REGULARITY OF LIPSCHITZ BOUNDARIES WITH PRESCRIBED SUB-FINSLER MEAN CURVATURE IN THE HEISENBERG GROUP $H^1$

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Abstract. For a strictly convex set $K \subset \mathbb{R}^2$ of class $C^2$ we consider its associated sub-Finsler $K$-perimeter $|\partial E|_K$ in $H^1$ and the prescribed mean curvature functional $|\partial E|_K - \int_E f$ associated to a function $f$. Given a critical set for this functional with Euclidean Lipschitz and intrinsic regular boundary, we prove that their characteristic curves are of class $C^2$ and that this regularity is optimal. The result holds in particular when the boundary of $E$ is of class $C^1$.

1. Introduction

The aim of this paper is to study the regularity of the characteristic curves of the boundary of a set with continuous prescribed mean curvature in the first Heisenberg group $H^1$ with a sub-Finsler structure. Such a structure is defined by means of an asymmetric left-invariant norm $\| \cdot \|_K$ in $H^1$ associated to a convex set $K \subset \mathbb{R}^2$ containing 0 in its interior, see [38]. We assume in this paper that $K$ has $C^2$ boundary with positive geodesic curvature.

Following De Giorgi [14], the authors of [38] defined a notion of sub-Finsler $K$-perimeter, see also [18]. Given a measurable set $E \subset H^1$ and an open subset $\Omega \subset H^1$, it is said that $E$ has locally finite $K$-perimeter in $\Omega$ if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup \left\{ \int_V \text{div}(U) \, dH^1 : U \in H^1_0(V), \|U\|_{K, \infty} \leq 1 \right\} < +\infty,$$

where $H^1_0(V)$ is the space of horizontal vector fields of class $C^1$ with compact support in $V$, and $\|U\|_{K, \infty} = \sup_{p \in V} \|U_p\|_K$. Both the divergence and the integral are computed with respect to a fixed left-invariant Riemannian metric $g$ on $H^1$.

When $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface the $K$-perimeter coincides with the area functional

$$A_K(S) = \int_S \|N_h\|_{K, \ast} \, dH^2,$$

where $H^2$ is the 2-dimensional Hausdorff measure associated to the left-invariant Riemannian metric $g$, $N$ is the outer unit normal to $S$, defined $H^2$-a.e on $S$, $N_h$ is

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the horizontal projection of $N$ to the horizontal distribution in $\mathbb{H}^1$ and $\|\cdot\|_{K,*}$ is the dual norm of $\|\cdot\|_K$.

We say that a set $E$ with Euclidean Lipschitz boundary has prescribed $K$-mean curvature $f \in C^0(\Omega)$ if, for any bounded open subset $V \subset \Omega$, $E$ is a critical point of the functional

$$A_K(S \cap B) - \int_{E \cap B} f \, d\mathbb{H}^1.$$ 

This notion extends the classical one in Euclidean space and the one introduced in [21] for the sub-Riemannian area. We refer the reader to the introduction of [21] for a brief historical account and references.

We say that a set $E$ has constant prescribed $K$-mean curvature if there exists $\lambda \in \mathbb{R}$ such that $E$ has prescribed $K$-mean curvature $\lambda$. In Proposition 2.2 we consider a set $E$ with Euclidean Lipschitz boundary and positive $K$-perimeter. We show that if $E$ is a critical point of the $K$-perimeter for variations preserving the volume up to first order then $E$ has constant prescribed $K$-mean curvature on any open set $\Omega$ avoiding the singular set $S_0$ and where $|\partial E|_K(\Omega) > 0$. This result can be applied to isoperimetric regions in $\mathbb{H}^1$ with Euclidean Lipschitz boundary.

The main result of this paper is Theorem 3.1, where we prove that the boundary $S$ of a set $E$ with prescribed continuous $K$-mean curvature is foliated by horizontal characteristic curves of class $C^2$ in its regular part. The minimal assumptions we require for the boundary $S$ of $E$ are to be Euclidean Lipschitz and $\mathbb{H}$-regular. The result holds in particular when the boundary of $E$ is of class $C^1$. As we point out in Remark 3.5, $C^2$ regularity is optimal since the Pansu-Wulff shapes obtained in [38] have prescribed constant mean curvature and their boundaries are foliated by characteristic curves with the same regularity as that of $\partial K$, that may be just $C^2$. In the proof of the Theorem 3.1 we exploit the first variation formula of the area following the arguments developed in [20, 21] and make use of the biLipschitz homeomorphism considered in [35]. One of the main differences in our setting is that the area functional strongly depends on the inverse $\pi_K$ of the Gauss map of $\partial K$. Therefore the first variation of the area depends on the derivative of the map that describes the boundary $\partial K$. In order to use the bootstrap regularity argument in [20, 21] we need to invert this map on the boundary $\partial K$, that is possible since the geodesic curvature of $\partial K$ is strictly positive, see Lemma 3.2. Moreover, the $C^2$ regularity of the characteristic curves implies that, on characteristic curves of a boundary with prescribed continuous $K$-mean curvature $f$, the ordinary differential equation

$$\langle D_Z \pi_K(\nu_h), Z \rangle = f,$$

is satisfied. In this equation $\nu_h = N_h/|N_h|$ is the classical sub-Riemannian horizontal unit normal, $Z$ is the unit characteristic vector field tangent to the characteristic curves and $D$ the Levi-Civita connection associated to the left-invariant Riemannian metric $g$ on $\mathbb{H}^1$. Equation (*) was proved to hold for $C^2$ surfaces in [38]. For regularity assumptions below $\mathbb{H}$-regular and Euclidean Lipschitz, equation (*) holds in a suitable weak sense, a result proved in [1] for the sub-Riemannian area, when $K$ coincides with the unit disk centered at 0.

Moreover, in Proposition 4.2 we stress that equation (*) is equivalent to

$$H_D = \kappa(\pi_K(\nu_h)) f,$$

where $H_D$ is the horizontal mean curvature of $\partial K$.
where $H_D = \langle D\nu, Z \rangle$ is the classical sub-Riemannian mean curvature introduced in [1] and $\kappa$ is the strictly positive Euclidean curvature of the boundary $\partial K$. A key ingredient to obtain equation (**) is Lemma 4.3, that exploits the ideas of Lemma 3.2 in an intrinsic setting.

This manuscript is a natural continuation of the many recent papers concerning sub-Riemannian area minimizers [22, 11, 8, 7, 5, 13, 2, 1, 26, 27, 28, 41, 29, 17, 4, 10, 30, 32, 24, 23, 6]. The sub-Riemannian perimeter functional is a particular case of the sub-Finsler functionals considered in this paper where the convex set is the unit disk $D$ centered at 0. In the pioneering paper [22] N. Garofalo and D.M. Nhieu showed the existence of sets of minimal perimeter in Carnot-Caratheodory spaces satisfying the doubling property and a Poincaré inequality. In [31] Leonardi and Rigot showed the existence of isoperimetric sets in Carnot groups. However the optimal regularity of the critical points of these variational problems involving the sub-Riemannian area is not completely understood. Indeed, even in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ there are several examples of non-smooth area minimizers: S. D. Pauls in [37] exhibited a solution of low regularity for the Plateau problem with smooth boundary datum; on the other hand in [8, 39, 34] the authors provided solutions of Bernstein’s problem in $\mathbb{H}^1$ that are only Euclidean Lipschitz.

In [36] P. Pansu conjectured that the boundaries of isoperimetric sets in $\mathbb{H}^1$ are given by the surfaces now called Pansu’s spheres, union of all sub-Riemannian geodesics of a fixed curvature joining two point in the same vertical line. This conjecture has been solved only assuming a priori some regularity of the minimizers of the area with constant prescribed mean curvature. In [41] the authors solved the conjecture assuming that the minimizers of the area are of class $C^2$, using the description of the singular set, the characterization of area-stationary surfaces, and the ruling property of constant mean curvature surfaces developed in [7]. Hence the a priori regularity hypothesis are central to study the sub-Riemannian isoperimetric problem. Motivated by this issue, it was shown in [9] that a $C^1$ boundary of a set with continuous prescribed mean curvature is foliated by $C^2$ characteristic curves. Regularity results for Lipschitz viscosity solutions of the minimal surface equation were obtained in [4]. Furthermore, in [21] the authors generalized the previous result when the boundary $S$ is immersed in a three-dimensional contact sub-Riemannian manifold. Finally M. Galli in [20] improved the result in [21] only assuming that the boundary $S$ is Euclidean Lipschitz and $\mathbb{H}$-regular in the sense of [19]. The Bernstein problem in $\mathbb{H}^1$ with Euclidean Lipschitz regularity was treated by S. Nicolussi and F. Serra-Cassano [35]. Partial solutions of the sub-Riemannian isoperimetric problem have been obtained assuming Euclidean convexity [33], or symmetry properties [12, 40, 32, 17]. An analogous sub-Finsler isoperimetric problem might be considered. Candidate solutions would be the Pansu-Wulff shapes considered in [38]. See [38, 18] for partial results in the sub-Finsler isoperimetric problem and [42] for earlier work.

We have organized this paper into several sections. In Section 2 we introduce sub-Finsler norms in the first Heisenberg group $\mathbb{H}^1$ and their associated sub-Finsler perimeter, the notion of $\mathbb{H}$-regular surfaces, intrinsic Euclidean Lipschitz graphs and the definition of sets with prescribed mean curvature. Moreover, at the end of this section we prove Proposition 2.2. Section 3 is dedicated to the proof of the main Theorem 3.1, that ensures that the characteristic curves are $C^2$. Finally in Section 4 we deal with the $K$-mean curvature equation, see Proposition 4.1 and Proposition 4.2.
2. Preliminaries

2.1. The Heisenberg group. We denote by \( \mathbb{H}^1 \) the first Heisenberg group, defined as the 3-dimensional Euclidean space \( \mathbb{R}^3 \) endowed with a product \( \ast \) defined by
\[
(x, y, t) \ast (\bar{x}, \bar{y}, \bar{t}) = (x + \bar{x}, y + \bar{y}, t + \bar{t} + xy - \bar{y}).
\]

A basis of left invariant vector fields is given by
\[
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.
\]

For \( p \in \mathbb{H}^1 \), the left translation by \( p \) is the diffeomorphism \( L_p(q) = p \ast q \). The horizontal distribution \( \mathcal{H} \) is the planar distribution generated by \( X \) and \( Y \), that coincides with the kernel of the (contact) one-form \( \omega = dt - ydx + xdy \).

We shall consider on \( \mathbb{H}^1 \) the left invariant Riemannian metric \( g = \langle \cdot, \cdot \rangle \), so that \( \{X, Y, T\} \) is an orthonormal basis at every point, and let \( D \) be the Levi-Civita connection associated to the Riemannian metric \( g \). The following relations can be easily computed
\[
\begin{align*}
D_X X &= 0, & D_Y Y &= 0, & D_T T &= 0 \\
D_X Y &= -T, & D_Y T &= Y, & D_Y T &= -X \\
D_Y X &= T, & D_T X &= Y, & D_T Y &= -X.
\end{align*}
\]

Setting \( J(U) = D_T T \) for any vector field \( U \) in \( \mathbb{H}^1 \) we get \( J(X) = Y \), \( J(Y) = -X \) and \( J(T) = 0 \). Therefore \( -J^2 \) coincides with the identity when restricted to the horizontal distribution. The Riemannian volume of a set \( E \) is, up to a constant, the Haar measure of the group and is denoted by \( |E| \). The integral of a function \( f \) with respect to the Riemannian measure by \( \int f d\mathbb{H}^1 \).

2.2. The pseudo-hermitian connection. The pseudo-hermitian connection \( \nabla \) is the only affine connection satisfying the following properties:

1. \( \nabla \) is a metric connection,
2. \( \text{Tor}(U, V) = 2(J(U), V)T \) for all vector fields \( U, V \).

In the previous line the torsion tensor \( \text{Tor}(U, V) \) is given by \( \nabla_U V - \nabla_V U - [U, V] \).

From the above definition and the Koszul formula it follows easily that \( \nabla X = \nabla Y = 0 \) and \( \nabla J = 0 \). For a general discussion about the pseudo-hermitian connection see for instance [15, § 1.2]. Given a curve \( \gamma : I \to \mathbb{H}^1 \) we denote by \( \nabla / ds \) the covariant derivatives induced by the pseudo-hermitian connection along \( \gamma \).

2.3. Sub-Finsler norms. Given a convex set \( K \subset \mathbb{R}^2 \) with \( 0 \in \text{int}(K) \) an associated asymmetric norm \( \| \cdot \| \) in \( \mathbb{R}^2 \), we define on \( \mathbb{H}^1 \) a left-invariant norm \( \| \cdot \|_K \) on the horizontal distribution by means of the equality
\[
(\|fX + gY\|_K)(p) = \|(f(p), g(p))\|,
\]
for any \( p \in \mathbb{H}^1 \). The dual norm is denoted by \( \| \cdot \|_{K^{**}} \).

If the boundary of \( K \) is of class \( C^\ell, \ell \geq 2 \), and the geodesic curvature of \( \partial K \) is strictly positive, we say that \( K \) is of class \( C^\ell_+ \). When \( K \) is of class \( C^2_+ \), the outer Gauss map \( N_K \) is a diffeomorphism from \( \partial K \) to \( S^1 \) and the map
\[
\pi_K fX + gY = N_K^{-1} \left( \frac{(f,g)}{\sqrt{f^2 + g^2}} \right),
\]
defined for non-vanishing horizontal vector fields \( U = fX + gY \), satisfies
\[
||U||_{K,*} = \langle U, \pi_K(U) \rangle.
\]

See § 2.3 in [38].

2.4. Sub-Finsler perimeter. Here we summarize some of the results contained in subsection 2.4 in [38].

Given a convex set \( K \subset \mathbb{R}^2 \) with \( 0 \in \text{int}(K) \), the norm \( || \cdot ||_K \) defines a perimeter functional: given a measurable set \( E \subset \mathbb{H}^1 \) and an open subset \( \Omega \subset \mathbb{H}^1 \), we say that \( E \) has locally finite \( K \)-perimeter in \( \Omega \) if for any relatively compact open set \( V \subset \Omega \) we have
\[
|\partial E|_K(V) = \sup \left\{ \int_E \text{div}(U) \, d\mathbb{H}^1 : U \in \mathcal{H}^1_0(V), ||U||_{K,\infty} \leq 1 \right\} < +\infty,
\]
where \( \mathcal{H}^1_0(V) \) is the space of horizontal vector fields of class \( C^1 \) with compact support in \( V \), and \( ||U||_{K,\infty} = \sup_{p \in V} ||U_p||_K \). The integral is computed with respect to the Riemannian measure \( d\mathbb{H}^1 \) of the left-invariant Riemannian metric \( g \). When \( K = D \), the closed unit disk centered at the origin of \( \mathbb{R}^2 \), the \( K \)-perimeter coincides with classical sub-Riemannian perimeter.

If \( K, K' \) are bounded convex bodies containing 0 in its interior, there exist constants \( \alpha, \beta > 0 \) such that
\[
\alpha ||x||_{K'} \leq ||x||_K \leq \beta ||x||_{K'}, \quad \text{for all } x \in \mathbb{R}^2,
\]
and it is not difficult to prove that
\[
\frac{1}{\beta} |\partial E|_{K'}(V) \leq |\partial E|_K(V) \leq \frac{1}{\alpha} |\partial E|_{K'}(V).
\]
As a consequence, \( E \) has locally finite \( K \)-perimeter if and only if it has locally finite \( K' \)-perimeter. In particular, any set with locally finite \( K \)-perimeter has locally finite sub-Riemannian perimeter.

Riesz Representation Theorem implies the existence of a \( |\partial E|_K \) measurable vector field \( \nu_K \) so that for any horizontal vector field \( U \) with compact support of class \( C^1 \) we have
\[
\int_{\Omega} \text{div}(U) \, d\mathbb{H}^1 = \int_{\Omega} \langle U, \nu_K \rangle \, d|\partial E|_K.
\]
In addition, \( \nu_K \) satisfies \( |\partial E|_K \)-a.e. the equality \( ||\nu_K||_{K,*} = 1 \), where \( || \cdot ||_{K,*} \) is the dual norm of \( || \cdot ||_K \).

Given two convex sets \( K, K' \subset \mathbb{R}^2 \) containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure \( |\partial E|_K \) and the vector field \( \nu_K \)
\[
|\partial E|_{K} = ||\nu_{K'}||_{K,*} |\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}.
\]
Indeed, for the closed unit disk \( D \subset \mathbb{R}^2 \) centered at 0 we know that in the Euclidean Lipschitz case \( \nu_D = \nu_h \) and \( |N_h| = ||N_h||_{D,*} \) where \( N \) is the outer unit normal. Hence we have
\[
|\partial E|_{K} = ||\nu_h||_{K,*} d|\partial E|_{D}, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}.
\]
Here \( |\partial E|_D \) is the standard sub-Riemannian measure. Moreover, \( \nu_h = N_h/||N_h| \) and \( |N_h\|^{-1} d|\partial E|_{D} = dS \), where \( dS \) is the standard Riemannian measure on \( S \). Hence
we get, for a set $E$ with Euclidean Lipschitz boundary $S$

$$\partial E|_K(\Omega) = \int_{S \cap \Omega} ||N_h||_{K,*} dS,$$

where $dS$ is the Riemannian measure on $S$, obtained from the area formula using a local Lipschitz parameterization of $S$, see Proposition 2.14 in [19]. It coincides with the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by $g$. We stress that here $N$ is the outer unit normal. This choice is important because of the lack of symmetry of $|| \cdot ||_K$ and $|| \cdot ||_{K,*}$.

2.5. Immersed surfaces in $\mathbb{H}^1$. Following [1, 19] we provide the following definition.

**Definition 2.1 (H-regular surfaces).** A real measurable function $f$ defined on an open set $\Omega \subset \mathbb{H}^1$ is of class $C^1_{\mathbb{H}}(\Omega)$ if the distributional derivative $\nabla_{\mathbb{H}} f = (Xf, Yf)$ is represented by a continuous function.

We say that $S \subset \mathbb{H}^1$ is an $\mathbb{H}$-regular surface if for each $p \in \mathbb{H}^1$ there exist a neighborhood $U$ and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{ f = 0 \}$. Then the continuous horizontal unit normal is given by

$$\nu_h = \frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}.$$

Given an oriented Euclidean Lipschitz surface $S$ immersed in $\mathbb{H}^1$, its unit normal $N$ is defined $\mathcal{H}^2$-a.e. in $S$, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by $g$. In case $S$ is the boundary of a set $E \subset \mathbb{H}^1$, we always choose the outer unit normal. We say that a point $p$ belongs to the singular set $S_0$ of $S$ if $p \in S$ is a differentiable point and the tangent space $T_p S$ coincides with the horizontal distribution $\mathcal{H}_p$. Therefore the horizontal projection of the normal $N_h$ at singular points vanishes. In $S \setminus S_0$ the horizontal unit normal $\nu_h$ is defined $\mathcal{H}^2$-a.e. by

$$\nu_h = \frac{N_h}{|N_h|},$$

where $N_h$ is the horizontal projection of the normal $N$. The vector field $Z$ is defined $\mathcal{H}^2$-a.e. on $S \setminus S'_0$ by $Z = J(\nu_h)$, and it is tangent to $S$ and horizontal.

$\mathbb{H}$-regularity plays an important role in the regularity theory of sets of finite sub-Riemannian perimeter. In [19], B. Franchi, R. Serapioni and F. Serra-Cassano proved that the boundary of such a set is composed of $\mathbb{H}$-regular surfaces and a singular set of small measure.

2.6. Sets with prescribed mean curvature. Consider an open set $\Omega \subset M$, and an integrable function $f \in L^1_{loc}(\Omega)$. We say that a set of locally finite $K$-perimeter $E \subset \Omega$ has *prescribed $K$-mean curvature $f$ in $\Omega* if, for any bounded open set $B \subset \Omega$, $E$ is a critical point of the functional

$$|\partial E|_K(B) - \int_{E \cap B} f d\mathbb{H}^1.$$ 

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then $S$ has prescribed $K$-mean curvature $f$ if it is a critical point of the functional

$$A_K(S \cap B) - \int_{E \cap B} f d\mathbb{H}^1,$$
for any bounded open set $B \subset \Omega$.

If $E$ has boundary $S = \partial E \cap \Omega$ of class $C^2$, standard arguments imply that $E$ has prescribed $K$-mean curvature $f$ in $\Omega$ if and only if $H_K = f$, where $H_K$ is the $K$-mean curvature

$$H_K = \langle D_Z \pi_K(v_h), Z \rangle,$$

and $v_h$ is the outer horizontal unit normal, see [38]. Since by [38, Lemma 2.1] the Levi-Civita connection $D$ and the pseudo-hermitian connection $\nabla$ coincide for horizontal vector fields, we obtain that

$$H_K = \langle D_Z \pi_K(v_h), Z \rangle = \langle \nabla_Z \pi_K(v_h), Z \rangle.$$

It is important to remark that the mean curvature $H_K$ strongly depends on the choice of $v_h$. When $K$ is centrally symmetric, $\pi_K(-u) = -\pi_K(u)$ and so the mean curvature changes its sign when we take $-v_h$ instead of $v_h$. When $K$ is not centrally symmetric, there is no relation between the mean curvatures associated to $v_h$ and $-v_h$.

A set $E \subseteq \mathbb{H}^1$ with Euclidean Lipschitz boundary has locally finite $K$-perimeter: we know that it has locally bounded sub-Riemannian perimeter by Proposition 2.14 in [19] and we can apply the perimeter estimates in §2.3. Letting $\mathcal{H}^2$ be the Riemannian 2-dimensional Hausdorff measure, the Riemannian outer unit normal $N$ is defined $\mathcal{H}^2$-a.e. in $\partial E$, and it can be proven that

$$|\partial E|_K(V) = \int_{\partial E \cap V} \|N_h\|_{K,*} \, d\mathcal{H}^2.$$

We say that a set $E$ of locally finite $K$-perimeter in an open set $\Omega \subseteq \mathbb{H}^1$ has constant prescribed $K$-mean curvature if there exists $\lambda \in \mathbb{R}$ such that $E$ has prescribed $K$-mean curvature $\lambda$. This means that $E$ is a critical point of the functional $E \mapsto |\partial E|_K(B) - \lambda |E \cap B|$ for any bounded open set $B \subseteq \Omega$.

Our next result implies that Euclidean Lipschitz isoperimetric boundaries (for the $K$-perimeter) have constant prescribed $K$-mean curvature.

**Proposition 2.2.** Let $E \subseteq \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary. Assume that $E$ a critical point of the $K$-perimeter for variations preserving the volume of $E$ up to first order. Let $\Omega \subseteq \mathbb{H}^1$ be an open set so that $\Omega \cap S_0 = \emptyset$ and $|\partial E|_K(\Omega) > 0$. Then $E$ has constant prescribed $K$-mean curvature in $\Omega$.

**Proof.** Since the $K$–perimeter of $E$ in $\Omega$ is positive there exists a horizontal vector field $U_0$ with compact support in $\Omega$ so that $\int_E \text{div} \, U_0 \, d\mathbb{H}^1 > 0$. Let $\{\psi_s\}_{s \in \mathbb{R}}$ be the flow associated to $U_0$ and define

$$H_0 = \frac{d}{ds}\big|_{s=0} A_K(\psi_s(S)) \bigg/ \frac{d}{ds}\big|_{s=0} \psi_s(E).$$

Let $W$ any vector field with compact support in $\Omega$ and associated flow $\{\varphi_s\}_{s \in \mathbb{R}}$. Choose $\lambda \in \mathbb{R}$ so that $W - \lambda U_0$ satisfies

$$\frac{d}{ds}\big|_{s=0} |\varphi_s(E)| - \lambda \frac{d}{ds}\big|_{s=0} |\psi_s(E)| = 0.$$

This means that the flow of $W - \lambda U_0$ preserves the volume of $E$ up to first order.

By our assumption on $E$ we get

$$Q(W - \lambda U_0) = 0,$$
where $Q$ is defined in (2.7). Now Lemma 2.3 implies $Q(W) = \lambda Q(U_0)$ and, from the definition of $H_0$, we get
\[ Q(W) = \lambda Q(U_0) = \lambda H_0 \frac{d}{ds} \bigg|_{s=0} |\psi_s(E)| = H_0 \frac{d}{ds} \bigg|_{s=0} |\varphi_s(E)|. \]
This implies that $E$ is a critical point of the functional $E \mapsto |\partial E| - H_0 |E|$ and so it has prescribed $K$-mean curvature equal to the constant $H_0$.

**Lemma 2.3.** Let $E \subset \mathbb{H}^1$ be a bounded set with Euclidean Lipschitz boundary $S$. Let $\Omega \subset \mathbb{H}^1$ be an open set such that $\Omega \cap S_0 = \emptyset$. Let $U$ be a vector field with compact support $\Omega$ and $\{\varphi_s\}_{s \in \mathbb{R}}$ the associated flow. Then the derivative
\[ Q(U) = \frac{d}{ds} \bigg|_{s=0} A_K(\varphi_s(S)) \]
exists and is a linear function of $U$.

**Proof.** For every $s \in \mathbb{R}$, the set $\varphi_s(E)$ has Euclidean Lipschitz boundary and so it has finite $K$-perimeter. By Rademacher’s Theorem, the set $B = \{p \in S : S$ is not differentiable at $p\}$ has $\mathcal{H}^2$-measure equal to 0.

For any $p \in S \setminus B$ we take the curve $\sigma(s) = \varphi_s(p)$. For every $s \in \mathbb{R}$ the surface $\varphi_s(S)$ is differentiable at $\sigma(p)$ and the vector field $W(s) = ((N_s)_h)_{\sigma(s)}$, where $N_s$ is the outer unit normal to $\varphi_s(\partial E)$, is differentiable along the curve $\sigma$. Let us estimate the quotient
\[ \frac{||W(s + h)||_{K,s} - ||W(s)||_{K,s}}{h}. \]
Writing $W(s) = f(s)X_{\sigma(s)} + g(s)Y_{\sigma(s)}$ we have $||W(s)||_{K,s} = ||(f(s), g(s))||$, where $|| \cdot ||$ is the planar asymmetric norm associated to the convex set $K$. We have
\[ \frac{||W(s + h)||_{K,s} - ||W(s)||_{K,s}}{h} \leq \frac{||f(s + h) - f(s), g(s + h) - g(s)||}{h} \]
\[ \leq C \left( ||f(s + h) - f(s)|| + ||g(s + h) - g(s)|| \right), \]
for a constant $C > 0$ that only depends on $K$. The derivatives of $f$ and $g$ can be estimated in terms of the covariant derivative $\frac{D}{ds}W = \frac{D}{ds}(N_s)_h$ along $\sigma$. Since
\[ \left| \frac{D}{ds}(N_s)_h \right| \leq |\text{div}_{\varphi_s}(U)| \]
we get an uniform estimate on the derivatives of $f$ and $g$ independent of $p$. So the quotient (2.8) is uniformly bounded above by a constant independent of $p$.

To compute the derivative of $A_K(\varphi_s(S))$ at $s = 0$ we write
\[ A_K(\varphi_s(S)) = \int_S \left( ||(N_s)_h||_{K,s} \circ \varphi_s \right) \text{Jac}(\varphi_s) d\mathcal{H}^2 \]
The uniform estimate of the quotient (2.8) allows us to apply Lebesgue’s dominated convergence theorem and Leibniz’s rule to compute the derivative of $A_K(\varphi_s(S))$, given by
\[ \int_S \frac{d}{ds} \bigg|_{s=0} \left( ||(N_s)_h|| \circ \varphi_s \right) \text{Jac}(\varphi_s) \right) d\mathcal{H}^2. \]
Given a point \( p \in (S \setminus B) \cap \text{supp}(U) \), since \( \text{supp}(U) \subset \Omega \) and \( \Omega \cap S_0 = \emptyset \) we get \( (N_h)_p \neq 0 \) and so

\[
\frac{D}{ds}\bigg|_{s=0} ||(N_s)_h||_{K,*}(\sigma(s)) = \frac{D}{ds}\bigg|_{s=0} ((N_s)_h, \pi_K((N_s)_h)(\sigma(s)) = \langle \frac{D}{ds}\bigg|_{s=0} (N_s)_h, (N_h)_p \rangle + \langle (N_h)_p, (d\pi_K)(\frac{D}{ds}\bigg|_{s=0} (N_s)_h) \rangle.
\]

Since

\[
\frac{D}{ds}\bigg|_{s=0} (N_s)_h = \frac{D}{ds}\bigg|_{s=0} N - \frac{D}{ds}\bigg|_{s=0} N, T \rangle T,
\]

and

\[
\frac{D}{ds}\bigg|_{s=0} N = \sum_{i=1}^2 \langle N_p, \nabla_{\epsilon_i} U \rangle e_i,
\]

where \( e_i \) is an orthonormal basis of \( T_p(\partial E) \), we get that

\[
\frac{D}{ds}\bigg|_{s=0} ||N_s||_{K,*}
\]

is a linear function \( L(U) \) of \( U \). \( \square \)

**Remark 2.4.** Proposition 2.2 can be applied to isoperimetric regions in \( \mathbb{H}^1 \) with Euclidean Lipschitz boundary. Of course, the regularity of isoperimetric regions in \( \mathbb{H}^1 \) is still an open problem.

### 2.7. Intrinsic Euclidean Lipschitz graphs on a vertical plane in \( \mathbb{H}^1 \)

We denote by \( \text{Gr}(u) \) the *intrinsic* graph (Riemannian normal graph) of the Lipschitz function \( u : D \to \mathbb{R} \), where \( D \) is a domain in a vertical plane. Using Euclidean rotations about the vertical axis \( x = y = 0 \), that are isometries of the Riemannian metric \( g \), we may assume that \( D \) is contained in the plane \( y = 0 \). Since the vector field \( Y \) is a unit normal to this plane, the intrinsic graph \( \text{Gr}(u) \) is given by \( \{ \exp_p(u(p)Y)_p : p \in D \} \), where \( \exp \) is the exponential map of \( g \), and can be parameterized by the map

\[
\Phi^u(x, t) = (x, u(x, t), t - xu(x, t)).
\]

The tangent plane to any point in \( S = \text{Gr}(u) \) is generated by the vectors

\[
\Phi^u_x = (1, u_x, -u - xu_x) = X + u_x Y - 2u T,
\]
\[
\Phi^u_t = (0, u_t, 1 - xu_t) = u_t Y + T
\]

and the characteristic direction is given by \( Z = \tilde{Z}/|\tilde{Z}| \) where

\[
(2.9) \quad \tilde{Z} = X + (u_x + 2uu_t) Y.
\]

A unit normal to \( S \) is given by \( N = \tilde{N}/|\tilde{N}| \) where

\[
\tilde{N} = \Phi^u_x \times \Phi^u_t = (u_x + 2uu_t)X - Y + u_t T
\]

and \( \text{Jac}(\Phi^u) = |\Phi^u_x \times \Phi^u_t| = |\tilde{N}| \). Therefore the horizontal projection of the unit normal to \( S \) is given by \( \tilde{N}_h = \tilde{N}_h/|\tilde{N}| \), where \( \tilde{N}_h = (u_x + 2uu_t)X - Y \). Observe that \( J(Z) = -\nu_h \).

We also assume that \( S = \text{Gr}(u) \) is an \( \mathbb{H} \)-regular surface, meaning that \( \tilde{N}_h \) and \( \tilde{Z} \) in (2.9) and are continuous. Hence also \( (u_x + 2uu_t) \) is continuous.
Remark 2.5. Let \( \gamma(s) = (x(t), t)(s) \) be a \( C^1 \) curve in \( D \) then
\[
\Gamma(s) = (x(s), u(x(s)), t - xu(x(s))) \subset \text{Gr}(u)
\]
is also \( C^1 \) and
\[
\Gamma'(s) = x'X + (x'ux + t'u)Y + (t' - 2ux')T.
\]
In particular horizontal curves in \( \text{Gr}(u) \) satisfy the ordinary differential equation
\[
(t') = 2u(x, t)x'.
\]

From (2.2), the sub-Finsler \( K \)-area for a Euclidean Lipschitz surface \( S \) is
\[
A_K(S) = \int_S \|N_h\|_{K,*} dS,
\]
where \( \|N_h\|_{K,*} = \langle N_h, \pi(N_h) \rangle \) with \( \pi = (\pi_1, \pi_2) = \pi_K \) and \( dS \) is the Riemannian area measure. Therefore when we consider the intrinsic graph \( S = \text{Gr}(u) \) we obtain
\[
A(\text{Gr}(u)) = \int_D \langle \tilde{N}_h, \pi(\tilde{N}_h) \rangle \ dxdt
\]
\[
= \int_D (u_x + 2uu_t)\pi_1(u_x + 2uu_t, -1) - \pi_2(u_x + 2uu_t, -1) \ dxdt.
\]
Observe that the \( K \)-perimeter of a set was defined in terms of the outer unit normal. Hence we are assuming that \( S \) is the boundary of the epigraph of \( u \).

Given \( v \in C^\infty_0(D) \), a straightforward computation shows that
\[
\left. \frac{d}{ds} \right|_{s=0} A(\text{Gr}(u + sv)) = \int_D (v_x + 2vu_t + 2vu_t)M \ dxdt,
\]
where
\[
(2.12) \quad M = F(u_x + 2uu_t),
\]
and \( F \) is the function
\[
(2.13) \quad F(x) = \pi_1(x, -1) + x \frac{\partial \pi_1}{\partial x}(x, -1) - \frac{\partial \pi_2}{\partial x}(x, -1).
\]
Since \( u_x + 2uu_t \) is continuous and \( \pi \) is at least \( C^1 \) the function \( M \) is continuous.

3. Characteristics curves are \( C^2 \)

Here we prove our main result, that characteristic curves in an intrinsic Euclidean Lipschitz \( \mathbb{H} \)-regular surface with continuous prescribed \( K \)-mean curvature are of class \( C^2 \). The reader is referred to Theorem 4.1 in [21] for a proof of the the sub-Riemannian case. The proof of Theorem 3.1 depends on Lemmas 3.2 and 3.3.

**Theorem 3.1.** Let \( K \) be a \( C^2 \) convex set in \( \mathbb{R}^2 \) with \( 0 \in \text{int}(K) \) and \( \| \cdot \|_K \) the associated left-invariant norm in \( \mathbb{H}^1 \). Let \( \Omega \subset \mathbb{H}^1 \) be an open set and \( E \subset \Omega \) a set of prescribed \( K \)-mean curvature \( f \in C^0(\Omega) \) with an Euclidean Lipschitz and \( \mathbb{H} \)-regular boundary \( S \). Then the characteristic curves of \( S \cap \Omega \) are of class \( C^2 \).

**Proof.** By the Implicit Function Theorem for \( \mathbb{H} \)-regular surfaces, see Theorem 6.5 in [19], given a point \( p \in S \), after a rotation about the vertical axis, there exists an open neighborhood \( B \subset \mathbb{H}^1 \) of \( p \) such that \( B \cap S \) is the intrinsic graph \( \text{Gr}(u) \) of a function \( u : D \to \mathbb{R} \), where \( D \) is a domain in the vertical plane \( y = 0 \), and \( B \cap E \) is
the epigraph of \( u \). The function \( u \) is Euclidean Lipschitz by our assumption. Since \( \text{Gr}(u) \) has prescribed continuous mean curvature \( f \), from equation (2.11) we get
\[
(3.1) \quad \int_D (v_x + 2uv_t + 2uu_t)M + f v \, dx \, dt = 0,
\]
for each \( v \in C_0^\infty(D) \). The function \( M \) is defined in (2.12). By Remark 4.3 in [21] implies that \((3.1)\) holds for each \( v \in C_0^\infty(D) \) for which \( v_x + 2uv_t \) exists and is continuous.

Let \( \Gamma(s) \) be a characteristic horizontal curve passing through \( p \) whose velocity is the vector field \( \tilde{Z} \) defined in (2.9), that only depends on \( u_x + 2uu_t \). Since \( S \) is \( \mathbb{H} \)-regular the function \( u_x + 2uu_t \) is continuous and \( \Gamma(s) \) is of class \( C^1 \). Let us consider the function \( F \) defined in (2.13) and define
\[
g(s) = (u_x + 2uu_t)_{\Gamma(s)}.
\]
Hence \( F(g(s)) = M(s) \). The function \( F \) is \( C^1 \) for any convex set \( K \) of class \( C^2_+ \) and, from Lemma 3.2, we obtain that \( F'(x) > 0 \) for each \( x \in \mathbb{R} \). Therefore \( F^{-1} \) is also \( C^1 \) and \( g(s) = F^{-1}(M(s)) \). Thanks to Lemma 3.3 we obtain that \( M \) is \( C^1 \) along \( \Gamma \) and we conclude that also \( g \) is \( C^1 \) along \( \Gamma \). So \( \tilde{Z} \) is \( C^1 \) and the curve \( \Gamma \) is \( C^2 \). \( \square \)

**Lemma 3.2.** Let \( K \subset \mathbb{R}^2 \) be a convex body of class \( C^2_+ \) such that \( 0 \in \text{int}(K) \). Then the function \( F \) defined in (2.13) is \( C^1 \) and \( F'(x) > 0 \) for each \( x \in \mathbb{R} \).

**Proof.** Parameterize the lower part of the boundary of the convex body \( K \) by a function \( \phi \) defined on a closed interval \( I \subset \mathbb{R} \). The function \( \phi \) is of class \( C^2 \) in \( I \) and the graph becomes vertical at the endpoints of \( I \). As \( K \) is of class \( C^2_+ \) we have \( \phi''(x) > 0 \) for each \( x \in \mathbb{R} \). Take \( x \in \mathbb{R} \), then we have
\[
\pi(x, -1) = N_K^{-1} \left( \frac{(x, -1)}{\sqrt{1 + x^2}} \right),
\]
where \( N_K \) is the outer unit normal to \( \partial K \). Let \( \varphi(x) \in \hat{I} \) be the point where
\[
(\varphi(x), \phi(\varphi(x))) = \pi(x, -1).
\]
Therefore, if we consider the normal \( N_K \) of the previous equality we obtain
\[
\frac{\phi'(\varphi(x))}{\sqrt{1 + (\phi'(\varphi(x)))^2}} \frac{1}{\sqrt{1 + x^2}} = \frac{(x, -1)}{\sqrt{1 + x^2}}.
\]
Hence \( \phi'(\varphi(x)) = x \) and so \( \varphi \) is the inverse of \( \phi' \), that is invertible since \( \phi''(x) > 0 \) for each \( x \in \mathbb{R} \). Notice that
\[
F(x) = \pi_1(x, -1) + x \frac{\partial \pi_1}{\partial x}(x, -1) - \frac{\partial \pi_2}{\partial x}(x, -1)
= \varphi(x) + x \varphi'(x) - \phi'(\varphi(x)) \phi'(x) = \varphi(x),
\]
since \( \phi'(\varphi(x)) = x \). Hence we obtain
\[
F'(x) = \varphi'(x) = \frac{1}{\phi''(\varphi(x))} > 0
\]
for each \( x \in \mathbb{R} \). \( \square \)

**Lemma 3.3.** Let \( \Omega \subset \mathbb{H}^1 \) be an open set and \( E \subset \Omega \) a set of prescribed \( K \)-mean curvature \( f \in C^0(\Omega) \) with Euclidean Lipschitz and \( \mathbb{H} \)-regular boundary \( S \). Then the
function $M$ defined in (2.12) is of class $C^1$ along characteristic curves. Moreover, the differential equation
\[
\frac{d}{ds} M(\gamma(s)) = f(\gamma(s))
\]
is satisfied along any characteristic curve $\gamma$.

Proof. Let $\Gamma(s)$ be a characteristic curve passing through $p$ in $\text{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the $xt$-plane, and $(a, b) \in D$ the projection of $p$ to the $xt$-plane. We parameterize $\gamma$ by $s \to (s, t(s))$. By Remark 2.5 the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$. For $\varepsilon$ small enough, Picard-Lindelöf’s theorem implies the existence of $r > 0$ and a solution $t_\varepsilon : [a-r, a+r] \to \mathbb{R}$ of the Cauchy problem
\[
\begin{cases}
t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\
t_\varepsilon(a) = b + \varepsilon.
\end{cases}
\]

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. Here we exploit an argument similar to the one developed in [35]. By Theorem 2.8 in [44] we gain that $t_\varepsilon$ is Lipschitz with respect to $\varepsilon$ with Lipschitz constant less than or equal to $e^{Lr}$. Fix $s \in [a-r, a+r]$, the inverse of the function $\varepsilon \to t_\varepsilon(s)$ is given by $\tilde{\chi}_t(-s) = \chi_t(-s) - b$ where $\chi_t$ is the unique solution of the following Cauchy problem
\[
\begin{cases}
\chi_t'(\tau) = 2u(\tau, \chi_t(\tau)) \\
\chi_t(a + s) = t.
\end{cases}
\]
Again by Theorem 2.8 in [44] we have that $\tilde{\chi}_t$ is Lipschitz continuous with respect to $t$, thus the function $\varepsilon \to t_\varepsilon$ is a locally biLipschitz homeomorphisms.

We consider the following Lipschitz coordinates
\[
G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi)) = (s, t)
\]
around the characteristic curve passing through $(a, b)$. Notice that, by the uniqueness result for (3.2), $G$ is injective. Given $(s, t)$ in the image of $G$ using the inverse function $\tilde{\chi}_t$ defined in (3.3) we find $\varepsilon$ such that $t_\varepsilon(s) = t$, therefore $G$ is surjective. By the Invariance of Domain Theorem [3], is a homeomorphism. The Jacobian of $G$ is defined by
\[
J_G = \det \begin{pmatrix} 1 & 0 \\ t'_\varepsilon & \frac{\partial t_\varepsilon}{\partial \varepsilon}(s) \end{pmatrix} = \frac{\partial t_\varepsilon}{\partial \varepsilon}(s)
\]
almost everywhere in $\varepsilon$. Any function $\varphi$ defined on $D$ can be considered as a function of the variables $(\xi, \varepsilon)$ by making $\tilde{\varphi}(\xi, \varepsilon) = \varphi(\xi, t_\varepsilon(\xi))$. Since the function $G$ is $C^1$ with respect to $\xi$ we have
\[
\frac{\partial \tilde{\varphi}}{\partial \xi} = \varphi_\xi + t'_\varepsilon \varphi_t = \varphi_x + 2u \varphi_t.
\]
Furthermore, by [16, Theorem 2 in Section 3.3.3] or [25, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of $v$ is contained in a sufficiently small neighborhood of $(a, b)$, we can express the integral (3.1) as
\[
\int_I \left( \int_{a-r}^{a+r} \left( \frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v} \tilde{u}_t \tilde{M} + \tilde{f}(\varepsilon) \frac{\partial t_\varepsilon}{\partial \varepsilon} \right) d\xi \right) d\varepsilon = 0,
\]
where $I$ is a small interval containing 0. Instead of $\tilde{v}$ in (3.6) we consider the function $\tilde{v}h/(t_{z+h} - t_z)$, where $h$ is a small enough parameter. Then we obtain
\[
\frac{\partial}{\partial \xi} \left( \frac{\tilde{v}h}{(t_{z+h} - t_z)} \right) = \frac{\partial \tilde{v}}{\partial \xi} \frac{h}{(t_{z+h} - t_z)} - \tilde{v}h \frac{t_{z+h}' - t_z'}{(t_{z+h} - t_z)^2} - 2\tilde{v}h \frac{u(\xi, t_{z+h}(\xi)) - u(\xi, t_z(\xi))}{(t_{z+h} - t_z)^2},
\]
that tends to
\[
\left( \frac{\partial t_z}{\partial \xi} \right)^{-1} \left( \frac{\partial \tilde{v}}{\partial \xi} - 2\tilde{v}u \right)
\]
a.e. in $\varepsilon$.

when $h$ goes to 0. Putting $\tilde{v}h/(t_{z+h} - t_z)$ in (3.6) instead of $\tilde{v}$ we gain
\[
\int_I \left( \int_{a-r}^{a+r} \frac{h \partial \tilde{v}}{\partial \xi} \left( \frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v} \left( \tilde{u} - \frac{\tilde{u}(\xi, \varepsilon + h) - \tilde{u}(\xi, \varepsilon)}{(t_{z+h} - t_z)} \right) \right) \hat{M} + \hat{f} \tilde{v} d\xi \right) d\varepsilon = 0.
\]
Using Lebesgue’s dominated convergence theorem and letting $h \to 0$ we have
\[
\int_I \left( \int_{a-r}^{a+r} \frac{\partial \tilde{v}}{\partial \xi} \hat{M} + \hat{f} \tilde{v} d\xi \right) d\varepsilon = 0.
\]
Let $\eta : \mathbb{R} \to \mathbb{R}$ be a positive function compactly supported in $I$ and for $\rho > 0$ we consider the family $\eta_{\rho}(x) = \rho^{-1} \eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_{\rho}(\varepsilon)\psi(\xi)$ in (3.7) and letting $\rho \to 0$ we get
\[
\int_{a-r}^{a+r} \psi(\xi) \hat{M}(\xi, 0) + \hat{f}(\xi, 0)\psi(\xi) d\xi = 0,
\]
for each $\psi \in C_0^\infty((a - r, a + r))$. Since $u_x + 2uu_t$ is continuous, $M$ in (2.12) is continuous, thus also $\hat{M}$. Hence thanks to Lemma 3.4 we conclude that $M$ is $C^1$ along $\gamma$, thus by Remark 2.5 is also $C^1$ along $\Gamma$.

Since $M$ is $C^1$ along the characteristic curve, we can integrate by parts in equation (3.8) to obtain
\[
\int_{a-r}^{a+r} \left( -\hat{M}'(0, \xi) + \hat{f}(0, \xi) \right) \psi(\xi) d\xi = 0,
\]
for each $\psi \in C_0^\infty((a - r, a + r))$. That means that $M$ satisfies the equation
\[
\frac{d}{ds} M(\gamma(s)) = f(\gamma(s))
\]
along characteristic curves. \hfill \Box

**Lemma 3.4** ([21, Lemma 4.2]). Let $J \subset \mathbb{R}$ be an open interval and $g, h \in C^0(J)$. Let $H \in C^1(J)$ be a primitive of $h$. Assume that
\[
\int_J \psi' g + h \psi = 0,
\]
for each $\psi \in C_0^\infty(J)$. Then the function $g - H$ is a constant function in $J$. In particular $g \in C^1(J)$.

**Remark 3.5.** Let $K$ be a convex body of class $C^2_+$ such that $0 \in K$. Following [38] we consider a clockwise-oriented $P$-periodic parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of $\partial K$. For a fixed $v \in \mathbb{R}$ we take the translated curve $s \to \gamma(s + v) - \gamma(v) = (x(s), y(s))$
and we consider its horizontal lifting \( \Gamma_v(s) \) to \( \mathbb{H}^1 \) starting at \( (0, 0, 0) \in \mathbb{H}^1 \) for \( s = 0 \), given by
\[
\Gamma_v(s) = \left( x(s), y(s), \int_0^s y(\tau)x'(\tau) - x(\tau)y'(\tau) d\tau \right).
\]
The Pansu-Wulff shape associated to \( K \) is defined by
\[
\mathcal{S}_K = \bigcup_{v \in [0, P]} \Gamma_v([0, P]).
\]
In [38, Theorem 3.14] it is shown that the horizontal liftings \( \Gamma_v \), for each \( v \in [0, P] \), are solutions for \( H_K = 1 \), therefore \( \mathcal{S}_K \) has constant prescribed \( K \)-mean curvature equal to 1. Since the curves \( \Gamma_v \) have the same regularity as \( \partial K \), the \( C^2 \) regularity result for horizontal curves obtained in Theorem 3.1 is optimal.

**Corollary 3.6.** Let \( K \) be a \( C^2 \) convex set in \( \mathbb{R}^2 \) with \( 0 \in \text{int}(K) \) and \( || \cdot ||_K \) the associated left-invariant norm in \( \mathbb{H}^1 \). Let \( \Omega \subset \mathbb{H}^1 \) be an open set and \( E \subset \Omega \) a set of prescribed \( K \)-mean curvature \( f \in C^0(\Omega) \) with \( C^1 \) boundary \( S \). Then the characteristic curves in \( S \setminus S_0 \) are of class \( C^2 \).

**Proof.** Since \( S \) is of class \( C^1 \), in the regular part \( S \setminus S_0 \) the horizontal normal \( \nu_h \) is a nowhere-vanishing continuous vector fields, thus \( S \setminus S_0 \) is an \( \mathbb{H} \)-regular surface. In particular a \( C^1 \) surface is Lipschitz, thus \( S \setminus S_0 \) verifies the hypotheses of Theorem 3.1 and the characteristic curves in \( S \setminus S_0 \) are of class \( C^2 \). \( \square \)

**Remark 3.7.** When \( S \) is of class \( C^1 \) the proof of Lemma 3.3 is is much easier. Indeed the solution \( t_\varepsilon \) of the Cauchy Problem (3.2) is differentiable in \( \varepsilon \), thus the function \( \partial t_\varepsilon / \partial \varepsilon \) satisfies the following ODE
\[
\left( \frac{\partial t_\varepsilon}{\partial \varepsilon} \right)'(s) = 2u_\varepsilon(s, t_\varepsilon(s)) \frac{\partial t_\varepsilon}{\partial \varepsilon}(s), \quad \frac{\partial t_\varepsilon}{\partial \varepsilon}(a) = 1.
\]
That implies that
\[
\frac{\partial t_\varepsilon}{\partial \varepsilon}(s) = e^{\int_0^s 2u_\varepsilon(\tau, t_\varepsilon(\tau)) d\tau} > 0.
\]

Since the Jacobian \( J_{t_\varepsilon} \) defined in (3.5) is equal to \( \partial t_\varepsilon / \partial \varepsilon > 0 \) the change of variables \( G(\xi, \varepsilon) \) is invertible. Hence the rest of the proof of Lemma 3.3 goes in the same way as before.

### 4. The sub-Finsler mean curvature equation

Given an Euclidean Lipschitz boundary \( S \) whose characteristic curves in \( S \setminus S_0 \) are of class \( C^2 \), for each point \( p \in S \setminus S_0 \) we can define the \( K \)-mean curvature \( H_K \) of \( S \) by
\[
H_K = \langle D_Z \pi_K(\nu_h), Z \rangle = \langle \nabla_Z \pi_K(\nu_h), Z \rangle,
\]
where \( \nu_h \) is the outer horizontal unit normal to \( S \). This definition was given in [38] for surfaces of class \( C^2 \).

**Proposition 4.1.** Let \( \Omega \subset \mathbb{H}^1 \) be an open set and \( E \subset \Omega \) a set of prescribed \( K \)-mean curvature \( f \in C^0(\Omega) \) Euclidean Lipschitz and \( \mathbb{H} \)-regular boundary \( S \). Then \( H_K(p) = f(p) \) for each \( p \in S \setminus S_0 \).
Proof. By the Implicit Function Theorem for $\mathbb{H}$-regular surfaces, Theorem 6.5 in [19], given a point $p \in S$, after a rotation about the $t$-axis, there exists an open neighborhood $B \subset \mathbb{H}^1$ of $p$ such that $B \cap S$ is the intrinsic graph of a function $u : D \to \mathbb{R}$ where $D$ is a domain in the vertical plane $y = 0$. The function $u$ is Euclidean Lipschitz by our assumption. We set $B \cap S = \text{Gr}(u)$. We assume that $E$ is locally the epigraph of $u$.

Let $\Gamma(s)$ be a characteristic curve passing through $p$ in $\text{Gr}(u)$ and $\gamma(s)$ its projection on the $xt$-plane. The characteristic vector $Z$ defined in (2.9) is given by

$$Z = \frac{X + (u_x + 2uu_t)Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}.$$  

Since $S$ is $\mathbb{H}$-regular, $Z$ and the horizontal unit normal

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{(1 + (u_x + 2uu_t)^2)^{\frac{1}{2}}}$$

are continuous vector fields. By Lemma 3.3 we have that $M = F(u_x + 2uu_t)$ defined in (2.12) satisfies the differential equation

$$\frac{d}{ds}M(\gamma(s)) = f(\gamma(s))$$

along the characteristic curves. Therefore we obtain

$$\frac{d}{ds}M(\gamma(s)) = F'(u_x + 2uu_t)\frac{d}{ds}[(u_x + 2uu_t)(\gamma(s))]$$

$$= \frac{1}{\phi''(u_x + 2uu_t)}\frac{d}{ds}[(u_x + 2uu_t)(\gamma(s))].$$

As in proof of Lemma 3.2, we parametrize the lower part of the boundary of the convex body $K$ by a function $\phi$ defined on a closed interval $I \subset \mathbb{R}$. Again by Lemma 3.2 we have

$$\pi_K(x, -1) = (\varphi(x), \phi(\varphi(x))),$$

where $\varphi$ is the inverse function of $\phi'$. Furthermore the $K$-mean curvature defined (4.1) is equivalent to

$$H_K = \langle D_2\pi_K(u_x + 2uu_t, -1), Z \rangle$$

$$= \langle \frac{D}{ds}[\varphi(u_x + 2uu_t)X_x + \phi(\varphi(u_x + 2uu_t))Y_x], Z \rangle$$

$$= \frac{\varphi'(u_x + 2uu_t)}{1 + (u_x + 2uu_t)^2}$$

$$= \frac{1}{\phi''(u_x + 2uu_t)}\frac{d}{ds}[(u_x + 2uu_t)(\gamma(s))].$$

Hence we obtain $H_K = \frac{d}{ds}M(\gamma(s))$ and so $H_K(p) = f(p)$ for each $p \in S \setminus S_0$. □

The following result allows us to express the $K$-mean curvature $H_K$ in terms of the sub-Riemannian mean curvature $H_D$.

**Proposition 4.2.** Let $K \subset \mathbb{R}^2$ be a convex body of class $C^2_+$ such that $0 \in \text{int}(K)$ and $\pi_K = N_K^{-1}$. Let $\kappa$ be the strictly positive curvature of the boundary $\partial K$. Let
\( \Omega \subset \mathbb{R}^3 \) be an open set and \( E \subset \Omega \) a set of prescribed \( K \)-mean curvature \( f \in C^0(\Omega) \) with Euclidean Lipschitz and \( \mathbb{H} \)-regular boundary \( S \). Then, we have
\[
H_D(p) = \kappa(\pi_K(\nu_h))(p) \quad \text{for each} \quad p \in S \setminus S_0,
\]
where \( H_D(p) = \langle D_Z\nu_h, Z \rangle \) is the sub-Riemannian mean curvature, \( \nu_h \) be the horizontal unit normal at \( p \) to \( S \in S_0 \) and \( Z = J(\nu_h) \) be the characteristic vector field.

**Proof.** By Proposition 4.1 we have \( H_K(p) = f(p) \) for each \( p \in S \setminus S_0 \). We remark that Theorem 3.1 implies that \( H_K \) is well-defined.

Let \( \gamma : (-\varepsilon, \varepsilon) \to S \setminus S_0 \) be the integral curve of \( Z \) passing through \( p \), namely \( \gamma'(s) = Z_{\gamma(s)} \) and \( \gamma(0) = p \). Let \( \nu_h(s) = -J(Z_{\gamma(s)}) \) be the horizontal unit normal along \( \gamma \) and let
\[
\pi(\nu_h(s)) = \pi_1(\nu_h(s))X_{\gamma(s)} + \pi_2(\nu_h(s))Y_{\gamma(s)}.
\]
Noticing that \( \nabla \pi = \nabla Y = 0 \) we gain
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \pi(\nu_h(s)) = \left( \frac{\partial}{\partial s} \right)_{s=0} \pi_1(\nu_h(s))X_{\gamma(0)} + \left( \frac{\partial}{\partial s} \right)_{s=0} \pi_2(\nu_h(s))Y_{\gamma(0)}.
\]
Setting \( \nu_h = aX + bY \) we obtain
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \pi(\nu_h(s)) = (d\pi)(a, b) \left( \frac{\partial}{\partial s} \right)_{s=0} \nu_h(s),
\]
where
\[
(d\pi)_{a, b} = \begin{pmatrix}
\frac{\partial \pi_1}{\partial a}(a, b) & \frac{\partial \pi_1}{\partial b}(a, b) \\
\frac{\partial \pi_2}{\partial a}(a, b) & \frac{\partial \pi_2}{\partial b}(a, b)
\end{pmatrix}.
\]
Moreover, by Corollary 1.7.3 in [43] we get \( \pi_K = \nabla h \), where \( h \) is a \( C^2 \) function. Thus by Schwarz's theorem the Hessian \( \text{Hess}_{(a, b)}(h) = (d\pi)_{(a, b)} \) is symmetric, i.e. \( (d\pi) = (d\pi)^* \). Equation (4.2) then implies
\[
H_K = \langle \nabla Z \pi_K(\nu_h), Z \rangle = \langle \nabla Z \nu_h, (d\pi)^*_{\nu_h} Z \rangle = \langle \nabla Z \nu_h, (d\pi)_{\nu_h} Z \rangle.
\]
Finally, by Lemma 4.3 we get
\[
H_K = \frac{1}{\kappa(\pi_K(\nu_h))} \langle \nabla Z \nu_h, Z \rangle.
\]
Hence we obtain \( \langle D_Z\nu_h, Z \rangle = \kappa(\pi_K(\nu_h)) \), since \( D_Z\nu_h = \nabla Z \nu_h \).

**Lemma 4.3.** Let \( K \subset \mathbb{R}^2 \) be a convex body of class \( C^2 \) such that \( 0 \in \text{int}(K) \) and \( N_K \) be the Gauss map of \( \partial K \). Let \( \kappa \) be the strictly positive curvature of the boundary \( \partial K \). Let \( S \) be an \( \mathbb{H} \)-regular surface with horizontal unit normal \( \nu_h \) and characteristic vector field \( Z = J(\nu_h) \). Then we have
\[
(d\pi)_{\nu_h} Z = \frac{1}{\kappa} Z \quad \text{and} \quad (d\pi)_{\nu_h} \nu_h = 0,
\]
where \( (d\pi)_{\nu_h} \) is the differential of \( \pi_K = N_K^{-1} \).

**Proof.** Let \( a(t) = (x(t), y(t)) \) be an arc-length parametrization of \( \partial K \) such that \( \dot{x}^2(t) + \dot{y}^2(t) = 1 \). Let \( \nu_h = aX + bY \) be the horizontal unit normal to \( S \), with \( a = \cos(\theta) \) and \( b = \sin(\theta) \) and \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Notice that \( \theta = \arctan\left(\frac{b}{a}\right) \). Then we have
\[
\pi_K(a, b) = N_K^{-1}((a, b)).
\]
Let $\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be the function satisfying
$$\pi_K(\cos(\theta), \sin(\theta)) = (x(\varphi(\theta)), y(\varphi(\theta))).$$
If we consider the normal $N_K$ of the previous equality we obtain
$$(\cos(\theta), \sin(\theta)) = (\dot{y}(\varphi(\theta)), -\dot{x}(\varphi(\theta))).$$
Therefore we have
$$\theta = \arctan\left(-\frac{\dot{x}}{\dot{y}}(\varphi(\theta))\right)$$
for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. That means that $\varphi$ is the inverse of the function $\arctan(-\frac{\dot{x}}{\dot{y}}(t))$, that is invertible since
$$\frac{d}{dt}\arctan(-\frac{\dot{x}}{\dot{y}}(t)) = \dot{x}\ddot{y} - \dot{y}\ddot{x} = \kappa(t) > 0.$$
Let $Z = J(\nu_h) = -bX + aY$ be the characteristic vector field, then we have
$$(d\pi)_{(a,b)} = (d\pi)_{(a,b)}^*$$
and
$$(d\pi)_{(a,b)}^*Z = \varphi'(\arctan(\frac{b}{a}))Z = \frac{1}{\kappa(\varphi(\theta))}Z.$$
A similar straightforward computation shows that $(d\pi)_{\nu_h} \nu_h = 0$. \hfill \Box

**References**

[1] L. Ambrosio, F. Serra Cassano, and D. Vittone. Intrinsic regular hypersurfaces in Heisenberg groups. *J. Geom. Anal.*, 16(2):187–232, 2006.

[2] V. Barone Adesi, F. Serra Cassano, and D. Vittone. The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations. *Calc. Var. Partial Differential Equations*, 30(1):17–49, 2007.

[3] L. E. J. Brouwer. Beweis des ebenen Translationssatzes. *Math. Ann.*, 72(1):37–54, 1912.

[4] L. Capogna, G. Citti, and M. Manfredini. Regularity of non-characteristic minimal graphs in the Heisenberg group $\mathbb{H}^1$. *Indiana Univ. Math. J.*, 58(5):2115–2160, 2009.

[5] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[6] J.-H. Cheng, H.-L. Chiu, J.-F. Hwang, and P. Yang. Umbilicity and characterization of Pansu spheres in the Heisenberg group. *J. Reine Angew. Math.*, 738:203–235, 2018.

[7] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang. Minimal surfaces in pseudohermitian geometry. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):129–177, 2005.

[8] J.-H. Cheng, J.-F. Hwang, and P. Yang. Existence and uniqueness for $p$-area minimizers in the Heisenberg group. *Math. Ann.*, 337(2):253–293, 2007.

[9] J.-H. Cheng, J.-F. Hwang, and P. Yang. Regularity of $C^1$ smooth surfaces with prescribed $p$-mean curvature in the Heisenberg group. *Math. Ann.*, 344(1):1–35, 2009.

[10] G. Citti and A. Sarti. A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vision*, 24(3):307–326, 2006.

[11] D. Danielli, N. Garofalo, and D. M. Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Adv. Math.*, 215(1):292–378, 2007.
[12] D. Danielli, N. Garofalo, and D.-M. Nhieu. A partial solution of the isoperimetric problem for the Heisenberg group. Forum Math., 20(1):99–143, 2008.

[13] D. Danielli, N. Garofalo, D. M. Nhieu, and S. D. Pauls. Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$. J. Differential Geom., 81(2):251–296, 2009.

[14] E. De Giorgi. Su una teoria generale della misura $(r - 1)$-dimensionale in uno spazio ad $r$ dimensioni. Ann. Mat. Pura Appl. (4), 36:191–213, 1954.

[15] S. Dragomir and G. Tomassini. Differential geometry and analysis on CR manifolds, volume 246 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2006.

[16] L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[17] V. Franceschi, G. P. Leonardi, and R. Monti. Quantitative isoperimetric inequalities in $\mathbb{H}^n$. Calc. Var. Partial Differential Equations, 54(3):3229–3239, 2015.

[18] V. Franceschi, R. Monti, A. Righini, and M. Sigalotti. The isoperimetric problem for regular and crystalline norms in $\mathbb{H}^1$. arXiv:2007.11384, 22 Jul 2020.

[19] B. Franchi, R. Serapioni, and F. Serra Cassano. Rectifiability and perimeter in the Heisenberg group. Math. Ann., 321(3):479–531, 2001.

[20] M. Galli. The regularity of Euclidean Lipschitz boundaries with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds. Nonlinear Anal., 136:40–50, 2016.

[21] M. Galli and M. Ritoré. Regularity of $C^1$ surfaces with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds. Calc. Var. Partial Differential Equations, 54(3):2503–2516, 2015.

[22] N. Garofalo and D.-M. Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Comm. Pure Appl. Math., 49(10):1081–1144, 1996.

[23] G. Giovannardi. Higher dimensional holonomy map for rules submanifolds in graded manifolds. Anal. Geom. Metr. Spaces, 8(1):68–91, 2020.

[24] G. Giovannardi, G. Citti, and M. Ritoré. Variational formulas for submanifolds of fixed degree. arXiv:1905.05131, 13 May 2019.

[25] P. Hajlasz. Change of variables formula under minimal assumptions. Colloq. Math., 64(1):93–101, 1993.

[26] R. K. Hladky and S. D. Pauls. Constant mean curvature surfaces in sub-Riemannian geometry. J. Differential Geom., 79(1):111–139, 2008.

[27] R. K. Hladky and S. D. Pauls. Variation of perimeter measure in sub-Riemannian geometry. Int. Electron. J. Geom., 6(1):8–40, 2013.

[28] A. Hurtado, M. Ritoré, and C. Rosales. The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. Adv. Math., 224(2):561–600, 2010.

[29] A. Hurtado and C. Rosales. Area-stationary surfaces inside the sub-Riemannian three-sphere. Math. Ann., 340(3):675–708, 2008.

[30] G. P. Leonardi and S. Masnou. On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. Ann. Mat. Pura Appl. (4), 184(4):533–553, 2005.

[31] G. P. Leonardi and S. Rigot. Isoperimetric sets on Carnot groups. Houston J. Math., 29(3):609–637, 2003.

[32] R. Monti. Heisenberg isoperimetric problem. The axial case. Adv. Calc. Var., 1(1):93–121, 2008.

[33] R. Monti and M. Rickly. Convex isoperimetric sets in the Heisenberg group. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 8(2):391–415, 2009.

[34] S. Nicolussi and M. Ritoré. Area-minimizing cones in $\mathbb{H}^1$. arXiv:2008.04027, 10 Aug 2020.

[35] S. Nicolussi and F. Serra Cassano. The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group. Calc. Var. Partial Differential Equations, 58(4):Paper No. 141, 28, 2019.

[36] P. Pansu. Une inégalité isopérimétrique sur le groupe de Heisenberg. C. R. Acad. Sci. Paris Sér. I Math., 295(2):127–130, 1982.

[37] S. D. Pauls. $H$-minimal graphs of low regularity in $\mathbb{H}^3$. Comment. Math. Helv., 81(2):337–381, 2006.

[38] J. Pozuelo and M. Ritoré. Pansu-Wulff shapes in $\mathbb{H}^1$. arXiv:2007.04683, 9 Jul 2020.

[39] M. Ritoré. Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity. Calc. Var. Partial Differential Equations, 34(2):179–192, 2009.
[40] M. Ritoré. A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$. 
Calc. Var. Partial Differential Equations, 44(1-2):47-60, 2012.

[41] M. Ritoré and C. Rosales. Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. Adv. Math., 219(2):633–671, 2008.

[42] A. P. Sánchez. Sub-finsler heisenberg perimeter measures, 2017.

[43] R. Schneider. Convex bodies: the Brunn-Minkowski theory, volume 151 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, expanded edition, 2014.

[44] G. Teschl. Ordinary differential equations and dynamical systems, volume 140 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.

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