A CONFORMAL GROUP APPROACH TO THE DIRAC-KÄHLER SYSTEM ON THE LATTICE

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Abstract. Starting from the representation of the $(n-1)+n-$dimensional Lorentz pseudo-sphere on the projective space $\mathbb{P}^{n-1,n}$, we propose a method to derive a class of solutions underlying to a Dirac-Kähler type equation on the lattice. We make use of the Cayley transform $\varphi(w) = \frac{1+w}{1-w}$ to show that the resulting group representation arise from the same mathematical framework as the conformal group representation in terms of the general linear group $GL(2,\Gamma(n-1,n-1) \cup \{0\})$. That allows us to describe such class of solutions as a commutative $n-$ary product, involving the quasi-monomials $\varphi(z_j)^{-\frac{1}{x_j}}$ ($x_j \in h\mathbb{Z}$) with membership in the paravector space $\mathbb{R} \oplus \mathbb{R}e_1 e_n$. 

1. Introduction

Discrete function-theoretical methods has become in an emerging topic in Clifford analysis, mainly due to the pioneering works of Faustino & Kähler (2007) [FK07], Faustino et al. (2007) [FKS07], De Ridder et al. (2010) [RSKS10] and the PhD dissertations of Faustino (2009) [F09] and De Ridder (2013) [R13]. This new research field is called discrete Clifford analysis and corresponds to a discrete counterpart of function theory towards the multivector representation of the null solutions for the discretized Dirac-Kähler equation, in the massless limit $m \to 0$:

$$ (d - \delta)\psi = m\psi. \tag{1} $$

In equation $\square$ $d$ stands for the exterior derivative, $\delta = \ast^{-1}d\ast$ for the codifferential form, and $m$ for the mass term (cf. [KK04, S14]). Hereby, the symbol $\ast$ represents the so-called De Rham operator.

Several approaches for finding systems of solutions associated to discretized versions of $\square$ through combinatorics (cf. [MF08, MT08]), Lie-algebraic representations (cf. [RSKS10, FR11, F13]), and a combination of both (cf. [RSS12, BBRST14, F14]) have been worked out successfully as a unifying point of view for the theory of orthogonal polynomials and special functions of discrete hypercomplex variables. However, there has not been shown yet in the context of Clifford algebras that a system of solutions for the Dirac-Kähler equation on the lattice may be built up from tools of group representation theory, although there seems to be appropriate to consider the Poincaré group, the (inhomogeneous) Lorentz group and its cousins.

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as faithful models for the study of discretized versions of the Dirac-Kähler equation (cf. [L97, LK99]).

The main purpose of this paper is to construct a class of null solutions associated to a multivector discretization of the Dirac-Kähler equation (1) on the lattice, as a continuation of the work developed in [F16]. We confirm that it can be derived in a natural way from the compactification of $\mathbb{R}^{n-1,n-1}$ at infinity by means of the Cayley transform $\varphi(w) = \frac{1+w}{1-w}$. Such interplay indicates possibly a remarkably beautiful amalgamation between Dirac-Kähler fermions on the lattice and Einstein’s theory on the Anti-de Sitter universe, yet still to be investigated in depth such as that proposed in [AFN14, C15].

The paper is organized as follows:

- In Section 2 we reformulate the construction considered in [F16] for an alternative discretization of the Dirac-Kähler equation (1). We also formulate the main result of this paper, Proposition 2.1.

- In Section 3 we provide the necessary background about Clifford algebras and the conformal group. Some references for this preliminary section are habilitation thesis of J. Cnops (1994) [C94] and the research paper of V.V. Kisil [K05] (2005); see also the books of H.B. Lawson & M.L. Michelsohn (1989) [LM89] and W.A. Rodrigues & E.C. de Oliveira (2007) [RO07] for further details.

- In Section 4 we make use of the mapping property $\text{spin}^+(n, n) \rightarrow \text{Spin}^+(n, n)$ to properly study the Cayley transform $w \mapsto \varphi(w)$. In particular, for each $j = 1, 2, \ldots, n$ we obtain a stereographic-like projection mapping property between the 2–vector subspaces $\mathbb{R}e_{j}e_{n+j}$ of $\text{spin}^+(n, n)$ and the (paravector) subspaces $H_{j}^{n-1,n}$ of the Lorentz pseudo-sphere $H^{n-1,n}$.

- In Section 5 we prove Proposition 2.1 in detail.

- In Section 6 we outlook the main contribution of the paper and discuss further directions of research.

### 2. Problem Setup and Main Result

The approach to be discussed throughout this paper is formulated in terms of the language of discrete multivector calculus carrying the lattice $h\mathbb{Z}^n$ with meshwidth $h > 0$, and from a class of finite difference counterparts of the Dirac-Kähler equation (1), constructed from the finite difference operator

$$D_h = \sum_{j=1}^{n} \left( e_j \frac{\partial_h^{-j} + \partial_h^{+j}}{2} + e_{n+j} \frac{\partial_h^{-j} - \partial_h^{+j}}{2} \right).$$

The notations are the following: $\partial_h^{\pm j}$ are the forward/backward finite difference operators

$$\partial_h^{+j} f(x) = \frac{f(x + h e_j) - f(x)}{h}, \quad \text{and} \quad \partial_h^{-j} f(x) = \frac{f(x) - f(x - h e_j)}{h},$$

and $e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ the generators of the Clifford algebra $C\ell_{n,n}$. This algebra is generated by the identity 1, the vectors $e_j, e_{n+k} (1 \leq j, k \leq n)$, and the
set of anti-commuting relations
\[
\begin{align*}
\epsilon_j \epsilon_k + \epsilon_k \epsilon_j &= -2\delta_{jk}, & 1 \leq j, k \leq n \\
\epsilon_j \epsilon_{n+k} + \epsilon_{n+k} \epsilon_j &= 0, & 1 \leq j, k \leq n \\
\epsilon_{n+j} \epsilon_{n+k} + \epsilon_{n+k} \epsilon_{n+j} &= 2\delta_{jk}, & 1 \leq j, k \leq n.
\end{align*}
\]

(3)

The Clifford algebra \( \mathcal{C}l_{n,n} \) is a linear associative algebra of dimension \( 2^{2n} \), that contains the field of real numbers \( \mathbb{R} \) and the Minkowski space \( \mathbb{R}^{n,n} \) as proper subspaces. Here one notice that \( \mathbb{R}^{n,n} \) is equipped by the quadratic form
\[
Q(u, v) = \sum_{j=1}^{n} (u_j^2 - v_j^2), \quad u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \quad \& \quad v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n.
\]

Next, we consider the following \( \mathcal{C}l_{n,n} \)-valued operator acting on \( h\mathbb{Z}^n \):
\[
\chi_h(x) = \prod_{j=1}^{n} (-1)^{\frac{x_j}{2}} \epsilon_{n+j} \epsilon_j.
\]

This operator may be rewritten as \( \chi_h(x) = (-1)^{\sum_{j=1}^{n} \frac{x_j}{2}} \gamma \), where \( \gamma = \prod_{j=1}^{n} \epsilon_{n+j} \epsilon_j \) stands for the pseudoscalar of \( \mathcal{C}l_{n,n} \).

The 1-vector representations \( x = \sum_{j=1}^{n} x_j \epsilon_j \) and \( x \pm he_j \) of \( \mathbb{R}^n \) will be used throughout this paper to describe the lattice point \( (x_1, x_2, \ldots, x_n) \in h\mathbb{Z}^n \) and the forward/backward shifts \( (x_1, x_2, \ldots, x_j \pm h, \ldots, x_n) \) over \( h\mathbb{Z}^n \), respectively. Also, the notation \( \frac{a}{b} := \frac{ab^{-1}}{b} \) will be adopted to get the analogy with the fractional-linear transformations on the complex plane \( \mathbb{C} \cong \mathcal{C}l_{0,1} \). Due to the non-commutativity of \( \mathcal{C}l_{n,n} \), one has \( \frac{ca}{bc} \neq \frac{ac}{bc} \) so that only the equality \( \frac{ac}{bc} = \frac{a}{b} \) holds for every \( a, b, c \) with membership in \( \mathcal{C}l_{n,n} \).

In [F16] Section 3 it was shown that the incorporation of the local unitary action \( \chi_h(x) \) on the lattice \( h\mathbb{Z}^n \) allows us to determine the null solutions of the Dirac-field operator \( D_h - m\chi_h(x) \) on \( h\mathbb{Z}^n \) as a direct sum involving the chiral and achiral spaces of discrete multivector functions, similar to the decomposition of \( \mathcal{C}l_{n,n} \) in terms of its even and odd parts. Nevertheless, the primitive idempotents \( \frac{1}{2} (1 \pm \chi_h(x)) \) only allows us to study discretizations of (1) on the Minkowski spacetime, but not in a general spacetime such as the conformal spacetime.

The conformal spacetime model on the lattice was roughly discussed by Lorente & Kramer in [LK99] when they treat Lorentz transformations modulo rotations by means of the Cayley transform. They also have shown that the Lorentz invariance only fulfills when the lattice parameter \( h \) tends to zero. However, it was not possible to fill the lattice fermion doubling on the Lorentz space when the energy-momentum relation on the \( n \)-cube \( [-\frac{1}{2}, \frac{1}{2}]^n \), as considered in [F16] Section 4:
\[
\sum_{j=1}^{n} \frac{4}{h^2} \sin^2 \left( \frac{h\xi_j}{2} \right) = m^2
\]

\(^1\)The so-called \( n \)-dimensional Brillouin zone.
was replaced by its counterpart on the Lorentz space (cf. [LK99 Section 5])

$$
\sum_{j=1}^{n} \frac{4}{\hbar^2} \tan^2 (h\xi_j) = m^2.
$$

A possible way to rid this gap was proposed by Kaplan (1992) in [K92], on which the mass \( m \) was replaced by a monotonic step function. With such approach it was able to renormalize and localize the chiral anomalies in the \textit{massless limit}, since the fermionic mass converges asymptotically to \( \pm m \), as the lattice parameter goes to zero. That gives in turn a valuable insight to study discrete function-theoretical methods from a mathematical physics perspective, such as the theory of finite difference potentials described on the paper [CKK15], and on the references given there.

On this paper we propose a scheme to compute the solutions underlying to a different discretization of the Dirac-Kähler equation. We consider for a fixed frame \((\omega_1, \omega_2, \ldots, \omega_n)\) of the \((n - 1)\)–sphere \(S^{n-1}\), the following discretization

$$
D_h - m\omega = \sum_{j=1}^{n} \left( e_j \frac{\partial^{-j}_h - \partial^{+j}_h}{2} + e_{n+j} \frac{\partial^{-j}_h - \partial^{+j}_h}{2} - m\omega_j e_{n+j} \right),
$$

carrying the mass term \( m \). Hereby \( \omega = \sum_{j=1}^{n} \omega_j e_{n+j} \) denotes the 1–vector representation of \((\omega_1, \omega_2, \ldots, \omega_n)\).

From the graded anti-commuting relations (3) it is straightforward to verify that \( D_h - m\omega \) satisfies the factorization property

$$
(D_h - m\omega)^2 = (2hm - 1) \Delta_h + m^2,
$$

where \( \Delta_h = \sum_{j=1}^{n} \partial^{-j}_h \partial^{+j}_h \) denotes the \textit{star-Laplacian} (cf. [FKS07, p. 455]).

The factorized operator \((2hm - 1) \Delta_h + m^2\) corresponds to a discretized version of the Klein-Gordon operator on the lattice, that differs from the one considered on [F16]. It has non-trivial solutions if and only if the energy-momentum condition

$$
\sum_{j=1}^{n} \frac{4}{\hbar^2} \sin^2 \left( \frac{h\xi_j}{2} \right) = \frac{m^2}{1 - 2hm}
$$

is fulfilled on \([-\pi \hbar, \pi \hbar]^{n}\).

This paper is centered around the construction of a class of null solutions for (4), through the ansatz

$$
\Psi_h(x, z) = \prod_{j=1}^{n} \left( \frac{1 + z_j}{1 - z_j} \right)^{- \frac{m_j}{2}},
$$

whereby each term \( z_j \in \mathbb{R} e_j e_{n+j} \) corresponds to a \( \text{spin}^+(n, n) \)–algebra representation of the Clifford algebra \( C\ell_{n,n} \). That corresponds to the following proposition:
Proposition 2.1. Let $\omega = \sum_{j=1}^{n} \omega_j e_{n+j}$ be a 1-vector representation for a point on the $n$-sphere $S^{n-1}$, and $z = \sum_{j=1}^{n} z_j$ a spin$^+(n,n)$-representation of $\mathbb{C}l_{n,n}$. Assuming that the function $\Psi_h(x,z)$ determined from the ansatz (5) satisfies the equation $D_h \Psi_h(x,z) = m \omega \Psi_h(x,z)$, we thus have the following:

(1) For $m \neq 0$ the set of points $z_j \in \mathbb{R} \mathbb{e}_j e_{n+j}$ ($j = 1, 2, \ldots, n$) is uniquely determined by

$$z_j = e_j e_{n+j} \frac{(h \omega_j)^2 - 2 h m \omega_j}{(h \omega_j)^2} \quad (j = 1, 2, \ldots, n).$$

(2) $\Psi_h(x,z)$ equals to

$$\left\{ \prod_{j=1}^{n} \frac{(h \omega_j)^2 + e_j e_{n+j} (h \omega_j)^2 - 2 h m \omega_j}{(h \omega_j)^2 - e_j e_{n+j} (h \omega_j)^2} \right\}^{-\frac{e_m}{2}} \prod_{j=1}^{n} (-1)^{\frac{e_m}{2}},$$

in the limit $m \rightarrow 0$.

Moreover, $\lim_{m \rightarrow 0} \Psi_h(x,z) \gamma = \chi_h(x)$, where $\gamma$ denotes the pseudoscalar of $\mathbb{C}l_{n,n}$, defined as above.

(3) $g(x) = \chi_h(x) \Psi_h(-x,z)$ is a null solution of $D_h - m \omega$, where $\chi_h(x)$ denotes the $\mathbb{C}l_{n,n}$-valued operator acting on $h\mathbb{H}^n$, defined as above.

3. Conformal group representation of the Lorentz pseudo-sphere

On this section one will introduce the conformal group representation of the $(n-1) + n$-dimensional Lorentz pseudo-sphere $H^{n-1,n} = \{(u,v) \in \mathbb{R}^{n,n} : Q(u,v) = 1\}$ of $\mathbb{R}^{n,n}$ from a multivector calculus perspective.

Let us first collect some basic facts about about the Clifford algebra $\mathbb{C}l_{n,n}$ introduced on the previous section. Starting from the basis graded anti-commuting relations (4), one can generate the basis elements of $\mathbb{C}l_{n,n}$. They consists on elements of the form $e_J = e_{j_1} e_{j_2} \ldots e_{j_r}$, associated to a subset $J = \{j_1, j_2, \ldots, j_r\}$ of $\{1, 2, \ldots, n, n+1, \ldots, 2n\}$ with cardinality $|J| = r$ so that $1 \leq j_1 < j_2 < \ldots < j_r \leq 2n$. For $J = \emptyset$ (empty set) one will use the convention $e_{\emptyset} = 1$ to denote the identity element of $\mathbb{C}l_{n,n}$.

Thus, any element $a$ of $\mathbb{C}l_{n,n}$ may be written as

$$a = \sum_{r=0}^{2n} [a]_r, \quad \text{with} \quad [a]_r = \sum_{|J|=r} a_J e_J.$$

Through the projection operator $[.]_r : \mathbb{C}l_{n,n} \rightarrow \Lambda^r(\mathbb{R}^{n,n})$, the algebra $\mathbb{C}l_{n,n}$ can thus be associated to the following multivector decomposition of the exterior
algebra $\Lambda^r(\mathbb{R}^{n,n})$ (cf. [RO07] Chapter 2) :

$$\Lambda^r(\mathbb{R}^{n,n}) = \bigoplus_{r=0}^{2n} \Lambda^r(\mathbb{R}^{n,n}),$$

leading in particular to the one-to-one identifications $a \in \mathbb{R} \leftrightarrow a e_0 \in \Lambda^0(\mathbb{R}^{n,n})$ and $(u, v) \in \mathbb{R}^{n,n} \leftrightarrow u + v \in \Lambda^1(\mathbb{R}^{n,n})$, with

$$u = \sum_{k=1}^n u_k e_{n+k} \quad \text{and} \quad v = \sum_{j=1}^n v_j e_j.$$

(6)

There is an automorphism $a \mapsto a'$ (main involution) and two anti-automorphisms, $a \mapsto a^*$ (reversion) and $a \mapsto a^\dagger$ (conjugation) respectively, that leave the structure of $C\ell_{n,n}$ invariant. They are defined recursively by the rules

$$\begin{align*}
(ab)' &= a'b' \\
(a_j e_j)' &= a_j e_j' e_j' \ldots e_j' \quad (1 \leq j_1 < j_2 < \ldots < j_r \leq 2n) \\
e_j' &= -e_j \quad \text{and} \quad e_{n+j}' = e_{n+j} \quad (1 \leq j \leq n)
\end{align*}$$  

(7)

$$\begin{align*}
(ab)^* &= b^* a^* \\
(a_j e_j)^* &= a_j e_j^* \ldots e_j^* e_j^* \quad (1 \leq j_1 < j_2 < \ldots < j_r \leq 2n) \\
e_j^* &= e_j \quad \text{and} \quad e_{n+j}^* = e_{n+j} \quad (1 \leq j \leq n)
\end{align*}$$  

(8)

$$\begin{align*}
(ab)^\dagger &= b^\dagger a^\dagger \\
(a_j e_j)^\dagger &= a_j e_j^\dagger \ldots e_j^\dagger e_j^\dagger \quad (1 \leq j_1 < j_2 < \ldots < j_r \leq 2n) \\
e_j^\dagger &= -e_j \quad \text{and} \quad e_{n+j}^\dagger = e_{n+j} \quad (1 \leq j \leq n).
\end{align*}$$  

(9)

In particular, for a given element $u + v$ of $\Lambda^1(\mathbb{R}^{n,n})$, $(u + v)' = (u + v)^\dagger = u - v$ and $(u + v)^* = u + v$. Notice also that for each $u + v \in \Lambda^1(\mathbb{R}^{n,n})$, $(u + v)^2 = u^2 + v^2$ equals to the quadratic form $Q(u, v)$. In case where $Q(u, v)$ is non-degenerate at $(u, v) \in \mathbb{R}^{2n}$, it readily follows that

$$(u + v)^{-1} = \frac{u + v}{u^2 + v^2},$$

the so-called Kelvin inverse of $u + v$.

From the relations established above it is clear that for each $n - p \geq 1$ and $n - q \geq 1$, the product of invertible 1-vectors lying on the subspaces $\mathbb{R}^{n-p,n-q}$ of $\mathbb{R}^{n,n}$ remains invertible. They form the so-called Clifford group $\Gamma(n - p, n - q)$ (cf. [RO07] Subsection 3.3.3).

The set of all products of vectors in the Minkowski space $\mathbb{R}^{n-p,n-q}$ will be denoted by $T(n - p, n - q)$ whereas the elements $a \in T(n - p, n - q)$ satisfying $aa^\dagger = 1$ form the connected subgroup $\text{Pin}^+(n - p, n - q)$ of

$$\text{Pin}(n - p, n - q) = \{ a \in T(n - p, n - q) : aa^\dagger = \pm 1 \}.$$

For the special choice $p = q = 1$ one can make use of the periodicity theorem (cf. [CG94] pp. 74]) to build up the correspondence between the general linear group $GL(2, \Gamma(n-1, n-1) \cup \{0\})$ of $2 \times 2$ matrices, and the Clifford group $\Gamma(n, n)$. Namely, from the one-to-one correspondence provided by the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{2} [(a + d') e_{2n} e_n + (b + c') e_{2n} + (-b + c')]$$  

(10)
one can describe every element of $\Gamma(n, n)$ through the $2\times2$ matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose entries $a, b, c \& d$ satisfy the following conditions:

1. $a, b, c, d \in \Gamma(n - 1, n - 1) \cup \{0\}.$
2. $bd^* - ac^* = 0.$
3. $ad^* - b^*c$ (the pseudo-determinant of $M$) belongs to $\mathbb{R} \setminus \{0\}.$

That corresponds to a natural extension of Waterman’s approach (cf. W93, Theorem 5 & Theorem 6) to the Minkowski space $\mathbb{R}^{n-1, n-1}$. In the shed of Filmore-Springer’s approach [FS90], one can define to each $M \in \Gamma(n, n)$ the representation of $\mathbb{R}^{n, n}$ the conformal group $\mathcal{M}(n - 1, n - 1)$ of $\mathbb{R}^{n-1, n-1} := \mathbb{R}^{n-1, n-1} \cup \{\infty\}^4$, the so-called Möbius transformations on $\mathbb{R}^{n-1, n-1}$ (cf. [C94, Subsection 5.1]):

$$\mu_M : z \mapsto \frac{az + b}{cz + d}.$$

It is well known that for every $M \in \text{GL}(2, \Gamma(n - 1, n - 1) \cup \{0\})$, the pseudo-orthogonal group $\mathbb{O}(n, n)$ may be conformally embedded on the projective space $\mathbb{P}\mathbb{R}^{n, n}$ through the mapping $S \mapsto \mathcal{M}(S(M')^{-1}$, whereby $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ denotes a $2 \times 2$ representation of $H^{n-1,n}$ for the inversion mapping $w \mapsto -w^{-1}$ on $C\ell_{n,n}$ (cf. W93 p. 93).

By employing the involution map (7) and the isomorphism (10), there holds $M' = \begin{pmatrix} a' & -b' \\ -c' & d' \end{pmatrix}$ (cf. [C94] p. 75). This leads to

$$MS = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \quad \text{and} \quad SM' = \begin{pmatrix} -c' & d' \\ -a' & b' \end{pmatrix}.$$

Therefore, the equation $MS(M')^{-1} = S$ is fulfilled whenever $M$ is of the form

$$(11) \quad M = \begin{pmatrix} a & b \\ b' & a' \end{pmatrix},$$

with $a, b \in T(n - 1, n - 1), ab^* \in \mathbb{R}^{n-1,n-1}$ and $|a|^2 - |b|^2 = 1$.

Mimicking [K05, Proposition 2.2.], the family of Möbius transformations that preserve the Lorentz pseudo-sphere $H^{n-1,n}$ are of the form

$$\mu'_M : z \mapsto \frac{az + b}{b'z + a'}.$$

They will be denoted throughout by $\mathcal{M}^+(n - 1, n - 1)$.

4. Compactification through the Cayley map

It is worth mentioning that, unlike to the sphere $S^{n-1}$ the Lorentz pseudo-sphere $H^{n-1,n}$ does not possess a group structure, and hence one cannot identify,

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2The notation $\mathbb{R}^{n-1,n-1} := \mathbb{R}^{n-1,n-1} \cup \{\infty\}$ means the compactification of $\mathbb{R}^{n-1,n-1}$ by the point at infinity.

3$O(n,n)$ is the group of isometries that preserves $Q(\cdot, \cdot)$ so that $SO(n,n)$ acts as an isometry on $H^{n-1,n}$.

4$S^{n-1}$ is an affine subspace of $H^{n-1,n}$. 
as in case of $S^{n-1}$, the manifold $H^{n-1,n}$ with the left coset $K \setminus \mathcal{M}^+(n-1, n-1)$, involving the maximal compact subgroup (cf. [K05 pp. 742-743])

$$K = \left\{ \begin{pmatrix} u & 0 \\ |u| & u' \\ 0 & 0 \end{pmatrix} : u \in \Gamma(n-1, n-1) \right\}. $$

However $\text{Pin}^+(n, n)$ gives a double covering of $\mathcal{M}^+(n-1, n-1)$ (cf. [C94 Subsection 5.3]), so that $H^{n-1,n}$ may be embedded in a group manifold of $\mathcal{M}^+(n-1, n-1)$. There is an alternative and more useful description for the conformal group $\mathcal{M}^+(n-1, n-1)$ that starts with the identification of the points of the Minkowski space $\mathbb{R}^{n-1,n-1}$ as elements with membership in a certain Lie algebra, on which the passage from the Lie algebra to the Lie group by means of the Cayley map

$$\varphi(w) = \frac{1 + w}{1 - w}$$

yields the compactification $\mathbb{R}^{n-1,n-1}$.

Here one notice that the associated Cayley transform (12) is encoded by the Spin$^+(n,n)$ representation

$$\frac{1}{\sqrt{2}}(I + S) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$ 

We easily check that

$$\varphi(w) + \varphi(w)^{-1} = 2 \frac{1 + w^2}{1 - w^2}$$

$$\varphi(w) - \varphi(w)^{-1} = \frac{1 - w^2}{4w}$$

$$w(\varphi(w) + \varphi(w)^{-1} + 2) = \varphi(w) - \varphi(w)^{-1},$$

hold for every $w^2 \neq 1$, that is $\frac{1}{7}(\varphi(w) \pm \varphi(w)^{-1})$ may be interpreted as coordinates of a stereographic-like projection.

In his habilitation thesis [C94] J. Cnops (1994) obtained, in particular, a generalization of the above description in terms of the Clifford group $\Gamma(n,n)$, following the insights of Filmore and Springer [FS90] (see [C94 Chapter 5]). In the shed of Cnops’s characterization provided by [C94 Corollary 5.2.9], the description of $\Gamma(n,n)$ may be represented in terms of $2 \times 2$-matrix representations of the group $\text{GL}(2, \Gamma(n-1, n-1) \cup \{0\})$, as depicted on Section 8. Since $\text{Pin}^+(n,n)$ is a subgroup of $\Gamma(n,n)$, it suffices (as it will be seen next) to describe $\mathbb{R}^{n-1,n-1}$ in terms of the spin group $\text{Spin}^+(n,n) = \text{Pin}^+(n,n) \cap C\ell^0_{n,n}$, where

$$C\ell^0_{n,n} = \{ a \in C\ell_{n,n} : a' = a \}$$

corresponds to the even subalgebra of $C\ell_{n,n}$.

Next, we examine the action of the Cayley map (12) on the (paravector) subspaces $H_j^{n-1,n}$ of $H^{n-1,n}$, defined as

$$H_j^{n-1,n} = \{ v_j + u_j e_{n+j} : v_j > 0, v_j^2 - u_j^2 = 1 \}.$$

Here we recall that $\text{spin}^+(n,n) = \text{spin}(n,n)$ coincides with the space of 2-vectors (cf. [LMS9 Proposition 6.1])

$$\Lambda^2(\mathbb{R}^{n,n}) = \text{span} \{ e_j e_k : 1 \leq j < k \leq 2n \}.$$
so that \( \frac{1}{\sqrt{2}} (I + S) : \text{spin}^+ (n, n) \rightarrow \text{Spin}^+ (n, n) \) yields the compactification \( \mathbb{R}^{n-1,n-1} \) in terms of the action \( w \in \Lambda^2 (\mathbb{R}^{n,n}) \rightarrow \varphi (w) \). The restriction of the Cayley map \( \bullet \) to the subspaces \( \mathbb{R} e_j e_{n+j} \) \( (j = 1, 2, \ldots, n) \) of \( \Lambda^2 (\mathbb{R}^{n,n}) \) will be of special interest in the proof of Proposition 2.1 mainly because for the (paravector) subspaces \( H_j^{n-1,n} \) of \( H_j ^{n-1,n} \) the map \( \varphi : \mathbb{R} e_j e_{n+j} \rightarrow H_j^{n-1,n} \) is evidently onto. This can be easily deduced from the stereographic-like parametrization

\[
\begin{align*}

v_j &= \frac{1 + z_j^2}{1 - z_j^2}, \quad z_j \in \mathbb{R} e_j e_{n+j}, \\
u_j e_j e_{n+j} &= \frac{2z_j}{1 - z_j^2}
\end{align*}
\]

(14)

5. PROOF OF Proposition 2.1

It is clear from (12) and (14) that for every \( \text{spin}^+ (n, n) \)-representation of the form \( z = \sum_{j=1}^{n} z_j (z_j \in \mathbb{R} e_j e_{n+j}) \), and for every \( x \in h\mathbb{Z}^n \), the function \( \Psi_h (x, z) \) may be rewritten as

\[
\Psi_h (x, z) = \prod_{j=1}^{n} \varphi (z_j)^{-\frac{z_j}{2}} = \prod_{j=1}^{n} (v_j + u_j e_j e_{n+j})^{-\frac{z_j}{2}}.
\]

(15)

This function corresponds to a commutative \( n \)-ary product. From the anti-commuting relations \( (3) \), it is clear that 2-vectors \( e_j e_{n+j} \) and \( e_k e_{n+k} \) commute \( (k = 1, 2, \ldots, n; j \neq k \) fixed). Thus, for every \( x_j, x_k \in h\mathbb{Z} \), one readily has

\[
(v_k + u_k e_k e_{n+k})^{-\frac{z_j}{2}} (v_j + u_j e_j e_{n+j})^{-\frac{z_k}{2}} = (v_j + u_j e_j e_{n+j})^{-\frac{z_j}{2}} (v_j + u_j e_j e_{n+j})^{-\frac{z_k}{2}},
\]

that is \( \varphi (z_k)^{-\frac{z_j}{2}} \varphi (z_j)^{-\frac{z_k}{2}} = \varphi (z_j)^{-\frac{z_j}{2}} \varphi (z_k)^{-\frac{z_k}{2}} \).

From now on, we assume that \( \Psi_h (x, z) \) is a solution of the discrete Dirac equation \( D_h g (x) = m \omega g (x) \). From (2) and (4), this is equivalent to

\[
\sum_{j=1}^{n} \left( e_j \Psi_h (x + he_j, z) - \Psi_h (x - he_j, z) \right) - e_{n+j} \Psi_h (x + he_j, z) + \Psi_h (x - he_j, z) = m_h \Psi_h (x, z),
\]

with \( m_h (\omega) = \sum_{j=1}^{n} \left( m \omega_j - \frac{1}{h} \right) e_{n+j} \).

Straightforward computations based on (15) gives rise to the set of finite difference equations

\[
\begin{align*}

e_j \Psi_h (x + he_j, z) - \Psi_h (x - he_j, z) &= -\frac{1}{h} e_j \frac{2z_j}{1 - z_j^2} \Psi_h (x, z) \\

\Psi_h (x + he_j, z) + \Psi_h (x - he_j, z) &= -\frac{1}{h} e_{n+j} \frac{1 + z_j^2}{1 - z_j^2} \Psi_h (x, z)
\end{align*}
\]

(\( j = 1, 2, \ldots, n \)).

Then, from the set of properties

\[
\left( \frac{1 + z_j^2}{1 - z_j^2} \right)^2 - \left( \frac{2z_j}{1 - z_j^2} \right)^2 = 1 \quad \text{for} \quad j = 1, 2, \ldots, n,
\]

A CONFORMAL GROUP APPROACH TO THE DIRAC-KÄHLER SYSTEM
the finding of the solution for the equation \( D_h \Psi(x, z) = m \omega \Psi_h(x, z) \) reduces to the problem of finding the set of points \( z_j \in \mathbb{R} e_{e_{n+j}} \) such that the system of equations

\[
\begin{cases}
2z_j \frac{e_j}{1-z_j} = \frac{1}{2} e_{n+j} \left[ (hm\omega_j - 1) - (hm\omega_j - 1)^{-1} \right] \\
\frac{1+z_j^2}{1-z_j} = \frac{1}{2} e_{n+j} \left[ (hm\omega_j - 1) + (hm\omega_j - 1)^{-1} \right]
\end{cases} \quad (j = 1, 2, \ldots, n)
\]

is fulfilled for each \( \omega = \sum_{j=1}^{n} \omega_j e_{n+j} \) with membership in \( S^{n-1} \).

From the above system of equations, the terms \( \varphi(z_j) \) of the right-hand of (15) are determined from the set of equations

\[
\varphi(z_j) = \frac{1+z_j^2}{1-z_j} + \frac{2z_j}{1-z_j} = \frac{1}{2} \left[ (hm\omega_j - 1) + (hm\omega_j - 1)^{-1} \right] + \frac{1}{2} e_j e_{n+j} \left[ (hm\omega_j - 1) - (hm\omega_j - 1)^{-1} \right].
\]

**Proof of Statement (1) of Proposition 2.1.** From the equation \( z_j \left( \frac{1+z_j^2}{1-z_j} + 1 \right) = \frac{2z_j}{1-z_j} \) provided by (13), one readily has

\[
z_j = e_j e_{n+j} \frac{1}{2} \left[ (hm\omega_j - 1) - (hm\omega_j - 1)^{-1} \right].
\]

A short computation gives rise to

\[
z_j = e_j e_{n+j} \frac{(hm\omega_j - 1)^2 - 1}{(hm\omega_j - 1)^2 + 2(hm\omega_j - 1) + 1} \quad (j = 1, 2, \ldots, n),
\]

or equivalently, to the simplified formula

\[
z_j = e_j e_{n+j} \frac{(hm\omega_j)^2 - 2hm\omega_j}{(hm\omega_j)^2} \quad (j = 1, 2, \ldots, n).
\]

This corresponds to the set of points for which \( D_h \Psi(x, z) = m \omega \Psi_h(x, z) \) is fulfilled, as desired.

**Proof of Statement (2) of Proposition 2.1.** In case where \( m \neq 0 \), the \( z_j \)'s determined as above satisfy \( z_j \neq -e_j e_{n+j} \) so that \( \varphi(z_j) = \frac{1+z_j^2}{1-z_j} + \frac{2z_j}{1-z_j} \) is well defined (see eq. (13)).

From a short computation, involving the identity \( \frac{1-ab^{-1}}{1+ab^{-1}} = \frac{b-a}{b+a} \) there holds

\[
\varphi(z_j) = \frac{(hm\omega_j)^2 + e_j e_{n+j} \left( (hm\omega_j)^2 - 2hm\omega_j \right)}{(hm\omega_j)^2 - e_j e_{n+j} \left( (hm\omega_j)^2 - 2hm\omega_j \right)} \quad (j = 1, 2, \ldots, n).
\]
In case where $m \to 0$, it readily follows from equation (16) that
$$\varphi(z) = -1.$$  

Finally, by inserting the $\varphi(z)$'s on the right-hand side of (15), we finish the proof of Statement (2).

Proof of Statement (3) of Proposition 2.1 First, we recall that the operator
$$\chi_h(x) = \prod_{j=1}^{n} (-1)^{\frac{x}{h}} e_{n+j} e_j$$
satisfy the unitary property, $\chi_h(x)^2 = 1$, the set of recursive equations $\chi_h(x \pm h e_j) = -\chi_h(x)$ $(j = 1, 2, \ldots, n)$ and the set of graded anti-commuting relations
$$\chi_h(x) e_j + e_j \chi_h(x) = \chi_h(x) e_{n+j} + e_{n+j} \chi_h(x) = 0 \quad \text{for} \quad j = 1, 2, \ldots, n.$$  

Then, from (2) and (4) the function $f(x)$ is a null solution for $D_h - m \omega$ if and only if the function $f(x)$ satisfies the equation
$$\sum_{j=1}^{n} \left( -e_j \chi_h(x) \frac{f(x + h e_j) - f(x - h e_j)}{2h} - e_{n+j} \chi_h(x) \frac{2f(x) - f(x + h e_j) - f(x - h e_j)}{2h} \right) = -m \omega \chi_h(x) f(x).$$

Hence, for $g(x) = \chi_h(x) f(x)$ the above equation is equivalent to
$$\sum_{j=1}^{n} \left( e_j \left. g(x + h e_j) - g(x - h e_j) \right|_{2h} - e_{n+j} \left. g(x + h e_j) + g(x - h e_j) \right|_{2h} \right) = -m_h(\omega) g(x),$$

with $m_h(\omega) = \sum_{j=1}^{n} \left( m \omega_j - \frac{1}{h} \right) e_{n+j}.$

Putting $f(x) = \Psi_h(-x, z)$, with $z = \sum_{j=1}^{n} z_j$ $(z_j \in \mathbb{R} e_{n+j})$, there holds from a straightforward computation that the solution of the above equation is determined from the system of equations (16). Therefore, one can easily infer that $g(x) = \chi_h(x) \Psi_h(-x, z)$ is a solution of the equation $D_h g(x) = m \omega g(x)$, as desired.

6. DISCUSSION

A scenario of modifying the lattice discretization of the Dirac-Kähler equation (11) was proposed throughout this paper. Two ingredients towards such modification were crucial. One of them consists on the treatment of the mass term $m$ as a 1-vector term of the Clifford algebra of signature $(n,0)$, say $m \omega$ with $\omega \in S^{n-1}$. The other was the construction of a one-to-one mapping between the subspaces $\mathbb{R} e_j e_{n+j}$ of $\text{spin}^+(n, n)$ and the (paravector) subspaces $H_j^{n-1,n}$ of the Lorentz pseudo-sphere $H^{n-1,n}$, by means of the stereographic-like projection (14). That was the main key on the proof of Proposition 2.1.

We believe that the same technique may also be applied to non-compact symmetric spaces that admit a horospherical or Iwasawa decomposition such as the Einstein static universe, carrying the universal cover of $\mathbb{R}^{n-1,n-1}$ (cf. [GS95]).

What we have tried to show throughout this paper is that the conformal group is encoded on the discretization of Dirac operators considered in a series of papers, and by several authors in the context of discrete Clifford analysis. This feature
is not tangible if we only consider projection operators to decouple the discrete Dirac-Kähler equation on its components (cf. [S14]).

Although this paper offers a different perspective to study discretizations of the Dirac-Kähler equation (1), the problem of lattice fermion doubling regarding the chiral operator $\chi_h(x)$ in $h\mathbb{Z}^n$ was not duly clarified. Here we would like to stress that $\chi_h(x) = \lim_{m \to 0} \Psi_h(x, z)\gamma$ is not a null solution of the discrete Dirac operator $D_h$.

One possible way to overcome the aforementioned problem may consists on the replacement of the mass term $m$ by the ratio $\frac{m}{\Lambda_h}$, where $\Lambda_h$ stands a cutoff term.

It should be recalled that in the case of the terms $\frac{m}{\Lambda_h} \omega_j$ of

$$\exp\left(e_j e_{n+j} \log \left(h \frac{m}{\Lambda_h} \omega_j - 1\right)\right) = \frac{1}{2} \left[ \left(h \frac{m}{\Lambda_h} \omega_j - 1\right) + \left(h \frac{m}{\Lambda_h} \omega_j - 1\right)^{-1} \right]$$

$$+ \frac{1}{2} e_j e_{n+j} \left[ \left(h \frac{m}{\Lambda_h} \omega_j - 1\right) - \left(h \frac{m}{\Lambda_h} \omega_j - 1\right)^{-1} \right]$$

satisfy the asymptotic condition $\frac{\omega_j}{\Lambda_h} \sim \frac{2}{hm}$, as $m \to 0$, one has

$$\exp\left(e_j e_{n+j} \log \left(h \frac{m}{\Lambda_h} \omega_j - 1\right)\right) \sim 1$$

in the massless limit. This is what Kaplan (1992) already did on the paper [K92] when he flipped the chirality gap by introducing a fermionic mass, parametrized in terms of hyperbolic coordinates.

There are other important questions that also deserve to be investigated as a whole and whether these results can be exploited in the shed of the Anti-De Sitter spacetime. In such model, the lattice term $\frac{m}{\Lambda_h}$ that appears on the eigenvalue-type equation $D_h f(x) = \frac{m}{\Lambda_h} \omega f(x)$, involving the discrete Dirac operator $D_h$, may play the role of the cosmological constant in Einstein’s equation.

We conjecture that the introduction of the cutoff term $\Lambda_h$ in our model will allow us to obtain the complete picture for the null solutions of $D_h$, near the conformal infinity. A case of special interest will be when the componentwise terms $\frac{m}{\Lambda_h} \omega_j$ of $D_h - \frac{m}{\Lambda_h} \omega$ satisfy the following asymptotic expansion, on the limit $\Lambda_h \to \pm \infty$:

$$\frac{m}{\Lambda_h} \omega_j \sim \frac{1 - \cosh(h\xi_j)}{h}, \quad \text{for} \quad \cosh(h\xi_j) = \frac{1 - z_j^2}{1 + z_j^2}.$$ 

In case that such condition is fulfilled, one gets an asymptotic expansion for the symmetric part of $D_h$, i.e. $D_h - \frac{m}{\Lambda_h} \omega \sim \sum_{j=1}^n e_j \frac{\partial_j^2 + \partial_h^2}{2}$ (cf. [E16 Section 4]).

Such characterization, if feasible, will provide an outstanding step on the theory of discrete monogenic functions.

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5 The resulting components of the right-hand side of (16) obtained from the change of variable $m \to \frac{m}{\Lambda_h}$. 
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