The algorithm for planning the trajectory of the 3-RPR robot, taking into account the singularity zones based on the method of non-uniform covering

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Abstract. The article describes the method for the trajectory planning of a 3-RPR robot, using the numerical solution of the forward kinematics. An algorithm for determining the position of a mobile robot platform based on the method of non-uniform coverings has been synthesized. The zones of singularity arising at some positions of the robot are investigated. The conditions for the emergence of singularity zones corresponding to its special provisions are revealed. It is shown that the solution of the forward kinematics is not unambiguous i.e. with the same change in the length of the rods there are different positions of the platform. The issues of planning the trajectory for a mobile platform are described while taking into account the zones of singularity. To implement the algorithm, a software package in the C++ language has been developed. The results of mathematical simulation are presented.

1. Introduction
Recently, parallel robots are increasingly used in the industry. Classical control system with feedback can be used to control such robots. However, to implement a mismatch control, a dynamic definition of the control result is required i.e., coordinates and orientation in the space of the output link of the robot. In most of the cases, the implementation of the position sensor of the mobile platform is difficult. So, it is necessary to organize the calculation of the feedback signal depending on the movement of the drive links i.e., solving the forward kinematics. The complexity of the forward kinematics is explained by the need to solve a system of nonlinear equations. Research in this area has led to the development of various ways to solve this problem [1-3]. It should be noted that there is an ambiguity of solving the forward kinematics for some of the parallel robots. This means that different positions of the output link can correspond to one state of the drive links. Thus, to find the “correct” solution using the forward kinematics is even more difficult for such mechanisms.

The article proposes a numerical method for solving the forward kinematics of a parallel robot based on optimization methods. For the approximation of the system of nonlinear coupling equations describing the geometry of the robot, an optimization algorithm based on the concept of non-uniform covering, described in [4,5], was applied. This method has been successfully applied to the approximation of the working area of some types of parallel robots [6]. In the article, this method is used to solve the forward kinematics of the 3-RPR robot i.e., finding the coordinates of its output link.
2. Formulation of the problem

Consider a 3-RPR robot (figure 1), in which three rods of variable length are pivotally attached to a fixed base at the vertices of an equilateral triangle \( A_1, A_2 \) and \( A_3 \) [7]. The other ends of the rods are hinged at the vertices of the equilateral triangle \( B_1, B_2 \) and \( B_3 \) on the movable platform, which allows it to move in the plane of the base. The input coordinates of the robot are the rod lengths \( l_1, l_2 \) and \( l_3 \), the output coordinates are the coordinates of the geometric center of the moving platform \( x, y \) and the angle of rotation of the platform \( \varphi \) relative to an axis perpendicular to the plane of the base.

![Figure 1. Flat 3-RPR robot](image)

Changing rod lengths \( l_i \) has the form

\[
l_i^2 = (x + r \cos(\gamma_i + \varphi) - x_{Ai})^2 + (y + r \sin(\gamma_i + \varphi) - y_{Ai})^2,
\]

where \( \gamma_i \) is the angle determining the geometry of the output link, \( x_{Ai}, y_{Ai} \) are the coordinates of the point \( A_i \) and \( A_i \) – the hinge attaching the rod to the fixed base, \( r \) is the distance from \( B_i \) – the hinge attaching the rod to the moving platform to \( O \) – the geometric center of the platform, \( l_i \) - the length of the \( i^{th} \) rod. If the points \( A_i \) and \( B_i \) are located at the vertices of equilateral triangles then the change in rod lengths is determined by

\[
l_1^2 = (x + 0,5r(\sin \varphi - \sqrt{3} \cos \varphi) + 0,5\sqrt{3}R)^2 + (y - 0,5r(\sqrt{3} \sin \varphi + \cos \varphi) + 0,5R)^2,
\]

\[
l_2^2 = (x + 0,5r(\sin \varphi + \sqrt{3} \cos \varphi) - 0,5\sqrt{3}R)^2 + (y + 0,5r(\sqrt{3} \sin \varphi - \cos \varphi) + 0,5R)^2,
\]

\[
l_3^2 = (x - r \sin \varphi)^2 + (y + r \cos \varphi - R)^2,
\]

where \( R \) is where the radii of the circumcircle of triangle \( A_1A_2A_3 \).

It should be noted that the solution of the forward kinematics of the robot is not unique, i.e. different positions of the movable platform correspond to the same rod lengths. In this case, an unambiguous determination of the position of the platform only by changing the lengths of the rods becomes impossible and there is an additional problem of identifying the "correct" solution of the forward kinematics to determine the current state of the control object. To solve the forward
kinematics and determine all the possible positions of the platform for the given rod lengths, we write the equations of communication

\[
\begin{align*}
(x_{B1} - x_{A1})^2 + (y_{B1} - y_{A1})^2 - l_1^2 &= 0 \\
(x_{B2} - x_{A2})^2 + (y_{B2} - y_{A2})^2 - l_2^2 &= 0 \\
(x_{B3} - x_{A3})^2 + (y_{B3} - y_{A3})^2 - l_3^2 &= 0 \\
(x_{B1} - x_{B2})^2 + (y_{B1} - y_{B2})^2 - l_2^2 &= 0 \\
(x_{B3} - x_{B2})^2 + (y_{B3} - y_{B2})^2 - l_2^2 &= 0 \\
(x_{B3} - x_{B1})^2 + (y_{B3} - y_{B1})^2 - l_2^2 &= 0
\end{align*}
\]

(4)

Denote in the system of equations (4) the functions of changing the coordinates \(g_i, i \in 1, \ldots, 6\) in the form:

\[
g_1 = (x_{B1} - x_{A1})^2 + (y_{B1} - y_{A1})^2 - l_1^2, \ldots, g_6 = (x_{B3} - x_{B1})^2 + (y_{B3} - y_{B1})^2 - l_2^2.
\]

When these coordinates change, singularity zones appear that correspond to the special positions of the robot in which the loads on the links increase and uncontrollable mobilities appear. For a 3-RPR robot, singularities of only the second type are characteristic for which the Jacobian \(J_A\) is zero [8]

\[
\begin{align*}
\frac{\partial F_1}{\partial x} &= 2x - 2x_{A1} + 2r \cos(\varphi + \gamma_i), \\
\frac{\partial F_1}{\partial y} &= 2y - 2y_{A1} + 2r \sin(\varphi + \gamma_i), \\
\frac{\partial F_i}{\partial \varphi} &= 2r(y - y_{A1} + r \sin(\varphi + \gamma_i)) \cos(\varphi + \gamma_i) - 2r(x - x_{A1} + r \cos(\varphi + \gamma_i)) \sin(\varphi + \gamma_i),
\end{align*}
\]

(5)

where \(\frac{\partial F_i}{\partial x} = 2x - 2x_{A1} + 2r \cos(\varphi + \gamma_i), \frac{\partial F_i}{\partial y} = 2y - 2y_{A1} + 2r \sin(\varphi + \gamma_i), \frac{\partial F_i}{\partial \varphi} = 2r(y - y_{A1} + r \sin(\varphi + \gamma_i)) \cos(\varphi + \gamma_i) - 2r(x - x_{A1} + r \cos(\varphi + \gamma_i)) \sin(\varphi + \gamma_i), \)

\(F_i\) is a function that characterizes the changes in rod lengths.

Then equation (5) takes the form:

\[
\det(J_A) = 12\sqrt{3} R r \sin \varphi (R^2 - 2 R r \cos \varphi + r^2 - x^2 - y^2).
\]

(6)

A singularity zones in the event that the Jacobian (6) is zero, that is, \(\det(J_A) = 0\), which is possible if the following conditions are fulfilled:

\[
\sin \varphi = 0
\]

(7)

\[
x^2 + y^2 = R^2 - 2 R r \cos \varphi + r^2
\]

(8)

We study the behavior of the robot at different positions of the platform taking into account the singularity zones described by conditions (7) and (8). Let us set the theoretically possible maximum boundaries for changing the coordinates of points of the platform \(B_1, B_2\) and \(B_3\)

\[
x_{A1} - l_{imax} - l \leq x_{B1} \leq x_{A1} + l_{imax} + l \\
y_{A1} - l_{imax} - l \leq y_{B1} \leq y_{A1} + l_{imax} + l
\]

(9)

(10)

where \(l_{imax}\) is the maximum length of the \(i\)th rod.

We can estimate the lengths of the rods \(l_1, l_2, l_3\) from the equations (1) - (3) for some center of the platform \(O\), given by the coordinates \(\varphi, x, y\). We take the value of the angle \(\varphi\) in the range from \(-119^\circ\) to \(+119^\circ\). Let us define the coordinates of the \(O\) - the center of the platform \(x\) and \(y\), located both inside and outside the circle described by equation (8), ranging from 0 mm to 140 mm. Next, we
consider the forward kinematics i.e. determine the coordinates of the point \( B_i: x_{B_i}, y_{B_i} \) for the obtained values of \( l_1, l_2, l_3 \).

3. Algorithm for solving the forward kinematics

To solve this problem in connection with its ambiguity, we will use the numerical method of approximation of the system of nonlinear equations (4), based on the method of non-uniform covering [4, 5], with some given approximation accuracy \( \delta \). The detailed analysis of this method is as follows.

If for some set of variable values of each functions \( g_i \) the following conditions are fulfilled:

\[
\min g_i \leq 0 \\
\max g_i \geq 0,
\]

then, therefore, each function takes the value 0 and there are solutions of nonlinear algebraic equations (4) within a given set of variables. It is necessary to obtain sets that have a diameter less than a given approximation accuracy \( \delta \), for which conditions (11), (12) are satisfied. The sets of values of six variables \( x_{B_i}, y_{B_i} \) are presented in the form of six-dimensional parallelepipeds. To calculate the minima and maxima of the functions, interval estimates are determined, which coincide with the extrema of the function. For this, the developed Snowgoose library in C++ is used.

An algorithm for determining the coordinates \( x_{B_i}, y_{B_i} \) of points of the moving platform \( B_1, B_2 \) and \( B_3 \) has been synthesized (figure 2). Let us introduce three lists of six-dimensional parallelepipeds \( \mathbb{P}, \mathbb{P}_A \) and \( \mathbb{P}_E \), where the list \( \mathbb{P} \) includes a set of parallelepipeds containing intervals of variable values, the list \( \mathbb{P}_A \) includes a set of parallelepipeds containing intervals of values of variables less than \( \delta \) for which the conditions (11), (12) are satisfied, \( \mathbb{P}_E \) includes a set of parallelepipeds containing intervals of values of variables for which at least one of the conditions (11), (12) is not satisfied. Each of the axes of the six-dimensional space corresponds to the changing parameters in the system of equations - the coordinates \( x_{B_i}, y_{B_i} \). At the first step of the algorithm, the lists \( \mathbb{P}_A \) and \( \mathbb{P}_E \) are empty, and the list \( \mathbb{P} \) consists of only one parallelepiped \( \mathbb{Q} \), which includes the entire range of theoretically maximum boundaries for finding points of the platform points \( B_1, B_2 \) and \( B_3 \) (9) and (10).

The algorithm works as follows:

1. Input of geometrical parameters of mechanism and approximation accuracy \( \delta \).
2. The lists \( \mathbb{P}_A \) and \( \mathbb{P}_E \) are empty, and the list \( \mathbb{P} \) consists of only one parallelepiped \( \mathbb{Q} \), which includes the entire range of theoretically maximum boundaries for finding points of the platform points \( B_1, B_2 \) and \( B_3 \) (9) and (10).
3. Extract from the list \( \mathbb{P} \) the parallelepiped \( \mathbb{Q}_j, j \in 1, n \).
4. We define for it a minimum of \( \min g_i \) and a maximum of \( \max g_i < 0 \) of functions \( g_i, i \in 1, \ldots, 6 \).
5. If at least one of the functions \( g_i \) satisfies at least one of the conditions \( \min g_i > 0 \) or \( \max g_i < 0 \), then the box is excluded from further consideration, falling into the list \( \mathbb{P}_E \), that is, \( \mathbb{P}_E := \mathbb{P}_E \cup \mathbb{Q}_j \).
6. If the parallelepiped has a diameter less than or equal to the given parameter \( \delta \), that is, \( d(\mathbb{Q}_j) \leq \delta \) characterizing the accuracy of approximation, then it is added to the \( \mathbb{P}_A \) list, that is, \( \mathbb{P}_A := \mathbb{P}_A \cup \mathbb{Q}_j \).
7. In other cases, the parallelepiped is divided into two equal parallelepipeds \( \mathbb{Q}_{m+1} \) and \( \mathbb{Q}_{m+2} \) along the edge with the greatest length. These parallelepipeds are entered at the end of the list \( \mathbb{P} \), that is, \( \mathbb{P} := \mathbb{P} \cup \mathbb{Q}_j \).
8. If the list \( \mathbb{P} \) becomes empty, that is \( \mathbb{Q}_j = \emptyset \), then the algorithm terminates its work.
9. In other cases, steps 3-9 are repeated.

The finiteness of the number of steps of the algorithm follows from the restriction on the minimum diameter of the parallelepiped.
4. Simulation results
The following geometric parameters were used for modeling: \( R = 100 \) mm, \( r = 50 \) mm. Figure 3 presents the simulation results for condition (7). A singularity zone arises at the angle of rotation of the platform \( \varphi = 0^\circ \), at which the forward kinematics has one solution (figure 3, a). When changing rod lengths, two solutions arise, thus with one control signal besides the “correct” platform can move to another position (figure 3, b).

Figure 3. Simulation results a) subject to (7) the occurrence of special provisions \((\varphi = 0^\circ)\), b) without being in a singularity zone \((\varphi = 5^\circ)\).

Figure 4 presents the simulation results for the singularity zone that arises when the center of the platform is located on the circle described by condition (8). If the center of the platform is inside a
circle, then the forward kinematics has two solutions (figure 4, a). When changing the lengths of the rods, providing for the exit of the center of the platform from the circle, the forward kinematics has four solutions. Thus, one control signal can move the platform to its “correct” or “incorrect” position (figure 4, b).

![Figure 4](image1.png)

**Figure 4.** Simulation results for finding the center of the platform a) inside the circle described by condition (7), b) outside the circle described by condition (7).

Figure 5 presents the simulation results for condition (7) with the position of the center of the platform outside the circle described by condition (8). A singularity zone occurs when the angle of rotation of the platform $\varphi = 0^\circ$, at which the forward kinematics has three solutions (figure 5, a). When changing rod lengths, four solutions appear (figure 5, b).

![Figure 5](image2.png)

**Figure 5.** Simulation results at the location of the center of the mobile platform outside the circle (8): a) if there are singularity zones ($\varphi = 0^\circ$), b) without singularity zones ($\varphi = 5^\circ$).

Thus, for the performance of technological operations, the center of the mobile platform must be within the circumference described by condition (8) and the angle $\varphi$ must vary either within $-120^\circ < \varphi < 0^\circ$ or within $0^\circ < \varphi < 120^\circ$. When passing through an angle of $0^\circ$, duality of solutions arises in the problem of positions. The exit of the center of the platform beyond the circumference determined by the condition (8) also leads to ambiguity of decisions and the impossibility of controlling such a robot.
Let us determine the position of the singularity zones corresponding to the specific positions within the working area of the robot when performing a specific trajectory of the working platform. To determine the working area of the robot, we will also use the method of irregular covering, discussed earlier for a certain range of variation of rod lengths $l_{\text{max}} = 130 \text{ mm}$, $l_{\text{min}} = 10 \text{ mm}$. The simulation results are presented in figure 6. It is clear from them that at the angle of rotation of the platform $\varphi = 20^\circ$ (figure 6, a) the singularity zones described by condition (8) and representing a circle are inside the working area, for the angle of rotation $\varphi = 40^\circ$ (figure 6, b) a significant part of the circle is located outside the working area and is unattainable.

![Figure 6. Results of modeling the working area: a) for the angle of rotation of the platform $\varphi = 20^\circ$, b) for the angle of rotation of the platform $\varphi = 40^\circ$.](image)

It should be noted that when solving management tasks, the position of the mobile platform should be determined regularly, and therefore the “correct” state of the output link always lies in a certain neighborhood from its previous state. The size of the neighborhood is known and depends on the speed of movement and the frequency of determining the position of the moving platform.

If, moreover, the trajectory of its movement passes far from the so-called special provisions of the robot, characterized by loss of degrees of freedom or the appearance of uncontrolled mobility, in such a neighborhood there will be only one solution to the forward kinematics, which is “correct”, reflecting current position of the mobile platform. Accordingly, it is necessary to supplement the system of communication equations (4) with the following inequalities:

$$\Delta x_{B_i}^2 + \Delta y_{B_i}^2 \leq \delta_1, i \in 1,\ldots,3,$$

where $\Delta x_{B_i}$ and $\Delta y_{B_i}$ are the change in the position of the coordinate point $B_i$ along the $X$ and $Y$ axes, respectively, between the states of the robot, $\delta_1$ is the size of the neighborhood, depending on the movement speeds and the frequency of determining the position of the moving platform. In fig. 6 shows the simulation results for: $l_1 = 70 \text{ mm}$, $l_2 = 60 \text{ mm}$, $l_3 = 80 \text{ mm}$. As a result of additional testing for finding a platform in the vicinity of some given previous (13) position, the “correct” solution is selected from two solutions of the position problem (figure 7, a) (figure 7, b).
5. Conclusion
In conclusion, we note that the method used to approximate nonlinear equation systems based on the concept of non-uniform covering can be successfully applied to achieve various tasks of the mechanics of parallel robots, including the problem of the positions of the robot and the definition of the working area. When such robots work, it is important to take into account the zones of singularities corresponding to singularity zones, when passing through which the ambiguity of solving the position problem takes place, which must be taken into account when constructing control algorithms for such robots. The method of non-uniform covering is applied to solve this problem. The effectiveness is characterized by the following indicators: the computation time for the accuracy of approximation $δ = 0.001$, depending on the proximity to the singularity zones, was from 3 to 30 seconds. This is explained by the fact that as one approaches singularity zones, the number of parallelepipeds $Q_j$ describing the coordinate of the point $B_i$, increases, for example, 74 times for the singularity zone described by condition (7). This may be one of the indicators for assessing the closeness to singularities. This question requires further more detailed research. The obtained results can also be used in the selection of constructive and geometrical parameters of 3-RPR, as well as in solving planning problems and optimizing its trajectory taking into account the zones of singularity.

Acknowledgments

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Figure 7. Simulation results a) without taking into account the condition (8), b) taking into account the condition (8).