FACTORABLE CONTINUITY OF RANDOM FIELDS,

with quantitative estimation

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Abstract.

We study in this paper the sufficient conditions for enhanced continuity of random fields, i.e. such that the modulus of its continuity allows the factorable representation by the product of random variable on the deterministic module of continuity.

We estimate also the ordinary and (possible) exponential moments of these random variables.

We consider also the case of random fields with heavy tails of distribution and the so-called rectangle its continuity.

Key words and phrases: Random variables (r.v.) and random fields (r.f.), Lebesgue-Riesz and other moment rearrangement invariant spaces, scaling function, distance, ordinary, factorable and rectangle factorable continuity, measure and measurable functions, Orlicz’s functions and spaces, $\Delta_2$ and $\nabla_2$ condition, light and heavy tails of distribution, entropy and majorizing measure conditions, modulus of continuity, sharp quantitative estimate.

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1 Introduction. Notations. Statement of problem.

Let $(X = \{x\}, d), d = d(x, y)$ be compact metric space relative the distance function $d(\cdot, \cdot), (\Omega, B, P)$ be a (sufficiently rich) probability space, $\xi = \xi(x) = \xi(x, \omega), x \in X, \omega \in \Omega$ be separable numerical valued random field (r.f.) (process).

Define as ordinary for each function $f : X \rightarrow R$, not necessary to be continuous, the modulus (module) of continuity $\Delta(f, \delta)$
\[
\Delta(f, d, \delta) = \Delta(f, \delta) \overset{def}{=} \sup_{d(x,y) \leq \delta} |f(x) - f(y)|, \ \delta \geq 0.
\]

(1.1)

In what follows the value \(\delta\) belongs to the closed segment

\[\delta \in [0, \text{diam}(d, X)], \quad \text{diam}(d, X) \overset{def}{=} \sup_{x,y \in X} d(x,y).\]

Obviously, \(\lim_{\delta \to 0^+} \Delta(f, d, \delta) = 0\) iff the function \(f(\cdot)\) is (uniformly) continuous.

**Definition 1.1.** The r.f. \(\xi = \xi(x)\) is said to be *factorable continuous* (FC), if there exists a continuous non-random non-negative function \(g = g(\delta)\), such that \(g(0) = g(0+) = 0\), and finite (non-negative) random variable (r.v.) \(\tau = \tau(\omega)\) such that

\[
\Delta(\xi(\cdot), d, \delta) \leq \tau(\omega) \cdot g(\delta),
\]

(factorable inequality).

Such a function \(g(\delta)\) in the relation (1.2) will be named by definition as a *scaling function*.

**Example 1.1.** Let \(X = \overline{N} \overset{def}{=} (1, 2, \ldots) \cup \{\infty\}\) (extended integer positive semi-axis) with the ordinary distance function

\[d_N(m, n) := |1/n - 1/m|, \ n, m < \infty; \ d_N(n, \infty) = d_N(\infty, n) := 1/n, \ n < \infty;\]

\[d_N(\infty, \infty) := 0.\]

(1.3)

Let also \(\{\xi_n\}, \ n = 1, 2, 3, \ldots\) be a sequence of random variables converging to zero almost everywhere:

\[\mathbf{P}\left( \lim_{n \to \infty} \xi_n = 0 \right) = 1.\]

We extend the definition of this sequence formally as follows: \(\xi_\infty = 0\); then the random field (random sequence) \(\{\xi_n\}, \ n \in \overline{N}\) is \(d_N\) continuous with probability one.

The factorable continuity of these sequences implies the existence of the non-negative r.v. \(\eta\) and the deterministic sequence \(\epsilon_n\) converging to null for which

\[|\xi_n| \leq \eta \cdot \epsilon_n.\]

Evidently, if the r.f. \(\xi(x)\) is factorable continuous, then it is continuous a.e. The inverse statement is also true, see [22], [23], [4], [21], chapter 4, Theorem 4.8.2, pages 219-221. For the random sequences \(\{\xi_n\}\) this conclusion there is in the textbook [14], chapter 2, section 3.

Thus, the ordinary continuity a.e. of the separable random field is quite equivalent to the factorable continuity.
Let $\Phi = \Phi(u), \ u \in R$ be an Orlicz function, i.e. convex, even, continuous, twice continuous differentiable in the domain $u \geq 2$, strictly decreasing in the right side half real line, and such that

$$\Phi(0) = \Phi(0+) = 0; \ \lim_{|u| \to \infty} \Phi(u) = \infty.$$  

Assume further that these Orlicz function $\Phi(\cdot)$ satisfies the so-called $\Delta_2$ condition:

$$\lim_{u \to \infty} \frac{\Phi(2u)}{\Phi(u)} < \infty.$$  

Briefly, $\Phi(\cdot) \in \Delta_2$.

It is proved in addition in [22], more detail investigation see in [21], chapter 4, Theorem 4.8.2, pages 219-221, that if the r.v. $\sup |\xi(x)|$ belongs to the Orlicz space $L(\Phi)$ builded on the our probability space $(\Omega, B, P)$, on the other words, has a light tail of distribution, then the function $g(\delta)$ in (1.2) may be picked such that the r.v. $\tau$ belongs also to the at the same Orlicz space $L(\Phi)$.

The generalization of this proposition on the case when this Orlicz’s function $\Phi(\cdot)$ does not satisfies the $\Delta_2$ condition (1.7), is investigated in [23].

The aim of this report is obtaining the quantitative estimation for the correspondent random variable and deterministic function appearing in the factorable inequality (1.2), up to the extent that to obtain the not significantly improve estimations.

Of course, the factorable estimate (1.2) is more convenient for application than the classical estimate for the some rearrangement invariant norm $||\Delta(\xi(\cdot), \delta)||$; see e.g. the recent publications [11], [26], [27].

For instance, we intend to derive the exact asymptotical behavior for the function $g(\delta)$ as $\delta \to 0+$ and the sharp classical moment estimates $|\tau|_p$ or exponential moments $\mathbb{E}\exp(\lambda \tau)$. We will denote hereafter as usually for arbitrary r.v. $\zeta$

$$|\zeta|_p \overset{def}{=} [\mathbb{E}|\zeta|^p]^{1/p}, \ p \geq 1.$$  

The Orlicz’s space over our probability triplet generated by the function $\Phi(\cdot)$ will be denoted by $L(\Phi)$, and the classical Luxemburg’s norm of the r.v. $\zeta$ in this Orlicz’s space will be denoted by $||\zeta||\Phi$.

If for example $\Phi(u) = \Phi_p(u) = |u|^p, \ p = \text{const} \geq 1$, then the space $L(\Phi_p)$ coincides with Lebesgue-Riesz space $L_p$. Evidently, this function $\Phi_p(\cdot)$ satisfies the $\Delta_2$ condition.

The detail representation of the theory of Orlicz’s spaces may be found in the classical monographs [16], [32], [33].
2 Previous works.

A. Metric entropy approach.

Let us introduce a new norm, the so-called moment norm, or equally Grand Lebesgue norm, on the set of r.v. defined on our probability space by the following way: the space $G(\psi)$ consist, by definition, on all the centered r.v. with finite norm

$$||\xi||_{G(\psi)} \overset{def}{=} \sup_{p \geq 2} [||\xi||_{p}/\psi(p)]. \quad (2.1)$$

Here $\psi = \psi(p)$, $1 \leq p < b$, $b = \text{const} \in (1, \infty]$ is some positive monotonically increasing continuous defined on the open interval $(1, b)$, bounded from below numerical function.

The detail investigation of these spaces, which are named as Grand Lebesgue Spaces (GLS), may be found in [15], [21], chapters 1,2.

Define the following functions

$$\phi(p) := \left[ \frac{p}{\psi(p)} \right]^{-1}, \quad \phi^*(\lambda) := \sup_{p} (|\lambda| p - \phi(p)), \ \lambda \in \mathbb{R}.$$ 

The transform (non-linear) $\phi \to \phi^*$ is named Legendre, or Young-Fenchel transform.

Let $b = \infty$; it is known [15] that the GLS space $G\psi$ coincides with a subspace of an exponential Orlicz’s space relative the Young-Orlicz function

$$N(u) = \exp(\phi^*(u)) - 1,$$

consisting only on all the centered (mean zero) random variables.

Note that if we choose as a capacity of the function $\psi(\cdot)$ a degenerate function

$$\psi_{(r)}(p) = 1, \ p = r; \ \psi_{(r)}(p) = \infty, \ p \neq r, \quad (2.2a)$$

where $r = \text{const} \in (1, \infty)$, we conclude formally

$$||\xi||_{G(\psi_{r})} = ||\xi||_{r}. \quad (2.2b)$$

Thus, the classical Lebesgue-Riesz spaces $L_{r}$ are particular, more precisely, extremal case of GLS.

Let again $\xi = \xi(x), \ x \in X$ be separable numerical random field. Suppose for some number $b > 1$ (finite or not)

$$\forall p \in [1, b) \Rightarrow \psi_{\xi}(p) := \sup_{x \in X} |\xi(x)|_{p} < \infty. \quad (2.3)$$

The introduced in (2.3) function is said to be natural function for the family of random variables $\{\xi(x)\}, \ x \in X$. It is clear that

$$\sup_{x \in X} ||\xi(x)||_{G\psi_{\xi}} = 1.$$
The natural distance \( d = d_\xi = d_\xi(x, y) \) generated by r.f. \( \xi(\cdot) \) satisfying the condition (2.3) one can introduced by the formula

\[
d(x, y) = d_\xi(x, y) = d_\xi(x, y) := ||\xi(x) - \xi(y)||G\psi_\xi.
\]  

(2.4)

Let us introduce for any subset \( V, V \subset X \) the so-called metric entropy

\[
H(V, d, \epsilon) = H(V, \epsilon) = \log N(V, \epsilon) = \log N(V) = N \]  

(2.5)

which cover all the set \( V \):

\[
S(V, x, \epsilon) \overset{def}{=} \{ y, y \in V, d(x, y) \leq \epsilon \},
\]

and we denote also

\[
H(V, d, \epsilon) = \log N; \quad S(x_0, \epsilon) \overset{def}{=} S(X, x_0, \epsilon), \quad H(d, \epsilon) \overset{def}{=} H(X, d, \epsilon).
\]

(2.6)

It follows from Hausdorff’s theorem that \( \forall \epsilon > 0 \Rightarrow H(V, d, \epsilon) < \infty \) iff the metric space \((V, d)\) is precompact set, i.e. is the bounded set with compact closure.

Denote for the function \( \psi(p) = \psi_\xi(p) \)

\[
v(z) := \ln \psi(1/y), \quad y > 0; \quad v_*(w) := \inf_{z \in (0,1)} (zw + v(z)).
\]

(2.7)

It is proved in particular in the monograph [21], p. 172-176 that

\[
||\Delta(\xi, \delta)||G\psi_\xi \leq 9 \cdot \int_0^\delta \exp[v_*(\ln 2 + H(X, d_\xi, \epsilon))] \, d\epsilon,
\]

if of course the integral in the right-hand side of inequality (2.7) convergent. Evidently, in this case the r.f. \( \xi(x) \) is continuous a.e.; moreover,

\[
\lim_{\delta \to 0^+} ||\Delta(\xi, \delta)||G\psi_\xi = 0.
\]

Analogous result see in [30], [18], [1], [5], [6], [9], [28], [36] etc.

**B. More modern majorizing measure approach.**

We note among the previous works the articles [2], [13]; see also the preprint [25].

It was imposed in some previous articles [17], [3] on the function \( \Phi(\cdot) \) the following \( \nabla^2 \) condition:

\[
\Phi(x)\Phi(y) \leq \Phi(K(x + y)), \quad \exists K = \text{const} \in (1, \infty), \quad x, y \geq 0
\]

or equally
\[
\sup_{x,y>0} \left[ \frac{\Phi^{-1}(xy)}{\Phi^{-1}(x) + \Phi^{-1}(y)} \right] < \infty. \tag{2.8}
\]

We do not suppose this condition. For instance, we can consider the function of a view \(\Phi(z) = |z|^p\), which does not satisfy (2.1).

Let us introduce the following constant (more exactly, functional)

\[
C_2 = C_2(\Phi) = \frac{\Phi^{-1}(1)}{54K^2}, \tag{2.9}
\]

if there exists. Under this assumption the distance \(d = d(x_1, x_2)\) may be constructively defined by the formula:

\[
d(x_1, x_2) = d_\Phi(x_1, x_2) := ||\xi(x_1) - \xi(x_2)||\Phi. \tag{2.10}
\]

We will use further the so-called method of majorizing (minorizing) measures. Let \(m = m(\cdot)\) be any probability measure on the set \(X\). Since the function \(\Phi = \Phi(z)\) is presumed to be continuous and strictly increasing, it follows from the relation (2.3) that \(V(d_\Phi) \leq 1\), where by definition

\[
V(d) := \int_X \int_X \Phi \left[ \frac{\rho(f(x_1), f(x_2))}{d(x_1, x_2)} \right] m(dx_1) m(dx_2). \tag{2.11}
\]

Let us define also the following important distance function: \(w(x_1, x_2) = w(x_1, x_2; V) = w(x_1, x_2; V, m) = w(x_1, x_2; V, m, \Phi) = w(x_1, x_2; V, m, \Phi, d) \overset{\text{def}}{=} \)

\[
6 \int_0^{d(x_1, x_2)} \left\{ \Phi^{-1} \left[ \frac{4V}{m^2(B(r, x_1))} \right] + \Phi^{-1} \left[ \frac{4V}{m^2(B(r, x_2))} \right] \right\} dr, \tag{2.12}
\]

where \(m(\cdot)\) is probabilistic Borelian measure on the set \(X\).

The triangle inequality and other properties of the distance function \(w = w(x_1, x_2)\) are proved in [17].

**Definition 2.1.** (See [17]). The measure \(m\) is said to be minorizing measure relative the distance \(d = d(x_1, x_2)\), if for each values \(x_1, x_2 \in X\) \(V(d) < \infty\) and moreover \(w(x_1, x_2; V(d)) < \infty\).

We will denote the set of all minorizing measures on the metric set \((X, d)\) by \(M = M(A')\).

Evidently, if the function \(w(x_1, x_2)\) is bounded, then the minorizing measure \(m\) is majorizing. Inverse proposition is not true, see [17], [2].

**Remark 2.1.** If the measure \(m\) is minorizing, then

\[
w(x_n, x; V(d)) \to 0 \Leftrightarrow d(x_n, x) \to 0, \; n \to \infty.
\]

Therefore, the continuity of a function relative the distance \(d_{\Phi}\) is equivalent to the continuity of this function relative the distance \(w\).
Remark 2.2. If
\[
\sup_{x_1, x_2 \in X} w(x_1, x_2; V(d)) < \infty,
\]
then the measure \( m \) is called **majorizing measure**. This classical definition with theory explanation and applications basically in the investigation of local structure of random processes and fields belongs to X. Fernique [8], [9] and M. Talagrand [34], [35].

See also [3], [5], [18], [23], [24], [25].

S. Kwapien and J. Rosinsky proved in [17] the following inequality:
\[
\mathbb{E} \Phi \left( 2 C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 1 + \sup_{t \neq s} \mathbb{E} \Phi \left( \frac{(\xi(t) - \xi(s))}{d(t, s)} \right). \tag{2.13}
\]
As long as we choose \( d(t, s) = d_{\phi}(t, s) \), we have
\[
\mathbb{E} \Phi \left( 2 C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 2.
\]
Recall that \( \Phi = \Phi(u) \) is convex function and \( \Phi(0) = 0 \); following
\[
\Phi \left( \frac{u}{2} \right) = \Phi \left( \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot u \right) \leq \frac{1}{2} \Phi(0) + \frac{1}{2} \Phi(u) = \frac{1}{2} \Phi(u),
\]
We conclude on the basis of this inequality
\[
\mathbb{E} \Phi \left( C_2 \sup_{t \neq s} \frac{(\xi(t) - \xi(s))}{w(t, s)} \right) \leq 1, \tag{2.15}
\]
or equally
\[
\| \sup_{w(x_1, x_2) \leq \delta} (\xi(x_1) - \xi(x_2)) \|_{\Phi} \leq \delta/C_2. \tag{2.15a}
\]

Suppose now the measure \( m \) and certain distance on the set \( X d = d(x_1, x_2) \) are such that
\[
| \xi(x_1) - \xi(x_2)|_p \leq d(x_1, x_2), \ p = \text{const} \geq 1, \tag{2.16}
\]
\[
m^2(B(r, x)) \geq r^\theta/C(\theta), \ r \in [0, 1], \ \theta = \text{const} > 0, \ C(\theta) \in (0, \infty). \tag{2.17}
\]
Let also \( p = \text{const} > \theta \).

It is proved in [25] that for the r.f. \( \xi = \xi(x) \) the following inequality holds: \( m \in M \) and
\[
|\xi(x_1) - \xi(x_2)| \leq 12 Z^{1/p} 4^{1/p} C^{1/p}(\theta) \frac{d^{1-\theta/p}(x_1, x_2)}{1 - \theta/p}, \tag{2.18}
\]
where the r.v. $Z$ has unit expectation: $E Z = 1$.

Note that in the estimate (2.18) the right-hand side is factorable. We improve its in the next section.

3 Main result. Ordinary Orlicz function.

1. Suppose the continuous a.e. random field $\xi = \xi(x)$, $x \in X$ is such that for some Orlicz function grounded over source probability space $\Phi = \Phi(u)$ and satisfying the $\Delta_2$ condition

$$|| \sup_{x \in X} \xi(x)|| \Phi < \infty. \quad (3.1)$$

The sufficient conditions for the continuity of $\xi(\cdot)$ and for the estimate (3.1) are aforementioned in the second section.

Denote

$$\theta(\delta) = \theta_{\Phi}(\delta) := ||\Delta(\xi, \delta)|| \Phi. \quad (3.2)$$

As long as

$$\Delta(\xi, \delta) \leq 2 \sup_x |\xi(x)|, \quad ||\Delta(\xi, \delta)|| \Phi \leq 2 \sup_x |\xi(x)| \quad ||\Phi||,$$

we conclude thanks to the Lebesgue dominated convergence theorem

$$\lim_{d_{\Phi}(x,y) \to 0^+} E \Phi(||\xi(x) - \xi(y)||) = 0. \quad (3.3)$$

The equality (3.3) implies the so-called on the language of the theory of Orlicz's spaces $\Phi$ – mean convergence. Since the Young-Orlicz function $\Phi(\cdot)$ satisfies the $\Delta_2$ condition, this convergence is completely equivalent to the ordinary Orlicz norm space convergence, see [16], chapter 2. Therefore

$$\lim_{\delta \to 0^+} \theta(\delta) = \lim_{\delta \to 0^+} \theta_{\Phi}(\delta) = 0. \quad (3.4)$$

2. We start from the relation (3.4). Let $a = \{a_n\}$, $n = 1, 2, \ldots$ be arbitrary positive strictly decreasing numerical non-random sequence tending to zero as $n \to \infty$. Let also $b = \{b_n\}$, $n = 1, 2, \ldots$ be arbitrary positive strictly decreasing numerical non-random sequence such that

$$\sum_{n=1}^{\infty} b_n = 1$$

and such that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$
Define the positive numerical values $\delta_n$ as a maximal solutions of the following equations

$$\theta(\delta_n) = ||\Delta(\xi, \delta_n)||_\Phi = a_n.$$  \hfill (3.5)

Let us consider the following random variable

$$\tau := \sum_{n=1}^{\infty} b_n \frac{\Delta(\xi, \delta_n)}{||\Delta(\xi, \delta_n)||_\Phi} = \sum_{n=1}^{\infty} b_n \frac{\Delta(\xi, \delta_n)}{a_n}.$$  \hfill (3.6)

The series in (3.6) converges in the $L(\Phi)$ norm: it follows from the triangle inequality ant from the completeness of the Orlicz spaces $\tau \in L(\Phi)$ and

$$||\tau||_\Phi \leq \sum_n b_n = 1.$$  \hfill (3.7)

3. We deduce from the definition (3.6) that

$$\Delta(\xi, \delta_n) \leq \tau \cdot \frac{a_n}{b_n}.$$  

As long as the function $\delta \to \Delta(\xi, \delta)$ is monotonically increasing, we derive that for the arbitrary positive value $\delta$ there exists an unique natural value $n = n(\delta)$ for which $\delta_{n+1} < \delta \leq \delta_n$ and hence

$$\Delta(\xi, \delta) \leq \tau \cdot \frac{a(n(\delta))}{b(n(\delta))}.$$  \hfill (3.8)

The second factor in the right-hand side (3.8) tends to zero by virtue of our choosing of both the sequences $\{a_n\}$ and $\{b_n\}$.

4. To obtain the continuous function as a capacity of the scaling function in the right-hand side of the inequality (3.8) instead $a(n(\delta))/b(n(\delta))$, we introduce the following function $\delta \to g_1(\delta)$ defined on some non-negative neighborhood of origin. We define for the values $\delta = \delta_n$, where $n = 1, 2, \ldots$

$$g_1(\delta) = g_1(\delta_n) \overset{\text{def}}{=} \frac{a(n)}{b(n)},$$  \hfill (3.9)

and define the values of this function inside the interval (open or closed) $[\delta_{n+1}, \delta_n]$ by means of a linear continuous interpolation (spline function). At last, put $g_1(0) := 0$; then the function $g_1(\cdot)$ is really certain scaling function, i.e. is non-negative, strictly increasing, continuous, $g_1(0) = g_1(0+) = 0$, and

$$\Delta(\xi, \delta) \leq \tau \cdot g_1(\delta).$$  \hfill (3.10)

5. The function $g_1$ may be redefined as follows.

$$g(\delta) := ||\tau||_\Phi \cdot g_1(\delta),$$
and define correspondingly the normed r.v. \( \tau_0 := \tau/||\tau||\Phi \). It is clear that it is non-trivial: \( 0 < ||\tau||\Phi < \infty \). Then we have

\[
\Delta(\xi, \delta) \leq \tau_0 \cdot g(\delta), \quad ||\tau_0||\Phi = 1.
\] (3.11)

To summarize.

**Theorem 3.1.** Suppose that all the conditions of this section imposed on the random field \( \xi = \xi(x) \) are satisfied. Then the modulus of its continuity \( \Delta(\xi, \delta) \) allows the factorable estimate (3.11).

Here is an example. Assume that \( \Phi(u) = \Phi_p(u) = |u|^p \). Suppose also the continuous a.e. random field \( \xi = \xi(x) \), \( x \in X \) is such that \( \sup_x |\xi(x)|^p = 1 \). The continuity of r.f. \( \xi(x) \) is understanding relative the natural (finite) distance

\[
d_p(x, y) = |\xi(x) - \xi(y)|^p, \; p = \text{const} \geq 1.
\]

Denote as before

\[
\theta_p(\delta) = ||\Delta(\xi, \delta)||^p.
\]

The sufficient conditions for the \( d_p(\cdot, \cdot) \) continuity of \( \xi(\cdot) \) and consistent as \( \delta \to 0^+ \) estimates for \( \theta_p(\delta) \) see, e.g. in [25], [30].

Suppose for definiteness

\[
\exists \alpha \in (0, 1] \Rightarrow \theta_p(\delta) \leq C_1 \delta^\alpha, \; \delta \in (0, 1/e).
\]

One can choose

\[
b_n = \nu \; n^{-1} \; \ln^{-1-\nu}(n+1), \; \nu > 0;
\]

\[
a_n = n^{-1-\theta}, \; \theta > 0.
\]

We deduce after some computations for at the same values \( \delta \)

\[
\Delta(\xi, \delta) \leq C_2(\alpha, p, C_1) \cdot \tau \cdot \delta^\alpha \theta/(1+\theta) \; (1 + \theta)^{-(1+\nu)} \; |\ln \delta|^{1+\nu},
\]

where \( \tau \) in the right-hand side is unique (non-negative) random variable for which \( E\tau^p = 1 \).

We conclude further after minimisation of the right-hand side over \( \theta \)

\[
\Delta(\xi, \delta) \leq C_3(\alpha, p, C_1) \cdot \tau \cdot \delta^\alpha.
\]

It is clear that the last estimate is essentially non-improvable.

**Remark 3.1.** The optimal choosing of the sequences \( \{a_n\}, \{b_n\} \) in general case is now an open question.
4 Main result. General case of an arbitrary Orlicz function.

We do not assume this section that the Young-Orlicz function $\Phi$ satisfies the $\Delta_2$ condition. In particular, it may be exponential Orlicz function.

We retain all the other suppositions (and notation) of previous sections.

Recall in the beginning of this section then the Orlicz function $\Psi(\cdot)$ is called weaker than ones function $\Phi$, if for all positive constant $v; v = \text{const} > 0$

$$\lim_{u \to \infty} \frac{\Psi(uv)}{\Phi(u)} = 0.$$  

Notation: $\Psi << \Phi$.

Theorem 4.1. We retain all the notations and conditions of the third section. Let also $\Psi(\cdot)$ be other Orlicz function weaker than $\Phi(\cdot)$. Then there exist a $L(\Psi)$ - normed r.v. $\zeta; \|\zeta\|_\Psi = 1$ and the non-negative strictly increasing continuous function $h(\cdot)$ with condition $h(0) = h(0+) = 0$, depending on $\Psi, \Phi$ such that

$$\Delta(\zeta, \delta) \leq \zeta \cdot h(\delta).$$  \hspace{1cm} (4.1)

Proof is at the same as one for theorem 3.1. The only new feature is following. Rewrite the equality (3.3)

$$\lim_{d_{\Phi}(x,y) \to 0^+} E\Phi(|\xi(x) - \xi(y)|) = 0.$$  \hspace{1cm} (3.3)

The equality (3.3) implies the so - called on the language of the theory of Orlicz’s spaces $\Phi$ – mean convergence. But since we consider now the case when the Young-Orlicz function $\Phi(\cdot)$ can not satisfy the $\Delta_2$ condition, this convergence means only that

$$\lim_{\delta \to 0^+} \theta_\Psi(\delta) = 0.$$

The scaling function $h = h(\delta)$ may be constructed as the function $g = g(\delta)$ in the third section.

Everything else on the-still, as before.

Example 4.1. Define the following family of Young-Orlicz functions

$$\Theta_p(u) := \exp(|u|^p) - 1, \quad p = \text{const} \in (0, \infty).$$

Note that if $0 < q < p < \infty$, then $\Theta_q(\cdot) << \Theta_p(\cdot)$.

Further, let the r.f. $\xi(x), x \in X$ be a given. Suppose the r.f. $\xi(x)$ satisfies all the conditions of theorem 4.1 relative the Young-Orlicz function $\Theta_p(\cdot)$. Assume also again that the number $q$ is arbitrary from the interval $0 < q < p < \infty$. We propose
\[ \Delta(\xi, d_{\Theta_q}, \delta) \leq \tau_{\Theta_q} \cdot g_{p,q}(\delta), \]

where the random variable \( \tau_{\Theta_q} \) belongs to the Orlicz space \( L(\Theta_q) \) and \( \|\tau_{\Theta_q}\|_{\Theta_q} = 1 \); obviously, the r.v. \( \tau_{\Theta_q} \) as well as the non-random scaling function \( g_{p,q} = g_{p,q}(\delta) \) dependent also on the variables \( p,q \).

**Example 4.2.** Consider as an exception the so-called Gaussian case, i.e. when the r.f. \( \xi(x) \) is (separable) centered Gaussian distributed. We can conclude that the correspondent Young-Orlicz function has a form

\[ \Phi(u) = \Phi_G(u) = \exp(u^2/2) - 1. \]

The (centered) random variables belonging to the Orlicz space \( L(\Phi_G) \) are named subgaussian.

The correspondent natural distance function \( d = d(x, y) = d_G(x, y) \) coincides here with the \( L_2(\Omega) \) distance between the random values \( \xi(x) \) and \( \xi(y) \):

\[ d_G(x, y) = \sqrt{\text{Var}(\xi(x) - \xi(y))}. \]

Suppose again that the r.f. \( \xi(\cdot) \) is \( r_G \) – continuous with probability one, or equally

\[ \theta_{\Phi_G}(\delta) := \|\Delta(\xi, r_G, \delta)\|_{L(\Phi_G)} \to 0, \delta \to 0+. \]

Here the \( r_G = r_G(x, y), x,y \in X \) is some \( d_G \) continuous distance at the same set \( X \).

It follows immediately from one of the main results in the famous work of X.Fernique [7] that the module of continuity \( \Delta(\xi, r_G, \delta) \) allows the ”good” factorization

\[ \Delta(\xi, r_G, \delta) \leq \tau_G \cdot \tilde{g}(\delta), \]

where \( \tilde{g}(\delta) \) is such that \( \lim_{\delta \to 0^+} \tilde{g}(\delta) = \tilde{g}(0) = 0 \) and \( \tau_G \) is normed subgaussian: \( \tau_G \in L(\Phi_G) \) and \( \|\tau_G\|_{\Phi_G} = 1 \).

This means in particular

\[ P(\tau_G > u) \leq e^{-u^2/2}, u \geq 1. \]

On the other hands, for some positive constant \( C = C(G), 0 < C(G) < 1 \)

\[ P(\tau_G > u) \geq e^{-C(G) u^2/2}, u \geq 1. \]

**(Sub)**-example 4.3. Let in addition to the example 4.2 \( \xi(t) = w(t), (\xi = w), t \in [0, 1/e] \) be an ordinary Brownian motion or equally Wiener process. It is well known that here

\[ g(\delta) = \sqrt{\delta \cdot |\ln \delta|}, \delta \in [0, 1/e]. \]

Denote
\[ \tau_w := \sup_{t,s \in (0,1/e)} \left[ \frac{|w(t) - w(s)|}{g(|t - s|)} \right], \]

then

\[ (2\pi)^{-1/2}e^{-u^2/2} \leq P(\tau_w > u) \leq 4.8 \ e^{-u^2/2+2u}, \quad u \geq 5. \]

see [29].

5 Heavy tailed fields.

We consider in this short section the "modified" fractional continuity for the random field \( \eta = \eta(x), x \in X \) with "very" heavy tails. More precisely, we do not suppose that \( \forall x \in \mathbb{R} \Rightarrow \eta(x) \in L_p(\Omega) \) for some \( p \geq 1 \). For instance, \( \eta(\cdot) \) may be stable distributed with parameter \( \alpha, \alpha \in (0,2); \) cf. [19], [20].

We will reduce the heavy tailed fields to the considered one. Namely, let us introduce the following transformation. \( \xi(x) := Z_m(\eta(x)) \), where

\[ Z_m(y) \stackrel{def}{=} \text{sign}(y) \cdot |\ln(1 + |y|)|^m, \quad (5.1) \]

so that \( Z_m(0) = 0 \), where the constant positive number \( m = \text{const} \) may be not integer.

Note that the function \( Z_m(x) \) is continuous, odd, strictly increasing.

Obviously, the tails of r.f. \( \xi(x) \) are much easier as ones of the r.f. \( \eta(x) \).

Suppose the transformed r.f. \( \xi(x) \) satisfies all the conditions of theorems 3.1 or 4.1. Then

\[ \Delta(Z_m(\eta), \delta) \leq \tau_{Z_m} \cdot g_{Z_m}(\delta), \quad (5.2) \]

which may be interpreted as a modified (weak) factorable modulus of continuity of the heavy tailed random field \( \eta(x) \).

6 Concluding remarks. Rectangle continuity of random fields.

In this last section the set \( X \) is convex closure of open non-empty subset of whole Euclidean space \( X = D \subset \mathbb{R}^d \).

We define as in [31], [12] the rectangle difference operator \( \Box[f](\vec{x}, \vec{y}) = \Box[f](x,y), \quad x,y \in D; \quad f : D \rightarrow \mathbb{R} \) as follows. \( \Delta^{(i)}[f](x,y) := \)

\[ f(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d) - f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d), \quad (6.0) \]
with obvious modification when \( i = 1 \) or \( i = d \);

\[
\Box[f](x, y) \overset{def}{=} \{ \otimes_{i=1}^{d} \Delta^{(i)} \} [f](x, y).
\]

For instance, if \( d = 2 \), then

\[
\Box[f](x, y) = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, x_2).
\]

If the function \( f : [0, 1]^d \to R \) is \( d \) times continuous differentiable, then

\[
\Box[f](\vec{x}, \vec{y}) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} \cdots \int_{x_d}^{y_d} \frac{\partial^d f}{\partial x_1 \partial x_2 \cdots \partial x_d} \, dx_1 \, dx_2 \cdots dx_d.
\]

The rectangle module of continuity \( \Omega(f, \vec{\delta}) = \Omega(f, \delta) \) for the (continuous a.e.) function \( f \) and vector \( \vec{\delta} = \delta = (\delta_1, \delta_2, \ldots, \delta_d) \in [0, 1]^d \) may be defined as well as ordinary module of continuity \( \Delta(f, \delta) \) as follows:

\[
\Omega(f, \vec{\delta}) \overset{def}{=} \sup \{ |\Box[f](x, y)|, \, (x, y) : |x_i - y_i| \leq \delta_i, \, i = 1, 2, \ldots, d \}.
\]

Let \( \xi = \xi(x) = \xi(x_1, x_2, ..., x_d) = \xi(\vec{x}) \), \( \vec{x} \in D \) be a separable random field (r.f), not necessary to be Gaussian. The sufficient condition for rectangle continuity of \( \xi(x) \) and Orlicz’s norm estimates

\[
\gamma(\xi, \delta) \overset{def}{=} ||\Omega(\xi, \vec{\delta})||\Phi,
\]

such that

\[
\lim_{||\delta|| \to 0} \gamma(\xi, \delta) = 0
\]

for it rectangle modulus of continuity are obtained in the articles [10], [31], [12], [24].

Recall that the first publication about fractional Sobolev’s inequalities [10] was devoted in particular to the such a problem.

It is not hard to obtain as before from (6.4) the sufficient conditions for factorable rectangle continuity of the r.f., i.e. the for the estimates of a form

\[
\Omega(\xi, \vec{\delta}) \leq \nu(\omega) \cdot g(\vec{\delta}),
\]

where \( g(\vec{\delta}) \) is continuous non-random scaling function such that

\[
\lim_{||\delta|| \to 0} g(\vec{\delta}) = 0,
\]

which may be constructed as before in the third section, and \( \nu(\cdot) \) is random variable for which \( ||\nu||\Psi = 1 \).
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