A POISSON TRANSFORM ADAPTED TO THE RUMIN COMPLEX

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Abstract. Let $G$ be a semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and $P \subset G$ a parabolic subgroup. Following ideas of P.Y. Gaillard, one may use $G$-invariant differential forms on $G/K \times G/P$ to construct $G$-equivariant Poisson transforms mapping differential forms on $G/P$ to differential forms on $G/K$. Such invariant forms can be constructed using finite dimensional representation theory. In this general setting, we first prove that the transforms that always produce harmonic forms are exactly those that descend from the de Rham complex on $G/P$ to the associated Bernstein-Gelfand-Gelfand (or BGG) complex in a well defined sense.

The main part of the article is devoted to an explicit construction of such transforms with additional favorable properties in the case that $G = SU(n+1,1)$. Thus $G/P$ is $S^{2n+1}$ with its natural CR structure and the relevant BGG complex is the Rumin complex, while $G/K$ is complex hyperbolic space of complex dimension $n+1$. The construction is carried out both for complex and for real differential forms and the compatibility of the transforms with the natural operators that are available on their sources and targets are analyzed in detail.

1. Introduction

The hyperbolic spaces over $\mathbb{K} := \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ together with the boundary spheres at infinity provide the simplest examples of compactifications of Riemannian symmetric spaces of the non–compact type. The realizations of the spheres $S^n$, $S^{2n+1}$, and $S^{4n+3}$ as the boundary at infinity correspond to certain geometric structures on the spheres. To describe these explicitly, let us denote by $G$ the special unitary group of a non–degenerate Hermitian form on $\mathbb{K}^{n+2}$ of Lorentzian signature $(n+1,1)$. Thus $G/P$ is $S^{2n+1}$ with its natural CR structure and the relevant BGG complex is the Rumin complex, while $G/K$ is complex hyperbolic space of complex dimension $n+1$. The construction is carried out both for complex and for real differential forms and the compatibility of the transforms with the natural operators that are available on their sources and targets are analyzed in detail.

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\( \mathbb{K}^{n+2} \) is well known to be the unique non-trivial parabolic subgroup of \( G \), so \( G/P \) realizes the boundary sphere as a generalized flag variety of \( G \).

The spaces \( G/P \) are the homogeneous models of so-called parabolic geometries of type \( (G, P) \), which are the geometric structures alluded to above. According to the different choices of \( \mathbb{K} \), this gives rise to the locally flat conformal structure on \( S^n \), the spherical CR structure on \( S^{2n+1} \) and the locally flat quaternionic contact structure on \( S^{4n+3} \). These structures can be realized as “infinities” of the hyperbolic metric, which is a Riemannian metric, a Kähler metric, or a quaternion Kähler metric of constant curvature according to the choice of \( \mathbb{K} \). The construction of such geometries at infinity extends to more general Einstein metrics of the types mentioned above, see \cite{1}.

These boundary spheres can be used to construct eigenfunctions for the Laplace operator on the corresponding Riemannian symmetric space. This is done by the Poisson transform, which is an integral operator assigning to every continuous function on the boundary \( G/P \) its average over the action of \( K \), c.f. \cite{13}, II.3.4. It turns out that each eigenfunction of the Laplace operator is obtained in this way and that its asymptotic behaviour towards the boundary recollects the initial data. Due to its importance the Poisson transform was generalized to map between sections of vector bundles in \cite{22} for the case of symmetric spaces of real rank 1 and independently in \cite{24} and \cite{20} for arbitrary symmetric spaces of noncompact type.

A variant of the Poisson transform in the case of realy hyperbolic space \( H_{\mathbb{R}}^{n+1} \) was introduced by P.Y. Gaillard in \cite{9}. Gaillard’s construction does not only work on functions (or densities) but on general differential forms. The basic idea of this construction is to start with differential forms on the product \( H_{\mathbb{R}}^{n+1} \times S^n \), for which there is a natural notion of bidegree. Gaillard used geometric ideas going back to Thurston to construct a \( G \)-invariant differential form \( \pi_k \) of bidegree \( (k, n-k) \) on this product for each \( k = 0, \ldots, n \). Given a \( k \)-form \( \alpha \in \Omega^k(S^n) \), one can pull it back to the product, wedge it with \( \pi_k \) to obtain a form of bidegree \( (k, n) \) and then integrate over the fibers to obtain a \( k \)-form on \( H_{\mathbb{R}}^{n+1} \), which is then defined to be the Poisson transform of \( \alpha \).

A simple computation shows that \( G \)-invariance of \( \pi_k \) implies that the resulting Poisson transform defines a \( G \)-equivariant map \( \Omega^{k}(S^n) \rightarrow \Omega^{k}(H_{\mathbb{R}}^{n+1}) \). The advantage of this construction is that one can directly relate the transform to the exterior derivatives on both factors and to the codifferential and the Laplacian on \( \Omega^*(H_{\mathbb{R}}^{n+1}) \). In particular, the transform always produces co-closed, harmonic forms on \( H_{\mathbb{R}}^{n+1} \). Gaillard’s construction was partly generalized to the case of complex hyperbolic space in \cite{10}, but things get much more involved there and the results are much less satisfactory.

In joint research of the first and third author, it was observed that there is a purely algebraic approach to the construction of invariant forms on the product space for each choice of \( \mathbb{K} \). This is based on the rather simple observation that \( G \) acts transitively on the product \( G/K \times G/P \), so this can be realized as \( G/M \), where \( M = K \cap P \). Consequently, describing \( G \)-invariant forms on the product reduces to questions of finite dimensional representation theory of \( M \). This not
only refers to the determination of invariant forms but also to the study of the relation between the Poisson transforms determined by invariant forms on $G/M$ and natural operations on differential forms on $G/P$ and $G/K$, respectively. This approach was worked out in a general setting (involving arbitrary generalized flag varieties and allowing also forms with values in certain vector bundles) in the PhD thesis of the second author [13], see also [12] and [14].

Considering this general setting, it quickly turns out that the case of real hyperbolic space as treated by Gaillard is deceptively simple. In this case, the invariant forms used to define the Poisson transform are essentially unique up to scale. Already in slightly more complicated cases, there is a large supply of invariant forms and thus of possible Poisson transforms, and the problem rather is to “design” Poisson transforms with favorable properties. An important ingredient in that direction is that apart from the exterior derivative, there are additional invariant differential operators on differential forms on $G/P$. These can be used to pass from the de Rham complex to the so-called Bernstein–Gelfand–Gelfand complex (or BGG complex), which also computes the (twisted) de Rham cohomology. While the BGG complex involves operators of order bigger than one, it has the advantage that all the bundles in the complex are associated to irreducible representations of $P$ (and thus of $M$). In contrast, in the (twisted) de Rham complex one typically meets bundles that are induced by representations that are indecomposable but not irreducible.

The CR–sphere provides the simplest example in which there is such a BGG refinement of the standard real or complex de Rham complex. It turns out that in this case the refinement does not really depend on the CR structure but only on the underlying contact structure. A construction of this complex on arbitrary contact manifolds (without using representation theory or touching the question of invariance) was given by M. Rumin in [21] and therefore it is known as the Rumin complex.

The relation between the de Rham complex and the BGG complex is slightly intricate, since the latter is a sub–quotient of the former. There is a calculus that can be used to construct BGG complexes (and also their analogs for curved parabolic geometries, see [5]) which allows to identify conditions on a Poisson transform which ensures that, while still acting on differential forms, it descends to a transform defined on the BGG complex. It turns out that in the general setting, this property is closely related to the fact that the values of the transform are harmonic forms. The main purpose of this article is to explicitly construct such a Poisson transform between the Rumin complex on the CR sphere and harmonic forms on complex hyperbolic space and study some of its properties. The construction of this transform does depend on the CR structure on the sphere (and not only on its contact structure) and require a rather careful analysis of the invariant differential forms available in this setting.

Poisson transforms defined on the Rumin complex play an important role in a long-term project of the third author. This aims at using the first part of a BGG complex together with carefully chosen Poisson transforms defined on the middle degree space in order to get an index one complex whose class in Kasparov
KK-theory is the so-called $\gamma$ element. This is a crucial step towards the proof of the Baum-Connes conjecture with coefficients in the case of simple Lie groups of rank one. See [17] for the case of $SU(n, 1)$ and [16] in general. Apart from their intrinsic interest, some of the results in this article should also be viewed as providing a basis for carrying out this program in the case under consideration.

2. Poisson transforms on differential forms

The first part of this section describes a general scheme for constructing Poisson transforms. For a semisimple Lie group $G$, these transforms map differential forms on a generalized flag variety $G/P$ to differential forms on the quotient $G/K$ by the maximal compact subgroup, which is a Riemannian symmetric space of the non-compact type. These transforms are induced by $G$-invariant differential forms on the homogeneous space $G/M$, where $M := K \cap P$, that admit a description in terms of finite dimensional representation theory. In this general setting, we describe a characterization of those transforms whose images consist of harmonic forms on $G/K$, which provides a link to the machinery of BGG sequences on $G/P$.

In the later parts of the article, we specialize to the case that $G = SU(n + 1, 1)$ and $P$ is the unique parabolic subgroup of $G$. We give elementary explicit descriptions of the structures needed to study Poisson transforms in that case later.

2.1. The general setup and Poisson transforms. In this general part we keep things rather short and abstract, they will be made explicit in the special case that $G = SU(n + 1, 1)$ soon. The basic idea to define Poisson transforms via invariant forms on a product goes back to [9], the general version was studied in [13]. Consider a non-compact semisimple Lie group $G$ with finite center, let $P \subset G$ be a parabolic subgroup and let $K \subset G$ be the maximal compact subgroup. Thus $G/P$ can be identified with $K/M$, where $M := K \cap P$, which in particular shows that $G/P$ is compact. On the other hand, $G/K$ is a Riemannian symmetric space of the non-compact type, and transitivity of the $K$-action on $G/P$ shows that $G$ acts transitively on $G/K \times G/P$. Thus, the product can be identified with the homogeneous space $G/M$.

Let us denote the resulting projections from $G/M$ onto the two factors by $\pi_K$ and $\pi_P$, respectively. Correspondingly, the tangent bundle of $G/M$ decomposes as $T' \oplus T''$ where $T' = \pi_K^* T(G/K)$ and $T'' = \pi_P^* T(G/P)$, implying that there is a well defined notion of bidegree for differential forms on $G/M$. Explicitly, a $k$-form $\alpha$ on $G/M$ is of bidegree $(i, j)$ with $i + j = k$ if for entries, which are either from $T'$ or from $T''$, it vanishes unless there are exactly $i$ entries from $T'$ and $j$ entries from $T''$. We will indicate the resulting decomposition of forms as $\Omega^k(G/M) = \bigoplus_{i+j=k} \Omega^{(i,j)}(G/M)$.

In particular, for $\alpha \in \Omega^k(G/P)$, the pullback $\pi_P^* \alpha$ lies in $\Omega^{(0,k)}(G/M)$. Given a form $\varphi \in \Omega^{(\ell,N-k)}(G/M)$, where $N = \dim(G/P)$, we can form the wedge product $\varphi \wedge \pi_P^* \alpha \in \Omega^{(\ell,N)}(G/M)$. This form can be integrated over the fibers of the
projection $G/M \to G/K$ to define an $\ell$-form on $G/K$. Thus we obtain an integral operator

$$\Phi : \Omega^\ell(G/P) \to \Omega^\ell(G/K) \quad \alpha \mapsto \int_{G/P} \varphi \wedge \pi_p^* \alpha.$$  

It is easy to show that $\Phi$ is $G$–equivariant if and only if the form $\varphi$ is $G$–invariant.

**Definition 2.1.** Let $\varphi \in \Omega^{(\ell,N-k)}(G/M)$ be a $G$–invariant differential form. Then the corresponding $G$–equivariant operator $\Phi : \Omega^k(G/P) \to \Omega^\ell(G/K)$ from (2.1) is called a Poisson transform and $\varphi$ is called its Poisson kernel.

Therefore, a Poisson transform in the sense of this definition is characterized by its $G$-invariant Poisson kernel $\varphi$. In turn, by Theorem 1.4.4 of [5], $\varphi$ is fully determined by its value $\varphi(eM)$ at the origin $eM \in G/M$, which is an $M$–invariant element in the corresponding finite dimensional $M$–representation $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$.

The basic advantage of working with differential forms is that it makes several natural operations available, which of course include the exterior derivatives on naturally are $\Lambda^k$–modules.

In the theory of parabolic geometries, this is traditionally denoted by $\partial^*$ and called the Kostant codifferential and the Laplace Beltrami operator available. On the generalized flag variety $G/P$ there are some less well know natural operations that we will discuss next.

### 2.2. The codifferential and BGG sequences.

It is well known that $T(G/P)$ is the homogeneous vector bundle $G \times_P (\mathfrak{g}/\mathfrak{p})$. Now the Killing form induces a $P$–equivariant duality between $\mathfrak{g}/\mathfrak{p}$ and the nilradical $\mathfrak{p}_+$ of $\mathfrak{p}$. Thus, the bundles $\Lambda^k T^*(G/P)$ of differential forms are the homogeneous bundles associated to the representations $\Lambda^k \mathfrak{p}_+$ of $P$, which are the chain spaces of the Lie algebra $\mathfrak{p}_+$ with coefficients in the trivial representation.

On these chain spaces, there is a natural Lie algebra homology differential. In the theory of parabolic geometries, this is traditionally denoted by $\partial^*$ and called the Kostant codifferential and we will stick to this tradition. Explicitly, $\partial^* : \Lambda^k \mathfrak{p}_+ \to \Lambda^{k-1} \mathfrak{p}_+$ is given by

$$\partial^* (Z_1 \wedge \cdots \wedge Z_k) = \sum_{i<j} (-1)^{i+j}[Z_i, Z_j] \wedge Z_1 \wedge \cdots \wedge \hat{Z}_i \wedge \cdots \wedge \hat{Z}_j \wedge \cdots \wedge Z_k,$$

where the $Z_\ell$ are in $\mathfrak{p}_+$ and hats denote omission. From the explicit formula it is immediate that $\partial^*$ is $P$–equivariant, so its kernel and its image are $P$–invariant subspaces in $\Lambda^k \mathfrak{p}_+$. Moreover, the homology spaces $H_k(\mathfrak{p}_+) := \ker(\partial^*)/\text{im}(\partial^*)$ naturally are $P$–modules.

The $P$–homomorphisms $\partial^*$ induce $G$–equivariant bundle maps $\Lambda^k T^*(G/P) \to \Lambda^{k-1} T^*(G/P)$. We use the symbol $\partial^*$ also for these bundle maps and for the induced tensorial operators on differential forms. The kernels and images of these bundle maps induce $G$-invariant subbundles of each $\Lambda^k T^*(G/P)$. In particular, we get $\text{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*(G/P)$ and we denote by $\mathcal{H}_k$ their quotient bundle, which by construction is associated to the $P$-representation $H_k(\mathfrak{p}_+)$. 


Let \( \pi_H : \Gamma(\ker(\partial^*)) \to \Gamma(\mathcal{H}_k) \) be the tensorial projection induced by the quotient projection. The machinery of BGG sequences introduced in [6] and [3] is based on the construction of a natural differential operator \( L : \Gamma(\mathcal{H}_k) \to \Gamma(\ker(\partial^*)) \) which splits the projection \( \pi_H \). Viewed as an operator to \( \Gamma(\mathcal{H}_k) \to \Omega^k(G/P) \), \( L \) is characterized by this splitting property (i.e. \( \partial^* \circ L = 0 \) and \( \pi_H \circ L = \text{id} \)) and the single condition that \( \partial^* \circ d \circ L = 0 \), where \( d \) denotes the exterior derivative.

For our purpose, the nicest description uses the operator \( \Box^R := \partial^* \circ d + d \circ \partial^* \), which defines an endomorphism of \( \Omega^k(G/P) \) for each \( k \), see Section 3 of [7]. It turns out that \( \ker(\Box^R) \subset \Gamma(\ker(\partial^*)) \subset \Omega^k(G/P) \) for each \( k \) and that \( \pi_H \) restricts to a linear isomorphism from this kernel onto \( \Gamma(\mathcal{H}_k) \) whose inverse is precisely the splitting operator. Moreover, for each \( k \), the inverse of this isomorphism can also be realized by applying an operator that can be written as a universal polynomial in \( \Box^R \) to any section of \( \ker(\partial^*) \) representing the given section of \( \mathcal{H}_k \). Having \( L \), one defines the \( k \)th BGG operator \( D_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1}) \) as \( D_k := \pi_H \circ d \circ L \) and by construction this is a \( G \)-equivariant differential operator. From the fact that \( d^2 = 0 \), one easily concludes that these operators form a differential complex, i.e. satisfy \( D_{k+1} \circ D_k = 0 \), which is called the BGG complex (associated to the trivial representation).

### 2.3. The action of the Casimir element.

The operators \( \Box^R \) we have met in the BGG construction have immediate relevance for the study of \( G \)-equivariant maps defined on \( \Omega^*(G/P) \). To explain this, let us recall that the Casimir element of the semisimple Lie algebra \( \mathfrak{g} \) induces a differential operator on the space of sections of any homogeneous vector bundle \( E \to G/H \), where \( G \) is any Lie group with Lie algebra \( \mathfrak{g} \) and \( H \subset G \) is any closed subgroup. Such a vector bundle is induced by a (finite dimensional) representation \( \mathcal{V} \) of \( H \), in the sense that \( E = G \times_H \mathcal{V} \). There is a natural action of \( G \) on \( \Gamma(E) \) by linear maps, which in the equivalent picture of \( H \)-equivariant smooth functions \( G \to \mathcal{V} \) is given by \( (g \cdot f)(g') = f(g^{-1}g') \). This shows that there is an induced action of the Lie algebra \( \mathfrak{g} \), for which \( X \in \mathfrak{g} \) acts on \( f \) as differentiation by the right invariant vector field \( R_X \in \mathfrak{X}(G) \) generated by \( X \).

This action naturally extends to an action of the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \). In particular, the Casimir element \( C \) acts by a differential operator \( D_C : \Gamma(E) \to \Gamma(E) \), which is \( G \)-equivariant by invariance of \( C \), compare with Section 2.4 of [8]. Returning to our setting, we conclude in particular that any \( G \)-equivariant linear map \( \Phi : \Omega^k(G/P) \to \Omega^k(G/K) \) that satisfies a weak continuity assumption has to be compatible with the actions of the two operators \( D_C \). Since the precise nature of these continuity conditions is not important, we state the result under a weak assumption, namely that \( \Phi \) is bounded for the natural bornologies on the spaces in question, see the book [18]. This condition on a linear map ensures that it is smooth in the sense of mapping smooth curves to smooth curves.

**Proposition 2.2.** Let \( \Phi : \Omega^k(G/P) \to \Omega^k(G/K) \) be a \( G \)-equivariant bounded linear operator. Then the following are equivalent:

(i) For all \( \alpha \in \Omega^k(G/P) \) the differential form \( \Phi(\alpha) \) is harmonic.

(ii) For all \( \alpha \in \Omega^k(G/P) \) we have \( \Phi(\Box^R \alpha) = 0 \).
(iii) We have \( \Phi \circ \partial^* = 0 \) and \( \Phi \circ d \circ \partial^* = 0 \).

Proof. As we have note above, \( \Phi \) is smooth in the sense that it maps smooth curves to smooth curves. Applying this to the action of a one-parameter subgroup in \( G \) on a fixed form and differentiating, one concludes that \( \Phi \) is equivariant for the infinitesimal actions of \( g \). The definition of the Casimir then directly implies that \( \Phi \) intertwines the actions of the Casimir operator on the two spaces.

Since the canonical Riemannian metric on \( G/K \) is induced by the Killing form on \( g \), the Casimir operator on \( \Omega^*(G/K) \) is a multiple of the Laplace Beltrami operator \( \Delta \), see e.g. p. 385 in [19]. On the other hand, it was shown in Corollary 1 of [8] that \( D_C = 2\square^R \) on \( \Omega^*(G/P) \). This readily implies the equivalence of (i) and (ii), and the definition of \( \square^R \) shows that (iii) implies (ii).

Thus it remains to show that (ii) implies (iii). To do this, we observe that Theorem 5.2 of [3] or Theorem 3.9 of [7] shows that \( \square^R \) is invertible on \( \Gamma(\im(\partial^*)) \). Thus we can write

\[
\Phi \circ \partial^* = \Phi \circ \square^R \circ (\square^R)^{-1} \circ \partial^*
\]

and this vanishes by (ii). Knowing that \( \Phi \circ \partial^* = 0 \), the definition of \( \square^R \) readily implies that \( \Phi \circ d \circ \partial^* = \Phi \circ \square^R = 0 \). \( \square \)

From this we can prove that an intertwining operator with harmonic values automatically descends to the BGG complex:

**Corollary 2.3.** Let \( \Phi : \Omega^k(G/P) \to \Omega^\ell(G/K) \) be an bounded linear intertwining operator which satisfies the equivalent conditions from Proposition 2.2, and let \( \mathcal{H}_k \to G/P \) be the \( k \)th homology bundle.

Then \( \Phi \) descends to a well defined, \( G \)-equivariant map

\[
\Phi : \Gamma(\mathcal{H}_k) \to \Omega^\ell(G/K),
\]

whose image is contained in the space of harmonic differential forms on \( G/K \). Explicitly, for \( \sigma \in \Gamma(\mathcal{H}_k) \), \( \Phi(\sigma) \) can be computed as \( \Phi(\alpha) \) for any \( \alpha \in \Gamma(\ker(\partial^*)) \) such that \( \pi_H \circ \alpha = \sigma \). Moreover, for \( \tau \in \Gamma(\mathcal{H}_{k-1}) \), we get \( \Phi(D_{k-1}(\tau)) = \Phi(d\beta) \) for any \( \beta \) such that \( \pi_H \circ \beta = \tau \).

Proof. From Proposition 2.2, we know that \( \Phi \circ \partial^* = 0 \), so the restriction of \( \Phi \) to \( \Gamma(ker(\partial^*)) \) descends to an operator \( \Phi_\beta \), which has the first claimed property.

Take \( \tau \in \Gamma(\mathcal{H}_{k-1}) \) and consider \( dL(\tau) \), where \( L \) is the splitting operator. This is a section of \( \ker(\partial^*) \) and by definition \( \pi_H \circ dL(\tau) = D_{k-1}(\tau) \). Thus we see that \( \Phi(D_{k-1}(\tau)) = \Phi(dL(\tau)) \). For \( \beta \in \Gamma(ker(\partial^*)) \) such that \( \pi_H \circ \beta = \tau \), we get \( L(\tau) - \beta \in \Gamma(\im(\partial^*)) \) and in the proof of Proposition 2.2 we have noted that this implies that it lies in the image of \( \square^R \). By definition, \( d^2 = 0 \) implies that \( \square^R \) commutes with \( d \), so \( dL(\tau) - d\beta \) lies in the image of \( \square^R \), and thus \( \Phi(dL(\tau)) = \Phi(d\beta) \). \( \square \)

3. The case of complex hyperbolic space

We next describe the machinery discussed in Chapter 2 in an elementary and explicit way in the case that \( G = SU(n+1,1) \) and \( P \) is the unique parabolic subgroup of \( G \). Thus \( G/K \) is complex hyperbolic space and \( G/P \) can be identified with the boundary sphere at infinity, which inherits a natural CR structure.
3.1. **Complex hyperbolic space.** From this point on, we will restrict our attention to the case that $G = SU(n+1,1)$ and $P$ is the unique parabolic subgroup of $G$. Fixing a Lorentzian Hermitian form $h$ on $V := \mathbb{C}^{n+2}$, and realizing $G$ as $SU(h)$, we can realize $K$ as the stabilizer of a complex line $\ell_\perp \subset V$ on which $h$ is negative definite. Then $K$ preserves the orthocomplement $\ell_\perp^\perp$ and acts unitarily on both $\ell_\perp^\perp$ and on $\ell_\perp$, which shows that $K \cong SU(n+1) \times U(1))$. This readily implies that $g$ is isomorphic to the space $L(\ell_\perp, \ell_\perp^\perp)$ of linear maps from $\ell_\perp$ to $\ell_\perp^\perp$ endowed with the natural action of $K$. The space $G/K$ can then be identified with the space of all complex lines in $V$ on which $h$ is negative definite, so this is complex hyperbolic space of dimension $n + 1$.

The complex structure on $L(\ell_\perp, \ell_\perp^\perp)$ clearly is $K$–invariant and there is a positive definite, Hermitian inner product on this space defined by $(X,Y) \mapsto \text{tr}(X^* \circ Y)$, which is $K$–invariant, too. Hence the homogeneous space $G/K$ carries a $G$–invariant almost complex structure $J$ and a $G$–invariant Hermitian metric. It is well known that these data actually make $G/K$ into a complete Kähler manifold. In particular, we obtain the usual decomposition of complex valued differential forms into $(p, q)$–types, which we indicate by $\Omega^p(G/K, \mathbb{C}) = \oplus_{0 \leq p, q \leq n+1} \Omega^{p,q}(G/K)$.

Using the (real valued) $G$–invariant Riemannian metric $g$, we obtain the “musical” operator $^g : T(G/K) \to T^*(G/K)$ via $\xi^g(\eta) = g(\xi, \eta)$ for all $\xi, \eta \in T(G/K)$, and we denote the induced map on the level of sections by the same symbol. The Kähler form $\omega \in \Omega^2(G/K)$ associated to $g$ is the $G$–invariant differential form characterized by $\omega(\xi, \eta) = g(J\xi, \eta)$ for all vector fields $\xi, \eta$ on $G/K$. In particular, the exterior power $\frac{1}{(n+1)!}\omega^{n+1}$ is the volume form $\text{vol}$ on $G/K$, which by construction is also $G$–invariant. We will denote the complex extensions of the Kähler form and the volume form by the same symbols. Note that these lie in $\Omega^{1,1}(G/K)$ and $\Omega^{n+1,n+1}(G/K)$, respectively. The Hermitian extension of $g$ can be used to identify the holomorphic part $T^{1,0}(G/K)$ in the complexified tangent bundle of $G/K$ with $T_{0,1}(G/K)$ and then in turn induces a complex bilinear dual pairing $T^{1,0}_*G/K \times T_{0,1}G/K \to \mathbb{C}$ that extends to exterior powers in the usual way. Denoting this by $\langle \ , \ \rangle$, we get the Hodge–$*-$operator. This is the $G$-equivariant complex linear map $* : \Lambda^{p,q}T^*(G/K) \to \Lambda^{n-1-q,n+1-p}T^*(G/K)$ characterized by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{ vol}$$

for all $\alpha \in \Lambda^{p,q}T^*(G/K)$, $\beta \in \Lambda^{p,q}T^*(G/K)$. We also denote the induced operator on forms by $*$. Denoting by $d$ the exterior derivative on complex valued forms, we then get the codifferential $\delta := - * d*$ and the Laplace-Beltrami operator $\Delta := d\delta + \delta d$.

The decomposition of the complexified tangent bundle into holomorphic and antiholomorphic vectors induces a splitting of the exterior derivative on $G/K$ into the sum $d = \partial + \overline{\partial}$, where the first operator maps a form of type $(p, q)$ to a form of type $(p+1, q)$ and the second is defined by $\overline{\partial}(\alpha) := (\overline{\partial\alpha})$. We say that a differential form $\alpha \in \Omega^k(G/K, \mathbb{C})$ is holomorphic (respectively, antiholomorphic) if $\overline{\partial}\alpha = 0$ (respectively $\partial\alpha = 0$). Similarly, we can decompose the codifferential $\delta = \partial^* + \overline{\partial}^*$ with $\partial^* = - * \overline{\partial}*$ and $\overline{\partial}^* = - * \partial*$. 
Finally, the wedge product with the Kähler form $\omega \in \Omega^{1,1}(G/K)$ defines the Lefschetz map $L : \Omega^{p,q}(G/K) \to \Omega^{p+1,q+1}(G/K)$, which is $G$-equivariant by construction. Its adjoint with respect to the Riemannian metric $g$ is the $G$-equivariant map $L^* : \Omega^{p,q}(G/K) \to \Omega^{p-1,q-1}(G/K)$, which can be computed as $L^* = -\ast \circ L \circ \ast$. We say that $\alpha \in \Omega^{p,q}(G/K)$ is primitive iff $L^* \alpha = 0$ and coprimitive iff $L \alpha = 0$. It is well known (c.f. Theorem 3.11 (b) in [23]) that the degree of a primitive differential form is at most half of the real dimension of $G/K$, while coprimitive forms exist only above that degree.

3.2. The CR sphere. In the notation introduced above, we define $P \subset G$ to be the stabilizer of a fixed complex line $\ell_0 \subset V$, which is isotropic for $h$. It is well known that $P$ is a minimal parabolic subgroup of $G$ and the space $G/P$ of complex isotropic lines in $V$ is diffeomorphic to the sphere $S^{2n+1}$. Such a diffeomorphism can be obtained by fixing a vector $v_- \in \ell_-$ such that $h(v_-, v_-) = -1$ and then sending each unit vector $v \in \ell_\perp$ to the line spanned by $v + v_-$, which is visibly isotropic. This also shows how $G/P$ can be viewed as the boundary at infinity of $G/K$, since mapping $v$ to the complex line spanned by $v + v_-$ also restricts to a diffeomorphism from the open unit ball in $\ell_\perp$ onto the space of negative lines in $V$, which can be identified with $G/K$.

Since $\ell_0$ is isotropic, the orthogonal space $\ell_0^\perp$ is a complex hyperplane in $V$ which contains $\ell_0$, so we can view this as defining a filtration $\ell_0 \subset \ell_1 \subset V$ of $V$ by complex subspaces. Let us write this filtration as $V^j = \{0\}$ for $j > 1$ and $V^0 = V$ for $j < -1$. Then we get an induced filtration of the Lie algebra $\mathfrak{g}$ of $G$ by defining $\mathfrak{g}^i = \{X \in \mathfrak{g} : \forall j : X \cdot V^j \subset V^{i+j}\}$. By definition, this filtration is compatible with the Lie bracket in the sense that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ for all $i, j$.

Since $P \subset G$ can be characterized as the stabilizer of the filtration of $V$, we see that $\mathfrak{g}^0 = \mathfrak{p}$ and that each $\mathfrak{g}^i$ is a $P$–invariant subspace in $\mathfrak{g}$. From the definition, it is evident that $\mathfrak{g}^i = \mathfrak{g}$ for $i \leq -2$ and $\mathfrak{g}^i = 0$ for $i > 2$, but for indices between $-2$ and $2$, we get a proper filtration. In particular, $\mathfrak{g}^{-1}/\mathfrak{p}$ is a $P$–invariant subspace in $\mathfrak{g}/\mathfrak{p}$, which is easily seen to be isomorphic to $L(\ell_0, \ell_0^\perp/\ell_0) \cong \mathbb{C}^n$. On the other hand, $\mathfrak{g}/\mathfrak{g}^{-1}$ has real dimension $1$ and the Lie bracket on $\mathfrak{g}$ induces a bilinear, skew symmetric map $\mathfrak{g}^{-1}/\mathfrak{p} \times \mathfrak{g}^{-1}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{g}^{-1}$. This is easily seen to be the imaginary part of a positive definite Hermitian form, compare with Section 4.2.4 of [5].

Since $T(G/P) = G \times_P (\mathfrak{g}/\mathfrak{p})$, we conclude that $G \times_P (\mathfrak{g}^{-1}/\mathfrak{p})$ defines a $G$–invariant corank-one subbundle $H \subset T(G/P)$ which carries a $G$–invariant complex structure. The considerations about the Lie bracket show that this is a contact structure, which makes $G/P$ into a strictly pseudoconvex partially integrable almost CR structure of hypersurface type. It is easy to see that this is indeed the standard (spherical) CR structure on $S^{2n+1}$ coming from the realization as the unit sphere in $\mathbb{C}^{n+1}$.

It is well known that the filtration on $\mathfrak{g}$ is actually induced by a grading, which is not $P$–invariant, however. This can either be realized by the choice of a complex isotropic line $\ell_0 \subset V$, which is transverse to $\ell_0^\perp$. Calling this line $V_{-1}$ and putting $V_0 := \ell_0^\perp \cap \ell_0$ and $V_1 = V^1$, we get $V = V_{-1} \oplus V_0 \oplus V_1$ such that $V^i = \oplus_{j \geq i} V_j$ for
all \(i\). Similarly as above, this induces a grading \(\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2\) compatible with the Lie bracket that induces the given filtration in the sense that \(\mathfrak{g}' = \oplus_{j \geq 0} \mathfrak{g}_j\).

Alternatively, the grading can be obtained by choosing a Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{g}\), which is contained in \(\mathfrak{g}^0\). Having made such a choice, \(\mathfrak{g}^2 := \mathfrak{g}_2\) becomes the highest root space, and one defines \(\mathfrak{g}_{-2}\) to be the lowest root space. Then there is a unique element \(E \in [\mathfrak{g}_{-2}, \mathfrak{g}_2] \subset \mathfrak{h}\), which fits into a standard \(\mathfrak{sl}_2\)–triple, and the grading of \(\mathfrak{g}\) is the decomposition into eigenspaces for \(\text{ad}(E)\). A crucial fact for our purposes is that the grading of \(\mathfrak{g}\) is invariant under the adjoint action of \(M := K \cap P\).

3.3. The Rumin complex on the CR–sphere. In the special case we consider, the BGG complex as discussed in 2.2 turns out to be a complex that is naturally defined on any contact manifold. This was first constructed in this general setting by M. Rumin, whence it is called the Rumin complex. Let us briefly discuss this direct construction and its relation to the general BGG construction. The construction is completely parallel for real and complex valued forms, and we do not distinguish between the two cases here.

Let \(H \subset T(G/P)\) be the contact subbundle from above and define \(Q := T(G/P)/H\). Then the short exact sequence \(0 \to H \to T(G/P) \to Q \to 0\) of homogeneous vector bundles dualizes to a short exact sequence \(0 \to Q^* \to T^*(G/P) \to H^* \to 0\). Since \(Q^*\) has rank 1, there is an induced short exact sequence for the exterior powers of order \(k = 1, \ldots, 2n\), which has the form

\[
0 \to \Lambda^{k-1}H^* \otimes Q^* \to \Lambda^k T^*(G/P) \to \Lambda^k H^* \to 0.
\]

The fact that the Lie bracket induces a tensorial map \(H \times H \to Q\) has a counterpart in the dual picture of the exterior derivative. Take a \(k\)–form, which is a section of the subbundle \(\Lambda^{k-1} H^* \otimes Q^*\), apply the exterior derivative and project to a section of \(\Lambda^k H^*\) (i.e. restrict the form to entries from \(H\)). The result of this operation is easily seen to be linear over smooth functions, thus defining a natural vector bundle map \(\Lambda^{k-1} H^* \otimes Q^* \to \Lambda^{k+1} H^*\), which is easily seen to be injective for \(k \leq n\) and surjective for \(k \geq n\).

This basically shows that it should be possible to pass to a “part” of the de Rham complex without changing the cohomology. Indeed, it should be possible to “leave out” a complement to the kernel of this bundle map in \(\Lambda^{k-1} H^* \otimes Q^*\) as well as its image without changing the cohomology, since these are just mapped isomorphically to each other by the exterior derivative. In M. Rumin’s original construction [21], naturality was not an issue and he proceeded by choosing splittings of the sequences (3.1) and then factoring by the irrelevant parts. With a bit more effort, one can use the exact sequences (3.1) and spectral sequence arguments (which can be made explicit as diagram chases in this simple case) to obtain a construction of a complex which is manifestly invariant under contactomorphisms, see [2] or [4].

To get to the setting of Section 2.2 we observe that the nilradical \(\mathfrak{p}_+\) of \(\mathfrak{p}\) in this simple case is the filtration component \(\mathfrak{g}^1\). (The duality between \(\mathfrak{g}/\mathfrak{p}\) and \(\mathfrak{g}^1\) easily follows from compatibility of the Killing form \(B\) with the filtration and the fact that \(\mathfrak{p} = \mathfrak{g}^0\).) As discussed in Section 2.2 we have the homology space \(H_k(\mathfrak{p}_+, \mathbb{K})\).
for $K = \mathbb{R}$ or $\mathbb{C}$ and the corresponding bundles $\mathcal{H}_k$. We can next describe these homology bundles explicitly.

**Proposition 3.1.** In terms of the exact sequences \((3.1)\), the homology bundle $\mathcal{H}_k$ satisfy the following. For $k \leq n$, $\mathcal{H}_k$ is a subbundle of $\Lambda^k H^*$, while for $k \geq n + 1$, $\mathcal{H}_k$ is a quotient bundle of $\Lambda^{k-1} H^* \otimes Q^*$.

*Proof.* From the definition of $\partial^*$ from \((2.2)\) and the simple structure of $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ in our case, we see that the bundle map $\partial^* : \Lambda^k T^*(G/P) \to \Lambda^{k-1} T^*(G/P)$ vanishes on the subbundle $\Lambda^{k-1} H^* \otimes Q^*$ and has values in $\Lambda^{k-2} H^* \otimes Q^*$. This it actually defines a bundle map $\partial^* : \Lambda^k H^* \to \Lambda^{k-2} H^* \otimes Q^*$. It is a well known result that this map is surjective for all $k \leq n + 1$ and injective for $k \geq n + 1$.

Thus we see that for $k \leq n$, $\mathcal{H}_k$ simply is $\ker(\partial^*) \subset \Lambda^k H^*$. For $k \geq n + 1$, the kernel of $\partial^*$ equals $\Lambda^{k-1} H^* \otimes Q^*$, so in these cases $\mathcal{H}_k$ is the quotient of $\Lambda^{k-1} H^* \otimes Q^*$ by $\mathrm{im}(\partial^*)$, which completes the proof. \(\square\)

As discussed in Section \(2.2\), we then obtain the BGG operators $D_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$, which are $G$-equivariant differential operators that form the BGG complex. It was shown in [2] that this complex coincides with the Rumin complex on the CR sphere.

### 3.4. Poisson transforms and natural operations.

We are now ready to invoke the machinery discussed in Section \(2.1\) in our special case. As discussed there, $G$-invariant forms on $G/M$ or, equivalently, $M$-invariant elements in $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ give rise to Poisson transforms mapping differential forms on $G/P$ to differential forms on $G/K$. We next describe the composition of such transforms with natural operations in terms of operations on the Poisson kernels.

First, observe that the decomposition of forms on $G/M$ according to bidegree induces a splitting of the exterior derivative as $d = d_K + d_P$, where $d_K$ and $d_P$ map forms of bidegree $(i, j)$ to forms of bidegree $(i + 1, j)$ and $(i, j + 1)$, respectively.

We call these operators, which are $G$-equivariant by construction, the $K$-derivative and the $P$-derivative. Since $d^2 = 0$, we immediately conclude that $d_K^2 = 0$, $d_P^2 = 0$ and $d_K d_P = -d_P d_K$. In view of the complex structure on $G/K$, we get an obvious splitting $d_K = \partial_K \oplus \overline{\partial}_K$.

Next, the Hodge–star operator on $G/K$ is induced by $K$–equivariant isomorphisms $\Lambda^k(\mathfrak{g}/\mathfrak{t})^* \to \Lambda^{2n+2-k}(\mathfrak{g}/\mathfrak{t})^*$. Of course, these isomorphisms are $M$–equivariant and tensorizing with appropriate identity maps and passing to the induced tensorial operator, we obtain tensorial maps

$$
*_{K} : \Omega_{(i,j)}(G/M) \to \Omega_{(2n+2-i,j)}(G/M)
$$

for all $i$ and $j$, which we call the $K$–Hodge–star. Observe that since $G/K$ has even dimension, the inverse of $*_{K}$ is $-*_{K}$. Thus we define the $K$–codifferential and the $K$–Laplacian on $\Omega^*(G/M)$ by $\delta_{K} := -*_{K} d_{K} *_{K}$ and $\Delta_{K} := d_{K} \delta_{K} + \delta_{K} d_{K}$, respectively.

Let $\omega_K \in \Omega_{(2,0)}(G/M)$ be the pullback of the Kähler form of $G/K$, which of course is $G$–invariant. Using this, we define the $K$–Lefschetz map

$$
\mathcal{L}_{K} : \Omega_{(i,j)}(G/M) \to \Omega_{(i+2,j)}(G/M)
$$

for all $i$ and $j$.
as the wedge product with $\omega_K$ and we consider its adjoint $L_K^* := - *_K \circ L_K \circ *_K$, which maps forms of bidegree $(i,j)$ to forms of bidegree $(i-2,j)$.

Finally, the Kostant codifferential $\partial^* : \Lambda^j(\mathfrak{g}/\mathfrak{p})^* \to \Lambda^{j-1}(\mathfrak{g}/\mathfrak{p})^*$ is $P$-equivariant, and thus as above, this gives rise to a tensorial operation

$$\partial^*_p : \Omega^{(i,j)}(G/M) \to \Omega^{(i,j-1)}(G/M),$$

which we call the $P$-codifferential. Here we fix the convention that for $\alpha \in \Omega^{(i,0)}(G/M)$ and $\beta \in \Omega^{(0,j)}(G/M)$, we put $\partial^*_p(\alpha \wedge \beta) = (-1)^i \alpha \wedge (\partial^*_p \beta)$, which by linearity defines the action on all of $\Omega^{(i,j)}(G/M)$.

To analyze the relation of the Kostant codifferential with Poisson transforms, we have to establish its compatibility with the wedge product on $\Lambda^k \mathfrak{p}_+$.

**Lemma 3.2.** For each $k = 1, \ldots, 2n$, $\alpha \in \Lambda^k \mathfrak{p}_+$ and $\beta \in \Lambda^{2n+2-k} \mathfrak{p}_+$ we get $(\partial^* \alpha) \wedge \beta = (-1)^k \alpha \wedge \partial^* \beta$.

**Proof.** From the grading property it readily follows that the Killing form $B$ vanishes on $\mathfrak{g}_i \times \mathfrak{g}_j$ unless $i + j = 0$, so $B$ has to restrict to a non-degenerate pairing on $\mathfrak{g}_i \times \mathfrak{g}_{-i}$ for each $i = 0, 1, 2$. Choose elements $\nu_\pm \in \mathfrak{g}_{\pm 2}$ such that $B(\nu_+, \nu_-) = 1$ as well as bases $\{\xi_+\}$ for $\mathfrak{g}_{-1}$ and $\{\eta_+\}$ for $\mathfrak{g}_1$ which are dual with respect to $B$. Then we claim that

$$\partial^* \alpha = \frac{1}{2} \sum_s \nu_+ \wedge (i_{[\eta_+, \nu_-]} i_{\xi_+} \alpha).$$

Here we view elements of $\Lambda^* \mathfrak{p}_+$ as multilinear maps on $\mathfrak{g}/\mathfrak{p}$ and for an element $X \in \mathfrak{g}_{-1}$ we denote by $i_X$ the usual insertion operator for the element $X + P \in \mathfrak{g}/\mathfrak{p}$.

By linearity of $\partial^*$, it suffices to prove (3.2) for decomposable elements $\alpha \in \Lambda^k \mathfrak{p}_+$, so we can take $\alpha = Z_1 \wedge \cdots \wedge Z_k$. Now $\mathfrak{g}_2$ is one-dimensional and the bracket on $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ has values in $\mathfrak{g}_2$ and vanishes if one of its entries lies in $\mathfrak{g}_2$. Using the definition of $\partial^*$, we conclude that $\partial^* \alpha = 0$ if one of the $Z_i$ lies in $\mathfrak{g}_2$. But since both $[\eta_+, \nu_-]$ and $\xi_+$ are in $\mathfrak{g}_{-1}$, the same is true for the right hand side of (3.2).

Thus we may restrict to the case that all $Z_i$ are in $\mathfrak{g}_1$.

Now we can write $[Z_i, Z_j] = B([Z_i, Z_j], \nu_-) \nu_+ + \nu_+ \wedge \nu_+ + \nu_+ \wedge \nu_+ + \nu_+ \wedge \nu_+ + \nu_+ \wedge \nu_+ + \nu_+ \wedge \nu_+ + \nu_+ \wedge \nu_+$ and invariance of $B$ shows that the numerical factor can be written as $B(Z_i, [Z_j, \nu_-])$. Now using the invariance of $B$ once more, we can in turn express $[Z_j, \nu_-] \in \mathfrak{g}_{-1}$ as $\sum_s -B(Z_j, [\eta_s, \nu_-]) \xi_s$. Inserting this, we obtain $[Z_i, Z_j] = - \sum_s B(Z_i, \xi_s) B(Z_j, [\eta_s, \nu_-]) \nu_+$, and observe that $B(Z, X)$ is the value of the linear map defined by $Z$ on $X \in \mathfrak{g}_{-1}$. Using this, the claimed formula follows from the definition of $\partial^*$ in (2.2) by a simple direct computation, thus proving the claim.

To complete the proof, we expand $(\partial^* \alpha) \wedge \beta$ according to (3.2) and move the wedge product with $\nu_+$ to obtain summands which are up to a constant multiple of the form $(-1)^k (i_{[\eta_+, \nu_-]} i_{\xi_+} \alpha) \wedge \nu_+ \wedge \beta$. Since $(i_{\xi_+} \alpha) \wedge \nu_+ \wedge \beta = 0$ and $B(\nu_+, [\eta_s, \nu_-]) = 0$, we see that this equals $(i_{\xi_+} \alpha) \wedge \nu_+ \wedge i_{[\eta_s, \nu_-]} \beta$. The same argument shows that the other insertion operator can be moved to $\beta$ at the expense of a sign $(-1)^{k+1}$ and exchanging the two insertion operators causes another sign change. Using (3.2) again, this completes the argument. 

Using this, we are ready to formulate the main compatibility result.
Proposition 3.3. Let $\Phi : \Omega^k(G/P) \to \Omega^\ell(G/K)$ be a Poisson transform with corresponding Poisson kernel $\varphi \in \Omega^{(2n+1-k)}(G/M)$.

(i) The compositions $d \circ \Phi$, $\ast \circ \Phi$, $\delta \circ \Phi$ and $\Delta \circ \Phi$ are again Poisson transforms with associated Poisson kernels $d_K\varphi$, $\ast_K\varphi$, $\delta_K\varphi$ and $\Delta_K\varphi$, respectively.

(ii) The compositions $\mathcal{L} \circ \Phi$ and $\mathcal{L}^* \circ \Phi$ are again Poisson transforms with associated Poisson kernels $\mathcal{L}_K\varphi$ and $\mathcal{L}^*_K\varphi$, respectively.

(iii) The compositions $\Phi \circ \delta$ and $\Phi \circ \partial^*$ are Poisson transforms with corresponding Poisson kernels $(-1)^{\ell-k}d_P\varphi$ and $(-1)^{\ell-k+1}\partial^*_P\varphi$, respectively.

Proof. (i) Recall from Proposition X in chapter VII of [11] that the exterior derivative commutes with the fiber integral. Thus, we obtain for all $\alpha \in \Omega^k(G/P)$ that

$$d\Phi(\alpha) = \int_{G/P} d(\varphi \wedge \pi_P^*\alpha) = \int_{G/P} (d_K\varphi) \wedge \pi_P^*\alpha,$$

where we used that $\varphi \wedge d\pi_P^*\alpha$ and $(d_P\varphi) \wedge \pi_P^*\alpha$ evidently vanish. Next, by tensoriality of the Hodge star it suffices to show the relation for the composition $\ast \circ \Phi$ at any point, where it can be deduced from the local description of the fiber integral. Combining those two, we obtain (i) and together with Proposition IX in chapter VII of [11], (ii) follows readily.

For $\alpha \in \Omega^{k-1}(G/P)$, we get $d(\varphi \wedge \pi_P^*\alpha) = d\varphi \wedge \pi_P^*\alpha + (-1)^{\ell-k+1}\varphi \wedge \pi_P^*d\alpha$. Applying $\int_{G/P}$ the left hand side vanishes, since $d$ commutes with $\int_{G/P}$. In the first summand in the right hand side, only $d_P\varphi$ leads to a form of the right bidegree, and we get $0 = \int_{G/P} d_P\varphi \wedge \pi_P^*\alpha + (-1)^{\ell-k+1}\Phi(d\alpha)$, which gives the first part of (iii).

For the second part of (iii), we observe that Lemma 3.2 and the definition of $\partial^*_P$ imply that for $\alpha \in \Omega^{k+1}(G/P)$ we get

$$\varphi \wedge \pi_P^*\partial^*\alpha = \varphi \wedge \partial^*_P\pi_P^*\alpha = (-1)^{2n+1-k+\ell}(\partial^*_P\varphi) \wedge \pi_P^*\alpha.$$

Applying $\int_{G/P}$, the left hand side gives $\Phi(\partial^*\alpha)$ and the result follows. \qed

Thus we see how to construct Poisson transforms that satisfy the equivalent conditions of Proposition 3.2 and hence descend to the Rumin complex as discussed in Corollary 3.3. We have to construct Poisson kernels $\varphi$ that satisfy $\partial^*_P\varphi = 0$ and $\partial^*_P d_P \varphi = 0$. To carry this out explicitly, it will be convenient to pass to complex valued differential forms. This allows us to decompose forms on $G/K$ into $(p,q)$–types. Similarly, splitting forms on $G/M$ into bidegrees, the first degree can be split further into $(p,q)$–types. It follows immediately from the definitions that $G$–invariance of $\varphi$ is equivalent to $G$–invariance of all $(p,q)$–components of $\varphi$ and similarly for vanishing of $\partial^*_P\varphi$ and $\partial^*_P d_P \varphi$.

3.5. The structure of $M$ and $g/m$. To make things concrete in our case, let us first describe the groups and Lie algebras we need explicitly. We realize $G = SU(n+1,1)$ as the group of all complex matrices $g \in GL(n+2, \mathbb{C})$ which satisfy
\( g^* Sg = S \) and \( \det(g) = 1 \), where \( S \) is the symmetric matrix

\[
S = \begin{pmatrix}
0 & 0 & 1 \\
0 & \text{id}_n & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

The maximal compact subgroup \( K \cong U(n+1) \) of \( G \) is given by the fixed points of the global Cartan involution \( g \mapsto (g^{-1})^* \). Writing elements in \( G \) as block matrices with the same block sizes as \( S \), the minimal parabolic subgroup \( P \subset G \) is given by

\[
P = \left\{ \begin{pmatrix}
a & -aY^*B & \frac{\alpha}{2}(b - |Y|^2) \\
0 & B & Y \\
0 & 0 & \overline{\alpha}^{-1}
\end{pmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C}, Y \in \mathbb{C}^n, B \in U(n), \det(B) = \overline{\alpha}a^{-1} \right\}.
\]

Let \( G = KAN \) be the Iwasawa decomposition of \( G \) with respect to the choices of \( K \) and \( P \). Then the group \( A \) is represented by all block–diagonal matrices in \( P \) with \( B = \text{id}_n, Y = 0, b = 0 \) and \( a \in \mathbb{R}^* \), whereas \( N \) corresponds to the elements with \( a = 1 \) and \( B = \text{id}_n \). Finally, the intersection \( M = K \cap P \) consists of all matrices in \( P \) with \( |a| = 1, Y = 0 \) and \( b = 0 \) and is therefore isomorphic to \( S(U(n) \times U(1)) \). In this way, \( P = MAN \) is the Langlands decomposition and \( G_0 := MA \) is the Levi subgroup of \( P \).

Turning to the infinitesimal picture, the Lie algebra \( \mathfrak{g} = \mathfrak{su}(n+1, 1) \) of \( G \) is

\[
\mathfrak{g} = \left\{ \begin{pmatrix}
b & -Y^* & y \\
x & X & Y \\
x & -X^* & -b
\end{pmatrix} : X, Y \in \mathbb{C}^n, b \in \mathbb{C}, x, y \in i\mathbb{R}, B \in \mathfrak{u}(n) \right\}.
\]

The block form of \( \mathfrak{g} \) defines the \( |2| \)-grading \( \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \), and the Lie algebra \( \mathfrak{p} \) of \( P \) is \( \oplus_{i \geq 0} \mathfrak{g}_i \). Writing \( \xi \in \mathfrak{g} \) as \( \xi = (x, X, (B, b), Y, y) \) according to this decomposition, the Lie algebra \( \mathfrak{k} \) of \( K \) is given by all elements \( (x, X, (B, b), X, x) \) with \( b \in i\mathbb{R} \). In particular, the Lie algebra \( \mathfrak{m} \) of \( M \) consists of all elements of the form \( (0, 0, (B, b), 0, 0) \) with \( b \in i\mathbb{R} \), whereas the Lie algebra \( \mathfrak{a} \) of \( A \) is generated by the grading element \( E := (0, 0, (0, 1), 0, 0) \in \mathfrak{g}_0 \).

Finally, the Killing form on \( \mathfrak{g} \) is a multiple of the trace form and therefore determined by its value on the grading element. For nice conventions, it is better to define \( B \) to be \( \frac{1}{2(n+2)} \) times the Killing form, which leads to the following non-degenerate pairings:

\[
\mathfrak{a} \times \mathfrak{a} \to \mathbb{R}, \quad B(E, E) = 2,
\]

\[
\mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad B((b_1, B_1), (b_2, B_2)) = (2b_1b_2 + \text{tr}(B_1B_2)),
\]

\[
\mathfrak{g}_{-i} \times \mathfrak{g}_i \to \mathbb{R}, \quad B(X, Y) = -2\langle X, Y \rangle,
\]

\[
\mathfrak{g}_{-2} \times \mathfrak{g}_2 \to \mathbb{R}, \quad B(x, y) = xy,
\]

where \( \langle , \rangle \) denotes the standard Hermitian inner product on \( \mathbb{C}^n \).

Since \( M \subset G_0 \), we get \( \mathfrak{m} \subset \mathfrak{g}_0 \) and the \( |2| \)-grading on \( \mathfrak{g} \) is invariant under the adjoint action of \( M \). In particular, as a representation of \( M \), the quotient \( \mathfrak{g}/\mathfrak{m} \) splits as \( \oplus_{i=2}^2 (\mathfrak{g}/\mathfrak{m})_i \). For \( i \neq 0 \) we get \( (\mathfrak{g}/\mathfrak{m})_i = \mathfrak{g}_i \), while \( (\mathfrak{g}/\mathfrak{m})_0 \) has real dimension one and is spanned by \( E + \mathfrak{m} \). Explicitly, writing elements of \( \mathfrak{g}/\mathfrak{m} \).
as \( \xi = (x, X, a, Y, y) \) with \( x, y \in i\mathbb{R}, a \in \mathbb{R} \) and \( X, Y \in \mathbb{C}^n \) according to this identification and viewing \( M \) as \( S(\mathbb{U}(n) \times \mathbb{U}(1)) \), the action of \( M \) on \( \mathfrak{g}/\mathfrak{m} \) is given by
\[
(B, b) \cdot \xi = (x, b^{-1}BX, a, b^{-1}BY, y).
\]

We know that \( \mathfrak{g}/\mathfrak{m} \) is the sum of the horizontal subspace \( \mathfrak{p}/\mathfrak{m} \) and vertical subspace \( \mathfrak{e}/\mathfrak{m} \) and it is easy to identify these subspaces. The space \( \mathfrak{p}/\mathfrak{m} \) consists of all elements of the form \((0, 0, a, X, x)\), while \( \mathfrak{e}/\mathfrak{m} \) consists of all elements of the form \((x, X, 0, X, x)\). As stated above, we will work with complex forms to use the decompositions into \((p, q)\)–types, so it will be helpful to deal with the complexification \((\mathfrak{g}/\mathfrak{m})_\mathbb{C}\). Since \( \mathfrak{p}/\mathfrak{m} \) is a complex subspace in \( \mathfrak{g}/\mathfrak{m} \), its complexification splits as \((\mathfrak{p}/\mathfrak{m})^{1,0} \oplus (\mathfrak{p}/\mathfrak{m})^{0,1}\). Explicitly, the complex structure on \( \mathfrak{p}/\mathfrak{m} \) maps \((0, 0, a, X, x)\) to \((0, 0, \frac{i\pi}{2}, JX, 2ia)\). Similarly, the complexification of the CR-subspace \( \mathbb{H} \subset \mathfrak{e}/\mathfrak{m} \) splits as \( \mathbb{H}^{1,0} \oplus \mathbb{H}^{0,1} \).

For further computations we fix some notation for elements of \((\mathfrak{g}/\mathfrak{m})_\mathbb{C}\). First, we put
\[
Z := \frac{1}{2}(0, 0, 1, 0, 2i) \in (\mathfrak{p}/\mathfrak{m})^{1,0} \quad I := (i, 0, 0, 0, i) \in (\mathfrak{e}/\mathfrak{m})_\mathbb{C}.
\]

On the other hand, for \( X \in \mathbb{C}^n \), we define
\[
F^1_0 := (0, 0, 0, X^{1,0}, 0) \in (\mathfrak{p}/\mathfrak{m})^{1,0} \quad G^1_0 := (0, X^{1,0}, 0, X^{1,0}, 0) \in \mathbb{H}^{1,0},
\]
and similarly we define \( F^{0,1}_X \) and \( G^{0,1}_X \), using \( X^{0,1} \) instead of \( X^{1,0} \). Of course, we also have \( \Xi \in (\mathfrak{p}/\mathfrak{m})^{0,1} \). Finally, one immediately verifies that the pullback \( g_K \) of the \( \mathbb{K} \)–invariant Hermitian inner product on \( \mathfrak{g}/\mathfrak{t} \) with the isomorphism to \( \mathfrak{p}/\mathfrak{m} \) corresponds to the standard pairing, i.e., the non–trivial pairings are given by
\[
g_K(F^{1,0}_X, F^{0,1}_Y) = \frac{i}{2}(X, Y) \quad g_K(Z, Z) = 1.
\]

3.6. Basic invariant forms on \( G/M \). It is now easy to construct several invariant one–forms on \( G/M \) via \( M \)–invariant elements in \((\mathfrak{g}/\mathfrak{m})_\mathbb{C}\) and analyze their exterior derivatives. First, we define \( I^* \in (\mathfrak{g}/\mathfrak{m})_\mathbb{C}^* \) by requiring that \( I^* \) vanishes on \((\mathfrak{p}/\mathfrak{m})_\mathbb{C} \) and on \( \mathbb{H}_\mathbb{C} \) and that \( I^*(I) = 1 \).

Recall from Section 2.1 that we split forms on \( G/M \) according to bidegree. Now we further decompose them according to \((p, q)\)–types. This is no problem with respect to the first degree, since this corresponds to multilinear functionals on \( \mathfrak{p}/\mathfrak{m} \), which is a complex subspace of \( \mathfrak{g}/\mathfrak{m} \). Accordingly, it makes sense to say that a form of bidegree \((k, \ell)\) has a certain \((p, q)\)–type with \( p + q = k \), which we will phrase as being of \( \mathbb{K} \)–type \((p, q)\). For the second degree, we say that a multilinear form \( \omega \) of bidegree \((k, \ell)\) has \( P \)–type \((r, s)\) if either \( r + s = \ell \), \( \omega(I) = 0 \) and \( \omega \in \Lambda^r*\mathbb{H}^*_\mathbb{C} \) or \( r + s = \ell – 1 \) and \( \omega = I^* \wedge \tilde{\omega} \) with \( \tilde{\omega} \in \Lambda^{\ell-r}\mathbb{H}^*_\mathbb{C} \). We then use the same wording for forms on \( G/M \).

In this language, \( I^* \) has bidegree \((0, 1)\), while its \( P \)–type is \((0, 0)\). On the other hand, there are obvious \( M \)–invariant linear functionals \( Z^* \) and \( Z \), which both have bidegree \((1, 0)\) and \( \mathbb{K} \)–type \((1, 0)\) and \((0, 1)\), respectively. Now we can of course form (partial) exterior derivatives and wedge products of these one forms. Recall that for an \( M \)–invariant \( k \)–linear, alternating functional \( \alpha \) on \((\mathfrak{g}/\mathfrak{m})_\mathbb{C}^* \), the exterior
derivative of the corresponding invariant \( k \)-form corresponds to the functional that sends \( X_0 + m, \ldots, X_k + m \) to
\[
\sum_{i<j} (-1)^{i+j} \alpha ([X_i, X_j] + m, X_1 + m, \ldots, i, \ldots, j, \ldots, X_k + m).
\]

Using this, we can easily compute the derivatives of \( I^* \), \( Z^* \) and \( Z^* \) and then correct by wedge products to obtain invariant two–forms into which \( I, Z, \) and \( Z \) all insert trivially. Explicitly, we define
\[
\omega_{2,0} := -i \partial K Z^* + i Z^* \wedge Z^*, \quad \omega_{1,1} := \frac{1}{2} d_P Z^* - i Z^* \wedge I^*, \quad \omega_{0,2} := \frac{1}{2} d_P I^*,
\]
whose properties are collected in the following table:

| form     | bidegree | \( K \)-type | \( P \)-type | explicit formula                |
|----------|----------|--------------|--------------|-------------------------------|
| \( \omega_{2,0} \) | (2, 0)   | (1, 1)       | (0, 0)       | \( \omega_{2,0}(F_{X}^{1,0}, F_{Y}^{0,1}) = -\frac{i}{2}\langle X, Y \rangle \) |
| \( \omega_{1,1} \) | (1, 1)   | (1, 0)       | (0, 1)       | \( \omega_{1,1}(F_{X}^{1,0}, G_{Y}^{0,1}) = \frac{i}{2}\langle X, Y \rangle \) |
| \( \omega_{1,1} \) | (1, 1)   | (0, 1)       | (1, 0)       | \( \omega_{1,1}(F_{0}^{1,1}, G_{0}^{0,0}) = \frac{i}{2}\langle X, Y \rangle \) |
| \( \omega_{0,2} \) | (0, 2)   | (0, 0)       | (1, 1)       | \( \omega_{0,2}(G_{X}^{1,0}, G_{Y}^{0,1}) = -\frac{i}{2}\langle X, Y \rangle \) |

**Table 1.** Invariant forms of degree 2.

Using these invariant 2-forms we can write the images of the invariant 1-forms under the partial derivatives as
\[
\partial K Z^* = 0, \quad d_P Z^* = 2 \omega_{1,1} + 2 i Z^* \wedge I^*,
\]
\[
\partial K Z^* = Z^* \wedge Z^* + i \omega_{2,0}, \quad d_P Z^* = 2 \omega_{1,1} - 2 i Z^* \wedge I^*,
\]
\[
\partial K I^* = Z^* \wedge I^*, \quad d_P I^* = 2 \omega_{0,2}.
\]

In particular, using \( d_K = \partial K + \overline{\partial K} \) it follows that
\[
d_K d_P(Z^*) = d_K d_P(\overline{Z}^*) = 0.
\]

For completeness, we also compute
\[
\partial K \omega_{2,0} = -Z^* \wedge \omega_{2,0}, \quad d_P \omega_{2,0} = 2 i Z^* \wedge \omega_{1,1} - 2 i \overline{Z}^* \wedge \omega_{1,1},
\]
\[
\partial K \omega_{1,1} = 0, \quad d_P \omega_{1,1} = -2 i I^* \wedge \omega_{1,1} + 2 i Z^* \wedge \omega_{0,2},
\]
\[
\partial K \omega_{1,1} = -I^* \wedge \omega_{2,0}, \quad d_P \omega_{1,1} = 2 i I^* \wedge \omega_{1,1} - 2 i \overline{Z}^* \wedge \omega_{0,2},
\]
\[
\partial K \omega_{0,2} = Z^* \wedge \omega_{0,2} - I^* \wedge \omega_{1,1}, \quad d_P \omega_{0,2} = 0.
\]

In particular, there are no new \( M \)-invariant forms obtained in this way.

### 4. Transforms adapted to the Rumin complex

Having constructed the basic invariant forms on \( G/M \), we can now proceed to construct higher degree forms with appropriate properties, and thus Poisson transforms for differential forms on a CR–sphere that descend to the Rumin complex.
4.1. The operator $\partial^*_p$. As discussed in the end of Section 3.4, a crucial role for verifying the conditions on a Poisson kernel from Corollary 2.3 is played by the operator $\partial^*_p$ on differential forms on $G/M$ defined in that section. For invariant forms this operator corresponds to a map on alternating multilinear forms on $(\mathfrak{g}/\mathfrak{m})^*_C$ that we denote by the same symbol. Thus we start by computing the latter operator using the notation from Section 3.6.

Proposition 4.1. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitian inner product. Then for $\alpha \in \Lambda^k(\mathfrak{g}/\mathfrak{m})^*_C$, and up to a non-zero multiple, we have

$$\partial^*_p \alpha = \sum_{s=1}^n \mathcal{I}^s \wedge \iota_{e_s^\#} e_s \alpha,$$

with $\iota$ denoting insertion operators. In particular, the operator $\partial^*_p$ vanishes on the ideal generated by $\mathcal{I}$.

Proof. The definition of $\partial^*_p$ in Section 3.4 was via the Kostant codifferential $\partial^*$ and the identification of $\mathfrak{k}$ with $\mathfrak{g}/\mathfrak{p}$. Thus we start by rewriting the formula (3.2) from the proof of Lemma 3.2 in terms of the basis of $\mathfrak{p}$ induced by $\{e_1, \ldots, e_n\}$.

For all $X \in \mathbb{C}^n$, we denote by $\xi_X$ and $\eta_X$ the corresponding elements in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, respectively. By $\nu_\pm$ we denote the elements in $\mathfrak{g}_{\pm 2}$ corresponding to $i$. Then $\{\xi_{es}, \xi_{ie}, \nu_+\}$ is a real basis of $\mathfrak{p}_+$, and from Section 3.5 we deduce that $\{-\frac{1}{2}\eta_{ie}, -\frac{1}{2}\eta_{es}, -\nu_-\}$ is the dual basis of $\mathfrak{g}_-$ with respect to $B$. Moreover, the Lie brackets $[\xi_{es}, \nu_-]$ and $[\xi_{ie}, \nu_-]$ equal $-\eta_{es}$ and $\eta_{ie}$, respectively. Thus, for all $\beta \in \Lambda^kp_+$ formula (3.2) says that $\partial^* \beta$ is a non-zero multiple of

$$\sum_{s=1}^n \nu_+ \wedge \iota_{\eta_{es}} \eta_{es} \beta.$$

To obtain the corresponding expression for $\partial^*_p$, we just have to interpret this in terms of the $M$-module $\mathfrak{g}/\mathfrak{m}$, so we identify $\mathfrak{g}_-$ with $\mathfrak{k}$ with $\mathfrak{p}_+$ with $(\mathfrak{k})^*$ via the Killing form. This readily shows that $\eta_{es}$ corresponds to the element $G_{es}$ while $\nu_+$ corresponds to $\mathcal{I}^s$ up to a non-zero factor. Therefore, we obtain for all $\alpha \in \Lambda^0k(\mathfrak{g}/\mathfrak{m})^*$ that $\partial^*_p \alpha$ is a nonzero multiple of

$$\sum_{s=1}^n \mathcal{I}^s \wedge \iota_{G_{es}} G_{es} \alpha,$$

and using the definition of $\partial^*_p$ on decomposable forms as well as linearity this continues to hold for all elements in $\Lambda^\ell k(\mathfrak{g}/\mathfrak{m})^*$. The claimed formula then holds by decomposing the basis vectors $G_{es}$ into holomorphic and antiholomorphic parts.

4.2. Invariant forms of higher degree. Now we can start building up forms of higher degree from the basic invariant two-planes introduced in Section 3.6. We do this in a notation that expresses the bidegree as well as the $K$-type and the $P$-type.

Definition 4.2. Let $p, q, k$ be non-negative integers such that $0 \leq p, q, k-p, k-q \leq n$. For all $\max\{0, p + q - k\} \leq j \leq \min\{p, q\}$ we define

$$\omega_{p,q}^{i,j} := \omega_{2,0}^{j} \wedge \omega_{0,2}^{k-p+q+j} \wedge \omega_{1,1}^{p-j} \wedge \omega_{1,1}^{1-i-j}.$$
By construction, each of these forms vanishes upon insertion of $I$, $Z$ and $\overline{Z}$. Moreover, from Table 2 we readily see that $\omega_{j}^{p,q;k}$ has bidegree $(p + q, 2k - p - q)$, $K$-type $(p, q)$, and $P$-type $(k - p, k - q)$. Since these types conversely determine $p$, $q$ and $k$, forms with different values of these parameters are automatically linearly independent (if non-zero). For later use, we next show that for fixed $p$, $q$ and $k$ with $k > n$, there is a linear relation between the forms $\omega_{j}^{p,q;k}$ for different values of $j$.

**Proposition 4.3.** For all $0 \leq p, q \leq n$ we define

$$\kappa_{j}^{p,q;k} := \binom{k}{p} \binom{k-p}{q-j},$$

where we agree that $\kappa_{j}^{p,q;k} = 0$ if one of the binomial coefficients is not defined. Then for $k > n$, we get

$$\sum_{j} \kappa_{j}^{p,q;k} \omega_{j}^{p,q;k} = 0.$$

**Proof.** Since $k > n$ the form $\omega_{0,2}^{k}$ is trivial. Hence for all $X_1, \ldots, X_p$ and $Y_1, \ldots, Y_q \in \mathbb{C}^n$, we obtain

$$(4.1) \quad t_{G_{X}^{1,0}} \ldots t_{G_{Y}^{1,0}} t_{G_{X}^{1,0}} \ldots t_{G_{Y}^{1,0}} \omega_{0,2}^{k} = 0.$$ 

Now our definitions easily imply that for all $X \in \mathbb{C}^n$ we get

$$t_{G_{X}^{1,0}} \omega_{0,2}^{k} = i t_{F_{X}^{0,1}} \omega_{1,1}^{2} \quad t_{G_{X}^{1,0}} \omega_{0,2}^{k} = -i t_{F_{X}^{0,1}} \omega_{1,1}^{2} \quad t_{G_{X}^{1,0}} \omega_{1,1}^{2} = -i t_{F_{X}^{0,1}} \omega_{2,0}^{k}.$$ 

Inductively, the first of these relations readily implies that, up to a non-zero constant, we get

$$t_{G_{X}^{1,0}} \ldots t_{G_{X}^{1,0}} \omega_{0,2}^{k} = t_{F_{X}^{0,1}} \ldots t_{F_{X}^{1,0}} \omega_{1,1}^{p,k-p}.$$ 

The remaining two relations then show that, up to a factor $(-i)$, we get

$$t_{G_{Y}^{1,0}} \omega_{0,2}^{k} = t_{F_{Y}^{0,1}} (p \omega_{0,2}^{0} \omega_{1,1}^{p,k-p} - (k - p) \omega_{1,1}^{p,k-p} \omega_{0,2}^{k-p}).$$

Since insertion operators always anti-commute, we conclude inductively, that (4.1) can be rewritten as

$$0 = t_{F_{Y}^{0,1}} \ldots t_{F_{X}^{0,1}} t_{F_{X}^{1,0}} \ldots t_{F_{X}^{1,0}} (\sum_{j} \kappa_{j}^{p,q;k} \omega_{j}^{p,q;k}).$$

But by construction $\sum_{j} \kappa_{j}^{p,q;k} \omega_{j}^{p,q;k}$ is a form that vanishes under insertion of $I$, $Z$, and $\overline{Z}$ and has $K$-type $(p, q)$, so the equation shows that it vanishes upon insertion of any $p + q$ tangent vectors from $T'$. But again by construction it has bidegree $(p + q, 2k - p - q)$, so the claim follows.

The coefficients showing up in these relations are characterized by a linear recursion that we discuss next.

**Lemma 4.4.** The coefficients $\kappa_{j} := \kappa_{j}^{p,q;k}$ defined in Proposition 4.3 satisfy the following system of linear relations, which determines them up to a non-zero constant.

$$\kappa_{j}(p - j)(q - j) = \kappa_{j+1}(j + 1)(k - (p + q) + j + 1) \quad \text{for all } j \in \mathbb{Z}.$$
Using the derivation property of insertion operators and the definitions, we readily have
\[ \omega_j^{p,q;k} \] is a constant multiple of \( \kappa_j^{p,q;k} \).

Finally, we can use Proposition 4.1 to compute the action of \( \partial_p \) on the forms \( \omega_j^{p,q;k} \):

**Lemma 4.5.** Up to a nonzero constant that is independent of \( p, q, k \) and \( j \), we have
\[
\partial_p \omega_j^{p,q;k} = (k - (p + q) + j)(k - j - (n + 1))I^* \wedge \omega_j^{p,q;k-1} + (p - j)(q - j)I^* \wedge \omega_j^{p,q;k-1}.
\]

**Proof.** In view of Proposition 4.1 we mainly have to compute \( \sum_s t_{G,P} G_{es} \omega_j^{p,q;k} \).
Using the derivation property of insertion operators and the definitions, we readily see that
\[
t_{G,P} G_{es} \omega_j^{p,q;k} = (k - p - q + j)(t_{G,P} G_{es} \omega_0^{q,2}) \omega_j^{p,q;k-1} + (q - j)(t_{G,P} G_{es} \omega_1^{q,1}) \omega_j^{p,q-1;k-1}.
\]
The image of the right hand side under \( t_{G,P} G_{es} \omega_0^{q,2} \) and \( t_{G,P} G_{es} \omega_1^{q,1} \) can again be computed using the derivation property of insertion operators and the definitions. This time, the expressions involve the one-forms like \( t_{G,P} G_{es} \omega_0^{q,2} \) and \( t_{G,P} G_{es} \omega_1^{q,1} \). After summation over \( s \) the wedge products of two one-forms occurring in the summands can be re-expressed in terms of the basic two-forms and the result follows from a direct computation.

**4.3. The case \( p + q \leq n \)**. We will now start the construction of Poisson kernels such that the associated Poisson transforms descend to the Rumin complex as described in Corollary 2.3. We focus on the case of transforms that preserve the degree of differential forms and then for a transform defined on \( \Gamma \) we can restrict to the case of a transform with values in \( \Omega^p,q(G/K) \) for some fixed type \( (p,q) \) such that \( p + q = \ell \). To obtain such a transform, we have to construct a Poisson kernel \( \varphi_{p,q} \) of bidegree \( (p + q, 2n + 1 - (p + q)) \) and \( K \)-type \( (p,q) \). To ensure that the transform descends to the Rumin complex, we need that \( \partial_p \varphi_{p,q} = 0 \) and \( \partial_p d_P \varphi_{p,q} = 0 \).

In Proposition 5.1 we have seen that the nature of the homology bundles \( H_\ell \) is different for \( \ell \leq n \) and \( \ell > n \). Thus it is not surprising that the appropriate choices for the kernels \( \varphi_{p,q} \) look rather different in the cases \( p + q \leq n \) and \( p + q > n \), and we will discuss these cases separately starting with the former. Denoting by \( H \subset T(G/P) \) the contact subbundle and putting \( Q := T(G/P)/H \), we know that in this case the homology bundle \( H_\ell \) is a subbundle of \( \Lambda^\ell H^* \). To get a transform that is non-zero on this homology bundle, we have to ensure that the kernel \( \varphi_{p,q} \) satisfies \( t \varphi_{p,q} \neq 0 \) and \( I^* \wedge \varphi_{p,q} = 0 \). Hence it is a natural idea to try constructing \( \varphi_{p,q} \) as a wedge product of \( I^* \) with some invariant form \( \pi \) of bidegree ...
(p + q, 2n − (p + q)) and K-type (p, q). By Proposition 4.1 this automatically implies that ∂_p^* φ_{p,q} = 0.

Since we also have to ensure that ∂_p d_P φ_{p,q} = 0, it is natural to try using a d_P-exact form π in the above construction. For 0 ≤ j ≤ min{p, q} define

$$\pi_j^{p,q;k} := (d_P I^*)^{k-(p+q)} \wedge (2d_K d_P \omega_{1,1})^{j} \wedge (d_P Z^*)^{p-j} \wedge (d_P Z^*)^{q-j},$$

which is of bidegree (p+q, 2k−(p+q)), of K-type (p, q) and d_P-exact. In particular, any linear combination of the forms π_j^{p,q;k} can be wedged with I^* to obtain a form φ_{p,q} as above. The definition readily implies that d_P(I^* \wedge π_j^{p,q;k}) = π_j^{p,q;k+1} (which explains why we do not restrict our considerations to the case k = n).

**Theorem 4.6.** For p + q ≤ n define φ_{p,q} := \sum_{j=0}^{\min{p,q}} k_j^{p,q,n+1} I^* \wedge π_j^{p,q;n}, where the constants k_j^{p,q,n+1} are defined in Proposition 4.3. Then this kernel gives rise to a Poisson transform Φ : Ω^{p,q}(G/P, C) → Ω^{p,q}(G/K) which satisfies the conditions of Proposition 2.3 and thus factorizes to the Rumin complex as described in Corollary 2.3 and has harmonic values.

**Proof.** It remains to prove that ∂_p d_P φ_{p,q} = 0. To do this, we first observe that by definition d_P(I^* \wedge π_j^{p,q;n}) = π_j^{p,q;n+1}. To compute ∂_p π_j^{p,q;n+1} we observe that for any form τ, we get ι_I I^* ∧ τ = I^* ∧ ι_I τ. Thus Proposition 4.1 shows that ∂_p I^* = ∂_p ι_I I^* (I^* ∧ τ). But the formulae in the end of Section 3.6 imply that, up to elements in the ideal generated by I^*, we can replace d_P Z^* by 2ω_{1,1} and d_P Z^* by 2ω_{1,1} and 2d_K d_P ω_{1,1} by 4ω_{1,1} and ω_{0,2}. This then shows that I^* \wedge π_j^{p,q;k} = 2^k I^* \wedge ω_j^{p,q;k} and thus ∂_p π_j^{p,q;k} = 2^k \partial_π ω_j^{p,q,k}. But this implies ∂_p d_P φ_{p,q} = 2^{n+1} \sum_j k_j^{p,q,n+1} \partial_π ω_j^{p,q,n+1}, which vanishes since \sum_j k_j^{p,q,n+1} = 0 by Proposition 4.3.

4.4. The case p + q > n. The case of high degree forms is significantly more complicated. For ℓ ≥ n + 1, the homology bundle H_ℓ is a quotient of Λ^{ℓ−1} H^* ⊗ Q^* by Proposition 3.4. Consequently, for an appropriate Poisson kernel φ_{p,q} we certainly must have I^* \wedge φ_{p,q} ≠ 0, so compared to the low-degree case, already satisfying ∂_p I^* \wedge φ_{p,q} = 0 becomes a non-trivial problem. The basic strategy to construct appropriate forms will again be to define a family of d_P-closed forms with non-trivial wedge product with I^* and construct φ_{p,q} as a linear combination of wedge products of these with appropriate invariant one-forms. Explicitly, for p + q − k ≤ j ≤ min{p, q}, we define

$$\pi_j^{p,q;k} := (2id_K Z^*)^j \wedge (d_P I^*)^{k-(p+q)+j} \wedge (d_P Z^*)^{p-j} \wedge (d_P Z^*)^{q-j}.$$

From Section 3.6, we know that d_P d_K Z^* = 0, so all these forms are d_P-closed and visibly π_j^{p,q;k} is of bidegree (p + q, 2k − (p + q)) and of K-type (p, q). Using these, we can now formulate our second main theorem.

**Theorem 4.7.** For p + q ≥ n + 1 and α, β ∈ C define

$$φ_{p,q}^{α,β} := φ_{p,q} := \sum_{j ∈ Z} \left( α_j Z^* \wedge π_j^{p,q-1,n} + β_j Z^* \wedge π_j^{p,q-1,n} + γ_j I^* \wedge π_j^{q,p;n} + 2δ_j I^* \wedge Z^* \wedge Z^* \wedge π_j^{p,q-1,n-1} \right),$$

where
where \( \alpha_j := \alpha_{p,q+1;n+1} \), \( \beta_j := \beta_{p,q+1;n+1} \), \( \gamma_j := \frac{\alpha(p+1) + \beta(q+1)}{p+q-n} \) and
\[
\delta_j := \frac{[n(a_{n+1}+\beta_{n+1}p)]\kappa_{p,q+1;n+1}}{p+q-n},
\]
with the constants \( \kappa \) defined in Proposition 4.4. \( \Phi \) thus gives rise to a 2-parameter family of Poisson transforms adapted to the Rumin complex 21 which both vanish by Lemma 4.4.

Proof. We have to prove that \( \partial_p \varphi_{p,q} = 0 \) and \( \partial_p d_P(\varphi_{p,q}) = 0 \). As noted in the proof of Theorem 4.6 we get \( \partial_p \tau = \partial_p i_l (I^* \wedge \tau) \) which will be used heavily. Using the fact that, up to elements in the ideal generated by \( Z^* \), the form \( 2id_K \) is congruent to \( 2\omega_{2,0} \), similar arguments as in the proof of Theorem 4.6 show that up to a multiple \( I^* \wedge Z^* \wedge \pi_j^{p-1,q;k} \) equals \( I^* \wedge Z^* \wedge \omega_j^{p-1,q;k} \) as well as that \( I^* \wedge \Theta^* \wedge \pi_j^{p,q-1:k} \) coincides with \( I^* \wedge \Theta^* \wedge \omega_j^{p,q-1;k} \). Next, from Proposition 4.1 we conclude that \( \partial_p (Z^* \wedge \omega_j^{p-1,q;k}) = -Z^* \wedge \partial_j \omega_j^{p-1,q;k} \) and similarly \( \partial_p (\pi_j^{p,q-1;k}) = -\pi_j^{p+1,q-1;k} \). Since we can ignore forms that contain \( I \) in computing the image of \( \varphi_{p,q} \) under \( \partial_p \) we conclude that \( \partial_p \varphi_{p,q} \) coincides with

\[
\sum_{j \in \mathbb{Z}} \left( \alpha_j Z^* \wedge \partial_p \omega_j^{p-1,q;n} + \beta_j \pi_j^{p,q-1,n} \right)
\]

up to a multiple. Now using Lemma 4.3 we see that the right hand side can be written as \( \sum_j a_j Z^* \wedge I^* \wedge \omega_j^{p-1,q;n-1} + b_j \pi_j^{p,q-1;n-1} \) and inserting for \( \alpha_j \) and \( \beta_j \) we see that, up to an overall constant,
\[
a_j = -\kappa_{p,q+1;n+1}(n-p-q+j)(j+1) + \kappa_{p,q+1;n+1}(p-j)(q-j+1),
\]
\[
b_j = -\kappa_{p,q+1;n+1}(n-p-q+j)(j+1) + \kappa_{p,q+1;n+1}(p-j)(q-j+1),
\]

which both vanish by Lemma 4.4.

To compute \( d_P \varphi_{p,q} \), we first observe that definition (4.4) readily implies that \( d_P (Z^* \wedge \pi_j^{p,q-1;n}) \), \( d_P (\pi_j^{p,q-1;n}) \) and \( d_P (I^* \wedge \pi_j^{p,q-1;n}) \) are all equal to \( \pi_j^{p,q+1;n+1} \). On the other hand, modulo the ideal generated by \( I^* \), \( d_P (I^* \wedge Z^* \wedge \pi_j^{p,q-1:n-1}) \) is congruent to \( Z^* \wedge \pi_j^{p-1,q-1;n} \), implying

\[
(4.5) \quad \partial_p d_P(\varphi_{p,q}) = \sum_j \left( (\alpha_j + \beta_j + \gamma_j) \partial_p \pi_j^{p,q+1;n} + 2i \delta_j \partial_p (Z^* \wedge \pi_j^{p,q-1;n}) \right).
\]

From Section 3.6 we know that \( d_K Z^* = Z^* \wedge \Theta^* - i \omega_{2,0} \) and using this we conclude as in the proof of Theorem 4.6 that

\[
(4.6) \quad I^* \wedge \pi_j^{p,q,n+1} = 2^{n+1} I^* \wedge (\omega_j^{p,q,n+1} + ij Z^* \wedge \pi_j^{p,q-1,n-1}) \wedge \omega_j^{p-1,q-1;n}).
\]

Thus in the right hand side of (4.5), we may replace \( \pi_j^{p,q,n+1} \) by \( 2^{n+1}(\omega_j^{p,q,n+1} + ij Z^* \wedge \Theta^* \wedge \omega_j^{p-1,q-1;n}) \). But (4.6) also implies that

\[
I^* \wedge Z^* \wedge \Theta^* \wedge \pi_j^{p-1,q,n-1} = 2^n I^* \wedge Z^* \wedge \Theta^* \wedge \omega_j^{p-1,q-1;n}.
\]

From Proposition 4.1 we next conclude that \( \partial_p (Z^* \wedge \Theta^* \wedge \pi) = Z^* \wedge \Theta^* \wedge \partial_p \tau \) for any form \( \tau \). Hence in the second term in the right hand side of (4.5), we can
replace $\tilde{\pi}_j^{p-1,q-1;n}$ by $2^n \tilde{\omega}_j^{p-1,q-1;n}$. Reordering the sum, we obtain

$$\partial_p d_p \varphi_{p,q} = 2^{n+1} \sum_j \left( \tilde{\gamma}_j \partial_p \tilde{\omega}_j^{p,q;n+1} + \tilde{\delta}_j i Z^* \wedge Z^* \wedge \partial_p \tilde{\omega}_j^{p-1,q-1;n} \right),$$

where $\tilde{\gamma}_j := \alpha_j + \beta_j + \gamma_j$ and $\tilde{\delta}_j := (j+1)\tilde{\gamma}_{j+1} + \delta_j$. Now a direct computation shows that the coefficients in the above sum are given by

$$\tilde{\gamma}_j = \frac{\alpha(n+1-q) + \beta(n+1-p)}{p+q-n} \kappa_j^{p,q;n+1}, \quad \tilde{\delta}_j = -(n+1-(p+q))\tilde{\gamma}_{j+1}.$$

Applying the formula for the $P$-codifferential from Lemma 4.3 to the first sum we deduce that the coefficient of $I^* \wedge \omega_j^{p,q;n+1}$ is given by

$$-(j+1)(n-(p+q)+j+2)\tilde{\gamma}_{j+1} + (p-j)(q-j)\tilde{\gamma}_j,$$

which is trivial due to Lemma 4.4. Similarly, using again the formula from Lemma 4.5 the coefficient of $I^* \wedge Z^* \wedge \omega_j^{p-1,q-1;n-1}$ is given by

$$-(j+2)(n-(p+q)+j+3)\tilde{\delta}_{j+1} + (p-j-1)(q-j-1)\tilde{\delta}_j,$$

so since $\tilde{\delta}_j$ is a constant multiple of $\kappa_j^{p,q;n+1}$ this is again trivial due to Lemma 4.4.

All in all we see that $\partial_p d_p \varphi_{p,q} = 0$. □

4.5. Properties of the image of the Poisson transforms. In the next step we analyze the properties of the images of the Poisson transforms $\Phi$ constructed in Theorems 4.6 and 4.7 with respect to several differential operators on $G/K$. By Proposition 3.3 it suffices to apply the corresponding $M$-equivariant maps to the underlying Poisson kernels $\varphi_{p,q}$. Since for their construction we had to distinguish the cases $p+q \leq n$ and $p+q > n$, their properties will also differ significantly in these cases.

**Proposition 4.8.** For $p+q \leq n$ let $\Phi: \Omega^{p+q}(G/P, \mathbb{C}) \to \Omega^p(G/K)$ be the Poisson transform constructed in Theorem 4.6. Then the image of $\Phi$ is contained in the space of harmonic, coclosed and primitive differential forms on $G/K$.

**Proof.** Recall from Theorem 4.6 that the Poisson kernel underlying $\Phi$ is given by $\varphi_{p,q} = \sum_{j=0}^{\min\{p,q\}} \kappa_j^{p,q;n+1} I^* \wedge \pi_j^{p,q;n}$, where the forms $\pi_j^{p,q;n}$ are defined by (4.3) and the constants $\kappa_j^{p,q;n+1}$ are defined in Proposition 4.3. By Proposition 3.3 it suffices to show that this kernel is contained in the kernel of $\delta_K$ and $L_K^\ast$.

We have seen in the proof of Theorem 4.6 that the forms $\pi_j^{p,q;k}$ satisfy the relation $I^* \wedge \pi_j^{p,q;k} = 2^k I^* \wedge \omega_j^{p,q;k}$. Hence, by linearity we can immediately apply the formulae for the adjoint of the $K$-Lefschetz map from Proposition A.3. After shifting the index appropriately we obtain that $L_K^\ast \varphi_{p,q} = 4 \sum_j \tilde{\kappa}_j I^* \wedge \pi_j^{p-1,q-1;n-1}$ with coefficients $\tilde{\kappa}_j = (j+1)(n+1-(p+q)+j+1)\kappa_j^{p,q;n+1} - (p-j)(q-j)\kappa_j^{p,q;n+1}$.

However, due to equation (4.4) these constants are trivial.

Similarly, applying the formulae for $\partial_p^\ast$ from Proposition A.3 to the Poisson kernel yields $\partial_p^\ast \varphi_{p,q} = 4i \sum_j \tilde{\kappa}_j I^* \wedge Z^* \wedge \pi_j^{p-1,q-1;n-1} = 0$. Furthermore, the complex
conjugate of $\varphi_{p,q}$ equals $\varphi_{q,p}$, which readily implies $\overline{\partial_K^* \varphi_{p,q}} = \overline{\partial_K^* \varphi_{q,p}} = 0$ and hence also $\partial_K \varphi_{p,q} = 0$.

Next, we determine properties of the 2-parameter family of Poisson transforms for the case $p + q > n$ constructed in Theorem 4.7.

**Proposition 4.9.** For $p + q > n$ and $\alpha, \beta \in \mathbb{C}$ the image of the Poisson transform $\Phi_{p,q}^{\alpha,\beta}$ constructed in Theorem 4.7 is contained in the space of coprimitive forms. Moreover, $\partial^* \circ \Phi_{p,q}^{\alpha,\beta}$ is independent of $\beta$, $\overline{\partial}^* \circ \Phi_{p,q}^{\alpha,\beta}$ is independent of $\alpha$ and these satisfy

$$\partial^* \circ \Phi_{p+1,q}^{\alpha,\beta} = \overline{\partial}^* \circ \Phi_{p,q+1}^{\alpha,\beta}.$$ 

Furthermore, the partial derivatives are related via

$$-2i(n-p)\partial \circ \Phi_{p,q}^{\alpha,\beta} = (p + q - n + 1)\Phi_{p+1,q}^{0,\beta} \circ d,$$

$$2i(n-q)\overline{\partial} \circ \Phi_{p,q}^{\alpha,\beta} = (p + q - n + 1)\Phi_{p,q+1}^{0,\alpha} \circ d.$$ 

**Proof.** For the first part we need to show that the wedge product of $\varphi_{p,q}$ with the pullback $\omega_M$ of the Kähler form on $G/K$ along the canonical projection is trivial. From (A.1) we know that $\omega_M = \frac{1}{2}(\omega_{2,0} + iZ^* \wedge \overline{Z}^*)$, which coincides with $\frac{1}{2}d_K Z^*$. Consequently, by definition of $\partial_n^{\phi,p,q,k}$ in (4.4) it readily follows that $\omega_M \wedge \partial_n^{\phi,p,q,k} = \frac{1}{4}(p+1,q+1;k+1)$. Therefore, we can write $\omega_M \wedge \varphi_{p,q}$ up to an overall constant as

$$\sum_{j \in \mathbb{Z}} \left( \alpha_j Z^* \wedge \partial_n^{\phi,p,q,k} + \beta_j \overline{Z}^* \wedge \partial_n^{\phi,p,q,k} \right)$$

with the coefficients defined in Theorem 4.7. Next, using the formulae for the basic invariant forms a direct computation shows that $Z^* \wedge \partial_n^{\phi,p,q,k}$ coincides with $2^{n+1}(Z^* \wedge \partial_n^{\phi,p,q,k} - i(q-j)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_{j+1}^{p,q,k})$. Now if we multiply the first summand by $\alpha_j = \alpha \delta_{j+1} + \beta_j \delta_{j+1}$ and sum over all $j \in \mathbb{Z}$ the result vanishes due to Proposition 3.3. Then, up to an overall constant we can replace in (4.8) the form $Z^* \wedge \partial_n^{\phi,p,q,k}$ with $-i(q-j)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_{j+1}^{p,q,k}$. Proceeding similarly for the other summands in (4.8) we can write $\omega_M \wedge \varphi_{p,q}$ up to an overall constant as $\sum_{j} a_j I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_{j+1}^{p,q,k}$ with coefficients

$$a_j := -\alpha_j(q-j) - \beta_j(p-j) + (j+2)\gamma_{j+1} + \delta_j.$$ 

But a direct computation yields $a_j = 0$ for all $j$ and therefore $\omega_M \wedge \varphi_{p,q} = 0$.

Next, we determine the image of $\varphi_{p,q}$ under $\partial_K$. Using the formulae for the partial derivatives of the basic invariant forms from Section 3.6 a direct computation yields

$$\partial_K \partial_n^{\phi,p,q,k} = (k - (p+q) + j)(Z^* \wedge \partial_n^{\phi,p,q,k} - I^* \wedge \partial_n^{\phi,p,q,k}).$$

Therefore, we directly compute that both $\partial_K(Z^* \wedge \partial_n^{\phi,p,q,k})$ and $\partial_K(I^* \wedge \partial_n^{\phi,p,q,k})$ coincide with $(n+1-(p+q)+j)(\partial_K I^*) \wedge \partial_n^{\phi,p,q,k}$, whereas

$$2i\partial_K(I^* \wedge Z^* \wedge \overline{Z}^* \wedge \partial_n^{\phi,p,q,k})$$

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Furthermore, using (4.12) and get
\[ \partial_K \varphi_{p,q} = \beta \sum_j (\kappa_{j+1}^{p,q+1,n+1} \partial_K (Z^* \wedge \tilde{\pi}_j^{p,q-1,n}) - (n+1) \kappa_{j+1}^{p,q,n+1} (\partial_K I^*) \wedge \tilde{\pi}_j^{p,q,n}) . \]

Our aim is to compare this with the $P$-derivative of $\varphi_{p,q}$ for $p+q > n+1$. For this, recall from the proof of Theorem 4.7 that $d_P(Z^* \wedge \tilde{\pi}_j^{p,q,n})$, $d_P(Z^* \wedge \tilde{\pi}_j^{p,q-1,n})$ as well as $d_P(I^* \wedge \tilde{\pi}_j^{p,q,n})$ all coincide with $\tilde{\pi}_j^{p,q,n+1}$. Furthermore, for the $P$-derivative of $I^* \wedge Z^* \wedge \tilde{\pi}_j^{p,q,n}$ we first apply the formulae for the derivatives of the basic invariant forms from Section 4.4 and then (4.9) to write it as a linear combination of the elements $\tilde{\pi}_j^{p,q,n}$ and (4.12). Now if we define $\tilde{\gamma}_j := \alpha_j + \beta_j + \gamma_j$ as in the proof of Theorem 4.7 a direct computation shows that
\[ \frac{\delta_i}{(n-(p+q)+j+2)} = -\tilde{\gamma}_{j+1}. \]

All in all, we obtain
\[ (4.10) \quad d_P \varphi_{p,q} = \sum_j \tilde{\gamma}_j \tilde{\pi}_j^{p,q,n+1} + 2i\tilde{\gamma}_j + 1 \tilde{Z}^* \wedge \partial_K \tilde{\pi}_j^{p,q-1,n} - 2i\delta_j (\partial_K I^*) \wedge \tilde{\pi}_j^{p,q,n}. \]

Next, we shift the index in the first summand and use that $\partial_K \tilde{Z}^* = -\partial_K Z^*$ to rewrite the first two summands as $\tilde{\gamma}_{j+1} \partial_K (\tilde{Z}^* \wedge \tilde{\pi}_j^{p,q-1,n})$. Inserting the explicit expressions for $\tilde{\gamma}_j$ and $\delta_j$ from Theorem 4.7 a direct computation yields
\[ (4.11) \quad d_P \varphi_{p,q} = -2ic_{\alpha,\beta} \sum_j \kappa_{j+1}^{p,q,n+1} \partial_K (\tilde{Z}^* \wedge \tilde{\pi}_j^{p,q-1,n}) - (n+1) \kappa_{j+1}^{p,q,n} (\partial_K I^*) \wedge \tilde{\pi}_j^{p,q,n} \]

with $c_{\alpha,\beta} = \frac{\alpha(n+1-q)+\beta(n+1-p)}{p+q-n}$. In particular, for $\alpha = 0$ we compare (4.9) with (4.11) and get
\[ (4.12) \quad -2i(n+1-p) \partial_K \varphi_{p,q}^{0,\beta} = (p+q-n) d_P \varphi_{p,q}^{0,\beta}. \]

Furthermore, using $\varphi_{p,q}^{0,\beta} = \varphi_{q,p}^{0,\beta}$ we obtain formulae for the operator $\partial_K$ similar to (4.9) and (4.12).

Finally, we determine the image of $\varphi_{p,q}$ under $\partial_K^*$. First, note that by (4.4) we can write $Z^* \wedge \tilde{\pi}_j^{p,q,n}$ as the sum of $2n(Z^* \wedge \omega_j^{p,q,n})$ and a multiple of $I^* \wedge Z^* \wedge \omega_j^{p,q-1,n-1}$. Now by Proposition A.3 the latter is contained in the kernel of $\partial_K^*$, implying that $\partial_K^*(Z^* \wedge \tilde{\pi}_j^{p,q,n})$ coincides with $2n \partial_K^*(Z^* \wedge \omega_j^{p,q,n})$. In the same way, we can argue for the other summands of $\varphi_{p,q}$, obtaining
\[ (4.13) \quad \partial_K^* \varphi_{p,q} = 2n \sum_j \partial_K^* \left( \alpha_j Z^* \wedge \omega_j^{p,q,n} + \beta_j \tilde{Z}^* \wedge \omega_j^{p,q-1,n} + \gamma_j I^* \wedge \omega_j^{p,q,n} \right). \]

In this form we can directly apply the formulae from Proposition A.3 to (4.13). After shifting the indices appropriately we have
\[ (4.14) \quad \partial_K^* \varphi_{p,q} = 2n \sum_j \alpha_j \omega_j^{p,q,n} + 2ib_j Z^* \wedge \tilde{Z}^* \wedge \omega_j^{p,q-1,n-1} + 2ic_j I^* \wedge \tilde{Z}^* \wedge \omega_j^{p,q-1,n-1} + 2id_j I^* \wedge \tilde{Z}^* \wedge \omega_j^{p,q-1,n-1}, \]
where the coefficients are given by
\begin{align*}
b_j &= (j + 1)(n - (p + q) + j + 2)\alpha_{j+1} - (p - j - 1)(q - j)\alpha_j \\
c_j &= (j + 1)(n - (p + q) + j + 2)\alpha_{j+1} - (p - j - 1)(q - j + 1)\alpha_j \\
d_j &= (j + 1)(n - (p + q)j + 2)(\beta_{j+1} + \gamma_{j+1}) + (p - j)(q - j)(\beta_j + \gamma_j)
\end{align*}
Inserting the definitions of the coefficients \(\alpha_j, \beta_j\) and \(\gamma_j\) we can exploit Lemma 4.3 to simplify these expressions. In particular, a direct computation shows that \(d_j\) does not depend on the parameter \(\beta\). Explicitly, a direct computation shows that
\begin{equation}
\partial^*_{K_p,q} := 2\alpha \sum_j \left( \kappa_{j+1}^{p,q+1;n+1} \omega_j^{p-1,q;n} - i(n + 1)\kappa_{j+2}^{p,q+1;n} \omega_j^{p-2,q-1;n-1} \right.
\end{equation}
\begin{align*}
&- i(q + 2)\kappa_{j+2}^{p,q+2;n+1} \omega_j^{p-2,q;n-1} \\
&+ i(p + 1)\kappa_{j+2}^{p+1,q+1;n+1} \omega_j^{p-2,q-1;n-1} \big). 
\end{align*}
In particular, we deduce that \(\partial^*_{K_p,q} \omega^{\alpha,\beta} = \partial^*_{K_p,q} \omega^{\alpha,0}\), and this vanishes if and only if \(\alpha = 0\). Furthermore, since \(\varphi_{p,q}^{\alpha,\beta} = \varphi_{q,p}^{\alpha,0}\), we easily obtain an analogous formula for \(\overline{\partial}^*_{K_p,q}\). In particular, applying the complex conjugation to (4.15) directly we deduce that
\begin{equation}
\partial^*_{K_p,q+1} = \overline{\partial}^*_{K_p,q+1}
\end{equation}
for all \(p + q \geq n\).

4.6. Real versions of Poisson transforms. In the final section we construct for all \(0 \leq k \leq 2n\) Poisson transforms \(\Phi^k : \Omega^k(G/P) \to \Omega^k(G/K)\) which factor to the Rumin complex. Their associated kernels will be defined as special linear combinations of the operators from Theorems 4.6 and 4.7 respectively. Explicitly, recall from Proposition 4.8 that for \(p + q \leq n\) the image of \(\Phi_{p,q}\) consists of coclosed differential forms, whereas from Proposition 4.9 that for \(p + q > n\) the 2-parameter family \(\Phi_{p,q}^{\alpha,\beta}\) satisfies relations between its compositions with the partial derivatives \(\partial\) and \(\overline{\partial}\) on \(G/K\) and the exterior derivative on \(G/P\).

Therefore, we will design the real operators \(\Phi^k\) so that their image consists of coclosed differential forms and that they satisfy a commutation relation with respect to the exterior derivatives on \(G/K\) and \(G/P\). Explicitly, for all \(0 \leq k \leq n\) we define the real differential form \(\varphi_k := \sum_{p+q=k} \lambda_{p,q} \varphi_{p,q}\) with coefficients
\[
\lambda_{p,q} := 2^{-k-1}i^{p-q}(n + 1 - p)!(n + 1 - q)!
\]
and for all \(n + 1 \leq k \leq 2n\) the form \(\varphi_k := \sum_{p+q=k} \varphi_{p,q}^{\alpha,\beta}\) with parameters
\[
\alpha_{p,q} := 2^{-k-1}i^{p-q}(n + 1 - p)!(n - q)!, \quad \beta_{p,q} := 2^{-k-1}i^{q-p}(n - p)!(n - q + 1)!. 
\]
These Poisson kernels satisfy \(\overline{\varphi_k} = \varphi_k\) for all \(0 \leq k \leq 2n\) and therefore induce Poisson transforms \(\Phi^k : \Omega^k(G/P) \to \Omega^k(G/K)\) which are \(G\)-equivariant and factor to the Rumin complex.
Theorem 4.10. For all $0 \leq k \leq 2n$ let $\Phi_k : \Gamma(H_k(G/P)) \to \Omega^k(G/K)$ be the $G$-equivariant linear operators induced by $\Phi_k$. Then these operators satisfy:

(i) The image of $\Phi_k$ is contained in the space of harmonic and coclosed differential forms, which are primitive for $0 \leq k \leq n$ and coprimitive for $n + 1 \leq k \leq 2n + 1$.

(ii) If $D_k : \Gamma(H_k) \to \Gamma(H_{k+1})$ denotes the $k$th BGG-operator, then

$$d \circ \Phi_k = c_k \Phi_{k+1} \circ D_k$$

for all $0 \leq k \leq 2n$, where the coefficient $c_k$ is defined by

$$c_k = \begin{cases} n - k + 1 & k \leq n \\ n - k - 1 & k \geq n + 1 \end{cases}$$

Proof. (i) The operators $\Phi_k$ are defined by $\Phi_k(\pi(\alpha)) = \Phi_k(\alpha)$ for all $\alpha \in \Gamma(\ker(\partial^*))$, where $\pi : \Gamma(\ker(\partial^*)) \to \Gamma(H_k)$ denotes the canonical projection. Therefore, it suffices to prove the claimed properties of the image of $\Phi_k$. But from Proposition 4.8 we already know that for $p + q \leq n$ the image of the operators $\Phi_{p,q}$ consists of harmonic, coclosed and primitive differential forms, so by linearity the same is true for the images of $\Phi_k$ for $0 \leq k \leq n$. Moreover, for $p + q > n$ we have shown in Proposition 4.9 that the image of $\Phi_{p,q}$ is contained in the space of harmonic and coprimitive differential forms, and thus the same holds for $\Phi_k$ for $n + 1 \leq k \leq 2n$. Hence it remains to show that $\delta \circ \Phi_k = 0$ for all $n + 1 \leq k \leq 2n + 1$, which by Proposition 5.3 amounts to proving that $\varphi_k$ is $K$-coclosed. But we have seen in Proposition 4.9 that $\partial_K \varphi_{p,q}^{\alpha,\beta} = \partial_K \varphi_{p,q}^{\alpha,0}$ for all parameter $\alpha, \beta \in \mathbb{C}$, and similarly for $\partial_K^*$. Therefore, using formula (4.16) we deduce that

$$\delta_K \varphi_k = \sum_{p+q=k} \partial_K^* \varphi_{p,q}^{\alpha,0} + \partial_K^* \varphi_{p,q}^{\alpha,0} = \sum_{p+q=k} \partial_K^* \varphi_{p,q}^{\alpha,0}$$

with parameter $\gamma_{p,q} := \alpha_{p,q} + \beta_{p-1,q+1}$. But a direct computation shows that $\gamma_{p,q} = 0$ and hence $\delta_K \varphi_k = 0$.

(ii) Using the definition of $\Phi_k$ and Proposition 2.2 we need to show that

$$d_K \varphi_k = c_k d_P \varphi_{k+1}$$

for all $0 \leq k \leq 2n$.

First, for $p + q \leq n$ the Poisson kernel $\varphi_{p,q}$ was given by $I^* \wedge \sum_j \kappa_j^{p,q} n_j^{p,q,n}$ with the forms $\pi_j^{p,q,n}$ being defined in (4.3). Using the formulae for the derivatives of the basic invariant forms from Section 3.6 a direct computation shows that $d_K(I^* \wedge \pi_j^{p,q,k}) = (k + 1 - (p + q))(Z^* + \overline{Z}^*) \wedge \pi_j^{p,q,k}$ and therefore

$$d_K \varphi_{p,q} = (n + 1 - (p + q))(Z^* + \overline{Z}^*) \wedge \varphi_{p,q}.$$ (4.18)

Similarly, the definition of $\pi_j^{p,q,n}$ readily implies that $d_P(I^* \wedge \pi_j^{p,q,n}) = \pi_j^{p,q,n+1}$, which we write as a linear combination of $\omega_j^{p,q,n+1}$, $I^* \wedge Z^* \wedge \omega_j^{p,q,n}$ and $I^* \wedge \overline{Z}^* \wedge \omega_j^{p,q,n}$. Inserting this into $d_P \varphi_{p,q}$ the sum $\sum_j \kappa_j^{p,q,n+1} \omega_j^{p,q,n+1}$ vanishes due to Proposition
For the other summands, a direct computation using the definition of the coefficients $\kappa$ implies that
\begin{equation}
(4.19) \quad d_P\varphi_{p,q} = 2i \left( (n + 2 - p)\mathcal{Z}^* \wedge \varphi_{p-1,q} - (n + 2 - q)\mathcal{Z}^* \wedge \varphi_{p,q-1} \right).
\end{equation}
In order to compute $d_K\varphi_k$ in the case $k < n$ we first use linearity of the $K$-differential and apply (4.18) to each of the summands of $\varphi_k$. Next, we apply the relation $\lambda_{p,q} = 2i\lambda_{p+1,q}(n+1-p)$ to the coefficients of the summands $\mathcal{Z}^* \wedge \varphi_{p,q}$ and $\lambda_{p,q} = -2i\lambda_{p,q+1}(n+1-q)$ to the coefficients of $\mathcal{Z}^* \wedge \varphi_{p,q}$. Shifting the summation indices of the resulting expression to $p + q = k + 1$ and comparing this to (4.19) yields (4.17) for all $0 \leq k < n$.

Next, if $p + q > n$ we have seen in the proof of Proposition 4.9 that $\partial_K\varphi_{p,q} = \partial_K\varphi_{p,q}^0$ and $\partial_K\varphi_{p,q}^0 = \partial_K\varphi_{p,q}^0$ for all $\alpha, \beta \in \mathbb{C}$. Since $p + q > n$ we know that both $p$ and $q$ are positive, so linearity of the $K$-differential and shifting the indices appropriately we obtain
\begin{equation}
(4.20) \quad d_K\varphi_k = \sum_{p+q=k} \partial_{K}\varphi_{p,q}^0 + \partial_{K}\varphi_{p,q}^0 = \sum_{p+q=k+1} \partial_{K}\varphi_{p,q-1}^0 + \partial_{K}\varphi_{p-1,q}^0.
\end{equation}

By definition, the parameters of $\varphi_k$ satisfy $\alpha_{p,q-1} = -2i(n + 1 - q)\alpha_{p,q}$ as well as $\beta_{p-1,q} = 2i(n+1-p)\beta_{p,q}$. Inserting this into (4.20) and afterwards applying formula (4.12) for the relation between the partial derivatives of $\varphi_{p,q}^\alpha$ a direct computation yields (4.17) for all $n < k < 2n$.

Finally, we need to show the formula in the special case $k = n$. In order to do so, using the definition of the forms $\pi_j$ a similar computation as in the proof of Proposition 4.9 show that the derivative $d_P\varphi_{p,q}$ for $p + q = n + 1$ can be written as
\begin{equation}
(4.21) \quad d_P\varphi_{p,q} = \sum_j \tilde{\gamma}_j \tilde{\pi}_j^{p,q,n+1} + 2i\delta_j d_P (\mathcal{I}^* \wedge \mathcal{Z}^* \wedge \bigwedge^k) \wedge \tilde{\pi}_j^{p-1,q-1,n-1},
\end{equation}
where the coefficients $\tilde{\gamma}_j$ and $\delta_j$ are given by $\tilde{\gamma}_j = (\alpha \beta)\kappa_j^{p,q,n+1}$ and $\delta_j = -(n+1)(\alpha \beta)\kappa_j^{p,q,n+1}$. Next, we expand the form $\tilde{\pi}_j^{p,q,n+1}$ in terms of wedge products of the $\omega$’s with invariant 1-forms. Inserting this into (4.21) we obtain the summand $\sum_j \tilde{\gamma}_j \tilde{\omega}_j^{p,q,n+1}$, which is trivial due to Proposition 4.3. Therefore, by definition of the forms $\pi_j^{p,q,k}$ we can replace $\tilde{\pi}_j^{p,q,n+1}$ in (4.21) by the expression
\begin{equation}
2i \mathcal{Z}^* \wedge \bigwedge^k \wedge \pi_j^{p-1,q-1,n} + 2i(p-j)\mathcal{I}^* \wedge \mathcal{I}^* \wedge \pi_j^{p-1,q,n} - 2i(q-j)\mathcal{Z}^* \wedge \mathcal{I}^* \wedge \pi_j^{p,q-1,n},
\end{equation}
and expanding the other summand as well we deduce that
\begin{equation}
(4.22) \quad d_P\varphi_{p,q} = 2i \sum_j \epsilon_j \mathcal{Z}^* \wedge \bigwedge^k \wedge \pi_j^{p-1,q-1,n} + \zeta_j \mathcal{Z}^* \wedge \mathcal{I}^* \wedge \pi_j^{p-1,q,n} - \eta_j \mathcal{Z}^* \wedge \mathcal{I}^* \wedge \pi_j^{p,q-1,n},
\end{equation}
with coefficients $\epsilon_j = (j+1)\tilde{\gamma}_j + \delta_j$, $\zeta_j = (p-j)\tilde{\gamma}_j - \delta_j$ and $\eta_j = (q-j)\tilde{\gamma}_j - \delta_j$. Inserting the definitions we obtain by a direct computation that $\epsilon_j = 0$, whereas $\zeta_j = (\alpha \beta)(q+1)\kappa_j^{p+1,q,n+1}$ and $\eta_j = (\alpha \beta)(p+1)\kappa_j^{p+1,q,n+1}$.

In order to determine the $P$-derivative of $\varphi_{n+1}$ we use linearity of $d_P$ and apply the formula (4.21). Next, we shift the summation to all $p, q$ with $p + q = n$, so that
we can write \( d_P \varphi_{n+1} \) as a linear combination of \( Z^* \wedge I^* \wedge \pi_{j+1}^{p,q;n} \) and \( \mathbb{Z}^* \wedge I \wedge \pi_{j}^{p,q;n} \).

Inserting the definition of the parameter \( \alpha \) and \( \beta \) and using that \( \lambda_{p,q} = 2^{-(p+1)}(p+1)(q+1)! \) for \( p+q = n \) a direct computation yields

\[
d_P \varphi_{n+1} = 2i \sum_{p+q=n+1} \sum_{j} \zeta_j Z^* \wedge I^* \wedge \pi_{j+1}^{p,1-q,n} - \eta_j \mathbb{Z}^* \wedge I^* \wedge \pi_{j}^{p,1-q,n}
\]

Finally, since \( p+q = n \) we conclude that \( \kappa_{j+1}^{p+1,1+n+1} = \kappa_{j}^{p,q;n+1} \) and thus

\[
d_P \varphi_{n+1} = \sum_{p+q=n} \lambda_{p,q}(Z^* + \mathbb{Z}^*) \wedge \varphi_{p,q},
\]

which coincides with \( d_K \varphi_n \) due to (4.18). □

**Appendix A. Some technical computations**

In this appendix we derive explicit formulae for the image of the Poisson kernels \( \omega_{j}^{p,q;k} \) defined in Section 4.2 under the \( K \)-Hodge star \( *_{K} \), the \( K \)-codifferential \( \delta_{K} \) and the map \( L_{K}^{+} \) corresponding to the adjoint of the Lefschetz map on \( G/K \).

**A.1. The \( K \)-Hodge star.** Recall from Section 4.2 that for all positive integers \( p, q \) and \( k \) with \( 0 \leq p, q, k - p, k - q \leq n \), the \( M \)-invariant elements \( \omega_{j}^{p,q;k} \) are defined by

\[
\omega_{j}^{p,q;k} = \omega_{2,0}^{j} \wedge \omega_{1,1}^{p-j} \wedge \omega_{1,1}^{q-j} \wedge \omega_{0,2}^{k-(p+q)+j}.
\]

In order to obtain formulae for \( \delta_{K} \) and \( L_{K}^{+} \) it will be necessary to write expressions of the form \( \omega \wedge *_{K} \omega_{j}^{p,q;k} \) again as an image of \( *_{K} \), where \( \omega \) is any of the \( M \)-invariant elements in \( \Lambda^{*}(g/m)^{*} \) of degree 2.

We will start by identifying the pullback \( \omega_{K} \) of the Kähler form on \( G/K \) along the canonical projection \( \pi_{K}: G/M \rightarrow G/K \).

By construction this is a real and non-degenerate \( M \)-invariant form on \( (g/m)_{C} \) of degree \((2,0)\) and \( K \)-type \((1,1)\) and therefore given by a multiple of \( iZ^* \wedge \overline{Z}^* + \omega_{2,0} \).

Furthermore, using the relation \( \omega_{K}(J\xi, \eta) = g_{K}(\xi, \eta) \), where \( g_{K} \) is the pullback of the \( K \)-invariant Hermitian inner product on \( g/k \), we deduce that

\[
(\text{A.1}) \quad \omega_{K} = \frac{1}{2}(iZ^* \wedge \overline{Z}^* + \omega_{2,0}).
\]

From this description we obtain that the volume form \( \text{vol}_{K} := *_{K}(1) \) on the horizontal subspace \( p/m \) of \( g/m \) is given by

\[
(\text{A.2}) \quad \text{vol}_{K} = \frac{1}{2^{n+1}n!}iZ^* \wedge \overline{Z}^* \wedge \omega_{2,0}^{n}.
\]

In the next step we determine a formula which expresses the \( M \)-invariant forms \( *_{K} \omega_{j}^{p,q;k} \) in terms of powers of the invariant 2-form \( \omega_{2,0} \).

In order to do so, note that \( \omega_{2,0} \) is of degree \((0, 2)\) and thus commutes with the \( K \)-Hodge star by construction.

Thus, it suffices to consider those \( M \)-invariant elements \( *_{K} \omega_{j}^{p,q;k} \) which do not
contain $\omega_{0,2}$ as a factor. By definition, this means that $j = p + q - k \geq 0$, and for any choice of $p$, $q$ and $k$ there is precisely one such $M$-invariant form.

For simplification of the following proofs we introduce some notation. By a holomorphic $k$-vector $X^{1,0}_k$ we mean a $k$-tuple $X^{1,0}_k := (X_1^{1,0}, \ldots, X_k^{1,0})$ whose entries are the holomorphic parts of vectors $X_1, \ldots, X_k \in \mathbb{C}^n$. Next, we denote by $F_{X}^{1,0}$ the $k$-tuple $(F_{X_1}^{1,0}, \ldots, F_{X_k}^{1,0})$ and abbreviate by $i_{F_{X}^{1,0}}$ the interior product $t_{F_{X}^{1,0}}$. In a similar way we define antiholomorphic $k$-vectors $Y^{0,1}$ and the corresponding tuples $F_{Y}^{0,1}$ and interior products $i_{F_{Y}^{0,1}}$. Finally, we also define $G_{X}^{1,0}$ and $G_{Y}^{0,1}$ as well as their interior products in a similar fashion.

**Lemma A.1.** Let $p$, $q$ and $k$ be integers so that $0 \leq p, q, k - p, k - q \leq n$. Let $X^{1,0}$ be a holomorphic $(k - p)$-vector and $Y^{0,1}$ an antiholomorphic $(k - q)$-vector, respectively. Then

$$
(A.3) \quad t_{G_{X}^{1,0} t_{G_{Y}^{0,1}} * K \omega_{p+q-k}^{p,q,k}} = \epsilon^{p,q,k} iZ^* \wedge Z' \wedge t_{F_{X}^{1,0} t_{F_{Y}^{0,1}}} o_{2,0}^{n-(p+q)+k},
$$

where the factor $\epsilon^{p,q,k}$ is defined by

$$
\epsilon^{p,q,k} = (-1)^{\frac{1}{2}(p+q)(p+q+1)-(p+q-k)2p+q-(n+1)(p+q-k)(k-p)(k-q)!}{(n-(p+q)+k)!}.
$$

**Proof.** We will prove this formula by induction on the $P$-type of $* K \omega_{p+q-k}^{p,q,k}$. If the $P$-type equals $(0,0)$, then $k = p = q$, so there are no interior products in (A.3). From the explicit description of the Hermitian product $g_K$ we compute $g_K(\omega_{2,0}^{k,1}, \omega_{2,0}^{k,1}) = 2^{2k} \frac{n!k!}{(n-k)!}$, which together with expression (A.2) for the volume form yields

$$
* K \omega_{k,k}^{k;k} = * K \omega_{2,0}^{k} = 2^{2k-(n+1)} \frac{k!}{(n-k)!} iZ^* \wedge Z' \wedge \omega_{2,0}^{n-k}.
$$

Note that the overall factor on the right hand side coincides with $\epsilon^{k,k,k}$, which completes the argument for $P$-type $(0,0)$.

Now assume that (A.3) is shown for all $M$-invariant elements $* K \omega_{p+q-k}^{p,q,k}$ of $P$-type $(k-p, k-q)$ and consider $* K \omega_{p+q+1, k}^{p+q+1;k} = * K (i_{F_{X}^{1,0} t_{F_{Y}^{0,1}}} \omega_{1,1}^{p,q,k})$, which by definition is of $P$-type $(k-p+1, k-q)$. Using that $t_{G_{X}^{1,0} \omega_{1,1}^{p,q}} = (F_{X}^{1,0})^{*}$ for all $X \in \mathbb{C}^n$ a direct computation yields

$$
* K \omega_{p+q+1;k+1}^{p,q+1:k+1} = (-1)^{p+q+1} 2(k-p+1) t_{F_{X}^{1,0}} * K \omega_{p+q-k}^{p,q,k}.
$$

We apply this equation to (A.3) for $X = X_1$ and afterwards move the resulting interior product with $F_{X_1}^{1,0}$ to the left. On the remaining term we insert the induction hypothesis and finally use the relation

$$
(-1)^{p+q+1} 2(k-p+1) \epsilon^{p,q,k} = \epsilon^{p,q+1,k+1},
$$

thereby obtaining the claim. Since our formula is symmetric in the parameters $p$ and $q$ the same line of arguments can also be applied to $* K \omega_{p+q+1;k+1}^{p+q+1;k+1}$, which is of $P$-degree $(k-p, k-q+1)$. \hfill \Box

Using Lemma A.1 we are now able to express the wedge product of an $M$-invariant 2-form and $* K \omega_{p,q,k}^{p,q,k}$ again as an image of the $K$-Hodge star operator. This
plays an essential role in the derivation of explicit formul\ae for the $K$-codifferential $\delta_K$ and the operator $\mathcal{L}^*_K$ in the following section.

**Theorem A.2.** Let \( p, q \) and \( k \) be integers so that \( 0 \leq p, q, k - p, k - q \leq n \). Then:

1. \( \omega_{1,1} \wedge *K\omega_{p,q;k} = (-1)^{p+q}2i \left( j *K\omega_{j-1,p-q-1;k} + (q - j) *K\omega_{p,q-1;k} \right) \)

2. \( \overline{\omega_{1,1}} \wedge *K\omega_{p,q;k} = (-1)^{p+q+1}2i \left( j *K\omega_{j-1,p-q+1;k} + (p - j) *K\omega_{p,q+1;k} \right) \)

3. \( \omega_{2,0} \wedge *K\omega_{p,q;k} = 4j(n + 1 - (p + q) + j) *K\omega_{p-1,q-1;k-1} - 4(p - j)(q - j) *K\omega_{p-1,q-1;k-1} \)

**Proof.** Since we can move the \((0,2)\)-form $\omega_{0,2}$ outside of the $K$-Hodge star it suffices to consider those forms $\omega_{p,q;k}$ in which $\omega_{0,2}$ does not appear as a factor. By definition, this corresponds to $j = p + q - k$ for which Lemma A.1 is applicable.

For part (i), let $X^{1,0}$ and $Z^{1,0}$ be holomorphic vectors on $\mathbb{C}^n$ of length $n - q + 1$ and $k - p$, respectively, and let $Y^{0,1}$ and $W^{0,1}$ be antiholomorphic vectors of length $n - p$ and $k + q + 1$, respectively. Then we rewrite the expression

\[
(A.4) \quad (\omega_{1,1} \wedge *K\omega_{p,q;k})(Z, Z, F_X^{1,0}, F_Y^{0,1}, F_Z^{1,0}, G_W^{1,0})
\]

as follows. First, we expand the wedge product by inserting the vectors $F_X^{1,0}$ and $G_W^{1,0}$ into $\omega_{1,1}$, where we pick up the signs \((-1)^{s+1}\) from the first vector and \((-1)^{j-k+q+1}\) from the second vector. Next, we use the relation $\omega_{1,1}(F_X^{1,0}, G_W^{1,0}) = -i\omega_{2,0}(F_X^{1,0}, W^{1,0})$ and apply Lemma A.1 to $*K\omega_{p,q;k}$, which adds the coefficient $i^{p+q}$. All in all we obtain that (A.4) equals

\[
\sum_{s,t} (-1)^{k-q+s+t} \omega_{p,q;k} \omega_{2,0}(F_X^{1,0}, F_Y^{0,1}, F_Z^{1,0}, F_W^{1,0}) \omega_{2,0}(F_X^{1,0}, F_Y^{0,1}, F_Z^{1,0}, F_W^{1,0})
\]

where the wedge product $F_X^{1,0}$ of length $(n - q)$ is obtained from $F_X^{1,0}$ by omitting the $s$th factor, and similarly for $F_W^{1,0}$. Using that the Poisson kernel $\omega_{2,0}$ is of $K$-type $(1,1)$, the expression (A.4) splits into the sum (A) + (B), where

\[
(A) := \frac{(-1)^{k-p+1}}{n-(p+q)+k+1} \omega_{p,q;k} \omega_{2,0}(F_X^{1,0}, F_Y^{0,1}, F_Z^{1,0}, F_W^{1,0})
\]

\[
(B) := \sum_{r=1}^{k-p+1} \sum_{t=1}^{k-q+1} (-1)^{k-p+r+t} \omega_{p,q;k} \omega_{2,0}(F_X^{1,0}, F_Y^{0,1}, F_Z^{1,0}, F_W^{1,0})
\]

Since (A) is given by a power of $\omega_{2,0}$ we can immediately apply Lemma A.1 and a direct computation yields

\[
(A) = (-1)^{p+q}2i(p + q - k) \left( *K\omega_{p,q-1;k-1} \right) (Z, Z, F_X^{1,0}, F_Y^{0,1}, G_Z^{1,0}, G_W^{1,0}).
\]

In order to simplify (B) we first apply Lemma A.1 to replace $\omega_{2,0}^{n-(p+q)+k}$ with the invariant form $*K\omega_{p+q-1;k-1}$ as well as change $\omega_{2,0}(F_X^{1,0}, F_W^{1,0})$ to $\omega_{0,2}(G_Z^{1,0}, G_W^{1,0})$. Finally, since the Poisson kernel $\omega_{0,2}$ is of degree $(0,2)$ it can be moved into the $K$-Hodge star. All in all, a direct computation yields

\[
(B) = (-1)^{p+q}2i(k - p) \left( *K\omega_{p+q-1;k-1} \right) (Z, Z, F_X^{1,0}, F_Y^{0,1}, G_Z^{1,0}, G_W^{1,0}).
\]
Since both expressions (A) and (B) have the same arguments as (A.1), the corresponding $M$-invariant forms have to coincide, proving (i). Furthermore, if we exchange $p$ and $q$ and apply complex conjugation we immediately obtain the claimed formula in (ii).

Finally, to prove (iii) let $X^{1,0}$ and $Z^{1,0}$ be holomorphic vectors on $\mathbb{C}^n$ of length $n - q + 1$ and $k - p$, respectively, and $Y^{0,1}$ and $W^{0,1}$ be antiholomorphic vectors on $\mathbb{C}^n$ of length $n - p + 1$ and $k - q$, respectively. We proceed similar as in the proof of (i) and rewrite

\[(A.5) \quad \left( \omega_{2,0} \wedge \ast_K \omega_{p,0}^{q,k} \right)(Z, \overline{Z}, F_{X}^{1,0}, F_{Y}^{0,1}, G_{Z}^{1,0}, G_{W}^{0,1})\]

as follows. First, we expand the wedge product by inserting the vectors $F_{X}^{1,0}$ and $F_{Y}^{0,1}$ into $\omega_{2,0}$, which adds the sign $(-1)^{n-q+r+s}$. Next, we apply Lemma A.1, thereby changing $\ast_K \omega_{p,0}^{q,k}$ to a multiple of $\omega_{2,0}^{n-(p+q)-k}$. All in all we obtain that (A.5) coincides with

\[
\sum_{r,s} (-1)^{n-q+r+s} i^{p,q,k}_{\ast_K} \omega_{2,0} \left( F_{X}^{1,0}, F_{Y}^{0,1} \right) \omega_{2,0}^{n-(p+q)-k} \left( F_{X}^{1,0}, F_{Y}^{0,1}, G_{Z}^{1,0}, G_{W}^{0,1} \right),
\]

which can be written as \((C) + (D)\), where

\[
(C) := \frac{n-q+1}{n(p+q)+k+1} i^{p,q,k}_{\ast_K} \omega_{2,0}^{n-(p+q)+k+1} \left( F_{X}^{1,0}, F_{Y}^{0,1}, F_{Z}^{1,0}, F_{W}^{0,1} \right),
\]

\[
(D) := \sum_{r,s} i^{p,q,k} (-1)^{k-q+r+s+1} \omega_{2,0} \left( F_{X}^{1,0}, F_{W_{r}}^{0,1} \right) \omega_{2,0}^{n-(p+q)+k} \left( F_{X}^{1,0}, F_{Y}^{0,1}, F_{Z}^{1,0}, F_{W_{r+1}}^{0,1} \right).
\]

For (C) we immediately apply Lemma A.1 which by a direct computation yields

\[
(C) = 4(n - q + 1)(p + q - k) \ast_K \omega_{p+q-k-1}^{1,q-k-1}(Z, \overline{Z}, F_{X}^{1,0}, F_{Y}^{0,1}, G_{Z}^{1,0}, G_{W}^{0,1}).
\]

On the other hand, the expression for (D) has the same form as (A.4), hence by a direct computation we obtain

\[
(D) = (-1)^{p+q+1} 2i(k - q) \left( \omega_{1,1} \wedge \ast_K \omega_{p+q-k-1}^{1,q-k-1} \right)(Z, \overline{Z}, F_{X}^{1,0}, F_{Y}^{0,1}, G_{Z}^{1,0}, G_{W}^{0,1}).
\]

All in all we have

\[
\omega_{2,0} \wedge \ast_K \omega_{p,0}^{q,k} = 4(n - q + 1)(p + q - k) \ast_K \omega_{p+q-k-1}^{1,q-k-1}
\]

\[
+ (-1)^{p+q+1} 2i(k - q) \omega_{1,1} \wedge \ast_K \omega_{p+q-k-1}^{1,q-k-1},
\]

so by applying formula (i) the claim follows. \qed

A.2. Formulae for $\delta_K$ and $\mathcal{L}_K^*$. Following the previous section we now determine explicit formulae for the images of the $M$-invariant forms $\omega_{p,0}^{q,k}$ under the $M$-equivariant maps $\delta_K$ and $\mathcal{L}_K^*$.

Recall from Section 3.4 that the decomposition of Poisson kernels into $K$-types induces a splitting $d_K = \partial_K + \overline{\partial}_K$, where the first and second operator map Poisson kernels of $K$-type $(p, q)$ to those of $K$-type $(p + 1, q)$ and $(p, q + 1)$, respectively. Indeed, the analogous arguments as in Proposition 3.3 show that if $\Phi$ is any Poisson transform with kernel $\varphi$, then $\partial \circ \Phi$ is again a Poisson transform with kernel $\partial K \varphi$, and similarly for its complex conjugate. Furthermore, the operators $\partial_K$ and $\overline{\partial}_K$ are again antiderivations, square to zero, anticommute with each other and are related via $\overline{\partial}_K \varphi = \overline{\partial}_K \varphi$ for all $M$-invariant forms $\varphi$ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$. 



Similarly, the $K$-codifferential decomposes as $\delta_K = \partial^*_K + \overline{\partial}_K$, where the first and second operator map Poisson kernels of $K$-type $(p, q)$ to those of $K$-type $(p - 1, q)$ and $(p - 1, q - 1)$, respectively. Explicitly, we have $\partial^*_K = -*_{K} \frac{\partial}{\partial K}$ and similarly for $\overline{\partial}_K$, so since the $K$-Hodge star is complex linear they are related via $\partial^*_K \overline{\varphi} = \overline{\partial}^*_K \varphi$ for all Poisson kernels $\varphi$.

**Proposition A.3.** Let $p$, $q$ and $k$ be integers so that $0 \leq p, q, k - p, k - q \leq n$. Then we have

\[
\partial^*_K(Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}) = 2(n - k)Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1} + 2ij(k - (p + q) + j)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1} - 2i(p - j)(n - k + p - j + 1)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1}.
\]

Furthermore, this implies

\[
\partial^*_K(Z^* \wedge \omega_j^{p,q;k}) = 2(n - 1 - k)\omega_j^{p,q;k} + 2ij(n - 1 - (p + q) + j)Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;k-1} - 2i(p - j)(q - j)Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1,k-1} + 2ij(k - (p + q) + j)I^* \wedge Z^* \wedge \omega_j^{p-1,q;k-1} - 2i(p - j)(n - k + q - j + 1)I^* \wedge Z^* \wedge \omega_j^{p-1,q;k-1},
\]

\[
\partial^*_K(\overline{Z}^* \wedge \omega_j^{p,q;k}) = 2ij(k - (p + q) + j)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1} - 2i(p - j)(n - k + q - j + 1)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1},
\]

as well as

\[
\partial^*_K(I^* \wedge \omega_j^{p,q;k}) = 2ij(n - 1 - (p + q) + j)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;k-1} - 2i(p - j)(q - j)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1,k-1},
\]

\[
\partial^*_K(I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}) = 2(k - n + 1)I^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}.
\]

**Proof.** In order to compute the image of $\omega := Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}$ under $\partial^*_K$ we have to determine the image of $*_{K} \omega$ under the partial derivative $\overline{\partial}_K$. Recall that for an $M$-invariant linear functional $\alpha$ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}^\ast$, the exterior derivative of the corresponding invariant differential form is induced by the functional $d\alpha$ which sends vectors $X_0 + m, \ldots, X_m + m \in \mathfrak{g}/\mathfrak{m}$ to

\[
(A.6) \quad \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j] + m, X_0 + m, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, X_m + m),
\]

where the hats denote omission. Furthermore, if $\alpha$ is of $K$-type $(r, s)$ and bidegree $(r + s, \ell)$, $\overline{\partial}_K \alpha$ is computed using (A.6) by inserting $\ell$ vectors in $(\mathfrak{f}/\mathfrak{m})_{\mathbb{C}}$ and $r + s + 1$ vectors in $(\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ of which $r$ are holomorphic and $s + 1$ are antiholomorphic.

In our case the element $\alpha = *_{K} \omega$ is trivial upon insertion of any invariant vector $Z, \overline{Z}$ and $I$. Furthermore, any representative of these vectors in $\mathfrak{g}$ is contained in the direct sum of all even grading components, whereas for $X, Y \in \mathbb{C}^n$ the representatives $\xi_X$ of $F_X$ and $\eta_Y$ of $G_Y$ with trivial $m$-part are contained in the odd grading components. Therefore, the vector $[X_i, X_j] + m$ is $M$-invariant unless
it is induced by a pairing of an invariant vector and one of the vectors \( \xi_X \) or \( \eta_Y \). In addition, using the bracket relations on \( g \) the pairing with \( Z \) preserves holomorphic and antiholomorphic vectors, hence by the \( K \)-type of \( \ast_K \omega \) the corresponding summands in (A.6) are trivial. All in all we deduce that

\[
(A.7) \quad \bar{\partial}_K \ast_K \omega = Z^\ast \wedge \iota_Z \bar{\partial}_K \ast_K \omega + \iota_I \wedge \iota_I \bar{\partial}_K \ast_K \omega.
\]

For the first summand in (A.7), let us denote the representative of \( Z \) in \( g \) with trivial \( m \)-part by the same symbol. Then the pairings of this vector with the holomorphic and antiholomorphic parts of \( \xi_X \) and \( \eta_Y \) are given by

\[
[Z, \xi_X^{1,0}] = \frac{1}{2} \xi_X^{1,0}, \quad [Z, \xi_X^{0,1}] = \frac{1}{2} \xi_X^{0,1}, \quad [Z, \eta_Y^{1,0}] = 0, \quad [Z, \eta_Y^{0,1}] = 2 \xi_Y^{0,1} - \frac{1}{2} \eta_Y^{0,1}.
\]

Therefore, since \( \ast_K \omega \) is of bidegree \( (2n - (p+q), 2k - (p+q)) \) a direct computation using (A.6) shows that

\[
(A.8) \quad \iota_Z \bar{\partial}_K \ast_K \omega = (k - n) \ast_K \omega.
\]

In order to compute the other summand of (A.7), we write \( \omega \) as the wedge product of \( \omega_{0,2}^{k-(p+q)+j} \) and \( \tilde{\omega} := Z^\ast \wedge Z^\ast \wedge \omega_j^{p,q,p+q-j} \), which does not contain \( \omega_{0,2} \) as a factor. Since both \( \bar{\partial}_K \) and \( \iota_I \) are antiderivations and \( \iota_I \ast_K \omega = 0 \) we get

\[
(A.9) \quad \iota_I \bar{\partial}_K \ast_K \omega = (\iota_I \bar{\partial}_K \ast_K \tilde{\omega}) \wedge \omega_{0,2}^{k-(p+q)+j} + (\ast_K \tilde{\omega}) \wedge \iota_I \bar{\partial}_K \omega_{0,2}^{k-(p+q)+j}.
\]

From Section 3.6 we know that \( \iota_I \bar{\partial}_K \omega_{0,2} = -\omega_{1,1} \), which we insert into the above expression and afterwards apply Theorem A.2(ii) to rewrite \( \omega_{1,1} \wedge \ast_K \tilde{\omega} \) as an image of the \( K \)-Hodge star operator.

Thus, it remains to compute \( \iota_I \bar{\partial}_K \ast_K \tilde{\omega} \), which will be done using formula (A.6). Let us denote the representative of \( I \) in \( g \) with trivial \( m \)-part by the same symbol. Regarding the degree of \( \ast_K \tilde{\omega} \) a moment of thought shows that the only nontrivial contribution in (A.6) is given by the summands containing the pairing of \( I \) with \( \xi_X \) for \( X \in \mathbb{C}^n \). Now a direct computation shows that

\[
[I, \xi_X^{0,1}] = i(\xi_X^{1,0} - \eta_X^{1,0}), \quad [I, \xi_X^{1,0}] = -i(\xi_X^{0,1} - \eta_X^{0,1}),
\]

for all \( X, Y \in \mathbb{C}^n \), so since the pairing with \( I \) respects holomorphic and antiholomorphic vectors the only nontrivial pairing in (A.6) is between \( I \) and \( \xi_X^{0,1} \). Therefore, using relation

\[
\iota_I \bar{\partial}_K \ast_K \tilde{\omega} = (-1)^{p+q+2(p-j)} i(p-j) \iota_I \bar{\partial}_K \ast_K \big( Z^\ast \wedge Z^\ast \wedge \ast_K \omega_j^{p-1,q,p+q-j-1} \big),
\]

which follows from Lemma A.1 a direct computation yields

\[
(A.9) \quad \iota_I \bar{\partial}_K \ast_K \tilde{\omega} = (-1)^{p+q+2} i(p-j)(n-q+1) \ast_K \big( Z^\ast \wedge Z^\ast \wedge \omega_j^{p-1,q,p+q-j-1} \big).
\]

Inserting this back into (A.9) and move the power of \( \omega_{0,2} \) back into the \( K \)-Hodge star we obtain by a direct computation that

\[
(A.10) \quad \iota_I \bar{\partial}_K \ast_K \omega = (-1)^{p+q+2} i(p-j)(k-n-1+p+j) \ast_K \big( Z^\ast \wedge Z^\ast \wedge \omega_j^{p-1,q,k-1} \big)
\]

\[\quad \quad \quad \quad \plus (-1)^{p+q+2} i(p-j)(k-n-1+p+j) \ast_K \big( Z^\ast \wedge Z^\ast \wedge \omega_j^{p-1,q,k-1} \big), \]

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Finally, we insert (A.8) and (A.10) into (A.7) and apply the negative of the $K$-Hodge star to the resulting equation. Then a direct computation yields the claimed formula.

The formulae for the images of the other Poisson kernels under $\partial_K^*$ follow by combining the above with the formulae for $K$-derivative of the Poisson kernels of low degree from 3.6 as well as the adjointness of the wedge product and the interior product with respect to the $K$-Hodge star. As an example, we compute the $K$-codifferential of $\iota_Z\omega = \overline{Z}^* \wedge \omega_j^{p,q;k}$. In order to do so, we first apply the $K$-Hodge star to this element, which by the relation $2Z^0 = \overline{Z}^*$ coincides with $\frac{1}{2}(-1)^{p+q}\overline{Z}^* \wedge \ast_K\omega$. Next, we apply the operator $\overline{\partial}_K$, where we use that $\overline{\partial}_K\overline{Z}^* = 0$ (c.f. Section 3.6). All in all,

$$\partial_K^* \left( \overline{Z}^* \wedge \omega_j^{p,q;k} \right) = \frac{1}{2}(-1)^{p+q} \ast_K \left( \overline{Z}^* \wedge \overline{\partial}_K \ast_K\omega \right) = -\iota_Z \partial_K^* \omega,$$

where we used again the adjointness of the interior and the wedge product for the last equality. □

For the rest of this section we derive formulae for the adjoint of the $K$-Lefschetz map. In order to do so, let $\omega_K$ be the pullback of the Kähler form on $G/K$ and as before denote the corresponding $M$-invariant element in $\Lambda^{2,0}(g/m)^*$ by the same symbol. Recall from Section 3.3 the definitions of the $K$-Lefschetz map $L_K$ and its adjoint $L_K^*$. Since these maps are $G$-equivariant, they induce $M$-equivariant linear maps on the level of the underlying representations, which we denote by the same symbols.

**Proposition A.4.** Let $p, q$ and $k$ be integers so that $0 \leq p, q, k - p, k - q \leq n$. Then

$$L_K^*\omega_j^{p,q;k} = 2j(n + 1 - (p + q) + j)\omega_j^{p-1,q-1;k-1} - 2(p - j)(q - j)\omega_j^{p-1,q-1;k-1}. $$

Furthermore, we have

$$L_K^*(Z^* \wedge \omega_j^{p,q;k}) = Z^* \wedge L_K^*\omega_j^{p,q;k}, \quad L_K^*(\overline{Z}^* \wedge \omega_j^{p,q;k}) = \overline{Z}^* \wedge L_K^*\omega_j^{p,q;k},$$

as well as

$$L_K^*(I^* \wedge Z^* \wedge \omega_j^{p,q;k}) = I^* \wedge Z^* \wedge \overline{Z}^* \wedge L_K^*\omega_j^{p,q;k} - 2iI^* \wedge \omega_j^{p,q;k}.$$

**Proof.** Recall that the pullback of the Kähler form is $\omega_K = \frac{1}{2}(iZ^* \wedge \overline{Z}^* + \omega_{2,0})$. Inserting this into the formula for $L_K^*\alpha$, where $\alpha \in \Lambda^*(g/m)_C^*$ is $M$-invariant, and using the relation $\left(Z^* \wedge \overline{Z}^* \wedge \ast_K\alpha \right) = 4 \ast_K \iota_Z \overline{\partial}_K \omega 2 \alpha$ we deduce

(A.11)$$L_K^*\alpha = 2iZ\iota_Z\alpha + \frac{1}{2} \ast_K^{-1} (\omega_{2,0} \wedge \ast_K\alpha).$$

In the case $\alpha = \omega_j^{p,q;k}$ the invariant vectors $Z$ and $\overline{Z}$ insert trivially into $\alpha$, so the first summand of (A.11) vanishes. For the second summand of (A.11) we can apply Theorem A.2(ii), which yields

$$L_K^*\omega_j^{p,q;k} = 2j(n + 1 - (p + q) + j)\omega_j^{p-1,q-1;k-1} - 2(p - j)(q - j)\omega_j^{p-1,q-1;k-1}. $$
The expressions for the other Poisson kernels follow analogously.

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