

Comments on: “Optical solitons in a parabolic law media with fourth order dispersion” [Appl. Math. Comput. 208 (2009) 209-302]

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Abstract. Recently, Biswas and Milovic [Appl. Math. Comput. 208 (2009) 209-302] have found optical one-soliton solutions of a fourth order dispersive cubic-quintic nonlinear Schrödinger equation. In this comment, we first show there are mistakes in the paper and demonstrate that the obtained solutions do not satisfy the considered equation. And then we reconstruct a series of analytical exact solutions by means of a direct ansatz method and F-expansion method. These solutions include solitary wave solutions of the bell shape, solitary wave solutions of the kink shape, and periodic wave solutions of Jacobian elliptic function.

1. Analysis of the solutions given in Ref.[1]

As is well known, the investigation for solition solutions of nonlinear Schrödinger equation is always an important and attractive topic. Very recently, Biswas and Milovic[1] considered the higher order dispersive cubic-quintic nonlinear Schrödinger equation,

\[ iq_t + a q_{xx} - b q_{xxxx} + c (|q|^2 + d |q|^4) q = 0, \] (1)

and obtained the optical soliton solution of Eq.(1). However, we find there are mistakes in the paper[1] and the obtained solution does not satisfy Eq.(1).

Biswas et al.[1] first introduced the transformation,

\[ q(x, t) = P(x, t) e^{i(-\kappa x + \omega t + \theta)}, \] (2)

where \( P(x, t) \) is a real function to be determined later, and \( \kappa, \omega \) are real constants. By using the transformation (2), Eq.(1) is converted into a complex differential equation of \( P(x, t), \)

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which the real and imaginary parts read,

$$\frac{\partial P}{\partial t} - 2\kappa (a + 2b\kappa^2) \frac{\partial P}{\partial x} + 4b\kappa \frac{\partial^3 P}{\partial x^3} = 0, \quad (3)$$

$$(\omega + a\kappa^2 + b\kappa^4) P - cP^3 - c d P^5 - (a + 6b\kappa^2) \frac{\partial^2 P}{\partial x^2} + b \frac{\partial^4 P}{\partial x^4} = 0. \quad (4)$$

Then the solution of Eqs. (3)-(4) was supposed as

$$P = \frac{A}{(\lambda + \cosh \tau)^p}, \quad \tau = B(x - \nu t). \quad (5)$$

Substituting Eq. (5) into Eqs. (3)-(4), the authors obtained two expressions with respect to \(\cosh \tau\) and \(\sinh \tau\). We have noticed that there are many mistakes about the expressions (12)-(13) given in Ref. [1]. Equating the coefficients of \(1/(\lambda + \cosh \tau)^{p+j}(j = 0, \cdots, 4)\) of the obtained expressions, the values of \(A, B, \omega, \lambda\) and \(\nu\) were determined.

At last, the authors obtained the optical soliton solution of Eq. (11) as follows,

$$q(x, t) = \frac{A}{\lambda + \cosh(B(x - \nu t))} e^{i(-\kappa x + \omega t + \theta)}, \quad (6)$$

where \(A, B, \omega, \lambda\) and \(\nu\) were given by Eqs. (16)-(21) of Ref. [1].

However, the “solution” (6) does not satisfy Eq. (11). We can note this fact without substituting (6) into Eq. (11). The solitons are the results of a delicate balance between dispersion and nonlinearity, thus it is impossible that the linear partial differential equation (3) admits the bell type solitary wave (5).

To be on the save side we have substituted Eq. (5) with \(p = 1\) into Eq. (3) and have obtained the following expression,

$$E_1 = \left[\frac{\nu + 2a\kappa + 4b\kappa^3 - 4b\kappa B^2}{(\lambda + \cosh \tau)^2} + \frac{24b\kappa B^2 \cosh \tau}{(\lambda + \cosh \tau)^3} - \frac{24b\kappa B^2 \sinh^2 \tau}{(\lambda + \cosh \tau)^4}\right] AB \sinh \tau. \quad (6)$$

We can see that this expression is equal to zero only in two cases. One is \(A = 0\) or \(B = 0\), and the other is \(\kappa = \nu = 0\). This means that the “solution” (6) obtained by Biswas et al. in [1] is not correct.

2. New optical solitary wave solution of Eq. (11)

In the following, we adopt the ansatz solution of Li et al. [2] in the form

$$q(x, t) = E(x, t) e^{i(k x - \omega t + \theta)}, \quad (7)$$

where
where $E(x,t)$ is the complex envelope function, and $k, \omega$ are real constants. Substituting Eq. (7) into Eq. (1) and removing the exponential term, we can rewrite Eq. (1) as

$$i E_t + 2i k (a + 2b k^2) E_x + (a + 6b k^2) E_{xx} - 4i b k E_{xxx} - b E_{xxxx}$$

$$+ (\omega - a k^2 - b k^4) E + c |E|^2 E + c d |E|^4 E = 0.$$  \hspace{1cm} (8)

We now take the complex envelope ansatz function $E(x,t)$ as

$$E(x,t) = i \beta + \lambda \tanh(\xi), \; \xi = p x + s t,$$ \hspace{1cm} (9)

where $\beta, \lambda, p, s$ are real constants. Substituting Eq. (9) into Eq. (8) and setting the coefficients of tanh($\xi$) to zero, one obtains the following algebraic equations:

$$\lambda (dc \lambda^4 - 24 p^4 b) = 0, \; \lambda (\lambda^3 \beta cd + 24 bp^3 k) = 0,$$

$$\lambda (s - 2 \lambda \beta^3 cd - \lambda \beta c + 2 kpa + 32 bp^3 k + 4 bp k^3) = 0,$$

$$\lambda (\lambda^2 c + 2 \lambda^2 \beta^2 cd + 40 p^4 b + 2 p^2 a + 12 k^2 p^2 b) = 0,$$

$$\lambda (2 p^2 a + k^2 a - \beta^4 cd - \beta^2 c + k^4 b + 16 p^4 b - \omega + 12 k^2 p^2 b) = 0,$$

$$\beta bk^4 - \beta \omega - \beta^3 c - 2 ap\lambda k - \beta^5 cd - s\lambda - 4 bp\lambda k^3 - 8 bp^3 k\lambda + \beta ak^2 = 0.$$

Solving it we obtain one set of nontrivial solution,

$$s = 8bpk(k^2 + p^2), \; \beta = -\frac{k \lambda}{p}, \; \omega = 2 p^2 a + 3 k^2 a + 37 k^4 b + 52 k^2 p^2 b + 16 p^4 b,$$

$$c = -\frac{2p^2(30bk^2 + 20bp^2 + a)}{\lambda^2}, \; d = -\frac{12bp^2}{\lambda^2(30bk^2 + 20bp^2 + a)}.$$ \hspace{1cm} (10)

From (7), (9) and (10), we obtain the optical solitary wave of Eq. (1),

$$q_1(x,t) = \left(-\frac{i k \lambda}{p} + \lambda \tanh(p x + 8bpk(k^2 + p^2) t)\right) e^{i \left(k x - (2 p^2 a + 3 k^2 a + 37 k^4 b + 52 k^2 p^2 b + 16 p^4 b) t + \theta\right)},$$

where $p,k$ are determined by the last two identities of Eq. (10). From (9) and (10), the amplitude of the complex envelope function $E(x,t)$ reads,

$$|E(x,t)| = \left\{ \frac{k^2 \lambda^2}{p^2} + \lambda^2 \tanh^2(p x + 8bpk(k^2 + p^2) t) \right\}^{1/2},$$

which may approach nonzero when the time variable approaches infinity.

3. A series of exact solutions for Eq. (11) by using F-expansion method

We suppose that the solution of (11) is of the form

$$q(x,t) = P(\tau) e^{i \eta}, \; \tau = B(x - \nu t), \; \eta = (-\kappa x + \omega t + \theta),$$ \hspace{1cm} (11)
where $P(\tau)$ is a real function, and $B, \nu, \kappa, \omega$ are real constants to be determined. Substituting Eq. (11) to Eq. (1) and separating the real and imaginary parts, one may obtain the following equations,

\begin{equation}
- B(\nu + 2a\kappa + 4b\kappa^3) P' + 4b\kappa B^3 P''' = 0, \tag{12}
\end{equation}

\begin{equation}
(\omega + a\kappa^2 + b\kappa^4) P - cP^3 - cdP^5 - B^2(a + 6b\kappa^2) P'' + bB^4 P'''' = 0. \tag{13}
\end{equation}

The linear ordinary differential equation (12) has no solitary wave solutions, thus we have to take $\kappa = \nu = 0$. In this case Eq. (12) is satisfied identically, and Eq. (13) becomes,

\begin{equation}
\omega P - cP^3 - cdP^5 - c d P^5 - B^2(a + 6b\kappa^2) P'' + bB^4 P'''' = 0. \tag{14}
\end{equation}

Eq. (14) can be solved by using the F-expansion method\cite{3-6}. According to the F-expansion method, we suppose,

\begin{equation}
P(\tau) = \sum_{i=0}^{n} A_i F^i(\tau), \quad A_n \neq 0, \tag{15}
\end{equation}

where $A_i(i = 0, \ldots, n)$ are real constants to be determined, the integer $n$ is determined by balancing the linear highest order term and nonlinear term. And $F(\tau)$ in (15) satisfies,

\begin{equation}
\left( \frac{dF(\tau)}{d\tau} \right)^2 = h_0 + h_2 F(\tau)^2 + h_4 F(\tau)^4, \tag{16}
\end{equation}

where $h_0, h_2, h_4$ are real constants. By balancing the linear highest order derivative term $P'''$ with nonlinear term $P^5$ in Eq. (14), we find $n = 1$. Thus Eq. (15) becomes,

\begin{equation}
P(\tau) = A_0 + A_1 F(\tau). \tag{17}
\end{equation}

Substituting Eq. (17) into Eqs. (14) along with Eq. (16), collecting all terms with the same power of $F^j(\tau)(j = 0, \ldots, 5)$, and equating the coefficients of these terms yields a set of algebraic equations with respect to $A_0, A_1, B, \omega, a, b, c, d, h_0, h_2, h_4$:

\begin{align*}
dcA_1^4A_0 &= 0, \quad \omega A_0 - cA_0^3 - dcA_0^5 = 0, \\
10dcA_1^2A_0^3 + 3cA_1^2A_0 &= 0, \quad 24bA_1B^4h_4^2 - dcA_1^5 = 0, \\
20bA_1B^4h_2h_4 - 10dcA_1^3A_0^2 - cA_1^3 - 2A_1B^2ah_4 &= 0, \\
\omega A_1 - 3cA_1A_0^2 - A_1B^2ah_2 + 12bA_1B^4h_4h_0 + bA_1B^4h_2^2 - 5dcA_1A_0^4 &= 0.
\end{align*}

Solving the above algebraic equations, we have a set of nontrivial solution,

\begin{align*}
A_0 &= 0, \quad A_1 = \pm \sqrt[4]{\frac{12bB^2h_4}{d(10bB^2h_2 - a)}}, \\
\omega &= B^2(ah_2 - 12bB^2h_4h_0 - bB^2h_2^2), \quad c = \frac{d(10bB^2h_2 - a)^2}{6b}. \tag{18}
\end{align*}
Special analytical solutions to Eq. (16) exists for certain choices of the constants $h_0$, $h_2$ and $h_4$. When $h_0 = 1$, $h_2 = -(1 + m^2)$, $h_4 = m^2$, Eq. (16) has the solution $F(\tau) = \text{sn}(\tau, m)$. From Eq. (11) and Eq. (17), Eq. (1) has the Jacobian elliptic sine function solution,

$$q_2(x, t) = \pm \sqrt{\frac{12B^2bm^2}{d(10B^2b + 10B^2bm^2 + a)}} \text{sn}(Bx, m) e^{i(-B^2(B^2b + B^2bm^2 + 14B^2bm^2 + am^2 + a)t + \theta)},$$

where $B$ is determined by $d(10bB^2 + 10m^2bB^2 + a)^2 - 6bc = 0$.

When $h_0 = 1 - m^2$, $h_2 = 2m^2 - 1$, $h_4 = -m^2$, Eq. (16) has the solution $F(\tau) = \text{cn}(\tau, m)$. Inserting it into (17) and using the transformation (11), Eq. (1) has the Jacobian elliptic cosine function solution,

$$q_3(x, t) = \pm \sqrt{\frac{12B^2bm^2}{d(10B^2b - 20B^2bm^2 + a)}} \text{cn}(Bx, m) e^{i(B^2(16B^2bm^2 - 16B^2bm^2 - B^2b - a + 2am^2)t + \theta)},$$

where $B$ is determined by $d(a + 10bB^2 - 20bB^2m^2)^2 - 6bc = 0$.

Some solitary wave solutions can be obtained if the modulus $m$ approaches to 1. For example, when $m \to 1$, the solution $q_2(x, t)$ degenerates to the kink type envelope wave solution,

$$q_4(x, t) = \pm \sqrt{\frac{12bB^2}{d(20B^2b + a)}} \tanh(Bx) e^{i(-2B^2(8B^2b + a)t + \theta)},$$

where $B$ is determined by $d(a + 20bB^2)^2 - 6bc = 0$.

When $m \to 1$, the solution $q_3(x, t)$ degenerates to the bell type envelope wave solution,

$$q_5(x, t) = \pm \sqrt{\frac{12B^2b}{d(a - 10B^2b)}} \text{sech}(Bx) e^{i(B^2(a - B^2b)t + \theta)},$$

where $B$ is determined by $d(a - 10bB^2)^2 - 6bc = 0$.

As pointed out in Ref. [3], Eq. (16) has many other Jacobi elliptic function solutions in terms of $\text{dn}(\xi)$, $\text{ns}(\xi)$, $\text{nd}(\xi)$, $\text{nc}(\xi)$, $\text{sc}(\xi)$, $\text{sd}(\xi)$, $\text{cd}(\xi)$, $\text{dc}(\xi)$ as well as the corresponding solitary wave and trigonometric function solutions. For simplicity, such types of solutions to Eq. (11) are not listed here.

With the aid of Maple, we have checked the solutions $q_j(x, t)(j = 1, \cdots, 5)$ by putting them back into Eq. (11).
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