SECOND-ORDER ELLIPTIC EQUATIONS WITH VARIABLY PARTIALLY VMO COEFFICIENTS

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Abstract. The solvability in $W^2_p(\mathbb{R}^d)$ spaces is proved for second-order elliptic equations with coefficients which are measurable in one direction and VMO in the orthogonal directions in each small ball with the direction depending on the ball. This generalizes to a very large extent the case of equations with continuous or VMO coefficients.

1. Introduction and main result

In this article we are concerned with the solvability in $W^2_p(\mathbb{R}^d)$ of the equation

$$Lu(x) - \lambda u(x) = f(x), \quad (1.1)$$

where $L$ is a uniformly nondegenerate elliptic differential operator with bounded coefficients of the form

$$Lu(x) = a^{ij}(x)u_{x^i x^j}(x) + b^i(x)u_{x^i}(x) + c(x)u(x)$$

in

$$\mathbb{R}^d = \{ x = (x^1, ..., x^d) : x^1, ..., x^d \in \mathbb{R} \}.$$

We generalize the main result of [7] where the solvability is established in the case that, roughly speaking, the coefficients $a^{ij}$ are measurable with respect to $x^1$ and are in VMO with respect to $(x^2, ..., x^d)$. Owing to a standard localization procedure, this result admits an obvious extension to the case in which for each ball $B \subset \mathbb{R}^d$ of a fixed radius there exists a sufficiently regular diffeomorphism that transforms equation (1.1) in $B$ into a similar equation with coefficients satisfying the conditions of [7] in $B$. In particular, one obtains the solvability if the matrix $a = (a^{ij})$ depends on $|x|$ in a measurable way, is in VMO with respect to the angular coordinates, and, say, is continuous at the origin.

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The main goal of the present article is to show that in the above described generalization the radius of balls need not be fixed. In the end of this section we give an example in which our result is applicable contrarily to the result of [7].

We develop a new technique which seems to be applicable in many situations for elliptic and parabolic equations with partially VMO coefficients as, for instance, in [6] and [5]. We only concentrate on elliptic equations in order to make simpler the presentation of the method. Generally, the theory of elliptic equations with partially VMO coefficients is quite new and originated in [7] in contrast with the case of completely VMO coefficients, which appeared in [4], or the classical case of equations with continuous coefficients treated in [1]. The reader can find further references to articles and books related to equations with VMO and partially VMO coefficients in the above cited articles and the references therein.

In [1] the main technical tool was the theory of singular integrals, in particular, the Calderón-Zygmund theorem. With development of Real Analysis later on in many sources the theory of singular integrals in applications to PDEs was replaced with using the John-Nirenberg theorem or Stampacchia interpolation theorem applied to sharp functions. However, the theory of singular integrals was used again in the paper [4], the results of which came as a real breakthrough in the theory of PDEs. Again later it turned out that using singular integrals can be replaced with appropriate other tools from Real Analysis such as the Fefferman-Stein theorem. To the author it seems highly unlikely that the theory of singular integrals can be used to obtain even the main auxiliary result of [7], which is the basis of the present paper along with a new inequality of the Fefferman-Stein type proved in Theorem 2.7.

In connection with this new development it is instructive to recall that L. Bers and M. Schechter said in 1964 (see [2]) that the linear theory of second order elliptic PDEs “is at present probably nearing completion”.

This paper deals with elliptic equations in nondivergence form. A different technique is developed in several articles by the authors of [3] for treating divergence type equations. It would be interesting to know if their methods could be applied to divergence or nondivergence type equations with coefficients satisfying our conditions. This could lead to extending our results to equations in domains. So far we can only deal with equations in the whole space or, for that matter, with interior estimates. Another restriction is that $p > 2$.

Now we state our assumptions rigorously.
Assumption 1.1. The coefficients \( a^{ij}, b^i, \) and \( c \) are measurable functions defined on \( \mathbb{R}^d \), \( a^{ij} = a^{ji} \) for all \( i, j = 1, \ldots, d \). There exist positive constants \( \delta \in (0, 1) \) and \( K \) such that

\[
|b^i(x)| \leq K, \quad i = 1, \ldots, d, \quad |c(x)| \leq K,
\]

\[
\delta |\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \leq \delta^{-1}|\xi|^2
\]

for any \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \).

To state the second assumption denote by \( A \) the set of \( d \times d \) symmetric matrix-valued measurable functions \( \bar{a} = (\bar{a}^{ij}(t)) \) of one variable \( t \in \mathbb{R} \) such that

\[
\delta |\xi|^2 \leq \bar{a}^{ij}(t)\xi^i\xi^j \leq \delta^{-1}|\xi|^2
\]

for any \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R}^d \).

Introduce \( \Psi \) as the set of mappings \( \psi : \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that

(i) the mapping \( \psi \) has an inverse \( \psi^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d \);

(ii) the mappings \( \psi \) and \( \phi = \psi^{-1} \) are twice continuously differentiable and

\[
|\psi_x| + |\psi_{xx}| \leq \delta^{-1}, \quad |\phi_y| + |\phi_{yy}| \leq \delta^{-1}.
\]

The following assumption contains a parameter \( \gamma > 0 \), which will be specified later. We denote by \( |B| \) the volume of a Borel set \( B \subset \mathbb{R}^d \).

Assumption 1.2 \((\gamma)\). There exists a constant \( R_0 > 0 \) such that for any ball \( B \subset \mathbb{R}^d \) of radius less than \( R_0 \) one can find an \( \bar{a} \in A \) and a \( \psi = (\psi^1, \ldots, \psi^d) \in \Psi \) such that

\[
\int_B |a(x) - \bar{a}(\psi^1(x))| \, dx \leq \gamma |B|.
\]  

(1.2)

Remark 1.3. Assumption 1.2 \((\gamma)\) is obviously satisfied with any \( \gamma > 0 \) if \( a \) is uniformly continuous as, for instance, in [1]. If Assumption 1.2 \((\gamma)\) is satisfied with any \( \gamma > 0 \) and constant \( \bar{a} \) (perhaps, changing with \( B \)), then one says that \( a \) belongs to VMO. This case was first treated in [1]. In [7] the solvability in \( W^{2,p}_p \) was proved under Assumption 1.2 \((\gamma)\) with a fixed function \( \psi \), which is not allowed to change with \( B \). (Actually, \( \psi = x \) in [7], but changing coordinates shows that the result holds for any \( \psi \in \Psi \).) By using partitions of unity the latter restriction on \( \psi \) can be easily somewhat relaxed to allow mappings such that in each ball \( B \) of radius exactly \( R_0 \) there is a mapping \( \psi \) which would suit all subballs inside \( B \).

As usual, by \( W^2_p = W^{2,p}_p(\mathbb{R}^d) \) we mean the Sobolev space on \( \mathbb{R}^d \). Set \( \mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^d) \).

Here is our main result.
Theorem 1.4. Take a $p \in (2, \infty)$. Then there exists a constant $\gamma = \gamma(d, \delta, p) > 0$ such that if Assumptions L.1 and L.2 ($\gamma$) are satisfied then for any $\lambda \geq \lambda_0(d, \delta, K, p, R_0) \geq 1$ and any $f \in \mathcal{L}_p$, there exists a unique $u \in W^2_p$ satisfying (1.1) in $\mathbb{R}^d$.

Furthermore, there is a constant $N$, depending only on $d$, $\delta$, $K$, $p$, and $R_0$, such that, for any $\lambda \geq \lambda_0$ and $u \in W^2_p$,

$$\lambda \|u\|_{\mathcal{L}_p} + \sqrt{\lambda} \|u_x\|_{\mathcal{L}_p} + \|u_{xx}\|_{\mathcal{L}_p} \leq N \|Lu - \lambda u\|_{\mathcal{L}_p}. \quad (1.3)$$

The proof of this theorem is given in Section 4 after we prepare the necessary auxiliary results in Section 3 which in turn require some general facts proved in Section 2.

We finish the section by giving the example we were talking about above. Let $f$ be a measurable function on $\mathbb{R}$ with support in the interval $(1/2, 1)$ and such that $|f| \leq 1$. Introduce $\xi(x) = \ln(|x| \wedge 1)$, $x \in \mathbb{R}$. It is well known that $\xi \in BMO$. Then for $\varepsilon > 0$ the function $\varepsilon \xi$ is also in $BMO$ and its $BMO$-norm can be made as small as we like on the account of choosing $\varepsilon$ small enough. The same is true for $\eta = \sin(\varepsilon \xi)$ and $\zeta(x) = \eta(4x - 3)$ with the latter function having support in the interval $(1/2, 1)$. Next take a large $\kappa \geq 4$ and for real $x$, $y$, and $z = (x, y)$ introduce

$$a(z) = \sum_{n=0}^{\infty} f(\kappa^n x) \zeta(\kappa^{2n} y) + \sum_{n=0}^{\infty} f(\kappa^{2n+1} x) \zeta(\kappa^{2n+1} x).$$

Notice that the support of $f(\kappa^r \cdot) \zeta(\kappa^r \cdot)$ belongs to $Q_r := (\kappa^{-r}/2, \kappa^{-r})^2$

Now, for a square $Q = I \times J \subset \mathbb{R}^2$ we are going to estimate the left-hand side of (1.2) with $Q$ and $z$ in place of $B$ and $x$, respectively, and with $\psi$ equal to either $x$ or $y$. For brevity we denote the modified left-hand side of (1.2) by $M$.

Define $\tau$ as the least integer $k \geq 0$ such that $Q \cap Q_k \neq \emptyset$. If there are no such $k$’s, then $M = 0$. If $\tau$ is an even number we set $\psi = x$ and $\widetilde{a}(x) = f(\kappa^{\tau} x) \bar{\zeta}$, where $\bar{\zeta}$ is the integral average of $\zeta(\kappa^r y)$ over $J$. Then

$$M \leq \int_I |f(\kappa^r x)| \, dx \int_J |\zeta(\kappa^r y) - \bar{\zeta}| \, dy + \sum_{i=\tau+1}^{\infty} |Q \cap Q_i|. \quad (1.4)$$

On the right, the first term is less than $|Q| \|\zeta\|_{BMO}$. Also observe that if $i \geq \tau + 1$ and $Q \cap Q_i \neq \emptyset$, then the lengths of $I$ and $J$ are at least $\kappa^{-\tau}/2 - \kappa^{-i}$, which is larger than $\kappa^{-\tau}/4$ since $\kappa \geq 4$. Hence, in that case $|Q \cap Q_i| \leq |Q_i| = 4^{-1} \kappa^{-2i} \leq 4\kappa^{2\tau-2i}|Q|$ implying that the infinite sum in (1.4) is less than $4(\kappa^2 - 1)^{-1}|Q|$. We see that in the case that $\tau$ is an even number $M \leq \gamma |Q|$ with any fixed $\gamma > 0$ provided that we choose sufficiently small $\varepsilon$ and sufficiently large $\kappa$. 


In case $\tau$ is odd interchanging $x$ and $y$ leads to the same conclusion and this easily shows that (1.2) holds indeed in its original form. Obviously, functions like the above $a$ cannot be treated by methods of [7] even modified in the way outlined in Remark 1.3.

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2. A partial version of the Fefferman-Stein theorem

First we recall a few standard notions and facts related to partitions and stopping times. All of them can be found in many books; we follow the exposition in [8].

Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space with a $\sigma$-finite measure $\mu$, such that $\mu(\Omega) = \infty$. Let $\mathcal{F}^0$ be the subset of $\mathcal{F}$ consisting of all sets $A$ such that $\mu(A) < \infty$. For $p \in [1, \infty)$ set $\mathcal{L}_p(\Omega) = \mathcal{L}_p(\Omega, \mathcal{F}, \mu)$. By $\mathcal{L}^0$ we denote a fixed dense subset of $\mathcal{L}_1(\Omega)$. For any $A \in \mathcal{F}$ we set $|A| = \mu(A)$.

For $A \in \mathcal{F}^0$ and functions $f$ summable on $A$ we use the notation

$$f_A = \frac{1}{|A|} \int_A f \, d\mu := \frac{1}{|A|} \int_A f(x) \, d\mu(x) \quad \left( \frac{0}{0} := 0 \right)$$

for the average value of $f$ over $A$.

**Definition 2.1.** Let $\mathbb{Z} = \{n : n = 0, \pm 1, \pm 2, \ldots\}$ and let $(\mathcal{C}_n, n \in \mathbb{Z})$ be a sequence of partitions of $\Omega$ each consisting of countably many disjoint sets $C \in \mathcal{C}_n$ and such that $\mathcal{C}_n \subset \mathcal{F}^0$ for each $n$. For each $x \in \Omega$ and $n \in \mathbb{Z}$ there exists (a unique) $C \in \mathcal{C}_n$ such that $x \in C$. We denote this $C$ by $C_n(x)$.

The sequence $(\mathcal{C}_n, n \in \mathbb{Z})$ is called a filtration of partitions if the following conditions are satisfied.

(i) The elements of partitions are “large” for big negative $n$’s and “small” for big positive $n$’s:

$$\inf_{C \in \mathcal{C}_n} |C| \to \infty \quad \text{as} \quad n \to -\infty, \quad \lim_{n \to -\infty} f_{C_n(x)} = f(x) \quad \text{(a.e.)} \quad \forall f \in \mathcal{L}^0.$$ 

(ii) The partitions are nested: for each $n$ and $C \in \mathcal{C}_n$ there is a (unique) $C' \in \mathcal{C}_{n-1}$ such that $C \subset C'$.

(iii) The following regularity property holds: for any $n$, $C$, and $C'$ as in (ii) we have

$$|C'| \leq N_0|C|,$$

where $N_0$ is a constant independent of $n, C, C'$. 
Observe that since the elements of partition $C_n$ become large as $n \to -\infty$, we have $N_0 > 1$.

The only example of a filtration of partitions important for this article in the case that $\Omega = \mathbb{R}^d$ with Lebesgue measure $\mu$ is given by dyadic cubes, that is, by

$$C_n = \{ C_n(i_1, ..., i_d), i_1, ..., i_d \in \mathbb{Z} \},$$

where

$$C_n(i_1, ..., i_d) = [i_1 2^{-n}, (i_1 + 1)2^{-n}) \times ... \times [i_d 2^{-n}, (i_d + 1)2^{-n}).$$

In this case, to satisfy requirement (i) in Definition 2.1, one can take $L^0$ as the set of continuous functions with compact support.

**Definition 2.2.** Let $C_n, n \in \mathbb{Z}$, be a filtration of partitions of $\Omega$.

(i) Let $\tau = \tau(x)$ be a function on $\Omega$ with values in $\{\infty, 0, \pm 1, \pm 2, ...\}$. The function $\tau$ is called a stopping time (relative to the filtration) if, for each $n = 0, \pm 1, \pm 2, ...$, the set

$$\{ x : \tau(x) = n \}$$

is either empty or else is the union of some elements of $C_n$.

(ii) For a function $f \in L_1(\Omega)$ and $n \in \mathbb{Z}$, we denote

$$f|_n(x) = \int_{C_n(x)} f(y) \mu(dy).$$

We read $f|_n$ as “$f$ given $C_n$”, continuing to borrow the terminology from probability theory. If we are also given a stopping time $\tau$, we let

$$f|_{\tau}(x) = f|_{\tau(x)}(x)$$

for those $x$ for which $\tau(x) < \infty$ and $f|_{\tau}(x) = f(x)$ otherwise.

The simplest example of a stopping time is given by $\tau(x) \equiv 0$. It is also known that if $g \in L_1(\Omega)$ and a constant $\lambda > 0$, then

$$\tau(x) = \inf \{ n \in \mathbb{Z} : g|_n(x) > \lambda \} \quad (\inf \emptyset := \infty)$$

is a stopping time and if, in addition, $g \geq 0$, then $g|_\tau \leq N_0 \lambda$ (a.e.).

For $f \in L_1(\Omega)$ we denote

$$\mathcal{M}f = \sup_{n \in \mathbb{Z}} |f|_n.$$

It is known that for any $f \in L_1(\Omega)$ and $p \in (1, \infty)$

$$\|\mathcal{M}f\|_{L_p(\Omega)} \leq q\|f\|_{L_p(\Omega)}, \quad (2.1)$$

where $q = p/(p - 1)$.

In the remaining part of the section we consider two functions $u, v \in L_1(\Omega)$ and a nonnegative measurable function $g$ on $\Omega$. 
Lemma 2.3. Assume that \(0 \leq u \leq v\) and for any \(n \in \mathbb{Z}\) and \(C \in \mathcal{C}_n\) we have
\[
\int_C (u - v_C)_+ \mu(dx) \leq \int_C g(x) \mu(dx). \tag{2.2}
\]
Then for any \(\lambda > 0\)
\[
|\{x : u(x) \geq \lambda\}| \leq 2\lambda^{-1} \int_\Omega g(x) I_{\mathcal{M}v(x) > \alpha \lambda} \mu(dx), \tag{2.3}
\]
where \(\alpha = (2N_0)^{-1}\).

Proof. Fix a \(\lambda > 0\) and define
\[
\tau(x) = \inf\{n \in \mathbb{Z} : v|_n(x) > \alpha \lambda\}.
\]
We know that \(\tau\) is a stopping time and if \(\tau(x) < \infty\), then
\[
v|_n(x) \leq \lambda/2, \quad \forall n \leq \tau(x).
\]
We also know that \(v|_n \to v\) (a.e.) as \(n \to \infty\). It follows that (a.e.)
\[
\{x : u(x) \geq \lambda\} = \{x : u(x) \geq \lambda, \tau(x) < \infty\} = \{x : u(x) \geq \lambda, v|_\tau \leq \lambda/2\} = \bigcup_{n \in \mathbb{Z}} \bigcup_{C \in \mathcal{C}_n^\tau} A_n(C),
\]
where
\[
A_n(C) := \{x \in C : u(x) \geq \lambda, v|_n \leq \lambda/2\},
\]
and \(\mathcal{C}_n^\tau\) is the family of disjoint elements of \(\mathbb{C}_n\) such that
\[
\{x : \tau(x) = n\} = \bigcup_{C \in \mathcal{C}_n^\tau} C.
\]
Next, for each \(n \in \mathbb{Z}\) and \(C \in \mathbb{C}_n\) on the set \(A_n(C)\), if it is not empty, we have \(v|_n = v_C\) and \(u - v_C \geq \lambda/2\), so that by Chebyshev's inequality and assumption (2.2)
\[
|A_n(C)| \leq 2\lambda^{-1} \int_C g \mu(dx),
\]
\[
|\{x : u(x) \geq \lambda\}| \leq 2\lambda^{-1} \sum_{n \in \mathbb{Z}} \sum_{C \in \mathcal{C}_n} \int_C g \mu(dx) = 2\lambda^{-1} \int_\Omega g \mathcal{M}v \mathcal{M}v > \alpha \lambda \mu(dx).
\]
It only remains to observe that \(\{\tau < \infty\} = \{\mathcal{M}v > \alpha \lambda\}\). The lemma is proved.

Remark 2.4. Obviously, the conditions of Lemma 2.3 are satisfied with \(g = (1/2)v^5\) if \(u = v\). One of nice features of the lemma is that under its conditions, for any measurable function \(a\) such that \(1 \leq a \leq 2\), the functions \(au, 2v,\) and \(2g\) also satisfy its conditions.
To give conditions to verify assumption (2.2) which are convenient in this article, we need the following.

**Assumption 2.5.** We have \(|u| \leq v\) and for any \(n \in \mathbb{Z}\) and \(C \in \mathbb{C}_n\) there exists a measurable function \(u^C\) given on \(C\) such that \(|u| \leq u^C \leq v\) on \(C\) and

\[
\left( \int_C |u - u^C| \mu(dx) \right) \wedge \left( \int_C |u^C - u^C_C| \mu(dx) \right) \leq \int_C g(x) \mu(dx). \tag{2.4}
\]

**Lemma 2.6.** Under Assumption 2.5 for any \(\lambda > 0\) we have

\[
|\{x : |u(x)| \geq \lambda\}| \leq 2\lambda^{-1} \int_{\Omega} g(x) I_{M_{\nu}(x) > \alpha \lambda} \mu(dx), \tag{2.5}
\]

where \(\alpha = (2N_0)^{-1}\). Moreover if \(u \geq 0\), then one can replace \(2\lambda^{-1}\) in (2.5) with \(\lambda^{-1}\).

**Proof.** First assume that \(u \geq 0\). Take an \(n \in \mathbb{Z}\) and a \(C \in \mathbb{C}_n\). If

\[
\int_C |u - u^C| \mu(dx) \leq \int_C g(x) \mu(dx).
\]

then, since \(u \leq v\), we have \(u^C \leq v^C\) and

\[
(u - u^C) + |u - u^C| = 2(u - u^C)_+ \geq 2(u - v^C)_+,
\]

implying that (2.2) is satisfied with \(g/2\) in place of \(g\). In case that

\[
\int_C |u^C - u^C_C| \mu(dx) \leq \int_C g(x) \mu(dx)
\]

we observe that \(u^C \geq u\), \(u^C_C \leq v^C\), so that

\[
(u^C - u^C_C) + |u^C - u^C_C| = 2(u^C - u^C_C)_+ \geq 2(u - v^C)_+,
\]

which again implies that (2.2) is satisfied with \(g/2\) in place of \(g\).

In the general case we need only show that condition (2.4) is almost preserved if we take \(|u|\) in place of \(u\). However, for any measurable set \(C\) we have

\[
\int_C |u(x)| - |u| \mu(dx) = \int_C \int_C (|u(x)| - |u(y)|) \mu(dy) |\mu(dx|
\]

\[
\leq \int_C \int_C |u(x) - u(y)| \mu(dy) \mu(dx) \leq 2 \int_C |u(x) - c| \mu(dx), \tag{2.6}
\]

where \(c\) is any constant. If we take \(c = u^C\), then we see that \(|u|\) satisfies (2.4) with \(2g\) in place of \(g\). The lemma is proved.

Now we are ready to prove a partial version of the Fefferman-Stein theorem about sharp functions.
**Theorem 2.7.** Under Assumption 2.5 for any $p \in (1, \infty)$ we have
\[
\|u\|_{L^p(\Omega)}^p \leq N(p, N_0)\|g\|_{L^p(\Omega)}\|v\|_{L^p(\Omega)}^{p-1}.
\] (2.7)

The same conclusion holds under the assumptions of Lemma 2.3.

**Proof.** We have
\[
\|u\|_{L^p(\Omega)}^p = \int_0^\infty \{|x : |u(x)| \geq \lambda^{1/p}\}| d\lambda 
\leq 2 \int_{\Omega} g(x) \left( \int_0^\infty \lambda^{-1/p} I_{Mv(x) > \alpha \lambda^{1/p}} d\lambda \right) \mu(dx) 
= 2q \alpha^{1-p} \int_{\Omega} g(Mv)^{p-1} \mu(dx),
\]
where $q = p/(p - 1)$. By using Hölder’s inequality and (2.1), we come to (2.7). The theorem is proved.

**Remark 2.8.** In the dyadic version of the original Fefferman-Stein theorem $u^C = u$, $v = |u|$, and $g$ is the sharp function $u^\#$ of $u$. In that case, assuming that $u \in L^p(\Omega)$, we get from (2.7) the Fefferman-Stein inequality $\|u\|_{L^p(\Omega)} \leq N\|u^\#\|_{L^p(\Omega)}$.

3. Auxiliary results

We denote by $B_r(x)$ the open ball in $\mathbb{R}^d$ of radius $r$ centered at $x$. Set $B_r = B_r(0)$ and introduce $\mathbb{B}$ as the family of balls in $\mathbb{R}^d$. For a Borel set $B \subset \mathbb{R}^d$ of nonzero Lebesgue measure and a measurable function $f$ we define
\[
f_B := \int_B f(x) \, dx := \frac{1}{|B|} \int_B f(x) \, dx,
\]
whenever the last integral is finite. The following is Lemma 4.8 of [7].

**Lemma 3.1.** Take an $\bar{a} \in A$ and set
\[
\bar{L}u(x) = \bar{a}^{ij}(x^1)u_{x^ix^j}(x).
\] (3.1)

There exists a constant $N = N(d, \delta)$ such that, for any $\kappa \geq 4$, $r > 0$, $u \in C_0^\infty(\mathbb{R}^d)$, and $i, j \in \{1, \ldots, d\}$ satisfying $ij > 1$ we have
\[
\int_{B_r} |u_{x^ix^j} - (u_{x^ix^j})_{B_r}|^2 \, dx \leq N \kappa^d \left( \|\bar{L}u\|_{L^2}^2 \right)_{B_{\kappa r}} + N \kappa^{-2} \left( \|u_{xx}\|_{L^2}^2 \right)_{B_{\kappa r}}.
\]

We need a version of this lemma for operators of a more general form.
Lemma 3.2. Take an \( \bar{a} \in A \) and a \( \psi \in \Psi \) and set
\[
Lu(x) = \bar{a}^{kn}(y^1)\phi_{y^n}(y)\phi_{y^n}(y)u_{x,x^n}(x),
\]
where \( y = \psi(x) \) and \( \phi = \psi^{-1} \). Then there exist constants \( N = N(d,\delta) \) and \( \chi = \chi(d,\delta) \geq 1 \) such that, for any \( \kappa \geq 4, \rho > 0, u \in C^\infty_0(\mathbb{R}^d), \) and \( i, j \in \{1,\ldots,d\} \) satisfying \( ij > 1 \) we have
\[
\int_{B_r} |u_{ij} - (u_{ij})_{B_r}|^2 dx \leq N\kappa^d (|\bar{L}u|^2)_{B_{\chi r}} + N\kappa^d (|u_x|^2)_{B_{\chi r}},
\]
where \( u_{ij}(x) \) are defined by
\[
u_{ij}(\phi(y)) = v_{y^iy^j}(y), \quad v(y) = u(\phi(y)), \quad \phi = \psi^{-1}.
\]
Proof. Without loss of generality we assume that \( \psi(0) = 0 \). Also set \( f = Lu \) and observe that
\[
\bar{a}^{kn}(y^1)v_{y^ky^n}(y) + \bar{b}^k(y)v_{y^k}(y) = f(\phi(y)),
\]
where
\[
\bar{b}^k(y) = \bar{a}^{kn}(y^1)\phi_{y^n}(y)\phi_{y^n}(y)\phi_{x,x^n}(x), \quad x = \phi(y).
\]
Next we apply Lemma 3.1 to the operator
\[
\bar{L}v(y) = \bar{a}^{kn}(y^1)v_{y^ky^n}(y)
\]
and for any \( \rho > 0 \) find
\[
\int_{B_\rho} |v_{y^iy^j} - (v_{y^iy^j})_{B_\rho}|^2 dy \leq N\kappa^d (|\bar{L}v|^2)_{B_{\chi \rho}} + N\kappa^d (|v_{yy}|^2)_{B_{\chi \rho}}.
\]
To transform this inequality we use the simple observation that there exist constants \( N, \chi < \infty \) depending only on \( d \) and \( \delta \) such that for any nonnegative measurable function \( g \) we have
\[
\int_{B_\rho} f(x) dx \leq N \int_{B_{\rho\chi}} f(\phi(y)) dy, \quad \int_{B_\rho} f(\phi(y)) dy \leq N \int_{B_{\rho\chi}} f(x) dx.
\]
Using this and closely following (2.10) we find
\[
\int_{B_r} |u_{ij} - (u_{ij})_{B_r}|^2 dx \leq \int_{B_r} |u_{ij}(x_1) - u_{ij}(x_2)|^2 dx_1 dx_2
\]
\[
\leq N \int_{B_r \chi} \int_{B_r \chi} |v_{y^iy^j}(y_1) - v_{y^iy^j}(y_2)|^2 dy_1 dy_2
\]
\[
\leq N \int_{B_r \chi} |v_{y^iy^j}(y) - (v_{y^iy^j})_{B_\rho}|^2 dy.
\]
Furthermore, for \( y = \psi(x) \) obviously \( |v_{yy}(y)| \leq N(|u_{xx}(x)| + |u_x(x)|) \) and by (3.5) also \( |\bar{L}v(y)| \leq |\bar{L}u(x)| + N|u_x(x)| \). By combining the
above observations we immediately obtain (3.3) from (3.6). The lemma is proved.

Set
\[ L_0 u(x) = a^{ij}(x) u_{x^i x^j}(x). \]

In the following lemma we prepare to check Assumption 2.5 for some functions to be introduced later and closely related to \( u_{ij} \). However, we still have \( B_r \) in place of \( C \).

**Lemma 3.3.** (i) Suppose that Assumptions 1.1 and 1.2 are satisfied.

(ii) Let \( \mu, \nu \in (1, \infty), \kappa \geq 4, \) and \( r > 0 \) be some numbers such that \( 1/\mu + 1/\nu = 1. \)

Then there exist a mapping \( \psi \in \Psi \) and constants \( N = N(d, \delta, \mu) \) and \( \chi = \chi(d, \delta) \geq 1 \) such that, for any \( C^0_0 \) function \( u \), vanishing outside a ball of radius \( R \leq R_0 \), and \( i, j \in \{1, \ldots, d\} \) satisfying \( ij > 1 \) we have
\[
\int_{B_r} |u_{ij} - (u_{ij})_{B_r}|^2 \, dx \leq N \kappa^d \left( |L_0 u|^2 \right)_{B_{\chi \kappa r}} + N \kappa^d \left( |u_{xx}|^2 \right)_{B_{\chi \kappa r}} + N (\kappa^d R^2 + \kappa^{-2}) \left( |u_{xx}|^2 \right)_{B_{\chi \kappa r}} + N \kappa^d \gamma^{1/\mu} \left( |u_{xx}|^{2\mu} \right)_{B_{\chi \kappa r}},
\]
where \( u_{ij}(x) \) are defined by (3.4).

Proof. We take \( \chi \) from Lemma 3.2 and split the proof into two parts.

**Case \( \chi \kappa r < R \).** Take a \( \psi \in \Psi \) and an \( \hat{a} \in A \) such that
\[
\int_{B_{\chi \kappa r}} |a(x) - \hat{a}(\psi^1(x))| \, dx \leq \gamma.
\]
Reducing \( \delta \) if necessary we may assume that, for an \( \bar{a} \in A \), we have
\[
\hat{a}^{ij}(t) = \bar{a}^{kn}(t) \phi_{y^n}(y_0) \phi_{y^m}(y_0).
\]
where \( y_0 = \psi(0) \). Then introduce \( \hat{L} \) by (3.2) and set
\[
\hat{L} u(x) = \hat{a}^{ij}(\psi^1(x)) u_{x^i x^j}(x).
\]

Observe that for \( y = \psi(x) \) and \( |x| \leq \chi \kappa r \) we have \( |y - y_0| \leq N(d, \delta) \chi \kappa r \) and
\[
|\hat{L} u(x)| = |\hat{a}^{kn}(y) (\phi_{y^n}(y) \phi_{y^m}(y) - \phi_{y^n}(y_0) \phi_{y^m}(y_0)) u_{x^i x^j}(x)|
\leq NR |u_{xx}(x)|.
\]

This and (3.3) yield
\[
\int_{B_r} |u_{ij} - (u_{ij})_{B_r}|^2 \, dx \leq N \kappa^d \left( |\hat{L} u|^2 \right)_{B_{\chi \kappa r}} + N \kappa^d R^2 \left( |u_{xx}|^2 \right)_{B_{\chi \kappa r}} + N \kappa^d \left( |u_{xx}|^2 \right)_{B_{\chi \kappa r}}.
\]
After that it only remains to notice that
\[
\left( |\hat{L}u|^2 \right)_{B_{\chi r}} \leq 2 \left( |L_0u|^2 \right)_{B_{\chi r}} + 2 \left( |(L_0 - \hat{L})u|^2 \right)_{B_{\chi r}},
\]
and by Hölder’s inequality and (3.8)
\[
\left( |(L_0 - \hat{L})u|^2 \right)_{B_{\chi r}} \leq N \left( |u_{xx}|^{2\mu} \right)_{B_{\chi r}}^{1/\mu} \gamma^{1/\nu},
\]
which yields (3.7).

**Case** $\chi r \geq R$. Let $u = 0$ outside $B_R(x_0)$. Take a $\psi \in \Psi$ and an $\hat{a} \in A$ such that
\[
\int_{B_R(x_0)} |a(x) - \hat{a}(\psi^1(x))| \, dx \leq \gamma,
\]
define $\bar{a}$ by (3.9) with $y_0 = \psi(x_0)$, and define $\hat{L}$ and $\bar{L}$ as above. Then on the support of $u$ we still have (3.10) and hence (3.11) holds again. Finally,
\[
\left( |(\hat{L} - L_0)u|^2 \right)_{B_{\chi r}} = \left( I_{B_R(x_0)} |(\hat{L} - L_0)u|^2 \right)_{B_{\chi r}}
\]
\[
\leq N \left( |u_{xx}|^{2\mu} \right)_{B_{\chi r}} J,
\]
where
\[
J' := \frac{1}{|B_{\chi r}|} \int_{B_{\chi r} \cap B_R(x_0)} |a(x) - \hat{a}(\psi^1(x))| \, dx
\]
\[
\leq \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |a(x) - \hat{a}(\psi^1(x))| \, dx \leq \gamma.
\]
It is seen that (3.12) is true again and the lemma is proved.

In the next lemma by $\mathbb{C}_n, n \in \mathbb{Z}$, we mean the filtration of dyadic cubes in $\mathbb{R}^d$ and by $\mathcal{M} f$ the classical maximal function of $f$ defined by
\[
\mathcal{M} f(x) = \sup_{B \in \mathbb{B} : B \ni x} \int_B |f(y)| \, dy.
\]

**Lemma 3.4.** (i) Suppose that Assumptions 1.1 and 1.2 ($\gamma$) are satisfied.

(ii) Let $\mu, \nu \in (1, \infty)$, and $\kappa \geq 4$ be some numbers such that $1/\mu + 1/\nu = 1$.

Then for any $n \in \mathbb{Z}$ and $C \in \mathbb{C}_n$ there exist a mapping $\psi \in \Psi$ and a constant $N = N(d, \delta, \mu)$ such that, for any $C^\infty_0$ function $u$, vanishing outside a ball of radius $R \leq R_0$, and $i, j \in \{1, \ldots, d\}$ satisfying $ij > 1$ we have
\[
\int_C |u_{ij} - (u_{ij})_C| \, dx \leq N \int_C g \, dx,
\]
(3.13)
where \( u_{ij}(x) \) are defined by (3.4) and \( g \) is a nonnegative function satisfying

\[
g^2 = \kappa^d (\langle |L_0 u|^2 \rangle + \langle |u_x|^2 \rangle)
+ (\kappa^d R^2 + \kappa^{-2} \langle |u_{xx}|^2 \rangle + \kappa^d \gamma^{1/\nu} (\langle |u_{xx}|^2 \rangle)^{1/\mu}.
\]

Furthermore,

\[
|u_{xx}| \leq N \sum_{ij>1} |u_{ij}| + N|u_x| + N|L_0 u|.
\]

(3.14)

Proof. Let \( B \) be the smallest ball containing \( C \) and let \( B' \) be the concentric ball of radius \( \chi \kappa r \), where \( r \) is the radius of \( B \) and \( \chi \) is taken from Lemma 3.3. One can certainly shift the origin in the situation of Lemma 3.3 and hence for \( ij > 1 \) and an appropriate \( \psi \in \Psi \)

\[
\int_B |u_{ij} - (u_{ij})_B|^2 \, dx \leq N_1 \kappa^d (\langle |L_0 u|^2 \rangle)_{B'} + N_1 \kappa^d (\langle |u_x|^2 \rangle)_{B'}
+ N_1 (\kappa^d R^2 + \kappa^{-2}) (\langle |u_{xx}|^2 \rangle)_{B'} + N_1 \kappa^d \gamma^{1/\nu} (\langle |u_{xx}|^2 \rangle)^{1/\mu},
\]

(3.15)

where \( N_1 = N(d, \delta, \mu) \). Obviously, the right-hand side of (3.15) is less than \( N_1 g^2(x) \) for any \( x \in C \) (and for that matter, for any \( x \in B' \)). In particular, the square root of the right-hand side of (3.15) is less than

\[
N_1^{1/2} \int_C g \, dx.
\]

After that, to finish proving the first assertion of the lemma, it only remains to use Hölder’s inequality showing that

\[
J := \int_B |u_{ij} - (u_{ij})_B| \, dx \leq \left( \int_B |u_{ij} - (u_{ij})_B|^2 \, dx \right)^{1/2}
\]

and observe that

\[
\int_C |u_{ij} - (u_{ij})_C| \, dx \leq \int_C \int_C |u_{ij}(x) - u_{ij}(y)| \, dxdy
\]

\[
\leq N(d) \int_B \int_B |u_{ij}(x) - u_{ij}(y)| \, dxdy \leq NJ.
\]

To prove the second assertion, define \( f = L_0 u, \, v(\psi(x)) = u(x) \), and by changing variables introduce an operator \( \hat{L} \) such that \( \hat{L} v(y) = f(\phi(y)) \). Then

\[
|v_{yy}| \leq N \sum_{ij>1} |v_{y'y'}| + N|\hat{L} v| + N|v_y|.
\]

By adding to this that \( |u_{xx}(x)| \leq N|v_{yy}(y)| + N|u_x(x)| \) for \( y = \psi(x) \), we come to (3.14). The lemma is proved.
Lemma 3.5. Let \( p \in (2, \infty) \). We assert that there exist constants \( \gamma = \gamma(d, \delta, p) > 0 \) and \( R = R(d, \delta, p, R_0) \in (0, R_0] \) such that if Assumptions 1.1 and 1.2 (\( \gamma \)) are satisfied, then for any \( C_0^\infty \) function \( u \) vanishing outside a ball of radius \( R \) we have

\[
\|u_{xx}\|_{L^p} \leq N (\|L_0 u\|_{L^p} + \|u_x\|_{L^p}),
\]

(3.16)

where \( N = N(d, \delta, p) \).

Proof. For the moment we suppose that Assumptions 1.1 and 1.2 (\( \gamma \)) are satisfied with a constant \( \gamma > 0 \) and will choose it appropriately near the end of the proof.

Take a number \( \kappa \geq 4 \) and set \( \mu = \frac{2 + p}{4} \) (\( \mu > 1, 2 \mu < p \)). Also take an \( n \in \mathbb{Z} \) and a \( C \in C_n \) and take a \( \psi \in \Psi \) from Lemma 3.4. Finally, take a \( C_0^\infty \) function \( u \) vanishing outside a ball of radius \( R \), introduce \( u_{ij} \) by (3.4), and set

\[
L_0 u = f, \quad U = |u_{xx}|, \quad U^C = \sum_{ij>1} |u_{ij}| + |u_x| + |f|, \quad V = |u_{xx}| + |u_x| + |f|.
\]

We want to apply Theorem 2.7. Estimate (3.14) says that

\[
U \leq N U^C.
\]

Furthermore, obviously \( U^C \leq NV \). Also, similarly to (2.6)

\[
\begin{align*}
\int_C |U^C - U^C_C| \, dx &\leq 2 \sum_{ij>1} \int_C |u_{ij} - (u_{ij})_C| \, dx \\
&\quad + 2 \int_C |u_x - (u_x)_C| \, dx + 2 \int_C |f - f_C| \, dx.
\end{align*}
\]

We estimate the sum over \( ij > 1 \) by using Lemma 3.4 and observe that

\[
\begin{align*}
\int_C |f - f_C| \, dx &\leq 2 |f|_C \leq 2 M f(x) \quad \forall x \in C, \\
\int_C |f - f_C| \, dx &\leq 2 \int_C M f \, dx, \quad \int_C |u_x - (u_x)_C| \, dx \leq 2 \int_C M |u_x| \, dx.
\end{align*}
\]

Hence

\[
\int_C |U^C - U^C_C| \, dx \leq N \int_C (g + M |u_x| + M f) \, dx,
\]

where \( g \) is defined in Lemma 3.4.

Since this holds for any \( n \in \mathbb{Z} \) and any \( C \in C_n \), by Theorem 2.7 we conclude

\[
\|u_{xx}\|_{L^p} = \|U\|_{L^p} \leq N (\|g + M |u_x| + M f\|_{L^p}^{1/p} \|V\|_{L^p}^{(p-1)/p}).
\]

By observing that

\[
\|V\|_{L^p} \leq \|u_{xx}\|_{L^p} + \|u_x\|_{L^p} + \|f\|_{L^p}
\]
and by Young’s inequality
\[ a^{1/p}b^{(n-1)/p} \leq N(\varepsilon, p)a + \varepsilon b, \quad \forall a, b, \varepsilon > 0, \]
we easily get that
\[ \|u_{xx}\|_{L^p} \leq N\|g + M|u_x| + Mf\|_{L^p} + \|u_x\|_{L^p} + \|f\|_{L^p}. \]
Next, by applying the Hardy-Littlewood maximal function theorem and using the fact that \( p/(2\mu) > 1 \) and \( p > 2 \) we find
\[ \|u_{xx}\|_{L^p} \leq N_1\kappa^{d/2}\|f\|_{L^p} + N_1\kappa^{d/2}\|u_x\|_{L^p} \]
\[ + N_1(\kappa^{d/2}R + \kappa^{-1} + \kappa^{d/2}\gamma^{1/(2\nu)})\|u_{xx}\|_{L^p}, \]
where \( \nu = \mu/(\mu - 1) \), \( N_1 = N(d, \delta, p) \), and \( \kappa \geq 4 \) is an arbitrary number. After choosing \( R = R(d, \delta, p) \in (0, R_0] \) and \( \kappa = \kappa(d, \delta, p) \geq 4 \) so that
\[ N_1\kappa^{-1} \leq 1/4, \quad N_1\kappa^{d/2}R \leq 1/4, \]
and finally choosing \( \gamma = \gamma(d, \delta, p) > 0 \) so that
\[ N_1\kappa^{d/2}\gamma^{1/(2\nu)} \leq 1/4, \]
we come to (3.16). The lemma is proved.

4. Proof of Theorem 1.4

We take a \( p \in (2, \infty) \) and take \( \gamma \) from Lemma 3.5 and suppose that Assumptions 1.1 and 1.2 (\( \gamma \)) are satisfied. As usual, bearing in mind the method of continuity, one sees that it suffices to prove the a priori estimate (1.3).

Notice that
\[ \|L_0u - \lambda u\|_{L^p} \leq \|Lu - \lambda u\|_{L^p} + N\|u_x\|_{L^p} + K\|u\|_{L^p}, \]
where \( N = N(d, K) \). Since we only consider large \( \lambda \), this shows that it suffices to prove (1.3) with \( L_0 \) in place of \( L \). Therefore, below we assume that \( b = c = 0 \).

In that case by using partitions of unity one easily derives from Lemma 3.5 that for any \( u \in W^2_p \)
\[ \|u_{xx}\|_{L^p} \leq N(\|Lu\|_{L^p} + \|u_x\|_{L^p} + \|u\|_{L^p}), \]
where \( N = N(d, \delta, p, R_0) \). Using the interpolation inequality
\[ \|u_x\|_{L^p} \leq \varepsilon\|u_{xx}\|_{L^p} + N(d, p)\varepsilon^{-1}\|u\|_{L^p}, \quad \varepsilon > 0, \]
shows that
\[ \|u_{xx}\|_{L^p} \leq N(\|Lu\|_{L^p} + \|u\|_{L^p}). \quad (4.1) \]
It follows that for any \( \lambda \geq 0 \)
\[ \lambda\|u\|_{L^p} + \sqrt{\lambda}\|u_x\|_{L^p} + \|u_{xx}\|_{L^p} \]
\[ \leq N(\|Lu - \lambda u\|_{L^p} + (\lambda + 1)\|u\|_{L^p}), \]

which implies that we only need to find \( \lambda_0(d, \delta, p, R_0) \geq 1 \) such that for \( \lambda \geq \lambda_0 \) we have

\[ \lambda \|u\|_{L^p} \leq N\|Lu - \lambda u\|_{L^p} \tag{4.2} \]

with \( N = N(d, \delta, p, R_0) \).

As is usual in such situations, we will follow an idea suggested by S. Agmon. Consider the space

\[ \mathbb{R}^{d+1} = \{ z = (x, y) : x \in \mathbb{R}^d, y \in \mathbb{R} \} \]

and the function

\[ \tilde{u}(z) = u(t, x)\zeta(y)\cos(\mu y), \]

where \( \mu = \sqrt{\lambda} \) and \( \zeta \) is a \( C_0^\infty(\mathbb{R}) \) function, \( \zeta \neq 0 \). Also introduce the operator

\[ \tilde{L}v(t, z) = a_{ij}(x)v_{x^i x^j}(z) + v_{yy}(z). \]

As is easy to see, the operator \( \tilde{L} \) satisfies Assumption 1.2 (\( \gamma' \)) (relative to \( \mathbb{R}^{d+1} \)) with \( \gamma' = N(d)\gamma \). Therefore, by reducing the \( \gamma \) taken from Lemma 3.5 if necessary, we may apply the above results to the operator \( \tilde{L} \) and in light of (4.1) applied to \( \tilde{u} \) and \( \tilde{L} \) we get

\[ \|\tilde{u}_{zz}\|_{L^p(\mathbb{R}^{d+1})} \leq N(\|\tilde{L}\tilde{u}\|_{L^p(\mathbb{R}^{d+1})} + \|\tilde{u}\|_{L^p(\mathbb{R}^{d+1})}). \tag{4.3} \]

It is not hard to see that

\[ \int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p \, dy \]

is bounded away from zero for \( \mu \in \mathbb{R} \). Therefore,

\[ \|u\|_{L^p(\mathbb{R}^d)}^p = \mu^{-2p} \left( \int_{\mathbb{R}} |\zeta(y)\cos(\mu y)|^p \, dy \right)^{-1} \int_{\mathbb{R}^{d+1}} \|\tilde{u}_{yy}(z)\|^p \]

\[ -u(x)[\zeta''(y)\cos(\mu y) - 2\mu\zeta'(y)\sin(\mu y)]^p \, dz \]

\[ \leq N\mu^{-2p} \left( \|\tilde{u}_{zz}\|_{L^p(\mathbb{R}^{d+1})}^p + (p^2 + 1)\|u\|_{L^p(\mathbb{R}^d)}^p \right). \]

This and (1.3) yield

\[ \mu^2\|u\|_{L^p} \leq N\|\tilde{L}\tilde{u}\|_{L^p(\mathbb{R}^{d+1})} + N(\mu + 1)\|u\|_{L^p}. \]

Since

\[ \tilde{L}\tilde{u} = \zeta\cos(\mu y)[Lu - \lambda u] + u[\zeta''\cos(\mu y) - 2\mu\zeta'\sin(\mu y)], \]

we have

\[ \|\tilde{L}\tilde{u}\|_{L^p(\mathbb{R}^{d+1})} \leq N\|Lu - \lambda u\|_{L^p} + N(\mu + 1)\|u\|_{L^p}, \]

so that

\[ \lambda\|u\|_{L^p} \leq N_1\|Lu - \lambda u\|_{L^p} + N_2(\sqrt{\lambda} + 1)\|u\|_{L^p}. \]
For $\lambda \geq \lambda_0 = 16N_2^2 + 4N_2$ we have

$$N_2\sqrt{\lambda} \leq \frac{1}{4}\lambda, \quad N_2 \leq \frac{1}{4\lambda}, \quad N_2(\sqrt{\lambda} + 1) \leq \frac{1}{2}\lambda$$

and we arrive at (4.2) with $N = 2N_1$. The theorem is proved.

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