SYMPLECTIC LEAVES OF 
CALOGERO-MOSER SPACES OF TYPE $G(\ell, 1, n)$ 

by 
RUSLAN MAKSIMAU

Abstract. — We study symplectic leaves of Calogero-Moser spaces of type $G(\ell, 1, n)$.
We prove that the normalization of the closure of each symplectic leaf is isomorphic to some
Calogero-Moser space. We also give a nice combinatorial parameterization of the symplectic leaves.

1. Introduction

This preprint is a part of an unfinished paper. This is a natural continuation of [4]. We
study symplectic leaves of Calogero-Moser spaces of type $G(\ell, 1, n)$ under the assumption
that the parameter $a$ is nonzero.

One of the main results of the paper is Theorem 3.19. There we prove that the nor-
malization of the closure of each symplectic leaf is isomorphic to some Calogero-Moser
space, which confirms a conjecture given in [3]. We also give in §3.11 a nice combinatorial
parameterization of the symplectic leaves.

Gwyn Bellamy and Travis Schedler informed me that they also proved Theorem 3.19
independently. It is expected that this preprint will become a part of a joint paper with
Gwyn Bellamy and Travis Schedler.

2. Combinatorics

2.A. Partitions. — Assume $\ell \in \mathbb{Z}_{\geq 0}\cup\{\infty\}$ and $n \in \mathbb{Z}_{\geq 0}$. A partition is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$
of positive integers (with no fixed length) such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, $r \geq 0$. Set $|\lambda| = \sum_{i=1}^r \lambda_i$. If $|\lambda| = n$, we say that $\lambda$ is a partition of $n$.

Denote by $\mathcal{P}$ (resp. $\mathcal{P}[n]$) be the set of all partitions (resp. the set of all partitions of $n$).
By convention, $\mathcal{P}[0]$ contains one (empty) partition (it has $r = 0$). We will identify parti-
tions with Young diagrams. The partition $\lambda$ corresponds to a Young diagram with $r$ lines
such that the $i$th line contains $\lambda_i$ boxes. For example the partition $(4, 2, 1)$ corresponds to
the Young diagram

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Let us use the following convention: for \( \ell = \infty \) we have \( \mathbb{Z}/\ell \mathbb{Z} = \mathbb{Z} \). We say that a box \( b \) of the Young diagram is at position \((r, s)\) if it is in the line \( r \) and column \( s \). The \( \ell \)-residue of the box \( b \) is the number \( s - r \mod \ell \). (We say that the integer \( s - r \) is the \( \infty \)-residue of the box \( b \)). Then we obtain a map

\[
\text{Res}_\ell : \mathcal{P} \to \mathbb{Z}/\ell \mathbb{Z}, \quad \lambda \mapsto \text{Res}_\ell(\lambda),
\]

such that for each \( i \in \mathbb{Z}/\ell \mathbb{Z} \) the number of boxes with \( \ell \)-residue \( i \) in \( \lambda \) is \( (\text{Res}_\ell(\lambda))_i \). (In particular, we obtain a map \( \text{Res}_\infty : \mathcal{P} \to \mathbb{Z} \).) For \( \ell = \infty \), we mean that \( \mathbb{Z}/\ell \mathbb{Z} = \mathbb{Z} \) is the direct sum (and not the direct product) of \( \mathbb{Z} \) copies of \( \mathbb{Z} \). In other words, our convention is that for an element \( d = (d_i)_{i \in \mathbb{Z}} \in \mathbb{Z}, \) only a finite number of integers \( d_i \) is nonzero.

**Example 2.1.** — For the partition \( \lambda = (4, 2, 1) \) and \( \ell = 3 \) the 3-residues of the boxes are

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
\end{array}
\]

In this case we have \( \text{Res}_3(\lambda) = (3, 2, 2) \) because there are three boxes with residue 0, two boxes with residue 1 and two boxes with residue 2. ■

We say that a box of a Young diagram is *removable* if it has no boxes on the right and on the bottom. In other words, a box \( b \) is removable for \( \lambda \) if \( \lambda \setminus b \) is still a Young diagram. We say that a box \( b \) is *addable* for \( \lambda \) if \( b \) is not a box of \( \lambda \) and \( \lambda \cup b \) is still a Young diagram. For \( i \in \mathbb{Z}/\ell \mathbb{Z} \), we say that a box is \( i \)-addable or respectively \( i \)-removable if it is an addable or respectively removable box with \( \ell \)-residue \( i \).

For \( \lambda, \mu \in \mathcal{P} \), we write \( \mu \leq \lambda \) if the Young diagram of \( \mu \) can be obtained from the Young diagram of \( \lambda \) by removing a sequence of removable boxes.

2.B. \( \ell \)-cores. — Assume \( \ell \in \mathbb{Z}_{>0} \).

**Definition 2.2.** — We say that the partition \( \lambda \) is an \( \ell \)-core if there is no partition \( \mu \leq \lambda \) such that the Young diagram of \( \mu \) differs from the Young diagram of \( \lambda \) by \( \ell \) boxes with \( \ell \) different \( \ell \)-residues.

See [2] for more details about the combinatorics of \( \ell \)-cores. Let \( \mathcal{C}_\ell \subset \mathcal{P} \) be the set of \( \ell \)-cores. Set \( \mathcal{C}_\ell[n] = \mathcal{P}[n] \cap \mathcal{C}_\ell \).

If a partition \( \lambda \) is not an \( \ell \)-core, then we can get a smaller Young diagram from its Young diagram by removing \( \ell \) boxes with different \( \ell \)-residues. We can repeat this operation again and again until we get an \( \ell \)-core. It is well-known, that the \( \ell \)-core that we get is independent of the choice of the boxes that we remove. Then we get an application

\[
\text{Core}_\ell : \mathcal{P} \to \mathcal{C}_\ell.
\]

If \( \mu = \text{Core}_\ell(\lambda) \), we will say that the partition \( \mu \) is the \( \ell \)-core of the partition \( \lambda \).

**Example 2.3.** — The partition \( (4, 2, 1) \) from the previous example is not a 3-core because it is possible to remove three bottom boxed. We get

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
\end{array}
\]

But this is still not a 3-core because we can remove three more boxes and we get

\[
\begin{array}{c}
0 \\
\end{array}
\]

This shows that the partition \( (1) \) is the 3-core of the partition \( (4, 2, 1) \). ■
Let $\delta_{t}$ denote the constant family $\delta_{t}=(1)_{i\in\mathbb{Z}/t\mathbb{Z}}\in\mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$.

**Remark 2.4.** Assume that we have $\mu=\text{Core}_{q}(\lambda)$ and $\mu$ is obtained from $\lambda$ by removing $rt$ boxes. Then we have $\text{Res}_{y}(\lambda)=\text{Res}_{y}(\mu)+r\delta_{t}$. In particular, if we have two partitions $\lambda_{1}$ and $\lambda_{2}$ with the same $\ell$-cores and such that $|\lambda_{1}|=|\lambda_{2}|$, then they have the same $\ell$-residues. More generally, if two partition $\lambda_{1}$ and $\lambda_{2}$ have the same $\ell$-cores then we have $\text{Res}_{y}(\lambda_{1})=\text{Res}_{y}(\lambda_{2})+r\delta_{t}$, where $r=(|\lambda_{1}|-|\lambda_{2}|)/\ell$.

For $u\in\mathcal{C}_{t}$, set $\mathcal{P}_{u}=\{\lambda \in \mathcal{P}; \text{Core}_{q}(\lambda)=u\}$ and $\mathcal{P}_{u}[n]=\mathcal{P}_{u} \cap \mathcal{P}[n]$.

**2.C. Action of the affine Weyl group.** Assume $\ell \in \mathbb{Z}_{\geq 0}$. Let $W_{t}^{\text{aff}}$ denote the affine Weyl group of type $A_{t-1}$. For $\ell \geq 2$ it is the Coxeter group with associated Coxeter system $(W_{\ell}^{\text{aff}}, S_{\ell}^{\text{aff}})$, where $S_{\ell}^{\text{aff}}=\{s_{i} | i \in \mathbb{Z}/t\mathbb{Z}\}$ and the Coxeter graph whose vertices are elements of $\mathbb{Z}/t\mathbb{Z}$ and we have an edge between $i$ and $i+1$ for each $i \in \mathbb{Z}/t\mathbb{Z}$. We also extend this notion to the case $\ell = 1$ by setting $W_{1}^{\text{aff}} = 1$. We denote by $l$ the length function $l: W_{t}^{\text{aff}} \to \mathbb{Z}_{\geq 0}$.

The non-affine Weyl group $W_{t}$ (isomorphic to the symmetric group $S_{t}$) is a parabolic subgroup of $W_{t}^{\text{aff}}$ generated by $s_{1}, \ldots, s_{t-1}$ (for $\ell = 1$ we mean that $W_{t} = 1$).

Consider the Lie algebra $\mathfrak{g}_{t} = \mathfrak{sl}_{t}((\mathbb{C})$ and its affine version $\tilde{\mathfrak{g}}_{t} = \tilde{\mathfrak{g}}_{t}(\mathbb{C}) = \mathfrak{sl}_{t}((\mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}^{1} \oplus \mathbb{C}\partial$. Let $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}_{t}$ be the Cartan subalgebra formed by the diagonal matrices and set $\tilde{\mathfrak{h}}' = \tilde{\mathfrak{h}} \oplus \mathbb{C}^{1} \oplus \mathbb{C}\partial$.

The $\mathbb{C}$-vector space $\tilde{\mathfrak{h}}^{*}$ has a basis $(a_{0}, a_{1}, \ldots, a_{t-1}, A_{0})$, where $a_{0}, a_{1}, \ldots, a_{t-1}$ are the simple roots of $\tilde{\mathfrak{h}}_{t}$ and $A_{0}$ is such that $A_{0}$ annihilates $\mathfrak{h}$ and $\partial$ and $A_{0}(1) = 1$. Denote by $R_{t}^{\text{aff}}$ and $R_{t}$ the affine and the non-affine root lattices respectively (i.e., $R_{t}^{\text{aff}}$ is the $\mathbb{Z}$-lattice generated by $a_{0}, a_{1}, \ldots, a_{t-1}$ and $R_{t}$ is the sublattice generated by $a_{1}, \ldots, a_{t-1}$).

Following [10], we define two actions of $W_{t}^{\text{aff}}$: a non-linear one on $\mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$, and a linear one on $\mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$. If $\ell = 1$, there is nothing to define so we may assume that $\ell \geq 2$. If $d = (d_{i})_{i\in\mathbb{Z}/t\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$ and if $j \in \mathbb{Z}/t\mathbb{Z}$, we set $s_{j}(d) = (d'_{i})_{i\in\mathbb{Z}/t\mathbb{Z}}$, where

$$
d'_{i} = \begin{cases} 
d_{i} & \text{if } i \neq j, \\
\delta_{i} & \text{if } i = j.
\end{cases}
$$

**Remark 2.5.** We can identify $\mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$ with the root lattice $R_{t}^{\text{aff}}$ by $d \mapsto \sum_{i\in\mathbb{Z}/t\mathbb{Z}} d_{i} a_{i}$. Under this identification the element $\delta_{t} \in \mathbb{Z}^{\mathbb{Z}/t\mathbb{Z}}$ corresponds to the imaginary root of $R_{t}^{\text{aff}}$ that we also denote by $\delta_{t}$.

Beware, the action considered here is not the usual action of $W_{t}^{\text{aff}}$ on the root lattice. When we have $w(d) = d'$ with respect to this action define above, this corresponds to $w(A_{0} = d) = A_{0} + d'$ for the usual action of $W_{t}^{\text{aff}}$ on $\tilde{\mathfrak{h}}^{*}$. \(\blacksquare\)

If $\theta = (\theta_{i})_{i\in\mathbb{Z}/t\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}/t\mathbb{Z}}$, we set $s_{j}(\theta) = (\theta'_{i})_{i\in\mathbb{Z}/t\mathbb{Z}}$, where

$$
\theta'_{i} = \begin{cases} 
\theta_{i} & \text{if } i \neq j \text{ or } i = j+1, \\
\theta_{i} + \theta_{j} & \text{if } i \in \{j-1, j+1\}, \\
-\theta_{i} & \text{if } i = j.
\end{cases}
$$
It is readily seen that these definitions on generators extend to an action of the whole group \( W^{\text{aff}}_\ell \). We also define a pairing \( \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \times \mathcal{C}^{\mathbb{Z}/\ell\mathbb{Z}} \to \mathcal{C}, (\mathbf{d}, \theta) \mapsto \mathbf{d} \cdot \theta \), where

\[
\mathbf{d} \cdot \theta = \sum_{i \in \mathbb{Z}/\ell\mathbb{Z}} d_i \theta_i.
\]

Then

\[
s_f(\mathbf{d}) \cdot s_f(\theta) = (\mathbf{d} \cdot \theta) - \delta_{f0} \theta_0.
\]

**Remark 2.6.** — [2] Sec. 3] defined an \( W^{\text{aff}}_\ell \)-action on \( \mathcal{C}_\ell \). Let us recall this construction. Fix \( i \in \mathbb{Z}/\ell\mathbb{Z} \) and \( v \in \mathcal{C}_\ell \).

1. Assume that \( v \) has neither \( i \)-removable boxes nor \( i \)-addable boxes, then we have \( s_i(v) = v \).
2. Assume that \( v \) has no \( i \)-removable boxes and has at least one \( i \)-addable box. Then \( s_i(v) \) is obtained from \( v \) by addition of all \( i \)-addable boxes.
3. Assume that \( v \) has no \( i \)-addable boxes and has at least one \( i \)-removable box. Then \( s_i(v) \) is obtained from \( v \) by removing of all \( i \)-removable boxes.
4. The situation when the \( \ell \)-core \( v \) has an \( i \)-addable box and an \( i \)-removable box at the same time is impossible.

By construction, the map \( \text{Res}_i : \mathcal{C}_\ell \to \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) is \( W^{\text{aff}}_\ell \)-invariant. Moreover, the \( \ell \)-residue of the empty partition is zero. The stabilizer of the empty partition in \( W^{\text{aff}}_\ell \) is \( W_\ell \) and the stabilizer of \( 0 \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) in \( W^{\text{aff}}_\ell \) is also \( W_\ell \). This implies that we have \( W^{\text{aff}}_\ell \)-invariant bijections

\[
W^{\text{aff}}_\ell / W_\ell \cong \mathcal{C}_\ell \cong W^{\text{aff}}_\ell \cdot 0 \subset \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}
\]

Since \( \text{Res}_i \) is a \( W^{\text{aff}}_\ell \)-invariant map and \( \text{Res}_i(0) = 0 \), then the bijection \( \mathcal{C}_\ell \cong W^{\text{aff}}_\ell \cdot 0 \) is given by the map \( \text{Res}_i \). In particular, we see that an element \( \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) is a residue of an \( \ell \)-core if and only if it is in the \( W^{\text{aff}}_\ell \)-orbit of 0.

Moreover, since we have \( w(\mathbf{d} + n\delta_\ell) = w(\mathbf{d}) + n\delta_\ell \) and since each \( W^{\text{aff}}_\ell \)-orbit in \( \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) contains exactly one element of the form \( n\delta_\ell \) (see [4] Lem. 2.8]), each element \( \mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) has a unique presentation in the form

\[
(2.6) \quad \mathbf{d} = \text{Res}_i(v) + n\delta_\ell, \quad v \in \mathcal{C}_\ell, n \in \mathbb{Z}.
\]

The following lemma is a reformulation of [2] Remark 3.2.3.

**Lemma 2.7.** — Fix \( v \in \mathcal{C}_\ell \) and \( i \in \mathbb{Z}/\ell\mathbb{Z} \). Let \( w \) be the unique element of \( W^{\text{aff}}_\ell \) such that \( w(0) = v \) and such that \( w \) is the shortest element in the coset \( w W_\ell \in W^{\text{aff}}_\ell / W_\ell \). The the situations (1), (2), (3) in Remark 2.6 are equivalent to the following situations (1), (2), (3) respectively:

1. \( s_i w \in w W_\ell \) and \( I(s_i w) > I(w) \),
2. \( s_i w \notin w W_\ell \) and \( I(s_i w) > I(w) \),
3. \( s_i w \notin w W_\ell \) and \( I(s_i w) < I(w) \).

For \( \mathbf{d} = (d_i)_{i \in \mathbb{Z}/\ell\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \) we set \( |\mathbf{d}| = \sum_{i \in \mathbb{Z}/\ell\mathbb{Z}} d_i \).
2.D. Another presentation of the affine Weyl group. — Recall that the affine Weyl group has another presentation. We have \( W_\text{aff}^\ell = W_\ell \ltimes R_\ell \). For each \( \alpha \in R_\ell \), denote by \( t_\alpha \) the image of \( \alpha \) in \( W_\text{aff}^\ell \). Each element of \( W_\text{aff}^\ell \) can be written in a unique way in the form \( w \cdot t_\alpha \), where \( w \in W_\ell \) and \( \alpha \in R_\ell \). We can also extend the notation \( t_\alpha \) to \( \alpha \in R_\text{aff}^\ell \) by setting \( t_\alpha := t_\pi(\alpha) \) for each \( \alpha \in R_\text{aff}^\ell \), where \( \pi \) is the following map \( \pi: R_\text{aff}^\ell \to R_\text{aff}^\ell / Z_{\ell} \cong R_\ell \).

In the following lemma we identify \( Z_{\ell} / Z_{\ell} \) with \( R_\text{aff}^\ell \).

**Lemma 2.8.** — Assume \( \alpha \in R_\ell \) and \( d \in Z_{\ell} / Z_{\ell} \). Then we have \( t_\alpha(d) \equiv d - \alpha \mod Z_{\ell} \).

**Proof.** — This statement is a partial case of [9, (6.5.2)] (see also Remark 2.5). \( \square \)

Consider the \( \mathbb{Z} \)-linear map \( R_\ell^\text{aff} \to \mathbb{C}^{Z_{\ell} / Z_{\ell}}, \ d \to \overline{d} \), given by

\[
(\overline{\alpha})_i = 2 \delta_{i,r} - \delta_{i,r+1} - \delta_{i,r-1}.
\]

The kernel of this map is \( Z_{\ell} \). Set

\[
\Sigma(\theta) = \sum_{i \in Z_{\ell} / Z_{\ell}} \theta_i.
\]

**Lemma 2.9.** — For each \( \alpha \in R_\ell \) and \( \theta \in \mathbb{C}^{Z_{\ell} / Z_{\ell}} \), we have \( t_\alpha(\theta) = \theta + \Sigma(\theta) \overline{\alpha} \).

**Proof.** — The \( W_\ell^\text{aff} \)-action on \( \mathbb{C}^{Z_{\ell} / Z_{\ell}} \) defined above coincides with the (usual) action of \( W_\ell^\text{aff} \) on the dual of the span of \( a_0, a_1, \ldots, a_{\ell-1} \) in \( \hat{h}^* \). The statement follows from [9, (6.5.2)]. \( \square \)

2.E. \( J \)-cores. — Fix a subset \( J \subset Z / \ell \mathbb{Z} \).

**Definition 2.10.** — We say that a box of a Young tableau is \( J \)-removable if it is removable and its residue is in \( J \). We say that a Young tableau is a \( J \)-core if it has no \( J \)-removable boxes. Denote by \( \mathcal{C}_J \) the set of all \( J \)-cores.

To each partition \( \lambda \in \mathcal{P} \) we can associate a partition \( \text{Core}_J(\lambda) \in \mathcal{C}_J \) obtained from it by removing \( J \)-removable boxes (probably in several steps). The result \( \text{Core}_J(\lambda) \) does not depend on the order of operations.

**Lemma 2.11.** — For each \( \mu \in \mathcal{C}_J \), we have \( \text{Core}_J(\mu) \in \mathcal{C}_J \).
Proof. — This statement is quite obvious when we see the partition \( \mu \) as an abacus, see for example [2, §2] for then definition of an abacus.

However we can give another proof based on the representation theory of quivers and the results of §3. Fix some \( J \)-standard \( \theta \in \mathbb{C}Z/\mathbb{Z} \). Then, since \( \nu \) is a \( J \)-core, the representation \( A_\mu \) constructed in §3.G is simple by Lemma 3.22. Then the dimension vector \( \text{Res}_\ell(\mu) \) of this representation is in \( E_0 \).

Now, let \( \nu \) be the \( \ell \)-core of \( \mu \). Assume that \( \nu \) is obtained from \( \mu \) by removing \( r \ell \) boxes. The we have \( \text{Res}_\ell(\mu) = \text{Res}_\ell(\nu) + r \ell \in E_0 \). Now, Lemma 3.30 implies that \( \nu \) is a \( J \)-core. \( \square \)

3. Preliminaries on quiver varieties

By an algebraic variety, we mean a reduced scheme of finite type over \( \mathbb{C} \).

3.A. Quiver varieties. — Assume \( \ell \in \mathbb{Z}_{>0} \cup \{ \infty \} \).

Let \( Q_\ell \) denote the cyclic quiver with \( \ell \) vertices, defined as follows:
- Vertices: \( i \in \mathbb{Z}/\ell \mathbb{Z} \) (recall that we use the convention that for \( \ell = \infty \) we have \( \mathbb{Z}/\ell \mathbb{Z} = \mathbb{Z} \)).
- Arrows: \( y_i : i \to i+1, i \in \mathbb{Z}/\ell \mathbb{Z} \).

We denote by \( Q_\ell \) the double quiver of \( Q_\ell \) that is, the quiver obtained from \( Q_\ell \) by adding an arrow \( x_i : i+1 \to i \) for all \( i \in \mathbb{Z}/\ell \mathbb{Z} \).

Now, let \( d = (d_i)_{i \in \mathbb{Z}/\ell \mathbb{Z}} \) be a family of elements of \( \mathbb{Z}_{>0} \). (For \( \ell = \infty \) we always assume additionally that \( d \) has a finite number of nonzero components.)

Let \( \text{Rep}(Q_\ell, d) \) be the variety of representations of \( Q_\ell \) in the family of vector spaces \( (\mathbb{C}^{d_i})_{i \in \mathbb{Z}/\ell \mathbb{Z}} \). More precisely, we have \( \text{Rep}(Q_\ell, d) = \bigoplus_{i \in \mathbb{Z}/\ell \mathbb{Z}} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_i}) \oplus \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_i}) \). An element of \( \text{Rep}(Q_\ell, d) \) is a couple \((X, Y)\) where

\[
X = (X_i)_{i \in \mathbb{Z}/\ell \mathbb{Z}}, \quad X_i \in \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_i}) \quad Y = (Y_i)_{i \in \mathbb{Z}/\ell \mathbb{Z}}, \quad Y_i \in \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_i}).
\]

We denote by \( \text{GL}(d) \) the direct product

\[
\text{GL}(d) = \prod_{i \in \mathbb{Z}/\ell \mathbb{Z}} \text{GL}_{d_i}(\mathbb{C}),
\]

The group \( \text{GL}(d) \) acts on \( \text{Rep}(Q_\ell, d) \). The orbits are the isomorphism classes of representations of \( Q_\ell \) of dimension vector \( d \). We denote by

\[
\mu_d : \text{Rep}(Q_\ell, d) \to \bigoplus_{i \in \mathbb{Z}/\ell \mathbb{Z}} \text{End}(\mathbb{C}^{d_i})
\]

\[
(X_i, Y_i)_{i \in \mathbb{Z}/\ell \mathbb{Z}} \mapsto (X_i Y_i - Y_{i-1} X_{i-1})_{i \in \mathbb{Z}/\ell \mathbb{Z}}
\]

the corresponding moment map. Finally, if \( \theta = (\theta_i)_{i \in \mathbb{Z}/\ell \mathbb{Z}} \) is a family of complex numbers, we denote by \( I_\theta(d) \) the family \((\theta_i \text{Id}_{\mathbb{C}^{d_i}})_{i \in \mathbb{Z}/\ell \mathbb{Z}} \). Finally, we set

\[
\mathcal{Y}_\theta^0(d) = \mu_d^{-1}(I_\theta(d)) \quad \text{and} \quad \mathcal{X}_\theta^0(d) = \mathcal{Y}_\theta^0(d) / \text{GL}(d).
\]

Note that the variety \( \mathcal{X}_\theta^0(d) \) is not empty only in the case \( d \cdot \theta = 0 \). Note that \( \mathcal{Y}_\theta^0(d) \) is endowed with a \( \mathbb{C}^\times \)-action: if \( \xi \in \mathbb{C}^\times \), we set

\[
\xi \cdot (X, Y) = (\xi^{-1} X, \xi Y).
\]

This action commutes with the action of \( \text{GL}(d) \) and the moment map is constant on \( \mathbb{C}^\times \) -orbits, so it induces a \( \mathbb{C}^\times \)-action on \( \mathcal{X}_\theta^0(d) \).
Now, we give a framed version $\mathcal{X}_\theta(d)$ of the variety $\mathcal{X}_\theta^0(d)$. Let $\widehat{Q}_\ell$ be the quiver obtained from $Q_\ell$ by adding a new vertex $\infty$ and arrows $0 \to \infty$ and $\infty \to 0$.

For each dimension vector $d$ for the quiver $Q$ we consider the dimension vector $\widehat{d}$ such that $\widehat{d}$ has dimension 1 at the vertex $\infty$ and the same dimension as $d$ for other vertices.

Let $\text{Rep}(\widehat{Q}_\ell, \widehat{d})$ be the variety of representations of $\widehat{Q}_\ell$ with dimension vector $\widehat{d}$. More precisely, we have

$$\text{Rep}(\widehat{Q}_\ell, \widehat{d}) = \text{Hom}(C^{\hat{d}_0}, C) \oplus \text{Hom}(C, C^{\hat{d}_0}) \oplus \bigoplus_{i \in \mathbb{Z}/\ell \mathbb{Z}} \text{Hom}(C^{\hat{d}_i}, C^{\hat{d}_{i+1}}) \oplus \text{Hom}(C^{\hat{d}_{i+1}}, C^{\hat{d}_i}).$$

An element of $\text{Rep}(\widehat{Q}_\ell, \widehat{d})$ is of the form $(X, Y, x, y)$ where

$$X = (X_i)_{i \in \mathbb{Z}/\ell, \mathbb{Z}}, \quad X_i \in \text{Hom}(C^{\hat{d}_{i+1}}, C^{\hat{d}_i}), \quad Y = (Y_i)_{i \in \mathbb{Z}/\ell, \mathbb{Z}}, \quad Y_i \in \text{Hom}(C^{\hat{d}_i}, C^{\hat{d}_{i+1}}),$$

$$x \in \text{Hom}(C, C^{\hat{d}_0}), \quad y \in \text{Hom}(C^{\hat{d}_0}, C).$$

The group $GL(d)$ acts on $\text{Rep}(\widehat{Q}_\ell, \widehat{d})$. The orbits are the isomorphism classes of representations of $\widehat{Q}_\ell$ of dimension vector $\widehat{d}$. We denote by

$$\widehat{\mu}_d: \text{Rep}(\widehat{Q}_\ell, \widehat{d}) \rightarrow \bigoplus_{i \in \mathbb{Z}/\ell \mathbb{Z}} \text{End}(C^{\hat{d}_i}) (X_i, Y_i, x, y)_{i \in \mathbb{Z}/\ell \mathbb{Z}} \rightarrow (X_i Y_i - Y_i X_{i-1} + \delta_i, 0, x, y)_{i \in \mathbb{Z}/\ell \mathbb{Z}}$$

the corresponding moment map. Finally, we set

$$\mathcal{Y}_\theta(d) = \widehat{\mu}_d^{-1}([I_\theta(d)]) \quad \text{and} \quad \mathcal{Y}_\theta^0(d) = \mathcal{Y}_\theta^0(d)/GL(d).$$

Note that in the case $d \cdot \theta = 0$ we have an obvious isomorphism $\mathcal{X}_\theta(d) = \mathcal{X}_\theta^0(d)$. Note that $\mathcal{Y}_\theta(d)$ is endowed with a $C^\times$-action: if $\xi \in C^\times$, we set

$$\xi \cdot (X, Y, x, y) = (\xi^{-1} X, \xi Y, x, y).$$

This action commutes with the action of $GL(d)$ and the moment map is constant on $C^\times$-orbits, so it induces a $C^\times$-action on $\mathcal{Y}_\theta^0(d)$.

**Remark 3.1.** — We extend the definition of $\mathcal{X}_\theta(d)$ to the case where $d \in \mathbb{Z}/\ell \mathbb{Z}$ by the convention that $\mathcal{X}_\theta(d) = \emptyset$ whenever at least one of the $d_j$’s is negative.

Let $\text{Rep}(\widehat{Q}_\ell)$ be the category of representations of the quiver $\widehat{Q}_\ell$. We can see each element of $\text{Rep}(\widehat{Q}_\ell, d)$ as an object in $\text{Rep}(\widehat{Q}_\ell)$ with dimension vector $\widehat{d}$. Now, assume $\ell \in \mathbb{Z}_{\geq 0}$.

**Definition 3.2.** — Consider the following map $\iota: \text{Rep}(\widehat{Q}_\infty) \rightarrow \text{Rep}(\widehat{Q}_\ell)$.

For each finite dimensional representation $(X, Y, x, y)$ of $\widehat{Q}_\infty$ in the vector space $V = \bigoplus_{j \in \mathbb{Z}} V_j$ we can associate a representation $(X', Y', x', y')$ of $\widehat{Q}_\ell$ in the vector space $V' = \bigoplus_{j \in \mathbb{Z}/\ell \mathbb{Z}} V'_j$ where

$$V'_i = \bigoplus_{j \equiv i \mod \ell} V_j, \quad X'_i = \bigoplus_{j \equiv i \mod \ell} X_j, \quad Y'_i = \bigoplus_{j \equiv i \mod \ell} Y_j,$$

$x'$ is the composition of $x$ with the natural map $V_0 \rightarrow V'_0$, $y'$ is the composition of $y$ with the natural map $V'_0 \rightarrow V_0$. 
3.B. Lusztig's isomorphism. — We use the $W^\text{aff}_t$-actions on $\mathbb{Z}/t\mathbb{Z}$ and $\mathbb{C}[\mathbb{Z}/t\mathbb{Z}]$ defined in Section 2.C.

It is proved in [10] Corollary 3.6 that

\[(3.3) \quad \mathcal{X}_{\delta_j(\theta)}(\mathcal{J}_j(d)) \cong \mathcal{X}_{\theta}(d) \quad \text{if } \theta_j \neq 0.\]

Note that this isomorphism takes into account the convention of Remark 3.1.

The isomorphism above motivates to consider the following equivalence relation on the set $\mathbb{Z}/t\mathbb{Z} \times \mathbb{C}[\mathbb{Z}/t\mathbb{Z}]$. Let $\sim$ be the transitive closure of

\[(d, \theta) \sim (s(d), s(\theta)), \quad \theta_j \neq 0.\]

The isomorphism (3.3) implies that if $(d, \theta) \sim (d', \theta')$, then we have an isomorphism of algebraic varieties $\mathcal{X}_d(d) \cong \mathcal{X}_d(d')$.

Remark 3.4. — Let $W_\theta$ be the stabilizer of $\theta$ in $W^\text{aff}_t$. Assume that $\theta$ is such that $W_\theta$ is a parabolic subgroup of $W^\text{aff}_t$. Then we can describe the set of couples that are equivalent to $(d, \theta)$ in the following way. They are of the form $(w(d), w(\theta))$ where $w$ is the element of $W^\text{aff}_t$ such that $w$ is the shortest element in the class $wW_\theta \in W^\text{aff}_t/W_\theta$.

3.C. Calogero-Moser space. — We fix a $\mathbb{C}$-vector space $V$ of finite dimension $n$ and a finite subgroup $W$ of $\mathbf{GL}_\mathbb{C}(V)$. We set

\[\text{Ref}(W) = \{ s \in W \mid \text{dim}_\mathbb{C} V^s = n - 1 \}\]

and we assume that $W = \langle \text{Ref}(W) \rangle$.

We set $\varepsilon : W \to \mathbb{C}^\ast$, $w \mapsto \det(w)$. If $s \in \text{Ref}(W)$, we denote by $\alpha_s^\vee$ and $\alpha_s$ two elements of $V$ and $V^\vee$ respectively such that $V^s = \ker(\alpha_s)$ and $V^\ast_s = \ker(\alpha_s^\vee)$, where $\alpha_s^\vee$ is viewed as a linear form on $V^\ast$.

Let us fix a function $c : \text{Ref}(W) \to \mathbb{C}$ which is invariant under conjugacy. We define the $\mathbb{C}$-algebra $\mathbf{H}_c$ to be the quotient of the algebra $T(V \oplus V^\ast) \rtimes W$ (the semi-direct product of the tensor algebra $T(V \oplus V^\ast)$ with the group $W$) by the relations

\[
\begin{cases}
[x, x'] = [y, y'] = 0, \\
[x, y] = \sum_{s \in \text{Ref}(W)} (e(s) - 1)c_s \frac{\langle y, \alpha_s \rangle (\alpha_s^\vee, x)}{\langle \alpha_s^\vee, \alpha_s \rangle} s,
\end{cases}
\]

for all $x, x', y, y' \in V^\ast$. $y, y' \in V$. The algebra $\mathbf{H}_c$ is called the rational Cherednik algebra at $t = 0$.

The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_c$ and $\mathbb{C}[V^\ast] \to \mathbf{H}_c$.

We denote by $Z_\mathbf{H}_c$ the center of $\mathbf{H}_c$: it is well-known [7] Lemma 3.5 that $Z_\mathbf{H}_c$ is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^\ast]^W$ as subalgebras (so it contains $P = \mathbb{C}[V]^W \otimes \mathbb{C}[V^\ast]^W$), and which is a free $P$-module of rank $|W|$. We denote by $\mathcal{Z}_c$ the algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_c]$ is $Z_\mathbf{H}_c$: this is the Calogero-Moser space associated with the datum $(V, W, c)$. If necessary, we will write $\mathcal{Z}_c(V, W)$ for $\mathcal{Z}_c$. 

3.D. Quiver varieties vs Calogero-Moser spaces. — Assume that $n \geq 2$, that $V = \mathbb{C}^n$ and that $W = G(\ell, 1, n)$. Recall that $G(\ell, 1, n)$ is the group of monomial matrices with coefficients in $\mu_\ell$ (the group of $\ell$-th root of unity in $\mathbb{C}^\times$).

We fix a primitive $\ell$-th root of unity $\zeta$. We denote by $s$ the permutation matrix corresponding to the transposition $(1, 2)$ and we set

$$t = \text{diag}(\zeta, 1, \ldots, 1) \in W.$$ 

Then $s, t, t^2, \ldots, t^{\ell-1}$ is a set of representatives of conjugacy classes of reflections of $W$. We set for simplification

$$a = c_s$$

and

$$k_j = \frac{1}{\ell} \sum_{i=1}^{\ell-1} \zeta^{-(j-1)} c_{t^i}$$

for $j \in \mathbb{Z}/\ell\mathbb{Z}$. Then

$$(3.5) \quad k_0 + \cdots + k_{\ell-1} = 0 \quad \text{and} \quad c_{t^i} = \sum_{j \in \mathbb{Z}/\ell\mathbb{Z}} \zeta^{(j-1)} k_j$$

for $1 \leq i \leq \ell - 1$. Finally, if $i \in \mathbb{Z}/\ell\mathbb{Z}$, we set

$$(3.6) \quad \theta_i = \begin{cases} k_{-i} - k_{1-i} & \text{if } i \neq 0, \\ -a + k_0 - k_1 & \text{if } i = 0. \end{cases}$$

and $\theta = (\theta_i)_{i \in \mathbb{Z}/\ell\mathbb{Z}}$.

The following result is proved in [8, Theorem 3.10]. (Note that our $k_i$ is related with Gordon’s $H_i$ via $H_i = k_{-i} - k_{1-i}$.)

**Proposition 3.7.** — There is a $\mathbb{C}^*$-equivariant isomorphism of varieties

$$\mathcal{X}_c \sim \mathcal{X}_\theta(n\delta_\ell).$$

In the isomorphism above, the parameter $a$ of the variety $\mathcal{X}_c$ corresponds to $-(\sum_{i \in \mathbb{Z}/\ell\mathbb{Z}} \theta_i)$ for $\mathcal{X}_\theta(n\delta_\ell)$. So, we will sometimes use the notation $a = -\Sigma(\theta) = -(\sum_{i \in \mathbb{Z}/\ell\mathbb{Z}} \theta_i)$ when we speak about an arbitrary quiver variety $\mathcal{X}_\theta(d)$. Note also that $a$ is invariant under the transformation of the parameter $\theta \mapsto s(\theta)$. In this paper, we will often assume $a \neq 0$.

**Remark 3.8.** — All statements in §3.D make also sense for $n = 1$ with the following modifications. We have no transposition $s$, so we have no parameter $a$. On the other hand, for $n = 1$, the variety $\mathcal{X}_\theta(n\delta_\ell)$ does not depend on $\theta_0$. Proposition 3.7 is true for an arbitrary choice of $a$ in (3.6).

We can also use the convention that for $n = 0$ the Calogero-Moser space is a point. Then Proposition 3.7 still holds.

Recall also from [7, §11] the following result, which follows from Proposition 3.7.

**Lemma 3.9.** — If $n > 0$, then $\mathcal{X}_\theta(n\delta_\ell)$ is normal and of dimension $2n$. 


3.E. Simple representations in \( \text{Rep}_\theta(\hat{Q}_\ell) \). — From now on we assume \( a \neq 0 \).

Denote by \( \text{Rep}_\theta(\hat{Q}_\ell) \) the additive category of representations \((X, Y)\) of \( \hat{Q}_\ell \) satisfying the moment map relations \( \mu_d(X, Y) = 1_\theta(d) \), where \( d \) is the dimension vector of the representation \((X, Y)\). In this section we give an explicit description of the set \( \Sigma_\theta \) of dimension vectors of simple representations in \( \text{Rep}_\theta(\hat{Q}_\ell) \). This description is done in much more generality in [5]. In this section, we precise how this description looks like in our particular case: the cyclic quiver and \( a \neq 0 \).

By [5] Theorem 5.8] there are two types of indecomposable representations in \( \text{Rep}_\theta(\hat{Q}_\ell) \):

- representations whose dimension vectors are positive roots,
- representations whose dimension vectors are of the form \( r \delta_\ell \) for \( r > 0 \).

Since we assume \( a \neq 0 \), the second situation is not possible. Now, let us give a precise description of the dimension vectors of simple representations.

Let \( R^+ \subset \mathbb{Z}^{2/\ell} \) be the set of positive real roots. Set \( R^+_\theta = \{ d \in R^+ ; d \cdot \theta = 0 \} \). The following proposition is the special case of [5] Theorem 1.2].

**Proposition 3.10.** — The dimension vectors of simple representations in \( \text{Rep}_\theta(\hat{Q}_\ell) \) are exactly the elements of \( R^+_\theta \) that are not presented as sums of (two or more) elements of \( R^+_\theta \).

**Corollary 3.11.** — For each dimension vector \( d \in \mathbb{Z}_e^{2/\ell} \), there exists at most one (up to isomorphism) semisimple representation in \( \text{Rep}_\theta(\hat{Q}_\ell) \) with dimension vector \( d \).

**Proof.** — The statement is equivalent to the fact that the variety \( X^0_\theta(d) \) contains at most one point.

First, assume \( d \in \Sigma_\theta \). Then, since \( a \neq 0 \), \( d \) is a positive root. This implies that there is exactly one (up to isomorphism) simple representation in \( \text{Rep}_\theta(\hat{Q}_\ell) \) with dimension vector \( d \) (see the introduction in [5]).

Now, consider an arbitrary \( d \in \mathbb{Z}_e^{2/\ell} \). Then there is a finite number of possibilities to decompose \( d \) in a sum of elements of \( \Sigma_\theta \). This implies that \( X^0_\theta(d) \) has a finite number of points. The variety \( X^0_\theta(d) \) is irreducible if it is non-empty by [6] Cor. 1.4]. So, the variety \( X^0_\theta(d) \) contains at most one point.

**Corollary 3.12.** — The elements of \( \Sigma_\theta \) are \( \mathbb{Z} \)-linearly independent.

**Proof.** — Since \( X^0_\theta(d) \) contains at most one point, there is at most one way (up to permutation) to decompose \( d \) in a sum of elements of \( \Sigma_\theta \). □

Denote by \( \Sigma \Sigma_\theta \) the set of sums of element of \( \Sigma_\theta \) (we also allow an empty sum, so we assume \( 0 \in \Sigma \Sigma_\theta \)). In other words, the set \( \Sigma \Sigma_\theta \) is the set of all dimension vectors \( d \) such that there exists a representation in \( \text{Rep}_\theta(\hat{Q}_\ell) \) of dimension vector \( d \). For each \( d \in \Sigma \Sigma_\theta \), denote by \( L(d) \) the unique semisimple representation in \( \text{Rep}_\theta(\hat{Q}_\ell) \).

3.F. Symplectic leaves. — Denote by \( \text{Rep}_\theta(\hat{Q}_\ell) \) the category of representations \((X, Y, x, y)\) of \( \hat{Q} \) whose dimension vector is of the form \( \bar{d} \) for some \( d \in \mathbb{Z}^{2/\ell} \) and satisfying the moment map relations \( \mu_d(X, Y) = 1_\theta(d) \). This category is not additive because we have imposed that the representations have dimension 1 at the vertex \( \infty \). However, it does make sense to add an object of \( \text{Rep}_\theta(\hat{Q}_\ell) \) and an object of \( \text{Rep}_\theta(\hat{Q}_\ell) \) getting an object of \( \text{Rep}_\theta(\hat{Q}_\ell) \).
An object $M$ of $\text{Rep}_\theta(\mathcal{Q}_t)$ is indecomposable as a representation of the quiver $\mathcal{Q}_t$ if and only if the only possible decomposition $M = M_0 \oplus M_1$ with $M_0 \in \text{Rep}_\theta(\mathcal{Q}_t)$ and $M_1 \in \text{Rep}_\theta(\mathcal{Q}_t)$ is $M = M \oplus 0$.

Denote by $E_\theta$ the set of all possible dimension vectors $\mathbf{d} \in \mathbb{Z}^{\mathcal{Q}_t}_{/\mathbb{Z}}$ such that there exists a simple representation in $\text{Rep}_\theta(\mathcal{Q}_t)$ with dimension vector $\mathbf{d}$. Sometimes we can write $E_{\theta,t}$ instead of $E_\theta$ to emphasize $t$.

**Remark 3.13.** Assume $\mathbf{d} \in E_\theta$. Then, by Lemma 3.15, the couple $(\mathbf{d}, \theta)$ is equivalent to a couple of the form $(n\delta_t, \theta')$ with $n > 0$. In particular, by Proposition 3.17 the variety $\mathcal{X}_\theta(\mathbf{d})$ is isomorphic to the Calogero-Moser space.

Each object $M \in \text{Rep}_\theta(\mathcal{Q}_t)$ has a unique decomposition $M = M_0 \oplus M_1$ such that $M_0 \in \text{Rep}_\theta(\mathcal{Q}_t)$, $M_1 \in \text{Rep}_\theta(\mathcal{Q}_t)$ and $M_0$ is indecomposable. Set $\dim_{\text{reg}} M = \dim M_0 \in \mathbb{Z}^{\mathcal{Q}_t}_{/\mathbb{Z}}$.

Take a point $[M] \in \mathcal{X}_\theta(\mathbf{d})$ presented by a semisimple representation $M \in \text{Rep}_\theta(\mathcal{Q}_t)$.

**Lemma 3.14.** Two points of $[M], [M'] \in \mathcal{X}_\theta(\mathbf{d})$ are in the same symplectic leaf if and only if we have $\dim_{\text{reg}}(M) = \dim_{\text{reg}}(M')$.

**Proof.** Let us decompose $M$ in a direct sum of simple representations $M = \bigoplus_{r=0}^k M_r$, where $M_0 \in \text{Rep}_\theta(\mathcal{Q}_t)$ and other summands are in $\text{Rep}_\theta(\mathcal{Q}_t)$.

Once we know the dimension vector $\mathbf{d}'$ of $M_0$, we know automatically $k$ and the dimension vectors of $M_1, M_2, \ldots, M_k$ (up to a permutation) because by Corollary 3.11 there is a unique semisimple representation in $\text{Rep}_\theta(\mathcal{Q}_t)$ of dimension vector $\mathbf{d} - \mathbf{d}'$. Then the statement follows from the description of symplectic leaves given in [1] Theorem 1.9. □

For two dimension vectors $\mathbf{d}$ and $\mathbf{d}'$ we set $\mathbb{L}^d_{\mathbf{d}} = \{[M] \in \mathcal{X}_\theta(\mathbf{d}) ; \dim_{\text{reg}}(M) = \mathbf{d}'\}$. By Lemma 3.14 $\mathbb{L}^d_{\mathbf{d}}$ is either a symplectic leaf of $\mathcal{X}_\theta(\mathbf{d})$ or is empty.

**Lemma 3.15.** The symplectic leaves $\mathbb{L}^d_{\mathbf{d}} \subset \mathcal{X}_\theta(\mathbf{d})$ define a finite stratification of $\mathcal{X}_\theta(\mathbf{d})$ into locally closed subsets. For two simplectic leaves $\mathbb{L}^d_{\mathbf{d}}$ and $\mathbb{L}^d_{\mathbf{d}'}$ of $\mathcal{X}_\theta(\mathbf{d})$ we have $\mathbb{L}^d_{\mathbf{d}} \subset \mathbb{L}^d_{\mathbf{d}'}$ if and only if $\mathbf{d}' - \mathbf{d} \in \Sigma_\theta$.

**Proof.** This statement is a special case of [1] Prop. 3.6.

Let us give some details. Let $M', M'' \in \text{Rep}_\theta(\mathcal{Q}_t)$ be simple representations with dimension vectors $\mathbf{d}'$ and $\mathbf{d}''$ respectively. Then we have $[L(\mathbf{d} - \mathbf{d}') \oplus M'] \in \mathbb{L}^d_{\mathbf{d}'}$ and $[L(\mathbf{d} - \mathbf{d}'') \oplus M''] \in \mathbb{L}^d_{\mathbf{d}''}$.

Assume that we have $\mathbf{d}'' - \mathbf{d}' \in \Sigma_\theta$. Then we have $L(\mathbf{d} - \mathbf{d}') \simeq L(\mathbf{d} - \mathbf{d}'') \oplus L(\mathbf{d}' - \mathbf{d}')$. Then the stabilizer of the representation $L(\mathbf{d} - \mathbf{d}'') \oplus M''$ in $\text{GL}(\mathbf{d})$ is clearly contained in the stabilizer of the representation $L(\mathbf{d} - \mathbf{d}') \oplus L(\mathbf{d}' - \mathbf{d}') \oplus M'$ in $\text{GL}(\mathbf{d})$. Then by [1] Prop. 3.6, we have $\mathbb{L}^d_{\mathbf{d}} \subset \mathbb{L}^d_{\mathbf{d}'}$.

Inversely, assume $\mathbb{L}^d_{\mathbf{d}} \subset \mathbb{L}^d_{\mathbf{d}'}$. Then, by [1] Prop. 3.6 there exists a semisimple representation $K \in \text{Rep}_\theta(\mathcal{Q})$ such that $[K] \in \mathbb{L}^d_{\mathbf{d}}$, and the stabilizer of $K$ in $\text{GL}(\mathbf{d})$ contains the stabilizer of $L(\mathbf{d} - \mathbf{d}'') \oplus M''$ in $\text{GL}(\mathbf{d})$. Let $g$ be the element of the stabilizer of $L(\mathbf{d} - \mathbf{d}'') \oplus M''$ that acts on $M''$ by multiplication by 1 and on $L(\mathbf{d} - \mathbf{d}'')$ by multiplication by 2. Let $K_1$ and $K_2$ be the eigenspaces of $K$ with respect to the eigenvalues 1 and 2. Then, since $g$ is in the stabilizer of $K$, we get a decomposition $K = K_1 \oplus K_2$ in a direct sum of subrepresentations. Moreover, we have $\dim K_1 = \dim M'' = \mathbf{d}'$. The representation $K_1$ can be decomposed as $K_1 = K_{10} \oplus K_{11}$, where $K_{10} \in \text{Rep}_\theta(\mathcal{Q}_t)$ is simple and $K_{11} \in \text{Rep}_\theta(\mathcal{Q}_t)$. We clearly have $\dim K_{10} = \mathbf{d}'$. Then we get $\dim K_{11} = \mathbf{d}'' - \mathbf{d}' = \mathbf{d}' - \mathbf{d}'$. This implies $\mathbf{d}'' - \mathbf{d}' \in \Sigma_\theta$. □
Proposition 3.16. —

a) For each dimension vector \( \mathbf{d} \) such that \( \mathcal{X}_\theta(\mathbf{d}) \neq \emptyset \), there is a decomposition \( \mathbf{d} = \mathbf{d}_0 + \mathbf{d}_1 \) such that \( \mathbf{d}_0 \in E_0 \) and \( \mathbf{d}_1 \in \Sigma \Sigma_\theta \) such that for any other decomposition \( \mathbf{d} = \mathbf{d}_0' + \mathbf{d}_1' \) with \( \mathbf{d}_0' \in E_0 \) and \( \mathbf{d}_1' \in \Sigma \Sigma_\theta \) we have \( \mathbf{d}_0 - \mathbf{d}_0' \in \Sigma \Sigma_\theta \).

b) \( \Sigma_{\mathbf{d}} \) is the unique open symplectic leaf in \( \mathcal{X}_\theta(\mathbf{d}) \).

c) We have an isomorphism of varieties

\[
\mathcal{X}_\theta(\mathbf{d}_0) \cong \mathcal{X}_\theta(\mathbf{d}), \quad [M] \mapsto [M \oplus L(\mathbf{d}_1)].
\]

Proof. — By Cor. 1.45, the smooth locus of \( \mathcal{X}_\theta(\mathbf{d}) \) is a symplectic leaf. Then it should be of the form \( \Sigma_{\mathbf{d}} \) for some \( \mathbf{d}_0 \). Since \( \mathcal{X}_\theta(\mathbf{d}) \) is irreducible by Cor. 1.4, we have \( \Sigma_{\mathbf{d}} = \mathcal{X}_\theta(\mathbf{d}) \). Then, by Lemma 3.15 for any other symplectic leaf \( \Sigma_{\mathbf{d}_0} \), we have \( \mathbf{d}_0 - \mathbf{d}_0' \in \Sigma \Sigma_\theta \). This proves a) and b).

Part c) follows from Theorem 1.1. \( \square \)

Now, we set \( \mathcal{X}_\theta(\mathbf{d})^{\text{reg}} = \Sigma_{\mathbf{d}} \). Assume that \( \mathbf{d} \) and \( \mathbf{d}' \) are such that \( \Sigma_{\mathbf{d}} \) is non-empty.

Lemma 3.17. — The normalization of the closure of \( \Sigma_{\mathbf{d}}^{\text{reg}} \) is isomorphic to \( \mathcal{X}_\theta(\mathbf{d}') \). The normalization map is bijective.

Proof. — Consider the following homomorphism of algebraic varieties:

\[
\phi : \mathcal{X}_\theta(\mathbf{d}') \to \Sigma_{\mathbf{d}}^{\text{reg}}, \quad [M] \mapsto [M \oplus L(\mathbf{d} - \mathbf{d}')].
\]

Let us show that \( \phi \) is bijective.

Fix a point \( [N] \in \Sigma_{\mathbf{d}}^{\text{reg}} \) presented by a semisimple representation \( N \). We can decompose \( N = M \oplus L(\mathbf{d} - \mathbf{d}') \) for some semisimple \( M \in \text{Rep}_\theta \). Then it is clear that the fibre \( \phi^{-1}([N]) \) contains a unique point: \( [M] \).

Moreover, the map \( \phi \) restricts to an isomorphism \( \mathcal{X}_\theta(\mathbf{d}')^{\text{reg}} \to \Sigma_{\mathbf{d}}^{\text{reg}} \), so \( \phi \) is birational. Now, since \( \mathcal{X}_\theta(\mathbf{d}') \) is normal, the map \( \phi \) is a normalization. \( \square \)

Corollary 3.18. — The normalization of the closure of each symplectic leaf \( \Sigma_{\mathbf{d}}^{\text{reg}} \) of the variety \( \mathcal{X}_\theta(\mathbf{d}) \) is isomorphic to a variety of the form \( \mathcal{X}_\theta(r \delta_\ell) \) for some \( r > 0 \) and some \( \theta' \in C_{Z/(2Z)} \).

Proof. — First of all, note that we have \( \mathbf{d}' \in E_0 \). By Remark 3.13, the pair \( (\mathbf{d}', \theta) \) is equivalent to some pair of the form \( (r \delta_\ell, \theta') \) where \( r > 0 \) and \( \theta' \in C_{Z/(2Z)} \). Then the isomorphism 3.3 yields \( \mathcal{X}_\theta(\mathbf{d}') \cong \mathcal{X}_\theta(r \delta_\ell) \).

Combining the corollary above with Proposition 3.7 yields the following theorem.

Theorem 3.19. — The normalization of the closure of each symplectic leaf of the Calogero-Moser space of type \( G(\ell, 1, n) \) with \( a \neq 0 \) is isomorphic to a Calogero-Moser space of type \( G(\ell, 1, r) \) for some \( r \in \{0; n\} \).

Remark 3.20. — Let us give an explicit relation between the parameters of the two Calogero-Moser spaces in the theorem above.

The original Calogero-Moser space is isomorphic to the quiver variety of the form \( \mathcal{X}_\theta(n \delta_\ell) \). Now, we consider the symplectic leaf \( \Sigma_{\mathbf{d}_0} \), the normalization of its closure is isomorphic to \( \mathcal{X}_\theta(\mathbf{d}) \). Then, by Remark 3.13, we can find \( w \in W_\ell^{\text{aff}} \) that realizes an equivalence between \( (\mathbf{d}', \theta) \) and \( (w(\mathbf{d}'), w(\theta)) \) and such that \( w(\mathbf{d}') \) is of the form \( r \delta_\ell \). Set
\[ \theta' = w(\theta). \] We have an isomorphism \[ \mathcal{X}(d') \cong \mathcal{X}(r \delta_i). \] Since we have \( w(d') = r \delta_i \), then, by Lemma 2.5 the element \( w \) should be of the form \( w = x^t q \), where \( x \in W_i \).

Then, by Lemma 2.9 we get \( \theta' = x^t q(\theta) = x(\theta - a \overline{\theta}) \). Moreover, the action the element \( x \in W_i \) on \( \mathbb{C}^2/\mathbb{Z} \) corresponds to some permutation of the parameters \( k_0, k_1, \ldots, k_{t-1} \) (see [4, Rem. 3.5]) and a permutation of the parameters does not change the Calogero-Moser space up to isomorphism, see [4, Cor. 3.6].

Now we see that the parameters \( a, k_0, k_1, \ldots, k_{t-1} \) (corresponding to \( \theta \)) of the original the Calogero-Moser space \( \mathcal{X}(n \delta_i) \) are related with the parameters \( a', k_0', k_1', \ldots, k_{t-1}' \) (corresponding to \( \theta' \)) of the new Calogero-Moser space \( \mathcal{X}(r \delta_i) \) in the following way (up to a permutation of the parameters \( k_i' \)):

\[
a' = a, \quad k_i' = k_i + (d_{i-1}' - d_{i-'}).
\]

In the case when \( n = r = 1 \), we can forget the parameter \( a \) or \( a' \) respectively. In the case \( r = 0 \), the variety \( \mathcal{X}(r \delta) \) is just a point.

3. G. \( \mathcal{C}^\times \)-fixed points. — For each \( J \subset \mathbb{Z}/t \mathbb{Z} \) we denote by \( W_J \) the parabolic subgroup of \( W^{\text{aff}}_t \) generated by \( s_i \) for \( i \in J \). Let us say that \( \theta \) is \( J \)-standard if the stabilizer \( W_\theta \) of \( \theta \) in \( W^{\text{aff}}_t \) is equal to \( W_J \). We say that \( \theta \in \mathbb{C}^2/\mathbb{Z} \) is standard if \( \theta \) is \( J \)-standard for some \( J \subset \mathbb{Z}/t \mathbb{Z} \).

For a standard \( \theta \), the set \( J \) is the set of indices \( i \in \mathbb{Z}/t \mathbb{Z} \) such that \( \theta_i = 0 \).

Now, let us describe the \( \mathcal{C}^\times \)-fixed points of \( \mathcal{X}(d, \theta) \). First of all, each couple \( (d, \theta) \) is equivalent to a couple whose \( \theta \) is standard.

The following lemma is obvious.

**Lemma 3.21.** — Assume that \( \theta \) is \( J \)-standard. Then we have \( \Sigma_\theta = \{ a_i; \ i \in J \} \).

Let us now assume that \( \theta \) is \( J \)-standard. For each partition \( \mu \), we construct a \( \mathcal{C}^\times \)-fixed point in \( \mathcal{X}(\text{Res}_J(\mu)) \). This construction is essentially the same as [12, Section 5], however [12] assumes that the variety \( \mathcal{X}(\text{Res}_J(\mu)) \) is smooth and we don’t need this assumption.

Each partition \( \mu \) can be described by some \( k \in \mathbb{Z}_{>0} \) and \( a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}_{>0} \) where \( k \) is maximal such that the Young diagram of \( \mu \) contains a box in position \( (k, k) \) and for each \( r \in [1; k] \) there are \( a_r \) boxes on the right of \( (r, r) \) and \( b_r \) boxes below \( (r, r) \). In other words, we see the Young diagram of the partition \( \mu \) as a union of \( k \) hooks. The box at position \( (i, j) \) is in the \( r \)th hook if \( \text{min}(i, j) = r \). The numbers \( a_r \) and \( b_r \) are the lengths of the arm and of the leg of \( r \)th hook respectively.

For \( i \in \mathbb{Z} \), we use the convention that \( \theta_i \) means \( \theta_{i \mod t} \). Set \( \beta_r = \sum_{i=-b_r}^{a_r} \theta_i \).

Let \( V \) be a complex vector space with basis \( \{ v_{r,j}; \ r \in [1; k]; \ j \in [-b_r, a_r] \} \). It has a \( \mathcal{Z} \)-grading \( V = \bigoplus_{j \in \mathbb{Z}} V_j \) such that \( v_{r,j} \in V_j \). Consider two endomorphisms \( X \) and \( Y \) of this vector space given by

\[
X(v_{r,j}) = \begin{cases} v_{r,j-1} & \text{if } j > -b_r, \\ 0 & \text{if } j = -b_r, \end{cases}
\]

and

\[
Y(v_{r,j}) = \begin{cases} \sum_{i=-b_r}^{j} \theta_i v_{r,j+1} + \sum_{i > j} \beta_i v_{r,j+1} & \text{if } j \in [-b_r, -1] \\ \sum_{i=0}^{a_r} \theta_i v_{r,j+1} - \sum_{i < r} \beta_i v_{r,j-1} & \text{if } j \in [0; a_r - 1], \\ - \sum_{i < r} \beta_i v_{r,j-1} & \text{if } j = a_r, \end{cases}
\]

Consider also the linear maps \( x: \mathbb{C} \to V_0 \) and \( y: V_0 \to \mathbb{C} \) given by

\[
x(1) = - \sum_{r=1}^{k} \beta_r v_{r,0} \quad \text{and} \quad y(v_{r,0}) = 1.
\]
Then \((X, Y, x, y)\) yields a representation \(A_\mu^\infty\) of the quiver \(\bar{Q}_\infty\). Applying the map \(\iota\) as in Definition \([3,2]\) we get a representation \(A_\mu\) of the quiver \(Q_l\). It satisfies the moment map relation \(\mu_d(A_\mu) = b_0(d)\).

**Lemma 3.22.** Assume that \(\theta\) is \(J\)-standard.

(a) If \(\mu\) is a \(J\)-core, then \(A_\mu\) is simple.

(b) Assume that \(b\) is a removable box of \(\mu\) with \(\ell\)-residue \(i \in J\). Then we either have a short exact sequence

\[0 \to L(\alpha_i) \to A_\mu \to A_{\mu \setminus b} \to 0\]

or we have a short exact sequence

\[0 \to A_{\mu \setminus b} \to A_\mu \to L(\alpha_i) \to 0.\]

**Proof.** First, we prove \((b)\). Assume that \(b\) is the box as in the statement. Assume that it is in the \(r\)th hook. Let \(j\) be the \(\infty\)-residue of \(b\).

Assume first \(j < 0\). We have \(X(v_{r,j}) = Y(v_{r,j}) = 0\). Then the vector \(v_{r,j}\) spans a subrepresentation isomorphic to \(L(\alpha_i)\). We get a short exact sequence

\[0 \to L(\alpha_j) \to A_\mu \to A_{\mu \setminus b} \to 0.\]

Now, assume \(j \geq 0\). Then we see that \(A_{\mu \setminus b}\) is a subrepresentation of \(A_\mu\). It is spanned by all basis vectors except \(v_{r,j}\). Then we have a short exact sequence

\[0 \to A_{\mu \setminus b} \to A_\mu \to L(\alpha_j) \to 0.\]

Now, let us prove \((a)\). First of all, we note that the assumption that \(\theta\) is \(J\)-standard implies that if for some \(a, b \in \mathbb{Z}, a < b\) we have \(\theta_a + \theta_{a+1} + \ldots + \theta_{b-1} + \theta_b = 0\), then we have \(\theta_a = \theta_{a+1} = \ldots = \theta_{b-1} = \theta_b = 0\). If \(\mu\) is a \(J\)-core, then the numbers \(\beta_1, \beta_2, \ldots, \beta_k\) are nonzero. Indeed, if some \(\beta_i\) is zero, then \(\beta_k\) is also zero. Then the \(\ell\)-residues of all boxes of the \(k\)th hook are in \(J\). In particular, the \(k\)th hook contains a removable box whose residue is in \(J\). This contradicts to the fact that \(\mu\) is a \(J\)-core.

In view of Lemma 3.21 if the representation \(A_\mu\) is not simple, then it must either contain a subrepresentation of the form \(L(\alpha_i)\), or it must have a quotient of the form \(L(\alpha_i)\). Let us show that both situations are impossible when \(\mu\) is a \(J\)-core.

Assume that \(A_\mu\) has a subrepresentation isomorphic to \(L(\alpha_i)\). Let \(v\) be a vector that spans this subrepresentation. We can write \(v = \sum_{j \in \mathbb{Z}, j \equiv i \mod \ell} v_j\), where \(v_j \in V_j\). Take \(j\) in this decomposition such that \(v_j \neq 0\). Then the vector \(v_j\) also spans a subrepresentation of \(A_\mu\) isomorphic to \(L(\alpha_i)\).

Let \(t\) be the number of boxes of \(\mu\) with the \(\infty\)-residue \(j\). Write \(v_j = \sum_{t=1}^\ell \lambda_{r,t} v_{r,t}\). Then \(X(v) = 0\) is only possible when \(\lambda_1 = \ldots = \lambda_{t-1} = 0\), so the vector \(v_{r,t}\) spans \(L(\alpha_i)\).

Assume \(j < 0\). Since the box \(b\) corresponding to the vector \(v_{r,t}\) cannot be removable, the diagram of \(\mu\) either contains the box below \(b\) or the box on the right of \(b\). In the first case we must have \(X(v_{r,j}) \neq 0\) and in the second case we must have \(Y(v_{r,j}) \neq 0\). This is a contradiction.

Assume \(j > 0\). Then \(X(v_{r,j}) \neq 0\). This is a contradiction.

Assume \(j = 0\). Then, since \(\beta_1 \neq 0\), \(Y(v_{r,0}) \neq 0\) is only possible for \(t = 1\). However, this implies that \(\mu\) contains only one hook (i.e., we have \(k = 1\)). Since the box \(b\) corresponding to the vector \(v_{r,0}\) cannot be removable, the diagram of \(\mu\) either contains the box below \(b\) or the box on the right of \(b\). The first case is not possible because it implies \(X(v_{r,0}) \neq 0\). In the second case we must have \(\theta_1 + \theta_2 + \ldots + \theta_{a_1} = 0\). However, this implies \(\theta_{a_1} = 0\) and then the unique box with \(\infty\)-residue \(a_1\) is removable. This is a contradiction.
Now, assume that $A_\mu$ has a quotient isomorphic to $L(a_i)$. Then the dual representation $A_\mu^*$ contains a submodule isomorphic to $L(a_i)$. An argument as above show that this is impossible if $A_\mu$ is a $J$-core.

Denote by $A'_\mu$ the semisimplification of $A_\mu$, i.e., $A'_\mu$ is the direct sum of the Jordan-Hölder subquotients of $A_\mu$.

**Corollary 3.23.** — Assume $\mu \in \mathcal{P}$ and set $\lambda = \text{Core}_J(\mu)$. Then the representation $A'_\mu$ has the following decomposition in a direct sum of simple representations

$$A'_\mu = A_\lambda \oplus \bigoplus_j L(a_j),$$

where the sum is taken by the multiset of $\ell$-residues of $\mu \smallsetminus \lambda$.

**Definition 3.24.** — We say that the representation $(X, Y, x, y)$ of $Q_\ell$ is $\mathbb{Z}$-gradable if it is isomorphic to the image by $\iota$ (see Definition 3.2) of some representation $L$ of $Q_{\infty}$. In this case we say that $L$ is a graded lift of $(X, Y, x, y)$.

A $\mathbb{Z}$-gradable representation yields a $C^\times$-fixed point in $\mathcal{X}_g(d)$.

**Lemma 3.25.** — Assume that $(X, Y, x, y)$ is simple and $\mathbb{Z}$-gradable. Then its $\mathbb{Z}$-grading is unique.

**Proof.** — Since we assume $a \neq 0$, the vector $v = x(1)$ must be nonzero (here 1 is a vector spanning the $\infty$-component of the representation, which is isomorphic to $\mathbb{C}$). Then $v$ should be in $\mathbb{Z}$-degree 0. Since the representation is simple, the vectors of the form $X^{a_1} Y^{b_1} \cdots X^{a_k} Y^{b_k}(v)$ and the vector 1 span the representation. But then vector $X^{a_1} Y^{b_1} \cdots X^{a_k} Y^{b_k}(v)$ must be in $\mathbb{Z}$-degree $b_1 - a_1 + \cdots + b_k - a_k$. This shows that the $\mathbb{Z}$-grading is unique.

**Example 3.26.** — If $\mu$ is a $J$-core, then the representation $A_\mu$ is simple. It is $\mathbb{Z}$-gradable by construction. Its graded lift $A_\mu^\infty$ is unique. The $\mathbb{Z}$-graded dimension of the graded lift $A_\mu^\infty$ is $\text{Res}_\infty(\mu)$.

**Corollary 3.27.** — For $\mu_1, \mu_2 \in \mathcal{P}_{\ell}[n\ell + |\psi|]$, the representations $A'_{\mu_1}$ and $A'_{\mu_2}$ are isomorphic if and only if $\mu_1$ and $\mu_2$ have the same $J$-cores.

**Proof.** — Let $\lambda_1$ and $\lambda_2$ be the $J$-cores of $\mu_1$ and $\mu_2$, respectively.

Assume that $A'_{\mu_1}$ and $A'_{\mu_2}$ are isomorphic. We see from Corollary 3.23 that the representations $A_{\lambda_1}$ and $A_{\lambda_2}$ are also isomorphic. Now, Example 3.26 implies $\text{Res}_\infty(\lambda_1) = \text{Res}_\infty(\lambda_2)$, this yields $\lambda_1 = \lambda_2$.

Now, assume that we have $\lambda_1 = \lambda_2$. Since we have $\mu_1, \mu_2 \in \mathcal{P}_{\ell}[n\ell + |\psi|]$, the partitions $\mu_1$ and $\mu_2$ have the same residues equal to $\text{Res}_\infty(\psi) + n\delta_\ell$. Then $\mu_1 \setminus \lambda_1$ and $\mu_2 \setminus \lambda_2$ have the same residues. Then Corollary 3.23 implies that $A'_{\mu_1}$ and $A'_{\mu_2}$ are isomorphic.

**Remark 3.28.** — For each partition $\mu$, we have a $C^\times$-fixed point $[A'_{\mu}] \in \mathcal{X}_g(\text{Res}_\ell(\mu))$ presented by the representation $A'_{\mu}$.

Set $d = \text{Res}_\ell(\psi) + n\delta_\ell$. Assume $d \in E_g$, see Remark 3.13. By [8, Prop. 8.3 (i)], the $C^\times$-fixed points in $\mathcal{X}_g(d)$ are parameterized by $J$-cores of elements of $\mathcal{P}_{\ell}[n\ell + |\psi|]$. On the other hand, we have already constructed the same number of $C^\times$-fixed points $[A'_{\mu}]$ for $\mu \in \mathcal{P}_{\ell}[n\ell + |\psi|]$, see Corollary 3.27.

This implies that each $C^\times$-fixed point in $\mathcal{X}_g(d)$ is of the form $[A'_{\mu}]$. 

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**Symplectic leaves of Calogero-Moser spaces of type $G(\ell, 1, n)$**

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3.H. Parameterization of symplectic leaves. —

Lemma 3.29. — The following conditions are equivalent.
(a) The pair \((\mathbf{d}, \theta)\) is equivalent to a pair of the form \((n \delta_i, \theta')\) with \(n \geq 0\).
(b) We have \(\mathbf{d} \in \mathcal{E}_\theta\).

Proof. — b) implies a) by Remark 3.13.

Now, let us prove that a) implies b). Assume that \((\mathbf{d}, \theta)\) satisfies a). Since, the isomorphism \((3.3)\) sends simple representations to simple representations by construction, it is enough to assume \(\mathbf{d} = n \delta_i\). Let \(\mathbf{d}_0\) be associated to \(\mathbf{d} = n \delta_i\) and \(\theta\) as in Proposition 3.16. Then \(b)\) is equivalent to \(\mathbf{d}_0 = \mathbf{d}\).

Assume that we have \(\mathbf{d}_0 \neq \mathbf{d}\). Since, the couple \((\mathbf{d}_0, \theta)\) satisfies \(b)\), it also satisfies a). So, it must be equivalent to some couple of the form \((n' \delta_i, \theta')\). Since we have \(\mathbf{d}_0 - n' \delta_i \in \mathbb{Z}_{\geq 0}^{\ell} \) and \(0 \neq n \delta_i - \mathbf{d}_0 \in \mathbb{Z}_{\geq 0}^{\ell}\), we get \(n > n'\).

Now, we get \(\mathcal{X}_\theta(n \delta_i) \cong \mathcal{X}_\theta(\mathbf{d}_0)\) by Proposition 3.16 c) and we have \(\mathcal{X}_\theta(\mathbf{d}_0) \cong \mathcal{X}_\theta(n' \delta_i)\) by \((3.3)\). This is impossible because by Lemma 3.9 we have \(\dim \mathcal{X}_\theta(n \delta_i) = 2n\), \(\dim \mathcal{X}_\theta(n' \delta_i) = 2n'\) and \(n' < n\).

Lemma 3.30. — Assume that \(\theta\) is \(J\)-standard. Then we have \(\mathbf{d} \in \mathcal{E}_\theta\) if and only if we have

\[
\mathbf{d} = \text{Res}_r(\nu) + r \delta_i
\]

with \(r \geq 0\) and \(\nu \in \mathcal{C}_i \cap \mathcal{C}_j\).

Proof. — The parabolic subgroup \(W_j\) of \(W_j^{\text{aff}}\) is the stabilizer of \(\theta\) in \(W_j^{\text{aff}}\). Write \(\mathbf{d} = \text{Res}_r(\nu) + r \delta_i\) as in \((2.6)\), we have \(r \in \mathbb{Z}\) and \(\nu \in \mathcal{C}_i\).

Assume \(\mathbf{d} \in \mathcal{E}_\theta\). Then Lemma 3.29 implies that \(r \geq 0\) and that we can find \(x \in W_j^{\text{aff}}\) (see Remark 3.4) such that \(x(\mathbf{d}) = r \delta_i\) and such that \(x\) is the shortest element in the coset \(x W_j \in W_j^{\text{aff}} / W_j\).

Let \(w\) be the shortest element in \(x^{-1} W_i\). We have \(\nu = x^{-1}(\theta) = w(\theta)\). Assume that \(\nu\) is not a \(J\)-core. Then we have \(|s_i(\nu)| < |\nu|\) for some \(i \in J\), this corresponds to the case \((3)\) in Remark 2.6. Then Lemma 2.7 implies \(I(s_i w) < I(w)\). Then we also have \(I(s_i x^{-1}) < I(x^{-1})\) or equivalently \(I(x s_i) < I(x)\). This contradicts to the fact that \(x\) is the shortest element in \(x W_j\). Then \(\nu\) must be a \(J\)-core.

Now, assume that we have \(\mathbf{d} = \text{Res}_r(\nu) + r \delta_i\) for \(r \geq 0\) and \(\nu \in \mathcal{C}_i \cap \mathcal{C}_j\). Let \(w\) be the element of \(W_i^{\text{aff}}\) such that \(w(\theta) = \nu\) and such that \(w\) is the shortest element in \(w W_i\). It is enough to prove that \(w\) in the shortest element in \(W_j w\). Indeed, if we prove this, then by Remark 3.4 we have \((\mathbf{d}, \theta) \sim (w^{-1}(\mathbf{d}), w^{-1}(\theta)) = (r \delta_i, w^{-1}(\theta))\) and then by Lemma 3.29 we have \(\mathbf{d} \in \mathcal{E}_\theta\).

Since \(\nu\) is a \(J\)-core, for each \(i \in J\) we have \(|s_i(\nu)| \geq |\nu|\). This means that for each \(i \in J\), we are either in the situation \((1)\) or in the situation \((2)\) of Remark 2.6. In both cases Lemma 2.7 yields \(I(s_i w) > I(w)\).

Remark 3.31. — Assume that \(\theta\) is \(J\)-standard and fix \(\mathbf{d} \in \mathcal{E}_\theta\). By the lemma above, we can write \(\mathbf{d}\) in the form \(\mathbf{d} = \text{Res}_r(\nu) + n \delta_i\) with \(n \geq 0\) and \(\nu \in \mathcal{C}_i \cap \mathcal{C}_j\). Then by Lemma 3.29 the couple \((\mathbf{d}, \theta)\) is equivalent to \((n \delta_i, \theta')\) for some \(\theta' \in \mathcal{C}^{\ell} / \mathbb{Z}\). Then Lemma 3.9 implies that the variety \(\mathcal{X}_\theta(\text{Res}_r(\nu) + n \delta_i)\) is normal of dimension \(2n\).
We see that the elements of $E_\theta$ are in bijection with the couples $(v, r)$ where $v$ is an $\ell$-core that is a $J$-core and $r \in \mathbb{Z}_{\geq 0}$.

Assume that $\theta$ is $J$-standard. Then we have a partial order $\succeq$ on $E_\theta$ given by $d \succeq d'$ if $d - d' \in \sum_{j \in I} \mathbb{Z}_{\geq 0} a_j$. In other words, we have $d \succeq d'$ if and only if $\mathbb{L}^{\theta}_d \neq \emptyset$. Using the bijection above, we may consider the order $\succeq$ as an order on the set $(C_{\ell} \cap C_j) \times \mathbb{Z}_{\geq 0}$.

**Lemma 3.32.** — We have $(v_1, r_1) \succeq (v_2, r_2)$ if and only if we have $r_1 \geq r_2$ and there exists a partition $\lambda \in P_{v_1}[|v_1| + \ell(r_1 - r_2)]$ such that $Core_J(\lambda) = v_2$.

**Proof.** — Assume $(v_1, r_1) \succeq (v_2, r_2)$. Then we have $\dim \mathcal{X}_\theta(Res_i(v_1) + r_1 \delta_\ell) = 2r_1$ and $\dim \mathcal{X}_\theta(Res_i(v_2) + r_2 \delta_\ell) = 2r_2$ by Remark 3.31. By Corollary 3.18 and its proof, the normalization of the closure of the symplectic leaf $\mathcal{L}^{Res_i(v_1) + r_1 \delta_\ell}_{Res_i(v_2) + r_2 \delta_\ell}$ is isomorphic to $\mathcal{X}_\theta(Res_i(v_2) + r_2 \delta_\ell)$. In particular,

$$\dim \mathcal{X}_\theta(Res_i(v_1) + r_1 \delta_\ell) \geq \dim \mathcal{L}^{Res_i(v_1) + r_1 \delta_\ell}_{Res_i(v_2) + r_2 \delta_\ell}$$

implies $r_1 \geq r_2$.

Now, $(v_1, r_1) \succeq (v_2, r_2)$ implies $Res_i(v_1) + r_1 \delta_\ell \succeq Res_i(v_2) + r_2 \delta_\ell$ and then $(v_1, r_1 - r_2) \succeq (v_2, 0)$. This means that the variety $\mathcal{X}_\theta(Res_i(v_1) + (r_1 - r_2) \delta_\ell)$ has a symplectic leaf $\mathcal{L}^{Res_i(v_1) + (r_1 - r_2) \delta_\ell}_{Res_i(v_2)}$. This symplectic leaf is 0-dimensional, so it is a $C^\infty$-fixed point. Then by 3.31 this should be a point of the form $[A']_\lambda$ for some $\lambda \in P_{v_1}[|v_1| + \ell(r_1 - r_2)]$. By Corollary 3.23 we have $\dim^{reg}(A'_\lambda) = Res_i(Core_j(\lambda))$. Then $[A'_\lambda] = \mathcal{L}^{Res_i(v_1) + (r_1 - r_2) \delta_\ell}_{Res_i(v_2)}$ implies $Core_J(\lambda) = v_2$.

Inversely, if $r_1 \geq r_2$ and if there exists such a partition $\lambda$, then the $C^\infty$-fixed point $[A'_\lambda]$ of $\mathcal{X}_\theta(Res_i(v_1) + (r_1 - r_2) \delta_\ell)$ is a simple symplectic leaf. Since $\dim^{reg}(A'_\lambda) = Res_i(Core_j(\lambda)) = Res_i(v_2)$, this is the symplectic leaf $\mathcal{L}^{Res_i(v_1) + (r_1 - r_2) \delta_\ell}_{Res_i(v_2)}$. Then we have $(v_1, r_1 - r_2) \succeq (v_2, 0)$. This implies $Res_i(v_1) + (r_1 - r_2) \delta_\ell \succeq Res_i(v_2)$ and then $(v_1, r_1) \succeq (v_2, r_2)$.

Assume that $\theta$ is $J$-standard and $d \in E_\theta$. Write $d = Res_i(v) + n \delta_\ell$, $v \in C_{\ell} \cap C_j$, $n \geq 0$.

**Corollary 3.33.** — For $d' \in \mathbb{Z}/\ell \mathbb{Z}$, the following conditions are equivalent.

(a) We have $\mathbb{L}^{\theta}_d \neq \emptyset$.

(b) There exists a partition $\lambda \in P_{v}[n' \ell + |v|]$ for some $n' \in [0; n]$ such that we have $d' = Res_i(Core_j(\lambda)) + (n - n') \delta_\ell$.

**Proof.** — Write $d' = Res_i(v') + r' \delta_\ell$. Then $\mathbb{L}^{\theta}_d \neq \emptyset$ is equivalent to $(v, n) \succeq (v', r')$. By the lemma above, this is equivalent to $n \geq r'$ and the existence of a partition $\lambda \in P_{v}[\ell(n - r') + |v|]$ such that $Core_j(\lambda) = v'$. Moreover, the condition $Core_j(\lambda) = v'$ is equivalent to $Res_i(Core_j(\lambda)) = Res_i(v') = d' - r' \delta_\ell$. Now we see that (a) is equivalent to (b) with $n' = n - r'$.

In particular, we see that the simplectic leaves of $\mathcal{X}_\theta(d)$ are parametrized by $\ell$-cores of $J$-cores of elements of $P_{v}[n' \ell + |v|]$ for $n' \in [0; n]$. Note that by Lemma 2.11, that $\ell$-cores of $J$-cores are also $J$-cores.

In other words, the symplectic leaves of $\mathcal{X}_\theta(d)$ are parametrized by a subset of the set $C_{\ell} \cap C_j$. This subset is the image of the set $\bigcup_{|v| \geq 0} P_{v}[n' \ell + |v|]$ by the map $Core_j \circ Core_i$.

Since each couple $(n \delta_\ell, \theta)$ is equivalent to some couple of the form $(d, \theta')$ such that $\theta'$ is $J$-standard for some $J$ and $d \in E_\theta$, (see Lemma 3.29), the description above gives a parameterization of the symplectic leaves of an arbitrary Calogero-Moser space of type $G(\ell, 1, n)$ with $a \neq 0$. 

**Example 3.34.** — Assume \( \ell = 2 \). In this case the set \( \mathcal{C}_2 \) of 2-cores is labelled by non-negative integers. We have \( \mathcal{C}_2 = \{ \nu_m, m \in \mathbb{Z}_{\geq 0} \} \) where \( \nu_m \) is the partition \( \nu_m = (m, m-1, m-2, \ldots, 2, 1) \) of \( m(m+1)/2 \). The two possible non-trivial examples of \( J \) are \( J_0 = \emptyset \) and \( J_1 = \{ 1 \} \). Then the 2-cores \( \nu_2, \nu_4, \nu_6, \ldots \) are \( J_0 \)-cores and not \( J_1 \)-cores, the 2-cores \( \nu_1, \nu_3, \nu_5, \ldots \) are \( J_1 \)-cores and not \( J_0 \)-cores, the 2-core \( \nu_0 = \emptyset \) is a \( J_0 \)-core and a \( J_1 \)-core.

Assume that \( \theta \) is \( J \)-standard and \( d \in E_0 \). Assume \( J = J_1 \) and write \( d = \text{Res}_2(\nu_m) + n\delta_2 \). Since \( \nu_m \) must be a \( J_1 \)-core, the number \( m \) must be odd or zero. Assume that \( m \) is odd.

Let us see which subset of \( \mathcal{C}_2 \cap \mathcal{C}_j \) parameterizes the symplectic leaves of \( \mathcal{X}_d \) in this case. If \( n \leq m + 1 \), then the only possible \( \nu' \) that we may get is \( \nu' = \nu_m \). This is the case where the variety \( \mathcal{X}_d \) is smooth. If \( n \geq m + 2 \) then it is also possible to get \( \nu' = \nu_{m+2} \). If \( n \geq 2(m+3) \) then it is also possible to get \( \nu' = \nu_{m+4} \), etc. If \( n \geq k(m+1+k) \) then it is also possible to get \( \nu' = \nu_{m+2k} \). Finally, we see that the symplectic leaves of \( \mathcal{X}_d \) are labelled by the following subset of \( \mathcal{C}_2 \cap \mathcal{C}_j \): \( \{ \nu_m, \nu_{m+2}, \nu_{m+4}, \ldots, \nu_{m+2k} \} \) where \( k \) is the maximal nonnegative integer such that \( n \geq k(m+1+k) \).

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**References**

[1] G. Bellamy & T. Schedler, *Symplectic resolutions of quiver varieties*, Selecta Mathematica 27, Article number: 56, 2021.

[2] C. Berg, B. Jones & M. Vazirani, *A bijection on core partitions and a parabolic quotient of the affine symmetric group*, Journal of Combinatorial Theory, Series A, 116(8), 1344-1360, 2009.

[3] C. Bonnafé, *Automorphisms and symplectic leaves of Calogero-Moser spaces*, preprint arXiv:2112.12405 2021.

[4] C. Bonnafé & R. Maksimau, *Fixed points in smooth Calogero-Moser spaces*, Annales de l’Institut Fourier 71(2), 643-678, 2021.

[5] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compo. Math. 126(3), 257-293, 2001.

[6] W. Crawley-Boevey, *Decomposition of Marsden-Weinstein reductions for representations of quivers*, Compo. Math. 130(2), 225-239, 2002.

[7] P. Etingof & V. Ginzburg. *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Inventiones Mathematicae 147(2), 243-348, 2002.

[8] I. Gordon, *Quiver varieties, category \( \mathcal{O} \) for rational Cherednik algebras, and Hecke algebras*, Int. Math. Res. Papers, 69 pages, 2008.

[9] V. G. Kac, *Infinite Dimensional Lie Algebras - An Introduction*, Birkhäuser, 1983.

[10] G. Lusztig, *Quiver varieties and Weyl group actions*, Ann. Inst. Fourier 50, 461-489, 2000.

[11] M. Martino, *Symplectic reflection algebras and Poisson geometry*, Doctoral dissertation, Pro-Quest Dissertations & Theses, 2006.

[12] T. Przedziacki, *The combinatorics of C*-fixed points in generalized Calogero-Moser spaces and Hilbert schemes*, Journal of Algebra, 556, 936-992, 2020.