ON THE COMBINATORICS OF RIGID OBJECTS IN 2-CALABI-YAU CATEGORIES

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Abstract. Given a triangulated 2-Calabi-Yau category $C$ and a cluster-tilting subcategory $T$, the index of an object $X$ of $C$ is a certain element of the Grothendieck group of the additive category $T$. In this note, we show that a rigid object of $C$ is determined by its index, that the indices of the indecomposables of a cluster-tilting subcategory $T'$ form a basis of the Grothendieck group of $T$ and that, if $T$ and $T'$ are related by a mutation, then the indices with respect to $T$ and $T'$ are related by a certain piecewise linear transformation introduced by Fomin and Zelevinsky in their study of cluster algebras with coefficients. This allows us to give a combinatorial construction of the indices of all rigid objects reachable from the given cluster-tilting subcategory $T$. Conjecturally, these indices coincide with Fomin-Zelevinsky’s $g$-vectors.

1. Introduction

This note is motivated by the representation-theoretic approach to Fomin-Zelevinsky’s cluster algebras [6] [7] [4] [8] developed by Marsh-Reineke-Zelevinsky [18], Buan-Marsh-Reineke-Reiten-Todorov [3], Geiss-Leclerc-Schröer [11] [12] and many others, cf. [2] for a survey. In this approach, a central role is played by certain triangulated 2-Calabi-Yau categories and by combinatorial invariants associated with their rigid objects (we refer to [14] [5] for different approaches). Here, our object of study is the index, which is a certain ‘dimension vector’ associated with each object of the given Calabi-Yau category.

More precisely, we fix a Hom-finite 2-Calabi-Yau triangulated category $C$ with split idempotents which admits a cluster-tilting subcategory $T$. It is known from [16] that for each object $X$ of $C$, there is a triangle

$$T_1 \to T_0 \to X \to \Sigma T_1$$

of $C$, where $T_1$ and $T_0$ belong to $T$. Following [19], we define the index of $X$ to be the difference $[T_0] - [T_1]$ in the split Grothendieck group $K_0(T)$ of the additive category $T$. We show that

- if $X$ is rigid (i.e. $C(X, \Sigma X) = 0$), then it is determined by its index up to isomorphism;
- the indices of the direct factors of a rigid object all lie in the same hyperquadrant of $K_0(T)$ with respect to the basis given by a system of representatives of the isomorphism classes of the indecomposables of $T$;
- the indices of the direct factors of a rigid object are linearly independent;
- the indices of a system of representatives of the indecomposable objects of any cluster-tilting subcategory $T'$ form a basis of $K_0(T)$. In particular, all cluster-tilting subcategories have the same (finite or infinite) number of pairwise non isomorphic indecomposable objects.

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Note that the last point was shown in Theorem I.1.8 of [1] under the additional assumption that \( C \) is a stable category. We then study how the index of an object transforms when we mutate the given cluster-tilting subcategory. We find that this transformation is given by the right hand side of Conjecture 7.12 of [8], cf. section 4. This motivates the definition of \( \mathbf{g}^\dagger \)-vectors as the combinatorial counterpart to indices. If, as we expect, Conjecture 7.12 of [loc. cit.] holds, then our \( \mathbf{g}^\dagger \)-vectors are identical with the \( \mathbf{g} \)-vectors of [loc. cit.], whose definition we briefly recall below. We finally show that if \( C \) has a cluster-structure in the sense of [1], then we have a bijection between \( \mathbf{g}^\dagger \)-vectors and indecomposable rigid objects reachable from \( T \) and between \( \mathbf{g}^\dagger \)-clusters and cluster-tilting subcategories reachable from \( T \).

Our results are inspired by and closely related to the conjectures of [8] and the results of section 15 in [10]. As a help to the reader not familiar with [8], we give a short summary of the notions introduced there which are most relevant for us: Let \( n \geq 1 \) be an integer and \( B \) a skew-symmetric integer matrix. Let \( \mathcal{F} \) be the field of rational functions \( \mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n) \) in \( 2n \) indeterminates. Let \( A \subset \mathcal{F} \) be the cluster algebra with principal coefficients associated with the initial seed \( (x, y, B) \), where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), cf. sections 1 and 2 of [8]. As shown in Proposition 3.6 of [8], each cluster variable of \( A \) lies in the ring \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n] \). Moreover, by Proposition 6.1 of [8], each cluster variable of \( A \) is homogeneous with respect to the \( \mathbb{Z}^n \)-grading on \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n] \) given by

\[
\deg(x_i) = e_i, \quad \deg(y_j) = -\sum_{i=1}^n b_{ij} e_i,
\]

where the \( e_i \) form the standard basis of \( \mathbb{Z}^n \). The \( g \)-vector associated with a cluster variable \( X \) is by definition the vector \( \deg(X) \) of \( \mathbb{Z}^n \). More generally, the \( g \)-vector of a cluster monomial \( M \) is \( \deg(M) \). Now we can state the conjectures of [8] which motivated the above statements on the combinatorics of rigid objects:

- different cluster monomials have different \( g \)-vectors (part (1) of Conjecture 7.10 of [8]);
- the \( g \)-vectors of the variables in a fixed cluster all lie in the same hyperquadrant of \( \mathbb{Z}^n \) (Conjecture 6.13 of [8]);
- the \( g \)-vectors of the variables in a fixed cluster form a basis of \( \mathbb{Z}^n \) (part (2) of Conjecture 7.10 of [8]);
- under a mutation of the initial cluster, the \( g \)-vector of a given cluster variable transforms according to a certain piecewise linear transformation, cf. section 4 (Conjecture 7.12 of [8]).

In [9], the results of this paper have been used to prove these conjectures for certain classes of cluster algebras.

2. A rigid object is determined by its index

Let \( k \) be an algebraically closed field and \( C \) a Hom-finite \( k \)-linear triangulated category with split idempotents. In particular, the decomposition theorem holds for \( C \): Each object decomposes into finite sum of indecomposable objects, unique up to isomorphism, and indecomposable objects have local endomorphism rings. We write \( \Sigma \) for the suspension functor of \( C \). We suppose that \( C \) is 2-Calabi-Yau, i.e. that the square of the suspension functor (with its canonical structure of triangle functor) is a Serre functor for \( C \). This implies that we have bifunctorial isomorphisms

\[
D_C(X,Y) \cong C(Y, \Sigma^2 X),
\]
where $X$ and $Y$ vary in $C$ and $D$ denotes the duality functor $\text{Hom}_k(?,k)$ over the ground field. Moreover, we suppose that $C$ admits a cluster-tilting subcategory $\mathcal{T}$ (called a maximal 1-orthogonal subcategory in [13]). Recall from [16] that this means that $\mathcal{T}$ is a full additive subcategory such that

- $\mathcal{T}$ is functorially finite in $C$, i.e. for all objects $X$ of $C$, the restrictions of the functors $C(X,?)$ and $C(?,X)$ to $\mathcal{T}$ are finitely generated, and
- an object $X$ of $C$ belongs to $\mathcal{T}$ iff we have $C(T,\Sigma X) = 0$ for all objects $T$ of $\mathcal{T}$.

We call an object $X$ of $C$ rigid if the space $C(X,\Sigma X)$ vanishes.

### 2.1. Rigid objects yield open orbits.

Let $X$ be a rigid object of $C$. From [17], we know that there is a triangle $T_1 \xrightarrow{f} T_0 \xrightarrow{h} X \xrightarrow{\Sigma} X$, where $T_0$ and $T_1$ belong to $\mathcal{T}$. The algebraic group $G = \text{Aut}(T_0) \times \text{Aut}(T_1)$ acts on $C(T_1, T_0)$ via $(g_0, g_1)f' = g_0f'g_1^{-1}$.

**Lemma.** The orbit of $f$ under the action of $G$ is open in $C(T_1, T_0)$.

**Proof.** It suffices to prove that the differential of the map $g \mapsto gf$ is a surjection from $\text{Lie}(G)$ to $C(T_1, T_0)$. This differential is given by $(\gamma_0, \gamma_1)f = \gamma_0 f - f\gamma_1$. Let $f'$ be an element of $C(T_1, T_0)$. Consider the following diagram

$$
\begin{array}{ccc}
\Sigma^{-1}X & \longrightarrow & T_1 \\
\gamma_1 & \nearrow & f' \\
T_1 & \xrightarrow{\gamma_0} & T_0 \xrightarrow{h} X \xrightarrow{\Sigma} T_1.
\end{array}
$$

Since $X$ is rigid, the composition $h f' e$ vanishes. So there is a $\beta_0$ such that $\beta_0 f = h f'$. Now $h$ is a right $\mathcal{T}$-approximation. So there is a $\gamma_0$ such that $h \gamma_0 = \beta_0$. It follows that we have $h(\gamma_0 f - f') = 0$

So there is a $\gamma_1$ such that

$$
\gamma_0 f - f' = f \gamma_1.
$$

This shows that the differential of the map $g \mapsto gf$ is indeed surjective. $\square$

### 2.2. Rigid objects have disjoint terms in their minimal presentations.

Let $F : C \rightarrow \text{mod } \mathcal{T}$ be the functor taking an object $Y$ of $C$ to the restriction of $C(?,Y)$ to $\mathcal{T}$. Let $X$ be a rigid object of $C$. Let $T_1 \xrightarrow{f} T_0 \xrightarrow{h} X \xrightarrow{\Sigma} T_1$ be a triangle such that $T_0$ and $T_1$ belong to $\mathcal{T}$ and $h$ is a minimal right $\mathcal{T}$-approximation.

**Proposition.** $T_0$ and $T_1$ do not have an indecomposable direct factor in common.

We give two proofs of the proposition. Here is the first one:
Proof. We know that

$$FT_1 \to FT_0 \to FX \to 0$$

is a minimal projective presentation of $FX$. Since $F$ induces an equivalence from $\mathcal{T}$ onto the category of projectives of $\mod \mathcal{T}$, it is enough to show that $FT_1$ and $FT_0$ do not have an indecomposable factor in common. For this, it suffices to show that no simple module $S$ occurring in the head of $FT_0$ also occurs in the head of $FT_1$. Equivalently, we have to show that if a simple $S$ satisfies $\text{Hom}(FX, S) \neq 0$, then we have $\text{Ext}^1(FX, S) = 0$. So let $S$ be a simple admitting a surjective morphism

$$p : FX \to S.$$

Let $f : FT_1 \to S$ be a map representing an element in $\text{Ext}^1(FX, S)$. Since $FT_1$ is projective, there is a morphism $f_1 : FT_1 \to FX$ such that $p \circ f_1 = f$. Now using the fact that $F$ is essentially surjective and full, we choose a preimage up to isomorphism $\tilde{S}$ of $S$ and preimages $\tilde{f}, \tilde{p}$ and $\tilde{f}_1$ of $f, p$ and $f_1$ in $\mathcal{C}$ as in the following diagram

$$\Sigma^{-1}X \xrightarrow{\Sigma^{-1}\epsilon} T_1 \xrightarrow{\tilde{f}_1} T_0 \xrightarrow{f} X \quad X \xrightarrow{\tilde{p}} \tilde{S}$$

Denote by $\mod \mathcal{T}$ the category of finitely presented $k$-linear functors from $\mathcal{T}^{op}$ to the category of $k$-vector spaces. Since $F$ induces a bijection

$$\mathcal{C}(T, Y) \to (\mod \mathcal{T})(FT, FY)$$

for all $Y$ in $\mathcal{C}$, we still have $\tilde{p} \circ \tilde{f}_1 = \tilde{f}$. The composition $\tilde{f}_1 \circ (\Sigma^{-1}\epsilon)$ vanishes since we have $\mathcal{C}(\Sigma^{-1}X, X) = 0$. Therefore, the composition

$$\tilde{f} \circ (\Sigma^{-1}\epsilon) = \tilde{p} \circ \tilde{f}_1 \circ (\Sigma^{-1}\epsilon)$$

vanishes. This implies that $\tilde{f}$ factors through the morphism $T_1 \to T_0$. But then $f$ factors through the morphism $FT_1 \to FT_0$ and $f$ represents $0$ in $\text{Ext}^1(FX, S)$.

Let us now give a second, more geometric, proof of the proposition:

Proof. Suppose that $T_0$ and $T_1$ have an indecomposable direct factor $T_2$ so that we have decompositions

$$T_0 = T'_0 \oplus T_2 \text{ and } T_1 = T'_1 \oplus T_2.$$  

For a morphism $f : T_1 \to T_0$, let

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

be the matrix corresponding to $f$ with respect to the given decompositions. Of course, up to isomorphism, the cone on $f$ only depends on the orbit of $f$ under the group $\text{Aut}(T_0) \times \text{Aut}(T_1)$. Suppose that the cone on $f$ is isomorphic to $X$, which is rigid. Then we know that the orbit of $f$ in $\mathcal{C}(T_1, T_0)$ is open. Hence there is some $f'$ in the orbit such that the component $f''_{22}$ is invertible. But then, using elementary operations on the rows and columns of the matrix of $f'$, we see that the orbit of $f$ contains a morphism $f''$ whose matrix is diagonal with invertible component $f''_{22}$. Clearly, the triangle on $f''$ is not minimal. This shows that $T_1$ and $T_0$ do not have a common indecomposable factor if they are the terms of a minimal triangle whose third term is the rigid object $X$.  

□
2.3. **A rigid object is determined by its index.** The (split) Grothendieck group $K_0(T)$ of the additive category $T$ is the quotient of the free group on the isomorphism classes $[T]$ of objects $T$ of $T$ by the subgroup generated by the elements of the form

$$[T_1 \oplus T_2] - [T_1] - [T_2].$$

It is canonically isomorphic to the free abelian group on the isomorphism classes of the indecomposable objects of $T$. It contains a canonical positive cone formed by the classes of objects of $T$. Each element $c$ of $K_0(T)$ can be uniquely written as

$$c = [T_0] - [T_1]$$

where $T_0$ and $T_1$ are objects of $T$ without common indecomposable factors. Let $X$ be an object of $C$. Recall that its index $|\text{ind}(X)|$ is the element

$$\text{ind}(X) = [T_0] - [T_1]$$

of $K_0(T)$ where $T_0$ and $T_1$ are objects of $T$ which occur in an arbitrary triangle

$$T_1 \rightarrow T_0 \rightarrow X \rightarrow \Sigma T_1.$$

Now suppose that $X$ is rigid. We know that if we choose the above triangle minimal, then $T_0$ and $T_1$ do not have common indecomposable factors. Thus they are determined by $\text{ind}(X)$. Moreover, since the $\mathcal{C}(T_1, T_0)$ is an irreducible variety (like any finite-dimensional vector space), each morphism $f : T_1 \rightarrow T_0$ whose orbit under the group $\text{Aut}(T_0) \times \text{Aut}(T_1)$ is open yields a cone isomorphic to $X$. Thus up to isomorphism, $X$ is determined by $\text{ind}(X)$. In fact, $X$ is isomorphic to the cone on a general morphism $f : T_1 \rightarrow T_0$ between the objects $T_0$ and $T_1$ without a common indecomposable factor such that $\text{ind}(X) = [T_0] - [T_1]$. We have proved the

**Theorem.** The map $X \mapsto \text{ind}(X)$ induces an injection from the set of isomorphism classes of rigid objects of $\mathcal{C}$ into the set $K_0(T)$.

This theorem was inspired by part (1) of conjecture 7.10 in [8].

2.4. **Direct factors of rigid objects have sign-coherent indices.** Let $A$ be a free abelian group endowed with a basis $e_i$, $i \in I$. A subset $X \subset A$ is **sign-coherent** if, for all elements $x, y \in X$ and for all $i \in I$, the sign of the component $x_i$ in the decomposition

$$x = \sum x_i e_i$$

agrees with the sign of $y_i$, cf. Definition 6.12 of [8]. This means that the set $X$ is entirely contained in a hyperquadrant of $A$ with respect to the given basis $e_i$, $i \in I$. Now consider the free abelian group $K_0(T)$ endowed with the basis formed by the classes of indecomposable objects of $T$. Suppose that $X$ is a rigid object of $\mathcal{C}$. We claim that the set of indices of the direct factors of $X$ is sign-coherent. Indeed, let $U$ and $V$ be direct factors of $X$. Choose minimal triangles

$$T^U_1 \rightarrow T^U_0 \rightarrow U \rightarrow \Sigma T^U_1$$

and

$$T^V_1 \rightarrow T^V_0 \rightarrow V \rightarrow \Sigma T^V_1,$$

where the $T^U_1$ and $T^V_1$ belong to $T$. Then the triangle

$$T^U_1 \oplus T^V_1 \rightarrow T^U_0 \oplus T^V_0 \rightarrow U \oplus V \rightarrow \Sigma (T^U_1 \oplus T^V_1)$$

is minimal. Since $U \oplus V$ is rigid, the two terms $T^U_1 \oplus T^V_1$ and $T^U_0 \oplus T^V_0$ do not have indecomposable direct factors in common. In particular, whenever an indecomposable object occurs in $T^U_0$ (resp. $T^V_1$), it does not occur in $T^V_1$ (resp. $T^U_0$). This shows that $\text{ind}(U)$ and $\text{ind}(V)$ are sign-coherent. This property is to be compared with conjecture 6.13 of [8].
2.5. **Indices of factors of rigid objects are linearly independent.** Let $X$ be a rigid object of $\mathcal{C}$ and let $X_i$, $i \in I$, be a finite family of indecomposable direct factors of $X$ which are pairwise non isomorphic. We claim that the elements $\text{ind}(X_i)$, $i \in I$, are linearly independent in $K_0(\mathcal{T})$. Indeed, suppose that we have a relation

$$\sum_{i \in I_1} c_i \text{ind}(X_i) = \sum_{j \in I_2} c_j \text{ind}(X_j)$$

for two disjoint subsets $I_1$ and $I_2$ of $I$ and positive integers $c_i$ and $c_j$. Then the rigid objects

$$\bigoplus_{i \in I_1} X_i^{c_i} \text{ and } \bigoplus_{j \in I_2} X_j^{c_j}$$

have equal indices. So they are isomorphic. Since $I_1$ and $I_2$ are disjoint, all the $c_i$ and $c_j$ have to vanish.

2.6. **The indices of the indecomposables of a cluster tilting subcategory form a basis.** The following theorem was inspired by part (2) of conjecture 7.10 of [8].

**Theorem.** Let $\mathcal{T}'$ be another tilting subcategory of $\mathcal{C}$. Then the elements $\text{ind}(\mathcal{T}')$, where $\mathcal{T}'$ runs through a system of representatives of the isomorphism classes of indecomposables of $\mathcal{T}'$, form a basis of the free abelian group $K_0(\mathcal{T})$.

**Proof.** Indeed, we already know that the $\text{ind}(\mathcal{T}')$ are linearly independent. So it is enough to show that the subgroup they generate contains $\text{ind}(T)$ for each indecomposable $T$ of $\mathcal{T}$. Indeed, let $T$ be an indecomposable of $\mathcal{T}$ and let

$$T \to T_1' \to T_0' \to \Sigma T$$

be a triangle with $T_1'$ in $\mathcal{T}'$ (this triangle allows to compute the index of $\Sigma T$ with respect to $\mathcal{T}'$). Then the map $FT_1' \to FT_0'$ is surjective and therefore, we have

$$\text{ind}(T) - \text{ind}(T_1') + \text{ind}(T_0') = 0$$

by Proposition 6 of [19]. Thus, $\text{ind}(T)$ is in the subgroup of $K_0(\mathcal{T})$ generated by the $\text{ind}(\mathcal{T}')$, where $\mathcal{T}'$ runs through the indecomposables of $\mathcal{T}'$. \qed

3. **How the index transforms under change of cluster-tilting subcategory**

Let $\mathcal{T}'$ be another cluster-tilting subcategory. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are related by a mutation, i.e. there is an indecomposable $S$ of $\mathcal{T}$ and an indecomposable $S^*$ of $\mathcal{T}'$ such that, if $\text{indec}$ denotes the set of isomorphism classes of indecomposables, we have

$$\text{indec}(\mathcal{T}') = \text{indec}(\mathcal{T}) \setminus \{S\} \cup \{S^*\},$$

and that there exist triangles

$$S^* \to B \to S \to \Sigma S^* \text{ and } S \to B' \to S^* \to \Sigma S$$

with $B$ and $B'$ belonging to $\mathcal{T} \cap \mathcal{T}'$, cf. e.g. [3] [12] [15]. We define two linear maps

$$\phi_+: K_0(\mathcal{T}) \to K_0(\mathcal{T}') \text{ and } \phi_-: K_0(\mathcal{T}) \to K_0(\mathcal{T}')$$

which both send each indecomposable $T''$ belonging to both $\mathcal{T}$ and $\mathcal{T}'$ to itself and such that

$$\phi_+(S) = [B] - [S^*] \text{ and } \phi_-(S) = [B'] - [S^*].$$

For an object $X$ of $\mathcal{C}$, we denote by $\text{ind}_\mathcal{T}(X)$ the index of $X$ with respect to $\mathcal{T}$ and by $[\text{ind}_\mathcal{T}(X):S]$ the coefficient of $S$ in the decomposition of $\text{ind}_\mathcal{T}(X)$ with respect to the basis given by the indecomposables of $\mathcal{T}$. The following theorem is inspired by Conjecture 7.12 of [8].
Theorem. Let $X$ be a rigid object of $\mathcal{C}$. We have
\[
\text{ind}_{T'}(X) = \begin{cases} 
\phi_+(\text{ind}_T(X)) & \text{if } [\text{ind}_T(X) : S] \geq 0 ; \\
\phi_-(\text{ind}_T(X)) & \text{if } [\text{ind}_T(X) : S] \leq 0.
\end{cases}
\]

Proof. Let
\[T_1 \to T_0 \to X \to \Sigma T_1\]
be a triangle with $T_0$ and $T_1$ in $\mathcal{T}$. Suppose first that $S$ occurs neither as a direct factor of $T_1$ nor of $T_0$. Then clearly the triangle yields both the index of $X$ with respect to $\mathcal{T}$ and with respect to $\mathcal{T}'$ and we have
\[
\phi_+(\text{ind}_T(X)) = \phi_-(\text{ind}_T(X)) = \text{ind}_{T'}(X).
\]

Now suppose that the multiplicity $[\text{ind}_T(X) : S]$ equals a positive integer $i \geq 1$. This means that $S$ occurs with multiplicity $i$ in $T_0$ but does not occur as a direct factor of $T_1$. Choose a decomposition $T_0 = T_0'' \oplus S^i$. From the octahedron constructed over the composition
\[T_0'' \oplus B^i \to T_0'' \oplus S^i \to X,
\]
we extract the following commutative diagram, whose rows and columns are triangles
\[
\begin{array}{cccccc}
\Sigma S^* & \to & \Sigma S^* \\
\| & & \| \\
T_1 & \to & T_0'' \oplus S^i & \to & X & \to & \Sigma T_1 \\
\| & & \| & & \| & & \| \\
T_1' & \to & T_0'' \oplus B^i & \to & X & \to & \Sigma T_1' \\
\| & & \| & & \| & & \| \\
S^* & \to & S^* \\
\end{array}
\]
Since there are no non zero morphisms from $T_1$ to $\Sigma S^*$ ($T_1$ and $S^*$ belong to $\mathcal{T}'$), the leftmost column is a split triangle and $T_1'$ is isomorphic to $S^* \oplus T_1$. Thus, the third line yields the index of $X$ with respect to $\mathcal{T}'$, which equals
\[
\text{ind}_{T'}(X) = [T_0'' \oplus B^i] - [T_1] = [T_0''] - [T_1] + i([B] - [S^*]) = \phi_+(\text{ind}_T(X)).
\]

Finally, suppose that the multiplicity $[\text{ind}_T(X) : S]$ is equals a negative integer $-i \leq -1$. This means that $S$ occurs with multiplicity $i$ in $T_1$ but does not occur in $T_0$. Choose a decomposition $T_1 = T_1'' \oplus S^i$. From the octahedron over the composition
\[\Sigma^{-1} X \to T_1'' \oplus S^i \to T_1'' \oplus B^i,
\]
we extract the following diagram, whose rows and columns are triangles

\[
\begin{array}{c}
\Sigma^{-1}S^{\ast i} \xrightarrow{1} \Sigma^{-1}S^{\ast i} \\
\Sigma^{-1}X \xrightarrow{1} T_1'' \oplus S^i \xrightarrow{1} T_0 \xrightarrow{1} X \\
\Sigma^{-1}X \xrightarrow{1} T_1'' \oplus B'^i \xrightarrow{1} T_0' \xrightarrow{1} X \\
S^{\ast i} \xrightarrow{1} S^{\ast i}
\end{array}
\]

Since there are no non zero morphisms from \(\Sigma^{-1}S^{\ast i}\) to \(T_0\) (\(S^{\ast}\) and \(T_0\) belong to \(T'\)), the object \(T_0'\) is isomorphic to \(T_0 \oplus S^i\) and we can read \(\text{ind}_{T'}(X)\) off the third line of the diagram:

\[
\text{ind}_{T'}(X) = [T_0'] - [T_1'' \oplus B'^i] = [T_0 \oplus S^{\ast i}] - [T_1'' - i[B'] + (B') - [S^{\ast}]) = \phi_-(\text{ind}_T(X)).
\]

\[\square\]

4. \(g^\dagger\)-vectors and \(g^\ddagger\)-clusters

In this section, we recall fundamental constructions from [8] in a language adapted to our applications. We will define \(g^\dagger\)-vectors using the right hand side of Conjecture 7.12 of [loc. cit.]. If, as we expect, this conjecture holds, then our \(g^\dagger\)-vectors are identical with the \(g\)-vectors of [loc. cit.].

Let \(Q\) be a quiver. Thus \(Q\) is given by a set of vertices \(I = Q_0\), a set of arrows \(Q_1\) and two maps \(s\) and \(t\) from \(Q_1\) to \(I = Q_0\) taking an arrow to its source, respectively its target. We assume that \(Q\) is locally finite, i.e. for each given vertex \(i\) of \(Q\) there are only finitely many arrows \(\alpha\) such that \(s(\alpha) = i\) or \(t(\alpha) = i\). Moreover, we assume that \(Q\) has no loops (i.e. arrows \(\alpha\) such that \(s(\alpha) = t(\alpha)\)) and no 2-cycles (i.e. pairs of distinct arrows \(\alpha \neq \beta\) such that \(s(\alpha) = t(\beta)\) and \(t(\beta) = s(\alpha)\)). The quiver \(Q\) is thus determined by the set \(I\) and the skew-symmetric integer matrix \(B = (b_{ij})_{I \times I}\) such that, whenever the coefficient \(b_{ij}\) is positive, it equals the number of arrows from \(i\) to \(j\) in \(Q\). Notice that if, for an integer \(x\), we write \([x]_+ = \max(x, 0)\), then the number of arrows from \(i\) to \(j\) in \(Q\) is \([b_{ij}]_+\).

The mutation \(\mu_k(Q)\) of \(Q\) at a vertex \(k\) is by definition the quiver with vertex set \(I\) whose numbers of arrows are given by the mutated matrix \(B' = \mu_k(B)\) as defined, for example, in definition 2.4 of [8]:

\[
b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k; \\
b_{ij} + \text{sgn}(b_{ik})b_{kj} & \text{otherwise.}
\end{cases}
\]

As in definition 2.8 of [8], we let \(T = T_I\) be the regular tree whose edges are labeled by the elements of \(I\) such that for each vertex \(t\) and each element \(k\) of \(I\), there is precisely one edge incident with \(t\) and labeled by \(k\). We fix a vertex \(t_0\) of \(T\) and define \(Q_{t_0} = Q\). Clearly, there is a unique map assigning a quiver \(Q_t\) to each vertex \(t\) such that if \(t\) and \(t'\) are linked by an edge labeled by \(k\), we have \(Q_{t'} = \mu_k(Q_t)\). In analogy with the terminology of [8], we call the map \(t \mapsto Q_t\) the quiver pattern associated with \(t_0\) and \(Q\).

Now for each vertex \(t\) of \(T\), we define \(K_t\) to be the free abelian group on the symbols \(e_i^t\), \(i \in I\). For two vertices \(t\) and \(t'\) linked by an edge labeled \(k\), we let \(\phi'^{t,t'}_{t'} : K_t \rightarrow K_{t'}\) respectively \(\phi'^{t,t}_t : K_t \rightarrow K_t\).
be the linear map sending $e_j'$ to $e_j''$ for each $j \neq k$ and sending $e_k$ to 

$$-e_k' + \Sigma_j [b'_{jk}] + e_j''$$

where $(b'_{ij})$ is the skew-symmetric matrix associated with the quiver $Q_t$. We define the piecewise linear transformation 

$$\phi_{t,t'}: K_t \to K_{t'}$$

to be the map whose restriction to the halfspace of elements with positive $\epsilon'_k$-coordinate is $\phi_{t,t}'$ and whose restriction to the opposite halfspace is $\phi_{t,t}'$. Thus, the image of an element $g$ with coordinates $g_j$, $j \in I$, is the element $g'$ with coordinates 

$$g'_j = \begin{cases} 
-g_j & \text{if } j = k; \\
 g_j + [b'_{jk}] + g_k & \text{if } j \neq k \text{ and } g_k \geq 0; \\
 g_j + [b'_{kj}] + g_k & \text{if } j \neq k \text{ and } g_k \leq 0.
\end{cases}$$

It is easy to check that this rule agrees with formula (7.18) in Conjecture 7.12 of \cite{8}.

If $t$ and $t'$ are two arbitrary vertices of $T$, there is a unique path 

$$t = t_1 \to t_2 \cdots \to t_N = t'$$

of edges leading from $t$ to $t'$ and we define $\phi_{t',t}$ to be the composition 

$$\phi_{t_N,t_{N-1}} \circ \cdots \circ \phi_{t_2,t_1}.$$ 

For a vertex $t$ of $T$ and a vertex $l$ of $Q$, the $g^t$-vector $g^t_{l,t}$ is the element of the abelian group $K_{l,t}$ defined by 

$$g^t_{l,t} = \phi_{l_0,t}(e^t_l).$$

The $g^t$-cluster associated with a vertex $t$ of $T$ is the set of $g^t$-vectors $g^t_{l,t}$, $l \in I$. If Conjecture 7.12 of \cite{8} holds for the cluster algebra with principal coefficients associated with the matrix $B$, then it is clear that in the notations of formula (6.4) of \cite{8}, we have 

$$g^t_{l,t} = g_{l,t}$$

for all vertices $t$ of $T$ and all $l \in I$, i.e. the $g^t$-vectors equal the $g$-vectors for the cluster algebra with principal coefficients associated with the skew-symmetric matrix $B$.

5. **Rigid objects in 2-Calabi-Yau categories with cluster structure**

Let $C$ be a Hom-finite 2-Calabi-Yau category with a cluster-tilting subcategory $T$. Let $Q = Q(T)$ be the quiver of $T$. Recall that this means that the vertices of $Q$ are the isomorphism classes of indecomposable objects of $T$ and that the number of arrows from the isoclass of $T_1$ to that of $T_2$ equals the dimension of the space of irreducible morphisms 

$$\text{irr}(T_1, T_2) = \text{rad}(T_1, T_2)/\text{rad}^2(T_1, T_2),$$

where $\text{rad}$ denotes the radical of $T$, i.e. the ideal such that $\text{rad}(T_1, T_2)$ is formed by all non isomorphisms from $T_1$ to $T_2$.

We make the following assumption on $C$: For each cluster-tilting subcategory $T'$ of $C$, the quiver $Q(T')$ does not have loops or 2-cycles. We refer to section 1, page 11 of \cite{11} for a list of classes of examples where this assumption holds. By theorem 1.6 of \cite{11}, the assumption implies that the cluster-tilting subcategories of $C$ determine a cluster structure for $C$. Let us recall what this means:
1) For each cluster-tilting subcategory $T'$ of $\mathcal{C}$ and each indecomposable $S$ of $T'$, there is a unique (up to isomorphism) indecomposable $S^*$ not isomorphic to $M$ and such that the additive subcategory $T'' = \mu_{S}(T')$ of $\mathcal{C}$ with
\[ \text{indec}(T'') = \text{indec}(T') \setminus \{S\} \cup \{S^*\} \]
is a cluster-tilting subcategory;
2) the space of morphisms from $S$ to $\Sigma S^*$ is one-dimensional and in the non-split triangles
\[ S^* \to B \to S \to \Sigma S^* \quad \text{and} \quad S \to B' \to S^* \to \Sigma S \]
the objects $B$ and $B'$ belong to $T' \cap T''$;
3) the multiplicity of an indecomposable $L$ of $T' \cap T''$ in $B$ equals the number of arrows from $L$ to $S$ in $Q(T')$ and that from $S^*$ to $L$ in $Q(T'')$; the multiplicity of $L$ in $B'$ equals the number of arrows from $S$ to $L$ in $Q(T')$ and that from $L$ to $S^*$ in $Q(T'')$;
4) finally, we have $Q(T'') = \mu_{S}(Q(T'))$.

Let $Q = Q(T)$ be the quiver of $T$. Notice that its set of vertices is the set $Q_0 = I$ of isomorphism classes of indecomposables of $T$. Let $\mathbb{T}$ be the regular tree associated with $Q$ as in section 4. We fix a vertex $t_0$ of $\mathbb{T}$ and put $\mathcal{T}_{t_0} = T$. For two cluster tilting subcategories $T'$ and $T''$ as above, let $\psi_{T'\to T'} : \text{indec}(T') \to \text{indec}(T'')$ be the bijection taking $S$ to $S^*$ and fixing all other indecomposables.

Thanks to point 1), with each vertex $t$ of $\mathbb{T}$, we can associate
- a unique cluster-tilting subcategory $\mathcal{T}_t$ and
- a unique bijection $\psi_{t,t_0} : \text{indec}(\mathcal{T}_{t_0}) \to \text{indec}(\mathcal{T}_t)$

such that $\mathcal{T}_{t_0} = T$ and that, whenever two vertices $t$ and $t'$ are linked by an edge labeled by an indecomposable $S$ of $T = \mathcal{T}_{t_0}$, we have
- $T_{t'} = \mu_{S'}(T_t)$, where $S' = \psi_{t,t_0}(S)$, and
- $\psi_{t',t_0} = \psi_{t',t} \circ \psi_{t,t_0}$.

Moreover, thanks to point 4), the map $t \mapsto Q(\mathcal{T}_t)$ is the quiver-pattern associated with $Q$ and $t_0$ in section 4. Notice that the group $K_0(\mathcal{T})$ with the basis formed by the isomorphism classes of indecomposables canonically identifies with the free abelian group $K_{t_0}$ of section 4. We define a cluster-tilting subcategory $T'$ to be reachable from $T$ if we have $T' = \mathcal{T}_t$ for some vertex $t$ of the tree $\mathbb{T}$. We define a rigid indecomposable $M$ to be reachable from $T$ if it belongs to a cluster-tilting subcategory which is reachable from $T$.

**Theorem.**
- a) The index $\text{ind}(M)$ of a rigid indecomposable reachable from $T$ is a $\mathbf{g}^\dagger$-vector and the map $M \mapsto \text{ind}(M)$ induces a bijection from the set of isomorphism classes of rigid indecomposables reachable from $T$ onto the set of $\mathbf{g}^\dagger$-vectors.
- b) Under the bijection $M \mapsto \text{ind}(M)$ of a), the cluster-tilting subcategories reachable from $T$ are mapped bijectively to the $\mathbf{g}^\dagger$-clusters.

**Proof.** a) By assumption, there is a vertex $t$ of $\mathbb{T}$ such that $M$ belongs to $\mathcal{T}_t$. Now we use theorem 3 and induction on the length of the path joining $t_0$ to $t$ in the tree $\mathbb{T}$ to conclude that
\[ \text{ind}(M) = \mathbf{g}^\dagger_{\mathcal{M}_t,t}, \text{ where } M = \psi_{t,t_0}(M'). \]
This formula shows that the map $M \mapsto \text{ind}(M)$ is a well-defined surjection onto the set of $\mathbf{g}^\dagger$-vectors. By theorem 2.3, the map $M \mapsto \text{ind}(M)$ is also injective. b) By assumption, a reachable cluster-tilting subcategory $T'$ is of the form $\mathcal{T}_t$ for some vertex $t$ of the tree $\mathbb{T}$. Thus its image is the $\mathbf{g}^\dagger$-cluster associated with $t$. This shows that the map is well-defined and surjective. It follows from a) that it is also injective. \(\square\)
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