ROBUSTLY UNSTABLE EIGENMODES OF THE MAGNETOSHEARING INSTABILITY IN ACCRETION DISKS

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ABSTRACT

The stability of nonaxisymmetric perturbations in differentially rotating astrophysical accretion disks is analyzed by fully incorporating the properties of shear flows. We verify the presence of discrete unstable eigenmodes with complex and pure imaginary eigenvalues, without any artificial disk edge boundaries, unlike Ogilvie & Pringle’s claim. By developing the mathematical theory of a non-self-adjoint system, we investigate the nonlocal behavior of eigenmodes in the vicinity of Alfvén singularities at \( \omega_D = \pm \omega_A \), where \( \omega_D \) is the Doppler-shifted wave frequency and \( \omega_A = k \cdot v_A \) is the Alfvén frequency. The structure of the spectrum of discrete eigenmodes is discussed, and the magnetic field and wavenumber dependence of the growth rate are obtained. Exponentially growing modes are present even in a region where the local dispersion relation theory claims to have stable eigenvalues. The velocity field created by an eigenmode is obtained, which explains the anomalous angular momentum transport in the nonlinear stage of this stability.

Subject headings: accretion, accretion disks — instabilities — MHD — plasmas

1. INTRODUCTION

Over the last several years, the presence of magnetic fields in a differentially rotating plasma has been proposed as a possible mechanism for accretion disk turbulence and its associated large anomalous angular momentum transport inside the disk. The presence of magnetic fields in a shear rotating gas cylinder makes the gas unstable against axisymmetric perturbations (Velikhov 1959; Chandrasekhar 1961). The normal mode analysis (Kumar, Coleman, & Kley 1994) of this local magnetoshearing instability showed the existence of unstable axisymmetric eigenmodes. The presence of this robust instability was recognized (Balbus & Hawley 1991) and confirmed by nonlinear ideal MHD simulations (Hawley & Balbus 1991, 1992; Hawley, Gammie, & Balbus 1995). The observational features of astrophysical accretion disks point to the need for large viscosity much beyond the one collisional mechanisms can yield. This robust instability has been invoked (Hawley & Balbus 1992) as a most promising candidate mechanism for the viscosity puzzle (e.g., Tajima & Shibata 1997).

Numerical investigation of nonaxisymmetric magnetoshearing modes has been carried out by adopting the shearing coordinates (e.g., Balbus & Hawley 1992). Matsumoto & Tajima (1995) analyzed nonaxisymmetric nonlocal eigenmodes that are sandwiched by two Alfvén singularities around the corotational point and are not influenced by the disk edge boundaries and grow exponentially in time. These modes are distinct from the modes discussed by Ogilvie & Pringle (1996), which are nonaxisymmetric modes contained within cylindrical boundaries and which depend strongly on the boundary conditions.

In this paper, we focus on the validity of the linear analysis of nonaxisymmetric eigenmodes questioned by Ogilvie & Pringle (1996). The analysis is performed in the frame rotating with the local angular velocity, which is adopted in nonaxisymmetric mode analysis (Ogilvie & Pringle 1996; Matsumoto & Tajima 1995), since eigenmodes evolve exponentially in time. The resolution of this question is important in the theory of accretion disks. Unless this magnetoshearing instability is a robust mode unaffected by the boundary conditions, the long search for candidate mechanisms of anomalous viscosity of accretion disks needs to be reopened. The criticism of Ogilvie & Pringle (1996) is interesting because it reflects the difficulty and the extreme mathematical and physical subtlety involved in the nature of this mode around the Alfvén singularity. The problem is marred by the non–self-adjointness of the differential equation that describes eigenmodes, arising from the presence of shear flows. We know of no systematic mathematical theory on non–self-adjoint differential equations. Thus, it takes the development of mathematical and physical theory of such a system in order to understand the argument by Ogilvie & Pringle and to respond aptly to it. In the end, such analysis has been developed, and, assuredly, we find that the magnetoshearing instability is both robust and insensitive to the boundary condition, as we thought originally.

In § 2, we derive the wave equation in a differentially rotating magnetized disk based on the analysis of Matsumoto & Tajima (1995), and basic properties of the wave equation are discussed. We show that discrete nonaxisymmetric eigenmodes exist, which are buffeted by the pair of Alfvén singularities where the Doppler-shifted wave frequency equals the Alfvén frequency. Our analysis finds that the eigenmodes oscillate indefinitely in the vicinity of the Alfvén singular points when the eigenvalue is real, whereas the eigenmodes are regular when the eigenvalue is complex. Numerical calculation of eigenmodes with the Alfvén frequency and wavenumber dependence is discussed...
in § 3. We compare the results with the local dispersion relation and show that these eigenmodes are discrete. Astrophysical implications and conclusions are discussed in § 4.

2. ANALYTICAL PROPERTIES OF THE NON–SELF-ADJOINT EQUATION NEAR THE ALFVEN SINGULARITY

We consider the MHD stability of magnetoshearing modes in the corotating frame of the fluid. The basic ideal MHD equations in the frame rotating with angular velocity \( \Omega \) are

\[
\left[ \frac{\partial}{\partial t} + (v \cdot \nabla) \right] v = -\frac{1}{\rho} \nabla P + \frac{\nabla \times B \times B}{4\pi \rho} + g + 2v \times \Omega + (\Omega \times r) \times \Omega , \tag{1}
\]

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) , \tag{2}
\]

where \( g \) is the gravitational acceleration and \( r \) is the position vector. We assume incompressibility for simplicity:

\[
v \cdot v = 0 . \tag{3}
\]

We also ignore self-gravity, which is not essential for the magnetoshearing instability.

We use the local Cartesian coordinates \((x, y, z)\) in the rotating frame where the \( x \)-axis is in the radial direction, the \( y \)-axis in the azimuthal direction, and the \( z \)-axis parallel to \( \Omega \). The uniform velocity shear \( v_s = -(3\Omega/2)x \) is assumed for the Keplerian disk, where \( x = 0 \) is the local corotating radial position. The wave equation is derived by linearizing the basic equations around the equilibrium state and assuming solution of the form \( \phi(x, t) \exp [i(k_y y + k_z z)] \). In the unperturbed state, the density, pressure, and magnetic field are assumed to be uniform. The assumption of \( v_s = v_s = B_z = 0 \) in the unperturbed state yields the unperturbed momentum equation

\[
g + 2v_0 \times \Omega + (\Omega \times r) \times \Omega = 0 . \tag{4}
\]

Next, the Laplace transform of the perturbation, \( \tilde{\phi}(x, \omega) \), is employed,

\[
\tilde{\phi}(x, \omega) = \int_0^\infty dt e^{i\omega t} \phi \, dt . \tag{5}
\]

Substitution of the Laplace transformed momentum and induction equations into the continuity equation yields (see Matsumoto & Tajima 1995 for detail) the initial value equation

\[
\frac{d^2 \tilde{v}_x}{dx^2} + \frac{3\Omega \omega_0^2 k_x}{\omega_0 (\omega_0 - \omega_\lambda^2)} \frac{d\tilde{v}_z}{dx} + \left[ -(k_y^2 + k_z^2) \right] \tilde{v}_x = \Gamma(x, \omega) , \tag{6}
\]

where \( \omega_0 \) is the Doppler-shifted frequency,

\[
\omega_0 = \omega + \frac{3}{2} \Omega k_y x , \tag{7}
\]

and \( \omega_\lambda \) is the Alfvén frequency,

\[
\omega_\lambda^2 = \frac{(k \cdot B)^2}{4\pi \rho} = k_0^2 v_\lambda^2 . \tag{8}
\]

The initial condition enters through the source function \( \Gamma(x, \omega) \).

The wave equation is derived by expressing the homogeneous part of equation (6) in terms of the normalized radial coordinate

\[
\zeta = \frac{3\Omega k_y}{2\omega_\lambda} x \tag{9}
\]

as

\[
\frac{d^2 \tilde{v}_x}{d\zeta^2} + \frac{2\omega_\lambda^2}{\omega_0 (\omega_0 - \omega_\lambda^2)} \frac{d\tilde{v}_z}{d\zeta} + \left[ 4 \left( 1 + \frac{1}{2} \left( \frac{\omega_\lambda}{\Omega} \right)^2 \right) - \frac{2\omega_\lambda^2}{\omega_0 (\omega_0 - \omega_\lambda^2)} + \frac{4 \omega_\lambda^4}{9} \frac{\omega_0^2 + 3\omega_\lambda^2}{(\omega_0 - \omega_\lambda^2)^2} \right] \tilde{v}_x = \Xi(\omega, \zeta) \tilde{v}_x = 0 , \tag{10}
\]

where the ratio of the squares of the azimuthal and vertical wavenumber is defined as

\[
q = \frac{k_y^2}{k_z^2} . \tag{11}
\]

Unstable eigenmodes may exist when the solution satisfies the boundary condition

\[
\lim_{|z| \to \infty} \tilde{v}_x = 0 \tag{12}
\]

in the upper half of the complex-\( \omega \) plane reference. This boundary condition makes our eigenmodes distinct from the modes found by Ogilvie & Pringle (1996), which are confined in rigid cylindrical boundaries and strongly dependent on the boundary condition. In order to have an overall angular momentum transport across the entire disk, it is imperative to have unstable modes within the disk, not just at the boundaries of the disk. To investigate the interior of the accretion disk, eigenmodes should not depend on the edge boundary conditions. The eigenmodes that arise from finite boundaries may contribute to the angular momentum only near the edge of the disk. Our mode, which is not affected by the edge, grows whenever the eigenfunction is only near the edge of the disk. Since the positions of those points are determined in corotating frame, the origin of the frame can be anywhere in the disk. The boundary condition then assures that the momentum transport resulted from the superposition of the growing eigenmodes, which can occur throughout the accretion disk.

The proper boundary condition interior of the disk can be easily examined by inspecting the asymptotic form of the basic differential equation (6) of the system. This indicates that the leading radial dependence of equation (6) leads to the exponential decay away from the corotation point and toward the Alfvén singularity. This mathematics is, of course, most reasonable and physical as well because the instability energy is provided within the two Alfvén singular layers to the mode, which dissipates the energy at (or near) the singularities.

Since the wave equation (10) is not self-adjoint due to the existence of the flow shear, the square of the eigenvalue \( \omega^2 \) is not guaranteed to be real, which any self-adjoint system always satisfies. The fact that the eigenvalue is not pure real or imaginary but in general complex prevents us from applying the Strum-Liouville theory to this system. Note that if \( k_y = 0 \) equation (6) is reduced to self-adjoint,

\[
\frac{d^2 \tilde{v}_x}{dx^2} + k_x^2 \left[ -1 + \Omega^2 \frac{\omega_0^2 + 3\omega_\lambda^2}{(\omega_0 - \omega_\lambda^2)^2} \right] \tilde{v}_x = 0 , \tag{13}
\]
and its eigenvalue constitutes the Alfvén continuum. The analysis of this mode has been done by Chandrasekhar (1961). We now assume $k_s \neq 0$.

To the best of our knowledge, no theory for non–self-adjoint operators exists, and we have to investigate the properties of non–self-adjoint systems in general. We find that the eigenmodes possess certain symmetry properties, which are originated in the character of the differential operator and radial symmetry. Since the differential operator $D(\omega, \xi)$ is invariant under the operation $(\omega, \xi) \to (-\omega, -\xi)$ because of radial symmetry, the eigenfunction $\vec{v}_s(\omega, \xi)$ is also the eigenfunction of $D(-\omega, -\xi)$,

$$D(-\omega, -\xi)\vec{v}_s(\omega, \xi) = 0 .$$  \hspace{1cm} (14)

Note that the operation $\xi \to -\xi$ is the same as changing the direction of the rotation $\Omega \to -\Omega$. Taking the conjugate of equation (14) and denoting $-\xi$ to $\xi$ yields

$$D(\omega^*, \xi)\vec{v}_s^*(\omega, -\xi) = D(\omega^*, -\xi)\vec{v}_s(\omega^*, \xi) = 0 ,$$  \hspace{1cm} (15)

where $\phi^*$ means the complex conjugate of $\phi$. Thus, if $\omega$ is an unstable eigenvalue of equation (10), $-\omega^*$ is another unstable eigenvalue whose eigenfunction $\vec{v}_s(-\omega^*, \xi)$ satisfies relation (15). Especially, when $\omega$ is pure imaginary, i.e., $-\omega^* = \omega$, the real part of the eigenfunction is symmetric and the imaginary part antisymmetric with respect to $\xi = 0$,

$$\vec{v}_s^*(\omega, -\xi) = \vec{v}_s(\omega, \xi) .$$  \hspace{1cm} (16)

These properties of the non–self-adjoint operator indicate that this system has complex eigenvalues in general, which differs from a self-adjoint system, and one unstable eigenvalue has another unstable and two stable companions. This symmetry of non–self-adjoint system and the comparison between self-adjoint and non–self-adjoint eigenvalues in the complex-$\omega$ plane is shown in Figure 1.

Next, we look for the solution of wave equation (10). The boundary conditions for these ideal MHD modes we are interested in are that the eigenmode resides and is differentiable around the corotational point and sandwiched by a pair of Alfvén singularities and decays toward $\xi \to \pm \infty$.

![Figure 1](image.png)

**Fig. 1.**—Symmetry of a non–self-adjoint system and the comparison between self-adjoint and non–self-adjoint eigenvalues in the complex-$\omega$ plane. Discrete modes are denoted by crosses. The eigenvalue of the self-adjoint operator should be purely real or imaginary, and the eigenfunctions are predictable by Strum-Liouville theory. However, there is no such restriction for the eigenvalue of the non–self-adjoint operator, and $\omega, \omega^*, -\omega$, and $-\omega^*$ make a group of solutions.
provided $\text{Re}(s)$ is a nonpositive integer,
\begin{equation}
\tilde{\nu}_s = (\xi - \xi_{\text{A,}S})^{\text{Re}(s)} \exp [\text{Im}(s) \log |\xi - \xi_{\text{A,}S}|].
\end{equation}

The singular points are on the real axis if and only if the eigenvalue $\omega$ is real. In general, the indices are given by
\begin{equation}
s = \frac{\omega(\omega' \pm 3i)}{2(1 \mp i\omega')}
\times \left[-1 \pm \sqrt{1 - \frac{16(4 \mp 2i\omega' - \omega^2)(1 \mp i\omega')^2}{9q \omega'^2(\omega' \mp 3i)^2}} \right],
\end{equation}
where the sign $\pm$ indicates that we take upper sign when $\xi = \xi_{\text{A,}+}$ and lower sign when $\xi = \xi_{\text{A,-}}$. Since $s$ is complex, the solution is not analytic at the Alfvén singular points.

Now, we investigate some special cases. First, when the eigenvalue is pure real, i.e., $\omega' = 0$, the indices are purely imaginary,
\begin{equation}
s = \pm \frac{2}{3\sqrt{q}} i,
\end{equation}
and the eigenfunction rapidly oscillates and is indefinite. Note that the boundary condition in this case becomes special in that we need to shoot outward from $\xi = \xi_c$. We can no longer shoot from the outer to the inner region. As the eigenmodes oscillate indefinitely around the Alfvén singularity, we find that the eigenvalue becomes continuous in this pure real case to form the Alfvén continuum. Second, Alfvén singularities, we find that the eigenvalue becomes continuous in the vicinity of the sin-Alfvén point, the physical eigenfunctions oscillate indefinitely in the vicinity of the sin-Alfvén point. Again, even though the eigenfunction is irregular at the corotation point, the physical eigenmodes on the real $\xi$-axis are regular. Moreover, since all the coefficients of differential equation (10) are real at the corotation point even when the eigenvalue is complex, the eigenfunction should be real at $\xi = \xi_c$.

Although irregular in the vicinity of the Alfvén singularities or the corotation point in most cases, eigenfunctions are not irregular in the physical sense unless the singularities are on the real axis. Instead, the oscillatory behavior and amplitude of the eigenfunction around the Alfvén singularities directly reflects the physical eigenfunction behavior on the real $\xi$-axis, especially if $\omega'$ is small, i.e., the magnetic field is strong.

When the eigenvalue $\omega$ and the index $s$ are both complex, the eigenfunction oscillates indefinitely in the vicinity of the singularity due to the imaginary component of the index $s$ (see eq. [20]). In this case, the physical eigenfunction on the real $\xi$-axis also oscillates very rapidly in the vicinity of the point that is the projection of the complex singular point to the real $\xi$-axis, but the physical eigenfunction oscillates only finite times because the projected point is not a singular point.

When the real component of $s$ is negative, the eigenfunction diverges at the singularity (eq. [20]). The amplitude of the physical eigenfunction on the real $\xi$-axis is large at the projected singular point on the real $\xi$-axis. However, since the projected point is not a singular point, the eigenfunction does not diverge at this point. It is also clear that the eigenvalue is regular even if the singular point is a branch point since we can choose the branch cut of the eigenvalue without crossing the real $\xi$-axis.

If the eigenvalue is real, the singularities are on the real $\xi$-axis and the eigenfunction is irregular in the physical sense. We will discuss pure real eigenvalue cases in § 3.

3. ROBUSTLY UNSTABLE MAGNETOSHEARING EIGENMODES

The eigenvalues of wave equation (10) are calculated numerically by the shooting method with the boundary condition discussed in § 2. Since equation (10) may have three singularities (corotation point and Alfvén resonances), we choose a complex initial value to avoid Alfvén singularities on the real axis and integrate (“shoot”) equation (10) on the real $\xi$-axis from the left and right asymptotic boundaries (which are far removed from any particular singular behavior) to the corotation point, where their value

\begin{equation} \frac{d^2}{dx^2} \psi + \frac{\nu_s^2}{\alpha^2} \psi = 0, \end{equation}
and first derivative of the eigenmode are to be matched. If they are not matched, we change the eigenvalue appropriately until they match. This iterative method is generally called the “shooting method” for eigenvalue problems. Spatial steps of the integration are smaller in the vicinity of the point where the Alfvén point is projected on the real $\xi$-axis than other regions, in case the eigenvalue is almost real but still complex. By using the Newton method to decrease errors of a trial function, we obtain higher accuracy and faster convergence than the previous shooting codes. This allows us to search for subtle singular and regular eigenfunctions over a wide range of parameter values.

In order to satisfy boundary condition (12), we impose $\bar{v}_x/k_\pm$ at the numerical boundaries $\xi = \pm 10$ (corresponding to the artificial infinity), where $k_\pm$ are the negative and positive solutions of the quadratic equation given by inserting the functional form $\bar{v}_x = \exp(k_\pm x)$ into equation (10). On the numerical boundaries, the leading term of equation (6) is the first term of the coefficient of $\bar{v}_x$, which is of the order of 1, and other terms are of the order of 0.01 or lower in our calculation. Our assumption for the boundary condition is justified as far as these estimations are valid.

Figure 2 shows examples of eigenfunctions $\bar{v}_x$ obtained by our shooting code when $\omega_A = 0.1\Omega$ and $q = 0.01$. The solid and dashed curves represent the real and imaginary parts of the eigenfunction, respectively. Figure 2a shows the fundamental pure imaginary eigenvalue, and Figure 2b shows the complex eigenvalue. Since the eigenvalue of Figure 2a is pure imaginary, the real part of the eigenfunction is symmetric and the imaginary part antisymmetric with respect to $\xi = 0$, which is consistent with equation (16). Figure 2b is the eigenfunction with a complex eigenvalue, which makes a pair with the eigenvalue $-\omega^*$ whose eigenfunction is derived from relation (15). These eigenfunctions are confined between two Alfvén singularities located at $\xi = \xi_{s1,2} \sim \pm 1$, and they are real at $\xi = \xi_{s}$.

Figure 3 shows the distribution of eigenvalues in the upper complex-$\omega$ plane when $\omega_A = 0.01$ and $q = 0.01$. It shows only the eigenvalues in the region $\text{Re}(\omega) \geq 0$, and all the complex eigenvalues have a paired eigenvalue $-\omega^*$ in the region $\text{Re}(\omega) < 0$. It is obvious that this non–self-adjoint system has complex eigenvalues, which never appear in a self-adjoint system (see Fig. 1). There are only two pure imaginary eigenvalues, which will be shown to merge by changing $\omega_A$ and $q$. We find that complex eigenvalues, which Matsumoto & Tajima (1995) did not find, exist and have a smaller imaginary part and grow slower in time than the fundamental eigenmode.

Figure 4 shows the dependence of unstable eigenvalues on $\omega_A$ when $q = 0.01$. The solid (dashed) curves show the imaginary (real) part of the eigenvalues. When $\omega_A$ is small, there exist two purely growing modes, which merge at $\omega_A \sim 0.66\Omega$ and form complex eigenvalues. These modes were found in Matsumoto & Tajima (1995), and the qualitative properties of this mode are about the same as found in Matsumoto & Tajima. However, the merging point is slightly greater than the earlier value and, more signifi-
spond to the eigenmodes in Figure 4, when \( q = 0.01 \). Eigenvalues calculated from local analysis are also shown in Figure 5, which we discuss later. Two pure imaginary modes are always distinct when \( \omega_A = 0.01 \Omega \) (two upper modes in Fig. 5a). However, when \( \omega_A = 0.66 \Omega \), these two modes merge and become complex at \( \log_{10} q \sim -1.8 \) and split to become pure imaginary again at \( \log_{10} q \sim 1.4 \). The eigenvalues of the other two complex modes become pure imaginary when \( q \) exceeds a certain value (log \( q = -1.25 \) for \( \omega_A = 0.01 \Omega \), log \( q = -0.75 \) for \( \omega_A = 0.66 \Omega \), and the growth rate for those modes saturates with increasing \( q \).

Next, we compare the nonlocal eigenfunction results with the local (Fourier) dispersion relation. By replacing \( d/dx \) in equation (6) with a constant \( k_x \) around \( x = 0 \) and assuming that the unperturbed magnetic field is toroidal \( (B_z = B_x = 0) \), the local solution in the regime

\[
|\omega| \sim \omega_A \ll \omega \equiv \sqrt{\Omega^2 k_x^2 / (k_x^2 + k_y^2 + k_z^2)}
\]

is (Matsumoto & Tajima 1995)

\[
\omega^2 = \frac{3}{2} \left[ 1 - \frac{3}{2} q \pm \frac{3}{2} \sqrt{(q - 2)(q - \frac{2}{9})} \right] \omega_A^2. \tag{28}
\]

This local dispersion relation (28) shows that pure real eigenmodes appear in the region \( q < \frac{2}{9} \), pure imaginary eigenmodes in \( q > 2 \), and complex in \( \frac{2}{9} < q < 2 \). However, when \( q \) is small, i.e., the perturbation is almost parallel to the magnetic field, nonlocal eigenmodes are unstable in both \( \omega_A = 0.01 \Omega \) and \( 0.66 \Omega \) (see Figs. 5a and 5b), and the growth rate of each mode does not have strong dependence on \( q \). We conclude that replacing \( d/dx \) by a single wavenumber \( k_x \) is invalid for these modes since such eigenmodes oscillate very rapidly in the vicinity of the Alfvén points in a pronounced fashion (see Fig. 2b). In other words, the spatial variation of the wavenumber in the radial direction is essential for the modal analysis of the magnetoshear instability. Radial dependency of the wavelength also prevents us from applying the WKB method to this model. The WKB method requires the wavelength of the eigenmodes \( L_r \) be smaller than the shear scale length \( L_s (L_s/L_r \ll 1) \), which may

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.pdf}
\caption{Alfvén frequency \( (\omega_A = k_x v_A) \) dependence of eigenvalues of magnetoshearing instability when \( q = 0.01 \). The dashed curves and solid curves show the growth rate \( \text{Im}(\omega) \) and the real frequency \( \text{Re}(\omega) \), respectively. The fundamental (F) and secondary (S) pure imaginary eigenmodes and two complex modes (1, 2) are shown, which correspond to the eigenmodes labeled F, S, 1, and 2 in Fig. 3, respectively, when \( \omega_A = 0.01 \). Two pure imaginary eigenvalues (F, S) merge at \( \log_{10} q \sim 0.01 \) to form a complex eigenvalue and split again to become imaginary at \( \log_{10} q \sim 0.66 \). The complex eigenmodes (1, 2) become pure imaginary with increasing \( q \), and the growth rate for those modes saturates with increasing \( q \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.pdf}
\caption{The \( \eta = k_x^2/k_z^2 \) dependence of eigenvalues of magnetoshearing instability. The dashed curves and solid curves show the growth rate \( \text{Im}(\omega) \) and \( \text{Re}(\omega) \), respectively. The fundamental (F) and secondary (S) pure imaginary eigenmodes and two complex modes (1, 2) are shown, which correspond to the eigenmodes labeled F, S, 1, and 2 in Fig. 4 when \( q = 0.01 \). Eigenvalues calculated from local mode analysis are also shown. (a) \( \omega_A = 0.01 \Omega \). Two pure imaginary eigenmodes (F, S) are always distinct, and the complex eigenmodes (1, 2) become pure imaginary, whose growth rate saturates, with increasing \( q \). (b) \( \omega_A = 0.66 \Omega \). Two pure imaginary eigenvalues (F, S) merge at \( \log_{10} q \sim -1.8 \) to form a complex eigenvalue and split again to become imaginary at \( \log_{10} q \sim 1.4 \). The complex eigenmodes (1, 2) become pure imaginary with increasing \( q \), and the growth rate for those modes saturates. In both cases, local modes are stable even in a region where nonlocal modes are unstable.}
\end{figure}
be satisfied around the corotational and Alfvén singularities, but the wavelength is comparable to the shear scale length in other regions ($L_s/L_s \approx 1$).

To show that these eigenmodes are discrete, let us first show the existence of the Alfvén continuum on the real $\omega$-axis. The wave equation (10) has a solution for any real $\omega$ for which $\omega^2 = \omega_a^2$ for some $x$. It follows that the spectrum of this mode is continuous, and the Alfvén continuum extends to the entire real $\omega$ by choosing some $B_z, k_x$, and $k_y$, which is different from the model chosen by Ogilvie & Pringle (1996) in which the Alfvén continuum is restricted by the boundary condition. The eigenmodes with the pure imaginary eigenvalue that we have shown are obviously not in this class.

When $B = 0$, the Kelvin-Helmholtz modes can be derived, which are stable in accretion disks. In this limit, equation (6) reduces to

$$\frac{d^2 \tilde{u}_x}{dx^2} + \left( -k_z^2 + \frac{\Omega^2 k_y^2}{\omega_B^2} \right) \tilde{u}_x = 0 \quad (29)$$

and when $k_z = 0$ it has a simple solution $\tilde{u}_x = \exp \left[ -k_x |x| \right]$, which satisfies boundary condition (12). The first derivative of this class of solutions is discontinuous at $x = 0$, which vanishes with introducing dissipation. The Kelvin-Helmholtz instability also has continuous eigenvalues, but this class of solutions is eliminated in our calculation because of the matching condition of the shooting method, which requires the eigenfunction and its first derivative to be continuous. We search for eigenvalues by choosing initial values in the region $0 < \omega/\omega_a < 1$ and $0 < \omega/\omega_a < 1$. We find that these initial values converge to one of the eigenvalues in Figure 3, and we conclude that all of the eigenmodes are discrete.

In Figure 6, we show another eigenmode whose eigenvalue gradually becomes real with increasing $q$, when $\omega_A = 0.66 \Omega$. However, when the eigenvalue is real we have already shown that the index of the eigenfunction is pure imaginary in the vicinity of the Alfvén singularities (eq. [22]) and that the eigenfunction has the form

$$\tilde{u}_x = \exp \left[ \frac{x}{s} \log (\zeta - \xi_A \pm) \right], \quad (30)$$

from equation (20). Such eigenfunctions oscillate indefinitely toward the singularity, and the function can take any value between $-1 < \tilde{u}_x < 1$. This indicates that the function in the inner region $\xi_A^- < \zeta < \xi_A^+$ and outer region $\zeta < \xi_A^-, \xi < \xi_A^+$ is discontinuous at the Alfvén singularities. Thus, the boundary condition for the continuum cannot be that of shooting from the inside $|\xi| = \infty$ toward the inside, but we should shoot from inside toward the singularities. Figure 7 shows an example of eigenfunction in the inner region when $\omega_A = 0.01 \Omega$, $\omega = 0.002 \Omega$, and $q = 0.01$ (see eq. [22] and arguments for detail of this mode). However, the eigenmode in Figure 6 is continuous even at the Alfvén singular points since the integration by a finite spatial step brings in an effective dissipation, which is not the case for the pure real eigenvalue. Instead, the eigenmode becomes continuous because of the numerical dissipation. Although this numerical eigenfunction is different from the theoretical eigenfunction beyond the passage of the singularity, the fact of continua remains the same for two different reasons. It should be pointed out that in a real physical situation there always exists a dissipation even for a nearly ideal MHD system. The dissipation prevents the eigenmode from blowing up on the Alfvén singular points, keeping the energy of the eigenmode finite. The numerical dissipation affects this eigenmode in the same manner mathematically as the real dissipation does, by passing oscillations through the singularity barrier and averaging oscillation in a finite spatial step. Thus, the numerically obtained eigenmode, though different from theoretically expected continua, may be regarded as realistic.

Finally, we describe the physical behavior of the eigenmodes in accretion disks. The expression of $\tilde{u}_y$ in terms of $\tilde{u}_x$ is derived from continuity equation (3),

$$\tilde{u}_y = \frac{i}{1 + q} \left[ \frac{q}{k_x} \frac{\partial}{\partial \xi} - \frac{\omega_B \Omega}{2(\omega_B^2 - \omega_A^2)} \left( \frac{3 \omega_A^2}{\omega_B^2} + 1 \right) \right] \tilde{u}_x \quad (31)$$

Figure 8a shows $\tilde{v}_y$ calculated from the fundamental pure imaginary mode (Fig. 2a), and the velocity field created in the $x$-$y$ plane by the fundamental eigenmode is shown in Fig. 8b. The eigenfunction of $\tilde{v}_y$ is also trapped between two Alfvén singularities. Since both $\tilde{u}_x$ and $\tilde{v}_y$ are almost out of
phase with each other, the velocity field created by the fundamental eigenmode consists of vortices in the $x$-$y$ plane that are the seeds of nonlinear instability (Matsumoto & Tajima 1995). Note that since we use the frame rotating with angular velocity $\Omega$ there is no unique origin $x$ in the $x$-$y$ plane. Thus, such unstable eigenmodes excited at various $x$-positions will overlap with each other to expand the unstable region in the $x$-direction.

4. SUMMARY

We have shown the existence of the discrete unstable nonaxisymmetric magnetoshearing instability eigenmodes. Since we assumed the exponentially decaying boundary condition (eq. [12]) for the radial component of velocity, our modes are independent of the effect of the boundary condition. In our accretion disk theory, this robust instability occurs without any unrealistic disk edge boundary condition in infinite linear shear flow. The scale length of a single eigenmode in the $x$-direction $\Delta x$ is determined by the local strength of the magnetic field, the direction and amplitude of the wavenumber, and the magnitude of the angular velocity. The eigenfunction is buffered by the Alfvén singular points $\omega_0 = \pm \omega_A$. If the magnetic field is pure toroidal, $\Delta x = 2\omega_A/3\Omega$, where $C_s$ is the sound speed and $\Omega$ is the thickness of the disk. When $v_\infty < C_s$, the mode is localized in the radial direction with the scale length smaller than the thickness of the disk. If the magnetic field has an azimuthal component, $\lambda x$ is proportional to $k_\|/k_y$ (Matsumoto & Tajima 1995). In both cases, our infinite boundary condition is sufficient if the scale length of the eigenmode is smaller than $H$. The curvature of the magnetic field is also small if $\Delta x \ll H$. However, for nearly axisymmetric perturbations ($k_\| \ll k_y$), the eigenmode has a large radial scale length, and the infinite boundary condition may not be valid. Density gradient and geometrical effects become important in this case.

Our result of the growth rate of unstable modes agrees with that of Matsumoto & Tajima (1995) in the region $\omega_A \lesssim \Omega$. We have found complex eigenvalues with smaller growth rates than the fundamental pure imaginary eigenmode. When $\omega_A$ is larger than $\Omega$, two pure imaginary eigenmodes merge, the results of which are the same as found in Matsumoto & Tajima. However, our result shows that the growth rate saturates with increasing $\omega_A$. This indicates that the accretion disk is unstable even if the Alfvén frequency is comparable to the angular velocity, a case of strong magnetic fields.

The comparison of the nonlocal and local dispersion relations demonstrates where and how the local Fourier modes change to nonlocal modes merge, the results of which are the same as found in Matsumoto & Tajima. However, our result shows that the growth rate saturates with increasing $\omega_A$. This indicates that the accretion disk is unstable even if the Alfvén frequency is comparable to the angular velocity, a case of strong magnetic fields.

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We have developed the mathematical and physical theory of a system of non–self-adjoint differential equations for the first time. In astrophysics, non–self-adjointness always appears whenever there is a shear flow, which is mathematically unsolved so far. We find the relationship of complex eigenvalues, which a self-adjoint system does not have, by a general approach to the non–self-adjoint system. A pair of eigenvalues $\omega$ and $-\omega^*$ relate to each other in our model since our model is symmetric with respect to the $x$-axis. Even if there is no such symmetry, we conclude that four eigenmodes make up a group in the non–self-adjoint system in general.

Although our model of the nonaxisymmetric mode has been linear in Cartesian coordinates and ignored the effect of diffusion, analysis in § 3 suggests how the eigenmode grows to enter a nonlinear stage and how it explains momentum transport in accretion disks. Since there is no particular sense in the radial direction in Cartesian coordinates, there exists no specific calibration of momentum transport in the linear stage. The eigenmodes can be excited, however, between any pair of Alfvén singularities in the radial direction (the $x$-axis) and create vortices in the disk.

**Fig. 8**.—Example of $\vec{e}_r$ calculated from the fundamental pure imaginary mode (Fig. 2a) and the velocity field in the $x$-$y$ plane by the fundamental mode. The solid curve and dashed curves show the real and imaginary part of $\vec{e}_r$, respectively. (a) $\vec{e}_r$ is almost out of phase with respect to $\vec{e}_x$. (b) Velocity field is created in the $x$-$y$ plane by the fundamental mode. Vortices are created between two Alfvén singularities, and they will overlap with other eigenmodes excited at various $x$-positions to expand the unstable region.
plane (the $x$-$y$ plane), as shown in Figure 8b. The eigenmode with the fundamental eigenvalue dominates in time for a given radial corotation point. For another (arbitrary) corotation point, the same applies. These eigenmodes can overlap with each other to form greater vortices. In this stage, the nonlinear effect gives rise to anomalous magnetic viscosity that underlies the momentum transport needed to explain astrophysical disks. Matsumoto & Tajima demonstrated nonlinear evolution of the eigenmode by a three-dimensional MHD simulation with the shearing-box model. The overlap of eigenmodes excited at various $x$'s was shown. They also calculated the magnetic viscosity parameter

$$a_{B} = - \frac{\langle \delta B_x \delta B_y \rangle}{4 \pi \rho C_s^2} < \frac{\langle \delta B_x^2 \rangle}{4 \pi \rho C_s^2} \approx \frac{\omega_c^2}{k_\parallel C_s^2} < \frac{1}{(k_\parallel H)^2}, \quad (32)$$

where the notation $\langle \delta B_x \delta B_y \rangle$, etc., denotes the spatial average. They found that when the poloidal field is dominant the magnetic viscosity is $a_{B} \sim O(0.1)$, which corresponds to $\varepsilon$ in dwarf novae during the bursting phase.

We conclude that the results of Matsumoto & Tajima are correct and that the robust mode in the magnetized accretion disks is of the magneto-shearing origin. This mode should be dominant in nonlinear theory, and our linear analysis supports the results from the three-dimensional simulation in Matsumoto & Tajima, which explained anomalous momentum transport in accretion disks.

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APPENDIX

EXISTENCE OF THE LOCALIZED GROWING MODE IN THE INFINITE SHEARING FLOW

Ogilvie & Pringle (1996) in their Appendix C claim to have proved the nonexistence of localized growing eigenfunctions. They discussed the real axis asymptotic behavior of a trial function as $e^{\pm ikx}$. No property other than this particular behavior for an eigenfunction is incurred in their paper. Their proof of nonexistence is based on an integral (Ogilvie & Pringle 1996, eq. [C8]),

$$I \equiv \int_{-\infty}^{+\infty} \left( \frac{dy}{d\lambda} \right)^2 \left( \frac{d\lambda}{dx} \right)^2 + (k^2 - f) |y|^2 \frac{dy}{d\lambda} d\lambda, \quad (A1)$$

where $y(x)$ is the eigenmode (or trial function). Note that $f(x)$ has two Alfvén singular points in the lower half-plane for an unstable mode and decays to zero as $|x| \to \infty$ in any direction in the complex plane. For the nature of nonexistence, they set out to probe for an unspecified function (other than the above asymptotic behavior) $y(x)$. However, with this little restriction it is impossible to demonstrate what they set out to do. In fact, we can show that under the specific case $x = x(\lambda) = \lambda$, their claim does not hold. From the boundary condition we know that for the eigenmode that is "trapped" between the two Alfvén singularities $y(x) \propto e^{\pm ikx}$ for $x \to -\infty$ and $y(x) \propto e^{-ikx}$ for $x \to +\infty$ (with $k > 0$). Between two Alfvén singularities, $y(x)$ takes a complex function form, as discussed in equation (17) with the index $s$ defined by equation (19). The function $y(x)$ is generally oscillatory in $x$ in this region. In the complex upper half-plane, there is no singularity. Thus, we might want to close an upper half-plane contour. This integral along the upper semicircle may be written as

$$I = I_1 + I_2 + I_3 = \lim_{r \to \infty} \lim_{\epsilon \to 0} \left( \int_{0}^{\pi/2 - \varepsilon} + \int_{\pi/2 + \varepsilon}^{\pi} \right) \left( \frac{dy}{dx} \right)^2 + (k^2 - f) |y|^2 \frac{dy}{d\lambda} d\lambda \quad (A2)$$

It is shown that this integral $I$ cannot be positive (or negative) definite. First, it is easy to show $I_1 = I_2 = 0$ because asymptotic solutions $e^{\pm ikx}$ vanish on the contour of these integrals. Note that in $I_1$ we should take the $e^{-ikx}$-like behavior, while in $I_2$ we need $e^{+ikx}$, as discussed above. This is the necessary feature to satisfy the boundary conditions for the bound mode. As for $I_2$, let us consider more in detail. The asymptotic form of the function $f(x)$ is $\eta/x^2 + \mu/x^4 \ (|x| \to \infty)$. The solution of the equation $y'' - (k^2 + \eta/x^2)y = 0$ is chosen as the trial function, which is given by a linear combination of two independent solutions $y(x) = \sqrt{x} Z_1(kx)$ and $y(x) = \sqrt{x} Z_2(kx)$ with $v^2 = 1/4 - \eta; Z_1(kx)$ is $I_1(kx)$ or $K_1(kx)$ when $v$ is real. When $v$ is pure imaginary, $Z_1$ is the usual series solutions of type of $I_1$ or $K_1$ by applying the Frobenius method, which has essential singularity at the origin. The integral $I_2$ is then given by

$$I_2 = \lim_{r \to \infty} \lim_{\varepsilon \to 0} \frac{i \mu}{v - 1} \frac{\sin [2\varepsilon (v - 1)] J_1^2(kr)}{r^2}. \quad (A3)$$

This integral (A3) vanishes as $\varepsilon \sim 1/r$ and taking $r \to \infty$. Thus, we proved that it cannot be said that $I$ is positive (or negative) definite. In fact, if Ogilvie & Pringle were correct the solution would exist no unstable eigenmode for arbitrary shear flows (as no particular shear flow rate form $u'$ is sorted) even for the Kelvin-Helmholtz problem. This is clearly not the case.

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