The Poincaré Group of Discrete Minkowskian Space-Time

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Abstract

The lattice of integral points of 4-dimensional Minkowski space, together with the inherited indefinite distance function, is considered as a model for discrete space-time. The Lorentz and Poincaré groups of this discrete space-time are identified as subgroups of the corresponding Lie groups. The lattice Lorentz group has irreducible projective (including linear) representations which are restrictions of (all) finite-dimensional irreducible projective representations of the Lorentz Lie group and hence can be used to describe all integral and half-odd-integral helicity. The (4-torus) momentum space has a well-defined “light cone” of null points and there are orbits of the lattice Lorentz group lying entirely in the torus light cone and having the lattice euclidean group of the plane as little group. Wigner’s method for the Poincaré Lie group can then be adapted to show, in the first instance, that the lattice Poincaré group has unitary representations describing lattice free fields of zero mass and an arbitrary Lorentz helicity, in particular chiral fermions. There are no representations with a nonzero invariant mass.
1. Introduction

The modelling of space and space-time by discrete sets or lattices has a long history, originating in efforts to regularise quantum field theory [1-3]. Over the past few decades, lattice methods have become effective tools in several areas of study: abelian and nonabelian gauge models [4,5], quantum gravity and strings [6,7], etc. etc. They are also an indispensable part of the machinery of nonperturbative numerical gauge theory calculations [8,9]. In almost all of this activity, the underlying “continuum” is euclidean space ($\mathbb{R}^3$) or space-time ($\mathbb{R}^4$) given a euclidean metric, or manifolds which are locally euclidean. Not infrequently, the lattices considered are not regular. These features are clearly inimical to the implementation and exploitation of the natural symmetries of realistic space-time: random lattices have no symmetries and lattice rotation groups in euclidean space, in any number of dimensions, are finite groups. This is an obvious handicap. First, the restoration of symmetries in the continuum limit becomes a nontrivial task. More seriously, the denial of natural symmetries can lead to various pathologies on the lattice. (We shall have occasion to refer to some troubles of this sort in the concluding section of this article).

The natural symmetries that are the concern of the present paper are those of flat minkowskian space-time $\mathbb{R}^4$. We describe here the first results of a study of lattices embedded in $\mathbb{R}^4$ with the inherited minkowskian distance, focusing on the corresponding (discrete) Poincaré group. Minkowskian lattices have seldom been considered in the extensive literature of lattice physics and never, to the author’s knowledge, from the viewpoint of their symmetries. Though the mathematical methods required in this endeavour are less widely known than the theory of representations of Lie groups and Lie algebras, there does exist a sufficiently rich and deep body of knowledge on discrete subgroups of Lie groups to make the effort worthwhile.

To see what we should be aiming at, we have only to recall the fundamental significance of relativistic symmetries in the context of quantum theory. Not only do they govern processes through the operation of conservation laws and selection rules, but, through the identification of certain irreducible unitary representations of the Poincaré group with 1-particle states, they actually *define* elementary particles; masses and helicities are invariants
of representations and the functions defining the space of a representation are the corre-
sponding (momentum space) free fields [10-13]. This is the ideal against which any proposal
for discretising space-time ought to be, ultimately, judged. Such a project will consist of
the following steps, at the least: i) identify and characterise the discrete counterpart of
the Poincaré group or, rather, its universal covering group; ii) find a general method for
constructing its irreducible unitary representations; iii) show that some at least of these
representations have a satisfactory physical interpretation and iv) derive the free-field dy-
namics (field equations and subsidiary conditions). The first three of these questions are
addressed in this paper and the answers are, in the context of lattice kinematics, more or
less satisfactory, with some surprises. The lattice Poincaré group is easy to determine and
(the “2-fold cover” of) its Lorentz subgroup turns out to be a discrete group of an especially
nice type. In particular, the latter has finite dimensional representations which, for purposes
of describing helicities, are as good as those of the Lorentz (Lie) group. The momentum
space properties of the group, however, are quite unlike those in the continuum case. We
shall find, apart from the inevitable momentum cutoff, that it is not possible to associate a
nonzero mass to a representation in a sensible way - the analogues of the massive represen-
tations of the Poincaré (Lie) group have the serious drawback that at certain values of the
momenta, all of them behave like tachyonic representations. Remarkably, massless represen-
tations (understanding the meaning of mass and masslessness is part of the work) are free
from this difficulty. (All statements regarding momenta are of course to be understood in
the general context of lattice momenta which are defined and conserved modulo the inverse
lattice spacing). The construction of the most general massless representation is, as is true
in general for discrete groups, much more demanding a task than in the continuum case.
This effort is only initiated here, for certain special types of “momentum shells” or orbits.
What is notable is one consequence of the combination of a good description of (arbitrary)
Lorentz helicity and of masslessness: chiral fields are automatic on the Minkowskian lattice,
in fact obligatory. The most serious drawback of euclidean lattice physics is thus obviated
by going minkowskian.

There are of course crucial differences in the physics of continuum and discrete quantum
relativity. These will be touched upon in the concluding section. But it is possible to argue, nevertheless, that they are not fatal to the tantalising possibility of the lattice model of special relativity being taken seriously as describing physics at a high enough energy scale, without actually going to the continuum limit.

2. The Discrete Poincaré Group

The group of isometries of 4-dimensional Minkowski space $\mathbb{R}^4_M$ is $O(3, 1, \mathbb{R})\mathcal{X}\mathbb{R}^4$, where $O(3, 1, \mathbb{R})$ is the full (or extended) Lorentz group, $\mathbb{R}^4$ is the translation group and $\mathcal{X}$ indicates the semidirect product, the arrow pointing from the quotient subgroup to the normal subgroup. Of this, only the subgroup connected to the identity appears to be an exact symmetry of nature. We shall denote the connected Lorentz group $SO(3, 1, \mathbb{R})_{\text{conn}} = SO(3, 1, \mathbb{R})/\{\pm 1 \in SO(S, 1, \mathbb{R})\}$ by $L(\mathbb{R})$, the corresponding inhomogeneous group $L(\mathbb{R})\mathcal{X}\mathbb{R}^4$ by $P(\mathbb{R})$ and refer to them simply as the (continuum or real) Lorentz and Poincaré group.

The discrete space-time of this work is the hypercubic lattice $\mathbb{Z}^4_M$ of points in $\mathbb{R}^4$ with integer coordinates (the lattice spacing is the unit of length and time), with a distance function given by the metric in $\mathbb{R}^4_M$: the (length)$^2$ of $X \in \mathbb{Z}^4_M = \{X_\mu \in \mathbb{Z} \mid \mu = 0, \cdots, 3\}$ is $X_0^2 - X_1^2 - X_2^2 - X_3^2 = X_\mu X_\mu = X^2$. The lattice Lorentz group is then the subgroup of $L(\mathbb{R})$ obtained by restricting every $4 \times 4$ matrix $\lambda \in L(\mathbb{R})$ to have integral entries: $L(\mathbb{Z}) = SO(3, 1, \mathbb{Z})/\{\pm 1 \in SO(3, 1, \mathbb{Z})\}$, where, as the notation makes clear, $SO(3, 1, \mathbb{Z})$ is $SO(3, 1)$ over the ring of integers. $L(\mathbb{Z})$ acts on the discrete translation group $\mathbb{Z}^4$ exactly as in the corresponding continuum case and the semidirect product $P(\mathbb{Z}) = L(\mathbb{Z})\mathcal{X}\mathbb{Z}^4$, the discrete Poincaré group, is our relativity group. It is an interesting remark that while $L(\mathbb{Z})$ is the trivial group in $1+1$ dimensions, it is an infinite group in all higher dimensional Minkowski lattices.

The representations of $P(\mathbb{Z})$ that are of interest in the context of quantum theory are its projective unitary representations, in accordance with Wigner’s general theorem on symmetries. In the continuum, such representations of $P(\mathbb{R})$ are found by first establishing [10] that every continuous projective unitary representation of $P(\mathbb{R})$ lifts to a continuous unitary representation of its universal covering group $\hat{P}(\mathbb{R}) = \hat{L}(\mathbb{R})\mathcal{X}\mathbb{R}^4 = SL(2, \mathbb{C})\mathcal{X}\mathbb{R}^4$, $SL(2, \mathbb{C})$
being the universal cover of $SO(3,1,\mathbb{R})_{\text{conn}}$. This key result, which is clearly very specific to Lie groups, has the following ingredients [10]: i) though $\mathbb{R}^4$ has nontrivial projective representations, they do not extend to $P(\mathbb{R})$ as nontrivial projective representations; ii) though semidirect product groups $G \rtimes A$ with $A$ abelian can have projective representations whose restrictions to $G$ and $A$ are linear representations (via 1-cocycles on $G$ with values in the character group of $A$) this does not happen for $P(\mathbb{R})$ because $L(\mathbb{R})$ is semisimple; and iii) every projective representation of $L(\mathbb{R})$ lifts to a linear representation of its universal cover, again because of semisimplicity. To deal with $P(\mathbb{Z})$ with the same degree of completeness will take us into difficult terrain and, for our purposes, it is unnecessary. Our first aim being the understanding of helicities, which are described by the representations of the Lorentz group, we ignore ingredients i) and ii) (the validity of point i) for $P(\mathbb{Z})$ is actually easy to establish) and concentrate on ingredient iii). The assertion iii) is a special case of a general result which says that, given any group $G$, we can construct a group $\hat{G}$ determined fully by $G$ with the property that every projective representation of $G$ lifts to a linear representation of $\hat{G}$ [14]. $\hat{G}$ is called a universal central extension of $G$ and is not always unique [14-16]. However, a connected semisimple Lie group has a unique universal central extension and it coincides with its universal cover, so that equivalence classes of its projective representations are classified by the character group of its fundamental group. This is the reason why $\hat{L}(\mathbb{R})$ is $SL(2,\mathbb{C})$. (Many Lie groups commonly occurring in physics have the property that their nontrivial projective representations have no relationship whatever with their fundamental groups [15,16]). A linear representation of $\hat{L}(\mathbb{R})$ restricting to its centre $\mathbb{Z}_2$ (= the fundamental group of $L(\mathbb{R})$) as the trivial (nontrivial) character passes to the quotient group $L(\mathbb{R})$ as a trivial projective, i.e. linear, (nontrivial projective) representation and these are the only classes of projective representations of $L(\mathbb{R})$. Keeping all this in mind, we shall refer to $\hat{L}(\mathbb{R})$ and $\hat{P}(\mathbb{R})$ themselves as the Lorentz and Poincaré groups.

The corresponding problem for $L(\mathbb{Z})$ is not so neatly settled. Indeed, lacking a physically significant criterion of continuity as is available for representations of Lie groups, the quest for “all” projective representations of $L(\mathbb{Z})$ is unduly ambitious, perhaps ill-defined. We shall content ourselves with showing the existence of certain finite dimensional (nonunitary)
projective representations of $L(\mathbb{Z})$ (and, eventually, of projective unitary representations of $P(\mathbb{Z})$) which are inherited naturally from those of $L(\mathbb{R})$, by restriction – in general, a nontrivial projective representation of a group need not restrict to a subgroup as a nontrivial projective representation.

To implement this aim, we look for a subgroup $\hat{L}(\mathbb{Z})$ of $SL(2, \mathbb{C}) = \hat{L}(\mathbb{R})$ which is such that every projective representation of $L(\mathbb{Z})$ that is a restriction of a projective representation of $L(\mathbb{R})$ lifts to a linear representation of $\hat{L}(\mathbb{Z})$; in other words, $\hat{L}(\mathbb{Z})$ should be such that its centre contains $\mathbb{Z}_2$, $\hat{L}(\mathbb{Z})/\mathbb{Z}_2 = L(\mathbb{Z})$ and $L(\mathbb{Z})$ is not a subgroup of $\hat{L}(\mathbb{Z})$. This can be done by following the standard treatment (as given, for example, in [17]) of $L(\mathbb{R})$ and $\hat{L}(\mathbb{R})$. Denote by $H(2, \mathbb{C})$ the (real) vector space of $2 \times 2$ complex matrices which are hermitian and by $\tau_i$ ($i = 1, 2, 3$) the Pauli spin matrices. The association of $x \in \mathbb{R}^4_M$ to $x_\mu \tau_\mu$ ($\tau_0 =$ unit matrix) is a bijection of $\mathbb{R}^4_M$ and $H(2, \mathbb{C})$ such that $x^2 = \det(x_\mu \tau_\mu)$. But $\{\tau_\mu\}$ are matrices whose elements are gaussian integers, i.e., complex numbers whose real and imaginary parts are integers. Therefore the restriction of $\mathbb{R}^4_M$ to $\mathbb{Z}^4_M$ gives a bijective map of $\mathbb{Z}^4_M$ into $H(2, \mathbb{Z}[i])$ where $\mathbb{Z}[i]$ stands for the ring of gaussian integers. The group $SL(2, \mathbb{Z}[i]) \subset SL(2, \mathbb{C})$ has an action on $H(2, \mathbb{Z}[i])$ by $X_\mu \tau_\mu \rightarrow A(X_\mu \tau_\mu)A^*$, preserving the determinant, exactly as $SL(2, \mathbb{C})$ acts on $H(2, \mathbb{C})$. Hence there is a discrete Lorentz transformation $\Lambda$ such that $(\Lambda X)_\mu \tau_\mu = A(X_\mu \tau_\mu)A^*$, $(\Lambda X)^2 = X^2$, with $A \rightarrow \Lambda$ defining a homomorphism of $SL(2, \mathbb{Z}[i])$ into $SO(3, 1, \mathbb{Z}) \subset SO(3, 1, \mathbb{R})$. But since $A \in SL(2, \mathbb{C})$, a continuity argument shows [17] that $\Lambda$, as an element of $SO(3, 1, \mathbb{R})$, belongs to its connected component and hence to $SO(3, 1, \mathbb{Z})/\{\pm 1 \in SO(3, 1, \mathbb{Z})\} = L(\mathbb{Z})$. It is easy to check now that the kernel of the homomorphism $A \rightarrow \Lambda$ is the centre of $SL(2, \mathbb{Z}[i]) = \mathbb{Z}_2 = \{\pm 1 \in SL(2, \mathbb{Z}[i])\}$ and that $L(\mathbb{Z})$ is the quotient of $SL(2, \mathbb{Z}[i])$ by its centre, but not a subgroup. All this is exactly as for $L(\mathbb{R})$ and $SL(2, \mathbb{C})$. (In the rest of this paper, $\Lambda$ will denote both an element of $L(\mathbb{Z})$ and the corresponding element(s) of $SL(2, \mathbb{Z}[i])$; no confusion will arise).

$SL(2, \mathbb{Z}[i])$ is the sought for group $\hat{L}(\mathbb{Z})$. A representation of $\hat{L}(\mathbb{Z})$ restricting to its centre $\mathbb{Z}_2$ as the trivial (nontrivial) character will pass to the quotient group $L(\mathbb{Z})$ as a trivial (nontrivial) projective representation.
3. Unitary Representations of $\hat{P}(\mathbb{R})$ - an Overview

For the eventual construction of unitary representations of $\hat{P}(\mathbb{Z})$, we shall try to follow Wigner’s method for $\hat{P}(\mathbb{R})$ [10-12,18] in a variant form described in [19], somewhat further streamlined. The generality of the method [11,20] allows room for hoping that, with suitable adjustments, it can be adapted to the discrete case. More importantly, the method naturally highlights the physical attributes, mass and (Lorentz) helicity, that permit a direct association of elementary quantum fields with irreducible unitary representations. The brief recapitulation below of this method of “inducing from little groups” is meant to highlight criteria for picking out certain unitary representations of $\hat{P}(\mathbb{R})$ as physical (or physically acceptable) especially in the case of massless representations. It will serve as a model for deciding which representations of $\hat{P}(\mathbb{Z})$ can be considered physical.

The momentum space is the dual group of the translation group $\mathbb{R}^4$, isomorphic to $\mathbb{R}^4$. We denote it by $M$. Let $O$ be an orbit of $\hat{L}(\mathbb{R})$ in $M$ (the action is defined to be the natural action of $L(\mathbb{R})$, lifted to $\hat{L}(\mathbb{R})$ by letting the central subgroup $\mathbb{Z}_2$ act trivially) and $S$ the stabiliser (little group) of any point in $O$: $O = \hat{L}(\mathbb{R})/S$. We suppose given, to begin with, a finite dimensional representation $\rho$ of $\hat{L}(\mathbb{R})$ on a Hilbert space $V$ with the property that the restriction of $\rho$ to $S$ is unitary. Denote by $\pi$ the projection of $\hat{L}(\mathbb{R})$ onto $O$ and let $\sigma$ be a section of $\pi$, i.e. any map $O \to \hat{L}(\mathbb{R})$ such that $\pi(\sigma(p)) = p$ for all $p \in O$ and $\omega$ the $\hat{L}(\mathbb{R})$-invariant measure on $O$.

On the space $\mathcal{H}_{O,V}$ of vector-valued functions $\phi, \psi : O \to V$, square-integrable with respect to $\omega$, define a bracket $\langle \phi, \psi \rangle$ by

$$\langle \phi, \psi \rangle = \int_0 d\omega(p) \langle \rho(\sigma(p)^{-1})\phi(p), \rho(\sigma(p)^{-1})\psi(p) \rangle_V.$$ 

If $\sigma$ and $\sigma'$ are two sections of $\pi$, it follows from $\pi(\sigma(p)) = \pi(\sigma'(p))$ ($= p$) that $\sigma(p)^{-1}\sigma'(p)$ is in $S$. And since $\rho$ restricts to $S$ as a unitary representation, $\langle \ , \ \rangle$ is independent of the section used to define it making it a scalar product. Moreover, $||\phi|| = 0$ if and only if $\rho(\sigma(p)^{-1})\phi(p) = 0$ for all $p \in O$ i.e., $\phi = 0$ identically. So (the completion of) $\mathcal{H}_{O,V}$ is a Hilbert space. Noting that $p \to \rho(\sigma(p)^{-1})\phi(p)$ is a section of a vector bundle over $O$ with fibre $V$, we can characterise $\mathcal{H}_{O,V}$ as the Hilbert space of such sections which are $L^2$ with
respect to \( \omega \).

On \( \mathcal{H}_O, \hat{P}(\mathbb{R}) \) has a unitary representation given by

\[
(U_{O,V}(\lambda, a)\phi)(p) = \chi_p(a)\rho(\lambda)\phi(\lambda^{-1}p), \quad \lambda \in \hat{L}(\mathbb{R}), \quad a \in \mathbb{R}^4,
\]

where \( \chi_p(a) = \exp(ipa) \) is the character of \( \mathbb{R}^4 \) corresponding to \( p \). We have

\[
||U_{O,V}(\lambda, a)\phi||^2 = \int d\omega(p)||\rho(\sigma(p)^{-1})\rho(\lambda)\phi(p)||^2_V
\]

using the invariance of the measure. But \( \sigma(p) \) and \( \lambda\sigma(p) \) have the same projection onto \( O \) and hence differ by an element of \( S \), \((\sigma(p)^{-1}\lambda^{-1}\sigma(p)) \) is a cocycle \( \hat{L}(\mathbb{R}) \times O \rightarrow S \) implying

\[
\rho(\sigma(p)^{-1})\rho(\lambda) = \rho(s)^{-1}\rho(\sigma(p)^{-1})
\]

for some \( s \in S \). Since \( \rho \) is unitary on \( S \) by assumption, the unitarity of \( U_{O,V} \) follows. We shall say that \( U_{O,V} \) is supported on \( O \) and ranges over \( V \). It is irreducible whenever \( V \) is an irreducible representation of \( \hat{L}(\mathbb{R}) \). In the language of induced representations, \( U_{O,V} \) is the representation induced by the (unitary) restriction of \( \rho \) to \( S \subset \hat{L}(\mathbb{R}) \).

When \( O \) is a positive (mass)\(^2 \) positive (or negative) energy mass shell of mass \( m \), i.e., the orbit \( O_m \) through \( p = (m, 0, 0, 0) \), the stabiliser \( S_{m,p} \) of \( p \in O_m \) is isomorphic to \( SU(2) \) and every irreducible representation \( \rho \) of \( \hat{L}(\mathbb{R}) \) restricts to \( S_{m,p} \) as an irreducible unitary representation. Consequently, the helicity spectrum of \( U_{m,V} \) is determined equivalently and alternatively by the \( \hat{L}(\mathbb{R}) \) or \( S_{m,p} \) transformation properties of the function \( \phi \); in particular, the number of helicity states in \( \mathcal{H}_{m,V} \) is \( \dim V \). Moreover, the condition that \( S_{m,p} \) fixes \( p \) translates as the condition

\[
(U_{m,V}(s, a)\phi)(p) = \chi_p(a)\rho(s)\phi(p),
\]

for all \( s \in S_{m,p} \), on the functions \( \phi \). This, or rather its Lie algebra version, is the invariant wave equation or the free field equation corresponding to the unitary representation \( U_{m,V} \) of \( \hat{P}(\mathbb{R}) \) [21].

In the light of the fact that all elementary particles have (mass)\(^2 \geq 0 \) and a finite set of helicities, we shall in general refer to unitary representations of \( \hat{P}(\mathbb{R}) \) with these properties (in
particular $\dim V < \infty$) as physical. The condition on the helicity spectrum is a powerful one - whenever $S$ is a noncompact group, it puts strong restrictions on the unitary representations of $S$ that can be used in the induction procedure. It will play a crucial role in sorting out physical representations of $\hat{P}(\mathbb{Z})$ as indeed it already does for massless representations of $\hat{P}(\mathbb{R})$.

There are three mass = 0 orbits: the vertex of the light cone in $M$ (a one-point orbit) and the open upper and lower half light cones. Consider representations supported on the upper half light cone $C^+$. The stabiliser $S_0$ is isomorphic to the subgroup of upper triangular matrices in $SL(2, \mathbb{C})$ (the representative point of $C$ which $S_0$ fixes is $(p_0, 0, 0, p_0)$ for any $p_0 > 0$) which we choose to parametrise as

$$s(\theta, z) = \begin{pmatrix} \exp(i\theta) & z \exp(-i\theta) \\ 0 & \exp(-i\theta) \end{pmatrix}, \quad 0 \leq \theta < 2\pi, \quad z \in \mathbb{C},$$

so that $s(\theta_1, z_1)s(\theta_2, z_2) = s(\theta_1 + \theta_2 \mod 2\pi), z_1 + z_2 \exp(2i\theta_1))$. So $S_0$ is the euclidean group of the plane $E(2, \mathbb{R}) = SO(2, \mathbb{R}) \times \mathbb{R}^2$, with $\mathbb{R}^2 = \{(\text{Re} z, \text{Im} z)\}$ on which $SO(2, \mathbb{R})$ acts as the two-fold cover of the rotation group.

Now, a finite dimensional unitary representation of $E(2, \mathbb{R})$ is necessarily nonfaithful; in fact the only such representations are characters of the subgroup $SO(2, \mathbb{R})$ and have the normal subgroup $\mathbb{R}^2$ as kernel [22]. Hence $\hat{L}(\mathbb{R})$, being simple, cannot have any finite dimensional representation restricting to $E(2, \mathbb{R})$ unitarily. This means that the straightforward induction procedure that works for massive representations is no longer valid and has to be modified suitably. The well-known way to do this [21,12] is to replace the Hilbert space $\mathcal{H}_{\mathcal{O}, V}$ of functions $\phi : C^+ \to V$ by a subspace $\mathcal{H}'_{\mathcal{O}, V}$ on which $U$ restricted to the subgroup $\mathbb{R}^2$ of $E(2, \mathbb{R})$ acts trivially:

$$\rho(r)\phi(r^{-1}p) = \phi(p), \quad r = (\text{Re} z, \text{Im} z) \in \mathbb{R}^2.$$

The Lie algebra form of this condition constitutes the subsidiary condition. A character of $SO(2, \mathbb{R}) = E(2, \mathbb{R})/\mathbb{R}^2$, $\rho(\theta)\phi(p) = \exp(im \theta/2)\phi(p), \theta/2 \in SO(2, \mathbb{R}), m \in \mathbb{Z}$, then induces a unitary representation of $\hat{P}(\mathbb{R})$ on $\mathcal{H}'_{\mathcal{O}, V}$. The fact to be emphasised, and relevant in the
context of $\hat{P}(\mathbb{Z})$, is that massless finite helicity representations of $\hat{P}(\mathbb{R})$ exist because $E(2, \mathbb{R})$ has a normal subgroup with compact quotient group. A massless physical irreducible unitary representation of $\hat{P}(\mathbb{R})$ has precisely one (Lorentz) helicity, namely the character of $SO(2, \mathbb{R})$ to which $\rho$ restricts. This helicity is not related to rotational spin which, in any case, is a meaningless notion for a state that cannot be transformed to rest.

For the sake of completeness, it should be remarked that all $(\text{mass})^2 < 0$ representations are doubly unphysical. In addition to the well-known causality problem, they suffer from unphysical helicities as well: the stabiliser, which is $SL(2, \mathbb{R})$, has no nontrivial finite dimensional unitary representations at all.

4. Physical Masses and Helicities for $\hat{P}(\mathbb{Z})$

This section is devoted to an examination of the extent to which the fundamental notions of mass and helicity can be carried over from $\hat{P}(\mathbb{R})$ to $\hat{P}(\mathbb{Z})$, as a prerequisite to the construction of unitary representations of $\hat{P}(\mathbb{Z})$ which are physically acceptable.

The momentum space of discrete space-time is the dual group of the discrete translation group $\mathbb{Z}^4$, namely the 4-torus $\mathbb{T}^4$, denoted simply by $T$ from now on. It is convenient for what follows to think of $T$ as $M/\mathbb{Z}^4$, where $M$ is the momentum space $\mathbb{R}^4$ of the continuum translation group and $\mathbb{Z}^4$ is the reciprocal lattice. Introducing coordinates $\{p_\mu\}$ in $M$, we identify $T$ as the hypercube $\{-\pi \leq P_\mu \leq \pi, \ P_\mu = p_\mu \ (\text{mod} \ 2\pi)\}$, i.e. as the fundamental region for the translation action of $\mathbb{Z}^4$ on $\mathbb{R}^4$, the unit cell of the reciprocal lattice. (Physically, of course, all this just means that momentum is defined and conserved modulo $2\pi$). We can then study the action of $\hat{L}(\mathbb{Z})$ on $T$ by starting with its action on $M$ and translating the image (of a point in $T \subset M$) back to $T$ by some integral multiples of $2\pi$. Under this projection $\tau: M \rightarrow T$, $P_\mu = \pi$ and $P_\mu = -\pi$ get identified for every $\mu$.

Thus every orbit $O_T$ of $\hat{L}(\mathbb{Z})$ on $T$, through a given point $P$, can be determined by first finding the orbit $O$ of $\hat{L}(\mathbb{Z}) \subset \hat{L}(\mathbb{R})$ in $M$ through $P$ and then projecting $O$ back to $T$. It is to be expected that, generically, such orbits will be quite wild [23]. $O$ being a subset of an orbit of $\hat{L}(\mathbb{R})$ in $M$, let us first determine the projection onto $T$ of a positive mass orbit $O_{T,m}$ of $\hat{L}(\mathbb{R})$ in $M$, the familiar mass shell. Figure \[\text{insert}\] shows such an orbit of $\hat{L}(\mathbb{R})$ in $T$,
projected further onto, say, the (0,1) plane. The corresponding orbit of $\hat{L}(\mathbb{Z})$ for a “mass” $< \pi$ is a subset of this.

The pathological nature of positive “mass” orbits $\mathcal{O}_{T,m}$ is made dramatically obvious by Figure 1. Despite their being the projections onto the unit cell of physical positive mass orbits of $\hat{L}(\mathbb{Z})$ in $M$, no fixed invariant mass can be associated to them. (This is just a reflection of the fact there are no $\hat{L}(\mathbb{Z})$-invariant (and hence $\hat{L}(\mathbb{R})$-invariant) nontrivial periodic functions on $M$ and stems from the periodicity of the momentum itself). On the contrary, $\mathcal{O}_{T,m}$ has points corresponding to arbitrarily small positive “$(mass)^2$” as well as tachyonic points with negative “$(mass)^2$” which are reached by large boosts. Representations supported on such orbits will violate (micro) causality and must be rejected.

On the other hand, consider the upper half light cone $C_+$ in $M$. Under projection onto $T$ (translation by multiples of $2\pi$), its image $C_T$ is the whole of the light cone lying in $T$, including the origin and the negative energy light cone inside $T$. $C_T$ is a hypersurface in $T$ which we shall refer to as the light cone of $T$. We can consistently associate a vanishing
mass to every point of $C_T$ – the polynomial $p_0^2 - p_1^2 - p_2^2 - p_3^2$ is periodic and invariant as long as it vanishes. The orbit through any point in $C_T$ of $\hat{L}(\mathbb{Z})$ is a set of discrete points in $C_T$ and every such orbit can be said, invariantly, to have zero mass.

Thus while no physically sensible meaning can be given to a non-zero mass, masslessness is a notion which makes sense on the lattice. Another way of understanding this distinction is to note that a massive state can be transformed to rest and then subjected to rotations in order to determine its helicity. But the lattice rotation group is a finite group and cannot possibly serve to define arbitrary helicities. A massless state is free from this paradox.

The last statement leads us naturally to the problem of defining helicities in terms of the lattice group $\hat{L}(\mathbb{Z})$. The situation here is as nice as it can be. We have the key result:

*Every finite dimensional irreducible representation of $L(\mathbb{R})$ or $\hat{L}(\mathbb{R})$ restricts to its lattice subgroup $L(\mathbb{Z})$ or $\hat{L}(\mathbb{Z})$ respectively as an irreducible representation.*

This is a special case of a general theorem, the density theorem of A. Borel [24], on representations of discrete subgroups of noncompact semisimple Lie groups [23-28]. A general formulation of the theorem is the following. Let $G$ be a semisimple Lie group none of whose factors is compact and $\Gamma$ a discrete subgroup of $G$ having the property that $G/\Gamma$ has finite volume. Then every finite dimensional irreducible representation of $G$ remains irreducible as a representation of $\Gamma$. The discrete groups $SO(3,1,\mathbb{Z})$ and $SL(2,\mathbb{Z}[i])$ have finite covolumes in $SO(3,1,\mathbb{R})$ and $SL(2,\mathbb{C})$ respectively and hence meet the conditions of the theorem.

Thus the helicity content of an irreducible finite dimensional representation of $\hat{L}(\mathbb{R})$ remains intact on restricting $\hat{L}(\mathbb{R})$ to $\hat{L}(\mathbb{Z})$; nothing is lost in this regard by discretising space-time as long as we do not try to define helicities through the rotation group. With this result in hand, we can attempt the construction of irreducible unitary representations of $\hat{P}(\mathbb{Z})$, characterised by a mass = 0 and a helicity identical to that corresponding to an irreducible representation of $\hat{L}(\mathbb{R})$. Moreover, from the density theorem, the relationship between finite dimensional representations of $L(\mathbb{Z})$ and $\hat{L}(\mathbb{Z})$ is exactly the same so that between representations of $L(\mathbb{R})$ and $\hat{L}(\mathbb{R})$: the linear representations of $\hat{L}(\mathbb{Z})$ which are trivial on its centre and hence pass to linear representations of $L(\mathbb{Z})$ are integral helicity represen-
tations and those which are not, and hence pass to nontrivial projective representations, are half-odd-integral helicity representations. In particular, \( \hat{L}(\mathbb{Z}) \) has “spin \( \frac{1}{2} \)” representations of both chirality and chiral Weyl spinor fields on \( \mathbb{Z}^4_M \), can naturally be associated with them.

The validity of the density theorem for the spin \( \frac{1}{2} \) representations, the most important in practice, is actually easy to establish. The defining (left-chiral) representation \( \rho_L \) of \( SL(2, \mathbb{C}) : \rho_L(\lambda) = \lambda \in SL(2, \mathbb{C}) \), restricts to \( SL(2, \mathbb{Z}[i]) \) as \( \rho_L(\Lambda) = \Lambda \in SL(2, \mathbb{Z}[i]) \). Let \( B \) be any operator on \( \mathbb{C}^2 \), the representation space of \( \rho_L \), commuting with \( \rho_L(\Lambda) \) for all \( \Lambda \in SL(2, \mathbb{Z}[i]) \). The Pauli matrices \( \{\tau_i\} \) multiplied by \( i \) are obviously in \( SL(2, \mathbb{Z}[i]) \). Hence \( B \) commutes with \( \rho_L(i\tau_i) = i\tau_i \) by assumption, which means that \( B \) is a multiple of the unit operator. By Schur’s lemma, \( \rho_L \) restricted to \( SL(2, \mathbb{Z}[i]) \) is thus irreducible. The argument for the conjugate right-chiral representation is the same. (We may note that any two of the Pauli matrices can be chosen to belong to a set of generators for \( SL(2, \mathbb{Z}[i]) \) [29]).

We shall consider as physical those and only those irreducible representations of \( \hat{L}(\mathbb{Z}) \) that are restrictions of (continuous) finite dimensional irreducible representations of \( \hat{L}(\mathbb{R}) \).
5. Unitary Representations of $\hat{P}(\mathbb{Z})$ - First Steps

The continuum Lorentz group $\hat{L}(\mathbb{R})$ acts transitively on $C_+$, the upper half light cone in $M$ (and likewise on $C_-$). Thus all of $C_+$ is one orbit of $\hat{L}$ and, consequently, there is an irreducible unitary representation of $\hat{P}(\mathbb{R})$ supported on $C_+$ and ranging over a given irreducible representation of $\hat{L}(\mathbb{R})$ (subject, of course, to the subsidiary conditions). This situation fails to hold for the action of $\hat{L}(\mathbb{Z})$ on the torus light cone $C_T$, there being a continuous infinity of disjoint orbits. A theory of unitary representations of $\hat{P}(\mathbb{Z})$ à la Mackey, physically acceptable in the sense explained earlier, would require a characterisation of the orbits of $\hat{L}(\mathbb{Z})$ in $C_T$, the determination of their stabilisers and, finally, a further classification according to the existence or otherwise of finite dimensional unitary representations of the latter. Needless to say, that will be a challenge and it is not undertaken here. We shall content ourselves in this preliminary look at the problem by exhibiting certain special types of orbits, of which one type does not admit physical representations and another type does, giving representations which are the discrete analogues of the physical massless representations of $\hat{P}(\mathbb{R})$.

The origin in $C_T$ is a one-point orbit whose stabiliser is the whole of $\hat{L}(\mathbb{Z})$; it need not be considered further.

In general, we can find the orbit of $\hat{L}(\mathbb{Z})$ in $C_T$ through a point $P \neq 0$ by first finding the orbit in $C_+$ (or $C_-$) through $P$ considered as point in $C_+$ and then mapping it to $C_T$ via the projection $\tau : C_+ \rightarrow C_T = C_+/\mathbb{Z}^4$ (see section 4). We call $P$ a rational point if its coordinates are all rational multiples of $\pi$: $P_\mu = (q_\mu/d_\mu)\pi$, $q_\mu, d_\mu \in \mathbb{Z}$ with $-|d_\mu| \leq q_\mu \leq |d_\mu|$ for $\mu = 0, 1, 2, 3$, and an irrational point otherwise. Every point in the orbit through a rational (irrational) point is rational (irrational).

Consider rational orbits first. Reexpress the coordinates $\{q_\mu/d_\mu\}$ (dropping the factor $\pi$ for the time being) in terms of the lowest positive common multiple $D$ of $\{d_\mu\}$, i.e., $P_\mu = Q_\mu/D$ with $-D \leq Q_\mu \leq D$, and no positive integer $D' < D$ exists such that $P_\mu = Q'_\mu/D'$ for any $\{Q'_\mu\}$ with $-D' \leq Q'_\mu \leq D'$ for all $\mu$; $\{Q_\mu/D\}$ will be referred as the standard expression for $P$. Then $\Lambda \in \hat{L}(\mathbb{Z})$ acts on $P$ by changing its standard expression to $\{\Lambda_{\mu\nu}Q_\nu/D\}$. Suppose
$\Lambda_{\mu\nu}Q_\nu$ has a common factor with $D$ for each $\mu$. Then the standard expression for $\Lambda P$ will have a denominator $D_\Lambda$ strictly less than $D$; otherwise $D_\Lambda = D$. But the same argument applies also to $\Lambda^{-1}$ acting on $\Lambda P$, implying that $D \leq D_\Lambda$. So $D_\Lambda = D$; $\hat{L}(\mathbb{Z})$ acts on every rational point without changing the denominator in its standard expression. Since the numerators $Q_\mu$ are bounded between $-D$ and $D$, we conclude that every rational orbit of $\hat{L}(\mathbb{Z})$ in $C_T$ is a finite set.

The stabiliser in $\hat{L}(\mathbb{Z})$ of any rational point of $C_T$ is therefore a subgroup of finite index, in other words almost all of $\hat{L}(\mathbb{Z})$. The situation is practically identical to that of the one-point orbit consisting of the origin - the stabiliser has no finite dimensional unitary representation from which a unitary representation of $\hat{P}(\mathbb{Z})$ with a finite helicity spectrum (subject to a finite set of subsidiary conditions, see remarks in section 3 on massless representations of $\hat{P}(\mathbb{R})$) can be induced. So rational orbits are to be discarded.

We turn now to irrational orbits. Consider in particular the orbit through a point with momentum along an axis, say $P = (P_0, P_0, 0, 0)$, with $P_0$ irrational. For the action of $\hat{L}(\mathbb{Z})$ on $T$, its stabiliser is

$$\Sigma_P = \{ \Lambda \in \hat{L}(\mathbb{Z}) \mid (\Lambda P)_\mu = P_\mu \mod \mathbb{Z} \},$$

i.e., the subgroup satisfying the conditions

$$(\Lambda_{00} - \Lambda_{01})P_0 = P_0 + N_0,$$
$$(\Lambda_{10} - \Lambda_{11})P_0 = P_0 + N_1,$$
$$(\Lambda_{20} + \Lambda_{21})P_0 = N_2,$$
$$(\Lambda_{30} + \Lambda_{31})P_0 = N_3,$$

for arbitrary integers $N_0, N_1, N_2, N_3$. These conditions can hold for irrational $P_0$ only if all the $N$ vanish. It follows that the stabiliser of $P = (P_0, P_0, 0, 0)$, $P_0$ irrational, for the $\hat{L}(\mathbb{Z})$ action on $C_T$ coincides with its stabiliser for the $\hat{L}(\mathbb{Z})$ action on the whole of $C_+$. The latter can be found exactly as in the case of $\hat{L}(\mathbb{R})$, leading to the conclusion that $\Sigma_P$ is the subgroup of $SL(2, \mathbb{Z}[i])$ consisting of upper triangular matrices

$$s(\zeta, Z) = \begin{pmatrix} \zeta & \zeta^{-1}Z \\ 0 & \zeta^{-1} \end{pmatrix}, \zeta, \zeta^{-1}, Z \in \mathbb{Z}[i].$$
The only elements of $\mathbb{Z}[i]$ with inverses in $\mathbb{Z}[i]$, namely its units, being $\zeta = \pm 1, \pm i$, the subgroup of diagonal matrices

$$s(\zeta, 0) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

is the cyclic group $\mathbb{Z}_4$. The subgroup of elements $s(1, Z)$ is the planar lattice $\mathbb{Z}^2$ whose points are identified with $(\text{Re } Z, \text{Im } Z)$. The multiplication in $\Sigma_P$ is given by

$$s(\zeta_1, Z_1)s(\zeta_2, Z_2) = s(\zeta_1 \zeta_2, Z_1 + \zeta_1^2 Z_2)$$

confirming that $\Sigma_P$ is indeed the discrete euclidean group $E(2, \mathbb{Z}) = \mathbb{Z}_4 \times \mathbb{Z}^2 = SO(2, \mathbb{Z}) \times \mathbb{Z}^2$, with $\zeta \in \mathbb{Z}_4$ acting on $\mathbb{Z}^2$ by $(\text{Re } Z, \text{Im } Z) \rightarrow (\text{Re } \zeta^2 Z, \text{Im } \zeta^2 Z)$ - the action by $\zeta^2$ is a reminder that $\hat{L}(\mathbb{Z})$ covers $L(\mathbb{Z})$ twice, again as in the continuum theory.

The reasoning given above covers all orbits containing a point of the form $P = (P_0, P_0, 0, 0)$, $(P_0, 0, P_0, 0)$ or $(P_0, 0, 0, P_0)$ with $P_0$ an arbitrary irrational number between $-1$ and $1$. Given such an orbit $O$, which is a discrete set, we can define (square summable) functions on it with values in a finite dimensional irreducible representation space $V$ of $\hat{L}(\mathbb{R})$ (and hence, by the density theorem, of $\hat{L}(\mathbb{Z})$) and carry through the Wigner-Mackey construction of physically acceptable unitary representations of $\hat{P}(\mathbb{Z})$. This is made possible precisely because the stabiliser $E(2, \mathbb{Z})$ has finite dimensional unitary representations, those that are trivial on its $\mathbb{Z}_2$ subgroup. However, it is not true in general that light-like momenta along different spatial directions, or along the same direction but are linearly independent over rationals, can be connected by discrete Lorentz transformations.

The actual construction of the unitary representation of $\hat{P}(\mathbb{Z})$ supported on an irrational orbit $O$ having $\Sigma = E(2, \mathbb{Z})$ as stabiliser and ranging over $V$ may, finally, be summarised as follows. Define the momentum space fields as functions $\phi : O \rightarrow V$ satisfying the subsidiary condition

$$\rho(R)\phi(R^{-1}P) = \phi(P)$$

where $\rho$ is the irreducible representation of $\hat{L}(\mathbb{R})$ (and hence, to repeat, of $\hat{L}(\mathbb{Z})$) on $V$ and $R$ is any element of $\mathbb{Z}^2 \subset E(2, \mathbb{Z})$. If $\sigma$ is a section $O \rightarrow \hat{L}(\mathbb{Z})$, such fields form a Hilbert
space $\mathcal{H}'_{\mathcal{O},V}$ with scalar product

$$\langle \phi, \psi \rangle = \sum_{\rho \in \mathcal{O}} \langle \rho(\sigma(P)^{-1})\phi(P), \rho(\sigma(P)^{-1})\psi(P) \rangle_V$$

if $\rho$ restricted to $E(2, \mathbb{Z})$ is a unitary character of $SO(2, \mathbb{Z})$, (assuming of course that $\langle \phi|\phi \rangle < \infty$). On $\mathcal{H}'_{\mathcal{O},V}$ we have a unitary representation of $\hat{P}(\mathbb{Z})$ given by the action

$$(U_{\mathcal{O},V}(\Lambda, A)\phi)(P) = \exp(iP_{\mu}A_{\mu})\phi(\Lambda^{-1}P), \quad \Lambda \in \hat{L}(\mathbb{Z}), \ A \in \mathbb{Z}^4.$$
its argument can only transform by the lattice group. The consequences of this inconsistent procedure become manifest in any attempt at its physical interpretation [34].

Next, still on the positive side, it is entirely straightforward to define gauge fields, abelian or nonabelian, on a minkowskian lattice, exactly as in the euclidean case [4], and to make models of massless chiral fermions interacting with them, e.g., a lattice version of the unbroken standard model. On the negative side, however, breaking gauge and chiral symmetry on the lattice, so as to generate gauge boson and fermion masses would appear to be a nontrivial task, in view of the exact masslessness of physically acceptable representations of $\hat{P}(\mathbb{Z})$. One possibility is that the generation of vacuum expectation values of scalar fields is a local phenomenon, i.e., that the vacuum becomes degenerate, or that some order parameter becomes nonzero, in the continuum limit. Alternatively, it may be that the breaking of gauge symmetry on the lattice is accompanied by the simultaneous breaking of strict $\hat{P}(\mathbb{Z})$ invariance so as to allow nonzero masses to emerge.

Apart form gauge model building, and even more speculatively, one can wonder whether minkowskian lattices can serve as a general model for space-time at a sufficiently small length scale (the Planck length?). Natural deviations from special relativistic symmetries will become operative at that scale, on account of gravitational effects - one cannot then talk meaningfully of discrete Poincaré invariance in isolation. Nonzero masses are, perhaps, a general-relativistic artifact and the “real world” a coarse-grained version of an underlying discrete space-time, the coarse-graining being accomplished by gravitational interactions.

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