HESSIAN STRUCTURES, EULER VECTOR FIELDS, AND THERMODYNAMICS

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Abstract. In this paper, a geometric structure which generalizes that of thermodynamics is presented; spaces of equilibrium states are portrayed as a particular case of the former. For this end, concepts like Euler vector field and extensive function, which are usual in thermodynamics, are introduced in a wider context.

1. Introduction

As first pointed out by Weinhold [25], an interesting consequence of the laws of thermodynamics (namely, the First and Second Laws, and the Entropy Maximum Principle) is that the space of equilibrium states of classical thermodynamic systems is endowed with a “degenerate metric tensor”, i.e., a symmetric positive 2-tensor field with one null direction. With the aid of this structure, many of the well-known equations of thermodynamics can be given a vector-geometric interpretation (see, e.g., Gilmore [7] and Torres del Castillo et al. [23]).

A far more useful application of the geometric structure of the space of equilibrium states was later proposed by Ruppeiner: he showed that in many different cases, critical states of thermodynamic systems (in the physical sense) correspond to points where the scalar curvature of a particular Riemannian submanifold of the space of equilibrium states diverges [17, 18, 20, 21, 16]. The geometric structure involved herein is not Weinhold’s, but a conformally equivalent one. Despite accurately predicting critical phenomena in several instances, the aforementioned relationship has not been proven to hold in general.

Other geometric approaches, both alternative and complementary to the Weinhold-Ruppeiner approach have also been proposed. The first alternative, relying on contact geometry, was suggested by Arnol’d [1]: a deeper insight was performed by Mrugala [10]. In contrast to the geometric structure that was described on the previous paragraphs, the contact-geometric approach to thermodynamics is not a straightforward consequence of the physical principles of the theory. It rather relies on the following mathematical fact: in certain coordinate charts, the meaningful geometric object of a contact structure—called a contact form—has an expression that resembles the differential statement of the conservation of energy in thermodynamics. The physical meaning of the results that this approach yields are somewhat unclear (cf. Bravetti et al. [3] and Mrugala [9]).

A third trend in the geometric approach to thermodynamics states that, if a 2-tensor field is supposed to describe a thermodynamic system, it has to possess the Legendre invariance of thermodynamics [12]. However, like the approaches described above, this standpoint does not provide a fundamental equation (which translates to the full description of a system). Instead, in this setting, the cases
under consideration have also exhibited a relationship between scalar curvature and critical behavior, reproducing the results in Ruppeiner’s approach rather than extending them in most situations [13, 14, 15, 11]. Unlike the Ruppeiner-Weinhold approach, this geometric theory of thermodynamics requires the introduction of an adequate Legendre-invariant metric structure by hand.

In this work, the “natural” geometric structure of thermodynamics described on the first paragraph is studied. The aim is to identify this structure and classify it according to other known geometric structures. To this end, many of the intuitive concepts of thermodynamics are restated in geometric terms. Furthermore, some well-known facts of the physical theory are presented in this setting.

This paper is organized as follows. Section 2 is dedicated to identify the key ingredients of the underlying geometric structure of thermodynamics, and translate them to a more suitable language. Section 3 presents some well-known results regarding the degeneracy of the geometric structure of thermodynamics in a more general setting. In Section 4, the Riemannian submanifolds of spaces of equilibrium spaces are characterized. The potential usefulness of the formalism is presented as suggested by Ruppeiner, by introducing the idea of critical points in this framework. Section 5 is dedicated to concluding remarks and perspectives.

2. Preliminary definitions

As mentioned before, the First and Second Laws of thermodynamics, together with the Maximum Entropy Principle, imply that the space of equilibrium states $E$ of any thermodynamic system is endowed with a positive semi-definite symmetric 2-tensor field $g$, called henceforth Ruppeiner’s tensor [1], which becomes a Riemannian metric in certain submanifolds of $E$. In contrast to other geometric approaches to thermodynamics, the existence of this structure is a consequence of the physical principles of the theory, and it requires no additional assumptions or definitions. For this reason, any manifold that represents the space of equilibrium states of a system must be endowed with such a structure. In this section, some definitions motivated by this geometric approach to thermodynamics are presented.

Let $E$ be the space of equilibrium states of a thermodynamic system. As any equilibrium state is fully characterized by a finite number of macroscopic parameters, viz., its internal energy and its deformation coordinates, $E$ can be (globally) identified with a subset of Euclidean space. Assuming that the latter is open endows $E$ with a smooth structure. The mapping between $E$ and its image, formed by internal energy and deformation coordinates, is known as the entropy representation of the system [4].

The components of Ruppeiner’s tensor with respect to the holonomic basis induced by the entropy representation are given by the Hessian matrix of the negative of entropy, $-S$. This means that Ruppeiner’s tensor can be written in a coordinate-free expression as the covariant derivative of $d(-S)$ with respect to a connection whose Christoffel symbol vanish in the entropy representation. In other words, the latter coordinate chart defines globally a symmetric, flat linear connection $\nabla$ on $E$ such that Ruppeiner’s tensor is $-\nabla dS$. Thus, the space of equilibrium states of a thermodynamic system is endowed with a structure that resembles a Hessian one.

1 Despite the existence of this tensor was first pointed out by Weinhold, it was Ruppeiner who linked geometric singularities to thermodynamic critical points, and hence the name. In the present work, both tensors can be equally considered at once, due to their conformal equivalence.
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[2] except for the fact that Ruppeiner’s tensor fails to be a Riemannian metric. Structures of the latter kind are introduced in the following definition, which also extends the concept of Hessian structure.[3]

**Definition 1.** Let $M$ be an $n$-dimensional smooth manifold, $g$ be a positive symmetric 2-tensor field and $\nabla$ a flat, symmetric linear connection on $M$. The pair $(\nabla, g)$ is a Hessian structure over $M$ if for each $p \in M$, there exist a neighborhood $U$ of $p$ and a function $\Phi \in C^\infty(U)$ such that

$$g|_U = \nabla \text{d}\Phi.$$ (1)

The function $\Phi$ is called a local potential of the Hessian structure.

If $g$ is degenerate, $(\nabla, g)$ will be called degenerate Hessian structure, whereas non-degenerate Hessian structures will be referred to as Riemannian Hessian structures.

**Remark 1.** In principle, local potentials need not be smooth. However, smoothness is often assumed in physics, as shall be done henceforth in this paper.

**Example 1.** Let $E$ denote the space of equilibrium states of a hydrostatic system. A global entropy representation on $E$ is given by $\phi = (U, V, N)$, where $U$, $V$, and $N$ represent the internal energy, volume, and number of particles of the system, respectively.

As is well-known, in the case of an ideal gas, a potential of the Hessian structure formed by the flat connection $\nabla$ induced by the entropy representation and Ruppeiner’s tensor is given by

$$-S = -NR \ln \left( KVU^c N^{-(c+1)} \right),$$ (2)

where $c$, $K$, and $R$ are constants. Hence, the matrix representation of $g$ in this coordinate system is given by

$$(g_{ij}) = R \begin{pmatrix} cNU^{-2} & 0 & -cU^{-1} \\ 0 & NV^{-2} & -V^{-1} \\ -cU^{-1} & -V^{-1} & (c+1)N^{-1} \end{pmatrix}.$$ (3)

It can readily be seen that $\det g = 0$. The null vectors of $g$ lie on the subspace spanned by

$$\xi = U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} + N \frac{\partial}{\partial N}.$$ (4)

As illustrated above, in the space of equilibrium states of any thermodynamic system, the reason that prevents Ruppeiner’s tensor from being a Riemannian metric is degeneracy: it has a null vector field. The degeneracy of this tensor is not a condition imposed over it, but rather a consequence of entropy’s being an extensive function, i.e., a degree-one homogeneous function of the entropy representation (cf. Equation (2)), which is often regarded as a key feature of any admissible thermodynamic entropy [4]. It shall be proven that the same result holds for Hessian structures in general. In order to do so, an adequate definition of extensive function is required in terms which are more suitable for this standpoint.

Let $E$ denote the space of equilibrium states of a thermodynamic system and $\varphi_t : E \to E$ be the one-parameter flow defined by its action on the entropy representation–denoted henceforth by $\phi_U$–as $\varphi_t^* \phi_U = e^t \phi_U$, for all $t$ belonging to

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2The traditional definition of a Hessian structure requires that $g$ be non-degenerate, which is not the case in thermodynamics. This makes the extension necessary.
an open subset of \( \mathbb{R} \). A function \( f \) defined over a subset of \( E \) is said to be extensive whenever \( \varphi_t^* f = e^t f \), for any value of \( t \).

The previous definition is suitable for spaces of equilibrium states since it relies on the fact that \( \varphi_U \) is globally defined. If one wishes to extend this concept to manifolds that either lack a global coordinate chart or for which there is no straightforward criterion to select a particular coordinate chart that plays the role of the entropy representation, the previous definition is certainly not useful.

The concept of extensive functions can be rephrased in terms of the infinitesimal generator of \( \{ \varphi_t \} \), denoted by \( \xi \): a function \( f \) is extensive if and only if \( \xi [ f ] = f \).

In order to use this result as a definition, an appropriate characterization of \( \xi \) is needed. First, notice that since \( \varphi_t^* \varphi_U = e^t \varphi_U \),

\[
\xi = U \frac{\partial}{\partial U} + V^i \frac{\partial}{\partial V^i},
\]

where, as in the forthcoming, the Einstein summation convention has been used for \( i \in \{ 1, \ldots, n - 1 \} \), \( U \) represents the internal energy, and \( V^1, \ldots, V^{n-1} \) are the deformation coordinates of the system. This means that \( \xi \) has the form of an Euler vector field in the entropy representation. The problem of an adequate choice of coordinates that was mentioned before persists if this should be taken as the definition of this vector.

In order to establish the concept of extensive functions in manifolds that lack a global coordinate chart, it is worth pointing out that in the space of equilibrium states of a system

\[
(3) \quad \nabla \xi = \text{Id},
\]

where \( \nabla \) is the flat connection induced by the entropy representation, as mentioned before. This is the case for any manifold admitting a flat connection, commonly known as flat manifold, as is now shown.

**Proposition 1.** Let \((M, \nabla)\) be a smooth \( n \)-dimensional flat manifold and \( \xi \in \mathfrak{X}(\mathcal{U}) \), where \( \mathcal{U} \) is an open subset of \( M \). If \( \xi \) satisfies Equation (3) then it locally has the form of an Euler vector field.

**Proof.** Let \((M, \nabla)\) be a flat \( n \)-dimensional smooth manifold, \( \xi \) a vector field defined in an open set \( \mathcal{U} \subset M \), and \( p \in \mathcal{U} \). Since \( M \) is flat, there exists a coordinate chart \((\mathcal{V}, (x^1, \ldots, x^n))\), with \( p \in \mathcal{V} \subset \mathcal{U} \) such that the Christoffel symbols of \( \nabla \) with respect to the coordinate frame \( \{ \partial_i \}_{i=1}^n \) (where \( \partial_i \) denotes \( \partial / \partial x^i \) on every in-line expression from now on) vanish [22]. If \( \xi \) satisfies Equation (3), then \( \xi|_{\mathcal{V}} = (x^i + c^i)\partial_i \), where \( c^1, \ldots, c^n \in \mathbb{R} \). Let \( \tilde{x}^k := x^k + c^k \), for all \( k \in \{ 1, \ldots, n \} \). Hence,

\[
\xi|_{\mathcal{V}} = \tilde{x}^i \frac{\partial}{\partial \tilde{x}^i}.
\]

Due to the latter result, vector fields satisfying Equation (3) will be called Euler vector fields. Notice that, in principle, the connection involved needs not be symmetric or flat. Nevertheless, in this paper, only flat connections will be considered.

Extensive functions may now be defined as follows.

**Definition 2.** A function \( f \) defined on an open set \( \mathcal{U} \) of a flat manifold \( M \) is said to be extensive if there exists an Euler vector field \( \xi \) defined on \( \mathcal{U} \) such that \( \xi [ f ] = f \).
Remark 2. The notion of extensive function may be weakened by requiring that
\( d(\xi[f]) = df \), for some Euler vector field \( \xi \). In what follows, functions that satisfy
this condition will be referred to as weakly extensive functions. It is important to
mention that in thermodynamics, only extensive functions are considered.

The concept of (weak) extensive functions is subordinate to the existence of an
Euler vector field. If two extensive functions are defined on the same domain, the
two corresponding Euler vectors that make them extensive might be different (see
below). For this reason, whenever necessary, it will be stressed that a function is
extensive with respect to the corresponding Euler vector field.

Notice that an Euler vector field is defined up to a \( \nabla \)-parallel vector field, i.e.,
if \( \xi \) is an Euler vector field and \( \nabla P = 0 \), then \( \xi + P \) is another Euler vector field.
Furthermore, if \( \xi \) and \( \zeta \) are two Euler vector fields, \( \zeta = \xi + Q \), where \( \nabla Q = 0 \).
In thermodynamics, this “gauge freedom” is fixed by subordinating the notion of
extensive functions to a unique prescribed Euler vector field \( \xi \).

Having defined extensive functions on flat manifolds, it is now possible to prove
in general that any Hessian structure having extensive potentials is degenerate.
This is done in the following section.

3. Thermodynamic structures

As mentioned before, in the context of thermodynamics, the degeneracy of Ruppeiner’s tensor is a consequence of entropy’s being extensive. With the aid of the
previous definitions, this fact may be proven in a more general setting.

Theorem 1. Let \( M \) be an \( n \)-dimensional smooth manifold endowed with a Hessian
structure \( (\nabla, g) \). Then \( g \) has a weakly extensive potential if and only if \( g \) has a null
Euler vector field.

Proof. Let \( M \) and \( (\nabla, g) \) be as above, and \( X^\flat \) denote the 1-form defined as
\( X^\flat(Y) = g(X, Y) \), for all \( Y \in \mathfrak{X}(M) \). If \( \Phi \) is a local potential of \( g \) that shares domain \( U \)
with an Euler vector field \( \xi \), it can readily be seen that
\( \xi^\flat = d(\xi[\Phi]) - d\Phi \).

Hence, if \( \xi \) is an Euler vector field that satisfies \( \xi^\flat = 0 \), then \( \Phi \) is weakly extensive.
Conversely, if \( \Phi \) is weakly extensive, then there exists a vector \( \xi \) defined on the
domain of \( \Phi \) such that the right-hand side of equation (4) vanishes, and the result
follows. \( \square \)

Theorem 1 is actually stronger than stated: all the potentials of a degenerate
Hessian structure having a null Euler vector field are weakly extensive, as is now proven.

Proposition 2. If a manifold endowed with a Hessian structure has a weakly ex-
tensive local potential, then any other local potential sharing domain with the latter
is weakly extensive.

Proof. Let \( M \) be an \( n \)-dimensional structure and \( (\nabla, g) \) a Hessian structure over
it. Suppose that there exists a local potential \( \Phi \in C^\infty(U) \), with \( U \) open in \( M \),
such that \( d(\xi d\Phi) = d\Phi \), for some Euler vector field \( \xi \in \mathfrak{X}(U) \). Let \( \Psi \in C^\infty(U) \) be
another local potential for \( g \). On one hand, \( d(\xi[\Phi - \Psi]) = d\Phi - d(\xi[\Psi]) \). On the
other, since \( \nabla d\Phi = \nabla d\Psi \),
\[ \Phi - \Psi = a_1 x^i + b, \]
where \((U, (x^1, \ldots, x^n))\) is an affine coordinate chart, and \(a^1, \ldots, a^n, b \in \mathbb{R}\). Hence,
\[
    d(\xi [\Phi - \Psi]) = d(\xi [a_i x^i + b]) = d(a_i x^i) = d(\Phi - \Psi),
\]
which implies that \(\Psi\) is weakly extensive. \(\square\)

The fact that entropy, the potential of the Hessian structure in a space of equilibrium states of a thermodynamic system, is an extensive function with respect to the globally prescribed notion of “extensivity”, determined by a vector field \(\xi\), can also be regarded as a consequence of \(\xi^b = 0\). In general, one can always find extensive potentials for Hessian structures defined on manifolds equipped with a global vector field satisfying the latter equation. Indeed, if \(\Phi\) is a local potential in such a manifold, then \(\Phi\) must be weakly extensive, i. e., \(\xi [\Phi] = \Phi + c\), for some constant \(c \in \mathbb{R}\), whence \(\Phi + c\) is an extensive potential. Spaces of equilibrium states can thus be considered a particular case of the class manifolds that is presented in the following definition.

**Definition 3.** Let \(M\) be an \(n\)-dimensional smooth manifold. A **thermodynamic structure** on \(M\) is a triad \((\bar{\nabla}, g, \xi)\) formed by a Hessian structure \((\bar{\nabla}, g)\) and a global Euler vector field \(\xi\) satisfying
\[
\xi^b = 0.
\]
The local potentials of \((\bar{\nabla}, g)\) are called **local thermodynamic potentials**.

In the context of thermodynamics, Equation (5) is known as the **Gibbs-Duhem equation** (cf. Weinhold [26]). Spaces of equilibrium states are manifolds equipped with a Hessian structure and an Euler vector field that satisfies the Gibbs Duhem equation which (commonly) have a global entropy representation. In this viewpoint, the latter may be regarded as extensive affine coordinate charts. This name will be adopted for any such chart on manifolds endowed with a thermodynamic structure. Notice that if two local entropy representations overlap, the transition function between them must be a linear transformation (cf. Weinhold [25]).

Notice that Theorem 1 and Proposition 2 imply that local thermodynamic potentials are weakly extensive, and thus local extensive potentials can always be chosen.

**Example 2.** Black holes are known to exhibit thermodynamic properties [24]. Those belonging to the so-called **Kerr-Newman family** are characterized by the values of their mass, \(M\), the magnitude of their angular momentum, \(L\), and their charge, \(q\) [5]. From the point of view of black hole thermodynamics, the family is regarded as a system and states are black holes contained herein. The entropy of this system is given by Smarr’s formula:

\[
S = \frac{1}{4} \left[ M^2 \left( 1 + \sqrt{1 - \frac{q^2}{M^2} - \frac{L^2}{M^4}} \right) - \frac{q^2}{2} \right],
\]
where \(q^2/M^2 + L^2/M^4 \leq 1\). The triad \((M, L, q)\) is commonly regarded as the straightforward analogue of the usual entropy representation in this context. Under this consideration, the space of equilibrium states of Kerr-Newman black holes fails to be endowed with a thermodynamic structure, as defined in this paper, since \(S\) is
not extensive with respect to the global Euler vector field \( \zeta = M\partial_M + L\partial_L + q\partial_q \) (cf. Equation (2)). This can be fixed by demanding a global entropy representation to be \((\text{sgn}(M))M^2, L, \text{sgn}(q)q^2\) \[6\].

In general, the same procedure can be followed for any quasi-homogeneous system \[2\], so that it fits into the mathematical framework described so far.

**Example 3.** An important class of thermodynamic systems are those described by
one deformation coordinate, which constitute the theoretical model of thermometers \[27\]. The space of equilibrium states of these systems is a two-dimensional manifold equipped with a thermodynamic structure. As it will be proven, there is a unique two-dimensional thermodynamic structure, up to certain conformal transformations.

Any open subset \( M \subset \mathbb{R}^2 \) endowed with a thermodynamic structure may be considered an archetypical example of a two-dimensional space of equilibrium states. In Cartesian coordinates \((x, y)\) (which happen to be a global entropy representation
on \( M \) with respect to the canonical flat connection \( \bar{\nabla} \) of \( \mathbb{R}^2 \) restricted hereto), the matrix representation of Ruppeiner’s tensor, with components \( g_1, g_2, \) and \( g_3 \), must satisfy
\[
(7) \quad \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_2 & g_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix},
\]
which is simply the matrix version of the Gibbs-Duhem equation.

Furthermore, since \((\bar{\nabla}, g)\) is supposed to be a Hessian structure on \( M \), the functions \( g_1, g_2, \) and \( g_3 \) are related by \[22\]
\[
(8) \quad \frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x},
(9) \quad \frac{\partial g_2}{\partial y} = \frac{\partial g_3}{\partial x}.
\]

Equations (7), (8), and (9) imply that
\[
(g_{ij}) = f \begin{pmatrix} -y/x & 1 \\ 1 & -x/y \end{pmatrix},
\]
where \((g_{ij})\) denotes the matrix representation of \( g \), and \( f \) satisfies \( f = -\partial_x f - \partial_y f \) (equivalently, \( f \) is implicitly defined by any equation of the form \( F(x/y, f x) = 0 \)).

As was mentioned before, any manifold endowed with a thermodynamic structure contains embedded Riemannian submanifolds. This result will be proven in the following section in a general setting.

### 4. Riemannian Submanifolds

The main interest in the study of the geometric structure of thermodynamics is its potential usefulness to describe critical behavior. The latter is understood as a feature of thermodynamic systems that may attain states that are qualitatively different from each other. For instance, in a hydrostatic system this feature is evident when some of its states correspond to a liquid phase, and some other to a gas phase.

A well-known geometric indication of critical behavior is related to the convexity of certain thermodynamic potentials that in the context of thermodynamics are said to have the intensive parameters of the system as their “natural variables” \[4, 8\].
The former are defined as \(-\partial_i S\), denoted henceforth as \(-S_i\), and yield important physical information of systems. For example, in the case of hydrostatic systems, \(-S_U = 1/T\), where \(T\) represents the temperature of the system; \(-S_V = p/T\), where \(p\) denotes its pressure; finally, \(-S_N = \mu/T\), where \(\mu\) represents the chemical potential of the system.

Following the physical idea above, one can define \textit{local intensive parameters} on any manifold \(M\) equipped with a thermodynamic structure \((\nabla, g, \xi)\). If \(\Phi \in C^\infty(\mathcal{U})\) is a local thermodynamic potential and \((\mathcal{U}, (x^1, \ldots, x^n))\) is a local entropy representation, \(\Phi_i := \partial_i \Phi\), with \(i \in \{1, \ldots, n\}\), will be called a \textit{local intensive parameter}. Notice that Equation (5) implies \(\xi[\Phi_i] = 0\), for all \(i\), which means that the local intensive parameters remain invariant under the flow of \(\xi\).

**Remark 3.** If \(\Phi \in C^\infty(\mathcal{U})\) is a local thermodynamic potential and \((\mathcal{U}, (x^1, \ldots, x^n))\) is an affine coordinate chart, then \(g|_{\mathcal{U}} = dx^i d\Phi_i\), where the juxtaposition of differential forms denotes the symmetrization of their tensor product. Let \(\{\varphi_t\}_{t \in (a, b)}\) denote the flow of \(\xi\), for some \(a, b \in \mathbb{R}\). Then, for every \(t \in (a, b)\), \(\varphi_t\) is a conformal mapping: \(\varphi_t^* g = e^{t^2} g\).

In general, the behavior of \(g\) under the flow of \(\xi\) as above does not guarantee the existence of a local extensive potential (and hence, of a thermodynamic structure). As an illustrative example of this statement, let \(M = \{(x, y, z) \in \mathbb{R}^2 : x, y > 0\}\) and \(g := x^{-1} (dx)^2 + y^{-1} (dy)^2\). The usual flat connection on \(\mathbb{R}^3\), denoted by \(\nabla\), together with \(g\) form a degenerate Hessian structure on \(M\), with a global potential given by \(\Phi = x \log(x) - x + y \log(y) - y\). Observe that if \(\varphi_t^* (x, y, z) := e^t (x, y, z)\), then \(\varphi_t\) is an isometry for any \(t \in \mathbb{R}\). Yet, \(\Phi\) is not an extensive potential.

The Gibbs-Duhem equation is meaningful in thermodynamics because it asserts that thermodynamic systems cannot be fully characterized by the values of their intensive parameters. This is in general true for any manifold endowed with a thermodynamic structure, since Equation (5) is written in affine coordinates as \(x^i d\Phi_i = 0\), implying that the functions \(\Phi_1, \ldots, \Phi_n\) are not functionally independent. Nevertheless, they constitute a coordinate chart in certain embedded submanifolds, which means that the latter are Riemannian. This fact gives rise to a geometric portrait of the thermodynamic potentials that are obtained through Legendre transformations and play an essential role in detecting critical behavior.

Riemannian submanifolds of a manifold \(M\) endowed with a thermodynamic structure are precisely those where Ruppeiner’s tensor does not vanish. This means that their tangent space must complete the space spanned by the null vectors of \(g\) on the tangent space of \(M\) at each point. This can be rephrased in terms of transversality owing to the following result.

**Theorem 2.** Let \(g\) be a 2-tensor field and \(\nabla\) a symmetric linear connection defined over an \(n\)-dimensional smooth manifold \(M\) satisfying

\[(10) \quad \nabla_X g(Y, Z) = \nabla_Y g(X, Z)\]

for all \(X, Y, Z \in \mathfrak{X}(M)\). If \(\Delta\) denotes the distribution defined by the null vectors of \(g\), then \(\Delta\) is integrable.
Proof. Let $X, Y \in \Delta$. It shall be proven that $[X, Y] \in \Delta$. For any $Z \in \mathfrak{x}(M)$,

$$[X, Y]^\flat(Z) = g([X, Y], Z)$$

$$= g(\nabla_X Y - \nabla_Y X, Z)$$

$$= X[g(Y, Z)] - \nabla_X g(Y, Z) - g(Y, \nabla_X Z) - Y[g(X, Z)] +$$

$$\nabla_Y g(X, Z) + g(X, \nabla_Y Z)$$

$$= \nabla_Y g(X, Z) - \nabla_X g(Y, Z)$$

$$= 0.$$

Hence, $[X, Y] \in \Delta$ and so the distribution is integrable. □

In a manifold $M$ endowed with a thermodynamic structure, any submanifold transversal to an integral manifold of $\Delta$ is Riemannian. Let $p \in M$. If $D$ is an integral manifold of $\Delta$ with $p \in D$, and $(\mathcal{U}, (x^1, \ldots, x^n))$ is an entropy representation chart containing $p$, then it can readily be seen that

$$N := \bigcap_{i \in I} (x^i)^{-1}(x^i(p))$$

is an $r$-dimensional embedded Riemannian submanifold of $M$ [23], where $I \subset \{1, \ldots, n\}$, $|I| = n - r$, and $r$ denotes the rank of $\flat$. This is true because $a^\alpha g_{\alpha\beta} = 0$ implies $a^\alpha = 0$, where $\alpha, \beta \in \{1, \ldots, n\} \setminus I$ and $g_{\alpha\beta}$ are the components of Ruppeiner’s tensor at any point $q \in \mathcal{U} \cap D$. Hence, the following result has been proven.

**Theorem 3.** Every point of a manifold $M$ endowed with a thermodynamic structure lies on a Riemannian submanifold of $M$.

If $N$ is a Riemannian submanifold as described above, then $\nabla|_{\mathfrak{x}(N) \times \mathfrak{x}(N)}$ is a flat connection defined hereon, which shall be denoted also by $\nabla$. Any entropy representation defines slice coordinates for $N$ which are affine. If $i : N \rightarrow M$ is the inclusion, then $i^*g$ is a Riemannian metric. Yet more, since $(\nabla, i^*g)$ satisfies Equation (10), this pair is a Riemannian Hessian structure. If $\Phi \in \mathcal{U}$ is a thermodynamic potential, then $i^*\Phi$ is a potential for the latter.

As is well known [22], if $\nabla$ denotes the Levi-Civita connection of $i^*g$, then $\nabla := 2\nabla - \nabla$ is another flat connection on $N$, for which $(N, (y_\alpha)_{\alpha \notin I})$ is an affine coordinate chart (here as in the forthcoming, $i^*y_\alpha$ is denoted by $y_\alpha$, for all $\alpha \in \{1, \ldots, n\} \setminus I$). The connection $\nabla$ is called the dual connection of $\nabla$ with respect to $i^*g$. Due to the fact that $d(x^\alpha \wedge dy_\alpha) = 0$, where $\alpha \in \{1, \ldots, n\} \setminus I$, $i^*g = \nabla d\Phi$. Any potential $\Phi$ of the dual Riemannian Hessian structure $(\nabla, i^*g)$ is related to $\Phi$ (up to a constant) by a Legendre transform [22]:

$$\Phi = x^\alpha y_\alpha - i^*\Phi,$$

or equivalently,

$$\tilde{\Phi} = -x^i y_i,$$

with $i \in I$.

**Example 4.** Let $E$ be the space of equilibrium states of a hydrostatic system. In this case, rank $\flat = 2$, which means that the manifolds defined by constant internal energy $U$, constant volume $V$, and constant particle number $N$ are Riemannian submanifolds of $E$. 
In the case of constant $N$, the system is called \textit{closed}. The corresponding dual potential of the induced Riemannian Hessian structure is simply $-\mu N/T$, which is the Massieu function proportional to the so-called \textit{Gibbs’ free energy} $G := \mu N$.

In the submanifold defined by constant volume, the induced geometric structure is known as \textit{Ruppeiner geometry}. The corresponding dual potential is given by the Massieu function proportional to the \textit{grand potential}, $-pV/T$, where $p$ is the pressure of the system.

At constant internal energy, the dual potential of the Hessian structure is given by $-U/T$.

Each one of the latter are thermodynamic potentials $\Phi$ whose “natural variables” are the intensive parameters. A system undergoes phase transition whenever these are not concave, \textit{i.e.}, if there exist states $p$ contained in the Riemannian submanifolds of such that $-(\nabla d\Phi)_p$ is not negative definite, which is equivalent to $(\iota^* g)_p$ not being positive definite. The points where the latter condition holds are called \textit{critical points}.

The concepts of Riemannian geometry have been applied to the Riemannian submanifolds of spaces of equilibrium states, and still lack a clear physical interpretation. Among these, scalar curvature stands out: it is \textit{conjectured} to be related to critical behavior. In many three-dimensional cases, it has been proven that the scalar curvature diverges at critical points [19, 21, 20, 16]. A proof of this relationship in a more general setting, like the one proposed in this paper, is still missing.

\section{Concluding remarks and future work}

A general geometric theory in which (homogeneous and quasi-homogeneous) thermodynamics fits as a particular case has been proposed. The meaningful geometric structure of thermodynamics is identified to be composed of two main ingredients: a degenerate Hessian structure and a global null Euler vector field.

The approach that has been presented herein solves the problem of the choice of a suitable entropy representation, which plays the essential role of defining a global flat connection that is in turn necessary for the definition of Ruppeiner’s tensor. Two different entropy representations yield different geometric structures, and the results thereby derived might differ significantly [6]. The existence of a global null Euler vector field fixes the family of admissible entropy representations.

It is worth pointing out that in this approach to thermodynamics, potentials and coordinates lose their intrinsic physical meaning. Not only are Weinhold’s and Ruppeiner’s approach treated at the same level, but any other conformally equivalent approach is considered as well.

The idea of critical point can be readily imported to manifolds endowed with a thermodynamic structure. Using standard methods of Riemannian geometry, these might be characterized (\textit{e.g.}, in terms of completeness). In this setting, the role of scalar curvature in thermodynamics might also be clarified.

From the mathematical point of view, the notion of Euler vector field can be extended to manifolds endowed with a linear connection which is not necessarily flat. This is also true for extensive functions. The condition over the degenerate Hessian structure of a thermodynamic structure can be weakened, by demanding that this be a (degenerate) Codazzi structure, instead. The usefulness of this extension in physics is not known, though.
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