The Stability of
the Non-Equilibrium
Steady States

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We show that the non-equilibrium steady state (NESS) of the free lattice Fermion model far from equilibrium is macroscopically unstable. The problem is translated to that of the spectral analysis of Liouville Operator. We use the method of positive commutators to investigate it. We construct a positive commutator on the lattice Fermion system, whose dispersion relation is $\omega(k) = \cos k - \gamma$. 
1 Introduction

The investigation of equilibrium states in statistical physics has a long history. In mathematical framework, an equilibrium state is defined as a state which satisfies the Kubo-Martin-Schwinger condition. This is a generalization of the Gibbs equilibrium state. Many researches to justify this definition have been made [HKT], [PW], [R1]. One of them is the investigation of "return to equilibrium"; an arbitrary initial state that is normal with respect to the equilibrium state will converge to the equilibrium state. Recall that the notion of normality represents the macroscopic equivalence in quasi-local system. Although it is a physically fundamental phenomenon, to prove it rigorously is not easy. A complete proof of return to equilibrium in the two-sided XY-model was given by H. Araki [A4]. The other example is an open quantum system, which consists of a finite subsystem and an infinitely extended reservoir in equilibrium [JP1], [JP2], [M]. The open quantum system converges to the asymptotic state, which is the equilibrium state of the coupled systems. The notable fact here is that this state is normal to the initial one, i.e., macroscopically equivalent to the initial state. This shows that the equilibrium state of the reservoir is macroscopically stable when a finite subsystem is connected to.

Recently non-equilibrium steady state (NESS) far from equilibrium has attracted considerable interests. The NESS is introduced as a state asymptotically realized from an inhomogeneous initial state [JP3], [JP4], [R2]. A question rises naturally here; is the NESS macroscopically stable? As an analogy of return to equilibrium in open system, we connect a finite small system to the NESS through a bounded interaction. Will the NESS converge to a state that is normal to the initial state or not? We consider this problem about a free Fermion model on one-dimensional lattice. The explicit form of the NESS is known on this model [HA], [AP]. We show if the NESS is far from equilibrium, it is macroscopically unstable. This results is due to the following fact: for the NESS far from equilibrium, the number of the particles with momentum $k$ is different from the number of the particles with momentum $-k$, although they have the same energy.

Technically, the investigation corresponds to the study of the spectral property about the Liouville operator, which represents the dynamics on the Gelfand Naimark Segal (GNS) Hilbert space of the initial state. We use the positive commutator method to analyze the spectrum. It is well-known that a radiation field system with energy dispersion relation $\omega(k) = |k|$ has a nice covariance property and the positive commutator is constructed with the aid of it. However, in our system, the dispersion relation is $\omega(k) = \cos k$, a new method for constructing a positive commutator is required. We shall construct such a positive commutator and investigate the macroscopic stability of the NESS in this paper.

The paper is organized as follows. In the next section, we introduce basic definitions and notations, then state the main theorems. In Section 3 we will explain the strategy of the proof. In Section 4 we review the role of the standard theory in the research of the NESS, and introduce the modular structure of our model. In Section 5 we introduce the rescaling group which is the key
to construct the positive commutator. In order for the positive commutator method to work, we have to introduce the cut off of the interaction. This is done in Section 6. Section 7 is devoted for the construction of positive commutator. Then in Section 8, we derive the spectral property of the Liouville operator using the method of M. Merkli on Virial Theorem. We complete the proof in Section 9. Below, for a self-adjoint operator $A$, we denote by $P(A \subset I)$ the spectral projection of $A$ onto the subset $I$.

2 Main Results

In this section, we introduce the basic definitions, and state the main results.

2.1 The $C^*$-algebraic Framework

A $C^*$-dynamical system is a pair $(\mathcal{O}, \tau)$ where $\mathcal{O}$ is a $C^*$-algebra, and $\tau$ is a strongly continuous one-parameter group of automorphisms of $\mathcal{O}$. The elements of $\mathcal{O}$ describe observables in the physical system and $\tau$ specifies their time evolution. Below we assume that $\mathcal{O}$ has an identity. A physical state is described as a positive linear functional with norm 1. Let $\omega$ be a state on $\mathcal{O}$ with GNS triple $(\mathcal{H}, \pi, \Omega)$. The notion of $\omega$-normal is defined as follows:

Definition 2.1 A state $\eta$ is said to be $\omega$-normal if there exists a density matrix $\rho$ on $\mathcal{H}$ such that $\eta(\cdot) = \text{Tr}\rho\pi(\cdot)$.

If a state is not $\omega$-normal, it is called $\omega$-singular. For quasi-local algebra, $\omega$-normality means that $\eta$ is approximated in norm topology by local perturbation of $\omega$. So in quasi-local algebra, we can make the following interpretation: if $\eta$ is $\omega$-normal and $\omega$ is $\eta$-normal, $\omega$ and $\eta$ are macroscopically equivalent.

The NESS of dynamics $\tau_t$ associated with the state $\omega$ are the weak\-"$*$ accumulation points of the set of states

$$\frac{1}{T} \int_0^T \omega \circ \tau_t dt$$

as $T \to \infty$. We denote the set of the NESS by $\Sigma_\tau(\omega)$. As the set of states on $\mathcal{O}$ is weak \-"$*$-compact by Alaoglu’s Theorem, $\Sigma_\tau(\omega)$ is a non-empty set whose elements are $\tau$-invariant.

2.2 Macroscopic instability

In this subsection, we introduce the notion of macroscopic instability. Let $(\mathcal{O}_m, \tau_m)$ be a $C^*$-dynamical system. Let $\omega_0$ be an initial state over $\mathcal{O}_m$ and let $\omega_m$ be the NESS corresponding to the pair $(\omega_0, \tau_m)$ i.e., $\omega_m \in \Sigma_{\tau_m}(\omega_0)$. Now we are interested in the stability of $\omega_m$. To investigate it, we add an external finite $C^*$-dynamical system $(\mathcal{O}_S, \tau_S)$. The combined system $(\mathcal{O}_S \otimes \mathcal{O}_m, \tau_S \otimes \tau_m)$ is also a $C^*$-dynamical system. Let us introduce a bounded interaction $V$ between $\mathcal{O}_S$ and $\mathcal{O}_m$, and denote by $\tau_V$ the perturbed dynamics. We shall define the macroscopic instability of $\omega_m$ as follows:
**Definition 2.2** The NESS $\omega_m \in \Sigma_{\tau_m}(\omega_0)$ is macroscopically unstable under a perturbation $V$, if for any state $\omega_S$ over $O_S$, no element in $\Sigma_{\tau_V}(\omega_S \otimes \omega_m)$ is $\omega_S \otimes \omega_m$-normal. The NESS $\omega_m \in \Sigma_{\tau_m}(\omega_0)$ is macroscopically unstable if $\omega_m$ is macroscopically unstable under some bounded perturbation $V$.

Let us explain it in detail for our system. First we divide the one-dimensional Fermion lattice to the left and the right, and consider a state that each side is in equilibrium at different temperature. This is the initial state $\omega_0$. The corresponding NESS $\Sigma_{\tau_m}(\omega_0)$ under the time evolution of free lattice Fermion $\tau_m = \alpha_f$ consists of only one point: $\Sigma_{\tau_m}(\omega_0) = \{\omega_{\rho}\}$. To investigate the stability of $\omega_m = \omega_{\rho}$, we prepare an external finite system described in a finite dimensional Hilbert space $C^d$. We connect it to $\omega_m$ through a bounded interaction $V$. If there is no $\omega_S \otimes \omega_m$-normal state in $\Sigma_{\tau_V}(\omega_S \otimes \omega_m)$, for any state $\omega_S$ over $O_S$, $\omega_m$ is macroscopically unstable.

### 2.3 The model

In this paper we consider the free lattice Fermion system in one dimension. The explicit form of the NESS is known for this system \cite{HA,AP}.

Let $\mathfrak{h} \equiv l^2(\mathbb{Z})$ be a Hilbert space of a single Fermion. By Fourier transformation, it is unitary equivalent to $L^2([-\pi, \pi])$. The Hamiltonian $h$ of a single Fermion is given by

$$
(hf)(n) = \frac{1}{2} (f(n-1) + f(n+1)) - \gamma f(n),
$$

on $\mathfrak{h}$ which is described in Fourier representation as

$$
\hat{hf}(k) = \omega(k) \hat{f}(k),
\omega(k) = \cos(k) - \gamma.
$$

The $\gamma$-term represents the interaction with the external field, and $\gamma$ is a parameter in $(-1, 1)$. The free lattice Fermi gas $\mathcal{R}$ is described as the CAR-algebra $O_f$ over $\mathfrak{h}$. And its dynamics is given by

$$
\alpha_f^t(a(f)) = a(e^{itf}).
$$

In the initial state $\omega_0$, the lattice is separated into the left and the right. And they are kept at different inverse temperature $\beta_-, \beta_+$, respectively. The NESS $\omega_{\rho}$ associated with $\omega_0$ is realized as the asymptotic state under the dynamics $\alpha_f^t \square 2$. The explicit form of $\omega_{\rho}$ was obtained in \cite{HA} and \cite{AP}, independently: $\omega_{\rho}$ is a state whose $n$-point functions have a structure

$$
\omega_{\rho}(a(f_1)^* \cdots a(f_l)^*a(g_1) \cdots a(g_m)) = \delta_{nm} \det(\langle f_i, pg_j \rangle),
$$
where \( \rho \) is represented as a multiplication operator,

\[
\rho(k) = \begin{cases} 
(1 + e^{\beta_+ \omega(k)})^{-1} & k \in [0, \pi) \\
(1 + e^{\beta_- \omega(k)})^{-1} & k \in [-\pi, 0) 
\end{cases}
\]

\( \omega(k) = \cos(k) - \gamma \),

in the Fourier representation. If \( \beta_+ \neq \beta_- \), we will say that the NESS \( \omega_0 \) is far from equilibrium. In this paper, the stability of this state is considered.

The observables of the small system are described as \( C^* \)-algebra \( \mathcal{O}_S \equiv B(H_S) \) on a finite \( d \)-dimensional Hilbert space \( H_S = \mathbb{C}^d \). We denote by \( H_S \) the free Hamiltonian of the system on \( H_S \). The free dynamics \( \alpha_t^S \) is given by

\[
\alpha_t^S(A) = e^{itH_S}Ae^{-itH_S}.
\]

The combined system \( S + R \) is described as the \( C^* \)-algebra

\[
\mathcal{O} \equiv \mathcal{O}_S \otimes \mathcal{O}_f.
\]

The free dynamics of the combined system is given by \( \alpha_t^0 = \alpha_t^S \otimes \alpha_t^f \). We denote by \( \delta_0 \) the derivation of \( \alpha_t^0 \).

Let us consider the dynamics including the interaction between \( S \) and \( R \). In this paper, we define the interaction term \( V \) by

\[
V = \lambda \cdot Y \otimes (a(f) + a^*(f)),
\]

where \( f \in \mathfrak{h} \) is called a form factor. Here \( \lambda \) is a coupling constant and \( Y \) is a self-adjoint operator on \( H_S \). Note that \( V \) is an element of \( \mathcal{O} \). The perturbed dynamics \( \alpha_t \) is generated by \( \delta = \delta_0 + i[V, \cdot] \) with \( D(\delta) = D(\delta_0) \). \( \alpha_t \) is expanded as follows;

\[
\alpha_t(A) \equiv \alpha_t^0(A) + \sum_{n \geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n [\alpha_{t_n}^0(V), \cdots, [\alpha_{t_1}^0(V), \alpha_t^0(A)]]
\]

The right hand side converges in norm topology in \( \mathcal{O} \). \( \alpha_t \) is strongly continuous one parameter group of automorphisms.

2.4 Main Theorem

In the analysis, we carry out the variable transformation from \( k \in [-\pi, \pi) \) to \( t \in \mathbb{R} \), by \( t(k) = \tan \frac{k}{2} \). Under this variable translation, we identify \( \mathfrak{h} = L^2((-\pi, \pi), dk) \) with \( L^2(\mathbb{R}, \frac{2}{\pi^2+1} dt) \). We need several assumptions on the small system and the interaction \( V \).
Assumption 2.1 Under the identification \( h = L^2([-\pi, \pi], dk) = L^2(\mathbb{R}, \frac{2}{\pi} dt) \), let \( u(\theta) \) be a strongly continuous one parameter unitary group on \( h \) defined by

\[
(u(\theta)g)(t) \equiv e^{\frac{\theta^2}{2} \sqrt{\frac{t^2 + 1}{c^2}} g(e^{\theta} t), \quad \theta \in \mathbb{R}, \quad g \in L^2((-\infty, \infty), \frac{2}{t^2 + 1} dt),
\]

and let \( p \) be the generator of \( u(\theta) = e^{i\theta p} \). For a constant \( 0 < v < 1 \), let \( \Lambda_v \) be the interval of \( \mathbb{R} \) defined by

\[
\Lambda_v = \{ t \in \mathbb{R} ; \quad s_1(t) = \frac{4t^2}{(1 + t^2)^2} \geq v \}.
\]

We assume that the form factor \( f \in h \) in (5) is in the domain \( D(p^3) \). Furthermore, we assume that there exists \( 0 < v < 1 \) such that the support of \( f \) satisfies

\[
\text{supp} f \subset \Lambda_v.
\]

Assumption 2.2 Let \( e_i, e_j \) be an arbitrary pair of eigenvalues of \( L_S = H_S \otimes 1 - 1 \otimes H_S \) such that \( e_i \neq e_j \). There exists \( k_{ij} \) such that

\[
\omega(\pm k_{ij}) = \cos k_{ij} - \gamma = e_i - e_j,
\]

\[
0 \leq k_{ij}, \quad t(k_{ij}) \in \Lambda_v.
\]

Assumption 2.3

1. The Hamiltonian \( H_S \) has no degenerated eigenvalue.

2. There exists \( a > 0 \) such that

\[
\min_{e_i \neq e_j} \| f(\pm k_{ij}) \| \geq a.
\]

Assumption 2.4 Let \( \varphi_n \) be the \( n \)-th eigenvector of \( H_S \) with eigenvalue \( E_n \), and let \( p_n \) be the spectral projection onto \( \varphi_n \) on \( H_S \). We denote the eigenvalue of \( L_S = H_S \otimes 1 - 1 \otimes H_S \) as \( E_{n,m} \equiv E_n - E_m \), and introduce the following subsets for each eigenvalue \( e \) of \( L_S \):

\[
N_l^{(i)} \equiv \{ i; E_{i,j} = e \}, \quad N_r^{(i)} \equiv \{ j; E_{i,j} = e \}
\]

\[
N_l \equiv \bigcup_j N_l^{(j)}, \quad N_r \equiv \bigcup_i N_r^{(i)}.
\]

For a set \( N \), we define a projection \( p_N \equiv \sum_{n \in N} p_n \).

1. For \( e \neq 0 \), we assume

\[
2\delta_0 \equiv \min_{n \in N_l} \inf \sigma \left( p_{N_l^{(n)}} \bar{Y} p_{N_l^{(n)}} \right) + \min_{m \in N_r} \inf \sigma \left( p_{N_r^{(m)}} Y p_{N_r^{(m)}} \right) > 0,
\]

where \( \sigma(A) \) represents the spectrum of \( A \), and \( \bar{Y} \) is the complex conjugation of \( Y \) with respect to the orthonormal basis consisting of eigenvectors of \( H_S \).
2. Let $Y_{mn} \equiv \langle \varphi_{m} | Y | \varphi_{n} \rangle$. We assume $|Y_{mn}| > 0$ for all $m \neq n$.

An example which satisfies all of the Assumptions is represented in Appendix A. Here is the main theorem of this paper.

**Theorem 2.3** Let $\omega_{\rho}$ be a NESS far from equilibrium, i.e., the state given by (3) with inverse temperatures $\beta_{+} \neq \beta_{-}$. Suppose that Assumption 2.1 to 2.4 are satisfied. Then there exists a $\lambda_{1} > 0$ s.t. if $0 < |\lambda| < \lambda_{1}$, then $\omega_{\rho}$ is macroscopically unstable under the perturbation $V (5)$. Especially, $\omega_{\rho}$ is macroscopically unstable.

Furthermore, assume $\beta_{0} < \beta_{+}, \beta_{-} < \beta_{1}, \|pf\|, \|f\| \leq b$ for any fixed $0 < \beta_{0} < \beta_{1} < \infty$ and $0 < b < \infty$. Here $p$ is the generator of $u(\theta) = e^{i\theta p}$. Then if we fix $\beta_{+}$ and $\beta_{-}$, we have $\lambda_{1} \sim O(v^{\frac{20}{11}})$ as $v$ goes to 0. On the other hand, if we fix $v$ then we have $\lambda_{1} \sim O(|\beta_{+} - \beta_{-}|^{\frac{20}{11}})$ as $\beta_{+} - \beta_{-} \to 0$.

**Theorem 2.4** Let $\omega_{\rho}$ be an equilibrium state i.e. the state given in (3) with $\beta \equiv \beta_{+} = \beta_{-}$. Then there exists a $\beta-KMS$ state $\omega_{\nu}$ w.r.t. the perturbed dynamics $\alpha_{t}$, which is normal to $\omega_{S} \otimes \omega_{\rho}$ for arbitrary faithful state $\omega_{S}$ of $O_{S}$. Suppose that Assumption 2.1 to 2.4 are satisfied. Then there exists a $\lambda_{1} > 0$ such that if $0 < |\lambda| < \lambda_{1}$, any $\omega_{\nu}$-normal state $\eta$ exhibits return to equilibrium in an ergodic mean sense, i.e.

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \eta(\alpha_{t}(A)) \, dt = \omega_{\nu}(A)$$

for all $A \in O$.

Furthermore, assume $\beta_{0} < \beta < \beta_{1}, \|pf\|, \|f\| \leq b$ for any fixed $0 < \beta_{0} < \beta_{1} < \infty$ and $0 < b < \infty$. Then we have $\lambda_{1} \sim O(v^{\frac{20}{11}})$ as $v$ goes to 0.

3 The Strategy of the Proof

In this section, we explain the strategy and the organization of the proof.

3.1 The kernel of Liouville operator

By the standard theory, the problem of macroscopic instability is translated into the spectral problem of so called Liouville operator $L$ (see Section 4 Proposition 4.3): if $\text{Ker} L = \{0\}$, $\omega$ is macroscopically unstable. In our model, the Liouville operator $L$ is an operator on $\mathcal{H} = (\mathcal{H}_{S} \otimes I_{S}) \otimes \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$, given by

$$L = (H_{S} \otimes 1) \otimes 1 - (1 \otimes H_{S}) \otimes 1 + 1 \otimes d\Gamma (h \oplus -h) + \lambda I_{0}. \quad (7)$$

Here $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ is a Fermi Fock space over $\mathfrak{h} \oplus \mathfrak{h}$ and $d\Gamma (h \oplus -h)$ is a second quantization of the multiplication operator $h \oplus -h$. $\lambda I_{0}$ is the interaction term.

The main part of this paper is to prove the following Theorem on the eigenvector of $L$: 
Theorem 3.1 For each eigenvalue $\tilde{e}$ of $L_S = H_S \otimes 1 - 1 \otimes H_S$, define an operator $\Gamma(\tilde{e})$ on the space $P(L_S = \tilde{e}) \cdot \mathcal{S}_S \otimes \mathcal{S}_S$ by

$$
\Gamma(\tilde{e}) = \int_{-\pi}^{\pi} dk \ m(k, 1)^* P (L_S \neq \tilde{e}) \delta (\omega (k) + L_S - \tilde{e}) m(k, 1) \\
+ \int_{-\pi}^{\pi} dk \ m(k, 2)^* P (L_S \neq \tilde{e}) \delta (-\omega (k) + L_S - \tilde{e}) m(k, 2)
$$

and

$$
m(k, i) \equiv Y \otimes 1 \cdot g_1^i(k) - 1 \otimes \overline{Y} \cdot g_2^i(k) \quad i = 1, 2.
$$

Here $g_i^1$ are defined by

$$
g_1^1 = (1 - \rho)^{\frac{1}{2}} f, \quad g_1^2 = \rho^{\frac{1}{2}} f, \quad g_2^1 = \rho^{\frac{1}{2}} \overline{f}, \quad g_2^2 = (1 - \rho)^{\frac{1}{2}} \overline{f},
$$

and $\overline{f}$ is the complex conjugation of $f$ in the Fourier representation. Let $\gamma_{\tilde{e}}$ be a strictly positive constant such that

$$
\Gamma(\tilde{e}) \geq \gamma_{\tilde{e}} \cdot (P(\Gamma(\tilde{e}) = 0))^\perp.
$$

Let $\tilde{P}_{e}$ be

$$
\tilde{P}_{e} = P(L_S = \tilde{e}) \cdot P(\Gamma(\tilde{e}) = 0) \otimes P_{\Omega_f},
$$

where $P_{\Omega_f}$ is the projection onto the vacuum $\Omega_f$ of $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$.

For $e \in \mathbb{R}$, let $\tilde{e}(e) \equiv e$ if $e$ is an eigenvalue of $L_S$, and let $\tilde{e}(e)$ be an eigenvalue of $L_S$ which is next to $e$, if $e$ is not an eigenvalue of $L_S$. Suppose that Assumption 2.1 and 2.4 are satisfied. Then there exists $\lambda_1 > 0$ s.t., if $0 < |\lambda| < \lambda_1$, then there is no eigenvector with eigenvalue $e$ which is orthogonal to $\tilde{P}_{e}(e)$. In particular, if $\tilde{P}_{e}(e) = 0$, $e$ is not an eigenvalue of $L$. Furthermore, if we assume $\beta_0 < \beta_+, \gamma_+ < \beta_1, ||pf||, ||f|| \leq b$ for any fixed $0 \leq \beta_0 < \beta_1 < \infty$ and $0 < b < \infty$, we can choose $\lambda_1$ as

$$
\lambda_1 = C \min \left\{ v^{\frac{100}{3}} \left( \frac{v}{\gamma_{\tilde{e}(e)}} \right)^{\frac{100}{3}}, (v^{\gamma_{\tilde{e}(e)}})^{\frac{100}{3}}, (v^{\gamma_{\tilde{e}(e)}})^{\frac{100}{3}} \right\}.
$$

Here $C$ is a constant which depends on $\beta_0, \beta_1$ and $b$, but is independent of $v$ and $\beta_+ - \beta_-$. By Theorem 3.1, the existence of the eigenvector of $L$ is determined by the kernel of $\Gamma(e)$. We have the following Theorem on it (Section 3):

Theorem 3.2 Suppose that Assumption 2.1 to 2.4 are satisfied. Then if $\beta_+ \neq \beta_-, \text{Ker}\Gamma(e) = \{0\}$ for any eigenvalue $e$ of $L_S$. If $\beta_+ = \beta_-, \text{Ker}\Gamma(e) = \{0\}$ for any non-zero eigenvalue $e \neq 0$ of $L_S$, but Ker$\Gamma(0)$ is non-trivial one-dimensional space.

The following fact will develop in the proof : the non-existence of the non-trivial kernel of $\Gamma(0)$ for the far from equilibrium case $\beta_+ \neq \beta_-$ is caused by the fact that the number of the particles with the momentum $k, -k$ are different although they have the same energy $\omega(k) = \omega(-k)$. Note that the same fact induces the existence of current for NESS.

Combining Theorem 3.1 and 3.2 we obtain the instability of NESS far from equilibrium.
3.2 The proof of Theorem 3.1

To prove the Theorem, we use the positive commutator method. The idea of the method is as follows: suppose that there exists anti-self-adjoint operator $A$ such that $[L, A] \geq c > 0$. If $L$ has an eigenvector $\psi$ with eigenvalue $e$, we have

$$0 = 2\text{Re} \langle (L - e)\psi, A\psi \rangle = \langle \psi, [L - e, A]\psi \rangle \geq c > 0,$$

which is a contradiction. Hence $L$ has no eigenvector. This is called Virial Theorem. However this argument works only formally and indeed we have to take care of the domain question.

The proof of Theorem 3.1 is divided into two steps: the first one is to construct the positive commutator (Section 5 to Section 7), and the second one is to justify the above arguments rigorously (Section 8).

To construct the positive commutator is a non-trivial problem, and we have to work out for each models. Now, our field has the dispersion relation $\omega(k) = \cos k - \gamma$. Under the variable translation $k \rightarrow t$, we show that there exists a strongly continuous one parameter group of unitaries $U(\theta)$ on $H$, which satisfies

$$U(-\theta) (1 \otimes d\Gamma (h \oplus -h)) U(\theta) = 1 \otimes d\Gamma (h^{-\theta} \oplus -h^0),$$

where $h^0$ is a multiplication operator on $L^2(\mathbb{R}, \frac{2}{\tau^2+1}dt)$ defined by $h^0(t) \equiv h(e^\theta t)$. This means $U(\theta)$ induces the rescaling of multiplication operator in $\mathfrak{g}$. Let $A_0$ be the generator of $U(\theta)$. This $A_0$ satisfies

$$[L_0, A_0] = S_1 = 1 \otimes d\Gamma (s_1 \oplus s_1) \geq 0$$

where $S_1$ is a second quantization of multiplication operator

$$(s_1 f)(t) = s_1(t) f(t), \quad s_1(t) = \frac{4t^2}{(1+t^2)^2}. $$

Hence by considering rescaling of multiplication operators with respect to $t$, we obtained positive commutator $[L_0, A_0] = S_1 \geq 0$. However, what is really needed is the strictly positive commutator. If $S_1$ has a spectral gap, following the well-known procedure, we can construct a strictly positive commutator. But $S_1$ does not have a gap now, and we need to overcome this problem.

Let $\Lambda_v^c$ be the complement of $\Lambda_v$, and let $N_{\Lambda_v^c}$ be the number operator of particles whose momentum is included in $\Lambda_v^c$. Furthermore, let $P$ be the spectral projection onto the subspace $N_{\Lambda_v^c} = 0$, and set $\bar{P} = 1 - P$. If we restrict ourselves to $P\mathcal{H}$, $PS_1P$ has a spectral gap $v > 0$, so if $L$ strongly commutes with $N_{\Lambda_v^c}$, we can construct the positive commutator for $L$ in $P\mathcal{H}$. Furthermore, if the subspace $P\mathcal{H}$ includes no eigenvector of $L$, we just need to analyze $L$ on $P\mathcal{H}$.

To make the spectral localization possible, we introduce a cut off of the form factor $f \in \mathfrak{g}$ with respect to $\Lambda_v$, i.e.,

$$\text{supp} f \subset \Lambda_v.$$
By this cut off, $L$ strongly commutes with $N_{\Lambda c}$, and furthermore, there is no eigenvector of $L$, in the subspace $PH$. Hence we obtain positive commutator which is sufficient to analyze the eigenvector of $L$.

To carry out the second step, the rigorous justification of the Virial Theorem, we used the new method introduced by M.Merkli. By approximating the eigenvector of $L$ by vectors in the domain of total number operator $N$ and $A_0$, we can carry out the arguments rigorously.

4 The Standard Theory and NESS

In this section, we explain the role of the standard theory (see [BR1]) in the investigation of the NESS and introduce the modular structure in our model.

4.1 The standard theory

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\Omega$. A positive linear functional $\omega$ on $\mathcal{M}$ is said to be normal if there exists a positive traceclass operator $\rho$ such that

$$\omega(\cdot) = \text{Tr}(\rho \cdot).$$

We denote by $\mathcal{M}_{++}$ the set of all normal positive linear functionals.

We can define an operator $S_0$ on a dense set $\mathcal{M}\Omega$ by

$$S_0 x\Omega = x^* \Omega.$$ 

$S_0$ is closable and we represent the polar decomposition of the closure $\bar{S}_0$ as

$$\bar{S}_0 = J \Delta^{\frac{1}{2}}.$$ 

$J$ is called the modular conjugation and $\Delta$ is called the modular operator. By the Tomita-Takesaki Theory we have $J\mathcal{M}J = \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. Let $j : \mathcal{M} \rightarrow \mathcal{M}'$ be the antilinear $\ast$-isomorphism defined by $j(x) = JxJ$. We define the natural positive cone $\mathcal{P}$ by the closure of the set $\{xj(x) \Omega; x \in \mathcal{M}\}$. For $\xi \in \mathcal{P}$, define the normal positive functional $\omega_\xi \in \mathcal{M}_{++}$ by

$$\omega_\xi (x) = \langle \xi, x\xi \rangle \quad \forall x \in \mathcal{M}.$$ 

We have the following Theorems;

**Theorem 4.1** For any $\omega \in \mathcal{M}_{++}$, there exists a unique $\xi(\omega) \in \mathcal{P}$ such that $\omega = \omega_{\xi(\omega)}$.

**Theorem 4.2** For any $\ast$-automorphism $\alpha$ of $\mathcal{M}$, there exists a unique unitary operator $U(\alpha)$ on $\mathcal{H}$ satisfying the following properties:

1. $U(\alpha)xU(\alpha)^* = \alpha(x), \quad x \in \mathcal{M}$;
2. \( U(\alpha) \mathcal{P} \subset \mathcal{P} \) and \( U(\alpha)\xi(\omega) = \xi(\alpha^{-1}(\omega)) \), \( \omega \in \mathcal{M}_{++} \),

where \( (\alpha^*\omega)(A) \equiv \omega(\alpha(A)) \),

3. \([U(\alpha), J] = 0\).

Let \( \alpha_t \) be a one parameter group of automorphisms and let \( U(t) \) be a one parameter group of unitaries associated with \( \alpha_t \). If \( U(t) \) is strongly continuous, it is written as \( U(t) = e^{itL} \) with self adjoint operator \( L \). We call \( L \) the Liouville operator of \( \alpha_t \).

4.2 The NESS and the Liouville operator

Now we move from \( W^* \)-dynamical systems to \( C^* \)-dynamical systems, and explain the role of the standard theory in the investigation of the NESS. Let \( \alpha_t \) be a one parameter group of automorphisms which describes the dynamics of a unital \( C^* \)-algebra \( \mathcal{O} \), and let \( \omega \) be a state of \( \mathcal{O} \). Let \( (\mathcal{H}, \pi, \Omega) \) be the GNS triple of \( \omega \). Suppose that \( \Omega \) is a cyclic and separating vector for von Neumann algebra \( \pi(\mathcal{O})'' \), and that there exists an extension \( \hat{\alpha}_t \) of \( \alpha_t \) to \( \pi(\mathcal{O})'' \). Then by Theorem \ref{thm:extension}, there is one parameter group of unitary operators \( U_t \) such that

\[
\hat{\alpha}_t(x) = U_t x U_t^* \quad x \in \pi(\mathcal{O})''
\]

\( U_t^* \mathcal{P} \subset \mathcal{P} \). (10)

In addition, suppose that \( U_t \) is strongly continuous and let \( L \) be the Liouville operator. If \( \omega_\infty \in \Sigma_\alpha(\omega) \) is \( \omega \)-normal, there exists a vector \( \xi_\infty \) in \( \mathcal{P} \) s.t.

\[
\omega_\infty(A) = \langle \xi_\infty, \pi(A) \xi_\infty \rangle, \quad A \in \mathcal{O},
\]

by Theorem \ref{thm:omega-normal}. As \( \omega_\infty \in \Sigma_\alpha(\omega) \), \( \omega_\infty \) is invariant under \( \alpha_t \): \( \omega_\infty \circ \alpha_t = \omega_\infty \).

Hence we have

\[
\langle U_t^* \xi_\infty, \pi(A) U_t^* \xi_\infty \rangle = \omega_\infty \circ \alpha_t(A) = \omega_\infty(A) = \langle \xi_\infty, \pi(A) \xi_\infty \rangle, \quad A \in \mathcal{O}.
\]

As \( \xi_\infty, U_t^* \xi_\infty \in \mathcal{P} \), we get

\[
U_t^* \xi_\infty = \xi_\infty,
\]

by Theorem \ref{thm:omega-normal}. This means

\[
\xi_\infty \in \text{Ker}L.
\]

In other words, we have the following proposition.

**Proposition 4.3** If \( \text{Ker}L = \{0\} \), any state in \( \Sigma_\alpha(\omega) \) is \( \omega \)-singular i.e., \( \omega \) is macroscopically unstable.
4.3 The modular structure of the model

Now we introduce the modular structure of our model. As the structure is the same as that of [JP3], we just state the results. Let $\omega_S$ be a state over $O_S$, defined by $\omega_S = \text{Tr}(\rho_S \cdot)$ with a density matrix

$$\rho_S = \sum_i p_i |\psi_i\rangle \langle \psi_i|,$$

on $\mathcal{H}_S$. Here $\{\psi_i\}$ are orthogonal unit vectors on $\mathcal{H}_S$. And let $\omega_\rho$ be a quasi-free state over $O_f$ defined in [3]. Let $\bar{A}$ be a complex conjugation of $A$ with respect to the orthogonal basis of $\mathcal{H}_S$, that is given by eigenvectors of $H_S$. We denote the Fermi Fock space over $\mathfrak{h} \oplus \bar{\mathfrak{h}}$ by $\mathcal{F}(\mathfrak{h} \oplus \bar{\mathfrak{h}})$. Then we have the following proposition.

**Proposition 4.4** Suppose that $\omega_S$ is faithful, and $0 < \rho < 1$. The GNS triple $(\mathcal{H}, \pi, \Omega)$ associated with the state $\omega = \omega_S \otimes \omega_\rho$ is given by

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_f, \quad \Omega = \Omega_S \otimes \Omega_f, \quad \pi = \pi_S \otimes \pi_f,$$

where

$$\mathcal{H}_S = \mathcal{H}_S \otimes \mathcal{H}_S, \quad \Omega_S = \sum \frac{1}{2} p_\frac{1}{2} |\psi_i\rangle \otimes \bar{\psi}_i, \quad \pi_S (A) = A \otimes 1, \quad A \in O_S,$$

$$\mathcal{H}_f = \mathcal{F}(\mathfrak{h} \oplus \bar{\mathfrak{h}}), \quad \Omega_f : \text{vacuum vector of } \mathcal{F}(\mathfrak{h} \oplus \bar{\mathfrak{h}}),$$

$$\pi_f (a (f)) = a \left( (1 - \rho)^{\frac{1}{2}} f \otimes 0 \right) + a^* \left( 0 \otimes \rho^{\frac{1}{2}} \bar{f} \right),$$

$$\pi_f (a^* (f)) = a^* \left( (1 - \rho)^{\frac{1}{2}} f \otimes 0 \right) + a \left( 0 \otimes \rho^{\frac{1}{2}} \bar{f} \right), \quad f \in \mathfrak{h},$$

and $\Omega$ is a cyclic and separating vector for the von Neumann algebra $\pi(O)'$. The dynamics $\alpha_t$ defined on $C^*\text{-algebra } O$ extends to a weakly continuous one parameter group of automorphisms $\tilde{\alpha}_t$ over $\pi(O)'$. $\tilde{\alpha}_t$ is implemented by a strongly continuous one parameter group of unitaries and the corresponding Liouville operator $L$ on $\mathcal{H}$ is

$$L = (H_S \otimes 1) \otimes 1 - (1 \otimes H_S) \otimes 1 + 1 \otimes d\Gamma \left( \tilde{h} \right)$$

$$+ \lambda (Y \otimes 1) \otimes (a (g_1) + a^* (g_1)) - \lambda (1 \otimes \bar{Y}) \otimes (-1)^N (a (g_2) - a^* (g_2)), \quad (11)$$

where

$$\tilde{h} = h \oplus -\bar{h},$$

$$g_1 = g_1^1 \oplus g_1^2, \quad g_2 = g_2^1 \oplus g_2^2,$$

$$g_1^1 = (1 - \rho)^{\frac{1}{2}} f, \quad g_1^2 = \rho^{\frac{1}{2}} \bar{f},$$

$$g_2^1 = \rho^{\frac{1}{2}} f, \quad g_2^2 = (1 - \rho)^{\frac{1}{2}} \bar{f}. \quad (14)$$

Here, $d\Gamma \left( \tilde{h} \right)$ is the second quantization of $\tilde{h}$. 

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We denote the Fermion number operator on $\mathcal{H}$ by $N$. We define the system Liouville operator $L_S$, the field Liouville operator $L_f$, the free Liouville operator $L_0$, and the interaction $I_0$ operator by

\begin{align*}
L_S &= (H_S \otimes 1) \otimes 1 - (1 \otimes H_S) \otimes 1,
L_f &= 1 \otimes d\Gamma \left(\tilde{h}\right), \\
L_0 &= L_S + L_f, \\
I_0 &= (Y \otimes 1) \otimes (a(g_1) + a^*(g_1)) - (1 \otimes \tilde{Y}) \otimes (-1)^N (a(g_2) - a^*(g_2)).
\end{align*}

(15)

5 Rescaling Group

In order to construct the positive commutator for the lattice system, we introduce a strongly continuous one parameter group of unitaries. By the variable transformation $k \to t$, we have

\begin{align*}
\mathfrak{h} &= L^2 ([−\pi, \pi), dk) = L^2 ((−\infty, \infty), d\mu),
\end{align*}

where $\mu$ is the positive measure on $\mathbb{R}$ given by

\begin{align*}
d\mu(t) = \frac{2}{t^2 + 1} dt.
\end{align*}

The multiplication operator $m(k)$ on $\mathfrak{h}$ is transformed as $m(k) \to m(k(t))$, where $k(t) \equiv 2 \tan^{-1} t$. In particular, the single particle energy $\mathfrak{h}$ is transformed to

\begin{align*}
\mathfrak{h}(k) = \cos k - \gamma \quad \to \quad \mathfrak{h}(k(t)) = \frac{2}{t^2 + 1} - 1 - \gamma.
\end{align*}

Now we introduce a unitary operator $u(\theta)$ on $\mathfrak{h}$ defined by

\begin{align*}
(u(\theta) f)(t) &= e^{\frac{\theta}{2} \sqrt{\frac{t^2 + 1}{e^{2\theta} t^2 + 1}}} f(e^{\theta} t), \quad \theta \in \mathbb{R}, \quad f \in L^2 ((−\infty, \infty), d\mu).
\end{align*}

By an elementary calculation, we have the following lemma;

**Lemma 5.1** $u(\theta)$ is a strongly continuous one parameter group of unitaries.

From this lemma, $u(\theta)$ is written as $u(\theta) = e^{ip\theta}$ with a selfadjoint operator $p$. The action of $p$ on $C_0^{\infty} (\mathbb{R})$ is

\begin{align*}
(pf)(t) &= -i \left( \frac{1}{2} f(t) - \frac{t^2}{t^2 + 1} f(t) + tf'(t) \right), \quad f \in C_0^{\infty} (\mathbb{R}).
\end{align*}

(16)

Let $\tilde{u}(\theta)$ be a unitary operator on $\mathfrak{h} \oplus \mathfrak{h}$ defined by

\begin{align*}
\tilde{u}(\theta) &= u(\theta) \oplus u(-\theta).
\end{align*}
The generator of \( \tilde{u}(\theta) \) is \( \tilde{p} \equiv p \oplus -p \). We define a unitary operator \( U(\theta) \) on \( \mathcal{H} \) by the second quantization \( \Gamma(\tilde{u}(\theta)) \) of \( \tilde{u}(\theta) \):

\[
U(\theta) \equiv 1 \otimes \Gamma(\tilde{u}(\theta)).
\]

Since \( U(\theta) \) is a strongly continuous one parameter unitary group, it is written as \( U(\theta) = e^{\theta A_0} \), where \( A_0 = 1 \otimes d\Gamma(i\tilde{p}) \) is an anti-selfadjoint operator.

Now we introduce the commutator of \( L_f \) and \( A_0 \) as a quadratic form on the dense set \( (D(N) \cap D(A_0)) \times (D(N) \cap D(A_0)) \). By straightforward calculation, the action of \( \tilde{u}(\theta) \) on a multiplication operator \( m_1 \oplus m_2 \) on \( h \oplus h \) is

\[
(\tilde{u}(\theta) (m_1 \oplus m_2) \tilde{u}(\theta))(t) = m_1 (e^{-\theta t}) \oplus m_2 (e^{\theta t}).
\]

Then, the action of \( U(\theta) \) on the second quantization \( d\Gamma(m_1 \oplus m_2) \) of \( m_1 \oplus m_2 \) is

\[
U(-\theta) (1 \otimes d\Gamma(m_1 \oplus m_2)) U(\theta) = 1 \otimes d\Gamma\left(m_1^θ \oplus m_2^θ\right), \tag{17}
\]

where

\[
(m_i^θ f)(t) = m_i(e^{\theta t}) f(t).
\]

We have the following proposition:

**Proposition 5.2** Suppose that \( m_1, m_2 \) are bounded multiplication operators on \( L^2(\mathbb{R}, d\mu) \) which are differentiable and satisfy

\[
\sup_{t \in \mathbb{R}} |qm_i(t)| < \infty,
\]

where

\[
qm_i(t) \equiv -tm'_i(t).
\]

Then we have

\[
\langle 1 \otimes d\Gamma(m_1 \oplus m_2) \varphi, A_0 \psi \rangle + \langle A_0 \varphi, 1 \otimes d\Gamma(m_1 \oplus m_2) \psi \rangle = \langle \varphi, 1 \otimes d\Gamma(qm_1 \oplus -qm_2) \psi \rangle, \tag{18}
\]

for all \( \varphi, \psi \in D(N) \cap D(A_0) \).

**Proof**

As \( U(\theta) \) preserves number, we have \( U(\theta) D(N) \subset D(N) \) and have the following relation for all \( \varphi, \psi \in D(N) \cap D(A_0) \):

\[
\lim_{\theta \to 0} \left\langle \varphi, 1 \otimes \frac{d\Gamma(m_1^θ \oplus m_2^θ) - d\Gamma(m_1 \oplus m_2)}{θ} \psi \right\rangle = \lim_{\theta \to 0} \left\langle (1 \otimes d\Gamma(m_1 \oplus m_2)) U(\theta) \varphi, \frac{U(θ) - 1}{θ} \psi \right\rangle + \left\langle \frac{U(θ) - 1}{θ} \varphi, 1 \otimes d\Gamma(m_1 \oplus m_2) \psi \right\rangle. \tag{19}
\]

Using dominated convergence theorem, we have the following statements:
1. For each $\psi \in D(N)$ and bounded operator $m$ on $\mathfrak{h} \oplus \mathfrak{h}$,
   \[
   \lim_{\theta \to 0} (1 \otimes d\Gamma (m)) U (\theta) \psi = (1 \otimes d\Gamma (m)) \psi,
   \]

2. For each $\psi \in D(A_0)$,
   \[
   \lim_{\theta \to 0} U (\theta) - \frac{1}{\theta} \psi = A_0 \psi,
   \]

3. For each $\psi \in D(N)$,
   \[
   \lim_{\theta \to 0} 1 \otimes \left( d\Gamma (m_1^{-\theta} \oplus m_2^{\theta}) - d\Gamma (m_1 \oplus m_2) - d\Gamma (qm_1 \oplus -qm_2) \right) \psi = 0.
   \]

Substituting these to (19), we obtain the statement of the proposition. □

Substituting $m_1 = h$, $m_2 = -h$ to (18), we obtain
   \[
   \langle L_f \varphi, A_0 \psi \rangle + \langle A_0 \varphi, L_f \psi \rangle = \langle \varphi, S_1 \psi \rangle,
   \]
for all $\varphi, \psi \in D(N) \cap D(A_0)$. Here $S_1$ is the second quantization $1 \otimes d\Gamma (s_1 \oplus s_1)$ of $s_1 \oplus s_1$ on $\mathfrak{h} \oplus \mathfrak{h}$, with $s_1$ defined by
   \[
   (s_1 f) (t) = s_1(t) f(t), \quad s_1(t) = \frac{4t^2}{(1 + t^2)^2}.
   \]

Note that $S_1$ is positive. This is the main point of this commutator.

Similarly, we have
   \[
   \langle S_1 \varphi, A_0 \psi \rangle + \langle A_0 \varphi, S_1 \psi \rangle = \langle \varphi, S_2 \psi \rangle,
   \]
\[
   \langle S_2 \varphi, A_0 \psi \rangle + \langle A_0 \varphi, S_2 \psi \rangle = \langle \varphi, S_3 \psi \rangle,
   \]
where
   \[
   S_2 = 1 \otimes d\Gamma (s_2 \oplus -s_2), \quad (s_2 f) (t) = s_2(t) f(t), \quad s_2(t) = \frac{8 (-1 + t^2) t^2}{(1 + t^2)^3},
   \]
\[
   S_3 = 1 \otimes d\Gamma (s_3 \oplus s_3), \quad (s_3 f) (t) = s_3(t) f(t), \quad s_3(t) = \frac{16 (t^2 - 4t^4 + t^6)}{(1 + t^2)^4}.
   \]

Note that $S_1, S_2, S_3$ are all $N$-bounded.

The commutator of $A_0$ with the interaction term can also be considered. As the form factor $f \in \mathfrak{h}$ satisfies $f \in D(p^3)$, $supp f \subset \Lambda_v$ and $\rho$ is differentiable in $\Lambda_v$, we have $g_1, g_2 \in D(\tilde{p}^3)$. We get
   \[
   \langle \varphi, I_0 A_0 \psi \rangle + \langle A_0 \varphi, I_0 \psi \rangle = \langle \varphi, I_1 \psi \rangle,
   \]
on $D(A_0) \times D(A_0)$. Here $I_1$ is defined by

$$I_1 \equiv (Y \otimes 1) \otimes (a (-i\bar{p} g_1) + a^* (-i\bar{p} g_1)) - (1 \otimes \bar{Y}) \otimes (-1)^N (a (-i\bar{p} g_2) - a^* (-i\bar{p} g_2)).$$

Similarly, we have

$$\langle \varphi, I_1 A_0 \psi \rangle + \langle A_0 \varphi, I_1 \psi \rangle = \langle \varphi, I_2 \psi \rangle, \quad \langle \varphi, I_2 A_0 \psi \rangle + \langle A_0 \varphi, I_2 \psi \rangle = \langle \varphi, I_3 \psi \rangle,$$

with

$$I_2 \equiv (Y \otimes 1) \otimes (a (-\bar{p}^2 g_1) + a^* (-\bar{p}^2 g_1)) - (1 \otimes \bar{Y}) \otimes (-1)^N (a (-\bar{p}^2 g_2) - a^* (-\bar{p}^2 g_2)),$$

$$I_3 \equiv (Y \otimes 1) \otimes (a (i\bar{p}^3 g_1) + a^* (i\bar{p}^3 g_1)) - (1 \otimes \bar{Y}) \otimes (-1)^N (a (i\bar{p}^3 g_2) - a^* (i\bar{p}^3 g_2)).$$

### 6 Cut Off of the Interaction and Eigenvector

In case the dispersion relation of the field $L_f$ is $\omega(k) = |k|$, $A_0$ is taken as the generator of the shift operator. And the commutator is given by the number operator $N : [L_f, A_0] = N$. Note that the dispersion relation of $N$ has a strictly positive spectral gap $1 > 0$. However in our case, the dispersion relation is $\omega(k) = \cos(k) - \gamma$, and the commutator is $[L_f, A_0] = S_1$. Note that the dispersion relation of $S_1$ is positive, but it does not have a spectral gap: it attains zero at $t = 0$ and $t = \pm \infty$. The existence of a spectral gap is essential in application of positive commutator method as seen in the arguments in [M].

So this situation causes a problem. To overcome this difficulty, we assume the Assumption [2.1] on the interaction. In this section, we see that as the result of the cut off $\text{supp} f \subset \Lambda_v$, any eigenvector of $L$ is in the range of $P = P(\Lambda_v = 0)$.

Now let $\hbar = \hbar \oplus \hbar_v$. We decompose $\hbar$ with respect to $R = \Lambda_v \oplus \Lambda_v^\perp$:

$$\hbar = \hbar_{\Lambda_v} \oplus \hbar_{\Lambda_v^\perp},$$

$$\hbar_{\Lambda_v} = L^2(\Lambda_v, d\mu) \oplus L^2(\Lambda_v, d\mu), \quad \hbar_{\Lambda_v^\perp} = L^2(\Lambda_v^\perp, d\mu) \oplus L^2(\Lambda_v^\perp, d\mu).$$

Note that $\hbar$ is decomposed into $\hbar = \hbar_{\Lambda_v} \oplus \hbar_{\Lambda_v^\perp}$, with respect to this decomposition of the Hilbert space, because it is a multiplication operator. Let $N_{\Lambda_v}$, $N_{\Lambda_v^\perp}$ be

$$N_{\Lambda_v} \equiv 1 \otimes d\Gamma(1_{\Lambda_v} \oplus 0), \quad N_{\Lambda_v^\perp} \equiv 1 \otimes d\Gamma(0 \oplus 1_{\Lambda_v^\perp}),$$

on $H$ with respect to this decomposition. Below for a Hilbert space $\mathcal{H}$, we denote by $\mathcal{U}(\mathcal{H})$ a set of unitary operators on $\mathcal{H}$, and by $\mathcal{F}(\mathcal{H})$ the Fock space over $\mathcal{H}$. Furthermore, $\Gamma(u)$ is the second quantization of $u \in \mathcal{U}(\mathcal{H})$, and for a self-adjoint operator $\kappa$ on $\mathcal{H}$, $d\Gamma(\mathcal{H}(\kappa)$ is the second quantization of $\kappa$. We have the following proposition:

**Proposition 6.1** There is a unitary operator $U : \mathcal{F}(\hbar_{\Lambda_v} \oplus \hbar_{\Lambda_v^\perp}) \to \mathcal{F}(\hbar_{\Lambda_v}) \otimes \mathcal{F}(\hbar_{\Lambda_v^\perp})$ which satisfies the following conditions;
1. For any $u_{\Lambda_v} \in \mathcal{U}(\tilde{h}_{\Lambda_v})$ and $u_{\Lambda_c} \in \mathcal{U}(\tilde{h}_{\Lambda_c})$, 
\[ U\Gamma(u_{\Lambda_v} \oplus u_{\Lambda_c})U^* = \Gamma(u_{\Lambda_v}) \otimes \Gamma(u_{\Lambda_c}), \]
2. For any $f \in \tilde{h}_{\Lambda_v}$ and $g \in \tilde{h}_{\Lambda_c}$ 
\[ Ua(f \oplus g)U^* = a(f) \otimes 1 + (-1)^{N_{\Lambda_v}} \otimes a(g). \]

By this proposition, we have 
\[ U\Gamma(e^{it\tilde{h}_{\Lambda_v}} \oplus e^{it\tilde{h}_{\Lambda_c}})U^* = \Gamma(e^{it\tilde{h}_{\Lambda_v}}) \otimes \Gamma(e^{it\tilde{h}_{\Lambda_c}}). \]

Recall that $L$ is given by (11). As the Assumption 2.1 \( \text{suppf} \subset \Lambda_v \) implies \( g_1, g_2 \in \tilde{h}_{\Lambda_v} \), we obtain the following unitary equivalence; 
\[ ULU^* = L_S + 1 \otimes d\Gamma_{\Lambda_v}(\tilde{h}_{\Lambda_v}) \otimes 1 + 1 \otimes d\Gamma_{\Lambda_c}(\tilde{h}_{\Lambda_c}) \]
\[ + \lambda(Y \otimes 1) \otimes (a(g_1) + a^*(g_1)) \otimes 1 - \lambda(1 \otimes Y) \otimes (-1)^{N_{\Lambda_v}}(a(g_2) - a^*(g_2)) \otimes (-1)^{N_{\Lambda_c}}. \]

By this equivalence, we have the following lemma,

**Lemma 6.2** $L$ and $N_{\Lambda_c}$ strongly commute.

Similarly, $N_{\Lambda_c}$ strongly commutes with $L_S, L_f, L_0, \lambda I_0, N, S_1$. Using this fact, we obtain the following proposition:

**Proposition 6.3** Suppose $\psi$ is an eigenvector of $L$. Then $\psi$ satisfies 
\[ P\left(N_{\Lambda_c} = 0\right) \psi = \psi, \]
where $P\left(N_{\Lambda_c} = 0\right)$ is the spectral projection of $N_{\Lambda_c}$ corresponding to $N_{\Lambda_c} = 0$.

**Proof**

Let $P_e$ (resp. $P_o$) be the spectral projection of $N_{\Lambda_c}$ onto $N_{\Lambda_c} = \text{even}$ (resp. $N_{\Lambda_c} = \text{odd}$). Let us decompose the Hilbert space $\mathcal{H}$ as 
\[ \mathcal{H} = P_e \mathcal{H} \oplus P_o \mathcal{H}. \]
Then by Lemma 6.2, $L$ is decomposed into 
\[ L = P_e L \oplus P_o L, \]
with respect to the subspaces $P_e \mathcal{H}$ and $P_o \mathcal{H}$. In particular, if $\psi$ is an eigenvector of $L$, $P_e \psi$ is an eigenvector of $P_e L$ and $P_o \psi$ is an eigenvector of $P_o L$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{F}(\tilde{h}_{\Lambda_v}) \otimes \mathcal{F}(\tilde{h}_{\Lambda_c})$, $P_e L$ on $P_e \mathcal{H}$ is 
\[ P_e L|_{P_e \mathcal{H}} = \left(L_S + 1 \otimes d\Gamma_{\Lambda_v}(\tilde{h}_{\Lambda_v}) + V_1 + V_2\right) \otimes 1 + (1 \otimes 1) \otimes d\Gamma_{\Lambda_c}(\tilde{h}_{\Lambda_c})|_{P_e \mathcal{H}} \]
with
\[ V_1 = \lambda (Y \otimes 1) \otimes (a(g_1) + a^*(g_1)) \]
\[ V_2 = -\lambda (1 \otimes \bar{Y}) \otimes (-1)^{N_{\Lambda^c}} (a(g_2) - a^*(g_2)) , \]
because
\[ P_e = 1 \otimes P(N_{\Lambda^c} = \text{even}). \]

On the other hand, \( P_o L \) on \( P_o H \) is
\[ P_o L|_{P_o H} = \left( L_S + 1 \otimes d\Gamma_{\Lambda^c} \left( \bar{h}_{\Lambda^c} \right) + V_1 - V_2 \right) \otimes 1 + (1 \otimes 1) \otimes d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \] \( P_o H \)
because
\[ P_o = 1 \otimes P(N_{\Lambda^c} = \text{odd}). \]

Note that \( d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \) and \( N_{\Lambda^c} \) strongly commute on \( F(\bar{h}_{\Lambda^c}) \). So \( d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \) is decomposed into \( P_e d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \) and \( P_o d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \). On the other hand, \( d\Gamma_{\Lambda^c} \bar{h}_{\Lambda^c} \) in \( F(\bar{h}_{\Lambda^c}) \) has unique eigenvector \( \Omega_{\Lambda^c} \) which is the vacuum vector of \( F(\bar{h}_{\Lambda^c}) \). So \( d\Gamma_{\Lambda^c} |_{P_o F(\bar{h}_{\Lambda^c})} \) has the unique eigenvector \( \Omega_{\Lambda^c} \), while \( d\Gamma_{\Lambda^c} |_{P_e F(\bar{h}_{\Lambda^c})} \) has no eigenvector. Hence by Theorem 7.1 if there exists an eigenvector \( \psi_e \) of \( P_e L \), it is of the form
\[ \psi_e = \varphi \otimes \Omega_{\Lambda^c}, \]
and \( P_o L \) has no eigenvector. Hence if \( \psi \) is the eigenvector of \( L \), we have
\[ P_e \psi = \varphi \otimes \Omega_{\Lambda^c}, \quad P_o \psi = 0. \]
That is,
\[ \psi = \varphi \otimes \Omega_{\Lambda^c}. \]
This means
\[ P(N_{\Lambda^c} = 0) \psi = \varphi \otimes P(N_{\Lambda^c} = 0) \Omega_{\Lambda^c} = \varphi \otimes \Omega_{\Lambda^c} = \psi. \]

\section{7 Positive Commutator}

In this section, we construct strictly positive commutator. First we define the operator \([L, A_0] \) on \( D([L, A_0]) \equiv D(N) \) by
\[ [L, A_0] \equiv S_1 + \lambda I_1. \]
Note that this is defined as an operator of the right hand side, and not as a commutator \( LA_0 - A_0 L \). However the arguments in Section 5 guarantees
\[ \langle L \varphi, A_0 \psi \rangle + \langle A_0 \varphi, L \psi \rangle = \langle \varphi, (S_1 + \lambda I_1) \psi \rangle = \langle \varphi, [L, A_0] \psi \rangle, \]
for all $\psi, \varphi \in D(A_0) \cap D(N)$.

As seen in the previous section, we consider the subspace $PH$. If we restrict ourselves to $P(1 \otimes P_{\Omega})^\perp$, we have

$$P(1 \otimes P_{\Omega})^\perp \|[L, A_0](1 \otimes P_{\Omega})^\perp P \geq (v - \lambda\|I_1\|)P(1 \otimes P_{\Omega})^\perp,$$

and $[L, A_0]$ is strictly positive on $P(1 \otimes P_{\Omega})^\perp \mathcal{H}$ for $\lambda$ small enough. But for $P(1 \otimes P_{\Omega}) \mathcal{H}$, we have

$$P(1 \otimes P_{\Omega})[L, A_0](1 \otimes P_{\Omega})P = 0.$$  

So we need to modify the commutator to make it strictly positive on $P(1 \otimes P_{\Omega}) \mathcal{H}$, too. This is done by introducing a bounded operator $b$:

$$b = b(e) = \theta\lambda(\bar{Q}R^2_1I_0Q - QI_0R^2_1\bar{Q}),$$

$$R_2 = R_2(e) = \left((L_0 - e)^2 + \epsilon^2\right)^{-\frac{1}{2}},$$

where $Q \equiv P(L_S = e) \otimes P_{\Omega}$, $\bar{Q} \equiv 1 - Q$ and $I_0$ is defined at (15). The parameters $\theta$ and $\epsilon$ will be determined later. We denote $R^2_1\bar{Q}$ by $R^2_2$. We define $[L, b(e)]$ on $D(N)$ by $[L, b(e)] = Lb(e) - b(e)L$. Note that $L_Sb(e) - b(e)L_S$ and $\lambda I_0b(e) - b(e)\lambda I_0$ are bounded. For all $\psi \in D(N)$,

$$[L, b(e)]\psi = \theta\lambda \left(\bar{Q}R^2_1[L, I_0]Q - Q[L, I_0]R^2_1\bar{Q}\right)\psi = \theta\lambda \left(\bar{Q}R^2_1\tilde{I}_1Q - Q\tilde{I}_1R^2_1\bar{Q}\right)\psi,$$

where

$$\tilde{I}_1 \equiv (Y \otimes 1) \otimes \left(-a \left(\tilde{h}g_1\right) + a^* \left(\tilde{h}g_1\right)\right) + (1 \otimes \tilde{Y}) \otimes (-1)^N \left(a \left(\tilde{h}g_2\right) + a^* \left(\tilde{h}g_2\right)\right).$$

As $\tilde{I}_1$ is bounded, it can be extended to the whole $\mathcal{H}$. We denote the extension by the same symbol $[L, b(e)]$. We define $[L, A]$ on $D(N)$ by

$$[L, A] = [L, A_0] + [L, b] = S_1 + \lambda I_1 + [L, b].$$

By the above choice of $b$, we have now

$$P(1 \otimes P_{\Omega})[L, A](1 \otimes P_{\Omega})P = 2\theta\lambda^2 P(L_S = e) \otimes P_{\Omega}I_0\bar{R}^2_1I_0(P(L_S = e) \otimes P_{\Omega})P \neq 0.$$  

In this section, we prove the following theorem:

**Theorem 7.1** Let $\Delta$ be the interval of $\mathbb{R}$ whose interior includes an eigenvalue $e$ of $L_S$, and no other eigenvalue is included in $\Delta$, i.e., $\sigma(L_S) \cap \Delta = \{e\}$. Let $\varsigma \in C^\infty_0(\mathbb{R})$ be a smooth function s.t. $\varsigma = 1$ on $\Delta$ and supp $\varsigma \cap \sigma(L_S) = \{e\}$. Let $\theta$ and $\epsilon$ be

$$\epsilon = \lambda^{\frac{1}{100}}, \quad \theta = \lambda^{\frac{-1}{100}}.$$
Let $P$ be the spectral projection of $N_{\Lambda_t}$ onto $\{0\}$. Suppose that Assumption 2.1 and 2.2 are satisfied. Then there exists $\lambda_1 > 0$ s.t.

$$P\varsigma(L)[L,A]\varsigma(L)P \geq \frac{1}{2} \cdot \lambda_1^{\frac{1}{10}} \cdot \gamma \varsigma(L) \left(1 - 14 \tilde{P}\gamma\right) \varsigma(L)P,$$

for any $0 < \lambda < \lambda_1$. Furthermore, if we assume $\beta_0 < \beta_1$, $\beta_- < \beta_1$, $\|p f\|, \|f\| \leq b$ for any fixed $0 < \beta_0 < \beta_1 < \infty$ and $0 < b < \infty$, we can choose $\lambda_1$ as

$$\lambda_1 = C \min\{v \frac{10}{\gamma e}, \left(\frac{v}{\gamma e}\right)^{\frac{10}{\gamma e}}, (\gamma e)^{\frac{10}{\gamma e}}, (v\gamma e)^{\frac{10}{\gamma e}}\}.$$

Here $C$ is a constant which depends on $\beta_0, \beta_1$ and $b$, but is independent of $v$ and $\beta_+ - \beta_-$. We carry out the proof in two steps. First, we show the strict positivity of $[L,A]$ with respect to the spectral localization in $L_0$. Second we derive the strict positivity of $[L,A]$ with respect to the spectral localization in $L$.

### 7.1 Positive commutator with respect to the spectral localization in $L_0$

In this subsection, we take the first step. Here is the main theorem of this subsection:

**Theorem 7.2** Let $\Delta$ be an interval of $\mathbb{R}$, containing exactly one eigenvalue $e$ of $L_S$ and let $E_{\Delta}^0$ be the spectral projection of $L_0$ onto $\Delta$. Suppose that Assumption 2.1 and 2.2 are satisfied. Then there exists $0 < t_i, i = 1, \cdots, 5$ such that, if $\lambda, \epsilon, \theta$ satisfy

$$\lambda + \theta \lambda^2 \epsilon^{-2} < t_1, \quad \theta < t_2, \quad \theta \lambda^2 \epsilon^{-1} < t_3, \quad \epsilon < t_4, \quad \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} < t_5,$$

then

$$P E_{\Delta}^0 [L,A] E_{\Delta}^0 P \geq \frac{\theta \lambda^2}{\epsilon} (1 - 14 \tilde{P} \gamma) E_{\Delta}^0 P.$$ 

The proof goes parallel to [M]. The difference emerges from the difference of the commutator $[L_0,A_0]$ in [M], it was $[L_0,A_0] = N$, while we have $[L_0,A_0] = S_1$. Whereas $N$ has a spectral gap, $S_1$ has no spectral gap. Since positive commutator method makes use of the finite spectral gap, the lack of the gap causes a difficulty. To overcome this, we introduced the cut off in Section 6.

To prove the theorem, we use the Feshbach map theorem [BFS1]:

**Theorem 7.3** Let $H$ be a closed operator densely defined on a Hilbert space $\mathcal{H}$, and let $P$ be a projection operator such that $\text{Ran}P \subset D(H)$. We set $\tilde{P} = 1 - P$. Then $H_\tilde{P} \equiv \tilde{P} H \tilde{P}$ is a densely defined operator on $\tilde{P} \mathcal{H}$. Let $z$ be an element in the resolvent set $\rho(H_\tilde{P})$ of $H_\tilde{P}$ on $\tilde{P} \mathcal{H}$. Assume

$$\|\tilde{P} (H_\tilde{P} - z)^{-1} \tilde{P}\| < \infty, \quad \|\tilde{P} (H_\tilde{P} - z)^{-1} \tilde{P} H \tilde{P}\| < \infty.$$
Then the Feshbach map
\[ f_P(H - z) \equiv P(H - z)P - PH\bar{P}(H - z)^{-1}\bar{P}HP \]
is well defined on \( PH \). \( f_P(H - z) \) and \( H \) have the isospectral property in the sense that
\[ z \in \sigma(H) \iff 0 \in \sigma(f_P(H - z)), \quad z \in \sigma_{PP}(H) \iff 0 \in \sigma_{PP}(f_P(H - z)). \]
Here \( \sigma \) and \( \sigma_{PP} \) represent spectrum and pure point spectrum respectively.

To apply the theorem, we introduce
\[ \chi_\nu \equiv P(N \leq \nu). \]
By this \( \chi_\nu \), we can consider \( B'_{ij} \) below as bounded operators. Projections \( \tilde{P}_e, \chi_\nu, P, Q, E_\Delta^0 \) are in relation that
\[ \tilde{P}_e \leq \chi_\nu, \quad \tilde{P}_e \leq P, \quad \tilde{P}_e \leq Q, \quad Q \leq E_\Delta^0, \quad Q \leq P. \]

Hence we have the followings;
\[ Q\tilde{P}_e = 0, \quad \tilde{Q}\tilde{P}_e = \tilde{Q}, \quad E_\Delta^0 P\chi_\nu \tilde{P}_e = \tilde{P}_e. \]

As \( L_0, N, L_S \) strongly commute, the projections \( E_\Delta^0, Q, P, \chi_\nu \) commute each other. So we can define the following projections
\[ Q_1 \equiv QE_\Delta^0 P\chi_\nu, \quad Q_2 \equiv \tilde{Q}E_\Delta^0 P\chi_\nu, \quad (22) \]
which satisfy
\[ Q_1\tilde{P}_e = \tilde{P}_e, \quad Q_2\tilde{P}_e = 0, \quad Q_2\tilde{P}_e^\perp = Q_2. \quad (23) \]

The relation \( NP = N_\Lambda, P \) implies
\[ N_\Lambda, Q_2 = N_\Lambda, P\tilde{Q}E_\Delta^0 \chi_\nu = NP\tilde{Q}E_\Delta^0 \chi_\nu = NQ_2. \]

As \( P(N = 0) = 1 \otimes P_{1_f} \), and \( \Delta \cap \sigma(L_0) = \{ e \} \), we have
\[ P(N = 0)\tilde{Q} = (1 - P(L_S = e)) \otimes P_{1_f} = \sum_{f \neq e} P(L_S = f) \otimes P_{1_f} \leq P(L_0 \subset \Delta^c), \]
which implies
\[ P(N = 0)Q_2 = P(N = 0)\tilde{Q}E_\Delta^0 P\chi_\nu = 0. \]

It follows that
\[ NQ_2 = N_\Lambda, Q_2 \geq Q_2. \quad (24) \]
Proof of Theorem 7.2
We apply Theorem 7.3 to the operator $B'$ defined by

$$B' ≡ [L, A] - δ_e \cdot \hat{P}_e^\perp.$$  

Here, $δ_e$ is a parameter which will be determined later. We define bounded operators $B'_{ij}$ by

$$B'_{ij} ≡ Q_i B' Q_j \quad i, j = 1, 2.$$  

If $ϑ \in \rho(B'_{22})$, then

$$\varepsilon (ϑ) ≡ B'_{11} - B'_{12}(B'_{22} - ϑ)^{-1} B'_{21},$$

is well-defined and

$$\varepsilon (ϑ) = f_{Q_1} (B' - ϑ) + ϑ Q_1.$$  

First we investigate the lower bound of $B'_{22}$ in order to study the resolvent set $ρ(B'_{22})$.

**Proposition 7.4** We have

$$B'_{22} ≥ (v - δ_e - C_1 (λ + θ λ^2 ε^{-2})) \cdot Q_2,$$

where

$$C_1 = \|I_1\| + 2 \cdot \|I_0\|^2.$$  

**Proof**
We estimate each term of

$$B'_{22} = Q_2 (S_1 + λ I_1 + [L, b] - δ_e \cdot \hat{P}_e^\perp) Q_2.$$  

From the inequality $S_1 ≥ v \cdot N_{λe}$ and the inequality (21), the first term is bounded below as $Q_2 S_1 Q_2 ≥ v Q_2$. The lower bound of the second term is given by $λ I_1 ≥ -λ \cdot \|I_1\|$, as $I_1$ is bounded. Let us estimate the norm of the third term. Substituting $L = L_0 + λ I_0$, $L_0 Q = Q L_0$ and $Q_2 Q = 0$, we obtain

$$Q_2 [L, b] Q_2 = -θ λ^2 Q_2 (I_0 Q L_0 R_e^2 Q + Q R_e^2 I_0 Q L_0) Q_2.$$  

Then it is estimated as $\|Q_2 [L, b] Q_2\| ≤ 2 \cdot \|I_0\|^2 θ λ^2 ε^{-2}$, since we have $\|R_e^2\| ≤ ε^{-2}$. The fourth term is $Q_2 \hat{P}_e^\perp Q_2 = 2 Q_2$ by (23). Hence we obtain

$$B'_{22} ≥ (v - C_1 (λ + θ λ^2 ε^{-2}) - δ_e) Q_2$$

where $C_1 = \|I_1\| + 2 \cdot \|I_0\|^2.$  

Suppose $λ, θ, ε$ satisfies

$$λ + θ λ^2 ε^{-2} < t_1,$$

where $t_1 = \frac{v}{2C_1}$. Then if $ϑ$ satisfies $ϑ ≤ \frac{1}{2} v - δ_e$, it is in the resolvent set $ρ(B'_{22})$ of $B'_{22}$ by this proposition.

Next we estimate the lower bound of $ ε (ϑ)$.  

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Proposition 7.5 Suppose \( \lambda, \theta, \epsilon \) satisfies

\[ \lambda + \theta \lambda^2 \epsilon^{-2} < t_1, \]

where \( t_1 = \frac{v}{\sqrt{v}} \). Then if \( \vartheta \) satisfies \( \vartheta \leq \frac{1}{2} v - \delta \epsilon \), it is in the resolvent set \( \rho(B'_{22}) \) of \( B'_{22} \) and the following inequality holds;

\[ \varepsilon(\vartheta) \geq 2\theta \lambda^2 Q_1 \left[ I_0 \bar{R}^2_0 I_0 - \frac{1}{2\theta \lambda^2} \delta_e \cdot \tilde{P}_e^\perp \right] Q_1 - C_2 \left( \theta^2 \lambda^2 Q_1 I_0 \bar{R}^2_0 I_0 Q_1 + (\lambda^2 + \theta^2 \lambda^4 \epsilon^{-4}) Q_1 \right) \]

where \( C_2 = 10v^{-1}\|I_1\|^2 + 20v^{-1} + 80v^{-1}\|I_0\|^4 \).

Proof
First \( B'_{11} \) is given by

\[ B'_{11} = Q_1 \left( S_1 + \lambda I_1 + [L, b] - \delta_e \cdot \tilde{P}_e^\perp \right) Q_1. \]

As \( S_1 Q = 0 \) and \( aQ = Qa^* = 0 \), the first and the second term vanish. Using \( \bar{Q} Q_1 = 0 \), \( L_0 Q = Q L_0 \), we obtain

\[ B'_{11} = 2\theta \lambda^2 Q_1 \left[ I_0 \bar{R}^2_0 I_0 - \frac{1}{2\theta \lambda^2} \delta_e \cdot \tilde{P}_e^\perp \right] Q_1. \]

Second, \( Q_2 (B'_{22} - \vartheta)^{-1} Q_2 \) is evaluated as follows;

Lemma 7.6 Suppose \( \lambda + \theta \lambda^2 \epsilon^{-2} < t_1 \). Then for all \( \vartheta \leq \frac{1}{2} v - \delta \epsilon \), \( \vartheta \) is in the resolvent set \( \rho(B'_{22}) \) of \( B'_{22} \) and

\[ 0 \leq \left\langle \psi, Q_2 (B'_{22} - \vartheta)^{-1} Q_2 \psi \right\rangle \leq 5\|S_1^{-\frac{1}{2}} Q_2 \psi\|^2, \quad \forall \psi \in \mathcal{H}. \]

Proof
As \( S_1 Q_2 \geq v N_{\lambda}, Q_2 \geq v Q_2, S_1 \) is invertible on \( Q_2 \mathcal{H} \). The proof is similar to that of equation (41) in \[M\]. In facts, it is easier because we are considering Fermion system, whose interaction terms are bounded. We omit the details. \( \square \)

Third, \( \|S_1^{-\frac{1}{2}} B'_{21} \psi\| \) is estimated as follows;

Lemma 7.7

\[ \left\| S_1^{-\frac{1}{2}} B'_{21} \psi \right\|^2 \leq C'_2 \left( \theta^2 \lambda^2 \| R_e I_0 Q_1 \psi \|^2 + \theta^2 \lambda^4 \epsilon^{-4} \| \psi \|^2 + \lambda^2 \| \psi \|^2 \right) \]

where \( C'_2 = 2v^{-1}\|I_1\|^2 + 4v^{-1} + 16v^{-1}\|I_0\|^4 \).

Proof
The proof is the same as that of \[M\] (p342). \( \square \)
Now let us complete the proof of Proposition 7.5. Combining Lemma 7.6 and 7.7, we obtain the following evaluation: if \( \lambda, \theta, \epsilon \) satisfies \( \lambda + \theta \lambda^2 \epsilon^{-2} < t_1 \), then for all \( \vartheta \leq \frac{1}{2} v - \delta_c \), we have \( \vartheta \in \rho(B_{22}') \) and

\[
0 \leq \langle \psi, B'_{12} (B'_{22} - \vartheta)^{-1} B'_{21} \psi \rangle \leq C_2 \left( \theta^2 \lambda^2 \| \hat{R}_\epsilon I_0 Q_1 \psi \|^2 + (\lambda^2 + \theta^2 \lambda^4 \epsilon^{-4}) \| \psi \|^2 \right)
\]

where \( C_2 = 10v^{-1} \| I_1 \|^2 + 20v^{-1} + 80v^{-1} \| I_0 \|^4 \). Combining this with (25), we obtain the required estimation. \( \Box \)

Now let us complete the proof of Theorem 7.2. Suppose that \( \lambda + \theta \lambda^2 \epsilon^{-2} < t_1 \). Then by Proposition 7.6 for all \( \vartheta \leq \frac{1}{2} v - \delta_c \), we have \( \vartheta \in \rho(B_{22}') \) and

\[ \epsilon (\vartheta) \geq D, \]

where

\[
D = 2\theta \lambda^2 Q_1 \left[ I_0 R_\epsilon^2 I_0 - \frac{1}{2\theta \lambda^2} \delta_c \cdot \hat{P}_\epsilon^\perp \right] Q_1 - C_2 \left( \theta^2 \lambda^2 Q_1 I_0 \hat{R}_\epsilon^2 I_0 Q_1 + (\lambda^2 + \theta^2 \lambda^4 \epsilon^{-4}) Q_1 \right).
\]

Note that \( D \) is \( \vartheta \)-independent.

Now let \( \vartheta_0 \equiv \inf \sigma(B'|(Q_1 \oplus Q_2)\mathcal{H}) \). Here we have two cases,

1. \( \vartheta_0 \geq \frac{1}{2} v - \delta_c \).
2. \( \frac{1}{2} v - \delta_c > \vartheta_0 \). In this case, \( \vartheta_0 \in \rho(B_{22}') \) and \( \epsilon (\vartheta_0) \) is well-defined. Then \( f_{Q_1}(B' - \vartheta_0) \) satisfies

\[
f_{Q_1}(B' - \vartheta_0) = \epsilon (\vartheta_0) - \vartheta_0 \cdot Q_1 \geq (\inf \sigma(D) - \vartheta_0) \cdot Q_1.
\]

On the other hand, note that \( \vartheta_0 \in \sigma(B'|(Q_1 \oplus Q_2)\mathcal{H}) \). By Feshbach Theorem 7.8, this implies \( 0 \in \sigma(f_{Q_1}(B' - \vartheta_0)) \). Hence \( f_{Q_1}(B' - \vartheta_0) \) is not invertible, and we get

\[ \vartheta_0 \geq \inf \sigma(D). \]

So we have

\[ \vartheta_0 \geq \min \{ \frac{1}{2} v - \delta_c, \inf \sigma(D) \}, \]

i.e.,

\[ B'|(Q_1 \oplus Q_2)\mathcal{H} \geq \min \{ \frac{1}{2} v - \delta_c, \inf \sigma(D) \}. \] (26)

The next problem is to investigate the lower bound of \( D \). For the purpose, we estimate \( Q_1 I_0 \hat{R}_\epsilon^2 I_0 Q_1 \). The annihilation operator with respect to \( g^1 \oplus g^2 \in \mathfrak{h} \oplus \mathfrak{h} \) is represented by operator valued distribution \( a^1_k, a^2_k \) as

\[ a(g^1 \oplus g^2) = \int dk \left( \hat{g}^1(k) a^1_k + \hat{g}^2(k) a^2_k \right). \]
Here, $a_k^i, a_k^{i*}$ satisfies \( \{ a_k^i, a_p^{i*} \} = \delta_{ij} \delta(k-p) \). By the pull-through formula,
\[
a_k^1 L_f = (L_f + \omega(k))a_k^1, \quad a_k^2 L_f = (L_f - \omega(k))a_k^2,
\]
we get
\[
a(g^1 \oplus g^2) R_\epsilon(e) = \int dk \left( \bar{g}^1(k) R_\epsilon^2(e - \omega(k)) a_k^1 + \bar{g}^2(k) R_\epsilon^2(e + \omega(k)) a_k^2 \right).
\]
Using this relation, we obtain the following bound;

**Proposition 7.8** Under Assumption 2.2 we have \( Q_1 I_0 \bar{R}_0^2 I_0 Q_1 \geq \pi \epsilon Q_1 \left[ \Gamma(e) \otimes 1 - C_3 \epsilon^{\frac{3}{4}} \right] Q_1 \).

Here \( C_3 \) is a positive constant which is independent of \( \beta_+ - \beta_- \) and \( v \).

Assume that \( \theta, \lambda, \epsilon \) satisfies the followings.

**Assumption 7.1**
\[
\lambda + \theta \lambda^2 \epsilon^{-2} < t_1, \quad \theta < t_2, \quad \theta \lambda^2 \epsilon^{-1} < t_3, \quad \epsilon < t_4, \quad \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} < t_5.
\]
where
\[
t_2 = \frac{1}{C_2}, \quad t_3 = \frac{v}{8\pi \gamma_e}, \quad t_4 = \left( \frac{\gamma_e}{4C_3} \right)^4, \quad t_5 = \frac{\pi \gamma_e}{4C_2}.
\]

The second assumption implies \( 2\theta \lambda^2 - C_2 \theta^2 \lambda^2 > \theta \lambda^2 \). Using this and the lower bound of \( I_0 \bar{R}_0^2 I_0 \) of Proposition 7.8, we obtain
\[
D \geq \theta \lambda^2 \frac{\pi}{\epsilon} Q_1 \left[ \Gamma(e) \otimes 1 - C_3 (\epsilon^{\frac{1}{2}}) \right] Q_1 - \delta_e Q_1 \bar{P}_e \epsilon Q_1 - C_2 (\lambda^2 + \theta^2 \lambda^4 \epsilon^{-1}) Q_1.
\]
(27)

Recall that \( \Gamma(e) \) is bounded below as \( \Gamma(e) \geq \gamma_e \cdot (P(\Gamma(e) = 0))^{\frac{1}{2}} \). Substituting this to (27), we get
\[
D \geq \theta \lambda^2 \frac{\pi}{\epsilon} Q_1 \left[ \left( \gamma_e - \delta_e \frac{\epsilon}{\pi \theta \lambda^2} \right) \cdot \bar{P}_e \epsilon - C_3 (\epsilon^{\frac{1}{2}}) - \frac{C_2}{\pi} (\theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3}) \right] Q_1.
\]
(28)

Now we determine \( \delta_e \) as
\[
\delta_e = \frac{\pi \theta \lambda^2}{\epsilon} (\gamma_e + a) \quad \text{with} \quad 0 < a < \gamma_e.
\]

Then we have
\[
D \geq \theta \lambda^2 \frac{\pi}{\epsilon} \left[ -a - C_3 (\epsilon^{\frac{1}{2}}) - \frac{C_2}{\pi} (\theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3}) \right] Q_1.
\]
Hence \( \inf \sigma(D) \) is bounded below by
\[
\inf \sigma(D) \geq \theta \lambda^2 \pi \epsilon \left[ -a - C_3(\epsilon^\frac{1}{2}) - \frac{C_2}{\pi} \left( \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} \right) \right] Q_1.
\]

Recall the lower bound (26) of \( B' \). As the third condition of Assumption 7.1 implies
\[
\frac{1}{2} v - \delta_c \geq \frac{1}{4} v > 0,
\]
and
\[
0 \geq \theta \lambda^2 \pi \epsilon \left[ -a - C_3(\epsilon^\frac{1}{2}) - \frac{C_2}{\pi} \left( \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} \right) \right],
\]
the lower bound is
\[
B'|_{(Q_1 \oplus Q_2)H} \geq \theta \lambda^2 \pi \epsilon \left[ -a - C_3(\epsilon^\frac{1}{2}) - \frac{C_2}{\pi} \left( \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} \right) \right].
\]

Substituting \([L, A] = B' + \delta_c \hat{P}_c \), and the last two conditions of Assumption 7.1, we get
\[
PE_\Delta^0[L, A]E_\Delta^0 P \geq \theta \lambda^2 \pi \epsilon PE_\Delta \left[ -a - C_3(\epsilon^\frac{1}{2}) - \frac{C_2}{\pi} \left( \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} \right) - (\gamma_c + a) \cdot \hat{P}_c \right] E_\Delta^0 P
\]
\[
= \theta \lambda^2 \pi \epsilon PE_\Delta \left[ \gamma_c - C_3(\epsilon^\frac{1}{2}) - \frac{C_2}{\pi} \left( \theta^{-1} \epsilon + \theta \lambda^2 \epsilon^{-3} \right) - (\gamma_c + a) \cdot \hat{P}_c \right] E_\Delta^0 P
\]
\[
\geq \theta \lambda^2 \epsilon^{-1} PE_\Delta^0 \gamma_c \left[ 1 - 7 \hat{P}_c \right] E_\Delta^0 P.
\]

Hence we obtain Theorem 7.2. \( \square \)

7.2 Positive Commutator with respect to the spectral localization in \( L \)

In this subsection we complete the proof of Theorem 7.1.

Let \( \Delta' \) be an interval s.t. \( \Delta \subset \Delta' \), \( \Delta' \cap \sigma(L_S) = \{ e \} \) and \( \text{supp} \varsigma \subset \Delta' \). Let \( F_\Delta' \in C_0^\infty(\mathbb{R}) \) be a smooth function \( 0 \leq F_\Delta' \leq 1 \) which satisfies \( F_\Delta' = 1 \) on \( \text{supp} \varsigma \) and \( \text{supp} F_\Delta' \subset \Delta' \). We set \( F_\Delta^0 \equiv F_\Delta'(L_0) \), and \( \tilde{F}_\Delta^0 \equiv 1 - F_\Delta^0 \). We evaluate each term of the following equation;
\[
P_\varsigma [L, A] \varsigma P = P_\varsigma F_\Delta^0[L, A]F_\Delta^0 \varsigma P
\]
\[
+ P_\varsigma \tilde{F}_\Delta^0[L, A]F_\Delta^0 \varsigma P + \text{h.c.}
\]
\[
+ P_\varsigma \tilde{F}_\Delta^0[L, A]F_\Delta^0 \varsigma P
\]
For the first part (29), we have the following lemma.
Lemma 7.9 Suppose that $\lambda, \epsilon, \theta$ satisfy Assumption 7.1. Then we have

$$\mathcal{L} = P \varsigma F_0^0 [L, A] F_0^0 \varsigma P \geq \frac{\theta \lambda^2}{\epsilon} \gamma \epsilon P \varsigma (1 - \lambda c_1 - 7 \tilde{P} \epsilon) \varsigma P,$$

where $c_1$ is a constant which depends only on $F_{\Delta'}$ and $\|I_0\|$.

Proof

The proof is the same as that of inequality (63) in [M]. □

Next we evaluate the second part (30).

Lemma 7.10

$$\mathcal{L} = P \varsigma F_0^0 [L, A] F_0^0 \varsigma P + h.c. \geq -c_2 \theta \lambda^2 \epsilon (\theta^{-1} + \epsilon + \lambda \epsilon^{-1}) P \varsigma^2 P$$

where $c_2$ is a constant which depends only on $F_{\Delta'}$ and $I_0$.

Proof

We divide (30) into three parts;

\begin{align*}
\mathcal{L} & = P \varsigma F_0^0 [L, A] F_0^0 \varsigma P + h.c. \quad (32) \\
& + \lambda P \varsigma F_0^0 [I_1, F_0^0 \varsigma P + h.c.] \quad (33) \\
& + P \varsigma F_0^0 [L, b] F_0^0 \varsigma P + h.c. \quad (34)
\end{align*}

Using the fact that $0 \leq F_{\Delta'} \leq 1$, we get (32) $\geq 0$ as in the case (57) of [M].

To evaluate (33), we note $\varsigma \bar{F}_0^0 \varsigma F_0^0 \varsigma P = \varsigma (L) \bar{F}_0^0 \varsigma (L_0) - \bar{F}_0^0 \varsigma (L)).$ By operator calculus, we have the estimation $\| \varsigma F_0^0 \| \leq c_1' |\lambda|$. As $I_1$ is bounded, we obtain $\mathcal{L} \geq -c_1' \lambda^2$. Here, $c_1', c_1''$ depends only on $I_0$ and $F_{\Delta'}$.

The last part (34) can be estimated in a manner of Proposition 5.2 in [M]:

$$\mathcal{L} = P \varsigma F_0^0 [L, b] F_0^0 \varsigma P + h.c. = P \varsigma \left( \sqrt{F_{\Delta'}(L_0)} - \sqrt{F_{\Delta'}(L)} \right) \sqrt{F_{\Delta'}(L, b) F_0^0 \varsigma P + h.c.} \geq -c_2'' (\theta \lambda^2 + \theta \lambda^3 \epsilon^{-2}) \varsigma^2 P = -c_2'' \frac{\theta \lambda^2}{\epsilon} (\epsilon + \lambda \epsilon^{-1}) \varsigma^2 P.$$

Hence we have

$$\mathcal{L} = (32) + (33) + (34) \geq -c_2 \frac{\theta \lambda^2}{\epsilon} (\theta^{-1} \epsilon + \epsilon + \lambda \epsilon^{-1}) \varsigma^2 P,$$

where $c_2$ depends only on $I_0$ and $F_{\Delta'}$. □

Finally, the third part (31) can be estimated as follows.

Lemma 7.11

$$P \varsigma F_0^0 [L, A] F_0^0 \varsigma P \geq -c_3 \frac{\theta \lambda^2}{\epsilon} (\theta^{-1} \epsilon + \lambda \epsilon^{-1}) \varsigma^2 P,$$

where $c_3$ depends only on $I_0$ and $F_{\Delta'}$. 

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Proof
The same as that of Proposition 5.2 in [M]. □

Now let us complete the proof of Theorem 7.1. By Lemma 7.9 to 7.11, if \(\lambda, \epsilon\) and \(\theta\) satisfy Assumption 7.1, we obtain

\[
P_\varsigma [L, A] \varsigma P = (29) + (30) + (31) \geq \theta \lambda^2/\epsilon \gamma e P_\varsigma \left(1 - 7 \tilde{P}_e - \left(\lambda C_4 + \frac{C_5}{\gamma e} (\epsilon \theta^{-1} + \epsilon + \lambda \epsilon^{-1})\right)\right) \varsigma P,
\]

where \(C_4, C_5\) depends only on \(F'\Delta\) and \(I_0\). Let \(\epsilon, \theta\) be

\[
\epsilon = \lambda^{44/100}, \quad \theta = \lambda^{26/100}.
\]

Then

\[
\epsilon \theta^{-1} = \lambda^{18/100}, \quad \lambda \epsilon^{-1} = \lambda^{16/100}, \quad \theta \lambda^2 \epsilon^{-1} = \lambda^{48/100}, \quad \theta \lambda^2 \epsilon^{-2} = \lambda^{118/100}, \quad \theta \lambda^2 \epsilon^{-3} = \lambda^{44/100}
\]

and if \(\lambda\) is sufficiently small, Assumption 7.1 is satisfied. Furthermore, for \(\lambda\) small enough,

\[
\lambda C_4 + \frac{C_5}{\gamma e} (\epsilon \theta^{-1} + \epsilon + \lambda \epsilon^{-1}) < \frac{1}{2},
\]

is satisfied. Hence we obtain

\[
P_\varsigma [L, A] \varsigma P \geq \frac{\theta \lambda^2}{\epsilon} \frac{\gamma e}{2} P_\varsigma \left(1 - 14 \tilde{P}_e\right) \varsigma P = \lambda^{182/100} \frac{\gamma e}{2} P_\varsigma \left(1 - 14 \tilde{P}_e\right) \varsigma P.
\]

Now let us estimate the range of \(\lambda\), which satisfies the Assumption 7.1 and (35). We assume \(\beta_0 < \beta_+ < \beta_- < \beta_1\) for any fixed \(0 < \beta_0 < \beta_1 < \infty\), and the bound of the form factor \(f \in \mathfrak{h} \|pf\|, \|f\| \leq b\) for fixed \(0 < b < \infty\). Under these bounds, one can easily check that there exists a constant \(C\) which is independent of \(v\) and \(\beta_+ - \beta_-\), such that if \(\lambda\) satisfies

\[
\lambda < C \min\{v^{100/26}, \left(\frac{v}{\gamma e}\right)^{100/192}, (\gamma e)^{100/192}, (v\gamma e)^{100/192}\},
\]

then the Assumption 7.1 and (35) are satisfied. Let

\[
\lambda_1 = C \min\left\{v^{100/26}, \left(\frac{v}{\gamma e}\right)^{100/192}, (\gamma e)^{100/192}, (v\gamma e)^{100/192}\right\},
\]

and we obtain Theorem 7.1. □

8 Virial Theorem

In this section, we complete the proof of Theorem 7.1. We apply the new method introduced by M.Merkli [M] to treat the domain question. He solved...
the problem by approximating the eigenvector of $L$ by vectors in the domain of $N$ and $A_0$.

Assume that $\psi$ is a normalized eigenvector of $L$ with eigenvalue $e$. Let $f$ be a bounded $C^\infty$-function such that $f' \geq 0$, $f'(0) = 1$ and let $g$ be a bounded $C^\infty$-function with support in the interval $[-1, 1]$. Define the operators $f_\alpha \equiv f(i\alpha A_0)$, $h_\alpha \equiv \sqrt{f'(i\alpha A_0)}$ and $g_\nu \equiv (\nu N)$. When $\alpha, \nu$ goes to zero, $h_\alpha, g_\nu$ strongly converges to 1. By approximating the eigenvector $\psi$ by $h_\alpha g_\nu \psi$, we can carry out the arguments regelously.

As the proof goes parallel to [M], we just comment on the differences. We define $\tilde{e}(e)$ as in Theorem 3.1. For simplicity, we use the notations $K \equiv [L, A_0] = S_1 + I^1, \langle A \rangle \equiv \langle \psi, A\psi \rangle, \psi_{\alpha, \nu} \equiv h_\alpha g_\nu \psi, \text{ and } I^\lambda \equiv \lambda I_j$. The proof is done by evaluating the upper and the lower bound of $\langle K \rangle_{\psi_{\alpha, \nu}} = \langle \psi_{\alpha, \nu}, K\psi_{\alpha, \nu} \rangle$.

The estimation of the upper bound is done by the expansion of commutators, using operator calculous:

$$
\langle K \rangle_{\psi_{\alpha, \nu}} = -\alpha^{-1} \nu \text{Im} \langle \psi | g_\nu f_\alpha I_0 \psi \rangle + \alpha^2 \langle \psi | g_\nu R g_\nu \psi \rangle + \text{Re} \langle \psi | g_\nu R' g_\nu \psi \rangle.
$$

(36)

Here, $I_0, R$ and $R'$ are given by

$$
\begin{align*}
I_0 &= 2 \int d\tilde{g}(z)(z - \nu N)^{-1} \alpha d^3_N(I^0_N)(z - \nu N)^{-1}, \\
R &= -\frac{1}{4} \int df''(z)(z - i\alpha A_0)^{-1} \alpha^2 d^3_{A_0}(L_0 + I^0_0)(z - i\alpha A_0)^{-1} \\
- \frac{1}{2} \int d\tilde{h}(z)(z - i\alpha A_0)^{-2} [h_\alpha, \alpha d^3_{A_0}(L_0 + I^0_0)] (z - i\alpha A_0)^{-1} \\
+ \frac{1}{2} \int d\tilde{h}(z)(z - i\alpha A_0)^{-1} h_\alpha \alpha d^3_{A_0}(L_0 + I^0_0)(z - i\alpha A_0)^{-1} \\
R' &= \int df(z)(z - i\alpha A_0)^{-3} \alpha d^3_{A_0}(L_0 + I^0_0)(z - i\alpha A_0)^{-1}
\end{align*}
$$

where $\alpha d^k_M$ is a $k$-fold commutator $[\cdots [\cdot, M], M, \cdots, M]$ with $M$. In the case of $M$, the commutators are given by

$$
\begin{align*}
\alpha d^3_{A_0}(L_0) &= N, \\
\alpha d^2_{A_0}(L_0) &= \alpha d^3_{A_0}(L_0) = 0,
\end{align*}
$$

and $\alpha d^k_{A_0}(I^0_0)$ and $\alpha d^k_N(I^0_N)$ are $N$-bounded. Hence $\langle K \rangle_{\psi_{\alpha, \nu}}$ is estimated as

$$
\langle K \rangle_{\psi_{\alpha, \nu}} \sim O(\alpha^2 \nu^{-\frac{1}{2}} + \nu^{\frac{1}{2}} \alpha^{-1}).
$$

(37)

In our case, the $k$-fold commutators don’t vanish:

$$
\begin{align*}
\alpha d^3_{A_0}(L_0) &= S_1, \\
\alpha d^2_{A_0}(L_0) &= S_2, \\
\alpha d^3_{A_0}(L_0) &= S_3.
\end{align*}
$$

On the other hand, the interaction terms $\alpha d^k_{A_0}(I_0)$, and $\alpha d^k_N(I^0_0)$ are bounded. Hence $\langle K \rangle_{\psi_{\alpha, \nu}}$ is estimated as

$$
\langle K \rangle_{\psi_{\alpha, \nu}} \sim O(\alpha^2 \nu^{-1} + \alpha^{-1} \nu).
$$

(38)
The difference emerges from two factors: the non-commutativity of $S_i$ with $A_0$, and the boundedness of the interaction. The first one makes things worse, while the second one makes it easier.

The estimation of the lower bound also goes parallel to [M]. As the positive commutator is localized with respect to the spectrum of $L$, we need to decompose the Hilbert space. Let $\Delta$ be an interval which contains $e$. Suppose that $\Delta$ contains exactly one eigenvalue $\bar{e}(e)$ of $LS$ i.e., $\Delta \cap \sigma(LS) = \{\bar{e}(e)\}$, and $\bar{e}(e)$ belongs to the interior of $\Delta$. In [M], M. Merkli introduced a partition of unity

$$\chi_\Delta^2 + \bar{\chi}_\Delta^2 = 1.$$ 

Here $\chi_\Delta \in C_0^\infty(\mathbb{R})$ is a smooth function s.t. $\chi_\Delta = 1$ on $\Delta$ and supp $\chi_\Delta \cap \sigma(LS) = \{\bar{e}(e)\}$. By Theorem 7.1, we have

$$P\chi_\Delta(L)\left[L, A\right] \chi_\Delta(L)P \geq b_{\bar{e}(e)}P\chi_\Delta(L) \left(1 - 14\bar{P}_{\bar{e}(e)}\right) \chi_\Delta(L)P, \quad (39)$$

for $0 < |\lambda| < \lambda_1$, where $b_{\bar{e}} = \frac{1}{2} \cdot \lambda^{\frac{30}{16}} \cdot \gamma_{\bar{e}}$. Another partition of unity is also needed:

$$\chi^2 + \bar{\chi}^2 = 1,$$

where $\chi \in C^\infty$ satisfies $\chi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\chi(t) = 0$ for $1 \leq |t|$. And he set $\chi_n = \chi(N/n)$, $\bar{\chi}_n^2 = 1 - \chi_n^2$ for $0 < n < 1/\nu$. With respect to the partition of unity, he obtained the lower bound

$$\langle K \rangle_{\psi_\alpha, \nu} \geq b_{\bar{e}(e)}\left\|P_{\alpha, \nu}\right\|^2 - Cb_{\bar{e}(e)}\delta_{\bar{e}(e), 0}\left\|\bar{P}_{\bar{e}(e)}\chi_n\psi_\alpha, \nu\right\|^2
- O(e\nu^{-1} + \eta + an^\frac{\alpha}{2} + n^{-\frac{\alpha}{2} - 1} - Cb_{\bar{e}(e)}(n^{-1} + n^{-3} + \alpha^2n^2),$$

where $C$ is a positive constant and $\eta, \epsilon > 0$ are arbitrary positive parameter which satisfy $\frac{\nu^2}{2} - C\eta^{-1}\epsilon^{-2} \geq b_{\bar{e}(e)}$. Here we represented the result in our notations.

The argument can be carried out parallel in our case. We just need to take care of the projection $P = P(N_{\Lambda^c} = 0)$, because the positive commutator is localized to the range of it. This is easily done by the strong commutativity of $N_{\Lambda^c}$ with and $K$ and $N$:

$$\langle K \rangle_{\psi_\alpha, \nu} \geq \langle K \rangle_{P\psi_\alpha, \nu} - \left\|P_{\alpha, \nu}\right\|^2.$$

Here we used $\bar{P}S_i\bar{P} \geq 0$. Then we obtain

$$O(\alpha^2\nu^{-1} + \alpha^{-1}\nu) \sim \langle K \rangle_{\psi_\alpha, \nu},$$

$$\geq b_{\bar{e}(e)}\left(\left\|\chi_\Delta P\chi_n\psi_\alpha, \nu\right\|^2 - 14\left\|\bar{P}_{\bar{e}(e)}\chi_\Delta P\chi_n\psi_\alpha, \nu\right\|^2 - \left\|P\chi_\Delta (\chi_\Delta - 1)\right\|^2
- C_3(n^{-1} + \nu + \alpha \cdot n + \alpha) - \left\|\bar{P}_{\alpha, \nu}\right\|^2
- \left\|\chi_\Delta (\chi_\Delta - 1)\right\|^2 - n^{-2} \cdot C_2.$$ 

(40)
Substituting $\nu = \alpha^{\frac{3}{2}}$ and $n = \alpha^{-\frac{1}{2}}$ and taking $\alpha \to 0$ limit, we obtain

$$0 = \lim_{\alpha \to 0} \langle K \rangle_{\psi_{\alpha,\nu}} \geq b_{\tilde{F}(e)} \left( \| \psi \|^2 - 14 \| \tilde{P}_{\tilde{F}(e)} \psi \|^2 \right).$$

If $\tilde{P}_{\tilde{F}(e)} \psi = 0$, the inequality (41) is a contradiction. Hence for $0 < |\lambda| < \lambda_1$, there is no eigenvector of $L$ with eigenvalue $\epsilon$ which is orthogonal to $\tilde{P}_{\tilde{F}(e)}$.

Recalling that $\lambda_1$ can be taken as (9), we obtain Theorem 3.1.

One might wonder if $\chi_n$ is necessary in our case where the interaction term is bounded. Note that the right hand side of (40) has a term of order $\alpha^n$, while the left hand side has a term of order $\alpha^{-1} \nu$. Without $\chi_n$, $\alpha^n$ is replaced by $\alpha^{n-1} \alpha^{-1} \nu$. We can’t make $\alpha^{n-1} \alpha^{-1} \nu$ converge to zero simultaneously, in any choice of $\nu$. So $\chi_n$ is still required.

9 The Stability of the NESS

In this section we investigate our physical interest: the stability of the NESS. By proving Theorem 4.2 we complete the proof of Theorem 2.3 and 2.4. Recall that the NESS of the free Fermion model is given by $n$-point functions (3) with a distribution function (4). We show if the NESS is far from equilibrium, i.e., the inverse temperature $\beta$ and $\beta +$ are different, it is macroscopically unstable (Theorem 2.3). This result is due to the following fact: for the NESS far from equilibrium, the number of the particles with momentum $k$ is different from the number of the particles with momentum $-k$, although they have the same energy. On the other hand, for a class of interaction, we show return to equilibrium (Theorem 2.4).

9.1 Instability of the NESS

In this subsection, we investigate the instability of NESS under the interaction with small system. For the purpose, we study the kernel of $\Gamma(e)$. Recall the definition of $\varphi_n, E_n, E_{nm, N_l^{(j)}, N_r^{(j)}}, N_l, N_r$, given in Assumption 2.4. As $H_S$ is finite dimensional, $N_l^e \neq \emptyset$ and $N_r^e \neq \emptyset$ for $e \neq 0$.

By a straightforward calculation, we obtain

$$\Gamma(e) = \int_{-\pi}^{\pi} dk \sum_{E_{n,m} \neq e} F_{n,m}^1(k)F_{n,m}^1(k) \delta(\omega(k) + E_{n,m} - e) + F_{n,m}^2(k)F_{n,m}^2(k) \delta(-\omega(k) + E_{n,m} - e)$$

where

$$F_{n,m}^1(k) = p_n Y p_{N_l^{(e)}} \otimes p_m g_1^1(k) - p_n \otimes p_m Y p_{N_l^{(e)}} g_2^1(k).$$

First let us consider $e \neq 0$ case. Note that each term of (42) is positive. So we take the sum over the following subset

$$(n, m) \in N_l \times N_r^e \cup N_l^e \times N_r.$$
and ignore the other contributions to estimate the lower bound. Note that
$N_l$, $N_r$, $N_l^c$ and $N_r^c$ are all non-empty in case $e \neq 0$. We have the following
relations: if $n \in N_l$, then $N_l^{(n)} \neq \emptyset$, if $m \in N_r$, then $N_r^{(m)} \neq \emptyset$, if $n \in N_l^c$, then $N_r^{(n)} = \emptyset$, and if $m \in N_r^c$, then $N_l^{(m)} = \emptyset$. Especially, $E_n,m \neq e$ for
$(n,m) \in N_l \times N_r^c \cup N_l^c \times N_r$. Hence for all $(n,m) \in N_l \times N_r^c$, we have
$$F_{n,m}(k) = -p_n \otimes p_m \bar{Y}_n Y_m g^*_n(k),$$
and for $(n,m) \in N_l^c \times N_r$, we have
$$F_{n,m}(k) = p_n Y_n Y_m p_{n,m}^* g^*_l(k).$$

Then under Assumption 2.3 and 2.4, we have
$$\Gamma(e) \geq b_0 \left( \sum_{n \in N_l} p_n \otimes p_{n^{(n)}} \bar{Y}_n Y_{n^{(n)}} \right) \geq b_0 \cdot P(L_S = e) \delta_0 = \gamma_e \cdot P(L_S = e),$$
with some $b_0 > 0$ and $\gamma_e = b_0 \delta_0 > 0$. Here, $b_0$ is a constant which is independent
of $\beta_+, \beta_-$ in the interval $(\beta_0, \beta_1)$ for fixed $0 < \beta_0 < \beta_1 < \infty$. Hence for $e \neq 0$, we have $\text{Ker}(e) = \{0\}$, and $P_e = 0$.

Next let us consider $e = 0$ case. In this case, we can not apply the above
arguments because $N_l^c = N_r^c = \emptyset$. By the first condition of Assumption 2.3, a
vector in $P(L_S = 0)$ is of the form
$$\varphi = \sum_i c_i \varphi_i \otimes \varphi_i.$$  
Note that Assumption 2.3 imply $N_l^{(n)} = N_r^{(n)} = \{n\}$. We have
$$\langle \varphi | \Gamma(0) | \varphi \rangle = \sum_{n \neq m} |Y_{nm}|^2 \left[ \begin{array}{c}
\left[ c_m \cdot (1 - \rho)^\frac{1}{2} (-q_{mn}) - c_n \cdot \rho^\frac{1}{2} (q_{mn}) \right]^2 \left| f(q_{nm}) \right|^2 \int_\pi^0 dk \delta (\omega (k) + E_{n,m}) \\
+ \left[ c_m \cdot (1 - \rho)^\frac{1}{2} (q_{mn}) - c_n \cdot \rho^\frac{1}{2} (-q_{mn}) \right]^2 \left| f(-q_{nm}) \right|^2 \int_\pi^0 dk \delta (\omega (k) + E_{n,m}) \\
+ \left[ c_m \cdot \rho^\frac{1}{2} (q_{nm}) - c_n \cdot (1 - \rho)^\frac{1}{2} (q_{mn}) \right]^2 \left| f(q_{nm}) \right|^2 \int_{-\pi}^\pi dk \delta (-\omega (k) + E_{n,m}) \\
+ \left[ c_m \cdot \rho^\frac{1}{2} (-q_{nm}) - c_n \cdot (1 - \rho)^\frac{1}{2} (-q_{mn}) \right]^2 \left| f(-q_{nm}) \right|^2 \int_{-\pi}^\pi dk \delta (-\omega (k) + E_{n,m}) 
\end{array} \right] \right] \left( 43 \right),$$
where $\omega(q_{mn}) = E_{nm}$. By Assumption 2.2 and Assumption 2.3, $f(q_{mn})$, $f(-q_{nm})$, $f(q_{nm})$, $f(-q_{nm})$ are non-zero, and the integrals including $\delta$-function give strictly
positive contributions. So there exist strictly positive $a_0$, such that

$$
\begin{align*}
\mathcal{E} & \geq \sum_{n \neq m} a_0 \left\{ |c_m \cdot (1 - \rho)^{\frac{1}{2}} (q_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (q_{mn})|^2 + |c_m \cdot (1 - \rho)^{\frac{1}{2}} (-q_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (-q_{mn})|^2 \\
& + |c_m \cdot \rho^{\frac{1}{2}} (q_{nm}) - c_n \cdot (1 - \rho)^{\frac{1}{2}} (q_{nm})|^2 + |c_m \cdot \rho^{\frac{1}{2}} (-q_{nm}) - c_n \cdot (1 - \rho)^{\frac{1}{2}} (-q_{nm})|^2 \right\}. \\
\end{align*}
$$

(44)

Hence the necessary condition for $\text{Ker} \Gamma(0)$ to be non-trivial is the existence of $\{c_n\}$ which satisfies

$$
|c_m \cdot (1 - \rho)^{\frac{1}{2}} (q_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (q_{mn})|^2 = |c_m \cdot (1 - \rho)^{\frac{1}{2}} (-q_{nm}) - c_n \cdot \rho^{\frac{1}{2}} (-q_{mn})|^2
$$

$$
= |c_m \cdot \rho^{\frac{1}{2}} (q_{nm}) - c_n \cdot (1 - \rho)^{\frac{1}{2}} (q_{nm})|^2 = |c_m \cdot \rho^{\frac{1}{2}} (-q_{nm}) - c_n \cdot (1 - \rho)^{\frac{1}{2}} (-q_{nm})|^2 = 0.
$$

This implies the following:

$$
\frac{c_n}{c_m} = \frac{(1 - \rho)^{\frac{1}{2}} (q_{mn})}{\rho^{\frac{1}{2}} (q_{mn})} = \frac{(1 - \rho)^{\frac{1}{2}} (-q_{mn})}{\rho^{\frac{1}{2}} (-q_{mn})} = \frac{\rho^{\frac{1}{2}} (q_{nm})}{(1 - \rho)^{\frac{1}{2}} (q_{nm})} = \frac{\rho^{\frac{1}{2}} (-q_{nm})}{(1 - \rho)^{\frac{1}{2}} (-q_{nm})}
$$

(45)

for $n \neq m$. Recall that the Fermion distribution $\rho$ in the NESS is given by

$$
\rho(k) \equiv \begin{cases} 
(1 + e^{\beta_- (\cos(k) - \gamma)})^{-1}, & k \in [0, \pi) \\
(1 + e^{\beta_+ (\cos(k) - \gamma)})^{-1}, & k \in [-\pi, 0).
\end{cases}
$$

The condition $\mathcal{E}$ requires

$$
e^{-\beta_- E_{nm}} = e^{-\beta_+ E_{nm}},
$$

i.e., $\beta_- = \beta_+$. That is, $\Gamma(0)$ has non trivial kernel only if the NESS is an equilibrium state indeed. Otherwise, we have $\Gamma(0) \geq \gamma_0 \cdot 1 > 0$. So, if $\beta_- \neq \beta_+$, we have $P_{\varepsilon} = 0$ for all eigenvalue $\varepsilon$ of $L_S$ and obtain the first statement of Theorem 3.2.

Now let us complete the proof of Theorem 3.3. Combining Theorem 3.1 and 3.2, the Liouville operator $L$ corresponding to the NESS does not have any eigenvector for $0 < |\lambda| < \lambda_1$. Hence the NESS is macroscopically unstable by Proposition 3.3. We fix the bound $0 < \beta_0 < \beta_+ < \beta_0$, $\| pf\|, \| f\| \leq b$ for any fixed $0 < \beta_0 < \beta_1 < \infty$ and $0 < b < \infty$. Let us estimate the dependence of $\lambda_1$ on $\beta_+ - \beta_-$ for fixed $\varepsilon$. For $\varepsilon \neq 0$, we have $\gamma_\varepsilon = b_0 \delta_0$ as seen in above, which is independent of $\beta_+ - \beta_-$. On the other hand, $\gamma_0$ converges to 0 as $\beta_+ - \beta_-$ goes to 0. Substituting

$$
c_n = \frac{e^{-\beta_- E_n}}{\sqrt{Z_{\beta_-}}}, \quad Z_{\beta_-} = \sum_n e^{-\beta_- E_n}
$$

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to (44), we can estimate \( \geq C(\beta_+ - \beta_-)^2 \geq \gamma_0 > 0 \), i.e. \( \gamma_0 \sim O(\beta_+ - \beta_-)^2 \).

Hence we have \( \lambda_1 \sim O(\beta_+ - \beta_-)^2 \). On the other hand, if we fix \( \beta_+ \) and \( \beta_- \), we have \( \lambda_1(v) \sim v^{\beta_+} \to 0 \) as \( v \to 0 \). Then, we obtain Theorem 2.3.

9.2 Return to equilibrium

In this section we investigate the equilibrium case, i.e., \( \beta_+ = \beta_- \). By the result of the previous subsection, we can show return to equilibrium for the class of interaction we introduced. The following Theorem was shown by H. Araki [A1, A3]:

**Theorem 9.1** Let \((\mathfrak{A}, \tau)\) be a \(C^*\)-dynamical system and let \(\omega\) be a \((\beta, \tau)\)-KMS state with GNS-representation \((\mathcal{H}, \pi, \Omega)\). Let \(L\) be the Liouville operator corresponding to \(\tau\). If \(P = P^* \in \mathfrak{A}\) then \(\Omega \in D(e^{\beta(L+\pi(P))}/2)\). Let \(\tau^P\) be the perturbed automorphism group by \(P\), and let \(\Omega^P \equiv e^{\beta(L+\pi(P))}/2\Omega\). Then the state \(\omega^P\) defined by

\[
\omega^P(A) = \frac{(\Omega^P, A\Omega^P)}{(\Omega^P, \Omega^P)}
\]

is a \((\beta, \tau^P)\)-KMS state.

On the other hand, the following theorem concerning return to equilibrium is known [BFS2]:

**Proposition 9.2** Let \((\mathfrak{A}, \tau)\) be a \(C^*\)-dynamical system and let \(\omega\) be a \((\beta, \tau)\)-KMS state with the GNS-representation \((\mathcal{H}, \pi, \Omega)\). Assume that the Liouville operator \(L\) of \(\tau\) has a simple eigenvalue 0 corresponding to the eigenvector \(\Omega\), and that the rest of the spectrum of \(L\) is continuous. Then for any \(\omega\)-normal state \(\eta\), we have the return to equilibrium in an ergodic mean sense:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(\alpha_t(A)) \, dt = \omega(A), \quad A \in \mathfrak{A}.
\]

Let us return to our system. Now we have \(\beta \equiv \beta_+ = \beta_-\). First let us investigate the kernel of \(\Gamma(e)\). As in the previous subsection, \(\text{Ker}(e) = \{0\}\) for \(e \neq 0\). On the other hand, the equation (43) implies that \(\text{Ker}\Gamma(0)\) is one-dimensional and is spanned by a vector of the form

\[
\varphi = \sum_{i} c_i \varphi_i \otimes \varphi_i,
\]

with

\[
c_n = \frac{e^{-\frac{2}{\sqrt{Z_\beta}}}}{Z_\beta} = \sum_n e^{-\beta E_n}.
\]

Hence we obtain Theorem 3.2.

Second, note that \(\omega_\rho\) is a \((\beta, \alpha_f)\)-KMS state of \(O_f\). We denote by \(\omega^S_\beta\) the \((\beta, \alpha_S)\)-KMS state over \(O_S\). The state \(\omega^S_\beta \otimes \omega_\rho\) is a \((\beta, \alpha_0)\)-KMS state over...
$O_s \otimes O_f$. As our perturbation $V$ is an element of $O$, the nontriviality of Ker$L$ is guaranteed by Theorem 9.1. The fact that 0 is the simple eigenvalue of $L$ can be derived as follows: let $\psi_1, \psi_2$ be eigenvectors of $L$ with eigenvalue 0. By Theorem 3.1, we have $\tilde{P}_0 \psi_i = c_i \phi \otimes \Omega$, with $c_i \neq 0$. Then $\frac{\psi_1}{c_1} - \frac{\psi_2}{c_2}$ is an eigenvector of $L$ with eigenvalue 0, which is orthogonal to $\tilde{P}_0$. By Theorem 3.1, this entails $\psi_1 = c_1 \psi_2$. Hence 0 is the simple eigenvalue of $L$. We denote the corresponding eigenvector by $\Omega_V$ and the state corresponding to $\Omega_V$ by $\omega_V$. By Theorem 9.1, $\omega_V$ is a $(\beta, \alpha)$-KMS state. On the other hand, as the kernel of $\Gamma(e)$ is trivial for $e \neq 0$, $L$ has no other eigenvalue. So the rest of the spectrum of $L$ is continuous. Accordingly, from the Proposition 9.2, we obtain the return to equilibrium Theorem 2.4.

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A An Example which satisfies Assumptions 2.1 to 2.4

We give an example in dimension $d = 2$. We consider $\gamma = 0$ case. We fix some $0 < v < 1$, and define $0 \leq k_v \leq \pi/2$ by $\sin^2 k_v = v$. Let $H_S = b \sigma_z$, with $0 < b < (\cos k_v)/4$, and define $0 \leq k_{4b} < k_{2b} < \frac{\pi}{2}$ by $4b = \cos k_{4b}$, $2b = \cos k_{2b}$. We have then $0 < k_v < k_{4b} < k_{2b} \leq \frac{\pi}{2}$, and $S = \{k_{4b}, -k_{4b}, \pi - k_{4b}, -\pi + k_{4b}, -k_{2b}, \pi - k_{2b}, -\pi + k_{2b}\}$ are all in $\Lambda_v$. We choose $f$ as a smooth function with support in $\Lambda_v$, which takes non-zero values on $S$. As the interaction, we take $Y$ as

$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Because $p$ acts on $C^\infty$ as $\tilde{14}$, $f$ is in $D(p^3)$ and Assumption 2.1 is satisfied. The eigenvalue of $L_S$ is $2b, 0, -2b$. We have $\cos(k_{4b}) = 4b, \cos(k_{2b}) = 2b, \cos(\pi - k_{4b}) = 2b, \cos(\pi - k_{2b}) = -2b, \cos(\pi - k_{4b}) = -2b$, and as $S$ is included in $\Lambda_v$, Assumption 2.2 is satisfied. Assumption 2.3 is trivial. Let us check the Assumption 2.4. Let $\varphi_0, \varphi_1$ be eigenvectors of $H_S$, corresponding to eigenvalue $-b, b$, respectively. For $e = 2b, N_l^{(0)} = \{1\}, N_l^{(1)} = \emptyset, N_r^{(0)} = \emptyset$ and $N_r^{(1)} = \{0\}$. Hence we have

$$p_{N_l^{(1)}} \tilde{Y} p_{N_l^{(1)}} \tilde{Y} p_{N_r^{(1)}} = |\varphi_0 \rangle \langle \varphi_0|$$

$$p_{N_l^{(0)}} Y p_{N_l^{(0)}} Y p_{N_r^{(0)}} = |\varphi_1 \rangle \langle \varphi_1|.$$ 

Then we obtain $\delta_0 = 1 > 0$, and Assumption 2.4 is satisfied. We can check for $e = -2b$ case in the same way.
B Tensor Product of Linear Operators

About tensor product of linear operators, we have the following Theorem. It can be proved using spectral theorem given in [RS].

**Theorem B.1** Let $\mathcal{H}_i (i = 1, 2)$ be separable Hilbert spaces, and let $L_i$ be self-adjoint operators on $\mathcal{H}_i$. Let $L$ be the self-adjoint operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, defined by

$$L \equiv L_1 \otimes 1 + 1 \otimes L_2.$$ 

Suppose that $L_2$ has a unique eigenvector $\Omega$. Then every eigenvector of $L$ is of the form

$$\varphi \otimes \Omega, \quad \varphi \in \mathcal{H}_1.$$ 

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