EXTENSION THEOREM AND REPRESENTATION FORMULA IN NON AXIALLY SYMMETRIC DOMAINS FOR SLICE REGULAR FUNCTIONS

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Abstract. Slice analysis is a generalization of the theory of holomorphic functions of one complex variable to quaternions. Among the new phenomena which appear in this context, there is the fact that the convergence domain of \( f(q) = \sum_{n \in \mathbb{N}} (q - p)^n a_n \), given by a \( \sigma \)-ball \( \Sigma(p, r) \), is not open in \( \mathbb{H} \) unless \( p \in \mathbb{R} \). This motivates us to investigate, in this article, what is a natural topology for slice regular functions. It turns out that the natural topology is the so-called slice topology, which is different from the Euclidean topology and nicely adapts to the slice structure of quaternions. We extend the function theory of slice regular functions to any domains in the slice topology. Many fundamental results in the classical slice analysis for axially symmetric domains fail in our general setting. We can even construct a counterexample to show that a slice regular function in a domain cannot be extended to an axially symmetric domain. In order to provide positive results we need to consider so-called path-slice functions instead of slice functions. Along this line, we can establish an extension theorem and a representation formula in a slice-domain.

1. Introduction

The richness of complex analysis makes it natural to look for generalizations to quaternions. Around the early thirties various people, among which Moisil and Fueter, considered possible definitions of analiticity over the quaternions. Since then, Fueter and his school started a systematic study so the notion of ‘regular’ quaternionic functions is the one associated with the so-called Cauchy-Riemann-Fueter equation, see [10]

\[
\frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} + j \frac{\partial f}{\partial x_3} + k \frac{\partial f}{\partial x_4} = 0.
\]

This theory has been widely studied, see e.g. [9, 19, 23] but also [6, 17] and the references therein. Unfortunately, the class of Fueter regular functions does not contain the identity function \( f(q) = q \) or any other polynomial in \( q \). However, Fueter [10] found a powerful approach to construct functions in higher dimensions based on holomorphic function of one complex variable.

This approach was further developed by Sce [21], Rinehart [20] and resulted in the theory of intrinsic or stem functions. Later on, Cullen [8] defined another
class of regular functions by intrinsic functions. Cullen regular functions contain quaternionic power series of the form \( \Sigma_{n \in \mathbb{N}} q^n a_n \).

Following Cullen’s approach another theory, called slice quaternionic analysis, was started by Gentili and Struppa [13, 14] based on more geometric formulation. This local theory has been well established first on balls centered at the origin [13,14] then over the axially symmetric slice domains [4,5]. Most of the local theory of holomorphic functions of one complex variable can be lifted to quaternions. It gives rise to the new notion of S-spectrum and has powerful applications in the quaternionic spectral theory see e.g. [1, 5], quaternionic Hilbert spaces [2, 3, 5, 16]. See [7, 12] and the references therein for other information.

In contrast to its full development in local theory, the global one remains to be developed. The challenging task of establishing the global theory over quaternions can lead to some new theories such as slice Riemann surfaces, slice regular domains, and slice Dolbeault complexes. Therefore, the first natural question to be answered is:

**What is the natural topology in slice analysis?**

It has been argued that any slice regular function on a domain of \( \mathbb{H} \) can be extended to an axially symmetric domain. But this is not true and we provide a counterexample in Example 8.10. This means that axially symmetric slice domains are not the maximal domains of definition of a slice regular function. In other words, axially symmetric domains do not play the role of the natural maximal domains in slice analysis. On the other hand, the convergence domain of the Taylor expansion of a slice regular function

\[
\sum_{n \in \mathbb{N}} (q - p)^n \frac{f_n(p)}{n!},
\]

completely described in terms of the \( \sigma \)-ball \( \Sigma(p, r) \) (see [11]), may not be an Euclidean domain. Hence the Euclidean topology is not a natural topology in slice analysis.

To seek a clue to the problem, one has to focus on the starting point of the slice theory, namely in the theory of intrinsic functions. We observe that the slice book structure of quaternions plays a key role which makes it feasible to lift the holomorphic theory of one variable based to quaternions. The slice book structure comes from the following decomposition of quaternions into complex planes,

\[
\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I,
\]

where \( \mathbb{C}_I = \mathbb{R} + I\mathbb{R} \) is the complex plane generated by the imaginary unit \( I \) and \( S \) consists of all imaginary unit \( I \) of quaternions. As a result, the slice book structure of quaternions is a natural structure in slice analysis.

Motivated by the slice book structure, we can answer the main question of this article. It turns out that the natural topology in slice analysis is the so-called slice topology, which adapts nicely to the book structure of quaternions. We prove that the slice topology is finer than the Euclidean topology and all of the \( \sigma \)-balls \( \Sigma(p, r) \) are domains in the slice topology.

With this slice topology, some natural questions arise. One can ask if the slice theory can be extended from the axially symmetric domains to any domains in slice topology, but the answer is negative in general. As an example, one can consider the representation formula. This formula is the most important feature of
the classical local theory of slice analysis. It states that any slice regular function over an axially symmetric slice domain is completely determined by its values on two pages, i.e. complex planes, of the book structures of \( \mathbb{H} \). This result cannot be directly extended to more general cases. Instead, we have to extend the theory of stem functions to a new one involving paths which produce path-slice functions.

Using the slice topology, one can also ask if any domain in the slice topology is a domain of holomorphy in some sense. Also the answer to this question is negative, in general, in contrast to the case of holomorphic functions in one variable. This leads to the study of the characterization of domains of holomorphy just like in the case of holomorphic functions of several variables. We provide conditions for a domain to be such a holomorphy domain.

The structure of the paper is the following. In Section 2, we introduce the slice topology on quaternions for slice regular functions and we describe our main results and ideas. In Section 3, we give some basic properties and examples for the slice topology. In Section 4, we prove an identity principle for slice regular functions on domains in the slice topology. In Section 5, we generalize the slice function to any subset of \( \mathbb{H} \) and give several equivalent definitions of slice functions. In Section 6, we prove a generalized extension formula. In Section 7, we define a class of functions, called path-slice functions. These functions play a similar role on slice-domains as the slice functions do on axially symmetric slice domains. We also give several equivalent definitions of path-slice functions and prove our main theorem, i.e. the Representation Formula 2.11. In Section 8, we give an example to show that the classical general representation formula [4, Theorem 3.2] does not work on non-axially symmetric s-domain, using the new Representation Formula 2.11. Section 9 is devoted to domains of holomorphy for slice regular function defined on slice-open set.

We will continue our further study on the global theory of slice analysis in some forthcoming articles.

2. Main results

In this section, we state our main results. To do this some notation and definitions from [13] are needed. Let

\[ \mathbb{S} := \{ q \in \mathbb{H} : q^2 = -1 \} \]

be the sphere of imaginary units of \( \mathbb{H} \). For any subset \( \Omega \) of \( \mathbb{H} \) and \( I \in \mathbb{S} \), we call

\[ \Omega_I := \Omega \cap \mathbb{C}_I \]

the \( I \)-slice (a slice) of \( \Omega \).

**Definition 2.1.** Assume that \( \Omega \) is an open set in \( \mathbb{C}_I \) for some \( I \in \mathbb{S} \). A function \( f : \Omega \to \mathbb{H} \) is said to be left \( \mathbb{C}_I \)-holomorphic (or, simply, holomorphic), if \( f \) has continuous partial derivatives and satisfies

\[
\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f(x + yI) = 0
\]

for any \( x, y \in \mathbb{R} \) with \( x + yI \in \Omega \).

The definition originally given in [13] is:

**Definition 2.2.** Let \( \Omega \) be a domain in \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is said to be (left) slice regular if \( f_I := f|_{\Omega_I} \) is left \( \mathbb{C}_I \)-holomorphic for any \( I \in \mathbb{S} \).
[11, Theorem 8] shows that the convergence domain of the series
\[ \sum_{n \in \mathbb{N}} (q-p)^n a_n \]
is the \( \sigma \)-ball
\[ \Sigma(p, r) := \{ q \in \mathbb{H} : \sigma(p, q) < r \} \]
with the \( \sigma \)-distance defined by
\[ \sigma(q, p) := \begin{cases} |q-p|, & \exists I \in \mathbb{S}, \text{ s.t. } p, q \in \mathbb{C}_I, \\ \sqrt{(\text{Re}(q-p))^2 + |\text{Im}(q)|^2 + |\text{Im}(p)|^2}, & \text{otherwise,} \end{cases} \]
for any \( p, q \in \mathbb{H} \). A \( \sigma \)-ball is not an Euclidean domain when \( p \in \mathbb{H} \setminus \mathbb{R} \). This illustrates the need to define 'slice regular' functions on more sets, such as above \( \sigma \)-balls. Note that the ‘holomorphic’ condition of \( f \) in Definition 2.2 is limited to each slice \( \mathbb{C}_I, I \in \mathbb{S} \). Thus in order to define ‘slice regular’, we just need to guarantee \( \Omega_I \) is open in \( \mathbb{C}_I \) for each \( I \in \mathbb{S} \).

**Definition 2.3.** A subset \( \Omega \) of \( \mathbb{H} \) is called slice-open, if \( \Omega_I \) is open in \( \mathbb{C}_I \) for any \( I \in \mathbb{S} \).

It is clear that the \( \sigma \)-ball \( \Sigma(p, r) \) is slice-open. Now we extend Definition 2.2 to slice-open sets.

**Definition 2.4.** Let \( \Omega \) be a slice-open set in \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is called (left) slice regular, if \( f_I \) is left holomorphic for any \( I \in \mathbb{S} \).

We note that, so far, slice quaternionic analysis has been developed over axially symmetric slice domains. Our goal is to generalize it to any slice-open set. Some properties can be proved as in the classical case, e.g. the following splitting lemma. Thus we state it without proof.

**Lemma 2.5.** (Splitting Lemma) Let \( f \) be a function on a slice-open set \( \Omega \). Then \( f \) is slice regular, if and only if for all \( I, J \in \mathbb{S} \) with \( I \perp J \), there are two complex-valued holomorphic functions \( F, G : \Omega_I \to \mathbb{C}_I \) such that \( f_I = F + GJ \).

The set of slice-open sets gives a topology on \( \mathbb{H} \), as proven in the following result.

**Lemma 2.6.**
\[
\tau_s(\mathbb{H}) := \{ \Omega \subset \mathbb{H} : \Omega \text{ is slice-open} \}
\]
is a topology of \( \mathbb{H} \).

**Proof.** (i) It is clear that \( \emptyset, \mathbb{H} \in \tau_s(\mathbb{H}) \).
(ii) Let \( U_1, \ldots, U_n \in \tau_s(\mathbb{H}) \). For each \( I \in \mathbb{S} \) and \( m \in \{1, 2, \ldots, n\} \),
\[
(U_m)_I \in \tau(\mathbb{C}_I).
\]
It follows that
\[
\left( \bigcap_{m=1}^n U_m \right) \bigcap \mathbb{C}_I = \bigcap_{m=1}^n (U_m)_I
\]
is open in \( \mathbb{C}_I \). Since it holds for an arbitrary \( I \in \mathbb{S} \), it follows that
\[
\bigcap_{m=1}^n U_m \in \tau_s(\mathbb{H}).
\]
(iii) As above, for each \( U_\lambda \in \tau_s(\mathbb{H}) \), \( \lambda \in \Lambda \), one can show that
\[
\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau_s(\mathbb{H}).
\]

\[\square\]

**Definition 2.7.** We call \( \tau_s(\mathbb{H}) \) the slice topology. Open sets, connected sets and paths in the slice topology are called slice-open sets, slice-connected and slice-paths.

**Remark 2.8.** In particular, a similar terminology will be used for all the other notions in the slice topology, with one remarkable exception. We will not use the terminology slice-domain to denote a domain in the slice topology, since this notion is already used in the literature to denote something different (see Definition 2.9 below). We will use instead the term slice topology-domain, in short, st-domain.

**Definition 2.9.** A set \( \Omega \) in \( \mathbb{H} \) is called classical slice domain, in short s-domain, if \( \Omega \) is a domain in the Euclidean topology, \( \Omega \cap \mathbb{R} \neq \emptyset \), and \( \Omega_I \) is a domain in \( \mathbb{C}_I \) for any \( I \in S \).

It is evident that s-domain must be a domain in the slice topology, i.e. an st-domain, but the converse statement is not true (see Example 3.12).

The classical slice quaternionic analysis is established on axially symmetric s-domains. The slice quaternionic analysis on st-domains shows differences with respect to the classical one, since it rely on the slice-connectedness. For example, the proof of the following generalized Identity Principle in Section 4, involves some properties of st-domains induced by slice-connectedness.

**Theorem 2.10.** (Identity Principle) Let \( f \) and \( g \) be two slice regular functions on an st-domain \( \Omega \) in \( \mathbb{H} \). If \( f \) and \( g \) coincide on a subset of \( \Omega_I \) with an accumulation point in \( \Omega_I \) for some \( I \in S \), then \( f = g \) on \( \Omega \).

Another fundamental result in the classical slice analysis is the general representation formula [4, Theorem 3.2]. We extend the result to the st-domains. Unfortunately, the classical general representation formula [4, Theorem 3.2] fails, in general, on st-domains, see Section 8.

To get the validity of the formula, we have to introduce the notion of path-slice functions, see Definition 7.1.

We consider the transform
\[
\mathcal{P}_I : \mathbb{C} \to \mathbb{C}_I \\
x + yi \mapsto x + yI
\]
for any \( x, y \in \mathbb{R} \) and \( I \in S \). For any path \( \gamma \) in \( \mathbb{C} \), we define its corresponding path in \( \mathbb{C}_I \) as
\[
\gamma^I := \mathcal{P}_I \circ \gamma
\]
for any \( I \in S \).

**Theorem 2.11.** (Representation Formula) Assume that \( \Omega \) is a slice-open set in \( \mathbb{H} \) and suppose \( \gamma \) is a path in \( \mathbb{C} \) satisfying the conditions
\[
\gamma(0) \in \mathbb{R}, \quad \gamma^I, \gamma^J, \gamma^K \subset \Omega
\]
for some $I, J, K \in \mathbb{S}$ with $J \neq K$. If $f$ is a slice regular function on $\Omega$, then

$$f \circ \gamma^I = (I - K)(J - K)^{-1}f \circ \gamma^J + (I - J)(K - J)^{-1}f \circ \gamma^K.$$  

**Remark 2.12.** Although we only assume the domain $\Omega$ in consideration is only slice-open, some restrictions related to slice-connectedness are implicitly involved as shown by the conditions

$$\gamma^I, \gamma^J, \gamma^K \subset \Omega.$$  

The path $\gamma^I$ in a slice can distinguish points of $\Omega$ more finely than $x + yI$ (by the Euclidean coordinate in $\mathbb{C}_I$), see Section 8. This ensures that the representation formula holds on non-axially symmetric domains.

The functions satisfying (2.2) is called path-slice in Section 7 based on an equivalent definition. It turns out that any slice regular function is a path-slice function. The proof of (2.2) shall depend on a new approach; see Proposition 7.2 (i) and (vi).

3. Slice topology

In this section, we study some properties of the slice topology $\tau_s(\mathbb{H})$. The slice structure induces the intricacy of the notion of slice-connectedness near the real axis. We tackle this issue in terms of slice-paths.

We denote by $\tau_s(\mathbb{H})$ and $\tau(\mathbb{H})$ the slice topology and the Euclidean topology of $\mathbb{H}$, respectively. Sometimes, we simply write $\tau_s$ and $\tau$, for short.

**Proposition 3.1.** $(\mathbb{H}, \tau_s)$ is a Hausdorff space and $\tau_s \subset \tau$.

**Proof.** Since every Euclidean open set in $\mathbb{H}$ is slice-open, we have $\tau \subset \tau_s$ and $\tau_s$ is Hausdorff. Note that $\Sigma(p, r)$ is slice-open and not open for any $p \in \mathbb{H} \setminus \mathbb{R}$ and $r \in \mathbb{R}_+$. It follows that the slice topology is strictly finer than the Euclidean topology.

We remark that the slice topology locally coincides with the Euclidean topology on a slice complex plane for any point away from the real axis $\mathbb{R}$, because for any $I \in \mathbb{S}$ the subspace topologies on $\mathbb{C}_I$ of $\tau_s(\mathbb{H})$ and $\tau(\mathbb{H})$ coincide, i.e.

$$\tau_s(\mathbb{C}_I) = \tau(\mathbb{C}_I).$$

However, $\tau_s(\mathbb{H})$ is quite different from the Euclidean topology $\tau(\mathbb{H})$ near $\mathbb{R}$ as demonstrated by the following example.

**Example 3.2.** Fix $I \in \mathbb{S}$. We construct a slice-open $\Omega$ in $\mathbb{H}$ as

$$\Omega := \bigcup_{J \in \mathbb{S}} \Omega_J,$$

where

$$\Omega_J := \begin{cases} 
\{x + yJ \in \mathbb{C}_J : x^2 + \frac{y^2}{\text{dist}(J, \mathbb{C}_I)} < 1\}, & J \neq \pm I, \\
\{x + yJ \in \mathbb{C}_J : x^2 + y^2 < 1\}, & J = \pm I.
\end{cases}$$

Here $\text{dist}(J, \mathbb{C}_I)$ is the Euclidean distance from $J$ to $\mathbb{C}_I$.

By the construction, we know that $\Omega$ is slice-open. But $\Omega$ is not open in $\mathbb{H}$ since $0 \in \Omega$ and 0 is not in the Euclidean interior of $\Omega$. This is because $\Omega_J$ is an ellipse whose minor semi-axis $\sqrt{\text{dist}(J, \mathbb{C}_I)}$ tends to 0, when $J$ approaches $I$ with $J \neq \pm I$. 

To deal with the difficulties of the topology near $\mathbb{R}$, a new notion, called real-connectedness, comes up. This provides an effective tool since the slice topology has a real-connected subbase.

**Definition 3.3.** A subset $\Omega$ of $\mathbb{H}$ is called real-connected, if
\[ \Omega_\mathbb{R} := \Omega \cap \mathbb{R} \]
is connected in $\mathbb{R}$. In particular, when $\Omega \cap \mathbb{R} = \emptyset$, $\Omega$ is real-connected.

**Proposition 3.4.** For any slice-open set $\Omega$ in $\mathbb{H}$ and $q \in \Omega$, there is a real-connected st-domain $U \subset \Omega$ containing $q$.

**Proof.** We take $U$ to be the slice-connected component of the set
\[ (\Omega \setminus \Omega_\mathbb{R}) \cup A \]
containing $q$. Here when $q \in \mathbb{R}$, we take $A$ to be the connected component of $\Omega_\mathbb{R}$ containing $q$ in $\mathbb{R}$; otherwise, we set $A := \emptyset$.

It is easy to check that $q \in U$ and $U$ is a real-connected st-domain. \qed

Now we describe slice-connectedness by means of slice-paths.

**Definition 3.5.** A path $\gamma$ in $(\mathbb{H}, \tau)$ is said to be on a slice, if $\gamma \subset C_I$ for some $I \in S$.

**Proposition 3.6.** Every path on a slice is a slice-path.

**Proof.** It follows directly from the fact that $\tau_s(C_I \setminus \mathbb{R}) = \tau(C_I)$ for any $I \in S$. \qed

**Proposition 3.7.** Assume that an st-domain $U$ is real-connected.

(i) If $U_\mathbb{R} = \emptyset$, then $U \subset C_I$ for some $I \in S$.

(ii) If $U_\mathbb{R} \neq \emptyset$, then for any $q \in U$ and $x \in U_\mathbb{R}$, there exists a path on a slice from $q$ to $x$.

**Proof.** (i). If $U_\mathbb{R} = \emptyset$, then
\[ U \subset \bigcup_{J \in S} \mathbb{C}_J^+, \]
where
\[ \mathbb{C}_J^+ := \{x + yJ \in \mathbb{H} : y > 0\} \]
is a slice-open set in $\mathbb{H}$ for any $J \in S$. This means that
\[ U \subset \mathbb{C}_I^+ \]
for some $I \in S$ since $U$ is slice-connected,

(ii). We fix $q \in U$ and $x \in U_\mathbb{R}$. Take $I \in S$ such that $q \in \mathbb{C}_I$. Since $U$ is an st-domain in $\mathbb{H}$, by definition $U_I$ is an open set in the plane $\mathbb{C}_I$. Let $V$ be the connected component of $U_I$ containing $q$.

By definition we have that $\mathbb{C}_I \setminus \mathbb{R}$ and $\bigcup_{J \in S \setminus \{\pm I\}} (\mathbb{C}_J \setminus \mathbb{R})$ are slice-open. If $V_\mathbb{R} = \emptyset$, then
\[ V = U \cap (\mathbb{C}_I \setminus \mathbb{R}) \quad \text{and} \quad U \setminus V = U \cap \left[ \bigcup_{J \in S \setminus \{\pm I\}} (\mathbb{C}_J \setminus \mathbb{R}) \right] \]
are slice-open. Since $U$ is slice-connected and nonempty, it follows from
\[ U = V \bigcup (U \setminus V) \]
that \( V = U \). This implies \( U_\mathbb{R} = V_\mathbb{R} = \emptyset \), which is a contradiction. We thus conclude

\[ V_\mathbb{R} \neq \emptyset. \]

We take a point \( x_0 \in V_\mathbb{R} \). Since \( V \) is the connected component of \( U_I \) containing \( q \), there exists a path \( \alpha \) in \( V \) from \( q \) to \( x_0 \). Because \( U \) is real-connected, we have a path \( \beta \) in \( U_\mathbb{R} \) from \( x_0 \) to \( x \). It is clear that \( \alpha \beta \) is a path on a slice from \( q \) to \( x \). \( \square \)

**Corollary 3.8.** Assume that an st-domain \( U \) is real-connected.

(i) \( U_I \) is a domain in \( \mathbb{C}_I \) for any \( I \in S \).

(ii) For any \( p, q \in U \), there exists two paths \( \gamma_1, \gamma_2 \) such that each of them is a path on a slice in \( U \), \( \gamma_1(1) = \gamma_2(0) \), and \( \gamma_1 \gamma_2 \) is a slice-path from \( p \) to \( q \).

**Proof.** This follows directly from Proposition 3.7. \( \square \)

**Proposition 3.9.** The topological space \( (\mathbb{H}, \tau_s) \) is connected, locally path-connected and path-connected.

**Proof.** It follows from Proposition 3.4 and Corollary 3.8 (ii) that \( (\mathbb{H}, \tau_s) \) is locally path-connected. Since \( \mathbb{H} \cap \mathbb{C}_I = \mathbb{C}_I \supset \mathbb{R} \) for any \( I \in S \), we have \( (\mathbb{H}, \tau_s) \) is path-connected so that it is also connected. \( \square \)

**Corollary 3.10.** A set \( \Omega \subset \mathbb{H} \) is an st-domain if \( \Omega_\mathbb{R} \neq \emptyset \) and \( \Omega_I \) is a domain in \( \mathbb{C}_I \) for any \( I \in S \).

**Proof.** If \( \Omega_I \) is open for any \( I \in S \), then by definition \( \Omega \) is slice-open. Since \( \Omega_\mathbb{R} \neq \emptyset \), we can take a fixed point \( x \in \Omega_\mathbb{R} \). By hypothesis, \( \Omega_I \) is a domain in \( \mathbb{C}_I \) for any \( I \in S \), there exits a path on a slice from \( x \) to each point of \( \Omega \). It implies that \( \Omega \) is slice-path-connected so that it is also slice-connected. Thus \( \Omega \) is an st-domain. \( \square \)

**Remark 3.11.** By Corollary 3.10, any s-domain is an st-domain. Therefore the notion of st-domain is a generalization of the notion of s-domain.

However not every st-domain \( \Omega \) is an s-domain, even when \( \Omega \) is a domain in \( \mathbb{H} \), as we show in the following example.

**Example 3.12.** We fix \( I \in S \) and consider a domain in \( \mathbb{H} \), defined by

\[ \Omega := B(0,2) \cup B(6,2) \cup U, \]

where

\[ U := \{ q \in \mathbb{H} : \text{dist}(q - I, [0,6]) < \frac{1}{2} \}. \]

It is easy to check that

\[ \Omega_J = B_J(0,2) \cup B_J(6,2) \]

for any \( J \in S \) with \( J \perp I \). Hence \( \Omega_J \) is not connected in \( \mathbb{C}_J \) so that \( \Omega \) is not an s-domain. However \( \Omega \) is slice-connected, because any point in \( \Omega \) can be connected to 0 or 6 by a path in a slice, and 0 can be connected to 6 by a path in \( \mathbb{C}_I \). And since \( \Omega_J \) is open in \( \mathbb{C}_J \) for any \( J \in S \), \( \Omega \) is an st-domain.
4. Identity Principle

In this section we provide an identity principle for slice regular functions defined on st-domains. Since the st-domains satisfy conditions weaker than those one required by s-domains, the proof of the identity principle 2.10 is more difficult than the one for s-domains. We need to reduce the problem to the special case where the domain is real-connected.

**Lemma 4.1.** Assume that an st-domain \( \Omega \) is real-connected. Let \( f \) and \( g \) be two slice regular functions on \( \Omega \). If \( f \) and \( g \) coincide on a subset of \( \Omega_I \) with an accumulation point in \( \Omega_I \) for some \( I \in \mathcal{S} \), then \( f = g \) on \( \Omega \).

**Proof.** By assumption, we have \( \Omega_I \neq \emptyset \) so that Corollary 3.8 (i) implies \( \Omega_I \) is a non-empty domain in \( \mathbb{C}_I \). Therefore, using the Splitting Lemma and the identity principle for classical holomorphic functions of a complex variable, we deduce that \( f \) and \( g \) coincide on \( \Omega_I \).

If \( \Omega_R = \emptyset \), then \( \Omega = \Omega_I \) due to Proposition 3.7 (i) so that \( f = g \) on \( \Omega \).

Otherwise, we have \( \Omega_R \neq \emptyset \). By Corollary 3.8 (i), \( \Omega_J \) is a domain in \( \mathbb{C}_J \) for all \( J \in \mathcal{S} \). Since \( f = g \) on \( \Omega_R(\subset \Omega_I) \), it follows that \( f = g \) on \( \Omega_J \) for any \( J \in \mathcal{S} \). Consequently, \( f = g \) on \( \Omega = \bigcup_{J \in \mathcal{S}} \Omega_J \).

Now we can give the proof of the identity principle for st-domains.

**Proof of Theorem 2.10.** We consider the set \( A := \{ x \in \Omega : \exists V \in \tau_s(\Omega), \text{ s.t. } x \in V \text{ and } f = g \text{ on } V \} \).

By definition, \( A \) is a slice-open set in \( \Omega \).

Next, we come to show that \( A \) is nonempty. Due to Proposition 3.4, there exists a real-connected st-domain \( U \) such that it contains the accumulation point \( p \) and \( U \subset \Omega \). It follows from Lemma 4.1 applied to \( U \) that \( f = g \) on \( U \). This means that \( p \in \Omega \) so that \( A \) is nonempty.

Finally, we claim that \( \Omega \setminus A \) is slice-open. From this claim and the fact that \( \Omega \) is slice-connected, we conclude that \( A = \Omega \) so that \( f = g \) on \( \Omega \).

It remains to prove the claim. Let \( q \in \Omega \setminus A \) be arbitrary. From Proposition 3.4, there exists a real-connected st-domain \( V \) containing \( q \) with \( V \subset \Omega \). We have already know that both \( A \) and \( V \) are slice-open, so is \( A \cap V \).

If \( A \cap V \neq \emptyset \), then \( A \cap V \) is a non-empty slice-open. Since \( f = g \) on \( A \cap V \), it follows from Lemma 4.1 that \( f = g \) on \( V \). This means that \( q \in A \), a contradiction.

Therefore, we have \( A \cap V = \emptyset \).

This implies that \( q \) is a slice-interior point of \( \Omega \setminus A \). Hence \( \Omega \setminus A \) is slice-open. This proves the claim and finishes the proof.

5. Slice Functions

Slice functions play a fundamental role in the theory of slice regular function. The related stem theory for slice analysis has been established in the case of real alternative \( * \)-algebras [15]. See [18] for a recent development.

In this section, we give several equivalent characterization of slice functions. For convenience, we consider slice functions on an arbitrary domain of definition.
We remark that our definition of the slice function is a different form of the classical one.

**Definition 5.1.** Let $\Omega$ be an arbitrary set in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called a slice function if there is a function $F : \mathbb{R}^2 \to \mathbb{H}^{2 \times 1}$ such that

\begin{equation}
(5.1) \quad f(x + yI) = (1, I)F(x, y)
\end{equation}

for any $x + yI \in \Omega$ such that $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $y \geq 0$.

The function $F$ is referred to as an upper stem function of the slice function $f$.

We note that we are not requiring, at this stage, any condition on $F$ and since it is defined in $\mathbb{R}^2$, for $x + iy \notin \Omega$, we set $F(x, y) = (0, 0)^T$. Let us denote

\[ S^2_1 := \{(I, J) \in S^2 : I \neq J \}. \]

For any $(J, K) \in S^2_1$ we have the noteworthy identity

\begin{equation}
(5.2) \quad (J - K)^{-1}J = -K(J - K)^{-1}.
\end{equation}

From this, it is easy to check that

\begin{equation}
(5.3) \quad \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} = \begin{pmatrix} (J - K)^{-1}J & (K - J)^{-1}K \\ (J - K)^{-1} & (K - J)^{-1} \end{pmatrix}.
\end{equation}

**Proposition 5.2.** For any function $f : \Omega \to \mathbb{H}$ with $\Omega \subset \mathbb{H}$, the following statements are equivalent:

(i) The function $f$ is a slice function.

(ii) There exists a function $F : \mathbb{R}^2 \to \mathbb{H}^{2 \times 1}$ such that

\begin{equation}
(5.4) \quad f(x + yI) = (1, I)F(x, y)
\end{equation}

for any $x + yI \in \Omega$ with $x, y \in \mathbb{R}$ and any $I \in \mathbb{S}$.

(iii) If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in S^2_1$ such that $x + yL \in \Omega$ for $L = I, J, K$,

\begin{equation}
(5.5) \quad f(x + yI) = (1, I) \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} \begin{pmatrix} f(x) \\ f(y) \end{pmatrix}
\end{equation}

(iv) If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in S^2_1$ such that $x + yL \in \Omega$ for $L = I, J, K$,

\begin{equation}
(5.6) \quad f(x + yI) = (J - K)^{-1}[Jf(x) + yJ] - Kf(x + yK) + I(J - K)^{-1}[f(x) - f(x + yK)].
\end{equation}

(v) If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in S^2_1$ such that $x + yL \in \Omega$ for $L = I, J, K$,

\begin{equation}
(5.7) \quad f(x + yI) = (I - K)(J - K)^{-1}f(x + yJ) + (I - J)(K - J)^{-1}f(x + yK).
\end{equation}

**Proof.** It follows from (5.2) and (5.3) that assertions (iii), (iv), (v) are equivalent.

(i) $\Rightarrow$ (ii). If $f$ is a slice function, then there is a function $G = (G_1, G_2)^T : \mathbb{R}^2 \to \mathbb{H}^{2 \times 1}$ such that

\[ f(x + yI) = (1, I)G(x, y) \]

for any $x + yI \in \Omega$ with $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $y \geq 0$.

Hence we can take function $F : \mathbb{R}^2 \to \mathbb{H}^{2 \times 1}$ defined by

\[ F(x, y) := \begin{cases} (G_1, G_2)^T(x, y), & y \geq 0, \\ (G_1, -G_2)^T(x, -y), & y < 0. \end{cases} \]
Directly calculation shows that (5.4) holds.

(ii) ⇒ (iii). According to (5.4), we have

\[
\begin{pmatrix}
  f(x + yJ) \\
  f(x + yK)
\end{pmatrix} = \begin{pmatrix}
  1 & J \\
  1 & K
\end{pmatrix} F(x, y)
\]

for any \( x, y \in \mathbb{R} \) and \((J, K) \in S_2^* \). This implies that

\[
(5.8) \quad F(x, y) = \begin{pmatrix}
  1 \\
  1
\end{pmatrix}^{-1} \begin{pmatrix}
  f(x + yJ) \\
  f(x + yK)
\end{pmatrix}.
\]

Combining this with (5.4), we deduce that (5.5) holds.

(iii) ⇒ (i). We consider the sets

\[
A := \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + yS) \cap \Omega| = 1\}
\]
and

\[
B := \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + yS) \cap \Omega| > 1\},
\]

where we denoted by \(|S|\) the cardinality of the set \(S\).

If \((x, y) \in B\), then there are at least two distinct points in the set \((x + yS) \cap \Omega\). Therefore, the axiom of choice shows that we can choose \((J_{x,y}, K_{x,y}) \in S_2^* \) such that

\[
x + yJ_{x,y}, \quad x + yK_{x,y} \in (x + yS) \cap \Omega
\]

for any \((x, y) \in B\).

From this, we can construct a function \(G : B \to \mathbb{H}^2 \times 1\) defined by

\[
G(x, y) := \begin{pmatrix}
  1 & J_{x,y} \\
  1 & K_{x,y}
\end{pmatrix}^{-1} \begin{pmatrix}
  f(x + yJ_{x,y}) \\
  f(x + yK_{x,y})
\end{pmatrix}.
\]

Finally, we can define our desired function \(F : \mathbb{R}^2 \to \mathbb{H}^2 \times 1\) via

\[
F(x, y) := \begin{cases}
  (f(x + yI_{x,y}), 0)^T, & (x, y) \in A, \\
  G(x, y), & (x, y) \in B, \\
  (0, 0)^T, & \text{otherwise},
\end{cases}
\]

where \(I_{x,y}\) is the unique imaginary unit \(I \in S\) such that \(x + yI \in \Omega\) for \((x, y) \in A\).

It is easy to check that \(F\) satisfies (5.1) so that \(f\) is a slice function. \(\square\)

Remark 5.3. By Proposition 5.2, the classical representation formula in [4, Theorem 3.2] can be interpreted in the formalism of slice function. That is, any slice regular function defined on an axially symmetric s-domain is a slice function.

6. Extension Theorem

In [4, Theorem 4.2], the extension theorem is generalized from balls centered on the real axis to axially symmetric s-domains. In this section, we consider its further generalization to non necessarily axially-symmetric st-domains.

For any \(I = (I_1, I_2) \in S_2^*\), we set

\[
\tau[I] := \{(U, V) : U \in \tau(C_{I_1}) \text{ and } V \in \tau(C_{I_2})\}.
\]

Associated with

\[
U = (U_1, U_2) \in \tau[I], \quad \text{with} \quad I = (I_1, I_2) \in S_2^*,
\]

\[
J := J_{I_1} \quad \text{and} \quad K := J_{I_2}
\]

we consider the sets

\[
A := \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + yS) \cap \Omega| = 1\}
\]
and

\[
B := \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + yS) \cap \Omega| > 1\},
\]

where we denoted by \(|S|\) the cardinality of the set \(S\).

If \((x, y) \in B\), then there are at least two distinct points in the set \((x + yS) \cap \Omega\). Therefore, the axiom of choice shows that we can choose \((J_{x,y}, K_{x,y}) \in S_2^* \) such that

\[
x + yJ_{x,y}, \quad x + yK_{x,y} \in (x + yS) \cap \Omega
\]

for any \((x, y) \in B\).

From this, we can construct a function \(G : B \to \mathbb{H}^2 \times 1\) defined by

\[
G(x, y) := \begin{pmatrix}
  1 & J_{x,y} \\
  1 & K_{x,y}
\end{pmatrix}^{-1} \begin{pmatrix}
  f(x + yJ_{x,y}) \\
  f(x + yK_{x,y})
\end{pmatrix}.
\]

Finally, we can define our desired function \(F : \mathbb{R}^2 \to \mathbb{H}^2 \times 1\) via

\[
F(x, y) := \begin{cases}
  (f(x + yI_{x,y}), 0)^T, & (x, y) \in A, \\
  G(x, y), & (x, y) \in B, \\
  (0, 0)^T, & \text{otherwise},
\end{cases}
\]

where \(I_{x,y}\) is the unique imaginary unit \(I \in S\) such that \(x + yI \in \Omega\) for \((x, y) \in A\).

It is easy to check that \(F\) satisfies (5.1) so that \(f\) is a slice function. \(\square\)
we introduce the following three sets:
\[ U^+_s := (U_1 \cup C^+_1) \setminus (U_2 \cap C^+_2) \setminus (U_1 \cap U_2 \cap \mathbb{R}), \]
\[ U^+_s := \{ x + yS : (x + yI_1, x + yI_2) \in U, y \in \mathbb{R}, y \geq 0 \}, \]
\[ U^{s+} := U^+_s \cup U^+_s. \]

Sometimes we also replace \( U^{s+} \) by \( U^{s+}_s \) to emphasize its dependence on \( \mathbb{I} \).

**Lemma 6.1.** Let \( \mathbb{I} \in \mathbb{S}^2 \) be fixed, then \( U^{s+}_s \) is slice-open.

**Proof.** We need to show that any \( q \in U^{s+}_s \) is a slice-interior point of \( U^{s+}_s \).

Case 1: \( q \in U^{s+}_s \setminus \mathbb{R} \):

If \( q \in U^+_s \setminus \mathbb{R} \), then \( q \) is an interior point of \( U_1 \cap C^+_1 \) or \( U_2 \cap C^+_2 \). Hence \( q \) is a slice-interior point of \( U^+_s \) as well as \( U^{s+}_s \).

If \( q \in U^{s+}_s \setminus \mathbb{R} \), it can be expressed as
\[ q = x + yJ \]
for some \( J \in \mathbb{S}, x, y \in \mathbb{R} \) with \( y > 0 \). By definition of \( U^+_s \),
\[ x + yI_1 \in U_1 \cap C^+_1, \quad x + yI_2 \in U_2 \cap C^+_2. \]
Hence there exists an \( r \in \mathbb{R}^+ \) such that
\[ B_1(x + yI_1, r) \subset U_1 \cap C^+_1, \quad B_2(x + yI_2, r) \subset U_2 \cap C^+_2. \]
This means
\[ B_J(x + yJ, r) \subset U^+_s \]
so that \( q \) is a slice-interior of \( U^{s+}_s \).

Case 2: \( q \in U^{s+}_s \cap \mathbb{R} \):

It is easy to check that
\[ U^{s+}_s \cap \mathbb{R} = U_1 \cap U_2 \cap \mathbb{R}. \]
Since \( q \in U^{s+}_s \cap \mathbb{R} \), there exists an \( r \in \mathbb{R}^+ \) such that
\[ B_1(q, r) \subset U_1, \quad B_2(q, r) \subset U_2, \]
which implies, by definition, that
\[ B_J(q, r) \subset U^+_s \]
for any \( J \in \mathbb{S} \). Hence \( B(q, r) \subset U^{s+}_s \).

**Theorem 6.2.** Let \( \mathbb{I} \in \mathbb{S}^2 \) and \( U = (U_1, U_2) \in \mathcal{J}[\mathbb{I}] \). If \( f : U_1 \cup U_2 \to \mathbb{H} \) is a function such that \( f|U_1 \) and \( f|U_2 \) are both holomorphic, then the function \( f|U^+_s \) admits a slice regular extension \( \tilde{f} \) to \( U^{s+}_s \).

Moreover, if \( W \) is an st-domain such that
\[ W \subset U^{s+}_s, \quad W \cap U^{s+}_s \neq \emptyset, \]
then \( \tilde{f}|_W \) is a slice function and it is the unique slice regular extension on \( W \) of \( f|_W \cap U^+_s \).
Proof. Define a function $g : \mathbb{U}_s^\Delta \to \mathbb{H}$ by

\[
(6.1) \quad g(x + yJ) := (J - I_2)(I_1 - I_2)^{-1}f(x + yI_1) + (J - I_1)(I_2 - I_1)^{-1}f(x + yI_2)
\]

for any $J \in \mathbb{S}$, $x, y \in \mathbb{R}$ with $y \geq 0$ and $x + yI_\lambda \in U_\lambda$, $\lambda = 1, 2$.

By direct calculation (see the proof of [4, Theorem 3.2]), we find that $g$ is slice regular on $\mathbb{U}_s^\Delta$ and $g = f$ on $\mathbb{U}_s^\Delta \cap \mathbb{U}_s^+$. Hence the function $\tilde{f} : \mathbb{U}_s^+ \to \mathbb{H}$, defined by

\[
(6.2) \quad \tilde{f} := \begin{cases} 
  g, & \text{on } \mathbb{U}_s^\Delta; \\
  f, & \text{on } \mathbb{U}_s^+,
\end{cases}
\]

is a slice regular extension of $f|_{\mathbb{U}_s^+}$.

If $h : W \to \mathbb{H}$ is a slice regular extension of $f|_{W \cap \mathbb{U}_s^+}$, then we have

\[
h = f = \tilde{f} \quad \text{on } W \cap \mathbb{U}_s^+
\]

so that the identity Principle 2.10 implies

\[
h = \tilde{f}|_W.
\]

Consequently, $\tilde{f}|_W$ is the unique slice regular extension on $W$ of $f|_{W \cap \mathbb{U}_s^+}$.

By (5.3), (6.2) and directly calculation, we rewrite (6.1) by

\[
\tilde{f}(x + yJ) = (1, J)F_{x,y}
\]

for any $x, y \in \mathbb{R}$ and $J \in \mathbb{S}$ with $y \geq 0$ and $x + yK \in W$, $K = J, I_1, I_2$, where

\[
F_{x,y} = \begin{pmatrix} I_1 & 1 \\ 1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} f(x + yI_1) \\ f(x + yI_2) \end{pmatrix}.
\]

Now we can introduce a function $G : \mathbb{R}^2 \to \mathbb{H}$ defined by

\[
G(x, y) := \begin{cases} 
  F_{x,y}, & x + yI_1, x + yI_2 \in W \\
  (f(x + yI_1), 0)^T, & x + yI_1 \in W \text{ and } x + yI_2 \notin W, \\
  (f(x + yI_2), 0)^T, & x + yI_1 \notin W \text{ and } x + yI_2 \in W, \\
  (0, 0)^T, & \text{otherwise},
\end{cases}
\]

It is easy to show that $G$ is a upper stem function of $f|_W$. This means that $\tilde{f}$ is slice on $W$ by definition. \qed

Corollary 6.3. If $f : B_1(q, r) \to \mathbb{H}$ is a holomorphic function with $I \in \mathbb{S}$, $q \in \mathbb{C}_I$ and $r \in \mathbb{R}_+$, then it can be uniquely extended to be a slice regular function over the $\sigma$-ball $\Sigma(q, r)$.

Proof. Case 1: $B_1(q, r) \cap \mathbb{R} = \emptyset$.

In this case, we have

\[
B_1(q, r) = \Sigma(q, r)
\]

so that $f = \tilde{f}$ is the unique slice regular extension of itself.

Case 2: $B_1(q, r) \cap \mathbb{R} \neq \emptyset$.

Now we take

\[
I := (I, -I) \in \mathbb{S}_s^2, \quad U := (B_1(q, r), B_1(q, r)) \in \tau(\mathbb{I}).
\]

It is easy to see

\[
\mathbb{U}_s^+ \Delta = \Sigma(q, r),
\]
which is an st-domain. By Proposition 6.2, \( f \) admits a unique slice regular extension \( \tilde{f} \) on \( \Sigma(q,r) \).

7. Path-slice functions and representation formula

In this section we extend the representation formula from axially symmetric domains to non-axially-symmetric domains. To this end, we introduce the new notion of path-slice functions. It turns out that any slice regular function on a slice-open set is path-slice, see Theorem 7.4. We can also prove the representation formula for path-slice functions.

We denote by \( \mathcal{P}(\mathbb{C}) \) the set of paths \( \gamma : [0,1] \to \mathbb{C} \) with initial point \( \gamma(0) \) in \( \mathbb{R} \) and we consider its subset \( \mathcal{P}(\mathbb{C}^+) := \{ \gamma \in \mathcal{P}(\mathbb{C}) : \gamma(0,1] \subset \mathbb{C}^+ \} \).

**Definition 7.1.** A function \( f : \Omega \to \mathbb{H} \) with \( \Omega \subset \mathbb{H} \) is called path-slice function if for any \( \gamma \in \mathcal{P}(\mathbb{C}) \), there is a function \( F_\gamma : [0,1] \to \mathbb{H}^{2 \times 1} \) such that

\[
(7.1) \quad f \circ \gamma^I = (1,I)F_\gamma
\]

for any \( I \in \mathbb{S} \) with \( \gamma^I \subset \Omega \).

We call \( \{ F_\gamma \}_{\gamma \in \mathcal{P}(\mathbb{C})} \) a (path-)stem system of the path-slice function \( f \).

Now, we provide equivalent characterizations for path-slice functions.

**Proposition 7.2.** For any function \( f : \Omega \to \mathbb{H} \) with \( \Omega \subset \mathbb{H} \), the following statements are equivalent:

(i) \( f \) is a path-slice function.

(ii) For any \( \gamma \in \mathcal{P}(\mathbb{C}) \), there is an element \( q_\gamma \in \mathbb{H}^{2 \times 1} \) such that

\[
(7.2) \quad f \circ \gamma^I = (1,I)q_\gamma
\]

for any \( I \in \mathbb{S} \) with \( \gamma^I \subset \Omega \).

(iii) For any \( \gamma \in \mathcal{P}(\mathbb{C}^+) \), there is an element \( p_\gamma \in \mathbb{H}^{2 \times 1} \) such that

\[
(7.3) \quad f \circ \gamma^I = (1,I)p_\gamma
\]

for any \( I \in \mathbb{S} \) with \( \gamma^I \subset \Omega \).

(iv) For any \( \gamma \in \mathcal{P}(\mathbb{C}) \) and \( I, J, K \in \mathbb{S} \) with \( J \neq K \) and \( \gamma^I, \gamma^J, \gamma^K \subset \Omega \), we have

\[
(7.4) \quad f \circ \gamma^I = (1,I) \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} \begin{pmatrix} f \circ \gamma^J \\ f \circ \gamma^K \end{pmatrix}.
\]

(v) For any \( \gamma \in \mathcal{P}(\mathbb{C}) \) and \( I, J, K \in \mathbb{S} \) with \( J \neq K \) and \( \gamma^I, \gamma^J, \gamma^K \subset \Omega \), we have

\[
f \circ \gamma^I = (J - K)^{-1}(Jf \circ \gamma^J - Kf \circ \gamma^K) + I(J - K)^{-1}(f \circ \gamma^J - f \circ \gamma^K).
\]

(vi) For any \( \gamma \in \mathcal{P}(\mathbb{C}) \) and \( I, J, K \in \mathbb{S} \) with \( J \neq K \) and \( \gamma^I, \gamma^J, \gamma^K \subset \Omega \), we have

\[
f \circ \gamma^I = (I - K)(J - K)^{-1}f \circ \gamma^J + (I - J)(K - J)^{-1}f \circ \gamma^K.
\]

**Proof.** From (5.2) and (5.3), one can deduce that assertions (iv), (v), and (vi) are equivalent.
(i) ⇒ (iv). Suppose that \( f \) is a path-slice function and let \( \{ F_\gamma \}_{\gamma \in \mathcal{P}(C)} \) be its stem system. By (7.1) it follows that

\[
(7.5) \quad \begin{pmatrix} f \circ \gamma^J \\ f \circ \gamma^K \end{pmatrix} = \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix} F_\gamma
\]

for any \( \gamma \in \mathcal{P}(C) \) and \( I, J, K \in \mathbb{S} \) with \( J \neq K \) and \( \gamma^J, \gamma^K \subset \Omega \). It follows from (7.1) and (7.5) that (7.4) holds.

(iv) ⇒ (iii) Suppose (iv) holds. We consider two sets

\[
(7.6) \quad A := \{ \gamma \in \mathcal{P}(C^+) : |\{ I \in \mathbb{S} : \gamma^I \subset \Omega \}| = 1 \}
\]

and

\[
(7.7) \quad B := \{ \gamma \in \mathcal{P}(C^+) : |\{ I \in \mathbb{S} : \gamma^I \subset \Omega \}| > 1 \}.
\]

By the axiom of choice, there is \((J_\gamma, K_\gamma) \in \mathbb{S}_2^* \) such that \( \gamma^{J_\gamma}, \gamma^{K_\gamma} \subset \Omega \) for any \( \gamma \in B \). We denote by \( I_\gamma \), the unique imaginary unit in \( \mathbb{S} \) such that \( \gamma^{I_\gamma} \in \Omega \) for any \( \gamma \in A \).

For any \( \gamma \in \mathcal{P}(C^+) \), we pick

\[
p_\gamma := \begin{cases} (f \circ \gamma^J, 0), & \gamma \in A, \\ \begin{pmatrix} 1 & J_\gamma \\ 1 & K_\gamma \end{pmatrix}^{-1} \begin{pmatrix} f \circ \gamma^{J_\gamma}(1) \\ f \circ \gamma^{K_\gamma}(1) \end{pmatrix}, & \gamma \in B, \\ 0, & \text{otherwise}, \end{cases}
\]

It is immediate to verify (7.3) holds.

(iii) ⇒ (ii) Let \( \gamma \in \mathcal{P}(C) \) be arbitrary. We define

\[
s := \max\{ t \in [0, 1] : \gamma(t) \in \mathbb{R} \}
\]

and construct the path \( \delta : [0, 1] \to \mathbb{C} \) defined by

\[
\delta(t) := \begin{cases} \gamma(1), & \gamma(1) \in \mathbb{R}, \\ \gamma((1-s)t + s), & \gamma(1) \in \mathbb{C}^+_f, \\ \gamma((1-s)t + s), & \text{otherwise}. \end{cases}
\]

By construction

\( \delta \in \mathcal{P}(\mathbb{C}^+) \),

moreover, if \( \gamma^I \subset \Omega \) for some \( I \in \mathbb{S} \), then

\( \delta^I \subset \Omega \),

where

\[
(7.8) \quad \epsilon := \begin{cases} 1, & \gamma(1) \in \mathbb{C}^+_f, \\ -1, & \text{otherwise}. \end{cases}
\]

We take

\[
q_\gamma := \begin{cases} p_\delta, & |\{ I \in \mathbb{S} : \gamma^I \subset \Omega \}| \geq 1, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( p_\delta \) is an element satisfying (7.3), i.e.,

\[
f \circ \delta^I(1) = (1, I)p_\delta
\]

for any \( I \in \mathbb{S} \) with \( \delta^I \subset \Omega \).

Obviously, \( q_\gamma \) satisfies (7.2) so that (ii) holds.
Let $\gamma \in \mathcal{P}(\mathbb{C})$ be arbitrary and fix a point $t \in [0,1]$. We consider the path $\delta : [0,1] \to \mathbb{C}$, defined by
$$\delta(s) := \gamma(ts).$$
Then $\delta$ is a path from $\gamma(0)$ to $\gamma(t)$ such that $\delta \in \mathcal{P}(\mathbb{C})$.

Let $q_\delta$ be an element satisfying (7.2), i.e.,
$$f \circ \delta^I(1) = (1, I) q_\delta$$
for any $I \in S$ with $\delta^I \subset \Omega$.

Now we can define a function $F_\gamma : [0,1] \to \mathbb{H}^{2 \times 1}$ via
$$F_\gamma(t) := \begin{cases} q_\delta, & \exists I \in S \text{ s.t. } \gamma^I \subset \Omega, \\ 0, & \text{otherwise}. \end{cases}$$
We remark that, by construction, the path $\delta$ depends on the parameter $t$.

It is direct to verify that
$$f \circ \gamma^I = (1, I) F_\gamma$$
for any $I \in S$ with $\gamma^I \subset \Omega$. This means that $f$ is path-slice since $\gamma$ is arbitrary. \hfill $\square$

**Proposition 7.3.** Every slice function is a path-slice function.

**Proof.** If $f$ is a slice function, then (5.4) holds. If we set
$$\gamma^I(t) := x(t) + y(t) I,$$
it is clear that (7.2) follows from (5.4). This implies $f$ is path-slice. \hfill $\square$

**Theorem 7.4.** Every slice regular function on a slice-open set is path-slice.

**Proof.** Let $\Omega$ be a slice-open set and $f : \Omega \to \mathbb{H}$ be a slice regular function. We show that $f$ is path-slice. To this end, we only need to verify (7.3), by Proposition 7.2, namely we need to choose $p_\gamma \in \mathbb{H}^{2 \times 1}$ such that
$$f \circ \gamma^I(1) = (1, I) p_\gamma$$
for any $\gamma \in \mathcal{P}(\mathbb{C}^+)$ and $I \in S$ with $\gamma^I \subset \Omega$. We have to treat three cases.

Case 1: Let $B$ be as in (7.7) and $\gamma \in B$.

In virtue of (7.7), there exist $J, K \in S$ such that
$$J \neq K, \quad \gamma^J, \gamma^K \subset \Omega.$$

Take $U_J$ and $U_K$ such that
$$\gamma^J \subset U_J, \quad \gamma^K \subset U_K,$$ and $U_J$ is a domain in $\Omega_J$ and $U_K$ is a domain in $\Omega_K$.

Let us set $\mathcal{J} = (J,K)$ and $\mathcal{U} = (U_J, U_K)$. We consider the function
$$g = f|_{\mathcal{U}}.$$ This function satisfies the conditions in Extension Formula 6.2. Therefore, $g|_{\mathcal{U}^+}$ has a slice regular extension $\widetilde{g}$ over the slice-connected component $W$ of $\mathcal{U}_s^+ \Delta \cap \Omega$ containing $\gamma(0)$. By the Identity Principle, see Theorem 2.10, we have
$$f = \widetilde{g} \text{ on } W.$$

Since $\widetilde{g}$ is slice on $W$, it follows that $f$ is slice on $W$.

Recall that $\gamma \in \mathcal{P}(\mathbb{C}^+)$. By construction we have
$$\gamma^J, \gamma^K \subset \mathcal{U}_s^+.$$
This implies that for any \( L \in S \)
\[ \gamma^L \subset U^+_{s,j}. \]

Then for any \( I \in S \) with \( \gamma^I \subset \Omega \),
\[ \gamma^I \subset U^+_{s,j} \cap \Omega. \]

Since \( \gamma^I(0) \in W \) and \( W \) is a slice-connected component of \( U^+_{s,j} \cap \Omega \), we thus conclude
\[ \gamma^I \subset W. \]

Due to the fact that \( f \) is slice on \( W \), Proposition 5.2 (ii) implies that there exists a function \( F_\gamma : \mathbb{R}^2 \to \mathbb{H}^2 \) such that
\[ f(x_\gamma + y_\gamma I) = (1, I) F_\gamma(x_\gamma, y_\gamma) \]
for any \( I \in S \) with \( \gamma^I \subset \Omega \), where we have written
\[ \gamma(1) = x_\gamma + y_\gamma i \]
for some \( x_\gamma, y_\gamma \in \mathbb{R} \).

Finally, we set
\[ p_\gamma := (f \circ \gamma^I(1), 0)^T, \quad \gamma \in \mathcal{A}. \]

Case 2. Let \( \mathcal{A} \) be as in (7.6) and \( \gamma \in \mathcal{A} \).

In this case, we take
\[ p_\gamma := (0,0)^T. \]

With the choice of \( p_\gamma \) above, it is clear that \( p_\gamma \) satisfies (7.3) as desired. □

**Proof of Theorem 2.11.** It is a direct consequence of Proposition 7.2 and Theorem 7.4. □

**Proposition 7.5.** The set of slice functions and the set of path-slice functions on axially symmetric path-slice-connected set which intersects with \( \mathbb{R} \) contain the same elements.

**Proof.** Let \( \Omega \) be an axially symmetric path-slice-connected set with \( \Omega \cap \mathbb{R} \neq \emptyset \). According to Proposition 7.3, we just need to prove that any path-slice function on \( \Omega \) is slice. Let \( f : \Omega \to \mathbb{H} \) be a path-slice function and let us prove that \( f \) is slice. Since \( \Omega \) is path-slice-connected, for any \( z \in \Omega \cap \mathbb{R} \) and \( q \in \Omega \), there is a slice-path \( \alpha \) from \( z \) to \( q \).

We write \( q = x + yI \) for some \( x, y \in \mathbb{R} \) and \( I \in S \). Since
\[ \mathbb{H} \setminus \mathbb{C} := \bigcup_{J \in S \setminus \{\pm I\}} \mathbb{C} \]
is slice-open, the preimage \( \alpha^{-1}(\mathbb{C}_I) \) is closed in \([0, 1]\). We write by \([t, 1]\) the slice-connected component of \( \alpha^{-1}(\mathbb{C}_I) \) containing \( 1 \) for some \( t \in [0, 1] \).

We consider the path \( \gamma : [0, 1] \to \mathbb{C} \), defined by
\[ \gamma(s) := P_I^{-1} \circ \alpha(t + (1 - t)s). \]
It is clear that $\gamma$ is in $\mathcal{P}(\mathbb{C})$ and $\gamma^J$ is a path from $\alpha(t)$ to $q$.

Since $\Omega$ is axially symmetric, we have

$$\gamma^J \subset \Omega$$

for any $J \in \mathbb{S}$. Since

$$\gamma^J(1) = x + yJ, \quad \forall \ J \in \mathbb{S},$$

it follows from Proposition 7.2 (iv) that

$$f(x + yL) = (1, L)^{-1} f(x + yJ) \mathbf{1} f(x + yK)$$

for any $L, J, K \in \mathbb{S}$ with $J \neq K$. This implies that $f$ is slice by Proposition 5.2 (iii).

**Remark 7.6.** Suppose that $\Omega$ in Theorem 2.11 is an axially symmetric $s$-domain in $\mathbb{H}$. For any $q = x + yI \in \Omega$, there exists a point $p \in \Omega$ and a path $\gamma$ in $\mathbb{C}$ such that $\gamma^I$ is a path from $p$ to $q$. Since $\Omega$ is axially symmetric, we know that $\gamma^K \subset \Omega$ and $\gamma^K(1) = x + yK$ for all $K \in \mathbb{S}$. By Theorem 2.11, we have

$$f(x + yI) = (I - K)(J - K)^{-1} f(x + yJ) + (I - J)(K - J)^{-1} f(x + yK)$$

for any $J, K \in \mathbb{S}$ with $J \neq K$. This means that Theorem 2.11 recovers the classical representation formula [4, Theorem 3.2].

## 8. Counterexample on non-axially symmetric domains

In this section, we give an example to illustrate that the classical representation formula may not hold for non-axially symmetric domains.

Let $s \in [0, 1]$ be fixed. Define a ray $\gamma_s : [0, 1) \rightarrow \mathbb{C}$ by

$$\gamma_s(t) := \frac{i}{2} + \frac{t}{1-t} e^{(\frac{\pi}{4} + \frac{s\pi}{2})}. $$

Geometrically, the ray starts from $\frac{i}{2}$ to $\infty$ and the angle between the ray and the positive real axis is $\frac{\pi}{4} + s\frac{\pi}{2}$.

For any continuous function $\varphi : \mathbb{S} \rightarrow [0, 1]$,

we define a continuous function $F : \mathbb{S} \times [0, 1) \rightarrow \mathbb{H}$ by

$$F(I, s) = \mathcal{P}_I \circ \gamma_{\varphi(I)}(s).$$

The complement of the image of $F$ is denoted by

$$\Omega_{\varphi} := \mathbb{H} \setminus F(\mathbb{S} \times [0, 1)).$$

**Proposition 8.1.** The set $\Omega_{\varphi}$ is a $s$-domain and

$$\Omega_{\varphi} \cap \mathbb{S} = \mathbb{S} \setminus \varphi^{-1}(\frac{1}{2}).$$

**Proof.** (i) For any $I \in \mathbb{S}$, we denote

$$\gamma_{\varphi}[I] := \mathcal{P}_I \circ \gamma_{\varphi(I)}([0, 1)).$$

Then $\gamma_{\varphi}[I]$ is an image of a ray in $\mathbb{C}_I$ from $\frac{i}{2}$ to $\infty$. And the angle between the ray and the positive real axis is

$$\frac{\pi}{4} + \varphi(I) \frac{\pi}{2}.$$
By definition,
\[ F(S \times [0, 1]) = \bigcup_{I \in S} \gamma_\varphi[I] \]
and
\( (8.1) \quad \Omega_\varphi = \mathbb{H} \setminus \bigcup_{I \in S} \gamma_\varphi[I]. \)

Since
\[ P_I \circ \gamma_\varphi(I)(t) := \frac{I}{2} + \frac{t}{1 - t} e^{\varphi(I)t/2 + \pi t}, \quad \forall \ t \in [0, 1), \]
we have
\[ \bigcup_{J \in S} B(\frac{J}{2}, \frac{t}{1 - t}) \cap P_I \circ \gamma_\varphi(I)[0, 1) = P_I \circ \gamma_\varphi(I) [0, t) \]
for any \( t \in (0, 1) \) and \( I \in S \). Taking the union for all \( I \in S \), we get
\( (8.2) \quad \bigcup_{J \in S} B(\frac{J}{2}, \frac{t}{1 - t}) \cap F(S \times [0, 1)) = F(S \times [0, t)). \)

Denote
\[ A_t := \bigcup_{J \in S} B(\frac{J}{2}, \frac{t}{1 - t}) \cap (\mathbb{H} \setminus F(S \times [0, t])). \]

Since \( F \) is continuous, we have \( F(S \times [0, t]) \) is compact so that \( A_t \) is open.

By virtue of (8.2), we have
\[ A_t = \bigcup_{J \in S} (B(\frac{J}{2}, \frac{t}{1 - t}) \setminus F(S \times [0, t])) \]
and
\[ \bigcup_{t \in (0, 1)} A_t = \bigcup_{t \in [0, 1)} \left( \bigcup_{J \in S} (B(\frac{J}{2}, \frac{t}{1 - t}) \setminus F(S \times [0, t])) \right) \]
\[ = \left( \bigcup_{t \in (0, 1)} \bigcup_{J \in S} (B(\frac{J}{2}, \frac{t}{1 - t}) \setminus F(S \times [0, 1)) \right) \]
This means that \( \Omega_\varphi \) is open.

Note that \((\Omega_\varphi) \cap \mathbb{C}_I = \mathbb{C}_I \) deleting two rays. One is emitting from \( I/2 \) lying in the upper space \( \mathbb{C}_I^+ \), while the other one emitting from \(-I/2 \) lying in the lower space \( \mathbb{C}_I^- := \mathbb{C}_I^+. \) Therefore, \((\Omega_\varphi)_I \) is a domain in \( \mathbb{C}_I \) and path-connected. And since \( \Omega_\varphi \cap \mathbb{R} = \mathbb{R}, \) \( \Omega_\varphi \) is a s-domain.

(ii) Note that for any \( I \in S, \) \( I \in \gamma_\varphi[I] \) if and only if \( \varphi(I) = \frac{1}{2} \). It follows that
\[ \Omega_\varphi \cap S = S \backslash \varphi^{-1}(\frac{1}{2}). \]

\( \square \)

We proceed to construct a desired function over \( \Omega_\varphi \). To this end, we first consider this function in the complex plane \( \mathbb{C}_J \), where \( J \in S \) is fixed.

The classical theory of holomorphic functions shows that the function
\( (8.3) \quad \Psi(z) = \sqrt{2z - J}, \quad \forall \ z \in \frac{J}{2} + \mathbb{R}_+ \),
admits a unique holomorphic extension $\Psi_s$ on
$$C_J \setminus (\gamma_s[J] \cup \gamma_s[-J]),$$
where
$$\gamma_s[J] \coloneqq \mathcal{P}_J \circ \gamma_s([0,1))$$
for any $s \in [0,1]$.

**Remark 8.2.** The function $\Psi_s$ has the following properties.

(i) For any $s, t \in [0,1]$, 
$$\Psi_s|_R = \Psi_t|_R.$$ 

(ii) For any $s \in [0,1]$, we have
$$\Psi_s(-J) = \sqrt{3}e^{-\frac{\pi}{4}}$$
and
$$\Psi_s(J) = \begin{cases} 
- e^{\frac{\pi}{4}}, & s \in [0, \frac{1}{2}], \\
 e^{\frac{\pi}{4}}, & s \in (\frac{1}{2}, 1]. 
\end{cases}$$

(iii) For any $s \in [0,1]$, denote $\alpha := \frac{\pi}{2} + \frac{s\pi}{2}$. Then for any $\lambda \in \mathbb{R}_+$
$$\lim_{\theta \to \alpha^-} \Psi_s\left(\frac{J}{2} + \lambda e^{\theta J}\right) = e^{-\frac{\pi}{4}}$$
and
$$\lim_{\theta \to \alpha^+} \Psi_s\left(\frac{J}{2} + \lambda e^{\theta J}\right) = -e^{-\frac{\pi}{4}}.$$ 

This implies that $\Psi_s$ cannot be extended continuously to any point in $\gamma_s[J]$.

**Proposition 8.3.** Let $\varphi : \mathbb{S} \to [0,1]$ be a continuous function and $\Psi$ is as in (8.3). Then the function $\Psi_\varphi : \Omega_\varphi \to \mathbb{H}$ defined by
\begin{equation}
\Psi_\varphi(x + yI) := \frac{1 - IJ}{2} \Psi_{\varphi(I)}(x + yJ) + \frac{1 + IJ}{2} \Psi_{\varphi(I)}(x - yJ),
\end{equation}
for $y \geq 0$, is the unique slice regular extension of $\Psi$ to $\Omega_\varphi$. In particular,
$$(\Psi_\varphi)_J = \Psi_{\varphi(J)}.$$ 

**Proof.** By direct calculation, $(\Psi_\varphi)_I$ is a holomorphic extension of $\Psi_{\varphi(I)}|_R$. And $\Psi_\varphi$ is well-defined by Remark 8.2 (i). It is clear that $\Psi_\varphi$ is the unique slice regular extension of $\Psi$. \qed

**Remark 8.4.** Consider $\varphi(K) = \frac{1}{2}|K - J|$ and choose $I \in \mathbb{S}$ such that $\frac{1}{2} < \varphi(I) < 1$. Then by Remark 8.2 (ii),
$$\Psi_{\varphi}(J) = \Psi_{\varphi(I)}(J).$$

It is easy to check from (8.4) that the equality
\begin{equation}
\Psi_{\varphi}(I) = \frac{1 - IJ}{2} \Psi_{\varphi}(J) + \frac{1 + IJ}{2} \Psi_{\varphi}(J),
\end{equation}
holds if and only if
$$\Psi_{\varphi}(J) = \Psi_{\varphi(I)}(J).$$

By Remark 8.2 (ii), to say that (8.4) is in force is equivalent to
$$(\varphi(I) - \frac{1}{2})(\varphi(J) - \frac{1}{2}) > 0.$$
The fact that (8.5) does not hold since \( \varphi(I) > \frac{1}{2} \) and \( \varphi(J) = 0 \) implies that (7.12) does not hold for non-axially symmetric domains.

**Definition 8.5.** Let \( \Omega \subset \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is called slice-Euclidean continuous if for any \( U \in \tau(\mathbb{H}) \), the preimage \( f^{-1}(U) \) is slice-open. In other words, \( f : (\Omega, \tau_s(\mathbb{H})) \to (\mathbb{H}, \tau(\mathbb{H})) \) is continuous.

**Proposition 8.6.** Let \( \Omega \subset \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is slice-Euclidean continuous if and only if for any \( I \in \mathbb{S} \), \( f_I \) is continuous.

**Proof.** Let \( I \in \mathbb{S} \) and \( U \in \tau(\mathbb{H}) \). If \( f_I \) is continuous, then \( (f_I)^{-1}(U) \) is open in \( \mathbb{C}_I \). Hence

\[
f^{-1}(U) = \bigcup_{I \in \mathbb{S}} (f_I)^{-1}(U)
\]

is slice-open.

Conversely, if \( f : \Omega \to \mathbb{H} \) is slice-Euclidean continuous, then for any \( U \in \tau(\mathbb{H}) \), \( f^{-1}(U) \in \tau_s(\mathbb{H}) \). This means \( (f|_{\mathbb{C}_I})^{-1}(U) \) is open in \( \mathbb{C}_I \) for any \( I \in \mathbb{S} \). Therefore, \( f_I \) is continuous. \( \square \)

**Proposition 8.7.** Every slice regular function is slice-Euclidean continuous.

**Proof.** This follows directly from Proposition 8.6. \( \square \)

**Proposition 8.8.** Let be \( \Psi_\varphi \) as in (8.4). For any continuous function \( \varphi : \mathbb{S} \to [0,1] \), there is a unique slice regular extension \( \widehat{\Psi}_\varphi \) of \( \Psi_\varphi \) on

\[
\hat{\Omega}_\varphi := \Omega_\varphi \bigcup \gamma_\varphi[-J].
\]

Moreover, \( \widehat{\Psi}_\varphi \) can not be extended slice-Euclidean continuously to any point in \( \mathbb{H} \setminus \hat{\Omega}_\varphi \).

**Proof.** Note that \( \Psi_\varphi(q) = \sqrt{2q - J} \) for any \( q \in \frac{I}{2} + \mathbb{R}_+ \). Form complex analysis, we know that \( \Psi_\varphi \) can be extended slice regularly to \( \gamma_\varphi[-J] \). This extension, denoted by \( \widehat{\Psi}_\varphi \), is unique by Identity Principle 2.10. For any \( \lambda \in \mathbb{R}_+ \), we have

\[
\lim_{\theta \to \beta} \Psi_\varphi(\frac{J}{2} + \lambda e^{-J\theta}) = \widehat{\Psi}_\varphi(\frac{J}{2} + \lambda e^{-J\beta})
\]

where

\[
\beta := \frac{\pi}{2} + \frac{\varphi(-J)\pi}{2}.
\]

For any \( I \in \mathbb{S} \setminus \{J\} \), denote

\[
\alpha := \frac{\pi}{2} + \frac{\varphi(I)\pi}{2}.
\]

It follows from (8.4) and (8.6) that for any \( \lambda \in \mathbb{R}_+ \)

\[
\lim_{\theta \to \alpha^-} \Psi_\varphi(\frac{J}{2} + \lambda e^{I\theta}) = \frac{1 - IJ}{2} \lim_{\theta \to \alpha^-} \Psi_\varphi(\frac{J}{2} + \lambda e^{I\theta}) + \frac{1 + IJ}{2} \widehat{\Psi}_\varphi(\frac{J}{2} + \lambda e^{-J\alpha})
\]

and

\[
\lim_{\theta \to \alpha^+} \Psi_\varphi(\frac{J}{2} + \lambda e^{I\theta}) = \frac{1 - IJ}{2} \lim_{\theta \to \alpha^+} \Psi_\varphi(\frac{J}{2} + \lambda e^{I\theta}) + \frac{1 + IJ}{2} \widehat{\Psi}_\varphi(\frac{J}{2} + \lambda e^{-J\alpha}).
\]

By Remark 8.2 (iii), we find

\[
\lim_{\theta \to \alpha^-} \widehat{\Psi}_\varphi(\frac{J}{2} + \lambda e^{I\theta}) \neq \lim_{\theta \to \alpha^+} \widehat{\Psi}_\varphi(\frac{J}{2} + \lambda e^{I\theta}).
\]
And according to Proposition 8.6, \( \widehat{\Psi}_\varphi \) cannot be extended slice-Euclidean continuously to any point in \( \gamma_\varphi[I] \). Since 
\[
\bigcup_{I \in \mathbb{S} \setminus \{-J\}} \gamma_\varphi[I] = \mathbb{H} \setminus \widehat{\Omega}_\varphi,
\]
it follows that \( \widehat{\Psi}_\varphi \) cannot be extended slice-Euclidean continuously to any point in \( \mathbb{H} \setminus \widehat{\Omega}_\varphi \). \( \square \)

**Proposition 8.9.** \( \widehat{\Psi}_\varphi \) cannot be slice regularly extended to any st-domain containing strictly \( \widehat{\Omega}_\varphi \).

**Proof.** This is a direct consequence of Propositions 8.7 and 8.8. \( \square \)

**Remark 8.10.** Notice that \( \widehat{\Omega}_\varphi \) is not axially symmetric when \( \varphi \) is not constant. By Remark 3.11 and Proposition 8.9, \( \widehat{\Psi}_\varphi \) cannot be slice regularly extended to any axially symmetric s-domain in \( \mathbb{H} \), when \( \varphi(K) = \frac{1}{2}|K - J| \) for each \( K \in \mathbb{S} \).

### 9. Domains of Slice Regularity

In this section, we consider domains of slice regularity for slice regular functions, analogous to holomorphic domains of homomorphic functions. It turns out that the \( \sigma \)-balls and axially symmetric slice-open sets are domains of slice regularity.

In contrast to complex analysis of one variable, an st-domain may fail to be a domain of slice regularity.

We also give a property of domains of slice regularity, see Proposition 9.4.

**Definition 9.1.** A slice-open set \( \Omega \subset \mathbb{H} \) is called a domain of slice regularity if there are no slice-open sets \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{H} \) with the following properties.

(i) \( \emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega \).

(ii) \( \Omega_2 \) is slice-connected and not contained in \( \Omega \).

(iii) For any slice regular function \( f \) on \( \Omega \), there is a slice regular function \( \tilde{f} \) on \( \Omega_2 \) such that \( f = \tilde{f} \) in \( \Omega_1 \).

**Example 9.2.** The \( \sigma \)-ball \( \Sigma(I, 2) \) is a domain of slice regularity for any \( I \in \mathbb{S} \).

The function \( f : \Sigma(I, 2) \to \mathbb{H} \), defined by
\[
f(q) = \sum_{n \in \mathbb{N}} q^{\alpha_n},
\]
does not extend to a slice regular function near any point of the boundary in any slice \( \mathbb{C} \cdot J, J \in \mathbb{S} \).

**Example 9.3.** Any axially symmetric st-domain is a domain of slice regularity.

**Proposition 9.4.** Let \( I \in \mathbb{S} \) and \( \Omega \) is a domain of slice regularity. If \( \gamma \in \mathcal{P}(\mathbb{C}) \) and \( (J, K) \in \mathbb{S}^2 \) with \( \gamma^J, \gamma^K \subset \Omega \), then \( \gamma^I \subset \Omega \) for any \( I \in \mathbb{S} \).

**Proof.** We shall prove this by contradiction. Suppose that \( \gamma^I \not\subset \Omega \) for some \( I \in \mathbb{S} \). Since \( \gamma^I \) is a slice-path, \( (\gamma^I)^{-1}(\Omega) \) is open in \([0, 1]\). Set
\[
t := \min \{ s \in [0, 1] : \gamma^I(s) \notin \Omega \}.
\]

By assumption, we have
\[
z_J := \gamma^J(t) \in \Omega, \quad z_K := \gamma^K(t) \in \Omega.
\]
so that

\[ B_J(z_J, r) \subset \Omega, \quad B_K(z_K, r) \subset \Omega \]

for some \( r \in \mathbb{R}_+ \).

Since \( \gamma^I(t) \) is continuous in \( C_I \), there is \( t' \in [0, t) \) such that \( \gamma^I(t') \in B_I(z_I, r) \), where \( z_I := \gamma^I(t) \).

For any slice regular function \( f \) on \( \Omega \), define a function \( g : B_I(z_I, r) \to \mathbb{H} \) by

\[ g(x + yI) = (l - K)(J - K)^{-1}f(x + yJ) + (I - J)(K - J)^{-1}f(x + yK) \]

for any \( x, y \in \mathbb{R} \) with \( x + yJ \in B_J(z_J, r) \).

It is direct to verify that \( g \) is holomorphic. And it is easy to check that \( g = f \) near \( \gamma^I(t') \).

By Corollary 6.3, there is a unique slice regular extension \( \tilde{g} \) on \( \Omega_1 := \Sigma(z_I, r) \). Since \( \Sigma(z_I, r) \) and \( \Omega \) are slice-open, it follows that \( \Sigma(z_I, r) \cap \Omega \) is slice-open. Hence the slice-connected component \( \Omega_2 \) of \( \Sigma(z_I, r) \cap \Omega \) containing \( \gamma^I(t') \) is an st-domain.

By Identity Principle 2.10, \( f = g \) on \( \Omega_2 \).

It is easy to check that \( \Omega, \Omega_1 \) and \( \Omega_2 \) satisfy (i-iii) in Definition 9.1. Hence \( \Omega \) is not a domain of slice regularity, which is a contradiction. \( \square \)

10. Final remarks

Let \( \tilde{\Omega}_\varphi \) and \( \tilde{\Psi}_\varphi \) be defined as in Proposition 8.8. Proposition 8.9 implies that \( \tilde{\Psi}_\varphi \) cannot be slice regularly extended to a large st-domain. However, according to Proposition 9.4, \( \Omega_\varphi \) is not a domain of slice regularity when \( \varphi \) is not constant. This suggests to establish an analogue of the theory of Riemann domains for quaternions and characterize the domain of existence of \( \tilde{\Psi}_\varphi \) which is an analogue of a Riemann domain. Since slice-topology is not Euclidean near \( \mathbb{R} \), we cannot consider quaternionic manifolds along the lines used in this paper. Instead, orbifolds over \( (\mathbb{H}, \tau_s) \) could be considered.

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