), we denote by \( C^*_\pi(G) \) the C*-algebra generated by the image of \( \pi \).

We will denote the Cantor set by \( X \).

Let \( \alpha \) be an action of a group \( G \) on \( X \) by homeomorphisms. The topological full group associated to \( \alpha \), denoted by \([\alpha] \), is the group of all homeomorphisms \( \gamma \) on \( X \) for which there exists a finite partition of \( X \) into clopen sets \( \{ A_i \}_{i=1}^n \) and \( g_1, \ldots, g_n \in G \) such that \( \gamma|_{A_i} = \alpha_{g_i}|_{A_i} \) for \( 1 \leq i \leq n \). That is, \([\alpha] \) consists of the homeomorphisms on \( X \) which are locally given by the action \( \alpha \).

Fix \( T \) a minimal homeomorphism on \( X \). We denote by \([T]\) the topological full group associated to the \( \mathbb{Z} \)-action induced by \( T \).

Let \( \mu \) be a \( T \)-invariant probability measure on \( X \). Note that \( \mu \) is also invariant under the action of \([T]\) on \( X \). Let \( \pi : \([T]\) \to B(L^2(X, \mu)) \) be given by \( \pi(g) (f) := f \circ g^{-1} \), for \( g \in \([T]\) \) and \( f \in L^2(X, \mu) \). This \( \pi \) is the so called Koopman representation associated to the action of \([T]\) on \((X, \mu)\).

We will use the faithful representation of \( C(X) \rtimes \mathbb{Z} \) in \( B(L^2(X, \mu)) \), with \( C(X) \) acting by multiplication operators, and, for \( n \in \mathbb{Z} \), \( \delta_n := \pi(T^n) \), so that \( C(X) \rtimes \mathbb{Z} := \text{span}\{ f \delta_n : f \in C(X), n \in \mathbb{Z} \} \).

Given \( g \in \([T]\) \) and \( \{ A_i \}_{i=1}^n \) a partition of \( X \) into clopen sets such that \( g|_{A_i} = T^{n_i}|_{A_i} \) for \( 1 \leq i \leq n \), notice that

\[
\pi(g) = \sum 1_{T^{n_i}(A_i)} \delta_{n_i}.
\]

In particular, \( C^*_\pi([T]) \subset C(X) \rtimes \mathbb{Z} \).

**Definition 3.1.** Given \( n \in \mathbb{N} \), we say that a subset \( A \subset X \) is \( n \)-disjoint if

\( A, T(A), \ldots, T^{n-1}(A) \)

are pairwise disjoint.

Suppose \( A \subset X \) is a clopen and \( n \)-disjoint set. Consider the symmetric group \( S_n \) acting on \( \{0, \ldots, n-1\} \). For \( \sigma \in S_n \), let \( \sigma_A \in \([T]\) \) be given by

\[
\sigma_A(x) = \begin{cases} T^{\sigma(i)-i}(x), & \text{if } 0 \leq i < n \text{ and } x \in T^i(A) \\ x, & \text{if } x \notin \bigcup_{i=0}^{n-1} T^i(A), \end{cases}
\]

Note that, for \( 0 \leq i < n \), \( \sigma_A(T^i(A)) = T^{\sigma(i)}(A) \).

**Lemma 3.2.** Let \( n \geq 4 \) and \( A \subset X \) be a clopen and \( n \)-disjoint set. For every \( \sigma \in S_n \), it holds that \( \pi(\sigma_A) \in C^*_\pi([T])' \).

**Proof.** Notice first that \( \{1_{T^{i}(A)} \delta_{i-j} \}_{0 \leq i, j < n} \) forms a system of matrix units in \( C(X) \rtimes \mathbb{Z} \) of type \( M_n(\mathbb{C}) \) (we see \( M_n(\mathbb{C}) \) as matrices indexed by the set \( \{0, \ldots, n-1\} \)).
Let \( B := (\bigcup_{i=0}^{n-1} T_i^A)^c \) and \( \varphi: \mathbb{C} \oplus M_n(\mathbb{C}) \to C(X) \rtimes \mathbb{Z} \) be the \(*\)-homomorphism given by \( \varphi(\alpha, e_i) := \alpha 1_B + 1_{T_i^A} \delta_{i,j} \), for \( \alpha \in \mathbb{C} \) and \( 0 \leq i, j \leq n - 1 \).

Let \( \rho: S_n \to \mathbb{C} \oplus M_n(\mathbb{C}) \) be the direct sum of the trivial representation and the permutation representation.

Given \( \sigma \in S_n \), by (1) and (2), it holds that \( \pi(\sigma_1) = 1_B + \sum 1_{T_{\pi(i)}(A)} \delta_{\sigma(i)-i} = \varphi(\rho(\sigma)) \).

Furthermore, since \( n \geq 4 \), the permutation representations of \( S_n^* \) and \( S_n \) decompose into the direct sum of a trivial representation and an irreducible representation of degree \( n - 1 \). Therefore, we have that \( C_\rho^*(S_n^*) = C_\rho^*(S_n) \).

Hence, \( \pi(\sigma_1) \in C_\rho^*(\|[T]\|') \) for any \( \sigma \in S_n \).

Given \( A \subset X \) clopen, consider the continuous function

\[
t_A: A \to \mathbb{N}
\]

\[
x \mapsto \min\{k \in \mathbb{N} : T^k(x) \in A\}.
\]

This is the so-called function of first return to \( A \).

Notice that, for \( j \in \mathbb{Z} \), it holds that

\[
t_{T^j(A)} = t^j_A = t_A.
\]

Let \( T_A \in \|[T]\| \) be defined by

\[
T_A(x) = \begin{cases} T^{t_A(x)}(x), & \text{if } x \in A \\ x, & \text{otherwise} \end{cases}, \quad x \in X.
\]

If \( B \subset X \) is a clopen set disjoint from \( A \), then \( T_A \) and \( T_B \) commute.

In order to prove Lemma 3.3, we will have to analyze the spectrum of \( C^* \)-algebras generated by certain commuting unitaries, and the next lemma will be useful for this.

We consider the circle \( T \) as a pointed space with basepoint 1.

**Lemma 3.3.** The universal \( C^* \)-algebra generated by commuting unitaries \( z_1, \ldots, z_n \) subject to the relations \( \{(z_i - 1)(z_j - 1) = 0 : 1 \leq i \neq j \leq n\} \) is \( C(\bigvee_{k=1}^n T) \), with \( z_k \) being given by

\[
z_k: \bigvee_{i=1}^n T \to \mathbb{C}
\]

\[
(x, i) \mapsto \begin{cases} x, & \text{if } i = k \\ 1, & \text{if } i \neq k \end{cases}.
\]

**Proof.** Consider the embedding \( F: \bigvee_{i=1}^n T \to \mathbb{T}^n \) which takes \( x \) in the \( i \)-th copy of \( T \) and sends it into \( (F(x)_i)_{1 \leq i \leq n} \in \mathbb{T}^n \) such that \( F(x)_i := x \) and \( F(x)_j := 1 \) if \( j \neq i \). Also let \( F': C(\mathbb{T}^n) \to C(\bigvee_{i=1}^n T) \) be given by \( F'(f) := \delta \circ f, \) for \( f \in C(\mathbb{T}^n) \).

For \( 1 \leq i \leq n, \) let \( w_i \in C(\mathbb{T}^n) \) be given by \( w_i(y) := y_i, \) for \( y \in \mathbb{T}^n \). Then \( F'(w_i) = z_i \).

Assume \( n > 1 \). Let \( A := C^*(\{(w_i - 1)^k(w_j - 1)^l : i \neq j \text{ and } k, l \in \mathbb{N}\}) \). We claim that \( \ker F' = A \). Clearly, \( A \subset \ker F' \).

Let \( Y := \mathbb{T}^n \setminus \text{im}(F) \). Notice that \( \ker F' = \{f \in C(\mathbb{T}^n) : f|_{\text{im}(F)} = 0\} \simeq C_0(Y) \).

By the Stone-Weierstrass Theorem, in order to show that \( A = C_0(Y) \), it is sufficient to show that, for every \( y \in Y \), there is \( f \in A \) such that \( f(y) \neq 0 \), and that \( A \)
separates the points of $Y$. The proof of the former condition is trivial, so we only show that $A$ separates the points of $Y$.

Take $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in Y$ distinct points. There is $i$ such that $x_i \neq y_i$. Without loss of generality, assume $x_i \neq 1$. Take $j \neq i$ such that $x_j \neq 1$. Then, by choosing $k \in \mathbb{N}$ appropriately, we get $(x_i - 1)^k(x_j - 1) \neq (y_i - 1)^k(y_j - 1)$.

Since $C(T^n)$ is the universal $C^*$-algebra generated by $n$ commuting unitaries and $C(\bigvee_{i=1}^n T)$ is generated by $\{z_1, \ldots, z_n\}$, the result follows.

$\square$

**Lemma 3.4.** Let $A \subset X$ be a clopen and $3$-disjoint set. Then $\pi(T_A) \in C^*([[T]])$.

**Proof.** Given $\sigma \in S_3$, $x \in A$ and $0 \leq i, j < 3$, we have that $\sigma_A T_{T^i(A)} T_{T^j(A)}^{-1} = T^{j}(x)$ if $j \neq \sigma(i)$ and

$$\sigma_A T_{T^i(A)} T_{T^j(A)}^{-1} (T^{\sigma(i)}(x)) = \sigma_A T_{T^i(A)} T^{j}(x)$$

where the equality in (*) is due to [3]. Hence, $\sigma_A T_{T^i(A)} T_{T^j(A)}^{-1} = T_{T^{\sigma(i)}(A)}$.

In particular, for $0 \leq i, j < 3$, we have that $T_{T^i(A)} T_{T^j(A)}^{-1} \in [[[T]]]'$.

If $0 \leq i \neq j < 3$, then $T^i(A)$ and $T^j(A)$ are disjoint, hence $(\pi(T_{T^i(A)}) - 1)(\pi(T_{T^j(A)}) - 1) = 0$.

Then, by Lemma 3.3, there is a $\ast$-homomorphism

$$\varphi: C\left(\bigvee_{i=1}^3 T\right) \to C^*(\{\pi(T_{T^i(A)}) : 0 \leq i < 3\})$$

$$z_i \mapsto \pi(T_{T^{i-1}(A)})$$

Furthermore, by the Stone-Weierstrass theorem, $C(\bigvee_{i=1}^3 T)$ is generated by

$$\{z_i z_j^* : 1 \leq i, j \leq 3\}$$

Hence, $\pi(T_A) \in C^*_\pi([[T]])$.

$\square$

**Theorem 3.5.** Let $(X, T, \mu)$ be a Cantor minimal system and $\pi$ the Koopman representation associated to the action of $[[T]]$ on $(X, \mu)$. Then $C^*_\pi([[T]]) = C^*_\pi([[T]])$.

**Proof.** By [3] Theorem 4.7], given $m \in \mathbb{N}$, $[[T]]$ is generated by

$$\bigcup_{n \geq m} \{T_A, \sigma_A : \sigma \in S_n, A \subset X \text{ is clopen and } n\text{-disjoint}\}$$

By Lemmas 3.2 and 3.4 the result follows.

$\square$

Notice that $1_X \in L^2(X, \mu)$ is invariant under $\pi([[T]])$. Therefore, $\pi$ contains the trivial representation. Denote by $\tau$ the associated character on $C^*_\pi([[T]])$. 
Lemma 3.6. Let \( \tau \) be the character on \( C^*_\pi([|T|]) \) coming from containment of the trivial representation. Then \( \ker \tau = \text{span}(1 - \pi(g) : g \in [|T|]) \).

Proof. Given \( d \in \ker \tau \) and \( \epsilon > 0 \), take \( d' \in \text{span} \pi([|T|]) \) such that \( \|d - d'\| < \frac{\epsilon}{2} \). Then \( \|d - (d' - \tau(d'))\| = \|d - d' + \tau(d' - d)\| < \epsilon \). Furthermore, \( d' - \tau(d') \in \ker \tau \cap \text{span} \pi([|T|]) \).

Since \( \ker \tau \cap \text{span} \pi([|T|]) = \text{span}(1 - \pi(g) : g \in [|T|]) \), the result follows. \( \square \)

Theorem 3.7. Let \( \tau \) be the character on \( C^*_\pi([|T|]) \) coming from containment of the trivial representation. Then \( \ker \tau \) is a hereditary C*-subalgebra of \( C(X) \rtimes Z \).

Proof. We are going to show that, for \( a \in C(X) \rtimes Z \) and \( b, c \in \ker \tau \), it holds that \( bac \in \ker \tau \).

Given \( A \subset X \) clopen and 2-disjoint, notice that \((\delta_0 - \delta_1)1_A(\delta_0 - \delta_{-1}) = \delta_0 - (1_{(A, T(A)})\delta_0 + 1_{T(A)}\delta_1 + 1_A\delta_{-1}) \in C^*_\pi([|T|]). \)

By using telescoping sums, it follows that, for \( n, m \in Z \) and \( A \subset X \) 2-disjoint and clopen, \((\delta_0 - \delta_n)1_A(\delta_0 - \delta_m) \in C^*_\pi([|T|]). \)

Given \( g, h \in [|T|] \), take a basis \( \mathcal{B} \) of 2-disjoint, clopen sets for the topology of \( X \). Moreover, assume that, for each \( A \in \mathcal{B} \), there is \( n(A), m(A) \in Z \) such that \( g|_A = T^n(A)|_A \) and \( h|_{h^{-1}(A)} = T^m(A)|_{h^{-1}(A)} \).

Then \((\delta_0 - \pi(g))1_A(\delta_0 - \pi(h)) = 1_A - \delta_{n(A)}1_A + 1_A\delta_{m(A)} - \delta_{m(A)}1_A\delta_{n(A)} = (\delta_0 - \delta_{n(A)})1_A(\delta_0 - \delta_{m(A)}) \in C^*_\pi([|T|]). \)

Since \( C(X) = \text{span}\{1_A : A \in \mathcal{B}\} \), we conclude that, for \( g, h \in [|T|] \) and \( f \in C(X) \), \((\delta_0 - \pi(g))f(\delta_0 - \pi(h)) \in C^*_\pi([|T|]). \)

By Lemma 3.6 and the fact that \( C(X) \rtimes Z = \text{span}\{f \delta_n : f \in C(X), n \in Z\} \), we conclude that, for \( b, c \in \ker \tau \) and \( a \in C(X) \rtimes Z \), \( bac \in C^*_\pi([|T|]). \)

Since \( \tau \) is a character, the result follows. \( \square \)

Corollary 3.8. Let \( \tau \) be the character on \( C^*_\pi([|T|]) \) coming from containment of the trivial representation. Then \( \ker \tau \) is stably isomorphic to \( C(X) \rtimes Z \). In particular, \( C^*_\pi([|T|]) \) has real rank zero and \( C^*([|T|]) \) is not AF.

Proof. By Theorem 3.7 and the fact that \( C(X) \rtimes Z \) is simple, it follows that \( \ker \tau \) is a full, hereditary C*-subalgebra of \( C(X) \rtimes Z \). Therefore, [2] Theorem 2.8 implies that \( \ker \tau \) is stably isomorphic to \( C(X) \rtimes Z \).

Furthermore, by Theorem 3.5 \( C^*_\pi([|T|]) = C^*([|T|]) \). Since \( C(X) \rtimes Z \) has real rank zero (see, for instance, [15] for proof of this fact), and \( K_1(C(X) \rtimes Z) \simeq Z \), and \( K_1(A) = 0 \) for any AF-algebra \( A \), the conclusion follows. \( \square \)

4. odometers

We start this section by giving a description of \( C^*_\pi([|T|]) \) when \( T \) is an odometer map.

Given \( m \in N \), let \( Z_m := Z/mZ \).

Example 4.1. Let \( (n_k) \) be a strictly increasing sequence of natural numbers such that, for every \( k, n_k|n_{k+1} \). Let \( \rho_k : Z_{n_{k+1}} \rightarrow Z_{n_k} \) be the surjective homomorphism such that \( \rho_k(1) = 1 \), and define \( X := \{(x_k) \in \prod_{k \in N} Z_{n_k} : \rho_k(x_{k+1}) = x_k, \forall k \in N\} \).
Consider \( T: X \to X \)
\[
(x_k) \mapsto (x_k + 1).
\]
Then \((X, T)\) is a Cantor minimal system, the so called odometer of type \((n_k)\).

For \(k \in \mathbb{N}\) and \(l \in \mathbb{Z}_{n_k}\), let \(U(k, l) := \{(x_m) \in X : x_k = l\} \).

Using the notation from [2] and [3], let, for \(k \in \mathbb{N}\), \(\Gamma_k := \{\{T_{U(k,l)}, \sigma_{U(k,0)} \in \mathbb{T} : l \in \mathbb{Z}_{n_k}, \sigma \in S_{n_k}\}\} \). As proven by Matui in [14 Proposition 2.1], \(\Gamma_k \subset \Gamma_{k+1}\), \(\Gamma_k \simeq \mathbb{Z}^n \times S_{n_k}\), and \(\bigcup_k \Gamma_k = \mathbb{T}\).

For \(k \in \mathbb{N}\), let \(A_k := \text{span}\{1_{U(k,l)}\delta_m : l \in \mathbb{Z}_{n_k}, m \in \mathbb{Z}\} \). Then \(A_k \subset A_{k+1}\), and \(C(X) \rtimes \mathbb{Z} = \bigcup_k A_k\).

Fix \(k \in \mathbb{N}\) and consider the isomorphism \(\varphi_k: A_k \to C(T, M_{\mathbb{Z}_{n_k}}(\mathbb{C}))\), such that \(\varphi_k(1_{U(k,0)}) = e_{1,l}\), for \(l \in \mathbb{Z}_{n_k}\), and, for \(z \in \mathbb{T}\),
\[
(\varphi_k(\delta_1)(z))_{i,j} := \begin{cases} 1, & \text{if } 0 < i \leq n_k - 1 \text{ and } j = i - 1 \\ z, & \text{if } i = 0 \text{ and } j = n_k - 1 \\ 0, & \text{otherwise.} \end{cases}
\]

Let \(\pi: \mathbb{T} \to U(C(X) \rtimes \mathbb{Z})\) be the homomorphism coming from the Koopman representation and \(B_k := \{b \in M_{\mathbb{Z}_{n_k}}(\mathbb{C}) : \forall i, j \in \mathbb{Z}_{n_k}, \sum_i b_{i,r} = \sum_j b_{s,j}\}\).

Then, for \(\sigma \in S_{n_k}\), we have that \(\varphi_k(\pi(\sigma_{U(k,0)})) = \sum c_{\sigma(i),i}\) and \(C^*(\{\varphi_k(\pi(\sigma_{U(k,0)})) : \sigma \in S_{n_k}\}) \simeq B_k\).

Furthermore, \(\varphi_k(C^*(\pi(\{T_{U(k,l)} : l \in \mathbb{Z}_{n_k}\}))) \simeq C(\bigvee_{l \in \mathbb{Z}_{n_k}} T)\) and \(\varphi_k(C^*_\Gamma(\Gamma_k)) = \{f \in C(\mathbb{T}, M_{\mathbb{Z}_{n_k}}(\mathbb{C})) : f(1) \in B_k\}\).

In [4], Dykema and Rørdam gave examples of non-locally finite groups \(G\) such that \(C^*_{\text{red}}(G)\) has real rank zero. As far as we are aware, there is no known example of non-locally finite group \(G\) such that \(C^*(G)\) has real rank zero.

**Question 4.2.** Let \((X, T)\) be an odometer as in Example 4.1. Does \(C^*(\mathbb{T})\) have real rank zero?

**Example 4.3.** Let \((X, T)\) be an odometer of type \((n_k)\) as in Example 4.1. Consider
\[
J: X \to X \\
(x_k) \mapsto (-x_k).
\]
Then \(J\) is an involutive homeomorphism on \(X\) such that \(JTJ = T^{-1}\). Hence, \(T\) and \(J\) induce an action \(\alpha\) of the infinite dihedral group \(\mathbb{Z} \rtimes \mathbb{Z}_2\) on \(X\). We will use Matui’s technique ([14 Proposition 2.1]) in order to compute \(\mathbb{T}\).

For every \(\gamma \in (\mathbb{Z} \rtimes \mathbb{Z}_2) \setminus \{e\}\), it holds that \(\{x \in X : \alpha_\gamma(x) = x\}\) has empty interior (it consists of at most two elements). Hence, given \(g \in \mathbb{T}\), there exists a unique continuous function \(c_g: X \to \mathbb{Z} \times \mathbb{Z}_2\) such that, for \(x \in X\), \(g(x) = \alpha_{c_g(x)}(x)\).

For \(k \in \mathbb{N}\) and \(l \in \mathbb{Z}_{n_k}\), let \(U(k, l)\) be as in Example 4.1 and
\[
\Gamma_k := \{g \in \mathbb{T} : c_g \text{ is constant on } U(k, l) \text{ for } l \in \mathbb{Z}_{n_k}\}\).

Define \(J_{k,l} \in \mathbb{T}\) by
\[
J_{k,l}(x) = \begin{cases} T^{2l}J(x), & \text{if } x \in U(k,l) \\ x, & \text{otherwise,} \end{cases}
\]
for \(x \in X\).
Then $\Gamma_k = \langle \{ T_{U(k,l)} \sigma_{U(k,0)} : l \in \mathbb{Z}_{n_k}, \sigma \in S_{n_k} \} \rangle$ and

\begin{equation}
\Gamma_k \cong (\mathbb{Z} \times \mathbb{Z}_2)^n \rtimes S_{n_k}, \quad \Gamma_k \subset \Gamma_{k+1}, \text{ and } \bigcup_k \Gamma_k = \{ [\alpha] \}.
\end{equation}

Notice that the constant sequence $(0) \in X$ is a fixed point for $J$. Hence, \cite{11} Theorem 3.5 implies that $C(X) \rtimes (\mathbb{Z} \times \mathbb{Z}_2)$ is AF (see also \cite{12}). Moreover, it follows from \cite{5} that the abelianization of $\{ [\alpha] \}$ is locally finite.

Therefore, the two obstructions that were used for ruling out the possibility of $C^*([T])$ and $C^*([T]'')$ being AF do not hold for $C^*([\alpha])$.

**Question 4.4.** Let $\alpha$ be as in Example 4.3. Is $C^*([\alpha])$ AF?

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Departamento de Matemática, Universidade Federal de Santa Catarina,
Campus Universitário Trindade, 88040-900, Florianópolis - SC, Brazil.
E-mail address: duduscarparo@gmail.com