The Lax pair structure for the spin Benjamin–Ono equation

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À la mémoire de Jean Ginibre

Abstract

We prove that the recently introduced spin Benjamin–Ono equation admits a Lax pair and deduce a family of conservation laws that allow proving global wellposedness in all Sobolev spaces $H^k$ for every integer $k \geq 2$. We also infer an additional family of matrix-valued conservation laws of which the previous family is just the traces.

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1 Introduction

In a recent paper [1], Berntson, Langmann, and Lenells have introduced the following spin generalization of the Benjamin–Ono equation on the line $\mathbb{R}$ or on the torus $\mathbb{T}$,

$$\partial_t U + \{U, \partial_x U\} + H \partial_x^2 U - i[U, H \partial_x U] = 0, \quad x \in X,$$

where $X$ denotes $\mathbb{R}$ or $\mathbb{T}$, the unknown $U$ is valued into $d \times d$ matrices, and $H$ denotes the scalar Hilbert transform on $X$; in fact, the authors chose the normalization $H = i \text{sign}(D)$ so that $H \partial_x = -|D|$, where $|D|$ denotes the Fourier multiplier associated to the symbol $|k|$. Notice that in front of the commutator term on the right-hand side, we take a different sign from the one used in [1]. However, passing to the other sign by applying the complex conjugation is easy. Consequently, the above equation reads

$$\partial_t U = \partial_x (|D|U - U^2) - i[U, |D|U]. \quad (1)$$

The purpose of this note is to prove that equation (1) enjoys a Lax pair structure and to infer the first consequences on the corresponding dynamics.

2 The Lax pair structure

Let us first introduce some more notation. Given operators $A, B$, we denote

$$[A, B] := AB + BA, \quad |A, B| := AB - BA$$
and \( A^* \) denote the adjoint of \( A \). We consider the Hilbert space \( \mathcal{H} := L^2(X, \mathbb{C}^{d \times d}) \) made of \( L^2 \) functions on \( X \) with Fourier transforms supported in nonnegative modes, and valued into \( d \times d \) matrices, endowed with the inner product \( \langle A | B \rangle = \int_X \text{tr}(AB^*) \, dx \). We denote by \( \Pi_{\geq 0} \) the orthogonal projector from \( L^2(X, \mathbb{C}^{d \times d}) \) onto \( \mathcal{H} \). According to the study of the integrability of the scalar Benjamin–Ono equation [2], given \( U \in L^2(X, \mathbb{C}^{d \times d}) \) valued into \( \mathbb{C}^{d \times d} \), we define on \( \mathcal{H} \) the unbounded operator

\[
L_U := D - T_U, \quad D := \frac{1}{i} \partial_x,
\]

where \( \text{dom}(L_U) := \{ F \in \mathcal{H} : DF \in \mathcal{H} \} \), and \( T_U \) is the Toeplitz operator of symbol \( U \) defined by \( T_U(F) := \Pi_{\geq 0}(UF) \). It is easy to check that \( L_U \) is self-adjoint if \( U \) is valued in Hermitian matrices. However, we do not need the latter property for establishing the Lax pair structure. If \( U \) is smooth enough (say belonging to the Sobolev space \( H^2 \)), we define the following bounded operator,

\[
B_U := i(T_{|D|U} - T_U^2),
\]

which is anti-self-adjoint if \( U \) is valued in Hermitian matrices. Our main result is the following.

**Theorem 1** Let \( I \) be a time interval and \( U \) be a continuous function on \( I \) valued into \( \mathcal{H} \) such that \( \partial_t U \) is continuous valued into \( L^2(X, \mathbb{C}^{d \times d}) \). Then \( U \) is a solution of (1) on \( I \) if and only if

\[
\partial_t L_U = [B_U, L_U].
\]

**Proof** Obviously, \( \partial_t L_U = -T_{\partial_t U} \). Since \( T_G = 0 \) implies classically \( G = 0 \), the claim is equivalent to the identity

\[
- T_{\partial_t (|D|U - U^2) - i(U, |D|U)} = [B_U, L_U].
\]

We have

\[
- T_{\partial_t (|D|U - U^2) - i(U, |D|U)} = [iT_{|D|U}, D] + T_{U\partial_t U + U^2} + iT_{|D|U, |D|U}
\]

\[
= [B_U, D] + T_{U\partial_t U + U^2} - T_{U^2} - T_{\partial_t U} T_U + iT_{|D|U, |D|U}
\]

\[
= [B_U, L_U] + T_{|D|U^2} - iT_{U^2} + iT_{|D|U} - iT_{|D|U} - iT_{U^2}.
\]

So, we have to check that

\[
T_{|D|U^2} - iT_{U^2} + iT_{U^2} = 0.
\] (2)

We need the following lemma, where we denote \( \Pi_{< 0} := Id - \Pi_{\geq 0} \).

**Lemma 1** Let \( A, B \in L^\infty(X, \mathbb{C}^{d \times d}) \). Then, for every \( F \in \mathcal{H} \),

\[
(T_{AB} - T_AT_B)F = \Pi_{\geq 0}(\Pi_{\geq 0}(A) \Pi_{< 0}(\Pi_{< 0}(B) F)).
\]
Let us prove Lemma 1. Write

\[ T_{AB}F = \Pi_{\geq 0}(ABF) = \Pi_{\geq 0}(A\Pi_{\geq 0}(BF)) + \Pi_{\geq 0}(A\Pi_{< 0}(BF)) = T_A T_B F + \Pi_{\geq 0}(A\Pi_{< 0}(BF)) \]

so that observing that the ranges of \( \Pi_{\geq 0} \) and of \( \Pi_{< 0} \) are stable through the multiplication,

\[ (T_{AB} - T_A T_B)F = \Pi_{\geq 0}(A\Pi_{< 0}(BF)) = \Pi_{\geq 0}(\Pi_{\geq 0}(A)\Pi_{< 0}(BF)). \]

This completes the proof of Lemma 1. Let us apply Lemma 1 to \( A = U, B = |D|U \). We get

\[ i(T_{|D|U} - T_{U|D|U})F = \Pi_{\geq 0}(\Pi_{\geq 0}(U)\Pi_{< 0}(i|D|U)F) \]

\[ = -\Pi_{\geq 0}(\Pi_{\geq 0}(U)\Pi_{< 0}(\partial_x U)F), \]

and similarly

\[ i(T_{|D|U} - T_{|D|U}T_{U})F = \Pi_{\geq 0}(\Pi_{\geq 0}(|D|U)\Pi_{< 0}(\Pi_{< 0}(U))F)) \]

\[ = \Pi_{\geq 0}(\Pi_{\geq 0}(\partial_x U)\Pi_{< 0}(\Pi_{< 0}(U))F)) \]

so that

\[ (iT_{|D|U} - iT_{|D|U}T_{U})F = -\Pi_{\geq 0}(\Pi_{\geq 0}(U)\Pi_{< 0}(\partial_x U)F) \]
\[ - \Pi_{\geq 0}(\Pi_{\geq 0}(\partial_x U)\Pi_{< 0}(\Pi_{< 0}(U))F) \]
\[ = -T_{|D|U}(F) + T_{U}(F), \]

using again Lemma 1. Hence, we have proved identity (2). \( \square \)

3 Conservation laws and global wellposedness

The following is an application of Theorem 1.

Corollary 1 Assume that \( U_0 \) belongs to the Sobolev space \( H^2(X, \mathbb{C}^{d \times d}) \) and is valued into Hermitian matrices. Then equation (1) has a unique solution \( U, \) depending continuously on \( t \in \mathbb{R}, \) valued into Hermitian matrices of the Sobolev space \( H^2(X), \) and such that \( U(0) = U_0. \)

Furthermore, the following quantities are conservation laws,

\[ \delta_k(U) = [L^k_\Omega(\Pi_{\geq 0}U)|\Pi_{\geq 0}U], \quad k = 0, 1, 2, \ldots \]

In particular, the norm of \( U(t) \) in the Sobolev space \( H^2(X) \) is uniformly bounded for \( t \in \mathbb{R}. \)

Proof The local wellposedness in the Sobolev space \( H^2 \) follows from an easy adaptation of Kato’s iterative scheme—see, e.g., Kato [3] for hyperbolic systems. Global wellposedness will follow if we show that conservation laws control the \( H^2 \) norm. Set \( U_t := \Pi_{\geq 0}U, U_- := \Pi_{< 0}U. \) Applying \( \Pi_{\geq 0} \) to both sides of (1), we get

\[ \partial_t U_+ = -i\partial_x^2 U_+ - 2T_{U} \partial_x U_+ - 2T_{U_-} U_+ = iL^2_\Omega(U_+) + B_{U}(U_+). \]
Therefore, from Theorem 1,

\[
\frac{d}{dt} \langle U_n^k(U_s)|U_s \rangle = \langle [B_{UL}L^k_U]U_s, |U_s \rangle + \langle L^k_U (iL^2_U(U_s) + B_{UL}(U_s)) |U_s \rangle \\
+ \langle L^k_U(U_s)iL^2_U(U_s) + B_{UL}(U_s) \rangle = 0,
\]

since \(B_{UL}\) and \(iL^2_U\) are anti-self-adjoint.

Now observe that \(\mathcal{E}_0(U) = \|U_s\|_{L^2}^2\). Since \(U\) is Hermitian, we have

\[
U = \begin{cases} 
U_s + U_s^* & \text{if } X = \mathbb{R}, \\
U_s + U_s^* - \langle U_s \rangle & \text{if } X = \mathbb{T},
\end{cases}
\]

where \(\langle F \rangle\) denotes the mean value of a function \(F\) on \(T\). We infer that \(\mathcal{E}_0(U)\) controls the \(L^2\) norm of \(U\). Let us come to \(\mathcal{E}_1(U)\). In view of the Gagliardo–Nirenberg inequality,

\[
\mathcal{E}_1(U) = \langle DU_s, |U_s \rangle - \langle T_{UL}(U_s) |U_s \rangle \geq \langle DU_s, |U_s \rangle - O(\|U_s\|_{L^2}^3) \\
\geq \langle DU_s, |U_s \rangle - O(DU_s, |U_s|^{1/2} \|U_s\|_{L^2}^2) - O(\|U_s\|_{L^2}^3).
\]

Consequently, \(\mathcal{E}_0(U)\) and \(\mathcal{E}_1(U)\) control \(\|U_s\|_{L^2}^2 + \langle DU_s, |U_s \rangle\), which is the square of the \(H^{1/2}\) norm of \(U_s\), since \(U_s\) only has nonnegative Fourier modes. Therefore, the \(H^{1/2}\) norm of \(U\) is controlled by \(\mathcal{E}_0(U)\) and \(\mathcal{E}_1(U)\).

Since \(\mathcal{E}_2(U)\) is the square of \(L^2\) norm of \(L_{UL}(U_s)\) and the \(L^2\) norm of \(T_{UL}(U_s)\) is controlled by the \(H^{1/2}\) norm of \(U\) by the Sobolev estimate, we infer that \(\mathcal{E}_0(U), \mathcal{E}_1(U), \text{ and } \mathcal{E}_2(U)\) control the \(L^2\) norms of \(U\) and of \(\partial_t U\), namely the Sobolev \(H^1\) norm of \(U\).

Finally, \(\mathcal{E}_3(U)\) is the square of the \(L^2\) norm of \(L_{UL}(U_s)\). Since \(L_{UL}(U_s)\) is already controlled in \(L^2\) and \(U\) is controlled in \(L^\infty\) by the Sobolev inclusion \(H^1 \subset L^\infty\), we infer that the \(H^1\) norm of \(L_{UL}(U_s)\) is controlled. But \(H^1\) is an algebra, so the \(H^1\) norm of \(T_{UL}(U_s)\) is also controlled. Finally, we infer that \(\{\mathcal{E}_n(U), n \leq 4\}\) control the \(H^1\) norms of \(U_s\) and \(\partial_t U_s\), namely the \(H^2\) norm of \(U_s\), and finally of \(U\).

\[\square\]

Remarks.

1. If the initial datum \(U\) belongs to the Sobolev space \(H^k\) for an integer \(k > 2\), a similar argument shows that the \(H^k\) norm of \(U\) is controlled by the collection \(\{\mathcal{E}_n(U), 0 \leq n \leq 2k\}\).

2. In [1], the evolution of multi-solitons for (1) is derived through a pole ansatz, and the question of keeping the poles away from the real line—or from the unit circle in the case \(X = \mathbb{T}\)—is left open. Since Corollary 1 implies that the \(L^\infty\) norm of the solution stays bounded as \(T\) varies, this implies a positive answer to this question, as far as the poles do not collide. In fact, we strongly suspect that such a collision does not affect the structure of the pole ansatz because it is likely that multisolitons have a characterization in terms of the spectrum of \(L_{UL}\), as it has in the scalar case [2].

Let us say a few more about conservation laws. The conservation laws \(\mathcal{E}_k\) can be explicitly computed in terms of \(U\). For simplicity, we focus on \(\mathcal{E}_0\) and \(\mathcal{E}_1\). In case \(X = \mathbb{R}\), we have...
exactly

\[ E_0(U) = \frac{1}{2} \int_{\mathbb{R}} \text{tr}(U^2) \, dx, \]

and

\[ E_1(U) = \text{tr}(D_U U) - \text{tr}(T_{U}(U)|U_+) \]
\[ = \int_{\mathbb{R}} \text{tr}\left( \frac{1}{2} U \frac{1}{2} D|U| - \frac{1}{3} U^3 \right) \, dx, \]

so we recover the Hamiltonian function derived in [1].

In case \( X = T \), the above formulæ must be slightly modified due to the zero Fourier mode. This leads us to a bigger set of conservation laws. Indeed, every constant matrix \( V \in \mathbb{C}^{d \times d} \) is a special element of \( \mathcal{H} \), and we observe that \( B_U(V) = -iL_2(V) \). Arguing exactly as in the proof of Corollary 1, we infer that, for every integer \( \ell \geq 1 \), for every pair of constant matrices \( V, W \), the quantity \( (L_\ell(U)|W) \) is a conservation law. Since \( V, W \) are arbitrary, this means that, if \( I \) denotes the identity matrix, all the matrix-valued functionals

\[ M_{\ell-2}(U) := \int_{T} L_\ell(U)(1) \, dx \]

for \( \ell \geq 1 \) are conservation laws. If the measure of \( T \) is normalised to 1, we have for instance

\[ M_{-1}(U) = -\langle U_+ \rangle = -\langle U \rangle, \]
\[ M_0(U) = \frac{1}{2} \langle U^2 - iUHU \rangle + \frac{1}{2} \langle U \rangle^2. \]

Then one can check that

\[ E_0(U) = \frac{1}{2} \text{tr}((U^2)) + \frac{1}{2} \text{tr}((U_+)^2), \]
\[ E_1(U) = \text{tr}\left( \frac{1}{2} U \frac{1}{2} D|U| - \frac{1}{3} U^3 \right) - \frac{5}{3} \text{tr}[\langle U \rangle^3] - \text{tr}[M_0(U)(U)]. \]

Observe again that the first term on the right-hand side of the expression of \( E_1(U) \) is the opposite of the Hamiltonian function in [1].

In the case \( X = \mathbb{R} \), all the matrix valued expressions \( M_k(U) \) make sense if \( k \geq 0 \) and are again conservation laws. For instance,

\[ M_0(U) = \frac{1}{2} \int_{\mathbb{R}} (U^2 - iUHU) \, dx. \]

Finally, notice that in both cases \( X = T \) and \( X = \mathbb{R} \), we have

\[ E_k(U) = \text{tr} \, M_k(U) \]

for every \( k \geq 0 \).
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Author contributions
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