Doubly periodic monopoles and $q$-difference modules

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Abstract

An interesting theme in complex differential geometry is to find a correspondence between algebraic objects and differential geometric objects. One of the most attractive is the non-abelian Hodge theory of Simpson. In this paper, pursuing an analogue of the non-abelian Hodge theory in the context of $q$-difference modules, we study Kobayashi-Hitchin correspondences between doubly periodic monopoles and parabolic $q$-difference modules, depending on twistor parameters.

MSC: 53C07, 58E15, 14D21, 81T13

1 Introduction

In [21], we studied Kobayashi-Hitchin correspondences between periodic monopoles and difference modules with parabolic structure depending on the twistor parameters. It is an interesting variant of Kobayashi-Hitchin correspondences for harmonic bundles pioneered by Corlette [7], Donaldson [10], Hitchin [12] and particularly Simpson [29, 30, 31, 32, 33]. See [21, §1] for more background.

In this paper, as another interesting variant, we shall study Kobayashi-Hitchin correspondences between doubly-periodic monopoles and $q$-difference modules, depending on the twistor parameters.

1.1 Meromorphic doubly periodic monopoles

Let $\Gamma$ be any lattice in $\mathbb{R}^2$. It naturally acts on $\mathbb{R}^2$ by the addition. We obtain the induced action of $\Gamma$ on $\mathbb{R}^2 \times \mathbb{R}^2$. Let $M$ denote the quotient space. It is naturally equipped with the metric $g_M$ induced by the Euclidean metric of $\mathbb{R}^3$. Let $Z$ be a finite subset in $M$.

Let $H$ be a complex vector bundle on $M \setminus Z$ equipped with a Hermitian metric $h$, a unitary connection $\nabla$, and an anti-self-adjoint endomorphism $\phi$ satisfying the Bogomolny equation

$$F(\nabla) = * \nabla \phi.$$ 

Here, $F(\nabla)$ denotes the curvature of $\nabla$, and $*$ denotes the Hodge star operator with respect to $g_M$. Such a tuple $(E, h, \nabla, \phi)$ is called a doubly periodic monopole because it can be regarded as a singular monopole on $\mathbb{R}^3$ with periodicity in two directions. It is called meromorphic in this paper if the following is satisfied:

1. Each point of $Z$ is Dirac type singularity of the monopole.
2. There exists a compact subset $C$ which contains $Z$ such that $F(\nabla)$ is bounded with respect to $h$ and $g_M$ on $M \setminus C$.

1.1.1 Examples

We use the coordinate system $(y_0, y_1, y_2)$ on $\mathbb{R} \times \mathbb{R}^2$. We may regard $\mathbb{R}^2(y_1, y_2)$ as $\mathbb{C}$ by the complex coordinate $z = y_1 + \sqrt{-1}y_2$, and we regard $T^0 := \mathbb{C}/\Gamma$ as an elliptic curve. It is equipped with the Euclidean metric $dz d\bar{z}$. The Riemannian manifold $M$ is naturally identified with the product $\mathbb{R} \times T^0$.

Take a holomorphic line bundle $L_m$ of degree $-m$, i.e., $\int_{T^0} c_1(L_m) = -m$. There exists a Hermitian metric $h_{L_m}$ such that the curvature of the Chern connection $\nabla_{h_{L_m}}$ is equal to $-\frac{2\pi m}{\text{Vol}(T^0)} dz d\bar{z}$. Let $p : \mathbb{R} \times T^0 \rightarrow T^0$ denote the projection. We obtain $(E_m, h_m, \nabla_m)$ as the pull back of $(L_m, h_{L_m}, \nabla_{L_m})$. Set $\phi_m := -\sqrt{-1} \frac{2\pi m}{\text{Vol}(T^0)} y_0$. Then, $(E_m, h_m, \nabla_m, \phi_m)$ is a meromorphic doubly periodic monopole.
Let $\Gamma' \subset \Gamma$ be a sub-lattice such that $|\Gamma/\Gamma'| = k$. We set $T' := \mathbb{C}/\Gamma'$. Let $T' \to T^0$ be the induced covering of degree $k$. Take a holomorphic line bundle $L_m'$ of degree $m$ on $T'$. Let $h_{L_m'}$ and $\nabla_{L_m'}$ be as above. We obtain a monopole $(E_m', h_m', \nabla_{L_m', \phi_m'})$ on $\mathbb{R} \times T'$. Set $\omega = m/k$. By taking the push-forward with respect to the induced covering $\mathbb{R} \times T' \to \mathbb{R} \times T^0$, we obtain a monopole $(E_\omega, h_\omega, \nabla_\omega, \phi_\omega)$ of rank $k$ on $\mathbb{R} \times T^0$.

Let $a = (a_0, a_1, a_2) \in \mathbb{R}^3$. Let $\mathbb{C}e$ be the product line bundle on $\mathcal{M}$ with a global frame $e$. Let $h$ be the metric determined by $h(e, e) = 1$. Let $\nabla_a$ and $\phi_a$ be determined by

$$\nabla_a e = e\sqrt{-1}(a_1 dy_1 + a_2 dy_2), \quad \phi_a = \sqrt{-1}a_0.$$ 

Then, $(\mathbb{C} e, h, \nabla_a, \phi_a)$ is a meromorphic monopole on $\mathcal{M}$.

### 1.2 Parabolic $q$-difference modules

#### 1.2.1 $q$-difference modules

Let $q \in \mathbb{C}^*$. Let $\Phi^*$ be the automorphism of the algebra $\mathbb{C}[y, y^{-1}]$ determined by $\Phi^*(f) = f(qy)$. A $q$-difference $\mathbb{C}[y, y^{-1}]$-module $V$ equipped with a $\mathbb{C}$-linear automorphism $\Phi^*$ such that $\Phi^*(fs) = \Phi^*(f)\Phi^*(s)$ for any $f \in \mathbb{C}[y, y^{-1}]$ and $s \in V$.

We set $\mathcal{A}_q := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}[y, y^{-1}]\langle \Phi^* \rangle^n$. It is a non-commutative algebra endowed with the multiplication induced by $(\Phi^*)^ny^k = y^kq^{nk}(\Phi^*)^n$. Then, $q$-difference modules are equivalent to $\mathcal{A}_q$-modules.

**Remark 1.1** The automorphism $\Phi^*$ is extended to automorphisms of $R := \mathbb{C}((y)), \mathbb{C}((y^{-1}))$ and $\mathbb{C}(y)$. The notion of $q$-difference $R$-modules are defined similarly.

In this section, we impose the following condition to $q$-difference $\mathbb{C}[y, y^{-1}]$-modules $V$ unless otherwise specified.

- It is torsion-free as $\mathbb{C}[y, y^{-1}]$-module.
- There exists a free $\mathbb{C}[y, y^{-1}]$-submodule $V \subset V$ of finite rank such that $V \otimes_{\mathbb{C}[y, y^{-1}]\langle \Phi^* \rangle} \mathbb{C}(y) = V \otimes_{\mathbb{C}[y, y^{-1}]\langle \Phi^* \rangle} \mathbb{C}(y)$ and $\mathcal{A}_q \cdot V = V$.

#### 1.2.2 Parabolic $q$-difference $\mathbb{C}[y, y^{-1}]$-modules

We introduce parabolic structure on $q$-difference $\mathbb{C}[y, y^{-1}]$-modules, which consists of good parabolic structure at infinity and parabolic structure at finite place.

**Good parabolic structure at infinity** Let $(\hat{\mathcal{V}}, \Phi^*)$ be a $q$-difference $\mathbb{C}((y))$-module, for which we always assume that $\dim_{\mathbb{C}((y))} \hat{\mathcal{V}} < \infty$. As known classically (see [25, 28, 35]), there exists a slope decomposition of $(\hat{\mathcal{V}}, \Phi^*) = \bigoplus_{\omega \in \mathbb{Q}} (\hat{\mathcal{V}}_\omega, \Phi^*)^\omega$ such that the following holds.

- Let $\omega = \ell/k$, where $\ell \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. Then, there exists a $\mathbb{C}[y]$-lattice $L_\omega \subset \hat{\mathcal{V}}_\omega$ such that $y^\ell(\Phi^*)^kL_\omega = L_\omega$.

Recall that a filtered bundle $\mathcal{P}_s\hat{\mathcal{V}}$ over $\hat{\mathcal{V}}$ means an increasing sequence of $\mathbb{C}[y]$-lattices $\mathcal{P}_s\hat{\mathcal{V}} \subset \hat{\mathcal{V}}$ ($s \in \mathbb{R}$) such that (i) $\mathcal{P}_{a+n}\hat{\mathcal{V}} = y^{-n}\mathcal{P}_a\hat{\mathcal{V}}$ for any $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, (ii) $\mathcal{P}_a\hat{\mathcal{V}} = \bigcap_{b > a} \mathcal{P}_b\hat{\mathcal{V}}$. A filtered bundle $\mathcal{P}_s\hat{\mathcal{V}}$ over $\hat{\mathcal{V}}$ is called good if the following holds.

- The filtration $\mathcal{P}_s\hat{\mathcal{V}}$ is compatible with the slope decomposition, i.e., $\mathcal{P}_s\hat{\mathcal{V}} = \bigoplus \mathcal{P}_s\hat{\mathcal{V}}_\omega$.
- $\Phi^*\mathcal{P}_a(\hat{\mathcal{V}}_\omega) = \mathcal{P}_{a+\omega}(\hat{\mathcal{V}}_\omega)$ holds.

Let $V$ be a $q$-difference $\mathbb{C}[y, y^{-1}]$-module. We set $V_{\bar{0}} := V \otimes \mathbb{C}((y))$ and $V_{\bar{\infty}} := V \otimes \mathbb{C}((y^{-1}))$. Then, a good parabolic structure of $V$ at infinity is defined to be good filtered bundles $\mathcal{P}_sV_{\bar{0}}$ and $\mathcal{P}_sV_{\bar{\infty}}$ over $V_{\bar{0}}$ and $V_{\bar{\infty}}$, respectively.
Parabolic structure at finite place  Set $y_{\alpha} := y - \alpha$ for any $\alpha \in \mathbb{C}^*$. For any subset $S \subset \mathbb{C}^*$, let $\mathbb{C}[y, y^{-1}]^{(*) S}$ denote the localization of $\mathbb{C}[y, y^{-1}]$ with respect to $(y_{\alpha} \mid \alpha \in S)$. For any $\mathbb{C}[y, y^{-1}]$-module $M$, we set $M^{(*) S} := M \otimes_{\mathbb{C}[y, y^{-1}]} \mathbb{C}[y, y^{-1}]^{(*) S}$.

A parabolic structure of $V$ at finite place is the following data:

- A free $\mathbb{C}[y, y^{-1}]$-submodule $V \subset V$ such that $V \otimes_{\mathbb{C}[y, y^{-1}]} \mathbb{C}(y) = V \otimes_{\mathbb{C}[y, y^{-1}]} \mathbb{C}(y)$ and $\mathcal{A}_q \cdot V = V$.
- A finite subset $D \subset \mathbb{C}^*$ such that $V^{(*) D} = (\Phi^*)^{-1}(V)^{(*) D}$ in $V$.
- A sequence $t_{\alpha} = (0 \leq t_{\alpha,0} < t_{\alpha,1} < \cdots < t_{\alpha,m(\alpha) - 1} < 1)$ and a tuple $\mathcal{L}_{\alpha} := (\mathcal{L}_{\alpha,i} \mid i = 1, \ldots, m(\alpha) - 1)$ of $\mathbb{C}[y_{\alpha}]$-lattices $\mathcal{L}_{\alpha,i}$ of $V \otimes_{\mathbb{C}[y, y^{-1}]} \mathbb{C}((y_{\alpha}))$ are attached to each $\alpha \in D$. We formally set $\mathcal{L}_{\alpha,0} := V \otimes \mathbb{C}[y_{\alpha}]$ and $\mathcal{L}_{\alpha,n(\alpha)} := (\Phi^*)^{-1}(V) \otimes \mathbb{C}[y_{\alpha}]$.

If we fix $D$ and $t_{\alpha}$ ($\alpha \in D$), it is called a parabolic structure at $(D, (t_{\alpha})_{\alpha \in D})$ or just $(t_{\alpha})_{\alpha \in D}$.

Parabolic $q$-difference modules and stability condition  A parabolic $q$-difference $\mathbb{C}[y, y^{-1}]$-module $V_*$ consists of a $q$-difference $\mathbb{C}[y, y^{-1}]$-module $V$ with a good parabolic structure at infinity $(P_\nu V_y, \mathcal{P}_\nu V_\infty)$ and a parabolic structure at finite place $(D, (t_{\alpha}, \mathcal{L}_{\alpha})_{\alpha \in D})$.

We define the parabolic degree of $V_*$. Note that we obtain a parabolic vector bundle $P_\nu V$ on $\mathbb{P}^1$ from $V$ and the filtered bundles $(P_\nu V_y, \mathcal{P}_\nu V_\infty)$. For each $\alpha \in D$ and $i = 0, \ldots, m(\alpha)$, we define

$$\deg(\mathcal{L}_{\alpha,i+1}, \mathcal{L}_{\alpha,i}) := \text{length}(\mathcal{L}_{\alpha,i+1}/\mathcal{L}_{\alpha,i} \cap \mathcal{L}_{\alpha,i}) - \text{length}(\mathcal{L}_{\alpha,i}/\mathcal{L}_{\alpha,i+1} \cap \mathcal{L}_{\alpha,i}).$$

Then, we set

$$\deg(V_*) := \deg(P_\nu V) + \sum_{\alpha \in D} \sum_{i=0}^{m(\alpha)} (1 - t_{\alpha,i}) \deg(\mathcal{L}_{\alpha,i+1}, \mathcal{L}_{\alpha,i}) - \sum_{\omega \in \mathbb{Q}} \frac{\omega}{2} \left( \dim_{\mathbb{C}[y^{-1}]}((V_\infty)_\omega) + \dim_{\mathbb{C}[y]}((V_0)_\omega) \right). \quad (1)$$

The stability condition is defined in a standard way. Let $\tilde{V}$ be a $q$-difference $\mathbb{C}(y)$-subspace of $\tilde{V} := V \otimes \mathbb{C}(y)$. We obtain a $q$-difference $\mathbb{C}[y, y^{-1}]$-submodule $V' := \tilde{V} \cap V$, which is equipped with the induced parabolic structure. We say that $V_*$ is stable (resp. semistable) if

$$\frac{\deg(V'_*)}{\dim_{\mathbb{C}(y)}(V')} < \frac{\deg(V_*)}{\dim_{\mathbb{C}(y)}(V)} \quad \text{(resp.} \quad \frac{\deg(V'_*)}{\dim_{\mathbb{C}(y)}(V')} \leq \frac{\deg(V_*)}{\dim_{\mathbb{C}(y)}(V)} \text{)}$$

for any $q$-difference $\mathbb{C}(y)$-subspace $0 \neq \tilde{V}' \subseteq \tilde{V}$. The polystability condition is also defined in the standard way.

1.3 Geometrization of parabolic $q$-difference $\mathbb{C}[y, y^{-1}]$-modules  It is the purpose in this paper to study the relationship between meromorphic doubly periodic monopoles and stable parabolic $q$-difference $\mathbb{C}[y, y^{-1}]$-modules of degree 0. As a bridge to connect them, let us explain geometric objects directly corresponding to parabolic $q$-difference $\mathbb{C}[y, y^{-1}]$-modules. We have already used a similar geometrization in the context of difference modules in [21].

1.3.1 Spaces  We consider the action of $\mathbb{Z}$ on $\mathcal{M}_q^\text{cov} := \mathbb{C}^* \times \mathbb{R}$ and $\mathcal{M}_q^\text{cov}$ := $\mathbb{P}^1 \times \mathbb{R}$ determined by $n \cdot (y, t) = (q^n y, t + n)$. We set $\mathcal{M}_q := \mathcal{M}_q^\text{cov}/\mathbb{Z}$ and $\overline{\mathcal{M}}_q := \mathcal{M}_q^\text{cov}/\mathbb{Z}$. For $\nu = 0, \infty$, we set $H^\text{cov}_{q,\nu} := \{ \nu \} \times \mathbb{R}$ and $H_q := H^\text{cov}_{q,0}/\mathbb{Z}$. We put $H^\text{cov}_q := H^\text{cov}_{q,0} \cup H^\text{cov}_{q,\infty}$ and $H_{\infty} := H_{q,0} \cup H_{q,\infty}$.

Let $\mathcal{O}_{\overline{\mathcal{M}}_q^\text{cov}}(\mathcal{H}^\text{cov}_q)$ denote the sheaf of algebras on $\overline{\mathcal{M}}_q^\text{cov}$ obtained as the pull back of $\mathcal{O}_{\mathbb{P}^1}(\{0, \infty\})$ via the natural projection $\overline{\mathcal{M}}_q \to \mathbb{P}^1$. It is naturally equivariant with respect to the $\mathbb{Z}$-action. Therefore, we obtain a sheaf of algebras $\mathcal{O}_{\overline{\mathcal{M}}_q^\text{cov}}(\mathcal{H}^\text{cov}_q)$ on $\overline{\mathcal{M}}_q$. For any subset $\mathcal{U} \subset \overline{\mathcal{M}}_q$, the restriction of $\mathcal{O}_{\overline{\mathcal{M}}_q^\text{cov}}(\mathcal{H}^\text{cov}_q)$ to $\mathcal{U}$ is denoted by $\mathcal{O}_{\mathcal{U}}(\mathcal{H}^\text{cov}_q)$. We use a similar notation for the restriction of $\mathcal{O}_{\overline{\mathcal{M}}_q^\text{cov}}(\mathcal{H}^\text{cov}_q)$ to subsets of $\overline{\mathcal{M}}_q^\text{cov}$.  

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1.3.2 Locally free sheaves with Dirac type singularity

Let $Z \subset \mathcal{M}_q$ be a finite subset. Let $Z^{\text{cov}}$ denote the subset of $\mathcal{M}_q^{\text{cov}}$ obtained as the pull back of $Z$. Let $\mathfrak{M}$ be a locally free $\mathcal{O}_{\mathcal{M}_q\setminus Z}(\ast H_q)$-module. Let $\mathfrak{M}^{\text{cov}}$ denote the $Z$-equivariant locally free $\mathcal{O}_{\mathcal{M}_q\setminus Z^{\text{cov}}}(H_q^{\text{cov}})$-module obtained as the pull back of $\mathfrak{M}$.

Let $U$ be an open subset in $\mathbb{P}^1$. If $U \times \{t\} \subset \mathcal{M}_q \setminus Z^{\text{cov}}$, the restriction $\mathfrak{M}^{\text{cov}}_{U \times \{t\}}$ is naturally a locally free $\mathcal{O}_U(\ast(U \cap \{0, \infty\}))$-module. Note that any local sections of $\mathfrak{M}^{\text{cov}}_{U \times \{t\}}$ are locally constant in the $t$-direction. Therefore, if $(U \times \{t_1, t_2\}) \cap Z^{\text{cov}} = \emptyset$, then there exists a naturally induced isomorphism $\mathfrak{M}^{\text{cov}}_{U \times \{t_1\}} \cong \mathfrak{M}^{\text{cov}}_{U \times \{t_2\}}$. We call it the scattering map by following [3].

Let $(\alpha_0, t_0) \in Z^{\text{cov}}$. Take a neighbourhood $U$ of $\alpha_0$ in $\mathbb{C}^*$ and small $\epsilon > 0$. Set $U^* := U \setminus \{\alpha_0\}$. We have the isomorphism of $\mathcal{O}_{U^*, -\text{modules}}$ $\mathfrak{M}^{\text{cov}}_{U^* \times \{t_0 - \epsilon\}} \cong \mathfrak{M}^{\text{cov}}_{U^* \times \{t_0 + \epsilon\}}$ induced by the scattering map. We say that $(\alpha_0, t_0)$ is Dirac type singularity if it is extended to an isomorphism of $\mathcal{O}_{U}(\ast\alpha_0)-\text{modules}$ $\mathfrak{M}^{\text{cov}}_{U \times \{t_0 - \epsilon\}}(\ast\alpha_0) \cong \mathfrak{M}^{\text{cov}}_{U \times \{t_0 + \epsilon\}}(\ast\alpha_0)$.

If any $(\alpha_0, t_0) \in Z^{\text{cov}}$ is Dirac type singularity, we say that $\mathfrak{M}$ is a locally free $\mathcal{O}_{\mathcal{M}_q \setminus Z}(\ast H_q)$-module with Dirac type singularity.

1.3.3 $q$-difference $\mathbb{C}[y, y^{-1}]$-modules with parabolic structure at finite place

Let $\mathfrak{M}$ be a locally free $\mathcal{O}_{\mathcal{M}_q \setminus Z}(\ast H_q)$-module with Dirac type singularity. Let $D$ denote the image of $Z^{\text{cov}} \cap (\mathbb{P}^1 \times [0, 1])$ by the projection $\mathbb{P}^1 \times [0, 1] \rightarrow \mathbb{P}^1$. For $\alpha \in D$, the sequence $0 \leq t_{\alpha, 0} < t_{\alpha, 1} < \cdots < t_{\alpha, m(\alpha)} < 1$ is determined by $\{(\alpha, t_{\alpha, i})\} = Z^{\text{cov}} \cap \{(\alpha) \times [0, 1]\}$. Let us observe that $\mathfrak{M}$ naturally induces a $q$-difference $\mathbb{C}[y, y^{-1}]$-module with parabolic structure at $(D, (t_{\alpha, i})_{\alpha \in D})$.

Take a sufficiently small $\epsilon > 0$ such that $(\mathbb{P}^1 \times [-\epsilon, 0]) \cap Z^{\text{cov}} = \emptyset$. The restriction of $\mathfrak{M}^{\text{cov}}$ to $\mathbb{P}^1 \times (-\epsilon)$ induces a locally free $\mathcal{O}_{\mathbb{P}^1}(\ast\{0, \infty\})$-module $\mathfrak{M}^{\text{cov}}_{-\epsilon}$. We obtain a $\mathbb{C}[y, y^{-1}]$-module $V := H^0(\mathbb{P}^1, \mathfrak{M}^{\text{cov}}_{-\epsilon})$. It is independent of a choice of $\epsilon$ up to canonical isomorphisms. Similarly, the restriction of $\mathfrak{M}^{\text{cov}}$ to $\mathbb{P}^1 \times \{1 - \epsilon\}$ induces a locally free $\mathcal{O}_{\mathbb{P}^1}(\ast\{0, \infty\})$-module $\mathfrak{M}^{\text{cov}}_{1 - \epsilon}$. We obtain a $\mathbb{C}[y, y^{-1}]$-module $V' := H^0(\mathbb{P}^1, \mathfrak{M}^{\text{cov}}_{1 - \epsilon})$.

Let $\Phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the morphism defined by $\Phi(y) = qy$. We have the natural isomorphism $\Phi^* : \mathfrak{M}^{\text{cov}}_{1 - \epsilon} \cong \mathfrak{M}^{\text{cov}}_{-\epsilon}$, which induces a $\mathbb{C}$-linear isomorphism

$$\Phi^* : V' \cong V$$

such that $\Phi^*(fS) = \Phi^*(f)\Phi^*(S)$ for any $f \in \mathbb{C}[y, y^{-1}]$ and $s \in V'$. The scattering map induces an isomorphism

$$V(\ast D) \cong V'(\ast D).$$

The isomorphisms [2] and [3] induce a $\mathbb{C}$-linear automorphism $\Phi^*$ on $\tilde{V} := V \otimes \mathbb{C}(y)$ such that $\Phi^*(fS) = \Phi^*(f)\Phi^*(S)$ for any $f \in \mathbb{C}(y)$ and $s \in \tilde{V}$. We set $V := \mathcal{A}_q \cdot V$ in $\tilde{V}$.

For $\alpha \in D$ and $t_{\alpha, i}$ ($1 \leq i \leq m(\alpha)$), we obtain the $\mathbb{C}[y_{\alpha}]-$lattices $\mathcal{L}_{\alpha, i}$ of $V \otimes \mathbb{C}(y_{\alpha})$ induced by the formal completion of the stalks of $\mathfrak{M}$ at $(\alpha, t_{\alpha, i} - \epsilon)$ for any sufficiently small $\epsilon > 0$. They induce a parabolic structure $\{(t_{\alpha, i}, \mathcal{L}_{\alpha})\}_{\alpha \in D}$ of $V$ at $(D, (t_{\alpha, i})_{\alpha \in D})$. The following lemma is easy to observe.

**Lemma 1.2** The above construction induces an equivalence between the following objects:

1. Locally free $\mathcal{O}_{\mathcal{M}_q \setminus Z}(\ast H_q)$-modules with Dirac type singularity.
2. $q$-difference $\mathbb{C}[y, y^{-1}]$-modules with parabolic structure at $(D, (t_{\alpha, i})_{\alpha \in D})$.

1.3.4 Good filtered bundles over equivariant $\mathcal{O}_{\mathcal{M}_q^{\text{cov}}}(\ast H_q^{\text{cov}})$-modules

We set $y_0 := y$ and $y_\infty := y^{-1}$. We also set $q_0 := q$ and $q_\infty := q^{-1}$. For $\nu = 0, \infty$, let $\mathcal{O}_{\mathcal{M}_q^{\text{cov}}}(\ast H_q^{\text{cov}})$ denote the sheaf of locally constant $\mathbb{C}(y_{\nu})$-valued functions on $H_q^{\text{cov}}$. It is $\mathbb{Z}$-equivariant by the action $n^*(f)(y_{\nu}) = f(q_{\nu}^n y_{\nu})$.

For any $\mathbb{Z}$-equivariant locally free $\mathcal{O}_{\mathcal{M}_q^{\text{cov}}}(\ast H_q^{\text{cov}})$-module $\mathfrak{M}^{\text{cov}}$, let $\mathfrak{M}^{\text{cov}}_t$ denote the restriction of $\mathfrak{M}^{\text{cov}}$ to $t \in \mathbb{R}$ which is naturally a $\mathbb{C}(y_{\nu})$-vector space. For any $t_1, t_2 \in \mathbb{R}$, we have the isomorphism called the scattering map:

$$\mathfrak{M}^{\text{cov}}_{|t_1} \cong \mathfrak{M}^{\text{cov}}_{|t_2}.$$
We define the stability condition in the standard way. The following is easy to see by the construction.

Therefore, $\hat{\mathcal{V}}_{\nu}^{\text{cov}}$ is naturally a $q$-difference $\mathbb{C}((y_\nu))$-module. It is easy to observe that this procedure induces an equivalence between $\mathbb{Z}$-equivariant locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}},Z}(\ast H_q)$-modules and $q_\nu$-difference $\mathbb{C}((y_\nu))$-modules.

Let $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$ be a $\mathbb{Z}$-equivariant locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}},Z}(\ast H_q)$-module. There exists a decomposition $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}} = \bigoplus_{\nu \in \mathbb{Q}} \hat{\mathcal{V}}_{\nu}^{\text{cov}}$ corresponding to the slope decomposition of the $q$-difference $\mathbb{C}((y_\nu))$-module $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$. A good filtered bundle $\mathcal{P}_t \hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$ over $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$ is defined to be a family of filtered bundles $(\mathcal{P}_t \hat{\mathcal{V}}_{\nu}^{\text{cov}}) | t \in \mathbb{R}$ such that the following holds.

- The isomorphism $(\mathcal{P}_t \hat{\mathcal{V}}_{\nu}^{\text{cov}})$ induces $\mathcal{P}_a(\hat{\mathcal{V}}_{\nu}^{\text{cov}}) \simeq \mathcal{P}_{a+\nu(t_2-t_1)}(\hat{\mathcal{V}}_{\nu}^{\text{cov}})$ for any $a \in \mathbb{R}$ and $t_1, t_2 \in \mathbb{R}$.
- The isomorphism $(\mathcal{P}_t \hat{\mathcal{V}}_{\nu}^{\text{cov}})$ induces $\mathcal{P}_a(\hat{\mathcal{V}}_{\nu}^{\text{cov}}) \simeq \mathcal{P}_{a+\nu(t+1)}(\hat{\mathcal{V}}_{\nu}^{\text{cov}})$ for any $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

Clearly, good filtered bundles over a $\mathbb{Z}$-equivariant $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}},Z}(\ast H_q)$-module $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$ are equivalent to good filtered bundles over $q_\nu$-difference $\mathbb{C}((y_\nu))$-module $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$.

### 1.3.5 Good parabolic structure at infinity

Let $\mathfrak{V}$ be a locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}}} \times Z(\ast H_q)$-module with Dirac type singularity. Let $\mathcal{V}_{\nu_0}^{\text{cov}}$ be the $\mathbb{Z}$-equivariant $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}}} \times Z^{\text{cov}}(\ast H_q)$-module obtained as the pull back of $\mathfrak{V}$. For any $t \in \mathbb{R}$ and $\nu = 0, \infty$, we obtain the formal completions $\hat{\mathcal{V}}_{\nu,t}^{\text{cov}}$ of $\hat{\mathcal{V}}_{(t,\{t\})}^{\text{cov}}$ at $(\nu, t)$. They induce $\mathbb{Z}$-equivariant locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}}}(\ast H_q)$-modules $\hat{\mathcal{V}}_{\nu,t}^{\text{cov}}$ ($\nu = 0, \infty$).

Let $\mathfrak{V}$ be a $q$-difference $\mathbb{C}((y,y^{-1}))$-module with a parabolic structure $(t_\alpha, \mathfrak{L}_\alpha)_{\alpha \in \mathcal{D}}$ at finite place corresponding to $\mathfrak{V}$ as in 1.3.2. Note that $\mathfrak{V}_{\nu_0}$ is naturally identified with $\hat{\mathcal{V}}_{\nu_0}^{\text{cov}}$. Under the identification, good filtered bundles $\mathfrak{P}_t \hat{\mathfrak{V}}_{\nu_0}^{\text{cov}} = (\mathfrak{P}_t \hat{\mathfrak{V}}_{\nu_0}^{\text{cov}}) | t \in \mathbb{R}$ over $\hat{\mathfrak{V}}_{\nu_0}^{\text{cov}}$ are equivalent to good filtered bundles $\mathfrak{P}_t \mathfrak{V}_{\nu,t}^{\text{cov}}$ over $\mathfrak{V}_{\nu,t}^{\text{cov}}$.

### 1.3.6 Geometrization of parabolic $q$-difference $\mathbb{C}((y,y^{-1}))$-modules

By the considerations in 1.3.3 and 1.3.5 we obtain the following.

**Proposition 1.3** The following objects are equivalent.

- $q$-difference $\mathbb{C}((y,y^{-1}))$-modules with a good parabolic structure at infinity and a parabolic structure at $(D, (t_\alpha)_{\alpha \in \mathcal{D}})$.
- Good filtered bundles with Dirac type singularity over $\mathbb{M}_q \times Z(\ast H_q)$, i.e., locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}}} \times Z(\ast H_q)$-modules $\mathfrak{V}$ with Dirac type singularity enhanced by good filtered bundles $\mathfrak{P}_t \hat{\mathcal{V}}_{\nu,t}^{\text{cov}}$ over $\hat{\mathcal{V}}_{\nu,t}^{\text{cov}}$.

Here, $Z$ and $(D, (t_\alpha)_{\alpha \in \mathcal{D}})$ are related as in 1.3.3.

Let $\mathfrak{V}$ be a locally free $\mathcal{O}_{\hat{\mathcal{M}}_q^{\text{cov}}} \times Z(\ast H_q)$-module with Dirac type singularity enhanced with good filtered bundles $\mathfrak{P}_t \hat{\mathfrak{V}}_{\nu,t}^{\text{cov}} = (\mathfrak{P}_t \hat{\mathfrak{V}}_{\nu,t}^{\text{cov}}) | t \in \mathbb{R}$ ($\nu = 0, \infty$). Let $\nu^{\text{cov}} : \hat{\mathcal{M}}_q^{\text{cov}} \to \mathbb{R}$ denote the projection. For any $t \in [0, 1]$, let $\mathfrak{P}_t \mathfrak{V}_{\nu,t}^{\text{cov}}$ denote the $\mathcal{O}_{\mathbb{P}^1}(\ast \{0, \infty\})$-module obtained as the restriction of $\mathfrak{V}_{\nu,t}^{\text{cov}}$ to $\mathbb{P}^1 \times \{t\}$. We obtain a filtered bundle $\mathfrak{P}_t \mathfrak{V}_{\nu,t}^{\text{cov}} = (\mathfrak{P}_t \mathfrak{V}_{\nu,t}^{\text{cov}}) | t \in \mathbb{R}$ and $(\mathfrak{P}_t \mathfrak{V}_{\nu,t}^{\text{cov}})_{\nu = 0, \infty}$.

We define the stability condition in the standard way. The following is easy to see by the construction.

**Lemma 1.4** The degree is preserved by the equivalence in Proposition 1.3. Therefore, the stability condition is also preserved by the equivalence.
1.4 From monopoles to q-difference modules

Let us explain how a meromorphic monopole on $\mathcal{M}$ induces geometric objects as in 1.3, and hence q-difference modules. More detailed explanation will be used later.

1.4.1 Space

Take $\mu_i \in \mathbb{C}$ ($i = 1, 2$) such that (i) $\mu_1$ and $\mu_2$ are linearly independent over $\mathbb{R}$, (ii) $\text{Im}(\mu_2/\mu_1) > 0$. Let $\Gamma$ denote the lattice of $\mathbb{C}$ generated by $\mu_1$ and $\mu_2$. Let $\text{Vol}(\Gamma)$ denote the volume of the quotient $\mathbb{C}/\Gamma$ with respect to the volume form $\sqrt{2} dz \, d\overline{z}$.

We set $X := \mathbb{C}_z \times \mathbb{C}_w$ with the Euclidean metric $dz \, d\overline{z} + dw \, d\overline{w}$. Let us consider the action of $\mathbb{R}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ on $X$ by $e_0(z, w) = (z, w + 1)$ and $e_i(z, w) = (z + \mu_i, w)$ ($i = 1, 2$).

Let $\mathcal{M}^{\text{cov}}$ be the quotient space of $X$ by the action of $\mathbb{R}e_0 \oplus \mathbb{Z}e_1$. It is equipped with an induced action of $\mathbb{Z}e_2$. The quotient space $\mathcal{M}^{\text{cov}}/\mathbb{Z}e_2$ is naturally identified with $\mathcal{M}$.

1.4.2 Mini-complex coordinate system

Let $\lambda$ be a complex number such that $\lambda \neq \pm \sqrt{-1}\mu_1|\mu_1|^{-1}$. As in Lemma 3.1 below, there exist $s_1 \in \mathbb{R}$ and $g_1 \in \mathbb{C}$ with $|g_1| = 1$ such that

$$-\lambda p_1 + s_1 = g_1(\mu_1 + \lambda s_1) \neq 0.$$ 

If $|\lambda| \neq 1$, there are two such choices. If $|\lambda| = 1$ and $\lambda \neq \pm \sqrt{-1}\mu_1|\mu_1|^{-1}$ there is a unique choice. We consider the complex coordinate system $(u, v)$ given as follows:

$$u = \frac{1}{1 - g_1 \lambda} (z + \lambda^2 \sigma + \lambda (w - w)), \quad v = \frac{1}{1 - g_1 \lambda} (-g_1 z - \lambda \sigma + w - \lambda g_1 w).$$

Note that

$$e_0(u, v) = (u, v) + (0, 1), \quad (e_1 + s_1 e_0)(u, v) = (u, v) + (\mu_1 + \lambda s_1, 0).$$

We define

$$U := \exp \left( \frac{2\pi \sqrt{-1}}{\mu_1 + \lambda s_1} u \right), \quad t := \text{Im}(v).$$

Then, $(U, t)$ induces an isomorphism $\mathcal{M}^{\text{cov}} \simeq \mathbb{C}^* \times \mathbb{R}_t$. We set

$$q^\lambda := \exp \left( \frac{2\pi \sqrt{-1}}{\mu_1 + \lambda s_1} p_2 \right), \quad t^\lambda := - \frac{\text{Vol} \Gamma}{\text{Re}(g_1 \mu_1)}.$$

The following holds:

$$e_2(U, t) = (q^\lambda U, t + t^\lambda).$$

Note that $t^\lambda$ is non-zero, but that $t^\lambda$ is not necessarily positive. We also remark that $|q^\lambda| = 1$ if and only if $|\lambda| = 1$.

When we consider the above coordinate system $(U, t)$, $\mathcal{M}^{\text{cov}}$ and $\mathcal{M}$ are also denoted by $\mathcal{M}^{\lambda, \text{cov}}$ and $\mathcal{M}^{\lambda}$, respectively.

1.4.3 Compactification

We set $\overline{\mathcal{M}}^{\text{cov}} := \mathbb{P}^1_\mathbb{R} \times \mathbb{R}_t$, which we regard a partial compactification of $\mathcal{M}^{\lambda, \text{cov}} \simeq \mathbb{C}^*_y \times \mathbb{R}_t$. It is equipped with the naturally induced $\mathbb{Z}e_2$-action. We put $\overline{\mathcal{M}}^{\lambda} := \overline{\mathcal{M}}^{\lambda, \text{cov}}/\mathbb{Z}e_2$ and $\mathcal{M}^{\lambda} := \mathcal{M}^{\lambda, \text{cov}}/\mathbb{Z}e_2$. Set $H^{\lambda, \text{cov}} := \overline{\mathcal{M}}^{\lambda, \text{cov}} \setminus \mathcal{M}^{\lambda, \text{cov}}$. We obtain $H^{\lambda} \subset \overline{\mathcal{M}}^{\lambda}$ as the quotient of $H^{\lambda, \text{cov}}$ by the $\mathbb{Z}e_2$-action.

We have the $\mathbb{Z}$-equivariant isomorphisms $\overline{\mathcal{M}}^{\lambda, \text{cov}} \simeq \overline{\mathcal{M}}^{\lambda, \text{cov}}$, $\mathcal{M}^{\lambda, \text{cov}} \simeq \mathcal{M}^{\lambda, \text{cov}}$, and $H^{\lambda, \text{cov}} \simeq H^{\lambda, \text{cov}}$ given by $U = y$ and $t = t^\lambda t$. It induces an isomorphism $\overline{\mathcal{M}}^{\lambda} \simeq \overline{\mathcal{M}}^{\lambda, \text{cov}}$, $\mathcal{M}^{\lambda} \simeq \mathcal{M}^{\lambda, \text{cov}}$, and $H^{\lambda} \simeq H^{\lambda}$. 

6
1.4.4 Mini-holomorphic bundles associated to monopoles

Let us explain how a meromorphic monopole induces \( \mathcal{O}_{\mathcal{M} \setminus \mathcal{Z}}((sH^\lambda))^\bullet \)-modules with Dirac type singularity enhanced with filtered bundles at infinity. It depends on the choice of \((\lambda, e_1, s_1)\).

Let \((E, h, \nabla, \phi)\) be a meromorphic monopole on \(\mathcal{M} \setminus \mathcal{Z}\). We have the naturally defined operators \(\partial_E, \nabla\) and \(\partial_s\), on \(E\) such that \([\partial_E, \partial_s] = 0\), which is a consequence of the Bogomolny equation. (Note that the vector fields \(\partial_E\) and \(\partial_s\) are not necessarily orthogonal.)

Let \(Z^\text{cov}\) denote the subset of \(\mathcal{M}^\text{cov}\) obtained as the pull back of \(Z\). Let \(E^\text{cov}\) denote the vector bundle on \(\mathcal{M}^\text{cov} \setminus Z^\text{cov}\) obtained as the pull back of \(E\). It is equipped with the induced operators \(\partial_{E^\text{cov}}, \nabla\) and \(\partial_{E^\text{cov}}, \nabla\).

We obtain a \(\mathcal{Z}\)-equivariant locally free \(\mathcal{O}_{\mathcal{M}^\text{cov} \setminus Z^\text{cov}}\)-module \(E^\text{cov}\) as the sheaf of \(\mathcal{C}^\infty\)-sections \(s\) of \(E^\text{cov}\) such that \(\partial_{E^\text{cov}}, s = \partial_{E^\text{cov}}, s = 0\). Each point of \(Z^\text{cov}\) is Dirac type singularity of \(E^\text{cov}\) under the assumption that each point of \(Z\) is Dirac type singularity of the monopole \((E, h, \nabla, \phi)\).

For \(t \in \mathcal{R}\), let \(E^\text{cov}(t)\) denote the restriction of \(E^\text{cov}\) to \((\mathcal{C}_t^\infty \times \{t\}) \setminus Z^\text{cov}\). Together with the operator \(\partial_{E^\text{cov}}, \nabla\), it is naturally a holomorphic vector bundle. The sheaf \(E^\text{cov}(t)\) of holomorphic sections of \(E^\text{cov}(t)\) is identified with the restriction of \(E^\text{cov}\) to \((\mathcal{C}_t^\infty \times \{t\}) \setminus Z^\text{cov}\).

Let \(h(t) = h(t)\) be the restriction of the metric \(h\) to \(E^\text{cov}(t)\). Because the monopole is meromorphic, it turns out that \((E^\text{cov}(t), \nabla_{E^\text{cov}}), h(t)\) is acceptable around \(U = 0, \infty\), i.e., the curvature of the Chern connection is bounded with respect to \(h(t)\) and the metric \(|\{u|1\}^{2/\pm} d\theta d\varphi\). (See Proposition 3.15, Lemma 3.19 and Corollary 6.3.) Therefore, \(E^\text{cov}(t)\) is extended to a locally free \(\mathcal{O}_{\mathcal{M}^\text{cov} \setminus \mathcal{Z}^\text{cov}}\)-modules \(E^\text{cov}(t)\). Moreover, we obtain filtered bundles \(\mathcal{P}, E^\text{cov}(t)|_\mathcal{P}(\nu = 0, \infty)\) over the formal completions \(E^\text{cov}(t)|_\mathcal{P}\) by considering the growth orders of the norms of sections with respect to \(h(t)\).

It is easy to see that the scattering map induces an isomorphism \(\mathcal{P}^\text{cov}(t_1) \simeq \mathcal{P}^\text{cov}(t_2)\) for \(t_1, t_2\) on neighbourhoods of \(U = 0, \infty\). Therefore, \(\mathcal{P}^\text{cov}(t)\) induces a \(\mathcal{Z}\)-equivariant locally free \(\mathcal{O}_{\mathcal{M}^\text{cov} \setminus \mathcal{Z}^\text{cov}}(sH^\lambda)^\bullet\)-module \(\mathcal{P}^\text{cov}\) with Dirac type singularity. We obtain a locally free \(\mathcal{O}_{\mathcal{M}^\text{cov} \setminus \mathcal{Z}^\text{cov}}(sH^\lambda)^\bullet\)-module \(\mathcal{P}\) with Dirac type singularity as the descent of \(\mathcal{P}^\text{cov}\). Moreover, the families of filtrations \(\mathcal{P}, E^\text{cov}(t)|_\mathcal{P}(\nu = 0, \infty)\) are good filtered bundles (Theorem 7.3). In this way, a meromorphic monopole on \(\mathcal{M} \setminus \mathcal{Z}\) induces a good filtered bundle with Dirac type singularity over \((\mathcal{M}^\text{cov}; H^\lambda, \mathcal{Z})\), and hence a parabolic \(q^\lambda\)-difference \(\mathcal{C}[y, y^{-1}]\)-module.

Then, the following theorem is the main result of this paper.

**Theorem 1.5 (Theorem 9.2)** The above construction induces an equivalence between meromorphic doubly periodic monopoles and polystable parabolic \(q^\lambda\)-difference modules of degree 0.

1.5 Filtered objects on elliptic curves

As the "Betti" side, we shall also give a minor complement on the parabolic version of the Riemann-Hilbert correspondence of \(q\)-difference modules \(|q| \neq 1\) and its relation with the Kobayashi-Hitchin correspondence in [110].

1.5.1 Riemann-Hilbert correspondence for \(q\)-difference modules with \(|q| \neq 1\)

Suppose that \(|q| \neq 1\). The Riemann-Hilbert correspondence for germs of analytic \(q\)-difference modules was established by van der Put and Reversat [21], and Ramis, Sauloy and Zhang [20]. The global Riemann-Hilbert correspondence for \(q\)-difference modules is due to Kontsevich and Soibelman.

Set \(q^\mathcal{Z} := \{q^n | n \in \mathbb{Z}\}\). Let \(\Phi : \mathbb{C}^\ast \rightarrow \mathbb{C}^\ast\) be defined by \(\Phi(y) = qy\). It induces a \(q^\mathcal{Z}\)-action on \(\mathbb{C}^\ast\). We set \(T := \mathbb{C}^\ast / q^\mathcal{Z}\). Clearly, \(q^\mathcal{Z}\)-equivariant coherent \(\mathcal{O}_{\mathcal{C}^\ast}\)-modules are equivalent to coherent \(\mathcal{O}_T\)-modules.

For a locally free \(\mathcal{O}_T\)-module \(E\), an anti-Harder-Narasimhan filtration of \(E\) is a filtration \(\mathcal{F}\) indexed by \((\mathbb{Q} \cup \{\infty\}, \leq\) such that (i) \(E_\mu := \text{Gr}_{\mu}^\mathcal{F}(E)\) is semistable with \(\text{deg}(E_\mu) / \text{rank}(E_\mu) = \mu\) if \(\mu \neq \infty\), (ii) \(\text{Gr}_{\infty}^\mathcal{F}(E)\) is torsion. If \(\text{Gr}_{s}^\mathcal{F}(E) = 0\) then we call it an anti-Harder-Narasimhan filtration indexed by \((\mathbb{Q}, \leq\).

According to [21] and [20], \(q^\mathcal{Z}\)-equivariant locally free \(\mathcal{O}_{\mathbb{C}^\ast}(s)\)-modules are equivalent to locally free \(\mathcal{O}_T\)-modules equipped with an anti-Harder-Narasimhan filtration indexed by \((\mathbb{Q}, \leq\). For \((E, \mathcal{F})\), let \(k_0(E, \mathcal{F})\) denote the corresponding \(q^\mathcal{Z}\)-equivariant locally free \(\mathcal{O}_{\mathbb{C}^\ast}(s)\)-module. It is equipped with the filtration induced by \(\mathcal{F}\) so that \(\text{Gr}_{s}^\mathcal{F}(k_0(E, \mathcal{F}))\) has pure slope \(\rho(q)\mu\), where \(\rho(q) \in \{\pm 1\}\) is the signature of \(\log |q| \neq 0\). Similarly, \(q^\mathcal{Z}\)-equivariant locally free \(\mathcal{O}_{\mathbb{C}^\ast-1}(s)\)-modules are also equivalent to locally free \(\mathcal{O}_T\)-modules equipped with
an anti-Harder-Narasimhan filtration indexed by $(\mathbb{Q}, \leq)$. For $(\mathbb{E}, \mathfrak{g})$, we have the $q^2$-equivariant locally free $\mathcal{O}_{C_y^{-1}}(\mu - \infty)$-module $K_{\mu} = (\mathbb{E}, \mathfrak{g})$. For the induced filtration, $\text{Gr}_{\mu}^0(K_{\mu}(\mathbb{E}, \mathfrak{g}))$ has pure slope $-\mu$. According to Kontsevich-Soibelman, $q$-difference $\mathbb{Q}[y, y^{-1}]$-modules are equivalent to locally free $\mathcal{O}_T$-modules $\mathbb{E}$ equipped with two anti-Harder-Narasimhan filtrations $\mathfrak{g}_{\pm}$ indexed by $(\mathbb{Q} \cup \infty, \leq)$.

### 1.5.2 Filtered objects on elliptic curves

The Riemann-Hilbert correspondence for $q$-difference modules in §1.5.1 is enhanced to the correspondence for filtered objects. Let us explain the filtered counterpart on the side of elliptic curves.

Let $\mathcal{D} \subset T$ be a finite subset. Let $\mathbb{E}$ be a locally free $\mathcal{O}_T(\mathcal{D})$-module. For each $P \in T$, let $\mathbb{E}_P$ denote the formal completion of the stalk of the $\mathbb{E}$ at $P$. A $q$-difference parabolic structure on $\mathbb{E}$ consists of the following data:

- A finite sequence $s_P = (s_{P,1} < s_{P,2} < \cdots < s_{P,m(P)})$ in $\mathbb{R}$ for each $P \in \mathcal{D}$.
- We formally set $s_{P,0} := -\infty$ and $s_{P,m(P)+1} := \infty$.
- A tuple of lattices $\mathcal{K}_P = (\mathcal{K}_{P,i} | i = 0, \ldots, m(P))$ of $\mathbb{E}_P$.

Note that we obtain the lattice $\mathbb{E}_- \subset \mathbb{E}$ determined by $\mathcal{K}_{P,0}$ ($P \in \mathcal{D}$) and the lattice $\mathbb{E}_+ \subset \mathbb{E}$ determined by $\mathcal{K}_{P,m(P)}$ ($P \in \mathcal{D}$).

- Let $\mathfrak{g}_{\pm}$ be anti-Harder-Narasimhan filtrations of $\mathbb{E}_\pm$ indexed by $(\mathbb{Q}, \leq)$.
- Filtrations $\mathcal{F}_\pm$ on $\text{Gr}^\pm_{\mathfrak{g}_\pm}(\mathbb{E}_\pm)$ ($\mu \in \mathbb{Q}$) indexed by $(\mathbb{R}, \leq)$ such that $\mathbb{E}_{a,\mu,\pm} := \text{Gr}^a_{\mu} \text{Gr}^\pm_{\mathfrak{g}_\pm}(\mathbb{E}_\pm)$ are also semistable with $\text{deg}(\mathbb{E}_{a,\mu,\pm}) = \frac{a}{\mu}$.

When we fix $(s_P)_{P \in \mathcal{D}}$, it is called $q$-difference parabolic structure at $(s_P)_{P \in \mathcal{D}}$.

We define the degree of $\mathbb{E}_*$ as $(\mathbb{E}, (s_P, \mathcal{K}_P)_{P \in \mathcal{D}}, (\mathfrak{g}_{\pm}, \mathcal{F}_\pm))$ as follows:

\[
\text{deg}(\mathbb{E}_*) := - \sum_{P \in \mathcal{D}} \sum_{i=1}^{m(P)} s_{P,i} \text{deg}(\mathcal{K}_{P,i}, \mathcal{K}_{P,i-1}) - \sum_{\mu \in \mathbb{Q}} \sum_{b \in \mathbb{R}} b \text{rank} \text{Gr}^\pm_{\mathfrak{g}_\pm} \text{Gr}^\pm_{\mathfrak{g}_\pm}(\mathbb{E}_-) - \sum_{\mu \in \mathbb{Q}} \sum_{b \in \mathbb{R}} b \text{rank} \text{Gr}^\pm_{\mathfrak{g}_\pm} \text{Gr}^\pm_{\mathfrak{g}_\pm}(\mathbb{E}_+). \tag{7}
\]

By using the degree, we define the stability, semistability and polystability conditions for filtered objects in the standard ways.

### Rescaling of $q$-difference parabolic structure

There is a rescaling of $q$-difference parabolic structure. For $t > 0$, we obtain a sequence $s^{(t)}_{P,i} := (ts_{P,i})$. We set $\mathcal{K}^{(t)}_P := \mathcal{K}_P$ and $\mathfrak{g}^{(t)}_{\pm} := \mathfrak{g}_{\pm}$. We also obtain filtrations $\mathcal{F}^{(t)}_\pm$ by $(\mathcal{F}^{(t)}_\pm)_{a} \text{Gr}^\pm_{\mathfrak{g}_{\pm}}(\mathbb{E}_\pm) := (\mathcal{F}_\pm)_a \text{Gr}^\pm_{\mathfrak{g}_{\pm}}(\mathbb{E}_\pm)$. We set

\[
\mathcal{H}^{(t)}(\mathbb{E}_*) := (\mathbb{E}, (s^{(t)}_P, \mathcal{K}^{(t)}_P)_{P \in \mathcal{D}}, (\mathfrak{g}^{(t)}_{\pm}, \mathcal{F}^{(t)}_\pm)).
\]

In the case $t < 0$, we set $s^{(t)}_{P,i} := ts_{P,m(P)-i+1}$, and $s^{(t)}_P := (s^{(t)}_{P,i})$. We set $\mathcal{K}^{(t)}_{P,i} := \mathcal{K}_{P,m(P)-i}$ and $\mathcal{K}^{(t)}_P := (\mathcal{K}^{(t)}_{P,i})$. We also set $$(\mathfrak{g}^{(t)}_{\pm})_{\mu} := \mathfrak{g}_{\pm,\mu}$$ and $$(\mathcal{F}^{(t)}_\pm)_{a} := \mathcal{F}_{\pm,a}$. Then, we define

\[
\mathcal{H}^{(t)}(\mathbb{E}_*) := (\mathbb{E}_*, (s^{(t)}_P, \mathcal{K}^{(t)}_P)_{P \in \mathcal{D}}, (\mathfrak{g}^{(t)}_{\pm}, \mathcal{F}^{(t)}_\pm)).
\]

It is easy to see $\text{deg}(\mathcal{H}^{(t)}(\mathbb{E}_*)) = |t| \text{deg}(\mathbb{E}_*)$. 

8
1.5.3 Equivalence

The natural projection $\mathcal{M}_q^{\text{cov}} \to \mathbb{C}^*$ induces $p : \mathcal{M}_q \to T$. Let $f : \mathcal{M}_q^{\text{cov}} \to \mathbb{R}$ be defined by

$$f(y, t) := t - \frac{\log |y|}{\log |q|}.$$  

It induces the map $f : \mathcal{M}_q \to \mathbb{R}$. Let $Z \subset \mathcal{M}_q$ be a finite subset. We set $D := p(Z)$. For each $P \in D$, we obtain

$$s_P = (s_{P, 1} < s_{P, 2} < \cdots < s_{P, m(P)}) := f(p^{-1}(P) \cap Z). \quad (8)$$

Let $\mathfrak{Y}$ be a locally free $\mathcal{O}_{\mathbb{P}^1 \setminus Z}(H_q)$-module with Dirac type singularity enhanced by good filtered bundles $\mathcal{P}, \hat{\mathfrak{Y}}_p (\nu = 0, \infty)$. Due to the scattering map, the restriction $\mathfrak{Y}|_{\mathcal{M}_q \setminus p^{-1}(D)}$ induces a locally free $\mathcal{O}_{T, \mathfrak{p}}$-module $E'$. For $P \in D$, we take $(\alpha_P, t_P) \in \mathcal{M}^{\text{cov}} \setminus Z^{\text{cov}}$ which is mapped to $P$. Take $U_P$ be a small neighbourhood of $\alpha_P$ in $\mathbb{C}^*$. Set $U_P := U_P \setminus \{\alpha_P\}$. There exists a natural isomorphism $E'_{|U_P \times \{t_P\}} \cong \mathfrak{Y}^{\text{cov}}_{|U_P \times \{t_P\}}$. By gluing $E'$ and $(\mathfrak{Y}^{\text{cov}}_{|U_P \times \{t_P\}})(\ast \alpha_P) (P \in D)$, we obtain a locally free $\mathcal{O}_T(\ast D)$-module $E$. It is independent of a choice of $(\alpha_P, t_P)$.

For $P \in D$, choose $\alpha_P \in \mathbb{C}^*$ which is mapped to $P$ by the projection $\mathbb{C}^* \to T$. We set

$$t_{P,i} = s_{P,i} + \frac{\log |\alpha_P|}{\log |q|}.$$  

Then, $Z^{\text{cov}} \cap \{\alpha_P \times \mathbb{R}\} = \{\alpha_P, t_{P,i}\}_{i = 1, \ldots, m(P)}$ holds. We formally set $t_{P,0} := -\infty$ and $t_{P,m(P)+1} := \infty$. We choose $t_{P,i} < t_{P,i+1} < t_{P,i+1}$ for $i = 0, \ldots, m(P)$. Let $K_{P,i} (i = 0, \ldots, m(P))$ denote the formal completion of $\mathfrak{Y}^{\text{cov}}$ at $(\alpha_P, t_{P,i})$. They induce lattices of $E_{|\hat{\mathfrak{Y}}}$.

We obtain a locally free $\mathcal{O}_T$-submodule $E_{\pm} \subset E$ determined by the lattices $K_{P,0} (P \in D)$. Similarly, we obtain a locally free $\mathcal{O}_T$-submodule $E_+$ of $E$ determined by the lattices $K_{P,m(P)} (P \in D)$. Note that we have the $q$-difference $\mathbb{C}[y, y^{-1}]$-module $V$ with a parabolic structure at finite place corresponding to $\mathfrak{Y}$. \[\bullet\] If $\log |q| > 0$, let $\tilde{\mathfrak{Y}}_{-}$ be the anti-Harder-Narasimhan filtration indexed by $(\mathfrak{Y}, \leq)$ on $E_-$ corresponding to the germ of $V$ at $y = 0$, and let $\tilde{\mathfrak{Y}}_{+}$ be the anti-Harder-Narasimhan filtration indexed by $(\mathfrak{Y}, \leq)$ on $E_+$ corresponding to the germ of $V$ at $y = \infty$. \[\bullet\] If $\log |q| < 0$, we replace $y = 0$ and $y = \infty$.

Moreover, good filtered bundles over $V|_{\hat{\mathfrak{Y}}}$ induce filtrations on $\text{Gr}^{\tilde{\mathfrak{Y}}_{\pm}}(E_{\pm})$ as in \ref{10.2}. (See \ref{10.5} for the relation with the growth order of the norms.) In this way, good filtered bundles with Dirac type singularity on $(\mathcal{M}_q; H_q, Z)$ induces locally free $\mathcal{O}_T(\ast D)$-modules with $q$-difference parabolic structure. The following is easy to see.

**Proposition 1.6** The above procedure induces an equivalence between good filtered bundles with Dirac type singularity on $(\mathcal{M}_q; H_q, Z)$ and locally free $\mathcal{O}_T(\ast D)$-modules with $q$-difference parabolic structure at $(s_P)_{P \in D}$. Here, $Z$ and $(s_P)_{P \in D}$ are related as in \ref{3}. Moreover, it preserves the degree.

1.5.4 Filtered objects associated to meromorphic monopoles

Let $(E, h, \nabla, \phi)$ be a meromorphic monopole on $\mathcal{M} \setminus Z$. We fix $\lambda \in \mathbb{C}$ such that $|\lambda| \neq 1$. Take $(\mathbf{e}_1, \mathbf{s}_1)$ as in \ref{1.4.2}. Let $q^\lambda(\mathbf{e}_1, \mathbf{s}_1)$ and $t^\lambda(\mathbf{e}_1, \mathbf{s}_1)$ denote $q^\lambda$ and $t^\lambda$ in \ref{6} to emphasize the dependence on $(\mathbf{e}_1, \mathbf{s}_1)$. Then, we have the associated parabolic $q^\lambda(\mathbf{e}_1, \mathbf{s}_1)$-difference module, and hence the associated filtered object $E_{(\mathbf{e}_1, \mathbf{s}_1)}$, on the elliptic curve $T^\lambda(\mathbf{e}_1, \mathbf{s}_1) = \mathbb{C}^*/q^\lambda(\mathbf{e}_1, \mathbf{s}_1)^{\mathbb{Z}}$. It is easy to observe that $T^\lambda(\mathbf{e}_1, \mathbf{s}_1)$ is independent of $(\mathbf{e}_1, \mathbf{s}_1)$ by the construction. Moreover, we obtain the following.

**Theorem 1.7** $H(t^\lambda(\mathbf{e}_1, \mathbf{s}_1))E_{(\mathbf{e}_1, \mathbf{s}_1)}$ are independent of the choice of $(\mathbf{e}_1, \mathbf{s}_1)$. 


1.6 Acknowledgement

I owe much to Carlos Simpson whose ideas on the Kobayashi-Hitchin correspondence are fundamental in this study. I have been stimulated by the works of Maxim Kontsevich and Yan Soibelman on q-difference modules. I am clearly influenced by the works of Benois Charbonneau and Jacques Hurtubise [3] and Sergey Cherkis and Anton Kapustin [4][5]. I am grateful to Claude Sabbah for his kindness and discussions on many occasions. A part of this study was done during my visits at the Tata Institute of Fundamental Research and the International Center for Theoretical Sciences. I appreciate Indranil Biswas for his excellent hospitality. I thank Yoshifumi Tsuchimoto and Akira Ishii for their constant encouragement. I thank Indranil Biswas, Sergey Cherkis, Jacques Hurtubise, Ko-ki Ito, Hisashi Kasuya, Maxim Kontsevich, Masa-Hiko Saito, Yota Shamoto, Carlos Simpson, and Masaki Yoshino for discussions.

I am partially supported by the Grant-in-Aid for Scientific Research (S) (No. 17H06127), the Grant-in-Aid for Scientific Research (S) (No. 16H06335), and the Grant-in-Aid for Scientific Research (C) (No. 15K04843), Japan Society for the Promotion of Science.

2 Good filtered formal q-difference modules

2.1 Formal q-difference modules

We review a classification of formal q-difference modules to prepare notations. See [24][25][27][28]. Some statements will be proved though they are standard and well known. It is just to explain that the statements are valid even in the case where q is a root of unity.

2.1.1 Preliminary

Take any non-zero complex number q. Set $q^\mathbb{Z} := \{q^n | n \in \mathbb{Z}\}$, which is a subgroup of $\mathbb{C}$. If q is not a root of 1, then $q^\mathbb{Z}$ is naturally isomorphic to $\mathbb{Z}$. If q is a primitive k-th root of 1, then $q^\mathbb{Z} = \{\mu \in \mathbb{C}^* | \mu^k = 1\}$. We fix $a \in \mathbb{C}$ such that $\exp(a) = q$, and we put $q_m := \exp(a/m)$ for any positive integer m.

We set $K := \mathbb{C}[q]$ and $R := \mathbb{C}[q]/\mathbb{C}$ where q is a variable. We fix m-th roots $y_m$ of q for any positive integers m such that $(y_m)^m = y_m$ for any $(m, n) \in \mathbb{Z}_{>0}^2$. We set $K_m := \mathbb{C}[q]/\mathbb{C}[q]$ and $R_m := \mathbb{C}[q]/\mathbb{C}$. Let $\Phi^* be the automorphisms of $K_m$ determined by $\Phi^*(f)(y_m) = f(q_my_m)$.

A q-m-difference $K_m$-module is a finite dimensional $K_m$-vector space $V$ equipped with a $\mathbb{C}$-linear isomorphism $\Phi^* : V \rightarrow V$ such that $\Phi^*(f) = \Phi^*(f)\Phi^*(s)$ for any $f \in K_m$ and $s \in V$.

A morphism of $q_m$-difference $K_m$-modules $g : (V_1, \Phi^*) \rightarrow (V_2, \Phi^*)$ is defined to be a morphism of $K_m$-vector spaces $g : V_1 \rightarrow V_2$ such that $g \circ \Phi^* = \Phi^* \circ g$.

Let $\text{Diff}_m(K, q)$ be the category of $q_m$-difference $K_m$-modules. If $m = 1$, it is also denoted by $\text{Diff}(K, q)$.

Let $(V_1, \Phi^*_1) \in \text{Diff}_m(K, q)$. The operators $\Phi^*_s$ on $V_1 \oplus V_2$ and $\Phi^*_1 \oplus V_2$ are defined by $\Phi^*_s(v_1 \oplus v_2) = \Phi^*(v_1) \oplus \Phi^*(v_2)$ and $\Phi^*_1(v_1 \oplus v_2) = \Phi^*(v_1) \oplus \Phi^*(v_2)$. Thus, we obtain the direct sum and the tensor product on $\text{Diff}_m(K, q)$. For $(V, \Phi^*) \in \text{Diff}_m(K, q)$, let $V^\vee := \text{Hom}_K(V, K)$. We define the operator $\Phi^*$ on $V^\vee$ by $\Phi^*(f)(v) := f(\Phi^*(v))$. We set $(V, \Phi^*)^\vee := (V^\vee, \Phi^*)$.

2.1.2 Pull back and push-forward

Let $(V, \Phi^*) \in \text{Diff}_m(K, q)$. For any $n \in \mathbb{Z}_{>0}$, we define a $\mathbb{C}$-automorphism $\Phi^*_n$ on $V \otimes_{K_m} K_{mn}$ by $\Phi^*_n(s \otimes g) = \Phi^*(s) \otimes \Phi^*(g)$. In this way, we obtain a $q_{mn}$-difference $K_{mn}$-module $(V \otimes_{K_m} K_{mn}, \Phi^*)$. It induces a functor $(p_{mn, m})^* : \text{Diff}_m(K, q) \rightarrow \text{Diff}_{mn}(K, q)$.

Let $(V, \Phi^*) \in \text{Diff}_{mn}(K, q)$ for $n, m \in \mathbb{Z}_{>0}$. We may naturally regard $V$ as a $q_m$-difference $K_m$-module. Thus, we obtain a functor $(p_{mn, m})_* : \text{Diff}_m(K, q) \rightarrow \text{Diff}_m(K, q)$.

For any $(V, \Phi^*) \in \text{Diff}_m(K, q)$, there exists a natural isomorphism $(p_{mn, m})_*((p_{mn, m})^*(V, \Phi^*)) \simeq (V, \Phi^*) \otimes (p_{mn, m})_*((K_{mn}, \Phi^*))$.

Let $\text{Gal}(nm, m)$ denote the Galois group of $K_{nm}/K_m$, which is naturally identified with $\{\mu \in \mathbb{C}^* | \mu^m = 1\}$ by the action $(\mu \cdot f)(y_{nm}) = f(\mu y_{nm})$. Note that $\Phi^*(\mu \cdot f) = \mu \cdot \Phi^*(f)$. 

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Let \((\mathcal{V}, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\). We set \(\mu^*(\mathcal{V}) := \mathcal{C}\)-vector space. Any element \(v \in \mathcal{V}\) is denoted by \(\mu^*(v)\) when we regard it as an element of \(\mu^*(\mathcal{V})\). We regard \(\mu^*\mathcal{V}\) as a \(\mathcal{K}_{nm}\)-vector space by \(f \cdot \mu^*(v) := \mu^*((\mu^{-1}f)v)\). Note that for any \((\mathcal{V}, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\), there exists a natural isomorphism

\[
(\mathcal{P}_{nm} \ast (\mathcal{P}_{nm})_*) (\mathcal{V}, \Phi^*) \cong \bigoplus_{\mu \in \text{Gal}(nm,m)} \mu^*(\mathcal{V}, \Phi^*).
\]

A \(\mathcal{K}_{nm}\)-vector space \(\mathcal{V}\) is called \(\text{Gal}(nm, m)\)-equivariant when a homomorphism \(\text{Gal}(nm, m) \to \text{Aut}_C(\mathcal{V})\) is given such that \(\mu \cdot (fv) = (\mu \cdot f) \cdot (\mu \cdot v)\) for any \(\mu \in \text{Gal}(nm, m)\), \(f \in \mathcal{K}_m\) and \(v \in \mathcal{V}\).

An object \((\mathcal{V}, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\) is called \(\text{Gal}(nm, m)\)-equivariant when \(\mathcal{V}\) is a finite dimensional \(\text{Gal}(nm, m)\)-equivariant \(\mathcal{K}_{nm}\)-vector space such that \(\Phi^* \circ (\mu \cdot v) = \mu \cdot \Phi^*(v)\) for any \(\mu \in \text{Gal}(nm, m)\) and \(v \in \mathcal{V}\).

For any \((\mathcal{V}, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\), \((\mathcal{P}_{nm} \ast (\mathcal{P}_{nm})_*) (\mathcal{V}, \Phi^*)\) is naturally \(\text{Gal}(nm, m)\)-equivariant. Conversely, let \((\mathcal{V}_1, \Phi^*)\) be a \(\text{Gal}(nm, m)\)-equivariant object in \(\text{Diff}_{nm}(\mathcal{K}, q)\). We set \(\mathcal{V}_0 := \{v \in \mathcal{V}_1 \mid \mu \cdot v = v \ (\forall \mu \in \text{Gal}(nm, m))\}\). Thus, we obtain \((\mathcal{V}_0, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\), which is called the descent of \((\mathcal{V}_1, \Phi^*)\). Then, \((\mathcal{P}_{nm} \ast (\mathcal{P}_{nm})_*) (\mathcal{V}_0, \Phi^*)\) is \(\text{Gal}(nm, n)\)-equivariantly isomorphic to \((\mathcal{V}_1, \Phi^*)\). In particular, \((\mathcal{V}_0, \Phi^*)\) is isomorphic to the descent of \((\mathcal{P}_{nm} \ast (\mathcal{P}_{nm})_*) (\mathcal{V}_0, \Phi^*)\).

### 2.1.3 A splitting lemma

Let \((\mathcal{V}, \Phi^*) \in \text{Diff}_{nm}(\mathcal{K}, q)\). For any \(\mathcal{R}_m\)-lattice \(\mathcal{L} \subset \mathcal{V}\) such that \(y_m \Phi^*(\mathcal{L}) \subset \mathcal{L}\), we have the induced endomorphism \(\sigma(y_m \Phi^*; \mathcal{L})\). \(\mathcal{L}_0 := \mathcal{L} / y_m \mathcal{L}\) obtained as follows: for any \(s \in \mathcal{L}_0\), we take \(\tilde{s} \in \mathcal{L}\) which induces \(s\), and let \(\sigma(y_m \Phi^*; \mathcal{L})(\tilde{s}) \in \mathcal{L}_0\) denote the element induced by \(y_m \Phi^* (\tilde{s}) \in \mathcal{L}\).

The following lemma is standard.

#### Proposition 2.1

Suppose that there exist an \(\mathcal{R}_m\)-lattice \(\mathcal{L} \subset \mathcal{V}\) and an integer \(\ell\) such that the following holds.

- \(y_m^\ell \Phi^*(\mathcal{L}) \subset \mathcal{L}\) holds. In particular, we obtain the induced endomorphism \(F := \sigma(y_m^\ell \Phi^*; \mathcal{L})\) of \(\mathcal{L}_0\).
- There exists a decomposition \(\mathcal{L}_0 = L_1 \oplus L_2\) such that \(F(L_i) \subset L_i\).
- Let \(Sp(F, L_i)\) be the set of eigenvalues of \(F|_{L_i}\). Then, \((q_m^\ell \cdot Sp(F, L_1)) \cap Sp(F, L_2) = \emptyset\).

Then, there exists a unique decomposition \(\mathcal{L} = L_1 \oplus L_2\) of \(\mathcal{R}_m\)-modules such that (i) \(y_m^\ell \Phi^*(\mathcal{L}) \subset \mathcal{L}_i\), (ii) \(\mathcal{L}_{i0} = L_i\).

#### Proof

We give only an indication. For any ring \(\mathcal{R}\) and a positive integer \(r\), let \(M_r(\mathcal{R})\) denote the space of \(r\)-square matrices with \(\mathcal{R}\)-coefficient. For any ring \(\mathcal{R}\) and positive integers \(r_i (i = 1, 2)\), let \(M_{r_1} \times M_{r_2}(\mathcal{R})\) denote the space of \((r_1 \times r_2)\)-matrices with \(\mathcal{R}\)-coefficient. For a decomposition \(r = r_1 + r_2 (r_i > 0)\), any element \(C\) of \(M_r(\mathcal{R})\) is expressed as

\[
C = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix},
\]

where \(C_{ij} \in M_{r_i}(\mathcal{R})\).

#### Lemma 2.2

Let \(r = r_1 + r_2 (r_i > 0)\) be a decomposition. Let \(A \in y_m^{-\ell} M_r(\mathcal{R}_m)\). We obtain \(A_{ij}\) \((1 \leq i, j \leq 2)\) as above, which have the expansions \(A_{ij} = \sum_{k=-\ell}^{\infty} A_{ij;k} y_m^k\). We assume the following.

- \(A_{ij;-\ell} = 0\) if \(i \neq j\).
- \((y_m^\ell Sp(A_{11;-\ell})) \cap Sp(A_{22;-\ell}) = \emptyset\), where \(Sp(A_{ii;-\ell})\) denote the sets of eigenvalues of \(A_{ii;-\ell}\).

Then, there exists \(G \in GL_r(\mathcal{R}_m)\) such that (i) \(G_{ii}\) are identity matrices in \(M_{r_i}(\mathcal{R}_m)\), (ii) \(G_{ij}|_{0} = 0 (i \neq j)\), (iii) \((G(y_m)^{-1} AG(q_my_m))_{ij} = 0 (i \neq j)\).

#### Proof

Let \(\tilde{A} \in y_m^{-\ell} M_r(\mathcal{R}_m)\) determined by (i) \(\tilde{A}_{ij} = 0 (i \neq j)\), (ii) \(\tilde{A}_{ii} = A_{ii}\). Let \(U\) denote a matrix in \(y_m^{-\ell} M_r(\mathcal{R}_m)\) such that (i) \(U_{ij} = 0 (i \neq j)\), (ii) \(U_{ii;-\ell} = 0\). We consider the following equation for \(G\) and \(U\):

\[
A(y_m) G(q_my_m) = G(y_m) (\tilde{A}(y_m) + U(y_m)).
\]
It is equivalent to the following equations:
\[ A_{12}(y_m)G_{21}(q_m y_m) = U_{11}(y_m), \quad A_{22}(y_m)G_{21}(q_m y_m) - G_{21}(y_m)A_{11}(y_m) + A_{21}(y_m) - G_{21}(y_m)U_{11}(y_m) = 0, \quad (9) \]
\[ A_{21}(y_m)G_{12}(q_m y_m) = U_{22}(y_m), \quad A_{11}(y_m)G_{12}(q_m y_m) - G_{12}(y_m)A_{22}(y_m) + A_{12}(y_m) - G_{12}(y_m)U_{22}(y_m) = 0. \quad (10) \]
From (9), we obtain the following equation for \( G_{21} \):
\[ A_{22}(y_m)G_{21}(q_m y_m) - G_{21}(y_m)A_{11}(y_m) + A_{21}(y_m) - G_{21}(y_m)A_{12}(y_m)G_{21}(q_m y_m) = 0. \]
It is equivalent to the following equations for \( G_{21;k} \) \( (k \in \mathbb{Z}_{\geq 0}) \).
\[ A_{22;\ell} G_{21;k} q_m^k - G_{21;k} A_{11;\ell} + \sum_{i+j=k-\ell \atop 0 \leq j < k} A_{22;i} G_{21;j} q_m^j + \sum_{i+j=k-\ell \atop 0 \leq j < k} G_{21;i} A_{11;j} G_{21;p} q_m^p = 0. \quad (11) \]
For \( k = 0 \), we have a solution \( G_{21;0} = 0 \). For \( k \geq 1 \), we can determine \( G_{21;k} \) in an inductive way by using (11). We obtain \( U_{11} \) from (9). Similarly, we obtain \( G_{12} \) and \( U_{22} \) from (10).

The following lemma is also standard and easy to see by using the power series expansions.

**Lemma 2.3** Let \((r_1, r_2) \in \mathbb{Z}_+^2 \) and \( A_i = \sum_{k \geq -\ell} A_{i;k} q_m^k \in y_m^0 M_{r_i}(\mathcal{R}_m) \). Assume the following.

- \((q^\ell \text{Sp}(A_{1;\ell})) \cap \text{Sp}(A_{2;\ell}) = \emptyset \), where \( \text{Sp}(A_{i;\ell}) \) denote the sets of the eigenvalues of \( A_{i;\ell} \).

Let \( H \in M_{r_1, r_2}(\mathcal{R}_m) \) such that \( A_2(y_m)H(q_m y_m) = H(y_m)A_1(y_m) \). Then, \( H = 0 \).

We obtain the claim of Proposition [2.1] from Lemma [2.2] and Lemma [2.3].

2.1.4 Fuchsian q-difference modules

We recall the Fuchsian (regular singular) condition of q-difference modules by following [26].

**Definition 2.4** A q\( _m \)-difference \( \mathbb{K}_m \)-module \((V, \Phi^*) \) is called Fuchsian if there exists an \( \mathcal{R}_m \)-lattice \( \mathcal{L} \) such that \( \Phi^*(\mathcal{L}) = \mathcal{L} \). Let \( \text{Diff}_m(\mathbb{K}, q; 0) \subset \text{Diff}_m(\mathbb{K}, q) \) denote the full subcategory of Fuchsian q\( _m \)-difference \( \mathbb{K}_m \)-modules.

Let \((V, \Phi^*) \in \text{Diff}_m(\mathbb{K}, q; 0) \). Let \( \mathcal{L} \) be an \( \mathcal{R}_m \)-lattice such that \( \Phi^*(\mathcal{L}) = \mathcal{L} \). We obtain the induced automorphism \( \sigma(\Phi^*; \mathcal{L}) \) of \( \mathcal{L}_0 \), and let \( \text{Sp}(\sigma(\Phi^*; \mathcal{L})) \subset \mathbb{C}^* \) denote the set of eigenvalues. Let \( [\text{Sp}(\sigma(\Phi^*; \mathcal{L})]) \) denote the image of \( \text{Sp}(\sigma(\Phi^*; \mathcal{L})) \) by \( \mathbb{C}^* \to \mathbb{C}^*/q_m^\mathbb{Z} \). There exists the decomposition \( \mathcal{L}_0 = \bigoplus_{\sigma \in \text{Sp}(\sigma(\Phi^*; \mathcal{L}))} \mathcal{L}_\sigma \) such that (i) \( \sigma(\Phi^*; \mathcal{L}) \mid_{\mathcal{L}_\sigma} = \mathcal{L}_\sigma \), (ii) the eigenvalues of \( \sigma(\Phi^*; \mathcal{L}) \mid_{\mathcal{L}_\sigma} \) are contained in \( \sigma \). We also obtain the following lemma from Lemma [2.2] and Lemma [2.3].

**Lemma 2.5** There exists a unique decomposition \((\mathcal{L}, \Phi^*) = \bigoplus_{\sigma \in \text{Sp}(\sigma(\Phi^*; \mathcal{L}))} (\mathcal{L}_\sigma, \Phi^*) \) such that \( \mathcal{L}_{\sigma_0} = \mathcal{L}_\sigma \). The set \([\text{Sp}(\sigma(\Phi^*; \mathcal{L})) \) is independent of the choice of an \( \mathcal{R}_m \)-lattice such that \( \Phi^*(\mathcal{L}) = \mathcal{L} \).

We set \([\text{Sp}(\sigma(\Phi^*; V)) := [\text{Sp}(\sigma(\Phi^*; \mathcal{L}))] \) for an \( \mathcal{R}_m \)-lattice \( \mathcal{L} \) such that \( \Phi^*(\mathcal{L}) = \mathcal{L} \), which is independent of the choice of \( \mathcal{L} \).

**Example 2.6** Let \( V \) be a finite dimensional \( \mathbb{C} \)-vector space. For any \( f \in \text{GL}(V) \), we set \( \varsigma_m(V, f) := V \otimes \mathbb{K}_m \), and we define the q\( _m \)-difference operator \( \Phi^* \) on \( \varsigma_m(V, f) \) by \( \Phi^*(s) = f(s) \) for any \( s \in V \). Then, \( (\varsigma_m(V, f), \Phi^*) \in \text{Diff}_m(\mathbb{K}, q; 0) \). We have \([\text{Sp}(\sigma(\Phi^*, \varsigma_m(V, f))) = [\text{Sp}(f)] \).

Similarly, for any \( r \in \mathbb{Z}_{>0} \) and \( A \in \text{GL}_r(\mathbb{C}) \), let \( \varsigma_m(A) \) denote the \( \mathbb{K}_m \)-vector space with a frame \( e = (e_1, \ldots, e_r) \) equipped with the q\( _m \)-difference operator defined by \( \Phi^*(e) = eA \). Then, \( \varsigma_m(A) \in \text{Diff}_m(\mathbb{K}, q; 0) \). We have \([\text{Sp}(\sigma(\Phi^*, \varsigma_m(A))) = [\text{Sp}(A)] \).

**Remark 2.7** Let \( S \subset \mathbb{C}^* \) be any subset such that the induced map \( S \to \mathbb{C}^*/q_m^\mathbb{Z} \) is a bijection. As proved in [25], if \( q_m \) is not a root of 1, for any \((V, \Phi^*) \in \text{Diff}_m(\mathbb{K}, q; 0) \), there exists \( A \in \text{GL}_r(\mathbb{C}) \) such that (i) \((V, \Phi^*) \cong \varsigma_m(A) \), (ii) \( \text{Sp}(A) \subset S \). If \( q_m \) is a root of 1, it does not hold in general.
2.1.5 Formal pure isoclinic \(q\)-difference modules

We recall the notion of pure isoclinic \(q\)-difference modules \([26]\).

**Definition 2.8** Let \(\omega\) be a rational number. We say that \((\mathcal{V}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q)\) is pure isoclinic of slope \(\omega\) if the following holds.

- Take any \(m_1 \in m\mathbb{Z}_{>0}\) such that \(\omega \in \frac{1}{m_1} \mathbb{Z}\). Then, there exists an \(\mathcal{R}_{m_1}\)-lattice \(\mathcal{L} \subset p^*_m \mathcal{V}\) such that \(y^{m_1}\omega \Phi^*(\mathcal{L}) = \mathcal{L}\).

Let \(\text{Diff}_m(\mathcal{K}, q; \omega) \subset \text{Diff}_m(\mathcal{K}, q)\) denote the full subcategory of pure isoclinic \(q_m\)-difference \(\mathcal{K}_m\)-modules of slope \(\omega\).

**Remark 2.9** Recall that \(|q| > 1\) is assumed in \([26]\). In the case \(|q| < 1\), it seems better to change the signature of the slope in the relation with the analytic classification of \(q\)-difference modules. However, because we also study the case \(|q| = 1\), we do not change the signature.

**Lemma 2.10** Let \((\mathcal{V}_i, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q; \omega_i)\) \((i = 1, 2)\). Let \(f : (\mathcal{V}_1, \Phi^*) \rightarrow (\mathcal{V}_2, \Phi^*)\) be a morphism in \(\text{Diff}_m(\mathcal{K}, q)\). If \(\omega_1 \neq \omega_2\), then \(f = 0\).

**Proof** It follows from Lemma 2.3.

**Lemma 2.11** Let \((\mathcal{V}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q)\). Suppose that there exist a rational number \(\omega\) and a finite family of subobjects \((\mathcal{V}_i, \Phi^*) \subset (\mathcal{V}, \Phi^*)\) in \(\text{Diff}_m(\mathcal{K}, q)\) such that (i) \(\mathcal{V} = \bigoplus_{i=1}^N \mathcal{V}_i\), (ii) \((\mathcal{V}_i, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q; \omega_i)\). Then, \(\mathcal{V} \subset \text{Diff}_m(\mathcal{K}, q; \omega)\).

**Proof** We may assume that \(\omega \in \frac{1}{m} \mathbb{Z}\). There exist \(\mathcal{R}_m\)-lattices \(\mathcal{L}_i \subset \mathcal{V}_i\) such that \(y^{m}\omega \Phi^*(\mathcal{L}_i) = \mathcal{L}_i\). We put \(\mathcal{L} := \bigoplus \mathcal{L}_i\), which is an \(\mathcal{R}_m\)-lattice of \(\mathcal{V}\). We have \(y^{m}\omega \Phi^*(\mathcal{L}) = \mathcal{L}\).

**Lemma 2.12** Let \(\omega = \ell/k \in \mathbb{Q}\), where \(\ell \in \mathbb{Z}\) and \(k \in \mathbb{Z}_{>0}\). Then, \((\mathcal{V}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q)\) is pure isoclinic of slope \(\omega\) if and only if there exists a lattice \(\mathcal{L} \subset \mathcal{K}\) such that \((\Phi^*)^k \mathcal{L} = y^\ell \mathcal{L}\).

**Proof** Set \(m_1 := km\). Suppose that \((\mathcal{V}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q; \omega)\). We obtain \((\mathcal{V}_1, \Phi^*) := \mathcal{P}_{m, m_1}(\mathcal{V}, \Phi^*)\). There exists a lattice \(\mathcal{L}_1 \subset \mathcal{V}_1\) such that \(y^{m_1}\omega \Phi^*(\mathcal{L}_1) = \mathcal{L}_1\). We set \(\mathcal{L}_2 := \bigoplus_{u \in \text{Gal}(m_1, m)} u^* \mathcal{L}_1\). Then, \(\mathcal{L}_2\) is \(\text{Gal}(m_1, m)\)-equivariant, and \(y^{m}\omega \Phi^* \mathcal{L}_2 = \mathcal{L}_2\) holds. Let \(\mathcal{L}\) be the \(\text{Gal}(m_1, m)\)-invariant part of \(\mathcal{L}_2\). We obtain \(y^{\ell} (\Phi^*)^k \mathcal{L} = \mathcal{L}\). Hence, we obtain a lattice with the desired property.

Suppose that a lattice \(\mathcal{L}\) of \(\mathcal{V}\) has the desired property. We set \(\mathcal{L}' := \mathcal{L} \otimes_{\mathcal{R}_m} \mathcal{R}_{m_1}\). We have \((y^{m_1} \Phi^*)^k \mathcal{L}' = \mathcal{L}'\). We set \(\mathcal{L}'' := \bigoplus_{j=0}^{k-1} (y^{m_1} \Phi^*)^j \mathcal{L}'\). Then, we obtain \((y^{m_1} \Phi^*)^k \mathcal{L}'' = \mathcal{L}''\).

2.1.6 Basic examples of pure isoclinic \(q_m\)-difference modules

Let \(\omega \in \mathbb{Q}\). If \(m \omega \in \mathbb{Z}\), we define \(\mathbb{L}_m(\omega) \subset \text{Diff}_m(\mathcal{K}, q; \omega)\) by the \(K_m\)-vector space \(K_m \cdot e_{m, \omega}\) with the operator \(\Phi^*(e_{m, \omega}) = y^{-m \omega} e_{m, \omega}\). For \(\omega \in \mathbb{Q} \setminus \frac{1}{m} \mathbb{Z}\), let \(\ell_0 \in \mathbb{Z}\) be the least common multiple of \(m_0 \in \mathbb{Z}\). We obtain \(\mathbb{L}_m(\omega) \subset \text{Diff}_m(\mathcal{K}, q)\). We set \(\mathbb{L}_m(\omega) := \bigoplus_{\mu \in \text{Gal}(m_1, m)} \mathcal{L}_m(\omega) \otimes \mathcal{V}_m(\mu^\ell)\). Then, the claim is clear.

Let \((\mathcal{V}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q; \omega)\). If \(\omega \in \frac{1}{m_1} \mathbb{Z}\) for \(m_1 \in m\mathbb{Z}_{>0}\), then there exist \((\mathcal{U}^{\text{res}}, \Phi^*) \subset \text{Diff}_m(\mathcal{K}, q; 0)\) and an isomorphism \((\mathcal{P}_{m, m_1})^*(\mathcal{V}, \Phi^*) \simeq \mathbb{L}_m(\omega) \otimes (\mathcal{U}^{\text{res}}, \Phi^*)\).
2.1.7 Slope decompositions

Definition 2.14 Let \((V, \Phi^*) \in \text{Diff}_m(K, q)\). A decomposition \((V, \Phi^*) = \bigoplus_{\omega \in Q}(V_\omega, \Phi^*)\) in \(\text{Diff}_m(K, q_m)\) is called a slope decomposition if \((V_\omega, \Phi^*) \in \text{Diff}_m(K, q)\).

We obtain the uniqueness of slope decompositions from Lemma 2.8.

Lemma 2.15 If \(V = \bigoplus V_\omega^{(i)} (i = 1, 2)\) are slope decompositions of \((V, \Phi^*) \in \text{Diff}_m(K, q)\), then \(V_\omega^{(1)} = V_\omega^{(2)}\) hold for any \(\omega \in Q\).

As a corollary, we obtain the following.

Corollary 2.16 Let \((V, \Phi^*) \in \text{Diff}_m(K, q)\). Let \(p_{m,m_1}^*(V, \Phi^*) = \bigoplus (V_\omega^{(m_1)}, \Phi^*)\) be a slope decomposition of \(p_{m,m_1}^*(V, \Phi^*)\). Then, the following holds.

\(\bullet\) \((V_\omega^{(m_1)}, \Phi^*)\) is \(\text{Gal}(m_1, m)\)-equivariant. In particular, we obtain a decomposition \((V, \Phi^*) = \bigoplus_{\omega \in Q}(V_\omega, \Phi^*)\) as the descent.

\(\bullet\) The decomposition is a slope decomposition of \((V, \Phi^*)\).

Proposition 2.17 Any \(q_m\)-difference \(K_m\)-module has a slope decomposition.

If \(q_m\) is not a root of 1, Proposition 2.17 is classically well known. (See [25, 35, 28] We give an outline of the proof only in the case \(q_m\) is a root of 1.

Notation 2.18 For any \((V, \Phi^*) \in \text{Diff}_m(K, q)\), let \(\text{Slope}(V)\) denote the set of \(\omega \in Q\) such that \(V_\omega \neq 0\).

2.1.8 Proof of Proposition 2.17 in the case where \(q\) is a root of 1

Cyclic vectors Let \((V, \Phi^*) \in \text{Diff}_m(K, q)\). For any \(v \in V\), we set

\[
\langle v \rangle := \sum_{j \in \mathbb{Z}} K_m \cdot (\Phi^*)^j(v), \quad \langle v \rangle := \sum_{j \geq 0} K_m \cdot (\Phi^*)^j(v).
\]

Note that \(\langle v \rangle = \langle \langle v \rangle \rangle\) holds. Indeed, we clearly have \(\Phi^*(\langle v \rangle) \subset \langle v \rangle\). Because \(\dim_{K_m} \Phi^*(\langle v \rangle) = \dim_{K_m} \langle v \rangle\), we obtain \(\Phi^*(\langle v \rangle) = \langle v \rangle\).

An element \(v \in V\) is called a cyclic vector if \(\langle \langle v \rangle \rangle = V\). The following lemma is standard.

Lemma 2.19 If \(V\) has a cyclic vector \(v\), there exist \(m_1 \in m\mathbb{Z}_{>0}\), \(\ell \in \mathbb{Z}\) and a decomposition \(p_{m,m_1}^*(V, \Phi^*) = (V_1, \Phi^*) \oplus (V_2, \Phi^*)\) such that \((i)\) \((V_1, \Phi^*)\) is pure isoclinic of slope \(\ell/m_1\), \((ii)\) \((V_1, \Phi^*) \neq 0\).

Proof We give only an indication. Set \(r := \dim_{K_m} V\). It is easy to see that \(v, \Phi^*(v), \ldots, (\Phi^*)^{r-1}(v)\) induce a frame of \(V\) over \(K_m\). There exists a relation \((\Phi^*)^r(v) = \sum_{j=0}^{r-1} a_j \cdot (\Phi^*)^j(v)\), where \(a_j \in K_m\). Note that one of \(a_j\) is not 0. We set

\[
\ell/s := \max \left\{ -\frac{\text{ord}_{q_m}(a_j)}{r-j} \mid j = 0, \ldots, r-1 \right\},
\]

where \((\ell, s) \in \mathbb{Z} \times \mathbb{Z}_{>0}\) such that g.c.d.\((\ell, s) = 1\). Note that \(\text{ord}_{q_m}(0) = \infty\). We set \(m_1 := sm\). Because 

\[
(y_{m_1}^{\ell}, \Phi^*)^j(v) = y_{m_1}^{\ell j} q_{m_1}^{(j-1)/2}(\Phi^*)^j(v),
\]

we obtain the following:

\[
(y_{m_1}^{\ell}, \Phi^*)^r(v) = \sum_{j=0}^{r-1} y_{m_1}^{\ell(r-j)} a_j q_{m_1}^{(\ell(r-j)-j(\ell-j-1))/2}(y_{m_1}^{\ell}, \Phi^*)^j(v).
\]

Note that \(b_j := y_{m_1}^{\ell(r-j)} a_j \in R_{m_1}\), and there exists \(j_0\) such that \(b_{j_0}(0) \neq 0\).

Let \(L \subset V \otimes K_m\) be the lattice generated by \((y_{m_1}^{\ell}, \Phi^*)^j(v) (j \in \mathbb{Z})\). Clearly, \(y_{m_1}^{\ell}, \Phi^*(L) \subset L\) holds. Moreover, the induced endomorphism \(F := \sigma(y_{m_1}^{\ell}, \Phi^*; L)\) of \(L_0\) is not nilpotent. There exists the decomposition \(L_0 = L_1 \oplus L_2\) such that \((i)\) \(F(L_1) \subset L_1\), \((ii)\) \(F|_{L_1}\) is invertible, \((iii)\) \(F|_{L_2}\) is nilpotent, \((iv)\) \(L_1 \neq 0\). There exists the decomposition \(L = L_1 \oplus L_2\) such that \((i)\) \(y_{m_1}^{\ell}, \Phi^*(L_1) \subset L_1\), \((ii)\) \(L_{1j=0} = L_1\). It induces a decomposition \(V = V_1 \oplus V_2\) with the desired property.
Remark 2.20 If $q_m$ is not a root of 1, it is classically known that any $(V, \Phi^*) \in \text{Diff}_m(K, q)$ has a cyclic vector. (See [3, 27].) If $q_m$ is a root of 1, a $q_m$-difference $K_m$-module does not necessarily have a cyclic vector.

Eigen decompositions As a preliminary to prove Proposition 2.17, we recall a standard result in linear algebra. Let $f$ be a $K_m$-automorphism of a $K_m$-vector space $U$. Set $rm$. Recall that the set of the eigenvalues $\mathcal{S}(f)$ is contained in $K_{rm}$. We obtain the decomposition

$$U \otimes_{K_m} K_{rm} = \bigoplus_{a \in \mathcal{S}(f)} U^{(r \mid m)}_a.$$ 

For any $\omega \in \mathcal{S}(p, \omega)$, we put $\mathcal{S}(f, \omega) := \{a \in \mathcal{S}(f) \mid \text{ord}_{\mathcal{S}(f)}(a) = r \mid m \omega\}$. We set

$$U^{(r \mid m)}_\omega := \bigoplus_{a \in \mathcal{S}(f, \omega)} U^{(r \mid m)}_a.$$ 

Let $G$ denote the Galois group of $K_{rm}$ over $K_m$. There is a natural $G$-action on $V \otimes_{K_m} K_{rm}$. Because $U^{(r \mid m)}_\omega$ is $G$-invariant, we have a subspace $U_\omega \subset U$ such that $U^{(r \mid m)}_\omega = U_\omega \otimes_{K_m} K_{rm}$, and we obtain the following decomposition:

$$U \otimes_{K_m} K_{rm} = \bigoplus_{\omega \in \mathcal{Q}} U^{(r \mid m)}_\omega.$$ (12)

Proof of Proposition 2.17 in the case $q$ is a root of 1 Let $(V, \Phi^*)$ be a $q_m$-difference $K_m$-module. Let us prove that $(V, \Phi^*)$ has a slope decomposition in the case where $q_m$ is an $s$-th root of 1 for some $s \in \mathbb{Z}_{>0}$. We use an induction of $\dim_{K_m} V$. We put $s_1 := r/s$. We set $\Psi^* := (\Phi^*)^{s_1}$. Note that $\Psi^* = \text{id}$ on $K_{m'}$ for any $m' \in \{m, 2m, \ldots, r \mid m\}$. Hence, $\Psi^*$ on $V$ is $K_m$-linear, and $\Psi^*$ on $V \otimes_{K_m} K_{m'}$ for $m' \in \{m, 2m, \ldots, r \mid m\}$ are the induced $K_{m'}$-linear automorphisms. We obtain the decomposition $V = \bigoplus_{\omega} V_\omega$ as in (12). Note that $(V_\omega, \Psi^*)$ has pure slope $\omega$. By using the commutativity of $\Phi^*$ and $\Psi^*$, and by the construction of (12), we obtain that $\Phi^*(V_\omega) = V_\omega$. Let us prove that $(V_\omega, \Phi^*)$ has pure slope $\omega/s_1$.

Suppose that $V_\omega$ does not have a cyclic vector. Take any $v \in V_\omega$. Note that $(\langle v, \Phi^* \rangle) \subsetneq (V_\omega, \Phi^*)$. Then, by the assumption of the induction, we may assume that there exists a decomposition $(v) = \bigoplus_{\mu \in \mathcal{Q}} (v)_\mu$, where $(\langle v \rangle_\mu, \Phi^*)$ has pure slope $\mu$. Because $(\langle v \rangle_\mu, \Psi^*)$ has pure slope $\mu s_1$, we obtain that $\langle v \rangle_\mu = 0$ unless $s_1 \mu = \omega$. Hence, we obtain that $(\langle v \rangle, \Phi^*)$ has pure slope $\omega/s_1$. By varying $v$, we obtain that $(V_\omega, \Phi^*)$ has pure slope $\omega/s_1$.

Suppose that $V_\omega$ has a cyclic vector. Then, there exist $m_1 \in m \mathbb{Z}_{>0}$ and a decomposition $V_\omega \otimes_{K_m} K_{m_1} = V^{(1)} \oplus V^{(2)}$ such that (i) $\Phi^*(V^{(1)}) = V^{(1)}$, (ii) $V^{(1)} \neq 0$, (iii) $V^{(1)}$ has pure slope. By using the hypothesis of the induction, we may assume that $V^{(2)}$ has a slope decomposition with respect to $\Phi^*$. Hence, $V \otimes_{K_m} K_{m_1}$ has a slope decomposition. As in the previous paragraph, we obtain that the slope of $(V^{(1)} \otimes_{K_m} K_{m_1}, \Phi^*)$ is $\omega/s_1$, and that $(V \otimes_{K_m} K_{m_1}, \Phi^*)$ has pure slope $\omega/s_1$. Hence, we can conclude that $(V_\omega, \Phi^*)$ has pure slope $\omega/s_1$.

2.2 Filtered formal bundles

We recall the notion of filtered bundles on $K_m$-vector spaces. Let $V$ be a finite dimensional vector space over $K_m$. A filtered bundle over $V$ is an increasing sequence $\mathcal{P}_a V = (\mathcal{P}_a V) | a \in \mathbb{R}$ of $R_m$-lattices of $V$ such that (i) $\mathcal{P}_a V = \bigcap_{a < b} \mathcal{P}_b V$ for any $a \in \mathbb{R}$, (ii) $\mathcal{P}_{a+n} V = y^n \mathcal{P}_a V$ for any $a \in \mathbb{R}$ and $n \in \mathbb{Z}$. We set $\text{Gr}_a^\mathcal{P}(V) := \mathcal{P}_a(V)/\mathcal{P}_{<a}(V)$. A morphism of filtered bundles $F : \mathcal{P}_a V_1 \rightarrow \mathcal{P}_b V_2$ is defined to be a $K_m$-homomorphism $F$ satisfying $F(\mathcal{P}_a V_1) \subset \mathcal{P}_b V_2$. Let $\text{Mod}(K_m)^{\text{Par}}$ denote the category of filtered bundles over finite dimensional $K_m$-vector spaces.

2.2.1 Pull back

Let $m_1 \in m \mathbb{Z}_{>0}$. Let $\mathcal{P}_a V_1 \in \text{Mod}(K_m)^{\text{Par}}$. Recall that we obtain the induced filtered bundle $\mathcal{P}_a(\mathcal{P}_{m, m_1} V)$ over $\mathcal{P}_{m, m_1} V$ given as follows:

$$\mathcal{P}_a(\mathcal{P}_{m, m_1} V) = \bigoplus_{b \in \mathcal{S}(m, m_1)} y_n \mathcal{P}_b(V) \otimes \mathcal{R}_m \mathcal{R}_{m_1},$$ (13)
where \( S(m_1, m) := \{(b, n) \in \mathbb{R} \times \mathbb{Z} | \frac{m_1}{m}b + n \leq a\} \). The filtered bundle \( \mathcal{P}_\ast(p_{m,m_1}^\ast V) \) is also denoted by \( p_{m,m_1}^\ast \mathcal{P}_\ast V \). Thus, we obtain the pull back functor

\[
p_{m,m_1}^\ast : \text{Mod}(K^{\text{Par}}_m) \rightarrow \text{Mod}(K^{\text{Par}}_{m_1}).
\]

Let \( \mathcal{P}_\ast V \in \text{Mod}(K^{\text{Par}}_m) \). We obtain the map \( \text{Gr}_a^P(V) \rightarrow \text{Gr}_a^P(m_1/m) + n(p_{m,m_1}^\ast V) \) as follows. For \( s \in \text{Gr}_a^P(V) \), take a lift \( \tilde{s} \in \mathcal{P}_\beta(V) \) of \( s \), then we obtain the element in \( \text{Gr}_a^P(m_1/m) + n(p_{m,m_1}^\ast V) \) induced by \( y_{m_1}^{-n}\tilde{s} \), which is independent of a choice of \( \tilde{s} \). This procedure induces an isomorphism

\[
\bigoplus_{(n,b) \in S_0(m_1,m,a)} \text{Gr}_b^P(V) \cong \text{Gr}_a^P(p_{m,m_1}^\ast V)
\]

where \( S_0(m_1,m,a) := \{(n,b) \in \mathbb{Z} \times \mathbb{R} | 0 \leq n < \frac{m_1}{m} + \frac{m_1}{m}b + n = a\} \). Each \( p_a(p_{m,m_1}^\ast V) \) is preserved by the \( \text{Gal}(m_1,m) \)-action, and hence we obtain the \( \text{Gal}(m_1,m) \)-action on \( \text{Gr}_a^P(p_{m,m_1}^\ast V) \). The decomposition (14) is identified with the canonical decomposition with respect to the \( \text{Gal}(m_1,m) \)-action, and \( \text{Gr}_b^P(V) \) is identified with the \( \text{Gal}(m_1,m) \)-invariant part.

### 2.2.2 Push-forward and descent

Let \( m \) and \( m_1 \) be positive integers such that \( m_1 \in m\mathbb{Z}_{\geq 0} \). Let \( \mathcal{P}_\ast V \in \text{Mod}(K^{\text{Par}}_m) \). Recall that the filtered bundle \( \mathcal{P}_\ast(p_{m,m_1}^\ast V) \) is induced as follows:

\[
\mathcal{P}_\ast(p_{m,m_1}^\ast V) = \mathcal{P}_\ast(m_1/m) V.
\]

The filtered bundle \( \mathcal{P}_\ast(p_{m,m_1}^\ast V) \) is also denoted by \( p_{m,m_1}^\ast \mathcal{P}_\ast V \). Thus, we obtain the push-forward \( p_{m,m_1}^\ast : \text{Mod}(K^{\text{Par}}_m) \rightarrow \text{Mod}(K^{\text{Par}}_{m_1}) \).

Let \( \mathcal{P}_\ast(V) \in \text{Mod}(K^{\text{Par}}_m) \). There exists the natural isomorphism

\[
\text{Gr}_a^P(p_{m,m_1}^\ast V) \cong \text{Gr}_a^P(m_1/m)(V).
\]

Let \( V \) be a finite dimensional \( \text{Gal}(m_1,m) \)-equivariant \( K_{m_1} \)-vector space. We say that a filtered bundle \( \mathcal{P}_\ast V \) over \( V \) is \( \text{Gal}(m_1,m) \)-equivariant if \( \mu \mathcal{P}_\ast(V) = \mathcal{P}_\ast(V) \) for any \( a \in \mathbb{R} \) and \( \mu \in \text{Gal}(m_1,m) \). We obtain the \( \mathcal{K}_m \)-vector space \( V^{\text{Gal}(m_1,m)} \) as the descent, i.e., as the \( \text{Gal}(m_1,m) \)-invariant part of \( V \). We have the induced filtered bundle \( \mathcal{P}_\ast(V^{\text{Gal}(m_1,m)}) \) over \( V^{\text{Gal}(m_1,m)} \) as

\[
\mathcal{P}_\ast(V^{\text{Gal}(m_1,m)}) := \big( \mathcal{P}_\ast(m_1/m) V \big)^{\text{Gal}(m_1,m)}.
\]

The filtered bundle is denoted by \( \mathcal{P}_\ast(V)^{\text{Gal}(m_1,m)} \), and called the descent of \( \mathcal{P}_\ast V \).

**Lemma 2.21**

- \( \mathcal{P}_\ast V \in \text{Mod}(K^{\text{Par}}_m) \) is naturally isomorphic to \( (p_{m,m_1}^\ast \mathcal{P}_\ast V)^{\text{Gal}(m_1,m)} \).

- Let \( \mathcal{P}_\ast V_1 \in \text{Mod}(K^{\text{Par}}_m) \). Then, \( p_{m,m_1}^\ast \mathcal{P}_\ast V_1 \) is naturally isomorphic to \( \bigoplus_{\mu \in \text{Gal}(m_1,m)} \mu^\ast \mathcal{P}_\ast V_1 \).

### 2.2.3 Reduction

For any \( \mathcal{P}_\ast V \in \text{Mod}(K^{\text{Par}}_m) \), we set

\[
G(\mathcal{P}_\ast V) := \bigoplus_{a \in \mathbb{R}} \text{Gr}_a^P(V).
\]

The multiplication of \( y_m \) induces \( \mathbb{C} \)-linear isomorphisms \( \text{Gr}_a^P(V) \rightarrow \text{Gr}_{a-1}^P(V) \) for any \( a \in \mathbb{R} \). Hence, we may naturally regard \( G(\mathcal{P}_\ast V) \) as a \( \mathbb{C}[y_m, y_m^{-1}] \)-module with an \( \mathbb{R} \)-grading. It is also \( \mathbb{R} \)-graded. For any \( a \in \mathbb{R} \), we set

\[
\mathcal{P}_a G(\mathcal{P}_\ast V) := \bigoplus_{b \leq a} \text{Gr}_b^P(V).
\]
It is a $\mathbb{C}[y_m]$-lattice of $\mathcal{G}(\mathcal{P}, \mathcal{V})$. By the construction, there exists a natural isomorphism

$$\text{Gr}_a\mathcal{G}(\mathcal{P}, \mathcal{V}) := \mathcal{P}_a\mathcal{G}(\mathcal{P}, \mathcal{V})/\mathcal{P}_{<a}\mathcal{G}(\mathcal{P}, \mathcal{V}) \simeq \text{Gr}_a^P(\mathcal{V}).$$

We set $\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V}) := \mathcal{G}(\mathcal{P}\mathcal{V}) \otimes_{\mathbb{C}[y_m, y_m^{-1}]} K_m$. We also set $\mathcal{P}_a\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V}) := \mathcal{P}_a\mathcal{G}(\mathcal{P}, \mathcal{V}) \otimes_{\mathbb{C}[y_m]} \mathcal{R}_m$ for any $a \in \mathbb{R}$. They give a filtered bundle over $\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V})$. In this way, we also regard $\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V})$ as a filtered bundle. For any $a \in \mathbb{R}$, there exist the natural isomorphisms:

$$\text{Gr}_a\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V}) \simeq \text{Gr}_a^P\mathcal{G}(\mathcal{P}, \mathcal{V}) \simeq \text{Gr}_a^P(\mathcal{V}). \quad (15)$$

**Remark 2.22** There exist a (non-unique) isomorphism of filtered bundles $\widehat{\mathcal{G}}(\mathcal{P}, \mathcal{V}) \simeq \mathcal{P}_*\mathcal{V}$ which induces the isomorphisms $\mathbb{C}[y, y^{-1}]$-modules.

### 2.3 Graded q-difference $\mathbb{C}[y, y^{-1}]$-modules

A $q_m$-difference free $\mathbb{C}[y_m, y_m^{-1}]$-module $(M, \Phi^*)$ is a free $\mathbb{C}[y_m, y_m^{-1}]$-module $M$ of finite rank equipped with a $\mathbb{C}$-linear automorphism $\Phi^*$ such that $\Phi^*(y_m s) = q_m y_m \Phi^*(s)$ for any $s \in M$. A $(\mathbb{Q}, \mathbb{R})$-grading is a decomposition

$$M = \bigoplus_{(\omega, a) \in \mathbb{R} \times \mathbb{Q}} M_{\omega, a}$$

such that the following holds:

- $y_m M_{\omega, a} = M_{\omega, a-1}$ for any $(\omega, a) \in \mathbb{Q} \times \mathbb{R}$.
- $\Phi^* M_{\omega, a} = M_{\omega, a+m \omega}$.

A morphism of $q_m$-difference free $\mathbb{C}[y_m, y_m^{-1}]$-modules with $(\mathbb{Q}, \mathbb{R})$-grading $(M_{1, \bullet}, \Phi^*_1) \rightarrow (M_{2, \bullet}, \Phi^*_2)$ is defined to be a morphism of $q_m$-difference $\mathbb{C}[y_m, y_m^{-1}]$-modules preserving the gradings. Let $\text{Diff}_m(\mathbb{C}[y, y^{-1}], q_m)_{(\mathbb{Q}, \mathbb{R})}$ denote the category of $q_m$-difference free $\mathbb{C}[y_m, y_m^{-1}]$-modules equipped with $(\mathbb{Q}, \mathbb{R})$-grading.

For each $\omega$, we have the expression $m \omega = \ell(m \omega)/k(m \omega)$, where $k(m \omega)$ and $\ell(m \omega)$ are uniquely determined by the conditions $k(m \omega) \in \mathbb{Z}_{>0}$, $\ell(m \omega) \in \mathbb{Z}$ and $\text{g.c.d.}(k(m \omega), \ell(m \omega)) = 1$. Let $\Lambda_{\omega} := \frac{1}{k(m \omega)} \mathbb{Z}$. Note that $\Lambda_{\omega}$ is the image of the map $\mathbb{Z}^2 \rightarrow \mathbb{R}$ defined by $(n_1, n_2) \mapsto n_1 m \omega - n_2$.

We obtain the automorphism $F_{\omega, a} := y_m^{\ell(m \omega)}(\Phi^*)^{k(m \omega)}$ on $M_{\omega, a}$ for any $(a, \omega) \in \mathbb{R} \times \mathbb{Q}$. We obtain the generalized eigen decomposition

$$(M_{\omega, a}, F_{\omega, a}) = \bigoplus_{\alpha \in \mathbb{C}^*} (M_{\omega, a, \alpha}, F_{\omega, a, \alpha})$$

where $F_{\omega, a, \alpha}$ has a unique eigenvalue $\alpha$. It is easy to see

$$y_m \cdot M_{\omega, a, \alpha} = M_{\omega, a-1, \alpha q_m^{k(m \omega)}}, \quad \Phi^* \cdot M_{\omega, a, \alpha} = M_{\omega, a+m \omega, \alpha q_m^{-k(m \omega)}}.$$

For $\omega \in \mathbb{Q}$, $-1/k(m \omega) < a \leq 0$ and $\alpha \in \mathbb{C}^*$, we set

$$M(\omega, a, \alpha) := \bigoplus_{b \in \Lambda_1} M_{\omega, a+b, \alpha q_m^{k(m \omega)b}}.$$

Then, we obtain a decomposition of $(\mathbb{Q}, \mathbb{R})$-graded $q_m$-difference $\mathbb{C}[y_m, y_m^{-1}]$-modules:

$$M = \bigoplus_{\omega \in \mathbb{Q}} \bigoplus_{-k(m \omega)-1 < a \leq 0} \bigoplus_{\alpha \in \mathbb{C}^*} M(\omega, a, \alpha).$$
2.3.1 The induced nilpotent endomorphism and the weight filtration

Each $M_{\omega,a,\alpha}$ is equipped with the nilpotent endomorphism $N_{\omega,a,\alpha}$ obtained as the logarithm of the unipotent part of $F_{\omega,a,\alpha}$. It induces the weight filtration $W(M_{\omega,a,\alpha})$. We obtain the nilpotent endomorphism $N = \bigoplus N_{\omega,a,\alpha}$ of $M$. It commutes with $y_m$ and $\Phi^*$. Hence, $N$ is a nilpotent endomorphism of $M$ in $\text{Diff}_m(\mathbb{C}[y,y^{-1}])_{(Q,R)}$. The weight filtration $W$ is a filtration of $M$ in $\text{Diff}_m(\mathbb{C}[y,y^{-1}])_{(Q,R)}$.

2.3.2 Classification

Let $\mathcal{C}$ denote the category of finite dimensional vector spaces $V$ equipped with a grading

$$V = \bigoplus_{\omega \in \mathbb{Q}} \bigoplus_{-k(\omega)-1 < a \leq 0} \bigoplus_{\alpha \in \mathbb{C}^*} V_{\omega,a,\alpha}$$

and a graded unipotent automorphism $u = \bigoplus_{\omega \in \mathbb{Q}} \bigoplus_{-k(\omega)-1 < a \leq 0} \bigoplus_{\alpha \in \mathbb{C}^*} u_{\omega,a,\alpha}$. A morphism

$$F : (V_{\bullet}^{(1)}, u_{\bullet}^{(1)}) \to (V_{\bullet}^{(2)}, u_{\bullet}^{(2)})$$

in $\mathcal{C}$ is a $\mathbb{C}$-linear map $F : V^{(1)} \to V^{(2)}$ such that (i) $F$ preserves the gradings, (ii) $F \circ u_{\bullet}^{(1)} = u_{\bullet}^{(2)} \circ F$.

For any $M \in \text{Diff}_m(\mathbb{C}[y,y^{-1}],q)_{(Q,R)}$, we obtain the finite dimensional graded vector space

$$\bigoplus_{\omega \in \mathbb{Q}} \bigoplus_{-k(\omega)-1 < a \leq 0} \bigoplus_{\alpha \in \mathbb{C}^*} M_{\omega,a,\alpha}$$

Let $u_{\omega,a,\alpha}$ denote the unipotent part of $F_{\omega,a,\alpha}$. We obtain an object $\bigoplus_{\omega,a,\alpha} (M_{\omega,a,\alpha}, u_{\omega,a,\alpha})$ in $\mathcal{C}$. Thus, we obtain a functor $\text{Diff}_m(\mathbb{C}[y,y^{-1}],q)_{(Q,R)} \to \mathcal{C}$. The following is easy to see.

**Lemma 2.23** The functor $\text{Diff}_m(\mathbb{C}[y,y^{-1}],q)_{(Q,R)} \to \mathcal{C}$ is an equivalence.

**Remark 2.24** Let $V_{\bullet} \in \mathcal{C}$. For each $(\omega,a,\alpha)$, we obtain the nilpotent endomorphism $N_{\omega,a,\alpha}$ of $V_{\omega,a,\alpha}$ as the logarithm of the unipotent automorphism $u_{\omega,a,\alpha}$. We obtain the weight filtration $W(V_{\omega,a,\alpha})$ with respect to $N_{\omega,a,\alpha}$. Note that the conjugacy classes of $u_{\omega,a,\alpha}$ are determined by the filtrations $W(V_{\omega,a,\alpha})$.

2.3.3 Tensor product

Let $M^{(i)} \in \text{Diff}_m(\mathbb{C}[y,y^{-1}],q)_{(Q,R)}$. We obtain the $q_m$-difference free $\mathbb{C}[y_m,y^{-1}]$-module $M^{(1)} \otimes M^{(2)}$ by the tensor product over $\mathbb{C}[y_m,y^{-1}]$. Let $(M^{(1)} \otimes M^{(2)})_{\omega,a}$ be the image of the injective map

$$\bigoplus_{\omega_1 + \omega_2 = \omega, a_1 + a_2 = a, -1 < a_1 \leq 0} M^{(1)}_{\omega_1,a_1} \otimes_{\mathbb{C}} M^{(2)}_{\omega_2,a_2} \to M^{(1)} \otimes_{\mathbb{C}[y_m,y_m^{-1}]} M^{(2)}. \tag{16}$$

Then, we obtain the grading $M^{(1)} \otimes M^{(2)} = \bigoplus (M^{(1)} \otimes M^{(2)})_{\omega,a}$. We have the automorphisms $F_{\omega,a}^{(i)}$ of $M^{(i)}$. We also have the automorphism $F_{\omega,a}$ of $(M^{(1)} \otimes M^{(2)})_{\omega,a}$.

**Lemma 2.25** Suppose that $M^{(1)}_{\omega,a} = 0$ unless $\omega = 0$. Under the identification

$$(M^{(1)} \otimes M^{(2)})_{\omega,a} = \bigoplus_{a_1 + a_2 = a, -1 < a_1 \leq 0} M^{(1)}_{0,a_1} \otimes_{\mathbb{C}} M^{(2)}_{\omega,a_2},$$

we have $F_{\omega,a} = \bigoplus (F_{0,a_2}^{(1)})_{k(m\omega)} \otimes F_{\omega,a_2}^{(2)}$. The nilpotent endomorphism $N$ of $M^{(1)} \otimes M^{(2)}$ is equal to $k(m\omega) N^{(1)} \otimes \text{id} + \text{id} \otimes N^{(2)}$, where $N^{(i)}$ are the nilpotent endomorphism of $M^{(i)}$. The filtration $W((M^{(1)} \otimes M^{(2)}))$ is equal to the filtration induced by $W(M^{(i)})$ ($i = 1,2$).
2.3.4 Pull back

Let \( m_1 \in m \mathbb{Z}_{>0} \). Let \( M \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}) \). We set \( p^*_{m,m_1}(M) := M \otimes_{\mathbb{C}[y, y^{-1}]} \mathbb{C}[y_1, y^{-1}_m] \) which is naturally a \( q_{m_1} \)-difference \( \mathbb{C}[y_1, y^{-1}_m] \)-module. Set \( S_0(m_1, m, a) := \{(n, b) \in \mathbb{Z} \times \mathbb{R} \mid 0 \leq n < \frac{m_1}{m}, \frac{m_1}{m}b + n = a \} \) as in \( \text{Lemma 2.26} \). We define \( p^*_{m,m_1}(M)_{\omega,a} \) as the image of the injection:

\[
\bigoplus_{(b,i) \in S_0(m_1, m, a)} y_1^{-i}M_{\omega,b} \rightarrow p^*_{m,m_1}(M).
\]

(17)

Then, we obtain the grading \( p^*_{m,m_1}(M) = \bigoplus p^*_{m,m_1}(M)_{\omega,a} \). Thus, we obtain

\[
p^*_{m,m_1} : \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}) \rightarrow \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}).
\]

Let \( F^{(1)}_{\omega,a} \) be the automorphism of \( p^*_{m,m_1}(M)_{\omega,a} \) induced by \( y_1^{\ell(m_1 \omega)}(\Phi^*)^{k(m_1 \omega)} \). Set \( d := k(m_1 \omega)/k(m_1) \in \mathbb{Z}_{>0} \).

\textbf{Lemma 2.26} Under the identification of \( p^*_{m,m_1}(M)_{\omega,a} \simeq \bigoplus_{(b,i) \in S_0(m_1, m, a)} y_1^{-i}M_{\omega,b} \), we have

\[
(F^{(1)}_{\omega,a})^d = \bigoplus_{(b,i) \in S_0(m_1, m, a)} \frac{1}{m_1} y_1^{\ell(m_1 \omega)}k(m_1 \omega)d(d-1)-ik(m_1 \omega)d F^{(1)}_{\omega,b}.
\]

Hence, \( dN^{(1)} = p^*_{m,m_1}N \) holds, where \( N^{(1)} \) and \( N \) are the nilpotent endomorphisms of \( p^*_{m,m_1}(M) \) and \( M \), respectively. We also obtain \( W(p^*_{m,m_1}M) = p^*_{m,m_1}W(M) \).

2.3.5 Push-forward

Let \( M \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}) \). It naturally induces a \( q_{m_1} \)-difference \( \mathbb{C}[y_1, y^{-1}_m] \)-module \( p^*_{m,m_1}(M) \). We set \( p^*_{m,m_1}(M)_{\omega,a} := M_{\omega,am/m_1} \). Thus, we obtain

\[
p^*_{m,m_1} : \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}) \rightarrow \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}).
\]

For any \( (\omega, b) \in \mathbb{Q} \times \mathbb{R} \), let \( F^{(1)}_{\omega,b} \) be the automorphism of \( p^*_{m,m_1}(M)_{\omega,b} \) induced by \( y_1^{\ell(m_1 \omega)}(\Phi^*)^{k(m_1 \omega)} \). Set \( d := k(m_1 \omega)/k(m_1) \in \mathbb{Z}_{>0} \).

\textbf{Lemma 2.27} We have

\[
(F^{(1)}_{\omega,a})^d = \frac{1}{m_1} y_1^{\ell(m_1 \omega)}k(m_1 \omega)d(d-1)-ik(m_1 \omega)d F^{(1)}_{\omega,am/m_1}.
\]

As a result, \( p^*_{m,m_1}(dN) = N^{(1)} \) and \( p^*_{m,m_1}W(M) = W(p^*_{m,m_1}M) \) hold.

2.3.6 Examples

Let \( \omega \in \mathbb{Q} \) and \( -k(m_1 \omega)^{-1} < a \leq 0 \). Let \( \mathcal{L}^a_m(\omega, a) \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{R}, \mathbb{Q}) \) be the object corresponding to

\[
(M_{\omega,a}', F_{\omega,a}') = \begin{cases} (\mathcal{C}, q_m^{-\ell(m_1 \omega)k(m_1 \omega)}k(m_1 \omega)) & (\omega', a') = (\omega, a) \\ (0, 1) & \text{(otherwise)} \end{cases}
\]

There exists a natural isomorphism \( p^*_{m,m_1}(\omega, ak(m_1)) \simeq \mathcal{L}^a_m(\omega, a) \).

For a finite dimensional vector space \( V \) with an automorphism \( F \), let \( \mathcal{V}^a_m(V, F) \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{R}, \mathbb{Q}) \) be the object corresponding to

\[
(M_{\omega,a}', F_{\omega,a}') = \begin{cases} (V, F) & (\omega', a') = (0, 0) \\ (0, 1) & \text{(otherwise)} \end{cases}
\]

Any \( M \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)(\mathbb{Q}, \mathbb{R}) \) is isomorphic to the object of the following form:

\[
\bigoplus_{i=1}^N \mathcal{L}^a_m(\omega_i, a_i) \otimes \mathcal{V}^a_m(V_i, F_i).
\]

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2.4 Good filtered formal q-difference modules

2.4.1 Good filtered bundles

Let \((V, \Phi^*) \in \text{Diff}_m(K, q)\).

**Definition 2.28** A filtered bundle \(P, V\) over \(V\) is called good if the following holds.

- The filtration \(P, V\) is compatible with the slope decomposition \(V = \bigoplus_{\omega \in \text{Slope}(V)} V_\omega\), i.e.,
  \[ P, V = \bigoplus_{\omega \in \text{Slope}(V)} P_a(V_\omega). \]

- Take \(m_1 \in m\mathbb{Z}_{\geq 0}\) such that \(m_1 \omega \in \mathbb{Z}\) for any \(\text{Slope}(V)\). Then, the following holds for any \(\omega \in \text{Slope}(V)\) and for any \(a \in \mathbb{R}\):
  \[ y_{m_1}^a \Phi^* \left( P_a(p_{m,1}^* V_\omega) \right) = P_a(p_{m,m_1}^* V_\omega). \]

Such \((P, V, \Phi^*)\) is called a good filtered \(q_m\)-difference \(K_m\)-module.

**Remark 2.29** As a special case, \((P, V, \Phi^*)\) is called unramifially good (resp. regular) if \(\text{Slope}(V) \subset \mathbb{Z}\) (resp. \(\text{Slope}(V) = \{0\}\)).

A morphism of good filtered \(q_m\)-difference \(K_m\)-modules \(F : (P, V_1, \Phi) \rightarrow (P, V_2, \Phi)\) is defined to be a morphism of \(q_m\)-difference \(K_m\)-modules \(F\) such that \(F(P_a V_1) \subset P_a V_2\) for any \(a \in \mathbb{R}\). Let \(\text{Diff}_m(K, q)_{\text{Par}}\) denote the category of good filtered \(q_m\)-difference \(K_m\)-modules. Let \(\text{Diff}_m(K, q; \omega)_{\text{Par}}\) denote the full subcategory of good filtered \(q_m\)-difference \(K_m\)-modules \((P, V, \Phi)\) such that \((V, \Phi) \in \text{Diff}_m(K, q; \omega)\).

**Lemma 2.30** Let \((V, \Phi^*) \in \text{Diff}_m(K, q)_{\text{Par}}\). Let \(P, V\) be a filtered bundle over \(V\). Then, \((P, V, \Phi^*)\) is good if and only if \(\Phi^*(P_a V) = P_{a+m, V}\) for any \(a \in \mathbb{R}\).

**Proof** It is clear if \(\omega \in \mathbb{Z}\). In general, we take \(m \in m\mathbb{Z}_{\geq 0}\) such that \(m_1 \omega \in \mathbb{Z}\). By definition, \(P, V\) is good if and only if \(\Phi^* P_a(p_{m,1}^* V) = P_{a+m_1}(p_{m,m_1}^* V)\). Because \(P_0(V) = P_0(m,1/m)(p_{m,m_1}^* V)^{\text{Gal}}\), we obtain the claim of the lemma.

2.4.2 Reduction to \((\mathbb{Q}, \mathbb{R})\)-graded q-difference modules

Let \((P, V, \Phi^*) \in \text{Diff}_m(K, q)_{\text{Par}}\). There exists the slope decomposition \(P, V = \bigoplus_{\omega \in \mathbb{Q}} P_a(V_\omega)\). We have the induced isomorphisms:

\[ \Phi^* : Gr^P_a(V_\omega) \simeq Gr^P_{a+m_1}(V_\omega) \]

for any \(a \in \mathbb{R}\). Thus, we obtain a \(\mathbb{C}\)-linear automorphism \(\Phi^*\) on \(G(P, V_\omega)\). It is easy to check \(\Phi^*(y_{m,s}) = q_m y_m \Phi^*(s)\) for any \(s \in G(V_\omega)\). Thus, \(G(P_a V, \Phi^*) = \bigoplus(G(P_a V_\omega, \Phi^*) \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)_{(\mathbb{Q}, \mathbb{R})}\). Thus, we obtain a functor \(G : \text{Diff}_m(K, q)_{\text{Par}} \rightarrow \text{Diff}_m(\mathbb{C}[y, y^{-1}], q)_{(\mathbb{Q}, \mathbb{R})}\). We have natural equivalences \(p_{m,m_1} \circ G \simeq G \circ p_{m,m_1}^*\).

For any \((P, V, \Phi^*) \in \text{Diff}_m(K, q)_{\text{Par}}\), by taking the formal completion of \(G(P, V, \Phi^*)\), we obtain \((\hat{G}(V), \Phi^*) \in \text{Diff}_m(K, q)\). Moreover, together with the induced filtered bundle \(P, \hat{G}(V)\) over \(\hat{G}(V)\), we obtain \((P, \hat{G}(V), \Phi^*) \in \text{Diff}_m(K, q)_{\text{Par}}\). Clearly, \(G(P, \hat{G}(V)) \simeq G(P, V)\).

2.4.3 The generalized eigen decomposition and the weight filtration

Let \((P, V, \Phi^*) \in \text{Diff}_m(K, q)_{\text{Par}}\). For each \((\omega, a) \in \mathbb{Q} \times \mathbb{R}\), \(Gr^P_a(V_\omega)\) is equipped with the automorphism \(F_{a,\omega}\) induced \(y_{m}(\text{m}(\omega))(\Phi^*)^{k(\text{m}(\omega))}\). We obtain the generalized eigen decomposition \(Gr^P_a(V_\omega) = \bigoplus_{\alpha \in \mathbb{C}, \epsilon} \text{E}_\alpha Gr^P_a(V_\omega)\). Let \(N_{a,\omega}^P\) denote the nilpotent endomorphism of the unipotent part of \(F_{a,\omega}\). We obtain the weight filtration \(W\) on \(Gr^P_a(V_\omega)\) with respect to \(N_{a,\omega}\). It is compatible with the generalized eigen decomposition.
2.4.4 Basic examples

For \( \alpha \in \mathbb{C}^* \), let \( \mathbb{V}_m(\alpha) = \mathcal{K}_m \alpha \) be a Fuchsian \( q_m \)-difference \( \mathbb{K}_m \)-module defined by \( \Phi^*(c) = \alpha c \), as in Example 2.6. For \( a \in \mathbb{R} \), we define the filtered bundle \( \mathcal{P}_a \mathbb{V}_m(\alpha) \) over \( \mathbb{V}_m(\alpha) \) as follows:

\[
\mathcal{P}_a \mathbb{V}_m(\alpha) = \mathcal{R}_m y^{-[b-a]} e.
\]

Here, we set \( [c] := \max\{n \in \mathbb{Z} | n \leq c \} \) for \( c \in \mathbb{R} \). Thus, we obtain \( (\mathcal{P}_x \mathbb{V}_m(\alpha), \Phi^*) \in \text{Diff}_m(\mathcal{K}, q; 0)^{\text{par}} \).

**Lemma 2.31** \( G(\mathcal{P}_x \mathbb{V}_m(\alpha), \Phi^*) \simeq \mathbb{L}^{\alpha}_m(0, a) \otimes \mathbb{V}^\alpha_m(\alpha). \)

Let \( V \) be a finite dimensional \( \mathbb{C} \)-vector space with a unipotent automorphism \( u \). For any \( a \in \mathbb{R} \), we define a filtered bundle \( \mathcal{P}_a \mathbb{V}_m(V, u) \) over \( \mathbb{V}_m(V, u) \) by

\[
\mathcal{P}_a \mathbb{V}_m(V, u) = \mathcal{R}_m y^{-[b-a]} \otimes_{\mathbb{C}} V.
\]

Thus, we obtain \( (\mathcal{P}_x \mathbb{V}_m(V, u), \Phi^*) \in \text{Diff}_m(\mathcal{K}, q; 0)^\text{par}. \)

**Lemma 2.32** \( G(\mathcal{P}_x \mathbb{V}_m(V, u), \Phi^*) \simeq \mathbb{L}^{\alpha}_m(0, a) \otimes \mathbb{V}^\alpha_m(V, u). \)

Take \( \omega \in \mathbb{L}^m \mathbb{Z} \). For any \( a \in \mathbb{R} \), we define the filtered bundle \( \mathcal{P}_a \mathbb{L}_m(\omega) \) as follows:

\[
\mathcal{P}_a \mathbb{L}_m(\omega) = \mathcal{R}_m y^{-[b-a]} e_{m, \omega}.
\]

Thus, we obtain \( (\mathcal{P}_x \mathbb{L}_m(\omega), \Phi^*) \in \text{Diff}_m(\mathcal{K}, q)^\text{par} \). More generally, for any \( \omega \in \mathbb{Q} \). Set \( m_1 := k(m \omega) \cdot m \). We define

\[
\mathcal{P}_a \mathbb{L}_m(\omega) := \mathbb{P}_{m, m_1} (\mathcal{P}_x^{[am_1/m]} \mathbb{L}_{m_1}(\omega)).
\]

The following is easy to see.

**Lemma 2.33** \( G(\mathcal{P}_x \mathbb{L}_m(\omega)) \simeq \mathbb{L}_m^{\alpha}(\omega, a). \)

For \( i = 1, \ldots, N \), we take \( \omega_i \in \mathbb{Q}, a_i \in \mathbb{R}, \alpha_i \in \mathbb{C}^*, \) and finite dimensional vector spaces \( V_i \) with a unipotent automorphism \( u_i \). Let us consider

\[
(\mathcal{P}_x V, \Phi^*) = \bigoplus_i \mathcal{P}_a \mathbb{L}_m(\omega_i) \otimes \mathcal{P}_a \mathbb{V}_m(\alpha_i) \otimes \mathcal{P}_a \mathbb{V}_m(V_i, u_i).
\]

Then, we have

\[
G(\mathcal{P}_x V, \Phi^*) \simeq \bigoplus_i \mathbb{L}_m^{\alpha}(\omega_i, a_i) \otimes \mathbb{V}_m^\alpha(\alpha_i) \otimes \mathbb{V}_m^\alpha(V_i, u_i).
\]

3 Mini-complex manifolds

3.1 A twistor family of mini-complex manifolds

3.1.1 A hyperkähler manifold \( X \) equipped with \( \mathbb{R} \times \mathbb{Z}^2 \)-action

Take \( \mu_1, \mu_2 \in \mathbb{C} \) which are linearly independent over \( \mathbb{R} \). We assume that \( \text{Im}(\mu_2/\mu_1) > 0 \). Let \( \Gamma \) denote the lattice of \( \mathbb{C} \) generated by \( \mu_1 \) and \( \mu_2 \). Let \( \text{Vol}(\Gamma) \) denote the volume of \( \mathbb{C}/\Gamma \) with respect to the volume form \( \frac{1}{2} dz \wedge d\bar{z} \), where \( z \) is the standard coordinate of \( \mathbb{C} \). The following holds:

\[
\text{Vol}(\Gamma) = \frac{1}{2 \sqrt{-1}} (\mu_2 \bar{\mu}_1 - \mu_2 \mu_1).
\]

We set \( X := \mathbb{C}_z \times \mathbb{C}_w \) with the Euclidean metric \( dz \wedge d\bar{z} + dw \wedge d\bar{w} \). It is a hyperkähler manifold. Let us consider the action of the group \( \mathbb{Z} \mathbf{e}_1 \oplus \mathbb{Z} \mathbf{e}_2 \) on \( X \) given by

\[
n_i \mathbf{e}_i(z, w) = (z, w) + n_i(\mu_i, 0).
\]

We also consider the action of \( \mathbb{R} \mathbf{e}_0 \) on \( X \) given by

\[
se_0(z, w) = (z, w + s).
\]

Thus, we obtain an action of \( \mathbb{R} \mathbf{e}_0 + \mathbb{Z} \mathbf{e}_1 \oplus \mathbb{Z} \mathbf{e}_2 \) on \( X \).
3.1.2 Complex manifold $X^\lambda$

For each $\lambda \in \mathbb{C}$, there exists the complex structure of $X$ given by the coordinate system

$$(\xi, \eta) = (z + \lambda \overline{w}, w - \lambda \overline{z}).$$

The complex manifold is denoted by $X^\lambda$. The action of $\Re e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is described as follows with respect to the coordinate system $(\xi, \eta)$:

$$s_0(\xi, \eta) = (\xi, \eta) + (\lambda s, s), \quad n_i e_i(\xi, \eta) = (\xi, \eta) + n_i(\mu_i - \lambda \overline{\mu}_i) \quad (i = 1, 2).$$

3.1.3 Some calculations

To introduce a more convenient complex coordinate system of $X^\lambda$, we make some calculations.

**Lemma 3.1** There exist $s_1 \in \mathbb{R}$ and $g_1 \in \mathbb{C}$ with $|g_1| = 1$ such that

$$- \lambda \overline{\mu}_1 + s_1 = g_1(\mu_1 + \lambda s_1) \neq 0. \quad (18)$$

- If $|\lambda| \neq 1$, there are two choices of $(s_1, g_1)$. One is contained in $\mathbb{R}_{>0} \times S^1$, and the other is contained in $\mathbb{R}_{<0} \times S^1$. Moreover, $1 - g_1 \lambda \neq 0$ holds.

- If $|\lambda| = 1$ and $\lambda \neq \pm \sqrt{-1} |\mu_1|^{-1}$, such $(s_1, g_1)$ is uniquely determined as $(s_1, g_1) = (0, -\lambda \overline{\mu}_1 \mu_1^{-1})$. Moreover, $1 - g_1 \lambda \neq 0$ holds.

- If $\lambda = \pm \sqrt{-1} |\mu_1|^{-1}$, the set of such $(s_1, g_1)$ is $\{ (s, \lambda^{-1}) \mid s \in \mathbb{R} \}$.

**Proof** Let us consider the condition $| - \lambda \overline{\mu}_1 + s_1 | = | \mu_1 + \lambda s_1 |$ for $s_1 \in \mathbb{R}$. It is equivalent to the following:

$$(1 - |\lambda|^2)s_1^2 - 2(\lambda \overline{\mu}_1 + \overline{\lambda} \mu_1)s_1 - (1 - |\lambda|^2)|\mu_1|^2 = 0. \quad (19)$$

If $|\lambda| \neq 1$, there exist two distinct solutions:

$$s_1 = \frac{\lambda \overline{\mu}_1 + \overline{\lambda} \mu_1}{1 - |\lambda|^2} \pm \left( \frac{(\lambda \overline{\mu}_1 + \overline{\lambda} \mu_1)^2}{(1 - |\lambda|^2)^2} + |\mu_1|^2 \right)^{1/2} = \frac{\lambda \overline{\mu}_1 + \overline{\lambda} \mu_1}{1 - |\lambda|^2} \pm \frac{|\mu_1 + \lambda^2 \overline{\mu}_1|}{1 - |\lambda|^2}.$$ 

Hence, we obtain

$$\mu_1 + \lambda s_1 = \frac{\mu_1 + \lambda^2 \overline{\mu}_1}{1 - |\lambda|^2} \pm \lambda \frac{|\mu_1 + \lambda^2 \overline{\mu}_1|}{1 - |\lambda|^2}.$$ 

Because $|\lambda| \neq 1$, we obtain $\mu_1 + \lambda s_1 \neq 0$. Once we choose $s_1$, we obtain a unique complex number $g_1$ satisfying $|g_1| = 1$ determined by the condition (18). Because $|\lambda| \neq 1$ and $|g_1| = 1$, we obtain $1 - g_1 \lambda \neq 0$.

If $|\lambda| = 1$ and $\lambda \neq \pm \sqrt{-1} |\mu_1|^{-1}$, we obtain $\lambda \overline{\mu}_1 + \overline{\lambda} \mu_1 \neq 0$, and hence the equation (18) has a unique solution $s_1 = 0$. In this case, $g_1$ is determined by $- \lambda \overline{\mu}_1 = g_1 \mu_1$, i.e., $g_1 = -\lambda \overline{\mu}_1 \mu_1^{-1}$. The following holds:

$$1 - \lambda g_1 = 1 + \lambda^2 \overline{\mu}_1 / \mu_1 = \lambda \mu_1^{-1}(\overline{\lambda} \mu_1 + \lambda \overline{\mu}_1) \neq 0.$$

If $\lambda = \pm \sqrt{-1} |\mu_1|^{-1}$, we can check the claim by a direct computation.

**Lemma 3.2** The following holds:

$$\Im \left( \frac{g_1 \mu_2 + \lambda \overline{\mu}_2}{1 - g_1 \lambda} \right) = \frac{\Vol(\Gamma)}{\Re \{g_1 \mu_1 \}} \neq 0. \quad (20)$$

In particular, $\Re \{g_1 \mu_1 \} \neq 0$. If $|\lambda| \neq 1$, we have $\Re \{g_1 \mu_1 \} \cdot (1 - |\lambda|^2)s_1 > 0$. 

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The following holds:

\[ \Im\left( \frac{g_1 \mu_2 + \lambda \mu_1}{1 - g_1 \lambda} \right) = \Im\left( \frac{(g_1 \mu_2 + \lambda \mu_1)s_1}{g_1 \mu_1 + \lambda \mu_1} \right) = \Im\left( \frac{(1 - |\lambda|^2)(\mu_2 \mu_1 - \mu_1 \mu_2)s_1}{2|g_1 \mu_1 + \lambda \mu_1|^2} \right) = \frac{(1 - |\lambda|^2) \text{Vol}(\Gamma)s_1}{|g_1 \mu_1 + \lambda \mu_1|^2}. \]

By using \((1 - g_1 \lambda)^{-1} = s_1(g_1 \mu_1 + \lambda \mu_1)^{-1}\) again, we obtain

\[ \frac{|g_1 \mu_1 + \lambda \mu_1|^2}{s_1(1 - |\lambda|^2)} = \frac{1}{1 - |\lambda|^2} \left( g_1 \mu_1 + \lambda \mu_1 - \lambda \mu_1 - |\lambda|^2 g_1 \mu_1 \right). \]

Because the left hand side of (21) is real, it is equal to the following:

\[ \frac{1}{2(1 - |\lambda|^2)} \left( g_1 \mu_1 + g_1 \mu_1 - |\lambda|^2 g_1 \mu_1 - |\lambda|^2 g_1 \mu_1 \right) = \Re(g_1 \mu_1). \]

Thus, we obtain (20).

Suppose \(|\lambda| = 1\). Because \(g_1 = -\lambda \mu_1^{-1}\), the following holds:

\[ \Im\left( \frac{g_1 \mu_2 + \lambda \mu_1}{1 - g_1 \lambda} \right) = \Im\left( \frac{\mu_2 \mu_1 - \mu_1 \mu_2}{\lambda \mu_1 + \lambda \mu_1} \right) = \frac{-2 \text{Vol}(\Gamma)}{(\lambda \mu_1 + \lambda \mu_1)} = \frac{-2 \text{Vol}(\Gamma)}{(g_1 \mu_1 + g_1 \mu_1)} = \frac{\text{Vol}(\Gamma)}{\Re(g_1 \mu_1)}. \]

Thus, we are done.

Lemma 3.3 \((g_1 - \lambda)(\mu_1 + \lambda s_1) = (1 + |\lambda|^2) \Re(g_1 \mu_1)\) holds.

Proof We have \((g_1 - \lambda)(\mu_1 + \lambda s_1) = -\lambda(\mu_1 + \lambda s_1) - \lambda \mu_1 + s_1 = (1 - |\lambda|^2)s_1 - \lambda \mu_1 - \lambda \mu_1\). In particular, it is a real number. We have the following:

\[ (g_1 - \lambda)(\mu_1 + \lambda s_1) = g_1(\mu_1 + \lambda s_1) - \lambda \mu_1 (-\lambda \mu_1 + s_1) = g_1 \mu_1 + |\lambda|^2 \mu_1 \mu_1 + s_1(g_1 \lambda - \lambda \mu_1). \]

Because it is a real number, it is equal to \(\Re(g_1 \mu_1 + |\lambda|^2 \mu_1 \mu_1) = (1 + |\lambda|^2) \Re(g_1 \mu_1)\).

Lemma 3.4 Suppose \(|\lambda| \neq 1\). Let \((s_1, g_1)\) and \((s'_1, g'_1)\) be two solutions of the equation (18). Then, the following holds:

\[ \Re(\mu_1 g_1) + \Re(\mu_1 g'_1) = 0. \]

Proof The following holds:

\[ \Re(\mu_1 g_1) = \Re\left( \frac{(-\lambda \mu_1 + s_1)\mu_1}{\mu_1 + \lambda s_1} \right) = \frac{|\mu_1 + \lambda \mu_1|^{2}|\lambda|^2}{(1 - |\lambda|^2)} \cdot \frac{s_1}{|\mu_1|^2 + (\lambda \mu_1 + \lambda \mu_1)s_1 + |\lambda|^2 s_1^2}. \]

The following holds:

\[ s_1(|\mu_1|^2 + (\lambda \mu_1 + \lambda \mu_1)s_1 + |\lambda|^2(s_1^2)) + s'_1(|\mu_1|^2 + (\lambda \mu_1 + \lambda \mu_1)s_1 + |\lambda|^2 s_1^2) = (s_1 + s'_1)(|\mu_1|^2 + s_1 s'_1 |\lambda|^2) + 2s_1 s'_1 (\lambda \mu_1 + \lambda \mu_1). \]

By using \(s_1 + s'_1 = 2(\lambda \mu_1 + \lambda \mu_1)(1 - |\lambda|^2)^{-1}\) and \(s_1 s'_1 = -|\mu_1|^2\), we obtain that (22) is 0. Then, we obtain the claim of the lemma by a direct calculation.

Lemma 3.5 We have \((1 + |\lambda|^2)|\Re(\mu_1 g_1)| = |\mu_1 + \lambda^2 \mu_1|\). If \(|\lambda| \neq 1\), we have the following more precise formula:

\[ (1 + |\lambda|^2) \Re(\mu_1 g_1) = \text{sign}(1 - |\lambda|^2) \cdot \text{sign}(s_1) \cdot |\mu_1 + \lambda^2 \mu_1|. \]
Proof If $|\lambda| = 1$, we have $\text{Re}(\mu_1 g_1) = \text{Re}(-\lambda \bar{\mu}_1)$. Because $|\lambda| = 1$, we also have $|\mu_1 + \lambda^2 \bar{\mu}_1| = |\bar{\lambda} \mu_1 + \lambda \bar{\mu}_1| = 2|\text{Re}(\lambda \bar{\mu}_1)|$. Hence, the claim of the lemma is clear.

Suppose $|\lambda| \neq 1$. We have the following:

$$|\mu_1|^2 + (\lambda \mu_1 + \lambda \bar{\mu}_1) s_1 + |\lambda|^2 s_1^2 = |\mu_1|^2(1 + |\lambda|^2) + (\lambda \mu_1 + \lambda \bar{\mu}_1) s_1 \left(1 + \frac{2|\lambda|^2}{1 - |\lambda|^2}\right)$$

$$= (1 + |\lambda|^2) \left(|\mu_1|^2 + \frac{\bar{\lambda} \mu_1 + \lambda \bar{\mu}_1}{1 - |\lambda|^2} s_1\right). \quad (24)$$

We also have the following:

$$\frac{|\mu_1|^2}{s_1} = \frac{\lambda \bar{\mu}_1 + \bar{\lambda} \mu_1}{1 - |\lambda|^2} = \frac{|\mu_1 + \lambda^2 \bar{\mu}_1|}{1 - |\lambda|^2}.$$  

Here, $\pm$ is equal to $\text{sign}(s_1)$. Because

$$\text{Re}(g_1 \mu_1) = \frac{|\mu_1 + \lambda^2 \bar{\mu}_1|}{(1 - |\lambda|^2)} \frac{s_1}{|\mu_1|^2 + (\lambda \mu_1 + \lambda \bar{\mu}_1) s_1 + |\lambda|^2 s_1^2} = \text{sign}(s_1) \frac{|\mu_1 + \lambda^2 \bar{\mu}_1|}{(1 - |\lambda|^2)} \frac{|1 - |\lambda|^2|}{|\mu_1 + \lambda^2 \bar{\mu}_1|},$$

we obtain the claim of the lemma.

3.1.4 Coordinate system $(u, v)$

We introduce a more convenient complex coordinate system of $X^\lambda$.

Assumption 3.6 In the following, we suppose $\lambda \neq \pm \sqrt{-1}|\mu_1| |\mu_1|^{-1}$.

We take $s_1$ and $g_1$ as in Lemma 3.1. We consider the $\mathbb{C}$-linear coordinate change $\mathbb{C}_u \times \mathbb{C}_v \simeq \mathbb{C}_\xi \times \mathbb{C}_\eta$ given by

$$(\xi, \eta) = (u + \lambda v, g_1 u + v), \quad (u, v) = \frac{1}{g_1 \lambda}(\xi - \lambda \eta, -g_1 \xi + \eta).$$

The action of $\text{Re} \oplus \mathbb{Z} \epsilon_1 + \mathbb{Z} \epsilon_2$ on $X^\lambda$ is described as follows in terms of $(u, v)$:

$$s \epsilon_0(u, v) = (u, v) + (0, s), \quad n_i \epsilon_i(u, v) = (u, v) + \frac{n_i}{1 - g_1 \lambda}(\mu_1 + \lambda^2 \bar{\mu}_1, -g_1 \mu_i - \lambda \bar{\mu}_i) \quad (i = 1, 2).$$

Lemma 3.7 The following holds:

$$(\epsilon_1 + s_1 \epsilon_0)(u, v) = (u, v) + \left(\frac{\mu_1 + \lambda^2 \bar{\mu}_1}{1 - g_1 \lambda}, 0\right) = (u, v) + (\mu_1 + \lambda s_1, 0). \quad (25)$$

Proof Note that the following holds by our choice of $s_1$ and $g_1$:

$$s_1 = g_1 \frac{\mu_1 + \lambda \bar{\mu}_1}{1 - g_1 \lambda}.$$ 

Hence, we obtain the first equality in (25). Note that

$$1 - g_1 \lambda = 1 - \frac{-\lambda^2 \bar{\mu}_1 + \mu_1 \lambda s_1}{\mu_1 + \lambda \bar{\mu}_1} = \frac{\mu_1 + \lambda^2 \bar{\mu}_1}{\mu_1 + \lambda \bar{\mu}_1}.$$ 

Hence, we obtain $(\mu_1 + \lambda^2 \bar{\mu}_1)(1 - g_1 \lambda)^{-1} = \mu_1 + \lambda s_1$, and the second equality in (25).

Remark 3.8 Let $(E, \nabla, h)$ is an instanton on $X^\lambda$. Let $F(\nabla) = F_{\xi \xi} d\xi d\xi + F_{\xi \eta} d\xi d\eta + F_{\eta \eta} d\eta d\eta + F_{\eta \xi} d\eta d\xi$ denote the curvature. For $\alpha \in \mathbb{C}$, set $H_\alpha := \{(u, v) \mid v = \alpha\} \subset X^\lambda$. Because $d\xi d\xi = d\eta d\eta$ and $d\eta d\xi = d\eta d\xi$ on $H_\alpha$, the restriction of $F(\nabla)$ to $H_\alpha$ is equal to the restriction of $F_{\xi \xi} d\xi d\xi + F_{\eta \eta} d\eta d\eta$.

In the study of doubly periodic monopoles, it is appropriate to assume the boundedness of $F(\nabla)$. In general, $F_{\xi \xi}$ and $F_{\eta \eta}$ are only bounded, but $F_{\xi \eta}$ and $F_{\eta \xi}$ decay more rapidly. We consider the above coordinate $(u, v)$ to obtain an appropriate curvature decay along $H_\alpha$.  


3.1.5 Partial quotient $Y_p^λ$ and its partial compactification

Take $p ∈ \mathbb{Z}_{>0}$. Let $Y_p^λ$ denote the quotient space of $X^λ$ by the action of $\mathbb{Z} \cdot p(e_1 + s_1 e_0)$. There exists the following induced holomorphic function on $Y_p^λ$:

$$u_p := \exp\left(2\pi \sqrt{-1} \frac{1 - g_1^λ}{p(\mu_1 + \lambda^2 \mu_1)} u\right) = \exp\left(2\pi \sqrt{-1} \frac{1}{p(\mu_1 + \lambda \mu_1)} u\right).$$

We obtain the holomorphic isomorphism $Y_p^λ \simeq \mathbb{C}^* × \mathbb{C}$ induced by $(u_p, v)$, with which we identify $Y_p^λ$ and $\mathbb{C}^* × \mathbb{C}$. We set $Y_p^λ := \mathbb{P}^1 × \mathbb{C}$, which is a partial compactification of $Y_p^λ$. We denote the naturally induced morphisms.

Rem 3.9 (\ref{eq3.9})\]\[
\begin{align*}
\text{Then, the action of } & \Re e_0 + \mathbb{Z} e_2 \text{ on } Y_p^λ \text{ is described as follows:} \\
& s(\Re e_0, v) = (u_p, v + s), \quad n(\mathbb{Z} e_2, v) = ((q_p^λ)^n u_p, v + \sqrt{-1} nt^λ).
\end{align*}
\]

We set $G_p := (\mathbb{Z} / p\mathbb{Z}) \cdot (e_1 + s_1 e_0)$. There exists the induced $G_p$-action on $Y_p$ described as follows:

$$(e_1 + s_1 e_0)(u_p, v) = \left(\exp(2\pi \sqrt{-1} / p) \cdot u_p, v\right).$$

The action naturally extends to an action on $Y_p^λ$.

\textbf{Remark 3.9} The following holds:

$$\text{Im} \left(\frac{\mu_2 + \lambda^2 \mu_2}{\mu_1 + \lambda^2 \mu_1}\right) = \frac{(1 + |λ|^2)(1 - |λ|^2) \text{Vol}(Γ)}{|μ_1 + λ^2 μ_1|^2}. \quad (26)$$

In particular, we obtain: $|q_p^λ| < 1$ in the case $|λ| < 1$; $|q_p^λ| > 1$ in the case $|λ| > 1$; $|q_p^λ| = 1$ in the case $|λ| = 1$.

3.1.6 Mini-complex manifolds $M_p^{λ_\text{cov}}$ and $\overline{M}_p^{λ_\text{cov}}$

Let $M_p^{λ_\text{cov}}$ be the quotient space of $Y_p^λ$ by the action of $\Re e_0$. By setting $t := \text{Im}(v)$, we obtain the mini-complex coordinate system $(u_p, t)$ of $M_p^{λ_\text{cov}}$. The coordinate system induces the identification $M_p^{λ_\text{cov}} \simeq \mathbb{C}^* × \mathbb{R}$. The induced action of $\mathbb{Z} e_2$ is described as follows:

$$e_2(u_p, t) = (q_p^λ u_p, t + t^λ).$$

Note that $M_p^{λ_\text{cov}}$ is naturally identified with the quotient space of $X^λ$ by the action of $\Re e_0 + \mathbb{Z} \cdot p e_1$.

Similarly, let $\overline{M}_p^{λ_\text{cov}}$ denote the quotient space of $Y_p^λ$ by $\Re e_0$. It is naturally a mini-complex manifold and naturally identified with $\mathbb{P}^1 × \mathbb{R}$.

We set $H_0^{λ_\text{cov}} := \{0\} × \mathbb{R}$ and $H_∞^{λ_\text{cov}} := \{∞\} × \mathbb{R}$ in $\mathbb{P}^1 × \mathbb{R}$. We set $H_p^{λ_\text{cov}} := H_0^{λ_\text{cov}} ∪ H_∞^{λ_\text{cov}}$.

The $G_p$-action on $Y_p$ induces $G_p$-actions on $M_p^{λ_\text{cov}}$ and $\overline{M}_p^{λ_\text{cov}}$. We identify $G_p$ and $(\mathbb{Z} / p\mathbb{Z}) \cdot e_1$ by $e_1 + s_1 e_0 \mapsto e_1$. Then, the action of $G_p$ on $\overline{M}_p^{λ_\text{cov}}$ is identified with $e_1(u_p, t) = (e^{2π \sqrt{-1} / p} u_p, t)$.

3.1.7 Mini-complex manifolds $M_p^{λ}$ and $\overline{M}_p^{λ}$

Let $M_p^{λ}$ be the mini-complex manifold obtained as the quotient space of $M_p^{λ_\text{cov}}$ by the action of $\mathbb{Z} e_2$. Similarly, let $\overline{M}_p^{λ}$ be the mini-complex manifold obtained as the quotient space of $\overline{M}_p^{λ_\text{cov}}$ by the action of $\mathbb{Z} e_2$. Let $H_{ν}^{λ}$ \quad (ν = 0, ∞) denote the quotient of $H_{ν}^{λ_\text{cov}}$ by $\mathbb{Z} e_2$. We set $H_0^{λ} := H_0^{λ_\text{cov}} ∪ H_∞^{λ_\text{cov}}$, which is the quotient of $H_0^{λ_\text{cov}}$ by $\mathbb{Z} e_2$. We have $M_p^{λ} = M_p^{λ} ∪ H_0^{λ}$, and it is compact. There exist the naturally induced $G_p$-actions on $M_p^{λ}$ and $\overline{M}_p^{λ}$.

Let $p_{ν}^{λ}$ : $\overline{M}_p^{λ_\text{cov}}$ $→$ $\overline{M}_p^{λ}$ denote the naturally induced morphisms.
Lemma 3.10 The following holds:

\[
\Psi(\mathcal{U}, \mathfrak{t}) = -\frac{p}{2\pi} \Re(g_1 \mu_1) \log |\mathcal{U}| + \frac{1 - |\lambda|^2}{1 + |\lambda|^2} \mathfrak{t}.
\]

Proof We have the following description of \( \Im(\Psi) \) in terms of \((u, v)\):

\[
\Im(w) = \frac{1}{1 + |\lambda|^2} \left( \Im((g_1 - \overline{\lambda})u) + (1 - |\lambda|^2) \Im(v) \right).
\]

The following holds:

\[
\log |\mathcal{U}| = \Re\left(2\pi \sqrt{-1} \frac{u}{p(\mu_1 + \lambda s_1)}\right) = -\frac{2\pi}{p} \Im\left(\frac{u}{(\mu_1 + \lambda s_1)}\right).
\]

Because

\[
-\frac{p}{2\pi} \left(\frac{\mu_1 + \lambda s_1}{u}\right) \times (g_1 - \overline{\lambda})u = -\frac{p}{2\pi}(1 + |\lambda|^2) \Re(g_1 \mu_1),
\]

the claim follows.

Corollary 3.11 If \( \Re(g_1 \mu_1) < 0 \), \( \mathcal{U}_{p, R} \cup H_{p, R}^\lambda \) is a neighbourhood of \( H_0^\lambda \). If \( \Re(g_1 \mu_1) > 0 \), \( \mathcal{U}_{p, R} \cup H_{\infty, p}^\lambda \) is a neighbourhood of \( H_{\infty, p}^\lambda \).

3.1.9 Complement in the case \(|\lambda| \neq 1\)

Suppose \(|\lambda| \neq 1\). For simplicity we assume \( p = 1 \). We use the notation \( \mathcal{U}, q, \) etc., instead of \( \mathcal{U}_p, q_p^\lambda, \) etc. According to Lemma 3.3 the following holds:

\[
t^\lambda = -\Vol(\Gamma) \frac{1 + |\lambda|^2}{|\mu_1 + \lambda^2 s_1|^2} \sign(s_1) \sign(1 - |\lambda|^2).
\]

By (27), the following holds:

\[
\log|q^\lambda| = -2\pi \frac{(1 + |\lambda|^2)(|\lambda|^2 - 1) \Vol(\Gamma)}{|\mu_1 + \lambda^2 s_1|^2}.
\]

We obtain the following:

\[
\frac{\log|q^\lambda|}{t^\lambda} = \frac{2\pi(1 - |\lambda|^2)}{|\mu_1 + \lambda^2 s_1|^2} \sign(s_1), \quad \frac{\log|q^\lambda|}{(t^\lambda)^2} = \frac{2\pi(1 - |\lambda|^2)}{\Vol(\Gamma)(1 + |\lambda|^2)}.
\]

In particular, we obtain the following.

Lemma 3.12 If \(|\lambda| \neq 1\), \( (t^\lambda)^{-2} \log|q^\lambda| \) is independent of the choice of \((e_1, s_1)\).

Let us rewrite (27). For simplicity, we assume \( p = 1 \).

Lemma 3.13 If \(|\lambda| \neq 1\), the following holds.

\[
\Psi(\mathcal{U}, \mathfrak{t}) = 1 - |\lambda|^2 \left( t - t^\lambda \frac{\log|\mathcal{U}|}{\log|q^\lambda|} \right).
\]

In particular, \( t - t^\lambda \frac{\log|\mathcal{U}|}{\log|q^\lambda|} \) is independent of the choice of \((e_1, s_1)\).

Proof By (28) and (27), we obtain

\[
\frac{t^\lambda}{\log|q^\lambda|} = \frac{1}{2\pi} \frac{1 + |\lambda|^2}{1 - |\lambda|^2} \Re(g_1 \mu_1).
\]

Together with (27), we obtain the claim of the lemma.
3.1.10 Two compactifications in the case $|\lambda| \neq 1$

If $|\lambda| \neq 1$, there are two solutions $(s_1, g_1)$ and $(s'_1, g'_1)$ of (13). We obtain two mini-complex coordinate systems $(U_p, t)$ and $(U'_p, t')$ on $M_{p}^{\lambda, \text{cov}}$. We obtain another partial compactification $M_{p}^{\lambda, \text{cov}}$ from $(U_p, t)$. Let $M_{p}^{\lambda}$ denote the quotient of $M_{p}^{\lambda, \text{cov}}$ by the action of $\mathbb{Z}_2$.

By the construction, we have $U_p = U'_p$. We have the relation:

$$t' = t - 2\lambda \frac{\log |u|}{\log |q|}$$

The identity on $M_{p}^{\lambda, \text{cov}}$ is not extended to an isomorphism $\overline{M}_{p}^{\lambda, \text{cov}}$ and $\overline{M}_{p}^{\lambda, \text{cov}}$.

We consider the automorphism $F$ of $M_{p}^{\lambda, \text{cov}}$ defined by $F^*(U_p) = U_p$ and $F^*(v') = -v$. Then, $F$ is equivariant with respect to the $\mathbb{Z}_2$-action by Lemma 3.1. Moreover $F$ is extended to an isomorphism $\overline{M}_{p}^{\lambda, \text{cov}} \simeq \overline{M}_{p}^{\lambda, \text{cov}}$. Hence, $F$ induces an isomorphism $\overline{M}_{p}^{\lambda} \simeq \overline{M}_{p}^{\lambda}$.

3.2 Curvature of mini-holomorphic bundles with Hermitian metric on $M^{\lambda}$

3.2.1 Mini-complex manifold $A^{\lambda}$

We set $A := X/\mathbb{R}e_0$. For each $\lambda$, it is equipped with the mini-complex structure induced by the complex structure of $X^{\lambda}$. (See [21] §2.6.) The mini-complex manifold is denoted by $A^{\lambda}$. There exists the naturally induced action of $\mathbb{Z}_e_1 \oplus \mathbb{Z}_e_2$ on $A^{\lambda}$. The quotient space of $A^{\lambda}$ by $p\mathbb{Z}_e_1$ is naturally isomorphic to $M_{p}^{\lambda, \text{cov}}$, and the quotient space of $A^{\lambda}$ by $p\mathbb{Z}_e_1 \oplus \mathbb{Z}_e_2$ is naturally isomorphic to $M_{p}^{\lambda}$.

3.2.2 Coordinate system $(\alpha, \tau)$ on $A^{\lambda}$

We have the complex coordinate system $(\alpha, \beta)$ on $X^{\lambda}$ determined by the following relation:

$$(\xi, \eta) = \alpha(1, -\bar{\lambda}) + \beta(\lambda, 1) = (\alpha + \beta\lambda, -\bar{\lambda}\alpha + \beta), \quad (\alpha, \beta) = \frac{1}{1 + |\lambda|^2}(\xi - \lambda\eta, \eta + \bar{\lambda}\xi).$$

We can check the following by direct computations.

Lemma 3.14 We have do $d\bar{\tau} + d\beta \ d\bar{\tau} = (1 + |\lambda|^2)^{-1}(d\xi \ d\bar{\xi} + d\eta \ d\bar{\eta}) = dz \ d\bar{z} + dw \ d\bar{w}$.

The actions of $\mathbb{R}e_0 \oplus \mathbb{Z}_e_1 \oplus \mathbb{Z}_e_2$ are described as follows with respect to $(\alpha, \beta)$:

$$s_{\alpha}(\alpha, \beta) = (\alpha, \beta) + (0, s), \quad e_{i}(\alpha, \beta) = (\alpha, \beta) + \frac{1}{1 + |\lambda|^2}(\mu_i + \lambda^2\bar{\mu}_i, -\lambda\mu_i + \bar{\lambda}\mu_i) \ (i = 1, 2).$$

Setting $\tau := \text{Im}(\beta)$, we obtain a mini-complex coordinate $(\alpha, \tau)$ on $A^{\lambda}$. We have the complex vector fields $\partial_\alpha$, $\partial_\tau$, and $\partial_r$ on $A^{\lambda}$. The induced complex vector fields on $M_{p}^{\lambda}$ are also denoted by the same notation.

We have the following relation:

$$\alpha = \frac{1 - \lambda g_i}{1 + |\lambda|^2} u, \quad \tau = \frac{\text{Im}((g_i + \bar{\lambda}) u)}{1 + |\lambda|^2} + t.$$

Hence, we have the following relation between the complex vector fields:

$$\partial_\tau = \frac{1 - \lambda g_i}{1 + |\lambda|^2} \partial_\tau + \frac{1}{2\sqrt{-1}} \frac{g_i + \lambda}{1 + |\lambda|^2} \partial_r, \quad \partial_u = \frac{1 - \lambda g_i}{1 + |\lambda|^2} \partial_u + \frac{1}{2\sqrt{-1}} \frac{g_i + \bar{\lambda}}{1 + |\lambda|^2} \partial_r, \quad \partial_t = \partial_r.$$
3.2.3 Monopoles and mini-holomorphic bundles

Let \((E, h, \nabla, \phi)\) be a monopole on an open subset \(U\) of \(\mathcal{M}_p^\lambda\), i.e., \(E\) is a vector bundle on \(U\) with a Hermitian metric \(h\), a unitary connection \(\nabla\), and an anti-self-adjoint endomorphism \(\phi\) of \(E\) satisfying the Bogomolny equation

\[
F(\nabla) = *\nabla \phi. \tag{29}
\]

Here, \(F(\nabla)\) denotes the curvature of \(\nabla\), and \(*\) denotes the Hodge star operator with respect to the Riemannian metric \(d\alpha d\tau + d\tau d\tau\). We have the expression \(F(\nabla) = F(\nabla)_{\alpha\tau} d\alpha d\tau + F(\nabla)_{\alpha\tau} d\alpha d\tau + F(\nabla)_{\pi\tau} d\alpha d\tau\). Then, the Bogomolny equation is equivalent to the pair of the following equations:

\[
[\nabla, \nabla] = 0, \tag{30}
\]

\[
F(\nabla)_{\alpha\tau} = \frac{\sqrt{-1}}{2} \nabla_{h,\tau} \phi. \tag{31}
\]

The equation (30) implies that \(\nabla\) and \(\nabla\) determine a mini-holomorphic structure on \(E\). (See [21, §2.2] for mini-holomorphic bundles.)

Conversely, Let \((E, \overline{\nabla} E)\) be a mini-holomorphic bundle on an \(U\) of \(\mathcal{M}_p^\lambda\). We have the differential operators \(\partial_{E,\nabla}E\) and \(\partial_{E,\nabla}E\). Let \(h\) be a Hermitian metric of \(E\). Recall that we obtain the Chern connection \(\nabla_h\) and the Higgs field \(\phi_h\). (See [21, §2.3].) Let \(F(h)\) denote the curvature of \(\nabla_h\). We have the expression \(F(h) = F(h)_{\alpha\tau} d\alpha d\tau + F(h)_{\alpha\tau} d\alpha d\tau + F(h)_{\pi\tau} d\alpha d\tau\). Then, \((E, h, \nabla_h, \phi_h)\) is a monopole if and only if

\[
F(h)_{\alpha\tau} = \frac{\sqrt{-1}}{2} \nabla_{h,\tau} h,\tau \phi_h. \tag{32}
\]

If \((E, h, \nabla_h, \phi_h)\) is a monopole, \((E, \overline{\nabla} E, h)\) is also called a monopole.

3.2.4 Contraction of curvature and the analytic degree

Let \((E, \overline{\nabla} E)\) be a mini-holomorphic bundle with a Hermitian metric \(h\) on an open subset \(U \subset \mathcal{M}_p^\lambda\). We obtain \((E, h, \nabla_h, \phi_h)\) as in [3.2.3]. We set

\[
G(h) := F(h)_{\alpha\pi} - \frac{\sqrt{-1}}{2} \nabla_{h,\tau} \phi_h. \tag{33}
\]

Note that the Bogomolny equation for \((E, h, \nabla_h, \phi_h)\) is equivalent to \(G(h) = 0\).

**Definition 3.15** Suppose that \(\text{Tr} G(h)\) is expressed as \(g_1 + g_2\), where \(g_1\) is an \(L^1\)-function on \(U\), and \(g_2\) is non-positive everywhere. Then, we set \(\deg(E, \overline{\nabla} E, h) := \int_U \text{Tr} G(h) \text{dvol}_U \in \mathbb{R} \cup \{-\infty\}\), which is called the analytic degree of \((E, \overline{\nabla} E, h)\).

Let us recall some formulas for \(G(h)\). See [21, §2.8] for more detail.

**Lemma 3.16** Let \(V\) be a mini-holomorphic bundle of \(E\). Let \(h_V\) be the induced metric of \(V\). Let \(p_V\) denote the orthogonal projection of \(E\) onto \(V\). Then, the following holds:

\[
\text{Tr} G(h_V) = \text{Tr} \left( G(h) p_V \right) - |\partial_{E,\nabla} p_V|_{h}^2 - \frac{1}{4} |\partial_{E,\nabla} p_V|_{h}^2.
\]

In particular, if \(|G(h)|_h\) is integrable, then \(\deg(V, h_V)\) is well defined for any mini-holomorphic subbundles \(V\) of \(E\).

**Lemma 3.17** Let \(h_1\) be another Hermitian metric of \(E\). Let \(s\) be the automorphism of \(E\) determined by \(h_1 = h \cdot s\). Then, the following holds.

\[
G(h_1) = G(h) - \partial_{E,\nabla}(s^{-1} \partial_{E,\nabla} s) - \frac{1}{4} \left[ \nabla_{h}, \sqrt{-1} \phi_h, [\nabla_{h}, \sqrt{-1} \phi_h, s] \right].
\]
As a consequence, we obtain the following equality:

\[-\left(\partial_\alpha \partial_\eta + \frac{1}{4} \partial_\eta^2\right) \text{Tr}(s) = \text{Tr}\left(s(G(h_1) - G(h))\right) - s^{-1/2} \partial_{\eta,E,h,\alpha}s_h^2 - \frac{1}{4} s^{-1/2} \partial_{\eta,E,h,\alpha}s_h^2.\]

The following equality also holds:

\[-\left(\partial_\alpha \partial_\eta + \frac{1}{4} \partial_\eta^2\right) \log(\text{Tr}(s)) \leq \left| G(h_1) \right|_{g_{h_1}} + \left| G(h) \right|_{g_h}.\]

If \(\text{rank}(E) = 1\), then \(G(h_1) - G(h) = 4^{-1} \Delta \log(s)\) holds on \(U\).

### 3.2.5 Another expression of \(G(h)\)

We introduce the following real vector fields on \(A^\lambda:\)

\[v := (g_1 \lambda + \overline{g_1} \lambda) \partial_\tau + \sqrt{-1}(g_1 - \lambda^2 g_1) \partial_\alpha - \sqrt{-1}(g_1 - \lambda^2 g_1) \partial_\tau.\]

The induced vector fields on \(\mathcal{M}_p^\lambda\) are also denoted by \(v\).

Let \((E, \overline{\nu}_E)\) be a mini-holomorphic bundle on an open subset \(U \subset \mathcal{M}_p^\lambda\) with a Hermitian metric \(h\).

**Proposition 3.18** We have the following equality:

\[G(h) = \left| 1 - g_1 \lambda \right|^{-2} (1 + |\lambda|^2) [\partial_{E,h,a}, \partial_{E,\pi}] + \left| 1 - g_1 \lambda \right|^{-2} \nabla_{h,v} \phi_h.\]

**Proof** We have the following formula for complex vector fields:

\[\partial_\tau = \frac{1 + |\lambda|^2}{1 - \lambda g_1} \partial_\tau + \frac{1}{2\sqrt{-1}(1 - \lambda g_1)} \partial_\tau, \quad \partial_\alpha = \frac{1 + |\lambda|^2}{1 - \lambda g_1} \partial_\alpha - \frac{1}{2\sqrt{-1}(1 - \lambda g_1)} \partial_\alpha, \quad \partial_\tau = \partial_\tau.\]

Hence, we have the following formulas:

\[\frac{1 + |\lambda|^2}{1 - \lambda g_1} \partial_{E,\pi} = \nabla_{h,\pi} - \frac{1}{2\sqrt{-1}(1 - \lambda g_1)} (\nabla_{h,\tau} - \sqrt{-1} \phi),\]

\[\frac{1 + |\lambda|^2}{1 - \lambda g_1} \partial_{E,h,a} = \nabla_{h,a} + \frac{1}{2\sqrt{-1}(1 - \lambda g_1)} (\nabla_{h,\tau} + \sqrt{-1} \phi).\]

We recall the formulas \([\nabla_{h,\pi}, \nabla_{h,\tau}] = -\sqrt{-1} \nabla_{h,\pi} \phi_h\) and \([\nabla_{h,a}, \nabla_{h,\tau}] = -\sqrt{-1} \nabla_{h,a} \phi_h\). (See [21 §2.8.2].) Then, we obtain the following:

\[\frac{(1 + |\lambda|^2)^2}{|1 - \lambda g_1|^2} [\partial_{E,h,a}, \partial_{E,\pi}] = [\nabla_{h,a}, \nabla_{h,\pi}] - \frac{g_1 + \lambda}{1 - g_1 \lambda} \nabla_{h,\pi} \phi + \frac{g_1 + \lambda}{1 - g_1 \lambda} \nabla_{h,a} \phi - \frac{\sqrt{-1}}{2} \frac{|g_1 + \lambda|^2}{|1 - \lambda g_1|^2} \nabla_{h} \phi = G(h) - \frac{g_1 + \lambda}{1 - g_1 \lambda} \nabla_{h,\pi} \phi + \frac{g_1 + \lambda}{1 - g_1 \lambda} \nabla_{h,a} \phi - \frac{\sqrt{-1}}{2} \frac{|g_1 + \lambda|^2}{|1 - \lambda g_1|^2} \nabla_{h} \phi.\]

Then, we obtain the desired formula. \(\blacksquare\)

Recall that \((z, w)\) is the complex coordinate system of \(X^0\). By setting \(y := \text{Im}(w)\), we obtain a mini-complex coordinate system \((z, y)\) of \(A^0\). We obtain the induced complex vector fields \(\partial_z\), \(\partial_{\tau}\) and \(\partial_y\) on \(\mathcal{M}_p^0\).

**Lemma 3.19** \(v = (1 + |\lambda|^2)(\sqrt{-1} g_1 \partial_z - \sqrt{-1} \overline{g_1} \partial_{\tau})\) holds.

**Proof** We obtain the following relations between complex vector fields:

\[(1 + |\lambda|^2) \partial_\alpha = \partial_z + \lambda^2 \partial_\tau + \sqrt{-1} \lambda \partial_y,\]


\[(1 + |\lambda|^2) \partial_\tau = \partial_\tau + \lambda^2 \partial_\tau - \sqrt{-1} \lambda \partial_\nu, \]

\[(1 + |\lambda|^2) \partial_\tau = 2\sqrt{-1} \lambda \partial_\tau - 2\sqrt{-1} \lambda \partial_\nu + (1 - |\lambda|^2) \partial_\nu. \]

Then, we obtain the claim of the lemma.

Let us give a consequence. Suppose \( U = \mathcal{M}_\lambda^\Lambda \setminus Z \), where \( Z \) is a finite set. We set \( S_1^\lambda := \mathbb{R}/\mathbb{Z} \). Let \( \pi_\nu^\text{cov} : \mathcal{M}_\lambda^\text{cov} \to \mathbb{R} \) be the map defined by \( \pi_\nu^\text{cov}(U_p, t) = t \). It induces a map \( \pi_p : \mathcal{M}_\lambda^\Lambda \to S_1^\Lambda \).

**Proposition 3.20** Suppose that \( \text{Tr} G(h) \) and \( \text{Tr} ([\partial_{E,h}, \partial_{E,p}]) \) are integrable on \( \mathcal{M}_\lambda^\Lambda \setminus Z \). Then, the following equality holds:

\[
\int_{\mathcal{M}_\lambda^\Lambda} \text{Tr} G(h) \, \text{dvol} = \int_{S_1^\Lambda} dt \int_{\pi_p^{-1}(t)} \text{Tr} ([\partial_{E,h}, \partial_{E,p}]) \frac{\sqrt{-1}}{2} \, du \, \text{d}\varphi. \tag{36}
\]

**Proof** By the assumption, the following holds:

\[
\int_{\mathcal{M}_\lambda^\Lambda} \text{Tr} G(h) \, \text{dvol} = \int_{\mathcal{M}_\lambda^\Lambda} |1 - g_1\lambda|^2 (1 + |\lambda|^2) \text{Tr} ([\partial_{E,h}, \partial_{E,p}]) \, \text{dvol} + \int_{\mathcal{M}_\lambda^\Lambda} \text{Tr} (|1 - g_1\lambda|^2 \nabla_{h,v} \phi_h) \, \text{dvol}. \tag{37}
\]

Because

\[
\text{dvol} = \frac{\sqrt{-1}}{2} du \, \text{d}\varphi \, d\tau = \frac{|1 - g_1\lambda|^2 \sqrt{-1}}{(1 + |\lambda|^2)^2} \frac{\sqrt{-1}}{2} du \, \text{d}\varphi \, dt,
\]

the first term of the right hand side of (37) is equal to the right hand side of (36). Let \( T_p \) denote the quotient of \( \mathbb{C} \) by \( p\mathbb{Z} + \mathbb{Z} \). Because \( \text{dvol} = \frac{\sqrt{-1}}{2} dz \, d\sigma \, dy \), the following holds:

\[
\int_{\mathcal{M}_\lambda^\Lambda} \text{Tr} (|1 - g_1\lambda|^2 \nabla_{h,v} \phi_h) \, \text{dvol} = \lim_{C \to \infty} \int_{-C}^C dy \int_{T_p \times \{y\}} \text{Tr} (|1 - g_1\lambda|^2 \nabla_{h,v} \phi_h) \frac{\sqrt{-1}}{2} dz \, d\sigma.
\]

Note that \( \int_{T_p \times \{y\}} \text{Tr} (\nabla_{h,v} \phi_h) dz \, d\sigma = 0 \). Hence, we obtain (36).

4 Good filtered bundles with Dirac type singularity on \((\overline{\mathcal{M}}^\Lambda; H^\lambda, Z)\)

4.1 Good filtered bundles on \((\hat{H}_\nu^\lambda, H_{\nu,p}^\lambda)\)

4.1.1 \( \mathcal{O}_{\hat{H}_\nu^\lambda} \) \((H_{\nu,p}^\lambda)\)-modules

For \( \nu = 0, \infty \), let \( \hat{H}_{\nu,p}^\lambda \) denote the formal completion of \( \overline{\mathcal{M}}^\Lambda \) along \( H_{\nu,p}^\lambda \). Similarly, let \( \hat{H}_{\nu,p}^\text{cov} \) denote the formal completion of \( \mathcal{M}_\lambda^\text{cov} \) along \( H_{\nu,p}^\text{cov} \). We have the natural \( \mathbb{Z} \)-equivariant action on \( \hat{H}_{\nu,p}^\text{cov} \), and \( \hat{H}_{\nu,p}^\lambda \) is naturally isomorphic to the quotient of \( \hat{H}_{\nu,p}^\text{cov} \). Hence, \( \mathcal{O}_{\hat{H}_{\nu,p}^\lambda} \) \((H_{\nu,p}^\lambda)\)-modules are equivalent to \( \mathbb{Z} \)-equivariant \( \mathcal{O}_{\hat{H}_{\nu,p}^\text{cov}} \) \((H_{\nu,p}^\text{cov})\)-modules. Let \( \mathcal{L} \text{FM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda) \) (resp. \( \mathcal{L} \text{FM}(\hat{H}_{\nu,p}^\text{cov}, H_{\nu,p}^\text{cov}) \)) denote the category of locally free \( \mathcal{O}_{\hat{H}_{\nu,p}^\lambda} \) \((H_{\nu,p}^\lambda)\)-modules (resp. \( \mathcal{O}_{\hat{H}_{\nu,p}^\text{cov}} \) \((H_{\nu,p}^\text{cov})\)-modules).

For \( \nu = 0, \infty \), let \( \hat{\nu}_p \) denote the formal completion of \( \overline{\mathcal{M}}^\Lambda \) at \( U_p = \nu \). We have the natural isomorphism \( \hat{H}_{\nu,p}^\text{cov} \simeq \hat{\nu}_p \times \mathbb{R} \). Set \( U_{0,p} := U_p \) and \( U_{\infty,p} := U_p^{-1} \). We also set \( q_0^\nu := q_\nu \), and \( q_\infty^\nu := (q_\nu)^{-1} \). The \( \mathbb{Z} \)-equivariant action on \( \hat{H}_{\nu,p}^\text{cov} \) is described as \( \mathcal{E}_1(U_{\nu,p}, t) = (q_\nu^{\nu,\nu} U_{\nu,p}, t + t^\lambda) \). The \( \mathcal{G}_p \)-action on \( \hat{H}_{\nu,p}^\text{cov} \) is described as \( \mathcal{E}_1(U_{\nu,p}, t) = (e^{2\pi i \frac{\nu}{p}} U_{\nu,p}, t) \), where the signature is + if \( \nu = 0 \), and − if \( \nu = \infty \).

Let \( \pi_\nu^\text{cov} : \hat{H}_{\nu,p}^\text{cov} \to \mathbb{R} \) denote the projection. We have the natural identification \( (\pi_{\text{cov}})^{-1}(t) \simeq \hat{\nu}_p \). We set \( S_1^\Lambda := \mathbb{R}/\mathbb{Z} \). We obtain the induced map \( \pi_\nu^\text{cov} : \hat{H}_{\nu,p}^\text{cov} \to S_1^\Lambda \). For each \( t \in S_1^\Lambda \), once we fix its lift to \( \mathbb{R} \), we obtain an isomorphism \( \pi_\nu^{-1}(t) \simeq \hat{\nu}_p \).

Set \( K_{\nu,p} := \mathcal{C}(U_{\nu,p}) \). Let us observe that locally free \( \mathcal{O}_{\hat{H}_{\nu,p}^\lambda} \) \((H_{\nu,p}^\lambda)\)-modules are equivalent to \( q_\nu^{\nu,\nu} \)-difference \( K_{\nu,p} \)-modules. Let \( q_{\nu,p} : \hat{H}_{\nu,p}^\text{cov} \to \hat{\nu}_p \) denote the projection. Let \((\mathcal{V}, \Phi^*) \) be a \( q_\nu^{\nu,\nu} \)-difference \( K_{\nu,p} \)-module. We
obtain the $\mathcal{O}_{\hat{H}_{\nu,p}^{\lambda\text{cov}}}(*H_{\nu,p}^{\lambda\text{cov}})$-module $q_{\nu,p}^*\mathcal{V}$. By the action of $\Phi^*$, $q_{\nu,p}^*\mathcal{V}$ is naturally $\mathbb{Z}\mathfrak{e}_2$-equivariant. Hence, we obtain an $\mathcal{O}_{\hat{H}_{\nu,p}^{\lambda\text{cov}}}(*H_{\nu,p}^{\lambda\text{cov}})$-module as the descent of $q_{\nu,p}^*\mathcal{V}$, which we denote by $\Upsilon_{\nu,p}^\lambda(\mathcal{V})$. The following is easy to see.

**Lemma 4.1** $\Upsilon_{\nu,p}^\lambda$ induces an equivalence $\text{Diff}_p(K_{\nu}, q_{\nu}^\lambda) \simeq \text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}})$. The quasi-inverse is induced by the restriction $\mathfrak{W} \mapsto (\Upsilon_{\nu,p}^\lambda)^{-1}(\mathfrak{W}) := \mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(0)}$, where $\mathfrak{W}^{\text{cov}}$ is the pull back of $\mathfrak{W}$ by $H_{\nu,p}^{\lambda\text{cov}} \rightarrow \hat{H}_{\nu,p}^{\lambda}$.

**Definition 4.2** We say that a locally free $\mathcal{O}_{\hat{H}_{\nu,p}^{\lambda\text{cov}}}(*H_{\nu,p}^{\lambda\text{cov}})$-module is pure isoclinic of slope $\omega$ if the corresponding $q_{\nu,p}^\lambda$-difference $K_{\nu,p}$-module is pure isoclinic of slope $\omega$. Let $\text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}}; \omega)$ denote the subcategory of pure isoclinic modules of slope $\omega$. A pure isoclinic module of slope 0 is also called Fuchsian or regular.

The following is a consequence of Proposition 2.17.

**Proposition 4.3** Any $\mathfrak{W} \in \text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}})$ has a decomposition $\mathfrak{W} = \bigoplus_{\omega \in \mathbb{Q}} \mathfrak{W}_\omega$ such that $\mathfrak{W}_\omega$ are pure isoclinic of slope $\omega$.

For $p_2 \in p_1\mathbb{Z}_{>0}$, we may regard $\hat{H}_{\nu,p_1}^{\lambda\text{cov}}$ as the quotient of $\hat{H}_{\nu,p_2}^{\lambda\text{cov}}$ by the action of the subgroup $(p_1\mathbb{Z}/p_2\mathbb{Z})\mathfrak{e}_1 \subset (\mathbb{Z}/p_2\mathbb{Z})\mathfrak{e}_1$. We have the naturally induced morphisms $\rho_{p_1,p_2} : \hat{H}_{\nu,p_2}^{\lambda\text{cov}} \rightarrow \hat{H}_{\nu,p_1}^{\lambda\text{cov}}$. We have the pull back and the push-forward:

$$\rho_{p_1,p_2}^* : \text{LFM}(\hat{H}_{\nu,p_1}^{\lambda\text{cov}}, H_{\nu,p_1}^{\lambda\text{cov}}) \rightarrow \text{LFM}(\hat{H}_{\nu,p_2}^{\lambda\text{cov}}, H_{\nu,p_2}^{\lambda\text{cov}}), \quad \rho_{p_1,p_2} : \text{LFM}(\hat{H}_{\nu,p_1}^{\lambda\text{cov}}, H_{\nu,p_1}^{\lambda\text{cov}}) \rightarrow \text{LFM}(\hat{H}_{\nu,p_2}^{\lambda\text{cov}}, H_{\nu,p_2}^{\lambda\text{cov}}).$$

They are compatible with the pull back and push-forwards for $\text{Diff}_p(K_{\nu}, q_{\nu}^\lambda)$ between $\text{Diff}_p(K_{\nu}, q_{\nu}^\lambda)$. We also have the descent of $(p_1\mathbb{Z}/p_2\mathbb{Z})\mathfrak{e}_1$-equivariant locally free objects in LFM$(\hat{H}_{\nu,p_2}^{\lambda\text{cov}}, H_{\nu,p_2}^{\lambda\text{cov}})$.

### 4.1.2 Filtered bundles on $\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}}$

**Definition 4.4** For any $\mathfrak{W} \in \text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}})$, a filtered bundle over $\mathfrak{W}$ is defined to be a family of filtered bundles $\mathcal{P}_*(\mathfrak{W}_{|\mathfrak{p}_{\nu,p}^\lambda(t)})(t \in S_\lambda^\lambda)$. Similarly, for any $\mathfrak{W}^{\text{cov}} \in \text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}})$, a filtered bundle over $\mathfrak{W}^{\text{cov}}$ is defined to be a family of filtered bundles $\mathcal{P}_*(\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t)})(t \in \mathbb{R})$. Such families are often denoted by $\mathcal{P}_*(\mathfrak{W})$ and $\mathcal{P}_*(\mathfrak{W}^{\text{cov}})$.

Let $p_2 \in p_1\mathbb{Z}_{>0}$. For any filtered bundle $\mathcal{P}_*(\mathfrak{W})$ over $\mathfrak{W} \in \text{LFM}(\hat{H}_{\nu,p_1}^{\lambda\text{cov}}, H_{\nu,p_1}^{\lambda\text{cov}})$, we obtain the induced filtered bundle $\mathcal{P}_*(\rho_{p_1,p_2}^*(\mathfrak{W}))$ over $\rho_{p_1,p_2}^*(\mathfrak{W})$. For any filtered bundle $\mathcal{P}_*(\mathfrak{W})$ over $\mathfrak{W} \in \text{LFM}(\hat{H}_{\nu,p_2}^{\lambda\text{cov}}, H_{\nu,p_2}^{\lambda\text{cov}})$, we obtain the induced filtered bundle $\mathcal{P}_*(\rho_{p_1,p_2}^*(\mathfrak{W}))$ over $\rho_{p_1,p_2}^*(\mathfrak{W})$. For any $(p_1\mathbb{Z}/p_2\mathbb{Z})$-equivariant locally free filtered bundle $\mathcal{P}_*(\mathfrak{W})$ over a $(p_1\mathbb{Z}/p_2\mathbb{Z})$-equivariant $\mathfrak{W} \in \text{LFM}(\hat{H}_{\nu,p_2}^{\lambda\text{cov}}, H_{\nu,p_2}^{\lambda\text{cov}})$, we obtain $\mathfrak{W}_1 \in \text{LFM}(\hat{H}_{\nu,p_1}^{\lambda\text{cov}}, H_{\nu,p_1}^{\lambda\text{cov}})$ as the descent of $\mathfrak{W}$, and we obtain a filtered bundle $\mathcal{P}_*(\mathfrak{W}_1)$ over $\mathfrak{W}_1$ as the descent of $\mathcal{P}_*(\mathfrak{W})$.

### 4.1.3 Good filtered bundles on $\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}}$

Let $\mathfrak{W}$ be a locally free $\mathcal{O}_{\hat{H}_{\nu,p}^{\lambda\text{cov}},(H_{\nu,p}^{\lambda\text{cov}})}$-module.

**Definition 4.5** A filtered bundle $\mathcal{P}_*(\mathfrak{W})$ over $\mathfrak{W}$ is pure isoclinic of slope $\omega$ if the following holds.

- Let $\mathfrak{W}^{\text{cov}} \in \text{LFM}(\hat{H}_{\nu,p}^{\lambda\text{cov}}, H_{\nu,p}^{\lambda\text{cov}})$ be the pull back of $\mathfrak{W}$. Take $t_1, t_2 \in \mathbb{R} \simeq H_{\nu,p}^{\lambda\text{cov}}$. Then, under the isomorphism $\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t_1)} \simeq \mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t_2)}$ induced by the parallel transport along the path,

$$P_\alpha(\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t_1)}) = P_{\alpha+p_\omega(t_2-t_1)/t_1}(\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t_2)})$$

holds for any $\alpha \in \mathbb{R}$. Note that the underlying $\mathfrak{W}$ is pure isoclinic of slope $\omega$. Note also that $\mathcal{P}_*(\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(t)})$ are uniquely determined by $\mathcal{P}_*(\mathfrak{W}^{\text{cov}}_{|\mathfrak{p}_{\nu,p}^\lambda(0)})$. 

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Let \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda; \omega)^{\text{par}} \) denote the category of filtered flat bundles over \((\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)\) which are pure isoclinic of slope \( \omega \).

**Remark 4.6** If \( \mathcal{P}_*, \mathcal{M} \) has pure slope 0, it is also called a regular filtered bundle.

**Definition 4.7** A filtered bundle \( \mathcal{P}_*, \mathcal{M} \) over \((\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)\) is called good if \( \mathcal{P}_*, \mathcal{M} = \bigoplus \mathcal{P}_*, \mathcal{M}_\omega \), where \( \mathcal{P}_*, \mathcal{M}_\omega \in \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda; \omega)^{\text{par}} \). Let \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)^{\text{par}} \) denote the category of good filtered bundles over \((\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)\).

For any \( \mathcal{P}_*, \mathcal{M} \in \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)^{\text{par}} \), the filtered bundle \( \mathcal{P}_*(\tau_{\nu,p}^{-1}(\mathcal{M})) \) is defined to be \( \mathcal{P}_*(\tau_{\nu,p}^{\text{cov}}|_{\tau_{\nu,p}^{-1}(0)}) \).

Conversely, for any \( (\mathcal{P}_*, \mathcal{M}, \Phi) \in \text{Diff}_*(\mathcal{K}_\nu, \mathcal{Q}_\nu^\lambda) \), the filtered bundle over \( \tau_{\nu,p}(\mathcal{V}) = \bigoplus \tau_{\nu,p}(\mathcal{M}_\omega) \) is defined by \( \mathcal{P}_*(\tau_{\nu,p}(\mathcal{V})|_{\tau_{\nu,p}^{-1}(0)}) = \mathcal{P}_* \mathcal{V} \). The following is clear.

**Lemma 4.8** \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)^{\text{par}} \) and \( \text{Diff}_*(\mathcal{K}_\nu, \mathcal{Q}_\nu^\lambda)^{\text{par}} \) are equivalent by \( \tau_{\nu,p}^\lambda \) and \( (\tau_{\nu,p}^{-1})^{-1} \). They also induce equivalences between \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda; \omega)^{\text{par}} \) and \( \text{Diff}_*(\mathcal{K}_\nu, \mathcal{Q}_\nu^\lambda; \omega)^{\text{par}} \).

For any \( \mathcal{P}_*, \mathcal{M} \in \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)^{\text{par}} \), we define \( \mathcal{G}(\mathcal{P}_*, \mathcal{M}) := \mathcal{G}(\tau_{\nu,p}^{-1}(\mathcal{P}_*, \mathcal{M})) \in \text{Diff}_m(\mathbb{C}[y, y^{-1}], \mathbb{Q}(\mathbb{Q}, \mathbb{Q})) \).

### 4.1.4 Basic examples

For any finite dimensional \( \mathbb{C} \)-vector space \( V \) with an automorphism \( f \), we set \( \mathcal{V}_{\nu,p}(V, f) := \tau_{\nu,p}(\mathcal{V}_p(V, f)) \). (See Example 2.6 for \( \mathcal{V}_p(V, f) \).) Recall that we have constructed filtered bundles \( \mathcal{P}^s(\mathcal{V}_p(V, f) \mathcal{V}_p(V, f)) \) in \([2.4.4]\) The \( \mathcal{R}_p \)-lattices \( \mathcal{P}^s(\mathcal{V}_p(V, f)) \) naturally define \( \mathcal{O}_\nu \)-lattices \( \mathcal{P}^s(a) \mathcal{V}_p(V, f) \) of \( \mathcal{V}_p(V, f) \). They induce a filtered bundle \( \mathcal{P}^s(a) \mathcal{V}_p(A) \) over \( \mathcal{V}_p(A) \) similarly.

**Lemma 4.9** \( \mathcal{P}^s(a) \mathcal{V}_p(V, f) \) and \( \mathcal{P}^s(a) \mathcal{V}_p(A) \) are objects in \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda; 0)^{\text{par}} \). We have the natural isomorphisms \( \mathcal{G}(\mathcal{P}^s(a) \mathcal{V}_p(V, f)) \simeq \mathcal{L}_s(0, a) \otimes \mathcal{V}_p(0, a) \otimes \mathcal{V}_p^0(0, a) \).

For any \( \omega \in \mathbb{Q} \), we set \( \mathcal{L}_p(\omega) := \tau_{\nu,p}(\mathcal{L}_p(\omega)) \). (See [2.4.6] for \( \mathcal{L}_p(\omega) \).) Set \( \mathcal{R}_{\nu,p} := \mathcal{C}[\mathcal{U}_{\nu,p}] \). If \( p \omega \in \mathbb{Z} \), the filtered bundle \( \mathcal{P}^s(0) \mathcal{L}_p(0) \) is given as follows:

\[
\mathcal{P}^s_b(\mathcal{Q}_p^s(\mathcal{L}_p(\omega)(\mathbb{Z}_{\nu,p}^{-1}(t))) = \mathcal{U}_p^{[0, [1-p \omega/t, t]} \mathcal{R}_{\nu,p} : \mathcal{Q}_p^{-1} \mathcal{C}_p(\mathcal{U}_p(\omega)).
\]

Here, we set \([c] := \max\{n \in \mathbb{Z} \mid n \leq c\}\) for any \( c \in \mathbb{R} \). Because it is naturally \( \mathbb{Z}_2 \)-equivariant, we obtain an induced filtered bundles \( \mathcal{L}_p(\omega) \) denoted by \( \mathcal{P}^s(a) \mathcal{L}_p(\omega) = \mathcal{P}^s(a) \mathcal{L}_p(\omega)(\mathbb{Z}_{\nu,p}^{-1}(t)) \mid t \in S^1 \).

For general \( \omega \in \mathbb{Q} \), we take set \( k_1 := k(p \omega) \), \( \ell_1 := \ell(p \omega) \) and \( p_1 := p \cdot k_1 \). A filtered bundle \( \mathcal{P}^s(a) \mathcal{L}_p(\omega) \) over \( \mathcal{L}_p(\omega) \) is obtained as the push-forward of \( \mathcal{P}^s(k_1) \mathcal{L}_p(\omega) \).

**Lemma 4.10** \( \mathcal{P}^s(a) \mathcal{L}_p(\omega) \) is an object in \( \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda) \). We have the natural isomorphism \( \mathcal{G}(\mathcal{P}^s(a) \mathcal{L}_p(\omega)) \simeq \mathcal{L}_s(\omega, a) \).

Let \( \mathcal{P}_*, \mathcal{M} \in \text{LFM}(\hat{H}_{\nu,p}^\lambda, H_{\nu,p}^\lambda)^{\text{par}} \). There exists the slope decomposition \( \mathcal{M} = \bigoplus_{\omega \in \text{Slope}(\mathcal{M})} \mathcal{M}_\omega \), where each \( \mathcal{M}_\omega \) has pure slope \( \omega \). We take \( \omega \in \mathbb{Z}_{p>0} \) such that \( \omega \in \mathbb{Z}_{p>0} \) for any \( \omega \in \text{Slope}(\mathcal{M}) \). There exists an isomorphism

\[
\mathcal{P}^s_{p_1}(\mathcal{M}) \simeq \bigoplus_{\omega \in \text{Slope}(\mathcal{M})} \mathcal{L}_{p_1}(\omega) \otimes \mathcal{U}_\omega^{(p_1)}, \tag{39}
\]

where \( \mathcal{U}_\omega^{(p_1)} \) are Fuchsian. Then, we have

\[
\mathcal{P}^s_{p_1}(\mathcal{M}_{p_1}) \simeq \bigoplus_{\omega \in \text{Slope}(\mathcal{M})} \mathcal{P}^s_{p_1}(\mathcal{M}(\mathbb{Z}_{p_1}^{-1}(t))) \otimes \mathcal{P}^s_{\omega}(\mathcal{U}_\omega^{(p_1)}), \tag{40}
\]

where \( \mathcal{P}^s_{\omega}(\mathcal{U}_\omega^{(p_1)}) \) are isoclinic of pure slope 0.
4.1.5 Decomposition and weight filtration on the associated graded vector spaces

Let $\mathcal{P}_* \mathfrak{W}$ be a good filtered bundle on $(\tilde{H}_{\nu}^*, H_{\nu}^*)$ with the slope decomposition $\mathcal{P}_* \mathfrak{W} = \bigoplus_{\omega \in \Omega} \mathcal{P}_* \mathfrak{W}_\omega$. Let $\mathfrak{W}^{\text{cov}} = \bigoplus \mathfrak{W}_{\text{cov}}^{\alpha}$ denote the locally free $\mathfrak{O}_{\tilde{H}_{\nu}^{*, \text{cov}}} \ast H_{\nu}^{*, \text{cov}}$-module obtained as the pull back of $\mathfrak{W}$. Let $\mathcal{P}_*(\mathfrak{W}^{\text{cov}}) = \bigoplus \mathcal{P}_* \mathfrak{W}_{\text{cov}}^{\alpha}$ denote the transport along the path connecting $t_1, t_2 \in \mathbb{R}$, we obtain the isomorphism

$$\text{Gr}_{a}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t_1)) \simeq \text{Gr}_{a+p\omega(t_2-t_1)/t_1}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t_2)). \quad (41)$$

Recall that $\mathcal{G}(\mathcal{P}_* \mathfrak{W})$ is $(\mathbb{Q}, \mathbb{R})$-graded $\mathcal{G}(\mathcal{P}_* \mathfrak{W}) = \bigoplus_{\omega, a} \mathcal{G}(\mathcal{P}_* \mathfrak{W})_{\omega, a}$. Each $\mathcal{G}(\mathcal{P}_* \mathfrak{W})_{\omega, a}$ is equipped with the automorphism $F_{\omega, a}$ and a generalized eigen decomposition $\mathcal{G}(\mathcal{P}_* \mathfrak{W})_{\omega, a} = \bigoplus_{r \in \mathbb{C}, \omega} \mathcal{G}(\mathcal{P}_* \mathfrak{W})_{\omega, r, a}$. Moreover, it is equipped with the nilpotent endomorphism $N_{\omega, a}$ and the weight filtration $W$. By the construction, $\mathcal{G}(\mathcal{P}_* \mathfrak{W})_{\omega, a}$ is naturally identified with $\text{Gr}_{a}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(0))$. Hence, each $\text{Gr}_{a}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(0))$ is equipped with the automorphism $F_{\omega, a}$ and the generalized eigen decomposition $\text{Gr}_{a}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(0)) = \bigoplus \mathfrak{E}_r \text{Gr}_{a}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(0))$. Moreover, it is equipped with the nilpotent endomorphism $N_{\omega, a}$ and the weight filtration $W$.

By the isomorphisms (31), each $\text{Gr}_{a+p\omega(t-1)/t_1}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t))$ is equipped with the automorphism $F_{\omega, a}$ and the generalized eigen decomposition $\text{Gr}_{a+p\omega(t-1)/t_1}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t)) = \bigoplus \mathfrak{E}_r \text{Gr}_{a+p\omega(t-1)/t_1}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t))$. Moreover, it is equipped with the nilpotent endomorphism $N_{\omega, a}$ and the weight filtration $W$.

4.1.6 The associated local systems

By using the isomorphisms (31), we obtain a local system $L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$ on $H_{\nu}^{*, \text{cov}}$ by setting

$$L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) = \text{Gr}_{a+p\omega(t-1)/t_1}^{P} (\mathfrak{W}^{\text{cov}}_{\omega|p_p} - 1(t)).$$

We obtain the automorphism $F_{\omega, a}$, the decomposition $L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) = \bigoplus \mathfrak{E}_r L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$, the nilpotent endomorphism $N_{\omega, a} = \bigoplus N_{\omega, a}$, and the weight filtration $W$.

The multiplication of $\mathfrak{U}_{\nu, \rho}$ induces isomorphisms $L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) \simeq L_{\omega, a+\rho}(\mathcal{P}_* \mathfrak{W})$. We also have the isomorphisms

$$e_2^{\text{cov}} L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) \simeq L_{\omega, a+\rho}(\mathcal{P}_* \mathfrak{W}).$$

Therefore, the multiplication of $e_2^{\text{cov}}(p_\omega)$ induces an isomorphism

$$(e_2^{\text{cov}}(p_\omega)) L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) \simeq L_{\omega, a}(\mathcal{P}_* \mathfrak{W}).$$

Hence, we obtain systems $L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$ on $S^{1}_{\omega, \nu} := H_{\nu}^{*, \text{cov}} / k(p_\omega) \mathfrak{E}_2$.

We obtain the monodromy $F_{\omega, a}$ on $L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$. We obtain the generalized eigen decomposition $L_{\omega, a}(\mathcal{P}_* \mathfrak{W}) = \bigoplus_{r \in \mathbb{C}, \omega} \mathfrak{E}_r L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$ with respect to $F_{\omega, a}$. Let $N_{\omega, a} = \bigoplus N_{\omega, a}$ be the nilpotent endomorphism obtained as the logarithm of the unipotent part of $F_{\omega, a}$. Let $W$ be the weight filtration of $N_{\omega, a}$.

Their pull back to $L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$ are equal to the automorphism the decomposition, the nilpotent endomorphism and the weight filtration on $L_{\omega, a}(\mathcal{P}_* \mathfrak{W})$.

4.1.7 Local filtrations by lattices

Let $\mathcal{P}_* \mathfrak{W}$ be a good filtered bundle on $(\tilde{H}_{\nu}^*, H_{\nu}^*)$. Take $t_0 \in \mathbb{R}$. Take $a \in \mathbb{R}$. Take a small $\epsilon > 0$. Set $I(\nu, t_0, \epsilon) := \{ t \mid |t - t_0| < \epsilon \} \subset H_{\nu}^{*, \text{cov}}$ and $\tilde{I}(\nu, \nu, t_0, \epsilon) := B_{p} \times I(\nu, t_0, \epsilon)$. We obtain $P_{a}^{(t_0)} \mathfrak{W}^{\text{cov}} \subset \mathfrak{W}^{\text{cov}}_{|I(\nu, t_0, \epsilon)}$ determined by the following for $t \in I(\nu, t_0, \epsilon)$:

$$P_{a}^{(t_0)} \mathfrak{W}^{\text{cov}}_{|I(\nu, t_0, \epsilon)} = \bigoplus_{\omega \in \text{Slope}(\mathfrak{W})} \mathcal{P}_{a+p\omega(t-t_0)/t_1}^{(t_0)} \mathfrak{W}^{\text{cov}}_{|I(\nu, t_0, \epsilon)}.$$
4.2 Good filtered bundles with Dirac type singularity on \( (M_\lambda^\alpha; H_\lambda^\alpha, Z) \)

Let \( \pi_{\lambda}^{\text{cov}} : M_\lambda^\alpha \to \mathbb{R} \) denote the projection \( \pi_{\lambda}^{\text{cov}}(u, t) = t \). It induces \( \pi_\lambda : M_\lambda^\alpha \to S_\lambda^1 \). The fibers \( (\pi_{\lambda}^{\text{cov}})^{-1}(t) \subset M_\lambda^\alpha \) are identified with \( \mathbb{P}^1 \). For each \( t \in S_\lambda^1 \), by fixing its lift to \( \mathbb{R} \), we obtain the isomorphism \( \pi_{\lambda}^{-1}(t) \simeq \mathbb{P}^1 \).

Let \( Z \subset M_\lambda^\alpha \) be a finite subset. Let \( \mathcal{U} \) be a locally free \( \mathcal{O}_{\overline{M}_\lambda^\alpha \setminus Z}(\ast H_\lambda^\alpha) \)-module. A filtered bundle over \( \mathcal{U} \) is a family of filtered bundles \( \mathcal{P}_*(\mathcal{U}) = \{ \mathcal{P}_*(\mathcal{U})_{\pi_{\lambda}^{-1}(t)} \mid t \in S_\lambda^1 \} \) over \( \mathcal{U}_{\pi_{\lambda}^{-1}(t)} \). It induces filtered bundles \( \mathcal{P}_*(\mathcal{U}|_{\overline{M}_\lambda^\alpha}) \) \((\nu = 0, \infty)\) over \( \mathcal{U}|_{\overline{M}_\lambda^\alpha} \).

**Definition 4.11** \( \mathcal{P}_*(\mathcal{U}) \) is called good if the induced filtered bundles \( \mathcal{P}_*(\mathcal{U}|_{\overline{M}_\lambda^\alpha}) \) are good. If moreover each point of \( Z \) is Dirac type singularity of \( \mathcal{U} \), we say that \( \mathcal{P}_*\mathcal{U} \) is a good filtered bundle with Dirac type singularity over \( (M_\lambda^\alpha; H_\lambda^\alpha, Z) \).

**4.2.1 Degree and stability condition**

Let \( \mathcal{P}_*(\mathcal{U}) \) be a good filtered bundle with Dirac type singularity on \( (M_\lambda^\alpha; H_\lambda^\alpha, Z) \). We define the degree of \( \mathcal{P}_*(\mathcal{U}) \) as follows:

\[
\deg(\mathcal{P}_*(\mathcal{U})) := \int_{S_\lambda^1} \deg(\mathcal{P}_*(\mathcal{U})_{\pi_{\lambda}^{-1}(t)}) \, dt.
\]

Let \( \mathcal{U}_1 \subset \mathcal{U} \) be an \( \mathcal{O}_{\overline{M}_\lambda^\alpha}(\ast H_\lambda^\alpha) \)-submodule. Then, it is also locally free, and each point of \( Z \) is with Dirac type singularity. The induced filtered bundle \( \mathcal{P}_*(\mathcal{U}_1) \) is good. We say that \( \mathcal{P}_*(\mathcal{U}) \) is stable if

\[
\deg(\mathcal{P}_*(\mathcal{U}_1)) / \text{rank}(\mathcal{U}_1) < \deg(\mathcal{P}_*(\mathcal{U})) / \text{rank}(\mathcal{U})
\]

for any saturated submodules \( \mathcal{U}_1 \) of \( \mathcal{U} \) such that \( \mathcal{U}_1 \neq 0, \mathcal{U} \). We say that \( \mathcal{P}_*(\mathcal{U}) \) is semistable if

\[
\deg(\mathcal{P}_*(\mathcal{U}_1)) / \text{rank}(\mathcal{U}_1) \leq \deg(\mathcal{P}_*(\mathcal{U})) / \text{rank}(\mathcal{U})
\]

for any non-trivial submodules \( \mathcal{U}_1 \) of \( \mathcal{U} \). We say that \( \mathcal{P}_*(\mathcal{U}) \) is polystable if it is semistable and a direct sum of stable ones.

4.3 Good filtered bundles on neighbourhoods of \( H_{\nu, p}^\lambda \)

For \( \nu = 0, \infty \), let \( \mathcal{U}_{\nu, p}^\lambda \) be a neighbourhood of \( H_{\nu, p}^\lambda \). We set \( \mathcal{U}_{\nu, p}^\lambda := \mathcal{U}_{\nu, p}^\lambda \setminus H_{\nu, p}^\lambda \). The induced map \( \mathcal{U}_{\nu, p}^\lambda \to S_\lambda^1 \) is denoted by \( \pi_\nu \). Let \( \mathcal{U} \) be a locally free \( \mathcal{O}_{\overline{M}_\nu^\lambda}(\ast H_{\nu, p}^\lambda) \)-module. A filtered bundle over \( \mathcal{U} \) is a family of filtered bundles \( \mathcal{P}_*(\mathcal{U}_{\nu, p}) \) over \( \mathcal{U}_{\nu, p} \) \((\nu = 0, \infty)\). The tuple \( \mathcal{P}_*(\mathcal{U}_{\nu, p}) \) is denoted by \( \mathcal{P}_*(\mathcal{U}) \). A filtered bundle \( \mathcal{P}_*(\mathcal{U}) \) over \( \mathcal{U} \) is called good if the induced filtered bundle over \( \mathcal{U}|_{\overline{M}_\nu^\lambda} \) is good.

4.3.1 Filtrations by local lattices

For \( t_0 \in S_\lambda^1 \), we set \( I(t_0, \epsilon) := \{ t \mid |t - t_0| < \epsilon \} \). For \( a \in \mathbb{R} \), we obtain the lattice \( P_{\alpha}^{(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))}) \subset \mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon))) \) from \( P_{\alpha}^{(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))}) \). Thus, we have the filtration \( P_{\alpha}^{(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))}) \). The induced local system \( \text{Gr}_a \comp P_{\alpha}^{(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))}) \) on \( I(t_0, \epsilon) \) is equipped with the weight filtration \( W \). We also have the decomposition

\[
\text{Gr}_a \comp P_{\alpha}^{(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))}) = \bigoplus \text{Gr}_a \comp P_{\alpha}^{\omega(t_0)}(\mathcal{U}_{\pi_{\lambda}^{-1}(I(t_0, \epsilon)))})
\]

induced by \( \mathcal{U}|_{\overline{M}_\nu^\lambda} = \bigoplus \mathcal{U}_\omega \). The decomposition and the filtration \( W \) are compatible.
4.3.2 Compatible frame

We continue to use the notation in §4.3. Set \( r := \text{rank}(\mathcal{V}) \).

Definition 4.12 Let \( \mathbf{v} = (v_i | i = 1, \ldots, r) \) be a frame of \( \text{P}^{(t_0)}(\mathcal{V}) \) on a neighbourhood of \( \pi_p^{-1}(t_0) \). We say that \( \mathbf{v} \) is compatible with the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \) and the slope decomposition if there exists a decomposition \( \{1, \ldots, r\} = \bigsqcup_{\omega \in \text{Slope}(\mathcal{V})} \bigsqcup_{a-1 < b \leq a} \gamma_{\omega,b} \) such that \( (v_i | i \in \gamma_{\omega,b}) \) induces a frame of \( \text{Gr}_k^W \text{P}_r^{(t_0)}(\mathcal{V}_\omega) \) for \( a-1 < b \leq a \).

We say that \( \mathbf{v} \) is compatible with the slope decomposition, the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \) and the filtration \( W \) if there exists a decomposition \( \{1, \ldots, r\} = \bigsqcup_{\omega \in \text{Slope}(\mathcal{V})} \bigsqcup_{a-1 < b \leq a} \bigsqcup_{k \in \mathbb{Z}} \bigsqcup_{t \in \mathbb{R}} \gamma_{\omega,b,k} \) such that \( (v_i | i \in \gamma_{\omega,b,k}) \) induces a frame of \( \text{Gr}_k^W \text{P}_r^{(t_0)}(\mathcal{V}_\omega) \).

Take a local frame \( \mathbf{v} \) of \( \text{P}^{(t_0)}(\mathcal{V}) \) compatible with the slope decomposition and the filtration \( \text{P}_r^{(t_0)} \). We set \( b(v_i) := b \) and \( \omega(v_i) := \omega \) if \( i \in \gamma_{\omega,b} \). If moreover \( \mathbf{v} \) is compatible with \( W \), we also set \( k(v_i) := k \) if \( i \in \gamma_{\omega,b,k} \).

4.3.3 Adaptedness and norm estimate

Let \( \mathcal{P}_*\mathcal{V} \) be a good filtered bundle over \( \mathcal{V} \). Let \( V \) be the mini-holomorphic bundle on \( \mathcal{U}_{\nu,p}^\lambda \). Let \( P \) be a point of \( H_{\nu,p}^\lambda \). Let \( U_P \) be a neighbourhood of \( P \) in \( \overline{U}_{\nu,p}^\lambda \). Let \( \mathbf{v} \) be a frame of \( \text{P}_r^{(t_0)}(\mathcal{V}) \) on \( U_P \) compatible with the slope decomposition and the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \). Let \( h_{P,v} \) be the Hermitian metric of \( V|_{U_P \setminus H_{\nu,p}^\lambda} \) determined by

\[
h_{P,v}(v_i, v_j) := \begin{cases} |\nu_{\nu,p}|^{-2b(v_i) - 2\omega(v_i)(t - t_0)/t^k} & (i = j) \\ 0 & (i \neq j). \end{cases}
\]

If moreover \( \mathbf{v} \) is compatible with the filtration \( W \), then let \( \tilde{h}_{P,v} \) be the Hermitian metric of \( V|_{U_P \setminus H_{\nu,p}^\lambda} \) determined by

\[
\tilde{h}_{P,v}(v_i, v_j) := \begin{cases} |\nu_{\nu,p}|^{-2b(v_i) - 2\omega(v_i)(t - t_0)/t^k} (-\log |\nu_{\nu,p}|)^{k(v_i)} & (i = j) \\ 0 & (i \neq j). \end{cases}
\]

The following is easy to see.

Lemma 4.13 Let \( \mathbf{v} \) and \( \mathbf{v}' \) be frames of \( \text{P}_r^{(t_0)}(\mathcal{V}) \) on \( U_P \) compatible with the slope decomposition and the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \). Take a relative compact neighbourhood \( U_P' \) of \( P \) in \( U_P \). Then, \( h_{P,v} \) and \( h_{P,v}' \) are mutually bounded on \( U_P' \setminus H_{\nu,p}^\lambda \). If moreover both \( \mathbf{v} \) and \( \mathbf{v}' \) are compatible with \( W \), then \( \tilde{h}_{P,v} \) and \( \tilde{h}_{P,v}' \) are mutually bounded on \( U_P' \setminus H_{\nu,p}^\lambda \).

Definition 4.14 A Hermitian metric \( h \) of \( V \) is called adapted to \( \mathcal{P}_*\mathcal{V} \) around \( P \) if the following holds.

- Let \( \mathbf{v} \) be a frame of \( \text{P}_r^{(t_0)}(\mathcal{V}) \) on a neighbourhood \( U_P \) of \( P \) compatible with the slope decomposition and the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \). Then, for any smaller neighbourhood \( U_P' \subset U_P \) and for any \( \epsilon \), there exists \( C_\epsilon > 1 \) such that

\[
C_\epsilon^{-1}|\nu_{\nu,p}|^{-\epsilon}h_{P,v} \leq h \leq C_\epsilon|\nu_{\nu,p}|^{-\epsilon}h_{P,v}
\]

on \( U_P' \setminus H_{\nu,p}^\lambda \).

We say that \( \mathcal{P}_*\mathcal{V} \) is adapted to \( h \) if it is adapted to \( h \) around any point of \( H_{\nu,p}^\lambda \).

Definition 4.15 Let \( h \) be a Hermitian metric of \( V \). We say that the norm estimate holds for \( \mathcal{P}_*\mathcal{V} \) and \( h \) around \( P \), if the following holds.

- Let \( \mathbf{v} \) be a frame of \( \text{P}_r^{(t_0)}(\mathcal{V}) \) on a neighbourhood \( U_P \) of \( P \) compatible with the slope decomposition, the filtration \( \text{P}_r^{(t_0)}(\mathcal{V}) \) and \( W \). Then, for any smaller neighbourhood \( U_P' \subset U_P \) there exists \( C > 1 \) such that

\[
C^{-1}\tilde{h}_{P,v} \leq h \leq C\tilde{h}_{P,v}
\]

on \( U_P' \setminus H_{\nu,p}^\lambda \).

We say that the norm estimate holds for \( \mathcal{P}_*\mathcal{V} \) and \( h \) if the norm estimate holds around any point of \( H_{\nu,p}^\lambda \).
4.4 Approximation

We use the notation in [13]. Let \( \mathcal{C}_\nu^\infty \) denote the sheaf of \( C^\infty \)-functions on \( \mathcal{B}_\nu^\infty \). For good filtered bundles \( \mathcal{P}_\nu^\lambda \mathcal{F}^{(i)} \) \( (i = 1, 2) \) over \( (\mathcal{U}_\nu^\lambda, H^\lambda_{\nu,p}) \), a \( C^\infty \)-isomorphism of \( f : \mathcal{P}_\nu^\lambda \mathcal{F}^{(1)} \simeq \mathcal{P}_\nu^\lambda \mathcal{F}^{(2)} \) means an isomorphism \( \mathcal{F}^{(1)} \otimes \mathcal{C}_\nu^\infty \simeq \mathcal{F}^{(2)} \otimes \mathcal{C}_\nu^\infty \) such that the restriction to \( \pi_p^{-1}(t) \) preserve the induced filtrations.

The following lemma is clear.

**Lemma 4.16** Let \( \mathcal{P}_\nu^\lambda \mathcal{F}^{(i)} \) \( (i = 1, 2) \) be good filtered bundles \( (\mathcal{U}_\nu^\lambda, H^\lambda_{\nu,p}) \). If there exists an isomorphism \( \hat{f} : \mathcal{P}_\nu^\lambda \mathcal{F}^{(1)} \simeq \mathcal{P}_\nu^\lambda \mathcal{F}^{(2)} \), then there exists an isomorphism

\[
f_{C^\infty} : \mathcal{P}_\nu^\lambda \mathcal{F}^{(1)} \otimes \mathcal{C}_\nu^\infty \simeq \mathcal{P}_\nu^\lambda \mathcal{F}^{(2)} \otimes \mathcal{C}_\nu^\infty
\]

whose restriction to \( \hat{H}_{\nu,p}^\lambda \) is equal to \( \hat{f} \).

**Lemma 4.17** Let \( \mathcal{P}_\nu^\lambda \mathcal{F}^{(i)} \) \( (i = 1, 2) \) be good filtered bundles \( (\mathcal{U}_\nu^\lambda, H^\lambda_{\nu,p}) \). If there exists an isomorphism \( f^a : \mathcal{G}(\mathcal{P}_\nu^\lambda \mathcal{F}^{(1)}) \simeq \mathcal{G}(\mathcal{P}_\nu^\lambda \mathcal{F}^{(2)}) \), there exists an isomorphism

\[
f : \mathcal{P}_\nu^\lambda \mathcal{F}^{(1)} \otimes \mathcal{C}_\nu^\infty \simeq \mathcal{P}_\nu^\lambda \mathcal{F}^{(2)} \otimes \mathcal{C}_\nu^\infty
\]

such that the following holds.

- For each \( t \in S^1_\lambda \), the restriction of \( f \) to \( \pi_p^{-1}(t) \) is holomorphic and preserves the filtrations.
- The induced morphism \( \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t)}) \) preserves the decomposition \( \text{Gr}_\mu^p(\mathcal{F}^{(i)}) = \bigoplus \text{Gr}_\mu^p(\mathcal{G}^{(i)}) \) induced by the slope decomposition \( \mathcal{F}^{(i)} = \bigoplus \mathcal{G}^{(i)} \). As a result, we obtain the decomposition \( \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t)}) = \bigoplus \text{Gr}_\mu^p(\mathcal{F}^{(i)}) \).
- If \( t_1 - t_2 \) is small, \( \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t_1)}) \omega \) and \( \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t_2)}) \omega \) are equal under the natural isomorphism

\[
\text{Gr}_\mu^p(f_{|\pi_p^{-1}(t_1)}) \omega \bigotimes \mathcal{G}^{(i)}(\mathcal{F}^{(i)}) \bigotimes \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t_2)}) \omega \bigotimes \mathcal{G}^{(i)}(\mathcal{G}^{(i)}) \bigotimes \text{Gr}_\mu^p(f_{|\pi_p^{-1}(t_1)}) \omega \bigotimes \mathcal{G}^{(i)}(\mathcal{G}^{(i)})
\]

**Proof** The isomorphism \( f^a \) induces an isomorphism \( f^a_{\pi_p^{-1}(t_1)} : \mathcal{G}(\mathcal{F}^{(1)})_{\omega p^{-1}(t_1)} \simeq \mathcal{G}(\mathcal{F}^{(2)})_{\omega p^{-1}(t_2)} \) for any \( a \in \mathbb{R}, \omega \in \mathbb{Q} \) and \( t \in S^1_\lambda \). For each \( t \in S^1_\lambda \), we take a small neighbourhood \( I(t_0) \) in \( S^1_\lambda \). We can take a holomorphic isomorphism \( f_{I(t_0)} : \mathcal{G}(\mathcal{F}^{(1)})_{\omega p^{-1}(t_0)} \simeq \mathcal{G}(\mathcal{F}^{(2)})_{\omega p^{-1}(t_0)} \) such that the following holds:

- For each \( t \in I(t_0) \), the restriction to \( \pi_p^{-1}(t) \) preserves the filtrations.
- The induced isomorphism \( \text{Gr}_\mu^p(\mathcal{F}^{(1)})_{\omega p^{-1}(t)} \simeq \text{Gr}_\mu^p(\mathcal{F}^{(2)})_{\omega p^{-1}(t)} \) is equal to \( f^a_{\pi_p^{-1}(t_2)} \).

We take a finite covering \( S^1_\lambda = \bigcup_{i=1}^N I(t_0(i)) \) and a partition of unity \( \{ \chi_i \} \) subordinate to the covering. We construct a \( C^\infty \)-isomorphism \( f \) as \( f = \sum_{i=1}^N \chi_i f_{I(t_0(i))} \). Then, \( f \) satisfies the conditions.

5 Basic examples of doubly periodic monopoles

5.1 Examples (1)

5.1.1 Construction

On \( \mathbb{A}^0 \), we have the mini-complex coordinate system \( (z, y) \), where \( y := \text{Im}(w) \). Let \( \mathbb{C} \cdot e \) denote the product line bundle on \( \mathbb{A}^0 \) with a global frame \( e \). Let \( h \) be the metric given by \( h(e, e) = 1 \). We consider the \( \mathbb{Z} e_1 \)-action on \( \mathbb{C} e \) given by \( e_1(e) = e \). It induces an action of \( \mathbb{Z}(m e_1) \) for any \( m \in \mathbb{Z}_{>0} \) as the restriction.
Take a positive integer $p$ and a rational number $\omega \in \frac{1}{p}\mathbb{Z}$. We have the expression $\omega = \ell(\omega)/k(\omega)$, where $k(\omega) \in \mathbb{Z}_{>0}$, $\ell(\omega) \in \mathbb{Z}$ and $\gcd(k(\omega), \ell(\omega)) = 1$. We set

$$\alpha(\omega) := \frac{2\pi \omega}{\text{Vol}(\Gamma)}.$$ 

We define the $\mathbb{Z}_e$-action on $\mathbb{C} \epsilon$ by

$$e^*_{\epsilon}(\epsilon) \mapsto \epsilon \cdot \exp\left(-\sqrt{1} \text{Vol}(\Gamma) \alpha(\omega) |\mu_1|^2 \text{Re}(\overline{\mu}_1 z)\right) = \epsilon \cdot \exp\left(-2\pi \sqrt{1} \omega |\mu_1|^2 \text{Re}(\overline{\mu}_1 z)\right).$$

**Lemma 5.1** The actions of $\mathbb{Z}(k(\omega)e_1)$ and $\mathbb{Z}e_2$ are commutative, i.e., the action of $\mathbb{Z}(k(\omega)e_1) \oplus \mathbb{Z}e_2$ on $\mathbb{C} \epsilon$ is well defined.

**Proof** It follows from $\exp\left(-2\pi \sqrt{1} \omega |\mu_1|^2 \text{Re}(\overline{\mu}_1 z)\right) = \exp\left(-2\pi \sqrt{1} k(\omega)\omega\right) = 1$.

Let $\phi_{p,\omega}$ be the Higgs field given as $\phi_{p,\omega} = \sqrt{-1} \omega y$. We define the connection $\nabla_{p,\omega}$ by

$$\nabla_{p,\omega} \epsilon = \epsilon \left(-\frac{\alpha(\omega)}{4}\right) |\mu_1|^2 (\overline{\mu}_1 z - \mu_1 \overline{z})(\overline{\mu}_1 d\overline{z} + \mu_1 dz).$$

**Lemma 5.2** The Bogomolny equation $F(\nabla_{p,\omega}) = *\nabla_{p,\omega} \phi_{p,\omega}$ is satisfied.

**Proof** We have $\nabla_{p,\omega} \phi_{p,\omega} = \sqrt{-1} \alpha(\omega) dy$, and hence $\nabla_{p,\omega} \phi_{p,\omega} = -\frac{1}{2} \alpha(\omega) dz d\overline{z}$. We also have

$$F(\nabla_{p,\omega}) = -\frac{\alpha(\omega)}{4} |\mu_1|^2 (\overline{\mu}_1 dz - \mu_1 d\overline{z})(\mu_1 d\overline{z} + \mu_1 dz) = -\frac{\alpha(\omega)}{2} dz d\overline{z}.$$ 

Hence, the Bogomolny equation is satisfied.

**Lemma 5.3** $(k(\omega)e_1)^* \nabla_{p,\omega} = \nabla_{p,\omega}$ and $e^*_2 \nabla_{p,\omega} = \nabla_{p,\omega}$.

**Proof** The claim $(k(\omega)e_1)^* \nabla_{p,\omega} = \nabla_{p,\omega}$ is clear. Because

$$e^*_2(\nabla_{p,\omega})e^*_2(\epsilon) = e^*_2(\epsilon) \cdot \left(-\frac{\alpha(\omega)}{4}\right) |\mu_1|^2 (\overline{\mu}_1 z - \mu_1 \overline{z} + \overline{\mu}_1 \mu_2 - \mu_1 \overline{\mu}_2)(\overline{\mu}_1 d\overline{z} + \mu_1 dz)

= e^*_2(\epsilon) \cdot \left(-\frac{\alpha(\omega)}{4}\right) |\mu_1|^2 (\overline{\mu}_1 z - \mu_1 \overline{z} + 2\sqrt{-1} \text{Vol}(\Gamma))(\overline{\mu}_1 d\overline{z} + \mu_1 dz)$$

we obtain $e^*_2 \nabla_{p,\omega} = \nabla_{p,\omega}$.

The monopole $(\mathbb{C} \epsilon, h, \nabla_{p,\omega}, \phi_{p,\omega})$ on $A^0$ is denoted by $L_p(\omega)$. Because it is equivariant with respect to $\mathbb{Z}k(\omega)e_1 \oplus \mathbb{Z}e_2$, we obtain a monopole $L^\text{cov}_p(\omega)$ on $M^0_p\text{cov}$, and a monopole $L_p(\omega)$ on $M^0_p$. Moreover, the monopoles are equivariant with respect to the $(k(\omega))\mathbb{Z}/p\mathbb{Z})e_1$-action.

Let $L^\lambda_p(\omega)$ be the mini-holomorphic bundle on $M^\lambda_p\text{cov}$ underlying $L^\text{cov}_p(\omega)$, which is naturally $\mathbb{Z}e_2$-equivariant. Let $L^\lambda_p(\omega)$ be the mini-holomorphic bundle on $M^\lambda_p$ underlying $L_p(\omega)$, which is obtained as the descent of $L^\lambda_p\text{cov}(\omega)$. The mini-holomorphic bundles are equivariant with respect to the $(k(\omega))\mathbb{Z}/p\mathbb{Z})e_1$-action.

### 5.1.2 Corresponding Instantons on $X$

Let $L_p(\omega) = (\mathbb{C} \epsilon, h, \nabla_{p,\omega})$ denote the $\mathbb{R} e_0 \oplus \mathbb{Z}k(\omega)e_1 \oplus \mathbb{Z}e_2$-equivariant instanton on $X$ corresponding to $L_p(\omega)$. We obtain $h(\epsilon, \epsilon) = 1$ and

$$\nabla_{p,\omega} \epsilon = \epsilon \left(-\frac{\alpha(\omega)}{4}\right) (|\mu_1|^2 (\overline{\mu}_1 z - \mu_1 \overline{z})(\overline{\mu}_1 d\overline{z} + \mu_1 dz) - (w - \overline{w})(dw + d\overline{w}))$$

Let $(\tilde{L}^\lambda_p(\omega), \tilde{F})$ be the underlying holomorphic vector bundle on $X^\lambda$, which is equivariant with respect to the $\mathbb{R} e_0 \oplus \mathbb{Z}k(\omega)e_1 \oplus \mathbb{Z}e_2$-action.
Lemma 5.4  The following holds:

\[ \overline{\mathcal{T}}^\chi = \frac{-\alpha(\omega)}{4} \frac{1}{(1 + |\lambda|^2)^2} \left( -(1 + |\lambda|^2) \xi d\xi^\ast + (1 + |\lambda|^2) \eta d\eta + (\overline{\mathcal{T}}^\chi_1|\mu_1|^{-2} \lambda - \overline{\nu}) \xi d\gamma \right. \]

\[ + (\mu_1^2|\mu_1|^{-2} \lambda + \lambda) \eta \xi d\xi + (\mu_1^2|\mu_1|^{-2} + \lambda^2) \xi d\xi^\ast - (\lambda^2 \overline{\mathcal{T}}^\chi_1|\mu_1|^{-2} + 1) \eta d\eta \bigg). \] (44)

Proof  In the proof, \( \alpha(\omega) \) is denoted by \( \alpha \). Because

\[ z = \frac{1}{1 + |\lambda|^2}(\xi - \lambda \eta), \quad w = \frac{1}{1 + |\lambda|^2}(\eta + \lambda \xi), \]

the following holds:

\[ \overline{\mathcal{T}}_1 z - \mu_1 \overline{\xi} = \frac{1}{1 + |\lambda|^2} (\overline{\mathcal{T}}_1 \xi - \lambda \overline{\mathcal{T}}_1 \eta - \mu_1 \xi + \mu_1 \overline{\lambda} \eta), \]

\[ \overline{\mathcal{T}}_1 dz + \mu_1 d\overline{\xi} = \frac{1}{1 + |\lambda|^2} (\overline{\mathcal{T}}_1 d\xi - \mu_1 \lambda d\gamma + \mu_1 d\xi^\ast - \mu_1 \lambda d\gamma). \]

Hence, we obtain

\[ (\overline{\mathcal{T}}_1 z - \mu_1 \overline{\xi})(\overline{\mathcal{T}}_1 dz + \mu_1 d\overline{\xi}) = \frac{1}{(1 + |\lambda|^2)^2} \left( \overline{\mathcal{T}}_1^2 \xi \xi d\xi^\ast + |\mu_1|^2 \xi \xi d\xi^\ast - \overline{\mathcal{T}}_1^2 \lambda \xi d\gamma - |\mu_1|^2 \lambda \xi d\gamma 

- |\mu_1|^2 \xi \xi d\xi^\ast - \overline{\mathcal{T}}^\chi \xi d\gamma 

- \lambda \overline{\mathcal{T}}_1 \eta d\xi - \lambda |\mu_1|^2 \overline{\nu} d\gamma + \lambda^2 \overline{\mathcal{T}}_1 \eta d\gamma + |\lambda|^2 |\mu_1|^2 \eta d\eta 

+ |\mu_1|^2 \overline{\nu} d\gamma + \mu_1 \lambda \overline{\nu} \xi d\xi^\ast - |\mu_1|^2 |\lambda|^2 \eta d\gamma - \mu_1^2 \lambda^2 \eta d\eta \right). \] (45)

Note that the following also holds:

\[ (w - \overline{w})(dw + d\overline{w}) = \frac{1}{(1 + |\lambda|^2)^2} \left( \eta d\eta + \lambda \eta d\xi^\ast + \eta d\overline{\lambda} d\xi^\ast 

+ \lambda^2 \eta d\xi^\ast + \lambda \xi d\xi^\ast + |\lambda|^2 \xi d\xi^\ast - \eta d\gamma - \lambda^2 \xi d\xi^\ast - \eta d\gamma - \lambda \xi d\xi^\ast \right). \] (46)

Hence, we obtain

\[ \left( \frac{-\alpha}{4} |\mu_1|^{-2} (\overline{\mathcal{T}}_1 z - \mu_1 \overline{\xi})(\overline{\mathcal{T}}_1 dz + \mu_1 d\overline{\xi}) + \frac{\alpha}{4} (w - \overline{w})(dw + d\overline{w}) \right)^{0.1} = \]

\[ \frac{\alpha}{4} \frac{1}{(1 + |\lambda|^2)^2} \left( -(1 + |\lambda|^2) \xi d\xi^\ast + (1 + |\lambda|^2) \eta d\eta + (\overline{\mathcal{T}}_1^2|\mu_1|^{-2} \lambda - \overline{\nu}) \xi d\gamma 

+ (\mu_1^2|\mu_1|^{-2} \lambda + \lambda) \eta \xi d\xi + (\mu_1^2|\mu_1|^{-2} + \lambda^2) \xi d\xi^\ast - (\lambda^2 \overline{\mathcal{T}}_1^2|\mu_1|^{-2} + 1) \eta d\gamma \right). \] (47)

Thus, we obtain the claim of the lemma. \( \blacksquare \)

5.1.3 Holomorphic frame of \( \hat{E}_p^\lambda(\omega) \)

We consider the following holomorphic frame of \( \hat{E}_p^\lambda(\omega) \) on \( \mathcal{X}^\lambda \):

\[ \hat{\varphi}_{\mu_1}^{\lambda, \omega} := \mathcal{E} \exp \left( \frac{-\alpha(\omega)}{4} \frac{1}{(1 + |\lambda|^2)^2} \left( (1 + |\lambda|^2) \xi d\xi^\ast - (1 + |\lambda|^2) \eta d\eta 

- (\overline{\mathcal{T}}_1^2|\mu_1|^{-2} \lambda - \overline{\nu}) \xi d\gamma - (\mu_1^2|\mu_1|^{-2} \lambda + \lambda) \eta \xi d\xi + (\mu_1^2|\mu_1|^{-2} + \lambda^2) \xi d\xi^\ast \right) \right). \] (48)
Lemma 5.5 We have $(e_0)^*\overline{\nu}_{p,\omega}^\lambda = \overline{\nu}_{p,\omega}^\lambda$ and $(k(\omega)e_1)^*\overline{\nu}_{p,\omega}^\lambda = \overline{\nu}_{p,\omega}^\lambda$. We also have

$$e_2^*\overline{\nu}_{p,\omega}^\lambda = \overline{E}_{p,\omega}^\lambda \cdot U_p^\omega \exp\left(\frac{\alpha(\omega)}{4}(\mu_1 + \lambda^2 \mathfrak{m}_1)^{-1}(\mu_1 - \lambda^2 \mathfrak{m}_1) \cdot |\mu_1|^{-2}(|\mu_1|^2|\mu_2|^2 - 2 \mathfrak{m}_1^2|\mu_2|^2 - 2 \mathfrak{m}_2^2|\mu_1|^2)/2\right).$$

(49)

Proof We can check $e_0^*\overline{\nu}_{p,\omega}^\lambda = \overline{\nu}_{p,\omega}^\lambda$ and $(k(\omega)e_1)^*\overline{\nu}_{p,\omega}^\lambda = \overline{\nu}_{p,\omega}^\lambda$ by direct computations. We give an indication to check the formula (49). We have

$$e_2^*\overline{\nu}_{p,\omega}^\lambda = \overline{E}_{p,\omega}^\lambda \exp\left(-\frac{\alpha(\omega)}{4}(\mu_1)^{-2}(\mathfrak{m}_1 + \mu_1 \mathfrak{m}_2)(\mathfrak{m}_1 + \mu_1 \mathfrak{m}_2)\right) \exp\left(\frac{\alpha(\omega)}{4(1 + |\lambda|^2)^2}F\right),$$

where

$$G = (1 + |\lambda|^2)(\xi + \mu_2)(\xi + \mu_2) - (1 + |\lambda|^2)(\eta(\lambda \mathfrak{m}_2(\mathfrak{m}_1 - \lambda \mathfrak{m}_2) - \eta(\lambda \mathfrak{m}_2(\mathfrak{m}_1 - \lambda \mathfrak{m}_2) - \eta|\lambda|)$$

$$- (\mu_2^2|\mu_1|^2 + \lambda^2)(\mathfrak{m}_1 + \mu_2^2 - \xi)(\mathfrak{m}_1 + \mu_2^2 - \xi) - (\lambda^2 \mathfrak{m}_1^2|\mu_1|^2 + 1)(\mathfrak{m}_1 + \mu_2^2 - \xi)$$

$$- (\mathfrak{m}_1^2 + \lambda^2)(\mathfrak{m}_1 + \mu_2^2 - \xi)^2 + (\xi + \mu_2)(\xi + \mu_2^2 - \xi)^2 - (\lambda^2 \mathfrak{m}_1^2|\mu_1|^2 + 1)(\mathfrak{m}_1 + \mu_2^2 - \xi)$$

$$- (\xi + \mu_2)(\xi + \mu_2^2 - \xi)^2 - (\xi + \mu_2)(\xi + \mu_2^2 - \xi)^2 - (\xi + \mu_2)(\xi + \mu_2^2 - \xi)^2 - (\xi + \mu_2)(\xi + \mu_2^2 - \xi)^2.$$ (50)

We set

$$F := -|\mu_1|^{-2}(\mathfrak{m}_1 + \mu_1 \mathfrak{m}_2)(1 + |\lambda|^2)(\mathfrak{m}_1 + \mu_1 \mathfrak{m}_2)$$

$$+ (1 + |\lambda|^2)(\mu_2^2 + \mu_2 \mathfrak{m}_1 + |\mu_2|^2 + \lambda \mathfrak{m}_2 \gamma + \mu_2 \mathfrak{m}_2 \gamma - |\lambda|^2 |\mu_2|^2) - \left(\mathfrak{m}_1^2 \lambda^2|\mu_1|^2 - \lambda(\mathfrak{m}_1 + \mu_2 \mathfrak{m}_2) - \mathfrak{m}_1^2 \lambda^2|\mu_1|^2 - \lambda(\mathfrak{m}_1 + \mu_2 \mathfrak{m}_2)\right)$$

$$- (\mu_2^2|\mu_1|^2 + \lambda) - (\mu_2^2|\mu_1|^2 + \lambda) - \left(\mathfrak{m}_1^2 \lambda^2|\mu_1|^2 + \lambda^2\right)$$

$$+ \left(\mathfrak{m}_1^2 \lambda^2|\mu_1|^2 + 1\right) - (\mu_2^2|\mu_1|^2 + 1) - (\mu_2^2|\mu_1|^2 + 1) - (\mu_2^2|\mu_1|^2 + 1) - (\mu_2^2|\mu_1|^2 + 1)$$

$$- (\mu_2^2|\mu_1|^2 + 1) - (\mu_2^2|\mu_1|^2 + 1) - (\mu_2^2|\mu_1|^2 + 1)$$

Then, we have

$$e_2^*\overline{\nu}_{p,\omega}^\lambda = \overline{E}_{p,\omega}^\lambda \exp\left(\frac{\alpha(\omega)}{4(1 + |\lambda|^2)^2}F\right).$$

We have the expression $F = A_1 \xi + A_2 \mathfrak{m}_1 + A_3 \xi + A_4 \eta + A_5$ for some constants $A_i$. Because $\overline{\nu}_{p,\omega}^\lambda$ and $e_0^*\overline{\nu}_{p,\omega}^\lambda$ are holomorphic and $e_0$-invariant, we have $A_1 = A_2 = 0$ and $A_4 = -\lambda A_3$. By a direct computation, we obtain that

$A_3 = 2(1 + |\lambda|^2)^2(\mu_1 \mathfrak{m}_2 - \mu_2 \mathfrak{m}_1)(\mu_1 \mathfrak{m}_2 - \mu_2 \mathfrak{m}_1)^{-1} = -4(1 + |\lambda|^2)^2 \sqrt{-1} \text{Vol}(\Gamma)(\mu_1 + \lambda^2 \mathfrak{m}_1)^{-1}.$

We can also obtain the following by a direct computation:

$A_5 = (1 + |\lambda|^2)^2(\mu_1 + \lambda^2 \mathfrak{m}_1)^{-1}(\mu_1 - \lambda^2 \mathfrak{m}_1)|\mu_1|^{-2}(|\mu_2|^2|\mu_1|^2 - \mu_2^2 \mathfrak{m}_1^2/2 - \mu_2^2 \mathfrak{m}_1^2/2).$

Then, we obtain the desired formula.

Let us study the growth order of $|\overline{\nu}_{p,\omega}^\lambda|$ as $U_p \to 0$ or $U_p \to \infty$. Recall $U_p = \exp(2\pi \sqrt{-1}p^{-1}(\mu_1 + \lambda \mathfrak{m}_1)^{-1})u$. We describe

$$u = p(\mu_1 + \lambda \mathfrak{m}_1)^{c + \sqrt{-1}\sigma} \sqrt{-1}$$

for real numbers $c$ and $\sigma$. 

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Lemma 5.6 We have
\[ |\tilde{\mathcal{V}}_{\mu,\omega}^\lambda| \sim \exp\left(\alpha(\omega) \Im(v) \Re(g_1\mu_1)pc\right) = \exp\left(p\omega \Im(v) \frac{\Re(g_1\mu_1)}{\Vol(\Gamma)}2\pi c\right) = \exp\left(-p\omega \Im(v)(t^\lambda - 1)2\pi c\right). \]

Proof We have
\[ |\tilde{\mathcal{V}}_{\mu,\omega}^\lambda| = \exp\left(\frac{\alpha(\omega)}{4(1 + |\lambda|^2)^2} \Re\left((1 + |\lambda|^2)(u + \lambda v)/(\overline{u} + \lambda \overline{v}) - (1 + |\lambda|^2)(g_1u + v)(\overline{g}_1\overline{u} + \overline{v}) - \overline{\lambda}^2(u + \lambda v)^2 + (g_1u + v)^2 - u(1 - g_1\lambda)/2\lambda(\overline{g}_1u + v) - (\mu_1 + \lambda^2\overline{\mu}_1)^{-1}(2|\lambda|^2\overline{\mu}_1 - \overline{\lambda}^2\mu_1 + \overline{\mu}_1)u^2(1 - g_1\lambda)^2\right). \] (52)

Let us look at the quadratic term with respect to \(u\).
\[ - \overline{\lambda}^2u^2 + g_1^2u^2 - 2\overline{\lambda}g_1u^2 + 2g_1^2|\lambda|^2u^2 - (\mu_1 + \lambda^2\overline{\mu}_1)^{-1}(2|\lambda|^2\overline{\mu}_1 - \overline{\lambda}^2\mu_1 + \overline{\mu}_1)(1 - g_1\lambda)^2u^2. \] (53)

We have
\[ g_1u = (-\lambda\overline{\mu}_1 + s_1)p(c + \sqrt{-1}\sigma)/\sqrt{-1}. \]

Hence, we can rewrite (53) as follows:
\[ \left(\overline{\lambda}^2(\mu_1 + \lambda s_1)^2 - (-\lambda\overline{\mu}_1 + s_1)^2 + 2\overline{\lambda}(\mu_1 + \lambda s_1)(-\lambda\overline{\mu}_1 + s_1) - 2|\lambda|^2(-\lambda\overline{\mu}_1 + s_1)^2 + (\mu_1 + \lambda^2\overline{\mu}_1)^{-1}(2|\lambda|^2\overline{\mu}_1 - \overline{\lambda}^2\mu_1 + \overline{\mu}_1)\right) \times p^2(c + \sqrt{-1}\sigma)^2. \] (54)

It is equal to \(p^2(c + \sqrt{-1}\sigma)^2(1 + |\lambda|^2)^2(1 - |\lambda|^2)s_1^2 + 2(\lambda\overline{\mu}_1 + \overline{\lambda}\mu_1)s_1 + (1 - |\lambda|^2)|\mu_1|^2\). By our choice of \(s_1\), it is 0.

Let us study the linear term with respect to \(u\) and \(\overline{u}\).
\[ \Re\left(1 + |\lambda|^2\right)\left(\overline{\lambda}u\overline{v} + \lambda uv - g_1u\overline{v} - g_1\overline{u}v\right) - 2|\lambda|^2uv + 2\overline{\lambda}|\lambda|^2uv - (1 - g_1\lambda)2\lambda uv \]
\[ = -2(1 + |\lambda|^2)\Re\left(\overline{u}(\overline{x} - g_1)(v - v)\right). \] (55)

Because \(u = p(\mu_1 + \lambda s_1)(c + \sqrt{-1}\sigma)/\sqrt{-1}\), it is rewritten as follows:
\[ -4(1 + |\lambda|^2)\Im(v)\Re(p(c + \sqrt{-1}\sigma)(\mu_1 + \lambda s_1)(\overline{x} - g_1)). \] (56)

We have the following:
\[ (\mu_1 + \lambda s_1)\overline{x} - g_1(\mu_1 + \lambda s_1) = \overline{x}(\mu_1 + \lambda s_1) - (-\lambda\overline{\mu}_1 + s_1) = (|\lambda|^2 - 1)s_1 + \overline{\lambda}\mu_1 + \lambda\overline{\mu}_1 \in \mathbb{R}. \]

We also have
\[ (\mu_1 + \lambda s_1)\overline{x} - g_1(\mu_1 + \lambda s_1) = \overline{g}_1(-\lambda\overline{\mu}_1 + s_1)\overline{x} - g_1(\mu_1 + \lambda s_1) = -|\lambda|^2\overline{g}_1\overline{\mu}_1 - g_1\mu_1 + s_1(\overline{\lambda}g_1 - \lambda g_1). \] (57)

Because (57) is real, it is equal to
\[ \frac{1}{2}\left(-|\lambda|^2\overline{g}_1\overline{\mu}_1 - |\lambda|^2g_1\mu_1 - g_1\mu_1 - \overline{g}_1\overline{\mu}_1\right) = -(1 + |\lambda|^2)\Re(g_1\mu_1). \]

Hence, (56) is rewritten as \(4(1 + |\lambda|^2)^2\Re(g_1\mu_1)\Im(v)pc\). Thus, we obtain the claim of the lemma. \(\blacksquare\)
5.1.4 Mini-holomorphic frames of $\mathcal{L}_p^\lambda$ (ω)

Because $e_0^*\tilde{v}_p^\lambda = \tilde{v}_p^\lambda$ and $(k(\omega)e_1)^*\tilde{v}_p^\lambda = \tilde{v}_p^\lambda$, we obtain a mini-holomorphic frame $\psi_p^\lambda_\omega$ of $\mathcal{L}_p^{\lambda(\omega)}$ on $\mathcal{M}_p^{\lambda(\omega)}$. By Lemma 5.5, we have

$$e_2^*\psi_p^\lambda_\omega = \psi_p^\lambda_\omega \cdot U_p^{pc} \exp\left(\frac{\lambda(\omega)}{4} (\mu_1 + \lambda^2 \tau_1)^{-1} (\mu_1 - \lambda^2 \tau_1) \cdot |\mu_1|^{-2} (|\mu_1|^2 |\mu_2|^2 - \mu_2^2 \tau_1^2 / 2 - \tau_2^2 \mu_1^2 / 2)\right).$$

By Lemma 5.6, we have

$$|\psi_p^\lambda_\omega| \sim \exp\left(\alpha(\omega) \cdot \text{Re}(g_1, \mu_1) pc\right) = \exp\left(\mu(\omega) \cdot \text{Re}(g_1, \mu_1) pc\right) = |U_p|^{-p/2} = |U_p|^{-p/2} \cdot \text{Re}(g_1, \mu_1) / \text{Vol}(\Gamma).$$

5.1.5 Associated filtered bundles

By using the frame $\psi_p^\lambda_\omega$, we extend $\mathcal{L}_p^{\lambda(\omega)}$ to a locally free $\mathcal{O}_{\mathcal{X}_p^{\lambda(\omega)}}(\ast H_p^{\lambda(\omega)})$-module $\mathcal{P}_p^{\lambda(\omega)}$. Because $\mathcal{L}_p^{\lambda(\omega)}$ is $\mathcal{O}(\mathcal{X}_p^{\lambda(\omega)})$-equivariant, we obtain the induced locally free $\mathcal{O}_{\mathcal{X}_p^{\lambda(\omega)}}(\ast H_p^{\lambda(\omega)})$-module $\mathcal{P}_p^{\lambda(\omega)}$, which is $(k(\omega)\mathcal{Z}/p\mathcal{Z})e_1$-equivariant.

We define a filtered bundle $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$ over $\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}}$ as follows: for $a = (a_0, a_\infty) \in \mathbb{R}^2$,

$$\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}}) = \mathcal{O}_{\mathcal{X}_p^{\lambda(\omega)}}(\{a_0 - p\omega \cdot t(\xi)^{-1}\} \cdot \{0\} + [a_\infty + p\omega \cdot t(\xi)^{-1}\} \cdot \{\infty\}) \psi_p^\lambda_\omega.$$

We obtain a filtered bundle $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$ over $\mathcal{P}_p^{\lambda(\omega)}$ as the descent, which is $(k(\omega)\mathcal{Z}/p\mathcal{Z})e_1$-equivariant.

Lemma 5.7 $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$ is isomorphic to $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$, where

$$\beta(\omega) := \exp\left(\frac{\lambda(\omega)}{4} (\mu_1 + \lambda^2 \tau_1)^{-1} (\mu_1 - \lambda^2 \tau_1) \cdot |\mu_1|^{-2} (|\mu_1|^2 |\mu_2|^2 - \mu_2^2 \tau_1^2 / 2 - \tau_2^2 \mu_1^2 / 2)\right).$$

Similarly, $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$ is isomorphic to $\mathcal{P}_a(\mathcal{L}_p^{\lambda(\omega)}|_{\mathcal{X}_p^{\lambda(\omega)}})$.

5.2 Examples (2)

5.2.1 Preliminary

We define the action of $\mathbb{R} e_3$ on $\mathbb{C}^2$ by

$$e_3(z, w) = (z, w + \sqrt{-1}).$$

It is described as follows in terms of $(\xi, \eta)$:

$$e_3(\xi, \eta) = (\xi, \eta) + (-\lambda \sqrt{-1}, \sqrt{-1}).$$

Let $(x, y)$ be the complex coordinate system determined by $(\xi, \eta) = x(-\lambda, 1) + y(1, \sqrt{-1})$. Note that $d\xi d\bar{\xi} + d\eta d\bar{\eta} = (1 + |\lambda|^2)(dx d\bar{x} + dy d\bar{y})$. The following holds:

$$\begin{cases}
\xi = -\lambda x + y, \\
\eta = x + \sqrt{-1} y,
\end{cases} \quad \begin{cases}
x = (1 + |\lambda|^2)^{-1} (n - \sqrt{-1} \xi), \\
y = (1 + |\lambda|^2)^{-1} (\xi + \lambda n).
\end{cases}$$

We have the following formulas:

$$e_3(x, y) = (x, y) + (\sqrt{-1}, 0),$$

$$e_0(x, y) = (x, y) + \frac{1}{1 + |\lambda|^2} (1 - |\lambda|^2, 2\lambda),$$

$$e_i(x, y) = (x, y) + \frac{1}{1 + |\lambda|^2} (-\lambda \tau_i - \lambda\mu_i, \mu_i + \lambda^2 \tau_i) \quad (i = 1, 2).$$

We have the following relations:

$$\begin{cases}
u = (1 - \lambda g_1)^{-1} (-2\lambda x + (1 - |\lambda|^2) y), \\
x = (1 + |\lambda|^2)^{-1} ((1 + \lambda g_1)x + (\sqrt{-1} - g_1)y), \\
y = (1 + |\lambda|^2)^{-1} ((1 + \lambda g_1)u + 2\lambda v).
\end{cases}$$
Lemma 5.8 There exists a unique solution \((A, B)\) \(\in \mathbb{C}^2\) of the equation
\[
e^0_i(\mathbf{Y} + Ax + By) - (\mathbf{Y} + Ax + By) = 0, \quad e^i(\mathbf{Y} + Ax + By) - (\mathbf{Y} + Ax + By) = 0.
\] (59)

Indeed, we have
\[
A = \frac{2(\lambda \mu_1 - \bar{\lambda}\mu_1)}{\mu_1 + \lambda^2 \mu_1}, \quad B = \frac{-(\mu_1 + \lambda^2 \mu_1)}{\mu_1 + \lambda^2 \mu_1}.
\] (60)

For such \(A\) and \(B\), the following holds:
\[
C := e^2_i(\mathbf{Y} + Ax + By) - (\mathbf{Y} + Ax + By) = -2\sqrt{-1}(1 + |\lambda|^2) \frac{\text{Vol}(\Gamma)}{\mu_1 + \lambda^2 \mu_1} \neq 0.
\] (61)

Proof The equation (59) is equivalent to the following equation:
\[
\left\{ \begin{array}{l}
2\mathbf{X} + A(1 - |\lambda|^2) + 2B\lambda = 0 \\
\mu_1 - \bar{\lambda}\mu_1 + A(-\lambda \mu_1 - \bar{\lambda}\mu_1) + B(\mu_1 - \lambda^2 \mu_1) = 0.
\end{array} \right.
\]

Because \((1 - |\lambda|^2)(\mu_1 - \lambda^2 \mu_1) - 2\lambda(-\lambda \mu_1 - \bar{\lambda}\mu_1) = (1 + |\lambda|^2)(\mu_1 + \lambda^2 \mu_1) \neq 0\), we have a unique solution \((A, B)\).

Recall \(U_0 = \exp\left(\frac{2\pi\sqrt{-1}}{p(1+\lambda s_1)}\right)\). We consider
\[
u = \frac{\mu}{\sqrt{-1}} e_0(c + \sqrt{-1}\sigma) \sim \mu_1 + \lambda s_1 p c.
\] (62)

We have
\[
x \sim \frac{1}{1 + |\lambda|^2} \left(\frac{\mu_1 + \lambda s_1}{\sqrt{-1}} p c\right), \quad y \sim \frac{1}{1 + |\lambda|^2} \left(\frac{\mu_1 - \lambda^2 \mu_1 + 2\lambda s_1}{\sqrt{-1}} p c\right).
\]

5.2.2 Construction

For \((a, b) \in \mathbb{R} \times \mathbb{C}\), let \(\tilde{L}(\lambda, a, b)\) be the line bundle on \(X^\lambda\) with a global frame \(\tilde{v}_0(a, b)\). Let \(\tilde{h}\) be the metric determined by \(\tilde{h}(\tilde{v}_0(a, b), \tilde{v}_0(a, b)) = 1\). Let \(\tilde{D}_{L(\lambda, a, b)}\) be the holomorphic structure determined by
\[
\partial_x \tilde{v}_0(a, b) = \tilde{v}_0(a, b) \sqrt{-1} a, \quad \partial_{\bar{x}} \tilde{v}_0(a, b) = \tilde{v}_0(a, b) b.
\]

The holomorphic bundle \(\tilde{L}(\lambda, a, b)\) with the metric is equivariant with respect to the action of \(\mathbb{R}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2\) by \(e^*_i(\tilde{v}_0(a, b)) = \tilde{v}_0(a, b)\). It induces a mini-holomorphic bundle \(L_p^{\text{cov}}(\lambda, a, b)\) of rank 1 with the induced metric \(h^{\text{cov}}\) on \(M_p^{\text{cov}}\), which is a \((\mathbb{Z} / p\mathbb{Z})e_1 \times \mathbb{Z}e_2\)-equivariant monopole. We also obtain a monopole \((L_p(\lambda, a, b), h)\) on \(M_p^\lambda\) as the descent, which is \((\mathbb{Z} / p\mathbb{Z})e_1\)-equivariant.

5.2.3 Underlying mini-holomorphic bundles

We have the holomorphic section \(\tilde{v}_1(a, b)\) of \(\tilde{L}(\lambda, a, b)\) given as follows:
\[
\tilde{v}_1(a, b) := \tilde{v}_0(a, b) \cdot \exp\left(\frac{1}{\sqrt{-1}}(x - \mathbf{X}) - (\mathbf{Y} + Ax + By) b\right).
\]

We have \(e^0_i \tilde{v}_1(a, b) = \tilde{v}_1(a, b)\) and \(e^i \tilde{v}_1(a, b) = \tilde{v}_1(a, b)\). We also have
\[
e^0_i \tilde{v}_1(a, b) = \tilde{v}_1(a, b) \cdot \exp\left(-Cb\right).
\]

We obtain the induced mini-holomorphic frame \(v_1(a, b)\) of \(L_p^{\text{cov}}(\lambda, a, b)\) on \(M_p^{\text{cov}}\) for which the following holds:
\[
e^0_i v_1(a, b) = v_1(a, b) \cdot \exp\left(-Cb\right).
\]
Because $|v_{1,(a,b)}|_b = \exp\left(\text{Re}((x - z)\sqrt{-1}a) - \text{Re}((y + Ax + By)b)\right)$, we have

$|\tilde{v}_{1,(a,b)}|_b \sim \exp\left(2\text{Re}(g_1)\mu_1 - \frac{2}{\sqrt{-1}\mu_1 + \lambda^2\mu_1}\left((|\lambda|^2 - 1)|\mu_1|^2 - (\overline{\lambda}\mu_1 + \lambda\overline{\mu}_1)s_1 + (\lambda\overline{\mu}_1 - \overline{\lambda}\mu_1)\text{Re}(g_1)\mu_1\right)\right)$, where $c$ is introduced as in \([12]\).

### 5.2.4 Associated filtered bundles

By using the frame $v_{1,(a,b)}$, we extend $L_p^{\text{cov}}(\lambda, a, b)$ to a locally free $\mathcal{O}_{X_p^{\text{cov}}}(*H^\lambda)^{\text{cov}}$-module $\mathcal{P}L_p^{\text{cov}}(\lambda, a, b)$. We set

$p(\lambda, a, b) := \frac{1}{2\pi} \text{Re}\left[2\text{Re}(g_1)\mu_1 - \frac{2}{\sqrt{-1}\mu_1 + \lambda^2\mu_1}\left((|\lambda|^2 - 1)|\mu_1|^2 - (\overline{\lambda}\mu_1 + \lambda\overline{\mu}_1)s_1 + (\lambda\overline{\mu}_1 - \overline{\lambda}\mu_1)\text{Re}(g_1)\mu_1\right)\right].$

We obtain the filtered bundle $\mathcal{P}_*L_p^{\text{cov}}(\lambda, a, b)$ over $\mathcal{P}L_p^{\text{cov}}(\lambda, a, b)$ determined as follows: for $a = (a_0, a_1) \in \mathbb{R}^2$,

$\mathcal{P}_a\left(L_p^{\text{cov}}(\lambda, a, b)|_{\pi_{p^{-1}}(t)}\right) := \mathcal{O}_t^a\left([a_0 - a\pi(\lambda, a, b)] \cdot \{0\} + [a_0 + a\pi(\lambda, a, b)] \cdot \{\infty\}\right) \cdot v_{1,(a,b)|_{\pi_{p^{-1}}(t)}}.$

Because $\mathcal{P}(L_p^{\text{cov}}(\lambda, a, b))$ and $\mathcal{P}_*(L_p^{\text{cov}}(\lambda, a, b))$ are equivariant with respect to the $\mathbb{Z}_2\mathfrak{e}_1$-action, we obtain a locally free $\mathcal{O}_{X_p^{\text{cov}}}(*H_p^\lambda)$-module $\mathcal{P}(L_p(\lambda, a, b))$ and a filtered bundle $\mathcal{P}_*(L_p(\lambda, a, b))$ over $\mathcal{P}(L_p(\lambda, a, b))$.

**Lemma 5.9** $\mathcal{P}_*L_p(\lambda, a, b)|_{\tilde{M}_p^{\lambda}}$ is isomorphic to $\mathcal{P}_*(pp(\lambda, a, b))\mathcal{V}_{p,0}(e^{-cb})$, and $\mathcal{P}_*L_p(\lambda, a, b)|_{\tilde{M}_p^{\lambda}}$ is isomorphic to $\mathcal{P}_*(-pp(\lambda, a, b))\mathcal{V}_{p,\infty}(e^{-cb})$.

### 5.2.5 Isomorphisms

For any $n = (n_1, n_2) \in \mathbb{Z}^2$, we set

$\chi_n(z) := \exp\left(\frac{\pi}{\text{Vol}(1)}\left(-n_1\mu_1z - n_2\mu_2z\right)\right).$

It induces a function $\chi_n$ on $\mathcal{M}_p^0 = \mathcal{M}_p^\lambda$. We have the isomorphism of monopoles $F_n : L_p(\lambda, a, b) \simeq L_p(\lambda, a', b')$ induced by $F_n(\chi_n\tilde{v}_{0,(a,b)}) = \tilde{v}_{0,(a',b')}$, where

$(a', b') := (a, b) + \frac{\pi}{1 + |\lambda|^2} \text{Vol}(1)\left[pm_1\left(-\sqrt{-1}(\mu_1\lambda - \mu_1\overline{\lambda}), \mu_1 + \lambda^2\mu_1\right) - n_2\left(-\sqrt{-1}(\mu_2\lambda - \mu_2\overline{\lambda}), \mu_2 + \lambda^2\mu_2\right)\right].$

We have $F(v_{1,(a,b)}) = U_p^{n_2}v_{1,(a',b')}.$

**Remark 5.10** The numbers $\exp(-cb)$ and $pp(\lambda, a, b)$ determine $(a, b)$ up to the induced action of $\mathbb{Z}_2\mathfrak{e}_1$.

### 5.2.6 Comparison with $\lambda = 0$

We define the bijection $F^\lambda : \mathbb{R} \times \mathbb{C} \simeq \mathbb{R} \times \mathbb{C}$ by

$F^\lambda(a, b) := \left(a + \frac{2\text{Im}(\overline{\lambda})}{1 + |\lambda|^2}, \ b + \lambda^2\overline{\lambda}\frac{\overline{\lambda}}{1 + |\lambda|^2}\right).$

**Lemma 5.11** $L_p(0, a, b) = L_p(\lambda, F^\lambda(a, b))$ holds on $\mathcal{M}_p^0 = \mathcal{M}_p^\lambda$.

**Proof** It is enough to compare the corresponding instantons on $X$. Let $\tilde{v}_{0,(a,b)}$ be the global frame of $\tilde{L}(0, a, b)$. The unitary connection is given as

$\tilde{\nabla}_{\tilde{v}_{0,(a,b)}} = \tilde{v}_{0,(a,b)}\left(\sqrt{-1}a\overline{d\overline{w}} - \sqrt{-1}a\overline{dw} + b\overline{dz} - \overline{d\overline{z}}\right)$.
We have the following relation:
\[ z = \frac{1}{1 + |\lambda|^2} (-\lambda x - \lambda x + y - \lambda^2 y), \quad w = \frac{1}{1 + |\lambda|^2} (x - |\lambda|^2 x + \lambda y + \lambda y). \]

By a direct computation, we obtain
\[
\sqrt{-1a\, dw} - \sqrt{-1a\, dw} + b\, d\bar{x} - \bar{b}\, dz = \sqrt{-1a\, (d\bar{x} - dx)} + \frac{1}{1 + |\lambda|^2} \left( 2\sqrt{-1} \operatorname{Im}(\bar{b}) dx + 2\sqrt{-1} \operatorname{Im}(\bar{b}) d\bar{x} + (b + \lambda^2 \bar{b}) dy - (\bar{b} + \bar{\lambda}^2 b) d\bar{y} \right). \tag{63}
\]

Thus, we obtain the claim of the lemma.

5.2.7 Twist

Recall that we constructed a monopole \( L_p(\omega) \) on \( M^0_p \) for \( \omega \in \frac{1}{p^2} \mathbb{Z} \) in §5.1.

Lemma 5.12 We set \( b_0 := -\frac{\omega}{\omega(1)} \). Then, \( e_1^* L_p(\omega) \) is isomorphic to \( L_p(\omega) \otimes L_p(0, 0, b_0) \). The isomorphism is induced by \( e_1^*(e) \mapsto e \otimes v_1(0, b_0) \).

Proof Note that \(-c b_0 = -2\pi -\omega\). Let \( v_1(0, b_0) \) be the mini-holomorphic frame of \( L^\text{cov}_p(0, 0, b_0) \) as in §5.2. Then, we have \( |v_1(0, b_0)| = 1 \) and \( e_2^* v_1(0, b_0) = v_1(0, b_0) \exp(-2\pi\omega) \). Because
\[
e_2^*(e_1^*) \mapsto e_1^*(e) \exp \left( -2\pi \sqrt{-1} \omega |\mu_1|^{-2} \operatorname{Re}(\mu_1 z) \right) \cdot \exp(-2\pi\omega),
\]
we obtain the claim of the lemma.

5.3 Examples (3)

5.3.1 Neighbourhoods

We continue to use the notation in §5.2. Let \( R > 0 \). We set \( \bar{U}_{-R} = \{ (x, y) \in X^\lambda \mid \operatorname{Im}(x) < -R \} \) and \( \bar{U}_{+R} = \{ (x, y) \in X^\lambda \mid \operatorname{Im}(x) > R \} \). Let \( U^\text{cov}_{p, \pm R} \) denote the quotient of \( \bar{U}_{\pm R} \) by the action of \( \mathbb{R} \mathfrak{e}_0 \oplus \mathbb{Z} \mathfrak{e}_1 \). Let \( U_{p, \pm R} \) denote the quotient of \( U^\text{cov}_{p, \pm R} \) by the action of \( \mathbb{Z} \mathfrak{e}_2 \).

If \( \operatorname{Re}(\mathfrak{g}_1 \mu_1) > 0 \), we set
\[
U^\lambda_{p, \pm R} := U_{p, \pm R}, \quad U^\lambda_{p, \pm R} := U^\text{cov}_{p, \pm R, \lambda}, \quad U^\lambda_{p, \pm R} := U_{p, \pm R}, \quad U^\lambda_{p, 0, R} := U^\text{cov}_{p, \pm R, \lambda, 0}.
\]

If \( \operatorname{Re}(\mathfrak{g}_1 \mu_1) < 0 \), we set
\[
U^\lambda_{p, \pm R} := U_{p, \pm R}, \quad U^\lambda_{p, \pm R} := U^\text{cov}_{p, \pm R, \lambda}, \quad U^\lambda_{p, \pm R} := U_{p, \pm R}, \quad U^\lambda_{p, 0, R} := U^\text{cov}_{p, \pm R, \lambda, 0}.
\]

Then, \( U^\lambda_{p, R} := U^\lambda_{p, R} \cup H^\lambda_{p, R} \) is a neighbourhood of \( H^\lambda_{p, R} \). Similarly, \( U^\lambda_{p, R} := U^\lambda_{p, R} \cup H^\lambda_{p, R} \) is a neighbourhood of \( H^\lambda_{p, R} \).

5.3.2 Examples of monopoles of rank 2 with unipotent monodromy

Let \( \tilde{V}_{}^{}(\lambda, 2) \) be the holomorphic vector bundle on \( \bar{U}_{\pm R} \) with a global frame \( \tilde{v} = (\tilde{v}_1, \tilde{v}_2) \) with the holomorphic structure determined by
\[
\partial_x \tilde{v} = 0, \quad \partial_y \tilde{v} = \tilde{v} N_2, \quad \text{where} \ N_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Let \( \tilde{h} \) be the metric of \( \tilde{V}_{}^{}(\lambda, 2) \) determined by \( \tilde{h}(\tilde{v}_1, \tilde{v}_2) = 0 \), \( \tilde{h}(\tilde{v}_1, \tilde{v}_2) = |\operatorname{Im}(x)| \) and \( \tilde{h}(\tilde{v}_2, \tilde{v}_2) = |\operatorname{Im} x|^{-1} \). The holomorphic bundles with a Hermitian metric are instantons on \( \bar{U}_{\pm R} \).

We have the holomorphic frame \( \tilde{u} := \tilde{v} \cdot \exp \left( -\frac{1}{2} (A x + B y) \right) N_2 \) of \( \tilde{V}_{}^{}(\lambda, 2) \). We have \( e_0^* \tilde{u} = \tilde{u} \) and \( e_1^* \tilde{u} = \tilde{u} \). We also have
\[
e_2^* \tilde{u} = \tilde{u} \exp(-CN_2).
\]
Lemma 5.13 Let $\tilde{h}_0$ be the metric of $V_{\pm}(\lambda, 2)$ determined by

$$
\tilde{h}_0(\tilde{u}_1, \tilde{u}_1) = |\text{Im}(x)|, \quad \tilde{h}_0(\tilde{u}_2, \tilde{u}_2) = |\text{Im}(x)|^{-1}, \quad \tilde{h}_0(\tilde{u}_1, \tilde{u}_2) = 0.
$$

Then, we have $\tilde{h}_0$ and $h$ are mutually bounded.

We define the action of $R \mathfrak{e}_0 \oplus Z \mathfrak{e}_1 \oplus Z \mathfrak{e}_2$ on $\bar{V}_{\pm}(\lambda, 2)$ by $e^{i}(\tilde{v}) = \tilde{v}$, and the holomorphic structure and the metric are preserved by the action. We obtain the corresponding mini-holomorphic bundles $V^{\text{cov}}_{p,\nu}(\lambda, 2)$ on $U^{\lambda}_{p,\nu,R}$ and $V_{p,\nu}(\lambda, 2)$ on $U^{\lambda}_{p,\nu,R}$ for $\nu = 0, \infty$. They are equipped with the induced metrics $h^{\text{cov}}$ and $h$, respectively. With the metrics, they are monopoles.

We obtain the induced mini-holomorphic frame $u$ of $V^{\text{cov}}_{p,\nu}(\lambda, 2)$, with which $V^{\text{cov}}_{p,\nu}(\lambda, 2)$ extends to a mini-holomorphic bundle $\mathcal{P}_{0}V^{\text{cov}}_{p,\nu}(\lambda, 2)$ on $U^{\lambda}_{p,\nu,R}$. It induces a filtered bundle $\mathcal{P}_{\nu}V^{\text{cov}}_{p,\nu}(\lambda, 2)$ over $(U^{\lambda}_{p,\nu,R}, H^{\lambda}_{p,\nu})$ such that $Gr_{a}^{\nu}V^{\text{cov}}_{p,\nu}(\lambda, 2) = 0$ unless $a \in Z$. We obtain the induced frame $|u|$ of $Gr_{0}^{\nu}(V^{\text{cov}}_{p,\nu}(\lambda, 2))$, for which $e^{i}_{0}|u| = |u| \exp(-\mathcal{C}N_{2})$ holds.

Because $\mathcal{P}_{\nu}V^{\text{cov}}_{p,\nu}(\lambda, 2)$ is $Z \mathfrak{e}_2$-equivariant, we obtain an induced filtered bundle $\mathcal{P}_{\nu}V^{\text{cov}}_{p,\nu}(\lambda, 2)$ on $(U^{\lambda}_{p,\nu,R}, H^{\lambda}_{p,\nu})$, which is an extension of $V_{p,\nu}(\lambda, 2)$. The conjugacy class of the monodromy of $Gr_{0}^{\nu}(V^{\text{cov}}_{p,\nu}(\lambda, 2))$ is $\exp(-\mathcal{C}N_{2})$.

5.3.3 Examples with any monodromy at infinity

Take $(a_{i}, b_{i}) \in R \times C$ $(i = 1, \ldots, m)$, and $\ell_{i} \in Z_{\geq 0}$ $(i = 1, \ldots, m)$. We obtain the following monopole:

$$
E = \bigoplus_{i=1}^{m} L(\lambda, a_{i}, b_{i}) \otimes \text{Sym}^{i} V(\lambda, 2).
$$

We have

$$
Gr_{a}^{\nu}(E) = \bigoplus_{i} Gr_{a}^{\nu}(L(\lambda, a_{i}, b_{i})) \otimes Gr_{0}^{\nu}(V(\lambda, 2)).
$$

The conjugacy class of the monodromy on $Gr_{a}^{\nu}(L(\lambda, a_{i}, b_{i})) \otimes Gr_{0}^{\nu}(V(\lambda, 2))$

$$
\exp(-\mathcal{C}b_{i}) \cdot \exp(-\mathcal{C}N_{\ell_{i}+1}).
$$

Here, $N_{\ell_{i}+1}$ is a $(\ell_{i} + 1)$-square matrix such that $(N_{\ell_{i}+1})_{j+1, j} = 1$ $(j = 1, \ldots, \ell_{i})$ and $(N_{\ell_{i}+1})_{i, j} = 0$ $(i \neq j + 1)$.

5.3.4 Another expression

Suppose that $A = (A_{1}, A_{2}, A_{3}) \in \mathfrak{su}(n)$ $(i = 1, 2, 3)$ satisfy $[A_{i}, A_{j}] + A_{k} = 0$ for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. Let $\bar{V}_{\pm}$ be a product bundle $\tilde{U}_{\pm,R} \times C^{n}$ on $\tilde{U}_{\pm,R}$ with a global frame $e = (e_{1}, \ldots, e_{n})$.

Let $h_{\bar{V}_{\pm}}$ be the Hermitian metric of $\bar{V}_{\pm}$ for which the frame $e$ is orthonormal. We define operators $\partial_{\bar{V}_{\pm},x}$ and $\partial_{\bar{V}_{\pm},\bar{y}}$ on $\bar{V}_{\pm}$ by

$$
\partial_{\bar{V}_{\pm},x} e = e \cdot \frac{1}{2 \text{Im}(x)} A_{3}, \quad \partial_{\bar{V}_{\pm},\bar{y}} e = e \cdot \frac{1}{2 \text{Im}(x)} (A_{1} + \sqrt{-1} A_{2}).
$$

Then, the operators give a holomorphic structure $\partial_{\bar{V}_{\pm}}$ of $\bar{V}_{\pm}$, and $(\bar{V}_{\pm}, \partial_{\bar{V}_{\pm}}, h_{\bar{V}_{\pm}})$ are instantons on $\tilde{U}_{\pm,R}$. It is naturally equivariant with respect to the action of $R \mathfrak{e}_0 \oplus Z \mathfrak{e}_1 \oplus Z \mathfrak{e}_2$ determined by $e^{i} e = e$. Hence, we obtain monopoles $V^{\text{cov}}_{p_{\pm},\nu}(\lambda, A)$ on $U^{\lambda}_{p_{\pm},R}$, and $V_{p_{\pm},\nu}(\lambda, A)$ on $U_{p_{\pm},R}$.

The following is easy to check.

Lemma 5.14 If $(k_{1}, \ldots, k_{m})$ be the weight decomposition of the $\mathfrak{su}(2)$-representation determined by $A$, then $V(\lambda, A)$ is naturally isomorphic to $\bigoplus_{i} \text{Sym}^{k_{i}} V(\lambda, 2)$.

The following is easy to see.

Lemma 5.15 $V(\lambda, A)$ is isomorphic to $V(0, A)$.
5.4 Example (4)

We set \( \Gamma^\nu := \{ b \in \mathbb{C} \mid \text{Im}(\chi b) \in \pi \mathbb{Z} \ (\forall \chi \in \Gamma) \} \). We set \( \mu_\nu := \pi \mu_\nu / \text{Vol}(\Gamma) \). Then, \( \Gamma^\nu := \mathbb{Z} \mu_\nu^\nu \oplus \mathbb{Z} \mu_\nu^\nu \). We set \( \Gamma_\nu := \mathbb{Z}(\mu_1) \oplus \mathbb{Z} \mu_2 \). We have \( (\Gamma_\nu^\nu)^\vee = \mathbb{Z} \mu_1^\vee \oplus \mathbb{Z} (\mu_2^\nu / \nu) \).

Let \( \omega \in \mathbb{Q} \). We set \( k(\omega) := \min \{ p \in \mathbb{Z}_{>0} \mid \nu \omega \in \mathbb{Z} \} \). We have the action of \((\mathbb{Z}/k(\omega)\mathbb{Z}) \cdot (\omega \mu_\nu^\nu) \) on \( \mathbb{C} / (\Gamma(\omega)) \) induced by the addition. It naturally induces an action of \((\mathbb{Z}/k(\omega)\mathbb{Z}) \cdot (\omega \mu_\nu^\nu) \) on \( \mathbb{R} \times \mathbb{C} / (\Gamma(\omega)) \).

Let \( I \subset \mathbb{Q} \) be a finite subset. For each \( \omega \in I \), let \( S_\omega \subset \mathbb{R} \times \mathbb{C} / (\Gamma(\omega)) \) which is preserved by the action of \((\mathbb{Z}/k(\omega)\mathbb{Z}) \cdot (\omega \mu_\nu^\nu) \). For each \((a, b) \in S_\omega \), let \( n(\omega, a, b) \in \mathbb{Z}_{\geq 0} \), and let

\[
A_{\omega, a, b} = (A_{1, \omega, a, b}, A_{2, \omega, a, b}, A_{b, \omega, a, b}) \in \text{su}(n(\omega, a, b))^3
\]
such that \([A_{i, \omega, a, b}, A_{j, \omega, a, b}] + A_{k, \omega, a, b} = 0 \) for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\). We assume

\[
A_{\omega, a, b} = A_{\omega, a, b + \nu \mu_\nu^\nu}.
\]

Let \( \tilde{S}_\omega \subset \mathbb{R} \times \mathbb{C} \) be a lift of \( S_\omega \), i.e., the projection \( \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{R} \times (\mathbb{C}/\Gamma(\omega)) \) induces a bijection \( \pi : \tilde{S}_\omega \simeq S_\omega \).

For each \((\tilde{a}, \tilde{b}) \in \tilde{S}_\omega \), we set \( A_{\omega, \tilde{a}, \tilde{b}} := A_{\omega, \pi(\tilde{a}, \tilde{b})} \).

We obtain the following monopole on \( \mathcal{U}_{k(\omega), \nu, \nu} \) (\( \nu = 0, \infty \)):

\[
M_{k(\omega), \nu}(\omega, S_\omega, \{A_{\omega, a, b}\}) := \bigoplus_{(\tilde{a}, \tilde{b}) \in \tilde{S}_\omega} L_{k(\omega)}(\omega) \otimes L_{0}(0, \tilde{a}, \tilde{b}) \otimes L_{0}(0, A_{\omega, \tilde{a}, \tilde{b}}). \tag{64}
\]

Recall that we have the isomorphism \( L_{0}(0, \tilde{a}, \tilde{b}) = \mathbb{Z}/(1, \tilde{a}, \tilde{b}) \) as explained in \( \S \). We also have the isomorphism \( e_{\Gamma_{k(\omega)}} : L_{k(\omega)}(\omega) \otimes L_{k(\omega)}(0, 0, -\mu_\nu^\nu) \) as in Lemma \( 5.12 \). By the isomorphisms, the monopole \( M_{k(\omega), \nu}(\omega, S_\omega, \{A_{\omega, a, b}\}) \) is naturally equivariant with respect to the action of \((\mathbb{Z}/k(\omega)\mathbb{Z}) e_{\Gamma_{k(\omega)}} \). We obtain monopoles

\[
M_{\nu}(\omega, S_\omega, \{A_{\omega, a, b}\})
\]
on \( \mathcal{U}_{k(\omega), \nu, \nu} \) as the descent of \( M_{k(\omega), \nu}(\omega, S_\omega, \{A_{\omega, a, b}\}) \). By taking the direct sum, we obtain a monopole

\[
M_{\nu}(I, \{S_\omega\}, \{A_{\omega, a, b}\}) := \bigoplus_{\omega \in I} M_{\nu}(\omega, S_\omega, \{A_{\omega, a, b}\})
\]
on \( \mathcal{U}_{k(\omega), \nu, \nu} \).

6 Asymptotic behaviour of doubly periodic monopoles

6.1 Statements

Let \((y_0, y_1, y_2)\) be the standard coordinate of \( \mathbb{R}^3 \). We consider the Euclidean metric \( \sum_{i=0,1,2} dy_i \). Let \( \Gamma \subset \{0\} \times \mathbb{R}^2 \) be a lattice. The volume of \( \mathbb{R}^2 / \Gamma \) is denoted by \( \text{Vol}(\Gamma) \). We may assume that \( \Gamma = \mathbb{Z} \cdot (0, a, 0) \oplus \mathbb{Z} \cdot (0, b, c) \), where \( a \) and \( c \) are positive numbers. We consider the action of \( \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \) on \( \mathbb{R}^3 \) by \( e_1 \cdot (y_0, y_1, y_2) = (y_0, y_1 + a, y_2 + c) \) and \( e_2 \cdot (y_0, y_1, y_2) = (y_0, y_1 + b, y_2 + c) \).

For any \( R \in \mathbb{R} \), we set \( \mathcal{U}_R := \{(y_0, y_1, y_2) \in \mathbb{R}^3 \mid y_0 < -R\} \). Let \( \mathcal{U}_R \) denote the quotient space of \( \mathcal{U}_R \) by the action of \( \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \).

Let \((E, h, \nabla, \phi)\) be a monopole on \( \mathcal{U}_R \) for some \( R_0 > 0 \). By the pull back, we obtain the \( \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \)-equivariant monopole \((\tilde{E}, \tilde{h}, \tilde{\nabla}, \tilde{\phi})\) on \( \tilde{U}_{R_0} \).

Assumption 6.1 We assume that the curvature \( F(\tilde{\nabla}) \) is bounded. It particularly implies \( |\phi|_h = O(|y_0|) \).
6.1.1 First reduction

We shall prove the following proposition in §6.3.1.

Proposition 6.2 There exists a finite subset \( I(\phi) \subset \mathbb{Q} \), and positive numbers \( R_1 > 0 \) and \( C_1 > 0 \) such that the following holds for \( (y_0, y_1, y_2) \in \mathcal{U}_{R_1} \):

- For any eigenvalue \( \alpha \) of \( \phi_{(y_0, y_1, y_2)} \), there exists \( \omega \in I(\phi) \) such that
  \[
  \left| \alpha - \frac{2\pi \sqrt{-1} \omega y_0}{\text{Vol}(\Gamma)} \right| < C_1.
  \]  

(65)

In particular, if \( R_1 > 0 \) is sufficiently large, we obtain the orthogonal decomposition

\[
(E, h, \phi)_{\mathcal{U}_{R_1}} = \bigoplus_{\omega \in I(\phi)} (E^\omega, h^\omega, \phi^\omega)
\]

such that any eigenvalue of \( \phi^\omega_{(y_0, y_1, y_2)} \) satisfies (65).

We obtain a decomposition \( \nabla = \nabla^\bullet + \rho \), where \( \nabla^\bullet \) is a direct sum of unitary connections \( \nabla^\omega \) on \( E^\omega \), and \( \rho \) is a section of \( \bigoplus_{\omega \neq \omega'} \text{Hom}(E^\omega, E^\omega') \otimes \Omega^1 \). The inner product of \( \rho \) and \( \partial_{y_i} \) are denoted by \( \rho_i \). Similarly, for any section \( s \) of \( \text{End}(E) \otimes \Omega^p \), we obtain a decomposition \( s = s^* + s^\top \), where \( s^* \) is a section of \( \bigoplus \text{End}(E^\omega) \otimes \Omega^p \), and \( s^\top \) is a section of \( \bigoplus_{\omega \neq \omega'} \text{Hom}(E^\omega, E^\omega') \otimes \Omega^p \). Note that \( (\nabla \phi)^* = \nabla^\bullet \phi \) and \( (\nabla \phi)^\top = [\rho, \phi] \).

We shall prove the following proposition in §6.3.2.

Theorem 6.3 There exist positive constants \( R_2, C_2 \) and \( \epsilon_2 \) such that \( |\rho|_h \leq C_2 \exp(-\epsilon_2 y_0^2) \) on \( \mathcal{U}_{R_2} \). Moreover, for any positive integer \( k \), there exist positive constants \( C_2(k) \) and \( \epsilon_2(k) \) such that

\[
|\nabla^\bullet_{\kappa_1} \circ \cdots \circ \nabla^\bullet_{\kappa_k} (F(\nabla^\bullet) - \nabla^\bullet \phi)|_{h, k} \leq C_2(k) \exp(-\epsilon_2(k) y_0^2)
\]

on \( \mathcal{U}_{R_2} \) for any \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \).

As a direct consequence, we obtain the following corollary.

Corollary 6.4 For any \( k \), there exist positive constants \( C_3(k) \) and \( \epsilon_3(k) \) such that

\[
|\nabla^\bullet_{\kappa_1} \circ \cdots \circ \nabla^\bullet_{\kappa_k} (F(\nabla^\bullet) - \nabla^\bullet \phi)|_{h, k} \leq C_3(k) \exp(-\epsilon_3(k) y_0^2)
\]

on \( \mathcal{U}_{R_2} \) for any \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \). Moreover,

\[
|\nabla^\bullet_{\kappa_1} \circ \cdots \circ \nabla^\bullet_{\kappa_k} (F(\nabla^\bullet) - \nabla^\bullet \phi)|_{h, k} \leq C_3(k) \exp(-\epsilon_3(k) y_0^2)
\]

is bounded on \( \mathcal{U}_{R_2} \) for any \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \).

For each \( \omega \in I(\phi) \), let \( p \) be determined by \( \min\{p' \in \mathbb{Z}_{>0} | p' \omega \in \mathbb{Z}\} \). For any \( R > 0 \), let \( \mathcal{U}_{p, R} \) denote the quotient of \( \tilde{U}_R \) by the action of \( p\mathbb{Z}_1 \oplus \mathbb{Z}_2 \). Let \( \varphi_p : \mathcal{U}_{p, R} \to \mathcal{U}_R \) denote the projection. On \( \mathcal{U}_{p, R} \), we set

\[
(E_w, h_w, \nabla_w, \phi_w) := \varphi_{p, R}^{-1}(E_w, h_w, \nabla_w, \phi_w) \otimes L_p(-\omega).
\]

(67)

Proposition 6.5 For any \( k \in \mathbb{Z}_{>0} \) and for any \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \), we obtain

\[
|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} (F(\nabla_{\omega}) - \nabla_{\omega} \phi)|_{h_{\omega}} \to 0
\]

as \( |y_0| \to \infty \).
6.1.2 Second reduction

For any $R > 0$, we set $\mathcal{H}_R := \{ y_0 \in \mathbb{R} | y_0 < -R \} \subset \mathbb{R}$. Let $\Psi : \mathcal{H}_p, R \rightarrow \mathcal{H}_R$ denote the projection. Let $A$ be the ring of the non-commutative polynomials of four variables. We obtain the following proposition from Proposition 6.20, Proposition 6.26, and Proposition 6.29 below.

Proposition 6.6 There exist a finite subset $S_\omega \subset \mathbb{R}^3$, a graded vector bundle $V_\omega = \bigoplus_{a \in S_\omega} V_{\omega, a}$ on $\mathcal{H}_R$, a graded Hermitian metrics $h_{V_\omega} = \bigoplus_{a \in S_\omega} h_{V_{\omega, a}}$, a graded unitary connection $\nabla_{V_\omega} = \bigoplus_{a \in S_\omega} \nabla_{V_{\omega, a}}$, graded anti-Hermitian endomorphisms $\phi_{i, \omega} = \bigoplus_{a \in S_\omega} \phi_{i, \omega, a}$ ($i = 1, 2, 3$), and an isomorphism $E_\omega \simeq \Psi^{-1}(V_\omega)$ such that the following holds:

1. Let $b_\omega$ be the automorphism of $E_\omega$ determined by $h_\omega = \Psi^{-1}(h_{V_\omega})b_\omega$. Then, for any $P \in A$, there exists $\epsilon(P) > 0$ such that
   \[ |P(\nabla_{\omega, y_0}, \nabla_{\omega, y_1}, \nabla_{\omega, y_2}, \phi_\omega)(b_\omega - \text{id})| = O\left(e^{\epsilon(P)y_0}\right). \]

2. For any $P \in A$, there exists $\epsilon(P) > 0$ such that
   \[ |P(\nabla_{\omega, y_0}, \nabla_{\omega, y_1}, \nabla_{\omega, y_2}, \phi_\omega)(\phi_\omega - \Psi^{-1}(\phi_{3, \omega}))| = O\left(e^{\epsilon(P)y_0}\right). \]

3. We set $R_{\omega, i} := \nabla_{\omega, y_i} - (\partial_{y_i} + \Psi^{-1}(\phi_{i, \omega, i}))$ ($i = 1, 2$), where $\partial_{y_i}$ are the naturally induced operators of $\Psi^{-1}(V_\omega)$. Then, for any $P \in A$, there exists $\epsilon(P) > 0$ such that
   \[ |P(\nabla_{\omega, y_0}, \nabla_{\omega, y_1}, \nabla_{\omega, y_2}, \phi_\omega)R_{\omega, i}| = O\left(e^{\epsilon(P)y_0}\right). \]

4. There exist anti-Hermitian endomorphisms $A_{i, \omega, a}$ ($i = 1, 2, 3$) of $V_{\omega, a}$ such that $\nabla_{V_{\omega, a}} A_{i, \omega, a} = 0$ and
   \[ \phi_{i, \omega, a} = \sqrt{-1}a_i \text{id}_{V_{\omega, a}} + y_0^{-1}A_{i, \omega, a} + O(y_0^{-2}). \]
   Moreover, $[A_{i, \omega, a}, A_{j, \omega, a}] + A_{k, \omega, a} = 0$ holds for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$.

6.1.3 A consequence

We obtain the following consequence.

Corollary 6.8 We obtain $|\nabla_{y_1} \phi| + |\nabla_{y_2} \phi|_h = O(y_0^{-2})$. Equivalently, we obtain $|F(\nabla)_{y_0, y_i}|_h = O(y_0^{-2})$ ($i = 1, 2$).

Remark 6.9 Note that $\nabla_{y_0} \phi$ is not necessarily $O(y_0^{-2})$. Equivalently, $|F(\nabla)_{y_0, y_i}|_h$ is not necessarily $O(y_0^{-2})$. See the examples in $[5.1.1]$. 

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6.2 Vector bundles with a connection on $S^1$

6.2.1 Statement

Let $r$ be a positive integer. Let $C_0 > 0$ be a constant. Let $A_0$ be an $r$-square Hermitian matrix. Set $S^1 := \mathbb{R}/\mathbb{Z}$. Let $A_1 : S^1 \to M_r(\mathbb{C})$ be a continuous function such that $|A_1| \leq C_0$. Let $V$ be a $C^\infty$-vector bundle of rank $r$ on $S^1$ with a frame $v$. We have the connection $\nabla$ determined by

$$\nabla v = v \cdot (A_0 + A_1) \, dt,$$

where $t$ is the standard coordinate of $\mathbb{R}$, and $dt$ is the induced 1-form on $S^1$. We have the monodromy $M(A_0 + A_1) : V_0 \to V_1 = V_0$ of the connection $\nabla$, and let $\mathcal{S}(M(A_0 + A_1))$ denote the set of eigenvalues. We shall prove the following proposition in [6.2.2–6.2.5].

**Proposition 6.10** There exists $R > 0$ depending only on $C_0$ such that the following holds.

- For any $\alpha \in \mathcal{S}(M(A_0 + A_1))$, there exists $\beta \in \mathcal{S}(M(A_0))$ such that $|\alpha \beta^{-1}| \leq R$ and $|\alpha^{-1} \beta| \leq R$.

Conversely, for any $\alpha \in \mathcal{S}(M(A_0))$, there exists $\beta \in \mathcal{S}(M(A_0 + A_1))$ such that $|\alpha \beta^{-1}| \leq R$ and $|\alpha^{-1} \beta| \leq R$.

6.2.2 Decomposition of a finite tuple of real numbers

We consider a finite tuple $(a_1, \ldots, a_N)$ of real numbers. We assume $a_i \leq a_j$ for $i < j$. We fix a positive number $c_0$ > 0. We take any $c_0 > 10N$.

**Lemma 6.11** There exist $k \geq 0$ and a decomposition $\{1, \ldots, N\} = \bigsqcup_{\ell = 1}^{m} J_\ell$ such that the following holds.

- If $i, j \in J_\ell$, then $|a_i - a_j| \leq 3Nc_1^k c_0$.

- If $i \in J_\ell$ and $j \in J_\ell$ with $\ell_1 \neq \ell_2$, then $|a_i - a_j| \geq \frac{1}{2} c_1^{k+1} c_0$.

**Proof** We set $m(0) := N$. We shall construct a finite decreasing sequence $m(0) > m(1) > \ldots > m(k)$, order preserving injective maps $G_n : \{1, \ldots, m(n)\} \to \{1, \ldots, N\}$ $(n = 0, \ldots, k)$, and order preserving surjective maps $F_n : \{1, \ldots, m(n)\} \to \{1, \ldots, m(n + 1)\}$ $(n = 0, \ldots, k - 1)$ by an inductive procedure. Suppose that we have already constructed $m(n)$, $G_n : \{1, \ldots, m(n)\} \to \{1, \ldots, N\}$. We set $J^{(n)} := \{i \mid a_{G_n(i)} > c_1^{n+1} c_0\} \cup \{m(n)\}$. If $J^{(n)} := \{1, \ldots, m(n)\}$, we stop the procedure. If $J^{(n)} \neq \{1, \ldots, m(n)\}$, we set $m(n + 1) := |J^{(n)}|$. We have the natural order preserving bijection $\varphi_{n+1} : \{1, \ldots, m(n + 1)\} \to J^{(n)}$. Because $J^{(n)} \subset \{1, \ldots, m(n)\}$, we obtain an injection $G_{n+1} : \{1, \ldots, m(n + 1)\} \to \{1, \ldots, N\}$ from $\varphi_{n+1}$ and $G_n$. For $i \in \{1, \ldots, m(n)\}$, there exists $j \in \{1, \ldots, m(n + 1)\}$ such that $\varphi_{n+1}(j - 1) < i \leq \varphi_{n+1}(j)$, where we formally set $\varphi_{n+1}(0) = 0$. We define $F_{n+1}(i) = j$ for such $i$ and $j$. Thus, we obtain the order preserving surjection $F_{n+1} : \{1, \ldots, m(n)\} \to \{1, \ldots, m(n + 1)\}$. The procedure will stop after finite steps.

By the construction, $|a_{G_{n}(i)} - a_{G_{n}(j)}| > c_1^{k+1} c_0$ holds for $i, j \in \{1, \ldots, m(k)\}$ with $i \neq j$. Let $F : \{1, \ldots, N\} \to \{1, \ldots, m(k)\}$ be the map obtained as the composite of $F_0, \ldots, F_{k-1}$. For $\ell \in F^{-1}(i)$, the following holds:

$$|a_\ell - a_{F_\ell}(i)| \leq N(c_1 + \cdots + c_1) c_0 = N(c_1^{k+1} - c_1)(c_1 - 1)^{-1} c_0.$$

Hence, if $\ell_1, \ell_2 \in F^{-1}(i)$, then

$$|a_{\ell_1} - a_{\ell_2}| \leq 2N(c_1^{k+1} - c_1)(c_1 - 1)^{-1} c_0 \leq 3Nc_1^k c_0.$$

For $\ell_1 \in F^{-1}(j)$ and $\ell_2 \in F^{-1}(i)$ with $i \neq j$, the following holds:

$$|a_{\ell_1} - a_{\ell_2}| \geq c_1^{k+1} c_0 - 2N(c_1^{k+1} - c_1)(c_1 - 1)^{-1} c_0 \geq \frac{1}{2} c_1^{k+1} c_0.$$

Thus, we are done.
6.2.3 An estimate

Let \(a\) be a non-zero real number. For any \(C^0\)-function \(g\) on \(S^1\), we have a unique \(C^1\)-function \(f\) such that \((\partial_\theta + a)f = g\).

**Lemma 6.12** We have \(\sup |f| \leq 2|a|^{-1}\sup |g|\).

**Proof** It is enough to consider the case \(a > 0\). Let \(f = \sum f_n e^{2\pi \sqrt{-1} \theta n}\) and \(g = \sum g_n e^{2\pi \sqrt{-1} \theta n}\) be the Fourier expansions. Because \((2\pi \sqrt{-1} \theta + a)f_n = g_n\), we obtain \(\int_0^1 |f|^2 dt = \sum |f_n|^2 \leq a^{-2} \sum |g_n|^2 = a^{-2} \int_0^1 |g|^2 dt\). Hence, there exists \(t_0 \in S^1\) such that \(|f(t_0)| \leq a^{-1}\sup |g|\). We may assume that \(t_0 = 0\) by a coordinate change. Because \(\partial_\theta(e^{at}f) = e^{at}g\), we have

\[ |e^{at}f(t) - f(0)| \leq \int_0^1 e^{as}|g(s)| ds \leq \sup |g| \cdot a^{-1}e^{at}. \]

Hence, we obtain the claim of the lemma.

6.2.4 Solving a non-linear equation

Let \(m\) be a positive integer. Let \(D_0\) be an \(m\)-square Hermitian matrix. Let \(C_{10}\) be a positive constant. Let \(B_0(t)\) be a \(C^0\)-map \(S^1 \to \mathbb{C}^m\) such that \(|B_0(t)| \leq C_{10}/3\). Let \(B_1(t)\) be a \(C^0\)-map \(S^1 \to M_m(\mathbb{C})\) such that \(|B_1(t)| \leq C_{10}/3\). Let \(B_2(t,x)\) be a \(C^0\)-map \(S^1 \times \mathbb{C}^m \to \mathbb{C}^m\) such that the following holds.

- \(|B_2(t,x)| = o(|x|)\) as \(|x| \to 0\).
- For any \(\epsilon > 0\), there exists \(\delta > 0\) such that \(|B_2(t,x) - B_2(t,y)| \leq \epsilon |x - y|\) if \(\max\{|x|, |y|\} \leq \delta\).

We take \(T > 1\) such that the following holds.

- If \(|x| < T^{-1}\), then \(|B_2(t,x)| \leq C_{10}/3\).
- If \(\max\{|x|, |y|\} < T^{-1}\), then \(|B_2(t,x) - B_2(t,y)| \leq C_{10}|x - y|/3\).

**Lemma 6.13** Assume that any eigenvalues \(a\) of \(D_0\) satisfies \(|a| \geq 10mTC_{10}\). Then, there exists \(f : S^1 \to \mathbb{C}^m\) such that (i) \((\partial_\theta + D_0)f(t) + B_0(t) + B_1(t) \cdot f(t) + B_2(t, f(t)) = 0, (ii) |f| \leq T^{-1}\). Such a function \(f\) is unique.

**Proof** We take any \(C^0\)-function \(f_0 : S^1 \to \mathbb{C}^m\) such that \(|f_0| \leq T^{-1}\). Inductively, we define \(f_i\) as a unique solution of \((\partial_\theta + D_0) f_i + B_0(t) + B_1(t) f_{i-1}(t) + B_2(t, f_{i-1}(t)) = 0\). Because \(|B_1(t)|f_{i-1}(t)| \leq C_{10}/3\) and \(|B_2(t, f_{i-1}(t))| \leq C_{10}/3\), we obtain \(|f_i| \leq (C_{10}T)^{-1}C_{10} \leq T^{-1}\) by Lemma 6.12. Note that

\[ (\partial_\theta + D_0)(f_{i+1}(t) - f_i(t)) + B_1(t)(f_i(t) - f_{i-1}(t)) + B_2(t, f_i(t)) - B_2(t, f_{i-1}(t)) = 0. \]

Because \(|B_1(t)(f_i(t) - f_{i-1}(t)) + B_2(t, f_i(t)) - B_2(t, f_{i-1}(t))| \leq C_{10}|f_i(t) - f_{i-1}(t)|\), we obtain sup \(|f_{i+1} - f_i| \leq T^{-1}\sup |f_i - f_{i-1}|\) by Lemma 6.12. Hence, the sequence \(f_i\) is convergent, and the limit \(f_\infty = \lim f_i\) satisfies the desired conditions. We also obtain the uniqueness.

6.2.5 Proof of Proposition 6.10

We may assume that \(A_0\) is diagonal. Let \(a_{i,i}\) denote the \((i,i)\)-th entries. We may assume that \(a_{i,i} \leq a_{j,i}\) for \(i \leq j\).

Take a sufficiently large constant \(C_1\). We have \(k \geq 0\) and a decomposition \(\{1, \ldots, r\} = \coprod \mathcal{J}_\ell\) as in Lemma 6.11. We choose \(i(\ell) \in \mathcal{J}_\ell\), and set \(\alpha_{\ell} := a_{i(\ell),i(\ell)}\). We put \(r(\ell) := |\mathcal{J}_\ell|\). We set \(\mathcal{A}_0 := \bigoplus \alpha_{i(\ell)} I_{r(\ell)}\) and \(\mathcal{A}_1 := A_0 - \mathcal{A}_0 + A_1\). We have \(|\mathcal{A}_1| \leq 4C_1^2 C_0\).

According to the decomposition \(\{1, \ldots, r\} = \coprod \mathcal{J}_\ell\), we have the decomposition \(\mathbb{C}^r = \bigoplus \mathbb{C}^{r(\ell)}\). It induces \(\text{End}(\mathbb{C}^r) = \bigoplus \text{End}(\mathbb{C}^{r(\ell)}) \oplus \bigoplus_{\ell_1, \ell_2} \text{Hom}(\mathbb{C}^{r(\ell_1)}, \mathbb{C}^{r(\ell_2)})\). For any matrix \(D \in \text{End}(\mathbb{C}^r)\), we have the decomposition \(D = D_+ + D_-\).

We consider the following equation for \(G : S^1 \to \text{GL}(r, \mathbb{C})\) and \(U : S^1 \to \bigoplus \text{End}(\mathbb{C}^{r(\ell)})\):

\[ G^{-1} \circ (\partial_\theta + \mathcal{A}_0 + \mathcal{A}_1) \circ G = \partial_\theta + \mathcal{A}_0 + \mathcal{A}_1 + U. \]  

(68)
We impose that $G^\omega = I_r$, and we regard \( \mathcal{S} \) as an equation for $G_+^\perp$ and $U$. It is equivalent to the following equations:

\[
(\tilde{A}_t^+G^\perp)^\circ = U, \quad \partial_tG^\perp + [\tilde{A}_0, G^\perp] + [\tilde{A}_t^+, G] + (\tilde{A}_t^+G^\perp)^\perp + G^\perp U = 0.
\]

By eliminating $U$, we obtain the following equation for $G^\perp$:

\[
\partial_tG^\perp + [\tilde{A}_0, G^\perp] + [\tilde{A}_t^+, G] + (\tilde{A}_t^+G^\perp)^\perp + G^\perp (\tilde{A}_t^+G^\perp)^\circ = 0.
\]

For a large $C_1$, we set $C_{10} := 400r^3C_0$ and $T := (1000r^3)^{-1}C_1$. By using Lemma 6.13 if $C_1$ is sufficiently large, we have a solution $G^\perp$ with $|G^\perp|^2 \leq T^{-1}$. We also obtain $U$ such that $|U| \leq C_{10}T^{-1}$.

By considering the eigenvalues of the monodromy of $\partial_t + A_0 + \tilde{A}_t^+ + U$, we obtain the claim of the proposition.

6.3 First reduction

6.3.1 Proof of Proposition 6.2

We take the mini-holomorphic structure determined by the decomposition $\mathbb{R}^3 = \mathbb{R} \cdot (0, a, 0) \times (\mathbb{R} \cdot (0, a, 0))^\perp$. We take $\mathbb{R}^3 \cong \mathbb{C} \oplus \mathbb{R}$ given by

\[
(y_0, y_1, y_2) \mapsto (2\pi e^{-1}(y_0 + \sqrt{-1}y_2), y_1).
\]

The action of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ on $\mathbb{C} \times \mathbb{R}$ are described as

\[
e_1(\tilde{\zeta}, y_1) = (\tilde{\zeta}, y_1 + a), \quad e_2(\tilde{\zeta}, y_1) = (\tilde{\zeta} + 2\pi\sqrt{-1}, y_1 + b).
\]

For any $R$, we set $\tilde{U}_R := \{ \tilde{\zeta} \in \mathbb{C} \mid \log(\tilde{\zeta}) < -R \}$. We have $\tilde{U}_R = \tilde{U}_R \times \mathbb{R}$ under the above identification $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$.

We have the associated mini-holomorphic bundle $(\tilde{E}, \tilde{\nabla})$ on $\tilde{U}_{R_0}$ with respect to the above mini-complex structure. By considering the flat sections along $\{ \tilde{\zeta} \} \times \mathbb{R}$ for each $\tilde{\zeta}$, we obtain a holomorphic vector bundle $(\tilde{V}, \tilde{\nabla})$ on $\tilde{U}_{R_0}$. The action of $\mathbb{Z}e_1$ induces a holomorphic automorphism $\tilde{F}$ of $\tilde{V}$. The action of $\mathbb{Z}e_2$ induces an isomorphism $\mathbb{Z}e_2 \tilde{V} \cong \tilde{V}$, where $\mathbb{Z}e_2 : \mathbb{C} \rightarrow \mathbb{C}$ is given by $\mathbb{Z}e_2(\tilde{\zeta}) = \tilde{\zeta} + 2\pi\sqrt{-1}$.

We identify $\mathbb{C}/(2\pi\sqrt{-1}\mathbb{Z}) \cong \mathbb{C}^*$ by $\tilde{\zeta} \mapsto \zeta = e^{\tilde{\zeta}}$. For any $R$, we set $U_R := \{ \zeta \in \mathbb{C}^* \mid c \log|\tilde{\zeta}| < -R \}$. We obtain the induced holomorphic bundle $(V, \nabla)_{U_R}$ on $U_{R_0}$. Because the actions of $e_1$ and $e_2$ are commutative, we obtain the induced automorphism $F$ of $(V, \nabla)$. We obtain the spectral curve $Sp(F)$ of $F$ contained in $U_{R_0} \times \mathbb{C}^*$. We set $\mathcal{S}p_{R_0} := U_{R_0} \cup \{ \emptyset \}$.

Lemma 6.14 The closure $\overline{Sp(F)}$ of $Sp(F)$ in $\mathcal{S}p_{R_0} \times \mathbb{P}^1$ is complex analytic.

Proof Let $\tilde{h}$ denote the Hermitian metric of $\tilde{E}$ induced by $h$. Let $s$ be a flat section of $\tilde{E}|_{\{ \tilde{\zeta} \} \times \mathbb{R}}$ with respect to $\nabla_{y_1} - \sqrt{-1}\phi$. Then, we have

\[
\partial_{y_1}\tilde{h}(s, s) = \tilde{h}(s, (\nabla_{y_1} + \sqrt{-1}\phi)s) = \tilde{h}(s, 2\sqrt{-1}\phi s).
\]

Hence, there exists $C > 0$, which is independent of $\tilde{\zeta}$, such that $|\partial_{y_1}\tilde{h}(s, s)| \leq C|\text{Re}(\tilde{\zeta})| \cdot \tilde{h}(s, s)$. It implies that

\[
|\log|\tilde{F}|\tilde{h}| = O\left( |\text{Re}(\tilde{\zeta})| \right).
\]

Then, we obtain the claim of the lemma.

By replacing $R_0$ with a larger number, we may assume to have the decomposition

\[
\mathcal{S}p_{R_0} = \prod_{\omega \in \Omega} \mathcal{S}p(\omega),
\]

where $\mathcal{S}p_{R_0}$ is a union of graphs of ramified meromorphic functions $g$ such that $|\zeta|\omega g$ are bounded.

The group $\mathbb{Z}e_1$ acts on $\{ \tilde{\zeta} \} \times \mathbb{R}$. Let $S^1_{-\tilde{\zeta}, a}$ denote the quotient space. For $\tilde{\zeta} \in \tilde{U}_{R_0}$, there exists a naturally induced injection $S^1_{-\tilde{\zeta}, a} \rightarrow \mathcal{U}_{R_0}$. We obtain the induced vector bundle $\tilde{E}^\zeta$ on $S^1_{\tilde{\zeta}}$ with the metric $h^\zeta$, the unitary
connection $\nabla^C$, and the anti-Hermitian endomorphism $\phi^C$. There exists an orthonormal frame $u = (u_1, \ldots, u_r)$ of $E^C$ such that the following holds.

- There exists a constant anti-Hermitian matrix $A$ such that $\nabla^C u = u \cdot A$. Moreover, the eigenvalues of $A$ are contained in $\{\sqrt{-1} \rho | 0 \leq \rho \leq 2\pi\}$.

Because $\nabla^C \phi^C$ is bounded independently from $\zeta$, there exists a constant $C_{10}$, which is independent of $\zeta$, and a decomposition $\phi^C = \psi_0 + \psi_1$, such that the following holds.

- There exists a constant anti-Hermitian matrix $\Psi_0$ such that $\psi_0 u = u \Psi_0$.
- $|\psi_1|_{h^i} \leq C_{10}$.

Let $S(\zeta)$ be the set of the eigenvalues of $\Psi_0$. Then, there exists $C_{11} > 0$, which is independent of $\zeta$, such that the following holds for any $y_1$.

- For any eigenvalue $\alpha$ of $\tilde{\phi}_{(\zeta, y_1)}$, there exists $\beta \in S(\zeta)$ such that $|\alpha - \beta| < C_{11}$. Conversely, for any $\beta \in S(\zeta)$, there exists an eigenvalue $\alpha$ of $\tilde{\phi}_{(\zeta, y_1)}$ such that $|\alpha - \beta| < C_{11}$.

By Proposition 6.10, there exists $C_{12} > 0$, which is independent of $\zeta$, such that the following holds.

- For any eigenvalue $\gamma$ of $F_{\zeta}$, there exists $\beta \in S(\zeta)$ such that $|\log | \gamma | + a \sqrt{-1} \beta | < C_{12}$.

Then, the claim of Proposition 6.2 follows from the decomposition (59).

6.3.2 Proof of Theorem 6.3

Lemma 6.15 For any $k \in \mathbb{Z}_{\geq 0}$ and any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$, $|\nabla_{\kappa_1} \cdots \nabla_{\kappa_k} (\nabla \phi)|_h$ is bounded on $U_{2R_0}$.

Proof Take a positive number $\epsilon_0 > 0$, and we take $\epsilon_1 > 0$ such that $2C_0 \epsilon_1^2 < \epsilon_0$. For any $(y_0, y_1, y_2) \in U_{2R_0}$, let $S_{y_0, y_1, y_2} = \{(z_0, z_1, z_2) | |z_0 - y_0| < \epsilon_1\}$. We have $\nabla_{y_0, y_1, y_2} : \{(x_0, x_1, x_2) \in |x| < 1\} \rightarrow S_{y_0, y_1, y_2}$ by $(x_0, x_1, x_2) \mapsto (y_0, y_1, y_2) + \epsilon_1(x_0, x_1, x_2)$. We have $|G_{y_0, y_1, y_2}^{-1} F(h)| \leq \epsilon_0$ and $|\epsilon_1 G_{y_0, y_1, y_2} \nabla \phi| \leq \epsilon_0$. Set $\nabla' := G_{y_0, y_1, y_2}^{-1} (\nabla)$. For any $k$ and $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$, there exists $B_1(k)$ which is independent of $(y_0, y_1, y_2)$, such that

$$\nabla_{x_{\kappa_1}} \cdots \nabla_{x_{\kappa_k}} G_{y_0, y_1, y_2}^{-1} (\nabla \phi)| h \leq B_1(k).$$

Then, we obtain the desired estimate for the derivatives of $\nabla \phi$.

We obtain the following lemma as in the case of [21 Lemma 6.15].

Lemma 6.16 We have $|\rho_{\kappa}| = O\left(|(\nabla_{\kappa} \phi)^T|_h\right)$ for $\kappa = 0, 1, 2$. We also have

$$|\nabla_{\kappa_1} \rho_{\kappa_2}|_h = O\left(|(\nabla_{\kappa_2} \phi)^T| + |\nabla^*_{{\kappa_1}} (\nabla_{\kappa_2} \phi)^T|\right)$$

for any $\kappa_1, \kappa_2 \in \{0, 1, 2\}$.

By the argument in [21 §6.3.3], we obtain the following estimates:

$$h \left(\nabla_{\kappa_1}^2 (\nabla_{\kappa_2} \phi)^T, (\nabla_{\kappa_2} \phi)^T\right) = O\left(|(\nabla_{\kappa_1} \phi)^T|_h + |\nabla^*_{{\kappa_1}} (\nabla_{\kappa_2} \phi)^T|\right) \cdot |(\nabla_{\kappa_2} \phi)^T|, \tag{70}$$

$$\sum_{\kappa_1=0,1,2} h \left(\nabla_{\kappa_1}^2 (\nabla_{\kappa_2} \phi)^T, (\nabla_{\kappa_2} \phi)^T\right) = \left|(\phi, (\nabla_{\kappa_2} \phi)^T)\right|^2_h + O\left|(\nabla \phi)^T|_h \cdot |(\nabla_{\kappa_2} \phi)^T|\right|. \tag{71}$$

By using the estimates (70), (71) and the argument in [21 §6.3.3], we obtain the following inequality on $U_{2R_0}$ for some $C_{20} > 0$ and $R_{20} > R_0$:

$$-(\partial y_0^2 + \partial y_1^2 + \partial y_2^2)|(\nabla \phi)^T|_h \leq -C_{20}|(\nabla \phi)^T|_h^2 y_0^2.$$
Let \( f_{T^2} |(\nabla \phi)^T|^2 \) denote the function on \( \mathcal{H}_{R_0} \) obtained as the fiber integral of \( |(\nabla \phi)^T|^2 \) with respect to \( \mathcal{U}_{R_0} \rightarrow \mathcal{H}_{R_0} \). We obtain the following inequality on \( \mathcal{H}_{R_{20}} \):

\[
-\partial^2_{y_0} \int_{T^2} |(\nabla \phi)^T|^2 |_{h^2} \leq -C_{20} y_0^2 \int_{T^2} |(\nabla \phi)^T|^2 |_{h^2}.
\]

We take \( B_1 > 0 \) such that \( f_{T^2} |(\nabla \phi)^T|^2 \leq B_1 e^{-(C_{11}/2) y_0^2} \) at \( y_0 = -R_{20} \). For any \( \delta > 0 \), we set \( F_\delta := B_1 e^{-(C_{11}/2) y_0^2} - \delta (y_0 + R_{20}) \). The following holds on \( \mathcal{H}_{R_{20}} \):

\[
-\partial^2_{y_0} F_\delta \geq -C_{11} F_\delta.
\]

We also have \( f_{T^2} |(\nabla \phi)^T|^2 \leq F_\delta \) at \( y_0 = -R_1 \). By an argument as in Ahlfors Lemma \[1, 30\], for any \( \delta > 0 \) we obtain

\[
\int_{T^2} |(\nabla \phi)^T|^2 |_{h^2} \leq F_\delta
\]
on \( \mathcal{H}_{R_1} \). Then, by taking the limit \( \delta \rightarrow 0 \), we obtain

\[
\int_{T^2} |(\nabla \phi)^T|^2 |_{h^2} \leq B_1 e^{-(C_{11}/2) y_0^2}.
\]

Then, by the argument in \[21 \S 6.3.4\], we obtain the Theorem \[6.3\].

**Corollary 6.17** For any \( k \), there exist positive constants \( C(k) \) and \( \epsilon(k) \) such that

\[
\left| \nabla^i \sigma \ldots \sigma \nabla^k \phi \left( (\nabla^i \sigma)^2 \nabla^i \phi - 4 [\nabla^i \sigma \nabla^i \phi + [\phi, [\phi, \nabla^i \phi]] \right) \right|_{h^2} \leq C(k) e^{-\epsilon(k) y_0^2}.
\]

Here, \((a, b, c)\) is a cyclic permutation of \((0, 1, 2)\).

**Proof** Recall the following equalities:

\[
(\nabla_2^2 + \nabla_1^2 + \nabla_2^2) \nabla_i \phi = 4 [\nabla_j \phi, \nabla_k \phi] - [\phi, [\phi, \nabla_i \phi]],
\]

(73) where \((i, j, k)\) is a cyclic permutation of \((0, 1, 2)\). (For example, see \[21 \text{ Lemma 6.16}\].) Then, the corollary follows from Theorem \[6.3\] and \[73\].

**6.3.3 Proof of Proposition 6.5**

Let \( \mathfrak{A} \) denote the set of the permutations of \((0, 1, 2)\). The following holds:

\[
\sum_{i} \nabla_{\omega, i} \nabla_{\omega, i} \phi_\omega + \sum_{\sigma \in \mathfrak{A}_3} \nabla_{\omega, \sigma(0)} \left( F(\nabla \omega)_{\sigma(1)} \sigma(2) - \nabla_{\omega, \sigma(0)} \phi_\omega \right) = \sum_{\sigma \in \mathfrak{A}_3} \nabla_{\omega, \sigma(0)} F(\nabla \omega)_{\sigma(1)} \sigma(2) = 0.
\]

(74)

Because \( \nabla_{\omega, \sigma(0)} \left( F(\nabla \omega)_{\sigma(1)} \sigma(2) - \nabla_{\omega, \sigma(0)} \phi_\omega \right) = O(e^{-\epsilon_{30} y_0^2}) \) for some \( \epsilon_{30} > 0 \), we obtain the following estimate for some \( \epsilon_{31} > 0 \):

\[
-\left( \partial^2_{y_0} + \partial^2_{y_1} + \partial^2_{y_2} \right) |\phi_\omega|^2 = -2 |\nabla \omega \phi_\omega|^2 + O(e^{-\epsilon_{31} y_0^2}).
\]

We obtain the following for some \( \epsilon_{32} > 0 \):

\[
-\partial^2_{y_0} \int_{T^2} |\phi_\omega|^2 + O(e^{-\epsilon_{32} y_0^2}) = -2 \int_{T^2} |\nabla \omega \phi_\omega|^2 \leq 0.
\]

Because \( |\phi_\omega| \) is bounded, we obtain \( \partial_{y_0} \int_{T^2} |\phi_\omega|^2 \rightarrow 0 \) as \( y_0 \rightarrow -\infty \). We also obtain

\[
\int_{-\infty}^{R_1} \partial_{y_0} \int_{T^2} |\nabla \omega \phi_\omega|^2 < \infty.
\]

By using \[72\] with \( k = 0 \), we obtain that \( |\nabla \omega \phi_\omega| \rightarrow 0 \) as \( y_0 \rightarrow -\infty \). By a standard bootstrapping argument, we obtain that the norms of the higher derivatives of \( \nabla \omega \phi_\omega \) also converge to 0 as \( y_0 \rightarrow -\infty \). We also obtain that the norms of \( F(\nabla \omega) \) and its higher derivatives converge to 0 as \( y_0 \rightarrow -\infty \).
6.4 Asymptotically spectral decomposition

6.4.1 Setting

Let $E$ be a $C^\infty$-vector bundle on $U_{R_0}$ with a Hermitian metric $h$, a unitary connection $\nabla$, and an anti-Hermitian endomorphism $\phi$ such that the following holds.

- For any $k \geq 0$, there exist $B(k) > 0$ and $\epsilon(k) > 0$ such that
  \[
  \left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} (F(\nabla) - \ast \nabla) \right| \leq B(k) e^{-\epsilon(k) y_0^2}
  \]
  for any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$.
- $|\phi|$ is bounded.
- For any $k \geq 0$, $|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} (\nabla \phi)| \to 0$ as $y_0 \to -\infty$.

6.4.2 Modification to mini-holomorphic structures

We set $z := y_1 + \sqrt{-1} y_2$. For any $k \geq 0$, there exists $\epsilon_1(k) > 0$ such that
\[
\left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} (\left[ \nabla_{\tau}, \nabla_{y_0} - \sqrt{-1} \iota \phi \right] ) \right|_h = O(e^{-\epsilon_1(k) y_0^2})
\]
for any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$.

**Lemma 6.18** There exists $A \in \text{End}(E)$ with the following property.

- $[\nabla_{\tau} + A, \nabla_{y_0} - \sqrt{-1} \iota \phi] = 0$.
- For any $k \in \mathbb{Z}_{\geq 0}$, there exists $\epsilon_2(k) > 0$ such that $\left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} A \right|_h = O(e^{-\epsilon_2(k) y_0^2})$ for any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$.

**Proof** It is enough to take the integral of $[\nabla_{\tau}, \nabla_{y_0} - \sqrt{-1} \iota \phi]$ along $y_0$ by using the parallel transport with respect to $\nabla_{y_0} - \sqrt{-1} \iota \phi$.

The bundle $E$ has the mini-holomorphic structure $\overline{\mathcal{D}}_E$ given by $\partial_{E, y_0} = \nabla_{y_0} - \sqrt{-1} \iota \phi$ and $\partial_{E, \tau} := \nabla_{\tau} + A$. By the construction, the following holds:
\[
G(h) = \left[ \nabla_{z} - A^1, \nabla_{\tau} + A \right] - \frac{\sqrt{-1}}{2} \nabla_{y_0} \iota \phi.
\]
Hence, for any $k \geq 0$, there exists $\epsilon_3(k) > 0$ such that
\[
\left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} G(h) \right|_h = O(e^{-\epsilon_3(k) y_0^2})
\]
for any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$.

6.4.3 Spectral decomposition

We have the decomposition of the mini-holomorphic bundle
\[
(E, \overline{\mathcal{D}}_E) = \bigoplus_{\alpha \in (T^2)^\vee} (E_\alpha, \overline{\mathcal{D}}_{E_\alpha}),
\]
where $\text{Spec}(E_\alpha, \overline{\mathcal{D}}_{E_\alpha}) = \{\alpha\}$. Let $\Psi : \pi_1 \to \mathcal{H}_R$ denote the projection for any $R$.

**Lemma 6.19** If $R_{41}$ is sufficiently large, there exists a vector bundle $V = \bigoplus V_\alpha$ on $\mathcal{H}_{R_{41}}$ with a graded connection $\nabla_V = \bigoplus \nabla_{V_\alpha}$, a graded endomorphism $f = \bigoplus f_\alpha$, and a graded isomorphism $\Psi^{-1}(V) = \bigoplus \Psi^{-1}(V_\alpha) \simeq E_{(\text{gR}_{41})} = \bigoplus E_\alpha$ such that the following holds.
\[ f_\alpha \text{ has a unique eigenvalue } \alpha. \]

\[ \nabla_{V_\alpha}(f_\alpha) = 0. \]

\[ \partial_{E \star} = \partial_\star + \Psi^{-1}(f), \text{ where } \partial_\star \text{ is the naturally defined operator on } \Psi^{-1}(V). \]

\[ \nabla_{s_0} - \sqrt{-1} \partial_\phi = \Psi^{-1}(\nabla_{V_{s_0}}). \]

**Proof** If \( R_{41} \) is sufficiently large, \( E|_{\Psi^{-1}(s_0)} \) is semistable of degree 0 for any \( s_0 < -R_{41} \). We may assume it from the beginning.

We set \( U^*_R := U_R \times \mathbb{R}_{s_0} \). We introduce the complex coordinate system \( z := y_1 + \sqrt{-1} y_2 \) and \( w := y_3 + \sqrt{-1} y_0 \). We also set \( H^*_R := H_R \times \mathbb{R}_{s_0} \), on which we have the complex coordinate \( w := y_3 + \sqrt{-1} y_0 \). Let \( Q_1 : U^*_R \rightarrow U_R \) and \( Q_0 : H^*_R \rightarrow H_R \) denote the projections. Let \( \Psi^* : U^*_R \rightarrow H^*_R \) denote the projection.

We set \( E^* := Q_1^{-1}(E) \), which is naturally \( \mathbb{R}_{s_0} \)-equivariant. Let \( \partial_{E^*} \) denote the derivative on \( E^* \) with respect to \( \partial_{s_0} \). Let \( \partial_{E^*, s_0} \) denote the derivative on \( E^* \) induced by \( \partial_{s_0} \). We obtain the Hermitian metric \( \partial_{E^*} \) on \( E^* \), which is \( \mathbb{R}_{s_0} \)-equivariant. We have the spectral decomposition \( (E^*, \partial_{E^*}) = \bigoplus (V^*_\alpha, \partial_{V^*_\alpha}) \) corresponding to the spectral decomposition of \( E \).

According to [19, §2.1], there exists an \( \mathbb{R}_{s_0} \)-equivariant graded holomorphic vector bundle \( (V^*, \partial_{V^*}) = \bigoplus (V^*_\alpha, \partial_{V^*_\alpha}) \) on \( H^* \mathcal{R}_{s_0} \) with an \( \mathbb{R}_{s_0} \)-equivariant holomorphic graded endomorphism \( f^* = \bigoplus f^*_\alpha \) and an \( \mathbb{R}_{s_0} \)-equivariant graded isomorphism

\[ (\Psi^*)^{-1}(V^*) = \bigoplus (\Psi^*)^{-1}(V^*_\alpha) \]

such that the following holds:

\[ f^*_\alpha \text{ has a unique eigenvalue } \alpha. \]

\[ \partial_{E^*, s_0} = \partial_\star + (\Psi^*)^{-1}(f^*), \text{ where } \partial_\star \text{ is the naturally induced derivative on } (\Psi^*)^{-1}(V^*). \]

\[ \partial_{E^*, s_0} \text{ is equal to the operator induced by } \partial_{V^*, s_0}. \]

By the \( \mathbb{R}_{s_0} \)-equivariance of \( V^* \), we obtain a graded \( C^\infty \)-vector bundle \( V = \bigoplus V_\alpha \) on \( \mathcal{R}_{s_0} \). The \( \mathbb{R}_{s_0} \)-equivariant holomorphic structure induces a graded flat connection \( \nabla_V = \bigoplus \nabla_{V_\alpha} \). The \( \mathbb{R}_{s_0} \)-equivariant holomorphic graded endomorphism \( f^* \) induces a flat graded endomorphism \( f = \bigoplus f_\alpha \). The \( \mathbb{R}_{s_0} \)-equivariant graded isomorphism induces a graded isomorphism \( E \simeq \Psi^{-1}(V) \). Then, it is easy to see that they have the desired property.

We obtain the Hermitian metric \( h_\alpha \) of \( V_\alpha \) as follows:

\[ h_\alpha = \frac{1}{\text{vol}(T^2)} \int_{T^2} h(\Psi^{-1}(u_1), \Psi^{-1}(u_2)) \, dy_1 \, dy_2. \]

We set \( h^\circ := \bigoplus h^{-1}(h_\alpha) \) on \( E \). We obtain the automorphism \( b \) which is self-adjoint with respect to both \( h \) and \( h^\circ \), determined by \( h = h^\circ \cdot b \). The following estimate can be proved by arguments in [19, 21].

**Proposition 6.20** For any \( P \in \mathcal{A} \), there exist \( C(P) > 0 \) and \( \epsilon(P) > 0 \) such that

\[ |P(\nabla_{s_0}, \nabla_{s_0}, d_{\phi})| \leq C(P)e^{\epsilon(P)s_0}. \]

**Proof** We give an outline of the proof. We use the notation in the proof of Lemma 6.19. Let \( h^* \) be the metric of \( E^* \) induced by \( h \). We obtain Hermitian metrics \( h^*_\alpha \) of \( V^*_\alpha \) in a way similar to the construction of \( h_\alpha \). We set \( h^* := \bigoplus (h^*)^{-1}(h^*_\alpha) \). We obtain \( b^* \) by \( h^* = h^* \cdot b^* \). The metrics \( h^*, h^*_\alpha \), and \( h^* \) are \( \mathbb{R}_{s_0} \)-equivariant, and hence \( b^* \) is also \( \mathbb{R}_{s_0} \)-equivariant.

Let \( F(h^*) \) denote the curvature of the Chern connection \( \nabla^* \) of \( (E^*, \partial_{E^*}, h^*) \). We have the expression

\[ F(h^*) = F_E d_{\Phi} d_{\Phi} + F_{\Phi^2} d_{\Phi} d_{\Phi} + F_{\Phi^2} d_{\Phi} d_{\Phi} + F_{\Phi^2} d_{\Phi} d_{\Phi} + F_{\Phi^2} d_{\Phi} d_{\Phi}. \]

Let \( U \) be any open subset of \( \mathcal{R}_{s_0} \). Let \( U^* := U \times \mathbb{R}_{s_0} \). The fiber integral induces the map

\[ C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^*_\alpha)) \rightarrow C^\infty(U^*, \text{End}(V^*_\alpha)). \]
Let $C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))$ denote the kernel. There exists the injection

$$C^\infty(U^*, \text{End}(V^\alpha_\alpha)) \rightarrow C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))$$

induced by the pull back. Thus, we obtain the decomposition

$$C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha)) = C^\infty(U^*, \text{End}(V^\alpha_\alpha)) \oplus C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\beta_\alpha))_0.$$

We set

$$C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))^\circ := \bigoplus_\alpha C^\infty(U^*, \text{End}(V^\alpha_\alpha)),$$

$$C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))^\perp := \bigoplus_\alpha C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))_0 \oplus \bigoplus_{\alpha \neq \beta} C^\infty((\Psi^*)^{-1}(U^*), \text{Hom}(E^\alpha_\alpha, E^\beta_\beta)).$$

We obtain the decomposition

$$C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha)) = C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))^\circ \oplus C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))^\perp.$$

For any sections $s \in C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))$, we obtain the decomposition $s = s^\circ + s^\perp$. We also obtain a function $||s||^2$ on $U^*$ by the fiber integral of $|s|^2$. There exists $C > 0$ such that $||\nabla^s_\alpha|| \geq C ||s^\perp||$ and $||\nabla^s_\alpha|| \geq C ||s^\perp||$ for any $s^\perp \in C^\infty((\Psi^*)^{-1}(U^*), \text{End}(E^\alpha_\alpha))^\perp$.

By using the argument in the proof of [19] §5.5.2, we obtain the following estimates for some small $\epsilon_i > 0$:

$$-\partial_\alpha \partial_{\bar{\alpha}} ||F^\perp_\alpha||^2 \leq -||\nabla^\perp_\alpha F^\perp_\alpha||^2 - ||\nabla^\perp_{\bar{\alpha}} F^\perp_\alpha||^2 - ||\nabla^\perp_w F^\perp_w||^2 + O \left( \epsilon_1 ||F^\perp_\alpha||^2 + \epsilon_1 ||F^\perp_{\bar{\alpha}}|| ||F^\perp_w|| + \epsilon_1 ||\nabla^\perp_{\bar{\alpha}} F^\perp_\alpha|| ||F^\perp_w|| + \epsilon_1 ||\nabla^\perp_w F^\perp_w|| ||F^\perp_\alpha|| \right) + O \left( \exp(-\epsilon_2 \gamma_0^2) \right). \quad (76)$$

$$-\partial_\alpha \partial_{\bar{\alpha}} ||F^\perp_{\bar{\alpha}}||^2 \leq -||\nabla^\perp_{\bar{\alpha}} F^\perp_{\bar{\alpha}}||^2 - ||\nabla^\perp_\alpha F^\perp_{\bar{\alpha}}||^2 - ||\nabla^\perp_w F^\perp_w||^2 + O \left( \epsilon_1 ||F^\perp_\alpha||^2 ||F^\perp_{\bar{\alpha}}|| + \epsilon_1 ||\nabla^\perp_\alpha F^\perp_{\bar{\alpha}}|| ||F^\perp_w|| + \epsilon_1 ||\nabla^\perp_w F^\perp_w|| ||F^\perp_{\bar{\alpha}}|| \right) + O \left( \exp(-\epsilon_2 \gamma_0^2) \right). \quad (77)$$

$$-\partial_\alpha \partial_{\bar{\alpha}} ||F^\perp_w||^2 \leq -||\nabla^\perp_\alpha F^\perp_w||^2 - ||\nabla^\perp_{\bar{\alpha}} F^\perp_w||^2 - ||\nabla^\perp_w F^\perp_w||^2 + O \left( \epsilon_1 ||F^\perp_{\bar{\alpha}}|| ||F^\perp_w|| + \epsilon_1 ||\nabla^\perp_{\bar{\alpha}} F^\perp_w|| ||F^\perp_w|| + \epsilon_1 ||\nabla^\perp_w F^\perp_w|| ||F^\perp_{\bar{\alpha}}|| \right) + O \left( \exp(-\epsilon_2 \gamma_0^2) \right). \quad (78)$$

From the estimate for $G(h)$, we obtain

$$-\partial_\alpha \partial_{\bar{\alpha}} ||F^\perp_\alpha||^2 = -\partial_\alpha \partial_{\bar{\alpha}} ||F^\perp_{\bar{\alpha}}||^2 + O \left( \exp(-\epsilon_2 \gamma_0^2) \right). \quad (79)$$

We set $g := ||F^\perp_\alpha||^2 + ||F^\perp_{\bar{\alpha}}||^2 + ||F^\perp_w||^2 + ||F^\perp_{\bar{\alpha}}||^2$. From these estimates, we obtain the following for some $C_1 > 0$:

$$-\partial_\alpha \partial_{\bar{\alpha}} g \leq -C_1 g + C_2 \exp(-\epsilon_2 \gamma_0^2).$$

Note that $g$ depends only on $y_0$ by the $\mathbb{R}_{y_0}$-equivariance. Hence, we obtain the following:

$$-\partial_{y_0}^2 g \leq -C_1 g + C_2 \exp(-\epsilon_2 \gamma_0^2).$$

By a standard argument of Ahlfors lemma [11] [20], we obtain that $g = O(e^{\epsilon_3 \gamma_0})$ for some $\epsilon_3 > 0$.

Set $F(h)^\perp := F^\perp_{\bar{\alpha}}dz d\bar{z} + F^\perp_\alpha dw d\bar{w} + F^\perp_w dw d\bar{w} + F^\perp_{\bar{\alpha}} dw d\bar{w}$. By using a standard bootstrapping argument as in the proof of [19] Proposition 5.8, we obtain the following.
• For any \( P \in A \), we have \( C(P) > 0 \) and \( \epsilon(P) > 0 \) such that
\[
|P(\nabla^s_z, \nabla^s_T, \nabla^s_w, \nabla^s_w)F(h^*)| \leq C(P) \exp(\epsilon(P)y_0).
\]

By [19] Lemma 10.13, we obtain the following.

• For any \( P \in A \), we have \( C(P) > 0 \) and \( \epsilon(P) > 0 \) such that
\[
|P(\nabla^s_z, \nabla^s_T, \nabla^s_w, \nabla^s_w)(b^* - \text{id})| \leq C(P) \exp(\epsilon(P)y_0).
\]

Note that for any \( \mathbb{R}_{g_3} \)-invariant section \( s \) of \( \text{End}(E) \), we have
\[
\nabla^s_w(s) = \frac{\sqrt{-1}}{2}(\nabla_{y_0} - \sqrt{-1}\phi)s, \quad \nabla^s_z(s) = \frac{\sqrt{-1}}{2}(\nabla_{y_0} + \sqrt{-1}\phi)s.
\]
We also have \( \nabla^s_z = (\nabla_z + A)s \) and \( \nabla^s_z = (\nabla_z - A^\dagger)s \). Hence, we obtain the desired estimate for \( b \).

6.4.4 Anti-Hermitian endomorphisms

We have \( \phi_{3,\alpha} \) determined by \( \nabla_{V_{\alpha}} = \nabla_{V_{\alpha}}^U - \sqrt{-1}\phi_{3,\alpha} dy_0 \), where \( \nabla^U_{V_{\alpha}} \) is a unitary connection of \( (V_{\alpha}, h_{\alpha}) \), and \( \phi_{3,\alpha} \) is an anti-Hermitian endomorphism of \( (V_{\alpha}, h_{\alpha}) \). Set \( \phi_3 := \bigoplus \phi_{3,\alpha} \).

**Proposition 6.21** For any \( P \in A \), there exist \( C(P) > 0 \) and \( \epsilon(P) > 0 \) such that
\[
|P(\nabla_{y_0}, \nabla_{y_1}, \nabla_{y_2}, \phi)(\phi - \Psi^{-1}(\phi_4))| \leq C(P)e^{\epsilon(P)y_0}.
\]

**Proof** It follows from Proposition 6.20.

We define the anti-Hermitian endomorphisms \( \phi_i = \bigoplus \phi_{i,\alpha} \) (\( i = 1, 2 \)) of \( (V, h_V) = \bigoplus(V_{\alpha}, h_{\alpha}) \) by \( f = \frac{1}{2}(\phi_1 + \sqrt{-1}\phi_2) \).

**Lemma 6.22** \( \nabla_{V_{y_0}}\phi_1 - [\phi_2, \phi_3] = 0 \) and \( \nabla_{V_{y_0}}\phi_2 - [\phi_3, \phi_1] = 0 \) hold.

**Proof** It follows from the flatness \( [\nabla_{V_{y_0}} - \sqrt{-1}\phi_3, f] = 0 \).

**Proposition 6.23** For any \( P \in A \), there exist \( C(P) > 0 \) and \( \epsilon(P) > 0 \) such that
\[
|P(\nabla_{V_{y_0}}, \phi_1, \phi_2, \phi_3)(\nabla_{V_{y_0}}\phi_3 - [\phi_1, \phi_2])| \leq C(P)e^{\epsilon(P)y_0}.
\]

**Proof** It follows from Proposition 6.20 and the estimate (75).

**Lemma 6.24** \( \phi_i \) (\( i = 1, 2 \)) are bounded.

**Proof** Let \( \nabla_{V^\ast}, + \partial_{V^\ast} \) be the Chern connection of \( (V^\ast, \nabla_{V^\ast}, h^*) \). Set \( \theta^* := f^* dh \), which is a Higgs field of \( (V^\ast, \nabla_{V^\ast}) \). Let \( (\theta^*)^\dagger \) denote the adjoint of \( \theta^\dagger \) with respect to \( h^* \). We obtain \( [\nabla_{V^\ast}, \partial_{V^\ast}] + [\theta^*, (\theta^*)^\dagger] = O(e^{\epsilon y_0}) \) for some \( \epsilon > 0 \). Note that the eigenvalues of \( \bar{f} \) are constant. Hence, as a variant of Simpson’s main estimate ([15] Proposition 2.10)), we obtain that \( |f_{h^*} \) is bounded. Then, the claim of the lemma follows.

6.5 Approximate solutions of Nahm equations

6.5.1 Reduction

Let \( V \) be a \( C^\infty \)-vector bundle on \( \mathcal{H}_R \) with a Hermitian metric \( h \), a unitary connection \( \nabla \), and bounded anti-self-adjoint endomorphisms \( \phi_i \) (\( i = 1, 2, 3 \)). We introduce a condition.
Condition 6.25 For any $P \in \mathcal{A}$, there exist $\epsilon(P) > 0$ and $B(P) > 0$ such that

$$\left| P(\nabla_{y_0}, \phi_1, \phi_2, \phi_3)(\nabla_{y_0} \phi_i - [\phi_j, \phi_k]) \right| \leq B(P)e^{\epsilon(P)y_0}, \quad (80)$$

where $(i, j, k)$ denotes any cyclic permutation of $(1, 2, 3)$. Moreover, $\nabla_{y_0} \phi_i \to 0$ as $y_0 \to -\infty$.

Proposition 6.26 There exist a finite subset $S \subset (\sqrt{-1}\mathbb{R})^3$, an orthogonal decomposition $V = \bigoplus_{b \in S} V_b$, a graded unitary connection $\nabla^v = \bigoplus \nabla_{\nabla_{y_0}}^v$, and graded anti-self-adjoint endomorphisms $\phi_i^v = \bigoplus \phi_i^{v,b}$ such that the following holds:

- The eigenvalues of $\phi_i^{v,b}$ converge to $b_i$ as $y_0 \to -\infty$.

- Set $\rho^v := \nabla - \nabla^v$. For any $k$, there exist $B(k) > 0$ and $\epsilon(k) > 0$ such that

$$\left\| (\nabla_{y_0})^k \rho^v \right\|_h + \sum \left\| (\nabla_{y_0})^k (\phi_i - \phi_i^v) \right\|_h \leq B(k)e^{\epsilon(k)y_0}.$$ 

As a result, $V_b$, the induced metric $h_b$, the induced connection $\nabla_{y_0}^v$, and the anti-Hermitian endomorphisms $\phi_i^{v,b}$ ($i = 1, 2, 3$) satisfy Condition 6.25.

Proof We begin with a preliminary.

Lemma 6.27 For each $i$, there exist a finite subset $S(\phi_i) \subset \sqrt{-1}\mathbb{R}$ such that the following holds:

- Let $Sp(\phi_{i|y_0})$ be the set of eigenvalues of $\phi_{i|y_0}$. For any $\alpha \in \mathbb{C}$ and $\delta > 0$, set $B_\alpha(\delta) := \{ \beta \in \mathbb{C} \mid |\alpha - \beta| < \delta \}$.

Then, for any $\delta_1 > 0$, there exists $R_2$ such that the following holds for any $y_0 < -R_2$:

$$Sp(\phi_{i|y_0}) \subset \bigcup_{\alpha \in S(\phi_i)} B_\alpha(\delta_1), \quad S(\phi_i) \subset \bigcup_{\alpha \in Sp(\phi_{i|y_0})} B_\alpha(\delta_1).$$

Proof We set $F := \phi_2 + \sqrt{-1}\phi_3$. Then, $(\nabla_{y_0} - \sqrt{-1}\phi_1)F = O(e^{\epsilon y_0})$. There exist $A \in \text{End}(V)$ such that $(\nabla_{y_0} - \sqrt{-1}\phi_1)A = (\nabla_{y_0} - \sqrt{-1}\phi_1)F$ and that $A = O(e^{\epsilon y_0})$ for some $\epsilon > 0$. We set $\tilde{F} := F - A$. Because $(\nabla_{y_0} - \sqrt{-1}\phi_1)\tilde{F} = 0$, the eigenvalues of $\tilde{F}$ are constant with respect to $y_0$. Then, we obtain the claim for $\phi_2$ and $\phi_3$. Similarly, we obtain the claim for $\phi_1$.

Let $(V, \phi_3) = \bigoplus_{\alpha \in S(\phi_3)} (V_\alpha, \phi_3, \alpha)$ be the decomposition satisfying the following condition:

- For any $\delta_1 > 0$, there exists $R_2$ such that eigenvalues $\beta$ of $\phi_{3,\alpha|y_0}$ ($y_0 < -R_2$) satisfy $|\alpha - \beta| \leq \delta_1$.

We obtain the decomposition $\nabla = \nabla^\bullet + \rho$, where $\nabla^\bullet = \bigoplus \nabla_{\alpha}^\bullet$ is the direct sum of unitary connections $\nabla_{\alpha}^\bullet$ on $V_\alpha$, and $\rho$ is a section of $\bigoplus_{\alpha \neq \beta} \text{Hom}(V_\alpha, V_\beta) d y_0$. We also obtain the decomposition $\phi_i = \phi_i^\bullet + \phi_i^\rho$ ($i = 1, 2, 3$) according to the decomposition $\text{End}(V) = \bigoplus \text{End}(V_\alpha) \oplus \bigoplus_{\alpha \neq \beta} \text{Hom}(V_\alpha, V_\beta)$. Clearly, $\phi_3^\bullet = \phi_3$ holds. We have the decomposition $\phi_i^\bullet = \bigoplus \phi_i^{\bullet, \alpha}$.

We obtain the following estimate by an argument similar to the proof of Theorem 6.3.

Lemma 6.28 For any $k$, there exists $\epsilon(k) > 0$ such that

$$\left| (\nabla_{y_0}^\bullet)^k \phi_1^\bullet \right|_h + \left| (\nabla_{y_0}^\bullet)^k \phi_2^\bullet \right|_h + \left| (\nabla_{y_0}^\bullet)^k \rho \right|_h = O(e^{\epsilon(k)y_0}).$$

Proof We give only an outline. By using an argument in the proof of $\nabla_{y_0}^k \phi_1$ are bounded for any $k$.

We obtain a bundle $\nabla := \Psi^{-1}(V)$ on $\mathcal{U}_{R_0}$, with the metric $\tilde{h} = \Psi^{-1}(h)$, the unitary connection $\nabla := \Psi^{-1}(\nabla + \phi_1 d y_1 + \phi_2 d y_2$ and the anti-Hermitian metric $\bar{\phi} := \Psi^{-1}(\phi_3)$. Let $F(\nabla)$ denote the curvature of $\nabla$. We have the following:

- For any $k \in \mathbb{Z}_{\geq 0}$, there exists $C(k) > 0$ and $\epsilon(k) > 0$ such that

$$\left| \nabla_{k_1} \circ \cdots \circ \nabla_{k_k} (F(\nabla) - \star \nabla \bar{\phi}) \right| \leq C(k)e^{\epsilon(k)y_0}$$

for any $(k_1, \ldots, k_k) \in \{0, 1, 2\}^k$.
• For any \((\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k\), \(|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k}(\nabla \phi)| \to 0\) as \(y_0 \to -\infty\).

Corresponding to the decomposition \(\text{End}(\tilde{V}) = \bigoplus \text{End}(\tilde{V}_a) \oplus \bigoplus_{\alpha \neq \beta} \text{Hom}(\tilde{V}_\alpha, \tilde{V}_\beta)\), we obtain \(\tilde{\nabla} = \tilde{\nabla}^* + \tilde{\rho}\). Note that \(\tilde{\rho} = \Psi^{-1}(\rho) + \Psi^{-1}(\phi_1^*)dy_0 + \Psi^{-1}(\phi_2^*)dy_2\). Any section \(s\) of \(\text{End}(\tilde{V})\) is decomposed into \(s^* + s^\top\).

By using the argument in the proof of [21, Lemma 6.16], we obtain the following:

\[
\sum_{i=0,1,2} \tilde{\nabla}_i^2(\nabla \phi) = 4[\tilde{\nabla}_b \phi, \tilde{\nabla}_c \phi] - [\tilde{\phi}, [\tilde{\phi}, \tilde{\nabla}_a \phi]] + O(e^{\epsilon_0y_0}),
\]

where \((a, b, c)\) is a cyclic permutation of \((0, 1, 2)\). By the argument in the proof of [21, Lemma 6.17], for any \(\delta > 0\) there exists \(R_{10}\) such that the following holds on \(U_{R_{10}}\):

\[
\tilde{h}(\tilde{\nabla}_{a_1}^2(\tilde{\nabla}_{a_2} \phi), (\tilde{\nabla}_{a_2} \phi)^\top) = O\left(\delta \cdot \left|\left|\tilde{\nabla}_{a_1} \phi\right|\right|_{h} + \left|\left|\tilde{\nabla}_{a_1}^* (\tilde{\nabla}_{a_1}^* \phi)^\top\right|\right|_{h} \right) + O(e^{\epsilon_0y_0}).
\]

By the argument in the proof of [21, Lemma 6.18], for any \(\delta > 0\) there exists \(R_{10}\) such that the following holds on \(U_{R_{10}}\):

\[
\sum_{a_1=0,1,2} \tilde{h}(\tilde{\nabla}_{a_1}^2(\tilde{\nabla}_{a_2} \phi), (\tilde{\nabla}_{a_2} \phi)^\top) = \left|\left|\tilde{\phi}, (\tilde{\nabla}_{a_2} \phi)^\top\right|\right|^2 + O\left(\delta \cdot \left|\left|\tilde{\nabla}_{a_2} \phi\right|\right|_{h} \cdot \left|\left|\tilde{\nabla}_{a_2}^* \phi\right|\right|_{h} \right) + O(e^{\epsilon_0y_0})
\]

By using the argument in the proof of [21, Lemma 6.19], we obtain

\[
- \sum_{i=0,1,2} \partial_i^2 \left|\left|\tilde{\nabla} \phi\right|\right|^2 = - \sum_{i=0,1,2} 2\left|\tilde{\nabla} \phi\right|^2 - 2\left|\phi, (\tilde{\nabla} \phi)^\top\right|^2 + O\left(\delta \cdot \left|\left|\tilde{\nabla} \phi\right|\right|^2 \right)
\]

\[
+ O\left(\delta \sum (\tilde{\nabla} i (\tilde{\nabla} \phi)^\top \cdot \left|\left|\tilde{\nabla} \phi\right|\right|_{h}) \right) + O(e^{\epsilon_0y_0}).
\]

Note that \(\left|\left|\tilde{\phi}, (\tilde{\nabla}_{a_2} \phi)^\top\right|\right|^2 \geq c \left|\left|\tilde{\nabla}_{a_2} \phi\right|\right|^2\) for some \(c > 0\), which we may assume to be independent of \(R_{10}\). Hence, we obtain the following if \(R_{10}\) is large enough:

\[
- \sum_{i=0,1,2} \partial_i^2 \left|\left|\tilde{\nabla} \phi\right|\right|^2 \leq -c_1 \left|\left|\tilde{\nabla} \phi\right|\right|^2 + O(e^{\epsilon_0y_0}).
\]

We set \(g := \left|\left|\nabla \phi_3\right|\right|^2 + \left|\left|\phi_3, \phi_1\right|\right|^2 + \left|\left|\phi_3, \phi_2\right|\right|^2\). Because \((\nabla, \tilde{h}, \tilde{\nabla}, \tilde{\phi})\) is equivariant with respect to the natural action of \(\mathbb{R}_{y_1} \oplus \mathbb{R}_{y_2}\), we obtain the following:

\[
-\partial^2_{y_0} g \leq -c_2 g + O(e^{\epsilon_0y_0}).
\]

By a standard argument, we obtain \(g = O(e^{\epsilon_1y_0})\) for some \(\epsilon_1 > 0\). We obtain \(|\phi_1^\top| + |\phi_2^\top| + |\rho^\top| = O(e^{\epsilon_1y_0})\). By a bootstrapping argument, we obtain the estimates for higher derivatives.

We obtain \((V_a, h_a)\) with a unitary connection \(\nabla_a^*\) and bounded anti-Hermitian endomorphisms \(\phi_i^*, \alpha\) \((i = 1, 2, 3)\) satisfying Condition [6.25]. Moreover, the eigenvalues of \(\phi_{3, \alpha}\) converges to \(\alpha\) as \(y_0 \to -\infty\). By an applying similar argument to \(\phi_{2, \alpha}\) and \(\phi_{1, \alpha}\) inductively, we obtain the claim of Proposition [6.26].

We shall study the behaviour of \(\phi_{i,b}^* - b_i \text{id} y_k\) in the next subsection.

6.5.2 Decay

Let \((V, h, \nabla, \{\phi_i\}_{i=1,2,3})\) be as in [6.5.1] satisfying Condition [6.25]. Moreover, we assume that the eigenvalues of \(\phi_i\) are convergent to 0 as \(y_0 \to -\infty\).

**Proposition 6.29** For any \(k \in \mathbb{Z}_{\geq 0}\), \([y_0^{k+1} \nabla_{y_0}^k \phi_i]\) are bounded. In particular, we obtain the expression \(\phi_i = y_0^{-1} A_i + O(y_0^{-2})\) for endomorphisms \(A_i\) such that \(\nabla A_i = 0\), and the tuple \((A_1, A_2, A_3)\) satisfies \([A_1, A_j] = A_k\), where \((i, j, k)\) are cyclic permutation of \((1, 2, 3)\).
Proof Let $F$, $\mathfrak{A}$ and $\tilde{F}$ be as in the proof of Lemma 6.27. Because the eigenvalues of $\phi_i$ converges to 0, we obtain that $\tilde{F}$ is nilpotent. By the construction, we have $\left[\nabla_{y_0} - \sqrt{-1}\phi_1, \tilde{F}\right] = 0$, and

$$\left[\tilde{F}'', \tilde{F}\right] + 2\sqrt{-1}\nabla_{y_0}\phi_1 = O(e^{y_0}).$$

We obtain the following:

$$\partial^2_{y_0} |\tilde{F}|^2_h = \left|\nabla_{y_0} + \sqrt{-1}\phi_1, \tilde{F}\right|_h^2 + \left|\tilde{F}'', \tilde{F}\right|_h^2 + O(e^{y_0} |\tilde{F}|^2_h).$$

Hence, we obtain

$$-\partial^2_{y_0} \log |\tilde{F}|^2_h \leq -\frac{|\tilde{F}'', \tilde{F}|^2_h}{|\tilde{F}|^2_h} + O(e^{y_0}).$$

Because $\tilde{F}$ is nilpotent, there exists a positive constant $c_1$ depending only on rank $E$ such that $|\tilde{F}', \tilde{F}|^2_h \geq c_1 |\tilde{F}|^2_h$. Hence, we obtain the following for some $c_2 > 0$:

$$-\partial^2_{y_0} \log |\tilde{F}|^2_h \leq c_2 |\tilde{F}|^2_h + O(e^{y_0}).$$

By a standard argument of Ahlfors lemma [1, 30] we obtain that $|\tilde{F}|^2_h = O(y_0^{-2})$. We obtain $|\phi_i| = O(y_0^{-1})$ ($i = 2, 3$). Similarly, we obtain $|\phi_1| = O(y_0^{-1})$. Then, we obtain $\nabla\phi_i = -[\phi_j, \phi_k] = O(y_0^{-2})$. By an inductive argument, we obtain the estimates for the higher derivatives of $\phi_i$. 

6.5.3 Norm estimate and the conjugacy class of the nilpotent map

Let $(V, h, \nabla)$ and $\phi_i$ ($i = 1, 2, 3$) be as in 6.5.2. Let $\tilde{F}$ be the endomorphism of $V$ as in the proof of Lemma 6.27. If is flat with respect to $\nabla_{y_0} - \sqrt{-1}\phi_1$. In this case, $\tilde{F}$ is nilpotent. We obtain the weight filtration $W$ of $V$ with respect to $\tilde{F}$, which is preserved by $\nabla_{y_0} - \sqrt{-1}\phi_1$.

Let $e = (e_1, \ldots, e_r)$ be a frame of $V$ satisfying the following conditions.

- $(\nabla_{y_0} - \sqrt{-1}\phi_1)e = 0$.
- $e$ is compatible with $W$, i.e., there is a decomposition $e = \bigcup_{k \in \mathbb{Z}} e_k$ such that $\bigcup_{k \leq \ell} e_k$ is a frame of $W_\ell$.

If $e_i \in e_k$, we set $k(i) := k$. Let $h_0$ be the Hermitian metric of $V$ defined by $h_0(e_i, e_i) = (-y_0)^{k(i)}$ and $h_0(e_i, e_j) = 0$ ($i \neq j$).

Proposition 6.30 $h$ and $h_0$ are mutually bounded.

Proof Set $\Delta_R^* := \{w \in \mathbb{C}^* \mid |w| < -R\}$. Let $Q : \Delta_R^* \longrightarrow \mathcal{H}_R$ be the map defined by $Q(w) = \log |w|$. We set $\widetilde{V} := Q^{-1}(V)$ and $\tilde{h} := Q^{-1}(h)$. They are naturally $S^1$-equivariant. We define the derivative $\partial_{\nabla', \nabla}$ on $\widetilde{V}$ with respect to $\nabla'$ by $\partial_{\nabla', \nabla'} Q^{-1}(s) = Q^{-1}((\nabla_{y_0} - \sqrt{-1}\phi_1)s)$. It induces an $S^1$-equivariant holomorphic structure $\partial_{\nabla'}$ on $\widetilde{V}$. Let $\tilde{f}$ be the holomorphic endomorphism of $\widetilde{V}$ induced by $\tilde{F}$. We set $\tilde{\theta} := \tilde{f} dw/w$. Let $\tilde{\nabla}$ denote the Chern connection of $(\tilde{V}, \partial_{\nabla', \nabla}, \tilde{h})$, and let $F(\tilde{\nabla})$ be the curvature of $\tilde{\nabla}$. Let $\tilde{\theta}$ denote the adjoint of $\tilde{\theta}$ with respect to $\tilde{h}$. Then, we have $F(\tilde{\nabla}) + [\tilde{\theta}, \tilde{\theta}] = O(|w|^{-2}) dw\partial_{\nabla}w$ for some $\epsilon > 0$. We also have $F(\tilde{\nabla}) = O(|w|^{-2}) dw\partial_{\nabla}w$.

Set $\Delta_R := \Delta_R^* \cup \{0\}$. We have the associated filtered bundle $\mathcal{F}_a \tilde{V}$ on $(\Delta_R, 0)$. Let us observe that $\text{Gr}_{a}^{\mathcal{F}}(\tilde{V}) = 0$ unless $a \in \mathbb{Z}$. Indeed, let $\tilde{e} = (\tilde{e}_i)$ denote the $S^1$-equivariant holomorphic frame of $\tilde{V}$ induced by $e$. Let $H(\tilde{h}, \tilde{e})$ be the Hermitian-matrix valued function whose $(i, j)$-entries are $h(\tilde{v}_i, \tilde{v}_j)$. Then, it is easy to see that $C^{-1}(-\log |w|)^N < H(\tilde{h}, \tilde{e}) < C(-\log |w|)^N$ for some $C > 1$ and $N > 0$. Thus, we obtain $\text{Gr}_{a}^{\mathcal{F}}(\tilde{V}) = 0$ unless $a \in \mathbb{Z}$. Then, the claim of Proposition 6.30 follows from the norm estimate in [30] follows.

Let $C_0$ be the matrix determined by $(C_0)_{i,i} = k(i)/2$ and $(C_0)_{i,j} = 0$ ($i \neq j$).

Proposition 6.31 The conjugacy class of $-\sqrt{-1}A_1$ is represented by $C_0$. 

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Proof Let $v$ be an orthonormal frame of $V$ such that $\nabla_{y_0} v = 0$. We obtain the matrix valued function $A_1$ determined by $\phi' v = v A_1$. There is a constant matrix $A_{1,0}$ such that $A_1 - y_0^{-1} A_{1,0} = O(y_0^{-2})$. We have $(\nabla_{y_0} - \sqrt{-1} \phi') v = v \cdot (-\sqrt{-1} A_1)$. We may assume that $A_{1,0}$ is diagonal.

We set $e'_i := (-y_0)^{-k(i)/2} e_i$. We obtain a frame $e' = (e'_i)$. Let $B$ be the GL($r$)-valued function determined by $v = e' \cdot B$. By Proposition 6.30 $B$ and $B^{-1}$ are bounded. Because $(\nabla_{y_0} - \sqrt{-1} \phi_1) e' = e' C_0 y_0^{-1}$, we obtain the relation

$$y_0 \partial_{y_0} B + C_0 B + \sqrt{-1} B \cdot A_{1,0} + \sqrt{-1} B \cdot (y_0 A_1 - A_{1,0}) = 0.$$ 

Note that the eigenvalues of $\sqrt{-1} A_{1,0}$ are contained in $\frac{i}{2} \mathbb{Z}$ because $(A_1, A_2, A_3)$ induces an $\mathfrak{su}(2)$-representation. It is easy to check the following lemma.

Lemma 6.32 Let $a \in \frac{i}{2} \mathbb{Z}$. Let $g$ be a bounded $C^\infty$-function on $\mathcal{H}_R$ satisfying $y_0 \partial_{y_0} g + ag = O(|y_0|^{-1})$. Then, the following holds.

- If $a = 1$, then $g = O(|y_0|^{-1} \log |y_0|)$.
- If $a = 1/2$, then $g = O(|y_0|^{-1/2})$.
- If $a = 0$, there exists $g_0 \in \mathbb{C}$ such that $g - g_0 = O(|y_0|^{-1})$.
- Otherwise, $g = O(|y_0|^{-1})$.

By Lemma 6.32 we obtain the following.

- There exists $B_{i,j,0} \in \mathbb{C}$ such that $B_{i,j} - B_{i,j,0} = O(|y_0|^{-1/2})$. Moreover, $B_{i,j,0} = 0$ unless $-\sqrt{-1} (A_{1,0})_{i,j} = k(i)/2$.

Then, the claim of Proposition 6.31 follows from the boundedness of $B$ and $B^{-1}$.

Proposition 6.33 The conjugacy class of $\tilde{F}$ is equal to the conjugacy class of $A_2 - \sqrt{-1} A_3$.

Proof Let $N_0$ be the matrix valued function determined by $\tilde{F} e' = e' \cdot N_0$. There exists a constant matrix $N_0$ such that $N_0 = N_0 y_0^{-1} = O(|y_0|^{-3/2})$. It is easy to observe that the conjugacy class of $\tilde{F}$ is represented by $N_0$.

Let $N_1$ be the matrix valued function determined by $(\phi_2 - \sqrt{-1} \phi_3) e' = e' \cdot N_1$. Because $\tilde{F} - (\phi_2 - \sqrt{-1} \phi_3) = O(e^{y_0})$, we obtain $N_1 - N_0 y_0^{-1} = O(|y_0|^{-3/2})$.

Let $v$ and $B$ be as in the proof of Proposition 6.31 Let $N_2$ be the matrix valued function determined by $(\phi_2 - \sqrt{-1} \phi_3) v = v \cdot N_2$. We have the constant matrix $N_2$ such that $N_2 - N_2 = O(|y_0|^{-1})$. The conjugacy class of $A_2 - \sqrt{-1} A_3$ is represented by $N_2$. We have the relation $N_2 = B^{-1} N_1 B$. Then, we obtain that $N_2$ and $N_0$ are conjugate.

7 Hermitian metrics and filtered prolongation

7.1 Prolongation of monopoles with bounded curvature

7.1.1 Prolongation of mini-holomorphic bundles with Hermitian metric

We use the notation in 3.14 Let $\nu$ denote 0 or $\infty$. Let $p$ be any positive integer. Let $\overline{U}_{\nu,p}$ be a neighbourhood of $H_{\nu,p}$ in $\overline{M}_{\nu,p}$. We set $\overline{U}_{\nu,p} := \overline{U}_{\nu,p} \setminus \overline{H}_{\nu,p}$. For any $t \in S_{\nu}$, we put $\overline{U}_{\nu,p}(t) := \pi_p^{-1}(t) \cap \overline{U}_{\nu,p}$ and $\overline{U}_{\nu,p}(t) := \pi_p^{-1}(t) \cap \overline{U}_{\nu,p}$. We also set $\overline{U}_{\nu,p} := p^{-1}(\overline{U}_{\nu,p})$ and $\overline{U}_{\nu,p} := p^{-1}(\overline{U}_{\nu,p})$. For any $t \in \mathbb{R}$, we put $\overline{U}_{\nu,p}(t) := \pi_{\nu}^{-1}(t) \cap \overline{U}_{\nu,p}$ and $\overline{U}_{\nu,p}(t) := \pi_{\nu}^{-1}(t) \cap \overline{U}_{\nu,p}$.

Let $(E, \nabla_E)$ be a Hermitian metric $h$. We have the Chern connection $\nabla_h$ and the Higgs field $\phi_h$. We set $(E, \overline{\nabla}_E, h) := p^{-1}(E, \overline{\nabla}_E, h)$ on $\overline{U}_{\nu,p}$. Suppose the following.

Condition 7.1 $[\partial_{E,\nu} \bar{\partial}_{E,\nu} \partial_{E,\nu}]_{h} = O(y_0^{-2})$ and $|\phi_{h}|_{h} = O(|y_0|)$ around any point of $H_{\nu,p}$. 


Note that \(|(\partial E, \eta, \partial E, h, u)|_p = O(y_0^{-2})\) and \(|\phi_h|_h = O(|y_0|)\) implies the acceptability of the holomorphic bundles with a Hermitian metric \((E, \mathcal{O}_E, h)|_{\mathcal{U}_{\nu,p}^0(t)}\) and \((E^{\mathrm{cov}}, \mathcal{O}_E^{\mathrm{cov}}, h^{\mathrm{cov}})|_{\mathcal{U}_{\nu,p}^0(t)}\). Hence, for any \(t \in \mathbb{R}, E^{\mathrm{cov}}|_{\mathcal{U}_{\nu,p}^0(t)}\) naturally extends to a filtered bundle \(\mathcal{P}_*(E^{\mathrm{cov}}|_{\mathcal{U}_{\nu,p}^0(t)})\) over a locally free \(\mathcal{O}_{\mathcal{U}_{\nu,p}^0}(\nu^*)\)-module \(\mathcal{P}(E^{\mathrm{cov}}|_{\mathcal{U}_{\nu,p}^0(t)})\). For any \(t \in S^1, E|_{\mathcal{U}_{\nu,p}^0(t)}\) naturally extends to a filtered bundle \(\mathcal{P}_*(E|_{\mathcal{U}_{\nu,p}^0(t)})\) over a locally free \(\mathcal{O}_{\mathcal{U}_{\nu,p}^0}(\nu^*)\)-module \(\mathcal{P}(E|_{\mathcal{U}_{\nu,p}^0(t)})\).

**Lemma 7.2** \((E^{\mathrm{cov}}, \mathcal{O}_E^{\mathrm{cov}})\) uniquely extends to a \(\mathbb{Z}_2\)-equivariant \(\mathcal{O}_{\mathcal{U}_{\nu,p}^{\mathrm{cov}}}(\nu^*)\)-module \(\mathcal{P}(E^{\mathrm{cov}})\) such that

\[ \mathcal{P}(E^{\mathrm{cov}})|_{\mathcal{U}_{\nu,p}^0(t)} = \mathcal{P}(E^{\mathrm{cov}}|_{\mathcal{U}_{\nu,p}^0(t)}) \]

for any \(t \in \mathbb{R}.\) Similarly, \((E, \mathcal{O}_E)\) uniquely extends to a locally free \(\mathcal{O}_{\mathcal{U}_{\nu,p}}(\nu^*)\)-module \(\mathcal{P}(E)\) such that

\[ \mathcal{P}(E)|_{\mathcal{U}_{\nu,p}^0(t)} = \mathcal{P}(E|_{\mathcal{U}_{\nu,p}^0(t)}) \]

for any \(t \in S^1.\)

**Proof** The uniqueness is clear. Because of \(|\phi_h|_h = O(|y_0|)\), the scattering map induces an isomorphism \(\mathcal{P}(E^{\mathrm{cov}})|_{\mathcal{U}_{\nu,p}^0(t_1)} \simeq \mathcal{P}(E^{\mathrm{cov}})|_{\mathcal{U}_{\nu,p}^0(t_2)}\) for any \(t_1, t_2 \in \mathbb{R}.\) Hence, the claim is clear.

In all, from \((E, \mathcal{O}_E, h)\) satisfying Condition 7.1, we obtain a locally free \(\mathcal{O}_{\mathcal{U}_{\nu,p}}(\nu^*)\)-module \(\mathcal{P}(E)\) and a filtered bundle \(\mathcal{P}_*(E)\) over \(\mathcal{P}(E).\) We also obtain a locally free \(\mathcal{O}_{\mathcal{U}_{\nu,p}^{\mathrm{cov}}}(\nu^*)\)-module \(\mathcal{P}(E^{\mathrm{cov}})\) and a filtered bundle \(\mathcal{P}_*(E^{\mathrm{cov}})\) over \(\mathcal{P}(E^{\mathrm{cov}}).\)

### 7.1.2 Statements

Let \((E, h, \nabla, \phi)\) be a monopole with bounded curvature on \(\mathcal{U}_{\nu,1}^\lambda.\) We obtain the mini-holomorphic bundle \((E, \overline{\nabla}_E, h)|_{\mathcal{U}_{\nu,1}^\lambda}\) with the metric \(h\) on \(\mathcal{U}_{\nu,1}^\lambda.\) According to Proposition 6.18, Lemma 6.19 and Corollary 6.7, \((E, \mathcal{O}_E, h)\) satisfies Condition 7.1. Hence, we obtain the locally free \(\mathcal{O}_{\mathcal{U}_{\nu,1}^\lambda}(\nu^*\mathcal{H}_\nu^1)\)-module \(\mathcal{P}E^\lambda\) and a filtered bundle \(\mathcal{P}_*(E^\lambda)\) over \(\mathcal{P}(E^\lambda).\) We shall prove the following theorem in [7.2] after some preliminaries.

**Theorem 7.3** The filtered bundle \(\mathcal{P}_*E^\lambda\) is good. Moreover, the norm estimate holds for \((\mathcal{P}_*E^\lambda, h).\)

There exist \(I(\phi) \subseteq Q\) and a decomposition [5.2] as in Proposition 6.2. For each \(\omega,\) there exist a finite subset \(S_\omega \subseteq \mathbb{R}\) as in Proposition 6.0. Moreover, for each \(a \in S_\omega,\) there exist the \(\mathfrak{su}(2)\)-representation \(H_{\omega, a}\) determined by \(A_{i, \omega, a} (i = 1, 2, 3)\) in Proposition 6.0. As in [5.4], we obtain a monopole

\[(E_0, h_0, \nabla_0, \phi_0) := \bigoplus_{\omega \in I(\phi)} \mathcal{M}(\omega, S_\omega, \{A_{\omega, a}\})\]

on \(\mathcal{U}_{\nu,1}^\lambda.\) We obtain a good filtered bundle \(\mathcal{P}_*E^\lambda_0.\)

**Theorem 7.4** There exists an isomorphism \(\mathcal{G}(\mathcal{P}_*E^\lambda) \simeq \mathcal{G}(\mathcal{P}_*E^\lambda_0).\)

### 7.2 Prolongation of asymptotically mini-holomorphic bundles

Let \(E\) be a \(\mathcal{C}^\infty\)-vector bundle on \(\mathcal{U}_{\nu,1}^\lambda\) with a Hermitian metric \(h,\) a unitary connection \(\nabla\) and an anti-Hermitian endomorphism \(\phi.\) Let \((\alpha, \tau)\) denote the local mini-complex coordinate system on \(\mathcal{U}_{\nu,1}^\lambda\) as in [3.2.2]. We define differential operators \(\partial_{E, \eta}, \partial_{E, h, a}\) and \(\partial_{E, \tau}\) by the following formula:

\[
\partial_{E, \eta} = \frac{1 - \lambda g_1}{1 + |\lambda|^2} \nabla_{\eta} - \frac{1}{2\sqrt{-1}} \frac{\bar{g}_1 + \lambda}{1 + |\lambda|^2} (\nabla_{\tau} - \sqrt{-1}\phi),
\]

\[
\partial_{E, h, a} = \frac{1 - \lambda g_1}{1 + |\lambda|^2} \nabla_{\alpha} + \frac{1}{2\sqrt{-1}} \frac{g_1 + \lambda}{1 + |\lambda|^2} (\nabla_{\tau} + \sqrt{-1}\phi),
\]

\[
\partial_{E, \tau} := \nabla_{\tau} - \sqrt{-1}\phi.
\]

We assume that \((E, h, \nabla, \phi)\) satisfies the following condition in [7.2.1, 7.2.2]

**Condition 7.5** \(\partial_{E, \eta, \partial_{E, h, a}} = O(y_0^{-2}),\) and \(|\phi|_h\) is bounded.
7.2.1 Case 1

In this subsection, we assume the following additional condition.

**Condition 7.6** For any \( k \geq 0 \), there exists \( \epsilon(k) > 0 \) such that the following holds for \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \):

\[
\left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} \left( [\partial E, \partial E, \tau] \right) \right|_h = O(e^{-\epsilon(k)}y_0^3).
\]

By taking the pull back by \( \mathfrak{P} \), we obtain \( (E^{\text{cov}}, h^{\text{cov}}) \) with the differential operators \( \partial_{E^{\text{cov}}} \pi \) and \( \partial_{E^{\text{cov}}, \tau} \). The restrictions \( (E^{\text{cov}}, \partial_{E^{\text{cov}}} \pi, h^{\text{cov}})_{\{u_{(\lambda, p)}\}(t)} \) and \( (E, \partial_{E^{\text{cov}}, \pi}, h)_{\{u_{(\lambda, p)}\}(t)} \) are holomorphic vector bundles with a Hermitian metric. By the assumption \( [\partial E, \partial E, \pi, h] = O(y_0^{-2}) \), \( E^{\text{cov}}_{\{u_{(\lambda, p)}\}(t)} \) extends to a filtered bundle \( P_\lambda^\tau(E^{\text{cov}}_{\{u_{(\lambda, p)}\}(t)}) \) for any \( t \in \mathbb{R} \). Similarly, for any \( t \in S_{\lambda, 1} \), \( E_{\{u_{(\lambda, p)}\}(t)} \) extends to a filtered bundle \( P_\lambda^\tau(E_{\{u_{(\lambda, p)}\}(t)}) \).

**Lemma 7.7**

- For each \( a \in \mathbb{R} \), \( E^{\text{cov}} \) uniquely extends to a \( C^\infty \)-bundle \( P_a(E^{\text{cov}}) \) on \( \mathcal{U}_{\nu, p}^{\text{cov}} \) such that \( P_a(E^{\text{cov}})_{\{u_{(\lambda, p)}\}(t)} = P_a(E^{\text{cov}}_{\{u_{(\lambda, p)}\}(t)}) \).
- \( \partial_{E^{\text{cov}}, \tau} \) and \( \partial_{E^{\text{cov}}} \pi \) extend to \( C^\infty \)-differential operators on \( P_a(E^{\text{cov}}) \).
- \( [\partial_{E^{\text{cov}}, \tau}, \partial_{E^{\text{cov}}} \pi]_{\{u_{(\lambda, p)}\}} = 0. \)

Similar claims hold for \( (E, \overline{\partial}_E, h) \) on \( \mathcal{U}_{\nu, p}^{\lambda} \).

**Proof** Take a holomorphic frame \( \mathfrak{v} \) of \( P_a(E^{\text{cov}}_{\{u_{(\lambda, p)}\}(0)}) \). We obtain a \( C^\infty \)-frame \( \mathfrak{v} \) of \( E^{\text{cov}} \) such that (i) \( \partial_{E^{\text{cov}}, \tau} \mathfrak{v} = 0 \), (ii) \( \mathfrak{v}_{\{u_{(\lambda, p)}\}(0)} = \mathfrak{v} \). We have the matrix valued function \( \mathfrak{A} \) on \( \mathcal{U}_{\nu, p}^{\text{cov}} \) determined by \( \partial_{E^{\text{cov}}, \pi} \mathfrak{v} = \mathfrak{v} \mathfrak{A} \). For each \( (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_{\geq 0}^3 \), there exists \( \epsilon(\ell_1, \ell_2, \ell_3) > 0 \) such that \( \partial_{E, \pi}^{\ell_1} \partial_{E, \pi}^{\ell_2} \mathfrak{A} = O(e^{-\epsilon(\ell_1, \ell_2, \ell_3)}y_0^3) \). It implies that for each \( (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_{\geq 0}^3 \), there exists \( \epsilon(\ell_1, \ell_2, \ell_3) > 0 \) such that \( \partial_{E, \pi}^{\ell_1} \partial_{E, \pi}^{\ell_2} \mathfrak{A} = O(e^{-\epsilon(\ell_1, \ell_2, \ell_3)}y_0^3) \). Hence, \( \mathfrak{A} \) extends to a \( C^\infty \)-function on \( \mathcal{U}_{\nu, p}^{\text{cov}} \). Moreover, we have \( \mathfrak{A}_{\{u_{(\lambda, p)}\}} = 0. \)

We extend \( E^{\text{cov}} \) to \( P_a(E^{\text{cov}}) \) by using the frame \( \mathfrak{v} \). The bundle \( P_a(E^{\text{cov}}) \) is independent of the choice of \( \mathfrak{v} \). The operator \( \partial_{E^{\text{cov}}, \tau} \) naturally induces a \( C^\infty \)-differential operator on \( P_a(E^{\text{cov}}) \). Because \( \mathfrak{A} \) extends to a \( C^\infty \)-function on \( \mathcal{U}_{\nu, p}^{\text{cov}} \), \( \partial_{E^{\text{cov}}} \pi \) also induces a \( C^\infty \)-differential operator on \( P_a(E^{\text{cov}}) \). Because \( \mathfrak{A}_{\{u_{(\lambda, p)}\}} = 0, \) we obtain \( [\partial_{E^{\text{cov}}, \tau}, \partial_{E^{\text{cov}}}, \pi]_{\{u_{(\lambda, p)}\}} = 0. \) It is easy to see that \( P_a(E^{\text{cov}})_{\{u_{(\lambda, p)}\}(t)} = P_a(E^{\text{cov}}_{\{u_{(\lambda, p)}\}(t)}) \) in a natural way for any \( t \in \mathbb{R} \).

**Corollary 7.8** If Condition 7.4 is satisfied, we obtain a locally free \( O_{\mathcal{H}_{\nu, p}} \)-module \( P_a(E)_{\mathcal{H}_{\nu, p}} \) for each \( a \in \mathbb{R} \), and hence a regular filtered bundle \( P_a(E)_{\mathcal{H}_{\nu, p}} \) over \( (\mathcal{H}_{\nu, p}, \mathcal{H}_{\nu, p}) \).

7.2.2 Case 2

In this subsection, we assume the following additional condition which is weaker than Condition 7.4.

**Condition 7.9** For any \( k \geq 0 \), there exists \( \epsilon(k) > 0 \) such that the following holds for \( (\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k \):

\[
\left| \nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k} \left( [\partial E, \partial E, \tau] \right) \right|_h = O(e^{-\epsilon(k)}y_0^3).
\]
We obtain the filtered bundles $\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)})$ (ex $\mathcal{S}_\lambda^1$), and the induced vector spaces for any $t \in \mathcal{S}_\lambda^1$ and $a \in \mathbb{R}$:

$$\text{Gr}^a(E, t) := \mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}) / \mathcal{P}_{<a}(E_{\mathcal{U}_{\nu,p}^\lambda(t)}).$$

Similarly, we obtain the vector spaces $\text{Gr}^a(E_{\text{cov}}, t) := \mathcal{P}_a(E_{\text{cov}, \mathcal{U}_{\nu,p}^\lambda(t)}) / \mathcal{P}_{<a}(E_{\text{cov}, \mathcal{U}_{\nu,p}^\lambda(t)})$ for any $a \in \mathbb{R}$ and $t \in \mathbb{R}$.

**Lemma 7.10** For any $a \in \mathbb{R}$, and for any $t_1, t_2 \in \mathbb{R}$, we have natural isomorphisms

$$\text{Gr}^a(E_{\text{cov}}, t_1) \simeq \text{Gr}^a(E_{\text{cov}}, t_2).$$

**Proof** We take a section $s^{t_1}$ of $\mathcal{P}_a(E_{\text{cov}, \mathcal{U}_{\nu,p}^\lambda(t_1)})$. By definition, we have $|s^{t_1}|_h = O(|\nu_p|^{-\alpha-\epsilon})$ for any $\epsilon > 0$. By the parallel transport with respect to $\nabla$, we obtain an induced $C^\infty$-section $s^{t_2}$ of $E_{\text{cov}, \mathcal{U}_{\nu,p}^\lambda(t_2)}$. By (86), we have $\partial^{E_{\text{cov}}}(s^{t_2}) = O(|\nu_p|^{-\alpha+\delta})$ for some $\delta > 0$, which implies $\partial^{E_{\text{cov}}}(s^{t_2}) = O(|\nu_p|^{-\alpha+\delta_1})$ for some $\delta_1 > 0$.

Thus, for any $a \in \mathbb{R}$, we obtain a local system $\text{Gr}^a(E_{\text{cov}})$ on $\mathbb{R}$, which is naturally $Z \text{e}_2$-equivariant. Thus, we obtain a local system $\text{Gr}^a(E)$ on $\mathcal{S}_\lambda^1$ for any $a \in \mathbb{R}$.

We obtain the filtration $W$ on $\text{Gr}^a(E)$ as the weight filtration of the nilpotent endomorphism obtained as the logarithm of the unipotent part of the monodromy.

Let $t \in \mathcal{S}_\lambda^1$. Let $\nu$ be a holomorphic frame of $\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)})$ compatible with the filtrations $\mathcal{P}$ and $W$. We obtain the numbers $b(v_i) := \deg^\mathcal{P}(v_i)$ and $k(v_i) := \deg^W(v_i)$. Let $h_0$ be the metric of $E_{\mathcal{U}_{\nu,p}^\lambda(t)}$ determined by $h_0(v_i, v_j) = 0$ $(i \neq j)$. By (85), we have $h_0(v_i, v_i) = |\nu_p|^{-\alpha-\epsilon}$. We say that the norm estimate holds for $(\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}), h)$ if $h_0$ and $h_{\nu,p}$ are mutually bounded. The following lemma is easy to see.

**Lemma 7.11** If the norm estimate holds for $(\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}), h)$ at some $t_0$, then the norm estimate holds for $(\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}), h)$ for any $t \in \mathcal{S}_\lambda^1$.

### 7.2.3 Comparison

Let $(E, h, \nabla, \phi)$ be as in (7.2.2). Let $E^\circ$ be a $C^\infty$-vector bundle on $\mathcal{U}_{\nu,p}^\lambda$ with a Hermitian metric $h^\circ$, a unitary connection $\nabla^\circ$ and an anti-Hermitian endomorphism $\phi^\circ$. Let $F : E \simeq E^\circ$ be a $C^\infty$-isomorphism. Let $b^\circ$ be the endomorphism of $E$ determined by $h = F^{-1}(h^\circ) b^\circ$. Assume the following condition on $F$.

**Condition 7.12** For any $k \in \mathbb{Z}_{\geq 0}$, there exists $c(k) > 0$ such that the following holds for any $(\kappa_1, \ldots, \kappa_k) \in \{0, 1, 2\}^k$:

$$|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k}(b^\circ - \text{id})|_h = O(e^{-c(k)|\nu_p|})$$

$$|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k}(\nabla - F^*\nabla^\circ)|_h = O(e^{-c(k)|\nu_p|})$$

$$|\nabla_{\kappa_1} \circ \cdots \circ \nabla_{\kappa_k}(\phi - F^*\phi^\circ)|_h = O(e^{-c(k)|\nu_p|}).$$

Note that $(E^\circ, h^\circ, \nabla^\circ, \phi^\circ)$ also satisfies Condition (7.3).

**Lemma 7.13** For any $a \in \mathbb{R}$, there exists a naturally induced isomorphism of the local systems $\text{Gr}^a(E) \simeq \text{Gr}^a(E^\circ)$. Moreover, if the norm estimate holds for $(\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}), h^\circ)$, then the norm estimate also holds for $(\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}), h)$.

**Proof** Let $s$ be a holomorphic section of $\mathcal{P}_a(E_{\mathcal{U}_{\nu,p}^\lambda(t)})$. Let $[s]$ be the induced element of $\text{Gr}^a(E_{\mathcal{U}_{\nu,p}^\lambda(t)})$. There exists a $C^\infty$-section $c$ of $E_{\mathcal{U}_{\nu,p}^\lambda(t)}^\circ$ such that (i) $\tilde{s} := F(s) - c$ is a holomorphic section of $\mathcal{P}_a(E^\circ)$, (ii) $|c| = O(|\nu_p|^{-\alpha+\epsilon})$ for some $\epsilon > 0$. Let $[\tilde{s}]$ denote the induced element of $\text{Gr}^a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}^\circ)$. Then, $[\tilde{s}]$ depends only on $[s]$. Thus, we obtain $\text{Gr}^a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}) \longrightarrow \text{Gr}^a(E_{\mathcal{U}_{\nu,p}^\lambda(t)}^\circ)$. This procedure induces the desired isomorphism.

The claim for the norm estimate is easy to check.
7.3 Prolongation to good filtered bundles

Let \((E, \overline{\nabla}_E)\) be a mini-holomorphic bundle on \(\mathcal{U}_p^\lambda\) with a Hermitian metric \(h\). The Chern connection \(\nabla\) and the Higgs field \(\phi\) are associated to \((E, \overline{\nabla}_E, h)\). Let \((y_0, y_1, y_2)\) be the local coordinate system of \(\mathcal{M}^0\) induced by \(z = y_1 + \sqrt{-1}y_2\) and Im\((w) = y_0\), as in \(\mathcal{M}\). Let \(\nabla_{h,i}\) denote \(\nabla_{h,y_i}\). Suppose that the following condition is satisfied.

**Condition 7.14** Condition \(\mathcal{M}\) is satisfied. Moreover, there exists an orthogonal decomposition

\[
(E, h, \phi) = \bigoplus_{\omega \in \mathbb{Z}^2} (E_\omega, h_\omega, \phi_\omega^*)
\]

such that the following holds.

- \(\phi_\omega - (2\pi\sqrt{-1}\omega / \text{Vol}(\Gamma))y_0 \text{id}_{E_\omega^*}\) are bounded.
- We have the decomposition \(\nabla = \nabla^* + \rho\), where \(\nabla^*\) is the direct sum of connections \(\nabla_\omega^*\) of \(E_\omega^*\), and \(\rho\) is a section of \(\bigoplus_{\omega \in \mathbb{Z}^2} \text{Hom}(E_\omega^*, E_\omega^*) \otimes \Omega^1\). Then, for any \(k \in \mathbb{Z}_{\geq 0}\), we have \(e(k) > 0\) such that the following holds for any \(\kappa_1, \ldots, \kappa_k \in \{0, 1, 2\}^k\):

\[
|\nabla_{\kappa_1} \cdots \nabla_{\kappa_k} \rho| = O(e^{-e(k)\omega^2}). \tag{87}
\]

**Proposition 7.15** \(\mathcal{P}(E)\) is a good filtered bundle over \(\mathcal{P}(E)\).

**Proof** Let \(E^*_\omega = \text{Hom}(E_\omega^*, E_\omega^*)\) denote the inclusion, and let \(p_\omega : E \to E^*_\omega\) denote the orthogonal projection. We set \(\partial_{E^*_\omega, \omega} := p_\omega \circ \partial_{E^*_\omega, \omega} \circ 1\omega\) and \(\partial_{E^*_\omega, \omega} := p_\omega \circ \partial_{E^*_\omega, \omega} \circ 1\omega\). Similarly, we obtain connection \(\nabla_{E^*_\omega}\) on \(E^*_\omega\).

We set \(E_\omega := L_p(\omega) \otimes E_\omega^*\). (See \((5.1.1)\) for the monopole \(L_p(\omega)\) and the \(O_{\mathcal{U}_p}^\lambda\)-(\(\ast H^\lambda_{v,p}\))-module \(\mathcal{P}L_p^\lambda(-\omega)\).) Let \(h_\omega\) be the induced metric on \(E_\omega\). We obtain the differential operators \(\partial_{E_\omega, \omega}\) and \(\partial_{E_\omega, \omega}\) from the mini-holomorphic structure of \(L_p^\lambda(-\omega)\), and the operators \(\partial_{E^*_\omega, \omega}\) and \(\partial_{E^*_\omega, \omega}\). We obtain the connection \(\nabla_{E_\omega}\) from \(\nabla_{E^*_\omega}\) and the connection of \(L_p^\lambda(-\omega)\). Similarly, we obtain the anti-Hermitian endomorphism \(\phi_\omega\) from \(\phi_\omega^*\) and the anti-Hermitian endomorphism of \(L_p^\lambda(-\omega)\). Then, \((E_\omega, h_\omega, \nabla_\omega, \phi_\omega)\) satisfies Condition 7.6 and the operators \(\partial_{E_\omega, \omega}\) and \(\partial_{E_\omega, \omega}\) are induced by \(\nabla_\omega\) and \(\phi_\omega\) as in \(\mathcal{M}\). We obtain \(C^\infty\)-bundles \(\mathcal{P}^\infty_a(E_\omega)\) for each \(a \in \mathbb{R}\). We may regard them as \(C^\infty_a\)\(-\text{modules}\).

Because \(E_\omega = L_p(\omega) \otimes E_\omega\), we have the following natural \(C^\infty\)-identification on \(\mathcal{U}_p^\lambda:\nolinebreak
\[
E \simeq \bigoplus_{\omega} L_p(\omega) \otimes E_\omega. \tag{88}
\]

**Lemma 7.16** The isomorphism \((58)\) extends to an isomorphism of \(C^\infty_{\mathcal{U}_p^\lambda}\)\(-\text{modules}:\nolinebreak
\[
F : \mathcal{P}(E) \otimes c_{\mathcal{U}_p^\lambda} \simeq \bigoplus \mathcal{P}(L_p^\lambda(\omega)) \otimes c_{\mathcal{U}_p^\lambda} \mathcal{P}_0^\infty(E_\omega).
\]

Moreover, \(F_{\mathcal{U}_p^\lambda}\) is mini-holomorphic.

**Proof** We take \(v_0 \in S^1\) and a neighbourhood \(I\) of \(v_0\) in \(S^1\). We set \(\mathcal{U}^\lambda_{v_0}(I) := \pi_p^{-1}(I) \cap \mathcal{U}^\lambda_{v_0}\), and \(\mathcal{U}^\lambda_{v_0}(I) := \mathcal{U}^\lambda_{v_0}(I) \setminus H^\lambda_{v_0}\). We also put \(\hat{H}^\lambda_{v_0}(I) := \hat{H}^\lambda_{v_0} \cap \pi_p^{-1}(I)\). We take a \(C^\infty\)-frame \(v_\omega\) of \(\mathcal{P}_0^\infty(E_\omega)\) such that \(v_\omega|_{\hat{H}^\lambda_{v_0}(I)}\) is mini-holomorphic. Fixing a lift of \(v_0\) to \(\mathbb{R}\), we obtain the mini-holomorphic frame \(v_{p,\omega}\) of \(\mathcal{P}L_p^\lambda(\omega)|_{\mathcal{U}^\lambda_{v_0}(I)}\). We obtain a \(C^\infty\)-frame \(v_{p,\omega} \otimes v_\omega\) of \(\mathcal{P}L_p^\lambda(\omega) \otimes_{\mathcal{U}_p^\lambda} \mathcal{P}_0^\infty(E_\omega)\). By the frame \(u\), we also obtain a \(C^\infty\)-vector bundle \(V\) on \(\mathcal{U}_p^\lambda(I)\) with an isomorphism \(V|_{\mathcal{U}_p^\lambda(I)} \simeq E\). We may naturally regard \(V\) as a \(\mathcal{P}L_p^\lambda(\omega) \otimes_{\mathcal{U}_p^\lambda} \mathcal{P}_0^\infty(E_\omega)\)\(-\text{submodule}\) of \(\bigoplus \mathcal{P}L_p^\lambda(\omega) \otimes_{\mathcal{U}_p^\lambda} \mathcal{P}_0^\infty(E_\omega)\).
Let $A\nu,p$ and $A_t$ be the matrix valued functions on $U\nu,p$, determined by $\partial E\nu,p\cdot u_{\nu,p} = u_{\nu,p} \cdot A\nu,p$ and $\partial E_t\cdot u_{\nu,p} = u_{\nu,p} \cdot A_t$. By the decay condition $|\tilde{b}|$, $A\nu,p$ and $A_t$ extend to $C^\infty$-functions on $\overline{U\nu,p}(I)$, and $A\nu,p|\pi^{-1}(t)\cap \tilde{H}_p = A_t|\pi^{-1}(t)\cap \tilde{H}_p = 0$. Hence, $\partial E\nu,p$ and $\partial E_t$ induce a mini-holomorphic structure on $V$.

There exists a mini-holomorphic frame $w = (w_i)$ on $\overline{U\nu,p}(I)$. We have $|w_i|_h = O(|u_{\nu,p}|^{-N})$ for some $N$. Hence, we obtain that $w_i$ induce mini-holomorphic sections of $P(E)$ on $\overline{U\nu,p}(I)$, which are also denoted by the same notation. Because $w_{i,\nu,p}(t)$ is a holomorphic frame of $E_{i,\nu,p}(t)$, we obtain that $w$ is a frame of $P(E)$.

Then, the claim of the lemma follows.

**Lemma 7.17** For each $t \in S\lambda$, $F$ induces an isomorphism of filtered bundles

$$P_*(E\nu,p(t)) \cong \bigoplus P_*(\mathcal{L}\nu,p(\omega) \otimes E\nu,p(t)) \otimes P_*(E\omega|U\nu,p(t)).$$

**Proof** Take $t \in S\lambda$. Let $\mathcal{L}\nu,p(\omega)$ denote the restriction of $\mathcal{L}\nu,p(\omega)$ to $U\nu,p$. We set $E\omega := E\nu,p(t)$. Let $E^i$ denote the restriction of $E$ to $U\nu,p(t)$.

Let $s$ be a holomorphic section of $P(E\nu,p(\omega) \otimes E\nu,p(t))$. In particular, $|s|_h = O(|u_{\nu,p}|^{-\alpha-\epsilon})$ for any $\epsilon > 0$. According to Lemma 7.16, $s$ induces a section of $P|\nu,p(s) = \mathcal{L}\nu,p(\omega) \otimes E\nu,p(t)$. Note that $\partial E\nu,p|\nu,p(s) = O(1)$ for some $\epsilon_1 > 0$. Hence, for any $N > 0$, there exists a $C^\infty$-section $b_N$ of $E^i$ such that $|b_N| = O(|u_{\nu,p}|^N)$ and $\partial E\nu,p(s - b_N) = 0$. Because $|s - b_N|_h = O(|u_{\nu,p}|^{-\alpha-\epsilon})$ for any $\epsilon > 0$, we obtain that $s - b_N$ is a section of $P|\nu,p$. Then, we obtain that $s$ is a $C^\infty$-section of $\bigoplus P_\nu(E\nu,p(t))$.

We take a lift of $t$ to $\mathbb{R}$, and we set

$$c(\omega, t) := \left\{ \begin{array}{ll}
\omega t / \lambda & (\nu = 0) \\
- \omega t / \lambda & (\nu = \infty).
\end{array} \right.$$  

For $\alpha \in \mathbb{R}$, we take a holomorphic frame $v^i\alpha$ of $P_{\alpha-c(\omega, t)}(E^i)$. Let $v^i\nu,\omega$ denote the restriction of $v^i\nu,\omega$ to $U\nu,p(t)$. We obtain a holomorphic frame $v^i\nu,\omega \otimes v^i\nu$ of $P(\mathcal{L}\nu,p(\omega) \otimes E\nu,p)$. We obtain an induced holomorphic frame $u^i$ of $\bigoplus P(\mathcal{L}\nu,p(\omega) \otimes E\nu,p)$. As observed above, $u^i$ induces a tuple of $C^\infty$-sections of $P\nu(E\nu,p)$, and $u^i$ are tuples of holomorphic sections of $P\nu(E\nu,p)$. Hence, we obtain that

$$\bigoplus P(\mathcal{L}\nu,p(\omega) \otimes E\nu,p)_{\nu,p} \subset P\nu(E\nu,p).$$

For each $u_t^i$, we have $\omega(i)$ such that $u_t^i$ is a section of $P_{\omega(i)}(E\nu,p(\omega(i)) \otimes E\nu,p(\omega(i)))$. Moreover, we obtain $a - 1 < b(i) \leq a$ such that $u_t^i$ is a section of $P_{b(i)}(E\nu,p(\omega) \otimes E\nu,p)$, and that the induced element in $\text{Gr}_{b(i)}(\mathcal{L}\nu,p(\omega) \otimes E\nu,p)$ is non-zero. We set $u_t^\nu := u_t^i|u_{\nu,p}(b(i))$. Then, for any $\epsilon > 0$, there exists $C(\epsilon) > 1$ such that

$$C(\epsilon)^{-1}|u_t^\nu|_h \leq C(\epsilon)|u_{\nu,p}|^{-\epsilon}.$$

We can take a holomorphic section $u_t^\nu$ of $P_{\omega(i)}(E)$ such that $u_t^\nu - u_t^\nu = O(|u_{\nu,p}|^{-b(i)+\epsilon})$ for some $N > 0$. We set $\tilde{u}_t^\nu := \tilde{u}_t^\nu|u_{\nu,p}(b(i))$. Then, for any $\epsilon > 0$, there exists $C(\epsilon) > 1$ such that

$$C(\epsilon)^{-1}|\tilde{u}_t^\nu|_h \leq C(\epsilon)|u_{\nu,p}|^{-\epsilon}.$$  

It implies that $(\tilde{u}_t^\nu)$ is a holomorphic frame of $P\nu(E\nu,p)$ compatible with the parabolic structure.

Then, we obtain the claim of Proposition 7.15. 

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7.4 Proof of Theorem 7.3 and Theorem 7.4

We obtain that \( \mathcal{P}_{\ast}E \) is a good filtered bundle from Proposition 6.2, Theorem 6.3, and Proposition 7.13. Let \((E_\omega, h_\omega, \nabla_\omega, \phi_\omega)\) be as in (7.7). Let \((V_\omega, h_{V_\omega}, \nabla_{V_\omega}, \phi_{V_\omega})\) be as in Proposition 6.6. We set \( E_\omega^\circ := \Psi^{-1}(V_\omega) \). Let \( h_\omega^\circ := \Psi^{-1}(h_{V_\omega}) \) be the induced metric. We set \( \nabla_\omega^\circ := \Psi'((\nabla_{V_\omega}) + \sum_{i=1,2} \phi_{V_\omega} dy_i \). We set \( \phi_\omega^\circ := \Psi^{-1}(\phi_{V_\omega}) \).

Note that \((E_\omega, h_\omega, \nabla_\omega, \phi_\omega)\) satisfies Condition 7.12 and Condition 7.14. Moreover, by modifying as in the proof of Lemma 6.27 we may assume that \((E_\omega^\circ, \partial E_{\omega^\circ}, \partial E_{\omega^\circ}, \nabla)\) is a mini-holomorphic bundle on \( U_{\nu, p} \).

**Lemma 7.18**

- The norm estimate holds for \((P_{\ast}(E_\omega^\circ|_{U_{\nu, p}(t)}), h_\omega^\circ)\).
- The isomorphism \( F : E_\omega \simeq E_\omega^\circ \) in Proposition 6.6 satisfies Condition 7.12.
- In particular, there exists the isomorphism of local systems \( Gr_\nu^P(E_\omega) \simeq Gr_\nu^P(E_\omega^\circ) \). Moreover, the norm estimate holds for \((P_{\ast}(E_\omega|_{U_{\nu, p}(t)}), h_\omega)\).

**Proof** We can check the first claim by using Lemma 6.9 and Proposition 6.30. The second claim is clear. The third claim follows from Lemma 7.13.

Take \( t_0 \in S^1_\lambda \). Let \( I(t_0) \) denote a small neighbourhood of \( t_0 \). We take a mini-holomorphic local frame \( \tilde{v} \) of \( P_{\ast}(t_0) \) \( \mathcal{P}_{\ast}(E_\omega^\circ|_{U_{\nu, p}(t_0)}) \) which is compatible with the slope decomposition and the filtrations \( P_{\ast}(t_0) \) and \( W \). We may regard \( \tilde{v} \) as a mini-holomorphic frame of \( (\bigoplus L_\omega(\omega) \otimes E_\omega)|_{U_{\nu, p}(t_0)} \), which is compatible with the slope decomposition and the filtrations \( P_{\ast}(t_0) \) and \( W \). There exists a \( C^\infty \)-frame \( v' \) of \( P_{\ast}(t_0) \) \( \left( \bigoplus L_\omega(\omega) \otimes E_\omega \right) \) such that \( v'|_{U_{\nu, p}(t_0)} = \tilde{v} \). We may assume that \( v' \) is compatible with the direct sum \( \bigoplus L_\omega(\omega) \otimes E_\omega \), i.e., \( v' = \bigcup v'_\omega \), where \( v'_\omega \) is a frame of \( P_{\ast}(t_0)(L_\omega(\omega) \otimes E_\omega) \). Let \( v''_\omega \) be the frame of \( E_\omega \) determined by \( v'_\omega = v''_\omega \otimes v''_\omega \). For each \( v''_{\omega, i} \), we have \( k(\omega, i) := \deg^W(v''_{\omega, i}) \) and \( b(\omega, i) := \deg^{P_{\ast}(t_0)}(v''_{\omega, i}) \). Let \( h''_{0, \omega} \) be the metric determined by \( h''_{0, \omega}(v''_{\omega, i}, v''_{\omega, j}) = 0 \) (\( i \neq j \)) and \( h''_{0, \omega}(v''_{\omega, i}, v''_{\omega, i}) = |v''_{\omega, i}||^2 b(\omega, i) \). Then, by Lemma 7.18 we obtain that \( h''_{0, \omega} \), and \( h''_{0, \omega} \), are mutually bounded. There exists a mini-holomorphic local frame \( v \) of \( P_{\ast}(t_0) \mathcal{P}_{\ast}^{-1}(E^\lambda) \) such that \( v - \hat{v} = O(U_{\nu, p}) \) for a sufficiently large \( N \). Then, by comparison of \( v \) and \( v' \), we easily obtain that the norm estimate holds for \( (\mathcal{P}_{\ast}(E^\lambda), h) \).

Let us prove Theorem 7.4. We have the induced \( C^\infty \)-isomorphism:

\[
\rho_{p, 1}^{-1}(E^\ast_\omega) \simeq L_\omega(\omega) \otimes E_\omega^\circ.
\]  

Note that \( L_\omega(\omega) \otimes E_\omega^\circ \) is naturally equivariant with respect to the action of \( Z/pZ \cdot e_1 \). We obtain a tuple \((E^\ast_\omega, h^\ast_\nu, \nabla^\ast_\nu, \phi^\ast_\omega)\) on \( U_{\nu, p} \) as the descent of \( L_\omega(\omega) \otimes (E_\omega^\circ, h_\omega^\circ, \nabla_\omega^\circ, \phi_\omega^\circ) \). Note that \( E^\ast_\omega \simeq (E^\ast_\omega, \partial E^\ast_\nu, \partial E^\ast_\nu, \nabla) \) is a mini-holomorphic bundle. We may assume that the isomorphism \( E^\ast_\omega \simeq (E^\ast_\omega, \partial E^\ast_\nu, \partial E^\ast_\nu, \nabla) \) is equivariant with respect to the action of \( Z/pZ \cdot e_1 \). We obtain the induced \( C^\infty \)-isomorphism:

\[
E^\ast_\omega \simeq E^\ast_\omega.
\]  

We set \( E' := \bigoplus E^\ast_\nu \). It is equipped with the induced metric \( h' \), and the induced mini-holomorphic structure. We obtain the mini-holomorphic bundle \( E^\lambda \). By Lemma 7.18 \( E^\lambda \) induces an isomorphism \( \mathcal{G}(\mathcal{P}, E^\lambda) \simeq \mathcal{G}(\mathcal{P}, E^\lambda) \).

Thus, we obtain the claim of Theorem 7.3 from Proposition 6.33. (See also 7.3.)

7.5 Initial metrics

Let \( \mathcal{U}_{\nu, p} \) be a neighbourhood of \( H^\lambda_{\nu, p} \) in \( \mathcal{M}^\lambda_{\nu, p} \). Let \( \mathcal{P} \mathcal{M} \) be a good filtered bundle on \( (\mathcal{U}_{\nu, p}, H^\lambda_{\nu, p}) \). Set \( \mathcal{U}^\lambda_{\nu, p} := \mathcal{U}^\lambda_{\nu, p} \setminus H^\lambda_{\nu, p} \). Let \( V \) be the mini-holomorphic bundle on \( \mathcal{U}^\lambda_{\nu, p} \) obtained as the restriction of \( \mathcal{P} \mathcal{M} \).

**Proposition 7.19** There exists a Hermitian metric \( h_0 \) of \( V \) with the following property.
The norm estimate holds for \((P, \mathfrak{D}, h_0)\).

- \(G(h_0)\) and its derivatives are \(O(e^{-|y_0|})\).
- \(F(h_0)\) is bounded.
- \([\partial_{V,h_0,\alpha}, \partial_{V,h_0,\alpha}] = O(y_0^{-2})\).

### 7.5.1 Approximation of regular filtered bundles

Let \(P, \mathfrak{D}\) be a regular filtered bundle over \((\mathcal{U}_p^\lambda, \mathcal{H}_p^\lambda)\). We have the monodromy \(\Gamma_a\) of the local system \(\text{Gr}_a^P (\mathfrak{D})\). For each \(a \in \text{Par}(P, \mathfrak{D})\), by using the results in §5.2-5.3 we can construct a monopole \((V_0, \alpha, \mathcal{D}_{V_0}, h_0, a)\) with the following property:

- \(F(h_0, a) = O(y_0^{-2})\). The associated Higgs field \(\phi_{0,a}\) is bounded.
- \(\text{Gr}_a^P (V_0, a) = 0\) unless \(b - a \in \mathbb{Z}\).
- We have an isomorphism of local systems \(\text{Gr}_a^P (V_0, a) \simeq \text{Gr}_a^P (V)\).

We set \(V_0 := \bigoplus V_0, a\). We obtain the metric \(h_0 = \bigoplus h_0, a\).

**Lemma 7.20** We have a \(C^\infty\)-isomorphism \(g : P_0 V_0 \simeq P_0 V\) with the following property.

- The induced isomorphism \(P_0 V_0|_{\mathcal{H}_p^\lambda} \simeq P_0 V|_{\mathcal{H}_p^\lambda}\) preserves the parabolic filtrations.
- The induced morphism \(\text{Gr}_a^P V \simeq \text{Gr}_a^P V_0\) is an isomorphism of local systems.
- \(g|_{\pi^{-1}(t)}\) are holomorphic.
- Let \(B\) be determined by \(B dt = \overline{\partial}_V - g^* \overline{\partial}_{V_0}\). Then, \(B\) and its derivatives are \(O(e^{-|y_0|})\) with respect to \(g^*(h_0)\).

**Proof** For each \(-1 < a \leq 0\), we have the decomposition \(\text{Gr}_a^P (V) = \bigoplus_{\alpha \in \mathbb{C}^*} \mathbb{E}_a \text{Gr}_a^P (V)\) obtained as the generalized eigen decomposition of the monodromy. For each \(\alpha \in \mathbb{C}^*\), we take \(\log \alpha \in \mathbb{C}\). We take a \(C^\infty\) frame \(u_{a,\alpha}\) of \(\mathbb{E}_a \text{Gr}_a^P (V)\) such that \(\partial_{V,a,\alpha} u_{a,\alpha} = u_{a,\alpha} \cdot A_{a,\alpha}\), where \(A_{a,\alpha}\) is a constant matrix with eigenvalues \((t^\lambda)^{-1} \log \alpha\).

We obtain a frame \(u_{a,\alpha}\) of \(\text{Gr}_a^P (V)\). We obtain a frame a frame \(u\) of \(\bigoplus_{-1 < a \leq 0} \text{Gr}_a^P (V)\).

Take \(t_0\). Let \(I(t_0, \epsilon)\) be a small neighbourhood of \(t_0\) in \(S^1\). We take a \(C^\infty\)-frame \(v(t_0)\) of \(P_0 \mathfrak{D}\) on \(\pi^{-1}(I(t_0, \epsilon))\) with the following property.

- \(v(t_0)\) induces \(u|_{I(t_0, \epsilon)}\).
- \(v(t_0)\) are holomorphic.

By using the partition of unity on \(S^1\), we can construct a \(C^\infty\)-frame \(v\) of \(P_0 \mathfrak{D}\) with the following property.

- \(v\) induces \(u\).
- \(v|_{\pi^{-1}(\epsilon)}\) are holomorphic.

We have \(B\) determined by \(\overline{\partial}_V v = v \cdot B dt\). We have \(\partial_{V} B = 0\). Let \(B^0\) be determined by \(B^0_{i,j} = B_{i,j}|_{\mathcal{H}_p^\lambda}\) if \(\deg^P (v_i) = \deg^P (v_j)\), and \(B^0_{i,j} = 0\) if \(\deg^P (v_i) \neq \deg^P (v_j)\). Then, the matrix \(B^0\) represents the monodromy of \(\bigoplus_a \text{Gr}_a^P (\mathfrak{D})\) with the frame \(u\).

We take a \(C^\infty\)-frame \(v_0\) of \(P_0 V_0\) with similar properties. We define \(g : P_0 \mathfrak{D}_0 \rightarrow P_0 \mathfrak{D}_0\) by \(g(v_0) = v\). It has the desired property.
7.5.2 Approximation of good filtered bundles

Let \( P_*\mathfrak{V} \) be a good filtered bundle. Suppose that we are given \( P_*\mathfrak{W}_0 \) for \( \omega \in \mathcal{S}(\mathfrak{V}) \), and an isomorphism

\[
P_*\mathfrak{W}_{|\mathbb{H}^\lambda_{\mathbb{V},p}} \simeq \bigoplus_{\omega \in \mathcal{S}(\mathfrak{W})} P_*\mathfrak{W}_{0,\omega|\mathbb{H}^\lambda_{\mathbb{V},p}}.
\]

(91)

We set \( P_*\mathfrak{W}_0 := \bigoplus P_*\mathfrak{W}_{0,\omega} \). The following is easy to see.

**Lemma 7.21** We have a \( C^\infty \)-isomorphism \( P_0\mathfrak{W} \simeq P_0\mathfrak{W}_0 \) which induces (91).

Set \( V_0 := \mathfrak{W}_0|_{U^\lambda_{\mathbb{V},p}\setminus H^\lambda_{\mathbb{V},p}} \). Let \( h_0 \) be a Hermitian metric of \( V_0 \) which has the properties in Proposition 7.19 for \( P_*\mathfrak{W}_0 \). By the isomorphism in Lemma 7.21, we may regard \( h_0 \) as a Hermitian metric of \( V := \mathfrak{W}|_{U^\lambda_{\mathbb{V},p}\setminus H^\lambda_{\mathbb{V},p}} \). Then, \( h_0 \) also has the properties in Proposition 7.19 for \( P_*\mathfrak{W} \).

7.5.3 Proof of Proposition 7.19

We can construct the desired metric by using the approximations in Lemma 7.20 and Lemma 7.21, and monopoles as in 5.7.4.

7.6 Boundedness of curvature and adaptedness

Let \( P_*\mathfrak{W} \) be a good filtered bundle on \( (U^\lambda_{\mathbb{V},p}, H^\lambda_{\mathbb{V},p}) \). Let \( V \) be the mini-holomorphic bundle on \( U^\lambda_{\mathbb{V},p}\setminus H^\lambda_{\mathbb{V},p} \) obtained as the restriction of \( \mathfrak{W} \). Let \( h \) be a Hermitian metric of \( V \) with the following property.

- \( G(h) = 0 \).
- \( h \) is adapted to \( P_*\mathfrak{W} \).

**Proposition 7.22** \( F(h) \) is bounded. Moreover, the norm estimate holds for \( (P_*\mathfrak{W}, h) \).

**Proof** We have a Hermitian metric \( h_0 \) as in Proposition 7.19. Let \( s \) be the automorphism of \( V \) determined by \( h = h_0 s \). We obtain

\[
\Delta \log \text{Tr}(s) \leq |G(h_0)|_{h_0} \leq C e^{-\epsilon|y_0|}.
\]

We obtain

\[
\Delta (\log \text{Tr}(s) - C_1 e^{-\epsilon|y_0|}) \leq 0
\]

for some \( C_1 > 0 \) and \( \epsilon_1 > 0 \). By the assumption, \( \log \text{Tr}(s) = O(|y_0|) \) holds. We take \( C_2 > 0 \) such that \( \log \text{Tr}(s) < C_2 \) on \( \{y_0 = R\} \). Note that \( \Delta (\delta|y_0|) = 0 \) for any \( \delta > 0 \). We obtain

\[
\Delta (\log \text{Tr}(s) - C_1 e^{-\epsilon|y_0|} - \delta|y_0| - C_2) \leq 0.
\]

Then, by a standard argument, we obtain that \( \log \text{Tr}(s) - C_1 e^{-\epsilon|y_0|} \leq C_2 + \delta|y_0| \) for any \( \delta > 0 \). Hence, \( \log \text{Tr}(s) - C_1 e^{-\epsilon|y_0|} \leq C_2 \). Thus, we obtain the boundedness of \( s \). Similarly, we obtain the boundedness of \( s^{-1} \). It implies that the norm estimate for \( (P_*\mathfrak{W}, h) \).

**Lemma 7.23** \( \int |\partial_{E,h_0,\tau}\pi s|^2 + \int |\partial_{E,h_0,\tau,s}|^2 < \infty \) and \( \int |\partial_{E,h_0,\alpha,s}|^2 + \int |\partial_{E,h_0,\tau,s}|^2 < \infty \) hold.

**Proof** The following holds:

\[
-(\partial_\alpha \partial_\tau + \frac{1}{4} \partial_\tau \partial_\tau) \text{Tr}(s) = -\text{Tr}(sG(h_0)) - |s^{-1/2}\partial_{E,h_0,\alpha,s}|^2 - \frac{1}{4} |s^{-1/2}\partial_{E,h_0,\tau,s}|^2.
\]

We set

\[
b_1 := \int_{T^2} \text{Tr}(s), \quad b_2 := \int_{T^2} \text{Tr}(sG(h_0)), \quad b_3 := \int_{T^2} |s^{-1/2}\partial_{E,h_0,\alpha,s}|^2 + \int_{T^2} \frac{1}{4} |s^{-1/2}\partial_{E,h_0,\tau,s}|^2.
\]

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Note that $\partial_{\tau}^2 + \frac{1}{4} \partial_{\tau} \partial_s = \frac{1}{4} (\partial_{y_0}^2 + \partial_{y_1}^2 + \partial_{y_2}^2)$. We obtain

$$-\partial_{y_0}^2 b_1 = -4b_2 - 4b_3.$$ 

Note that $|b_2| = O(e^{-ct}|y_0|)$. Hence, there exists $c_2$ such that $|c_2| = O(e^{-ct}|y_0|)$ and $-\partial_{y_0}^2 (b_1 - c_2) = -4b_3$. Note that $b_3 \geq 0$. Because $b_3 - c_2$ is bounded and subharmonic, we obtain that there exists $\lim_{y_0 \to \infty} (\partial_{y_0} (b_3 - c_2))$. Then, we obtain the existence of $\lim_{R \to \infty} J^R b_3$. It implies the claim of the lemma.

As in [20], there exists $C > 0$ such that

$$\Delta |s^{-1} \partial_{E,h_0}s|^2_{h_0} \leq C \left(1 + |s^{-1} \partial_{E,h_0}s|^2_{h_0}\right).$$

By using [11] Theorem 9.20 and Lemma [23] we obtain the boundedness of $s^{-1} \partial_{E,h_0}s$. By using the equation for the monopole, we also obtain that $s$ and its derivatives are bounded.

## 8 Rank one monopoles

### 8.1 Preliminary

#### 8.1.1 Ahlfors type lemma

Let $R > 0$. Let $g$ be a $C^\infty$-function $\{t \geq R\} \to \mathbb{R}_{\geq 0}$ such that $g = O(t^N)$ for some $N > 0$. Suppose $-\partial_t^2 g \leq -C_0 g + C_1 e^{-at}$ for some $C_1 > 0$ and $a > 0$.

**Lemma 8.1** We obtain $g = O\left(\exp(-\epsilon t)\right)$ for some $\epsilon > 0$.

**Proof** By making $C_0$ smaller, we may assume that $C_0 < a^2$. We set $C_2 := C_1 (a^2 - C_0)^{-1}$. The following holds:

$$-\partial_t^2 C_2 e^{-at} = -(a^2 C_2 - C_1) e^{-at} - C_1 e^{-at} = -C_2 e^{-at} - C_1 e^{-at}.$$ 

We obtain

$$-\partial_t^2 (g + C_2 e^{-at}) \leq -C_0 (g + C_2 e^{-at}).$$

For $C_3 > 0$ and $\delta > 0$, we set $F_{C_3,\delta}(t) := C_3 \exp(-\epsilon t) + \delta \exp(\epsilon t)$. There exists $C_3 > 0$ such that $F_{C_3,\delta}(R) > (g + C_2 e^{-at})_{t=R}$ for any $\delta > 0$. Then, the set $\{t \mid F_{C_3,\delta}(t) < g(t)\}$ is relatively compact in $\{t \geq R\}$. Set $\epsilon := C_0^{1/2}$. By using $-\partial^2 F_{C_3,\delta} = -C_0 F_{C_3,\delta}$ with a standard argument, we obtain that $F_{C_3,\delta} > g + C_2 e^{-at}$ on $\{t \geq R\}$ for any $\delta > 0$. By taking the limit $\delta \to 0$, we obtain the desired estimate.

#### 8.1.2 Global subharmonic functions on $X \times \mathbb{R}$

Let $(X, g_X)$ be a compact Riemannian manifold. The Riemannian metric $g_X + dt^2$ on $X \times \mathbb{R}$ is induced.

**Lemma 8.2** Let $f$ be a bounded function $X \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\Delta f \leq 0$. Then, $f$ is constant. In particular, $\Delta f = 0$.

**Proof** We obtain the decomposition $f = f_0 + f_1$, where $f_0$ is constant on $X \times \{t\}$, and $\int_{X \times \{t\}} f_1 = 0$ holds for any $t$. We obtain $-\partial_t^2 f_0 \leq 0$. Because $f_0$ is bounded, we obtain that $f_0$ is constant. Let $d_X$ denote the exterior derivative in the $X$-direction. We obtain

$$\Delta |f|^2 \leq -|d_X f|^2 = -|d_X f_1|^2.$$ 

We obtain

$$-\partial_t^2 \int_{X \times \{t\}} |f|^2 = -\partial_t^2 \int_{X \times \{t\}} |f_1|^2 \leq -\int_{X \times \{t\}} |d_X f_1|^2 \leq -C_1 \int_{X \times \{t\}} |f_1|^2.$$ 

By Lemma [8.1] we obtain $\int_{X \times \{t|} |f_1|^2 = O\left(\exp(-\epsilon |t|)\right)$ for some $\epsilon > 0$. Because $\int_{X \times \{t\}} |f_1|^2 \geq 0$ is subharmonic, we obtain $\int_{X \times \{t\}} |f_1|^2$ is constantly 0. It implies $f_1 = 0$. 

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8.1.3 Poisson equation on $X \times \mathbb{R}$

Let $a$ be a $C^\infty$-function on $X \times \mathbb{R}$ such that $a = O(\exp(-|t|))$, and that $\int_{X \times \mathbb{R}} a = 0$. For any $t \in \mathbb{R}$, we set $X_t := X \times \{t\}$.

**Lemma 8.3** There exists a $C^\infty$-function $b$ on $X \times \mathbb{R}$ such that (i) $\Delta b = a$, (ii) $|b| = O(\exp(\epsilon_1 t))$ as $t \to -\infty$, (iii) there exists the limit $\lim_{t \to \infty} b = b_\infty$, and $|b - b_\infty| = O(\exp(-\epsilon_1 t))$ as $t \to \infty$.

**Proof** Let $a = a_0 + a_1$ be the decomposition such that (i) $a_0$ is constant on $X_t$ for any $t$, (ii) $\int_{X_t} a_1 = 0$ for any $t$. We may regard $a_0$ as a $C^\infty$-function on $\mathbb{R}$ such that $a_0 = O(\exp(-\epsilon_1 t))$. It is easy to see that there exists a function $b_0$ on $\mathbb{R}$ such that (i) $-\partial^2_t b_0 = a_0$, (ii) $b_0 = O(\exp(\epsilon_1 t))$ for some $\epsilon_1 > 0$ as $t \to -\infty$, (iii) there exists $b_\infty := \lim_{t \to \infty} b_0(t)$, and $b_0 - b_\infty = O(\exp(-\epsilon_2 t))$ for some $\epsilon_2 > 0$ as $t \to \infty$.

There exists a complete orthonormal set $\{\varphi\}$ in $C^\infty(X)$ such that $\Delta X \varphi = \lambda(\varphi)\varphi$, where $\lambda(\varphi) \in \mathbb{R}_{\geq 0}$. Let $a_1 = \sum_{\lambda(\varphi) > 0} a_{1,\varphi}(t)\varphi$ be the expansion. We set

$$b_{1,\varphi}(t) := e^{-\lambda(\varphi)\epsilon_1^{1/2}} \int_{-\infty}^t e^{2\lambda(\varphi)\epsilon_1^{1/2}s} ds \int_s^\infty e^{-\lambda(\varphi)\epsilon_1^{1/2}} a_{1,\varphi}(u) du.$$

Then, $(-\partial^2_t + \lambda(\varphi)b_{1,\varphi})b_{1,\varphi} = a_{1,\varphi}$ holds. Set $\|a_{1,\varphi}\|_{L^2} := \left(\int_X |a_{1,\varphi}(t)|^2 dt\right)^{1/2}$. We obtain $|b_{1,\varphi}(t)| \leq C\|a_{1,\varphi}\|_{L^2}$ for some $C > 0$. Because $\sum_{\varphi} \|a_{1,\varphi}\|^2_{L^2} < \infty$, we obtain the locally $L^2$-function $b_1 := \sum b_{1,\varphi}\varphi$ on $X \times \mathbb{R}$, and $\Delta b_1 = a_1$ holds in the sense of distributions. By the ellipticity, $b_1$ is $C^\infty$. Set $f(t) := \int_{X_t} |b_1|^2$, and then $|f(t)| \leq C \sum \|a_{1,\varphi}\|_{L^2}$. The following holds:

$$\int_{X_t} a_{1,\varphi} b_1 = \int_{X_t} (-\partial^2_t + \Delta X) b_{1,\varphi} b_1 = -\partial^2_t f + \int_{X_t} |d_X b_1|^2.$$

There exists $\epsilon_2 > 0$ such that $\int_{X_t} |d_X b_1|^2 \geq \epsilon_1 f$. There exist $C_i$ ($i = 1, 2$) and $\epsilon_i > 0$ ($i = 2, 3, 4$) such that

$$-\partial^2_t f \leq C_1 e^{-\epsilon_2 |t|} f^{1/2} - \epsilon_1 f \leq C_2 e^{-\epsilon_3 |t|} - \epsilon_4 f.$$

on $\{|t| > R\}$ for some $R > 0$. Then, we obtain that $|f| = O(\exp(-\epsilon_5 |t|))$ for some $\epsilon_5 > 0$. Thus, we are done.

8.2 Examples of monopoles of rank 1 with Dirac type singularity

8.2.1 Filtered bundles of rank 1

Suppose that $g^v > 0$. Take a small $\epsilon > 0$. We set $\mathcal{W} := P^1 \times (-\epsilon, 1]$. We have the open embedding $\mathcal{W} \hookrightarrow \mathcal{M}^{\text{cov}}$ induced by $(U, t) \mapsto (U, t^{v+1})$. It induces the surjection $\mathcal{W} \twoheadrightarrow \mathcal{M}^\lambda$. We have the isomorphism $\Phi : \mathcal{M}^\lambda \cong P^1 \times (-\epsilon, 0) \cong P^1 \times (-\epsilon, 1]$ given by $\Phi(U, t) = (a^U(p)U, t+1)$. We regard $\mathcal{M}^\lambda$ as the quotient space of $\mathcal{W}$ by identifying $P^1 \times (-\epsilon, 0]$ and $P^1 \times (-\epsilon, 1]$.

Let $(A_0, t_0) \in \mathcal{C}^* \times [0, 1]$. We set $\mathcal{Z}_{\mathcal{W}, (A_0, t_0)} := \{A_0\} \times t_0, 1]$. We set

$$\mathcal{V}^\lambda_{(A_0, t_0), n} := \mathcal{Z}_{\mathcal{W}, (A_0, t_0)} \left( \{0, \infty\} \times [-\epsilon, 1]\right) \cdot n.$$

Let $\pi : \mathcal{W} \to (-\epsilon, 1]$ denote the projection. We define the filtered bundles $P_\tau^\lambda(\mathcal{V}^\lambda_{\pi^{-1}(t)})$ by the following conditions:

- The parabolic degree of $v_{\pi^{-1}(t)}$ at $\infty$ is constantly 0.
- The parabolic degree of $v_{\pi^{-1}(t)}$ at 0 is $a - nt$.

We define the isomorphism $\Phi^*\left(\mathcal{V}^\lambda_{\pi^{-1}(t)}\right) \cong \mathcal{V}^\lambda_{\pi^{-1}(t)} \times (-\epsilon, 0]$ by

$$\Phi^*\left(\pi - A_0\right)^n \pi^{-n} v_{\pi^{-1}(t)} \cong v_{\pi^{-1}(t)} \times (-\epsilon, 0].$$

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It induces an isomorphism of filtered bundles for $t \in [-\epsilon, 0]$:

$$\Phi^* (\mathcal{P}_t^{(a)} \mathcal{V}_{|\pi^{-1}(t+1)}(t)) \simeq \mathcal{P}_t^{(a)} \mathcal{V}_{|\pi^{-1}(t)}.$$  

We set $t_0 := t^\lambda t_0$. We obtain an induced $O_{\mathcal{M} \backslash \{(A_0, t_0)\}}(sH_p)$-module $\mathcal{L}(A_0, t_0, n)$, and a filtered bundle $\mathcal{P}_t^{(a)} \mathcal{L}(A_0, t_0, n)$ over $\mathcal{L}(A_0, t_0, n)$. We obtain the following lemma by a direct computation.

**Lemma 8.4** deg($\mathcal{P}_t^{(a)} \mathcal{L}(A_0, t_0, n)$) = $|t\lambda|(-a - n/2 + nt_0/t^\lambda)$ holds. In particular,

$$a(t_0, n) = n(-1/2 + t_0/t^\lambda),$$

we obtain deg $\mathcal{P}_t^{(a(t_0, n))} \mathcal{L}(A_0, t_0, n) = 0$.

### 8.2.2 Monopoles

Set $\mathcal{U}(A_0, t_0) := \mathcal{M}^\lambda \backslash \{(A_0, t_0)\}$.

**Proposition 8.5** There exists a Hermitian metric $h$ of $\mathcal{L}(A_0, t_0, n)_{|\mathcal{U}(A_0, t_0)}$ such that the following holds.

- $(\mathcal{L}(A_0, t_0, n)_{|\mathcal{U}(A_0, t_0)}, h)$ is a monopole with Dirac type singularity on $\mathcal{U}(A_0, t_0)$.

- The norm estimate holds for $\mathcal{P}_t^{(a(t_0, n))} \mathcal{L}(A_0, t_0, n)$ with $h$.

Such $h$ is unique up to the positive constant multiplications.

**Proof** Set $\mathcal{L} := \mathcal{L}(A_0, t_0, n)$. There exists a Hermitian metric $h_0$ of such that (i) $G(h_0)$ and its derivatives are $O(e^{-r|\omega|})$, (ii) $(\mathcal{L}, h_0)$ is a monopole with Dirac type singularity on $U_{A_0, t_0} \backslash \{(A_0, t_0)\}$, where $U_{A_0, t_0}$ denotes a neighbourhood of $(A_0, t_0)$, (iii) the norm estimate holds for $h_0$. For another metric $h_0 e^\varphi$, we have $G(h_0 e^\varphi) = G(h_0) + 4^{-1} \Delta \varphi$. Because deg($\mathcal{P}_t^{(a(t_0, n))} \mathcal{L}$) = 0, we obtain $\int G(h_0) = 0$, and hence there exists a bounded $C^\infty$-function $\varphi$ such that $G(h_0 e^\varphi) = 0$ according to Lemma 8.3. The uniqueness is clear.

### 8.3 Classification of rank one monopoles

Let $Z_0 = \{(A_i, t_i)\} \subset \mathcal{M}^{\lambda, \text{cov}}$ be a finite subset such that $0 \leq t_i/t^\lambda < 1$. Let $Z \subset \mathcal{M}^\lambda$ be the induced subset. For each $i$, we set $a_i := -1/2 + t_i/t^\lambda$. The following lemma is clear.

**Lemma 8.6** Let $\mathcal{P}_t \mathcal{L}$ be a good filtered bundle with Dirac type singularity of degree 0 on $\mathcal{M}^\lambda; Z \cup H^\lambda$. Then, there exist $\ell \in \mathbb{Z}$, $(a, b) \in \mathbb{R} \times \mathbb{C}$ and an isomorphism

$$\mathcal{P}_t \mathcal{L} \simeq \mathcal{P}_t(\mathcal{L}_1(\ell)) \otimes \mathcal{P}_t(\mathcal{L}_1(\lambda, a, b)) \otimes \bigotimes_{i=1}^m \mathcal{P}_t^{(a_i)} \mathcal{L}(A_i, t_i, 1).$$

Here, see [5.1.5] for $\mathcal{P}_t \mathcal{L}_1(\ell)$, and [5.2.3] for $\mathcal{P}_t \mathcal{L}_1(\lambda, a, b)$.

**Proposition 8.7** There exists an equivalence between the following objects:

- Monopoles of rank one $(E, h, \nabla, \phi)$ on $\mathcal{M}^\lambda \backslash Z$ such that (i) each point of $Z$ is Dirac type singularity, (ii) $F(\nabla)$ is bounded.

- Filtered bundles with Dirac type singularity of rank one with degree 0 on $\mathcal{M}^\lambda; H^\lambda, Z$.

The correspondence is induced by $(E, h, \nabla, \phi) \mapsto \mathcal{P}_t^h E$. 

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9 Kobayashi-Hitchin correspondence for doubly-periodic monopoles

9.1 Main statement

Let \( Z \) be a finite subset of \( \mathcal{M}^\lambda \).

**Definition 9.1** A monopole \((E, h, \nabla, \phi)\) on \( \mathcal{M}^\lambda \setminus Z \) is called meromorphic if the following holds.

- Any points of \( Z \) are Dirac type singularity of \((E, h, \nabla, \phi)\).
- There exists a compact subset \( C \) of \( \mathcal{M}^\lambda \) such that (i) \( Z \subset C \), (ii) \( F(\nabla) \) is bounded on \( \mathcal{M}^\lambda \setminus C \).

For any meromorphic monopole \((E, h, \nabla, \phi)\), we have the associated good filtered bundle with Dirac type singularity \( \mathcal{P}_* E^\lambda \) on \( (\mathcal{M}^\lambda; H^\lambda, Z) \), as explained in [21, Proposition 9.4]. We shall prove the following theorem in [9.2–9.3]

**Theorem 9.2** The above procedure induces the bijection of the isomorphism classes of the following objects:

- Meromorphic monopoles on \( \mathcal{M}^\lambda \setminus Z \).
- Polystable good filtered bundle with Dirac type singularity of degree 0 on \( (\mathcal{M}^\lambda; H^\lambda, Z) \).

9.2 Preliminary

9.2.1 Ambient good filtered bundles with appropriate metric

Let \( Z \) be a finite subset in \( \mathcal{M}^\lambda \). Let \( \mathcal{P}_* \mathcal{E}^\lambda \) be a good filtered bundle with Dirac type singularity on \( (\mathcal{M}^\lambda; H^\lambda, Z) \). Let \((E, \overline{\nabla}_E)\) denote the mini-holomorphic bundle with Dirac type singularity on \( \mathcal{M}^\lambda \setminus Z \) obtained as the restriction of \( \mathcal{P}_* \mathcal{E}^\lambda \).

Let \( h_1 \) be a Hermitian metric of \( E \) adapted to \( \mathcal{P}_* \mathcal{E} \) such that the following holds.

(A1) Around \( H^\lambda \), we have \( G(h_1) = O(e^{-\epsilon|y_0|}) \) for some \( \epsilon > 0 \), and \((E, \overline{\nabla}_E, h_1)\) satisfies the norm estimate with respect to \( \mathcal{P}_* \mathcal{E} \). Moreover, we have
\[
[\partial_{E, \overline{\nabla}_E, h_1, u}] = O(y_0^{-2}).
\]

(A2) Around each point of \( Z \), \((E, \overline{\nabla}_E, h_1)\) is a monopole with Dirac type singularity. In particular, it induces a \( C^\infty \)-metric of the Kronheimer resolution of \( E \).

9.2.2 Degree of filtered subbundles

Let \( \mathcal{P}_* \mathcal{E}_1 \subset \mathcal{P}_* \mathcal{E} \) be a filtered subbundle on \( (\mathcal{M}^\lambda; H^\lambda, Z) \). Let \( E_1 \) be the mini-holomorphic bundle with Dirac type singularity on \( (\mathcal{M}^\lambda, Z) \). Let \( h_{1, E_1} \) denote the metric of \( E_1 \) induced by \( h_1 \). By the Chern-Weil formula, the analytic degree \( \deg(E_1, h_{1, E_1}) \in \mathbb{R} \cup \{ -\infty \} \) makes sense.

**Proposition 9.3** There exists \( C > 0 \) such that \( C \deg(\mathcal{P}_* \mathcal{E}_1) = \deg(E_1, h_{1, E_1}) \) for any \( \mathcal{P}_* \mathcal{E}_1 \).

**Proof** Because the argument is essentially the same as the proof of [21, Proposition 9.4], we give only an outline. We take a metric \( h_{0, E_1} \) of \( E_1 \) which satisfies the conditions (A1,2) for \( \mathcal{P}_* \mathcal{E}_1 \). Because \( G(h_{0, E_1}) = O(e^{-\epsilon|y_0|}) \) \((\epsilon > 0)\) around \( H^\lambda \), and because \( G(h_{0, E_1}) = 0 \) around each point of \( Z \), \( G(h_{0, E_1}) \) is \( L^1 \). Let \( \nabla_0 \) and \( \phi_0 \) be the Chern connection and the Higgs field associated to \((E_1, \overline{\nabla}_E)\) with \( h_{0, E_1} \). Because \((E_1, \overline{\nabla}_E, h_{0, E_1})\) is a monopole with Dirac type singularity around each point \( P \) of \( Z \), we have \( (\nabla_0 \phi_0)_x = O(d(x, P)^{-2}) \) around \( P \), and hence \( \nabla_0 \phi_0 \) is \( L^1 \) around \( P \). Let \( \partial_{E_1, u} \) denote the operator induced by \( \partial_{E_1, \overline{\nabla}_E} \) and \( h_{0, E_1} \). Because \( [\partial_{E_1, u}, \partial_{E_1, \overline{\nabla}_E}] = O(y_0^{-2}) \) around \( H^\lambda \), \( [\partial_{E_1, u}, \partial_{E_1, \overline{\nabla}_E}] \) is \( L^1 \) around \( H^\lambda \). Hence, by Proposition 8.20 we obtain the following equality:
\[
\int \text{Tr} G(h_{0, E_1}) \, \text{dvol} = C \int_0^{|t|} \text{par-deg}(\mathcal{P}_* \mathcal{E}_1|_{\pi^{-1}(t)}) \, dt = C \deg(\mathcal{P}_* \mathcal{E}_1).
\]
Let us prove the following equality:

$$\int \text{Tr} G(h_{1,E_1}) \, \text{dvol} = \int \text{Tr} G(h_{0,E_1}) \, \text{dvol}. \quad (93)$$

By considering $\det E_1 \subset \wedge^{\text{rank } E_1} E$, it is enough to consider the case $\text{rank } E_1 = 1$. We have a Hermitian metric $h'_{E_1}$ of $E_1$ such that (i) $(E_1, \mathcal{O}_{E_1}, h'_{E_1})$ is a meromorphic monopole, (ii) the meromorphic extension $\mathcal{P}^{h'_{E_1}} E_1$ is equal to $\mathcal{P} E_1$. We have $\deg(\mathcal{P}^{h'_{E_1}} E_1) = 0$. By considering $(E, \mathcal{O}_E, h) \otimes (E_1, \mathcal{O}_{E_1}, h'_{E_1})^{-1}$ and $\mathcal{P} \mathcal{E} \otimes (\mathcal{P}^{h'_{E_1}} E_1)^\vee$, we may reduce the issue to the case where there exists an isomorphism $\mathcal{P} E_1 \simeq \mathcal{O}_{\mathcal{M}^a}(*H^\lambda)$. Let $g$ be a section of $\mathcal{P} E_1$ corresponding to $1 \in \mathcal{O}_{\mathcal{M}^a}(\mathcal{P} E_1)$ under the isomorphism. We have the number $a_0$ such that $g \in \mathcal{P}_{a_0} E_1$ and $f \notin \mathcal{P}_{<a_0} E_1$ around $H^\lambda_0$, and the number $a_\infty$ such that $g \in \mathcal{P}_{a_\infty} E_1$ and $g \notin \mathcal{P}_{<a_\infty} E_1$ around $H^\lambda_\infty$. By considering the metric $h_1 e^{-a_0 y_0}$ on around $H^\lambda_0$ and $h_1 e^{a_\infty y_0}$ on around $H^\lambda_\infty$, it is enough to consider the case $a_0 = a_\infty = 0$.

**Lemma 9.4** Let $\mathcal{B}^\lambda$ be a neighbourhood of $H^\lambda_\infty$ in $\mathcal{M}^\lambda$. Let $E$ be a mini-holomorphic bundle on $\mathcal{B}^\lambda = \mathcal{B}^\lambda \setminus H^\lambda_\infty$ with a metric $h$ such that $G(h)$ is $L^1$. Let $f$ be a mini-holomorphic section of $E$ such that

$$C_1^{-1} \leq |f|_{h_0}^{1-k} \leq C_1$$

for some $C_1 > 0$ and $k \in \mathbb{R}$. Then, $|\nabla \alpha f|_{h}^{1-k} \cdot |f|_{h}^{-1}$ and $|((\nabla \alpha + \sqrt{-1} \phi)) f|_{h}^{1-k} \cdot |f|_{h}^{-1}$ are $L^2$.

**Similar claim** holds on a neighbourhood of $H^\lambda_0$.

**Proof** It is enough to prove that $|\nabla \alpha f|_{h}^{1-k} \cdot |f|_{h}^{-1}$ and $|((\nabla \alpha + \sqrt{-1} \phi)) f|_{h}^{1-k} \cdot |f|_{h}^{-1}$ are $L^2$. Because $f$ is mini-holomorphic, we have $\nabla \alpha f = 0$ and $|((\nabla \alpha + \sqrt{-1} \phi)) f|_{h}^{1-k} \cdot |f|_{h}^{-1}$ are $L^2$. Because $f$ is mini-holomorphic, we may assume that $\mathcal{B}^\lambda = \{ y_0 > R \}$.

We take a $C^\infty$-function $\rho : \mathbb{R} \to \{ 0 \leq a \leq 1 \} \subset \mathbb{R} \geq 0$ such that, (i) $\rho(t) = 0$ if $t \geq 1$, (ii) $\rho(t) = 1$ if $t \leq 1/2$, (iii) $\rho(t)^{1/2} \rho'(t)/\rho(t)^{1/2}$ give $C^\infty$-functions.

For any large positive integer $N$, we set $\chi_N(y_0) := \rho(N^{-1} y_0)$ in $\mathcal{B}^\lambda$. We obtain $C^\infty$-functions $\chi_N : \mathcal{B}^\lambda \to \mathbb{R} \geq 0$ such that $\chi_N(y_0) = 0$ if $y_0 > N$ and $\chi_N(y_0) = 1$ if $y_0 < N/2$. Let $\mu : \mathcal{B}^\lambda \to \mathbb{R} \geq 0$ be a $C^\infty$-function such that $\mu(y_0) = 1 - \rho(y_0 - R)$. We set $\tilde{\chi}_N := \mu \cdot \chi_N$. We have

$$\partial_{y_0} \tilde{\chi}_N(y_0) = \partial_{y_0} \mu(y_0) \chi_N(y_0) + \mu(y_0) \rho'(N^{-1} y_0) N^{-1},$$

By the assumption on $\rho$, $\partial_{y_0} \tilde{\chi}_N(y_0)/\tilde{\chi}_N(y_0)^{1/2}$ naturally give $C^\infty$-functions on $\mathcal{B}^\lambda$, and there exists $C_2 > 0$, which is independent of $N$, such that the following holds:

$$|\partial_{y_0} \tilde{\chi}_N(y_0)/\tilde{\chi}_N(y_0)^{1/2}| \leq C_2 y_0^{-1}.$$

Because $\partial_{y_0} y_0$ is constant, we have $C_3 > 0$, which is independent of $N$, such that the following holds:

$$|\partial_{\alpha} \tilde{\chi}_N(y_0)/\tilde{\chi}_N(y_0)^{1/2}| \leq C_3 y_0^{-1}.$$

We consider the following integral:

$$\int_{\mathcal{B}^\lambda} \tilde{\chi}_N(y_0) \cdot h(\nabla \alpha f, \nabla \alpha f) y_0^{-2k} \, \text{dvol} = \int_{\mathcal{B}^\lambda} \partial_{\alpha} \tilde{\chi}_N(y_0) \cdot h(f, \nabla \alpha f) y_0^{-2k} \, \text{dvol}$$

$$- \int_{\mathcal{B}^\lambda} \tilde{\chi}_N(y_0) \cdot h(f, \nabla \alpha f) y_0^{-2k} \, \text{dvol} + \int_{\mathcal{B}^\lambda} \tilde{\chi}_N(y_0) \cdot h(f, \nabla \alpha f) (-2k) y_0^{-2k-1} \partial_{\alpha} y_0 \, \text{dvol} \quad (94)$$

We have the following inequality:

$$|\partial_{\alpha} \tilde{\chi}_N \cdot h(f, \nabla \alpha f) y_0^{-2k}| \leq C_3 C_1 y_0^{-1} \cdot \left( \tilde{\chi}_N(y_0)^{1/2} \cdot |\nabla \alpha f|_{h}^{1-k} \right).$$

We also have the following inequality:

$$|\tilde{\chi}_N \cdot h(f, \nabla \alpha f) y_0^{-2k-1} \partial_{\alpha} y_0| \leq 2 \left( C_1 \tilde{\chi}_N^{1/2} \cdot y_0^{-1} \right) \cdot \left( |\tilde{\chi}_N|^{1/2} |\nabla \alpha f|_{h}^{1-k} \right).$$

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Note that $\nabla_\pi \nabla_\alpha f = (\nabla_\pi \nabla_\alpha - \nabla_\alpha \nabla_\pi) f = -F_{\alpha, \pi}(h)f$. We have $C_4, C_5 > 0$ which are independent of $N$, such that the following holds:

$$\int_{B^\lambda} \nabla_\pi \nabla_\alpha f |\nabla_\alpha f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol \leq C_4 + C_5 \left( \int_{B^\lambda} \nabla_\pi \nabla_\alpha f |\nabla_\alpha f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol \right)^{1/2} + \int_{B^\lambda} \nabla_\pi \nabla_\alpha f (f, F_{\alpha, \pi} f) y_0^{-2k} \, \mathrm{d}vol. \quad (95)$$

Similarly, we have the following:

$$\int_{B^\lambda} \nabla_\pi \nabla_\alpha f |(\nabla_\tau + \sqrt{-1} \mathrm{ad} \phi) f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol \leq C_4 + C_6 \left( \int_{B^\lambda} \nabla_\pi \nabla_\alpha f |(\nabla_\tau + \sqrt{-1} \mathrm{ad} \phi) f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol \right)^{1/2} + \int_{B^\lambda} \nabla_\pi \nabla_\alpha f (f, -2\sqrt{-1} \nabla_\tau \phi \cdot f) y_0^{-2k} \, \mathrm{d}vol. \quad (96)$$

Here, $C_i$ ($i = 5, 6$) are positive constants, which are independent of $N$. Because $G(h)$ is $L^1$, we have a constant $C_7 > 0$, which is independent of $N$, such that the following holds:

$$\int_{B^\lambda} \nabla_\pi \nabla_\alpha f h(f, F_{\alpha, \pi} f) y_0^{-2k} \, \mathrm{d}vol + \frac{1}{4} \int_{B^\lambda} \nabla_\pi \nabla_\alpha f (f, (\sqrt{-1} \nabla_\tau \phi \cdot f) y_0^{-2k} \, \mathrm{d}vol \leq C_7.$$  

We put 

$$A_N := \int_{B^\lambda} \nabla_\pi \nabla_\alpha f |\nabla_\alpha f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol + \frac{1}{4} \int_{B^\lambda} \nabla_\pi \nabla_\alpha f |(\nabla_\tau + \sqrt{-1} \mathrm{ad} \phi) f|^2_{h, y_0}^{2 - 2k} \, \mathrm{d}vol.$$  

We have constants $C_i > 0$ ($i = 8, 9$), which are independent of $N$, such that the following holds:

$$A_N \leq C_8 + C_9 A_N^{1/2}.$$  

Hence, we obtain that $A_N$ are bounded. By taking $N \to \infty$, we obtain the claim of the lemma.

Let $h_{2, E_1}$ be a Hermitian metric of $E_1$ such that the following holds.

- We have a neighbourhood $N_1$ of $Z$ and that $h_{2, E_1} = h_{0, E_1}$ on $\mathcal{M}^\lambda \setminus N_1$.
- We have a neighbourhood $N_2$ of $Z$ contained in $N_1$ such that $h_{2, E_1} = h_{1, E_1}$ on $N_2 \setminus Z$.

We have the function $s$ determined by $h_{1, E_1} = h_{2, E_1} \cdot s$. We have the relation $G(h_{1, E_1}) - G(h_{2, E_1}) = 4^{-1} \Delta \log s$. The support of $\log s$ is contained in $\mathcal{M}^\lambda \setminus N_2$. By using the previous lemma, we obtain $\int \Delta \log s = 0$. Hence, we have $\int G(h_{1, E_1}) = \int G(h_{2, E_1})$. By using the argument in the proof of [21] Proposition 9.4, we obtain $\int G(h_{0, E_1}) = \int G(h_{2, E_1})$.

### 9.2.3 Analytic degree of subbundles

Let $E_2 \subset E$ be a mini-holomorphic subbundle. Let $h_{1, E_2}$ denote the metric of $E_2$ induced by $h_1$. By the Chern-Weil formula, $\deg(E_2, h_{1, E_2}) \in \mathbb{R} \cup \{-\infty\}$ makes sense.

**Proposition 9.5** Suppose that $\deg(E_2, h_{1, E_2}) \neq -\infty$. Then, there exists a good filtered subbundle $\mathcal{P}_s E_2 \subset \mathcal{P}_s E$ such that $\mathcal{P}_s E_2|_{\mathcal{M}^\lambda \setminus Z} = E_2$. Moreover, $\deg(E_2, h_{1, E_2}) = C \deg(\mathcal{P}_s E_2)$ holds, where $C$ is the constant in Proposition 9.3

**Proof** By [22] and [29] Lemma 10.6, $E_2|_{\mathcal{M}^\lambda \setminus \{t_1\}}$ are extended to a locally free $\mathcal{O}_{\mathcal{M}^\lambda}|_{\mathcal{M}^\lambda \setminus \{t_1\}}$-submodules of $\mathcal{P}_s E|_{\mathcal{M}^\lambda \setminus \{t_1\}}$. We take $P \in H^\lambda_{\infty} \times \mathcal{M}^\lambda$ and a small neighbourhood $\mathcal{U}_P$ of $P$ in $\mathcal{M}^\lambda$. On $\mathcal{U}_P$, we use a local minicomplex coordinate system $(\mathbb{U}^{-1}, t)$. On $\tilde{\mathcal{U}}_P := \mathbb{R} \times \mathcal{U}_P$, we use the complex coordinate system $(\mathbb{U}^{-1}, v) = (\mathbb{U}^{-1}, s + \sqrt{-1}t)$ as in [35, 35]. We set $D := \mathbb{R} \times (\mathcal{U}_P \cap H^\lambda_{\infty})$. Then, we have the locally free $\mathcal{O}_{\tilde{\mathcal{U}}_P}(\ast D)$-module $\tilde{\mathcal{P}}\mathcal{E}$ induced by $\mathcal{P}\mathcal{E}$. We also have the holomorphic vector subbundle $\tilde{E}_2$ of $\tilde{\mathcal{P}}\mathcal{E}|_{\tilde{\mathcal{U}}_P \setminus D}$ induced by $E_2$. Let $p : \tilde{\mathcal{U}}_P \to D$ be the projection given by $p(U^{-1}, v) = v$. By the above consideration, $\tilde{E}_2|_{\tilde{\mathcal{U}}_P \setminus D}$
extends $\mathcal{O}_{\mathcal{M}}^{\langle \lambda \rangle}(\ast \infty)$-submodule of $\mathcal{PE}_{\mathcal{M}}^{\langle \lambda \rangle}$. By using [21, Theorem 4.5], we obtain that $E_2$ extends $\mathcal{P}_{\mathcal{U}_P}(\ast D)$-submodule $\mathcal{PE}_2$ of $\mathcal{PE}$. By the construction, $\mathcal{PE}_2$ is naturally $\mathbb{R}$-equivariant, we obtain that $E_2|_{\mathcal{U}_P \setminus H^\lambda}$ extends to a locally free $\mathcal{O}_{\mathcal{U}_P}(\ast (H^\lambda_\infty \cap \mathcal{U}_P))$-submodule of $\mathcal{PE}|_{\mathcal{U}_P}$. Hence, we obtain that $E_2$ is extended to a locally free $\mathcal{O}_{\mathcal{M}}(\ast H^\lambda_\infty)$-module $\mathcal{PE}_2$. We have the naturally induced good filtered bundle $\mathcal{P}_E$ over $\mathcal{PE}_2$. The claim for the degree follows from Proposition 9.3.

As a consequence, we obtain the following.

**Corollary 9.6** $\mathcal{P}_E$ is stable if and only if $(E, h_1)$ is analytic stable.

### 9.3 Proof of Theorem 9.2

#### 9.3.1 Associated filtered bundles

Let $Z$ be a finite subset of $\mathcal{M}^\lambda$. Let $(E, \overline{\omega}_E, h)$ be a meromorphic monopole on $\mathcal{M}^\lambda \setminus Z$. Let $\mathcal{P}_E$ be the associated filtered bundle with Dirac type singularity on $(\mathcal{M}^\lambda; H^\lambda, Z)$.

**Proposition 9.7** The good filtered bundle $\mathcal{P}_E$ is polystable with $\text{deg}(\mathcal{P}_E) = 0$. If the monopole $(E, \overline{\omega}_E, h)$ is irreducible, $\mathcal{P}_E$ is stable.

**Proof** By Corollary 6.3 $(E, \overline{\omega}_E, h)$ satisfies the condition in 9.2.1. We obtain

$$C \text{deg}(\mathcal{P}_E) = \text{deg}(E, h) = 0.$$ 

Let $\mathcal{P}_E E_1$ be a good filtered subbundle of $\mathcal{P}_E$. We have $C \text{deg}(\mathcal{P}_E E_1) = \text{deg}(E_1, h_{E_1}) \leq 0$. Moreover, if $\text{deg}(\mathcal{P}_E E_1) = 0$, $E_1$ is flat with respect to the Chern connection, and the orthogonal decomposition $E = E_1 \oplus E_1^\perp$ is mini-holomorphic. Hence, we have the decomposition $\mathcal{P}_E = \mathcal{P}_E E_1 \oplus \mathcal{P}_E E_1^\perp$. We also have that $E_1$ and $E_1^\perp$ with the induced metrics are monopoles. Hence, we obtain the poly-stability of $\mathcal{P}_E$ by an easy induction on the rank of $E$.

#### 9.3.2 Uniqueness

**Proposition 9.8** Let $h'$ be another metric of $E$ such that (i) $(E, \overline{\omega}_E, h')$ is a monopole, (ii) any points of $Z$ are Dirac type singularity, (iii) $h'$ is adapted to $\mathcal{P}_E$. Then, the following holds.

- There exists a mini-holomorphic decomposition $(E, \overline{\omega}_E) = \bigoplus_{i=1}^m (E_i, \overline{\omega}_{E_i})$, which is orthogonal with respect to both $h$ and $h'$.
- There exist positive numbers $a_i$ $(i = 1, \ldots, m)$ such that $h_{E_i} = a_i h'_{E_i}$.

**Proof** By the norm estimate, $h$ and $h'$ are mutually bounded. Hence, we obtain the claim from [20, Proposition 2.4, Proposition 3.16].

#### 9.3.3 Construction of monopoles

Let $Z$ be a finite subset. Let $\mathcal{P}_E$ be a stable good filtered bundle with Dirac type singularity on $(\mathcal{M}^\lambda; H^\lambda_\infty, Z)$ with $\text{deg}(\mathcal{P}_E) = 0$. Set $E := \mathcal{P}_E|_{\mathcal{M}^\lambda \setminus Z}$.

The following proposition is similar to [21, Proposition 9.10].

**Proposition 9.9** There exists a Hermitian metric $h$ such that (i) $(E, \overline{\omega}_E, h)$ is a meromorphic monopole, (ii) $(E, \overline{\omega}_E, h)$ satisfies the norm estimate with respect to $\mathcal{P}_E$.

**Proof** We give only an outline. By Proposition 8.19 there exists a Hermitian metric $h_0$ of $E$ such that (i) $(\mathcal{P}_E, h_0)$ satisfies the norm estimate, (ii) $(E, \overline{\omega}_E, h_0)$ is a monopole with Dirac type singularity on a neighbourhood of each $P \in Z$, (iii) $G(h_0) = O(e^{-|h_0|})$. By Proposition 8.7 we may assume that $(\text{det}(E), \overline{\omega}_{\text{det}(E)}, \text{det}(h_0))$ is a monopole with Dirac type singularity such that $\text{det}(h_0)$ is adapted to $\mathcal{P}_E(\text{det}(E))$. By Corollary 9.6 $(E, \overline{\omega}_E, h_0)$ is analytically stable. By [20, Theorem 2.5, Proposition 3.16], there exists a Hermitian metric $h$ of $E$ such that the following holds:
• \( \det(h) = \det(h_0) \).
• \( G(h) = 0 \), i.e., \((E, \overline{\nabla}_E, h)\) is a monopole.
• Let \( s \) be the automorphism of \( E \) which is self-adjoint with respect to \( h \) and \( h_0 \), determined by \( h = h_0 s \). Then, \( s \) and \( s^{-1} \) are bounded with respect to \( h_0 \), and \( \overline{\nabla}_E s \) is \( \mathcal{L}^2 \).

By Proposition 2.22, there exists a compact subset \( C \subset \mathcal{M}^\lambda \) such that (i) \( Z \subset C \), (ii) \( F(h) \) is bounded on \( \mathcal{M}^\lambda \setminus C \). By Proposition 2.10], each point of \( Z \) is Dirac type singularity of \((E, \overline{\nabla}_E, h)\). Because \( s \) and \( s^{-1} \) are bounded, \((P, \mathcal{E}, h)\) satisfies the norm estimate. Thus, we obtain Proposition 3.9.

The claim of Theorem 9.2 follows from Proposition 9.7 Proposition 9.8 and Proposition 9.9.

10 Riemann-Hilbert correspondences of filtered objects \((|\lambda| \neq 1)\)

We give a complement on the Riemann-Hilbert correspondence for good filtered bundles with Dirac type singularity on \((\mathcal{M}^\lambda; H^\lambda, Z)\) for a finite subset \( Z \subset \mathcal{M}^\lambda \) in the case \( |\lambda| \neq 1 \). It is a parabolic version of the Riemann-Hilbert correspondence for local analytic \( q \)-difference modules, due to Ramis, Sauloy and Zhang [20] and van der Put and Reversat [24], and for the global \( q \)-difference modules due to Kontsevich and Soibelman, where \(|q| \neq 1 \).

As a result, from meromorphic doubly periodic monopoles, for each \( \lambda \) with \( |\lambda| \neq 1 \), we obtain filtered objects on the elliptic curve \( \mathbb{C}^*/(q^\lambda)Z \). They are constructed through the associated good filtered bundles on \((\mathcal{M}^\lambda; H^\lambda, Z)\). Recall that \((\mathcal{M}^\lambda; H^\lambda, Z)\) depends on the choice of \( e_1 \) and \( s_1 \). However, the induced filtered objects on \( \mathbb{C}^*/(q^\lambda)Z \) are essentially independent of the choice of \( e_1 \) and \( s_1 \) (Theorem 10.12).

10.1 Analytic \( q \)-difference modules

Let \( \mathcal{K}^\text{an} \) denote the field of the convergent Laurent power series \( \mathbb{C}(\{ y \}) \). Let \( \mathcal{R}^\text{an} \) denote the ring of the convergent power series \( \mathbb{C}[[ y ]] \). Let \( q \in \mathbb{C}^* \). Suppose that \(|q| \neq 1 \). Let \( \Phi^* : \mathcal{K}^\text{an} \to \mathcal{K}^\text{an} \) be determined by \( \Phi^*(f)(y) := f(qy) \). A \( q \)-difference \( \mathcal{K}^\text{an} \)-module is a finite dimensional \( \mathcal{K}^\text{an} \)-vector space \( \mathcal{V}^\text{an} \) equipped with a \( \mathbb{C} \)-linear automorphism \( \Phi^* \) such that \( \Phi^*(fs) = \Phi^*(f) \cdot \Phi^*(s) \) for any \( f \in \mathcal{K}^\text{an} \) and \( s \in \mathcal{V}^\text{an} \). Let \( \text{Diff}(\mathcal{K}^\text{an}, q) \) denote the category of \( q \)-difference \( \mathcal{K}^\text{an} \)-modules. By taking the formal completion

\[ \mathcal{E}(\mathcal{V}^\text{an}, \Phi^*) := (\mathcal{V}^\text{an} \otimes_{\mathcal{K}^\text{an}} \mathcal{K}, \Phi^*) \]

we obtain the functor \( \mathcal{E} : \text{Diff}(\mathcal{K}^\text{an}, q) \to \text{Diff}(\mathcal{K}, q) \).

10.1.1 Pure isoclinic modules

Let \( \omega \in \mathbb{Q} \). A \( q \)-difference \( \mathcal{K}^\text{an} \)-module \((\mathcal{V}^\text{an}, \Phi^*)\) is called pure isoclinic of slope \( \omega \) if \( \mathcal{E}(\mathcal{V}^\text{an}, \Phi^*) \) is pure isoclinic of slope \( \omega \). Let \( \text{Diff}(\mathcal{K}^\text{an}, q; \omega) \) denote the full subcategory of pure isoclinic \( q \)-difference \( \mathcal{K}^\text{an} \)-modules of slope \( \omega \). It is known that \( \mathcal{E} \) induces an equivalence

\[ \mathcal{E} : \text{Diff}(\mathcal{K}^\text{an}, q; \omega) \simeq \text{Diff}(\mathcal{K}, q; \omega) \].

10.1.2 Slope filtrations

Any \((\mathcal{V}, \Phi^*) \in \text{Diff}(\mathcal{K}, q)\) has a slope decomposition \((\mathcal{V}, \Phi^*) = \bigoplus_{\omega} (\mathcal{V}_\omega, \Phi^*)\), where \((\mathcal{V}_\omega, \Phi^*) \in \text{Diff}(\mathcal{K}, q)\). We define the slope filtration \( \mathfrak{F} \) of \((\mathcal{V}, \Phi^*)\) indexed by \((\mathbb{Q}, \leq)\) as follows:

\[ \mathfrak{F}_\mu \mathcal{V} := \bigoplus_{\phi(\omega) \leq \mu} \mathcal{V}_\omega \]

where we put \( \phi(\omega) := 1 \) (\(|\omega| > 1\) or \( \phi(\omega) := -1 \) (\(|\omega| < 1\)). We naturally have \( \text{Gr}_\mu^\phi(\mathcal{V}) = \mathcal{V}_{\phi(\mu) \omega} \).

According to Sauloy [28], any \((\mathcal{V}^\text{an}, \Phi^*) \in \text{Diff}(\mathcal{K}^\text{an}, q)\) has a unique filtration \( \mathfrak{F} \) indexed by \((\mathbb{Q}, \leq)\) such that \( \mathcal{E}(\mathfrak{F}) \) (\( \mathcal{V}^\text{an} \)) is pure isoclinic of slope \( \phi(\omega) \mu \). The filtration is functorial, i.e., for any morphism \( f : \mathcal{V}^\text{an}_\omega \to \mathcal{V}^\text{an}_\omega \), we have \( f(\mathfrak{F}_\mu \mathcal{V}^\text{an}_\omega) \subset \mathfrak{F}_\mu \mathcal{V}^\text{an}_\omega \), and more strongly \( f(\mathfrak{F}_\mu \mathcal{V}^\text{an}_\omega) \subset \mathfrak{F}_\mu \mathcal{V}^\text{an}_\omega \cap f(\mathcal{V}^\text{an}_\omega) \).
10.1.3 Equivalences

We set $T := \mathbb{C}^*/q^\mathbb{Z}$. Let $\text{Vect}(T)$ denote the category of locally free $O_T$-modules of finite rank. For any $\mu \in \mathbb{Q}$, let $\text{Vect}^{ss}(T; \mu) \subset \text{Vect}(T)$ denote the full subcategory of semistable sheaves of slope $\mu$, i.e., $E \in \text{Vect}(T)$ such that $\text{deg}(E)/\text{rank}(E) = \mu$.

For $E \in \text{Vect}^{ss}(T)$, a $\mathbb{Q}$-anti-Harder-Narasimhan filtration of $E$ is a filtration $\mathcal{F}$ of $E$ in $\text{Vect}(T)$ indexed by $(\mathbb{Q}, \leq)$ such that $\text{Gr}_q^\mathbb{Z}(E) \in \text{Vect}^{ss}(T; \mu)$. Let $\text{Vect}^{\mathbb{Q} \text{AHN}}(T)$ denote the category of locally free $O_T$-modules $E$ equipped with a $\mathbb{Q}$-anti-Harder-Narasimhan filtration $\mathcal{F}$.

Let us recall that there exists a natural equivalence

$$K : \text{Vect}^{\mathbb{Q} \text{AHN}}(T) \simeq \text{Diff}(\mathbb{C}^*, q)$$

due to van der Put, Reversat \[24\] and Ramis, Sauloy and Zhang \[26\]. Let $(E, \mathcal{F}) \in \text{Vect}^{\mathbb{Q} \text{AHN}}(T)$. We obtain the $q^\mathbb{Z}$-equivariant locally $O_{\mathbb{C}^*}$-module $E$ by the pull back $\mathbb{C}^* \rightarrow T$. It is equipped with a $q^\mathbb{Z}$-equivariant filtration $\mathcal{F}$. There exists a canonical extension of $E$ to a $q^\mathbb{Z}$-equivariant locally free $O_{\mathbb{C}^*(0)}$-module $\tilde{E}$ equipped with a $q^\mathbb{Z}$-equivariant filtration $\tilde{\mathcal{F}}$ such that the formal completion of $\text{Gr}_\mu^\mathbb{Z}(\tilde{E})$ are pure isoclinic of slope $\nu(q)\mu$. By taking the stalk of $\tilde{E}$ at $0$, we obtain $K(E, \mathcal{F}) \in \text{Diff}(\mathbb{C}^*, q)$. The same procedure induces $K : \text{Vect}^{ss}(T; \mu) \simeq \text{Diff}(\mathbb{C}^*, q)$.

For any $\mu \in \mathbb{Q}$, we take $L_1(\mu) \in \text{Vect}^{ss}(T; \mu)$ with an isomorphism $K(L_1(\mu)) \simeq L_1(\nu(q)\mu)$ in $\text{Diff}(\mathbb{C}^*, q)$. (See Example 2.6 for $L_m(\omega)$.) For any $A \in \text{GL}_r(\mathbb{C})$, we take $V_1(A) \in \text{Vect}^{ss}(T, 0)$ with an isomorphism $K(V_1(A)) \simeq V_1(A)$ in $\text{Diff}(\mathbb{C}^*, q; 0)$. (See Example 2.6 for $V_{m,n}(A)$.) Similarly, for any finite dimensional $\mathbb{C}$-vector space $V$ equipped with an automorphism $f$, we take $V_1(V, f) \in \text{Vect}^{ss}(T, 0)$ with an isomorphism $K(V_1(V, f)) \simeq V_1(V, f)$.

10.2 Classification of good filtered formal $q$-difference modules in the case $|q| \neq 1$

Let $\text{Vect}^{ss}(T; \mu)^{\text{Par}}$ denote the category of $E \in \text{Vect}^{ss}(T; \mu)$ equipped with a filtration $\mathcal{F}_\bullet(E)$ indexed by $(\mathbb{Q}, \leq)$ such that (i) $\mathcal{F}_a(E) = \bigcap_{a < b} \mathcal{F}_b(E)$, (ii) $\text{Gr}_a^\mathbb{Z}(E) := \mathcal{F}_a(E)/\mathcal{F}_{<a}(E) \in \text{Vect}^{ss}(T, \mu)$ for any $a \in \mathbb{R}$. Note that $\{a \in \mathbb{R} | \text{Gr}_a^\mathbb{Z}(E) \neq 0\}$ is finite. For any $C > 0$, let us construct an equivalence $K^C : \text{Vect}^{ss}(T; \mu)^{\text{Par}} \simeq \text{Diff}(\mathbb{C}^*, q)$ depending on $C$.

10.2.1 The case $\mu = 0$

Take $A_\alpha \in \text{GL}_r(\mathbb{C})$ which has a unique eigenvalue $\alpha$. Let $\mathcal{F}$ be a filtration of $V_1(A_\alpha)$ such that $(V_1(A_\alpha), \mathcal{F}) \in \text{Vect}^{ss}(T, 0)^{\text{Par}}$. We obtain the induced filtration $\mathcal{F}$ on $K(V_1(A_\alpha))$ in $\text{Diff}(\mathbb{C}^*, q; 0)$. For $a \in \mathbb{R}$, we set

$$b(q, \alpha, a) := C \cdot \left(a + \frac{\log |\alpha|}{\log |q|}\right).$$

We define the filtration $F$ of $K(V_1(A_\alpha))$ in $\text{Diff}(\mathbb{C}^*, q; 0)$ indexed by $\mathbb{R}$ as follows:

$$F_a K(V_1(A_\alpha)) = \mathcal{F}_{b(q, \alpha, a)} K(V_1(A_\alpha)).$$

There exists a frame $v$ of $K(V_1(A_\alpha))$ such that (i) $\Phi v = v A_\alpha$, (ii) $v$ is compatible with $F$, i.e., there exists a decomposition $v = \bigoplus_{c \in \mathbb{Z}} v_c$ such that $\bigoplus_{c \leq a} v_c$ is a frame of $F_a K(V_1(A_\alpha))$. For each $v_i$, let $c(v_i)$ be determined by $v_i \in v_{c(v_i)}$. We define

$$P_d K(V_1(A_\alpha)) = \bigoplus \mathcal{R} \cdot y^{-[d-c(v_i)]} v_i.$$

In this way, we obtain the filtered bundle $K^C(V_1(A_\alpha), \mathcal{F}) := P_d K(V_1(A_\alpha))$.

In general, for any $(E, \mathcal{F}) \in \text{Vect}^{ss}(T, 0)^{\text{Par}}$, there exist a partition $r = \sum r_i$, matrices $A_{\alpha_i} \in \text{GL}_{r_i}(\mathbb{C})$ with a unique eigenvalue $\alpha_i$, objects $(V(A_{\alpha_i}), \mathcal{F}) \in \text{Vect}^{ss}(T; 0)^{\text{Par}}$, and an isomorphism

$$(E, \mathcal{F}) \simeq \bigoplus_{i=1}^{N} (V(A_{\alpha_i}), \mathcal{F}).$$
We obtain the filtered bundle $\mathcal{P}_s K(E)$ over $K(E)$ induced by the isomorphism $K(E) \simeq \bigoplus K(\overline{V}(A_\alpha))$ and the filtered bundle $\bigoplus K^C(\overline{V}(A_\alpha), \mathcal{F})$. It is easy to check that $\mathcal{P}_s K(E)$ is independent of the choice of $A_\alpha$, and an isomorphism $K$. We define $K^C(E, \mathcal{F}) := \mathcal{P}_s K(E)$. Thus, we obtain a functor

$$K^C : \text{Vect}^{ss}(T; 0)^{Par} \to \text{Diff}(K; 0)^{Par}.$$ 

**Lemma 10.1** $K^C$ induces an equivalence $\text{Vect}^{ss}(T; 0)^{Par} \simeq \text{Diff}(K; q; 0)$.

**Proof** Let $\mathcal{L}$ be a lattice of $\mathcal{V} = \mathcal{V}(A_\alpha)$ such that $\Phi^*(\mathcal{L}) = \mathcal{L}$. We obtain the automorphism $\sigma(\Phi^*; \mathcal{L})$ of $\mathcal{L}_0$, and the generalized eigen decomposition

$$\mathcal{L}_0 = \bigoplus_{i \in \mathbb{Z}} E_{\alpha q_i}(\mathcal{L}_0).$$

We set $i_0 := \max\{i | E_{\alpha q_i}(\mathcal{L}_0) \neq 0\}$ and $i_1 := \min\{i | E_{\alpha q_i}(\mathcal{L}_0) \neq 0\}$.

If $i_0 > 0$, we define $\mathcal{L}'$ as the kernel of the $\mathcal{L} \to E_{\alpha q_i-1}(\mathcal{L}_0)$. Then, it is easy to see that $\Phi^*(\mathcal{L}') = \mathcal{L}'$. We have the natural inclusion $\mathcal{L}' \to \mathcal{L}$. It induces $E_{\alpha q_i-1}(\mathcal{L}'_0) = E_{\alpha q_i-1}(\mathcal{L}_0)$ for $i < i_0 - 1$, and the following exact sequence:

$$0 \to E_{\alpha q_i-1}((y\mathcal{L})_0) \to E_{\alpha q_i-1}(\mathcal{L}'_0) \to E_{\alpha q_i-1}(\mathcal{L}_0) \to 0. \quad (100)$$

Moreover, we have $E_{\alpha q_i-1}(\mathcal{L}'_0) = 0$ for $i \geq i_0$.

If $i_1 < 0$, we define $\mathcal{L}''$ as the kernel of the following:

$$y^{-1} \mathcal{L} \to \bigoplus_{i > i_1 + 1} E_{\alpha q_i}(((y^{-1}\mathcal{L})_0).$$

We have $\Phi^*(\mathcal{L}'') = \mathcal{L}''$. We have the natural inclusion $\mathcal{L} \to \mathcal{L}''$. It induces $E_{\alpha q_i-1}(\mathcal{L}_0) \simeq E_{\alpha q_i-1}(\mathcal{L}''_0)$ for $i > i_1 + 1$, and the following exact sequence:

$$0 \to E_{\alpha q_i-1}((\mathcal{L}_0) \to E_{\alpha q_i-1}(\mathcal{L}''_0) \to E_{\alpha q_i-1}(y^{-1}\mathcal{L}_0) \to 0. \quad (101)$$

Suppose that each $E_{\alpha q_i-1}(\mathcal{L}_0)$ is equipped with a filtration $F$ satisfying the following conditions.

- For $i_1 < i < i_0$, $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ is indexed by $\mathbb{R}_{\leq 0}$.
- $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ is indexed by $\mathbb{R}_{\geq 0}$.
- $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ is indexed by $\mathbb{R}_{> 0}$.
- $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ is indexed by $\mathbb{R}_{< 0}$.

Note that $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ induces a filtration $F_\bullet E_{\alpha q_i-1}((y\mathcal{L})_0)$ indexed by $\mathbb{R}_{\leq 0}$, and that $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ induces a filtration $F_\bullet E_{\alpha q_i-1}((y^{-1}\mathcal{L})_0)$ indexed by $\mathbb{R}_{> 0}$.

We obtain a filtration $F_\bullet E_{\alpha q_i-1}(\mathcal{L}'_0)$ for $i < i_0 - 1$ by using the isomorphism $E_{\alpha q_i-1}(\mathcal{L}_0) \simeq E_{\alpha q_i-1}(\mathcal{L}_0)$. We obtain the filtration $F_\bullet E_{\alpha q_i-1}(\mathcal{L}'_0)$ for $i > i_1 + 1$ by using the isomorphism $E_{\alpha q_i-1}(\mathcal{L}_0) \simeq E_{\alpha q_i-1}(\mathcal{L}_0)$. We obtain the filtration $F_\bullet E_{\alpha q_i-1}(\mathcal{L}_0)$ for $i > i_1 - 1$ by using the exact sequence (101).

Let $\mathcal{E}$ be a frame of $\mathcal{V}$ such that $\Phi^*(\mathcal{E}) = \mathcal{E} \cdot A_\alpha$. Let $\mathcal{L}(A_\alpha)$ be the lattice of $\mathcal{V}$ generated by $\mathcal{E}$. Note that such $\mathcal{L}(A_\alpha)$ is independent of a choice of $\mathcal{E}$. Starting from a regular filtered bundle $\mathcal{P}_s \mathcal{V}$, by applying the above procedure inductively, we obtain a filtration $F$ on $\mathcal{L}(A_\alpha)_0$. There exists a unique filtration $\tilde{F}$ of $\mathcal{L}(A_\alpha)$ such that (i) it is preserved by $\Phi^*$, (ii) it induces $F(\mathcal{L}(A_\alpha)_0)$. We define $\mathcal{F}(\mathcal{V})$ from $\tilde{F}$ by using (107) and (108). It is easy to see that this gives a quasi-inverse of $K$.

10.2.2 The case of general $\mu$

Let $\mu \in \mathbb{Q}$. Let $(E_\mu, F) \in \text{Vect}^{ss}(T; \mu)^{Par}$. There exists $(E_\mu', F') \in \text{Vect}^{ss}(T; 0)^{Par}$ with an isomorphism $\mathcal{F}_s E_\mu \simeq \mathcal{F}_s E_\mu'$. We define

$$K^C(E_\mu, F) := P^-_{\Phi(q)/2} \mathcal{L}_1(\Phi(q)/\mu) \otimes K^C(E_\mu', F) \in \text{Diff}(K, q; \Phi(q)/\mu)^{Par}.$$ 

Thus, we obtain a functor $K^C : \text{Vect}^{ss}(T; \mu)^{Par} \to \text{Diff}(K, q; \Phi(q)/\mu)^{Par}$. As a consequence of Lemma 10.1, we obtain the following lemma.
Lemma 10.2 $K^C$ induces an equivalence $\text{Vect}^{ss}(T; \mu)^{Par} \simeq \text{Diff}(\mathcal{K}, q; \varrho(q)\mu)$.

Remark 10.3 Let $(E, F) \in \text{Vect}^{ss}(T; \mu)^{Par}$. We define a new filtration $\mathcal{F}^{C}$ on $E$ by $\mathcal{F}^{C}_{a}(E) := \mathcal{F}_{a\mu}(E)$. The correspondence $(E, F) \mapsto (E, \mathcal{F}^{C})$ induces an equivalence $\text{Vect}^{ss}(T; \mu)^{Par} \rightarrow \text{Vect}^{ss}(T; \mu)^{Par}$. The following is commutative by the construction.

\[
\begin{array}{ccc}
\text{Vect}^{ss}(T; \mu)^{Par} & \xrightarrow{\eta^{C}} & \text{Diff}(\mathcal{K}, q; \varrho(q)\mu) \\
\downarrow & & \downarrow \\
\text{Vect}^{ss}(T; \mu)^{Par} & \xrightarrow{K^{C}} & \text{Diff}(\mathcal{K}, q; \varrho(q)\mu).
\end{array}
\]

10.2.3 Graded objects

A $(\mathbb{Q}, \mathbb{R})$-grading of $E \in \text{Vect}(T)$ is a decomposition

\[E = \bigoplus_{\mu \in \mathbb{Q}} \bigoplus_{a \in \mathbb{R}} E_{\mu, a}\]

such that $E_{\mu, a} \in \text{Vect}^{ss}(T, \mu)$. Let $\text{Vect}(T)(\mathbb{Q}, \mathbb{R})$ be a category of $E \in \text{Vect}(T)$ with a $(\mathbb{Q}, \mathbb{R})$-grading. Let us construct a functor $\text{Vect}(T)(\mathbb{Q}, \mathbb{R}) \rightarrow \text{Diff}(\mathbb{C}[y, y^{-1}], q)/(\mathbb{Q}, \mathbb{R})$.

For any $E \in \text{Vect}^{ss}(T; \mu)$, we obtain $\mathbb{Q}^{2}$-equivariant $\mathcal{O}_{\mathbb{C}_{y}}$-module $\check{E}$ as the pull back of $E$ by $\mathbb{C}^{*} \rightarrow T$. It is extended to a locally free $\mathbb{Q}^{2}$-equivariant $\mathcal{O}_{\mathbb{C}_{y}}(\{0\})$-module $\check{E}$ such that the formal completion $\check{E}_{0} \otimes \mathbb{C}((y))$ is naturally an isoclinic $\mathbb{Q}$-difference $\mathcal{C}(y)$-module of slope $\varrho(q)\mu$. Similarly, $\check{E}$ is extended to a locally free $\mathbb{Q}^{2}$-equivariant $\mathcal{O}_{\mathbb{C}_{y^{-1}}}(\{0\})$-module $\check{E}_{\infty}$ such that $\check{E}_{\infty} \otimes \mathcal{C}(y^{-1})$ is naturally an isoclinic $\mathbb{Q}$-difference $\mathcal{C}(y^{-1})$-module of slope $-\varrho(q)\mu$. By gluing $\check{E}_{0}$ and $\check{E}_{\infty}$, we obtain a locally free $\mathbb{Q}^{2}$-equivariant $\mathcal{O}_{\mathbb{P}^{1}}(\{0, \infty\})$-module $\check{E}$. Note that $\check{E}$ has an $\mathcal{O}_{\mathbb{P}^{1}}$-lattice. Hence, $\check{E}$ is an algebraic. By taking the global section on $\mathbb{P}^{1}$, we obtain a $\mathbb{Q}$-difference $\mathcal{C}[y, y^{-1}]$-module $\mathcal{K}(E)$.

Suppose that $E = \mathcal{V}(A_{\alpha})$, where $A_{\alpha}$ has a unique eigenvalue $\alpha$. Let $b \in \mathbb{R}$ Let $\mathcal{F}^{(b)}$ be the filtration on $E$ determined by $\mathcal{F}_{b}(E) = E$ and $\mathcal{F}_{<b}(E) = 0$. We obtain the filtered bundle $P_{*}(\check{E}_{0} \otimes \mathcal{C}(y))$ over $\check{E}_{0} \otimes \mathcal{C}(y))$ induced by $K^{C}(E, \mathcal{F}^{(b)})$. We set

\[a_{0} := C^{-1}b + \frac{\log |\alpha|}{\log |q|} \]

Then, $\text{Gr}^{P}_{c}(\check{E}_{0}) = 0$ unless $c \in a_{0} - \varrho(q)\mu/2 + \mathbb{Z}$. We also obtain the filtered bundle $P_{*}(\check{E}_{\infty} \otimes \mathcal{C}(y^{-1}))$ over $\check{E}_{\infty} \otimes \mathcal{C}(y^{-1}))$. We set

\[a_{\infty} := C^{-1}b + \frac{\log |\alpha|}{\log |q|} \]

Then, $\text{Gr}^{P}_{c}(\check{E}_{0}) = 0$ unless $c \in a_{\infty} + \varrho(q)\mu/2 + \mathbb{Z}$. Set $a := a_{0} - \varrho(q)\mu/2$. For any $n \in \mathbb{Z}$, let $\mathcal{L}_{(\mu, a+n)} \subset \check{E}$ be the lattice determined by $P_{a+n-\varrho(q)\mu/2}\check{E}_{0}$ and $P_{a+n+\varrho(q)\mu/2}\check{E}_{\infty}$. Then, it turns out that $\mathcal{L}_{(\mu, a+n)}$ is isomorphic to $\mathcal{O}\text{rank }E$. We set $K^{C}(E)_{(\mu, a+n)} := H^{0}(P^{1}, \mathcal{L}_{(\mu, a+n)})$ for $n \in \mathbb{Z}$. We also set $K^{C}(E)_{(\mu, a+n)} := 0$ unless $c - a \in \mathbb{Z}$.

Let $E = \bigoplus E_{\mu, b} \in \text{Vect}(T)(\mathbb{Q}, \mathbb{R})$. For each $E_{\mu, b}$, we apply the above construction, and we obtain $\mathcal{K}^{C}(E_{\mu, b}) \in \text{Diff}(\mathbb{C}[y, y^{-1}], q)/(\mathbb{Q}, \mathbb{R})$. We define $K^{C}(E) := \bigoplus \mathcal{K}^{C}(E_{\mu, b}) \in \text{Diff}(\mathbb{C}[y, y^{-1}], q)/(\mathbb{Q}, \mathbb{R})$.

For each $(E_{\mu}, F) \in \text{Vect}^{ss}(T; \mu)^{Par}$, we obtain $\text{Gr}_{\sigma}^{F}(E_{\mu}) = \bigoplus \text{Gr}_{\sigma}^{F}(E_{\mu}) \in \text{Vect}(T)(\mathbb{Q}, \mathbb{R})$. The following is easy to see by the construction.

Lemma 10.4 For $(E_{\mu}, F) \in \text{Vect}^{ss}(T; \mu)^{Par}$, we have the natural isomorphism $G(K^{C}(E_{\mu}, F)) \simeq K^{C}(\text{Gr}_{\sigma}^{F}(E_{\mu}))$.
10.2.4 Weight filtration

Let $E \in \text{Vect}^{ss}(T; \mu)$. There exists an isomorphism $E \simeq \bigoplus_i \tilde{L}(\mu) \otimes \tilde{V}(A_{\alpha_i})$, where each $A_{\alpha_i}$ has a unique eigenvalue $\alpha_i$. We obtain the logarithm $N_{\alpha_i}$ of the unipotent part of $A_{\alpha_i}$, and the nilpotent endomorphism $N := \bigoplus N_{\alpha_i}$ of $E$. It is independent of the choice of an isomorphism $E \simeq \bigoplus_i \tilde{L}(\mu) \otimes \tilde{V}(A_{\alpha_i})$ and $\alpha_i$. We obtain the weight filtration $W$ of $E$ with respect to $N$.

Let $(E, \mathcal{F}) \in \text{Vect}^{ss}(T; \mu)^{par}$. Each $\text{Gr}_{\mu}^\mathcal{F}(E)$ is equipped with the nilpotent endomorphism $N$ and $W$. The following is clear by the construction.

**Lemma 10.5** The functor $K^\mathcal{C}$ preserves the nilpotent endomorphism and the weight filtrations.

10.2.5 Analytic case

Let $\text{Vect}^{\mathcal{Q}\text{AHN}}(T)^{par}$ denote the category of $(E, \mathcal{F}) \in \text{Vect}^{\mathcal{Q}\text{AHN}}(T)$ equipped with filtrations $\mathcal{F}$ of $\text{Gr}_{\mu}^\mathcal{F}(E)$ for any $\mu \in \mathbb{Q}$ such that $(\text{Gr}_{\mu}^\mathcal{F}(E), \mathcal{F}) \in \text{Vect}^{ss}(T; \mu)$.

For any $(\mathcal{V}^{an}, \Phi^*) \in \text{Diff}(\mathcal{K}^{an}, q)$, a good filtered bundle over $(\mathcal{V}^{an}, \Phi^*)$ means a good filtered bundle over $\mathcal{C}(\mathcal{V}^{an}, \Phi^*)$. Note that $\mathcal{R}^{an*}$-lattices of $\mathcal{V}^{an}$ are equivalent to $\mathcal{R}$-lattices of $\mathcal{C}(\mathcal{V}^{an})$. Let $\text{Diff}(\mathcal{K}^{an}, q)^{par}$ denote the category of good filtered $q$-difference $\mathcal{K}^{an}$-modules.

We obtain an equivalence
\[ K^\mathcal{C} : \text{Vect}^{\mathcal{Q}\text{AHN}}(T)^{par} \simeq \text{Diff}(\mathcal{K}^{an}, q)^{par} \]
from the equivalence $K : \text{Vect}^{\mathcal{Q}\text{AHN}}(T) \simeq \text{Diff}(\mathcal{K}^{an}, q)$ and $K^\mathcal{C} : \text{Vect}^{ss}(T; \mu)^{par} \simeq \text{Diff}(\mathcal{K}, q; \mu)^{par}$.

10.3 $q$-difference parabolic structure of sheaves on elliptic curves

Let $\mathcal{D} \subset T = \mathbb{C}^*/q^2$ be a finite subset.

**Definition 10.6** Let $\bar{E}$ be a locally free $\mathcal{O}_T(*\mathcal{D})$-module of finite rank. A $q$-difference parabolic structure of $\bar{E}$ is data as follows:

- A sequence $s_{P,1} < s_{P,2} < \cdots < s_{P,m(P)}$ in $\mathbb{R}$ for each $P \in \mathcal{D}$.
  
  We formally set $s_{P,0} := -\infty$ and $s_{P,m(P)+1} := \infty$.

- A tuple of lattices $\mathcal{K}_P = (\mathcal{K}_{P,i} \mid i = 0, \ldots, m(P) + 1)$ of $\bar{E}_P$.

Note that we obtain the lattice $\mathcal{E}_- \subset \bar{E}$ determined by $\mathcal{K}_{P,0}$ ($P \in \mathcal{D}$) and the lattice $\mathcal{E}_+ \subset \bar{E}$ determined by $\mathcal{K}_{P,m(P)+1}$ ($P \in \mathcal{D}$).

- Objects $(\mathcal{E}_\pm, \mathcal{F}_\pm) \in \text{Vect}^{\mathcal{Q}\text{AHN}}(T)^{par}$.

When we fix $(s_P)_{P \in \mathcal{D}}$, it is called a $q$-difference parabolic structure at $(\mathcal{D}, (s_P)_{P \in \mathcal{D}})$.

Let $\bar{E}_i = (\bar{E}_i, (s_P, \mathcal{K}_P^{(i)})_{P \in \mathcal{D}}, (\mathcal{F}_\pm^{(i)})_{i})$ be locally free $\mathcal{O}_T(*\mathcal{D})$-modules of finite rank with $q$-difference parabolic structure at $(s_P)_{P \in \mathcal{D}}$. A morphism $\bar{E}_1^{(1)} \rightarrow \bar{E}_2^{(2)}$ is defined to be a morphism $f : \bar{E}_1^{(1)} \rightarrow \bar{E}_2^{(2)}$ of locally free $\mathcal{O}_T(*\mathcal{D})$-modules such that the following holds:

- $f(\mathcal{K}_{P,i}^{(1)}) \subset \mathcal{K}_{P,i}^{(2)}$.

- The induced morphisms $f : \mathcal{E}_\pm^{(1)} \rightarrow \mathcal{E}_\pm^{(2)}$ are compatible with the filtrations $(\mathcal{G}_\pm, \mathcal{F}_\pm)$, i.e., they induce $f : (\mathcal{E}_\pm^{(1)}, \mathcal{G}_\pm^{(1)}, \mathcal{F}_\pm^{(1)}) \rightarrow (\mathcal{E}_\pm^{(2)}, \mathcal{G}_\pm^{(2)}, \mathcal{F}_\pm^{(2)})$ in $\text{Vect}^{\mathcal{Q}\text{AHN}}(T)$.

Let $\text{Vect}^q(T, (s_P)_{P \in \mathcal{D}})$ denote the category of locally free $\mathcal{O}_T(*\mathcal{D})$-modules of finite rank with $q$-difference parabolic structure at $(\mathcal{D}, (s_P)_{P \in \mathcal{D}})$.
We define the degree of $\tilde{E}_s = (\tilde{E}, (s_P, K_P)_{P \in D}, (\tilde{\mathcal{S}}_{\pm}, F_{\pm}))$ as follows:

$$
\deg(\tilde{E}_s) := -\sum_{P \in D} \sum_{i=1}^{n(P)} s_{P,i} \deg(K_{P,i}, K_{P,i-1}) \\
- \sum_{\omega \in Q} \sum_{b \in R} b \text{rank } Gr^F_{\omega} \mathcal{G}_{\omega}^-(\tilde{E}_s) - \sum_{\omega \in Q} \sum_{b \in R} b \text{rank } Gr^F_{\omega} \mathcal{G}_{\omega}^+(\tilde{E}_s). 
$$

(102)

10.3.1 Rescaling of parabolic structure

Let $D \subset T$ be a finite subset. Let $\tilde{E}_s = (\tilde{E}, (s_P, K_P)_{P \in D}, (\tilde{\mathcal{S}}_{\pm}, F_{\pm}))$ be a locally free $\mathcal{O}_T(\ast D)$-module with q-difference parabolic structure.

Let $t > 0$. We obtain a sequence $s^{(t)}_P := (ts_{P,i})$. We set $K^{(t)}_P := K_P$ and $\tilde{\mathcal{S}}^{(t)}_{\pm} := \tilde{\mathcal{S}}_{\pm}$. By setting $(F^{(t)}_\pm)_{t,a} : = (F_\pm)_{a} \text{Gr}^{\tilde{\mathcal{S}}_{\pm}}(\tilde{E}_\pm)$, we obtain filtrations $F^{(t)}_\pm$. We set

$$
\mathcal{H}^{(t)}(\tilde{E}_s) := (\tilde{E}, (s^{(t)}_P, K^{(t)}_P)_{P \in D}, (\tilde{\mathcal{S}}^{(t)}_{\pm}, F^{(t)}_{\pm})).
$$

Let $t < 0$. We set $s^{(t)}_P := ts_{P,a(P)-i+1}$. We obtain a sequence $s^{(t)}_P$. We set $K^{(t)}_{P,i} := K_{P,a(P)+i-1}$, and we obtain a sequence of lattices $K^{(t)}_P$. We set $E^{(t)}_\pm := E_\pm$. Let $\tilde{\mathcal{S}}^{(t)}_{\pm}(E^{(t)}_\pm)$ denote the filtration induced by $\tilde{\mathcal{S}}_{\pm}(E_\mp)$. We set $(F^{(t)}_{\pm})_{t,a} : = (F_{\pm})_{a} \text{Gr}^{\tilde{\mathcal{S}}_{\pm}}(E^{(t)}_\pm)$. Thus, we obtain

$$
\mathcal{H}^{(t)}(\tilde{E}_s) := (\tilde{E}, (s^{(t)}_P, K^{(t)}_P)_{P \in D}, (\tilde{\mathcal{S}}^{(t)}_{\pm}, F^{(t)}_{\pm})).
$$

The following is easy to check.

Lemma 10.7 \(\deg(\mathcal{H}^{(t)}(\tilde{E}_s)) = |t| \deg(\tilde{E}_s)\).

10.4 Global correspondence for parabolic q-difference modules

10.4.1 Parabolic q-difference modules

Let $D \subset \mathbb{C}^*$ be a finite subset. A parabolic structure of q-difference $\mathbb{C}[y, y^{-1}]$-module is defined as in §1.2. Let $V^{(i)} = (V^{(i)}, V^{(i)}(\tau), (\alpha_\alpha)_{\alpha \in D}, P, V^{(1)}, V^{(i)}(\zeta))$ for $i = 1, 2$ be q-difference $\mathbb{C}[y, y^{-1}]$-modules with good parabolic structure at infinity and parabolic structure at $(\omega, (\alpha))_{\alpha \in D}$. A morphism $\mathcal{V}_s^{(i)} \rightarrow \mathcal{V}_s^{(2)}$ is defined to be a morphism of q-difference $\mathbb{C}[y, y^{-1}]$-module $f : V^{(1)} \rightarrow V^{(2)}$ such that the following holds:

- $f(V^{(1)}) \subset V^{(2)}$.
- $f(L^{(1)}_{\hat{\alpha}}) \subset L^{(2)}_{\hat{\alpha}}$.
- $f : P_\nu V^ {\nu(1)}_{\nu} \rightarrow P_\nu V^{\nu(2)}_{\nu}$ for $\nu = 0, \infty$.

Let Diff$(\mathbb{C}[y, y^{-1}], q, (\tau)_{\alpha \in D})^{Par}$ be the category of q-difference $\mathbb{C}[y, y^{-1}]$-modules with good parabolic structure at infinity and parabolic structure at $(\tau)_{\alpha \in D}$.

10.4.2 An equivalence

Let $\pi : \mathbb{C}^* \rightarrow T := \mathbb{C}^*/q^\mathbb{Z}$ denote the projection. Let $D \subset T$ be a finite subset. For each $P \in D$, let $s_P = (s_{P,1} < \cdots < s_{P,m(P)})$ be a sequence in $\mathbb{R}$. For each $s_{P,i}$, there exists $\alpha_{P,i} \in \pi^{-1}(P) \subset \mathbb{C}^*$ determined by the following conditions:

$$
0 \leq s_{P,i} + \frac{\log|\alpha_{P,i}|}{\log|q|} < 1.
$$

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We set \( u_{p,i} := s_{p,i} + \log |\alpha_{p,i}| \). We set \( D := \prod_{p \in \mathbb{P}} \{ \alpha_{p,i} \mid i = 1, \ldots, m(P) \} \subseteq \mathbb{C}^* \). For each \( \alpha \in \pi^{-1}(P) \cap D \), we set 
\( Z(\alpha) := \{ u_{p,i} \mid \alpha_{p,i} = \alpha \} \subseteq [0, 1] \). We obtain the sequence 
\( t_\alpha = (0 \leq t_{\alpha,0} < t_{\alpha,1} < \cdots < t_{\alpha,m(\alpha)} < 1) \) by ordering the elements of \( Z(\alpha) \). Let \( i(\alpha) \) be determined by \( u_{p,i(\alpha)} = t_{\alpha,0} \).

Let us construct an equivalence \( K : \text{Vect}^3(T, (s_p)_{p \in \mathbb{P}}) \simeq \text{Diff}(\mathbb{C}[y, y^{-1}], q, (t_\alpha)_{\alpha \in D})^{\text{par}}. \)

Let \( \widetilde{E}_n = (\mathbb{E}_n, (t_p, K_p, P_{i,n}), (\mathbb{G}, F_{\mathbb{G}})). \) Let \( \mathbb{E} \) be the locally free \( \mathcal{O}_{\mathbb{C}}(\pi^{-1}(D)) \)-module obtained as the pull back of \( \mathbb{E} \). For each \( \alpha \in D \cap \pi^{-1}(P) \), we obtain a lattice \( \mathcal{L}_\alpha \) of \( \mathbb{E}_\alpha \). We obtain a locally free \( \mathcal{O}_{\mathbb{C}} \)-submodule \( \mathcal{V} \subseteq \mathbb{E} \) determined by \( \mathcal{L}_\alpha (\alpha \in \pi^{-1}(D)). \) It is extended to a filtered bundle \( \mathcal{P}_\nu \mathcal{V} \) on \((\mathbb{P}, (0, \infty))\) by \((\mathbb{G}, F_{\mathbb{G}})) \) using the functors \( \mathbb{K}^1 \). We set \( V := H^0(\mathbb{P}, \mathcal{P}_\nu \mathcal{V}), \) which is \( \mathbb{C}[y, y^{-1}] \)-free module of finite rank. We set \( \mathcal{V} = V \otimes \mathcal{C}(y) \), which is naturally a \( \mathbb{Q}(y) \)-module. Let \( \mathcal{V} \) be the \( \mathbb{Q}(y) \)-difference \( \mathbb{C}[y, y^{-1}] \)-submodule of \( \mathcal{V} \) generated by \( V \). For each \( \alpha \in D \cap \pi^{-1}(P) \), we obtain the lattices \( \mathcal{L}_{\alpha,j} \) \( (1 \leq j \leq m(\alpha) - 1) \) of \( \mathcal{V}(\alpha) \) induced by \( K_{\nu,i(\alpha)} \). We also obtain good filtered bundles \( \mathcal{P}_\nu \mathcal{V}_{\nu} \) \( (\nu = 0, \infty) \) over \( \mathcal{V}_{\nu} \) from \( \mathcal{P}_\nu \mathcal{V} \). Thus, we obtain

\[
\mathcal{V} = (\mathbb{V}, \mathcal{V}, (\mathcal{V}, \mathcal{L}_\alpha)_{\alpha \in D}, (\mathcal{P}_\nu \mathcal{V}_{\nu}, \mathcal{P}_\nu \mathcal{V})(\nu = \infty) \in \text{Diff}(\mathbb{C}[y, y^{-1}], q, (t_\alpha)_{\alpha \in D})^{\text{par}}.
\]

The following is clear by the construction.

**Proposition 10.8** \( K \) induces an equivalence \( \text{Vect}^3(T, (s_p)_{p \in \mathbb{P}}) \simeq \text{Diff}(\mathbb{C}[y, y^{-1}], q, (t_\alpha)_{\alpha \in D})^{\text{par}}. \) Moreover, \( \deg(K(\mathcal{E}_n)) = \deg(\mathcal{E}_n) \) holds. As a result, the equivalence preserves the stable objects, semistable objects and polystable objects.

### 10.5 Filtrations and growth orders of norms

Let us consider the action of \( \text{Z}_2 \) on \( \mathcal{M}_\mathbb{C}^{\text{cov}} := \mathbb{C} \times \mathbb{R} \) by \( \mathbb{C}\mathfrak{e}(y,t) = (qy, t+1) \). It is extended to the action of \( \text{Z}_2 \) on \( \mathcal{M}_\mathbb{C}^{\text{cov}} := \mathbb{P} \times \mathbb{R} \). Let \( \mathcal{M}_\mathbb{C} \) and \( \mathcal{M}_\mathbb{Q} \) denote the quotient spaces of \( \mathcal{M}_\mathbb{C}^{\text{cov}} \) and \( \mathcal{M}_\mathbb{Q}^{\text{cov}} \) by the action of \( \text{Z}_2 \), respectively. For \( \nu = 0, \infty \), let \( H_\nu \) denote the image of \( H^{\text{cov}} := \{ \nu \} \times \mathbb{R} \rightarrow \mathcal{M}_\mathbb{Q} \).

Let \( \nu \) denote \( 0 \) or \( \infty \). Let \( \mathcal{U}_\nu \) be a neighbourhood of \( H_\nu \) in \( \mathcal{M}_\mathbb{Q} \). We set \( \mathcal{U}_\nu := \mathcal{U}_\nu \setminus H_\nu \). Let \( \mathcal{U}_\nu^{\text{cov}} \) denote the pull back of \( \mathcal{U}_\nu \) by \( \mathcal{M}_\mathbb{Q} \rightarrow \mathcal{M}_\mathbb{Q} \). Similarly, let \( \mathcal{U}_\nu^{\text{cov}} \) denote the pull back of \( \mathcal{U}_\nu \) by \( \mathcal{M}_\mathbb{Q} \rightarrow \mathcal{M}_\mathbb{Q} \). We set \( y_0 := y \) and \( y_\infty := y^{-1} \). We set \( q_0 := q \) and \( q_\infty := q^{-1} \).

#### 10.5.1 Equivalences

Let \( \text{LFM}(\mathcal{U}_\nu, H_\nu) \) denote the category of locally free \( \mathcal{O}_{\mathcal{U}_\nu}(\mathcal{H}_\nu) \)-modules. We obtain an equivalence \( \mathbb{F} : \text{Diff}(\mathcal{M}_\mathbb{C}^{\text{cov}}, q_0) \simeq \text{LFM}(\mathcal{U}_\nu, H_\nu) \) as in the formal case. (See \( \mathbb{F} \)) Hence, we obtain the following equivalence:

\[
K_{\nu} : \text{Vect}^{\text{AHN}}(T) \simeq \text{LFM}(\mathcal{U}_\nu, H_\nu).
\]

Let \( \text{LFM}(\mathcal{U}_\nu, H_\nu)^{\text{par}} \) denote the category of good filtered bundles over \( \mathcal{U}_\nu \). By the definition of good filtered bundles, we obtain the following equivalence:

\[
K_{\nu} : \text{Vect}^{\text{AHN}}(T)^{\text{par}} \simeq \text{LFM}(\mathcal{U}_\nu, H_\nu)^{\text{par}}.
\]

#### 10.5.2 Metrics and slope filtrations

Let \( \mathcal{E}_n \in \text{Vect}^{\text{AHN}}(T) \). We obtain \( \mathcal{M}_\nu := K_{\nu}(\mathcal{E}_n) \in \text{LFM}(\mathcal{U}_\nu, H_\nu) \). Let \( h_\nu \) be a Hermitian metric of \( \mathcal{M}_\nu \) such that the following holds.

- Let \( P \) be any point of \( H_\nu \). Let \( v \) be a frame of \( \mathcal{M}_\nu \) on a neighbourhood \( U_\nu \) of \( P \) in \( \mathcal{U}_\nu \). Let \( H(v) \) be the Hermitian matrix valued function on \( U_\nu \setminus H_\nu \) determined by \( H(v)_{i,j} = h_\nu(v_i, v_j) \). Then, there exists \( C > 1 \) and \( N > 0 \) such that \( C^{-1}|y_\nu|^N \leq H(v) \leq C|y_\nu|^{-N} \).

It is easy to construct such a Hermitian metric \( h_\nu \).

Let \( \mathcal{M}^{\text{cov}} \) be the pull back of \( \mathcal{M}_\nu \) by \( \mathcal{M}_\nu^{\text{cov}} \rightarrow \mathcal{U}_\nu \). Let \( h_\nu^{\text{cov}} \) be the metric of \( \mathcal{M}_\nu^{\text{cov}} \) induced by \( h \). Let \( Q \) be any point of \( T \). We take \( \alpha_0 \in \mathbb{C}^* \), which is mapped to \( Q \) by \( \pi : \mathbb{C}^* \rightarrow T = \mathbb{C}^* / \mathbb{Q}^* \). Set \( \alpha_\infty = (q_\infty^\nu/q_0^\nu)^{-1} \) for an appropriate \( n \in \mathbb{Z} \). We may assume that the half line \( 1_\nu := \{ (\alpha_\nu, t) \mid q(\alpha_\nu)t \geq 0 \} \) is contained in \( \mathcal{U}_\nu^{\text{cov}} \). For each \( s \in \mathcal{E}_n \), we obtain a flat section \( s_\nu \) of \( \mathcal{M}_\nu^{\text{cov}} \) along \( 1_\nu \). The following is easy to see.
Lemma 10.9 s is contained in $\mathcal{S}_\mu(E)|Q$ if and only if the following holds for any $\epsilon > 0$:

$$\log |s|_{K^{\infty}_\nu} = O \left( \frac{\mu}{2} \left| \log |q_\nu| \right| \left( t - \frac{\log |\alpha_\nu|}{\log |q_\nu|} \right)^2 + \epsilon \left( t - \frac{\log |\alpha_\nu|}{\log |q_\nu|} \right)^2 \right).$$

More strongly, for any $s \in \mathcal{S}_\mu \setminus \mathcal{S}_< \mu$, the following holds:

$$\log |s|_{K^{\infty}_\nu} = \frac{\mu}{2} \left| \log |q_\nu| \right| \left( t - \frac{\log |\alpha_\nu|}{\log |q_\nu|} \right)^2 + O \left( t \log |q_\nu| - \log |s_\nu| \right).$$

10.5.3 Refinement

Let $(E, \mathfrak{S}, \mathcal{F}) \in \text{Vect}_{\text{HNN}}(T)^{\text{Par}}$. We set $\mathcal{P}_\nu \mathcal{S}_\nu := \mathcal{K}_1(E, \mathfrak{S}, \mathcal{F})$. Suppose that $h_\nu$ is adapted to $\mathcal{P}_\nu \mathcal{S}_\nu$. For $s \in \mathcal{S}_\mu(E)|Q$, let $[s]$ denote the induced element of $\text{Gr}^\mu_\mu(E)|Q$.

Lemma 10.10 $[s] \in \mathcal{F}_b \text{Gr}^\mu_\mu(E)|Q$ if and only if the following holds for any $\epsilon > 0$:

$$\log |s|_{K^{\infty}_\nu} = O \left( \frac{\mu}{2} \left| \log |q_\nu| \right| \left( t - \frac{\log |\alpha_\nu|}{\log |q_\nu|} \right)^2 + (b + \epsilon) \left( t \log |q_\nu| - \log |s_\nu| \right) \right).$$

Let $W_k \mathcal{F}_b \text{Gr}^\mu_\mu(E)$ denote the inverse image of $W_k \text{Gr}^\nu_\nu \text{Gr}^\mu_\mu(E)$ by the surjection $\mathcal{F}_b \text{Gr}^\mu_\mu(E) \to \text{Gr}^\nu_\nu \text{Gr}^\mu_\mu(E)$.

Lemma 10.11 Suppose moreover that the norm estimate holds for $(\mathcal{P}_\nu \mathcal{S}_\nu, h)$. Then, $[s] \in W_k \text{Gr}^\nu_\nu \text{Gr}^\mu_\mu(E)$ if and only if the following holds:

$$\log |s|_{K^{\infty}_\nu} = O \left( \frac{\mu}{2} \left| \log |q_\nu| \right| \left( t - \frac{\log |\alpha_\nu|}{\log |q_\nu|} \right)^2 + b \left( t \log |q_\nu| - \log |s_\nu| \right) + \frac{k}{2} \log \left( t \log |q_\nu| - \log |s_\nu| \right) \right).$$

10.6 Filtered objects on elliptic curves associated to monopoles

10.6.1 Induced filtered objects on the elliptic curve

We use the notation in $\mathbf{K}$. Suppose that $|\lambda| \neq 1$. We set $T^\lambda := \mathbb{C}_0^*/(q^\lambda)^2$. Let $\pi : \mathcal{M}^\lambda \to T^\lambda$ denote the morphism induced by $\mathbb{C}_0^* \times \mathbb{R}_+ \to \mathbb{C}_0^*$. Let $Z \subseteq \mathcal{M}^\lambda$ be a finite subset. We set $\mathcal{D} := \pi(Z) \subseteq T^\lambda$. Note that the function $U$ on $\mathcal{M}^\lambda$ is independent of the choice of $(e_1, s_1)$, but $\mathcal{T}$ depends on $(e_1, s_1)$. Hence, we use the notation $\mathcal{T}(e_1, s_1)$ to emphasize the dependence on $(e_1, s_1)$. Similarly, we use the notation $\overline{\mathcal{M}}_{(e_1, s_1)}^\lambda$ to denote $\overline{\mathcal{M}}^\lambda$ in $\mathbf{K}$ to emphasize the dependence on $(e_1, s_1)$. The sets $H^\lambda$ are also denoted by $H^\lambda_{(e_1, s_1)}$. The number $t^\lambda$ is denoted by $t^\lambda(e_1, s_1)$. Let us denote $q^\lambda$ by $q^\lambda(e_1)$ to emphasize the dependence on $e_1$.

Let $(E, h, \nabla, \phi)$ be a meromorphic monopole on $\mathcal{M}^\lambda \setminus Z$. We obtain a good filtered bundle with Dirac type singularity $\mathcal{P}_+ E_{(e_1, s_1)}$ on $\overline{\mathcal{M}}_{(e_1, s_1)}^\lambda; H^\lambda_{(e_1, s_1)}; Z$. It is equivalent to a parabolic $q^\lambda(e_1)$-difference $\mathbb{C}[U, U^{-1}]$-module. Let $\mathcal{E}_{(e_1, s_1)}$ denote the corresponding locally free $\mathcal{O}_{\mathcal{T}^\lambda}(*\mathcal{D})$-module with a $q^\lambda(e_1)$-difference parabolic structure. (See Proposition 10.8) By rescaling the parabolic structure, we obtain a locally free $\mathcal{O}_{\mathcal{T}^\lambda}(*\mathcal{D})$-module with a $q^\lambda(e_1)$-difference parabolic structure $\mathcal{H}^{t^\lambda(e_1, s_1)}(\mathcal{E}_{(e_1, s_1)})$.

Theorem 10.12 $\mathcal{H}^{t^\lambda(e_1, s_1)}(\mathcal{E}_{(e_1, s_1)})$ is independent of the choice of $(e_1, s_1)$.

Proof Recall that the filtered object $\mathcal{E}_{(e_1, s_1)}^*$ consists of

- a locally free $\mathcal{O}_{\mathcal{T}^\lambda}(*\mathcal{D})$-module $\mathcal{E}_{(e_1, s_1)}$,
• a tuple \((s_P(e_1, s_1), \mathcal{L}_P(e_1, s_1))\) for \(P \in D\),
• filtrations \(\tilde{\mathcal{F}}_{e_1, s_1}\) on \(\tilde{E}_{(e_1, s_1)}\),
• filtrations \(\mathcal{F}_{e_1, s_1}\) on \(\text{Gr}^{\pm}(e_1, s_1)(\tilde{E}_{(e_1, s_1)})\).

(See §10.3) We have the isomorphism \(f_{(e_1, s_1)}: \mathcal{M}(\mathcal{M}_q(e_1)) \sim \mathcal{M}_q(e_1)\) induced by
\[
(U, t(e_1, s_1)) \mapsto (U, t(e_1, s_1)/t^\lambda(e_1, s_1)).
\]

Note that \(\tilde{E}_{(e_1, s_1)}\) depend only on the mini-holomorphic bundle \((E, \overline{U}_E)\) on \(\mathcal{M}(\mathcal{M}_q(e_1)) \sim \mathcal{M}_q(f_{(e_1, s_1)}(Z))\) underlying the monopole \((E, h, \nabla, \phi)\). Hence, they are independent of \((e_1, s_1)\). According to Lemma 3.13 and Lemma 3.14, the filtrations \((\tilde{\mathcal{F}}_{e_1, s_1}(), \mathcal{F}_{e_1, s_1}(t^\lambda))\) are independent of \((e_1, s_1)\). Therefore, we obtain that the sequence \((s(e_1, s_1)e_1\lambda)^{(t^\lambda(e_1, s_1))}\) and \(\mathcal{L}_P(e_1, s_1)(t^\lambda)\) are independent of \((e_1, s_1)\). According to Lemma 10.9 and Lemma 10.10, the filtrations \(\tilde{\mathcal{F}}_{e_1, s_1}(), \mathcal{F}_{e_1, s_1}(t^\lambda)\) are characterized by the growth order of the norms of the \(\partial_h\)-flat sections with respect to \(h\). Then, Lemma 5.12 and Lemma 3.13 imply that the filtrations \((\tilde{\mathcal{F}}_{e_1, s_1}(t^\lambda), \mathcal{F}_{e_1, s_1}(t)\) are independent of \((e_1, s_1)\). Thus, we obtain Theorem 10.12.

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