A model with generalized Y-junction

P. N. Bibikov and L. V. Prokhorov

August 11, 2009

Abstract

The Klein-Fock-Gordon equation is studied on the generalized Y-junction of $N$ strings with a massive center. The corresponding formulas for wave scattering and normal modes are obtained.

Key words differential equations on networks, transmission rules, Klein-Fock-Gordon equation.

1 Introduction.

Dynamical systems on manifolds are standard objects studied both in classical and quantum mechanics. Here we continue to study mechanics not on manifolds. The simplest example of such a system gives a 3-ray star studied in [1] (see also [2] and [3]). This space is not a manifold. The problem became of special importance a couple decades ago after the discovery of D-branes. Problems of this type arise in different branches of Physics.

String theory. Effectively strings are ordered sets of interacting harmonic oscillators and end of a string can belong to another string (D1-brane). Problem: describe the evolution of excitations of the system.

Nanoelectronics and theory of polymers. Nanoelectronics and theory of polymers are important for modern technologies.

The importance of this problem for strings is self evident. Gradually it becomes clear that at the Planck scale matter manifests itself in form of strings, and a 3D-network of superstrings can model the physical space [4], clearly leading to unification of all interactions, including gravitation.

There are two types of problems (both in classical and quantum mechanics): (i) determination of normal coordinates, (ii) description of scattering (i.e. description of evolution
of waves given on the rays). It turns out that here normal coordinates are given by linear
transformations, i.e. these systems in fact give examples of free field theories. The rel-
ativistic Schrödinger equation on a tail (i.e. the Klein-Fock-Gordon equation) coincides
with the equation of motion of a free field. Thus the problems of scattering on the junc-
tion are in fact identical both in classical and quantum cases. The only difference is that
in classical physics one usually considers scattering of a wave packet, while in quantum
mechanics it is the scattering of a particle with certain momentum \( k \). It is significant that
though the fields are formally free there exists nontrivial scattering: a wave on a ray may
either be reflected or pass to an other tail [1].

A bosonic string is in fact an ordered set of harmonic oscillators with identical masses.
In nanoelectronics there are 3-ray stars made of 3 quantum wires and a quantum well (or
gate). The latter is modeled by a central oscillator. That is why here we study the case of
a nontrivial junction when the mass of the central oscillator is different from the mass of
the others. The problem of the 2-ray star (oscillators on an axis), one oscillator with an
arbitrary mass, was studied in [5]. This problem is close to the problem of one-dimensional
scattering on a \( \delta \)-potential.

We solve here the problem of scattering on such a junction in both classical and
quantum cases (Sec. 2) and find normal coordinates in the general case (\( N \)-tail star)
(Sec. 3).

\section{Harmonic network approximation}

The Klein-Fock-Gordon equation in 1+1 dimensions has a natural discretization,

\[ \ddot{\varphi}_n = \frac{1}{\Delta^2} \left( \varphi_{n+1} + \varphi_{n-1} - 2\varphi_n \right) - m^2 \varphi_n. \]  

(1)

Here \( \Delta \) is the lattice constant.

A solution of this system of differential equations corresponding to a monochromatic
wave with positive energy has the following form,

\[ \varphi_n(t) = e^{-i(\omega_k t - kn\Delta)}, \]  

(2)

where the dispersion,

\[ \omega_k^2 = \frac{4}{\Delta^2} \sin^2 \frac{k\Delta}{2} + m^2, \]  

(3)

in the limit \( \Delta \to 0 \) pass to the well-known relativistic form.
The simplest way to describe the junction is to introduce for it a corresponding new variable which is denoted by \( u \). If \( \varphi_n^{(j)} \), \( j = 1, 2, 3 \), are components of \( \varphi \) related to the corresponding rays we may propose the following Lagrangian,

\[
L = \sum_j \left( L^{(j)} - \frac{1}{2\Delta^2}(u - \varphi_1^{(j)})^2 \right) + \frac{1}{2}(M\ddot{u}^2 - m^2u^2),
\]

(4)

where

\[
L^{(j)} = \frac{1}{2} \sum_{n=1}^\infty \left( \dot{\varphi}_n^{(j)} - \frac{1}{\Delta^2}(\varphi_{n+1}^{(j)} - \varphi_n^{(j)})^2 - m^2\varphi_n^{(j)} \right).
\]

(5)

The Lagrangian (4), (5) produces a nontrivial realization of a one-dimensional harmonic lattice. It gives the following equations of motion:

\[
M\ddot{u} = \frac{1}{\Delta^2} \sum_j \left( \varphi_1^{(j)} - u \right) - m^2u,
\]

(6)

\[
\ddot{\varphi}_1^{(j)} = \frac{1}{\Delta^2} \left( \varphi_2^{(j)} + u - 2\varphi_1^{(j)} \right) - m^2\varphi_1^{(j)},
\]

(7)

\[
\ddot{\varphi}_n^{(j)} = \frac{1}{\Delta^2} \left( \varphi_{n+1}^{(j)} + \varphi_{n-1}^{(j)} - 2\varphi_n^{(j)} \right) - m^2\varphi_n^{(j)}, \quad n > 1.
\]

(8)

Here \( \Delta \) is the lattice parameter while \( m \) is the quasiparticle (excitation) mass. The parameter \( M \) is dimensionless.

We now study scattering on the junction which is described by the following general solution [1],

\[
\varphi_n^{(x)}(t) = e^{-i(\omega_k t + kn)} + R(k)e^{-i(\omega_k t - k\Delta n)},
\]

(9)

\[
\varphi_n^{(y)}(t) = \varphi_n^{(z)}(t) = (R(k) + 1)e^{-i(\omega_k t - k\Delta n)},
\]

(10)

\[
u(t) = (R(k) + 1)e^{-i\omega_k t}.
\]

(11)

Our task is to obtain \( R(k) \). The Eqs. (6)-(7) lead to

\[
2M(1 - \cos k\Delta)(R(k) + 1) = 3(R(k) + 1) - e^{-ik\Delta} - (3R(k) + 2)e^{ik\Delta},
\]

(12)

or

\[
R(k) = \frac{1}{3}e^{i\theta(k)} - \frac{2}{3},
\]

(13)

where

\[
e^{i\theta(k)} = \frac{(2M - 3)\sin \frac{k\Delta}{2} - 3i \cos \frac{k\Delta}{2}}{(2M - 3)\sin \frac{k\Delta}{2} + 3i \cos \frac{k\Delta}{2}}.
\]

(14)
Now we may suppose that the parameter $M$ is a function of $\Delta$ and take the limit $\Delta \to 0$. Defining a new parameter

$$k_1 = \lim_{\Delta \to 0} \frac{3i}{\Delta \cdot M(\Delta)}$$

we obtain the following equation,

$$e^{j\theta(k)} = \frac{k - k_1}{k + k_1}.$$  \hfill (16)

Eq. (16) is the main result of this section.

### 3 Normal modes for the N-star model

In the discrete model (4) we put for simplicity $\Delta = 1$. This is equivalent to the normalization $|k| \leq \pi$ of the wave number. Now $j = 1, 2, ..., N$.

The complex coordinates,

$$\varphi^{(j)}(k) = \sum_{n=1}^{\infty} e^{ikn} \varphi_n^{(j)},$$

satisfy the following equations:

$$\ddot{\varphi}^{(j)}(k) = -\omega_k^2 \varphi^{(j)}(k) + u e^{ik} - \varphi_1^{(j)},$$

It is convenient to pass to their real and imaginary parts,

$$\varphi_c^{(j)}(k) = \sum_{n=1}^{\infty} \varphi_n^{(j)} \cos kn, \quad \varphi_s^{(j)}(k) = \sum_{n=1}^{\infty} \varphi_n^{(j)} \sin kn.$$  \hfill (19)

They satisfy the equations:

$$\ddot{\varphi}_c^{(j)}(k) = -\omega_k^2 \varphi_c^{(j)}(k) + u \cos k - \varphi_1^{(j)};$$

$$\ddot{\varphi}_s^{(j)}(k) = -\omega_k^2 \varphi_s^{(j)}(k) + u \sin k.$$  \hfill (21)

If we define

$$\xi_0(k) = \sum_{j=1}^{N} \varphi_c^{(j)}(k) + Mu,$$

then

$$\ddot{\xi}_0(k) = -\omega_k^2 \xi_0(k) - (N - 2)u(1 - \cos k).$$  \hfill (23)
The normal modes are:

\[ \xi_j(k) = \sin k\xi_0(k) + [(N - 2M)(1 - \cos k) + (1 - M)m^2]\varphi^{(j)}_s(k). \] (24)

They satisfy the following system of equations:

\[ \ddot{\xi}_j(k) = -\omega_k^2\xi_j(k). \] (25)

In the case \( M = 1 \) we may divide (24) by \( 2\sin k/2 \) and come to a simpler formula,

\[ \xi_j(k) = \cos \frac{k}{2}\xi_0(k) + (N - 2)\sin \frac{k}{2}\varphi^{(j)}_s(k). \] (26)

In this case we may express \( u \) and \( \varphi^{(j)}_n \) from \( \xi_m(k) \).

4 Inverse transformation from normal modes for \( M = 1 \)

In order to express \( u \) and \( \varphi^{(j)}_n \) from \( \xi_j(k) \) we note that according to (24) for \( M = 1 \),

\[ \sum_{j=1}^{N} \xi_j(k) = Nu \cos \frac{k}{2} \]

\[ + \sum_{n=1}^{\infty} \sum_{j=1}^{N} \varphi^{(j)}_n \left( N \cos k\gamma \cos \frac{k}{2} + (N - 2) \sin k\gamma \sin \frac{k}{2} \right), \] (27)

\[ \xi_j(k) - \xi_{j-1}(k) = \sin \frac{k}{2} \sum_{n=1}^{\infty} (\varphi^{(j)}_n - \varphi^{(j-1)}_n) \sin k\gamma. \] (28)

Defining new variables,

\[ Q_0 = Nu, \quad Q_n = \sum_{j=1}^{N} \varphi^{(j)}_n, \quad n = 1, ..., \] (29)

one may rewrite Eq. (27) in the following form:

\[ \sum_{j=1}^{N} \xi_j(k) = \sum_{n=0}^{\infty} (Q_n + (N - 1)Q_{n+1}) \cos k\left( n + \frac{1}{2} \right). \] (30)

Then according to (28) and (30)

\[ \Delta Q_{j,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\xi_{j+1}(k) - \xi_j(k)}{\sin \frac{k}{2}} \sin k\gamma dk, \] (31)

\[ \eta_n = Q_n + (N - 1)Q_{n+1}, \] (32)

\[ \text{5} \]
where

\[ \Delta Q_{j,n} = \varphi^{(j+1)}_n - \varphi^{(j)}_n. \]  

(33)

and

\[ \eta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{N} \xi_j(k) \cos k \left(n + \frac{1}{2}\right) dk. \]  

(34)

The recurrent system (32) may be represented in the following matrix form:

\[
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\vdots
\end{pmatrix} = A 
\begin{pmatrix}
Q_1 \\
Q_2 \\
\vdots
\end{pmatrix},
\]  

(35)

where

\[
A = \begin{pmatrix}
1 & N - 1 & 0 & 0 & \ldots \\
0 & 1 & N - 1 & 0 & \ldots \\
0 & 0 & 1 & N - 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]  

(36)

It can be easily proved that

\[
A^{-1} = \begin{pmatrix}
1 & 1 - N & (1 - N)^2 & (1 - N)^3 & \ldots \\
0 & 1 & (1 - N) & (1 - N)^2 & \ldots \\
0 & 0 & 1 & (1 - N) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(37)

and from Eqs. (35) and (37) it follows that

\[ Q_n = \sum_{m=n}^{\infty} \eta_m (1 - N)^m \]  

(38)

or, according to (34),

\[ Q_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{N} \xi_j(k) \sum_{m=0}^{\infty} \left[(1 - N)^m \cos k(n + m + \frac{1}{2})\right] dk. \]  

(39)

Using an equality

\[
\sum_{m=0}^{\infty} (1 - N)^m \cos k(n + m + \frac{1}{2}) = \text{Re} \frac{\exp(ik(n + \frac{1}{2}))}{1 + (N - 1) \exp(ik)}
\]  

(40)

one can readily obtain

\[ Q_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{N} \xi_j(k) \frac{\cos k(n + \frac{1}{2}) + (N - 1) \cos k(n - \frac{1}{2})}{N^2 - 4(N - 1) \sin^2 \frac{k}{2}} dk. \]  

(41)

Now according to (29) and (33) one can readily write the transverse transformation

\[ \varphi^{(j)}_n(k) = \frac{1}{N} \left(Q_n - \sum_{m=0}^{N-1} (N - 1 - m) \Delta Q_{j+m,n} \right). \]  

(42)
5 Conclusions

In the present paper we have considered the discrete version the Klein-Fock-Gordon equation in the Y-junction with arbitrary mass of the central oscillator. We obtained the corresponding formulas for wave propagation. The normal modes were obtained in the general case of N-rays. In the special case when the junction point mass is equal to unity the explicit formulas for inverse transformations were also obtained.

The authors are grateful to B. S. Pavlov for his interest to the paper.

References

[1] P.N. Bibikov, L.V. Prokhorov 2008, to be published in J. Phys. A

[2] B. Bellazzini, M. Burrello, M. Mintchev, P. Sorba, Quantum field theory on star graphs, [arXiv:0801.2852], to be published in ”Analysis on Graphs and its Applications”, Proc. Symp. Pure. Math. (AMS) (2008)

[3] P. Exner, O. Post, math-ph 0706.0481

[4] L.V. Prokhorov, Phys. Part. Nucl. 38 (3) 364 (2007).

[5] S. Fedorov, B. Pavlov, J. Phys. A: Math. Gen. 39, 2657, (2006)