A Growth Behavior of Szegö Type Operators

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Abstract
We define new integral operators on the Haydy space similar to Szegö projection. We show that these operators map from $H^p$ to $H^2$ for some $1 \leq p < 2$, where the range of $p$ is depending on a growth condition. To prove that, we generalize the Hausdorff-Young Theorem to multi-dimensional case.

Keywords
Szegö Projection, Hausdorff-Young Theorem, Coefficient Multiplier, Stein Interpolation Theorem

1. Introduction
Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension $n$. The inner product on $\mathbb{C}^n$ is given by

$$\langle z, w \rangle := z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and the associated norm is $|z| := \sqrt{\langle z, z \rangle}$. The unit ball in $\mathbb{C}^n$ is the set

$$B_n := \{ z \in \mathbb{C}^n : |z| < 1 \}$$

and its boundary is the unit sphere

$$S_n := \{ z \in \mathbb{C}^n : |z| = 1 \}.$$
Let \( f \) be a function in \( H^p(B_n) \), it is known that \( f \) has a radial limit \( f^* \) almost everywhere on \( S_n \). Here, the radial limit \( f^* \) of \( f \) is defined by

\[
f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)
\]

provided that the limit exists for \( \zeta \in S_n \). Moreover, the mapping \( f \mapsto f^* \) is an isometry from \( H^p(B_n) \) into \( L^p(S_n, d\sigma_n) \). Consequently, each \( H^p(B_n) \) can be identified with a closed subspace of \( L^p(S_n, d\sigma_n) \).

Since \( H^2(B_n) \) can be identified with a closed subspace of \( L^2(S_n, d\sigma_n) \), there exists an orthogonal projection from \( L^2(S_n, d\sigma_n) \) onto \( H^2(B_n) \). By using a reproducing kernel function, which is called the Szegö kernel, we also obtain a function \( f \) from its radial function \( f^* \). More precisely,

\[
f(z) = T[f](z) := \int_{S_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^m} d\sigma_n(\zeta)
\]

for \( f \in H^2(B_n) \). We usually call this integral operator as the Szegö projection. It is well known that for \( 1 < p < \infty \) the Szegö projection maps \( L^p(S_n, d\sigma_n) \) boundedly onto \( H^p(B_n) \). For more details, we refer the classical text books \cite{1,2}.

In this paper we consider a class of integral operators defined by

\[T_{m,N}[f](z) := \int_{S_n} \frac{\langle z, \zeta \rangle^{m+N}}{(1 - \langle z, \zeta \rangle)^m} f^*(\zeta) d\sigma_n(\zeta)\]  

for \( m = 1, 2, \ldots, n \) and a positive integer \( N \). Compared with the Szegö projection, the growth condition in the denominator factor is better. Thus these operators are bounded on \( H^2(B_n) \).

Interestingly these operators map from \( H^1(B_n) \) to \( H^2(B_n) \) for any positive integer \( N \) when \( 1 \leq m < \frac{n}{2} \). More precisely we have the following result.

**Theorem 1.1.** Let \( m \) be a positive integer with \( 1 \leq m < \frac{n}{2} \). Then there exists a constant \( C = C(n) > 0 \) such that

\[\|T_{m,N}[f]\|_{H^2} \leq CN^{m-\frac{n}{2}} \|f\|_{H^1}\]

for any positive integer \( N \).

For \( \frac{n}{2} \leq m < n \), the operator \( T_{m,N} \) maps from \( H^p(B_n) \) to \( H^1(B_n) \), but the range of \( p \) is depending on \( m \), which determines the growth condition of the kernel function. Explicitly we have the following theorem.

**Theorem 1.2.** Let \( m \) be a positive integer with \( \frac{n}{2} \leq m < n \) and \( \frac{2m}{3n-2m} < p < 2 \). Then there exists a constant \( C = C(n,p) > 0 \) such that

\[\|T_{m,N}[f]\|_{H^2} \leq CN^{p'} \|f\|_{H^p}\]

for any positive integer \( N \). Here \( p' \) is a negative number defined by \( p' := m + n\left(\frac{1}{p} - \frac{3}{2}\right) \).

To prove the Theorem 1.2, we generalize the Hausdorff-Young Theorem to the multi-dimensional case using the Stein interpolation theorem.
2. Preliminary Results

We use the conventional multi-index notation. For a multi-index

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \]

with nonnegative integers \( \alpha_i \), the following are common notations;

\[ |\alpha| := \alpha_1 + \cdots + \alpha_n, \]
\[ \alpha! := \alpha_1! \cdots \alpha_n!. \]

For \( z \in \mathbb{C}^n \), the monomial is defined as

\[ z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \]

At first, we show that the Szegő type operators \( T_{m,N} \) defined in (1.1) are actually coefficient multipliers.

**Lemma 2.1.** Let \( m, N \) be positive integers with \( 1 \leq m \leq n \). For a multi-index \( \alpha \), there exists \( \lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|) \) such that

\[ T_{m,N}[\zeta^\alpha](z) = \lambda_\alpha z^\alpha. \]

**Proof.** From the definition of \( T_{m,N} \), we have

\[ T_{m,N}[\zeta^\alpha](z) = \int_{\mathbb{S}_n} \frac{(z, \zeta)^{m+N}}{(1 - (z, \zeta))^m} \, d\sigma_n(\zeta) \]

for a multi-index \( \alpha \). Note that

\[ \frac{1}{(1 - (z, \zeta))^m} = \sum_{k=0}^{\infty} \binom{k + m - 1}{k} (z, \zeta)^k. \]

Since the monomials are orthogonal on \( L^2(\mathbb{S}_n, d\sigma_n) \); see ([1] Proposition 1.4.8), we have \( T_{m,N}[\zeta^\alpha](z) = 0 \) if \( |\alpha| < m + N \). In case of \( |\alpha| \geq m + N \), we have

\[ T_{m,N}[\zeta^\alpha](z) = \int_{\mathbb{S}_n} \sum_{k=0}^{\infty} \binom{k + m - 1}{k} (z, \zeta)^{k+m+N} \zeta^\alpha \, d\sigma_n(\zeta) \]
\[ = \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \int_{\mathbb{S}_n} (z, \zeta)^{|\alpha|} \zeta^\alpha \, d\sigma_n(\zeta). \]

Expanding the term inside the above integral as

\[ (z, \zeta)^{|\alpha|} = \sum_{|\beta| = |\alpha|} \frac{|\alpha|!}{|\beta|!} z^\beta \zeta^\alpha, \]

we obtain that

\[ T_{m,N}[\zeta^\alpha](z) = \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{S}_n} |\zeta^\alpha|^2 \, d\sigma_n(\zeta) \]
\[ = \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} z^\alpha, \]

see ([1] Proposition 1.4.9) for the last equality. Putting \( \lambda_\alpha \) as

\[ \lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|) = \begin{cases} 0 & \text{for } |\alpha| < m + N \\ \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} & \text{for } |\alpha| \geq m + N, \end{cases} \]

we conclude the lemma. \( \square \)
To prove the main theorems, we need the Hausdorff-Young Theorem for the multi-dimensional Hardy space. For a holomorphic function $f$ in the unit disk, we have the Taylor series expansion as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$  

For the Hardy space defined in the unit disk, a relationship between the functions in $H^p(D)$ and the growth condition of their coefficients are given by the Hausdorff-Young Theorem, see ([3] p.76, Theorem A).

**Theorem 2.2** (Hausdorff-Young Theorem for $H^p(D)$). For $1 \leq p \leq \infty$, let $q$ be the conjugate exponent, with $\frac{1}{p} + \frac{1}{q} = 1$.

1) If $1 \leq p \leq 2$, then $f \in H^p(D)$ implies $\{a_n\} \in l^q$, and $\|\{a_n\}\|_{l^q} \leq \|f\|_{H^p}$.

2) If $2 \leq p \leq \infty$, then $\{a_n\} \in l^q$ implies $f \in H^p(D)$, and $\|f\|_{H^p} \leq \|\{a_n\}\|_{l^q}$.

Before proceeding, we introduce some notation. Let $\mathbb{N}_0^n$ be the product set of nonnegative integers. Define a weight function $w_n$ on $\mathbb{N}_0^n$ by

$$w_n(\alpha) := \frac{(n-1)! |\alpha|!}{(n-1 + |\alpha|)!}$$

for $\alpha \in \mathbb{N}_0^n$. Using the weight $w_n$, we define a norm on $\mathbb{N}_0^n$ by

$$\|c\|_{p,t}^p := \sum_{\alpha \in \mathbb{N}_0^n} |c(\alpha)|^p w_n^t(\alpha)$$

for $1 \leq p < \infty$ and a positive real number $t$. For $p = \infty$, we define

$$\|c\|_{\infty,t} := \sup_{\alpha \in \mathbb{N}_0^n} |c(\alpha)| w_n^t(\alpha).$$

Let $l^{p,t}$ be the collection of all function $c$ defined on $\mathbb{N}_0^n$ with the norm $\|c\|_{p,t} < \infty$.

For a holomorphic function $f$ on $B_n$, whose Taylor series is given by

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

we define the coefficient function $c_f$ of $f$ define on $\mathbb{N}_0^n$ by

$$c_f(\alpha) := c_\alpha.$$

**Proposition 2.3** (Hausdorff-Young Theorem for $H^p(B_n)$). For $1 \leq p < \infty$, let $q$ be the conjugate exponent, with $\frac{1}{p} + \frac{1}{q} = 1$.

1) If $p = 1$, then $f \in H^1(B_n)$ implies $c_f \in l^{\infty,1}$, and $\|c_f\|_{l^{\infty,1}} \leq \|f\|_{H^1}$.

2) If $1 < p \leq 2$, then $f \in H^p(B_n)$ implies $c_f \in l^{q,q-1}$, and $\|c_f\|_{l^{q,q-1}} \leq \|f\|_{H^p}$.

3) If $2 \leq p < \infty$, then $c_f \in l^{q,q-1}$ implies $f \in H^p(B_n)$, and $\|f\|_{H^p} \leq \|c_f\|_{l^{q,q-1}}$.
Proof. For a multi-index \( \alpha \), we note that
\[
\int_{S_n} |\zeta^\alpha|^2 \, d\sigma_n(\zeta) = w_n(\alpha).
\] (2.2)

From the orthogonality of monomials on \( S_n \), we get
\[
\int_{S_n} f(\zeta) \overline{\zeta^\alpha} \, d\sigma_n(\zeta) = c_f(\alpha) w_n(\alpha),
\]
for \( f \in H^1(B_n) \). Thus we obtain
\[
\| c_f \|_{\infty, 1} \leq \| f \|_{H^1},
\]
which prove the Proposition (1).

\[
\| c_f \|_\infty, 1 \leq \| f \|_{H^1}, \quad \| c_f \|_{2, 1} \leq \| f \|_{H^2},
\] (2.3)
and proved (1). Now we apply an interpolation theorem in the Equations (2.3). Since the defined norms have weight functions, we use the Stein interpolation theorem; see (4 Theorem 3.6). Then we have
\[
\| c_f \|_{q, q-1} \leq \| f \|_{H^p}
\]
for \( 1 < p \leq 2 \) and proved (2).

Now we prove (3). For \( 1 < q \leq 2 \) we let \( c_f \) be given with \( \| c_f \|_{q, q-1} < \infty \) and define
\[
f_k(z) := \sum_{|\alpha| \leq k} c_{\alpha} z^\alpha
\]
for a positive integer \( k \). Since each \( f_k \in H^p(B_n) \), for any \( g \in H^q(B_n) \) with coefficient function \( c_g \) we have
\[
\left| \int_{S_n} f_k g \, d\sigma_n \right| = \left| \sum_{|\alpha| \leq k} c_f(\alpha) c_g(\alpha) w_n(\alpha) \right|
\leq \left( \sum_{\alpha} |c_f(\alpha)|^q w_n(\alpha)^{q-1} \right)^{1/q} \left( \sum_{\alpha} |c_g(\alpha)|^p w_n(\alpha)^{p-1} \right)^{1/p}
\leq \| c_f \|_{q, q-1} \| g \|_{H^q}
\]
by above proved Proposition (2). Since for \( 1 < p < \infty \) the dual space of \( H^p(B_n) \) is \( H^q(B_n) \), we have
\[
\| f_k \|_{H^p} \leq \| c_f \|_{q, q-1}
\]
for any positive integer \( k \). Moreover \( w_n \leq 1 \) and \( q \leq 2 \) implies that
\[
\| c_f \|_{q, q/2} \leq \| c_f \|_{q, q-1} < \infty,
\]
so we have
\[
\| f \|_{H^2} = \| c_f \|_{2, 1} < \infty.
\]
Consequently \( \| f - f_k \|_{H^2} \) goes to zero as \( k \) increase. Hence \( f_k \) converges to \( f \) pointwise and by applying Fatou’s lemma we finish the proof. \( \square \)
3. Proofs

For a holomorphic function $f$ on $B_n$ with Taylor series

$$f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha},$$

we define $j^{th}$ partial sum of $f$ by

$$S_{j} f(z) := \sum_{|\alpha| \leq j} c_{\alpha} z^{\alpha} \quad (3.1)$$

for a positive integer $j$.

Now we are ready to prove the Theorem 1.1. Here we restate the Theorem 1.1 for convenience.

**Theorem 3.1.** For $1 \leq m < \frac{n}{2}$, there exists a constant $C = C(n)$ depending only on $n$ such that

$$\|T_{m,N}[f]\|_{H^2} \leq CN^{m-\frac{n}{2}} \|f\|_{H^1}$$

for any positive integer $N$.

**Proof.** For a given $z \in B_n$, the kernel function $\frac{(z,\zeta)^{m+N}}{(1-\zeta)^{m}}$ is bounded in $\zeta \in S_n$. So for $p > 1$, we have

$$|T_{m,N}[f](z) - T_{m,N}[S_{j} f](z)|^p \leq \int_{S_n} \frac{(\zeta,\zeta)^{m+N}}{(1-\zeta)^{m}} |f(\zeta) - S_{j} f(\zeta)|^p \, d\sigma_n(\zeta) \leq C_{z} \int_{S_n} |f(\zeta) - S_{j} f(\zeta)|^p \, d\sigma_n(\zeta) = C_{z} \|f - S_{j} f\|^p_{H^p},$$

where $C_{z}$ is a constant depending on $z \in B_n$. However the Taylor series $S_{j} f$ converges to $f$ in the norm $H^p$ when $p > 1$; we refer ([5] Theorem 1.1). Thus for $f \in H^p(B_n)$ with $p > 1$ we have

$$T_{m,N}[f](z) = \lim_{j \to \infty} T_{m,N}\left[ \sum_{|\alpha| \leq j} c_{\alpha}(\alpha) \zeta^\alpha \right] (z) = \lim_{j \to \infty} \sum_{|\alpha| \leq j} c_{\alpha}(\alpha) T_{m,N}[\zeta^\alpha](z) = \sum_{\alpha} c_{\alpha}(\alpha) \lambda(m,n,N,|\alpha|) z^\alpha,$$

where we used the Lemma 2.1. So we have

$$\|T_{m,N}[f]\|^2_{H^2} = \sum_{|\alpha| \geq m+N} \lambda(m,n,N,|\alpha|) c_{\alpha}(\alpha)^2 w_n(\alpha) \quad (3.2)$$

$$\leq \|c_{\alpha}\|^2_{H^1} \cdot \sum_{|\alpha| \geq m+N} \lambda(m,n,N,|\alpha|) w_n(\alpha)^{-1}$$

By Proposition 2.3 and the Equation (2.1), we have

$$\|T_{m,N}[f]\|^2_{H^2} \leq C(n) \|f\|^2_{H^1} \cdot \sum_{|\alpha| \geq m+N} \frac{1}{|\alpha|^{2(n-m)}} \frac{(n-1 + |\alpha|)!}{\alpha!}$$

DOI: 10.4236/apm.2020.109030
where $C(n)$ is a constant depending on $n$. By the multinomial theorem we know that
\[
\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{2^n}{k!},
\]
so we get
\[
\sum_{|\alpha| \geq m+N} \frac{1}{\alpha^{2(n-m)}} \frac{(n-1+|\alpha|)!}{\alpha!}
\leq C(n) \sum_{k=m+N}^{\infty} \frac{1}{k^{2(n-m)}} \frac{(n-1+k)!}{k!}
\leq C(n) \sum_{k=N}^{\infty} \frac{1}{k^{n-2m+1}}
\leq C(n)N^{2m-n},
\]
where the series is bounded by the assumption $m < \frac{n}{2}$. And the constant $C(n)$ is also depending only on $n$. Thus we have
\[
\|T_{m,N}[f]\|_{H^2} \leq C(n)N^{m-\frac{n}{2}} \|f\|_{H^1},
\]
for $f \in H^p(\mathcal{B}_n)$ with $p > 1$.

Now we fix $p_0 > 1$. Since $H^{p_0}$ is dense in $H^1$ and $T_{m,N}$ is uniformly continuous from $H^{p_0}$ to $H^2$, there exists the unique continuous extension $\overline{T}_{m,N}$ from $H^1$ to $H^2$ defined as
\[
\overline{T}_{m,N}[f] := \lim_{k \to \infty} T_{m,N}[f_k],
\]
for any $f_k \to f$ with each $f_k$ in $H^{p_0}$. In particular, $\overline{T}_{m,N}[f] = \lim_{r \to 1^-} T_{m,N}[f^r]$ where $f^r(z) := f(rz)$ for $0 < r < 1$. By Fatou’s lemma we have $\overline{T}_{m,N}[f](z) = \lim_{r \to 1^-} T_{m,N}[f^r](z)$ for any $z \in \mathcal{B}_n$.

Moreover using the similar argument in the beginning of the proof, for any $z \in \mathcal{B}_n$ we have
\[
|\overline{T}_{m,N}[f](z) - T_{m,N}[f^r](z)| \leq C_z \|f - f^r\|_{H^1},
\]
where $C_z$ is a constant depending on $z$. Since $\|f - f^r\|_{H^1} \to 0$ as $r \to 1^-$, we also have $T_{m,N}[f](z) = \lim_{r \to 1^-} T_{m,N}[f^r](z)$. Consequently we have $\overline{T}_{m,N}[f](z) = T_{m,N}[f](z)$ for any $f \in H^1$ and
\[
\|T_{m,N}[f]\|_{H^2} \leq C(n)N^{m-\frac{n}{2}} \|f\|_{H^1},
\]
for any $f \in H^1(\mathcal{B}_n)$.

Since $T_{m,N}$ is an integral operator with conjugate symmetric kernel, we have the following corollary by using its adjoint operator.

**Corollary 3.2.** For $1 \leq m < \frac{n}{2}$, there exists a constant $C = C(n)$ depending only on $n$ such that
\[
\|T_{m,N}[f]\|_{BMO} \leq C N^{m-\frac{n}{2}} \|f\|_{H^2}
\]
for any positive integer $N$. Here $\| \cdot \|_{BMO}$ means the BMOA norm.

**Remark.** For $0 < p_1 < p_2 < \infty$, we have
\[
\|f\|_{H^{p_1}} \leq \|f\|_{H^{p_2}} \leq \|f\|_{BMO}.
\]
By Theorem 3.1 and Corollary 3.2 we obtain that if \(1 \leq m < \frac{n}{2}\),
\[
\|T_{m,N}[f]\|_{H^p(\text{or } BMO)} \lesssim \frac{\|f\|_{H^p(\text{or } BMO)}}{N^{\frac{n}{2} - m}},
\]
for \(1 \leq p < \infty\). That is, the Szegö type operators makes the norm decrease quickly as \(N\) goes large when \(m < \frac{n}{2}\).

We prove the Theorem 1.2. Here we restate the Theorem 1.2 for convenience.

**Theorem 3.3.** Let \(\frac{n}{2} \leq m < n\) and \(\frac{2n}{3n-2m} < p < 2\). Then there exists a constant \(C = C(n, p)\) such that
\[
\|T_{m,N}[f]\|_{H^2} \leq C N^p \|f\|_{H^p}
\]
for any positive integer \(N\). Here \(p'\) is a negative number defined by \(p' := m + n(\frac{1}{p} - \frac{3}{2})\).

**Proof.** We begin with the Equation (3.2). Then we have
\[
\|T_{m,N}[f]\|_{H^2}^2 = \sum_{|\alpha| \geq m+N} \lambda(m,n,|\alpha|)^2 |c_f(\alpha)|^2 w_n(\alpha)
\]
\[
\leq \left( \sum_{|\alpha| \geq m+N} |c_f(\alpha)|^q w_n(\alpha)^{q-1} \right)^{2/q}
\]
\[
\left( \sum_{|\alpha| \geq m+N} \lambda(m,n,|\alpha|)^{2r} w_n(\alpha)^{-1} \right)^{1/r},
\]
where \(q\) is the conjugate index of \(p\) and \(r\) is of \(q=2\). By Proposition 2.3 and the Equation (2.1), we have
\[
\|T_{m,N}[f]\|_{H^2}^2 \leq C(n) \|f\|_{H^p}^2 \cdot \left( \sum_{|\alpha| \geq m+N} \frac{1}{|\alpha|^{2r(n-m)}} \frac{(n - 1 + |\alpha|)!}{\alpha!} \right)^{1/r},
\]
where \(C(n)\) is a constant depending on \(n\). By the multinomial theorem, we know that
\[
\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{2^n}{k!}.
\]
So we get
\[
\sum_{|\alpha| \geq m+N} \frac{1}{|\alpha|^{2r(n-m)}} \frac{(n - 1 + |\alpha|)!}{\alpha!} \leq C(n) \sum_{k=m+N}^\infty \frac{1}{k^{2r(n-m)} k!} \frac{(n - 1 + k)!}{\alpha!} \leq C(n) \sum_{k=N}^\infty \frac{1}{k^{2r(n-m)-n+1}},
\]
where a constant \(C(n)\) is depending on \(n\). Since \(p > \frac{2n}{3n-2m}\), we have \(r > \frac{n}{2(n-m)}\). So the above series converges and bounded by
\[
\frac{N^{-2r(n-m)+n}}{2r(n-m) - n}.
\]

Thus we prove
\[
\|T_{m,N}[f]\|_{H^2} \leq C(n, p) N^{m+n(\frac{1}{p} - \frac{3}{2})} \|f\|_{H^p}.
\]
Funded
This work was supported by the 2018 New Professor Research Grant funded by Korea National University of Education.

Conflicts of Interest
The author declares no conflicts of interest regarding the publication of this paper.

References
[1] Rudin, W. (2008) Function Theory in the Unit Ball of \( \mathbb{C}^n \). Springer-Verlag, Berlin.

[2] Zhu, K. (2005) Spaces of Holomorphic Functions in the Unit Ball. Springer-Verlag, New York.

[3] Duren, P. and Schuster, A. (2004) Bergman Spaces. American Mathematical Society, Providence, RI. https://doi.org/10.1090/surv/100

[4] Bennett, C. and Sharpley, R. (1988) Interpolation of Operators. Academic Press, Boston.

[5] Yang, J. (2015) Norm Convergent Partial Sums of Taylor Series. Bulletin of the Korean Mathematical Society, 52, 1729-1735. https://doi.org/10.4134/BKMS.2015.52.5.1729