Linear Response of Thin Superconductors in Perpendicular Magnetic Fields: An Asymptotic Analysis

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Abstract

The linear response of a thin superconducting strip subjected to an applied perpendicular time-dependent magnetic field is treated analytically using the method of matched asymptotic expansions. The calculation of the induced current density is divided into two parts: an “outer” problem, in the middle of the strip, which can be solved using conformal mapping; and an “inner” problem near each of the two edges, which can be solved using the Wiener-Hopf method. The inner and outer solutions are matched together to produce a solution which is uniformly valid across the entire strip, in the limit that the effective screening length $\lambda_{\text{eff}}$ is small compared to the strip width $2a$. From the current density it is shown that the perpendicular component of the magnetic field inside the strip has a weak logarithmic singularity at the edges of the strip. The linear Ohmic response, which would be realized in a type-II superconductor in the flux-flow regime, is calculated for both a sudden jump in the magnetic field and for an ac magnetic field. After a jump in the field the current propagates in from the edges at a constant velocity $v = 0.772D/d$ (with $D$ the diffusion constant and $d$ the film thickness), rather than diffusively, as it would for a thick sample. The ac current density and the high frequency ac magnetization are also calculated. The long time relaxation
of the current density after a jump in the field is found to decay exponentially with a time constant $\tau_0 = 0.255 aD/D$. The method is extended to treat the response of thin superconducting disks, and thin strips with an applied current. There is generally excellent agreement between the results of the asymptotic analysis and the recent numerical calculations by E. H. Brandt [Phys. Rev. B 49, 9024; 50, 4034 (1994)].

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I. INTRODUCTION

When a thin superconductor is placed in a perpendicular magnetic field there are large demagnetizing fields which produce an enhanced response to the applied field. For instance, the induced magnetic moment in the perpendicular geometry for a sample with thickness $d$ and a width $a \gg d$ is $O(a/d)$, while in for a longitudinal magnetic field it is only $O(1)$ \cite{1}. These demagnetizing fields are important not only for static properties, but also for dynamic properties, such as the response of the sample to an applied time-dependent magnetic field. Surprisingly, there has been little theoretical work on this subject until recently. Brandt has devised an efficient numerical method for calculating the linear or nonlinear response of superconducting strips \cite{1} or disks \cite{2} in time-dependent magnetic fields. His studies have unveiled a number of interesting properties of the sheet current and magnetic moment in these geometries, including dynamic scaling properties of the current density at the sample edges at high frequencies or short times.

The present paper is an analytic treatment of the linear electrodynamics of thin superconducting strips and disks in perpendicular magnetic fields, and as such is complementary to Brandt’s numerical work. Many of the features of the linear response which were extracted numerically by Brandt emerge naturally from this analysis; the analytic results obtained here agree quite well with Brandt’s numerical results. The calculational technique employed here is the method of matched asymptotic expansions \cite{3,4}, a technique originally devised to treat boundary layer problems in fluid mechanics \cite{4}, and used recently to study several interesting problems in nonequilibrium and inhomogeneous superconductivity \cite{5,6}. The idea is to split the problem into two pieces; an “outer problem” in the middle of the strip (which is straightforward to solve), and an “inner problem” near each of the two edges of the strip (which requires a bit more ingenuity). The solutions are then matched together, and a “uniform” solution, valid across the entire strip, is constructed. The time dependence of the applied field is treated using Laplace transforms, which provides a unified framework for treating the transient response after the applied field is suddenly switched on, as well as
the steady-state ac response. It should be noted that there is an allusion to such a matching procedure for the same problem in a paper by Larkin and Ovchinnikov [7]; these authors simply quote a result for the behavior of the current near the edge of a strip in a perpendicular field. The present work goes well beyond that of Larkin and Ovchinnikov, by explicitly constructing the inner, outer, and uniform solutions, and using these solutions to study the nonequilibrium response. It also appears that the result quoted by Larkin and Ovchinnikov is incorrect in detail (see Appendix B).

As this paper is somewhat long, the primary results are collected in Table I. The organization of the paper is as follows. In Sec. II the integro-differential equation for the current density in the strip is derived, and a small parameter $\epsilon = 2\lambda_{\text{eff}}/a$ is identified, with $\lambda_{\text{eff}}$ the effective penetration depth and $2a$ the strip width. Sec. III treats the large-$\epsilon$ limit, which is helpful for anticipating some of the features of the small-$\epsilon$ solution. The asymptotic analysis for small-$\epsilon$ is constructed in Sec. IV; the outer problem is essentially solved by conformal mapping, while the inner problem is solved using the Wiener-Hopf method. The solutions are matched using formal asymptotic matching, and the uniform solution is constructed. The uniform solution is used to study the perpendicular magnetic field in Sec. V, and the current density and magnetization in Sec. VI. In Sec. VII the method is extended to treat the response of thin superconducting disks, and thin strips in the presence of an applied current. In Sec. VIII the long-time behavior is studied via an eigenfunction expansion of the current density, and the fundamental relaxation time is calculated using a variation on the analysis of the previous sections. The results are summarized in Sec. IX. Some of the more algebra intensive parts of the calculation are relegated to Appendices A–C.

**II. DERIVATION OF THE INTEGRO-DIFFERENTIAL EQUATION**

To begin, we will derive the integro-differential equation which determines the current distribution in the strip. The geometry is illustrated in Fig. I. The strip is in the $x - y$ plane, being infinite in the $y$-direction, and having a width $2a$ such that the strip occupies
the region $-a < x < a$. The applied field $\mathbf{H}_a(t) = H_a(t)\hat{z}$ is normal to the strip and in the $z$-direction. The vector potential $\mathbf{A}$ satisfies

$$- \nabla^2 \mathbf{A} = 4\pi \mathbf{J}, \quad (2.1)$$

in the transverse gauge in which $\nabla \cdot \mathbf{A} = 0$. We will focus here on situations in which the current density is invariant along the $y$-direction (along the length of the strip), so that both $\mathbf{J}$ and $\mathbf{A}$ are along the $y$-direction. In the thin film approximation, the current density $J_y(x, z, t)$ is averaged over the thickness $d$ of the film; the averaged current will be denoted by $j(x, t)$, so we have

$$J_y(x, z, t) = d \frac{d}{dx} \delta(z) \Theta(a^2 - x^2). \quad (2.2)$$

Eq. (2.1) is solved by introducing the Green’s function for the two dimensional Laplacian, $G(x - x', z - z')$:

$$A_y(x, z, t) = A_{0,y} - 4\pi \int G(x - x', z - z') J_y(x, z, t) \, dx' \, dz'$$

$$= A_{0,y} - 4\pi d \int_{-a}^{a} G(x - x', z) j(x', t) \, dx'. \quad (2.3)$$

Differentiating both sides with respect to $x$, and using $\partial A_{0,y}/\partial x = H_a(t)$, we obtain

$$H_z(x, z, t) = \frac{\partial A_y(x, z, t)}{\partial x}$$

$$= H_a(t) - 4\pi d \int_{-a}^{a} \frac{\partial G(x - x', z)}{\partial x} j(x', t) \, dx'. \quad (2.4)$$

Finally, if we specialize Eq. (2.4) to $z = 0$, and use $\partial G(x - x', 0)/\partial x = 1/2\pi(x - x')$, we obtain for the magnetic field normal to the strip,

$$H_z(x, t) = H_a(t) + 2d(P) \int_{-a}^{a} \frac{j(x', t)}{x' - x} \, dx', \quad (2.5)$$

where $(P)$ indicates a principle value integral.

To complete the description, we require a constitutive relation between the averaged current and the fields. This paper will concentrate on the linear response of the current, so that the general time-dependent response is
\[ j(x, t) = \int_{-\infty}^{t} \sigma(t - t') E_y(x, t') \, dt', \quad (2.6) \]

with \( \sigma \) the conductivity. In most of this paper we shall be interested in the solution of the initial value problem; i.e., the time evolution of the sheet current after an applied current has been switched on. Then it is natural to Laplace transform the currents and the fields with respect to time \( t \):

\[ j(x, s) = \int_{0}^{\infty} e^{-st} j(x, t) \, dt, \quad (2.7) \]

and so on. The inverse Laplace transform is

\[ j(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} j(x, s) \, ds, \quad (2.8) \]

with the integration contour chosen to pass to the right of any singularities. After Laplace transforming, Eq. (2.10) becomes

\[ j(x, s) = \sigma(s) E_y(x, s) = -s \sigma(s) A_y(x, s), \quad (2.9) \]

where \( E_y(x, t) = -\partial A_y(x, t)/\partial t \) and an integration by parts has been used to obtain the last line. Now Laplace transform Eq. (2.5) with respect to time, and use Eq. (2.9) to eliminate the fields in favor of the currents, to obtain the following equation of motion for the current:

\[ -4\pi \lambda_{\text{eff}}(s) d \frac{\partial j(x, s)}{\partial x} = H_a(s) + 2d (P) \int_{-a}^{a} \frac{j(x', s)}{x' - x} \, dx', \quad (2.10) \]

where \( \lambda_{\text{eff}}(s) \) is an effective screening length defined by

\[ \lambda_{\text{eff}}(s) = \frac{1}{4\pi d s \sigma(s)}, \quad (2.11) \]

For a superconductor, \( \sigma(s) = 1/4\pi \lambda^2 s \), with \( \lambda \) the London penetration depth, so that in this case \( \lambda_{\text{eff}} = \lambda^2 / d \). For an Ohmic conductor, \( \lambda_{\text{eff}} = 1/4\pi \sigma(0) ds = D / ds \), with \( D \) the diffusion constant for the magnetic flux. An Ohmic response would be realized in a normal metal or in a type-II superconductor in the flux-flow regime. Eq. (2.10) is essentially identical to the
equations of motion derived by Larkin and Ovchinnikov [7], Eq. (33), and Brandt [1], Eq. (3.6) the only difference being that Laplace rather than Fourier transforms are being used here. This equation of motion can be conveniently expressed in dimensionless variables by writing \( x' = x/a \), \( f(x', s) = j(x, s)/(H_a(s)/2\pi d) \), and \( \epsilon = 2\lambda_{\text{eff}}(s)/a \):

\[
- \epsilon f'(x, s) = 1 + \frac{1}{\pi} \left( P \right) \int_{-1}^{1} \frac{f(x', s)}{x' - x} \, dx'
\]  

(2.12)

(the \( s \) dependence of \( \epsilon \) will be suppressed for notational simplicity; it will be reinstated later when the transforms are inverted). The current density is the primary quantity of interest in this paper. A related, and experimentally accessible, quantity is the magnetic moment per unit length of the strip [1]:

\[
M(s) = \int_{-a}^{a} x \, j(x, s) \, dx = -\frac{a^2 H_a}{4d} m(s)
\]  

(2.13)

where \( m(s) \) is a dimensionless moment defined as

\[
m(s) = -\frac{2h(s)}{\pi} \int_{-1}^{1} x f(x, s) \, dx,
\]  

(2.14)

where \( h(s) \) is defined through \( H_a(s) = H_a h(s) \).

There are no known analytical solutions of Eq. (2.12); its solution is the subject of the remainder of this paper. However, for a typical sample \( \epsilon \ll 1 \), in which case the left hand side of Eq. (2.12) constitutes a singular perturbation (the highest derivative is multiplied by the small parameter), and we can bring to bear all of the techniques of asymptotic analysis to solve this problem [4]. Before attempting this, we will first develop a perturbative analysis which is valid for \( \epsilon \gg 1 \), which is physically less interesting but mathematically simpler.

III. EXPANSION FOR LARGE \( \epsilon \)

To study the behavior of Eq. (2.12) for large \( \epsilon \), it is useful to first rescale by introducing a new function \( g(x, s) = \epsilon f(x, s) \). We then expand \( g(x, s) \) in powers of \( \epsilon^{-1} \):
\[ g(x, s) \sim g_0(x, s) + \epsilon^{-1} g_1(x, s) + \ldots \] 

Substituting into Eq. (2.12) and matching terms of the same order, we have

\[ g'_0(x, s) = -1, \tag{3.2} \]

\[ g'_1(x, s) = -\frac{1}{\pi}(P) \int_{-1}^{1} \frac{g_0(x', s)}{x' - x} dx', \tag{3.3} \]

and so on with the higher order terms. Assuming that there is no net current in the strip (i.e., no applied current), then the current is odd in \( x \), so that \( g(0) = 0 \), and we have

\[ g_0(x, s) = -x, \tag{3.4} \]

\[ g_1(x, s) = \frac{1}{\pi} \left[ x + \frac{1 - x^2}{2} \ln \left( \frac{1 + x}{1 - x} \right) \right]. \tag{3.5} \]

The current increases linearly across the sample, except near the edges. Near the left edge \( (x = -1) \), we have

\[ g(x, s) \sim 1 - \frac{1}{\pi \epsilon} \left[ 1 + (1 + x) \ln(2/(1 + x)) \right]. \tag{3.6} \]

As we shall see below, a similar behavior near the edge will also emerge in the small-\( \epsilon \) limit. From the expansion for \( f(x, s) \) we can calculate the magnetic moment from Eq. (2.14):

\[ m(s) = h(s) \left[ 4 \frac{1}{3 \pi \epsilon(s)} - \frac{2}{(\pi \epsilon(s))^2} + O(\epsilon^{-3}) \right]. \tag{3.7} \]

For an Ohmic conductor, \( \epsilon(s) = 2D/ads \), so the magnetic moment vanishes linearly with frequency at low frequency. This result is essentially equivalent to Eq. (5.6) of Ref. [1].

IV. ASYMPTOTIC ANALYSIS FOR SMALL \( \epsilon \)

We now turn to the solution of Eq. (2.12) for small \( \epsilon \). An asymptotic solution can be obtained using the method of matched asymptotic expansions. We break up the strip into an “outer region,” which is the interior of the strip, and two “inner regions,” one near each edge. This is illustrated schematically in Fig. 2. The solutions are then matched in common overlap regions. A detailed discussion of the method can be found in Ref. [4].
A. Outer solution

The expansion in the outer region (away from the edges) is obtained by expanding $f$ as a series in $\epsilon$:

$$f(x; \epsilon) \sim f_0(x) + \epsilon f_1(x) + \ldots.$$  \hfill (4.1)

Substituting this expansion into Eq. (2.12) and collecting terms of the same order, we find for $f_0(x)$

$$\frac{1}{\pi} (P) \int_{-1}^{1} \frac{f_0(x')}{x' - x} \, dx' = -1.$$  \hfill (4.2)

The solution of this singular integral equation which is odd is $x$ can be found in Ref. [8]:

$$f_0(x) = -\frac{x}{(1 - x^2)^{1/2}},$$  \hfill (4.3)

which coincides with the usual solution obtained from conformal mapping techniques. Near the edges at $x = \pm 1$, $f_0$ behaves as

$$f_0 \sim \pm \frac{1}{[2(1 \mp x)]^{1/2}},$$  \hfill (4.4)

so the current has a square root divergence at the edges. This is due to the fact that the outer solution corresponds to complete screening of the applied field, which can only be achieved by having an infinite current density at the edges. The outer solution therefore breaks down at distances of order $\epsilon$ of the edges; the current at the edges is thus of order $\epsilon^{-1/2}$. In order to remedy this problem, we proceed to the solution of the inner problem near each of the edges.

B. Inner solution

We will first study the inner problem at the left edge, $x = -1$. The outer solution breaks down at $x + 1 \sim \epsilon$, suggesting that the appropriate variable in the inner region is

$$X = (x + 1)/\epsilon.$$  \hfill (4.5)
Also, since \( f(-1) \sim \epsilon^{-1/2} \), we rescale \( f(x) \) in the inner region as

\[
F(X) = \epsilon^{1/2} f(x),
\]

so that \( F(0) = O(1) \). In terms of these inner variables, Eq. (2.12) becomes

\[
F' = -\epsilon^{1/2} - \frac{1}{\pi}(P) \int_0^{2/\epsilon} \frac{F(X')}{X' - X} dX'.
\]

Next, expand \( F(X; \epsilon) \) in powers of \( \epsilon^{1/2} \), as suggested by the rescaled form of the integro-differential equation:

\[
F(X; \epsilon) \sim F_0(X) + \epsilon^{1/2} F_1(X) + \ldots.
\]

The lowest order term satisfies

\[
F'_0(X) = -\frac{1}{\pi}(P) \int_0^{2/\epsilon} \frac{F_0(X')}{X' - X} dX'
\]

\[
= -\frac{1}{\pi}(P) \int_0^\infty \frac{F_0(X')}{X' - X} dX' + \frac{1}{\pi}(P) \int_{2/\epsilon}^\infty \frac{F_0(X')}{X' - X} dX'.
\]

For small \( \epsilon \) the second integral will be dominated by the large \( X \) behavior of \( F_0(X) \); it will be shown below that in order to match onto the outer solution this is necessarily of the form \( F_0(X) \sim (2X)^{-1/2} \). Therefore, we see that the second integral is of order \( \epsilon^{1/2} \), and can be dropped at this order of the calculation. Our final integral equation for \( F_0(X) \) is then

\[
F'_0(X) = -\frac{1}{\pi}(P) \int_0^\infty \frac{F_0(X')}{X' - X} dX'.
\]

The problem in the inner region consists of solving a homogeneous integro-differential equation on a semi-infinite interval. This is equivalent to finding the current distribution in a semi-infinite strip in zero applied magnetic field.

Eq. (4.10) can be solved using the Wiener-Hopf method [8], as follows. The function \( F_0(X) \rightarrow 0 \) as \( X \rightarrow \infty \), and \( F_0(X) = 0 \) for \( X < 0 \); introduce a second unknown function \( G(X) \) such that \( G(X) = 0 \) for \( X > 0 \) and \( G(X) \rightarrow 0 \) as \( X \rightarrow -\infty \). We then introduce the complex Fourier transforms of \( F_0(X) \) and \( G(X) \),

\[
\Phi_+(k) = \int_0^\infty F_0(X) e^{ikX} dX,
\]

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\[ G_-(k) = \int_{-\infty}^{0} G(X)e^{ikX}dX, \]  

(4.12)

such that \( \Phi_+(k) \) is analytic for \( \text{Im}(k) > -\beta \) and \( G_-(k) \) is analytic for \( \text{Im}(k) < \alpha \), for some \( \alpha > \beta \). Then Fourier transforming Eq. (4.10), and integrating by parts, we obtain

\[ G_-(k) - F_0(0) - ik\Phi_+(k) = isgn(k)\Phi_+(k). \]  

(4.13)

We see that as \( k \to \infty \),

\[ \Phi_+(k) \sim -\frac{F_0(0)}{ik} + O(k^{-2}). \]  

(4.14)

To take care of the ambiguities in defining \( sgn(k) \), replace it by \( k/(k^2 + \delta^2)^{1/2} \), with the real part > 0 for \( \text{Re}(k) > 0 \), and choose the branch cuts to run between \((-i\infty, -i\delta)\) and \((i\delta, i\infty)\). We can then take \( \delta \to 0 \) at some later point in the calculation. Rearranging Eq. (4.13) a bit, we have

\[ ikK(k)i\Phi_+(k) = G_-(k) - F_0(0), \]  

(4.15)

where

\[ K(k) = 1 + \frac{1}{(k^2 + \delta^2)^{1/2}}. \]  

(4.16)

Now, if we can factor \( K(k) \) into the form \( K(k) = K_+(k)/K_-(k) \), with \( K_+(k) \) analytic for \( \text{Im}(k) > -\delta \) and \( K_-(k) \) analytic for \( \text{Im}(k) < \delta \), then we may rewrite Eq. (4.13) as

\[ ikK_+(k)i\Phi_+(k) = K_-(k)\left[G_-(k) - F_0(0)\right]. \]  

(4.17)

Both sides are now analytic in their respective regions of analyticity; we can then use analytic continuation arguments to note that both sides must then equal an entire function \( E(k) \). By examining the limiting behavior of the left hand side as \( k \to \infty \), we see that this function must be chosen to be a constant \( C \) (any positive power of \( k \) would produce non-integrable singularities in \( F_0(X) \)), so that we finally have

\[ \Phi_+(k) = \frac{C}{ikK_+(k)}. \]  

(4.18)
The function $F_0(X)$ is then obtained by inverting the Fourier transform,

$$F_0(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_+(k) e^{-ikX} dk,$$

with the integration path indented so as to pass above any singularities on the real axis.

The only remaining task is the decomposition of $K(k)$, which is carried out in Appendix A. Using Eqs. (A4) and (A5), we have

$$F_0(X) = \frac{C}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\varphi(k)}}{k(1 + 1/|k|)^{1/2}} e^{-ikX} dk = -\frac{C}{\pi} \int_0^{\infty} \frac{\sin [kX + \varphi(k)]}{(k^2 + k)^{1/2}} dk,$$

where the phase $\varphi(k)$ is given by

$$\varphi(k) = \frac{\pi}{4} + \frac{1}{\pi} \int_0^k \frac{\ln u}{1 - u^2} du.$$

The constant $C$ is determined from the matching conditions, which are discussed below.

**C. Asymptotic matching**

The inner and outer solutions may now be matched together in a suitable overlap region. This is done by expressing the outer solution $f_0(x)$ in terms of the inner variable $X = (x + 1)/\epsilon$, and then taking $X \to 0$ while holding $\epsilon$ fixed:

$$f_0(X) = -\frac{\epsilon X - 1}{[(2 - \epsilon X)\epsilon X]^{1/2}} \sim \frac{1}{(2\epsilon X)^{1/2}} \quad (X \to 0).$$

The inner solution $F_0(X)$ must match onto this outer solution as $X \to \infty$, so the asymptotic behavior of $F_0(X)$ must be

$$F_0(X) \sim \frac{1}{(2X)^{1/2}} \quad (X \to \infty).$$

Now expand Eq. (4.20) for large-$X$; the integral is dominated by the small-$k$ behavior of the integrand, and we find
\[ F_0(X) \sim -\frac{C}{(\pi X)^{1/2}}, \quad (4.24) \]

so that we have

\[ C = -\left(\frac{\pi}{2}\right)^{1/2} \]

in order to match the inner and outer solutions. Therefore our final expression for the inner solution is

\[ F_0(X) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{\sin [kX + \varphi(k)]}{(k^2 + k)^{1/2}} dk. \quad (4.26) \]

Comparing Eqs. (4.14) and (4.18), we see that

\[ F_0(0) = -C = (\pi/2)^{1/2}. \]

By rotating the integration contour, it is possible to show that for \( X < 0 \) the integral vanishes (as it should), while for \( X > 0 \) the integral may be written as

\[ F_0(X) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{e^{-Xy-g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} dy, \quad (4.27) \]

with

\[ g(y) = \frac{1}{\pi} \int_0^y \frac{\ln u}{1 + u^2} du. \quad (4.28) \]

This function has the limiting behaviors

\[ g(y) = \begin{cases} -y \ln(e/y)/\pi + O(y^3), & y \ll 1; \\ -\ln(ey)/\pi y + O(y^{-3}), & y \gg 1. \end{cases} \quad (4.29) \]

The square-root singularity in the integrand at \( y = 0 \) can be removed by changing variables to \( y = z^2 \), so that

\[ F_0(X) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \frac{e^{-Xz^2-g(z^2)}}{(z^4 + 1)^{3/4}} dz, \quad (4.30) \]

which is particularly convenient for numerical evaluation. As shown in Appendix B, for small-\( X \), \( F_0(X) \) has the expansion

\[ F_0(X) = \frac{1}{(2\pi)^{1/2}} \left[ \pi - 1.4228X + X \ln X + O(X^2) \right] \quad (4.31) \]
(which disagrees with Eq. (37) of Larkin and Ovchinnikov, Ref. [7]; see the remarks in Appendix B). Although the current is finite at $X = 0$, it has a slope which diverges as $\ln(X)$. The result of a numerical evaluation of the integral, along with a comparison of the numerical results to the asymptotic expansions, is shown in Fig. 3.

So far we have only discussed the matching procedure at the left edge of the strip $x = -1$. The same procedure can be carried out at the right edge, $x = 1$, as follows. At the right edge, the inner variable will be $\bar{X} = (1 - x)/\epsilon$. As before, we will also rescale $f(x)$ as $\bar{F}(\bar{X}) = \epsilon^{1/2} f(x)$. Substituting these expressions into Eq. (2.12), we obtain

$$\bar{F}'(\bar{X}) = \epsilon^{1/2} \frac{1}{\pi} \frac{1}{\epsilon} \int_{0}^{2/\epsilon} \frac{\bar{F}(\bar{X}')}{\bar{X}' - \bar{X}} d\bar{X'},$$

which is the same as Eq. (4.7) except for the minus sign in front of the $\epsilon^{1/2}$. Expanding in powers of $\epsilon^{1/2}$, the $O(1)$ term, $\bar{F}_0(\bar{X})$, satisfies Eq. (2.12), and the method of solution is identical. To match onto the outer solution, we write $f_0(x)$ in terms of $\bar{X}$, and then take $\bar{X} \to 0$ while holding $\epsilon$ fixed:

$$f_0(\bar{X}) = -\frac{1 - \epsilon\bar{X}}{[(2 - \epsilon\bar{X})\epsilon\bar{X}]^{1/2}} \sim -\frac{1}{(2\epsilon\bar{X})^{1/2}} \quad (\bar{X} \to 0).$$

Therefore, the asymptotic behavior of $\bar{F}_0(\bar{X})$ must be

$$\bar{F}_0(\bar{X}) \sim -\frac{1}{(2\bar{X})^{1/2}} \quad (\bar{X} \to \infty).$$

Therefore, $\bar{F}_0(\bar{X}) = -F_0(\bar{X})$, which could have been surmised from the symmetry of the problem.

**D. Uniform solution**

We are now in a position to construct an asymptotic solution which is uniformly valid across the entire width of the strip; i.e., valid for all $x$ as $\epsilon \to 0$. To do this we simply add the inner and outer solutions; however, this would produce a result which was $2f_{\text{match}}(x)$ in
the matching region, so we also need to subtract the $f_{\text{match}}(x)$ for each of the two matching regions $[1]$. The result is

$$f_{\text{unit}}(x, s) = -\frac{x}{(1-x^2)^{1/2}} + \frac{1}{[2(1-x)]^{1/2}} - \frac{1}{[2(1+x)]^{1/2}}$$

$$+ \epsilon(s)^{-1/2} \left\{ F_0[(1+x)/\epsilon(s)] - F_0[(1-x)/\epsilon(s)] \right\}. \quad (4.35)$$

The uniform solution is plotted in Fig. 4 for $\epsilon = 0.1$.

With the uniform solution we can calculate the magnetic moment, given in Eq. (2.14). The result is

$$m(s) = h(s) \left\{ 1 - \frac{8}{3\pi} + \left( \frac{2}{\pi} \right)^{3/2} \int_0^\infty \frac{e^{-\eta y}[(y+1)e^{-2y} + y - 1]}{y^{5/2}[1 + \epsilon^2 y^2]^{3/4}} dy \right\}. \quad (4.36)$$

For $\epsilon = 0$, the integral is $8/3\pi$, so that $m(s) = h(s)$, which is the ideal screening limit. Determining the leading $\epsilon$ behavior of the integral is rather subtle; the details are relegated to Appendix C. The result is

$$m(s) = h(s) \left[ 1 - \frac{6}{\pi^2} \ln \left( \frac{8e^{\gamma - 5/3}}{\epsilon} \right) \epsilon \right]. \quad (4.37)$$

This expression should give the correct high-frequency ($s \to \infty$) behavior of the magnetization. Note that for $s = 0$ ($\epsilon \to \infty$), $m(s) = 1 - 8/3\pi = 0.151$; however, we know that the correct limiting behavior is $m(0) = 0$ (see Sec. III). Therefore the uniform approximation does not reproduce the correct low frequency behavior of the magnetization for an Ohmic conductor.

**V. MAGNETIC FIELD WITHIN THE STRIP**

By using the constitutive relation, Eq. (2.9), it is also possible to calculate the magnetic field perpendicular to the strip. Using $H_z(x, s) = \partial A_y(x, s)/\partial x$, going to our dimensionless variables, and using the uniform approximation from the section above, we have

$$H_z(x, s) = -H_a(s)\epsilon(s) \frac{\partial f(x, s)}{\partial x}$$

$$= H_a(s) \left\{ \epsilon(s) \left[ -\frac{1}{(1-x^2)^{3/2}} - \frac{1}{[2(1-x)]^{3/2}} - \frac{1}{[2(1+x)]^{3/2}} \right] \right.$$  

$$-\epsilon(s)^{-1/2} \left\{ F_0[(1+x)/\epsilon(s)] + F_0[(1-x)/\epsilon(s)] \right\} \right\}. \quad (5.1)$$
The magnetic field is plotted in Fig. 5. From the results in Appendix B, close to the edges 
\((1 \pm x = O(\epsilon))\) we have
\[
F'_0(X) = -\frac{1}{(2\pi)^{1/2}} \ln(1/X) + O(1),
\]
(5.2)
so that it would appear that the field diverges logarithmically at the edges, in agreement with
the numerical work in Ref. [1]. However, as we get even closer to the edge \((1\pm x = O(\epsilon^3))\), this
log divergence is swamped by a square-root divergence from the outer and overlap terms.
This latter behavior is most likely an artifact of the approximation, and would probably
disappear in a higher-order calculation.

VI. DYNAMICS OF THE CURRENT DENSITY AND MAGNETIZATION

Having obtained a uniformly asymptotic solution to the equation of motion for the av-
eraged current density, we can now examine its evolution in the time domain by inverting
the Laplace transform for \(j(x, s)\), Eq. (2.8). The details of the inversion process will depend
upon the time dependence of the applied field, and the model chosen for \(\epsilon(s)\). Two different
models for \(\epsilon(s)\) will be considered: (1) \(\epsilon(s) = 2\lambda^2/ad\) a constant, corresponding to a super-
conductor; (2) \(\epsilon(s) = (2D/ad)(1/s)\), with \(D = 1/4\pi\sigma(0)\) the diffusion constant for flux in
the normal phase, corresponding to an Ohmic conductor (a type-II superconductor in the
flux-flow regime, for instance [1]).

A. Superconductor

For a superconductor the inversion of the Laplace transform is particularly simple. In
conventional units we have (recall that \(\lambda_{\text{eff}} = \lambda^2/d\))
\[
\dot{j}_{\text{unit}}(x, t) = \frac{H_0(t)}{2\pi d} \left\{ -\frac{x}{(1-x^2)^{1/2}} + \left[ \frac{1}{2(1-x)} \right]^{1/2} - \left[ \frac{1}{2(1+x)} \right]^{1/2} \right. \\
\left. \epsilon^{-1/2} \left[ F_0 \left( \frac{1+x}{\epsilon} \right) - F_0 \left( \frac{1-x}{\epsilon} \right) \right] \right\},
\]
(6.1)
In this case the induced current is in phase with the applied field.
B. Ohmic conductor: penetration of a jump in the applied field

We will first treat the case in which the field is suddenly switched on, so that $H_a(t) = H_a \theta(t)$, and thus $H_a(s) = H_a/s$. Inverting the Laplace transform, we have

$$j_{\text{unif}}(x,t) = \frac{H_a(t)}{2\pi d} \left\{ \frac{-x}{(1 - x^2)^{1/2}} + \left[ \frac{1}{2(1 - x)} \right]^{1/2} - \left[ \frac{1}{2(1 + x)} \right]^{1/2} \right\} + j_{\text{edge}}(1 + x,t) - j_{\text{edge}}(1 - x,t),$$

(6.2)

where the edge current $j_{\text{edge}}(\bar{x},t)$ is given by

$$j_{\text{edge}}(\bar{x},t) = \frac{H_a}{2\pi d} \left( \frac{ad}{4D} \right)^{1/2} \int_0^\infty e^{-g(y)} y^{1/2} (y^2 + 1)^{3/4} dy \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left[ st - \left( \frac{ad \bar{x}}{2D} \right) ys / (\pi s)^{1/2} \right] ds. \quad (6.3)$$

The Laplace transform can be calculated by closing the contour in the left half plane (for $t > d\bar{x}/2D$), wrapping the contour around the branch cut along the negative $s$ axis, with the result

$$j_{\text{edge}}(\bar{x},t) = \frac{H_a}{2\pi d} \left( \frac{ad}{4Dt} \right)^{1/2} \mathcal{F}_1 \left( \frac{ad \bar{x}}{2D t} \right), \quad (6.4)$$

where $\mathcal{F}_1(u)$ is a scaling function given by

$$\mathcal{F}_1(u) = \frac{1}{\pi} \int_0^{1/u} \frac{e^{-g(y)}}{(1 - uy)^{1/2} y^{1/2} (y^2 + 1)^{3/4}} dy. \quad (6.5)$$

The scaling function has been normalized so that $\mathcal{F}_1(0) = 1$; for large-$u$, $\mathcal{F}_1(u) \sim u^{-1/2}$.

For numerical purposes, it is useful to transform the integral by making the substitution $1 - uy = \sin^2 \theta$, so that

$$\mathcal{F}_1(u) = \frac{2u}{\pi} \int_0^{\pi/2} \exp \left[ -g(\cos^2 \theta/u) \right] \frac{d\theta}{(\cos^4 \theta + u^2)^{3/4}}, \quad (6.6)$$

which removes the square-root singularities from the integrand at the endpoints of integration.

The scaling function $\mathcal{F}_1(u)$ is plotted in Fig. [ ] There is a maximum at $u = 0.386$, with $\mathcal{F}_{1,\text{max}} = 1.1078$. From the position of this peak we can define a velocity $v$ of flux penetration:
\[ v = \left( \frac{\bar{x}}{t} \right)_{\text{max}} = 0.772 \frac{D}{d}. \]  

(6.7)

These results agree exactly with the numerical work of Brandt [1]. As noted by Brandt, in the thin film geometry the flux entry is ballistic rather than diffusive, as it would be in a bulk sample with no demagnetizing fields.

We can also calculate the time dependent magnetization after a jump in the field. By using the small-\( \epsilon \) expansion in Eq. (4.37), and inverting the Laplace transform by wrapping the integration contour around the branch cut along the negative-\( s \) axis, we find at short times

\[ m(t) = 1 - \frac{6}{\pi^3} \left( t/\tau \right) \ln\left( 8 \pi e^{-2/3} t/\tau \right) + O(t^2), \]  

(6.8)

where \( \tau = ad/2\pi D \) is a characteristic relaxation time. Similar behavior was found by Brandt [1] in his numerical studies of the magnetization; for the prefactor of the log he obtained 0.205, compared to the present value of \( 6/\pi^3 = 0.194 \); for the constant inside the log, he obtained 25, compared to our value of \( 8 \pi e^{-2/3} = 12.9 \). It is not clear whether these small discrepancies are the result of the approximations in this paper or uncertainties in Brandt’s numerical work.

C. Ohmic conductor: ac response

Next, we consider the response of the strip to an ac magnetic field, \( H_a(t) = H_a \exp(i\omega t) \), so that \( H_a(s) = H_a/(s - i\omega) \). When inverting the Laplace transform, there will be contributions both from the pole at \( i\omega \) and from the square-root branch cut along the negative-\( s \) axis. The branch cut contribution decays as \( t^{-1/2} \); since we are concerned here with the steady-state behavior, we will neglect this term, keeping only the pole contribution. Writing \( j_{\text{edge}}(\bar{x}, t) = j_{\text{edge}}(\bar{x}, \omega) \exp(i\omega t) \), we have for the current near the edge

\[ j_{\text{edge}}(\bar{x}, \omega) = \frac{H_a}{2\pi d} \left( \frac{\omega \pi ad}{8D} \right)^{1/2} F_2 \left( \frac{d\omega \bar{x}}{2D} \right), \]  

(6.9)

where the ac scaling function \( F_2(u) \) is given by
\[ F_2(u) = \frac{\sqrt{2}}{\pi} \int_{0}^{\infty} \frac{e^{-g(y) - iuy + i\pi/4}}{y^{1/2}(y^2 + 1)^{3/4}} \, dy. \]  

(6.10)

The scaling function is defined so that \( \text{Re}F_2(0) = \text{Im}F_2(0) = 1 \). The real and imaginary parts of the ac scaling function are plotted in Fig. 7. The real part of the scaling function has a maximum at \( u = 0.232 \) \((\bar{x}\omega = 0.0738 a/\tau \text{ in conventional units})\), with \( \text{Re}F_{2,\text{max}} = 1.0787 \); the imaginary part changes sign at \( u = 2.43 \) \((\bar{x}\omega = 0.773 a/\tau \text{ in conventional units})\).

These results are once again very close to Brandt’s numerical results [1,2]. The uniform approximation to the current density is

\[
j_{\text{unit}}(x, \omega) = \frac{H_a}{2\pi d} \left\{ -\frac{x}{(1-x^2)^{1/2}} + \left[ \frac{1}{2(1-x)} \right]^{1/2} - \left[ \frac{1}{2(1+x)} \right]^{1/2} \right\} + j_{\text{edge}}(1+x, \omega) - j_{\text{edge}}(1-x, \omega).
\]

(6.11)

Using the small-\( \epsilon \) expansion of the magnetization, Eq. (4.37), we can also calculate the high frequency magnetization; the result is

\[
m(\omega) = 1 - \frac{6}{\pi^3} \frac{\ln(8\pi e^\gamma - 5/3 i\omega \tau)}{i\omega \tau} + O(\omega^{-2}),
\]

(6.12)

which is quite similar to the numerical result obtained by Brandt [1]. The constants differ slightly; here the prefactor is 0.194, while Brandt obtained \( 2/\pi^2 = 0.203 \); for the constant inside the log, Brandt obtained 16.2, compared to the present value of \( 8\pi e^{\gamma - 5/3} = 8.45 \).

VII. EXTENSIONS OF THE METHOD

With minor modifications it is also possible to treat two related problems, the current distribution in a thin superconducting disk in a perpendicular field, and the current distribution in a thin strip in the presence of an applied current. Rather than discussing these cases in detail, only a brief sketch of the results will be provided.

A. Disk geometry

Rather than a strip we now have a superconducting disk centered at the origin of the \( x - y \) plane, of radius \( a \). The current density and the vector potential are both in the \( \hat{\phi} \)
direction. Using a Green’s function method similar to that in Sec. II, the $z$-component of the magnetic field inside the disk satisfies

$$H_z(r, z = 0, t) = H_a(t) + 2d(P) \int_0^a P(r, r') j(r', t) dr',$$  \hspace{1cm} (7.1)

where the kernel $P(r, r')$ is given by

$$P(r, r') = \frac{K(k)}{r + r'} + \frac{E(k)}{r' - r}, \quad k^2 = \frac{4rr'}{(r + r')^2},$$  \hspace{1cm} (7.2)

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind. We again Laplace transform, use the constitutive relation between $j$ and $A$, and go to the dimensionless variables $r' = r/a$, $f(r, s) = j(r, s)/(H_a(s)/\pi^2d)$, and $\epsilon(s) = 2\lambda_{eff}/a$, to arrive at the following integro-differential equation:

$$-\frac{1}{r} \frac{\partial}{\partial r}[rf(r, s)] = \frac{\pi}{2} + \frac{1}{\pi} (P) \int_0^1 P(r, r') f(r', s) dr'.$$  \hspace{1cm} (7.3)

The outer solution is

$$f_0(r) = \frac{r}{(1 - r^2)^{1/2}},$$  \hspace{1cm} (7.4)

which has a square root singularity at the edge. Near the edge we construct the inner solution by defining the inner variables $R = (1 - r)/\epsilon$, $F(R) = \epsilon^{1/2} f(r)$; for small $\epsilon$ the kernel behaves as

$$P(1 - \epsilon R, 1 - \epsilon R') = \frac{1}{\epsilon R - R'} + O(\ln \epsilon).$$  \hspace{1cm} (7.5)

After expanding $F(R; \epsilon)$ in $\epsilon$, we find that the lowest order term satisfies Eq. (4.10), so that the inner solution is the same as before. Finally, the inner and outer solutions are matched as before; the uniform solution is then

$$f_{\text{unif}}(r, s) = \frac{r}{(1 - r^2)^{1/2}} - \frac{1}{[2(1 - r)]^{1/2}} + \epsilon(s)^{-1/2} F_0[(1 - r)/\epsilon(s)].$$  \hspace{1cm} (7.6)

The magnetic moment of the disk is

$$M(s) = \pi \int_0^a r^2 j(r, s) dr = \frac{2a^3H_a}{3\pi d} m(s),$$  \hspace{1cm} (7.7)
where the dimensionless moment in the disk geometry is

\[ m(s) = \frac{3h(s)}{2} \int_0^1 r^2 f(r, s) \, dr. \]  

(7.8)

Using the uniform approximation, this becomes

\[ m(s) = h(s) \left\{ 1 - \frac{4\sqrt{2}}{5} + \frac{3}{2\sqrt{2}\pi} \int_0^\infty \frac{e^{-g(y)} \left[ y^2 - 2y + 2 - 2e^{-y} \right]}{y^{7/2}(1 + \epsilon^2 y^2)^{3/4}} \, dy \right\}. \]  

(7.9)

For small \( \epsilon \) the integral can be expanded using the same method as for the strip (see Appendix C), with the result

\[ m(s) = h(s) \left[ 1 - \frac{2\sqrt{2}}{\pi} \ln \left( \frac{4e^{\gamma-5/3}}{\epsilon} \right) \epsilon + O(\epsilon^2) \right]. \]  

(7.10)

For an Ohmic conductor \( \epsilon(s) = 2D/\alpha d = 1/\pi \tau s \), with \( \tau = \alpha d/2\pi D \). After inverting the Laplace transform, we find that after a jump in the magnetic field, the magnetization for small \( t \) is

\[ m(t) = 1 - \frac{2\sqrt{2}}{\pi^2} \frac{t}{\tau} \ln \left( 4\pi e^{-2/3} t/\tau \right), \]  

(7.11)

while the ac magnetization at high frequencies is

\[ m(\omega) = 1 - \frac{2\sqrt{2}}{\pi^2} \frac{\ln(4\pi e^{\gamma-5/3} i\omega \tau)}{i\omega \tau}. \]  

(7.12)

The behavior is essentially the same as for the strip, with slightly different constants. Similar behavior has been found by Brandt [2] in his numerical studies. He finds a prefactor of \( 3/\pi^2 = 0.304 \), compared to the present value of 0.286; for the constant inside the log, he obtains 11.3, compared to our 4.23. As in the strip geometry, the source of the discrepancy is unclear.

**B. Current distribution in the presence of an applied current**

The effect of an applied transport current is to modify the \( O(1) \) outer solution, which now becomes
where \( f_a = I(s)/(H_a(s)a/2) \), with \( I(s) \) the total current (not the current density) in the strip. Writing the outer solution in terms of the inner variable \( X \), we see that the matching conditions are

\[
f_0(X) = \pm (1 \pm f_a) \frac{1}{(2\epsilon X)^{1/2}},
\]

with the + corresponding to the left edge and the − to the right edge. The solution to the inner problem is the same as before. After matching the inner and outer solutions, we have for the uniform solution

\[
f_{\text{unif}}(x,s) = -\frac{x-f_a}{(1-x^2)^{1/2}} + \frac{1-f_a}{[2(1-x)]^{1/2}} - \frac{1+f_a}{[2(1+x)]^{1/2}} + \epsilon(s)^{-1/2} \{(1+f_a)F_0[(x+1)/\epsilon(s)] - (1-f_a)F_0[(1-x)/\epsilon(s)]\}.
\]

With this expansion it is possible to study the time-dependent response, just as in the zero current case.

**VIII. LONG TIME BEHAVIOR AFTER A JUMP IN THE APPLIED FIELD**

All of the previous sections have been concerned with the short-time or high frequency response of a strip to an applied field. In this section we will treat the long-time relaxation of the current density in an Ohmic strip after a jump in the perpendicular field. We will again use the method of matched asymptotic expansions, but in a slightly different form.

We start with the equation of motion for the magnetic field, Eq. (2.5), and differentiate with respect to time \( t \). The time derivative of the magnetic field can be related to the current density for an Ohmic conductor with conductivity \( \sigma \) through \( \partial H_z/\partial t = -\partial E_y/\partial x = -\sigma^{-1}\partial j/\partial x \). Since the applied field is a step function, \( \partial H_a(t)/\partial t = H_a\delta(t) \). Therefore, for \( t > 0 \) the current density satisfies (using \( a \) as the unit of length)

\[
-\frac{1}{a\sigma} \frac{\partial j(x,t)}{\partial x} = 2d(P) \int_{-1}^{1} \frac{1}{x'-x} \frac{\partial j(x',t)}{\partial t} \, dx'.
\]
Following Brandt [1,2], write the current density as an eigenfunction expansion:

\[ j(x,t) = \frac{H_a}{2\pi d} \sum_n c_n \psi_n(x) e^{-t/\tau_n}, \quad (8.2) \]

where the relaxation times \( \tau_n = ad/2D\lambda_n \) are related to the eigenvalues \( \lambda_n \), which follow from the solution of

\[ \frac{d\psi_n(x)}{dx} = \frac{\lambda_n}{\pi (P)} \int_{-1}^{1} \frac{\psi_n(x')}{x' - x} \, dx'. \quad (8.3) \]

This equation is similar to the homogeneous version of the integral equation for the current density, Eq. (2.12), but with an important sign difference on the left hand side. As a result, we can expect the eigenfunctions to be oscillatory, rather than decaying, in the middle of the strip. The long time behavior of the current density will be controlled by the smallest eigenvalue \( \lambda_0 \), which produces the longest relaxation time \( \tau_0 \).

We can develop an asymptotic analysis of the eigenvalue spectrum by first assuming that \( \lambda_n \gg 1 \), so that \( 1/\lambda_n \) serves as our small parameter. The consistency of this assumption should be checked at the end of the calculation. As before, we break the problem up into an outer problem in the middle of the strip, and two inner problems near each of the two edges.

First we treat the outer problem. Define the outer variables \( X_o = \lambda_n x, \, \Psi_n^{(o)}(X_o) = \psi_n(x) \).

Then the integral equation for the outer function is

\[ \frac{d\Psi_n^{(o)}(X_o)}{dX_o} = \frac{1}{\pi (P)} \int_{-\lambda_n}^{\lambda_n} \frac{\Psi_n^{(o)}(X_o')}{X_o' - X_o} \, dX_o'. \quad (8.4) \]

Taking \( \lambda_n \rightarrow \infty \), we then have an integro-differential equation which relates the derivative of a function to its Hilbert transform. The solutions are \( \cos(X_o) \) and \( \sin(X_o) \); however, in the absence of an applied current the current density must be odd in \( x \), so the physically acceptable solution is

\[ \Psi_n^{(o)}(X_o) = A_n \sin(X_o), \quad (8.5) \]

with \( A_n \) a constant which can in principle depend upon \( n \). This outer solution must be matched onto the inner solution, which we turn to next.
Let’s first consider the inner problem at the left edge. Define the inner variables $X_i = \lambda_n (1 + x)$ and $\Psi_n^{(i)}(X_i) = \psi_n(x)$. Writing Eq. (8.3) in terms of the inner variables, and taking $\lambda_n \to \infty$, we have

$$\frac{d\Psi_n^{(i)}(X_i)}{dX_i} = \frac{1}{\pi} P \int_0^\infty \frac{\Psi_n^{(i)}(X_i')}{X_i' - X_i} dX_i'. \quad (8.6)$$

which is once again an integral equation of the Wiener-Hopf type. To solve, introduce an unknown function $\tilde{G}(X_i)$ such that $\tilde{G}(X_i) = 0$ for $X_i > 0$, and introduce the complex Fourier transforms

$$\tilde{\Phi}^+(k) = \int_0^\infty \Psi_n^{(i)}(X_i)e^{ikX_i}dX_i, \quad (8.7)$$

$$\tilde{G}^+(k) = \int_0^\infty G(X_i)e^{ikX_i}dX_i, \quad (8.8)$$

such that $\tilde{\Phi}^+(k)$ is analytic for $\text{Im}(k) > -\beta$ and $\tilde{G}^+(k)$ is analytic for $\text{Re}(k) < \alpha$, for some $\alpha > \beta$. Fourier transforming Eq. (8.6), we then obtain

$$\tilde{G}^+(k) - \Psi_n^{(i)}(0) = ik\tilde{K}(k) \left( \frac{k^2 - k_0^2}{k^2 + \delta^2} \right) \tilde{\Phi}^+(k), \quad (8.9)$$

where

$$\tilde{K}(k) = \frac{k^2 + \delta^2}{k^2 - k_0^2} \left[ 1 - \frac{1}{(k^2 + \delta^2)^{1/2}} \right], \quad (8.10)$$

with $\delta$ a small parameter which is taken to zero at some convenient point of the calculation, and $k_0 = (1 - \delta^2)^{1/2}$. The kernel $\tilde{K}$ has been constructed so that it is free of zeros in the strip $-\delta < \text{Im}(k) < \delta$, and $\tilde{K}(k) \to 1$ as $|k| \to \infty$. The kernel can be factored into the quotient form $\tilde{K} = \tilde{K}^+ / \tilde{K}^-$ using the general factorization procedure (see Appendix A), with the result that

$$\tilde{K}^+(k) = [\tilde{K}(k)]^{1/2} e^{i\tilde{\phi}(k)}, \quad (8.11)$$

$$\tilde{\phi}(k) = -\frac{k}{\pi} \int_0^\infty \ln[\tilde{K}(x)/\tilde{K}(k)] \frac{dx}{x^2 - k^2} dk. \quad (8.12)$$
With the factorization, Eq. (8.9) can be written as
\[(k - i\delta)\tilde{K}_-(k)[\tilde{G}_+(k) - \Psi^{(i)}_n(0)] = \imath k \left(\frac{k^2 - k_0^2}{k + i\delta}\right) \tilde{K}_+(k) \tilde{\Phi}_+(k).\] (8.13)

Both sides are analytic in their respective regions of analyticity; analytic continuation allows us to set both sides equal to an entire function \(E(k)\). To choose \(E(k)\), we require that \(\Psi^{(i)}_n(0)\) be finite, so that \(\tilde{\Phi}_+(k) \sim 1/k\) for large \(k\). This can only be achieved by taking \(E(k) = B_n k/2^{1/2}\), with \(B_n\) a constant (the \(2^{1/2}\) has been added to simplify some of the resulting expressions). Therefore,
\[\tilde{\Phi}_+(k) = \frac{B_n}{2^{1/2}} \frac{k + i\delta}{i(k^2 - k_0^2)K_+(k)}.\] (8.14)

Now we invert the Fourier transform, with the integration path passing above the poles on the real axis. Closing the contour in the lower half plane, we pick up contributions from the poles and from a branch cut which runs along the \(\text{Im}(k) < 0\) axis, with the final result
\[\Psi^{(i)}_n(X_i) = -B_n \cos(X_i - \pi/8) + B_n \psi_{\text{cut}}(X_i),\] (8.15)

where \(\psi_{\text{cut}}(X_i)\) is the contribution from the branch cut, which is \(O(X_i^{-3/2})\) for large \(X_i\).

We must now match together our outer solution, Eq. (8.5), and our inner solution, Eq. (8.13). Taking the outer limit of the inner solution, and rewriting \(X_o\) and \(X_i\) in terms of \(x\), we find that
\[\Psi^{(i)}_n(X_i) = -B_n \cos[\lambda_n(1 + x) - \pi/8] + A_n \sin(\lambda_n x).\] (8.16)

This is only satisfied if \(\lambda_n - \pi/8 = (n + 1/2)\pi\) and \(A_n = (-1)^n B_n\). Therefore, the eigenvalue spectrum is
\[\lambda_n = \frac{5\pi}{8} + n\pi, \quad n = 0, 1, 2, \ldots\] (8.17)

The same matching procedure must also be carried out at the right edge, and the analysis is identical. From the inner and outer solutions we can construct a uniform solution, which is
\[ \psi_{n,\text{unif}}(x) = \sin\left[ (n + 5/8)\pi x \right] + (-1)^n \left\{ \psi_{\text{cut}}[\lambda_n(1 + x)] - \psi_{\text{cut}}[\lambda_n(1 - x)] \right\}, \quad (8.18) \]

where an overall constant has been dropped.

This asymptotic expansion should be accurate for large \( n \). To compare these results to Brandt’s numerical work \([1,2]\), first note that Brandt calculates \( \Lambda_n = \lambda_n / \pi \), so we find \( \Lambda_0 = 5/8 = 0.625 \). Brandt obtains the numerical value of \( \Lambda_0 = 0.638 \), which is within 2\% of the result of our asymptotic analysis. The approximation only improves for large \( n \), so our result appears to be quite accurate for all \( n \). This due in part to the fact that \( \lambda_0 = 5\pi/8 \) is large enough for the asymptotic analysis to be effective. With regard to the eigenfunctions, for large \( n \) the cut contributions become less significant (their contribution is localized near the edges), and so \( \psi_n(x) = \sin\left[ (n + 5/8)\pi x \right] \) becomes an accurate approximation to the eigenfunctions. Based on his numerical results, Brandt quotes a similar result, but with \( 5/8 \) replaced by \( 1/2 \); our result should provide a better approximation, and even appears to resemble the numerical result for \( n = 0 \). The cut contributions have derivatives which diverge logarithmically at the edges, similar to the numerical solutions.

\section*{IX. Discussion and Summary}

By applying the method of matched asymptotic expansions to the integro-differential equation for the current density in the strip, Eq. (2.12), we have been able to derive a uniform approximation for the current density, which has been used to study the nonequilibrium response of the current in the strip, as well as the ac current density and the magnetization. Most of the effort has gone into understanding the response of an Ohmic strip. However, the method is easily generalized to more complicated dispersive conductivities \( \sigma(s) \); the only difficulties arise in inverting the Laplace transform to obtain the temporal response. For the purely Ohmic response, we found that after a jump in the perpendicular magnetic field the current propagates in from the edges at a constant velocity, in contrast to the longitudinal case, where the current propagates diffusively \([1]\). The difference is due to the demagnetizing effects in the perpendicular geometry; initially the field lines must bend around the edges.
of the sample, resulting in a large magnetic “pressure” which drives the current into the sample.

A number of simplifying assumptions were made in order to make this problem analytically tractable. The most important is the assumption of linear response. For many type-II superconductors, however, the current-voltage characteristics are highly nonlinear due to the collective pinning of the flux lines. In the perpendicular geometry there has recently been some progress in incorporating pinning (nonlinear response) into calculations of the current and field patterns for thin superconducting strips [9,10]. It is possible that the asymptotic methods used in this paper would be useful for studying the nonequilibrium, nonlinear response in the perpendicular geometry. A second assumption is that the current in the strip does not vary in the $y$-direction, so that we have an essentially one-dimensional problem. It would be interesting to include small variations of the current along the $y$-direction in order to determine the stability of the current fronts which enter after a jump in the perpendicular field; this problem might also be amenable to the type of analysis discussed in this paper.

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APPENDIX A: DECOMPOSITION OF THE KERNEL $K$

We want to decompose the kernel $K(k)$ given in Eq. (4.16) into $K_+(k)/K_-(k)$, with $K_+$ analytic in the upper half plane and $K_-$ analytic in the lower half plane. Since $K(k)$ is free of zeros in the strip $-\delta < \text{Im}(k) < \delta$ and approaches 1 as $k \to \pm\infty$, $\ln K(k)$ is analytic in this strip and approaches 0 as $k \to \pm\infty$. We can therefore perform an additive decomposition of

$$\ln K(k) = \ln K_+(k) - \ln K_-(k)$$  \hspace{1cm} (A1)
by writing

\[
\ln K_+(k) = \frac{1}{2\pi i} \int_{-\infty - i\alpha}^{\infty - i\alpha} \ln K(z) \frac{dz}{z - k},
\]

(A2)

with \(\alpha > \delta\); there is an analogous expression for \(K_-(k)\). This integral will be analytic for \(\text{Im}(k) > -\delta\). Now if \(k\) can be taken to be real, and the integration path coincides with the real axis (indented to pass under the pole at \(k\)), then

\[
\ln K_+(k) = \frac{1}{2} \ln K(k) + \frac{k}{\pi i} \int_0^\infty \frac{\ln[K(x)/K(k)]}{x^2 - k^2} dx,
\]

(A3)

where \(K(-x) = K(x)\) has been used to simplify the integral. Therefore, our decomposition of \(K = K_+ / K_-\) is (letting \(\delta \to 0\))

\[
K_\pm(k) = \left[ 1 + \frac{1}{|k|} \right]^{\pm 1/2} e^{i\varphi(k)},
\]

(A4)

with

\[
\varphi(k) = \frac{\text{sgn}(k)}{\pi} \int_0^\infty \ln \left[ \frac{u(|k| + 1)}{|k|u + 1} \right] \frac{du}{u^2 - 1}.
\]

(A5)

The last integral may be rewritten in a more convenient form by first differentiating with respect to \(k\), integrating with respect to \(u\), and finally integrating with respect to \(k\):

\[
\varphi(k) = \frac{\text{sgn}(k)}{\pi} \int_0^{[k]} \ln \frac{u}{1 - u^2} du + \frac{\pi}{4} \text{sgn}(k).
\]

(A6)

This integral can be expressed in terms of dilogarithm functions, but this is not particularly useful for our purposes.

For large \(|k|\), \(\varphi(k)\) has the asymptotic expansion

\[
\varphi(k) \sim +\frac{1}{\pi} \ln(e|k|) + O(k^{-3}),
\]

(A7)

while for small \(k\), \(\varphi(k)\) has the series expansion

\[
\varphi(k) = \frac{\pi}{4} \text{sgn}(k) - \frac{1}{\pi} \ln(e/|k|)k + O(k^3).
\]

(A8)
In this Appendix the behavior of $F_0(X)$ for small $X$ will be derived by using the method of matched asymptotic expansions to derive a uniformly valid expansion for the integrand of $F_0(X)$. Call the integrand in Eq. (4.27) $I(y, X)$:

$$I(y, X) = \frac{e^{-Xy - g(y)}}{y^{1/2}(y^2 + 1)^{3/4}}. \tag{B1}$$

Expanding for small $X$,

$$I_i(y, X) = \frac{e^{-g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} - \frac{y^{1/2}e^{-g(y)}}{(y^2 + 1)^{3/4}}X + O(X^2). \tag{B2}$$

This constitutes our inner expansion, which breaks down when $Xy = O(1)$. To derive the outer expansion, define an outer variable $Y = Xy$, rewrite $I(y, X)$ in terms of $Y$, and expand to lowest order in $X$:

$$I_o(Y, X) = \frac{e^{-Y - g(Y/X)}}{Y^{1/2}(Y^2 + X^2)^{3/4}}X^2$$

$$= \frac{e^{-Y}}{Y^2}X^2 + O(X^3). \tag{B3}$$

In order to match the two expansions, express the inner expansion $I_i(y, X)$ in terms of the outer variable $Y$, and expand for small $X$:

$$I_i(Y/X, X) = \frac{e^{-g(Y/X)}}{Y^{1/2}(Y^2 + X^2)^{3/4}}X^2 - \frac{Y^{1/2}e^{-g(Y/X)}}{(Y^2 + X^2)^{3/4}}X^2$$

$$= \left[ \frac{1}{Y^2} - \frac{1}{Y} \right] X^2. \tag{B4}$$

On the other hand, if we express the outer expansion $I_o(Y, X)$ in terms of the inner variable $y$, we have

$$I_o(Xy, X) = \frac{e^{-Xy}}{y^2}$$

$$= \frac{1}{y^2} - \frac{X}{y}. \tag{B5}$$

We see that the 1 term outer expansion of the 2 term inner expansion is equal to the 2 term inner expansion of the 1 term outer expansion, in agreement with the van Dyke matching
principle [3]. To obtain the uniform expansion, add the inner and outer expansions, and subtract the overlap:

\[
I_{\text{unif}}(y, X) = \frac{e^{-g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} - \frac{y^{1/2}e^{-g(y)}}{(y^2 + 1)^{3/4}}X + \frac{e^{-Xy}}{y^2} - \frac{1}{y^2} + \frac{X}{y} + O(X^2)
\]

\[
= \frac{e^{-g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} - \frac{y^{1/2}[e^{-g(y)} - 1]}{(y^2 + 1)^{3/4}}X
\]

\[
+ \frac{e^{-Xy} - 1}{y^2} + \frac{X}{y} - \frac{Xy^{1/2}}{(y^2 + 1)^{3/4}} + O(X^2).
\]  \hspace{1cm} (B6)

To obtain the small \(X\) behavior of \(F_0(X)\), we can integrate \(I_{\text{unif}}(y, X)\) on \(y\). From the arguments given in Sec. IV.C, we know that \(F_0(0) = (\pi/2)^{1/2}\), so the first integral is

\[
\int_0^\infty \frac{e^{-g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} dy = \pi.
\]  \hspace{1cm} (B7)

The second integral must be evaluated numerically, with the result

\[
- X \int_0^\infty \frac{y^{1/2}[e^{-g(y)} - 1]}{(y^2 + 1)^{3/4}} dy = -0.7457X.
\]  \hspace{1cm} (B8)

The last three integrals require some care. The third integral is logarithmically divergent for small \(y\), so integrate down to a cutoff \(a\), and take \(a \to 0\) at a later point:

\[
\int_a^\infty \frac{e^{-Xy} - 1}{y^2} dy = -(1 - \gamma)X + X \ln(a) + X \ln(X) + O(a),
\]  \hspace{1cm} (B9)

with \(\gamma = 0.5772 \ldots\). The last two integrals are logarithmically divergent for large \(y\) (the two logs will cancel), so integrate up to \(A\) and then take \(A \to \infty\). For the fourth integral we then have

\[
X \int_a^A \frac{dy}{y} = X[\ln(A) - \ln(a)].
\]  \hspace{1cm} (B10)

The fifth integral converges for small \(y\), so we can extend the lower limit of integration to 0. By integrating by parts, we can extract the leading \(\ln(A)\) behavior; there is one remaining integral which must be calculated numerically, with the result

\[
- X \int_0^A \frac{y^{1/2}}{(y^2 + 1)^{3/4}} dy = -X[\ln(A) + \ln(2) - 0.4388].
\]  \hspace{1cm} (B11)
Adding Eqs. (B7)–(B11) together, we see that the dependence upon $a$ and $A$ drops out, as it should, and we finally obtain

$$ F_0(X) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{e^{-Xy - g(y)}}{y^{1/2}(y^2 + 1)^{3/4}} \, dy $$

$$ = \frac{1}{(2\pi)^{1/2}} \left[ \pi - 1.4228X + X \ln X + O(X^2) \right]. \quad (B12) $$

This is slightly different from Eq. (37) of Larkin and Ovchinnikov [7], which in our notation is

$$ F_{0,\text{Larkin}}(X) = \frac{1}{(2\pi)^{1/2}} \left[ \pi - \ln(4e^{-\gamma})X + X \ln X + O(X^2) \right] $$

$$ = \frac{1}{(2\pi)^{1/2}} \left[ \pi - 0.8091X + X \ln X + O(X^2) \right]. \quad (B13) $$

The difference between these two expressions becomes significant for values of $X$ near 1. For instance, numerically we find $F_0(1) = 1.732$, while Eq. (B12) gives 1.718, a difference of only 0.8%; the Larkin and Ovchinnikov expression gives 2.332, an error of 35%. The reason for the discrepancy is not clear, since the derivation of their result appears not to have been published.

**APPENDIX C: SMALL $\epsilon$ BEHAVIOR OF THE MAGNETIZATION**

In this Appendix we will determine the small-$\epsilon$ behavior of the dimensionless magnetization, $m(s)$, given by Eq. (4.36), using the method of matched asymptotic expansions. The derivation is quite analogous to derivation of the small $X$ behavior of $F_0(X)$ which was discussed in Appendix B. Call the integrand in Eq. (4.36) $J(y, \epsilon)$:

$$ J(y, \epsilon) = \frac{e^{-g(\epsilon y)}[(y + 1)e^{-2y} + y - 1]}{y^{5/2}[1 + \epsilon^2 y^2]^{3/4}}. \quad (C1) $$

Expand for small-$\epsilon$, using the small-$x$ behavior of $g(x)$ given in Eq. $(g(y))$, to obtain the inner expansion:

$$ J_i(y, \epsilon) = \frac{(y + 1)e^{-2y} + y - 1}{y^{5/2}} \left[ 1 + (y/\pi) \ln(e/\epsilon y) + O(\epsilon^2) \right]. \quad (C2) $$
This expansion breaks down when \( \epsilon y = O(1) \). To derive the outer expansion, define an outer variable \( Y = \epsilon y \), rewrite \( J(y, \epsilon) \) in terms of \( Y \), and expand to lowest order in \( \epsilon \):

\[
J_o(Y, \epsilon) = \frac{e^{-g(Y)}}{Y^{3/2}(1 + Y^2)^{3/4}} \epsilon^{3/2} + O(\epsilon^{5/2}). \tag{C3}
\]

To match the two expressions, write the inner expansion \( J_i(y, \epsilon) \) in terms of the outer variable \( Y \) and expand for small \( \epsilon \):

\[
J_i(Y/\epsilon, \epsilon) = Y^{-3/2} \left[ 1 + (Y/\pi) \ln(e/Y) \right] \epsilon^{3/2}. \tag{C4}
\]

Next, take the outer expansion \( J_o(Y, \epsilon) \) and write it in terms of the inner variable \( y \) and expand for small \( \epsilon \):

\[
J_o(\epsilon y, \epsilon) = y^{-3/2} \left[ 1 + (y/\pi) \ln(e/\epsilon y) \right]. \tag{C5}
\]

Again, we see that the 2 term inner expansion of the 1 term outer expansion is equal to the 1 term outer expansion of the 2 term inner expansion [3]. To obtain the uniform expansion (i.e., an expansion valid for arbitrary \( y \) and small-\( \epsilon \)), add the inner and outer expansions, and subtract the overlap:

\[
J_{\text{unif}}(y, \epsilon) = \frac{(y + 1)e^{-2y} + y - 1}{y^{5/2}} + \frac{(1/\pi) \ln(e/\epsilon y)}{y^{3/2}} \frac{(y + 1)e^{-2y} - 1}{y^{3/2}} \epsilon
\]

\[
+ \frac{e^{-g(\epsilon y)} - 1}{y^{3/2}(1 + \epsilon^2 y^2)^{3/4}} + y^{-3/2} \left[ \frac{1}{(1 + \epsilon^2 y^2)^{3/4}} - 1 \right]. \tag{C6}
\]

We must now integrate \( J_{\text{unif}}(y, \epsilon) \) over \( y \). For the first term we have

\[
\int_0^\infty \frac{(y + 1)e^{-2y} + y - 1}{y^{5/2}} dy = \frac{2(2\pi)^{1/2}}{3}. \tag{C7}
\]

For the second term we have two integrals:

\[
\frac{\ln(e/\epsilon)}{\pi} \int_0^\infty \frac{(y + 1)e^{-2y} - 1}{y^{3/2}} dy = -\frac{3}{(2\pi)^{1/2}} \epsilon \ln(e/\epsilon), \tag{C8}
\]

\[-\frac{\epsilon}{\pi} \int_0^\infty \ln y \frac{(y + 1)e^{-2y} - 1}{y^{3/2}} dy = -\left( \frac{2}{\pi} \right)^{1/2} \left[ -4 + \frac{9}{2} \ln 2 + \frac{3}{2} \gamma \right] \epsilon. \tag{C9}
\]
The fourth and fifth integrals are both $O(\epsilon^{1/2})$, as can be seen by rescaling the integration variable. The fourth integral was performed numerically, with the result

$$\epsilon^{1/2} \int_0^\infty \frac{e^{-g(x)} - 1}{x^{3/2}(1 + x^2)^{3/4}} \, dx = 2\epsilon^{1/2}, \quad (C10)$$

where the factor of 2 was determined to an accuracy of 1 part in $10^8$. The last integral is

$$\epsilon^{1/2} \int_0^\infty x^{-3/2} \left[ \frac{1}{(1 + x^2)^{3/4}} - 1 \right] \, dx = -2\epsilon^{1/2}. \quad (C11)$$

We see that for all purposes the last two integrals sum to zero, although this has not been proven analytically. Collecting together the other terms, we finally have

$$\int_0^\infty \frac{e^{-g(\epsilon y)}[(y + 1)e^{-2y} + y - 1]}{y^{5/2}[1 + \epsilon^2 y^2]^{3/4}} \, dx = \frac{2(2\pi)^{1/2}}{3} \left[ 1 - \frac{9}{4\pi} \ln \left( \frac{8\epsilon^{7/3}}{\epsilon} \right) \epsilon + O(\epsilon^2) \right]. \quad (C12)$$

A numerical evaluation of the integral for $\epsilon = 1$ gives 0.49577; the expansion at $\epsilon = 1$ gives 0.48624, an error of about 2%. We see that the expansion is quite accurate even for relatively large values of $\epsilon$. 
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FIGURES

FIG. 1. Illustration of the film geometry considered in this paper. The film has a width $2a$ in the $x$-direction, and a thickness $d$. The applied field $H_a$ is in the $z$-direction.

FIG. 2. Schematic diagram of the outer, inner, and matching regions used in the small-$\epsilon$ asymptotic analysis.

FIG. 3. The inner solution $F_0(X,s)$, calculated from Eq. (4.30) (solid line). Also shown are the asymptotic expansions for small $X$ (dotted line) and large $X$ (dashed line).

FIG. 4. Uniform approximation $f_{\text{unif}}(x,s)$ to the current density for $\epsilon = 0.1$, from Eq. (4.35) (solid line). Since the current density is odd in $x$ only the current density over half the strip is shown. For comparison the ideal screening case (the outer solution $f_0(x) = -x/(1-x^2)^{1/2}$) is also shown (dashed line). At the left edge we have $f_{\text{unif}}(x = -1) = 3.948$.

FIG. 5. Perpendicular component of the magnetic field $H_z(x)$ within the strip for $\epsilon = 0.1$, from Eq. (5.1). The field is even in $x$ so only half the strip is shown. Note the weak logarithmic singularity at $x = -1$.

FIG. 6. Scaling function $F_1(u)$ for the current density near an edge after the magnetic field is suddenly switched on, from Eq. (6.6). There is a maximum at $u = 0.386$, with $F_{1,\text{max}} = 1.1078$.

FIG. 7. Scaling functions for the response to an ac magnetic field, from Eq. (6.10); $\text{Re} F_2$ is the solid line and $\text{Im} F_2$ is the dotted line. The real part has a maximum at $u = 0.232$, with $\text{Re} F_{2,\text{max}} = 1.0787$; the imaginary part changes sign at $u = 2.43$. The integral which defines the scaling function converges quite slowly, resulting in some numerical inaccuracies which are reflected in the small amplitude oscillations in the plots.
TABLE I. Summary of the primary results. The small parameter for the asymptotic expansion is $\epsilon = 2\lambda_{\text{eff}}/a$, where $\lambda_{\text{eff}}$ is the effective penetration depth of the magnetic field and $a$ is either half the width of a strip or the radius of a disk. The strip thickness is $d$, $D = 1/4\pi\sigma(0)$ is the diffusion constant for the magnetic field, and $\tau = ad/2\pi D$ is a relaxation time. For a strip, $c_1 = 0.194$, $c_2 = 8.45$; for a disk, $c_1 = 0.286$, $c_2 = 4.23$.

| Outer solution in center (conformal mapping) | $f_0(x)$ |
|---------------------------------------------|----------|
| Inner solution near the edges (Wiener-Hopf method) | $F_0(X)$ |
| Uniform solution for current density | $f_{\text{unif}}(x, s)$ |
| Uniform solution for magnetic field | $H_z(x, z = 0)$ |
| Time-dependent magnetization | $m(t) = 1 - c_1(t/\tau) \ln(1.526c_2\tau/t) + O(t^2)$ |
| Ac magnetization | $m(\omega) = 1 - c_1 \ln(c_2i\omega\tau)/(i\omega\tau) + O(\omega^{-2})$ |
| Scaling of current at edge after jump in the field | $t^{-1/2}F_1(ad\bar{x}/2Dt)$ |
| Velocity of current propagation after jump in the field | $v = 0.772D/d$ |
| Scaling of ac current at edge | $\omega^{1/2}F_2(ad\bar{x}\omega/2D)$ |
| Fundamental relaxation time for a strip | $\tau_0 = (8/10\pi)ad/D = 0.255ad/D$ |
Scaling function $F_1(u)$
Real and imaginary parts of scaling function $F_2(u)$