There is no Diophantine $D(-1)$-quadruple

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Abstract

A set of positive integers with the property that the product of any two of them is the successor of a perfect square is called Diophantine $D(-1)$-set. Such objects are usually studied via a system of generalized Pell equations naturally attached to the set under scrutiny. In this paper, an innovative technique is introduced in the study of Diophantine $D(-1)$-quadruples. The main novelty is the uncovering of a quadratic equation relating various parameters describing a hypothetical $D(-1)$-quadruple with integer entries. In combination with extensive computations, this idea leads to the confirmation of the conjecture according to which there is no Diophantine $D(-1)$-quadruples.

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1 | THE STRATEGY

In the third century, Diophantus of Alexandria found four positive rationals such that the product of any two of them increased by unity is a square, see, for instance, [6–8, 23]. Fermat found in the 17th century the quadruple consisting of positive integers 1, 3, 8, 120 with the same property. As Euler remarked, Fermat’s set can be enlarged by inserting $777480/8288641$ without losing the defining property. It was in 1969 that Baker and Davenport [3] proved that there is no quintuple of positive integers containing Fermat’s set and still having the property of interest. On this occasion the authors introduced an important tool, nowadays referred to as the Baker–Davenport lemma, for the effective resolution of Diophantine equations.

Diophantus also studied a problem that turned out to be closely related to that mentioned before. Namely, he asked for numbers such that the product of any two of them increased by
the sum of these two is a square. Since \( ab + a + b = (a + 1)(b + 1) - 1 \), the question boils down to finding sets with the property that the product of any two of its elements is one more a square. The essence of both problems is captured by the next definition.

Let \( m \geq 2 \) and \( n \) be integers. A set of \( m \) positive integers is called Diophantine \( D(n)\)-\( m \)-set if the product of any two distinct elements increased by \( n \) is a perfect square. In this terminology, Fermat’s example is a \( D(1)\)-quadruple, and the set \( \{4, 9, 28\} \) presented by Diophantus himself as an answer to the second problem gives rise to the \( D(−1)\)-triple \( \{5, 10, 29\} \). A more general notion is obtained by considering elements of any commutative ring instead of positive integers. However, many difficult, interesting problems already occur in the setting fixed by the given definition. In the rest of the paper we will refer only to this definition, even when we omit the adjective ‘Diophantine’.

It is worth mentioning that the objects produced by this definition with \( n = 0 \) are not particularly interesting — for each positive \( m \) there exist infinitely many \( D(0)\)-\( m \)-sets and even infinite \( D(0)\)-sets. Therefore, when speaking of \( D(n)\)-\( m \)-sets we will always assume \( n \) is nonzero.

A natural question is how large a \( D(n)\)-set can be. It is known [10] that for \( |n| < 400 \) one has \( m < 31 \), and for other \( n \) the cardinality of any \( D(n)\)-\( m \)-tuple is at most \( 16 \log |n| \). Better bounds are known for particular values of \( n \). As noted in several papers (among which [5, 21, 30]), for \( n \equiv 2 \pmod{4} \) there is no \( D(n)\)-quadruple. On the opposite side, in [9] it is shown that if \( n \not\equiv 2 \pmod{4} \) and \( n \notin S := \{-4, −3, −1, 3, 5, 8, 12, 20\} \), then there exists at least one \( D(n)\)-quadruple. In the same paper Dujella expressed his confidence that this is all one can hope for.

**Conjecture.** There exists no \( D(n)\)-quadruple for \( n \in \{-4, −3, −1, 3, 5, 8, 12, 20\} \).

According to a remarkable result of Dujella and Fuchs [12], in any \( D(−1)\)-quadruple \((a, b, c, d)\) with \( a < b < c < d \) one has \( a = 1 \). This readily implies the nonexistence of \( D(−1)\)-quintuples. The same authors together with Filipin proved in [11] that there are at most finitely many \( D(−1)\)-quadruples. The present authors obtained in [4] the bound \( 10^{71} \) for the number of \( D(−1)\)-quadruples, thus improving on the previous bound \( 10^{356} \) found in [15]. Better estimates have been given lately: in [14] one finds the upper bound \( 5 \cdot 10^{60} \), successively strengthened to \( 3.01 \cdot 10^{58} \) in [36], to \( 4.7 \cdot 10^{58} \) in [24]. The best bound we are aware of is \( 3.677 \cdot 10^{58} \) found in [25].

A basic technique in the study of \( D(n)\)-sets exploits a connection with systems of generalized Pell equations. We explain the main ideas of this approach in the framework of \( D(−1)\)-quadruples.

Suppose \((1, b, c, d)\) is a \( D(−1)\)-quadruple with \( 1 < b < c < d \). Then there are positive integers \( r, s, t, x, y, z \) satisfying

\[
\begin{align*}
  b - 1 &= r^2, & c - 1 &= s^2, & bc - 1 &= t^2, \\
  d - 1 &= x^2, & bd - 1 &= y^2, & cd - 1 &= z^2.
\end{align*}
\]

Eliminating \( d \) in Equation (1.2), one obtains a system of three generalized Pell equations

\[
\begin{align*}
  z^2 - cx^2 &= c - 1, & (1.3) \\
  bz^2 - cy^2 &= c - b, & (1.4) \\
  y^2 - bx^2 &= b - 1. & (1.5)
\end{align*}
\]
By Theorem 1.2 in [4], we may assume \( c < 2.5b^6 \). Then, according to [11, Lemmas 1 and 5], the positive integer solutions of each of the above Pell equations are, respectively, given by

\[
\begin{align*}
    z + x \sqrt{c} &= s(s + \sqrt{c})^{2m}, \quad m \geq 0, \\
    z \sqrt{b} + y \sqrt{c} &= (s \sqrt{b} + \rho r \sqrt{c})(t + \sqrt{bc})^{2n}, \quad n \geq 0, \\
    y + x \sqrt{b} &= r(r + \sqrt{b})^{2l}, \quad l \geq 0,
\end{align*}
\]

for fixed \( \rho \in \{-1, 1\} \). Therefore, the triples \((x, y, z)\) of positive integers that simultaneously satisfy Equations (1.3)–(1.4) are those such that

\[
z = v_m = w_n,
\]

where the integer sequences \((v_p)_{p \geq 0}, (w_p)_{p \geq 0}\) are given by explicit formulas

\[
v_p = \frac{s}{2} \left( (s + \sqrt{c})^{2p} + (s - \sqrt{c})^{2p} \right)
\]

and, respectively,

\[
w_p = \frac{s \sqrt{b} + \rho r \sqrt{c}}{2 \sqrt{b}} (t + \sqrt{bc})^{2p} + \frac{s \sqrt{b} - \rho r \sqrt{c}}{2 \sqrt{b}} (t - \sqrt{bc})^{2p}.
\]

These formulas give rise in the usual way to linear forms in the logarithms of three algebraic numbers, for which upper bounds are obtained directly, while lower bounds are given by a general theorem of Matveev [29]. Comparison of these bounds results in inequalities for indices \( m \) and \( n \) in terms of elementary functions in \( b \) and \( c \). In order to get reverse inequalities, Dujella and Pethő introduced in [13] the congruence method. Their idea is to consider the recurrent sequences modulo \( 8c^2 \) and prove that suitable hypotheses entail that these congruences are actually equalities. The best result obtained by this approach is due to Dujella, Filipin and Fuchs [11].

**Theorem 1.1.** Let \((1, b, c, d)\) with \(1 < b < c < d\) be a \( D(-1)\)-quadruple. Then \( b > 100 \) and \( c < \min\{11 b^6, 10^{491}\}\). More precisely:

(a) if \(b^3 \leq c < 11 b^6\), then \(c < 10^{238}\);  
(b) if \(b^{1.1} \leq c < b^3\), then \(c < 10^{491}\);  
(c) if \(3b \leq c < b^{1.1}\), then \(c < 10^{94}\);  
(d) if \(b < c < 3b\), then \(c < 10^{74}\).

A variant of the congruence method has been introduced in [4]. The new idea is to interpret an equivalence \( L \equiv R \) (mod \( c \)) as an equality \( L - R = j c \) for a suitable integer \( j \). Instead of striving to get \( j = 0 \), as did the predecessors, all possibilities for the sign of \( j \) have been analyzed. As a result of this study, inequalities of the form \( n > f(b, c)^{a(j)} \) have been established. Combined with another new idea, called smoothification in [4], and large-scale computations, always performed with the help of the package PARI/GP [33], this yields much better results.
Theorem 1.2. Let \((1, b, c, d)\) with \(1 < b < c < d\) be a \(D(-1)\)-quadruple. Then \(b > 1.024 \cdot 10^{13}\) and \(\max\{10^{14} b, b^{1.16}\} < c < \min\{2.5 b^6, 10^{148}\}\). More precisely:

(i) if \(b^5 \leq c < 2.5 b^6\), then \(c < 10^{100}\);
(ii) if \(b^4 \leq c < b^5\), then \(c < 10^{82}\);
(iii) if \(b^{3.5} \leq c < b^4\), then \(c < 10^{66}\);
(iv) if \(b^3 \leq c < b^{3.5}\), then \(c < 10^{57}\);
(v) if \(b^2 \leq c < b^3\), then \(c < 10^{111}\);
(vi) if \(b^{1.5} \leq c < b^2\), then \(c < 10^{109}\);
(vii) if \(b^{1.4} \leq c < b^{1.5}\), then \(c < 10^{128}\);
(viii) if \(b^{1.3} \leq c < b^{1.4}\), then \(c < 10^{148}\);
(ix) if \(b^{1.2} \leq c < b^{1.3}\), then \(c < 10^{133}\);
(x) if \(b^{1.16} \leq c < b^{1.2}\), then \(c < 10^{107}\).

More recently, Filipin and Fujita obtained an even better relative bound for the third element of a hypothetical \(D(-1)\)-quadruple. In \([16]\), they proved the remarkable result quoted below. The proof is based on an improved variant of Rickert’s theorem \([34]\).

Theorem 1.3. Any \(D(-1)\)-quadruple \((1, b, c, d)\) with \(1 < b < c < d\) satisfies \(c < 9.6 b^4\).

Figures 1 and 2 give a graphical representation of these results. The outer (inner) polygon in Figure 1 represents the region where \(c\) is confined according to Theorem 1.1 (Theorem 1.2). From Theorem 1.3 we see that in the search for \(D(-1)\)-quadruples we can restrict ourselves to the part of polygon sitting in the half-plane \(c < 9.6 b^4\). Figure 2 contains an approximate illustration of Theorem 1.2 in polar coordinates. It is seen that \(c\) is to be found in a region whose shape looks like a nonstandard fan. We interpret the presence of inlets as a strong hint that actually there are no \(D(-1)\)-quadruples whose third entry is located in the nonconvex blades. Eliminating these blades has the effect of ‘partially closing the fan’. The aim of this paper is to ‘completely close the fan’.
Theorem 1.4. There is no Diophantine $D(-1)$-quadruple.

Since, by [9, Remark 3], all elements of a $D(-4)$-quadruple are even, from Theorem 1.4 we get for free another result that provides partial confirmation of Dujella’s conjecture.

Theorem 1.5. There is no Diophantine $D(-4)$-quadruple.

Our strategy is based on the following interpretation of Equation (1.1): the initial triple $(1, b, c)$ of a hypothetical $D(-1)$-quadruple with $1 < b < c < d$ is associated to a member of a two-parameter family of integers $b = r^2 + 1$, $c = s^2 + 1$ and $t$ witnesses the fact that we deal with a $D(-1)$-triple. We would like to handle all these parameters simultaneously. This goal is achieved by considering the integer

$$f = t - rs,$$

which is easily seen to be positive. It already appeared in several proofs available in literature, see, for instance, [12, 19, 20, 22]. Up to our work it always had a secondary role, sitting in background. Focusing on $f$ turned out to open new prospects for the study of $D(-1)$-quadruples.

The developments below seem to have been overlooked in the literature. Squaring $f + rs = t$, one gets $f^2 + 2frs + r^2s^2 = (r^2 + 1)(s^2 + 1) - 1$, whence

$$r^2 + s^2 = 2frs + f^2. \tag{1.9}$$

Our approach hinges on the study of solutions in positive integers to the master equation (1.9) in its various disguises. This study is much easier than the examination of solutions to the system of generalized Pell equations (1.3)–(1.5). As will be seen in Section 2, rather strong results are obtained by elementary proofs relying on properties of solutions to Equation (1.9). Besides the emphasis on $f$ already mentioned, our treatment introduces here yet another variant of the congruence method by considering modulus $8f^6$ instead of the ‘classical’ $8c^2$. Also, we revisit published results and use them in a novel way. The study is straightforward if $gcd(r, s) = f$. In the general case, we complement it with considerations along the lines described in the next paragraphs.
Let us denote by $J$ the set of pairs of integers $(r, s)$ such that there exists a $D(-1)$-quadruple $(1, b, c, d)$ with $1 < b < c < d$ and $b = r^2 + 1, c = s^2 + 1$. For $(r, s) \in J$ we put $s = r^\theta$ and define

$$\theta^- = \inf_{(r, s) \in J} \theta, \quad \theta^+ = \sup_{(r, s) \in J} \theta.$$ 

These numbers measure the size of $c$ with respect to $b$. ‘Closing the fan’ means showing that for a putative $D(-1)$-quadruple one has $\theta^+ < \theta^-$. The main result of [4] (quoted in Theorem 1.2) gives in particular $c \leq 3b^6$. Using this upper bound, a computer verification described in [4, Section 2] led to the conclusion that $r > 32 \times 10^5$, so that $b > 10^{13}$. Hence

$$\theta^+ \leq 6.1.$$ 

The approach followed in [4] used, among other things, the inequalities $c > 3.999 f^2 b$ and $f > 10^7$ in order to increase the lower bound $\theta^-$ from 1 to 1.16. Pursuing this idea requires the examination of much higher values of $f$, a process that becomes prohibitively time-consuming. To give the reader a feeling for the difficulties encountered, we mention that computations that allow exclusion of values $f \leq 10^7$ needed about two weeks (measured by wall-clock) on a personal computer; to reach the level $f > 10^8$, our program ran on a network of up to 6 computers for about three months; further computations were performed over six months on as many as 30 computers. Therefore, another course has been chosen: instead of shortening the interval $[\theta^-, \theta^+]$, we looked for methods of splitting it in such a way that parallel processing is possible. The alternative approach was devised after it was observed that there is no $D(-1)$-quadruple with $b^2 \leq c \leq 16b^3$ even when $\gcd(r, s) \neq f$. The breakthrough was realized after making the choice $F := s - 2rf$. A short study revealed that it is very helpful in separating values $c < 4b^2$ from values $c > 4b^2$. It was a matter of days to reach the conclusion that around $c = 4b^2$ there is a large gap (a comparatively long interval in which there is no third entry of a $D(-1)$-quadruple).

Completion of the proof requires new explicit computations. Besides those needed for use of results providing bounds for linear forms in logarithms, a considerable amount of them were devoted to solving many quadratic Diophantine equations and then applying the reduction procedure based on the Baker–Davenport lemma. Further explanations and full details are given in Section 4. A successful implementation of the strategy just sketched requires one to pay attention to several aspects which will be described in Section 3. Here we mention only one point. It is clear that the smaller the upper bounds on $b$ and $c$ are, the faster the subsequent computations depending on them are. To this end, Matveev’s general theorem [29] was first replaced by a strengthening of it due to Aleksentsev [1] and next by an older result of Matveev [28]. This course of action is determined by our experience, according to which a giant step is better replaced by successive small steps.

2 | A SUFFICIENT CONDITION FOR THE NONEXISTENCE OF $D(-1)$-QUADRUPLES

The aim of this section is to revisit results of our previous work [4] in the light of the new guiding strategy. As it turns out, a lot of information already available can be exploited in a novel manner, producing unexpected results and suggesting further developments.
The starting point of our study of solutions in positive integers to the equation
\[ r^2 + s^2 = 2frs + f^2, \]
is the observation that \( \gcd(r, s) \) is a divisor of \( f \). As extremal elements / circumstances generally are very interesting, we consider solutions to Equation (1.9) such that
\[ f = \gcd(r, s). \tag{2.1} \]
Then one has
\[ r = fu, \quad s = fv, \]
for some positive integers \( u, v \) satisfying
\[ (v - fu)^2 - eu^2 = 1, \]
with \( e = f^2 - 1 \).

Let \( \gamma = f + \sqrt{e} \) be the fundamental solution to the Pell equation \( V^2 - eU^2 = 1 \) and \( \overline{\gamma} = f - \sqrt{e} \) its algebraic conjugate. According to Lemma 3.5 from [4], \( v > 3.9991/2fu \), so that all positive solutions to Equation (1.9) have the form
\[ r = \frac{f(y^k - \overline{\gamma}^k)}{\gamma - \overline{\gamma}}, \quad s = \frac{f(y^{k+1} - \overline{\gamma}^{k+1})}{\gamma - \overline{\gamma}}, \quad k \in \mathbb{N}, \tag{2.2} \]
whence
\[ b = \frac{f^2(y^k + \overline{\gamma}^k)^2 - 4}{(\gamma - \overline{\gamma})^2}, \quad c = \frac{f^2(y^{k+1} + \overline{\gamma}^{k+1})^2 - 4}{(\gamma - \overline{\gamma})^2}. \tag{2.3} \]
Note that the main result of He–Togbé [22] assures \( f \geq 2 \), a piece of information we will repeatedly use without explicitly mentioning it. Later on, a more stringent restriction, checked computationally, will be preferred.

With this notation fixed, we proceed to examine the properties of solutions to Equation (1.9) under condition (2.1).

**Proposition 2.1.** One has \( c < b\gamma^2 \). Moreover, for \( k \geq 2 \) it holds
\[ \gamma^2 - \frac{1}{2} < \frac{c}{b}. \]

**Proof.** The numerator of \( by^2 - c \) is \( (\gamma^2 - 1)[f^2(2 + \gamma^{2k} + \overline{\gamma}^{2k+2}) - 4] \), which is manifestly positive. The numerator of \( 2c - (2\gamma^2 - 1)b \) is found to be
\[ 4(2\gamma^2 - 3) + f^2[\gamma^{2k} - 2(2\gamma^2 - 3) + \overline{\gamma}^{2k}(1 - 2\gamma^2 + 2\overline{\gamma}^2)]. \]
Since $k \geq 2$ and $0 < \gamma < 1$, it is sufficient to prove

$$\gamma^4 > 2(2\gamma^2 - 3) + 2\gamma^2 - 1.$$ 

This inequality is obvious on noticing the identity $\gamma^4 = 4(f^2 - 1)\gamma^2 + 2\gamma^2 - 1$. □

We can bound $b$ from below by a power of $\gamma$.

**Proposition 2.2.** If $(1, b, c, d)$ is a $D(−1)$-quadruple with $10^{13} < b < c < d$ and $b, c$ given by formula (2.3), then

$$\gamma^{2k - 1} < b.$$ 

**Proof.** The desired inequality is equivalent to $f^2(\gamma^{2k} + 2 + \gamma^{2k}) - 4 > 4(f^2 - 1)\gamma^{2k - 1}$, which follows from $f^2\gamma > 4(f^2 - 1)$. □

We are now in a position to show that the third entry in a $D(−1)$-quadruple restricted as in (2.1) is much closer to the second one than was previously known.

**Proposition 2.3.** Any $D(−1)$-quadruple $(1, b, c, d)$ with $10^{13} < b < c < d$ and $b, c$ given by (2.3) satisfies $c < b^3$.

**Proof.** Assuming $b^3 \leq c$, we deduce with the help of the previous results

$$\gamma^{4k - 2} < b^2 \leq \frac{c}{b} < \gamma^2,$$

that is, $k < 1$. Thus $k = 0$, when $r = 0$ and $b = 1$, which is not possible. □

Propositions 2.1 and 2.2 have other important consequences drawn from information made available by our previous work. For the sake of convenience, we recall an experimental result obtained after two weeks of computer calculations for the needs of [4, Lemma 3.5]. Its proof is based on the well-known structure of solutions to a Pellian equation of the type

$$W^2 - DU^2 = N, \quad D > 0 \text{ nonsquare and } N \neq 0. \quad (2.4)$$

The most familiar reference is Nagell’s book [31] in its various editions but the results have been published already in the 19th century by Chebyshev [35]. More details are available in the proof of [4, Lemma 2.9].

**Proposition 2.4.** There are no $D(−1)$-quadruples $(1, b, c, d)$ with the corresponding $f$ less than or equal to $10^7$.

This result is used below in conjunction with the fact that for any hypothetical $D(−1)$-quadruple one has $n < 10^{19}$ (see [4, Table 1]).

After these preparations, we proceed with the study of positive solutions to Equation (1.9) given by (2.2).
Proposition 2.5. There are no $D(-1)$-quadruples $(1, b, c, d)$ with $10^{13} < b < c < d$, $b^2 \leq c < b^3$ and $b, c$ given by (2.3).

Proof. Suppose, by way of contradiction, that the thesis is false. From

$$y^{2k-1} < b \leq \frac{c}{b} < y^2$$

we get $k < 2$. Since $k \geq 1$, we conclude that $k = 1$.

Eliminating $d$ in Equation (1.2) yields the system of generalized Pell equations (1.3)–(1.5). It is well known that $z$ appears in two second-order linearly recurrent sequences. Thus (see, for example, [11, 12]) $z = v_m = w_n$, with $m, n$ positive integers of the same parity, and

$$v_0 = s, \quad v_1 = (2c - 1)s, \quad v_{m+2} = (4c - 2)v_{m+1} - v_m,$$  \hspace{1cm} (2.5)

$$w_0 = s, \quad w_1 = (2bc - 1)s + 2\rho rt, \quad w_{n+2} = (4bc - 2)w_{n+1} - w_n,$$  \hspace{1cm} (2.6)

where $\rho = \pm 1$. Since $k = 1$, one has $r = f$, $b = f^2 + 1$, $s = 2f^2$, $c = 4f^4 + 1$, $t = 2f^3 + f$, and therefore

$$v_{m+2} = 2(8f^4 + 1)v_{m+1} - v_m,$$

$$w_{n+2} = 2(8f^6 + 8f^4 + 2f^2 + 1)w_{n+1} - w_n.$$

Taken modulo $8f^6$, these recurrent relations readily give

$$v_m \equiv 2f^2 \pmod{8f^6}$$

and

$$w_n \equiv 2(\rho n + 1)f^2 + \left(\frac{4n^3 + 8n}{3}\rho + 4n^2\right)f^4 \pmod{8f^6}.$$

Together with $v_m = w_n$, this implies

$$\rho n + \left(\frac{2n^3 + 4n}{3}\rho + 2n^2\right)f^2 \equiv 0 \pmod{4f^4}. \hspace{1cm} (2.7)$$

Note that $n$ must be even and use this information to deduce $n \equiv 0 \pmod{4f^2}$, so that $n = 4f^2u$ for some positive integer $u$. Replace $n$ by $4f^2u$ in (2.7) to get $u \equiv 0 \pmod{f^2}$, and therefore $n \geq 4f^4$.

Using this inequality and Proposition 2.4 one gets $n > 10^{28}$, in contradiction with the fact $n < 10^{19}$ proved in [4, Proposition 4.3]. \hfill $\Box$

Proposition 2.6. There are no $D(-1)$-quadruples $(1, b, c, d)$ with $10^{13} < b < c < d$, $b^{1.5} \leq c < b^2$, and $b, c$ given by (2.3).
Proof. We reason by reduction to absurd. Suppose that $(1, b, c, d)$ is a $D(-1)$-quadruple satisfying $10^{13} < b < c < d$, $b^{1.5} \leq c < b^2$, and $b, c$ given by (2.3). It is easy to prove the upper bound $2k < 5$ as above. Assuming $k = 1$, from $c < b^2$ it results $4f^4 + 1 < (f^2 + 1)^2$, whence $f < 1$, which is impossible.

So it is established that $k = 2$. Therefore, one has $r = 2f^2$, $s = 4f^3 - f$, and $t = 8f^5 - 2f^3 + f$. The solutions to the system of Pellian equations (1.3)–(1.5) verify $y = U_n = u_l$, where $n, l$ are positive integers and

$$u_{l+2} = (4b - 2)u_{l+1} - u_l, \quad u_0 = r, \quad u_1 = (2b - 1)r,$$

$$U_{n+2} = (4bc - 2)U_{n+1} - U_n, \quad U_0 = \rho r, \quad U_1 = (2bc - 1)\rho r + 2bفت.$$  

Considering these recurrence relations modulo $r^3$, one readily gets

$$u_l \equiv r \pmod{r^3}.$$  

A short inductive reasoning that takes into account the explicit formulas giving $r, s, t$ in terms of $f$ results in the congruence

$$U_n \equiv (n^2r^2 + r)\rho + \frac{(10n - n^3)}{3}r^2 - nr \pmod{r^3},$$

so that $u_l = U_n$ implies $r\rho - nr \equiv r \pmod{r^3}$. Therefore, there exists an integer $\lambda$ such that

$$n = \rho - 1 + \lambda r.$$  

Since $n \geq 7$ by [4, Proposition 2.2], it follows that $\lambda$ is positive. Introducing this formula for $n$ in the congruence for $U_n$, one sees that in fact one has $\lambda \geq r - 1$, so that $n > 0.9r^2 > 3f^4 > 10^{19}$. As explained previously, this contradicts [4, Proposition 4.3]. The contradiction is due to the assumption that there exists a $D(-1)$-quadruple satisfying all the hypotheses of the present proposition.

Proposition 2.7. There are no $D(-1)$-quadruples $(1, b, c, d)$ with $10^{13} < b < c < d$, $b^{1.4} \leq c < b^{1.5}$, and $b, c$ given by (2.3).

Proof. As above, we reason by reduction to absurd. Suppose that $(1, b, c, d)$ is a $D(-1)$-quadruple satisfying $10^{13} < b < c < d$, $b^{1.4} \leq c < b^{1.5}$, and $b, c$ given by (2.3). As seen in the proof of the previous result, one then has $k \geq 2$. For $k = 2$, from $c < b^{1.5}$ one gets

$$c = 16f^6 - 8f^4 + f^2 + 1 < (4f^4 + 1)^{1.5} < 9f^6 + 1,$$

that is, $f < 1$, a contradiction. Therefore, we conclude that $k \geq 3$. This and Proposition 2.2 yield

$$\frac{c}{b} \geq b^{0.4} > \gamma^{0.4(2k-1)} \geq \gamma^2,$$

which contradicts Proposition 2.1.
**Proposition 2.8.** There are no \( D(-1) \)-quadruples \((1, b, c, d)\) with \(10^{13} < b < c < d, b^{1.3} < c < b^{1.4},\) and \(b, c\) given by (2.3).

**Proof.** From the last proof we retain that \(k\) is at least 3, while from the chain of inequalities

\[
\gamma^2 > \frac{c}{b} \geq b^{0.3} > \gamma^{0.3(2k-1)}
\]

we deduce that \(k \leq 3\). To obtain a bound for \(f\), we follow the reasoning in the proof of Proposition 2.5.

Since \(k = 3\), one has

\[
v_m \equiv -4f^2 + 8f^4 \pmod{8f^6},
\]

\[
w_n \equiv -(2\rho n + 4)f^2 + \left(\frac{4n^3 - 4n^3}{3}\rho - 8n^2 + 8\right)f^4 \pmod{8f^6},
\]

whence again it follows that \(n \geq 4f^4\). As already seen, this leads to a contradiction. \(\square\)

**Proposition 2.9.** There are no \( D(-1) \)-quadruples \((1, b, c, d)\) with \(10^{13} < b < c < d, b^{1.2} < c < b^{1.3},\) and \(b, c\) given by (2.3).

**Proof.** We adapt the reasoning used to establish Proposition 2.6. So let \((1, b, c, d)\) be a \(D(-1)\)-quadruple satisfying \(10^{13} < b < c < d, b^{1.2} < c < b^{1.3},\) and \(b, c\) given by (2.3).

In the proof of Proposition 2.7 it was shown that \(k \geq 3\). For \(k = 3\), from \(c < b^{1.3}\) it results

\[
16f^4(2f^2 - 1)^2 + 1 < (f^2(4f^2 - 1)^2 + 1)^{1.3} < 40f^8 + 1,
\]

whence \(f^2 < 3\), a contradiction. Therefore, one has \(k \geq 4\). From

\[
\gamma^2 > \frac{c}{b} \geq b^{0.2} > \gamma^{0.2(2k-1)},
\]

we deduce that \(k \leq 5\). Assuming \(k = 5\), one obtains \(n \geq 4f^4\) as in the proof of Proposition 2.5, so a contradiction appears in this case. It remains to examine what happens when \(k = 4, r = 8f^4 - 4f^2, s = 16f^5 - 12f^3 + f,\) and \(t = 128f^9 - 160f^7 + 56f^5 - 4f^3 + f.\)

A short study of the sequences \((u_i)_i, (U_n)_n\) introduced in the proof of Proposition 2.6 gives

\[
U_n \equiv ((8 - 8n^2)f^4 - 4f^2)\rho + \left(\frac{4n^3 - 100n}{3}\right)f^4 + 2nf^2 \pmod{8f^6}.
\]

Proceeding as in the proof of Proposition 2.6, one gets \(n > 3f^4\), whence the same contradiction emerges. \(\square\)

**Proposition 2.10.** There are no \( D(-1) \)-quadruples \((1, b, c, d)\) with \(10^{13} < b < c < d, b^{1.16} < c < b^{1.2},\) and \(b, c\) given by (2.3).
Proof. Assume the contrary. By the previous result, \( k \geq 4 \). For \( k = 4 \) one gets

\[
f^2(16f^4 - 12f^2 + 1)^2 + 1 < (16f^4(2f^2 - 1)^2 + 1)^{1.2} < 169f^{10} + 1,
\]

which is false for \( f > 1 \). For \( k = 5 \) one adapts the reasoning introduced in the proof of Proposition 2.5 to obtain \( n > 10^{28} \), in contradiction with [4, Proposition 4.3].

As \( 0.16(2k - 1) < 2 \) yields \( k \leq 6 \), it remains to consider the possibility \( k = 6 \). The argument indicated at the end of the proof of Proposition 2.6 can be adapted to the present context. One finally obtains \( n > 3f^4 \), which is not compatible with the existence of a \( D(-1) \)-quadruple subject to all constraints from the hypothesis of the present proposition. \( \square \)

Summing up what has been done in this section and noting that condition (2.1) holds if \( f \) has no prime divisor congruent to 1 modulo 4, we get the next result.

**Theorem 2.11.** There are no \( D(-1) \)-quadruples \( (1, b, c, d) \) with \( 10^{13} < b < c < d \) and \( b, c \) given by (2.3). In particular, there exists no \( D(-1) \)-quadruple for which the corresponding \( f \) has no prime divisor congruent to 1 modulo 4.

An alternative proof, more familiar to experts in Diophantine equations, is based on linear forms in logarithms. Here is the sketch of such a reasoning.

For the rest of the paragraph we put

\[
\alpha = s + \sqrt{c}, \quad \beta = r + \sqrt{b}, \quad \text{and} \quad \gamma = \sqrt[4]{\frac{s\sqrt{b}}{r\sqrt{c}}}.
\]

As in [4, Section 4; 22], to a putative \( D(-1) \)-quadruple \( (1, b, c, d) \) it is associated a linear form in logarithms

\[
\Lambda := m \log \alpha - l \log \beta + \log \gamma.
\]

We put

\[
\Delta := (k + 1)m - kl.
\]

Then

\[
(k + 1)\Lambda = \log(\gamma^{k+1}\beta^{\Delta}) - l \log(\alpha^{k+1}\beta^{-k})
\]

can be considered as a linear form in the logarithms of two algebraic numbers. An elementary study shows that one has

\[
|\log(\alpha^{k+1}\beta^{-k})| \leq \frac{k + 1}{f^2} \quad \text{and} \quad |\Delta| \leq \frac{2l}{f^2 \log f}.
\]

Then Theorem 2.11 follows from Laurent’s estimates on linear forms in two logarithms given in [26] and our computations which showed that \( f > 10^7 \) and \( b > 10^{13} \) for each \( D(-1) \)-quadruple.
Each approach has its own advantages over the other. The former is more ‘human-friendly’ (and consequently longer), provides insight and has explanatory power, while the latter is computer-intensive and therefore shorter yet less enlightening. Since the former approach involves ideas which proved to be pivotal for subsequent developments, we decided to expound it extensively.

The idea at the basis of the proof of Theorem 2.11 can be succinctly stated as ‘reduce the master equation to a Pellian equation’. The same paradigm can be applied for an arbitrary $D(-1)$-quadruple.

Write $f = f_1 f_2$, with $f_1$ the product of all the prime divisors of $f$ which are congruent to 1 modulo 4, multiplicity included. Then in any solution $(r, s)$ to (1.9) one has

$$r = f_2 u, \quad s = f_2 v,$$

for some positive integers $u, v$ satisfying

$$u^2 + v^2 = 2 f uv + f_1^2 \iff (v - fu)^2 - (f^2 - 1)u^2 = f_1^2. \quad (2.8)$$

Below is a specialization of Frattini’s theorems from [17, 18] giving a representation for the non-negative solutions to the equation relevant for us

$$W^2 - (f^2 - 1)U^2 = f_1^2. \quad (2.9)$$

**Proposition 2.12.** The nonnegative solutions of Equation (2.9) are given by

$$w + u \sqrt{f^2 - 1} = (w_0 + u_0 \sqrt{f^2 - 1})(f + \sqrt{f^2 - 1})^k, \quad k \geq 0,$$

or

$$w + u \sqrt{f^2 - 1} = (w_0 - u_0 \sqrt{f^2 - 1})(f + \sqrt{f^2 - 1})^k, \quad k \geq 1,$$

where $(w_0, u_0)$ runs through the nonnegative solutions of Equation (2.9) with

$$f_1 \leq w_0 \leq f_1 \sqrt{\frac{f + 1}{2}}, \quad 0 \leq u_0 \leq \frac{f_1}{\sqrt{2(f + 1)}}.$$

Let $\gamma = f + \sqrt{f^2 - 1}$ be the fundamental solution to the Pell equation $X^2 - (f^2 - 1)Y^2 = 1$ and $\varepsilon = w_0 + \zeta u_0 \sqrt{f^2 - 1}$, where $\zeta \in \{-1, 1\}$, a fundamental solution as described above. According to [4, Lemma 3.5], $v > 3.999^{1/2}fu$, so that all positive solutions to Equation (2.8) have the form

$$v - fu + u \sqrt{f^2 - 1} = \varepsilon \gamma^k, \quad k \geq (1 - \zeta)/2. \quad (2.10)$$

Introducing the algebraic conjugates $\overline{\gamma} = f - \sqrt{f^2 - 1}$, $\overline{\varepsilon} = w_0 - \zeta u_0 \sqrt{f^2 - 1}$, one readily obtains

$$r = \frac{f_2(\varepsilon \gamma^k - \overline{\varepsilon} \overline{\gamma}^k)}{\gamma - \overline{\gamma}}, \quad (2.11)$$
\[ s = \frac{f_2(\varepsilon \gamma^{k+1} - \bar{\varepsilon} \bar{\gamma}^{k+1})}{\gamma - \bar{\gamma}}, \quad (2.12) \]

whence

\[ b = \frac{f_2^2(\varepsilon \gamma^k + \bar{\varepsilon} \bar{\gamma}^k)^2 - 4}{(\gamma - \bar{\gamma})^2}, \quad (2.13) \]

\[ c = \frac{f_2^2(\varepsilon \gamma^{k+1} + \bar{\varepsilon} \bar{\gamma}^{k+1})^2 - 4}{(\gamma - \bar{\gamma})^2}. \quad (2.14) \]

Note that when \( u_0 = 0 \) one has \( \varepsilon = \bar{\varepsilon} = f_1 \), so that (2.11) and (2.12) coincide with (2.2) and (2.3), respectively.

One major source of difficulties with this approach is the fact that the components \( w_0, u_0 \) of a fundamental solution are known only approximately, being confined to a box defined by the inequalities stated in the last line of Proposition 2.12. Another reason for complexity is the existence of positive solutions to Equation (2.9) for which \( \zeta = -1 \). We have succeeded to overcome all such complications and prove Theorem 1.4 along these lines. Our attempts to simplify the proof and avoid intricate arguments were successful as soon as we changed once more the underlying paradigm.

Multiplication by a power of the minimal solution for the associated Pell equation can be viewed as a vehicle to move from a fundamental solution to Equation (2.9) to a solution of interest. Metaphorically speaking, one can say that in the proof for Theorem 1.4 presented in Section 4 we travel backwards — we examine to what extent information about a specific solution is transferred to associated solutions.

Before making explicit the explanations alluded to above, we present in the next section strengthened versions of some technical results from [4].

### 3.1 Bounds for Linear Forms in Logarithms

Recall that for a nonzero algebraic number \( \gamma \) of degree \( D \) over \( \mathbb{Q} \), with minimal polynomial \( A \prod_{j=1}^{D}(X - \gamma^{(j)}) \) over \( \mathbb{Z} \), the absolute logarithmic height is defined by

\[ h(\gamma) = \frac{1}{D} \left( \log A + \sum_{j=1}^{D} \log^+ |(\gamma^{(j)})| \right), \]

where \( \log^+ x = \log \max(x, 1) \).

Next we quote a theorem from [1] giving very good lower bounds for linear forms in the logarithms of three algebraic numbers under hypotheses that are easily checked in the context of interest here.

**Theorem 3.1** (Aleksentsev). Let \( \Lambda_1 \) be a linear form in logarithms of \( n \) multiplicatively independent totally real algebraic numbers \( \beta_1, \ldots, \beta_n \), with rational coefficients \( b_1, \ldots, b_n \). Let \( h(\beta_j) \) denote the absolute logarithmic height of \( \beta_j \) for \( 1 \leq j \leq n \). Let \( D \) be the degree of the number field \( K = \mathbb{Q}(\beta_1, \ldots, \beta_n) \),
and let $B_j = \max(Dh(\beta_j), |\log \beta_j|, 1)$. Finally, let

$$E = \max \left( \max_{1 \leq i, j \leq n} \left\{ \frac{|b_i|}{B_j} + \frac{|b_j|}{B_i} \right\}, 3 \right). \quad (3.1)$$

Then

$$\log |\Lambda_1| \geq -5.3n^{-n+1/2}(n + 1)^{n+1}(n + 8)^2(n + 5)(31.44)^n D^2(\log E) \log (3nD) \prod_{j=1}^{n} B_j.$$

We apply Theorem 3.1 for $D = 4$, $n = 3$, and

$$\Lambda_1 = 2m \log \beta_1 - 2l \log \beta_2 + \log \beta_3, \quad (3.2)$$

with the choices

$$\beta_1 = s + \sqrt{c}, \quad \beta_2 = r + \sqrt{b}, \quad \beta_3 = \frac{s\sqrt{b}}{r\sqrt{c}}, \quad b_1 = 2m, \quad b_2 = -2l, \quad b_3 = 1. \quad (3.3)$$

The required multiplicative independence readily follows by noting that $\beta_1$ and $\beta_2$ are algebraic units while $\beta_3$ is not. Indeed, any possible relation of multiplicative dependence has the shape $\beta_1^u = \beta_2^v$ for some positive integers $u$, $v$. Note that $\mathbb{Q}(\beta_1) \cap \mathbb{Q}(\beta_2) = \mathbb{Q}$, as otherwise $b$ and $c$ would have the same square-free part, so that $bc$ would be a perfect square, in contradiction with $bc - 1 = r^2$. One concludes that it holds $\beta_1^u \in \mathbb{Q}$, which is not possible because $\beta_1$ is not a root of unity.

For compatibility with [4], we introduce the notation $\alpha = s + \sqrt{c}, \beta = r + \sqrt{b}$. It is clear that it holds

$$h(\beta_1) = \frac{1}{2} \log \alpha, \quad h(\beta_2) = \frac{1}{2} \log \beta,$$

so that

$$B_1 = 2 \log \alpha, \quad B_2 = 2 \log \beta. \quad (3.4)$$

The minimal polynomial for $\beta_3$ is $r^2cX^2 - s^2b$ divided by $\gcd(r^2c, s^2b)$, so that

$$h(\beta_3) = \frac{1}{2} \log \left( \frac{s^2b}{\gcd(r^2c, s^2b)} \right).$$

As the lower bound for $\log |\Lambda_1|$ given by Aleksentsev’s theorem decreases when $B_3$ increases, we can take

$$B_3 = 4 \log s \sqrt{b}.$$

Combining the obvious relations $\alpha > \beta > \beta_3$, $B_3 > B_1 > B_2$ with $m \log \alpha < l \log \beta$ (proved in [22, Lemma 3.3]) and its consequence $l > m$, one obtains

$$E = \max \left( \frac{2l}{\log \beta}, 3 \right).$$
Having in view that by Theorem 1.2 one has \( b < 10^{148/1.16} \), for \( l \geq 250 \) one gets

\[
E = \frac{2l}{\log \beta}.
\]

Since \( \log \Lambda_1 < -4l \log \beta + \log(b) - \log(b - 1) \) by [22, Lemma 3.1], one has

\[
l < 6.005171 \cdot 10^{11} \log \alpha \log(s \sqrt{b}) \log \left( \frac{2l}{\log \beta} \right).
\]

Most of the previous work on \( D(-1) \)-quadruples has focused on the \( z \)-component of the solutions to systems (1.3)–(1.5). In order to use the information already available in the literature, we will derive from the inequality above one involving \( m \) and subsequently another one in terms of \( n \). In a first step toward this goal we employ the elementary fact that the function \( x \mapsto x / \log x \) is increasing for \( x > 3 \). By [22, Lemma 3.3], we thus get

\[
m < 6.005171 \cdot 10^{11} \log \beta \log(s \sqrt{b}) \log \left( \frac{2m \log \alpha}{\log \beta} \right).
\]

A slight simplification is possible thanks to the following result.

**Lemma 3.2.** \( \frac{\log(s + \sqrt{\gamma})}{\log(r + \sqrt{b})} < \frac{\log c}{\log b} \).

*Proof.* Consider the real functions \( f_1(x) = \log(\sqrt{x} + \sqrt{x - 1}) \) and \( f_2(x) = \log x \) defined for \( x \geq 1 \). As \( f'_2(x) > 0 \) and \( f'_1(x)/f'_2(x) \) is decreasing for \( x > 1 \), by [2] we know that

\[
\frac{\log(\sqrt{x} + \sqrt{x - 1})}{\log x} = \frac{f_1(x) - f_1(1)}{f_2(x) - f_2(1)}
\]

is decreasing as well. \( \square \)

Using this observation together with the obvious inequality \( bs^2 < bc - 1 \), we get

\[
2m < 6.005171 \cdot 10^{11} \log \beta \log(bc - 1) \log \left( \frac{2m \log c}{\log b \log \beta} \right).
\]

As explained above, for \( m \geq 250 \) one has \( 2m > 3 \log \beta \), so one can apply the same reasoning to pass from \( m \) to \( n \) with the help of the inequality \( m \log(4c) > n \log(bc - 1) \) proved in [4, Lemma 2.7]. The resulting formula is

\[
2n < 6.005171 \cdot 10^{11} \log \beta \log(4c) \log \left( \frac{2n \log(bc - 1) \log c}{\log b \log(4c) \log \beta} \right).
\]

Since \( 2r < \beta < 2 \sqrt{b} \), Theorem 3.1 yields the following corollary.
Corollary 3.3. If $n \geq 250$, then

$$n < 1.5002 \cdot 10^{11} \log(4b) \log(4c) \log\left(\frac{4n \log(bc)}{\log b \log(4b)}\right).$$

Upper bounds of this type are complemented by reverse inequalities. Our next immediate goal is to sharpen some lower bounds for $n$ in terms of $b$ and $c$ established in [4]. To this end, we will use the positive integer $A$ introduced in [12] via the formula

$$A = (2b - 1)c - 2rst.$$  

Routine calculations lead to the simpler statement $A = f^2 + b$. For the proof of our next results we recall from [4, Lemma 3.4] that $A$ satisfies the double inequality

$$\frac{c - 5}{4b} + b < A < \frac{1}{3.9999} \left(\frac{c}{b} + 4b\right)$$  

as well as the congruence

$$2(bn^2 - m^2) \equiv \pm An \pmod{c}. \quad (3.6)$$

Occasionally we will rewrite this as

$$2(bn^2 - m^2) + \rho An = jc \quad (3.7)$$

for a fixed $\rho \in \{-1, 1\}$ and a certain integer $j$.

Actually, slightly stronger upper bounds on $A$ are valid in the context of interest in this paper.

Lemma 3.4. Let $(1, b, c, d)$ with $1 < b < c < d$ be a $D(-1)$-quadruple. Then

(a) For $c < b^3$ one has

$$A < \left(\frac{c}{4b} + b\right)\left(1 + \frac{1}{b}\right).$$

(b) $A < 2b$ for $c < 4b^2$.

(c) $A < c/(3.9999b)$ for $c > b^3$.

Proof. For part (a) we use the inequality

$$A < b + \frac{1}{4} \left(\frac{b - 1}{c - 1} + \frac{c - 1}{b - 1}\right) + \frac{1}{2}$$

established in the proof of [4, Lemma 3.4]. Since

$$\frac{b - 1}{c - 1} < \frac{b}{c}$$
because \( b < c \), and

\[
\frac{c - 1}{b - 1} < \frac{c}{b - 1} = \frac{c}{b} + \frac{c}{b^2} + \frac{c}{b^2(b - 1)},
\]

the result follows from the hypothesis \( c < b^3 \) and the estimate \( b > 10^{13} \).

When \( c < 4b^2 \), part (a) yields \( A < 2b + 2 \). The assumption \( c < 4b^2 \) implies \( s < 2b \), so for \( s = 2b - 1 \) one sees from the definition of \( A \) that \( 2b - 1 \) divides \( A \), so one necessarily has \( A = 2b - 1 \). Hence, \( c \) is odd, which is not possible with \( s \) odd.

When \( s = 2b - 2 = 2r^2 \), one readily gets \( t = 2r^3 + r \) and \( f = t - sr = r \). As we already know that there is no \( D(-1) \)-quadruple with \( \gcd(r, s) = f \), we conclude that \( s \leq 2b - 3 \). Therefore,

\[
A < \frac{c}{4b} + b + 2 \leq 2b - 1 + \frac{9}{4b} < 2b,
\]
as claimed in (b).

The inequality (c) follows as soon as we show that it holds

\[
b + 1 + \frac{c - 1}{4(b - 1)} < \frac{c}{3.9999b}.
\]

This is a corollary of the slightly stronger inequality

\[
b + 1 < \frac{bc - 20000c}{15996b(b - 1)},
\]

which is valid because \( c > b^3 \) and \( b > 10^{13} \).

Now we are in a position to give a simplified list of lower bounds for \( n \) in terms of \( b \) and \( c \). More precisely, the constants appearing in these bounds improve upon those provided by [4, Lemmas 3.6, 3.9 and 4.2].

**Proposition 3.5.** Let \((1, b, c, d)\) with \( 1 < b < c < d \) be a \( D(-1) \)-quadruple with \( c > b^3 \). Then \( n > \min\{b, 0.125\sqrt{c/b}\} \).

**Proof.** We reason by reduction to absurd. Assuming that the conclusion is false, with the help of Lemma 3.4 we get

\[
0 < An \leq \frac{c}{3.9999} < 0.2501c,
\]

\[
0 < 2(bn^2 - m^2) < 2bn^2 \leq \frac{c}{32} < 0.0313c.
\]

Therefore, congruence (3.6) is actually an equality \( An = 2(bn^2 - m^2) \), whence \( A < 2bn \leq 0.25 \sqrt{bc} \). This inequality is not compatible with \( A > c/(4b) \) when \( c > b^3 \).

**Proposition 3.6.** Let \((1, b, c, d)\) with \( 1 < b < c < d \) be a \( D(-1) \)-quadruple with \( 4b^2 < c < b^3 \). Then \( n > 0.125c/b^2 \).
Proof. As before, we suppose that the conclusion is false. Note that part (a) of Lemma 3.4 entails $A < 0.6c/b$, whence $An < 0.075c^2/b^3$. Since $2bn^2 < 0.032c^2/b^3$, we conclude yet again that congruence (3.6) is actually an equality. We therefore get

\[ \frac{c}{4b} < A < 2bn \leq \frac{c}{4b}, \]

a blatant contradiction.

The result just proved is not useful when $c$ is close to $4b^2$. One way to eliminate this inconvenient follows. In the statement below we refer to Equation (3.7).

**Proposition 3.7.** Suppose $(1, b, c, d)$ with $1 < b < c < d$ is a $D(-1)$-quadruple with $c < b^3$.

(a) If $\rho = 1$, then $n > 0.5\sqrt{c/b}$.
(b) Let $\rho = -1$. Then $j$ is nonnegative. If $j$ is positive, then $n > 0.5\sqrt{2c/b}$. If $j = 0$, then $c > 7164532b^2 > b^{2.155}$ and

\[ n > \begin{cases} 
\frac{c^{2/11}}{11} & \text{for } c \geq \max\{b^{2.5}, 10^{50}\}, \\
0.214(c/b)^{1/3} & \text{for } c < b^{2.5}. 
\end{cases} \]

Proof. The result is very close to [4, Lemma 3.8]. There are two differences: the hypothesis $c < b^3$ (instead of $c < b^{2.75}$) which allows one to employ part (a) of the above Lemma 3.4 and the conclusion $c > 7164532b^2$ (instead of $c > 51.99b^2$).

(a) The proof given in [4, Lemma 3.8] is valid under the present hypotheses.
(b) In [4, Lemma 3.8] it was shown that for $\rho = -1$ and $j = 0$ one has $n > 0.214\left(\frac{c}{b}\right)^{1/3}$ when $c < b^{2.5}$. Since $b > 20^{10}$, one gets

\[ A = 2bn - 2m^2/n > 2(b - 4)n > 0.4279 \left(51.99 \cdot 20^{10}\right)^{1/3}b. \]

As a consequence of Lemma 3.4(a) one also has

\[ A < 1.0001\left(\frac{c}{4b} + b\right). \]

Comparison of the two bounds on $A$ results in the inequality $c > 138703b^2$.

Resume the reasoning from the previous paragraph with this new information instead of $c > 51.99b^2$. The outcome is an improved lower bound on $c$. After fourteen more iterations one obtains $c > 7164532b^2$. According to [4, Proposition 4.1], for $b^2 < c < b^3$ one has $b < 10^{44}$. This readily gives $7164532 > b^{0.155}$, which ends the proof.

For hypothetical $D(-1)$-quadruples with $c < 4b^2$ we also offer two kinds of lower bounds for $n$.

**Proposition 3.8.** Let $(1, b, c, d)$ with $1 < b < c < d$ be a $D(-1)$-quadruple. If $c < 4b^2$, then $n > 0.707\sqrt{c/b}$.
Proof. The previous proposition gives $n > 0.5 \sqrt{2/c/b}$ when $\rho = -1$, which is slightly stronger than the claimed inequality. When $\rho = 1$ then one has $2(bn^2 - m^2) + An \geq c$. In view of our previous Lemma 3.4, we get $2bn^2 + 2bn - c > 0$, so that

$$2bn > -b + \sqrt{b^2 + 2bc} > 1.414 \sqrt{bc},$$

where the last inequality holds because $c > 3.999 f^2 b > 3.999 \cdot 10^{14} b$ by [4, Lemma 3.5].

Proposition 3.9. Suppose $n \geq 1000$.

(i) If $b^{1.32} \leq c < b^{1.40}$, then $n \geq \left(\frac{15.927 b^2}{c}\right)^{1/4}$.

(ii) If $b^{1.27} \leq c < b^{1.32}$, then $n \geq \left(\frac{15.830 b^2}{c}\right)^{1/4}$.

(iii) If $b^{1.22} \leq c < b^{1.27}$, then $n \geq \left(\frac{15.387 b^2}{c}\right)^{1/4}$.

(iv) If $b^{1.16} \leq c < b^{1.22}$, then $n \geq \left(\frac{12.850 b^2}{c}\right)^{1/4}$.

Proof. The new idea is to use the observation that for any integers $L, M, R$ with $L \equiv R \pmod{2M}$ it follows that $L^2 \equiv R^2 \pmod{4M}$.

We put $e = f^2 - 1$, $\Delta = f^2$, then $A = b + \Delta$ and, as seen in the proof of Lemma 3.4, it follows that

$$\Delta < \frac{c}{4b} + \frac{c}{4b^2} + 1 < 0.2501 \frac{c}{b}.$$  

Note that $b + e - 1 = 2ft - c$, so that

$$b^2 + 2(e - 1)b + e^2 - 2e + 1 \equiv c^2 + 4f^2 t^2 \equiv c^2 - 4f^2 = c^2 - 4(e + 1) \pmod{4c}.$$  

It follows that

$$b^2 + 2(e - 1)b + e^2 + 2e + 5 \equiv c^2 \pmod{4c}.$$  

The congruence method introduced in [13] is based on the relation $s(m^2 - bn^2) \equiv \rho rt n \pmod{4c}$. Multiplying both sides by $2s$ one obtains

$$2(bn^2 - m^2) \equiv -\rho An + cn \pmod{2c},$$  

equivalently

$$b(2n^2 + \rho n) \equiv 2m^2 - \rho \Delta n + cn \pmod{2c}.$$  

From this we get

$$b^2(2n^2 + \rho n)^2 \equiv (2m^2 - \rho \Delta n + cn)^2 \pmod{4c}$$

as well as

$$2(e - 1)b(2n^2 + \rho n)^2 \equiv 2(e - 1)(2n^2 + \rho n)(2m^2 - \rho \Delta n + cn) \pmod{4c}.$$
By summation we get that \((b^2 + 2(e - 1)b)(2n^2 + \rho n)^2\) is congruent modulo \(4c\) to
\[
(2m^2 - \rho \Delta n + cn)^2 + 2(e - 1)(2n^2 + \rho n)(2m^2 - \rho \Delta n + cn),
\]
equivalently, again modulo \(4c\),
\[
(c^2 - e^2 - 2e - 5)(2n^2 + \rho n)^2 \equiv (2m^2 - \rho \Delta n + cn)^2 + 2(e - 1)(2n^2 + \rho n)(2m^2 - \rho \Delta n + cn).
\]
Considering separately even and odd values of \(n\), it is seen that
\[-(e^2 + 2e + 5)(2n^2 + \rho n)^2 \equiv (2m^2 - \rho \Delta n)^2 + 2(e - 1)(2n^2 + \rho n)(2m^2 - \rho \Delta n) \quad \text{(mod 4c)}.
\]
Now we want to find an upper bound for the expression
\[
\Phi := (e^2 + 2e + 5)(2n^2 + \rho n)^2 + (2m^2 - \rho \Delta n)^2 + 2(e - 1)(2n^2 + \rho n)|2m^2 - \rho \Delta n|.
\]
We proceed piece by piece, taking into account the relative size of \(b\) and \(c\) as well as the upper bound on \(c\). We give all details for part (i), leaving the other cases to the reader.

According to [4, Lemma 2.5], we have
\[
\frac{2m - 1}{2n} < \log(\frac{4 \cdot 10^{147.43 \cdot 58/33}}{3.996 \cdot 10^{147.43}}) < 1.7545,
\]
whence
\[
m < 1.7545 n + 1/2 \leq 1.7555 n.
\]

Therefore,
\[
|2m^2 - \rho \Delta n| < 2 \times 1.7555^2 n^2 + 0.2501 \frac{cn}{b} = \left(6.16005 + 0.2501 \frac{c}{nb}\right)n^2,
\]
\[
|2m^2 - \rho \Delta n|^2 < \left(6.16005 + 0.2501 \frac{c}{nb}\right)^2 n^4,
\]
\[
2(e - 1)(2n^2 + \rho n)|2m^2 - \rho \Delta n| < \frac{0.5002c}{b} \times 2.001 n^2 \times \left(6.16005 + 0.2501 \frac{c}{nb}\right)n^2,
\]
hence
\[
2(e - 1)(2n^2 + \rho n)|2m^2 - \rho \Delta n| < 1.001 \frac{c}{b} \left(6.16005 + 0.2501 \frac{c}{nb}\right)n^4.
\]
By
\[
e^2 + 2e + 5 = \Delta^2 + 4 < \left(0.2501^2 + \frac{4}{b^{0.64}}\right)\frac{c^2}{b^2},
\]
\[
< \left(0.2501^2 + \frac{4}{10^{13 \cdot 0.64}}\right)\frac{c^2}{b^2} < 0.062551 \frac{c^2}{b^2}
\]
we also have

\[(e^2 + 2e + 5)(2n^2 + \rho n)^2 < 0.062551 \times \frac{c^2}{b^2} \times 2.001^2 n^4 < 0.250455 \frac{n^4 c^2}{b^2}.
\]

Collecting all these estimates we get that \(\Phi\) is less than

\[
\left(0.250455 + \left(6.16005 \frac{b}{c} + \frac{0.2501}{n}\right)^2 + 1.001 \left(6.16005 \frac{b}{c} + \frac{0.2501}{n}\right)\right) n^4 \frac{c^2}{b^2}.
\]

Using the known lower bounds on \(n\) and \(c/b\), it is easy to verify that

\[
\Phi < 0.251134 n^4 \frac{c^2}{b^2}
\]

and we see that

\[
n < \left(\frac{15.927 b^2 c}{c}\right)^{1/4} \implies \Phi < 4c.
\]

But, by a previous congruence, the nonnegative integer \(\Phi\) is a multiple of \(4c\), so for \(\Phi < 4c\) it holds

\[(e^2 + 2e + 5)(2n^2 + \rho n)^2 + (2m^2 - \rho \Delta n)^2 + 2(e - 1)(2n^2 + \rho n)(2m^2 - \rho \Delta n) = 0.
\]

The left-hand side of the last equation is of the form

\[(e^2 + 2e + 5)X^2 + 2(e - 1)XY + Y^2, \quad \text{with } X = (2n^2 + \rho n), \ Y = (2m^2 - \rho \Delta n),
\]

a quadratic form whose discriminant (equal to \(-4f^2\)) is negative, so that \(\Phi\) is always positive when \(n > 0\), a contradiction which implies \(n \geq (15.927 b^2 c)^{1/4}\). □

The results just proved serve to improve Theorem 1.2. To this end we combine them with Aleksentsev’s theorem in conjunction with a similar result due to Matveev [28, Theorem 2.1] applicable in the following context.

Let \(\beta_1, \beta_2, \beta_3\) be real algebraic numbers and denote \(K := \mathbb{Q}(\beta_1, \beta_2, \beta_3)\). Put \(D := [K : \mathbb{Q}]\). Assume that \(\beta_1, \beta_2, \beta_3\) satisfy the Kummer condition, that is,

\[[K(\sqrt[\beta_1]{1}, \sqrt[\beta_2]{2}, \sqrt[\beta_3]{3}) : K] = 8.
\]

Consider a linear form \(\Lambda_1 := b_1 \log \beta_1 + b_2 \log \beta_2 + b_3 \log \beta_3\), where \(b_1, b_2, b_3\) are integers with \(b_3 \neq 0\). Put \(A_j := h(\beta_j)\) for \(1 \leq j \leq 3\). We take \(E, E_1, C_3, C_1, C_2\) as follows:

\[
E \geq \frac{1}{3D} \max \left\{ \left| \frac{\log \beta_1}{A_1} \pm \frac{\log \beta_2}{A_2} \pm \frac{\log \beta_3}{A_3} \right| \right\},
\]

\[
E_1 = \frac{1}{2D} \left( \frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} \right).
\]
\[ C_3^* \exp(C_3^* E e) \geq e^3, \quad C_3 = \max\{C_3^*, 3\}, \]
\[ C_1 = \left( 1 + \frac{e^{-6}}{148} \right)(3 \log 2 + 2) \frac{4}{3C_3^*}, \]
\[ C_2 = 16 \left( 6 + \frac{5}{3 \log 2 + 2} \right) \frac{e^6}{3^{1/2} C_3^*}. \]

We also put
\[ \Omega := A_1 A_2 A_3, \]
\[ \omega := \Omega \left( \frac{DC_1}{e} \right)^3 C_3 \exp(C_3) E e. \]

Let \( C_0 \) be a real number satisfying
\[ C_0 \geq \max \left\{ 2C_3, \log \left( 4C_2 \max \left\{ \frac{C_0 \omega}{4C_1 A_3}, C_0, \frac{2E_1 C_3}{C_1} \right\} \right) \right\}. \]

Furthermore, put
\[ B_0 := \sum_{j=1}^{2} \frac{|b_3| + |b_j|}{8 \gcd(b_j, b_3) C_0 C_2 \omega}, \]
\[ B_1 := \sum_{j=1}^{2} \frac{1}{24 \gcd(b_j, b_3) C_1} \left( \frac{|b_3|}{A_j} + \frac{|b_j|}{A_3} \right), \]
\[ B_2 = \sum_{j=1}^{2} \frac{1}{8 |b_3| C_0 C_2 \omega} \left| \log \beta_j \right| \left( |b_3| + |b_j| \right), \]
\[ B_3 = \sum_{j=1}^{2} \frac{1}{24 |b_3| C_1} \left( \frac{|b_3|}{A_j} + \frac{|b_j|}{A_3} \right), \]

and take a real number \( W_0 \) satisfying
\[ W_0 \geq \max\{2C_3, \log(e(1 + B_0 + B_1 + B_2 + B_3))\}. \]

Now we are ready to state [28, Theorem 2.1] in a form applicable to our situation.

**Theorem 3.10.** (Matveev) Suppose that
\[ 2 \omega \min\{C_0, W_0\} \geq C_3, \]
\[ \omega \min\{C_0, W_0\} \geq 2C_1 C_3 \max\{A_1, A_2, A_3\}, \]
\[ 3(4C_1)^2 4C_0 \Omega \geq C_3 \max\{A_1, A_2, A_3\}. \]
Then,

$$\log |\Lambda_1| > -11648C_2C_0W_0\omega.$$  

In order to apply this result to the linear form (3.2), we need to check the Kummer condition is valid.

First, we show that $\sqrt{\beta_1} \notin K$. Assume on the contrary that $\sqrt{\beta_1} \in K$. Then one may write $\sqrt{\beta_1} = l_0 + l_1\sqrt{b} + l_2\sqrt{c} + l_3\sqrt{bc}$ with $l_0, l_1, l_2, l_3 \in \mathbb{Q}$. Squaring both sides yields

$$s + \sqrt{c} = l_0^2 + bl_1^2 + ccl_2^2 + 2(l_0l_1 + cl_2l_3)\sqrt{b}$$

$$+ 2(l_0l_2 + bl_1l_3)\sqrt{c} + 2(l_0l_3 + cl_1l_2)\sqrt{bc},$$

whence $s = l_0^2 + bl_1^2 + ccl_2^2, \quad 1 = 2(l_0l_2 + bl_1l_3)$. The arithmetic mean–geometric mean inequality yields

$$s \geq 2|l_0l_2|\sqrt{c} + 2|l_1l_3|b\sqrt{c} \geq 2(l_0l_2 + bl_1l_3)\sqrt{c} = \sqrt{c} > s,$$

a contradiction.

Similarly, one proves that $\sqrt{\beta_2} \notin K$. To check $\sqrt{\beta_3} \notin K$, we suppose the contrary and get $0 = l_0^2 + bl_1^2 + ccl_2^2 + bcl_3^2$ for some $l_0, l_1, l_2, l_3 \in \mathbb{Q}$. Since $b, c > 0$, it follows that all $l_j$ are zero, so $\beta_3 = 0$, which is absurd.

Second, assume that $\sqrt{\beta_1} \in K(\sqrt{\beta_2})$. Then one may write $\sqrt{\beta_1} = k_0 + k_1\sqrt{\beta_2}$ for some $k_0, k_1 \in K$, equivalently $s + \sqrt{c} = k_0^2 + k_1^2(r + \sqrt{b}) + 2k_0k_1\sqrt{\beta_2}$. If $k_0k_1 \neq 0$, then this equation shows that $\sqrt{\beta_2} \in K$, which is impossible as seen above. If $k_1 = 0$, then $s + \sqrt{c} = k_0^2$, which contradicts $\sqrt{\beta_1} \notin K$. It remains $k_0 = 0$, so that $s + \sqrt{c} = k_1^2(r + \sqrt{b})$ with $k_1 = l_0 + l_1\sqrt{b} + l_2\sqrt{c} + l_3\sqrt{bc}$ and $l_0, l_1, l_2, l_3 \in \mathbb{Q}$. Identification of coefficients of $\sqrt{b}$ on the two sides of this equation followed by application of the arithmetic mean–geometric mean inequality results in

$$0 = l_0^2 + bl_1^2 + ccl_2^2 + bcl_3^2 + 2(l_0l_1 + cl_2l_3)r$$

$$\geq 2(|l_0l_1| + c|l_2l_3|)\sqrt{b} + 2(l_0l_1 + cl_2l_3)r$$

$$\geq 2(|l_0l_1| + l_0l_1 + c(|l_2l_3| + l_2l_3))r \geq 0.$$

The middle inequality is strict unless $l_0l_1 = l_2l_3 = 0$, in which case all $l_j$ are zero. In either case we reached a contradiction. Similarly, one shows that $\sqrt{\beta_2} \notin K(\sqrt{\beta_1})$.

It remains only to show that $\sqrt{\beta_3} \notin K(\sqrt{\beta_1}, \sqrt{\beta_2})$. Assume the contrary and put

$$\sqrt{\beta_3} = k_0 + k_1\sqrt{\beta_1} + k_2\sqrt{\beta_2} + k_3\sqrt{\beta_1\beta_2}$$

with some $k_0, k_1, k_2, k_3 \in K$. Squaring both sides, one has

$$\beta_3 = k_0^2 + k_1^2\beta_1 + k_2^2\beta_2 + k_3^2\beta_1\beta_2 + 2(k_0k_1 + k_2k_3\beta_2)\sqrt{\beta_1}$$

$$+ 2(k_0k_2 + k_1k_3\beta_1)\sqrt{\beta_2} + 2(k_0k_3 + k_1k_2)\sqrt{\beta_1\beta_2}.$$ (3.8)
If \( k_0k_1 + k_2k_3\beta_2 \neq 0 \), with the help of \( \sqrt{\beta_2} \notin K \) one deduces first that \( k_0k_1 + 2(k_0k_3 + k_1k_2)\sqrt{\beta_2} \neq 0 \) and next that \( \sqrt{\beta_1} \in K(\sqrt{\beta_2}) \). A similar contradiction is reached assuming either \( k_0k_3 + k_1k_2 \neq 0 \) or \( k_0k_2 + k_1k_3\beta_1 \neq 0 \). So it holds

\[
k_0k_1 = -k_2k_3\beta_2, \quad k_0k_2 = -k_1k_3\beta_1, \quad k_0k_3 = -k_1k_2,
\]

whence

\[
k_0k_1(k_2^2 - \beta_1k_2^2) = 0, \quad k_0k_2(k_1^2 - \beta_2k_1^2) = 0, \quad k_1k_2(k_0^2 - \beta_1\beta_2k_0^2) = 0.
\]

Having in view what we already proved, it is readily seen that the last three equations imply that precisely one of \( k_j \) is nonzero. Note that \( k_0 \neq 0 \) gives \( \sqrt{\beta_3} \in K \), which is absurd. For \( k_1 \neq 0 \) one has \( s\sqrt{b} = k_1^2(s + \sqrt{c})r\sqrt{c} \). Passing to \( \mathbb{Q} \) and comparing the coefficients of 1 and \( \sqrt{c} \) on both sides, one gets a linear system of equations \( sX + cY = 0, X + sY = 0 \), with \( X = l_0^2 + bl_2^2 + cl_3^2 + bcl_3^2, Y = 2(l_0l_2 + bl_1l_3) \), and \( l_0, l_1, l_2, l_3 \in \mathbb{Q} \). Since the determinant of this system is \( s^2 - c = -1 \), it has only the trivial solution, which gives the contradiction \( k_1 = 0 \). Similarly, one can conclude that neither \( k_2 \neq 0 \) nor \( k_3 \neq 0 \) is possible.

Now the verification that the Kummer condition holds for our \( \Lambda \) is complete, so we can proceed with choosing suitable values for the parameters in the statement of Theorem 3.10.

As discussed in connection with Theorem 3.1, we take

\[
A_1 = \frac{1}{2} \log(s + \sqrt{c}), \quad A_2 = \frac{1}{2} \log(r + \sqrt{b}), \quad A_3 = \log(s\sqrt{b}).
\]

Then

\[
E \geq \frac{1}{12} \max \left\{ \left\{ \pm 2 \pm 2 \pm \frac{\log(s\sqrt{b}/r\sqrt{c})}{\log(s\sqrt{b})} \right\} \right\}.
\]

From

\[
\frac{s\sqrt{b}}{r\sqrt{c}} = \sqrt{1 + \frac{s^2 - r^2}{(s^2 + 1)r^2}} < \sqrt{1 + \frac{1}{r^2}} < 1 + \frac{1}{2r^2},
\]

we see that we can take

\[
E = \frac{4 + 10^{-15}}{12}.
\]

Thus, we may take \( C_3^* = 2.8 \) and \( C_3 = 3 \).

In order to fix a value for \( E_1 \), we need lower bounds for \( \log \beta_j \). Using Theorem 1.2, it is readily seen that a suitable value is

\[
E_1 = 0.033653.
\]

It is easy to see that \( C_0 \) should satisfy

\[
C_0 \geq \log \left( \frac{C_0C_2\omega}{C_1A_3} \right) = \log(C_0T)
\]
with \( T = 96Ee C_1^2 C_2 A_1 A_2 \), which allows us to take

\[
C_0 = \log T + \log(\log T) + \log(\log(\log T)) + 2\log(\log(\log(\log T)))
\]

(note that \( \log(\log(\log T)) > 0 \)). Since \( m \log \beta_1 < l \log \beta_2 \) and \( A_1 > A_2 \), one has

\[
B_0 + B_1 + B_2 + B_3 < \left( \frac{l}{2C_0 C_2 \omega} + \frac{1}{6C_1} \left( \frac{1}{\log \beta_2} + \frac{l}{A_3} \right) \right) (1 + \log \beta_1).
\]

We therefore take

\[
W_0 = 1 + \log \left( 1 + \left( \frac{l}{2C_0 C_2 \omega} + \frac{1}{6C_1} \left( \frac{1}{\log \beta_2} + \frac{l}{A_3} \right) \right) (1 + \log \beta_1) \right).
\]

Hence, combining the estimate in Theorem 3.10 with \( 0 < \Lambda_1 < (8ac/(b - 1)) \beta^{-2a} \) one gets

\[
l < 69 888 C_0 C_2 W_0 E e \log(s + \sqrt{c}) \log(s \sqrt{b}) + 0.25 \log \left( \frac{b}{b - 1} \right). \tag{3.9}
\]

As previously, we pass from this inequality to one involving \( m \) and subsequently to one in terms of \( n \). Assuming that it holds \( b^{del} < c < b^{Del} \) for some real numbers \( del \) and \( Del \) satisfying \( 1.16 < del < Del < 4.1 \), we finally get

\[
n < 17 472 C_0^3 C_2^3 W_0 S e \log(4b) \log(4c), \tag{3.10}
\]

with

\[
S = 1 + \log \left( 1 + \left( Del + 1 \right) \left( \frac{1}{2C_0 C_2 \omega} + \frac{1}{6C_1 A_3} n + \frac{1}{6C_1 \log \beta_2} \right) (1 + \log \beta_2) \right).
\]

At this moment we have all ingredients for the proof of the main result of this section.

**Theorem 3.11.** Let \((1, b, c, d)\) with \( 1 < b < c < d \) be a \( D(−1) \)-quadruple. Then \( b > 1.024 \cdot 10^{13} \) and \( \max\{10^{14} b, b^{1.233}\} < c < \min\{b^{2.93}, 10^{99}\} \). More precisely:

(i) If \( b^{2} < c < b^{2.93}, \) then \( b < 6.89 \cdot 10^{32} \) and \( c < 8.48 \cdot 10^{70}. \)

(ii) If \( b^{1.5} < c < b^{2}, \) then \( b < 1.26 \cdot 10^{69} \) and \( c < 4.48 \cdot 10^{73}. \)

(iii) If \( b^{1.4} < c < b^{1.5}, \) then \( b < 2.07 \cdot 10^{62} \) and \( c < 1.77 \cdot 10^{97}. \)

(iv) If \( b^{1.3} < c < b^{1.4}, \) then \( b < 6.26 \cdot 10^{73} \) and \( c < 10^{99}. \)

(v) If \( b^{1.233} < c < b^{1.3}, \) then \( b < 10^{69} \) and \( c < 4.85 \cdot 10^{99}. \)

**Proof.** Each interval \( b^{y} < c < b^{5} \) has been covered by subintervals \( b^{\mu} < c < b^{\mu+0.0001} \). On each subinterval, Corollary 3.3 and Proposition 3.5 produce an upper bound on \( b \), which in turn leads to a bound on \( S \). When \( n < 1000 \), instead of Proposition 3.5 we apply similar results from [4] valid for \( n \geq 7 \), which results in much sharper bounds on \( b \). Using the estimate on \( S \) in (3.10), an improved upper bound on \( b \) is obtained. From our computations we learned that \( b < 10^{13} \) for \( c > b^{2.928} \), whence the conclusion that no \( D(−1) \)-quadruple has \( c > b^{2.928} \).
We also bound $f$ from above with the help of the master equation, which shows that

$$f < \frac{s}{2r} + \frac{r}{2s}.$$  

Since our computations yield that for $c \leq b^{1.233}$ one has $f < 10^7$, by Proposition 2.4 we conclude that there exists no $D(-1)$-quadruple with $c$ so close to $b$.

Comparison with Theorems 1.2 reveals the superiority of Theorem 3.11. However, it is also apparent that a lot of work is required to confirm the nonexistence of $D(-1)$-quadruples using tools already employed. Therefore, completely different ideas are required for further advancements. The next section details and clarifies the change in viewpoint on the problem.

## 4 A PROOF FOR THE MAIN THEOREM

Recall that we have denoted by $J$ the set of pairs of integers $(r, s)$ such that there exists a $D(-1)$-quadruple $(1, b, c, d)$ with $1 < b < c < d$ and $b = r^2 + 1$, $c = s^2 + 1$. For $(r, s) \in J$ we put $s = r^\theta$ and define

$$\theta^- = \inf_{(r, s) \in J} \theta, \quad \theta^+ = \sup_{(r, s) \in J} \theta.$$  

Using the upper bound $c \leq 2.5b^6$ (see Theorem 1.2), a computer-aided search described in [4, Section 2] led to the conclusion that any hypothetical $D(-1)$-quadruple satisfies $r > 32 \times 10^5$, so that $b > 10^{13}$. Hence

$$\theta^+ \leq 6.04.$$  

Based on a refinement of a Diophantine approximation result of Rickert [34], Filipin and Fujita proved in [16] the inequality $c \leq 9.6b^4$, which yields

$$\theta^+ \leq 4.08.$$  

What we just proved in Theorem 3.11 entails

$$\theta^+ < 3.$$  

The lower bound $\theta^- > 1.16$ has been obtained in [4] and improved in Theorem 3.11 above to

$$\theta^- > 1.23.$$  

Further shortening of the interval $[\theta^-, \theta^+]$ along these lines becomes hopeless, so we introduce the new approach mentioned in the introduction. Our next concern is to have a closer look at solutions of the master equation compatible with the information gathered so far. A convenient tool was suggested by the fact that any solution $(x, y)$ to a Diophantine equation of the type $X^2 - 2fXY + Y^2 = C$ gives rise to two other solutions, namely $(-x + 2fy, y)$ and $(x, 2fx - y)$; see [32].
In this section we will see how changes of variables indeed allow one to transfer information regarding one specific solution to an associated solution.

Introduce a new variable

\[ F := s - 2rf. \]

As we will show shortly, it satisfies

\[ F \sim \begin{cases} \frac{1}{4}r^{\theta - 2}, & \text{if } \theta > 2, \\ -r^{2-\theta}, & \text{if } \theta < 2. \end{cases} \tag{4.1} \]

Therefore, we hope to exploit this variable in order to split the interval \([\theta^-, \theta^+]\) into two subintervals having a common end-point about 2.

We study \(F\) with the help of the equation

\[ r^2 + s^2 = 2frs + f^2 \]

or its equivalent forms

\[ sf = f^2 - r^2, \tag{4.2} \]

\[ F^2 + 2frF + r^2 - f^2 = 0. \tag{4.3} \]

From the master equation one obtains

\[ f = -rs + (r^2s^2 + s^2 + r^2)^{1/2} = \frac{s^2 + r^2}{rs + (r^2s^2 + s^2 + r^2)^{1/2}} \sim \frac{1}{2}r^{\theta - 1}. \]

It follows

\[ F = \frac{f^2 - r^2}{s} \sim \frac{1}{4}r^{\theta - 2} - r^{2-\theta}, \]

whence estimate (4.1).

The first properties of \(F\) are almost obvious.

**Lemma 4.1.**

(a) \( F = 0 \iff s = 2rf \iff f = r \iff s = 2r^2 \iff s = 2f^2. \)

(b) \( F > 0 \iff s > 2rf \iff f > r \iff s > 2r^2 \iff s < 2f^2. \)

(c) \( F < 0 \iff s < 2rf \iff f < r \iff s < 2r^2 \iff s > 2f^2. \)

**Proof.** (b) To prove \( 'F > 0 \iff s < 2f^2', \) note that from Equation (1.9) one gets

\[ r = sf - \sqrt{s^2f^2 - s^2 + f^2} = \frac{s^2 - f^2}{sf + \sqrt{s^2f^2 - s^2 + f^2}}, \]

and the last expression is smaller than \(s/(2f)\) precisely when \((s^2 - f^2)(s^2 - 4f^4) < 0.\)

For \( 'F > 0 \iff s > 2r^2', \) use \( f = -rs + \sqrt{r^2s^2 + r^2 + s^2}. \) \( \square \)
Observe that there are no $D(-1)$-quadruples for which the corresponding $F$ is zero.

**Lemma 4.2.** $F \neq 0$.

**Proof.** Suppose, by way of contradiction, that the statement is false. Since $F = 0$ if and only if $s = 2f^2$ and $r = f$, we are in a situation we have dealt with in Section 2. There it was found that this is possible for no $D(-1)$-quadruple. □

From these results it readily follows that if $f \neq r$, then $f$ is comparatively far away from $r$. The quantitative expression is given by the next lemma.

**Lemma 4.3.**

(a) If $f > r$, then $f > 2rF \geq 2r$.

(b) If $f < r$, then $0 > F > -2fr$.

**Proof.**

(a) Part (a) follows from Equation (4.3) rewritten as $F^2 + r^2 = f^2 - 2rfF$.

(b) In view of Lemma 4.1, $F$ is negative when $f < r$. The lower bound for $F$ follows from (4.3) rewritten as $F^2 + 2rfF = f^2 - r^2$. □

Now we have all the ingredients to show that the newly introduced variable $F$ indeed serves to separate values of $c$ smaller than $4b^2$ from those bigger than this threshold.

**Lemma 4.4.** $f < r \iff c < 4b^2$.

**Proof.** We know that $f < r$ holds if and only if $s \leq 2r^2 - 1 = 2b - 3$, which in turn is equivalent to $c \leq 4b^2 - 12b + 10$. Hence, $c < 4(b - 1)^2$ for $f < r$. To prove the converse implication, note that $c < 4b^2$ is tantamount to $s \leq 2r^2 + 1$. For $s = 2r^2 + 1$, Equation (1.9) becomes a quadratic in $f$ without integer roots, having discriminant $4r^6 + 8r^4 + 6r^2 + 1 = (2r^3 + 2r)^2 + 2r^2 + 1 = (2r^3 + 2r + 1)^2 - 4r^3 + 2r^2 - 4r$. Thus one has $s \leq 2r^2$, with equality prohibited by Lemmas 4.1 and 4.2. Thus $s < 2r^2$, which, according to the last part of Lemma 4.1, means $f < r$. □

Our next result shows that the existence of $D(-1)$-quadruples is not compatible with small values of $F$.

**Proposition 4.5.** There is no $D(-1)$-quadruple with $|s - 2rf| \leq 2 \cdot 10^6$.

**Proof.** This claim can be obtained by the following algorithm.

Start by rewriting Equation (4.3) in the form

\[ X^2 - (F^2 + 1)Y^2 = F^2, \quad \text{where} \quad X = f - rF, Y = r. \] (4.4)

For any $F$, one obvious solution is $(F^2 - F + 1, F - 1)$. A conjecture of Dujella predicts that an equation $X^2 - (a^2 + 1)Y^2 = a^2$ has at most one positive solution with $0 < Y < |a| - 1$ (this readily implies the nonexistence of $D(-1)$-quadruples; see [27]). In [27], this claim is checked for
\[ |a| < 2^{50}, \text{ so, for each } F \text{ with absolute value up to } 2 \cdot 10^6 \text{ we can find at most one exceptional solution } (x_0, y_0) \text{ with } 0 < y_0 < |F| - 1. \]

Next, we consider solutions \((x, y)\) to Equation (4.4) associated to either the obvious solution or to the exceptional one. Invert the relations \(x = f - rF, y = r, F = s - 2fr\) to obtain \(r = y, f = x + rF, s = F + 2rf\). Check if the resulting values for \(r, s, f\) satisfy the necessary conditions \(r > 20^5, r^{1.23} < s < r^3, f > 10^7\).

Finally, apply the Baker–Davenport lemma (we preferred the version from [13, Lemma 5]) for each solution surviving the sieving step and produce a contradiction with a known fact.

We use this procedure for \(|F| \leq 2 \cdot 10^6\). For the last step, we performed computations with real numbers of 173 decimal digits. In all cases, the outcome of the reduction step is \(n = 1\). This contradicts [4, Proposition 2.2], where it was shown that for no \(D(-1)\)-quadruple is \(n < 7\) possible.

Now we are in a position to halve the region where \(\theta\) is confined.

**Proposition 4.6.** There is no \(D(-1)\)-quadruple with \(4b^2 \leq c < b^3\).

**Proof.** The key new ingredient is the observation that for \(c < b^3\) one has

\[ 2bn > A. \]

Lemma 4.3 together with Lemma 4.4 imply \(A = f^2 + b > 4r^2F^2\). By Proposition 4.5, the right-hand side is greater than \(15 \cdot 10^{12}b\). We have thus obtained \(n > 7 \cdot 10^{12}\). However, explicit computations find that for \(c \geq 4b^2\) one has \(n < 10^{12}\).

Coming to the proof of the claim, we note that it was explicitly established in case \(\rho = 1\) during the proof of [4, Lemma 3.8]. It is also shown there that if \(\rho = -1\) then \(j \geq 0\), so Equation (3.7) gives \(2(bn^2 - m^2) \geq An\), whence \(2bn > A\).

Further compression of the interval \([\theta^-, \theta^+]\) is possible by examining other solutions of the master equation. Let us define a recurrent sequence by the relation

\[ F_{i+1} = 2fF_i - F_{i-1}, \quad i \geq 0, \quad F_{-1} = -s, \quad F_0 = -r. \]

It is readily seen that \(F_1 = F\) and for any \(i \geq -1\) it holds

\[ F_i^2 - 2fF_iF_{i+1} + F_{i+1}^2 = f^2. \]  

(4.5)

Moreover, for \(i \geq 0\) one has

\[ F_i = P_{i-1}s - P_{i}r, \]  

(4.6)

where \(P_{-1} = 0, P_0 = 1,\) and \(P_{i+1} = 2fP_i - P_{i-1}\) for any nonnegative \(i\).

As we will see shortly, all terms of the sequence \((F_i)_{i \geq 1}\) have properties similar to those established above for \(F = F_1\). First, we argue that all \(F_i\) are nonzero. In view of Theorem 2.11, it is sufficient to prove the next result.
Lemma 4.7. Assume $F_i = 0$ for some $i \geq 1$. Then $r = P_{i-1}f$ and $s = P_if$.

**Proof.** Since $F_i = 0$ is tantamount to $sP_{i-1} = rP_i$, Equation (1.9) becomes

\[ (P_i^2 + P_{i-1}^2 - 2fP_{i-1}P_i)r^2 = P_{i-1}^2f^2. \] (4.7)

As the expression within parentheses is

\[ P_i^2 + P_{i-1}^2 - 2fP_{i-1}P_i = P_i^2 - P_{i-1}P_{i+1} = P_0^2 - P_{-1}P_1 = 1, \]

the equality (4.7) implies $r = P_{i-1}f$, whence $s = P_if$. □

Note that from $F_i^2 - F_{i-1}F_{i+1} = f^2$, for $i \geq 1$ one gets by induction

\[ F_i \sim \frac{1}{4} r^i \theta^{-i-1} - r^{i+1-i} \theta, \]

so that

\[ F_i \sim \begin{cases} 
\frac{1}{4} r^i \theta^{-i-1}, & \text{if } \theta > (i+1)/i, \\
-r^{i+1-i} \theta, & \text{if } \theta < (i+1)/i.
\end{cases} \]

A reasoning similar to the proof of Lemma 4.3 yields the following result.

Lemma 4.8. Assume $i \geq 0$. If $F_iF_{i-1} < 0$, then $f > -2F_iF_{i-1}$. If $F_iF_{i+1} > 0$, then $F_i^2 = F_{i-1}F_{i+1} + f^2 > F_{i-1}F_{i+1} + 10^{14}$.

**Proof.** The desired inequalities are obtained by rewriting the master equation in the equivalent forms $F_i^2 + F_{i-1}^2 = f^2 + 2fF_iF_{i-1}$ and $F_i^2 - F_{i-1}F_{i+1} = f^2$ and taking into account Proposition 2.4. □

Proposition 4.9. For any $D(-1)$-quadruple it holds $|F_i| > 2 \cdot 10^6$ for $2 \leq i \leq 5$.

**Proof.** The algorithm described in the proof of Proposition 4.5 can be adapted for the present context. One necessary modification is in the second step, now the inversion of the equations $x = f + F_{i-1}F_i$, $y = F_{i-1}$, $F_i = P_{i-1}s - P_ir$ gives $r = P_{i-2}F_i - P_{i-1}y$, $s = P_{i-1}F_i - P_iy$, $f = x - F_iy$, because $P_{i-1}^2 - P_{i-2}P_i = 1$. The resulting values have to satisfy $r^{1.23} < s < r^{2.05}$ by Theorem 3.11 and Proposition 4.6. □

The last two results have the following consequence.

Corollary 4.10. If $c < 4b^2$, then $F_i < -10^7$ for $-1 \leq i \leq 5$.

**Proof.** For $F_{-1} = -s < -r^{1.23}$ and $F_0 = -r$, the desired conclusion follows from Proposition 2.4 in conjunction with Lemma 4.1(c). Explicit computations show that for any hypothetical $D(-1)$-quadruple one has $f < 3 \cdot 10^{12}$. Together with Lemma 4.8 and Proposition 4.9, this implies $F_i < -10^7$ for $1 \leq i \leq 5$. □
The last new ingredient in the proof of our main result is obtained by applying a specialization of the binomial theorem

\[ 0 < u < 1 \implies \sqrt{1 + u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{L}{16}u^3, \quad \text{where} \quad 0 < L < 1, \]

to the formula

\[ f = rs\left(-1 + \sqrt{1 + r^{-2} + s^{-2}}\right). \]

Maybe it is worth mentioning that \( L \) depends on \( u \) but the only property of the function \( L \) used below is its boundedness.

When one uses the resulting expression

\[ f = \frac{s}{r} + \frac{r}{s} - \frac{s}{4r^3} - \frac{1}{2rs} - \frac{r}{4s^3} + \left(\frac{s}{8r^5} + \frac{3}{8r^3s} + \frac{3}{8s^5} + \frac{r}{8s^5}\right)L \]

in \( F_1 = s - 2fr \), it gives

\[ F_1 = \frac{s}{4r^2} - \frac{r^2}{s} + \frac{1}{2s} + \frac{r^2}{4s^3} + \left(-\frac{s}{8r^4} - \frac{3}{8r^2s} - \frac{3}{8s^3} - \frac{r^2}{8s^5}\right)L. \]

Similarly, from \( F_2 = 2f F_1 + r \) one gets

\[ F_2 = \frac{s^2}{4r^3} - \frac{r^3}{s^2} - \frac{s^2}{16r^5} + \frac{1}{r} - \frac{1}{4r^3} + \frac{5r}{4s^2} - \frac{3}{8rs^2} + \frac{r^3}{2s^4} - \frac{r}{4s^4} - \frac{r^3}{16s^6} + LH_2 + L^2J_2, \]

with

\[ -\frac{s^2}{r^3} < H_2 < 0, \quad -\frac{s^2}{r^9} < J_2 < 0. \]

The recurrence relation \( F_{i+1} = 2fF_i - F_{i-1} \) together with the chain of inequalities \( s > r^{1.23} > 20^{0.23}r > 31r \) give polynomial expressions in \( L \) of the form

\[ F_3 = \frac{(16r^4 - 8r^2 + 1)s^3}{64r^8} + \frac{(32r^4 - 22r^2 + 3)s}{32r^6} + \frac{128r^4 - 96r^2 + 15}{64r^4s} + \frac{-16r^6 + 32r^4 - 26r^2 + 5}{16r^2s^3} + \frac{48r^4 - 56r^2 + 15}{64s^5} + \frac{-6r^4 + 3r^2}{32s^7} + \frac{r^4}{64s^9} + LH_3 + L^2J_3 + L^3K_3 \]

with

\[ \frac{17s^3}{128r^6} < H_3 < 0, \quad -\frac{9s^3}{256r^{10}} < J_3 < 0, \quad -\frac{s^3}{256r^{14}} < K_3 < 0, \]
and

\[
F_4 = \frac{(64r^6 - 48r^4 + 12r^2 - 1)s^4}{256r^{11}} + \frac{(32r^6 - 36r^4 + 11r^2 - 1)s^2}{32r^9} + \frac{128r^6 - 192r^4 + 69r^2 - 7}{64r^7} + \frac{96r^6 - 144r^4 + 60r^2 - 7}{32r^5s^2} + \frac{-128r^8 + 504r^6 - 250r^4 - 35}{128r^3s^4} + \frac{32r^6 - 60r^4 + 39r^2 - 7}{32rs^6} + \frac{64r^7 + 96r^6 - 144r^4 + 60r^2 - 7}{32r^9} + \frac{128r^9 + 352r^6 - 504r^4 - 35}{128r^3s^4} + \frac{32r^6 - 60r^4 + 39r^2 - 7}{32rs^6} + LH_4 + L^2J_4 + L^3K_4 + L^4M_4
\]

with

\[
-\frac{17s^4}{128r^7} < H_4 < 0, \quad -\frac{25s^4}{256r^{11}} < J_4 < 0,
\]

\[
-\frac{s^4}{128r^{15}} < K_4 < 0, \quad -\frac{s^4}{2048r^{19}} < M_4 < 0.
\]

In view of Corollary 4.10, it is clear that Theorem 1.4 is established as soon as we prove the next result.

**Proposition 4.11.** Let \((1, b, c, d)\) be a \(D(-1)\)-quadruple with \(1 < b < c < d\), \(b = r^2 + 1\), \(c = s^2 + 1\), and \(s = r^\theta\). Then the following statements hold:

(a) if \(1.64 \leq \theta < 2.05\), then \(|F_1| < 7 \cdot 10^6\);
(b) if \(1.40 \leq \theta < 1.64\), then \(|F_2| < 3 \cdot 10^6\);
(c) if \(1.30 \leq \theta < 1.40\), then \(|F_3| < 6 \cdot 10^6\);
(d) if \(1.23 \leq \theta < 1.30\), then \(|F_4| < 2 \cdot 10^6\).

In its proof we use an elementary fact, proved here for the sake of completeness.

**Lemma 4.12.** Keep the notation from Proposition 4.11. For \(1.2 < \theta < 2.05\) one has

\[
0 < \theta - \frac{\log c}{\log b} < \frac{1}{r^2}.
\]

**Proof.** The left inequality follows directly from Bernoulli’s inequality. Indeed, if \(\eta = \log c / \log b\), then

\[
c = r^{2\theta} + 1 = (r^2 + 1)^\eta = r^{2\eta}(1 + r^{-2})^\eta > r^{2\eta}(1 + \eta r^{-2}) > r^{2\eta} + 1.
\]

In view of the well-known fact \(b^{-1} < \log b < \log r^2 < r^{-2}\), the right inequality is consequence of \(h(r^2) > \theta\), where

\[
h(x) = \frac{1}{x} + \log x + \frac{x}{x^3 + 1}.
\]
The numerator of $h'$ is found to be $g(x) = (x - 1)(x^2 + 1)^2 + (1 - \theta)x^3 + 1 + x^2$, so that

$$g''(x) = (4\theta^2 + 2\theta)x^{2\theta-1} - (4\theta^2 - 2\theta)x^{2\theta-2} - (\theta^3 + 2\theta^2 - \theta - 2)x^\theta$$

$$+ (2\theta^2 + 4\theta)x^{\theta-1} - (2\theta^2 - 2\theta)x^{\theta-2} + 2.$$ 

The sum of the last three terms in the above expression is obviously positive and it is easily checked that the same is true for the sum of the other terms. Therefore, for $x > 2$ and $\theta < 2.05$ one has $g'(x) > g'(2) > g'(1) = 13 - \theta - \theta^2 > 0$ and $g(x) > g(2) > (6 - 3\theta^2)2^\theta + \theta + 6 > 0$. We conclude that the function $h$ is increasing, so $h(r^2) > 13 \log 10 > \theta$. \hfill $\square$

**Proof of Proposition 4.11.**

(a) As $\theta < 2.05$, we can bound from above $F_1$ as follows:

$$F_1 < \frac{s}{4r^2} - \frac{r^2}{s} + \frac{1}{2s} + \frac{r^2}{4s^3} < \frac{s}{4r^2} < 0.25 r^{0.05}.$$ 

By Theorem 3.11, for $1.64 \leq \theta < 2.05$ it holds $b < 10^{50}$. Therefore,

$$F_1 < 0.25 \left(10^{25}\right)^{0.05} < 5.$$ 

We bound from below $F_1$ quite similarly:

$$F_1 > \frac{s}{4r^2} - \frac{r^2}{s} + \frac{1}{2s} + \frac{r^2}{4s^3} - \frac{s}{8r^4} - \frac{3}{8r^2s} - \frac{3}{8s^3} - \frac{r^2}{8s^5} > \frac{-r^2}{s} > -r^{0.36}.$$ 

Our program for computation of an absolute upper bound for $b$ iterates over $\log c/\log b$ not over $\theta$, which explains the need for Lemma 4.12. The computations show $b < 10^{38}$ when $\theta$ is in the range $1.639 < \theta < 2.05$, so that

$$F_1 > -(10^{19})^{0.36} > -7 \cdot 10^6.$$ 

(b) Under the current hypothesis we get

$$F_2 < \frac{s^2}{4r^3} - \frac{r^3}{s^2} + \frac{2}{r} < \frac{s^2}{4r^3} = 0.25 r^{2\theta-3} < 0.25 r^{0.28}.$$ 

Using the bound on $b$ stated in Theorem 3.11 results in a bound on $F_2$ outside the desired range. Therefore we split the interval where $\theta$ takes its values. From the output of our program for computation of an absolute upper bound for $b$ we see that $b < 10^{49.3}$ when $1.499 < \theta < 1.64$, so that

$$F_2 < 0.25 \left(10^{24.65}\right)^{0.28} < 3 \cdot 10^6 \quad \text{when} \ 1.5 \leq \theta < 1.64.$$ 

For $1.40 \leq \theta < 1.50$ one gets at once

$$F_2 < 0.25 r^0 < 1.$$
We similarly bound $F_2$ from below:

$$F_2 > \frac{s^2}{4r^3} - \frac{r^3}{s^2} - \frac{s^2}{16r^5} - \frac{s^2}{r^9} > \frac{-r^3}{s^2} = -r^{3-2\theta}.$$ 

When $1.5 \leq \theta < 1.64$, this gives

$$F_2 > -1,$$

while on the subinterval $1.40 \leq \theta < 1.50$ it implies

$$F_2 > -r^{0.2} > -(10^{31.25})^{0.2} > -2 \cdot 10^6$$

because $b < 10^{62.5}$ on this subinterval.

(c) From the expression for $F_3$ we first obtain

$$F_3 < \frac{s^3}{4r^4} + \frac{s}{r^2} + \frac{2}{s} - \frac{r^4}{s^3} + \frac{2r^2}{s^3} + \frac{3r^4}{4s^5} < \frac{s^3}{4r^4} + \frac{2s}{r^2} - \frac{r^4}{s^3} < \frac{s^3}{4r^4}.$$ 

As seen from Theorem 3.11, one has $b < 6.26 \cdot 10^{73}$ when $\theta \geq 1.299$. Therefore, the upper bound for $F_3$ just obtained can be bounded from above as follows:

$$0.25 r^{0.2} < 0.25 (62.6^{0.5} \cdot 10^{36})^{0.2} < 6 \cdot 10^6.$$ 

To obtain a lower bound for $F_3$, we can ignore all fractions but the first, the fourth and the sixth in its free term and replace the coefficients of positive powers of $L$ by their respective lower bounds. We thus get

$$F_3 > \frac{3s^3}{16r^4} + \frac{s}{2r^2} - \frac{r^4}{s^3} - \frac{17s^3}{128r^6} - \frac{9s^3}{256r^{10}} - \frac{s^3}{256r^{14}} + \frac{s}{2r^2} - \frac{r^4}{s^3} - \frac{3r^4}{16s^7} > \frac{-r^4}{3s^3},$$

whence

$$F_3 > -r^{0.1} > -(62.6^{0.5} \cdot 10^{36})^{0.1} > -5000.$$ 

(d) We similarly see that it holds

$$F_4 < \frac{s^4}{4r^5} < 0.25 r^{0.2} < 0.25 (10^{34.5})^{0.2} < 2 \cdot 10^6$$

and

$$F_4 > -\frac{r^5}{s^4} > -r^{0.08} > -(10^{34.5})^{0.08} > -600.$$ 

Proposition 4.11 being established, the proof of the nonexistence of $D(−1)$-quadruples is complete. □
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