Representation Theory of Compact Metric Spaces and Computational Complexity of Continuous Data\footnote{Supported by the National Research Foundation of Korea (grant NRF-2017R1E1A1A03071032) and the International Research \& Development Program of the Korean Ministry of Science and ICT (grant NRF-2016K1A3A7A03950702). We thank Florian Steinberg for helpful discussions. Example\textsuperscript{17} is due to Gleb Pogudin.}

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\textbf{Abstract.} Choosing an encoding over binary strings for input/output to/by a Turing Machine is usually straightforward and/or inessential for discrete data (like graphs), but delicate — heavily affecting computability and even more computational complexity — already regarding real numbers, not to mention more advanced (e.g. Sobolev) spaces. For a general theory of computational complexity over continuous data we introduce and justify ‘quantitative admissibility’ as requirement for sensible encodings of arbitrary compact metric spaces, a refinement of qualitative ‘admissibility’ due to [Kreitz& Weihrauch’85]:

An \textit{admissible} representation of a $T_0$ space $X$ is a (i) \textit{continuous} partial surjective mapping from the Cantor space of infinite binary sequences which is (ii) maximal w.r.t. \textit{continuous} reduction. By the Kreitz-Weihrauch (aka “Main”) Theorem of computability theory for continuous data, a function $f : X \to Y$ with admissible representations of co/domain is \textit{continuous} if it admits \textit{continuous} a code-translating mapping on Cantor space, a so-called \textit{realizer}. We require a \textit{linearly/polynomially} admissible representation of a compact metric space $(X,d)$ to have (i) asymptotically optimal modulus of continuity, namely close to the entropy of $X$, and (ii) be maximal w.r.t. reduction having optimal modulus of continuity in a similar sense.

Careful constructions show the category of such representations to be Cartesian closed, and non-empty: every compact $(X,d)$ admits a linearly-admissible representation. Moreover such representations give rise to a tight quantitative correspondence between the modulus of continuity of a function $f : X \to Y$ on the one hand and on the other hand that of its realizer: a “Main Theorem” of computational \textit{complexity}.

This suggests (how) to take into account the entropies of the spaces under consideration when measuring/defining algorithmic cost over continuous data; and to follow [Kawamura&Cook’12] considering and classifying \textit{generalized} representations with domains ‘larger’ than Cantor space for (e.g. function) spaces of exponential entropy.
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1 Motivation, Background, and Summary of Contribution

Arguably most computational problems in Science and Engineering are concerned with continuous rather than with discrete data [BC06, Bra13]. Here the Theory of Computability exhibits new topological — and continuous complexity theory furthermore metric — aspects that trivialize, and are thus invisible, in the discrete realm. In particular input and output require rather careful a choice of the underlying encoding as sequences of bits to be read, processed, and written by a Turing machine. For example,

- encoding real numbers via their binary expansion \( x = \sum_{n=0}^{\infty} b_n 2^{-n-1} \), and thus operating on the Cantor space of infinite binary sequences \( b = (b_0, b_1, \ldots, b_n, \ldots) \), renders arithmetic averaging \([0; 1]^2 \ni (x, y) \mapsto (x+y)/2 \in [0; 1]\) discontinuous and uncomputable [Tur37, Wei00, Exercise 7.2.7].
- Encoding real numbers via a sequence of (numerator and denominator, in binary, of) rational approximations up to absolute error \( \leq 2^{-n} \) does render averaging computable [Wei00, Theorem 4.3.2], but admits no worst-case bound on computational cost [Wei00, Examples 7.2.1+7.2.3].
- The dyadic representation encodes \( x \in [0; 1] \) as any integer sequence \( a_n \in \{0, \ldots, 2^n\} \) (in binary without leading 0) s.t. \( |x - a_n/2^n| \leq 2^{-n} \); and similarly encode \( y \in [0; 1] \) as \( (b_n) \). Then the an integer \( c_n \) closest to \( (a_{n+1} + b_{n+1})/4 \) satisfies \( |(x+y)/2 - c_n/2^n| \leq 2^{-n} \), and is easily computed — but requires first reading/writing \( a_m, b_m, c_m \) for all \( m < n \); a total of \( \Theta(n^2) \) bits.
- Encoding \( x \in [0; 1] \) as signed binary expansion \( x = \sum_{n \geq 0} (2b_n + b_{n+1} - 1) \cdot 2^{-n-1} \) with \( b_n \in \{0, 1\} \) s.t. \( 2b_n + b_{n+1} \in \{-1, 0, 1\} \), and similarly \( y \), renders averaging computable in linear time \( \mathcal{O}(n) \) [Wei00, Theorem 7.3.1].

The signed binary expansion is thus asymptotically ‘optimal’ up to a constant factor, the dyadic representation is still optimal up to a quadratic polynomial, the rational representation is ‘unbounded’, and the binary expansion is unsuitable.
But how to choose and quantitatively assess complexity-theoretically appropriate encodings of spaces $X$ other than $[0; 1]$, such as those common in the analysis and solution theory of PDEs [Tri06]?

The present work refines the existing classification of encodings from the computability theory of general continuous data while guided by and generalizing the well-established theory of computational complexity over real numbers. There, the binary expansion is known to violate the technical condition of admissibility; and we introduce and investigate quantitative strengthenings linear admissibility (satisfied by the signed binary, but neither by the dyadic nor by the rational representation) and polynomial admissibility (satisfied by the signed binary and by the dyadic, but not by the rational representation).

1.1 Computability over Continuous Data, Complexity in Real Computation

Here we review established notions and properties of computability and complexity theory over more general abstract spaces; as guideline to the sensible complexity theory of more general abstract spaces developed in the sequel.

**Definition 1.** A Type-2 Machine $\mathcal{M}$ is a Turing machine with dedicated one-way output tape and infinite read-only input tape [Wei00] Definitions 2.1.1+2.1.2]

Naturally operating on infinite sequences of bits, $\mathcal{M}$ computes a partial function $F : \subseteq C \rightarrow C$ on the Cantor space $C = \{0, 1\}^\omega$ of infinite binary sequences if, when run with any input $b \in \text{dom}(F)$ on its tape, $\mathcal{M}$ keeps printing the symbols of $F(b)$ one by one; while its behaviour on other inputs may be arbitrary.

$\mathcal{M}$ computes $F$ in time $t : \mathbb{N} \rightarrow \mathbb{N}$ if it prints the $n$-th symbol of $F(b)$ after at most $t(n)$ steps regardless of $b \in \text{dom}(F)$.

For a fixed predicate $\varphi : C \rightarrow \{0, 1\}$, a Type-2 Machine with oracle $\varphi$ can repeatedly query $\varphi(\bar{z}) \in \{0, 1\}$ for finite strings $\bar{z}$ during its computation.

Concerning topological spaces $X$ of continuum cardinality beyond real numbers, the Type-2 Computability Theory systematically studies and compares encodings, formalized as follows [Wei00, §3]:

**Definition 2.**

a) A representation of a set $X$ is a partial surjective mapping $\xi : \subseteq C := \{0, 1\}^\mathbb{N} \rightarrow X$ on the Cantor space of infinite streams of bits.

b) The product of representations $\xi : \subseteq C \rightarrow X$ and $\upsilon : \subseteq C \rightarrow Y$ is $\xi \times \upsilon : \subseteq C \ni (b_0, b_1, \ldots, b_n, \ldots) \mapsto (\xi(b_0, b_2, b_4, \ldots), \upsilon(b_1, b_3, \ldots)) \in X \times Y$.

c) For representations $\xi : \subseteq C \rightarrow X$ and $\upsilon : \subseteq C \rightarrow Y$, a $(\xi, \upsilon)$-realizer of a function $f : X \rightarrow Y$ is a partial function $F : \text{dom}(\xi) \rightarrow \text{dom}(\upsilon) \subseteq C$ on Cantor space such that $f \circ \xi = \upsilon \circ F$ holds; see Figure 7.

d) $(\xi, \upsilon)$-computing $f$ means to compute some $(\xi, \upsilon)$-realizer $F$ of $f$ in the sense of Definition 1.

e) A reduction from representation $\xi \rightarrow X$ to $\xi' \rightarrow X$ is a $(\xi, \xi')$-realizer of the identity $\text{id} : X \rightarrow X$; that is, a partial function $F : \text{dom}(\xi) \rightarrow \text{dom}(\xi')$ on Cantor space such that $\xi = \xi' \circ F$. We write $\xi \preceq_T \xi'$ to express that a continuous reduction exists, where $C$ is equipped with the Cantor space metric $d_C(b, \bar{a}) = 2^{-\min\{n; b_n \neq a_n\}}$. 


Examples [13, 14, 15, and 16] below formalize the above binary, rational, dyadic, and signed encodings of the reals as representations $\beta, \rho, \delta,$ and $\sigma,$ respectively. It is well-known that the latter three, but not $\beta,$ are pairwise continuously reducible [Wei00, Theorem 7.2.5] and thus equivalent with respect to the notions of computability they induce on reals; but only $\delta$ and $\sigma$ admit mutual reductions with polynomial modulus of continuity.

Remark 3 Recall [Wei03, §6] that a modulus of continuity of a function $f : X \to Y$ between compact metric spaces $(X,d)$ and $(Y,e)$ is a non-decreasing mapping $\mu : \mathbb{N} \to \mathbb{N}$ such that $d(x,x') \leq 2^{-\mu(n)}$ implies $e(f(x),f(x')) \leq 2^{-n}$. Every uniformly continuous function has a (pointwise minimal) modulus of continuity; Lipschitz-continuity corresponds to moduli $\mu(n) = n + \mathcal{O}(1),$ and Hölder-continuity to linear moduli $\mu(n) = \mathcal{O}(n)$; see Fact 18c).

According to the sometimes so-called Main Theorem of Computable Analysis, a real function $f$ is continuous iff $f$ is computable by some oracle Type-2 Machine w.r.t. $\rho$ and/or $\delta$ and/or $\sigma$ [Wei00, Definitions 2.1.1+2.1.2]. For spaces beyond the reals, Kreitz and Weihrauch [KW85] have identified admissibility as central condition on 'sensible' representations in that these make the Main Theorem generalize [Wei00, Theorem 3.2.11]:

Fact 4 Let $X$ and $Y$ denote second-countable $T_0$ spaces equipped with admissible representations $\xi$ and $\upsilon,$ respectively. A function $f : X \to Y$ is continuous iff it admits a continuous $(\xi,\upsilon)$-realizer.

Recall [Wei00 Theorem 3.2.9.2] that a representation $\xi : C \to X$ is admissible iff (i) it is continuous and (ii) every continuous representation $\zeta : C \to X$ satisfies $\zeta \preceq_T \xi.$

Computability-theoretically 'sensible' representations $\xi$ are thus those maximal, among the continuous ones, with respect to continuous reducibility. The present work refines these considerations and notions from qualitative computability to complexity. For representations satisfying our proposed strengthening of admissibility, Theorem 27 below asymptotically optimally translates quantitative continuity between functions $f : X \to Y$ and their realizers $F.$ Such translations heavily depend on the co/domains $X,Y$ under consideration:

Example 5 a) A function $f : [0; 1] \to [0; 1]$ has polynomial modulus of continuity iff it has a $(\delta,\delta)$-realizer w.r.t. the dyadic and/or signed binary expansion (itself having a polynomial modulus of continuity) iff $f$ has a $(\sigma,\sigma)$-realizer of polynomial modulus of continuity; cmp. [Ko91, Theorem 2.19], [Wei00, Exercise 7.1.7], [KSZ16a, Theorem 14].

b) For the compact space $[0; 1]_1 := \text{Lip}_1([0; 1],[0; 1])$ of non-expansive (=Lipschitz-continuous with constant 1) $f : [0; 1] \to [0; 1]$ equipped with the supremum

\footnote{From the general logical perspective this constitutes a Skolemization as canonical quantitative refinement of qualitative continuity [Koh08]. We consider a modulus as an integer function due to its close connection with asymptotic computational cost: Fact 21a+b). For the continuous conception of a modulus common in Analysis see Lemma 20 below.}
norm, the application functional \([0; 1]^2 \times [0; 1] \ni (f, r) \mapsto f(r) \in [0; 1]\) is non-expansive, yet admits no realizer with sub-exponential modulus of continuity for any product representation (Definition 2b) of \([0; 1]^2 \times [0; 1]\) \cite{KMRZ15, Example 6g}.

![Diagram](https://via.placeholder.com/150)

**Fig. 1.** Realizers of \(f : X \to Y\) with respect to different representations.

It is naturally desirable that, like in the discrete setting, every computation of a total function \(f : X \to Y\) admit some (possibly fast growing, but pointwise finite) worst-case complexity bound \(t = t(n)\); however already for the real unit interval \(X = [0; 1] = Y\) this requires the representation \(\xi\) of \(X\) to be chosen with care, avoiding both binary \(\beta\) and rational \(\rho\) \cite[Example 7.2.3]{Wei00}. Specifically, one wants its domain \(\text{dom}(\xi) \subseteq C\) to be compact \cite{Sch95,Wei03,Sch04}; cmp. Example 14.

**Fact 6** a) Suppose Type-2 Machine \(M\) (with/out some fixed oracle) computes a function \(F : C \to C\) with compact \(\text{dom}(F)\). Then \(M\) admits a bound \(t(n) \in \mathbb{N} = \{0, 1, 2, \ldots\}\) on the number of steps it takes to print the first \(n\) output symbols of the value \(F(b)\) regardless of the argument \(b \in \text{dom}(F)\); see \cite[Exercise 7.1.2]{Wei00}.

b) If \(F : C \to C\) is computable (with/out some fixed oracle) in time \(t\), then \(n \mapsto t(n)\) constitutes a modulus of continuity of \(F\).

c) Conversely to every continuous \(F : C \to C\) with modulus of continuity \(\mu\) there exists an oracle \(\varphi\) and Type-2 Machine \(M^\varphi\) computing \(F\) in time \(O(n + \mu(n))\); cmp. \cite[Theorem 2.3.7.2]{Wei00}.

Item b) expresses that, on Cantor space, quantitative continuity basically coincides with time-bounded relativized computability. Item a) requires a continuous representation \(\xi\) to map closed subsets of Cantor space to closed subsets of \(X\) — hence one cannot expect \(\xi\) to be an open mapping, as popularly posited in computability \cite[Lemma 3.2.5.(2+3+5)]{Wei00} and ingredient to (the proof of) the aforementioned Main Theorem; cmp. \cite[Theorem 3.2.9]{Wei00}.

\footnote{In our model, query tape and head are not altered by an oracle query.}
1.2 Summary of Contribution

We establish a quantitative refinement of the Main Theorem for arbitrary compact metric spaces, tightly relating moduli of continuity of functions \( f : X \to Y \) to those of their realizers \( F \) relative to the entropies of co-domains \( X \) and \( Y \):

Recall [KT59], [Wei03], §6 that the \( \text{entropy}\) of a compact metric space \((X, d)\) is the mapping \( \eta : \mathbb{N} \to \mathbb{N} \) such that \( X \) can by covered by \( 2^{\eta(n)} \), but not by \( 2^{\eta(n)-1} \), closed balls of radius \( 2^{-n} \). The real unit interval \([0; 1]\) has entropy \( \eta(n) = n - 1 \); whereas \([0; 1]' = \text{Lip}_1([0; 1], \{0; 1\})\) has entropy \( \eta_1'(n) = \Theta(2^n) \); see Example 13 for further spaces.

**Remark 7** By Example 12f), for any modulus of continuity \( \kappa \) of a representation \( \xi : \subseteq C \to X \), the space \( X \) has entropy \( \eta \leq \kappa \); and we require a linearly admissible \( \xi \) to (i) have modulus \( \kappa(n) \) of \( \eta(n+O(1)) \) almost optimal: permitting asymptotic ‘slack’ a constant factor in value and constant shift in argument.

Moreover a linearly admissible \( \xi \) must satisfy that, (ii) to every representation \( \xi : \subseteq C \to X \) with modulus \( \nu \) there exists a mapping \( F : \text{dom}(\xi) \to \text{dom}(\xi) \) with modulus of continuity \( \mu \) such that \( \xi = \xi \circ F \) and \( (\mu \circ \kappa)(n) \leq \nu(O(n)) \).

This new Condition (ii) strengthens previous qualitative continuous reducibility “\( \zeta \preceq_T \xi^C \)” to what we call linear metric reducibility “\( \zeta \preceq_\text{linear} \xi \)”\(^1\), requiring a \( (\zeta, \xi)\)-realizer \( F \) with almost optimal modulus of continuity: For functions \( \varphi : X \to Y \) and \( \psi : Y \to Z \) with respective moduli of continuity \( \mu \) and \( \kappa \), their composition \( \psi \circ \varphi : X \to Z \) is easily seen to have modulus of continuity \( \mu \circ \kappa \).

Abbreviating \( \text{lin}(\nu) = O\left(\nu(O(n))\right) \) and with the semi-inverse \( \nu^{-1}(n) := \min\{m : \nu(m) \geq n\} \), our results are summarized as follows:

a) Let \((X, d)\) and \((Y, e)\) denote infinite compact metric spaces with entropies \( \eta \) and \( \theta \) and equipped with linearly admissible representations \( \xi \) and \( \nu \). If \( f : X \to Y \) has modulus of continuity \( \mu \), it admits a realizer \( F \) with modulus of continuity \( \text{lin}(\eta) \circ \mu \circ \text{lin}(\theta^{-1}) \). Conversely if \( F \) is a realizer of \( f \) with modulus \( \nu \), then \( f \) has modulus \( \text{lin}(\eta^{-1}) \circ \nu \circ \text{lin}(\theta) \).

b) Every compact metric space \((X, d)\) admits a linearly admissible representation \( \xi \). For ‘popular’ spaces \( X, Y \) having linear/polynomial entropy \( \eta, \theta \), the moduli of continuity of functions and their realizers are thus linearly/polynomially related; yet according to Examples 13l+e) there exist both spaces of entropy growing arbitrarily slow and arbitrarily fast. Still, estimates (a) are asymptotically tight in a sense explained in Remark 28.

c) The category of quantitatively admissible representations is Cartesian closed: Given linearly admissible representations for spaces \( X_j \) \((j \in \mathbb{N})\), we construct one for the product space \( \prod_j X_j \) w.r.t. the ‘Hilbert Cube’ metric \( d_H = \sup_j d_j/2^j \) (cmp. Example 13i) such that the canonical projections \( (x_0, \ldots x_j, \ldots) \mapsto x_j \) and embeddings \( x_j \mapsto (x_0, \ldots x_j, \ldots) \) admit realizers.

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1From the general logical perspective, entropy constitutes a Skolemization as canonical quantitative refinement of qualitative precompactness [Koh08].
with optimal modulus of continuity in the sense of (a). For the compact space $\mathcal{K}(X)$ of non-empty compact subsets w.r.t. Hausdorff Distance we construct a canonical polynomially admissible representation; and for the compact space $X'_\mu := C_\mu(X, [0; 1])$ of functions $f : X \to [0; 1]$ having modulus of continuity $\mu$, equipped with the supremum norm, one such that the application functional $X'_\mu \times X \ni (f, x) \mapsto f(x)$ has a realizer with optimal modulus of continuity in the sense of (a).

See Theorems 27 and 29 and 30 and 31 for the precise statements. Example 16 verifies that the signed digit expansion of the interval $[0; 1]$ is linearly admissible; hence our quantitative “Main Theorem” (a) indeed generalizes the real case as well as quantitatively refining the qualitative Fact 4.

Note in (a) the typical form of transition maps, similar for instance to change-of-basis in linear algebra or change-of-chart in differential geometry. It thus captures the information-theoretic ‘external’ influence of the co/domain according to Fact 6b), and allows to separate that from the ‘inherent’ recursion-theoretic complexity of a computational problem: Informally speaking, an algorithm operating on continuous data is not to be ‘blamed’ for incurring large cost if the underlying domain has large entropy, as in Example 5b); see Remark 36 below.

1.3 Previous and Related Work, Current Ideas and Methods

Computability theory of real numbers was initiated by Alan Turing (1936), then generalized to real functions by Grzegorczyk (1957), to Euclidean subsets by Kreisel and Lacombe (1957), to Banach Spaces by Pour-El and Richards (1989), to topological $T_0$ spaces by Weihrauch [Wei00, §3.2], and furthermore to so-called QCB spaces by Schröder [Sch02]: These works had introduced the fundamental notions which in turn have enabled the abundance of investigations that constitute the contemporary computability theory over continuous data.

Computational (bit-)complexity theory of real numbers and functions was introduced by Harvey Friedman and Ker-I Ko [KF82,Bra05]. It differs from the discrete setting for instance by measuring computational cost in dependence on the output approximation error $2^{-n}$. Some effort, a careful choice of representation, and the hypothesis of a compact domain is needed to prove that any total computable real function actually admits a finite runtime bound depending only on $n$ [Sch04]. It took even more effort, as well as guidance from discrete Implicit Complexity Theory [KC96,Lam06], to proceed from this Complexity Theory of real functions [Ko91] to a suitable definition of computational complexity for real operators [KC12].

The latter involves a modified model of computation discussed in Subsection 5.1. Again, only the notions introduced in the above works have enabled the present plethora of investigations and rigorous classifications of common numerical problems, such as [Ko91,IRW02,BBY06,BBY07,BGP11,KMRZ15,KSZ17]. And their sensible further generalization to abstract function (e.g. Sobolev) spaces common in analysis is still in under development and debate [KSZ16b,Ste17].
Indeed the real co/domain is special in that it has linear entropy; hence the impact of co/domain on the computational complexity of problems had been hidden before our quantitative “Main Theorem” (a).

In [KSZ16a] we had picked up from [Wei03] towards a general theory of computational complexity for compact metric spaces \((X,d)\): exhibiting its entropy \(\eta\) as a lower bound on the bit-cost of real 1-Lipschitz functions \(f : X \to [0; 1]\), and constructing a generic representation with modulus of continuity \(\kappa(n) \leq O(n \cdot \eta(n))\) that allows an appropriate (oracle) Type-2 Machine to compute any fixed such \(f\) in time polynomial in \(\eta\).

The present work generalizes and extends this as follows:

- Theorem 25 constructs a generic representation \(\xi\) with (i) modulus of continuity \(\kappa \leq \text{lin}(\eta)\) linear in the entropy \(\eta\)
- and (ii) establishes said \(\xi\) maximal-complete w.r.t. linear metric reduction “\(\zeta \lessdot_0 \xi\)” among all uniformly continuous representations \(\zeta\) of \(X\).
- We propose (i) and (ii) as axioms and quantitative strengthening of classical, qualitative admissibility for complexity-theoretically sensible representations.
- Theorem 27 strengthens the classical, qualitative Main Theorem by quantitatively characterizing (moduli of) continuous functions \(f : X \to Y\) in terms of (moduli of) their \((\xi,\upsilon)\)-realizers and the entropies of co/domain \(X,Y\).
- Theorem 29d) confirms the classical categorical binary Cartesian product of representations [Wei00, Definition 3.3.3.1] to maintain linear admissibility (i) and (ii).
- The classical categorical countable Cartesian product of representations [Wei00, Definition 3.3.3.2] does not maintain linear (nor polynomial) admissibility; but our modified construction exhibited in Theorem 30 does. In particular (ii) it is maximal/complete w.r.t. linear metric reductions.
- Moreover, as opposed to some linearly admissible representation constructed from ‘scratch’ by invoking Theorem 25, the representation from Theorem 30 additionally guarantees the canonical componentwise projections and embeddings to admit continuous realizers with optimal moduli of continuity.
- Theorem 31 constructs from any linearly admissible representation \(\xi\) of compact metric \((X,d)\) a polynomially admissible representation \(\xi'_{\mu}\) of \(X'_{\mu} = \mathcal{C}_\mu(X,[0;1])\). In particular (ii) it is maximal/complete w.r.t. polynomial metric reductions.
- Moreover said representation \(\xi'_{\mu}\) guarantees the application functional \(X'_{\mu} \times X \ni (f,x) \mapsto f(x) \in [0;1]\) to have a continuous realizer with optimal modulus of continuity.

Revolving around notions like entropy and modulus of continuity, our considerations and methods are mostly information-theoretic: carefully constructing representations and realizers, analyzing the dependence of their value on the argument, and comparing thus obtained bounds on their modulus of continuity to bounds on the entropy of the space under consideration, estimated from above by constructing coverings with ‘few’ balls of given radius \(2^{-n}\) as well as bounded from below by constructing subsets of points of pairwise distance \(> 2^{-n}\).
1.4 Overview

Section 2 formally introduces our conception of quantitatively (polynomially/linearly) admissible representations. Subsection 2.1 re-analyzes the above three representations of \([0; 1]\) from this new perspective. And Subsection 2.2 collects metric properties of other popular spaces; including new ones constructed via binary and countable Cartesian products (w.r.t. Hilbert Cube metric), the hyperspace of non-empty compact subsets w.r.t. Hausdorff metric, and function spaces. Section 3 recalls and rephrases our previous results \([KSZ16a, §3]\) from this new perspective: constructing a generic polynomially-admissible representation for any compact metric space \((X, d)\). And Subsection 3.1 improves that to linear admissibility. Section 4 finally formally states our complexity-theoretic Main Theorem 27 in quantitative detail; and presents categorical constructions for new quantitatively admissible representations from given ones, paralleling the considerations from Subsection 2.2: binary and countable Cartesian products, Hausdorff hyperspace of non-empty compact subsets, and function spaces. Section 5 collects some questions about possible refinements/strengthenings, such as improving/dropping constant factors in our results. Subsection 5.1 finally extends our considerations to generalized representations for higher types in the sense of \([KC12]\); and Subsection 5.2 puts them into the larger context of quantitatively-universal compact metric spaces.

2 Intuition and Definition of Quantitative Admissibility

In order to refine computability to a sensible theory of computational complexity we propose in this section two quantitative refinements of qualitative admissibility formalized in Definition 11 below. But first let us briefly illustrate how a reasonable representation can be turned into an unreasonable one, and how that affects the computational complexity of a function: to get an impression of what quantitative admissibility should prohibit.

Consider ‘padding’ a given representation \(\xi\) with some fixed strictly increasing \(\varphi : \mathbb{N} \to \mathbb{N}\) in order to obtain a new representation \(\xi_\varphi\) defined by
\[
(\xi_\varphi)(\vec{b}) := \xi((b_\varphi(n))_n) \text{ for } \vec{b} = (b_n)_n \in \mathcal{C}.
\]
Then \(\text{dom}(\xi_\varphi)\) is compact whenever \(\text{dom}(\xi)\) is; but computing some \((\xi_\varphi, \nu)\)-realizer now may require ‘skipping’ over \(\varphi(n)\) bits of any given \(\xi_\varphi\)-name before reaching/collecting the same information as contained in only the first \(n\) bits of a given \(\xi\)-name when computing a \((\xi, \nu)\)-realizer: possibly increasing the time complexity from \(t\) to \(t \circ \varphi^{-1}\), definitely increasing its optimal modulus of continuity. On the other end computing a \((\xi, \nu_\varphi)\)-realizer might become easier, as now as many as \(\varphi(n)\) bits of the padded output can be produced from only \(n\) bits of the unpadded one: possibly decreasing the time complexity from \(t\) to \(t \circ \varphi^{-1}\), see Definition 11 below.

Definition 8.  a) We abbreviate \(\vec{x}|_{<n} := (x_0, \ldots x_{n-1})\) and
\[
(x_0, \ldots x_{n-1}) \circ \mathbb{Z}^n = \{(x_0, \ldots x_n, z_n, z_{n+1}, \ldots) : z_n, z_{n+1} \ldots \in \mathbb{Z}\}.
\]
\(\{0, 1\}^n \ni \vec{x} = (x_0, \ldots x_{n-1}) \mapsto \langle \vec{x} \rangle := (0 \underbrace{x_0 \ 0 x_1 \ 0 x_2 \ldots \ 0 x_{n-1} 1}_{2n+1}) \in \{0, 1\}^{2n+1}\)
Lemma 10. Let \( |r| \in \mathbb{Z} \) mean the integer closest to \( r \in \mathbb{R} \) with ties broken towards 0: 
\[
\left\lfloor \pm n + \frac{1}{2} \right\rfloor = \pm n \text{ for } n \in \mathbb{N}.
\]
b) Let \( \text{Reg} \) denote the set of all non-decreasing unbounded mappings \( \nu : \mathbb{N} \to \mathbb{N} \).
The lower and upper semi-inverse of \( \nu \in \text{Reg} \) are
\[
\nu^{-\ominus}(n) := \min\{m : \nu(m) \geq n\}, \quad \nu^{-\uplus}(n) := \min\{m : \nu(m+1) > n\}.
\]
c) Extend Landau’s class of asymptotic growth
\[
\nu \leq O(\mu) \iff \exists C \in \mathbb{N} \forall n \in \mathbb{N} : \nu(n) \leq C \cdot \mu(n) + C
\]
\[
\nu \leq P(\mu) := O(\mu(n))^{O(1)} \iff \exists C \forall n : \nu(n) \leq (C + C \cdot \mu(n))^{C}
\]
\[
\nu \leq S(\mu) := \mu + O(1) \iff \exists C \forall n : \nu(n) \leq \mu(n) + C
\]
\[
\nu \leq o(\mu) := \mu \circ O(id) \iff \exists C \forall n : \nu(n) \leq \mu(C \cdot n + C)
\]
\[
\nu \leq \nu(\mu) := \mu \circ P(id) \iff \exists C \forall n : \nu(n) \leq \mu(C \cdot n) + C
\]
\[
\nu \leq s(\mu) := \mu \circ S(id) \iff \exists C \forall n : \nu(n) \leq \mu(n) + C
\]
\[
\nu \leq \text{poly}(\mu) := P(\nu(\mu)) \iff \exists C \forall n : \nu(n) \leq (n + C \cdot C \cdot n) + C
\]

to denote sequences bounded linearly/polynomially/additively by, and/or after linearly/polynomially/additively growing the argument to, \( \mu \).

In Item c), classes \( \text{lin}(\mu) \leq \text{poly}(\mu) \) capture ‘relative’ asymptotics in increasing granularity. They are transitive and compositional in the following sense:

Observation 9  a) \( \mu \leq O(\nu) \) and \( \nu \leq O(\kappa) \) implies \( \mu \leq O(\kappa) \).
\[
\mu \leq P(\nu) \text{ and } \nu \leq P(\kappa) \text{ implies } \mu \leq P(\kappa).
\]
\[
\mu \leq S(\nu) \text{ and } \nu \leq S(\kappa) \text{ implies } \mu \leq S(\kappa).
\]
b) \( \mu \leq o(\nu) \) and \( \nu \leq o(\kappa) \) implies \( \mu \leq o(\kappa) \).
\[
\mu \leq \nu(\nu) \text{ and } \nu \leq \nu(\kappa) \text{ implies } \mu \leq \nu(\kappa).
\]
\[
\mu \leq s(\nu) \text{ and } \nu \leq s(\kappa) \text{ implies } \mu \leq s(\kappa).
\]
c) \( \mu \leq O(\nu) \) implies \( \mu \circ \kappa \leq O(\nu \circ \kappa) \).
\[
\mu \leq P(\nu) \text{ implies } \mu \circ \kappa \leq P(\nu \circ \kappa).
\]
\[
\mu \leq S(\nu) \text{ implies } \mu \circ \kappa \leq S(\nu \circ \kappa).
\]
d) \( \mu \leq o(\nu) \) implies \( \kappa \circ \mu \leq o(\kappa \circ \nu) \).
\[
\mu \leq \nu(\nu) \text{ implies } \kappa \circ \mu \leq \nu(\kappa \circ \nu).
\]
\[
\mu \leq s(\nu) \text{ implies } \kappa \circ \mu \leq s(\kappa \circ \nu).
\]
e) \( o(\mu) \circ O(\nu) = o(\mu) \circ O(\nu) = \mu \circ O(\nu) \).
\[
\nu(\mu) \circ P(\nu) = \nu(\mu) \circ P(\nu) = \mu \circ P(\nu).
\]
\[
s(\mu) \circ S(\nu) = s(\mu) \circ S(\nu).
\]

In particular, polynomial ‘absolute’ growth means \( \text{poly}(id) = P(id) = \nu(id) \); and linear means \( \text{lin}(id) = O(id) = o(id) \). We also collect some properties of the semi-inverses:

Lemma 10. a) For \( \mu \in \text{Reg} \), \( \mu^{-\ominus} \) and \( \mu^{-\uplus} \) are again in \( \text{Reg} \) and \( \mu^{-\ominus} \leq \mu^{-\uplus} \).
b) Every \( \mu \in \text{Reg} \) satisfies \( \mu \circ \mu \leq \text{id} \leq \mu^{-1} \circ \mu \), with equality in case \( \mu \) is injective (necessarily growing at least linearly); and every \( \mu \in \text{Reg} \) satisfies \( \mu \circ \mu^{-1} \leq \text{id} \leq \mu \circ \mu^{-1} \), with equality in case \( \mu \) is surjective (necessarily growing at most linearly).

c) For \( \nu, \kappa \in \text{Reg} \) it holds

\[
\mu \circ \nu \leq \kappa \Leftrightarrow \nu \leq \mu^{-1} \circ \kappa \Leftrightarrow \mu \leq \kappa \circ \nu^{-1}
\]

d) Suppose \( a, b, c, d \in \mathbb{N} \) with \( b, d \geq 1 \) and \( \mu, \nu \in \text{Reg} \) satisfy \( \forall n : \nu(n) \leq a + b \cdot \mu(c + d \cdot n) \). Then it holds \( \forall m : \mu^{-1}(m) \leq c + d \cdot \nu^{-1}(a + b \cdot m) \) and \( \mu^{-1}(m) \leq d \cdot \nu^{-1}(a + b \cdot m) + c + d - 1 \). In particular,

\[
O(\mu^{-1}) = o(\mu^{-1}) \quad \text{and} \quad O(\nu^{-1}) = O(\mu^{-1}) \quad \text{and} \quad \lim (\mu^{-1}) = \lim (\nu^{-1});
\]

similarly \( O(\mu^{-1}) = o(\mu^{-1}) \quad \text{and} \quad O(\mu^{-1}) = O(\mu^{-1}) \quad \text{and} \quad \lim (\mu^{-1}) = \lim (\mu^{-1}) \).

e) There are \( \mu, \nu \in \text{Reg} \) s.t. \( \mu \leq \text{poly}(\nu) \) but \( \mu^{-1} \nsubseteq \text{poly}(\nu^{-1}) \) and \( \nu^{-1} \nsubseteq \text{poly}(\mu^{-1}) \). There are \( \mu, \nu \in \text{Reg} \) s.t. \( \mu \circ \mu^{-1} \nsubseteq \text{poly}(\text{id}) \) and \( \nu^{-1} \circ \nu \nsubseteq \text{poly}(\text{id}) \) and \( \nu^{-1} \circ O(\nu) \nsubseteq \text{poly}(\text{id}) \) and \( \mu \circ \mu^{-1} \nsubseteq \text{poly}(\text{id}) \).

f) \( \max \{ \mu^{-1}, \nu^{-1} \} = \min \{ \mu, \nu \}^{-1} \) and \( \min \{ \mu^{-1}, \nu^{-1} \} = \max \{ \mu, \nu \}^{-1} \) and \( \max \{ \mu^{-1}, \nu^{-1} \} = \min \{ \mu, \nu \}^{-1} \) and \( \min \{ \mu^{-1}, \nu^{-1} \} = \max \{ \mu, \nu \}^{-1} \).

Here finally comes our formal definition of quantitative admissibility:

**Definition 11.** a) Consider a compact subset \( K \) of a metric space \((X, d)\). Its relative entropy is the non-decreasing integer mapping \( \eta = \eta_{X, K} : \mathbb{N} \to \mathbb{N} \) such that some \( 2^{\eta(n)} \), but no \( 2^{\eta(n)-1} \), closed balls \( B(x, r) \) of radius \( r := 2^{-n} \) with centers \( x \in X \) can cover \( K \). If \( X \) itself is compact, we write \( \eta_X := \eta_{X, X} \) for its (intrinsic) entropy.

b) Consider a uniformly continuous representation \( \xi \) of the compact metric space \((X, d)\) and uniformly continuous mapping \( \zeta : \mathcal{C} \to X \). Defining Definition 11, call a reduction \( F : \text{dom}(\zeta) \to \text{dom}(\xi) \) polynomial ("\( \zeta \propto_{\xi} \xi \)") if it has a modulus of continuity \( \mu \) and \( \xi \) has a modulus \( \kappa \) satisfying \( \mu \circ \kappa \leq \circ(\nu) \) for every modulus \( \nu \) of \( \zeta \).

\( F \) is linear ("\( \zeta \preceq_{\xi} \xi \)") if it holds \( \mu \circ \kappa \leq \circ(\nu) \).

c) A representation \( \xi \) of the compact metric space \((X, d)\) is polynomially admissible iff (i) it has a modulus of continuity \( \kappa \leq \text{P}(\eta) \), i.e., bounded polynomially in the entropy with linearly transformed argument, and (ii) every uniformly continuous representation \( \zeta : \mathcal{C} \to X \) satisfies \( \zeta \propto_{\xi} \xi \) in the sense of (b).

d) Representation \( \xi \) of \((X, d)\) is linearly admissible iff (i) it has a modulus of continuity \( \kappa \leq O(\eta) \), i.e., not exceeding the entropy by more than a constant factor in value and constant shift in argument, and (ii) every uniformly continuous representation \( \zeta : \mathcal{C} \to X \) satisfies \( \zeta \preceq_{\xi} \xi \).

According to Example 19f) below, any representation’s modulus of continuity satisfies \( \kappa \geq \eta \), i.e., is bounded from below by the entropy; and Condition (i) in Definition 11d requires a complexity-theoretically appropriate representation to be close to that optimum — which itself can be arbitrarily small/large according to Example 19f+e). The converse Condition (ii) in Definition 11d)
similarly requires that $\mu \circ \kappa$, a modulus of continuity of $\xi \circ F = \zeta$ be ‘close’ to that of $\zeta$. Note that linear admissible representations may (i) exceed the entropy by a constant factor in value and by an additive constant in the argument while (ii) linear reduction only allows for the latter: because (i) is what we can achieve in Theorem 25 while (ii) guarantees transitivity; similarly for the polynomial case.

**Remark 12** We record that relations “$\preceq_F$” and “$\preceq_O$” are transitive: Fix $\alpha$ with modulus of continuity $\lambda$, $\beta$ with modulus $\mu$, and $\gamma$ with $\nu$; and linear reduction $F : \text{dom}(\alpha) \to \text{dom}(\beta)$ with modulus of continuity $\iota$ such that $\alpha = \beta \circ F$ and $\iota \circ \mu \leq o(\lambda)$; as well as linear reduction $G : \text{dom}(\beta) \to \text{dom}(\gamma)$ with modulus of continuity $\kappa$ such that $\beta = \gamma \circ G$ and $\kappa \circ \nu \leq o(\mu)$. Then $\alpha = \gamma \circ G \circ F$, where reduction $G \circ F : \text{dom}(\alpha) \to \text{dom}(\gamma)$ has modulus $\iota \circ \kappa$ satisfying $(\iota \circ \kappa) \circ \nu \leq \iota \circ o(\mu) \leq o(\lambda)$ by Observation (3).

Also note that, according to Lemma 10d), condition $\mu \circ \kappa \leq o(\nu)$ is equivalent to $\mu \leq \nu \circ O(\kappa^{-1}) = \circ(\nu) \circ \kappa^{-1}$; similarly for $\mu \circ \kappa \leq \nu$. Definition 11d) coincides with [KSZ16a, Definition 18] and strengthens the computability-theoretically common qualitative notion of admissibility from [Wei10] Definitions 2.1.1+2.1.2; while Definition 11d) in turn strengthens (c) from polynomial to linear.

**Proof (Lemma 10).** Most claims are immediate, we thus only expand on Item d):

1. First suppose $c = 0$ and $d = 1$. Then
   
   $$\mu^{-1}(m) = \min \left\{ n : a + b \cdot \mu(n) \geq a + b \cdot m \right\}$$
   
   $$\leq \min \left\{ n : \nu(n) \geq a + b \cdot m \right\} = \nu^{-1}(a + b \cdot m)$$
   
   $$\mu^{-1}(m) = \min \left\{ n : a + b \cdot \mu(n + 1) > a + b \cdot m \right\}$$
   
   $$\leq \min \left\{ n : \nu(n + 1) > a + b \cdot m \right\} = \nu^{-1}(a + b \cdot m).$$

2. Next, $\mu^{-1}(m) = \min \left\{ n : \mu(n) \geq m \right\}$
   
   $$\leq \min \left\{ c + d \cdot n' : \mu(c + d \cdot n') \geq m \right\} \leq \min \left\{ c + d \cdot n' : \nu(n') \geq m \right\}$$
   
   $$= c + d \cdot \nu^{-1}(m)$$
   
   in case $a = 0$ and $b = 1$. If additionally $d = 1$, then
   
   $$\mu^{-1}(m) = \min \left\{ n : \mu(n + 1) > m \right\} \leq \min \left\{ c + n : \mu(c + n + 1) > m \right\}$$
   
   $$\leq \min \left\{ c + n : \nu(n + 1) > m \right\} = c + \nu^{-1}(m).$$

Finally in case $a = 0 = c$ and $b = 1 \neq d$,

$$\mu^{-1}(m) = \min \left\{ n : \mu(n + 1) > m \right\} \leq \min \left\{ d \cdot [n/d] : \mu(n + 1) > m \right\}$$

$$\leq \min \left\{ d \cdot n' + (d - 1) : \mu(c \cdot n' + 1) > m \right\}$$

$$\leq \min \left\{ d \cdot n' + (d - 1) : \nu(n' + 1) > m \right\}$$

$$= d \cdot \nu^{-1}(m) + (d - 1).$$

\[\square\]
2.1 Real Examples

Here we formally recall, and analyze from the perspective of admissibility, the three representations of the real unit interval mentioned in the introduction: binary, dyadic and signed binary. Let us record that the real unit interval $[0; 1]$ has entropy $\eta_{[0;1]}(n) = n - 1$ for all integers $n \geq 1$.

**Example 13 (Binary Representation)** The binary representation of the real unit interval

$$\beta : C \ni b \mapsto \sum_n b_n 2^{-n+1} \in [0; 1]$$

is surjective and 1-Lipschitz, i.e., has the identity $id : \mathbb{N} \ni n \mapsto n \in \mathbb{N}$ as modulus of continuity: coinciding with the entropy up to shift 1, i.e., optimal! However it is not (even qualitatively) admissible \cite{Wei00} Theorem 4.1.13.6, does not admit a continuous realizer of, e.g., the continuous mapping $[0; 1/3] \ni x \mapsto 3x \in [0; 1]$; cmp. \cite{Wei00} Example 2.1.4.7.

**Example 14 (Rational Representation)** Consider the binary encoding of non-negative integers without leading 0:

$$\text{bin} : \mathbb{N} \ni 2^n - 1 + \sum_{0 \leq j < n} b_j 2^j \mapsto (b_0, \ldots b_{n-1}) \in \{0, 1\}^n . \quad (1)$$

The rational representation of $[0; 1]$ is the mapping $\rho : \subseteq C \to [0; 1]$ with

$$\left(\langle \text{bin}(a_0)\rangle \langle \text{bin}(c_0)\rangle \langle \text{bin}(a_1)\rangle \langle \text{bin}(c_1)\rangle \ldots \langle \text{bin}(a_n)\rangle \langle \text{bin}(c_n)\rangle \ldots \right) \mapsto \lim_j a_j / c_j ,$$

$$\text{dom}(\rho) = \left\{ \ldots \langle \text{bin}(a_n)\rangle \langle \text{bin}(c_n)\rangle \ldots \right\} : \exists x \in [0; 1] |a_n / c_n - x| \leq 2^{-n} \right\} .$$

Representation $\rho$ is continuous, but not uniformly continuous (its domain is not compact) and thus has no modulus of continuity.

**Proof.** Consider a $\rho$-name of $r = 1/2$ starting with any $a_0 \in \mathbb{N}$ and $b_0 := 2a_0$. Increasingly long $a_0$ thus give rise to a sequence of $\rho$-names of $r$ with no converging subsequence. Moreover $a_0 / b_0 = 1/2$ fixes $r$ up to error $2^{-n}$ only for $n := 0$; but requires ‘knowing’ the first $\mu(0) \geq \log_2(a_0) \to \infty$ bits of its $\rho$-name. \hfill $\square$

**Example 15 (Dyadic Representation)** The dyadic representation of the real unit interval $[0; 1]$

$$\delta : \subseteq C \ni \left(\langle \text{bin}(a_0)\rangle \langle \text{bin}(a_1)\rangle \langle \text{bin}(a_2)\rangle \ldots \langle \text{bin}(a_n)\rangle \ldots \right) \mapsto \lim_j a_j / 2^j ,$$

$$\text{dom}(\delta) = \left\{ \ldots \langle \text{bin}(a_n)\rangle \ldots \right\} : 2^n \geq a_n \in \mathbb{N} , |a_n / 2^n - a_m / 2^m| \leq 2^{-n} + 2^{-m} \right\}$$

i) has a quadratic modulus of continuity $\nu(n) := 2 \cdot (n + 1) \cdot (n + 2)$ but no sub-quadratic one and in particular is not Hölder-continuous.

ii) To every partial function $\zeta : \subseteq C \to [0; 1]$ with modulus of continuity $\nu$ there exists a mapping $F : \text{dom}(\zeta) \to \text{dom}(\delta)$ with modulus of continuity $\nu$ such that $\zeta = \delta \circ F$ holds. In particular $\delta$ is polynomially admissible.
iii) To every $m \in \mathbb{N}$ and every $r, r' \in [0;1]$ with $|r - r'| \leq 2^{-m-1}$, there exist $\delta$-names $\bar{y}_r$ and $\bar{y}'_r$ of $r = \delta(\bar{y}_r)$ and $r' = \delta(\bar{y}'_r)$ with $d_C(\bar{y}_r, \bar{y}'_r) \leq 2^{-m-1}$.

iv) If $(Y, e)$ is a compact metric space and $f : [0;1] \to Y$ such that $f \circ \delta : \text{dom}(\delta) \subseteq C \to Y$ has modulus of continuity $\nu$, then $f$ has modulus of continuity $\nu$.

Here and as opposed to Definition 11 (ii) applies also to non-surjective $\zeta$.

Proof.  

i) Record that $0 < a_n \leq 2^n$ implies $(\text{bin}(a_n)) \in \{0,1\}^*$ to have length between 1 and $2n + 1$; and $(\text{bin}(a_0)) \ldots (\text{bin}(a_n))$ has binary length between $n + 1$ and $\kappa(n)$. Therefore permuting a $\delta$-name $\bar{y}$ of some $r \in [0;1]$ to $\bar{y}'$ with $d_C(\bar{y}, \bar{y}') \leq 2^{-\kappa(n)}$ will keep (the binary expansions of) $a_0, \ldots, a_n$ unmodified; and thus satisfies $|\delta(\bar{y}) - \delta(\bar{y}')| \leq 2^{-n}$. Hence $\kappa(n)$ is a modulus of continuity. On the other hand consider the $\delta$-name $\bar{g}$ of $r := 3/4$ with $a_n := 3 \cdot 2^{n-2}$ for $n \neq 2$ has $(\text{bin}(a_n))$ of length $2n - 1$ and starts at bit position $\sum_{m<n}(2m - 1) \geq \Omega(n^2)$; yet changing $a_m' := 3 \cdot 2^{m-2} + 2^{m-n}$ for all $m > n$ turns it into a $\delta$-name $\bar{g}'$ of $r' := r + 2^{-n}$ with $d_C(\bar{g}, \bar{g}') \leq 2^n\Omega(n^2)$. So $\bar{\omega}$ has no sub-quadratic modulus of continuity.

ii) We construct $F : \text{dom}(\zeta) \to \text{dom}(\bar{\omega})$ as limit $F(\bar{x}) = \lim_n F_n(\bar{x} |_{\nu(n+1)})$ of a sequence of partial functions $F_n : \{0,1\}^{\nu(n+1)} \to \{0,1\}^{\kappa(n)}$ which is monotone in that $F_{n+1}(\bar{x} |_{\nu(n+2)})$ contains $F_n(\bar{x} |_{\nu(n+1)})$ as initial segment. To every $n$ and each (of the finitely many) $\bar{x} |_{\nu(n+1)}$ with $\bar{x} \in \text{dom}(\zeta)$, fix some $r_n = r_n(\bar{x} |_{\nu(n+1)}) \in \zeta(\bar{x} |_{\nu(n+1)}) \cap C \subseteq [0;1]$. Then given $\bar{x} \in \text{dom}(\bar{\omega})$ and iteratively for $n = 0, 1, \ldots$ let

$$F_n(\bar{x} |_{\nu(n+1)}) := F_{n-1}(\bar{x} |_{\nu(n)}) \circ (\text{bin}(a_n)), \quad a_n := |r_n \cdot 2^n| \in \{0, \ldots, 2^n\}.$$ 

Since $\nu$ is a modulus of continuity of $\zeta$, it follows

$$\zeta(\bar{x} |_{\nu(n+1)}) \subseteq [r_n - 2^{n-1}, r_n + 2^{n-1}] \subseteq \left[\frac{a_n}{2^n} - 2^{-n}, \frac{a_n}{2^n} + 2^n\right]$$

as $|r_n - a_n/2^n| \leq 2^{-n-1}$. Thus it holds $|a_n/2^n - r| \leq 2^{-n}$ for $r := \zeta(\bar{x})$ since $|r - r_n| \leq 2^{-n-1}$; $F(\bar{x}) = \lim_n F_n(\bar{x} |_{n+1}) = (\text{bin}(a_0)) \ldots (\text{bin}(a_n)) \ldots$ is a $\delta$-name of $r$; and fixing the first $\nu(n+1)$ symbols of $\bar{x} \in \text{dom}(\zeta)$ fixes $\bar{y} := F_n(\bar{x} |_{\nu(n+1)})$ as well as $(a_0, \ldots, a_n)$ and therefore also (at least) the first $n + 1$ symbols of $\bar{y} := F(\bar{x})$: hence $F$ has modulus of continuity $\nu(n)$.

As recorded above, $[0;1]$ has entropy $\eta(n) = n - 1$; hence (i) and (ii) imply polynomial admissibility according to Definition 11.

iii) To $r \in [0;1]$ consider the $\delta$-name $\bar{y}_r := \ldots (\text{bin}(a_n)) \ldots \in C$ of $r$ with $a_n := |r \cdot 2^n|$. For every $m \in \mathbb{N}$, its initial segment $(\text{bin}(a_0)) \ldots (\text{bin}(a_m))$ has binary length between $m + 1$ and $\kappa(m)$; and, for every $r' \in [0;1]$ with $|r - r'| \leq 2^{-m-1}$, can be extended to a $\delta$-name $\bar{y}'_r$ via $a_m' := |r' \cdot 2^m|$ for all $m' > m$.

iv) Applying (iii) to $m := \nu(n)$, the hypothesis implies $e(f(r), f(r')) = e(f \circ \delta(\bar{y}), f \circ \delta(\bar{y}')) \leq 2^{-n}$. □

The dyadic representation in Example 15 has modulus of continuity quadratic (i.e. polynomial), but not linear, in the entropy. This overhead comes from the
‘redundancy’ of the precision-$n$ approximation $a_n/2^m$ of binary length $O(n)$ superseding all previous $a_m/2^m$, $m < n$. The signed binary representation on the other hand achieves precision $2^{-n}$ by appending one ‘signed’ digit $b_{n-2} \in \{-1, 0, 1\}$, encoded as two binary digits $(b_{2n-4}, b_{2n-3}) \in \{00, 01, 10\}$ via $\tilde{b}_{n-2} = 2b_{2n-4} + b_{2n-3} - 1$, to the previous approximation up to error $2^{-n+1}$, yielding a modulus of continuity linear in the entropy:

**Example 16 (Signed Binary Represent.)** The signed binary representation, considered as total mapping

$$\sigma \subseteq \{00, 01, 10\}^\mathbb{N} \subseteq \mathcal{C} \ni \tilde{b} \mapsto \frac{1}{2} + \sum_{m=0}^{\infty} (2b_{2m} + b_{2m+1} - 1) \cdot 2^{-m-2} \in [0; 1] \quad (2)$$

i) is surjective and has modulus of continuity $\kappa(n) = 2n$, i.e., is Hölder-continuous.

ii) To every partial function $\zeta : \subseteq \mathcal{C} \rightarrow [0; 1]$ with modulus of continuity $\nu$ there exists a mapping $F : \text{dom}(\zeta) \rightarrow \text{dom}(\sigma)$ with modulus of continuity $\kappa : 2m \mapsto \nu(m + 1)$ such that $\zeta = \sigma \circ F$ holds.

In particular $\sigma$ is linearly admissible.

iii) To every $n \in \mathbb{N}$ and every $r, r' \in [0; 1]$ with $|r - r'| \leq 2^{-n}$, there exist $\sigma$-names $\bar{y}_r$ and $\bar{y}_r'$ of $r = \sigma(\bar{y}_r)$ and $r' = \sigma(\bar{y}_r')$ with $d_C(\bar{y}_r, \bar{y}_r') \leq 2^{-2n}$.

iv) If $(Y, d)$ is a compact metric space and $f : [0; 1] \rightarrow Y$ such that $f \circ \sigma : \text{dom}(\sigma) \subseteq \mathcal{C} \rightarrow Y$ has modulus of continuity $2\nu$, then $f$ has modulus of continuity $\nu$.

**Proof (Example 16).**

i) Note that appropriate choice of $\tilde{b}_0, \tilde{b}_1, \ldots \in \{-1, 0, 1\}$ yields precisely any possible value $[-1/2; +1/2] \ni \sum_{m=0}^{\infty} \tilde{b}_{2m-2} \cdot 2^{-m}$; hence $\sigma$ is total and surjective. In one worst case, changing $(b_{2m}, b_{2m+1}) = 00$ (encoding the signed digit $-1$) to $(b_{2m}, b_{2m+1}) = 10$ (encoding $+1$) for all $m \geq n$ changes the real number $r$ from Equation (2) to $r' = r + \sum_{m \geq n} (2m - 2m - 2) \cdot 2^{-m-2} = r + 2^{-n-1}$, while $\tilde{b}'$ agrees with $\tilde{b}$ up to position $2n - 1$: This asserts $\kappa(n) = 2n$ to be a modulus of continuity and, in view of $[0; 1]$ having entropy $\eta(n) = n - 1$, establishes Condition (i) of Definition (11b).

ii) Similarly to the proof of Example 15 for every $n$ and every $\bar{x}|_{<\nu(n+1)}$ with $\bar{x} \in \text{dom}(\zeta)$, consider the compact set $\zeta([\bar{x}|_{<\nu(n+1)} \circ \mathcal{C}] \subseteq [0; 1]$: Having diameter $\leq 2^{-(n+1)}$ by the definition of $\nu$, it is contained in $[r_n - 2^{-n-2}; r_n + 2^{-n-2}]$ for $r_n := (\min \zeta([\bar{x}|_{<\nu(n+1)} \circ \mathcal{C}]) + \max \zeta([\bar{x}|_{<\nu(n+1)} \circ \mathcal{C}]) / 2 \in [0; 1]$; hence $|r_{n+1} - r_n| \leq 2^{-n-2}$. Now let $r'_1 := \frac{1}{2}$ capture the constant term $\tilde{b}_{-1} := -1$ in Equation (2) such that $|r_1 - r'_1| \leq \frac{1}{4}$ with $r_1 \in \left(\frac{1}{4}; \frac{3}{4}\right)$; and for $n = 2, 3, \ldots$ inductively append one additional signed digit

$$2b_{2n-4} + b_{2n-3} - 1 = \tilde{b}_{n-1} := [2^n \cdot (r_n - r'_n)] \in \{-1, 0, +1\} \leq \pm 3 \cdot 2^{-n-1}$$

such that $r'_n := \frac{1}{2} + \sum_{m=1}^{n} \tilde{b}_{m-2} \cdot 2^{-m}$ again satisfies $|r_n - r'_n| \leq 2^{-n-1}$ and $|r_{n+1} - r'_n| \leq |r_{n+1} - r_n| + |r_n - r'_n| \leq 3 \cdot 2^{-n-2}$ and $|r - r'_n| \leq |r - r_n| + |r_n - r'_n| \leq 3 \cdot 2^{-n-2}$.
2^{-n} for \( r = \zeta(\bar{x}) \in \zeta[\bar{x} \lhd \nu(n + 1) \circ \mathcal{C}] \): Hence \( F(\bar{x}) := (b_0, \ldots, b_{2n-4}, b_{2n-3}, \ldots) \) is a \( \sigma \)-name of \( r \), and the thus defined function \( F \) has modulus of continuity \( 2^n \mapsto \nu(n + 1) \).

iii) To \( r \in [0; 1] \) and \( n \in \mathbb{N} \) consider signed digits \( \tilde{b}_0, \ldots, \tilde{b}_{n-2} \in \{-1, 0, 1\} \) and \( r'_n := \frac{1}{2} + \sum_{m=1}^{n} \tilde{b}_{m-2} \cdot 2^{-m} \) with \(|r - r'_n| \leq 2^{-n-1}\) as in (ii). As in (i), appropriate choice of \( \tilde{b}_{n-1}, \tilde{b}_{n}, \ldots \in \{-1, 0, 1\} \) yields any possible value \([-2^n, +2^n]\) \( \ni \sum_{m=n-1}^{\infty} \tilde{b}_{m-2} \cdot 2^{-m} \); hence every \( r' \in [0; 1] \) with \(|r - r'| \leq 2^{-n}\) admits a signed binary expansion \( r' = \frac{1}{2} + \sum_{m=1}^{\infty} \tilde{b}_{m-2} \cdot 2^{-m} \) extending \( (\tilde{b}_0, \ldots, \tilde{b}_{n-2}) \), and \( \sigma \)-name \( \bar{y}_r \) coinciding on the first \( 2^n \) binary symbols.

iv) Applying (iii) to \( n := \nu(m) \), the hypothesis implies \( e(f(r), f(r')) = e(f \circ \sigma(\bar{y}_r), f \circ \sigma(\bar{y}_r')) \leq 2^{-m} \).

The signed binary representation renders real addition computable by a finite-state transducer:

Starting off in state \( \mathcal{C} \), in each round \( \#n = 0, 1, \ldots \) it reads the next signed digits \( a_n, b_n \in \{-1, 0, 1\} \) in the respective expansions of real arguments \( x = \sum_{n} a_n 2^{-n} \) and \( y = \sum_{n} b_n 2^{-n} \), and follows that edge whose first label agrees with \( a_n + b_n \) while outputting the second label \( c_{n-2} \in \{-1, 0, 1\} \) of said edge such that \( x + y = \sum_{n} c_n 2^{-n} \). The transducer works by storing for each state the accumulated value from previous input except those already output.

The signed-digit expansion’s modulus of continuity leaves a constant-factor gap to the entropy, attained by the binary expansion. One can trade between both, namely permit \( \bar{I} \) only at asymptotically fewer positions (i) while incurring possible ‘carry ripples’ between them over asymptotically longer ranges (ii):
Example 17 Fix a strictly increasing function \( \varphi : \mathbb{N} \to \mathbb{N} \) with \( \varphi(0) = 0 \). Representation \( \sigma_\varphi \) ‘interpolates’ between Examples 13 and 16 by considering signed binary expansions \( \sum_{m=1}^{\infty} \tilde{c}_m \cdot 2^{-m} \) with \( \tilde{c}_m \in \{ \bar{1}, 0, 1 \} \) for every \( m \) in range \( \varphi = \varphi[\mathbb{N}] \) but \( \tilde{c}_m \in \{ 0, 1 \} \) for all \( m \in \mathbb{N} \setminus \varphi[\mathbb{N}] \). Each \( \tilde{c}_m \) with \( m \in \mathbb{N} \setminus \varphi[\mathbb{N}] \) is encoded as one bit, each \( \tilde{c}_m \) with \( m \in \varphi[\mathbb{N}] \) as two bits. Thus \( \sigma_{\text{id}} = \sigma \) recovers Example 16.

i) \( \sigma_\varphi \) is surjective and has modulus of continuity \( n \mapsto n + \varphi^{-1}(n) \).

ii) There exists a mapping \( F_\varphi : \text{dom}(\sigma) \to \text{dom}(\sigma_\varphi) \) with modulus of continuity \( 2\varphi \circ (\text{id} + \varphi)^{-1} \) such that \( \sigma = \sigma_\varphi \circ F_\varphi \) holds.

In particular \( \sigma_{n^2} := \sigma_{m \mapsto n^2} \) has modulus of continuity \( n + O(\sqrt{n}) \); and to every partial function \( \zeta : \subseteq \mathcal{C} \to [0;1] \) with modulus of continuity \( \nu \) there exists a mapping \( F_{n^2} : \text{dom}(\zeta) \to \text{dom}(\sigma_{n^2}) \) with modulus of continuity \( n \mapsto \nu(n+1) + O(\sqrt{\nu(n+1)}) \) such that \( \zeta = \sigma_{n^2} \circ F_{n^2} \).

Note that \( \varphi(n) := 2^n \) has \( \text{id} + \varphi^{-1}(n) \geq \log_2(n) + 1 \) infinitely often and therefore \( \varphi \circ (\text{id} + \varphi)^{-1}(n) \geq 2 \cdot n \): yielding a linear reduction \( \zeta \preceq_0 \sigma_\varphi \), but no better.

Proof (Example 17).

i) Similarly to the proof of Example 16, the first \( n \) digits \( \tilde{c}_0, \ldots, \tilde{c}_{n-1} \) of an expansion fix the value up to absolute error \( < 2^{-n} \). Differing from Example 16 this initial segment of the expansion occupies not \( 2n \) but \( n + \varphi^{-1}(n) \) bits since ‘signed’ digits (permitted) only at the \( \varphi^{-1}(n) \) positions \( \varphi[\mathbb{N}] \) within \( \{0, \ldots, n-1\} \).

ii) We describe a transformation \( F_\varphi \) converting a given signed-digit expansion \( r = \frac{1}{2} + \sum_{m=1}^{\infty} \tilde{b}_m \cdot 2^{-m} \) with \( \tilde{b}_m \in \{ \bar{1}, 0, 1 \} \) to the required form \( r = \sum_{m=1}^{\infty} \tilde{c}_m \cdot 2^{-m} \) with \( \tilde{c}_m \neq \bar{1} \) except for positions \( m \in \varphi[\mathbb{N}] \). Reflecting the constant term in Equation 2, initially let \( c_0 := 1 \), tentatively. Now iteratively for \( k = 1, 2, \ldots \) re-code the signed integer

\[
-2^{\varphi(k)} + 1 = 0 - 2^{\varphi(k)-1} - 2^{\varphi(k)-2} - \ldots - 2 - 1 \\
\leq 2^{\varphi(k)} \cdot c_{\varphi(k-1)} + 2^{\varphi(k)-1} \cdot \tilde{b}_{\varphi(k-1)} + 2^{\varphi(k)-2} \cdot \tilde{b}_{\varphi(k-1)+1} + \ldots \\
\ldots + 2 \cdot \tilde{b}_{\varphi(k)-2} + \tilde{b}_{\varphi(k)-1} \\
=: 2^{\varphi(k)} \cdot \tilde{c}_{\varphi(k-1)} + 2^{\varphi(k)-1} \cdot \tilde{c}_{\varphi(k-1)+1} + 2^{\varphi(k)-2} \cdot \tilde{c}_{\varphi(k-1)+2} + \ldots \\
\ldots + 2 \cdot \tilde{c}_{\varphi(k)-1} + c_{\varphi(k)}
\]

uniquely with \( \tilde{c}_{\varphi(k-1)} \in \{ \bar{1}, 0, 1 \} \) and \( \tilde{c}_{\varphi(k-1)+1}, \ldots, \tilde{c}_{\varphi(k)-1}, c_{\varphi(k)} \in \{ 0, 1 \} \); the latter again only tentatively. Thus the \( \varphi(k) + k \) bits of \( (\tilde{c}_0, \ldots, \tilde{c}_{\varphi(k)-1}) \) depend precisely on the \( 2\varphi(k) \) bits of \( (\tilde{b}_0, \ldots, \tilde{b}_{\varphi(k)-1}) \); the transformation \( F_\varphi \) on Cantor space thus has modulus of continuity \( \varphi(k) + k \mapsto 2\varphi(k) \) for all \( k \in \mathbb{N} \), and \( 2\varphi \circ (\text{id} + \varphi)^{-1} \) in general. \( \square \)
2.2 Abstract Examples

This subsection collects some properties, relations, and examples of moduli of continuity and entropies of spaces.

Fact 18 a) Every compact metric space \((X, d)\) can be covered by finitely many open balls \(B(x, r)\); therefore its entropy \(\eta\) is well-defined. If \(X\) is infinite then \(\eta \in \text{Reg}\).

b) Every continuous function \(f : X \rightarrow Y\) between compact metric spaces \((X, d)\) and \((Y, e)\) is uniformly continuous and therefore has a modulus \(\mu\) of continuity.

c) Lipschitz-continuous functions have moduli of continuity \(\mu(n) = n + \mathcal{O}(1)\), and vice versa. Hölder-continuous functions have linear moduli of continuity \(\mu(n) = \mathcal{O}(n)\), and vice versa.

d) Proceeding from metric \(d\) on \(X\) to a metric \(d'\) with \(d' \leq 2^{-n}\) whenever \(d \leq 2^{-\nu(n)}\) changes the entropy \(\eta\) to \(\eta' \leq \eta \circ \nu\). Additionally proceeding from \(e\) on \(Y\) to \(e'\) satisfying \(e' \leq 2^{-\kappa(n)}\) \(\Rightarrow\) \(e \leq 2^{-n}\) turns a modulus of continuity \(\mu\) of \(f : X \rightarrow Y\) into \(\mu'\) with \(\mu \leq \nu \circ \mu' \circ \kappa\).

e) Fix a compact metric space \((X, d)\) and equip it with the Hausdorff metric \(\mathcal{D}(X) = \max\{\sup\{d_Y(w) : w \in W\}, \sup\{d_Y(v) : v \in V\}\}\), where

Example 19 (Entropy) a) The real unit interval \([0; 1]\) has entropy \(\eta_{[0; 1]}(n) = n - 1\) for all integers \(n \geq 1\). Cantor space has entropy \(\eta_{C} = \text{id}\). The Hilbert Cube \(\mathcal{H} = \prod_{j \geq 0}[0; 1]\) with metric \(d_{\mathcal{H}}(\bar{x}, \bar{y}) = \sup_{j} |x_j - y_j|/2^j\) has entropy \(\eta_{\mathcal{H}}(n) = \Theta(n^2)\).

b) Let compact \((X, d)\) and \((Y, e)\) have entropies \(\eta_{X}\) and \(\eta_{Y}\), respectively. Then the entropy \(\eta_{X \times Y}\) of compact \((X \times Y, \max\{d, e\})\) satisfies

\[
\forall n:\; \eta_{X}(n) + \eta_{Y}(n) \leq \eta_{X \times Y}(n + 1) + 1 \leq \eta_{X}(n + 1) + \eta_{Y}(n + 1) + 1.
\]

c) Let compact \((X_j, d_j)\) all have diameter \(\leq 1\) and entropy \(\eta_j, j \in \mathbb{N}\). Then \((\prod_j X_j, \sup_j d_j/2^j)\) is compact and has entropy \(\eta\) satisfying

\[
\forall n:\; \sum_{j \leq n} \eta_j(n) \leq \eta(n+1) + \lceil n/2 \rceil \leq \sum_{j \leq n} \eta_j(n+1-j) + \lfloor n/2 \rfloor.
\]

For \(X_j \equiv [0; 1]\) this recovers the Hilbert Cube \(\prod_j[0; 2^{-j}]\).

d) Let \(X\) be a compact space with metric \(d \leq 1\) and entropy \(\eta\). Then \(D(x, y) := 1/\left(\log_2 2/d(x, y)\right) \leq 1\) constitutes a topologically equivalent metric yet inducing entropy \(H(n) = \eta(2^n - 1)\).

e) Fix an arbitrary non-decreasing unbounded \(\varphi : \mathbb{N} \rightarrow \mathbb{N}\) and re-consider Cantor space, now equipped with \(d_{\varphi} : (\bar{a}, \bar{b}) \mapsto 2^{-\varphi(\min(m: a_m \neq b_m))}\) \(\in [0; 1]\). This constitutes a metric, topologically equivalent to \(d_C = d_{id}\) but with entropy \(\eta_{\varphi} = \varphi^{-1}\).

f) For \(K\) a closed subset of compact \((X, d)\), it holds \(\eta_{X,K} \leq \eta_{X,X}\) but not necessarily \(\eta_K \leq \eta_X\). The image \(Z := f[X] \subseteq Y\) has entropy \(\eta_Z \leq \eta_X \circ \mu\), where \(\mu\) denotes a modulus of continuity of \(f\). Every connected compact metric space \(X\) has entropy at least linear \(\eta(n) \geq n + \Omega(1)\).

g) Fix a compact metric space \((X, d)\) with entropy \(\eta\). Let \(K(X)\) denote the set of non-empty closed subsets of \(X\) and equip it with the Hausdorff metric \(D(V, W) = \max\{\sup\{d_Y(w) : w \in W\}, \sup\{d_Y(v) : v \in V\}\}\), where
\(d_V : X \ni x \mapsto \inf\{d(x,v) : v \in V\}\) denotes the distance function. Then \((K(X),D)\) constitutes a compact metric space \([\text{Wei07}],\) Exercise 8.1.10. It has entropy \(H \leq 2^n\) with \(2^{(n+1)} < H(n+1)\).

h) Fix a connected compact metric space \((X,d)\) with entropy \(\eta\), and consider the convex metric space \(X' := C(X,\{0;1\})\) of continuous real functions equipped with the supremum norm \(|f| = \sup_{x \in X} |f(x)|\). Its subset \(X_1' := \text{Lip}_1(X,\{0;1\})\) of non-expansive functions \(f : X \to \{0;1\}\) is compact by Arzelà-Ascoli; it has relative entropy \(\eta_1(n) := \eta_{X',X_1'}(n) = \Theta(2^{(n+O(1))})\); more precisely: \(2^{(n-1)} < \eta_1(n) \leq O(2^{(n+2)})\).

Item (d) yields spaces with asymptotically large entropy; and Item (e) does similarly for small entropy — of a totally disconnected space in view of Example 19. The connection between modulus of continuity and entropy (Item f) had been observed in [Ste16, Lemma 3.1.13]. We emphasize that Item h) refers to \(X_1'\) as subset of \(X'\), not as metric space of its own: see also Question 32 below. According to Item (d) of the following Lemma, analyses of function spaces \(C_\mu(X,\{0;1\})\) may indeed w.l.o.g. suppose \(\mu = \text{id}\), i.e., the consider the non-expansive case.

**Lemma 20.** For non-decreasing unbounded \(\mu : \mathbb{N} \to \mathbb{N}\) let

\[
\omega_\mu : [0;\infty) \ni t \mapsto \inf \left\{ \sum_{j=1}^J 2^{-n_j} : J, n_1, \ldots, n_J \in \mathbb{N}, t \leq \sum_{j=1}^J 2^{-\mu(n_j)} \right\} \in [0;\infty)
\]

a) It holds \(\omega_\mu(0) = 0\) and \(\omega_\mu(t) > 0\) for \(t > 0\). \(\omega_\mu\) is subadditive: \(\omega(s+t) \leq \omega(s) + \omega(t)\). \(\omega_\mu\) has modulus of continuity \(\mu\).

b) If \(\mu\) is strictly increasing, then \(\mu(n) = \min\{m \in \mathbb{N} : \omega_\mu(2^{-m}) \leq 2^{-n}\}\).

c) For a compact convex metric space \((X,d)\) and any \(x,y \in X\), there exists an isometry \(i : [0;d(x,y)] \to X\) with \(i(0) = x\) and \(i(d(x,y)) = y\).

d) If \((X,d)\) is compact convex and \(\mu\) a modulus of continuity of \(f : X \to \mathbb{R}\), then \(|f(x) - f(x')| \leq \omega_\mu(d(x,x'))\) for all \(x,x' \in X\).

In particular \(C_\mu((X,d),\mathbb{R}) = \text{Lip}_1((X,\omega_\mu \circ d),\mathbb{R})\) holds for every strictly increasing \(\mu\).

Recall that a (not necessarily linear) metric space \(X\) is called convex if, to any distinct \(x,y \in X\), there exists a \(z \in X\setminus\{x,y\}\) with \(d(x,y) = d(x,z) + d(z,y)\). Examples include compact convex subsets of Euclidean space with its inherited metric, but also connected compact subsets when equipped with the intrinsic (=shortest-path) distance, while Cantor space is not convex.

**Example 21 (Modulus of Continuity)** a) The function \([0;1] \ni t \mapsto 1/\ln(e/t)\) in \((0;1]\) extends uniquely continuously to 0 and has an exponential, but no polynomial, modulus of continuity.

b) Picking up on Example 19, let \(\xi_j : \mathcal{C} \to X_j\) have modulus of continuity \(\kappa_j\) and fix some injective ‘pairing’ function \(\mathbb{N} \times \mathbb{N} \ni (n,m) \mapsto \langle n,m \rangle \in \mathbb{N}\).
such as Cantor’s $\langle n, m \rangle = (n + m) \cdot (n + m + 1)/2 + n$. Then  
\[
\prod_j x_j : \leq C \ni \bar{b} \mapsto (\xi_j(b_{(j,0)}, \ldots b_{(j,n)}, \ldots)) \in \prod_j X_j
\]
has modulus of continuity $n \mapsto \sup_{j < n} \langle j, \kappa_j(n - j) \rangle$.

c) If $\xi$ is a representation of $X$ with modulus of continuity $\mu$, then the following $2^\xi$ is a representation of $K(X)$ with modulus of continuity $m \mapsto 2^{\mu(m) + 1} - 1$:

$(b_0, b_1, \ldots b_m, \ldots) \in C$ is a $2^\xi$-name of $A \in K(X)$ iff, for every $n \in \mathbb{N}$ and every $\bar{v} \in \{0, 1\}^{\leq \mu(n)}$ it holds:

\[
\begin{aligned}
(\bar{v} \circ C) \cap \text{dom}(\xi) \neq \emptyset &\land b_{\text{bin}(\bar{v})} = 1 \Rightarrow \xi[\bar{v} \circ C] \cap \overline{B}(A, 2^{-n}) \neq \emptyset \\
(\bar{v} \circ C) \cap \text{dom}(\xi) \neq \emptyset &\land b_{\text{bin}(\bar{v})} = 0 \Rightarrow \xi[\bar{v} \circ C] \cap \overline{B}(A, 2^{-n-1}) = \emptyset 
\end{aligned}
\]

where $\bar{v} \circ C := \{\bar{v}w : \bar{w} \in C\} \subseteq C$ and $\text{bin}(v_0, \ldots v_{n-1}) = v_0 + 2v_1 + 4v_2 + \ldots + 2^{n-1}v_n + 2^n - 1$ and $\overline{B}(A, r) := \bigcup_{a \in A} \overline{B}(a, r)$.

Note that $\mu \leq O(\eta)$ implies $2^\mu \leq P(2^\eta)$; reflected in Theorem 30(1) and Theorem 31 below. According to Example 19, Example 21 does not preserve linear admissibility already in case of spaces $X_j$ with quadratic entropy: a more sophisticated construction is needed in Theorem 30.

2.3 Proofs

Proof (Example 21).

\(c\) $2^\xi$ is a representation, as $A \in K(X)$ can be recovered from any name $\bar{b}$: On the one hand, for every $\xi$-name $\bar{v}$ of $a \in A$ and every $n \in \mathbb{N}$, $b_{\text{bin}(v_0, \ldots, v_{\mu(n)-1})} = 1$; on the other hand, for every $\xi$-name $\bar{v}$ of $a \notin A$, $B(a, 2^{-n}) \cap A = \emptyset$ implies $b_{\text{bin}(v_0, \ldots, v_{\mu(n)-1})} = 0$. Since $\text{bin}(v_0, \ldots, v_{\mu(n)-1}) < 2^{v_{\mu(n)}+1} - 1$, this also establishes $2^{\mu+1} - 1$ as modulus of continuity of $2^\xi$.

$\square$

Proof (Lemma 31A).

\(a\) Regarding subadditivity, $\omega_\mu(s + t) =$

\[
= \inf \left\{ \sum_{j=1}^J 2^{-n_j} + \sum_{k=1}^K 2^{-m_k} : J, K, n_1, \ldots n_j, m_1, \ldots m_K \in \mathbb{N}, s + t \leq \sum_{j=1}^J 2^{-\mu(n_j)} + \sum_{k=1}^K 2^{-\mu(m_k)} \right\}
\]

\[
\leq \inf \left\{ \sum_{j=1}^J 2^{-n_j} + \sum_{k=1}^K 2^{-m_k} : J, K, n_1, \ldots n_j, m_1, \ldots m_K \in \mathbb{N}, s \leq \sum_{j=1}^J 2^{-\mu(n_j)} \land t \leq \sum_{k=1}^K 2^{-\mu(m_k)} \right\}
\]

\[
= \omega_\mu(s) + \omega_\mu(t)
\]
By definition \((J := 1)\) it holds \(0 \leq \omega_\mu(t) \leq 2^{-n}\) for \(t \leq 2^{-n}\), and in particular \(\omega_\mu(0) = 0\). By subadditivity and whenever \(0 \leq \delta \leq 2^{-\mu(n)}\), we have both \(\omega_\mu(t) \leq \omega_\mu(t+\delta) \leq \omega_\mu(t) + \omega_\mu(\delta) \leq \omega_\mu(t)+2^{-n}\) and \(\omega_\mu(t) - 2^{-n} \leq \omega_\mu(t) - \omega_\mu(\delta) = \omega_\mu(t - \delta + \delta) - \omega_\mu(\delta) \leq \omega_\mu(t - \delta) \leq \omega_\mu(t)\).

b) For every \(t \leq 2^{-\mu(n)}\) it holds \(\omega_\mu(t) \leq 2^{-n}\) by definition, and hence \(\bar{\mu}(n) := \min \{m \in \mathbb{N} : \omega_\mu(2^{-m}) \leq 2^{-n}\} \leq \mu(n)\). Conversely for \(m \leq n_1, \ldots, n_j \in \mathbb{N}\),

\[
\sum_j 2^{-\mu(n_j)} \leq \sum_j 2^{-\mu(m)-n_j+m} = 2^{-\mu(m)} \cdot 2^m \cdot \sum_j 2^{-n_j}
\]

from strict monotonicity \(\mu(n_j) = \mu(m + n - m) \geq \mu(m) + (n_j - m)\) by induction. So \(\sum_j 2^{-\mu(n_j)} \geq 2^{-\mu(m)}) \) implies \(\sum_j 2^{-n_j} \geq 2^{-m}\) and \(\bar{\mu}(n) \geq \mu(n)\).

c) Using transfinite induction and completeness, Exercise 5.1.17 constructs a \(z \in X\) with \(d(x, z) = d(x, y) = d(x, y)/2\). Now iterating with both \((x, z)\) and \((z, y)\) in place of \((x, y)\) yields a sequence of refinements \(z_n \in X, n = 0, \ldots, N = 2^k z_0 = x\) and \(z_N = y\) and \(d(z_n, z_{n+1}) = d(x, y)/N\). Again by completeness, \(t \in (0, 1)\) converges uniformly to the claimed isometry.

d) For \(t := d(x, x')\), and to any \(r \in \mathbb{N}\) and \(n_1, \ldots, n_j \in \mathbb{N}\) with \(t \leq \sum_j 2^{-\mu(n_j)}\), c) yields \(x_0, x_1, \ldots, x_j = x' \in X\) with \(d(x_{j-1}, x_j) \leq 2^{-\mu(n_j)}\). Hence

\[
|f(x) - f(x')| \leq \sum_j |f(x_{j-1}) - f(x_j)| \leq \sum_j 2^{-n_j} \leq \omega_\mu(t)
\]

Fact 22 Fix a compact metric space \((X, d), \) non-empty \(Z \subseteq X\) and \(L > 0\). For \(L\)-Lipschitz \(f : Z \rightarrow \mathbb{R}\), the functions

\[
f_\ast : x \mapsto \sup_{z \in Z} f(z) - L \cdot d(z, x), \quad f^\ast : x \mapsto \inf_{z \in Z} f(z) + L \cdot d(z, x) \quad (4)
\]

extend \(f\) to \(X\) while preserving \(L\)-Lipschitz continuity. Moreover every \(L\)-Lipschitz extension \(\hat{f} : X \rightarrow \mathbb{R}\) of \(f\) to \(X\) satisfies \(f_\ast \leq \hat{f} \leq f^\ast\), where \(f^\ast - f_\ast \leq 2L d_Z = 2L \sup_z d_Z(x)\). The extension operator \(\text{Lip}_L(Z, \mathbb{R}) \ni f \mapsto f^\ast := (f_\ast + f^\ast)/2 \in \text{Lip}_L(X, \mathbb{R})\) is a well-defined isometry of compact metric spaces w.r.t. the supremum norm.

\((f_\ast, f^\ast)\) is known as McShane-Whitney pair [Pet18]. For the purpose of self-containment, we include a proof:

Proof (Fact 22). W.l.o.g. \(L = 1\). For \(x \in Z\), choosing \(z := x\) shows \(f_\ast(x) \geq f(x)\) while \(f(z) - d(z, x) \geq f(z) - |f(z) - f(x)| \geq f(x)\) implies \(f_\ast(x) \leq f(x)\).

Moreover, for every \(z \in Z\), we have

\[-\sup \left\{f(z')-d(z', x') : z' \in Z\right\} \leq -f(z)+d(z, x) \leq -f(z)+d(z, x)+d(x, x')\]

and hence

\[f_\ast(x) - f_\ast(x') \leq \sup \left\{f(z)+d(z, x)-f(z)+d(z, x)+d(x, x') : z \in Z\right\} = d(x, x').\]
The estimates for \( f^* \) proceed similarly. For \( x \in X \) and with \( z, z', w, w' \) ranging over \( Z \), \( (f_\ast(x) + f'(x)) - (g_\ast(x) + g'(x)) = \\
= \sup_z f(z) - d(z, x) - \sup_w g(w) - d(w, x) - \inf_{w'} g(w') + d(w', x) + \inf_z f(z') + d(z', x) \\
= \sup_z f(z) - d(z, x) + \inf_w - g(w) + d(w, x) + \sup_{w'} - g(w') - d(w', x) + \inf_z f(z) + d(z', x) \\
\leq \sup_z f(z) - d(z, x) - g(z) + d(z, x) + \sup_{w'} - g(w') - d(w', x) + f(w') + d(w', x) \\
= 2 \cdot \sup_z f(z) - g(z) \leq 2 \cdot \sup_z |g(z) - f(z)|. \square

Proof (Example 15). Let \( \mathcal{H}_X(n) \) denote the least number of closed balls of radius \( 2^{-n} \) covering \( X \), so that \( \eta_X(n) = \lceil \log_2 \mathcal{H}(n) \rceil \). Let \( \mathcal{C}_X(n) \) denote the largest number of points in \( X \) of pairwise distance \( > 2^{-n} \), also known as capacity. (Since \( X \) is not an integer function, there is not danger of confusion this notation with that of space of continuous functions.) Then \( \mathcal{C}_X(n) \leq \mathcal{H}_X(n + 1) \): To cover \( X \) requires covering the \( \mathcal{C}_X(n) \) points as above; but any closed ball of radius \( 2^{-(n+1)} \) can contain at most one of those points having distance \( > 2^{-n} \). On the other hand \( \mathcal{H}_X(n) \leq \mathcal{C}_X(n) \), since balls of radius \( 2^{-n} \) whose centers form a maximal set \( X_n \) of pairwise distance \( > 2^{-n} \) cover \( X \): if they missed a point, that had distance \( > 2^{-n} \) to all centers in \( X_n \) and thus could be added to \( X_n \): contradicting its maximality.

a) Cover \([0; 1]\) by \( 2^{n-1} \) closed intervals \( I_{n,j} := [j \cdot 2^{-(n-1)}; (j + 1) \cdot 2^{-(n-1)}] \), \( j = 0, \ldots, 2^{n-1} - 1 \), of radius \( 2^{-n} \) around centers \( (2j + 1)2^{-n} \); optimally. Cover \( \mathcal{C} \) by \( 2^n \) closed balls \( \mathcal{O} \circ \mathcal{C} \) of radius \( 2^{-n} \) around centers \( \mathcal{O} \in \{0, 1\}^n \); optimally. Cover \( \mathcal{H} \) by \( 2^{n-1} \cdot 2^{n-2} \cdot 2 \cdot 1 = 2^{n(n+1)/2} \) closed balls

\[
I_{n,j_0} \times I_{n-1,j_1} \times \cdots \times I_{1,j_{n-1}} \times \prod_{j \geq n} [0; 1]
\]

of radius \( 2^{-n} \) with indices ranging as follows: \( 0 \leq j_0 < 2^{-n-1} \), \( 0 \leq j_1 < 2^{-n-2} \), \( \ldots \), \( j_{n-1} = 0 \).

b) Obviously \( \mathcal{H}_{X \times Y} \leq \mathcal{H}_X \cdot \mathcal{H}_Y \) and \( \mathcal{C}_{X \times Y} \geq \mathcal{C}_X \cdot \mathcal{C}_Y \): \( \mathcal{H}_X(n) \cdot \mathcal{H}_Y(n) \leq \mathcal{C}_X(n) \cdot \mathcal{C}_Y(n) \leq \mathcal{C}_{X \times Y}(n) \leq \mathcal{H}_{X \times Y}(n+1) \leq \mathcal{H}_X(n+1) \cdot \mathcal{H}_Y(n+1)

Also record \( \lceil s \rceil + \lceil t \rceil \geq \lceil s + t \rceil \geq \lceil s \rceil + \lceil t \rceil - 1 \) for all \( s, t > 0 \).

c) Abbreviating \( \mathcal{H}_j := \mathcal{H}_X_j \) and \( \mathcal{C}_j := \mathcal{C}_X_j \), we have \( \mathcal{H}(n) \leq \prod_{j \leq n} \mathcal{H}_j(n - j) \) and \( \mathcal{C}(n) \geq \prod_{j < n} \mathcal{C}_j(n - j) \): note that \( \mathcal{H}(0) = 1 = \mathcal{C}(0) \) as \( X_j \) has diameter \( \leq 1 \). Finally \( \sum_{j=0}^{n-1} \lfloor t_j \rfloor \geq \lceil \sum_{j=0}^{n-1} t_j \rceil \geq \sum_{j=0}^{n-1} \lfloor t_j \rfloor - \lfloor n/2 \rfloor \).

d) For a counterexample to \( \eta_X \leq \eta_X \) consider a circle/hyper-/sphere with and without center.

Regarding the lower bound for connected compact metric spaces, consider \( N := 2^{n(n)} \) and \( x_1, \ldots, x_N \in X \) such that balls with centers \( x_j \) and radius \( 2^{-n} \) cover \( X : X \subseteq \bigcup_{n=1}^{N} B(x_n, 2^{-n}) \). Consider the finite undirected graph \( G_n = (V_n, E_n) \) with vertices \( V_n = \{1, \ldots, N\} \) and edges \( \{i, j\} \in E_n \iff B(x_i, 2^{-n-1}) \cap B(x_j, 2^{-n-1}) \neq \emptyset \) whenever the two open balls with centers \( x_i, x_j \) and radius twice \( 2^{-n} \) intersect. This graph is connected: If \( I, J \subseteq V_n \) were distinct connected components, then \( \bigcup_{n \in I} B(x_n, 2^{-n-1}) \) and
Definition 15: the construction of a standard representation which 
Lemma 3.2.5] then shows to satisfy properties (i) and (ii) from Fact 4. Here we first recall from [KSZ16a, Definition 3.2.2 + 3.2.7] introduce qualitative admissibility in terms of
representation which [Wei00, Lemma 3.2.5] then shows to satisfy (1)

\[ H_X(n) \leq 2^{\mathcal{H}_X} \quad \text{and} \quad C_X(n) \geq 2^{C_X} \]

Obviously \( H_X(n) \leq 2^{\mathcal{H}_X} \) and \( C_X(n) \geq 2^{C_X} \).

h) Fix \( n \in \mathbb{N} \) and consider an maximal set \( X_n \subseteq X \) of \( N := C_X(n) \) points of pairwise distance \( > 2^{-n} \). There are \( 2^{C_X(n)} \) different \( f : X_n \rightarrow \{0, 2^{-n}\} \); each is 1-Lipschitz, and extends to \( f_+^* : X \rightarrow [0;1] \); and, according to Fact 22

different such \( f \) give rise to \( f_+^* \) of mutual supremum distance \( \geq 2^{-n} \); hence \( C_{X_1}(n) \geq 2^{C_X(n)} \), and \( X_1' \subseteq X' \) has intrinsic entropy \( \eta_1'(n) \geq \log_2 C_{X_1}(n - 1) \geq C_X(n - 1) \geq H_X(n - 1) > 2^{\mathcal{H}_X(n - 1)} - 1 \)

Conversely, for any 1-Lipschitz \( f : X \rightarrow [0;1] \), consider \( f_+^* := \left| 2^n \cdot f \right|_{\mathcal{X}_n} / 2^n \); still \( (1 + 1/2) \)-Lipschitz since rounding affects the value by at most \( 2^{\mathcal{H}_X(n - 1)} - 1 \) on arguments of distance \( > 2^{-n} \). As argued before, maximality of \( X_n \) implies that the closed balls around centers \( x \in X_n \) of radius \( 2^{-n} \) cover \( X \) (hence \( d_{X_n} \leq 2^{-n} \)); consequently so do the open balls with radius \( 2^{-n+1} \). Similarly to the proof of (f), consider the finite undirected and connected graph \( G_n = (X_n, E_n) \) with edge \( \{x, y\} \in E_n : \iff B(x, 2^{-n+1}) \cap B(y, 2^{-n+1}) \neq \emptyset \). Any vertex \( y \) of \( G_n \) adjacent to some \( x \) has distance \( d(x, y) < 2^{-n+2} \); and since \( f_+^* : X_n \rightarrow \mathbb{D}_n \) is \( \frac{3}{2} \)-Lipschitz, this implies \( \mathbb{D}_n \ni \left| f_+^*(x) - f_+^*(y) \right| \leq \frac{3}{2} \cdot 2^{-n+2} \) leaving no more than 13 possible values for \( f_+^*(x) - f_+^*(y) \in \{-6 \cdot 2^{-n}, \ldots, 0, \ldots, 6 \cdot 2^{-n}\} \). Connectedness of \( G_n \) with \( N \) vertices thus limits the number of different \( \frac{3}{2} \)-Lipschitz \( f_+^* : X_n \rightarrow \mathbb{D}_n \) to \( \leq (1 + 2^n) \cdot 13^{N-1} \leq 2^{O(C_X(n))} \) in view of (1). And by Fact 22 each such \( f_+^* \) extends to some \( \frac{3}{2} \)-Lipschitz \( f_+^* : X \rightarrow [0;1] \). Moreover \( |d_{X_n}| \leq 2^{-n} \) implies \( |f - f_+^*| \leq |f_+^* - f_n| / 2 \leq \frac{3}{2} \cdot 2^{-n} \) for the \( \frac{3}{2} \)-Lipschitz (!) extension of the restriction \( f_n := f_+^* \). Since \( g \Rightarrow g_+^* \) is an isometry, this implies \( |f - f_+^*| \leq |f - f_n^*| + |f_n - f_+^*| \leq \frac{3}{2} \cdot 2^{-n} + 2^{\mathcal{H}_X - 1} = 2^{-n+1} \). The \( 2^{O(C_X(n))} \) closed balls of radius \( 2^{-n+1} \) around centers \( f_+^* \in \text{Lip}_{3/2}(X, [0;1]) \subseteq X' \) thus cover \( \text{Lip}_1(X, [0;1]) = X_1' \): \( \eta_1'(n - 1) = \eta_{X',X_1'}(n - 1) \leq O(C_X(n)) \leq O(H_X(n + 1)) \leq O(2^{2(n+1)}) \).}

3 Concise Standard Representations

[Wei00] Definitions 3.2.2 + 3.2.7 introduce qualitative admissibility in terms of a standard representation which [Wei00, Lemma 3.2.5] then shows to satisfy properties (i) and (ii) from Fact 4. Here we first recall from [KSZ16a, Definition 15] the construction of a concise standard representation of any fixed compact metric space \((X,d)\) that generalizes Example 15. For each \( n \), fix a covering of \( X \) by \( \leq 2^{\mathcal{H}_X(n)} \) balls of radius \( 2^{-n} \) according to the entropy; assign to each ball a binary string \( \tilde{a}_n \) of length \( \eta(n) \); then every \( x \in X \) can be approximated by the center of some of these balls; finally define a name of \( x \) to be such a sequence \((\tilde{a}_n)_n\) of binary strings. Theorem 24 establishes that this representation is polynomially admissible, provided the balls’ radius is reduced to \( 2^{-n-1} \). Subsection 3.1 improves the construction to yield a linearly admissible standard representation.
Definition 23. Let \((X, d)\) denote a compact metric space with entropy \(\eta\). For each \(n \in \mathbb{N}\) fix some partial mapping \(\xi_n : \{0, 1\}^{\eta(n+1)} \to X\) such that \(X = \bigcup_{\xi \in \text{dom}(\xi_n)} B(\xi_n(\bar{a}_n), 2^{-n-1})\), where \(B(x, r) = \{x' \in X : d(x, x') \leq r\}\) denotes the closed ball around \(x\) of radius \(r\). The standard representation of \(X\) (with respect to the family \(\xi_n\) of partial dense enumerations) is the mapping

\[
\xi : \mathcal{C} \ni ((\bar{a}_0) (\bar{a}_1) \ldots (\bar{a}_n) \ldots) \mapsto \lim_n \xi_n(\bar{a}_n) \in X,
\]

where \(\text{dom}(\xi) := \{(\ldots(\bar{a}_n)\ldots) : \bar{a}_n \in \text{dom}(\xi_n), d(\xi_n(\bar{a}_n), \xi_m(\bar{a}_m)) \leq 2^{-n} + 2^{-m}\}\).

Fact 18 asserts such \(\xi_n\) to exist. The real Example 15 a special case of this definition with \(\eta_{[0,1]}(n+1) = n\) according to Example 19 and

\[
\delta'_\eta : \{0,1\}^n \ni \bar{a} \mapsto \left(\frac{1}{2} + a_0 + a_1 a_2 + \ldots + 2^{n-1} a_{n-1}\right)/2^n.
\]

The covering balls’ radius being \(2^{-n-1}\) instead of \(2^{-n}\) is exploited in the following theorem:

Theorem 24. i) The standard representation \(\xi\) of \((X, d)\) w.r.t. \((\xi_n)\) according to Definition 23 has modulus of continuity \(\kappa(n) := \sum_{m=0}^{n} \eta(m+1)\).

ii) To every partial function \(\zeta : \mathcal{C} \to X\) with modulus of continuity \(\nu\) there exists a mapping \(F : \text{dom}(\zeta) \to \text{dom}(\xi)\) with modulus of continuity \(\mu = \nu \circ (1 + \kappa^{-1}) : \kappa(n) \to \nu(n+1)\) such that \(\zeta = \xi \circ F\).

In particular \(\kappa\) is polynomially admissible, provided that the entropy grows at least with some positive power \(\eta(n) \geq \Omega(n^\epsilon), \epsilon > 0\).

iii) To every \(m \in \mathbb{N}\) and every \(x, x' \in X\) with \(d(x, x') \leq 2^{-m-1}\), there exist \(\xi\)-names \(\bar{y}_x\) and \(\bar{y}'_{x'}\) of \(x = \xi(\bar{y}_x)\) and \(x' = \xi(\bar{y}'_{x'})\) with \(d_C(\bar{y}_x, \bar{y}_x') \leq 2^{-\kappa(m)}\).

iv) If \((Y, e)\) is a compact metric space and \(f : X \to Y\) such that \(f \circ \xi : \text{dom}(\xi) \subseteq \mathcal{C} \to Y\) has modulus of continuity \(\kappa \circ \nu\), then \(f\) has modulus of continuity \(\nu + 1\).

Again (ii) strengthens Definition 11 in applying to not necessarily surjective \(\zeta\). In view of Lemma 10c), (iv) can be rephrased as follows: \(f \circ \xi\) with modulus of continuity \(\nu\) implies \(f\) to have modulus of continuity \(1 + \kappa^{-1} \circ \nu\). However (ii) is not saying that \(\zeta\) with modulus of continuity \(\nu \circ \mu\) yields \(F\) with modulus of continuity \(n \mapsto \nu(n+1)\).

Proof (Theorem 24).

i) First observe that \(\xi\) is well-defined: as compact metric space, \(X\) is complete and the dyadic sequence \(\xi_n(\bar{a}_n) \in X\) therefore converges. Moreover \(\xi\) is surjective: To every \(x \in X\) and \(n \in \mathbb{N}\) there exists by hypothesis some \(\bar{a}_n \in \text{dom}(\xi_n)\) with \(x \in B(\xi_n(\bar{a}_n), 2^{-n-1})\); hence \(d(\xi_n(\bar{a}_n), \xi_m(\bar{a}_m)) \leq 2^{-n} + 2^{-m}\) and \(\lim_n \xi_n(\bar{a}_n) = x\). Furthermore, \(\bar{a}_n\) has binary length \(\eta(n+1)\); hence \((\bar{a}_0 \ldots \bar{a}_n)\) has length \(\kappa(n)\) as above; and fixing this initial segment of a \(\xi\)-name \(\bar{x}\) implies \(d(\xi(\bar{x}), \xi_n(\bar{a}_n)) \leq 2^{-n}\) by Equation 5.

Well, almost: \(\text{bin}(\bar{a}_n)\) has length between 1 and \(2n+1\) while here we make all strings in \(\text{dom}(\xi_n)\) have the same length \(= \eta(n+1)\): Using strings of varying length \(< \eta(n+1)\) would additionally require encoding delimiters.
ii) To every $n$ and each (of the finitely many) $\bar{x}|_{\nu(n+1)}$ with $\bar{x} \in \text{dom}(\xi)$, fix some $\bar{a}_n = a_n(\bar{x}|_{\nu(n+1)}) \in \text{dom}(\xi_n)$ such that

$$\xi_n(\bar{a}_n) \in \xi(\bar{x}|_{\nu(n+1)} \circ C) \subseteq \bigcup_{\bar{a} \in \text{dom}(\xi_n)} B(\xi_n(\bar{a}), 2^{-n-1}).$$

Then iteratively for $n = 0, 1, \ldots$ let similarly to the proof of Example [15] $F_n(\bar{x}|_{\nu(n+1)}) := F_{n-1}(\bar{x}|_{\nu(n)}) \circ (\bar{a}_n) \in \{0, 1\}^{\kappa(n-1) + \eta(n+1) = \kappa(n)}.$

This makes $F(\bar{x}) := \lim_n F_n(\bar{x}|_{\nu(n+1)}) \in C$ well-defined with modulus of continuity $\kappa : \kappa(n) \mapsto \nu(n + 1).$ Moreover it holds $F(\bar{x}) \in \text{dom}(\xi)$ and $\xi(F(\bar{x})) = \xi(\bar{x})$ since $\xi_n(\bar{a}_n), \xi(\bar{x}) \in \xi(\bar{x}|_{\nu(n+1)} \circ C) \subseteq B(\xi_n(\bar{a}_n), 2^{-n})$ because $\nu$ is a modulus of continuity of $\xi$.

Finally observe $\eta(n+1) \leq \kappa(n) \leq (n+1) \cdot \eta(n+1) \in \text{poly}(\eta)$; hence (i) and (ii) of Definition [14] hold.

iii) To $x \in X$ consider the $\xi$-name $\bar{y}_x := (\ldots(\bar{a}_m)\ldots)$ of $x$ with $d(\xi_m(\bar{a}_m), x) \leq 2^{-m-1}.$ For $m \in \mathbb{N}$ its initial segment $(\bar{a}_0\ldots \bar{a}_m)$ has binary length $\kappa(m)$; and, for every $x' \in X$ with $d(x, x') \leq 2^{-m-1}$, can be extended to a $\xi$-name $\bar{y}_{x'}$.

iv) Applying (iii) to $m := \nu(n)$, the hypothesis implies $e(f(x), f(x')) = e(f \circ \xi(\bar{y}_x). f(\xi(\bar{y}_{x'}))) \leq 2^{-n}$. \qed

3.1 Improvement to Linear Admissibility

The generic representation $\xi$ of a compact metric space $(X, d)$ according to Definition [23] being ‘only’ polynomially admissible, this subsection improves the construction to achieve linear admissibility. Note that $\kappa(n) = \sum_{m=0}^{n} \eta(m+1)$ according to Theorem [24] already is in $O(\eta(m+1))$ whenever $\eta(m) \geq 2^{O(m)}$ grows at least exponentially; hence we focus on spaces with sub-exponential entropy. To this end fix some unbounded non-decreasing $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and define a representation $\xi^{\varphi}$ of $X$ (with respect to the family $\xi_n$ of partial dense enumerations) based on the subsequence $\xi_{\varphi(n)}$ of $\xi_n$:

$$\xi^{\varphi} :\subseteq C \ni (\bar{a}_0 \ldots \bar{a}_n \ldots) \mapsto \lim_n \xi_{\varphi(n)}(\bar{a}_n) \in X, \quad \text{dom}(\xi^{\varphi}) := (6) \{ (\bar{a}_0 \ldots \bar{a}_n \ldots) : \bar{a}_n \in \text{dom}(\xi_{\varphi(n)}), d(\xi_{\varphi(n)}(\bar{a}_n), \xi_{\varphi(m)}(\bar{a}_m)) \leq 2^{-n+2^{-m}} \}
$$

Intuitively, proceeding to a subsequence $\xi_{\varphi(n)}$ amounts to ‘skipping’ intermediate precisions/error bounds and ‘jumping’ directly from $2^{-\varphi(n-1)}$ to $2^{-\varphi(n)}$.

It formalizes a strategy implemented for instance by the iRRAM C++ library for Exact Real Computation [Mil01] which starts with $\varphi(0) = 50$ bits double precision and in phases $n = \#1, \#2, \ldots$ increases to $\varphi(n) = \lceil \frac{6}{5} \cdot \varphi(n-1) \rceil + 20$.

The proof of Theorem [24] carries over literally to see:

i) Representation $\xi^{\varphi}$ has modulus of continuity $\kappa^{\varphi}(n) := \sum_{m=0}^{\varphi(n)} \eta(\varphi(m) + 1)$.
ii) To every partial function \( \zeta : \subseteq C \rightarrow X \) with modulus of continuity \( \nu \) there exists a mapping \( F : \text{dom}(\zeta) \rightarrow \text{dom}(\xi^\varphi) \) with modulus of continuity \( \nu \circ (1 + \kappa^\varphi - 1) \) such that \( \zeta = \xi^\varphi \circ F \) holds.

iii) To every \( m \in \mathbb{N} \) and every \( x, x' \in X \) with \( d(x, x') \leq 2^{-m-1} \), there exist \( \xi^\varphi \)-names \( y_x \) and \( y'_x \) of \( x = \xi^\varphi(y_x) \) and \( x' = \xi^\varphi(y'_x) \) with \( d_{\mathcal{C}}(y_x, y'_x) \leq 2^{-\kappa^\varphi(m)} \).

iv) If \((Y, e)\) is a compact metric space and \( f : X \rightarrow Y \) such that \( f \circ \xi^\varphi : \text{dom}(\xi) \subseteq C \rightarrow Y \) has modulus of continuity \( \kappa^\varphi \circ \nu \), then \( f \) has modulus of continuity \( \nu + 1 \).

**Theorem 25.** Let \((X, d)\) denote a compact metric space of entropy \( \eta \), equipped with partial mappings \( \xi_n : \subseteq \{0, 1\}^{\eta(n+1)} \rightarrow X \) such that \( X = \bigcup_{\bar{a} \in \text{dom}(\xi_n)} \overline{B(\xi_n(\bar{a}), 2^{-n-1})} \).

There exists an unbounded non-decreasing \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) such that the representation \( \xi^\varphi \) from Equation (40) has modulus of continuity \( \kappa^\varphi(n) \leq \frac{27}{4} \cdot \eta(n+1) \) and \( \kappa^\varphi(n) \geq \eta(n+1) \). In particular \( \xi^\varphi \) is linearly admissible.

The proof of Theorem 25 follows immediately from Item d) of the following lemma, applied to \( c := 3/2 \) with \( \eta(n+1) \) in place of \( \eta(n) \).

**Lemma 26.** Let \( \eta : \mathbb{N} \rightarrow \mathbb{N} \) be unbounded and non-decreasing and fix \( c > 1 \).

Then there exists a strictly increasing mapping \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) such that it holds

a) \( \forall m \in \mathbb{N} : \eta(\varphi(m + 1)) \leq c^2 \cdot \eta(\varphi(m) + 1) \).

b) \( \forall m \in \mathbb{N} : c \cdot \eta(\varphi(m)) \leq \eta(\varphi(m + 1)) \).

c) \( \forall m \in \mathbb{N} : c \cdot \eta(\varphi(m) + 1) \leq \eta(\varphi(m + 1) + 1) \).

d) \( \sum_{m=0}^{\varphi^{-1}(n)} \eta(\varphi(m)) \leq \frac{c^3}{c^2 - 1} \cdot \eta(n) \).

Think of an infinite roll of toilet papers with numbers \( \eta(0), \eta(1), \ldots \) printed on them. We shall cut this roll into appropriate runs from sheet \( \#\varphi(m) \) to \( \#\varphi(m+1) - 1 \). Item a) asserts that integers on sheets within the same run differ by no more than factor \( c^2 \). Items b) and c) formalizes that labels on consecutive runs grow at least exponentially.

**Proof (Lemma 26).** We will construct an infinite subset of \( \mathbb{N} \) by picking elements one by one. Its elements in increasing order will then constitute the sequence \( \varphi \).

First, in case there exists \( x \in \mathbb{N} \) such that \( \varphi(x) = 0 \), pick the largest such \( x \). And pick all those \( x \in \mathbb{N} \) satisfying

\[ 0 < \varphi(x) \cdot c \leq \varphi(x + 1) \]

Possibly we have picked only finitely many elements. Let \( m \) be the largest number picked so far. Pick \( x > m + 1 \) such that

\[ c \leq \frac{\varphi(x)}{\varphi(m+1)} < c^2 \]

Such \( x \) is guaranteed to exist so that we can choose. Now take \( m = x \) and repeat this process infinitely. We can mechanically check that conditions (b) and (c)
are met now. What remains is to pick more numbers so that (a) be satisfied while maintaining (b) and (c).

We will pick some more numbers for each \( i \in \mathbb{N} \) that fails condition (a). Suppose that \( i \in \mathbb{N} \) fails (a). Denote for convenience \( a := f(i) + 1 \) and \( b := f(i + 1) \). There are two cases.

**Case i)** Suppose that \( \phi(b) \leq c2^k \) for some \( k \). Pick \( x_1, x_2, \ldots, x_k \) such that the followings hold for \( j = 1, 2, \ldots, k \):

\[
\frac{\varphi(x_j)}{\varphi(a)} < \frac{c2^{j-1}}{\varphi(a)} \leq \frac{\varphi(x_j + 1)}{\varphi(a)}.
\]

**Case ii)** Suppose that \( \phi(b) > c2^k \) for some \( k \). Pick \( x_1, x_2, \ldots, x_k \) such that the followings hold for \( j = 1, 2, \ldots, k \):

\[
\frac{\varphi(x_j)}{\varphi(a)} < \frac{c2^j}{\varphi(a)} \leq \frac{\varphi(x_j + 1)}{\varphi(a)}.
\]

It is now mechanical to check that all conditions (a), (b), and (c) are fulfilled.

\( \square \)

4 Quantitative Main Theorem and Categorical Constructions

We can now establish the quantitative Main Theorem strengthening the classical qualitative one [Wei00, Theorem 3.2.11].

**Theorem 27 (Main Theorem of Type-2 Complexity Theory).** Let \((X, d)\) be compact with entropy \( \eta \) and linearly admissible representation \( \xi \) of modulus of continuity \( \kappa \). Let \((Y, e)\) be compact with entropy \( \theta \) and linearly admissible representation \( \upsilon \) of modulus of continuity \( \lambda \).

a) If \( f : X \to Y \) has modulus of continuity \( \mu \), then it admits a \((\xi, \upsilon)\)-realizer \( F \) with modulus of continuity

\[
\nu = \kappa \circ (1 + \mu) \circ (\lambda^{-1} + \mathcal{O}(1)) \in \text{lin}(\eta) \circ \mu \circ \text{lin}(\theta^{-1})
\]

b) If \( f : X \to Y \) has \((\xi, \upsilon)\)-realizer \( F \) with modulus of continuity \( \nu \), then \( f \) has modulus

\[
\mu = \kappa^{-1} \circ \nu \circ \lambda(1 + \text{id}) + \mathcal{O}(1) \in \text{lin}(\eta^{-1}) \circ \nu \circ \text{lin}(\theta)
\]

The estimated moduli of continuity are (almost) tight:

**Remark 28** Applying first (a) and then (b) always recovers \( f \) to have modulus of continuity \( \mu' : n \mapsto \mu(n + \mathcal{O}(1)) + \mathcal{O}(1) \) in place of \( \mu \), that is, optimal up to a constant shift; recall Lemma 10c).

On the other hand applying first (b) and then (a) in general recovers \( F \) only to have modulus of continuity \( \nu' = \kappa \circ \kappa^{-1} \circ \nu \circ \lambda \circ (\lambda^{-1} + \mathcal{O}(1)) + \mathcal{O}(1) \) which simplifies to \( m \mapsto \nu(m + \mathcal{O}(1)) + \mathcal{O}(1) \) under additional hypotheses such as

- Both \( \kappa \) and \( \lambda \) being surjective (and hence growing at most linearly), or
− ν being of the form κ ◦ ν′ ◦ λ−1.

Since the real unit cube [0; 1]d has linear modulus of continuity (Example 19a+b) and the signed binary representation is linearly admissible (Example 16), Theorem 27 yields the following strengthening of Example 3(b):

For any fixed d, e ∈ N and non-decreasing µ : N → N, a function f : [0; 1]d → [0; 1]e has modulus of continuity lin(µ) iff it admits a (σ, σ)-realizer with modulus of continuity lin(µ).

Recall (Remark 7) that linear metric reducibility “ζ ≻_O ξ” refines continuous reducibility “ζ ≻_T ξ” by requiring a reduction F : dom(ζ) → dom(ξ) with ζ = ξ ◦ F to have modulus of continuity µ ◦ O(O−1) = o(ν) ◦ κ−1 for every modulus of continuity ν of ζ and some κ of ξ.

Theorem 29. a) Every infinite compact metric space (X, d) of diameter ≤ 1 admits a linearly admissible representation ξ of X. Linear metric reducibility is transitive. A representation ζ is linearly admissible iff (i) it has a modulus of continuity in lin(η), and (ii) admits a linear metric reduction ξ ≺_O ζ.

b) If ξ : C → X is a linearly admissible representation of the same compact space X with respect to two metrics d and d′, then it holds d ≤ ν ≤ d′ ≤ d for some ν ∈ lin(η−1) ◦ lin(δ) ◦ lin(η).

c) Let ξ and ν be linearly admissible representations for compact (X, d) and (Y, e), respectively. Then ξ × ν : C → (ξ × ν, ξ) is linearly admissible for (X × Y, max{d, e}). Moreover it satisfies the following universal properties: The projections π_1 : X × Y ↦ (x, y) ↦ x ∈ X has a (ξ × ν, ξ)-realizer with linear modulus of continuity n ↦ 2n; π_2 : X × Y ↦ (x, y) ↦ y ∈ Y has a (ξ × ν, ν)-realizer with modulus of continuity n ↦ 2n + 1. Conversely for every fixed y ∈ Y the embedding π_2,y : X ↦ x ↦ (x, y) ∈ X × Y has a (ξ, ξ × ν)-realizer with modulus of continuity 2n ↦ n; and for every fixed x ∈ X the embedding π_1,x : Y ↦ y ↦ (x, y) ∈ X × Y has a (ν, ξ × ν)-realizer with modulus of continuity 2n + 1 ↦ n.

d) Let ξ be a linearly admissible representation of connected compact (X, d). Then the representation 2ξ of K(X) from Example 27(e) is polynomially admissible.

Note that linear ‘slack’ in a modulus of continuity of ξ translates to polynomial one in 2ξ. Item (c) justifies [Wei00, Definition 3.3.3.1] constructing a representation of X × Y from such of X and Y that it (as opposed one created one ‘from scratch’ by invoking a) is compatible with the canonical morphisms X × Y. However [Wei00, Definition 3.3.3.2] for countable products does not preserve linear admissibility; and neither does Example 21(b), already in case of spaces X_j with quadratic entropy according to Example 19(c)). For that purposes a more careful construction is needed:

Theorem 30. Fix compact metric spaces (X_j, d_j) of entropies η_j and diameters between 1/2 and 1, j ∈ N. Let ξ_j : C → X_j be uniformly linearly admissible in that (i) it has modulus of continuity κ_j(n) ≤ c + c · η_j(c + c · n)
and (ii) to every representation \( \zeta_j : \subseteq C \rightarrow X_j \) with modulus of continuity \( \nu_j \)
there exists a mapping \( F_j : \text{dom}(\zeta_j) \rightarrow \text{dom}(\zeta_j) \) with modulus of continuity \( \mu_j \)
with \( \mu_j(\kappa_j(n)) \leq \nu_j(c + c \cdot n) \) for some \( c \in \mathbb{N} \) independent of \( j \).

Let a name of \((x_0, x_1, \ldots, x_j, \ldots) \in \prod_j X_j \) be any infinite binary sequence

\[
\tilde{b}(0)|_{\kappa_0(0):\kappa_0(1)}, \quad \tilde{b}(0)|_{\kappa_0(1):\kappa_0(2)}, \quad \tilde{b}(1)|_{\kappa_1(0):\kappa_1(1)}, \\
\tilde{b}(0)|_{\kappa_0(2):\kappa_0(3)}, \quad \tilde{b}(1)|_{\kappa_1(1):\kappa_1(2)}, \quad \tilde{b}(2)|_{\kappa_2(0):\kappa_2(1)}, \quad \ldots \ldots \\
\ldots \ldots \tilde{b}(0)|_{\kappa_0(n-1):\kappa_0(n)}, \quad \tilde{b}(1)|_{\kappa_1(n-2):\kappa_1(n-1)}, \quad \ldots \\
\ldots \tilde{b}(n)|_{\kappa_j(n-j-1):\kappa_j(n-j)}, \ldots \tilde{b}(n-1)|_{\kappa_{n-1}(0):\kappa_{n-1}(1)}, \quad \ldots 
\]

such that \( \tilde{b}(j) \) is a \( \xi_j \)-name of \( x_j \). Here \( \tilde{b}|_{k, \ell} \) abbreviates the finite segment
\( \tilde{b}_{k, \ldots, b_{\ell-1}} \) of \( \tilde{b} \). The thus defined representation \( \xi := \prod_j \zeta_j : \subseteq C \rightarrow \prod_j X_j =: X \)
has modulus of continuity \( \kappa : n \mapsto \sum_{j<n} \kappa_j(n-j) \) and is linearly admissible for
\((\prod_j X_j, \sup_j d_j/2^j)\).

Moreover the projection \( \pi_j : \prod_j X_j \ni (x_0, \ldots, x_j, \ldots) \mapsto x_j \in X_j \) has a \((\prod_j \xi_j, \xi_j)\)-
realizer with modulus of continuity \( m \mapsto \kappa(\kappa_j^{-1}(m) + j) \); and for every fixed \( \bar{x} \in X \) and \( j \in \mathbb{N} \), the embedding \( \eta_{j, \bar{x}} : X_j \ni x \mapsto (x_0, \ldots, x_j, \ldots) \in X \) has a
\((\xi_j, \prod_n \xi_n)\)-realizer with modulus of continuity \( m \mapsto \kappa_j(\max\{0, \kappa_j^{-1}(m) - j\}) \).

Note that the derived moduli of continuity of the canonical morphisms \( \pi_j \) and
\( \eta_{j, \bar{x}} \) agree linearly with those predicted by Theorem 27 are thus optimal in the
sense of Remark 28. For Cartesian closure we finally treat function spaces, in view of Lemma 20w.l.o.g. the 1-Lipschitz case:

**Theorem 31.** Fix convex compact metric space \((X, d)\). To any linearly admissible representation \( \xi \) of \( X \) there exists a polynomially admissible representation
\( \xi'_1 \) of the convex compact space \( X'_1 = \text{Lip}_1(X, [0; 1]) \) of non-expansive functions
\( f : X \rightarrow [0; 1] \). It is canonical in that it asserts the application functional
\( X'_1 \times X \ni (f, x) \mapsto f(x) \in [0; 1] \) admit a \((\xi'_1, \mu, \xi, \sigma)\)-realizer \( F \) with asymptotically optimal modulus of continuity \( \leq \text{lin}(\eta'_1) \).

Recall that Condition (ii) of polynomial admissibility means polynomial metric reducibility \( \sim_p \xi_1 \) to \( \xi_1' \) of any other continuous representation \( \zeta \) of \( X'_1 \).

**4.1 Proofs**

**Proof (Theorem 27).**

a) First suppose \( \xi = \xi^\sharp \) the representation from Theorem 25 with modulus of
continuity \( \kappa(n) \leq O(\theta(n+1)) \) and similarly \( v \) with \( \lambda(n) \leq O(\theta(n+1)) \).

Applying (ii) to \( \zeta := f \circ \xi : \text{dom}(\xi) \subseteq C \rightarrow Y \) with modulus of continuity
\( \kappa \circ \mu \) yields a \((\xi, v)\)-realizer \( F \) with modulus \( \nu = \kappa \circ \mu \circ (1 + \lambda^{-1}) \).

Next consider arbitrary linearly admissible \( \xi' \) with modulus \( \kappa' \) and \( v' \) with \( \lambda' \).

Applying (ii) to \( \xi' \) yields \( G : \text{dom}(\xi) \rightarrow \text{dom}(\xi) \) with modulus \( \kappa' \circ (1 + \lambda^{-1}) \)
such that \( \xi' = \xi \circ G \); and applying (ii) of Definition 11 to \( v \) yields \( H : \text{dom}(\xi) \rightarrow \text{dom}(\xi') \) with modulus \( \lambda \circ (\lambda^{-1} + O(1)) \) such that \( v = v' \circ H \).
Together, \( F' := H \circ F \circ G \) constitutes a \((\xi', \nu')\)-realizer of \( f \) with modulus of continuity

\[
\nu' = \kappa' \circ (1 + \lambda^{-1}) \circ \kappa \circ \mu \circ (1 + \lambda^{-1}) \circ \lambda \circ (\lambda^{-1} + \mathcal{O}(1)) \\
\leq \kappa' \circ (1 + \mu) \circ (\lambda^{-1} + \mathcal{O}(1)) \in \text{lin}(\eta) \circ \mu \circ \text{lin}(\theta^{-1})
\]

by Lemma \( \text{(10)+e} \) since \( \eta \leq \kappa' \in \text{lin}(\eta) \) and \( \theta \leq \lambda' \in \text{lin}(\theta) \) according to Definition \( \text{(11)i} \) and Example \( \text{(19)} \).

b) As in (a) first suppose \( F \) is a \((\xi, \nu)\)-realizer of \( f \) with modulus \( \nu \), for \( \xi = \xi' \) from Theorem \( \text{25} \) with modulus of continuity \( \kappa(n) \leq \mathcal{O}(\eta(n+1)) \) and similarly \( \nu \) with \( \lambda(n) \leq \mathcal{O}(\theta(n+1)) \). Then \( f \circ \xi = \nu \circ F : \text{dom}(\xi) \subseteq \mathcal{C} \rightarrow Y \) has modulus of continuity \( \nu \circ \lambda \leq \kappa \circ \kappa' \circ \nu \circ \lambda \) by Lemma \( \text{(10)+e} \); and (iv) implies \( f \) to have modulus \( \mu = 1 + \kappa^{-1} \circ \nu \circ \lambda \).

Next consider arbitrary linearly admissible \( \xi' \) with modulus \( \kappa' \) and \( \nu' \) with \( \lambda' \); and let \( F' \) be a \((\xi', \nu')\)-realizer of \( f \) with modulus \( \nu' \). (iii) yields \( H' \) with \( \nu' = \nu \circ H' \) of modulus \( \lambda' \circ (1 + \lambda^{-1}) \); and \( G' \) with \( \xi' = \xi' \circ G' \) of modulus \( \kappa \circ (\kappa^{-1} + \mathcal{O}(1)) \) according to Definition \( \text{(11)i} \). Together, \( F := H' \circ F' \circ G' \) constitutes a \((\xi, \nu)\)-realizer of \( f \) with modulus \( \nu = \kappa \circ (\kappa^{-1} + \mathcal{O}(1)) \circ \nu' \circ \lambda' \circ (1 + \lambda^{-1}) \). So our initial consideration implies \( f \) to have modulus

\[
\mu' = 1 + \kappa^{-1} \circ \kappa \circ (\kappa^{-1} + \mathcal{O}(1)) \circ \nu' \circ \lambda' \circ (1 + \lambda^{-1}) \circ \lambda \\
\leq \mathcal{O}(1) + \kappa^{-1} \circ \nu' \circ \lambda'(1 + \text{id}) \in \text{lin}(\eta^{-1}) \circ \nu' \circ \text{lin}(\theta)
\]

by Lemma \( \text{(10)+e} \) since \( \eta \leq \kappa' \in \text{lin}(\eta) \) and \( \theta \leq \lambda' \in \text{lin}(\theta) \) according to Definition \( \text{(11)i} \) and Example \( \text{(19)} \).

**Proof (Theorem \( \text{25} \)).**

a) Theorem \( \text{25} \) asserts the first claim. For the second let \( \zeta \) have modulus \( \nu \) and \( \zeta' \) have modulus \( \nu' \) and \( \zeta'' \) have modulus \( \nu'' \); let \( F : \text{dom}(\zeta) \rightarrow \text{dom}(\zeta') \) with \( \zeta = \zeta' \circ F \) have modulus of continuity \( \nu \circ \mathcal{O}(\nu'^{-1}) \) and \( F' : \text{dom}(\zeta') \rightarrow \text{dom}(\zeta'') \) with \( \zeta' = \zeta'' \circ F' \) have modulus of continuity \( \nu' \circ \mathcal{O}(\nu''^{-1}) \). Then \( \zeta = \zeta'' \circ F' \circ F \), where \( F' \circ F \) has modulus

\[
\nu \circ \mathcal{O}(\nu'^{-1}) \circ \nu' \circ \mathcal{O}(\nu''^{-1}) = \nu \circ \mathcal{O}(\nu''^{-1})
\]

Finally, \( \tilde{\xi} \leq_{\mathcal{O}} \xi \) is necessary according to Definition \( \text{(11)ii} \); while any other representation \( \tilde{\xi} \leq_{\mathcal{O}} \tilde{\xi} \leq_{\mathcal{O}} \zeta \) since \( \tilde{\xi} \) is admissible.

b) First consider \( \xi = \xi' \) the representation from Theorem \( \text{25} \) with modulus \( \tilde{\kappa} \).

By (iii), to every \( x, x' \in X \) with \( d(x, x') \leq 2^{-m-1} \), there exist \( \xi \)-names \( \tilde{y}_{x'} \) and \( \tilde{y}_{x'}' \) of \( x = \tilde{\xi}(\tilde{y}_{x}) \) and \( x' = \tilde{\xi}(\tilde{y}_{x}') \) with \( d_{C}(\tilde{y}_{x}, \tilde{y}_{x}') \leq 2^{-\tilde{\kappa}(m)} \). Hence \( d'(x, x') = d'(\tilde{\xi}(\tilde{y}_{x}), \tilde{\xi}(\tilde{y}_{x}')) \leq 2^{-n} \) by (i) for \( \tilde{\kappa}' \in \text{lin}(\eta) \) modulus of continuity of \( \xi \) w.r.t. \( d' \) and \( \tilde{\kappa}'(m) \geq \tilde{\kappa}'(n) \), that is for \( m = \tilde{\kappa}'^{-1} \circ \tilde{\kappa}'(n) \) by Lemma \( \text{(10)} \): \( d' \leq \mu, d' \leq \nu, d' \leq \nu, d' \leq \nu, d' \leq \nu \)

Now let \( \xi \) be linearly admissible with modulus \( \kappa \). Then \( \xi = F \circ \tilde{\xi} \) for some \( F : \text{dom}(\xi) \rightarrow \text{dom}(\tilde{\xi}) \) with modulus \( \mu \leq \kappa \circ \mathcal{O}(\tilde{\kappa}^{-1}) \) Hence \( \tilde{z}_{x} := F(\tilde{y}_{x}) \) and \( \tilde{z}_{x}' = F(\tilde{y}_{x}') \) have \( d_{C}(\tilde{z}_{x}, \tilde{z}_{x}') \leq 2^{-m} \) for \( \tilde{\kappa}(m) \geq \mu(m) \); and again \( d'(x, x') \leq 2^{-n} \) for \( m' \geq \tilde{\kappa}'(n) \) modulus of continuity of \( \xi \) w.r.t. \( d' \) for \( \nu := 1 + \tilde{\kappa}^{-1} \circ \mu \circ \tilde{\kappa}' \in \text{lin}(\eta^{-1}) \circ \text{lin}(\eta) \circ \text{lin}(\eta) \).
c) Let $X$ and $Y$ have entropies $\eta$ and $\theta$, respectively; $\xi$ and $\nu$ moduli of continuity $\mu \leq \text{lin}(\eta)$ and $\nu \leq \text{lin}(\theta)$. (i) $X \times Y$ has entropy at least $\eta(n-1) + \theta(n-1) - 1$ by Example (19); and $\xi \times \nu$ has modulus of continuity $2 \cdot \max\{\mu, \nu\} \leq \text{lin}(n \mapsto \eta(n-1) + \theta(n-1) - 1)$. (ii) Let $\zeta : \subseteq C \mapsto X \times Y$ be any representation with modulus $\kappa$. Then its projection onto the first component $\zeta_1$ constitutes a continuous representation of $X$ with some modulus $\kappa$. Hence there exist by hypothesis $F : \text{dom}(\zeta) \rightarrow \text{dom}(\xi)$ of modulus $\kappa \circ O(\mu^{-1})$ such that $\zeta_1 = \xi \circ F$; similarly $\zeta_2 = \nu \circ G$ with $G : \text{dom}(\zeta) \rightarrow \text{dom}(\nu)$ of modulus $\kappa \circ O(\nu^{-1})$. Then $F \times G : = (F_0, G_0, F_1, G_1, F_2, \ldots) : \text{dom}(\zeta) \rightarrow \text{dom}(\xi \times \nu)$ satisfies $\zeta = (\xi \times \nu) \circ (F \times G)$ and has modulus
\[
2 \cdot \max \left\{ \kappa \circ O(\mu^{-1}), \kappa \circ O(\nu^{-1}) \right\} = 2 \cdot \kappa \circ O(\min\{\mu, \nu\}^{-1})
\]
\[
\leq \kappa \circ O(2 \cdot \max\{\mu, \nu\}^{-1}) \quad \text{by Lemma (19)}.
\]
Finally, $b(n, b_1, b_2, b_3, b_4, \ldots) \mapsto (b_0, b_2, b_4, \ldots)$ is a $(\xi \times \nu, \xi)$-realizer of $\pi_1$ with modulus of continuity $n \mapsto 2n$; and $(b_0, b_1, b_2, b_3, b_4, \ldots) \mapsto (b_1, b_3, \ldots)$ a $(\xi \times \nu, \nu)$-realizer of $\pi_2$ with modulus of continuity $n \mapsto 2n + 1$. For any fixed $\nu$-name $(b_0, b_3, \ldots)$ of $y$, $(b_0, b_2, \ldots) \mapsto (b_0, b_1, b_3, b_4, \ldots)$ is a realizer of $\pi_2 y$ with modulus of continuity $2n \mapsto n$; and any fixed $\xi$-name $(b_0, b_2, \ldots)$ of $x$, $(b_1, b_3, \ldots) \mapsto (b_0, b_1, b_2, b_3, b_4, \ldots)$ is a realizer of $\pi_1 x$ with modulus of continuity $2n + 1 \mapsto n$.

d) Combine Example (21) with Example (19). $\square$

**Proof (Theorem 20).** Since $X_j$ has diameter between 1/2 and 1, w.l.o.g. $\eta_j(0) \leq 0$ and $\eta_j(n) \geq 1$ for $n \geq 2$ and w.l.o.g. $\kappa_j(0) = 0$. $X := \prod_j X_j$ has entropy $\eta(n) \geq \sum_{j<n} \eta_j(n-1-j) - [n/2]$ by Example (19). On the other hand the initial segment $\bar{b}^{(j)}(\nu_{j,n-j})$ of a $\xi_j$-name $\bar{b}^{(j)}$ determines $x_j = \xi_j(\bar{b}^{(j)})$ up to error $2^{-(n-j)}$ w.r.t. $d := d_j/2^j$; and is located among the first $\kappa_0(n) + \kappa_1(n-1) + \cdots + \kappa_n(0) = \kappa(n)$ symbols of a $\xi$-name of $(x_j)_j$ according to Equation (7); recall $\kappa_0(0) = 0$. Therefore (i) $\xi$ has modulus of continuity $\kappa(n)$, which is $\leq \text{lin}(\eta(n))$ since $\kappa_j(n) \leq c + c \cdot \eta_j(c + c \cdot n)$ and $\eta_j(n) \geq 1$ for $n \geq 2$ ‘covers’ the $[n/2]$.

Regarding (ii), let $\zeta : \subseteq C \mapsto X$ have modulus of continuity $\nu$. The projection $\pi_j : X \ni (x_0, \ldots x_j, \ldots) \mapsto x_j \in X_j$ has modulus of continuity $n \mapsto n + j$ since $X$ is equipped with metric $d = \sup_j d_j/2^j$. The representation $\zeta_j := \pi_j \circ \zeta : \text{dom}(\zeta) \subseteq C \mapsto X_j$ thus has modulus $\nu_j : n \mapsto \nu(n + j)$. By hypothesis (ii) on $\xi_j$, there exists a mapping $F_j : \text{dom}(\zeta) \rightarrow \text{dom}(\xi_j)$ whose modulus of continuity $\mu_j$ satisfies $\mu_j(\kappa_j(n)) \leq \nu_j(c + c \cdot n)$ such that it holds $\pi_j \circ \zeta = \xi_j \circ F_j$. Now let $F :=$
so that $\zeta = \xi \circ F$. Moreover $F_j|_{\kappa_j(n-j)}$ depends on the first $\mu_j(\kappa_j(n-j))$ symbols of its argument; hence $F$ has modulus of continuity $\mu$ with

$$\mu(\kappa(n)) = \sup_{j<n} \mu_j(\kappa_j(n-j)) \leq \sup_{j<n} \nu_j(c+c \cdot (n-j)) = \nu(c+c \cdot n).$$

Mapping an infinite binary sequence according to Equation (7) to $\bar{b}^{(j)}$ constitutes a $\left(\prod_i \xi_i, \xi_j\right)$-realizer of $\pi_j$; and for $n \geq j$ the first $\kappa_j(n-j)$ bits of $\bar{b}^{(j)}$ are determined by the first $\kappa(n)$ bits of the given $\xi$-name; hence $\kappa_j(n-j) \Rightarrow \kappa(n)$ a modulus of continuity. Conversely mapping $\xi_j$-name $\bar{b}^{(j)}$ of $x_j$ to the infinite binary sequence from Equation (7) constitutes a $\left(\xi_j, \prod_i \xi_i\right)$-realizer of $\nu_j, x$ with modulus of continuity $\kappa(n) \Rightarrow \kappa_j(\max\{0, n-j\})$.

Proof (Theorem [77]). Let $\eta$ denote the entropy of $X$ and $\kappa \leq \ln(\eta)$ a modulus of continuity of $\xi$. Only for notational simplicity, consider the case $\mu = \text{id}$ of $1$-Lipschitz functions $X_1':=\text{Lip}_1(X,[0,1])$. We pick up on, and refine, the entropy analysis from the proof of Example [19h]). Fix $n \in \mathbb{N}$ and, for every $\bar{w} \in \{0,1\}^{\kappa(n)}$ with $\xi[\bar{w}\check{C}] \neq \emptyset$, choose some $x_{\bar{w}} \in \xi[\bar{w}\check{C}]$. Record that, as centers of closed balls of radius $2^{-n}$, these cover $X$.

Next choose some subset $W_n \subseteq \{0,1\}^{\kappa(n)}$ such that any two distinct $\bar{w} \in W_n$ satisfy $d(x_{\bar{w}}, x_{\bar{w}'}) > 2^{-n}$ while the closed balls $\overline{B}(x_{\bar{w}}, 2^{-n+1})$ of double radius still cover $X$. Such $W_n$ can be created greedily by repeatedly and in arbitrary order weeding out one of $(\bar{w}, \bar{w}')$ whenever $d(x_{\bar{w}}, x_{\bar{w}'}) \leq 2^{-n}$; observe $\overline{B}(x_{\bar{w}}, 2^{-n+1}) \supseteq \overline{B}(x_{\bar{w}'}, 2^{-n})$ and abbreviate $X_n := \{x_{\bar{w}} : \bar{w} \in W_n\}$.

We now formalize the idea that a $\xi_1'$-name of $f \in X_1'$ encodes a sequence $f_n : X_n \to \mathbb{D}_n$ of $\frac{3}{2}$-Lipschitz functions whose $\frac{3}{2}$-Lipschitz extensions $f_n^*$ approximate $f$ up to error $2^{-n+1}$: As in the proof of Example [19h]), $f_n^* := [2^n \cdot f |_{X_n}] / 2^n$ satisfies this condition, hence asserting every $f \in X_1'$ to have a $\xi_1'$-name, i.e. $\xi_1'$ be surjective.

In order to encode the $f_n$ succinctly, and make $\xi_1'$ have modulus of continuity $\kappa_1' \leq \ln(\eta_1')$ for the entropy $\eta_1'$ of $X_1'$, recall the connected undirected graph $G_n = (X_n, E_n)$ from the proof of Example [19h] with edge $(x,x')$ present iff the open balls of radius $2^{-n+2}$ around centers $x, x'$ intersect. Choose some directed spanning tree $F_n \subseteq E_n$ of $G_n$ with root $x_{n,0}$ and remaining nodes $x_{n,1}, \ldots, x_{n,N_n-1}$ in some topological order, where $N_n := \text{Card}(W_n) \leq 2^{\kappa(n)}$.

As in the proof of Example [19h]), every $\frac{3}{2}$-Lipschitz $f_n : X_n \to \mathbb{D}_n$ is uniquely described by $f_n(x_{n,0}) \in \mathbb{D}_n \cap [0;1]$ together with the sequence

$$f_n(x_{n,m}) - f_n(x_{n,m-1}) \in \{-6 \cdot 2^{-n}, \ldots, 0, \ldots, +6 \cdot 2^{-n}\}, \ 1 \leq m < N_n .$$

The first takes $n+1$ bits to describe, the latter $(N_n - 1) \times \log_2(13)$ bits: in view of Example [19c] a total of $O(2^{\kappa(n)})$ to encode $f_n$ in some $\bar{u}_n$. So the initial segment $(\bar{u}_0, \ldots, \bar{u}_{n+1})$ of a thus defined $\xi_1'$-name $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots)$ of $f$, encoding $f_0, \ldots, f_{n+1}$ and thus determining $f$ up to error $2^{-n}$, has length $\kappa_1'(n) = O(2^{\kappa(0)}) + \cdots + O(2^{\kappa(n+1)}) \leq O(2^{\kappa(n+2)}) \leq \ln(\eta_1')$ again by Example [19h])—thus establishing Condition (i).

Regarding the application functional $(f, x) \mapsto f(x)$, consider the following ‘algorithm’ and recall Theorem [20b]): Given a $(\xi_j \times \xi)$-name $(u_0, u_1, v_1, \ldots, u_m, v_m, \ldots)$

[9]Different choices lead to different representations $\xi_1'$ of $X_1'$. 
of \((f, x)\) and \(n \in \mathbb{N}\), let \(\bar{v} := \bar{v}_{<\kappa(n)}\) and ‘find’ some \(\bar{w} \in W_n\) with \(d(x_{\bar{w}}, x_{\bar{v}}) \leq 2^{-n+1}\); then ‘trace’ the path in spanning tree \((W_n, F_n)\) from its root \(x_{n,0}\) to \(x_{\bar{w}} = x_{n,M}\): all information contained within the first \(\kappa_1'(n)\) bits of \(\bar{u}\) encoding \(\frac{3}{2}\)-Lipschitz \(f_n : X_n \to \mathbb{D}_n\) whose extension approximates \(f\) up to error \(2^{-n+1}\), sufficient to recover the value

\[
y_n := f_n(x_{\bar{w}}) = f_n(x_{n,0}) + \sum_{m=1}^M \left( f_n(x_{n,m}) - f_n(x_{n,m-1}) \right) \in \mathbb{D}_n
\]
satisfying \(|y_n - f(x)| \leq |y_n - f_n(x)| + |f_n(x) - f(x)| \leq \frac{3}{2} \cdot 2^{-n+1} + 2^{-n+1} \leq 2^{-n+3}\). The (initial segment of length \(n + 3\) of) sequence \(y_0, \ldots, y_n, \ldots\) in turn is easily converted to (an initial segment of length \(n\) of) a signed binary expansion of \(f(x)\) \cite[Lemma 7.3.5]{Wei00}: yielding a \((\xi_1' \times \xi, \sigma)\)-realizer of \((f, x) \mapsto f(x)\) with asymptotically optimal modulus of continuity \(\mu(n) = \max\{2\kappa(n+3), 2\kappa_1'(n + 3)\} \leq \text{lin} (\eta_1'(n + 3))\) by Example \cite[19]{19}.

\(\Box\)

5 Conclusion and Perspective

For an arbitrary compact metric space \((X, d)\) we have constructed a generic representation \(\xi\) with optimal modulus of continuity, namely agreeing with the space’s entropy up to a constant factor. And we have shown this representation to exhibit properties similar to the classical standard representation of a topological \(T_0\) space underlying the definition of qualitative admissibility, but now under the quantitative perspective crucial for a generic resource-bounded complexity theory for computing with continuous data: \(\xi\) is maximal with respect to optimal metric reduction among all continuous representations. The class of such metrically optimal representations is closed binary and countable Cartesian products, and gives rise to metrically optimal representations of the Hausdorff space of compact subsets and of the space of non-expansive real functions. Moreover, with respect to such linearly admissible representations \(\xi\) and \(\nu\) of compact metric spaces \(X\) and \(Y\), optimal moduli of continuity of functions \(f : X \to Y\) and their \((\xi, \nu)\)-realizers \(F : \text{dom}(\xi) \to \text{dom}(\nu)\) are linearly related up to composition with (the lower semi-inverse) of the entropies of \(X\) and \(Y\), respectively.

All our notions (entropy, modulus of continuity) and arguments are information-theoretic: according to Fact \ref{fact:b} these precede, and under suitable oracles coincide with, complexity questions. They thus serve as general guide to investigations over concrete advanced spaces of continuous data, such as of integrable or weakly differentiable functions employed in the theory of Partial Differential Equations \cite{Ste17}.

In order to strengthen our Main Theorem \ref{main theorem} namely to further decrease the gap between (a) and (b) according to Remark \ref{remark} we wonder:

**Question 32** Which infinite compact metric spaces \((X, d)\) with entropy \(\eta\) admit

\(a)\) a representation with \(\eta(n + \mathcal{O}(1)) + \mathcal{O}(1)\) as modulus of continuity?

\(b)\) an admissible representation with modulus of continuity \(\eta(n + \mathcal{O}(1)) + \mathcal{O}(1)\)?

\(c)\) a linearly admissible representation with modulus \(\eta(n + \mathcal{O}(1)) + \mathcal{O}(1)\)?
d) If \( \xi \) is a linearly admissible representation of \((X,d)\) and \( Z \subseteq X \) closed, is the restriction \( \xi|_Z \) then again linearly admissible? [Wei00] Lemma 3.3.2

e) In view of Example [22], how large can be the asymptotic gap between intrinsic \( \eta_{K,K} \) and relative entropy \( \eta_{X,K} \)?

f) (How) does Theorem [21] generalize from real \([0;1]\) to other compact metric codomains \( Y \)?

g) How do the above considerations carry over from the Type-2 setting of computation on streams to that of oracle arguments [Ko91, KC12]?

5.1 Algorithmic Cost and Representations for Higher Types

It seems counter-intuitive that the application functional should be as hard to compute as Example [3] predicts. To ‘mend’ this artefact, the Type-2 Machine model from Definition [11] — computing on ‘streams’ of infinite sequences of bits \( \bar{b} = (b_0, b_1, \ldots b_n, \ldots) \in \{0,1\}^\mathbb{N} \) with sequential/linear-time access — is commonly modified for problems involving continuous subsets or functions as arguments to allow for ‘random’/logarithmic-time access [Ko91, KC12]:

**Definition 33.** a) Abbreviate with \( C' := \{0,1\}^{0,1} \) the set of total finite string predicates \( \varphi : \{0,1\}^* \rightarrow \{0,1\} \), equipped with the metric \( d_C(\varphi, \psi) = 2^{-\min\{|\bar{u}|; \varphi(\bar{u}) \neq \psi(\bar{u})\}} \), where \( |\bar{u}| \in \mathbb{N} \) denotes the length \( n \) of \((u_0, \ldots u_{n-1}) \in \{0,1\}^* \).

b) A Type-3 Machine \( M^\varphi \) is an ordinary oracle Turing machine with variable oracle. It computes the partial function \( F \subseteq C' \rightarrow C' \) if, for every \( \varphi \in \text{dom}(F) \subseteq C' \), \( M^\varphi \) computes \( F(\varphi) \in C' \) in that it accepts all inputs \( \bar{v} \in (F(\varphi))^{-1}[1] \) and rejects all inputs \( \bar{v} \in (F(\varphi))^{-1}[0] \); The behaviour of \( M^\varphi \) for \( \varphi \notin \text{dom}(F) \) may be arbitrary.

c) For \( \psi \in C' \), \( M^{\psi \varphi} \) denotes a Type-3 Machine with fixed oracle \( \psi \) and additional variable oracle in that it operates, for every \( \varphi \in \text{dom}(F) \subseteq C' \), like \( M^{\psi \otimes \varphi} \), where

\[
\psi \otimes \varphi : (0 \bar{u}) \mapsto \psi(\bar{u}), \quad (1 \bar{u}) \mapsto \varphi(\bar{u}).
\]

d) \( M^\varphi \) computes \( F \) in time \( t : \mathbb{N} \rightarrow \mathbb{N} \) if \( M^\varphi \) on input \( \bar{v} \in \{0,1\}^n \) stops after at most \( t(n) \) steps regardless of \( \varphi \in \text{dom}(F) \); similarly for \( M^{\psi \otimes \varphi} \).

Encoding a continuous space using oracles \( \psi \in C' \) rather than sequences \( \bar{b} \in \mathcal{C} \) gave rise to a different, new notion of representation [KC12 §3.4]. Here we shall call it hyper-representation, and avoid confusion with the previous conception now referred to as stream representation.

**Definition 34.** a) A hyper-representation of a space \( X \) is a partial surjective mapping \( \Xi : \subseteq C' \rightarrow X \).

b) For hyper-representations \( \Xi : \subseteq C' \rightarrow X \) and \( \Upsilon : \subseteq C' \rightarrow Y \), a \((\Xi,\Upsilon)\)-realizer of a function \( f : X \rightarrow Y \) is a partial function \( F : \text{dom}(\Xi) \rightarrow \text{dom}(\Upsilon) \) such that \( f \circ \Xi = \Upsilon \circ F \) holds.

c) A reduction from \( \Xi : \subseteq C' \rightarrow X \) to \( \Xi' : \subseteq C' \rightarrow X \) is a \((\Xi,\Xi')\)-realizer of the identity \( \text{id} : X \rightarrow X \).
d) \((ξ,Υ)\)-computing \(f\) means to compute some \((ξ,Υ)\)-realizer \(F\) of \(f\) in the sense of Definition 35d).

e) The product hyper-representation of \(ξ : C → X\) and \(Υ : C' → Y\) is

\[ξ × Υ : C → X\times Y\]

\(\varphi → (ξ(\bar{v}) → Ψ(0 \bar{v})), Υ(\bar{v}) → Ψ(1 \bar{v}))\) ∈ \(X × Y\)

f) Consider the hyper-representation (sic!) \(υ : C → Y\) of \(f\).

For stream representation \(ξ : C → X\), let \(ξ \circ υ : C' → X\) denote its induced unary hyper-representation.

g) Abusing notation, consider the hyper-representation

\[\text{bin} : C' → ϕ → (ϕ(\text{bin}(n)))_n \in \mathcal{C} \]

For stream representation \(ξ : C → X\), let \(ξ \circ \text{bin} \subseteq C' → X\) denote its induced binary hyper-representation.

Indeed, according to [KC12, §4.3], appropriate hyper-representations now allow to compute the application functional in time more reasonable than in Example 5b); cmp. Item d) of the following Proposition. Items a) to c) correspond to Fact [6]

**Proposition 35.**

a) \(C'\) is compact of entropy \(η_{C'} = 2^{id} - 1\). If \(F : C → C'\) is computed by \(M\) and if \(\text{dom}(F)\) is compact, then this computation admits a time bound \(t : \mathbb{N} → \mathbb{N}\) in the sense of Definition 35d); similarly for \(M^{ψ,\#:\#}\).

b) If \(M^{ψ,\#:\#}\) computes \(F : C → C'\) in time \(O(n + 2^k(n))\).

c) If \(F : C → C'\) has modulus of continuity \(μ : \mathbb{N} → \mathbb{N}\), then there exists an oracle \(ψ ∈ C'\) and a Type-3 Machine \(M^{ψ,\#:\#}\) computing \(F\) in time \(O(n + 2^k(n))\).

d) The compact space \([0; 1]_1\) from Example 5b) admits a hyper-representation \(Δ_1\) such that application \([0; 1]_1 × [0; 1] \ni (f, r) → f(r) \in [0; 1]\) is \((Δ_1 × δ, δ \circ υ_{C'}\#:\#)-computable in polynomial time for the unary hyper-representation \(δ \circ υ_{C'}\#:\#\) induced by the aforementioned dyadic stream representation \(δ\) of \([0; 1]_1\).

e) Hyper-representation (sic!) \(υ : C → C\) is an isometry. Stream representation \(ξ : C → X\) has modulus of continuity \(κ\) iff induced unary hyper-representation \(ξ \circ υ_{C'}\#:\#\) does.

f) For \((ξ,υ)\)-realizer \(F : C → C\) of \(f : X → Y\), \(F := υ_{C'}^{-1} \circ F \circ υ_{C} : C → C'\) is a \((ξ \circ υ_{C}, υ \circ υ_{C'})\)-realizer of \(f : X → Y\). \(F\) has modulus of continuity \(μ\) iff \(F\) does.

g) Hyper-representation \(\text{bin} : C' → C\) has logarithmic modulus of continuity.

Its inverse, the stream representation \(\text{bin} : C → C'\) has exponential modulus of continuity. Both are optimal. Stream representation \(ξ : C → X\) has modulus of continuity \(O(2^κ)\) iff induced binary hyper-representation \(ξ \circ \text{bin} : C' → X\) has modulus of continuity \(κ\).

Note the gap between Type-3 Proposition 35b+c), absent in the Type-2 Fact 6b+c).

From a high-level perspective, the exponential lower complexity bound to the application functional in Example 5b) is due to \([0; 1]'\) having exponential entropy while stream representations’ domain \(C\) has only linear entropy: see
Example [19h+h]. Proposition [35d]) avoids that information-theoretic bottleneck by proceeding to hyper-representations with domain $C'$ also having exponential entropy.

In fact [KCT12 §4.3] extends the representation $\Delta_1$ of $[0;1]_1 = \text{Lip}_1([0;1],[0;1])$ from Proposition [35d]) to entire $C([0;1],[0;1]) = \bigcup_\mu C_\mu([0;1],[0;1])$, where the union ranges over all strictly increasing integer output precision parameter. In view of Proposition [35a) one cannot expect a time bound on entire $\mu([0;1])$ depending only on the output precision $n$. Instead [KCT12, §3.2] considers runtime polynomial if bounded by some term $P = P(n,\mu)$ in both the integer output precision parameter $n$ and a modulus of continuity $\mu : \mathbb{N} \to \mathbb{N}$ of the given function argument $f$: a higher-type parameter $\text{NS18}$. The considerations in the present work suggest a natural

**Remark 36** According to Example [19h]) $C_\mu([0;1],[0;1])$ has entropy $\eta = \Theta(2^n)$ such that $\log \eta = \Theta(\mu)$ `recovers’ the modulus of continuity. Moreover our complexity-theoretic “Main” Theorem [27] confirms that even metrically well-behaved (e.g. 1-Lipschitz) functionals $\varphi : X \to [0;1]$ can only have realizers with modulus of continuity/time complexity growing in $X$’s entropy $\eta$. This suggests generalizing second-order polynomial runtime bounds $P = P(n,\mu)$ from spaces of continuous real functions [KCT12 §3.2] to $P(n,\log \eta)$ for compact metric spaces $X$ beyond $C_\mu([0;1],[0;1])$. The logarithm is consistent with the quantitative properties of hyper-representations expressed in Proposition [35].

**Proof (Proposition 35).**

a) Compactness of $C'$ follows from König’s Lemma: it is an infinite finitely (only, as opposed to $C$, increasingly) branching tree. Cover $C'$ by $2^{2^{n-1}}$ closed balls $\{\varphi : \varphi|_{[0,1]^{<n}} = \psi\}, \psi : \{0,1\}^n \to \{0,1\}$ of radius $2^{-n}$: optimally. Consider the number $N := t_M(\varphi, \vec{u}) \in \mathbb{N} \cdot \mathcal{M}_\varphi$ makes on input $\vec{u} \in \{0,1\}^n$ for $\varphi \in \text{dom}(\varphi) \subseteq C'$. During this execution, $\mathcal{M}_\varphi$ can construct and query oracle $\varphi$ only on strings $\vec{u}$ of length $|\vec{u}| < N$. Replacing $\varphi$ with some $\psi \in C'$ of distance $d_{C'}(\varphi, \psi) \leq 2^{-N}$ will remain undetected, that is, $\mathcal{M}_\psi$ on input $\vec{u}$ will behave the same way, and in particular still terminate after $N$ steps. This establishes continuity of $t_M(\cdot, \vec{u})$. By compactness of $\text{dom}(\varphi)$, the following maxima thus exist:

$$t_M(\vec{u}) := \max \{ t_M(\varphi, \vec{u}) : \varphi \in \mathcal{F} \} $$

$$t_M(n) := \max \{ t_M(\vec{u}) : \vec{u} \in \{0,1\}^n \} $$

b) As mentioned in the proof of (a), $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$ will behave identically on all inputs $\vec{u} \in \{0,1\}^n$ for $\varphi, \psi \in \text{dom}(\varphi)$ with $d_{C'}(\varphi, \psi) \leq 2^{-t(n)}$: Meaning $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$ have distance $\leq 2^{-n}$.

c) By hypothesis, $\mathcal{F}(\varphi)(\vec{u}) \in \{0,1\}$ depends only on the restriction $\varphi|_{[0,1]^{<m}}$ for $m := \mu(n)$ and $n := |\vec{u}|$. Thus

$$\psi(\vec{u}, \varphi(0), \varphi(1), \varphi(00), \ldots \varphi(1^{m-1})) := \mathcal{F}(\varphi)(\vec{u})$$

is well-defined an oracle. And, for given $\vec{u}$ and $\varphi$, making this query of length $O(n + 2^m)$ recovers the value $\mathcal{F}(\varphi)(\vec{u})$. □
5.2 Representation Theory of Compact Metric Spaces

Both stream and hyper representations translate (notions of computability and bit-complexity) to their co-domains from ‘universal’ \([\text{Ben}98]\) compact structures (which are already naturally equipped with formal conceptions of computing, namely) \(C\) and \(C'\), respectively. Matthias Schröder [personal communication 2017] has suggested a third candidate domain of generalized representations: the Hilbert Cube \(H = \prod_{j \geq 0} [0; 1] \) equipped with metric \(d_H(x, y) = \sup_j |x_j - y_j|/2^j\); recall Example \([19h]\). This parallels earlier developments in continuous computability theory considering equilogical [BBS04] and quotients of countably-based topological (=QCB) spaces [Sch06]. And its suggests the following generalization:

**Definition 37.** Fix a compact metric space \((X, D)\) with entropy \(\Theta\).

a) A \(X\)-representation of another compact metric space \((X, d)\) is a surjective partial mapping \(\xi : \subseteq X \rightarrow X\).

b) Let \(\eta\) denote the entropy of \((X, d)\).

Call \(\xi\) linearly admissible if it has a modulus of continuity \(\kappa\) such that
(i) \(\Theta \circ \kappa \leq O_\Theta(\eta)\), i.e., \(\exists C \in \mathbb{N} \forall n \in \mathbb{N} : (\Theta \circ \kappa)(n) \leq C + C \cdot \eta(n + C)\)
and (ii) for every uniformly continuous partial surjection \(\xi : \subseteq X \rightarrow X\) it holds \(\xi \leq_\Theta \xi\), meaning: There exists a map \(F : \subseteq \text{dom}(\xi) \rightarrow \text{dom}(\xi)\) such that \(\xi = \xi \circ F\) and, for every modulus of continuity \(\nu\) of \(\xi\), \(F\) has a modulus of continuity \(\mu\) satisfying \(\mu \circ \kappa \leq O_\Theta(\nu)\).

c) Call \(\xi\) polynomially admissible if it has a modulus of continuity \(\kappa\) such that
(i) \(\Theta \circ \kappa \leq P_\Theta(\eta)\), and
(ii) for every uniformly continuous partial surjection \(\xi : \subseteq X \rightarrow X\) it holds \(\xi \leq_\Theta \xi\), meaning: There exists a map \(F : \subseteq \text{dom}(\xi) \rightarrow \text{dom}(\xi)\) such that \(\xi = \xi \circ F\) and, for every modulus of continuity \(\nu\) of \(\xi\), \(F\) has a modulus of continuity \(\mu\) satisfying \(\mu \circ \kappa \leq P_\Theta(\nu)\).

d) \((X, D)\) is linearly/polynomially universal if every compact metric space \((X, d)\) admits a linearly/polynomially admissible \(X\)-representation.

Note how (b) and (c) boil down to Definition \([11d+e]\) in case \((X, D) = (C, d_C)\) of linear entropy \(\Theta = \text{id}\). And Theorem \([25]\) now means that Cantor space is linearly universal. It seems worthwhile to identify and classify linearly/polynomially universal compact metric spaces.

**References**

BBS04. Andrej Bauer, Lars Birkedal, and Dana S. Scott. Equilogical spaces. *Theor. Comput. Sci.*, 315(1):35–59, 2004.

BBY06. I. Binder, Mark Braverman, and M. Yampolsky. On computational complexity of Siegel Julia sets. *Comm. Math. Phys.*, 264(2):317–334, 2006.

BBY07. Ilia Binder, Mark Braverman, and Michael Yampolsky. On the computational complexity of the Riemann mapping. *Ark. Mat.*, 45(2):221–239, 2007.

BC06. Mark Braverman and Stephen Cook. Computing over the reals: Foundations for scientific computing. *Notices of the AMS*, 53(3):318–329, 2006.

Ben98. Yoav Benyamini. Applications of the universal surjectivity of the cantor set. *The American Mathematical Monthly*, 105(9):832–839, 1998.
BGP11. Olivier Bournez, Daniel S. Graça, and Amaury Pouly. Solving analytic differential equations in polynomial time over unbounded domains. In *Mathematical foundations of computer science 2011*, volume 6907 of *Lecture Notes in Comput. Sci.*, pages 170–181, Heidelberg, 2011. Springer.

Bra05. Mark Braverman. On the complexity of real functions. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’05*, pages 155–164, Washington, DC, USA, 2005. IEEE Computer Society.

Bra13. Mark Braverman. Computing with real numbers, from Archimedes to Turing and beyond. *Communications of the ACM*, 56(9):74–83, September 2013.

Kap72. Irving Kaplansky. *Set Theory and Metric Spaces*. Allyn and Bacon Boston, 1972.

KC96. Bruce M. Kapron and Stephen A. Cook. A new characterization of type-2 feasibility. *SIAM J. Comput.*, 25(1):117–132, 1996.

KC12. Akitoshi Kawamura and Stephen Cook. Complexity theory for operators in analysis. *ACM Transactions on Computation Theory*, 4:2(5), 2012.

KF82. Ker-I Ko and H. Friedman. Computational complexity of real functions. *Theoretical Computer Science*, 20:323–352, 1982.

KMRZ15. Akitoshi Kawamura, Norbert Müller, Carsten Rösnick, and Martin Ziegler. Computational benefit of smoothness: parameterized bit-complexity of numerical operators on analytic functions and Gevrey’s hierarchy. *J. Complexity*, 31(5):689–714, 2015.

Ko91. Ker-I Ko. *Complexity Theory of Real Functions*. Progress in Theoretical Computer Science. Birkhäuser, Boston, 1991.

Koh08. Ulrich Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer, Berlin, 2008.

KSZ16a. Akitoshi Kawamura, Florian Steinberg, and Martin Ziegler. Complexity theory of (functions on) compact metric spaces. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS ’16*, New York, NY, USA, July 5-8, 2016, pages 837–846, 2016.

KSZ16b. Akitoshi Kawamura, Florian Steinberg, and Martin Ziegler. Towards computational complexity theory on advanced function spaces in analysis. In Arnold Beckmann, Laurent Bienvenu, and Nataša Jonoska, editors, *Pursuit of the Universal*, volume 9709 of *Lecture Notes in Computer Science*, pages 142–152, Switzerland, 2016. Springer. 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 - July 1, 2016.

KSZ17. Akitoshi Kawamura, Florian Steinberg, and Martin Ziegler. On the computational complexity of the dirichlet problem for poisson’s equation. *Mathematical Structures in Computer Science*, 27:8:1437–1465, 2017.

KT59. A.N. Kolmogorov and V.M. Tikhomirov. $\mathcal{E}$-entropy and $\mathcal{E}$-capacity of sets in functional spaces. *Uspekhi Mat. Nauk*, 14(2):3–86, 1959.

KW85. Christoph Kreitz and Klaus Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35–53, 1985.

Lam06. Branimir Lambov. The basic feasible functionals in computable analysis. *Journal of Complexity*, 22(6):909–917, 2006.

Müll01. Norbert Th. Müller. The iRRAM: Exact arithmetic in C++. In Jens Blanck, Vasco Brattka, and Peter Hertling, editors, *Computability and Complexity in Analysis*, volume 2064 of *Lecture Notes in Computer Science*, pages 222–252, Berlin, 2001. Springer. 4th International Workshop, CCA 2000, Swansea, UK, September 2000.

NS18. Elke Neumann and Florian Steinberg. Parametrised second-order complexity theory with applications to the study of interval computation. In *Proc. DICE: Developments in Implicit Computational Complexity*, 2018. arXiv:1711.10530.

Pet18. Iosif Petrakis. Meshane-whitney extensions in constructive analysis. *ArXiv e-prints*, April 2018.

RW02. Robert Rettinger and Klaus Weihrauch. The computational complexity of some Julia sets. In Vasco Brattka, Matthias Schröder, and Klaus Weihrauch, editors, *CCA 2002 Computability and Complexity in Analysis*, volume 66 of *Electronic Notes in Theoretical Computer Science*, Amsterdam, 2002. Elsevier. 5th International Workshop, CCA 2002, Málaga, Spain, July 12–13, 2002.
Sch95. Matthias Schröder. Topological spaces allowing type 2 complexity theory. In Ker-I Ko and Klaus Weihrauch, editors, \emph{Computability and Complexity in Analysis}, volume 190 of \emph{Informatik Berichte}, pages 41–53. FernUniversität Hagen, September 1995. CCA Workshop, Hagen, August 19–20, 1995.

Sch02. Matthias Schröder. Extended admissibility. \emph{Theoretical Computer Science}, 284(2):519–538, 2002.

Sch04. Matthias Schröder. Spaces allowing type-2 complexity theory revisited. \emph{Mathematical Logic Quarterly}, 50(4,5):443–459, 2004.

Sch06. Matthias Schröder. Admissible representations in computable analysis. In \emph{Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, June 30-July 5, 2006, Proceedings}, pages 471–480, 2006.

Ste16. Florian Steinberg. \emph{Computational Complexity Theory for Advanced Function Spaces in Analysis}. PhD thesis, TU Darmstadt, 2016.

Ste17. Florian Steinberg. Complexity theory for spaces of integrable functions. \emph{Logical Methods in Computer Science}, 13(3), 2017.

Tri06. Hans Triebel. \emph{Theory of Function Spaces I, II, III}. Birkhäuser, 1983, 1992, 2006.

Tur37. Alan M. Turing. On computable numbers, with an application to the “Entscheidungsproblem”. \emph{Proc. London Mathematical Society}, 42(2):230–265, 1937.

Wei00. Klaus Weihrauch. \emph{Computable Analysis}. Springer, Berlin, 2000.

Wei03. Klaus Weihrauch. Computational complexity on computable metric spaces. \emph{Mathematical Logic Quarterly}, 49(1):3–21, 2003.