An Efficient Planar Bundle Adjustment Algorithm

Lipu Zhou* Daniel Koppel† Hui Ju‡ Frank Steinbruecker§ Michael Kaess ¶
Magic Leap Magic Leap Magic Leap Magic Leap Carnegie Mellon University

ABSTRACT
This paper presents an efficient algorithm for the least-squares problem using the point-to-plane cost, which aims to jointly optimize depth sensor poses and plane parameters for 3D reconstruction. We call this least-squares problem Planar Bundle Adjustment (PBA), due to the similarity between this problem and the original Bundle Adjustment (BA) in visual reconstruction. As planes ubiquitously exist in the man-made environment, they are generally used as landmarks in SLAM algorithms for various depth sensors. PBA is important to reduce drift and improve the quality of the map. However, directly adopting the well-established BA framework in visual reconstruction will result in a very inefficient solution for PBA. This is because a 3D point only has one observation at a camera pose. In contrast, a depth sensor can record hundreds of points in a plane at a time, which results in a very large nonlinear least-squares problem even for a small-scale space. Fortunately, we find that there exist a special structure of the PBA problem. We introduce a reduced Jacobian matrix and a reduced residual vector, and prove that they can replace the original Jacobian matrix and residual vector in the generally adopted Levenberg-Marquardt (LM) algorithm. This significantly reduces the computational cost. Besides, when planes are combined with other features for 3D reconstruction, the reduced Jacobian matrix and residual vector can also replace the corresponding parts derived from planes. Our experimental results verify that our algorithm can significantly reduce the computational time compared to the solution using the traditional BA framework. Besides, our algorithm is faster, more accurate, and more robust to initialization errors compared to the start-of-the-art solution using the plane-to-plane cost [4].

Index Terms: Bundle Adjustment—Nonlinear Optimization—SLAM—Depth Sensor

1 INTRODUCTION
A 3D map is essential for an Augmented Reality (AR) system, and many other computer vision and robotics applications. The 3D structure of the scene can be derived from 2D visual feature matching among multiple images through technologies such as Structure from Motion (SfM) or Simultaneous Localization and Mapping (SLAM) [16, 19, 22]. Bundle Adjustment (BA) is a nonlinear least-squares problem which aims to jointly refine camera poses and 3D structure to minimize the visual reconstruction cost. It is a crucial component in the visual reconstruction to generate a high quality and globally consistent 3D map. It is known that this problem has been well-studied in the computer vision and robotics communities [3, 7, 16, 20, 21, 23, 37, 39].

Generally, SfM or SLAM algorithms yield sparse or semi-dense maps. Recently, the emergence of commercial high-quality depth sensors (e.g., LiDAR and RGBD camera) has made feasible the building of dense 3D maps, which are desirable, not only for providing the information for localization, but also for the information for environment understanding, such as 3D object classification and semantic segmentation [26], required in most of today's AR applications. As planes ubiquitously exist in man-made scenes, they are generally exploited as features in the SLAM algorithms for various depth sensors [10–13, 15, 17, 24, 27, 30, 34, 35, 38].

Similar to the visual reconstruction, we face a nonlinear least-squares problem whose goal is to estimate optimal sensor poses and plane parameters. Although BA is a specific terminology related to visual reconstruction, here we name this least-squares problem as Planar Bundle Adjustment (PBA), due to the similarity between the two least-squares problems. Although BA for visual reconstruction has been well studied in the literature, little research has been done on the PBA. Therefore, this raises the demand for studying the PBA problem for depth sensors.

The PBA is the problem of jointly optimizing parameters of planes and sensor poses. Although the plane is seemingly the counterpart of the point in BA for visual reconstruction, there exists a significant difference between them. In the literature, the plane-to-plane distance based on the rigid-body transformation for plane parameters is generally employed to construct the cost function. However, this cost function may introduce bias which may result in a suboptimal solution as described in Sect. 2. Thus this paper adopts the point-to-plane distance to construct the cost function. A 3D point can only yield one observation for each camera pose. However, as a plane is an infinite object, one recording of a depth sensor can provide many points as a partial observation of a plane. Thus a single depth sensor recording can generate many constraints on planes and poses. Therefore, directly adopting the original visual BA framework to the planar case will result in a large-scale nonlinear least-square problem even for a small scene, which incurs high computational cost and memory consumption. This paper addresses this problem.

The main contribution of this paper is an efficient planar bundle adjustment algorithm. The key point of our algorithm is to explore the special structure of PBA. Based on the special structure of PBA, we introduce a reduced Jacobian matrix and a reduced residual vector. We prove that no matter how many points of a plane are recorded by a depth sensor, the derived Jacobian matrix and residual vector can be replaced by the reduced ones in the Levenberg-Marquardt (LM) algorithm [18, 21]. The reduced Jacobian matrix and residual vector have fixed sizes that are much smaller than the original ones. This significantly reduces computational cost and memory usage. More generally, when 3D reconstruction applications exploit planes together with other features such as points [6, 9, 29, 22, 16], the same reduction technique can be applied to the blocks inside the Jacobian and the residual, which correspond to the planar constraints.

2 RELATED WORK AND THEORETICAL BACKGROUND
Planes are widely adopted as landmarks in SLAM algorithms for depth sensors. The related topics include plane detection, matching, 3D registration and joint optimization with poses and planes, etc. This research on the joint optimization problem is relative small. This paper focuses on this problem. In this section, we introduce the related work and the theoretical background for the optimization problem. In the following description, we use italic, boldfaced lowercase and boldfaced uppercase letters to represent scalars, vectors...
and matrices, respectively.

2.1 Cost Functions for the Plane Correspondence

One prerequisite step of formulating an optimization problem is to construct the cost function. For the plane correspondence, there are two cost functions generally used in the literature, i.e., plane-to-plane and point-to-plane.

Plane-to-plane cost is based on the rigid-body transformation for the plane parameters. Specifically, suppose a plane is represented by the Hesse normal form as \( \mathbf{p} = [\mathbf{n}, d] \), where \( \mathbf{n} \) is the plane normal with \( ||\mathbf{n}||_2 = 1 \) and \( d \) is the negative distance from the coordinate system origin to the plane. Assume the rotation and translation from a depth sensor coordinate system to a global coordinate system is \( \mathbf{R} \) and \( \mathbf{t} \). Let \( \mathbf{p}_i \), represent the parameters of a plane in the depth sensor coordinate system, which are estimated from a set of points \( \mathcal{P} \) in the depth sensor measurements, and \( \mathbf{p}_g \) denote its parameters in the global coordinate system. Then the relation between \( \mathbf{p}_i \) and \( \mathbf{p}_g \) can be described as [8]

\[
\mathbf{p}_g = \mathbf{T}^T \mathbf{p}_i, \quad \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}.
\] (1)

\( \mathbf{p}_i \) and \( \mathbf{T} \) are the variables that we want to estimate. We can have a general form of the plane-to-plane residual as below

\[
\mathbf{p}_i - \mathbf{T}^T \mathbf{p}_g
\] (2)

where \( \odot \) is a function to measure the difference between \( \mathbf{p}_i \) and \( \mathbf{T}^T \mathbf{p}_g \). Kaess [13] represents planes as quaternions, and defines \( \odot \) by measuring the difference between two quaternions. Geneva et al. [4] introduce the closest point (CP) vector, i.e., \( \mathbf{d} \), to parameterize a plane. They define \( \odot \) as the difference between two CP vectors.

The above equation (2) can be directly used to construct cost function for jointly tuning poses and planes [12]. But the relative plane formulation introduced by Kaess [13] converges faster. In the relative plane formulation, a plane is expressed relative to the first pose that observes it. This formulation is adopted later in [10, 11] for global joint optimization of poses and planes when loop closure occurs. Geneva et al. [4] present a similar relative plane formulation as [13]. They introduce the CP vector to represent the plane, which shows improved accuracy and faster convergence compared to the plane parameterization using quaternion introduced in [13].

Point-to-plane cost is the squared distance from a point to a plane. As mentioned above, \( \mathcal{P} \) is the point set of a plane observed by a depth sensor. Assume \( \mathbf{p}_i \in \mathcal{P} \) and \( \mathbf{p}_g = [\mathbf{n}_j; d_j] \). The signed distance from \( \mathbf{p}_i \) to \( \mathbf{p}_g \) has the form as

\[
\delta = \mathbf{n}_j \cdot (\mathbf{R}\mathbf{p}_i + \mathbf{t}) + d_j.
\] (3)

where \( \cdot \) represents the dot product. The point-to-plane cost \( \delta^2 \) is generally employed to calculate the pose between a local depth sensor point cloud and a global point cloud [11, 23, 39] rather than to jointly optimize poses and plane parameters. This is because there generally exists a large number of points in \( \mathcal{P} \), which leads to a very large-scale least-squares problem. Thus, it is seldom adopted in the SLAM algorithm for global poses and planes joint optimization.

plane-to-plane vs. point-to-plane As described in [8], the cost function will impact the accuracy of the solution. The point-to-plane cost is well defined. It is the squared distance from \( \mathbf{p}_i \) to \( \mathbf{p}_g \), which is invariant to rigid transformations. This means the cost is invariant to the choice of the global coordinate system. It only depends on estimation errors of poses and planes. However, the plane-to-plane cost based on plane parameter transformation [1] does not have this property. As demonstrated in [10] (a) changing the coordinate system will vary the plane-to-plane cost. Thus the accuracy of the result of minimizing the plane-to-plane cost may depend on the particular choice of coordinate systems, which introduces uncertainty to the solution.

Furthermore, the plane-to-plane cost may introduce bias. For example, given the ground truth of the two poses and \( \mathbf{p}_g \), the two local point clouds in (b) generate different plane-to-plane costs, although they yield the same point-to-plane costs. Plane \( \mathbf{p}_g^1 \) has smaller cost than \( \mathbf{p}_g^2 \), merely because of the certain choice of the global coordinate system rather than the second pose error is larger. However, as plane \( \mathbf{p}_g^1 \) generates smaller cost than \( \mathbf{p}_g^2 \), this may cause the optimization algorithm to change the second pose of the sensor to push plane \( \mathbf{p}_g^2 \) toward plane \( \mathbf{p}_g^1 \) to minimize the plane-to-plane cost, which is prone to increase errors.

Our experimental results verify that minimizing the point-to-plane cost results in a more accurate solution than minimizing the plane-to-plane cost. Besides, our results show that the point-to-plane cost is more robust to initialization errors. However, using the point-to-plane cost yields a much larger scale least-squares problem. This paper presents an efficient solution to minimize the point-to-plane cost.

2.2 The Bundle Adjustment Problem

3D reconstruction is a fundamental problem with a large number of applications. BA is a crucial step for 3D reconstruction. Let us first consider the well-established BA framework for the visual reconstruction.

The BA framework for the visual reconstitution is essentially a nonlinear least-squares problem with sparse structure [3, 20, 21, 33]. The LM algorithm [15, 21] is generally adopted to solve this problem. Let us begin with a brief introduction of the LM algorithm for a general least-squares problem.

Suppose that we have an \( n \)-dimensional measurements \( \mathbf{y} = [y_1; y_2; \ldots; y_m] \) modeled by function \( \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}); f_2(\mathbf{x}); \ldots; f_n(\mathbf{x})] \), where \( \mathbf{x} \in \mathbb{R}^m \) is an \( m \)-dimensional model parameters which we seek to estimate. The least-squares problem is to find the optimal \( \mathbf{x} \) which minimizes the sum of squared errors of \( \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{y} \), i.e.,

\[
\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \frac{1}{2} \| \hat{\mathbf{y}}(\mathbf{x}) \|_2^2.
\] (4)

We use \( \mathbf{J}(\mathbf{x}) \) to represent the Jacobian matrix of \( \mathbf{f}(\mathbf{x}) \) at \( \mathbf{x} \), whose \( i \)-th row and \( j \)-th column element is \( J_{ij}(\mathbf{x}) = \frac{\partial f_i(\mathbf{x})}{\partial x_j} \), where \( x_j \) is the
which can make the size of the resulting nonlinear least-squares problem prohibitively large. It would be time-consuming to just compute the Jacobian matrix in (6). But plane parameters have extra constraints. The parameterization of both entities has been well-studied. For instance, $\rho_i$ can be parameterized by a quaternion, angle-axis or Euler angles. The plane parameters $\pi_j$ can be represented by homogeneous coordinates or the minimal parameterization based on quaternion introduced in [13]. Our algorithm does not depend on a special parameterization. We define $\theta \mapsto R(\theta)$ and $\omega \mapsto \pi(\omega)$ to represent arbitrary parameterization for rotation and plane parameters. $	heta$ and $\omega$ are related to the sensor pose. According to the convention of visual BA introduced above, we combine them as $\rho_i = \{\theta_i; \omega_i\}$. Generally, $\rho_i$ has 6 or 7 unknowns, depending on the parameterization of the rotation matrix (6 for the cases of any minimal representations of the rotation and 7 for the quaternion). $\omega_i$ generally has 3 or 4 unknowns (3 for minimal representations of a plane [13 36] and 4 for the homogeneous coordinates of a plane [8]).

Using the above notations, $\delta_{ijk}$ is a function of $\rho_i$ and $\omega_j$. The PBA is the problem of jointly refining all $\rho_i$ ($i \neq 1$) and $\omega_j$ by minimizing the following nonlinear least-squares problem

$$\min_{\rho_i; \omega_j} \sum_i \sum_j \sum_k \delta_{ijk}^2 (\rho_i; \omega_j).$$

Here the first pose $\rho_1$ is fixed during the optimization to anchor the coordinate system rigidly.

### 3.2 Structure of the Jacobian Matrix

One crucial step of the LM algorithm is to calculate the Jacobian matrix. The Jacobian matrix of planar BA has a special structure. The observation of the $j$th plane at the $i$th pose is a point set $P_{ij}$. Let us consider the Jacobian matrix $J_{ij}$ derived from $P_{ij}$. The whole Jacobian matrix $J$ is the stack of all $J_{ij}$. Assume there are $K_i$ points in $P_{ij}$ and $J_{ij}$ ($i \neq 1$) has the following form

$$J_{ij} = \begin{bmatrix}
0 & \cdots & \frac{\partial \delta_{i1}}{\partial \rho_i} & \cdots & 0 & \cdots & \frac{\partial \delta_{i1}}{\partial \omega_j} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \frac{\partial \delta_{iK_i}}{\partial \rho_i} & \cdots & 0 & \cdots & \frac{\partial \delta_{iK_i}}{\partial \omega_j} & \cdots & 0
\end{bmatrix}$$

(10)

To calculate $\frac{\partial \delta_{ij}}{\partial \rho_i}$ and $\frac{\partial \delta_{ij}}{\partial \omega_j}$, we begin by considering the form of $\delta_{ijk}$ in (8). Let us define

$$R_i = \begin{bmatrix}
R_i^{1,1} & R_i^{1,2} & R_i^{1,3} \\
R_i^{2,1} & R_i^{2,2} & R_i^{2,3} \\
R_i^{3,1} & R_i^{3,2} & R_i^{3,3}
\end{bmatrix},
t_i = \begin{bmatrix}
t_i^1 \\
t_i^2 \\
t_i^3
\end{bmatrix},
n_j = \begin{bmatrix}
n_j^1 \\
n_j^2 \\
n_j^3
\end{bmatrix},
p_{ijk} = \begin{bmatrix}
x_{ijk} \\
y_{ijk} \\
z_{ijk}
\end{bmatrix}$$

(11)

Figure 2: Bayes Net of PBA with $N$ poses and $M$ planes. The first pose $\rho_1$ is fixed during the optimization. $P_{ij}$ is the set of measurements of the $j$th plane $\pi_j$ recorded at the $i$th pose $\rho_i$. PBA is the problem of jointly refining $\rho_i$ ($i \neq 1$) and $\pi_j$.

Figure 3: The geometric entities involved in PBA. $p_{ijk}$ is the $k$th point of the point set $P_{ij}$ that are captured from the $j$th plane $\pi_j$ at the $i$th pose $\rho_i$. $\delta_{ijk}$ is the signed distance from $p_{ijk}$ to $\pi_j$ defined in (8).
Note that the elements of $\mathbf{R}$ defined above are functions of $\mathbf{\theta}$, and also that $d_j$ and the elements of $\mathbf{n}_i$ are functions of $\mathbf{\omega}_j$. Substituting (11) into (8) and expanding it, we have

$$
\begin{aligned}
\delta_{ijk} &= x_{ijk}R_i^{1,1}n_j + y_{ijk}R_i^{1,2}n_j + z_{ijk}R_i^{1,3}n_j + \\
x_{ijk}R_i^{2,1}n_j + y_{ijk}R_i^{2,2}n_j + z_{ijk}R_i^{2,3}n_j + \\
x_{ijk}R_i^{3,1}n_j + y_{ijk}R_i^{3,2}n_j + z_{ijk}R_i^{3,3}n_j + \\
n_{ij1}^T + n_{ij2}^T + n_{ij3}^T + d_j,
\end{aligned}
\tag{12}
$$

We can rewrite (12) as

$$
\delta_{ijk} = c_{ijk} \cdot \mathbf{v}_{ij},
\tag{13}
$$

where $c_{ijk}$ and $\mathbf{v}_{ij}$ are 13-dimensional vectors as

$$
c_{ijk} = \left[ x_{ijk}, y_{ijk}, z_{ijk}, x_{ijk}, y_{ijk}, z_{ijk}, x_{ijk}, y_{ijk}, z_{ijk}, 1, 1, 1 \right]^T,
$$

$$
\mathbf{v}_{ij} := \left[ R_i^{1,1}n_j, R_i^{1,2}n_j, R_i^{1,3}n_j, R_i^{2,1}n_j, R_i^{2,2}n_j, R_i^{2,3}n_j, R_i^{3,1}n_j, R_i^{3,2}n_j, R_i^{3,3}n_j, n_{ij1}^T, n_{ij2}^T, n_{ij3}^T, d_j \right]^T.
\tag{14}
$$

The elements in $c_{ijk}$ are from the observation $p_{ijk}$ or 1. They are constants. On the other hand, the elements in $\mathbf{v}_{ij}$ are functions of $\mathbf{\rho}_i$ and $\mathbf{\omega}_j$. They are related to the unknowns which we want to estimate.

Let us calculate the partial derivative of $\delta_{ijk}$. Assume that $\mathbf{p}_i$ has $n_p$ unknowns and $\mathbf{\omega}_j$ has $n_\omega$ unknowns. We define

$$
\zeta_{ij} := \left\{ \frac{\partial}{\partial \mathbf{\rho}_i} \right\}_n \text{ unknowns}
\tag{15}
$$

Suppose $c_{ijk}^d$ is the $d$th element of $\zeta_{ij}$. According to (13), the partial derivative of $\delta_{ijk}$ with respect to $c_{ijk}^d$ has the following form

$$
\frac{\partial \delta_{ijk}}{\partial c_{ijk}^d} = \frac{\partial c_{ijk} \cdot \mathbf{v}_{ij}}{\partial c_{ijk}^d} = c_{ijk} \cdot \frac{\partial \mathbf{v}_{ij}}{\partial c_{ijk}},
\tag{16}
$$

where $\frac{\partial \mathbf{v}_{ij}}{\partial c_{ijk}}$ is a 13-dimensional vector whose elements are the partial derivatives of the elements of $\mathbf{v}_{ij}$ with respect to $c_{ijk}^d$.

Then we consider $\frac{\partial \delta_{ijk}}{\partial \mathbf{p}_i}$ and $\frac{\partial \delta_{ijk}}{\partial \mathbf{\omega}_j}$. According to (16), $\frac{\partial \delta_{ijk}}{\partial \mathbf{p}_i}$ has the following form

$$
\frac{\partial \delta_{ijk}}{\partial \mathbf{p}_i} = \left[ \frac{\partial \delta_{ijk}}{\partial \mathbf{v}_{ij}}, \ldots, \frac{\partial \delta_{ijk}}{\partial \mathbf{v}_{ij}} \right] = \left[ c_{ijk} \cdot \frac{\partial \mathbf{v}_{ij}}{\partial \zeta_{ij}}, \ldots, c_{ijk} \cdot \frac{\partial \mathbf{v}_{ij}}{\partial \zeta_{ij}} \right] = c_{ijk}^T \frac{\partial \mathbf{v}_{ij}}{\partial \zeta_{ij}},
\tag{17}
$$

Similarly, we can calculate $\frac{\partial \delta_{ijk}}{\partial \mathbf{\omega}_j}$ as

$$
\frac{\partial \delta_{ijk}}{\partial \mathbf{\omega}_j} = c_{ijk}^T \left[ \frac{\partial \mathbf{v}_{ij}}{\partial \zeta_{ij}}, \ldots, \frac{\partial \mathbf{v}_{ij}}{\partial \zeta_{ij}} \right],
\tag{18}
$$

Typically, $\mathbf{v}_{ij}^\omega$ is a $13 \times 3$ or $13 \times 4$ matrix ($13 \times 3$ for minimal representations of a plane $\mathbf{I}_3$ and $13 \times 4$ for homogeneous coordinates of a plane $\mathbf{I}_4$).

Now we consider the form of $\mathbf{J}_{ij}$. Let us define

$$
\mathbf{C}_{ij} := \begin{bmatrix} c_{ij1}^T \\ c_{ij2}^T \\ \vdots \\ c_{ijK_i}^T \end{bmatrix},
\tag{19}
$$

where the $k$th row $c_{ijk}^T$ is defined in (14). $\mathbf{C}_{ij}$ is a matrix of size $K_i \times 13$. Substituting (17) and (18) into (10) and using the definition of $\mathbf{C}_{ij}$ in (19), we have

$$
\mathbf{J}_{ij} := \begin{bmatrix} 1 & c_{ij1} \mathbf{y}_{ij}^\rho & \cdots & 0 & \cdots & c_{ij2} \mathbf{y}_{ij}^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & c_{ijK_i} \mathbf{y}_{ij}^\rho & \cdots & 0 \\ c_{ij1} \mathbf{v}_{ij}^\rho & c_{ij2} \mathbf{v}_{ij}^\rho & \cdots & \cdots & \cdots & c_{ijK_i} \mathbf{v}_{ij}^\rho & \cdots & \cdots & \cdots \end{bmatrix},
\tag{20}
$$

**Jacobian Matrix for the First Pose** Suppose $\mathbf{P}_{ij}$ is the set of measurements from the $j$th plane $\mathbf{\omega}_j$ at the first pose $\mathbf{\rho}_1$. As we fix $\mathbf{\rho}_1$ during the optimization, the Jacobian matrix $\mathbf{J}_{ij}$ derived from $\mathbf{P}_{ij}$ has a special form as

$$
\mathbf{J}_{ij} := \begin{bmatrix} 0 & \cdots & c_{ij1} \mathbf{y}_{ij}^\rho & \cdots & 0 & \cdots & c_{ij2} \mathbf{y}_{ij}^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & c_{ijK_i} \mathbf{y}_{ij}^\rho & \cdots & 0 \\ c_{ij1} \mathbf{v}_{ij}^\rho & c_{ij2} \mathbf{v}_{ij}^\rho & \cdots & \cdots & \cdots & c_{ijK_i} \mathbf{v}_{ij}^\rho & \cdots & \cdots & \cdots \end{bmatrix}.
\tag{21}
$$

### 3.3 Factorization of $\mathbf{C}_{ij}$

$\mathbf{C}_{ij}$ has a special structure. It has duplicated columns, according to the structure of $c_{ijk}$ defined in (14). We have the following lemma about $\mathbf{C}_{ij}$:

**Lemma 1.** $\mathbf{C}_{ij}$ can be written in the following form

$$
\mathbf{C}_{ij} = \mathbf{Q}_{ij} \mathbf{M}_{ij},
\tag{22}
$$

where $\mathbf{M}_{ij}$ has the size $4 \times 13$ and $\mathbf{Q}_{ij} \mathbf{Q}_{ij}^T = \mathbf{I}_4$, where $\mathbf{I}_4$ is the $4 \times 4$ identity matrix.

**Proof.** As shown in the definition of $c_{ijk}$ in (14), $x_{ijk}$, $y_{ijk}$, $z_{ijk}$ and 1 are duplicated several times to form $c_{ijk}$. Therefore, there are only 4 unique columns among the 13 columns of $\mathbf{C}_{ij}$, which contains the constant 1 and the $x, y, z$ coordinates of points within $\mathbf{P}_{ij}$. We denote them as

$$
\mathbf{F}_{ij} := \begin{bmatrix} x_{ij1} & y_{ij1} & z_{ij1} & 1 \\ x_{ij2} & y_{ij2} & z_{ij2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{ijK_i} & y_{ijK_i} & z_{ijK_i} & 1 \end{bmatrix},
\tag{23}
$$

$$
= \begin{bmatrix} x_{ij} & y_{ij} & z_{ij} & 1 \end{bmatrix}.
\tag{24}
The 13 columns in \( C_{ij} \) are simply a copy of the 4 columns in \( E_{ij} \). Let us define the thin QR decomposition \( \mathbf{E}_{ij} = \mathbf{Q}_{ij} U_{ij} \) as

\[
\mathbf{E}_{ij} = \mathbf{Q}_{ij} U_{ij}
\]  

(24)

where \( \mathbf{Q}_{ij}^T \mathbf{Q}_{ij} = \mathbf{I}_4 \) and \( U_{ij} \) is an upper triangular matrix. \( \mathbf{Q}_{ij} \) is of size \( K_{ij} \times 4 \) and \( U_{ij} \) is of size \( 4 \times 4 \). Here we use the thin QR decomposition, since the number of points \( K_{ij} \) is generally much larger than 4. The thin QR decomposition can reduce computational time. We partition \( U_{ij} \) into its columns as

\[
\mathbf{U}_{ij} = \begin{bmatrix}
\mathbf{u}_{ij}^1 & \mathbf{u}_{ij}^2 & \mathbf{u}_{ij}^3 & \mathbf{u}_{ij}^4
\end{bmatrix}.
\]  

(25)

Substituting (25) into (23), we have

\[
\mathbf{E}_{ij} = \mathbf{Q}_{ij} \begin{bmatrix}
\mathbf{u}_{ij}^1 & \mathbf{u}_{ij}^2 & \mathbf{u}_{ij}^3 & \mathbf{u}_{ij}^4
\end{bmatrix}.
\]  

(26)

Comparing (23) and (26), we get

\[
x_{ij} = \mathbf{Q}_{ij} \mathbf{u}_{ij}^1, \quad y_{ij} = \mathbf{Q}_{ij} \mathbf{u}_{ij}^2,
\]

\[
z_{ij} = \mathbf{Q}_{ij} \mathbf{u}_{ij}^3, \quad 1 = \mathbf{Q}_{ij} \mathbf{u}_{ij}^4.
\]  

(27)

As the columns of \( C_{ij} \) are copies of the columns of \( E_{ij} \), according to the form of \( c_{ij,k} \) in (14) and the definition of \( \mathbf{E}_{ij} \) in (25), \( C_{ij} \) can be written as

\[
C_{ij} = \begin{bmatrix}
x_{ij}, y_{ij}, z_{ij}, x_{ij}, y_{ij}, z_{ij}, x_{ij}, y_{ij}, z_{ij}, 1, 1, 1, 1
\end{bmatrix}
\]  

(28)

Substituting (27) into (28), we finally have

\[
C_{ij} = \mathbf{Q}_{ij} \begin{bmatrix}
\mathbf{u}_{ij}^1, \mathbf{u}_{ij}^2, \mathbf{u}_{ij}^3, \mathbf{u}_{ij}^4,
\mathbf{u}_{ij}^1, \mathbf{u}_{ij}^2, \mathbf{u}_{ij}^3, \mathbf{u}_{ij}^4,
\mathbf{u}_{ij}^1, \mathbf{u}_{ij}^2, \mathbf{u}_{ij}^3, \mathbf{u}_{ij}^4,
\mathbf{u}_{ij}^1, \mathbf{u}_{ij}^2, \mathbf{u}_{ij}^3, \mathbf{u}_{ij}^4
\end{bmatrix}
\]  

\[
= \mathbf{Q}_{ij} \mathbf{M}_{ij}
\]  

(29)

The factorization of \( C_{ij} \) can be used to significantly reduce the computational cost as described below.

### 3.4 Reduced Jacobian Matrix

According to Lemma 1, \( C_{ij} \) can be factorized as \( C_{ij} = \mathbf{Q}_{ij} \mathbf{M}_{ij} \). We define the reduced Jacobian matrix \( J'_{ij} \) of \( J_{ij} \) as

\[
J'_{ij} = \begin{cases}
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix} & i = 1 \\
\begin{bmatrix}
0 & \cdots & \mathbf{M}_{ij} V_{ij}^o & \cdots & 0 & \cdots & 0
\end{bmatrix} & i \neq 1
\end{cases}
\]  

(30)

We call it a reduced Jacobian matrix, because \( \mathbf{M}_{ij} \) is a much smaller matrix than \( C_{ij} \). \( C_{ij} \) has size \( K_{ij} \times 13 \). According to Lemma 1, \( \mathbf{M}_{ij} \) has size \( 4 \times 13 \). Generally, \( K_{ij} \) is much larger than 4.

We stack \( J_{ij} \) and \( J'_{ij} \) to form the Jacobian matrix \( J \) and the reduced Jacobian matrix \( J' \) for the cost function \( \omega \), as

\[
\begin{bmatrix}
\mathbf{J}_{ij} \\
\vdots \\
\mathbf{J}_{ij}
\end{bmatrix}, \quad \begin{bmatrix}
\mathbf{J}'_{ij} \\
\vdots \\
\mathbf{J}'_{ij}
\end{bmatrix}
\]  

(31)

The following lemma shows that \( J' \) can replace \( J \) to calculate \( J^T J \) in the LM algorithm.

**Lemma 2.** For the planar BA, we have \( J^T J = J'^T J' \).

**Proof.** \( J \) and \( J' \) are block vectors in terms of \( J_{ij} \) and \( J'_{ij} \) as defined in (31). According to block matrix multiplication, we have

\[
\begin{bmatrix}
\mathbf{J}_i^T \mathbf{J}_i \\
\vdots \\
\mathbf{J}_i^T \mathbf{J}_i
\end{bmatrix} = \sum_{i,j} \mathbf{J}_{ij}^T \mathbf{J}_{ij}, \quad \begin{bmatrix}
\mathbf{J}'_i^T \mathbf{J}'_i \\
\vdots \\
\mathbf{J}'_i^T \mathbf{J}'_i
\end{bmatrix} = \sum_{i,j} \mathbf{J}'_{ij}^T \mathbf{J}'_{ij}
\]  

(32)

For \( i \neq 1 \), using the expression in (20), \( J_{ij}^T J_{ij} \) has the form

\[
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]  

Similarly, using the expression in (30), \( J'_{ij}^T J'_{ij} \) has the form

\[
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]  

(33)

Substituting (27) into \( \mathbf{V}_{ij}^T \mathbf{C}_{ij}^T \mathbf{C}_{ij} \mathbf{V}_{ij}^o \) and using the fact \( \mathbf{Q}_{ij}^T \mathbf{Q}_{ij} = \mathbf{I}_4 \), we have

\[
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]  

Similarly, we have

\[
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]  

(34)

For \( i = 1 \), according to (21), the only non-zero term for \( \mathbf{J}_{ij}^T \mathbf{J}_{ij} \) is \( \mathbf{V}_{ij}^o \). On the other hand, according to (30), \( \mathbf{J}'_{ij}^T \mathbf{J}'_{ij} \) has only one corresponding non-zero term \( \mathbf{V}_{ij}^o \mathbf{M}_{ij}^T \mathbf{M}_{ij} \mathbf{V}_{ij}^o \). Similar to the derivation in (35), we have

\[
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]  

(36)

In summary, using (35), (36) and (37), we have \( J_{ij}^T J_{ij} = J'_{ij}^T J'_{ij} \). According to (32), consequently we have \( J^T J = J'^T J' \).

### 3.5 Reduced Residual Vector

Let us define the residual vector for the \( K_{ij} \) points in \( \mathbf{p}_{ij} \) as \( \mathbf{d}_{ij} = \begin{bmatrix} \delta_{ij,1}, \delta_{ij,2}, \cdots, \delta_{ij,K_{ij}} \end{bmatrix} \). According to (13) and (19), \( \mathbf{d}_{ij} \) can be written as

\[
\mathbf{d}_{ij} = \mathbf{C}_{ij} \mathbf{v}_{ij}.
\]  

(38)
We define the reduced residual vector \( \delta'_{ij} \) of \( \delta_{ij} \) as
\[
\delta'_{ij} = M_{ij}v_{ij} \tag{39}
\]
Stacking all \( \delta_{ij} \) and \( \delta'_{ij} \), we get the residual vector \( \delta \) and the reduced residual vector \( \delta' \) as
\[
\delta = \begin{bmatrix} \vdots \\ \delta_{ij} \\ \vdots \\ \delta_{ij} \\ \vdots \end{bmatrix}, \quad \delta' = \begin{bmatrix} \vdots \\ \delta'_{ij} \\ \vdots \end{bmatrix} \tag{40}
\]
The following lemma shows that \( \delta' \) can replace \( \delta \) in the LM algorithm.

**Lemma 3.** For the planar BA, we have \( J^T \delta = J'^T \delta' \).

**Proof.** \( J, J', \delta \) and \( \delta' \) are block vectors with elements \( J_{ij}, J'_{ij}, \delta_{ij} \) and \( \delta'_{ij} \) as defined in (31) and (40), respectively. Applying the block matrix multiplication, we have
\[
J^T \delta = \sum J^T_{ij} \delta_{ij}, \quad J'^T \delta' = \sum J'^T_{ij} \delta'_{ij} \tag{41}
\]
For \( i \neq 1 \), using the expression of \( J_{ij} \) in (29) and \( J'_{ij} \) in (38), and the expression of \( \delta_{ij} \) in (38) and \( \delta'_{ij} \) in (39), \( J^T_{ij} \delta_{ij} \) and \( J'^T_{ij} \delta'_{ij} \) have the forms as
\[
J^T_{ij} \delta_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]
\[
J'^T_{ij} \delta'_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]
Substituting (22) into \( V_{ij}^T C_{ij}^T C_{ij} v_{ij} \) and using the fact \( Q^T_{ij} Q_{ij} = I_4 \), we have
\[
V_{ij}^T C_{ij}^T C_{ij} v_{ij} = V_{ij}^T M_{ij}^T M_{ij} v_{ij} \tag{42}
\]
Similarly, we have
\[
V_{ij}^T C_{ij}^T C_{ij} v_{ij} = V_{ij}^T M_{ij}^T M_{ij} v_{ij} \tag{44}
\]
For \( i = 1 \), substituting (21) and (38) into \( J^T_{1j} \delta_{1j} \) and applying the block matrix multiplication, we find the only non-zero term of \( J^T_{1j} \delta_{1j} \) is \( V_{1j}^T C_{1j}^T C_{1j} v_{1j} \). On the other hand, substituting (30) and (39) into \( J'^T_{1j} \delta'_{1j} \), we find that \( J'^T_{1j} \delta'_{1j} \) only has one non-zero term \( V_{1j}^T M_{1j}^T M_{1j} v_{1j} \). Similar to the derivation in (43), we have
\[
V_{ij}^T C_{ij}^T C_{ij} v_{ij} = V_{ij}^T M_{ij}^T M_{ij} v_{ij} \tag{45}
\]
In summary, from (43), (44) and (45), we have \( J^T_{ij} \delta_{ij} = J'^T_{ij} \delta'_{ij} \). According to (41), we finally get \( J^T \delta = J'^T \delta' \).

### 3.6 Planar Bundle Adjustment Algorithm

According to Lemma [1] Lemma [2] and Lemma [3] we have the following theorem.

**Theorem 1.** For the planar BA, \( J' \) and \( \delta' \) can replace \( J \) and \( \delta \) in (3) to compute the step in the LM algorithm, and each block \( J'_{ij} \) and \( \delta'_{ij} \) in \( J' \) and \( \delta' \) has 4 rows.

**Proof.** LM algorithm uses (5) to calculate the step for each iteration. According to Lemma [2], we have \( J^T J = J'^T J' \). Besides, based on Lemma [3], we have \( J^T \delta = J'^T \delta' \). Consequently, we have \( J^T + \lambda I \xi = J'^T \delta' \) equivalent to \( (J^T J + \lambda I) \xi = J^T \delta \). Thus \( J' \) and \( \delta' \) can replace \( J \) and \( \delta \) for computing the step in the LM algorithm.

According to the definition of \( J'_{ij} \) in (30) and \( \delta'_{ij} \) in (39), we know that the number of rows of \( J'_{ij} \) and \( \delta'_{ij} \) is the same as the number of rows of \( M_{ij} \). According to Lemma [1], \( M_{ij} \) has 4 rows. Consequently, we have that \( J'_{ij} \) and \( \delta'_{ij} \) have 4 rows.

As mentioned in Theorem 1, no matter how may points are in \( P_{ij} \), the reduced \( J'_{ij} \) and \( \delta'_{ij} \) always have 4 rows. This significantly reduces the computational cost in the LM algorithm. Specifically, we have the following corollary.

**Corollary 1.** The runtime for computing \( J'_{ij} \) and \( \delta'_{ij} \) of computing the original \( J_{ij} \) and \( \delta_{ij} \) is \( \frac{1}{16} \) relative to computing the original \( J_{ij} \), \( \delta_{ij} \), \( J'_{ij} \), \( \delta'_{ij} \) and \( J'_{ij} \delta'_{ij} \), respectively.

**Proof.** From the expression of \( J'_{ij} \) in (30) and \( J_{ij} \) in (10) and (21), we know that the difference between \( J'_{ij} \) and \( J_{ij} \) is that we use \( M_{ij} \) to replace \( C_{ij} \). \( M_{ij} \) has 4 rows and \( C_{ij} \) has \( K_{ij} \) rows. Thus, the runtime of computing \( J'_{ij} \) is \( \frac{4}{K_{ij}} \) of computing \( J_{ij} \). Similarly, the runtime of computing \( \delta'_{ij} \) is \( \frac{4}{K_{ij}} \) of computing \( \delta_{ij} \).

According to Theorem 1, \( J'_{ij} \) and \( \delta'_{ij} \) has 4 rows, and \( J_{ij} \) and \( \delta_{ij} \) has \( K_{ij} \) rows. According to the matrix multiplication rule, the runtime of computing \( J'_{ij} \) \( J'_{ij} \) and \( \delta'_{ij} \) is \( \frac{1}{K_{ij}} \) of computing \( J_{ij} \) and \( \delta_{ij} \), respectively.

The additional cost here is to calculate \( C_{ij} = Q_{ij} M_{ij} \). As \( C_{ij} \) keeps constant during the iteration, we only need to compute it once before the iteration. As shown in our experimental results, this step only slightly increases the computational time for the pose graph initialization step. We summarize our PBA in Algorithm 1.
WiFi signals. It provides an accurate trajectory and a dense point cloud.

We evaluate the accuracy, computational time, convergence speed with standard deviation (STD) $\delta$ angles which are sampled from a zero-mean Gaussian distribution rotation matrix by an error rotation matrix represented by the Euler angles. We add zero-mean Gaussian noise to the translation. We consider three noise levels as listed below:

- Noise Level 1: $\delta_R = 0.1^\circ$, $\delta_t = 0.01m$
- Noise Level 2: $\delta_R = 0.5^\circ$, $\delta_t = 0.03m$
- Noise Level 3: $\delta_R = 1.0^\circ$, $\delta_t = 0.05m$

Although the noise is small in terms of one pose, the noise will accumulate along the trajectory which may yield a larger error. Generally, a longer trajectory will lead to a larger error. Fig. 5 shows the perturbed as well as the original trajectories for the 3 datasets. We use the perturbed trajectory to initialize the poses. For planes, we first calculate their local parameters from the local point clouds at the first poses that observe them. We initialize the global plane parameters by transforming the local parameters into the global coordinate system using the perturbed poses.

**Experiment Setup** In the experiment, all the algorithms use the same parameterizations for the rotation matrix and the plane, i.e., angle-axis and CP parameterization. We ran the experiments on a computer with an Intel(R) Xeon(R) ES-2620 2.10GHz CPU and 80G memory. We adopted Ceres as the nonlinear least-squares solver. We used the Schur complement trick to solve the linear system. We set the function and parameter tolerance to $10^{-10}$, and set the maximum number of iterations to 1000, except for the dataset3 at noise level 3. In this case, the initial error is very large, and more iterations are required. We set the maximum number of iterations to 15,000 for dataset3 at noise level 3.

We use the perturbed trajectory to initialize the LM algorithm, and compare the original trajectory with the results from the LM algorithm. We employ the absolute trajectory error (ATE) to quantify the accuracy of the result. Specifically, for the $k$th pose, given the ground truth $R_k$ and $t_k$ and the estimation $\tilde{R}_k$ and $\tilde{t}_k$, we calculate $\Delta R_k$ and $\Delta t_k$ as

$$
\Delta R_k = R_k \tilde{R}_k^T,
$$

$$
\Delta t_k = t_k - \Delta R_k \tilde{t}_k.
$$

The ATE is calculated as

$$
\text{ATE}_R = \left( \frac{1}{N} \sum_{k=1}^{N} \| \angle(\Delta R_k) \|^2 \right)^{\frac{1}{2}},
$$

$$
\text{ATE}_t = \left( \frac{1}{N} \sum_{k=1}^{N} \| \Delta t_k \|^2 \right)^{\frac{1}{2}},
$$

where $\angle(\cdot)$ represents the angle of the angle-axis representation of $\Delta R_k$.

**Results** The quantitative results are listed in Table 2. Fig. 5 shows the qualitative results. We find the algorithms in Table 2 generally converge to a larger cost and obtain less accurate results when the initialization error enlarges. This may be caused by the function tolerance stop condition. The iteration in Ceres stops when $\Delta \text{cost} < \text{function tolerance}$, where $\Delta \text{cost}$ is the change in cost function value in the current iteration, and $\text{cost}^0$ is the initial cost. Thus, Ceres may stop at a larger $\Delta \text{cost}$ for a larger $\text{cost}^0$. This may result in the algorithms in Table 2 converge to larger cost and get less accurate results as the noise level increases.

It is clear that our algorithm is more accurate and more robust to the initialization error than PL2PL. In addition, our algorithm converges faster which means it allows for fewer iterations, and has lower computational complexity. One point-to-plane cost in PL2PL only involves 2 variables. But one relative plane cost in PL2PL involves 3 variables (2 poses and 1 plane). Thus the Jacobian matrix of PL2PL has roughly 50% more non-zero items than ours. This increases the runtime for calculating the Jacobian matrix. For each iteration, our algorithm is more than 2 times faster than PL2PL on dataset2 and dataset3. In the minimization process, our algorithm is about 74, 49 and 107 times faster than DPT2PL on the three datasets, respectively. The extra computational cost for the factorization $C_{ij} = Q_{ij}M_{ij}$ in DPT2PL is marginal compared to the significant gain from the optimization process.

### 4 Experiments

In this section, we evaluate the performance of our algorithm. We compare our algorithm against the direct solution for the point-to-plane cost using the traditional BA framework (DPT2PL), and the state-of-the-art solution using plane-to-plane cost (PL2PL). We evaluate the accuracy, computational time, convergence speed and the robustness to errors of the initialization in the compared algorithms.

**Databases** We collected 3 indoor datasets using the NavVis M6 device. The NavVis M6 estimates the device pose by 1 multi-layer Velodyne LiDAR, 3 single-layer LiDAR, IMU as well as WiFi signals. It provides an accurate trajectory and a dense point cloud with an accuracy of centimeters. We use the recordings of the Velodyne LiDAR to build the datasets. We sampled the trajectory so that the distance between two poses is larger than 5cm. Planes are detected for each recording of the Velodyne LiDAR by the region-growing method introduced in [25]. Then we use the known pose to get the plane-plane data association. Specifically, the global planes are initialized by the planes detected in the first pose. We track and grow the global planes frame by frame. Local planes of the latest frame are first transformed into the global coordinate system using the known pose. We calculate the distances between points of a local plane and the global planes, then we find out the global plane that has the shortest mean point-to-plane distance. A match happens when a local plane and the closest global plane has a mean point-to-plane distance smaller than 5cm, and the angle between the normals of the two planes is less than 10°. The points of the local plane whose distance are less than 5cm are then added into the global plane, and the parameters of the global plane are then updated. Unmatched local planes that have more than 50 points are recognized as new planes, and are added into the global plane list for tracking during future frames. Fig. 4 shows the 3 datasets. The characteristics of the 3 datasets are listed in Table 1.

| dataset  | #poses | #planes | #points | length (m) |
|---------|--------|---------|---------|------------|
| dataset1 | 695    | 154     | 6.98 x 10^6 | 43.2       |
| dataset2 | 1781   | 370     | 16.82 x 10^6 | 104.4      |
| dataset3 | 6547   | 591     | 68.99 x 10^6 | 403.5      |

**Combine Planes with Other Features** Planes are sometimes together with other features, such as points. In the cost function derived from multiple features, the Jacobian matrix from the plane cost would have the same form as (20) and (21), and the residual vector would also have the same form as (38). Therefore, they can be replaced by our reduced Jacobian matrix and reduced residual vector in the BA with multiple features.

**Initialization Error** The LM algorithm requires an initial estimation. We consider how the error of the initial estimation affects the performance of different algorithms. We add zero-mean Gaussian noise to each pose of the trajectory. Specifically, we perturb the rotation matrix by an error rotation matrix represented by the Euler angles which are sampled from a zero-mean Gaussian distribution with standard deviation (STD) $\delta R$. Additionally, we add zero-mean Gaussian noise with STD $\delta t$ to the translation. We consider three noise levels as listed below:

- Noise Level 1: $\delta_R = 0.1^\circ$, $\delta_t = 0.01m$
- Noise Level 2: $\delta_R = 0.5^\circ$, $\delta_t = 0.03m$
- Noise Level 3: $\delta_R = 1.0^\circ$, $\delta_t = 0.05m$

1. https://www.navvis.com/m6
2. https://velodynelidar.com/
5 Conclusion

In this paper, we present an efficient solution for the PBA problem using the point-to-plane cost. Although the point-to-plane cost involves a large number of constraints, we find the resulting least-squares problem has special structure. We prove that we can use a reduced Jacobian matrix and residual vector with 4 rows to replace the original Jacobian matrix and residual vector with $K_{ij}$ rows in the LM algorithm. This reduces the runtime for computing $J_{ij}$, $\delta_{ij}$, $J_{ij}^T J_{ij}$ and $J_{ij}^T \delta_{ij}$ by a factor of $\frac{4}{K_{ij}}$ relative to the brute force implementation. Our experimental results show that the extra cost of the one-time factorization is marginal compared to the significant gain from this new formulation. Furthermore, we have verified that our algorithm is faster, more accurate, and more robust to initial errors compared to the start-of-the-art formulation using the plane-to-plane cost for joint poses and planes optimization [4].

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Figure 5: Experimental results of our algorithm and PL2PL [4] with varying initialization noise levels. PL2PL is not stable to large initial errors.

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Table 2: Experimental results for different algorithms. The column QR shows the time for the QR decomposition of our algorithm. The column Init. presents the time for initializing the Ceres solver. The column Optimization describes the total time for the LM algorithm. The column Per Iter lists the average runtime for each LM iteration. Our algorithm has a lower initialization time. The computational cost for QR decomposition is ignorable compared to the significant gain from the optimization process. Specifically, our algorithm is about 74, 49 and 107 times faster than PL2PL on the three datasets, respectively. Here we only evaluate PL2PL on Noise level 3 for the 2 datasets, as we only consider the computational time of DPT2Pl. For a specific dataset, the runtime of one iteration for different noise levels is similar. Our algorithm is more accurate, has lower computational complexity and converges faster than PL2PL [4].

| Dataset | Noise Level | Method | #Iter | ATE Initial (m) | ATE Final (m) | Cost in Eq. [9] | QR | Time (s) |
|---------|-------------|--------|-------|----------------|--------------|----------------|----|----------|
| dataset1 | level 1     | Ours   | 81    | 4.44 × 10^{-2} | 3.07 × 10^{-4} | 2.38 × 10^7 | 1.16 | 0.016 | 11.06 | 0.14 |
|         |             | DPT2Pl | 81    | 4.44 × 10^{-2} | 3.07 × 10^{-4} | 2.38 × 10^7 | 0 | 6.26 | 823.31 | 10.16 |
|         |             | PL2PL  | 384   | 0.58         | 0.28          | 2.38 × 10^5 | 0.026 | 69.53 | 0.18 |
|         | level 2     | Ours   | 176   | 4.68 × 10^{-2} | 3.98 × 10^{-4} | 1.78 × 10^6 | 1.18 | 0.017 | 22.86 | 0.13 |
|         |             | PL2PL  | 1000  | 0.58         | 0.28          | 1.78 × 10^6 | 0.025 | 185.97 | 0.19 |
|         | level 3     | Ours   | 224   | 4.96 × 10^{-2} | 4.22 × 10^{-4} | 1.78 × 10^6 | 1.15 | 0.016 | 28.56 | 0.13 |
|         |             | PL2PL  | 1000  | 0.58         | 0.28          | 1.78 × 10^6 | 0.024 | 165.64 | 0.17 |
| dataset2 | level 1     | Ours   | 75    | 4.88 × 10^{-2} | 8.57 × 10^{-5} | 7.27 × 10^3 | 2.88 | 0.060 | 37.90 | 0.51 |
|         |             | DPT2Pl | 75    | 4.88 × 10^{-2} | 8.57 × 10^{-5} | 7.27 × 10^3 | 0 | 16.46 | 1843.87 | 24.58 |
|         |             | PL2PL  | 203   | 1.09         | 0.20          | 4.38 × 10^6 | 2.89 | 0.068 | 63.62 | 0.53 |
|         | level 2     | Ours   | 119   | 5.76 × 10^{-2} | 2.32 × 10^{-4} | 4.38 × 10^6 | 2.86 | 0.062 | 365.70 | 0.51 |
|         |             | PL2PL  | 384   | 1.09         | 0.20          | 4.38 × 10^6 | 0 | 11 | 413.91 | 1.07 |
|         | level 3     | Ours   | 723   | 7.61 × 10^{-2} | 1.02 × 10^{-2} | 8.38 × 10^3 | 8.38 | 0.07 | 75.62 | 1.05 |
|         |             | PL2PL  | 186   | 2.56         | 0.97          | 8.38 × 10^3 | 0 | 12 | 195.56 | 1.05 |
| dataset3 | level 1     | Ours   | 503   | 4.28 × 10^{-2} | 3.17 × 10^{-4} | 2.81 × 10^7 | 11.14 | 0.17 | 4.87 × 10^8 | 0.97 |
|         |             | DPT2Pl | 503   | 4.28 × 10^{-2} | 3.17 × 10^{-4} | 2.81 × 10^7 | 0 | 7.03 | 5.22 × 10^8 | 0.97 |
|         |             | PL2PL  | 1000  | 0.90         | 0.14          | 2.81 × 10^7 | 0 | 3.5 | 2.45 × 10^3 | 2.45 |
|         | level 2     | Ours   | 883   | 6.56 × 10^{-2} | 2.32 × 10^{-2} | 5.99 × 10^6 | 10.89 | 0.16 | 8.38 × 10^3 | 0.95 |
|         |             | PL2PL  | 1000  | 0.95         | 0.47          | 5.99 × 10^6 | 0 | 3.6 | 2.49 × 10^3 | 2.49 |
|         | level 3     | Ours   | 13,385 | 8.85 × 10^{-2} | 6.22 × 10^{-2} | 3.97 × 10^9 | 11.32 | 0.17 | 1.42 × 10^9 | 1.06 |
|         |             | PL2PL  | 15,000 | 8.66      | 1.34          | 3.97 × 10^9 | 0 | 0.33 | 3.77 × 10^3 | 2.51 |

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