Finite-dimensional global and exponential attractors for the reaction–diffusion problem with an obstacle potential

Antonio Segatti\textsuperscript{1} and Sergey Zelik\textsuperscript{2}

\textsuperscript{1} Dipartimento di Matematica ‘F.Casorati’, Universit\`a di Pavia, Via Ferrata 1, 27100 Pavia, Italy
\textsuperscript{2} Department of Mathematics, University of Surrey, Guildford, GU2 7XH, UK

E-mail: antonio.segatti@unipv.it and s.zelik@surrey.ac.uk

Received 16 February 2009, in final form 17 September 2009
Published 13 October 2009
Online at stacks.iop.org/Non/22/2733

Recommended by C-Q Cheng

Abstract

A reaction–diffusion problem with an obstacle potential is considered in a bounded domain of $\mathbb{R}^N$. Under the assumption that the obstacle $\mathcal{K}$ is a closed convex and bounded subset of $\mathbb{R}^n$ with smooth boundary or it is a closed $n$-dimensional simplex, we prove that the long-time behaviour of the solution semigroup associated with this problem can be described in terms of an exponential attractor. In particular, the latter means that the fractal dimension of the associated global attractor is also finite.

Mathematics Subject Classification: 37L30, 35B41, 35K57

1. Introduction

This paper is devoted to the long-time behaviour of solutions of the following reaction–diffusion system with an obstacle potential in a bounded and regular domain $\Omega \subset \mathbb{R}^N$

$$
\begin{cases}
\partial_t u - \Delta u + \partial I_\mathcal{K}(u) - \lambda u \ni 0, \\
u|_{t=0} = u_0, \quad u|_{\partial \Omega} = 0.
\end{cases}
$$

(1.1)

Here $u = (u_1(t, x), \ldots, u_n(t, x))$ is an unknown vector-valued function, $\Delta$ is the Laplacian with respect to the variable $x$, $\lambda > 0$ is a given constant, $\mathcal{K}$ is a given bounded closed convex set in $\mathbb{R}^n$ containing zero and $\partial I_\mathcal{K}$ stands for the subdifferential of its indicator function $I_\mathcal{K}$:

$$
I_\mathcal{K}(u) := \begin{cases} 0, & u \in \mathcal{K}, \\ \infty, & u \notin \mathcal{K}.
\end{cases}
$$

(1.2)

We recall that the subdifferential $\partial I_\mathcal{K}$ consists of the set of vectors in $\mathbb{R}^n$ such that $y \in \partial I_\mathcal{K}(x)$ if and only if $(y, x - w)_{\mathbb{R}^n} \geq 0$ for any $w \in K$, where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the scalar product in $\mathbb{R}^n$. 

0951-7715/09/112733+28$30.00 © 2009 IOP Publishing Ltd and London Mathematical Society Printed in the UK 2733
It is also well known that $\partial I_{\mathcal{K}}$ turns out to be a multivalued maximal monotone operator (see [6, p 25]; see also section 2 for the rigorous definitions).

Equations and systems of the type (1.1) appear quite often in the mathematical analysis of phase transitions models or in reaction–diffusion processes with constraints. In the first physical situation, (1.1) rules the evolution of the so-called order parameter, which is an $n$-dimensional vector in the case of multicomponent systems (see, e.g. [7]). Moreover, since the order parameter $u$ is usually related to the pointwise proportions of the $n$ independent phases shown by system under study, it is physically reasonable the fact that it attains values only in a bounded (convex) subset of $\mathbb{R}^n$, usually an $n$-dimensional simplex

$$\mathcal{K} := \left\{ p = (p_1, \ldots, p_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} p_i \leq 1, \quad p_i \geq 0, \quad i = 1, \ldots, n \right\}.\quad (1.3)$$

This is motivated by the requirement that neither void nor overlapping should appear between the phases. In particular, equation (1.1) with $n = 2$ and $\lambda = 0$ appears in the Frémond models of shape memory alloys (see [16] also for other models of phase change showing the ubiquity of subdifferential operators in this framework) where the two components $u_1$ and $u_2$ denote the pointwise proportions of the two martensitic variants. Finally, in the simpler scalar case, i.e. $n = 1$ and $\mathcal{K} = [0, 1]$ (for instance), equation (1.1) is usually referred to as Allen–Cahn equation with double obstacle.

The mathematical analysis of equations of the type (1.1) (and more general equations associated with maximal monotone operators in Hilbert spaces) has attracted the attention of researchers for many years. In the particular case of equation (1.1), results concerning existence, approximation and long-time behaviour of solutions (e.g. in terms of global attractors) are known and by now classic (without any sake of completeness and referring only to results in the Hilbert space framework, we quote [3, 5, 6, 17, 26, 28]).

However, to the best of our knowledge, the finite/infinite dimensionality of the global attractor associated with the obstacle problems has not been yet understood (even in the simplest case of Allen–Cahn equation with double obstacle). Indeed, the classical machinery for proving the finite dimensionality (in terms of fractal or/and Hausdorff dimension) of the global attractor (which perfectly works in many cases of dissipative systems generated by nonlinear PDEs with regular nonlinearities, see [2, 28] and references therein) is based on the so-called volume contraction arguments and requires the associated solution semigroup to be (uniformly quasi-) differentiable with respect to the initial data at least on the attractor.

Unfortunately, this differentiability condition is usually violated if the underlying PDE has singularities or/and degenerations and, in particular, it is clearly violated for the obstacle problems such as (1.1). Thus, the classical scheme is not applicable here and this makes the problem much more difficult and interesting. In fact, it has been recently shown that, contrary to the usual regular case, the singular/degenerate dissipative systems can easily generate infinite-dimensional attractors even in bounded domains. For instance, the global attractor of the degenerate analogue of the real Ginzburg–Landau equation

$$\partial_t u = \Delta x (u^3) + u - u^3, \quad u|_{\Omega} = 0 \quad (1.4)$$

is infinite-dimensional for any bounded domain $\Omega$ of $\mathbb{R}^N$ (thanks to the degeneration at $u = 0$), see [11]. On the other hand, in recent years the finite dimensionality of the global attractor result has been established for many important classes of degenerate/singular dissipative systems including Cahn–Hilliard equations with logarithmic potentials (see [22]), porous media equations (under some natural restrictions which exclude the example of (1.4), see [11]), doubly nonlinear parabolic equations of different types (see [12, 23, 25]), etc. In these papers, the finite dimensionality of the global attractor is typically a consequence of the existence of
a more refined object called exponential attractor, whose existence proof is often based on
proper forms of the so-called squeezing/smoothing property for the differences of solutions.

We recall that the concept of exponential attractor has been introduced in [8] in order
to overcome two major drawbacks of the global attractors: the slow (uncontrollable) rate of
attraction and the sensitivity to perturbations. Roughly speaking, an exponential attractor
(which always contains the global one) is a compact finite-dimensional set in the phase space
which attracts exponentially fast the images of all bounded sets as time tends to infinity (see
section 3 for the rigorous definition). Thus it turns out that, in contrast to global attractors, the
exponential attractors are much more robust to perturbations (usually, Hölder continuous with
respect to the perturbation parameter). Moreover, the rate of convergence to the exponential
attractor can be controlled in terms of physical parameters of the system considered, see [8]
and the more recent survey [24] (and also references therein) for more details. Finally, the
finite dimensionality of the global attractor immediately follows from the finite dimensionality
of the exponential attractor.

The main aim of this paper is to extend the exponential attractors theory to some classes
of reaction–diffusion problems with obstacle potentials. Although the methods based on
the proper squeezing/smoothing property for the differences of solutions do not require the
differentiability with respect to the initial data and, in principle, can be applied also to the
obstacle problem (1.1), this application is far from being straightforward in our situation since,
to this end, one needs to produce estimates for the difference between the Lagrange multipliers
(namely, the selection of the subdifferential $\partial I_K(u)$ which turns the differential inclusion (1.1)
into an equation) associated with two solutions $u_1(t)$ and $u_2(t)$. This kind of estimates, which
form assumption $\mathcal{L}$ (see (2.20) and (3.10) for the rigorous formulation), roughly speaking,
look as follows:

$$\int_0^1 \|\partial I_K(u_1(t)) - \partial I_K(u_2(t))\|_{L^1(\Omega)} \, dt \leq C \|u_1(0) - u_2(0)\|_{L^2},$$

(1.5)

where $u_1$ and $u_2$ denote two solutions starting from the proper absorbing set and, with a little
abuse of notation, we refer to $\partial I_K$ as it were single valued. Such kind of estimates, to the best
of our knowledge, do not seem to be already known.

This paper is organized as follows. In section 2, we recall the basic results related to the
wellposedness and regularity of solutions of the obstacle problem (1.1). As usual, the results
on the singular problem (1.1) are obtained by approximating the singular obstacle potential
by more regular ones. The usual regularization used for this kind of problems is the Moreau–
Yosida approximation (see, e.g. [6]). However, it turns out that, in order to prove estimates of
the type (1.5), it is more convenient to implement different kinds of approximation schemes.
Thus, in section 2 we also give a sketch of the wellposedness result for (1.1) (which is well
known, see [6] and [3]) by introducing another approximation scheme. This scheme becomes
also very useful to prove an $L^\infty$-estimate for the approximation of $\partial I_K$ (independent of the
approximation parameter) via the maximum principle. This kind of estimate at the $\varepsilon$-level will
guarantee that the same bound remains valid also for $\partial I_K$. This fact will be rather crucial in
proving (1.5).

Under the assumption that the estimate (1.5) for the Lagrange multipliers is known, in
section 3 we give the conditional proof of the existence of an exponential attractor for the
solution semigroup $S(t)$ associated with equation (1.1). This result is obtained using some
modification of the so-called method of $l$-trajectories which was originally introduced by
Málek and Nečas in [19] and is widely used nowadays in the attractors theory, see [20, 24] and
references therein.

Section 4 is the key part of the paper and it is devoted to verifying estimate (1.5) for the
case in which the boundary $\partial \mathcal{K}$ of the convex set $\mathcal{K}$ is regular enough. The proof is based
on the maximum principle and the Kato inequality together with rather delicate construction of the approximating potentials. Unfortunately, the most relevant (from the applications point of view) choice of $K$ is that of the simplex (1.3) which does not fit with the regularity assumptions on the boundary of $K$. Thus, we have to consider this particular case separately using a different approximation scheme for the subdifferential $\partial I_K$. As a result, the existence of an exponential attractor is proved for the case in which the boundary of $K$ is smooth and for the particular choice of $K$ as a simplex (1.3) (although we expect that this result should be true for any closed and bounded convex set $K$). The section closes with some discussion on possible extensions of our result to more general classes of reaction–diffusion equations.

Finally, in the last section 5, we study the convergence of the exponential attractors $M_\varepsilon$ for the regular approximating problems to the limit exponential attractor $M_0$ of the singular problem (1.1). For simplicity, we restrict ourselves there only to the case of the simplex (1.3) and verify that

$$\text{distsym}_{L^\infty(\Omega)}(M_\varepsilon, M_0) \leq C\varepsilon^\kappa,$$

where distsym stands for the symmetric Hausdorff distance and the positive constants $C$ and $\kappa$ are independent of $\varepsilon$. The result is based on the proper application of the abstract theorem on perturbations of exponential attractors proved in [14] to the obstacle problem (1.1).

2. Wellposedness and regularity

The aim of this section is to recall some known facts about the solutions of reaction–diffusion problem (1.1) with obstacle potential and to formulate some additional estimates which will be crucial for what follows. We start with

A word on the notation. The unknown function $u$ is actually a vector valued function but, for the sake of simplicity, will be denoted as a scalar valued function. Consequently, also the functional spaces we will use in the course of the paper we will have a ‘scalar’ notation. This means that a notation like, e.g., $L^2$ will be preferred to a (more precise) notation like $(L^2(\Omega))^n$. The same applies to dualities $(\langle \cdot, \cdot \rangle)$ and scalar products $(\langle \cdot, \cdot \rangle)$. Moreover, we will indicate with same symbols $K$ and $I_K$ the convex set in $\mathbb{R}^n$ and its indicator function and their realization in $L^2$ (see the next proposition 2.4). Thus, the definition of (weak) solutions of our problem is as follows.

**Definition 2.1.** A function $u = u(t, x)$ is a solution of the obstacle problem (1.1) if $u(t, x) \in K$ for almost all $(t, x) \in [0, T] \times \Omega$.

$$u \in C([0, T]; L^2) \cap L^2(0, T; H_0^1), \quad \partial_t u \in L^2(0, T; H^{-1}),$$

and the following variational inequality holds for almost every $t \in (0, T]$

$$\langle \partial_t u(t), u(t) - z \rangle + \langle \nabla u, \nabla (u - z) \rangle \leq \lambda(u, u - z), \quad \text{for any } z \in H_0^1 \cap K. \quad (2.2)$$

The next theorem is a standard result in the theory of the evolution equations associated with maximal monotone operators (see the seminal references [3, 5, 6]).

**Theorem 2.2 (Wellposedness).** Let $K$ be a closed and bounded convex set containing the origin in $\mathbb{R}^n$. Then, for any given measurable $u_0$ such that $u_0(x) \in K$ for almost all $x \in \Omega$ there exists a unique global solution $u$ of problem (1.1) in the sense of definition 2.1. Moreover, $u(t) \in H^2$ and $\partial_t u(t) \in L^2$ for $t > 0$ and the following estimate holds:

$$\|\partial_t u(t)\|_{L^2} + \|u(t)\|_{H^2} \leq C \frac{t + 1}{t}, \quad (2.3)$$
where the constant $C$ depends on $\lambda$ and $\mathcal{K}$, but is independent of $u_0$. If, in addition, $u_0 \in H^2 \cap H^1_0$, then the term with $\frac{1}{2}z$ in the right-hand side of (2.3) can be removed.

**Proof.** Although this result is well known, for the convenience of the reader, we briefly recall one of its possible proof and deduce the regularity estimate (2.3).

We first note that the uniqueness follows immediately from the variational inequality (2.2). Indeed, let $u$ and $v$ be two solutions of (1.1). Then, taking a sum of (2.2) for $u(t)$ and $\zeta = v(t)$ with (2.2) for $v(t)$ with $\zeta = u(t)$, we have

$$\partial_t u(t) - \partial_t v(t), u(t) - v(t) + \|\nabla u(t) - v(t)\|_{L^2}^2 \leq \lambda \|u(t) - v(t)\|_{L^2}^2.$$  

Denoting now $w(t) = u(t) - v(t)$ we arrive at the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \leq \lambda \|w(t)\|_{L^2}^2. \quad (2.4)$$

Applying the Gronwall inequality and using that, by definition $u, v \in C_0([0, T]; L^2)$ for any $T > 0$, we finally have

$$\|w(t)\|_{L^2}^2 + \int_0^t \|w(s)\|_{H^1_0}^2 ds \leq C e^{\lambda s} \|w(0)\|_{L^2}^2, \quad (2.5)$$

where the constant $C$ is independent of $u$ and $v$. Thus, the uniqueness holds.

The existence part is slightly more delicate and requires the approximations of the singular convex potential $I_{\mathcal{K}}$ by suitable regular ones $F_\varepsilon$. In fact, thanks to the uniqueness result this approximation can be done in several ways. The typical choice is usually the Moreau–Yosida approximation (see [6]). However, in view of the next results (see in particular proposition 2.5 and theorems 4.1 and 4.6), it is more convenient to adopt another kind of approximation. To this end, we let $M(u)$ to be the distance from the point $u \in \mathbb{R}^n$ to the convex set $\mathcal{K}$, namely, the real valued function

$$M(u) = \text{dist}(u, \mathcal{K}). \quad (2.6)$$

Then, $M$ is convex, globally Lipschitz continuous and smooth outside $\mathcal{K}$. In addition, there holds that

$$M(u) \geq 0, \quad M(u) = 0, \quad \text{if } u \in \mathcal{K},$$

$$|\nabla M(u)| = 1, \quad \text{if } u \notin \mathcal{K}. \quad (2.7a)$$

Now for any $\varepsilon > 0$, we introduce the real function

$$f_\varepsilon(z) := \begin{cases} 0, & z \leq \varepsilon, \\ \varepsilon^{-1}(z - \varepsilon)^2, & z \geq \varepsilon. \end{cases} \quad (2.8)$$

Finally, the desired approximation is defined as

$$F_\varepsilon(u) := f_\varepsilon(M(u)). \quad (2.9)$$

Then, obviously, $F_\varepsilon(u)$ is convex and smooth (at least $C^{1,1}$) and $F_\varepsilon(u) = F_\varepsilon'(u) = 0$ for all $u \in \mathcal{K}$, where $F_\varepsilon'$ denotes the gradient of $F_\varepsilon$. We thus consider the following approximation of the problem (1.1):

$$\begin{cases} \partial_t u - \Delta_x u + F_\varepsilon'(u) = \lambda u, \\ u|_{\partial \Omega} = 0, \quad u|_{t=0} = u_0, \end{cases} \quad (2.10)$$

where the initial data $u_0 \in L^\infty(\Omega)$ and $u_0(x) \in \mathcal{K}$ a.e. in $\Omega$. 


Our next step is to obtain a number of uniform (with respect to \( \varepsilon \to 0 \)) estimates for the solutions of the regular problems (2.10). We start with the usual \( L^2 \)-estimate. Indeed, multiplying equation (2.10) by \( u \), integrating over \( \Omega \) and using the obvious facts that

\[
\frac{1}{2} (F'_\varepsilon(u), u) - \lambda \|u\|_{L^2}^2 \geq -C, \quad (F'_\varepsilon(u), u) = (F'_\varepsilon(u) - F'_\varepsilon(0), u) \geq 0
\]

where the constant \( C \) depends only on \( \lambda \) and on \( \Omega \) but is independent of \( \varepsilon \), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla_x u(t)\|_{L^2}^2 + \frac{1}{2} (F'_\varepsilon(u), u) \leq C.
\]

Applying the Gronwall inequality to this relation, we see that

\[
\|u(t)\|_{L^2}^2 + \int_t^{t+1} \|u(s)\|_{L^2}^2 + (F'_\varepsilon(u(s)), u(s)) \, ds \leq C.
\] (2.11)

Here we have implicitly used that \( K \) is bounded and, therefore, \( \|u_0\|_{L^2} \) is uniformly bounded for all admissible initial data \( u_0 \).

Next, we obtain the uniform energy estimate for the solutions of (2.10). To this end, we multiply equation (2.10) by \( \partial_t u(t) \) and integrate over \( x \):

\[
\frac{d}{dt} (\|\nabla_x u(t)\|_{L^2}^2 + 2 (F_\varepsilon(u(t), 1)) + \|\partial_t u(t)\|_{L^2}^2 \leq \lambda \|u(t)\|_{L^2}^2.
\]

Integrating this estimate with respect to time, using (2.11) together with the obvious estimate

\[
0 \leq (F_\varepsilon(u), 1) \leq (F'_\varepsilon(u), u)
\]

(which is an immediate consequence of the convexity of \( F_\varepsilon \)) and arguing in a standard way (see, e.g. [29]), we deduce the desired \( H^1 \)-energy estimate

\[
\|u(t)\|_{H^1}^2 + (F_\varepsilon(u(t)), 1) + \int_t^{t+1} \|\partial_t u(s)\|_{L^2}^2 \, ds \leq C \left( \frac{1 + t}{t} \right), \quad t > 0,
\] (2.12)

where the constant \( C \) is independent of \( u_0 \) and \( \varepsilon \).

We are now ready to verify the analogue of (2.3) for the approximate solutions. We start with the estimate of \( \partial_t u(t) \). To this end, we differentiate (2.10) with respect to time and denote \( v(t) := \partial_t u(t) \). Then, we have

\[
\partial_t v - \Delta_x v + F''_\varepsilon(u) v = \lambda \partial_t u.
\]

Multiplying this equation by \( v(t) \) integrating with respect to time, using (2.12) together with the monotonicity of \( F'_\varepsilon(u) \) and arguing in a standard way (see again [29] for details), we have

\[
\|\partial_t u(t)\|_{L^2}^2 \leq C \left( \frac{1 + t}{t^2} \right)
\] (2.13)

for some constant \( C \) independent of \( u_0 \) and \( \varepsilon \).

Now, in order to obtain the \( H^2 \)-part of estimate (2.3), it is sufficient to rewrite equation (2.10) in the form of elliptic equation (for every fixed \( t \))

\[
\Delta_x u(t) - F'_\varepsilon(u(t)) = \partial_t u(t) - \lambda u(t), \quad u(t)|_{\partial \Omega} = 0,
\] (2.14)

multiply it by \( \Delta_x u(t) \), integrate over \( \Omega \) and use the fact that, thanks to the convexity of \( F_\varepsilon \), the term \( (F'_\varepsilon(u), -\Delta_x u) \) is non-negative. Then, due to (2.13), (2.12) and the elliptic regularity result for the Laplacian (recall that the domain \( \Omega \) is assumed to be smooth enough),

\[
\|u(t)\|_{H^2} \leq C \|\Delta_x u(t)\|_{L^2} \leq C (\|\partial_t u(t)\|_{L^2} + \|u(t)\|_{L^2}^2) \leq C' \left( \frac{t + 1}{t} \right),
\] (2.15)

where the constants \( C' \) and \( C \) are independent of \( \varepsilon \) and \( u_0 \). Furthermore, from equation (2.14), we now conclude that

\[
\|F''_\varepsilon(u(t))\|_{L^2} \leq C \left( \frac{t + 1}{t} \right).
\]
Finally, we are now ready to pass to the limit $\epsilon \to 0$ in equations (2.10) and verify the existence of a solution $u$ for the limit obstacle problem. Indeed, let $\epsilon_m > 0$ be a sequence of positive numbers such that $\epsilon_m \to 0$ as $m \to \infty$. Assume for the first that the initial data $u_0$ is smooth enough and verify the constraint, namely,

$$u_0 \in H^2 \cap H^1_0$$ and $u_0(x) \in \mathcal{K}$, a.e. in $\Omega$.

Let $u_m(t) = u_{\epsilon_m}(t)$ be the corresponding solution of the approximating problem (2.10). Then, analogously to the derivation of (2.3) for $u_m$, but using in addition that $u_0$ is regular, we have

$$\| \partial_t u_m(t) \|_{L^2} + \| u_m(t) \|_{H^1} + \| F_{\epsilon}(u_m(t)) \|_{L^2} + \| F_{\epsilon}(u_m(t)) \|_{L^1} \leq Q(\| u_0 \|_{H^2}).$$

(2.16)

where the function $Q$ is independent of $m$ and $t \geq 0$. Thus, up to not relabelled subsequence, we have that $u_m$ converges weakly star in

$$X := L^\infty(0, T; H^2) \cap W^{1,\infty}(0, T; L^2)$$

and strongly (thanks to [27, corollary 4]) in $C^0([0, T]; H^1_0)$ to some function $u \in X \cap C^0([0, T]; H^1_0)$. So, we only need to check that $u$ is a desired solution of (1.1) in the sense of definition 2.1. Indeed, the regularity of $u$ is obvious. The fact that $u(t, x) \in K$ for almost $(t, x) \in [0, T] \times \Omega$ follows in a standard way from the uniform bounds for $F_{\epsilon_m}(u_m(t))$ in the $L^1$-norm and from the fact that $F_{\epsilon_m}(w) \to \infty$ for all $w \notin \mathcal{K}$. So, we only need to check the variational inequality (2.2). Let $z = z(x)$ be any admissible test function. Then, using the monotonicity of $F_{\epsilon}'(u)$ and the fact that $F_{\epsilon}'(z) = 0$ for all $z \in \mathcal{K}$, we see that

$$(F_{\epsilon}'(u(t)), u(t) - z) - (F_{\epsilon}'(u(t)) - F_{\epsilon}'(z), u - z) \geq 0.$$ 

Multiplying now (2.10) by $u(t) - z$, integrating over $\Omega$ and using the last inequality, we arrive at

$$\partial_t u_n(t), u_n(t) - z + (\nabla u_n(t), \nabla_x (u_n(t) - z)) \leq \lambda(u_n(t), u_n(t) - z).$$

Passing to the limit $m \to \infty$ in this inequality and using the above convergences, we see that (2.2) holds for almost any $t$ and all admissible test functions $z$. Thus, we have proved the existence of the desired solution $u$ for the case of smooth initial data $u_0$.

For general initial data $u_0$, the corresponding solution can be now constructed by approximating the non-smooth initial data $u_0$ by smooth ones $u_0^n$ and passing to the limit $m \to \infty$. To conclude, we finally recall that the continuity of the limit function near zero ($u \in C([0, T], L^2)$) follows from the global Lipschitz continuity (2.5)). Theorem 2.2 is proved.

Remark 2.3. The above proof reveals that the concrete form of the functionals $F_{\epsilon}(u)$ is not essential for the proof of the existence result. In fact, the solutions $u_\epsilon$ of the approximating problems (2.10) will converge to the unique solution $u$ of the obstacle problem if the following assumptions are satisfied:

1. The functions $F_\epsilon$ is convex and regular enough
2. $F_\epsilon' \to \partial I_\mathcal{K}$ as $\epsilon \to 0$.

Where the last condition means that

(i) $F_\epsilon'(v) \to 0$, $\forall v \in \mathcal{K}$,
(ii) $F_\epsilon'(v) \to +\infty$, $\forall v \notin \mathcal{K}$.

In other words, we only need the subdifferential $F_\epsilon'(u)$ of $F_\epsilon$ to be an approximation, via graph convergence, of the subdifferential $\partial I_\mathcal{K}$ (see, e.g. [1, proposition 3.60 and theorem 3.66]). However, as we have already mentioned, the choice of suitable approximating functionals $F_\epsilon(u)$ turns out to be crucial for our method of estimating the dimension of the global attractor. Thus, we will construct this approximation in a rather specific way depending on the structure of the convex set $\mathcal{K}$ (see the next sections).
For the long-time analysis it is actually more convenient to reformulate the variational inequality (2.2) in terms of an equation coupled with a differential inclusion for the subdifferential of the indicator function of $\mathcal{K}$ (actually of its realization). This subdifferential, which a priori should be understood with respect to the duality $H^1_0 - H^{-1}$, actually makes sense in $L^2$ and thus almost everywhere in $\Omega$ (see theorem 2.2 and the next proposition 2.4). Thus, we introduce the function (named Lagrange multiplier in what follows)

$$h_u(t) := -\partial_t u(t) + \Delta_x u(t) + \lambda u, \quad h_u(t) \in \partial I_{\mathcal{K}}(u).$$

Then, due to theorem 2.2, $h_u \in L^\infty(\tau, T; L^2)$, for any $\tau > 0$. Moreover, the definition of subdifferential (w.r.t. the $L^2$ scalar product) gives (2.2),

$$(h_u(t), u(t) - z) \geq 0, \quad \text{for almost all } t \in [0, T] \quad (2.18)$$

and any admissible test function $z = z(x)$. The last inequality can be also written in a pointwise form.

**Proposition 2.4.** Let the assumptions of theorem 2.2 hold and let $h_u = h_u(t, x)$ be the Lagrange multiplier associated with the solution $u(t)$ of problem (1.1). Then

$$(h_u(t, x), u(t, x) - Z)_{\mathbb{R}^n} \geq 0, \quad \forall Z \in \mathcal{K} \quad (2.19)$$

and almost all $(t, x) \in [0, T] \times \Omega$, which means

$$h_u(t, x) \in \partial I_{\mathcal{K}}(u(t, x)), \quad \text{a.e. in } [0, T] \times \Omega.$$

**Proof.** Indeed, arguing as in the proof of theorem 2.2, we see that

$$h_{un}(t) := F'_{\epsilon_n}(u_n(t)) \rightharpoonup h_u(t)$$

weakly star in the space $L^\infty(\tau, T; L^2)$, for any $\tau > 0$. Thus, it is sufficient to verify (2.19) for the functions $h_{un}$ only. But these inequalities are immediate due to the monotonicity of $F'_{\epsilon}$ and the fact that $F'_{\epsilon}(Z) = 0$ if $Z \in \mathcal{K}$. Indeed,

$$(h_{un}(t, x), u_n(t, x) - Z)_{\mathbb{R}^n} = (F'_{\epsilon_n}(u_n(t, x)) - F'_{\epsilon_n}(Z), u_n(t, x) - Z)_{\mathbb{R}^n} \geq 0.$$

Passing to the limit $n \to \infty$ in that inequalities, we deduce (2.19) and complete the proof of the proposition.

Thus, the obstacle problem (1.1) can be rewritten in terms of functions $u$ and $h_u$ as follows:

$$\begin{cases}
\partial_t u - \Delta_x u + h_u = \lambda u, & \text{in the sense of distributions,} \\
h_u(t, x) \in \partial I_{\mathcal{K}}(u(t, x)), & \text{for almost all } (t, x) \in \mathbb{R}_+ \times \Omega, \\
u|_{t=0} = u_0, & u|_{\partial \Omega} = 0.
\end{cases} \quad (2.20)$$

The next proposition shows that the function $h_u$ is, in a fact, globally bounded in the $L^\infty$-norm.

**Proposition 2.5.** Let the assumptions of theorem 2.2 hold and let $h_u(t)$ be the Lagrange multiplier associated with the solution $u(t)$ of problem (1.1). Then, $h_u \in L^\infty(\mathbb{R}_+ \times \Omega)$ and

$$\|h_u(t)\|_{L^\infty} \leq C, \quad (2.21)$$

where the constant $C$ depends only on $\mathcal{K}$ and $\lambda$ (and is independent of $u$ and $t \geq 0$).

**Proof.** Thanks to the lower semicontinuity of norms with respect to the weak convergence, we will verify (2.21) only in the case in which $h_u$ in equation (2.20) is replaced by its approximation $h_{un}(t) := F'_{\epsilon_n}(u_n(t))$. To this end, we test equation (2.10) in the scalar product of $\mathbb{R}^n$ with
\[ \nabla M(u) \] (where the function \( M \) is the same as in the proof of theorem 2.2), and use that, due to the convexity of \( M \),
\[ (-\Delta_t u, \nabla M(u))_{\mathbb{R}^n} = -\Delta_t (M(u)) + (H(M(u)) \nabla u, \nabla u)_{\mathbb{R}^n} \geq -\Delta_t (M(u)), \]
where \( H(M) \) denotes the Hessian matrix of \( M \) (actually, \( H(M)(u) \) does not exist if \( u \in \partial \mathcal{K} \), but the \textit{inequality} still holds and can be easily verified, say, by approximating the non-smooth convex function \( M \) by the smooth convex ones). Then, we arrive at the differential inequality for the function \( M(u) \)
\[ \partial_t M(u) - \Delta_t M(u) + f'_{e_n}(M(u)) \leq \lambda(u, \nabla M(u)), \quad M(u)|_{t=0} = 0, \]
where we have implicitly used that \( |\nabla M(u)|^2 = 1 \) for \( u \in \mathcal{K} \) (see (2.7)). Furthermore, since \( |u| \leq M(u) + C_{\mathcal{K}} \), where \( C_{\mathcal{K}} := \text{diam}(\mathcal{K}) \), we finally have
\[ \partial_t M(u) - \Delta_t M(u) + f'_{e_n}(M(u)) \leq \lambda M(u) + \lambda C_{\mathcal{K}}, \quad M(u)|_{t=0} = 0. \] (2.22)

Applying the comparison principle to the scalar parabolic equation (2.22), we see that \( M(u(t, x)) \leq v_{e_n} \), for a.e. \( (t, x) \in \mathbb{R}_+ \times \Omega \), where the constant (w.r.t. \( x \) and \( t \)) \( v_{e_n} > 0 \) is the solution of the following equation:
\[ f'_{e_n}(v_{e_n}) = \lambda v_{e_n} + \lambda C_{\mathcal{K}}. \]
In addition, from (2.8), we see that \( v_{e_n} \to 0 \) as \( e_n \to 0 \) (i.e. \( n \to +\infty \)). For this reason
\[ f'_{e_n}(M(u)) \leq f'_{e_n}(v_{e_n}) = \lambda v_{e_n} + \lambda C_{\mathcal{K}} \leq C_{\lambda}, \]
and, therefore,
\[ |h_{u_n}(t)| = f'_{e_n}(M(u_n(t))) \leq C_{\lambda} \]
uniformly with respect to \( n \to \infty \). Finally, passing to the limit \( n \to \infty \), we arrive at (2.21) (with \( C = \lambda \text{diam}(\mathcal{K}) \)) and complete the proof of the proposition.

**Remark 2.6.** In contrast to theorem 2.2 and proposition 2.4 which are based only on the energy type estimates which are valid for much more general equations, e.g. with non-scalar diffusion matrix, etc, the \( L^{\infty} \)-estimate obtained in proposition 2.5 is based on the maximum/comparison principle and requires the diffusion matrix to be scalar. In particular, we do not know whether or not this estimate remains true even for the case of diagonal, but non-scalar diffusion matrix.

As direct consequence of the previous proposition, we have that (2.20) could be understood as the heat equation
\[ \partial_t u - \Delta u = \lambda u - h_u \]
with the external forces belonging to \( L^{\infty} \). Thus, the parabolic interior regularity estimates give (see, e.g. [18]) the following corollary.

**Corollary 2.7.** Let the assumptions of theorem (2.2) hold and let \( u(t) \) be a solution of problem (1.1). Then, for every \( \nu > 0 \), \( u(t) \in C^{2+\nu, 1}^{\nu}(\Omega) \) for \( t > 0 \) and the following estimate holds:
\[ \|u(t)\|_{C^{2+\nu}} \leq C_{\nu} \frac{1 + t^\alpha}{t^\nu}, \] (2.23)
where the positive constants \( C_{\nu} \) and \( \alpha \) are independent of \( t \) and \( u \). If, in addition, \( u_0 \in C^{2+\nu} \cap H^1_0 \), then (2.23) holds with \( \alpha = 0. \)
3. Global and exponential attractors

The aim of this section is to study the long-time behaviour of solutions of problem (1.1) in terms of global and exponential attractors. We first recall that, due to theorem 2.2, equation (1.1) generate a (dissipative) semigroup \( \{S(t), t \geq 0\} \) in the phase space

\[
\Phi = \Phi, := \{u \in L^\infty : u(x) \in K \text{ for almost all } x \in \Omega\},
\]

i.e.

\[
S(t) : \Phi \to \Phi, \quad S(t)u_0 := u(t),
\]

where \( u(t) \) is the solution to (1.1) at time \( t \). Moreover, due to estimate (2.5) this semigroup is globally Lipschitz continuous in the \( L^2 \)-metric

\[
\|u_1(t) - u_2(t)\|_{L^2} + \int_0^t \|u_1(s) - u_2(s)\|_{H^1}^2 \, ds \leq C e^{\mu t} \|u_1^0 - u_2^0\|_{L^2}^2,
\]

where the positive constants \( C \) and \( \mu \) are independent of \( t \) and of the initial data \( u_0^1, u_0^2 \in \Phi \).

In addition, due to corollary 2.7, we have the following regularization estimate:

\[
\|\partial_t u(t)\|_{L^2} + \|u(t)\|_{C^2 - \nu(\Omega)} \leq C_\nu 1 + t^\nu,\]

where \( \nu > 0 \) is arbitrary and the positive constants \( C_\nu \) and \( M \) are independent of the initial condition \( u_0 \) and of \( t \). These two estimates immediately imply the existence of a global attractor \( A \) for the semigroup \( S(t) \) associated with the obstacle problem (1.1). We recall that, by definition, a set \( A \subset \Phi \) is a global attractor for the semigroup \( S(t) \) if

1. The set \( A \) is compact in \( \Phi \);
2. It is strictly invariant: \( S(t)A = A, t \geq 0 \);
3. For every neighborhood \( \mathcal{O} = \mathcal{O}(A) \) of \( A \) in \( \Phi \) there exists a time \( T = T(\mathcal{O}) \) such that

\[
S(t) \Phi \subset \mathcal{O}(A), \quad t \geq T.
\]

**Remark 3.1.** The attraction property (3.5) is usually formulated not for the whole phase space \( \Phi \), but only for its bounded subsets. In our case, however, the whole phase space \( \Phi \) is automatically bounded (since \( \mathcal{H} \) is bounded in \( \mathbb{R}^n \)), so we do not need to use bounded sets to define the attractor. Note also that the attraction property (3.5) can be reformulated as follows:

\[
\text{dist}_{L^\infty}(S(t)\Phi, A) \to 0 \quad \text{as } t \to \infty,
\]

where \( \text{dist}_V(X, Y) := \sup_{x \in X} \inf_{y \in Y} d_V(x, y) \) is the non-symmetric Hausdorff distance between the subsets \( X \) and \( Y \) of the metric space \( V \).

**Theorem 3.2 (Global Attractor).** Under the assumptions of theorem 2.2, the semigroup \( S(t) \) associated with the obstacle equation (1.1) possesses the global attractor \( A \) in \( \Phi \) which is bounded in \( C^{\nu, \alpha}(\Omega) \) for every \( \nu > 0 \). This attractor is generated by all the trajectories of the semigroup \( S(t) \) defined for all \( t \in \mathbb{R} \):

\[
A = \mathcal{K} |_{t=0},
\]

where \( \mathcal{K} \subset L^\infty(\mathbb{R}, \Phi) \) is a set of all solutions of (1.1) defined for all \( t \in \mathbb{R} \).

**Proof.** The proof of this theorem is standard and well known (see, for instance [2, 28] or [17]). Indeed, due to estimate (3.4) the semigroup \( S(t) \) possesses an absorbing set which is bounded in \( C^{\nu, \alpha}(\Omega) \) and, therefore, compact in \( L^\infty(\Omega) \) and estimate (3.3) guarantees that the semigroup has a closed graph. Thus, all assertions of the theorem follow from the abstract attractor existence criterion, see [2]. 

\[\square\]
Recall that the semigroup $S(t)$ associated with the obstacle problem (1.1) possesses a global Lyapunov function of the form

$$L(u) := \| \nabla_x u \|^2_{L^2} - \lambda \| u \|^2_{L^2}.$$  \hspace{1cm} (3.8)

Indeed, using the test function $z = u(t - h)$ in the variational inequality (2.2), dividing it by $h > 0$ and passing to the limit $h \to 0$, we arrive at

$$\| \partial_t u(t) \|^2_{L^2} + \frac{d}{dt} L(u(t)) \leq 0.$$  

Therefore, according to the general theory (see, e.g. [2]), every trajectory $u(t) = S(t)u_0$ tends as $t \to \infty$ to the set $\mathcal{R}$ of all equilibria of problem (1.1)

$$\text{dist}(S(t)u_0, \mathcal{R}) \to 0, \quad \text{as } t \to \infty.$$  

However, in contrast to the case of regular systems, the equilibria set $\mathcal{R}$ is generically not discrete for the obstacle type singular problems. Thus, in our situation, the existence of a Lyapunov function does not allow us to obtain the stabilization of every trajectory to a single equilibrium even in ‘generic’ situation. In addition, the semigroup $S(t)$ is not differentiable with respect to the initial data (it is in fact only globally Lipschitz continuous), so we are not able to construct the stable/unstable manifolds associated with an equilibrium. Thus, the so-called theory of regular attractors is not applicable to equations of the type (1.1). Moreover, due to the above-mentioned non-differentiability, the standard way of proving the finite dimensionality of the global attractor based on the volume contraction method does not work here. So, the existence of the finite-dimensional reduction for the associated long-time dynamics becomes a non-trivial problem which, to the best of our knowledge, has not been yet tackled. In this paper we will prove that the global attractor for (1.1) has finite fractal dimension by using the concept of the so-called exponential attractor and the estimation of the dimension based on the proper chosen squeezing/smoothing property for the difference between two solutions. This method has the advantage that it does not require the differentiability with respect to the initial data The existence of an exponential attractor is of interest in itself. In fact, we recall once more that the global attractor represents the first (although extremely important) step in the understanding of the long-time dynamics of a given evolutive process. However, it may also present some severe drawbacks. Indeed, as simple examples show, the rate of convergence to the global attractor may be arbitrarily slow. This fact makes the global attractor very sensitive to perturbations and to numerical approximation. In addition, it is usually extremely difficult to estimate the rate of convergence to the global attractor and to express it in terms of the physical parameters of the system. In particular, it may even be reduced to a single point, thus failing in capturing the very rich and most interesting transient behaviour of the system considered. In order to overcome these drawbacks, the concept of exponential attractor has then been proposed in [8]) to possibly overcome this difficulty. We recall below the definition of an exponential attractor adopted for our case. We refer to [8, 24] for more detailed exposition.

**Definition 3.3.** A compact subset $M$ of the phase space $\Phi$ is called an exponential attractor for the semigroup $S(t)$ if the following conditions are satisfied:

1. The set $M$ is positively invariant, i.e. $S(t)M \subset M$ for all $t \geq 0$;
2. The fractal dimension (see, e.g. [21, 28]) of $M$ in $\Phi$ is finite;
3. The set $M$ attracts exponentially fast the image the phase space $\Phi$. Namely, there exist $C, \beta > 0$ such that

$$\text{dist}_{L^\infty}(S(t)\Phi, M) \leq Ce^{-\beta t}, \quad \forall t \geq 0.$$  \hspace{1cm} (3.9)
Thanks to the control of the convergence rate \((E3)\) it follows that, compared with the global attractor, an exponential attractor is much more robust to perturbation (usually it is Hölder continuous with respect to the perturbation parameter, see section 5). However, since the exponential attractor \(\mathcal{M}\) is only positively invariant (see (E1)), it is obviously not unique. Thus, the concrete choice of an exponential attractor and its explicit construction becomes essential. We recall also that, in the original paper [8] the construction was extremely implicit (involving the Zorn lemma) and this fact did not allow us to develop a reasonable perturbation theory. This drawback has been overcome later in [9, 10] where an alternative and relatively simple and explicit construction for the exponentially attractor has been suggested. Note also that the construction of [10] gives an exponential attractor which is automatically Hölder continuous with respect to the reasonable perturbations of the semigroup considered and this somehow resolves the non-uniqueness problem. We refer the reader to the recent survey [24] for the detailed information on the exponential attractors theory.

The proof of the existence of an exponential attractor for our obstacle problem (1.1) will be organized as follows. First of all, we establish (in a quite standard way) the existence of an exponential attractor under a crucial additional (for this moment only) assumption on the differences of the Lagrange multiplies \(h_{u_1}\) and \(h_{u_2}\) of two different solutions \(u_1(t)\) and \(u_2(t)\) of (1.1). This assumption is the core of the exponential attractor existence theorem and it is the main result of the paper. Section 4 will be dedicated to its proof in two different situations: the convex set \(\mathcal{K}\) is smooth or the convex set is an \(n\)-dimensional simplex.

Assumption \(L\). There exists a closed positively invariant absorbing set \(B_0 \subset \Phi\) for the semigroup \(S(t)\) such that, for every two solutions \(u_1\) and \(u_2\) of (2.20) starting from \(B_0\) (i.e. \(u_i(0) \in B_0\)), the corresponding Lagrange multipliers \(h_{u_1}\) and \(h_{u_2}\) satisfy the following estimates:

\[
\|h_{u_1} - h_{u_2}\|_{L^1([0,1] \times \Omega)} \leq C \|u_1(0) - u_2(0)\|_{L^2}, \tag{3.10}
\]

where the constant \(C\) is independent of \(u_1\) and \(u_2\).

Under assumption \(L\), in the next theorem we prove the existence of an exponential attractor (and thus the finite dimensionality of the global attractor).

**Theorem 3.4.** Let the assumption of theorem 2.2 hold and let, in addition, the semigroup \(S(t)\) associated with equation (1.1) satisfy assumption \(L\). Then, \(S(t)\) possesses an exponential attractor \(\mathcal{M} \subset \Phi\) in the sense of definition 3.3. Moreover, \(\mathcal{M}\) is a bounded subset of \(C^{2-v}\), for all \(v > 0\). Finally, the global attractor \(\mathcal{A}\) constructed in theorem 3.2 has finite fractal dimension.

**Proof.** Recall that, since \(B_0\) is a semi-invariant absorbing set for \(S(t)\), it is sufficient to verify the existence of the exponential attractor for the restriction of \(S(t)\) on \(B_0\) only. As usual, we first verify the existence of such attractor for the discrete semigroup generated by the map \(S = S(1)\) and then extend to the continuous time. To this end, we will use the following abstract exponential attractor existence theorem suggested in [9].

**Lemma 3.5.** Let \(\mathcal{H}\) and \(\mathcal{H}_1\) be two Banach spaces such that \(\mathcal{H}_1\) is compactly embedded to \(H\). Let \(\mathbb{B}_0\) be a bounded closed subset of \(\mathcal{H}\) and a map

\[
S : \mathbb{B}_0 \rightarrow \mathbb{B}_0
\]

be such that

\[
\|Sb_1 - Sb_2\|_{\mathcal{H}_1} \leq K \|b_1 - b_2\|_{\mathcal{H}}, \quad b_1, b_2 \in \mathbb{B}_0, \tag{3.11}
\]
where the constant $K$ is independent of $b_1$ and $b_2$. Then, the discrete semigroup $\{S(n), n \in \mathbb{N}\}$ generated on $B_0$ by the iterations of the map $S$ possesses an exponential attractor, i.e. there exists a compact set $M_d \subset B_0$ such that

(E1) $M_d$ is positively invariant: $S M_d \subset M_d$;

(E2) The fractal dimension of $M_d$ in $H$ is finite:

$$\dim_f (M_d, H) \leq M < \infty;$$

and

(E3) $M_d$ attracts exponentially the images of $B_0$ under the iterations of the map $B_0$:

$$\text{dist}_{H}(S(n)B_0, M_d) \leq C e^{-kn}.$$ 

Moreover, the positive constants $M$, $C$ and $k$ can be expressed explicitly in terms of the squeezing constant $K$, the size of the set $B_0$ and the entropy of the compact embedding $H_1 \subset H$.

We will use the so-called method of $l$-trajectories (introduced by Málek and Nečas in [19], see also [20, 29]) in order to construct the proper spaces $H$ and $H_1$ and to verify the assumptions of lemma 3.5.

Namely, let us consider the trajectory space $B_0$ consisting of the pieces of trajectories of the solution semigroup $S(t)$ of length one starting from $B_0$:

$$B_0 := \{ u \in L^\infty([0, 1], \Phi), \ u(0) = u_0 \in B_0, \ u(t) = S(t)u_0, \ t \in [0, 1] \}. \quad (3.12)$$

Then, there is a one-to-one correspondence between $B_0$ and $B_0$ generated by the solution map

$$T : B_0 \to B_0, \quad (Tu_0)(t) := S(t)u_0$$

and, therefore, we may lift the semigroup $S(t) : B_0 \to B_0$ to the conjugated semigroup $S(t)$ acting on the trajectory space $B_0$:

$$S(t) : B_0 \to B_0, \quad S(t) := T \circ S(t) \circ T^{-1}. \quad (3.13)$$

We intend to apply lemma 3.5 to the map $S = S(1)$ acting on the trajectory space $B_0$. To this end, we define the spaces $H$ and $H_1$ as follows:

$$H := L^2(0, 1; L^2), \quad H_1 := L^2(0, 1; H^1_0) \cap W^{1,1}(0, 1; H^{-s}), \quad (3.14)$$

where $s = \max\{1, N/2\}$ is a fixed exponent. Then, obviously, the embedding $H_1 \subset H$ is compact and we only need to check the smoothing property (3.11). To this end, we need (together with assumption $\mathcal{L}$ and estimate (3.3)) the following additional regularization property of the semigroup $S(t)$.

**Lemma 3.6.** Let $u_1(t)$ and $u_2(t)$ be two solutions of problem (1.1). Then, the following estimate holds:

$$\|u_1(1) - u_2(1)\|_{L^2}^2 \leq (2\lambda + 1) \int_0^1 \|u_1(t) - u_2(t)\|_{L^2}^2 \, dt. \quad (3.15)$$

Indeed, multiplying the differential inequality (2.4) by $t$ and integrating $t \in [0, 1]$, we arrive at (3.15).

We are now ready to verify the smoothing property (3.11).

**Lemma 3.7.** Let the assumptions of theorem 2.2 hold and, in addition, assumption $\mathcal{L}$ be satisfied. Then the following estimate holds for every two solutions $u_1(t)$ and $u_2(t)$ such that $u_0^i \in B_0$, $i = 1, 2$:

$$\|u_1 - u_2\|_{L^2(1, 2; H^1_0)} + \|\partial_t u_1 - \partial_t u_2\|_{L^2(1, 2; H^{-s})} \leq L \|u_1 - u_2\|_{L^2([0, 1]; L^2)}, \quad (3.16)$$

where the constant $L$ is independent of $u_1$ and $u_2$. 

Proof. First of all we recall that, for $v \in L^1(1, 2; H^{-1})$, 
\[ \|v\|_{L^1(1, 2; H^{-1})} = \sup_{\varphi} \left| \int_1^2 (v, \varphi) \, dr \right|, \]
where the sup is taken over the $\varphi \in L^\infty(1, 2; H^1_0)$ such that $\|\varphi\|_{L^\infty(1, 2; H^1_0)} = 1$ and the duality pairing is of course between $H^{-1}$ and $H^1_0$. Consequently, there holds 
\[ \int_1^2 \|\partial_t u_1(t) - \partial_t u_2(t)\|_{H^{-1}} \, dt \leq \int_1^2 \|u_1(t) - u_2(t)\|_{H^1_0} \, dt \]
\[ + \int_1^2 \|h_{u_1}(t) - h_{u_2}(t)\|_{L^1} \, dt + \lambda \int_1^2 \|u_1(t) - u_2(t)\|_{L^2} \, dt \]
(here we have implicitly used that $s = \max\{1, N/2\}$ which implies that $H^s \subset L^\infty$). Using now assumption $\mathcal{L}$, together with the global Lipschitz continuity (3.3), we have 
\[ \|u_1 - u_2\|_{L^1(1, 2; H^1_0)} + \|\partial_t u_1 - \partial_t u_2\|_{L^1(1, 2; H^{-1})} \leq C \|u_1(1) - u_2(1)\|_{L^2}. \]
Combining this estimate with (3.15), we arrive at (3.16) and complete the proof of the lemma.

We are now ready to complete the proof of theorem 3.4. Indeed, we have verified that the map $\mathcal{S} := \mathcal{S}(1)$ satisfies all of the assumptions of the abstract lemma 3.5 and, therefore, the semigroup $\mathcal{S}(n), n \in \mathbb{N}$ possesses an exponential attractor $\mathcal{M}_d$ in the trajectory space $\mathcal{B}_0$ endowed with the topology of $\mathcal{H} = L^2(0, 1; L^2)$. Projecting it back to the phase space $B_0 \subset \Phi$ via 
\[ \mathcal{M}_d := \mathcal{M}_d|_{t=1} \]
and using estimate (3.15), we see that $\mathcal{M}_d$ is indeed the exponential attractor for the discrete semigroup $\{S(n), n \in \mathbb{Z}\}$ acting on $B_0$ (endowed with the topology of $L^2$). In addition, we see that 
\[ \mathcal{M}_d \subset S(1)B_0 \subset S(1)\Phi \]
which implies that $\|\partial_t u(t)\|_{L^2} \leq C$ for every trajectory $u(t)$ starting from $\mathcal{M}_d$ (due to estimate (2.3)). Thus, thanks to (3.3), the map $(t, u_0) \rightarrow S(t)u_0$ is globally Lipschitz continuous on $[0, 1] \times \mathcal{M}_d$ (in the $\mathbb{R} \times L^2(\Omega)$-metric). Thus, the desired exponential attractor for the solution semigroup $S(t)$ with continuous time can now be constructed by following the standard expression (see [8] for the details):
\[ \mathcal{M} := \bigcup_{t \in [0, 1]} \mathcal{M}_d. \]

Note that, up to now, we have verified that $\mathcal{M}$ has the finite fractal dimension and possesses the exponential attraction property in the topology of $L^2$ only. In order to verify that these properties actually hold in the $L^\infty$-topology of the phase space $\Phi$, we use (in a standard way) the additional regularity of $\mathcal{M}$ and a proper interpolation inequality. Indeed, by our construction, $\mathcal{M} \subset S(1)\Phi$ and, therefore, due to estimate (2.23), $\mathcal{M}$ is globally bounded in $C^{2+\kappa}$. Using now the following interpolation inequality:
\[ \|u - v\|_{L^\infty} \leq C_{\kappa} \|u - v\|_{L^\infty}^{1-\kappa} \|u - v\|_{C^{2+\kappa}}^\kappa, \quad 0 < \kappa < 1, \]
we see that the dimension of $\mathcal{M}$ is finite not only in $L^2$ but also in $L^\infty$ and that the attraction property holds in $L^\infty$ as well. Thus, the desired exponential attractor $\mathcal{M}$ in the phase space $\Phi$ is constructed and theorem 3.4 is proved.

Remark 3.8. Note once more that the method of proving the conditional result of theorem 3.4 (which is more or less standard variation of the $l$-trajectories method) is widely used nowadays in the exponential attractors theory, see, e.g. [24] and the references therein. Thus, the major difficulty here (and the major novelty of the paper) is related to the verification of assumption L1 for the solution semigroup $S(t)$ associated with the obstacle problem (1.1).
4. Estimates on the difference of the Lagrange multipliers and verification of assumption \( L \)

The aim of this section is to verify assumption L1 under some additional assumptions on the structure of the convex set \( \mathcal{K} \). We start with the case in which \( \mathcal{K} \) has a smooth boundary.

4.1. The case of regular \( \mathcal{K} \).

The main result of this subsection is the following theorem.

**Theorem 4.1.** Let the assumptions of theorem 2.2 hold and let, in addition, the boundary \( S = \partial \mathcal{K} \) be smooth enough (at least, \( C^{2,1} \)). Then, the solution semigroup \( S(t) \) associated with equation (1.1) satisfies assumption \( L \).

**Proof.** The proof is based on an argument that combines a proper choice of an approximation scheme and the maximum principle similar to the one devised to prove proposition 2.5. However, since the function \( M(u) = \text{dist}(u, \mathcal{K}) \) used to define \( F_\varepsilon \) in the existence theorem 2.2 is not smooth enough near the boundary, we should introduce another approximation of the singular potential. The smooth correction of \( M \) is given by the following lemma.

**Lemma 4.2.** Let \( \mathcal{K} \) be a bounded convex set with the \( C^{2,1} \)-smooth boundary \( S \). Then, there exists a function \( M : \mathbb{R}^n \to \mathbb{R} \) with the following properties:

\[
\begin{align*}
M &\in C^{2,1}(\mathbb{R}^n); & (4.1a) \\
M &\text{is convex;} & (4.1b) \\
S &= \partial \mathcal{K} = \{z \in \mathcal{K} \text{ such that } M(z) = 0\}; & (4.1c) \\
|\nabla M(z)| &= \vartheta(M(z)) & (4.1d)
\end{align*}
\]

where \( \vartheta = \vartheta(z) \) is a monotone increasing function which is smooth near \( z = 0 \) and such that \( \vartheta(0) \neq 0 \).

**Proof.** The function \( M \) can be constructed as follows. First of all, for any \( \delta > 0 \), we introduce the following set:

\[
S_{-\delta} := \{z_0 \in \mathcal{K} : \text{dist}(z_0, S) = \delta\}. \tag{4.2}
\]

Then, we consider the subset \( \mathcal{K}_{-\delta} \subset \mathcal{K} \) whose boundary is \( S_{-\delta} \). It turns out that, being \( \mathcal{K} \) convex, the domain \( \mathcal{K}_{-\delta} \) is convex too. Moreover, by possibly taking \( \delta \) small, the boundary \( S_{-\delta} \) has the same regularity of \( S \). The candidate for \( M \) is thus the function

\[
M(z) := [\text{dist}(z, \mathcal{K}_{-\delta})]^3 - \delta^3. \tag{4.3}
\]

In fact, \( M \) is clearly convex and regular. Moreover, by choosing \( \vartheta = \vartheta(w) = 3(w + \delta^3)^{2/3} \) and using the identity

\[
|\nabla (\text{dist}(z, \mathcal{K}_{-\delta}))| = 1,
\]

one sees that condition (4.1d) is satisfied.

Finally, by possibly taking a small \( \delta \), the smoothness of the boundary \( S \) entails the validity of (4.1c). Note also that \( M \) has the additional property of qualifying the fact that \( v \in \mathcal{K} \), namely, it turns out that \( v \in \mathcal{K} \) if and only if \( -\delta^3 \leq M(v) \leq 0 \). ■
Now, we introduce the approximations $F_\varepsilon(u)$ of the indicator function $I_{\mathcal{X}_\varepsilon}$ as in \eqref{eq:2.9} using the above defined function $M$. Let now $u(t)$ and $v(t)$ be two solutions of the singular equation \eqref{eq:2.20} with initial conditions

$$u_0, v_0 \in C^{2-v} \cap H^1_0,$$

with sufficiently small positive $v$ (actually, $v = 1$ is sufficient for what follows). Then, let $u_\varepsilon(t)$ and $v_\varepsilon(t)$ be their approximations, namely, the solutions of the approximate problems \eqref{eq:2.20} with initial data $u_0$ and $v_0$. As one can easily check, proposition 2.5 and corollary 2.7 still hold for this new approximation entailing the validity of the following two estimates:

$$
\|u_\varepsilon(t)\|_{C^{2-v}} + \|v_\varepsilon(t)\|_{C^{2-v}} \leq Q(\|u_0\|_{C^{2-v}} + \|v_0\|_{C^{2-v}}), \quad t \geq 0, \quad (4.4)
$$

and

$$
\|F'_{\varepsilon}(u_\varepsilon(t))\|_{L^\infty} + \|F'_{\varepsilon}(v_\varepsilon(t))\|_{L^\infty} \leq C, \quad (4.5)
$$

where the monotone function $Q$ and the constant $C$ are both independent of $\varepsilon$. Thus, up to not relabelled subsequence,

$$
\begin{align*}
\varepsilon(u_\varepsilon) & \rightharpoonup u, & \varepsilon(v_\varepsilon) & \rightharpoonup v
\end{align*}
$$

weakly star in $L^\infty([0, T] \times \Omega)$. Thus, in order to prove the theorem, we only need to verify that

$$
\|F'_{\varepsilon}(u_\varepsilon) - F'_{\varepsilon}(v_\varepsilon)\|_{L^1([0, T] \times \Omega)} \leq Q(\|u_0\|_{C^{2-v}} + \|v_0\|_{C^{2-v}})\|u_0 - v_0\|_{L^2} \quad (4.6)
$$

with the function $Q$ not depending on $\varepsilon$ and then by semicontinuity we obtain the desired estimate \eqref{eq:3.10} in the limit $\varepsilon \searrow 0$. This estimate will hold for every two trajectories $u$ and $v$ such that $u_0$ and $v_0$ are bounded in $C^{2-v}$. In order to construct the desired positively invariant absorbing set $B_0$, it is sufficient to take a ball

$$
B := \{ u \in C^{2-v/2} \cap H^1_0 \cap \Phi, \quad \|u\|_{C^{2-v}} \leq R \}
$$

with a sufficiently large radius $R$. Then, thanks to corollary 2.7, $B$ will be an absorbing set for the semigroup $S(t)$ associated with equation \eqref{eq:1.1} and, therefore, the closure $B_1 := [B]_\Phi$ of the set $B$ in the topology of $\Phi$ will be a bounded in $C^{2-v}$ and closed in $\Phi$ absorbing set for the semigroup $S(t)$. Applying corollary 2.7 again, we see that the set

$$
B_0 := \bigcup_{i \geq 0} B_i \quad (4.7)
$$

will be the desired positively invariant, closed in $\Phi$ and bounded in $C^{2-v}(\Omega)$ absorbing set for semigroup $S(t)$ and estimate \eqref{eq:4.6} will guarantee that \eqref{eq:3.10} will hold uniformly with respect to all trajectories starting from $B_0$.

Thus, it only remains to verify the uniform estimate \eqref{eq:4.6}. The first step is, roughly speaking, to reduce \eqref{eq:4.1} to an equation with scalar constraint (actually to an approximation of). This is done using the function $M$ as we did in \eqref{eq:2.22}. Indeed, testing equation \eqref{eq:2.20} in the scalar product $(\cdot, \cdot)_{\mathbb{R}^n}$ of $\mathbb{R}^n$ with $\nabla M(u)$ and using condition \eqref{eq:4.1d}, we obtain

$$
\partial_t (M(u_\varepsilon)) - \Delta (M(u_\varepsilon)) + F'_{\varepsilon}(M(u_\varepsilon)) + D(u_\varepsilon) = 0, \quad (4.8)
$$

with $F'_{\varepsilon}(z) := f'(z)\theta^2(z)$ and

$$
D(u) := \sum_{i=1}^N \sum_{j,k=1}^n M'_{j,k}(u)\partial_u u_j \partial_u u_k - \lambda \sum_{j} M_j(u)u_j, \quad (4.9)
$$

where $\{M_{j,k}\}_{j,k=1}^n$ denotes the entries of the Hessian matrix $\mathcal{H}(M)$ of the function $M$ (and the analogous equation holds for $v_\varepsilon$). Let now $w_\varepsilon := M(u_\varepsilon) - M(v_\varepsilon)$. Then

$$
\partial_t w_\varepsilon - \Delta w_\varepsilon + [F'_{\varepsilon}(M(u_\varepsilon)) - F'_{\varepsilon}(M(v_\varepsilon))] + [D(u_\varepsilon) - D(v_\varepsilon)] = 0. \quad (4.10)
$$
We now multiply this equation by $\text{sgn} w_{\varepsilon}$ and integrate over $\Omega$ obtaining
\[
\frac{d}{dt} \| u_{\varepsilon} - v_{\varepsilon} \|_{L^1} + \| F'(M(u_{\varepsilon})) - F'(M(v_{\varepsilon})) \|_{L^1} = -\int_{\Omega} (D(u_{\varepsilon}) - D(v_{\varepsilon})) \text{sgn} w_{\varepsilon} \, dx =: R,
\]
where we have used the monotonicity of $F'(z)$ (recall in particular that $\vartheta(z) = (z + \delta^3)^{2/3}$) and the Kato inequality. To estimate the term with $R$ we start to rewrite it in the more explicit form
\[
R = \sum_{i=1}^{N} \int_{\Omega} ((\mathcal{H}(M(u_{\varepsilon})) - (\mathcal{H}(M(v_{\varepsilon}))) \partial_x u_{\varepsilon}, \partial_x u_{\varepsilon})_{\mathbb{R}^n} \text{sgn}(w_{\varepsilon}) \, dx
\]
\[
+ \sum_{i=1}^{N} \int_{\Omega} ((\mathcal{H}(M(v_{\varepsilon})) \partial_x u_{\varepsilon} - \partial_x v_{\varepsilon}), \partial_x u_{\varepsilon})_{\mathbb{R}^n} \text{sgn}(w_{\varepsilon}) \, dx
\]
\[
+ \sum_{i=1}^{N} \int_{\Omega} ((\mathcal{H}(M(v_{\varepsilon})) \partial_x u_{\varepsilon} - \partial_x v_{\varepsilon}), \partial_x v_{\varepsilon})_{\mathbb{R}^n} \text{sgn}(w_{\varepsilon}) \, dx
\]
\[
+ \lambda \int_{\Omega} (\nabla M(u_{\varepsilon}), u_{\varepsilon})_{\mathbb{R}^n} \text{sgn}(w_{\varepsilon}) \, dx - \lambda \int_{\Omega} (\nabla M(v_{\varepsilon}), v_{\varepsilon})_{\mathbb{R}^n} \text{sgn}(w_{\varepsilon}) \, dx. \tag{4.12}
\]
Thus, thanks to the assumed $C^{2,1}$ regularity on the function $M$ (which follows from the smoothness of the boundary of $\mathcal{K}$) and the $C^1$-regularity of $u_{\varepsilon}$ and $v_{\varepsilon}$ which follows from (4.4), it is not difficult to realize that
\[
\| R(t) \|_{L^1(\Omega)} \leq C \| u_{\varepsilon}(t) - v_{\varepsilon}(t) \|_{H^1} \text{ for any } t \geq 0, \tag{4.13}
\]
where the constant $C$ depends on the $C^1$-norms of $u_0$ and $v_0$, but is independent of $\varepsilon > 0$. Integrating the differential inequality (4.11) with respect to $t \in [0, 1]$ and using (4.13) and the analogue of (3.3) for the solutions $u_{\varepsilon}$ and $v_{\varepsilon}$ of the approximate problems (2.10), we arrive at (recall also that $u_{\varepsilon}(0) = u_0$ and $v_{\varepsilon}(0) = v_0$)
\[
\int_0^1 \| F'(M(u_{\varepsilon})) - F'(M(v_{\varepsilon})) \|_{L^1} \, dt \leq \| u_0 - v_0 \|_{L^1} + \int_0^1 R(t) \, dt
\]
\[
\leq \| u_{\varepsilon}(0) - v_{\varepsilon}(0) \|_{L^1} + C \int_0^1 \| u_{\varepsilon}(t) - v_{\varepsilon}(t) \|_{H^1} \, dt
\]
\[
\leq C_1 \| u_0 - v_0 \|_{L^2}, \tag{4.14}
\]
where the constants $C$ and $C_1$ depend on the $C^1$-norms of $u(0)$ and $v(0)$, but are independent of $\varepsilon$.

The next step is to estimate the difference $f_{\varepsilon}'(M(u_{\varepsilon})) - f_{\varepsilon}'(M(v_{\varepsilon}))$ in terms of the difference of $F'(M(u_{\varepsilon}))$ and $F'(M(v_{\varepsilon}))$. To this end, we note that $f_{\varepsilon}(z) \equiv 0$ if $z \leq 0$, so, without loss of generality, we may assume that, say, $M(u_{\varepsilon}) \geq 0$. Then, if $M(v_{\varepsilon}) \leq 0$
\[
| f_{\varepsilon}'(M(u_{\varepsilon})) - f_{\varepsilon}'(M(v_{\varepsilon})) | = f_{\varepsilon}'(M(u_{\varepsilon})) = \theta^{-2}(M(u_{\varepsilon}))(F_{\varepsilon}'(M(u_{\varepsilon}))
\]
\[
\leq \theta^{-2}(0)F_{\varepsilon}'(M(u_{\varepsilon})) = C | F_{\varepsilon}'(M(u_{\varepsilon})) - F_{\varepsilon}'(M(v_{\varepsilon})) |. \tag{4.15}
\]
Let now $M(v_\varepsilon) \geq 0$. Then,
\[
|f'_c(M(u_\varepsilon)) - f'_c(M(v_\varepsilon))| = |F'_c(M(u_\varepsilon))\theta^{-2}(M(u_\varepsilon)) - F'_c(M(v_\varepsilon))\theta^{-2}(M(v_\varepsilon))|
\leq \theta^{-2}(M(u_\varepsilon)) \cdot |F'_c(M(u_\varepsilon)) - F'_c(M(v_\varepsilon))| + F'_c(M(v_\varepsilon))\theta^{-2}(M(v_\varepsilon))
\leq -\theta^{-2}(M(v_\varepsilon))| + \theta^{-2}(0)|F'_c(M(u_\varepsilon)) - F'_c(M(v_\varepsilon))|
+ \|F'_c(u_\varepsilon)\|_{L^\infty} \theta^{-3}(0)\theta^2(M(u_\varepsilon)) - \theta^2(M(v_\varepsilon))
\leq C \left(|F'_c(M(u_\varepsilon)) - F'_c(M(v_\varepsilon))| + |u_\varepsilon - v_\varepsilon|\right), \tag{4.16}
\]
where the constant $C$ is independent of $\varepsilon$ (here we have implicitly used that the $L^\infty$-norms of $F'_c(u_\varepsilon)$ are uniformly bounded, thanks to (4.5)). Thus, due to (4.14), (4.16) and (4.15)
\[
\int_0^1 \|f'_c(M(u_\varepsilon(t))) - f'_c(M(v_\varepsilon(t)))\|_{L^1} \text{ d}t \leq C\|u_\varepsilon(0) - v_\varepsilon(0)\|_{L^2}.
\]
Finally, since $M$ is at least $C^2$,
\[
\|F'_c(u_\varepsilon) - F'_c(v_\varepsilon)\|_{L^1} = \|f'_c(M(u_\varepsilon))M'(u_\varepsilon) - f'_c(M(v_\varepsilon))M'(v_\varepsilon)\|_{L^1}
\leq \|M'(u_\varepsilon)\|_{L^\infty} \|f'_c(M(u_\varepsilon)) - f'_c(M(v_\varepsilon))\|_{L^1} + \|f'_c(M(u_\varepsilon))\|_{L^\infty} \|M'(u_\varepsilon) - M'(v_\varepsilon)\|_{L^1}
- M'(v_\varepsilon)\|_{L^1} \leq C(\|f'_c(M(u_\varepsilon)) - f'_c(M(v_\varepsilon))\|_{L^1} + \|u_\varepsilon - v_\varepsilon\|_{L^1}) \tag{4.17}
\]
and
\[
\int_0^1 \|F'_c(M(u_\varepsilon(t))) - F'(M(v_\varepsilon(t)))\|_{L^1} \text{ d}t \leq C\|u_\varepsilon(0) - v_\varepsilon(0)\|_{L^2}.
\]
Thus, estimate (4.6) is verified and theorem 4.1 is proved. $\blacksquare$

**Corollary 4.3.** Let the assumptions of theorem 2.2 hold and let boundary $S = \partial \mathcal{X}$ be of class $C^{2,1}$. Then, the semigroup $S(t)$ associated with the obstacle problem (1.1) possesses an exponential attractor $\mathcal{M}$ in the sense of definition 3.3 in the phase space $\Phi$. Moreover, the global attractor has finite fractal dimension.

Indeed, this assertion is an immediate corollary of theorems 3.4 and 4.1.

### 4.2. The case of an irregular convex $\mathcal{X}$: the simplex (1.3)

As we saw in the former section, the smoothness of the boundary $S = \partial \mathcal{X}$ seems crucial for the method of verifying assumption $\mathcal{Z}$ suggested in the proof of theorem 4.1. First of all, the $C^{2,1}$ smoothness of $M$ is necessary in order to obtain estimate (4.13). Secondly, the fact that $\theta(0) \neq 0$ in (4.1d), which is crucial to obtain estimates (4.16) and (4.15) implies that $\nabla M(z) \neq 0$ for all $z \in S$. Thus, by the implicit function theorem, the boundary $S$ also must be at least $C^{2,1}$-smooth. However, for the possible applications to phase transition problems, one of the most important examples for the set $\mathcal{X}$ is the $n$-dimensional simplex, namely, the set
\[
\mathcal{X} := \left\{ p = (p_1, \ldots, p_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n p_i \leq 1, \quad p_i \geq 0, \quad i = 1, \ldots, n \right\} \tag{4.18}
\]
which is a polyhedron and its boundary is only piecewise smooth. Thus, theorem 4.1 is not directly applicable here. Nevertheless, as we will show below, assumption $\mathcal{Z}$ remains valid for the non-regular case (4.18). Again, the verification of assumption $\mathcal{Z}$ will rely
on an approximation argument and, to this purpose, we will consider a slightly different approximation for the indicator function $I_K$, namely, let us introduce
\[ F_\varepsilon(u) := f_\varepsilon \left( \sum_{i=1}^{n} u_i - 1 \right) + \sum_{i=1}^{n} f_\varepsilon(-u_i), \]  
(4.19)
where the function $f_\varepsilon$ is defined by (2.8). Obviously, $F_\varepsilon(u)$ is convex and satisfies the conditions of remark 2.3. Therefore, we can indeed use it for approximating the singular problem (1.1). Consequently, all of the estimates and convergences obtained in the proof of theorem 2.2 hold for this new approximation.

On the other hand, an inspection in the proof of the $L^\infty$ estimate in proposition 2.5 reveals that one of keys point was the structure of the approximations. It is also evident that the new defined approximation for the simplex is structurally different from the approximation introduced before; thus the $L^\infty$ bound of proposition 2.5 need a different proof. In particular, having in mind also the study of the approximation of the exponential attractor (see the next section), we formulate the analogue of proposition 2.5 in a slightly stronger form by indicating also the invariant regions at the $\varepsilon$-level. Thus, the global bound result on the Lagrange multipliers takes this form

**Theorem 4.4.** Let $\varepsilon > 0$ be small enough and let the function $F_\varepsilon(u)$ be defined by (4.19). Then, there exists a positive constant $p$ (independent of $\varepsilon$) such that the set
\[ \mathcal{K}_\varepsilon := \{ u \in \mathbb{R}^n, F_\varepsilon(u) \leq p \varepsilon \} \]  
(4.20)
is an invariant region for the solution semigroup $S_\varepsilon(t)$ of the approximate problems (2.10), namely,
\[ S_\varepsilon(t) : \Phi_\varepsilon \rightarrow \Phi_\varepsilon, \quad t \geq 0, \]  
(4.21)
where
\[ \Phi_\varepsilon := \{ u \in L^\infty(\Omega), \ u(x) \in \mathcal{K}_\varepsilon \ \text{for almost all} \ x \in \Omega \}. \]  
(4.22)
Finally, the approximations $h_\varepsilon(t) := F_\varepsilon'(u_\varepsilon(t))$ (where $u_\varepsilon(t) := S_\varepsilon(t)u_0$, $u_0 \in \Phi_\varepsilon$) are uniformly bounded in the $L^\infty$-norm:
\[ \| h_\varepsilon(t) \|_{L^\infty(\Omega)} \leq C \]  
(4.23)
where the constant $C$ is independent of $\varepsilon$ and of the concrete choice of $u_0 \in \Phi_\varepsilon$.

**Proof.** First of all, we need the following lemma which clarifies the relations between $|F_\varepsilon'(u)|$ and $F_\varepsilon(u)$.

**Lemma 4.5.** Let the function $F_\varepsilon$ be defined by (4.19) and the function $f_\varepsilon(z)$ be given by (2.8). Then, there exist two positive constants $\kappa_1$ and $\kappa_2$ (independent of $\varepsilon$) such that
\[ \frac{\kappa_2}{\varepsilon} F_\varepsilon(u) \leq |F_\varepsilon'(u)|^2 \leq \frac{\kappa_1}{\varepsilon} F_\varepsilon(u) \]  
(4.24)
for all $\varepsilon > 0$ and all $u \in \mathbb{R}^n$.

**Proof.** Indeed, let $f_\varepsilon' := -f_\varepsilon'(-u_i)$ and $f_0' := f_\varepsilon'\left(\sum_{i=1}^{n} u_i - 1\right)$. Then,
\[ |F_\varepsilon'(u)|^2 = \sum_{i=1}^{n} (f_\varepsilon' + f_0')^2 = n|f_0'|^2 + \sum_{i=1}^{n} |f_\varepsilon'|^2 + 2 \left( \sum_{i=1}^{n} f_\varepsilon' \right) f_0'. \]  
(4.25)
Let us consider two cases:

**Case I:** \( f'_{0} = f'_{0}(\sum u_{i} - 1) = 0 \). Then, (4.25) simply reads as

\[
|F'_{\epsilon}(u)|^{2} = \sum_{i=1}^{n} |f'_{i}|^{2} = \sum_{i=0}^{n} |f'_{i}|^{2}.
\]

**Case II:** \( f'_{0} = f'_{0}(\sum u_{i} - 1) \neq 0 \). Then, keeping in mind the definition of the function \( f_{\epsilon} \), we conclude that at least one of \( u_{i} \), \( i = 1, \ldots, n \) (say, \( u_{n} \) for definiteness) must be positive and, therefore, without loss of generality, we may assume that \( u_{n} > 0 \) and, consequently, \( f'_{n} = -f'_{\epsilon}(-u_{n}) = 0 \). Then, using the elementary inequality \( ab \geq -\frac{1}{4}(\frac{a^{2}}{\alpha} + \alpha b^{2}) \), \( \alpha > 0 \), we arrive at

\[
n|f'_{0}|^{2} + \sum_{i=1}^{n-1} |f'_{i}|^{2} + \sum_{i=1}^{n} f'_{0} f'_{i} \geq (n - \alpha(n - 1))|f'_{0}|^{2} + (1 - \alpha^{-1}) \sum_{i=1}^{n-1} |f'_{i}|^{2}.
\]

Choosing the positive \( \alpha = \alpha(n) \) in an optimal way as a solution of the equation

\[
1 - \alpha^{-1} = n(1 - \alpha) + \alpha,
\]
we see that, in the second case

\[
|F'_{\epsilon}(u)|^{2} \geq \theta(n) \sum_{i=0}^{n} |f'_{i}|^{2}
\]

with \( \theta(n) := \frac{n+1-\sqrt{(n+1)^{2}-4}}{4} > 0 \). Thus, in both cases

\[
|F'_{\epsilon}(u)|^{2} \geq \theta(n) \sum_{i=0}^{n} |f'_{i}|^{2} = \theta(n) \left(|f'_{0}|^{2} + \sum_{i=1}^{n} |f'_{i}|^{2}\right).
\]

Since the upper bound is obvious, we arrive at the following inequality:

\[
(4.26) \quad \theta(n) \left(\sum_{i=1}^{n} |f'_{i}(-u_{i})|^{2} + \left|f'_{\epsilon}\left(\sum_{i=1}^{n} u_{i} - 1\right)\right|^{2}\right) \leq |F'_{\epsilon}(u)|^{2}
\]

\[
\leq \sum_{i=1}^{n} |f'_{i}(-u_{i})|^{2} + \left|f'_{\epsilon}\left(\sum_{i=1}^{n} u_{i} - 1\right)\right|^{2}.
\]

It only remains to note that, due to our choice (2.8) of the function \( f_{\epsilon} \),

\[
|f'_{\epsilon}(z)|^{2} = \frac{4}{\varepsilon} f_{\epsilon}(z)
\]

and, consequently, (4.26) implies (4.24) and completes the proof of the lemma. ■

With the help of lemma 4.5, it is now not difficult to complete the proof of theorem 4.4. Indeed, testing equation (2.10) in the scalar product of \( \mathbb{R}^{n} \) with \( F'_{\epsilon}(u) \) and using that \( F''_{\epsilon}(u) \geq 0 \), we have (compare with (2.22))

\[
\partial_{u} F_{\epsilon}(u) - \Delta_{\epsilon}(F_{\epsilon}(u)) + |F'_{\epsilon}(u)|^{2} \leq \lambda(u, F_{\epsilon}(u))_{\mathbb{R}_{+}}.
\]

Using now lemma 4.5 together with the obvious fact that

\[
|u| \leq F_{\epsilon}(u) + C,
\]

where \( C \) is independent of \( \varepsilon \to 0 \), we arrive at the differential inequality for the scalar function \( V(t) := F'(u(t)) \):

\[
\partial_t V - \Delta_x V + \left( \frac{\kappa_2}{2\varepsilon} - \lambda^2 \right) V \leq C_1, \quad V|_{\partial\Omega} = 0,
\]

(4.27)

where the positive constants \( C_1 \) and \( \kappa \) are independent of \( \varepsilon \) and the concrete choice of the solution \( u \). Applying the comparison principle for the heat equations to (4.27), we see that the region \( \{ u, V(u) \leq p\varepsilon \} \) will be invariant if

\[
p\varepsilon \geq \frac{\kappa_2}{2\varepsilon} - \lambda^2 = \frac{2C_1\varepsilon}{\kappa_2 - \lambda^2\varepsilon}.
\]

Thus, taking \( p := \frac{3C_1}{\kappa_2} \), we see that the region \( K_{\varepsilon} \) defined by (4.20) will be indeed invariant with respect to the semi-flow \( S_{\varepsilon}(t) \) if \( \varepsilon > 0 \) is small enough. Thus, we only need to check estimate (4.23). To this end, it remains to observe that, due to lemma 4.5

\[
|F'(u(t))|^2 \leq \frac{\kappa_1}{\varepsilon} F(u(t)) \leq \frac{\kappa_1}{\varepsilon} \cdot p\varepsilon = \kappa_1 p.
\]

Theorem 4.4 is proved. ■

We are now ready to formulate the main result of this subsection.

**Theorem 4.6.** Let the assumptions of theorem 4.4 hold and \( \varepsilon > 0 \) be small enough. Then, for any two solutions \( u_\varepsilon(t) := S_\varepsilon(t)u_0 \) and \( v_\varepsilon(t) := S_\varepsilon(t)v_0 \) of the approximate problems (2.10) starting from \( \Phi_\varepsilon(u_0, v_0) \in \Phi_{\varepsilon} \), the associated Lagrange multipliers \( h_\varepsilon(t) := F'(u_\varepsilon(t)) \) and \( h_{\varepsilon}(t) := F'(v_\varepsilon(t)) \) satisfy the following estimate:

\[
\int_0^1 \| h_\varepsilon(t) - h_{\varepsilon}(t) \|_{L^1(\Omega)} \, dt \leq L\| u_0 - v_0 \|_{L^2(\Omega)},
\]

(4.28)

where the constant \( L \) is independent of \( \varepsilon, u_0 \) and \( v_0 \). In particular, the limit semigroup \( S(t) \) associated with the obstacle problem (1.1) satisfies assumption \( \mathcal{L} \) with \( B_0 := \Phi \).

**Proof.** In order to simplify the notations we forget for a while the \( \varepsilon \) dependence and will write \( u(t) := S(t)u_0 \) and \( v(t) := S(t)v_0 \). Then, the function \( w(t) := u(t) - v(t) \) solves

\[
\partial_t w_i - \Delta_x w_i + [f'(v_i) - f'(u_i)] + \left[ \sum_{i=1}^n u_i - 1 \right] \cdot \lambda w_i,
\]

\[
i = 1, \ldots, n.
\]

(4.29)

Multiply now the \( i \)th equation of (4.29) by \( \text{sgn}w_i \) and integrate over \( \Omega \). Then, taking the sum over \( i \) and using the Kato inequality, we arrive at

\[
\partial_t \| w \|_{L^1} - \lambda \| w \|_{L^1} + \sum_{i=1}^n \| f'(u_i) - f'(v_i) \|_{L^1}
\]

\[
\leq - \left( \left[ \sum_{i=1}^n u_i - 1 \right] \cdot \lambda \right) \cdot \sum_{i=1}^n \text{sgn}w_i \leq L\| u_0 - v_0 \|_{L^2(\Omega)}.
\]

(4.30)

Let us estimate the right-hand side of this inequality. To this end, we first note that

\[
\sum_{i=1}^n \text{sgn}w_i \leq n.
\]
In addition, in the worst case where the equality holds, all \( w_i \) are of the same sign. Therefore, at that point the term \( f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \) will also have the same sign and its product with \( \sum_{i=1}^n \text{sgn} w_i \) will be non-negative. Thus,
\[
- \left[ f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right] 
\cdot \sum_{i=1}^n \text{sgn} w_i \leq (n - 1) \left| f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right|
\]
and
\[
\partial_t \|w\|_{L^1} + \sum_{i=1}^n \left\| f'_\varepsilon(-u_i) - f'_\varepsilon(-v_i) \right\|_{L^1}
\leq \lambda \|w\|_{L^1} + (n - 1) \left\| f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right\|_{L^1}. \tag{4.31}
\]

Next, in order to estimate the right-hand side of (4.31), we sum all the equations (4.29) and multiply the obtained relation by \( \text{sgn}(\sum_{i=1}^n u_i - \sum_{i=1}^n v_i) \). Then, using again the Kato inequality, we have
\[
\partial_t \|\sum_{i=1}^n w_i\|_{L^1} + \sum_{i=1}^n \left\| f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right\|_{L^1}
\leq \lambda \|\sum_{i=1}^n w_i\|_{L^1} + \sum_{i=1}^n \left\| f'_\varepsilon(-u_i) - f'_\varepsilon(-v_i) \right\|_{L^1}. \tag{4.32}
\]

Multiplying inequality (4.32) by \( \frac{n+1}{n+1} \) and summing it to inequality (4.31), we finally arrive at
\[
\partial_t \left( \|w\|_{L^1} + \frac{n}{n+1} \left\| \sum_{i=1}^n w_i \right\|_{L^1} \right) + \frac{1}{n+1} \left( \left\| f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right\|_{L^1}
+ \sum_{i=1}^n \left\| f'_\varepsilon(-u_i) - f'_\varepsilon(-v_i) \right\|_{L^1} \right) \leq \lambda \left( \|w\|_{L^1} + \frac{n}{n+1} \left\| \sum_{i=1}^n w_i \right\|_{L^1} \right). \tag{4.33}
\]

Applying the Gronwall inequality to this relation, we infer
\[
\int_0^1 \left\| f'_\varepsilon(\sum_{i=1}^n u_i(t) - 1) - f'_\varepsilon(\sum_{i=1}^n v_i(t) - 1) \right\|_{L^1}
+ \sum_{i=1}^n \left\| f'_\varepsilon(-u_i(t)) - f'_\varepsilon(-v_i(t)) \right\|_{L^1} \, dt
\leq (n + 1)e^{\lambda} \left( \|w(0)\|_{L^1} + \frac{n}{n+1} \left\| \sum_{i=1}^n w_i(0) \right\|_{L^1} \right) \leq C \|u(0) - v(0)\|_{L^2} \tag{4.34}
\]
which together with the obvious inequality
\[
\|h_{w_i}(t) - h_{v_i}(t)\|_{L^1}
\leq C \left( \left\| f'_\varepsilon(\sum_{i=1}^n u_i - 1) - f'_\varepsilon(\sum_{i=1}^n v_i - 1) \right\|_{L^1}
+ \sum_{i=1}^n \left\| f'_\varepsilon(-u_i) - f'_\varepsilon(-v_i) \right\|_{L^1} \right)
\]
completes the proof of estimate (4.28). Passing now to the limit \( \varepsilon \to 0 \) in that estimate (analogously to the proof of theorem 4.1), we see that the limit semigroup \( S(t) \) generated by the obstacle problem (1.1) satisfies indeed assumption \( \mathcal{L} \) with \( B_0 := \Phi \). Theorem 4.6 is proved.
Corollary 4.7. Let the assumptions of theorem 2.2 hold and let the set $\mathcal{K}$ be an $n$-dimensional simplex (4.18). Then, the semigroup $S(t)$ associated with the obstacle problem (1.1) possesses an exponential attractor $\mathcal{M}$ in the sense of definition 3.3. Moreover, the global attractor $A$ has finite fractal dimension.

Indeed, this assertion is an immediate corollary of theorems 3.4 and 4.6.

4.3. Some generalizations

We now discuss the applications of our method to more general problems. We start with the obvious observation that all the above results remain valid if we replace the term $\lambda u$ in the left-hand side of equation (1.1) by any sufficiently regular interaction function $g(u, x)$. Namely, consider the problem

$$\partial_t u - \Delta u + \partial I_K(u) + g(x, u) \ni 0,$$

(4.35)

where $g \in C(\Omega, C^1(\mathbb{R}^n, \mathbb{R}^n))$ is an arbitrary interaction function. Then, the following result holds.

Theorem 4.8. Let $\mathcal{K}$ be a convex bounded set of $\mathbb{R}^n$ containing zero with a smooth boundary (or let $\mathcal{K}$ be an $n$-dimensional simplex (4.18)) and let $g \in C(\Omega, C^1(\mathbb{R}^n, \mathbb{R}^n))$ be an arbitrary (not necessarily a gradient!) nonlinear interaction function. Then, the solution semigroup $S(t)$ associated with equation (4.35) possesses an exponential attractor $\mathcal{M}$ in the sense of definition 3.3. Moreover, the fractal dimension of the global attractor is finite.

Indeed, since $\mathcal{K}$ is bounded, the solution $u$ is also automatically bounded in $L^\infty$ (and the same will be true for the solutions $u_\varepsilon$ of the approximate problems (2.10) if $\varepsilon > 0$ is small enough no matter how the regular interaction function $g$ looks like). So, the term $g(u, x)$ can be treated as a perturbation and the proof of theorem 4.8 repeats word by word the given proof for the particular case $g(u, x) = -\lambda u$.

Remark 4.9. The function $g(x, u)$ may even depend explicitly on the gradient $\nabla x u$: $g = g(x, u, \nabla x u)$. However, in that case we also need to impose some restrictions on the growth of $g$ with respect to $\nabla x u$ (since the obstacle potential controls only the $L^\infty$-norm of the solution, then the control of its $W^{1,\infty}$-norm should be additionally obtained). In particular, if $g$ does not grow with respect to $\nabla x u$, i.e.

$$|g(x, u, \nabla x u)| \leq Q(|u|)$$

for some monotone function $Q$ independent of $x$ and $\nabla x u$, the proof of theorem 4.8 still repeats word by word the case of $g(u) = -\lambda u$. However, our conjecture here is that the result remains true under the standard sub-quadratic growth restriction

$$|g(x, u, \nabla x u)| \leq Q(|u|)(1 + |\nabla x u|^p)$$

with $p < 2$.

Remark 4.10. As we have already pointed out, our method of estimating the Lagrange multipliers is strongly based on the maximum principle for the leading linear part of equation (1.1). For this reason, we are unable in general to extend it to the case of non-scalar diffusion matrix. However, we point out that for some particular convex sets assumption $\mathcal{L}$ can be verified also for diffusion matrices (say diagonal). This is the case of the convex $\mathcal{K} := [0, L]^n$, $L > 0$. 
In this case, assumption $\mathcal{L}$ can be easily verified (just multiplying the $k$th equation of (2.10) by $\text{sgn}(u^1_k - u^2_k)$ even in the case of diagonal diffusion matrix

$$\partial_t u - a\Delta_x u + \partial I_k(u) - \lambda u \ni 0,$$

(4.36)

with $a = \text{diag}(a_1, \ldots, a_k)$, $a_i > 0$ for $i = 1, \ldots, n$.

One more generalization can be obtained replacing the Laplacian $\Delta_x u$ in equations (1.1) by the quasi-linear second order differential operator

$$A(u) := \text{div}(a(|u|^2) \nabla u), \quad u = (u_1, \ldots, u_n), \quad |u|^2 := u_1^2 + \cdots + u_n^2$$

with some natural assumptions on the scalar diffusion function $a$. Then, it is not difficult to verify that the proofs of assumption $\mathcal{L}$ given above remain true and the associated solution semigroup possesses an exponential attractor under the assumptions of theorem 4.8.

5. Approximations of exponential attractors: the case of a simplex

The aim of this section is to show that the exponential attractor $M$ of the singular problem (1.1) can be approximated by the sequence of exponential attractors $M_{\varepsilon}$ of the regular equations (2.10). For simplicity, we consider below only the case in which $\mathcal{K}$ is an $n$-dimensional simplex (the case of an arbitrary bounded convex set with smooth boundary can be considered analogously). To be more precise, the main result of this section is the following theorem.

**Theorem 5.1.** Let the assumptions of theorem 4.4 hold. Then, for any sufficiently small $\varepsilon \geq 0$, there exists an exponential attractor $M_{\varepsilon}$ for the semigroup $S_\varepsilon(t) : \Phi_1 \varepsilon \rightarrow \Phi_1 \varepsilon$ associated with the approximation problem (2.10) (and the case $\varepsilon = 0$ corresponds to the semigroup $S(t) = S_0(t) : \Phi \rightarrow \Phi$ associated with the limit singular problem (1.1)). Moreover, the following conditions hold:

- The exponential attractors $M_{\varepsilon}$ are uniformly bounded in $C^{2-\nu}$ for any $\nu > 0$.
- $S_\varepsilon(t)M_{\varepsilon} \subset M_{\varepsilon}$ and the rate of convergence is uniform with respect to $\varepsilon \rightarrow 0$:

$$\text{dist}_{L^\infty}(S_\varepsilon(t)\Phi_1 \varepsilon, M_{\varepsilon}) \leq C e^{-\alpha t}, \quad \varepsilon \geq 0,$$

(5.1)

where the positive constants $C$ and $\alpha$ are independent of $\varepsilon$.
- The fractal dimension also remains bounded as $\varepsilon \rightarrow 0$:

$$\text{dim}_{f}(M_{\varepsilon}, L^\infty) \leq C < \infty,$$

(5.2)

where $C$ is independent of $\varepsilon$.
- The family $M_{\varepsilon}$ is Hölder continuous at $\varepsilon = 0$:

$$\text{dist}_{L^\infty}^{H^\kappa}(M_{\varepsilon}, M_0) \leq C \varepsilon^\kappa,$$

(5.3)

where the positive constants $C$ and $\kappa$ are independent of $\varepsilon$ and $\text{dist}_{L^\infty}^{H^\kappa}$ stands for the symmetric Hausdorff distance between sets.

**Proof.** The proof of this theorem is based on the following abstract result taken from [14] (see also [10]).

**Proposition 5.2.** Let $\mathcal{H}$ and $\mathcal{H}_1$ be two Banach spaces such that $\mathcal{H}_1$ is compactly embedded in $\mathcal{H}$ and let, for any $\varepsilon > 0$ there exists a bounded closed set $B_\varepsilon \subset \mathcal{H},$ a map $S_\varepsilon : B_\varepsilon \rightarrow B_\varepsilon$ and two (nonlinear) projectors $\Pi_\varepsilon : B_\varepsilon \rightarrow B_0$ and $Q_\varepsilon : B_0 \rightarrow B_\varepsilon$ such that the following
properties hold:

1) Uniform smoothing property: for every \( h_1, h_2 \in \mathbb{B}_\varepsilon \),
\[
\| S_\varepsilon h_1 - S_\varepsilon h_2 \|_{\mathcal{H}} \leq L \| h_1 - h_2 \|_{\mathcal{H}},
\] (5.4)
where \( L > 0 \) is independent of \( \varepsilon \geq 0 \) and \( h_1, h_2 \in \mathbb{B}_\varepsilon \).

2) The maps \( S_\varepsilon \) and \( S_0 \) are close in the following sense:
\[
\begin{align*}
\| S_\varepsilon \circ \mathcal{Q}_\varepsilon h - S_0 h \|_{\mathcal{H}} &\leq C \varepsilon, & \forall h \in \mathbb{B}_0, \\
\| S_0 \circ \mathcal{P}_\varepsilon h - S_\varepsilon h \|_{\mathcal{H}} &\leq C \varepsilon, & \forall h \in \mathbb{B}_\varepsilon,
\end{align*}
\] (5.5)
where the constant \( C \) is independent of \( \varepsilon \) and \( h \).

Then, the discrete semigroups \( S_\varepsilon(n) \) generated by the maps \( S_\varepsilon \) possess a uniform family of exponential attractors \( \mathcal{M}_\varepsilon \subset \mathbb{B}_\varepsilon \) such that the following properties hold:

1) Uniform rate of attraction:
\[
\text{dist}_{\mathcal{H}}(S_\varepsilon(n)\mathbb{B}_\varepsilon, \mathcal{M}_\varepsilon) \leq C \varepsilon^{-an}
\] (5.6)
for some positive \( C \) and \( a \) independent of \( \varepsilon \).

2) Uniform bounds for the fractal dimension:
\[
\dim_f(\mathcal{M}_\varepsilon, \mathcal{H}) \leq C,
\] (5.7)
where \( C \) is independent of \( \varepsilon \).

3) Hölder continuity at \( \varepsilon = 0 \):
\[
\text{dist}^\alpha_{\mathcal{H}}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C \varepsilon^\alpha
\] (5.8)
for some positive \( C \) and \( \alpha \) independent of \( \varepsilon \).

We can now prove theorem 5.1 by combining (as we did in the proof of theorem 3.4) the proposition above with the \( \ell \) trajectory approach. Namely, we define the spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \) by (3.14) and introduce the trajectory phase spaces
\[
\mathbb{B}_\varepsilon := \{ u \in \mathcal{H}_1, \ u(t) = S_\varepsilon(t)u_0, \ t \in [0, 1], \ u_0 \in \Phi_\varepsilon \}
\] (5.9)
and, using the lifting solution operator
\[
T_\varepsilon : \Phi_\varepsilon \to \mathbb{B}_\varepsilon, \quad (T_\varepsilon u_0)(t) := S_\varepsilon(t)u_0,
\]
we lift the solution semigroup \( S_\varepsilon(t) \) to the trajectory phase space \( \mathbb{B}_\varepsilon \):
\[
S_\varepsilon(t) := T_\varepsilon \circ S_\varepsilon(t) \circ T_\varepsilon^{-1}, \quad S_\varepsilon(t) : \mathbb{B}_\varepsilon \to \mathbb{B}_\varepsilon
\]
and set, finally, \( S_\varepsilon := S_\varepsilon(1) \).

Let us verify the assumptions of the abstract proposition for the maps thus defined. To this end, we note that, thanks to theorem 4.6, we have the uniform estimate (4.28) for the approximations to the Lagrange multipliers \( h_{u^1} - h_{u^2} \) associated with two trajectories \( u^i_\varepsilon \) := \( S_\varepsilon(t)u_{0^i} \), \( i = 1, 2 \). Therefore, arguing exactly as in the proof of lemma 3.7, we conclude that
\[
\| u_{1}^1 - u_{2}^1 \|_{L^1(1, 2; \mathcal{H})} + \| \partial_t u_{1}^1 - \partial_t u_{2}^1 \|_{L^1(1, 2; \mathcal{H})} \leq L \| u_{1}^1 - u_{2}^1 \|_{L^1(0, 1; \mathcal{H})}
\] (5.10)
for some positive \( L \) which is independent of \( \varepsilon \) and \( u_{i}^1, i = 1, 2 \). Thus, assumption (5.4) is verified.

In order to verify (5.5), we first fix an arbitrary interior point \( w_0 \in \mathbb{R}^d \) of \( K_0 := K \) and introduce the linear contraction map \( \tilde{E}_\varepsilon : K_\varepsilon \to K \) (where \( K_\varepsilon \) is defined by (4.20)) by the following expression:
\[
\tilde{E}_\varepsilon(u) := \frac{1}{1 + r\varepsilon}(u - w_0) + w_0 = \frac{1}{1 + r\varepsilon}u + \frac{r\varepsilon}{1 + r\varepsilon}w_0.
\] (5.11)
where \( r \) is a sufficiently large (but independent of \( \varepsilon \)) positive constant. Indeed, it is not difficult to show using the explicit expression for the set \( K_\varepsilon \) and for the function \( F_\varepsilon \), that \( E_\varepsilon(K_\varepsilon) \subset K_0 \) if \( r > 0 \) is large enough. Finally, we define the map \( E_\varepsilon : \Phi_\varepsilon \to \Phi \) as follows:

\[
E_\varepsilon(u)(x) := \varphi_\varepsilon(x)\tilde{E}_\varepsilon(u(x)),
\]

where the cut-off function \( \varphi_\varepsilon(x) \in C^1(\mathbb{R}^N) \) is such that \( 0 \leq \varphi_\varepsilon(x) \leq 1 \), \( \varphi_\varepsilon(x) = 0 \) if \( x \in \partial \Omega \) and \( \varphi_\varepsilon(x) = 0 \) if \( \operatorname{dist}(x, \partial \Omega) > \varepsilon \). Then, obviously,

\[
\|(1 - E_\varepsilon)v\|_{L^2} \leq C\varepsilon \|v\|_{L^2}, \quad \forall \ v \in \Phi_\varepsilon
\]

and for some positive \( C \) independent of \( \varepsilon \) and \( v \).

We now need the following crucial lemma.

**Lemma 5.3.** Let the assumptions of theorem 5.1 and let \( E_\varepsilon \) be defined by (5.11). Then, for every \( u_0^\varepsilon \in \Phi_\varepsilon \) and every \( u_0 \in K_0 \), the following estimate holds:

\[
\|S_\varepsilon(t)u_0^\varepsilon - S_0(t)u_0\|_{L^2} \leq C(\varepsilon^{1/2} + \|u_0 - u_0^\varepsilon\|_{L^2})e^{Kt},
\]

where the positive constants \( C \) and \( K \) are independent of \( \varepsilon \), \( u_0^\varepsilon \in \Phi_\varepsilon \), and \( u_0 \in \Phi \).

**Proof.** Let \( \tilde{u}(t) := S_0(t)u_0, u_\varepsilon(t) := S_\varepsilon(t)u_0^\varepsilon \) and \( v_\varepsilon(t) := E_\varepsilon(u_\varepsilon(t)) \). Then, since \( v_\varepsilon(t, x) \in K \) for all \( t \geq 0 \), it is an admissible test function for the variational inequality (2.2). Therefore,

\[
(\partial_t \tilde{u}(t), \tilde{u}(t) - v_\varepsilon(t)) + (\nabla_x \tilde{u}(t), \nabla_x \tilde{u}(t) - \nabla_x v_\varepsilon(t)) \leq \lambda(\tilde{u}(t), \tilde{u}(t) - v_\varepsilon(t))
\]

for almost all \( t > 0 \). Introducing the function \( w_\varepsilon(t) := u_\varepsilon(t) - v_\varepsilon(t) = (1 - E_\varepsilon)u_\varepsilon(t) \) and (5.12), we have

\[
(\partial_t \tilde{u}(t), \tilde{u}(t) - u_\varepsilon(t)) + (\nabla_x \tilde{u}(t), \nabla_x \tilde{u}(t) - \nabla_x u_\varepsilon(t)) - \lambda(\tilde{u}(t), \tilde{u}(t) - u_\varepsilon(t))
\]

\[
= (\partial_t \tilde{u}(t), \tilde{u}(t) - v_\varepsilon(t)) + (\nabla_x \tilde{u}(t), \nabla_x \tilde{u}(t) - \nabla_x v_\varepsilon(t)) - \lambda(\tilde{u}(t), \tilde{u}(t) - v_\varepsilon(t))
\]

\[
+ (\partial_t \tilde{u}(t), u_\varepsilon(t) - v_\varepsilon(t)) - (\Delta_x \tilde{u}(t), u_\varepsilon(t) - v_\varepsilon(t)) - \lambda(\tilde{u}(t), u_\varepsilon(t) - v_\varepsilon(t))
\]

\[
\leq -(h_{u\varepsilon}(t), w_\varepsilon(t)) \leq C\varepsilon,
\]

where we have implicitly used that the Lagrange multiplier \( h_{u\varepsilon}(t) \) associated with the solution \( \tilde{u}(t) \) of the singular problem (2.20) is uniformly bounded (see proposition 2.5). Moreover, multiplying equation (2.10) by \( u_\varepsilon(t) - \tilde{u}(t) \), integrating over \( x \in \Omega \) and using the monotonicity of \( F_\varepsilon(u) \) and the fact that \( F_\varepsilon(\tilde{u}(t)) \equiv 0 \), we arrive at

\[
(\partial_t u_\varepsilon(t), u_\varepsilon(t) - \tilde{u}(t)) + (\nabla_x u_\varepsilon(t), \nabla_x u_\varepsilon(t) - \nabla_x \tilde{u}(t)) - \lambda(u_\varepsilon(t), u_\varepsilon(t) - \tilde{u}(t)) \leq 0
\]

which holds for almost all \( t > 0 \). Taking a sum of (5.15) and (5.16), we finally have

\[
\frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t) - \tilde{u}(t)\|_{L^2}^2 + \|\nabla_x u_\varepsilon(t) - \nabla_x \tilde{u}(t)\|_{L^2}^2 - \lambda \|u_\varepsilon(t) - \tilde{u}(t)\|_{L^2}^2 \leq C\varepsilon
\]

which, together with the Gronwall inequality, gives (5.13) and completes the proof of the lemma.

It is now not difficult to complete the proof of the theorem. To this end, we set \( \Pi_\varepsilon : \mathbb{B}_\varepsilon \to \mathbb{B}_0 \) and \( \mathcal{Q}_\varepsilon : \mathbb{B}_0 \to \mathbb{B}_\varepsilon \) by

\[
\Pi_\varepsilon := T_0 \circ E_\varepsilon \circ T_\varepsilon^{-1}, \quad \mathcal{Q}_\varepsilon := T_\varepsilon \circ T_0^{-1},
\]

where \( T_0 \) is the lifting operator related to \( S_0(t) \). Then, it is not difficult to see using lemma 5.3 and estimate (5.12) that the projectors \( \Pi_\varepsilon \) and \( \mathcal{Q}_\varepsilon \) satisfy estimates (5.5). Thus, all of the assumptions of the abstract proposition 5.2 are verified and, consequently, the discrete
semigroups $S_\varepsilon(n)$ acting on the trajectory phase spaces $\mathcal{B}_\varepsilon$ possess the uniform family of exponential attractors $\mathcal{M}_\varepsilon$ which satisfies conditions (5.6), (5.7) and (5.8) of proposition 5.2.

The rest of the proof can be made exactly as in theorem 3.4. Indeed, arguing as in lemma 3.6, we see that
\begin{equation}
\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^2} \leq (2\lambda + 1) \int_0^1 \|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^2}^2 \, dt
\end{equation}
for any two solutions $u_1^\varepsilon$ and $u_2^\varepsilon$ of the approximation problems (2.10). In addition, multiplying (5.17) by $t$ and integrating over $t \in [0, 1]$, we see that
\begin{equation}
\|u_\varepsilon(1) - \bar{u}(1)\|_{L^2}^2 \leq (2\lambda + 1) \int_0^1 \|u_\varepsilon(t) - \bar{u}(t)\|_{L^2}^2 \, dt.
\end{equation}
Thus, projecting the constructed exponential attractors $\mathcal{M}_\varepsilon$ back to the physical phase spaces $\Phi_1\varepsilon$ by
\[ \mathcal{M}_\varepsilon^d := \mathcal{M}_\varepsilon|_{t=1}, \]
we obtain the uniform family of exponential attractors for the discrete semigroups $S_\varepsilon(n)$ : $\Phi_\varepsilon \to \Phi_\varepsilon$, which satisfy (5.1), (5.2) and (5.3) in a weaker space $L^2$ instead of $L^\infty$. In addition, we also have that
\[ \mathcal{M}_\varepsilon^d \subset S_\varepsilon(1)\Phi_\varepsilon \]
and, therefore, due to estimate (4.23) and to corollary 2.7 these attractors are uniformly bounded in $C^{2-v}(\Omega)$ for all $v > 0$ and the maps $(t, u_0) \to S_\varepsilon(t)u_0$ are uniformly Lipschitz continuous on $[0, 1] \times \mathcal{M}_\varepsilon^d$ (with the Lipschitz constant independent of $\varepsilon$ as well). Thus, the standard formula
\[ \mathcal{M}_\varepsilon := \bigcup_{t \in [0, 1]} S_\varepsilon(t)\mathcal{M}_\varepsilon^d \]
gives the desired uniform family of exponential attractors for the semigroups $S_\varepsilon(t)$ : $\Phi_\varepsilon \to \Phi_\varepsilon$ with continuous time. This family of exponential attractors satisfies (5.1), (5.2) and (5.3) in a weaker topology of $L^2(\Omega)$ instead of $L^\infty(\Omega)$ but, using the fact that $\mathcal{M}_\varepsilon$ are uniformly bounded in $C^{2-v}(\Omega)$ together with the interpolation inequality (3.19), we see that (5.1), (5.2) and (5.3) hold in the initial topology of $L^\infty$ as well. Theorem 5.1 is thus completely proved. ■

Remark 5.4. It is not difficult to show that, under the assumptions of theorem 5.1, the solution semigroups $S_\varepsilon(t)$ are actually defined not only on the space $\Phi_\varepsilon$, but in much lager phase spaces, namely, on $L^2$. Moreover, arguing analogously to the proof of theorem 4.4, one can see that the set $\Phi_\varepsilon$ is an absorbing set for the semigroup $S_\varepsilon(t)$. This means that, for every bounded $B \subset L^2(\Omega)$ there exists $T = T(\|B\|)$ (independent of $\varepsilon$) such that
\[ S_\varepsilon(t)B \subset \Phi_\varepsilon, \quad t \geq T. \]
Thus, the constructed exponential attractors $\mathcal{M}_\varepsilon$ attract (uniformly with respect to $\varepsilon$) the bounded sets of $L^2(\Omega)$ as well.

Finally, one may even extend the solution semigroup $S(t)$ associated with the obstacle equation (1.1) to the whole space $L^2$ as well. To this end, we just need to pass to the limit $\varepsilon \to 0$ in the solutions of the approximate problems (2.10). It is not difficult to show that the associated solution $u(t) := S(t)u_0$ will belong to $\Phi$ for every $t > 0$ and, in the case $u_0 \notin \Phi$, it will have a jump at $t = 0$ and the $L^2$-limit
\[ \tilde{u}_0 := \lim_{{t \to 0^+}} u(t) \in \Phi \]
will exist. Thus, we factually have
\[ u(t) = S(t)u_0 = S(t)\Pi_{\Phi}(u_0), \]
where $\Pi_{\Phi} : L^2(\Omega) \to \Phi$ is a non-linear ‘projector’ to the set $\Phi$. 

However, in contrast to the semigroup $S(t): \Phi \to \Phi$ which is uniquely defined by the singular equation (1.1), the above nonlinear ‘projector’ $\Pi_\rho$ essentially depends on the particular choice of the approximation of the obstacle potential and, therefore, is not canonically defined by the problem (1.1) itself. For this reason, we have preferred not to use such projectors in our paper and to define the solution $u(t)$ for the initial data $u(0) \in \Phi$ only.

References

[1] Attouch H 1984 Variational Convergence for Functions and Operators (London: Pitman)
[2] Babin A and Vishik M 1992 Attractors of Evolution Equations (Amsterdam: North-Holland)
[3] Barbu V 1976 Nonlinear semigroups and differential equations in Banach spaces (Leyden: Noordhoff)
[4] Bonetti E 2001 Global solution to a Frémond model for shape memory alloys with thermal memory Nonlinear Anal. 46 535–65
[5] Brézis H 1972 Problèmes unitaires J. Math. Pures Appl. 51 1–168
[6] Brézis H 1973 Opérateurs Maximaux Monotones et Semi-groupes de Contraction dans les Espaces de Hilbert (North-Holland Mathematics Studies vol 5) (Amsterdam: North-Holland)
[7] Colli P and Hoffmann K H 1993 A nonlinear evolution problem describing multi-component phase change with dissipation Numer. Funct. Anal. Optim. 14 275–97
[8] Eden A, Foias C, Nicolaenko B and Temam R 1994 Exponential Attractors for Dissipative Evolution Equations (Research in Applied Mathematics) (New York: Wiley)
[9] Efendiev M, Miranville A and Zelik S 2000 Exponential attractors for a nonlinear reaction–diffusion system in $\mathbb{R}^3$ C. R. Acad. Sci. Paris Sér. I Math. 330 713–8
[10] Efendiev M, Miranville A and Zelik S 2005 Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems Proc. R. Soc. Edinb. A 135 703–30
[11] Efendiev M and Zelik S 2008 Finite and infinite dimensional attractors for porous media equations Proc. R. Soc. Lond. A 36 107–35
[12] Efendiev M and Zelik S 2009 Finite-dimensional attractors and exponential attractors for degenerate doubly nonlinear equations Math. Methods Appl. Sci. 32 1638–68
[13] Fabrie P and Galusinski C 1996 Exponential attractor for a partially dissipative reaction–diffusion system Asymptotic Anal. 12 329–54
[14] Fabrie P, Galusinski C, Miranville A and Zelik S 2004 Uniform exponential attractors for a singularly perturbed damped wave equation Discrete Cont. Dyn. Sys. 10 211–38
[15] Federer H 1969 Geometric Measure Theory (New York: Springer)
[16] Frémond M 2002 Non-smooth Thermomechanics (Berlin: Springer)
[17] Laurençot Ph 1994 Semi continuité supérieure des attracteurs associés à l’approximation d’une équation de réaction-diffusion avec contraintes C. R. Acad. Sci. Paris Sér. I Math. 318 37–41
[18] Ladyzhenskaya O A, Solonnikov V A and Ural’ceva N N 1968 Linear and Quasi-linear Equations of Parabolic type (Translations of Mathematical Monographs) (Providence, RI: American Mathematical Society)
[19] Málek J and Nečas J 1996 A finite-dimensional attractor for three-dimensional flow of incompressible fluids J. Diff. Eqns 127 498–518
[20] Málek J and Prážek D 2002 Large time behaviour via the Method of l-trajectories J. Diff. Eqns 181 243–79
[21] Mañé R 1981 On the dimension of the compact invariant sets of certain nonlinear maps Dynamical Systems and Turbulence (Warwick 1980 (Coventry, 1979/1980)) (Lecture Notes in Mathematics vol 898) (Berlin: Springer) pp 230–42
[22] Miranville A and Zelik S 2004 Robust exponential attractors for Cahn–Hilliard type equations with singular potentials Math. Methods Appl. Sci. 27 545–82
[23] Miranville A and Zelik S 2007 Finite-dimensionality of attractors for degenerate equations of elliptic-parabolic type Nonlinearity 20 1773–97
[24] Miranville A and Zelik S 2008 Attractors for dissipative partial differential equations in bounded and unbounded domains Handbook of Differential Equations: Evolutionary Equations (Handbook of Differential Equations vol IV) (Amsterdam: Elsevier/North-Holland) pp 103–200
[25] Miranville A and Zelik S 2009 Doubly nonlinear Cahn–Hilliard–Gurtin equations Hokkaido Math. J. 38 315–60
[26] Nochetto R, Savaré G and Verdi C 2000 A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations Commun. Pure Appl. Math. 53 525–89
[27] Simon J 1987 Compact sets in the space $L^p(0, T; B)$ Ann. Mat. Pura Appl. 146 65–96
[28] Temam R 1997 Infinite-Dimensional Dynamical Systems in Mechanics and Physics (New York: Springer)
[29] Zelik S 2000 The attractor for a nonlinear reaction–diffusion system with a supercritical nonlinearity and its dimension Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 24 1–25