Some calculations on type II$_1$ unprojection

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August 2007

Abstract

The type II$_1$ unprojection is, by definition, the generic complete intersection type II unprojection, in the sense of [P] Section 3.1, for the parameter value $k = 1$, and depends on a parameter $n \geq 2$. Our main results are the explicit calculation of the linear relations of the type II$_1$ unprojection for any value $n \geq 2$ (Theorem 3.16) and the explicit calculation of the quadratic equation for the case $n = 3$ (Theorem 4.1). In addition, Section 5 contains applications to algebraic geometry, while Section 6 contains the Macaulay 2 code for the type II$_1$ unprojection for the parameter value $n = 3$.

1 Introduction

The first appearance of the type II unprojection was in the study of elliptic involutions between Fano 3-fold hypersurfaces in [CPR], while the theoretical foundations were developed in [P] using valuations. Valuations were useful to prove the existence of certain relations, but didn’t provide any explicit formulas for them. The present paper is an effort towards the explicit calculations for the type II unprojection.

We define the type II$_k$ unprojection to be the generic complete intersection type II unprojection, in the sense of [P] Section 3.1, for the parameter value $k \geq 1$. According to [P], it depends on a parameter $n \geq 2$, increases the codimension from $nk - 1$ to $nk - 1 + (k + 1)$ and preserves Gorensteiness. To our knowledge, the only previously done explicit calculation for the type II unprojection was for the type II$_1$ case for $n = 2$ ([CPR], [R]), which is reproduced for completeness in Subsection 4.4 below.

Section 3 contains the calculation, using homological and multilinear algebra, of the linear equations of the type II$_1$ unprojection for any value $n \geq 2$. The main result is Theorem 3.16, which provides explicit formulas for the linear relations.
Section 4 contains the calculation of the quadratic equation of the type II\textsubscript{1} unprojection for the case $n = 3$. The main result is Theorem 4.1 which provides an explicit symmetric – in the sense of $\text{SL}_3$ invariance, see Subsection 4.2 – formula for the quadratic relation for this case. The proof of Theorem 4.1 is based on the explicit equality (4.6) which was verified using the computer algebra program Macaulay 2 [GS93-08]. In Subsection 4.3 we briefly sketch how we arrived to the formulas contained in Theorem 4.1 by computer assisted calculations (using the computer algebra program Maple).

As an application of the above results, we sketch in Section 5 the construction of two codimension 4 Fano 3-folds. The quasismoothness checking, using the explicit equations obtained in Sections 3 and 4, was done by the computer algebra program Singular [GPS01].

The calculation of the quadratic equation for the type II\textsubscript{1} unprojection for $n \geq 4$ remains open. We believe that the explicit formulas of the linear relations obtained in the present work together with representation and invariant theoretic techniques could, perhaps, lead to their calculation, and also to a better understanding of the complicated looking formula (4.2). To our knowledge, the problem of calculating the type II\textsubscript{k} unprojection for $k \geq 2$ is also open.

Acknowledgements: I wish to thank Janko Boehm, Stephen Donkin, Sébastien Jansou, Nondas Kechagias, Frank–Olaf Schreyer and Bart Van Steirteghem for useful discussions. Parts of this work were financially supported by the Deutschen Forschungsgemeinschaft Schr 307/4-2 and by the Portuguese Fundação para a Ciência e a Tecnologia through Grant SFRH/BPD/22846/2005.

2 Notation

As already mentioned in Section 1 by type II\textsubscript{1} unprojection we mean the generic complete intersection type II unprojection in the sense of [P] Section 3.1 with fixed parameter value $k = 1$. According to [P], the type II\textsubscript{1} unprojection depends on an integral parameter $n \geq 2$, the initial data for the unprojection is specified by a triple

$$I_X \subset I_D \subset \mathcal{O}_{amb}$$

(2.1)

(where $I_X$ and $I_D$ are ideals of $\mathcal{O}_{amb}$), and the unprojection constructs an ideal

$$I_Y \subset \mathcal{O}_{amb}[s_0, s_1],$$

(2.2)

where $s_0, s_1$ are new variables.
Fix \( n \geq 2 \). The ambient ring \( \mathcal{O}_{amb} \) will be the polynomial ring
\[
\mathcal{O}_{amb} = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n, z, A_{ij}, B_{lm}]
\]
where \( 1 \leq p \leq n - 1, \ 1 \leq i < j \leq n \) and \( 1 \leq l \leq m \leq n \). The ring \( \mathcal{O}_{amb} \) corresponds to the ring denoted by \( \mathcal{O}_{amb}^p \) in [P] Section 3.1, the variable \( x_i \) corresponds to the variable \( a_{1i} \) in [P] Section 2, and the variable \( y_i \) corresponds to the variable \( a_{2i} \) in [P] Section 2. (In Section 4 \( \mathcal{O}_{amb} \) will change slightly to allow 2 to be invertible.)

We denote by \( M \) the \( 2 \times 2 \) matrix
\[
M = \begin{pmatrix}
  y_1 & \cdots & y_n & zx_1 & \cdots & zx_n \\
  x_1 & \cdots & x_n & y_1 & \cdots & y_n \\
\end{pmatrix}
\] (2.3)
which corresponds to the matrix with the same name in [P] Equation (2.1).

The ideal \( I_D \subset \mathcal{O}_{amb} \) in (2.1) is the ideal generated by the \( 2 \times 2 \) minors of \( M \) and corresponds to the ideal \( I_D^p \) in [P] Section 3.1.

For \( 1 \leq p \leq n - 1 \), we denote by \( A^p \) the \( n \times n \) skew-symmetric matrix with \((ij)\)-entry (for \( i < j \)) equal to \( A_{ij}^p \) (and zero diagonal entries), and by \( B^p \) the \( n \times n \) symmetric matrix with \((ij)\)-entry (for \( i \leq j \)) equal to \( B_{ij}^p \).

For \( 1 \leq p \leq n - 1 \) we set
\[
f_p = \sum_{1 \leq i < j \leq n} A_{ij}^p F_{ij} + \sum_{i=1}^{n} B_{ii}^p G_{ii} + \sum_{1 \leq i < j \leq n} B_{ij}^p G_{ij},
\]
where,
\[
F_{ij} = x_i y_j - x_j y_i, \quad G_{ij} = 2(y_i y_j - z x_i x_j), \quad G_{mm} = y_m^2 - z x_m^2
\]
for \( 1 \leq i < j \leq n \) and \( 1 \leq m \leq n \). In matrix notation
\[
f_p = x A^p y^t + y B^p y^t - z x B^p x^t.
\] (2.4)

The ideal \( I_X \) in (2.1) is the ideal
\[
I_X = (f_1, \ldots, f_{n-1})
\]
and corresponds to the ideal \( I_X^p \) in [P] Section 3.1.

In addition, we set
\[
x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).
\] (2.5)
Remark 2.1 According to [P] Proposition 2.16, the ideal \( I_Y \) of \( \mathcal{O}_{amb}[s_0, s_1] \) contains \( I_X \) as a subset and is generated by \( I_X \) together with \( n \) polynomials \( l_1, \ldots, l_n \in \mathcal{O}_{amb}[s_0, s_1] \) affine linear in \( s_0, s_1 \) of the form
\[
l_i = zx_is_0 + y_is_1 + \sigma^1_i,
\]
where \( 1 \leq i \leq n \) and \( \sigma^1_i \in \mathcal{O}_{amb} \), together with \( n \) polynomials \( l_{n+1}, \ldots, l_{2n} \in \mathcal{O}_{amb}[s_0, s_1] \) affine linear in \( s_0, s_1 \) of the form
\[
l_{i+n} = y_is_0 + x_is_1 + \sigma^2_i,
\]
where \( 1 \leq i \leq n \) and \( \sigma^2_i \in \mathcal{O}_{amb} \), together with a single affine quadratic polynomial \( q \in \mathcal{O}_{amb}[s_0, s_1] \) of the form
\[
q = s_1^2 - zs_0^2 + \text{lower terms},
\]
where 'lower terms' means an affine linear polynomial in \( s_0, s_1 \), i.e., of the form \( e_1s_0 + e_2s_1 + e_3 \) with \( e_1, e_2, e_3 \in \mathcal{O}_{amb} \).

In other words, we have the equality
\[
I_Y = (f_1, \ldots, f_{n-1}) + (l_1, \ldots, l_{2n}) + (q)
\]
(2.6)
of ideals of \( \mathcal{O}_{amb}[s_0, s_1] \). Calculating \( I_Y \) means providing explicit formulas for \( l_1, l_2, \ldots, l_{2n} \) and \( q \).

For simplicity, we call the elements of \( I_X \) the original relations, we call \( l_1, \ldots, l_{2n} \) the linear relations and, finally, we call \( q \) the quadratic relation.

For the parameter value \( n = 2 \) the ideal \( I_Y \) has been explicitly calculated by Reid in [R] Section 9.5. For completeness, we reproduce his results in Subsection 4.4.

In Theorem 3.16 we calculate, for any \( n \geq 2 \), the linear relations \( l_1, \ldots, l_{2n} \). In addition, in Theorem 4.1 we calculate the quadratic polynomial \( q \) for the parameter value \( n = 3 \).

The calculation of the quadratic polynomial \( q \) for \( n \geq 4 \) remains open.

3 Linear relations of Type II₁ unprojection

In this section we use the notations of Section 2.
3.1 Generalities

Assume $R$ is a ring, $L$ is an $R$-module, and $h: L \to R$ a homomorphism of $R$-modules. For $p \geq 1$, [BH] p. 43 defines an $R$-homomorphism $d^p h : \wedge^p L \to \wedge^{p-1} L$ by

$$d^p h(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} h(v_i) v_1 \wedge \cdots \hat{v}_i \wedge \cdots \wedge v_p,$$

(3.1)

for all $v_1, \ldots, v_p \in L$. (For $p = 1$ we set $\wedge^0 L = R$ and $d^1 h = h$.) The maps $d^p h$ define the Koszul complex

$$\cdots \to \wedge^p L \xrightarrow{d^p h} \wedge^{p-1} L \to \cdots \to \wedge^2 L \xrightarrow{d^2 h} L \xrightarrow{h} R \to 0$$

associated to the homomorphism $h$.

**Proposition 3.1** Assume $h_1, h_2 : L \to R$ are two homomorphisms of $R$-modules. For $p \geq 2$ we have the equality

$$d^{p-1} h_1 \circ d^p h_2 + d^{p-1} h_2 \circ d^p h_1 = 0,$$

of maps $\wedge^p L \to \wedge^{p-2} L$.

**Proof** Indeed,

$$0 = d^{p-1} (h_1 + h_2) \circ d^p (h_1 + h_2) = d^{p-1} h_1 \circ d^p h_2 + d^{p-1} h_2 \circ d^p h_1,$$

since

$$d^{p-1} h_1 \circ d^p h_1 = d^{p-1} h_2 \circ d^p h_2 = 0.$$

QED

3.2 The second complex

Assume $L$ is a free $O_{amb}$-module of rank $n$, and let $e_1, \ldots, e_n$ be a fixed basis of $L$. We define four $O_{amb}$-homomorphisms $h_1, \ldots, h_4 : L \to O_{amb}$, with

$$h_1(e_i) = y_i, \quad h_2(e_i) = x_i, \quad h_3(e_i) = zx_i, \quad h_4(e_i) = y_i,$$

for $1 \leq i \leq n$. In addition, we define, for $1 \leq p \leq n$, homomorphisms

$$\phi_p : \wedge^p L \oplus \wedge^p L \to \wedge^{p-1} L \oplus \wedge^{p-1} L$$
where by definition $\wedge^0 L = \mathcal{O}_{amb}$ as follows:

$$
\phi_p(a, b) = \begin{pmatrix}
  d^p h_1 & d^p h_3 \\
  d^p h_2 & d^p h_4
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix} = (d^p h_1(a) + d^p h_3(b), d^p h_2(a) + d^p h_4(b)),
$$

for $a, b \in \wedge^p L$.

**Proposition 3.2** For $p \geq 2$ we have

$$
\phi_{p-1} \circ \phi_p = 0.
$$

**Proof** Using the equalities $h_4 = h_1$ and $h_3 = zh_2$ the result follows from Proposition 3.1. QED

The proof of the following proposition will be given in the Subsection 3.3.

**Proposition 3.3** The complex

$$
L^* : \mathcal{O}_{amb} \oplus \mathcal{O}_{amb} \xrightarrow{\phi_1} L \oplus L \xrightarrow{\phi_2} \wedge^2 L \oplus \wedge^2 L \xrightarrow{\phi_3} \wedge^3 L \oplus \wedge^3 L \leftarrow \ldots \quad (3.2)
$$

is exact.

### 3.3 Proof of Proposition 3.3

The proof of Proposition 3.3 will be based on the Buchsbaum–Eisenbud acyclicity criterion as stated in [BH] Theorem 1.4.13. We first need the following combinatorial lemma.

**Lemma 3.4** Let $p$ be an integer with $1 \leq p \leq n$. Then

$$
\sum_{j=p}^{n} (-1)^{j-p} \binom{n}{j} = \binom{n-1}{p-1}.
$$

**Proof** We have

$$
\sum_{j=p}^{n} (-1)^{j-p} \binom{n}{j} = (-1)^{n-p} + \sum_{j=p}^{n-1} (-1)^{j-p} \binom{n}{j}
$$

$$
= (-1)^{n-p} + \sum_{j=p}^{n-1} (-1)^{j-p} \left( \binom{n-1}{j-1} + \binom{n-1}{j} \right)
$$

$$
= (-1)^{n-p} + \binom{n-1}{p-1} + (-1)^{n-1-p} = \binom{n-1}{p-1},
$$

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where we used the Pascal’s rule \( \binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j} \) which is valid whenever \( 1 \leq j \leq n - 1 \). QED

For the following we fix an integer \( p \) with \( 1 \leq p \leq n \). We set
\[
r_p = \sum_{j=p}^{n} (-1)^{j-p} \text{rank} (\wedge^p L \oplus \wedge^p L),
\]
where for a finitely generated free \( \mathcal{O}_{amb} \)-module \( N \) we denote by \( \text{rank} N \) the minimal number of generators of \( N \) as \( \mathcal{O}_{amb} \)-module. Using Lemma 3.4 we get
\[
r_p = 2 \binom{n-1}{p-1}.
\]
(3.3)

Denote by \( I_{r_p}(\phi_p) \) the \( r_p \)-th Fitting ideal of the map \( \phi_p \), that is the ideal of \( \mathcal{O}_{amb} \) generated by the \( r_p \times r_p \) minors of any matrix representation of \( \phi_p \). (For more information about Fitting ideals see, for example, \([BH]\) Section 1.4.)

We will need the following general property of Koszul complexes.

**Lemma 3.5** Fix \( t \in \{1, 2\} \) and \( s, p \) with \( 1 \leq s, p \leq n \). Denote by \( M_t^p \) the matrix representation of the map
\[
d^p h_t : \wedge^p L \rightarrow \wedge^{p-1} L
\]
with respect to the bases \( e_{i_1} \wedge \cdots \wedge e_{i_p} \) of \( \wedge^p L \) and \( e_{i_1} \wedge \cdots \wedge e_{i_{p-1}} \) of \( \wedge^{p-1} L \) induced by the basis \( (e_1, \ldots, e_n) \) of \( L \). For a suitable ordering of the two bases there exists an \( (r_p/2) \times (r_p/2) \) submatrix of \( M_t^p \) which is diagonal with diagonal entries equal to \( a_t \) or \(-a_t\), where \( a_1 = y_s \) and \( a_2 = x_s \).

**Proof** Consider the subset of the above mentioned basis of \( \wedge^p L \) consisting of the elements where \( e_s \) appears on the wedge. This subset has \( \binom{n-1}{p-1} \) elements, hence \( (r_p/2) \) elements using (3.3). The result follows by restricting the homomorphism \( d^p h_t \) to the \( \mathcal{O}_{amb} \)-submodule of \( \wedge^p L \) generated by this subset and noticing that the corresponding matrix has a diagonal submatrix of the desired form. QED

**Proposition 3.6** Fix \( s, p \) with \( 1 \leq s, p \leq n \). There exists a polynomial \( g_s \in I_{r_p}(\phi_p) \) of the form
\[
g_s = y^r_p + g^1_s,
\]
(3.4)
where \( g^1_s \) involves only the variables \( y_s, z, x_s \) and has degree in \( y_s \) strictly less than \( r_p \).
Proof  It follows immediately from the definition of $\phi_p$ and Lemma 3.5. QED

Proposition 3.7 Let $F$ be either the field $\mathbb{Q}$ of rational numbers or a finite field $\mathbb{Z}/(q)$, where $q$ is a prime. The ideal $\tilde{I}$ of $O_{amb} \otimes_{\mathbb{Z}} F$ generated by the image of $I_r(\phi_p)$ under the natural map $O_{amb} \to O_{amb} \otimes_{\mathbb{Z}} F$ has codimension greater or equal than $n$.

Proof  Using Proposition 3.6 it is clear that there exists a monomial ordering for the variables $y_j, z, x_j$ (with indices $1 \leq j \leq n$) such that the leading term of the polynomial $g_s$ appearing in (3.4) is equal to $y_r^p$, for $1 \leq s \leq n$. Since over the field $F$ an ideal $\tilde{I}$ and the leading term ideal of $\tilde{I}$ have the same codimension the result follows. QED

Recall (cf. [BH] Section 1.2) that the grade of a proper ideal $I \subset O_{amb}$ is the length of a maximal $O_{amb}$-regular sequence contained in $I$.

Proposition 3.8 The ideal $I_r(\phi_p)$ has grade greater or equal than $n$.

Proof  Combining Proposition 3.7 and [L] Theorem 3.12 we get that $I_r(\phi_p)$ has codimension in $O_{amb}$ greater or equal than $n$. Being a polynomial ring over the integers, the ring $O_{amb}$ is Cohen–Macaulay. Using [BH] Corollary 2.14 the result follows. QED

We now finish the proof of Proposition 3.3. Combining Lemma 3.4 and Proposition 3.8, Proposition 3.3 follows from the Buchsbaum–Eisenbud acyclicity criterion as stated in [BH] Theorem 1.4.13.

3.4 The first complex

Fix $n \geq 2$. Let $N$ be a free $O_{amb}$-module of rank $n - 1$, with basis $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$.

We define an $O_{amb}$-homomorphism $\psi : N \to O_{amb}$, with $\psi(\tilde{e}_p) = f_p$, for $1 \leq p \leq n - 1$, where $f_p$ was defined in (2.4). As in Subsection 3.1, we have the Koszul complex

$$N^* : O_{amb} \xleftarrow{\psi} N \xrightarrow{d^2} \wedge^2 N \xrightarrow{d^3} \wedge^3 N \leftarrow \ldots$$  \hspace{1cm} (3.5)

Proposition 3.9 The sequence $f_1, \ldots, f_{n-1}$ is an $O_{amb}$-regular sequence. As a consequence, the complex $N^*$ is exact.
Proof For an element $g$ of $\mathcal{O}_{amb}$ we denote by $\eta(g)$ the result of substituting to $g$ zero for $z$, zero for $A_{ij}^p$ with $1 \leq p \leq n - 1$ and $1 \leq i < j \leq n$, and zero for $B_{ij}^p$ for $1 \leq p \leq n - 1$ and $1 \leq i \leq j \leq n$ with $(i, j) \neq (p, p)$. That is, we set zero the variable $z$ and all possible $A_{ij}^p$ and all possible $B_{ij}^p$ with the exception of $B_{pp}^p$ (for $1 \leq p \leq n - 1$).

It is clear that for $1 \leq p \leq n - 1$ we have

$$\eta(f_p) = B_{pp}^p(y_p)^2.$$ 

Therefore, the ideal $(\eta(f_1), \ldots, \eta(f_{n-1})) \subset \mathcal{O}_{amb}$ has codimension in $\mathcal{O}_{amb}$ equal to $n - 1$. As a consequence, the ideal $(f_1, \ldots, f_{n-1})$ of $\mathcal{O}_{amb}$ has also codimension $n - 1$. Since $\mathcal{O}_{amb}$ is Cohen-Macaulay the sequence $f_1, \ldots, f_{n-1}$ is an $\mathcal{O}_{amb}$-regular sequence. The exactness of the Koszul complex $N^*$ follows from [BH] Corollary 1.6.14. QED

3.5 The first commutative square

We define two $\mathcal{O}_{amb}$-homomorphisms $u_1, u_2: N \rightarrow L$, that will make the following diagram

$$\begin{array}{ccc}
\mathcal{O}_{amb} & \xrightarrow{\psi} & N \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \downarrow & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
\mathcal{O}_{amb} \oplus \mathcal{O}_{amb} & \xleftarrow{\phi} & L \oplus L
\end{array}$$

commutative. That is, they will satisfy the two conditions

$$\psi = h_1 \circ u_1 + zh_2 \circ u_2 \quad (3.6)$$

$$0 = h_1 \circ u_2 + h_2 \circ u_1 \quad (3.7)$$

We set, for $1 \leq j \leq m$,

$$u_1(\tilde{e}_j) = \sum_{i=1}^n c_{ij} e_i \quad \text{and} \quad u_2(\tilde{e}_j) = \sum_{i=1}^n c'_{ij} e_i,$$

with

$$c_{ij} = -\sum_{l=1}^{i-1} A_{il}^j x_l + \sum_{l=i+1}^n A_{il}^j x_l + \sum_{l=1}^i B_{il}^j y_l + \sum_{l=i+1}^n B_{il}^j y_l,$$

and

$$c'_{ij} = -\sum_{l=1}^i B_{il}^j x_l - \sum_{l=i+1}^n B_{il}^j x_l$$
for $1 \leq i \leq n$ and $1 \leq j \leq n - 1$.

In matrix notation, $u_1$ and $u_2$ are given, with respect to the above defined bases $(\bar{e}_j)$ of $N$ and $(e_i)$ of $L$, by the following two $n \times (n - 1)$ matrices:

$$
u_1 = (-A^1 x^t + B^1 y^t \mid - A^2 x^t + B^2 y^t \mid \ldots \mid - A^{n-1} x^t + B^{n-1} y^t)$$
$$
u_2 = (-B^1 x^t \mid - B^2 x^t \mid \ldots \mid - B^{n-1} x^t). \tag{3.8}$$

That is, for $1 \leq j \leq n - 1$, $u_1$ is the matrix with $j$-th column equal to $-A^j x^t + B^j y^t$, while $u_2$ is the matrix with $j$-th column equal to $-B^j x^t$.

**Proposition 3.10** The maps $u_1$ and $u_2$ satisfy the equations (3.6) and (3.7).

**Proof** An easy calculation using (3.8). QED

### 3.6 The basic formula

The key result of the present subsection is Proposition 3.12, essentially a formal consequence of (3.7), which relates the differentials and the alternating products of the homomorphisms $h_i$ and $u_j$. We need the following definitions.

Assume $a_1, a_2 : N \to L$ are two $\mathcal{O}_{amb}$-homomorphisms.

For $p \geq 0$ we have a natural induced homomorphism $\wedge^p a_1 : \wedge^p N \to \wedge^p L$. By definition, $\wedge^0 a_1$ is the identity map $\mathcal{O}_{amb} \to \mathcal{O}_{amb}$. For $p \geq 1$, the map $\wedge^p a_1$ is uniquely specified by the property

$$\wedge^p a_1(c_1 \wedge \cdots \wedge c_p) = a_1(c_1) \wedge \cdots \wedge a_1(c_p)$$

for all $c_1, \ldots, c_p \in N$. We denote the map $\wedge^p a_1$ also by $\text{alt}(a_1^p)$.

We will now define for $p, q \geq 1$ an $\mathcal{O}_{amb}$-homomorphism

$$\text{alt}(a_1^p, a_2^q) : \wedge^{p+q} N \to \wedge^{p+q} L.$$  

We set

$$\text{alt}(a_1^p, a_2^q) = \sum_{I} w_I,$$

where the sum is over all subsets $I \subseteq \{1, \ldots, p + q\}$, with $|I| = p$ and, by definition, $w_I = w_1 \wedge \cdots \wedge w_{p+q}$ with $w_i = a_1$ if $i \in I$ and $w_i = a_2$ if $i \notin I$.

More precisely, even though the tensor product map

$$w_1 \otimes \cdots \otimes w_{p+q} : N^\otimes(p+q) \to L^\otimes(p+q)$$
does not induce a map $\wedge^{p+q}N \to \wedge^{p+q}L$, taking the sum of those maps over the family of subsets $I$ as above induces a well-defined map $\wedge^{p+q}N \to \wedge^{p+q}L$, and this is the map denoted by $\text{alt}(a^p_1, a^q_2)$.

**Remark 3.11** For $p = q = 1$ we have

$$\text{alt}(a_1, a_2)(c_1 \land c_2) = a_1(c_1) \land a_2(c_2) + a_2(c_1) \land a_1(c_2),$$

for all $c_1, c_2 \in N$. We will also denote $\text{alt}(a_1, a_2)$ by $a_1 \land a_2 + a_2 \land a_1$.

For $p = 2, q = 1$ we have

$$\text{alt}(a^2_1, a_2) = a_1 \land a_1 \land a_2 + a_1 \land a_2 \land a_1 + a_2 \land a_1 \land a_1,$$

in the sense that

$$\text{alt}(a^2_1, a_2)(c_1 \land c_2 \land c_3) = a_1(c_1) \land a_1(c_2) \land a_2(c_3) + a_1(c_1) \land a_2(c_2) \land a_1(c_3) + a_1(c_1) \land a_2(c_2) \land a_1(c_3) + a_2(c_1) \land a_1(c_2) \land a_1(c_3)$$

for all $c_1, c_2, c_3 \in N$.

For the following proposition, which will be proved in Subsection 3.7, recall that $u_1, u_2: N \to L$ were defined in Subsection 3.5, and $h_1, h_2: L \to \mathcal{O}_{amb}$ were defined in Subsection 3.2. We also use the notational conventions $\text{alt}(u^0_1, u^0_2) = \text{alt}(u^1_1), \text{alt}(u^0_1, u^2_2) = \text{alt}(u^2_1), \text{alt}(u^{p-2}_1, u^2_2) = 0$ for $p = 1$ and $q \geq 1$, and $\text{alt}(u^p_1, u^{q-2}_2) = 0$ for $q = 1$ and $p \geq 1$.

**Proposition 3.12** a) Let $u: M \to L$ and $h: L \to \mathcal{O}_{amb}$ be two (arbitrary) homomorphisms of $\mathcal{O}_{amb}$-modules. Assume $p \geq 2$. We have

$$d^{p-1}h \circ \text{alt}(u^{p-1}) = \text{alt}(u^{p-2}) \circ d^{p-1}(h \circ u).$$

b) For $p, q \geq 1$ we have

$$(d^{p+q-1}h_2) \circ \text{alt}(u^p_1, u^{q-1}_2) + (d^{p+q-1}h_1) \circ \text{alt}(u^{p-1}_1, u^q_2) = \text{alt}(u^{p-2}_1, u^{q}_2) \circ d^{p+q-1}(h_1 \circ u_1) + \text{alt}(u^{p}_1, u^{q-2}_2) \circ d^{p+q-1}(h_2 \circ u_2),$$

(3.9)

where the equality is, of course, as maps $\wedge^{p+q-1}N \to \wedge^{p+q-2}L$.

**3.7 Proof of Proposition 3.12**

We will use the following two lemmas.
Lemma 3.13 Assume $p, q \geq 1$ and $u_1, u_2 : N \to L$ are two $O_{amb}$-module homomorphisms. We have

$$\text{alt}(u_1^p, u_2^q) = u_1 \wedge \text{alt}(u_1^{p-1}, u_2^q) + u_2 \wedge \text{alt}(u_1^p, u_2^{q-1}),$$

in the sense that

$$\text{alt}(u_1^p, u_2^q)(c \wedge c_1) = u_1(c) \wedge \left[\text{alt}(u_1^{p-1}, u_2^q)(c_1)\right] + u_2(c) \wedge \left[\text{alt}(u_1^p, u_2^{q-1})(c_1)\right]$$

for all $c \in N$ and $c_1 \in \wedge^{p+q-1}N$.

**Proof** Immediate from the definitions. QED

Lemma 3.14 Assume $p, q \geq 1$, $u_1, u_2 : N \to L$ are two $O_{amb}$-module homomorphisms, and $h : N \to O_{amb}$ is an $O_{amb}$-homomorphism. We have

$$\text{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h) = h \wedge \text{alt}(u_1^p, u_2^q) - u_1 \wedge \left[\text{alt}(u_1^{p-1}, u_2^q) \circ d^{p+q}h\right] - u_2 \wedge \left[\text{alt}(u_1^p, u_2^{q-1}) \circ d^{p+q}h\right],$$

in the sense that

$$\left[\text{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h)\right](c \wedge c_1) = h(c)\left[\text{alt}(u_1^p, u_2^q)(c_1)\right] - u_1(c) \wedge \left(\text{alt}(u_1^{p-1}, u_2^q) \circ d^{p+q}h\right)(c_1) - u_2(c) \wedge \left(\text{alt}(u_1^p, u_2^{q-1}) \circ d^{p+q}h\right)(c_1)$$

for all $c \in N$ and $c_1 \in \wedge^{p+q}N$.

**Proof** Since

$$d^{p+q+1}h(c \wedge c_1) = h(c)c_1 - c \wedge d^{p+q}h(c_1),$$

we have, using Lemma 3.13, that

$$\text{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h)(c \wedge c_1) = h(c)\left(\text{alt}(u_1^p, u_2^q)(c_1)\right) - \left[u_1 \wedge \text{alt}(u_1^{p-1}, u_2^q) + u_2 \wedge \text{alt}(u_1^p, u_2^{q-1})\right](c \wedge d^{p+q}h(c_1))$$

and the result follows. QED
We will now prove part a) of Proposition 3.12. Assume \( p \geq 2 \). For \( c_1, \ldots, c_{p-1} \in N \) we have

\[
d^{p-1} h \circ \text{alt}(u^{p-1})(c_1 \land \cdots \land c_{p-1}) = d^{p-1} h(u(c_1) \land \cdots \land u(c_{p-1}))
\]

\[
= \sum_{i=1}^{p-1} (-1)^{i-1} h(u(c_i))(u(c_1) \land \cdots \land u(c_i) \cdots \land u(c_{p-1}))
\]

\[
= \text{alt}(u^{p-2}) \circ d^{p-1}(h \circ u)(c_1 \land \cdots \land c_{p-1}).
\]

We will now prove part b) of Proposition 3.12. Assume that \( p, q \geq 1 \), that \( N, L \) are \( \mathcal{O}_{amb} \)-modules and that \( u_1, u_2 : N \to L \) and \( h_1, h_2 : L \to \mathcal{O}_{amb} \) are four \( \mathcal{O}_{amb} \)-module homomorphisms with the property

\[
h_1 \circ u_2 + h_2 \circ u_1 = 0,
\]

(3.11)

cf. (3.7). We will show by induction on \( p + q \) that (3.9) holds. If \( p = q = 1 \) the result is clear, since it is exactly our hypothesis (3.11).

Assume now that \( p + q \geq 3 \) and that (3.9) holds for the \((p-1, q)\) and \((p, q-1)\) cases.

Using Lemma 3.13 we have

\[
d^{p+q-1}h_2 \circ \text{alt}(u_1^p, u_2^q) + d^{p+q-1}h_1 \circ \text{alt}(u_1^{p-1}, u_2^q) = \]

\[
d^{p+q-1}h_2 \circ [u_1 \land \text{alt}(u_1^{p-1}, u_2^q) + u_2 \land \text{alt}(u_1^p, u_2^{q-2})] \]

\[
+ d^{p+q-1}h_1 \circ [u_1 \land \text{alt}(u_1^{p-2}, u_2^q) + u_2 \land \text{alt}(u_1^{p-1}, u_2^{q-1})]
\]

which, using (3.10), is equal to

\[
(h_2 \circ u_1) \land \text{alt}(u_1^{p-1}, u_2^q) - u_1 \land [d^{p+q-2}h_2 \circ \text{alt}(u_1^{p-1}, u_2^{q-1})]
\]

\[
+ (h_2 \circ u_2) \land \text{alt}(u_1^p, u_2^{q-2}) - u_2 \land [d^{p+q-2}h_2 \circ \text{alt}(u_1^p, u_2^{q-2})]
\]

\[
+ (h_1 \circ u_1) \land \text{alt}(u_1^{p-2}, u_2^q) - u_1 \land [d^{p+q-2}h_1 \circ \text{alt}(u_1^{p-2}, u_2^q)]
\]

\[
+ (h_1 \circ u_2) \land \text{alt}(u_1^{p-1}, u_2^{q-1}) - u_2 \land [d^{p+q-2}h_1 \circ \text{alt}(u_1^{p-1}, u_2^{q-1})]
\]

which, using (3.11), is equal to

\[
-u_1 \land [d^{p+q-2}h_2 \circ \text{alt}(u_1^{p-1}, u_2^q) + d^{p+q-2}h_1 \circ \text{alt}(u_1^{p-2}, u_2^q)]
\]

\[
-u_2 \land [d^{p+q-2}h_2 \circ \text{alt}(u_1^p, u_2^{q-2}) + d^{p+q-2}h_1 \circ \text{alt}(u_1^{p-1}, u_2^{q-1})]
\]

\[
+ (h_2 \circ u_2) \land \text{alt}(u_1^p, u_2^{q-2}) + (h_1 \circ u_1) \land \text{alt}(u_1^{p-2}, u_2^q)
\]
which, using the inductive hypothesis, is equal to

\[
(h_2 \circ u_2) \land \text{alt}(u_1^p, u_2^{q-2}) - u_1 \land [\text{alt}(u_1^{p-1}, u_2^{q-2}) \circ d^{p+q-2}(h_2 \circ u_2)]
\]

\[
- u_2 \land [\text{alt}(u_1^p, u_2^{q-3}) \circ d^{p+q-2}(h_2 \circ u_2)]
\]

\[
+(h_1 \circ u_1) \land \text{alt}(u_1^{p-2}, u_2^q) - u_1 \land [\text{alt}(u_1^{p-3}, u_2^q) \circ d^{p+q-2}(h_1 \circ u_1)]
\]

\[
- u_2 \land [\text{alt}(u_1^{p-2}, u_2^{q-1}) \circ d^{p+q-2}(h_1 \circ u_1)]
\]

which, using Lemma 3.14, is equal to

\[
\text{alt}(u_1^{p-2}, u_2^q) \circ d^{p+q-1}(h_1 \circ u_1) + \text{alt}(u_1^p, u_2^{q-2}) \circ d^{p+q-1}(h_2 \circ u_2)
\]

which finishes the proof of Proposition 3.12.

3.8 The connecting homomorphisms

For an integer \( p \), with \( 1 \leq p \leq n - 1 \), we set

\[
T_p = \left( \begin{array}{c}
\text{alt}(u_1^p) + z \text{alt}(u_1^{p-2}, u_2^q) + z^2 \text{alt}(u_1^{p-4}, u_2^q) + \ldots \\
\text{alt}(u_1^{p-1}, u_2) + z \text{alt}(u_1^{p-3}, u_2^q) + z^2 \text{alt}(u_1^{p-5}, u_2^q) + \ldots 
\end{array} \right),
\]

with the summation as long as all exponents are nonnegative. More precisely, using the notational convention \( z^0 = 1 \), and denoting, for \( j = 1, 2 \), by \( T_p^j \) the \( j \)-th row of \( T_p \), we have

\[
T_p^1 = \sum_{i=0}^{\left\lfloor \frac{p}{2} \right\rfloor} z^i \text{alt}(u_1^{p-2i}, u_2^{2i}), \quad T_p^2 = \sum_{i=0}^{\left\lfloor \frac{p-1}{2} \right\rfloor} z^i \text{alt}(u_1^{p-2i-1}, u_2^{2i+1}).
\]

We consider \( T_p \) as an \( \mathcal{O}_{amb} \)-homomorphism \( \wedge^p N \to \wedge^p L \oplus \wedge^p L \). Taking the two rows of \( T_p \) we get two \( \mathcal{O}_{amb} \)-homomorphisms \( T_p^1, T_p^2 : \wedge^p N \to \wedge^p L \).

For example,

\[
T_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \wedge^2 u_1 + z \wedge^2 u_2 \\ u_1 \land u_2 + u_2 \land u_1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \wedge^3 u_1 + z \text{alt}(u_1, u_2^q) \\ \text{alt}(u_1^3, u_2) + z \text{alt}(u_1, u_2^3) \end{pmatrix},
\]

\[
T_4 = \begin{pmatrix} \wedge^4 u_1 + z \text{alt}(u_1^3, u_2^q) + z^2 \wedge^4 u_2 \\ \text{alt}(u_1^3, u_2^q) + z \text{alt}(u_1, u_2^3) \end{pmatrix}, \quad T_5 = \begin{pmatrix} \wedge^5 u_1 + z \text{alt}(u_1^3, u_2^q) + z^2 \text{alt}(u_1, u_2^4) \\ \text{alt}(u_1^4, u_2) + z \text{alt}(u_1^3, u_2^q) + z^2 \wedge^5 u_2 \end{pmatrix}.
\]

For the following proposition recall that the map \( \phi_p \) was defined in Subsection 3.2 while the map \( \psi \) was defined in Subsection 3.4.
Proposition 3.15 For any $p$, with $2 \leq p \leq n - 1$, we have a commutative diagram

$$
\begin{array}{ccc}
\wedge^{p-1}N & \overset{d^p \psi}{\longrightarrow} & \wedge^{p}N \\
T_{p-1} \downarrow & & \downarrow T_p \\
\wedge^{p-1}L \oplus \wedge^{p-1}L & \overset{\phi_p}{\longrightarrow} & \wedge^{p}L \oplus \wedge^{p}L
\end{array}
$$

That is,

$$
\phi_p \circ T_p = T_{p-1} \circ d^p \psi
$$
as maps

$$
\wedge^{p}N \rightarrow \wedge^{p-1}L \oplus \wedge^{p-1}L.
$$

Proof Fix $p \geq 2$. For $i \in \mathbb{Z}$ we set

$$
C_i = \text{alt}(u_1^i, u_2^{p-i})
$$
if $0 \leq i \leq p$ and $C_i = 0$ otherwise, and we set

$$
D_i = \text{alt}(u_1^i, u_2^{p-1-i})
$$
if $0 \leq i \leq p - 1$ and $D_i = 0$ otherwise.

For the rest of the proof all sums are for $i \in \mathbb{Z}$.

We have

$$
T_{p-1}^1 = \sum z^i C_{p-2i}, \quad T_p^1 = \sum z^i C_{p-2i-1},
$$

$$
T_{p-1}^2 = \sum z^i D_{(p-1)-2i}, \quad T_p^2 = \sum z^i D_{(p-1)-2i-1}.
$$

By Proposition 3.12, for any $i \in \mathbb{Z}$ we have

$$
d^p h_1 \circ C_{i-1} + d^p h_2 \circ C_i = D_{i-2} \circ d^p (h_1 \circ u_1) + D_i \circ d^p (h_2 \circ u_2). \quad (3.12)
$$

Using (3.6), we need to show that

$$
d^p h_1 \circ T_p^1 + zd^p h_2 \circ T_p^2 = T_{p-1}^1 \circ d^p (h_1 \circ u_1 + zh_2 \circ u_2)
$$

and that

$$
d^p h_2 \circ T_p^1 + d^p h_1 \circ T_p^2 = T_{p-1}^2 \circ d^p (h_1 \circ u_1 + zh_2 \circ u_2).
$$

Using (3.12) we get

$$
d^p h_1 T_p^1 + zd^p h_2 T_p^2 = \sum (z^i (d^p h_1 C_{p-2i} + d^p h_2 C_{p-2i+1})).
$$
\[
= \sum [z^i (D_{p-2i-1} d^p (h_1 u_1) + D_{p-2i+1} d^p (h_2 u_2))]
\]
\[
= \sum [z^i (D_{(p-1)-2i} (d^p (h_1 u_1) + zd^p (h_2 u_2)))] = T_{p-1}^1 d^p (h_1 u_1 + zh_2 u_2).
\]
Similarly,
\[
d^p h_2 T_p^1 + d^p h_1 T_p^2 = \sum [z^i (d^p h_2 C_{p-2i} + d^p h_1 C_{p-2i-1})]
\]
\[
= \sum [z^i (D_{p-2i-2} d^p (h_1 u_1) + D_{p-2i} d^p (h_2 u_2))]
\]
\[
= \sum [z^i (D_{(p-1)-2i-1} (d^p (h_1 u_1) + zd^p (h_2 u_2)))] = T_{p-1}^2 d^p (h_1 u_1 + zh_2 u_2).
\]
QED

3.9 Linear equations of type \(\Pi_1\) unprojection

Assume \(n \geq 2\) is given. We use the notations of the previous subsections. In Subsection 3.8 we defined two \(O_{amb}\)-module homomorphisms \(T_{n-1}^1, T_{n-1}^2 : \Lambda^{n-1} N \rightarrow \Lambda^{n-1} L\).

There exist, for \(j = 1, 2\) and \(1 \leq i \leq n\), unique element \(\sigma_i^j \in O_{amb}\) such that
\[
T_{n-1}^j (\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-1}) = \sum_{i=1}^{n} (-1)^{i-1} \sigma_i^j e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n.
\]

For \(1 \leq i \leq n\) we define the polynomials \(l_i, l_{i+n} \in O_{amb}[s_0, s_1]\) with
\[
l_i = zx_i s_0 + y_i s_1 + \sigma_i^1
\]
\[
l_{i+n} = y_i s_0 + x_i s_1 + \sigma_i^2
\]
(3.13)

Using Proposition 3.15 and arguing as in [R] Section 9.5 we get the following Theorem.

**Theorem 3.16** Fix a parameter value \(n \geq 2\). The polynomials \(l_1, \ldots, l_{2n} \in O_{amb}[s_0, s_1]\) specified in (3.13) are the linear relations of the type \(\Pi_1\) unprojection in the sense of Remark 2.1.
4 Quadratic relation of Type $\text{II}_1$ for $n = 3$

Assume now that we are in the type $\text{II}_1$ unprojection with $n = 3$. We follow the notations of Section 2 with only one change: Since our symmetric formula (4.2) for the quadratic relation will need $1/2$ as coefficient, our ambient ring $\mathcal{O}_{amb}$ will now be

$$\mathcal{O}_{amb} = \mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_3, y_1, \ldots, y_3, z, A_{ij}^p, B_{lm}^p],$$

with indices $1 \leq p \leq 2$ and $1 \leq i < j \leq 3$, $1 \leq l \leq m \leq 3$.

Recall that in Section 2 we defined two symmetric $3 \times 3$ matrices $B^1$ and $B^2$ by

$$B^j = \begin{pmatrix} B_{11}^j & B_{12}^j & B_{13}^j \\ \text{sym} & B_{22}^j & B_{23}^j \\ \text{sym} & B_{32}^j & B_{33}^j \end{pmatrix},$$

for $j = 1, 2$, two skew–symmetric $3 \times 3$ matrices $A^1$ and $A^2$ by

$$A^j = \begin{pmatrix} 0 & A_{12}^j & A_{13}^j \\ -\text{sym} & 0 & A_{23}^j \\ \text{sym} & -\text{sym} & 0 \end{pmatrix},$$

for $j = 1, 2$, and two $1 \times 3$ matrices $x, y$ with

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3).$$

We now define three induced $3 \times 3$ symmetric matrices $\text{ad} B^1$, $\text{ad} B^2$ and $\text{ad} B^{1,2}$.

For a square $n \times n$ matrix $M$ with entries in a commutative ring $R$ with unit, we denote by $\text{ad} M$ the $n \times n$ matrix with $kl$-th entry, for $1 \leq k, l \leq n$, equal to $(-1)^{k+l}$ times the determinant of the submatrix of $M$ obtained by deleting the $l$-th row and the $k$-th column. It is well-known that we have the identities $M(\text{ad} M) = (\text{ad} M)M = (\det M)I_n$, where $I_n$ is the identity $n \times n$ matrix over $R$.

In our situation, since $B^1$ and $B^2$ are symmetric $3 \times 3$ matrices we have that $\text{ad} B^1$ and $\text{ad} B^2$ are symmetric $3 \times 3$ matrices, such that for $j = 1, 2$

$$\text{ad} B^j = \begin{pmatrix} B_{22}^j B_{33}^j - (B_{23}^j)^2 & -(B_{12}^j B_{33}^j - B_{13}^j B_{23}^j) & B_{12}^j B_{23}^j - B_{13}^j B_{22}^j \\ \text{sym} & B_{11}^j B_{33}^j - (B_{13}^j)^2 & -(B_{11}^j B_{23}^j - B_{13}^j B_{12}^j) \\ B_{11}^j B_{22}^j - (B_{12}^j)^2 & \text{sym} & \text{sym} \end{pmatrix}.$$
Notice that for $1 \leq k \leq 3$, $1 \leq l \leq 3$, we have

$$(\text{ad} B^1)_{kl} = (-1)^{k+l}(B^1_{ij}B^1_{mn} - B^1_{im}B^1_{mj})$$

where $\{i, k, m\} = \{j, l, n\} = \{1, 2, 3\}$ and $i < m$, $j < n$, and similarly for $\text{ad} B^2$.

We define $\text{ad} B^{1,2}$ to be the $3 \times 3$ symmetric matrix, such that, for $1 \leq k \leq 3$, $1 \leq l \leq 3$,

$$(\text{ad} B^{1,2})_{kl} = (-1)^{k+l}(B^1_{ij}B^2_{mn} + B^2_{ij}B^1_{mn} - B^1_{im}B^2_{mj} - B^2_{im}B^1_{mj}),$$

where $\{i, k, m\} = \{j, l, n\} = \{1, 2, 3\}$ and $i < m$, $j < n$. By the above definition we have

$$\begin{align*}
\text{ad} B^{1,2}_{11} &= B^1_{22}B^2_{33} + B^2_{22}B^1_{33} - 2B^1_{23}B^2_{23} \\
\text{ad} B^{1,2}_{12} &= -B^1_{12}B^2_{33} - B^2_{12}B^1_{33} + B^1_{13}B^2_{23} + B^2_{13}B^1_{23} \\
\text{ad} B^{1,2}_{13} &= B^1_{12}B^2_{23} + B^2_{12}B^1_{23} - B^1_{13}B^2_{22} - B^2_{13}B^1_{22} \\
\text{ad} B^{1,2}_{22} &= B^1_{11}B^2_{33} + B^2_{11}B^1_{33} - 2B^1_{13}B^2_{13} \\
\text{ad} B^{1,2}_{23} &= -B^1_{11}B^2_{23} - B^2_{11}B^1_{23} + B^1_{12}B^2_{12} + B^2_{12}B^1_{12} \\
\text{ad} B^{1,2}_{33} &= B^1_{11}B^2_{22} + B^2_{11}B^1_{22} - 2B^1_{12}B^2_{12}
\end{align*}$$

and the rest of the entries of $\text{ad} B^{1,2}$ are determined by symmetry.

We now define two polynomials $a, b \in O_{amb}$. We set

$$a = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ A^1_{23} & -A^1_{13} & A^1_{12} \\ A^2_{23} & -A^2_{13} & A^2_{12} \end{pmatrix} \quad \text{(4.1)}$$

$$= x_1(A^1_{12}A^2_{13} - A^2_{12}A^1_{13}) + x_2(A^1_{12}A^2_{23} - A^2_{12}A^1_{23}) + x_3(A^1_{13}A^2_{23} - A^2_{13}A^1_{23}),$$

and

$$b = \frac{1}{2} \left[ yB^1(\text{ad} B^2)B^1y^t + yB^2(\text{ad} B^1)B^2y^t \\ + \ 2xA^1(\text{ad} B^2)(A^1)^t x^t + 2xA^2(\text{ad} B^1)(A^2)^t x^t \\ - \ 2xA^1(\text{ad} B^{1,2})(A^2)^t x^t \\ - \ zxB^1(\text{ad} B^2)(A^1)^t x^t - zxB^2(\text{ad} B^1)(A^2)^t x^t \\ + \ 4yB^1(\text{ad} B^2)(A^1)^t x^t + 4yB^2(\text{ad} B^1)(A^2)^t x^t \right]$$

$$- \left( \sum_{1 \leq i < j \leq 3} A^1_{ij}(x_i y_j - x_j y_i))( \sum_{1 \leq i, j \leq 3} B^1_{ij} \text{ad} B^2_{ij}) \right)$$

$$- \left( \sum_{1 \leq i < j \leq 3} A^2_{ij}(x_i y_j - x_j y_i))( \sum_{1 \leq i, j \leq 3} B^2_{ij} \text{ad} B^1_{ij}) \right).$$
The two polynomials $f_1, f_2$ corresponding to the ones in (2.4), are

$$f_j = x A^j y^j + y B^j y^j - z x B^j x^j,$$  

(4.3)

for $j = 1, 2$.

We denote by $u_1$ the $3 \times 2$ matrix

$$u_1 = (-A_1 x + B_1 y | - A_2 x + B_2 y)$$

and by $u_2$ the $3 \times 2$ matrix

$$u_2 = (-B_1 x | - B_2 x).$$

These matrices correspond to the maps with the same name defined in (3.8).

For $j = 1, 2$, we set $\tilde{\phi}_1(u^j)$ to be the $3 \times 1$ matrix

$$\tilde{\phi}_1(u^j) = \begin{pmatrix}
    u_1^{j} u_2^{j} - u_2^{j} u_3^{j} \\
    -u_1^{j} u_2^{j} + u_1^{j} u_3^{j} \\
    u_1^{j} u_2^{j} - u_1^{j} u_3^{j}
\end{pmatrix}$$

and we also define the $3 \times 1$ matrix

$$\tilde{\phi}_2(u^1, u^2) = \begin{pmatrix}
    u_1^{1} u_2^{1} + u_2^{1} u_3^{1} - u_2^{2} u_3^{1} - u_2^{2} u_3^{1} \\
    -u_1^{1} u_2^{1} - u_2^{1} u_3^{1} + u_1^{2} u_3^{1} + u_2^{2} u_3^{1} \\
    u_1^{1} u_2^{2} + u_1^{2} u_2^{2} - u_1^{2} u_2^{1} - u_1^{2} u_2^{1}
\end{pmatrix}.$$

The matrix $\tilde{\phi}_1(u^j)$ corresponds to the map $\wedge^2 u_j$ and the matrix $\tilde{\phi}_2(u^1, u^2)$ corresponds to the map $\text{alt}(u_1, u_2)$ which were defined in Subsection 3.6.

We denote, for $1 \leq i \leq 3$,

$$l_i = z x_i s_0 + y_i s_1 + [\tilde{\phi}_1(u^1)]_{1i} + z [\tilde{\phi}_1(u^2)]_{1i}$$

(4.4)

$$l_{i+3} = y_i s_0 + x_i s_1 + [\tilde{\phi}_2(u^1, u^2)]_{1i},$$

where we use the usual notation $M_{ij}$ for the $ij$-th entry of a matrix $M$. Clearly, for $1 \leq i \leq 6$, $l_i \in \mathcal{O}_{\text{amb}}[s_0, s_1]$ is affine linear with respect to $s_0$ and $s_1$. The polynomials $l_1, \ldots, l_6$ correspond to the polynomials with the same name in (3.13), hence by Theorem 3.16 they are the linear relations of the unprojection in the sense of Remark 2.1.

The main result of this section is the following theorem, which specifies the quadratic relation of the unprojection ring for the parameter value $n = 3$. It will be proved in Subsection 4.1.
Theorem 4.1 The polynomial
\[ q = s_1^2 - zs_0^2 - as_0 + b \in \mathcal{O}_{amb}[s_0, s_1], \tag{4.5} \]
where \( a \) was defined in (4.1) and \( b \) was defined in (4.2), is the quadratic relation of the type \( \Pi_1 \) unprojection for the parameter value \( n = 3 \) in the sense of Remark 2.1.

In other words, the ideal \( I_Y \subset \mathcal{O}_{amb}[s_0, s_1] \) defining, in the sense of Section 2, the type \( \Pi_1 \) unprojection is equal to
\[ I_Y = (f_1, f_2) + (l_1, \ldots, l_6) + (q), \]
where the two polynomials \( f_1, f_2 \) generating \( I_X \) were defined in (4.3) and the six polynomials \( l_1, \ldots, l_6 \) were calculated in (4.4).

4.1 Proof of Theorem 4.1

We first prove the following reduction lemma.

Lemma 4.2 To prove Theorem 4.1 it is enough to prove that the element \((x_1 + x_2 + x_3)q\) of \( \mathcal{O}_{amb}[s_0, s_1] \) is inside the ideal \( (f_1, f_2, l_1, \ldots, l_6) \) of \( \mathcal{O}_{amb}[s_0, s_1] \).

Proof Indeed, by [P] Proposition 2.16 there exists \( \tilde{q} \in \mathcal{O}_{amb}[s_0, s_1] \) of the form
\[ \tilde{q} = s_1^2 - zs_0^2 + \tilde{u} \]
where \( \tilde{u} \) is affine linear in \( s_0, s_1 \), such that the ideal of the unprojection ring is equal to \( J \), with
\[ J = (f_1, f_2, l_1, \ldots, l_6, \tilde{q}) \subset \mathcal{O}_{amb}[s_0, s_1]. \]

Since by the assumptions of the lemma \((x_1 + x_2 + x_3)q \in J\), and by [P] Section 3.1 \( J \) is a prime ideal of \( \mathcal{O}_{amb}[s_0, s_1] \) we get \( q \in J \). Therefore, the affine linear with respect to \( s_0, s_1 \) element \( q - \tilde{q} \) is in \( J \), as a consequence (compare proof of [P] Proposition 2.13) we get
\[ q - \tilde{q} \in (f_1, f_2, l_1, \ldots, l_6), \]

hence
\[ J = (f_1, f_2, l_1, \ldots, l_6, q) \]
which finishes the proof of Lemma 4.2. QED
For the rest of the proof we define some more notation:

We define two polynomials \( \tilde{g}_1, \tilde{g}_2 \), such that, for \( j \in \{1, 2\} \), we have

\[
\tilde{g}_j = (x_1 B^j_{11} + x_2 B^j_{12} + x_3 B^j_{13})(ad B^p_{11} + ad B^p_{12} + ad B^p_{13}) \\
+ (x_1 B^j_{21} + x_2 B^j_{22} + x_3 B^j_{23})(ad B^p_{21} + ad B^p_{22} + ad B^p_{23}) \\
+ (x_1 B^j_{31} + x_2 B^j_{32} + x_3 B^j_{33})(ad B^p_{31} + ad B^p_{32} + ad B^p_{33})
\]

where \( p \in \{1, 2\} \) is uniquely specified by \( \{j, p\} = \{1, 2\} \).

Moreover, we define two 1 \times 3 matrices

\[
L_1 = (l_1, l_2, l_3), \quad L_2 = (l_4, l_5, l_6)
\]

and two 3 \times 3 matrices \( M_1 \) and \( M_2 \) by

\[
(M_1)_{ij} = \tilde{\phi}_3(B^1_{i1}, B^1_{i2}, B^1_{i3}, B^1_{j1}, B^1_{j2}, B^1_{j3}, B^2_{i1}, B^2_{i2}, B^2_{i3}, B^2_{j1}, B^2_{j2}, B^2_{j3})
\]

and

\[
(M_2)_{ij} = \tilde{\phi}_3(A^1_{i1}, A^1_{i2}, A^1_{i3}, A^1_{j1}, A^1_{j2}, A^1_{j3}, A^2_{i1}, A^2_{i2}, A^2_{i3}, A^2_{j1}, A^2_{j2}, A^2_{j3}) \\
- \phi_3(A^3_{i1}, A^3_{i2}, A^3_{i3}, A^3_{j1}, A^3_{j2}, A^3_{j3})
\]

where

\[
\tilde{\phi}_3(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3) = (a_1(d_3 - d_2) + a_2(d_1 - d_3)) \\
+ a_3(d_2 - d_1)) - (c_1(b_3 - b_2) + c_2(b_1 - b_3) + c_3(b_2 - b_1)).
\]

Finally, we define the polynomial

\[
\tilde{w} = xM_1(L_1)^t + yM_1(L_2)^t + xM_2(L_2)^t.
\]

**Proposition 4.3** We have the following equality

\[
2(x_1 + x_2 + x_3)q = -\tilde{w} + \tilde{g}_1 f_1 + \tilde{g}_2 f_2 - 2s_0(l_1 + l_2 + l_3) + 2s_1(l_4 + l_5 + l_6) \quad (4.6)
\]

of elements of \( \mathcal{O}_{\text{amb}}[s_0, s_1] \).

**Proof** The verification was done using the computer algebra program Macaulay 2 [GS93-08]. QED

The proof of Theorem 4.1 follows now by combining Proposition 4.3 with Lemma 4.2.
4.2 Invariance under $\text{SL}_3$

Let $P \in \text{SL}_3(\mathbb{Z})$ be a $3 \times 3$ matrix with integer coefficients and determinant one. Define two $1 \times 3$ matrices $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ by

$$\tilde{x} = xP^{-t}, \quad \tilde{y} = yP^{-t},$$

(where $P^{-t}$ is, of course, the transpose matrix of the inverse of $P$), two $3 \times 3$ skew–symmetric matrices $\tilde{A}^j$ and two $3 \times 3$ symmetric matrices $\tilde{B}^j$, for $j = 1, 2$, by

$$\tilde{A}^j = P^tA^jP, \quad \tilde{B}^j = P^tB^jP$$

and finally set $\tilde{z} = z$. Moreover, we denote by $\tilde{A}^j_{lm}$ the $lm$-th entry of the matrix $\tilde{A}^j$ and by $\tilde{B}^j_{lm}$ the $lm$-th entry of the matrix $\tilde{B}^j$.

It can be checked that if we denote by $\tilde{a}$ and $\tilde{b}$ the polynomials of Expressions (4.1) and (4.2) respectively, with the variables without tilde replaced by the corresponding expressions with tilde, we have $\tilde{a} = a$ and $\tilde{b} = b$.

4.3 How we got to Theorem 4.1

In this subsection we briefly sketch the method that allowed us to calculate, using the computer program Maple, the expressions for $a$ and $b$ appearing in Theorem 4.1.

**Step 1.** Recall the explicit expressions for $l_i$ in (4.4). We started from the expression

$$2 \left[ -s_0(l_1 + l_2 + l_3) + s_1(l_4 + l_5 + l_6) \right]$$

which is equal to

$$2(x_1 + x_2 + x_3)(s_1^2 - zs_0^2) + 2H,$$

where

$$H = -s_0 \sum_{i=1}^{3} [\phi_1(u^1) + z\phi_1(u^2)]_{i1} + s_1 \sum_{i=1}^{3} [\phi_2(u^1, u^2)]_{i1}.$$

It is easy to see that $2H$ is a homogeneous quadratic polynomial with respect to the variables $A^t_{ij}, B^s_{kl}$. More precisely, $2H$ has a (unique) natural representation as a sum of terms of the form $A^1_{ij}A^2_{kl}c^3_{ijkl}$ plus sum of terms of the form $A^1_{ij}B^2_{kl}c^{A}_{ijkl}$ plus sum of terms of the form $A^2_{ij}B^1_{kl}c^{B}_{ijkl}$.

**Step 2.** We first handle the ‘subpart’ of $2H$ which is sum of terms of the form $A^1_{ij}A^2_{kl}c^A_{ijkl}$. These term appear only as contributions from $s_0(l_1 + l_2 + l_3)$ and, moreover, this subpart is equal to $2asa_0$, where $a$ was defined in (4.1).
Step 3. We now handle the 'subpart' of \(2H\) which is sum of terms of the form \(A_{ij}^1 B_{kl}^2 c_{ijkl}^B\). We notice that there is a kind of complementarity of terms for the pairs \((l_1, l_4), (l_2, l_5), (l_3, l_6)\) in the sense of the following example:

A partial sum of the coefficient of \(A_{13}^1 B_{21}^2\) coming from \(l_1\) and \(l_4\) is \(2y_1x_1s_0 + 2x_1^2s_1\) which can be written as \(2x_1(y_1s_0 + x_1s_1)\). Using \(l_4\) we get an expression not involving \(s_0\) and \(s_1\). Similarly, another partial sum coming from \(l_2\) and \(l_5\) is \(2x_1(y_2s_0 + x_2s_1)\) and we can use \(l_5\) to get an expression not involving \(s_0\) and \(s_1\). A similar procedure can be done for the partial sum coming from \(l_3\) and \(l_6\).

By symmetry, the handling of the 'subpart' of \(2H\) which is sum of terms of the form \(B_{ij}^1 B_{kl}^2 c_{ijkl}^B\) is similar.

The handling of the 'subpart' of \(2H\) which is sum of terms of the form \(B_{ij}^1 B_{kl}^2 c_{ijkl}^B\) is similar, but we need to take care of preserving the symmetries. This is the only part where we actually need the coefficient 2 in \(2H\).

To give an example, a partial sum of the coefficient of \(B_{13}^1 B_{23}^2\) coming from \(l_1\) and \(l_4\) is

\[
2 \left[ -s_0(-x_1x_3z - y_1y_3) + s_1(y_3x_1 + y_1x_3) \right]
\]

which we first write as

\[
y_3(y_1s_0 + x_1s_1) + x_1(zzs_0 + y_3s_1) + y_1(y_3s_0 + x_3s_1) + x_3(zzs_0 + y_1s_1)
\]

and then we use the appropriate linear equations \(l_i\) to get an expression not involving \(s_0\) and \(s_1\).

Step 4. At the end of step 3 we arrived to an expression not involving \(s_0, s_1\). By suitable subtraction (in a symmetric way) of multiples of \(f_1\) and \(f_2\) we get a polynomial divisible by \(x_1 + x_2 + x_3\). The quotient is \(b\), which after some further term by term effort can be written in the form (4.2).

4.4 Type \(\Pi_1\) unprojection for \(n = 2\)

This subsection contains the explicit equations obtained by Miles Reid for the type \(\Pi_1\) unprojection with parameter value \(n = 2\). It is taken from [R] Section 9.5.

For \(n = 2\) the ambient ring is

\[
\mathcal{O}_{\text{amb}} = \mathbb{Z}[x_1, x_2, y_1, y_2, z, A_{11}, B_{11}, B_{12}, B_{22}],
\]

the matrix \(M\) corresponding to the one in (2.3) is equal to

\[
M = \begin{pmatrix} y_1 & y_2 & z & x_1 & z & x_2 \\ x_1 & x_2 & z & y_1 & z & x_2 \end{pmatrix},
\]

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there is a unique polynomial
\[ f = A_{12}(xy_2 - x_2y_1) + B_{11}(y_1^2 - zx_1^2) + 2B_{12}(y_1y_2 - zx_1x_2) + B_{22}(y_2^2 - zx_2^2) \]
corresponding to (2.4), the linear equations \( l_1, \ldots, l_4 \) are given by
\[
\begin{align*}
  l_1 &= zx_1s_0 + y_1s_1 + (x_1A_{12} + y_1B_{12} + y_2B_{22}) \\
  l_2 &= zx_2s_0 + y_2s_1 + (x_2A_{12} - y_1B_{11} - y_2B_{12}) \\
  l_3 &= y_1s_0 + x_1s_1 + (-x_1B_{12} - x_2B_{22}) \\
  l_4 &= y_2s_0 + x_2s_1 + (x_1B_{11} + x_2B_{12})
\end{align*}
\]
and the quadratic equation \( q \) is given by
\[ q = s_1^2 - zs_0^2 - A_{12}s_0 - (B_{12})^2 + B_{11}B_{22}. \]

In other words, using the notations of Section 2 we have the equality of ideals of \( \mathcal{O}_{amb}[s_0, s_1] \)
\[ I_Y = (f) + (l_1, \ldots, l_4) + (q). \]

**Remark 4.4** Since \( f = y_2l_1 - y_1l_2 + zx_2l_3 - zx_1l_4 \), we even get
\[ I_Y = (l_1, \ldots, l_4) + (q). \]

**Remark 4.5** By [P] the ideal \( I_Y \) is Gorenstein codimension 3. It is easy to see that it is equal to the ideal generated by the 5 submaximal Pfaffians of the \( 5 \times 5 \) skew-symmetric matrix (with entries in \( \mathcal{O}_{amb}[s_0, s_1] \))
\[
\begin{pmatrix}
  0 & x_1 & x_2 & y_1 & y_2 \\
  0 & -s_0 & -B_{22} & s_1 + B_{12} & \\
  0 & -s_1 + B_{12} & -B_{11} & \\
  \overset{-\text{sym}}{0} & 0 & \overset{-s_0 - A_{12}}{0} & \overset{-z\text{sym}}{0}
\end{pmatrix}
\]
(For a discussion about the Pfaffians of a skew-symmetric matrix see, for example, [BH] Section 3.4.)

### 5 Applications to algebraic geometry

We believe that the explicit formulas of Theorem 4.1 can be used together with Gavin Brown’s online database of graded rings [Br] for the proof of the existence of a number of (singular) Fano 3-folds with anticanonical ring of
codimension 4. We discuss below two such examples, the first of which is due to Selma Altınok, and has the interesting property $|−K_X| = \emptyset$. We also expect that the explicit formulas of Theorem 4.1 together with the orbifold Riemann–Roch theorem obtained in [BS] can lead to the construction of new codimension 4 Calabi–Yau 3-folds.

5.1 The example of Altınok

This example is taken from [R] Example 9.14 and is due to Altınok. Consider the weighted projective space $\mathbb{P}^2(1, 3, 5)$ with coordinates $u, v, w$ and the weighted projective space $\mathbb{P}^5(2, 3, 4, 5, 6, 7)$ with coordinates $x, v, y, w, z, t$. We define the map $\mathbb{P}(1, 3, 5) \to \mathbb{P}(2, 3, 4, 5, 6, 7)$ given by

\[
x = u^2, \quad v = v, \quad y = uw, \quad w = w, \quad z = uw, \quad t = u^7.
\]

The equations of the image $D$ of the map are

\[
\operatorname{rank} \begin{pmatrix} y & z & t & xv & xw & x^4 \\ v & w & x^3 & y & z & t \end{pmatrix} \leq 1,
\]

that is,

\[
yw = vz, \quad yx^3 = vt, \quad zx^3 = wt, \quad y^2 = v^2x, \quad yz = vw, \quad yt = vx^4, \quad z^2 = w^2x, \quad zt = wx^4, \quad t^2 = x^7.
\]

A general complete intersection $X_{12,14}$ containing $D$ is obtained by choosing two general combinations, one of degree 12 and the other of degree 14, of the above equations of $D$.

We perform a type II$_1$ unprojection of the pair $D \subset X_{12,14}$ to get a codimension 4 3-fold

\[
Y \subset \mathbb{P}^7(2, 3, 4, 5, 6, 7, 8, 9).
\]

Notice that the new variable $s_0$ has degree 8 and the new variable $s_1$ has degree 9.

After substituting to the formulas (3.13) and (4.5) we checked the quasismoothness of $Y$ using the computer algebra program Singular [GPS01].

5.2 The second Fano example

In this subsection we sketch a construction, suggested by Brown’s online database of graded rings [Br], of a codimension 4 Fano 3-fold

\[
Y \subset \mathbb{P}^7(1, 1, 2, 2, 2, 2, 3, 3)
\]

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starting from a (nongeneric) codimension 2 complete intersection Fano 3-fold
\(X_{4,6} \subset \mathbb{P}^5(1,1,2,2,2,3)\).

Write \(x_1, x_2, y_1, y_2, y_3, w\) for the coordinates of \(\mathbb{P}^5(1,1,2,2,2,3)\), and consider the subscheme \(D \subset \mathbb{P}^5(1,1,2,2,2,3)\) with equations

\[
\text{rank} \begin{pmatrix}
  y_1 & y_2 & w & y_3 x_1 & y_3 x_2 & y_3^2 \\
  x_1 & x_2 & y_3 & y_1 & y_2 & w
\end{pmatrix} \leq 1.
\]

By definition, the ideal \(I_D\) of \(D\) is generated by the \(2 \times 2\) minors of the above matrix. Denote by \(X_{1,6} \subset \mathbb{P}^5(1,1,2,2,2,3)\) a general codimension 2 complete intersection (with equations of degrees 4 and 6) containing \(D\). The equations of \(X_{4,6}\) are obtained by choosing two general combinations, one of degree 4 and the other of degree 6, of the above equations of \(D\).

We perform a type \(\Pi_1\) unprojection of the pair \(D \subset X_{4,6}\) to get a codimension 4 3-fold \(Y \subset \mathbb{P}^5(1,1,2,2,2,2,3,3)\). Notice that here the new variable \(s_0\) has degree 2 and the new variable \(s_1\) has degree 3.

After substituting to the explicit formulas (3.13) and (4.5) we checked the quasismoothness of \(Y\) using the computer algebra program Singular [GPS01].

## 6 Appendix: Macaulay 2 code

This section contains the Macaulay 2 code for the type \(\Pi_1\) unprojection with \(n = 3\).

```m2
-- M2 code for type II_1 unprojection for n=3

kk = QQ
S = kk [A1p12,A1p13,A1p23,A2p12,A2p13,A2p23, B1p11,B1p12,B1p13, B1p22,B1p23,B1p33, B2p11,B2p12,B2p13,B2p22,B2p23,B2p33, y1,y2,y3,x1,x2,x3,z,s0,s1,
  Degrees => {6:2, 12:1, 3:2, 3:1,2,3,4}]

--adjointMatrix: input: M 3x3 matrix,
-- output: classical adjoint of M (which is 3x3)

adjointMatrix = (M) -> transpose matrix {
  {det submatrix(M,{1,2},{1,2}), -det submatrix(M,{1,2},{0,2}),
  det submatrix(M,{1,2},{0,1})},
  {-det submatrix(M,{0,2},{1,2}), det submatrix(M,{0,2},{0,2}),
  -det submatrix(M,{0,2},{0,1})},
```

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\{\det \text{submatrix}(M,\{0,1\},\{1,2\}), -\det \text{submatrix}(M,\{0,1\},\{0,2\}), \\
\det \text{submatrix}(M,\{0,1\},\{0,1\})}\}

--doubleAdjointMatrix: input: M1, M2 are 3x3 matrices
-- output: 3x3 matrix

doubleAdjointMatrix = (M1,M2) -> ( 
    M111 := M1_0_0; M112 := M1_1_0; M113 := M1_2_0; 
    M121 := M1_0_1; M122 := M1_1_1; M123 := M1_2_1; 
    M131 := M1_0_2; M132 := M1_1_2; M133 := M1_2_2; 
    M211 := M2_0_0; M212 := M2_1_0; M213 := M2_2_0; 
    M221 := M2_0_1; M222 := M2_1_1; M223 := M2_2_1; 
    M231 := M2_0_2; M232 := M2_1_2; M233 := M2_2_2; 
    transpose matrix { 
        {M122*M233-M123*M232+ M222*M133-M223*M132, 
        -(M121*M233-M123*M231+ M221*M133-M223*M131 ), 
        M121*M232-M122*M231+ M221*M132-M222*M131}, 
        {- (M112*M233-M113*M232+M212*M133-M213*M132), 
        M111*M233-M113*M231+ M211*M133-M213*M131, 
        - (M111*M232-M112*M231+ M211*M132-M212*M131)}, 
        { M112*M223-M113*M222+ M212*M123-M213*M122, 
        -(M111*M223-M113*M221+ M211*M123-M213*M121), 
        M111*M222-M112*M221+M211*M122-M212*M121}}}

-- wedge2: input: single 3x2 matrices u, 
-- output: single 3x1 matrix \wedge^2 (u)

wedge2 = (u) -> 
    matrix {{\det(submatrix (u,\{1,2\},\{0,1\}))}, 
    { -\det(submatrix (u,\{0,2\},\{0,1\})) }, 
    {\det(submatrix (u,\{0,1\},\{0,1\}))}}

-- mixedwedge: input two 3x2 matrices u1 and u2, output 
-- single 3x1 matrix u1 \wedge u2 + u2 \wedge u1

mixedwedge = (u1,u2) ->
    matrix{
        {\det (submatrix(u1,\{1\},\{0,1\}) || submatrix(u2,\{2\},\{0,1\}))+ 
        \det (submatrix(u2,\{1\},\{0,1\}) || submatrix(u1,\{2\},\{0,1\}))}, 
        { - (\det (submatrix(u1,\{0\},\{0,1\}) || 
            submatrix(u2,\{2\},\{0,1\})) + 
            \det (submatrix(u2,\{0\},\{0,1\}) || 
            submatrix(u1,\{2\},\{0,1\}))}}}
\[ \text{submatrix}(u_1,\{2\},\{0,1\}))}, \quad \{\text{det} \ (\text{submatrix}(u_1,\{0\},\{0,1\})) \mid \text{submatrix}(u_2,\{1\},\{0,1\})) \} + \text{det} \ (\text{submatrix}(u_2,\{0\},\{0,1\})) \mid \text{submatrix}(u_1,\{1\},\{0,1\}))} \]  
\[ x = \text{matrix} \ \{\{x_1,x_2,x_3\}\} \]  
\[ y = \text{matrix} \ \{\{y_1,y_2,y_3\}\} \]  
\[ A_1 = \text{matrix} \ \{\{0,A_1p_{12},A_1p_{13}\},\{-A_1p_{12},0,A_1p_{23}\},\{-A_1p_{13},-A_1p_{23},0\}\} \]  
\[ A_2 = \text{matrix} \ \{\{0,A_2p_{12},A_2p_{13}\},\{-A_2p_{12},0,A_2p_{23}\},\{-A_2p_{13},-A_2p_{23},0\}\} \]  
\[ B_1 = \text{matrix} \ \{\{B_{1p_{11}},B_{1p_{12}},B_{1p_{13}}\},\{B_{1p_{12}},B_{1p_{22}},B_{1p_{23}}\},\{B_{1p_{13}},B_{1p_{23}},B_{1p_{33}}\}\} \]  
\[ B_2 = \text{matrix} \ \{\{B_{2p_{11}},B_{2p_{12}},B_{2p_{13}}\},\{B_{2p_{12}},B_{2p_{22}},B_{2p_{23}}\},\{B_{2p_{13}},B_{2p_{23}},B_{2p_{33}}\}\} \]  

\text{-- A}_1,\text{A}_2 \text{ are generic 3x3 skew, } B_1,B_2 \text{ are generic 3x3 symmetric}

\[ \text{ID} = \text{ minors} (2, \text{ matrix} \ \{\{ y_1,y_2,y_3, z*x_1,z*x_2, z*x_3\}, \{ x_1,x_2,x_3, y_1,y_2, y_3\}\}) \]  
\[ \text{adjB}_1 = \text{adjointMatrix} (B_1); \]  
\[ \text{adjB}_{1p_{11}} = \text{adjB}_1\_0\_0; \text{adjB}_{1p_{22}} = \text{adjB}_1\_1\_1; \text{adjB}_{1p_{33}} = \text{adjB}_1\_2\_2; \]  
\[ \text{adjB}_{1p_{12}} = \text{adjB}_1\_1\_0; \text{adjB}_{1p_{13}} = \text{adjB}_1\_2\_0; \text{adjB}_{1p_{23}} = \text{adjB}_1\_2\_1; \]  
\[ \text{adjB}_2 = \text{adjointMatrix} (B_2); \]  
\[ \text{adjB}_{2p_{11}} = \text{adjB}_2\_0\_0; \text{adjB}_{2p_{22}} = \text{adjB}_2\_1\_1; \text{adjB}_{2p_{33}} = \text{adjB}_2\_2\_2; \]  
\[ \text{adjB}_{2p_{12}} = \text{adjB}_2\_1\_0; \text{adjB}_{2p_{13}} = \text{adjB}_2\_2\_0; \text{adjB}_{2p_{23}} = \text{adjB}_2\_2\_1; \]  
\[ a = \text{det} \ \text{matrix} \ \{\{x_1,x_2,x_3\},\{A_{1p_{23}},-A_{1p_{13}},A_{1p_{12}}\}, \{A_{2p_{23}},-A_{2p_{13}},A_{2p_{12}}\}\}; \]  
\[ \text{invofo2} = \text{substitute} ((\frac{1}{S}/2_S), S); \]  
\[ b = (\text{invofo2} \ast (y\ B_1\ \text{adjB}_2\text{B}_1\text{transpose}(y)) + \ y\ B_2\ \text{adjB}_1\text{B}_2\text{transpose}(y)) + \ 2\ x\ A_1\ \text{adjB}_2\text{transpose}(A_1)\text{transpose}(x) + \ 2\ x\ A_2\ \text{adjB}_1\text{transpose}(A_2)\text{transpose}(x) - \ 2\ x\ A_1\ \text{doubleAdjointMatrix} (B_1,B_2)\text{transpose}(A_2)\text{transpose}(x) - \ z\ x\ B_1\text{adjB}_2\text{B}_1\text{transpose}(x) - \ z\ x\ B_2\text{adjB}_1\text{B}_2\text{transpose}(x) + \ 4\ y\ B_1\text{adjB}_2\text{transpose}(A_1)\text{transpose}(x) + \ 4\ y\ B_2\text{adjB}_1\text{transpose}(A_2)\text{transpose}(x) - \ (A_{1p_{12}}(x_1*y_2-x_2*y_1)+A_{1p_{13}}(x_1*y_3-x_3*y_1)+\ ]
\[ A1p23*(x2*y3-x3*y2) * \\
(B1p11* adjB2p11+B1p22* adjB2p22 + B1p33*adjB2p33 + \\
2*(B1p12* adjB2p12+B1p13* adjB2p13+ B1p23*adjB2p23)) - \\
(A2p12*(x1*y2-x2*y1)+A2p13*(x1*y3-x3*y1)+ \\
A2p23*(x2*y3-x3*y2))* \\
(B2p11* adjB1p11+B2p22* adjB1p22+ B2p33*adjB1p33 + \\
2*(B2p12* adjB1p12+B2p13* adjB1p13+B2p23*adjB1p23)))_0_0; \\
\]

\[ u1 = (-A1*(transpose x) + B1 *(transpose y)) | \\
(-A2*(transpose x) + B2 *(transpose y)) \\
u2 = (-B1*(transpose x)) | (-B2*(transpose x)) \]

\[ \sigma_1 = \text{wedge2} (u1) + z * \text{wedge2} (u2) \]

\[ \sigma_2 = \text{mixedwedge} (u1,u2) \]

\[ f1 = (x*A1*transpose(y)+y*B1*transpose(y)-z*x*B1*transpose(x))_0_0; \]

\[ f2 = (x*A2*transpose(y)+y*B2*transpose(y)-z*x*B2*transpose(x))_0_0; \]

\[ l1 = z*x1*s0+y1*s1+\sigma_1_{0,0}; \]

\[ l2 = z*x2*s0+y2*s1+\sigma_1_{0,1}; \]

\[ l3 = z*x3*s0+y3*s1+\sigma_1_{0,2}; \]

\[ l4 = y1*s0 + x1*s1+ \sigma_2_{0,0}; \]

\[ l5 = y2*s0 + x2*s1+ \sigma_2_{0,1}; \]

\[ l6 = y3*s0 + x3*s1+ \sigma_2_{0,2}; \]

\[ q = s1^2-z*s0^2-a*s0+b; \]

\[ \text{IY} = \text{ideal} (f1,f2,l1,l2,l3,l4, l5,l6,q); \]

\[ -- The following is an example of substitution to a nongeneric case \]

\[ \text{subslist} = \{ x1 => x1,x2 => 0,x3 => x2, \]

\[ y1 => y1,y2 => y2,y3 => 0, z => 5, \]

\[ A1p12 => 1,A1p13=>2 ,A1p23=>0, \]

\[ A2p12 => 1,A2p13=> 3 ,A2p23=> 1, \]

\[ B1p11 =>1, B1p12 => 5, B1p13=> 0, \]

\[ B1p22 =>1, B1p23=>1 ,B1p23=> 1, \]

\[ B2p11=> -1,B2p12 => 2,B2p13=> 1, \]

\[ B2p22=>1,B2p23=>7 ,B2p33=>1\}; \]

\[ \text{specificIYbefore} = \text{substitute} ( \text{IY},\text{subslist}); \]

\[ \text{specificS} = \text{kk} [ \ y1,y2,y3,x1,x2,x3,s0,s1, Degrees => \{8:1\}] \]

\[ \text{specificIY} = \text{sub} (\text{specificIYbefore},\text{specificS}) \]

\[ \text{isHomogeneous specificIY} -- true \]

\[ \text{codim specificIY} \]

\[ \text{betti res specificIY} \]
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