Topological properties of the Cosmic Microwave Background polarization map

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Rapid progress has been made in observations of the temperature anisotropies of the Cosmic Microwave Background (CMB). These observations have enabled cosmologists to characterize the state of the universe at recombination, and observational efforts are now being directed towards obtaining the polarization map of the CMB. Here we draw an analogy between the CMB polarization map and nematic liquid crystals, pointing out that the two have similar defects. Making use of known results in the theory of defect formation we predict the statistical distribution of defects in the CMB polarization map and provide a novel tool for probing the CMB at large angular scales.

A polarization map of the CMB will consist of headless vectors laid on the surface of last scattering. One can view the polarization map as rod-shaped molecules laid out on a two-dimensional sphere ($S^2$), the sky. This picture immediately connects the CMB polarization map with nematics where the molecules are also rod-shaped. The topological properties of nematics have been studied for many decades [1]. The very topological properties that are relevant to nematics will also apply to the CMB polarization map, as first examined in Ref. [2].

At first sight, it might appear that the analogy is doomed because neighboring molecules in nematics interact, while CMB polarization vectors do not. However, the processes at last scattering that produce the polarization are described by interacting field theory and so neighboring polarization vectors, just as neighboring liquid crystal molecules, tend to align. The CMB polarization map is just like a snapshot of a thin film of nematic liquid crystal.

The mathematical description of the CMB polarization map in a local patch of the sky is in terms of a $2 \times 2$ symmetric traceless matrix which we denote by $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$

We define a normalized “order parameter” as: $\mathbf{M} = \frac{1}{2} \mathbf{P}/\sqrt{Q^2+U^2}$. The matrix $\mathbf{M}$ can also be written in terms of a unit vector $\mathbf{n}$ as:

$$M_{ij} = \hat{n}_i \hat{n}_j - \frac{1}{2} \delta_{ij}$$

So the space of all matrices $\mathbf{M}$ is given by the space ($S^1$) of all two-dimensional unit vectors but with the $Z_2$ identification under $\mathbf{n} \rightarrow -\mathbf{n}$. The normalized field $\mathbf{M}$ therefore defines a unit vector field with no arrows, also known as a “line field.”

Since $\mathbf{M}$ lies in $S^1/Z_2$, a space that has incontractable loops, the line field will have singularities, also called “defects.” Some of these singularities are shown in Fig. 1.

![FIG. 1: Fundamental defects of charge ±1/2. Each dash represents the linear polarization of the CMBR at that point. The x marks the position of the singularity where the linear polarization vanishes for topological reasons.](image1)

![FIG. 2: Defects of charge ±1. These can be constructed by combining two fundamental defects shown in Fig. 1.](image2)

The radial and tangential singularities commonly discussed in the CMB literature and shown in Fig. 2 can be constructed by combining two of the fundamental singularities.

In Euclidean space, each singularity has a degree $W$ which is given by:

$$W_\Gamma = \frac{1}{2\pi} \oint_{\Gamma} dx^i \ Tr(\mathbf{M} \partial_i \mathbf{M})$$

where $\Gamma$ is a contour around the location of the singular-
ity, and
\[ \epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  
(4)

In terms of the functions \( Q \) and \( U \), we simply have
\[ W_T = \frac{1}{2\pi} \int_{\Gamma} dx^i \partial_i \alpha \]  
(5)

where \( \alpha = (1/2)\tan^{-1}(U/Q) \). The fundamental defects shown in Fig. 1 have degrees \( \pm 1/2 \), while the defects in Fig. 2 have degrees \( \pm 1 \). In general we can take \( \Gamma \) to be any closed contour on which the polarization is everywhere non-vanishing, and then Eq. (5) tells us the number of singularities within that contour. (We assume that all the singularities are isolated.)

Equation (5) can also be used on the surface of a sphere provided we first parallel transport \( M \) along a geodesic from points on \( \Gamma \) to a point within \( \Gamma \). In other words, to compute the degree, we have to parallel transport \( M \) to a common tangent space.

The Poincare-Hopf theorem \cite{1} tells us that the algebraic sum of degrees of all singularities, also called the “index” of a vector field, on a compact manifold is equal to the Euler number of the manifold. By explicit evaluation we can show that the Poincare-Hopf result also applies to a line field on \( S^2 \). Hence the CMB polarization map must necessarily have index 2 since the Euler number of \( S^2 \) is 2.

The distribution of defects produced at a phase transition has received much attention over the past few decades \cite{2}. A key result is that if we consider the length, \( L \), of \( \Gamma \) to be much larger than the correlation scale, \( \xi \), of the matrix \( M \), then we can write \cite{2}
\[ W_T,\text{rms} \equiv \langle W_T^2 \rangle^{1/2} = c \left( \frac{L}{\xi} \right)^\beta \]  
(6)

where \( c \) is a system-dependent constant but the exponent \( \beta \) is universal with the value \( \beta = 1/2 \). The reasoning leading to this scaling is very simple. \( W_T \) is the sum of random fluctuations in \( \alpha \) and the number of elements in the sum goes like \( L/\xi \). Therefore the root mean square value of \( W_T \) is proportional to \( \sqrt{L/\xi} \).

Satellite observations of the CMB \cite{4,5,6} show that the temperature anisotropy correlation function, \( C(\theta) \), falls off very steeply with increasing angular separation, and becomes very small above an angular scale \( \theta = \chi \sim \) a few degrees. If this lack of large-scale temperature correlations is indeed real, and not some misunderstood systematic effect, it is natural to expect that one should find a corroborating lack of correlation for polarization on all angular scales larger than \( \chi \). After all, thermal and polarization fluctuations are expected to arise from similar processes during recombination. Hence, on angular scales larger than \( \chi \), the matrix \( M \) would be essentially uncorrelated. This tells us that the areal density of defects on the sky will be approximately one per \( \chi^2 \sim 10 \) square degrees. Further, we expect \( \beta = 1/2 \) on angular scales larger than \( \chi \). If the data shows \( \beta \neq 1/2 \) or exponential decay with \( L \), then that would indicate the presence of large-scale correlations in the CMB polarization. This observation would be equivalent to defect formation with “bias” in condensed matter systems.

If the statistical distribution of the polarization is Gaussian, then all correlation functions of the polarization matrix \( P \) can be calculated in terms of the two point correlation function. However, correlators such as \( \langle W_T^2 \rangle \) that are central to our discussion, cannot be written in terms of two point correlators of \( P \), since they are given by correlators of the normalized matrix \( M \) which involves dividing \( P \) by \( \sqrt{Q^2 + U^2} \). Alternately, \( (W_T^2) \) depends on the two point correlation function of the angle \( \alpha = (1/2)\tan^{-1}(U/Q) \) and this is not given by the two point correlation functions of \( U \) and \( Q \). Hence the information provided by the distribution of defects is complementary to that provided by the two point correlators of the polarization matrix usually discussed.

There is a global constraint on the CMB polarization map, namely that “B-modes” are absent or highly suppressed \cite{4,10,11,12} and the reader may worry that this would affect the distribution of defects. However, the B-mode relates the variation of the polarization intensity, \( \sqrt{Q^2 + U^2} \), to the distribution of the unit vector \( \hat{n} \). The topology, however, does not depend on the polarization intensity, only on the direction \( \hat{n} \). Hence for a fixed distribution of \( \hat{n} \), we would need to find \( \sqrt{Q^2 + U^2} \) such that the B-modes vanish. One can show that even pure E-modes exhibit the defects shown in Figs. 1 and 2 \cite{12}.

As polarized CMB radiation propagates from the last scattering surface to us, intervening effects such as weak lensing and Faraday rotation can affect the polarization map. As long as these effects are continuous, they can only move the locations of the existing defects or, create or destroy defect-antidefect pairs. This is similar in spirit to the coarsening of defects observed in condensed matter systems, though the actual dynamics of polarization defects is likely to be very different.

In conclusion, we have drawn an analogy between the CMB polarization map and nematic liquid crystals, classified the defects in the polarization map, and made a prediction for the scaling of the index of the CMB polarization map. Ongoing observations of the CMB will be able to find the exponent \( \beta \) and compare it to the theoretical prediction of 1/2. Further statistical measures of the distribution of defects can provide additional novel probes of the CMB.

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