ABSTRACT

Previous results on quasi-classical limit of the KP and Toda hierarchies are now extended to the BKP hierarchy. Basic tools such as the Lax representation, the Baker-Akhiezer function and the tau function are reformulated so as to fit into the analysis of quasi-classical limit. Two subalgebras $W_{1+\infty}^B$ and $w_{1+\infty}^B$ of the W-infinity algebras $W_{1+\infty}$ and $w_{1+\infty}$ are introduced as fundamental Lie algebras of the BKP hierarchy and its quasi-classical limit, the dispersionless BKP hierarchy. The quantum W-infinity algebra $W_{1+\infty}^B$ emerges in symmetries of the BKP hierarchy. In quasi-classical limit, these $W_{1+\infty}^B$ symmetries are shown to be contracted into $w_{1+\infty}^B$ symmetries of the dispersionless BKP hierarchy.
1. Introduction

The notion of quasi-classical limit of integrable hierarchies [1] has been one of attractive topics in recent studies of nonlinear integrable systems. Of particular interest are its relation to two dimensional quantum gravity and topological conformal field theory [2] and the underlying algebraic structure, so called W-infinity algebras [3].

This note is a sequel to recent papers [4] on quasi-classical (or dispersionless) limit of the KP and Toda hierarchies. We here consider a KP hierarchy of B type, the BKP hierarchy [5] from the same point of view. The BKP hierarchy is a kind of reduction of the ordinary KP hierarchy, but the machinery of reduction is somewhat distinct from the KdV and generalized KdV reductions. Our main concern lies in the fate of W-infinity symmetries of the KP and dispersionless KP hierarchies in the course of reduction.

2. Lax and dressing operators of BKP hierarchy

Let $L$ be the Lax operator of the KP hierarchy,

$$L = \partial_x + \sum_{n=1}^{\infty} u_{n+1}(t) \partial_x^{-n}, \quad \partial_x = \partial/\partial x,$$

(1)

where the coefficients $u_n$ are unknown functions of a set of flow variables $t$. Through this note, the space variable $x$ is identified with the first flow variable

$$x = t_1.$$  

(2)

Given a pseudo-differential operator $P$, let $P^*$ denote the formal adjoint:

$$P = \sum p_n \partial_x^n \Rightarrow P^* = \sum (-\partial_x)^n \cdot p_n.$$  

(3)

The BKP hierarchy is a reduction of the KP hierarchy by the constraint [5]

$$\partial_x^{-1} L^* \partial_x = -L.$$  

(4)
This constraint is preserved only by “odd” flows, \( t_3, t_5, \ldots \), of the KP hierarchy. These flows are given by the Lax equations

\[
\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L], \quad B_{2n+1} = (L^{2n+1})_{\geq 0},
\]

(5)

where, as usual, \(( \quad )_{\geq 0}\) denotes the projection onto nonnegative powers of \( \partial_x \).

The above constraint can be translated into the language of a dressing operator. As in the KP hierarchy, one can find a dressing operator

\[
W = 1 + \sum_{n=1}^{\infty} w_n(t) \partial_x^{-n}
\]

(6)

that satisfies the dressing relation

\[
L = W \partial_x W^{-1}
\]

(7)

and the equations of flows

\[
\frac{\partial W}{\partial t_{2n+1}} = B_{2n+1} W - W \partial_x^{2n+1}.
\]

(8)

In terms of \( W \), the constraint to \( L \) can rewritten

\[
[W^* \partial_x W, \partial_x] = 0.
\]

(9)

This implies that \( W^* \partial_x W \) is a pseudo-differential operator with constant coefficients of the form

\[
W^* \partial_x W = \partial_x + (\text{lower order terms})
\]

(10)

One can eliminate (or “gauge away”) the tail of lower order terms by a residual gauge transformation

\[
W \rightarrow W(1 + c_1 \partial_x^{-1} + c_2 \partial_x^{-2} + \cdots), \quad c_n = \text{const.},
\]

(11)
which leaves (7) and (8) invariant. One can thus single out a dressing operator that satisfies the condition

$$W^* \partial_x W = \partial_x.$$  \hspace{1cm} (12)

This gives an ultimate form of the constraint that $W$ has to satisfy.

Having this choice of $W$, we now define the second Lax operator

$$M = W \left( \sum_{n=1}^{\infty} (2n+1)t_{2n+1} \partial_x^{2n+1} + x \partial_x \right) W^{-1}$$

$$= \sum_{n=1}^{\infty} (2n+1)t_{2n+1} L^{2n+1} + xL + \sum_{n=1}^{\infty} v_{n+1}(t)L^{-n}. \hspace{1cm} (13)$$

By construction, $M$ satisfies the constraint

$$\partial_x^{-1} M^* \partial_x = -M.$$  \hspace{1cm} (14)

Furthermore, as in the KP hierarchy, $M$ satisfies the Lax equations

$$\frac{\partial M}{\partial t_{2n+1}} = [B_{2n+1}, M]$$  \hspace{1cm} (15)

and the canonical commutation relation

$$[L, M] = L.$$  \hspace{1cm} (16)

It should be noted that these $L$ and $M$ amount to $L$ and $ML^{-1}$ in the KP hierarchy.
3. Quasi-classical (or dispersionless) limit

We now introduce a parameter $\hbar$ ("Planck constant") and assume that the coefficients $u_n$ and $v_n$ of the Lax operators depend on $\hbar$ as well as $t$ and behave smoothly as $\hbar \to 0$:

$$u_n(\hbar, t) = u_n^0(t) + O(\hbar),$$
$$v_n(\hbar, t) = v_n^0(t) + O(\hbar).$$  \hfill (17)

Furthermore, we modify the previous Lax formalism of the BKP hierarchy by replacing

$$\partial_x \to \hbar \partial_x, \quad \frac{\partial}{\partial t_{2n+1}} \to \hbar \frac{\partial}{\partial t_{2n+1}}.$$  \hfill (18)

The Lax operators are now written

$$L = \hbar \partial_x + \sum_{n=1}^{\infty} u_{n+1}(\hbar, t)(\hbar \partial_x)^{-n},$$
$$M = \sum_{n=1}^{\infty} (2n + 1) t_{2n+1} L^{2n+1} + xL + \sum_{n=1}^{\infty} v_{n+1}(\hbar, t) L^{-n},$$  \hfill (19)

and the Lax equations and the canonical commutation relation become

$$\hbar \frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L], \quad \hbar \frac{\partial M}{\partial t_{2n+1}} = [B_{2n+1}, M],$$  \hfill (20)

and

$$[L, M] = \hbar L$$  \hfill (21)

respectively.

In quasi-classical ($\hbar \to 0$) limit, commutators of pseudo-differential operators are replaced by Poisson bracket by the rule

$$[\hbar \partial_x, x] = \hbar \to \{k, x\} = 1,$$  \hfill (22)
where $k$ is a conjugate variable of $x$. Lax equations (20) and canonical commutation relation (21) thereby turn into the quasi-classical (or dispersionless) Lax equations

\[
\frac{\partial L}{\partial t_{2n+1}} = \{B_{2n+1}, L\}, \quad \frac{\partial M}{\partial t_{2n+1}} = \{B_{2n+1}, M\},
\]

and the Poisson bracket relation

\[
\{L, M\} = L
\]

of the Laurent series

\[
L = k + \sum_{n=1}^{\infty} u_{n+1}(t) k^{-n},
\]

\[
M = \sum_{n=1}^{\infty} (2n+1)t_{2n+1} L^{2n+1} + x L + \sum_{n=0}^{\infty} u_{2n+2}(t) L^{-2n-1},
\]

where $B_n$ are given by

\[
B_{2n+1} = (L^{2n+1})_{\geq 0},
\]

( )$_{\geq 0}$ being the projection onto nonnegative powers of $k$. Similarly, the previous constraints to $L$ and $M$ are replaced by constraints to $L$ and $M$:

\[
L(t, -k) = -L(t, k), \quad M(t, -k) = -M(t, k).
\]

In other words, $L$ and $M$ are odd Laurent series of $k$; this is a reason why the Laurent expansion of $M$ in (24) contains only odd powers of $L$. These $L$ and $M$, like $L$ and $M$, correspond to $L$ and $M L^{-1}$ of the dispersionless KP hierarchy. Following the terminology in the KP and Toda hierarchies [1], we call the above hierarchy the “dispersionless BKP hierarchy.”
4. Asymptotics of Baker-Akhiezer function and tau function

In the presence of \( \hbar \), the Baker-Akhiezer function \( \Psi \) is given by the (formal) Laurent series

\[
\Psi(\hbar, t, \lambda) = \left( 1 + \sum_{n=1}^{\infty} w_n(\hbar, t) \lambda^{-n} \right) \exp \hbar^{-1} t(\lambda),
\]

\[
t(\lambda) = \sum_{n=0}^{\infty} t_{2n+1} \lambda^{2n+1} = \sum_{n=1}^{\infty} t_{2n+1} \lambda^{2n+1} + x \lambda,
\]

of a parameter \( \lambda \) (“spectral parameter”). By construction, \( \Psi \) satisfies the linear equations

\[
\lambda \Psi = L \Psi, \quad \hbar \lambda \frac{\partial \Psi}{\partial \lambda} = M \Psi,
\]

\[
\hbar \frac{\partial \Psi}{\partial \zeta_{2n+1}} = B_{2n+1} \Psi.
\]

This implies that \( \Psi \) has a WKB asymptotic form as \( \hbar \to 0 \):

\[
\Psi = \exp[\hbar^{-1} S(t, \lambda) + O(\hbar^0)].
\]

The phase function \( S \) becomes an odd function of \( \lambda \):

\[
S(t, -\lambda) = -S(t, \lambda).
\]

As in the case of the KP and Toda hierarchies \([4]\), the phase function satisfy a set of Hamilton-Jacobi equations, and the Hamilton-Jacobi equations reproduce the previous Lax formalism of the dispersionless BKP hierarchy by means of a Legendre transformation. In particular, one can find a direct relation between \( S \) and \( \mathcal{M} \):

\[
\lambda \frac{\partial S}{\partial \lambda} = \mathcal{M}(\lambda) = \sum_{n=0}^{\infty} (2n+1)t_{2n+1} \lambda^{2n+1} + \sum_{n=0}^{\infty} v_{2n+2}^{(0)} \lambda^{-2n-1}.
\]
The tau function $\tau^B$ of the BKP hierarchy is known to coincide with the square root of the KP tau function $\tau$ [5]. In the present setting, $\tau^B$ is a function that satisfies the relation

$$\Psi = \frac{\exp[-2\bar{h}\tilde{\partial}(\lambda^{-1})]\tau^B(h, t)}{\tau^B(h, t)} \exp h^{-1}t(\lambda),$$

$$\tilde{\partial}(\lambda^{-1}) = \sum_{n=0}^{\infty} \frac{\lambda^{-2n-1}}{2n+1} \frac{\partial}{\partial t_{2n+1}}.$$  \hspace{1cm} (32)

To be consistent with WKB asymptotic form (29) of the Baker-Akhiezer function in quasi-classical ($\bar{h} \to 0$) limit, the tau function turns out to behave as

$$\tau^B(h, t) = \exp[\bar{h}^{-2}F^B(t) + O(\bar{h}^{-1})]$$ \hspace{1cm} (33)

with a suitable scaling function $F^B(t)$ ("free energy"). Since $\tau^B = \sqrt{\tau}$, the free energy is related to the KP free energy $F$ as $F^B = F/2$. The phase function $S$ can now be written

$$S(t, \lambda) = t(\lambda) - 2\sum_{n=0}^{\infty} \frac{\lambda^{-2n-1}}{2n+1} \frac{\partial F^B}{\partial t_{2n+1}}.$$ \hspace{1cm} (34)

5. W-INFINITY SYMMETRIES IN TERMS OF LAX AND DRESSING OPERATORS

Let us first consider W-infinity symmetries of the BKP hierarchy, putting $\bar{h} = 1$. Following the case of the KP hierarchy [6], we seek to construct W-infinity symmetries in such a form as

$$\delta_A W = A(L, M)_{\leq -1} W,$$ \hspace{1cm} (35)

where $A(L, M)$, the data of a symmetry, is a non-commutative Laurent series of $L$ and $M$,

$$A(L, M) = \sum_{i \in \mathbb{Z}, j \geq 0} a_{ij} L^i M^j,$$ \hspace{1cm} (36)
and \((\quad)_{\leq -1}\) the projection onto negative powers of \(\partial_x\). If \(A\) is arbitrary, this gives a generic \(W_{1+\infty}\) symmetry of the KP hierarchy. The Lax operators are transformed as

\[
\delta_A L = [A(L,M)_{\leq -1},L], \quad \delta_A M = [A(L,M)_{\leq -1},M].
\] (37)

We now have to find a condition under which \(\delta_A\) preserves the constraint to \(W\) to the effect that

\[
\partial_x^{-1}W^*\partial_x = W^{-1} \Rightarrow \delta_A(\partial_x^{-1}W^*\partial_x) = \delta_A(W^{-1}).
\] (38)

Here \(\delta_A\), by definition, is understood to act on both sides of the last relation as:

\[
\delta_A(\partial_x^{-1}W^*\partial_x) = \partial_x^{-1} \cdot (\delta_A W)^* \cdot \partial_x,
\]

\[
\delta_A(W^{-1}) = -W^{-1} \cdot \delta_A W \cdot W^{-1}.
\] (39)

One can then prove, after somewhat lengthy technical calculations of pseudo-differential operators, that \(\delta_A\) fulfills the above condition if

\[
\partial_x^{-1}A(L,M)^*\partial_x = -A(L,M).
\] (40)

Actually, this condition is equivalent to

\[
\partial_x^{-1}A(\partial_x,x\partial_x)^*\partial_x = -A(\partial_x,x\partial_x).
\] (41)

Pseudo-differential operators satisfying the above condition form a Lie subalgebra \(W^B_{1+\infty}\) of \(W_{1+\infty}\). The \(W\)-infinity algebra \(W_{1+\infty}\) is now realized by general pseudo-differential operators. The Lie algebra homomorphism \(A \rightarrow -\partial_x^{-1}A^*\partial_x\) of this Lie algebra into itself is obviously an involution. The positive eigenspace, \(W^B_{1+\infty}\), thus becomes a Lie subalgebra:

\[
W_{1+\infty} \supset W^B_{1+\infty} = \{A(\partial_x, x\partial_x) \mid \partial_x^{-1}A^*\partial_x = -A\}.
\] (42)
The Lax operators $L$ and $M$ are elements of this Lie subalgebra. The dressing operator $W$ may be thought of as an element of an associated “W-infinity group.” Such a group actually does not exist in a mathematically rigorous sense, but may be realized as a kind of formal group whose generic elements are written $g = \exp \epsilon A$, where $A$ lies in $W_{1+\infty}^B$ and $\epsilon$ is a formal parameter, and satisfy the condition

$$\partial_x^{-1} g^* \partial_x = g^{-1}. \quad (43)$$

These $W_{1+\infty}^B$ symmetries give rise to symmetries of the dispersionless BKP hierarchy as follows. Let $\hbar$ now take a generic value. As $\hbar \to 0$, the Lax operators $L$ and $M$ are replaced by their quasi-classical counterparts $\mathcal{L}$ and $\mathcal{M}$, and we are left with symmetries of the dispersionless BKP hierarchy of the form

$$\delta A \mathcal{L} = \{ A(\mathcal{L}, \mathcal{M}) \leq -1, \mathcal{L} \}, \quad \delta A \mathcal{M} = \{ A(\mathcal{L}, \mathcal{M}) \leq -1, \mathcal{M} \}, \quad (44)$$

where $A$ is now a Laurent series of $\mathcal{L}$ and $\mathcal{M}$,

$$A(\mathcal{L}, \mathcal{M}) = \sum_{i \in \mathbb{Z}, j \geq 0} a_{ij} \mathcal{L}^i \mathcal{M}^j, \quad (45)$$

and $(\quad)_{\leq -1}$ the projection onto negative powers of $k$. In place of (40), $A$ has to satisfy the condition

$$A(-\lambda, -\mu) = -A(\lambda, \mu). \quad (46)$$

This defines a Lie subalgebra $w_{1+\infty}^B$ of the classical W-infinity algebra $w_{1+\infty}$,

$$w_{1+\infty} \supset w_{1+\infty}^B = \{ A(\lambda, \mu) \mid A(-\lambda, -\mu) = -A(\lambda, \mu) \}, \quad (47)$$

with respect to the Poisson bracket

$$\{ A, B \} = \lambda \frac{\partial A}{\partial \lambda} \frac{\partial B}{\partial \mu} - \mu \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial \lambda}. \quad (48)$$
6. **W-infinity symmetries in terms of tau function**

The above W-infinity symmetries can also be derived from a vertex operator. The vertex operator for the BKP hierarchy \((\hbar = 1)\) discovered by Date et al. [5] can be written

\[
Z^B(\tilde{\lambda}, \lambda) = \frac{1}{2} \frac{\tilde{\lambda} - \lambda}{\lambda + \lambda} \left( \exp[t(\tilde{\lambda}) + t(\lambda)] \exp[-2\tilde{\partial}(\tilde{\lambda}^{-1}) - 2\partial(\lambda^{-1})] - 1 \right). \tag{49}
\]

This gives a two-parameter family of infinitesimal transformation of the BKB tau function: \(\tau^B \to \tau^B + \epsilon Z^B(\tilde{\lambda}, \lambda) \tau^B\). Furthermore, by a boson-fermion correspondence, \(Z^B(\tilde{\lambda}, \lambda)\) corresponds to the fermion bilocal operator \(\phi(\tilde{\lambda})\phi(\lambda)\) of a neutral fermion field \(\phi(\lambda)\) [5]. The previous \(W^B_{1+\infty}\) symmetry \(\delta_A\) can be identified with

\[
W_A = \oint A(\tilde{\lambda} \frac{\partial}{\partial \tilde{\lambda}}, \tilde{\lambda}) Z^B(\tilde{\lambda}, \lambda)|_{\tilde{\lambda} = -\lambda} \frac{d \log \lambda}{2\pi i} \tag{50}
\]

in the bosonic language, and with

\[
O_A = \oint : A(\tilde{\lambda} \frac{\partial}{\partial \tilde{\lambda}}, \tilde{\lambda}) \phi(\tilde{\lambda})\phi(\lambda)|_{\tilde{\lambda} = -\lambda} : \frac{d \log \lambda}{2\pi i} \tag{51}
\]

in the fermionic language. If \(\hbar\) takes a generic value, one has to replace

\[
t_{2n+1} \to \hbar^{-1} t_{2n+1}, \quad \frac{\partial}{\partial t_{2n+1}} \to \hbar \frac{\partial}{\partial t_{2n+1}}, \quad \frac{\lambda}{\partial \lambda} \to \hbar \frac{\lambda}{\partial \lambda} \tag{52}
\]

in the above construction. Let \(W_A(\hbar)\) and \(O_A(\hbar)\) denote the corresponding symmetry generators.

We now show how these \(W^B_{1+\infty}\) symmetries of the tau function \(\tau^B\) can be reduced to \(w^B_{1+\infty}\) symmetries of the free energy \(F^B\). To this end, we consider the special generators

\[
A_{ij} = \frac{1}{2} (L^i M^j + (-1)^{i+j} M^j L^i) \tag{53}
\]
of $W_{1+\infty}^B$ and the corresponding bosonic symmetry operators $W_{ij}^B(\hbar) = W_{A_{ij}}^B(\hbar)$. Recalling asymptotic behavior (33) of the tau function, one can easily calculate the action by $W_{ij}^B(\hbar)$ to the lowest order of $\hbar$-expansion as:

$$\frac{W_{ij}^B(\hbar) \tau^B(\hbar, t)}{\tau^B(\hbar, t)} = -\hbar^{-j-1} \left( \frac{1}{j+1} \text{Res} \: \lambda^i \mathcal{M}(\lambda)^{j+1} d\log \lambda + O(\hbar) \right), \quad (54)$$

where “Res” means the formal residue,

$$\text{Res} \: \lambda^n d\log \lambda = \delta_{n,0}. \quad (55)$$

Picking out the most singular term ($\propto \hbar^{-j-1}$), we define

$$w_{ij}^B F^B = -\frac{1}{j+1} \text{Res} \: \lambda^i \mathcal{M}(\lambda)^{j+1} d\log \lambda$$

$$= -\frac{1}{j+1} \text{Res} \: \mathcal{L}^i \mathcal{M}^{j+1} d\log \mathcal{L}. \quad (56)$$

This gives a $w_{1+\infty}^B$ symmetry of the dispersionless BKP hierarchy associated with the element

$$A_{ij} = \lambda^i \mu^j \quad (57)$$

of $w_{1+\infty}^B$. The symmetry is indeed realized as the infinitesimal transformation $F^B \to F^B + \epsilon w_{ij}^B F^B$ of the free energy.

Note that $w_{ij}^B F$ vanishes if $i+j$ is an even number. Thus only half of the above $W_{1+\infty}^B$ generators $A_{ij}$ (i.e., those with $i+j$ being odd) correspond to nontrivial $w_{1+\infty}^B$ symmetries of the BKP hierarchy. This is quite natural, because $A_{ij}$ belong to $w_{1+\infty}^B$ if and only if $i+j$ is odd.
7. Conclusion

Our previous results on quasi-classical limit of the KP and Toda hierarchies can thus be extended to the BKP hierarchy. We have been able to identify the two W-infinity algebras $W_{1+\infty}^B$ and $w_{1+\infty}^B$ as fundamental Lie algebras of the BKP and dispersionless BKP hierarchies. These W-infinity algebras indeed emerge in both the Lax formalism of the hierarchies and the construction of symmetries. As in the case of the KP and Toda hierarchies, we have seen that these structures of the quantum W-infinity algebra $W_{1+\infty}^B$ are smoothly contracted to those of the classical W-infinity algebra $w_{1+\infty}^B$.

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