Texture zeroes and discrete flavor symmetries in light and heavy Majorana neutrino mass matrices: a bottom-up approach

Amol Dighe$^{1,*}$ and Narendra Sahu$^{2,†}$

$^1$Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, INDIA

$^2$Department of Physics, Lancaster University, Lancaster, LA1 4YB, UK

Abstract

Texture zeroes in neutrino mass matrix $M_\nu$ may give us hints about the symmetries involved in neutrino mass generation. We examine the viability of such texture zeroes in a model independent way through a bottom-up approach. Using constraints from the neutrino oscillation data, we develop an analytic framework that can identify these symmetries and quantify deviations from them. We analyze the textures of $M_\nu$ as well as those of $M_M$, the mass matrix of heavy Majorana neutrinos in the context of Type-I seesaw. We point out how the viability of textures depends on the absolute neutrino mass scale, the neutrino mass ordering and the mixing angle $\theta_{13}$. We also examine the compatibility of discrete flavor symmetries like $\mu-\tau$ exchange and $S_3$ permutation with the current data. We show that the $\mu-\tau$ exchange symmetry for $M_\nu$ can be satisfied for any value of the absolute neutrino mass, but for $M_\nu$ to satisfy the $S_3$ symmetry, neutrino masses have to be quasi-degenerate. On the other hand, both these symmetries are currently allowed for $M_M$ for all values of absolute neutrino mass and both mass orderings.

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$^*$Electronic address: amol@theory.tifr.res.in

$^†$Electronic address: n.sahu@lancaster.ac.uk
I. INTRODUCTION

The current low energy neutrino oscillation data \[1\] indicate that all three of the physical left-handed neutrinos have different masses and they mix among themselves. If the neutrinos are Majorana, the neutrino mass matrix \( M_\nu \) in the flavor basis is diagonalized by the unitary Pontecorvo-Maki-Nakagawa-Sakata matrix \( U_{\text{PMNS}} \) \[2, 3\] through

\[
M_\nu^{\text{diag}} = U_{\text{PMNS}}^\dagger M_\nu U_{\text{PMNS}}^*, \quad \text{i.e.,} \quad M_\nu = U_{\text{PMNS}} M_\nu^{\text{diag}} U_{\text{PMNS}}^T .
\]  

(1)

For Majorana neutrinos, \( U_{\text{PMNS}} \) is given by

\[
U_{\text{PMNS}} = U_\chi \cdot \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13} \text{e}^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23} \text{e}^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13} \text{e}^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13} \text{e}^{i\delta} & c_{12}s_{23} - s_{12}s_{23}s_{13} \text{e}^{i\delta} & c_{33}c_{13}
\end{pmatrix} \cdot U_\phi ,
\]  

(2)

where \( c_{ij} \) and \( s_{ij} \) stand for \( \cos \theta_{ij} \) and \( \sin \theta_{ij} \) respectively. Here \( U_\phi = \text{diag}(\text{e}^{i\phi_1}, \text{e}^{i\phi_2}, 1) \), with the Majorana phases \( \phi_1 \) and \( \phi_2 \) defined in such a way that the diagonal elements of \( M_\nu^{\text{diag}} \) are given by

\[
M_\nu^{\text{diag}} = \text{diag}(m_1, m_2, m_3) .
\]  

(3)

Here \( m_i \) \((i = 1, 2, 3)\) correspond to the neutrino masses, which are chosen to be real and positive. The Dirac phase \( \delta \) accounts for the charge parity (CP) violation in the lepton number conserving processes. The phase matrix \( U_\chi \equiv \text{diag}(\text{e}^{i\chi_e}, \text{e}^{i\chi_\mu}, \text{e}^{i\chi_\tau}) \) consists of the three “flavor phases”\(^1\) \( \chi_\alpha \) that correspond to the multiplication of a neutrino flavor eigenstate \( \nu_\alpha \) by \( \text{e}^{i\chi_\alpha} \). Note that once the mixing angles \( \theta_{ij} \) have been defined to be in the first quadrant, the Dirac phase \( \delta \) can take values in \([0, 2\pi]\) and the phases \( \chi_e, \phi_i \) can take values between \([0, \pi]\). All the angles and phases are then uniquely defined.

A global analysis of the current neutrino oscillation data at 3\(\sigma\) C.L. yields \[1\]

\[
0.25 < \sin^2 \theta_{12} < 0.37 , \quad 0.36 < \sin^2 \theta_{23} < 0.67 , \quad \sin^2 \theta_{13} < 0.056 .
\]  

(4)

\(^1\) These phases are often referred to in the literature as “unphysical phases”. Though these phases have no relevance for the low energy neutrino phenomenology and cannot be determined through low energy measurements, their values may be predictable within the context of specific models with new physics at the high scale.
While the absolute mass scale of the neutrinos is not yet fixed, the two mass-squared differences have already been determined to a good degree of accuracy:

\[
\Delta m^2_{\odot} \equiv m_2^2 - m_1^2 = (7.06 \pm 8.34) \times 10^{-5} \text{ eV}^2,
\]

\[
\Delta m^2_{\text{atm}} \equiv m_3^2 - \left(\frac{m_1 + m_2}{2}\right)^2 = \pm(2.07 \pm 2.75) \times 10^{-3} \text{ eV}^2.
\]

(5)

It is not known whether the neutrino mass ordering is normal \((m_1 < m_2 < m_3)\) or inverted \((m_3 < m_1 < m_2)\). The Dirac phase \(\delta\) and Majorana phases \(\phi_{1,2}\) are completely unknown. The absolute values of neutrino masses cannot be probed by oscillation experiments, the direct limit on the neutrino mass scale \(m_0\) is obtained by the tritium beta decay experiments \([4]\) as \(m_0 < 2.2\) eV. The most stringent constraint on \(m_0\) however comes from cosmology: the WMAP data implies \([5]\)

\[
\sum m_i \lesssim 1\text{ eV}.
\]

(6)

In this paper, we shall take the upper bound on each neutrino mass conservatively to be \(m_i < 0.5\) eV.

Given the absolute values of the neutrino masses and the complete matrix \(U_{\text{PMNS}}\), the neutrino mass matrix \(M_\nu\) in the flavor basis can be reconstructed through Eq. (1). The structure of this matrix may reveal the presence of flavor symmetries in the neutrino sector. In this paper, we consider multiple texture zeroes of \(M_\nu\) \([6, 7, 8]\) as well as symmetries like the \(\mu - \tau\) exchange \([9]\) and \(S_3\) permutation \([10]\), which predict certain relations between the elements of \(M_\nu\).

The symmetry-based relations among the elements of \(M_\nu\) and texture zeroes of \(M_\nu\) have been explored earlier mainly by adopting a top-down approach \([11]\). In this approach an appropriate symmetry is imposed on the neutrino mass matrix, which in turn gives a prediction for the neutrino mixing parameters that can then be checked against the available data. We take the bottom-up approach, starting with our current knowledge about neutrino masses and mixings, and checking if a certain texture zero combination or a symmetry-based relation is allowed. This allows us to test in a model independent manner the symmetries present in neutrino mass generation mechanisms. It also enables us to determine which future measurements can act as tests of these symmetries.

In the present approach the elements of \(M_\nu\) are expressed as functions of the absolute neutrino masses, the mixing angles as well as the Dirac, Majorana and flavor phases.
current complete ignorance about these phases allows a lot of freedom for the elements of $M_{\nu}$ in spite of the relatively well measured values of the masses and mixing angles. Even with this freedom, some of the texture zero combinations and symmetries are clearly forbidden, as has been numerically verified [12]. We develop an analytical treatment, using perturbative expansion in appropriate small parameters, and demonstrate the analytical rationale behind the ruling out of some of these relations. This also leads us to the result that the additional knowledge of the absolute mass scale of the neutrinos and the mixing angle $\theta_{13}$ will be crucial in testing for these relations in near future.

The seesaw mechanism [13] is one of the most favored and explored mechanisms for neutrino mass generation, which gives rise to light Majorana neutrinos that can satisfy the low energy neutrino oscillation data, as well as to heavy Majorana neutrinos that may play an important role in leptogenesis [14]. If the neutrino masses are generated from a Type-I seesaw mechanism where three singlet heavy Majorana neutrinos are added to the Standard Model (SM), then we have the effective neutrino mass matrix

$$M_{\nu} = -m_D M_M^{-1} (m_D)^T,$$

where $m_D$ is the Dirac mass matrix of neutrinos, and $M_M$ is the Majorana mass matrix for the right-handed heavy Majorana neutrinos. If the heavy Majorana neutrinos are written in a basis where $m_D$ is diagonal, the texture zeroes as well as symmetry relations between elements of $M_M$ can be related to those of the inverse neutrino matrix $M_{\nu}^{-1}$ in a straightforward manner [15]. The same analytical treatment developed for $M_{\nu}$ can then be extended to test the symmetry relations for $M_M$ in this basis. We perform this analysis, with a particular emphasis on the dependence of these relations on the absolute masses of the light neutrinos.

The paper is organized as follows. In Sec. II, we introduce our formalism and set up the analytical framework under which the symmetry relations may be examined. In Sec. III and Sec. IV we test the texture zeroes of $M_{\nu}$ and $M_M$ respectively, numerically as well as analytically. In V and VI we examine the $\mu - \tau$ exchange symmetry and $S_3$ permutation symmetry for $M_{\nu}$ and $M_M$ respectively. Sec. VII concludes.
II. THE ANALYTICAL FRAMEWORK

A. parameterization of neutrino masses and mixing

We parameterize the absolute values of neutrino masses in terms of three parameters $m_0$, $\epsilon$ and $\rho$ as [16]

$$m_1 = m_0(1 - \rho)(1 - \epsilon), \quad m_2 = m_0(1 - \rho)(1 + \epsilon), \quad m_3 = m_0(1 + \rho),$$

where $m_0$ sets the overall mass scale of neutrinos, while the dimensionless parameters $\rho$ and $\epsilon$ can be expressed in terms of the solar and atmospheric mass scales as

$$\rho = \frac{\Delta m^2_{\text{atm}}}{4m_0^2}, \quad \epsilon = \frac{\Delta m^2_{\odot}}{4m_0^2(1 - \rho)^2}.$$  \hspace{1cm} (9)

Clearly, $\rho$ is positive (negative) for normal (inverted) mass ordering of neutrinos. The sum of neutrino masses may be expressed in terms of the above parameters as

$$\sum_i m_i = 3m_0 \left(1 - \frac{\rho}{3}\right) \lesssim 1 \text{ eV}.$$  \hspace{1cm} (10)

The condition $0 < m_i < 0.5 \text{ eV}$ then yields

$$m_0 \gtrsim 0.025 \text{ eV}, \quad 2.43 \times 10^{-3} < |\rho| < 1, \quad 8 \times 10^{-5} < \epsilon < 1.$$  \hspace{1cm} (11)

The value of $|\rho|$ approaches unity as $m_0$ approaches its lowest allowed value. The value of $\epsilon$ can be $> 0.01$ only for normal mass ordering and $m_0 < 0.06$ eV, whereas $\epsilon \ll |\rho|$ everywhere except for $m_0 \approx 0.025$ eV. Taking the best-fit values of solar and atmospheric neutrino masses, in Fig. 1 we show the values of $\rho$ and $\epsilon$ as functions of $m_0$ for normal as well as inverted hierarchies. For the purpose of this paper, we divide the neutrino parameter space into three scenarios:

(i) Normal mass ordering with hierarchical masses (NH), where $m_1 \ll m_2 \ll m_3$. The current data give $\rho \approx 0.85$ and $\epsilon \approx 0.92$ in the extreme limit, however these values decrease rather rapidly as $m_0$ increases, as can be seen from Fig. 1. This scenario can then be analyzed through a perturbative expansion in the set of the small parameters $\tilde{\rho} \equiv 1 - \rho$ and $\tilde{\epsilon} \equiv 1 - \epsilon$ in the extreme limit, however one has to be careful while treating quantities like $(1 - \rho^2)$, which stays higher than 0.3.

(ii) Inverted mass ordering with hierarchical masses (IH), such that $m_3 \ll m_1 < m_2$. In this
FIG. 1: The parameters $\rho$ and $\epsilon$ as functions of $m_0$ for (a) normal ordering and (b) inverted ordering of neutrino masses, for best-fit values of $\Delta m^2_{\text{atm}}$ and $\Delta m^2_{\odot}$.

In addition, at appropriate places we shall also consider $\theta_{13}$ and $\tilde{\theta}_{23} \equiv \theta_{23} - \pi/4$ as small parameters in order to facilitate a perturbative expansion.

B. Elements in $M_\nu$, $M_\nu^{-1}$, $\vec{M}_\nu$ and $M_M$

Let the low energy neutrino mass matrix in the flavor basis be written as

$$M_\nu = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}. \quad (12)$$

Since the neutrinos are Majorana, $M_\nu$ is symmetric. In terms of the parameterization of neutrino masses in Sec. [IIA] mixing angles and CP violating phases, the elements of $M_\nu$
may be written as

\[ a = m_0 e^{2i\chi_e} \left[ e^{2i\phi_1} (-1 + \rho) (-1 + \epsilon) c_{12}^2 c_{13}^2 \\
- e^{2i\phi_2} (-1 + \rho) (1 + \epsilon) \right] \]

\[ b = m_0 e^{i(\chi_e + \chi_\mu)} c_{13} \left[ e^{-i\delta} (1 + \rho) s_{23} s_{13} \\
- e^{2i\phi_1} (-1 + \rho) (-1 + \epsilon) c_{12} (-s_{23} s_{12} + e^{i\delta} c_{12} s_{23} s_{13}) \\
+ e^{2i\phi_2} (-1 + \rho) (1 + \epsilon) s_{12} (-c_{12} c_{23} + e^{i\delta} s_{12} s_{23} s_{13}) \right] \]

\[ c = m_0 e^{i(\chi_e + \chi_\tau)} c_{13} \left[ e^{-i\delta} (1 + \rho) c_{23} s_{13} \right. \\
- e^{2i\phi_1} (-1 + \rho) (-1 + \epsilon) c_{12} (-s_{12} s_{23} + e^{i\delta} c_{12} c_{23} s_{13}) \]

\[ d = m_0 e^{2i\chi_\mu} \left[ (1 + \rho) c_{13}^2 s_{23} \right. \\
+ e^{2i\phi_1} (-1 + \rho) (-1 + \epsilon) \left( c_{23}s_{12} + e^{i\delta} c_{12} s_{23} s_{13} \right)^2 \]

\[ e = m_0 e^{i(\chi_\mu + \chi_\tau)} \right] \]

\[ f = m_0 e^{2i\chi_\tau} \left[ (1 + \rho) c_{13}^2 c_{23}^2 \right. \\
+ e^{2i\phi_1} (-1 + \rho) (-1 + \epsilon) c_{12} s_{23} - e^{i\delta} c_{12} c_{23} s_{13})^2 \]

\[ - e^{2i\phi_2} (-1 + \rho) (1 + \epsilon) c_{12} s_{23} + e^{i\delta} c_{23} s_{12} s_{13})^2 \right] \]

The inverse of the neutrino mass matrix, \( M_\nu^{-1} \), can be written as

\[ M_\nu^{-1} = \frac{\tilde{M}}{\text{det}(M_\nu)} = \frac{1}{\text{det}(M_\nu)} \left( \begin{array}{ccc} A & B & C \\ B & D & E \\ C & E & F \end{array} \right) \]

where

\[ \text{det}(M_\nu) = m_0^3 (1 - \rho)^2 (1 + \rho) (1 - \epsilon^2) e^{2i \sum \chi} \]
is the determinant of $M_\nu$, with $\sum \chi \equiv \chi_e + \chi_\mu + \chi_\tau$. Here $\tilde{M}$ is the adjoint neutrino mass matrix. From Eq. (14) it is obvious that texture zeroes in $M_\nu^{-1}$ are the same as those in $\tilde{M}$. The elements of $\tilde{M}$ can be written in terms of the masses and the elements of $U \equiv U_{\text{PMNS}}$ matrix as

\begin{align*}
A &= m_0^2 e^{2i(\chi_e + \chi_\tau)} [(1 - \rho)^2(1 - \epsilon^2) (U_{21}U_{32} - U_{22}U_{31})^2 \\
  &\quad + (1 - \rho^2)(1 - \epsilon) (U_{21}U_{33} - U_{23}U_{31})^2 + (1 - \rho^2)(1 + \epsilon) (U_{22}U_{33} - U_{23}U_{32})^2] \\
B &= m_0^2 e^{i(\chi_e + \chi_\mu + 2\chi_\tau)} [(1 - \rho)^2(1 - \epsilon^2) (U_{21}U_{32} - U_{22}U_{31}) (U_{31}U_{12} - U_{11}U_{32}) \\
  &\quad + (1 - \rho^2)(1 - \epsilon) (U_{31}U_{13} - U_{33}U_{11}) (U_{21}U_{33} - U_{23}U_{31}) \\
  &\quad + (1 - \rho^2)(1 + \epsilon) (U_{22}U_{33} - U_{23}U_{32}) (U_{32}U_{13} - U_{12}U_{33})] \\
C &= m_0^2 e^{i(\chi_e + 2\chi_\mu + \chi_\tau)} [(1 - \rho)^2(1 - \epsilon^2) (U_{21}U_{32} - U_{22}U_{31}) (U_{22}U_{11} - U_{12}U_{21}) \\
  &\quad + (1 - \rho^2)(1 - \epsilon) (U_{11}U_{23} - U_{13}U_{21}) (U_{21}U_{33} - U_{31}U_{23}) \\
  &\quad + (1 - \rho^2)(1 + \epsilon) (U_{12}U_{23} - U_{13}U_{22}) (U_{22}U_{33} - U_{32}U_{23})] \\
D &= m_0^2 e^{2i(\chi_e + \chi_\tau)} [(1 - \rho)^2(1 - \epsilon^2) (U_{11}U_{32} - U_{12}U_{31})^2 + (1 - \rho^2)(1 - \epsilon) (U_{11}U_{33} - U_{13}U_{31})^2 \\
  &\quad + (1 - \rho^2)(1 + \epsilon) (U_{12}U_{33} - U_{32}U_{13})^2] \\
E &= m_0^2 e^{i(2\chi_e + \chi_\mu + \chi_\tau)} [(1 - \rho)^2(1 - \epsilon^2) (U_{11}U_{32} - U_{12}U_{31}) (U_{12}U_{21} - U_{11}U_{22}) \\
  &\quad + (1 - \rho^2)(1 - \epsilon) (U_{11}U_{33} - U_{13}U_{31}) (U_{21}U_{13} - U_{11}U_{23}) \\
  &\quad + (1 - \rho^2)(1 + \epsilon) (U_{12}U_{33} - U_{13}U_{32}) (U_{22}U_{13} - U_{12}U_{23})] \\
F &= m_0^2 e^{2i(\chi_e + \chi_\mu)} [(1 - \rho)^2(1 - \epsilon^2) (U_{11}U_{22} - U_{12}U_{21})^2 + (1 - \rho^2)(1 - \epsilon) (U_{11}U_{23} - U_{13}U_{21})^2 \\
  &\quad + (1 - \rho^2)(1 + \epsilon) (U_{12}U_{23} - U_{13}U_{22})^2]. \tag{16}
\end{align*}

In order to analyze the heavy majorana neutrino mass matrix $M_M$, we invert Eq. (7) to obtain

\begin{equation}
M_M = -m_D^T M_\nu^{-1} m_D. \tag{17}
\end{equation}

Following [15], we choose the “flavor” basis for heavy Majorana neutrinos in which the Dirac mass matrix is real and diagonal,

\begin{equation}
m_D = \text{diag}(x, y, z). \tag{18}
\end{equation}
In this basis, $M_M$ may be written as

$$M_M = -\frac{1}{\text{Det}(M_\nu)} \begin{pmatrix} x^2 A & xy B & xz C \\ xy B & y^2 D & yz E \\ xz C & yz E & z^2 F \end{pmatrix}.$$  \hspace{1cm} (19)

Again, the texture zeroes of $M_M$ are same as the texture zeroes of $M_\nu^{-1}$, and hence those of $\tilde{M}$, which have relatively tractable analytical expressions. The discrete symmetries like $\mu - \tau$ exchange or $S_3$, on the other hand, also depend on the values of the Dirac masses. However even in that case, we can test for certain relations between elements of $M_M$ that are independent of these Dirac masses, as we shall see in Sec. VI.

C. Quantifying deviation from exact symmetry in the bottom-up approach

In the traditional top-down approach, discrete flavor symmetries like $\mu - \tau$ exchange, $S_3$-permutation, etc. are assumed in the neutrino mass matrix, which predict the mixing parameters measured in the low energy neutrino oscillation data. The main purpose of these symmetries is to understand why the (1-3) family mixing is small, while the (2-3) family mixing is almost maximal and the (1-2) family mixing is large but not maximal. Traditionally these symmetries are employed to set $U_{13} = 0$. The dynamical breaking of these symmetries then may predict a non-zero $U_{13}$, which is then compared with the experimental value. Since the current low energy neutrino oscillation data have large uncertainties, the data allow an enormous freedom to propose such discrete flavor symmetries in the top-down approach.

However it is also crucial to examine, by starting with the available low energy neutrino oscillation data, the parameter space of neutrino mass matrix where such discrete flavor symmetries can be realized. We call it the bottom-up approach. In this approach, since the low energy data have intrinsic uncertainties, the symmetry relations can only be said to be satisfied approximately. One therefore needs to quantify when one may declare the relevant symmetry to be allowed.

In $M_\nu$, the magnitudes of all the elements are expected to be $\sim m_0$, as can be seen from Eqs. (13), taking into account that the sine and cosine of $\theta_{12}, \theta_{23} \sim O(1)$ and also assuming that $\phi_i \sim O(1)$ in the absence of any symmetry principle. If an element $|M_\nu(i, j)|$ is $\ll m_0$, it is either an accidental cancellation or the signature of a discrete symmetry at work. We take the position that for a sufficiently small value of $\xi$, the observation $|M_\nu(i, j)|/m_0 < \xi$ would
indicate that the symmetry that would make $M_{\nu}(i,j) = 0$ is present. We choose $\xi = 10^{-2}$, which is motivated by the accuracy to which the mixing angles are currently known. It also indicates the extent to which we tolerate the breaking of the discrete symmetries under consideration. In other words, when $|M_{\nu}(i,j)|/m_0 < 10^{-2}$, we consider $M_{\nu}(i,j)$ to be effectively zero. Thus, we declare a texture zero viable if

$$\text{Min} \left( \frac{|M_{\nu}(i,j)|}{m_0} \right) < 10^{-2},$$

(20)

where the minimization is over all the allowed values ($3\sigma$) of the mixing parameters at a particular value of $m_0$. If this condition is not satisfied, then the symmetry that would lead to $M_{\nu}(i,j) = 0$ is ruled out.

Similarly, from Eq. (16) one can see that the elements of $\tilde{M}$ are expected to be $\sim m_0^2$ in the absence of any cancellations. Hence if

$$\text{Min} \left( \frac{|\tilde{M}(i,j)|}{m_0^2} \right) < 10^{-2},$$

(21)

then we conclude that the symmetry that requires $\tilde{M}(i,j) = 0$ is still allowed. In the case of the discrete symmetries like $\mu - \tau$ exchange or $S_3$, certain ratios are expected to be equal to unity. Here, we demand that the deviation of such ratios from unity to be less than $10^{-2}$ for the symmetry to be acceptable. Note that the right hand side of Eqs. (20) and (21) can be changed to any small number of one’s choice, depending on how much deviation from the exact symmetry one is willing to allow. Our numerical results cover the complete relevant range, so the required numbers can be read off from our figures.

Note that our criteria give the necessary conditions for a particular symmetry to hold. Further considerations may disallow some of the symmetry relations that are permitted by conditions in Eqs. (20) and (21).

III. TEXTURE ZEROES IN $M_{\nu}$

A. Individual zeroes in $M_{\nu}$

Whether a particular element in the neutrino mass matrix $M_{\nu}$ can potentially vanish can be checked analytically from Eqs. (13). To simplify the expressions, we define three
FIG. 2: The minima of certain $M_\nu(i,j)/m_0$ are shown against $m_0$ for (a) normal and (b) inverted ordering of neutrino masses. The minima of the remaining elements of $M_\nu/m_0$ are less than 0.001 for both normal as well as inverted ordering. The neutrino mixing parameters are varied in their 3σ allowed range in this and the subsequent figures.

quantities

\[ \zeta_1 \equiv (1 - \epsilon)e^{2i\phi_1}, \]
\[ \zeta_2 \equiv (1 + \epsilon)e^{2i\phi_2}, \]
\[ \zeta_3 \equiv s_{13}e^{i\delta}. \]

Note that since $\epsilon > 0$, we have $|\zeta_1| < |\zeta_2|$ in case of NH, while $|\zeta_1| \approx |\zeta_2| \approx 1$ in the IH and QD scenarios where $\epsilon \ll 1$.

The following observations may be made from the analytic expressions for $a, b, c, d, e, f$:

(i) $a \equiv M_\nu(1,1) = m_0(1 - \rho)e^{2i\chi} \left[ \zeta_1 c_{12}^2 + \zeta_2 s_{12}^2 + (1 + \rho)\zeta_3^* c_{12}^2 \right].$

In NH, while the $(1 - \rho)$ provides one suppression factor, the other suppression is provided by the terms involving $\zeta_i$, all of which have phases that can be adjusted so as to cause a cancellation among the three terms inside the square bracket. Thus, $M_\nu(1,1)$ can vanish for NH as can be seen from Fig. 2. Note that if $\theta_{13}$ were extremely small so as to make the $\zeta_3$ term negligible, the cancellation could not have been achieved since current data implies $|\zeta_1 c_{12}^2| < |\zeta_2 s_{12}^2|$ in this extreme hierarchical limit. So a significantly non-vanishing $\theta_{13}$ is crucial for allowing a texture zero of $M_\nu(1,1)$ for NH.

In IH and QD, since $|\zeta_1| \approx |\zeta_2| \approx 1$ and $c_{12}^2 - s_{12}^2 > 0.26$, the minimum magnitude
of the sum of the first two terms in the square bracket is \( \approx 0.26 \). On the other hand, \( |\zeta_3^\prime| < 0.04 \), so this term cannot cancel the first two. Thus, the smallness of \( \theta_{13} \), combined with the non-maximality of \( \theta_{12} \), ensures that \( M_\nu(1,1) \) cannot be a texture zero for IH and QD scenarios.

In the normal mass ordering, extreme hierarchy allows the texture zero of \( M_\nu(1,1) \) whereas quasidegeneracy prevents it. The transition between these two extremes as a function of \( m_0 \) may be observed in Fig. 2.

(ii) 
\[
\begin{align*}
 b &\equiv M_\nu(1,2) \approx -(m_0/\sqrt{2})e^{i(x_\nu+x_\tau)}[\Omega_b + O(\theta_{13}^2, \tilde{\theta}_{23})] , \\
 c &\equiv M_\nu(1,3) \approx +(m_0/\sqrt{2})e^{i(x_\nu+x_\tau)}[\Omega_c + O(\theta_{13}^2, \tilde{\theta}_{23})] , \\
 &\text{where} \\
 \Omega_b = \Omega_c = (1-\rho)[c_{12}s_{12}(\zeta_1 - \zeta_2) - \zeta_3(\zeta_1c_{12}^2 + \zeta_2s_{12}^2)] + (1+\rho)\zeta_3^* . 
\end{align*}
\]

In NH, if \( \theta_{13} = 0 \) then \( \Omega_b \) and \( \Omega_c \) cannot vanish, since \( |\zeta_1| < |\zeta_2| \). However, with a non-zero \( \zeta_3 \), the phases \( \phi_1, \phi_2 \) and \( \delta \) may be chosen properly to make \( \Omega_b \) and \( \Omega_c \) vanish. Note that the \((1+\rho)\) coefficient of the \( \zeta_3^* \) term makes that term comparable to the terms involving \( \zeta_1, \zeta_2 \), which are suppressed by the coefficient \((1-\rho)\) for NH. Thus, \( M_\nu(1,2) \) and \( M_\nu(1,3) \) can be texture zeroes in NH as long as \( \theta_{13} \) is not extremely small.

In IH and QD, \( |\zeta_1| \approx |\zeta_2| \), so that \( \phi_1 \approx \phi_2 \) can make \( \zeta_1 - \zeta_2 \approx 0 \). Thus even with \( \theta_{13} \) vanishing, \( M_\nu(1,2) \) and \( M_\nu(1,3) \) can be texture zeroes of the neutrino mass matrix.

(iii) 
\[
\begin{align*}
 d &\equiv M_\nu(2,2) \approx (m_0/2)e^{2ix_\nu}[\Omega_d + O(\theta_{13}^2, \tilde{\theta}_{23})] , \\
 f &\equiv M_\nu(3,3) \approx (m_0/2)e^{2ix_\nu}[\Omega_f + O(\theta_{13}^2, \tilde{\theta}_{23})] , \\
 &\text{where} \\
 \Omega_d = \Omega_f = (1+\rho) + (1-\rho)[(\zeta_1s_{12}^2 + \zeta_2c_{12}^2) + 2\zeta_3s_{12}c_{12}(\zeta_1 - \zeta_2)] . 
\end{align*}
\]

In NH, \((1+\rho)\) is as high as 1.9. This cannot be cancelled by the other terms involving \( \zeta_1, \zeta_2 \) since these terms are already suppressed by \((1-\rho)\). As a result, \( M_\nu(2,2) \) and \( M_\nu(3,3) \) cannot be texture zeroes for NH. This can be seen from Fig. 2(a). In IH, \((1+\rho) \approx 0\). While the coefficient of \((1-\rho)\) in \( \Omega_d \) and \( \Omega_f \) cannot vanish for \( \theta_{13} = 0 \), it can be made to vanish for \( \theta_{13} \neq 0 \) with proper choices of \( \phi_1, \phi_2 \) and \( \delta \). On the other hand in QD, the choice of \( \phi_1 = \phi_2 = \pi/2 \) would make \( \Omega_d \) and \( \Omega_f \) vanish even when \( \theta_{13} = 0 \). Thus \( M_\nu(2,2) \) and \( M_\nu(3,3) \) can be texture zeroes in IH and QD scenarios, the former scenario requiring a nonzero \( \theta_{13} \).

(iv) 
\[
\begin{align*}
 e &\equiv M_\nu(2,3) \approx (m_0/2)e^{i(x_\nu+x_\tau)}[(1+\rho) - (1-\rho)(\zeta_1s_{12}^2 + \zeta_2c_{12}^2) + O(\theta_{13}^2, \tilde{\theta}_{23})] . 
\end{align*}
\]
\[ M(ij - kl) \equiv \text{Min} \left( \sqrt{|M_\nu(i,j)|^2 + |M_\nu(k,l)|^2} \right). \] (25)

There are fifteen \( M(ij - kl) \) constructed out of six \( M_\nu(i,j) \) elements. From the discussion in Sec. IIIA it is clear that the five \( M(ij - kl) \) involving \( M_\nu(1,1) \) cannot vanish unless the mass ordering is normal and \( m_0 \lesssim 0.035 \text{ eV} \). Moreover, even in such a case the other element of the pair cannot be \( M_\nu(2,2), M_\nu(2,3) \) or \( M_\nu(3,3) \). The only two-zero textures including vanishing \( M_\nu(1,1) \) are therefore

\[
\begin{pmatrix}
0 & 0 & X \\
0 & X & X \\
X & X & X
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & X & 0 \\
X & X & X \\
0 & X & X
\end{pmatrix},
\] (26)

where \( X \) denotes an element that may or may not vanish. That both these patterns correspond to NH can be seen from Fig. 3. The above two-zero textures with normal ordering

\begin{center}
\begin{tabular}{cccccc}
a & b & c & d & f & e \\
\hline
NH & \Theta & \Theta & \Theta & \otimes & \otimes \\
IH & \otimes & \sqrt{} & \sqrt{} & \Theta & \Theta & \otimes \\
QD & \otimes & \sqrt{} & \sqrt{} & \sqrt{} & \sqrt{} & \sqrt{}
\end{tabular}
\end{center}

TABLE I: Viability of individual texture zeroes for \( M_\nu \). Here \( \sqrt{} \) indicates that the texture is allowed even for \( \theta_{13} \) vanishing, \( \Theta \) indicates that the texture is allowed but needs a nonzero \( \theta_{13} \), whereas \( \otimes \) indicates that the texture is not allowed.

In NH, arguments similar to above can be used to show that the large value of \( (1 + \rho) \) cannot be cancelled by the other terms that are suppressed by \( (1 - \rho) \). As a result, \( M_\nu(2,3) \) cannot be a texture zero for NH. In IH, while \( (1 - \rho) \) vanishes, the coefficient of \( (1 - \rho) \) term cannot vanish owing to the non-maximal nature of \( \theta_{12} \), preventing \( M_\nu(2,3) \) from being a texture zero for IH. In QD, however, \( \phi_1 = \phi_2 = \pi/2 \) would make \( e \) vanish, so that \( M_\nu(2,3) \) can be a texture zero.

In Table I we summarize our results on the viability of the texture zeroes in various scenarios.

B. Two zeroes in \( M_\nu \)

In order to quantify the two-zero textures in \( M_\nu \) in the bottom-up approach, we define

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\[ M(ij - kl) \equiv \text{Min} \left( \sqrt{|M_\nu(i,j)|^2 + |M_\nu(k,l)|^2} \right). \] (25)

There are fifteen \( M(ij - kl) \) constructed out of six \( M_\nu(i,j) \) elements. From the discussion in Sec. IIIA it is clear that the five \( M(ij - kl) \) involving \( M_\nu(1,1) \) cannot vanish unless the mass ordering is normal and \( m_0 \lesssim 0.035 \text{ eV} \). Moreover, even in such a case the other element of the pair cannot be \( M_\nu(2,2), M_\nu(2,3) \) or \( M_\nu(3,3) \). The only two-zero textures including vanishing \( M_\nu(1,1) \) are therefore

\[
\begin{pmatrix}
0 & 0 & X \\
0 & X & X \\
X & X & X
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & X & 0 \\
X & X & X \\
0 & X & X
\end{pmatrix},
\] (26)

where \( X \) denotes an element that may or may not vanish. That both these patterns correspond to NH can be seen from Fig. 3. The above two-zero textures with normal ordering
FIG. 3: $M(11-12)/m_0$ and $M(11-13)/m_0$ as functions $m_0$ in (a) normal and (b) inverted ordering of neutrino masses.

imply that the neutrinoless double beta decay $(0\nu\beta\beta)$, whose rate is proportional to

$$|M_{\nu(1,1)}|^2 = \left| \sum_i m_i U_{1i}^2 \right|^2,$$

(27)

is unobservable since $|M_{\nu(1,1)}| < 10^{-2}m_0$ \[17\]. In other words, if the neutrino masses are normally ordered and the $0\nu\beta\beta$ is observed then the corresponding symmetry which drives the cancellation in the neutrino mass matrix leading to these textures should be forbidden.

The five textures

$$
\begin{pmatrix}
X & 0 & 0 \\
0 & X & X \\
0 & X & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & 0 & X \\
0 & X & 0 \\
X & 0 & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & X & 0 \\
X & X & 0 \\
0 & 0 & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & X & X \\
X & 0 & 0 \\
X & 0 & X \\
\end{pmatrix},
$$

(28)

cannot satisfy the neutrino data since the first two lead to $\theta_{12} = 0$, the second and the third lead to $\theta_{23} = 0$, while the last two are inconsistent with a small $\theta_{13}$ and large $\theta_{23}$ simultaneously.

The remaining five pairs of $M_{\nu(i,j)}$ textures are allowed by both normal and inverted ordering of neutrino mass pattern at large $m_0$, i.e. they are allowed in the QD scenario. These textures are

$$
\begin{pmatrix}
X & 0 & X \\
0 & 0 & X \\
X & X & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & 0 & X \\
0 & X & X \\
0 & X & 0 \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & 0 & X \\
X & X & 0 \\
0 & X & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & X & X \\
X & 0 & X \\
0 & 0 & X \\
\end{pmatrix}, \quad
\begin{pmatrix}
X & X & X \\
X & 0 & X \\
X & 0 & X \\
\end{pmatrix}.
$$

(29)
FIG. 4: $M(ij - kl)/m_0$ as functions of $m_0$ for (a) normal and (b) inverted ordering of neutrino masses. The fluctuations seen in this and some of the subsequent figures are numerical artifacts, a consequence of the inability of the randomly chosen neutrino parameters to find the actual minimum.

The $m_0$ values at which these textures start becoming viable can be read off from Fig. 4.

Notice that the seven allowed two-zero textures, given by Eqs. (26) and (29), have already been predicted in the top-down scenario [11]. Here in addition to obtaining them through a bottom-up approach, we have also correlated them with the measured values of $m_0$ and $\theta_{13}$.

C. Three zeroes in $M_\nu$

In order to quantify the possible three-zero textures in $M_\nu$ in the bottom-up approach, we define

$$M(ij - kl - mn) \equiv \text{Min} \left( \frac{\sqrt{|M_\nu(i,j)|^2 + |M_\nu(k,l)|^2 + |M_\nu(m,n)|^2}}{3} \right).$$

(30)
From the two-zero textures obtained in the last subsection, one can deduce that the only possible three-zero textures of $M_\nu$ are the combinations $(12-22-33)$ and $(13-22-33)$. In Fig. 5, we show these two combinations $M(ij-kl-mn)/m_0$ as functions of $m_0$ for normal as well as inverted ordering of neutrino masses. It can be seen that $M(ij-kl-mn)/m_0$ does not vanish in any region of the neutrino parameter space. Thus, there are no viable three-zero textures for $M_\nu$.

![FIG. 5: $M(ij-kl-mn)/m_0$ as functions of $m_0$ for selected values of $(ij, kl, mn)$ for (a) normal and (b) inverted ordering of neutrino masses. For all the rest $(ij, kl, mn)$ values, $M(ij-kl-mn)$ is always greater than $0.1m_0$ and hence the corresponding symmetry is forbidden.](image)

**IV. TEXTURE ZEROES IN $M_\nu^{-1}, \tilde{M}$ AND $M_M$**

The texture zeroes in $M_\nu^{-1}$, $\tilde{M}$ and $M_M$ are identical, as can be seen from Eqs. (14) and (19). The quantification of “smallness” of the matrix elements is even more arbitrary for $M_M$ due to the presence of the unknown Dirac masses $x, y, z$. However for the sake of uniformity, we determine the texture zeroes of $\tilde{M}$ through the quantitative criteria described in Sec. II C and apply the same criteria for the zeroes of $M_M$. This correspondence preserves all the zeroes in $\tilde{M}$, and does not add any additional ones due to the hierarchy of Dirac masses, for example.

We thus continue with determining the zeroes of $\tilde{M}$ on lines similar to the last section.
where we determined zeroes of $M_\nu$. For convenience, we define
\[
\tilde{M}(ij - kl) \equiv \operatorname{Min} \left( \frac{\sqrt{\tilde{M}(i,j)^2 + \tilde{M}(k,l)^2}}{2} \right), \tag{31}
\]
\[
\tilde{M}(ij - kl - mn) \equiv \operatorname{Min} \left( \frac{\sqrt{\tilde{M}(i,j)^2 + \tilde{M}(k,l)^2 + \tilde{M}(m,n)^2}}{3} \right). \tag{32}
\]

The quantities $\operatorname{Min}(|\tilde{M}(i,j)|)$, $\tilde{M}(ij - kl)$ and $\tilde{M}(ij - kl - mn)$ need to be less than $10^{-2}m_0^2$ in order to qualify as one-, two- or three-zero textures respectively.

**A. Individual zeroes in $\tilde{M}$**

![Figure 6](image_url)

**FIG. 6:** Individual elements of $\operatorname{Min}(\tilde{M}(i,j))/m_0^2$ as functions of $m_0$ for (a) normal and (b) inverted ordering of neutrino mass spectrum. The minima of the remaining elements which are not shown in the figure are less than 0.001 for all values of $m_0$.

(i) \[ A \equiv \tilde{M}(1, 1) \approx m_0^2 e^{2i(x_u + x_\tau)} [(1 - \rho^2)(\zeta_1 c_{12}^2 + \zeta_2 c_{12}^2) + (1 - \rho)^2 \zeta_1 \zeta_2 \zeta_3^2]. \]

In NH, the last term is suppressed quadratically in $(1 - \rho)$ as well as in $\zeta_3$, whereas the $(1 - \rho^2)$ term does not have such a strong suppression. Moreover, the coefficient of $(1 - \rho^2)$ cannot vanish since $|\zeta_1 c_{12}^2| \ll |\zeta_2 s_{12}^2|$. As a result, $\tilde{M}(1, 1)$ does not vanish for NH. In IH, $\rho^2 \approx 1$, so that the leading term vanishes. Further, the $(1 - \rho^2)$ dependent term vanishes at $\theta_{13} = 0$. $\tilde{M}(1, 1)$ as a texture zero is therefore allowed. In QD on the other hand, the $(1 - \rho^2)$ term and its coefficient are nonzero, while the second term is suppressed by $(1 - \rho)^2 \theta_{13}^2$ and is unable to cancel the first term. As a result, $\tilde{M}(1, 1)$ cannot vanish for QD.
In the inverted mass ordering, the hierarchical limit allows a texture zero of \( \tilde{M}(1,1) \) whereas the quasi-degenerate limit prevents it. The transition between these two limits as a function of \( m_0 \) can be seen in Fig. 6(b).

(ii) \[ B \equiv \tilde{M}(1,2) \approx +\left(m_0^2/\sqrt{2}\right)e^{i(x_e+x_\mu+2x_\tau)}[\Omega_B + O(\theta_{13}^2, \bar{\theta}_{23})] , \]
\[ C \equiv \tilde{M}(1,3) \approx -(m_0^2/\sqrt{2})e^{i(x_e+2x_\mu+x_\tau)}[\Omega_C + O(\theta_{13}^2, \bar{\theta}_{23})] , \]
where
\[ \Omega_B = \Omega_C = (1 - \rho^2)[c_{12}s_{12}(\zeta_1 - \zeta_2) + \zeta_3(c_{12}s_{12}^2 + \zeta_2c_{12}^2)] - (1 - \rho^2)\zeta_1\zeta_2\zeta_3 . \]

In NH, the last term is suppressed by \((1 - \rho)^2\theta_{13}\), and is therefore rather insignificant. However, the coefficient of the \((1 - \rho^2)\) term cannot vanish with any choice of the Majorana and Dirac phases, so \(M_\nu(1,2)\) and \(M_\mu(1,3)\) cannot be texture zeroes in NH. This feature can be seen from Fig. 6(a). In IH and QD, since \(|\zeta_1| \approx |\zeta_2|\), one may choose \(\phi_1 \approx \phi_2\) to make \(\zeta_1 - \zeta_2 \approx 0\), so that \(\tilde{M}(1,2)\) and \(\tilde{M}(1,3)\) can be texture zeroes.

(iii) \[ D \equiv \tilde{M}(2,2) \approx m_0^2e^{2i(x_e + x_\tau)}[\Omega_D + O(\theta_{13}^2, \bar{\theta}_{23})] , \]
\[ F \equiv \tilde{M}(2,3) \approx m_0^2e^{2i(x_e + x_\mu)}[\Omega_F + O(\theta_{13}^2, \bar{\theta}_{23})] , \]
\[ E \equiv \tilde{M}(2,3) \approx m_0^2e^{2i(x_e + x_\mu + x_\tau)}[\Omega_E + O(\theta_{13}^2, \bar{\theta}_{23})] , \]
where
\[ \Omega_D = \left(1 - \rho^2\right)^2\zeta_1\zeta_2 + \left(1 - \rho^2\right)^2\zeta_1c_{12}^2 + \zeta_3(c_{12}s_{12}^2 - 2\zeta_3^*c_{12}s_{12}) , \]
\[ \Omega_F = \left(1 - \rho^2\right)^2\zeta_1\zeta_2 + \left(1 - \rho^2\right)^2\zeta_1c_{12}^2 + \zeta_3(c_{12}s_{12}^2 + 2\zeta_3^*c_{12}s_{12}) , \]
\[ -2(1 - \rho^2)c_{12}s_{12}\zeta_1\zeta_2\zeta_3 \]
\[ \Omega_E = -\left(1 - \rho^2\right)^2\zeta_1\zeta_2 + \left(1 - \rho^2\right)^2\zeta_1c_{12}^2 + \zeta_3(c_{12}s_{12}^2 + 2\zeta_3^*(\zeta_1 - \zeta_2)c_{12}s_{12}) . \]

In NH, the first term in each of the expressions \(\Omega_D, \Omega_F, \Omega_E\) is suppressed by \((1 - \rho^2)\zeta_1\), whereas the coefficient of \((1 - \rho^2)\) can be made to vanish with an appropriate choice of phases, as long as \(\theta_{13}\) is finite and \(\zeta_3\) can participate in the cancellation. So \(\tilde{M}(2,2), \tilde{M}(2,3)\) and \(\tilde{M}(3,3)\) can be texture zeroes for NH. In IH, though the \((1 - \rho^2)\) term vanishes, the coefficient of the \((1 - \rho^2)\) term is of the order of unity and as a result, none of the above three can be texture zeroes. In QD, the first term is suppressed by \((1 - \rho^2)\zeta_1\), while the coefficient of \((1 - \rho^2)\) becomes
\[ (\zeta_1c_{12}^2 + \zeta_2s_{12}^2 \pm \zeta_1\zeta_2) \]
TABLE II: Viability of individual texture zeroes for $M_M$. Here √ indicates that the texture is allowed even for $\theta_{13}$ vanishing, Θ indicates that the texture is allowed but needs a nonzero $\theta_{13}$, whereas ⊗ indicates that the texture is not allowed.

|     | A | B | C | D | F | E |
|-----|---|---|---|---|---|---|
| NH  | ⊗ | ⊗ | ⊗ | Θ | Θ | Θ |
| IH  | √ | √ | √ | ⊗ | ⊗ | ⊗ |
| QD  | ⊗ | √ | √ | √ | √ | √ |

when $\theta_{13} = 0$. The choices of the Majorana phases $\phi_1 = \phi_2 = \pi/2$ and $\phi_1 = \phi_2 = 0$ make the above expression vanish for the + and - sign respectively. Thus, $M_\nu(2, 2)$, $M_\nu(2, 3)$ and $M_\nu(3, 3)$ are allowed as texture zeroes for QD.

In the inverted mass ordering, the hierarchical limit allows a texture zero of the above three matrix elements, whereas the quasidegenerate limit prevents it. The transition between these two limits can be seen in Fig. 6(b).

In Table II we summarize our results on the viability of the texture zeroes in various scenarios.

B. Two zeroes in $\tilde{M}$

Out of fifteen elements in $\tilde{M}(ij - kl)$, five two-zero textures involve a zero of $\tilde{M}(1, 1)$. These are clearly not allowed for NH and QD, since the value of $\tilde{M}(1, 1)$ itself is high in these scenarios. In IH, $\tilde{M}(1, 1)$ can be small, however $\tilde{M}(2, 2)$, $\tilde{M}(2, 3)$ and $\tilde{M}(3, 3)$ are large, so that $\tilde{M}(11 - 22)$, $\tilde{M}(11 - 23)$ and $\tilde{M}(11 - 33)$ cannot be texture zeroes. This leaves us with the two possible textures

$$
\begin{pmatrix}
0 & 0 & X \\
0 & X & X \\
X & X & X
\end{pmatrix},
\begin{pmatrix}
0 & X & 0 \\
X & X & X \\
0 & X & X
\end{pmatrix}.
$$

(33)

However, these two inverse neutrino mass matrices correspond to the last two two-zero textures of $M_\nu$ matrices in (28), which are forbidden. Thus, even the textures in (33) are not allowed.
FIG. 7: Minima of $\tilde{M}(22 - 23)$ and $\tilde{M}(23 - 33)$ as functions of $m_0$ for (a) normal and (b) inverted ordering of neutrino masses.

The textures

$$
\begin{pmatrix}
X & 0 & 0 \\
0 & X & X \\
0 & X & X 
\end{pmatrix},
\begin{pmatrix}
X & 0 & X \\
0 & X & 0 \\
X & 0 & X 
\end{pmatrix},
\begin{pmatrix}
X & X & 0 \\
X & X & 0 \\
0 & 0 & X 
\end{pmatrix},
$$

are ruled out since in each of these, one of the neutrinos is completely decoupled, whereas we need large values for $\theta_{12}$ as well as $\theta_{23}$.

Textures involving zeroes of $\tilde{M}(2, 2)$, $\tilde{M}(2, 3)$ and $\tilde{M}(3, 3)$ can be allowed only for NH and QD. It is observed that $\tilde{M}(22 - 23)$ and $\tilde{M}(23 - 33)$ can vanish in the NH scenario, but not in the rest. The dependence of the vanishing of these two textures,

$$
\begin{pmatrix}
X & X & X \\
X & X & 0 \\
X & 0 & 0 
\end{pmatrix},
\begin{pmatrix}
X & X & X \\
X & X & 0 \\
X & 0 & X 
\end{pmatrix},
$$

can be seen from Fig. 7 where we have plotted $\tilde{M}(ij - kl)/m_0^2$ as a function of $m_0$ for normal as well as inverted ordering of neutrino masses.

The remaining five two-zero textures,

$$
\begin{pmatrix}
X & 0 & X \\
0 & 0 & X \\
X & X & X 
\end{pmatrix},
\begin{pmatrix}
X & 0 & X \\
0 & X & X \\
X & X & 0 
\end{pmatrix},
\begin{pmatrix}
X & X & 0 \\
X & X & X \\
X & X & 0 
\end{pmatrix},
\begin{pmatrix}
X & X & 0 \\
X & X & X \\
0 & X & 0 
\end{pmatrix},
\begin{pmatrix}
X & X & X \\
X & X & 0 \\
0 & 0 & X 
\end{pmatrix},
$$

(36)
are allowed by both normal and inverted ordering of neutrino mass pattern, though only in the quasi-degenerate limit. This can be seen quantitatively in Fig. 8. Note that the textures given in Eqs. (35), and (36) are already predicted in the top-down scenario [11]. Here in addition to obtaining them through a bottom-up approach, we have also correlated them with the measured values of $m_0$ and $\theta_{13}$.

C. Three zeroes in $\tilde{M}$

Using the two-zero textures of $\tilde{M}$ predicted in the last section, there are only two possible three-zero textures that are viable: $(12 - 22 - 33)$ and $(13 - 22 - 33)$. Fig. 9 shows the value of $\tilde{M}(ij - kl - mn)/m_0^2$ for these two cases. As can be seen from the figure, there are no three-zero textures of the adjoint neutrino mass matrix $\tilde{M}$, and hence of $M_M$.

FIG. 8: Selected $\tilde{M}(ij - kl)/m_0^2$ as functions of $m_0$ for (a) normal and (b) inverted ordering of neutrino masses.
FIG. 9: $\tilde{M}(ij - kl - mn)/m_0^2$ as functions of $m_0$, for potentially viable values of $(ij, kl, mn)$ in case of (a) normal as well as (b) inverted ordering of neutrino masses.

V. $\mu - \tau$ AND $S_3$ SYMMETRIES IN $M_\nu$

A. $\mu - \tau$ exchange symmetry in $M_\nu$

A $\mu - \tau$ symmetry in $M_\nu$ implies that, in the notation of (12), we have $d = f$ and $b = c$. The deviations from $\mu - \tau$ symmetry may be then quantified in terms of the dimensionless parameter

$$\Delta_{\mu\tau} = \left| 1 - \frac{d}{f} \right|^2 + \left| 1 - \frac{b}{c} \right|^2. \quad (37)$$

The existence of a $\mu - \tau$ symmetry in $M_\nu$, as per our convention in Sec. II C, requires $\text{Min}(\Delta_{\mu\tau}) < 10^{-2}$. In order to analytically understand the constraints on various parameters, we expand $\Delta_{\mu\tau}$ in terms of the relevant set of small parameters.

For NH, using Eq. (13) one can calculate

$$(1 - d/f)_{\text{NH}} = 1 - e^{2i(\chi_\mu - \chi_\tau)} + O(\tilde{\rho}, \tilde{\epsilon}, \theta_{13}, \tilde{\theta}_{23}),$$

$$(1 - b/c)_{\text{NH}} = 1 - e^{i(\chi_\mu - \chi_\tau)} \left( \frac{2\theta_{13} + e^{i(2\phi_2 + \delta)} \tilde{\rho} \sin(2\theta_{12})}{2\theta_{13} - e^{i(2\phi_2 + \delta)} \tilde{\rho} \sin(2\theta_{12})} \right) + O(\tilde{\rho}, \tilde{\epsilon}, \theta_{13}, \tilde{\theta}_{23}). \quad (38)$$

For $\Delta_{\mu\tau} \approx 0$, we need both the above expressions to vanish, which can only happen if $|\chi_\mu - \chi_\tau| \approx \pi$ and $\theta_{13} \ll \tilde{\rho} \sin(2\theta_{12})$. 

22
In the IH scenario, we get
\begin{align}
(1 - d/f)_{\text{IH}} &= 1 - e^{2i(\chi_\mu - \chi_\tau)} + \mathcal{O}(\tilde{\rho}, \bar{\epsilon}, \theta_{13}, \tilde{\theta}_{23}) , \\
(1 - b/c)_{\text{IH}} &= 1 + e^{i(\chi_\mu - \chi_\tau)} + \mathcal{O}(\tilde{\rho}, \epsilon, \theta_{13}, \tilde{\theta}_{23}) ,
\end{align}
(39)
so that \(|\chi_\mu - \chi_\tau| \approx \pi\) is enough to get \(\Delta_{\mu\tau} \approx 0\) to zeroth order in the relevant small parameter.

![Graph](image1)

**FIG. 10:** The contours of \(\Delta_{\mu\tau}\), the deviation from the \(\mu - \tau\) exchange symmetry in \(M_\nu\), in the plane of \(\sin \theta_{23}\) versus \(\sin \theta_{13}\) for (a) normal and (b) inverted mass ordering, for \(m_0 = 0.03\) eV.

The QD scenario also gives
\begin{align}
(1 - d/f)_{\text{QD}} &= 1 - e^{2i(\chi_\mu - \chi_\tau)} + \mathcal{O}(\tilde{\rho}, \bar{\epsilon}, \theta_{13}, \tilde{\theta}_{23}) , \\
(1 - b/c)_{\text{QD}} &= 1 + e^{i(\chi_\mu - \chi_\tau)} + \mathcal{O}(\tilde{\rho}, \epsilon, \theta_{13}, \tilde{\theta}_{23}) ,
\end{align}
(40)
so that we expect \(\Delta_{\mu\tau} \approx 0\) to be satisfied if \(|\chi_\mu - \chi_\tau| \approx \pi\).

In Fig. 10 we show the numerical results for \(\text{Min}(\Delta_{\mu\tau})\) for hierarchical neutrino masses with both mass orderings. The current data allow for exact \(\mu - \tau\) symmetry whatever the value of \(m_0\) or nature of mass ordering, it will be possible to rule out this symmetry if a significant value of \(\theta_{13}\) is measured at experiments.

### B. \(S_3\) permutation symmetry in \(M_\nu\)

The \(S_3\) permutation symmetry in \(M_\nu\) demands that, in addition to the conditions \(d = f\) and \(b = c\) satisfied by the \(\mu - \tau\) exchange symmetry, one also needs \(b = e\) and \(a = f\). The
deviation from $S_3$ symmetry then can be quantified by the dimensionless parameter

$$\Delta_{S_3} \equiv \left| 1 - \frac{d}{f} \right|^2 + \left| 1 - \frac{b}{c} \right|^2 + \left| 1 - \frac{a}{f} \right|^2 + \left| 1 - \frac{b}{e} \right|^2. \quad (41)$$

Clearly, the $\mu - \tau$ symmetry is a subset of the $S_3$ permutation symmetry. Then the discussion in Sec. [V A] shows that we need the conditions in Eqs. (38), (39) and (40) satisfied in the NH, IH and QD scenarios respectively, to ensure $|1 - d/f| \ll 1$ and $|1 - b/c| \ll 1$. In addition, we need to check if the terms $|1 - a/f|$ and $|1 - b/e|$ can be small quantities.

We shall again analyze the relevant parameter space by considering the analytic expansion in the relevant small parameters as done in the previous section.

In the NH scenario,

$$(1 - a/f)_{\text{NH}} = 1 + O(\bar{\rho}, \bar{\varepsilon}, \theta_{13}, \tilde{\theta}_{23}), \quad (42)$$

$$(1 - b/e)_{\text{NH}} = 1 + O(\rho^2, \epsilon^2, \theta_{13}^2, \tilde{\theta}_{23}^2).$$

None of the above conditions can be satisfied in this region of parameter space, where $\bar{\rho} \ll 1$. Therefore, one cannot have $S_3$ symmetry.

In the IH scenario,

$$(1 - a/f)_{\text{IH}} = 1 - 2e^{2i(\chi_e - \chi_r)} \left( \frac{e^{2i\phi_1} c_{12}^2 + e^{2i\phi_2} s_{12}^2}{e^{2i\phi_1} s_{12}^2 + e^{2i\phi_2} c_{12}^2} \right) + O(\bar{\rho}, \varepsilon, \theta_{13}, \tilde{\theta}_{23}). \quad (43)$$

$$(1 - b/e)_{\text{IH}} = 1 - \sqrt{2}e^{i(\chi_e - \chi_r)} \left( \frac{(e^{2i\phi_1} - e^{2i\phi_2}) c_{12} s_{12}}{e^{2i\phi_1} s_{12}^2 + e^{2i\phi_2} c_{12}^2} \right) + O(\bar{\rho}, \varepsilon, \theta_{13}, \tilde{\theta}_{23}). \quad (44)$$

These conditions correspond to strict correlations between the flavor phases, the Majorana phases and the mixing angle $\theta_{12}$. However, the above two equations cannot be satisfied simultaneously. This can be seen as follows: the simultaneous conditions $a/f \approx 1$ and $b/e \approx 1$ would imply $(b/e)^2 \approx a/f$, or $(b/e)^2 - a/f \approx 0$. However, one gets

$$[(b/e)^2 - a/f]_{\text{IH}} = \frac{2e^{2i(\chi_e - \chi_r + \phi_1 - \phi_2)}}{(e^{2i\phi_2} c_{12}^2 + e^{2i\phi_1} s_{12}^2)^2} + O(\bar{\rho}, \varepsilon, \theta_{13}, \tilde{\theta}_{23}), \quad (45)$$

which does not vanish even to the zeroth order of the small parameters. Thus, the current data do not allow $S_3$ symmetry to hold in the IH scenario.

On the other hand, when neutrinos are quasi-degenerate in mass,

$$(1 - a/f)_{\text{QD}} = 1 - 2e^{2i(\chi_e - \chi_r)} \left( \frac{e^{2i\phi_1} c_{12}^2 + e^{2i\phi_2} s_{12}^2}{1 + e^{2i\phi_1} s_{12}^2 + e^{2i\phi_2} c_{12}^2} \right) + O(\bar{\rho}, \bar{\varepsilon}, \theta_{13}, \tilde{\theta}_{23}). \quad (46)$$

$$(1 - b/e)_{\text{QD}} = 1 + \sqrt{2}e^{i(\chi_e - \chi_r)} \left( \frac{(e^{2i\phi_1} - e^{2i\phi_2}) c_{12} s_{12}}{1 - e^{2i\phi_1} s_{12}^2 - e^{2i\phi_2} c_{12}^2} \right) + O(\bar{\rho}, \bar{\varepsilon}, \theta_{13}, \tilde{\theta}_{23}). \quad (47)$$
FIG. 11: The deviation of $S_3$ symmetry in $M_\nu$ as a function of $m_0$ for (a) normal and (b) inverted mass ordering.

FIG. 12: The contours of $\Delta S_3$, deviation of $S_3$ permutation symmetry in $M_\nu$, is shown in the $\tan \theta_{12} - \sin \phi_1$ plane for (a) normal and (b) inverted mass ordering, and $m_0 = 0.2$ eV.

which also constrain $\theta_{12}, \phi_1, \phi_2, \chi_\epsilon, \chi_\tau$. The above two equations have simultaneous solutions, so the current data allow $S_3$ symmetry to hold in the QD scenario.

Thus, the $m_0$ value determines whether $S_3$ is an allowed symmetry in the normal as well as inverted ordering. Fig. 11 shows the transition between hierarchical and degenerate scenarios where $S_3$ becomes viable. In Fig. 12 we show the deviation from $S_3$ symmetry, which we have quantified in terms of $\Delta S_3$, as a function of $\theta_{12}$ and $\phi_1$. Since these two parameters can
be directly constrained from the neutrinoless double beta decay, these experiments will be crucial for testing the $S_3$ symmetry in neutrinos.

VI. $\mu - \tau$ AND $S_3$ SYMMETRIES IN $M_M$

A. $\mu - \tau$ symmetry in $M_M$

A $\mu - \tau$ exchange symmetry in $M_M$ implies, from (19), that $xyB = xzC$ and $y^2D = z^2F$. Since we are completely ignorant about the Dirac masses $x, y, z$, we cannot use these two conditions separately. However, we can use the Dirac-mass independent combination of these conditions, which gives $(B/C)^2 = (D/F)$. The deviation from this $\mu - \tau$ symmetry relation may be quantified by

$$\tilde{\Delta}_{\mu\tau} \equiv \frac{|B^2}{C^2} - \frac{D}{F}|^2.$$  \hspace{1cm} (48)

Since this condition is less restrictive than that for the $\mu-\tau$ symmetry in $M_\nu$, which itself is consistent with the current data irrespective of the value of $m_0$, one would expect that the $\mu-\tau$ symmetry will hold also for $\tilde{M}$. Indeed, it is found that $\text{Min}(\tilde{\Delta}_{\mu\tau}) \ll 10^{-2}$ for all allowed values of the neutrino parameters. This can be understood as follows. In the NH scenario, the expansion in terms of the small parameters $\theta_{13}, \tilde{\theta}_{23}, \tilde{\epsilon}, \tilde{\rho}$ gives

$$(B^2/C^2 - D/F)_{\text{NH}} = \mathcal{O}(\tilde{\rho}^2, \tilde{\epsilon}^2, \theta_{13}^2, \tilde{\theta}_{23}^2),$$ \hspace{1cm} (49)

where we have used Eq. (16). Note that even terms linear in the small parameters $\theta_{13}, \tilde{\theta}_{23}, \tilde{\epsilon}, \tilde{\rho}$ are absent. This indicates that in this scenario, the $\mu - \tau$ exchange symmetry is easily satisfied. In the IH scenario,

$$(B^2/C^2 - D/F)_{\text{IH}} = -8e^{-2i(\chi_\mu - \chi_\tau)}\tilde{\theta}_{23} + \frac{8\theta_{13}e^{i(2\chi_\mu - 2\chi_\tau - 2\phi_1 - 2\phi_2 - \delta)}}{c_{12}s_{12}\tilde{\rho}(e^{2i\phi_1} - e^{2i\phi_2})},$$ \hspace{1cm} (50)

which vanishes for $\tilde{\theta}_{23}, \theta_{13} \to 0$. In the QD scenario also, $\mu - \tau$ exchange symmetry is easily satisfied since

$$(B^2/C^2 - D/F)_{\text{QD}} = \mathcal{O}(\tilde{\rho}, \epsilon, \theta_{13}, \tilde{\theta}_{23}).$$ \hspace{1cm} (51)

Thus the $\mu - \tau$ symmetry is consistent with the current data irrespective of $m_0$ or the mass ordering of neutrinos.

In the IH scenario, the deviation from the $\mu-\tau$ symmetry would manifest itself in the angles $\theta_{13}$ and $\theta_{23}$. Clearly the symmetry is valid if $\theta_{13} = 0 = \tilde{\theta}_{23}$. If $\theta_{13} = 0$ exactly and

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FIG. 13: The contours of $\tilde{\Delta}_{\mu\tau}$, the deviation from the $\mu-\tau$ permutation symmetry in $M_M$, in the plane of $\sin \theta_{23}$ versus $\sin \theta_{13}$ for (a) normal and (b) inverted mass ordering, for $m_0 = 0.025$ eV.

$\tilde{\theta}_{23} \neq 0$, Eq. (50) indicates that $(B^2/C^2 - D/F)_{\text{NH}}$ cannot vanish, thus the symmetry is not obeyed. However, if the upper bound on $\theta_{13}$ is significantly nonzero, the two terms in Eq. (50) can cancel each other since the $\theta_{13}$ term is enhanced by the factor $(1/\tilde{\rho})$. Thus to satisfy $\mu-\tau$ symmetry in $M_M$, either $\theta_{13} \approx 0 \approx \tilde{\theta}_{23}$, or the upper bound on $\theta_{13}$ should be large. In the latter case, the actual value of $\sin \theta_{13}$ would have to be much smaller than $\tilde{\rho}$ to prevent the second term from becoming too large. These features can be seen from Fig. 13. Indeed, for $\sin \theta_{13} \gtrsim 0.01$, it is possible for $M_M$ to satisfy the $\mu-\tau$ symmetry for any value of $\tilde{\theta}_{23}$. For smaller values of $\theta_{13}$, one needs $\tilde{\theta}_{23} \lesssim 1.5^\circ$.

B. $S_3$ symmetry in $M_M$

The $S_3$ symmetry in $M_M$ implies, from (19), that $xyB = xzC = yzE, x^2A = y^2D = z^2F$. The Dirac-independent conditions that can be obtained are $(B/C)^2 = (D/F)$ as in the case of the $\mu-\tau$ exchange symmetry, and the additional condition $(B/E)^2 = (A/F)$ is required for $S_3$ symmetry to be satisfied in $M_M$. Therefore, the deviation from $S_3$ symmetry can be quantified as:

$$
\tilde{\Delta}_{S_3} = \left| \frac{B^2}{C^2} - \frac{D}{F} \right|^2 + \left| \frac{B^2}{E^2} - \frac{A}{F} \right|^2.
$$

In the NH scenario,

$$(B^2/E^2 - A/F)_{\text{NH}} = \mathcal{O}(\tilde{\rho}, \tilde{\epsilon}, \theta_{13}, \tilde{\theta}_{23}),$$

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so that, combined with Eq. (49), it is conceivable that $S_3$ symmetry holds.

In the IH scenario, we get

\[
(B^2/E^2 - A/F)_{\text{IH}} = \mathcal{O}(\hat{\rho}, \epsilon, \theta_{13}, \tilde{\theta}_{23}).
\]

The extent to which $S_3$ is satisfied then depends only upon the extent to which the $\mu$–$\tau$ symmetry is obeyed. The effect of the deviation from this symmetry on the mixing angles is exactly the same as given in the $\mu$–$\tau$ case. The $S_3$ permutation symmetry in $M_M$ can thus be satisfied for either $\sin \theta_{13} > 0.01$, or $|\tilde{\theta}_{23}| \lesssim 1.5^\circ$ and $\sin \theta_{13} \lesssim 0.01$, as can be seen in Fig. 13. In the latter case, the actual value of $\sin \theta_{13}$ needs to be much smaller than $\hat{\rho}$.

For degenerate neutrino masses,

\[
(B^2/E^2 - A/F)_{\text{QD}} = \frac{2e^{2i(\chi_\nu - \chi_r)}(e^{2i\phi_1} - e^{2i\phi_2})^2 c_{12}^2 s_{12}^2}{(e^{2i\phi_1} c_{12}^2 + e^{2i\phi_2} s_{12}^2 - e^{2i(\phi_1 + \phi_2)})^2} - \frac{(e^{2i\phi_2} c_{12}^2 - e^{2i\phi_1} s_{12}^2)^2}{e^{2i\phi_1} c_{12}^2 + e^{2i\phi_2} s_{12}^2 - e^{2i(\phi_1 + \phi_2)}}
\]

Since at the moment we have no restrictions on the Majorana phases, proper choice of the values of these phases would allow us to make $(B^2/E^2 - A/F)_{\text{QD}}$ vanish. Combining this with Eq. (51), one can claim that it is possible to have $\tilde{\Delta} S_3 \approx 0$ for quasi-degenerate neutrinos.

VII. CONCLUSIONS AND OUTLOOK

We examine texture zeroes and discrete flavor symmetries in the neutrino mass matrix through a bottom-up approach. We develop a formalism that uses the low energy data, checks its consistency with the desired texture or symmetry, and quantifies the deviation from this symmetry. To this end, we parameterize the neutrino mass matrix in terms of the masses, mixing angles, CP violating Dirac and Majorana phases, as well as the “flavor” phases that have no phenomenological implication but may be fixed by the mechanism of neutrino mass generation. We consider both the normal as well as inverted mass ordering of neutrinos, and the cases when the neutrino masses are hierarchical and quasidegenerate.

The results can be described analytically in the three scenarios: normal ordering with hierarchical neutrino masses (NH), inverted ordering with hierarchical neutrino masses (IH), and any mass ordering, but quasidegenerate neutrinos (QD). In each independent scenario, we identify parameters that are small and hence can be used in a perturbative expansion.
This provides a universal framework to analytically understand many results that were only known numerically before. In addition, it allows us to make new predictions about symmetries that can be verified by numerical means.

In the bottom-up approach, the extent to which the symmetries can be tested is limited by the accuracy of the experimental data on neutrino parameters, and hence one can only talk about the symmetry being approximately satisfied, or it being consistent with the data. In order to achieve this, we quantify the deviation from such a symmetry in each case through a quantity that vanishes in the limit of exact symmetry. The minimum value of such a quantity that is consistent with the current data is an indication of the extent to which the symmetry is valid. If this minimum value is zero, the symmetry is clearly obeyed. However, our formalism allows even for quantification of the breaking of the symmetry in a model independent, unified way. For illustration, in each scenario we consider the symmetry to be viable if the relevant quantity is less than $10^{-2}$.

It is found that the viability of texture zeroes strongly depends on the absolute neutrino mass scale, the nature of mass ordering, and the angle $\theta_{13}$. For the neutrino mass matrix $M_\nu$, all the six possible one-zero textures are allowed, though only in certain scenarios. For example, in normal mass ordering one can have only three possible one-zero textures when the neutrino masses are hierarchical, and five one-zero textures if they are quasidegenerate. The inverted ordering allows for four one-zero textures in the hierarchical limit (IH) and five in the quasidegenerate limit. We find that seven two-zero textures are allowed in $M_\nu$, while no three-zero texture is permitted.

The texture zeroes of the inverse mass matrix $M_\nu^{-1}$ correspond to those of the Majorana mass matrix $M_M$ of the heavy neutrinos in the Type-I seesaw mechanism, so we can treat them together. We find that all six one-zero textures are allowed for $M_M$, though only five of them are allowed for normal ordering. For inverted ordering, only three are allowed if the neutrinos have hierarchical masses, while five are allowed when the neutrinos are quasidegenerate. Seven two-zero textures are found to be consistent with the current data, while no three-zero texture is permitted.

All the results with texture zeroes can be understood analytically in terms of the expansion in small parameters that we have introduced in each scenario. In most of the scenarios, the allowed freedom in choosing Dirac and Majorana phases helps us to take the mass matrix closer to the exact symmetry. Wherever one cannot do that, there is a special reason...
that can be understood in terms of our analytic framework. Since the viability of the texture zeroes is very sensitive to the absolute mass scale $m_0$ and the mixing angle $\theta_{13}$, future experiments aimed at determining these quantities will have a large impact, which can be gauged from our results.

We use the same technique to examine flavor symmetries like $\mu-\tau$ exchange and $S_3$. We define quantities involving elements of the $M_\nu$ and $M_M$ matrices that vanish identically for the exact symmetry. In order for the symmetry to be viable, we require this quantity to vanish to the zeroth order of the relevant small parameters.

In $M_\nu$, we find that the $\mu-\tau$ symmetry is viable in IH and QD scenarios as long as the unphysical phases are related by $|\chi_\mu - \chi_\tau| \approx \pi$. In the NH scenario however, one needs $|\chi_\mu - \chi_\tau| \approx \pi$ and $\theta_{13} \ll \left(1 - \Delta m^2_{\text{atm}}/4m_0^2\right)$. As far as the $S_3$ symmetry is concerned, the current data allows it only if the neutrinos are quasidegenerate.

Since the Dirac masses of neutrinos in Type-I seesaw mechanism are unknown, the constraints put by the discrete flavor symmetries on $M_M$ are slightly weaker. As a result, the $\mu-\tau$ exchange symmetry is viable for NH and QD, while in the IH scenario one needs $\theta_{13} \ll |1 + \Delta m^2_{\text{atm}}/(4m_0^2)|$. The $S_3$ symmetry in $M_M$ also holds under these conditions.

This paper provides a universal formalism to analyze the symmetries of neutrino mass matrices and illustrates its applications to a set of symmetries, viz. texture zeroes and discrete flavor symmetries. Currently many of the texture zeroes and flavor symmetries considered here seem to be viable. However, improved data on the mixing angles and masses expected in near future will be able to rule out some of these. Moreover, the formalism developed here can be applied to a much wider class of problems related to the structure of neutrino mass matrix, that can give us more insights into the mechanism of neutrino mass generation.

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