Integrable systems with quadratic nonlinearity in Fourier space

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Abstract

The Lax pair representation in Fourier space is used to obtain a list of integrable scalar evolutionary equations with quadratic nonlinearity. The famous systems of this type such as KdV, intermediate long-wave equation (ILW), Camassa-Holm and A. Degasperis systems are represented in this list. Some new systems are obtained as well. The generalizations on two-dimensional and discrete systems are discussed.

An important class of the integrable evolution systems is a class of systems with quadratic nonlinearity. Traditionally, the evolution integrable systems are classified by power of dispersion law. We don’t assume a concrete form of dispersion law beforehand. Moreover, we consider dispersionless systems also.

We shall consider the systems of following type

\[
\frac{d}{dt} u_q(t) = \omega(q) u_q(t) + \sum_{p_1 + p_2 = q} w(p_1, p_2) u_{p_1}(t) u_{p_2}(t)
\]

(1)

The summation in (1) has a symbolic meaning. One can consider a finite set of values of “momentum” \(p\) - this case corresponds to the finite-dimensional systems. The other case \(p \in \mathbb{R}\) corresponds to the one-dimensional evolution equations.

In fact, one can apply the inverse Fourier transform (we will omit an imaginary unit) to equation (1) to obtain the systems in a coordinate space.

For example, one can substitute \(\omega(k) = k^3, w(p, q) = p + q\) to the equation (1) and apply the inverse Fourier transform to obtain the KdV equation. Another example: the intermediate long-wave water (ILW) equation corresponds to a choice \(\omega(k) = k^3 \frac{1 + e^{hk}}{1 - e^{hk}}, w(p, q) = p + q\) in equation (1).

One of the most effective tools used in the theory of integrable systems is Lax pair representation. One can try to obtain Lax representation of (1) in the following form

\[
L_t = [A, L], \quad L = \alpha + \sum_p U^p u_p, \quad A = \beta + \sum_p V^p u_p, \quad [\alpha, \beta] = 0
\]

(2)

where \(\alpha, \beta, U^p, V^p\) - some constant operators. It is easy to see that (2) is Lax pair for (1) if and only if

\[
[V^p, U^q] + [V^q, U^p] = 2w(p, q) U^{p+q}, \quad [\beta, U^p] + [V^p, \alpha] = \omega(p) U^p
\]

(3)

One can use a following matrix representation for the operators in Fourier space

\[
\alpha_{k,k'} = \alpha(k) \delta_{k,k'}, \quad \beta_{k,k'} = \beta(k) \delta_{k,k'}, \quad U^p_{k,k'} = l(k, p) \delta_{k,k'+p}, \quad V^p_{k,k'} = a(k, p) \delta_{k,k'+p}
\]

(4)

One can substitute (4) to (3) to obtain

\[
a(k, q) l(k - q, p) + a(k, p) l(k - p, q) - a(k - q, p) l(k, q) - a(k - p, q) l(k, p) = 2w(p, q) l(k, p + q)
\]

(5)

and

\[
[\alpha(k) - \alpha(k - p)] a(k, p) = [\beta(k) - \beta(k - p) - \omega(p)] l(k, p)
\]

(6)

It is easy to obtain a particular solution of (4),

\[
\alpha(k) = l(k, 0), \quad \beta(k) = a(k, 0), \quad \omega(k) = 2w(k, 0).
\]

(7)

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This solution is compatible with (3), but is trivial because it corresponds to a shift \( u_p(t) \rightarrow u_p(t) + \delta_{p,a} \) in (1).

We shall consider \( l(k, p) = \frac{1}{p(k)} \) case now, then equation (3) transforms to

\[
a(k, q) \frac{\rho(k)}{\rho(k - q)} + a(k, p) \frac{\rho(k)}{\rho(k - p)} - a(k - q, p) - a(k - p, q) = 2w(p, q)
\]  (8)

It is natural to differentiate (3) on \( p \) and on \( q \) to obtain

\[
a(k, p) = k\rho(A) + b(p) + c(k), \quad 2w(p, q) = p\rho(A) + q\rho(A) + B(p) + B(q), \quad b(-p) = b(p) + p\rho(A) + B(p)
\]  (9)

where \( A(p) = A(-p), \quad B(p) = -B(-p), \quad \omega(p) = -\omega(-p) \).

One can substitute (9) to (8) to obtain a functional equation

\[
U(x, y) = \rho(x)[A(x - y) + b(x - y) + c(x)]
\]  (10)

that defines a whole class of integrable dispersionless \((\omega_p = 0)\) systems with quadratic nonlinearity because one can choose \( \alpha = 0, \beta = 0 \) in this case.

One can rewrite (3) to obtain another functional equation

\[
\beta(x) - \beta(y) - \omega(x - y) = (\alpha(x) - \alpha(y))U(x, y)
\]  (11)

where \( U(x, y) \) is some solution of (10).

The general solution of (11) defines a class of integrable systems (1) with nonzero dispersion \((\omega_p \neq 0)\). Evidently, this solution doesn’t exist for all \( U(x, y) \) from (10).

1 Classification of dispersionless systems

It is possible to obtain the general solution of (10).

The first step is to apply operators \( D = \partial_x + \partial_y, \quad D^2 D^3 \) to equation (10) to obtain three more equations - it is important that \( Df(x - y) = 0 \). Therefore, equation (10) transforms to

\[
W_D(x\rho(x) - y\rho(y), x, \rho(y), \rho(x)c(x) - \rho(y)c(y)) = 0
\]  (12)

where \( D\)-Wronskian is defined by formula

\[
W_{D}(f_1(x, y), f_2(x, y), ..., f_N(x, y)) = \det(W_{ij}^k), \quad W_{ij}^k = D^j-1 f_i(x, y), \quad i, j, k = 1, 2, ..., N
\]  (13)

The second step is to apply \( y \to x \) limit to equation (12) to obtain a necessary condition

\[
c(x) = \mu_1 \frac{R(x)}{\rho(x)} + \mu_2 x + \mu_3, \quad \lambda_1 \rho''(x) + \lambda_2 x\rho'(x) + \lambda_3 \rho'(x) + \lambda_4 \rho(x) = 0, \quad R'(x) = \rho(x)
\]  (14)

There are two cases - case \( \lambda_2 = 0 \) is “trivial” - any solution of (14) corresponds to a solution of (12):

\[
\rho(x) = c_1 e^{R_{1}x} + c_2 e^{R_{2}x} \quad \text{or} \quad \rho(x) = (x - a)e^{R_{1}x} \quad \text{or} \quad \rho(x) = e^{R_{1}x} \quad \text{or} \quad \rho(x) = x \quad \text{or} \quad \rho(x) = 1.
\]

In \( \lambda_2 \neq 0 \) case one can substitute \( x\rho(x) = -\frac{1}{2}(\lambda_1 \rho'(x) + (\lambda_3 - \lambda_2)\rho(x) + \lambda_4 R(x)) \) to (12), to take the \( y \to x \) limit as well to obtain the second necessary condition

\[
\nu_4 \rho''(x) + \nu_2 \rho'''(x) + \nu_1 \rho''(x) + \nu_0 = 0
\]  (15)

In this case we have only three solutions different from \( \lambda_2 = 0 \) case: \( \rho(x) = (x - a)(x - a - b)(x - a + b) \) or \( \rho(x) = (x - a)(x - b) \) - Degasperis [1],[2] and Camassa-Holm [3] systems respectively and \( \rho(x) = 1/x \) case.

So we have obtain following classification of dispersionless systems with quadratic nonlinearity:

1. \( \rho(k) = 1 \)

   In this class one can found the hydrodynamic type systems

\[
u_t = m u_x, \quad m_p = A(p)u_p, \quad L_{kk'} = u_{k - k'}, \quad A_{kk'} = \frac{1}{2}(k + k')A(k - k')u_{k - k'}.
\]  (16)

2. \( \rho(k) \neq 1, \quad A(k) = 0 \)

   - The system from this class have a general form in a \( \rho(x) = e^{bx} \) case

\[
u_t = m u, \quad m_p = B(p)u_p, \quad L_{kk'} = e^{-bk}u_{k - k'}, \quad A_{kk'} = b(k - k')u_{k - k'}, \quad B(p) = b(p)(e^{bp} - 1).
\]  (17)

It is interesting that a choice \( b(x) = \frac{1}{1 + e^{ax}} \) corresponds to Hilbert-Hopf equation \( m_t = L(\partial_k)m^2, \quad L(k) = \frac{th k^2}{2} \) where \( u_k = m_k(e^{hk} + 1) \).
• The case $\rho(x) = (x - a) e^{hx}$ corresponds to

$$u_t = m \dot{u}, \quad m_p = B(p) u_p, \quad L_{kk'} = (k - a)^{-1} e^{-hx} u_{k-k'}, \quad A_{kk'} = \left( \frac{1}{k - a} + \frac{e^{h(k'-k)}}{k - k'} \right) u_{k-k'}.$$  \hspace{1cm} (18)

where $B(p) = \frac{1}{2p}(2 - e^{hp} - e^{-hp})$

• Two-exponent case $\rho(x) = e^{hix} + e^{h2x}$ corresponds to

$$u_t = m \dot{u}, \quad m_p = B(p) u_p, \quad B(p) = b(-p) - b(p), \quad b(x) = \frac{b_1(1 - e^{hix}) - b_2(1 - e^{h2x})}{e^{hix} - e^{h2x}}$$

where $L_{kk'} = (a_1 e^{hix} + a_2 e^{h2x})^{-1} u_{k-k'}, \quad A_{kk'} = (b(k - k') + c(k)) u_{k-k'}, \quad c(k) = \frac{b_1 a_1 e^{hix} + b_2 a_2 e^{h2x}}{a_1 e^{hix} + a_2 e^{h2x}}$.

Note that the systems (18) and (19) have a second Lax pair representation (17).

3. $\rho(k) \neq 1$, $A(k) \neq 0$

• First example in this class is a system with $\rho(x) = x^2 - 1/4$ : Lax pair has a form

$$L_{kk'} = \frac{1}{k^2 - 1/4} u_{k-k'}, \quad A_{kk'} = \frac{1}{2} \left( \frac{k - 3k'}{k^2 - 1} + \frac{k}{k^2 - 1/4} \right) u_{k-k'},$$

that corresponds to Camassa-Holm [3] equation

$$u_t = 2f_x u + f u_x, \quad u = \frac{1}{2} (f_{xx} - f)$$  \hspace{1cm} (20)

• Second example is a system with $\rho(x) = x(x^2 - 1)$ : Lax pair in this case has a form

$$L_{kk'} = \frac{1}{k(k^2 - 1)} u_{k-k'}, \quad A_{kk'} = \left( \frac{k - 2k'}{k^2 - 1} + \frac{k}{k^2 - 1} \right) u_{k-k'},$$

that corresponds to A. Degasperis system [1], [2]

$$u_t = 3f_x u + f u_x, \quad u = \frac{1}{2} (f_{xx} - f)$$  \hspace{1cm} (21)

The other cases are

• Systems with $\rho(x) = x$ and $A_{kk'} = k A(k - k')$ have a form

$$u_t = \partial_x (mu), \quad m_p = A(p) u_p$$  \hspace{1cm} (22)

• Systems with $\rho(x) = \frac{1}{x}$ and $A_{kk'} = \frac{1}{2} A(k - k')$ have a form

$$u_t = m u_x - m u, \quad m_p = A(p) u_p$$  \hspace{1cm} (23)

2 The systems with dispersion

One can try to solve equation (11) starting from the general solution of (10) to obtain a list of systems with dispersion. One can verify that there are no solutions of (10) for systems (20) and (21). So the statement is: It is impossible to obtain a generalization of Camassa-Holm [3] and A. Degasperis [2] on the systems with dispersion. This statement is valid for the systems (18) and (19) as well.

The rest of possibilities are systems with $\rho(x) = 1$, $\rho(x) = e^{hx}$ or $\rho(x) = x$.

1. In a case $\rho(x) = 1$ it is easy to obtain $U(x, y) = \frac{1}{2} (x + y) A(x - y) + d(x - y)$, where $A(x)$, $d(x)$ are even functions. One can substitute given $U(x, y)$ to (11), apply the operator $D$ to equation to obtain a necessary condition

$$W_D(\alpha(x) - \alpha(y), (x + y)(\alpha(x) - \alpha(y)), \beta(x) - \beta(y), 1) = 0$$  \hspace{1cm} (24)

The method of solution of equation (24) is similar to that of solution of (12): Limit $x \to y$ gives $\beta'(x) = c \alpha'(x)$. Substitution $\beta(x)$ to (24) and taking $x \to y$ limit give a necessary condition $\alpha''(x) + c_1 x \alpha'(x) + c_2 \alpha'(x) + c_3 = 0$. The only possible choice to fulfill (24) is $\alpha(x) = x^2$, $\beta(x) = 4x^3$ (one can choose $c = 6$).

It is easy to obtain $\omega(x) = x^3$ in this case, that corresponds to KdV equation $u_t = u_{xxx} + 6uu_x$ with

$$L_{kk'} = k^2 \delta_{kk'} + u_{k-k'}, \quad A_{kk'} = 4k^3 \delta_{kk'} + 3(k + k') u_{k-k'}.$$  \hspace{1cm} (25)
2. In \( \rho(x) = e^{hx} \) case one have \( U(x,y) = e^{\frac{1}{2}(x+y)}d(x-y) \), where \( d(x) \) is even function.

The necessary condition in this case

\[
W_D((\alpha(x) - \alpha(y))e^{\frac{1}{2}(x+y)}, \beta(x) - \beta(y), 1) = 0
\]  

(26)

One have \( \beta'(x) = c_1\alpha'(x)e^{hx} + c_2 \). The only non-trivial solution of (26) \( \alpha(x) = x e^{-hx}, \beta(x) = x^2 \) gives intermediate long-wave (ILW) equation

\[
u_t = \Gamma(u_{xx}) - 2uu_x, \quad \Gamma(p) = \coth\left(\frac{ph}{2}\right)
\]

(27)

with Lax pair

\[
L_{kk'} = k e^{-hk}\delta_{kk'} + e^{-hk}u_{k-k'}, \quad A_{kk'} = k^2\delta_{kk'} + 2\frac{k - k'}{1 - e^{h(k-k')}}u_{k-k'}
\]

(28)

3. In \( \rho(x) = x \) case we just obtain alternative Lax pair for KdV equation:

\[
L = \delta_{kk'}\left(\frac{k^2}{2} + \frac{\mu}{k} + \frac{1}{k}u_{k-k'}, \quad A = \delta_{kk'}\left(\frac{k^3}{6} - \mu k\right) + k' u_{k-k'}, \quad L_t = [A, L] \iff u_{p,t} = \frac{p^3}{6}u_p + p \sum_q u_q u_{p-q}. \]

(29)

### 3 Generalizations

It is possible to extend our approach to high-dimensional and multi-component systems. Consider two-dimensional scalar systems with quadratic nonlinearity. Unlike an one-dimensional case, the ”wave” variables \( k, p \) and \( q \) in the functional equations (3) and (1) are vectors with two components. To obtain the functional equations in this case one can use following trick: One can write \( \vec{k} = (k_x, k_y) = (k, \lambda) \), where \( k \) is an argument of the functional equations and \( \lambda \) is parameter. For example, one can rewrite equation (10) as follows

\[
\rho(x,\lambda)[xA(x-y,\lambda-\mu) + \lambda A_2(x-y,\lambda-\mu) + b(x-y,\lambda-\mu) + c(x,\lambda)] = \\ 
\rho(y,\mu)[yA(y-x,\mu-\lambda) + \mu A_2(y-x,\mu-\lambda) + b(y-x,\mu-\lambda) + c(y,\mu)]
\]

(30)

The method of solution of (31) is analogous to one-dimensional ones. One can use notations \( \rho(x) = \rho(x,\lambda), c(x) = c(x,\lambda), \tilde{\rho}(y) = \rho(y,\mu), \tilde{c}(y) = c(y,\mu) \) to obtain

\[
W_D(x\rho(x) - y\tilde{\rho}(y), \rho(x), \tilde{\rho}(y), \rho(x)c(x) - \tilde{\rho}(y)\tilde{c}(y)) = 0
\]

(31)

in this case (compare with (12)).

Next step is to take \( \mu \to \lambda \) limit in equation (31) to obtain equation (12) exactly! Note that one can’t apply this limit to functional equation (11) because possible divergence in the functions \( A, A_2, b \).

Then the only possibility to obtain two-dimensional systems with quadratic nonlinearity is to start from one-dimensional systems and to try to extend some solutions of (11) to (31). We have no full classification in two-dimensional case at the moment, but a couple of examples are in agree with our approach:

First example is a famous Kadomtsev-Petviashvili (KP) equation

\[
u_t = u_{xxx} + 3\theta^{-1}(u_y) + 6uu_x, \quad \theta^{-1} = \partial_y\partial_x^{-1}
\]

has a Lax pair in Fourier space

\[
l_{kk'} = (k_y + k_x^2)\delta_{kk'} + u_{k-k'}, \quad a_{kk'} = 4k_x^3\delta_{kk'} + 3(k_x + k_y')\delta_{kk'} + \frac{k_y - k_y'}{k_x - k_x'}u_{k-k'}
\]

(32)

that is an agreement with the general form of Lax pair (1) in our method.

Second example is Veselov-Novikov (VN) equation

\[
u_t = u_{xxx} + u_{yyy} + u\theta(u_x) + u_x\theta(u) + u\theta^{-1}(u_y) + u_y\theta^{-1}(u), \quad \theta = \partial_u\partial_y^{-1}
\]

(33)

One can consider VN equation (33) as a sum of two symmetries \( u_t = u_{t+} + u_{t-} \) where \( u_{t+} = u_{xxx} + u\theta(u_x) + u_x\theta(u) \) and \( u_{t-} = u_{yyy} + u\theta^{-1}(u_y) + u_y\theta^{-1}(u) \).

Lax pair in Fourier space for "+" symmetry has a form

\[
L_{kk'} = 3k_y\delta_{kk'} + \frac{1}{k_x}u_{k-k'}, \quad A_{kk'} = k_x^3\delta_{kk'} + k_x^2\delta_{kk'} + \frac{1}{k_y}u_{k-k'}
\]

(34)
One can introduce new operators:

\[ L_1 = L, \quad L_2 = L[x \leftrightarrow y], \quad A_1 = A, \quad A_2 = A[x \leftrightarrow y], \quad L_{i,t_i} = [L_i, A_1], \quad L = L_1 + L_2, \quad A = A_1 + A_2, \]  

(35)

It is easy to obtain the Manakov’s triada representation for Veselov-Novikov equation \([5]\):

\[ L_t = [A, L] + BL, \quad B = A^{(1)}_1 + A^{(2)}_2 - A_1 - A_2 \]  

(36)

where \(A^{(1)}_{1,kk'} = k_x A_{1,kk'} \frac{1}{k_{k'}}\), \(A^{(2)}_{kk'} = k_y A_{2,kk'} \frac{1}{k_{k'}} \Rightarrow B_{k} = \frac{(k_x-k')^2}{k_x-k_y} + \frac{(k_x-k')^2}{k_x-k_y} \).

A case of multi-component systems corresponds to a choice \(k \rightarrow (k, n)\) where \(k \in \mathbb{R}\) and \(n = 1, 2, \ldots, N\). A classification of those systems is a subject of forthcoming papers, but in a case of two-component systems we have two interesting classes:

First example is so called Toda-type systems

\[ z_{q,t} = -\omega(q)z_q + \gamma(q)p_q + \sum_s \Gamma^{(1)}(s, q - s)z_s z_{q-s}, \quad p_{q,t} = \omega(q)p_q + \sum_s \Gamma^{(2)}(s, q - s)z_s p_{q-s} \]

with Lax pair

\[ L = \alpha + \sum_q U^q z_q + \sum_q W^q p_q \quad A = \beta + \sum_q V^q z_q, \quad [\alpha, \beta] = 0 \]  

(37)

Note that Toda system has Lax pair representation in coordinate space

\[ L_t = [A, L], \quad L = \partial + z - \lambda + p\partial^{-1}, \quad A = \partial^2 + 2\partial z \iff z_t = \partial[-z' + z^2 + 2p], \quad p_t = \partial[p' + 2pz]. \]  

(38)

This dynamical system can be transformed to nonlinear Schrödinger equation \((p = \psi^*\psi, \quad z = -\partial_z \log(\psi))\) and Bäcklund transformation of this system have a form of Toda lattice (see \([4]\) for more details). It is easy to see that Fourier transform of \([5]\) has a form \([3]\).

The other example is a system with 3-wave interaction

\[ i a_{p,t} = \omega(p)a_p + \sum w^{(1)}(p, q)a_{p+q}a_{p-q}^* + \sum w^{(2)}(p, q)a_qa_{p-q} \]

\[ -i a_{p,t}^* = \omega(p)a_{p}^* + \sum w^{(1)}(p, q)a_{p+q}^*a_{p-q} + \sum w^{(2)}(p, q)a_{p}^*a_{p-q} \]  

(39)

The problem is to find all possible functions \(W^{(1)}\), \(W^{(2)}\) and \(\omega\) in case when system \([5]\) is integrable. This question was discussed in \([4]\) by the use of perturbative approach.

Lax pair in this case has a form

\[ L = \alpha + \sum_{p>0} U^p a_p + \sum_{p>0} (U^p)^* a_p^*, \quad A = \beta + \sum_{p>0} V^p a_p + \sum_{p>0} (V^p)^* a_p^*, \quad [\alpha, \beta] = 0. \]  

(40)

One can derive a system of functional equations as in scalar systems case, but a general solution of these equations is unknown as yet. It is interesting that system \([3]\) is equivalent to scalar system \([1]\) in a "real" case:

\[ u_p = \theta(p)a_p + \theta(-p)a_{-p}^*, \quad u_p^* = u_{-p}, \quad W^{(1)}(p, q) = w(p, q), \quad W^{(2)}(p, q) = w(p, q). \]

In summary, we have use a Fourier representation of Lax pairs to classify scalar integrable systems with quadratic nonlinearity. The similar (but not the same) approach is so called symbolic method. This method was applied to theory of integrable systems by Gelfand and Dickii \([6]\) and was improved in the recent papers \([7]\), \([8]\). The common feature and main advantage of both methods is that all coefficients of equations in Fourier space are the functions (not operators!) of wave numbers \(k\). Because this fact, all necessary conditions can be formulated as the functional equations. The irony of it is that in some cases it is better to solve functional equations rather then differential ones.

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