MANIN’S CONJECTURE FOR CERTAIN SPHERICAL THREEFOLDS

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Abstract. We prove Manin’s conjecture on the asymptotic behavior of the number of rational points of bounded anticanonical height for a spherical threefold with canonical singularities and two infinite families of spherical threefolds with log terminal singularities. Moreover, we show that one of these families does not satisfy a conjecture of Batyrev and Tschinkel on the leading constant in the asymptotic formula. Our proofs are based on the universal torsor method, using Brion’s description of Cox rings of spherical varieties.

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1. INTRODUCTION

1.1. Spherical varieties and Manin’s conjecture. Manin’s conjecture [FMT89, BM90, Pey95, BT98b, Pey03, Pey17] makes a precise prediction for the asymptotic behavior of the number of rational points of bounded anticanonical height on (almost) Fano varieties over number fields whose set of rational points is Zariski dense. For a smooth Fano variety over $\mathbb{Q}$ with a Zariski dense set of rational points, one may introduce an anticanonical height function $H: X(\mathbb{Q}) \to \mathbb{R}_{>0}$ and ask for the asymptotic behavior of the number of rational points of bounded height, as the height bound tends to infinity. The total number might be dominated by points on accumulating subvarieties (or, more generally, accumulating thin subsets, see Peyre [Pey03, §8]), and hence it is more interesting to restrict to their complement $U$. By [BM90, Conjecture B'], we lead to the expectation that

$$N_{U,H}(B) := \# \{ x \in U(\mathbb{Q}) : H(x) \leq B \} \sim c B (\log B)^{\rho - 1}$$

as $B \to \infty$, where $\rho$ is the Picard number of $X$. A conjecture for the leading constant $c$ is given by Peyre in [Pey95]. If $X$ is a singular Fano variety with a crepant resolution $\pi: \tilde{X} \to X$ (i.e., a desingularization with $\pi^*(-K_X) = -K_{\tilde{X}}$), then [BM90, Conjecture C'] and [Pey03, 5.1] tell us that such an asymptotic formula
should hold with $\rho$ and $c$ computed on $\tilde{X}$. If $X$ has worse singularities, Conjecture C' and Peyre03 3.6 predict

$$N_{U,H}(B) \sim cB^a(\log B)^{b-1},$$

where we may have $a > 1$; Batyrev and Tschinkel BT98b give a prediction for $c$.

Manin’s conjecture has been proved for some classes of varieties and several individual examples. Most of the known cases are proved using either harmonic analysis on adelic points or the universal torsor method combined with various analytic techniques.

Many of them are spherical varieties, i.e., normal $G$-varieties containing a dense $B$-orbit, where $G$ is a connected reductive group and $B \subseteq G$ is a Borel subgroup. Spherical varieties are a huge class of varieties that admit a combinatorial description by spherical systems (Luna’s program Lun01) and colored fans (Luna–Vust theory LV83) generalizing the combinatorial description of toric varieties.

In particular, harmonic analysis has been used to prove Manin’s conjecture for some classes of equivariant compactifications of algebraic groups, for example flag varieties FMT89, toric varieties BT98a, horospherical varieties ST99, and wonderful compactifications of semi-simple groups GMO05 STBT07. All these varieties are spherical varieties; more precisely, flag varieties and toric varieties are special cases of horospherical varieties (which are toric bundles over flag varieties, at least after blow-ups); wonderful compactifications of semi-simple groups are special cases of wonderful varieties. This approach has also been applied to some non-spherical varieties, namely equivariant compactifications of vector groups CLT02 and Cayley’s singular ruled cubic surface BBS16.

The universal torsor method for Manin’s conjecture was initiated by Salberger Sal98, who gave a new proof of Manin’s conjecture for split toric varieties over $\mathbb{Q}$, which are spherical. Moreover, estimating rational points on a projective variety $X \subseteq \mathbb{P}^n$ by counting integral points on its affine cone in $\mathbb{A}^{n+1}$, e.g., by the circle method Bir62, can be interpreted as an instance of the universal torsor method. However, all other applications of the universal torsor method seem to concern non-spherical varieties. In dimension 2, there are many examples of smooth and singular del Pezzo surfaces with a crepant resolution; see Bre02 BBP12 BBD07 BB13, for example. In higher dimension, only three cases are known so far: Segre’s singular cubic threefold Bre07, a singular cubic fourfold BBS14 and a singular biprojective cubic threefold BBS18; in all three cases, the singularities have a crepant resolution. Hence all results proved by the universal torsor method are explained by Peyre’s relatively classical version of Manin’s conjecture Pey03 5.1.

The goal of our project is to start the investigation of Manin’s conjecture for spherical varieties by the universal torsor method. For this method, an explicit description of the universal torsors is needed; this can be obtained from the Cox rings of the underlying varieties (for details, see DPL14, for example). Cox rings of spherical varieties were determined by Brion Bri07. Also note that our results below are the first applications of the universal torsor method to varieties without a crepant resolution, where the more general conjectures of Batyrev and Tschinkel BT98b are relevant.

1.2. A singular weighted cubic threefold and $(2 \times 2)$-determinants that are cubes. One of the simplest spherical varieties that is neither horospherical nor wonderful has the following nice and easy description: It is the singular weighted cubic threefold

$$X_2 := V(ad - bc - z^3) \subseteq Y_2 := \mathbb{P}_Q(1,2,1,2,1)$$
in the weighted projective space $Y_2$ with weighted homogeneous coordinates $(a : b : c : d : z)$. It is closely related to the following Diophantine problem: How often is the determinant of a $(2 \times 2)$-matrix a cube? The question of representing a fixed number as a determinant over $\mathbb{Z}$ is considered in [DRS93].

The action of the reductive group $SL_2 \times G_m$ defined by

$$
\left(\left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right), \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) \cdot \left(\begin{array}{cc} t \\ 0 \end{array}\right) = \left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right) \cdot \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \cdot \left(\begin{array}{cc} t \\ 0 \end{array}\right),
$$

turns $X_2$ into a spherical variety. Its geometry can be analyzed by the combinatorial theory of spherical varieties, which allows us to determine its Picard number and its anticanonical divisor, for example; we will do this in Section 2. For this introduction, we emphasize a weighted-projective point of view; see [Dol82].

Since $-K_{X_2} = \mathcal{O}_{X_2}(4)$, we obtain an anticanonical height

$$
H : X_2(\mathbb{Q}) \to \mathbb{R}_{>0}
$$

defined by

$$
H(a : b : c : d : z) := \frac{\max\{|a^4|, |b^2|, |c^4|, |d^2|, |z^4|\}}{\gcd(a^4, b^2, c^4, d^2, z^4)}
$$

for $a, b, c, d, z \in \mathbb{Z}$; note that in weighted projective space, we may not assume that the coordinates are coprime. See also Section 3.

Blowing up its singular locus $V(a, c, z) \cong \mathbb{P}_\mathbb{Q}^1$ gives a crepant resolution $\pi : \widetilde{X}_2 \to X_2$ (in particular, $X_2$ has at worst canonical singularities), with $\text{Pic}(\widetilde{X}_2)$ free of rank 2. This means that we are in the situation of Peyre’s relatively classical version [Pey03] 5.1 of Manin’s conjecture. Our first main result (see Theorem 7.6 for its proof) is compatible with this prediction (see Section 4).

**Theorem 1.1.** We have

$$
N_{X_2, H}(B) = \epsilon B \log B + O(B),
$$

where

$$
\epsilon = \frac{1}{8} \cdot \frac{1}{\zeta(2)\zeta(3)} \left(2 \int_{[a,c]} \int_{[a,c]} \int_{[a,c]} \int_{[a,c]} \frac{1}{\abs{c}} \abs{a} \abs{d} \abs{c} \abs{d} \right)
$$

is Peyre’s constant.

1.3. A family of spherical threefolds. Our weighted cubic threefold $X_2 \subseteq \mathbb{P}_\mathbb{Q}(1, 2, 1, 2, 1)$ can be generalized as follows. For any positive integer $n$, consider the weighted hypersurface

$$
X_n := \mathcal{V}(ad - bc - z^{n+1}) \subseteq Y_n := \mathbb{P}_\mathbb{Q}(1, n, 1, n, 1)
$$

of degree $n + 1$ in the weighted projective space $Y_n$ with weighted homogeneous coordinates $(a : b : c : d : z)$. With an action of $SL_2 \times G_m$ that has the same description as above for $X_2$, each $X_n$ is a spherical threefold that is neither horospherical (see the beginning of Section 2) nor wonderful (because it is not smooth).

Let $n \geq 3$. By choosing sections of the very ample $\frac{n}{n+2}$-th power of the $\mathbb{Q}$-Cartier divisor $-K_{X_n} = \mathcal{O}_{X_n}(n + 2)$, we obtain an anticanonical height

$$
H : X_n(\mathbb{Q}) \to \mathbb{R}_{>0}
$$

defined by

$$
H(a : b : c : d : z) = \left(\frac{\max\{|a^n|, |b^n|, |c^n|, |d^n|, |z^n|\}}{\gcd(a^n, b^n, c^n, d^n, z^n)}\right)^{\frac{n+2}{4}}
$$

for $a, b, c, d, z \in \mathbb{Z}$; see also Section 3.

Naive heuristic considerations ignoring the denominator of the height function (analogous to the ones in [HB07, Heuristic principle] and [BT98b] §5.1) lead to

...
the expectation that $N_{X_n,H}(B)$ might grow linearly. However, in our second main result, we show (see Theorem 7.4 for its proof):

**Theorem 1.2.** Let $n \geq 3$. We have

$$N_{X_n,reg,H}(B) = \left( \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus V(a,c)} c_x \right) B \frac{\zeta(2)}{2} + O(B),$$

where $X_{n,reg}$ denotes the smooth locus of $X_n$. The values in the leading constant are

$$c_x = \frac{1}{2} \cdot \frac{1}{\zeta(2)} \cdot \omega_{\infty,x},$$

where (assuming that $a, c, z$ are coprime integral coordinates for $x$)

$$\omega_{\infty,x} = \left\{ \int \frac{1}{|a|} db \, dw \, \text{ for } a \neq 0, \right.$$

$$\left. \int \frac{1}{|c|} dd \, dw \, \text{ for } c \neq 0. \right\}$$

We will see that $X_{n,reg}$ is covered by rational curves, each of which contains $\sim c_x B^{2n/(n+2)}$ rational points of height at most $B$. Therefore, we cannot obtain linear growth by removing a closed or thin subset.

Instead, we discuss in the next part of this introduction how our result is explained by the predictions of Batyrev–Tschinkel [BT98b]; see Section 5 for more details. Note that the singular locus $X_{n,sing}$ is a weakly accumulating subvariety, with $N_{X_n,sing,H}(B) \sim \frac{2}{\zeta(2)} B^{2n/(n+2)}$ (see Remark 5.5), we exclude it in Theorem 1.2 to obtain a result that is compatible with [BT98b].

**1.4. The predictions of Batyrev–Tschinkel.** Let $X$ be a Fano variety over $\mathbb{Q}$ with at worst log terminal singularities and a Zariski dense set of rational points. Let $H : X(\mathbb{Q}) \to \mathbb{R}_{>0}$ be an anticanonical height function. Let $\pi : \tilde{X} \to X$ be a desingularization and $L := \pi^*(-K_X)$. By [BM90, Conjecture C'] and [Pey03, 3.6], we expect

$$N_{U,H}(B) := \# \{ x \in U(\mathbb{Q}) : H(x) \leq B \} \sim c B^b (\log B)^{b-1}$$

as $B \to \infty$, where $U$ is the complement of the closed (or thin) subset consisting of the accumulating subvarieties, $a := \inf \{ t \in \mathbb{R} : t \cdot L + K_{\tilde{X}} \text{ is effective} \}$ and $b$ is the codimension of the minimal face of the effective cone of $\tilde{X}$ containing $a \cdot L + K_{\tilde{X}}$. Note that the effective cone of a Fano variety with log terminal singularities is simplicial by [BCHM10, Corollary 1.3.2]. If $X$ has at worst canonical singularities, then $L + K_{\tilde{X}}$ is effective, hence $a \leq 1$. On the other hand, for varieties with worse singularities, we may have $a > 1$, in which case more than linear growth is expected.

A prediction for the leading constant $c$ is given in [BT98b]. Here, one considers the $\mathcal{L}$-primitive fibration (see [BT98b, Definition 2.4.2])

$$\phi : X \dasharrow P := \text{Proj} \left( \bigoplus_{i \geq 0} \Gamma \left( \tilde{X}, \left( a \cdot L + K_{\tilde{X}} \right)^{\otimes i} \right) \right),$$

and, for some restriction to open subsets $\phi : U \to V$, the constant $c$ is given by

$$\sum_{x \in V} c_x,$$

where $c_x$ is the expected constant in the asymptotic formula for the fiber $\phi^{-1}(x)$. The sum should be taken over the fibers that contain a positive proportion of the rational points (these are called $\mathcal{L}$-targets, see [BT98b, Definition 3.2.4]). If
the divisor $a \cdot L + K_X$ is rigid ([BT98b] Definition 2.3.1, e.g., if $X$ has a crepant resolution), then the variety $P$ is a point.

Batyrev and Tschinkel make the following prediction in [BT98b, Conjecture 3.5.1]:

**Conjecture 1.3.** Let $\overline{H}$ be a height on $P$ relative to the line bundle $\mathcal{O}_P(-1) \otimes \omega_P$. Then there exist positive constants $c_1, c_2$ and an open subset $V \subseteq P$ such that for every $x \in V$ we have

$$c_1 \overline{H}(x) \leq c_x \leq c_2 \overline{H}(x).$$

We apply the conjectures of [BT98b] to our family $X_n$ of spherical varieties; see Section 5.3 for their geometry. Blowing up the singular locus $V(a, c, z) \cong \mathbb{P}^1_Q$ gives a desingularization $\pi: \tilde{X}_n \rightarrow X_n$, and we will see that we have

$$a = \frac{2n}{n+2} \quad \text{and} \quad b = \begin{cases} 2, & \text{for } n = 2, \\ 1, & \text{for } n \geq 3. \end{cases}$$

For $n \geq 3$, the singularities of $X_n$ are not canonical, but log terminal. The divisor $a \cdot \pi^*(-K_{X_n}) + K_{\tilde{X}_n}$ is not rigid, and the $\mathcal{L}$-primitive fibration turns out to be a map $\phi: \tilde{X}_n \rightarrow P_n \cong \mathbb{P}^2_Q$ with $\mathcal{O}_{P_n}(1) \cong \mathcal{O}_{\mathbb{P}^2}(n-2)$ such that the constants $c_x$ appearing in Theorem 1.2 are Peyre’s constant for the fibers $\phi^{-1}(x)$.

In the proofs of Theorems 1.1 and 1.2 we work with universal torsors over a further blow-up $\tilde{X}_n \rightarrow \tilde{X}_n \rightarrow X_n$ because this leads to more convenient coprimality conditions in the associated counting problem (see Remark 3.4). This seems surprising to us because proofs of cases of Manin’s conjecture for singular del Pezzo surfaces usually use universal torsors of their minimal desingularizations.

It turns out that Conjecture 1.3 of Batyrev–Tschinkel is true for $X_n$ (see Theorem 5.3).

**Theorem 1.4.** Let $\overline{H}: \mathbb{P}^2(Q) \rightarrow \mathbb{R}_{>0}$ be a height relative to $\mathcal{O}_{\mathbb{P}^2}(-n-1) \cong \mathcal{O}_{P_n}(-1) \otimes \omega_{P_n}$.

There exist positive constants $c_1, c_2$ such that for every $x \in \mathbb{P}^2(Q) \setminus V(a, c)$ we have

$$c_1 \overline{H}(x) \leq c_x \leq c_2 \overline{H}(x).$$

This implies that the sum over the constants $c_x$ in Theorem 1.2 converges.

1.5. A second family of spherical threefolds. Since the varieties $X_n$ considered above are equivariant compactifications of $G_a^3$, Manin’s conjecture is already known for them by [CLT02] (for heights corresponding to smooth adelic metrics; note that we work with a height corresponding to an adelic metric that is not smooth). To illustrate that our approach can also be applied to spherical varieties without such a structure, we consider a family of varieties $X'_n$ for $n \geq 2$ that do not belong to any of the classes of varieties for which Manin’s conjecture is known.

A comparison of the geometric description, the shape of the main results and their proofs for the family $X_n$ with the family $X'_n$ will reveal many similarities, but also several additional complications for $X'_n$. In particular, we will see that Conjecture 1.3 fails for $X'_n$. Hence the family $X_n$ can be regarded as a warm-up for the family $X'_n$.

Fix an integer $n \geq 2$. Consider the weighted projective space $Y_{n-1}$ with Cox coordinates $(a : b : c : d : y)$ and the toric modification $Y'_n \rightarrow Y_{n-1}$ obtained by first blowing up the singular locus of $Y_{n-1}$, then blowing up the two torus invariant curves in the resulting exceptional divisor, and finally contracting the exceptional divisor from the first step. With Cox coordinates $(a : b : c : d : y : z : t)$, where $z$ corresponds to the torus invariant curve in $Y'_{n-1}$ contained in $V(y)$ and $t$ to the other one, we consider the hypersurface

$$X'_n := V(ad - bc - y^n z^{n+1}) \subseteq Y'_n.$$
Equipped with a suitable action of the reductive group $\text{SL}_2 \times \text{G}_m$, it is a singular spherical threefold that is neither horospherical (see the beginning of Section 2) nor wonderful (because it is not smooth); moreover it is not isomorphic to an equivariant compactification of $\text{G}_m^3$ since its effective cone can be shown not to be simplicial.

In Section 2, we will construct a desingularization $\pi: X'_n \to X'_n$, and we will see that we have

$$a = \frac{2n + 2}{n + 3} \quad \text{and} \quad b = 1.$$ 

In Section 3, we will construct an anticanonical height

$$H': X'_n \to \mathbb{R}_{>0}$$

by choosing sections of a very ample power of the $\mathbb{Q}$-Cartier divisor $\pi^*(-K_{X'_n})$ on $\tilde{X}'_n$.

The singularities of $X'_n$ are log terminal, and the divisor $a \cdot \pi^*(-K_{X'_n}) + K_{X'_n}$ is not rigid. We will find the $\mathcal{L}$-primitive fibration $\phi': X'_n \to P'_n \cong \mathbb{P}^2_Q$, where we denote the homogeneous coordinates of $\mathbb{P}^2_Q$ by $(\hat{a}: \hat{c}: \hat{y})$. Again, our main result (see Theorem 3.1) is compatible with the predictions of [BT98b] (see Section 4).

**Theorem 1.5.** Let $n \geq 2$ and $U' := X'_n \setminus \mathcal{V}(yz)$. For every $\epsilon > 0$, we have

$$N_{U', H'}(B) = \left( \sum_{x \in \mathbb{P}^2_Q \setminus (\mathcal{V}(\hat{a}, \hat{c}) \cup \mathcal{V}(\hat{y}))} \epsilon_x \right) B^{\frac{n+2}{2n+3}} + O_\epsilon(B^{1+\epsilon}),$$

where each summand $\epsilon_x$ in the leading constant is Peyre’s constant for the rational fiber $\phi'^{-1}(x)$. Its value is

$$\epsilon_x = \frac{1}{2} \cdot \left( \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right) \omega_{p, x}\right) \cdot \omega_{\infty, x},$$

with (assuming that $\hat{a}, \hat{c}, \hat{y}$ are coprime integral coordinates for $x$ and $\epsilon := \frac{n+1}{n+3}$)

$$\omega_{p, x} = \left(1 - \frac{1}{p}\right) \cdot \left(1 - \frac{(p^\epsilon)^{\nu_p(\hat{y})+1}}{1 - p^\epsilon} + \frac{(p^\epsilon)^{\nu_p(\hat{y})}}{p}\right) \cdot (p^{\epsilon+1}\min(\nu_p(\hat{a}), \nu_p(\hat{c}))),$$

and

$$\omega_{\infty, x} = \left\{ \begin{array}{ll} \int \max \left\{ \mathcal{M}'(\hat{a}, \hat{b}, \hat{c}, (bc+\hat{y}^nw)/\hat{a}, \hat{c}, 1, 1, w) \right\} \leq 1 \frac{1}{[\hat{y}]} \; db \; dw & \text{for } \hat{a} \neq 0, \\ \int \max \left\{ \mathcal{M}'(\hat{a}, (\hat{a}d-\hat{y}^nw)/\hat{c}, \hat{c}, 1, 1, w) \right\} \leq 1 \frac{1}{[\hat{c}]} \; dd \; dw & \text{for } \hat{c} \neq 0, \end{array} \right.$$ 

where $\mathcal{M}'(\ldots)$ denotes the set of 13 monomials from Remark 3.8.

In particular, the expressions for the $p$-adic densities $\omega_{p, x}$ are apparently much more complicated than in previous applications of the universal torsor method for Manin’s conjecture. Also note that $\omega_{p, x}$ depends on the base point $x$, while the $p$-adic densities in Theorem 1.2 are independent of $x$.

Finally, Conjecture 1.3 of Batyrev–Tschinkel is not true for $X'_n$. In fact, even a weaker “up to $\epsilon$”-version of this conjecture fails (see Theorem 6.3). Roughly, the reason is that $\gcd(\hat{a}, \hat{c})^{\epsilon+1}$ appears in the product of the $p$-adic densities in $\epsilon_x$:

**Theorem 1.6.** Let $\overline{H}: \mathbb{P}^2(Q) \to \mathbb{R}_{>0}$ be a height relative to an arbitrary line bundle. Then there are $\epsilon > 0$ such that there does not exist an open subset $V \subseteq \mathbb{P}^2(Q)$ with positive constants $c_1, c_2$ such that for every $x \in V$ we have

$$c_1\overline{H}(x)^{1-\epsilon} \leq \epsilon_x \leq c_2\overline{H}(x)^{1+\epsilon}.$$
Nevertheless, we can show that the sum over the constants $c_v$ in Theorem 1.5 converges (see Proposition 8.3).

See [PLL17 §4.2] for a second example where Conjecture 1.3 fails; in that case of a certain conic bundle over $\mathbb{P}^1$, however, the upper bound of the conjecture holds “up to $c$”. For an investigation of the behavior of Peyre’s constant for families of diagonal quartic threefolds, see [PLL07 Theorem 1.6].

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2. Two families of spherical hypersurfaces in toric varieties

Let $G$ be a connected reductive group over $\overline{\mathbb{Q}}$, and let $B \subseteq G$ be a Borel subgroup. A normal $G$-variety $X$ over $\overline{\mathbb{Q}}$ is called spherical if it contains a dense $B$-orbit. Over an algebraically closed field of characteristic 0 (such as $\overline{\mathbb{Q}}$), there is a complete combinatorial description of spherical varieties. First, spherical homogeneous spaces are described by a program initiated by Luna [Lun01], which has been recently completed [BP16, CFT11, Los09]. Then, given a spherical homogeneous space $G/H$, the Luna–Vust theory [LV83, Kno91] describes all spherical embeddings, i.e., $G$-equivariant open embeddings $G/H \to X$ into a normal irreducible $G$-variety $X$, in terms of colored fans, which generalize the fans of toric varieties. For further details, we refer to the general references [BL11, Perl14, Tim11].

For $G := SL_2$ the spherical $G$-varieties are at most 2-dimensional, where each complete one is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or the blow-up of $\mathbb{P}^2$ in one point. The next possible step is to consider $G := SL_2 \times G_m$. Let $\varepsilon : G_m \to G_m$ be a primitive character and consider the spherical subgroup

$$H := \{ (\lambda, \varepsilon(\lambda)) : \lambda \in T_{SL_2} \} \subseteq G,$$

where $T_{SL_2}$ denotes a maximal torus in $SL_2$. Then $T := T_{SL_2} \times G_m$ is maximal torus in $G$. Let $B \subseteq G$ be a Borel subgroup containing $T$. Let $\alpha \in X(T) = X(B)$ be the unique simple root corresponding to these choices.

We can now briefly introduce the central combinatorial objects associated to $G/H$ by the Luna-Vust theory. The weight lattice $M \subseteq X(B)$, i.e., the lattice of weights of $B$-semi-invariants (or $B$-eigenvectors) occurring in $\overline{Q}(G/H)$, has basis $(\frac{1}{2}\alpha + \varepsilon, \frac{1}{2}\alpha - \varepsilon)$. The set of colors $D$, i.e., the set of $B$-invariant prime divisors in $G/H$, contains two elements, which we denote by $D'$ and $D''$. The set $D$ is equipped with the map $\rho : D \to N := Hom(M, \mathbb{Z})$ defined by $(\rho(D), \chi) := \nu_D(f_\chi)$ where $\nu_D$ is the valuation on $\overline{Q}(G/H)$ which is induced by the prime divisor $D$ and $f_\chi \in \overline{Q}(G/H)$ is a $B$-semi-invariant of weight $\chi \in M$ (which is defined up to a constant factor because of the open $B$-orbit). We can choose $D'$ and $D''$ such that $(\rho(D'), \rho(D''))$ is the dual basis to $(\frac{1}{2}\alpha + \varepsilon, \frac{1}{2}\alpha - \varepsilon)$ of $N$. Finally, the valuation cone $V \subseteq N_Q := Hom(M, Q)$, which can be identified with the $Q$-valued $G$-invariant discrete valuations on $\overline{Q}(G/H)$, is given by $V = \{ v \in N_Q : \langle v, \alpha \rangle \leq 0 \}$. Spherical varieties with $V = N_Q$ are called horospherical. In particular, because we have $V \neq N_Q$, no embedding of our example $G/H$ is horospherical. The situation inside the vector space $N_Q$ is illustrated in the following picture.
A spherical embedding $G/H \hookrightarrow X$ is now described by a \textit{colored fan}, which is a set of \textit{colored cones}, which are pairs $(C, F)$ where $C$ is a polyhedral cone in $\mathbb{Q}^n$ and $F$ is a subset of $D$, and where moreover certain properties and compatibility conditions are satisfied. Similarly to the case of toric varieties, the colored cones are in bijection with the $G$-orbits in $X$. The colored cones corresponding to $G$-orbits of codimension $1$ are easier to describe: they have the form $(\rho, \emptyset)$ where $\rho$ is a ray in $V$, which means that we have $\rho = \operatorname{cone}(u)$ for a uniquely determined primitive element $u \in V \cap \mathbb{N}$. 

Now let $G/H \hookrightarrow X$ be a spherical embedding, and let $u_1, \ldots, u_n \in V \cap \mathbb{N}$ be the primitive elements corresponding to (the open orbits in) the $G$-invariant prime divisors $D_1, \ldots, D_n$ in $X$. According to [Bri07, Proposition 4.1.1], the divisor class group $\operatorname{Cl}(X)$ is generated by divisor classes $[D_1], \ldots, [D_n]$ and the divisor classes of the colors $D$, and the relations can be computed from the relative position of the $u_1, \ldots, u_n \in \mathbb{N}$ similarly to the toric case. Moreover, the Cox ring $\mathcal{R}(X)$ of $X$ can be obtained explicitly using [Bri07, Theorem 4.3.2] or [Gag14, Theorem 3.6]:

**Proposition 2.1.** Let $r_i := -\langle u_i, \alpha \rangle$. Then we have

$$\mathcal{R}(X) = \mathbb{Q}[a, b, c, d, z_1, \ldots, z_n]/\langle ad - bc - z_1^{r_1} \cdots z_n^{r_n} \rangle$$

with $\deg(a) = \deg(c) = [D']$, $\deg(b) = \deg(d) = [D'']$, and $\deg(z_i) = [D_i]$. 

For every $n \geq 2$, we consider the spherical embedding $G/H \hookrightarrow X_n$ with exactly one $G$-invariant prime divisor corresponding to the primitive element

$$u_z := -\rho(D') - n\rho(D'') \in V \cap \mathbb{N}.$$ 

It can be shown that $X_n$ is isomorphic to an equivariant compactification of $G_3$. We therefore also consider the spherical embedding $G/H \hookrightarrow X'_n$ with two additional $G$-invariant prime divisors corresponding to the primitive elements

$$u_y := -\rho(D') - (n-1)\rho(D'') \in V \cap \mathbb{N},$$

$$u_t := \rho(D') - \rho(D'') \in V \cap \mathbb{N}.$$ 

It can be shown that the effective cone of $X'_n$ is not simplicial, hence $X'_n$ is not isomorphic to an equivariant compactification of $G_3$. The colored fans of $X_2$ and $X'_2$ are illustrated in the following pictures.
Using [DP14], we consider $X_n$ and $X'_n$ as varieties over $\mathbb{Q}$. According to Proposition 2.1, we have

$$\mathcal{R}(X_n) = \mathbb{Q}[a, b, c, d, z]/\langle ad - bc - z^{n+1} \rangle$$

with $\text{Cl}(X_n) \cong \mathbb{Z}$ where $\deg(a) = \deg(c) = \deg(z) = 1$ and $\deg(b) = \deg(d) = n$. Moreover, the graded ring $\mathbb{Q}[a, b, c, d, z]$, where we ignore the relation, is identified as the Cox ring of the weighted projective space $Y_n := \mathbb{P}_{\mathbb{Q}}(1, n, 1, n, 1)$. It follows that $X_n$ is a hypersurface in $Y_n$ defined by $ab - cd - z^{n+1} = 0$.

Similarly, we have

$$\mathcal{R}(X'_n) = \mathbb{Q}[a, b, c, d, y, z, t]/\langle ad - bc - y^n z^{n+1} \rangle$$

with $\text{Cl}(X'_n) \cong \mathbb{Z}^3$ where $\deg(a) = \deg(c) = (1, 1, -1)$, $\deg(b) = \deg(d) = (n, n - 1, 1)$, $\deg(z) = (1, 0, 0)$, $\deg(y) = (0, 1, 0)$, and $\deg(t) = (0, 0, 1)$. Again, the variety $X'_n$ is a hypersurface in a toric variety $Y'_n$ with graded Cox ring $\mathbb{Q}[a, b, c, d, y, z, t]$.

According to [Bri97] 4.1 and 4.2 or [ADHL15] Proposition 3.3.2.2, we have the anticanonical divisor classes $-K_{X_n} = n + 2$ and $-K_{X'_n} = (n + 2, n + 1, 1)$. Moreover, according to [GH15] Theorem 1.9 or [ADHL15] 3.3.2.9, the varieties $X_n$ and $X'_n$ are Fano for every $n \geq 2$, and the variety $X_2$ is Gorenstein.

The singular loci are $X_n,_{\text{sing}} = X \cap \mathbb{V}(a, c, z)$ and $X'_n,_{\text{sing}} = X' \cap \mathbb{V}(z, t)$. We construct desingularizations $\pi : \tilde{X}_n \to X_n$ and $\pi' : \tilde{X}'_n \to X'_n$ by subdividing their colored fans. We add a $G$-invariant prime divisor corresponding to the primitive element $u_w := -\rho(D^n) \in \mathcal{V} \cap \mathcal{N}$. The resulting colored fans of the spherical varieties $\tilde{X}_2$ and $\tilde{X}'_2$ are illustrated in the following pictures.

According to Proposition 2.1, we have

$$\mathcal{R}(\tilde{X}_n) = \mathbb{Q}[a, b, c, d, z, w]/\langle ad - bc - z^{n+1}w \rangle$$

with $\text{Pic}(\tilde{X}_n) \cong \text{Cl}(\tilde{X}_n) \cong \mathbb{Z}^2$ where $\deg(a) = \deg(c) = \deg(z) = (1, 0)$, $\deg(b) = \deg(d) = (n, 1)$, and $\deg(w) = (0, 1)$. Moreover, we have

$$\mathcal{R}(\tilde{X}'_n) = \mathbb{Q}[a, b, c, d, y, z, t, w]/\langle ad - bc - y^n z^{n+1}w \rangle$$

with $\text{Pic}(\tilde{X}'_n) \cong \text{Cl}(\tilde{X}'_n) \cong \mathbb{Z}^4$ where $\deg(a) = \deg(c) = (1, 1, -1, 0)$, $\deg(b) = \deg(d) = (n, n - 1, 1, 1)$, $\deg(z) = (1, 0, 0, 0)$, $\deg(y) = (0, 1, 0, 0)$, $\deg(t) = (0, 0, 1, 0)$, and $\deg(w) = (0, 0, 0, 1)$.

In order to obtain explicit descriptions of $\tilde{X}_n$ and $\tilde{X}'_n$, we use [ADHL15] Theorem 2.2.2.2, Proposition 3.3.2.9, and Construction 3.2.1.3 and [DP14], according to which the quasi-affine varieties

$$\mathcal{T}_n := \text{Spec}(\mathcal{R}(\tilde{X}_n)) \setminus (\mathcal{V}(a, c, z) \cup \mathcal{V}(b, d, w)),$$

$$\mathcal{T}'_n := \text{Spec}(\mathcal{R}(\tilde{X}'_n)) \setminus (\mathcal{V}(a, c) \cup \mathcal{V}(b, d, z) \cup \mathcal{V}(b, d, w) \cup \mathcal{V}(y, w) \cup \mathcal{V}(y, t) \cup \mathcal{V}(z, t))$$

are universal torsors $\mathcal{T}_n \to \tilde{X}_n$ and $\mathcal{T}'_n \to \tilde{X}'_n$ with respect to the natural actions of the tori

$$\text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}_n)]) \cong G_m^2$$

and

$$\text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}'_n)]) \cong G_m^4$$

respectively.
According to [Bri07] 4.1 and 4.2 or [ADHL15] Proposition 3.3.3.2, we have
\[ -K_{\tilde{X}^e_n} = (n + 2, 2), \quad \pi^*(-K_{X_n}) = (n + 2, \frac{n+2}{n}), \]
\[ -K_{\tilde{X}^e_n} = (n + 2, n + 1, 2), \quad \pi^*(-K_{X_n}) = \left(n + 2, n + 1, 1, \frac{n+4}{n+2}\right). \]
In particular, the resolution \( \pi: \tilde{X}_n \to X_n \) is crepant and \( X_n \) has at worst canonical singularities if and only if \( n = 2 \) (see, for instance, [AB04]).

3. Parameterization of rational points via universal torsors

Using the universal torsors \( T_n \) and \( T'_n \) from Section 2, we parameterize the rational points on \( X_n \) and \( X'_n \), respectively.

Consider the line bundles
\[ L := \frac{n}{n+2} \cdot \pi^*(-K_{X_n}) = (n, 1), \]
\[ L' := (n + 1) \cdot \pi^*(-K_{X_n'}) = (n^2 + 3n + 2, n^2 + 2n + 1, n + 1, n + 3). \]

We define
\[ \mathcal{M}_n(a, b, c, d, z, w) := \{ \text{monomials in } \mathcal{R}(\tilde{Y}_n) \text{ of degree } L \text{ restricted to } \tilde{X}_n \}, \]
\[ \mathcal{M}'_n(a, b, c, d, y, z, t, w) := \{ \text{monomials in } \mathcal{R}(\tilde{Y}_n') \text{ of degree } L' \text{ restricted to } \tilde{X}_n' \}. \]

Then we have
\[ H(\pi(a : b : c : d : z : w)) := \left( \frac{\max |\mathcal{M}_n(a, b, c, d, z, w)|}{\gcd \mathcal{M}_n(a, b, c, d, z, w)} \right)^{(n+2)/n}, \]
\[ H'(\pi(a : b : c : d : y : z : t : w)) := \left( \frac{\max |\mathcal{M}'_n(a, b, c, d, y, z, t, w)|}{\gcd \mathcal{M}'_n(a, b, c, d, y, z, t, w)} \right)^{1/(n+1)} \]
for anticanonical heights \( H \) and \( X_n \) and \( H' \) on \( X'_n \).

We are now going to state the counting problem for \( X_n \). We consider the open subset
\[ U := \tilde{X}_n \setminus \mathcal{V}(w) = X_n \setminus \mathcal{V}(a, c, z). \]

**Proposition 3.1.** There is a natural 4-to-1 correspondence between
\[ \mathcal{U} := \left\{ (a, b, c, d, z, w) \in \mathbb{Z}^6 : w \neq 0; \text{ ad } - \text{ bc } - z^{n+1}w = 0 \right\} \]
and the set \( U(Q) \). Moreover, for \((a, b, c, d, z, w) \in \mathcal{U}\), we have
\[ H(\pi(a : b : c : d : z : w)) = \max |\mathcal{M}_n(a, b, c, d, z, w)|^{(n+2)/n}. \]

**Proof.** The toric variety \( \tilde{Y}_n \) comes from a regular fan. According to [Sal98] Section 8], we may construct a toric scheme \( \mathfrak{Y}_n \) over \( \text{Spec}(\mathbb{Z}) \), together with a map
\[ \mathfrak{M}_n := \text{Spec}(\mathbb{Z}[a, b, c, d, z, w]) \setminus (\mathcal{V}(a, c, z) \cup \mathcal{V}(b, d, w)) \to \mathfrak{Y}_n, \]
which is a model for the universal torsor \( \mathcal{W}_n \to \tilde{Y}_n \), obtain a 4-to-1 quotient
\[ \mathfrak{M}_n(\mathbb{Z}) \to \mathfrak{Y}_n(\mathbb{Z}) = \tilde{Y}_n(Q) \]
for the \( \mathbb{G}_m^2(\mathbb{Z}) \cong \{ \pm 1 \}^2 \)-action as well as the claim on the height function. As we have
\[ \mathfrak{M}_n(\mathbb{Z}) = \left\{ (a, b, c, d, z, w) \in \mathbb{Z}^6 : \text{gcd}(a, c, z) = \text{gcd}(b, d, w) = 1 \right\}, \]
the result follows after restricting to the equation \( \text{ad } - \text{ bc } - z^{n+1}w = 0 \).

Alternatively, the claims can easily be verified by elementary manipulations of the defining equation. \( \square \)
Corollary 3.2. We have that $N_{U,H}(B^{(n+2)/n})$ is equal to
\[
\frac{1}{4} \left\{ \begin{array}{l}
w \neq 0; \ ad - bc - z^{n+1}w = 0 \\
(a,b,c,d,z,w) \in \mathbb{Z}^6: \ \gcd(a,c,z) = \gcd(b,d,w) = 1 \\
\max |M_n(a,b,c,d,z,w)| \leq B
\end{array} \right\}.
\]

As the Diophantine equation $ad - bc = z^{n+1}w$ is easier to solve for $d$ or $b$ under the additional condition $\gcd(a,c) = 1$, we will use the following counting problem, which introduces an additional variable.

Corollary 3.3. We have that $N_{U,H}(B^{(n+2)/n})$ is equal to
\[
\frac{1}{8} \left\{ \begin{array}{l}
w t \neq 0; \ ad - bc - z^{n+1}w = 0 \\
(a,b,c,d,z,w,t) \in \mathbb{Z}^7: \ \gcd(a,c) = \gcd(b,d,w) = \gcd(z,t) = 1 \\
|a^n w t^{n+1}|, |c^n w t^{n+1}|, |z^n w t|, |b|, |d| \leq B
\end{array} \right\}.
\]

Remark 3.4. Corollary 3.3 can be interpreted as a version of Corollary 3.2, where instead of the desingularization $\tilde{X}_n \to X_n$, we use a further blow-up $X_n \to \tilde{X}_n \to X_n$.

The colored fan of $\tilde{X}_n$ is illustrated in the following picture (for $n = 2$).

According to Proposition 2.1, we have
\[
\mathcal{R}(\tilde{X}_n) = \mathbb{Q}[a,b,c,d,z,w,t]/(ad - bc - z^{n+1}w)
\]
with $\text{Pic}(\tilde{X}_n) \cong \mathbb{Z}^3$ where $\text{deg}(z) = (1,0,0)$, $\text{deg}(w) = (0,1,0)$, $\text{deg}(t) = (0,0,1)$, $\text{deg}(a) = \text{deg}(c) = (1,0,-1)$, and $\text{deg}(b) = \text{deg}(d) = (n,1,1)$. Moreover, the quasi-affine variety
\[
\mathcal{T}_n := \text{Spec}(\mathcal{R}(\tilde{X}_n)) \setminus (V(a,c) \cup V(b,d,w) \cup V(z,t))
\]
amits a torsor $\mathcal{T}_n \to \tilde{X}_n$ for the action of the torus $\text{Spec}(\mathbb{Q}[\text{Pic}(\tilde{X}_n)]) \cong G_m^3$.

Remark 3.5. On the singular locus $X_{n,\text{sing}} = V(a,c,z) \cong \mathbb{P}^1_Q$ with coordinates $(b : d)$, the height $H$ is the $\frac{2n+2}{2n}$-th power of the standard anticanonical height on $\mathbb{P}^1_Q$. Therefore, we have
\[
N_{X_{n,\text{sing}},H}(B) = \frac{2}{\zeta(2)} B^{\frac{2n+2}{2n}} + O(B^{\frac{2n+2}{2n}} \log B).
\]

In particular,
\[
N_{X_{2,\text{sing}},H}(B) = N_{U,H}(B) + O(B).
\]

We are now going to state the counting problem for $X_n'$. We consider the open subset
\[
U' := \tilde{X}_n \setminus V(yzw) = X_n' \setminus V(yzt).
\]
Proposition 3.6. There is a natural 16-to-1 correspondence between
\[ U' := \left\{ (a, b, c, d, y, z, t, w) \in \mathbb{Z}^8 : \begin{cases} yzw \neq 0; \quad ad - bc - y^n z^{n+1} w = 0 \\ \gcd(a, c) = \gcd(z, t) = \gcd(y, t) = 1 \\ \gcd(b, d, z) = \gcd(b, d, w) = \gcd(y, w) = 1 \end{cases} \right\} \]
and the set \( U'(\mathbb{Q}) \). Moreover, for \((a, b, c, d, y, z, t, w) \in U'\), we have
\[ H'(\pi(a : b : c : d : y : z : t : w)) = \max |\mathcal{M}_n(a, b, c, d, y, z, t, w)|^{1/(n+1)}. \]
Proof. As Proposition 3.1. \( \square \)

Corollary 3.7. We have that \( N_{U', H'}(B^{1/(n+1)}) \) is equal to
\[ \frac{1}{16} \# \left\{ (a, b, c, d, y, z, t, w) \in \mathbb{Z}^8 : \begin{cases} yzw \neq 0; \quad ad - bc - y^n z^{n+1} w = 0 \\ \gcd(a, c) = \gcd(z, t) = \gcd(y, t) = 1 \\ \gcd(b, d, z) = \gcd(b, d, w) = \gcd(y, w) = 1 \\ \max |\mathcal{M}_n(a, b, c, d, y, z, t, w)| \leq B \right\}. \]

Remark 3.8. It is not difficult to see that we may assume that \( \mathcal{M}_n(a, b, c, d, y, z, t, w) \) only contains the 13 monomials
\[ \begin{align*} 
\{b, d\}^{n+3} & \cdot \{a, c\}^2 \cdot y^2, \\
\{b, d\}^{n+1} & \cdot \{a, c\}^2 \cdot z^{2n+2} \cdot t^{2n+2} \cdot u^2, \\
\{a, c\}^{n^2+2n+1} & \cdot \{y, w\}^{n+1} \cdot \{z, t\}^{n^2+3n+2} \cdot \{t, w\}^{n+1} \cdot \{u, w\}^{n+3}, \\
\end{align*} \]
where the notation \( \{b, d\} \) resp. \( \{a, c\} \) means \( b \) or \( d \) resp. \( a \) or \( c \).

4. The expected formula for \( X_2 \)

The aim of this section is to determine the expected asymptotic formula for \( N_{U, H}(B) \) where
\[ U := \widetilde{X}_2 \setminus V(w) = X_2 \setminus V(a, c, z). \]
The resolution \( \pi : \widetilde{X}_2 \to X_2 \) is crepant, hence the pullback of \( H \) is an anticanonical height on \( \widetilde{X}_2 \). According to [BM90, Conjecture C] and [Pey03, 5.1], we have the predicted asymptotic formula
\[ N_{U, H}(B) \sim \alpha \beta \tau B \log B. \]
with
\[ \alpha = \text{rk} \text{Pic}((\widetilde{X}_2)^\vee) \cdot \text{vol} \left\{ t \in \text{Eff}((\widetilde{X}_2)^\vee) : (t, -K_{\widetilde{X}_2}) \leq 1 \right\} \]
where the volume is normalized such that \( \text{Pic}((\widetilde{X}_2)^\vee) \) has covolume 1 in \( \text{Pic}((\widetilde{X}_2)^\vee) \).
Under the identification \( \text{Pic}(\widetilde{X}_2) \cong \mathbb{Z}^2 \) from Section 2 we have
\[ \alpha = 2 \cdot \text{vol} \left\{ (t_1, t_2) \in \mathbb{R}_{\geq 0}^2 : 4t_1 + 2t_2 \leq 1 \right\} = \frac{1}{8}. \]
The cohomological constant \( \beta \) is
\[ \beta = \# H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic}((\widetilde{X}_2)_{\overline{\mathbb{Q}}})) = 1 \]
since \( \widetilde{X}_2 \) is split. Finally, we determine the Tamagawa number \( \tau \). Consider the chart
\[ \mathbb{A}^4_{\overline{\mathbb{Q}}} \to \widetilde{X}_2, \]
\[ (a, d, z) \mapsto (a : ad - z^3 : 1 : d : z : 1). \]
It follows from [Pey03, 4.6] and [Pey95, 2.2.1] that we have
\[ \tau = \omega_\infty \left( \prod_{p \text{ prime}} \lambda_p \omega_p \right) \]
with \( \lambda_p = (1 - p^{-1})^2 \) and
\[ \omega_\nu := \int \int \int_{Q_2} \frac{1}{\max \{ \mathcal{M}(a, ad - z^3, 1, d, z, 1) \}_{\nu}} \, da \, dd \, dz, \]
for \( \nu = p \) and \( \nu = \infty \), where we have used the isomorphism
\[ \omega_{\tilde{X}_2} \cong \mathcal{O}_{\tilde{X}_2}(-4, -2) \]
identifying the section \( da \wedge db \wedge dz \) from the chart with the section \( 1/c^4 w^2 \) from the Cox ring. Note that \( \omega_p \) and \( \omega_\infty \) (but not the product \( \tau \)) depend on the choice of such an isomorphism.

**Lemma 4.1.** We have
\[ \prod_{p \text{ prime}} \lambda_p \omega_p = \frac{1}{\zeta(2) \zeta(3)}. \]

**Proof.** A direct calculation of the \( p \)-adic integral yields
\[ \omega_p = \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{1}{p} + \frac{1}{p^2} \right), \]
from which the result follows. Alternatively, we may compute \( \omega_p \) using an integral model of \( \tilde{X}_2 \). The toric variety \( \tilde{Y}_2 \) comes from a regular fan, which can be used to construct a toric scheme \( \tilde{Y}_2 \) over \( \text{Spec}(\mathbb{Z}) \), together with a map
\[ \text{Spec}(\mathbb{Z}[a, b, c, d, z, w]) \setminus (V(a, c, z) \cup V(b, d, w)) \to \tilde{Y}_2, \]
which is a model for the universal torsor over
\[ \tilde{X}_2 = \tilde{Y}_2 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}). \]
For details, we refer to [Sal98, Section 8]. It can now be verified that the equation \( ab - cd - z^{n+1}w = 0 \) defines a closed subscheme \( \tilde{X}_2 \to \tilde{Y}_2 \), which is smooth and has integral fibers over \( \text{Spec}(\mathbb{Z}) \) such that
\[ \tilde{X}_2 = \tilde{X}_2 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}). \]
We have the chart
\[ \mathbb{A}^3_\mathbb{Z} \to \tilde{X}_2, \quad (a, d, z) \mapsto (a : ad - z^3 : 1 : d : z : 1) \]
and see that the isomorphism of line bundles
\[ \omega_{\tilde{X}_2} \cong \mathcal{O}_{\tilde{X}_2}(-4, -2) \]
identifying \( da \wedge db \wedge dz \) and \( 1/c^4 w^2 \) can be defined over \( \text{Spec}(\mathbb{Z}) \). Hence, according to [Pey17] Lemme 6.1, we have
\[ \omega_p = \frac{\# \tilde{X}_2(F_p)}{p^2} = \frac{(p + 1)(p^2 + p + 1)}{p^2} = \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{1}{p} + \frac{1}{p^2} \right). \]

Finally, we compute the real density.

**Lemma 4.2.** We have
\[ \omega_\infty = 2 \int \int \int \int_{\mathbb{Z}} \frac{1}{|c|} da \, dc \, dd \, dz. \]
We introduce an additional integration over $c$ using the identity
\[
\frac{1}{s} = \frac{1}{2} \int_{|c| \geq s} \frac{1}{|c^2|} \, dc.
\]
for $s \in \mathbb{R}_{>0}$ and obtain
\[
\omega_\infty = \frac{1}{2} \iiint_{|a^2|, |1|, |z|, |ad - z^2|, |d| \leq |c|^{1/2}} \frac{1}{|c^2|} \, da \, dc \, dz.
\]
Now the transformation $c \mapsto \frac{1}{c}$ (with $dc \mapsto \frac{1}{|c^2|} \, dc$) yields
\[
\omega_\infty = 2 \iiint_{|a^2|, |1|, |z^2|, |ad - z^2|, |d| \leq |c|^{1/2}} |c^3| \, da \, dc \, dz,
\]
and, finally, the transformation $(a, d, z) \mapsto (\frac{a}{z}, \frac{d}{z}, z)$ yields
\[
\omega_\infty = 2 \iiint_{|a^2|, |1|, |z^2|, |(ad - z^2)/c|, |d| \leq 1} \frac{1}{|c|} \, da \, dc \, dz.
\]
\[\square\]

5. The expected formula for $X_n$ in the case $n \geq 3$

The aim of this section is to determine, for $n \geq 3$, the expected asymptotic formula for $N_{U,H}(B)$, where
\[
U := \tilde{X}_n \setminus \mathcal{V}(w) = X_n \setminus \mathcal{V}(a,c,z)
\]
and, moreover, to prove Theorem 1.4.

Recall from Section 3 that we consider
\[
L := \frac{n}{n+2} \cdot \pi^*(-K_{X_n}) = (n, 1) \in \mathbb{Z}^2 \cong \text{Pic}(\tilde{X}_n),
\]
and that the pullback of $H^{n/(n+2)}$ is a height relative to $L$. According to [BM90 Conjecture C'] (see also [Pey03, 3.6]), the predicted asymptotic formula is
\[
N_{U,H}(B) \sim c B^a (\log B)^{b-1},
\]
where
\[
a := \frac{n}{n+2} \cdot \inf \left\{ t \in \mathbb{R} : t \cdot L + K_{\tilde{X}_n} \text{ is effective} \right\} = \frac{2n}{n+2}
\]
and $b = 1$ is the codimension of the minimal face of the effective cone of $\tilde{X}_n$ containing $a \cdot L + K_{\tilde{X}_n}$.

Next, we compute the prediction of [BT98b] for $c$. The divisor
\[
\frac{n+2}{n} \cdot a \cdot L + K_{\tilde{X}_n} = (n-2, 0)
\]
is not rigid, hence, according to [BT98b] Remark 2.4.4, we consider the natural fibration
\[
\phi: \tilde{X}_n \to P_n := \text{Proj} \left( \bigoplus_{\nu \geq 0} \Gamma \left( \tilde{X}_n, \mathcal{O}_{\tilde{X}_n}(n-2, 0)^{\otimes \nu} \right) \right),
\]
where we have an isomorphism $\mathbb{P}^2_{\mathbb{Q}} \cong P_n$ such that
\[
\phi: \tilde{X}_n \to \mathbb{P}^2_{\mathbb{Q}}, \quad (a : b : c : d : z : w) \mapsto (a : c : z).
\]
As we have $\phi^{-1}(x) \subseteq \mathcal{V}(w)$ if and only if $x \in \mathcal{V}(a,c)$, we only consider points $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathcal{V}(a,c)$ and determine the predicted asymptotic formula
\[
N_{\phi^{-1}(x), \pi^*_H(B)} \sim c_x B^{a_x} (\log B)^{b_x-1}
\]
for the fiber $\phi^{-1}(x)$. We have isomorphisms

\[
P_1^Q \rightarrow \phi^{-1}(x), \quad (b : w) \mapsto \left( a : b : c : \frac{bc + zn+1}{a} : z : w \right), \quad \text{for } a \neq 0,
\]

\[
P_1^Q \rightarrow \phi^{-1}(x), \quad (d : w) \mapsto \left( a : \frac{ad - zn+1}{c} : c : d : z : w \right), \quad \text{for } c \neq 0,
\]

which depend on the choice of $a, c, z \in \mathbb{Q}$ such that $x = (a : c : z)$. We now see that $\pi^*H^{2n/(n+2)}$ restricted to $\phi^{-1}(x)$ is an anticanonical height on $P_1^Q$, which means that the predicted asymptotic formula is

\[
N_{\phi^{-1}(x), \pi^*H} (B^{n+2}) \sim \frac{1}{2} \omega_{\infty, x} \left( \prod_{p \text{ prime}} \lambda_p \omega_{p, x} \right) B,
\]

where $\lambda_p = 1 - p^{-1}$. Now, consider the charts

\[
\mathbb{A}_1^Q \rightarrow \phi^{-1}(x), \quad b \mapsto \left( a : b : c : \frac{bc + zn+1}{a} : z : 1 \right), \quad \text{for } a \neq 0,
\]

\[
\mathbb{A}_1^Q \rightarrow \phi^{-1}(x), \quad d \mapsto \left( a : \frac{ad - zn+1}{c} : c : d : z : 1 \right), \quad \text{for } c \neq 0.
\]

According to [Pey95, 2.2.1], we have

\[
\omega_{\nu, x} := \left\{ \begin{array}{ll}
\int_{Q_x} \frac{1}{a} \max |c_n(a, b, c, (bc + zn+1)/a, z, 1)|_{\nu}^2 \, db & \text{for } a \neq 0, \\
\int_{Q_x} \frac{1}{c} \max |c_n(a, (ad - zn+1)/c, z, 1)|_{\nu}^2 \, dd & \text{for } c \neq 0
\end{array} \right.
\]

for $\nu := p$ and $\nu := \infty$, where we have used the isomorphism

\[
\omega_{\phi^{-1}(x)} \cong \mathcal{O}_{X_n}(-2L)|_{\phi^{-1}(x)}
\]

identifying the section $db$ from the first chart (resp. the section $dd$ from the second chart) with the section $a/w^2$ from the Cox ring (resp. the section $c/w^2$ from the Cox ring). Imposing the conditions $a, c, z \in \mathbb{Z}$ and $\gcd(a, c, z) = 1$, the integrals $\omega_{\nu, x}$ only depend on $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbf{V}(a, c)$.

It follows that we have $a_x = \frac{2n}{nt^2} = a$, $b_x = 1 = b$, and

\[
\epsilon_x = \frac{1}{2} \omega_{\infty, x} \prod_{p \text{ prime}} \lambda_p \omega_{p, x}.
\]

Summing over all the fibers, we obtain the expected constant

\[
\epsilon = \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbf{V}(a, c)} \epsilon_x.
\]

in the asymptotic formula for $N_{U, H}(B)$. We show in Corollary 5.4 that this sum converges.

**Lemma 5.1.** For every $x \in \mathbb{P}^2(\mathbb{Q}) \setminus \mathbf{V}(a, c)$ we have

\[
\prod_{p \text{ prime}} \lambda_p \omega_{p, x} = \frac{1}{\zeta(2)}.
\]

**Proof.** A straightforward calculation of the $p$-adic integrals yields

\[
\omega_{p, x} = 1 + \frac{1}{p},
\]

from which the result follows. \qed
Lemma 5.2. We have
\[
\omega_{\infty, x} = \begin{cases} 
\frac{1}{|a|} \int_{|b|, |(bc+zn+1)|/a| \leq 1} |a^n w|, |c^n w|, |z^n w| \leq 1 \, db \, dw & \text{for } a \neq 0, \\
\frac{1}{|c|} \int_{|(ad-zn+1)|/c| \leq 1} |a^n w|, |c^n w|, |z^n w| \leq 1 \, dd \, dw & \text{for } c \neq 0.
\end{cases}
\]

Proof. We consider the case \(a \neq 0\) (the case \(c \neq 0\) is similar). In our expression for \(\omega_{\nu, x}\) above, with \(\nu = \infty\), we introduce an additional integration over \(w\) as in Lemma 5.2 and obtain
\[
\omega_{\infty, x} = \frac{1}{2} \int_{|b|, |(bc+zn+1)|/a| \leq 1} \frac{1}{|a|} \int |a^n w|, |c^n w|, |z^n w| \leq 1 \, db \, dw.
\]
Now the transformations \(w \mapsto \frac{1}{w}\) and then \(b \mapsto \frac{b}{w}\) give the result.

The following result proves Theorem 1.4.

Theorem 5.3. Let \(\overline{H} : P^2(Q) \to R_+\) be a height relative to \(O_{P^2(Q)}(-n-1) \cong O_{P_n}(-1) \otimes \omega_{P_n}\). There exist positive constants \(c_1, c_2\) such that for every \(x \in P^2(Q) \setminus V(a, c)\) we have
\[
c_1\overline{H}(x) \leq \xi_2 \leq c_2\overline{H}(x).
\]

Proof. Let \(x := (a : c : z) \in P^2(Q)\) with integral coordinates and \(\gcd(a, c, z) = 1\). Without loss of generality, we can define the height \(\overline{H}\) as
\[
\overline{H}(x) := \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}}.
\]
First, assume \(\max\{|a|, |c|, |z|\} = |a|\). Then we have
\[
\omega_{\infty, x} \leq \int_{|a^n w|, |b| \leq 1} \frac{1}{|a|} \, db \, dw = \int_{|a^n w| \leq 1} \frac{2}{|a|} \, dw \ll \frac{1}{|a|^{n+1}}.
\]
Moreover, the conditions \(|a^n w|, |b| \leq \frac{1}{2}\) imply all the conditions on the integral \(\omega_{\infty, x}\), so that we obtain
\[
\omega_{\infty, x} \geq \int_{|a^n w|, |b| \leq \frac{1}{2}} \frac{1}{|a|} \, db \, dw = \frac{1}{|a|^{n+1}}.
\]
The case \(\max\{|a|, |c|, |z|\} = |c|\) is similar. It remains to consider \(\max\{|a|, |c|, |z|\} = |z|\). Assume \(|a| \geq |c|\). We have
\[
\omega_{\infty, x} \leq \int_{|b|, |(bc+zn+1)|/a| \leq 1} \frac{1}{|a|} \, db \, dw = \int_{|b| \leq 1} \frac{2|a|}{|az|^{n+1}} \, db \ll \frac{1}{|z|^{n+1}}.
\]
Moreover, the conditions \(|b| \leq \frac{|a|}{2|c|}\) and \(|w| \leq \frac{|a|}{2|z|^{n+1}}\) imply all the conditions on the integral \(\omega_{\infty, x}\), so that we obtain
\[
\omega_{\infty, x} \geq \int_{|b| \leq \frac{|a|}{2|c|}, |w| \leq \frac{|a|}{2|z|^{n+1}}} \frac{1}{|a|} \, db \, dw = \frac{|a|}{|cz^{n+1}|} \geq \frac{1}{|z|^{n+1}}.
\]
The case \(|c| \geq |a|\) is similar. Together, we obtain \(\overline{H}(x) \asymp \omega_{\infty, x} \asymp \xi_2\).

Corollary 5.4. We have
\[
\sum_{x \in P^2(Q) \setminus V(a, c)} \xi_2 < \infty.
\]
Proof. Since \( n \geq 3 \), we have
\[
\sum_{x \in \mathbb{P}^2(Q) \setminus V(a, c)} c_x \ll \sum_{a, c, z} \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}} \ll 1. \tag*{\square}
\]

6. The expected formula for \( X'_n \) in the case \( n \geq 2 \)

The aim of this section is to determine, for \( n \geq 2 \), the expected asymptotic formula for \( N_{U', H'}(B) \) where
\[
U' := \tilde{X}'_n \setminus V(yztw) = X'_n \setminus V(yzt)
\]
and, moreover, to prove Theorem 1.6.

Recall from Section 3 that we consider
\[
P(\tilde{X}'_n) \cong \mathbb{Z}^4,
\]
and, moreover, to prove Theorem 1.6. According to [BM90, Conjecture C%; see also Pey03, 3.6], the predicted asymptotic formula is
\[
N_{U', H'}(B) \sim c B^a (\log B)^{b-1},
\]
where
\[
a := (n + 1) \cdot \inf \left\{ t \in \mathbb{R} : t \cdot L' + K_{\tilde{X}'_n} \in \text{Pic}(\tilde{X}'_n) \text{ is effective} \right\} = \frac{2n+2}{n+3},
\]
and \( b = 1 \) is the codimension of the minimal face of the effective cone of \( \tilde{X}'_n \) containing \( a \cdot L' + K_{\tilde{X}'_n} \). A prediction for \( c \) can be found in [BT98b]. The \( Q \)-divisor
\[
\frac{1}{n+3} \cdot a \cdot L' + K_{\tilde{X}'_n} = \frac{n+1}{n+3} \cdot (n + 2, n + 1, 1, 0)
\]
is not rigid, hence, according to [BT98b, Remark 2.4.4], we consider the natural fibration
\[
\phi' : \tilde{X}'_n \to P'_n := \text{Proj} \left( \bigoplus_{\nu \geq 0} \Gamma \left( \tilde{X}'_n, O_{\tilde{X}'_n}(n + 2, n + 1, 1, 0) \otimes \frac{1}{n+3} \nu \right) \right),
\]
where we have an isomorphism \( P'_n \cong P'_n \) such that \( \phi' \) extends to
\[
\phi' : \tilde{X}'_n \to P'_n, \quad (a : b : c : d : y : z : t : w) \mapsto (\tilde{a} : \tilde{c} : \tilde{y}) := (at : ct : yz).
\]
As we have \( \phi'^{-1}(x) \subseteq V(yztw) \) if and only if \( x \in V(\tilde{a}, \tilde{c}) \cup V(\tilde{y}) \), we only consider points \( x \in P'_n \setminus (V(\tilde{a}, \tilde{c}) \cup V(\tilde{y})) \) and determine the predicted asymptotic formula
\[
N_{\phi'^{-1}(x), \pi', H'}(B) \sim c_x B^a (\log B)^{b-1}
\]
for the fiber \( \phi'^{-1}(x) \). We have isomorphisms
\[
P_1^1 \to \phi'^{-1}(x), \quad (b : w) \mapsto \left( \tilde{a} : b : \tilde{c} : \frac{bw + \tilde{y}w}{\tilde{c}} : \tilde{y} : 1 : 1 : w \right), \quad \text{for } \tilde{a} \neq 0,
\]
\[
P_1^1 \to \phi'^{-1}(x), \quad (d : w) \mapsto \left( \tilde{a} : \frac{\tilde{a}d - \tilde{y}w}{\tilde{c}} : \tilde{c} : d : \tilde{y} : 1 : 1 : w \right), \quad \text{for } \tilde{c} \neq 0,
\]
which depend on the choice of \( \tilde{a}, \tilde{c}, \tilde{y} \in \mathbb{Q} \) such that \( x = (\tilde{a} : \tilde{c} : \tilde{y}) \). We now see that \( (\pi' H')^{(2n+2)/(n+3)} \) restricted to \( \phi'^{-1}(x) \) is an anticanonical height on \( P_1^1 \), which means that the predicted asymptotic formula is
\[
N_{\phi'^{-1}(x), \pi', H'}(B \frac{n+1}{n+3}) \sim \frac{1}{2} \omega_{\infty, x} \left( \prod_{\text{prime } p} \lambda_p \omega_p, x \right) B,
\]
where \( \lambda_p \) is the Tamagawa number at \( p \).
where $\lambda_p = 1 - p^{-1}$. Now, consider the charts

$$A^1_Q \to \phi'^{-1}(x), \quad b \mapsto \left(\hat{a} : b : \hat{c} : \frac{bc + \hat{y}^n}{\hat{a}} : \hat{y} : 1 : 1 : 1\right),$$

for $\hat{a} \neq 0$,

$$A^1_Q \to \phi'^{-1}(x), \quad d \mapsto \left(\hat{a} : \frac{\hat{a}d - \hat{y}^n}{\hat{c}} : \hat{c} : d : \hat{y} : 1 : 1 : 1\right),$$

for $\hat{c} \neq 0$.

According to [Pey95, 2.2.1], we have

$$\omega_{\nu,x} := \begin{cases} 
1 & \text{for } \nu \neq 0, \\
\int_{Q_{\nu}} |\hat{a}|_{\nu} \max |\mathcal{M}'_n(\hat{a}, \hat{b}, \hat{c}, (\hat{b}c + \hat{y}^n)/\hat{a}, \hat{y}, 1, 1, 1)|^{2/(n+3)} db & \text{for } \hat{a} \neq 0, \\
\int_{Q_{\nu}} |\hat{c}|_{\nu} \max |\mathcal{M}'_n(\hat{a}, \hat{d} - \hat{y}^n)/\hat{c}, \hat{c}, \hat{d}, \hat{y}, 1, 1, 1)|^{2/(n+3)} dd & \text{for } \hat{c} \neq 0 
\end{cases}$$

for $\nu := p$ and $\nu := \infty$, where we have used the isomorphism

$$\omega_{\nu^{-1}(x)} \cong \mathcal{O}_{\mathcal{X}_n}'(-2L'_n)|_{\phi'^{-1}(x)}$$

identifying the section $(db)^{\otimes(n+3)}$ from the first chart (resp. the section $(dd)^{\otimes(n+3)}$ from the second chart) with the section $\mathcal{L}^{n+3}/w^{2n+6}$ from the Cox ring (resp. the section $\mathcal{L}^{n+3}/w^{2n+6}$ from the Cox ring). Imposing the conditions $\hat{a}, \hat{c}, \hat{y} \in \mathbb{Z}$ and $\gcd(\hat{a}, \hat{c}, \hat{y}) = 1$, the integrals $\omega_{\nu,x}$ only depend on $x \in \mathbb{P}^2(Q) \setminus (V(\hat{a}, \hat{c}) \cup V(\hat{y}))$.

It follows that we have $a_x = \frac{2n}{n+2} = a$, $b_x = 1 = b$, and

$$c_x = \frac{1}{2} \omega_{\infty,x} \prod_{p \text{ prime}} \lambda_p \omega_{p,x}.$$

Summing over all the fibers, we obtain the expected constant

$$\epsilon = \sum_{x \in \mathbb{P}^2(Q) \setminus (V(\hat{a}, \hat{c}) \cup V(\hat{y}))} c_x,$$

in the asymptotic formula for $N_{U', H'(B)}$. We show in Proposition~\ref{prop:asympformula} that $\epsilon < \infty$.

**Lemma 6.1.** Let $\epsilon := \frac{-n+1}{n+3}$. For every $x \in \mathbb{P}^2(Q) \setminus (V(\hat{a}, \hat{c}) \cup V(\hat{y}))$ we have

$$\omega_{p,x} = \left(1 - \frac{1}{p}\right) \cdot \left(1 - \frac{(p^x)^{\nu(\hat{y})} + 1}{1 - p^x} + \frac{1}{p} \cdot \frac{(p^x)^{\nu(\hat{y})}}{p}\right).$$

**Proof.** A lengthy, but straightforward calculation of the $p$-adic integrals yields

$$\omega_{p,x} = \begin{cases} 
1 + \left(1 - \frac{1}{p}\right) \cdot \sum_{j=1}^{\nu(\hat{y}) - 1} (p^x)^j & \text{for } \nu(\hat{y}) > 0, \\
\left(1 + \frac{1}{p}\right) \cdot (p^{x+1})^{\min(\nu(\hat{a}), \nu(\hat{c}))} & \text{otherwise,}
\end{cases}$$

from which the result follows.

**Lemma 6.2.** We have

$$\omega_{\infty,x} = \begin{cases} 
\int_{\max \left|\mathcal{M}'_n(\hat{a}, \hat{b}, \hat{c}, (\hat{b}c + \hat{y}^n)/\hat{a}, \hat{y}, 1, 1, w)/\mathcal{L}_n\right| \leq 1} \frac{1}{|\hat{a}|} db dw & \text{for } \hat{a} \neq 0, \\
\int_{\max \left|\mathcal{M}'_n(\hat{a}, (\hat{a}d - \hat{y}^n)/\hat{c}, \hat{c}, \hat{d}, \hat{y}, 1, 1, 1, w)/\mathcal{L}_n\right| \leq 1} \frac{1}{|\hat{c}|} dd dw & \text{for } \hat{c} \neq 0.
\end{cases}$$

**Proof.** This is completely analogous to Lemma~\ref{lem:asympformula}.

The following result proves Theorem~\ref{thm:mainthm}.
Theorem 6.3. Let \( \overline{\mathcal{H}} : \mathbb{P}^2(\mathbb{Q}) \to \mathbb{R}_{\geq 0} \) be a height relative to an arbitrary line bundle. Then there are \( \epsilon > 0 \) such that there does not exist an open subset \( V \subseteq \mathbb{P}^2(\mathbb{Q}) \) with positive constants \( \epsilon_1, \epsilon_2 \) such that for every \( x \in V \) we have

\[
c_1 \overline{\mathcal{H}}(x)^{1-\epsilon} \leq \epsilon x \leq c_2 \overline{\mathcal{H}}(x)^{1+\epsilon}.
\]

Proof. Let \( x = (\hat{a} : \hat{c} : \hat{y}) \in \mathbb{P}^2(\mathbb{Q}) \) with integral coordinates and \( \gcd(\hat{a}, \hat{c}, \hat{y}) = 1 \). Without loss of generality, we can define the height \( \overline{\mathcal{H}} \) as \( \overline{\mathcal{H}}(x) := \max\{ |\hat{a}|, |\hat{c}|, |\hat{y}| \}^r \) for some \( r \in \mathbb{Z} \). We define

\[
\omega^-_{\infty,x} := \int_{\mathbb{R}} \frac{1}{|\hat{a}|} \max |\mathcal{H}_n'(\hat{a}, 2\hat{c}, \hat{y}, 0, 1, 1)|^{2/(n+3)} \, db,
\]

\[
\omega^+_{\infty,x} := \int_{\mathbb{R}} \frac{1}{|\hat{a}|} \max |\mathcal{H}_n'(\hat{a}, \hat{c}, \hat{y}, 0, 1, 1)|^{2/(n+3)} \, db.
\]

Clearly \( \omega^-_{\infty,x} \leq \omega^+_{\infty,x} = 2\omega^-_{\infty,x} \). For \( \hat{a} \geq \max\{2, |\hat{c}|, |\hat{y}| \} \), we have \( \omega^-_{\infty,x} \leq \omega^-_{\infty,x} \). Now we choose \( \hat{a}_0, \hat{c}_0, \hat{y}_0 \in \mathbb{Z} \) with \( \gcd(\hat{a}_0, \hat{c}_0, \hat{y}_0) = 1 \) and \( \max\{ |\hat{a}_0|, |\hat{c}_0|, |\hat{y}_0| \} = |\hat{a}_0| \), and a prime \( p \) with \( p \nmid \hat{a}_0, \hat{c}_0, \hat{y}_0 \) such that

\[
x_0(m) := (\hat{a}_0p^m : \hat{c}_0 : \hat{y}_0),
\]

\[
x'_0(m) := (\hat{a}_0p^m : \hat{c}_0p^m : \hat{y}_0)
\]

lie in a given \( V \) for all \( m \gg 1 \). Here and in the following, all implicit constants may depend on \( \hat{a}_0, \hat{c}_0, \hat{y}_0 \), and \( r \), but not on \( m \). We have

\[
\overline{\mathcal{H}}(x'_0(m)) = \overline{\mathcal{H}}(x_0(m)) = (p^m|\hat{a}_0|)^r.
\]

For \( m \gg 1 \), we have

\[
\epsilon_{x_0(m)} \asymp \omega^-_{\infty,x_0(m)} \asymp \omega^-_{\infty,x'_0(m)} \asymp \omega^-_{\infty,x'_0(m)},
\]

since \( \omega^-_{\infty,x} \) does not depend on \( \hat{c} \) for \( |\hat{a}| \geq \hat{c} \). We also have

\[
\epsilon_{x'_0(m)} \asymp (p^m)^{\epsilon+1}\omega^-_{\infty,x'_0(m)} \asymp (p^m)^{\epsilon+1}\omega^-_{\infty,x'_0(m)} \asymp (p^m)^{\epsilon+1}\epsilon_{x_0(m)}.
\]

Assume that, for all \( m \geq 0 \), we have

\[
\epsilon_{x_0(m)} \gg \overline{\mathcal{H}}(x_0(m))^{1-\epsilon}, \quad \epsilon_{x'_0(m)} \ll \overline{\mathcal{H}}(x'_0(m))^{1+\epsilon}.
\]

Then

\[
(p^m)^{\epsilon+1}(p^m|\hat{a}_0|)^{(1-\epsilon)} \ll (p^m)^{\epsilon+1}\epsilon_{x_0(m)} \ll (p^m)^{\epsilon+1}\epsilon_{x'_0(m)} \ll (p^m|\hat{a}_0|)^{(1+\epsilon)},
\]

which implies

\[
(p^m)^{\epsilon+1-2\epsilon} \ll |\hat{a}_0|^{2\epsilon}.
\]

For \( \epsilon + 1 - 2\epsilon > 0 \) (i.e., \( \epsilon < \frac{2}{3(n+3)} \)) and \( m \to \infty \), we arrive at a contradiction. \( \square \)

7. Estimating integral points on the universal torsor of \( X_n \)

We are going to prove Theorem 7.1 by showing

\[
N_{U,H}(B^2) = cB^2 \log(B^2) + O(B^2),
\]

where \( c \) is as in Section 4 and we are going to prove Theorem 7.2 by showing

\[
N_{U,H}(B^{\frac{n+2}{2}}) = cB^2 + O(B^{\frac{n+2}{2}})
\]

for \( n \geq 3 \), where \( c \) is as in Section 5.

We use [Der09, Lemma 3.1] and [DF14, Lemma 3.6] repeatedly to approximate sums by integrals. Note that we have

\[
\max |\mathcal{H}_n(a, b, c, d, z, w, t)| = \max\{ |b|, |d|, |a^n w t^{n+1}|, |c^n w t^{n+1}|, |z^n w t| \}.
\]
We define
\[ V_1(a, c, z, w; B) := \begin{cases} \int_{|d|, |(ad - z^{n+1}w)/c| \leq B} \frac{1}{|c|} \, dd & \text{for } c \neq 0, \\ \int_{|b|, |(bc + z^{n+1}w)/a| \leq B} \frac{1}{|a|} \, db & \text{for } a \neq 0. \end{cases} \]

Note that for \( ac \neq 0 \) the two cases coincide. Moreover, we have
\[ V_1(a, c, z, w; B) \leq \frac{B}{\max\{|a|, |c|\}}. \]

We also define
\[ V_2(a, c, z; B) := \int_{|w| \leq B} V_1(a, c, z, w; B) \, dw. \]

**Proposition 7.1.** For \( n \geq 2 \), we have
\[ N_{U,H}(B^{\frac{n+2}{n}}) = \frac{1}{4\zeta(2)} \sum_{|a|, |c|, |z| \geq 0} V_2(a, c, z; B) + O(B^{\frac{n+2}{n}}). \]

**Proof.** By Corollary 3.3 we have
\[ N_{U,H}(B^{\frac{n+2}{n}}) = \frac{1}{8} \sum_{|a|, |c|, |z| \geq 0} \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{c} ad - bc = z^{n+1}w \\ |b|, |d| \leq B \\ \gcd(b, d, w) = 1 \end{array} \right\}. \]

We apply a Möbius inversion to the condition \( \gcd(b, d, w) = 1 \) and obtain
\[ N_{U,H}(B^{\frac{n+2}{n}}) = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8} \sum_{|a|, |c|, |z| \geq 0} \# \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{c} ad - bc = z^{n+1}w \\ |b|, |d| \leq B/\alpha \\ \gcd(a, c) = \gcd(z, t) = 1 \end{array} \right\} \]
\[ = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8} \sum_{|a|, |c|, |z| \geq 0} \left( V_1(a, c, z, w; B/\alpha) + O(1) \right) \]
\[ = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{|a|, |c|, |z| \geq 0} \left( \frac{1}{\alpha} V_1(a, c, z, \alpha w; B) + O(1) \right) \]
\[ = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{|a|, |c|, |z| \geq 0} \left( \frac{1}{\alpha} V_1(a, c, z, \alpha w; B) + O(B^{-\frac{n+2}{n}}) \right), \]

where the condition \( \gcd(a, c) = 1 \) is used for the second equality, the transformation \( ab \mapsto b \) or \( ad \mapsto d \) is applied inside the integral \( V_1 \) for the third equality, and the
fourth equality follows from the estimate

\[
\sum_\alpha \sum_{a,c,z,w,t} 1 \leq \sum_\alpha \sum_{a,c,z,t} \frac{B}{\alpha \max\{|a|,|c|\}^n |t|^{n+1}} \\
\leq \sum_\alpha \sum_{a,c,z,t} \frac{B^{(n+1)/n}}{\alpha^{(n+1)/n} \max\{|a|,|c|\}^n |t|^{n+2/n}} \\
\leq \begin{cases} 
B(n+1)/n \log B & \text{for } n = 2 \\
B(n+1)/n & \text{for } n \geq 3
\end{cases} \leq B^{n+2}.
\]

Replacing the sum over \(w\) by an integral, we obtain that \(N_{U,B}(B^{n+2})\) is equal to

\[
\sum_{\alpha \geq 1} \frac{\mu(\alpha)}{8\alpha} \sum_{|a|,|c|,|z| \geq 0} \left( \frac{1}{\gcd(a,c)=\gcd(z,t)=1} \frac{1}{\alpha \|a\|^n + 1} \sum_{|a|,|c|,|z| \geq 0} \frac{1}{\alpha \|a\|^n + 1} \right) + O(B^{n+2})
\]

where we have applied the transformation \(\alpha w \mapsto w\) for the first equality and the second equality follows from the estimates

\[
R_1 = \max_w V_1(a,c,z,w) \leq \frac{B}{\max\{|a|,|c|\}}
\]

and

\[
\sum_\alpha \frac{1}{\alpha} \sum_{a,c,z,t} \frac{B}{\alpha^{1/n} \max\{|a|,|c|\}^{1/n} |t|^{1/n}} \\
\leq \sum_\alpha \frac{1}{\alpha} \sum_{a,c,z,t} \frac{B^{(n+1)/n}}{\alpha^{(n+1)/n} \max\{|a|,|c|\}^{1/n} |t|^{1/n}} \\
\leq \sum_\alpha \frac{1}{\alpha} \sum_{a,c,z,t} \frac{B^{n+2/n}}{\alpha^{2/n} |t|^{n+2/n}} \leq B^{n+2}.
\]

Next, we replace the condition \(\alpha \leq |w|\) by the condition \(|t^{-1}| \leq |w|\). For \(0 < \epsilon < \frac{1}{n}\), we have

\[
\sum_\alpha \frac{1}{\alpha^2} \sum_{a,c,z,t} \left( \frac{1}{\|a\|^n + 1} \sum_{|a|,|c|,|z| \geq 0} \frac{1}{\|a\|^n + 1} \right) V_1(a,c,z,w) dw \\
\leq \sum_\alpha \frac{1}{\alpha^2} \sum_{a,c,z,t} \left( \frac{1}{\|a\|^n + 1} \sum_{|a|,|c|,|z| \geq 0} \frac{1}{\|a\|^n + 1} \right) \frac{B}{\max\{|a|,|c|\}} dw
\]
\[ \ll \sum_{\alpha} \frac{1}{\alpha^2} \sum_{a,c,z,t \text{ s.t. } z \neq 0 \atop |at|^n, |ct|^n, |z|^n \leq B} \max\{ |a|, |c| \} |z^n| |t|^{\epsilon} + \sum_{\alpha} \frac{1}{\alpha^2} \sum_{a,c,z \text{ s.t. } z \neq 0 \atop |at|^n, |ct|^n, |z|^n \leq B} \max\{ |a|, |c| \} (1 + \epsilon |t|^{(n+1)}) \ll \sum_{\alpha} \frac{1}{\alpha^{1+\epsilon}} \sum_{t} B^{(n+2)/\alpha} \ll B^{\frac{n+2}{\alpha}}, \]

so that we obtain that \( N_{U,H}(B^{\frac{n+2}{\alpha}}) \) is equal to

\[ \frac{1}{8\zeta(2)} \sum_{|a|, |c|, |z| \geq 0 \atop |t| \geq 1 \atop \gcd(a,c) = \gcd(z,t) = 1} \int \Ll|w| \geq 1 \atop |a^n w^{n+1}| \leq B \atop |c^n w^n| \leq B \atop |z^n w| \leq B \Rr| \frac{1}{|t|} V_1(a, c, z, w; B) \, dw + O(B^{\frac{n+2}{\alpha}}) = \frac{1}{8\zeta(2)} \sum_{|a|, |c|, |z| \geq 0 \atop |t| \geq 1 \atop \gcd(a,c) = \gcd(z,t) = 1} \int \Ll|w| \geq 1 \atop |a^n w^n| \leq B \atop |c^n w^n| \leq B \atop |z^n w| \leq B \Rr| \frac{1}{|t|} V_1(a, c, z, w/t; B) \, dw + O(B^{\frac{n+2}{\alpha}}) = \frac{1}{8\zeta(2)} \sum_{|a|, |c|, |z| \geq 0 \atop |t| \geq 1 \atop \gcd(a,c) = \gcd(z,t) = 1} \int \Ll|w| \geq 1 \atop |a^n w^n| \leq B \atop |c^n w^n| \leq B \atop |z^n w| \leq B \Rr| \frac{1}{|t|} V_1(a, c, z, w; B) \, dw + O(B^{\frac{n+2}{\alpha}}), \]

where the transformation \( tw \mapsto w \) is applied for the first equality and the 2-to-1 substitution \((ta, tc) \mapsto (a, c)\) is applied for the third equality. Finally, adding the condition \( |w| \leq B \) leaves the integral unchanged. \( \square \)

We define

\[ V_2'(a, c, z) := \int_{a^n w^n \leq 1 \atop |c^n w^n| \leq 1 \atop |z^n w| \leq 1} V_1(a, c, z, w; 1) \, dw. \]

**Corollary 7.2.** For \( n \geq 2 \), we have

\[ N_{U,H}(B^{\frac{n+2}{\alpha}}) = \frac{1}{4\zeta(2)} \sum_{|a|, |c|, |z| \leq B^{1/n} \atop \gcd(a,c,z) = 1 \atop (a,c) \neq (0,0)} V_2'(a, c, z) B^2 + O(B^{\frac{n+2}{\alpha}}). \]

**Proof.** In the formula from Proposition 7.1, we may restrict the sum to \( |a|, |c|, |z| \leq B^{1/n} \) since \( V_2(a, c, z; B) \) vanishes otherwise. The transformations \((b, w) \mapsto (Bb, Bw)\) for \( a \neq 0 \) and \((d, w) \mapsto (Bd, Bw)\) for \( c \neq 0 \) show that we have

\[ \int_{a^n w \leq B} V_1(a, c, z, w; B) \, dw = V_2'(a, c, z) B^2. \]

Comparing the left side with \( V_2(a, c, z; B) \), we see that the condition \( |w| \leq B \) in the definition of \( V_2(a, c, z; B) \) follows from the other conditions, and it remains to remove the condition \( |w| \geq 1 \). The corollary now follows from the computation

\[ \sum_{|a|, |c|, |z| \leq B^{1/n}} \int_{|w| \leq 1} V_1(a, c, z, w; B) \, dw \ll \sum_{|a|, |c|, |z| \leq B^{1/n}} \frac{B}{\max\{ |a|, |c| \}} \ll B^{\frac{n+2}{\alpha}}. \]
**Remark 7.3.** For \( n \geq 3 \), we have \( V_2'(a, c, z) = \omega_{\infty,(a,c):z} \) according to Lemma 5.2 and hence
\[
V_2'(a, c, z) \asymp \frac{1}{\max\{|a|, |c|, |z|\}^{n+1}},
\]
but this also holds for \( n = 2 \). Theorem 5.3 and Corollary 7.2 now yield
\[
N_{U,H}(B^{\frac{n+2}{2}}) \asymp \sum_{|a|,|c|,|z| \leq B^{1/n}} \frac{B^2}{\max\{|a|, |c|, |z|\}^{n+1}} \asymp \begin{cases} B^2 \log B, & n = 2, \\ B^2, & n \geq 3. \end{cases}
\]
In the following, we turn these upper and lower bounds into asymptotic formulas.

We begin with the case \( n \geq 3 \).

**Theorem 7.4.** For \( n \geq 3 \), we have
\[
N_{U,H}(B^{\frac{n+2}{2}}) = \mathcal{C} B^2 + O(B^{\frac{n+2}{2}}).
\]

**Proof.** We remove the conditions \(|a|, |c|, |z| \leq B^{1/n}\) from the sum in Corollary 7.2 with a satisfactory error term. Since \( \omega_{\infty,(a,c):z} \ll \max\{|a|, |c|, |z|\}^{-n-1} \) by Theorem 5.3, we have
\[
\sum_{\max\{|a|, |c|, |z| > B^{1/n}\}} \omega_{\infty,(a,c):z} B^2 \ll \sum_{|a| \leq |b| \leq |z| > B^{1/n}} \frac{B^2}{|z|^{n+1}} \ll \sum_{|z| > B^{1/n}} \frac{B^2}{|z|^{n-1}} \ll B^{\frac{n+2}{2}}.
\]
It follows that we have
\[
N_{U,H}(B^{\frac{n+2}{2}}) = \frac{1}{4\zeta(2)} \sum_{|a|, |c|, |z| \geq 0} \omega_{\infty,(a,c):z} B^2 + O(B^{\frac{n+2}{2}})
\]
\[
= \left( \sum_{x \in \mathbb{P}^2(Q) \setminus V(a,c)} \frac{\omega_{\infty, x}}{2\zeta(2)} \right) B^2 + O(B^{\frac{n+2}{2}}),
\]
as predicted in Section 5. \( \square \)

We now turn to the case \( n = 2 \). Note that we already have a result on the order of magnitude in Remark 7.3 following from Corollary 7.2. In order to obtain an asymptotic formula, we resume our calculation from Proposition 7.1. We define
\[
V_3(B) := \iiint V_2(a, c, z; B) \, da \, dc \, dz.
\]

**Lemma 7.5.** For \( n = 2 \), we have
\[
V_3(B) = \omega_{\infty} B^2 \log B.
\]

**Proof.** We have
\[
V_3(B) = \int \cdots \int_{\substack{|a|^2w, |c|^2w, |z|^2w \leq B \\text{ and } (ad-z^2)w/c, |d| \leq B \}} \frac{1}{|c|} \, da \, dc \, dz \, dw.
\]
Applying the transformations \((a, c, z) \mapsto B^{1/2}/w^{1/2}(a, c, z)\) and \(b \mapsto Bb\), we obtain
\[
V_3(B) = B^2 \int \cdots \int_{\substack{1 \leq |w| \leq B \\text{ and } (ad-z^2)/c, |d| \leq B \}} \frac{1}{|cw|} \, da \, dc \, dz \, dw.
\]
We apply a Möbius inversion to the condition for Theorem 7.6.

Now the transformation \(|w| \mapsto B^{|w|}\) (with \(dw \mapsto B^{|w|} \log B \, dw\)) yields

\[
V_3(B) = B^2 \log B \int_{0 \leq |w| \leq 1} \int_{(0,0)} \frac{1}{|e|} \, da \, dc \, dd \, dz
\]

\[
= 2B^2 \log B \int_{0 \leq |w| \leq 1} \int_{(0,0)} \frac{1}{|e|} \, da \, dc \, dd \, dz
\]

\[
= \omega_B B^2 \log B,
\]

where the last equality follows from Lemma 4.2.

\[\square\]

**Theorem 7.6.** For \(n = 2\), we have

\[
N_{U,H}(B^2) = cB^2 \log(B^2) + O(B^2).
\]

**Proof.** According to Proposition 7.1 we have

\[
N_{U,H}(B^2) = \frac{1}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\
\gcd(a,c,z) = 1 \\
(a,c) \neq (0,0)}} V_2(a, c, z; B) + O(B^2).
\]

We apply a Möbius inversion to the condition \(\gcd(a, c, z) = 1\) and replace the sum over \(z\) by an integral to obtain

\[
N_{U,H}(B^2) = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)} \sum_{\substack{|a|, |c|, |z| \geq 0 \\
\gcd(a,c,z) = 1 \\
(a,c) \neq (0,0)}} V_2(aa, ac, az; B) + O(B^2)
\]

\[
= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)} \sum_{\substack{|a|, |c| \geq 0 \\
\gcd(a,c) = 1 \\
(a,c) \neq (0,0)}} \left( \int V_2(aa, ac, az; B) \, dz + O(R_2) \right) + O(B^2)
\]

\[
= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{\substack{|a|, |c| \geq 0 \\
\gcd(a,c) = 1 \\
(a,c) \neq (0,0)}} \int V_2(aa, ac, z; B) \, dz + O(B^2)
\]

\[
= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{\substack{|a|, |c| \geq 1}} \int V_2(aa, ac, z; B) \, dz + O(B^2),
\]

where the third equality follows from the transformation \(az \mapsto z\) and the estimates

\[
R_2 \leq \max_{z} \int_{1 \leq |w| \leq B} \frac{B}{\max\{|aa|, |ac|\}} \, dw \ll \frac{B^2}{\alpha^3 \max\{|a|, |c|\}^3}
\]

and

\[
\sum_{\alpha, a, c} \frac{B^2}{\alpha^3 \max\{|a|, |c|\}^3} \ll B^2.
\]

Note that for every \(a, c \in \mathbb{R}\) with \((a, c) \neq (0,0)\) we have

\[
\int V_2(a, c, z; B) \, dz \leq \int \frac{B}{|a^2w| \leq B} \frac{1}{\max\{|a|, |c|\}} \, dw \, dz
\]

\[
\int \frac{B}{|c^2w| \leq B} \frac{1}{\max\{|a|, |c|\}} \, dw \, dz
\]

\[
\int \frac{B}{|z^2w| \leq B} \frac{1}{\max\{|a|, |c|\}} \, dw \, dz.
\]
which in particular implies the last equality above. We now successively replace the
sums over \( a \) and \( c \) by integrals to obtain

\[
N_{U,H}(B^2) = \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha} \sum_{|c| \geq 1} \left( \int_{|a| \geq 1} V_2(\alpha a, \alpha c, z; B) \, da \, dz + O(R_3) \right) + O(B^2)
\]

\[
= \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{4\zeta(2)\alpha^3} \left( \int_{|a|,|c| \geq 1} V_2(\alpha a, \alpha c, z; B) \, da \, dc \, dz + O(R_4) \right) + O(B^2)
\]

since we have

\[
R_3 = \max_a \int V_2(\alpha a, \alpha c, z; B) \, dz \ll \frac{B^2}{\alpha^2 |c|^2}
\]

and

\[
R_4 = \max_c \int V_2(\alpha a, \alpha c, z; B) \, da \, dz \ll \frac{B^2}{\alpha^2}
\]

as well as

\[
\sum_{\alpha} \frac{1}{\alpha} \sum_{c} \frac{B^2}{\alpha^2 |c|^2} \ll B^2
\]

and moreover the transformations \( \alpha a \mapsto a \) as well as \( \alpha b \mapsto b \) have been applied for the last equality. Finally, we can remove the conditions \( |a|, |c| \geq \alpha \) from the integral in order to obtain \( V_3(B) \) since

\[
\sum_{\alpha} \frac{1}{\alpha^2} \int_{|a|,|c| \leq \alpha} V_2(a, c, z; B) \, da \, dc \, dz \\
\ll \sum_{\alpha} \frac{1}{\alpha^3} \int_{|a|,|c| \leq \alpha} \frac{B^2}{\max\{|a|,|c|\}} \, da \, dc \\
\ll \sum_{\alpha} \frac{B^2}{\alpha^2} \ll B^2,
\]

as well as

\[
\sum_{\alpha} \frac{1}{\alpha^3} \int_{|a| \leq \alpha, |c| \geq \alpha} V_2(a, c, z; B) \, da \, dc \\
\ll \sum_{\alpha} \frac{1}{\alpha^3} \int_{|a| \leq \alpha, |c| \geq \alpha} \frac{B^2}{|c|^2} \, da \, dc \\
\ll \sum_{\alpha} \frac{1}{\alpha^3} \int_{|a| \leq \alpha} \frac{B^2}{\alpha} \, da \ll \sum_{\alpha} \frac{B^2}{\alpha^3} \ll B^2,
\]

The case \( |a| \geq \alpha, |c| \leq \alpha \) is handled similarly. It follows that we obtain

\[
N_{U,H}(B^2) = \frac{1}{4\zeta(2)} V_3(B) + O(B^2)
\]

\[
= \frac{\omega}{4\zeta(2)\zeta(3)} B^2 \log B + O(B^2)
\]

\[
= \frac{\omega}{8\zeta(2)\zeta(3)} B^2 \log(B^2) + O(B^2),
\]

by Lemma 7.5 as predicted in Section 4.
We also define

\[ N_{U', H'}(B^{|\lambda|}) = c B^{|\lambda|} + O_c(B^{|\lambda| + \epsilon}) \]

for \( n \geq 2 \) and any \( \epsilon > 0 \), where \( c \) is as in Section 6.

As in the preceding section, we repeatedly use [Der09, Lemma 3.1] and [DF12, Lemma 3.6] to approximate sums by integrals. We define

\[ \mathcal{H}(b, d, w, a, c, y, z, t, w) := \max \{ \mathcal{H}_n(a, b, c, d, y, z, t, w) \}. \]

Moreover, we define

\[ V_{1, \lambda}(a, c, y, z, t, w; B) := \begin{cases} \int_{\mathcal{H}(\lambda b, (\lambda bc + y^n z^{n+1} w) / a, ..., \lambda b, (\lambda bc + y^n z^{n+1} w) / c, \lambda d, ..., \lambda d, \leq B} \frac{1}{|a|} \, db & \text{for } a \neq 0, \\ \int_{\mathcal{H}(\lambda d - y^n z^{n+1} w) / c, \lambda d, ..., \lambda d, \leq B} \frac{1}{|c|} \, dd & \text{for } c \neq 0. \end{cases} \]

Note that for \( ac \neq 0 \) the two cases coincide and that we have

\[ V_{1, 1}(a, c, y, z, t, w; B) \ll \frac{B^{1/(n+3)}}{\max \{|a|, |c|\} (n+5)/(n+3) |y|^{2/(n+3)}}. \]

We also define

\[ V_2(a, c, y, z, t; B) := \int V_{1, 1}(a, c, y, z, t, w; B) \, dw. \]

**Theorem 8.1.** Let \( n \geq 2 \). For any \( \epsilon > 0 \), we have

\[ N_{U', H'}(B^{|\lambda|}) = c B^{|\lambda|} + O_c(B^{|\lambda| + \epsilon}). \]

**Proof.** By Corollary 3.7 we have that \( 16 \cdot N_{U', H'}(B^{|\lambda|}) \) is equal to

\[
\sum_{|a|, |c| \geq 0} \sum_{|\alpha|, |\beta|, |\delta|, |\epsilon|, |\lambda|, |\mu| |\nu|, |\omega| \geq 1} \frac{1}{\alpha \beta} \mu(\alpha) \mu(\beta) \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{c}
\mathcal{H}(\lambda b, (\lambda bc + y^n z^{n+1} w) / a, ..., \lambda b, (\lambda bc + y^n z^{n+1} w) / c, \lambda d, ..., \lambda d, \leq B \\
gcd(b, d, z) = \gcd(b, d, w) = 1
\end{array} \right\} \]

We apply a Möbius inversion to the conditions

\[ \gcd(b, d, z) = \gcd(b, d, w) = 1, \]

so that, after using the transformation \((b, d) \mapsto ([\alpha, \beta]b, [\alpha, \beta]d)\), we obtain that

\[ 16 \cdot N_{U', H'}(B^{|\lambda|}) \text{ is equal to} \]

\[
\sum_{|a|, |c| \geq 0} \sum_{|\alpha|, |\beta|, |\delta|, |\epsilon|, |\lambda|, |\mu| |\nu|, |\omega| \geq 1} \frac{1}{\alpha \beta} \mu(\alpha) \mu(\beta) \left\{ (b, d) \in \mathbb{Z}^2 : \begin{array}{c}
\mathcal{H}([\alpha, \beta]b, [\alpha, \beta]d, ..., \leq B \\
gcd([\alpha, \beta]b, [\alpha, \beta]d, w) = 1
\end{array} \right\} \]

\[
= \sum_{|a|, |c| \geq 0} \sum_{|\alpha|, |\beta|, |\delta|, |\epsilon|, |\lambda|, |\mu| |\nu|, |\omega| \geq 1} \frac{1}{\alpha \beta} \mu(\alpha) \mu(\beta) \left\{ V_{1, [\alpha, \beta]}(a, c, y, z, t, w; B) + O(1) \right\} \]

\[
= \sum_{|a|, |c| \geq 0} \sum_{|\alpha|, |\beta|, |\delta|, |\epsilon|, |\lambda|, |\mu| |\nu|, |\omega| \geq 1} \frac{1}{\alpha \beta} \mu(\alpha) \mu(\beta) \left( \frac{1}{1} V_{1, 1}(a, c, y, z, t, w; B) + O(1) \right) \]
we obtain the estimate
\[
\sim \sum_{|a|,|c| \geq 0} \sum_{\alpha,\beta > 0 \atop \gcd(a,c) = \gcd(y,w) = 1} \frac{\mu(\alpha) \mu(\beta)}{[\alpha,\beta]} V_{1,1}(a, c, y, z, t, w; B) + O(B^{\frac{n+1}{n+3}})
\]
for any \( \epsilon > 0 \), where we have used the fact that \( H(0, 0, \ldots) \leq B \) implies
\[
\max \{|a|,|c|\} |n^2+n| |z| t |2n+2| |w|^{n+3} \leq B
\]
to obtain the estimate
\[
\sum_{a,c,y,z,t,w \atop |a|=|c|, |y|, |z|, |t|, |w| \geq 1 \atop \gcd(a,c) = \gcd(y,w) = 1} 1 \ll \sum_{a,c,y,z,t} 2^{\omega(z) + \omega(w)} B^{1/(n+3)} \log B \max \{|a|,|c|\}^{(n+5)/(n+3)} y^{(n+1)/(n+3)} |zt|^{(2n+2)/(n+3)}
\]
\[
\ll \sum_{a,c} \mu(\alpha) \mu(\beta) \mu(\gamma) \alpha,\beta,\gamma > 0 \atop |a|,|c|, |y|, |z|, |t|, |w| \geq 1 \atop \gcd(a,c) = \gcd(y,z,t) = 1} \frac{B^{1/(n+1)} \log B \max \{|a|,|c|\}^{(n+3n)/(n+3)}}{B^{1/(n+1)} \log B \max \{|a|,|c|\}^{(n+3n)/(n+3)} \ll B^{\frac{n}{n+1}} (\log B)^2
\]

We now apply a Möbius inversion to the condition \( \gcd(y,w) = 1 \), so that, after using the transformation \( w \mapsto [\beta, \gamma]w \), we obtain that 16 \( \cdot \) \( N_{U,H}(B^{\frac{n}{n+1}}) \) is equal to
\[
\sum_{|a|,|c| \geq 0} \sum_{\alpha,\beta,\gamma > 0 \atop |a|,|c|, |y|, |z|, |t|, |w| \geq 1 \atop \gcd(a,c) = \gcd(y,z,t) = 1} \frac{\mu(\alpha) \mu(\beta) \mu(\gamma)}{[\alpha,\beta]} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) + O(B^{\frac{n}{n+1}} + \epsilon).
\]
Replacing the sum over \( w \) by an integral, we obtain
\[
\sum_{|w| \geq 1} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) = \int_{|w| \geq 1} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) \, dw + O(R_1),
\]
where
\[
R_1 = \max_{w} V_{1,1}(a, c, y, z, t, [\beta, \gamma]w; B) \ll \frac{B^{1/(n+3)}}{\max \{|a|,|c|\}^{(n+5)/(n+3)} y^{2/(n+3)}},
\]
Using the fact that \( H(0, 0, [\beta, \gamma]w, \ldots) \leq B \) and \( |w| \geq 1 \) imply
\[
\max \{|a|,|c|\}^{n^2+n+1} |z| |n+1| |\beta|^{n+3} |t|^{n^2+3n+2} \leq B,
\]
\[
|y|^{n^2+2n+1} |z|^{n^2+3n+2} |\beta|^{n+3} |t|^{n+1} \leq B,
\]
we obtain the estimate
\[
\sum_{a,c,y,z,t} R_1 \ll \sum_{a,c,y,z,t} \sum_{\alpha,\beta} \frac{2^{\omega(z) + \omega(w)} B^{1/(n+3)}}{\beta \max \{|a|,|c|\}^{(n+5)/(n+3)} y^{2/(n+3)}}
\]
\[
\ll \sum_{z,t} \sum_{\beta} \frac{2^{\omega(z) + \omega(w)} B^{1+2/(n+1)/(n+3)} \log B}{\beta \max \{|a|,|c|\}^{(n+5)/(n+3)} |zt|} \ll B^{\frac{1}{n+3}} (\log B)^4
\]
Hence we obtain that 16 \( \cdot \) \( N_{U,H}(B^{\frac{n}{n+1}}) \) is equal to
\[
\sum_{|a|,|c| \geq 0} \sum_{\alpha,\beta,\gamma > 0 \atop |a|=|c|, |y|, |z|, |t|, |w| \geq 1 \atop \gcd(a,c) = 1 \atop \gcd(y,z,t) = 1} \frac{\mu(\alpha) \mu(\beta) \mu(\gamma)}{[\alpha,\beta]} \int_{|w| \geq |\beta, \gamma|} V_{1,1}(a, c, y, z, t, w; B) \, dw + O(B^{\frac{n}{n+1}} + \epsilon)
\]
Removing the condition $|w| \geq \lfloor \beta, \gamma \rfloor$, we obtain
\[
\int_{|w| \geq \lfloor \beta, \gamma \rfloor} V_{1,1}(a, c, y, z, t, w; B) \, dw = V_2(a, c, y, z, t; B) + O(R_2),
\]
where, using the geometric mean of the conditions
\[
\max\{|a|, |c|\} \leq \frac{2^{2n-1}}{3} |y|^{\frac{2^{2n}}{3}} |z|^{\frac{2^{2n+1}}{2}} |w|^n \leq B,
\]
(see by \(A(0, 0, \ldots) \leq B\)) with weight \(2 - \delta = (\frac{2}{n+1} + \epsilon(n + 3))\) and \(|w| \leq \lfloor \beta, \gamma \rfloor\) with weight \(1 - \delta\),
\[
R_2 = \int_{|w| \leq \lfloor \beta, \gamma \rfloor} V_{1,1}(a, c, y, z, t, w; B) \, dw
\]
\[
\ll \frac{\lfloor \beta, \gamma \rfloor^{1-\delta} B^{1/(n+1)+\epsilon}}{\max\{|a|, |c|\} \max\{|ay|, |cy|\}^{1+\epsilon(n^2+2n+1)/2} |zt|^{1+\epsilon(n^2+4n+3)/2}}
\]
for every sufficiently small \(\epsilon > 0\). Summing \(R_2\) over the remaining variables gives the error term
\[
\sum_{a, \beta, c, y, z, t} \sum_{\substack{|a|, |c| \geq 0 \cr \gcd(a, c) = \gcd(y, z, t) = 1 \cr \alpha|z}} \frac{\mu(a)\mu(\beta)\mu(\gamma)}{[\alpha, \beta, c]} V_2(a, c, y, z, t; B) + O_e(B^{\frac{1}{n+1}+\epsilon})
\]
\[
\ll \epsilon \sum_{\beta} \frac{B^{1/(n+1)+\epsilon}}{\beta^{1+\delta}} = B^{\frac{1}{n+1}+\epsilon}.
\]

Hence \(16 \cdot N_{U, H'}(B^{\frac{1}{n+1}})\) is equal to
\[
\sum_{|a|, |c| \geq 0 \atop \alpha|z} \sum_{\substack{|a|, |c| \geq 0 \atop \gcd(a, c) = \gcd(y, z, t) = 1 \atop \alpha|z}} \frac{\mu(a)\mu(\beta)\mu(\gamma)}{[\alpha, \beta, c]} V_2(a, c, y, z, t; B) + O_e(B^{\frac{1}{n+1}+\epsilon})
\]
\[
= \sum_{|a|, |c| \geq 0 \atop \alpha|z} \sum_{\substack{|a|, |c| \geq 0 \atop \gcd(a, c) = \gcd(y, z, t) = 1 \atop \alpha|z}} \frac{\mu(a)\mu(\beta)\mu(\gamma)}{[\alpha, \beta, c]} V_2(a, c, y, z, t; B) + O_e(B^{\frac{1}{n+1}+\epsilon}),
\]
where we have applied the transformation
\((b, w) \mapsto B^{\frac{1}{n+1}}(b, w)\) or \((d, w) \mapsto B^{\frac{1}{n+1}}(d, w)\)
inside the integral.

Next, we apply the transformation,
\((at, ct, yz, z, t) \mapsto (\hat{a}, \hat{c}, \hat{y}, z, t)\)
and then
\((b, w) \mapsto ((zt)^{\frac{1}{n+1}}b, (zt)^{\frac{n-1}{n+1}}w)\) or \((d, w) \mapsto ((zt)^{\frac{1}{n+1}}d, (zt)^{\frac{n-1}{n+1}}w)\)
inside the integral to obtain that \(16 \cdot N_{U, H'}(B^{\frac{1}{n+1}})\) is equal to
\[
\sum_{|a|, |c| \geq 0 \atop \alpha|z} \sum_{\substack{|a|, |c| \geq 0 \atop \gcd(\hat{a}, \hat{c}) = \gcd(\hat{a}, \hat{c}, t) = 1 \atop \alpha|z}} \frac{\mu(a)\mu(\beta)\mu(\gamma)}{[\alpha, \beta, c]} V_2(\hat{a}/t, \hat{c}/t, \hat{y}/z; z, t; B) + O_e(B^{\frac{1}{n+1}+\epsilon})
\]
\[
= \sum_{|a|, |c| \geq 0 \atop \alpha|z} \sum_{\substack{|a|, |c| \geq 0 \atop \gcd(\hat{a}, \hat{c}, \hat{y}) = 1 \atop \alpha|z}} \theta(\hat{a}, \hat{c}, \hat{y}) B^{\frac{1}{n+1}} + O_e(B^{\frac{1}{n+1}+\epsilon}),
\]
where
\[ \vartheta(\hat{a}, \hat{c}, \hat{y}) := \sum_{|z|, |t| \geq 1 \atop |t| = \gcd(\hat{a}, \hat{c}) \atop \alpha, \beta, \gamma > 0} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha, \beta][\beta, \gamma]} |z|^{-\frac{n+1}{n+3}} |t|^{-\frac{1}{n+3}} V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \]
\[ = 2|\gcd(\hat{a}, \hat{c})|^{\frac{1}{n+3}} \sum_{|z|, |t| \geq 1 \atop |t| = \gcd(\hat{a}, \hat{c}) \atop \alpha, \beta, \gamma > 0} \frac{\mu(\alpha)\mu(\beta)\mu(\gamma)}{[\alpha, \beta][\beta, \gamma]} |z|^{-\frac{n+1}{n+3}} V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \]
\[ = 4\omega_{\infty,(\hat{a}, \hat{c}, \hat{y})} \prod_{p \text{ prime}} \lambda_p \omega_p, (\hat{a}, \hat{c}, \hat{y}). \]

In total,
\[ N_{U', H'}(B^{\frac{1}{n+1}}) = \sum_{|\hat{a}|, |\hat{c}| \geq 1 \atop |\hat{y}| \geq 1 \atop \gcd(\hat{a}, \hat{c}, \hat{y}) = 1 \atop \hat{a} \neq (0, 0)} \left( 1 + \frac{1}{2} \omega_{\infty,x} \prod_{p \text{ prime}} \lambda_p \omega_p, x \right) \left( 1 + \frac{1}{2} \omega_{\infty,x} \prod_{p \text{ prime}} \lambda_p \omega_p, x \right) B^{\frac{2}{n+1}} + O_x(B^{\frac{1}{n+1} + \epsilon}), \]
as predicted in Section 8.

**Remark 8.2.** We have omitted the details of the calculation of \( \vartheta(\hat{a}, \hat{c}, \hat{y}) \) since, according to [Pey95, Corollaire 6.2.18], Manin’s conjecture is true with Peyre’s constant for all heights on \( P_1^1(\mathbb{Q}) \cong \phi^{-1}(x) \) and hence it follows that \( \vartheta(\hat{a}, \hat{c}, \hat{y}) \) is equal to \( 2\epsilon(\hat{a}, \hat{c}, \hat{y}) \).

**Proposition 8.3.** We have
\[ \sum_{x \in \mathbb{P}^2(\mathbb{Q}) \setminus (V(\hat{a}, \hat{c}) \cup V(\hat{y}))} \epsilon_x < \infty. \]

**Proof.** In the case \( \hat{a} \neq 0 \), the condition \( \max |\mathcal{M}_a(\hat{a}, \hat{b}, \hat{c}, (\hat{b} + \hat{y}^n)w)/\hat{a}, \hat{y}, 1, 1, w)| \leq 1 \) implies
\[ \|b\|^{n+1}\|\hat{a}\|^{2n+2}\|w\|^2 \leq 1 \text{ and } \|\hat{a}\|^{n^2+2n+1}\|w\|^{n+3} \leq 1, \]
hence we obtain
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \leq \int \int \frac{1}{|\hat{a}|} \, db \, dw \ll \int \int \frac{1}{|\hat{a}|^{3}|w|^{2/(n+1)}} \, dw \ll \frac{1}{|\hat{a}|^{3+(n^2-1)/(n+3)}}. \]
Similarly, we obtain
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \leq \frac{1}{|\hat{c}|^{3+(n^2-1)/(n+3)}}. \]
The condition \( \max |\mathcal{M}_a(\hat{a}, \hat{b}, \hat{c}, (\hat{b} + \hat{y}^n)w)/\hat{a}, \hat{y}, 1, 1, w)| \leq 1 \) also implies
\[ \|b\|^{n+1}\|\hat{c}\|^{2}\|\hat{y}\|^2 \leq 1 \text{ and } \|\hat{b} + \hat{y}^nw\|/\hat{a}^{n+3}\|\hat{a}\|^{2}\|\hat{y}\|^2 \leq 1, \]
hence we obtain
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1, 1; 1) \leq \int \int \frac{1}{|\hat{a}|} \, db \, dw = \int \int \frac{1}{\max\{|b|, |w|\}^{n^2+3}|\hat{a}|^2|\hat{y}|^n} \, db \, dw \]
\[ \ll \frac{1}{|\hat{y}|^n} \cdot \frac{1}{|\hat{a}|^{3+(n^2-1)/(n+3)}}. \]
Similarly, we obtain
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1; 1) \ll \frac{1}{[\hat{c}]^{4/(n+3)}[\hat{y}]^{n+4/(n+3)}}. \]

Together, we obtain
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1; 1) \ll \frac{1}{\max \{|\hat{a}|, |\hat{c}|\}^{4/(n+3)} \max \{|\hat{a}|, |\hat{c}|, |\hat{y}|\}^{(n^2+3n+4)/(n+3)}}. \]

There exist \( \lambda_1, \lambda_2 > 0 \) with
\[ \lambda_1 > \frac{2n + 2}{n^2 + 3n + 4}, \quad \lambda_2 > \frac{n + 3}{n^2 + 3n + 4}, \quad \lambda_1 + \lambda_2 = 1, \]
so that using \( \max \{|\hat{a}|, |\hat{c}|, |\hat{y}|\} \geq \max \{|\hat{a}|, |\hat{c}|\} \lambda_1 |\hat{y}| \lambda_2 \), we obtain that there exists \( \mu > 0 \) such that
\[ V_2(\hat{a}, \hat{c}, \hat{y}, 1; 1) \ll \frac{1}{\max \{|\hat{a}|, |\hat{c}|\}^{2+\mu} |\hat{y}|^{1+\mu}}. \]

With \( \vartheta(\hat{a}, \hat{c}, \hat{y}) \) from the proof of Theorem 8.1, we have
\[ \sum_{\hat{a}, \hat{c}, \hat{y} \geq 0 \atop \langle \hat{a}, \hat{c}, \hat{y} \rangle \neq (0,0)} \vartheta(\hat{a}, \hat{c}, \hat{y}) = \sum_{\hat{a}, \hat{c}} \gcd(\hat{a}, \hat{c}) \frac{\max \{\hat{a}, \hat{c}\}^{2+\mu} \sum_{\hat{y}} 4^{\omega(\hat{y})} d(\hat{y})}{\hat{y}^{1+\mu}}. \]

Our aim is to show that this sum converges. We have
\[
\sum_{\hat{a}, \hat{c} \leq M} \frac{\gcd(\hat{a}, \hat{c})}{\max \{\hat{a}, \hat{c}\}^{2+\mu}} \sum_{\hat{a} \leq \hat{c}} \frac{1}{\hat{c}^{1+\mu}} \sum_{\hat{a} \leq \hat{c}} \frac{\gcd(\hat{a}, \hat{c})}{\hat{c}^{1+\mu}} = \sum_{\hat{c} \leq M} \frac{1}{\hat{c}^{1+\mu}} \sum_{d \mid \hat{c}} \vartheta(\hat{c})
\ll \sum_{\hat{c} \leq M} \frac{d(\hat{c})}{\hat{c}^{1+\mu}} \ll \sum_{\hat{c} \leq M} \frac{d(\hat{c})}{M^{1+\mu}} + \int_1^M \sum_{\hat{c} \leq \lambda} \frac{d(\hat{c})}{\lambda^{2+\mu}} d\lambda
\ll \frac{\log M}{M^\mu} + \int_1^M \frac{\log \lambda}{\lambda^{1+\mu}} d\lambda \ll 1.
\]

Note that we have
\[
\sum_{\hat{y} \leq M} 4^{\omega(\hat{y})} d(\hat{y}) = \sum_{\hat{y} \leq M} \sum_{\hat{y} \leq y} 4^{\omega(\hat{y})} \ll \sum_{y, z \geq 1} \sum_{y \leq M} \sum_{z \leq M/y} 4^{\omega(z)} \ll \sum_{y \leq M} \frac{4^{\omega(y)} M (\log M)^3}{y} \ll M (\log M)^7.
\]

It follows that we have
\[
\sum_{\hat{y} \leq M} \frac{4^{\omega(\hat{y})} d(\hat{y})}{\hat{y}^{1+\mu}} = \sum_{\hat{y} \leq M} \frac{4^{\omega(\hat{y})} d(\hat{y})}{M^{1+\mu}} + \int_1^M \sum_{\hat{y} \leq \lambda} \frac{4^{\omega(\hat{y})} d(\hat{y})}{\lambda^{2+\mu}} d\lambda
\ll \frac{(\log M)^7}{M^\mu} + \int_1^M \frac{(\log M)^7}{M^{1+\mu}} d\lambda \ll 1.
\]
\[ \square \]
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