London’s limit for the lattice superconductor

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A stability problem for the current state of the strong coupling superconductor has been considered within the lattice Ginzburg-Landau model. The critical current problem for a thin superconductor film is solved within the London limit taking into account the crystal lattice symmetry. The current dependence on the order parameter modulus is computed for the superconductor film for various coupling parameter magnitudes. The field penetration problem is shown to be described in this case by the one-dimensional sine-Gordon equation. The field distribution around the vortex is described at the same time by the two-dimensional elliptic sine-Gordon equation.

I. INTRODUCTION

An influence of the external magnetic field on the superconducting state near the upper critical field is intensively studied during the last decade. It was observed that a strong magnetic field near the upper critical field point makes superconducting order parameter fluctuations quasione-dimensional and, therefore, strong. Nonrenormalizability of the theory at a strong field leads to impossibility to apply the renormalization group technique. The lattice translation symmetry is understood to be crucial for the renormalizability of the theory. The Ginzburg - Landau effective action is shown to acquire the Harper’s operator form of the kinetic term that restores a three-dimensional nature of fluctuations and renormalizability of the theory. It is important to consider, what physical consequences follow from this form of the kinetic term in the Ginzburg-Landau theory. However, a sophisticated mathematical structure of this operator makes working with it rather difficult. Here we consider an alternative approach to the problem using the quasiclassical approach in the London’s limit of extremely small coherence length.

II. EQUATIONS OF ELECTRODYNAMICS FOR THE NARROW-BAND SUPERCONDUCTOR

We start from the lattice Ginzburg-Landau functional which was derived in and used for a study of the critical behaviour in:

\[ F = \int d^3x \left[ \psi^*(x) \epsilon(-i\hbar \nabla - \frac{2e}{c} A(x)) \psi(x) + \tau \psi^*(x) \psi(x) + \frac{g}{2} (\psi^*(x) \psi(x))^2 + \frac{B^2}{8\pi} \right], \tag{1} \]

where \( B = \text{rot} A \) is the magnetic induction., \( \tau = \alpha (T - T_c) / T_c \) is the mean-field approximation superconducting transition temperature, \( \epsilon(p) \) is a 3d-perodic function of the quasimomentum with periods \( 2\pi/a_1, 2\pi/a_2, 2\pi/a_3 \). Operators \( \nabla \) and \( A(x) \) commute if we choose the Coulomb-London gauge \( \text{div} A = 0 \). This functional can be rewritten in the Wannier

\[ F = \sum_{m, m'} \sum_{m, m'} J_{m, m'} \exp \left[ \frac{2e}{\hbar c} \int_m^{m'} dA(x) \right] \phi^*_m \phi_m + \sum_m \left[ \tau \phi^*_m \phi_m + g (\phi^*_m \phi_m)^2 \right] + \int d^3x \frac{B^2}{8\pi} \tag{2} \]

or Bloch

\[ F = \sum_q \left[ \varphi^*_q \epsilon(q - \frac{2e}{\hbar c} A(i \frac{\partial}{\partial q})) \varphi_q + \tau \varphi^*_q \varphi_q \right] + \sum_{q_1 + q_2 + q_3 + q_4 = 0} \frac{g}{2} \varphi^*_q \varphi_q \tau \varphi^*_q \varphi_q + \int d^3x \frac{B^2}{8\pi} \tag{3} \]

representations. The lattice vectors \( m \) numerate the lattice cites in the plane perpendicular to the magnetic field. Varying Eq. (1) over the vector potential \( A(x) \), we obtain the Maxwell equation

\[ \frac{c}{4\pi} \text{rot} B = j \tag{4} \]
with the following current density

\[ j = \frac{e}{2} \left[ \psi^*(x) v(-i\hbar\nabla - \frac{2e}{c} A(x))\psi(x) + \psi(x) v(i\hbar\nabla - \frac{2e}{c} A(x))\psi^*(x) \right], \tag{5} \]

where \( v(p) = \partial \epsilon / \partial p \) is the group velocity of the order parameter wave packet. Let us present the complex order parameter \( \psi(x) \) in the form

\[ \psi(x) = R(x) \exp(i\phi(x)), \tag{6} \]

where \( R^2 = n_s \) is the density of superconducting electrons. When the coherence length is small in comparison with the field penetration length, \( \tau \) is large that is equivalent to a presence of the small parameter in the kinetic term. Then the order parameter modulus \( R \) can be taken as a spatially uniform solution of Eq. (1):

\[ \tau(T) R_0 = -g R_0^3 \tag{7} \]

almost everywhere. A small parameter before the higher derivative does not guarantee here a complete spatial uniformity: this is just a loophole for the vortex core solution to appear. Assuming the order parameter modulus to be constant and the phase gradient to be slow we come to the expression for the current in the London limit:

\[ j = e n_0^s v(\nabla \phi - \frac{2e}{\hbar c} A(x)), \tag{8} \]

where \( n_0^s = R_0^2 \). Introducing the gauge invariant vector field

\[ A = A - \frac{\hbar c}{2e} \nabla \phi, \tag{9} \]

we can rewrite the Maxwell equation Eq. (4) in the form

\[ \frac{c}{4\pi} \text{rot rot } A = -e R_0^2 v(\frac{2e}{\hbar c} A). \tag{10} \]

Eq. (10) can be easily reduced to the standard London equation

\[ \delta^2 \nabla^2 A - A = 0 \tag{11} \]

with \( \delta^2 = mc^2 / (4\pi e^2 R^2) \) in the continuum approximation limit \( v(p) = p/m \). Eq. (10) will be applied below to some classical problems of superconductivity in order to see, what physical consequences follow from an explicit taking into account the crystal translation symmetry.

III. CRITICAL CURRENT IN THE THIN FILM

We analyze the case of the simple tetragonal lattice with the \( \epsilon(p) \) function of the form

\[ \epsilon(p) = \Delta_\perp (2 - \cos \frac{p_x a}{\hbar} - \cos \frac{p_y a}{\hbar}) + \Delta_\parallel (1 - \cos \frac{p_z b}{\hbar}). \tag{12} \]

Let us consider the film of thickness \( d \) with the current \( j \) parallel to the \( x \)-axis. We assume \( d \ll \xi(T), d \ll \delta(T) \), where \( \xi(T) \) is the coherence length that secure uniformity across the film thickness respectively of the order parameter \( R \) and the current density \( j \). Using Eqs. (7), (11), (6) and (9), we can write the following free energy minimum conditions (neglecting the magnetic field effect as it is usually done in such problems)

\[ j_x = -e R^2 v_x (2e / (\hbar c) A), \tag{13} \]
\[ \epsilon_x(2e/(hc)A)R + \tau R + gR^3 = 0. \]  

Here

\[ \epsilon_x(q_x) = \Delta_\perp (1 - \cos q_xa), \]  

\[ v_x(q_x) = \frac{\partial \epsilon_x(q_x)}{\partial q_x} = a\Delta_\perp/(h) \sin q_xa. \]  

Eliminating \( v_x \) and \( \epsilon_x \) from Eqs. (15) and with a use of the relation

\[ \epsilon_x = \Delta_\perp \left[ 1 - \sqrt{1 - \left( \frac{\hbar v_x}{a\Delta_\perp} \right)^2} \right] \]  

we obtain the following relation between \( j_x \) and \( R \):

\[ \Delta_\perp R \left( 1 - \sqrt{1 - (h/(ea\Delta_\perp))^2 j_x^2/R^4} \right) - |\tau| R + gR^3 = 0. \]  

Introducing dimensionless variables \( J \) and \( f \) according to formulae

\[ j_x = ea\Delta_\perp R_0^2 J/h, \]  

\[ R = fR_0, \]  

where \( R_0^2 = |\tau|/g \) is a non-trivial solution of eq. (7), we arrive at the following result

\[ 1 - \sqrt{1 - J^2/f^4} = k (1 - f^2), \]  

where \( k = |\tau|/\Delta_\perp \). In the limit of \( \hbar v_x/(a\Delta_\perp) \rightarrow 0 \) and, therefore, \( \lambda \rightarrow 0 \), Eq. (24) reduces to the classical form for the continuum approximation:

\[ J^2 = 2kf^4(1 - f^2) \]  

Two plots for the dimensionless current \( J \) dependence on the dimensionless order parameter \( f \) are presented in fig. 1 for the cases of \( k \rightarrow 0 \) and \( k = 1 \). A dependence of the maximum position \( f_m \) on \( k \) is depicted in fig. 2.

IV. FIELD PENETRATION PROBLEM

Let us return to Eq. (10). It can be used in order to determine the vortex structure outside the core. For the case of the kinetic term in the form of Eq. (12) we can express Eq. (10) as following

\[ \frac{c}{4\pi} \text{rotrot}\mathbf{A} = -eR_0^2 \nu(e\mathbf{A}/(hc)), \]  

or, in London’s gauge \( \text{div}\mathbf{A} = 0 \):

\[ \frac{c}{4\pi} \nabla^2 \mathbf{A} - eR_0^2 \nu(A2e/(hc)) = 0. \]  

Assuming the dispersion law Eq. (12) to be valid, we can write Eq. (24) for the \( x \)-component:

\[ \nabla^2 A_x - 4\pi eR_0^2 a\Delta/(hc) \sin(2\pi A_x a/\Phi_0) = 0, \]
where \( \Phi_0 = \frac{2\pi \hbar c}{2e} \) is the Onsager flux quantum. This equation can be reduced for the weak field case to (11). It is convenient to introduce the variable

\[
2\pi A_x a / \Phi_0 = \chi. \tag{26}
\]

Then Eq. (25) can be rewritten in the form

\[
\nabla^2 \chi - \frac{1}{\delta^2} \sin \chi = 0. \tag{27}
\]

Let us consider the classical one-dimensional field penetration problem basing on Eq. (25). In the one-dimensional case it reads

\[
\frac{d^2 \chi}{dx^2} - \frac{1}{\delta^2} \sin \chi = 0. \tag{28}
\]

Eq. (28) is the well-known equation for the pendulum (but with different sign); the one-dimensional penetration is described by the solution

\[
\chi = \arcsin \left\{ \pm cn \left[ \frac{x - x_0}{\kappa \delta}; \kappa \right] \right\}, \tag{29}
\]

where \( cn(x) \) is the elliptic cosine; \( \kappa \) is the elliptic module. Notice that the field penetration problem for the lattice model of a superconductor has a similarity with the fluxon problem in the one-dimensional Josephson contact.

**V. VORTEX SOLUTIONS**

The field distribution around vortices is described by Eq. (27), which is better to rewrite in the form

\[
\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = \frac{1}{\delta^2} \sin \chi. \tag{30}
\]

This is the elliptic sine-Gordon equation. It is well investigated now. Screw dislocations in the elasticity theory, magnetization distribution in easy plane magnets, Berezinskii-Kosterlitz-Thouless transition and other systems can be described by this equation. It has a rich variety of solutions including arrays of vortices in the London limit. Vortex solutions are determined at the plane, punctured in the vicinity of the vortex cores. The N-vortex solution for arbitrary positions of vortices can be written in the extreme London limit \( \delta \to \infty \) as follows

\[
\chi = \sum_k N_k \arctan \frac{x - x_k}{y - y_k}. \tag{31}
\]

Solutions of the nonlinear equation (30) was found, for instance, in. A structure of the solitary vortex can be analyzed if we note that the radial equation

\[
\frac{d^2 \chi}{dr^2} + \frac{1}{r} \frac{d \chi}{dr} = \frac{1}{\delta^2} \sin \chi \tag{32}
\]

belongs to the third Painlevé class. This equation was thoroughly investigated in. Its solutions are shown to have logarithmic asymptotics at \( r \to 0 \) and oscillate at \( r \to \infty \). In conclusion, we have considered an effect of the crystal lattice symmetry and boundness of the kinetic term operator spectrum on electrodynamic properties of the strong coupling superconductor.
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FIG. 1: Current – order parameter relation for the thin film. Thick line – the continuum limit; thin line – the narrow band case with $k = 1$. The maximum position gives the critical current value.
FIG. 2: Current maximum position as a function of $k$.

FIG. 3: Maximum dimensionless current $J$ as a function of $k$. 

FIG. 3: Maximum dimensionless current $J$ as a function of $k$. 