Estimate of the Cosmological Bispectrum from the MAXIMA-1 Cosmic Microwave Background Map

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Introduction: All theories of structure formation in the universe predict the properties of the probability distribution function (PDF) of cosmological perturbations. In all cases of interest, the PDF can be completely described in terms of its spatial n-point correlation functions, which are the expectation values of all possible products of the random field with itself at different points in space. Under the assumption of statistical isotropy and homogeneity, it is normally more useful to characterize the PDF in terms of higher order moments of the Fourier transform of the field. Most readers are familiar with the 2-point moment, the power spectrum of fluctuations ($C_l$). Indeed current efforts in the analysis of Cosmic Microwave Background (CMB) data have focused mainly on increasingly precise estimates of the angular power spectrum. The theoretical bias for this is clear: for Inflation induced perturbations, which is the current favourite model of structure formation, the statistics are Gaussian and all non-zero moments of order $n > 2$ can be expressed in terms of the $C_l$.

In this letter we present the first estimate of the bispectrum of the CMB on degree, and sub-degree, angular scales. The bispectrum is the cubic moment of the Fourier transform of the temperature field and it can be seen as a scale dependent decomposition of the skewness of the fluctuations (in much the same way as the $C_l$ is a scale dependent decomposition of the variance of fluctuations). The bispectrum can be used to look for the presence of a non-Gaussian signal in the CMB sky. We use the data collected with the MAXIMA-1 experiment [1] to quantify the bispectrum of the CMB. The Gaussianity of this data set has already been analysed using complementary methods in [2], including the methods of moments, cumulants, the Kolmogorov test, the $\chi^2$ test, and Minkowski functionals in eigen, real, Wiener-filtered and signal-whitened spaces.

In the past few years, interest in the bispectrum has grown in the scientific community. Estimates of the bispectrum in the COBE data proved the statistic to be extremely sensitive to some non-Gaussian features in the data, be they cosmological or systematic [3]; the quality of galaxy surveys has made it possible to test for the hypothesis that the matter overdensity is a result of non-linear gravitational collapse of Gaussian initial conditions [4]. On the other hand a serious effort has been undertaken to calculate the expected bispectrum from various cosmological effects; secondary anisotropies (such as the Ostriker-Vishniac effect, lensing, Sunyaev-Zel’dovich effect) [4], as well as primordial sources (such as non-linear corrections to inflationary perturbations or cosmic seeds) may lead to observable signatures in the bispectrum of the CMB [5].

Let us establish some notation. We shall be working in the small sky approximation where a map of the CMB can be considered approximately flat [6]. The anisotropy of the CMB, $\Delta T(x)$, can then be expanded in terms of 2-dimensional Fourier modes as follows:

$$\Delta T(x) = \int \frac{d^2k}{(2\pi)^2} a(k)e^{ikx} \tag{1}$$

As stated above, the complete statistical properties of $\Delta T$ can be encoded in the expectation values of products of the $a(k)$. The power spectrum is defined to be $a(k)a(k') = (2\pi)^2 C(k)\delta^2(k-k')$. On small angular scales, the correspondence between the flat sky power spectrum and the full sky angular power spectrum...
Given a map, we Fast Fourier Transform it and construct the following bispectrum estimator:

\[ \hat{B}_{\ell_1 \ell_2 \ell_3} = \frac{1}{N_{\ell_1, \ell_2, \ell_3; \Delta \ell}} \sum_{\mathbf{k} \in S(\ell, \Delta \ell)} \text{Re} \{a(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3)\} \]  

(3)

with

\[ \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0 \]  

(4)

where \( S(\ell, \Delta \ell) \) is a ring in Fourier space centered at \( \mathbf{k} = 0 \) and with radial coordinates \( \mathbf{k} \in [\ell_i - \Delta \ell/2, \ell_i + \Delta \ell/2] \). \( N_{\ell_1, \ell_2, \ell_3; \Delta \ell} \) are the number of modes which satisfy this condition and \( \text{Re} \{A\} \) is the real part of \( A \). For a given choice of \( \ell_i \) (with \( i = 1, 2, 3 \)) we obtain an estimate of the bispectrum averaged in a bin of width \( \Delta \ell \). We correct for the finite resolution of the experiment and the pixelization of the map by replacing the quantity \( a(\mathbf{k}) \) (that is estimated directly from the map) by \( a(\mathbf{k})/|B(\mathbf{k})W(\mathbf{k})| \), where \( B(\mathbf{k}) \) and \( W(\mathbf{k}) \) are the beam and pixel window functions, respectively (see [11] for a detailed Fourier space description of the beam).

There are a number of approximations in our analysis. We do not discuss any systematic effects that may have come into play when generating the map; a detailed description of these effects is presented in [12]. The flat sky approximation in the estimate of the power spectrum is valid to within 1% for the MAXIMA-1 100-square-degrees map. The fact that we are not considering a full sky map leads to two further complications [13]. Firstly, there will be a finite correlation length in Fourier space between adjacent modes. In Maximum Likelihood Methods this is automatically taken into account when constructing the correlation matrix, but in our case we must take care in assessing how our results depend on the width of the bins, \( \Delta \ell_i \), in which we estimate our bispectrum. Secondly, the map we are working with does not have periodic boundary conditions, an essential underlying assumption when performing a Fast (or Discrete) Fourier Transform. We correct for this by multiplying the map by a Welch window function which suppresses the mismatch at the border of the map thus reducing the leakage between neighboring scales in Fourier space.
ance. We should note however that there are two limitations to this approximation. On the one hand the sky signal is not uniformly distributed in Fourier space, i.e. there may be weak correlations between different Fourier modes. On the other hand the noise is anisotropic and correlated which means that the noise covariance matrix is not diagonal in Fourier space. Both of these effects may lead to correlations between the M approximately independent samples but for large enough \(\Delta \ell\) they should be negligible. Given that the bootstrap method is the only non-parametric (or model independent) method which one can apply in this situation, we choose to neglect these correlations [4].

Our approach to test for the Gaussianity is to generate \(10^5\) Monte Carlo realizations of the MAXIMA-1 data set, assuming a Gaussian signal with the power spectrum of the best fit model to the band powers estimated in [1]. Note that each of these mock data sets will have a realization of the noise which obeys the full anisotropic, non-diagonal correlation matrix; moreover the effect of pixelization and finite beam are taken into account. We then compare our estimate of the real data with the Gaussian ensemble and quantify a goodness of fit.

**Results:** We present the results we have obtained analysing a square patch in the center of the MAXIMA-1 map, with \(50^2\) pixels. Given the dimensions of the map, we consider \(\Delta \ell = 75\); these correspond to the bin-widths of the estimates of the \(C_\ell\) in [1] and lead to correlations of order a few percent between adjacent bins. In Fig. 1 we present the diagonal elements \((\ell_1 = \ell_2 = \ell_3 = \ell)\) of the estimate of the bispectrum (see also Table [1]). Note that all values of the bispectrum are of order \((0.001 - 0.01)C_\ell^{3/2}\), and the fact that \(B_{\ell\ell\ell}\mid_{\ell=224}\) is so large is mostly due to the fact that this corresponds to the peak value of \(C_\ell\).

The bootstrap errors are evaluated from resamplings with replacement of the approximately independent samples within each ring; the errors correspond to the 68\% confidence regions with these simulated distributions. We find that the average bootstrap errors, \(\sigma_{bs}\) over an ensemble of Gaussian maps to be between 4\% and 8\% lower than the true underlying variance. This bias is due to the correlations between adjacent samples within each ring. Moreover the number of approximately independent samples ranges from \(M = 2\) at \(\ell = 148\) to \(M = 10\) at \(\ell = 748\) and one should therefore bear in mind that, for low \(\ell\) the variance in the estimate of the bootstrap errors is large.

We have performed a number of tests to evaluate how robust the result is on the parameters of our estimator. We have taken a larger patch of the MAXIMA-1 map and considered maps of \(50^2\) pixels with different locations within the MAXIMA-1 map. The estimated bispectra vary by a few percent. Alternatively we have considered different bin-widths (with \(\Delta \ell = 60\) and \(\Delta \ell = 90\)) and found that estimates of the bispectrum vary smoothly and are consistent within different binnings. The use of the Welch window function turns out to be essential for small \(\ell\); this is to be expected as it should be the values of \(B_{\ell\ell\ell}\) for low \(\ell\) which are most affected by finite size effects. A different choice of window function (such as the Bartlett window function) changes the estimate of \(B_{\ell\ell\ell}|_{\ell=148\text{ and }\ell=224}\) by an order of 15\% but leaves the remaining values of \(B_{\ell\ell\ell}\) unaffected. One final test we have undertaken was to rotate the ring considered in Fourier space, this way displacing the \(M\) angular slices; we have found that the results vary by at most 10\% in the lowest \(\ell\) bin.

In Fig. 2 we plot the diagonal estimate of the MAXIMA-1 bispectrum compared to the 68\% and 95\% contour values if the sky was indeed Gaussian. We have checked that our statistic is unbiased even in the presence of anisotropic Gaussian noise and, as can be seen, the MAXIMA-1 \(B_{\ell}\) seem to be consistent with the Gaussian assumption. The obvious way to quantify this is to use a goodness of fit. For the Monte Carlo realizations of the Gaussian sky signal, we find that most of the histograms of the \(B_{\ell_1\ell_2\ell_3}\) are well approximated by Gaussians and we therefore define the standard \(\chi^2 = \sum_{\ell_1\ell_2\ell_3} C_{\ell_3}^{-1} \left(B_{\ell_1\ell_2\ell_3}^{obs} - B_{\ell_1\ell_2\ell_3}^{th}\right)^2\) where \(B_{\ell_1\ell_2\ell_3}^{th} = 0\) and \(C\) is the covariance matrix of the estimators evaluated from the Monte-Carlo realizations. In all we have 115 values and we find \(\chi^2 = 130\). From \(10^4\) realizations we construct the expected distribution of this \(\chi^2\); we find that 70\% of the distribution is contained to the left of the measured value. Even if we remove the outlier from the set of bins centered at \(\ell = 224\) we still find that 52\% of the distribution lies to the left of the measured \(\chi^2\).

**Cosmological Implications:** One can roughly divide the two possible sources of non-gaussianity in the CMB into primordial and late time. The latter have been extensively studied in [1] and typically give rise to non-zero...
bispectra on very small angular scales ($\ell > 1000$). We do not expect to find any evidence for such signatures in the MAXIMA-1 map. Moreover, the observed bispectrum limits point source contribution to the MAXIMA power spectrum as it shows no significant rise at high $\ell$. Primordial effects may give rise to non-Gaussianity on degree scales and we shall focus on a few possibilities now. Inflation predicts almost Gaussian fluctuations to a very good approximation; there is however the possibility that second order corrections in the evolution of the inflaton field may lead to mild non-Gaussianity. Komatsu and Spergel have parameterized this non-linearity in terms of a "non-linear coupling constant", $f_{NL}$, which can be related to dynamical parameters in a variety of models of inflation. For example, $f_{NL} \approx (3\epsilon - 2\eta)$ where $\epsilon$ and $\eta$ are the slow roll parameters of single field inflation; one expects from slow roll models that at most $f_{NL} \approx O(1)$. An order of magnitude estimate gives

$$B_{\ell_1,\ell_2,\ell_3} \approx b_{\ell_1,\ell_2} + b_{\ell_2,\ell_3} + b_{\ell_3,\ell_1},$$

and $\Delta T_\ell^2 \approx \ell (\ell + 1) C_\ell / (2\pi)$. Using the Monte Carlo realizations described before it is possible to estimate the smallest amplitude, $|f_{NL}|$, distinguishable from the Gaussian hypothesis; we find that $|f_{NL}| < 944$ is indistinguishable from a Gaussian signal at the 95% confidence level. Note that the use of lower multipoles (as measured by COBE) should narrow this interval. A fit to the measured values using the Gaussian covariance matrix gives $|f_{NL}| \approx 900$. ($\chi^2 = 122$).

More exotic possibilities can be considered, such as for example, global topological defects. A semi-analytic framework exists which allows one to calculate the statistical effects using the $O(N)$ non-linear $\sigma$-model. Different values of $N$ will correspond to different types of localized objects, with, for example, $N = 2$ corresponding to global strings, $N = 3$ monopoles and $N = 4$ corresponding to textures (taking $N$ to infinity we recover gaussianity). Verde et al. (see also [3]) have estimated the bispectrum and found that

$$B_{\ell_1,\ell_2,\ell_3} \approx \frac{2 \times 10^5}{T_{CMB}} \alpha \left( \frac{\Delta T_\ell_1}{\ell_1} \frac{\Delta T_\ell_2}{\ell_2} \frac{\Delta T_\ell_3}{\ell_3} \right)^3,$$

where $\alpha = N^{-1/2}$ (we should point out that this expression was derived for large angles). In what follows we shall extrapolate this expression to subdegree scales. The current sensitivity is such that models with $\alpha \approx 2.4$ are indistinguishable from Gaussian theories; this range of $\alpha$ corresponds to any value of $N$. We find the best fit $\alpha$ to be $\alpha = 2.2$. Current estimates of the bispectrum do not therefore constrain global topological defects.

The bispectrum analysis of the MAXIMA data indicates that the data is consistent with Gaussianity. This reinforces the conclusions obtained in [2] and validates the assumptions that go in to the data-analysis pipeline, namely the assumption of Gaussianity of the sky signal which goes into both Maximum-Likelihood and Monte-Carlo estimates of the power spectra.

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**Table I.** Measured bispectrum values and corresponding errors. The first column has the bandwidths, the second column the estimate of the bispectrum, the third column has an estimate of its variance using bootstrap methods, the fourth column has an estimate of its variance assuming the signal is Gaussian and the fifth column has its variance just due to noise. Columns 2-5 are in units of ($\mu K$)$^3$.

| $\ell_{\text{min}}, \ell_{\text{max}}$ | $\ell^3 B_\ell$ | $\ell^3 \sigma_\ell$ | $\ell^3 \sigma_{\ell\sigma}$ | $\ell^3 \sigma_N$ |
|-----------------|----------------|----------------|----------------|----------------|
| [111, 185]      | -5455          | 4477           | 16329          | 38             |
| [186, 260]      | 79622          | 55440          | 41363          | 145            |
| [261, 335]      | -13167         | 15798          | 17590          | 183            |
| [336, 410]      | -1373          | 7687           | 8504           | 366            |
| [411, 485]      | -5208          | 1977           | 7593           | 1071           |
| [486, 560]      | 3298           | 8939           | 8801           | 1815           |
| [561, 635]      | 3199           | 6213           | 9387           | 2892           |
| [636, 710]      | 16952          | 12518          | 13997          | 5939           |
| [711, 785]      | -2802          | 18725          | 26058          | 14197          |
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