Space-time anisotropy: theoretical issues and the possibility of an observational test.

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Abstract

The specific astrophysical data collected during the last decade causes the need for the modification of the expression for the Einstein-Hilbert action, and several attempts sufficing this need are known. The modification suggested in this paper stems from the possible anisotropy of space-time and this means the natural change of the simplest scalar in the least action principle. To provide the testable support to this idea, the optic-metrical parametric resonance is regarded - an experiment on the galactic scale based on the interaction between the electromagnetic radiation of cosmic masers and periodical gravitational waves emitted by close double systems or pulsars. Since the effect depends on the space-time metric, the possible anisotropy could reveal itself through observations. To give the corresponding theory predicting the corrections to the expected results of the experiment, the specific mathematical formalism of Finsler geometry was chosen. It was found that in case the anisotropy of the space-time exists, the orientation of the astrophysical systems suitable for observations would show it. In the obtained geodesics equation there is a direction dependent term.

1 Introduction

The amount of astrophysical data collected during the last decade and contradicting the mainstream solutions of general relativity makes one think that the least action principle based on the "simplest scalar" in the form of Ricci scalar curvature does not work properly in some important cases (e.g. rotation curves for spiral galaxies [1]). The efforts to improve the situation were undertaken in the following directions: complicating of the existing scalar (including
$f(R)$ theories); change of the scalar (for example, the use of Weyl tensors); introduction of additional scalar fields (including MOND theory); passing to a non-symmetrical metric.

An unusual, but fundamental assumption of the anisotropy of the space-time means that in this case its geometry must be not Riemannian, but Finslerian. It means that the "simplest scalar" will become more complicated in a natural way. Obviously, this brings a lot of consequences which means that, first of all, we need a way to test the validity of this theoretical approach for real physics. The corresponding test can be performed with the help of observations based on the effect of the optic-metrical parametric resonance (OMPR). In this paper we give the brief theory of the OMPR effect in order to understand what kind of corrections due to anisotropy might be of theoretical and experimental interest. After that we introduce the mathematical formalism needed to construct simple models of the anisotropic space and then regard two types of metrics. Every time we present the modifications of the OMPR conditions that could make it possible to discover the space-time anisotropy (if any) on the galactic scale. In the Discussion the main results are given.

2 Brief theory of the OMPR effect

Let us regard a two-level atom in the strong monochromatic quasi-resonant field. The system of Bloch’s equations for the components of the density matrix components is

$$\frac{d}{dt}\rho_{22} = -\gamma \rho_{22} + 2i\alpha_1 \cos(\Omega t - ky)(\rho_{21} - \rho_{12})$$

$$\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial y}\right)\rho_{12} = -(\gamma_{12} + i\omega)\rho_{12} - 2i\alpha_1 \cos(\Omega t - ky)(\rho_{22} - \rho_{11})$$

$$\rho_{22} + \rho_{11} = 1$$

Here $\rho_{22}$ and $\rho_{11}$ are the populations of the levels, $\rho_{12}$ and $\rho_{21}$ are the polarization terms, $\gamma$ and $\gamma_{12}$ are the longitudinal and transversal decay rates of the atom (if level 1 is the ground level, $\gamma_{12} = \gamma/2$); $\alpha_1 = \frac{\mu E}{\hbar}$ is the Rabi parameter (Rabi frequency) proportional to the intensity of the electromagnetic wave (EMW), $\mu$ is the dipole momentum, $E$ is the electric stress, $k$ is the $y$-component of the wave vector of the EMW, $v$ is the atom velocity along the $Oy$-axis pointing at the detector, $\gamma << \alpha_1$ is the condition of the strong field. The dynamics of this system in cases when various parametric resonances are possible was investigated in[2].

Let this atom belong to a saturated space maser in the field of the periodic gravitational wave (GW) emitted by a compact binary star or by a pulsar and propagating anti-parallel to the $Ox$-axis pointing at the GW-source. The GW acts on the atomic levels, on the maser radiation and on the geometrical location of the atom. In [3] it was shown that the first effect is much smaller than the
other two effects. The action of the GW on the monochromatic EMW could be accounted for by the solution of the eikonal equation

$$g^{ik} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^k} = 0; \quad i, k = 1 \div 4$$ (2)

The atom velocity, \(v\), could be obtained from the solution of the geodesic equation

$$\frac{d^2 x^i}{ds^2} + \Gamma^{i}_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0; \quad i, k, l = 1 \div 4$$ (3)

(and not from the solution of the geodesic declination equation as in the calculations of the displacements of the parts of the laboratory setups, designed for the detection of the GW).

The equations (1-3) are basic for the theory of the OMPR effect. Such a signal being registered could provide the possibility to detect the GW in a principally new way which differs from the other 18 known ones [4] by the fact that the OMPR effect is of the zero order and not of the first order in the non-dimensional amplitude of the GW.

If we use the regular Riemann geometry, the solution gives the following.

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The weak gravitational field in the empty space (far from masses) is described by the linearized Einstein equations. Then the perturbation \(h^{ki}\) of the flat space metric tensor (Minkowski metric) suffices the wave equation, which has the solution [8]

$$h^{ki} = \text{Re}[A^{ki} \exp(iK_{\alpha}x^{\alpha})]$$ (4)

where \(K^{\alpha}\) is a light-like vector, i.e. it satisfies the equation \(K_{\alpha}K^{\alpha} = 0\). Then the perturbed metric tensor in the isotropic case can be written as

$$g^{ik} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 + h \cos \frac{D}{c}(x^1 - x^2) & 0 \\
0 & 0 & 0 & -1 - h \cos \frac{D}{c}(x^1 - x^2)
\end{pmatrix}$$ (5)

where \(h\) is the dimensionless (small) amplitude of the GW, \(D\) is the frequency of the GW, \(x^i\), \((i = 1 \div 4)\) are the coordinates.

The solution of (2) with regard to (5)

$$\psi = \frac{\omega}{c} x^1 + \sum_{i=2}^{4} k_i x^i - h c^2 k_3^2 - k_4^2 \sin \frac{D}{c} (x^1 - x^2)$$ (6)

shows that the action of the GW causes a phase modulation of the EMW. Since \(h\) is very small, the phase modulated EMW can be presented (in Cartesian coordinates) as a superposition [9]

$$E(t) = E \cos(\Omega t - ky) + E \frac{\omega}{8D} h \cos((\Omega - D)t - ky) - \cos((\Omega + D)t - ky)$$ (7)
The solution of (3) with regard to (5) gives

\[ y(t) \sim h \frac{c}{D} \sin(Dt + Kx) \] (8)

where \( K \) is the GW wave vector. The expression (8) makes it possible to get the component of the atom velocity directed towards the Earth

\[ v = v_0 + v_1 \cos Dt \] (9)

\[ v_1 = \frac{hc}{8} \]

Substituting (9) and (7) into (1), one gets

\[ \frac{d}{dt} \rho_{22} = -\gamma \rho_{22} + 2i[\alpha_1 \cos(\Omega t - ky) + \alpha_2 \cos((\Omega - D)t - ky)](\rho_{21} - \rho_{12}) \] (10)

\[ \frac{d}{dt} \rho_{12} = -\gamma_{12} - \omega \rho_{12} - 2i[\alpha_1 \cos(\Omega t - ky) + \alpha_2 \cos((\Omega - D)t - ky)](\rho_{22} - \rho_{11}) \] (11)

\[ \rho_{22} + \rho_{11} = 1 \]

where \( \alpha_2 = \frac{\omega h}{8D} \alpha_1 \), and (9) was used in the the expression for the full time derivative \( \frac{d}{dt} = \frac{\partial}{\partial t} + kv \). The solution of the system (10) is performed by the asymptotical expansion method, the small parameter being \( \varepsilon = \frac{\alpha_2}{\alpha_1} \) (notice, that \( \frac{\alpha_2}{\alpha_1} \sim \varepsilon \) too). The essential point is the OMPR conditions

\[ \frac{\gamma}{\alpha_1} = \Gamma \varepsilon; \Gamma = O(1); \varepsilon << 1 \] (13)

\[ \frac{\alpha_2}{\alpha_1} = \frac{\omega h}{8D} = b \varepsilon; b = O(1); \varepsilon << 1 \] (14)

\[ \frac{k \nu_1}{\alpha_1} = \frac{\omega h}{8D} = \kappa \varepsilon; \kappa = O(1); \varepsilon << 1 \] (15)

\[ (\omega - \Omega + k \nu_0)^2 + 4\alpha_1^2 = D^2 + O(\varepsilon) \Rightarrow D \sim 2\alpha_1 \] (16)

If they are fulfilled, then the principal term of the asymptotic expansion for \( \text{Im}(\rho_{21}) \) which characterizes the scattered radiation energy flow can be calculated explicitly. The effect of the OMPR is that at a frequency shifted by \( D \) from the central peak of the EMW (that is from the signal of the space maser), the energy flow is proportional to \( \varepsilon^0 \), i.e. has zero order in the powers of the small parameter of the expansion, and has the form

\[ \text{Im}(\rho_{21}) \sim \frac{\alpha_1}{D} \cos 2Dt + O(\varepsilon) \] (17)

It means that the energy flow at this frequency is periodically amplified and attenuated with the (doubled) frequency of the GW. The OMPR signal can be
registered with the help of the special statistical processing of the radio telescope signal. The registration of such a signal would give a new experimental evidence of the GW existence and since the value of the OMPR signal is comparable to that of the regular maser peak, such observations could lead to the design of a GW map of the sky.

Turning back to the problems mentioned in the Introduction, one can suggest that such kind of measurements could be able to give an evidence of the space-time anisotropy (if any) on the galactic scale - that is on the scale where problems appear. In this case the use of the Finslerian Hilbert-Einstein action instead of the Riemannian one would be grounded. In order to develop a theory supporting such forthcoming observational results, we have first of all to show explicitly that Einstein equations in empty space for the anisotropic case still have the form of the wave equation though its solution might become dependent on the direction. (In the qualitative analysis presented in [5], [6], [7] it was presumed so). Then we have to perform the corresponding modifications of the eikonal and geodesics equations and use the results to describe the changes in the OMPR signal.

3 Mathematical formalism and basic equations

Let \( M = \mathbb{R}^4 \) be regarded as a differentiable 4-dimensional manifold of class \( C^\infty \), \( TM \) its tangent bundle, and \( (x, y) = (x^i, y^i) \); \( i = 1, \ldots, 4 \) the coordinates in a local frame. We call locally Minkowskian a metric with the property that there exists a system of local coordinates on \( TM \) in which it does not depend on the positional variables, \( x^i \), but may depend on the directional variables, \( y^i = \frac{\partial x^i}{\partial t} \) (\( t \) is a parameter), \( i = 1 \div 4 \). Let us regard a metric tensor \( g_{ij}(x, y) = \gamma_{ij}(y) + \varepsilon_{ij}(x, y) \), \( \forall (x, y) \in TM \), where \( \gamma = \gamma(y) \) is a Finsler-locally Minkowskian undeformed metric tensor on \( M \) and \( \varepsilon = \varepsilon(x, y) \) is a small anisotropic deformation of \( \gamma \). In anisotropic spaces, the tangent spaces \( T_x M, x \in M \) are generally curved. The general approach we shall use was developed by R. Miron and M. Anastasiei [10], [11] and is known as h-v metric model formalism. Some specific models close to that under discussion here were given in [12] and [13]. In these models, \( TM \) turns to be a Riemannian manifold of dimension 8.

The Ricci scalar \( \mathcal{R} = G^{\alpha\beta} R_{\alpha\beta} \) used in the Einstein-Hilbert action in the case of h-v models [10], [13] leads to the expression \( \mathcal{R} = R + S \), where \( S \) is the Ricci scalar of the tangent (Riemannian) space \( T_x M \), for \( x \in M \). In our approach, we shall choose for simplicity a model in which \( S = 0 \) while \( R \) depends on the direction, \( y \), through the \( y \)-dependence of the Christoffel symbols, \( \Gamma^i_{jk} \), calculated with regard to \( g(x, y) \), that is \( \Gamma^i_{jk} = \frac{1}{2} \gamma^{ij} (\frac{\partial \varepsilon_{hk}}{\partial x^k} + \frac{\partial \varepsilon_{hk}}{\partial x^j} - \frac{\partial \varepsilon_{jk}}{\partial x^h}) \).

The only components of the Ricci tensor \( R_{\alpha\beta} \) (\( R_{jk}, P_{bj}, P_{jb}, S_{ab} \)) that do not identically vanish for our model are \( R_{jk} = \frac{\partial \Gamma^i_{jk}}{\partial x^i} - \frac{\partial \Gamma^i_{ji}}{\partial x^k} + \Gamma^h_{jk} \Gamma^i_{hi} - \Gamma^h_{ji} \Gamma^i_{hk} \) and the "mixed" component \( P_{jb} = P^i_{j,ib} = \frac{1}{2} (\delta^i_j g^{li} - \gamma^{il} \gamma_{sj}) \frac{\partial \Gamma^i_{li}}{\partial y^b} \). Hence, the
Ricci scalar in this simple model turns to be $R = \gamma^{jk} R_{jk}$.

3.1 Einstein equations

The Einstein equations for the empty space [10], [11], [13] appear to be, for our linearized model:

$$R_{ij} - \frac{1}{2}R \gamma_{ij} = 0 \quad (18)$$

$$(\delta^i_l \delta^j_s - \gamma^i_l \gamma^j_s) \frac{\partial \Gamma^s_{li}}{\partial y^b} = 0 \quad (19)$$

The first set of equations (18) involves only the $x$-derivatives of the deformation, $\varepsilon$, while the second ones, (19), contain mixed derivatives of $\varepsilon$ of the second order. In order to integrate the first equations (18), we apply the same procedure as in the classical Riemannian case and look for the solutions satisfying the harmonic (Lorentz) gauge conditions $\gamma^{ij} \Gamma^k_{ij} = 0$, which are actually

$$\frac{\partial \varepsilon^i}{\partial x^j} - \frac{1}{2} \frac{\partial \varepsilon}{\partial x^j} = 0. \quad (20)$$

Consequently, the first set of equations (18) becomes

$$\square \varepsilon_{ij} = 0; \quad (21)$$

which demonstrates explicitly the existence of the GW in the anisotropic space. We look for a wave solution in the form

$$\varepsilon_{jh} = \text{Re}(a_{jh}(y)e^{iK_m(y)x^m}), \quad (22)$$

where $i$ denotes the imaginary unit. Strictly speaking, we should take the perturbation as a series each term of which corresponds to a wave. But for simplicity, we regard only one term of this series. Both the amplitude $a_{jh}(y)$ and the wave vector $K(y)$ of the GW are no longer isotropic, but may depend on direction. Substituting (22) into (21) we see that either $\varepsilon$ itself is zero, or we must have $\gamma^{hl} K_h K_l = 0$. By (18), and (20), we infer that in the anisotropic space $a_{jh}(y)$ and $K_m(y)$ should obey the algebraic system

$$\gamma^{hl} K_h K_l = 0$$

$$a^i_{j} K_i = \frac{1}{2} a^i_{i} K_j \quad (23)$$

Thus, the wave solutions (22) obeying (21) of the Einstein equations (18) must suffice:

$$\gamma^{hl} K_h K_l = 0$$

$$a^i_{j} K_i = \frac{1}{2} a^i_{i} K_j$$

$$\frac{\partial}{\partial y^b}(\frac{1}{2} a^i_{i} K_j) = C^i_{lb}(2a^l_{j} K_i - a^l_{i} K_j). \quad (24)$$
where the third equation comes from the "mixed Einstein equation" \( [19] \), \( C^i_{\; lb} = \frac{1}{2} \gamma^{ib} \frac{\partial \gamma^{hl}}{\partial y^i} \). We also see that the amplitude \( a_{ij} \) and the wave vector \( K_i \) now depend on each other.

Such solutions will behave tensorially under coordinate transformations \( \tilde{x}^i = \Lambda^i_{\; j} x^j \) \((\Lambda^i_{\; j} \in \mathbb{R})\) on the manifold \( M = \mathbb{R}^4 \) (which include homogeneous Lorentz transformations, \([14]\), in the Minkowski case and the group \( G_1(P_{k+2m}) \), \([13]\), in the more complicated Berwald-Moor case to be regarded below).

### 3.2 Eikonal equation

In case when the space is anisotropic, the first approximation of the eikonal, \( \psi(x^i, y^i) \), corresponding to some wave (e.g. to the EMW), can be written as

\[
\psi = \psi_0 + \frac{\delta \psi}{\delta x^i} dx^i + \frac{\partial \psi}{\partial y^a} \delta y^a.
\]

Then, (2) becomes:

\[
g^{ij} \frac{\delta \psi}{\delta x^i} \frac{\delta \psi}{\delta x^j} = 0,
\]

where the "adapted derivative" \( \frac{\delta}{\delta x^i} \) is characteristic for Finsler and Lagrange geometries, \([11], [10]\) and insures the tensorial character of \( \frac{\delta \psi}{\delta x^i} \). Under our assumptions on the metric structure and on coordinate transformations, this equation becomes simply:

\[
g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = 0.
\]

Let us look for the eikonal in the form \( \psi(x, y) = f(x, y) + h \tilde{g}(x, y) \) \((h << 1, h^2 \approx 0)\) that satisfies \((20)\). With \( g^{ij} = \gamma^{ij} - h \hat{a}^{ij} \cos(K_m x^m) \), for our model, we get

\[
\gamma^{ij} \hat{f}_i \hat{f}_j + h(2 \gamma^{ij} \hat{f}_i \hat{g}_j - \hat{a}^{ij} \cos(K_m x^m) \hat{f}_i \hat{f}_j) = 0.
\]

In search for the solutions sufficing

\[
\begin{cases}
\gamma^{ij} \hat{f}_i \hat{f}_j = 0 \\
2 \gamma^{ij} \hat{f}_i \hat{g}_j - \hat{a}^{ij} \cos(K_m x^m) \hat{f}_i \hat{f}_j = 0
\end{cases}
\]

we get the following class of solutions:

\[
\hat{f} = k_i(y) x^i + \Phi_1(y)
\]

\[
\gamma^{ij} k_i k_j = 0,
\]

where \( \Phi_1 = \Phi_1(y) \) is arbitrary. Here \( k_i \) is the wave 4-vector of the EMW, \( k_1 = -\frac{\omega}{c} \), and \( K_m \) is the wave 4-vector of the GW. Considering \( k^i = \gamma^{ij} k_j \), we get

\[
\tilde{g} = \frac{1}{2} \hat{a}^{ij} k^i k^j \sin(K_i x^i) + \Phi_2(y, x);
\]

\[
\gamma \approx h \frac{\delta}{\delta x^i} \frac{\delta \psi}{\delta x^j}.
\]
If we choose, for simplicity, $\Phi_1 = \Phi_2 = 0$, we obtain the eikonal
\[
\psi = k_i(y)x^i + h\tilde{A}(y)\sin\left(K_i x^i\right), \quad \gamma^{ij} k_i k_j = 0, \quad k_i = k_i(y)
\] (32)
where $\tilde{A}(y) = \frac{1}{2} \tilde{a}_{ij} k^i k^j = \frac{1}{2} \tilde{a}_{ij} k_i k_j$. In anisotropic spaces, the components, $k_i$, generally also depend on the directional variables $y^i$.

### 3.3 Generalized geodesics

The Finslerian function, $F$, corresponding to the deformed locally Minkowskian metric, namely, $F^2 = (\gamma_{kl}(y) + \varepsilon_{kl}(x, y))y^l y^l$, leads to the Euler-Lagrange equations
\[
\frac{\partial F^2}{\partial x^i} - \frac{d}{ds}\left(\frac{\partial F^2}{\partial y^i}\right) = 0,
\] (33)
that are equivalent to
\[
g^*_{ij} \frac{dy^j}{ds} + \frac{1}{2} \left(\frac{\partial^2 F^2}{\partial y^i \partial y^j} y^j - \frac{\partial F^2}{\partial x^i}\right) = 0,
\] (34)
where $s$ is the arclength $s = \int_0^t F(x(\tau), y(\tau)) d\tau$, and $g^*_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$. Performing the computations, we get, in linear approximation,
\[
\frac{dy^i}{ds} + \gamma^{it}(\Gamma_{tls} + \frac{1}{2} \frac{\partial \varepsilon_{st}}{\partial y^l} y^l) y^s y^t = 0
\] (35)
This equation has a physical meaning. For the locally Minkowskian space with small anisotropic deformation, the force potentials consist of two terms. The second term in brackets, originating from the anisotropy of the deformation, is associated with the velocity and provides an analogue to the second term in the expression for the Lorentz force in electrodynamics. This illustrates the ideas discussed in the end of [5], [6].

If $\varepsilon_{ij}(x, y) = h\tilde{a}_{ij}(y) \cos(K_m(y)x^m)$, then, performing the derivations, we obtain the geodesic equations:
\[
\frac{dy^i}{ds} + hA^i(y) \sin(K_m x^m) + hB^i_p(y) x^p \cos(K_m x^m) = 0
\] (36)
where the tensorial coefficients
\[
A^i = -\frac{1}{2} \gamma^{ij} y^j y^s \frac{\partial (K_i \tilde{a}_s)}{\partial y^t} + (K_s \tilde{a}_t + K_t \tilde{a}_s - K_i \tilde{a}_s),
\] (37)
\[
B^i_p = -\frac{1}{2} \gamma^{ij} y^j y^s K_j \tilde{a}_s \frac{\partial K_p}{\partial y^i} = -\frac{1}{2} \gamma^{ij} K_0 \tilde{a}_0 \frac{\partial K_p}{\partial y^i},
\]
depend only on the directional variables $y^i$. Here $\tilde{a}_{00} \equiv \tilde{a}_{nm} y^n y^m$ and $K_0 \equiv K_i y^i$. In particular, if $K_i$ are constant, then $B^i_p = 0$, $i = 1 \div 4$ and the equations
of geodesics simplify. Solving one can find that unit-speed geodesics of the perturbed metric \( g_{ij}(x, y) = \gamma_{ij}(x, y) + h\tilde{a}_{ij}(y) \cos(K_m y^m) \) are described by

\[
x^i(s) = \alpha^i s + \beta^i - \frac{h}{2} \gamma^{it} \frac{\partial}{\partial y^t} \left( \frac{\tilde{a}_{00}}{K_0} \right) \sin(K_m x^m) - \frac{h x^p}{2} \gamma^{it} \frac{\partial}{\partial y^t} \left( \frac{\tilde{a}_{00}}{K_0} \right) \cos(K_m x^m),
\]

(38)

where \( \alpha^i \) and \( \beta^i \) depend on the initial conditions. In particular, if \( K_m \) are constant, geodesics of the perturbed metric obey

\[
x^i(s) = \alpha^i s + \beta^i - \frac{h}{2} \gamma^{it} \frac{\partial}{\partial y^t} \left( \frac{\tilde{a}_{00}}{K_0} \right) \sin(K_m x^m).
\]

(39)

From (38), we get that along geodesics \( hx^i(s) \simeq h(\alpha^i s + \beta^i) \), and \( hy^i(s) = h \frac{dx^i}{ds} \simeq h\alpha^i \).

4 Weak anisotropic perturbation of the flat Minkowski metric

In order to find explicitly the way in which the anisotropy could reveal itself in observations, we will first search for those solutions which are as close to the solutions given in Section 2 as possible. Let the initial metric be the flat Minkowskian one \( \gamma = \text{diag}(1, -1, -1, -1) \); then the system (24) becomes

\[
\begin{align*}
\gamma^{hl} K_l K_l &= 0 \\
a_i^j K_i &= \frac{1}{2} a^i_j K_j \\
\left( \frac{1}{2} a^i_j K_j \right)_b &= 0.
\end{align*}
\]

(40)

Let us choose \( K_1 = -K_2 = \frac{D}{c} \), where \( D, c \in \mathbb{R} \), \( K_3 = K_4 = 0 \), and \( a_3^i = -a(y), a_4^i = a(y), a^i_j = 0 \) for all other \( (i, j) \). Then (40) is identically satisfied and the perturbed Minkowski metric in the weakly anisotropic case can be expectedly presented as

\[
g_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 + a(y) \cos\left( \frac{D}{c} (x^1 - x^2) \right) & 0 \\
0 & 0 & 0 & -1 - a(y) \cos\left( \frac{D}{c} (x^1 - x^2) \right)
\end{pmatrix},
\]

(41)

where \( a(y) \) is an arbitrary scalar 0-homogeneous function, small enough such that \( a^2 \simeq 0 \). When \( a(y) \) is a constant, this metric reduces to the perturbed Minkowski metric for the isotropic empty space and the results of Section 2 are valid.
4.1 Eikonal

Let \( a(y) \) in the (41) be equal to \( \tilde{a}(y') \), where \( \tilde{a}(y') \) is an arbitrary scalar 0-homogeneous function, \( h^2 \approx 0 \). Then, \( \hat{A} = \frac{1}{2} \tilde{a}(y') \) and the eikonal (32) takes the form

\[
\psi = -\frac{\omega}{c} x^1 + k_2 x^2 + k_3 x^3 + k_4 x^4 + \frac{h c^2 \tilde{a}(y')(k_3^2 - k_4^2)}{2 D(c k_2 - \omega)} \sin(K_i x^i), \tag{42}
\]

Example:

Let \( v \in X(M) \) be an arbitrary vector field and \( \tilde{a} = \frac{K_i y^i}{\gamma_{ij} v^j} \), where \( K \) is the wave vector. Thus, \( \tilde{a} \) is globally defined. In the given local frame, in which \( K_1 = \frac{D}{c} \), \( K_2 = -\frac{D}{c} \), \( K_3 = 0 \), \( K_4 = 0 \) (which is, chosen such that the GW propagates antiparallel to the \( Ox \) axis), \( \tilde{a} \) is equal to \( \frac{D}{c} (y^1 - y^2) \). In a simple case, \( v^i = 0 \), \( i = 1, 2, 3 \) and \( v^4 = -\frac{D}{c} \), we get

\[
\tilde{a} = \frac{y^1 - y^2}{y^4}
\]

hence,

\[
\psi = -\frac{\omega}{c} x^1 + k_2 x^2 + k_3 x^3 + k_4 x^4 + \frac{h c^2 (y^1 - y^2)(k_3^2 - k_4^2)}{2 D y^4 (c k_2 - \omega)} \sin(K_i x^i) \tag{43}
\]

4.2 Geodesics

In the case of a small anisotropic perturbation of the Minkowski metric \( \text{diag}(1, -1, -1, -1) \), geodesics are described by (39):

\[
x^i(s) = \alpha^i s + \beta^i - \frac{h}{2} \gamma^{ij} \frac{\partial}{\partial y^j} \left( \tilde{a}_{00} \right) \sin(K_m x^m).
\]

In particular, for \( \tilde{a} = \tilde{a}(\frac{y^1 - y^2}{\phi(y^3, y^4)}) \) and \( K_1 = \frac{D}{c} \), \( K_2 = -\frac{D}{c} \), \( K_3 = K_4 = 0 \), \( D, c \in \mathbb{R} \), we substitute it into eq. (39) and get \( x^1 - x^2 = \alpha s + \beta; y^1 - y^2 = \alpha \in \mathbb{R} \). Particularly, the Cartesian \( Oy \) coordinate is

\[
x^3 = \nu + u_0 \left\{ \frac{x^1 - x^2 - \beta}{\alpha} + \frac{hc}{2 D \alpha v^3} \sin\left( \frac{D}{c} (x^1 - x^2) \right) \right\}.
\tag{44}
\]

where \( u_0, \nu \in \mathbb{R} \).
Examples: 1) For \( a = h = \text{const} \), where \( \tilde{a} = 1 \), we get the expression obtained in [3] which immediately leads to (8).

2) If \( a = h \frac{y_1^1 - y_2^2}{y^4} \) as earlier, we get \( a = \frac{ha}{y^4} = \frac{1 + ha_1}{y^4} \) along geodesics, and, with \( x^i(0) = 0 \),

\[
x^3 = u_0 \frac{x^1 - x^2}{1 + ha_1} + u_0 \frac{hc}{y^4D} \sin(\frac{D}{c}(x^1 - x^2)).
\] (45)

Then the \( Oy \)-component of the atom velocity will contain a term of the form

\[
y^3 \sim u_0 \frac{hc}{y^4} \cos(\frac{D}{c}(x^1 - x^2))
\]
and the amplitude factor in front of the cosine depends on the velocity component, \( y^4 \), orthogonal to \( Ox \) and \( Oy \) axes.

4.3 OMPR modification

The physical interpretation of the obtained solutions leading to the modifications in the OMPR effect is the following. One can see that the anisotropy does not destroy the solution of the OMPR equations [7-10]. For a simple anisotropic deformation of the Minkowski metric, we get the dependence of eqs. (43, 45) on the directional variable orthogonal to \( Ox \) and \( Oy \), i.e. to the plane containing the Earth, the space maser and the GW source.

Geodesics describe the trajectory of the particle, and the sample eq. (45) means that the amplitude of the oscillations of the space maser atom velocity component oriented at the Earth, \( y^3 \), depends on \( y^4 \). This means that when the system "Earth-space maser-GW source" is located close to the periphery of the galaxy, the orientation of this system might affect the OMPR conditions. In our example two of the OMPR conditions (13) must be modified and take the form

\[
\frac{\alpha_2}{\alpha_1} = \frac{\omega h}{8Dy^4} = b\varepsilon; b = O(1); \varepsilon << 1
\] (46)

\[
\frac{k\nu_1}{\alpha_1} = \frac{\omega hc}{\alpha_1 y^4} = \kappa\varepsilon; \kappa = O(1); \varepsilon << 1
\] (47)

that illustrates the qualitative analysis given in [6]. This means that the experimental investigation of the astrophysical systems with various orientations might provide the information on the quantitative characteristics of the geometrical anisotropy (if any) of our galaxy.
5 Weak perturbation of the anisotropic Berwald-Moor metric

Instead of the anisotropic correction to the isotropic (Minkowski) metric, we could try an originally anisotropic but still locally Minkowskian (i.e. spatial variables independent) metric on $\mathbb{R}^4$. Let us consider the Finslerian Berwald-Moor metric

$$\gamma_{ij}(y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

in which $F = \sqrt[4]{y^1 y^2 y^3 y^4}$. The explicit form of the unperturbed metric is provided by the matrices

$$\begin{pmatrix} \gamma_{ij} \end{pmatrix} = \begin{cases} -\frac{1}{8} F^2, & i = j \\ \frac{1}{8} \frac{F^2}{y^i y^j}, & i \neq j \end{cases}; \quad (48)$$

$$\begin{pmatrix} \gamma^{ij} \end{pmatrix} = \begin{cases} -\frac{2}{2} F^2 (y^i)^2, & i = j, \\ \frac{2}{2} F^2 y^i y^j, & i \neq j \end{cases}. \quad (49)$$

The wave solutions, $\varepsilon_{ij} = a_{ij}(y) \cos(K_i x^i)$, for Einstein’s equations in vacuum for the anisotropic case are given by the solutions of the system (24), where the coefficients,

$$0_{C_{jd}} = \frac{1}{2} \gamma^{ih} \frac{\partial \gamma_{hj}}{\partial y^d}$$

will be given by

$$0_{C_{jd}} = \frac{p}{8} \frac{y^i}{y^2 y^d}, \quad p = \begin{cases} 3 & \text{if } i = j = d \\ \frac{1}{8} & \text{if } i = j \neq d \text{ or } i \neq j = d \text{ or } i = d \neq j. \end{cases} \quad (50)$$

If we choose the coordinate system such that

$$K_3 = K_4 = 0, \quad (51)$$

then the light-like condition $\gamma^{ij} K_i K_j$ leads to $K_2 = \frac{y^1 y^2}{y^3} K_1$. Moreover, $a_{ij} = h \lambda(y) K_i K_j$ (here $\lambda(y)$ is an arbitrary scalar 0-homogeneous function and $h$ is a small constant $h^2 \simeq 0$) defines a solution of (24) obeying the transverse traceless conditions $a^i_i = 0, \quad a^i_j K_1 = 0$. We got the following solution for the Einstein equations

$$\varepsilon_{ij}(x, y) = h \lambda K_i K_j \cos(K_i x^i), \quad i, j = 1, ..., 4, \quad (52)$$

where the first component $K_1 = K_1(y)$ of the wave vector is an arbitrary 0-homogeneous function of the directional variables.
Let us denote: \( K_1 = \frac{D}{y^1} \) and \( n_i = \frac{c}{D} K_i, \ i = 1, ..., 4. \) With these, the solution can be written as:

\[
\epsilon_{ij} = h \frac{\lambda D^2}{c^2} n_in_j \cos\left(\frac{D}{c}(n_ix^i)\right). \tag{53}
\]

In the given frame, we have \( K_2 = \frac{D}{y^2}, \ n_3 = n_4 = 0. \)

If, moreover, \( x^1 = ct \) and the preferred direction is \( y^i = \frac{dx^i}{dt} = c \frac{dx^i}{dx^1} \) (the time derivatives of positional variables), then \( y^1 = c, \ n_1 = 1, \ K_1 = \frac{D}{c} \) and the cosine in the perturbation is \( \cos\left(\frac{D}{c}(x^1 + n_2x^2)\right). \)

**Examples:**

1) If \( \lambda \) and \( D \) are constant, \( n_i = n_i(y), \ i = 1, 2 \)

\[
(a_{ij}) = h \frac{\lambda D^2}{c^2} \begin{pmatrix} (n_1)^2 & n_1 n_2 & 0 & 0 \\ n_1 n_2 & (n_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{54}
\]

2) A less simple example which is interesting because of its symmetry, is \( \lambda D^2 = y^1 y^2. \) Then,

\[
(a_{ij}) = h\lambda D^2 \begin{pmatrix} \frac{1}{y^1 y^2} & 0 & 0 & 0 \\ 0 & \frac{1}{y^1 y^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h \begin{pmatrix} n_1 & 1 & 0 & 0 \\ n_2 & 1 & 0 & 0 \\ 0 & n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{55}
\]

and we see that in this case the perturbation, \( \epsilon_{ij}, \) of the metric becomes

\[
\epsilon_{ij} = h \frac{n_in_j}{n_1 n_2} \cos\left[\frac{D}{c}(n_1 x^1 + n_2 x^2)\right], \ i,j = 1, 2. \tag{56}
\]

### 5.1 Eikonal equation

With the help of \( \tilde{A} = \frac{1}{2} \tilde{a}^{ij} k_ik_j = \frac{1}{2} \lambda (K^i k_i)^2 = \frac{\lambda(y)}{2} K^i k_i, \) eq. (32) yields the solution for the eikonal (32). Rewriting \( \tilde{A} = \frac{\lambda D}{2c} n_i k_i, \) one obtains the solution for the eikonal as:

\[
\psi = k_i(y)x^i + h \frac{\lambda D}{2c} n_i k_i \sin\left[\frac{D}{c}(n_1 x^1 + n_2 x^2)\right] \tag{57}
\]
Example: if $\lambda D^2 = y_1 y_2$, then
\[
\psi = k_i(y)x^i + \frac{c}{2D} \frac{(n_1 k^1 + n_2 k^2)}{n_1 n_2} \sin[\frac{D}{c} (n_1 x^1 + n_2 x^2)]
\]

Equation (57) describes the eikonal of the wave propagating in the model anisotropic space-time with the Berwald-Moor metric perturbed by the GW. In the Berwald-Moor case, the components $k_i(y)$ cannot be constant, since the equation $\gamma^{ij} k^i k^j = 0$ does not have any constant solutions (except the trivial one $k_i = 0, i = 1, \ldots, 4$).

5.2 Geodesics

In the case under discussion we get $K_0 = K_1 y^1 + K_2 y^2 = 2D$, $a_{00} = \lambda K_0^2 = 4 \lambda D^2$, $\frac{a_{00}}{K_0} = 2 \lambda D$, and the unit-speed geodesics ($F = 1$) in eq. (58) obey
\[
x^i(s) = \alpha^i s + \beta^i - \hbar \gamma^{ij} \frac{\partial (\lambda D)}{\partial y^j} \sin(K_m x^m) - \hbar \gamma^{ij} \frac{\partial K_p}{\partial y^j} x^p \lambda D \cos(K_m x^m). \quad (58)
\]

The simplest solutions are obtained for $D = D(y^1, y^2)$; in this case, performing the calculations, we find
\[
\frac{K_0}{K_1} = 2 \frac{K_0^{1/2}}{2^{1/2}}.
\]
Calculating the derivative and using the initial conditions, $x^i(0) = 0 \Rightarrow \beta^i = 0, i = 1, 2, 3$, we find that the last term in (58) disappears for $y^3, y^4$ and get
\[
y^3 = \alpha^3 - 4 \frac{\hbar \lambda D^2 y^3}{F^2} \cos(K_m x^m). \quad (59)
\]

Example: if $\lambda D^2 = y_1 y_2$, then
\[
y^3 = \alpha^3 - 4 \frac{\hbar F^2}{y^4} \cos(K_m x^m) \quad F = \frac{1}{2} \frac{1}{y^4} \cos(K_m x^m) \quad (60)
\]

5.3 OMPR modification

As in the previous case, the anisotropy does not destroy the OMPR effect itself, but now the modifications are more pronounced. Eq. (57) for the eikonal also gives a trichromatic EMW, but the amplitudes of the sidebands and their frequencies are now different from the isotropic case. The geodesics in the form (59) shows that the amplitude of the atomic oscillations is now also different.

All this would affect the OMPR conditions (13) and they would be modified in the following way
\[
\frac{\alpha_2}{\alpha_1} = \frac{\hbar \lambda D}{4 \epsilon} n_1 k^i = b \epsilon; b = O(1); \epsilon << 1 \quad (61)
\]
\[
4 \hbar \omega \frac{\lambda D^2}{c^2} \sqrt{\frac{n_1 n_2 n_4}{n_3}} = \kappa \epsilon; \kappa = O(1); \epsilon << 1 \quad (62)
\]
\[
(\omega - \Omega + k v_0)^2 + 4 \alpha_1^2 = D^2 n_1^2 + O(\epsilon) \Rightarrow D n_1 \sim 2 \alpha_1 \quad (63)
\]
or, for the sample example, \( \lambda D^2 = y^1 y^2 \),

\[
\frac{\alpha_2}{\alpha_1} = h \frac{c}{4D} \frac{(n_1 k^1 + n_2 k^2)}{n_1 n_2} = b \varepsilon; \ 0 = O(1); \ \varepsilon << 1
\]  

(64)

\[
4h \omega \frac{\omega}{\alpha_1} \sqrt{\frac{n_1}{n_1 n_2 n_3}} = \kappa \varepsilon; \ \kappa = O(1); \ \varepsilon << 1
\]  

(65)

\[
(\omega - \Omega + kv_0)^2 + 4\alpha_1^2 = D^2 n_1^2 + O(\varepsilon) \Rightarrow Dn_1 \sim 2\alpha_1
\]  

(66)

As in the previous Section, we find that the orientation of the system (see Fig.1) would affect the observations. Calculating the left hand sides of the second condition in (5.18) for the systems I and II, one can see that their ratio is equal to the ratio of the star velocity corresponding to the galactic rotation and the star velocity in the direction of the galaxy axis. Therefore, if we take two equivalent astrophysical systems that initially suffice the OMPR conditions and differ only by their orientation with regard to the galactic plane, only one of them will produce an observable OMPR signal.

6 Discussion

The main results obtained in this paper are the following. In search for the modifications of the Einstein-Hilbert action due to the anisotropy of the space-time, we have constructed two simple models of the anisotropic space-time with the metrics containing small perturbations. The additional terms lead to a change in the Einstein equations. We have shown that in the anisotropic case Einstein equations for the empty space still have wave solutions (gravitational waves), but now they become direction dependent and their amplitudes, \( a_{jk}(y) \), and wave numbers, \( K_j(y) \), can become coupled. We also performed the corresponding generalizations of the equations for the eikonal and for the geodesics and used them to find how the OMPR conditions would change in the anisotropic space-time. It turned out that the orientation of the astrophysical system (taking part in the OMPR) with regard to the galactic plane causes changes in the observable effect, thus, giving one the possibility to experimentally investigate the space-time geometrical properties on the galactic scale.

The expression for the “simplest scalar” which can be used in the variation principle based on the Einstein-Hilbert expression for the action was particularized for our model. If the perturbed locally Minkowskian metric can be presented as \( g_{ij}(x, y) = \gamma_{ij}(y) + \varepsilon_{ij}(x, y) \), then the space-time anisotropy produces additional terms in Ricci tensor \( R_{jk} \) which is to be calculated with regard to \( \Gamma^i_{jk} \) equal to

\[
\Gamma^i_{jk} = \frac{1}{2} \gamma^i_l \left( \frac{\partial \varepsilon_{lj}}{\partial x^k} + \frac{\partial \varepsilon_{lk}}{\partial x^j} - \frac{\partial \varepsilon_{jk}}{\partial x^l} \right) = \frac{1}{2} \gamma^i_l (a_{lj} K_k + a_{lk} K_j - a_{jk} K_l) \sin(K_m x^m),
\]  

(67)
It turned out that the generalized equations of geodesics for the anisotropic space-time contain the "force potentials" consisting of two terms. The second term is associated to the directional variable ("velocity") and provides an analogue to the corresponding term in the expression for the Lorentz force in electrodynamics.

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