ASYMPTOTICS OF SUMMANDS I: SQUARE INTEGRABLE INDEPENDENT RANDOM VARIABLES

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ABSTRACT. This paper is part of series on self-contained papers in which a large part, if not the full extent, of the asymptotic limit theory of summands of independent random variables is exposed. Each paper of the series may be taken as review exposition but specially as a complete exposition expect a few exterior resources. For graduate students and for researchers (beginners or advanced), any paper of the series should be considered as a basis for constructing new results. The contents are taken from advanced books but the organization and the proofs use more recent tools, are given in more details and do not systematically follow previous one. Sometimes, theorems are completed and innovated.

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1. Introduction

The largest part of the asymptotic theory of partial sums of random variables concentrated on independent random variables over at least two centuries. Almost all the greatest scientists in probability Theory (Lévy, Kolmogorov, Lyapounov, Lynderberg, Gnedenko, Feller, etc.) engaged themselves in such an enterprise. Besides, a very large part of the current theory on dependent sequence of random variables is based on transformations of independence structures, for example on notions of nearness of the dependence to independence (weak dependence, φ-mixing, associated sequence, independent increments, etc.).

So it is important to have the deepest knowledge of that past. In Lo (2018b), we introduced some important elements of that theory (Central limit theorems, laws of the large numbers, law of the iterated logarithm, zero-one laws, etc.).

We are beginning a series on self-contained papers in which a large part of the central limit theorem, if not the full extent, of the asymptotics of summands of independent random variables will exposed. Each paper of the series may be taken as review exposition but specially as a complete exposition expect a few exteroir ressources. For graduate students and for researchers (beginners or advanced), any paper of the series should be considered as a basis for constructed new results. The contents are taken from advanced books but the organization and the proofs used more recent tools, are given in more details and do not systematically follow previous one. Sometimes, theorem are completed and innovated.

In this first paper of the series, we focus of the full characterization of the CLT problem of independent summands for square integrable random variables. The main material is extracted from Loève (1997) as a general guide. But we use arguments from our previous works (Lo (2017b), Lo (2018a), Lo (2018b), Lo et al. (2016), etc.) to have unified and a self-contained Lo (2018b). In particular, the text on the weak convergence of bounded measures and its exppression on $\mathbb{R}^k$ ($k \geq 1$) provides tools to make the conclusions in Loève (1997) more clear, as we expect as least.

Papers of this series focus on complete mathematical texts rather than on a historical review of contributions of different authors. We refer to Loève
(1997) for that aspect.

Since the set of weak limits of independent summands for square integrable coincide with the set of infinitely decomposable laws, this paper will deal with the most important properties of such probability laws to the extent needed here. More developments, including the study of stable laws, will be given in the other papers of the series.

Let us introduce the problem, after we provide some notations. below, all sequences and all arrays of random variables have all their elements defined on a same probability space. So, we set a generic probability space \((\Omega, \mathcal{A}, \mathbb{P})\).

Following Loève (1997), we transform the study of sums of independent and centered random variables $S_n = X_1 + \ldots + X_n$, $n \geq 1$, (with the notation $\sigma^2_i = \mathbb{E}X_i^2$, $i \geq 1$, if they exist), by summands

$$S_n = \sum_{1 \leq k \leq k_n} X_{k,n}, n \geq 1,$$

where for each $n \geq 1$, the family \(\{X_{k,n}, 1 \leq k \leq k_n = k(n)\}\) is a family of independent and centered random variables such that $F_{k,n}$ stands for the cdf of $X_{k,n}$ and $\sigma^2_{k,n} = \mathbb{E}X_{k,n}^2$, $1 \leq k \leq k_n$. We suppose also that $k_n \to +\infty$ as $n \to +\infty$.

**Notations.** The notation already given and completed by the notation $f_{kn}$ for the characteristic function of $X_{k,n}$, are fixed for once.

In the case of simple summands, we have for each $n \geq 1$, $k_n = n$ and $X_{k,n} = X_k$ for $k \in [1,n]$. Here, the rows $(X_{k,n})_{1 \leq k \leq k_n}$ are such that each of them is obtained by adding one element to the predecessor. But, in the general case, no relation between families $\mathcal{E}_n = \{X_{k,n}, 1 \leq k \leq k(n)\}$ is required. Also, in the case of the simple sequence $(X_k)_{k \geq 1}$, the studied array for each $n \geq 1$, is $\{X_1/s_n, \ldots, X_n/s_n\}$ where $s_n^2 = \text{Var}(X_1 + \cdots + X_n)$.

Here, we are going to investigate the general problem of finding all the possible weak limits of $S_n$. Without restrictions, we this may lead to trivial results. So we have to fix a general frame in which the study will be done. In
doing, the best way seems to go back to the complete theory of Feller-Levy-Lynderberg and there, discover the following two fundamental hypotheses.

2. The Bounded Variance Hypothesis (BVH) and the Uniformly Asymptotic Negligibility (UAN)

Given a sequence \((X_k)_{k \geq 1}\) of independent, centered and square-integrable random variables, we set \(\sigma_k^2 = \mathbb{E}X_k^2, k \geq 1\), for \(n \geq 1\)

\[ s_n^2 = \sum_{k=1}^{n} \sigma_k^2, \quad t_n^2 = \max\{\sigma_k^2, 1 \leq k \leq n\} \quad \text{and} \quad B_n = t_n^2 / s_n^2, \]

\(k(n) = n\) for each \(n \geq 1\) and for each \(n \geq 1\)

\[ \{X_{k,n}, 1 \leq k \leq k(n)\} = \left\{\frac{X_k}{s_n}, 1 \leq k \leq k(n)\right\} \]

and

\[ S_n = \sum_{k=1}^{k(n)} X_{k,n}. \]

The Feller-Levy-Lynderberg (F2L) theorem (see Loève (1997), or Lo (2018b), Chapter 7, Section 2, Part B) ensures that:

\[ S_n \xrightarrow{\text{N}(0,1)} [WC] \quad \text{and} \quad B_n \rightarrow 0 \quad [NG]\]

if and only if, for any \(\varepsilon > 0\),

\[ L_n(\varepsilon) = \sum_{1 \leq k \leq k(n)} \int_{|X_{k,n}| \geq \varepsilon} X_{k,n}^2 \, d\mathbb{P} \rightarrow 0. \]

Let us see how behave the two following important quantities in that frame:

\[ U(n, \varepsilon) = \sup_{1 \leq k \leq n} \mathbb{P}(|X_{k,n}| \geq \varepsilon) \quad \text{and} \quad MV(n) = \sum_{1 \leq k \leq k(n)} \text{Var}(X_{k,n}). \]
We have, by Markov inequality,

\[ U(n, \varepsilon) \leq \sup_{1 \leq k \leq n} \frac{E X_{k,n}^2}{\varepsilon s_n^2} = \varepsilon^{-2} B_n, \]

and \( MV(n) = 1 \) for all \( n \geq 1 \) and for \( c = 1 \)

\[ \sup_{n \geq 1} MV(n) = c < +\infty. \]

The theory we are going to develop in a more general case needs the properties we just introduced with specific names.

**Definition.** Under the notation given above, we say that:

(i) the **Bounded Variance Hypothesis (BVH)** holds if and only if

\[ \sup_{n \geq 1} MV(n) = c < +\infty; \]

(ii) the **Variance Convergence Hypothesis (VCH)** holds if and only if

\[ MV(n) \to c \in ]0, +\infty[; \]

(iii) the **Uniformly Asymptotic Negligibility (UAN)** holds if and only if, for any \( \varepsilon > 0 \),

\[ U(n, \varepsilon) \to 0. \]

We express the F2L theorem as follows: If

(a) the (UAN) condition holds;

(b) the (BVH) holds, then

\[ S_n \rightsquigarrow \mathcal{N}(0,1) \text{ if and only if } L_n(\varepsilon) \to 0, \text{ for any } \varepsilon > 0. \]

In this particular case, the characteristic function of the weak law \( \mathcal{N}(0,1) \)

\[ \psi_\infty(u) = \exp(-u^2/2) \]

is such that, for any \( p \in \mathbb{N}, p > 0, \psi_\infty^{1/p} \) defined by
\[ \psi_\infty^{1/p}(u) = \exp(-p^{-1}u^2/2) \]
is still a characteristic function, actually of a \( \mathcal{N}(0, p^{-1}) \) law. Let us denote by \( C_f \) the class of all characteristic functions \( f : \mathbb{R} \to \mathbb{C} \) on \( \mathbb{R} \).

Now we may set our general task to be done: \((Task)\). Given the UAN and the BVH conditions, what is the class of all possible limits \( Z_\infty \) of characteristic function \( \psi_\infty \). By the particular case of F2L theorem, we may think that the searched class can be

\[ C_{fid} = \{ \psi \in C_f : (\forall p > 1), \psi^{1/p} \in C_f \}. \]

We define \( C_{fid} \) as the class of infinitely divisible characteristic functions. At least, in the current version of the central limit theorem, the Gaussian weak limit is in \( C_{fid} \).

We are going to see that the suggestion in the description of the task is effectively the global solution.

In the sequel, we will devote Section 3 on infinitely divisible (or decomposable) laws. In Section 4, we finish the task we have given to ourselves under the UAN Condition and the BV. Finally in 5, rediscover the characterization of the CLT to a Gaussian law and that of the CLT to a Poisson law. In the next element of the series, we proceed to a general theory with non-necessarily square-integrable random variables.

3. Class of infinitely divisible (or decomposable) laws on \( \mathbb{R} \)

3.1. Definitions and examples.

The basic definition is the following.

**Definition 1.** A characteristic function \( \psi \in C_f \) is infinitely decomposable (idecomp), denoted by \( \psi \in C_{fid} \) if and only if for all positive integer \( p \), \( \psi^{1/p} \) is still a characteristic function.

Let us explain the notion of idecomp in terms of random variables. Suppose that \( \psi \in C_f, p \geq 1 \) and \( \psi_p = \psi^{1/p} \). Suppose that \( \psi_p \) is the characteristic function (ch.f) of a probability measure \( \mathbb{P}_p \). By the Kolmogorov theorem, it
is possible to construct a probability space \((\Omega_p, \mathcal{A}_p, \mathbb{P}_p)\) holding independent real-valued random variables \(Z_p, Z_{1,p}, \ldots, Z_{p,p}\) having all the characteristic function \(\psi_p\), i.e.,

\[
\psi_p(u) = \mathbb{E}_{\mathbb{P}_p} \exp(iuZ_p) = \int \exp(iux) \, d\mathbb{P}_p.
\]

It is clear that \(\psi_p\) is the characteristic function of \(S_p = Z_{1,p} + \cdots + Z_{p,p}\). As well, \(\psi\) is the characteristic function of a probability measure \(\mathbb{P}_\psi\) on \(\mathbb{R}\) and let us denote by \(Z_\psi\) a random variable with \(\mathbb{P}_\psi\) as probability law.

We easily see that we may extend the definition as follows. In the definition below, we use the notion of idecomp probability law at the place of idecomp characteristic function or idecom random variable.

**Definition 2.** Let \(Z\) be a real-valued random variable with probability law \(\mathbb{P}_Z\) and characteristic function \(\psi_Z\). \(\mathbb{P}_Z\) is idecom (equivalently \(\psi_Z\) is idecomp or \(Z\) is idecomp) if and only if one of the following assertions holds:

1. For all \(p \in \mathbb{N} \setminus \{0\}\), \(\psi_Z \downarrow p\) is a characteristic function.
2. For all \(p \in \mathbb{N} \setminus \{0\}\), there exists a characteristic function \(\psi_p\) such that \(\psi_Z = \psi_p^p\).
3. For all \(p \in \mathbb{N} \setminus \{0\}\), there exists a probability \(\mathbb{P}_p\) on \(\mathbb{R}\) such that

\[
\mathbb{P}_Z = \mathbb{P}_p^ \otimes p,
\]

that is, \(\mathbb{P}_Z\) is the convolution product of \(\mathbb{P}_p\) by itself \(p\) times.
4. For all \(p \in \mathbb{N} \setminus \{0\}\), there exists a sequence \(Z_{1,p}, \ldots, Z_{p,p}\) of independent and identically distributed real-valued random variables such that

\[
Z =_d Z_{1,p} + \cdots + Z_{p,p}.
\]

**Examples.** Let us give some quick examples.

**Example 1.** (Degenerate random variable). Let \(Z = a\), p.s of characteristic function

\[
\psi_Z(t) = e^{iat}, \quad t \in \mathbb{R}.
\]
For $p \geq 1$, $\psi_Z(t)^{1/p} = e^{i(a/p)t}$, which is the \textit{cha.f} of the degenerate r.v $Z_p = a/p$.

**Example 2.** (Gaussian random variables). Let $Z \sim \mathcal{N}(m, \sigma^2)$, $m \in \mathbb{R}$, $\sigma \in \mathbb{R}_+ \setminus \{0\}$ with \textit{cha.f}

$$
\psi_Z(t) = \exp(i mt - \sigma^2 t^2 / 2), \ t \in \mathbb{R}.
$$

For $p \geq 1$, we have

$$
\psi_Z(t)^{1/p} = \exp(i (m/p)t - (\sigma/\sqrt{p})^2 t^2 / 2), \ t \in \mathbb{R},
$$

which is the \textit{cha.f} of a $\mathcal{N}(m/p, \sigma^2/p)$ r.v.

**Example 3.** (Translated Poisson random variables). Let $Z \sim \mathcal{P}(a, \lambda) \equiv +\mathcal{P}(\lambda)$, $a \in \mathbb{R}$, $\lambda \in \mathbb{R}_+ \setminus \{0\}$ with \textit{cha.f}

$$
\psi_Z(t) = \exp \left( iat + \lambda(e^{it} - 1) \right).
$$

For $p \geq 1$, we have

$$
\psi_Z(t)^{1/p} = \exp \left( i(a/p)t + (\lambda/p)(e^{it} - 1) \right), \ t \in \mathbb{R},
$$

which is the \textit{cha.f} of a $\mathcal{P}(a/p, \lambda/p)$ r.v.

**Example 4.** (Gamma random variables). Let $Z \sim \gamma(a, b)$, $a > 0$, $b > 0$, with \textit{cha.f}

$$
\psi_Z(t) = (1 - it/b)^{-a}, \ t \in \mathbb{R}.
$$

For $p \geq 1$, we have

$$
\psi_Z(t)^{1/p} = (1 - it/b)^{-(a/p)}, \ t \in \mathbb{R},
$$

which is the \textit{cha.f} of a $\gamma(a/p, b)$ r.v.

**Example 5.** (Cauchy random variables). Let $Z \sim Ca(a, b)$, $a \in \mathbb{R}$, $b > 0$ with \textit{cha.f}

$$
\psi_Z(t) = \exp(iua - b|t|), \ t \in \mathbb{R}.
$$
For \( p \geq 1 \), we have
\[
\psi_Z(t)^{1/p} = \exp(iu(a/p) - (b/p)|t|), \ t \in \mathbb{R},
\]
which is the \( \text{cha.f} \) of a \( Ca(a/p, b/p) \) r.v.

Now, let us focus on properties of such laws.

### 3.2. Properties.

**Property 1.** If \( \psi_1 \) and \( \psi_2 \) are two idecomp \( \text{cha.f} \), then \( \psi = \psi_1 \psi_2 \) is an idecomp \( \text{cha.f} \).

**Proof.** Suppose that \( \psi_1 \) and \( \psi_2 \) are two idecomp \( \text{cha.f} \) and let \( p \geq 1 \). Thus \( \psi^{1/p} = \psi_1^{1/p} \psi_2^{1/p} \) is the \( \text{cha.f} \) of the convolution product of the probability measures associated to the \( \text{cha.f} \) \( \psi_i^{1/p} \ (i \in \{1, 2\}) \).

**Property 2.** If \( \psi \) is an idecomp \( \text{cha.f} \), the conjugate \( \overline{\psi} \) is also an idecomp \( \text{cha.f} \) and the complex square norm \( \|\psi\|^2 \) is an idecomp \( \text{cha.f} \).

**Proof.** Let \( \psi \) be the \( \text{cha.f} \) of \( X \), i.e., \( \psi(t) = \psi_X(t) = \mathbb{E}(e^{itX}) \), it is clear that
\[
\mathbb{E}(e^{-itX}) = \mathbb{E}(e^{itX}) = \mathbb{E}(e^{itX}) = \overline{\psi}(t).
\]
This and \( \mathbb{E}(e^{-itX}) = \psi_X(t) \) for \( t \in \mathbb{R} \) show that \( \psi \) is a \( \text{cha.f} \). It is also direct to see that \( X \) and \( -X \) are idecomp or non-idecomp at the same time or not. Finally by Property 1, \( \|\psi\|^2 = \psi \overline{\psi} \) is idecomp if \( \psi \) is.

**Property 3.** If \( \psi \) is an idecomp \( \text{cha.f} \), then \( \psi^{1/n} \) converges to 1 everywhere, as \( n \to +\infty \).

**Proof.** Suppose that \( \psi \) is an idecomp \( \text{cha.f} \). Let us denote, for all \( n \geq 1 \), \( \psi_n = \psi^{1/n} \), that is a \( \text{cha.f} \). But \( \|\psi\| \leq 1 \) and \( \|\psi_n\|^2 = \|\psi\|^{2/n} \) converges to \( g \) with \( g = 0 \) on \( \psi = 0 \) and \( g = 1 \) on \( \psi \neq 0 \). Let us show that \( \psi \) cannot take the null value. Indeed \( \psi \) is continuous (at zero in particular) and \( \psi(0) = 1 \). So \( \psi > 1/2 \) on an interval \( ]-r, r[ \), \( r > 0 \) and next \( g = 1 \) on \( ]-r, r[ \). But the function \( h \equiv 1 \) is the \( \text{cha.f} \) of the random variable \( Z = 0 \). By Proposition in Billingsley (1968) (see page 388), we get that \( g = h \) and then \( g = 1 \).
everywhere, so \( \| \psi_n \|^2 \to 1 \). This ensures that \( \psi \) does not take the null value. Finally, we get rid of the norm by

\[
\psi^{1/n} = \exp \left( \frac{1}{n} \log \psi \right) \to 1 \quad \text{as } n \to +\infty.
\]

**Property 4.** Let \((\psi_n)_{n \geq 1}\) be a sequence of idecomp cha.f’s such that \(\psi_n \to \psi\) and \(\psi\) is continuous at zero. Then \(\psi\) is an idecomp cha.f.

**Proof.** Let \(C_{\text{fid}} \ni \psi_p \to \psi\) and \(\psi\) is continuous at zero. For any fixed \(q \geq 1\), \(|\psi_p|^{2/q} \to |\psi|^{2/q}\). Since the \(|\psi_p|^{2/q}\) are cha.f and \(|\psi|^{2/q}\) is continuous at zero, it comes that \(|\psi|^{2/q}\) is a cha.f for any \(q \geq 1\). So \(|\psi|^2\) is an idecomp cha.f and by property 3, \(\psi\) is nowhere zero and next

\[
\psi_p^{1/q} = \exp \left( \frac{1}{q} \log \psi_p \right) \to \exp \left( \frac{1}{q} \log \psi \right) = \psi^{1/q}
\]

is a cha.f by the Levy continuity theorem.

**Property 5.** An cha.f \(\psi\) is idecomp if and only if it is limit of a sequence of products Poisson type cha.f.

**Proof.** If \(\psi\) is a limit of a sequence of products Poisson type cha.f, it is idecomp by Property 4, since products of Poisson type cha.f are idecomp cha.f.

Conversely, let us be given an idecomp cha.f \(\psi\). Since \(\psi\) is non-where equal to zero (Property 3), we have

\[
\log \psi = \lim_{p \to +\infty} p(\psi^{1/p} - 1).
\]

For \(p \geq 1\), let us denote by \(F_p\) the cdf associated with the cha.f \(\psi_p = \psi^{1/p}\). So we have

\[
\Psi_p(t) = p(f^{1/p} - 1) = \int p \left( e^{itx} - 1 \right) dF_p(x).
\]

Since the function \(p \left( e^{itx} - 1 \right)\) is bounded on \(\mathbb{R}\), it is locally integrable and \(\lambda_{F_p}\) is a finite measure, we may apply Lebesgue Dominated theorem and we can conclude that for any fixed \(p \geq 1\),
\[ \Psi_p(t) = p(f^{1/p} - 1) = \lim_{0<a\to+\infty} \int_{-a}^a p(e^{itx} - 1) \ dF_p(x) =: \lim_{0<a\to+\infty} \Psi_{p,a}. \]

By continuity of the integrand, the integral \( \Psi_{p,a} \) is limit of Riemann-Stieltjes, which are of the form

\[ \sum_{1 \leq j \leq k(p,a)} pb_{j,p} \left( e^{ic_{j,p}u} - 1 \right), \]

which are sums of logarithms of Poisson type cha.f. Hence \( \exp(\Psi_{p,a}) \) are cha.f and next \( \exp(\Psi_p) \) is a cha.f as limit of the sequence \( \exp(\Psi_{p,a}) \).

Finally \( \psi \) is limit of cha.f of the form \( \exp(\Psi_p) \), which is a sequence of products of Poisson type cha.f.

**Property 6.** A cha.f is idecomp if and only if it is limit of a sequence of products of Poisson type cha.f.

**Proof** Let \( \psi \) be a cha.f. Let \( \psi_p \) a product of cha.f of type Poisson laws

\[ \psi_p(t) = \prod_{j=1}^{k(p)} \exp \left( ia_{j,p}t + b_{j,p} \left( e^{ic_{j,p}t} - 1 \right) \right), \ t \in \mathbb{R}, \]

where the \( a_{j,p} \)'s and \( c_{j,p} \)'s are real numbers and the \( b_{j,p} \)'s positive numbers. We have, for \( p \) fixed and for \( q \geq 1 \)

\[ \psi_p(t)^{1/q} = \exp \left( i \left\{ \frac{1}{q} \sum_{j=1}^{k(p)} a_{j,p} \right\} t + \frac{1}{q} \sum_{j=1}^{k(p)} b_{j,p} \left( e^{ic_{j,p}t} - 1 \right) \right), \ t \in \mathbb{R}. \]

This is still a product of cha.f's of type Poisson type laws and is a cha.f. If \( \psi_p \to \psi \), thus by Property 4, \( \psi \) is idecomp.

We will need more facts on cha.f's that we will introduce when needed,

We begin by studying the case of bounded variances. First, we deal with three important results that constitute the pillars of the current theory.
3.3. **The three pillars of that theory.** In this subsection, we assume that both the UAN and the BVH hold.

**Lemma 1.** (Comparison Lemma) The complex function \( \log f_{k,n} \) is well-defined and for any \( u \in \mathbb{R} \)

\[
\sum_{k=1}^{k(n)} \{ \log f_{k,n}(u) - (f_{k,n}(u) - 1) \} \to 0,
\]
as \( n \to +\infty \).

**Proof.** Let \( u \in \mathbb{R} \) fixed and \( n \geq 1 \). Then for any \( k \in \{1, \ldots, k(n)\} \), we have the one order expansion

\[
f_{k,n}(u) = 1 + \theta_{k,n} u^2 \sigma_{k,n}^2 / 2,
\]
with \( |\theta_{k,n}| < 1 \) and \( |\circ| \) stands for the norm in \( \mathbb{C} \) or the absolute value when applied to real numbers. In all this chapter, numbers of the form \( \theta_{\circ} \), possibly written with primes or double primes, are only required to have norms less than one and their values are not important. So, we get

\[
\max_{1 \leq k \leq k(n)} |f_{k,n}(u) - 1| \leq \frac{u^2 B_n}{2} \to 0 \text{ as } n \to +\infty.
\]

Next for \( v_{k,n} = \theta_{k,n} u^2 \sigma_{k,n}^2 / 2 \), we surely have that \( \max_{1 \leq k \leq k(n)} |v_{k,n}| \leq (u^2 B_n)/2 \) goes to zero. We also have for all \( u \in \mathbb{R} \),

\[
\log f_{k,n}(u) = \log(1 + (f_{k,n}(u) - 1)) = \log(1 + v_{k,n}) = v_{k,n} + \theta'_{k,n} v_{k,n}^2
\]
\[
= (f_{k,n}(u) - 1) + \theta'_{k,n} v_{k,n}^2,
\]
which leads to, as \( n \to +\infty \),
\[ \sum_{k=1}^{k(n)} \{ \log f_{k,n}(u) - (f_{k,n}(u) - 1) \} \leq \sum_{k=1}^{k(n)} |\theta_{k,n}^2| v_{k,n}^2 \]

\[ \leq \sum_{k=1}^{k(n)} \frac{u^4}{4} |\theta_{k,n}|^2 \sigma_{k,n}^4 (L3) \]

\[ \leq \frac{u^4 B_n}{4} \sum_{k=1}^{k(n)} \sigma_{k,n}^2 (L4) \]

\[ \leq \frac{c u^4 B_n}{4} \to 0. \]

The proof of Lemma 1 is over. \( \square \)

Now, let us use new expressions of the results in Lemma 1. Since the variables \( X_{k,n} \) are centered, we have

\[ \forall n \geq 1, \forall 1 \leq k \leq k(n), \int X_{k,n} \, d\mathbb{P} = \int x \, dF_{k,n}(x) = 0 \text{ and } \int x \, dF_{k,n}(x) = \sigma_{k,n}^2. \]

Let us set, for \( n \geq 1, \)

\[ \psi_n(u) = \sum_{k=1}^{k(n)} (f_{k,n}(u) - 1) = \sum_{k=1}^{k(n)} \int \left( e^{iux} - 1 \right) \, dF_{k,n}(x), \quad u \in \mathbb{R}. \]

By using the remark that \( \mathbb{E} X_{k,n} = 0, \) i.e. \( \int x \, dF_{k,n}(x) = 0, \) we get

\[ \psi_n(u) = \sum_{k=1}^{k(n)} \int \left( e^{iux} - 1 - iux \right) \, dF_{k,n}(x) \]

\[ = \int \left( e^{iux} - 1 - iux \right) \sum_{k=1}^{k(n)} dF_{k,n}(x) \]

\[ = \int \frac{1}{x^2} \left( e^{iux} - 1 - iux \right) x^2 \sum_{k=1}^{k(n)} dF_{k,n}(x), \quad u \in \mathbb{R}. \]

But, by putting
\[ dK_n(x) = x^2 \sum_{k=1}^{k(n)} dF_{k,n}(x), \]

we get

\[ \psi_n(u) = \int \frac{1}{x^2} (e^{iux} - 1 - iux) \ dK_n(x). \]

Finally, Lemma 1 can be expressed as

**Lemma 2.**

\[ \forall u \in \mathbb{R}, \log \left( \prod_{k=1}^{k(n)} f_{k,n}(u) \right) - \psi_n(u) \to 0, \text{ as } n \to +\infty, \]

where

\[ dK_n(x) = x^2 \sum_{k=1}^{k(n)} dF_{k,n}(x) \]

and

\[ \psi_n(u) = \int \frac{1}{x^2} (e^{iux} - 1 - iux) \ dK_n(x). \]

This lemma becomes the second pillar. The third is the following

**Lemma 3.** For any \( n \geq 1 \), \( \exp(\Psi_n) \) is an idecomp cha.f and is the cha.f of a centered random variable of variance

\[ \int dK_n(x) = s_n^2. \]

**Proof.** Let \( n \geq 1 \) be fixed. We have

\[ \Psi_n(u) = \int g(u, x) \ dK_n(x) \text{ with } g(u, x) = \frac{e^{iux} - 1 - iux}{x^2}, \ x \in \mathbb{R}. \]

Clearly \( g \) is continuous on \( \mathbb{R} \times \mathbb{R}^* \) (with \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \)) and for \( u \) fixed, \( g(u, 0) \) is the extension of \( g(u, x) \) by limit, since an expansion at zero gives
\[ g(u, x) = \frac{1 + iux - u^2x^2/2 - 1 - iux + O(x^3)}{x^2} \to -u^2/2 \text{ as } x \to 0. \]

So, for \( u \) fixed, \( x \mapsto g(u, x) \) is continuous everywhere. Moreover we have

\[ \forall u \in \mathbb{R}, \forall x \in \mathbb{R}^*, \ |g(u, x)| \leq \frac{2}{x^2} + \frac{|u|}{x} \]

and

\[ \int x^{-2} \, dK_n(x) = \sum_{k=1}^{k(n)} \int dF_{k,n}(x) = k(n), \]

and, by using \((|x| \leq 1 + x^2)\)

\[ \int |x|^{-1} \, dK_n(x) = \sum_{k=1}^{k(n)} \int \frac{x^2}{|x|} \, dF_{k,n}(x) \]
\[ = \sum_{k=1}^{k(n)} \int |x| \, dF_{k,n}(x) \]
\[ \leq \sum_{k=1}^{k(n)} \int (1 + x^2) \, dF_{k,n}(x) \]
\[ = k(n) + s_n^2. \]

We conclude that \( g(u, x) \) is bounded by \( g_0(x) = 2x^{-2} + |ux^{-1}| \) which is \( K_n \)-integrable. So by the dominated convergence theorem, \( \Psi_n \) is continuous at zero. Also, as an improper Riemann-Stieltjes integral, for \( \varepsilon > 0 \) fixed, we can find a number \( A > 0 \) such that for \( a \geq A \)

\[ |\psi_n(u) - \Psi_{n,a}(x)| < \varepsilon \text{ with } \Psi_{n,a}(u) = \int_{-a}^{a} g(u, x) \, dK_n(x). \]

Now, since \( \Psi_{n,a}(u) \) is continuous, it a limit of a sequence of of Riemann-Stieltjes sums: there exists a partition of \([-a, a]\)

\[-a = x_{0,p} < \cdots < x_{j-1,p} < x_{j,p} < \cdots < x_{\ell(p),p} = a \]
and a sequence of points $c_{j,p} \in (x_{j,p}, x_{j+1,p})$, $0 \leq j \leq \ell(p) - 1$,

$$S_p(u) = \sum_{j=0}^{\ell(p)-1} \{ K_n(x_{j+1,p}) - K_n(x_{j,p}) \} g(u, c_{j,p}) \to \Psi_{n,a}(u),$$

as $\max\{x_{j+1,p} - x_{j,p}, 1 \leq j \leq \ell(p) - 1\} \to 0$ as $p \to +\infty$. We may choose all the $c_{j,p}$ not null from the interior of $(x_{j,p}, x_{j+1,p})$ $x_{j,p} < x_{j+1,p}$). We have

$$S_p(u) = \sum_{j=0}^{\ell(p)-1} \frac{\lambda K_n([x_{j,p}, x_{j+1,p}])}{c_{j,p}^2} \left( e^{ic_{j,p}u} - 1 - ic_{j,p}u \right)$$

$$= \sum_{j=0}^{\ell(p)-1} -i \frac{\lambda K_n([x_{j,p}, x_{j+1,p}])}{c_{j,p}^2} c_{j,p} u + \frac{\lambda K_n([x_{j,p}, x_{j+1,p}])}{c_{j,p}^2} \left( e^{ic_{j,p}u} - 1 \right)$$

$$=: \sum_{j=0}^{\ell(p)-1} -i \mu_{j,p} u + \lambda_{j,p} \left( e^{ic_{j,p}u} - 1 \right),$$

with

$$\lambda_{j,p} = \frac{\lambda K_n([x_{j,p}, x_{j+1,p}])}{c_{j,p}^2} \text{ and } \mu_{j,p} = \frac{\lambda K_n([x_{j,p}, x_{j+1,p}])}{c_{j,p}}.$$

We clearly see that $\exp(S_p)$ is the product of Poisson type cha.f converging to $\exp(\Psi_n)$ as $p \to +\infty$ and $a \to +\infty$. But we also have that $\exp(\Psi_n)$ is continuous. So by the Lévy continuity theorem, $\exp(\Psi_n)$ is a cha.f and it is idecomp by Property 4 (see page 12).

Let us study the differentiability of $\Psi_n$. We have

$$\left| \frac{\partial g(u, x)}{\partial x} \right| = \left| \frac{ix(e^{iux} - 1)}{x^2} \right| \leq \frac{2}{|x|} \in L^1(K_n),$$

and hence

$$\Psi'_n(u) = \int \frac{ix(e^{iux} - 1)}{x^2} dK_n(x) \text{ and } \Psi_n'(0) = 0.$$
\[ \left| \frac{\partial^2 g(u, x)}{\partial^2 x} \right| = \left| \frac{-x^2 e^{iux}}{x^2} \right| = 1 \in L^1(K_n), \]

and hence
\[
\Psi''_n(u) = -\int e^{iux} dK_n(x) \quad \text{and} \quad \Psi''_n(0) = -s_n^2.
\]

Finally, let \( Z_n \) be a r.v. with \( cha.f \) \( \exp(\Psi_n) \). The first and second derivatives of \( \exp(\Psi_n) \) are
\[
\Psi'_n(u) \exp(\Psi_n(u)) \quad \text{and} \quad \{ \Psi''_n(u) \exp(\Psi_n(u)) + (\Psi'_n(u))^2 \exp(\Psi_n(u)) \}
\]

taking the values
\[
\Psi'_n(0) \exp(\Psi_n(0)) = 0 \quad \text{and} \quad \{ \Psi''_n(0) \exp(\Psi_n(0)) + (\Psi'_n(0))^2 \exp(\Psi_n(0)) \} = -s_n^2.
\]

We conclude that \( \mathbb{E}Z_n = 0 \) and \( \mathbb{V}ar(Z_n) = s_n^2 \). The relation
\[
\Psi''_n(u) = -\int e^{iux} dK_n(x) \quad (C)
\]
shows that \( \Psi''_n(u) \) characterizes \( K_n \) and vice-versa. Now, for two functions \( \Psi_n \) and \( \Phi_n \) such that \( \Psi''_n = \Phi''_n \) with \( \Psi_n(0) = \Phi_n(0) = 0 \) and \( \Psi'_n(0) = \Phi'_n(0) = 0 \), we have
\[
\forall u \in \mathbb{R}, \quad \Psi'_n(u) = \Phi'_n(u) + d_1,
\]
and by applying this for \( u = 0 \), we get \( d_1 = 0 \). Next, we have
\[
\forall u \in \mathbb{R}, \quad \Psi_n(u) = \Phi_n(u) + d_2,
\]
and by applying this for \( u = 0 \), we get \( d_2 = 0 \). So \( \Psi = \Phi \) and we have the following fact.

**Fact 1.** \( K_n \) characterizes \( \Psi_n \) and vice-versa. \( \blacksquare \)
4. The weak convergence theorem of summands under the BVH and the UAN Condition

4.1. The Central limit theorem for centered, independent and square integrable random variables.

We are going to conclude the discussion above to find solutions of the CLT problem under the BVH and the UAN Condition. We will have two studies from which of them we draw a final conclusion.

**Study (A).** From Lemma 2 and from the notations above, we have

\[ \forall \ t \in \mathbb{R}, \ \Psi_{S_n}(t) - \exp(\Psi_n(t)) \to 0 \text{ as } n \to +\infty. \]

But \( \exp(\Psi_n(\circ)) \) is an idecomp cha.f for any \( n \geq 1 \) and is linked to

\[
(4.1) \quad \Psi_n(u) = \int g(u, x) \, d\lambda_{K_n}(x) \text{ with } g(u, x) = \frac{e^{iux} - 1 - iux}{x^2}, \ x \in \mathbb{R},
\]

where \( \lambda_{K_n} \) is the Lebesgue-Stieltjes measure associated with the \( df \ K_n \). Now, we are using the weak convergence theory of bounded measures on \( \mathbb{R} \) as exposed Chapter 6 in *Lo et al. (2016)*.

**Direct part.** Let us suppose that \( \lambda_{K_n} \) pre-weakly converges to some \( df \ \lambda_K \), i.e., (for \( C(K) \) standing for the set continuity points of \( K \)),

\[ \forall x \in C(K), \ K_n(x) \to K(x) \text{ as } n \to +\infty. \]

By Part (i) of Proposition 37 in Chapter 6 in *Lo et al. (2016)*, we have

\[ \lambda_K(\mathbb{R}) \leq \lim \inf_{n \to +\infty} \lambda_{K_n}(\mathbb{R}) \leq c, \]

since, for any \( n \geq 1 \),

\[ \lambda_{K_n}(\mathbb{R}) = \sum_{k=1}^{k(n)} \int x^2 dF_{k,n}(x) = \sum_{k=1}^{k(n)} \text{Var}(X_{k,n}) \leq c \]

from the BVH. Hence the pre-weak limit \( \lambda_K \) is a bounded measure. Now we apply the integral Helly-Bray theorem as in Theorem 30 in Chapter 6 in *Lo et al. (2016)* to (4.1) (See above). By (3.2), for any fixed real number
$u$, the function $g(u, x)$ (in $s$) in (4.1) is continuous and satisfies $g(\pm \infty) = 0$. So by the cited Helly-Bray integral theorem, we have

$$\forall u \in \mathbb{R}, \Psi_n(u) \to \Psi_{K}(u) = \int g(u, x) \, d\lambda_{K}(x) =: \int \frac{e^{ix} - 1 - iux}{x^2} \, d\lambda_{K}(x).$$

Now, from the expression of $\Psi_{K}(u)$ and from (3.2), we see that $\Psi_{K}(u)$ is a parametrized (in $u$) integral and by the dominated convergence theorem, $\Psi_{K}(u)$ is continuous and $\Psi_{K}(0) = 0$. Therefore,

$$\forall u \in \mathbb{R}, \exp(\Psi_n(u)) \to \exp(\Psi_{K}(u)) =: f_{K}(u).$$

Since $f_{K}\circ$ is continuous at zero and $f_{K}(0) = 1$, we get by the Lévy continuity theorem (See Theorem 11 in Chapter 3 in Lo et al. (2016)), we conclude that $f_{K}$ is cha.f and by designating by $K_{K}$ the probability law associated to the cha.f $f_{K}$, we have

$$S_n \Rightarrow K_{K}.$$ 

By Property 4 (see page 11 above), $K$ is an idecomp probability law, following the fact that each $\exp(\Psi_{n}(\circ)), n \geq 1$, is an idecomp cha.f.

**Indirect Part.** Suppose that for some $df \, K_0$,

$$S_n \Rightarrow K_{K_0},$$

where $K_{K_0}$ is the probability law associated to $K_0$. We are going to use a Prohorov's type argument. By the asymptotic tightness theorem (See Theorem 29 in Lo et al. (2016)), any sub-sequence $(\lambda_{K_{n_j}})_{j \geq 1}$ of $(\lambda_{K_n})_{n \geq 1}$ contains a sub-sequence $(\lambda_{K_{n_j\ell}})_{\ell \geq 1}$ pre-weakly converging to some $\lambda_{K^*}$. By the direct part, $S_{n_{j\ell}} \Rightarrow K_{K^*},$

where $K_{K^*}$ is associated to a cha.f $f_{K^*} = \exp(\Psi_{K^*})$, with

$$\forall u \in \mathbb{R}, \exp(\Psi_{n_{j\ell}}(u)) \to \exp(\Psi_{K^*}(u)) =: f_{K}(u)$$

and
∀u ∈ ℜ, Ψ_{\mathcal{K}^*}(u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, d\lambda_{\mathcal{K}^*}(x).

By uniqueness of the weak limit, \mathcal{K}_{\mathcal{K}^*} =_{a} \mathcal{K}_{0}. Then each sub-sequence of \( \mathcal{K}^*_n \) contains a sub-sequence converging to \( \mathcal{K}_0 \). We conclude that by Prohorov theorem

\[ \mathcal{K}_n \overset{\text{pre}}{\Rightarrow} \mathcal{K}_0. \]

In both parts, \( \text{Var}(\mathcal{K}) < +\infty \) and by Fact 1 applied to \( \mathcal{K} \), we may conclude that \( \mathcal{K} \) and \( \Psi_{\mathcal{K}} \) characterizes one the other.

We conclude as follows.

**Theorem 1.** Under the BVH and the UAN Condition for summands of independent, centered and square integrable real valued random variables, we have:

(a) If

\[ S_n \overset{\text{pre}}{\Rightarrow} \mathcal{K}, \]

where \( \mathcal{K} \) is a probability law, then is \( \mathcal{K} \) is idecomp.

(b) For any idecomp probability law \( \mathcal{K} \) of a centered and square integrable random variable \( Z \), for which for any \( n \geq 1 \), there exists \( X_{1,n}, \ldots, X_{n,n} \) independent and of same law (they are necessarily centered and square integrable) such that

\[ Z = X_{1,n} + \cdots + X_{n,n} =: S_n. \]

Then clearly, \( \mathcal{K} \) is a weak limit of summands of independent, centered and square integrable real valued random variables under the BVH and the UAN Condition.

(3) We have, under the BVH and the UAN Condition,

\[ S_n \overset{\text{pre}}{\Rightarrow} \mathcal{K}_{\mathcal{K}}, \]

for some df \( \mathcal{K} \), if and only if (using the notation stated above)
Moreover

\[ \Psi_K(u) = \int \frac{e^{ixu} - 1 - iux}{x^2} dK \]

and \( K \) characterize each other, and \( \exp(\Psi_K(\circ)) \) is the characteristic function of \( K \).

**Study (B).** Here, we suppose that the VCH and the UAN Condition hold. We begin by remarking that the Comparison Lemma 1 holds since formula (3.1) (page 14) holds with the use of the VCH in Line (L4).

**Direct part.** Let \( K_n \sim K \). In particular \( K_n \sim_{pre} K \). By the direct part of Study (A), we still have

\[
\lambda_K(\mathbb{R}) \leq \liminf_{n \to +\infty} \lambda_{K_n}(\mathbb{R}) = c,
\]

and

\[ S_n \sim K_K. \]

Actually, by weak convergence, we exactly have

\[ \lambda_K(\mathbb{R}) = \lim_{n \to +\infty} \lambda_{K_n}(\mathbb{R}) = c; \]

but this not play any role for the direct part.

**Indirect part.** Let

\[ S_n \sim K_K, \]

for some df \( K \). By the indirect part of Study (A), we still have

\[ K_n \to_{pre} K. \]

Now if, for \( Z \sim K_K \) such that \( \text{Var}(Z) = c \), we have that \( \lambda_K(\mathbb{R}) = \text{Var}(Z) \) and then
\[
\lim_{n \to +\infty} \lambda_{K_n}(\mathbb{R}) = \lambda_K(\mathbb{R}) \quad \text{and} \quad K_n \to_{pre} K.
\]

By Proposition 37 in Chapter 6 in *Lo et al. (2016)*, we conclude that \( K_n \sim K \) as \( n \to +\infty \).

We conclude as follows.

**Theorem 2.** Under the VCH and the UAN Condition for summands of independent, centered and square integrable real valued random variables, we have the following characterization. If \( K_K \) is associated with a random variable \( Z \) such that \( \text{Var}(Z) = c \), where \( c \) is the limit in the VCH, then we have

\[
S_n \sim K_K,
\]

if and only if

\[
K_n \sim K.
\]

**4.2. The Central limit theorem for non-centered, independent and square integrable random variables.**

Let us re-conduct all the notations in Subsection 4.1. Let us denote

\[
\left( \forall n \geq 1, \forall 1 \leq k \leq k(n), \mathbb{E}X_{k,n} = a_{k,n} \right) \quad \text{and} \quad \left( \forall n \geq 1, \sum_{k=1}^{k(n)} a_{k,n} = a_n \right)
\]

Let us write

\[
\forall n \geq 1, \quad S_n = (S_n - a_n) + a_n = \sum_{k=1}^{k(n)} (X_{k,n} - a_{k,n}) + a_n =: S^*_n + a_n.
\]

Let us denote by \( F_{k,n}^* \) the cdf of \( (X_{k,n} - a_{k,n}) \) for \( n \geq 1 \) and \( 1 \leq k \leq k(n) \),

\[
\forall u \in \mathbb{R}, \Psi_{K^*}(u) = \int \frac{e^{iux} - 1 - iux}{\pi^2} dK^*(x)
\]

and
\[
\forall n \geq 1, \forall u \in \mathbb{R}, \quad \Psi_{K_n^*}(u) = \int e^{iux} \frac{1 - iux}{x^2} dK_n^*(x)
\]

with

\[
\forall n \geq 1, \forall x \in \mathbb{R}, \quad K_n^*(x) = \int_{-\infty}^{x} y^2 \sum_{k=1}^{k(n)} dF_{k,n}^*(y).
\]

**Direct part.** If \( K_n^*(x) \rightsquigarrow_{pre} K^* \) and \( a_n \to a \), then

\[
S_n \rightsquigarrow K_K^* + a = K_0.
\]

Moreover, the *chaf* of \( K_K^* \) is \( \exp(\Psi_{K^*}(\cdot)) \) and next the *chaf* of \( K_0 \) is

\[
\forall u \in \mathbb{R}, \quad \Psi(u) = \exp \left( iau + \Psi_{K^*}(u) \right).
\]

**Indirect part.** Suppose that

\[
S_n \rightsquigarrow K_0,
\]

where \( K_0 \) is associated with an a.s finite random variable \( Z \). Then \( b = \limsup_{n \to +\infty} a_n \) is finite. Otherwise consider a sub-sequence \( a_{n_{\ell}} \to +\infty \) as \( \ell \to +\infty \). So \( S_{n_{\ell}} = S^*_{n_{\ell}} + a_{n_{\ell}} \) necessarily weakly converges to \( Z \), where by Theorem 1, \( S^*_{n_{\ell}} \rightsquigarrow Z^* \), of law \( K_K^* \) and \( Z^* \) finite a.s. and hence \( Z \) is a.s infinite. Hence \( b = \limsup_{n \to +\infty} a_n \) if finite. Now, each sub-sequence of \( (a_n)_{n \geq 1} \) contains a sub-sequence \( (a_{n'})_{n' \geq 1} \) is converging to \( a \) finite. By the argument given above, \( S_{n'}^* \) weakly converges to some \( K_{K^*} \). By prohorov’s criteria, \( S_n^* \) weakly converges to \( K_K^* \) and \( S_n \) weakly converges to \( K_K^* + a = d K_0 \). The later inequality shows that all converging subsequences of \( (a_n)_{n \geq 1} \) converge to the same number \( a \). Finally

\[
S_n \rightsquigarrow K_K^* + a,
\]

with \( a_n \to a \). Let us summarize the discussions as follows.
Theorem 3. Under the BVH and the UAN Condition for summands of independent and square integrable real valued random variables, we have the following characterization. Let us denote

\[
\left( \forall n \geq 1, \forall 1 \leq k \leq k(n), \ E X_{k,n} = a_{k,n} \right) \text{ and } \left( \forall n \geq 1, \sum_{k=1}^{k(n)} a_{k,n} = a_n \right);
\]

\[
\forall n \geq 1, \ S_n = (S_n - a_n) + a_n = \sum_{k=1}^{k(n)} (X_{k,n} - a_{k,n}) + a_n =: S_n^* + a_n;
\]

\(F^*_{k,n}\) the cdf of \((X_{k,n} - a_{k,n})\) for \(n \geq 1\) and \(1 \leq k \leq k(n)\):

\[
\forall u \in \mathbb{R}, \Psi_{K^*}(u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, dK^*(x)
\]

and, finally,

\[
\forall n \geq 1, \forall u \in \mathbb{R}, \Psi_{K^*_n}(u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, dK^*_n(x)
\]

with

\[
\forall n \geq 1, \forall x \in \mathbb{R}, \ K^*_n(x) = \int_{-\infty}^{x} y^2 \sum_{k=1}^{k(n)} dF^*_n(y).
\]

We have the following facts.

(i) If \(K^*_n \overset{pre}{\to} K^*\) and \(a_n \to a\), then

\(S_n \overset{d}{\to} K^* + a\).

(ii) If

\(S_n \overset{d}{\to} K_0\)

where \(K_0\) is associated to an a.s finite random variable \(Z\), then the sequence \((a_n)_{n \geq 1}\) converges to a real number \(a\) and

\(K_0 =_{d} K^* + a\).
and

\[ K^*_n \overset{\text{pre}}{\sim} K^*. \]

Moreover if the VCH holds at the place of the BVH and the variance of \( K^*_n \) is equal to \( c \), we have

\[ K^*_n \overset{\text{pre}}{\sim} K^* \]

in both parts (i) and (ii).

5. Characterizations of two important examples

The two important limits of Gaussian law and Poisson law are very important. In stochastic analysis, these laws allow to represent some stochastic process into a discontinuous process (Poisson component) and a continuous process (Gaussian part).

5.1. Gaussian limit.

Let us suppose that the weak limit of the summands \((S_n)_{n \geq 1}\) is the standard Gaussian law

\[ \exp(\Psi_K(u)) = \exp(-u^2/2), \quad u \in \mathbb{R}, \]

i.e.

\[ \Psi_K(u) = \int \frac{e^{ixu} - 1 - iux}{x^2} \, dK(x) = -\frac{u^2}{2}, \quad u \in \mathbb{R}. \]  

But, for \( \lambda_K = \delta_0 \), that is, \( K = 1_{[0, +\infty[} \), we have

\[ \int \frac{e^{ixu} - 1 - iux}{x^2} \, d\delta_0(x) = \left[ \frac{e^{ixu} - 1 - iux}{x^2} \right]_{x=0} = -\frac{u^2}{2}. \]

We are going to rediscover the Lévy-Lynderberg-Feller (L2F) theorem as stated in Lo (2018b) (See Theorem 20 in page ...).
Theorem 4. Let $S_n = X_{1,n} + \cdots + X_{k(n),n}$ summands of centered and square integrable random variables as denoted above such that

\begin{equation}
\sum_{1 \leq k \leq k(n)} \text{Var}(X_{k,n}) = \sum_{1 \leq k \leq k(n)} \sigma^2_{k,n} = 1.
\end{equation}

For $\varepsilon > 0$ and $n \geq 1$, let us denote the Lynderberg function as

\begin{equation}
g_n(\varepsilon) = \sum_{1 \leq k \leq k(n)} \int_{(|x| \geq \varepsilon)} x^2 \, dF_{k,n}(x).
\end{equation}

We have the following characterization:

\begin{equation}
S_n \Rightarrow N(0, 1) \text{ as } n \to +\infty
\end{equation}

and

\begin{equation}
\max_{1 \leq k \leq k(n)} \text{Var}(X_{k,n}) \to 0 \text{ as } n \to +\infty
\end{equation}

if and only if, for any $\varepsilon > 0$, the following Lynderberg criterion holds:

\begin{equation}
g_n(\varepsilon) \to 0 \text{ as } n \to +\infty.
\end{equation}

Proof. Let us begin by linking the Lynderberg function as \((5.3)\) with \((5.5)\). We have

\begin{align*}
\max_{1 \leq k \leq k(n)} \text{Var}(X_{k,n}) &= \max_{1 \leq k \leq k(n)} \int x^2 \, dF_{k,n}(x) \\
&= \sum_{1 \leq k \leq k(n)} \int x^2 \, dF_{k,n}(x) \\
&= \sum_{1 \leq k \leq k(n)} \int_{(|x| \leq \varepsilon)} x^2 \, dF_{k,n}(x) + \sum_{1 \leq k \leq k(n)} \int_{(|x| > \varepsilon)} x^2 \, dF_{k,n}(x) \quad \text{(L3)} \\
&= \varepsilon^2 + g_n(\varepsilon),
\end{align*}

\begin{equation}
(5.7)
\end{equation}
where we used (5.2) in the first summation in Line (L3). By letting $n \to +\infty$ first and next, by letting $\varepsilon \to 0$, we get that the Lynderberg criterion implies (5.5). We have:

**Fact 2.** The Lynderberg criterion (5.6) implies (5.5), which in turn implies the UAN hypothesis.

Now, let us prove both implications.

**Direct implication.** Suppose that (5.4) and (5.5) hold. So the $BVH$ (by (5.2)) and the UAN Condition holds (by Fact 2). Actually the $BVH$ (5.2) is also a $VCH$ Conditions. So may apply both Theorems 1 and 2. By applying Theorem 1, we have

$$
\forall x \in C(K), \quad K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{-\infty}^{x} y^2 \ dF_{k,n}(y) \to 1_{(x \geq 0)}
$$

since $1_{(x \geq 0)}$ is the df associated with $\delta_0$. Any $x > 0$ is in $C(K)$ and then

$$
K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{-\infty}^{x} y^2 \ dF_{k,n}(y) \to 1
$$

$$
\Leftrightarrow \sum_{1 \leq k \leq k(n)} \int_{-\infty}^{x} y^2 \ dF_{k,n}(y) - \sum_{1 \leq k \leq k(n)} \int_{(y>x)} y^2 \ dF_{k,n}(y) \to 1
$$

$$
\Leftrightarrow 1 - \sum_{1 \leq k \leq k(n)} \int_{(y>x)} y^2 \ dF_{k,n}(y) \to 1.
$$

Hence

$$
\forall x > 0, \ g_{n,1}(x) := \sum_{1 \leq k \leq k(n)} \int_{(y>x)} y^2 \ dF_{k,n}(y) \to 0.
$$

Next, any $x < 0$ is in $C(K)$ and then
\[ K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{-\infty}^{x} y^2 \, dF_{k,n}(y) \to 0 \]
\[ \iff \sum_{1 \leq k \leq k(n)} x^2 \lambda_{F_{k,n}}(\{x\}) + \sum_{1 \leq k \leq k(n)} \int_{(y < x)} y^2 \, dF_{k,n}(y) \to 0 \]
\[ (5.8) \iff \lambda_{K_n}(\{x\}) + \sum_{1 \leq k \leq k(n)} \int_{(y < x)} y^2 \, dF_{k,n}(y) \to 0. \]

But, by Portmanteau Theorem (see Criterion (vi) of Theorem 2, page 47 in Lo et al. (2016)), \( \lambda_{K_n}(\{x\}) \to \lambda_K(\{x\}) \) since \( \partial \{x\} = \{x\} \) and hence \( \lambda_K(\{x\}) = K(x) - K(x-0) = 0 \) since \( x \in C(K) \).

Hence
\[ \forall x < 0, \ g_{n,2}(x) := \sum_{1 \leq k \leq k(n)} \int_{(y < x)} y^2 \, dF_{k,n}(y) \to 0. \]

By putting together the two last results, for any \( \varepsilon > 0 \)
\[ g_n(\varepsilon) = \sum_{1 \leq k \leq k(n)} \int_{(|y| > \varepsilon)} y^2 \, dF_{k,n}(y) \]
\[ = \sum_{1 \leq k \leq k(n)} \int_{(y > \varepsilon)} y^2 \, dF_{k,n}(y) + \sum_{1 \leq k \leq k(n)} \int_{(y < -\varepsilon)} y^2 \, dF_{k,n}(y) \]
\[ = g_{n,1}(\varepsilon) + g_{n,2}(\varepsilon) \]
\[ \to 0 \text{ as } n \to +\infty. \]

**Proof of the indirect implication.** Let (5.6) holds. So, by Fact 2, (5.5) holds and then the UAN is satisfied and the BVH is already satisfied as an hypothesis of the theorem. Still by Theorem 1, (5.4) holds whenever
\[ (5.9) \quad \forall x \in C(K), \ K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{-\infty}^{x} y^2 \, dF_{k,n}(y) \to 1_{(x \geq 0)}. \]

Let us prove (5.9), by exploiting (5.6). We have \( C(F) = (x < 0) + (x > 0) \). For \( x > 0 \), we have
\[ K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{(y \leq x)} y^2 \, dF_{k,n}(y) \]

\[ = 1 - \sum_{1 \leq k \leq k(n)} \int_{(y > x)} y^2 \, dF_{k,n}(y) \]

\[ = 1 - \sum_{1 \leq k \leq k(n)} \int_{(|y| > x)} y^2 \, dF_{k,n}(y) \]

\[ = 1 - g_n(x) \]

\[ \to 1 \text{ as } n \to +\infty. \]

For \( x < 0 \), we have

\[ K_n(x) = \sum_{1 \leq k \leq k(n)} \int_{(y \leq x)} y^2 \, dF_{k,n}(y) \]

\[ = \sum_{1 \leq k \leq k(n)} x^2 \lambda_{F_{k,n}}(\{x\}) + \sum_{1 \leq k \leq k(n)} \int_{(y < x)} y^2 \, dF_{k,n}(y) \]

\[ = \sum_{1 \leq k \leq k(n)} x^2 \lambda_{F_{k,n}}(\{x\}) + \sum_{1 \leq k \leq k(n)} \int_{(|y| > x)} y^2 \, dF_{k,n}(y) \]

\[ = \lambda_{K_n}(\{x\}) + \sum_{1 \leq k \leq k(n)} \int_{(|y| > -x)} y^2 \, dF_{k,n}(y) \]

\[ = \lambda_{K_n}(\{x\}) + g_n(-x). \]

Now, by (5.6), \( g_n(-x) \to 0 \) and by using a similar technical in line (5.8), \( \lambda_{K_n}(\{x\}) \to 0 \).

So (5.9) holds and we have proved (5.4) and (5.5). \( \blacksquare \)

5.2. Poisson limit.

The searched limit here is a translated Poisson law \( P(b, \lambda) = b + P(\lambda) \), with \( b \in \mathbb{R} \) and \( \lambda > 0 \) of characteristic function
\[ \exp(\Psi_K(u)) = \exp(ibu + \lambda(e^{iu} - 1)) = \exp(i(b + \lambda)u + \lambda(e^{iu} - 1 - iu)), \quad u \in \mathbb{R}, \]

with

\[ \Psi_K(u) = i(b + \lambda)u + \Psi_K^*(u), \quad \Psi_K^*(u) = \lambda(e^{iu} - 1 - iu), \]

where \( \exp(\Psi_K^*(\circ)) \) is the characteristic function of the centered Poisson law \( P^*(\lambda) = (P(\lambda) - \lambda) \).

Let us state the characterization theorem.

**Theorem 5.** Let \( S_n = X_{1,n} + \cdots + X_{k(n),n} \) be summands of independent and square integrable random variables. As above, let \( a_{k,n} = \mathbb{E}X_{k,n} \) and let \( F_{k,n}^* \) be the cdf of \( X_{k,n} - a_{k,n} \). Let us introduce Lynderberg-type functions, for \( \varepsilon > 0 \) and \( n \geq 1 \), as

\[
(5.10) \quad g_{n,\text{pois}}(\varepsilon) = \sum_{k=1}^{k(n)} \int_{|x-1| > \varepsilon} x^2 \, dF_{k,n}^*(x).
\]

Suppose that, as \( n \to +\infty \) with \( (\text{MVP}(n) = \sum_{1 \leq k \leq k(n)} \sigma_{k,n}^2) \),

\[
(5.11) \quad B_n = \max_{1 \leq k \leq k(n)} \sigma_{k,n}^2 \to 0 \quad \text{and} \quad \text{MVP}(n) \to \lambda.
\]

Let \( b \in \mathbb{R} \). We have the following characterization.

\[
(5.12) \quad S_n \xrightarrow{\text{d}} P(b, \lambda) \quad \text{as} \quad n \to +\infty
\]

if and only if,

\[
(5.13) \quad \sum_{k=1}^{k(n)} \mathbb{E}(X_{k,n}) = a_n \to a = b + \lambda \quad \text{as} \quad n \to +\infty
\]

and for any \( \varepsilon > 0 \), the following Lynderberg Poisson-type criterion holds:

\[
(5.14) \quad g_{n,\text{pois}}(\varepsilon) \to 0 \quad \text{as} \quad n \to +\infty.
\]
Proof. Based (5.11), the CVH and the UAN condition hold. We can apply Theorem 3. We study the limit of \( \Psi_{K_n^*}(u) \), for any \( u \in \mathbb{R} \) to

\[
\Psi_{K^*}(u) = \int e^{iux} \frac{1 - iux}{x^2} \, dK^*(x) = \lambda(e^{iu} - 1 - iu).
\]

Let \( \lambda_{K^*} = \lambda \delta_1 \), i.e., \( K^*(x) = \lambda 1_{(x \geq 1)} \). Thus

\[
\int e^{iux} \frac{1 - iux}{x^2} \, dK^*(x) = \lambda \left[ \frac{e^{iux} - 1 - iux}{x^2} \right]_{x=1} = \lambda(e^{iu} - 1 - iu), \quad u \in \mathbb{R}.
\]

Proof of the direct part. Suppose that (5.12) holds. Applying Theorem 3, where the probability law limit is associated with an a.s finite random variable, leads to

\[
a_n \to b + \lambda \quad \text{and} \quad K_n^* \rightsquigarrow K^*.
\]

So (5.13) holds. We also have that \( K_n \rightsquigarrow K \) means:

\[
\forall x \in C(K^*), \quad K_n^*(x) \to \lambda 1_{(x \geq 1)}, \quad \text{as } n \to +\infty,
\]

since \( C(K^*) = (x < 1) + (x > 1) \) and \( \lambda_{K_n}(\mathbb{R}) \to \lambda_{K^*}(\mathbb{R}) = \lambda \). For \( x > 1 \), we have

\[
\sum_{k=1}^{k(n)} \int_{y \leq x} y^2 \, dF^*_k(y) \to \lambda
\]

\[
\Leftrightarrow \sum_{k=1}^{k(n)} \int y^2 \, dF^*_k(y) + \sum_{k=1}^{k(n)} \int_{y > x} y^2 \, dF^*_k(y) \to \lambda
\]

\[
\Leftrightarrow \sum_{k=1}^{k(n)} \sigma_{k,n}^2 + \sum_{k=1}^{k(n)} \int_{y > x} y^2 \, dF^*_k(y) \to \lambda,
\]

where we use that \( \int y^2 \, dF^*_k(y) = \text{Var}(X_{k,n} - a_{k,n}) = \sigma_{k,n}^2 \) in the last line. Hence
\[ \sum_{k=1}^{k(n)} \int_{y \leq x} y^2 \, dF_{k,n}(y) \to \lambda \]

\[ \Leftrightarrow \sum_{k=1}^{k(n)} \int_{y > x} y^2 \, dF_{k,n}(y) \to 0. \quad (L22) \]

Let \( \varepsilon = x - 1 > 0 \), (L22) is equivalent to

\[ \sum_{k=1}^{k(n)} \int_{(y-1 > x-1)} y^2 \, dF_{k,n}(y) \to 0, \]

which is

\[ \sum_{k=1}^{k(n)} \int_{(|y-1| > x-1)} y^2 \, dF_{k,n}(y) \to 0, \]

that is

\[ g_{n,\text{pois}}(x - 1) \to 0. \]

Next, For \( x < 1 \), we have

\[ \sum_{k=1}^{k(n)} \int_{y \leq x} y^2 \, dF_{k,n}(y) \to 0 \]

\[ \Leftrightarrow \sum_{1 \leq k \leq k(n)} x^2 \lambda_{F_{k,n}}(\{x\}) + \sum_{k=1}^{k(n)} \int_{(y < x)} y^2 \, dF_{k,n}(y) \to 0 \]

\[ \Leftrightarrow \lambda_{K_\ast}(\{x\}) + \sum_{k=1}^{k(n)} \int_{(1-y > 1-x)} y^2 \, dF_{k,n}(y) \to 0 \]

\[ \Leftrightarrow \sum_{k=1}^{k(n)} \int_{(|1-y| > 1-x)} y^2 \, dF_{k,n}(y) \to 0, \]

where we use that \( \lambda_{K_\ast}(\{x\}) \to 0 \) (as shown in line (5.8) above) in the last line. Hence
\( g_{n,\text{pois}}(1 - x) \to 0. \)

By combining these results, we have for any \( \varepsilon > 0 \), by taking either \( x - 1 = \varepsilon \) (for \( x > 1 \)) or \( 1 - x = \varepsilon \) (for \( x < 1 \)), we arrive at (5.14).

**Proof of the indirect implication.** Suppose that (5.13) and (5.14) are satisfied. Let us exploit (5.14). For \( x > 1 \),

\[
K_n(x) = \sum_{k=1}^{k(n)} \int_{(y \leq x)} y^2 dF_{k,n}^*(y)
\]

\[
= \sum_{k=1}^{k(n)} \sigma_{k,n}^2 - \sum_{k=1}^{k(n)} \int_{(y - 1 > x - 1)} y^2 dF_{k,n}^*(y)
\]

\[
= \sum_{k=1}^{k(n)} \sigma_{k,n}^2 - g_{n,\text{pois}}(x - 1) \to \lambda.
\]

For \( x < 1 \)

\[
K_n(x) = \sum_{k=1}^{k(n)} \int_{(y \leq x)} y^2 dF_{k,n}^*(y)
\]

\[
= \lambda K_n(\{x\}) + \sum_{k=1}^{k(n)} \int_{(|y - 1| > 1 - x)} y^2 dF_{k,n}^*(y)
\]

\[
= \lambda K_n(\{x\}) + g_{n,\text{pois}}(1 - x).
\]

So, \( \lambda K_n(\{x\}) \to 0 \) is shown exactly as in the lines (5.8) above. Next \( g_{n,\text{pois}}(1 - x) \to 0 \) is Assumption . Hence \( K_n \sim_{\text{pre}} \lambda 1_{(\text{c} \geq 1)} \). To complete the proof, we remark that for any \( n \geq 1 \), \( K_n(-\infty) = 0 \) and \( K_n(+\infty) = \sum_{k=1}^{k(n)} \sigma_{k,n}^2 \). So

\[
\lambda K_n(\mathbb{R}) = K_n(+\infty) = \sum_{k=1}^{k(n)} \sigma_{k,n}^2 \to \lambda = \lambda K(\mathbb{R}).
\]

By Theorem 3, we conclude that (5.12) holds. ■
6. Conclusion

We hope that we have given a complete exposition of the theory of the weak limits of independent summands of square integrable random variables on \( \mathbb{R} \). The next step will be a re-do of the same theory when the existence of the moments, even the first moment, is not required.

References

Billingsley, P.(1968). *Convergence of Probability measures*. John Wiley, New-York.

Lo, G. S. (2017) Measure Theory and Integration By and For the Learner. SPAS Books Series. Saint-Louis, Senegal - Calgary, Canada. Doi : http://dx.doi.org/10.16929/sbs/2016.0005, ISBN : 978-2-9559183-5-7.

Lo, G.S.(2016). Weak Convergence (IA). Sequences of random vectors. SPAS Books Series. Saint-Louis, Senegal - Calgary. Canada. Doi : 10.16929/sbs/2016.0001. Arxiv : 1610.05415. ISBN : 978-2-9559183-1-9

Lo, G.S.(2016). A Course on Elementary Probability Theory. SPAS Editions. Saint-Louis, Calgary, Abuja. Doi : 10.16929/sbs/2016.0003.

Lo, G.S.(2018). *Mathematical Foundation of Probability Theory*. SPAS Books Series. Saint-Louis, Senegal - Calgary, Canada. Doi : http://dx.doi.org/10.16929/sbs/2016.0008. Arxiv : arxiv.org/pdf/1808.01713

Michel Loève (1997). *Probability Theory I*. Springer Verlag. Fourth Edition.