Quantum solitons of the nonlinear $\sigma$-model with broken chiral symmetry

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Abstract

It is proved that the quantum-mechanical consideration of global breathing of a hedgehog-like field configuration leads to the dynamically stable soliton solutions in the nonlinear $\sigma$-model without the Skyrme term. Such solutions exist only when chiral symmetry of the model is broken.

1 Introduction.

It is well known that due to the Derric theorem there are no topologically non-trivial classical soliton solutions in the nonlinear $\sigma$-model when the Skyrme term is omitted.

However, recently it was assumed \cite{1, 2, 3} that quantum consideration of global breathing of hedgehog-like field configuration in the nonlinear $\sigma$-model without the Skyrme term leads to stable stationary soliton solutions. They are supposed to vanish in the classical limit ($\hbar \to 0$) and in this sense are called “quantum solitons”.


Unfortunately, in the above papers the chiral angle was not determined from
the field equations. It either was approximated by the variation method on
some more or less suitable trial function family \[3\], or was postulated by some
phenomenological reasons \[2, 4\].

In the present work the field equation for hedgehog-like quantum globally
breathing and rigidly rotating soliton in the nonlinear $\sigma$-model with broken
chiral symmetry and without the Skyrme term is studied and numerical solution
of this equation is obtained\[1\]. We show that the model has quantum soliton
solutions, but only in the case when the chiral symmetry is broken. So the
numerical estimations done in Refs.\[2, 3, 4\], where the chiral symmetry limit
was assumed, are incorrect.

The paper is organized as follows.

Sec.2 contains a formulation of the nonlinear spectral integro-differential
problem describing in terms of collective coordinates a rotating and breath-
ing soliton with topological charge 1. This problem consists of a Schrödinger
equation describing the dynamics of the collective coordinates and a differential
equation for the chiral angle with boundary conditions.

Sec.3 is devoted to the analysis of the boundary-value problem for t he chiral
angle and its exact numerical solutions.

In Sec.4 the Schrödinger equation spectrum is constructed, the exact solution
of the integro-differential problem corresponding to lowest-lying breathing states
with isotopic spin $1 \over 2$ and $3 \over 2$ are obtained. The latter are considered as models
of lightest baryons.

An analysis of the results and discussions are given in Sec.5.

2 The nonlinear integro-differential spectral problem
for the rigidly rotating and globally breathing soliton.

The initial Lagrangian density of the nonlinear $\sigma$-model with broken chiral symmetry is given by

$$\mathcal{L} = - \frac{f^2}{4} \text{Tr}(J_\mu J^\mu) + \frac{1}{16} f_\pi^2 m_\pi^2 \text{Tr}(\overline{\tau} U \tau U^+ - 3). \quad (1)$$

The following notations are used: $J_\mu \equiv U + \frac{\partial U}{\partial x^\mu}$, $U = U(t, \vec{x})$ is the $2 \times 2$ chiral
field matrix; $x^\alpha = (t, \vec{x})$, $\alpha = 0, 1, 2, 3$ are the coordinates in the Minkowski space
with the signature (+ − − −), $\overline{\tau} = (\tau_1, \tau_2, \tau_3)$ are the isotopic Pauli matrices,

\[\text{Footnote 1: Similar equation was also discussed early} \[3\], \text{but now we do not include so-called quantum corrections arising from quantum-mechanical treatment of collective coordinates in the initial Lagrangian.}\]
\( f_\pi = 93 \text{MeV} \) stands for the pion decay constant, \( m_\pi \) is a parameter of the theory and, as it will be discussed later, need not to be equal to physical pion mass.

Instead of the term breaking chiral symmetry commonly used

\[
\frac{1}{4} f^2_\pi m^2_\pi \text{Tr}(U + U^+ - 2),
\]

we choose the term

\[
\frac{1}{16} f^2_\pi m^2_\pi \text{Tr}(\bar{U}U - \bar{U}U^+) - 3),
\]

which considerably simplifies the further analysis. It can be shown that the term \( (3) \) corresponds to chiral symmetry breaking term \( (2.10) \) of Ref. \( \[6\] \) with \( k = 2 \). For the weak pion fields (which take place, e.g., at the large distances from the soliton center) both terms \( (2) \) and \( (3) \) are equivalent.

Here we shall consider the solution in the form of the rigidly rotating and globally breathing hedgehog ansatz used in Refs. \( \[2, 5\] \):

\[
U = A(t) \exp\{i\vec{\tau}(\theta(z))\} A^+(t),
\]

where \( \vec{x} = \frac{x}{|x|} \), \( A \in SU(2) \), \( A = a_0(t) + i\vec{\tau}\vec{a}(t) \), \( a_0^2 + \vec{a}^2 = 1 \), \( r = |\vec{x}| \),

\[
z = \frac{r}{\lambda(t)}.
\]

The chiral angle \( \theta(z) \) satisfies the boundary conditions:

\[
\theta(z)|_{z=0} = \pi, \quad \theta(z)|_{z=\infty} = 0,
\]

which correspond to the field configuration with the topological charge 1. The quantities \( a_p = (a_0, \vec{a}) \), \( p = 0, 1, 2, 3 \) and the homogeneous scale transformation parameter \( \lambda(t) \) are the quantum collective coordinates describing rotation and breathing of the soliton, respectively.

In terms of these collective coordinates the dynamical system Lagrangian

\[
\Lambda = \int d^3x \mathcal{L}
\]

is written as:

\[
\Lambda = \frac{1}{2} m[\theta]q^2 - v(q) + \frac{1}{2} \mu[\theta]q^2 \bar{W}^2
\]

\[
v(q) = \eta[\theta]q^{2/3} + \frac{3}{4} m^2_\pi \mu[\theta]q^2,
\]

where \( q = \lambda^{3/2} \), \( \bar{W} = 2(a_0 \vec{a} - \vec{a}a_0 + \vec{a} \times \vec{a}) \) is the rotation velocity. The quantities

\[
m[\theta] = \frac{16}{9} f_\pi^2 \int_0^\infty dz z^4 \left( \frac{d\theta}{dz} \right)^2,
\]

3
\[
\mu[\theta] = \frac{8}{3} \pi f^2 \int_\theta^\infty dz z^2 \sin^2 \theta \tag{11}
\]

and
\[
\eta[\theta] = 2 \pi f^2 \int_0^\infty dz \left( z^2 \left( \frac{d\theta}{dz} \right)^2 + 2 \sin^2 \theta \right) \tag{12}
\]

are functionals on the chiral angle \( \theta (z) \).

Introducing the new generalized coordinates \( q_p = qa_p \), one obtains [2, 5] the
Hamiltonian operator, corresponding to the Lagrangian (8)
\[
H = -\frac{1}{2m} q^2 + \frac{1}{2} \frac{\partial}{\partial q} \left( q^2 \frac{\partial}{\partial q} \right) + \frac{\vec{J}^2}{2\mu q^2} + v(q) \tag{13}
\]
( \( \vec{J} \) stands for the angular momentum operator). In this case the normalization
condition for the wave function satisfying the Schrödinger equation
\[
H\psi = E\psi \tag{14}
\]
is
\[
\int d\Omega \int_0^\infty dq q^3 |\psi|^2 = 1. \tag{15}
\]
(the first integral denotes integration over the angular variables in the space of
generalized coordinates \( q_p \) ). The wave function \( \psi(q) \) is supposed to satisfy the boundary conditions
\[
\psi|_{q=0} = \psi|_{q \to \infty} = 0. \tag{16}
\]
Separating in Eq.(14) the rotational and the breathing variables \( \psi = \psi_{nj}(q)\psi_j(a_p) \) one gets the Schrödinger equation for the new radial wave function \( \varphi_{nj}(q) = q^{3/2}\psi_{nj}(q) \) in the following form
\[
-\frac{1}{2m} \frac{d^2}{dq^2}\varphi_{nj}(q) + (v_j(q) - E_{nj})\varphi_{nj}(q) = 0, \tag{17}
\]
\[
v_j(q) = \frac{j(j+1)}{2\mu q^2} + \frac{3}{8m_\pi^2} + \mu q^{2/3} + \frac{3}{4} m_\pi^2 \mu q^2 \tag{18}
\]
with the normalization condition
\[
\int_0^\infty dq |\varphi_{nj}(q)|^2 = 1. \tag{19}
\]

\( j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \) and \( n = 0, 1, 2, \ldots \) are quantum numbers of isotopic and
radial excitation respectively.

When \( m_\pi \neq 0 \), it is convenient to reduce Eqs.(17)-(19) to the form
\[
\frac{d^2}{d\chi^2}\varphi_{nj}(\chi) - \left( \frac{\beta^2 - 1}{4\chi^2} + \chi^2 + \gamma \chi^{2/3} - \varepsilon_{nj} \right) \varphi_{nj}(\chi) = 0, \tag{20}
\]
\[ \int_0^\infty d\chi |\tilde{\phi}_{nj}(\chi)|^2 = 1 \]  
(21)

by the replacement

\[ \chi = q \sqrt{\frac{m}{2}} \sqrt{\frac{\mu}{m}} \]  
(22)

Here we use the following notation

\[ \beta_j^2 = 4 \left( \frac{m}{\mu} \right) (j + 1) + 1 \]  
(23)

\[ \gamma = \frac{2}{m^{2/3}} \left( \frac{4m}{9\mu} \right)^{1/3} \eta^{1/3} \mu^{1/3} \]  
(24)

\[ \varepsilon_{nj} = \frac{2E_{nj}}{m} \sqrt{\frac{2m}{3\mu}} \]  
(25)

To solve Eq.(20) the quantities \( \beta_j \) and \( \gamma \) must be determined. They are expressed in terms of \( m, \mu \) and \( \eta \) defined by (10)–(12), and thus are functionals on the chiral angle \( \theta(z) \).

It was shown [5] that \( \theta(z) \) satisfies the following equation

\[ \langle nj| \delta \theta \Lambda |nj \rangle = 0 \]  
(26)

where \( |nj \rangle \) is the eigenvector of the Hamiltonian operator (13). The Lagrangian (8) depends on \( \theta(z) \) through three functionals \( m, \mu \) and \( \eta \), therefore one can rewrite Eq.(26) in the following form

\[ a \delta \theta m + b \delta \theta \mu + c \delta \theta \eta = 0. \]  
(27)

The quantities

\[ a = \langle nj| \frac{1}{2} q^2 |nj \rangle, \]  
(28)

\[ b = \langle nj| \frac{1}{2} T^2 W^2 - \frac{3}{4} m^2 q^2 |nj \rangle, \]  
(29)

\[ c = -\langle nj| q^{2/3} |nj \rangle \]  
(30)

are functionals of the quantum system wave function.

Using the notations

\[ z_0 = \sqrt{\frac{9}{8} \frac{|c|}{a}}, \]  
(31)

\[ \alpha = \frac{3b}{4a} \]  
(32)
and introducing the new variables
\[ \theta(z) = \theta(z_0) = F(\xi) \] (33)
one obtains the differential equation for the chiral angle:
\[ (\xi^2(1-\xi^2)F')' - (1 - \alpha \xi^2) \sin 2F = 0. \] (34)
It should be supplemented by the boundary conditions
\[ F|_{\xi=0} = \pi, \] (35)
\[ F|_{\xi\to\infty} = 0. \] (36)
Eqs. (34) and (14) are related by the integral expressions for their coefficients (10)–(12) and (32), (28), (29). Thus the rotating and breathing soliton is described by the nonlinear integro-differential spectral problem.

When the condition \( m_\pi \neq 0 \) is fulfilled, it is convenient to modify the integro-differential problem replacing the Eq. (14) by (20) and to express the functionals (28)–(30) in terms of wave function \( \tilde{\phi}(\chi) \):
\[ a = \frac{m_\pi}{2m} \sqrt{\frac{3}{2m}} \int_0^\infty \tilde{\phi}^* \left( \frac{3}{4} \frac{1}{\chi^2} - \frac{d^2}{d\chi^2} \right) \tilde{\phi} d\chi, \] (37)
\[ b = \frac{m_\pi}{2m} \sqrt{\frac{3}{2m}} \int_0^\infty \left( \frac{m}{\mu} \frac{j(j+1)}{\chi^2} - \chi^2 \right) |\tilde{\phi}|^2 d\chi, \] (38)
\[ c = -\frac{1}{(m_\pi \sqrt{\frac{3}{2} m \mu})^{1/3}} \int_0^\infty \chi^{2/3} |\tilde{\phi}|^2 d\chi. \] (39)
Applying the quantum virial theorem to the Schrödinger equation (20) one has
\[ \int_0^\infty \tilde{\phi}^* \frac{d^2}{d\chi^2} \tilde{\phi} d\chi = \int_0^\infty \left( \frac{\beta^2}{4\chi^2} - \chi^2 - \frac{1}{3} \gamma \chi^{2/3} \right) |\tilde{\phi}|^2 d\chi. \] (40)
Substituting the last expression into (37) and taking into account (28) one gets
\[ a = -\frac{m_\pi}{2m} \sqrt{\frac{3}{2m}} \int_0^\infty \left( \frac{m}{\mu} \frac{j(j+1)}{\chi^2} - \chi^2 - \frac{1}{3} \gamma \chi^{2/3} \right) |\tilde{\phi}|^2 d\chi. \] (41)

3 Properties of the chiral angle.

Let us investigate the behavior of the solution of Eq. (34) with different \( \alpha \). This equation has three singular points \( \xi = 0, 1, \infty \).
Taking into account (35) one obtains the following solution for $\xi \ll 1$ region

$$F = \pi + \xi F'|_{\xi=0} + O(\xi^3).$$

On the other hand, the chiral angle will be regular at $\xi = 1$ only when the following condition is fulfilled

$$F'|_{\xi=1} + \frac{1}{2} (1 - \alpha) \sin 2F|_{\xi=1} = 0. \quad (43)$$

One has to make such a choice of the quantities $F'|_{\xi=0}$ and $F|_{\xi=1}$ that the two solutions of Eq.(34) starting from points $\xi = 0$ and $\xi = 1$ with boundary conditions (42) and (43), respectively, will be smoothly joint in arbitrary point of the interval $(0, 1)$ (See Appendix). The numerical analysis shows that for every $\alpha$ there exist two sets of these values. Although the existence of another set is not excluded completely, we have not succeeded in obtaining it.

So on $[1, \infty)$ the boundary-value problem (34)–(36) is reduced to two initial-value problems. Their initial conditions $F'|_{\xi=1}$ and $F|_{\xi=1}$ and, consequently, their solutions are uniquely determined by $\alpha$ value. So $\alpha$ is the spectral parameter of boundary-value problem, because the condition (36) can be satisfied only by a special choice of $\alpha$.

The two obtained sets of values $F'|_{\xi=0}$ and $F|_{\xi=1}$ are turned into each other by the transformation $F \rightarrow 2\pi - F$. Eq.(34) is also invariant under this transformation. So the two solutions of the initial-value problems corresponding to these two sets have the same symmetry.

One of them tends to the following values at large $\xi$

$$F|_{\xi \to \infty} = \begin{cases} 
\frac{\pi}{2} & \text{if } \alpha > 0 \\
\sim 0.86 & \text{if } \alpha = 0 \\
0 & \text{if } \alpha < 0.
\end{cases} \quad (44)$$

Using the transformation $F \rightarrow 2\pi - F$ one gets that the other solution does not satisfy (33) for any $\alpha$. (For $\alpha < 0$ it leads to the solution with topological charge $-1$.) So the boundary-value problem (34)–(36) has solution under the condition

$$\alpha < 0 \quad (45)$$

only. It is equivalent to

$$m_\pi^2 > \frac{2}{3} \frac{\langle n_j | g^2 W^2 | n_j \rangle}{\langle n_j | q^2 | n_j \rangle}.$$

The last expression has been obtained by comparing the expression (45) with (28), (29) and (32).

Note that right hand side of (44) is always positive, so the boundary-value problem (34)–(36) has no solution in the chiral approximation ($m_\pi = 0$).
It should be noted here that some authors [7, 8], using variational method, have also concluded that the globally breathing and rigidly rotating hedgehog-like quantum soliton does not exist in the chiral limit of the nonlinear $\sigma$-model without the Skyrme term.

Provided that the condition (45) is fulfilled, Eq.(34) is asymptotically reduced to

$$\xi^2 F'' + 4\xi F' - 2\alpha F = 0$$

at $\xi \gg 1$. The solution of this equation, coinciding with the solution of boundary-value problem (34)–(36) for $\xi \gg 1$, can be expressed in terms of elementary functions

$$F(\xi) = \begin{cases} 
C_1\xi^{\kappa_1} + C_2\xi^{\kappa_2}, & \text{if } -\frac{9}{8} < \alpha < 0 \\
C_1\xi^{-3/2} (\ln \xi + C_2), & \text{if } \alpha = -\frac{9}{8} \\
C_1\xi^{-3/2} \sin (\kappa_0 \ln \xi + C_2), & \text{if } \alpha < -\frac{9}{8}
\end{cases}$$

where

$$\kappa_{1,2} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + 2\alpha},$$

$$\kappa_0 = \sqrt{2|\alpha| - \left(\frac{3}{2}\right)^2},$$

$C_1$ and $C_2$ are arbitrary constants. It was shown numerically that for any $\alpha$ from the interval $(-9/8, 0)$ $C_1$ is nonequal to zero. So, it is easy to mention that for all $\alpha < 0$ the functionals [10] and [11] are divergent, because the chiral angle decrease too slowly at infinity. In spite of that, the soliton parameters are finite. Actually, according to the formulas (23)–(25) Eq. (20) contains $m$, $\mu$ and $\eta$ through the ratios $\frac{m}{\mu}$ and $\frac{\eta}{\mu^{1/3}}$ only. Considering the ratios of the divergent integrals in the sense of principal value, for instance:

$$\frac{m}{\mu} = \lim_{Z \to \infty} \frac{16}{9} \pi^2 f_\pi^2 \int_0^Z dz z^4 \left(\frac{d\theta}{dz}\right)^2 ,$$

and taking into account (48), (49) and (50) one gets finite numbers

$$\frac{m}{\mu} = \begin{cases} 
\frac{2}{3} \kappa_1^2 & \text{if } -\frac{9}{8} < \alpha < 0 \\
\frac{4}{3} |\alpha| & \text{if } \alpha \leq -\frac{9}{8}
\end{cases}$$
\[ \frac{n}{\mu^{1/3}} = 0 \quad \text{for all } \alpha < 0. \quad (53) \]

4 The exact soliton solutions for lowest-lying quantum states.

Substituting (38) and (41) into (32) and taking into account (53) and (24) one has

\[ \alpha = -\frac{3m}{4\mu}. \quad (54) \]

Comparing the last expression with (52) one obtains the algebraic equation for \( \alpha \). For \(-9/8 < \alpha < 0\) this equation is written as

\[ \frac{3}{2} \sqrt{\left(\frac{3}{2}\right)^2 + 2\alpha} = \left(\frac{3}{2}\right)^2 + 2\alpha \quad (55) \]

and has no solution, but for \( \alpha \leq -9/8 \) it can be reduced to

\[ \alpha = -|\alpha|, \quad (56) \]

i.e. it is fulfilled identically. Hence the integro-differential problem formulated in Sec.2 has a solution for any \( \alpha \) from \((-\infty, -9/8]\). Still, for further calculations we have chosen \( \alpha = -9/8 \), because, as seen in (48), for \( \alpha < -9/8 \) the chiral angle becomes oscillating at large distances and the physical meaning of such solutions is unclear.

Taking into account (53) and (24) the Eq.(20) can be reduced to

\[ \frac{d^2}{d\chi^2} \tilde{\phi}_{nj}(\chi) - \left( \frac{\beta_j^2 - 1}{4\chi^2} + \chi^2 - \epsilon_{nj} \right) \tilde{\phi}_{nj}(\chi) = 0. \quad (57) \]

The eigenvalue spectrum of this equation is as follows:

\[ \epsilon_{nj} = 2(2n + 1) + \beta_j, \quad n = 0, 1, 2, \ldots \quad (58) \]

The solution of Eq.(57) corresponding to the ground breathing state has the form

\[ \tilde{\phi}_{0j}(\chi) = \frac{2}{\sqrt{\beta_j 1(\beta_j/2)}} \chi^{\frac{1}{2}(\beta_j+1)} \exp\left(-\frac{\chi^2}{2}\right) \quad (59) \]

where \( \Gamma(y) \) is a Gamma function.

Comparing the formulas (25),(58) and (23) one gets the energy spectrum expressed in terms of \( \alpha \):

\[ E_{nj} = \left( n + \frac{1}{2} + \frac{\beta_j}{4} \right) \frac{3m_\pi}{\sqrt{2|\alpha|}}, \quad (60) \]

\[ \beta_j = 2\sqrt{\frac{4}{3}|\alpha|j(j + 1) + 1. \quad (61) \]
It was shown in Ref. [3] that the mean square radius of the breathing soliton can be evaluated by
\[
\langle r^2 \rangle_{nj} = \int_0^\infty d\chi |\tilde{\varphi}_{nj}(\chi)|^2 
\]
\[
\times \int_0^\infty dr r^2 \left( -\frac{2}{\pi} \sin^2 \left( \theta \left( \frac{r}{\lambda(\chi)} \right) \right) \frac{d\theta \left( \frac{r}{\lambda(\chi)} \right)}{dr} \right) \quad (62)
\]

Using (5), (33) and (59) one obtains after straightforward calculations:
\[
\langle r^2 \rangle_{nj} = \frac{2|\alpha|}{m^2} \frac{\Gamma\left( \frac{\beta_j}{2} + \frac{1}{2} \right) \Gamma\left( \frac{\beta_j}{2} + \frac{2}{3} \right) \left( \beta_j + \frac{2}{3} \right)}{\left( \Gamma\left( \frac{\beta_j}{2} \right) \right)^2} \langle \xi^2 \rangle, \quad (63)
\]
where
\[
\langle \xi^2 \rangle = -\frac{2}{\pi} \int_0^\infty d\xi \xi^2 \sin^2 F(\xi) \frac{dF(\xi)}{d\xi}. \quad (64)
\]

It should be mentioned that the soliton parameters \( E_{nj} \) and \( \langle r^2 \rangle_{nj} \) depend on the parameter \( m_\pi \) only, and do not on the pion decay constant \( f_\pi \).

Considering the solutions for the three lowest-lying breathing states with the isotopic spin numbers \( \frac{1}{2} \) and \( \frac{3}{2} \) as models of the particles \( N(940) \), \( N(1440) \), \( N(1710) \), \( \Delta(1232) \), \( \Delta(1600) \) and \( \Delta(1920) \), respectively, we have calculated some relations between their static properties (the masses and the mean squared radius of the nucleon), which are independent on \( m_\pi \) value. The results of the calculations and the corresponding experimental values are shown in the Table 1.

## 5 Conclusions and discussion.

In the present paper we have proved that the quantum-mechanical consideration of the global breathing of hedgehog-like field configuration leads to the dynamically stable soliton solutions of the nonlinear \( \sigma \)-model. Such solutions exist only when chiral symmetry is broken.

The relations between some static properties of the solitons in ground breathing state with isotopic spin \( \frac{1}{2} \) and \( \frac{3}{2} \) appear to be close with accuracy 10–15% to those of nucleon and delta, respectively. However for the excited states the predictions of the model are somewhat worse.

Nevertheless, the proposed model has some difficulties.

Firstly, the soliton masses and the mean squared radius depend on the parameter \( m_\pi \) and in order to reproduce the experimental values one has to use \( m_\pi \approx 2.7 \times (\text{the physical pion mass}) \). We believe that this problem can be
Table 1:
The relations between some soliton static properties being compared with corresponding experimental values.

|                  | Present model | Experiment |
|------------------|---------------|------------|
| $E_{0\frac{1}{2}} \langle r^2 \rangle_{0\frac{1}{2}}^{1/2}$ | 3.82          | 3.38       |
| $\frac{E_{1\frac{1}{2}}}{E_{0\frac{1}{2}}}$   | 1.83          | 1.53       |
| $\frac{E_{2\frac{1}{2}}}{E_{0\frac{1}{2}}}$   | 2.66          | 1.82       |
| $\frac{E_{0\frac{3}{2}}}{E_{0\frac{1}{2}}}$   | 1.43          | 1.31       |
| $\frac{E_{1\frac{3}{2}}}{E_{0\frac{1}{2}}}$   | 2.26          | 1.70       |
| $\frac{E_{2\frac{3}{2}}}{E_{0\frac{1}{2}}}$   | 3.09          | 2.04       |
solved in the framework of the quantum soliton model for the pion \[9\]. In such approach the physical pion mass arises as an eigenvalue of corresponding Hamiltonian operator, but not as a parameter in the initial Lagrangian.

Secondly, the slow asymptotic falloff of the profile function on the large distances from the soliton center \((F(\xi) \sim \xi^{-3/2} \ln \xi)\) contradicts the Yukawa law. This means that the assumption about homogeneous global breathing and rigid rotation is too rough and does not describe the soliton external part correctly. We hope, this problem may be solved by localization of the collective coordinate proposed in Refs.\[8, 10\].

Thirdly, the physical meaning of oscillating solutions \((\alpha < -\frac{9}{8})\) is unclear.

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Appendix: The Numerical Calculation Scheme.

Within the interval \([0, 1]\) the chiral angle \(F\) can be obtained as a solution of the boundary-value problem consisting of the differential equation \([34]\) and boundary conditions \([35]\) and \([43]\).

Because \(\xi = 0\) and \(\xi = 1\) are singular points of Eq.\([34]\), the conventional shooting method cannot be used in the present case. So it was modified as follows.

For any value \(F'_0\) one obtains an initial-value problem with initial conditions

\[
\frac{dF}{d\xi}\bigg|_{\xi=0} = F'_0 \tag{65}
\]

and \([35]\). For arbitrarily chosen point \(\xi_p \in (0, 1)\) this problem can be solved on the interval \([0, \xi_p]\) by, for instance, the Runge-Kutta method.

Similarly, for any \(F_1\) the problem with initial conditions

\[
F_{\xi=1} = F_1 \tag{66}
\]

\[
\frac{dF}{d\xi}\bigg|_{\xi=1} = -\frac{1}{2}(1 - \alpha) \sin 2F_1 \tag{67}
\]

can be solved from point \(\xi = 1\) to \(\xi = \xi_p\).

Let us introduce the following notations

\[
\Delta(F'_0, F_1) = F^{(0)}_{\xi=\xi_p} - F^{(1)}_{\xi=\xi_p} \tag{68}
\]

\[
\Delta'(F'_0, F_1) = \frac{dF^{(0)}}{d\xi}\bigg|_{\xi=\xi_p} - \frac{dF^{(1)}}{d\xi}\bigg|_{\xi=\xi_p}, \tag{69}
\]
where \( F^{(0)} \) and \( F^{(1)} \) are solutions of the first and the second initial-value problem, respectively.

To solve the original boundary-value problem, we have to join smoothly both solutions, i.e. the conditions

\[
\begin{align*}
\Delta(F_0', F_1) &= 0 \\
\Delta'(F_0', F_1) &= 0
\end{align*}
\]  

(70)

should be satisfied. Eq.(70) can be considered as a system of two coupling algebraic equations and can be solved by the Newton method.

We have found numerically two solutions of Eq.(70). Substituting them into Eqs.(66) and (67) one obtains two pairs of initial condition for the chiral angle on the interval \([1, \infty)\). Corresponding initial-value problems can be also solved numerically.

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