Abstract—In this paper, we focus on the “positive” $l_2$ induced norm of discrete-time linear time-invariant systems where the input signals are restricted to be nonnegative. To cope with the nonnegativity of the input signals, we employ copositive programming as the mathematical tool for the analysis. Then, by applying an inner approximation to the copositive cone, we derive numerically tractable semidefinite programming problems for the upper and lower bound computation of the “positive” $l_2$ induced norm. This norm is typically useful for the stability analysis of feedback systems constructed from an LTI system and nonlinearities where the nonlinear elements provide only nonnegative signals. As a concrete example, we illustrate the usefulness of the “positive” $l_2$ induced norm for the stability analysis of recurrent neural networks with activation functions being rectified linear units.

Keywords: $l_2$ induced norm, nonnegative input signals, copositive programming, stability, recurrent neural networks.

I. INTRODUCTION

The $l_2$ ($L_2$) induced norm plays a central role in stability and performance analysis of discrete-time (continuous-time) feedback systems [1]. As is well known, small-gain stability criterion allows us to assess the stability of feedback systems constructed from two subsystems by evaluating their $l_2$ induced norms. One of the key discoveries on the $l_2$ induced norm of (finite-dimensional) linear time-invariant (LTI) systems would be KYP lemma [2], which characterizes the induced norm by a semidefinite programming problem (SDP). It should be noted that, even though the $l_2$ induced norm is defined in time-domain, the core in deriving KYP lemma is the treatments of LTI systems in frequency-domain.

In the standard $l_2$ induced norm analysis of LTI systems, we of course presume that the input signals are sign indefinite in time-domain. On the contrary, in this paper, we focus on the “positive” $l_2$ induced norm of discrete-time LTI systems where the input signals are restricted to be nonnegative. As clarified in this paper, this norm is typically useful for the stability analysis of feedback systems constructed from an LTI system and nonlinearities where nonlinear elements provide only nonnegative signals. It is also true that the norm analysis is partly motivated from our preceding studies on positive systems [3], [4], where the treatments of nonnegative signals are essentially important.

The analysis of the “positive” $l_2$ induced norm for discrete-time LTI systems is mathematically challenging due to the following reasons:

(i) The nonnegativity constraint on the input signals is a genuine time-domain constraint and hence it does not allow us to carry out the analysis in frequency-domain.

(ii) Even though SDP is commonly used for LTI system analysis, the positive semidefinite cone employed in SDP has no functionality to distinguish nonnegative vectors or signals.

To get around these difficulties and cope with the nonnegativity of the input signals, we loosen the positive semidefinite cone to the copositive cone and employ copositive programming (COP) [5] as a mathematical tool for the analysis. COP is a convex optimization problem on the positive semidefinite cone, but unfortunately known to be numerically intractable [5]. Therefore, by further applying an inner approximation to the copositive cone, we derive numerically tractable SDPs for upper and lower bound computation of the “positive” $l_2$ induced norm. We illustrate the usefulness of the “positive” $l_2$ induced norm for the stability analysis of recurrent neural networks (RNNs) with activation functions being rectified linear units (ReLU).

Recently, the usefulness of RNNs is widely recognized for analysis and estimation of time series generated by (hidden) dynamical systems. This is achieved by incorporating feedback loops in the networks. However, the existence of feedback loops could be a source of instability, and stability analysis of RNNs remains to be an outstanding issue in the fields of neural network and machine learning [6], [7], [8]. By making use of the fact that ReLUs only provide nonnegative signals, we derive novel small-gain type stability conditions for RNNs on the basis of the “positive” $l_2$ induced norm. We illustrate the effectiveness of the new stability tests by numerical examples.

We use the following notation in this paper. The set of natural numbers is denoted by $\mathbb{N}$. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n \times m}$, and the set of $n \times m$ entrywise nonnegative (strictly positive) matrices is denoted by $\mathbb{R}_{+}^{n \times m}$ ($\mathbb{R}_{++}^{n \times m}$). For a matrix $A$, we also write $A \geq 0$ ($A > 0$) to denote that $A$ is entrywise nonnegative (strictly positive). For $A \in \mathbb{R}^{n \times n}$, we define $\text{He}(A) := A + A^T$.

We denote the set of $n \times n$ real symmetric matrices by $S_n$, and define the inner product of $A, B \in S_n$ by $\langle A, B \rangle :=$
tracing. A set \( K_n \subset S_n \), the interior of \( K_n \) is denoted by \( K_n^\circ \). The dual cone of \( K_n \subset S_n \) is defined by \( K_n^\circ := \{ A \in S_n | \forall B \in K_n, \langle A, B \rangle \geq 0 \} \). If \( K_n \subset S_n \) is a closed convex cone which has a nonempty interior and pointed (i.e., \( x, -x \in K_n \) \( \Rightarrow \) \( x = 0 \)), then \( K_n \) is called proper [9]. The interior of a proper cone \( K_n \) is defined as follows [10]:
\[
K_n^\circ = \{ A \in S_n | \forall B \in K_n^\circ \setminus \{0\}, \langle A, B \rangle > 0 \}.
\]

For a proper cone \( K_n \subset S_n \) and \( P \in S_n \), by defining [9], we define a partial ordering \( P \preceq_K 0 \) (\( P \preceq_K 0 \)) which means \( P \in K_n \) \((-P \in K_n)\). We also write \( P \succeq_K 0 \) (\( P \preceq_K 0 \)) if \( P \in K_n^\circ \) \((-P \in K_n^\circ)\). The subscript \( K \) is often omitted if \( K \) is the positive semidefinite cone. For an \( S_n \)-valued affine function \( F(x) \), we call the inequalities of the form \( F(x) \preceq_K 0 \) and \( F(x) \succeq_K 0 \) (\( F(x) \preceq_K 0 \)) the linear matrix inequalities (LMIs) on the cone \( K \).

For a discrete-time signal \( w \) defined over the time interval [0, \( \infty \)], we define
\[
\|w\|_2 := \sqrt{\sum_{k=0}^{\infty} |w(k)|^2}
\]
where \( v \in \mathbb{R}^{n^2} \) we define \( \|v\|_2 := \sqrt{\sum_{j=1}^{n^2} v_j^2} \). We also define
\[
l_2 := \{ w : \|w\|_2 < \infty \},
\]
\[
l_2^+ := \{ w : w \in l_2, w(k) \geq 0 (\forall k \geq 0) \}
\]
and
\[
l_{2e} := \{ w : w_\tau \in l_2, \forall \tau \in [0, \infty) \}
\]
where \( w_\tau \) is the truncation of the signal \( w \) up to the time instant \( \tau \) and defined by
\[
w_\tau(k) = \begin{cases} w(k) & (k \leq \tau), \\ 0 & (k > \tau). \end{cases}
\]

For an operator \( H : l_{2e} \ni w \to z \in l_{2e} \), we define its \( l_2 \) induced norm by
\[
\|H\|_2 := \sup_{w \in l_{2e}, \|w\|_2 = 1} \|z\|_2.
\]
The main interest of the present paper concerns a variant of the \( l_2 \) induced norm with input being restricted to be nonnegative which is defined by
\[
\|H\|_{2^+} := \sup_{w \in l_{2e^+}, \|w\|_2 = 1} \|z\|_2.
\]
With a little abuse of notation, for a matrix \( M \in \mathbb{R}^{n \times m} \), we define
\[
\|M\|_2 := \sup_{v \in \mathbb{R}^n, \|v\|_2 = 1} |Mv|_2 \quad (= \sigma_{\text{max}}(M)),
\]
\[
\|M\|_{2^+} := \sup_{v \in \mathbb{R}^{n^2}, \|v\|_2 = 1} |Mv|_2
\]
where \( \sigma_{\text{max}}(M) \) is the maximal singular value of \( M \).

II. COPOSITIVE PROGRAMMING PROBLEM (COP)

Cooperative Programming Problem (COP) is a convex optimization problem in which we minimize a linear objective function over the LMI constraints on the copositive cone [5]. In this section, we summarize its basics.

A. Convex Cones Related to COP

Let us review the definition and the property of convex cones related to COP.

**Definition 1:** [11] The definition of proper cones \( \mathcal{PSD}_n, \mathcal{COP}_n, \mathcal{CP}_n, \mathcal{N}_n \), and \( \mathcal{DN}_n \) in \( S_n \) are as follows.\n
1) \( \mathcal{PSD}_n := \{ P \in S_n : \forall x \in \mathbb{R}^n, x^TPx \geq 0 \} \) = \{ \text{max } \{ \exists B \text{ s.t. } P = B B^T \} \} is called the positive semidefinite cone.
2) \( \mathcal{COP}_n := \{ P \in S_n : \forall x \in \mathbb{R}^n, x^TPx \geq 0 \} \) is called the copositive cone.
3) \( \mathcal{CP}_n := \{ P \in S_n : \exists B \geq 0 \text{ s.t. } P = B B^T \} \) is called the completely positive cone.
4) \( \mathcal{N}_n := \{ P \in S_n : \forall x \in \mathbb{R}^n, x^TPx \geq 0 \} \) is called the nonnegative cone.
5) \( \mathcal{PSD}_n + \mathcal{N}_n := \{ P + Q : P \in \mathcal{PSD}_n, Q \in \mathcal{N}_n \} \)
This is the Minkowski sum of the positive semidefinite cone and the nonnegative cone.
6) \( \mathcal{DN}_n := \mathcal{PSD}_n \cap \mathcal{N}_n \) is called the doubly nonnegative cone.

From Definition 1 we clearly see that the following inclusion relationships hold:
\[
\mathcal{CP}_n \subset \mathcal{DN}_n \subset \mathcal{PSD}_n \subset \mathcal{PSD}_n + \mathcal{N}_n \subset \mathcal{COP}_n,
\]
\[
\mathcal{CP}_n \subset \mathcal{DN}_n \subset \mathcal{N}_n \subset \mathcal{PSD}_n + \mathcal{N}_n \subset \mathcal{COP}_n.
\]

In particular, when \( n \leq 4 \), it is known that \( \mathcal{COP}_n = \mathcal{PSD}_n + \mathcal{N}_n \) and \( \mathcal{CP}_n = \mathcal{DN}_n \) hold [11] On the other hand, on the duality of these cones, \( \mathcal{CP}_n \) and \( \mathcal{CP}_n \) are dual to each other, \( \mathcal{PSD}_n + \mathcal{N}_n \) and \( \mathcal{DN}_n \) are dual to each other, and \( \mathcal{PSD}_n \) and \( \mathcal{N}_n \) are self-dual.

When dealing with analysis problems of LTI systems, we often need to consider strict LMI conditions. With this fact in mind, let us review the characterization of the interiors of the convex cones in Definition 1.

**Proposition 1:** [12], [13] The interiors of the proper cones given in Definition 1 are characterized as follows.

1) \( \mathcal{PSD}_n^\circ = \{ P \in S_n : \forall x \in \mathbb{R}^n \setminus \{0\}, x^TPx > 0 \} \) = \{ \text{max } \{ \exists B \text{ s.t. } P = B B^T, \text{ rank}(B) = n \} \}.
2) \( \mathcal{COP}_n^\circ = \{ P \in S_n : \forall x \in \mathbb{R}^n \setminus \{0\}, x^TPx > 0 \} \) = \{ \text{max } \{ \exists B \text{ s.t. } P = B B^T, \text{ rank}(B) = n \} \}.
3) \( \mathcal{CP}_n^\circ = \{ P \in S_n : \exists B \text{ s.t. } P = B B^T, \text{ rank}(B) = n \} \).
4) \( \mathcal{N}_n^\circ = \{ P \in S_n : \forall x \in \mathbb{R}^n, x^TPx > 0 \} \).
5) \( \mathcal{PSD}_n + \mathcal{N}_n^\circ = \mathcal{PSD}_n^\circ + \mathcal{N}_n^\circ \).
6) \( \mathcal{DN}_n^\circ = \mathcal{PSD}_n^\circ \cap \mathcal{N}_n^\circ \).

B. Basic Properties of COP

COP is a convex optimization problem on the copositive cone and its dual is a convex optimization problem on the completely positive cone. As mentioned in [5], the problem to determine whether a given symmetric matrix is copositive or not is a co-NP complete problem, and the problem to determine whether a given symmetric matrix is completely positive or not is an NP-hard problem. Therefore, it is hard to solve COP numerically in general. However, since the problem to determine whether a given matrix is in \( \mathcal{PSD} + \mathcal{N} \)
\( N N \) or in \( D N N \) can readily be reduced to SDPs, we can numerically solve the convex optimization problems on the cones \( \mathcal{PSD} + N N \) and \( D N N \) easily. Moreover, when \( n \leq 4 \), it is known that \( \mathcal{COP}_n = \mathcal{PSD}_n + N N_n \) and \( \mathcal{COP}_n = D N N_n \) as stated above, and hence those COPs with \( n \leq 4 \) can be reduced to SDPs.

III. \( l_2 \) Induced Norm Analysis of LTI Systems for Nonnegative Inputs

A. Problem Description

Let us consider the discrete-time LTI system \( G \) given by

\[
G: \begin{cases}
    x(k+1) = Ax(k) + Bw(k), \quad x(0) = 0,
    \\
    z(k) = Cx(k) + Dw(k)
\end{cases}
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times n} \), \( C \in \mathbb{R}^{n \times n} \), and \( D \in \mathbb{R}^{n \times n} \). We assume that the system \( G \) is stable, i.e., the matrix \( A \) is Schur-Cohn stable. It is well known that the \( l_2 \) induced norm \( \|G\|_2 \) defined by (2) coincides with the \( H_\infty \) norm for stable LTI systems and plays an essential role in stability analysis of feedback systems. In this paper, we are interested in computing the \( l_2 \) induced norm where the input signal \( w \) is constrained to be nonnegative. Namely, we focus on the computation of the “positive” \( l_2 \) induced norm \( \|G\|_{2+} \) defined by (3). From the definition (3), it is very clear that \( \|G\|_{2+} \leq \|G\|_2 \). Here, note that a discrete-time LTI system of the form (5) is said to be externally positive if its output is nonnegative for any nonnegative input under zero initial state [14]. Then, in this case, it is well known that \( \|G\|_{2+} = \|G\|_2 \), see, e.g., [15], [16].

B. Basic Results

The next result forms an important basis of this study.

**Theorem 1:** For the stable LTI system \( G \) described by (5) and given \( \gamma > 0 \), suppose there exist \( P \in \mathcal{PSD}_n \) and \( Q \in \mathcal{COP}_{n+w} \) such that

\[
L(A, B, C, D, P, Q, \gamma) < 0
\]

where

\[
L(A, B, C, D, P, Q, \gamma) := \begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I_{n+w} + Q \end{bmatrix} + \begin{bmatrix} A & B^T \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I_{n+z} \end{bmatrix} \begin{bmatrix} A & B^T \\ C & D \end{bmatrix}.
\]

Then we have \( \|G\|_{2+} < \gamma \).

**Proof of Theorem 1:** We first note that if (6) holds with \( P \) and \( Q \) then there exists a sufficiently small \( \varepsilon > 0 \) such that the next condition holds with exactly the same \( P \), \( Q \) and \( \tilde{\gamma} := \gamma - \varepsilon \):

\[
\begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I_{n+w} + Q \end{bmatrix} + \begin{bmatrix} A & B^T \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I_{n+z} \end{bmatrix} \begin{bmatrix} A & B^T \\ C & D \end{bmatrix} < 0.
\]

With this fact in mind, let us consider the trajectory of the state \( x \) corresponding to the input signal \( w \in \ell_2 \) with \( \|w\|_2 = 1 \) for the system \( G \). Then, from (7), we have

\[
\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I_{n+w} + Q \end{bmatrix} + \begin{bmatrix} A & B^T \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I_{n+z} \end{bmatrix} \begin{bmatrix} A & B^T \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \leq 0,
\]

or equivalently,

\[
\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I_{n+w} + Q \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \leq 0,
\]

or equivalently,

\[
\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I_{n+w} + Q \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \leq 0,
\]

\( (k = 0, 1, \cdots) \).

By summing up the above inequalities up to \( k = N \), we have

\[
x(N+1)^T P x(N+1) - \tilde{\gamma}^2 \sum_{k=0}^{N} \|w(k)\|^2_2
\]

\[
+ \sum_{k=0}^{N} \|w(k)\|^2_2 \leq 0.
\]

We see \( x(N+1)^T P x(N+1) \geq 0 \) since \( P \in \mathcal{PSD}_n \). On the other hand, since \( w \in \ell_{2+} \) and since \( Q \in \mathcal{COP}_{n+w} \), we see that

\[
\sum_{k=0}^{N} \|w(k)\|^2_2 \leq 0.
\]

Therefore, by letting \( N \to \infty \), we have \( \|x\|_2^2 \leq \tilde{\gamma}^2 \|w\|_2^2 = \gamma^2 \). Since this condition holds for arbitrary \( w \in \ell_{2+} \), we can conclude that

\[
\|G\|_{2+} = \sup_{\|w\|_2 = 1} \|z\|_2 \leq \tilde{\gamma} < \gamma.
\]

This completes the proof.

On the basis of Theorem 1, let us consider the COP:

\[
\mathcal{COP} := \inf_{\gamma, P, Q} \text{subject to (5)}, \quad P \in \mathcal{PSD}_n, \quad Q \in \mathcal{COP}_{n+w}.
\]

In relation to this COP, recall that

\[
\|G\|_2 = \inf_{\gamma, P} \text{subject to (4)}, \quad P \in \mathcal{PSD}_n, \quad Q = 0.
\]

It follows that \( \|G\|_{2+} \leq \mathcal{COP} \leq \|G\|_2 \). Unfortunately, as we have already mentioned, it is hard to solve the COP (9) in general. However, an upper bound of \( \mathcal{COP} \) can be computed efficiently by replacing \( \mathcal{COP} \) in (9) by \( \mathcal{PSD} + N N \) as follows:

\[
\mathcal{COP} := \inf_{\gamma, P, Q} \text{subject to (10)}, \quad P \in \mathcal{PSD}_n, \quad Q \in \mathcal{PSD}_{n+w} + N N_{n+w}.
\]

Note that this problem is essentially an SDP and hence tractable. We can readily see that \( \|G\|_{2+} \leq \mathcal{COP} \leq \|G\|_2 \).

Up to this point, we have described the basic idea of the (upper bound) computation of \( \|G\|_{2+} \). However, in the case where \( n+w = 1 \), i.e., if the system \( G \) has only a single disturbance input, then it is very clear that \( \mathcal{COP} = \|G\|_2 \).

This is because, since \( \mathcal{COP}_1 = \mathcal{PSD}_1 = \mathbb{R}_+ \), and since the variable \( Q \) enters in block-diagonal part in (5), we see that the optimal value of \( Q \) in COP (9) is zero. Namely, if \( n+w = 1 \), it is impossible to obtain an upper bound of \( \|G\|_{2+} \) which is better than the trivial upper bound \( \|G\|_2 \) if we directly work on (9). In addition, we also deduce from this fact that the improvement of \( \mathcal{COP} \) over \( \|G\|_2 \) might not be significant if \( G \) has a few number of disturbance inputs.

To get around this difficulty, in the next section, we employ the discrete-time system lifting [17].

C. Better Upper Bound Computation by System Lifting

By applying the \( N \)-th order discrete-time lifting [17] to \( \mathcal{COP} \) with \( N \in \mathbb{N} \), we can obtain another discrete-time LTI system \( \tilde{G}_N \) of the form...
\[ \hat{G}_N : \begin{cases} \hat{x}(k+1) = \hat{A}_N x(k) + \hat{B}_N \hat{w}(k), \quad \hat{x}(0) = 0, \\ \hat{z}(k) = \hat{C}_N x(k) + \hat{D}_N \hat{w}(k) \end{cases} \]  
(11)

where  
\[ \hat{A}_N := A^N, \quad \hat{B}_N := [A^{N-1}B \ldots AB B], \]
\[ \hat{C}_N := \begin{bmatrix} C \\ CA \\ \vdots \\ C A^{N-1} \end{bmatrix}, \quad \hat{D}_N := \begin{bmatrix} D \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}. \]  
(12)

and 
\[ \hat{w}(k) = \begin{bmatrix} w(kN) \\ w(kN + 1) \\ \vdots \\ w((k+1)N - 1) \end{bmatrix}, \quad \hat{z}(k) = \begin{bmatrix} z(kN) \\ z(kN + 1) \\ \vdots \\ z((k+1)N - 1) \end{bmatrix}. \]  
(13)

It is very clear that \( G \) is stable (i.e., \( \hat{A}_N \) is Schur-Cohn stable) if and only if \( \hat{G}_N \) is stable (i.e., \( \hat{A}_N \) is Schur-Cohn stable). In addition, from \((13)\), we can readily see that \( \| G \|^2 = \| \hat{G}_N \|^2 \) and \( \| G \|^2 = \| \hat{G}_N \|^2 \). With these facts in mind, for given \( N \in \mathbb{N} \), let us define 
\[ \gamma_{N+} := \inf_{\gamma, P, Q} \text{ subject to } \]
\[ L(\hat{A}_N, \hat{B}_N, \hat{C}_N, \hat{D}_N, P, Q, \gamma) < 0, \]
\[ P \in \mathcal{PSD}_n, \quad Q \in \mathcal{COP}_{n \times n}, \]
\[ \gamma_{N+} := \inf_{\gamma, P, Q} \text{ subject to } \]
\[ P \in \mathcal{PSD}_N, \quad Q \in \mathcal{PSD}_{N \times n} + \mathcal{NN}_{n \times n}. \]  
(14)

Then, we have \( \forall N \in \mathbb{N} \) that 
\[ \| G \|^2 = \| \hat{G}_N \|^2 \leq \gamma_{N+} \leq \gamma_{N+} \leq \| \hat{G}_N \|^2 = \| G \|^2. \]

Namely, \( \gamma_{N+} \) and \( \gamma_{N+} \) are upper bounds of \( \| G \|^2 \), and the latter is easy to compute. In particular, it is worth mentioning that we can obtain better (no worse) upper bounds by increasing \( N \) as shown in the next theorem. The proof of this theorem is given at appendix section.

**Theorem 2:** For given \( N_1, N_2 \in \mathbb{N} \) with \( N_2 = pN_1 \) (\( \exists p \in \mathbb{N} \)), we have 
\[ \gamma_{N_2+} \leq \gamma_{N_1+}, \quad \gamma_{N_2+} \leq \gamma_{N_1+}. \]  
(16)

We demonstrate the effectiveness of the lifting-based treatment in Subsection

**D. Lower Bound Computation by System Lifting**

In the preceding subsection, we consider the upper bound computation of \( \| G \|^2 \) for the discrete-time LTI system \( G \). However, if we merely compute upper bounds, it is inherently impossible to evaluate their accuracy. To remedy this, in the section, we consider a method for lower bound computation.

In \((12)\), recall that the matrix \( \hat{D}_N \) captures the input-output behavior of the system \( G \) up to time instant \( N - 1 \). Namely, we have 
\[ \begin{bmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{bmatrix} = \hat{D}_N \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{bmatrix}. \]

Therefore we can readily see that 
\[ \| G \|^2 \geq \| \hat{D}_N \|^2 \quad (\forall N \in \mathbb{N}). \]  
(17)

It is also true that \( \| \hat{D}_N \|^2 \) is monotonically non-decreasing with respect to \( N \in \mathbb{N} \) and \( \| G \|^2 = \lim_{N \to \infty} \| \hat{D}_N \|^2 \).

Therefore, if we can compute \( \| \hat{D}_N \|^2 \) for each \( N \in \mathbb{N} \) exactly, we can construct a monotonically nondecreasing sequence of the lower bounds that converges to \( \| G \|^2 \).

As is well known, “the maximal singular value” \( \| \hat{D}_N \|^2 \) is characterized by the SDP:
\[ \| \hat{D}_N \|^2 = \inf_{\gamma_{N+}} \gamma_{N+} \text{ subject to } \gamma_{N+} I_{N \times n} - \hat{D}_N^T \hat{D}_N > 0. \]

Similarly, we see that the “positive” maximal singular value \( \| \hat{D}_N \|^2 \) is characterized by the COP:
\[ \| \hat{D}_N \|^2 = \inf_{\gamma_{N+}} \gamma_{N+} \text{ subject to } \gamma_{N+} I_{N \times n} - \hat{D}_N^T \hat{D}_N \succ 0. \]

This is essentially an SDP. Nevertheless, it is quite important to note that the SDP \((18)\) provides an upper bound of (the square of) \( \| \hat{D}_N \|^2 \) and hence this is not fully fitted to our purpose here. Note that our goal here is to compute a lower bound of \( \| G \|^2 \) by way of \((17)\) and hence what is required is to compute a lower bound of \( \| \hat{D}_N \|^2 \).

To compute a lower bound of \( \| \hat{D}_N \|^2 \) in numerically tractable fashion, let us consider the dual of the SDP \((18)\) which is given as follows:
\[ \sup_{Z_N} \text{ trace}(\hat{D}_N^T \hat{D}_N Z_N) \text{ subject to } \]
\[ \text{ trace}(Z_N) = 1, \quad Z_N \in \mathcal{PSD}_{N \times n}. \]  
(19)

Again, this is essentially an SDP and hence numerically tractable. Here, it is very clear that the (primal) SDP \((18)\) has an interior point solution. Therefore, there is no duality gap between the SDPs \((18)\) and \((19)\), and in particular the SDP \((19)\) has an optimal solution \((18)\). It follows from the zero duality gap that the SDP \((19)\) again provides an upper bound of \( \| \hat{D}_N \|^2 \) and hence we have to go further.

With the above mentioned facts in mind, let us denote by \( Z_N^* \) an optimal solution of the SDP \((19)\). Moreover, let \( v_N^* \in \mathbb{R}_+^{n^2} \) denote the unit eigenvector corresponding to the maximal eigenvalue of \( Z_N^* \). It should be noted that, from Perron-Frobenius theorem \((19)\), we can confirm that \( v_N^* \) is certainly nonnegative since \( Z_N^* \geq 0 \). Then, if we define \( \gamma_{N} := |D_N v_N^*|^2 \), it is very clear that \( \gamma_{N} \leq \| D_N \|^2 \).

This idea of lower bound computation comes from the rank-one exactness verification test for LMI relaxation, which is frequently employed in the literature, see, e.g., \([20]\). Namely, it is straightforward to see that if \( \text{rank}(Z_N^*) = 1 \) then \( \gamma_{N} = \| \hat{D}_N \|^2 \). We finally note that \( \gamma_{N} = \| \hat{D}_N \|^2 \) always holds if \( \hat{D}_N^T \hat{D}_N \geq 0 \). This is a direct consequence again from Perron-Frobenius theorem.
E. Exact Computation

In some special cases, we can compute \( \|G\|_{2+} \) exactly by solving an SDP. For instance, let us consider the case where \( G \) is “static” and its input-output property is given by \( z(k) = Dw(k) \). Then, we can readily see that

\[
\|G\|_{2+}^2 = \|D\|_{2+}^2 = \inf_{\gamma_{\text{eq}}} \gamma_{\text{eq}} - D^TD \geq \text{coer} 0.
\]

Since this COP is essentially an SDP if \( n_w \leq 4 \), we arrive at the conclusion that we can compute \( \|G\|_{2+} \) exactly by solving an SDP in the above special case.

F. Numerical Examples

We consider the case where the coefficient matrices of the system (5) are given by

\[
A = \begin{bmatrix}
0.27 & 0.06 & -0.24 & 0.19 \\
-0.26 & -0.18 & 0.35 & 0.43 \\
0.06 & -0.88 & -0.78 & 0.27 \\
-0.07 & 0.11 & -0.25 & -0.01
\end{bmatrix},
B = \begin{bmatrix}
0.68 \\
1.46 \\
-0.22 \\
0.45
\end{bmatrix},
C = \begin{bmatrix}
0.33 & -2.06 & 1.22 & 1.12
\end{bmatrix},
D = 0.05.
\]

By applying the discrete-time system lifting and following the ideas in Subsections III-C and III-D, we computed upper and lower bounds of \( \|G\|_{2+} \). The results are shown in Fig. 1. The \( l_2 \) induced norm of \( G \) turned out to be \( \|G\|_2 = 9.0797 \). On the other hand, the best upper bound obtained by lifting is \( \gamma_{2+} = 6.9034 \) and the best lower bound obtained by lifting is \( \gamma_{2-} = 5.9453 \). The obtained upper bounds are NOT monotonically decreasing, but we can confirm that (16) is surely satisfied.

IV. STABILITY ANALYSIS OF RNN WITH RELU

In this section, we demonstrate the usefulness of the “positive” \( l_2 \) induced norm in stability analysis of Recurrent Neural Networks (RNNs).

A. Basics of RNN and Stability

Let us consider the dynamics of the discrete-time RNNs typically described by

\[
x(k+1) = Ax(k) + W_{in}w(k) + v(k), \\
z(k) = W_{out}x(k), \\
w(k) = \Phi(z(k) + s(k))
\]

where \( x \in \mathbb{R}^n \) is the state and \( A \in \mathbb{R}^{n \times n} \), \( W_{in} \in \mathbb{R}^{m \times n} \), \( W_{out} \in \mathbb{R}^{m \times n} \) are constant matrices with \( A \) being Schur-Cohn stable. On the other hand, note that \( s : [0, \infty) \rightarrow \mathbb{R}^m \) and \( v : [0, \infty) \rightarrow \mathbb{R}^n \) are external input signals and \( \Phi : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is the static activation function typically being nonlinear. The matrices \( W_{in} \) and \( W_{out} \) are constructed from the weightings of the edges in RNN. We assume \( x(0) = 0 \).

In relation to (20), let us define

\[
G := \begin{bmatrix}
\Lambda \\
W_{out} \\
W_{in}
\end{bmatrix}
\]

\[
G_0 := \begin{bmatrix}
\Lambda \\
W_{out} \\
W_{in}
\end{bmatrix},
G_1 := \begin{bmatrix}
\Lambda \\
W_{out} \\
W_{in}
\end{bmatrix}
\]

where the initial states of these three systems are all zeros. Then, the dynamics of the RNN given by (20) can be represented by the block-diagram shown in Fig. 2. We consider the typical case where the activation function is Rectified Linear Unit (ReLU) whose input-output property is given by

\[
\Phi(\xi) = [\phi(\xi_1) \cdots \phi(\xi_m)]^T,
\phi : \mathbb{R} \rightarrow \mathbb{R},
\phi(\eta) = \begin{cases} 
\eta & (\eta \geq 0), \\
0 & (\eta < 0).
\end{cases}
\]

Since here we are dealing with nonlinear systems, it is of prime importance to clarify the definition of “stability.” The definition we employ for the analysis of RNN is as follows.

**Definition 2:** [1] (Finite Gain \( l_2 \) Stability) An operator \( H : l_{2e} \ni u \rightarrow y \in l_{2e} \) is said to be finite gain \( l_2 \) stable if there exists a non-negative constant \( \gamma \) such that \( \|y\|_2 \leq \gamma \|u\|_2 \) holds for any \( u \in l_{2e} \) and \( \tau \in [0, \infty) \).

In the following, we analyze the finite gain \( l_2 \) stability of the operator in RNN shown in Fig. 2 with respect to input \( [s^T \quad v^T]^T \in l_{2e} \) and output \( [z^T \quad w^T]^T \in l_{2e} \). Note that the feedback connection in Fig. 2 is well-posed since its dynamics is given by the state-space equation (20). We also note that we implicitly use the causality of \( G \) and \( \Phi \).

B. Small Gain Type Stability Conditions

The next theorem provides a small gain type stability condition with “positive” \( l_2 \) induced norm.

![Block-Diagram Representation of RNN](image_url)
Theorem 3: The RNN given by (20) with the activation function ReLU given by \( \Phi(z + s) \) is finite-gain \( l_2 \) stable if \( \| G_0 \|_{2+} < 1 \), where \( G_0 \) is given by (21).

Proof of Theorem 3: We can readily see that \( \| \Phi \|_2 = 1 \).

It is also true from \( \Phi(z + s) \) is nonnegative whatever \( z \) and \( s \) are. Therefore, for any \( sT \in l_2 \) and \( \tau \in [0, \infty) \), we have

\[
\parallel z_\tau \parallel_2 = \parallel \left( \begin{array}{c} G(w) \\ v \end{array} \right) \tau \parallel_{l_2} \leq \parallel \left( \begin{array}{c} G(w) \\ v \end{array} \right) \parallel_{l_2} + \| G(0) \|_{l_2} \| (v) \|_{l_2} = \parallel G(\Phi(z + s)\tau) \parallel_{l_2} \leq \parallel G_0(\Phi(z + s)\tau) \parallel_{l_2} + \| G_1 \|_{l_2} \| v \|_{l_2} \leq \parallel G_0 \|_{2+} \parallel \Phi \parallel_2 \| z_\tau \|_2 + \| G_1 \|_{l_2} \| v \|_{l_2} \leq \parallel G_0 \|_{2+} \parallel \Phi \parallel_2 \| z_\tau \|_2 + \| G_1 \|_{l_2} \| v \|_{l_2} \] (23)

\[
\parallel w_\tau \parallel_2 = \parallel \left( \begin{array}{c} G(\Phi(z + s)\tau) \end{array} \right) \parallel_{l_2} \leq \parallel \Phi \parallel_2 \parallel z_\tau \parallel_2 + \| G_1 \| \| v \|_2 \leq \parallel \Phi \parallel_2 \parallel z_\tau \parallel_2 + \| G_1 \| \| v \|_2 \] (24)

If we define

\[
\gamma_0 := \| G_0 \|_2, \quad \gamma_0 := \| G_0 \|_{2+}, \quad \gamma_1 := \| G_1 \|_2
\]

and assume \( \gamma_0 < 1 \), then we readily obtain

\[
\parallel z_\tau \parallel_2 \leq (1 - \gamma_0)^{-1} \| z_\tau \|_2 + \| G_0 \| \| v \|_2, \quad \parallel w_\tau \parallel_2 \leq (1 - \gamma_0)^{-1} \| z_\tau \|_2 + \| G_1 \| \| v \|_2.
\]

It follows from Lemma 3 given in the appendix section that

\[
\parallel \left( \begin{array}{c} z_\tau \\ w_\tau \end{array} \right) \parallel_2 \leq \sqrt{2} \left( (1 - \gamma_0)^{-1} \| z_\tau \|_2 + (1 - \gamma_0)^{-1} \| z_\tau \|_2 + (1 - \gamma_0)^{-1} \| z_\tau \|_2 \right)^{1/2} = \sqrt{2} \quad (25)
\]

holds for all \( sT \in l_2 \) and \( \tau \in [0, \infty) \). Therefore we can conclude that RNN given by (20) with ReLU given by \( \Phi(z + s) \) is finite-gain \( l_2 \) stable if \( \gamma_0 < 1 \).

The stability condition \( \| G_0 \|_{2+} < 1 \) in Theorem 3 is of course a milder condition than the “standard” small gain condition that requires \( \| G_0 \|_2 < 1 \). From Theorem 1, we see that \( \| G_0 \|_{2+} < 1 \) holds if there exist \( \in PSD_n \) and \( Q \in COP_m \) such that

\[
\begin{bmatrix} -P + 0 \\ 0 -I_n \end{bmatrix} + \begin{bmatrix} \Lambda \in PSD_n \\ W_{out} \end{bmatrix} \begin{bmatrix} P \in COP_m \\ 0 -I_n \end{bmatrix} < 0.
\]

C. Scaled Small Gain Type Stability Condition

It is not hard to see that ReLU \( \Phi \) satisfies \( \phi(\xi) = (D^{-1} \Phi D)(\xi) \) for any \( D \in \mathbb{D}^n_{++} \) where \( \mathbb{D}^n_{++} \subset \mathbb{R}^{n \times n} \) stands for the set of diagonal matrices with strictly positive diagonal entries. Therefore we readily deduce that the RNN given by (20) with ReLU given by \( \Phi(z + s) \) is finite-gain \( l_2 \) stable if there exists \( D \in \mathbb{D}^n_{++} \) such that \( \| D^{-1} G_0 D \|_{2+} < 1 \). From Theorem 1, this condition holds if there exist \( \in PSD_n \), \( D \in \mathbb{D}^n_{++} \), and \( Q \in COP_m \) such that

\[
\begin{bmatrix} -P + 0 \\ 0 -I_n \end{bmatrix} + \begin{bmatrix} \Lambda \in PSD_n \\ W_{out} \end{bmatrix} \begin{bmatrix} P \in COP_m \\ 0 -I_n \end{bmatrix} < 0.
\]

We can equivalently translate this nonconvex condition to the convex condition that there exist \( \in PSD_n \), \( D \in \mathbb{D}^n_{++} \), and \( Q \in COP_m \) such that

\[
\begin{bmatrix} -P + 0 \\ 0 -S + Q \end{bmatrix} + \begin{bmatrix} \Lambda \in PSD_n \\ W_{out} \end{bmatrix} \begin{bmatrix} P \in COP_m \\ 0 -S \end{bmatrix} < 0.
\]

We note that the “standard” small gain condition is recovered if we let \( Q = 0 \) in (25).

D. Numerical Examples

In (20), let us consider the case \( \Lambda = 0, W_{out} = I_n \) and

\[
W_{in} = \begin{bmatrix} 0.29 -0.04 \quad 0.02 -0.35 \quad -0.05 -0.12 \\ -0.29 -0.24 \quad -0.01 \quad 0.12 -0.13 -0.17 \quad -0.15 -0.17 -0.18 \quad -0.50 -0.23 -0.40 -0.28 -0.08 \quad 0.14 -0.27 -0.15 -0.13 -0.47 -0.28 \quad 0.14 -0.10 -0.08 -0.14 -0.22 -0.50 \quad -0.11 -0.28 -0.21 -0.14 -0.09 -0.20 \end{bmatrix}
\]

For \( (a, b) = (0, 0) \) we see \( \| G_0 \|_2 = 0.9605 \). Here we examined the finite-gain \( l_2 \) stability over the (time-invariant) parameter variation \( a \in [-8, 8] \) and \( b \in [-8, 8] \). We tested the following stability conditions.

SSG:

\[
\text{Find } P \in PSD_n, S \in \mathbb{D}^n_{++}, \text{ and } Q \in PSD_m + \mathbb{N} \times m \text{ such that (25) holds where } Q = 0.
\]

SSG+COP:

\[
\text{Find } P \in PSD_n, S \in \mathbb{D}^n_{++}, Q \in PSD_m + \mathbb{N} \times m \text{ such that (25) holds.}
\]

It is very clear that if (26) is feasible then (27) is. In Fig. 3, we plot \( (a, b) \) for which the RNN is proved to be stable by the above stability conditions. Both LMIs (26) and (27) turned out to be feasible for \( (a, b) \) with green plot, whereas only (27) turned out to be feasible for \( (a, b) \) with magenta plot. We can clearly see the effectiveness of the present new stability condition with the “positive” \( l_2 \) induced norm.

V. CONCLUSION AND FUTURE WORKS

In this paper, we newly introduced the “positive” \( l_2 \) induced norm of discrete-time LTI systems where the input signals are restricted to be nonnegative. On the basis of the above formulations, we introduce tractable methods for the upper and lower bound computation of the “positive” \( l_2 \) induced norm. We illustrate its usefulness in stability analysis of recurrent neural networks with activation functions being rectified linear units.

The present paper just described basic treatments of the “positive” \( l_2 \) induced norm and its application. In closing, we summarize outstanding issues to be investigated.

1) Treatment of COP: In the present paper, we converted a COP to an SDP by simply replacing COP by \( PSD + \mathbb{N} \times m \). However, this treatment is primitive and hence conservative. In this respect, Lasserre [21] has already shown how to construct a hierarchy of SDPs to solve COP in an asymptotically exact fashion. Nevertheless, this approach does not allow us to handle practical size problems since the size of SDPs grows very rapidly. We need further effort to reduce computational burden for instance by finding out sparsity structure. We plan to rely on efficient first-order methods to solve the specific conic relaxations arising from polynomial optimization problems with sphere constraints [22].
2) Stability Analysis of Lur'e Systems with COP Multipliers: Our (scaled) small-gain type treatment for the stability analysis of RNN might be too shallow in view of advanced integral quadratic constraint (IQC) theory [23]. Namely, for the stability analysis of feedback systems constructed from an LTI system and nonlinear elements (i.e., Lur'e systems), the effectiveness of the IQC approach with Zames-Falb multipliers [24] is widely recognized, see, e.g., [25], [26]. Therefore it is strongly preferable if we can build a new COP-based approach on the basis of powerful IQC-based framework. To this end, we need to explore sound ways to capture the properties of nonlinear elements exhibiting positivity (such as ReLU) by introducing copositive multipliers and incorporate them into existing IQC conditions. It is also important to seek for possible ways to introduce copositive multipliers to deal with saturated systems on the basis of the techniques developed for their analysis and synthesis [27]. These topics are currently under investigation.

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APPENDIX

PROOF OF THEOREM 2

For the proof we need the next two lemmas.

Lemma 1: For the system $G$ given by (5) and given $N_1, N_2 \in \mathbb{N}$, let us define $(\tilde{A}_{N_1}, \tilde{B}_{N_1}, \tilde{C}_{N_1}, \tilde{D}_{N_1})$ and $(\tilde{A}_{N_2}, \tilde{B}_{N_2}, \tilde{C}_{N_2}, \tilde{D}_{N_2})$ by (12). Then, we have

$$
\tilde{A}_{N_2} \tilde{A}_{N_1} = \tilde{A}_{N_1+N_2}, \quad \tilde{A}_{N_2} \tilde{B}_{N_1} \tilde{B}_{N_2} = \tilde{B}_{N_1+N_2},
$$

$$
\tilde{C}_{N_1} = \tilde{C}_{N_1+N_2}, \quad \tilde{C}_{N_2} \tilde{B}_{N_1} \tilde{D}_{N_2} = \tilde{D}_{N_1+N_2}.
$$

Lemma 2: For given $A_1 \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times m_1}$, $C_1 \in \mathbb{R}^{l_1 \times n}$, $D_1 \in \mathbb{R}^{l_1 \times m_1}$ and $A_2 \in \mathbb{R}^{n \times n}$, $B_2 \in \mathbb{R}^{n \times m_2}$, $C_2 \in \mathbb{R}^{l_2 \times n}$, $D_2 \in \mathbb{R}^{l_2 \times m_2}$ and $\gamma > 0$, suppose there exist $P \in S_{n}, Q_1 \in S_{m_1}$ and $Q_2 \in S_{m_2}$ such that

$$
L(A_1, B_1, C_1, D_1, P, Q_1, \gamma) \prec 0,
$$

$$
L(A_2, B_2, C_2, D_2, P, Q_2, \gamma) \prec 0.
$$

Then we have

$$
L(A, B, C, D, P, Q, \gamma) \prec 0,
$$

where $(A, B, C, D, P, Q, \gamma)$ is given by (5).
\[ A := A_2 A_1, \quad B := [A_2 B_1 \ B_2], \]
\[ C := [C_1 \ C_2 A_1], \quad D := [D_1 \ D_2 \ 0] \]
\[ [Q_1 \ 0] \quad (31). \]

We can confirm the validity of Lemma 1 by direct calculation. The proof of Lemma 2 is given as follows.

**Proof of Lemma 2**: From [28], we see that (28) holds if and only if there exists \( G_1 \in \mathbb{R}^{n \times n} \) such that
\[ L_e(A_1, B_1, C_1, D_1, P, Q_1, G_1, \gamma) < 0. \]

By multiplying the above inequality by
\[ \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & I_{m_1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & I_{m_2}
\end{bmatrix}
\]
from left and its transpose from right, we have
\[ \begin{bmatrix}
-P + C_1^T C_1 & 0 & 0 & 0 & 0 \\
0 & D_1^T D_1 - 2 \gamma I_{m_1} + Q_1 & 0 & 0 & 0 \\
0 & 0 & C_2^T C_2 & 0 & 0 \\
0 & 0 & 0 & D_2^T D_2 - 2 \gamma I_{m_2} + Q_2 & 0 \\
0 & 0 & 0 & 0 & I_n
\end{bmatrix}
+ \begin{bmatrix}
0 & A_1^T & B_1^T & 0 & -I \\
0 & 0 & 0 & A_2^T & B_2^T \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
< 0. \]

Since
\[ \begin{bmatrix}
A_1^T & 0 & 0 & 0 & 0 \\
B_1^T & 0 & 0 & 0 & 0 \\
-I & A_2^T & 0 & 0 & 0 \\
0 & B_2^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I
\end{bmatrix}
= \begin{bmatrix}
I_0 & 0 & A_1^T & 0 & A_2^T \\
0 & I_{m_1} & B_1^T & 0 & B_2^T \\
0 & 0 & 0 & I_{m_2} & B_2^T \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
= J,
\]
the inequality (32) implies
\[ \begin{bmatrix}
-P + C_1^T C_1 & 0 & 0 & 0 & 0 \\
0 & D_1^T D_1 - 2 \gamma I_{m_1} + Q_1 & 0 & 0 & 0 \\
0 & 0 & C_2^T C_2 & 0 & 0 \\
0 & 0 & 0 & D_2^T D_2 - 2 \gamma I_{m_2} + Q_2 & 0 \\
0 & 0 & 0 & 0 & I_n
\end{bmatrix}
J^T < 0. \]

or equivalently,
\[ \begin{bmatrix}
-P & 0 & C_2^T & 0 & 0 \\
0 & -D_1^T D_1 & 0 & 0 & 0 \\
0 & 0 & C_2^T & 0 & 0 \\
0 & 0 & 0 & -D_2^T D_2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
< 0. \]

From (31), this can be rewritten equivalently as
\[ \begin{bmatrix}
-P & 0 & C_1^T & 0 & 0 \\
0 & -D_1^T D_1 & 0 & 0 & 0 \\
0 & 0 & C_1^T & 0 & 0 \\
0 & 0 & 0 & -D_2^T D_2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
< 0. \]

This clearly shows that (30) holds.

We are now ready to prove Theorem 3.

**Proof of Theorem 3**: We prove \( \mathcal{N}_3 + \leq \mathcal{N}_1 + \). The proof of \( \mathcal{N}_3 + \leq \mathcal{N}_1 + \) follows similarly. For the proof of \( \mathcal{N}_3 + \leq \mathcal{N}_1 + \), it suffices to show that if there exist \( P_N \in \mathcal{P} D_n \) and \( Q_N \in \mathcal{C} OP_{N_1 +} \), such that
\[ L(\tilde{A} N_1, \tilde{B} N_2, \tilde{C} N_1, \tilde{D} N_2, P, Q, N_1, \gamma) < 0 \]
for given \( \gamma > 0 \), then there exist \( P_N \in \mathcal{P} D_n \) and \( Q_N \in \mathcal{C} OP_{N_1 +} \) such that
\[ L(\tilde{A} N_1, \tilde{B} N_2, \tilde{C} N_1, \tilde{D} N_2, P, Q, N_1, \gamma) < 0. \]

To this end, we first note from (33) and Lemma 2 that
\[ L(\tilde{A} N_1, \tilde{B} N_2, \tilde{C} N_1, \tilde{D} N_2, P, Q, N_1, \gamma) < 0. \]

By repeating this procedure \( p - 1 \) times, we can conclude that (33) holds with
\[ P_N = P_N, \quad Q_N = \text{diag}(Q_1, \cdots, Q_1). \]

This completes the proof.

**Lemma in the Proof of Theorem 3**: In the proof of Theorem 3 we use the next lemma.

**Lemma 3**: For given \( \tilde{z} \in \mathbb{R}^{n_1}, \tilde{w} \in \mathbb{R}^{n_2}, \tilde{s} \in \mathbb{R}^{n_3}, \tilde{v} \in \mathbb{R}^{n_4} \) and \( a, b, c, d \in \mathbb{R} \), suppose
\[ \| \tilde{z} \|_2 \leq a \| \tilde{s} \|_2 + b \| \tilde{v} \|_2, \]
\[ \| \tilde{w} \|_2 \leq c \| \tilde{s} \|_2 + d \| \tilde{v} \|_2. \]

Then, we have
\[ \| \tilde{z} \|_2 \leq \sqrt{2} \left[ \begin{array}{c} a \ b \\ c \ d \end{array} \right] \| \tilde{w} \|_2. \]

**Proof of Lemma 3**: From (36), we first obtain
\[ \| \tilde{z} \|_2 \leq \| a \ b \|_2 \| \tilde{s} \|_2 + \| c \ d \|_2 \| \tilde{v} \|_2. \]

It follows that
\[ \| \tilde{z} \|_2 \leq \| a \ b \|_2 \| \tilde{s} \|_2 + \| c \ d \|_2 \| \tilde{v} \|_2. \]

Similarly, we have from (37) that
\[ \| \tilde{w} \|_2 \leq \| a \ b \|_2 \| \tilde{s} \|_2 + \| c \ d \|_2 \| \tilde{v} \|_2. \]

Therefore we have
\[ \| \tilde{z} \|_2 \leq \| a \ b \|_2 \| \tilde{s} \|_2 + \| a \ b \|_2 \| \tilde{v} \|_2. \]

This clearly shows that (38) holds.