Dichotomic probability representation of quantum states

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Abstract. We present systematic proofs of statements about probability representations of qudit density states in terms of standard probability distributions of dichotomic random variables. New relations and new entropic-information inequalities are derived. The examples of 3- and 4- level states are explicitly worked out.

Keywords: Quantum tomography, dichotomic probability distributions, qudit density matrices, entropic inequalities.

Introduction

Quantum states are associated with rays of a Hilbert space, or, in general, with density operators acting on it [1–9]. In either case, quantum states do replace classical probability distributions but they do not represent fair probabilities on any sample space. However, they can be associated with true probability distributions in the tomographic picture of quantum mechanics. There, states are identified with tomographic-probability distributions of homodyne quadrature components for continuous variables or spin tomographic-probability distributions for discrete spin-projection variables, see the discussion e.g. in [10–15].

Optical tomograms for systems with continuous variables were measured in experiments [16,17], where the Wigner function of photon states was reconstructed, by means of the relation between tomograms and Wigner functions through the Radon transform [18] found in [19,20].
In [10], symplectic-tomography for photon quadrature was introduced, as an alternative to optical tomography. Also in [10], tomographic-probability distributions were suggested as a primary notion of photon states, providing a “classical-like” description of quantum states in a statistical mechanics approach. The tomographic characterization of spin states was introduced in [11,12]; see also the review [13].

Recently the possibility to parameterise density matrices of qudit states (d-level atom states, spin states) by sets of dichotomic probability distributions has been proposed and developed in [21–26]. The aim of this work is to provide general statements about the probability description of qudit states by means of dichotomic probabilities and to prove new properties of nonnegative trace-one Hermitian matrices.

The approach is quite simple, out of the d-dimensional Hilbert space \( \mathbb{C}^d \), we identify \( d(d-1) \) complex planes, in each one of them we consider a group of unitary matrices isomorphic with \( U(2) \), and matrices corresponding to the associated \( \mathfrak{u}(2) \) Lie algebra. Clearly this family of \( \mathfrak{u}(2) \) Lie algebras is sufficient to describe the whole \( \mathfrak{u}(d) \) algebra, in some redundant way. In some sense we could say that a generic quantum system may be studied by means of properly chosen qubit systems (this should not be confused with the description of the total system by means of qubit subsystems. Indeed, in the latter situation, the associated composite system would be a tensor product of qubits, which is not the case with our generic Hilbert space). We then use the tomographic description of each \( \mathfrak{u}(2) \) subalgebra: because of dimensionality, the tomograms will represent dichotomic probability distributions. We shall prove that an arbitrary \( d \times d \) density matrix can be parameterized by \( (d^2 - 1) \) probability distributions of dichotomic random variables. Having elaborated these tools, we use them to decompose a given d-state, with \( d = nm \), into \( n \)-states and \( m \)-states. Using this result, we obtain new entropic-information inequalities among matrix elements of an arbitrary matrix \( \rho \), associated with a state. In addition, we obtain new inequalities for characteristic polynomials associated with such matrices. We illustrate some of these claims in the case of qubit and qutrit density matrices. This proposal will be worked out in details in the coming sections. We shall use a pedagogical style and spell out all computations.

The paper is organized as follows. In section 1, we exhibit the dichotomic probability representation of qubit and qutrit states. In section 2, we generalize the dichotomic probability representation to qudit states. In section 3, we illustrate a reduction procedure to construct two types of new density states from a \( d = nm \)-dimensional starting one, with dimension \( n \) and \( m \), respectively. In section 4, we obtain some new entropic inequalities for the matrix elements of density states, as spin off of the approach developed. Finally, we give our conclusions in section 5.
Quantum states and probability vectors of dichotomic observables can be considered within the probability representation of quantum mechanics, where the states are usually considered to replace fair probabilities.

For simplicity, we restrict our considerations to finite-dimensional Hilbert spaces. In what follows we shall take a pedagogical attitude and spell out all details so that general statements are always illustrated by the example.

Let $H$ be the Hilbert space associated with our quantum system. If $H$ and $|e_1angle, |e_2angle, \ldots, |e_dangle$ is an orthonormal basis, we can associate a matrix with $|\psi\rangle$, by first defining a rank-one projector and then using a specific basis of orthonormal vectors:

\[
\|\psi_{jk}^{(e)}\| = \left\| \frac{\langle e_j|\psi\rangle\langle \psi|e_k\rangle}{\langle \psi|\psi\rangle} \right\|.
\]

The diagonal elements $\{\psi_{jj}^{(e)}\}$ represent a probability distribution with $n$ components; $\psi_{jj}^{(e)} \geq 0$ and $\sum_j \psi_{jj}^{(e)} = 1$. It is a probability distribution on the set $\{1, 2, \ldots, d\}$, we call it a probability vector.

If we select a different orthonormal basis, say, $|f_1\rangle, |f_2\rangle, \ldots, |f_n\rangle$, we associate with $|\psi\rangle$ a different matrix

\[
\|\psi_{jk}^{(f)}\| = \left\| \frac{\langle f_j|\psi\rangle\langle \psi|f_k\rangle}{\langle \psi|\psi\rangle} \right\|.
\]

Again, the diagonal elements provide a new probability vector, a new probability distribution on the set $\{1, 2, \ldots, d\}$.

Thus, with a given vector, depending on the chosen basis, we associate a family of probability distributions on the set $\{1, 2, \ldots, d\}$. As a matter of fact, with a given $|\psi\rangle$ but using different orthonormal bases, one can obtain all possible probability distributions. As a matter of fact, given a probability vector, say $(p_1, p_2, \ldots, p_d)$, it is possible to find a whole family of states corresponding to the same probability distribution; indeed, in the given basis we would have $|\psi\rangle\langle \psi| = \sum_{j,k} \sqrt{p_j p_k} e^{i(\theta_j - \theta_k)} |e_j\rangle\langle e_k|$, where $\varphi_j$ is completely arbitrary, by using different bases we would get different states $|\psi_f\rangle\langle \psi_f| = \sum_{j,k} \sqrt{p_j p_k} e^{i(\varphi_j - \varphi_k)} |f_j\rangle\langle f_k|$. We notice that states which we build are rank-one Hermitian operators of trace one.

The identification of the probability distribution with the diagonal of the matrix associated with a given vector $|\psi\rangle$ shows also that the association disregards all “off-diagonal” elements, i.e., not only rank-one operators but also states of higher rank, as long as the diagonal is unchanged, will give rise to the same probability distribution.

The probability distributions associated with every vector by means of different bases are called tomograms; indeed, tomography consists of reconstructing
the state when a sufficient set (a “quorum”) of tomograms is provided. Such
tomograms of spin states were studied, e.g., in [28, 29]. The spin tomography was
discussed in [10, 11]; see also the review [30].

Having stressed that alternative states, both pure and mixed, may give rise to
the same probability distribution, it is quite surprising and highly not triv-
ial that by giving a sufficient set of probability distributions thought of as related to the
same state, we are able to reconstruct uniquely the state, be it pure or mixed.

Let us identify the mathematical ingredients of previous construction. We have
first the association of a rank-one projector with every vector, say, $|\psi\rangle \rightarrow |\psi\rangle\langle\psi| = \rho_\psi$. Next, the selection of a basis in $\mathcal{H}$ provides a resolution of the identity, say,
$1 = \sum_j |e_j\rangle\langle e_j|$ and, moreover, allows for the construction of a basis of Hermitian
matrices, specifically,

$$
|e_j\rangle\langle e_j|, \ |e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|, \ i(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|), \ |e_k\rangle\langle e_k|
$$

when j and k go from 1 to n. However, when we restrict to given pair of j and
k, in each “(j,k)-plane” we build a basis of a $u(2)$ Lie algebra. Thus we obtain
a different ”placement” of an abstract $u(2)$ Lie algebra, for any choice of a (j,k)
plane. The unitary transformation taking from one basis to a different one changes
the placement of the $u(2)$ Lie algebra.

The Weyl basis $\{|e_j\rangle\langle e_k|\}$ allows for the construction of the matrix associated
with any vector $|\psi\rangle$; we have $\psi_{jk} = \text{Tr}\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} |e_j\rangle\langle e_k|\right)$.

The association of a probability distribution with $\rho_\psi$ only uses diagonal el-
ments $\{|e_j\rangle\langle e_j|\}$; thus, we need a sufficient number of independent bases so
that by means of the diagonal elements associated with the various bases, say,
$[\{|e_j\rangle\langle e_j|\}, \{|f_j\rangle\langle f_j|\}, \ldots, \{|k_j\rangle\langle k_j|\} \ldots]$, we may generate a basis of rank-one
operators.

In order to fully reconstruct a state we need $d^2 - 1 = (d - 1)(d + 1)$ parameters.
On using resolutions of the identity

$$
\sum_j |f_j\rangle\langle f_j| = \sum_j |e_j\rangle\langle e_j| = \cdots = \sum_j |k_j\rangle\langle k_j| = 1,
$$

the independent diagonal elements associated with every basis are $(d - 1)$ in num-
ber; therefore we need $(d + 1)$ of such independent families.

Since orthonormal bases may be constructed by means of normalized eigenvectors of a generic observable $A$ with simple eigenvalues, to obtain full information
on the quantum state, we can measure $(d + 1)$ independent families of $(d - 1)$ pair-
wise commuting observables, which are independent. From each family, it would
be enough to measure just one observable which has a non degenerate spectrum.
Remark: By using the expectation value functions, \( e_A(\psi) = \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} \), we may define the independence to be the functional independence of the expectation value functions associated with every observable of the pairwise commuting family.

To nail down these general considerations, we consider two examples, namely, a qubit and a qutrit.

1.1. The qubit case. Here, \( d = 2 \) and \( \mathcal{H} = \mathbb{C}^2 \). We have to measure \( d + 1 = 3 \) independent families of \( d - 1 = 1 \) commuting observables, which we choose to be one of the Pauli matrices \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), which are Hermitian operators in the space of \( 2 \times 2 \) matrices. Clearly, each Pauli matrix, together with the identity matrix \( \sigma_0 \) will define a basis of \( \mathfrak{u}_2 \).

For the observable associated to \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) we have

\[
|f_3^+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |f_3^-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The two vectors \( |f_3^+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |f_3^-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), being orthonormal eigenvectors for \( \sigma_3 \), determine rank-one projectors \( \hat{\Pi}_3^+, \hat{\Pi}_3^- \)

\[
\hat{\Pi}_3^+ = |f_3^+\rangle \langle f_3^+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\Pi}_3^- = |f_3^-\rangle \langle f_3^-| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

besides the matrices

\[
\hat{\Pi}_3^\pm = |f_3^+\rangle \langle f_3^-| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\Pi}_3^\mp = |f_3^-\rangle \langle f_3^+| = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

 Altogether they form a basis in the linear space of \( 2 \times 2 \)-matrices.

For the observable associated to \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) the orthonormal eigenvectors are

\[
|f_1^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |f_1^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

They determine the rank-one projectors \( \hat{\Pi}_1^+, \hat{\Pi}_1^- \)

\[
\hat{\Pi}_1^+ = |f_1^+\rangle \langle f_1^+| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \hat{\Pi}_1^- = |f_1^-\rangle \langle f_1^-| = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}
\]

and the matrices

\[
\hat{\Pi}_1^\pm = |f_1^\pm\rangle \langle f_1^\mp| = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}, \quad \hat{\Pi}_1^\mp = |f_1^-\rangle \langle f_1^+| = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix}
\]

yielding another basis in the linear space of \( 2 \times 2 \)-matrices.
Finally, for the observable associated to the Pauli matrix \(\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\), the orthonormal basis of eigenvectors is

\[
|f_2^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |f_2^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.
\]

They determine the rank-one projectors \(\hat{\Pi}_2^+, \hat{\Pi}_2^-\);

\[
\hat{\Pi}_2^+ = |f_2^+\rangle \langle f_2^+| = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}, \quad \hat{\Pi}_2^- = |f_2^-\rangle \langle f_2^-| = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}
\]

and the matrices

\[
\hat{\Pi}_2^\mp = |f_2^\mp\rangle \langle f_2^\mp| = \begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}, \quad \hat{\Pi}_2^\pm = |f_2^\pm\rangle \langle f_2^\pm| = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & -1/2 \end{pmatrix}.
\]

which form yet another basis in the linear space of \(2\times2\)-matrices. We introduce for future convenience a compact notation for all rank one-projectors, namely

\[
\hat{\Pi}^+_a = \frac{1}{2}(\sigma_0 + \sigma_a); \quad \hat{\Pi}^-_a = \frac{1}{2}(\sigma_0 - \sigma_a).
\]

For each one of these bases, by using just rank-one projectors over positive eigenstates, we may associate two-dimensional probability vectors, say, \((p_1, 1 - p_1); (p_2, 1 - p_2); (p_3, 1 - p_3)\) to a given state \(\rho\), be it pure or mixed. We have indeed

\[
p_1(\rho) = \text{Tr}\rho \hat{\Pi}^+_1, \quad p_2(\rho) = \text{Tr}\rho \hat{\Pi}^+_2, \quad p_3(\rho) = \text{Tr}\rho \hat{\Pi}^+_3,
\]

and analogous relations for \(1 - p_i\) in terms of rank-one projectors of negative eigenstates. Eqs. (13) define genuine probabilities \(0 \leq p_i \leq 1\) because \(\rho\) is a positive Hermitean matrix while \(\hat{\Pi}^+_a\) are rank-one projectors.

In order to discuss the dependence of the dichotomic probability representation on the choice of rank-one projectors, let us observe that Eq. (12) for \(\hat{\Pi}^+_a\) may be rewritten in the following form

\[
\hat{\Pi}^+_a = \frac{1}{2}(\sigma_0 + \bar{x}_a \cdot \vec{\sigma}), \quad (\bar{x}_a)_j = \delta_{aj}, \quad a, j = 1, 2, 3.
\]

This implies that, upon rotation of the three vectors \(\bar{x}_a\), we obtain rotated projectors \((\hat{\Pi}^+_a)'\). By means of the standard double covering of \(SO(3)\) by \(SU(2)\), we have indeed

\[
\bar{x}_a' = R\bar{x}_a \rightarrow (\bar{x}_a' \cdot \vec{\sigma})' = U(\bar{x}_a \cdot \vec{\sigma})U^\dagger, \quad R \in SO(3), U \in SU(2)
\]

so that

\[
(\hat{\Pi}^+_a)' = U\hat{\Pi}^+_a U^\dagger
\]

and

\[
p_a'(\rho) = \text{Tr}\rho(\hat{\Pi}^+_a)' = \text{Tr}\rho U^\dagger U\hat{\Pi}^+_a
\]

\]
yielding the transformation law of the dichotomic probabilities under rotation of rank-one projectors. This result can be easily generalized to the \( d \)-dimensional case, as we shall see in next section.

By means of these dichotomic probabilities it can be easily checked by direct computation that we can reconstruct the state by setting

\[
\rho = (\sigma_0/2) + (p_1 - 1/2)\sigma_1 + (p_2 - 1/2)\sigma_2 + (p_3 - 1/2)\sigma_3. 
\]

We notice, although trivial for \( d = 2 \), that the latter is equivalent to the tomographic approach, where, given a state

\[
\rho = \frac{1}{2}(\sigma_0 + \vec{y} \cdot \vec{\sigma}) 
\]

we have the tomographic relation (see for example [14])

\[
y_j = 2(\mathcal{W}_j - \frac{1}{2}) 
\]

with \( \mathcal{W}_j = p_j, j = 1, \ldots, 3 \), the tomographic probabilities.

Conversely, given a family of dichotomic probabilities \( (p_j, 1 - p_j), j = 1, \ldots, 3 \), Eq. (18) can be chosen as a definition of a mixed state \( \rho \). Indeed, the latter is Hermitian and can be checked to verify \( \text{Tr}\rho = 1 \). Moreover, its determinant is nonnegative if the coefficients satisfy the inequality

\[
(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 \leq 1/4
\]

If the dichotomic variables are not correlated, we have

\[
(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 \leq 3/4,
\]

the only constraint being \( 0 \leq p_j \leq 1, \quad j = 1, 2, 3 \). Notice that the inequalities are also satisfied if we use \( (1 - p_j) \) instead of \( p_j \).

Eq. (18) describes a mixed state in terms of tomograms. If the inequality is saturated, we are dealing with pure states, i.e., \( \rho^2 = \rho \).

Thus, out of three dichotomic probability distributions, we have been able to reconstruct a state.

Finally, in order to make contact with the coming sections and make it easier to generalize the results to higher dimensions, it is useful to rewrite Eq. (18) in the Weyl basis for \( u_2 \). To this, let us introduce an orthonormal basis in \( \mathbb{C}^2 \), say \( \{|e_1\}, |e_2\} \). The Weyl basis is represented by four rank-one operators

\[
E_{jk} = |e_j\rangle\langle e_k|, \quad j, k = 1, 2.
\]

By expressing the Hermitian \( u_2 \) generators in terms of the latter

\[
\frac{1}{2}\sigma_0 = \frac{1}{2}(E_{11} + E_{22}), \quad \frac{1}{2}\sigma_3 = \frac{1}{2}(E_{11} - E_{22})
\]

\[
\frac{1}{2}\sigma_1 = \frac{1}{2}(E_{12} + E_{21}), \quad \frac{1}{2}\sigma_2 = \frac{i}{2}(E_{12} - E_{21})
\]
Eq. (18) may be rewritten according to
\[
\rho = E_{11}p_3 + E_{22}(1 - p_3)
\]
\[
+ E_{12} \left[ (p_1 - \frac{1}{2}) - i(p_2 - \frac{1}{2}) \right] + E_{21} \left[ (p_1 - \frac{1}{2}) + i(p_2 - \frac{1}{2}) \right].
\]

We notice, for future convenience, that the diagonal elements are directly expressed in terms of the diagonal rank-one projectors associated to \(\sigma_3\), it being
\[
E_{11} = \hat{\Pi}_3^+; \quad E_{22} = \hat{\Pi}_3^-.
\]
Moreover the pertinent matrix entries \(\rho_{jj}\) are completely determined by either the positive-eigenvalue projector or the negative one, it being
\[
p_3 = \text{Tr}(\rho\hat{\Pi}_3^+); \quad 1 - p_3 = \text{Tr}(\rho\hat{\Pi}_3^-).
\]
Therefore we can rewrite the state \(\rho\) as follows
\[
\rho = p_3E_{11} + (1 - p_3)E_{22} + \sum_{j\neq k} \rho_{jk}E_{jk}
\]
with
\[
\rho_{jk} = \text{Tr}(\rho^TE_{jk}).
\]
This remark will be relevant for higher level systems. This example shows very well how pure states may give rise to all probability vectors in the ”classical simplex”, indeed the terms with \(j\) not equal to \(k\) play no role in the expression of the probability vector and their role is simply to change the rank of the state which is being represented.

1.2. The qutrit case. According to our previous considerations and notation, here \(d = 3\), the Hilbert space is \(\mathcal{H} \equiv \mathbb{C}^3\) and we have to measure \(d + 1 = 4\) independent families of \(d - 1 = 2\) commuting observables, which, as for the two-levels system, can be chosen to be a Cartan subalgebra of the relevant Lie algebra, here \(u(3)\), in four different realizations. For each choice of Cartan subalgebra, to which we add the identity, their joint diagonalization yields three eigenvectors, which play the role of the eigenvectors of Pauli matrices in the previous subsection. This is the tomographic approach, which allows to reconstruct the state as it is detailed in [14].

Such a procedure becomes however difficult to apply in practice with increasing number of levels. The approach we want to pursue in this paper is, instead, to use our knowledge of the two-levels system and characterize all parameters of a qudit state in terms of dichotomic probabilities which are amenable to the \(u(2)\) subalgebras of the relevant \(u(n)\).

To this, let us consider an orthonormal basis in \(\mathbb{C}^3\), say \(|e_1\rangle, |e_2\rangle, |e_3\rangle\) and let us construct for \(u(3)\) the Weyl basis of nine rank-one operators, \(E_{jk} = |e_j\rangle\langle e_k|, j, k = 0, 1, 2\).
1, \ldots, 3

\begin{align}
|e_1\rangle\langle e_1| & \quad |e_1\rangle\langle e_2| \quad |e_1\rangle\langle e_3| \\
|e_2\rangle\langle e_1| & \quad |e_2\rangle\langle e_2| \quad |e_2\rangle\langle e_3| \\
|e_3\rangle\langle e_1| & \quad |e_3\rangle\langle e_2| \quad |e_3\rangle\langle e_3| \\
\end{align}

(30)

all of them providing a representation of a pair groupoid \([34,35]\). Let us notice that an alternative basis for \(u(3)\) is represented by the eight Gell–Mann matrices \(\lambda_i\) to which we add the identity. The latter, which was used in \([14]\), is however not convenient for the present purposes, and, once again, not immediately generalizable to higher dimensions.

It is now easy to see that, in a natural way, we have the possibility to define three different \(u(2)\) bases, namely,

\begin{align}
\mathbf{u}(2)^{12} & : \{ |e_1\rangle\langle e_1|, \quad |e_1\rangle\langle e_2|, \quad |e_2\rangle\langle e_1|, \quad |e_2\rangle\langle e_2| \} \\
\mathbf{u}(2)^{13} & : \{ |e_1\rangle\langle e_1|, \quad |e_1\rangle\langle e_3|, \quad |e_3\rangle\langle e_1|, \quad |e_3\rangle\langle e_3| \} \\
\mathbf{u}(2)^{23} & : \{ |e_2\rangle\langle e_2|, \quad |e_2\rangle\langle e_3|, \quad |e_3\rangle\langle e_2|, \quad |e_3\rangle\langle e_3| \}
\end{align}

which are obtained by the array \((30)\) removing, in the order, the third row and third column, the second row and the second column, the first row and the first column.

For each \(u(2)\), namely, for each choice of \((jk)\), \(j, k \in (1, 2, 3)\) and \(j < k\), we can realize Hermitean \(u_2\) generators \(S_{\mu}, \mu = 0, \ldots, 3\) acting on the \((jk)\) plane, according to

\begin{align}
S_{0}^{(jk)} & = \frac{1}{2}(E_{jj} + E_{kk}) \quad S_{3}^{(jk)} = \frac{1}{2}(E_{jj} - E_{kk}) \\
S_{1}^{(jk)} & = \frac{1}{2}(E_{jk} + E_{kj}) \quad S_{2}^{(jk)} = -\frac{i}{2}(E_{jk} - E_{kj})
\end{align}

(31)

(32)

Since

\begin{align}
\text{Tr}E_{jk}^T E_{mn} = \delta_{km}\delta_{jn}
\end{align}

(33)

the Hermitean \(u(2)\) generators \(S_{\mu}^{(jk)}\) are orthonormal with respect to the scalar product \(\langle A|B \rangle = \text{Tr} A^T B\).

For each \(u(2)\) we can apply the procedure described in previous section to obtain rank-one projectors. We consider the eigenvector \(|f^+\rangle\) of positive eigenvalue, for each Hermitean generator of each \(u(2)\) algebra, namely: \(|f_1^{(12)}\rangle, |f_2^{(12)}\rangle, |f_3^{(12)}\rangle, |f_1^{(13)}\rangle, |f_2^{(13)}\rangle, |f_3^{(13)}\rangle, |f_1^{(23)}\rangle, |f_2^{(23)}\rangle, |f_3^{(23)}\rangle\) (where we have omitted the superscript +; we could have chosen to work with the eigenvectors of negative eigenvalue, as shown in the previous section) and we construct rank one-projectors

\begin{align}
\Pi_{a}^{(jk)} = |f_{a}^{(jk)}\rangle\langle f_{a}^{(jk)}| = S_{0}^{(jk)} + S_{a}^{(jk)}, \quad a = 1, \ldots, 3
\end{align}

(34)
which explicitly read

\[
\hat{\Pi}^{(12)}_1 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\hat{\Pi}^{(12)}_2 = \begin{pmatrix}
\frac{1}{2} & -\frac{i}{2} & 0 \\
\frac{i}{2} & 1/2 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\hat{\Pi}^{(12)}_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\hat{\Pi}^{(13)}_1 = \begin{pmatrix}
\frac{1}{2} & 0 & 1/2 \\
0 & 0 & 0 \\
1/2 & 0 & 1/2
\end{pmatrix}, \\
\hat{\Pi}^{(13)}_2 = \begin{pmatrix}
1/2 & 0 & -i/2 \\
0 & 0 & 0 \\
i/2 & 0 & 1/2
\end{pmatrix}, \\
\hat{\Pi}^{(13)}_3 = \hat{\Pi}^{(12)}_3,
\]

\[
\hat{\Pi}^{(23)}_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2
\end{pmatrix}, \\
\hat{\Pi}^{(23)}_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1/2 & -i/2 \\
0 & i/2 & 1/2
\end{pmatrix}, \\
\hat{\Pi}^{(23)}_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

To these we associate the dichotomic probabilities \((p^{(jk)}_a, 1 - p^{(jk)}_a)\) given by

\[
(35) \quad p^{(jk)}_a = \text{Tr} \rho \hat{\Pi}^{(jk)}_a = \langle f^{(jk)}_a | \rho | f^{(jk)}_a \rangle.
\]

These are indeed real positive numbers not greater than one, because \(\rho\) is a positive Hermitian matrix (a state), whereas \(\hat{\Pi}^{(jk)}_a\) are rank-one projectors (pure states).

Notice that these dichotomic probabilities refer to \(\hat{\Pi}^+\) projectors. Only for the qubit case the second component of the probability vector, namely \(1 - p\), can be obtained by projecting the density state on \(\hat{\Pi}^-\). In general we have to choose either positive or negative projectors to work. In this paper we use the positive ones.

In order to fully determine the state \(\rho\) we have to invert (35) for the matrix elements of \(\rho\). As for the diagonal entries, we observe that for the diagonal projectors it holds

\[
(36) \quad \hat{\Pi}^{(jk)}_3 = E_{jj}, \quad j < k
\]

namely, they are given by the diagonal elements of the Weyl basis, hence independent on the index \(k\), in the \((jk)\) plane, as we can verify in the table above, where \(\hat{\Pi}^{(13)}_3 = \hat{\Pi}^{(12)}_3 = E_{11}\). This implies that the probabilities \(p^{(jk)}_3, j < k \leq 3\) only depend on the first of the two indices, \((jk)\), labelling the plane. We shall therefore use the notation \(p^{(jk)}_3 \rightarrow p^{(jj)}_3\), and, we derive, from (35), (36)

\[
(37) \quad \rho^{(jj)}_j = p^{(jj)}_3, \quad j = 1, 2
\]

\[
(38) \quad \rho^{(33)} = 1 - \sum_j^2 p^{(jj)}_3
\]

where \(\text{Tr} \rho = 1\) has been used.

As for the off-diagonal entries of the matrix \(\rho\), according to Eq. (35) we have to consider dichotomic probabilities associated to the off-diagonal projectors \(\hat{\Pi}^{(jk)}_a, a = 1, 2\). These allow to determine the 6 off-diagonal entries \(\rho_{jk}\) by
means of the relation

$$\rho_{jk} = \text{Tr}\rho^{(jk)}E_{jk}^T; \quad j < k$$

where we have introduced auxiliary qubit states $\rho^{(jk)}$ as follows

$$\rho^{(jk)} = S_0^{(jk)} + \left[ 2p_1^{(jk)} - (p_3^{(jj)} + p_3^{(kk)}) \right] S_1^{(jk)} - i \left[ 2p_2^{(jk)} - (p_3^{(jj)} + p_3^{(kk)}) \right] S_3^{(jk)}$$

Matrix elements $\rho_{jk}$, $j > k$ are obtained by complex conjugation. Explicitly in terms of the Weyl basis we have then

$$\rho = E_{11}p_3^{(1)} + E_{22}p_3^{(2)} + E_{33}(1 - p_3^{(1)} - p_3^{(2)})$$

$$+ \left[ E_{12} \left( p_1^{(12)} - \frac{1}{2}(p_3^{(11)} + p_3^{(22)}) - i(p_2^{(12)} - \frac{1}{2}(p_3^{(11)} + p_3^{(22)})) \right) \right]$$

$$+ E_{13} \left( p_1^{(13)} - \frac{1}{2}(p_3^{(11)} + p_3^{(33)}) - i(p_2^{(13)} - \frac{1}{2}(p_3^{(11)} + p_3^{(33)})) \right)$$

$$+ E_{23} \left( p_1^{(23)} - \frac{1}{2}(p_3^{(22)} + p_3^{(33)}) - i(p_2^{(23)} - \frac{1}{2}(p_3^{(22)} + p_3^{(33)})) \right)$$

$$+ \text{h.c.}.$$  \hspace{1cm} (41)

Similarly to the two-level system, the diagonal elements are associated to rank-one projectors of positive eigenvalue of the observable $S_3^{(jk)}$, except for $\rho_{33}$ which is obtained by the others through the constraint $\text{Tr}\rho = 1$.

2. QUDIT GENERALIZATION

For a system with $d$ levels the Hilbert space $\mathcal{H} = \mathbb{C}^d$ is spanned by $d$ orthonormal vectors $|e_1\rangle, \ldots, |e_d\rangle$. The Lie algebra $\mathfrak{u}(d)$ can be described in terms of the Weyl basis $E_{jk}, j, k = 1, \ldots, d$. As previously, we have $d! / 2$ different $\mathfrak{u}(2)$ subalgebras, labelled by $(jk)$, $j \neq k$, and associate with each of them the Hermitian generators

$$S_0^{(jk)} = \frac{1}{2}(E_{jj} + E_{kk}) \quad S_3^{(jk)} = \frac{1}{2}(E_{jj} - E_{kk})$$

$$S_1^{(jk)} = \frac{1}{2}(E_{jk} + E_{kj}) \quad S_2^{(jk)} = -\frac{i}{2}(E_{jk} - E_{kj})$$

acting on the $(j, k)$ plane. The eigenvalues of the operators $S_a^{(jk)}$ are equal to $+1/2$ and $-1/2$. They can be interpreted as spin projections along the $x, y, z$ axes. For $d$-level atoms, these eigenvalues and the corresponding eigenvectors may be related to $j$th and $k$th levels when other levels are not excited.

Hence we construct the rank-one projectors relative to the positive eigenvalue of each $S_a^{(jk)}$, $a = 1, \ldots, 3$ generator according to

$$\hat{\Pi}_a^{(jk)} = |f_a^{(jk)}\rangle \langle f_a^{(jk)}| = S_0^{(jk)} + S_3^{(jk)}, \quad a = 1, \ldots, 3.$$

Out of the $d! / 2$ different $\mathfrak{u}(2)$ subalgebras, we select those labelled by $(jk)$, with $j < k$. They are $d(d - 1) / 2$. Hence we compute

$$p_a^{(jk)} = \text{Tr}\rho \hat{\Pi}_a^{(jk)}$$
obtaining explicitly

\begin{align}
\rho^{(jk)}_1 &= \frac{1}{2}(\rho_{jj} + \rho_{kk}) + \text{Re}\rho_{jk} \\
\rho^{(jk)}_2 &= \frac{1}{2}(\rho_{jj} + \rho_{kk}) - \text{Im}\rho_{jk} \\
\rho^{(jk)}_3 &= \rho_{jj}.
\end{align}

They define dichotomic probability vectors \((p^{(jk)}_a, 1 - p^{(jk)}_a)\) for each \(a = 1, ..., 3\), and each couple \((j, k)\), \(j < k\), corresponding to \(3d(d-1)/2\) probabilities. Then we observe, as in the previous, two- and three-dimensional cases, that the diagonal projectors \(\hat{\Pi}^{(jk)}_3\) are independent of the second index in any \((jk)\) plane and coincide with the diagonal elements of the Weyl basis:

\begin{align}
\hat{\Pi}^{(jk)}_3 = \hat{\Pi}^{(jh)}_3 = E_{jj}, \quad k \neq h, \quad j = 1, ..., N - 1.
\end{align}

This implies that, out of the \(d(d-1)/2\) probabilities \(p^{(jk)}_3\), only \(d - 1\) are different. Thus the total number of independent parameters is \(2d(d-1)/2 + d - 1 = (d+1)(d-1)\). In other words, our choice of the \(u(2)^{(jk)}\) subalgebras with \(j < k\) provides us with a quorum.

Summarizing, we are ready to state the following:

**Theorem 2.1.** Let \(\rho\) be a qudit state and \(p^{(jk)}_a, a = 1, 2, 3, j < k = 1, ..., N\) dichotomic probabilities, defined by Eq. (45).

(i) We have, for the diagonal elements

\begin{align}
\rho_{jj} = \text{Tr}\rho E_{jj} = p^{(jj)}_3, \quad j = 1, ..., N - 1; \quad \rho_N = 1 - \sum_{j}^{N-1} p^{(jj)}_3
\end{align}

where we have re-labeled as previously \(p^{(jk)}_3 \rightarrow p^{(jj)}_3\).

(ii) The off-diagonal elements are obtained by

\begin{align}
\rho_{jk} = \text{Tr}\rho^{(jk)} E_{jk}^T, \quad j < k
\end{align}

with

\begin{align}
\rho^{(jk)} = S_0^{(jk)} + [2p^{(jk)}_1 - (p^{(jj)}_3 + p^{(kk)}_3)]S_1^{(jk)} - i[2p^{(jk)}_2 - (p^{(jj)}_3 + p^{(kk)}_3)]S_2^{(jk)} \quad 1 \leq j < k \leq N
\end{align}

auxiliary qubit states. The matrix elements \(\rho_{kj}\) are given by complex conjugation.

**Proof.** The first statement is an immediate consequence of Eq. (49). The second statement can be checked by direct computation of the RHS of Eq. (51), on using the auxiliary qubits (52) and Eqs. (44), (45). \(\square\)

In terms of the matrix elements obtained above in Eqs. (50), (51), the explicit form of the density state \(\rho\) in the Weyl basis can be readily written down and the result is a straightforward generalisation of Eq. (41).
Let us now discuss in full generality the dependence of the dichotomic probability representation on the choice of rank-one projectors. In the present case, for any \((j, k)\)-plane we have

\[
\hat{\Pi}^{(jk)}_a = S^{(jk)}_0 + \vec{x}_a \cdot \vec{S}^{(jk)},
\]

with \(\vec{S}^{(jk)} = (S^{(jk)}_1, S^{(jk)}_2, S^{(jk)}_3)\) and \((\vec{x}_a)_b = \delta_{ab}, \ a, b = 1, 2, 3.\)

As before, a rotation of the three vectors \(\vec{x}_a\) entails rotated projectors \((\hat{\Pi}^{(jk)}_a)'\). We have indeed

\[
\vec{x}_a' = R^{(jk)} \vec{x}_a - \vec{x}_a \cdot \vec{S}^{(jk)} \vec{S}^{(jk)\dagger}, \quad R^{(jk)} \in SO(3), U^{(jk)} \in SU(2)^{(jk)}
\]

so that

\[
(\hat{\Pi}^{(jk)}_a)' = U^{(jk)} \hat{\Pi}^{(jk)}_a U^{(jk)\dagger}
\]

and

\[
(p^{(jk)}_a)'(\rho) = \text{Tr} \rho (\hat{\Pi}^{(jk)}_a)' = \text{Tr} U^{(jk)\dagger} \rho U^{(jk)} \hat{\Pi}^{(jk)}_a
\]

yielding the transformation law of the of dichotomic probabilities under rotation of rank-one projectors. Notice that, in order to preserve Eqs. (49) and (51) one has to choose one and the same rotation, \(R^{(jk)} = R\), in any \((j, k)\)-plane.

3. Reduction of the density matrix

We have shown in previous section that qudit states can be described in terms of a set of different \((d^2 - 1)\) dichotomic probabilities \((p^{(jk)}_{1,2,3}, 1 - p^{(jk)}_{1,2,3})\) of classical-like random variables. These probability distributions must satisfy the Silvester criterion of nonnegativity of the density operator, \(\rho \geq 0\), i.e., eigenvalues of this operator must be nonnegative. Moreover the principal minors of the operator \(\rho\) in an arbitrary orthogonal basis must be nonnegative.

In this section we shall illustrate how these inequalities give rise to quantum correlations for the auxiliary qubits associated to qudit states.

To be definite, let us start with a qudit state, \(\rho\), represented by a \(d \times d\) matrix, \(d = n \cdot m\). Let us consider two orthonormal bases, \(
\{ |e_j\rangle, j = 1, ..., n \}, \quad \{ |f_j\rangle, j = 1, ..., m \},
\)
for the complex vector spaces \(\mathbb{C}^n, \mathbb{C}^m\) respectively, and let us introduce in the space of \(n \times n\), respectively \(m \times m\) complex matrices, the natural bases

\[
E_{jk} = |e_j\rangle \langle e_k|, j, k = 1, ..., n \quad F_{jk} = |f_j\rangle \langle f_k|, j, k = 1, ..., m.
\]

Hence, \(\rho\) may be rewritten as follows

\[
\rho = E_{11} \otimes R_{11} + E_{12} \otimes R_{12} + ... + E_{nn} \otimes R_{nn}
\]

with \(R_{jk}\) \(m \times m\) complex matrices defined by

\[
R_{jk} = \sum_{p,q=1}^{m} (e_j \otimes f_p |\rho| e_k \otimes f_q) F_{pq}
\]
so to have \( \rho \) rearranged into \( n^2 \) blocks, each one of \( m \times m \) dimension

\[
\rho = \begin{pmatrix}
R_{11} & R_{12} & \ldots & R_{1n} \\
\vdots & \ddots & \ddots & \vdots \\
R_{n1} & R_{n2} & \ldots & R_{nn}
\end{pmatrix}
\]

We then define a \( n \times n \) matrix \( \rho_1 \) by taking the partial trace over the second element of the tensor product

\[
\rho_1 := \text{Tr}_2 \rho = \sum_{j,k=1}^{n} E_{jk} \text{Tr} R_{jk}
\]

Alternatively, we can trace over the first element of the tensor product. Since \( \text{Tr} E_{jk} = \delta_{jk} \), we obtain a \( m \times m \) matrix, \( \rho_2 \)

\[
\rho_2 := \text{Tr}_1 \rho = \sum_{j,k=1}^{n} \delta_{jk} R_{jk} = R_{11} + \ldots + R_{nn}.
\]

We can actually exchange the role of the two bases, \( E_{jk}, F_{jk} \) and express \( \rho \) as follows

\[
\rho = \sum_{p,q=1}^{m} \tilde{R}_{pq} \otimes F_{pq}
\]

with \( \tilde{R}_{pq} n \times n \) complex matrices defined by

\[
\tilde{R}_{pq} = \sum_{j,k=1}^{n} \langle f_p \otimes e_j | \rho | f_q \otimes e_k \rangle E_{jk}
\]

so to have \( \rho \) rearranged into \( m^2 \) blocks of \( n \times n \) dimension

\[
\rho = \begin{pmatrix}
\tilde{R}_{11} & \tilde{R}_{12} & \ldots & \tilde{R}_{1m} \\
\vdots & \ddots & \ddots & \vdots \\
\tilde{R}_{m1} & \tilde{R}_{m2} & \ldots & \tilde{R}_{mm}
\end{pmatrix}
\]

We then define a \( n \times n \) matrix, \( \tilde{\rho}_1 \), by taking the partial trace over the second element of the tensor product in Eq. (63)

\[
\tilde{\rho}_1 = \text{Tr}_2 \rho = \sum_{p,q=1}^{m} \delta_{pq} \tilde{R}_{pq} = \tilde{R}_{11} + \ldots + \tilde{R}_{mm}.
\]

By tracing over the first element we get instead

\[
\tilde{\rho}_2 = \text{Tr}_1 \rho = \sum_{p,q=1}^{m} (\text{Tr} \tilde{R}_{pq}) F_{pq}.
\]

Before showing that \( \rho_{1,2}, \tilde{\rho}_{1,2} \) are all density states, namely nonnegative, Hermitian, trace-one complex matrices, for any value of \( n, m \), let us see how the construction works for the simple case of qu-quart states with \( n = m = 2 \),

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix}
= \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix},
\]
with \( R_{11} = R_{11}^\dagger, R_{22} = R_{22}^\dagger, R_{21} = R_{12}^\dagger \), \( 2 \times 2 \)-block matrices. The general expressions given above reduce therefore to

\[
\rho_1 = \tilde{\rho}_2 = \begin{pmatrix} \text{Tr}R_{11} & \text{Tr}R_{12} \\ \text{Tr}R_{21} & \text{Tr}R_{22} \end{pmatrix} \quad \text{and} \quad \rho_2 = \tilde{\rho}_1 = R_{11} + R_{22}.
\]

(69)

It is readily seen that the latter are Hermitian and trace-one matrices

\[
\rho_1^\dagger = \rho_1, \quad \text{Tr} \rho_1 = 1; \quad \rho_2^\dagger = \rho_2, \quad \text{Tr} \rho_2 = 1.
\]

(70)

Nonnegativity is proven below, directly for the general case of a qudit, with \( d = nm \).

To this aim, we shall need the following well known results (see for example [31–33]):

**Proposition 3.1.** Let \( \mathcal{B}(\mathcal{H}) \) denote the bounded operators on a Hilbert space \( \mathcal{H} \). For any positive operator \( B \in \mathcal{B}(\mathcal{H}) \) there exists an operator \( A \in \mathcal{B}(\mathcal{H}) \) such that

\[ B = A^\dagger A. \]

**Proposition 3.2.** A density state, \( \rho \), is a positive linear functional over \( \mathcal{B}(\mathcal{H}) \) iff it is nonnegative when evaluated on the positive elements of \( \mathcal{B}(\mathcal{H}) \), that is,

\[ \rho(A^\dagger A) \geq 0 \]

with \( A \in \mathcal{B}(\mathcal{H}) \).

The we can state the following

**Theorem 3.1.** For a given \( d \times d \) nonnegative trace-one Hermitian matrix, with \( d = nm \), the reduced matrices \( \rho_{1,2}, \tilde{\rho}_{1,2} \), defined in Eqs. (61), (62), (66), (67), are trace-one Hermitian nonnegative matrices, i.e. they are quantum states.

**Proof.** Hermiticity and trace-one property are an immediate consequence of \( \rho = \rho^\dagger, \text{Tr} \rho = 1 \).

In order to prove nonnegativity of \( \rho_1 \) we advocate the two propositions quoted above. Let us take \( \rho \) in the form (58) and evaluate it over the positive operator \((A \otimes 1)^\dagger (A \otimes 1) = A^\dagger A \otimes 1 \). According to Prop. 3.2 we have

\[
0 \leq \rho(A^\dagger A \otimes 1) = \sum_{j,k} E_{jk}(A^\dagger A) \text{Tr}R_{jk} = \rho_1(A^\dagger A)
\]

that is, according to Prop. 3.1, \( \rho_1 \) is nonnegative. Nonnegativity of \( \tilde{\rho}_2 \) can be proven in the same way, by representing \( \rho \) in the form (63).

Analogously, to prove nonnegativity of \( \rho_2 \) we take again \( \rho \) in the form (58) but evaluate it over the positive operator \((1 \otimes A)^\dagger (1 \otimes A) = 1 \otimes A^\dagger A \). According to Prop. 3.2 we have

\[
0 \leq \rho(1 \otimes A^\dagger A) = \sum_{j,k} E_{jk}(1) R_{jk}(A^\dagger A) = \rho_2(A^\dagger A)
\]

that is, according to Prop. 3.1, \( \rho_2 \) is nonnegative. Nonnegativity of \( \tilde{\rho}_1 \) can be proven in the same way, by representing \( \rho \) in the form (63). □
3.1. Polynomial roots of probabilities. As a direct consequence of Theorem 3.1 we may derive new interesting inequalities. To this, let us consider the characteristic polynomial in $\lambda$, associated to $\rho$, $d \times d$, Hermitean, positive, trace-one matrix, which may be written as
\[
\det (\rho - \lambda I) = \sum_{k=1}^{d} c_k \lambda^k = \prod_{k=1}^{d} (\lambda - \lambda_k),
\]
where $\lambda_k \geq 0, k = 1, ..., d$ are the eigenvalues of $\rho$, and $\sum \lambda_k = 1$. Then, the solution of the eigenvalues equation
\[
(\lambda - \lambda_1) = 0
\]
yields a probability vector $(\lambda_1, ..., \lambda_d)$. By virtue of Theorem 3.1, the following Corollary holds.

**Corollary 3.1.1.** Let $d = nm$, $\rho$ a qudit and $\rho_1, \rho_2$ respectively $n$-and $m$-dimensional states defined in Eq. (61), (62). Let us consider the associated characteristic polynomials

(72) \[
\det (\rho_1 - \lambda_1 I_{n \times n}) = \prod_{s_1=1}^{n} (\lambda - \Lambda_{s_1}),
\]

(73) \[
\det (\rho_2 - \lambda_1 I_{m \times m}) = \prod_{s_2=1}^{m} (\lambda - \bar{\Lambda}_{s_2}).
\]

Then,
\[
0 \leq \Lambda_{s_1} \leq 1, 0 \leq \bar{\Lambda}_{s_2} \leq 1.
\]
Moreover, the map which associates to the probability distribution $\lambda_1, \lambda_2, ..., \lambda_d$ the probability distributions $\Lambda_1, \Lambda_2, ..., \Lambda_n$ and $\bar{\Lambda}_1, \bar{\Lambda}_2, ..., \bar{\Lambda}_m$, is bijective.

By explicitly computing the LHS of Eqs. (72),(73) one can derive bounds on the determinant of the states $\rho_1, \rho_2$. Similar inequalities can be obtained starting with $\tilde{\rho}_1, \tilde{\rho}_2$, defined in Eqs. (66), (67).

As an example, let us consider the case of $d = 4$. With a straightforward calculation we find

\[
\Lambda_{1,2} = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4 \det \rho_1} \right] \implies 0 \leq \det \rho_1 \leq 1/4,
\]

(75) \[
\bar{\Lambda}_{1,2} = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4 \det \rho_2} \right] \implies 0 \leq \det \rho_2 \leq 1/4.
\]

The inequality for the determinant of the state $\rho_1$ can be easily checked to be true for the general case $d = 2m$ ($\rho_1$ being two-dimensional again). We have then
\[
0 \leq (\text{Tr} R_{11}) (\text{Tr} R_{22}) - (\text{Tr} R_{12}) (\text{Tr} R_{21}) \leq 1/4,
\]

(76) or
\[
(\text{Tr} R_{11}) (\text{Tr} R_{22}) \geq (\text{Tr} R_{12}) (\text{Tr} R_{21}), \quad (\text{Tr} R_{12}) (\text{Tr} R_{21}) + 1/4 \geq (\text{Tr} R_{11}) (\text{Tr} R_{22}).
\]

The new inequalities are susceptible to be checked experimentally for density matrices obtained within the framework of quantum tomography approach.
4. New information-entropic inequalities for nonnegative trace-one Hermitian matrices

As an application of the results of previous sections, we derive in this section new information-entropic inequalities for density states. By using Eqs. (46)-(48), we can express the dichotomic probabilities \( (p^{(jk)}_a, 1 - p^{(jk)}_a), a = 1, 2, 3 \) in terms of the matrix elements of \( \rho \). Upon substituting them in expressions like von Neumann or Tsallis relative entropy we get the desired inequalities as follows.

For dichotomic variables relative von Neumann entropy reads

\[
S_{vN} = -[p \ln p + (1 - p) \ln(1 - p)] \geq 0
\]

so that

\[
- [p^{(jk)}_a \ln \frac{p^{(jk)}_a}{1 - p^{(jk)}_a} + \ln(1 - p^{(jk)}_a)] \geq 0
\]

Analogously, for Tsallis relative entropy we have

\[
S_{Ts} = (1 - q)^{-1} \left\{ (p^{(jk)}_a)^q (p^{(jk)}_b)^{1-q} + (1 - p^{(jk)}_a)^q (1 - p^{(jk)}_b)^{1-q} - 1 \right\} \geq 0.
\]

In the particular qubit case, we can drop the \( (jk) \) index, and get

\[
\begin{align*}
p_1 &= \frac{1}{2} + \Re \rho_{12} \\
p_2 &= \frac{1}{2} - \Im \rho_{12} \\
p_3 &= \rho_{11} \\
1 - p_1 &= \frac{1}{2} - \Re \rho_{12} \\
1 - p_2 &= \frac{1}{2} + \Im \rho_{12} \\
1 - p_3 &= 1 - \rho_{11}
\end{align*}
\]

and Eq. (79) becomes

\[
\begin{align*}
\ln \sqrt{\frac{1}{4} - (\Re \rho_{12})^2} + \Re \rho_{12} \ln \frac{\frac{1}{2} + \Re \rho_{12}}{\frac{1}{2} - \Re \rho_{12}} &\leq 0 \\
\ln \sqrt{\frac{1}{4} - (\Im \rho_{12})^2} + \Im \rho_{12} \ln \frac{\frac{1}{2} + \Re \rho_{12}}{\frac{1}{2} - \Re \rho_{12}} &\leq 0
\end{align*}
\]

Assuming \( a = 1, b = 2 \) Eq. (80) for the Tsallis relative entropy becomes in turn,

\[
(1 - q)^{-1} \left\{ \left(\frac{1}{2} + \Re \rho_{12}\right)^q \left(\frac{1}{2} - \Im \rho_{12}\right)^{1-q} + \left(\frac{1}{2} - \Re \rho_{12}\right)^q \left(\frac{1}{2} + \Im \rho_{12}\right)^{1-q} - 1 \right\} \geq 0.
\]

Finally, in the qudit case, with \( N = 2n \), we can apply our findings to the \( 2 \times 2 \) state

\[
\rho_1 = \begin{pmatrix}
\text{Tr} R_{11} & \text{Tr} R_{12} \\
\text{Tr} R_{21} & \text{Tr} R_{22}
\end{pmatrix},
\]

and obtain, from inequalities (82),(83), (84), new inequalities by substituting \( \rho_{12} \) with \( \text{Tr} R_{12} \).

5. Conclusions

To conclude, we point out the main results of our study.

We proved that a \( d \)-dimensional density state, \( \rho \), has matrix elements which can be parameterized in terms of dichotomic probability distributions and we discussed the dependence of such a representation on the chosen basis of rank-one projectors. The expression of matrix elements \( \rho_{jk} \) of the qudit in terms of dichotomic probabilities is
the argument of Theorem 2.1. The probabilities $p_{a}^{(jk)}$ satisfy the Silvester criterion of nonnegativity of the density matrix $\rho$. These rigorously proven expressions for the density matrix of qudit states in terms of dichotomic probabilities $p_{a}^{(jk)}$ are the main result of this study.

It is worth noting that a possibility to reconstruct the matrix elements of the density operator in discrete basis was suggested in [36] without obtaining the dichotomic probability representation of the density matrix; it was related to experiments where photon-number distributions were measured to determine the density matrices of photon states.

Upon elaborating on previous claims [21–26] we proved that it is possible to define reduced matrices from the original qudit, where $d = nm$, and obtain smaller $n \times n$- and $m \times m$-dimensional matrices, which keep the properties of the initial matrix $\rho = \rho^{\dagger}$, $\text{Tr} \rho = 1$, and $\rho \geq 0$ of being states. The theorem can be extended iteratively to matrices with $d = n_{1}n_{2} \cdots n_{m}$.

We obtained new relations for the determinants and eigenvalues of reduced states. We derived new inequalities, including entropic inequalities for the matrix elements of the qudit, which provide new relations for its matrix elements. These inequalities can be employed to control the accuracy of experiments where density matrix elements are reconstructed, in particular by using tomographic methods.

The description of quantum states in terms of dichotomic probabilities amounts to decompose a point in the equilateral triangle by using an orthogonal decomposition with respect to the edges instead of using the vertices. This decomposition may be used to describe the evolution on the space of quantum systems in terms of evolution in their dichotomic probability distributions. For example, for systems coupled to an environment (open systems) the Markovian or non-Markovian evolution of qudit states studied in [37] could be mapped onto the time evolution of the associated dichotomic probabilities. It is conceivable that one has to consider higher order ordinary differential equations in order to describe the first order differential equations GKLS on the space of quantum states. We plan to perform such an analysis in a forthcoming publication.

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