Asymptotic Behaviour of Solutions of a Lotka-Volterra System

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Abstract—In this paper, we study a Lotka-Volterra model of two prey species and one predator species presented in [3]. First, we give existence of an invariant set under a set of sufficient conditions of parameters in the model. These conditions also guarantee that the growth of all the species is permanent. Second, we present a condition under which the predator species falls into decay. Finally, by using Lyapunov functions, we prove global asymptotic stability of solutions of the model.

I. INTRODUCTION

In this paper, we study a Lotka-Volterra model of two prey and one predator, which is presented in [3]:

\[
\begin{align*}
  x'_1 &= x_1(a_1(t) - b_{11}(t)x_1 - b_{12}(t)x_2) - \frac{c_{1}(t)x_1x_3}{\alpha(t) + \beta(t)x_1 + \gamma(t)x_3}, \\
  x'_2 &= x_2(a_2(t) - b_{21}(t)x_1 - b_{22}(t)x_2) - \frac{c_{2}(t)x_2x_3}{\alpha(t) + \beta(t)x_2 + \gamma(t)x_3}, \\
  x'_3 &= x_3\left[-a_3(t) + \frac{d_{1}(t)x_1}{\alpha(t) + \beta(t)x_1 + \gamma(t)x_3} + \frac{d_{2}(t)x_2}{\alpha(t) + \beta(t)x_2 + \gamma(t)x_3}\right].
\end{align*}
\]

Here, \(x_i(t)\) represents the population density of species \(X_i\) at time \(t\) \((i \geq 1)\). \(X_3\) is a predator species and \(X_1, X_2\) are competitive prey species. At time \(t\), \(a_i(t)\) is the intrinsic growth rate of \(X_i\) \((i = 1, 2)\) and \(a_3(t)\) is the death rate of \(X_3\); \(b_{ij}(t)\) measures the amount of competition between \(X_1\) and \(X_j\) \((i \neq j, i, j \leq 2)\), and \(b_{ii}(t)\) \((i \leq 2)\) measures the inhibiting effect of environment on \(X_i\). The predator population consumes prey with amounts described by the Beddington-DeAngelis functional responses (see [11, 22]), say

\[
\begin{align*}
  \frac{c_{1}(t)x_1x_3}{\alpha(t) + \beta(t)x_1 + \gamma(t)x_3}, \\
  \frac{c_{2}(t)x_2x_3}{\alpha(t) + \beta(t)x_2 + \gamma(t)x_3}.
\end{align*}
\]

These amounts contribute to the predator species’ growth rate with amounts of the same form

\[
\begin{align*}
  \frac{d_{1}(t)x_1}{\alpha(t) + \beta(t)x_1 + \gamma(t)x_3}, \\
  \frac{d_{2}(t)x_2}{\alpha(t) + \beta(t)x_2 + \gamma(t)x_3}.
\end{align*}
\]

In our previous work [3], we assumed that the parameters \(a_i(t), b_{ij}(t), c_{i}(t), d_{i}(t), \alpha(t), \beta(t), \gamma(t)(1 \leq i, j \leq 3)\) are periodic in time. Then we investigated the periodicity of positive solutions of (1) and the stability of boundary periodic solutions.

In this paper, we will consider the system (1) with these parameters to be continuous and bounded above and below on \(\mathbb{R}\) by some positive constants. First, we shall show existence of an invariant set of (1). Second, we present a condition under which the predator species falls into decay. Finally, we give a sufficient condition for global asymptotic stability of the system.

The remainder of this paper is organized as follows. Section II provides some definitions. Section III gives our main results.

II. DEFINITIONS

Let us introduce some basic definitions related to the system (1). Denote by \(\mathbb{R}^+_1\) the positive cone of \(\mathbb{R}^3\), i.e.

\[
\mathbb{R}^+_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i > 0, i \geq 1\}.
\]

Let \(g(t)\) be a function, for a brevity, instead of writing \(g(t)\) we write \(g\). If \(g\) is a bounded function on \(\mathbb{R}\), we denote

\[
\begin{align*}
  g^u &= \sup_{t \in \mathbb{R}} g(t), \\
  g^l &= \inf_{t \in \mathbb{R}} g(t).
\end{align*}
\]

Let \(x(t) = (x_1(t), x_2(t), x_3(t))\) be a solution of (1) with an initial value \(x^0 = (x_{10}^0, x_{20}^0, x_{30}^0) = (x_1(t_0), x_2(t_0), x_3(t_0))\), \(t_0 \geq 0\). For biological reasons, throughout this paper, \(x^0\) is taken only in \(\mathbb{R}^3_+\).

Definition 2.1: The system (1) is said to be permanent if there exist \(\delta_i \in \mathbb{R}(i = 1, 2)\) such that

\[
\delta_i \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq \delta_2 \quad (i \geq 1)
\]

for all \(x^0 \in \mathbb{R}^3_+\).

Definition 2.2: A set \(A \subset \mathbb{R}^3_+\) is called an ultimately bounded region of (1) if for any solution \(x(t)\) of (1), there exists \(T_1 > 0\) such that \(x(t) \in A\) for every \(t \geq t_0 + T_1\).
Definition 2.3: A nonnegative solution $x^*(t)$ of (1) is called a global asymptotic stable solution if it attracts any other solution $x(t)$ of (1) in the sense that
$$\lim_{t \to \infty} \sum_{i=1}^{3} |x_i(t) - x^*_i(t)| = 0.$$

Remark 2.4: It is seen that if a solution of (1) is globally asymptotically stable, then so are all solutions of (1). In this case, the system (1) is also said to be globally asymptotically stable.

III. MAIN RESULTS

In this section, we will study existence of an invariant set, the decay of the predator species and the asymptotic stability of (1). It is not difficult to verify that (1) has a global and unique positive solution (for the proof, we estimate the right hand sides of (1) and use the comparison theorem).

For $\epsilon \geq 0$ we put
$$M_1^\epsilon = a_1^0 \epsilon^\frac{1}{13} + \epsilon, \quad M_2^\epsilon = a_2^0 \epsilon^\frac{1}{22} + \epsilon,$$
$$M_3^\epsilon = \frac{d_1^0 M_1^\epsilon + d_3^0 M_2^\epsilon - a_3^0 \epsilon}{a_3^0},$$
$$m_1^\epsilon = \frac{(a_1^0 - b_1^a M_2^\epsilon)(\alpha^\epsilon + \gamma^\epsilon M_3^\epsilon) - c_1^0 M_3^\epsilon}{b_1^\epsilon (\alpha^\epsilon + \gamma^\epsilon M_3^\epsilon)},$$
$$m_2^\epsilon = \frac{(a_2^0 - b_2^a M_1^\epsilon)(\alpha^\epsilon + \gamma^\epsilon M_3^\epsilon) - c_2^0 M_3^\epsilon}{b_2^\epsilon (\alpha^\epsilon + \gamma^\epsilon M_3^\epsilon)},$$
$$m_3^\epsilon = \frac{(d_1^0 - \alpha^\epsilon \beta^\epsilon) m_1^\epsilon + (d_2^0 - \beta^\epsilon \gamma^\epsilon) m_2^\epsilon - 2 a_3^0 \alpha^\epsilon}{2 a_3^0 \gamma^\epsilon}.$$

Let us first show existence of an invariant set of (1).

Theorem 3.1: Assume that $M_3^0 > 0, m_1^0 > 0$ for $i \geq 1$. Then for any sufficient small $\epsilon > 0$ such that $M_3^\epsilon > 0$ and $m_i^\epsilon > 0 (i \geq 1)$, the set $\Gamma_{\epsilon}$ defined by
$$\Gamma_{\epsilon} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid m_i^\epsilon < x_i < M_i^\epsilon, \ i \geq 1\}$$
is positively invariant with respect to (1).

Proof: Since $M_3^0 > 0, m_1^0 > 0$, there exists $\epsilon > 0$ such that $M_i^\epsilon > m_i^0 > 0$ for $i \geq 1$. Therefore, $\Gamma_{\epsilon}$ is well-defined and is a subset of the positive cone of $\mathbb{R}^3$.

To prove the theorem, we have to show that $x(t) \in \Gamma_{\epsilon}$ for $x^0 \in \Gamma_{\epsilon}$ and $t \geq t_0$. We shall use the fact that the equation
$$\begin{cases}
X'(t) = A(t)X(t)[B - X(t)] \quad (B \neq 0), \\
X(t_0) = X^0
\end{cases}$$
has the explicit solution
$$X(t) = \frac{BX^0 e^{\int_{t_0}^{t} A(s)Bds}}{X_0 e^{\int_{t_0}^{t} A(s)Bds} - 1} + B.$$

First, let us give upper estimates for $x_i(t)$. From the first equation of (1), we observer that
$$x_1'(t) \leq x_1(t)[a_1^0 - b_1^a x_1(t)] \leq x_1(t)[a_1^0 - b_1^a x_1(t)] = b_1^a x_1(t)(M_1^0 - x_1).$$

By using the comparison theorem and the inequality $0 < x_1^0 < M_1^0$, we then obtain
$$x_1(t) \leq \frac{x_1^0 M_1^0 e^{a_1^0 (t-t_0)}}{x_1^0 e^{a_1^0 (t-t_0)} - 1} + M_1^0 \leq \frac{x_1^0 M_1^0 e^{a_1^0 (t-t_0)}}{x_1^0 e^{a_1^0 (t-t_0)} - 1} + M_1^0 < M_1^0, \quad t \geq t_0.$$

Similarly, we have
$$x_2(t) < M_2^\epsilon, \quad t \geq t_0.$$  \hfill (4)

Substituting the estimates (3) and (4) into the third equation of (1), it follows that
$$x_3' \leq \frac{-a_3^0 x_3 + \frac{d_1^0 x_1 x_3}{\alpha^\epsilon + \beta^\epsilon x_1 + \gamma^\epsilon x_3} + \frac{d_2^0 x_2 x_3}{\alpha^\epsilon + \beta^\epsilon x_1 + \gamma^\epsilon x_3}}{x_3}\leq \frac{x_3}{\alpha^\epsilon + \gamma^\epsilon x_3} = \frac{a_3^0 \gamma^\epsilon}{\alpha^\epsilon + \gamma^\epsilon x_3} x_3 (M_3^\epsilon - x_3).$$

Hence, by using the comparison theorem, we conclude that for $t \geq t_0$
$$x_3(t) \leq \frac{x_3^0 e^{M_3^\epsilon \int_{t_0}^{t} C_1(s)ds}}{x_3^0 e^{M_3^\epsilon \int_{t_0}^{t} C_1(s)ds} - 1} + M_3^\epsilon < M_3^\epsilon$$
(since $0 < x_3^0 < M_3^\epsilon$), where
$$C_1(t) = \frac{a_3^0 \gamma^\epsilon}{\alpha^\epsilon + \gamma^\epsilon x_3(t)}.$$  \hfill (7)

Let us now give lower estimates for $x_i(t)$. By (1), (3), (4) and (6), we have
$$x_1'(t) \geq x_1 \left[ a_1^0 - b_1^a x_1 - b_1^a M_2^\epsilon - \frac{c_1^0 M_3^\epsilon}{\alpha^\epsilon + \beta^\epsilon x_1 + \gamma^\epsilon M_3^\epsilon} \right] \geq x_1 \left[ (a_1^0 - b_1^a M_2^\epsilon)(\alpha^\epsilon + \gamma^\epsilon M_3^\epsilon) - c_1^0 M_3^\epsilon \right] - b_1^a x_1 = b_1^a x_1 (m_1^\epsilon - x_1).$$

Applying the comparison theorem again, we obtain
$$x_1(t) > m_1^\epsilon, \quad t \geq t_0.$$  \hfill (8)

Similarly, we have
$$x_2(t) > m_2^\epsilon$$
for every $t \geq t_0.$
Finally, from the last equation of (1) it yields that $x_3$ completes the proof.

$$x_3 = -a_3x_3 + \frac{d_1x_1x_3}{\alpha + \beta x_1 + \gamma x_3} + \frac{d_2x_2x_3}{\alpha + \beta x_2 + \gamma x_3} \geq -a_3^*x_3 + \frac{d_1^*m_1^*x_3}{\alpha^* + \beta^*m_1^* + \gamma^*x_3} + \frac{d_2^*m_2^*x_3}{\alpha^* + \beta^*m_2^* + \gamma^*x_3} \geq x_3 \left[-a_3^* + \frac{d_1^*m_1^* + d_2^*m_2^*}{2\alpha^* + \beta^*(m_1^* + m_2^*) + 2\gamma^*x_3}\right] = \frac{2\alpha^* + \beta^*(m_1^* + m_2^*) + 2\gamma^*x_3}{2\alpha^* + \beta^*(m_1^* + m_2^*) + 2\gamma^*x_3} \times \left[\left[\left\{d_1^*m_1^* + d_2^*m_2^* - a_3^*\right\} \left\{2\alpha^* + \beta^*(m_1^* + m_2^*)\right\}\right] - 2a_3^*\gamma^*x_3\right] = \frac{2\alpha^* + \beta^*(m_1^* + m_2^*) + 2\gamma^*x_3}{2\alpha^* + \beta^*(m_1^* + m_2^*) + 2\gamma^*x_3} \times \frac{2a_3^*\gamma^*}{m_3^* - x_3}.

The comparison theorem then provides that $x_3(t) > m_3^*$ for every $t \geq t_0$.

We have thus verified that $x(t) \in \Gamma_c$ for every $t \geq t_0$. It completes the proof.

**Corollary 3.2**: Let the assumptions in Theorem 3.1 be satisfied. Then

$$m_i^* \leq \liminf_{t \to \infty} x_i(t) \leq \limsup_{t \to \infty} x_i(t) \leq M_i^* \quad (i \geq 1).$$

This means that the system (1) is permanent and $\Gamma_c$ is an ultimately bounded region.

**Proof**: According to the proof of Theorem 3.1 we have

$$x_i(t) \leq \frac{x_0^*e^{\alpha_i^*(t-t_0)} - 1}{x_i^0[e^{\alpha_i^*(t-t_0)} - 1] + M_i^0} \quad (i = 1, 2).$$

Therefore,

$$\limsup_{t \to \infty} x_i(t) \leq M_i^0.$$ Hence, there exists $t_1 \geq t_0$ such that

$$x_i(t) < M_i^* \quad \text{for } t \geq t_1 \text{ and } i = 1, 2.$$ Repeating the argument for proving the estimate (5), we observe that

$$x_3(t) \leq \frac{M_3^*x_3}{x_3^0[e^{\alpha_3^*(t-t_0)} - 1] + M_3^0} \leq \max\{M_3^*, x_3^0\} \quad (9)$$

for every $t \geq t_1$ where $x_3^1 = x_3(t_1)$. Combining (7) and (9), it implies that $\inf_{t \geq t_1} C_i(t) > 0$. Therefore, by taking the limit as $t \to \infty$ in (6), we conclude that

$$\limsup_{t \to \infty} x_3(t) \leq M_3^*.$$ Similarly, we have

$$\liminf_{t \to \infty} x_i(t) \geq m_i^*, \quad i \geq 1.$$ Thus, the proof is complete.

Let us next give a sufficient condition under which the predator species falls into decay.

**Theorem 3.3**: Assume that $M_3^0 < 0$. Then $\lim_{t \to \infty} x_3(t) = 0$. This means that the predator species falls into decay.

**Proof**: It follows from $M_3^0 < 0$ that $M_3^* < 0$ with a sufficiently small $\epsilon > 0$. Similarly to the proof of Theorem 3.1 we have

$$x_3(t) \leq \frac{a_3^*\gamma^*}{\alpha^* + \gamma^*x_3} x_3(M_3^* - x_3) < 0. \quad (10)$$

Hence, $x_3(t)$ is a decreasing function in $t$. Consequently, there exists $C > 0$ such that $\lim_{t \to \infty} x_3(t) = C$ and $C \leq x_3(t) \leq x_3^0$ for every $t \geq 0$. If $C > 0$ then by (10) there exists $\mu > 0$ such that $x_3'(t) < -\mu$. Therefore, $x_3(t) < -\mu(t-t_0) + x_3^0$ for every $t \geq t_0$. This makes a contradiction

$$0 \leq \lim_{t \to \infty} x_3(t) = -\infty.$$ Thus, $\lim_{t \to \infty} x_3(t) = 0$.

Finally, let us show the asymptotic stability of solutions of (1). We start with the lemma (see (4)).

**Lemma 3.4**: Let $h$ be a real number and $f$ be a nonnegative function defined on $[h, \infty)$ such that $f$ is integrable and uniformly continuous on $[h, \infty)$. Then $\lim_{t \to \infty} f(t) = 0$.

**Proof**: Suppose the contrary that $f(t) \neq 0$ as $t \to \infty$. Then there exists a sequence $(t_n)$ such that $t_n \to \infty$ as $n \to \infty$ and $f(t_n) \geq \varepsilon$ for every $n \in \mathbb{N}$. By the uniform continuity of $f$, there exists a $\delta > 0$ such that $|f(t_n) - f(t)| \leq \frac{\varepsilon}{2}$ for every $n \in \mathbb{N}$ and $t \in [t_n, t_n + \delta]$. Thus,

$$f(t) = |f(t_n) - [f(t_n) - f(t)]| \geq |f(t_n)| - |f(t_n) - f(t)| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \quad (11)$$

Therefore, for each $n \in \mathbb{N}$

$$\int_{t_n}^{t_n + \delta} f(t) dt = \int_{t_n}^{t_n + \delta} f(t) dt \geq \frac{\varepsilon}{2}.$$ On the other hand, since the Riemann integral $\int_{t_n}^{t_n + \delta} f(t) dt$ exists, the integral $\int_{t_n}^{t_n + \delta} f(t) dt$ converges to 0 as $n \to \infty$. We then arrive at a contradiction

$$0 \geq \frac{\varepsilon}{2}.$$ Thus, $\lim_{t \to \infty} f(t) = 0$.

**Theorem 3.5**: Let $x^*$ be a solution of (1). Let $\epsilon > 0$ be sufficiently small such that $m_i^* > 0$ ($i \geq 1$) and

$$\begin{cases}
\sup_{t \geq t_0} \left\{b_{21} + \alpha d_1 + (\gamma d_1 + \beta d_1)M_1^* - b_{11}\right\} < 0, \\
\sup_{t \geq t_0} \left\{b_{12} + \gamma d_2 + (\beta d_2 + \beta d_2)M_2^* - b_{22}\right\} < 0, \\
\sup_{t \geq t_0} \left\{c_1(\alpha + \beta M_1^*) + c_2(\alpha + \beta M_2^*) - \frac{\gamma d_1 m_1^*}{u_1(M_1^*, M_2^*)} - \frac{\gamma d_2 m_2^*}{u_2(M_2^*, M_3^*)}\right\} < 0,
\end{cases} \quad (11)$$

where $u_i(a, b) = (\alpha + \beta x_i^* + \gamma x_i^*)(\alpha + \beta a + \gamma b)$ ($i = 1, 2$). Then $x^*$ is globally asymptotically stable.

**Proof**: Let $x$ be any other solution of (1). On the account of Corollary 3.2, $\Gamma_c$ is an ultimately bounded region of (1). Therefore, there exists $T_1 > 0$ such that $x(t)$ and $x^*(t)$ belong.
to $\Gamma_e$ for every $t \geq t_0 + T_1$. Consider a Lyapunov function defined by

$$V(t) = \sum_{i=1}^{3} |\ln x_i - \ln x_i^*|, \quad t \geq t_0.$$ 

A direct calculation of the right derivative $D^+ V(t)$ of $V(t)$ along the solutions of (11) gives

$$D^+ V(t) = \sum_{i=1}^{3} \text{sgn}(x_i - x_i^*) \left( \frac{x_i}{x_i^*} - \frac{x_i^*}{x_i} \right)$$

$$= \text{sgn}(x_1 - x_1^*) \left( -b_{11}(x_1 - x_1^*) - b_{22}(x_2 - x_2^*) \right)$$

$$- c_1 \left( \frac{x_3 - x_3^*}{\alpha + \beta x_1 + \gamma x_3} - \frac{x_3^*}{\alpha + \beta x_1^* + \gamma x_3^*} \right)$$

$$+ \text{sgn}(x_2 - x_2^*) \left( -b_{21}(x_1 - x_1^*) - b_{22}(x_2 - x_2^*) \right)$$

$$- c_2 \left( \frac{d_1 x_1}{\alpha + \beta x_2 + \gamma x_3} - \frac{d_2 x_2}{\alpha + \beta x_2^* + \gamma x_3^*} \right)$$

$$+ \text{sgn}(x_3 - x_3^*) \left( \frac{d_2 x_2}{\alpha + \beta x_2 + \gamma x_3^*} - \frac{d_1 x_1}{\alpha + \beta x_2^* + \gamma x_3} \right)$$

$$\leq (b_{21} - b_{11}) |x_1 - x_1^*| + (b_{22} - b_{22}) |x_2 - x_2^*|$$

$$- c_1 \text{sgn}(x_1 - x_1^*) \frac{(\alpha + \beta x_1)(x_3 - x_3^*) - \beta x_3(x_1 - x_1^*)}{u_1(x_1, x_3)}$$

$$- c_2 \text{sgn}(x_2 - x_2^*) \frac{(\alpha + \beta x_2)(x_3 - x_3^*) - \beta x_3(x_2 - x_2^*)}{u_2(x_2, x_3)}$$

$$+ \text{sgn}(x_3 - x_3^*) \frac{d_1 (x_1 - x_1^*) + \gamma d_1 x_1 (x_3 - x_3^*)}{u_1(x_1, x_3)}$$

$$+ \text{sgn}(x_3 - x_3^*) \frac{d_2 (x_2 - x_2^*) + \gamma d_2 x_2 (x_3 - x_3^*)}{u_2(x_2, x_3)}$$

Using the expression

$$x_i x_i^* - x_i^* x_3 = x_3 (x_3 - x_i) + x_3 (x_i - x_i^*) \quad (i = 1, 2),$$

we observe that

$$D^+ V(t) \leq \sum_{i=1}^{3} \text{sgn}(x_i - x_i^*) \left( \frac{x_i}{x_i^*} - \frac{x_i^*}{x_i} \right)$$

$$\leq \sum_{i=1}^{3} \text{sgn}(x_i - x_i^*) \left( \frac{x_i}{x_i^*} - \frac{x_i^*}{x_i} \right)$$

$$\leq \left( b_{21} - b_{11} \right) |x_1 - x_1^*| + \left( b_{22} - b_{22} \right) |x_2 - x_2^*|$$

$$- c_1 \text{sgn}(x_1 - x_1^*) \frac{(\alpha + \beta x_1)(x_3 - x_3^*) - \beta x_3(x_1 - x_1^*)}{u_1(x_1, x_3)}$$

$$- c_2 \text{sgn}(x_2 - x_2^*) \frac{(\alpha + \beta x_2)(x_3 - x_3^*) - \beta x_3(x_2 - x_2^*)}{u_2(x_2, x_3)}$$

$$+ \text{sgn}(x_3 - x_3^*) \frac{d_1 (x_1 - x_1^*) + \gamma d_1 x_1 (x_3 - x_3^*)}{u_1(x_1, x_3)}$$

$$+ \text{sgn}(x_3 - x_3^*) \frac{d_2 (x_2 - x_2^*) + \gamma d_2 x_2 (x_3 - x_3^*)}{u_2(x_2, x_3)}$$

\[ \leq \left[ b_{21} + \frac{(\alpha + \gamma M_1^*) d_1 + \beta c_1 M_1^*}{u_1(m_1^*, M_1^*)} - b_{11} \right] |x_1 - x_1^*| \]

\[ + \left[ b_{22} + \frac{(\alpha + \gamma M_2^*) d_2 + \beta c_2 M_2^*}{u_2(m_2^*, M_2^*)} - b_{22} \right] |x_2 - x_2^*| \]

\[ + \left[ c_1 (\alpha + \beta M_1^*) + c_2 (\alpha + \beta M_2^*) \right] \frac{d_1 m_1^*}{u_1(m_1^*, M_1^*)} \]

$$- \frac{\gamma d_2 m_2^*}{u_2(m_2^*, M_2^*)} |x_3 - x_3^*|, \quad t \geq t_0 + T_1.$$ 

Substituting (11) into (12), we have

$$D^+ V(t) \leq -\mu \sum_{i=1}^{3} |x_i - x_i^*| \quad \text{for every } t \geq t_0 + T_1,$$

where $\mu$ is some positive constant. Integrating both the hand sides of (12) from $t_0 + T_1$ to $t \geq t_0 + T_1$, we obtain

$$V(t) + \mu \int_{t_0 + T_1}^{t} \sum_{i=1}^{3} |x_i - x_i^*| \, ds \leq V(t_0 + T_1) < \infty.$$ 

Hence, for every $t \geq t_0 + T_1$

$$\int_{t_0 + T_1}^{t} \sum_{i=1}^{3} |x_i - x_i^*| \, ds \leq \mu^{-1} V(t_0 + T_1) < \infty.$$ 

This means that $\sum_{i=1}^{3} |x_i - x_i^*| \in L^1([t_0 + T_1, \infty)).$

On the other hand, by the ultimate boundedness of $x_i^*$ and $x_i$ and $x_i^*$ (i > 1) have bounded derivatives for $t \geq t_0 + T_1$. As a consequence, $\sum_{i=1}^{3} |x_i - x_i^*|$ is uniformly continuous on $[t_0 + T_1, \infty)$. Thanks to Lemma 3.4 we conclude that $\lim_{t \to \infty} \sum_{i=1}^{3} |x_i - x_i^*| = 0$. This completes the proof. 

\[ \Box \]

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