STATISTICAL PROPERTIES OF ONE-DIMENSIONAL EXPANDING MAPS WITH SINGULARITIES OF LOW REGULARITY

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Abstract. We investigate the statistical properties of piecewise expanding maps on the unit interval, whose inverse Jacobian may have low regularity near singularities. The method is new yet simple: instead of directly working with the 1-d map, we first lift the 1-d expanding map to a hyperbolic map on the unit square, and then take advantage of the functional analytic method developed by Demers and Zhang in [21, 22, 23] for hyperbolic systems with singularities. By projecting back to the 1-d map, we are able to prove that it inherits nice statistical properties, including the large deviation principle, the exponential decay of correlations, as well as the almost sure invariance principle for the expanding map on a large class of observables. Moreover, we are able to prove that the projected SRB measure has a piecewise continuous density function. Our results apply to rather general 1-d expanding maps, including some $C^1$ perturbations of the Lorenz-like map and the Gauss map whose statistical properties are still unknown as they fail all other available methods.

1. Introduction.

1.1. Settings and background. In this paper, we study the one-dimensional piecewise expanding maps on the unit interval, with possibly wild behavior near singularities. More precisely, we consider a piecewise-defined map $F : I = [0, 1] \to I$, with a set of finite or countably many singular points

$$1 = c_0 > c_1 > \cdots > c_j > \cdots > c_N = 0,$$

where $N \leq \infty$ could be a finite number or $\infty$. For each $0 \leq j < N$, the restriction $F|_{I_j} : I_j = (c_{j+1}, c_j) \to F(I_j)$ is of class $C^3$ and strictly monotonic, that is, the derivative $F'(x) \neq 0$ for any $x \in I_j$. We shall call $\mathcal{S}_0 = \{c_0, c_1, \ldots, c_N\}$ the singular set of $F$, at points of which $F$ may be discontinuous, or $F'$ does not exist, or $F'$ blows up to $\pm \infty$.

For simplicity, we assume that $F|_{I_j}$ can be extended to a continuous map on the closure $\overline{I_j} = [c_{j+1}, c_j]$, however, it may not be $C^1$ or even Hölder extension. We further assume that:

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(E1) $F$ is uniformly expanding, that is, $\inf_{0 \leq j < N} \inf_{x \in I_j} |F'(x)| > 1$.

(E2) $F$ is surjective, and $F$ is topologically exact in the following sense: for any subinterval $J \subset I \setminus S_0$, there is $n = n(J) \geq 1$ such that $F^n(J) = I$.

Among various methods in the study of the statistical properties for dynamical systems, the classical functional approach is to analyze the spectrum of the associated Ruelle-Perron-Frobenius (RPF) transfer operators on some suitable function spaces. In the setting of a one-dimensional expanding map $F$, a crucial factor is the regularity of the inverse Jacobian $g = |F'|^{-1}$, where $g(x) = |F'(x)|^{-1}$ for $x \in I_j$ and $g(x) = 0$ elsewhere. If the singularity set $S_0$ is finite and each $g_j = g|I_j$ has bounded variation, then it is well known that the corresponding RPF transfer operator $\mathcal{L}_F$ is well-defined and has spectral gap on the space of functions with bounded variation, from which the existence of absolutely continuous invariant measures (acim), the exponential correlation decay and the central limit theorem follow. The same results hold in the case when $S_0$ is infinite, if there is a constant $C > 0$ such that $\var(g_j)/\sup(g_j) \leq C$ and $\sum_j \sup(g_j) \leq C$ (see e.g. [32, 33, 12, 26, 37, 30, 5, 9, 41, 1, 38, 19, 39, 36] for references). The bounded variation condition could be extended to weaker regularity conditions, such as the generalized bounded variation [31, 42, 35, 14], and the quasi-Hölder condition [27, 28], etc.

However, it is very often that the inverse Jacobian may not have any of the above good regularities, especially when the singular set $S_0$ is infinite. For instance, we consider a class of piecewise linear maps defined as follows. We set the singular set $S_0 = \{c_j\}_{0 \leq j \leq \infty}$ by letting $c_0 = 1$, $c_\infty = 0$ and choosing a sequence of decreasing numbers $\{c_j\}_{j \geq 1} \subset (0, 1)$. Also, we pick a sequence of positive numbers $\{\lambda_j\}_{j \geq 0}$ such that $\lambda_j \geq 2$. Let $F : I = [0, 1] \supset$ be given by

$$F(x) = \begin{cases} 0, & x = 0, \\ \lambda_j(x - c_{j+1}), & x \in (c_{j+1}, c_j], \ j \geq 0. \end{cases}$$

(1)

We further require that

1. the sequence $\{\lambda_j^{-1}\}_{j \geq 0}$ is not summable, i.e., $\sum_{j \geq 0} \lambda_j^{-1} = \infty$;
2. $c_j \leq (c_j - c_{j+1})\lambda_j \leq 1$ for all $j \geq 0$. In addition, $(c_j - c_{j+1})\lambda_j = 1$ for $j = 0, 1$.

A particular choice of these parameters would be $c_j = \frac{1}{(j+1)^2}$ and $\lambda_j \asymp j + 1$. The above condition (2) guarantees that $F$ satisfies the assumption (E2), that is, $F$ is topologically exact since $F(c_{j+1}, c_j] \supset (c_j, c_{j-1}]$ and thus $F^{j+1}(c_{j+1}, c_j] = [0, 1]$.

For expanding maps with singularities of low regularities, no much is known on how to explicitly construct the function spaces that are invariant under the RPF transfer operator $\mathcal{L}_F$, except some remarkable framework developed by Liverani in [35], as well as a generalization of Keller’s result [31] by Butterley [15] in the case of countable inverse branches. However, our example in (1) does not fit in any of the above references. Indeed, the inverse Jacobian of the map in (1) is exactly given by $g(x) = \lambda^{-1}$ for $x \notin S_0$ and $g(x) = 0$ whenever $x \in S_0$. By Condition (1), the inverse Jacobian $g(x)$ is of low regularity due to the following non-summable condition

$$\sum_j \sup_{x \in (c_{j+1}, c_j]} |g(x)| = \infty,$$
which fails the crucial assumption (3.1) in [35], as well as the crucial assumption (4.4) in [15]. Indeed, the original formula of (3.1) in [35] is
\[ \sum_j \sup_{x \in (c_{j+1}, c_j)} |\xi(x)F'(x)\gamma|^{1/(1-\gamma)} < \infty. \]

To study the standard RPF transfer operator \( L_F \), we need to take \( \xi = 1/F' \); as for [15], it was required that \( \sum_j \|1/F'\|_{L^\infty(c_{j+1}, c_j)} \|\xi\|_{L^\infty((c_{j+1}, c_j))} < \infty \), where \( \xi \equiv 1 \) for the standard RPF transfer operator \( L_F \). Both conditions lead to the summable condition \( \sum_j \sup_{x \in (c_{j+1}, c_j)} |1/F'(x)| < \infty \), which fails for the example (1).

In order to overcome the troubles that are caused by low regularity, one needs to propose a weaker condition related to the One-step Expansion. See Remark 2 for more details. Combining our one-step expansion condition and the fact that \( x = 0 \) is a fixed point, we are able to show the statistical properties for Example (1), and other expanding maps with low regularity.

1.2. Assumptions on singularities. In this paper, we shall establish the statistical properties of the one-dimensional maps based on the following assumptions on the singularity set \( S_0 \): for every \( c_i \in S_0 \),

(h1) There are \( \sigma_i^+ \geq 0 \) such that
\[ \limsup_{x \to c_i^+} \frac{|F''(x)|}{|F'(x)|^{\sigma_i^+ + 2}} < \infty, \]
\[ \limsup_{x \to c_i^+} \frac{|F^{(3)}(x)|}{|F'(x)|^{\sigma_i^+ + 2}} < \infty. \]

(h2) There is \( 0 \leq a_i < 1 \) such that
\[ 0 < \liminf_{x \to c_i^+} |(x - c_i)^{a_i}F'(x)| \leq \limsup_{x \to c_i^+} |(x - c_i)^{a_i}F'(x)| < \infty. \]

(h3) There are only finitely many essential singularities, that is, those \( c_i \in S_0 \) satisfying \( \sigma_i^+ > 0 \). For such \( c_i \) which are not fixed points, we further assume that \( a_i^+ \sigma_i^+ < 1 \).

Remark 1. Condition (2) in (h1) is a generalization of the Adler’s condition with \( \sigma_i^- = 0 \) (see the Afterword in [12]), which guarantees the bounded distortion of the one-dimensional map \( F \).

It is easy to check that the map \( F \) in (1) satisfies (h1) with \( \sigma_i^+ = 0 \) since \( F \) is piecewise linear. Moreover, (h2) automatically holds with \( a_j = 0 \) for any nonessential singularities \( c_j \) with \( j \geq 1 \). The only essential singularity \( c_\infty = 0 \) satisfies (h2) if and only if \( \lambda_j \approx c_j^{-a} \) for some \( a \in [0, 1) \). For instance, if \( c_j = \frac{1}{(j+1)^2} \) and \( \lambda_j \approx j + 1 \), then we could take \( a = \frac{1}{2} \).

We also mention two concrete examples that satisfy our assumptions.

(1) The Lorenz-like map, i.e., a map \( F : [-1, 1] \to [-1, 1] \) has a discontinuity at \( x = 0 \) and is piecewise-defined on two branches \([-1, 0) \) and \((0, 1] \). \( F \) is an increasing \( C^{1+} \) diffeomorphism, respectively, from \([-1, 0) \) onto \([y_L, 1) \) and from \((0, 1] \) onto \((-1, y_R) \), where \(-1 < y_L < 0 < y_R < 1 \). We assume that \( F'(x) \geq \sqrt{2} \) for any \( x \in [-1, 1] \setminus \{0\} \). Moreover, \( F \) can be continuously extended to \( x = 0 \) from either side, that is, \( \lim_{x \to 0^-} F(x) = 1 \) and \( \lim_{x \to 0^+} F(x) = -1 \).
Figure 1. Lorenz-like Map

Figure 2. Gauss Map

(2) The Gauss map $F : [0, 1] \rightarrow [0, 1]$ given by

$$F(x) = \begin{cases} 
\frac{1}{x} - \lfloor \frac{1}{x} \rfloor, & 0 < x \leq 1, \\
0, & x = 0.
\end{cases}$$

Note that $F$ has countably infinitely many branches, and $F^2$ is uniformly expanding.

It is not hard to check that $x = 0$ is the only singularity with $\sigma > 0$ for both the Lorenz-like map and the Gauss map, and Assumptions (h1) and (h2) hold for both maps at $x = 0$. Indeed many $C^1$ perturbations of these two maps still satisfy Conditions (h1) - (h3), including the perturbations given by (1).

1.3. Statement of the main results. Throughout this paper, we assume that the one-dimensional expanding map $F : I \subset \mathbb{R}$ satisfies Conditions (E1) - (E2) and Assumptions (h1) - (h3).

Our first main result concerns about the physical measures of $F$.

**Theorem 1.1.** The map $F$ preserves a unique SRB measure $\mu$. Moreover, the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure on $I$, and the density function $\rho = \frac{d\mu}{dx}$ is positive and continuous except at a countable set $S^*_\infty \subset I$.

The precise definition of the sets $S^*_K$, where $0 \leq K \leq \infty$, will be given by (11). We denote $P^*_K$ the partition of the unit interval $I$ divided by points in $S^*_K$.

Next we introduce a class of observables. Let $P$ be a sub-interval of $I$. A function $\phi : P \rightarrow \mathbb{R}$ is said to be H"older continuous on $P$ with an exponent $\lambda \in (0, 1]$, if

$$|\phi|_{C^\lambda(P)} := \sup_{x, y \in P, x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\lambda} < \infty.$$ 

Furthermore, a function $\phi : I \rightarrow \mathbb{R}$ is said to be piecewise H"older continuous with respect to a partition $\mathcal{P}$ of $I$, with an exponent $\lambda \in (0, 1]$, if $\phi$ is H"older continuous on each interval $P \in \mathcal{P}$, and

$$|\phi|_{C^\lambda(\mathcal{P})} := \sup_{P \in \mathcal{P}} |\phi|_{C^\lambda(P)} < \infty. \quad (3)$$

We denote $C^\lambda(\mathcal{P})$ the space of such piecewise H"older continuous functions. For our purpose, we shall consider the space $C^\lambda(P^*_K)$ for some integer $K \geq 0$.

We also introduce the space of dynamical H"older functions on $I$. More precisely, we first define the separation time $s(x, x')$ for every $x, x' \in I$ to be the the smallest
integer \( n \geq 0 \) such that \( x \) and \( x' \) belongs to distinct connected component of \( \mathcal{P}^*_n \). A function \( \phi : I \to \mathbb{R} \) is said to be dynamically Hölder continuous, if there is \( \theta \in (0, 1) \) such that
\[
|\phi|_\theta := \sup_{x, x' \in I, x \neq x'} \frac{|\phi(x) - \phi(x')|}{d(x, x')} < \infty.
\]
We denote by \( \mathcal{H}_\theta \) the space of all such functions \( \phi \). Note that for any finite \( K \geq 0 \), there is some \( \theta \in (0, 1) \) such that \( C^\lambda(\mathcal{P}^*_K) \subset \mathcal{H}_\theta \).

Our second main result, which is about the statistical properties of the expanding system \((F, \mu)\) for the class of piecewise Hölder functions, is stated as following.

**Theorem 1.2.** Let \( \mu \) be the acim of \( F \) given by Theorem 1.1. We have the following statistical properties for the one-dimensional system \((F, I, \mu)\).

1. **Exponential decay of multiple correlations:** Let \( \theta \in (0, 1) \). For any observables \( \phi_0, \phi_1, \ldots, \phi_k \in \mathcal{H}_\theta \cap L^\infty(I) \) such that \( \|\phi_i\|_{L^\infty} = \|\phi_0\|_{L^\infty} \) and \( \|\phi_i\|_\theta = \|\phi_0\|_\theta \) for all \( 1 \leq i \leq k \), and observables \( \psi_0, \psi_1, \ldots, \psi_l \in L^\infty(I) \) such that \( \|\psi_i\|_{L^\infty} = \|\psi_0\|_{L^\infty} \) for all \( 1 \leq i \leq l \), we set \( \Phi = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_k \circ \Psi = \psi_0 \circ \psi_1 \circ \cdots \circ \psi_l \circ F^k \). Then there are constants \( C > 0 \) and \( \theta \in (0, 1) \) such that
\[
\left| \int_I \Phi \cdot \Psi \cdot F^n d\mu - \int_I \Phi d\mu \int_I \Psi d\mu \right| \leq C \theta^n B_{\Phi, \Psi},
\]
for any \( n \geq 0 \), where
\[
B_{\Phi, \Psi} := \left\| \phi_0 \right\|_{L^\infty} \left\| \psi_0 \right\|_{L^\infty} \left[ \left\| \phi_0 \right\|_{L^\infty} \left\| \psi_0 \right\|_{L^\infty} + \frac{\left\| \phi_0 \| \right\| \psi_0 \|_{L^\infty}}{1 - \theta} \right].
\]

In the remaining of this theorem, we fix \( K \geq 0 \), \( \lambda > \max\{p, \beta/(1 - \beta)\} \) and let \( \nu \in \Pi, \mathcal{B} \) be a probability measures on \( I \), where constants \( p, \beta \) and the space \( \mathcal{B} \) will be given in Section 2.3, and \( \Pi \) will be given by (12). 

2. **Local large deviation estimate:** let \( \phi \in C^\lambda(\mathcal{P}^*_K) \) and \( S_n^F \phi = \sum_{k=0}^{n-1} \phi \circ F^k \).

Then
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \nu \left\{ x \in I : \frac{1}{n} S_n^F \phi(x) \in [t - \varepsilon, t + \varepsilon] \right\} = -I(t),
\]
where the rate function \( I(t) \geq 0 \) is independent of any measure \( \nu \in \Pi, \mathcal{B} \), and \( t \) is in a neighborhood of \( \int \phi d\mu \).

3. **Vector-valued almost-sure invariance principle:** consider a \( \mathbb{R}^d \)-valued function \( \Phi = (\phi_1, \ldots, \phi_d) \) such that \( \phi_i \in C^\lambda(\mathcal{P}^*_K) \) and \( \int \phi_i d\mu = 0 \) for each \( i = 1, \ldots, d \), and distribute \( \Phi \circ F^j \) according to the probability measure \( \nu \). Then there is a probability space \( \Omega \) with random variables \( \{X_n\} \) satisfying \( S_n^F \Phi \stackrel{\text{dist.}}{=} X_n \), and a Brownian motion \( W \) with zero mean and covariance matrix \( \Sigma^2 \) such that for any \( r \in (1/4, 1/2] \),
\[
X_n = W(n) + o(n^r), \text{ almost surely in } \Omega.
\]

The rest of the paper is organized as follows. In Section 2, we first introduce the method of lifting our 1-d expanding map to a 2-d hyperbolic skew product, and then state the known results in \([21, 22, 23]\) on the spectral and statistical properties for the 2-d map under standard assumptions \((H1)-(H4)\). In Section 3, we first verify \((H1)-(H4)\) under the Conditions \((E1), (E2), (h1)-(h3)\) for the 1-d map, and then prove our main theorems.
2. Properties of the lifted 2-d map.

2.1. Lifting to 2-d hyperbolic map. In this paper, instead of working directly on the RPF transfer operator for the expanding map, we invent a method that enables us to obtain new results for these expanding maps, by taking advantage of the results obtained for the 2-d hyperbolic systems. Indeed, there have been significant progresses recently made for the 2-d hyperbolic systems with very general singularities, including Sinai billiards, see \[10, 34, 2, 24, 7, 8, 20, 25, 3, 4, 40, 6\], especially by \[21, 22, 23\].

More precisely, we first lift the 1-d expanding map \( F \) of the unit interval to be a 2-d hyperbolic skew product \( \hat{F} \) of the unit square, which preserves the vertical stable foliation. This geometric structure allows us to construct certain function spaces on the unit square that only requires the smoothness along stable curves. The spectral analysis of the transfer operator on such function spaces, associated to the 2-d hyperbolic map, has been systematically developed by Demers and Zhang in \[21, 22, 23\]. Applying the techniques therein, one obtains that under some standard assumptions, the lifted 2-d map admits an SRB measure. We are able to prove that the projected measure is an absolutely continuous invariant measure for the 1-d map. Furthermore, we establish exponential decay of correlations, large deviation principle and the almost sure invariance principle on a large class of observables for the 1-d expanding map.

We first lift the expanding map \( F : I \to I \) to a 2-d hyperbolic map of the unit square using the scheme described below.

Let \( Q = I \times I = [0, 1] \times [0, 1] \) and \( Q_j = (c_j, c_{j+1}) \times I \), where \( 0 \leq j < N \). Here \( N \leq \infty \) could be a finite number or \( \infty \). We assume that \( G : Q \to I \) is a piecewise-defined function such that for each \( j \), \( G|_{Q_j} \) is of class \( C^2 \), and it can be continuously extended on the closure of \( Q_j \). Furthermore, we assume that if \( x_1, x_2 \notin S_0 \) such that \( F(x_1) = F(x_2) \), then
\[
G(\{x_1\} \times I) \cap G(\{x_2\} \times I) = \emptyset. \tag{4}
\]

We then define a two-dimensional skew product \( \hat{F} : Q \to Q \) by
\[
\hat{F}(x) = (F(x), G(x, y)), \quad \text{where } x = (x, y).
\]

By Condition (4), it is clear that \( \hat{F} \) is invertible on each connected component \( Q_j \) of \( Q \setminus \hat{S}_0 \), where \( \hat{S}_0 = \partial Q \cup (S_0 \times I) \). Note that \( \hat{F} \) preserves the vertical foliation \( \{\{x\} \times I : x \in I\} \), or equivalently,
\[
D_x\hat{F} = \begin{pmatrix}
F'(x) & 0 \\
G_x(x, y) & G_y(x, y)
\end{pmatrix}, \quad \text{for any } x = (x, y) \in Q \setminus \hat{S}_0,
\]

preserves the vertical line field \( \{(0, v) : v \in \mathbb{R}\} \).

2.2. Assumptions on the 2-d hyperbolic systems with singularities. In the rest of this subsection, we will review the assumptions on the two-dimensional lift \( \hat{F} : Q \to Q \), according to \[21, 23\], which guarantees the exponential decay of correlations as well as other statistical properties that we are interested. For notational conveniences, we shall repeatedly use \( C \) for universal constants.

Recall that \( \hat{S}_0 = \partial Q \cup (S_0 \times I) \), and denote \( \hat{S}_{\pm n} = \bigcup_{i=0}^{n} \hat{F}^i \hat{S}_0 \) for any \( n \geq 1 \).

We require that \( \hat{F}^{\pm n} : Q \setminus \hat{S}_{\mp n} \to Q \setminus \hat{S}_{\mp n} \) is a \( C^2 \) diffeomorphism on each connected component. Notice that for all \( n \geq 1 \), \( \hat{S}_n \setminus \partial Q \) are unions of vertical lines in \( Q \).
Suppose that there are two families of cones $C^u(x)$ (unstable) and $C^s(x)$ (stable) in the tangent space $T_xQ = \mathbb{R}^2$, which are continuous on the closure of each connected component of $Q \setminus S_1$, such that for all $x \in Q$ the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero. Due to the skew-product structure of $\hat{F}$, we may choose unstable and stable cones of the form

$$C^u(x) = \{(u, v) \in \mathbb{R}^2 : |v| \leq \Theta^+ |u|\},$$

$$C^s(x) = \{(u, v) \in \mathbb{R}^2 : |v| \geq \Theta^- |u|\},$$

for some numbers $\Theta^- > \Theta^+ > 0$. Now we state the first assumption.

**Uniform hyperbolicity of the map $\hat{F}$.**

1. $D_x\hat{F}(C^u(x)) \subset C^u(\hat{F}(x))$ whenever $D_x\hat{F}$ exists, and $D_x\hat{F}^{-1}(C^s(x)) \subset C^s(\hat{F}^{-1}(x))$ whenever $D_x\hat{F}^{-1}$ exists.

2. There is $\Lambda > 1$ such that $\|D_x\hat{F}v\|_* \geq \Lambda \|v\|_*$ for any $v \in C^u(x)$, and $\|D_x\hat{F}^{-1}v\|_* \geq \Lambda \|v\|_*$ for any $v \in C^s(x)$, where $\|\cdot\|_*$ is an adapted norm that is equivalent to the Euclidean norm on $\mathbb{R}^2$.

3. There is a positive $C^1$ function $\hat{f} : Q \setminus \hat{S}_1 \to \mathbb{R}^+$ such that

$$\frac{C^{-1}}{\hat{f}(x)} \leq \frac{\|D_x\hat{F}v\|}{\|v\|} \leq \frac{C}{\hat{f}(x)},$$

for some constant $C > 1$ which is independent of $x$.

Note that Assumption (H1)(2) actually implies the uniform expansion condition (E1) for the one-dimensional map $F$, due to the skew product structure of $\hat{F}$.

The function $\hat{f}$ is usually defined on the whole unit square $Q$ with its zero set $\hat{f}^{-1}(0) \subset S_1 \times I$. In order to control the possibly blow-up behavior of $D\hat{F}$ near the vicinity of $\hat{f}$, we introduce the concept of homogeneity regions that was first proposed by Sinai [13] in the study of billiards. To be precise, fix an exponent $r_h > 1$, and pick a sufficiently large integer $k_0^j$ for each $0 \leq j < N$. We then choose a countable family of vertical line segments $\left\{S^H_{j,k}\right\}_{j,k}$, called homogeneous stripes, in each component $Q_j = (c_j, c_{j+1}) \times I$ with $|k| > k_0^j$, such that $\hat{f}(x) \asymp |k|^{-r_h + 1}$ for any $x \in S^H_{j,k}$. For convenience, we suppose that each curve $S^H_{j,k}$ lies closely on the right-hand side of $\{c_j\} \times I$ if $k > k_0^j$ and on the left-hand side of $\{c_{j+1}\} \times I$ if $k < -k_0^j$. The choices of $\{k_0^j\}_{0 \leq j < N}$ will be determined by Assumption (H4). We set the extended singularity as

$$S^H_{0} = S^H \cup \left(\cup_{0 \leq j < N} \cup_{|k| \geq k_0^j} S^H_{j,k}\right).$$

The region between $S^H_{j,k}$ and $S^H_{j,k+1}$ is called a homogeneity region with index $k$ in $Q_j$, and denoted as $\mathbb{H}^k_j$ in $Q_j$, and denoted as $\mathbb{H}^k_j$ in $Q_j$. We denote by $\mathbb{H}^0_{\ast}$ the connected components of $Q_j \setminus \cup_{|k| \geq k_0^j} \mathbb{H}^k_j$. We also define for $n \geq 1$,

$$\mathbb{S}^H_n = \mathbb{S}^H_0 \cup \hat{F}^{-1}\mathbb{S}^H_0 \cup \cdots \cup \hat{F}^{-n}\mathbb{S}^H_0,$$

and $\mathbb{S}^H_{\ast,n} = \mathbb{S}^H_0 \cup \hat{F}\mathbb{S}^H_0 \cup \cdots \cup \hat{F}^n\mathbb{S}^H_0$.

We say that a smooth curve $W \subset Q$ is a stable (or unstable) curve if at every point $x \in W$ the tangent line $T_xW$ lies in the stable cone $C^s(x)$ (or unstable cone $C^u(x)$). We call a stable (or unstable) curve homogeneous if it lies entirely in one homogeneity region, either $\mathbb{H}^k_j$ or $\mathbb{H}^0_{\ast}$. 


Let $W^s$ (or $W^u$) denote a family of $C^2$ homogeneous stable (or unstable) curves with length less than some positive constant $\delta_0$ and with curvature bounded above by some uniform constant $B > 0$. Due to the skew product structure of $\hat{F}$, we choose the stable family $W^s$ to be the family of homogeneous local stable manifolds, all of which are vertical line segments.

**H2** Invariance and Distortion. We assume that the unstable family $W^u$ is invariant under $\hat{F}$ in the sense that for any $W \in W^u$, all connected components of $\hat{F}W$ belong to $W^u$. Moreover, there exist $p_0 \in (0, 1]$ and $C \geq 1$ such that if $W \in W^u$ is such that $\hat{F}W \in W^u$, then for any $x_1, x_2 \in W$, we have

$$|\ln J_{x_1}W - \ln J_{x_2}W| \leq C d_{\hat{F}(W)}(\hat{F}(x_1), \hat{F}(x_2))^{p_0},$$

where $J_W$ is the Jacobian of $\hat{F}$ along the unstable curve $W$ evaluated at $x$.

We also require the similar invariance and distortion control for $W^s$ under $\hat{F}^{-1}$.

**H3** Structures of Singularities.

1. There exists $0 < a_0 < 1$ such that

$$\sup_{x \in Q \setminus \hat{S}_1} \frac{d(x, \hat{S}_1)^{a_0}}{\hat{f}(x)} < \infty. \tag{9}$$

2. If $D$ is a connected component of $Q \setminus \hat{S}_1$, then $\partial D$ consists of finitely many smooth compact curves. Moreover, for each $\varepsilon > 0$, there are at most finitely many connected components of $Q \setminus \hat{S}_1$ containing unstable curves of length greater than $\varepsilon$.

3. We assume the following weak transversality condition: there exist $C > 0$ and $0 < t_0 < 1$ such that for any unstable curve $W$ and any smooth curve $S \subseteq \hat{S}_n^u$, we have

$$m_W(N_\varepsilon(S) \cap W) \leq C \varepsilon^{t_0}$$

for all $\varepsilon > 0$ sufficiently small, where $N_\varepsilon(\cdot)$ denotes the $\varepsilon$-neighborhood of a set in $Q$, and $m_W(\cdot)$ denotes the Euclidean length on the curve $W$.

The assumptions on $\hat{S}_n^u$ are made in a similar fashion.

Note that our assumption (H3) is slightly different from that of [23]. For (H3)(1), one can check that together with (6), our condition (9) implies that

$$\|D_{x} \hat{F}v\| \leq C\|v\|d(x, \hat{S}_1)^{-a},$$

for all $x \in Q \setminus \hat{S}_1$ and $v \in C^u(x)$,

which was the first condition in (H3) of [23]. Also, we can check that (H3)(4)(5) of [23] are automatically satisfied for our set $\hat{S}_n^u$ due to our special choice of $W^s$. So we no longer need to assume them in this paper.

**H4** One-step expansion. Let $W \in W^u$ and $\{V_i, i \geq 1\}$ be the connected components of $\hat{F}W$ in $W^u$. Let $|J_{W_i} \hat{F}|_*$ denote the minimum expansion on $W_i = \hat{F}^{-1}V_i$ under $\hat{F}$ in the metric induced by some adapted norm $\|\cdot\|_*$. We assume that

$$\limsup_{\delta \to 0} \sup_{W \in W^u} \sum_{i} |J_{W_i} \hat{F}|_*^{-1} < 1.$$
Remark 2. It is worth noting that Condition (3.1) in [35] is indeed a stronger version of our Assumption (H4) in the sense that (3.1) requires the one-step expansion over all branches is summable, which could be regarded as a quantitative form of the Big Image Property. In our Assumption (H4), we only require such condition over homogeneous components of some family of unstable curves, whose images may be of arbitrarily small length. In other words, here we only require a local version of one-step expansions.

2.3. Transfer operator and function spaces. The definitions of the transfer operator and the function spaces are analogous to those in [23].

The Ruelle-Perron-Frobenius (RPF) transfer operator \( \mathcal{L}_F \) associated to the two-dimensional hyperbolic lift \( \tilde{F} \) is given by

\[
\mathcal{L}_F \tilde{\phi}(x) = \frac{\tilde{\phi}(\tilde{F}^{-1}(x))}{|\det(D_{\tilde{F}^{-1}(x)}\tilde{F})|}, \text{ for any } \tilde{\phi} \in L^1(Q, dx).
\]

We shall extend the definition of \( \mathcal{L}_F \) onto some distribution spaces as follows. Given \( 0 < p \leq 1 \) and a complex-valued function \( \tilde{\phi} : Q \to \mathbb{C} \), we define \( H^p_Q(\tilde{\phi}) \) to be the Hölder constant of \( \tilde{\phi} \) on \( Q \) with the exponent \( p \). Let

\[
\tilde{C}^p(Q) = \{ \tilde{\phi} : Q \to \mathbb{C} : |\tilde{\phi}|_{C^p(Q)} := |\tilde{\phi}|_{C^0(Q)} + H^p_Q(\tilde{\phi}) < \infty \}
\]

and let \( C^p(Q) \) be the closure of \( \tilde{C}^1(Q) \) in the \( \tilde{C}^p(Q) \)-norm.

Similarly, given \( 0 < p \leq 1 \), a stable curve \( W \in \mathcal{W}^s \) and a complex-valued function \( \tilde{\psi} : W \to \mathbb{C} \), we define \( H^p_W(\tilde{\psi}) \) to be the Hölder constant of \( \tilde{\psi} \) on \( W \) with the exponent \( p \) measured in the Riemannian metric \( d_W \). Let

\[
\tilde{C}^p(W) = \{ \tilde{\psi} : Q \to \mathbb{C} : |\tilde{\psi}|_{\tilde{C}^p(W)} := |	ilde{\psi}|_{C^0(W)} + H^p_W(\tilde{\psi}) < \infty \}\quad (10)
\]

and let \( C^p(W) \) be the closure of \( \tilde{C}^1(W) \) in the \( \tilde{C}^p(W) \)-norm. We further define

\[
\tilde{C}^p(\mathcal{W}^s) = \{ \tilde{\psi} : Q \to \mathbb{C} : |\tilde{\psi}|_{\tilde{C}^p(\mathcal{W}^s)} := \sup_{W \in \mathcal{W}^s} \left| \tilde{\psi} \right|_{\tilde{C}^p(W)} < \infty \}
\]

and let \( C^p(\mathcal{W}^s) \) be the closure of \( \tilde{C}^1(\mathcal{W}^s) \) in the \( \tilde{C}^p(\mathcal{W}^s) \)-norm.

Given \( \alpha, p \geq 0 \) and a stable curve \( W \in \mathcal{W}^s \), we define the following norm

\[
|\tilde{\psi}|_{W, \alpha, p} := |W|^\alpha \tilde{f}(W) |\tilde{\psi}|_{\tilde{C}^p(W)}
\]

for a test function \( \tilde{\psi} \in C^p(W) \), where \( |W| \) is the Euclidean arclength of \( W \), \( \tilde{f} \) is given in Assumption (H1)(3) with \( \tilde{f}(W) \) being the average value of \( \tilde{f} \) on \( W \), and \( |\tilde{\psi}|_{\tilde{C}^p(W)} \) is given by (10).

We now fix the following choices of parameters based on Assumptions (H1)-(H4):

First choose \( \alpha, \gamma > 0 \) such that \( 0 < \gamma < \alpha < 1/r_h \), where \( r_h \) determines the spacing of \( \mathbb{H}^k \); next choose \( 0 < q < p \leq p_0 \) such that \( p < \gamma \), where \( p_0 \) is the Hölder exponent from Assumption (H2); finally, choose

\[
0 < \beta < \min \{ p - q, \, (1 - a)(\alpha - \gamma), \, 1/r_h - \alpha \},
\]

where \( a \) is from Assumption (H3)(1).

Given a test function \( \tilde{\phi} \in C^1(Q) \), we define the weak norm of \( \tilde{\phi} \) by

\[
|\tilde{\phi}|_w := \sup_{W \in \mathcal{W}^s} \sup_{|\tilde{\psi}|_{W, \gamma, p} \leq 1} \int_W \tilde{\psi} \tilde{\phi} dm_W.
\]
We define the strong stable norm of $\hat{\phi}$ as
\[
\|\hat{\phi}\|_u := \sup_{W \in \mathcal{W}} \sup_{\phi \in C^\beta(W)} \int_W \hat{\phi} \, dm_W,
\]
and the strong unstable norm of $\hat{\phi}$ as
\[
\|\hat{\phi}\|_s := \sup_{\varepsilon \leq \varepsilon_0} \sup_{W_1, W_2 \in \mathcal{W}} \sup_{\phi_1, \phi_2 \in C^\beta(W_i)} \frac{1}{\varepsilon^\beta} \left| \int_{W_1} \hat{\phi}_1 dm_W - \int_{W_2} \hat{\phi}_2 dm_W \right|,
\]
where $\varepsilon_0$ is chosen less than $\delta_0$ in Assumption (H4), and the definitions of $d_{\mathcal{W}}(W_1, W_2)$ and $d_q(\psi_1, \psi_2)$ are similar to those given by Section 3.1 in [23]. We then define the strong norm of $\hat{\phi}$ by
\[
\|\hat{\phi}\|_B = \|\hat{\phi}\|_s + c_u \|\hat{\phi}\|_u,
\]
where $c_u$ is sufficiently small constant. Finally, we define the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ to be the completion of $C^1(Q)$ in the strong and weak norm respectively.

2.4. Statistical properties of the two-dimensional lift $\hat{F}$. To state the statistical properties of $\hat{F}$, we introduce the following natural partitions that are characterized by singularities:

\[
\hat{S}_K = \text{vertical line segments of } \bigcup_{n=0}^{K} \hat{F}^n \hat{S}_0^\text{II}, \quad \text{and } \hat{S}_K^* := \Pi \hat{S}_K,
\]
for any $0 \leq K \leq \infty$, where $\hat{S}_0^\text{II}$ is the extended singularity set of $\hat{F}$ given by (7), and

\[
\Pi : Q \ni x = (x, y) \mapsto x \in I
\]
(12)
is the vertical projection from $Q$ onto $I$. For any finite $K \geq 0$, we denote $\hat{P}_K^*$ the partition of the unit square $Q$ divided by vertical line segments in $\hat{S}_K^*$. Note that $\hat{P}_K^*$ satisfies the conditions in Lemma 5.3 of [23]. Similar to (3), we denote $C^\lambda(\hat{P}^*)$ the space of piecewise Hölder functions on $Q$ with exponent $\lambda$ and with respect to the partition $\hat{P}^*$.

Using the functional analytic scheme in [21, 23], it can be shown that under the assumptions (H1)-(H4), the embeddings $C^1(Q) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w$ are continuous and injective. Moreover, $\mathcal{L}_\hat{F}$ is well defined as a continuous operator on both $\mathcal{B}$ and $\mathcal{B}_w$, and it satisfies certain Lasota-Yorke-type estimates. The following spectral properties of $\mathcal{L}_\hat{F}$ and the consequential statistical properties were obtained in [21, 23] under Assumptions (H1)-(H4).

**Lemma 2.1.** The operator $\mathcal{L}_\hat{F} : \mathcal{B} \rightarrow \mathcal{B}$ is quasi-compact, that is, the spectral radius of $\mathcal{L}_\hat{F}$ in $\mathcal{B}$ is 1 while its essential spectral radius is bounded by some $\sigma_0 < 1$. Moreover,

1. Let $\mathcal{V}_\theta$ be the eigenspace of $\mathcal{L}_\hat{F}$ in $\mathcal{B}$ corresponding to the peripheral eigenvalue $e^{2\pi i \theta}$. The maps $(\hat{F}^n)_{n \in \mathbb{N}}$ admit only finitely many physical measures, which form a basis for $\mathcal{V} := \oplus_\theta \mathcal{V}_\theta$. All the positive elements of $\mathcal{V}_0$ are ergodic SRB measures.}

\[\text{1See Remark 2.10 in [20].}\]
For the rest of the lemma, we assume that 1 is the only peripheral eigenvalue, that is, $V = V_0$. We fix a mixing SRB measure $\hat{\mu}$ in $V_0$. Choose an integer $K \geq 0$ and a constant $\lambda > \max\{p, \beta/(1-\beta)\}$, where $p$ and $\beta$ are given in Section 2.3.

(2) Exponential decay of correlations: there are $C > 0$ and $\theta \in [0,1)$ such that for any $\hat{\phi} \in C^\lambda(\hat{P}_K)$ and $\hat{\psi} \in C^p(W^s)$, we have

$$\left| \int_Q \hat{\phi} \cdot \hat{\psi} \circ \hat{F}^n d\hat{\mu} - \int_Q \hat{\phi} d\hat{\mu} \int_Q \hat{\psi} d\hat{\mu} \right| \leq C \theta^n |\hat{\phi}|_{C^\lambda(\hat{P}_K)} |\hat{\psi}|_{C^p(W^s)}.$$

(3) Local large deviation estimate: for any probability measure $\hat{\nu} \in \mathcal{B}$,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \hat{\nu}\left\{ x \in Q : \frac{1}{n} S_n^\hat{F} \hat{\phi}(x) \in [t - \varepsilon, t + \varepsilon] \right\} = -I(t),$$

where the rate function $I(t) \geq 0$ is independent of $\hat{\nu} \in \mathcal{B}$, and $t$ is in a neighborhood of $\int \hat{\phi} d\hat{\mu}$.

(4) Vector-valued almost-sure invariance principle: consider a $\mathbb{R}^d$-valued function $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_d)$ such that $\hat{\phi}_i \in C^\lambda(\hat{P}_K)$ and $\int \hat{\phi}_i d\hat{\mu} = 0$ for each $i = 1, \ldots, d$, and distribute $(\hat{\phi} \circ \hat{F})_{j \in \mathbb{N}}$ according to a probability measure $\hat{\nu} \in \mathcal{B}$. Then there is a probability space $\Omega$ with $\mathbb{R}^d$-valued random variables $\{X_n\}_{n \in \mathbb{N}}$ satisfying $S_n^\hat{F} \hat{\phi} \overset{\text{dist.}}{=} X_n$, and a Brownian motion $W$ with zero mean and covariance matrix $\Sigma$ such that for any $r \in (1/4, 1/2)$,

$$X_n = W(n) + o(n^r),$$

almost surely in $\Omega$.

By Item (2) of Lemma 2.1, the system $(\hat{F}, \hat{\mu})$ is exponentially mixing for the observables $\hat{\phi} \in C^\lambda(\hat{P}_K)$ against $\hat{\psi} \in C^p(W^s)$. Following the arguments in [18], we can further obtain exponential decay of correlations for the pair $(\hat{\phi}, \hat{\psi})$ of observables in slightly more general classes, once a mixing SRB measure $\hat{\mu}$ is fixed.

More precisely, we first define the forward separation time $s_+(x, x')$ for every $x, x' \in Q$ to be the the smallest integer $n \geq 0$ such that $x$ and $x'$ belongs to distinct connected component of $Q \setminus \bigcup_{i=0}^{n} \hat{F}^{-i} S_n^\hat{F}$, where $S_n^\hat{F}$ is given by (7). A function $\hat{\phi} : Q \to \mathbb{R}$ is said to be dynamically forward Hölder continuous, if there are $\vartheta^+ \in (0,1)$ such that

$$|\hat{\phi}_i|_{\vartheta^+} := \sup_{W} \sup_{x, x' \in W} \frac{\left| \hat{\phi}(x) - \hat{\phi}(x') \right|}{\left| (\vartheta^+)^{s_+(x, x')} \right|} < \infty,$$

where the first supremum is taken over all the homogeneous unstable manifolds. We denote by $\mathcal{H}_{\vartheta^+}$ the space of all such functions $\hat{\phi}$. Note that $C^\lambda(\hat{P}_K) \subset \mathcal{H}_{\vartheta^+}$ for some $\vartheta^+ \in (0,1)$. Similarly we define the backward separation time $s_-(\cdot, \cdot)$, dynamically backward Hölder continuous functions and the space $\mathcal{H}_{\vartheta^-}$.

The following lemma follows directly from Theorem 3 in [18].

**Lemma 2.2.** Given $\vartheta^\pm \in (0,1)$, there exist $C > 0$ and $\theta \in (0,1)$ such that for every pair of dynamical Hölder continuous functions $\hat{\phi} \in \mathcal{H}_{\vartheta^+} \cap L^\infty(Q, \hat{\mu})$ and $\hat{\psi} \in \mathcal{H}_{\vartheta^-} \cap L^\infty(Q, \hat{\mu})$ and $n \geq 0$,

$$\left| \int_Q \hat{\phi} \cdot \hat{\psi} \circ \hat{F}^n d\hat{\mu} - \int_Q \hat{\phi} d\hat{\mu} \int_Q \hat{\psi} d\hat{\mu} \right| \leq C \theta^n B_{\vartheta^+, \vartheta^-},$$
where

\[ B_{\hat{\phi}, \hat{\psi}} := \| \hat{\phi} \|_{L^{\infty}} \| \hat{\psi} \|_{L^{\infty}} + \| \hat{\phi}_{\partial^+} \|_{L^{\infty}} + \| \hat{\phi} \|_{L^{\infty}} \| \hat{\psi}_{\partial^-} \|. \]

The above exponential correlation decay can be extended to variables made at multiple times. More precisely, let \( \hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_k \in \mathcal{H}^+_{\partial^+} \) such that \( \| \hat{\phi}_i \|_{L^{\infty}} = \| \hat{\phi}_0 \|_{L^{\infty}} \) and \( \hat{\phi}_{i+} = \hat{\phi}_0 \|_{\partial^+} \) for all \( 1 \leq i \leq k \), and define the product \( \hat{\Phi} = \hat{\phi}_0 \cdot (\hat{\phi}_1 \circ \hat{F}) \cdots (\hat{\phi}_k \circ \hat{F}^k) \). Similarly, given \( \hat{\psi}_0, \hat{\psi}_1, \ldots, \hat{\psi}_k \in \mathcal{H}^-_{\partial^-} \) such that \( \| \hat{\psi}_i \|_{L^{\infty}} = \| \hat{\psi}_0 \|_{L^{\infty}} \) and \( \hat{\psi}_{i-} = \hat{\psi}_0 \|_{\partial^-} \) for all \( 1 \leq i \leq \ell \), we set the product \( \hat{\Psi} = \hat{\psi}_0 \cdot (\hat{\psi}_1 \circ \hat{F}) \cdots (\hat{\psi}_\ell \circ \hat{F}^\ell) \). Then the exponential decay of multiple correlation follows from Theorem 4 in [18].

**Lemma 2.3.** Let \( C \) and \( \theta \) be the same as in Lemma 2.2, then for any \( n \geq 0 \),

\[ \left| \int_Q \hat{\Phi} \cdot \hat{\Psi} \circ \hat{F}^n d\hat{\mu} - \int_Q \hat{\Phi} d\hat{\mu} \int_Q \hat{\Psi} d\hat{\mu} \right| \leq C \theta^n B_{\hat{\Phi}, \hat{\Psi}}, \]

where

\[ B_{\hat{\Phi}, \hat{\Psi}} := \| \hat{\phi}_0 \|_{L^{\infty}} \| \hat{\psi}_0 \|_{L^{\infty}} \left[ \| \hat{\phi}_0 \|_{L^{\infty}} \| \hat{\psi}_0 \|_{L^{\infty}} + \| \hat{\phi}_{\partial^+} \|_{L^{\infty}} \| \hat{\psi}_{\partial^-} \|_{L^{\infty}} \cdot \frac{1 - \theta^+}{1 - \theta^-} \right]. \]

3. Proofs of main results.

3.1. Verification of Assumptions (H1) - (H4). In this section, we shall show that under Conditions (E1) (E2) and Assumptions (h1) - (h3), the one-dimensional expanding map \( F \) admits a two-dimensional hyperbolic skew product lift \( \hat{F} : Q \subset \mathbb{R} \)

that satisfies Assumptions (H1)-(H4).

For computational simplicity, we mainly consider the expanding maps with finite inverse branches and only one essential singularity, as the general cases can be dealt with in a similar fashion. Moreover, it is no harm to assume that \( F \) is linear on the branches which do not contain the essential singularities. More precisely, we study the following model:

\[ F(x) = \begin{cases} T(x), & 0 \leq x \leq \frac{1}{3}, \\ 3x - 1, & \frac{1}{3} < x \leq \frac{2}{3}, \\ 3x - 2, & \frac{2}{3} < x \leq 1. \end{cases} \]

where \( T \in C^0([0, \frac{1}{3}]) \cap C^3((0, \frac{1}{3}]) \) such that \( \inf_{x \in [0, \frac{1}{3}]} T'(x) \geq 2 \). Moreover,

(1) The endpoint \( x = 0 \) is the only essential singularity near which \( |F'|^{-1} \) may have low regularity, that is, \(|F'|^{-1} \) is of class \( C^2 \) but may have unbounded variation on \( I_0 = (0, \frac{1}{3}] \).

(2) For convenience, we further assume that \( F \) is a surjective affine map on the other two branches \( I_1 = (\frac{1}{3}, \frac{2}{3}] \) and \( I_2 = (\frac{2}{3}, 1] \).

Note that \( T(\frac{1}{3}) = T(\frac{2}{3}) \geq \frac{2}{3} \), which implies that \( F \) satisfies the topological mixing property in (E2). In general, \( F \) is non-Markov unless \( T(\frac{1}{3}) = \frac{2}{3} \) or \( 1 \).

We first deal with the fixed point case, that is, \( T(0) = 0 \). The non-fixed point situation will be taken care of in Section 3.1.3. Conditions (h1) and (h2) in this model are then rephrased as follows.

(h1) There is \( \sigma > 0 \) such that

\[ \limsup_{x \to 0^+} \frac{|T''(x)|}{|T'(x)|^\sigma+2} < \infty, \quad (13) \]
\[ \lim_{x \to 0^+} \frac{|T^{(3)}(x)|}{|T'(x)|^{\alpha + 2}} < \infty. \]

(h2) There is \( 0 < \alpha < 1 \) such that

\[ 0 < \liminf_{x \to 0^+} x^\alpha T'(x) \leq \limsup_{x \to 0^+} x^\alpha T'(x) < \infty. \]

3.1. Construction of the two-dimensional Lift. We denote by \( c_i \in I_i, i = 1, 2 \), the points such that \( F(c_i) = F(\frac{i}{3}) \), and then further divide \((0, 1] \) into the following subintervals:

\[ I_{00} = (0, 1/3], \quad I_{01} = (1/3, c_1], \quad I_{11} = (c_1, 2/3], \quad I_{02} = (2/3, c_2], \quad I_{12} = (c_2, 1]. \]

The singularity set of \( F \) is then extended to be \( S_0 = \{0, \frac{1}{3}, c_1, \frac{2}{3}, c_2, 1\} \). We associate each \( x \in I_{00} \cup I_{01} \cup I_{02} \) with a unique star-point \( x^* \in I_{00} \) such that \( F(x^*) = F(x) \).

Recall that \( Q = I \times I = [0, 1]^2 \), and set \( Q_{ij} = I_{ij} \times I \), for \( i, j \in \{0, 1, 2\} \). It is clear that the rectangles \( \{Q_{ij}\} \) form a partition of \( Q \setminus \{(0) \times I\} \).

Let \( \sigma > 0 \) be given by Condition (h1). We define a two-dimensional lift \( \hat{F}(x) = \hat{F}(x, y) = (F(x), G(x, y)) : Q \to Q \) by setting \( G(0, y) = (0, 0) \) for all \( y \in I, \) and

\[
G(x, y) = \begin{cases} 
\frac{y}{|T'(x)|^\alpha}, & (x, y) \in Q_{00}, \\
\frac{1}{|T'(x^*)|^\alpha} + \frac{y + j - 1}{2} \left(1 - \frac{1}{|T'(x^*)|^\alpha}\right), & (x, y) \in Q_{0j}, \\
y + j - 1, & (x, y) \in Q_{1j}, 
\end{cases}
\]

\[ j = 1, 2, \quad (x, y) \in Q_{0j}, \quad (x, y) \in Q_{1j}, \]

The two-dimensional singular set of \( \hat{F} \) is given by \( \hat{S}_0 = \partial Q \cup (S_0 \times I) \). We then set \( \hat{S}_1 = \hat{S}_0 \cup \hat{F}^{-1}S_0 \) and \( \hat{S}_{-1} = \hat{S}_0 \cup \hat{F} \hat{S}_0 \) (see Figure 3).

**Figure 3.** The two dimensional lifting map \( \hat{F} : Q \setminus \hat{S}_1 \to Q \setminus \hat{S}_{-1} \)

The matrix representation of derivative \( D_x \hat{F} \) in the Euclidean coordinate is of the form

\[
D_x \hat{F} = \begin{pmatrix} F'(x) & 0 \\
G_x(x, y) & G_y(x, y) \end{pmatrix}, \quad (x, y) \in Q \setminus \hat{S}_1,
\]
where \((G(x, y), G(y, x))\) is given by

\[
\begin{cases}
-\sigma y T''(x) \frac{T''(x)^{\sigma+1}}{[T'(x)]^\sigma}, & (x, y) \in Q_{00}, \\
\frac{3\sigma(y+j-3)}{2} T''(x^*) \frac{1}{[T'(x^*)]^{\sigma+2}} \left(1 - \frac{1}{[T'(x^*)]^\sigma}\right), & (x, y) \in Q_{0j}, j = 1, 2, \\
0, & (x, y) \in Q_{1j}, j = 1, 2.
\end{cases}
\]

In the computation of \(G_x\) on \(Q_{0j}, j = 1, 2\), we have used the chain rule that

\[
\frac{dx^*}{dx} = \frac{d(F(x^*))}{dx} = \frac{3}{T'(x^*)}.
\]

Now we shall verify Assumptions \((H1) - (H4)\) for the lifting map \(\hat{F}\).

**Proposition 1.** The two-dimensional lift \(\hat{F}\) in Subsection 3.1.1 satisfies Assumptions \((H1) - (H4)\) if Conditions \((h1)\) and \((h2)\) hold.

Proposition 1 is decomposed into the following lemmas, which will be proven in Subsection 3.1.2.

**Lemma 3.1.** \(\hat{F}\) satisfies Assumption \((H1)\) if \((13)\) in Condition \((h1)\) holds. In particular,

1. The unstable cone \(C^u(x)\) and stable cone \(C^s(x)\) are constructed in the form of \((5)\) with \((\Theta^+, \Theta^-) = (\Theta, t\Theta)\) for some \(\Theta > 0\) and \(t \gg 1\).
2. The function \(\hat{f} : Q \setminus \hat{S}_i \to \mathbb{R}^+\) in Assumption \((H1)(3)\) could be chosen as

\[
\hat{f}(x) := \frac{1}{T'(x^*)}, \quad \text{for any } x = (x, y) \in Q_{00},
\]

and \(\hat{f}(x) = 1\) elsewhere.

There is no need to introduce the homogeneous strips if \(\lim_{x \to 0^+} T'(x) < \infty\). In the case when \(\lim_{x \to 0^+} T'(x) = \infty\), the function \(\hat{f}(x, y)\) defined in \((16)\) vanishes at \(x = 0\).

If Condition \((h2)\) also holds, we choose a constant \(b > 0\) such that

\[
\frac{1}{a} + \frac{1}{b} > \sigma + 1.
\]

Further, we choose a sequence \(\{c^k_H\}_{k \geq k_0}\) such that

\[
c^k_H = k^{-\frac{b}{a}},
\]

for sufficiently large \(k_0 \in \mathbb{N}\), then we have \(T'(x) \asymp k^b\) for any \(x \in (c_{k+1}^{\tilde{H}}, c_k^{\tilde{H}}]\).

**Remark 3.** In our particular case when \(x = 0\) is a fixed point, \(b > 0\) is sufficient for the one-step expansion condition \((H4)\). See the verification of Assumption \((H4)\) in the proof of Lemma 3.2.

In the case when \(x = 0\) in not a fixed point, we do require \(b > 1\) to guarantee \((H4)\). Thus by \((17)\), we need an additional condition \(a\sigma < 1\) See Section 3.1.3 for more details.
Set the homogeneity curves to be vertical line segments \( S_k^H = \{ c_k^H \} \times I \) for all \( k \geq k_0 \). In other words, let \( I \) be the collection of intervals

\[
I_{01}, I_{11}, I_{02}, I_{12}, I_{00}^k := (\frac{c_k^H}{k}, 1/3], \quad \text{and} \quad I_{00}^k := (\frac{c_{k+1}^H}{k+1}, \frac{c_k^H}{k}], \quad k \geq k_0,
\]

then each homogeneity region is of the form \( J \times I \) for some \( J \in I \). We denote the regions \( Q_{00}^k = I_{00}^k \times I \) and \( Q_{10}^k = I_{00}^k \times I \). With such choice of homogeneity regions, we have

**Lemma 3.2.** \( \hat{F} \) satisfies Assumptions (H2), (H3) and (H4) if Conditions (h1) and (h2) hold.

### 3.1.2. Proof of Lemmas

We now prove lemmas in the previous subsection.

**Proof of Lemma 3.1.** Given \((u, v) \in \mathbb{R}^2 \) and \((u_1, v_1) = D_x \hat{F}(u, v) \in \mathbb{R}^2\), we have that

\[
v_1 \geq \frac{G_x(x, y)u + G_y(x, y)v}{F'(x)u} = \frac{G_x(x, y) + G_y(x, y)v}{F'(x)} u.
\]

To guarantee the cone inclusion condition \((H1)(1)\), it is sufficient to have that

\[
\sup_{(x, y) \in Q \setminus \hat{S}_0} \left| \frac{G_x(x, y)}{F'(x)} \right| + \Theta \sup_{(x, y) \in Q \setminus \hat{S}_0} \left| \frac{G_y(x, y)}{F'(x)} \right| \leq \Theta. \tag{20}
\]

By the explicit formula of \( D_x \hat{F} \) in (15) and the fact that \( F'(x) \geq 2 \), we obtain that

\[
\inf_{x \in \mathcal{T}(x)} F'(x) \geq 2, \quad \sup_{(x, y) \in Q \setminus \hat{S}_0} \left| G_y(x, y) \right| \leq \frac{1}{2},
\]

and

\[
\sup_{(x, y) \in Q \setminus \hat{S}_0} \left| \frac{G_x(x, y)}{F'(x)} \right| \leq \sup_{x \in (0, \frac{1}{2})} \frac{\sigma |T''(x^*)|}{|T'(x^*)|^\sigma + 2}.
\]

By (13) and that \( T \in C^2((0, 1/3]) \), we have that

\[
\Theta_0 := \sup_{x \in (0, \frac{1}{2})} \frac{\sigma |T''(x)|}{|T'(x)|^\sigma + 2} < \infty. \tag{21}
\]

Then (20) and thus Assumption \((H1)(1)\) holds by choosing \( \Theta \geq 10 \Theta_0 \).

Once the angle \( \Theta \) is established, we define the following adapted norm

\[
\|(u, v)\|_* = \left| u \right| + \frac{1}{t_1 \Theta} |v|, \quad \text{for} \quad (u, v) \in \mathcal{T}(x, y) Q,
\]

for some \( t_1 > 1 \). If \((u, v) \in \mathcal{C}^u(x) \) and \((u_1, v_1) = D_x \hat{F}(u, v) \) for \( x \in Q \setminus \hat{S}_1 \), then

\[
\frac{\|(u_1, v_1)\|_*}{\|(u, v)\|_*} \leq \frac{|u_1|}{|u|} + \frac{1}{t_1 \Theta} |v| \geq \frac{|F'(x)u|}{(1 + 1/t_1)|u|} \geq \frac{2}{1 + 1/t_1} \geq \Lambda; \tag{22}
\]

and if \((u_1, v_1) \in \mathcal{C}^s(x_1) \) and \((u, v) = D_x \hat{F}^{-1}(u_1, v_1) \) for \( x_1 \in Q \setminus \hat{S}_{-1} \), then

\[
\frac{\|(u, v)\|_*}{\|(u_1, v_1)\|_*} \geq \frac{1}{t_1 \Theta} \frac{|v|}{|u_1| + \frac{1}{t_1 \Theta} |v|} \geq \frac{1}{t_1 \Theta} \frac{1 - G_y(x, y) |u_1|}{G_x(x, y) |v|} \geq \frac{1}{1 + 1/t_1} \frac{1 - \Theta_0 |v|}{|u_1|} \geq \frac{2(1 - 0.1/t)}{1 + 1/t_1} \geq \Lambda,
\]

where we choose \( t \gg t_1 \gg 1 \) and \( \Lambda = 1.9 \). Therefore, \( D \hat{F} \) is uniformly expanding along unstable cones and \( D \hat{F}^{-1} \) is uniformly contracting along stable cones, and
thus Assumption (H1)(2) holds. We notice that the expansion in (22) can be larger than 2.9 if \( x \notin Q_{00} \).

It is easy to see that \( \| D_x \hat{F} v \| / \| v \| \), for \( x \in Q \setminus S_1 \) and \( v \in C^u(x) \setminus \{ 0 \} \), may blow up only for those points \( x = (x, y) \in Q_{00} \) near \( \{ 0 \} \times I \). More precisely, we have that

\[
D_x \hat{F} = \begin{pmatrix}
T'(x) & 0 \\
-\frac{\sigma y T''(x)}{T'(x)^{\sigma+1}} & \frac{1}{T'(x)^{\sigma}}
\end{pmatrix}, \quad (x, y) \in Q_{00},
\]

and note that by (13), there is \( C > 0 \) such that

\[
\left| \frac{\sigma y T''(x)}{T'(x)^{\sigma+1}} \right| \leq \sigma \left| \frac{T''(x)}{T'(x)^{\sigma+1}} \right| \leq CT'(x) \text{ for any } x \in I_{00} = (0, 1/3].
\]

Hence we can choose the function \( \hat{f}(x) \) as in (16) for Assumption (H1)(3).

\[ \Box \]

**Proof of Lemma 3.2.** Under Conditions (h1) and (h2), we prove that Assumptions (H2)-(H4) hold for the unstable curves \( W^u \). The proof for stable curves in \( W^s \) can be proceeded in a similar fashion.

First we prove that Assumption (H2) holds. Recall that \( W \in W^u \) is a homogeneous unstable curve with size less than sufficiently small \( \delta_0 > 0 \) and curvature bounded above by sufficiently large \( B > 0 \), which can be regarded as the graph of a local \( C^2 \) function. Furthermore, there is \( C_0 > 0 \) such that

\[
1 \leq \frac{dm_W(x)}{dx} \leq C_0, \quad x = (x, y) \in W,
\]

where \( m_W \) is the length on \( W \) induced by the adapted norm. Consequently, \( d_W(x_1, x_2) \sim |x_1 - x_2| \) for any \( x_1 = (x_i, y_i) \in W, \ i = 1, 2 \). If further \( \hat{F}(W) \in W^u \),

\[
d_{\hat{F}(W)}(\hat{F}(x_1), \hat{F}(x_2)) \sim |F(x_1) - F(x_2)|. \tag{23}
\]

We now show the distortion condition (8) along all homogeneous unstable curve \( W \in W^u \) with \( \hat{F}(W) \in W^u \), with the choice of the exponent

\[
p_0 = 1 - \frac{\sigma}{a + 1} - \frac{b}{a + 1} \in (0, 1),
\]

where \( \sigma, a \) and \( b \) are given by (h1), (h2) and (17) respectively. Recall that in (19), the homogeneity region is of the form \( J \times I \), where \( J = I_{01}, I_{11}, I_{02}, I_{12}, I_{00}, \) or \( I_{k0}, k \geq k_0 \). In what follows, we only prove that (8) holds for \( W \subset I_{00}^k \times I, k \geq k_0 \). The other cases can be shown in a similar way.

By the explicit formula of \( D_x \hat{F} \) in (15), given \( x_1 = (x_1, y_1), x_2 = (x_2, y_2) \in W, \) the matrix \( D_x \hat{F}(D_{x^2} \hat{F})^{-1} - \text{Id} \) is

\[
\begin{pmatrix}
\frac{T'(x_1)}{T'(x_2)} - 1 & 0 \\
-\sigma \left[ \frac{y_1 T''(x_1)}{T'(x_1)^{\sigma+1} T'(x_2)} - \frac{y_2 T''(x_2)}{T'(x_1)^{\sigma} T'(x_2)^2} \right] & \left[ \frac{T'(x_2)}{T'(x_1)} \right]^{\sigma} - 1
\end{pmatrix}. \tag{24}
\]

To obtain (8), we need to show that each entry of the above matrix is bounded by \( Cd_{\hat{F}(W)}(\hat{F}(x_1), \hat{F}(x_2))^{p_0} \) for some \( C > 0 \). By (23), this upper bound is equivalent to \( C|T(x_1) - T(x_2)|^{p_0} \).
Note that $x_1, x_2 \in I_{k_0}$ for some $k \geq k_0$. We pick $x_3, x_4 \in [x_1, x_2]$ such that

$$T(x_1) - T(x_2) = T^\prime(x_3)(x_1 - x_2), \quad \frac{1}{T^\prime(x_1)} - \frac{1}{T^\prime(x_2)} = \frac{T''(x_4)}{T'(x_2)^2}(x_1 - x_2).$$

By (18), we have $x_i \in ((k + 1)^{-\frac{1}{2}}, k^{-\frac{1}{2}}]$, and there exists $C > 0$ (independent of $k$) such that

$$C^{-1}k^b \leq T'(x_i) \leq Ck^b, \quad i = 1, 2, 3, 4.$$

By Conditions (h1), (h2) and (21),

$$\left| \frac{T'_{(x_1)} - 1}{T(x_1) - T(x_2)^{p_0}} \right| \leq \frac{T'(x_1)}{T'(x_3)^{p_0}} \left| \frac{1}{T'(x_2)^{p_0}} - \frac{1}{T'(x_1)^{p_0}} \right| \left| x_1 - x_2 \right|^{\sigma} \leq \frac{T'(x_1)}{T'(x_3)^{p_0}} \left| x_1 - x_2 \right|^{\sigma} \leq C_1(k^b)^{1+\sigma-p_0}(k^{-\frac{1}{2}}-1)^{1-p_0} \leq C_1.$$

Thus, the first entry of the matrix given by (24) is bounded by $C_1|T(x_1) - T(x_2)|^{p_0}$.

The last entry $\left[ \frac{T'(x_1)T'(x_4)^\sigma}{T'(x_3)^{p_0}} \right]_{x_1} - x_2 \leq \left( \frac{T'(x_2)^{p_0}}{T'(x_1)^{p_0}} \right)_{x_1} - x_2 \leq C_3|T(x_1) - T(x_2)|^{p_0} + \Theta_0C_4|y_1 - y_2| + 2\Theta_0C_1|T(x_1) - T(x_2)|^{p_0}$.

Next we verify Assumption (H3). Recall that $\tilde{S}_1$ consists of vertical singularity lines and $\partial Q$, and thus $d(x, \tilde{S}_1) = \min \{ |x, y, 1 - y| \}$ for $x = (x, y)$ with $x \in I_{k_0}$. By the definition of $\tilde{f}(x)$ in (16), we obtain that $\tilde{f}(x) = f(x) = \frac{1}{T'(x)}$. Therefore, (9) in Assumption (H3)(1) holds if Condition (h2) is true. Assumptions (H3)(2)(3) are automatically true due to the skew-product structure of $\tilde{F}$ and our special choices of homogeneity regions.

Finally, we show Assumption (H4). For any homogeneous unstable curve $W \subset \mathcal{W}^u$, if $W \not\subset Q_{k_0} = I_{k_0} \times I$, $k \geq k_0$, then $\tilde{F}(W)$ crosses at most two regions out of
the five regions \( Q_{00}, Q_{01}, Q_{02}, Q_{11} \) and \( Q_{12} \) by choosing \( \delta_0 \) small enough. Hence,

\[
\sum_i |J_{W_i} \tilde{F}|^{-1}_* < \frac{1}{1.9} + \frac{1}{2.9} < 1.
\]

So we only need to show the one-step expansion for unstable curves \( W \subset Q_{00}^i \) with \( |W| < \delta_0 \). Denote by \( \{V_i\} \) the set of homogeneous components of \( \tilde{F}(W) \), and set \( W_i = \tilde{F}^{-1}V_i. \) Let \( V_{i_0} \) be the possible component such that \( V_{i_0} \subset Q_{00}^i. \) Any other homogeneous components \( V_i \) must lie inside \( Q_{00}^i \) for \( i_1 \leq i \leq i_2, \) where

\[
i_1 := \max\{i : c_i \leq T(c_{k+1})\}, \quad \text{and} \quad i_2 := \min\{i : c_i \geq T(c_k)\}.
\]

By our choice of \( c_k \) in (18), Condition (H2) and the mean value theorem,

\[
i_2 - i_1 \leq \left[T(k^{-\frac{a}{b}})^{\frac{a}{b}} + 1\right] - \left[T((k+1)^{-\frac{a}{b}})^{\frac{a}{b}} - 1\right]
\]

\[
\leq 2 + T(k^{-\frac{a}{b}})^{\frac{a}{b}} - T((k+1)^{-\frac{a}{b}})^{\frac{a}{b}}
\]

\[
\leq 2 + T(z)^{-\frac{a}{b}} - \sum b \frac{a}{b} - (k+1)^{-\frac{a}{b}}
\]

\[
\leq 2 + C'(k^{-\frac{a}{b}})^{(1-a)(\frac{a}{b})} - a k^{-\frac{a}{b}} - 1
\]

\[
\leq 2 + C'k^{-a} \leq 2 + C',
\]

thus we have

\[
\sum_i |J_{W_i} \tilde{F}|^{-1}_* = \sum_{i_1 \leq i \leq i_2} |J_{W_i} \tilde{F}|^{-1}_* \leq \frac{1}{1.9} + C'' \sum_{i_1 \leq i \leq i_2} \min_{x \in \{i_0\}} T'(x)
\]

\[
\leq \frac{1}{1.9} + C''k^{-b}(i_2 - i_1 + 1)
\]

\[
\leq \frac{1}{1.9} + C''(2 + C')k_0^{-b} < 1,
\]

provided that \( k_0 \) is sufficiently large. This completes the verification of Assumption (H4). \( \square \)

3.1.3. Remarks for general cases. In the above subsections, we assume that the only essential singularity \( x = 0 \) is a fixed point for \( F \), and \( F \) only has finitely many inverse branches. In general, we consider the following cases:

(1) \( F \) could have countably infinitely many branches, but the subset \( \mathcal{S}_0 \subset \mathcal{S}_0 \) of essential singularities is a finite set. Further, singularities need not be fixed points;

(2) On each branch, \( F \) need not be linear or surjective.

Let \( \mathcal{J} \) be the collection of all branches of \( F \), and \( \mathcal{J}_* \subset \mathcal{J} \) be the sub-collection such that \( \partial J \cap \mathcal{S}_0 \neq \emptyset \) for any \( J \in \mathcal{J}_* \). By taking high iterates of \( F \) and adding extra singularities, we can assume that \( F(J_1) = F(J_2) \) or \( \text{int}(F(J_1)) \cap \text{int}(F(J_2)) = \emptyset \) for any two branches \( J_1, J_2 \in \mathcal{J}_* \). Furthermore, we assume that for any \( y \in I \), there are at least two branches in \( \mathcal{J} \setminus \mathcal{J}_* \) whose images contain \( y \), that is,

\[
\#\{J \in \mathcal{J} \setminus \mathcal{J}_* : y \in F(J)\} \geq 2.
\]
Then we can construct the two-dimensional lift \( \hat{F}(x, y) = (F(x), G(x, y)) \), where \( G(x, y) \) is defined in a similar style as in (14) such that \( G(x, y) \) is linear in \( y \)-coordinate on each component, and \( G(x, y) = \frac{y}{|F'(x)|} \) for all \( x = (x, y) \) that belong to components having boundaries in \( S^*_0 \times I \).

Using similar arguments as in the previous subsections, we can show that Conditions \((E_1), (E_2), (h_1)\) and \((h_2)\) imply the Assumptions \((H_1)-(H_3)\). To verify the one-step expansion assumption \((H_4)\), we need an additional assumption \((h_3)\), that is, \( a\sigma < 1, \) for any \( c \in S^*_0 \) which is not a fixed point. Indeed, Assumption \((h_3)\) implies that we can choose \( b > 1 \) in (17), and thus

\[
\sum_i |\mathcal{J}_{W_i} \hat{F}^{-1}|_s = |\mathcal{J}_{W_0} \hat{F}^{-1}|_s + \sum_{i: W_i \subset Q_{00}^k} |\mathcal{J}_{W_i} \hat{F}^{-1}|_s \\
\leq \frac{1}{1.9} + C'' \sum_{i: W_i \subset Q_{00}^k} \min_{x \in I_{00}^k} T'(x) \\
\leq \frac{1}{1.9} + C'' \sum_{k=k_0}^\infty k^{-b} < 1,
\]

provided that \( k_0 \) is sufficiently large.

3.2. Proof of Theorem 1.1. Before we prove Theorem 1.1, let us first explain why we do not project the functional spaces directly. Indeed, one wishes to project the functional space \( \mathcal{B} \) over \( Q = I \times I \) to some space over \( I \) on which the one-dimensional RPF transfer operator \( \mathcal{L}_F \) has spectral gap. However, the definition of \( \mathcal{B} \) heavily relies on the family of homogeneous stable and unstable curves, which makes no sense for the one-dimensional expanding map \( F \).

Nevertheless, using the techniques by Chernov and Korepanov in [16] (see also Appendix A in [11]), we can project an SRB measure for \( \hat{F} : Q \to Q \) along the vertical stable foliation down to an absolutely continuous invariant probability measure \( \mu \), from which the statistical properties of the one-dimensional system \( (F, \mu) \), such as the exponential decay of correlations, the large deviation principle and the almost-sure invariance principle follow immediately.

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. By Item (1) in Lemma 2.1, if \( 1 \) is the only peripheral eigen-space, that is, \( \mathcal{V} = \mathcal{V}_0 \), then we can choose a mixing SRB measure \( \hat{\mu} \) in the space \( \mathcal{V} \). We shall deal with the case when \( \mathcal{V} \supseteq \mathcal{V}_0 \) at the end of the proof.

We now set \( \mu = \Pi_* \hat{\mu} \), where \( \Pi \) is the vertical projection from \( Q \) onto \( I \). Note that \( \mu \) is independent of our choice for \( \hat{\mu} \). This is due to Condition \((E_2)\), that is, the map \( F \) is surjective and topologically mixing, which guarantees the uniqueness of the acim for \( F \). It is easy to see that the projection measure \( \mu = \Pi_* \hat{\mu} \) is an \( F \)-invariant probability measure on \( I \). It then remains to show the absolute continuity of \( \mu \), whose density function is positive and continuous except on the set \( S^*_\infty \) given by (11).

To this end, we first recall some preliminary results about geometric structures of SRB measure for the two-dimensional lift \( \hat{F} \). See [17] for more details.

It is well known that unstable manifolds are \( C^2 \) smooth (See e.g. [29]). More precisely, a homogeneous unstable manifold \( W \) is the graph of some \( C^2 \) function \( \varphi_W \) with uniformly bounded derivatives up to second order, which is defined on an open subinterval of \( I \) (the length of the subinterval could be small though). We
define
\[ J_W(x) := \frac{dm_W(x)}{dx} = \sqrt{1 + (\varphi'_W(x))^2}, \]
then the locally defined function \( J_W \) is \( C^1 \) with uniformly bounded derivatives up to first order. In particular, by our choice of unstable cones in (5), \( |\varphi'_W(x)| \leq \Theta^+ \) for any \( x = (x, y) \in W \), and as a result, \( 1 \leq J_W(x) \leq \sqrt{1 + (\Theta^+)^2} \).

Let \( \Gamma \) be the measurable partition of the unit square \( Q \) into homogeneous unstable manifolds (up to subdivisions by intersections with \( \cup_{n \geq 0} F^n S^0_H \) if needed). The SRB measure \( \hat{\mu} \) induces a conditional probability measure \( \hat{\mu}_W \) on each \( W \in \Gamma \) and a factor measure \( \eta \) on \( \Gamma \) with a standard \( \sigma \)-algebra. In other words, we have the following disintegration
\[ \hat{\mu}(B) = \int_{W \in \Gamma} \hat{\mu}_W(B) d\eta(W), \]
for any Borel measurable set \( B \subset Q \). Moreover, the conditional probability measure \( \hat{\mu}_W \) is absolutely continuous with respect to the arclength \( m_W(\cdot) \) on \( W \) with a \( C^1 \) smooth density \( \rho_W := \frac{\hat{\mu}_W}{m_W} \) satisfying the bounded distortion condition,\(^2\) i.e., there is \( C > 0 \) such that
\[ \left| \frac{d}{dx} \ln \rho_W(x) \right| \leq C |W|^{1/r_h - 1}, \text{ for any } x \in W. \]  

Consequently, there is \( C > 0 \) such that \( \frac{1}{C} \leq \frac{\rho_W(x)}{\rho_W(x')} \leq C \) for any \( x, x' \in W \), and hence \( \rho_W \) is proportional to its average \( \frac{1}{|W|} \int \rho dm_W = |W|^{-1} \), that is,
\[ C^{-1} |W|^{-1} \leq \rho_W(x) \leq C |W|^{-1}, \text{ for any } x \in W. \]  

For every \( W \in \Gamma \) and \( x \in W \), the point \( x \) divides the curve \( W \) into two pieces, and we denote by \( r_W(x) \) the length of the shorter one (in the Euclidean metric of \( W \)). The size of a homogeneous unstable manifold \( W \) is then described by the following lemma.

**Lemma 3.3.** There is \( C > 0 \) such that for any \( W \in \Gamma \) and \( \varepsilon > 0 \),
\[ \hat{\mu} \{ x \in Q : r_W(x) < \varepsilon \} \leq C \varepsilon. \]

The proof of the above lemma is similar to Theorem 5.17 in [17], which we leave to the reader as an exercise. This lemma immediately implies that
\[ \int_{\Gamma} \frac{d\eta(W)}{|W|} < \infty, \]
and hence allows us to renormalize the conditional measures and the factor measure of \( \hat{\mu} \) as follows:
\[ d\hat{\mu}_W' = |W| d\hat{\mu}_W, \quad \text{and} \quad d\eta^#(W) = \frac{d\eta(W)}{|W|}. \]

In this way, the density function of the new conditional measure \( \hat{\mu}_W' \) is given by \( \rho_W' = |W| \rho_W \), and by (26) and (27), \( \rho_W' \) is uniformly bounded above and away from zero, and \( \left| \frac{d}{dx} \rho_W'(x) \right| \leq C |W|^{1/r_h - 1} \) for some constant \( C > 0 \). Moreover, the new factor measure \( \eta^# \) is a finite measure, and further, as shown by Lemma 4.1 in [16], \( \eta^#(W) = 0 \) for any \( W \in \Gamma \).

\(^2\)The proof is similar to that in Section 5.6, [17] for the case when \( r_h = 3 \).
Given \( W \in \Gamma \), we define \( \mu_W(A) := \hat{\mu}_W^*(A \times I) \) for any Borel measurable subset \( A \subset I = [0,1] \). It is obvious that \( \mu = \Pi_* \hat{\mu} \) has disintegration with the conditional measure \( \mu_W \) on each \( W \in \Gamma \) and the factor measure \( \eta^\# \). Indeed,

\[
\mu(A) = \hat{\mu}(\Pi^{-1}(A)) = \hat{\mu}(A \times I) = \int_{W \in \Gamma} \hat{\mu}_W^*(A \times I) d\eta^\#(W) = \int_{W \in \Gamma} \mu_W(A) d\eta^\#(W).
\]

Recall that \( \hat{F}^{-n}(W) \cap S^m_0 = \emptyset \) for any \( n \geq 0 \) and each \( W \in \Gamma \), and hence \( \Pi(W) \subset I \setminus S^\infty_\varepsilon \), where \( S^\infty_\varepsilon \) is defined by (11). Then it is not hard to see that \( I \setminus S^\infty_\varepsilon = \bigcup_{W \in \Gamma} \Pi(W) \). Given \( x \in I \setminus S^\infty_\varepsilon \) that lies in \( \Pi(W) \) for some \( W \in \Gamma \), let \( x = (x',y) \in W \), then we have

\[
\lim_{\varepsilon \to 0} \frac{\mu_W(x - \varepsilon, x + \varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0} \int_{x' \in \Pi^{-1}(x - \varepsilon, x + \varepsilon) \times I} \rho_W^\#(x') dm_W(x')
\]

where \( J_W(\cdot) \) is given by (25). The last step used the fact that \( \rho_W^\# \) and \( J_W \) are \( C^1 \) bounded functions with bounded first derivative (These bounds might depend on \( W \) but not \( \varepsilon \)). In other words, the conditional measure \( \mu_W \) is absolutely continuous with respect to the Lebesgue measure on the unit interval \( I \), and its density function

\[
p_W(x) = \rho_W^\#(x) J_W(x)
\]

is well-defined on \( I \setminus S^\infty_\varepsilon \) such that it is uniformly bounded above and away from zero.

Therefore, the measure \( \mu = \Pi_* \hat{\mu} \) is absolutely continuous with respect to the Lebesgue measure on \( I \), and its density function is given by

\[
\rho(x) = \lim_{\varepsilon \to 0} \frac{\mu(x - \varepsilon, x + \varepsilon)}{2\varepsilon} = \int_{W \in \Gamma} p_W(x) d\eta^\#(W),
\]

for all \( x \in I \setminus S^\infty_\varepsilon \). By the bounded convergence theorem and the fact that \( \eta^\# \) is a finite measure, we immediately get that \( \rho(x) \) is positive and continuous except on \( S^\infty_\varepsilon \).

Finally, let us discuss the case when \( \forall \supseteq \forall_0 \), that is, there are peripheral eigenvalues for \( \hat{L}_F \) other than 1. By Item (1) in Lemma 2.1, since \( \dim(\forall) \) is finite, there is \( n_0 \) such that the peripheral eigenspace for \( \hat{F}^{n_0} \) only contains 1. Take a mixing SRB measure \( \hat{\mu} \) for \( \hat{F}^{n_0} \), and the projection \( \mu = \Pi_* \hat{\mu} \). Due the Condition (E2) and all the above arguments, we obtain that \( \mu \) is the unique acim for \( F^{n_0} \). We claim that \( \mu \) is in fact the unique acim for \( F \). Indeed, if we let \( \mu' = \frac{1}{n_0} \sum_{n=0}^{n_0-1} \mu \circ F^n \), then it is clear that \( \mu' \) is absolutely continuous and \( F \)-invariant (and of course, still \( F^{n_0} \)-invariant). By the uniqueness of acim for \( F^{n_0} \), we must have \( \mu = \mu' \), which implies that \( \mu \) is \( F \)-invariant. \( \square \)
3.3. Proof of Theorem 1.2. We are now ready to show Theorem 1.2.

Proof of Theorem 1.2. For simplicity, let us assume that 1 is the only peripheral eigenspace for $\mathcal{L}_\vartheta$. Otherwise, we can apply similar arguments in the last paragraph of the proof of Theorem 1.1.

All the statistical properties of the expanding map $(F, \mu)$ follow directly from Lemma 2.1, Lemma 2.3 and Theorem 1.1, together with the following observations: we can regard a function $\phi \in C^\infty (P^c_\vartheta )$ (or $\hat{\phi} (x) := \phi \circ \Pi (x) = \phi (x)$), which belongs to $C^\infty (\hat{P}^c_\vartheta )$ (or $\hat{\mathcal{H}}^\vartheta_+ )$. Similarly, a function $\psi \in L^\infty (I)$ can be viewed as $\hat{\psi} (x) := \psi \circ \Pi (x) = \psi (x)$, which belongs to $\mathcal{H}^- \cap L^\infty (Q, \hat{\mu})$ for any $\vartheta \in (0, 1)$. Furthermore, we have that $S_n^F \hat{\phi} (x) = S_n^F \phi (x)$ for any $x \in Q$, and

$$\int_Q \hat{\phi} \cdot \hat{\psi} \circ F^nd\hat{\mu} = \int_I \phi \cdot \psi \circ F^nd\mu, \quad \int_Q \hat{\phi} d\hat{\mu} = \int_I \phi d\mu, \quad \int_Q \hat{\phi} d\nu = \int_I \phi d\nu.$$

This completes the proof of Theorem 1.2. \qed

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