Weighted gradient estimates for the class of singular $p$-Laplace system

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Abstract

Let $n \in \{2, 3, 4, \ldots\}$, $N \in \{1, 2, 3, \ldots\}$ and $p \in (1, 2 - \frac{1}{n}]$. Let $\beta \in (1, \infty)$ be such that

$$\frac{np}{n - p} < \beta' < \frac{n}{n(2 - p) - 1}$$

and $f \in L^{\beta}(\mathbb{R}^n; \mathbb{R}^N)$. Consider the $p$-Laplace system

$$-\Delta_p u = -\text{div} \left(|Du|^{p-2} Du\right) = f \quad \text{in} \quad \mathbb{R}^n.$$

We obtain a weighted gradient estimate for distributional solutions of this system.

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1 Introduction

Calderon-Zygmund theory is undoubtedly classical to linear partial differential equations. In the last few years, its extension to non-linear settings has become an active area of research. For a comprehensive survey on this account, cf. [Min10] and also the references therein. Our paper continues this trend with a gradient estimate for the solutions of a $p$-Laplace system.

Specifically, let $n \in \{2, 3, 4, \ldots\}$, $N \in \{1, 2, 3, \ldots\}$ and $p \in \left(1, 2 - \frac{1}{n}\right]$. Consider the $p$-Laplace system

$$- \Delta_p u = - \text{div} \left(|Du|^{p-2} Du\right) = f \quad \text{in} \quad \mathbb{R}^n,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ belongs to some appropriate Lebesgue space.

Our aim is to derive a general Muckenhoupt-Wheeden-type gradient estimate for (1.1). This result inherits the spirit of [KM18], [NP19], [NP] and [NP20]. Specifically, in [NP19], [NP] the authors obtained such estimates when $N = 1$ and $1 < p \leq 2 - \frac{1}{n}$. If in addition $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$, pointwise gradient estimates with measure data are also available (cf. [NP20]). In a system setting (i.e. $N \geq 1$) with measure data, pointwise gradient bounds via Riesz potential and Wolff potential for $p > 2 - \frac{1}{n}$ were obtained in [KM18]. Regarding the method of proof, we follow the general frameworks presented in these papers. Our main contribution involves the reconstructions of a comparison estimate and a good-\(\lambda\)-type bound peculiar to the setting in this paper.
To state our main result, we need some definitions.

**Definition 1.1.** A function $u : \mathbb{R}^n \to \mathbb{R}^N$ is a distributional (or weak) solution to (1.1) if

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du : D\varphi \, dx = \int_{\mathbb{R}^n} f \varphi \, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$.

Here $Du$, which is a counterpart of $\nabla u$ in the equation setting, is understood in the sense of tensors. See Section 2 for further details.

Next recall the notion of Muckenhoupt weights.

**Definition 1.2.** A positive function $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be an $A_\infty$-weight if there exist constants $C > 0$ and $\nu > 0$ such that

$$\omega(E) \leq C \left( \frac{|E|}{|B|} \right)^\nu \omega(B),$$

for all balls $B \subset \mathbb{R}^n$ and all measurable subset $E$ of $B$. The pair $(C, \nu)$ is called the $A_\infty$-constants of $\omega$ and is denoted by $[\omega]_{A_\infty}$.

In what follows, we will also make use of the maximal function defined by

$$M_\beta(f)(x) = \sup_{\rho > 0} \rho^\beta \int_{B_\rho(x)} |f(y)| \, dy$$

for all $x \in \mathbb{R}^n$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\beta \in [0, n]$, where

$$\int_{B_\rho(x)} |f(y)| \, dy := \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} f(y) \, dy.$$  

When $\beta = 0$, the Hardy-Littlewood maximal function $M = M_0$ is recovered.

Our main result is as follows.

**Theorem 1.3.** Let $n \in \{2, 3, 4, \ldots\}$, $N \in \{1, 2, 3, \ldots\}$ and $p \in (1, 2 - \frac{1}{n}]$. Let $\beta \in (1, \infty)$ be such that

$$\frac{np}{n-p} < \beta' < \frac{n}{n(2-p)-1}$$

and $f \in L^\beta(\mathbb{R}^n, \mathbb{R}^N)$. Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function that satisfies

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi(t) = \infty.$$
Furthermore assume that there exists a $c > 1$ such that

$$\Phi(2t) \leq c \Phi(t)$$

for all $t \geq 0$. Then for all $\omega \in A_\infty$ there exist a $C > 0$ and a $\delta \in (0, 1)$, both depending on $n, p, \Phi$ and $[\omega]_A$ only, such that

$$\int_{\mathbb{R}^n} \Phi(|Du|) \omega \, dx \leq C \int_{\mathbb{R}^n} \Phi \left( M_p(|f|^\delta) \right)^{\frac{1}{p-1+\delta}} \omega \, dx$$

for all distributional solution $u$ of $(1.1)$.

Note that in our setting all functions are vector fields. For short we will write, for instance, $C^\infty_c(\mathbb{R}^n)$ in place of $C^\infty_c(\mathbb{R}^n, \mathbb{R}^N)$ hereafter. When scalar-valued functions are in use, we will explicitly write $C^\infty_c(\mathbb{R}^n, \mathbb{R})$. This convention applies to all function spaces in the whole paper.

When $n = 1$ it has been known that the distributional solution $u$ is locally $C^{1,a}$ for some exponent $a = a(n, N, p) > 0$, whose result is due to [Uh77]. Hence we only consider $n \geq 2$ in this project. We also remark that the function $\Phi$ in the above theorem is quite general. In particular, we do not require $\Phi$ to be convex or to satisfy the so-called $\nabla_2$ condition: $\Phi(t) \geq \frac{1}{2a} \Phi(at)$ for some $a > 1$ and for all $t \geq 0$. As such one can take, for examples, $\Phi(t) = t^a$ or $\Phi(t) = [\log(1 + t)]^a$ for any $a > 0$.

The outline of the paper is as follows. Section 2 collects definitions and basic facts about tensors and $p$-harmonic maps. In Sections 3 and 4 we derive a comparison estimate and a good-$\lambda$-type bound respectively. Lastly Theorem 1.3 is proved in Section 5.

Notations. Throughout the paper the following set of notation is used without mentioning. Set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^n = \{1, 2, 3, \ldots\}$. For all $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For all ball $B \subset \mathbb{R}^d$ we write $w(B) := \int_B w$. The constants $C$ and $c$ are always assumed to be positive and independent of the main parameters whose values change from line to line. Given a ball $B = B_r(x)$, we let $tB = B_{tr}(x)$ for all $t > 0$. If $p \in [1, \infty)$, then the conjugate index of $p$ is denoted by $p'$.

Throughout assumptions. In the entire paper, we always assume that $n \in \{2, 3, 4, \ldots\}$, $N \in \{1, 2, 3, \ldots\}$ and $p \in \left(1, 2 - \frac{1}{n}\right]$ without explicitly stated.
2 Tensors and $p$-harmonic maps

This section briefly summarizes definitions and basic facts regarding tensors and $p$-harmonic maps. Further details are available in [KM18, Sections 2 and 3]. These will be used frequently in subsequent sections without mentioning.

Let $\{e_j\}_{j=1}^n$ and $\{e^a\}_{a=1}^N$ be the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively. Let $\zeta$ and $\xi$ be second-order tensors of size $(N, n)$, that is,

$$\zeta = \zeta_j^a e^a \otimes e_j \quad \text{and} \quad \xi = \xi_j^a e^a \otimes e_j
$$

in which repeated indices are summed. Note that the linear space of all second-order tensors is isomorphic to $\mathbb{R}^{N \times n}$.

The Frobenius product of $\zeta$ and $\xi$ is given by

$$\zeta : \xi = \zeta_j^a \xi_j^a,$$

from which we also obtain the Frobenius norm of $\zeta$ as $|\zeta|^2 = \zeta : \zeta$. The divergence of $\zeta$ is defined by

$$\text{div} \, \zeta = (\partial_j \zeta^a_j) e^a.$$

Also the gradient of a first-order tensor $u = u^a e^a$ is the second-order tensor

$$Du = (\partial_j u^a) e^a \otimes e_j.$$

Next consider the tensor field

$$A_q(z) := |z|^{q-2} z = |z|^{q-2} z_j^a e^a \otimes e_j$$

defined on the linear space of all second-order tensors, where $q \in (1, \infty)$. The differential of $A_q$ is defined as a fourth-order tensor

$$\partial A_q(z) = |z|^{q-2} \begin{pmatrix} \delta_{a\beta} \delta_{ij} + (q-2) \frac{z_i^a z_j^\beta}{|z|^2} \end{pmatrix} (e^a \otimes e_i) \otimes (e^\beta \otimes e_j).$$

Here $\delta_{a\beta}$ is the Kronecker’s delta. This leads to

$$\partial A_q(z) : \xi = |z|^{q-2} \begin{pmatrix} \xi + (q-2) \frac{\xi_j^a z_i^\beta}{|z|^2} \end{pmatrix}$$
and

$$\langle \partial A_\alpha(z) : \xi \rangle : \xi = |z|^{q-2} \left( |\xi|^2 + (q-2) \frac{(z : \xi)^2}{|z|^2} \right).$$

Regarding second-order tensors, the following inequality is well-known (cf. [KM18, (4.51)]).

**Lemma 2.1.** Let $q \in (1, \infty)$. There exists a $c = c(n; N; p) \leq 1$ such that

$$\left( |z_2|^{q-1}z_2 - |z_1|^{q-1}z_1 \right) : (z_2 - z_1) \geq c \left( |z_2|^2 + |z_1|^2 \right)^{(q-2)/2} |z_2 - z_1|^2$$

for all second-order tensors $z_1$ and $z_2$.

We end this section with the definition of a $q$-harmonic map.

**Definition 2.2.** Let $q \in (1, \infty)$. A function $v \in W^{1,q}(\mathbb{R}^n)$ is said to be $q$-harmonic if

$$\int_{\mathbb{R}^n} |Dv|^{q-2} Dv : D\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

3 A comparison estimate

In this section we prove a comparison estimate between the weak solutions of (1.1) and a $p$-harmonic map, which is the content of Proposition 3.1.

In what follows it is convenient to denote

$$q_0 = \frac{\beta' (p-1) n}{\beta' (n-1) - n}. \quad (3.1)$$

Note that $q_0 \in (1, p)$. Also set $B_\sigma = B_\sigma(0)$ for all $\sigma \in (0, 1]$.

**Proposition 3.1.** Let $\varepsilon > 0$, $M \geq 1$ and $\beta \in (1, \infty)$ be such that $\frac{np}{n-p} < \beta' < \frac{n}{m(2p-1)}$. Let $1 < q < q_0$ and

$B = B_\sigma(x_0)$ be a ball in $\mathbb{R}^n$. Suppose $u \in W^{1,p}(B)$ satisfy

$$\int_B |u| \, dx \leq Mr. \quad (3.2)$$

Then there exists a positive constant $\delta = \delta(n, N, p, q, M, \varepsilon) \in (0, 1)$ such that if

$$\left| \int_B |Du|^{p-2} Du : D\varphi \, dx \right| \leq \frac{\delta}{r} \left( \int_B |\varphi(x)|^{p} \, dx \right)^{1/p'} \quad (3.3)$$
for all \( \varphi \in W^{1,p}_0(B) \cap L^p(B) \), then there exist a constant \( c = c(n, N, p, q) > 0 \) and a \( p \)-harmonic map \( v \in W^{1,p}(\frac{1}{2}B) \) such that
\[
\left( \int_{\frac{1}{2}B} |Dv|^q \, dx \right)^{1/q} \leq \varepsilon
\]
as well as
\[
\int_{\frac{1}{2}B} |v| \, dx \leq M 2^n \quad \text{and} \quad \left( \int_{\frac{1}{2}B} |Dv|^q \, dx \right)^{1/q} \leq cM.
\]

We divide the proof of Proposition 3.1 into several parts. To begin with, recall the following self-improving property of reverse H"older inequalities (cf. [HK, Lemma 3.38]).

**Lemma 3.2.** Let \( 0 < q < a < \gamma < \infty, \xi \geq 0 \) and \( M \geq 0 \). Let \( v \) be a non-negative Borel measure with finite total mass and \( B \subset \mathbb{R}^n \) be a ball. Suppose \( 0 \leq g \in L^p(U, v) \) satisfies the following: there exists a \( c_0 > 0 \) such that
\[
\left( \int_{\sigma_1 B} g^\gamma \, dv \right)^{1/\gamma} \leq \frac{c_0}{(\sigma - \sigma_1)^\xi} \left( \int_{\sigma B} g^a \, dv \right)^{1/a} + M
\]
for all \( \kappa \leq \sigma_1 < \sigma \leq 1 \), where \( \kappa \in (0, 1) \). Then there exists a \( c = c(c_0, \xi, \sigma, a, q) > 0 \) such that
\[
\left( \int_{\sigma_1 B} g^\gamma \, dv \right)^{1/\gamma} \leq \frac{c}{(1 - \sigma)^\xi} \left[ \left( \int_{\sigma B} g^q \, dv \right)^{1/q} + M \right]
\]
for all \( \sigma \in (\kappa, 1) \), where
\[
\xi := \frac{p (\gamma - q)}{q (\gamma - a)}.
\]

Next we will establish suitable a priori estimates for (scaled) weak solutions of (1.1) under the assumptions in Proposition 3.1.

**Lemma 3.3.** Let \( M \) and \( \beta \) be as in Proposition 3.1. Let \( \delta \in (0, 1) \). Suppose \( \overline{u} \in W^{1,p}(B_1) \) satisfies
\[
\int_{B_1} |\overline{u}| \, dx \leq 1 \quad \text{(3.4)}
\]
and
\[
\left| \int_{B_1} |D\overline{u}|^{p-2} D\overline{u} : \nabla \eta \, dx \right| \leq M^{1-p} \delta \| \eta \|_{L^p(B_1)} \quad \text{(3.5)}
\]
for all \( \eta \in W^{1,p}_0(B_1) \cap L^p(B_1) \). Then there exists a \( c = c(n, N, p, q) \) such that
\[
\| \overline{u} \|_{W^{-1,\infty}(B_2, q)} \leq c
\]
for all \( q \in (1, q_0) \).
\textbf{Proof.} The main idea is to test (3.5) with suitable test functions. Following [KMT18] Proof of Theorem 4.1 consider for each $t > 0$ the truncation operator $T_t : \mathbb{R}^N \mapsto \mathbb{R}^N$ defined by

$$T_t(z) := \min \left\{ 1, \frac{t}{|z|} \right\} z. \quad (3.6)$$

By direct calculations, $DT_t : \mathbb{R}^N \mapsto \mathbb{R}^N \otimes \mathbb{R}^N$ is given by

$$DT_t(z) = \begin{cases} I & \text{if } |z| \leq t \\ \frac{t}{|z|} \left( I - \frac{z \otimes z}{|z|^2} \right) & \text{if } |z| > t, \end{cases} \quad (3.7)$$

where $I : \mathbb{R}^N \mapsto \mathbb{R}^N \otimes \mathbb{R}^N$ denotes the identity operator.

Now let $\phi \in C_c^\infty(B_1; \mathbb{R})$ be such that $0 \leq \phi \leq 1$ and then choose

$$\eta := \phi^p T_t \left( \frac{\partial}{\partial x} \right)$$

as a test function in (3.5). We have

$$D\eta = 1_{(\frac{\partial}{\partial x})} \left( \phi^p D\frac{\partial}{\partial x} + p\phi^{p-1} \frac{\partial}{\partial x} D\phi \right) + 1_{(\frac{\partial}{\partial x})} \frac{t}{|\partial|} \left( \phi^p (I - P) D\left( \frac{\partial}{\partial x} \right) + p\phi^{p-1} \frac{\partial}{\partial x} D\phi \right),$$

where $P := \frac{\partial}{\partial x}$. Also notice that

$$D\frac{\partial}{\partial x} : \left( (I - P) D\frac{\partial}{\partial x} \right) = |D\frac{\partial}{\partial x}|^2 - \frac{u^\mu D\frac{\partial}{\partial x} u^\mu D\frac{\partial}{\partial x}}{|u|^2} = |D\frac{\partial}{\partial x}|^2 - \frac{\sum_{j=1}^{n} (D_j \cdot u)^2}{|u|^2} \geq 0 \quad (3.8)$$

and

$$\|\eta\|_{L^p(B_1)} = \left( \int_{B_1} \left| T_t \left( \frac{\partial}{\partial x} \right)^{1/\theta} \phi^{p\theta} d\chi \right|^{1/\theta} \right)^{1/\theta} = \left( \int_{B_1} \left| T_t \left( \frac{\partial}{\partial x} \right)^{\theta \theta} \phi^{p\theta} d\chi \right|^{1/(1-\theta)} \phi^{p(1-\theta)} d\chi \right)^{1/(1-\theta)} \leq t^\theta \left\| \phi^{1/\theta} \right\|_{L^{p/(1-\theta)}(B_1)}^{1-\theta},$$

where $0 < \theta < 1$.

Substituting these into (3.5) and using Young’s inequality we obtain

$$\int_{B_1 \cap \{ \frac{\partial}{\partial x} \}} |D\frac{\partial}{\partial x}|^p \phi^p d\chi \leq c \int_{B_1 \cap \{ \frac{\partial}{\partial x} \}} |\frac{\partial}{\partial x}|^p |D\phi|^p d\chi + c M^{1-p} \|\phi\| L^{p/(1-\theta)}(B_1) \|\phi^{1/\theta}\|_{L^{p/(1-\theta)}(B_1)}^{1-\theta}$$

$$+ ct \int_{B_1 \cap \{ \frac{\partial}{\partial x} \}} |D\frac{\partial}{\partial x}|^{p-1} |D\phi|^p d\chi \quad (3.9)$$

for some $c = c(n, N, p) > 0$. 

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For the rest of the proof we use \( c = c(n, N, p) \) whose value may vary from line to line.

Next let \( \gamma \in (0, 1) \). Multiplying (3.9) by \( (1 + t)^{-1 - \gamma - \theta} \) and then integrating on \((0, \infty)\) with respect to \( t \) give

\[
\frac{1}{\theta + \gamma} \int_{B_1} \frac{|D\overline{u}|^p \phi^p}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx \leq \frac{c}{\gamma + \theta} \int_{B_1} (1 + |\overline{u}|)^{\rho - \gamma - \theta} |D\phi|^p \, dx \\
+ \frac{c}{\gamma} \left\| \phi \right\|_{L^{\rho(1-\theta)}(B_1)}^{1-\theta} + c \int_{B_1} \frac{|\overline{u}| |D\overline{u}|^{p-1} |D\phi| \phi^p}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx.
\]

It follows from Young’s inequality that

\[
\int_{B_1} \frac{|D\overline{u}|^{p-1} |D\phi| \phi^{p-1}}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx \leq \frac{1}{2c(\gamma + \theta)} \int_{B_1} \frac{|D\overline{u}|^p \phi^p}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx + c(\gamma + \theta)^{p-1} \int_{B_1} (1 + |\overline{u}|)^{-(\gamma + \theta)} |D\phi|^p |\overline{u}|^p \, dx.
\]

Consequently

\[
\int_{B_1} \frac{|D\overline{u}|^{p-1} \phi^p}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx \leq c \int_{B_1} (1 + |\overline{u}|)^{\rho - \gamma - \theta} |D\phi|^p \, dx + \frac{c}{\gamma} \left\| \phi \right\|_{L^{\rho(1-\theta)}(B_1)}^{1-\theta} \tag{3.10}
\]

The pointwise inequality \(|D| |D\overline{u}| \leq |D\overline{u}|^p\) implies

\[
|D((1 + |\overline{u}|)^{1 - \frac{\rho\gamma}{p}} \phi)|^p \leq \frac{c |D\overline{u}|^p}{(1 + |\overline{u}|)^{1 + \gamma}} \phi^p + c(1 + |\overline{u}|)^{\rho - \theta - \gamma} |D\phi|^p.
\]

Combining with (3.10), we obtain

\[
\int_{B_1} |D((1 + |\overline{u}|)^{1 - \frac{\rho\gamma}{p}} \phi)|^p \, dx \leq c \int_{B_1} (1 + |\overline{u}|)^{\rho - \gamma - \theta} |D\phi|^p \, dx + \frac{c}{\gamma} \left\| \phi \right\|_{L^{\rho(1-\theta)}(B_1)}^{1-\theta} \tag{3.11}
\]

Applying Sobolev’s inequality to (3.11) and combining the derived estimate with (3.10) yield

\[
\int_{B_1} \frac{|D\overline{u}|^{p-1} \phi^p}{(1 + |\overline{u}|)^{\gamma + \theta}} \, dx + \left( \int_{B_1} (1 + |\overline{u}|)^{\frac{\rho - \gamma - \theta}{\rho - p}} \phi^{\frac{p}{\rho - p}} \, dx \right) \leq c \int_{B_1} (1 + |\overline{u}|)^{\rho - \gamma - \theta} |D\phi|^p \, dx \\
+ \frac{c}{\gamma} \left\| \phi \right\|_{L^{\rho(1-\theta)}(B_1)}^{1-\theta} \tag{3.12}
\]

Next let \( 7/8 \leq \sigma_1 < \sigma \leq 1 \) and \( \psi \in C^\infty_c(B_{\sigma}) \) be such that

\[
0 \leq \psi \leq 1, \quad \psi|_{B_{\sigma_1}} = 1 \quad \text{and} \quad |D\psi| \leq \frac{100}{\sigma - \sigma_1}.
\]

With this choice of test function, we deduce from (3.12) that

\[
\left( \int_{B_{\sigma_1}} (1 + |\overline{u}|)^{\frac{\rho - \gamma - \theta}{\rho - p}} \phi^{\frac{p}{\rho - p}} \, dx \right) \leq \frac{c}{\sigma - \sigma_1} \int_{B_{\sigma}} (1 + |\overline{u}|)^{\rho - \gamma - \theta} \, dx + \frac{c}{\gamma} \left\| \phi \right\|_{L^{\rho(1-\theta)}(B_{\sigma})}^{1-\theta} \tag{3.13}
\]

for all \( \gamma, \theta \in (0, 1) \).
Now we choose $\theta, \gamma \in (0, 1)$ such that $p - \theta - \gamma \geq 1$. Then thanks to Lemma (3.2) and (3.4), we get

$$
\left( \int_{B_{r_1}} \left(1 + |u| \right)^{\frac{p - \theta - \gamma}{n-p}} \, dx \right)^{\frac{n-p}{p}} \leq \frac{c}{1-\sigma} + \frac{c}{\gamma} \|u\|_{L^{p/(1-\theta)(B)}_x} \leq \frac{c}{1-\sigma} + \frac{c}{\gamma} \left(1 + |u| \right)^{\frac{1-\theta}{p/(1-\theta)(B)}_x}.
$$

The lemma can now be achieved by iterating (3.14) multiple times. Indeed if we denote $b = \frac{n}{n-p}$ then (3.14) reads

$$
\|1 + |u|\|_{L^{p/(1-\theta)(B)}_x} \leq \frac{c}{1-\sigma} + \frac{c}{\gamma} \left(1 + |u| \right)^{\frac{1-\theta}{p/(1-\theta)(B)}_x}.
$$

(3.14)

For each $k \in \mathbb{N}^*$ set $\gamma_k = (2\beta')^{-k}$ and $\theta_k$ such that

$$
\left\{ \begin{array}{l}
\theta_1 = 1 - \frac{1}{\beta'}, \\
\theta_{k+1} = 1 - \frac{b}{\beta'}(p - \theta_k - \gamma_k) \in (0, 1).
\end{array} \right.
$$

Using (3.14), (3.3), we obtain

$$
\|1 + |u|\|_{L^{p/(1-\theta_k)(B_{r/2})_x}} + \|1 + |u|\|_{L^{p/(1-\theta_k)(B_{r/2})_x}} \leq c_k
$$

(3.15)

for all $k \in \mathbb{N}^*$, where $c_k = c_k(n, N, p, k)$.

By extracting a subsequence when necessary, we may assume without loss of generality that $\lim_{k \to \infty} \theta_k = \theta_0$.

Then

$$
\beta' (1 - \theta_0) = \frac{(p - \theta_0)n}{n - p}
$$

or equivalently

$$
\theta_0 = \frac{\beta' (n - p) - p n}{\beta' (n - p) - n}.
$$

Observe that for all $a_1 > 0$ there exists a $k_1 \in \mathbb{N}^*$ such that $\theta_0 + \frac{a_1}{b} \geq \theta_{k_1} + \gamma_{k_1}$. Therefore (3.15) implies

$$
\int_{B_{r/2}} \left(1 + \|u\| \right)^{\frac{(p-\theta_0)n}{n-p}} \, dx \leq c(n, N, p, a_1)
$$

(3.16)

for all $a_1 > 0$. Choosing a suitable test function in (3.10) leads to

$$
\int_{B_{r/4}} \frac{|D\bar{u}|^p}{(1 + |u|)^{p+\gamma}} \, dx \leq c \int_{B_{r/2}} \left(1 + |u| \right)^{p+\gamma} \, dx + \frac{c}{\gamma} \|u\|_{L^{p/(1-\theta)(B_{r/2})_x}}^{\frac{1-\theta}{p/(1-\theta)(B_{r/2})_x}}.
$$

Then (3.15) in turn implies

$$
\int_{B_{r/4}} \frac{|D\bar{u}|^p}{(1 + |u|)^{p+\gamma}} \, dx \leq c_k,
$$

(3.17)
for all \( k \in \mathbb{N}^* \), where \( c_k = c_k(n, N, p, k) \).

Analogously for all \( a_2 > 0 \) there exists a \( k_2 \in \mathbb{N}^* \) such that \( \theta_0 + a_2 > \theta_{k_2} + \gamma_{k_2} \). Therefore (3.17) gives

\[
\int_{B_{3/4}} \frac{|Dn|^p}{(1 + |n|)^{\theta_0 + a_2}} \, dx \leq c(n, N, p, a_2)
\]

(3.18)

for all \( a_2 > 0 \).

Now let \( a = \frac{p'(p-1)n}{\beta'(n-1)-n} \) and apply Hölder’s inequality for the exponent \( \frac{p}{a-a_2} \) to arrive at

\[
\int_{B_{3/4}} |Dn|^{\frac{p'(p-1)n}{\beta'(n-1)-n}-a_2} \, dx = \int_{B_{3/4}} |Dn|^{a-a_2} (1 + |n|)^{-(\theta_0+a_2)/(p'(p-1)n)}(1 + |n|)^{(\theta_0+a_2)/(a-a_2)} \, dx
\]

\[
\leq \left( \int_{B_{3/4}} \frac{|Dn|^p}{(1 + |n|)^{\theta_0 + a_2}} \, dx \right)^{(a-a_2)/p}
\times \left( \int_{B_{3/4}} (1 + |n|)^{(\theta_0+a_2)/n-\alpha_2} \, dx \right)^{(p-a+a_2)/p}.
\]

(3.19)

Since \( (a-a_2)/(p-a+a_2) < a/(p-a) \) and \( \beta' > np/(n-p) \), one has

\[
\frac{\theta_0 a}{p - a} < \frac{(p - \theta_0)n}{n-p}
\]

and so

\[
\frac{(\theta_0 + a_2)(a - a_2)}{p - a + a_2} < \frac{(\theta_0 + a_2)a}{p - a} < \frac{(p - \theta_0)n}{n-p} - a_1
\]

for all \( a_1, a_2 > 0 \) small enough.

By putting (3.16), (3.18) and (3.19) together,

\[
\int_{B_{3/4}} |Dn|^{\frac{p'(p-1)n}{\beta'(n-1)-n}-a_2} \, dx \leq c(n, N, p, a_2)
\]

(3.20)

for sufficiently small \( a_2 > 0 \).

We now combine (3.13) and (3.20) to conclude that

\[
\int_{B_{3/4}} |n|^{\frac{p'(p-1)n}{\beta'(n-1)-n}-\alpha_1} \, dx \leq c(n, N, p, a_1) \quad \text{and} \quad \int_{B_{3/4}} |Dn|^{\frac{p'(p-1)n}{\beta'(n-1)-n}-a_2} \, dx \leq c(n, N, p, a_2)
\]

for all sufficiently small \( a_1, a_2 > 0 \) (and so trivially for all larger values of \( a_1 \) and \( a_2 \)).

This verifies our claim.

\[\text{Lemma 3.4. Let } M \text{ and } \beta \text{ be as in Proposition 3.1. Let } \{u_j\}_{j \in \mathbb{N}^*} \subset W^{1,p}(B_1) \text{ satisfy}
\]

\[
\int_{B_1} |u_j| \, dx \leq 1
\]

(3.21)
and
\[
\left| \int_{B_1} |Du_j|^{p-2} Du_j : D\varphi \, dx \right| \leq M^{1-p} 2^{-j} \left( \int_{B_1} |\varphi(x)|^{\beta'} \, dx \right)^{1/\beta'}
\] (3.22)
for all \( \varphi \in W^{1,p}_0(B_1) \cap L^{\beta'}(B_1) \). Then there exists a \( \tilde{u} \in W^{1,q}(B_{3/4}) \) such that
\[
\lim_{j \to -\infty} u_j = \tilde{u} \quad \text{in } W^{1,q}(B_{3/4})
\]
for all \( q \in (1, q_0) \). Moreover,
\[
\int_{B_{1/2}} |\tilde{D}\tilde{u}|^{p-2} \tilde{D}\varphi \, dx = 0
\] (3.23)
for all \( \varphi \in C_c^\infty(B_{1/2}) \).

**Proof.** Let \( 1 < q < q_0 \) and \( q_1 = (q + q_0)/2 \). By Lemma 3.3, there exists a \( c = c(n, N, p, q) \) such that
\[
\int_{B_{3/4}} |\tilde{D}u_j|^q \, dx \leq c \quad \text{and} \quad \int_{B_{3/4}} |\tilde{D}u_j|^{q_1} \, dx \leq c
\] (3.24)
uniformly in \( j \in \mathbb{N}^n \).

For convenience we will constantly use \( c = c(n, N, p, q) \) without mentioning further, the value of which may vary from line to line.

By passing to a subsequence if necessary, we may assume there exist \( \tilde{u} \in W^{1,q}(B_{3/4}) \), \( b \in L^{q/(p-1)}(B_{3/4}) \) and \( h \in L^q(B_{3/4}) \) such that
\[
\int_{B_{3/4}} |\tilde{D}u_j|^q \, dx + \sup_j \int_{B_{3/4}} |\tilde{D}u_j|^{q_1} \, dx + \sup_j \int_{B_{3/4}} |\tilde{D}u_j|^{q_1} \, dx < \infty,
\] (3.25)
\[
\tilde{D}u_j \rightharpoonup \tilde{D}u, \quad |\tilde{D}u_j - \tilde{D}u| \rightarrow h \quad \text{weakly in } L^q(B_{3/4}),
\] (3.26)
\[
|\tilde{D}u_j|^{p-2} \tilde{D}u_j \rightharpoonup b \quad \text{weakly in } L^{q/(p-1)}(B_{3/4}) \quad \text{and}
\] (3.27)
\[
\tilde{u}_j \longrightarrow \tilde{u} \quad \text{strongly in } L^q(B_{3/4}) \quad \text{and pointwise in } B_{3/4}.
\] (3.28)

As a consequence of (3.21) and (3.24) we have
\[
\int_{B_{3/4}} |\tilde{u}| \, dx \leq 2^q \quad \text{and} \quad \int_{B_{3/4}} |\tilde{D}u|^q \, dx \leq c.
\] (3.29)

Next we aim to prove that \( h = 0 \) almost everywhere, from which the lemma follows at once. To this end it suffices to show that
\[
h(\overline{x}) = 0
\] (3.30)
for all $\bar{x} \in B_{3/4}$ which is a Lebesgue point simultaneously for $\bar{u}$, $D\bar{u}$, $h$ and $b$, that is,

$$\lim_{\theta \to 0} \int_{B_{\theta}(\bar{x})} \left[ |\bar{u} - \bar{u}(\bar{x})| + |D\bar{u} - D\bar{u}(\bar{x})| + |h - h(\bar{x})| + |b - b(\bar{x})|^{1/(p-1)} \right]^q \, dx = 0 \quad (3.31)$$

and

$$|\bar{u}(\bar{x})| + |D\bar{u}(\bar{x})| + |h(\bar{x})| + |b(\bar{x})| < \infty. \quad (3.32)$$

To see this, with (3.30) in mind, $D\bar{u}_j \to D\bar{u}$ strongly in $L^1(B_{3/4})$. Whence the second bound in (3.29) and interpolation yield

$$\left\| D\bar{u}_j - D\bar{u}\right\|_{L^q(B_{3/4})} \leq \left\| D\bar{u}_j - D\bar{u}\right\|_{L^1(B_{3/4})} \left\| D\bar{u}_j - D\bar{u}\right\|_{L^{q^*}(B_{3/4})}^{1-\theta} \to 0,$$

where $\theta$ is such that $1/q = \theta + (1 - \theta)/q_1$.

Now back to the proof of (3.30), let $\bar{x} \in B_{3/4}$ be a simultaneous Lebesgue point for $\bar{u}$, $D\bar{u}$, $h$ and $b$. Set

$$\alpha_\sigma(x) := (\bar{u})_{B_\sigma(\bar{x})} + D\bar{u}(\bar{x}) \cdot (x - \bar{x})$$

for all $\sigma \in (0, 3/4)$. Poincare’s inequality for $\alpha_\sigma$ implies

$$\lim_{\sigma \to 0} \int_{B_\sigma(\bar{x})} \left| \frac{\bar{u} - \alpha_\sigma}{\sigma} \right|^q \, dx \leq c \lim_{\sigma \to 0} \int_{B_\sigma(\bar{x})} |D\bar{u} - D\bar{u}(\bar{x})|^q \, dx = 0. \quad (3.33)$$

By (3.26) we have

$$h(\bar{x}) = \lim_{\sigma \to 0} \lim_{j \to \infty} \int_{B_{\sigma/2}(\bar{x})} |D\bar{u}_j - D\bar{u}| \, dx$$

$$= \lim_{\sigma \to 0} \lim_{j \to \infty} \int_{B_{\sigma/2}(\bar{x})} 1_{|\bar{u}_j - \alpha_\sigma| < \sigma} |D\bar{u}_j - D\bar{u}| \, dx + \lim_{\sigma \to 0} \lim_{j \to \infty} \int_{B_{\sigma/2}(\bar{x})} 1_{|\bar{u}_j - \alpha_\sigma| \geq \sigma} |D\bar{u}_j - D\bar{u}| \, dx$$

$$=: I + II. \quad (3.34)$$

We aim to show that $I = II = 0$. For this we estimate each term separately. Term $II$ turns out to be easier to estimate so we do it first.

**Term $II$:** We first show that

$$\lim_{j \to \infty} \int_{B_{\sigma/2}(\bar{x})} 1_{|\bar{u}_j - \alpha_\sigma| \geq \sigma} |D\bar{u}_j - D\bar{u}| \, dx \leq \int_{B_{\sigma/2}(\bar{x})} 1_{|\bar{u} - \alpha_\sigma| \geq \sigma} h \, dx. \quad (3.35)$$
To this end note that
\[
\int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma]} |D\vec{u}_j - D\vec{u}| \, dx \leq \int_{\mathcal{B}_1(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma/2]} |D\vec{u}_j - D\vec{u}| \, dx
\]
\[
+ \int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma/2]} |D\vec{u}_j - D\vec{u}| \, dx.
\]

By invoking (3.25) and (3.28) one has
\[
\int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma]} |D\vec{u}_j - D\vec{u}| \, dx \leq \left( \int_{\mathcal{B}_1(\mathcal{X})} |D\vec{u}_j - D\vec{u}|^q \, dx \right)^{1/q} \left( \frac{\{ |x \in B_{3/4} : |\vec{u}_j - \vec{u}| | \geq \sigma/2 \}}{|B_{\sigma/2}(\mathcal{X})|} \right)^{1/q'} \quad \text{as } j \to \infty
\]
\[
\leq 0.
\]

This justifies (3.35).

Next we use (3.31), (3.32) and (3.33) to obtain
\[
\int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma/2]} |h| \, dx \leq \left( \int_{\mathcal{B}_1(\mathcal{X})} |h|^q \, dx \right)^{1/q} \left( \int_{\mathcal{B}_1(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| \geq \sigma/2]} \, dx \right)^{1/q'}
\]
\[
\leq \left[ \left( \int_{\mathcal{B}_1(\mathcal{X})} |h - h(\mathcal{X})|^q \, dx \right)^{1/q} + h(\mathcal{X}) \right] \left( \int_{\mathcal{B}_1(\mathcal{X})} \left| \frac{\vec{u} - \alpha_\sigma}{\sigma} \right|^q \, dx \right)^{1/q'} \quad \text{as } \sigma \to 0
\]
\[
\leq 0.
\]

Hence $II = 0$.

**Term I:** One has
\[
\int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u}_j - D\vec{u}| \, dx \leq \int_{\mathcal{B}_1(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u}_j - D\alpha_\sigma| \, dx
\]
\[
+ 2^n \int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u} - D\alpha_\sigma| \, dx.
\]

Since
\[
\lim_{\sigma \to 0} \lim_{j \to \infty} \int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u} - D\alpha_\sigma| \, dx \leq \lim_{\sigma \to 0} \int_{\mathcal{B}_1/2(\mathcal{X})} |D\vec{u} - D\vec{u}(\mathcal{X})| \, dx = 0
\]
by (3.31), it remains to show that
\[
\lim_{\sigma \to 0} \lim_{j \to \infty} \int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u}_j - D\alpha_\sigma| \, dx = 0. \tag{3.36}
\]

By Holder’s inequality,
\[
\int_{\mathcal{B}_1/2(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} |D\vec{u}_j - D\alpha_\sigma| \, dx \leq \left( \int_{\mathcal{B}_1(\mathcal{X})} 1_{[|\vec{u}_j - \vec{u}| < \sigma]} \left( |D\vec{u}_j| + |D\alpha_\sigma| \right)^{-p} \, dx \right)^{1/p} \left( \int_{\mathcal{B}_1(\mathcal{X})} |D\vec{u}_j - D\alpha_\sigma|^2 \, dx \right)^{1/2}
\]
\[
\times \left( \int_{\mathcal{B}_1(\mathcal{X})} \left( |D\vec{u}_j| + |D\alpha_\sigma| \right)^{-2p} \, dx \right)^{1/2}.
\]
The second integral on the right-hand side is bounded uniformly in \( j \) due to (3.24) and (3.29). Hence to achieve (3.36), it suffices to show that

\[
\lim_{\sigma \to 0} \lim_{j \to \infty} \int_{B_{\sigma/2}(\overline{x})} 1_{|\overline{\nu}_j - \alpha_\sigma| < \sigma} \left( |D\overline{\nu}_j| + |D\alpha_\sigma| \right)^{p-2} |D\overline{\nu}_j - D\alpha_\sigma|^2 \, dx = 0.
\]

To this end, let \( \phi \in C_c^\infty(B_{\sigma}(\overline{x})) \) be such that

\[
0 \leq \phi \leq 1, \quad \phi|_{B_{\sigma/2}(\overline{x})} = 1 \quad \text{and} \quad |D\phi| \leq \frac{4}{\sigma}.
\]

Set \( \eta := \phi T_\sigma(\overline{u}_j - \alpha_\sigma) \), where \( T_\sigma \) is defined by (3.6). It follows from (3.7) that

\[
\left( |D\overline{\nu}_j|^{p-2}D\overline{\nu}_j - |D\alpha_\sigma|^{p-2}D\alpha_\sigma \right) \cdot D\eta
\]

\[
= 1_{|\overline{\nu}_j - \alpha_\sigma|} \left( \left( |D\overline{\nu}_j|^{p-2}D\overline{\nu}_j - |D\alpha_\sigma|^{p-2}D\alpha_\sigma \right) \cdot D(\overline{u}_j - \alpha_\sigma) \right) \phi
\]

\[
+ 1_{|\overline{\nu}_j - \alpha_\sigma|} \left( \left( |D\overline{\nu}_j|^{p-2}D\overline{\nu}_j - |D\alpha_\sigma|^{p-2}D\alpha_\sigma \right) \cdot (I - P_j)D(\overline{u}_j - \alpha_\sigma) \right) \phi
\]

\[
+ \left( |D\overline{\nu}_j|^{p-2}D\overline{\nu}_j - |D\alpha_\sigma|^{p-2}D\alpha_\sigma \right) \cdot [T_\sigma(\overline{u}_j - \alpha_\sigma) \otimes D\phi]
\]

\[
=: G_{j,\sigma}^1(x) + G_{j,\sigma}^2(x) + G_{j,\sigma}^3(x),
\]

where

\[
P_j := \frac{(\overline{u}_j - \alpha_\sigma) \otimes (\overline{u}_j - \alpha_\sigma)}{|\overline{u}_j - \alpha_\sigma|^2} \quad \text{and} \quad P := \frac{(\overline{u} - \alpha_\sigma) \otimes (\overline{u} - \alpha_\sigma)}{|\overline{u} - \alpha_\sigma|^2}.
\]

Since \( \alpha_\sigma \) is affine, one has

\[
\int_{B_i} |D\alpha_\sigma|^{p-2}D\alpha_\sigma \cdot D\eta \, dx = 0.
\]

Therefore

\[
0 \leq \int_{B_\sigma(\overline{x})} G_{j,\sigma}^1(x) \, dx \leq 2^{-j} a^{1-n} - \int_{B_\sigma(\overline{x})} G_{j,\sigma}^2(x) \, dx - \int_{B_\sigma(\overline{x})} G_{j,\sigma}^3(x) \, dx,
\]

(3.37)

where we used the monotonicity of the vector field \( z \mapsto |z|^{p-2}z \) in the first step.

Next we estimate the two integrals on the right-hand side of the above inequality.

**Integral of \( G_{j,\sigma}^3 \):** First we deduce from (3.24) that \( \{|D\overline{\nu}_j|^{p-2}D\overline{\nu}_j\}_{j \in \mathbb{N}^*} \) is bounded in \( L^{\frac{q}{q-p-1}} \). This together with (3.24) and (3.28) imply that

\[
\lim_{j \to \infty} \int_{B_\sigma(\overline{x})} G_{j,\sigma}^3(x) \, dx = \int_{B_\sigma(\overline{x})} (b - |D\alpha_\sigma|^{p-2}D\alpha_\sigma) \cdot [T_\sigma(\overline{u} - \alpha_\sigma) \otimes D\phi] \, dx.
\]
Holder’s inequality then gives

\[
\left| \int_{B_{r}(\Omega)} (b - |D\alpha|)^{p-2} D\alpha \right| \leq c \left( \int_{B_{r}(\Omega)} b(b(x) |q/(p-1) + |b(x) |q/(p-1) + |D\tilde{x}(x)|^{q} dx \right)^{\frac{p}{q}} \times \left( \int_{B_{r}(\Omega)} \left( \frac{\min\{|\tilde{\alpha}_{\Omega} - \alpha_{\sigma}|\}}{\sigma} \right)^{\frac{q}{q-(p-1)}} dx \right)^{\frac{p}{q}}.
\]

Note that the first integral on the right-hand side is bounded. For the second integral, we have

\[
\int_{B_{r}(\Omega)} \left( \frac{\min\{|\tilde{\alpha}_{\Omega} - \alpha_{\sigma}|\}}{\sigma} \right)^{q/(q-(p-1))} dx \leq \int_{B_{r}(\Omega)} \left( \frac{\min\{|\tilde{\alpha}_{\Omega} - \alpha_{\sigma}|\}}{\sigma} \right)^{q} dx \leq \int_{B_{r}(\Omega)} \left( \frac{\tilde{\alpha}_{\Omega} - \alpha_{\sigma}}{\sigma} \right)^{q} dx \rightarrow 0,
\]

where we used the fact that \(\frac{q}{q-(p-1)} > q\) and \((3.33)\) in the first and second steps respectively.

Consequently

\[
\lim_{\sigma \to 0} \lim_{j \to \infty} \left| \int_{B_{r}(\Omega)} G_{j,\sigma}^{3} (x) dx \right| = 0.
\]

**Integral of** \(G_{j,\sigma}^{2}\): We have \(D\tilde{u}_{j} : (I - P_{j})D\tilde{u}_{j} \geq 0\) by a similar argument to that of \((3.8)\). Therefore

\[
|D\tilde{u}_{j}|^{p-2} D\tilde{u}_{j} - |D\alpha_{\sigma}|^{p-2} D\alpha_{\sigma} \geq (I - P_{j})D(\tilde{u}_{j} - \alpha_{\sigma}).
\]

\[
(3.38)
\]

Observe also that \(1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}| \geq \alpha_{\sigma}} P_{j} \to 1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}| \geq \alpha_{\sigma}} P\) a.e. and hence strongly in \(L^{s}(B_{3/4})\) for every \(s \geq 1\). The same also applies to the convergence \(1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|} |\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|^{-1} \to 1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|} |\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|^{-1}\). These in combination with \((3.38)\) and \((3.27)\) yield that

\[
\limsup_{j \to \infty} \left( - \int_{B_{r}(\Omega)} G_{j,\sigma}^{2} (x) dx \right) \leq \int_{B_{r}(\Omega)} b : (I - P)D\alpha_{\sigma} \frac{\sigma 1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|}}{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|} dx + \int_{B_{r}(\Omega)} |D\alpha_{\sigma}|^{p-2} D\alpha_{\sigma} : (I - P)D(\tilde{\alpha}_{\sigma} - \alpha_{\sigma}) \frac{\sigma 1_{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|}}{|\tilde{\alpha}_{\sigma} - \alpha_{\sigma}|} dx.
\]

Next we estimate each on the right-hand side separately. As \(q > p - 1\) there exists an \(s > 1\) such that
At the same time,

\[ \frac{q(s-1)}{q-p+1} \leq q. \]

Keeping in mind (3.33) one has

\[
\left| \int_{B_r(x)} b : (I - P)D\sigma \frac{1_{[\tilde{u} - \alpha_\sigma, \sigma\tilde{u}]} d\sigma}{|\tilde{u} - \alpha_\sigma|} \right| \leq c \int_{B_r(x)} |b| \frac{\left| \tilde{u} - \alpha_\sigma \right|^{s-1}}{\sigma} \, d\sigma
\]

\[
\leq c \left( \int_{B_r(x)} |b|^{q/(p-1)} \, d\sigma \right)^{(p-1)/q} \left( \int_{B_r(x)} \frac{|\tilde{u} - \alpha_\sigma|^{q}}{\sigma} \, d\sigma \right)^{(1-1/q)}
\]

\[
\sigma \to 0.
\]

At the same time,

\[
\int_{B_r(x)} |D\alpha_\sigma|^p \, d\sigma : (I - P)D(\tilde{u} - \alpha_\sigma) \frac{1_{[\tilde{u} - \alpha_\sigma, \sigma\tilde{u}]} d\sigma}{|\tilde{u} - \alpha_\sigma|}
\]

\[
\leq c \int_{B_r(x)} |D(\tilde{u} - \alpha_\sigma)| \frac{|\tilde{u} - \alpha_\sigma|^{q-1}}{\sigma} \, d\sigma
\]

\[
\leq c \left( \int_{B_r(x)} |D\tilde{u} - D\tilde{u}(x)|^q \, d\sigma \right)^{1/q} \left( \int_{B_r(x)} \frac{|\tilde{u} - \alpha_\sigma|^{q}}{\sigma} \, d\sigma \right)^{1-1/q}
\]

\[
\sigma \to 0.
\]

As a consequence,

\[
\limsup_{\sigma \to 0} \limsup_{j \to \infty} \left( - \int_{B_r(x)} G_{j,\sigma}^2(x) \, dx \right) \leq 0.
\]

This finishes our estimate for the integral of \( G_{j,\sigma}^2 \).

Continuing with (3.37), we conclude that

\[
\limsup_{\sigma \to 0} \limsup_{j \to \infty} \int_{B_r(x)} G_{j,\sigma}^1(x) \, dx = 0. \quad (3.39)
\]

We proceed with the proof of (3.36). It follows from (3.39) and Lemma 2.1 that

\[
\limsup_{\sigma \to 0} \limsup_{j \to \infty} \int_{B_r(x)} \frac{1_{[\tilde{u} - \alpha_\sigma, \sigma\tilde{u}]} (|D\tilde{u} - D\sigma_\sigma|)^{p-2} |D\tilde{u}_j - D\alpha_\sigma|^{2} \phi \, d\sigma = 0.
\]

Hence \( I = 0 \).

That \( h(x) = 0 \) now follows from (3.34), whence \( Du \in L^q(B_{3/4}) \). Lastly, we let \( j \to \infty \in (3.22) \) to obtain (3.25). This completes our proof.

We now have enough preparation to derive Proposition 3.1.
Proof of Proposition 3.1. We proceed via a proof by contradiction. Our arguments follow \[\text{KMT18}^\text{Step 5 in Proof of Theorem 4.1}\] closely.

For a contradiction, assume that there exist an \(\epsilon > 0\) and sequences of balls \(\{B_{r_j}(x_j)\}_{j \in \mathbb{N}^*}\) and \(\{u_j\}_{j \in \mathbb{N}^*} \subseteq W^{1,p}(B_{r_j}(x_j))\) such that

\[
\int_{B_{r_j}(x_j)} |u_j| \, dx \leq M r_j \quad \text{and} \quad \left| \int_{B_{r_j}(x_j)} |Du_j|^{p-2} Du_j : D\phi \, dx \right| \leq \frac{2^{-j}}{r_j} \|\phi\|_{L^p(B_{r_j}(x_j))} \tag{3.40}
\]

for all \(\phi \in W^{1,p}_0(B_{r_j}(x_j)) \cap L^{p^*}(B_{r_j}(x_j))\), whereas

\[
\left( \frac{\int_{B_{r_j/2}(x_j)} |Du_j - Dv|^q \, dx}{r_j} \right)^{1/q} > \epsilon
\]

for all \(v \in W^{1,p}(B_{r_j/2}(x_j))\) being \(p\)-harmonic in \(B_{r_j}(x_j)\) and satisfying

\[
\int_{B_{r_j/2}(x_j)} |v| \, dx \leq 2^n M r_j \quad \text{and} \quad \left( \frac{\int_{B_{r_j/2}(x_j)} |Dv|^q \, dx}{r_j} \right)^{1/q} \leq \left( \frac{2^n \epsilon}{|B_1|} \right)^{1/q} M
\]

for all \(q \in (1, q_0)\), where \(c = c(n, N, p, q)\).

For the rest of the proof, \(c\) will always denote a constant depending on \(n, N, p, q\) only whose value may vary from line to line.

We first perform a scaling on \(u_j\) for all \(j \in \mathbb{N}\). For convenience, we denote \(u_0 = u\). For each \(j \in \mathbb{N}\) and \(\varphi \in W^{1,p}_0(B) \cap L^{p^*}(B)\) let

\[
\bar{u}_j(x) = \frac{u_j(x_0 + r x)}{M r} \quad \text{and} \quad \eta(x) = \frac{\varphi(x_0 + r x)}{r}.
\]

Then (3.2), (3.3) and (3.40) become

\[
\int_{B_1} |\bar{u}_j| \, dx \leq 1 \tag{3.41}
\]

and

\[
\left| \int_{B_1} |D\bar{u}_j|^{p-2} D\bar{u}_j : D\eta \, dx \right| \leq M^{1-p} \delta_j \|\eta\|_{L^{p^*}(B_1)}, \tag{3.42}
\]

where

\[
\delta_j := \begin{cases} 
\delta & \text{if } j = 0, \\
2^{-j} & \text{otherwise}. 
\end{cases}
\]

It follows from Lemma 3.3 that

\[
\|\bar{u}_j\|_{W^{1,q}(B_{1/2})} \leq c
\]
for all \( q \in (1, q_0) \) and \( j \in \mathbb{N} \).

Using Lemma [3.4] there exists a \( \tilde{u} \in W^{1,q}(B_{3/4}) \) such that

\[
\lim_{j \to \infty} u_j = \tilde{u} \quad \text{in } W^{1,q}(B_{3/4})
\]

for all \( q \in (1, q_0) \) with the property that

\[
\int_{B_{1/2}} |D\tilde{u}|^{q-2} D\tilde{u} : D\varphi \, dx = 0
\]

for all \( \varphi \in C_c^\infty(B_{1/2}) \).

We aim to show that \( \tilde{u} \) is \( p \)-harmonic. In particular, we will show that \( D\tilde{u} \in L^p(B_{1/2}) \).

Let \( \phi \in C_c^\infty(B_{3/4}) \) be such that \( 0 \leq \phi \leq 1 \) and \( \phi \vert_{B_{1/2}} = 1 \). It follows from (3.9) that

\[
\int_{B_1 \cap \{ |\tilde{u}| < t \}} |D\overline{\tilde{u}}|^p \phi \, dx \leq c \int_{B_1 \cap \{ |\tilde{u}| < t \}} |\overline{\tilde{u}}|^p \vert D\phi \vert^p \, dx + c M^{1-p} \delta_j \rho^0 \left\| \overline{\tilde{u}} \phi^{p-1} \right\|_{L^{p/(1-\theta)}(B_1)}^{1-\theta} + ct \int_{B_1 \cap \{ |\tilde{u}| \geq t \}} |D\overline{\tilde{u}}|^{p-1} \vert D\phi \vert \phi^{p-1} \, dx.
\]

By taking the inferior limit both sides of this inequality when \( j \to \infty \) and then referring to Fatou’s lemma for the left-hand side, one has

\[
\int_{B_{3/4} \cap \{ |\tilde{u}| < t \}} |D\tilde{u}|^p \phi \, dx \leq c \int_{B_{3/4} \cap \{ |\tilde{u}| < t \}} |\tilde{u}|^p \vert D\phi \vert^p \, dx + ct \int_{B_{3/4} \cap \{ |\tilde{u}| \geq t \}} |D\tilde{u}|^{p-1} \vert D\phi \vert \phi^{p-1} \, dx
\]

for all \( t > 0 \).

Next let \( \gamma \in (0, 1) \). By multiplying the above inequality by \( (1 + t)^{-1-\gamma} \), integrating over \( (0, \infty) \) with respect to \( t \) and then invoking Fubini’s theorem we arrive at

\[
\frac{1}{\gamma} \int_{B_{3/4}} \frac{|D\tilde{u}|^p \phi \rho}{(1 + |\tilde{u}|)^\gamma} \, dx \leq \frac{c}{\gamma} \int_{B_{3/4}} (1 + |\tilde{u}|)^{-\gamma} \vert D\phi \vert^p \, dx
\]

\[
+ c \int_0^\infty \frac{1}{(1 + t)^\gamma} \int_{B_{3/4} \cap \{ |\tilde{u}| \geq t \}} |D\tilde{u}|^{p-1} \vert D\phi \vert \phi^{p-1} \, dx \, dt.
\]

To handle the second integral on the right-hand side of this inequality, an application of Fubini’s theorem and
Young’s inequality gives

\[ c \int_0^\infty \frac{1}{(1 + t)^\gamma} \int_{B_{3/4} \cap \{|\tilde{u}| \geq t\}} |D\tilde{u}|^{p-1} |D\phi|^{p-1} dx dt \leq \frac{c}{1 - \gamma} \int_{B_{3/4}} |D\tilde{u}|^{p-1} (1 + |\tilde{u}|)^{1-\gamma} |D\phi|^{p-1} dx \]

\[ \leq \frac{1}{2\gamma} \int_{B_{3/4}} \frac{|D\tilde{u}|^{p} \phi}{(1 + |\tilde{u}|)^{\gamma}} dx \]

\[ + \frac{c\gamma^{p-1}}{(1 - \gamma)^p} \int_{B_{3/4}} (1 + |\tilde{u}|)^{p-\gamma} |D\phi|^p dx. \]

Hence

\[ \int_{B_{3/4}} \frac{|D\tilde{u}|^{p} \phi}{(1 + |\tilde{u}|)^{\gamma}} dx \leq \frac{c}{(1 - \gamma)^p} \int_{B_{3/4}} (1 + |\tilde{u}|)^{p-\gamma} |D\phi|^p dx. \] (3.43)

From this there are two possibilities. If \( n < p^2 \) then \( p < q_0 \), from which it follows that \( u \in L^p(B_{3/4}) \). So taking \( \gamma \to 0 \) in (3.43) yields \( Du \in L^p(B_{1/2}) \). It remains to consider \( p^2 \leq n \). In this case choose \( \gamma \geq \frac{n-p^2}{n-p} \).

Using the fact that \( \tilde{u} \in W^{1,q}(B_{3/4}) \) for all \( q \in (1, q_0) \) we deduce that right-hand side in (3.43) is finite.

Since

\[ |D((1 + |\tilde{u}|)^{\frac{\gamma}{p}})\phi|^p \leq \left(1 - \frac{\gamma}{p}\right)^p |D\tilde{u}|^{p} (1 + |\tilde{u}|)^{-\gamma}, \]

(3.43) implies that

\[ \int_{B_{3/4}} \left|D\left((1 + |\tilde{u}|)^{\frac{\gamma}{p}} \phi\right)\right|^p dx \leq \frac{c}{(1 - \gamma)^p} \int_{B_{3/4}} (1 + |\tilde{u}|)^{p-\gamma} |D\phi|^p dx. \] (3.44)

Set \( \theta = \frac{n}{n-p} = \frac{p^*}{p} \), where \( p^* \) denotes the Sobolev’s exponent. Using Sobolev’s inequality and (3.44), we obtain

\[ \left( \int_{B_{3/4}} (1 + |\tilde{u}|)^{1-\gamma/p} \phi \right)^{\theta_p} dx \right)^{1/\theta} \leq \frac{c}{(1 - \gamma)^p} \int_{B_{3/4}} (1 + |\tilde{u}|)^{p-\gamma} \phi |D\phi|^p dx. \] (3.45)

Next we use an iterating argument in the spirit of (finite) Moser’s interation to derive the claim. Define

\[ q_j = \theta^j(p - \gamma), \quad \gamma_j = p - q_j, \quad B_j = B_{3/8 + 1/(j+1)} \]

and correspondingly choose \( \{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(B') \) such that

\[ 0 \leq \phi_j \leq 1, \quad \phi_{j+1} \leq \phi_j \quad \text{and} \quad \phi_j|_{B_{3/4}} = 1 \]

for all \( j \in \mathbb{N} \). Note that \( \{\gamma_j\}_{j \in \mathbb{N}} \) is decreasing.
Now (3.45) reads
\[
\left( \int_{B_{3/4}} (1 + |\tilde{u}|)^{\theta(p-\gamma_j)} \phi_j^{\theta_p} \right)^{1/\theta} \leq c \int_{B_{3/4}} (1 + |\tilde{u}|)^{p-\gamma_j} |D\phi_j|^\rho \, dx
\]
for all \( j \in \mathbb{N} \), provided that \( \gamma_j > 0 \). In other words \( u \in L^\rho(B/r) \) implies \( u \in L^\rho_0(B_1) \) for all \( j \in \mathbb{N} \) such that \( \gamma_j > 0 \).

Let \( j_0 \in \mathbb{N} \) be the smallest number such that \( \gamma_{j_0+1} \leq 0 \). Then \( u \in L^\rho(B_0) \). This in particular yields \( \tilde{u} \in L^\rho(B_{3/8}) \). Combining this with (3.43) and then taking the limit when \( r \to 0 \) give \( D\tilde{u} \in L^\rho(B_{1/2}) \).

The claim now follows by reversing the scaling process at the beginning of the proof. \( \blacksquare \)

The following lemmas are direct consequences of Proposition 3.1.

**Lemma 3.5.** Let \( \beta \in (1, \infty) \) be such that
\[
\frac{np}{n - p} < \beta' < \frac{n}{n(2 - p) - 1}.
\]

Let \( B = B_r(x_0) \) be a ball and \( f \in L^\beta(B) \). Let \( u \in W^{1,\beta}(B) \) be a weak solution to (1.1) in \( B \). Let \( \epsilon \in (0, 1) \) and \( q \in (1, q_0) \), where \( q_0 \) is defined in (3.1). Then there exist \( \delta = \delta(n, \delta, \beta, \epsilon) \) \((0, 1) \) and a \( p \)-harmonic map \( v \) in \( \frac{1}{2}B \) such that
\[
\left( \int_{\frac{1}{2}B} |Du - Dv|^q \, dx \right)^{1/q} \leq \frac{\epsilon}{r} \int_B |u - (u)_B| \, dx + \frac{\epsilon}{\delta^{1/(p-1)}} \left[ r \left( \int_B |f|^{\beta} \, dx \right)^{1/\beta} \right]^{1/(p-1)}.
\]  
(3.46)

**Proof.** We use a scaling argument with
\[
\bar{u} := \frac{u - (u)_B}{\lambda} \quad \text{and} \quad \bar{f} := \frac{f}{\lambda^{p-1}},
\]  
(3.47)

where
\[
\lambda := \frac{1}{r} \int_B |u - (u)_B| \, dx + \left[ \frac{r}{\delta} \left( \int_B |f|^{\beta} \, dx \right)^{1/\beta} \right]^{1/(p-1)}
\]

and \( \delta = \delta(n, \rho, p, q, \epsilon) \) is given in Proposition 3.1 with \( M = 1 \).

It follows that
\[
\int_B |\bar{u}| \, dx \leq r \quad \text{and} \quad -\Delta_\rho \bar{F} = \bar{f} \quad \text{in} \ B.
\]

If \( \lambda = 0 \) then \( u \) is constant and so we can choose \( v = u \).
Next assume that $\lambda > 0$. We have
\[
\left| \int_B |D\overline{u}|^{p-2}D\overline{u} : D\varphi \, dx \right| \leq \frac{1}{2^{p-1}} \left( \int_B |\varphi|^{\theta'} \, dx \right)^{1/\theta'} \left( \int_B |f|^{\theta} \, dx \right)^{1/\theta} \leq \frac{\delta}{r} \left( \int_B |\varphi|^{\theta'} \, dx \right)^{1/\theta'},
\]
for all $\varphi \in W^{1,\theta}_0(B) \cap L^\theta(B)$. Therefore by Proposition 3.1 there exists a $p$-harmonic map $\overline{v}$ in $\frac{1}{2}B$ such that
\[
\left( \int_{\frac{1}{2}B} |D\overline{u} - D\overline{v}|^q \, dx \right)^{1/q} \leq \varepsilon.
\]
Scaling back to $u$ with $v = \lambda \overline{v}$ we obtain (3.46). To finish note that $v$ is $p$-harmonic.

**Proposition 3.6.** Adopt the assumptions and notation in Lemma 3.5 Then there exist constants
\[
\delta = \delta(n, N, p, q, \varepsilon) \in (0, 1), C = C(n, p, q) > 0
\]
and a $p$-harmonic map $v \in W^{1,\theta}(\frac{1}{2}B)$ such that
\[
\left( \int_{\frac{1}{2}B} |Du - Dv|^q \, dx \right)^{1/q} \leq \frac{\varepsilon}{\delta^{1/(p-1)}} \left[ r \left( \int_B |f|^{\theta} \, dx \right)^{1/\theta} \right]^{1/(p-1)} + \varepsilon \left( \int_B |Du|^q \, dx \right)^{1/q}
\]
and
\[
\|Dv\|_{L^\infty(\frac{1}{2}B)} \leq \frac{CE}{\delta^{1/(p-1)}} \left[ r \left( \int_B |f|^{\theta} \, dx \right)^{1/\theta} \right]^{1/(p-1)} + C(1 + \varepsilon) \left( \int_B |Du|^q \, dx \right)^{1/q}.
\]

**Proof.** Using Lemma 3.5 Poincare’s and Holder’s inequalities, there exists a $p$-harmonic map $v \in W^{1,\theta}(\frac{1}{2}B)$ such that
\[
\left( \int_{\frac{1}{2}B} |Du - Dv|^q \, dx \right)^{1/q} \leq \varepsilon \left( \int_B |Du|^q \, dx \right)^{1/q} + \frac{\varepsilon}{\delta^{1/(p-1)}} \left[ r \left( \int_B |f|^{\theta} \, dx \right)^{1/\theta} \right]^{1/(p-1)}.
\]
Next it follows from [KM18, (3.6)] that
\[
\|Dv\|_{L^\infty(\frac{1}{2}B)} \leq C \int_{\frac{1}{2}B} |Dv| \, dx \leq C \left( \int_{\frac{1}{2}B} |Dv|^q \, dx \right)^{1/q}
\]
for a constant $C = C(n, p, q)$.

The claim now follows by combining these two estimates together.
4 Good-$\lambda$ type bounds

In this section we present a good-$\lambda$-type estimate - Proposition 4.3. In order to do this, we need two auxiliary results.

The first one can be viewed as a (weighted) substitution for the Calderon-Zygmund-Krylov-Safonov decomposition (cf. [MP11]).

Lemma 4.1. Let $\omega$ be an $A_\infty$-weight and $B$ be a ball of radius $R$ in $\mathbb{R}^n$. Let $E \subseteq F \subseteq B$ be measurable and $\varepsilon \in (0, 1)$ satisfy the following property:

(i) $\omega(E) < \varepsilon \omega(B)$.

(ii) $\omega(E \cap B_{\rho}(x)) \geq \varepsilon \omega(B_{\rho}(x))$ implies $B_{\rho}(x) \cap F \subset F$ for all $x \in B$ and $\rho \in (0, R)$.

Then there exists a $C = C(n, [\omega]_{A_\infty})$ such that $\omega(E) \leq C\varepsilon \omega(F)$.

The next result is a variation of Lemma 4.1.

Lemma 4.2. Let $\omega$ be an $A_\infty$-weight. Let $E \subseteq F$ be measurable and $\varepsilon \in (0, 1)$ satisfy the following property:

For all $x \in \mathbb{R}^n$ and $R \in (0, \infty)$, one has

$$\omega(E \cap B_R(x)) \geq \varepsilon \omega(B_R(x)) \quad \text{implies} \quad B_R(x) \subset F.$$ \hspace{1cm} (4.1)

Then there exists a $C = C(n, [\omega]_{A_\infty})$ such that $\omega(E) \leq C\varepsilon \omega(F)$.

Proof. Without loss of generality, we may assume that $\omega(E) \vee \omega(F) < \infty$. Let $x_0 \in \mathbb{R}^n$ and $R$ be sufficiently large such that $\omega(E) < \varepsilon \omega(B_R(x_0))$. Set $S = E \cap B_R(x_0)$ and $T = F \cap B_R(x_0)$. The claim follows directly from Lemma 4.1 with $S$, $T$, $B_R(x_0)$ and $\varepsilon$.

Indeed, we have $\omega(S) \leq \omega(E) < \varepsilon \omega(B_R(x_0))$. Assume that $x \in B_R(x_0)$ and $\rho \in (0, R]$ satisfy

$$\omega(S \cap B_{\rho}(x)) \geq \varepsilon \omega(B_{\rho}(x)).$$

Obviously we also have

$$\omega(E \cap B_{\rho}(x)) \geq \varepsilon \omega(B_{\rho}(x)).$$
Now we let \( R \) tend to infinity to complete the proof.

Recall the maximal function defined by

\[
M_\beta(f)(x) = \sup_{\rho>0} \rho^\beta \int_{B_\rho(x)} |f(y)| \, dy
\]

for all \( x \in \mathbb{R}^n \), \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \beta \in [0, n] \). The case \( \beta = 0 \) corresponds to the usual Hardy-Littlewood maximal function \( M = M_0 \).

We now turn to the aforementioned good-\( \lambda \)-type estimate.

**Proposition 4.3.** Let \( \omega \in \Lambda_\infty \), \( \epsilon > 0 \) and \( q \in (1, q_0) \). Let \( \beta \in (1, \infty) \) be such that \( \frac{np}{n-p} < \beta' < \frac{n}{n-2} \) and \( f \in L^\beta(\mathbb{R}^n) \). Then there exist constants

\[
C = C(n, [\omega]_{\Lambda_\infty}), \quad \Lambda_0 = \Lambda_0(n, p, q) > 3^n/q \quad \text{and} \quad \delta = \delta(n, p, q, \epsilon, [\omega]_{\Lambda_\infty}) \in (0, 1),
\]

such that

\[
\omega \left\{ x \in \mathbb{R}^n : \left( M(|Du|^q)(x) \right)^{1/q} > \Lambda_0 \lambda, \quad \left( M_\beta(|f|^\beta)(x) \right)^{1/(p-1)\beta} \leq \delta^{1/(p-1)} \lambda \right\} \leq C \epsilon \omega \left( \left\{ x \in \mathbb{R}^n : \left( M(|Du|^q)(x) \right)^{1/q} > \lambda \right\} \right)
\]

for all \( \lambda > 0 \).

**Proof.** Set

\[
E_{\lambda, \delta} = \left\{ y \in \mathbb{R}^n : \left( M(|Du|^q)(y) \right)^{1/q} > \Lambda_0 \lambda, \quad \left( M_\beta(|f|^\beta)(y) \right)^{1/(p-1)\beta} \leq \delta^{1/(p-1)} \lambda \right\}
\]

and

\[
F_\lambda = \left\{ y \in \mathbb{R}^n : \left( M(|Du|^q)(y) \right)^{1/q} > \lambda \right\}
\]

for each \( \delta \in (0, 1) \) and \( \lambda > 0 \). Here \( \Lambda_0 = \Lambda_0(n, p, q) \) is to be chosen later.

We will use Lemma 4.2 for \( E_{\lambda, \delta} \) and \( F_\lambda \). That is, we will verify that

\[
\omega(E_{\lambda, \delta} \cap B_r(x)) \geq \epsilon \omega(B_r(x)) \quad \Longrightarrow \quad B_r(x) \subset F_\lambda
\]
for all $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\lambda > 0$, provided that $\delta$ is sufficiently small.

Indeed, let $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\lambda > 0$. To avoid triviality, we consider $E_{\lambda, \delta} \cap B_r(x) \neq \emptyset$. By contraposition, assume that $B_r(x) \cap F_{\delta}^r \neq \emptyset$. Then there exist $x_1, x_2 \in B_r(x)$ such that

$$
\left( M(|Du|^q)(x_1) \right)^{1/q} \leq \lambda \quad \text{and} \quad \left( M(1_{B_{\rho}(x)}|Du|^q)(y) \right)^{1/q} \leq \delta^{1/(p-1)} \lambda.
$$

(4.2)

We aim to show that

$$
\omega(E_{\lambda, \delta} \cap B_r(x)) < \epsilon \omega(B_r(x)).
$$

First note that

$$
\left( M(|Du|^q)(y) \right)^{1/q} \leq \max \left\{ \left( M\left(1_{B_{2r}(x)}|Du|^q\right)(y)\right)^{\frac{1}{q}}, 3^{n/q} \lambda \right\}
$$

(4.3)

for all $y \in B_r(x)$. Indeed, if $\rho \leq r$ then

$$
\int_{B_\rho(y)} |Du|^q dx = \int_{B_\rho(y)} 1_{B_{2r}(x)}|Du|^q dx \leq M\left(1_{B_{2r}(x)}|Du|^q\right)(y).
$$

Otherwise $B_\rho(y) \subset B_{2r+\rho}(x_1)$ and we have

$$
\int_{B_\rho(y)} |Du|^q dx \leq \frac{1}{|B_\rho(y)|} \int_{B_{2\rho}(y)} |Du|^q dx = 3^n \int_{B_{2\rho}(y)} M(|Du|^q)(x_1) \leq 3^n \lambda^q.
$$

It follows from (4.3) that

$$E_{\lambda, \delta} \cap B_r(x) = \left\{ y \in \mathbb{R}^n : \left( M\left(1_{B_{2r}(x)}|Du|^q\right)(y)\right)^{\frac{1}{q}} \leq 3^{n/q} \lambda \right\} \cap B_r(x)
$$

for all $\lambda > 0$ and $\lambda_0 \geq 3^{n/q}$.

Applying Proposition 3.6 to $u \in W^{1,p}_0(\mathbb{R}^n)$, $f, B = B_{8r}(x)$ and $\eta \in (0, 1)$, there exist constants $\delta = \delta(n, p, q, \epsilon, |\omega|_{A_\infty}) \in (0, 1)$, $C_0 = C_0(n, p, q) > 0$ and a $p$-harmonic map $v \in W^{1,p}(B_{4r}(x))$ such that

$$
\|Du\|_{L^\infty(B_{2r}(x))} \leq \frac{C_0\eta}{\delta^{1/(p-1)}} \left[ r \left( \int_{B_{8r}(x)} |f|^\beta \, dy \right)^{1/\beta} \right]^{1/(p-1)} + C_0(1 + \eta) \left( \int_{B_{8r}(x)} |Du|^q \, dy \right)^{1/q}
$$

and

$$
\left( \int_{B_{2r}(x)} |Du - Dv|^q \, dx \right)^{\frac{1}{q}} \leq \frac{\eta}{\delta^{1/(p-1)}} \left[ r \left( \int_{B_{8r}(x)} |f|^\beta \, dx \right)^{1/\beta} \right]^{1/(p-1)} + \eta \left( \int_{B_{8r}(x)} |Du|^q \, dx \right)^{1/q}.
$$

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Using (4.2) we deduce that
\[
\|Du\|_{L^\infty(B_{2r}(x))} \leq \frac{C_0 \eta}{\delta^{1/(p-1)}} \left( \mathcal{M}_\beta(|f|^\beta)(x_2) \right)^{1/(p-1)} + C_0(1 + \eta) \left[ \mathcal{M}(\|Du\|^q)(x_1) \right]^{1/q}
\]
\[
\leq C_0(1 + \eta)\lambda \leq 2C_0\lambda
\]
and
\[
\left( \int_{B_{\delta r}(x)} |Du - Dv|^q \, dx \right)^{1/q} \leq \frac{\eta}{\delta^{1/(p-1)}} \left[ R \left( \int_{B_{\delta r}(x)} |f|^\beta \, dx \right)^{1/\beta} \right]^{1/(p-1)} + \eta \left[ \int_{B_{\delta r}(x)} |Du|^q \, dx \right]^{1/q}
\]
\[
\leq \eta \lambda.
\]
Clearly
\[
\left[ \mathcal{M}\left( \left| \sum_{j=1}^3 f_j \right|^q \right) \right]^{1/q} \leq 3 \sum_{j=1}^3 \left[ \mathcal{M}(\|f_j\|)^q \right]^{1/q}.
\]
Hence
\[
|E \cap B_r(x)| \leq \left| \left\{ y \in \mathbb{R}^n : \mathcal{M} \left( \mathbf{1}_{B_{2r}(x)} |D(u - v)|^q(y) \right)^{1/q} > \Lambda_0 \lambda/9 \right\} \cap B_r(x) \right|
\]
\[
+ \left| \left\{ y \in \mathbb{R}^n : \mathcal{M} \left( \mathbf{1}_{B_{2r}(x)} |Dv|^q(y) \right)^{1/q} > \Lambda_0 \lambda/9 \right\} \cap B_r(x) \right|.
\]
In view of (4.4) there holds
\[
\left| y \in \mathbb{R}^n : \left( \mathcal{M} \left( \mathbf{1}_{B_{2r}(x)} |Dv|^q(y) \right)^{1/q} > \Lambda_0 \lambda/9 \right) \cap B_r(x) \right| = 0,
\]
provided that \( \Lambda_0 \geq \max\{3^{n/q}, 30C_0\} \).

Combining (4.5) and (4.6) yields
\[
|E \cap B_r(x)| \leq \left| \left\{ y \in \mathbb{R}^n : \mathcal{M} \left( \mathbf{1}_{B_{2r}(x)} |D(u - v)|^q(y) \right)^{1/q} > \Lambda_0 \lambda/9 \right\} \cap B_r(x) \right|
\]
\[
\leq \frac{C}{\Lambda_0^{1/q}} \int_{B_{2r}(x)} |D(u - v)|^q \, dx \leq C \eta^{q} \rho^n,
\]
where we used the fact that \( \mathcal{M} \) is of weak type \((1, 1)\) in the second step.

Thus
\[
\omega(E \cap B_r(x)) \leq c \left( \frac{|E \cap B_r(x)|}{|B_r(x)|} \right)^\nu \omega(B_r(x)) \leq c(C \eta^q)^\nu \omega(B_r(x)) < \varepsilon \omega(B_r(x)),
\]
where we chose \( \eta \) small enough such that \( c(C\eta^q)^v < e \).

This completes our proof. \( \blacksquare \)

5 Global weighted gradient estimates

With the knowledge from the previous sections, we are now ready to tackle the main theorem.

**Proof of Theorem 1.3.** By Theorem 4.3, for all \( \epsilon > 0 \) and \( q \in (1, q_0) \), where \( q_0 \) is defined in (3.1) there exist constants \( C = C(n, [\omega]_{A_{\infty}}) \), \( \delta = \delta(n, p, q, \epsilon, [\omega]_{A_{\infty}}) \in (0, 1) \) and \( \Lambda_0 = \Lambda_0(n, p, q) > 3^{q/4} \) such that

\[
\omega \left( \left\{ x \in \mathbb{R}^n : (M(|Du|^q)(x))^{1/q} > \Lambda_0 \lambda, \left( \frac{1}{\delta^{1/(p-1)}} \right)^\frac{1}{\omega} \leq \delta^{1/(p-1)} \lambda \right\} \right) \\
\leq C\epsilon \omega \left( \left\{ x \in \mathbb{R}^n : \left( M_{\beta}(|f|^\beta)(x) \right)^{1/q} > \lambda \right\} \right)
\]

for all \( \lambda > 0 \).

By hypothesis \( \Phi \) is invertible and \( \Phi^{-1} : [0, \infty) \to [0, \infty) \). Therefore

\[
\omega \left( \left\{ x \in \mathbb{R}^n : (M(|Du|^q)(x))^{1/q} > \Phi^{-1}(t) \right\} \right) \leq \omega \left( \left\{ x \in \mathbb{R}^n : \left( M_{\beta}(|f|^\beta)(x) \right)^{1/q} > \frac{\delta^{1/(p-1)}}{\Lambda_0} \Phi^{-1}(t) \right\} \right) \\
+ C\epsilon \omega \left( \left\{ x \in \mathbb{R}^n : \left( M(|Du|^q)(x) \right)^{1/q} > \frac{\Phi^{-1}(t)}{\Lambda_0} \right\} \right)
\]

for all \( t > 0 \). This in turn implies

\[
\int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt \\
\leq C\epsilon \int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > \Lambda_0 \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt \\
+ \int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M_{\beta}(|f|^\beta)(x) \right)^{\frac{1}{\omega}} \right) > t \right\} \right) dt \\
\leq C\epsilon \int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : H_1 \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt \\
+ \int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : H_2 \Phi \left( \left( M_{\beta}(|f|^\beta)(x) \right)^{\frac{1}{\omega}} \right) > t \right\} \right) dt,
\]

where we used the fact that \( \Phi(2t) \leq c \Phi(t) \) and \( \Phi \) is increasing in the second step. Here \( T > 0 \), \( H_1 = c^{\log_2(\Lambda_0)} \) and \( H_2 = c^{\log_2(\frac{\Lambda_0}{\delta^{1/(p-1)}})} \), in which \( \lceil \cdot \rceil \) denotes the ceiling function.
Using a change of variables we arrive at

$$
\int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt
$$

$$
\leq H_1 C \epsilon \int_0^{\frac{T}{H_1 \epsilon}} \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt
$$

$$
+ H_2 \int_{\frac{T}{H_1 \epsilon}}^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M_\beta(|f|^\beta)(x) \right)^{\frac{1}{(p-1)p}} \right) > t \right\} \right) dt.
$$

Now we choose \( \epsilon = \frac{1}{2H_1 C} \) so that the first integral on the right is absorbed by the left-hand term, which yields

$$
\int_0^T \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) > t \right\} \right) dt
$$

$$
\leq 2H_2 \int_0^{\frac{T}{H_1 \epsilon}} \omega \left( \left\{ x \in \mathbb{R}^n : \Phi \left( \left( M_\beta(|f|^\beta)(x) \right)^{\frac{1}{(p-1)p}} \right) > t \right\} \right) dt.
$$

Recall that

$$
\int_{\mathbb{R}^n} \Phi(|f|) \omega dx = \int_0^\infty \omega(\{ x \in \mathbb{R}^n : \Phi(|f(x)|) > t \}) dt.
$$

Thus by letting \( T \to \infty \) in the above inequality we arrive at

$$
\int_{\mathbb{R}^n} \Phi \left( \left( M(|Du|^q)(x) \right)^{\frac{1}{q}} \right) \omega dx \leq 2H_2 \int_{\mathbb{R}^n} \Phi \left( \left( M_\beta(|f|^\beta)(x) \right)^{\frac{1}{(p-1)p}} \right) \omega dx
$$

as required.

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