COMMUNICATION NETWORK THEORY

Propagation of Chaos and Poisson Hypothesis

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Abstract—We establish the strong Poisson hypothesis for symmetric closed networks. In particular, we prove asymptotic independence of nodes as the size of the system tends to infinity.

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1. INTRODUCTION

In this paper we consider simple symmetric closed networks consisting of \(N\) servers and \(M\) customers. Each server has its own infinite buffer, where the customers are queuing for service; thus, there are \(N\) queues. The service discipline in all queues is FIFO with i.i.d. service times with the distribution function \(F(x), 0 \leq x < \infty\). We list the conditions on \(F\) in Section 2.1. They include the continuity of its density \(p(x)\) and finiteness of the second moment.

The network is maximally symmetric; each customer that has finished its service at some server is placed afterwards at the end of one of the \(N\) queues with probability \(1/N\). It can be described by a Markov process. Namely, for each queue \(i = 1, \ldots, N\) let us consider the elapsed service time \(t_i\) of the customer which is currently in service. The state of the Markov process \(A_{N,M}(t)\) is the set of lengths of \(N\) queues (which are integers) and the set of elapsed service times of all customers which are under the service now.

The general algebraic structure governing the symmetric network is the symmetry group of the Markov process. In our case it is the permutation group of \(N\) elements. If some group \(G\) acts on the phase space \(X\) of the Markov process and the transition probabilities are \(G\)-invariant, then we can pass to the factor-process and consider the Markov process on the space \(X/G\) of orbits of the group \(G\).

In our case the state of the Markov process is the sequence \((q_1, q_2, \ldots, q_N)\) where \(q_i\) is a pair, \(q_i = (z_i, k_i)\), consisting of the length \(k_i\) of the \(i\)th queue and the elapsed service time \(z_i\) for the \(i\)th

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server. The orbit of this state can be interpreted as an atomic probability measure that assigns the mass $1/N$ to each $q_i$ (empirical measure). Thus, our Markov process can be factorized to a process on empirical measures. The general fact proved for a broad class of symmetric queueing systems is the convergence of the process on empirical measures to the deterministic evolution of measures $m(t)$ as $N \to \infty$ (and $M \to \infty$ in our case); see [1–3].

For given $N$ and $M$, the service process is ergodic. Indeed, let us register the following event: all the customers are collected in the same queue and the first one just starts its service. This renewal event will happen with probability 1 and with a finite mean waiting time. Hence the ergodicity follows. The same argument is valid for many other networks, but we do not consider them here, since we rely on previous results proved only for specific networks.

The equilibrium distribution $Q_{N,M}$ of this Markov process is symmetric, i.e., the states of queues in equilibrium are exchangeable random variables. Our problem is to study the asymptotic properties of this equilibrium distribution $Q_{N,M}$ as $M$ and $N$ tend to infinity. (In the general case there is no explicit formula for the joint distribution of queues as well as for its marginal distributions.)

We want to find conditions under which the limit $\lim_{N \to \infty} Q_{N,aN}$ exists and obeys the “Propagation of Chaos” (PoC) property. The PoC property means that under $Q_{N,M}$ different nodes of our network are asymptotically independent. We prove the PoC in Section 6. A similar but different case of the PoC was addressed in [4,5].

The PoC property for the stationary measure is a part of the strong Poisson hypothesis (SPH) formulated below. To formulate this hypothesis, let us consider a single server. Let the inflow to this server be a stationary Poisson flow with intensity $\lambda < 1$. The outflow is a stationary (non-Poisson) flow with the same intensity $\lambda$. The distribution of the state of the server is the stationary measure of an $M/GI/1/\infty$ system with input intensity $\lambda$. The parameter $\lambda$ is found from the following argument: the expectation of the length of the queue in the stationary $M/GI/1/\infty$ system with input intensity $\lambda$ is equal to $a$.

**Strong Poisson hypothesis.** The SPH claims that in the stationary state in the limit as $N \to \infty$, $M/N \to a$, the empirical measure (of each of the server states $q_1, \ldots, q_N$) tends weakly in probability to the distribution of the state of the server described above. Moreover, all the servers in the limit as $N \to \infty$ become asymptotically independent. For the nonrandom service time, this hypothesis was proved in a seminal paper by A. Stolyar [6].

Now we will remind the reader about nonlinear Markov processes (NLMP) which describe the limiting properties of the processes $A_{N,M}(t)$ as $M,N \to \infty$. This NLMP (in the sense of [7]), denoted below by $A(t)$, has the following structure. There is a single server $M(t)/GI/1/\infty$ with a nonstationary Poisson input flow of rate $\lambda(t)$. This rate equals the expected value of the output intensity of the server in the state corresponding to the measure $m(t)$, which is the distribution of the state of the queue at time $t$. This condition defines the NLMP $m(t)$ in a unique way if the initial measure $m(0)$ is given; see [1].

A theorem about the existence and uniqueness of NLMP was proved in [1]. It was generalized to a broad class of symmetric queueing systems in [2,3]. The convergence of empirical measures for any finite time $t$ can be interpreted as a functional law of large numbers: on any finite time interval $[0,T]$, the random evolution of empirical measures converges in probability to the deterministic evolution $m(t)$ if the initial empirical measure converges to $m(0)$.

From the convergence of empirical measures and from the general properties of exchangeable random variables, the weak Poisson hypothesis (WPH) follows:

**Weak Poisson hypothesis.** Assume that the initial distribution of queues is symmetric and initial empirical measures converge in probability to the measure $m(0)$ as $N \to \infty$ and $M \to \infty$. 

PROBLEMS OF INFORMATION TRANSMISSION Vol. 54 No. 3 2018
Then at any time $t$ the servers are asymptotically independent (i.e., the PoC holds) and the queue distribution at any server is close to the measure $m(t)$ determined by the nonlinear Markov process.

In comparison, the SPH in terms of empirical measures claims that in the stationary regime the empirical measures satisfy the law of large numbers, i.e., converge to some nonrandom measure as $N \to \infty$, $M \to \infty$, $M/N \to a$. In other words, the limiting invariant measure of the Markov processes on empirical measures—which is a measure on measures—is in fact concentrated at one single measure as $N \to \infty$, $M \to \infty$, $M/N \to a$.

From the general argument (known as Khasminski lemma; see, e.g., [8, Proposition 2.14]) it follows that if the sequence of Markov processes converges to some deterministic evolution, then any limit point of the sequence of invariant measures of these processes is an invariant measure of the limit dynamical system. Thus, in order to prove the SPH it is good to know the invariant measures of the NLMP. This problem was considered in [2]: for any given value of the parameter $a$, the mean queue length, the limiting dynamical system (NLMP) $\mathfrak{A}(t)$ has a unique fixed point.

Summarizing, our strategy to prove the SPH for the measures $Q_{N,M}$ is the following:

1. First we check the convergence of the processes $A_{N,M}(t)$ to the NLMP $\mathfrak{A}(t)$ as $N \to \infty$ (this is the statement of the WPH);
2. Further, we check that the dynamical system $\mathfrak{A}(t)$ has a unique stationary point, $\nu_a$, which is a global attractor on any “leaf” where the mean queue length equals $a$;
3. Finally, we check that the limit points of the (precompact) family $Q_{N,M}$ as $N \to \infty$ and $M/N \to a$, which, in principle, could be mixtures of measures $\nu_{\tilde{a}}$ with $\tilde{a} \leq a$, are in fact just the measure $\nu_a$ itself.

A difficult part of the program is to show that the invariant measures of the dynamical system (NLMP) can be only fixed points. This was established in [2] in some cases by using the flow smoothing property of the $M(t)/GI/1/\infty$ queueing system. This fact is not true in general: for some symmetric queueing systems (with several types of servers and several types of customers) there exist nonatomic invariant measures of the NLMP supported by nontrivial attractors. In [9] the corresponding example with three types of servers and three types of customers was presented. In the case of the simple symmetric closed network considered here, an example of a nontrivial attractor (but with unbounded function $\beta(x)$) was constructed in [10].

At the end of the article we present a general scheme connecting the asymptotic independence of exchangeable random variables with the law of large numbers for empirical measures (a de Finetti type theorem; see also [11–13]).

2. SINGLE NODE

2.1. State Space

The basic element of our model is a server with a queue. Customers arrive to the queue and are served in the order of arrivals (FIFO service discipline). The random service times $\eta$ of each customer are i.i.d. with the distribution $F(x)$, $0 \leq x < \infty$. This paper relies on our previous results concerning the NLMPs [2], which require some restrictions on $\eta$. We list now the properties of $\eta$ needed:

1. The density function $p(t)$ of the random variable $\eta$ is defined on $t \geq 0$ and is uniformly bounded from above; moreover, it is differentiable in $t$, with $p'(t)$ continuous;
2. $p(t)$ satisfies the following strong Lipschitz condition: for some $C < \infty$ and for all $t \geq 0$,

$$|p(t + \Delta t) - p(t)| \leq C p(t) |\Delta t|,$$  \quad (1)

provided that $t + \Delta t > 0$ and $|\Delta t| < 1$;
3. For some $\delta > 0,$
\[ M_\delta \equiv E(\eta^{2+\delta}) < \infty; \]  
(2)

4. Defining the density
\[ p_\tau(t) = \frac{p(t+\tau)}{1-F(\tau)}, \quad t \geq 0, \]
we require that the function $p_\tau(0)$ is bounded uniformly in $\tau \geq 0,$
\[ p_\tau(0) \leq B < \infty; \]  
(3)

5. The function $\frac{d}{dt}p_\tau(0)$ is continuous and bounded uniformly in $\tau \geq 0;$

6. The limits $\lim_{\tau \to \infty} p_\tau(0), \lim_{\tau \to \infty} \frac{d}{dt}p_\tau(0)$ exist and are finite.

7. Also, without loss of generality we assume that
\[ E(\eta) = 1. \]  
(4)

In particular, random variables $\eta$ with power law decay of the density are allowed.

The state space $Q$ of queues at a single server is the set of pairs $(z,k),$ where $z \geq 0$ is the elapsed service time of the customer under the service, $z \in \mathbb{R}^1,$ and $k$ is an integer. Of course, the empty state $\emptyset$ is also included in $Q.$

2.2. Dynamics

The dynamics is defined by the following simple relations. Assume that we are in a state
\[ q(t) = (z(t),k(t)) \in Q. \]

While the time goes and nothing happens, $k$ stays constant, and $z$ grows linearly: $\dot{z}(t) = 1.$ If a customer arrives at moment $t,$ then we have a jump:
\[ (z(t),k(t)) \to (z(t),k(t) + 1). \]

If a customer leaves at moment $t,$ then we have another jump:
\[ (z(t),k(t)) \to (0,k(t) - 1). \]

3. MEAN-FIELD NETWORK

3.1. Definition

The mean-field network consists of $N$ nodes described above. Their collective behavior is defined as follows. As soon as a customer finishes its service at some node, it is routed to one of $N$ nodes with equal probability $1/N$ (this is why the network is called a mean-field network). Upon arrival to this node, the customer joins the queue and waits for its turn to be served. Thus, the total number of customers,
\[ M = \sum_{i=1}^{N} k_i, \]
is conserved by the dynamics. The resulting Markov process is denoted by $A_{N,M}(t).$
3.2. Empirical Measures

For each pair \((N, M)\), we denote by \(Q_{N,M}\) the unique equilibrium distribution of the process \(A_{N,M}(t)\). This is a probability measure on \(Q^{N}\).

A point \((q_1, \ldots, q_N)\) of the space \(Q^{N}\) can be conveniently identified with a probability measure \(\mu = \frac{1}{N} \sum_{n=1}^{N} \delta_{q_n}\) on \(Q\), i.e., with an element of \(\mathcal{M}(Q)\). In fact, it is an element of the subspace \(\mathcal{M}_N(Q) \subset \mathcal{M}(Q)\) of the atomic measures with atom weights equal to \(\frac{k}{N}\), \(k = 1, \ldots\). Hence the distributions of all the processes \(A_{N,M}(t)\), as well as the measures \(Q_{N,M}\), are elements of \(\mathcal{M}(\mathcal{M}_N(Q))\), i.e., measures on the space of measures.

4. SOME FACTS ABOUT NLMPs

4.1. Nonlinear Markov Processes

The NLMPs \(\mathfrak{A}_a\) define the semigroup of nonlinear transformations on \(\mathcal{M}(Q)\). Under \(\mathfrak{A}_a\), the measures evolve, informally speaking, in the same way as under \(A_{N,aN}\) with \(N\) very large, but in a deterministic way. This is an analog of the LLN for the considered system. For details of NLMPs, see [1–3].

Here is a description of the corresponding dynamical systems. Each of them acts on the space of states \(\mu \in \mathcal{M}(Q)\) of a single server. Every initial measure \(\mu(0)\) defines a certain function \(\lambda_{\mu(0)}(t) \geq 0\), which is the expected rate of the output flow, and coincides with the rate of the input Poisson flow (to the same node). This rate \(\lambda_{\mu(0)}(t)\) of the Poissonian inflow defines the evolution \(\mu(t)\) of the measure \(\mu(0)\).

The definition of this rate \(\lambda(t)\) is somewhat complicated. Assume that we know the value of \(\lambda(t)\) at \(t\). Then we know the exit flow (non-Poisson, in general) from our server. This flow also has some rate, \(b(t)\), which is defined by \(\mu(t)\) as follows. Given \(\mu(t)\), let \(\nu_{\mu(t)}\) be the probability distribution of the remaining service time of the customer currently served. Then \(b(t) = \lim_{\Delta \to 0} \frac{\nu_{\mu(t)}([0, \Delta])}{\Delta}\). The function \(\lambda_{\mu(0)}(t)\) is the solution of the following (nonlinear integral) equation:

\[
\lambda_{\mu(0)}(t) = b(t).
\]

The subscript \(a\) in the notation \(\mathfrak{A}_a\) refers to the fact that our dynamics conserves the average number \(N(\mu)\) of customers:

\[
N(\mu) = \int_{Q} k \, d\mu(q) = a, \quad q = (z, k) \in Q.
\]  \hspace{1cm} (5)

Below we will need the following property of our NLMPs, which is proved in [2, Lemma 7].

**Lemma 1.** Let \(\Delta > 0\) and the parameter \(a\) of (5) be fixed. There exists a function \(T(\Delta, a)\) such that for any \(T > T(\Delta, a)\) and for any initial state \(\mu(0)\) satisfying (5) the corresponding rate function \(\lambda_{\mu(0)}\) satisfies the estimate

\[
\int_{\tau}^{\tau+T} \lambda_{\mu(0)}(t) \, dt < T - \Delta
\]  \hspace{1cm} (6)

for any \(\tau > 0\).
4.2. Convergence to NLMP as $N \to \infty$

Here we formulate a theorem about finite time convergence of processes $A_{N,M}(t)$ to $\mathfrak{A}_a(t)$, where $a = \lim_{N \to \infty} M/N$. More precisely, for any finite time interval $[0, T]$, the evolution under $A_{N,M}$ with the initial state $\mu_N(0)$ converges to the evolution under $\mathfrak{A}_a$ with the same initial state $\mu_N(0)$. The convergence here is the weak convergence of measures.

Let $P_{N,M}(t)$ be the semigroup corresponding to the Markov process $A_{N,M}(t)$.

**Theorem 1.** Let $T > 0$, and let $f$ be a bounded continuous functional on the set of measures $\{\mu(t), t \in [0, T]\}$. Assume that for any $N$ the number $M_N$ of customers is chosen in such a way that the sequence $a_N = M_N/N \to a$. Then for any family of initial states $\{\mu_N(0) \in \mathcal{M}_N(\mathcal{Q}), N = 1, 2, \ldots\}$ with $M_N$ customers we have

$$|(P_{N,M}(t)f)(\mu_N) - f(\mathfrak{A}_a\mu_N)| \to 0 \quad \text{as} \quad N \to \infty,$$

(7)

uniformly in $\{\mu_N(0) \in \mathcal{M}_N(\mathcal{Q}), N = 1, 2, \ldots\}$ and in $t$, $0 \leq t \leq T$.

The meaning of this theorem is that the random trajectories of the Markov process $A_{N,M}(t)$ converge to the deterministic trajectories $\mathfrak{A}_a(t)$ of the dynamical system.

**Proof.** See [1, Theorem 7.1]. (Of course, the convergence in (7) is not uniform in $T$.) △

Actually, we will only need a special case of this theorem, applied to certain functionals $f_{n,T}$. Let us consider our network of $N$ servers and fix one of them, $s$. Consider the random variable $C_{N,T,\mu}$ defined as the number of customers coming to the server $s$ during the time interval $[0, T]$ in the process $A_{N,M}\mu$ (started from the state $\mu \in \mathcal{M}_N(\mathcal{Q})$). In the same way we define the random variable $C_{T,\mu}$ as the number of customers coming to the server $s$ during the time interval $[0, T]$ in the NLMP $\mathfrak{A}_{\infty,\mu}$.

**Corollary.** Under the conditions of Theorem 1, we have the following: for each $n$ and $T$,

$$|P(C_{N,T,\mu} = n) - P(C_{T,\mu} = n)| \to 0$$

(8)

as $N \to \infty$.

4.3. Convergence of NLMPs as $T \to \infty$

In this section we formulate the convergence properties of our NLMPs, which will be crucially used later. In words, they state that the trajectories of our dynamical system $(\mathfrak{A}_a\mu)(t)$ go to the limit point as $t \to \infty$ (and not to some more complicated limit set), this point being a global attractor. This statement is the content of the main result (Theorem 1) in [2].

**Proposition.** Assume that the measure $\mu$ on $\mathcal{Q}$ satisfies

i) $\mathcal{N}(\mu) = a$;

ii) $\int z \, d\mu(q) < \infty$ (where $q = (z,k) \in \mathcal{Q}$).

Then the limit $\lim_{t \to \infty} (\mathfrak{A}_a\mu)(t)$ exists and equals the measure $\nu_a$, which is the unique stationary point of the evolution $\mathfrak{A}_a$.

The stationary measures $\nu_a$ satisfy $\mathcal{N}(\nu_a) = a$; they are uniquely defined by $a$ and the random service time distribution $\eta$. They are stationary distributions of a single server with the corresponding Poisson input flow of constant rate $\lambda_a$. 

PROBLEMS OF INFORMATION TRANSMISSION Vol. 54 No. 3 2018
Here we will prove the convergence:

$$\lim_{N \to \infty} M_N = a, \quad \lim_{N \to \infty} Q_{N,M} = \nu_a,$$

(9)

where the equilibrium measures $\nu_a$ were introduced in the preceding section.

First we show that the family $Q_{N,M}$ is compact. Indeed, the expectations $\mathcal{N}(Q_{N,M}) \to a$. Let us show that for the mean of the elapsed times we have

$$T(Q_{N,M}) = \int z dQ_{N,M}(q) \leq E(\eta^2).$$

(10)

That would be sufficient, since the function $N + T$ is a compact function on $Q$. Relation (10) follows from the fact that the distribution of the duration of the service time of the customer who is currently served has density $\int tp(t) dt$ in the stationary state.

To prove (9), we first show that for every limit point $\tilde{Q}$ of the family $Q_{N,M}$ we have

$$\mathcal{N}(\tilde{Q}) = a.$$  

(11)

In general one can only claim that $\mathcal{N}(\lim_{N \to \infty} Q_{n,M}) \leq a$, but in our case the equality holds. To show this, we will define a process $B_T$ which dominates the processes $A_{N,M}$ for all $N$ large enough, as well as their stationary states $Q_{N,M}$. Since its stationary distribution $\bar{B}$ has finite expected number of customers $\mathcal{N}(\bar{B})$, the family $Q_{N,M}$ is uniformly integrable, so (11) will follow.

In order to define the process $B_T$ we will use the notion of measure dominance. We introduce some notation.

Let $\xi$ and $\zeta$ be two probability measures on $Z_1^+ = \{0, 1, 2, \ldots\}$.

1. We say that $\zeta$ dominates $\xi$, i.e., $\xi \preceq \zeta$, if and only if for any $n > 0$

$$\xi([0,n]) \geq \zeta([0,n]).$$

(12)

(In words, $\zeta$ is to the right of $\xi$.)

2. For $l > 0$ we say that $\xi \preceq_l \zeta$ if and only if (12) holds for $n \leq l$.

For any $\xi \preceq \zeta$ and every $k \in Z_1^+$ we now define measures $\xi \circ_k \zeta$ so that $\zeta = \xi \circ_0 \zeta \geq \xi \circ_1 \zeta \geq \ldots \geq \xi$. The probability measure $\xi \circ_k \zeta$ is uniquely defined by the following properties:

1. $\xi \circ_k \zeta \preceq \zeta$;
2. For every $n < k$ we have $(\xi \circ_k \zeta)([0,n]) = \xi([0,n])$;
3. For every $n > K(\xi, \zeta) > k$ we have $(\xi \circ_k \zeta)([n, +\infty)) = \zeta([n, +\infty))$, where the integer $K(\xi, \zeta)$ satisfies

$$\zeta([k, K(\xi, \zeta) - 1]) = 0.$$

4. With this notation we have a simple lemma:

**Lemma 2.** Assume that $\xi \preceq \zeta$ and for a random variable $\tau \geq 0$ and some $k > 0$ we have

$$\tau \preceq_k \zeta.$$

Then

$$\tau \preceq \xi \circ_{k+1} \zeta.$$
Now we will construct a stationary process which dominates the queue length distributions of all the processes $A_{N,M}$ with $N$ large enough, as well as their stationary states $Q_{N,M}$.

Let us fix some value $\Delta > 0$ (compare with Lemma 1; for example, $\Delta = 1$ would do), and fix some $T > T(\Delta, a)$. Consider a discrete random variable $\chi_{T-\Delta/2}$ which has Poisson distribution with parameter $T - \Delta/2$. Let us pick a small positive $\varepsilon > 0$ to be specified later, and define the integer $K$ as the one satisfying

$$P(\chi_{T-\Delta/2} > K) < \varepsilon.$$ 

According to Theorem 1, its corollary, and Lemma 1 we know that for all $N$ large enough and for any initial state $\mu \in M_N(Q)$ we have

$$C_{N,T,\mu} \preceq K \chi_{T-\Delta/2}.$$ 

We have also a straightforward relation

$$C_{N,T,\mu} \preceq \chi_{TB};$$

see (3). Applying Lemma 2, we conclude that

$$C_{N,T,\mu} \preceq \chi_{T-\Delta/2} \circ K+1 \chi_{TB}$$

(provided $B > 1$). What is very important for us is that

$$E(\chi_{T-\Delta/2} \circ K+1 \chi_{TB}) \leq (T - \Delta/2) + \varepsilon TB < T - \Delta/4$$

once $\varepsilon$ is small enough.

Consider now the random queueing process $B_T$ where the customers are arriving in groups only at discrete moments $kT$, $k = 0, 1, 2, \ldots$, while the number of customers in groups is i.i.d., with distribution $\chi_{T-\Delta/2} \circ K+1 \chi_{TB}$. Because of (14), the process $B_T$ is ergodic, and because of (13) it dominates all the processes $A_{N,M}$; see [14]. Therefore, the stationary distribution $\tilde{B}$ of $B_T$ dominates all the states $Q_{N,M}$.

To conclude, we note that any limit point $\tilde{Q}$ of the family $Q_{N_n,M_n}$ is a stationary point of the process $\tilde{A}$. Since for any $a$ the unique stationary state of $\tilde{A}$ is $\nu_a$ and $N(\tilde{Q}) = a$, we have $\tilde{Q} = \nu_a$.

6. PROPAGATION OF CHAOS

Here we prove finally the Propagation of Chaos property: under $Q_{N,M}$, different nodes of our network are asymptotically independent.

This result follows from the general theorem which will be presented now. Let $k$ be fixed, and assume that for every set of integers $n_1, \ldots, n_k$ a collection of random variables $\xi_1^1, \xi_2^1, \xi_3^1, \ldots, \xi_1^k, \xi_2^k, \ldots, \xi_n^k$ is given. Assume that the joint distribution $P_{n_1,\ldots,n_k}$ of this collection is invariant under the action of the product of the permutation groups $S_{n_1} \times \ldots \times S_{n_k}$, where each group $S_{n_i}$ permutes the random variables $\xi_1^i, \ldots, \xi_{n_i}^i$. Assume that for each $i = 1, \ldots, k$ the law of large numbers (LLN) holds for $\xi_1^i, \ldots, \xi_{n_i}^i$, which means that for every bounded measurable function $f$ we have the convergence of the average

$$\frac{1}{n_i} \sum_{j=1}^{n_i} f(\xi_j^i) \to \mu^i(f)$$

in probability, where $\mu^i(\ast)$ is some (nonrandom) functional. Then the collection $\xi_1^1, \ldots, \xi_{n_1}^1, \xi_2^1, \ldots, \xi_{n_2}^1, \ldots, \xi_1^k, \ldots, \xi_{n_k}^k$ is asymptotically independent.
Theorem 2. For any \( m_1, \ldots, m_k \) and any collection \( f^j_i \) of bounded measurable functions, \( j = 1, \ldots, m_i, \ i = 1, \ldots, k \), we have the convergence of the expectation

\[
E_{P_{n_1, \ldots, n_k}} \left( \prod_{i=1}^{k} \prod_{j=1}^{m_i} f^j_i(\xi^i_j) \right) \to \prod_{i=1}^{k} \prod_{j=1}^{m_i} \mu^i(f^j_i) \tag{15}
\]

when all \( n_i \to \infty \). Also,

\[
\prod_{i=1}^{k} \prod_{j=1}^{m_i} E_{P_{n_1, \ldots, n_k}} [f^j_i(\xi^i_j)] \to \prod_{i=1}^{k} \prod_{j=1}^{m_i} \mu^i(f^j_i).
\]

Proof. The second claim follows immediately from the LLN for the collections \( \xi^i_j, j = 1, \ldots, n_i \), since it claims that \( E_{P_{n_1, \ldots, n_k}} [f^j_i(\xi^i_j)] \to \mu^i(f^j_i) \) as \( n_i \to \infty \). To see (15), let us save on the notation and consider the case \( k = 2, m_1 = m_2 = 2 \). Thus, we are dealing with the random variables \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m \), while their joint distribution \( P_{n,m} \) is \((S_n \times S_m)\)-invariant. Due to the symmetry, we have

\[
E[f_1(\xi_1)f_2(\xi_2)g_1(\eta_1)g_2(\eta_2)] = \frac{1}{n(n-1)m(m-1)} \left( \sum_{i \neq j} f_1(\xi_i)f_2(\xi_j)g_1(\eta_k)g_2(\eta_l) \right).
\]

Since the functions \( f \) and \( g \) are bounded,

\[
\frac{1}{n(n-1)m(m-1)} \left| \sum_{i \neq j} f_1(\xi_i)f_2(\xi_j)g_1(\eta_k)g_2(\eta_l) \right| \to 0
\]

as \( m, n \to \infty \). But

\[
\frac{1}{n(n-1)m(m-1)} \left( \sum_{i,j,k,l} f_1(\xi_i)f_2(\xi_j)g_1(\eta_k)g_2(\eta_l) \right) = \frac{nm}{(n-1)(m-1)} \left[ \left( \frac{1}{n} \sum_i f_1(\xi_i) \right) \left( \frac{1}{m} \sum_j g_1(\eta_k) \right) \left( \frac{1}{m} \sum_l g_2(\eta_l) \right) \right].
\]

Due to the LLN, the product in the square brackets goes to \( \mu^1(f_1) \mu^1(f_2) \mu^2(g_1) \mu^2(g_2) \) in probability, so the theorem follows. \( \triangle \)

This theorem is also contained in [11] and, for the one-type system, in [13]. We have it here in the general form since we will use it later for the multi-type systems of the type treated in [9].

7. CONCLUSIONS

The stochastic dominance technique introduced originally by A. Stolyar [6] for the deterministic service time was extended here to the case of the general service time distribution. Similar methods can hopefully be used for the analysis of other mean-field models.

REFERENCES

1. Karpelevich, F.I. and Rybko, A.N., Asymptotic Behavior of the Thermodynamical Limit for Symmetric Closed Queueing Networks, *Probl. Peredachi Inf.*, 2000, vol. 36, no. 2, pp. 69–95 [Probl. Inf. Transm. (Engl. Transl.), 2000, vol. 36, no. 2, pp. 154–179].
2. Rybko, A.N. and Shlosman, S.B., Poisson Hypothesis for Information Networks. I, II, *Mosc. Math. J.*, 2005, vol. 5, no. 3, pp. 679–704; no. 4, pp. 927–959.

3. Baccelli, F., Rybko, A.N., and Shlosman, S.B., Queueing Networks with Mobile Servers: The Mean-Field Approach, *Probl. Peredachi Inf.*, 2016, vol. 52, no. 2, pp. 86–110 [Probl. Inf. Transm. (Engl. Transl.), 2016, vol. 52, no. 2, pp. 178–199].

4. Bramson, M., Lu, Y., and Prabhakar, B., Asymptotic Independence of Queues under Randomized Load Balancing, *Queueing Syst.*, 2012, vol. 71, no. 3, pp. 247–292.

5. Bramson, M., Lu, Y., and Prabhakar, B., Randomized Load Balancing with General Service Time Distributions, in *Proc. ACM SIGMETRICS Int. Conf. on Measurement and Modeling of Computer Systems (SIGMETRICS'2010)*, June 14–18, 2010, New York, USA, pp. 275–286.

6. Stolyar, A.L., Asymptotics of Stationary Distribution for a Closed Queueing System, *Probl. Peredachi Inf.*, 1989, vol. 25, no. 4, pp. 80–92 [Probl. Inf. Transm. (Engl. Transl.), 1989, vol. 25, no. 4, pp. 321–331].

7. McKean, H.P., Jr., An Exponential Formula for Solving Boltzmann’s Equation for a Maxwellian Gas, *J. Combin. Theory*, 1967, vol. 2, no. 3, pp. 358–382.

8. Liggett, T.M., *Interacting Particle Systems*, New York: Springer, 1985.

9. Rybko, A., Shlosman, S., and Vladimirov, A., Spontaneous Resonances and the Coherent States of the Queueing Networks, *J. Stat. Phys.*, 2008, vol. 134, no. 1, pp. 67–104.

10. Rybko, A. and Shlosman, S., Phase Transitions in the Queueing Networks and the Violation of the Poisson Hypothesis, *Mosc. Math. J.*, 2008, vol. 8, no. 1, pp. 159–180.

11. Graham, C., Chaoticity for Multiclass Systems and Exchangeability within Classes, *J. Appl. Probab.* 2008, vol. 45, no. 4, pp. 1196–1203.

12. Petrova, E.N. and Pirogov, S.A., On “Asymptotic Independence . . .” by V.M. Gertsik, *Markov Process. Related Fields*, 2014, vol. 20, no. 2, pp. 381–384.

13. Sznitman, A.-S., Topics in Propagation of Chaos, *École d’Été de Probabilités de Saint-Flour XIX—1989*, Hennequin, P.-L., Ed., Lect. Notes Math., vol. 1464, Berlin: Springer, 1991, pp. 165–251.

14. Baccelli, F. and Foss, S., Ergodicity of Jackson-type Queueing Networks, *Queueing Systems Theory Appl.*, 1994, vol. 17, no. 1–2, pp. 5–72.