A Monopole Metric

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Abstract

We calculate explicitly in terms of complete elliptic integrals the metric on the moduli space of tetrahedrally-symmetric, charge four, $SU(2)$ monopoles. Using this we verify that in the asymptotic regime the metric of Gibbons and Manton is exact up to exponentially suppressed corrections.
1 Introduction

The moduli space $\mathcal{M}_n$ of charge $n$ $SU(2)$ BPS monopoles is a $4n$-dimensional manifold, whose metric is of interest for three main reasons. First, it is known to be hyperkähler, and explicit examples of such metrics are rare. Second, the dynamics of $n$ slowly moving monopoles can be approximated by geodesic motion on $\mathcal{M}_n [10, 13]$. Finally, following the work of Sen [12], predictions of S-duality in the $N = 4$ supersymmetric quantum theory can be tested by an analysis of the harmonic forms on $\mathcal{M}_n$.

There is an isometric splitting

$$\widetilde{\mathcal{M}}_n = \mathbb{R}^3 \times S^1 \times \mathcal{M}_n^0$$

(1.1)

where $\widetilde{\mathcal{M}}_n$ is an $n$-fold covering of $\mathcal{M}_n$. Thus the interesting structure of the moduli space is contained in the $4(n - 1)$-dimensional hyperkähler manifold $\mathcal{M}_n^0$. The simplest case $\mathcal{M}_2^0$ is the Atiyah-Hitchin manifold [2], where the metric can be written explicitly in terms of complete elliptic integrals. Unfortunately, for $n > 2$ the metric on $\mathcal{M}_n^0$ is not known. Indeed until recently the exact metric was unknown on any submanifold of $\mathcal{M}_n^0$. However, recent results [9, 3] have shown that for $n > 2$, $\mathcal{M}_n^0$ contains a totally geodesic submanifold which is the Atiyah-Hitchin manifold. Thus, in this sense, the metric on a 4-dimensional submanifold of $\mathcal{M}_n^0$ is now known, but at the present time this is the full extent of explicit results for the exact metric. Using a point particle approximation the asymptotic metric on parts of the moduli space representing well-separated monopoles has been explicitly computed by Gibbons and Manton [4].

Imposing tetrahedral symmetry upon charge four monopoles gives, after fixing the centre of mass and orientation, a totally geodesic 1-dimensional submanifold $\mathcal{N} \subset \mathcal{M}_4^0$. The associated four monopole scattering has been investigated in detail [8] and the metric on $\mathcal{N}$ computed numerically [14]. In this letter we calculate this metric exactly, and in closed form, in terms of complete elliptic integrals. As noted above, this is the first explicit calculation of the metric on any submanifold of $\mathcal{M}_n^0$, except for the Atiyah-Hitchin submanifolds. These latter submanifolds arise through the embedding of $n$ collinear monopoles and thus it is clear that the intersection of the submanifold $\mathcal{N}$ with the above Atiyah-Hitchin submanifold is empty.

The approach taken is to construct the metric on the moduli space of Nahm data, with the tangent vectors obtained by direct differentiation. Comparisons with numerical results and the asymptotic metric are then made.

2 Tetrahedral charge four monopoles

The Nahm data and spectral curve for the one-parameter family of 4-monopoles described by $\mathcal{N}$ have been calculated [8]. The spectral curve for this family is

$$\eta^4 + i36\omega^3\eta\zeta(\zeta^4 - 1) + 3\omega^4(\zeta^8 + 14\zeta^4 + 1) = 0$$

(2.1)
where \( a \in (-a_c, a_c), \) \( a_c = 3^{-5/4} \sqrt{2}, \) and \( \omega \) is the real half-period of the elliptic curve
\[
y^2 = 4(x^3 - x + 3a^2).
\] (2.2)

For the special point \( a = 0 \) the monopole actually has cubic symmetry and was discovered by Hitchin, Manton and Murray \[7\]. In the limit as \( a \to a_c \), we have \( \omega \to \infty \). Recalling that a single monopole with position \((x_1, x_2, x_3)\) has spectral curve \( \eta - (x_1 + i x_2) + 2x_3 \zeta + (x_1 - i x_2) \zeta^2 = 0 \)
(2.2), the product of the four spectral curves corresponding to monopoles positioned on the vertices \( \{(-l, -l, -l)\sqrt{3}, (-l, +l, +l)\sqrt{3}, (+l, +l, -l)\sqrt{3}, (+l, -l, +l)\sqrt{3}\} \)
of a large regular tetrahedron then has asymptotic spectral curve
\[
\eta^4 + i \frac{16}{3^2} l^3 \eta \zeta (\zeta^4 - 1) + \frac{4}{9} l^4 (\zeta^8 + 14 \zeta^4 + 1) = 0.
\] (2.3)

By comparing (2.1) and (2.3) we see that we can make the identification
\[
l = \Lambda a^{1/3} \omega, \quad \text{where} \quad \Lambda = 3^{7/6} 2^{-2/3}.
\] (2.4)

This \( l \in R \) is a good global coordinate on \( \mathcal{N} \): it is zero when the monopoles coincide to form the cubic monopole and may be identified with the coordinates of the vertices of the tetrahedron when \( l \) is large. The task at hand is to compute the metric on \( \mathcal{N} \) in terms of \( l \). It is known that the transformation between the monopole moduli space metric and the metric on Nahm data is an isometry \[11\] and so we may construct the metric on \( \mathcal{N} \) by computing the metric on the Nahm data.

The Nahm data for this family of monopoles has the form
\[
T_i(s) = x(s)X_i + y(s)Y_i + z(s)Z_i \quad i = 1, 2, 3
\] (2.5)

where \( x, y, z \) are the real functions
\[
(x(s), y(s), z(s)) = \left( \frac{\omega}{9} \left( -2 \sqrt{\bar{\wp}(u)} + \frac{1}{4\wp(u)} \frac{d\wp(u)}{du} \right), \frac{\omega}{20} \left( \sqrt{\bar{\wp}(u)} + \frac{1}{2\wp(u)} \frac{d\wp(u)}{du} \right), \frac{a\omega}{2\wp(u)} \right).
\] (2.6)

Here \( u = \omega s \) and \( \bar{\wp} \) is the Weierstrass function satisfying
\[
\left( \frac{d\bar{\wp}}{du} \right)^2 = 4\bar{\wp}^3 - 4\bar{\wp} + 12a^2.
\] (2.7)

The tetrahedrally symmetric Nahm triplets \( X_i, Y_i, \) and \( Z_i \) are constant \( 4 \times 4 \) matrices; explicit expressions for these may be found in \[14\]. The spectral curve \([2.1]\) is then related to this data by setting \( \det(\eta I + (T_1 + iT_2) - 2\zeta T_3 + (T_1 - iT_2)\zeta^2) = 0 \).

Let \( V_i = dT_i/dl \) be the tangent vector corresponding to the point with Nahm data \( T_i \). The metric on Nahm data is then given by
\[
g = -\Omega \int_0^2 \sum_{i=1}^3 \text{tr}(V_i^2) \ ds,
\] (2.8)
where \( tr \) denotes trace and \( \Omega \) is a normalization constant. In general a fourth Nahm matrix and its corresponding tangent vector needs to be introduced to ensure orthogonality to gauge orbits, but in this particular case the tetrahedral symmetry of the Nahm data implies that this can be ignored and the resulting tangent vectors are automatically orthogonal to the gauge orbits. (See [14] for a discussion and proof of this fact.) After substituting the explicit expressions for the matrices \( X_i, Y_i \) and \( Z_i \) and performing the traces we obtain

\[
g(l) = 12\Omega \int_0^2 \{ 5(\frac{dx}{dl})^2 + 80(\frac{dy}{dl})^2 + 3(\frac{dz}{dl})^2 \} ds. \tag{2.9}
\]

We calculate the quantity \( g(l) \) in the next section. Before turning to this however we need to make few remarks about normalizations.

For a single monopole the length of the Higgs field \( |\Phi| \) has the asymptotic behaviour

\[
|\Phi| = v - \frac{g}{4\pi r} + O(e^{-8\pi vr/g}) \tag{2.10}
\]

where \( v \) is the vacuum expectation value of the Higgs field and \( g \) is the magnetic charge. The monopole mass is the product of these two constants, \( m = vg \). Performing the ADHMN construction for a single monopole we find

\[
|\Phi| = 1 - \frac{1}{2r} + .. \tag{2.11}
\]

from which we can read off our units to be \( m = g = 2\pi \).

## 3 Determining the metric

In this section we use several identities from the theory of elliptic functions. These can be found in, for example, reference [1] and we follow their notation throughout.

As a first step towards evaluating \( g(l) \) it is helpful to disentangle the \( l \) dependence in the arguments of the functions appearing in the integrand by exploiting the homogeneity properties of the \( \wp \)-function. This enables us to express Nahm data in a more convenient form. We have

\[
\wp(\omega s) = \omega^{-2} \wp(s) \tag{3.1}
\]

where \( \wp(s) \) now satisfies the equation (throughout ‘ denotes differentiation with respect to \( s \))

\[
\wp'^2 = 4\wp^3 - g_2\wp - g_3 \tag{3.2}
\]

with the parameters

\[
g_2 = 4(m^2 - m + 1)K^4/3 \nonumber \\
g_3 = 4(m - 2)(2m - 1)(m + 1)K^6/27 \nonumber \\
\omega = [(m^2 - m + 1)/3]^{1/4}K. \tag{3.3}
\]
Here $K$ denotes the complete elliptic integral with parameter $m$. The functions in the Nahm data now take the simplified form

$$
(x(s), y(s), z(s)) = \left( \frac{1}{5} \left( -2\sqrt{\varphi(s)} + \frac{1}{4} \frac{\varphi'(s)}{\varphi(s)} \right), \frac{1}{20} \left( \sqrt{\varphi(s)} + \frac{1}{2} \frac{\varphi'(s)}{\varphi(s)} \right), \frac{\sqrt{-g_3}}{2\sqrt{12\varphi(s)}} \right).
$$

(3.4)

All of the $l$ dependence resides in the parameters $g_2$ and $g_3$. In terms of the elliptic parameter $m$ the geodesic coordinate $l$ may now be expressed as

$$
l^6 = \frac{3^7}{2^3} a^2 \omega^6 = \frac{3^3}{2^4} (2-m)(2m-1)(m+1) K^6.
$$

(3.5)

Because we have expressed all the parameters in terms of $m$ it is convenient to perform differentiations with respect to $m$ (which we denote by a dot) rather than $l$. Accordingly we will evaluate

$$
\dot{g} = g \dot{l}^2,
$$

(3.6)

where, upon using (3.5), we find that

$$
\dot{l} = \frac{\sqrt{3} [(2-m)(2m-1)(m+1) E - K (1-m)(m^2 + 2m - 2)]}{m(1-m) [4(2-m)(2m-1)(m+1)]^{5/6}}.
$$

(3.7)

Here $E$ is the complete elliptic integral of the second kind with parameter $m$.

Upon substituting (3.4) into (2.9) and noting (3.6) our task is now to evaluate

$$
\dot{g} = 3\Omega^{3/4} \int_0^2 J ds.
$$

(3.8)

Using standard expressions for the partial derivatives of the $\varphi$-function with respect to the invariants $g_2$ and $g_3$ we obtain

$$
\dot{\varphi} = \varphi' H + F, \quad \text{where} \quad H = a \zeta + b s \quad \text{and} \quad F = \alpha \varphi^2 + \beta \varphi + \gamma.
$$

(3.9)

Here we have set

$$
\alpha = (3g_2\dot{g}_3 - 9g_3\dot{g}_2)/\Delta, \quad \beta = (g_2^2\dot{g}_2/4 - 9g_3\dot{g}_3)/\Delta, \quad \gamma = (3g_2g_3\dot{g}_2/2 - g_2^2\dot{g}_3)/\Delta.
$$

(3.10)

Of course, $\zeta$ is the Weierstrass $\zeta$-function satisfying $\zeta' = -\varphi$ and $\Delta$ is the discriminant $\Delta = g_3^3 - 27g_2^2$.

Substituting (3.9) into (3.8) and using (3.2) together with its derivative we find

$$
\dot{g} = \frac{3\Omega}{4} \int_0^2 J ds.
$$

(3.11)
with the integrand $J$ given by
\begin{equation}
\varphi^4 J = H^2[20\varphi^6 - 2g_2\varphi^4 - 4g_3\varphi^3 + \frac{1}{4}g_2^2\varphi^2 + 2g_2g_3\varphi + 2g_3^2] + 2H\varphi'\left[\frac{1}{3}g_3\varphi + F(4\varphi^3 - g_3) + (2\varphi^3 + \frac{1}{3}g_2\varphi + g_3)((\alpha - a)\varphi^2 + b\varphi - \gamma)\right] + F(4\varphi^3 - g_3) + (4\varphi^3 - 2g_2\varphi - g_3)((\alpha - a)\varphi^2 + b\varphi - \gamma)^2 + \dot{g}_3\varphi(F - \frac{1}{4}\dot{g}_3\varphi/g_3).
\end{equation}

Now the potentially problematic terms are those involving $H$, since this contains the $\zeta$-function. However, after some calculation, we can express all the terms involving $H$ as total derivatives ie.
\begin{equation}
J = \frac{d}{ds}\left\{H^2\left[\frac{10}{3}\varphi' + \frac{2}{3}g_3\varphi'/\varphi^3 + \frac{1}{6}g_2\varphi'/\varphi^2\right] + H\left[(6\alpha + \frac{4}{3}a)\varphi^2 + 8(\beta - \frac{1}{3}b)\varphi\right] - \frac{2}{3}(g_2b - ag_3)/\varphi - \frac{1}{2}(\dot{g}_3 - 2\beta g_3 - g_2\gamma + \frac{2}{3}bg_3)/\varphi^2 + \frac{4}{3}\gamma g_3/\varphi^3\right\}
+ P/\varphi^3
\end{equation}
where $P$ is a $7^{th}$ order polynomial in $\varphi$. Using the identity
\begin{equation}
\frac{1}{\varphi^r} = \frac{d}{ds}\left(\frac{1}{g_3(r-1)\varphi^{r-1}}\right) + \frac{1}{g_3\varphi^{r-3}}\left(4 - \frac{6}{r-1}\right) + \frac{g_2}{g_3\varphi^{r-3}}\left(\frac{1}{2(r-1) - 1}\right)
\end{equation}
valid for $r \geq 2$ we then recursively remove the terms $1/\varphi^4$, $1/\varphi^3$ and $1/\varphi^2$ and we find the difficult $1/\varphi$-term vanishes. We are then left with a polynomial of degree 3 in $\varphi$ which is readily integrated. At this stage we have expressed the integrand $J$ as the total derivative of a density $j$, ie. $J = dj/ds$. It now remains to evaluate the density at the limits of integration, $s = 0, 2$.

As $s \to 0$ the required asymptotic limits are
\begin{equation}
\varphi \sim \frac{1}{s^2} + \frac{1}{20}g_2s^2, \quad \zeta \sim \frac{1}{s^2} - \frac{1}{60}g_2s^3.
\end{equation}

We find that the pole terms in the density cancel, which is a highly non-trivial check on our calculation, and furthermore everything is proportional to $s$, giving the result $j|_{s=0} = 0$.

Finally, as $s \to 2$, we have
\begin{equation}
\varphi \sim \frac{1}{(s-2)^2} + \frac{1}{20}g_2(s-2)^2, \quad \zeta \sim \frac{1}{s-2} + \frac{2}{3}K(3E + (m-2)K) - \frac{1}{60}g_2(s-2)^3.
\end{equation}

Again the pole terms in the density cancel, and we are left with a finite value for $j|_{s=2}$ which is non-zero. Using (3.14) and (3.7) and choosing the normalization constant $\Omega = \pi$ we obtain the final result
\begin{equation}
g(l) = 8\pi \left(\frac{f(m)}{2}\right)^{2/3} \frac{2f(m)(2-m)E^2 - 2f(m)E^3/K - 2f(m)(1-m)EK + m^2(1-m)^2K^2}{[f(m)E-(m^2+2m-2)(1-m)K]^2}
\end{equation}
where we have set
\begin{equation}
f(m) = (2 - m)(1 + m)(2m - 1)
\end{equation}
and we recall that
\begin{equation}
l = \frac{\sqrt{3}}{2^{3/2}}[(2 - m)(2m - 1)(m + 1)]^{1/6}K.
\end{equation}
4 Analysis of the metric

In figure 1 we plot (solid curve) the metric (3.16) as a function of $l$ for $l \in [0, 6]$. Using a point particle approximation, Gibbons and Manton \cite{[1]} have calculated the asymptotic metric on regions of $\mathcal{M}_n$ which describe well-separated monopoles. For pure monopoles i.e. with zero electric charge, the asymptotic metric for $n$ monopoles with positions $\mathbf{x}_i$ is given by $ds^2 = g_{ij} \dot{x}_i \cdot \dot{x}_j$,

\begin{align}
  g_{jj} &= m - \frac{g^2}{4\pi} \sum_{i\neq j} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}, \\
  g_{ij} &= \frac{g^2}{4\pi} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}.
\end{align} \tag{4.1}

Here $m$ and $g$ are the mass and magnetic charge of a single monopole. As stated earlier, in the normalization we have chosen these values are $m = g = 2\pi$. Note that the numerical construction of the metric in reference \cite{[4]} involves a renormalisation of $l \mapsto 2l$ in the plot of the metric. From equation (2.10) it can easily be seen that the effect of this scaling of the space coordinates is to convert to the more standard normalization of $m = g = 4\pi$.

For four monopoles on the vertices of a regular tetrahedron, as described earlier, the asymptotic metric (4.1) becomes

$g_{GM} = 8\pi (1 - \frac{\sqrt{6}}{2l}).$ \tag{4.2}

In figure 1 we also plot this metric (dashed curve) for comparison with the true metric.

It is known that in the asymptotic region of large $l$ the correction to the metric (4.2) is exponentially small. Using the exact metric (3.16) and taking the asymptotic limit, which corresponds to $m \to 1$, we can calculate the leading order correction to (4.2).

Introducing $m_1 = 1 - m$ and using the standard expansions

\begin{align}
  E &= E(m) = 1 + \frac{m_1}{4} \left( \ln \frac{16}{m_1} - 1 \right) + \frac{3m_1^2}{32} \left( \ln \frac{16}{m_1} - \frac{13}{6} \right) + \ldots \tag{4.3} \\
  K &= K(m) = \frac{1}{2} \ln \frac{16}{m_1} + \frac{m_1}{4} \left( \frac{1}{2} \ln \frac{16}{m_1} - 1 \right) + \frac{9m_1^2}{64} \left( \frac{1}{2} \ln \frac{16}{m_1} - \frac{7}{6} \right) + \ldots \tag{4.4}
\end{align}

we find that (working to quadratic order in $m_1$),

\begin{align}
  E &= 1 + \frac{1}{4}(2K - 1)m_1 + \frac{1}{64}(4K - 5)m_1^2 \tag{4.5} \\
  K &= \frac{l\sqrt{2}}{\sqrt{3}} (1 + \frac{1}{4}m_1 + \frac{15}{32}m_1^2). \tag{4.6}
\end{align}

Upon substituting these expressions into (3.16) the terms linear in $m_1$ cancel and, after replacing $m_1$ by its leading order approximation

\begin{equation}
  m_1 \sim 16 \exp(-l\sqrt{8}/\sqrt{3}), \tag{4.7}
\end{equation}
the leading order correction to (4.2) is found to be
\[
\tilde{g} = 8\pi (1 - \frac{\sqrt{6}}{2l} - 192\sqrt{6} l \exp(-2l\sqrt{8/\sqrt{3}})). \tag{4.8}
\]

The first two terms in this expansion are readily identified as coming from the first two terms in the numerator of (3.16).

It is perhaps of interest to compare this exponential correction with that in the charge two case, where there is an exponential correction to the Taub-NUT metric to obtain the asymptotic Atiyah-Hitchin metric.

In the normalization we are using the asymptotic Atiyah-Hitchin metric is
\[
\tilde{g}_2 = 4\pi (1 - \frac{2}{r} + 8re^{-r}) \tag{4.9}
\]
where \(r\) is the distance of each monopole from the centre of mass. To compare this with the charge four case we write \(l = r\sqrt{6}/4\), after which (4.8) becomes
\[
\tilde{g}/2 = 4\pi (1 - \frac{2}{r} - 288re^{-2r}). \tag{4.10}
\]

We see that the exponential correction is a higher order effect in the charge four case, which stems from the vanishing of the linear terms in \(m_1\). Presumably this arises since we are considering a configuration which is particularly symmetric, leading to a cancellation between the naive collection of two monopole pairs. Note also that the correction to the Taub-NUT metric in the charge four case has opposite sign to that in the charge two case.

Finally, in figure 1 we also plot some numerical values for the metric (diamonds) computed using the algorithm introduced in [14]. This demonstrates that the numerical scheme is accurate, in fact the computed values are all within \(\frac{1}{2}\%\) of the true values, and can be reliably used to compute the metric in other less tractable cases.

5 Conclusion

By direct calculation, and the use of explicit Nahm data, we have computed the metric on the moduli space of tetrahedrally symmetric charge four BPS monopoles. An important ingredient was the ability to write a combination of tangent vectors to Nahm data as a total derivative. Note that Hitchin has remarked that a similar situation may exist in general [8], with the charge \(n\) metric being determined by boundary values of a Riemann theta function on a surface of genus \((n - 1)^2\). The difficulty in this formulation appears to be the implementation of suitable boundary conditions on the general theta function solution of Nahm’s equation. For the situation considered in this letter the tetrahedral symmetry implies that it is the quotient surface under the tetrahedral group which is relevant, and this has genus 1, thus allowing an explicit computation in terms of elliptic functions and integrals.
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Figure 1: The exact metric (solid curve), asymptotic metric (dashed curve) and numerical results (diamonds).