On the Spectral Analysis of Quantum Electrodynamics with Spatial Cutoffs. I

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Abstract. In this paper, we consider the spectrum of a model in quantum electrodynamics with a spatial cutoff. It is proven that (1) the Hamiltonian is self-adjoint; (2) under the infrared regularity condition, the Hamiltonian has a unique ground state for sufficiently small values of coupling constants. The spectral scattering theory is studied as well and it is shown that asymptotic fields exist and the spectral gap is closed.

1 Introduction

The present paper investigates the existence and uniqueness of the ground state of a model in quantum electrodynamics (QED) in the Coulomb gauge. QED has, of course, been studied so far from the physical point of view. Nevertheless, it is interesting to investigate it purely from the mathematical standpoint. Indeed it is not so well understood in mathematical rigor.

The first task is to realize the Hamiltonian of QED as a self-adjoint operator on an appropriate Hilbert space, which means that the Hamiltonian generates a unique unitary time evolution. What we need to do mathematically is to specify conditions under which the Hamiltonian is a self-adjoint operator. The second task is to study the spectral properties of the Hamiltonian defined as a self-adjoint operator. The eigenvector associated with the bottom of the spectrum of a self-adjoint operator is called ground state if it exists. We are concerned with the ground states of our self-adjoint operator associated with a model in QED. Note that it is not trivial to show the existence and uniqueness of the ground state, since the ground state for zero coupling is embedded in the continuous spectrum and then the regular perturbation theory [22] does not work directly even for nonzero but sufficiently weak couplings. Nevertheless, we can give a sufficient condition such that the unique ground state exists.

Before starting a rigorous discussion, we roughly review our model for readers’ convenience. Informally, the standard Hamiltonian of QED in the Coulomb gauge without external potentials is given by

\[ H = H_{el} + H_{ph} + \alpha \int_{\mathbb{R}^3} \psi^+(x) \alpha^i \psi(x) A_j(x) dx + \alpha^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\psi^+(x) \psi(x) \psi^+(y) \psi(y)}{|x-y|} dx dy, \tag{1} \]

where \( \alpha \in \mathbb{R} \) denotes the coupling constant, \( \psi \) the Dirac field, \( A_j \) the quantized radiation field, \( \alpha^i, j = 1, 2, 3, 4 \times 4 \) Dirac matrices satisfying canonical anticommutation relations, and \( H_{el} \) and \( H_{ph} \) the
kinetic term of the Dirac field and the photon field, respectively. Here, ultraviolet cutoffs $\chi_{\text{ph}}$ and $\chi_{\text{el}}$ are imposed on $A^j$ and $\psi$, respectively. All the definitions are given rigorously in Section 2. The first two terms on the right-hand side of (1),

$$H_{\text{el}} + H_{\text{ph}},$$

(2)
describe the zero coupling Hamiltonian which is a well defined nonnegative self-adjoint operator on the tensor product of a Fermion Fock space and a Boson Fock space, $\mathcal{F}_{\text{QED}}$, and the bare vacuum of $\mathcal{F}_{\text{QED}}$ is its unique ground state. While the third term describes the minimal coupling between the Dirac field and the quantized radiation field, the last term is derived from the Coulomb gauge condition.

Although (1) is a standard Hamiltonian in QED in the Coulomb gauge, it is not clear if $H$ with $\alpha \neq 0$ is well defined and can be realized as a self-adjoint operator bounded from below. One of the simplest ways to realize $H$ as a well defined self-adjoint operator is to introduce a real spatial cutoff function $\chi$, and thus, we define the spatial cutoff Hamiltonian by replacing the Dirac field $\psi$ with $\chi \psi$. Namely our Hamiltonian turns out to be of the form

$$H_{\chi} = H_{\text{el}} + H_{\text{ph}} + \alpha \int \mathbb{R}^3 \chi(x) \psi^+(x) \alpha^j \psi(x) A^j(x) dx$$

$$+ \alpha^2 \int \mathbb{R}^3 \chi(x) \chi(y) \frac{\psi^+(x) \psi(x) \psi^+(y) \psi(y)}{|x - y|} dxdy. \quad (3)$$

Suppose that $\chi$ satisfies that

$$\int \mathbb{R}^3 |\chi(x)| dx < \infty \quad \text{and} \quad \int \mathbb{R}^3 \frac{|\chi(x)\chi(y)|}{|x - y|} dxdy < \infty. \quad (4)$$

By virtue of (4) it can be seen that the interaction term in (3) is a well defined symmetric operator and moreover $H_{\chi}$ is a self-adjoint operator bounded from below for all $\alpha \in \mathbb{R}$. $H_{\chi}$ is the main object in this paper.

Next, we study the spectral properties of $H_{\chi}$.

(Translation invariance) Since $H$ has no external potential, it is translation-invariant, i.e., $H$ is invariant under transformation

$$\psi(x) \rightarrow \psi(x + a), \quad A^j(x) \rightarrow A^j(x + a)$$

for arbitrary $a \in \mathbb{R}^3$. It is crucial, however, that the spatial cutoff breaks this translation invariance. Thus, the cutoff could be regarded as an external potential.

(Ground state) In addition to (4) supposing integrability:

$$\int \mathbb{R}^3 |x| |\chi(x)| dx < \infty, \quad (5)$$

we can show that $H_{\chi}$ has a unique ground state for a sufficiently small coupling constant under the infrared regularity condition:

$$\int \mathbb{R}^3 \frac{|\chi_{\text{ph}}(k)|^2}{\omega(k)^3} dk < \infty, \quad (6)$$

where $\omega(k) = |k|$ denotes the dispersion relation of photons.

(Total charge) Let

$$Q = N_+ - N_- \quad (7)$$
be the total charge of the Dirac field, where \( N_+ \) (resp. \( N_- \)) denotes the number operator for electrons (resp. positron). Since \( \psi^\dagger(x)\psi(x) \) leaves the total charge invariant for each \( x \in \mathbb{R}^3 \); \( e^{itQ}H_xe^{-itQ} = H_x \), \( H_x \) leaves the total charge invariant. Then, \( \mathcal{F}_{\text{QED}} \) is decomposed with respect to the spectrum of the total charge as

\[
\mathcal{F}_{\text{QED}} = \bigoplus_{z \in \mathbb{Z}} \mathcal{F}_z.
\]  

(8)

It can be shown that the unique ground state of \( H_x \) belongs to \( \mathcal{F}_0 \), i.e., the total charge of the ground state is zero.

Finally, we also establish that \( H_x \) has no spectral gap, i.e., the gap between the bottom of the spectrum and that of the continuum is closed. This is established by constructing asymptotic fields.

The main roles of the spatial cutoff are:

1. It ensures well defined self-adjointness of the Hamiltonian;
2. It leaves the total charge invariant;
3. It breaks translation invariance and serves as an external potential.

The spectral analysis of this kind of system has been developed in the last decade, and many results have been obtained. In particular, this paper is inspired by Arai-Hirokawa [4], where generalized spin-boson (GSB) models are studied. Dimassi-Guillot [11] and Barbaroux-Dimassi-Guillot [8] also study QED, in which the Hamiltonian has an external potential. In [11], [8] the self-adjointness of the Hamiltonian, and the existence and uniqueness of the ground state are established under certain conditions. Furthermore, Bach-Fröhlich-Sigal [7], Sphon [26], Gérard [13], and Grisemer-Lieb-Loss [15] and references therein discuss related models.

Finally, we provide several technical comments comparing with the GSB models studied in [4].

(Existence of the ground state) In order to prove the existence of the ground state, we use the momentum lattice approximation. References [7], [19] prove the existence of the ground state by combining the spatial localization of nonrelativistic electrons and the momentum lattice approximation. In our case, the spatial localization is converted into the assumption \( \int_{\mathbb{R}^3} |x|\chi(x)dx < \infty \). Note that the interaction terms in GSB models are of the simple form \( \sum_{j=1}^N A_j \otimes B_j \). Thus, the localization argument is not needed in GSB models.

(Uniqueness of the ground state) The physically realistic dispersion relation is \( \omega(k) = |k| \). Namely, photons are massless. Thus, in showing the uniqueness of the ground state, the min-max principle applied in [4] for massive boson is not applicable. Instead, we apply the strategy given in [16].

(Non-compact resolvent) In [4], the fermion term of the zero coupling Hamiltonian is assumed to have a compact resolvent. In our case, however, \( H_{d1} \) has no compact resolvent since \( \sigma(H_{d1}) = \{0\} \cup [m, \infty) \). So, we apply the strategy given in [7], [18], [19] for nonrelativistic QED.

(Asymptotic fields) In [21], the asymptotic field is constructed for massive cases. However, since we have to cover massless cases, we construct it through the stationary phase method discussed in [12], [18].
This paper is organized as follows: Section 1 is devoted to defining the Hamiltonian with spatial cutoffs, which is a slight generalization of $H_\chi$ mentioned above, and stating the main results. In Section 2, we prove that for sufficiently small values of the coupling constant, a unique ground state of the Hamiltonian exists under the infrared regularity condition. In section 3, the spectral scattering theory is considered and it is shown that the spectral gap is closed. Section 4 proves that the total charge of the ground state is zero.

1.1 Boson Fock Spaces and Fermion Fock spaces

Let $X$ and $Y$ be Hilbert spaces over $\mathbb{C}$. We denote by $\otimes^n_X$ the n-fold symmetric tensor product of $X$ and by $\otimes^a_Y$ the n-fold anti-symmetric tensor product of $Y$. The boson Fock space over $X$ is defined by

$$\mathcal{F}_b(X) := \oplus_{n=0}^\infty (\otimes^n_X),$$

and the fermion Fock space over $Y$ by

$$\mathcal{F}_f(Y) := \oplus_{n=0}^\infty (\otimes^a_Y),$$

$\mathcal{F}_b(X)$ is the Hilbert space with the inner product $\langle \Phi, \Psi \rangle = \sum_{n=0}^\infty \langle \Phi(n), \Psi(n) \rangle_{\otimes^n_X}$, and also $\mathcal{F}_f(Y)$ is the Hilbert space with the same inner product. In this paper, the inner product $(g,f)_X$ of Hilbert space $X$ is linear in $f$ and anti-linear in $g$. Let $S_n : \otimes^n_X \to \otimes^n_X$ and $A_n : \otimes^n_Y \to \otimes^n_Y$ be orthogonal projections. For $\xi \in X$, the creation operator $A^*(\xi)$ on $\mathcal{F}_b(X)$ is defined by

$$(A^*(\xi)\Psi)(n) = \sqrt{n+1}S_{n+1}(\xi \otimes \Psi(n)), \quad n \geq 1,$$

and $(A^*(\xi)\Psi)(0) = 0$, while the creation operator $B^*(\eta)$ on $\mathcal{F}_f(Y)$ is defined by

$$(B^*(\eta)\Psi)(n) = \sqrt{n+1}A_{n+1}(\eta \otimes \Psi(n)), \quad n \geq 1,$$

and $(B^*(\eta)\Psi)(0) = 0$. The annihilation operators $A(\xi)$ and $B(\eta)$ are defined by the adjoint operators of $A^*(\xi)$ and $B^*(\eta)$, respectively. Let $\Omega_b = \{1,0,\cdots\} \in \mathcal{F}_b(X)$ and $\Omega_f = \{1,0,\cdots\} \in \mathcal{F}_f(Y)$. We denote the boson-finite particle subspace over $M \subset X$ by

$$\mathcal{F}_b^\text{fin}(M) = \mathcal{L}\{A^*(\xi_1)\cdots A^*(\xi_n)\Omega_b, \Omega_b \mid \xi_j \in M, j=1,\cdots,n, n \in \mathbb{N}\},$$

and the fermion-finite particle subspace over $N \subset Y$ by

$$\mathcal{F}_f^\text{fin}(N) = \mathcal{L}\{B^*(\eta_1)\cdots B^*(\eta_n)\Omega_f, \Omega_f \mid \eta_j \in N, j=1,\cdots,n, n \in \mathbb{N}\}.$$

For simplicity, we call $\mathcal{F}_b^\text{fin}(X)$ the finite particle subspace. The domain of $A^*(\xi)$ is given by

$$\mathcal{D}(A^*(\xi)) = \{\Psi = \{\Psi(n)\}_{n=0}^\infty \in \mathcal{F}_b(X) \mid \sum_{n=0}^\infty \|A^*(\xi)\Psi(n)\|^2_{\otimes^n_X} < \infty\}.$$

It is seen that the domains of $A^*(\xi)$ and $A(\xi')$ include the finite particle subspace, and leave it invariant. They satisfy the canonical commutation relations on the finite particle space:

$$[A(\xi), A^*(\xi')] = \langle \xi', \xi \rangle_X, \quad [A(\xi), A(\xi')] = [A^*(\xi), A^*(\xi')] = 0,$$
where \([A, B] = AB - BA\). On the other hand, \(B^\ast(\eta)\) and \(B(\eta')\) are bounded on \(\mathcal{F}_f(\eta)\), and satisfy the canonical anti-commutation relations on \(\mathcal{F}_f(\eta)\):
\[
\{ B(\eta), B^\ast(\eta') \} = \langle \eta, \eta' \rangle_{\eta}, \quad \{ B(\eta), B(\eta') \} = \{ B^\ast(\eta), B^\ast(\eta') \} = 0,
\]
where \(\{ A, B \} = AB + BA\).

Let \(X\) be an operator on \(X\). The second quantization operator \(d\bar{\Gamma}_b(X)\) on \(\mathcal{F}_b(X)\) is defined by
\[
d\bar{\Gamma}_b(X) = \oplus_{n=0}^\infty \left( \sum_{j=1}^n (I \otimes \cdots \otimes \underbrace{X \otimes I \cdots \otimes I}_{j\text{th}}) \right).
\]
In the same way as \(d\bar{\Gamma}_b(X)\), we define \(d\bar{\Gamma}_f(Y)\) on \(Y\) for an operator \(Y\).

### 1.2 Radiation Fields

In this paper, we consider the photon field quantized in the Coulomb gauge.

Let
\[
\mathcal{F}_\text{ph} = \mathcal{F}_b(L^2(\mathbb{R}^3; \mathbb{C}^2)),
\]
and \(e_r(k) = (e_r^j(k))_{j=1}^2\), \(r = 1, 2\), be the polarization vector satisfying
\[
e_r(k) \cdot e_r'(k) = \delta_{r,r'}, \quad k \cdot e_r(k), \quad \text{a.e. } k \in \mathbb{R}^3.
\]
For \(f \in L^2(\mathbb{R}^3)\), let \(a_r^\ast(f) = A^\ast((f, 0))\) and \(a_2^\ast(f) = A^\ast((0, f))\). Then, \(a_r(f)\) and \(a_r^\ast(g)\) satisfy the canonical commutation relations:
\[
[a_r(f), a_{r'}^\ast(g)] = \delta_{r,r'}(f, g),
\]
on the finite particle subspace. The energy of a photon with momentum \(k \in \mathbb{R}^3\) is given by
\[
\omega(k) = |k|.
\]
The free Hamiltonian of the photon field is defined by
\[
H_{\text{ph}} = d\bar{\Gamma}_b(\omega).
\]
Let
\[
f_{rX}^j(k) = \frac{\chi_{\text{ph}}(k)e_r^j(k)}{\sqrt{2(2\pi)^3\omega(k)}},
\]
where \(e_r^j(k) = e_r^j(k)e^{-ik\cdot x}\). The quantized radiation field is defined by
\[
A_j(x) = \sum_{r=1,2} (a_r(f_{rX}^j) + a_r^\ast(f_{rX}^j)).
\]
We assume the following conditions.
The energy of an electron with momentum has negative energy part with spin. Next, we define the Dirac field. Let

\[ \| \alpha_r(f) \| \leq \| \frac{f}{\sqrt{\omega}} \| \| H_{ph}^{1/2} \|, \]

\[ \| \alpha_r^*(f) \| \leq \| \frac{f}{\sqrt{\omega}} \| \| H_{ph}^{1/2} \| + \| f \| \| \Psi \|. \]

Then \( A_j(x) \) is also relatively bounded with respect to \( H_{ph}^{1/2} \):

\[ \| A_j(x) \| \leq \sum_{r=1,2} (2 M_{ph,j,r}^{1/2} \| H_{ph}^{1/2} \| + M_{f,j,r}^{ph} \| \Psi \|), \]

where \( M_{k,j,r}^{ph} = \left\| \frac{\chi_{k,j,r}^f}{\sqrt{(2\pi)^3 \omega}} \right\| \) for \( k = 1, 2, r = 1, 2 \) and \( j = 1, 2, 3 \).

### 1.3 Dirac Fields

Next, we define the Dirac field. Let

\[ \mathcal{F}_{el} = \mathcal{F}_f(L^2(\mathbb{R}^3; \mathbb{C}^4)). \]

The energy of an electron with momentum \( \mathbf{p} \) is given by

\[ E_M(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}, \quad M > 0, \]

where \( M \) denotes the mass of an electron and we fix it. The free Hamiltonian of the Dirac field is defined by

\[ H_{el} = d\Gamma_f(E_M), \]

Let

\[ h_D(\mathbf{p}) = \alpha \cdot \mathbf{p} + \beta M, \quad s(\mathbf{p}) = \mathbf{s} \cdot \mathbf{p}, \]

where \( \alpha^j, j = 1, 2, 3 \) and \( \beta \) be the \( 4 \times 4 \) matrix satisfying the canonical anti-commutation relation:

\[ \{ \alpha^j, \alpha^{j'} \} = 2 \delta_{jj'}, \quad \{ \alpha_j, \beta \} = 0, \quad \beta^2 = I, \]

and \( \mathbf{s} = (s_j)_{j=1}^3 \) denotes the angular momentum of spin. Throughout this paper, we fix them. The spinors \( u_s(\mathbf{p}) = (u_s^j(\mathbf{p}))_{j=1}^4 \) describe the positive energy part with spin \( s \) and \( v_s(\mathbf{p}) = (v_s^j(\mathbf{p}))_{j=1}^4 \) the negative energy part with spin \( s, s = \pm 1/2 : \)

\[ h_D(\mathbf{p}) u_s(\mathbf{p}) = E_M(\mathbf{p}) u_s(\mathbf{p}), \quad s(\mathbf{p}) u_s(\mathbf{p}) = s|\mathbf{p}| u_s(\mathbf{p}), \]

\[ h_D(\mathbf{p}) v_s(\mathbf{p}) = -E_M(\mathbf{p}) v_s(\mathbf{p}), \quad s(\mathbf{p}) v_s(\mathbf{p}) = s|\mathbf{p}| v_s(\mathbf{p}). \]
These form an orthogonal base of $\mathbf{C}^4$

\[ u_s(p)^* u_\tau(p') = v_s(p)^* v_\tau(p') = \delta_{\tau, \tau} \sqrt{E_M(p)} \sqrt{E_M(p')}, \quad u_s(p)^* v_\tau(p') = v_s(p)^* u_\tau(p') = 0. \]

Moreover, the completeness condition is satisfied

\[ \sum_{s = \pm 1/2} \left( u_s'(p) u_s'(p')^* + v_s'(p) v_s'(p')^* \right) = \delta_{\tau, \tau}. \]

Let us set the creation operators by

\[ b_{1/2}^+(f) = B^+(l(f, 0, 0, 0)), \quad b_{-1/2}^+(f) = B^+(l(0, f, 0, 0)), \]
\[ d_{1/2}^+(f) = B^+(l(0, 0, f, 0)), \quad d_{-1/2}^+(f) = B^+(l(0, 0, 0, f)), \]

for $f \in L^2(\mathbb{R}^3)$. Then, the creation and annihilation operators satisfy the CAR:

\[ \{ b_s(f), b^*_\tau(g) \} = \{ d_s(f), d^*_\tau(g) \} = \delta_{\tau, \tau}(f, g), \]
\[ \{ b_s(f), b_\tau(g) \} = \{ d_s(f), d_\tau(g) \} = 0, \]
\[ \{ b_s(f), d_\tau(g) \} = \{ b_\tau(f), d^*_s(g) \} = 0. \]

Let

\[ g_{s,x}^l(p) = \frac{\chi_{\text{el}}(p) u_{s,x}^l(p)}{\sqrt{(2\pi)^3 E_M(p)}}, \quad h_{s,x}^l(p) = \frac{\chi_{\text{el}}(p) v_{s,x}^l(p)}{\sqrt{(2\pi)^3 E_M(p)}}, \tag{20} \]

where $u_{s,x}^l(p) = u_s^l(p)e^{-ip \cdot x}$ and $v_{s,x}^l(p) = v_s^l(-p)e^{-ip \cdot x}$.

The field operator for electron is defined by

\[ \psi_l(x) = \sum_{s = \pm 1/2} (b_s(g_{s,x}^l) + d^*_s(h_{s,x}^l)). \tag{21} \]

We assume the following conditions.

\textbf{(A.2) (Ultraviolet cutoff for the Dirac field)}

\[ \int_{\mathbb{R}^3} \left[ \frac{\chi_{\text{el}}(p)}{\sqrt{E_M(p)}} \right]^2 dp < \infty. \]

It is seen that $b_s(f)$ and $d_s(f)$ are bounded with $\| b_s(f) \| = \| d_s(f) \| = \| f \|$. Then, we can see that

\[ \| \psi_l(x) \| \leq M^l_{\text{el}}, \quad l = 1, \ldots, 4, \tag{22} \]

where $M^l_{\text{el}} = \sum_{s = \pm 1/2} \left( \left\| \frac{\chi_{\text{el}} u^l_s}{\sqrt{(2\pi)^3 E_M}} \right\| + \left\| \frac{\chi_{\text{el}} v^l_s}{\sqrt{(2\pi)^3 E_M}} \right\| \right)$. 

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1.4 Total Hamiltonian

The total Hilbert space is defined by

$$\mathcal{F}_{QED} = \mathcal{F}_{el} \otimes \mathcal{F}_{ph},$$

and the decoupled Hamiltonian on $$\mathcal{F}_{QED}$$ by

$$H_0 = H_{el} \otimes I + I \otimes H_{ph}.$$  \hfill (24)

In order to define the interaction, we introduce spatial cutoff functions $$\chi_I$$ and $$\chi_{II}$$ satisfying the following properties.

(A.3)

$$\int_{\mathbb{R}^3} |\chi_I(x)| \, dx < \infty, \quad \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\chi_{II}(x) \, \chi_{II}(y)|}{|x - y|} \, dx \, dy < \infty.$$

If $$\chi_{II} \in L^{6/5}(\mathbb{R}^3)$$, it follows from the Hardy-Littlewood-Sobolev inequality ([23]; 4.3 Theorem) that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\chi_{II}(x) \, \chi_{II}(y)|}{|x - y|} \, dx \, dy \leq \text{const.} \|\chi_{II}\|_{L^{6/5}}^2.$$

Let $$\Psi \in \mathcal{D}(I \otimes H^{1/2}_{ph})$$. Then, we can define the functional $$\ell_\Psi : \mathcal{F}_{QED} \to \mathbb{C}$$ by

$$\ell_\Psi(\Phi) = \sum_j \int_{\mathbb{R}^3} \chi_I(x)(\psi^\dagger(x) \alpha^j \psi(x) \otimes A_j(x) \Psi, \Phi)_{\mathcal{F}_{QED}} \, dx.$$

Since $$\|A_j(x)\Psi\|$$ and $$\|\psi(x)\|$$ are uniformly bounded with respect to $$x$$ by (17) and (22),

$$|\ell_\Psi(\Phi)| \leq \left( L_1 \| (I \otimes H^{1/2}_{ph}) \Psi \| + R_1 \| \Psi \| \right) \| \Phi \|$$

follows, where

$$L_1 = 2 \|\chi_I\|_{L^1} \sum_{j,l,l',r} |\alpha_{j,l'}^l| M_{l'}^j M_{l'}^1 M_{l,l'_j,r}^1,$$

$$R_1 = \|\chi_I\|_{L^1} \sum_{j,l,l',r} |\alpha_{j,l'}^l| M_{l'}^j M_{l'}^3 M_{l,l'_j,r}^{ph}.$$  \hfill (26, 27)

Here, we used $$\|\chi_I\|_{L^1} < \infty$$ in (A.3). By the Riesz representation theorem, there exists a unique vector $$\Xi_\Psi \in \mathcal{F}_{QED}$$ such that

$$\ell_\Psi(\Phi) = \langle \Xi_\Psi, \Phi \rangle \quad \text{for all} \quad \Phi \in \mathcal{F}_{QED}.$$

Let us define $$H'_I : \mathcal{F}_{QED} \to \mathcal{F}_{QED}$$ by

$$H'_I : \Psi \mapsto \Xi_\Psi.$$  \hfill (28)

It is seen from (25) that

$$\|H'_I \Psi\| \leq L_1 \| (I \otimes H^{1/2}_{ph}) \Psi \| + R_1 \| \Psi \|.$$  \hfill (29)
We may denote $H'_I$ formally by

$$H'_I = \int_{\mathbf{R}^3} \chi_I(x)\psi^*(x)\alpha^j\psi(x)\otimes A_j(x)\,dx.$$  

In the similar as $H'_I$, let us define the functional $q_\Psi : \mathcal{F}_{\text{QED}} \to \mathbf{C}$ by

$$q_\Psi(\Phi) = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(x)\chi_{II}(y)}{|x-y|} (\psi^*(x)\psi(x)\psi^*(y)\psi(y) \otimes I\Psi, \Phi)_{\mathcal{F}_{\text{QED}}} \,dx\,dy.$$  

It is seen that by (A.3)

$$|q_\Psi(\Phi)| \leq \left( M_{II} \sum_{l,v} (M_l^{cl}M_v^{cl})^2 \right) \|\Psi\|\|\Phi\|,$$  

where $M_{II} := \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{II}(x)\chi_{II}(y)|}{|x-y|} \,dx\,dy$. Then by the Riesz representation theorem, there exists a unique vector $\Upsilon_\Psi \in \mathcal{F}_{\text{QED}}$ such that

$$q_\Psi(\Phi) = (\Upsilon_\Psi, \Phi) \quad \text{for all} \quad \Phi \in \mathcal{F}_{\text{QED}}.$$  

Then we can define $H'_{II}$ by

$$H'_{II} : \Psi \mapsto \Upsilon_\Psi.$$  

By (30), it is seen that $H_{II}$ is bounded with

$$\|H'_{II}\| \leq M_{II} \sum_{l,v} (M_l^{cl}M_v^{cl})^2.$$  

We may also denote $H'_II$ formally by

$$H'_{II} = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{II}(x)\chi_{II}(y)}{|x-y|} \psi^*(x)\psi(x)\psi^*(y)\psi(y) \otimes I\,dx\,dy.$$  

Now let us define the total Hamiltonian under consideration by

$$H = H_0 + H'(\kappa_I, \kappa_{II}),$$  

where

$$H'(\kappa_I, \kappa_{II}) = \kappa_I H'_I + \kappa_{II} H'_{II}, \quad \kappa_I, \kappa_{II} \in \mathbf{R}.$$  

**Lemma 1.1 (Self-adjointness)**

Assume that (A.1)-(A.3) hold. Then, $H$ is self-adjoint on $\mathcal{D}(H_0)$. Moreover, $H$ is essentially self-adjoint on any core of $H_0$ and bounded from below.

**Remark 1.1** By the previous lemma, it can be seen that $H$ is essentially self-adjoint on

$$\mathcal{D}_0 := \mathcal{F}_{el}^{\text{fin}}(\mathcal{D}(E_M)) \hat{\otimes} \mathcal{F}_{ph}^{\text{fin}}(\mathcal{D}(\omega)),$$

where $\hat{\otimes}$ denotes the algebraic tensor product.
Next let us consider the spectrum of $H_{el}$ and $H_{ph}$. It is well known that $\sigma(H_{el}) = \{0\} \cup [M, \infty)$, $\sigma(H_{ph}) = [0, \infty)$ and $\sigma_p(H_{el}) = \sigma_p(H_{ph}) = \{0\}$. Then, the spectrum of $H_0 = H_{el} \otimes I + I \otimes H_{ph}$ is $\sigma(H_0) = [0, \infty)$ and the point spectrum is $\sigma_p(H_0) = \{0\}$. It is also seen that $H_0 \Omega_0 = 0$, where $\Omega_0 = \Omega_{el} \otimes \Omega_{ph}$. Since the ground state energy 0 of $H_0$ is embedded in $[0, \infty)$, it is not trivial to see that $H$ has the ground state for nonzero $\kappa_I$ and $\kappa_{II}$.

To prove the existence of the ground state of $H$, we introduce following assumptions.

(A.4) It holds that $\chi_{ph} \in L^1_{loc}(\mathbb{R}^3)$.

(A.5) It holds that $\int_{\mathbb{R}^3} |x| |\chi_I(x)| dx < \infty$.

(A.6) (Infrared regularity condition)
It holds that $\int_{\mathbb{R}^3} \left\| \frac{\chi_{ph}(k)}{\sqrt{\omega(k)}} \right\|^2 dk, \quad j = 1, 2, 3, \ r = 1, 2$.

Theorem 1.2 (Existence of ground state)
Assume that (A.1)-(A.6) hold. Then for sufficiently small $|\kappa_I|$ and $|\kappa_{II}|$, $H$ has a ground state.

Next let us investigate the multiplicity of the ground states. In order to show the uniqueness of the ground state and the existence of the asymptotic field, we make a stronger assumption than (A.4).

(A.7) There exists a closed set $O_{ph} \subset \mathbb{R}^3$ with the zero Lebesgue measure such that $\chi_{ph} \in C(\mathbb{R}^3 \setminus O_{ph})$ and $\phi_j^r \in C(\mathbb{R}^3 \setminus O_{ph}), \quad j = 1, 2, 3, \ r = 1, 2$.

Proposition 1.3 (Uniqueness of ground state)
Assume (A.1)-(A.7). Then, for sufficiently small $|\kappa_I|$ and $|\kappa_{II}|$, dimker $(H - E_0(H)) = 1$.

In addition, we investigate the spectral scattering theory. Let us assume the following condition.

(A.8) There exists a closed set $O_{el} \subset \mathbb{R}^3$ with the Lebesgue measure zero such that $\chi_{el}, \ u_s, \ v_s \in C^\infty(\mathbb{R}^3 \setminus O_{el}), \quad s = \pm 1/2.$ (36)

Example :
Let us take the standard representation

$$\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

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where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, the angular momentum of the spin is $s = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. Let us set $O = \{ p = (p_1, p_2, p_3) \mid p_1 \neq 0 \}$. Then, the asymptotic field
\[
\phi_+(p) = \begin{cases}
\frac{1}{\sqrt{2[|p|^2 - p_3]}} \begin{pmatrix} p_1 - ip_2 \\ p_2 \end{pmatrix} & \text{if } p \notin O,
\end{cases}
\]
\[
\phi_-(p) = \begin{cases}
\frac{1}{\sqrt{2[|p|^2 - p_3]}} \begin{pmatrix} p_3 - |p| \\ p_1 + ip_2 \end{pmatrix} & \text{if } p \notin O,
\end{cases}
\]
\[
\phi_0(p) = \begin{cases}
\frac{1}{\sqrt{2[|p|^2 - p_3]}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } p \in O.
\end{cases}
\]

Let $\lambda_\pm(p) = \frac{1}{\sqrt{2}} \sqrt{1 \pm E_M(p)}$. Then it is seen that
\[
\begin{split}
\psi_{\pm 1/2}(p) &= \begin{pmatrix} \lambda_+(p) \phi_+(p) \\ \pm \lambda_-(p) \phi_-(p) \end{pmatrix}, \\
\psi_{\pm 1/2}(p) &= \begin{pmatrix} \mp \lambda_-(p) \phi_+(p) \\ \pm \lambda_+(p) \phi_-(p) \end{pmatrix}.
\end{split}
\]

Then $u_{\pm s}, u_{\pm s} \in C^\infty(R^3 \setminus O)$.

**Theorem 1.4 (Asymptotic photon fields)**
Suppose (A.1)-(A.3), (A.6) and (A.7). Let $\eta = \mathcal{D}(\omega^{-1/2})$. Then for $\Psi \in \mathcal{D}(H)$, the asymptotic field
\[
a^{\pm}_{\psi}(\xi) \Psi := s - \lim_{t \to \pm \infty} e^{iH} e^{-it \phi_0} (I \otimes a^{\pm}_{\psi}(\xi)) e^{it \phi_0} e^{-iH} \Psi,
\]
effects.

**Theorem 1.5 (Asymptotic Dirac fields)**
Suppose (A.1)-(A.3) and (A.8). Let $\eta = \mathcal{D}(\omega^{-1/2})$. Then, the asymptotic fields
\[
b^{\pm}_{\psi}(\eta) := s - \lim_{t \to \pm \infty} e^{iH} e^{-it \phi_0} (b^{\pm}_{\psi}(\eta) \otimes I) e^{it \phi_0} e^{-iH},
\]
\[
a^{\pm}_{\psi}(\xi) := s - \lim_{t \to \pm \infty} e^{iH} e^{-it \phi_0} (a^{\pm}_{\psi}(\xi) \otimes I) e^{it \phi_0} e^{-iH}
\]
exist.

By using the asymptotic fields, we can obtain the following theorem.

**Theorem 1.6 (Absence of spectral gap)**
Suppose that (A.1)-(A.8) hold. Then $\sigma(H) = [E_0(H), \infty)$.

Finally, we consider the total charge of the ground state. The number operators of the electron and the positron
\[
N_+ = d\Gamma_+(1,1,0,0), \quad N_- = d\Gamma_-(0,0,1,1),
\]
respectively, and the total charge
\[
Q = N_+ - N_-.
\]
Since $\psi^\dagger(x) \psi(x)$ leaves the total charge invariant $[\psi^\dagger(x) \psi(x), Q] = 0$, it is proven in Lemma 4.2 that $H$ also leaves the total charge invariant $e^{iQ_{\psi}} H e^{-iQ_{\psi}} = H$. Then, $\mathcal{F}_{\text{QED}}$ is decomposed with respect to the spectrum of the total charge as
\[
\mathcal{F}_{\text{QED}} = \bigoplus_{z \in \mathbb{Z}} \mathcal{F}_{z},
\]
We will prove that the total charge of the ground state is zero.
2 Ground States

2.1 Self-adjointness

It is noted that, by the spectral decomposition theorem, for all \( \varepsilon > 0 \), there exists a positive number \( c_\varepsilon > 0 \) such that for all \( \Psi \in \mathcal{D}(H_{\text{ph}}) \),

\[
\|H_{\text{ph}}^{1/2}\Psi\| \leq \varepsilon \|H_{\text{ph}}\Psi\| + c_\varepsilon \|\Psi\|. \tag{40}
\]

(Proof of Lemma 1.1)

By (40) and (29), we see that for \( \Psi \in \mathcal{D}(H_0) \),

\[
\|H_1\Psi\| \leq \varepsilon L_4 \|H_0\Psi\| + (c_\varepsilon L_4 + R_1) \|\Psi\|. \tag{41}
\]

From (32) and (41), it follows that for \( \Psi \in \mathcal{D}(H_0) \),

\[
\|H'(\kappa_1, \kappa_2)\Psi\| \leq \varepsilon |\kappa_1| L_4 \|H_0\Psi\| + (|\kappa_1|(c_\varepsilon L_4 + R_1) + \kappa_2 \|H'_2\|) \|\Psi\|. \tag{42}
\]

Let us take \( \varepsilon > 0 \) such as \( \varepsilon |\kappa_1| L_4 < 1 \). Then the Kato-Rellich theorem reveals that \( H \) is self-adjoint on \( \mathcal{D}(H_0) \), essentially self-adjoint on any core of \( H_0 \), and bounded from below. \( \blacksquare \)

2.2 Existence of Ground State

To prove the existence of a ground state of \( H \), we introduce some Hamiltonians approximating \( H \). For \( m > 0 \), let \( \omega_m(k) = \omega(k) + m \) and \( H_{\text{ph}}(m) = d\Gamma_b(\omega_m) \). Let

\[
H(m) = H_0(m) + H'(\kappa_1, \kappa_2), \tag{43}
\]

where \( H_0(m) = H_{\text{el}} \otimes I + I \otimes H_{\text{ph}}(m) \).

In order to prove the existence of a ground state of \( H(m) \), we apply the momentum lattice approximation (e.g. [4], [7], [8], [11], [14], [17], [18]). For \( V > 0 \) and \( L > 0 \), we set

\[
\Gamma_V = \frac{2\pi}{V} \mathbb{Z}^3 = \{ \mathbf{q} = (q_1, q_2, q_3) | q_j = \frac{2\pi}{V} n_j, \ n_j \in \mathbb{Z}, \ j = 1, 2, 3 \},
\]

\[
\Gamma_{V,L} = \{ \mathbf{q} = (q_1, q_2, q_3) \in \Gamma_V \mid |q_j| + \frac{\pi}{V} \leq L, \ j = 1, 2, 3 \},
\]

and \( \mathcal{F}_{\text{ph},V} = \ell^2(\Gamma_V) \otimes \ell^2(\Gamma_V) \). We can identify \( \mathcal{F}_{\text{ph},V} \) with a closed subspace of \( \mathcal{F}_{\text{ph}} \). For a lattice point \( \mathbf{q} \in \Gamma_V \), we set \( C_{\mathbf{q},V} = \prod_{j=1}^3 [q_j - \frac{\pi}{V}, q_j + \frac{\pi}{V}] \subset \mathbb{R}^3 \).

Let

\[
\omega_{m,V}(k) = \sum_{\mathbf{q} \in \Gamma_V} \omega_m(\mathbf{q}) \chi_{C_{\mathbf{q},V}}(k), \quad (f_{r,x}^L)^{L,V}(k) = \sum_{\mathbf{q} \in \Gamma_{V,L}} f_{r,x}^L(\mathbf{q}) \chi_{C_{\mathbf{q},V}}(k),
\]

\[
12
\]
where $\chi_{C_qV}$ is the characteristic function on $C_qV$. In addition let us set $(f_{rX})^L(k) = \chi_{L}(k)(f_{rX})(k)$ for $\chi_L(k) = \chi_{[-L,L]}(k_1)\chi_{[-L,L]}(k_2)\chi_{[-L,L]}(k_3)$. Let $\mathcal{F}_V = \mathcal{F}_{el} \otimes \mathcal{F}_{ph,V}$. We introduce the operators

$$H_{0,V}(m) = H_{el} \otimes I + I \otimes H_{ph,V}(m),$$

where $H_{ph,V}(m) = d\Gamma_b(\omega_{m,V})$, and

$$H'_{L,V} = \sum_j \int_{\mathbb{R}^3} \chi_1(x) \left( \psi^*(x) \alpha_j \psi(x) \otimes A^L_j(x) \right) dx,$$

$$H'_{L} = \sum_j \int_{\mathbb{R}^3} \chi_1(x) \left( \psi^*(x) \alpha_j \psi(x) \otimes A^L_j(x) \right) dx,$$

where $A^L_j(x) = \sum_{r=1,2} \left( a_r((f_{rX})^L) + a^*_r((f_{rX})^L) \right)$, $A^L_j(x) = \sum_{r=1,2} \left( a_r((f_{rX})^L) + a^*_r((f_{rX})^L) \right)$.

Let us set

$$H_{L,V}(m) = H_{0,V}(m) + \kappa_I H'_{L,V} + \kappa_{II} H'_{II},$$

$$H_{L}(m) = H_{0}(m) + \kappa_I H'_{L} + \kappa_{II} H'_{II}.$$  \hspace{1cm} (44)

$$\text{In similar fashion to the proof of Lemma 1.1, it can be proven that } H_{L,V}(m) \text{ and } H_L(m) \text{ are self-adjoint, and essentially self-adjoint on any core of } H_{0,V} \text{ and } H_0(m), \text{ respectively. In particular } H_{L,V}(m) \text{ is essentially self-adjoint on } \mathcal{D}_{0,V}^m := \mathcal{F}_{el}^\infty(\mathcal{D}(E_M)) \otimes \mathcal{F}_{ph}^\infty(\mathcal{D}(\omega_{m,V})), \text{ and } H_L(m) \text{ on } \mathcal{D}_0^m := \mathcal{F}_{el}^\infty(\mathcal{D}(E_M)) \otimes \mathcal{F}_{ph}^\infty(\mathcal{D}(\omega_{m,V})).}$$

\textbf{Lemma 2.1} \quad \text{Assume (A.1)-(A.3). Then}

$$E_0(H) \leq |\kappa_{II}| \|H'_{II}\|$$

\text{(46)}

holds, where $E_0(H) = \inf \sigma(H)$.

\textbf{(Proof)} Let $\Psi_{el} \in \mathcal{D}(H_{el})$ and $\|\Psi_{el}\| = 1$. For $\Psi = \Psi_{el} \otimes \Omega_{ph}$,

$$(\Psi, H\Psi) = (\Psi_{el}, H_{el}\Psi_{el}) + \kappa_{II}(\Psi_{el}, H'_{II}\Psi_{el}) \leq (\Psi_{el}, H_{el}\Psi_{el}) + |\kappa_{II}| \|H'_{II}\|. $$

Here we used that $(\Omega_{ph}, H_{ph}\Omega_{ph}) = 0$ and $(\Omega_{ph}, A_j(x)\Omega_{ph}) = 0$. Then, $E_0(H) \leq (\Psi_{el}, H_{el}\Psi_{el}) + |\kappa_{II}| \|H'_{II}\|$, and hence $E_0(H) \leq E_0(H_{el}) + |\kappa_{II}| \|H'_{II}\| = |\kappa_{II}| \|H'_{II}\|$. \hspace{1cm} \blacksquare

\textbf{Lemma 2.2} \quad \text{Assume (A.1)-(A.3). Then}

(1) $H_{L,V}(m)$ is reduced by $\mathcal{F}_V$,

(2) For sufficiently small $m$, $|\kappa_I|$ and $|\kappa_{II}|$, $H_{L,V}(m)$ has a purely discrete spectrum in $[E_0(H_{L,V}(m)), E_0(H_{L,V}(m)) + m]$.

\textbf{(Proof)} (1) Let $\Psi = \Psi_{el} \otimes \Psi_{ph} \in \mathcal{D}_{0,V}^m$ with $\Psi_{ph} = a^*_r(f_j) \cdots a^*_r(f_j) \Omega_{ph}, j = 1, \cdots, n$. Let $q_V : L^2(\mathbb{R}^3) \rightarrow \ell^2(\Gamma_V)$ and $Q_V : \mathcal{F}_{ph} \rightarrow \mathcal{F}_{ph,V}$ be the orthogonal projections. It is seen that

$$H_{0,V}(m)(I \otimes Q_V)\Psi = (I \otimes Q_V)H_{0,V}(m)\Psi.$$  \hspace{1cm} (47)
Since $q_{V}x_{q_{V}} = x_{q_{V}}$, we also see that, by the definition of $A_{j}^{LV}(x)$,

$$Q_{V}A_{j}^{LV}(x)\Psi_{ph} = A_{j}^{LV}(x)Q_{V}\Psi_{ph},$$  \hspace{1cm} (48)

follows. Then, for $\Phi \in \mathcal{F}$,

$$(\Phi, (I \otimes Q_{V})H_{1,L,V}^{I}\Psi) = \sum_{j} \int_{R^{3}} \chi_{I}(x) (\Phi, \psi^{*}(x)\alpha^{j}(x) \otimes (Q_{V}A_{j}^{LV}(x))\Psi) \, dx$$

$$= \sum_{j} \int_{R^{3}} \chi_{I}(x) (\Phi, \psi^{*}(x)\alpha^{j}(x) \otimes (A_{j}^{LV}(x)Q_{V})\Psi) \, dx$$

$$= (\Phi, H_{1,L,V}^{I}(I \otimes Q_{V})\Psi).$$

Hence, we have $(I \otimes Q_{V})H_{1,L,V}^{I}\Psi = H_{1,L,V}^{I}(I \otimes Q_{V})\Psi$ for $\Psi \in \mathcal{D}_{0,V}^{m}$. It is trivial to see that $(I \otimes Q_{V})H_{II}^{I}\Psi = H_{II}^{I}(I \otimes Q_{V})\Psi$ for $\Psi \in \mathcal{D}_{0,V}^{m}$. Then

$$(I \otimes Q_{V})H_{L,V}(m) = H_{L,V}(m)(I \otimes Q_{V}),$$  \hspace{1cm} (49)

on $\mathcal{D}_{0,V}^{m}$. Since $\mathcal{D}_{0,V}^{m}$ is a core of $H_{L,V}(m)$, we can extend (49) for all $\Psi \in \mathcal{D}(H_{0,V}(m))$. Therefore, $H_{L,V}(m)$ is reduced by $\mathcal{F}_{V}$.

(2) By (29), there exist $c_{m,L,V} > 0$ and $d_{m,L,V} > 0$ such that

$$|\langle \Psi, H_{L,V}(m)\Psi \rangle - \langle \Psi, H_{0,V}(m)\Psi \rangle| \leq \kappa_{1}c_{m,L,V}\langle \Psi, I \otimes H_{ph,V}(m)\Psi \rangle$$

$$+ (\kappa_{d_{m,L,V}} + \kappa_{II}\|H_{II}^{I}\|\|\Psi\|^{2}).$$

Therefore, it can be seen that

$$H_{el} \otimes I + (1 - \kappa_{1}c_{m,L,V}) I \otimes H_{ph,V}(m) - (\kappa_{d_{m,L,V}} + \kappa_{II}\|H_{II}^{I}\|) \leq H_{L,V}(m),$$  \hspace{1cm} (50)

$$H_{L,V}(m) \leq H_{el} \otimes I + (1 + \kappa_{1}c_{m,L,V}) I \otimes H_{ph,V}(m) + (\kappa_{d_{m,L,V}} + \kappa_{II}\|H_{II}^{I}\|),$$  \hspace{1cm} (51)

where $A \leq B$ denotes that $(f,Af) \leq (f, Bf)$ for $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$. Let $X_{L,V}(m) := H_{L,V}(m) - E_{0}(H_{L,V}(m)) - m$.

We shall show that $X^{V}_{L,V}(m)$ has a purely discrete spectrum in $[-m,0)$, and hence $H_{L,V}(m)$ has a purely discrete spectrum in $[E_{0}(H_{L,V}(m)), E_{0}(H_{L,V}(m)) + m]$. By (46) and (50), we see that

$$X_{L,V}(m)$$

$$\geq H_{el} \otimes I + I \otimes \{(1 - \kappa_{1}c_{m,L,V})H_{ph,V}(m) - (\kappa_{d_{m,L,V}} + \kappa_{II}\|H_{II}^{I}\| + E_{0}(H_{m,L,V}) + m)\}$$

$$\geq H_{el} \otimes I + I \otimes \{(1 - \kappa_{1}c_{m,L,V})H_{ph,V}(m) - (\kappa_{d_{m,L,V}} + 2\kappa_{II}\|H_{II}^{I}\| + m)\}.$$  \hspace{1cm} (52)

It is noted that $I = E_{H_{el}}([0,M]) + E_{H_{el}}([M,\infty))$ and $H_{el} \geq ME_{H_{el}}([M,\infty))$, where $E_{H_{el}}$ is the spectral projection of $H_{el}$. Let $\kappa_{1}$, $\kappa_{II}$ and $m$ be small such that $-(\kappa_{d_{m,L,V}} + 2\kappa_{II}\|H_{II}^{I}\| + M) > 0$. Then we have $X_{L,V}(m) \geq S_{L,V}(m)$, where

$$S_{L,V}(m) = E_{H_{el}}([0,M]) \otimes \{(1 - \kappa_{1}c_{m,L,V})H_{ph,V}(m) - (\kappa_{d_{m,L,V}} + 2\kappa_{II}\|H_{II}^{I}\| + m)\}.$$
Let \( \{ e_n^+ \}_{n=0}^{N_+} \) and \( \{ e_n^- \}_{n=0}^{N_-} \) be complete orthonormal systems of \( \mathcal{F}_+ := E_{X_L,V}(m)([0,\infty)) \) and \( \mathcal{F}_- := E_{X_L,V}(m)((-\infty,0]) \) \( \mathcal{F}_{\text{QED}} \), respectively. For a self-adjoint operator \( X \), we set

\[
X^+ = E_X([0,\infty))XE_X([0,\infty)), \quad X^- = E_X((-\infty,0])XE_X((-\infty,0]),
\]

where \( E_X \) is the spectral projection of \( X \). Then we have

\[
0 \geq \text{Tr} X_{L,V}(m) |_{\mathcal{F}_-} \geq \sum_{n=1}^{N_-} (e_n^-, S_{L,V}(m)e_n^-) \geq \sum_{n=1}^{N_-} (e_n^-, S_{L,V}(m^-)e_n^-).
\]

(53)

Here we used \( S_{L,V}(m) = S_{L,V}(m)^+ + S_{L,V}(m)^- \) and \( S_{L,V}(m)^+ \geq 0 \). Since \( E_{H_0} \geq 0 \), we see that

\[
S_{L,V}(m)^- = E_{H_0}([0,M]) \otimes ((1 - \kappa_\text{c}_{m,L,V})H_{\text{ph},V}(m) - (\kappa_\text{d}_{m,L,V} + 2\kappa_\text{II}}||H_{\text{II}}^L|| + m)^-.
\]

Then it follows from (53) that

\[
| \text{Tr}(H_{L,V}(m) - E_0(H_{L,V}(m)) - m)| \leq \text{Tr} E_{H_0}([0,M]) \times | \text{Tr} ((1 - \kappa_\text{c}_{m,L,V})H_{\text{ph},V}(m) - (\kappa_\text{d}_{m,L,V} + 2\kappa_\text{II}}||H_{\text{II}}^L|| + m)^- | < \infty.
\]

Hence \( X_{L,V}(m) \) has a purely discrete spectrum in \([-m,0)\).

Let \( M_V = \ell^2(\Gamma_V) \otimes \ell^2(\Gamma_V) \). We can decompose \( \mathcal{F}_{\text{ph}} \) as \( \mathcal{F}_{\text{ph}} \simeq \bigoplus_{n=0}^{\infty} (\mathcal{F}_{\text{ph},V} \otimes (\otimes_x M_V^\perp)) \), and hence \( \mathcal{F}_{\text{QED}} \simeq \mathcal{F}_V \otimes \mathcal{F}_V^\perp \), where \( \mathcal{F}_V^\perp = \bigoplus_{n=1}^{\infty} (\mathcal{F}_V \otimes (\otimes_x M_V^\perp)) \). For \( n \geq 1 \),

\[
H_{L,V} |_{\mathcal{F}_V \otimes (\otimes_x M_V^\perp)} \simeq H_{L,V} |_{\mathcal{F}_V} \otimes I_{\bigoplus_x M_V^\perp} + \int_{\mathcal{F}_V} d\Gamma_V(\omega_V) |_{\bigoplus_x M_V^\perp} \geq E_0(H_{L,V}) + nm.
\]

Hence, \( H_{L,V} |_{\mathcal{F}_V} \geq E_{H_0} + m \). ■

**Lemma 2.3** Assume (A.1)-(A.5). For sufficiently large \( L \), there exist constants \( a_1(m) > 0 \) and \( b_1(m) > 0 \) independent of \( L \) such that

\[
\|H_0(m)\Psi\| \leq a_1(m)\|H_L(m)\Psi\| + b_1(m)\|\Psi\|, \quad \Psi \in \mathcal{D}(H_0(m)).
\]

(Proof) Let \( \Psi \in \mathcal{D}(H_0). \) It is seen that

\[
\|H_0(m)\Psi\| = \|H_L(m)\Psi - (\kappa_\text{c}H_{L}^L(m) + \kappa_\text{II}^L)\Psi\|
\leq \|H_L(m)\Psi\| + |\kappa_\text{c}||H_L^L\Psi| + |\kappa_\text{II}||H_{\text{III}}^L||\Psi||.
\]

Note that

\[
\|A_j^L(x)\Psi\| \leq 2\left\|\frac{\sqrt{2\pi}e^{i\omega_m t}}{L}\right\|\|H_{\text{ph}}(m)^{1/2}\Psi\| + \left\|\frac{\sqrt{2\pi}e^{i\omega_m t}}{L}\right\|\|H_{\text{ph}}(m)^{1/2}\Psi\|
\]

and \( \lim_{L \to \infty} \|\frac{\sqrt{2\pi}e^{i\omega_m t}}{L}\| = \|\frac{\sqrt{2\pi}e^{i\omega_m t}}{\omega_m}\| \) and \( \lim_{L \to \infty} \|\frac{\sqrt{2\pi}e^{i\omega_m t}}{\omega_m}\| = \|\frac{\sqrt{2\pi}e^{i\omega_m t}}{\omega_m}\| \). Hence, for sufficiently large \( L \), \( \|A_j^L(x)\Psi\| \) is bounded uniformly in \( L \). By (40), it is seen that for all \( \varepsilon > 0 \), there exists a constant \( \tilde{c}_\varepsilon \) such that \( \|H_{L}^L\Psi\| \leq \varepsilon\|H_0(m)\Psi\| + \tilde{c}_\varepsilon\|\Psi\| \). Hence, we have

\[
\|H_0(m)\Psi\| \leq \frac{1}{1 - \varepsilon}\|H_L(m)\Psi\| + \frac{\tilde{c}_\varepsilon + \kappa_\text{II}||H_{\text{III}}^L||}{1 - \varepsilon}\|\Psi\|,
\]

and the proof is completed. ■
Lemma 2.4 Assume (A.1)-(A.5). For all \( z \in \mathbb{C} \setminus \mathbb{R} \), it follows that
\[
\lim_{V \to \infty} \| (H_{L,V}(m) - z)^{-1} - (H_L(m) - z)^{-1} \| = 0, \quad (55)
\]
\[
\lim_{L \to \infty} \| (H_L(m) - z)^{-1} - (H(m) - z)^{-1} \| = 0. \quad (56)
\]

(Proof)
We see that
\[
(H_{L,V}(m) - z)^{-1} - (H_L(m) - z)^{-1} = (H_{L,V}(m) - z)^{-1}(I \otimes (H_{ph}(m) - H_{ph,V}(m)))(H_L(m) - z)^{-1}
\]
\[
+ \kappa_i(H_{L,V}(m) - z)^{-1}(H'_{LL} - H'_{LL,V})(H_L(m) - z)^{-1}. \quad (57)
\]
Let \( C_V(m) = \sqrt{3} \left( \frac{3}{m} \right)^3 \left( \frac{1}{m} + 1 \right) \). Then, it is seen (\([3] \); Lemma 3.1) that for \( \Psi \in \mathcal{D}(H_{ph}(m)) \),
\[
\| (H_{ph}(m) - H_{ph,V}(m))\Psi \| \leq \frac{2C_V(m)}{1 - C_V(m)} \| H_{ph}(m)\Psi \|
\]
hence, we obtain
\[
\| (I \otimes (H_{ph}(m) - H_{ph,V}(m)))(H_L(m) - z)^{-1} \| \leq \frac{2C_V(m)}{1 - C_V(m)} \| (I \otimes H_{ph}(m))(H_L(m) - z)^{-1} \| \to 0,
\]
as \( V \to \infty \). The second term on the right-hand side of (57) can be estimated as
\[
\| (H'_{LL} - H'_{LL,V})(H_L(m) - z)^{-1}\Psi \| \leq \sum_{j,l,p} |\alpha_{j,p}^l| M_j^1 M_p^1 \int_{\mathbb{R}^3} |\chi_l(x)\| |I \otimes \left( A_j^l(x) - A_j^L(x) \right) (H_L(m) - z)^{-1}\Psi \| dx.
\]
By (15) and (16) we see that for \( \Xi \in \mathcal{D}(H_{ph}(m))^{1/2} \)
\[
\| (A_j^l(x) - A_j^L(x)\Xi) \| \leq \frac{1}{2(2\pi)^3} \sum_{r} \left( \| (X_{ph}e_x^j)^L_{\sqrt{\omega}} - (X_{ph}e_x^j)^L_{\sqrt{\omega'}} \| H_{ph}(m))^{1/2} \Xi \| \right.
\]
\[
\left. + \| (X_{ph}e_x^j)^{L,V}_{\sqrt{\omega}} - (X_{ph}e_x^j)^{L,V}_{\sqrt{\omega'}} \| \Xi \| \right).
\]
Hence, in order to prove \( \lim_{V \to \infty} \| (H'_{LL} - H'_{LL,V})(H_L(m) - z)^{-1} \| = 0 \), it is enough to show that
\[
\lim_{V \to \infty} \int_{\mathbb{R}^3} |\chi_l(x)\| \left( (X_{ph}e_x^j)^L_{\sqrt{\omega}} - (X_{ph}e_x^j)^L_{\sqrt{\omega'}} \right) dx = 0. \quad (58)
\]
It is seen that,
\[
\| (X_{ph}e_x^j)^L_{\sqrt{\omega}} - (X_{ph}e_x^j)^L_{\sqrt{\omega'}} \|^2 \leq 2 \int_{L} |X_{ph}(k)e_x^j(k) - \sum_{q \in \Gamma_{L,V}} X_{ph}(q)e_x^j(q) \chi_{q_{L,V}}^j(k) \|^2 dk
\]
\[
+ 2 \int_{L} \sum_{q \in \Gamma_{L,V}} \frac{X_{ph}(q)e_x^j(q)}{\sqrt{\omega(q)}} (e^{ik \cdot x} - e^{q \cdot x}) \chi_{q_{L,V}}^j(k) \|^2 dk.
\]
By the inequality $|e^{ik \cdot x} - e^{iq \cdot x}| \leq |x||k - q|$, we obtain

$$\left\| \frac{(X_{ph} e^{i} \cdot x)^{L}}{\sqrt{\omega}} - \frac{(X_{ph} e^{i} \cdot x)^{L,V}}{\sqrt{\omega}} \right\|^{2} \leq X_{L,V}^{j} + |y|^{2} Y_{L,V}^{j},$$

where

$$X_{L,V} = 2 \int_{\mathbb{R}^{d}} \left| \frac{X_{ph}(k) e^{i} \cdot x(k)}{\sqrt{\omega}(k)} - \sum_{q \in \Gamma_{V,\kappa}} \frac{X_{ph}(q) e^{i} \cdot x(q)}{\sqrt{\omega}(q)} \chi_{C_{q,V}}(k) \right|^{2} dk,$$

$$Y_{L,V} = 2 \int_{\mathbb{R}^{d}} \left( \sum_{q \in \Gamma_{V,\kappa}} \frac{|X_{ph}(q) e^{i} \cdot x(q)|}{\sqrt{\omega}(q)} |k - q| \chi_{C_{q,V}}(k) \right)^{2} dk.$$

Hence, we have

$$\int_{\mathbb{R}^{d}} |\chi_{1}(x)| \left\| \frac{(X_{ph} e^{i} \cdot x)^{L}}{\sqrt{\omega}} - \frac{(X_{ph} e^{i} \cdot x)^{L,V}}{\sqrt{\omega}} \right\| dx \leq \|\chi_{1}\| L_{1} \sqrt{X_{L,V}^{j}} + \int_{\mathbb{R}^{d}} |\chi_{1}(x)| dx \sqrt{Y_{L,V}^{j}}.$$

Here, we used the assumption $\int_{\mathbb{R}^{d}} |\chi_{1}(x)| dx < \infty$ of (A.6). Then, by the Lebesgue dominated convergence, we see that $X_{L,V}^{j} \to 0$ and $Y_{L,V}^{j} \to 0$ as $V \to \infty$. Then (58) is obtained, and hence (55) follows. We can prove (56) similarly to (55) by using Lemma 2.3.

**Proposition 2.5**
Assume (A.1) - (A.5). Then $H(m)$ has a purely discrete spectrum in $[E_{0}(H(m)), E_{0}(H(m)) + m]$. In particular $H(m)$ has a ground state.

**(Proof)** By Lemma 2.2, $H_{L,V}(m)$ has a purely discrete spectrum in $[E_{0}(H_{L,V}(m)), E_{0}(H_{L,V}(m)) + m]$. In addition, $H_{L,V}(m)$ converges to $H_{L}(m)$ in the norm resolvent sense as $V \to \infty$ by Lemma 2.4. Hence, by the general theorem ([25]; Lemma 4.6), $H_{L}(m)$ has a purely discrete spectrum in $[E_{0}(H_{L}(m)), E_{0}(H_{L}(m)) + m]$. It is also seen that $H_{L}(m)$ converges to $H$ in the norm resolvent sense as $L \to \infty$ by Lemma 2.4. Hence, $H(m)$ has a purely discrete spectrum in $[E_{0}(H(m)), E_{0}(H(m)) + m]$ by the same theorem ([25]; Lemma 4.6).

By proposition 2.5, $H(m)$ has a ground state $\Psi_{m}$:

$$H(m) \Psi_{m} = E_{0}(H(m)) \Psi_{m}.$$

The number operator of $\mathcal{F}_{b}(L^{2}(\mathbb{R}^{3}; C^{2}))$ is defined by

$$N_{ph} = d\Gamma_{b}(I).$$

**Lemma 2.6** Suppose that (A.1) - (A.6). Then

$$\| (I \otimes N_{ph}^{1/2}) \Psi_{m} \| \leq |\kappa_{l}| \sum_{j,l',r} \cos^{j} v^{l,l',p} \left\| \frac{X_{ph} e^{i} \cdot x_{r}}{\sqrt{2(2\pi)^{d} \omega_{m}^{2}} \omega} \right\| \|\Psi_{m}\|,$$

where $v^{l,l',p} = \|\chi_{1}\| L_{1} |\alpha_{l,p}| M_{l}^{e} M_{p}^{e}$. 

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(Proof) Since $\omega_m > 0$, $\mathcal{D}(H_{\text{ph}}(m)) \subset \mathcal{D}(N_{\text{ph}})$ follows. Hence we see that $\Psi_m \in \mathcal{D}(I \otimes N_{\text{ph}})$. Let

$$T^{r,j}(f) = -I \otimes a_r(\omega f) - i \sum_{k} \int_{\mathbb{R}^3} \chi_l(x) \left( f, \frac{\mathcal{X}_m f}{\sqrt{2(2\pi)^3}} \right) \alpha^j (\Psi^*)(x) \Psi(x) \otimes I \, dx. \quad (60)$$

By the commutation relation of $H$ and $a_r(f)$, we see that for all $\Phi \in \mathcal{D}_0$, $(I \otimes a_r(f)\Psi_m, (H - E_0(H(m)))\Phi) = (T^{r,j}(f)\Psi_m, \Phi)$. Hence, $I \otimes a_r(f)\Psi_m \in \mathcal{D}(H^*) = \mathcal{D}(H)$ and

$$(H - E_0(H(m))) I \otimes a_r(f)\Psi_m = T^{r,j}(f)\Psi_m \quad (61)$$

follow. Then by (61),

$$0 \leq (I \otimes a_r(f)\Psi_m, (H(m) - E_0(H(m)))(I \otimes a_r(f)\Psi_m) = - (I \otimes a_r(f)\Psi_m, I \otimes a_r(\omega_m f)\Psi_m)$$

$$- i \sum_{k} \int_{\mathbb{R}^3} \chi_l(x) \left( f, \frac{\mathcal{X}_m f}{\sqrt{2(2\pi)^3}} \right) (I \otimes a_r(f)\Psi_m, (\Psi^*)(x)\alpha^j \Psi(x) \otimes I)\Psi_m \right) \, dx. \quad (62)$$

Let $\{g_k\}_{k=1}^\infty$ be a complete orthonormal system of $L^2(\mathbb{R}^3)$ such that $g_k \in \mathcal{D}(\omega_m^{1/2})$, $k \geq 1$. By (4; Lemma 4.2), it is seen that for all $\Psi \in \mathcal{D}(I \otimes H_{\text{ph}}(m))$,

$$\sum_{k=1}^\infty \sum_{r=1,2} (I \otimes a_r(\frac{g_k}{\sqrt{\omega_m}})\Psi, I \otimes a_r(\sqrt{\omega_m}g_k)\Psi) = \|I \otimes N_{\text{ph}}^{1/2}\Psi\|^2. \quad (63)$$

By (62), it follows that for $N < \infty$,

$$\sum_{k=1}^N \sum_{r=1,2} \left( (I \otimes a_r(\frac{g_k}{\sqrt{\omega_m}})\Psi_m, I \otimes a_r(\sqrt{\omega_m}g_k)\Psi_m) \right) + i \sum_{j=1}^3 \int_{\mathbb{R}^3} \chi_l(x) \left( I \otimes a_r(\eta_{r,j,x}^k)\Psi_m, (\Psi^*)(x)\alpha^j \Psi(x) \otimes I)\Psi_m \right) \, dx \right) \leq 0, \quad (64)$$

where $\eta_{r,j,x}^k = \frac{1}{\sqrt{\omega_m}}(g_k, \frac{\mathcal{X}_m f}{\sqrt{2(2\pi)^3}} \omega_m \omega) g_k$. For $N \in \mathbb{N}$, we define

$$\lambda_{r,j}^k(x) := \sum_{k=1}^N \chi_l(x) \left( I \otimes a_r(\eta_{r,j,x}^k)\Psi_m, (\Psi^*)(x)\alpha^j \Psi(x) \otimes I)\Psi_m \right), \quad j = 1, 2, 3,$$

and let

$$\lambda_{r,j}^l(x) := \chi_l(x) \left( I \otimes a_r(\frac{\mathcal{X}_m f}{\sqrt{2(2\pi)^3}} \omega_m \omega)\Psi_m, (\Psi^*)(x)\alpha^j \Psi(x) \otimes I)\Psi_m \right), \quad j = 1, 2, 3.$$

Since $\{g_k\}_{k=1}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R}^3)$, we have

$$\lim_{N \to \infty} \left\| \sum_{k=1}^N \eta_{r,j,x}^k - \frac{\mathcal{X}_m f}{\sqrt{2(2\pi)^3}} \omega_m \omega \right\| = 0,$$
Lemma 2.7 Suppose independent of It can be proven, in similar manner to Lemma 2.3, that there exist constants such that

\[
\lim_{N \to \infty} \int_{\mathbb{R}^3} |\lambda_N^j(x) - \lambda^j(x)| \, dx = 0
\]

as \( N \to \infty \). We also see that

\[
\left\| \int_{\mathbb{R}^3} \lambda_N^j(x) \, dx \right\| \leq \left\| \int_{\mathbb{R}^3} \lambda(x) \, dx \right\| = \left\| \chi(x) \right\|_{L^1} \sum_{i, i'} |\alpha_i | \, M_{i i'} \left\| \frac{\lambda(x)}{\sqrt{(2\pi)^3 \omega_m}} \right\| \left\| \left( I \otimes N_{ph}^{1/2} \right) \psi_m \right\| \psi_m
\]

Then \( \lim_{N \to \infty} \int_{\mathbb{R}^3} \lambda_N^j(x) \, dx = \int_{\mathbb{R}^3} \lambda^j(x) \, dx \) by the Lebesgue dominated convergence theorem. Therefore, by taking \( N \to \infty \) in (64),

\[
\sum_{k=1}^{\infty} \sum_{l=1, 2} \left( I \otimes \alpha_l \left( \frac{g_k}{\sqrt{\omega_m}} \right) \psi, I \otimes \alpha_l \left( \frac{g_k}{\sqrt{\omega_m}} \right) \psi \right)
\]

Then for \( \delta > 0 \),

\[
\left\| \left( I \otimes N_{ph}^{1/2} \right) \psi_m \right\|^2 \leq \left\| \chi(x) \right\|_{L^1} \sum_{j=1}^{\delta} \left| \left( I \otimes \alpha_j \left( \frac{\lambda(x)}{\sqrt{(2\pi)^3 \omega_m}} \right) \psi_m, \left( \psi^*(x) \alpha_j \psi(x) \otimes \right) \psi_m \right) \right| dx
\]

By (63),

\[
\left\| \left( I \otimes N_{ph}^{1/2} \right) \psi_m \right\|^2 \leq \left\| \chi(x) \right\|_{L^1} \sum_{j, j', \mu} \left| \alpha_j^j, \mu | M_{j j'}^{1, \mu} \right| \left\| \frac{\lambda(x)}{\sqrt{(2\pi)^3 \omega_m}} \right\| \left\| \left( I \otimes N_{ph}^{1/2} \right) \psi_m \right\| \psi_m
\]

Thus, the proof is completed. 

Let

\[
\mathcal{I}_{el, \delta} := E_{H_0}([0, \delta)) \mathcal{I}_{el}, \quad \delta > 0.
\]

We define the orthogonal projections by

\[
P_\delta = E_{H_0}([0, \delta)), \quad P_\delta^\perp = I - P_\delta, \quad P_{\Omega_{ph}} : \mathcal{I}_{ph} \to \mathcal{L}\{z \Omega_{ph} \mid z \in \mathbb{C} \}.
\]

It can be proven, in similar manner to Lemma 2.3 that there exist constants \( a_2 > 0 \) and \( b_2 > 0 \) independent of \( m \) such that

\[
\left\| H_0(m) \psi \right\| \leq a_2 \left\| H(m) \psi \right\| + b_2 \left\| \psi \right\|, \quad \psi \in \mathcal{D}(H(m)).
\]

Lemma 2.7 Suppose (A.1) - (A.6). Let \( \kappa_m \) be sufficiently small such that \( \kappa_m \left\| H_0' \right\| < \delta \). Then for \( \varepsilon > 0 \),

\[
\left\| (P_\delta^\perp \otimes P_{\Omega_{ph}}) \psi_m \right\| \leq \frac{|\kappa_m| \varepsilon + |\kappa_m| \left\| H_0' \right\|}{\delta - E_0(H(m))} \left\| \psi_m \right\|,
\]

where \( \varepsilon L_0(a_2 E_0(H(m)) + b_2) + c_\varepsilon L_0 + R_1 \) and \( c_\varepsilon \) is the constant in (40).
Remark 2.1 It is noted that $E_0(H(m)) < \delta$ follows for sufficiently small $\kappa_H$ as $|\kappa_H||H_r'| < \delta$ by Lemma 2.1.

(Proof) It is similar to (4); Lemma 4.7. □

(Proof of Theorem 1.2)

By the general theorem (4); Lemma 4.9, it is enough to show that $\lim_{m \to 0} E_0(H_m) = E_0(H)$ and there exists a nonzero weak limit of $\Psi_m$. We see that $E_0(H(m)) = (\Psi_m, H(m)\Psi_m) = (\Psi_m, H\Psi_m) + m(\Psi_m, I \otimes \mathcal{N}_{\text{ph}} \Psi_m)$. Then, we have $\liminf_{m \to 0} E_0(H(m)) \geq E_0(H)$. Since $\lim_{m \to \infty} \|H(m)\Psi - H\Psi\| = 0$ for $\Psi \in \mathcal{D}_0$, it is seen that $H(m)$ converges to $H$ as $m \to \infty$ in the strong resolvent sense. Hence, $\limsup_{m \to 0} E_0(H(m)) \leq E_0(H)$. Thus, we see that $\lim_{m \to 0} E_0(H(m)) = E_0(H)$. We next show that there exists a nonzero weak limit of $\Psi_m$. We assume that $\|\Psi_m\| = 1$ for all $m > 0$. Then, there exists a subsequence $\{\Psi_{m_j}\}$ such that $\Psi := w - \lim_{j \to \infty} \Psi_{m_j}$ exists. It is seen that $P_\delta \otimes P_{\Omega_{\text{ph}}} = I - I \otimes \mathcal{N}_{\text{ph}} - P_\delta \otimes P_{\Omega_{\text{ph}}}$ follows. By this inequality, Lemma 2.6 and Lemma 2.7 yield that

$$(\Psi_{m_j}, (P_\delta \otimes P_{\Omega_{\text{ph}}})\Psi_{m_j}) \geq 1 - \left| \kappa_l \right| \|\Psi_{m_j}\| \delta - E_0(H(m_j)) \right|^2.$$

Assume that $|\kappa_l||H_j'| < \delta < M$. Then, $P_\delta \otimes P_{\Omega_{\text{ph}}}$ is a finite rank operator, since $\sigma(H_{el}) = \{0\} \cup [M, \infty)$. Taking the limit of $\Psi_{m_j}$ as $j \to \infty$ in the above inequality, it follows that $(\Psi_0, (P_\delta \otimes P_{\Omega_{\text{ph}}})\Psi_0) > 0$ for sufficiently small $\kappa_l$ and $\kappa_H$. Then $\Psi_0 \neq 0$, and the proof is completed. □

2.3 Uniqueness of Ground States

Lemma 2.8 Assume (A.7). Then, for $\xi \in C^\infty(\mathbb{R}^3)$, there exist $C_{r,j}^1 > 0$ and $C_{r,j}^2 > 0$ such that

$$
\left| \left( \xi, \frac{\chi_{\text{ph}} e^{i\omega(k) k}}{\sqrt{2(2\pi)^3 \omega(k)}} \right) \right| \leq \frac{c_{r,j}^1}{t(1+t)} + |\xi| \frac{c_{r,j}^2}{t(1+t)}.
$$

(Proof) It is seen that $e^{-i\omega(k)} = \frac{1}{i\omega(k)} \frac{\partial}{\partial k_v} e^{-i\omega(k)}$. Using integration by parts, we obtain

$$
\int \mathbb{R}^3 \frac{\xi(k)}{\sqrt{2(2\pi)^3 \omega(k)}} \frac{\chi_{\text{ph}} e^{i\omega(k)} e^{i(k \cdot x - t\omega(k))} d\mathbf{k}}{\sqrt{2(2\pi)^3 \omega(k)}} = \frac{1}{it} \int \mathbb{R}^3 \left( \frac{\partial}{\partial k_v} K_{r,j}(k) \right) e^{i(k \cdot x - t\omega(k))} d\mathbf{k} - \frac{\lambda}{t} \int \mathbb{R}^3 K_{r,j}(k) e^{i(k \cdot x - t\omega(k))} d\mathbf{k},
$$

where $K_{r,j}(k) = \frac{\xi(k)}{\sqrt{2(2\pi)^3 \omega(k)}} \frac{\chi_{\text{ph}} e^{i\omega(k)} d\mathbf{k}}{\sqrt{2(2\pi)^3 \omega(k)}}$. Since $e^{i(k \cdot x - t\omega(k))} \in C^\infty(\mathbb{R}^3 \setminus O_{\text{ph}})$ and $\chi_{\text{ph}} \in C^\infty(\mathbb{R}^3)$, it follows that $\chi_{\text{ph}} e^{i\omega(k)} \in C^\infty(\mathbb{R}^3 \setminus O_{\text{ph}})$. Hence for $\xi \in C^\infty(\mathbb{R}^3 \setminus \{0\})$, $K_{r,j}$, $\partial_{k_v} K_{r,j} \in C^\infty(\mathbb{R}^3 \setminus O_{\text{ph}})$. By (24; Theorem XI.19), there exist $c_{r,j}^1 > 0$ and $c_{r,j}^2 > 0$ such that

$$
\sup_{x \in \mathbb{R}^3} \left| \int \mathbb{R}^3 (\partial_{k_v} K_{r,j}(k)) e^{i(k \cdot x - t\omega(k))} d\mathbf{k} \right| \leq \frac{c_{r,j}^1}{1+t}, \ \sup_{x \in \mathbb{R}^3} \left| \int \mathbb{R}^3 K_{r,j}(k) e^{i(k \cdot x - t\omega(k))} d\mathbf{k} \right| \leq \frac{c_{r,j}^2}{1+t},
$$

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and hence the proof is completed. ■

(Proof of Proposition 1.3)
Let $\Phi, \Psi \in \mathcal{D}(H)$ and $\xi \in C^\infty(\mathbb{R}^3)$. Let us define the bilinear form $[X,Y]^0 : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by $[X,Y]^0(\phi, \psi) = (X^* \phi, Y \psi) - (Y^* \phi, X \psi)$. Then, we see that

$$
[H'(\kappa_1, \kappa_{11}), I \otimes \alpha_r(\xi)]^0(\Phi, \Psi) = \kappa_1 \sum_j \int_{\mathbb{R}^3} \chi_1(x)(\xi, \frac{X_{ph}e_j^r}{\sqrt{2(2\pi)^3}} \alpha_r(\xi), (\psi^*(x) \alpha^l \psi(x) \otimes I) \Psi) dx
$$

$$
= \int_{\mathbb{R}^3} \xi(k)(\Phi, T(r,k) \Psi) dk,
$$

where

$$
T(r,k) \Psi = \sum_j \int_{\mathbb{R}^3} \chi_1(x) \frac{X_{ph}e_j^r(k)e^{ikx}}{\sqrt{2(2\pi)^3}} (\psi^*(x) \alpha^l \psi(x) \otimes I) \Psi dx.
$$

It is sufficient to show that $H = H_0 + H'(\kappa_1, \kappa_{11})$ and $T(r,k)$ satisfy the assumptions (H.1)-(H.6) in the appendix with $X = X_0 + qX'$ and $S(r,k)$ replaced by $H = H_0 + H'(\kappa_1, \kappa_{11})$ and $T(r,k)$, respectively. We immediately see that $H = H_0 + H'(\kappa_1, \kappa_{11})$ and $T(r,k)$ satisfy (H.1)-(H.3) and (H.5). Then the remaining task is to show (H.4) and (H.6). Let $\Psi, \Phi \in \mathcal{D}(H)$. Then we see that

$$
\int_{\mathbb{R}^3} \xi(k)(\Phi, e^{-it(H-E_0(H)+\omega(k))} T(r,k) \Psi) dk
$$

$$
= \kappa_1 \sum_j \int_{\mathbb{R}^3} \chi_1(x)(\xi, \frac{X_{ph}e_j^r}{\sqrt{2(2\pi)^3}} \alpha_r(\xi), (\psi^*(x) \alpha^l \psi(x) \otimes I) \Psi) dx.
$$

Then by Lemma 2.8 and (A.3), (A.6), we have

$$
\left| \int_{\mathbb{R}^3} \xi(k)(\Phi, e^{-it(H-E_0(H)+\omega(k))} T(r,k) \Psi) dk \right|
$$

$$
\leq \sum_{j,l,l'} |\alpha_{l,j}^r| |M_l^r M_{l'}^l| \|\Phi\| \|\Psi\| \left( c_{l,j}^r \|\chi_1\|_{L^1} + c_{l,j}^r \int_{\mathbb{R}^3} |x| |\chi_1(x)| dx \right) \frac{1}{t(1+t)}.
$$

Hence, we obtain

$$
\int_{\mathbb{R}^3} \xi(k)(\Phi, e^{-it(H-E_0(H)+\omega(k))} T(r,k) \Psi) dk \in L^1([0, \infty), dt).$$

It is also seen that

$$
\|T(r,k)\Psi\| \leq \|\chi_1\|_{L^1} \left( \sum_{j,l,l'} |\alpha_{l,j}^r| |M_l^r M_{l'}^l| \frac{|X_{ph}(k)e_j^r(k)|}{\sqrt{2(2\pi)^3} \omega(k)} \right) \|\Psi\|,
$$

and hence, $\int_{\mathbb{R}^3} \|T(r,k)\Psi\|^2 dk < \infty$. Therefore, (H.4) follows. It is seen that for any ground state $\Psi_g$ of $H$,

$$
\|(H - E_0(H) + \omega(k))^{-1} T(r,k) \Psi_g\| \leq \left( \|\chi_1\|_{L^1} \sum_{j,l,l'} |\alpha_{l,j}^r| |M_l^r M_{l'}^l| \frac{|X_{ph}(k)e_j^r(k)|}{\sqrt{2(2\pi)^3} \omega(k)} \right) \|\Psi_g\|.
$$

Hence (H.6) follows. ■
3 Asymptotic fields

3.1 Existence of asymptotic fields

Let

\[ a_{r,t}^\xi(\xi) = e^{itH} e^{-itH_0} (I \otimes a_r^\xi(\xi)) e^{itH_0} e^{-itH}, \]

and

\[ b_{r,t}^\xi(\eta) = e^{itH} e^{-itH_0} (b_r^\xi(\eta) \otimes I) e^{itH_0} e^{-itH}, \]

\[ d_{s,t}^\xi(\zeta) = e^{itH} e^{-itH_0} (d_s^\xi(\zeta) \otimes I) e^{itH_0} e^{-itH}, \]

where \( X^s = X \) or \( X^* \).

It is proven, in a manner similar to that used in the proof of Lemma 2.3, that there exist \( a_0 > 0, b_0 > 0 \) such that

\[ ||H_0 \Psi|| \leq a_0 ||H \Psi|| + b_0 ||\Psi||. \]  \( (66) \)

(Proof of Theorem 1.4)

Let \( \xi \in C^\infty_0(\mathbb{R}^3 \setminus O_{ph}) \). We see that \( e^{-itH_0} I \otimes a_r(\xi) e^{itH_0} = I \otimes a_r(e^{-it\omega} \xi) \). Let \( \Phi(t) = e^{-itH} \Phi \) and \( \Psi(t) = e^{-itH} \Psi \) for \( \Phi, \Psi \in \mathcal{D}(H) \). By the strong differentiability of \( e^{itH} \Psi \) and \( e^{itH_0} \Psi \) with respect to \( t \),

\[ (\Phi, a_{r,T}(\xi) \Psi - (\Phi, a_{r,T_0}(\xi) \Psi) \]

\[ = \int_{T_0}^{T} \left\{ -\kappa_1 \sum_j \int_{\mathbb{R}^3} \chi_j(x) (e^{-it\omega} \xi, f_j \Psi)_{L^2} (\Phi(t), (\Psi(x) \alpha j \Psi(x) \otimes I) \Psi(t)) dx \right\} dt. \]

By Lemma 2.8 and (A.6),

\[ \left| \kappa_1 \sum_j \int_{\mathbb{R}^3} \chi_j(x) (e^{-it\omega} \xi, f_j \Psi)_{L^2} (\Phi(t), (\Psi(x) \alpha j \Psi(x) \otimes I) \Psi(t)) dx \right| \]

\[ \leq |\kappa_1| ||\Phi|| ||\Psi|| \sum_{j, l, l'} |\alpha_{j,l'}| |M_{j,l}^{1, \mathbb{R}^3}| \left( c_{j,l} ||\chi_j||_{L^1} + c_{j,l'} \sum_{j', l'} \int_{\mathbb{R}^3} |x| |\chi_j(x)| dx \right) \frac{1}{t(1 + t)}. \]  \( (67) \)

Hence \( ||a_{r,T}(\xi) \Psi - a_{r,T_0}(\xi) \Psi|| \leq \text{const.} f_{T_0}^{T} \frac{1}{t(1 + t)} dt \to 0 \), as \( T, T' \to \infty \). Thus, for \( \xi \in C^\infty_0(\mathbb{R}^3 \setminus O_{ph}) \),

\[ a_{r,\infty}(\xi) \Psi = \text{const.} f_{T_0}^{T} \frac{1}{t(1 + t)} dt \to 0 \), as \( n \to 0 \). Then for \( t' < t \),

\[ ||a_{r,(f)}(\xi) \Psi - a_{r,T}(f)\Psi|| \leq ||a_r(e^{-it\omega}(f - f_n))e^{-itH} \Psi|| + ||a_{r,T}(f) \Psi - a_{r,T}(f_n) \Psi|| \]

\[ + ||a_r(e^{-it\omega}(f - f_n))e^{-itH} \Psi||. \]  \( (68) \)

By (15), (40), and (66),

\[ ||a_r(e^{-it\omega}(f - f_n))e^{-itH} \Psi|| \leq \frac{1}{f_{T_0}^{T}} ||(\epsilon a_0 ||H \Psi|| + (\epsilon b_0 + c_\epsilon) ||\Psi||) + ||f - f_n ||||\Psi|| \to 0, \]  \( (69) \)

as \( n \to \infty \). Hence by (68), \( ||a_{r,T}(f) \Psi - a_{r,T'}(f) \Psi|| \to 0 \), as \( t, t' \to \infty \). □
Lemma 3.1 Let $\eta, \zeta \in L^2(\mathbb{R}^3)$. Then

\begin{align*}
[\psi^*_t(x) \psi_t(x), b_\tau(\eta)] &= -(\eta, g^l_{s,x}) \psi_t(x), \\
[\psi^*_t(x) \psi_t(x), d_\tau(\zeta)] &= (\zeta, h^l_{s,x}) \psi^*_t(x).
\end{align*}

(Proof of Lemma 3.1)

We see that by the anti-commutation relations

\begin{align*}
[\psi^*(x) \alpha^j \psi(x), b_\tau(\eta)] &= -\sum_{l,l'} \alpha^j_{l,l'}(\eta, g^l_{s,x}) \psi_l(x),
[\psi^*(x) \alpha^j \psi(x), d_\tau(\zeta)] &= \sum_{l,l'} \alpha^j_{l,l'}(\zeta, h^l_{s,x}) \psi^*_l(x),
[\rho(x) \rho(y), b_\tau(\eta)] &= -\sum_l \left((\eta, g^l_{s,x}) \rho(x) \psi_l(y) + (\eta, g^l_{s,x}) \psi_l(x) \rho(y)\right),
[\rho(x) \rho(y), d_\tau(\zeta)] &= \sum_l \left((\zeta, h^l_{s,x}) \rho(x) \psi^*_l(y) + (\zeta, h^l_{s,x}) \psi^*_l(x) \rho(y)\right).
\end{align*}

By Lemma 3.1, it follows that

\begin{align*}
[\psi^*(x) \alpha^j \psi(x), b_\tau(\eta)] &= -\sum_{l,l'} \alpha^j_{l,l'}(\eta, g^l_{s,x}) \psi_l(x), \tag{72}
[\psi^*(x) \alpha^j \psi(x), d_\tau(\zeta)] &= \sum_{l,l'} \alpha^j_{l,l'}(\zeta, h^l_{s,x}) \psi^*_l(x), \tag{73}
[\rho(x) \rho(y), b_\tau(\eta)] &= -\sum_l \left((\eta, g^l_{s,x}) \rho(x) \psi_l(y) + (\eta, g^l_{s,x}) \psi_l(x) \rho(y)\right), \tag{74}
[\rho(x) \rho(y), d_\tau(\zeta)] &= \sum_l \left((\zeta, h^l_{s,x}) \rho(x) \psi^*_l(y) + (\zeta, h^l_{s,x}) \psi^*_l(x) \rho(y)\right). \tag{75}
\end{align*}

It is known by (24; Theorem XI.15) that for $\eta \in C_0^0(\mathbb{R}^3)$, there exist constants $v_l(s) > 0$ and $\tilde{v}_l(s) > 0$ such that

\begin{align*}
\sup_{x \in \mathbb{R}^3} |(e^{itEM} \eta, g^l_{s,x})|_{L^2} \leq \frac{v_l(s)}{(1+t)^{3/2}},
\sup_{x \in \mathbb{R}^3} |(e^{itEM} \eta, h^l_{s,x})|_{L^2} \leq \frac{\tilde{v}_l(s)}{(1+t)^{3/2}}.
\end{align*}

We also see from (40) and (17), that for $\epsilon > 0$,

\begin{align*}
\|I \otimes A_j(x) \Psi\| \leq L_j(\epsilon)\|H \Psi\| + R_j(\epsilon)\|\Psi\|,
\end{align*}

(77)
where \( L_1^j(\epsilon) = 2\epsilon a_0 \sum_r M_{2,j,r}^{ph}, \) \( R_1^j(\epsilon) = \sum_r (2M_{2,j,r}^{ph}(\epsilon b_0 + c_\epsilon) + M_{1,j,r}^{ph}). \)

**Proof of Theorem 1.5**

Let \( \eta \in C_0^\infty(\mathbb{R}^3 \setminus O_{cl}). \) It is seen that \( e^{-itH_0} (b_s(\eta) \otimes I) e^{itH_0} = b_s(e^{-itH_\eta} \otimes I). \) Let \( \Phi(t) = e^{-itH} \Phi \) and \( \Psi(t) = e^{-itH} \Psi, \) for \( \Phi, \Psi \in \mathcal{D}(H), \) respectively. As in the case of the photon fields,

\[
(\Phi, b_s,T(\eta)\Psi) - (\Phi, b_s,T_0(\eta)\Psi) = \int_{T_0}^T \{ \kappa_1 [H'_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] + \kappa_2 [H''_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] \} dt. \tag{78}
\]

By (72),

\[
[H'_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] = - \sum_{j,l,p} \alpha_{j,l,p}^2 \int_{\mathbb{R}^3} (e^{-itH_\eta} \Phi_t)^j \eta_
^j(x) \Phi_t^l \eta_
^l(x) \Psi_t^p \eta_
^p(x) d^3x. \tag{79}
\]

We also see that by (77),

\[
| (\Phi(t), \psi_I(x) \otimes A_j(x)\Psi(t)) | \leq M_I^2 \| \Phi \| (L_1^j(\epsilon) \| H\Psi \| + R_1^j(\epsilon) \| \Psi \|),
\]

and hence from (76)

\[
| [H'_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] | \leq |\kappa_2| \| \Phi \| \sum_{j,l,p} \| \alpha_{j,l,p} \| \nu M_I^2 (L_1^j(\epsilon) \| H\Psi \| + R_1^j(\epsilon) \| \Psi \|) \cdot \frac{1}{(1+t)^{3/2}}. \tag{80}
\]

In addition, we see that by (74),

\[
[H''_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] = -i \sum_{l} \int_{\mathbb{R}^3} \frac{\chi_{l}(x) \chi_{l}(y)}{|x-y|} \left\{ (e^{-itH_\eta} \Phi_t \otimes I) (\rho \phi(y) \otimes I \Phi^\prime(t)) + (e^{-itH_\eta} \Phi_t \otimes I) (\rho \psi(y) \otimes \Phi^\prime(t)) \right\} d^3x d^3y.
\]

By (77),

\[
| (\Phi(t), \rho(x) \psi_I(y) \otimes I\Phi^\prime(t)) | \leq M_I^2 \sum_{l} (M_I^M)^2 \| \Phi \| \| \Psi \| \tag{81}
\]

follows. Then by (76), we have

\[
| [H''_1, b_s(e^{-itH_\eta} \otimes I)\Phi(t),\Psi(t)] | \leq 2M_I \left( \sum_{l} (M_I^M)^2 \| \Phi \| \| \Psi \| \right) \cdot \frac{1}{(1+t)^{3/2}}. \tag{82}
\]

Thus, (80) and (82) yield

\[
\| b_{s,T}(\eta)\Psi - b_{s,T'}(\eta)\Psi \| \leq \text{const.} \int_{T'}^T \frac{1}{(1+t)^{3/2}} dt \to 0,
\]

as \( T, T' \to \infty. \) Hence, we obtain the asymptotic fields \( b_{s,\infty}(\eta) := s - \lim_{r \to \infty} b_{s,T}(\eta) \Psi \) for \( \eta \in C_0^\infty(\mathbb{R}^3). \)

Since \( C_0^\infty(\mathbb{R}^3) \) is dense in \( L^2(\mathbb{R}^3) \) and \( \| b_{s,T}(\eta) \| \leq \| \eta \|, \) we can extend the asymptotic fields \( b_{s,\infty}(\eta) \) for \( \eta \in L^2(\mathbb{R}^3). \) The proof is thus completed. \( \blacksquare \)
3.2 Basic Properties of the Asymptotic Fields

Lemma 3.2 Assume (A.1)-(A.3), (A.5), (A.7) and (A.8).
(1) Let $\eta, \zeta \in L^2(\mathbb{R}^3)$. Then for $\Phi, \Psi \in \mathcal{D}(H)$,
\[
(\Phi, b_{s,\pm \infty}(\eta)\Psi) = (b_{s,\pm \infty}^*(\eta)\Phi, \Psi), \quad (\Phi, d_{s,\pm \infty}(\zeta)\Psi) = (d_{s,\pm \infty}^*(\zeta)\Phi, \Psi).
\]

(2) Let $\zeta \in \mathcal{D}(\omega^{-1/2})$. Then for $\Phi, \Psi \in \mathcal{D}(H)$,
\[
(\Phi, a_{r,\pm \infty}(\zeta)\Psi) = (a_{r,\pm \infty}^*(\zeta)\Phi, \Psi).
\]

(Proof) It is seen that
\[
(\Phi, b_{s,\pm \infty}(\eta)\Psi) \lim_{t \to \pm \infty} (\Phi, b_{s,t}(\eta)\Psi) = (b_{s,\pm \infty}^*(\eta)\Phi, \Psi).
\]

Hence we have (1). Similarly, we can prove (2). ■

Lemma 3.3 Assume that (A.1)-(A.3), (A.6), (A.7). Let $\eta, \eta', \zeta, \zeta' \in L^2(\mathbb{R}^3)$. It follows that,
\[
\{b_{s,\pm \infty}(\eta), b_{s',\pm \infty}^*(\eta')\} = \delta_{s,s'}(\eta, \eta'),
\]
\[
\{d_{s,\pm \infty}(\zeta), d_{s',\pm \infty}^*(\zeta')\} = \delta_{s,s'}(\zeta, \zeta'),
\]
\[
\{b_{s,\pm \infty}(\eta), b_{s',\pm \infty}^*(\eta')\} = \{d_{s,\pm \infty}(\zeta), d_{s',\pm \infty}^*(\zeta')\} = 0,
\]
\[
\{b_{s,\pm \infty}(\eta), d_{s',\pm \infty}^*(\zeta')\} = \{b_{s,\pm \infty}^*(\eta), d_{s',\pm \infty}(\zeta')\} = 0.
\]

(Proof) It is seen that for $\Phi, \Psi \in \mathcal{F}_{\text{QED}}$
\[
(\Phi, \{b_{s,t}(\eta), b_{s',t}(\eta')\}\Psi) = (e^{-i\hat{H}\Phi}, I \otimes \{b_{s,t}(e^{-i\hat{E}_\text{M}}\eta), b_{s',t}(e^{-i\hat{E}_\text{M}}\eta')\}e^{-i\hat{H}\Psi})
= \delta_{s,s'}(\eta, \eta')(\Phi, \Psi).
\]

Hence we obtain $\{b_{s,\pm \infty}(\eta), b_{s',\pm \infty}^*(\eta')\} = \delta_{s,s'}(\eta, \eta')$. Similarly, it can be proven in other cases. ■

Lemma 3.4 Assume (A.1)-(A.3), (A.5) and (A.7). Let $\xi, \xi' \in \mathcal{D}(\omega^{1/2})$, $k = -1, 1, 2$. Then, on $\mathcal{D}(H)$,

1. $[a_{r,\pm \infty}(\xi), a_{r',\pm \infty}^*(\xi')] = \delta_{r,r'}(\xi, \xi')$,

2. $[a_{r,\pm \infty}(\xi), a_{r',\pm \infty}(\xi')] = [a_{r,\pm \infty}^*(\xi), a_{r',\pm \infty}^*(\xi')] = 0$.

(Proof) It is similar to Lemma 3.3. ■

Lemma 3.5 Assume (A.1)-(A.3), (A.5), (A.7) and (A.8).

1. Let $\eta, \zeta \in L^2(\mathbb{R}^3)$. Then, for $\Psi \in \mathcal{D}(H)$,
\[
e^{i\hat{H}}b_{s,\pm \infty}^*(\eta)\Psi = b_{s,\pm \infty}^*(e^{i\hat{E}_\text{M}}\eta)e^{i\hat{H}}\Psi, \quad e^{i\hat{H}}d_{s,\pm \infty}(\zeta)\Psi = d_{s,\pm \infty}(e^{i\hat{E}_\text{M}}\zeta)e^{i\hat{H}}\Psi.
\]

2. Let $\zeta \in \mathcal{D}(\omega^{-1/2})$. Then, for $\Psi \in \mathcal{D}(H)$,
\[
e^{i\hat{H}}a_{r,\pm \infty}^*(\zeta)\Psi = a_{r,\pm \infty}^*(e^{i\hat{\omega}}\zeta)e^{i\hat{H}}\Psi.
\]
(proof)
We see that
\[ e^{itH} b_s,\xi' (\xi) \Psi = e^{i(t + t')} H e^{-i(t + t')} H_0 b_s (e^{itE_M} \eta) \otimes I e^{i(t + t')} H_0 e^{-i(t + t')} H e^{itH} \Psi. \]
By taking \( t' \to \pm \infty \), we obtain (1). We can prove (2) similarly to (1). \( \blacksquare \)

Since \( a_r^* (\xi) \) maps \( \mathcal{D}(H^3_{ph}) \) to \( \mathcal{D}(H_0) \), it can be proven that \( a_r^* (\xi) \) maps \( \mathcal{D}(|H|^{3/2}) \) to \( \mathcal{D}(H) \) in the similar way as ([13]; Lemma 4.10, Lemma 4.11). Then by the strong differentiability of \( e^{itH} \Psi \) and Lemma 3.5 we obtain the following lemma.

**Lemma 3.6**
Assume (A.1)-(A.3), (A.5), (A.7) and (A.8).

(1) Let \( \eta, \xi \in \mathcal{D}(E_M) \). It follows that on \( \mathcal{D}(H) \),
\[
[H, b_{s,\pm \infty} (\eta)] = -b_{s,\pm \infty} (E_M \eta), \quad [H, b_{s,\pm \infty}^* (\eta)] = b_{s,\pm \infty}^* (E_M \eta),
\]
\[
[H, d_{s,\pm \infty} (\xi)] = -d_{s,\pm \infty} (E_M \xi), \quad [H, d_{s,\pm \infty}^* (\xi)] = d_{s,\pm \infty}^* (E_M \xi).
\]

(2) Let \( \xi \in \mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega) \). It follows that on \( \mathcal{D}(|H|^{3/2}) \),
\[
[H, a_{r,\pm \infty} (\xi)] = -a_{r,\pm \infty} (\omega \xi), \quad [H, a_{r,\pm \infty}^* (\xi)] = a_{r,\pm \infty}^* (\omega \xi).
\]

**Lemma 3.7**
Assume (A.1)-(A.3), (A.6)-(A.8). Let \( \Psi_E \) be an eigenvector of \( H \) with the eigenvalue \( E \). Then

(1) for \( \eta, \xi \in L^2(\mathbb{R}^3) \),
\[
b_{s,\pm \infty} (\eta) \Psi_E = 0, \quad d_{s,\pm \infty} (\xi) \Psi_E = 0,
\]
(2) for \( \xi \in \mathcal{D}(\omega^{-1/2}) \),
\[
a_{r,\pm \infty} (\xi) \Psi_E = 0.
\]

**(Proof)**
Let \( \eta \in C_0^\infty (\mathbb{R}^3 \setminus O_{cl}) \). We see that
\[
\| b_{s,\xi} (\eta) \Psi_E \| = \| e^{itH} b_s (e^{-itE_M} \eta) \otimes I e^{-itH} \Psi_E \| = \| b_s (e^{-itE_M} \eta) \otimes I \Psi_E \|.
\]
Let \( \Phi = b_{s_1} (\eta_1) \cdots b_{s_n} (\eta_n) d_{\xi_1}^* (\xi_1) \cdots d_{\xi_n}^* (\xi_n) \Omega_{cl} \otimes \Phi_{ph} \in \mathcal{T}_{el} (C_0^\infty (\mathbb{R}^3 \setminus O_{cl})) \otimes \mathcal{T}_{ph}^{\text{fin}} (\mathcal{D}(\omega)) \). By the anti-canonical commutation relation,
\[
\| b_s (e^{-itE_M} \eta) \otimes I \Phi \|^2 \leq \sum_{j=1}^n \| (e^{-itE_M} \eta_j) L^2 \| \| b_{s_1} (\eta_1) \cdots b_{s_j} (\eta_j) \cdots b_{s_n} (\eta_n) d_{\xi_1}^* (\xi_1) \cdots d_{\xi_j}^* (\xi_j) \cdots d_{\xi_n}^* (\xi_n) \Omega_{cl} \otimes \Phi_{ph} \|.
\]
By ([24]; Theorem XI.19) there exists a constant \( F_1 \) such that \( \| (e^{-itE_M} \xi, \xi) \| L^2 \| \leq \frac{F_1}{(1 + t)^{1/2}} \). Hence, we have\( \lim_{t \to \infty} \| I \otimes a_r (e^{-it\omega} \xi) \Phi \| = 0 \). Since \( \mathcal{F}_{el} (C_0^\infty (\mathbb{R}^3 \setminus O_{cl})) \otimes \mathcal{T}_{ph}^{\text{fin}} (\mathcal{D}(\omega)) \) is a core of \( H \), we obtain \( \| b_{s,\xi} (\eta) \Psi_E \| = 0 \) for \( \eta \in C_0^\infty (\mathbb{R}^3 \setminus O_{cl}) \). Since \( C_0^\infty (\mathbb{R}^3 \setminus O_{cl}) \) is dense in \( L^2(\mathbb{R}^3) \) and \( E_{M-1/2} \) is a bounded operator, we can extend for all \( \eta \in L^2(\mathbb{R}^3) \). Thus, we can complete the proof of (1). (2) is proven similarly to (1). \( \blacksquare \)
Let $\Psi_g$ be a ground state of $H$. We next consider the asymptotic in/out-going Fock space. Let
\[
\mathcal{G}^n_{\pm \infty} = \mathcal{L} \left\{ \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \right\}.
\]
where $\mathcal{D}$ denotes the closure of $\mathcal{D}$. We set $\mathcal{G}^{\pm,0,0} = \{ z\Psi_g \mid z \in \mathbb{C} \}$. Let us define the asymptotic in/out-going Fock space by $\mathcal{G}_{\pm \infty} = \oplus_{n,l,m} \mathcal{G}^{n,l,m}_{\pm \infty}$. Let
\[
\mathcal{G}^{n,l,m} = \mathcal{L} \left\{ \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \right\}.
\]
We define the wave operator by $W^{n,l,m}_{\pm \infty} : \mathcal{G}^{n,l,m} \rightarrow \mathcal{G}^{n,l,m}_{\pm \infty}$ by
\[
W^{n,l,m}_{\pm \infty} = \mathcal{L} \left\{ \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \right\}.
\]
By the commutation relations given by Lemma 3.3 and Lemma 3.4, $W^{n,l,m}_{\pm \infty}$ can be extended to the unitary operator from $\mathcal{G}^{n,l,m}$ onto $\mathcal{G}^{n,l,m}_{\pm \infty}$. Let $W_{\pm \infty} = \oplus_{n,l,m} W^{n,l,m}_{\pm \infty}$.

(Proof of Theorem 1.6)
Let $\xi_i \in \mathcal{D}(\omega^{-1/2})$, $i = 1, \ldots, n$, $\eta_j \in L^2(\mathbb{R}^3)$, $j = 1, \ldots, l$, and $\zeta_k \in L^2(\mathbb{R}^3)$, $k = 1, \ldots, m$. By Lemma 3.5,
\[
e^{\hat{H}t} a^*_\xi \cdots a^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot a^*_\xi \cdots a^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \Psi_g = e^{\hat{H}t} \sum_{n,l,m} \left( e^{\hat{H}t} a^*_\xi \cdots a^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \right) \Psi_g.
\]
Then, $e^{\hat{H}t}$ leaves $\mathcal{G}_{\pm \infty}$ invariant, and hence $H$ is reduced by $\mathcal{G}_{\pm \infty}$. Then,
\[
W^{n,l,m}_{\pm \infty} = e^{\hat{H}t - \hat{E}_{\eta_j}t} a^*_\xi \cdots a^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \Psi_g = e^{\hat{H}t - \hat{E}_{\eta_j}t} \sum_{n,l,m} \left( e^{\hat{H}t - \hat{E}_{\eta_j}t} a^*_\xi \cdots a^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \cdot \alpha^*_\xi \cdots \alpha^*_\eta \cdot \beta^*_\eta \cdot \beta^*_\xi \right) \Psi_g.
\]
Thus, we obtain $W^{n,l,m}_{\pm \infty} = e^{\hat{H}t} W^{n,l,m}_{\pm \infty}$, on $\mathcal{G}_{\pm \infty}$. Then we have $H_0 + E_0(H) = W^{n,l,m}_{\pm \infty} H_{\pm \infty} W^{n,l,m}_{\pm \infty}$. Thus, we obtain $\sigma(H_0 + E_0(H)) \subset \sigma(H)$, and hence $[E_0(H), \infty) \subset \sigma(H)$. On the other hand, it is trivial to see $\sigma(H) \subset [E_0(H), \infty)$. Hence, the proof is completed. 

4 Total Charge of Ground States
It is seen that for $\eta, \zeta \in L^2(\mathbb{R}^3)$,
\[
\begin{align*}
[N_+, b_\eta(\eta)] &= -b_\eta(\eta), & [N_+, b^*_\eta(\eta)] &= -b^*_\eta(\eta), \\
[N_-, b_\zeta(\zeta)] &= -b_\zeta(\zeta), & [N_-, b^*_\zeta(\zeta)] &= -b^*_\zeta(\zeta),
\end{align*}
\]
on $\mathcal{G}_{cl}^{n,l,m}(L^2(\mathbb{R}^3; \mathbb{C}^4))$. 

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Lemma 4.1  It follows that on $\mathcal{F}_f^0(L^2(R^3;C^4))$

\[
[Q, \psi_I(x)^*\psi_I(x)] = 0, \quad [Q, \psi_I(x)^*\alpha^I\psi_I(x)] = 0, \quad \text{(85)}
\]

for each $x \in R^3$.

(Proof)

By (83) and (84), it is seen that

\[
[N_+, b_s^*(g_{s,x}^l) d_r^*(g_{r,x}^s)] = b_s^*(g_{s,x}^l) d_r^*(g_{r,x}^s), \quad [N_+, d_s(h_{s,x}^l) b_r(g_{r,x}^s)] = -d_s(h_{s,x}^l) b_r(g_{r,x}^s),
\]

and

\[
[N_+, b_s^*(g_{s,x}^l) b_r(g_{r,x}^s)] = [N_+, d_s(h_{s,x}^l) d_r(h_{r,x}^s)] = 0.
\]

We also see that

\[
[N_-, b_s^*(g_{s,x}^l) d_r^*(g_{r,x}^s)] = b_s^*(g_{s,x}^l) d_r^*(g_{r,x}^s), \quad [N_-, d_s(h_{s,x}^l) b_r(g_{r,x}^s)] = -d_s(h_{s,x}^l) b_r(g_{r,x}^s),
\]

and

\[
[N_-, b_s^*(g_{s,x}^l) b_r(g_{r,x}^s)] = [N_-, d_s(h_{s,x}^l) d_r(h_{r,x}^s)] = 0.
\]

Hence, $[N_+, \psi_I(x)^*\psi_I(x)] = [N_-, \psi_I(x)^*\psi_I(x)] = \sum_{\alpha} b_s^*(g_{s,x}^l) d_r^*(g_{r,x}^s) - d_s(h_{s,x}^l) b_r(g_{r,x}^s)$ follows.

Lemma 4.2  Assume (A.1)-(A.3). Then $e^{-uQ}$ leaves $\mathcal{D}(H)$ onto itself and

\[
e^{\hat{Q} \otimes I} He^{-\hat{Q} \otimes I} = H, \quad \text{(86)}
\]

on $\mathcal{D}(H)$.

(Proof)

Let $\Psi \in \mathcal{D}_0$ with $\Psi = a_{r_1}^* (\xi_1) \cdots a_{r_n}^* (\xi_n) b_{s_1}^* (\eta_1) \cdots b_{s_l}^* (\eta_l) d_{r_1}^* (\zeta_1) \cdots b_{s_m}^* (\zeta_m) \Omega_{\text{ci}} \otimes \Omega_{\text{ph}}$. It is trivial to see $[Q \otimes I, H_0] = 0$. For $\Phi \in \mathcal{D}(H)$, it is seen from Lemma[4.1]that

\[
(\Phi, [Q \otimes I, H_0]) \Psi = \int_{R^3} \chi_I(x) (\Phi, [Q, \psi_I(x)^*\alpha^I\psi_I(x)] \otimes A_I(x)) \Psi dx = 0,
\]

\[
(\Phi, [Q \otimes I, H_0]) \Psi = \int_{R^3} \chi_I(x) (\Phi, [Q, \psi_I(x)\psi_I(y)\psi_I(y)] \otimes I) \Psi dx dy = 0.
\]

Hence, we obtain $[Q \otimes I, H] \Psi = 0$. Since $\Psi$ is an analytic vector of $Q$, we get

\[
e^{\hat{Q} \otimes I} He^{-\hat{Q} \otimes I} \Psi = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} ad_Q^n H \Psi = H \Psi,
\]

where $ad_Q^0 H := H$, $ad_Q^n := [Q, ad_Q^{n-1} H]$, $n \geq 1$. Since $\mathcal{D}_0$ is the core of $H$, we obtain $e^{\hat{Q} \otimes I} He^{-\hat{Q} \otimes I} \Psi = H \Psi$ for $\mathcal{D}(H)$. 



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(Proof of Theorem 1.7)
From (72) and (73) it is seen that for $\Phi, \Psi \in \mathcal{D}(H)$,

$$[H'_I, b_s(\eta)]^0(\Phi, \Psi) = \int_{\mathbb{R}^3} \overline{\eta(p)}(\Phi, F_+(s, p))\Psi dp,$$

$$[H'_I, d_s(\zeta)]^0(\Phi, \Psi) = \int_{\mathbb{R}^3} \overline{\zeta(p)}(\Phi, F_-(s, p))\Psi dp,$$

where

$$F_+(s, p)\Psi = -\sum_{j,l} a^l_{s,j} \int_{\mathbb{R}^3} \chi_1(x) g^l_{s,x}(p) \psi_i(x) \otimes A_j(x) \Psi dx,$$

$$F_-(s, p)\Psi = \sum_{j,l} a^l_{s,j} \int_{\mathbb{R}^3} \chi_1(x) h^l_{s,x}(p) \psi_i(x) \otimes A_j(x) \Psi dx,$$

and

$$[H''_I, b_s(\eta)]^0(\Phi, \Psi) = \int_{\mathbb{R}^3} \overline{\eta(p)}(\Phi, G_+(s, p))\Psi \mathcal{D}_{\text{QED}} dp,$$

$$[H''_I, d_s(\zeta)]^0(\Phi, \Psi) = \int_{\mathbb{R}^3} \overline{\zeta(p)}(\Phi, G_-(s, p))\Psi \mathcal{D}_{\text{QED}} dp,$$

where

$$G_+(s, p)\Psi = -\sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\chi_{2}(x)\chi_{2}(y)}{|x-y|} \left( g^l_{s,y}(p) \rho(x) \psi_i(y) + g^l_{s,x}(p) \psi_i(x) \rho(y) \right) \otimes I \, dx \, dy,$$

$$G_-(s, p)\Psi = \sum_{j,l} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\chi_{2}(x)\chi_{2}(y)}{|x-y|} \left( h^l_{s,y}(p) \rho(x) \psi_i(y) + h^l_{s,x}(p) \psi_i(x) \rho(y) \right) \otimes I \, dx \, dy.$$

Let $\Psi_g$ be the ground state of $H$. In a manner similar to [19], we can obtain the pull-through formula

$$||(N_{1/2} \otimes I)\Psi_g||^2 = \sum_{s = \pm 1/2} \int_{\mathbb{R}^3} \|(H - E_0 + E_M(p))^{-1} (\kappa_1 F_{\pm}(s, p) + \kappa_1 G_{\pm}(s, p)) \Psi_g||^2 dp.$$  (95)

It is seen that from (A.3) and (77)

$$||F_+(s, p)\Psi_g|| \leq \sum_{j,l} |a^l_{s,j}| |g^l_{s,0}(p)|||\chi_{2}||_{L^3} M^j L^j \epsilon E_0(H) + R^j \epsilon ||\Psi_g||,$$  (96)

$$||G_+(s, p)\Psi_g|| \leq 2 \sum_{v,l} |g_{s,0}(p)||M^l \epsilon^2 ||\Psi_g||.$$  (97)

Then, there exist constants $\mu_+(s) > 0$ and $\nu_+(s) > 0$ such that

$$||(H - E_0(H) + E_M(p))^{-1} (\kappa_1 F_+(s, p) + \kappa_1 G_+(s, p)) \Psi_g|| \leq (\kappa_1 \mu_+(s) + \kappa_1 \nu_+(s)) \frac{|u^l_0(p)|}{\sqrt{E_M(p)}} ||\Psi_g||.$$  (98)

Similarly, there exist constants $\mu_-(s) > 0$ and $\nu_-(s) > 0$ such that

$$||(H - E_0(H) + E_M(p))^{-1} (\kappa_1 F_-(s, p) + \kappa_1 G_-(s, p)) \Psi_g|| \leq (\kappa_1 \mu_-(s) + \kappa_1 \nu_-(s)) \frac{|v^l_0(p)|}{\sqrt{E_M(p)}} ||\Psi_g||.$$  (99)
Note that $\mathcal{D}(d\Gamma(E_M)) \subset \mathcal{D}(N_{\pm})$. By (95), (98) and (99), for sufficiently small $\kappa_I$ and $\kappa_{II}$, it follows that
\[
(\Psi_g, N_{+}\Psi_g) + (\Psi_g, N_{-}\Psi_g) < 1. \tag{100}
\]
Now let us consider $\Psi_g \in \mathcal{F}_n$, $n \neq 0$. Then it follows that
\[
(\Psi_g, N_{+}\Psi_g) - (\Psi_g, N_{-}\Psi_g) \geq 1 \quad \text{or} \quad (\Psi_g, N_{+}\Psi_g) - (\Psi_g, N_{-}\Psi_g) \leq -1. \tag{101}
\]
But this contradicts (100). Hence, $\Psi_g \in \mathcal{F}_0$ follows. ■

Appendix (Uniqueness of Ground States; [19])
Let $\mathcal{K}$ be a Hilbert space. We consider an abstract Hilbert space $\mathcal{H}$ as
\[
\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_b(L^2(\mathbb{R}^3; \mathbb{C}^2)).
\]
Let
\[
X_0 = K \otimes I + I \otimes H_{\text{ph}},
\]
and
\[
X(q) = X_0 + qX', \quad q \in \mathbb{R},
\]
The operator $K$ satisfies the following conditions:

(H.1) The operator $K$ is self-adjoint and bounded from below.

(H.2) $X'$ is a symmetric operator on $\mathcal{H}$, and there exist constants $a > 0$ and $b > 0$ such that
\[
\|X'\Psi\| \leq a\|X_0\Psi\| + b\|\Psi\|, \quad \Psi \in \mathcal{D}(H_0).
\]

(H.3) There exists an operator $S(r,k) : \mathcal{H} \rightarrow \mathcal{H}$, $k \in \mathbb{R}^3$, $r = 1, 2$, such that for $\Phi, \Psi \in \mathcal{D}(H_0)$,
\[
(I \otimes a_r^*(f)\Phi, X'\Psi) - (X'\Phi, I \otimes a_r(f)\Psi) = \int_{\mathbb{R}^3} \overline{f(k)}(\Phi, S(r,k)\Psi)d\mathbf{k}.
\]

Assume that $X(q) = X_0 + qX'$ has a ground state $\Psi_0(q) : X(q)\Psi_0(q) = E_0(X(q))\Psi_0(q)$.

(H.4) Let $\Phi \in \mathcal{D}(X_0)$. Then for $f \in C^\infty(\mathbb{R}^3)$, $S(r,k)$ in (H.3) satisfies
\[
\int_{\mathbb{R}^3} \overline{f(k)} \left( \Phi, e^{i\omega_0(q-\omega)dt}S(r,k)\Psi_0(q) \right) d\mathbf{k} \in L^1([0, \infty), dt),
\]
and $\int_{\mathbb{R}^3} \|S(r,k)\Psi_0(q)\|^2 d\mathbf{k} < \infty$. 

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Theorem A.1 ([19]; Theorem 2.9) Assume that (H.1) - (H.4). Let $\Psi_0(q)$ be an arbitrarily ground state. Then (a) and (b) are equivalent.

(a) $\Psi_0(q) \in \mathcal{D}(I \otimes N^{1/2}_b)$.

(b) $\int_{\mathbb{R}} \|(X(q) - E_0(X(q)) + \omega(k))^{-1} S(r,k)\Psi_0(q)\|^2 dk < \infty$.

In particular, if (a) or (b) holds,

$$\|(I \otimes N^{1/2}_b)\Psi(q)\|^2 = q^2 \sum_{r=1,2} \int_{\mathbb{R}^3} \|(X(q) - E_0(q) + \omega(k))^{-1} S(r,k)\Psi_0(q)\|^2 dk.$$

\textbf{(H.5) (Spectral gap of $K$)} $\inf \sigma_{ess}(K) - E_0(K) > 0$.

\textbf{(H.6)} Let $N_q = \ker(X(q) - E_0(X(q)))$. Then it follows that

$$\lim_{q \to 0} \sup_{\Psi(q) \in N_q \backslash \{0\}} q^2 \sum_{r=1,2} \int_{\mathbb{R}^3} \|(X(q) - E_0(q) + \omega(k))^{-1} S(r,k)\Psi_0(q)\|^2 dk / \|\Psi_0(q)\|^2 = 0.$$

Theorem A.2 ([19]; Theorem 4.2)
Assume that (H.1) -(H.6). Then there exists a constant $\tilde{q} > 0$ such that for $|q| < \tilde{q}$,

$$\dim \ker(X(q) - E_0(X(q))) \leq \dim \ker(K - E_0(K)).$$

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