Research Article

On the Convergence Ball and Error Analysis of the Modified Secant Method

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We aim to study the convergence properties of a modification of secant iteration methods. We present a new local convergence theorem for the modified secant method, where the derivative of the nonlinear operator satisfies Lipchitz condition. We introduce the convergence ball and error estimate of the modified secant method, respectively. For that, we use a technique based on Fibonacci series. At last, some numerical examples are given.

1. Introduction

A large number of nonlinear dynamic systems and scientific engineering problems can be concluded to the form of nonlinear equation

\[ f(x) = 0, \] (1)

where \( f \) is a nonlinear operator defined on a convex subset \( D \) of a complex dimension space \( C \). Hence, finding the roots of the nonlinear (1) is widely required in both mathematical physics and nonlinear dynamic system. Iterative methods are considerable methods. There are many iterative methods for solving the nonlinear equation.

Secant method [1, 2], which uses divided differences instead of the first derivative of the nonlinear operator, is one of the most famous iterative methods for solving the nonlinear equation. Secant method reads as follows:

\[ x_{n+1} = x_n - f(x_n)(x_n - x_{n-1}) \]
\[ n \geq 0 \] \( (x_0, x_{-1} \in D) \) (2)

where the operator \([x, y; f]\) is called a divided difference of first-order for the operator \( f \) on the points \( x \) and \( y(x \neq y) \) if the following equality holds:

\[ [x, y; f](x - y) = f(x) - f(y). \] (3)

Due to the well performance of the secant method, secant method and secant-like methods have been widely studied by many authors [3–11]. The authors [12] proposed a new method for solving the nonlinear equation.

Convergence ball is a very important issue in the study of the iterative procedures. When nonlinear operator \( f \) is first-order differentiable convex subset \( D \) can be open or closed, suppose \( x_* \) is the root of the equation \( f(x) = 0 \), an open area \( B(x_*, R) \) is called the convergence ball of the iterative algorithm. Authors [13–17] have discussed the convergence of the iterative methods using a convergence ball \( B(x_*, R) \) with center \( x_* \) and radius \( R \). For example, Ren and Wu [15] discussed the convergence of the secant method under Hölder continuous divided differences using a convergence ball.

In this study, we consider the modified secant method with the below form based on [12]

\[ x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{3f(x_n) + f(x_{n-1}) - 4f((x_n + x_{n-1})/2)} \]
\[ n \geq 0 \] (4)

and we will establish the convergence ball and give the error analysis of the modified secant method for the nonlinear equation.
2. Convergence Ball Study

**Theorem 1.** Suppose $x_*$ is the root of the equation $f(x) = 0$ and $f'(x) \neq 0$. $f$ is first-order differentiable, where the derivative of $f$ satisfies the Lipschitz condition: $|f'(x_*)^{-1}(f'(x) - f'(y))| \leq K|x - y|$ for all $x, y \in D$ and $K > 0$. Then, the sequence $\{x_n\}$ generated by the modified secant method (4), starting from any two initial points $x_0, x_1 \in B(x_*, R)$, converges to the solution $x_*$. $x_*$ is the unique solution in $B(x_*, 2/K)$, where $B(x_*, R) \subset B(x_*, 2/K)$. Moreover, the following error estimate holds:

$$|x_n - x_*| \leq R \left( \frac{|x_n - x_0|}{R} \right)^{F_n} \left( \frac{|x_n - x_{n-1}|}{R} \right)^{F_{n-1}}.$$  

$$x_n - x_{n+1} = x_n - x_{n-1} + \frac{f(x_n)(x_n - x_{n-1})}{3f(x_n) + f(x_{n-1}) - 4f\left(\frac{x_n + x_{n-1}}{2}\right)}$$

$$= x_n - x_{n-1} + \frac{(f(x_n) - f(x_*))(x_n - x_{n-1})}{3f(x_n, x_{n+1})/2 + f(x_{n-1}, x_{n+1})/2}$$

$$= x_n - x_{n-1} + \frac{f[x_n, x_*(x_n - x_*)](x_n - x_{n-1})}{2f[x_n, x_*](x_n - x_*)}$$

$$= x_n - x_{n-1} + \frac{2f[x_n, x_*](x_n - x_*)}{(x_n - x_{n-1})}$$

$$= (x_n - x_*)(1 - \frac{2f[x_n, x_*]}{3f[x_n, x_{n+1}]/2 - f[x_{n-1}, x_{n+1}]/2})$$

$$= (x_n - x_*)\left(3\frac{f[x_n, x_{n+1}]/2 - f[x_{n-1}, x_{n+1}]/2}{2f[x_n, x_*]}\right)$$

$$- f'\left(tx_{n-1} + (1-t)\frac{x_n + x_{n-1}}{2}\right)$$

$$- 2f'\left(tx_n + (1-t)x_*\right)\right|,$$

$$= \left[3f[x_n, x_{n+1}]/2 - f[x_{n-1}, x_{n+1}]/2\right]$$

$$- f\left(x_{n-1}, \frac{x_n + x_{n-1}}{2}\right)$$

$$- 3f\left(x_{n-1}, x_{n+1}/2\right) + f\left(x_{n-1}, \frac{x_n + x_{n-1}}{2}\right)$$

$$= \left[3f[x_n, x_{n+1}]/2 - f[x_{n-1}, x_{n+1}]/2\right]$$

$$- f\left(x_{n-1}, \frac{x_n + x_{n-1}}{2}\right)$$

$$+ f\left(tx_{n-1} + (1-t)\frac{x_n + x_{n-1}}{2}\right)\right|.$$
Using Lipschitz condition with the above (9) and (10), we have
\[
\left| f'(x_s)^{-1} \left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right) - (tx_n + (1 - t)x_s) \right| dt \\
\leq K \left| x_n - x_s + x_s - x_{n-1} \right| \left( 3 \right) \\
+ 2K \left| x_n - x_s + (x_n - x_s) \right| \leq K \left| x_s - x_n \right| \\
+ K \left| x_s - x_{n-1} \right| ,
\]
(11)
and
\[
\left| 2 - f'(x_s)^{-1} \left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right) \right| \\
\leq \int_0^1 \left( 2K \left| x_s - \left( tx_n + (1 - t) \frac{x_n + x_{n-1}}{2} \right) \right| + K \left| \left( tx_{n-1} + (1 - t) \frac{x_n + x_{n-1}}{2} \right) - \left( tx_n + (1 - t) \frac{x_n + x_{n-1}}{2} \right) \right| \right) dt \\
\leq 2K \int_0^1 \left| 1 + \frac{t}{2} (x_s - x_n) + \frac{1 - t}{2} (x_s - x_{n-1}) \right| dt + K \int_0^1 \left| t(x_{n-1} - x_n) \right| dt \leq 2K \left( \frac{3}{4} \left| x_s - x_n \right| + \frac{1}{4} \left| x_s - x_{n-1} \right| \right) \\
+ K \left| (x_s - x_n) + (x_{n-1} - x_s) \right| \leq 2K \left| x_s - x_n \right| + K \left| x_s - x_{n-1} \right| .
\]
(12)
We divide above inequality (12) number 2, so
\[
\left| 1 - f'(x_s)^{-1} \left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right) \right| \\
\leq f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \leq K \left| x_s - x_n \right| + \frac{K}{2} \left| x_s - x_{n-1} \right| ,
\]
(13)
According to the definition of R and \( x_n, x_{n-1} \in B(x_s, R) \), we get \( K \left| x_s - x_n \right| + (K/2)\left| x_s - x_{n-1} \right| < \frac{3}{5} < 1 \), by Banach Lemma, so \( 3f \left[ x_n, (x_n + x_{n-1})/2 \right] - f \left[ x_{n-1}, (x_n + x_{n-1})/2 \right] \) is reversible and also
\[
\left| 2 - f'(x_s) \right| \\
\leq \left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right)^{-1} \\
\leq \frac{1}{1 - K \left| x_s - x_n \right| - (K/2) \left| x_s - x_{n-1} \right|} .
\]
(14)
Dividing (14) inequality number 2, we can get
\[
\left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right)^{-1} \\
\leq \frac{1}{2 - 2K \left| x_s - x_n \right| - K \left| x_s - x_{n-1} \right|} ,
\]
and with (11) and (15), we get following estimate formula:
\[
\left| f'(x_s)^{-1} \left( 3f \left[ x_n, \frac{x_n + x_{n-1}}{2} \right] - f \left[ x_{n-1}, \frac{x_n + x_{n-1}}{2} \right] \right) \right| \\
\leq \frac{K \left| x_s - x_n \right| + K \left| x_s - x_{n-1} \right|}{2 - 2K \left| x_s - x_n \right| - K \left| x_s - x_{n-1} \right|} .
\]
(16)
and with (6) and (16) and $x_n, x_{n-1} \in B(x_*, R)$, we get following estimate formula:

$$
|x_* - x_{n+1}| \leq |x_* - x_n| \left| \frac{3 f \left[ x_n (x_n + x_{n-1}) / 2 \right] - f \left[ x_{n-1}, (x_n + x_{n-1}) / 2 \right] - 2 f \left[ x_n, x_* \right] / 2}{3 f \left[ x_n (x_n + x_{n-1}) / 2 \right] - f \left[ x_{n-1}, (x_n + x_{n-1}) / 2 \right]} \right| \leq |x_* - x_n| \left| \frac{KR + KR}{2 - 2KR - KR} \right| = |x_* - x_n| < R.
$$

From the definition of $\rho_n$ and above formulation, we can get

$$
|x_* - x_n| \leq R \left( \frac{|x_* - x_0|}{R} \right)^{F_n} \left( \frac{|x_* - x_{n-1}|}{R} \right)^{F_{n-1}} = \frac{2}{5K} \left( \frac{5K}{2} \right)^{F_n} \left( \frac{5K}{2} \right)^{F_{n-1}} (n \geq 1).
$$

At last, we show the uniqueness of the solution in the area $B(x_*, 2/K)$. Assume that there exists another solution $y_* \in B(x_*, 2/K)$, $y_* \neq x_*$. We consider the operator

$$
A = \int_0^1 f'(tx_* + (1-t)y_*) \, dt.
$$

Since $A[y_* - y_1] = f(y_1) - f(x_1) = 0$, if operator $A$ is invertible, then $y_* = x_*$. Indeed from (24), we have

$$
\left| 1 - f' (x_*) \right|^{-1} A
$$

$$
= \left| f' (x_*) \right|^{-1} \int_0^1 \left( f' (x_*) - f' (tx_* + (1-t)y_*) \right) \, dt
$$

$$
\leq K \int_0^1 |(1-t)(x_* - y_*)| \, dt = \frac{K}{2} |x_* - y_*| < 1.
$$

Then, by Banach lemma, we can tell that operator $A$ is invertible. From the definition of radius $R$, it is easy to verify that the ball $B(x_*, 2/K)$ is bigger than $B(x_*, R)$.

That completes the proof of Theorem 1. □

3. Numerical Examples

In this section, the convergence ball results were applied to numerical examples.

**Example 1.** Let us consider

$$
f(x) = x^2 - 1, \quad x \in [0, 2].
$$

It is obviously that $f'(f(x)) = 2x$, $f(x) = 0$ has a root $x_1 = 1$ and $f'(x_*) = 2$. It is easy to know

$$
\left| f' (x_*) \right|^{-1} \left( f' (x) - f' (y) \right) \leq |x - y|.
$$

According to Theorem 1, we can obtain the fact that the radius of the convergence ball of the modified secant method is $R = 2/5 = 0.4$ at least.
Example 2. Let us consider the following numerical problem which has been studied in [4, 11, 13]:

\[ f(x) = e^x - 1, \]
\[ D = [-1, 1]. \] (28)

\[ f'(x) = e^x, x_0 = 0, \text{ and } f'(x_0) = 1. \]

We know \(|e^x - e^y| \leq e|y - x|\); hence,

\[ \left| f'(x_0) \right|^{-1} \left| f'(x) - f'(y) \right| \leq e \left| |x - y| \right|. \] (29)

So \(K = e\) in this problem.

By Theorem 1, we can obtain the fact that the radius of the convergence ball of the modified secant method is \(R = 2/5K = 7/5 \approx 1.4\) at least.

Example 3. Let us consider the nonlinear equation

\[ f'(x) = \frac{2}{5} \sin x + x, \quad x \in [0, 2]. \] (30)

Here, \(f'(x) = (2/5) \cos x + 1, x_0 = 0, \text{ and } f'(x_0) = 7/5. \)

We know that \(|\cos x - \cos y| \leq |x - y|\); then it is obvious that

\[ \left| f'(x_0) \right|^{-1} \left| f'(x) - f'(y) \right| \leq \frac{2}{7} \left| x - y \right|. \] (31)

In this case, the radius of the convergence ball of the modified secant method is \(R = 2/5K = 7/5 \approx 1.4\) at least, according to Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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