A Note on Estimation in Hilbertian Linear Models

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ABSTRACT. We study estimation and prediction in linear models where the response and the regressor variable both take values in some Hilbert space. Our main objective is to obtain consistency of a principal component-based estimator for the regression operator under minimal assumptions. In particular, we avoid some inconvenient technical restrictions that have been used throughout the literature. We develop our theory in a time-dependent setup that comprises as important special case the autoregressive Hilbertian model.

Key words: adaptive estimation, consistency, dependence, functional regression, Hilbert spaces, infinite dimensional data, prediction

1. Introduction

In this paper, we are concerned with a regression problem of the form

\[ Y_k = \Psi(X_k) + \epsilon_k, \quad k \geq 1, \]

where \( \Psi \) is a bounded linear operator mapping from space \( H_1 \) to space \( H_2 \). This model is fairly general, and many special cases have been intensively studied in the literature. Our main objective is the study of this model when the regressor space \( H_1 \) is infinite dimensional. Then model (1) can be seen as a general formulation of a functional linear model, which is an integral part of functional data literature. Its various forms are introduced in Chapters 12–17 of Ramsay & Silverman (2005). A few recent references are Cuevas et al. (2002), Malfait & Ramsay (2003), Cardot et al. (2003), Chiuou et al. (2004), Müller & Stadtmüller (2005), Yao et al. (2005), Cai & Hall (2006), Li & Hsing (2007), Hall & Horowitz (2007), Reiss & Ogden (2007), Febrero-Bande et al. (2010), Crambes et al. (2009), Yuan & Cai (2011), Ferraty et al. (2011) and Crambes & Mas (2013).

From an inferential point of view, a natural problem is the estimation of the ‘regression operator’ \( \Psi \). Once an estimator \( \hat{\Psi} \) is obtained, we can use it in an obvious way for prediction of the responses \( Y \). Both the estimation and the prediction problems are addressed in this paper. In existing literature, these problems have been discussed from several angles. For example, there is the distinction between the ‘functional regressors and responses’ model (e.g. Cuevas et al. (2002)) and the perhaps more widely studied ‘functional regressor and scalar response model’ (e.g. Cardot et al. (1999)). Other papers deal with the effect when random functions are not fully observed but are obtained from sparse, irregular data measured with error (e.g. Yao et al. (2005)). More recently, the focus was on establishing rates of consistency (e.g. Cai & Hall (2006), Cardot & Johannes (2010)). The two most popular methods of estimation are based on principal component (PC) analysis (e.g. Bosq (1991), Cardot et al. (1999), Hall & Horowitz (2007)) or spline smoothing estimators (e.g. Hastie & Mallows (1993), Eilers & Marx (1996), Crambes et al. (2009)).

In this paper, we address the estimation and prediction problems for this model when the data are fully observed, using the PC approach. Let us explain what is the new contribution and what distinguishes our paper from previous work.
(i) The crucial difficulty for this type of problems is that the infinite dimensional operator $\Psi$ needs to be approximated by a sample version $\hat{\Psi}_K$ of finite dimension $K$, say. Clearly, $K = K_n$ needs to depend on the sample size and tends to $\infty$ in order to obtain an asymptotically unbiased estimator. In existing papers, determination of $K$ and proof of consistency require, among others, unnecessary moment assumptions and artificial restrictions concerning the spectrum of the covariance operator of the regressor variables $X_k$. As our main result, we will complement the current literature by showing that the PC estimator remains consistent without such technical constraints. We provide a data-driven procedure for the choice of $K$, which may even be used as a practical alternative to cross-validation (CV).

(ii) We allow the regressors $X_k$ to be dependent. This is important for two reasons. First, many examples in functional data analysis (FDA) literature exhibit dependencies as the data stem from a continuous time process, which is then segmented into a sequence of curves, for example, by considering daily data. Examples of this kind include intra-day patterns of pollution records, meteorological data, financial transaction data or sequential functional magnetic resonance imaging recordings. See, for example, Horváth & Kokoszka (2012).

Second, our framework detailed in the succeeding text will include the important special case of a functional autoregressive model that has been intensively investigated in the functional literature and is often used to model autoregressive dynamics of a functional time series. This model is analysed in detail in Bosq (2000). We can not only greatly simplify the assumptions needed for consistent estimation but also allow for a more general setup. For example, in our theorem 2, we show that it is not necessary to assume that $\Psi$ is a Hilbert–Schmidt operator if our intention is prediction. This quite restrictive assumption is standard in existing literature, although it even excludes the identity operator.

(iii) As we already mentioned before, the literature considers different forms of functional linear models. Arguably, the most common are the scalar response and functional regressor and the functional response and functional regressor case. We will not distinguish between these cases but work with a linear model between two general Hilbert spaces.

In the next section, we will introduce notation, assumptions, the estimator and our main results. In Section 3, we provide a small simulation study that compares our data-driven choice of $K$ with CV. As we will see, this procedure is quite competitive with CV in terms of mean squared prediction error, while it is clearly favourable to the latter in terms of computational costs. Finally, in Section 5, we give the proofs.

2. Estimation of $\Psi$

2.1. Notation

Let $H_1, H_2$ be two (not necessarily distinct) separable Hilbert spaces. We denote by $\mathcal{L}(H_i, H_j), (i, j \in \{1, 2\})$, the space of bounded linear operators from $H_i$ to $H_j$. Further, we write $\langle \cdot, \cdot \rangle_H$ for the inner product on Hilbert space $H$ and $\|x\|^2_H = \langle x, x \rangle_H$ for the corresponding norm. For $\Phi \in \mathcal{L}(H_i, H_j)$, we denote by $\|\Phi\|_{\mathcal{L}(H_i, H_j)} = \sup_{\|x\|_{H_i} \leq 1} \|\Phi(x)\|_{H_j}$ the operator norm and by $\|\Phi\|_{\mathcal{S}(H_i, H_j)}^2 = \sum_{k=1}^{\infty} \|\Phi(e_k)\|_{H_j}^2$, where $e_1, e_2, \ldots \in H_i$ is any orthonormal basis (ONB) of $H_i$, the Hilbert–Schmidt norm of $\Phi$. It is well known that this norm is independent of the choice of the basis. Furthermore, with the inner product $\langle \Phi, \Theta \rangle_{\mathcal{S}(H_1, H_2)} = \sum_{k \geq 1} \langle \Phi(e_k), \Theta(e_k) \rangle_{H_2}$ the space $\mathcal{S}(H_1, H_2)$ is again a separable Hilbert space.
Assume \( X, Y \in \mathcal{L} \), and \( X \) called \( L \) processes fit in this framework. More precisely, if \( X \) possesses a first moment, then \( X \) possesses a mean \( \mu \), determined as the unique element for which \( E(X, x)_H = \langle \mu, x \rangle_H, \forall x \in H \). For \( x \in H_1 \) and \( y \in H_j \), let \( x \otimes y : H_i \to H_j \) be an operator defined as \( x \otimes y(v) = \langle x, v \rangle y \). If \( X \in L^2_H \), then it possesses a covariance operator \( C \), given by \( C = E[(X - \mu) \otimes (X - \mu)] \). It can be easily seen that \( C \) is a Hilbert–Schmidt operator. Assume \( X, Y \in L^2_H \). Following Bosq (2000), we say that \( X \) and \( Y \) are orthogonal \((X \perp Y)\) if \( EX \otimes Y = 0 \). A sequence of orthogonal elements in \( H \) with a constant mean and constant covariance operator is called \( H \)-white noise.

2.2. Setup

We consider the general regression problem (1) for fully observed data. Let us collect our main assumptions.

(A): We have \( \Psi \in \mathcal{L}_{12} \). Further, \( \{\varepsilon_k\} \) and \( \{X_k\} \) are zero mean variables which are assumed to be \( L^4 - m \)-approximable in the sense of Hörmann & Kokoszka (2010) (see below). In addition \( \{\varepsilon_k\} \) is \( H_2 \)-white noise. For any \( k \geq 1 \) we have \( X_k \perp \varepsilon_k \).

Here is the weak dependence concept that we impose.

Definition 1 (Hörmann & Kokoszka (2010)). A random sequence \( \{X_n\}_{n \geq 1} \) with values in \( H \) is called \( L^p - m \)-approximable, if it can be represented as

\[
X_n = f(\delta_n, \delta_{n-1}, \delta_{n-2}, \ldots),
\]

where the \( \delta_i \) are independent and identically distributed (i.i.d.) elements taking values in a measurable space \( S \) and \( f \) is a measurable function \( f : S^\infty \to H \). Moreover, if \( \delta'_i \) are independent copies of \( \delta_i \) defined on the same probability space, then for

\[
X^{(m)}_n = f(\delta_n, \delta_{n-1}, \delta_{n-2}, \ldots, \delta_{n-m+1}, \delta'_{n-m}, \delta'_{n-m-1}, \ldots)
\]

we have

\[
\sum_{m=1}^{\infty} v_{p,H} \left( X_m - X^{(m)}_m \right) < \infty.
\]

Evidently, i.i.d. sequences with finite \( p \)-th moments are \( L^p - m \)-approximable. This leads to the classical functional linear model. But it is also easily checked that functional linear processes fit in this framework. More precisely, if \( X_n \) is of the form

\[
X_n = \sum_{k \geq 0} b_k(\delta_{n-k}),
\]

where \( b_k : H_0 \to H_1 \) are bounded linear operators such that \( \sum_{m \geq 1} \sum_{k \geq m} \|b_k\|_{C_{01}} < \infty \), and \( (\delta_n) \) is i.i.d. noise with \( v_{p,H_0}(\delta_0) < \infty \), then \( \{X_n\} \) is \( L^p - m \)-approximable. Other (also non-linear) examples of functional time series covered by \( L^p - m \)-approximability can be found in Hörmann & Kokoszka (2010).
A very important example included in our framework is the autoregressive Hilbertian model of order 1 (ARH(1)) given by the recursion $X_{k+1} = \Psi(X_k) + \varepsilon_{k+1}$. It will be treated in more detail in Section 2.4.

The notion of $L^4$–m–approximability implies that the process is stationary and ergodic and that it has finite fourth moments. The latter is in line with existing literature. We are not aware of any article that works with less than four moments. In contrast, for several consistency results, finite moments of all orders (or even bounded random variables) are assumed. Because our estimator in the succeeding text is a moment estimator, on the basis of second-order moments, one could be tempted to believe that some of our results may be deduced directly from the ergodic theorem under finite second moment assumptions. We will explain in the next section, after introducing the estimator, why this line of argumentation is not working.

Our weak dependence assumption implies that a possible non-zero mean of $X_k$ can be estimated consistently by the sample mean. Moreover, we have (Hörmann & Kokoszka, 2012)\[
\sqrt{n} \| \bar{X} - \mu \|_{H_1} = O_P(1).
\]

We conclude that the mean can be accurately removed in a preprocessing step and that $EX_k = 0$ is not a stringent assumption. Because by lemma 2.1 in Hörmann & Kokoszka (2010), $\{Y_k\}$ will also be $L^4$–m–approximable, the same argument justifies that we study a linear model without intercept.

### 2.3. The estimator

The PC-based estimator for $\Psi$ described in the succeeding text was first studied by Bosq (1991) and is based on a finite basis approximation. To achieve optimal approximation in finite dimension, one chooses eigenefunctions of the covariance operator $C = E[X_1 \otimes X_1]$ as a basis. Let $\Delta = E[X_1 \otimes Y_1]$. By Assumption (A), both $\Delta$ and $C$ are Hilbert–Schmidt operators. Let $(\lambda_i, v_i)_{i \geq 1}$ be the eigenvalues and corresponding eigenefunctions of the operator $C$, such that $\lambda_1 \geq \lambda_2 \geq \ldots$. The eigenfunctions are orthonormal and those belonging to a non-zero eigenvalue form an ONB of $\text{Im}(C)$, the closure of the image of $C$. Note that, with probability one, we have $X \in \text{Im}(C)$. Because $\text{Im}(C)$ is again a Hilbert space, we can assume that $H_1 = \text{Im}(C)$, that is, that the operator is of full rank. In this case, all eigenvalues are strictly positive. Using linearity of $\Psi$ and the requirement $X_k \perp \varepsilon_k$ from (A), we obtain

\[
\Delta(v_j) = E\langle X_1, v_j \rangle_{H_1} Y_1 = E\langle X_1, v_j \rangle_{H_1} \Psi(X_1) + E\langle X_1, v_j \rangle_{H_1} \varepsilon_1
\]

\[
= \Psi(E\langle X_1, v_j \rangle_{H_1} X_1) = \Psi(C(v_j)) = \lambda_j \Psi(v_j).
\]

Then, for any $x \in H_1$, the derived equation leads to the representation

\[
\Psi(x) = \Psi \left( \sum_{j=1}^{\infty} \langle v_j, x \rangle v_j \right) = \sum_{j=1}^{\infty} \frac{\Delta(v_j)}{\lambda_j} \langle v_j, x \rangle. \tag{2}
\]

Here we assume implicitly that $\dim(H_1) = \infty$. If $\dim(H_1) = M < \infty$, then (2) still holds with $\infty$ replaced by $M$. This case is well understood and will therefore be excluded.

Equation (2) gives a core idea for estimation of $\Psi$. We will estimate $\Delta, v_j$ and $\lambda_j$ from our sample $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and substitute the estimators into formula (2). The estimated eigenelements $\left(\hat{\lambda}_{j,n}, \hat{v}_{j,n}; 1 \leq j \leq n\right)$ will be obtained from the empirical covariance operator

\[
\hat{C}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \otimes X_k.
\]

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In a similar straightforward manner, we set

\[ \hat{\Delta}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \otimes Y_k. \]

For ease of notation, we will suppress in the sequel the dependence on the sample size \( n \) of these estimators.

Apparently, from the finite sample, we cannot estimate the entire sequence \( (\lambda_j, v_j) \), rather we have to work with a truncated version. This leads to

\[ \hat{\Psi}_K(x) = \sum_{j=1}^{K} \frac{\hat{\Delta}(\hat{v}_j)}{\hat{\lambda}_j} \{\hat{v}_j, x\}. \]  

(3)

where the choice of \( K = K_n \) is crucial. Because we want our estimator to be consistent, \( K_n \) has to grow with the sample size to infinity. On the other hand, we know that \( \lambda_j \to 0 \). Hence, it will be a delicate issue to control the behaviour of \( \frac{1}{\lambda_j} \). A small error in the estimation of \( \lambda_j \) can have an enormous impact on (3).

Define \( \Psi_K(x) = \sum_{j=1}^{K} \frac{\Delta(v_j)}{\lambda_j} \{v_j, x\} \). Via the ergodic theorem, one can show that the individual terms \( \hat{\lambda}_j, \hat{v}_j \) and \( \hat{\Delta} \) in (3) converge to their population counterparts. It follows that

\[ \| \Psi_K - \Psi_K \|_{L_{12}} \to 0 \text{ a.s., as long as } K \text{ is fixed.} \]

In fact, this holds true under finite second moments. However, as it is well known, the ergodic theorem does not assure rates of convergence. Even if the underlying random variables were bounded, convergence can be arbitrarily slow. Consequently, we cannot let \( K \) grow with the sample size in this approach. We need to impose further structure on the dynamics of the process and existence of higher order moments. Both are combined in the concept of \( L^4-m\) approximability.

In most existing papers, determination of \( K_n \) is related to the decay rate of \( (\lambda_j) \). For example, Cardot et al. (1999) assumed that \( n \lambda_n^4 \to \infty \) and \( n^2 \lambda_n^2 / \left( \sum_{j=1}^{K_n} \frac{1}{\alpha_j} \right)^2 \to \infty \), when

\[ \alpha_1 = \lambda_1 - \lambda_2 \quad \text{and} \quad \alpha_j = \min\{\lambda_{j-1} - \lambda_j, \lambda_j - \lambda_{j+1}\}, \quad j > 1. \]  

(4)

Similar requirements are used in Bosq (2000) (theorem 8.7) or Yao et al. (2005) (assumption (B.5)). Hall & Horowitz (2007) assumed in the scalar response model that \( \alpha_j \geq C^{-1} j^{-\alpha-1} \), \( \left| \Delta(v_j) \lambda_j^{-1} \right| \leq C j^{-\beta} \) for some \( \alpha > 1 \) and \( \frac{1}{2} \alpha + 1 < \beta \). Here \( C \) is a constant arising from the additional assumption \( E(X_1, v_j)^4 \leq C \lambda_j^2 \). They emphasize the importance of a sufficient separation of the eigenvalues for their result. Then, within this setup, optimal minimax bounds are proven to hold for \( K = n^{1/(\alpha+2\beta)} \). Of course, in practice, this choice of \( K \) is only possible under the unrealistic assumption that we know \( \alpha \) and \( \beta \). Cai & Zhou (2008) modified the approach by Hall & Horowitz (2007) by proposing an adaptive choice of \( K \) that is based on a block thresholding technique. They recover the optimal rates of Hall & Horowitz (2007) but need to impose further technical assumptions. Among others, the assumptions in Hall & Horowitz (2007) are strengthened to \( E\|X_k\|^p < \infty \) for all \( p > 0 \), \( j^{-\alpha} \ll \lambda_j \ll j^{-\alpha} \), and \( \alpha_j \gg j^{-\alpha-1} \). Here \( a_n \ll b_n \) means that \( \limsup n |a_n|/b_n | < \infty \). Rates of convergence are also obtained in Cardot & Johannes (2010). They propose a new class of estimators that are based on projecting on some fixed ONB instead on empirical eigenfunctions. Again, the accuracy of the estimator relies on a thresholding technique, and similar as to the aforecited papers, the very strong results are at the price of several technical constraints.
2.4. Consistency results

The papers cited in the previous paragraph focus on rates of consistency for the estimator \( \hat{\Psi}_K \). These important and interesting results require technical assumptions on the operator \( \Psi \) and the spectrum of \( C \). In practice, such technical conditions cannot be checked and may be violated. Furthermore, because we have no knowledge of \( \alpha_j \) and \( \lambda_j \), determination of \( K \) has to be performed heuristically. It then remains open if the widely used PC-based estimation methods stay consistent in the case where some of these conditions are violated. Our theorems in the succeeding text show that the answer to this question is affirmative, even if data are dependent. We propose a selection of \( K_n \), which is data driven and can thus be practically implemented. The \( K_n \) we use in the first result, theorem 1, is given as follows:

\[ (K): \quad \text{Let } m_n \to \infty \text{ such that } m_n^6 = o(n). \text{ Then we define } K_n = \min(B_n, E_n, m_n) \text{ where } B_n = \arg\max \left\{ j \geq 1 \mid \hat{\lambda}_j^{-1} \leq m_n \right\} \text{ and } E_n = \arg\max \left\{ k \geq 1 \mid \max_{1 \leq j \leq k} \hat{\alpha}_j^{-1} \leq m_n \right\}. \text{ Here } \hat{\lambda}_j \text{ and } \hat{\alpha}_j \text{ are the estimates for } \lambda_j \text{ and } \alpha_j \text{ (given in (4))}, \text{ respectively, obtained from } \hat{C}. \]

A discussion on the tuning parameter \( m_n \) is given at the end of this section. The choice of \( K_n \) is motivated by a ‘bias variance trade-off’ argument. If an eigenvalue is very small (in our case, \( \leq 1/m_n \)) it means that the direction it explains has only small influence on the representation of \( X_k \). Therefore, excluding it from the representation of \( \Psi \) will not cause a big bias, whereas it will considerably reduce the variance. It will be only included if the sample size is big enough, in which case we can hope for a reasonable accuracy of \( \hat{\lambda}_j \). In practice, it is recommended to replace \( \frac{1}{\hat{\lambda}_j} \) in the definition of \( B_n \) by \( \frac{\hat{\lambda}_j}{\hat{\lambda}_j} \) and \( \frac{1}{\hat{\alpha}_j} \) in the definition of \( E_n \) by \( \frac{\hat{\lambda}_j}{\hat{\alpha}_j} \) to adapt for scaling. For the asymptotics, such a modification has no influence.

**Theorem 1.** Consider the linear Hilbertian model (1) and assume that Assumptions (A) and (K) hold. Suppose further that the eigenvalues \( \{\lambda_j\} \) are mutually distinct and \( \Psi \) is a Hilbert–Schmidt operator. Then the estimator described in Section 2.3 is weakly consistent, that is,

\[ \| \hat{\Psi}_{K_n} - \Psi \|_{L_{12}} \to 0, \text{ if } n \to \infty. \]

It is not hard to see that consistent estimation of \( \Psi \) via the principal component approach requires compactness of the operator. As a simple example, suppose that \( \Psi \) is the identity operator, which is not Hilbert–Schmidt anymore. Then, for any ONB \( \{v_i\} \), we have \( \Psi = \sum_{i \geq 1} v_i \otimes v_i \). Even if from the finite sample, our estimators for \( v_1, \ldots, v_K \) would be perfect (\( v_i = \hat{v}_i \)), we have \( \| \Psi - \hat{\Psi}_K \|_{L_{12}} = 1 \) for any \( K \geq 1 \). This is easily seen by evaluating \( \Psi \) and \( \hat{\Psi}_K \) at \( v_{K+1} \).

In our next theorem, we show that if our target is prediction, then we can further simplify the assumptions. In this case, we will be satisfied if \( \| \Psi(X_n) - \hat{\Psi}(X_n) \|_{H_2} \) is small. For example, if \( (X_n, v) = 0 \) with probability one, then the direction \( v \) plays no role for describing \( X_n \) and a larger value of \( \| \Psi(v) - \hat{\Psi}(v) \|_{H_2} \) is not relevant.

**Theorem 2.** Let Assumption (A) hold and define the estimator \( \hat{\Psi}_{K_n} \) as in Section 2.3 with \( K_n = \arg\max \left\{ j \geq 1 \mid \hat{\lambda}_j / \hat{\lambda}_1 \leq m_n \right\} \), where \( m_n \to \infty \) and \( m_n = o(\sqrt{n}) \). Then

\[ \| \Psi(X_n) - \hat{\Psi}_{K_n}(X_n) \|_{H_2} \overset{P}{\to} 0. \]

**Remark 1.** For our proof, it will not be important to evaluate \( \Psi \) and \( \hat{\Psi} \) at \( X_n \). We could equally well use \( X_1 \), or \( X_{n+1} \), or some arbitrary variable \( X \overset{d}{=} X_1 \).
Theorem 2 should be compared with theorem 3 in Crambes & Mas (2013) where an asymptotic expansion of \(E\left\|\Psi(X_{n+1}) - \hat{\Psi}_k(X_{n+1})\right\|_{H_2}^2\) is obtained (for fixed \(k\)). Their result implies consistency but requires again assumptions on the decay rate of \(\{\lambda_i\}\), an operator \(\Psi\) that is Hilbert–Schmidt, and \(E\left\|X_k\right\|^p < \infty\) for all \(p > 0\). In our theorem, we need no assumptions on the eigenvalues anymore, not even that they are distinct.

In the last theorem, we saw that whenever \(m_n = o\left(\sqrt{n}\right)\) and \(m_n \to \infty\), convergence holds. This leaves open what is a good choice of the tuning parameter \(m\). Indeed, denoting \(\hat{\Psi}_K\), the practitioner can be sure that his or her approach leads to a consistent estimator under very general assumptions. In Section 3, we use for the simulations \(m\) such that \(K\) has the form \(n \to \infty\).

The practitioner can be sure that his or her approach leads to a consistent estimator under very general assumptions. In Section 3, we use for the simulations \(m_n = \sqrt{n}/\log n\). The performance of this estimator is in all tested setups comparable with CV.

To address the optimality issue from a theoretical point of view seems to be very difficult and depends on our final objective: is it prediction or estimation. In both cases, we believe that the importance of this question should not be overrated. Most applied researchers will use CV or some comparable method, which usually will give a \(K_{alt}\) that is presumably close to optimal. Hence, if we suppose that

\[
E\left\|\Psi(X_n) - \hat{\Psi}_K(X_n)\right\|_{H_2} \ll E\left\|\Psi(X_n) - \hat{\Psi}_K_n(X_n)\right\|_{H_2} \quad (n \to \infty),
\]

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2.5. Applications to functional time series

Functional time series analysis has seen an upsurge in FDA literature, in particular the forecasting in a functional setup (see, e.g. Hyndman & Shang (2009) or Sen & Klüppelberg (2010)). We sketch here two possible applications in this context.

**FAR(1).** Of particular importance in functional time series is the ARH(1) model of Bosq (2000). We show now that our framework covers this model. With i.i.d. innovations \(\delta_k \in L_2^H\), the process \(\{X_k\}\) defined via \(X_{k+1} = \Psi(X_k) + \delta_{k+1}\) is \(L_2^H\)-approximable if \(\Psi \in L(H, H)\) such that \(\|\Psi\|_{L(H, H)} < 1\), see Hörmann & Kokoszka (2010). The stationary solution for \(X_k\) has the form

\[X_k = \sum_{j \geq 0} \Psi^j(\delta_{k-j}).\]

Setting \(\varepsilon_k = \delta_{k+1} + Y_k = X_{k+1}\), we obtain the linear model (1). Independence of \(\{\delta_k\}\) implies that \(X_k \perp \varepsilon_k\), and hence, Assumption (A) holds. Bosq (2000) has obtained a (strongly) consistent estimator of \(\Psi\), if \(\Psi\) is Hilbert–Schmidt and again by imposing assumptions on the spectrum of \(C\).

In our approach, we do not even need that the innovations \(\{\delta_k\}\) are i.i.d. As long as we can assure that \(\{\delta_k\}\) and \(\{X_k\}\) are \(L^\infty\)-approximable, we only need that \(\{\delta_k\}\) is \(H\)-white noise. Indeed, denoting \(A^*\) the conjugate of operator \(A\), we have for any \(x \in H_1\) and \(y \in H_2\) that

\[
E\left(\langle X_k, x \rangle_{H_1}, \langle \varepsilon_k, y \rangle_{H_2}\right) = \sum_{j \geq 0} E\left(\langle \Psi^j(\delta_{k-j}), x \rangle_{H_1}, \langle \delta_{k+1}, y \rangle_{H_2}\right) = 0.
\]

This shows \(X_k \perp \varepsilon_k\), and Assumption (A) follows.
We obtain the following:

**Corollary 1.** Let \( \{X_n\}_{n \geq 1} \) be an ARH(1) process given by the recurrence equation \( X_{n+1} = \Psi(X_n) + \epsilon_{n+1} \). Assume \( \|\Psi\|_{L_{12}} < 1 \). If \( \{\epsilon_i\} \) is \( H \)--white noise and Assumption (A) holds, then for the estimator \( \hat{\Psi}_K \) given in theorem 2, we have \( \|\Psi(X_n) - \hat{\Psi}_K(X_n)\|_{H_2} \overset{P}{\to} 0 \). In particular, if \( \{\epsilon_i\} \) is i.i.d. in \( L^4_H \), Assumption (A) will hold.

**Corollary 2.** Let \( \{X_n\}_{n \geq 1} \) be an ARH(1) process given by the recurrence equation \( X_{n+1} = \Psi(X_n) + \epsilon_{n+1} \). Assume \( \|\Psi\|_{S_{12}} < 1 \) and that the covariance operator related to \( X_1 \) has distinct eigenvalues. If \( \{\epsilon_i\} \) is \( H \)--white noise and (A) and (K) hold, then the estimator \( \hat{\Psi}_K \) is consistent.

We remark that employing the usual state-space representation for FAR(p) processes, these results are easily generalized to higher order FAR models.

**FARCH(1).** Another possible application of our result refers to a recently introduced functional version of the celebrated ARCH model (Hörmann et al. (2012)), which plays a fundamental role in financial econometrics. It is given by the two equations

\[
y_k(t) = \epsilon_k(t)\sigma_k(t), \quad t \in [0, 1], \ k \in \mathbb{Z},
\]

and

\[
\sigma_k^2(t) = \delta(t) + \int_0^1 \beta(t, s)y_{k-1}^2(s)ds, \quad t \in [0, 1], \ k \in \mathbb{Z}.
\]

Without going into details, let us just mention that one can write the squared observations of a functional ARCH model as an autoregressive process with innovations \( v_k(t) = y_k^2(t) - \sigma_k^2(t) \). The new noise \( \{v_k\} \) is no longer independent, and hence, the results of Bosq (2000) are not applicable to prove consistency of the involved estimator for the operator \( \beta \). But it is shown in Hörmann et al. (2012) that the innovations of this new process form Hilbertian white noise and that the new process is \( L^4-m \)--approximable. This allows us to obtain a consistent estimator for \( \beta \).

### 3. Simulation study

We consider a linear model of the form \( Y_n = \Psi(X_n) + \epsilon_n \), where \( X_1, \epsilon_1, X_2, \epsilon_2, \ldots \) are mutually independent. We are testing the performance of the estimator in context of prediction; that is, we work under the setting of theorem 2. For the simulation study, we obviously have to work with finite dimensional spaces \( H_1 \) and \( H_2 \). However, because of the asymptotic nature of our results, we set the dimension relatively high and define \( H_1 = H_2 = \text{span}\{f_j : 0 \leq j \leq 34\} \), where \( f_0(t) = 1 \), \( f_{2k-1}(t) = \sin(2\pi kt) \) and \( f_{2k}(t) = \cos(2\pi kt) \) are the first 35 elements of a Fourier basis on \( [0, 1] \). We work with Gaussian curves \( X_i(t) \) by setting

\[
X_i(t) = \sum_{j=0}^{34} A_i^{(j)} f_{j-1}(t), \quad (5)
\]

where \( \left(A_i^{(0)}, A_i^{(1)}, \ldots, A_i^{(34)}\right)' \) are independent Gaussian random vectors with mean zero and covariance \( \Sigma \). This setup allows us to easily manipulate the eigenvalues \( \{\lambda_k\} \) of a covariance operator \( C_X = EX \otimes X \). Indeed, if we define \( \Sigma = \text{diag}(a_1, \ldots, a_{35}) \), where \( a_1 \geq a_2 \geq \cdots \geq a_{35} \), then \( \lambda_k = a_k \) and \( v_k = f_{k-1} \) is the corresponding eigenfunction. We test three sets of eigenvalues \( \{\lambda_k\}_{1 \leq k \leq 35} \):
• $\lambda_1 : \lambda_k = c_1 \rho^{k-1}$ with $\rho = 1/2$; [geometric decay],
• $\lambda_2 : \lambda_k = c_2 / k^2$ [fast polynomial decay],
• $\lambda_3 : \lambda_k = c_3 / k^{1.1}$ [slow polynomial decay].

To bring our data on the same scale and make results under different settings comparable, we set $c_1$, $c_2$ and $c_3$ such that $\sum_{k=1}^{35} \lambda_k = 1$. This implies $E\|X_i\|^2 = 1$ in all settings. The noise \{\$\varepsilon_k\$\} is also assumed to be of the form (5), but now with $E\|\varepsilon_i\|^2 = \sigma^2 \in \{0.25, 1, 2.25, 4\}$.

We test three operators, all of the form $\Psi(x) = \sum_{i=1}^{35} \sum_{j=1}^{35} \psi_{ij}(x, v_j) v_j$.

• $\Psi_1$: for $1 \leq i, j \leq 35$, we set $\psi_{ij} = 1$ and $\psi_{ij} = 0$ when $i \neq j$;
• $\Psi_2$: the coefficients $\psi_{ij}$ are generated as i.i.d. standard normal random variables;
• $\Psi_3$: for $1 \leq i, j \leq 35$, we set $\psi_{ij} = \frac{1}{y}$.

We standardize the operators such that the operator norm equals one. The operators $\Psi_2$ are generated once and then fixed for the entire simulation. We generate samples of size $n + 1 = 80 \times 4^\ell + 1, \ell = 0, \ldots, 4$. Estimation is based on the first $n$ observations. We run 200 simulations for each setup $(\lambda, \Psi, \sigma, n)$. As a performance measure for our procedure, the mean squared error (MSE) on the $(n + 1)$–st observation

$$MSE = \frac{1}{200} \sum_{k=1}^{200} \left\| \Psi \left( X_{n+1}^{(k)} \right) - \hat{\Psi} \left( X_{n+1}^{(k)} \right) \right\|_{H_2}^2,$$  \hspace{1cm} (6)

is used. Here $X_{n+1}^{(k)}$ is the $i$–th observation of the $k$–th simulation run.

Now we compute the median truncation level $K$ obtained from our data-driven procedure described in theorem 2 with $m_n = n^{1/2 \log n}$. We compare it with the median truncation level obtained by CV ($K^{CV}$) on the same data. To this end, we divide the sample into training and test sets in proportion $(n-n_{test}) : n_{test}$, where $n_{test} = \max(n/10, 100)$. The estimator is obtained from the training set for different truncation levels $k = 1, 2, \ldots, 35$. Then, from the test set, we determine $K^{CV} = \arg\min_{k \in \{1, \ldots, 35\}} \|Y_{n+1} - \hat{\Psi}_k(X_{n+1})\|_{H_2}^2$.

### Table 1. Truncation levels obtained by theorem 2 ($K$) and by CV ($K^{CV}$) and corresponding MSE. For each constellation, we present med($K$) of 200 runs

|        | $\psi_1$ | $\psi_2$ | $\psi_3$ |
|--------|----------|----------|----------|
| $\lambda_1$ |
| 80     | 1.10     | 1.06     | 0.64     |
| 320    | 0.48     | 0.32     | 0.21     |
| 1280   | 0.21     | 0.14     | 0.09     |
| 5120   | 0.08     | 0.07     | 0.05     |
| 20480  | 0.03     | 0.03     | 0.02     |
| $\lambda_2$ |
| 80     | 1.00     | 1.02     | 0.56     |
| 320    | 0.56     | 0.26     | 0.20     |
| 1280   | 0.26     | 0.14     | 0.07     |
| 5120   | 0.13     | 0.08     | 0.04     |
| 20480  | 0.06     | 0.04     | 0.02     |
| $\lambda_3$ |
| 80     | 1.60     | 1.78     | 1.71     |
| 320    | 0.85     | 0.84     | 0.72     |
| 1280   | 0.55     | 0.22     | 0.08     |
| 5120   | 0.25     | 0.16     | 0.04     |
| 20480  | 0.08     | 0.07     | 0.02     |

CV, cross-validation; MSE, mean squared error.
The MSE and the size of \( K \) and \( K^{CV} \) are shown for different constellations in Table 1. We display the results only for \( \sigma = 1 \). Not surprisingly, the bigger the variance of the noise, the bigger MSE, but otherwise our findings were the same across all constellations of \( \sigma \). The table shows that the choice of \( K \) proposed by our method results in an MSE, which is competitive with CV. We also see that an optimal choice of \( K \) cannot be solely based on the decay of the eigenvalues as it is the case in our approach. It clearly also depends on the unknown operator itself. Not surprisingly, the best results are obtained under settings \( \Lambda_1 \) (exponentially fast decay of eigenvalues) and \( \Psi_3 \) (which is the smoothest among the three operators).

4. Conclusion

Estimation of the regression operator in functional linear models has obtained much interest over the last years. Our objective in this paper was to show that one of the most widely applied estimators in this context remains consistent, even if several of the synthetic assumptions used in previous papers are removed. If our intention is prediction, we can further simplify the technical requirements. Our approach comes with a data-driven choice of the parameter that determines the dimension of the estimator. While our main intention is to show that this choice leads to a consistent estimator, we have seen in simulations that our method is performing remarkably well when compared with CV.

5. Proofs

Throughout this entire section, we assume the setup and notation of Section 2.2.

5.1. Proof of theorem 1

We work under Assumptions (A) and (K) and assume distinct eigenvalues of the covariance operator \( C \) and that \( \Psi \) is Hilbert–Schmidt. The first important lemma that we use in the proof of theorem 1 is an error bound for the estimators of the operators \( \Delta \) and \( C \). In the succeeding text, we extend results in Hörmann & Kokoszka (2010).

Lemma 1. There is a constant \( U \) depending only on the law of \( \{(X_k, Y_k)\} \) such that

\[
\max \left\{ E \left\| \Delta - \hat{\Delta}_n \right\|_{s_{12}}^2, E \left\| C - \hat{C}_n \right\|_{s_{11}}^2 \right\} < U.
\]

Proof of lemma 1. We only prove the bound for \( \Delta \), and the one for \( C \) is similar. First, note that by lemma 2.1 in Hörmann & Kokoszka (2010) and Assumption (A), \( \{Y_k\} \) is also \( L^{4–m} \)-approximable. Next, we observe that

\[
nE \left\| \Delta - \hat{\Delta}_n \right\|_{s_{12}}^2 = nE \left\| \frac{1}{n} \sum_{k=1}^{n} Z_k \right\|_{s_{12}}^2,
\]

where \( Z_k = X_k \otimes Y_k - \Delta \). Set \( Z^{(r)}_k = X^{(r)}_k \otimes Y^{(r)}_k - \Delta \). Using the stationarity of the sequence \( \{Z_k\} \), we obtain

\[
nE \left\| \frac{1}{n} \sum_{k=1}^{n} Z_k \right\|_{s_{12}}^2 = \sum_{|r| < n} \left( 1 - \frac{|r|}{n} \right) E(Z_0, Z_r)_{s_{12}},
\]

\[
\leq E\|Z_0\|_{s_{12}}^2 + 2 \sum_{r=1}^{\infty} |E(Z_0, Z_r)_{s_{12}}|.
\]
By the Cauchy–Schwarz inequality and the independence of $Z_r^{(r-1)}$ and $Z_0$, we derive the following:
\[
|E(Z_0, Z_r)_{S_{12}}| = |E(Z_0, Z_r - Z_r^{(r-1)})_{S_{12}}| \leq \left( E\|Z_0\|_{S_{12}}^2 \right)^{1/2} \left( E\|Z_r - Z_r^{(r-1)}\|_{S_{12}}^2 \right)^{1/2}.
\]
Using $\|X_0 \otimes Y_0\|_{S_{12}} = \|X_0\|_{H_1} \|Y_0\|_{H_2}$ and again the Cauchy–Schwarz inequality, we obtain
\[
E\|Z_0\|_{S_{12}}^2 = E\|X_0\|_{H_1}^2 \|Y_0\|_{H_2}^2 \leq v_{4,H_1}^2(X_0) v_{4,H_2}^2(Y_0) < \infty.
\]

To finish the proof, weshow that $\sum_{r=1}^{\infty} \left( E\|Z_r - Z_r^{(r-1)}\|_{S_{12}}^2 \right)^{1/2} < \infty$. By using an inequality of the type $|ab - cd|^2 \leq 2|a|^2|b| + 2|c|^2|d|$, we obtain
\[
E\|Z_r - Z_r^{(r-1)}\|_{S_{12}}^2 = E\|X_r \otimes Y_r - X_r^{(r-1)} \otimes Y_r^{(r-1)}\|_{S_{12}}^2 \leq 2E\|X_r\|_{H_1}^2 \|Y_r - Y_r^{(r-1)}\|_{H_2}^2 \leq 2v_{4,H_1}^2(X_r) v_{4,H_2}^2(Y_r - Y_r^{(r-1)}) + 2v_{4,H_2}^2(Y_r^{(r-1)}) v_{4,H_1}^2(X_r - X_r^{(r-1)}).
\]

Convergence of (7) follows now directly from $L^4$-approximability. \(\square\)

Application of this lemma leads also to bounds for estimators of eigenvalues and eigenfunctions of $C$ via the following two lemmas (Hörmann & Kokoszka, 2010).

**Lemma 2.** Suppose $\lambda_i, \hat{\lambda}_i$ are the eigenvalues of $C$ and $\hat{C}$, respectively, listed in decreasing order. Let $v_i, \hat{v}_i$ be the corresponding eigenvectors, and let $\hat{e}_i = (v_i, \hat{v}_i)$. Then for each $j \geq 1$,
\[
\hat{\alpha}_j \|v_j - \hat{e}_j \|_{H_1} \leq 2\sqrt{2} \|\hat{C} - C\|_{L^{11}},
\]
where $\hat{\alpha}_j = \min(\hat{\lambda}_{j-1} - \hat{\lambda}_j, \hat{\lambda}_j - \hat{\lambda}_{j+1})$ and $\hat{\alpha}_1 = \hat{\lambda}_2 - \hat{\lambda}_1$.

**Lemma 3.** Let $\lambda_j, \hat{\lambda}_j$ be defined as in lemma 2. Then for each $j \geq 1$,
\[
|\lambda_j - \hat{\lambda}_j| \leq \|C - \hat{C}\|_{L^{11}}.
\]

In the following calculations, we work with finite sums of the representation in (2):
\[
\Psi_K(x) = \frac{K}{\lambda_j} \sum_{j=1}^{K} \Delta(v_j) \langle v_j, x \rangle.
\]  
(8)

In order to prove the main result, we consider the term $\|\Psi - \hat{\Psi}_K\|_{L^{12}}$ and decompose it using the triangle inequality into four terms
\[
\|\Psi - \hat{\Psi}_K\|_{L^{12}} \leq \sum_{i=1}^{4} \|S_i(K)\|_{L^{12}},
\]
Lemma 4. Let $n$. Notice that because

\begin{equation}
\Delta \left( \hat{e}_j \hat{v}_j \right) = \hat{e}_j \hat{v}_j - \hat{e}_j \hat{v}_j \qquad \text{and} \qquad \Delta \left( \hat{e}_j \hat{v}_j \right) = \frac{\hat{e}_j \hat{v}_j}{\lambda_j} - \frac{\hat{e}_j \hat{v}_j}{\lambda_j},
\end{equation}

we can find $m$ such that $S_n(K) = \Psi - \Psi_K$. The following simple lemma gives convergence of $S_n(K_n)$, provided $K_n \xrightarrow{P} \infty$.

**Lemma 4.** Let $\{K_n, n \geq 1\}$ be a random sequence taking values in $\mathbb{N}$, such that $K_n \xrightarrow{P} \infty$ as $n \to \infty$. Then $\Psi_{K_n}$ defined by (8) converges to $\Psi$ in probability.

**Proof.** Notice that because $\| \Psi \|_{S_{12}}^2 = \sum_{j=1}^{\infty} \| \Psi(v_j) \|_{H_2}^2 < \infty$ for some orthonormal base $\{v_j\}$, we can find $m \in \mathbb{N}$ such that $\| \Psi - \Psi_m \|_{S_{12}}^2 = \sum_{j=m}^{\infty} \| \Psi(v_j) \|_{H_2}^2 \leq \varepsilon$, whenever $m > m_\varepsilon$. Hence,

\begin{align*}
P(\| \Psi - \Psi_{K_n} \|_{S_{12}}^2 > \varepsilon) &= \sum_{m=1}^{\infty} P\left(\| \Psi - \Psi_m \|_{S_{12}}^2 > \varepsilon \cap K_n = m\right) \\
&= P(K_n \leq m_\varepsilon).
\end{align*}

The next three lemmas deal with terms (9)–(11).

**Lemma 5.** Let $S_1(K)$ be defined by (9) and $U$ the constant derived in lemma 1. Then

\begin{equation}
P(\| S_1(K_n) \|_{L_{12}} > \varepsilon) \leq \frac{Um_n^2}{\varepsilon^{2/n}}.
\end{equation}

**Proof.** Note that for an orthonormal system $\{e_i \in H_1 | i \geq 1\}$ and any sequence $\{x_i \in H_2 | i \geq 1\}$, the following identity holds:

\begin{equation}
\left\| \sum_{i=1}^{K} e_i \otimes x_i \right\|_{S_{12}}^2 = \sum_{j=1}^{\infty} \left\| \sum_{i=1}^{K} (e_i, e_j) x_i \right\|_{H_2}^2 = \sum_{j=1}^{K} \| x_j \|_{H_2}^2.
\end{equation}

Using this and the fact that the Hilbert–Schmidt norm bounds the operator norm, we derive

\begin{align*}
P\left(\| S_1(K_n) \|_{L_{12}}^2 > \varepsilon\right) &\leq P\left(\left\| \sum_{j=1}^{K_n} \hat{e}_j \hat{v}_j \otimes \frac{1}{\lambda_j} \left( \hat{\Delta} - \Delta \right) \hat{e}_j \hat{v}_j \right\|_{S_{12}}^2 > \varepsilon\right) \\
&\leq P \left(\frac{1}{\lambda_{K_n}^2} \sum_{j=1}^{K_n} \left\| \left( \hat{\Delta} - \Delta \right) \hat{e}_j \hat{v}_j \right\|_{H_2}^2 > \varepsilon\right) \\
&\leq P \left(m_n^2 \left\| \hat{\Delta} - \Delta \right\|_{S_{12}}^2 > \varepsilon\right).
\end{align*}
By the Markov inequality,

$$P \left( \| S_1(K_n) \|_{\ell_1^2}^2 > \varepsilon \right) \leq E \left\| \hat{\Delta} - \Delta \right\|_{S_1^2}^2 \frac{m_n^2}{\varepsilon} \leq U \frac{m_n^2}{\varepsilon n},$$

where the last inequality is obtained from lemma 1.

\[\square\]

**Lemma 6.** Let $S_2(K)$ be defined by (10) and $U$ the constant from lemma 5. Then

$$P(\| S_2(K_n) \|_{\ell_1^2}^2 > \varepsilon) \leq 4 U \| \Delta \|_{S_1^2}^2 \frac{m_n^4}{b^2 n^2}.$$  

**Proof.** Assumption $K_n \leq B_n$ and identity (13) imply that

$$P(\| S_2(K_n) \|_{\ell_1^2}^2 > \varepsilon) = P \left( \max_{1 \leq j \leq K_n} \left| \hat{\lambda}_j - \hat{\lambda}_j \right| > \frac{b}{m_n^2 \| \Delta \|_{S_1^2}} \right),$$

For simplifying the notation let $b^2 = \frac{m_n^2 \| \Delta \|_{S_1^2}}{\varepsilon}$, then

$$P(\| S_2(K_n) \|_{\ell_1^2}^2 > \varepsilon) \leq P \left( \max_{1 \leq j \leq K_n} \left| \hat{\lambda}_j - \lambda_j \right| > \frac{b}{m_n^2 \| \Delta \|_{S_1^2}} \right).$$

The first summand vanishes because

$$P \left( \frac{1}{\lambda_{K_n}} \max_{1 \leq j \leq K_n} \left| \hat{\lambda}_j - \lambda_j \right| > b \cap \max_{1 \leq j \leq K_n} \left| \hat{\lambda}_j - \lambda_j \right| \leq \frac{b}{2m_n} \right),$$

and

$$\leq P \left( \frac{b}{2 \lambda_{K_n} m_n} > b \cap \left| \hat{\lambda}_{K_n} - \lambda_{K_n} \right| \leq \frac{b}{2m_n} \right),$$

$$\leq P \left( \frac{b}{2m_n} > \lambda_{K_n} \cap \left| \hat{\lambda}_{K_n} - \lambda_{K_n} \right| \leq \frac{\sqrt{\varepsilon}}{2m_n^2 \| \Delta \|_{S_1^2}} \right),$$

which is equal to 0 for $n$ large enough, because $\hat{\lambda}_{K_n} \geq \frac{1}{m_n}$ and the distance between $\lambda_{K_n}$ and $\hat{\lambda}_{K_n}$ shrinks faster than $\frac{1}{m_n}$. For the second term, we use lemma 3 and the Markov inequality:

$$P \left( \| S_2(K_n) \|_{\ell_1^2}^2 > \varepsilon \right) \leq P \left( \max_{1 \leq j \leq K_n} \left| \hat{\lambda}_j - \lambda_j \right| > \frac{b}{2m_n} \right),$$

$$\leq P \left( \left\| \hat{C} - C \right\|_{\ell_1^2} > \frac{b}{2m_n} \right),$$

$$\leq \frac{4m_n^2}{b^2} \left( \left\| \hat{C} - C \right\|_{\ell_1^2} \right)^2,$$

$$\leq 4 U \| \Delta \|_{S_1^2}^2 \frac{m_n^4}{b^2 n},$$

\[\square\]
Lemma 7. Let $S_3(K)$ be defined by (11) and $U$ be the constant defined in lemma 5, then

$$P(\|S_3(K_n)\|_{L_{12}} < \varepsilon) \leq U \left(128\|\Delta\|^2_{L_{12}} + 4\varepsilon^2\right) \frac{m_n^6}{\varepsilon^6 n}.$$  

Proof. By adding and subtracting the term $\hat{c}_j \hat{v}_j \Delta(v_j)$ and using the triangle inequality, we derive

$$P(\|S_3(K_n)\|_{L_{12}} > \varepsilon) = P \left(\left\|\sum_{j=1}^{K_n} \frac{1}{\lambda_j} (\hat{c}_j \hat{v}_j \otimes \Delta(\hat{c}_j \hat{v}_j) - v_j \otimes \Delta(v_j))\right\|_{L_{12}} > \varepsilon\right),$$

$$= P \left(\sum_{j=1}^{K_n} \frac{1}{\lambda_j} \left\|\hat{c}_j \hat{v}_j \otimes \Delta(\hat{c}_j \hat{v}_j - v_j) + (\hat{c}_j \hat{v}_j - v_j) \otimes \Delta(v_j)\right\|_{L_{12}} > \varepsilon\right),$$

$$\leq P \left(\sum_{j=1}^{K_n} \frac{1}{\lambda_j} \left(\|\Delta\|_{L_{12}} \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} + \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} \|\Delta\|_{L_{12}}\right) > \varepsilon\right).$$

Now we split $\Omega = A \cup A^c$ where $A = \left\{\frac{1}{\lambda K_n} > 2m_n\right\}$ and obtain

$$P(\|S_3(K_n)\|_{L_{12}} > \varepsilon) \leq P \left(\frac{1}{\lambda K_n} \sum_{j=1}^{K_n} \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} > \frac{\varepsilon}{2\|\Delta\|_{L_{12}}}\right),$$

$$\leq P \left(\sum_{j=1}^{K_n} \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} > \frac{\varepsilon}{4m_n\|\Delta\|_{L_{12}}}\right) + P \left(\frac{1}{\lambda K_n} > 2m_n\right).$$

(14)

For the first term in the inequality (14), by lemma 2, definition of $E_n$ and the Markov inequality, we obtain

$$P \left(\sum_{j=1}^{K_n} \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} > \frac{\varepsilon}{4m_n\|\Delta\|_{L_{12}}}\right) \leq P \left(m_n \max_{1 \leq j \leq E_n} \|\hat{c}_j \hat{v}_j - v_j\|_{H_1} > \frac{\varepsilon}{4m_n\|\Delta\|_{L_{12}}}\right),$$

$$\leq P \left(\max_{1 \leq j \leq E_n} \frac{2\sqrt{2}}{\alpha_j} \left\|\hat{C} - C\right\|_{L_{12}} > \frac{\varepsilon}{4m_n^2\|\Delta\|_{L_{12}}}\right),$$

$$\leq P \left(\left\|\hat{C} - C\right\|_{L_{12}} > \frac{\varepsilon}{8 \sqrt{2} m_n^3\|\Delta\|_{L_{12}}}\right),$$

$$\leq 128\|\Delta\|^2_{L_{12}} m_n^6 \frac{E \left\|\hat{C} - C\right\|^2_{L_{12}}}{\varepsilon^2},$$

$$\leq 128U\|\Delta\|^2_{L_{12}} \frac{m_n^6}{\varepsilon^6 n}.$$
Because \( \hat{\lambda}_{K_n} \geq \frac{1}{m_n} \), the second term in the inequality (14) is bounded by

\[
P\left( \lambda_{K_n} < \frac{1}{2m_n} \right) \leq P\left( \lambda_{K_n} < \frac{1}{2m_n} \right) \leq P\left( |\hat{\lambda}_{K_n} - \lambda_{K_n}| \leq \frac{1}{2m_n} \right) + P\left( |\hat{\lambda}_{K_n} - \lambda_{K_n}| > \frac{1}{2m_n} \right).
\]

By lemmas 4, 5, 6, 7 and assumption \( m_n \) large enough, we have, by combining lemmas 1 and 3, that

\[
P\left( \|\hat{\mathcal{C}} - C\|_{\mathcal{L}_{12}} > \frac{1}{2m_n} \right).
\]

Thus, we derive

\[
P(\|S_3(K_n)\|_{\mathcal{L}_{12}} > \varepsilon) \leq 128U\|\Delta\|_{\mathcal{L}_{12}}^2 \frac{m_n^6}{\varepsilon^6 m} + 4U \frac{m_n^2}{n} \leq U \left( 128\|\Delta\|_{\mathcal{L}_{12}}^2 + 4\varepsilon^2 \right) \frac{m_n^6}{\varepsilon^6 m}.
\]

Finally, we need a lemma that assures that \( K_n \) tends to infinity.

**Lemma 8.** Let \( K_n \) be defined as in (K), then \( K_n \overset{p}{\to} \infty \).

**Proof.** We have to show that \( P(\min\{B_n, E_n\} < p) \to 0 \) for any \( p \in \mathbb{N} \). Because \( \frac{1}{m_n} \to 0 \), for \( n \) large enough, we have, by combining lemmas 1 and 3, that

\[
P(B_n < p)^* = P\left( \hat{\lambda}_p < \frac{1}{m_n} \right) = P\left( \lambda_p - \hat{\lambda}_p > \lambda_p - \frac{1}{m_n} \right)
\]

\[
\leq P\left( |\hat{\lambda}_p - \lambda_p| > \lambda_p - \frac{1}{m_n} \right) \to 0.
\]

Now we are ready to prove the main result

**Proof of theorem 1.** First, by the triangle inequality, we obtain

\[
\|\Psi - \Psi_{K_n}\|_{\mathcal{L}_{12}} \leq \|\Psi - \hat{\Psi}_{K_n}\|_{\mathcal{L}_{12}} + \|\Psi - \Psi_{K_n}\|_{\mathcal{L}_{12}},
\]

\[
\leq \|S_1(K_n)\|_{\mathcal{L}_{12}} + \|S_2(K_n)\|_{\mathcal{L}_{12}} + \|S_3(K_n)\|_{\mathcal{L}_{12}} + \|\Psi - \Psi_{K_n}\|_{\mathcal{L}_{12}}.
\]

By lemmas 4, 5, 6, 7 and assumption \( m_n^6 = o(n) \), we finally obtain for large enough \( n \) that

\[
P\left( \|\Psi - \hat{\Psi}_{K_n}\|_{\mathcal{L}_{12}} > \varepsilon \right) \leq 4^3 U \frac{m_n^2}{\varepsilon^2 m} + 4^2 U \|\Delta\|_{\mathcal{L}_{12}}^2 \frac{m_n^4}{\varepsilon^4 m} + 4^2 U \left( 128\|\Delta\|_{\mathcal{L}_{12}}^2 + 4\varepsilon^2 \right) \frac{m_n^6}{\varepsilon^6 m}
\]

\[
+ P(\|\Psi - \Psi_{K_n}\|_{\mathcal{L}_{12}} > \varepsilon/4) \overset{n \to \infty}{\longrightarrow} 0.
\]

**5.2. Proof of theorem 2**

In order to simplify the notation, we will denote \( K = K_n \). This time as a starting point, we take a representation of \( \Psi \) in the basis \( \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m\} \). Let \( M_m = \overline{sp}\{v_1, v_2, \ldots, v_m\}, \hat{M}_m = \overline{sp}\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m\} \) where \( \overline{sp}\{x_i, i \in I\} \) denotes the closed span of the elements \( \{x_i, i \in I\} \). If \( \text{rank} \left( \hat{C} \right) = \ell \), then \( \{\hat{v}_i, i > \ell\} \) can be any ONB of \( \hat{M}_{\ell}^\perp \). We write \( P_A \) for the projection operator that maps on a closed linear space \( A \). As usual, \( A^\perp \) denotes the orthogonal complement of \( A \). Because for any \( m \geq 1 \), we can write \( x = P_{M_m}(x) + P_{M_m^\perp}(x) \), the linearity of \( \Psi \) and the projection operator give
\[ 
\Psi(x) = \Psi(P_{M_m}(x)) + \Psi(P_{\hat{M}_m}(x)), \\
= \sum_{j=1}^{m} \langle \hat{v}_j, x \rangle_{H_1} \Psi(\hat{v}_j) + \Psi(P_{\hat{M}_m}(x)).
\]

Now we evaluate \( \Psi \) in some \( \hat{v}_j \), which is not in the kernel of \( \hat{C} \). By definitions of \( \Psi, \hat{C} \) and again by linearity of the involved operators,

\[ 
\Psi(\hat{v}_j) = \frac{1}{\lambda_j} \Psi(\hat{C}(\hat{v}_j)), \]
\[ 
= \frac{1}{\lambda_j} \frac{1}{n} \sum_{i=1}^{n} \langle X_i, \hat{v}_j \rangle_{H_1} \Psi(X_i), \]
\[ 
= \frac{1}{\lambda_j} \frac{1}{n} \sum_{i=1}^{n} \langle X_i, \hat{v}_j \rangle_{H_1} (Y_i - \epsilon_i), \]
\[ 
= \frac{1}{\lambda_j} \left( \hat{\Lambda}(\hat{v}_j) + \hat{\Lambda}(\hat{v}_j) \right). 
\]

where \( \hat{\Lambda} = -\frac{1}{n} \sum_{i=1}^{n} X_i \otimes \epsilon_i \). Hence, if \( m \) is such that \( \hat{\lambda}_m > 0 \) (which will now be implicitly assumed in the sequel), \( \Psi \) can be expressed as

\[ 
\Psi(x) = \sum_{j=1}^{m} \langle \hat{v}_j, x \rangle_{H_1} \frac{1}{\lambda_j} \hat{\Lambda}(\hat{v}_j) + \sum_{j=1}^{m} \langle \hat{v}_j, x \rangle_{H_1} \frac{1}{\lambda_j} \Lambda(\hat{v}_j) + \Psi(P_{\hat{M}_m}(x)).
\]

Note that the first term on the right-hand side is just \( \hat{\Psi}_m(x) \). Therefore, for any \( x \), the distance between \( \Psi(x) \) and \( \hat{\Psi}_m(x) \) takes the following form:

\[ 
\left\| \Psi(x) - \hat{\Psi}_m(x) \right\|_{H_2} = \left\| \sum_{j=1}^{m} \langle \hat{v}_j, x \rangle_{H_1} \frac{1}{\lambda_j} \hat{\Lambda}(\hat{v}_j) + \Psi(P_{\hat{M}_m}(x)) \right\|_{H_2}. \tag{15}
\]

To assess (15), we need the following four lemmas.

**Lemma 9.** Let \((\lambda_i, \psi_i)_{i \geq 1}\) and \((\hat{\lambda}_i, \hat{\psi}_i)_{i \geq 1}\) be eigenvalues and eigenfunctions of \( C \) and \( \hat{C} \), respectively. Set \( j, m \in \mathbb{N} \) such that \( j \leq m \leq n \), then

\[ 
\left\| v_j - P_{\hat{M}_m}(v_j) \right\|_{H_1}^2 \leq 4 \frac{\left\| C - \hat{C} \right\|_{L_{11}}^2}{(\hat{\lambda}_{m+1} - \hat{\lambda}_j)^2}. 
\]

**Proof.** Note that by using Parseval’s identity, we obtain

\[ 
\left\| v_j - P_{\hat{M}_m}(v_j) \right\|_{H_1}^2 = \sum_{k=1}^{\infty} \left\| v_j - P_{\hat{M}_m}(v_j), \hat{v}_k \right\|_{H_1}^2 = \sum_{k>m} \left\| v_j, \hat{v}_k \right\|_{H_1}^2.
\]

Now
\[ 
(\hat{\lambda}_{m+1} - \hat{\lambda}_j)^2 \sum_{k>m} \left\| v_j, \hat{v}_k \right\|_{H_1}^2 \leq \sum_{k>m} \left( \hat{\lambda}_k \left\langle v_j, \hat{v}_k \right\rangle_{H_1} - \hat{\lambda}_j \left\langle v_j, \hat{v}_k \right\rangle_{H_1} \right)^2, 
\]
\[ 
= \sum_{k>m} \left( \left\langle v_j, \hat{C}(\hat{v}_k) \right\rangle_{H_1} - \hat{\lambda}_j \left\langle v_j, \hat{v}_k \right\rangle_{H_1} \right)^2.
\]

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Because \( \hat{C} \) is a self-adjoint operator, simple algebraic transformations yield

\[
(\hat{\lambda}_{m+1} - \hat{\lambda}_j)^2 \sum_{k > m} \langle v_j, \hat{v}_k \rangle^2_{H_1} \leq \sum_{k > m} \left( (\hat{C}(v_j), \hat{v}_k)_{H_1} - \hat{\lambda}_j \langle v_j, \hat{v}_k \rangle_{H_1} \right)^2,
\]

\[
= \sum_{k > m} \left( \left( (\hat{C} - C)(v_j), \hat{v}_k \right)_{H_1} - \left( \hat{\lambda}_j - \hat{\lambda}_j \right) \langle v_j, \hat{v}_k \rangle_{H_1} \right)^2,
\]

\[
\leq 2 \sum_{k > m} \left| \left( \hat{C} - C \right)(v_j), \hat{v}_k \right|_{H_1}^2 + 2 \sum_{k > m} \left( \hat{\lambda}_j - \hat{\lambda}_j \right) \langle v_j, \hat{v}_k \rangle_{H_1}^2.
\]

By Parseval’s inequality and lemma 3,\(^{46}\)

\[
(\hat{\lambda}_{m+1} - \hat{\lambda}_j)^2 \sum_{k > m} \langle v_j, \hat{v}_k \rangle^2_{H_1} \leq 2 \left\| (\hat{C} - C)(v_j) \right\|_{H_1}^2 + 2 \left| \hat{\lambda}_j - \hat{\lambda}_j \right|^2 \leq 4 \left\| C - C \right\|_{L_{11}}^2.
\]

Lemma 10. Let \( \Psi \) be defined as in lemma 2 and \( K = K_n \xrightarrow{P} \infty \). Then \( P_M(X_n) \xrightarrow{P} 0 \).

Proof. We write here and in the sequel \( X = X_n \). We first remark that for any \( \varepsilon > 0 \),

\[
P \left( \left\| P_M(X) \right\|_{H_2}^2 > \varepsilon \right) = P \left( \sum_{i = K+1}^{\infty} |\langle v_i, X \rangle_{H_1}|^2 > \varepsilon \right).
\]

Because \( \sum_{i = 1}^{\infty} |\langle v_i, X \rangle_{H_1}|^2 = \| X \|_{H_1}^2 \), there exists a random variable \( J_\varepsilon \in \mathbb{R} \) such that \( \sum_{i = J_\varepsilon}^{\infty} |\langle v_i, X \rangle_{H_1}|^2 < \varepsilon \). Because by assumption \( E \| X \|_{H_1}^2 < \infty \), we conclude that \( J_\varepsilon \) is bounded in probability. Hence, we obtain

\[
P \left( \left\| P_M(X) \right\|_{H_2}^2 > \varepsilon \right) \leq P \left( \sum_{i = K+1}^{\infty} |\langle v_i, X \rangle_{H_1}|^2 > \varepsilon \cap K > J_\varepsilon \right) + P(K \leq J_\varepsilon),
\]

where the last term converges to zero as \( n \to \infty \). \( \square \)

Lemma 11. Let \( L_n = \arg \max \left\{ r \leq K : \sum_{i = 1}^{r} (\hat{\lambda}_{K+1} - \hat{\lambda}_i)^2 \leq \xi_n \right\} \), where \( K = K_n \) is given as in theorem 2 and \( \xi_n \to \infty \). Then \( L_n \xrightarrow{P} \infty \).

Proof. Let \( r \in \mathbb{N} \) such that for all \( 1 \leq i \leq r \), we have \( \lambda_{r+1} \neq \hat{\lambda}_i \). Note that \( E \| X \|_{H_1}^2 < \infty \) implies \( \lambda_i \to 0 \), and because \( \lambda_i > 0 \), we can find infinitely many \( r \) satisfying this condition. We choose such \( r \) and obtain

\[
P(L_n < r) \leq P \left( \sum_{i = 1}^{r} \left( \frac{1}{(\hat{\lambda}_{K+1} - \hat{\lambda}_i)^2} \right) > \xi_n \cap K \geq r \right) + P(K < r).
\]
Lemma 8 implies that $P(K < r) \to 0$. The first term is bounded by $P\left(\sum_{i=1}^{r} \frac{1}{(\lambda_{r+1} - \lambda_i)^2} > \xi_n\right)$. Since $\lambda_i \xrightarrow{P} \lambda_i$ and $r$ is fixed while $\xi_n \to \infty$, it follows that $P(L_n < r) \to 0$ if $n \to \infty$. Because $r$ can be chosen arbitrarily large, the proof is finished.

**Lemma 12.** Let $\Psi$ be defined as in lemma 2, then $\|P_{M_K}(X) - P_{\hat{M}_K}(X)\|_{H_1} \xrightarrow{P} 0$.

**Proof.** Let us define two variables $X^{(1)} = \sum_{i=1}^{L} (X, v_i)_{H_1} v_i$, $X^{(2)} = \sum_{i=L+1}^{\infty} (X, v_i)_{H_1} v_i$ and $L$ as in lemma 11. Again for simplifying the notation, we will write $L$ instead of $L_n$. Because $X = X^{(1)} + X^{(2)}$, we derive

$$
\left\|P_{M_K}(X) - P_{\hat{M}_K}(X)\right\|_{H_1} \leq \left\|P_{M_K}\left(X^{(1)}\right) - P_{\hat{M}_K}\left(X^{(1)}\right)\right\|_{H_1} + \left\|P_{M_K}\left(X^{(2)}\right)\right\|_{H_1}.
$$

(16)

The last two terms are bounded by $2\|X^{(2)}\|_{H_1}$. For the first summand in (16), we obtain

$$
\left\|P_{M_K}\left(X^{(1)}\right) - P_{\hat{M}_K}\left(X^{(1)}\right)\right\|_{H_1} = \left\|\sum_{i=1}^{L} (X, v_i)_{H_1} \left( v_i - P_{\hat{M}_K}(v_i) \right) \right\|_{H_1}.
$$

Let us choose $\xi_n = o(n)$ in lemma 11. The triangle inequality, the Cauchy–Schwarz inequality, lemma 9 and the definition of $L$ entail

$$
P_{M_K}\left(X^{(1)}\right) - P_{\hat{M}_K}\left(X^{(1)}\right) \leq \sum_{i=1}^{L} \left\|(X, v_i)_{H_1}\right\| v_i - P_{\hat{M}_K}(v_i) \right\|_{H_1},
$$

$$
\leq \left(\sum_{i=1}^{L} \left\|(X, v_i)_{H_1}\right\|^2\right)^{1/2} \left(\sum_{i=1}^{L} \left\|v_i - P_{\hat{M}_K}(v_i)\right\|_{H_1}^2\right)^{1/2},
$$

$$
\leq \|X\|_{H_1} \left(\sum_{i=1}^{L} \left\|v_i - P_{\hat{M}_K}(v_i)\right\|_{H_1}^2\right)^{1/2},
$$

$$
\leq 2\|X\|_{H_1} C - \hat{C} \sqrt{\xi_n},
$$

This implies the inequality

$$
P_{M_K}(X) - P_{\hat{M}_K}(X) \leq 2\|X\|_{H_1} \left|C - \hat{C}\right| \sqrt{\xi_n} + 2\|X^{(2)}\|_{H_1}.
$$

(17)

Hence, by lemma 1, we have $2\|X\|_{H_1} \left|C - \hat{C}\right| \sqrt{\xi_n} = o_P(1)$. Furthermore, we have that $\|X^{(2)}\| = (\sum_{j > L} \|X, v_j\|^2)^{1/2} \xrightarrow{P} 0$. This follows from the proof of lemma 10.

**Lemma 13.** Let $\Psi$ be defined as in lemma 2, then $\left\|\Psi\left(P_{\hat{M}_K}(X)\right)\right\|_{H_2} \xrightarrow{P} 0$. 

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Proof. Some simple manipulations show
\[
\left\| \Psi \left( P_{M_{\hat{K}}}(X) \right) \right\|_{H_2} = \left\| \Psi \left( X - P_{M_K}(X) \right) \right\|_{H_2}.
\]
\[
= \left\| \Psi \left( P_{M_K}(X) + P_{M_{\hat{K}}}(X) - P_{M_K}(X) \right) \right\|_{H_2},
\]
\[
\leq \left\| \Psi \left( P_{M_K}(X) \right) \right\|_{H_2} + \left\| \Psi \left( P_{M_{\hat{K}}}(X) \right) \right\|_{H_2},
\]
\[
\leq \Psi \left\|_{L_{12}} \left( \left\| P_{M_K}(X) - P_{M_K}(X) \right\|_{H_1} + \left\| P_{M_{\hat{K}}}(X) \right\|_{H_1} \right) \right. \right. \right.
\]
Direct applications of lemmas 10 and 12 finish the proof.

Proof of theorem 2. Set
\[
\Theta_n(x) = \sum_{j=1}^{K_n} \frac{\hat{\lambda}_j \left( \tilde{v}_j, x \right)}{\hat{\lambda}_j} \left( \tilde{v}_j, x \right) \right]_{H_1},
\]
By the representation (15) and the triangle inequality,
\[
\left\| \Psi(X) - \tilde{\Psi}(X) \right\|_{H_2} \leq \left\| \Theta_n(X) \right\|_{H_2} + \left\| \Psi \left( P_{M_{\hat{K}_n}}(X) \right) \right\|_{H_2}.
\]
Lemma 13 shows that the second term tends to zero in probability.
If in lemma 1, we define \( \Psi \equiv 0 \), then \( \hat{\Lambda} = \hat{\Lambda} \), and by independence of \( \varepsilon_k \) and \( X_k \), we obtain \( \Lambda = 0 \). By the arguments of lemma 5, we infer \( P(\left\| \Theta_n \right\|_{L_{12}} > \varepsilon) \leq Um_n^2/\varepsilon^2n \), which implies that \( \left\| \Theta_n(X) \right\|_{H_2} \xrightarrow{P} 0 \).

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