Odd character degrees for $\text{Sp}(2n, 2)$

Degrés de caractères impairs sur $\text{Sp}_{2n}(2)$

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1. Introduction

If $G$ is a finite group and $\ell$ is a prime number, denote by $\text{Irr}_{\ell'}(G)$ the set of irreducible characters of $G$ with degree prime to $\ell$. The McKay conjecture asserts that

$$|\text{Irr}_{\ell'}(G)| = |\text{Irr}_{\ell'}(N_G(P))|$$

for $P$ a Sylow $\ell$-subgroup of $G$. This conjecture has gained new interest since appearance of Isaacs–Malle–Navarro’s theorem reducing it to a related conjecture on quasi-simple groups (see [7]). The latter has been checked for all quasi-simple groups not of Lie type.

Among groups of Lie type and for $\ell$ being the defining prime, the group $\text{Sp}_{2m}(2)$ remained open (see [13]). This is the main purpose of this note (see Corollary 4 below). The method is by use of the Jordan decomposition of characters for the $\text{Irr}_{\ell'}(G)$ side (see Proposition 2), while, for the $|\text{Irr}_{\ell'}(N_G(P))|$ side, we compute the abelian quotient of the Sylow $2$-subgroup (Proposition 3), the latter an exception pointed by [6].

In a joint work with B. Späth, we developed some general methods which also cover the case $\text{Sp}_{2m}(2^m)$ (see [2]) for $n > 2$, $m > 1$. Here, we present however the case of $\text{Sp}_{2m}(2^m)$ which requires some ad hoc analysis (see Section 3).

Notations. When $\ell$ is a prime and $n > 1$ an integer, one denotes by $n_\ell$ the greatest power of $\ell$ dividing $n$ and $n_{\ell'} := n/n_\ell$. If $H$ is a finite group and $X \subseteq \text{Irr}(H)$, one denotes $X_{\ell'} := X \cap \text{Irr}_{\ell'}(H)$.

If $H$ acts on a set $Y$, one denotes by $Y^H$ the subset of fixed points. For finite reductive groups $G^F$ and their characters, we follow the notations of [4].
2. Odd character degrees for $\text{Sp}_{2n}(2)$

Let us denote by $\mathbb{F}$ the algebraic closure of $\mathbb{F}_2$ the field with 2 elements. Let $n \geq 2$ be an integer, let $G = \text{Sp}_{2n}(\mathbb{F})$ with Frobenius endomorphism $F_0 : G \to G$ squaring matrix entries. Let $G = G^0 = \text{Sp}_{2n}(\mathbb{F}_2)$, also denoted by $\text{Sp}_{2n}(2)$ or $\text{Sp}(2n, 2)$.

2.1. The global case

We refer to [4] for the notion of unipotent characters. Let $n \geq 2$ be an integer. For our first lemma, see [9] 6.8.

**Lemma 1.** $\text{Sp}_{2n}(2)$ has five unipotent characters of odd degrees.

**Proposition 2.** $\text{Sp}_{2n}(2)$ has $2^{n+1}$ characters of odd degrees.

**Proof.** Recall $G = \text{Sp}_{2n}(\mathbb{F})$ with Frobenius endomorphism $F_0 : G \to G$ squaring matrix entries. Let $G = G^0 = \text{Sp}_{2n}(2)$ (part of case (a) in [8] Section 8). Note that $G$ has (trivial) connected center.

By [8] p. 164, $\mathbb{F}$ being of characteristic 2, there is an isogeny between $G$ and its dual $G^*$ inducing a bijection between rational semi-simple elements with isomorphism of centralizers of corresponding elements. This, along with property (A) of [8] 7.8 shows that $\text{Irr}(G)$ is in bijection with the disjoint union of the $\mathcal{E}(C_G(s), 1)$'s for $s$ ranging over the semi-simple conjugacy classes of $G$ (see [8] 8.7.6). Through this Jordan decomposition, the degrees are multiplied by $|G^F|_2^2 |C_G(s)|_2^{-1}$, so $|\text{Irr}_2(G)| = \sum s |C_G(s), 1|_2$, a sum over the semi-simple classes of $G$.

Characteristic polynomials provide a bijection between the classes of semi-simple elements of $\text{Sp}_{2n}(2)$ and the set of self dual polynomials $f \in \mathbb{F}_2[X]$ of degree $2n$. If $s$ corresponds with $f$, then $C_G(s) \cong \text{Sp}_{2n}(2) \times C_2$ where $C_2$ is a product of finite linear groups and $2m$ is the multiplicity of $(X - 1)$ in $f$. For a given $m < n$, the number of such classes is $2^{n-m-1}$. This is because one has to count the polynomials $f = (X - 1)^{2m}$ with a self dual $g(X) = 1 + a_1X + \cdots + a_{n-m}X^{n-m-1} + a_{n-m}X^{n-m} + a_{n-m-1}X^{n-m+1} + \cdots + a_1X^{2n-2m-1} + X^{2n-2m}$ such that $g(1) \neq 0$. Such $g$'s are $2^{n-m-1}$, corresponding to the choice of coefficients at degrees $1, 2, \ldots, n - m - 1$ since $g(1) = a_{n-m}$ has to be $= 1$. For $m = n$ (central element) there is 1 conjugacy class ($s = 1$).

The unipotent characters of finite reductive groups of type $A$ in characteristic 2 are of even degrees except the trivial character (see for instance [6] or [9] 6.8). Then Lemma 1 implies that each semi-simple class $s$ corresponding with $m$ as above satisfies $|\mathcal{E}(C_G(s), 1)|_2 = 5$ for $m \geq 2$, $|\mathcal{E}(C_G(s), 1)|_2 = 1$ otherwise. So the above indeed implies $|\text{Irr}_2(G)| = 5 \sum_{m=2}^{n} 2^{n-m-1} + 5 + 2^{n-2} + 2^{n-1} = 5 \cdot 2^{n-2} + 3 \cdot 2^{n-2} = 2^{n+1}$. □

2.2. The local case

We use the description of $\text{Sp}_{2n}(\mathbb{F}_2) \subset G = \text{GL}_{2n}(\mathbb{F}_2)$ as the subgroup of matrices $u$ such that $u(\begin{smallmatrix}0 & J \\ J & 0 \end{smallmatrix})u^T = \begin{smallmatrix}0 & J \\ J & 0 \end{smallmatrix}$ where $J$ denotes the matrix with coefficients $\delta_{i,j+1}1 \leq i, j \leq n$ and $u^T$ denotes transposition (see [4] 15.2). Let $U := \{\begin{smallmatrix}x & s \\ 0 & s^T \end{smallmatrix} \mid x \in G, s \in \text{Sym}_n\}$ where $\text{Sym}_n$ (resp. $V$) is the set of symmetric (resp. upper triangular unipotent) matrices of order $n$ with coefficients in $\mathbb{F}_2$, and one denotes $s = x^{-1}$. We have

**Proposition 3.** $U$ is a Sylow $2$-subgroup of $G = \text{Sp}_{2n}(2)$ for $n \geq 2$. Moreover $N_G(U) = U$ and $U/[U, U]$ is of order $2^{n+1}$.

**Corollary 4.** McKay conjecture (on character degrees) is satisfied in $G = \text{Sp}_{2n}(2)$ for the prime $2$ ($n \geq 2$). That is, the normalizer of any Sylow $2$-subgroup of $G$ has the same number of characters of odd degrees as $G$ itself.

**Proof.** By Proposition 3, the irreducible characters of $N_G(U)$ of odd degrees are exactly the linear characters of $U$. So their number is the cardinality of $U/[U, U]$, that is $2^{n+1}$ thanks to Proposition 3 again. Combining with Proposition 2 gives our claim. □

**Proof of Proposition 3.** Note that $U$ equals the group of elements over $\mathbb{F}_2$ of a rational Borel subgroup (see [4] 15.2), so it equals its normalizer by the axioms of finite BN-pairs which are satisfied by this group. Thus our first claim.

Note also the semi-direct decomposition $U \cong \text{Sym}_n \rtimes V$ for the action of $V$ on $\text{Sym}_n$ given by $x.s = xsx^T$ for $x \in V$, $s \in \text{Sym}_n$. Since $\text{Sym}_n$ is abelian and since the Sylow $2$-subgroup $V$ of $\text{GL}_{2n}(\mathbb{F}_2)$ is known to satisfy $|V|/|V : V| = 2^{n-1}$ (see for instance [4] p. 129 and [6]), our claim about $U/[U, U]$ reduces to show that $\text{Sym}_n/\text{Sym}_n \cdot V$ is of order 4. So we have to prove that the sum $S' = \sum_{x \in V} \theta_s(\text{Sym}_n)$ of images of endomorphisms $\theta_s : s \mapsto xsx^T - s$ of $\text{Sym}_n$ has codimension 2.

For $1 \leq i, j \leq n$, let us denote by $E_{ij}$ the usual elementary matrix of order $n$. We have $E_{ij} + E_{ji} \in S'$ for any $1 \leq i < j \leq n$, by computing $\theta_s(s)$ for $s = E_{ij}$, $i = I_n + E_{ij}$. We also have $E_{ij} + E_{ji} \in S'$ for any $1 \leq i < j \leq n$ with $(i, j) \neq (n-1, n)$ (taking $s = E_{ij} + E_{ji}$ and $x = I_n + E_{ik}$ for some $k > i, k \neq j$). This shows that $S'$ contains the $E_{ij}$ plus $E_{ji}$'s for $1 \leq i < j \leq n$ with $(i, j) \neq (n-1, n)$, along with $E_{11}, E_{22}, \ldots, E_{n-2,n-2}$ and $E_{n-1,n} + E_{n,n-1} + E_{n-1,n-1}$. This makes a subspace of codimension

$2^{n+1} - 1 = 2^{n+1}$.
2 in $\text{Sym}_n$, a supplement subspace being generated by $E_{n-1,n-1}$ and $E_{n,n}$. The action of $V$ on the quotient is easily checked to be trivial (one just has to check the images of $E_{n-1,n-1}$ and $E_{n,n}$ by $\theta_k$ for $x = I_n + E_{ij}$ — which we just did above — since the latter generate $V$ as a group, using again the fact that the field has two elements). So this subspace is indeed the sum of the images of all the $\theta_k$’s for $x \in V$. □

**Theorem 5.** Let $n \geq 3$ be an integer. Then $\text{Sp}_{2n}(2)$ is a simple group that satisfies the conditions of [7] Section 10 for all prime numbers.

**Proof.** When $n = 3$, $\text{Sp}_6(2)$ satisfies the theorem by [10] 4.1. When $n > 3$, $\text{Sp}_{2n}(2)$ has trivial Schur multiplier and trivial outer automorphism group (see [5]), so the checking required by [7] just amounts to the McKay conjecture itself (see [7] 10.3). For $\ell = 2$, it is Corollary 4. In the case of other primes, this is a consequence of Malle’s parametrization [9] 7.8 along with Späth’s extensibility results (see [11] 1.2, [12] 1.2, 8.4). □

3. $\text{Sp}_4(2^m)$

**Theorem 6.** Let $m \geq 2$ be an integer. Then $\text{Sp}_4(2^m)$ is a simple group that satisfies the conditions of [7] Section 10 for all prime numbers.

We keep $F$ as above and let $G = \text{Sp}_4(F)$. We denote by $T_0$ its diagonal torus and $(F, +) \rightarrow G$, $t \mapsto x_0(t)$ its minimal unipotent $F$-stable torus, and latter Levi subgroups, we refer to [4] p. 113). Arguing as in the proof of [9] 5.14, any Sylow $\ell$-subgroup $P$ has a unique maximal toral elementary abelian subgroup whose normalizer $N$ in $G$ is then also $N := N_G(S_\ell) = N_G(T_\ell)$. It is stable by any automorphism $\sigma$ such that $\sigma(P) = P$. From what has been said about possible $\sigma$’s, and noting that $N$ has an abelian normal subgroup $T_\ell'$ with $\ell'$ index, we see that we must just prove that

$$|\text{Irr}_{\ell'}(G)^{F_\ell}| = |\text{Irr}(N)^{x_{F_\ell'}}| \quad (E)$$

for any $F'$ a power of $F_0$ and some $x \in G$ is such that $F'(S_\ell) = S_\ell^x$.

Bringing $(T_\ell, F)$ to $(T_0, w_\ell F)$ by conjugacy with some $g \in G$ such that $g^{-1}F(g) \in w_\ell T_0$, we may rewrite the above as

$$|\text{Irr}_{\ell'}(G)^{F'_\ell}| = |\text{Irr}_0(N_0T_0)^{w_\ell F'}|^\ell \quad (E')$$

when $F''$ is an isogeny commuting with $w_\ell F$ and is in the same class as $F'$ mod inner automorphisms of $G$.

Recall Malle’s bijection $\text{Irr}_{\ell'}(G) \rightarrow \text{Irr}_{\ell'}(N)$ which, among other properties, sends components of $R_\ell^G \theta$ to components of $\text{Ind}_{T_\ell}^G \theta$ for relevant $\theta \in \text{Irr}(T_\ell^G)$ (see [9] Section 7.1).

Let us first look at regular characters $\pm R_\ell^G(\theta)$. They are of degree $\ell^r$ if and only if $T_\ell = C_G(S_\ell)$ (see [9] 4.6). Such a character is fixed by $F_0$ if and only if $F'(T_\ell, \theta)$ and $(T_\ell, \theta)$ are $G^F$-conjugate (see [1] Section 2.1.2). This is equivalent to $x_{F'}(\theta)$ being $N_0T_0$-conjugate to $\theta$ ([9] 5.11). This is also the criterion for $\text{Ind}_{T_\ell}^G(\theta)$ being $x_{F'}$-fixed as can be seen easily from the definition of induced characters. Thus our claim (E).

Let us now turn to unipotent characters. From [9] 6.5, we know that they have to be in $E(G^F, T_\ell)$, the set of irreducible characters occurring in the generalized character $R_\ell^G$. So we have to check that $E(G^F, T_\ell)^{F'}$ and $\text{Irr}(N/T_\ell^F)^{F'}$ have same cardinality.

As for the first set, one knows that among the six unipotent characters of $\text{Sp}_4(2^m)$, only the two that are of generic degree $\frac{1}{2}q(q^2 + 1)$ are not fixed by $F_0$ (see [9] 3.9.a). Those are among unipotent characters of degree prime to $\ell$ only when $\ell = 1$ or 2. So it suffices to check that all characters of $N_0/T_\ell^F$ but 2 are fixed by $x_{F'}$ in case $e = 1$ or 2 and $F'$ is an odd power of $F_0$, and that all are fixed otherwise.

In cases $e = 1$ or 2, $w_1 = 1$, $w_2 = s_1s_2s_1$, both are fixed by $F_0$, so one may take $F'' = F'$ in $(E')$ above. Recall that $F_0$ acts on $W$ by permuting $s_1$ and $s_2$. The group $W$ is dihedral of order 8, so $F_0$ induces an automorphism of order two of $W_{ab}$, so two linear characters out of four are $F_0$-fixed, while the character of degree two is fixed. Hence our claim for any...
odd power of $F'$. In the case of an even power, the action is trivial, as expected. In the case $e = 4$, one may take $w_4 = s_{152}$ and $F'' = (s_1 F'_0)^a$ when $F' = (F'_0)^a$. Then the action of $F''$ on $(N_G(T_0)w_4^F)w_4 = C_W(w_4)$ is trivial.

We now assume $\mathcal{E}(G, s)^F \neq \emptyset$ for an $s$ that is neither central nor regular. The group $C_G(s)$ is always a Levi subgroup of $G$ (see proof of Proposition 2 above) and by [9] 6.5 it must contain a Sylow $\phi_1$-torus. A proper $F$-stable Levi subgroup of $G$ can contain a $\phi_1$-Sylow for types $(L_{s_1}, F)$ and $(L_{s_2}, F)$ and a $\phi_2$-Sylow for types $(L_{s_1}, s_2s_1s_2F)$ and $(L_{s_2}, s_1s_2s_1F)$. In each case the corresponding finite group has two unipotent characters, the trivial and the Steinberg characters, of distinct degrees, so that for an $s$ whose class is $F'$-stable with such a centralizer in the dual, $\mathcal{E}(G, s)$ has two elements with distinct degrees, so $F'$ acts trivially on $\mathcal{E}(G, s)$.

The corresponding statement on the local side is as follows: if $\theta$ is a non-regular non-central linear character of $T_0^wF$, then $\text{Ind}_{T_0^wF}^{N_G(T_0)^wF} \theta$ has two elements both $F''$-fixed if $F''(\theta) \in N_G(T_0)^wF. \theta$. This holds because non-regularity implies $(N_G(T_0)^wF)_{\theta}/T_0^wF$ is of order 2, but then $F''$ can act only trivially on it. 

References

[1] O. Brunat, On the inductive McKay condition in the defining characteristic, Math. Z. 263 (2) (2009) 411–424.
[2] M. Cabanes, B. Späth, Equivariance and extendibility in finite reductive groups with connected center, in preparation, 2011.
[3] R. Carter, Simple Groups of Lie Type, Wiley, New York, 1972.
[4] F. Digne, J. Michel, Representations of Finite Groups of Lie Type, Cambridge University Press, 1991.
[5] D. Gorenstein, R. Lyons, R. Solomon, The Classification of the Finite Simple Groups, Math. Surveys Monogr., vol. 3, Amer. Math. Soc., Providence, 1998.
[6] R. Howlett, On the degrees of Steinberg characters of Chevalley groups, Math. Z. 135 (1974) 125–135.
[7] M. Isaacs, G. Malle, G. Navarro, A reduction theorem for McKay conjecture, Invent. Math. 170 (2007) 33–101.
[8] G. Lusztig, Irreducible representations of finite classical groups, Invent. Math. 43 (1977) 125–175.
[9] G. Malle, Height 0 characters of finite groups of Lie type, Represent. Theory 11 (2007) 192–220.
[10] G. Malle, The inductive McKay condition for simple groups not of Lie type, Comm. Algebra 36 (2) (2008) 455–463.
[11] B. Späth, Sylow $d$-tori of classical groups and the McKay conjecture II, J. Algebra 323 (2010) 2494–2509.
[12] B. Späth, Inductive McKay condition in defining characteristic, preprint, arXiv:1009.0463, 2010.