Advectional enhancement of eddy diffusivity under parametric disorder

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Received 30 October 2009
Accepted for publication 24 March 2010
Published 31 December 2010
Online at stacks.iop.org/PhysScr/T142/014050

Abstract

Frozen parametric disorder can lead to the appearance of sets of localized convective currents in an otherwise stable (quiescent) fluid layer heated from below. These currents significantly influence the transport of an admixture (or any other passive scalar) along the layer. When the molecular diffusivity of the admixture is small in comparison to the thermal one, which is quite typical in nature, disorder can enhance the effective (eddy) diffusivity by several orders of magnitude in comparison to the molecular diffusivity. In this paper, we study the effect of an imposed longitudinal advection on the delocalization of convective currents, both numerically and analytically, and report a subsequent drastic boost of the effective diffusivity for weak advection.

PACS numbers: 05.40.−a, 44.30.+v, 47.54.−r, 72.15.Rn

Disorder in operation conditions of a dynamic system is known to be able to play not only a trivial destructive role, distorting the system behavior, but also a constructive one, inducing a certain degree of order and leading to various non-trivial effects: Anderson localization (AL) [1–4], stochastic [5] and coherence resonances [6], noise-induced synchronization [7], etc. One of the most distinguished and fair effects is the AL, which is the localization of states in spatially extended linear systems subject to a frozen parametric disorder (random spacial inhomogeneity of parameters). AL was first discovered and discussed for quantum systems [1]. Later on, investigations were extended to diverse branches of semiclassical and classical physics: wave optics [3], acoustics [4], etc. The phenomenon was comprehensively studied and well understood mathematically for the Schrödinger equation and related mathematical models [2, 8, 9]. The role of nonlinearity in these models was addressed in the literature as well (e.g. [9, 10]).

While extensively studied for conservative media, the localization phenomenon did not receive comparable attention for active/dissipative ones, like in problems of thermal convection or reaction–diffusion. The main reason is that the physical interpretation of formal solutions to the Schrödinger equation is essentially different from that of governing equations for active/dissipative media; therefore, the theory of AL may be extended to the latter only under certain strong restrictions (see [11, 12] for reference). Nevertheless, effects similar to AL can be observed in fluid dynamical systems [11, 12]. In [12], we addressed the problem where localized thermoconvective currents excited in a horizontal porous layer under frozen parametric disorder (spacial inhomogeneity of the macroscopic permeability, the heat diffusivity, etc) drastically influence the process of transport of a passive scalar (e.g. a pollutant) along the layer. Below the threshold of instability of the disorder-free system, the effective diffusivity quantifying this transport has been found to be faithfully determined by localization properties of patterns. Meanwhile, in [11] these properties have been revealed to be greatly affected by a weak imposed longitudinal advection. Hence, one can expect a weak advection to lead to significant enhancement of the effective diffusivity of a nearly indiffusive pollutant. The treatment of this effect is the subject of this paper.

The role of irregular sets of localized flow patterns in the transport of pollution is not only interesting from the viewpoint of mathematical physics, but also important due...
to ecological (transport of pollution by underground water) and technical (cooling reactors, filtration, etc.) applications. Although the currents’ irregularity is actually not a ‘classical’ turbulent one, approaches well developed for the problems of turbulence (eddy diffusivity, statistical theories, etc [13, 14]) are fruitful for the study we present.

The paper is organized as follows. In section 1, we formulate the specific physical problem we deal with and introduce the relevant mathematical model; in particular, we discuss the disorder-induced excitation of localized currents below the instability threshold of the disorder-free system and advectional delocalization of these patterns. Section 2 presents the results of our numerical simulation and calculation of effective diffusivity. In section 3, we develop an analytical theory for effective diffusivity in the presence of an imposed advection. Section 4 ends the paper with the conclusions.

1. Problem formulation and the current state of research

In nature and technology, a wide variety of active media where pattern selection occurs are governed by Kuramoto–Sivashinsky-type equations. In the presence of an imposed advectional transport \( u \) in the \( x \)-direction, the modified Kuramoto–Sivashinsky equation reads

\[
\dot{\theta}(x, t) = -\left(u\theta(x, t) + \theta_{xx}(x, t)\right) + q(x)\theta_t(x, t) - (\theta_t(x, t))^3 \tag{1}
\]

This equation describes two-dimensional large-scale natural thermal convection in a horizontal fluid layer heated from below [15, 16] and is still valid for a turbulent fluid [17], a binary mixture at small Lewis number [18], a porous layer saturated with a fluid [19], etc. In these fluid dynamical systems, except for the turbulent one [17], the plates bounding the layer should be nearly thermally insulating (in comparison to the fluid) for a large-scale convection to arise. In the problems mentioned, equation (1) governs the evolution of temperature perturbations \( \theta \) that are nearly uniform along the vertical coordinate \( z \) and determine fluid currents.

The origin of such a frequent occurrence of equation (1) is its general validity, which may be argued as follows. Basic laws in physics are conservation ones. This often results in final governing equations having the form \( \dot{\theta}[\text{quantity}] + \nabla \cdot [\text{flux of quantity}] = 0 \). Such conservation laws lead to Kuramoto–Sivashinsky type-equations. Whereas the original Kuramoto–Sivashinsky equation has a quadratic nonlinear term (cf [20]), this term should be replaced by a cubic one for systems with sign inversion symmetry of the fields, which is widespread in nature, or for the description of a spatiotemporal modulation of an oscillatory mode. Thus, the governing equation takes the form (1). On these grounds, we state that equation (1) describes pattern formation in a wide variety of physical systems.

We restrict this paper to the case of convection in a porous medium [12, 19] for the sake of definiteness; nonetheless, most of our results may be extended in a straightforward manner to the other physical systems mentioned. Equation (1) is already dimensionless, and below we introduce all parameters and variables in appropriate dimensionless forms.

In the large-scale (or long-wavelength) approximation, which we use, the characteristic horizontal scales are assumed to be large against the layer height \( h \). In equation (1), \( q(x) \) represents the local supercriticality: \((21/2)h^3q(x)\) is the sum of relative deviations of the heating intensity and of the macroscopic properties of the porous matrix (porosity, permeability, heat diffusivity, etc) from the critical values for the spatially homogeneous case [19]. For positive spatially uniform \( q \), convection is set up, whereas for negative \( q \), all the temperature perturbations decay. In a porous medium [19], the macroscopic fluid velocity field is

\[
\bar{v} = \frac{\partial \Psi}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial x}{\partial z},
\]

(2)

\[
\Psi = \frac{3\sqrt{3}}{h^3} z(h - z) \theta_t(x, t) \equiv f(z) \psi(x, t),
\]

where \( \psi(x, t) = \theta_t(x, t) \) is the stream function amplitude, and the reference frame is such that \( z = 0 \) and \( z = h \) are the lower and upper boundaries of the layer, respectively (figure 1(b)). Although the temperature perturbations obey equation (1) for diverse convective systems, the function \( f(z) \), which determines the relation between the flow pattern and the temperature perturbation, is specific to each case. Note that \( u \) is not presented in expression (2) owing to its smallness in comparison to the excited convective currents \( \bar{v} \).

The impact of a weak imposed advective flow on the evolution of temperature perturbations is caused by its symmetry properties: the gross advective flux through the vertical cross-section is \( u \), while the convective flow \( \bar{v} \) possesses zero gross flux and therefore yields a less effective heat transfer along the layer [19].

Although equation (1) is valid for a large-scale inhomogeneity \( q(x) \), which means \( h|q_x|/|q| \ll 1 \), one can set such a hierarchy of small parameters, namely \( h \ll (|q_x|/|q|)^2 \ll 1 \), that a frozen random inhomogeneity may be represented by white Gaussian noise \( \xi(x) \):

\[
q(x) = q_0 + \xi(x), \quad \langle \xi(x) \rangle = 0, \quad \langle \xi(x)\xi(x') \rangle = 2\sigma^2 \delta(x - x'),
\]

where \( \sigma^2 \) is the disorder intensity and \( q_0 \) is the mean supercriticality (i.e. departure from the instability threshold of the disorder-free system). Numerical simulation reveals only time-independent solutions to settle in (1) with \( u = 0 \) and such \( q(x) \) [11]; for a small non-zero \( u \), stable oscillatory regimes are of low probability by continuity.

In the stationary case for \( u = 0 \) the linearized form of equation (1), i.e.

\[
-\theta_{xx}(x) - \xi(x) \theta_t(x) = q_0 \theta_t(x) \tag{3}
\]

is a stationary Schrödinger equation for \( \psi = \theta_t \) with \( q_0 \) instead of the state energy and \( -\xi(x) \) instead of the potential. Therefore, similarly to the case of the Schrödinger equation (see [2, 8, 9]), all the solutions \( \psi(x) \) to the stationary linearized equation (1) are spatially localized for arbitrary \( q_0 \); asymptotically,

\[
\psi(x) \propto \exp(-\gamma|x|),
\]

where \( \gamma \) is the localization exponent. Such a localization can be readily seen for the solution to the nonlinear problem (1) in figure 1(a) for \( q_0 = -2.5, u = 0 \), which is a solitary vortex.
Figure 1. (a) The steady solutions to equation (1) are sets of exponentially localized patterns (shown for one and the same realization of random inhomogeneity $\xi(x)$ and $\varepsilon = 1$; $q_0(x)$ is represented by $q_0(x) = \pi^{-1} \int_{-\pi/2}^{x} q(x') dx'$). (b) The stream lines corresponding to the solutions in (a) are plotted for the case of convection in a porous layer (see equation (2) for an exact relation).

For $u \neq 0$, the solitary patterns are still exponentially localized (figure 1(a), $u = 0.3$). However, their localization properties change drastically because, instead of the second-order linear ODE with respect to $\psi = \theta_x$, equation (3), one finds a third-order equation:

$$-u \psi(x) - \theta_{x,x}(x) - \xi(x) \theta_x(x) = q_0 \theta_x(x). \quad (4)$$

The two symmetric modes $\theta \propto \exp(\pm \gamma x)$ and the trivial solution $\theta = \text{const}$ ($\psi = 0$) turn into three modes $\theta \propto \exp(\gamma_1 x)$, $\gamma_3 < 0 < \gamma_2 < \gamma_1$, of equation (4) with $u > 0$, $q_0 < 0$ (see a sample spectrum of $\gamma_1$ in figure 2; cf [11] for details). Specifically, $\gamma_1$-mode is the successor of $+\gamma$, $\gamma_2$-mode is the successor of the trivial homogeneous mode and $\gamma_3$-mode is that of $-\gamma$. Thus, the upstream flank of the localized pattern is now composed of two modes decaying in distance from the pattern:

$$\theta(x) \approx \Theta_1(x) e^{\gamma_1 x} + \Theta_2(x) e^{\gamma_2 x} \quad \text{or} \quad \psi(x) \approx \gamma_1 \Theta_1(x) e^{\gamma_1 x} + \gamma_2 \Theta_2(x) e^{\gamma_2 x},$$

where the functions $\Theta_i(x)$ and $\tilde{\Theta}_i(x)$ neither grow nor decay over large distances. The $\gamma_2$-mode, which disappears for $u = 0$, i.e. $\gamma_2 = 0$, decays slowly for a small finite $u$, prevails...
over the $\gamma_1$-mode decaying rapidly and thus determines the upstream localization properties of the pattern. The upstream localization length $1/\gamma_1$ can become remarkably large, leading to upstream delocalization of patterns, which can be seen in figure 1.

One should keep in mind that consideration of solitary patterns makes sense where such patterns can be distinguished, i.e. are sparse enough in space. This is the case of negative $q_0 \lesssim -1$. In figure 1, for a sample realization of $\xi(x)$, one can see that localized patterns can be discriminated for $\varepsilon = 1$, $q_0 = -1.5$ and the localization properties are very well pronounced for $q_0 = -2.5$.

Here, we would like to emphasize the fact of existence of convective currents below the instability threshold of the disorder-free system. These currents considerably and nontrivially affect the transport of a pollutant (or other passive scalars), especially when its molecular diffusivity is small in comparison to the thermal one, which is quite typical in nature (for instance, in standard conditions, the molecular diffusivity of NaCl in water is $1.1 \times 10^{-9}$ m$^2$ s$^{-1}$ as against the heat diffusivity of water, which is $1.3 \times 10^{-7}$ m$^2$ s$^{-1}$). The transport of a nearly indifferent passive scalar, quantified by the effective (or eddy) diffusivity coefficient, is the object of our research, as a ‘substance’ that is essentially influenced by these localized currents and thus provides an opportunity to observe a manifestation of disorder-induced phenomena discussed in [11].

In [12], we studied the problem for the case of no advection ($u = 0$) and calculated (both numerically and analytically) the enhancement of effective diffusivity by disorder-induced currents; this enhancement is especially strong for low molecular diffusivity deep below the instability threshold of the disorder-free system (see figure 3). In this paper, we address the role of an imposed advection in this problem. The interest in advection was provoked by its dramatic influence on localization properties, i.e. the upstream delocalization of convective currents that is described above. We expect this delocalization to result in a giant increase of the effective diffusivity for a nearly indifferent pollutant and, particularly, in the lowering of the mean supercriticality ($\tilde{q}_0$) value at which a transition from sets of localized convective currents to an almost everywhere intense ‘global’ flow occurs.

2. Effective diffusivity

In this section, we describe the transport of a passive pollutant by a steady convective flow (2); ‘passive’ means that the flow is not influenced by the pollutant. The assumption of passiveness is practically relevant, because (biologically/chemically) significant concentrations of a pollutant can be very small and mechanically negligible. The flux $\vec{j}$ of the pollutant concentration $C$ is

$$\vec{j} = u \overrightarrow{C} - D \nabla C,$$

where the first term describes the convective transport, the second term represents the molecular diffusion and $D$ is the molecular diffusivity. The time-independent distributions of the pollutant obey

$$\nabla \cdot \vec{j} = 0.$$  

Equation (6) (taking into account (2)) yields a distribution of $C$ that is uniform along $z$ and is determined by

$$\frac{dC(x)}{dx} \equiv -\frac{J}{D + [21 \psi^2(x)/2 h^2 D]}.$$  

where $J$ is the constant pollutant flux along the layer. A detailed derivation of equation (7) for $u = 0$ can be found in [12], where it was performed in the spirit of the standard multiscale method (interested readers can consult e.g. [14, 21]). Remarkably, advection velocity $u$ is not presented in the last equation: its direct contribution to convective currents transferring the pollutant is small in comparison to the contribution of excited convective currents. Instead, it influences the heat transfer and, consequently, excited flows, drastically changing the properties of the field $\psi(x)$. Note that, for the other convective systems mentioned in section 1, the result differs only in the factor ahead of $\psi^2/D$.

Thus, we come to introducing the effective diffusivity for the system under consideration (general ideas about the effective diffusivity in systems with irregular currents can be found e.g. in [13, 14]). Let us consider the domain $x \in [0, L]$ with the imposed concentration difference $\delta C$ at the ends. Then the steady-state pollutant flux $J$ is defined by the integral (cf (7))

$$\delta C = -J \int_0^L \frac{dx}{D + [21 \psi^2(x)/2 h^2 D]}.$$  

For a lengthy domain, the specific realization of $\xi(x)$ becomes insignificant:

$$\delta C = -J L \left( \frac{D + 21 \psi^2(x)}{2 h^2 D} \right)^{-1} \equiv -\sigma^{-1} J L.$$

![Figure 3. Dependences of effective diffusivity $\tilde{\sigma}$ on mean supercriticality $\tilde{q}_0$ in the presence of an imposed advection ($u = 0.2$) and without it ($u = 0$). The bold black solid line indicates the analytical dependence for $u = 0$ and the dashed line indicates the one for $u = 0.2$ (see section 3).](image-url)
Hence

$$J = -\sigma \frac{\delta C}{L},$$

which means that $\sigma$ can be treated as an effective diffusivity.

The effective diffusivity

$$\sigma = \left( D + \frac{2\psi^2(x)}{2h^2D} \right)^{-1}$$

(8)

turns into $D$ for vanishing convective flow. For small $D$ the regions of the layer where the flow is damped, $\psi \ll 1$, make a large contribution to the mean value appearing in (8) and diminish $\sigma$, thus leading to the locking of spreading of the pollutant.

The disorder strength $\varepsilon^2$ can be excluded from equations by appropriate rescaling of the parameters and fields. As a consequence, the results on the effective diffusivity can be comprehensively presented in terms of $\tilde{D}$, $\tilde{\sigma}$, $\tilde{q}_0$ and $\tilde{u}$:

$$\tilde{D} = \frac{2}{21} \varepsilon^{4/3} D, \quad \tilde{\sigma} = \frac{2}{21} \varepsilon^{4/3} h \sigma, \quad \tilde{q}_0 = \frac{q_0}{\varepsilon^4 \tilde{u}}, \quad \tilde{u} = \frac{u}{\varepsilon^3}.$$

Figure 3 provides calculated dependences of effective diffusivity $\tilde{\sigma}$ on $\tilde{q}_0$ for moderate and small values of molecular diffusivity $\tilde{D}$. Concerning these dependences, the following is worth noting:

1. For small $\tilde{D}$, quite a sharp transition in effective diffusivity $\tilde{\sigma}$ between moderate values and values comparable with $\tilde{D}$ occurs near $q_0 = 0$ (note the logarithmic scale of the vertical axis), suggesting that a transition from an almost everywhere intense ‘global’ flow to a set of localized currents takes place.

2. In the presence of a weak imposed advection, $\tilde{u} = 0.2$, the transition to ‘global’ flow occurs at a value of $\tilde{q}_0$ that is considerably lower than that without advection.

3. Below the instability threshold of the disorder-free system, where only sparse localized currents are excited, effective diffusion can be significantly enhanced by these currents.

4. The disorder-induced enhancement of effective diffusivity is especially drastic in the presence of an imposed advection; for example, for $\tilde{D} = 10^{-4}$, $\tilde{q}_0 = -1$, effective diffusivity is increased by one order of magnitude compared to the molecular diffusivity without advection ($\tilde{u} = 0$) and by two orders of magnitude for $\tilde{u} = 0.2$.

3. Analytical theory

3.1. Transport through time-independent patterns

The effective diffusivity can be analytically evaluated for a small molecular diffusivity ($\tilde{D} \ll 1$) and sparse domains of excitation of convective currents (the spacial density of the excitation domains $v \ll 1$). In [12], it was evaluated for the case of no advection,

$$\tilde{\sigma}_{u=0} \approx \tilde{D} \left( \frac{2}{\tilde{D}} \right)^{2v/\gamma},$$

(9)

where one can use the asymptotic expressions for the density of the excitation domains $v$,

$$v \approx \frac{1}{4\sqrt{1.95 \pi \varepsilon^{2/3}|\tilde{q}_0|}} \exp \left( -\frac{1.95 \tilde{q}_0^2}{4} \right)$$

(10)

and

$$\gamma(\tilde{u} = 0) = \varepsilon^{-2/3} \left( |\tilde{q}_0|^{1/2} - \frac{1}{2} |\tilde{q}_0|^{-1} - \frac{5}{32} |\tilde{q}_0|^{-5/2} + \cdots \right),$$

which are valid for $\tilde{q}_0 < -1$. The latter expression is known from the classical theory of AL (see e.g. [8, 9]). In the following, we advance the evaluation procedure realized in [12] in order to account for the asymmetry between upstream and downstream localization exponents.

Now we calculate the average

$$\frac{1}{\tilde{\sigma}} = \left( \tilde{D} + \frac{\psi^2(x)}{\tilde{D}} \right)^{-1}.$$  

Due to ergodicity, this average over $x$ for a given realization of $\xi(x)$ coincides with the average over realizations of $\xi(x)$ at a certain point $x_0$. We set the origin of the $x$-axis at $x_0$ and find that $\tilde{\sigma}^{-1} = \left( \tilde{D} + \psi^2(0)/\tilde{D} \right)^{-1}_{\tilde{\xi}}$.

When the two closest to the origin excitation domains are distant and localized near $x_1 > 0$ and $x_2 < 0$ (see figure 4),

$$\psi(0) \approx \gamma_1 \theta_1 e^{-\gamma_1 x_1} + \gamma_2 \theta_2 e^{-\gamma_2 x_2} + \gamma_3 \theta_3 e^{-\gamma_3 x_2},$$

(11)

where $\theta_1, 2$ and $\theta_3$ characterize the amplitudes of temperature perturbation modes excited around $x_1$ and $x_2$, respectively. For small $\tilde{D}$ and density $v$, the contribution of the excitation domains to $\tilde{\sigma}^{-1}$ is negligible as against that of the extensive regions, where the flow is weak, but $|\psi|$ is still larger than $\tilde{D}$. Therefore, one may be not very subtle with ‘cores’ of excitation domains and may utilize expression (11) even for small $x_1, 2$:

$$\frac{1}{\tilde{\sigma}} = \left( \frac{1}{\tilde{D} + \psi^2(0)/\tilde{D}} \right)_{\tilde{\xi}} = \int_{0}^{\infty} dx_1 \frac{p(x_1) p(x_2)}{\tilde{D} + \psi^2(0)/\tilde{D}} \left( \gamma_1 \theta_1 e^{-\gamma_1 x_1} + \gamma_2 \theta_2 e^{-\gamma_2 x_2} + \gamma_3 \theta_3 e^{-\gamma_3 x_2} \right)^{2_{\theta_1, \theta_2}} x_2,$$

(12)

where $p(x_1) (p(x_2))$ is the density of the probability to observe the nearest right (left) excitation domain at $x_1 (-x_2)$. For probability distribution $P(x_1 > x)$, one finds that $\frac{dP(x_1 > x)}{dx} = \frac{P(x_1 > x) (1 - v dx)}{dx}$, i.e. $(d/dx) P(x_1 > x) = -v P(x_1 > x)$. Hence, $P(x_1 > x) = e^{-vx}$ and probability density $p(x) = \frac{d}{dx} P(x_1 > x) = v e^{-vx}$. As regards averaging over $\theta_i$, it is important that the
multiplication of $\theta_i$ by a factor $F$ is effectively equivalent to the shift of the excitation domain by $|\gamma_i|^{-1}\ln F$, which is insignificant for $F \sim 1$ in the limit case that we consider. Hence, one can assume that $\theta_i = \pm 1$ (the topological difference between different combinations of signs of $\theta_i$ is not to be neglected) and rewrite equation (12) as

$$\frac{1}{\tilde{\sigma}} = \frac{1}{2} \left( 1 - \frac{1}{D + \tilde{D}^{-1}(\gamma_1 e^{-\gamma_1 x_1} + \gamma_2 e^{-\gamma_2 x_2} + \gamma_3 e^{\gamma_3 x_2})^2} + \frac{1}{D + \tilde{D}^{-1}(\gamma_1 e^{-\gamma_1 x_1} + \gamma_2 e^{-\gamma_2 x_2} - \gamma_3 e^{\gamma_3 x_2})^2} \right).$$

For $v/\gamma_i \ll 1$ and $\gamma_2$-mode dominating over $\gamma_1$-mode (which is the case in figure 1), the last formula yields

$$\tilde{\sigma} \approx \tilde{D} \left( \frac{\gamma_2}{D} \right)^{v/\gamma_2} \left( \frac{|\gamma_1|}{D} \right)^{v/|\gamma_1|}. \tag{13}$$

Here, we assume that advection is weak and suppresses thermal convection in a negligible fraction of the excitation centers and the asymptotic expression (10) is still valid.

Noticeable differences between equation (13) for $\tilde{u} = 0$, i.e. $\gamma_2 = -\gamma_2 = \gamma$, and equation (9) are actually not seen in our approximations, because the moderate number $\gamma/2$ risen to the small power $2v/\gamma$, namely $(\gamma/2)^{2v/\gamma}$, which is the ratio of these equations, approximately equals 1. For a instance, in figure 3, the curves given by analytic expressions (9) and (13) with $\gamma_2 = -\gamma_1 = \gamma$ are visually indistinguishable.

For a small finite $\tilde{u}$, the smallness of $\gamma_2$ in equation (13) gives rise to a significant enhancement of effective diffusivity $\tilde{\sigma}$, which is in agreement with the results of the numerical simulation presented in figure 3. Figure 5 shows that for $\tilde{u} = 0.2$ the effective diffusivity in the presence of advection is always stronger than that without advection; the larger the difference between the effective and the molecular diffusivity the stronger the advective enhancement of the effective diffusivity. For instance, for $D \approx 10^{-4}$, $\tilde{q}_0 = -1$ the effective diffusivity in the presence of advection $\tilde{u} = 0.2$ is by a factor of 10 larger than that without advection, and this factor grows as $\tilde{D}$ decreases.

It is noteworthy that, for $\tilde{u} = 0.2$, expression (13) provides a slightly overestimated value of the effective diffusivity, while for $\tilde{u} = 0$ the analytical estimation is accurate. The inaccuracy appears because in our analytical theory we ignore three factors: (i) decrease of the spatial density of the excitation centers owing to advectional suppression; (ii) for small $\tilde{u}$ the rapidly decaying upstream $\gamma_1$-mode is significant because of the smallness of the slowly decaying $\gamma_2$-mode; and (iii) as the advection strengthens, currents in some excitation domains disappear via a Hopf bifurcation [11] and thus there is non-zero probability to observe oscillatory flows for small $\tilde{u}$ even though there are no stable time-dependent solutions for $\tilde{u} = 0$. Unfortunately, inaccuracies caused by these three assumptions cannot be minimized simultaneously: the first and third assumptions require $\tilde{u} \to 0$, whereas the second one needs $\tilde{u}$ to be small but finite.

3.2. Discussion of transport through oscillatory patterns

Oscillatory localized patterns discovered in the dynamic system (1) for non-zero $\tilde{u}$ (figure 6(a); see [11] for details) are statistically improbable and rare when $\tilde{u}$ is small. Notably, their relative contribution to the effective diffusivity is much larger than their fraction among the excited localized patterns. In figure 6(b), one can see the soaring of the effective diffusivity along a finite region as a localized pattern turns oscillatory ($\tilde{q}_0 = -1.6$) before disappearing ($\tilde{q}_0 \lesssim -1.7$). Figure 6(a) reveals the origin of this soaring: the oscillatory pattern is not so well localized as the time-independent one. Indeed, the localization properties of the oscillatory pattern of frequency $\omega$ are determined by the following linearization of equation (1):

$$\text{io} \theta = -(u \theta + \theta_x x + (q_0 + \xi(x)) \theta_t)_x. \tag{14}$$

In contrast to (4), this is already a fourth-order differential equation, which yields four localization exponents. The newly appeared fourth mode possesses $\gamma \propto \omega \propto u$ i.e. decays slowly, and contributes to the downstream flank of localized patterns (evidence of these facts is beyond the scope of this paper and will be presented elsewhere). Not only the $\gamma_1$-mode results in upstream delocalization of time-independent patterns, but also the new mode leads to downstream delocalization of oscillatory patterns, which appear to be weakly localized both upstream and downstream, as one can see from figure 6(a).

Nevertheless, owing to the smallness of the fraction of the oscillatory patterns among all the localized patterns at small $\tilde{u}$, their contribution to the effective diffusivity over large domains is still negligible. This is additionally confirmed by the accuracy of our analytical theory disregarding oscillatory currents (equation (13)). Meanwhile, the analytical theory accounting for the oscillatory patterns should involve the distribution of frequencies of excited patterns, which are to be determined only from the non-linear problem (1): this is not an analytically solvable problem.

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1 Equation (9) is more accurate than equation (13) because the case of $\tilde{u} = 0$ is simpler than the case we consider here and admits analytical evaluation of integrals with a fewer number of approximations.
employed and has proved itself to be fruitful. We have calculated numerically the dependence of effective diffusivity on molecular diffusivity and the mean supercriticality for a non-zero advection strength (see figure 3). The results reveal that advectional delocalization of convective currents greatly assists the transfer of a nearly indifferentusive pollutant ($D \ll 1$) below the instability threshold of the disorder-free system: the effective diffusivity can become by several orders of magnitude larger in comparison to that without advection.

The analytical theory focusing on advectional delocalization of localized current patterns yields results that are in fair agreement with the results of numerical simulation. This correspondence confirms our treatment of the importance of disorder-induced patterns and their localization properties in active/dissipative media, which provoked the works [11, 12].

4. Conclusion

We have studied the transport of a pollutant in a horizontal fluid layer by spatially localized two-dimensional thermoconvective currents appearing under frozen parametric disorder in the presence of an imposed longitudinal advection. Although we have considered a specific physical system, a horizontal porous layer saturated with a fluid and confined between two nearly thermally insulating plates, our results can be in a straightforward manner extended to a wide variety of fluid dynamical systems (like the ones studied in [15–18]).

The effective (or eddy) diffusivity approach for the description of the transport through an irregular flow, which is well-established for ‘classical’ turbulence [13, 14], has been

Figure 6. (a) A sample of an oscillatory flow appearing for $\alpha = 0.2$, $\varepsilon = 1$: at $q_0 = -1.5$ the pattern is time independent; further, as $q_0$ decreases, it turns oscillatory ($q_0 = -1.6$, shown at a certain moment of time) and then decays for $q_0 \lesssim -1.7$. (b) The effective diffusivity coefficients calculated over a short domain (length $L = 100$), with the sample $\xi(x)$ and the patterns plotted in (a), are compared with that calculated in the limit of an infinite domain ($L = 10^5$).

Acknowledgments

DSG thanks M Zaks and E Shklyaeva for useful discussions and acknowledges support from the BRHE program (CRDF grant no. Y5-P-09-01 and MESRF grant no. 2.2.2.3.8038).

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