A BLOWUP PROBLEM OF REACTION DIFFUSION EQUATION RELATED TO THE DIFFUSION INDUCED BLOWUP PHENOMENON

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Abstract

This work studies nonnegative solutions for the Cauchy, Neumann, and Dirichlet problems for the logistic type equation $u_t = \Delta u + \mu u^p - a(x)u^q$ with $p, q > 1, \mu > 0$. The finite time blowup results for nonnegative solutions under various restrictions on $a(x), p, q, \mu$ are presented. Applying the results allows one to construct some reaction diffusion systems with
the so-called diffusion-induced blowup phenomenon, particularly in the case of equal diffusion rates.

**Key Words:** reaction-diffusion system, blowup, global existence, diffusion-induced blowup

1 Introduction

This study considers the reaction diffusion equation

\[
\begin{cases}
  u_t = \Delta u + \mu u^p - a(x)u^q & \text{in } D \times (0, T^*), \\
  u(x, 0) = u_0(x) \geq 0 & \text{on } D,
\end{cases}
\]

where \( D \subset \mathbb{R}^n \) is a bounded smooth domain, or \( D = \mathbb{R}^n, u_0 \in C(D), \) and \( T^* \) is the maximum life time of a solution defined below. When \( D \) is a bounded smooth domain, the homogeneous Neumann boundary condition or Dirichlet boundary condition is imposed:

\[
\begin{align*}
  \frac{\partial u}{\partial n} &= 0, & \text{on } \partial D \times (0, T^*), \\
  u &= 0, & \text{on } \partial D \times (0, T^*),
\end{align*}
\]

where \( n \) denotes the unit exterior normal vector on \( \partial D \). It is assumed that \( \mu > 0, p, q > 1, \) and \( a : D \rightarrow R \) is a smooth function which satisfies some additional conditions (see the conditions in the main theorems).
on the blowup problem of $u_t = \Delta u + \mu u^p - a(x)u^q$.

Two motivations exist for studying the above problem.

First, as well known, a classical solution of a reaction diffusion system may blow up in a finite time (see Levine [17]). Previous literature usually adopt a spatially homogeneous reaction term. Therefore elucidating the spatial influence on the blowup problem for a reaction diffusion equation is a worthwhile task. Fujita [8], Hayakawa [12], Kobayashi, Sario, and Tanaka [15], and Aronson and Weinberger [1] considered the Cauchy problem for

$$u_t = \Delta u + u^p, \quad 1 < p \leq 1 + \frac{2}{n}.$$  

According to their results, the above problem has no nonnegative nontrivial solution satisfying

$$||u(\cdot, t)||_{L^\infty} < \infty, \quad \text{for all } t \geq 0.$$  

Thus, the solutions of $u_t = \Delta u + \mu u^p - a(x)u^q$ blow up in finite time when $a \equiv 0$. On the other hand, if $q > p$ and $a$ is uniformly positive, the solutions exist all of the time. Therefore, what happens when $a$ is nonnegative but vanishes at one or more points is worth exploring.

Second, as an application of our blowup result for (1.1) (Theorem 1.2), a reaction diffusion system, (1.6) below, with the so-called diffusion-induced blowup phenomenon (see [20]) can be constructed. For a given reaction diffusion system: $u_t = D\Delta u + f(x, u)$, with the Cauchy, Neumann or Dirichlet problem, the diffusion term can usually smooth solutions. However, recent works
On the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

[4,20,21,25,31,32] indicated that diffusion terms may cause a blowup in finite time. Such a phenomenon is called *diffusion-induced blowup*. Notably, for a single equation, no such phenomenon exists by the comparison principle. Churbanov [4] provided an example for the Cauchy problem. Meanwhile, Morgan [21], M. X. Wang [31], Mizoguchi, Ninomiya, and Yanagida [20], and Weinberger [32] constructed examples for the Neuman and Dirichlet problems.

More precisely, Morgan gave the following example:

\[
\begin{aligned}
&v_t = (\frac{1}{\pi^2})v_{xx} + v, \quad 0 < x < 1, \quad t > 0, \\
u_t = u^2 - |3 - v|u^3, \quad 0 < x < 1, \quad t > 0, \\
v_x(0, t) = v_x(1, t) = 0, \quad t > 0, \\
v(x, 0) = a \cos(\pi x), \quad u(x, 0) = u_0(x), \quad 0 < x < 1,
\end{aligned}
\]  

(1.4)

where \( u_0 \) is a smooth function such that \( u_0(\frac{1}{\pi \arccos(3/a)}) > 0 \) with \( |a| \geq 3 \).

Morgan proved that the solutions of (1.4) blow up in finite time. However, if the diffusion term is excluded, the solutions of the corresponding ordinary differential equations of (1.4) will exist globally for all initial data. Later, M. X. Wang [31] modified the above example by adding the diffusion term in the \( u \)-equation and letting \( a = 3 \). In fact, since solution \( v \) can be solved explicitly, under appropriate rescaling of variables \( u \) and \( t \), Wang considered the blowup
and global existence problem of the following scalar equation:

\[
\begin{cases}
  u_t = d u_{xx} + u^p - (1 - \cos \pi x) u^q, & 0 < x < 1, \ t > 0, \\
  u_x = 0, & x = 0, 1; t > 0, \\
  u(x, 0) = u_0(x) > 0, & 0 < x < 1,
\end{cases}
\]  \tag{1.5}

where \( q > p, \ d > 0 \). He proved that if \( 1 < p < q \leq 2p - 1 \), the solution blows up in finite time whenever \( d \) is small and \( u_0(x) \) is large. Moreover, the blowup point occurs at 0 only. Notably, Wang did not prove the global existence of the corresponding ordinary equation in this general \( p, q \) setting. Therefore, whether or not Wang’s example contains the diffusion-induced blowup phenomenon (\( p = 2, q = 3 \) is OK from Morgan’s result) will be verified later. Recently, Mizoguchi et al. [20] provided an example which has the diffusion-induced blowup phenomenon with unequal diffusion rates and asked whether examples with equal diffusion rates which have such phenomena exist. Weinberger [32] confirmed that such example exists. This study considers the Cauchy, homogeneous Neumann, and Dirichlet problems of the following example which has the diffusion-induced blowup phenomenon (see Sec.4):

\[
\begin{cases}
  v_t = d_1 \Delta v + f(x, v), & \text{in } D \times (0, \infty), \\
  u_t = d_2 \Delta u + \mu u^p - |m - v|^\sigma u^q, & \text{in } D \times (0, \infty), \\
  v(x, 0) = v_0(x) \geq 0, \ u(x, 0) = u_0(x) \geq 0, & x \in D.
\end{cases}
\]  \tag{1.6}

where \( D = \mathbb{R}^n \) or a smooth bounded domain, \( d_1, d_2, \mu \) are nonnegative constants, \( f \) is a smooth function, \( v(x,t) = v_0(x) \) independent of \( t \) is an entire
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solution of \( \mathbb{R}^n \) or a solution for the corresponding homogeneous elliptic Neumann or Dirichlet problem, and \( m = v_0(x_0) \), for some \( x_0 \in D \). Notice that since the \( v \) equation in (1.6) does not couple with \( u \), by solving \( v \) and substituting it into \( u \) equation we can get a scalar equation in the form as in (1.1). Therefore our blowup results about (1.1) are related to the above reaction diffusion system (1.6). The corresponding ordinary differential equation is as follows:

\[
\begin{cases}
  v_t = f(x_0, v), \\
  u_t = \mu u^p - |m - v|^\sigma u^q, \ t \geq 0, \\
  v(0) = \xi \geq 0, u(0) = \eta \geq 0.
\end{cases}
\] (1.7)

In sum, our blowup results generalize Wang’s results into a more general equation (in fact, the condition “\( 1 < p < q \leq 2p - 1 \)” in Wang’s results is the special case “\( \sigma = 2 \)” in Theorem 1.2 of this paper) and hold for the Cauchy, Neumann, and Dirichlet problems. Also, the example herein which exhibits the diffusion-induced blowup phenomenon allows for equal diffusion rates. Therefore, the question in [20] is also answered positively.

Now, some preparations are made for stating our main results. As well known, (1.1) with (1.2) or (1.3) in the bounded smooth domain \( D \) has an unique local classical solution \( u(x, t) \) defined in some maximal interval of existence \( 0 < t < T^*(u_0) \) such that

\[
\lim_{t \to T^*(u_0)} ||u(\cdot, t)||_{\infty} = \infty
\]
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

whenever \( T^*(u_0) < \infty \) (see Henry [13]). \( T^*(u_0) \) is called the blowup time if \( T^*(u_0) < \infty \). If there exist sequences \( \{x_k\}, \{t_k\} \) with \( t_k \in (0, T^*) \), \( x_k \in \overline{D} \) satisfying

\[
\lim_{k \to \infty} x_k = x_0, \quad \lim_{k \to \infty} t_k = T^*, \quad \lim_{k \to \infty} |u(x_k, t_k)| = \infty,
\]

this \( x_0 \) is called a blowup point. As for the Cauchy problem (i.e., \( D = \mathbb{R}^n \)), when \( u_0 \) is bounded uniformly continuous, a local solution in the class of bounded uniformly continuous functions is guaranteed (see [18, Sect. 1.4.]), and the blowup time can be similarly defined as in the bounded smooth domain case.

This work considers solutions for the Cauchy problem in the more general space:

\[
|u(x, t)| \leq M|x|^\frac{q}{p-1}, \quad (x, t) \in \mathbb{R}^n \times (0, T^*), \quad M > 0.
\]

(1.8)

Where the Cauchy problem for (1.1) is denoted by (I), and the homogeneous Neumann and Dirichlet problem is represented by (II) and (III) respectively. The solution of (I) is always assumed to satisfy (1.8).

Before Proposition 1.1, i.e., the key lemma in this study, is stated, some heuristic reasons for this proposition are explained. Notably, the equation in (1.1) with \( a(x) = |x|^\sigma \) is invariant under the scaling: \( t \to \lambda t, \quad x \to \lambda^{\frac{1}{p-1}} x, \quad u \to \lambda^{\frac{1}{p-1}} u \), if \( \sigma = 2(q - p)(p - 1)^{-1} \) (or \( q = (\frac{\sigma}{2} + 1)p - \frac{\sigma}{2} \)). This feature suggests finding the self-similar lower solution and using it as a blowing up lower solution to obtain the blowup results. The following problem (1.9) is obtained by using similarity variables (see Lemma 3.1).
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

**Proposition 1.1.** Let \( q \geq p > 1 \) and \( \sigma \geq 0 \). When \( n \geq 3 \), further assume that \( p < \frac{n+2}{n-2} \). Then, there exists \( \mu_0 > 0 \) such that the Dirichlet problem

\[
\begin{aligned}
\Delta w(y) - \frac{\gamma}{2} y \cdot \nabla w(y) - \frac{1}{p-1} w(y) + \mu_0 w^p(y) - |y|^q w^q(y) = 0, \\
w|_{\partial B(0,r_0)} = 0,
\end{aligned}
\]

has a radially symmetry positive weak solution \( w_0 \in H^1_0(B(0,r_0), \rho) \), where \( \rho(y) = \exp(-|y|^2) \), \( B(0,r_0) \) is the open ball centered at 0 with radius \( r_0 \), and \( H^1_0(B(0,r_0), \rho) \) is the \( H^1_0 \) Sobolev space with weight \( \rho(y) \) (see Escobedo and Ka-vian [5]). Moreover, if \( 1 < q < \frac{n+2}{n-2} \) when \( n \geq 3 \), then \( w_0 \in C^{2+\alpha}(B(0,r_0)) \) for some \( 0 < \alpha < 1 \).

Proposition 1.1 can be proved similarly to Proposition 2 of [9] (variational method) and Theorem 3.12 of [5] (using the bootstrap method for the regularity). Therefore the proof of Proposition 1.1 is omitted here.

Define

\[
w(r) \equiv \begin{cases} 
  w_0(r), & 0 \leq r \leq r_0, \\
  0, & r > r_0.
\end{cases}
\]

The following theorem is the main result concerning blowup:

**Theorem 1.2.** Assume that \( u(x,t) \) is a nonnegative classical upper solution of (I). Also assume \( q \geq p > 1 \) and \( a(x) \) is a real valued smooth function
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satisfying

\[
a(x) \leq M|x - x_0|^{\sigma}, \quad x \in \mathbb{R}^n, \quad M > 0, \quad \sigma \geq 2(q - p)(p - 1)^{-1},
\]

where \( a(x_0) = 0 \). When \( n \geq 3 \), it is further assumed that \( q < \frac{n+2}{n-2} \). Let \( \mu \geq M^\frac{p-1}{q-1} \rho_0 \) in (1.1) and \( t_0 \) be any negative number where \( \sigma = 2(q - p)(p - 1)^{-1} \) or \(-1 \leq t_0 < 0 \) where \( \sigma > 2(q - p)(p - 1)^{-1} \). If \( u_0(x) \geq M^{\frac{1}{q-1}}(-t_0)^{\frac{1}{q-1}}w\left(\frac{|x-x_0|}{\sqrt{-t_0}}\right)\)

for all \( x \in \mathbb{R}^n \), then \( u(x, t) \) will blow up at a finite time which is before or equal to \( -t_0 \). Similarly, if \( u(x, t) \) is a nonnegative classical upper solution of (II), or (III) in \( D = B(x_0, r_0\sqrt{-t_0}) \) with \( r_0 \) as in Proposition 1.1, then the above result also holds under the same conditions as in (I).

**Remark.** (i) The proof of Theorem 1.2 indicates that the lower bound of blowup rate is \((T^* - t)^{\frac{1}{p-1}}\). If \( a(x) \) is a nonnegative term and \( u(x, t) \) is a nonnegative classical solution of the Cauchy problem (I), then the upper bound of the blowup rate is also \((T^* - t)^{\frac{1}{p-1}}\) by [10, Theorem 3.7] and the comparison principle. Therefore, the blowup rate is \((T^* - t)^{\frac{1}{p-1}}\).

(ii) The result obtained in Theorem 1.2 is natural from a biological perspective. Since the term, \([\mu a(x)^{-1}]^{\frac{1}{p-1}}\), is explained as the saturation level of the species, the conditions of \( \mu, a(x) \) in Theorem 1.2 imply \([\mu a(x)^{-1}]^{\frac{1}{p-1}}\) will become very large, particularly at the zero points of \( a(x) \), and the population density tends to increase rapidly at such points, as stated in Theorem 1.2.

The structure of the blowup set is as follows:
The blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

**Theorem 1.3.** Assume that \( q > p > 1 \) and \( a(x) \) is a nonnegative smooth function. For problem (III), if \( \sigma \geq 2(q-p)(p-1)^{-1} \), where \( \sigma \) is the zero order of \( a(x) \) at any zero point, then the blowup points occur at the zero points of \( a(x) \) only. For problem (II), if \( a(x) \) satisfies the following condition: for every \( x \in \partial D \), either \( \frac{\partial a(x)}{\partial n} \leq 0 \), or \( a(x) = 0 \), then the blowup points occur at the zero points of \( a(x) \) only.

The methods applied herein to verify the accuracy of these results are the similarity variables method and comparison principle in the weak or classical sense. The similarity variables method is widely applied to study the blow-up or global existence phenomenon (see [5, 9, 10, 24, 28]). Generally, two kinds of self-similar solutions exist, backward and forward. The backward self-similar solution is relevant to the blowup problem, while the forward self-similar solution is relevant to the global existence problem. Proposition 1.1 provides a backward self-similar classical solution of the original Eqn. (1.1) with \( a(x) = |x|^\sigma \) in some special domain. To apply a comparison principle in the weak sense for the Cauchy problem (Lemma 2.3) or the Neumann and Dirichlet problems (Lemma 2.2) to the subject domain \( D \), in Lemma 3.2 the above backward self-similar solution is suitably extended to the whole domain \( D \) via Lemma 2.4, which is a connecting lemma for producing weak lower solution in a larger domain. Thus the blowup result (Theorem 1.2) follows from comparison with the weak lower solution in \( D \).

The rest of this paper is organized as follows. Section 2 lists some comparison
principles in a weak sense. Section 3 proves the blowup results, Theorem 1.2 and Theorem 1.3. Finally, Section 4 presents some examples that illustrate the diffusion-induced blowup phenomenon.

2 The Comparison Principle

The main tools applied within for proving the main results are the comparison principles in a weak sense (see [7, 16, 19, 30]). In this section, these comparison principles are stated or derived in the suitable form convenient for the application to the Dirichlet, Neumann and Cauchy problems herein.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded smooth domain. The following linear equation is considered.

$$
\frac{\partial u}{\partial t} \equiv \Delta u + \sum_{i=1}^{n} a_i(x, t) \frac{\partial u}{\partial x_i} + a(x, t)u \quad \text{in} \quad \Omega,
$$

(2.1)

where $a_j, \frac{\partial a_j}{\partial x_k}, a$ are Hölder continuous in $(x, t) \in \Omega$ and $a \leq 0$ in $\Omega$. The so-called Strong Maximum Principle in a weak sense is as follows (see [7]).

**Lemma 2.1.** Assume that $u(x, t)$ is a bounded measurable function in $\Omega$ satisfying

$$
\int_{E} u(x, t)(\Delta v - \sum_{i} \frac{\partial (a_i v)}{\partial x_i} + \frac{\partial v}{\partial t} + av)dxdt \geq 0,
$$

(2.2)

for every compact subset $E \subset \Omega$ and nonnegative test function $v$ with $\text{supp}(v) \subset E$. Assume that $u(x^0, t^0) = \text{esssup}_{(x, t) \in \Omega} u(x, t) \geq 0$ with $(x^0, t^0) \in \Omega$. Use $C(x^0, t^0)$
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

to denote the set of all point \((x^1, t^1)\) in \( \Omega \) that can be connected to \((x^0, t^0)\) by a smooth curve in \( \Omega \) along which the \( t \)-coordinate is increasing. If \( u|_{C(x^0, t^0)} \) is a continuous function, then \( u \equiv M \) almost everywhere in \( C(x^0, t^0) \).

Lemma 2.1 is used herein to prove a comparison principle for the Neumann and Dirichlet problems. Let \( \Omega \equiv B(0, r_0 \sqrt{-t_0}) \times (t_0, t_1) \) with \( t_0 < t_1 < 0 \) fixed, \( S \equiv \partial B(0, r_0 \sqrt{-t_0}) \times (t_0, t_1], \) and \( I \equiv \{(x, t) \in \Omega : |x| > r_0 \sqrt{-t}\}. \)

**Lemma 2.2.** Assume that \( u(x, t) \) is a continuous function in \( \Omega \). Let \( h(x, t) \in C^{\alpha, \beta}(\Omega) \) be a bounded function. Assume that \(-u\) satisfies (2.2) with \( a_i \) and \( a \) replaced by 0 and \(-h\) respectively. For the Dirichlet problem, if \( u|_S \geq 0 \) and \( u(x, t_0) \geq 0 \), then \( u(x, t) \geq 0 \) in \( \Omega \). For the Neumann problem, further assume that \( u|_I \) is smooth. Then under \( \frac{\partial u}{\partial n}|_S \geq 0 \) and \( u(x, t_0) \geq 0 \), solution \( u(x, t) \) is also nonnegative in \( \Omega \).

**Proof.** Since \(-u\) satisfies (2.2), we have

\[
\int_{\Omega} u(-\Delta v - v_t + h(x, t)v) \geq 0
\]

for all nonnegative test function with \( \text{supp}(v) \subset \subset \Omega \). Let \( \nu \equiv ve^{lt} \) with \( l > -h \). Then

\[
-\Delta v - v_t + h(x, t)v = [-\Delta \nu + (h + l)\nu - \nu_t]e^{-lt}.
\]

Hence

\[
\int_{\Omega} u e^{-lt}[-\Delta \nu - \nu_t + (h + l)\nu]dxdt \geq 0.
\]
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

Set \( u^*(x,t) \equiv u(x,t)e^{-lt} \). To prove the result, it suffices to show \( u^*(x,t) \geq 0 \). By \( u(x,t_0) \geq 0 \), \( u^*(x,t_0) \geq 0 \) is obtained. Assume that the minimum of \( u^* \) over \( \Omega \) is negative, say \( m \), and \( u^*(\vec{x},\vec{t}) = m \) for some point \( (\vec{x},\vec{t}) \in \Omega \).

First, if \( (\vec{x},\vec{t}) \in \Omega \cup B(0,r_0\sqrt{-t_0}) \times \{t_1\} \), then Lemma 2.1 can be used to obtain \( u^*(x,t_0) \equiv m \). This contradicts the fact “\( u^*(x,t_0) \geq 0 \)” Similarly, \( \vec{t} \) is not equal to \( t_0 \). For the Dirichlet problem, \( (\vec{x},\vec{t}) \) does not occur at \( S \), producing a contradiction. Therefore we get a contradiction. For the Neumann problem, if \( (\vec{x},\vec{t}) \in S \), then since \( \min_{\mathcal{F}} u^* = m \), \( u^*|_S \) is smooth, and \( u^*(x,t)|_S > m \) (by the previous argument), \( \frac{\partial u^*}{\partial \nu}(\vec{x},\vec{t}) < 0 \) on \( S \) is obtained by the classical Hopf maximum principle. This finding also contradicts “\( \frac{\partial u}{\partial \nu}|_S \geq 0 \)” The proof is completed.

Consider the following problem

\[
\begin{align*}
& u_t = \Delta u + f(x,t,u) \quad \text{in} \quad \mathbb{R}^n \times (0,T^*), \\
& u|_{t=0} = u_0(x) \quad \text{in} \quad \mathbb{R}^n,
\end{align*}
\]

where \( T^* > 0 \) and \( u_0(x) \) is a continuous function. For the above Cauchy problem, the corresponding comparison principle is deduced in [30]:

**Lemma 2.3.** Assume that \( \overline{\omega}(x,t) \) is a continuous function on \( \mathbb{R}^n \times [0,T^*) \) satisfying

\[
\overline{\omega}|_{t=0} \geq (\leq) u_0(x)
\]
on the blowup problem of $u_t = \Delta u + \mu u^p - a(x)u^q$

and

$$\int_{R^n} \pi(x,t)\eta(x,t)dx|_{t=T_1}^{t=0} \geq (\leq) \int_0^{T_1} \int_{R^n} [\pi(x,s)(\Delta \eta + \eta_t)(x,s) + \eta(x,s)f(x,s,\pi)]dxds$$

for $T_1 \in [0,T^*)$,

where $\eta$ is a nonnegative, smooth function on $R^n \times [0,T^*)$ with supp($\eta(\cdot,t)$)$\subset\subset R^n$ for all $t \in [0,T^*)$. Assume $(\pi - u)(x,t) \geq -B\exp(\beta|x|^2)$ on $R^n \times (0,T^*)$ and

$$f(x,t,\pi(x,t)) - f(x,t,\underline{u}(x,t)) \geq c(x,t)(\pi - \underline{u})(x,t),$$

where $B, \beta > 0$, $c \in C_{\text{loc}}^{0,\frac{\alpha}{2}}(R^n \times (0,T^*))$, and $c(x,t) \leq c_0(|x|^2 + 1)$ on $R^n \times (0,T^*)$ for some $c_0$. Then, $\pi \geq u$ on $R^n \times [0,T^*)$.

Remark. The $-u$ ($u$) in Lemma 2.2 is called a continuous weak lower (upper) solution of the Dirichlet problem, or the Neumann problem. Also, $\pi$ and $\underline{u}$ in Lemma 2.3 will be called the continuous weak upper and lower solutions of the Cauchy problem (2.3) respectively.

For later convenience, we now prove the following “connecting lemma” which is used to combine two classical lower solutions into one continuous weak lower solution. Let $\Omega_1$ be a bounded domain in $R^n \times (t_0,t_1)$, $\Omega_2 \equiv R^n \times (t_0,t_1) - \overline{\Omega_1}$, $D_1(t) \equiv \{x \in R^n : (x,t) \in \Omega_1\}$ for $t \in (t_0,t_1)$, and $D \equiv \bigcup_{t \in (t_0,t_1)} D_1(t)$.
Lemma 2.4. Let \( \underline{u}(x,t) \) be a smooth function and satisfy

\[
\underline{u}_t \leq \Delta \underline{u} + f(x,t,\underline{u})
\]

on \( \Omega_1, \Omega_2 \) respectively. If \( \underline{u} \) is continuous on \( \mathbb{R}^n \times [t_0, t_1] \) and \( \frac{\partial \underline{u}(\cdot,t)}{\partial n} |_{\partial D_i(t)} + \frac{\partial \underline{u}(\cdot,t)}{\partial n} |_{\partial D_2(t)} \leq 0 \), then \( \underline{u} \) satisfies (2.4) in the case “\( \leq \)” with \((0,T^\ast)\) replaced by \((t_0,t_1)\), where \( n_i \) is the unit outer-normal vector of \( D_i(t) \) for all \( t \in (t_0,t_1) \).

Proof. For a nonnegative, smooth function \( \phi \) on \( \mathbb{R}^n \times [t_0, t_1] \) with \( \text{supp}(\phi(\cdot,t)) \subset \subset \mathbb{R}^n \) for all \( t \in [t_0, t_1] \), \((\Delta)\) denotes the following term

\[
(\Delta) = \int_{t_0}^{t_1} \int_{\Omega_1} [\underline{u}_t - \Delta \underline{u} - f(x,t,\underline{u})] \phi \ dt + \int_{\Omega_2} [\underline{u}_t - \Delta \underline{u} - f(x,t,\underline{u})] \phi \ dt.
\]

Let \((A) = \int_{\Omega_1} \underline{u} \phi, \ (B) = \int_{\Omega_2} \underline{u} \phi, \ (C) = \int_{\Omega_1} \Delta \underline{u} \phi, \) and \((D) = \int_{\Omega_2} \Delta \underline{u} \phi.\)

\[
(A) = \int_D dx \int_{a(x)}^{b(x)} \underline{u}_t \phi \ dt = \int_D \underline{u}_t \phi \ |_{a(x)}^{b(x)} dx - \int_{\Omega_1} \underline{u} \phi_t,
\]

\[
(B) = \int_D dx \int_{[t_0,t_1]-[a(x),b(x)]} \underline{u}_t \phi dt + \int_{\mathbb{R}^n-D} dx \int_{t_0}^{t_1} \underline{u} \phi dt
\]

\[
= \int_D (\underline{u}_t^{a(x)} + \underline{u}_t^{-b(x)}) dx + \int_{\mathbb{R}^n-D} \underline{u}_t^{a(x)} dx - \int_{\Omega_2} \underline{u} \phi_t.
\]
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

Adding (A) and (B) leads to

\[
(A) + (B) = -\int_{R^n \times [t_0, t_1]} u \phi_t + \int_{R^n} u \phi_1^{t_0} dx.
\]

\[
(C) = \int_{t_0}^{t_1} dt \int_{D_1(t)} \Delta u \phi dx
= \int_{t_0}^{t_1} dt \int_{\partial D_1(t)} \phi \frac{\partial u}{\partial n_1} d\sigma - \int_{t_0}^{t_1} dt \int_{\partial D_1(t)} \frac{\partial \phi}{\partial n_1} d\sigma + \int_{t_0}^{t_1} dt \int_{D_1(t)} u \Delta \phi dx,
\]

\[
(D) = \int_{t_0}^{t_1} dt \int_{D_2(t)} \Delta u \phi dx
= \int_{t_0}^{t_1} dt \int_{\partial D_2(t)} \phi \frac{\partial u}{\partial n_2} d\sigma - \int_{t_0}^{t_1} dt \int_{\partial D_2(t)} \frac{\partial \phi}{\partial n_2} d\sigma + \int_{t_0}^{t_1} dt \int_{D_2(t)} u \Delta \phi dx.
\]

Substituting the above results into (\( \Delta \)) produces

\[
(\Delta) = \int_{R^n} u \phi_1^{t_0} dx - \int_{R^n \times [t_0, t_1]} u \phi_t + u \Delta \phi + f(x, t, u) \phi
+ \int_{t_0}^{t_1} dt \left( \int_{\partial D_1(t)} \phi \frac{\partial u}{\partial n_1} d\sigma + \int_{\partial D_2(t)} \frac{\partial \phi}{\partial n_2} d\sigma \right)
- \int_{t_0}^{t_1} dt \left( \int_{\partial D_1(t)} \phi \frac{\partial u}{\partial n_1} d\sigma + \int_{\partial D_2(t)} \frac{\partial u}{\partial n_2} d\sigma \right).
\]

Since \( u(x, t) \) is a classical lower solution on \( \Omega_1, \Omega_2 \), respectively and \( \phi \geq 0 \),

\( (\Delta) \leq 0 \) is obtained. By the smoothness of \( \phi \) and \( \frac{\partial u}{\partial n_1} |_{\partial D_1(t)} + \frac{\partial u}{\partial n_2} |_{\partial D_2(t)} \leq 0 \), the proof is completed.

\( \blacksquare \)

**Remark.** Lemma 2.4 can also be proved by modifying the argument of the
proof of Lemma 3.3 in [19]. A slightly different proof is presented herein for completeness.

3 Proofs of Blowup Results

In this section, the main results about the blowup problem, Theorem 1.2 and Theorem 1.3, will be proved. Since the equation in (1.1) with \( a(x) = |x|^\sigma \) has some scaling invariant property under the condition \( q = (\frac{\sigma}{2} + 1)p - \frac{\sigma}{2} \), the following lemma states the relation between the original equation and the equation which is satisfied by the self-similar lower or upper solution.

**Lemma 3.1.** Assume that \( \mu > 0, \sigma \geq 0, p > 1, \) and \( q = (\frac{\sigma}{2} + 1)p - \frac{\sigma}{2} + m, \) for some \( m \in \mathbb{R} \). Let \( \Omega_1 \) be a domain of \( \mathbb{R}^n \times (t_1, 0) \) for some \( t_1 < 0 \) and \( Q_1 \) be defined by \( Q_1 \equiv \{ \frac{|x|}{\sqrt{-t}} : (x, t) \in \Omega_1 \} \). Assume that \( w(r) \) is a smooth function defined on \( Q_1 \) and \( u(x, t) \) is defined by \( u(x, t) \equiv (-t)^{-1} w(\frac{|x|}{\sqrt{-t}}) \) on \( \Omega_1 \). The following is then obtained:

\[
\frac{\partial u}{\partial t} \leq \Delta u + \mu w^p - |x|^\sigma u^q \quad \text{on} \quad \Omega_1
\]  

(3.1)

if and only if

\[
w_{rr} + \frac{n-1}{r} w_r - \frac{1}{2} r w_r - \frac{1}{p-1} w + \mu w^p - r^\sigma (-t)^{-\frac{\sigma}{2}} w^q \geq 0
\]

on \( Q_1 \), where \( r = \frac{|x|}{\sqrt{-t}} \).

(3.2)
Proof. The proof is straightforward. ■

From the above lemma, the backward self-similar solution satisfies the equation in (1.9) or (3.2) when the solution is radially symmetric.

To apply \( w_0 \) in Proposition 1.1 to our domain, \( w_0 \) is extended by (1.10) and then the following lemma is obtained:

**Lemma 3.2.** Suppose \( q \geq p > 1 \) and \( \sigma \geq 2(q-p)(p-1)^{-1} \). When \( n \geq 3 \), it is further assumed that \( 1 < q < \frac{n+2}{n-2} \). Let \( w(r) \) be the function defined by (1.10) and then \( \underline{u} \) is defined as follows:

\[
\underline{u}(x, t) \equiv (-t)^{-rac{n-2}{n-2}} w\left( \frac{|x|}{\sqrt{-t}} \right) \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times [t_0, t_1],
\]

where \( t_0 < t_1 < 0 \), if \( \sigma = 2(q-p)(p-1)^{-1} \) or \(-1 \leq t_0 < t_1 < 0\), if \( \sigma > 2(q-p)(p-1)^{-1} \). If \( u_0(x) \geq \underline{u}(x, t_0) \) and \( \mu \geq \mu_0 \), then \( \underline{u}(x, t) \) is a continuous weak lower solution of problems (I), (II), and (III) with \((0, T^*)\), \( a(x) \), and \( D \) replaced by \((t_0, t_1)\), \(|x|^\sigma\), and \( B(0, r_0 \sqrt{-t_0}) \), respectively, where \( r_0 \) is defined in Proposition 1.1.

Proof. As mentioned in Proposition 1.1, \( w_0(y) \) is a smooth, nonnegative, radially symmetric, and nonconstant solution of (1.9). By the maximal principle,
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

\( w_0(y) \) is positive on \( B(0,r_0) \) and \( \frac{\partial w_0}{\partial n}\big|_{\partial B(0,r_0)} = w_0'(r_0) < 0 \). Thus, our result follows from (1.10), Lemma 2.4, and Lemma 3.1 whenever \( \mu \geq \mu_0 \).

Applying the comparison principle for the Cauchy problem (Lemma 2.3) to problem (I) allows us to obtain the following lemma:

**Lemma 3.3.**

(a) Assume that \( \overline{u}(x,t) \) is a continuous weak upper solution of (I). If \( \overline{u}(x,t) \in C^{1,2}_{loc}(R^n \times (t_0,t_1)) \), then \( \overline{u}(x,t) \geq 0 \) on \( R^n \times [t_0,t_1] \).

(b) If \( u(x,t) \) is a classical upper solution of (I) and \( \underline{u}(x,t) \) is the continuous weak lower solution in Lemma 3.2, then \( u(x,t) \geq \underline{u}(x,t) \) on \( R^n \times [t_0,t_1] \).

Similarly, applying the comparison principle for the Neumann and Dirichlet problems (Lemma 2.2) to (II),(III) allows us to obtain the following lemma:

**Lemma 3.4.** Let \( u \) be a nonnegative, bounded classical upper solution of (II) or (III) and \( \underline{u}(x,t) \) be defined in Lemma 3.2. Then, \( u(x,t) \geq \underline{u}(x,t) \) in \( B(0,r_0\sqrt{-t_0}) \times (t_0,t_1) \).

Now, the special case for Theorem 1.2 is proved, i.e., \( a(x) = |x|^\sigma \) in (1.1).
Lemma 3.5. Assume that \( u(x,t) \) is a nonnegative classical upper solution of (I). Suppose \( q \geq p > 1 \) and \( a(x) = |x|^\sigma \) with \( \sigma \geq 2(q-p)(p-1)^{-1} \). When \( n \geq 3 \), it is further assumed that \( 1 < q < \frac{n+2}{n-2} \). Let \( \mu \geq \mu_0 \) in (1.1) and \( t_0 \) be any negative number when \( \sigma = 2(q-p)(p-1)^{-1} \) or \(-1 \leq t_0 < 0 \) when \( \sigma > 2(q-p)(p-1)^{-1} \). If \( u_0(x) \geq (t_0)^{\frac{q-1}{p-1}} w\left( \frac{|x|}{\sqrt{-t_0}} \right) \), then \( u(x,t) \) will blow up at finite time which is before or equal to \( -t_0 \).

Similarly, if \( u(x,t) \) is a nonnegative classical upper solution of (II) or (III) in \( D = B(0, r_0 \sqrt{-t_0}) \), then the above result also holds under the same conditions.

Proof. First, problem (I) is considered. By translation, the result of Lemma 3.3 also holds for nonnegative time, i.e., \( u(x,t) \geq u(x,t + t_0) \) for \((x,t) \in \mathbb{R}^n \times [0,-t_0) \). Since

\[
\bar{u}(x,t + t_0) = (-t + t_0)^{\frac{q-1}{p-1}} w\left( \frac{|x|}{\sqrt{-t + t_0}} \right),
\]

\[
\bar{u}(0,t + t_0) = (-t + t_0)^{\frac{q-1}{p-1}} w(0), \quad \text{and}
\]

\[
\lim_{t \to -t_0} \bar{u}(0,t + t_0) = \infty,
\]

\( u(x,t) \) will blow up at some finite time which is before or equal to \( -t_0 \).

For problems (II), (III), the results can be proved by Lemma 3.4 similarly to the above proof. ■

Proof of Theorem 1.2. By rescaling: \( u^*(x,t) \equiv M^{-\frac{1}{p-1}} u(x,t) \) and applying
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

Lemma 3.5 to \( u^* \), the theorem can easily be proved.

\[ \blacksquare \]

**Proof of Theorem 1.3.** First the problem (II) is considered. Without loss of generality, \( n = 1 \) is assumed. Let

\[ w(x, t) \equiv a^l(x)u(x, t), \quad (3.3) \]

where \( l \) is some positive constant to be determined later. Then the equation in (1.1) is transferred into the following:

\[ w_t - w_{xx} = -2la^{-1}a_xw_x + \mu a^{(1-p)l}u^p - a^{1+l(1-q)}u^q \]

\[ + [l^2a^2a^{-2} + la^2a^{-2} - la_{xx}a^{-1}]w. \quad (3.4) \]

Given a number \( \alpha \equiv (p - 1)l \), then the following is obtained:

\[ a^\alpha(w_t - w_{xx}) = -2la^{(p-1)l-1}a_xw_x + \mu w^p - w^{q-p-l-1} \]

\[ + a^{(p-1)l-1}[l^2a^2a^{-1} + la^2a^{-1} - la_{xx}]w. \quad (3.5) \]

If a sufficiently large \( l \) is chosen, then the coefficients of \( w_x, w^{p+l-1} \), and \( w \) are smooth. Let the coefficients of \( w \) be bounded by \( M \). Since \( u \) blows up in a finite time, \( u \not= \text{constant} \). Meanwhile, since \( u_0(x) \geq 0 \), then \( u(x, t) > 0 \) for \( (x, t) \in D \times (0, T^*) \) by the weak and Hopf maximum principles. Without
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

a loss of generality, \( u(x,t) > 0 \) is assumed for \((x,t) \in \overline{D} \times [0,T^*)\). Define 
\[ \beta \equiv \min_{(x,t) \in \overline{D} \times [0,T^*)} u(x,t) > 0 \] 
Let \( w^+ \) be the largest root of the following equation:
\[ \mu w^{p-1} - \beta^{q-p-l-1} w^{p+l-1-1} + M = 0 \]
and \( w^* \) be the following number:
\[ w^* \equiv \max\{w^+, \max_{x \in \overline{D}} w(x,0)\} > 0. \]

In the following, it will be shown that
\[ w(x,t) \leq w^*, \text{ for } (x,t) \in \overline{D} \times [0,T^*) \tag{3.6} \]
and then the conclusion of the theorem can be implied.

If (3.6) is not true, then there exists \((x_0,t_0) \in \overline{D} \times (0,T^*)\) such that \( w_0 = w(x_0,t_0) > w^* > 0 \) and \( w_0 = \max_{(x,t) \in \overline{D} \times [0,t_0]} \). Therefore
\[ \mu w_0^{p-1} - \beta^{q-p-l-1} w_0^{p+l-1-1} + M < 0. \tag{3.7} \]

If \( x_0 \in D \), then we have
\[ a^\alpha(x_0)[w_t(x_0,t_0) - w_{xx}(x_0,t_0)] \geq 0 \text{ and } w_x(x_0,t_0) = 0. \]
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

Therefore from (3.5) we have

\[
\mu a_0^{p-1} - u^{q-p-l^{-1}} a_0^{p+l^{-1}-1} + ap^{q-1} l^{-1} [l^2 a_x^2 a^{-1} + l a_x^2 a^{-1} - l a_{xx}] \geq 0.
\]

However, this contradicts (3.7) by the definitions of \( \beta, M \).

If \( x_0 \in \partial D \), then \( a(x_0) > 0 \) and so \( \frac{\partial a(x_0)}{\partial n} \leq 0 \). Using (3.7) and the continuity of \( w(x, t) \) and \( a(x) \), there exists a \( \delta > 0 \) small enough such that

\[
h(x, t) \equiv \mu a_0^{p-1} - u^{q-p-l^{-1}} a_0^{p+l^{-1}-1} + ap^{q-1} l^{-1} [l^2 a_x^2 a^{-1} + l a_x^2 a^{-1} - l a_{xx}]
\]

\[
\leq \mu a_0^{p-1} - \beta^{q-p-l^{-1}} a_0^{p+l^{-1}-1} + M < 0 \text{ on } B(x_0, \delta) \cap D \times [t_0 - \delta, t_0].
\]

and

\[
a(x) > 0 \text{ on } B(x_0, \delta) \cap D.
\]

Because \( w \neq \text{constant} \) and (3.8), (3.9), \( \frac{\partial w(x_0, t_0)}{\partial n} > 0 \) is obtained. However, the following conditions: \( \frac{\partial u}{\partial n} \bigg|_{\partial D} = 0 \) and \( \frac{\partial a(x_0)}{\partial n} \leq 0 \), lead to a contradiction.

Second, problem (III) is considered. Let \( l \equiv (q - p)^{-1} \) and then (3.5) becomes the following:

\[
a^2 (w_t - w_{xx}) = -2la(p-1)l^{-1} a_x w_x + \mu w^p - w^{p+l^{-1}} + a^{(p-1)l^{-1}} [l^2 a_x^2 a^{-1} + l a_x^2 a^{-1} - l a_{xx}] w.
\]
on the blowup problem of $u_t = \Delta u + \mu u^p - a(x)u^q$

Under the condition: $\sigma \geq 2(q-p)(p-1)^{-1}$, the coefficients of $w_x$, $w$ in (3.10) are smooth. The result is obtained by proceeding as in problem (II) and noting that $x_0$ above does not occur at $\partial D$.

4 Application: Diffusion-Induced Blowup

This section attempts to demonstrate that the reaction diffusion system (1.6) has the diffusion-induced blowup phenomenon.

**Proposition 4.1.** Assume that $q \geq p > 1$ and $\sigma \geq 2(q-p)(p-1)^{-1}$. When $n \geq 3$, it is further assumed that $1 < q < \frac{n+2}{n-2}$. If $\mu$ and $u_0(x)$ are large enough, then the solutions $u(x,t)$ for the Cauchy, Neumann, and Dirichlet problems of (1.6) blow up in finite time.

**Proof.** This follows from Theorem 1.2.

**Proposition 4.2.** Assume that $f(x_0,v)$ is a smooth function such that the solution $v(t)$ for (1.7) exists globally for any $\xi \geq 0$ (e.g., $f(x_0,0) \geq 0$, $f(x_0,v) \leq 0$ when $v$ is large, or $f(x_0,v)$ is linear in $v$), $q > p$, $q > 1$, and $\sigma \geq 0$. If $f(x_0,m) \neq 0$, then the solution $u$ for the kinetic system (1.7) is global, bounded, and nonnegative for any nonnegative initial data $\eta$.

**Proof.** The argument in [18] is modified to obtain the global existence and boundedness of $u$. The solutions $v,u$ of (1.7) are denoted by $v(t;\xi)$ and $u(t;\eta)$. 

If there exist \( \xi_0, \eta_0 \) such that \( u(t; \eta_0) \) blows up at finite time \( T^* \), from the phase plane analysis, then \( v(T^*; \xi_0) = m \). Notably, the \( v \)-equation in (1.7) is autonomous. Thus, only three type of orbits of \( v \) can occur: equilibrium, closed orbit, and strictly monotone orbit. Since \( v \) is scalar, the \( \omega \)-limit set of \( v \) consists of just one equilibrium point; thus the orbit can not be the closed one. By \( f(x_0, m) \neq 0 \), \( v \) is strictly monotone. Hence, \( T^* \) is the only finite time such that \( u \) blows up. By the comparison principle of the ordinary differential equation, \( u(t; \eta) \) also blows up at time \( T^* \) for \( \eta \geq \eta_0 \). Now, we want to demonstrate the following fact: \( u(t; \eta) \) tends to infinity uniformly on \([0, T^*]\), as \( \eta \) tends to infinity. Contrarily, assume that there exists a large number \( M > \eta_0 \) satisfying: for any \( \eta > M \), a finite time \( t(\eta) > 0 \) exists such that \( u(t(\eta); \eta) < M \). Let \( t_M(\eta) \equiv \min\{t : u(t; \eta) = M\} \) for any \( \eta > M \). Because \( u(t; \eta) \) blows up at \( T^* \), by means of the comparison principle of ordinary differential equation and sequentially compact property, an increasing sequence \( \{t_M(\eta_i)\}_i \) which tends to some \( t_M \in (0, T^*) \) exists. The solution \( u \) which passes through the point \((M, t_M)\) in the \( u - t \) plane will blow up at some finite time less than \( T^* \). This contradicts the fact: “\( T^* \) is the only finite time such that \( u \) blows up.”

Set \( w := u^{-(q-1)} \). The following equation for \( w \) is then obtained:

\[
w_t = -(q - 1) \mu w^{\frac{q-1}{q-2}} + (q - 1) |m - v(t; \xi_0)|^r. \tag{4.1}
\]

From above, a family of \( w(t; \eta^{-(q-1)}) \) is obtained, which tends to zero uniformly on \([0, T^*]\), as \( \eta \) tends to infinity. Since \( f(x_0, m) \neq 0 \), some \( \tilde{t} \in [0, T^*] \)
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

exists, such that \( v(\tilde{t}; \xi_0) \neq m \) and then \( w_t > 0 \) at the point \((\tilde{t}, 0)\) in the t-w plane by (4.1). This will contradict the continuous dependence property of \( w(t; \eta^{-(q-1)}) \) near the point \((\tilde{t}, 0)\) in the t-w plane. Therefore \( v \) exists globally.

From \( f(x_0, m) \neq 0 \), it can be easily obtained that \( v(t_k; \xi) \not\to m \) for any sequence \( t_k \to \infty \). Thus, \( u \) is also bounded.

The proof is finished.

Some examples illustrating the diffusion-induced blowup phenomenon are as follows:

( i ). \( f(x, v) : = \lambda v \), where \( \lambda > 0 \) is an eigenvalue of \(-\Delta\) for the homogeneous Dirichlet, or Neumann problem in \( D \). Clearly, the solution \( v(t) \) of (1.7) exists globally but is not bounded. In addition, based on the eigenfunction’s property, it is possible to find \( x_0 \in D \) such that \( f(x_0, v_0(x_0)) = \lambda v_0(x_0) \neq 0 \), where \( v_0 \) is the eigenfunction with respect to \( \lambda \).

( ii ). \( f(x, v) : = \lambda v - h(x)v^l \), where \( \lambda > 0 \), \( l > 1 \) are constants, and \( h \), not identical to zero, is a nonnegative smooth function with isolated zero points. From [22, Theorem 2] and maximum principle, an unique nonnegative solution \( v_0 \) of the elliptic Dirichlet problem and \( x_0 \in D \) can be found such that \( f(x_0, v_0(x_0)) \neq 0 \), for \( \lambda > \lambda_0 \), where \( \lambda_0 \) is the first eigenvalue of \(-\Delta\) for the homogeneous Dirichlet problem in \( D \). Clearly, the solution \( v(t) \) of (1.7) exists globally, and is also bounded with the further restriction that “\( h(x) \) is positive.”

( iii ). Take the same \( f(x, v) \) as in ( ii ), but where “\( h \) is not a constant.” Also from [22, Theorem 3] and maximum principle, a unique nonnegative solution \( v_0 \)
on the blowup problem of \( u_t = \Delta u + \mu u^p - a(x)u^q \)

of the elliptic Neumann problem and \( x_0 \in D \) can be found such that \( f(x_0, v_0(x_0)) \neq 0 \), for \( \lambda > 0 \). The conclusion for \( v(t) \) is the same as in (ii).

Therefore, under the conditions “\( q > p > 1, 1 < q < \frac{n+2}{n-2} (n \geq 3) \) or \( n = 1, 2, \) and \( \sigma \geq 2(q-p)(p-1)^{-1} \)”, the above examples have the so-called diffusion-induced blowup phenomenon, particularly in the case \( f(x,v) = f(v) \).

**Remark.** When we take \( d_1 = d_2 \) in the examples herein, the question in [20] as mentioned in the Introduction has been resolved.

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