Efficient Distributed Computations in Anonymous Dynamic Congested Systems with Opportunistic Connectivity*

Dariusz R. Kowalski  
Augusta University,  
Augusta, GA, USA,  
dkowalski@augusta.edu

Miguel A. Mosteiro  
Pace University,  
New York, NY, USA,  
mmosteiro@pace.edu

Abstract

In this work we address the question of efficiency of distributed computing in anonymous, congested and highly dynamic and not-always-connected networks/systems. More precisely, the system consists of an unknown number of anonymous nodes with congestion on links and local computation. Links can change arbitrarily from round to round, with only limitation that the union of any $T$ consecutive networks must form a temporarily connected (multi-)graph on all nodes (knowledge of $T$ is the only information the nodes require, otherwise the communication would not be feasible). Nodes do not have any IDs, only some number $\ell$ of them have a bit distinguishing them from nodes without such a bit. In each round a node can send and receive messages from its current neighbors. Links and nodes are congested, in the sense that the length of messages and local cache memory for local computation is (asymptotically) logarithmic.

All-to-all communication is a fundamental principle in distributed computing – it assumes that each node has an input message to be delivered to all other nodes. Without loss of generality, the size of each input message is logarithmic to fit in the link and node congestion assumption; otherwise, they could be split in logarithmic batches and considered one-by-one. Because of anonymity, each node needs to receive only a set of all input messages, each accompanied by a number of initiating nodes (message multiplicity). We prove that this task can be done in time polynomial in the (initially unknown) number of nodes $n$ and in the lower bound on the isoperimetric numbers of dynamically evolving graphs. This allows to efficiently emulate a popular Congested Clique model on top of Anonymous Dynamic Congested Systems (ADCS) with Opportunistic Connectivity, even if the number of nodes may arbitrarily change in the beginning of emulation.

1 Introduction

The Congested Clique [40] is a standard synchronous message passing model of distributed computation. In such model there are $n$ labeled nodes that synchronously, in rounds, communicate among them. In each round, each node may send a message of $O(\log n)$ bits to each of the other nodes and perform some local computations. Performance is measured in rounds of communication. But can we efficiently run Congested Clique algorithms if the nodes are anonymous (e.g., for the sake of privacy), of restricted capacity (e.g., logarithmic local memory) and their number is unknown? Even more, what if links may be available only occasionally, depending on an adversarial schedule, and the only necessary requirement is that the union of any $T$ consecutive networks, for some known parameter $T$, forms a temporally connected multi-graph? Could deterministic distributed computing on such Congested Clique be made efficient? In this work we address and answer this question in the affirmative by presenting an ALL-TO-ALL COMMUNICATION algorithm to emulate one round of Congested Clique under a harsh Dynamic Network model that includes all those restrictions.

Distributed computations in the Congested Clique model have attracted a lot of attention recently [4, 8, 9, 16, 17, 21, 23, 22, 30, 32, 39, 43, 50], the reason being that all-to-all communication is becoming a frequent feature of modern

*This work was partially supported by Pace University SRC grant and Kenan fund.
distributed systems. However, algorithms developed for such environments are not compatible with many systems where not all nodes are directly connected. Thus, in order to apply the wealth of Congested Clique research to those systems, protocols to emulate a single round of all-to-all communication in congested multi-hop networks are needed.

System restrictions may go far beyond multi-hop communication. Indeed, in some environments network connectivity may be highly dynamic due to mobility or unreliability (e.g. Dynamic Networks [25][36][38]). Also, node identifiers may not be feasible in massive low-cost platforms, or one may not want to reveal the identifiers due to privacy concerns (e.g. Anonymous Dynamic Networks [12][13][33][35][41]). Additionally, in a system where message size is limited (as in the Congested Clique) it is natural to apply the same limitation to memory access, specially when nodes are expected to be low-cost devices (e.g. Weak Sensor Model [18][19]).

Following up on the CONGEST model [51], in the Congested Clique nodes initially know their neighbors. Being a clique implies that they know the total number of nodes. However, when implementing the Congested Clique in a multi-hop topology, nodes may not be connected to all other nodes. Moreover, in face of dynamicity and anonymity, each node does not even know the number of its neighbors before receiving messages from them. Thus, in this work we assume that neither the exact number, nor even an upper bound on the total number of nodes, is initially known. Even more, the network does not need to be connected at all, as long as a union of $T$ consecutive networks is temporally connected, for some given parameter $T$.

To the best of our knowledge, our Anonymous Dynamic Congested Systems model described above is the most challenging for distributed computing among the existing models in the Dynamic Networks literature.

Our Contributions and Approach. In this work, we present a deterministic protocol to emulate a single round of the Congested Clique on Anonymous Dynamic Congested Systems (ADCS) with Opportunistic Connectivity. That is, on systems where nodes lack identifiers, message size and memory access are limited to $O(\log n)$ bits, where $n$ is the number of nodes (initially unknown), and the communication network is multi-hop and adversarially dynamic, allowing even disconnection with some limitations. The overhead introduced by our emulator is a polynomial function of $n$ and, if known, a lower bound $i_{min}$ on the isoperimetric numbers of dynamic networks\(^1\), making our bounds tighter for networks with good expansion – more precisely, the $i_{min}$ in the denominator of the time complexity formula could be as large as $\Theta(n)$ for networks with good expansion. See Table 1 for details and comparison with most closely related work. We also prove that the knowledge of the connectivity parameter $T$ and the number $\ell$ of distinguished nodes (they have additional distinguishing bit, and are called supervisors throughout the paper) is necessary for ALL-TO-ALL COMMUNICATION in ADCS with Opportunistic Connectivity. Similarly, using $o(\log n)$ local memory bits are not enough for ALL-TO-ALL COMMUNICATION in the model.

| Ref | Known $n$ | Connected | Message size | Memory access | IDs | Time complexity |
|-----|-----------|-----------|--------------|---------------|-----|----------------|
| this work | no | no | $O(\log n)$ | $O(\log n)$ | no | $O\left(\frac{2^{1+2T(1+\varepsilon)}}{i_{min}^2} \log n \log \left(\frac{n}{i}\right) + \frac{n^{\ell}}{i_{min}(1+i_{min})} \log^2 n\right)$ |
| [37] | no | yes | $O(\log n)$ | no limit | yes | $O(n^2)$ |
|       | no | yes | $O(\log n)$ | no limit | yes | $\Omega(n \log n)$ |
| [26] | upper bound | yes | $b \geq \log n$ | no limit | yes | $O\left(\frac{dn^2}{b^2}\right)$ ($b \geq d \geq \log n$) |

Table 1: Comparison of most relevant deterministic ALL-TO-ALL COMMUNICATION results in Dynamic Networks. Message size and memory access are in bits, and time complexity is in rounds of communication. $n$ denotes the number of nodes (initially unknown), $n'$ – the number of different input messages, $\ell$ – the number of nodes with distinguishing bit (supervisors), $T$ – the connectivity parameter, $\varepsilon > 0$ – any chosen constant, $i_{min}$ – the lower bound on the isoperimetric numbers of the ADCS as defined in Section 4.2 (if unknown, the formula holds after substituting $2/n$ for $i_{min}$). $d$ is the size of the input message in [26].

Our algorithmic approach is not a simple classic gossip-based technique, which is well-known in Distributed Systems.

\(^1\) Formally, it is a lower bound on an isoperimetric number of the product of $T$ consecutive networks, as defined formally in Section 4.2.
Computing \([20,24,31,33,41,43]\). In such algorithms initially nodes hold some values to be shared with neighboring nodes repeatedly until some stopping rule is met. In fact, the ADCS model yields such techniques incorrect, as multiplicities of the same input message may not be counted correctly due to anonymity and dynamic behavior of the underlying networks. Therefore, we combine a spreading-with-stopping technique with coding messages and distribution of potential, c.f., some of their applications in less-demanding systems \([20,33,43]\). Combining them in a carefully selected way results in a complex algorithm, which occurs surprisingly efficient in very demanding systems such as ADCS with Opportunistic Connectivity.

One of the critical components of our ALL-TO-ALL COMMUNICATION algorithm is based on distribution of potential. We use it when counting all active nodes or nodes with an identified input message. The main challenge is to show that the potential distribution process stabilizes (with negligible deviations at nodes) at a desired value – unlike a typical mass-distribution process, c.f., \([2,43,54,55]\), our process has to accommodate (1) lack of connectivity (only due to congested links/nodes), and (3) lack of any knowledge of \(T\)

We use it when counting all active nodes or nodes with an identified input message. The main challenge is to show that the potential distribution process stabilizes (with negligible deviations at nodes) at a desired value – unlike a typical mass-distribution process, c.f., \([2,43,54,55]\), our process has to accommodate (1) lack of connectivity (only due to congested links/nodes), and (3) lack of any knowledge of \(n\) (in the beginning).

**Roadmap.** In Section 2 we discuss the related work. Section 3 presents the model and useful definitions, while Section 4 states preliminary results on imposibility of all-to-all communication and properties of dynamic networks. The main ALL-TO-ALL COMMUNICATION algorithm is presented and analyzed in Section 5 while its major components, MULTIPLICITY and RESTRICTED METHODOLOGICAL COUNTING, are given and analyzed in Sections 6 and 7 respectively.

## 2 Related Work

To the best of our knowledge, there is no previous study of distributed computations under a Dynamic Networks model that include all the restrictions of Anonymous Dynamic Congested Systems. That is, worst-case dynamicity with connectivity that is only opportunistic, communication limited to \(O(\log n)\) bits per round, local memory access also limited to \(O(\log n)\) bits per round, and nodes without ID’s. We overview in this section the Dynamic Networks extant work on related models.

Two closely related works on ALL-TO-ALL COMMUNICATION in Dynamic Networks are \([26,37]\). In both, the model includes node ID’s, continuous connectivity, and unbounded memory access. Nevertheless, lower bounds under weaker conditions apply to the ADCS stricter model. For deterministic algorithms, a lower bound of \(\Omega(n \log n)\) rounds was proved in \([37]\), even for centralized algorithms.

The most frequent model of worst-case dynamicity still assumes continuous connectivity. That is, even though the set of links may change arbitrarily from round to round, it is guaranteed that in every round of communication there is a path between every pair of nodes. For instance, in the population protocol model in \([1]\) and the continuous connectivity model in \([49]\). Later on, in \([37]\), Kuhn, Lynch and Oshman parameterized the continuous connectivity assumption in the \(T\)-interval connectivity model where the topology may change, but changes are restricted to maintain an underlying connected static graph over each sequence of \(T\) rounds. Instantiating \(T = 1\) the model is equivalent to continuous connectivity.

In none of the above models disconnection is allowed, not even temporarily. In a more recent work \([42]\), Michail, Chatzigiannakis, and Spirakis do consider disconnections, with limitations that yield opportunistic connectivity. Specifically, in at most \(k\) rounds every node influences at least one other node that has not been influenced yet (meaning for example that the status of a node is learned by some other node in at most \(k\) rounds). Since the limitation holds for every node, the model is more restrictive than simply parameterizing the overhead on dissemination of one message to a factor of \(k \geq 1\). In fact the authors show that for \(k = 1\) it is not possible to influence only one node in every round. In the same paper, the authors study a second model of opportunistic connectivity where communication between neighboring nodes must be allowed within some time window, but the underlying topology is fixed. The ALL-TO-ALL COMMUNICATION problem is not studied in that work.

Communication congestion has been thoroughly studied in the Congested Clique model and others \([4,8,9,16,17,21,23,27,30,32,39,48,50]\), but to the best of our knowledge ours is the first model to combine communication and memory access congestion.

Finally, with respect to Dynamic Networks with anonymous nodes, the Anonymous Dynamic Networks (ADN) model has attracted also a lot of attention recently \([10,12,15,33,35,41,44]\). A comprehensive overview of work
related to ADNs can be found in a survey by Casteigts, Flocchini, Quattrociocchi, and Santoro [7]. In all these works continuous connectivity is assumed.

Other studies also dealing with the time complexity of information gathering exist [3, 5, 11, 46, 52, 53], but include in their model additional assumptions, such as the network having the same topology frequently enough or node identifiers.

3 Problem and Model

To emulate a single round of communication in the Congested Clique, we study the ALL-TO-ALL COMMUNICATION problem defined as follows. Initially all nodes hold an input message, and to solve the problem all nodes must receive the input message of all other nodes. For each input message initially held by more than one node, nodes must receive the number of copies of that input message as well.

We study ALL-TO-ALL COMMUNICATION in Anonymous Dynamic Congested Systems (ADCS) formed by \( n \) processing nodes. Nodes lack identifiers and the set of communication links among nodes is adversarially dynamic, with some limits on disconnection as specified below. When two nodes are able to communicate (that is, they are the endpoints of a communication link), we say that they are neighbors. We assume that \( n \) is initially unknown. Moreover, due to dynamicity, each ADCS node does not know even the number of its neighbors before receiving messages from them. That is, a node knows the number of its neighbors in any one round after receiving messages, but it does not who they are. Nonetheless, although we refer to the set of neighbors of a node and we label nodes in the presentation of algorithms and the analysis, we do it only for the sake of clarity – the nodes have only access to the messages sent by neighbors. Throughout the paper, references to algorithms lines are given as \((\text{algorithm}\#), (\text{line}\#)\) for succinctness.

Time is discretized in communication rounds. In each round of communication, each node may send a message to all its neighbors (the same to all\(^2\)), receive messages from all its neighbors, access local memory, and perform local computations. We evaluate time complexity in rounds of communication, given that in comparison local memory access and local computations take negligible time.

Each message is limited to \( O(\log n) \) bits, as in the Congested Clique [40] and CONGEST [51] models. Additionally, each memory access is limited to \( O(\log n) \) bits as well – this is the reason we call nodes congested as well. Specifically, each memory access is limited to \( O(\log n) \) bits as well – this is the reason we call nodes congested as well. The additional internal memory for local computation also holds \( O(\log n) \) bits. It implies, in particular, that a local computation algorithm may process a constant number of received messages (in their buffers) at a time, aggregating them somehow in local memory, and continuing with other buffers. (In other words, in a single round the algorithm may check all messages, but cannot upload them from the buffers all at once but instead needs to process them in constant batches). At the end of the round, the algorithm may store some \( O(\log n) \)-bit information in external memory, e.g., the learned input message together with its multiplicity. We show below that this restriction is tight. That is, if memory access is restricted to \( o(\log n) \) bits, some instances of ALL-TO-ALL COMMUNICATION cannot be solved.

We assume the presence of \( \ell \) distinguished nodes, called supervisors, where \( 0 < \ell < n \). That is, the set of nodes is partitioned in two classes: supervisors and supervised nodes. The number of supervisors \( \ell \) is known to all nodes. As we show later, these assumptions are necessary to solve ALL-TO-ALL COMMUNICATION deterministically without knowing \( n \). Within each class, nodes are indistinguishable.

The communication network topology model is a \( T \)-connected time-evolving graph, defined as follow.\(^3\)

Given a fixed set \( V \) of \( n \) nodes, let a time-evolving graph (or evolving graph for short) be an infinite sequence of graphs \( G = \{ G^{(t)} \}_{t \in \mathbb{N}} \) such that \( G^{(t)} = (V, E^{(t)}) \), where each \( E^{(t)} \) is the (possibly different) set of links of the graph \( G^{(t)} \). We call each \( G^{(t)} \) a constituent graph.

For any pair of nodes \( u, v \in V \) in an evolving graph \( G \), let an opportunistic path of length \( k > 0 \) from \( u \) to \( v \) be a sequence of links \( (x_1, x_2), (x_2, x_3), \ldots, (x_{k-1}, x_k) \) where \( x_i \in V \) for every \( i \in [k] \), \( x_1 = u, x_k = v \), and for every consecutive pair of links \( (x_{i-1}, x_i), (x_i, x_{i+1}) \) in the sequence, such that \( 1 < i < k \), if \( (x_{i-1}, x_i) \in E^{(t)} \) and

\(^2\)Dynamicity and anonymity prevent the nodes from sending destination-oriented messages.

\(^3\)There are many formalisms in the literature to specify graphs that change with time. Names include temporal graphs, dynamic graphs, evolving graphs, time-varying graphs, and others. We adopt the notation that provides more clarity to our analysis.
For the sake of contradiction, assume there exists such algorithm, call it Proof. ℓ nodes

Observation 4. Given a \( T \)-connected evolving graph \( G \), for any sequence of constituent graphs of \( G^{(t+1)}, G^{(t+2)}, \ldots, G^{(t+T)} \), \( t \geq 0 \), the union graph \( G_{\cup t} = \left( V, \bigcup_{i=1}^{T} E^{(t+i)} \right) \) is connected.\(^4\)

4Notice that the one or more paths between each pair of nodes \( u, v \in V \) in \( G_{\cup t} \) are derived from the opportunistic path from \( u \) to \( v \) and the opportunistic path from \( v \) to \( u \), which exist due to \( T \)-connectivity.

4.1 Impossibility Results

We start this section establishing some impossibility facts that validate the assumptions of our model.

Observation 1. For each ALL-TO-ALL COMMUNICATION deterministic algorithm \( A \), there exists an ADCS with \( T \)-connected evolving graph topology such that, if \( A \) does not use \( T \), the problem cannot be solved.

Proof. For the sake of contradiction, assume there exists such algorithm \( A \). Let \( T \) be the worst-case running time of \( A \) on a 1-connected ADCS of \( n \) nodes. Consider an ADCS with \( n + 1 \) nodes that during the first \( T \) rounds of execution of \( A \) is formed by a clique of \( n \) nodes and an isolated node \( v \), which connects to the clique for round \( T + 1 \). The nodes in the clique are not able to communicate with \( v \) during the \( T \) execution steps of \( A \). Hence they stop, but then the ALL-TO-ALL COMMUNICATION problem was not solved (as nodes in the clique do not know the message of the other node), and the described execution is feasible for \( T \)-connected ADCS, where \( T = T + 1 \) is unknown to the algorithm.

Observation 2. There is no deterministic algorithm to solve ALL-TO-ALL COMMUNICATION in an ADCS without at least one distinguished node, and without knowledge of the total number of nodes \( n \) and the number of distinguished nodes \( \ell \).

Proof. For the sake of contradiction, assume there exists such algorithm, call it \( A \). Then, we can use \( A \) to compute \( n \) by simply assigning a message 1 to each node and counting how many 1’s are received. However, this is a contradiction because, even for Anonymous Dynamic Networks without node and edge congestion it has been shown in [41] that counting the number of nodes deterministically is not possible without some distinguished node. It was also shown in [55] that the number of distinguished nodes needs to be known.

Observation 3. There exist applications of ALL-TO-ALL COMMUNICATION where, if memory access is limited to \( o(\log n) \) bits, the problem cannot be solved.

Proof. Consider an ALL-TO-ALL COMMUNICATION algorithm used to compute the number of nodes \( n \) in an ADCS, where \( n \) is unknown (i.e., each node has initially the same message 1). For each \( n \) there should be at least one final state of a node in which the node stops and outputs \( n \). On the other hand, nodes with internal memory \( \mu = o(\log n) \) may result in only \( 2^\mu = o(n) \) states; hence, for some \( n \) there will be no terminating state outputting \( n \) and the ALL-TO-ALL COMMUNICATION fails.

4.2 Expansion in Evolving Graphs with Opportunistic Connectivity

The relevant property that \( T \)-connectivity provides to our analysis is the following.

Observation 4. Given a \( T \)-connected evolving graph \( G \), for any sequence of constituent graphs of \( G^{(t+1)}, G^{(t+2)}, \ldots, G^{(t+T)} \), \( t \geq 0 \), the union graph \( G_{\cup t} = \left( V, \bigcup_{i=1}^{T} E^{(t+i)} \right) \) is connected.
Consider a $\mathcal{T}$-connected evolving graph $\mathcal{G}$ conceptually divided in subsequences of $\mathcal{T}$ consecutive constituent graphs. That is, $\mathcal{G} = \{G_i\}_{i=0}^\infty$, where $G_i = \{G_{i\mathcal{T}+1}\}_{i=1}^\mathcal{T}$. By Observation[4] for each $i \geq 0$, the union graph $G_{i\mathcal{T}+1}$ defined on $\mathcal{G}_i$ is connected, whereas each constituent graph of $\mathcal{G}$ may be not connected. Thus, rather than analyzing the potential distribution process as a Markov chain on the evolving graph $\mathcal{G}$, we study the process on the evolving graph $\mathcal{G}^* = \{G_{i\mathcal{T}+1}\}_{i=0}^\infty$, where each constituent union graph $G_{i\mathcal{T}+1}$ is connected. We do so taking into account that, in fact, each union graph corresponds to a sequence of graphs from $\mathcal{G}$.

The potential distribution process can be seen as a multiplication of a vector of values, one component for each node, by a matrix of shares, where component $(u,v)$ corresponds to the fraction of potential shared by node $u$ with neighboring node $v$. Let $P^{(t)}$ be the matrix of shares used by a gossip-based algorithm in round $t$ (corresponding to a constituent graph $G^{(t)}$). Then, for each $i \geq 0$, the matrix of shares corresponding to the potential distribution on the evolving graph $\mathcal{G}_i$ is $P_i = \prod_{t=1}^T P^{(t\mathcal{T}+1)}$. Each $P^{(t)}$ is doubly stochastic, hence, $P_i$ is also doubly-stochastic.

For each $i \geq 0$, consider a time-homogeneous Markov chain $X_i$ with state space $V$ and transition matrix $P_i$. $X_i$ is finite, irreducible and aperiodic, and given that each $G_{i\mathcal{T}+1}$ is connected, it is ergodic. Thus, the stationary distribution of $X_i$ is unique [47]. The uniform distribution, that is, $\pi_v = 1/n$ for all $v \in V$, is a solution of $\pi P_i = \pi$ because $P_i$ is doubly stochastic. Thus, the stationary distribution of $X_i$ is uniform.

Classic bounds on mixing time of Markov chains require the transition matrix to be symmetric [55], but the $P_i$’s above may not be. Thus, we apply instead a bound by Mihail [43] that is applicable to arbitrary irreducible Markov chains, as long as they are strongly aperiodic (as in our case where nodes keep at least half of the potential), and it is a function of transition matrix conductance.

The conductance of a transition matrix $P = (p_{uv})$ of a Markov chain $X_i$ over state space $V$ with stationary distribution $\pi$ is defined as follows:

$$\phi(P) = \min_{S \subseteq V : |S| \leq n/2} \phi_P(S), \quad \text{where} \quad \phi_P(S) = \frac{\sum_{u \in S} \sum_{v \in V \setminus S} p_{uv} \pi_u}{\sum_{v \in S} \pi_v}$$

and $w_{uv} = \pi_u p_{uv} \pi_v$.

We instantiate this definition on $P_i = (p_{uv})$ and $\pi = \frac{1}{n}$ as follows:

$$\phi(P_i) = \min_{S \subseteq V : |S| \leq n/2} \phi_{P_i}(S), \quad \text{where} \quad \phi_{P_i}(S) = \frac{1}{|S|} \sum_{u \in S} \sum_{h \in V \setminus S} p_{uh} \pi_h.$$ 

**Theorem 1.** (derived from Theorem 3.1 in [43].) For an irreducible strongly aperiodic Markov chain $X_i$ with state space $V$, such that $|V| = n$, with a unique uniform stationary distribution and transition matrix $P_i$, it is

$$\left\| \Pi_{i+1} - \frac{I}{n} \right\|_2^2 \leq (1 - \phi(P_i)^2) \left\| \Pi_i - \frac{I}{n} \right\|_2^2.$$ 

Where $\Pi_i$ is the distribution after $t \geq 0$ steps of $X_i$.

Notice that the above analysis applies to each $i \geq 0$. That is, each $P_i$ may be different for each $G_{i\mathcal{T}+1}$, but all of the $X_i$ converge to a uniform stationary distribution and the bounds in Theorem[1] apply to each $X_i$. Thus, by application of these bounds to each sequence of $\mathcal{T}$ rounds, the convergence time for the evolving graph $\mathcal{G}$ can be obtained.

We define the minimum conductance $\phi_{\min}$ corresponding to the transition matrices $P_i = \prod_{t=1}^T P^{(i\mathcal{T}+1)}$ used by our algorithms on the evolving graph topology $\mathcal{G}$ as follows:

$$\phi_{\min} = \min_{i=0,1,2,...} \phi(P_i).$$

Conductance is a useful expansion characteristic to provide tighter time bounds, but being a function of the probabilities of transition it is specific for each algorithm. To obtain bounds that depend on network characteristics only, we will use the isoperimetric number[5]. The isoperimetric number of a static graph $G = (V,E)$ is defined as follows:

$$i(G) = \min_{X : |X| \leq |V|/2} \frac{|\partial X|}{|X|},$$

The isoperimetric number of a graph (a.k.a. graph Cheeger constant) is the discrete analogue of the Cheeger isoperimetric constant, c.f., [6].
where \( \partial X \) denotes the set of links of \( G \) that have one end in \( X \) and the other end in \( V \setminus X \). We apply this definition to each constituent union graph \( G_{\cup j} \) of the evolving graph \( G^{*} = \{G_{\cup j}\}_{j=0}^{\infty} \), and obtain:

\[
i(G_{\cup j}) = \min_{X:|X| \leq |V|/2} \frac{\partial X}{|X|} \quad \text{for each } j = 0, 1, \ldots \quad \text{and then } \quad i_{\min} = \min_{j=0,1,2,\ldots} i(G_{\cup j}).
\]

The following relation between conductance and isoperimetric number will be used. For each \( P(t) \), let each non-zero entry be at least \( 1/d \) for some \( d > 0 \). Given that \( P_j = \prod_{t=1}^{T} P(jT + t) \), we have that each non-zero entry of \( P_j \) is at least \( 1/d^T \). Thus, for each \( j = 0, 1, \ldots \) we have

\[
\phi(P_j) \geq \min_{X \subseteq V:|X| \leq |V|/2} \frac{1}{|X|} \sum_{u_i \in X} \sum_{u_j \in V \setminus X} \frac{1}{d^T} = \frac{1}{d^T} \min_{X \subseteq V:|X| \leq |V|/2} \frac{\partial X}{|X|} = \frac{i(G_{\cup j})}{d^T}.
\]

### 5 All-to-all Communication

Assume each node \( v \in V \) has an input message to send to every other node consisting of at most \( \ell_v \leq \lceil \log n \rceil \) bits (recall that \( n \) is unknown to nodes, but this is only an upper bound on the input message length). This length could be scaled by any constant \( c > 0 \) w.l.o.g. The set of neighbors \( N \) and nodes are labeled only for presentation, but nodes do not have and do not use such knowledge; instead, they can only send and receive messages to/from them, see the pseudo-codes.

Our **ALL-TO-ALL COMMUNICATION** Algorithm [1] starts with counting the number of participants \( n \) using the **RESTRICTED METHODICAL COUNTING** algorithm (refer to Section [7]), and then proceeds in subsequent epochs. Each epoch is dedicated to finding a new input message (see the Discovery Part) and counting how many copies of this input message are at nodes (see the Processing Part). The Discovery Part proceeds in phases. Each phase parameterized by \( i \) corresponds to the discovery of the index bit of some input message, whose first index \( i - 1 \) bits have been discovered in preceding phases (but of an input message not been discovered and counted in previous epochs). This discovery is done by spreading and updating variables \( \text{match}_1 \) at each node, and if no node is discovered, then spreading and updating variables \( \text{match}_0 \). The phases’ parameter \( i \) is numbered from 1 to \( \lceil \log n \rceil \) (the latter being an upper bound on the input message’s length). In each phase, a sufficiently long broadcast is run to update information on whether the index bit being matched is 1 or 0. The length of the broadcast \( r' = \lceil T \ln n/ \ln(1 + i_{\min}) \rceil \) is a bound on temporal diameter of the evolving graph \( G^{*} \) based on the initially computed \( n \), the connectivity parameter \( T \), and, if known, a bound \( i_{\min} \) on the isoperimetric number, which otherwise can be lower bounded by a function of \( n \) (refer to the first line in Algorithm [1] and Section [4.2]). Nodes whose input messages have been discovered in previous epochs, or whose input message does not match the bits already discovered in the current epoch (which is encoded by setting the variable \( \text{match} \) to \( \text{false} \)) only forward the received information in the main algorithm and the procedure **MULTIPLICITY** (see Algorithm [2] in Section [5]) run in the Processing Part. All nodes whose input message has been discovered (and so, also delivered to all nodes) in the Discovery Part of the current epoch (which is encoded by setting the variable \( \text{delivered} \) to \( \text{true} \)) are counted using the procedure **MULTIPLICITY** by all nodes. The output—the discovered input message and its multiplicity—is stored in external storage at the end of the epoch. The algorithm finishes when no new input message is discovered.

#### 5.1 Analysis of ALL-TO-ALL COMMUNICATION

In this section we present the main theorem of this work, showing the correctness and running time of our **ALL-TO-ALL COMMUNICATION** algorithm. For the purpose of the analysis, we conceptually divide time into blocks of \( T \) rounds. The analysis of **RESTRICTED METHODICAL COUNTING** () and **MULTIPLICITY** () is included in the sections that follow. The following lemma will be used.

**Lemma 1.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( n \) nodes, each holding a true/false value. Then, if all nodes broadcast true values (initially held or received from others) for at least \( r' = T \ln n/ \ln(1 + i_{\min}) \) rounds, all nodes know whether there was initially some true value in the system or not.
Algorithm 1: ALL-TO-ALL COMMUNICATION algorithm for each node. input message is the input message initially held by this node. \( N \) is the set of neighbors of this node in the current round -- the node does not know them, but can send and receive short messages to/from them. \( r' = \lceil T \ln n / \ln(1 + i_{\text{min}}) \rceil \) as in Lemma 1, where \( i_{\text{min}} \) is the minimum isoperimetric number of the evolving graph topology as defined in Section 4.2.

1. \( n \leftarrow \text{RESTRICTED METHODOICAL COUNTING}() \) // Counting \( n \)
2. delivered \( \leftarrow \text{false} \) // Own input message not delivered yet
3. while true do // Iterating Epochs -- as long as there are undelivered input messages
   4. match \( \leftarrow \neg \text{delivered} \) // match will indicate if discovered bits match own input message
   5. new input message \( \leftarrow \text{empty string of bits} \) // Discovery Part
   6. for index \( \leftarrow 1 \) to \( \lceil \log n \rceil \) do // Iterating Phases -- from most to least
      7. bit \( \leftarrow \text{extract bit index from input message} \)
      8. match0 \( \leftarrow \text{match} \land (\text{bit} = 0) \) // Indicates matching so far, with 0 in index bit
      9. match1 \( \leftarrow \text{match} \land (\text{bit} = 1) \) // Indicates matching so far, with 1 in index bit
   10. for round \( \leftarrow 1 \) to \( r' \) do // Broadcasting match1
       11. Send \( \langle \text{match1} \rangle \) and Receive \( \langle \text{match1}_v \rangle, \forall v \in N \) // To/from neighbors in \( N \)
       12. match1 \( \leftarrow \bigvee_{v \in N} \text{match1}_v \bigvee \text{match1} \) // Incorporating neighbors' knowledge about match1
       13. if match1 = true then // Discovered index bit 1
          14. append '1' to new input message
          15. if bit = 0 then match \( \leftarrow \text{false} \)
       16. else
          17. for round \( \leftarrow 1 \) to \( r' \) do // Broadcasting match0
             18. Send \( \langle \text{match0} \rangle \) and Receive \( \langle \text{match0}_v \rangle, \forall v \in N \) // To/from neighbors in \( N \)
             19. match0 \( \leftarrow \bigvee_{v \in N} \text{match0}_v \bigvee \text{match0} \) // Incorporating neighbors' knowledge on match0
             20. if match0 = true then // Discovered index bit 0
                 21. append '0' to new input message
                 22. if bit = 1 then match \( \leftarrow \text{false} \)
             23. else return // No more matches, all input messages delivered
       24. end if
   25. end for
   26. end for
   27. end for
   28. end for
   29. end if
30. end if
31. count \( \leftarrow \text{MULTIPlicity} \langle \text{new input message} \rangle \) // Move newly discovered info to external storage
32. output \( \langle \text{new input message}, \text{count} \rangle \) // Processing Part
33. if match = true then delivered \( \leftarrow \text{true} \) // Own input message was delivered in this epoch
Proof. Let \( \{S_i, V \setminus S_i\} \) be a partition of the set of nodes at the beginning of some block \( i \) with transition matrix \( P_i = \prod_{t=1}^i P_t^{(T+t)} \). By definition of isoperimetric number (refer to Section 4.2) we know that \( i_{\min} \leq |\partial S_i|/|S_i| \). That is, the number of links crossing the partition \( \{S_i, V \setminus S_i\} \) is larger than \( |S_i|i_{\min} \). Hence, adding a 1-hop neighbourhood to any initial subset of nodes \( |S_i| \), the new subset at the beginning of block \( i+1 \) is such that \( |S_{i+1}| \geq (1 + i_{\min})|S_i| \).

Consider any node that did not receive true yet, call it \( x \), and let \( S_i = \{x\} \). The question of how many blocks are needed for \( x \) to receive a true (if there is any) is equivalent to ask what is the minimum \( t \) such that the \( t \)-hop neighbourhood of \( x \) (in the evolving graph) includes all nodes. That is, we want to find what is the minimum \( r' \) such that \( (1 + i_{\min})^r' \geq n \). Manipulating the latter equation, and taking into account that each block has \( T \) rounds, the claimed \( r' = T \ln n/\ln(1 + i_{\min}) \) follows.

Theorem 2. Consider an ADCS with a \( T \)-connected evolving graph topology, \( T \in O(1) \), with minimum isoperimetric number \( i_{\min} \), formed by \( \ell \geq 1 \) supervisor nodes and \( n - \ell \) supervised nodes, \( n \geq 2 \), each holding an input message, running the ALL-TO-ALL COMMUNICATION algorithm. Then, the ALL-TO-ALL COMMUNICATION problem is solved in

\[
O\left(\frac{n^{1+2T(1+\epsilon)}}{\ell i_{\min}^2} \log n \log \left(\frac{n}{\ell}\right) + \frac{n^{r'}}{\ln(1 + i_{\min})} \log^2 n\right)
\]

rounds,

where \( n' \leq n \) is the number of different input messages.

Proof. The correctness of RESTRICTED METHODOICAL COUNTING () and MULTIPLECTITY () is proved in Theorems 4 and 3 respectively. Thus, to complete the proof of correctness, it is enough to prove that all input messages are discovered, that every discovered input message is an input message held by some node, and that the algorithm runs under the restrictions of the ADCS model. Refer to Algorithm 1.

Assume first for the sake of contradiction that there is some node \( v \) with input message \( m \) that is not discovered. Each bit of each newly discovered input message \( m' \) is matched to the input messages of all other nodes holding \( m' \) in Lines 1.13 (for a 1 bit) or 1.20 (for a 0 bit). Thus, \( m \) will be discovered as long as the number of rounds \( r' \) is enough to broadcast the match status of \( v \) to all other nodes. Lemma 1 proves that \( r' \) is large enough for \( v \) to disseminate its match. Therefore \( m \) is discovered, which is a contradiction.

Assume now, again for the sake of contradiction, that there is some input message \( m \) that is discovered but it is not an input message of any node in the system. However, an input message is discovered when there is a match of the input message of some node (or nodes). Therefore, if \( m \) is not the input message of any node it cannot be discovered.

RESTRICTED METHODOICAL COUNTING () and MULTIPLECTITY () are proved to run under the restrictions of the ADCS model in Theorems 4 and 3 respectively, and the rest of the messages and calculations in the rest of the algorithm are on Boolean variables. Thus, RESTRICTED METHODOICAL COUNTING does not violate the \( O(\log n) \) bits limit on message size and local computation. All the analysis of the various parts of the algorithm apply to \( T \)-connected evolving graph topologies. Therefore, the proof of correctness is complete.

Regarding the running time complexity, recall that performance is measured in rounds of communication. Theorems 4 and 3 prove the time complexity of RESTRICTED METHODOICAL COUNTING () and MULTIPLECTITY () respectively. The running time of the remaining parts can be obtained by simple inspection. Namely, communication is carried out in Lines 1.11 and 1.18, each inside a loop of \( r' \) iterations, each nested in another loop of \( \phi_{\min}^2 \) iterations (Line 1.6), and the latter nested in a loop of \( n' \) iterations (Line 1.5), where \( n' \) is the number of different input messages. Thus, the total number of rounds excluding RESTRICTED METHODOICAL COUNTING () and MULTIPLECTITY () is

\[
2r' \ln n \ln n' = \frac{2T \ln n \ln n' n'}{\ln(1 + i_{\min})}.
\]

Combining the latter with the running times \( O(\ln n/\phi_{\min}^2) \) proved in Theorem 3 and \( O(n \ln n \log(n/\ell)/(\ell \min \{\phi_{\min}^2, \ln(1 + i_{\min})\})) \) in Corollary 1 (to Theorem 3), the total time complexity is

\[
O\left(\frac{n \ln n \log(n/\ell)}{\ell \min \{\phi_{\min}^2, \ln(1 + i_{\min})\}} + \frac{n' \ln n \log n}{\ln(1 + i_{\min})}\right).
\]

Using the bound in Equation 1 to replace \( \phi_{\min} \), we obtain the claimed time complexity.
6 Counting the Number of Copies of an input message

In the main ALL-TO-ALL COMMUNICATION Algorithm at the end of each epoch we count the number of nodes that have a discovered input message as its own input message. We do counting using the following procedure MULTIPLICITY (refer to Algorithm). It tries to locally balance potential truncated to \( c \log d \) bits for a sufficiently long time, and once it (almost) stabilizes, returns it after scaling by the number of all nodes \( n \) (recall that in the main algorithm ALL-TO-ALL COMMUNICATION all nodes compute \( n \) in the very beginning).

Algorithm 2: MULTIPLICITY algorithm for each node. \( \text{input message} \) is the input message initially held by this node and \( \text{new input message} \) is the input message whose multiplicity has to be counted. \( N \) is the set of neighbors of this node in the current round - the node does not know them, but can send and receive short messages to/from them. Parameters \( d, r'' \), and \( c \) are as defined in Theorem.

```plaintext
1 if \text{input message} = \text{new input message} then \Phi ← 1 else \Phi ← 0 // Assign initial potential
2 for \text{round} = 1 to \( r'' \) do
3 \quad Send \langle \Phi \rangle and Receive \langle \Phi_{v} \rangle, \forall v ∈ N // To/from neighbors in \( N \)
4 \quad \Phi ← \Phi + \sum_{v \in N}[d^{c-1}\Phi_{v}] / d^{c} - |N|[d^{c-1}\Phi] / d^{c} // Share potential truncated to \( c \log d \) bits
5 return \( \Phi \cdot n \) rounded to the closest integer (up or down)
```

6.1 Analysis of MULTIPLICITY

We analyze the evolution of potentials in the MULTIPLICITY algorithm as a Markov chain on the \( T \)-connected evolving graph \( G = \{ G_{j} \}_{j=0}^{\infty} \), where \( G_{j} = \{ G_{j,T+1} \}_{t=1}^{T} \), as defined in Section. We adjust the potentials distribution after each \( T \) rounds by the error produced by the truncation of potentials. In different nodes such error may delay the convergence to the stationary distribution. To upper bound the total time, we upper bound the delay (that is, the additional error with respect to the stationary distribution due to truncation). As a worst case we assume that the convergence may be delayed at all nodes.

For the purpose of the analysis, we conceptually divide time into blocks of \( T \) rounds. Let \( b = r'' / T \) be the number of blocks in MULTIPLICITY. We denote the vector of potentials at the beginning of round \( i \) of block \( j \) as \( \Phi_{i,j} \). For clarity, we will sometimes refer to round \( i \) of block \( j \) as round \( (i,j) \). For inductive arguments we may refer to the round that follows (resp. precedes) as \( (i+1,j) \) (resp. \( (i-1,j) \)), omitting the fact that such round may correspond to a different block (i.e., \( (1,j+1) \) and \( (T,j-1) \), resp.) - this is only for notation and will not influence technical arguments. Also, we denote the potential right after the potential distribution is stopped (i.e. after the loop in Line as \( \Phi_{1,b+1} \), meaning the potential after the last block of rounds \( b \).

Theorem 3. Consider an ADCS with a \( T \)-connected evolving graph topology with \( n \geq 2 \) nodes running the MULTIPLICITY algorithm with parameters \( d \geq 2n, c \geq 5\alpha + 2T + 4, \) and \( r'' = Tb \), where \( b \geq 4\alpha \ln n / \phi_{\min}^{2}, \alpha \geq \max \{ \log_{n}(3T), 3 \} \). Let \( 1 \leq \delta \leq n \) be the number of copies being counted. Then, all nodes return the correct count \( \delta \).

Proof. The algorithm runs under the ADCS restrictions because the potentials are truncated to \( O(\log n) \) bits.

Consider the initial distribution \( \Pi_{1} \) on the overall potential \( \| \Phi_{1,1} \|_{1} = \delta \). Given that \( d \geq 2n \), the transition matrix is strongly aperiodic. Then, using Theorem we know that after \( T \) rounds the distribution \( \Pi_{T+1} \) at the beginning of round \( T + 1 \) would be such that

\[
\left\| \Pi_{T+1} - \frac{1}{n} \Pi_{1} \right\|_{2}^{2} \leq \frac{1}{n} \left( 1 - \phi(P_{0})^{2} \right)^{n} \left\| \Pi_{1} - \frac{1}{n} \Pi_{1} \right\|_{2}^{2}.
\]

(2)

Where \( \phi(P_{0}) \) is the conductance of \( P_{0} = \prod_{t=1}^{T} P^{(t)} \), and \( P^{(t)} \) is the matrix of shares used by the algorithm in round \( t \).
On the other hand, due to truncation, the vector of potentials is such that

$$\left\| \frac{\Phi_{1,2}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \sum_{v \in V} \left( \left| \Pi_{T+1}(v) - \frac{1}{n} \right| + \xi(v) \right)^2,$$

where $\xi(v)$ is the error introduced by the truncation at node $v$ during the $T$ rounds. As a worst case we have assumed that $\xi(v)$ contributes to the deviation with respect to the stationary distribution. Then, we have that

$$\left\| \frac{\Phi_{1,2}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \sum_{v \in V} \left( \left( \Pi_{T+1}(v) - \frac{1}{n} \right)^2 + 2\xi(v) \left| \Pi_{T+1}(v) - \frac{1}{n} \right| + \xi(v)^2 \right)$$

$$= \left\| \Pi_{T+1} - \frac{1}{n} \right\|_2^2 + \sum_{v \in V} \left( 2\xi(v) \left| \Pi_{T+1}(v) - \frac{1}{n} \right| + \xi(v)^2 \right)$$

$$\leq \left\| \Pi_{T+1} - \frac{1}{n} \right\|_2^2 + \sum_{v \in V} (2\xi(v) + \xi(v)^2).$$

(3)

The potential received from each neighboring node is truncated to $c \log d$ bits after dividing by $d$. Thus, the error introduced in the calculation of its new potential in each round is at most $(n - 1)/d^c \leq 1/d^{c-1}$. Therefore, we have that after $T$ rounds it is $\xi(v) \leq T/c^{c-1}$, which given that $c \geq 5\alpha + 2T + 4 \geq \log_d T + 1$ it is $\xi(v) \leq 1$. Then,

$$\sum_{v \in V} (2\xi(v) + \xi(v)^2) \leq \sum_{v \in V} 3\xi(v) \leq \sum_{v \in V} \frac{3Tn}{d^{c-1}} = \frac{3Tn}{d^{c-1}} \leq \frac{3T}{d^{c-2}}. \quad (4)$$

Replacing Eqs. 2 and 4 in 3, we have that

$$\left\| \frac{\Phi_{1,2}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \left(1 - \phi(R_0)^2\right) \left\| \Pi_1 - \frac{1}{n} \right\|_2^2 + \frac{3T}{d^{c-2}} \leq \left(1 - \phi(R_0)^2\right) \left\| \frac{\Phi_{1,1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 + \frac{3T}{d^{c-2}}.$$

The distribution of normalized potential at the beginning of round $T + 1$ (which is the same as round $\langle 1, 2 \rangle$) is also a probability distribution. Thus, the above analysis applies inductively to every subsequent block of $T$ rounds.

Then, the vector of potentials after $b$ blocks is such that

$$\left\| \frac{\Phi_{1,b+1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \left(1 - \phi_{\text{min}}^2\right)^b \left\| \frac{\Phi_{1,1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 + \frac{3T}{d^{c-2}} \sum_{i=1}^{b-1} \left(1 - \phi_{\text{min}}^2\right)^i$$

$$\leq \left(1 - \phi_{\text{min}}^2\right)^b \left\| \frac{\Phi_{1,1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 + \frac{3T}{d^{c-2}} \frac{1}{\phi_{\text{min}}^2} \leq \exp \left(-b\phi_{\text{min}}^2\right) + \frac{3T}{d^{c-2}\phi_{\text{min}}^2}.$$

In potential distribution algorithms where each non-zero entry of $P(jT+i)$ is at least $1/d$, for any $j > 0$, $i \in [1, T]$, and some $d \geq 1$, as it is the case in the MULTIPLICITY algorithm, it is $\phi_{\text{min}} \geq 2/(nd^T) \geq 2/d^{T+1}$. Then, it is

$$\left\| \frac{\Phi_{1,b+1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \exp \left(-b\phi_{\text{min}}^2\right) + \frac{3T}{4d^{c-2}T-4}. \quad (5)$$

Given that $b \geq 4\alpha \ln n/\phi_{\text{min}}^2$, it is

$$\left\| \frac{\Phi_{1,b+1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \frac{1}{n^{4\alpha}} + \frac{3T}{4d^{c-2}T-4}.$$

Replacing $\alpha \geq \log_d(3T)$ and $d \geq n$, it is

$$\left\| \frac{\Phi_{1,b+1}}{\|\Phi_{1,1}\|} - \frac{1}{n} \right\|_2^2 \leq \frac{1}{n^{4\alpha}} + \frac{1}{4n^{c-2}T-4-\alpha}.$$
And for $c \geq 5\alpha + 2T + 4$ we have that
\[
\left\| \frac{\Phi_{1,b+1}'}{\Phi_{1,1}'} - \frac{1}{n} \right\|^2_2 \leq \frac{5}{4n^{2\alpha}} \leq \frac{1}{n^{2\alpha}}, \text{ for } \alpha \geq 1/2 \text{ and } n \geq 5/4.
\]

Given that \( (\Phi_{1,b+1}'[v]/\|\Phi_{1,1}'\|_1 - 1/n)^2 \leq \left\| \Phi_{1,b+1}'/\|\Phi_{1,1}'\|_1 - \frac{1}{n} \right\|^2_2 \) for any node \( v \), we have that \( (\Phi_{1,b+1}'[v]/\|\Phi_{1,1}'\|_1 - 1/n)^2 \leq 1/n^{2\alpha} \) and hence \( 1/n - 1/n^{\alpha} \leq \Phi_{1,b+1}'[v]/\|\Phi_{1,1}'\|_1 \leq 1/n + 1/n^{\alpha} \).

Then, we have \( (\delta/n - 1/n^{\alpha-1}) \leq \Phi_{1,b+1}'[v] \leq (\delta/n + 1/n^{\alpha-1}) \) and so \( \delta - 1/n^{\alpha-2} \leq \Phi_{1,b+1}'[v]n \leq \delta + 1/n^{\alpha-2} \). Replacing \( \alpha \geq 3 \) and \( n \geq 2 \) we have that \( \delta - 1/2 \leq \Phi_{1,b+1}'[v]n \leq \delta + 1/2 \). Thus, after rounding to the closest integer in Line 25 the returned value is \( \delta \) as claimed.

7 Counting the Number of Nodes in the System

The first step of our ALL-TO-ALL COMMUNICATION algorithm is to compute the unknown number of nodes \( n \). Counting the number of nodes in Anonymous Dynamic Networks has attracted a lot of attention recently [12,13,33,35,41]. Most notably, the METHODICAL COUNTING (MC) protocol [33] was the first one to achieve polynomial time after a flurry of papers improving bounds that started with doubly-exponential time [41]. The ADN model is challenging due to lack of node identifiers and arbitrary topology changes, but still lags behind real-world constraints such as limited bandwidth, disconnection of the network, and/or limited computational resources.

In this work, as an integral part of our ALL-TO-ALL COMMUNICATION algorithm, but also of independent interest, we present the RESTRICTED METHODICAL COUNTING (RMC) algorithm which computes the number of nodes under the harsh restrictions of Anonymous Dynamic Congested Systems with opportunistic \( T \)-connectivity. RMC is inspired on an extension of MC called METHODICAL MULTI-COUNTING (MMC) [35], but carefully adapted to cope with limited messages, memory accesses, and connectivity. Our analysis relies on a bound on mixing time for \( T \)-connected evolving graphs. The analysis and design of the algorithm handle the errors produced by truncation of the calculations, purposely introduced to stay within the limits on message size and memory access. We describe RMC in the following section and analyze its properties afterwards. The pseudocode of RMC can be found in Algorithms 5 and 4.

7.1 The Restricted Methodical Counting Algorithm

RMC is executed synchronously by all nodes. Given that no preliminary knowledge of the network size is available, nodes maintain a running estimate of \( n \) that starts with the minimum possible, that is, one more than the number of supervisors \( \ell \). By means of various alarms, the estimate is detected to be either low, high, or correct. Starting with the initial value, the estimate is updated by exponential search until it is correct. If the estimate becomes high (that is, if the correct value is skipped) the correct estimate is found by binary search in the last range between low and high detection.

The evaluation of each estimate is implemented as follows. In each round of communication nodes share some potential values with neighbors in a gossip-based fashion. The fraction of potential shared, as well as the truncation of messages and calculations to adapt to the limits in communication and memory access, are functions of the running estimate, given that \( n \) is not known. This gossiping continues for a number of rounds after which supervisors move their potential to a separate accumulator completing what is called conceptually a phase. The process repeats for a number of phases to complete the evaluation of the current estimate. The number of rounds and phases are functions of the running estimate, the errors produced by truncation, and the disconnection characteristic \( T \).

If the estimate is correct, the number of rounds and phases are such that supervisors have moved almost all the initial potential to their accumulators. Then, supervisors univocally decide that the estimate is correct comparing the value in their accumulators with some carefully designed range, and they disseminate the news to all other nodes by simple broadcast.

If on the other hand the estimate is incorrect, that is, if it is either low or high, the algorithm detects by one or more of the following alarms. If nodes receive messages from more neighbors than the running estimate, the estimate is obviously low. If early in the execution potentials are above some carefully calculated threshold, the estimate is low as
our analysis shows. Finally, if the supervisors accumulated potential is outside the abovementioned range, or a node receives a notification of wrong estimate from other nodes, the estimate is either low or high (whatever corresponds).

A crucial difference between RMC and MMC is the way that potentials are shared and updated in each round to cope with limits in communication and memory access. Specifically, in MMC the potential $\Phi$ of each node $u$ is updated in round $r$ of phase $p$ of an epoch with estimate $k$ as

$$\Phi_{p,r+1}[u] = \Phi_{p,r}[u] + \sum_{v \in N_{p,r}[u]} \frac{\Phi_{p,r}[v]}{d} - \frac{|N_{p,r}[u]|}{d} \Phi_{p,r}[u],$$

where $N_{p,r}[u]$ is the set of neighbors of $u$ in round $r$ of phase $p$, $d = 2k^{1+\epsilon}$ and $\epsilon > 0$ is an arbitrarily small constant.

In words, each node shares a fraction $1/d$ with its neighbors. So, node $u$ adds to its current potential a $1/d$ fraction of the potential of each neighbor, and subtracts a $1/d$ fraction of its own potential for each neighbor. The potential $\Phi$ is transmitted as a whole rather than the fraction since all nodes know $d = 2k^{1+\epsilon}$ because the algorithm is synchronous.

However, after successive rounds of dividing by $d$, the potential of some nodes may require $\omega(\log d)$ bits of precision. That is, for $k \in \omega(\sqrt[4]{n})$, they would require $\omega(\log n)$ bits of precision, violating the limits on communication and memory access. To avoid it, in RMC shared potentials are truncated to the most significant $c \log d$ bits, for some constant $c$ defined later. More precisely, in RMC the potential $\Phi$ of each node $u$ is updated in round $r$ of phase $p$ of an epoch with estimate $k$ as

$$\Phi_{p,r+1}[u] = \Phi_{p,r}[u] + \sum_{v \in N_{p,r}[u]} \frac{|d^{-1}\Phi_{p,r}[v]|}{d} - \frac{|N_{p,r}[u]|}{d} \frac{|d^{-1}\Phi_{p,r}[u]|}{d}.$$

Also, to attain strong aperiodicity as required by Theorem 1, we set $d = 2k^{1+\epsilon}$. Given that throughout the execution $k < 2n$, RMC does not violate the limits on communication and memory access.

### 7.2 Analysis of Restricted Methodical Counting

We analyze the evolution of potentials as a Markov chain on the $T$-connected evolving graph $G = \{G_j\}_{j=0}^{\infty}$, where $G_j = \{G_jT+t\}_{t=1}^{T}$, as defined in Section 4.2. We adjust the potentials distribution after each $T$ rounds by the error produced by the truncation of potentials. In different nodes such error may delay the convergence to the stationary distribution. To upper bound the total time, we upper bound the delay (that is, the additional error with respect to the stationary due to truncation). As a worst case we assume that the convergence is delayed at all nodes.

For the purpose of the analysis, we conceptually divide each phase in blocks of $T$ rounds. (The number of rounds in each phase will be a multiple of $T$.) We denote the vector of potentials at the beginning of round $i$ of block $j$ of phase $h$ as $\Phi_{i,j,h}$. For clarity, we will sometimes refer to round $i$ of block $j$ of phase $h$ as round $(i,j,h)$, and for inductive arguments we may refer to the round that follows (resp. preceeds) as $(i+1,j,h)$ (resp. $(i-1,j,h)$), omitting the fact that such round may correspond to a different block. Also, we refer to the potential right before the beginning of phase $h > 1$, denoted as $\Phi_{0,0,h}$, meaning the potential after the last round of phase $h - 1$, before the potential is reset at the beginning of phase $h$.

First, we prove the following two claims about properties of the potential during the execution of RMC, for later use.

**Claim 1.** Given an ADCS of $n$ nodes running RMC with parameter $d$, for any round $(i,j,h)$ of phase $h$, if $d$ was larger than the number of neighbors of each node $x$ for every round $(i',j',h)$ before round $(i,j,h)$, then $||\Phi_{i,j,h}||_1 = ||\Phi_{1,1,h}||_1$.

**Proof.** For any given round $(i,j,h)$ and any given node $u$, if $d/2$ is larger than the number of neighbors of $u$, the potential is updated as (refer to Lines 3.10 and 5.2)

$$\Phi_{i+1,j,h}[u] = \Phi_{i,j,h}[u] + \sum_{v \in N_{i,j,h}[u]} \frac{|d^{-1}\Phi_{i,j,h}[v]|}{d} - \frac{|N_{i,j,h}[u]|}{d} \frac{|d^{-1}\Phi_{i,j,h}[u]|}{d},$$

Notice that nodes do not know $N_{p,r}[u]$, but they know $|N_{p,r}[u]|$ after receiving messages. Also, recall that nodes are labeled only for the presentation, but nodes do not have identifiers.
Algorithm 3: **Restrict Methodical Counting** algorithm for each *supervisor node*. $N$ is the set of neighbors of this node in the current round – the node does not know them, but can send and receive short messages to/from them. $\ell$ is the number of supervisor nodes. The parameters $d, p, r, \tau$ and $c$ are as defined in Theorem 4.

1. $k \leftarrow \ell + 1, \min \leftarrow k, \max \leftarrow \infty$  
   // initial size estimate and range
2. repeat  
   // iterating epochs
3.   status $\leftarrow$ probing  
4.   $\Phi \leftarrow 0$  
5.   $\rho \leftarrow 0$  
6.   for phase = 1 to $p$ do  
7.     for round = 1 to $r$ do  
8.       Send $\langle \Phi, \text{status} \rangle$ and Receive $\langle \Phi_i, \text{status}_i \rangle, \forall i \in N$  
9.       if status = probing and $|N| < d/2$ and $\forall i \in N : \text{status}_i = \text{probing}$ then  
10.          $\Phi \leftarrow \Phi + \sum_{v \in N} \left( d^{-1} \Phi_v \right) / d^\ell - |N| \left( d^{-1} \Phi \right) / d^\ell$  
11.              // Share potential truncated to $c \log d$ bits
12.       else  
13.          // $k < n$
14.          status $\leftarrow$ probing, $\Phi \leftarrow \ell$
15.       if phase = 1 and $\Phi > \tau$ then status $\leftarrow$ low, $\Phi \leftarrow \ell$  
16.          // $k < n$
17.       if status = probing then  
18.          $\rho \leftarrow \rho + \Phi$  
19.              // consume potential
20.          $\Phi \leftarrow 0$
21.       if status = probing then  
22.          if $(k - \ell)(1 - k^{-\gamma}) \leq (k - \ell)(1 + k^{-\gamma})$ then  
23.              status $\leftarrow$ done  
24.              // $k = n$
25.          if $\rho < (k - \ell)(1 - k^{-\gamma})$ then status $\leftarrow$ high  
26.              // $k > n$
27.          if $\rho > (k - \ell)(1 + k^{-\gamma})$ then status $\leftarrow$ low  
28.              // $k < n$
29.       for round = 1 to $d$ do  
30.          Broadcast $\langle \text{status} \rangle$ and Receive $\langle \text{status}_i \rangle, \forall i \in N$  
31.          if status = low then  
32.              $\min \leftarrow k + 1$
33.          if max = $\infty$ then $k \leftarrow 2k$ else $k \leftarrow \lfloor (\min + max) / 2 \rfloor$
34.          else  
35.              if status = high then  
36.                  $\max \leftarrow k - 1$
37.                  // $k < n$
38.              $k \leftarrow \lfloor (\min + max) / 2 \rfloor$
39.       until status = done
40. return $k$
Algorithm 4: Restricted Methodical Counting algorithm for each supervised node. \( N \) is the set of neighbors of this node in the current round – the node does not know them, but can send and receive short messages to/from them. \( \ell \) is the number of supervisor nodes. The parameters \( d, p, r, \tau \) and \( c \) are as defined in Theorem 4.

\[
k \leftarrow \ell + 1, \min \leftarrow k, \max \leftarrow \infty \quad \text{// initial size estimate and range}
\]

\[
\text{repeat} \\
\text{// iterating epochs}
\]

\[
\text{status} \leftarrow \text{probing} \\
\Phi \leftarrow \ell \quad \text{// current potential}
\]

\[
\text{for phase = 1 to } p \text{ do} \\
\text{// iterating phases}
\]

\[
\text{for round = 1 to } r \text{ do} \\
\text{// iterating rounds}
\]

\[
\text{Send } \langle \Phi, \text{status} \rangle \text{ and Receive } \langle \Phi_i, \text{status}_i \rangle, \forall i \in N \quad \text{// To/from neighbors in } N
\]

\[
\text{if status = probing and } |N| < d/2 \text{ and } \forall i \in N : \text{status}_i = \text{probing} \text{ then}
\]

\[
\Phi \leftarrow \Phi + \sum_{v \in N} \left\lfloor \frac{d^{c-1} \Phi_v}{d^{c-1} \Phi} \right\rfloor / |N| \left\lfloor \frac{d^{c-1} \Phi}{d^{c-1} \Phi} \right\rfloor \quad \text{// Share potential truncated to } c \log d \text{ bits}
\]

\[
\text{else} \\
\text{// } k < n
\]

\[
\text{if phase = 1 and } \Phi > \tau \text{ then } \text{status} \leftarrow \text{low}, \Phi \leftarrow \ell \quad \text{// } k < n
\]

\[
\text{for round = 1 to } d \text{ do} \\
\text{// disseminate status}
\]

\[
\text{Broadcast } \langle \text{status} \rangle \text{ and Receive } \langle \text{status}_i \rangle, \forall i \in N
\]

\[
\text{if status = low then} \\
\min \leftarrow k + 1
\]

\[
\text{if max = } \infty \text{ then } k \leftarrow 2k \text{ else } k \leftarrow \lfloor (\min + \max) / 2 \rfloor
\]

\[
\text{else}
\]

\[
\text{if status = high then}
\]

\[
\max \leftarrow k - 1
\]

\[
\text{k} \leftarrow \lfloor (\min + \max) / 2 \rfloor
\]

\[
\text{until status = done}
\]

\[
\text{return } k
\]
where $N_{i,j,h}[u]$ is the set of neighbors of $u$ in round $\langle i, j, h \rangle$. Inductively, assume that the claimed overall potential holds for some round $\langle i, j, h \rangle$, we want to show that consequently it holds for $\langle i + 1, j, h \rangle$. The potential for round $\langle i + 1, j, h \rangle$ is

$$||\Phi_{i+1,j,h}||_1 = ||\Phi_{i,j,h}||_1 + \frac{1}{d^c} \sum_{u,v \in V: v \in N_{i,j,h}[u]} \left( |d^{-c} \Phi_{i,j,h}[v]| - |N_{i,j,h}[u]| |d^{-c} \Phi_{i,j,h}[u]| \right). \quad (6)$$

In the ADCS model, communication is symmetric. That is, for every pair of nodes $x, y \in V$, and round $\langle i, j, h \rangle$ it is $x \in N_{i,j,h}[y] \iff y \in N_{i,j,h}[x]$. Fix a pair of nodes $x', y' \in V$ such that in round $\langle i, j, h \rangle$ it is $y' \in N_{i,j,h}[x']$ and hence $x' \in N_{i,j,h}[y']$. Consider the summations in Equation $6$. Due to symmetric communication, we have that the potential $\Phi_{i,j,h}[y']$ appears with positive sign when the indices of the summations are $x = x'$ and $y = y'$, and with negative sign when the indices are $x = y'$ and $y = x'$. (Notice that it is truncated the same way at both nodes.) This observation applies to all pairs of nodes that communicate in any round. Therefore, we can re-write Equation $6$ as

$$||\Phi_{i+1,j,h}||_1 = ||\Phi_{i,j,h}||_1 + \frac{1}{d^c} \sum_{u,v \in V: v \in N_{i,j,h}[u]} \left( |d^{-c} \Phi_{i,j,h}[v]| - |d^{-c} \Phi_{i,j,h}[u]| + |d^{-c} \Phi_{i,j,h}[v]| - |d^{-c} \Phi_{i,j,h}[u]| \right) = ||\Phi_{i,j,h}||_1.$$

Thus, the claim follows.

**Claim 2.** Given an ADCS with $\ell > 0$ supervisors and $n - \ell > 0$ supervised nodes running RMC with parameter $c \geq 2$, for any round $\langle i, j, h \rangle$ and any node $u \in V$, it is $\Phi_{i,j,h}[u] \geq 0$.

**Proof.** At the beginning of the first round the potential of the supervisor nodes is 0 and the potential of any supervised node $x$ is $\ell$. Thus, the claim follows.

Inductively, for any round $\langle i, j, h \rangle$ after the first round, we consider two cases according to node status. If a node $x$ is in alarm status “low” at the beginning of the round, then it is $\Phi_{i,j,h}[x] = \ell$ because, whenever the status of a node is updated to “low”, its potential is set to $\ell$ and will not change until the next epoch (refer to Algorithms 3 and 4).

In the second case, if a node $u$ is in “probing” status at the beginning of round $\langle i, j, h \rangle$, it means that it had its potential updated in all previous rounds $\langle i', j', h' \rangle$ as (refer to Lines 310 or 319)

$$\Phi_{i',j',h'}[u] + \sum_{v \in N_{i',j',h'}[u]} \frac{|d^{-c} \Phi_{i',j',h'}[v]| - |N_{i',j',h'}[u]| |d^{-c} \Phi_{i',j',h'}[u]|}{d^c} \leq \Phi_{i',j',h'}[u] + \sum_{v \in N_{i',j',h'}[u]} \frac{\Phi_{i',j',h'}[v]}{d} - |N_{i',j',h'}[u]| \frac{\Phi_{i',j',h'}[u]}{d} + |N_{i',j',h'}[u]| \frac{1}{d^c}.$$ \quad (7)

Where the last term is the maximum extra potential kept by node $u$ due to truncation to $c \log d$ bits by nodes in $N_{i,j,h}[u]$. For all rounds $\langle i', j', h' \rangle$, node $u$ exchanged potential with less than $d/2$ neighbors, because otherwise it would have been changed to alarm status (refer to Lines 312 and 314). Therefore it is $|N_{i',j',h'}[u]| \Phi_{i',j',h'}[u]/d < \Phi_{i',j',h'}[u]$ which implies $\Phi_{i,j,h}[u] \geq 0$.

The general structure of the rest of the proof is the following. We divide the analysis in 4 cases according to the relation between the estimate $k$ and the network size $n$.

| $k^{1+\varepsilon}$ | $k < n \leq k^{1+\varepsilon}$ | $k = n$ | $k > n$ |
|------------------|------------------|----------|----------|
| then $\rho > (k-\ell) \left( 1 + \frac{1}{\ell^2} \right)$ | $\rho > (k-\ell) \left( 1 + \frac{1}{\ell^2} \right)$ | $\rho < (k-\ell) \left( 1 + \frac{1}{\ell^2} \right)$ | $\rho < (k-\ell) \left( 1 + \frac{1}{\ell^2} \right)$ |
| proved in | $\text{Lemma 6 \& auxiliary 4 and 5}$ | $\text{Lemma 3}$ | $\text{Lemma 2}$ |

| proved in | $\text{Lemma 7}$ |

We begin the analysis considering the case $k = n$, as follows.
Lemma 2. Consider an ADCS with a $\mathcal{T}$-connected evolving graph topology with $\ell > 0$ supervisors and $n - \ell > 0$ supervised nodes running the RMC protocol with parameters $d \geq 2k$, $p \geq (2\gamma \ln k) / (\ell (\frac{1}{k} + \frac{1}{k^n}))$, $r \geq 4\mathcal{T} \alpha \ln k/\phi_{\min}^2$, and $c \geq 5\alpha + 2\mathcal{T} + 4$, where $\gamma > 0$ and $\alpha \geq \max\{2, 1 + \gamma + \log_k 3, \log_k (3\mathcal{T})\}$. Then, if $k = n$, the potential $\rho$ consumed by each of the $\ell$ supervisor nodes is such that

$$(k - \ell) \left(1 - \frac{1}{k^\gamma}\right) \leq \rho \leq (k - \ell) \left(1 + \frac{1}{k^\gamma}\right).$$

Proof. Given that $d \geq 2n$ and $c > \log_d \mathcal{T} + 1$, we have from Eq. 5 in the proof of Theorem 3 (changing the notation appropriately) that right before the beginning of any phase $h > 1$ it is

$$
\left\| \frac{\Phi_{0,0,2}}{\Phi_{1,1,1}} - \frac{1}{n} \right\|_2^2 \leq \exp \left(-b\phi_{\min}^2\right) + \frac{3\mathcal{T}}{4d^{c-2}T^{4}}.
$$

The above applies to any phase as long as $d \geq n$. Thus, we have that right before the beginning of any phase $h > 1$:

$$
\left\| \frac{\Phi_{0,0,h}}{\Phi_{1,1,h-1}} - \frac{1}{n} \right\|_2^2 \leq \exp \left(-b\phi_{\min}^2\right) + \frac{3\mathcal{T}}{4d^{c-2}T^{4}}.
$$

(8)

Given that $r \geq 4\mathcal{T} \alpha \ln k/\phi_{\min}^2$ and each block has $\mathcal{T}$ rounds, it is $b \geq 4\alpha \ln k/\phi_{\min}^2$. Thus,

$$
\left\| \frac{\Phi_{0,0,h}}{\Phi_{1,1,h-1}} - \frac{1}{n} \right\|_2^2 \leq \frac{1}{k^{4\alpha}} + \frac{3\mathcal{T}}{4d^{c-2}T^{4}}.
$$

Replacing $\alpha \geq \log_k (3\mathcal{T})$ and $d \geq k$, it is

$$
\left\| \frac{\Phi_{0,0,h}}{\Phi_{1,1,h-1}} - \frac{1}{n} \right\|_2^2 \leq \frac{1}{k^{4\alpha}} + \frac{1}{4d^{c-2}T^{4} - 4 - \alpha}.
$$

And for $c \geq 5\alpha + 2\mathcal{T} + 4$ we have that

$$
\left\| \frac{\Phi_{0,0,h}}{\Phi_{1,1,h-1}} - \frac{1}{n} \right\|_2^2 \leq \frac{5}{4k^{4\alpha}} \leq \frac{1}{k^{2\alpha}}, \text{ for } \alpha \geq 1/2 \text{ and } k \geq 2.
$$

Given that $(\Phi_{0,0,h}(v)) / \Phi_{1,1,h-1}(1 - 1/n)^2 \leq \left\| \Phi_{0,0,h} / \Phi_{1,1,h-1} \right\|_2$ for any node $v$ and phase $h > 1$, we have that $(\Phi_{0,0,h}(v)) / \Phi_{1,1,h-1}(1 - 1/n)^2 = (\Phi_{0,0,h}(v)) / (1/k)^2 \leq 1/k^{2\alpha}$ and hence $\Phi_{0,0,h}(v) \geq (1/k - 1/k^\alpha) \Phi_{1,1,h-1}$. Notice that the latter is true for any initial distribution. Therefore, after each phase a supervisor node consumes between $1/k - 1/k^\alpha$ and $1/k + 1/k^\alpha$ fraction of the total potential in the system, and the total potential in the system drops by at least $\ell(1/k - 1/k^\alpha)$ and by at most $\ell(1/k + 1/k^\alpha)$ fraction. Recall that the initial overall potential in the system is $\ell(n - \ell) = \ell(k - \ell)$, and that by Claim 1 if $d > n$, the overall potential in the system is the same throughout each phase.

Using the latter observations, we first find conditions on the number of phases $p$ to obtain the desired bounds on $\rho$, as follows. After $p$ phases a supervisor node consumes at least

$$
\rho \geq \ell(k - \ell) \left(1 - \frac{1}{k^\alpha}\right) \sum_{i=0}^{p-1} \left(1 - \ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right)\right)^i,
$$

(9)

and at most

$$
\rho \leq \ell(k - \ell) \left(1 + \frac{1}{k^\alpha}\right) \sum_{i=0}^{p-1} \left(1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right)\right)^i.
$$

(10)
Given that $0 < \ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right) < 1$ for $\alpha \geq 2$ and $k > \ell$, Equation 10 is

$$\rho \geq \ell(k - \ell) \frac{1}{k} \left(1 - \left(1 - \ell \frac{1}{k} + \frac{1}{k^\alpha}\right)^p\right) \left(1 - \left(1 - \ell \frac{1}{k} + \frac{1}{k^\alpha}\right)^p\right).$$

Given that $0 < \ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right) < 1$ for $\ell < k$ and $\alpha \geq 2$, it is

$$\rho \geq (k - \ell) \frac{k^\alpha - k}{k^\alpha + k} \left(1 - \exp(-p\ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right))\right).$$

Thus, to prove the lower bound on $\rho$, it is enough to find values of $p$ and $\alpha$ such that

$$\frac{k^\alpha - k}{k^\alpha + k} \left(1 - \exp(-p\ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right))\right) \geq 1 - \frac{1}{k^\gamma}.$$

We note first that for

$$p \geq \frac{2\gamma \ln k}{\ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right)},$$

it is

$$1 - \exp\left(-p\ell \left(\frac{1}{k} + \frac{1}{k^\alpha}\right)\right) \geq 1 - \frac{1}{k^{2\gamma}}.$$

Replacing, it is enough to prove

$$\frac{k^\alpha - k}{k^\alpha + k} \left(1 + \frac{1}{k}\right) \geq 1 \quad \alpha = 2k^{1-\gamma}.$$

Thus, for $\gamma > 0$ it is enough to prove $k^{\alpha - \gamma} \geq 3k$, which is true for $\alpha \geq 1 + \gamma + \log_k 3$. We show now the upper bound on $\rho$ starting from Equation 11

$$\rho \leq \ell(k - \ell) \left(\frac{1}{k} + \frac{1}{k^\alpha}\right)^{p-1} \sum_{i=0}^{p-1} \left(1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right)\right)^i.$$

Given that $1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right) < 1$ for $\alpha \geq 2 > 1$, it is

$$\rho \leq \ell(k - \ell) \left(\frac{1}{k} + \frac{1}{k^\alpha}\right)^{1 - \left(1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right)^p\right)} \left(1 - \left(1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right)^p\right)\right).$$

Given that $0 < \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right) < 1$ and $p > 0$, it is $\left(1 - \left(1 - \ell \left(\frac{1}{k} - \frac{1}{k^\alpha}\right)^p\right)\right) < 1$. Then, replacing, we get

$$\rho \leq (k - \ell) \frac{k^\alpha + k}{k^\alpha - k}.$$

Thus, to prove the upper bound on $\rho$, it is enough to show that

$$\frac{k^\alpha + k}{k^\alpha - k} \leq 1 + \frac{1}{k^\gamma}.$$

This is the same as Equation 11 and hence the claim follows.

The previous lemma shows that, after running RMC enough time, if for some supervisor node it is $\rho > (k - \ell) \left(1 + \frac{1}{k^\alpha}\right)$ or $\rho \leq (k - \ell) \left(1 - \frac{1}{k^\alpha}\right)$, for some $\gamma > 0$, we know that the estimate $k$ is wrong. However, the complementary case, that is, $(k - \ell) \left(1 - \frac{1}{k^\alpha}\right) \leq \rho \leq (k - \ell) \left(1 + \frac{1}{k^\alpha}\right)$, may occur even if the estimate is $k \neq n$ and hence the error has to be detected by other means. To prove correctness in this case we further separate the range of $k$ in three cases. The first one, when $k < n \leq k^{1+\epsilon}$, for some $\epsilon > 0$, in the following lemma, which is based on upper bounding the potential left in the system after running RMC long enough.
Lemma 3. Consider an ADCS with a $T$-connected evolving graph topology with $\ell > 0$ supervisors and $n - \ell > 0$ supervised nodes running the RMC protocol with parameters $d \geq k^{1+\epsilon}$, $p \geq 2\delta (ln k)/(\ell (1/n + 1/k^\delta))$, $r \geq 4T \beta \ln k/\phi^2_{\text{min}}$, and $c \geq 5\beta + 2T + 4$, where $\beta \geq \log_{k^\gamma} \max\{(n(2k^\delta + 1), 3T), \beta > 2, \delta > \log_k(nk^\gamma/(nk^\gamma - (n - 1)(k^\gamma + 1))), \gamma > \log_k(n - 1), \text{ and } \epsilon > 0$. Then, if $k < n \leq k^{1+\epsilon}$, the potential $\rho$ consumed by any supervisor node is $\rho > (k - \ell)(1 + 1/k^\gamma)$.

Proof. Given that $d \geq n$ and $c > \log_d T + 1$, we have from Eq. 8 that right before the beginning of any phase $h > 1$

$$\left\| \frac{\Phi_{0,0,h}}{\|\Phi_{1,1,h-1}\|} - \frac{1}{n} \right\|_2^2 \leq \exp(-b\phi^2_{\text{min}}) + \frac{3T}{4d^2-2T+4}.$$  

Given that $r \geq 4T \beta \ln k/\phi^2_{\text{min}}$ and each block has $T$ rounds, it is $b \geq 4\beta \ln k/\phi^2_{\text{min}}$. Thus,

$$\left\| \frac{\Phi_{0,0,h}}{\|\Phi_{1,1,h-1}\|} - \frac{1}{n} \right\|_2^2 \leq \exp(-4\beta \ln k) + \frac{3T}{4d^2-2T+4} = \frac{1}{k^{4\beta}} + \frac{3T}{4d^2-2T+4}.$$  

Given that $c \geq 5\beta + 2T + 4$ and $d > k$ we have that

$$\left\| \frac{\Phi_{0,0,h}}{\|\Phi_{1,1,h-1}\|} - \frac{1}{n} \right\|_2^2 \leq \frac{1}{k^{4\beta}} + \frac{3T}{4d^{2\beta}}.$$  

And for $d > k$ and $\beta \geq \log_k(3T)$, it is

$$\left\| \frac{\Phi_{0,0,h}}{\|\Phi_{1,1,h-1}\|} - \frac{1}{n} \right\|_2^2 \leq \frac{5}{4k^{4\beta}} \leq \frac{1}{k^{2\gamma}}, \text{ for } \beta \geq 1/2 \text{ and } k \geq 2.$$  

Given that $\langle \Phi_{0,0,h}[v]/\|\Phi_{1,1,h-1}\|1 - 1/n \rangle^2 \leq \left\| \frac{\Phi_{0,0,h}}{\|\Phi_{1,1,h-1}\|} - \frac{1}{n} \right\|_2^2$ for any node $v$, we have that $\langle \Phi_{0,0,h}[v]/\|\Phi_{1,1,h-1}\|1 - 1/n \rangle^2 \leq 1/k^{2\beta}$ and hence $\Phi_{0,0,h}[v] \geq (1/n - 1/k^{2\beta})\|\Phi_{1,1,h-1}\|1$. The latter is true for any initial distribution and any phase $h$. Therefore, after each phase a supervisor node consumes at least $1/n - 1/k^{2\beta}$ fraction of the total potential in the system, and the total potential in the system drops by at most $\ell(1/n + 1/k^{2\beta})$ fraction. Recall that the initial overall potential in the system is $\ell(n - \ell)$, and that by Claim 1 if $d > n$, the overall potential in the system does not change during each phase.

Using the latter observations, after $p$ phases, any given supervisor node consumes at least

$$\rho \geq \ell(n - \ell) \left( 1 - \frac{1}{k^{2\beta}} \right)^{p-1} \sum_{i=0}^{p-1} (1 - \ell(1/n + 1/k^{2\beta}))^i.$$  

Given that $0 < \ell(1/n + 1/k^{2\beta}) < 1$ for $\beta \geq 2$ and $k > \ell$, we have that

$$\rho \geq \ell(n - \ell) \left( 1 - \frac{1}{k^{2\beta}} \right) \frac{1 - \ell(1/n + 1/k^{2\beta})^p}{1 - \ell(1/n + 1/k^{2\beta})} = (n - \ell) \frac{k^{2\beta} - n}{k^{2\beta} + n} \left( 1 - \left( 1 - \ell(1/n + 1/k^{2\beta}) \right)^p \right).$$  

Again using that $0 < \ell(1/n + 1/k^{2\beta}) < 1$ for $\beta \geq 2$ and $k > \ell$, and given that $1 - x \leq e^{-x}$ for any $0 < x < 1$ [45], we have

$$\rho \geq (n - \ell) \frac{k^{2\beta} - n}{k^{2\beta} + n} \left( 1 - \exp\left( -p\ell\left( 1 + \frac{1}{k^{2\beta}} \right) \right) \right), \text{ replacing } p \geq \frac{2\delta \ln k}{\ell(1/n + 1/k^{2\beta})};$$

$$\geq (n - \ell) \frac{k^{2\beta} - n}{k^{2\beta} + n} \left( 1 - \frac{1}{k^{2\beta}} \right) \geq (n - \ell) \frac{k^{2\beta} - n}{k^{2\beta} + n} \left( 1 + \frac{1}{k^{2\beta}} \right) \left( 1 - \frac{1}{k^{2\beta}} \right) \geq (n - \ell) \left( 1 - \frac{1}{k^{2\beta}} \right).$$  

The latter inequality holds for $\beta \geq \log_k(n(2k^{2\beta} + 1))$. Then, to complete the proof, it is enough to show that

$$(n - \ell) \left( 1 - \frac{1}{k^{2\beta}} \right) > (k - \ell) \left( 1 + \frac{1}{k^{2\beta}} \right).$$  

Which is true for $k < n$, $\delta > \log_k(nk^{2\beta}/(nk^{2\beta} - (n - 1)(k^{2\beta} + 1)))$ and $\gamma > \log_k(n - 1).$.  

19
We now consider the case \( k^{1+\epsilon} < n \). We focus on the first phase. We define a threshold \( \tau \) and a number of rounds \( r \) such that, after the phase is completed, all nodes that have potential above \( \tau \) can send an alarm to the leader, as such potential indicates that the estimate is low.

In order to do that, we first establish an upper bound of at most \( k^{1+\epsilon} \) nodes with potential at most \( \tau \) at the end of the first phase (Lemma 3). Given that \( k^{1+\epsilon} < n \), using this lemma we know that there is at least one node with potential above \( \tau \) at the end of the first phase. Second, we show that if the estimate is not low, that is \( k \geq n \), then all nodes have potential at most \( \tau \) at the end of the first phase (Lemma 5). That is, a potential above \( \tau \) can only happen when indeed the estimate is low. Finally, we show that if \( k^{1+\epsilon} < n \) an alarm “low” initiated by nodes with potential above \( \tau \) must be received after \( k^{1+\epsilon} \) further rounds of communication (Lemma 6).

**Lemma 4.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( \ell > 0 \) supervisors and \( n - \ell > 0 \) supervised nodes running the RMC protocol. For \( \epsilon > 0 \), after running the first phase, there are at most \( k^{1+\epsilon} \) nodes that have potential at most \( \tau = (1 - \ell/k^{1+\epsilon}) \).

**Proof.** We define the slack of node \( x \) at the beginning of round \( (i, j, h) \) as \( s_{i,j,h}[x] = \ell - \Phi_{i,j,h}[x] \) and the vector of slacks at the beginning of round \( (i, j, h) \) as \( s_{i,j,h} \). In words, the slack of a node is the “room” for additional potential shared by nodes in \( i, j, h \). That is, the slack of a node is the “room” for additional potential at the beginning of the first phase. Second, we show that if the estimate is not low, that is \( k \geq n \), then all nodes have potential at most \( \tau \) at the end of the first phase (Lemma 5). That is, a potential above \( \tau \) can only happen when indeed the estimate is low. Finally, we show that if \( k^{1+\epsilon} < n \) an alarm “low” initiated by nodes with potential above \( \tau \) must be received after \( k^{1+\epsilon} \) further rounds of communication (Lemma 6).

**Lemma 5.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( \ell > 0 \) supervisors and \( n - \ell > 0 \) supervised nodes running the RMC protocol with parameters \( d > k \), \( p > 0 \), \( r \geq T b \), \( b \geq \)
(5 + 2\epsilon - 2\log_k(k^{\epsilon} - 1)) \ln k/\phi_{\text{min}}^2, \text{and } c \geq 5 + 2\epsilon - 2\log_k(k^{\epsilon} - 1) + 2T + 4 + \alpha, \text{where } \epsilon > 0, \text{and } \alpha \geq \log_k(3T).

Then, if \( k \geq n \), at the end of the first phase no individual node has potential larger than \( \tau = \ell(1 - \ell/k^{1+\epsilon}). \)

**Proof.** Given that \( d \geq n \) and \( c > \log_d T + 1 \), we have from Eq.\( \ref{eq:potential} \) that right before the beginning of phase 2 it is

\[
\| \Phi_{0,2} \|_{\Phi_1,1,1} - \frac{1}{n} \|_2^2 \leq \exp \left( -2\phi_{\text{min}}^2 \right) + \frac{3T}{4d^{k-2T-4}}.
\]

Given that \( b \geq (5 + 2\epsilon - 2\log_k(k^{\epsilon} - 1)) \ln k/\phi_{\text{min}}^2, \) it is

\[
\| \Phi_{0,2} \|_{\Phi_1,1,1} - \frac{1}{n} \|_2^2 \leq \frac{1}{k^{5+2\epsilon-2\log_k(k^{\epsilon} - 1)}} + \frac{3T}{4d^{k-2T-4}}.
\]

Replacing \( \alpha \geq \log_k(3T) \) and \( d \geq k \) we have that

\[
\| \Phi_{0,2} \|_{\Phi_1,1,1} - \frac{1}{n} \|_2^2 \leq \frac{1}{k^{5+2\epsilon-2\log_k(k^{\epsilon} - 1)}} + \frac{1}{4d^{k-2T-4-\alpha}}.
\]

For \( c \geq 5 + 2\epsilon - 2\log_k(k^{\epsilon} - 1) + 2T + 4 + \alpha \) we have that

\[
\| \Phi_{0,2} \|_{\Phi_1,1,1} - \frac{1}{n} \|_2^2 \leq \frac{5}{4k^{5+2\epsilon-2\log_k(k^{\epsilon} - 1)}} \leq \frac{1}{k^{4+2\epsilon-2\log_k(k^{\epsilon} - 1)}}.
\]

The latter is true because \( k \geq 2 \).

Given that for any node \( j \), it is \( (\Phi_{0,2}[j]/\| \Phi_1,1,1 \|_1 - 1/n)^2 \leq \| \Phi_{0,2} \|_{\Phi_1,1,1} - \frac{1}{n} \|_2^2 \), we have that

\[
(\Phi_{0,2}[j]/\| \Phi_1,1,1 \|_1 - 1/n)^2 \leq 1/k^{4+2\epsilon-2\log_k(k^{\epsilon} - 1)/\ln k}.
\]

Hence, it is \( \Phi_{0,2}[j] \leq (1/n + 1/k^{2+\epsilon-\log_k(k^{\epsilon} - 1)})/\| \Phi_1,1,1 \|_1 \) for any node \( j \). Moreover, given that \( d > k \geq n \) the total potential in the network is \( \ell(n - \ell) \) (Claim\( \ref{claim:potential} \)). Thus, no individual node should have potential larger than \( \ell(n - \ell)(1/n + 1/k^{2+\epsilon-\log_k(k^{\epsilon} - 1)}). \)

We show that the latter is at most \( \tau = \ell(1 - \ell/k^{1+\epsilon}) \) as follows. We want to prove

\[
\ell(n - \ell) \left( \frac{1}{n} + \frac{1}{k^{2+\epsilon-\log_k(k^{\epsilon} - 1)}} \right) \leq \ell \left( 1 - \frac{\ell}{k^{1+\epsilon}} \right)
\]

\[
\frac{n - \ell}{k^{2+\epsilon-\log_k(k^{\epsilon} - 1)}} \leq \frac{n - \ell}{k^{1+\epsilon}}.
\]

Given that \( k \geq n \), it is enough to show that

\[
\frac{k - \ell}{k^{2+\epsilon-\log_k(k^{\epsilon} - 1)}} \leq \frac{\ell}{k} - \frac{\ell}{k^{1+\epsilon}}
\]

\[
k - \ell \leq \ell(k^{\epsilon} - 1)k^{1-\log_k(k^{\epsilon} - 1)}
\]

\[
k - \ell \leq \ell k.
\]

And the latter is true because \( \ell \geq 1 \).

The previous lemma shows that, if the estimate is “not-low” \( (k \geq n) \), at the end of the first phase all nodes must have “low” potential \( \Phi_{0,0,2} \leq \tau \). (Notice the inverse relation between estimate and potential.) So, to complete the proof of the case \( k^{1+\epsilon} < n \) (i.e. low estimate) we show in the following lemma that if \( k^{1+\epsilon} < n \) (i.e. low estimate) there are some nodes with \( \Phi_{0,0,2} > \tau \) (i.e. high potential), and that all the other nodes will know this within the following phase.

**Lemma 6.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( \ell > 0 \) supervisors and \( n - \ell > 0 \) supervised nodes running the RMC protocol with parameters \( d > k, p > 1 \) and \( r \geq T/4 \), where \( b \geq (5 + 2\epsilon - 2\log_k(k^{\epsilon} - 1)) \ln k/\min \{ \phi_{\text{min}}^2, \ln(1 + t_{\text{min}}) \} \), and \( \epsilon > 0 \). Then, if \( k^{1+\epsilon} < n \) within the second phase all nodes receive an alarm status “low”.

21
Proof. Consider a partition \( \{L, H\} \) of the set of nodes, where \( L \) is the set of nodes with potential at most \( \tau \) at the end of the first phase. As shown in Lemma \( 4 \), the size of \( L \) is at most \( k^{1+\epsilon} \), and because \( k^{1+\epsilon} < n \) the size of \( H \) is at least 1.

Based on their “high” potential (above \( \tau \)), and the property proved in Lemma \( 5 \), that in case of not-low estimate, \( k \geq n \) there would not be any node like them (notice that \( b \) fulfills the condition of such lemma), all nodes in \( H \) move to alarm status “low” at the end of phase 1 (refer to Lines \( 3.13 \) or \( 4.12 \)). (Notice the inverse relation between potential and status, which in turn indicates whether the estimate is low or not.) We want to compute the number of blocks until \( k \geq 1 \) to alarm status “low” at the end of phase \( h \).

Proof. Consider a partition \( \{L, H\} \) it can be proved that \( \ln d / \ln(1+i_{min}) \) blocks are enough to disseminate the alarm throughout the network. Therefore, within the following \( (1+\epsilon) \ln k / \ln(1+i_{min}) \) blocks after the beginning of the second phase any \( x \in L \) receives the alarm. Given that \( b \) is larger, the claim follows.

Finally, to complete the proof of correctness, we show in the following lemma that if \( k > n \), supervisor nodes detect that the potential consumed is too low for the estimate to be correct.

**Lemma 7.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( \ell > 0 \) supervisors and \( n - \ell > 0 \) supervised nodes running the RMC protocol with parameters \( d > k \),

\[
p \leq 2\delta \ln k \frac{1 - \ell (\frac{n}{k} - \frac{1}{k^n})}{\ell (\frac{n}{k} - \frac{1}{k^n})},
\]

\[
r \geq T b, \ b \geq 4\beta \ln k / \phi_{\min}^2, \text{ and } c \geq 5\beta + 2T + 4, \text{ where } \beta \geq \log_k(n(2k^\delta - 1)), \gamma > \log_k(n - \ell + 1), \text{ and}
\]

\[
\delta > \log_k \frac{k^\gamma(n - \ell)}{k^\gamma - (n - \ell) - 1}.
\]

Then, if \( k > n \), the potential \( \rho \) consumed by any supervisor node is \( \rho < (k - \ell)(1 - 1/k^\gamma) \).

Proof. Given that \( d > k > n \) and \( c \geq 5\beta + 4T + 2 > \log_d 2T + 1 \), we have from Eq. \( 8 \) that right before the beginning of any phase \( h > 1 \) it is

\[
\left\| \frac{\Phi_{0,0,h}}{||\Phi_{1,1,h-1}||_1} - \frac{I}{n} \right\|_2^2 \leq \exp (-b\phi_{\min}^2) + \frac{3T}{4d^{c-2T-4}}.
\]

For a number of blocks \( b \geq 4\beta \ln k / \phi_{\min}^2 \), it is

\[
\left\| \frac{\Phi_{0,0,h}}{||\Phi_{1,1,h-1}||_1} - \frac{I}{n} \right\|_2^2 \leq \exp (-4\beta \ln k) + \frac{3T}{4d^{c-2T-4}} = \frac{1}{k^{4\beta}} + \frac{3T}{4d^{c-2T-4}}.
\]

Given that \( c \geq 5\beta + 2T + 4 \) and \( d > k \) we have that

\[
\left\| \frac{\Phi_{0,0,h}}{||\Phi_{1,1,h-1}||_1} - \frac{I}{n} \right\|_2^2 \leq \frac{1}{k^{4\beta}} + \frac{3T}{4d^{5\beta}}.
\]

And for \( d \geq k \) and \( \beta \geq \log_k(3T) \), it is

\[
\left\| \frac{\Phi_{0,0,h}}{||\Phi_{1,1,h-1}||_1} - \frac{I}{n} \right\|_2^2 \leq \frac{5}{4k^{4\beta}} \leq \frac{1}{k^{2\beta}}, \text{ for } \beta \geq 1/2 \text{ and } k \geq 2.
\]

For any node \( j \), given that \( \left( \Phi_{0,0,h}[j]/||\Phi_{1,1,h-1}||_1 - 1/n \right)^2 \leq \left\| \frac{\Phi_{0,0,h}}{||\Phi_{1,1,h-1}||_1} - \frac{I}{n} \right\|_2^2 \) we have that

\[
\left( \Phi_{0,0,h}[j]/||\Phi_{1,1,h-1}||_1 - 1/n \right)^2 \leq \frac{1}{k^{2\beta}} \text{ and hence } \Phi_{0,0,h}[j] \geq (1/n - 1/k^\beta)||\Phi_{1,1,h-1}||_1. \text{ The latter is true for any initial distribution. Therefore, after each phase a supervisor node consumes at most } 1/n + 1/k^\beta \text{ fraction of the total potential in the system, and the total potential in the system drops by at least } \ell(1/n - 1/k^\beta) \text{ fraction. Recall that}
\]

22
the initial overall potential in the system is \( \ell(n - \ell) \), and that by Claim[1] if \( d > n \), the overall potential in the system does not change during each phase.

Using the latter observations, after \( p \) phases, any given supervisor node consumes at most

\[
\rho \leq \ell(n - \ell) \left( \frac{1}{n} + \frac{1}{k^\beta} \right) \frac{p - 1}{1 - \ell (\frac{1}{n} - \frac{1}{k^\beta})} \sum_{i=0}^{p-1} \left( 1 - \ell \left( \frac{1}{n} - \frac{1}{k^\beta} \right) \right)^i.
\]

Given that \( 0 < \ell \left( \frac{1}{n} - \frac{1}{k^\beta} \right) < 1 \) for \( \beta \geq 1 \) and \( k > n > \ell \), we have that

\[
\rho \leq \ell(n - \ell) \left( \frac{1}{n} + \frac{1}{k^\beta} \right) \frac{1 - (1 - \ell (\frac{1}{n} - \frac{1}{k^\beta}))^p}{1 - (1 - \ell (\frac{1}{n} - \frac{1}{k^\beta}))} (n - \ell) \frac{k^\beta + n}{k^\beta - n} \left( 1 - \ell (\frac{1}{n} - \frac{1}{k^\beta}) \right)^p.
\]

Again using that \( 0 < \ell \left( \frac{1}{n} - \frac{1}{k^\beta} \right) < 1 \) for \( \beta \geq 1 \) and \( k > n > \ell \), we have that

\[
\rho \leq \frac{(n - \ell) k^\beta + n}{k^\beta - n} \left( 1 - \exp \left( -p \frac{\ell (\frac{1}{n} - \frac{1}{k^\beta})}{1 - \ell (\frac{1}{n} - \frac{1}{k^\beta})} \right) \right),
\]

\[
\leq (n - \ell) \frac{k^\beta + n}{k^\beta - n} \left( 1 - \frac{1}{k^\beta} \right) = (n - \ell) \frac{k^\beta + n}{k^\beta - n} \left( 1 + \frac{1}{k^\beta} \right) \left( 1 - \frac{1}{k^\beta} \right) \leq (n - \ell) \left( 1 + \frac{1}{k^\beta} \right).
\]

The latter inequality holds for \( \beta \geq \log_k(n(2k^\delta - 1)) \) and \( \delta \geq \log_k(3/2) \). The second inequality is true because \( k^{\log_k(n(2k^\delta - 1))} > \log_k(3/2) \) for \( k > n > \ell > 0 \). Then, to complete the proof, it is enough to show that

\[
(n - \ell) \left( 1 + \frac{1}{k^\gamma} \right) < (k - \ell) \left( 1 - \frac{1}{k^\gamma} \right),
\]

which is true for \( k > n \), \( \delta > \log_k \frac{k^{\gamma(n - \ell)}}{k^{\gamma(n - \ell)} - 1} \) and \( \gamma > \log_k(n - \ell + 1) \). Hence, the claim follows. \( \qed \)

We establish the correctness and running time of RMC in the following theorem.

**Theorem 4.** Consider an ADCS with a \( T \)-connected evolving graph topology with \( \ell > 0 \) supervisors and \( n - \ell > 0 \) supervised nodes running the RMC protocol with parameters:

\[
d = 2k^{1 + \epsilon},
\]

\[
p = \left[ \frac{2 \ln k}{\ell} \max \left\{ \frac{\gamma}{1/k + 1/k^\alpha}, \frac{\delta}{1/k + 1/k^\beta} \right\} \right],
\]

\[
r = \lceil \frac{T \beta}{2} \rceil,
\]

\[
\tau = \ell(1 - 1/k^{1 + \epsilon}),
\]

\[
c \geq 2T + 4 + \max\{5\beta, 5 + \alpha + 2\epsilon - 2\log_k(k^\epsilon - 1)\},
\]

where the number of blocks \( b \) is the following.

(i) If the isoperimetric number \( i_{\min} \) is known:

\[
b = \max\{\alpha, \beta, 5 + 2\epsilon - 2\log_k(k^\epsilon - 1)\} 2^{2T(2+\epsilon)} \frac{n^{2T(1+\epsilon)}}{i_{\min}^2} \ln k.
\]

(ii) Otherwise:

\[
b = \max\{\alpha, \beta, 5 + 2\epsilon - 2\log_k(k^\epsilon - 1)\} 2^{2T(2+\epsilon)} n^{2+2T(1+\epsilon)} \ln k.
\]
Then, under the following conditions:

\[
\alpha \geq \max\{1 + \gamma + \log_k 3, \log_k (3T)\}, \\
\beta \geq \log_k \max\{d(2^\delta + 1), 3T\}, \\
\gamma > \log_k (d - 1), \\
\delta > \log_k \frac{d^\gamma}{k^\gamma + 1 - d}, \\
\epsilon > 0, \\
T \in O(1),
\]

all nodes stop after at most \(\sum_{k \in E \cup B} (pr + d)\) rounds of communication and output \(n\), for \(E = \{2^i(\ell + 1) : i = 0, 1, \ldots, \log\lceil n/(\ell + 1) \rceil\}\), and \(B = \{2^{\log\lceil n/(\ell + 1) \rceil} - 2^i(\ell + 1) : i = 0, 1, \ldots, \log\lceil n/(\ell + 1) \rceil - 2\}\).

**Proof.** The proof of correctness of the computation and running time for Part (i) is similar to [35], adapted to our parameters and applying the bound on conductance in Eq. 1, and Part (ii) is obtained applying the lower bound \(i_{\text{min}} \geq 2/n\) to Equation 12.

About the ADCS limitations, messages sent by nodes are only their status and potential. The status requires only 4 bits. Potentials are truncated to \(c \log d \leq c(1 + (1 + \epsilon) \log(2n))\) bits (refer to Lines 3.10 and 4.9) fulfilling the restrictions on message size and memory access at the same time as long as there exists a \(c \in O(1)\), which can be seen replacing tight bounds on the conditions above.

In the following corollary we relate the knowledge of network characteristics to the asymptotic running time of RMC.

**Corollary 1.** The time complexity of RMC on an ADCS with \(\ell > 0\) supervisor nodes and \(n - \ell > 0\) supervised nodes is the following.

(i) If the isoperimetric number \(i_{\text{min}}\) is known:

\[
\tilde{O}\left(\frac{n^{1+2T(1+\epsilon)}}{\ell i_{\text{min}}^2}\right).
\]

(ii) Otherwise:

\[
\tilde{O}\left(\frac{n^{3+2T(1+\epsilon)}}{\ell}\right).
\]

**Proof.** Fixing \(\gamma = \log_k d\) it would be \(\log_k \frac{d^\gamma}{k^\gamma + 1 - d} = \log_k d^2 = 2(\log_k 2 + 1 + \epsilon)\). That is, it is enough to set \(\gamma = \log_k d\) and \(\delta = 2(2 + \epsilon)\) to meet the conditions on those parameters in Theorem 4. Replacing these and \(d = 2^{k^{1+\epsilon}} < 2(2n)^{1+\epsilon}\),

\[
p < \left\lceil \frac{2d \ln d}{\ell} \right\rceil \in O\left(\frac{n \ln n}{\ell}\right).
\]

Replacing \(\gamma\) and \(\delta\) in \(\alpha\) and \(\beta\), and \(T \in O(1)\) it is

\[
r \in O\left(\frac{1}{\min\{\phi_{\text{min}}^2, \ln(1 + i_{\text{min}})\}}\right).
\]

Then, it is

\[
pr + d \in O\left(\frac{n \ln n}{\ell \min\{\phi_{\text{min}}^2, \ln(1 + i_{\text{min}})\}}\right).
\]

The total number of terms in the summation of the running time in Theorem 4 is \(O(\log \frac{n}{T})\), hence the claim follows.
8 Discussion and Open Problems

Although the presented algorithm is the first that guarantees a polynomial emulation of ALL-TO-ALL COMMUNICATION on the top of anonymous congested highly-dynamic (and not necessarily always connected) systems, the main challenges are to further shrink the polynomials and/or provide lower bounds better than long-time known $\Omega(D + n \log n)$, where $D$ stands for temporal diameter. We hypothesize that shrinking this complexity gap may depend on computational power, for instance, being able to process history trees may allow provably faster ALL-TO-ALL COMMUNICATION.

References

[1] Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. Distributed Computing, 18(4):235–253, March 2006.

[2] Chen Avin, Michal Koucký, and Zvi Lotker. How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). In Automata, languages and programming, pages 121–132. Springer, 2008.

[3] Siddhartha Banerjee, Aditya Gopalan, Abhik Kumar Das, and Sanjay Shakkottai. Epidemic spreading with external agents. IEEE Transactions on Information Theory, 60(7):4125–4138, 2014.

[4] Florent Becker, Antonio Fernández Anta, Ivan Rapaport, and Eric Reémila. Brief announcement: A hierarchy of congested clique models, from broadcast to unicast. In Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, pages 167–169, 2015.

[5] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah. Randomized gossip algorithms. IEEE Transactions on Information Theory, 52(6):2508–2530, 2006.

[6] Peter Buser. On the bipartition of graphs. Discrete applied mathematics, 9(1):105–109, 1984.

[7] Arnaud Casteigts, Paola Flocchini, Walter Quattrociocchi, and Nicola Santoro. Time-varying graphs and dynamic networks. International Journal of Parallel, Emergent and Distributed Systems, 27(5):387–408, 2012.

[8] Keren Censor-Hillel, Petteri Kaski, Janne H Korhonen, Christoph Lenzen, Ami Paz, and Jukka Suomela. Algebraic methods in the congested clique. Distributed Computing, 32(6):461–478, 2019.

[9] Keren Censor-Hillel, Merav Parter, and Gregory Schwartzman. Derandomizing local distributed algorithms under bandwidth restrictions. Distributed Computing, 33(3):349–366, 2020.

[10] Maitri Chakraborty, Alessia Milani, and Miguel A. Mosteiro. Counting in practical anonymous dynamic networks is polynomial. In Proceedings of the 4th International Conference on Networked Systems, volume 9944 of Lecture Notes in Computer Science, pages 131–136, 2016.

[11] Yuxin Chen, Sanjay Shakkottai, and Jeffrey G Andrews. On the role of mobility for multimessage gossip. IEEE Transactions on Information Theory, 59(6):3953–3970, 2013.

[12] Giuseppe Antonio Di Luna and Roberto Baldoni. Non Trivial Computations in Anonymous Dynamic Networks. In Emmanuelle Anceaume, Christian Cachin, and Maria Potop-Butucaru, editors, Proceedings of the 19th International Conference on Principles of Distributed Systems, volume 46 of Leibniz International Proceedings in Informatics (LIPIcs), pages 1–16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[13] Giuseppe Antonio Di Luna, Roberto Baldoni, Silvia Bonomi, and Ioannis Chatzigiannakis. Conscious and unconscious counting on anonymous dynamic networks. In Proceedings of the 15th International Conference on Distributed Computing and Networking, volume 8314 of Lecture Notes in Computer Science, pages 257–271. Springer Berlin Heidelberg, 2014.
[14] Giuseppe Antonio Di Luna, Roberto Baldoni, Silvia Bonomi, and Ioannis Chatzigiannakis. Counting in anonymous dynamic networks under worst-case adversary. In Proceedings of the 34th International Conference on Distributed Computing Systems, pages 338–347. IEEE, 2014.

[15] Giuseppe Antonio Di Luna, Silvia Bonomi, Ioannis Chatzigiannakis, and Roberto Baldoni. Counting in anonymous dynamic networks: An experimental perspective. In Proceedings of the 9th International Symposium on Algorithms and Experiments for Sensor Systems, Wireless Networks and Distributed Robotics, volume 8243 of Lecture Notes in Computer Science, pages 139–154. Springer Berlin Heidelberg, 2014.

[16] Danny Dolev, Christoph Lenzen, and Shit Peled. “tri, tri again”: Finding triangles and small subgraphs in a distributed setting. In International Symposium on Distributed Computing, pages 195–209. Springer, 2012.

[17] Andrew Drucker, Fabian Kuhn, and Rotem Oshman. On the power of the congested clique model. In Proceedings of the 2014 ACM symposium on Principles of distributed computing, pages 367–376, 2014.

[18] Martin Farach-Colton, Rohan J. Fernandes, and Miguel A. Mosteiro. Bootstrapping a hop-optimal network in the weak sensor model. ACM Trans. Algorithms, 5(4), nov 2009.

[19] A. Fernández Anta, M. A. Mosteiro, and C. Thraves. Deterministic communication in the weak sensor model. In Proc. of the 11th International Conference On Principles Of Distributed Systems, volume 4878 of Lecture Notes in Computer Science, pages 119–131. Springer-Verlag, Berlin, 2007.

[20] Antonio Fernández Anta, Miguel A. Mosteiro, and Christopher Thraves. An early-stopping protocol for computing aggregate functions in sensor networks. J. Parallel Distrib. Comput., 73(2):111–121, 2013.

[21] Mohsen Ghaffari. An improved distributed algorithm for maximal independent set. In Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms, pages 270–277. SIAM, 2016.

[22] Mohsen Ghaffari. Distributed mis via all-to-all communication. In Proceedings of the ACM symposium on principles of distributed computing, pages 141–149, 2017.

[23] Mohsen Ghaffari and Merav Parter. Mst in log-star rounds of congested clique. In Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing, pages 19–28, 2016.

[24] George Giakkoupis. Tight bounds for rumor spreading in graphs of a given conductance. In Symposium on Theoretical Aspects of Computer Science (STACS2011), volume 9, pages 57–68, 2011.

[25] Seth Gilbert, Nancy A Lynch, and Alexander A Shvartsman. Rambo: a robust, reconfigurable atomic memory service for dynamic networks. Distributed Computing, 23(4):225–272, 2010.

[26] Bernhard Haeupler and David Karger. Faster information dissemination in dynamic networks via network coding. In Proceedings of the 30th annual ACM SIGACT-SIGOPS symposium on Principles of distributed computing, pages 381–390, 2011.

[27] James W Hegeman, Gopal Pandurangan, Sriram V Pemmaraju, Vivek B Sardeeshmulk, and Michele Scquizzato. Toward optimal bounds in the congested clique: Graph connectivity and mst. In Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, pages 91–100, 2015.

[28] James W Hegeman and Sriram V Pemmaraju. Lessons from the congested clique applied to mapreduce. Theoretical Computer Science, 608:268–281, 2015.

[29] James W Hegeman, Sriram V Pemmaraju, and Vivek B Sardeeshmulk. Near-constant-time distributed algorithms on a congested clique. In International Symposium on Distributed Computing, pages 514–530. Springer, 2014.

[30] Monika Henzinger, Sebastian Krinninger, and Danupon Nanongkai. A deterministic almost-tight distributed algorithm for approximating single-source shortest paths. SIAM Journal on Computing, 50(3):STOC16–98, 2019.
[31] D. Kempe, A. Dobra, and J. Gehrke. Gossip-based computation of aggregate information. In Proc. of the 44th IEEE Ann. Symp. on Foundations of Computer Science, pages 482–491, 2003.

[32] Janne H. Korhonen. Deterministic MST sparsification in the congested clique. CoRR, abs/1605.02022, 2016.

[33] Dariusz R Kowalski and Miguel A Mosteiro. Polynomial counting in anonymous dynamic networks with applications to anonymous dynamic algebraic computations. Journal of the ACM (JACM), 67(2):1–17, 2020.

[34] Dariusz R. Kowalski and Miguel A. Mosteiro. Supervised average consensus in anonymous dynamic networks. In Proceedings of the 33rd ACM Symposium on Parallelism in Algorithms and Architectures, SPAA ’21, page 307–317, New York, NY, USA, 2021. Association for Computing Machinery.

[35] Dariusz R. Kowalski and Miguel A. Mosteiro. Polynomial anonymous dynamic distributed computing without a unique leader. J. Comput. Syst. Sci., 123:37–63, 2022.

[36] Fabian Kuhn, Nancy Lynch, and Rotem Oshman. Distributed computation in dynamic networks. In Proceedings of the forty-second ACM symposium on Theory of computing, pages 513–522, 2010.

[37] Fabian Kuhn, Nancy Lynch, and Rotem Oshman. Distributed computation in dynamic networks. In Proceedings of the 42nd ACM Symposium on Theory of Computing, pages 513–522. ACM, 2010.

[38] Fabian Kuhn and Rotem Oshman. Dynamic networks: models and algorithms. ACM SIGACT News, 42(1):82–96, 2011.

[39] Christoph Lenzen. Optimal deterministic routing and sorting on the congested clique. In Proceedings of the 2013 ACM symposium on Principles of distributed computing, pages 42–50, 2013.

[40] Zvi Lotker, Boaz Patt-Shamir, Elan Pavlov, and David Peleg. Minimum-weight spanning tree construction in o (log log n) communication rounds. SIAM Journal on Computing, 35(1):120–131, 2005.

[41] Othon Michail, Ioannis Chatzigiannakis, and Paul G Spirakis. Naming and counting in anonymous unknown dynamic networks. In Stabilization, Safety, and Security of Distributed Systems, pages 281–295. Springer, 2013.

[42] Othon Michail, Ioannis Chatzigiannakis, and Paul G Spirakis. Causality, influence, and computation in possibly disconnected synchronous dynamic networks. Journal of Parallel and Distributed Computing, 74(1):2016–2026, 2014.

[43] Milena Mihail. Conductance and convergence of markov chains-a combinatorial treatment of expanders. In FOCs, volume 89, pages 526–531, 1989.

[44] Alessia Milani and Miguel A. Mosteiro. A faster counting protocol for anonymous dynamic networks. In Proceedings of the 19th International Conference on Principles of Distributed Systems, volume 46 of Leibniz International Proceedings in Informatics, pages 1–13, 2015.

[45] D. S. Mitrinović. Elementary Inequalities. P. Noordhoff Ltd. - Groningen, 1964.

[46] Damon Mosk-Aoyama and Devavrat Shah. Fast distributed algorithms for computing separable functions. IEEE Transactions on Information Theory, 54(7):2997–3007, 2008.

[47] R. Motwani and P. Raghavan. Randomized Algorithms. Cambridge University Press, 1995.

[48] Danupon Nanongkai. Distributed approximation algorithms for weighted shortest paths. In Proceedings of the forty-sixth annual ACM symposium on Theory of computing, pages 565–573, 2014.

[49] Regina O’Dell and Rogert Wattenhofer. Information dissemination in highly dynamic graphs. In Proceedings of the 2005 Joint Workshop on Foundations of Mobile Computing, DIALM-POMC ’05, page 104–110, New York, NY, USA, 2005. Association for Computing Machinery.
[50] Boaz Patt-Shamir and Marat Teplitsky. The round complexity of distributed sorting. In *Proceedings of the 30th annual ACM SIGACT-SIGOPS symposium on Principles of distributed computing*, pages 249–256, 2011.

[51] David Peleg. *Distributed computing: a locality-sensitive approach*. SIAM, Philadelphia, PA, USA, 2000.

[52] Sujay Sanghavi, Bruce Hajek, and Laurent Massoulié. Gossiping with multiple messages. *IEEE Transactions on Information Theory*, 53(12):4640–4654, 2007.

[53] Atish Das Sarma, Anisur Rahaman Molla, and Gopal Pandurangan. Distributed computation in dynamic networks via random walks. *Theoretical Computer Science*, 581:45–66, 2015.

[54] Thomas Sauerwald and Luca Zanetti. Random Walks on Dynamic Graphs: Mixing Times, Hitting Times, and Return Probabilities. In *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 93:1–93:15. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019.

[55] Alistair Sinclair and Mark Jerrum. Approximate counting, uniform generation and rapidly mixing markov chains. *Information and Computation*, 82(1):93–133, 1989.