Diffusion Asymptotics for Sequential Experiments

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Abstract

We propose a new diffusion-asymptotic analysis for sequentially randomized experiments. Rather than taking sample size $n$ to infinity while keeping the problem parameters fixed, we let the mean signal level scale to the order $1/\sqrt{n}$ so as to preserve the difficulty of the learning task as $n$ gets large. In this regime, we show that the behavior of a class of methods for sequential experimentation converges to a diffusion limit. This connection enables us to make sharp performance predictions and obtain new insights on the behavior of Thompson sampling. Our diffusion asymptotics also help resolve a discrepancy between the $\Theta(\log(n))$ regret predicted by the fixed-parameter, large-sample asymptotics on the one hand, and the $\Theta(\sqrt{n})$ regret from worst-case, finite-sample analysis on the other, suggesting that it is an appropriate asymptotic regime for understanding practical large-scale sequential experiments.  

Keywords: Multi-arm bandit, Thompson sampling, Stochastic differential equation.

1 Introduction

Sequential experiments, pioneered by Wald [1947] and Robbins [1952], involve collecting data over time using a design that adapts to past experience. The promise of sequential experiments is that, relative to classical randomized trials, they can effectively concentrate power on studying the most promising alternatives and save on costs by helping us avoid repeatedly taking sub-optimal actions. Such adaptivity, however, does not come for free, and sequential experiments induce intricate dependence patterns in the data that result in delicate practical considerations; see Bubeck and Cesa-Bianchi [2012] for a review and discussion.

To illustrate the complexity sequential experiments, consider the following $K$-arm setting, also known as a $K$-armed bandit. There is a sequence of decision points $i = 1, 2, \ldots$ at which an agent chooses which action $A_i \in \{1, \ldots, K\}$ to take and then observes a reward $Y_i \in \mathbb{R}$ whose distribution $P_{A_i}$ depends on the selected action, where the action $A_i$ may depend on past observations. A standard goal is for the agent to choose actions with the highest possible expected reward, and to minimize expected regret $R_n$ relative to always taking the best action, where

$$ R_n = n \sup_{1 \leq k \leq K} \{\mu_k\} - \sum_{i=1}^{n} A_i \mu_{A_i}, \quad \mu_k = \mathbb{E}_{P_k}[Y]. \quad (1) $$

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In this setting, Lai and Robbins [1985] show that given any fixed set of arms \( \{P_k\}_{k=1}^K \), a well-designed sequential algorithm can achieve regret that scales logarithmically with the number \( n \) of time steps considered, i.e., \( R_n = O((\log(n)) \). Conversely, given any fixed time horizon \( n \), it is possible to choose probability distributions \( \{P_k\}_{k=1}^K \) such that the expected regret \( \mathbb{E}[R_n] \) of any sequential algorithm is lower-bounded to order \( \sqrt{K/n} \) [Auer et al., 2002]. A similar difficulty arises when we want to use \( n \) samples to choose an action \( \hat{k} \) to deploy, and we want to control the cost of mistakes \( \sup_k \{\mu_k\} - \mathbb{E}[\mu_k] \). Here, for a fixed problem setting, the cost of mistakes can be made to decay exponentially in \( n \) [Russo, 2020], but for fixed \( n \) it’s possible to choose problems for which the regret of any algorithm is lower bounded by \( \sqrt{K/n} \).

This discrepancy between the regret behavior of fixed-\( P \), \( n \rightarrow \infty \) asymptotics, and worst-case, fixed-\( n \) analysis complicates the use of formal results to guide practical design of sequential experiments. When running a large sequential experiment in practice with, say, \( n = 100,000 \) samples, should one expect to pay regret on the order of \( \sqrt{n} \) or \( \log(n) \)? Should we expect the fraction of times we pull each arm to concentrate, as is suggested by fixed-\( P \), \( n \rightarrow \infty \) asymptotics, or should we expect it to show genuine variability based on analyses targeted at the fixed-\( n \) setting? The discrepancy further highlights the importance of a “moderate data” regime in sequential experiments that is not addressed by existing asymptotic analysis, one in which the analyst has enough data to perform meaningful inference, but not so much data that taking optimal actions becomes asymptotically trivial.

In this paper, we study a new way of performing asymptotic analysis for sequential experiments that eliminates this discrepancy. Rather than taking sample size \( n \) to infinity while otherwise keeping the problem setting fixed, we also let the reward distributions \( \{P^n_k\}_{k=1}^K \) change with \( n \) such as to preserve the difficulty of the learning task as \( n \) gets large, and to keep the resulting asymptotic regret in line with worst-case results from Auer et al. [2002]. Specifically, we consider a sequence of systems, indexed by \( n = 1, 2, \ldots \), and for each \( n \) we consider a \( K \)-arm sequential experiment with sample size \( n \) and reward distribution \( \{P^n_k\}_{k=1}^K \). Furthermore, we assume that the reward distributions satisfy

\[
\mu^n_k = \mathbb{E} P^n_k \left[ Y \right] = \mu_k / \sqrt{n}, \quad (\sigma^n_k)^2 = \text{Var}_{P^n_k} \left[ Y \right] = \sigma^2_k
\]

for \( k = 1, \ldots, K \), where \( \mu_k \in \mathbb{R} \) and \( \sigma^2_k \geq 0 \) remain fixed across \( n \). The main feature of this scaling is that, even as \( n \) becomes large, we cannot hope to estimate \( \mu_k \) with arbitrarily high accuracy. Rather, even after \( n \) samples are collected, there remains genuine uncertainty about the relative merits of each arm. This type of scaling may be particularly relevant in scientific settings where the sample size \( n \) used by the analyst is tailored to the scale of the effects they are expecting to encounter.

In this regime, we show that the behavior of a class of methods for sequential experimentation converges to a diffusion limit. This result applies to a wide variety of adaptive experimentation rules arising in statistics, machine learning and behavioral economics. We subsequently use this connection to derive new insights about Thompson sampling, a popular Bayesian heuristic for sequential experimentation [Thompson, 1933]. We show that, in this diffusion regime, it is essential to use what we refer to as “asymptotically undersmoothed” Thompson sampling, i.e., to use a prior whose regularizing effect becomes vanishingly small in the limit. Without asymptotic undersmoothing, the regret of Thompson sampling can become unbounded as \( |\mu_k| \) grows, a rather counter-intuitive finding given that one would expect the learning task to become easier as the signal strength increases.

\[\text{If we consider translation-invariant algorithms, only the arm differences need to scale with } n, \text{ and we could consider } \mu^n_k = \mu_0 + \delta_k / \sqrt{n}; \text{ see Section 5 for an example.}\]
1.1 Related Work

The choice of the diffusion scaling (2) and the ensuing functional limit are motivated by insights from both queueing theory and statistics. The scaling (2) plays a prominent role in heavy-traffic diffusion approximation in queueing networks [Gamarnik and Zeevi, 2006, Harrison and Reiman, 1981, Reiman, 1984]. Here, one considers a sequence of queueing systems in which the excessive service capacity, defined as the difference between arrival rate and service capacity, decays as $1/\sqrt{T}$, where $T$ is the time horizon. Under this asymptotic regime, it is shown that suitably scaled queue-length and workload processes converge to reflected Brownian motion. Like in our problem, the diffusion regime here is helpful because it captures the most challenging problem instances, where the system is at once stable and exhibiting non-trivial performance variability.

The diffusion scaling (2) is further inspired by a recurring insight from statistics that, in order for asymptotic analysis to yield a normal limit that can be used for finite-sample insight, we need to appropriately down-scale the signal strength as the sample size gets large. One concrete example of this phenomenon arises when we seek to learn optimal decision rules from (quasi-)experimental data. Here, in general, optimal behavior involves regret that decays as $1/\sqrt{n}$ with the sample size [Athey and Wager, 2021, Kitagawa and Tetenov, 2018]; however, this worst-case regret is only achieved if we let effect sizes decay as $1/\sqrt{n}$. For any fixed sampling design, it’s possible to achieve faster than $1/\sqrt{n}$ rates asymptotically [Luedtke and Chambaz, 2017].

The multi-arm bandit problem is a popular framework for studying sequential experimentation; see Bubeck and Cesa-Bianchi [2012] for a broad discussion. The $\Theta(\log(n))$ regret scaling in the fixed-$P$ asymptotic regime was established in the seminal work of Lai and Robbins [1985]. The worst-case regret of $\Theta(\sqrt{Kn})$ for a fixed problem horizon of $n$ was first shown in Auer et al. [2002]. It is worth-noting that the problem instance that achieves the $\sqrt{Kn}$ regret lower bound in Auer et al. [2002] involves the same mean reward scaling as (2), further affirming the idea that the diffusion scaling proposed here captures the most challenging sub-family of learning tasks.

Thompson sampling [Thompson, 1933] has gained considerable popularity in recent years thanks to its simplicity and impressive empirical performance [Chapelle and Li, 2011]. Regret bounds for Thompson sampling have been established in the frequentist [Agrawal and Goyal, 2017] and Bayesian [Bubeck and Liu, 2014, Lattimore and Szepesvári, 2019, Russo and Van Roy, 2016] settings; the setup here belongs to the first category. None of the existing instance-dependent regret bounds, however, appears to have sufficient precision to yield meaningful characterization in our regime. For example, the upper bound in Agrawal and Goyal [2017] contains a constant to the order of $1/\Delta^4$, where $\Delta$ is the gap in mean reward.

3At a higher level, a similar phenomenon also arises when considering optimal estimation of $\theta$ from $n$ samples drawn from a probability distribution parametrized by $\theta$. One might conjecture that popular estimators (such as the maximum likelihood estimator) should be asymptotically optimal in general, but this is unfortunately not true: one can design “superefficient” estimators as counterexamples. In response, Le Cam [1960] proposed local asymptotic normality as a framework for studying estimation in large samples. The key insight is that if we focus on a sequence of models parameterized by $\theta_n = \theta_0 + h/\sqrt{n}$, i.e., $1/\sqrt{n}$-scale perturbations of a known $\theta_0$, then the problem of estimating $h$ using $n$ samples from a distribution parametrized by $\theta_n$ becomes asymptotically equivalent to the problem of estimating $h$ from a single Gaussian random variable with mean $h$ and variance depending only on $\theta_0$. This fact can then be used to provide a clean theory of asymptotically optimal estimation, and to reveal to role of Gaussian in a wide variety of statistical problems; see van der Vaart [1998] for a modern textbook treatment. In our setting, we analogously find that running a sequential experiment is asymptotically equivalent to controlling a diffusion process once we down-scale effect sizes by $1/\sqrt{n}$, and this provides a natural generalization of Gaussian approximation theory to sequential problems.
between optimal and sub-optimal arms. This constant would have led to a trivial bound of $O(n^2)$ in our regime, where mean rewards scale as $1/\sqrt{n}$. Furthermore, most of the existing finite-time bounds seem to require delicate assumptions on the reward distributions (e.g., bounded support, exponential family). In contrast, the diffusion asymptotics adopted in this paper are universal in the sense that they automatically allow us to obtain approximations for a much wider range of reward distributions, requiring only a bounded fourth moment.

Diffusion approximations have been also been used for optimal stopping in sequential experiments [Siegmund, 1985]. In this literature, the randomization is typically fixed throughout the horizon. In contrast, in our multi-arm bandit setting the probabilities in the randomization depend on the history which creates a qualitatively different limit object. Our work is also broadly related, in spirit, to recent work on models of learning and experimentation using diffusion processes in the operations research literature [Araman and Caldentey, 2019, Harrison and Sunar, 2015, Wang and Zenios, 2020].

2 Sequentially Randomized Markov Experiments

As discussed above, the goal of this paper is to establish a diffusion limit for a class of sequential experiments under triangular array asymptotics characterized by (2). For this purpose, we focus on sequentially randomized experiments whose sampling probabilities depend on past observations only through the state variables

$$Q_{k,i} = \sum_{j=1}^{i} 1\{\{A_j = k\}\}, \quad S_{k,i} = \sum_{j=1}^{i} 1\{\{A_j = k\}\}Y_j,$$

(3)

where $Q_{k,i}$ counts the cumulative number of times arm $k$ has been chosen by the time we collect the $i$-th sample, and $S_{k,i}$ measures its cumulative reward. When useful, we use the convention $Q_{k,0} = S_{k,0} = 0$. As shown via examples below, the abstraction of sequentially randomized Markov experiments covers many popular ways of running sequential experiments.

**Definition 1.** A $K$-arm sequentially randomized Markov experiment chooses the $i$-th action $A_i$ by taking a draw from a distribution

$$A_i \mid \{A_1, Y_1, \ldots, A_{i-1}, Y_{i-1}\} \sim \text{Multinomial}(\pi_i),$$

(4)

where the sampling probabilities are computed using a measurable sampling function $\psi$,

$$\psi : [0, 1]^K \times \mathbb{R}^K \rightarrow \Delta^K, \quad \pi_i = \psi(Q_{i-1}, S_{i-1}),$$

(5)

where $(Q_{i-1}) = (Q_{k,i-1})_{k=1, \ldots, K}, S_{i-1} = (S_{k,i-1})_{k=1, \ldots, K}$, and $\Delta^K$ is the $K$-dimensional unit simplex.

**Example 1.** Thompson sampling is a popular Bayesian heuristic for running sequential experiments [Thompson, 1933]. In Thompson sampling an agent starts with a prior belief distribution $G_0$ on the reward distributions $\{P_k\}_{k=1}^K$. Then, at each step $i$, the agent draws the $k$-th arm with probability $\rho_{k,i}$ corresponding their posterior belief $G_{i-1}$ that $P_k$ has the highest mean, and any so-gathered information to update the posterior $G_i$ using Bayes’ rule. The motivation behind Thompson sampling is that it quickly converges to pulling the best arm, and thus achieves low regret [Agrawal and Goyal, 2017, Chapelle and Li, 2011].
Thompson sampling does not always satisfy Definition 1. However, widely used modeling choices involving exponential families for the $\{P_k\}_{k=1}^K$ and conjugate priors for $G_0$ result in these posterior probabilities $\rho_{k,i}$ satisfying the Markov condition (5) [Russo et al., 2018], in which case Thompson sampling yields a sequentially randomized Markov experiment in the sense of Definition 1. See Sections 4 and 5 for further discussion.

**Example 2.** Exploration sampling is a variant of Thompson sampling where, using notation from the above example, we pull each arm with probability $\pi_{k,i} = \rho_{k,i}(1-\rho_{k,i}) / \sum_{l=1}^K \rho_{l,i}(1-\rho_{l,i})$ instead of $\pi_{k,i} = \rho_{k,i}$ [Kasy and Sautmann, 2021]. Exploration sampling is preferred to Thompson sampling when the analyst is more interested in identifying the best arm than simply achieving low regret [Kasy and Sautmann, 2021, Russo, 2020]. Exploration sampling satisfies Definition 1 under the same conditions as Thompson sampling.

**Example 3.** A greedy agent may be tempted to always pull the arm with the highest apparent mean, $S_{i,k}/Q_{i,k}$; however, this strategy may fail to experiment enough and prematurely discard good arms due to early unlucky draws. A tempered greedy algorithm instead chooses

$$\pi_{i,k} = \exp \left[ \alpha \frac{S_{k,i}}{Q_{k,i} + c} \right] / \sum_{i=1}^K \exp \left[ \alpha \frac{S_{i,i}}{Q_{i,i} + c} \right],$$

where $\alpha, c > 0$ are tuning parameters that serve to govern the strength of the extent to which the agent focuses on the greedy choice and to protect against division by zero respectively. The selection choices (6) satisfy (5) and thus Definition 1 by construction.

**Example 4.** Similar learning dynamics arise in human psychology and behavioral economics where an agent chooses future actions with a bias towards those that have accrued higher (un-normalized) cumulative reward [Erev and Roth, 1998, Luce, 1959, Xu and Yun, 2020]. A basic example of these policies, known as Luce’s rule, uses sampling probabilities

$$\pi_{i,k} = (S_{k,i} \lor \alpha) / \sum_{i=1}^K (S_{i,i} \lor \alpha),$$

where $\alpha > 0$ is a tuning parameter governing the amount of baseline exploration. More generally, the agent may ascribe to arm $k$ a weight $f(S_{k,i})$, where $f$ is a non-negative potential function, and sample actions with probabilities proportional to the weights. The decision rule in (7) only depends on $S$ and thus satisfies (5).

**Example 5.** The Exp3 algorithm, proposed by Auer et al. [2002], uses sampling probabilities

$$\pi_{i,k} = \exp \left[ \alpha \sum_{j=1}^{i-1} \frac{1(\{A_j = k\}) Y_j}{\pi_{k,j}} \right] / \sum_{i=1}^K \exp \left[ \alpha \sum_{j=1}^{i-1} \frac{1(\{A_j = l\}) Y_j}{\pi_{l,j}} \right],$$

where again $\alpha > 0$ is a tuning parameter. The advantage of Exp3 is that it can be shown to achieve low regret even when the underlying distributions $\{P_k\}_{k=1}^K$ may be non-stationary and change arbitrarily across samples. The sampling probabilities (8) do not satisfy (5), and so the Exp3 algorithm is not covered by the results given in this paper; however, it is plausible that a natural extension of our approach to non-stationary problems could be made accommodate it. We leave a discussion of non-stationary problems to further work.
Diffusion scaling  We now consider a sequence of experiments indexed by \( n \). In order for sequentially randomized Markov experiments to admit a limit distribution in the triangular array setting (5), we further need the sampling functions \( \psi^n \) used in the \( n \)-th experiment to converge in an appropriate sense. As discussed further below, the natural scaling of the the \( Q_{k,i} \) and \( S_{k,i} \) state variables defined in (3) is

\[
Q_{k,i}^n = \frac{1}{n} \sum_{j=1}^{i} 1(\{ A_j = k \}), \quad S_{k,i}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^{i} 1(\{ A_j = k \}) Y_j.
\]

(9)

We then say that a sequence of sampling functions \( \psi^n \) is convergent if it respects this scaling.

**Definition 2.** Writing sampling functions in a scale-adapted way as follows,

\[
\tilde{\psi}^n(q, s) = \psi^n(nq, \sqrt{n}s), \quad q \in [0, 1]^K, s \in \mathbb{R}^K,
\]

(10)

we say that a sequence sampling functions \( \psi^n \) satisfying (5) is convergent if, for all values of \( q \in [0, 1]^K \) and \( s \in \mathbb{R}^K \), we have

\[
\lim_{n \to \infty} \tilde{\psi}^n(q, s) = \psi(q, s)
\]

(11)

for a limiting sampling function \( \psi : [0, 1]^K \times \mathbb{R}^K \to \Delta^K \).

Our main result is that, when performed on a sequence of reward distributions satisfying (2) and under a number of regularity conditions discussed further below, the sample paths of the scaled statistics \( Q_{k,i}^n \) and \( S_{k,i}^n \) of a sequentially randomized Markov experiments with convergent sampling functions converge in distribution to the solution to a stochastic differential equation

\[
\begin{align*}
& dQ_{k,t} = \psi_k(Q_t, S_t) dt, \\
& dS_{k,t} = \mu_k \psi_k(Q_t, S_t) dt + \sigma_k \sqrt{\psi_k(Q_t, S_t)} dB_{k,t},
\end{align*}
\]

(12)

where \( B_{k,t} \) is a standard \( K \)-dimensional Brownian motion, \( \mu_k \) and \( \sigma_k \) and the mean and variance parameters given in (2), and the time variable \( t \in [0, 1] \) approximates the ratio \( i/n \). A formal statement is given in Theorem 2.

We end this section by commenting briefly on conditions under which we may hope for sampling functions \( \psi^n \) to be convergent in the sense of Definition 2. The tempered greedy method from Example 3 can immediately be seen to be convergent, provided we use a sequence of tuning parameters \( \alpha_n \) and \( c_n \) satisfying

\[
\lim_{n \to \infty} \sqrt{n} \alpha_n = \alpha \quad \text{and} \quad \lim_{n \to \infty} nc_n = c
\]

for some \( \alpha, c \in \mathbb{R}_+ \), resulting in a limiting sampling function

\[
\psi(q, s) = \exp \left[ \frac{\alpha s_k}{q_k + c} \right] / \sum_{l=1}^{K} \exp \left[ \frac{\alpha s_l}{q_l + c} \right].
\]

(13)

For tempered greedy sampling to be interesting, we in general want the limit \( \alpha \) to be strictly positive, else the claimed diffusion limit (12) will be trivial. Conversely, for the second parameter, both the limits \( c > 0 \) and \( c = 0 \) may be interesting, but working in the \( c = 0 \) limit may lead to additional technical challenges due to us getting very close to dividing by 0.

Meanwhile, as discussed further in Sections 4 and 5, variants of Thompson sampling as in Examples 1 and 2 can similarly be made convergent via appropriate choices of the prior \( G_0 \);
and we will again encounter questions regarding whether a scaled parameter analogous to $c_n$ in Example 3 converges to 0 or has a strictly positive limit. Finally, similar convergence should hold for Luce’s rule in Example 4 provided that $\sqrt{n}c_n$ converges to a positive limit.

**Remark 1 (Continuity of the Sampling Function).** With some appropriate adjustments, one may also express the upper-confidence bound (UCB) and $\epsilon$-greedy algorithms in a form that is consistent with Definition 1. Unfortunately, our main results on diffusion approximation do not currently cover these two algorithms. The main reason is that the sampling functions $\psi$ for these algorithms are discontinuous with respect to the underlying state $(Q, S)$. This causes a problem because the convergence to a diffusion limit, as well as the well-posed-ness of the limit stochastic integral, requires $\psi$ to be appropriately continuous (Assumption 1). Modifying the UCB and $\epsilon$-greedy in such a manner as to ensure some smoothness in the sampling probability should resolve this issue. Whether a well defined diffusion limit exists even under a discontinuous sampling function, such as that of vanilla UCB or $\epsilon$-greedy, remains an open question.

### 3 Convergence to Diffusion Limit

We state our main results in this section: Given any Lipschitz sampling functions, we will show that a suitably scaled version of the process $(Q^n_t, S^n_t)$ converges to an Itô diffusion process. We will make the following assumptions on the sampling functions:

**Assumption 1.** Assume the following is true:

1. The limiting sampling function $\psi$ is Lipschitz-continuous.
2. The convergence of $\bar{\psi}_n$ to $\psi$ (Definition 2) occurs uniformly over compact sets.

Define $\overline{Q}^n_t$ to be the linear interpolation of $Q^n_{[tn]}$,

$$\overline{Q}^n_{k,t} = (1 - tn + [tn])Q^n_{k,[tn]} + (tn - [tn])Q^n_{k,[tn] + 1}, \quad t \in [0, 1], k = 1, \ldots, K,$$

and define the process $\overline{S}^n_t$ analogously. Let $C$ be the space of continuous functions $[0, 1] \rightarrow \mathbb{R}^2$ equipped with the uniform metric: $d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|, x, y \in C$. We have the following result.

**Theorem 2.** Fix $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$ and $\sigma \in \mathbb{R}^K_+$. Suppose that Assumption 1 holds, and $(Q^0_0, S^0_0) = 0$. Then, as $n \rightarrow \infty$, $(\overline{Q}^n_t, \overline{S}^n_t)_{t \in [0, 1]}$ converges weakly to $(Q_t, S_t)_{t \in [0, 1]} \in C$, which is the unique solution to the following stochastic differential equation over $t \in [0, 1]$:

$$dQ_{k,t} = \psi_k(Q_t, S_t)dt,$$

$$dS_{k,t} = \psi_k(Q_t, S_t)\mu_k dt + \sqrt{\psi_k(Q_t, S_t)}\sigma_k dB_{k,t},$$

where $(Q_0, S_0) = 0$, and $B_t$ is a standard Brownian motion in $\mathbb{R}^K$. Furthermore, for any bounded continuous function $f : \mathbb{R}^{2K} \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(\overline{Q}^n_t, \overline{S}^n_t)] = \mathbb{E}[f(Q_t, S_t)], \quad \forall t \in [0, 1].$$

(16)
The following theorem gives a more compact representation of the stochastic differential equations in Theorem 2, which will be useful, for instance, in our subsequent analysis of Thompson sampling.\footnote{The result below does not guarantee the Brownian motions \( W_k \) to be mutually independent. Whether this result can be strengthened to guarantee mutual independence is an interesting open question; see Section 6 for further discussion.}

**Theorem 3.** The stochastic differential equation in (20) can be equivalently written as

\[ dQ_{k,t} = \psi_k(\mu Q_t + \sigma W_Q, Q_t) \, dt, \quad k = 1, \ldots, K, \tag{17} \]

with \( Q_0 = 0 \), where \( \{W_k\}_{k=1,\ldots,K} \) is a set of one-dimensional standard Brownian motions. Here, \( Q_t \mu \) and \( \sigma W_Q \) are understood to be vectors of element-wise products: \( Q_t \mu = (Q_k \mu_k)_{k=1,\ldots,K} \), and \( \sigma W_Q = (\sigma_k W_{k,Q_k})_{k=1,\ldots,K} \). In particular, we may also represent \( S_t \) explicitly as a function of \( Q \) and \( W \): \( S_t = Q_t \mu + \sigma W_Q \).

### 3.1 Connections to Partial Differential Equations

We discuss in this section an immediate consequence of Theorem 2: We can associate the distribution of the diffusion process as a solution to a certain partial differential equation, known commonly as the Kolmogorov backward equation. To this end, let us first introduce some notation to streamline the presentation of the result. Define \( Z_t = (Q_t, S_t) \). Denote by \( I_S \) and \( I_Q \) the indices in \( Z_t \) corresponding to the coordinates of \( S \) and \( Q \), respectively. Both sets are understood to be an ordered set of \( K \) elements, where the subscript is for distinguishing whether the ordering is applied to \( S \) versus \( Q \). For \( z \in \mathbb{R}_+^{K} \times \mathbb{R}^{K} \), define the functions \((b_k)_{k \in I_Q \cup I_S} \),

\[ b_k(z) = \begin{cases} \psi_k(z), & k \in I_Q, \\ \psi_k(z) \mu_k, & k \in I_S. \end{cases} \tag{18} \]

For \( 1 \leq k, l \leq K \), define

\[ \eta_{k,l}(z) = \begin{cases} \sqrt{\psi_k(z)} \sigma_k, & \text{if } k = l \in I_S, \\ 0, & \text{otherwise}. \end{cases} \tag{19} \]

Then, the Itô diffusion SDE in (15) can be written more compactly as

\[ dZ_t = b(Z_t) \, dt + \eta(Z_t) \, dB_t, \quad t \in [0, 1], \tag{20} \]

with \( Z_0 = 0 \). Let \( \mathcal{R} = \mathbb{R}_+^{K} \times \mathbb{R}^{K} \). Define the infinitesimal generator \( L \) of the Itô diffusion \( Z_t \) as

\[ Lf(z) = \lim_{t \downarrow 0} \frac{\mathbb{E}[f(Z_t) \mid Z_0 = z] - f(z)}{t}, \quad z \in \mathcal{R}, \tag{21} \]

and let \( \mathcal{D}_L \) be the set of functions \( f \) for which the above limit exists for all \( z \in \mathcal{R} \). Denote by \( C_0^2(\mathcal{R}) \) the set of twice continuously differentiable functions on \( \mathcal{R} \) with a compact support. We have the following theorem; the proof is relatively standard and achieved by recognizing that \( Z_t \) is an Itô diffusion and applying Dynkin’s formula. See Oksendal [2013, Theorems 7.3.3, 8.1.1] and Durrett [1996, Theorem 7.3.4].
Theorem 4 (Kolmogorov Backward Equation). Let \((Z_t)_{t\in[0,1]}\) be the limiting diffusion in Theorem 2. The following holds:

1. If \(f \in C^2_0(\mathbb{R})\), then \(f \in \mathcal{D}_L\), and the generator can be expressed explicitly as:

\[
Lf(z) = \sum_k b_k(z) \frac{\partial}{\partial z_k} f(z) + \frac{1}{2} \sum_{k,l}(\eta^T)_{k,l}(z) \frac{\partial^2}{\partial z_k \partial z_l} f(z), \quad z \in \mathbb{R}. \tag{22}
\]

2. Fix \(f \in C^2(\mathbb{R}^{2K})\), and let

\[
\eta(z,t) = \mathbb{E} \left[ f(Z_t) \mid Z_0 = z \right]. \tag{23}
\]

The function \(\eta(\cdot,t)\) belongs to \(\mathcal{D}_L\) for all \(t \in [0,1]\). Furthermore, \(\eta\) satisfies the following partial differential equation:

\[
\frac{\partial}{\partial t} \eta(z,t) = (Lu(\cdot,t))(z), \quad t \in (0,1), z \in \mathbb{R},
\]

\[
\eta(\cdot,0) = f,
\]

with \(L\) defined in (21).

3. Conversely, let \(w(z,t) : \mathbb{R} \times [0,1] \to \mathbb{R}\) be a continuous bounded function that is twice continuously differentiable in the first argument. Suppose that \(w\) is a solution to (24), with \(L\) given explicitly as in (22), then we must have \(w = \eta\), i.e.,

\[
w(z,t) = \mathbb{E} \left[ f(Z_t) \mid Z_0 = z \right]. \tag{25}
\]

Remark 5. Note that the PDE in (24) is given respect to the implicitly defined \(L\) in (21). Unfortunately, when \(L\) is explicitly given as in (22), the theorem does not guarantee that the solution to (24) exists; see Section 6 for further discussion.

4 One-Arm Bandit under Thompson Sampling

As a first application of our framework, we consider the following one-arm sequential experiment. In periods \(i = 1, \ldots, n\), an agent has an option to draw from a distribution \(P^n\) with (unknown) mean \(\mu^n\) and (known) variance \((\sigma^n)^2\), or do nothing and receive zero reward. As discussed above, we are interested in the regime where \(n \to \infty\), and \(\mu^n = \mu/\sqrt{n}\) for some fixed \(\mu \in \mathbb{R}\) while \(\sigma^n = \sigma^2\) remains constant.

Following the paradigm of Thompson sampling, the agent starts with a prior belief distribution \(G^n_0\) on \(P^n\). Then, at each step \(i\), the agent draws a new sample with probability \(\pi_i^n = \mathbb{P}_{G^n_{i-1}}(\mu^n > 0)\), and uses any so-gathered information to update the posterior \(G^n_i\) using Bayes’ rule. Furthermore, we assume that the agent takes \(P^n\) to be a Gaussian distribution with (unknown) mean \(\mu^n\) and (known) variance \(\sigma^2\), and sets \(G^n_0\) to be a Gaussian prior on \(\mu^n\) with mean 0. Thus, writing \(I_i\) for the even that we draw a sample in the \(i\)-th period and \(Y_i\) for the observed outcome, we get

\[
\begin{align*}
\mu^n \mid G^n_i & \sim \mathcal{N}\left( \frac{\sigma^{-2} \sum_{j=1}^i Y_j}{\sigma^{-2} \sum_{j=1}^i I_j + (\nu^n)^{-2}}, \frac{1}{\sigma^{-2} \sum_{j=1}^i I_j + (\nu^n)^{-2}} \right), \\
\pi_i & = \Phi \left( \frac{\sigma^{-2} \sum_{j=1}^i Y_j}{\sqrt{\sigma^{-2} \sum_{j=1}^i I_j + (\nu^n)^{-2}}} \right), \tag{26}
\end{align*}
\]

9
where \((\nu^\prime)^2\) is the prior variance and \(\Phi\) is the standard Gaussian cumulative distribution function. Qualitatively, one can motivate this sampling scheme by considering an agent gambling at a slot machine: Here, \(\mu^n\) represents the expected reward from playing, and the agent’s propensity to play depends on the strength of their belief that this expected reward is positive.

An interesting question in this case is how we scale the prior variance \(\nu_n\) used in the Thompson sampling heuristic. A first choice is to choose a scaling such that

\[ \lim_{n \to \infty} \nu_n^{-2}/n = c > 0, \]  

in which case Theorem 2 immediately implies that the scaled sample paths of \(S^n\) and \(Q^n\) converge weakly to the solution to the stochastic differential equation

\[ dQ_t = \pi_t dt, \quad dS_t = \mu \pi_t dt + \sqrt{\pi_t} dB_t, \quad \pi_t = \Phi \left( \frac{S_t}{\sigma \sqrt{Q_t + \sigma^2 c}} \right). \]  

We refer to Thompson sampling with this scaling of the prior variance as “smoothed” Thompson sampling. As a direct corollary of Theorem 3, we obtain the following more compact characterization of the diffusion limit for the one-arm smoothed Thompson sampling:

**Theorem 6.** Let \(W_t\) be a one-dimensional standard Brownian motion. Define

\[ \Pi(c, q) = \Phi \left( \frac{q \mu + W_q}{\sigma \sqrt{q + \sigma^2 c}} \right). \]  

The stochastic differential equation in (28) can be equivalently written as

\[ dQ_t = \Pi(c, Q_t) dt, \quad Q_0 = 0. \]  

Alternatively, one could also consider a setting where \(\nu_n = \nu > 0\) does not scale with \(n\), or where \(\nu_n\) decays slowly enough that \(\lim_{n \to \infty} \nu_n^{-2}/n = 0\). This is the scaling of Thompson sampling that is most commonly considered in practice; for example, \(\nu_n\) is simply set to 1 in [Agrawal and Goyal, 2017]. Such “undersmoothed” Thompson sampling is not covered by Theorem 2, because the drift function \(\psi\) now violates the Lipschitz condition, and it is not clear whether the stochastic differential equation (15) admits a solution. Fortunately, the following theorem ensures that the limiting diffusion process almost surely admits a unique limit as \(c \to 0\); the proof makes use of the law of the iterated logarithm (LLL) of Brownian motion to show that the \(\Pi(0, Q_t)\) does not exhibit too wild of an oscillation near \(t = 0\).

**Theorem 7 (Diffusion Limit for Undersmoothed Thompson Sampling).** The diffusion limit \((Q_t)_{t \in [0,1]}\) under Thompson sampling converges uniformly to a limit \(\tilde{Q}\) as \(c \to 0\) almost surely. Furthermore, \(\tilde{Q}\) is a strong solution to the stochastic differential equation:

\[ d\tilde{Q}_t = \Pi(0, \tilde{Q}_t) dt, \quad \tilde{Q}_0 = 0. \]  

The next result combines Theorems 2 and 7 to show that that if \(\nu_n\) tends to infinity at an appropriate rate, then the pre-limit sample path of one-arm Thompson sampling converges to the diffusion limit (31) with \(c = 0\); the claim follows immediately by taking a triangulation limit across \(\nu_n\) and \(n\).
\[ \mu_n = 10/\sqrt{n}, \quad \nu_n = 1/\sqrt{n} \]

\[ \mu_n = -10/\sqrt{n}, \quad \nu_n = 1/\sqrt{n} \]

Figure 1: Convergence to the diffusion limit under one-arm Thompson sampling. The plots show the evolution of the scaled cumulative reward \( S_n^t \), and its limiting process \( S^t \), with \( n = 1500 \) over 1000 simulation runs. The lines and shades represent the empirical mean and one empirical standard deviation from the mean, respectively. The two columns corresponding to the smoothed (left) and undersmoothed (right) Thompson sampling, respectively.

**Corollary 8** (Diffusion Approximation of Undersmoothed Thompson Sampling). There exists a sequence \((\nu_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} \nu_n^{-2} = 0 \), such that \((\tilde{Z}^n)_{t \in [0,1]}\) converges weakly to the solution to the stochastic differential equation in (31) as \( n \to \infty \).
4.1 Regret Profiles for One-Arm Thompson Sampling

As discussed in the introduction, one standard metric for evaluating the performance of bandit algorithms is regret (1). The guarantee (16) from Theorem 2 immediately implies that the large-sample behavior of regret is captured by the limiting stochastic differential equation under our diffusion asymptotic setting, and that (appropriately scaled) regret converges in distribution:

\[ \frac{R_n}{\sqrt{n}} \Rightarrow R := (\mu)_+ - \mu Q_1. \]  

Thus, given access to the distribution of the final state \( Q_1 \) in our diffusion limit, we also get access to the distribution of regret. In Figure 2, we plot both expected regret \( \mathbb{E}[R] \) and \( \mathbb{E}[Q_1] \) as a function of \( \mu \), across for several choices of smoothing parameter \( c \) (throughout, we keep \( \sigma^2 = 1 \)). This immediately highlights several properties of Thompson sampling; some well known, and some that are harder to see without our diffusion-based approach.

First, we see that when \( \mu = 0 \) we have \( \mathbb{E}[Q_1] < 0.5 \), meaning that bandits are biased towards being pessimistic about the value of an uncertain arm. This is in line with well established results about the bias of bandit algorithms [Nie et al., 2018, Shin et al., 2019].

Second, we see that these regret profiles are strikingly asymmetric: In general, getting low regret when \( \mu > 0 \) appears easier than when \( \mu < 0 \). This again matches what one might expect: When \( \mu < 0 \), there is a tension between learning more about the data-generating distribution (which requires pulling the arm), and controlling regret (which requires not pulling the arm), resulting in a tension between exploration and exploitation. In contrast, when \( \mu > 0 \), pulling the arm is optimal both for learning and for regret, and so as soon as the bandit acquires a belief that \( \mu > 0 \) they will pull the arm more and more frequently—thus reinforcing this belief.
Third, Figure 2 highlights an intriguing relationship between the regularization parameter $c$ and regret. As predicted in Theorem 7, we see that regret in fact converges as $c \to 0$. What’s particularly interesting, and perhaps more surprising, is that setting $c = 0$ is very close to being optimal regardless of the true value of $\mu$. When $\mu < 0$, any deviations from instance-wise optimality are not visible given the resolution of the plot, whereas for some values of $\mu > 0$ the choice $c = 0$ is sub-optimal but not by much.

Furthermore, the Bayesian heuristic behind Thompson sampling appears to be mostly uninformative about which choices of $c$ will perform well in terms of regret. For example, when $\mu = -2$, one might expect a choice $c = 1/4$ to be well justified, as $c = 1/4$ arises by choosing a prior standard deviation of $\nu = 2$, in line with the effect size. But this is in fact a remarkably poor choice here: By setting $c = 0$ we achieve expected scaled regret of 0.44, but with $c = 1/4$ this number climbs by 41\% up to 0.62. In other words, while Bayesian heuristics may be helpful in qualitatively motivating Thompson sampling, our diffusion-based analysis gives a much more precise understanding of the instance-based behavior of the method.

4.2 Thompson Sampling in the Super-Diffusive Limit

Motivated by the previous observations, we now pursue a a more formal analysis of the interplay between $c$ and $\mu$ when $\mu$ is large. Specifically, we study the regret scaling of one-arm Thompson sampling in what we refer to as the super-diffusive regime: We first take the diffusion limit as $n \to \infty$ for a fixed $\mu$, and subsequently look at how the resulting limiting process behaves in the limit as $|\mu| \to \infty$. In other words, this is a regime of diffusion processes where the magnitude of the scaled mean-rewards, $\mu$, is relatively large.

A striking finding here is that we see a sharp separation in the regret performance of one-arm Thompson sampling, depending on whether it is smoothed ($c > 0$) or undersmoothed ($c = 0$). Below, we will use the following asymptotic notation: We write that $f(x) \prec g(x)$, if for any $\beta \in (0,1)$

$$\frac{f(x)}{g(x)^{\beta}} \to 0,$$

as $x$ tends to a certain limit. We have the following theorem:

**Theorem 9.** Consider the diffusion limit associated with one-arm Thompson sampling, where $\nu^2/n \downarrow c$ as $n \to \infty$. Then, the following holds almost surely:

1. If $c > 0$, then
   $$R \to \infty, \quad \text{as } \mu \to -\infty,$$
   $$R \prec 1/\mu, \quad \text{as } \mu \to +\infty. \quad (34)$$

2. If $c = 0$, then
   $$R \prec 1/|\mu|, \quad \text{as } |\mu| \to \infty. \quad (35)$$

Here, $R$ is the scaled regret defined in (32).

We discuss below two interesting implications of Theorem 9:
The value of undersmoothing  We see that undersmoothed Thompson sampling always achieves vanishing regret in the super-diffusive limit, regardless of whether \( \mu \) tends to positive or negative infinity. In contrast, the regret of smoothed Thompson sampling explodes as \( \mu \to -\infty \). This is because, as made clear in the proof, Thompson sampling with any non-trivial smoothing will lead to the algorithm being too slow in shutting down a bad arm, thus leading to inferior regret in the super-diffusive limit. The asymmetry between the two smoothing methods can also be observed numerically in Figures 1 and 2. From a practical perspective, this result suggests that using smoothed Thompson sampling should be considered risky, as this algorithm may have arbitrarily scaled regret—and furthermore may have arbitrarily high excess regret relative to undersmoothed Thompson sampling.

Minimax optimality of Thompson sampling  Theorem 9 also provides a quantitative rate of convergence for the regret: With undersmoothed Thompson sampling, regret decays faster than \( 1/|\mu|^{1-\epsilon} \) for any \( \epsilon > 0 \) as \( |\mu| \to \infty \). It is interesting to compare this against known lower bounds in the frequentist stochastic bandit literature. In this setting, it is known that across a family of \( K \)-arm bandit problems where all but one arm has the same mean reward, the minimax regret is bounded from below by [Mannor and Tsitsiklis, 2004]

\[
CK \frac{1}{\Delta} \log \left( \frac{\Delta^2 n}{K} \right),
\]

(36)

where \( n \) is the horizon and \( \Delta \) the gap between the mean reward of the optimal arm versus the rest and \( C \) is a universal constant. In our setting, with \( \Delta = |\mu_n| = |\mu|/\sqrt{n} \) and \( K = 2 \), (36) would suggest that

\[
\frac{\mathbb{E} [R_n]}{\sqrt{n}} \geq 2C \frac{1}{\sqrt{n}|\mu_n|} \log \left( \frac{n|\mu_n|^2}{2} \right) = 2C \log\left( \frac{|\mu|/2}{|\mu|} \right).
\]

(37)

Comparing this with (35) in Theorem 9 shows that the regret of undersmoothed Thompson sampling almost matches the minimax finite-sample regret lower bound (up to an arbitrarily small polynomial factor). To the best of our knowledge, this behavior of Thompson sampling in the super-diffusive regime has not been reported in the literature.

Furthermore, it is interesting to notice that most known algorithms that attain regret upper bounds on the order of (36) rely on substantially more sophisticated mechanisms, such as adaptive arm elimination and time-dependent confidence intervals [Auer and Ortner, 2010]. It is thus both surprising and encouraging that such a simple and easily implementable heuristic as Thompson sampling should achieve near-optimal minimax regret. Admittedly, Theorem 9 is restricted to the relatively simple one-arm setting. We are hopeful that similar insights can be generalized to Thompson sampling applied to general \( K \)-arm bandits; see Section 6 for further discussion.

4.3 The Evolution of Sampling Probabilities

One final insight given by the diffusion limit in Theorems 6 and 7 is that they give us a sharp characterization of the evolution of the sampling probabilities \( \pi_t = \Pi(c, \mathcal{Q}_t) \) over time. Tracking the sampling probabilities \( \pi_t \) for Thompson sampling is particularly interesting, since \( \pi_t \) corresponds to the subjective time-\( t \) belief that \( \mu > 0 \) held by the “agent” running the algorithm. In the limiting regime with \( c = 0 \), an application of the law of the iterated logarithm to the representation (29) for the sampling probabilities immediately implies the following.
Figure 3: Sample paths of the sampling probability $\pi_t$ in one-arm Thompson sampling as defined in (28), in the undersmoothed regime $c = 0$.

Corollary 10. In the setting of Theorem 8, regardless of the effect size $\mu \in \mathbb{R}$, we have

$$
\lim_{n \to \infty} \mathbb{P} \left( \sup_{1 \leq i \leq n} \pi_n \geq 1 - \eta \right) = \lim_{n \to \infty} \mathbb{P} \left( \inf_{1 \leq i \leq n} \pi_n \geq \eta \right) = 1,
$$

for any $\eta > 0$.

In other words, in the undersmoothed limit, Thompson sampling will almost always at some point be arbitrarily convinced about $\mu$ having the wrong sign; and this holds no matter how large $|\mu|$ really is. However, Thompson sampling will eventually recover, thus achieving low regret. We further illustrate sample paths in the case with $c = 0$ in Figure 3. At the very least, this finding again challenges a perspective that would take Thompson sampling literally as a principled Bayesian algorithm (since in this case we’d expect belief distributions to follow martingale updates), and instead highlights that Thompson sampling has subtle and unexpected behaviors that can only be elucidated via dedicated methods.

5 Two-Arm Thompson Sampling

We now extend the discussion from the previous section to the case of a two-armed bandit. In periods $i = 1, \ldots, n$, an agent chooses which of two distributions $P_{1n}$ or $P_{2n}$ to draw from a distribution $P$, each with (unknown) mean $\mu_{kn}$ and (known) variance $\sigma_{kn}^2$. The agent uses a variant of translation-invariant Thompson sampling, whereby they start with one forced draw from each arm, and subsequently pull arm 1 in period $i$ with probability

$$
\pi_i = \Phi \left( \frac{\alpha_{i-2} \Delta_i}{\alpha_{i-2}^2 + \nu_n^{-2}} \right), \quad \Delta_i = \left( \frac{S_{1,i}}{Q_{1,i}} - \frac{S_{2,i}}{Q_{2,i}} \right), \quad \alpha_i^2 = \frac{\sigma_{2i}^2}{Q_{1,i} Q_{2,i}},
$$

(39)
where $\nu_n$ is interpreted as the prior standard deviation for the arm difference $\delta_n = \mu_{1,n} - \mu_{2,n}$. Here, we note that $Q_{1,i} + Q_{2,i} = i$, and so from here on out we define a single variable $Q_i$ such that $Q_{1,i} = Q_i$ and $Q_{2,i} = i - Q_i$.

As usual, we focus on asymptotics along a sequence of problems with $\mu_{kn} = \mu_k / \sqrt{n}$ and $\sigma_{kn}^2 = \sigma^2$. We also need to choose a scaling for $\nu_n$. Again, one option is to use non-vanishing smoothing, $\lim_{n \to \infty} n \nu_n^{-2} = c$. In this case, Theorem 2 immediately implies that our system converges in distribution to the solution of the stochastic differential equation

$$
\begin{align*}
    dS_{1,t} &= \mu_1 \pi_t dt + \sqrt{\pi_t} dB_{1,t}, \\
    dS_{2,t} &= \mu_2 (1 - \pi_t) dt + \sqrt{1 - \pi_t} dB_{2,t}, \\
    dQ_t &= \pi_t dt, \\
    \pi_t &= \Phi \left( \frac{\sigma^{-2} Q_t(t - Q_t) (S_{1,t}/Q_t - S_{2,t}/(t - Q_t))}{\sqrt{\sigma^{-2} Q_t(t - Q_t) + t^2 c}} \right).
\end{align*}
$$

(40)

Meanwhile, in the undersmoothed case $\lim_{n \to \infty} n \nu_n^{-2} = 0$, our findings from Theorem 7 and Corollary 8 suggest convergence to a version of (39) with $c = 0$.

Figure 4 shows the limiting regret of two-arm Thompson sampling in the diffusion limit, with $\sigma^2 = 1$. At a high level, the qualitative implications closely mirror those from the one-arm as reported in Figure 2. The behavior of Thompson sampling converges as $c \to 0$, and its regret properties with $c = 0$ are in general very strong. If anything, the $c = 0$ choice is now even more desirable than before: With the one-arm case, this choice was modestly but perceptibly dominated by other choices of $c$ for some values of $\mu > 0$, but here $c = 0$ is effectively optimal across all $\delta$ to within the resolution displayed in Figure 4.

Another interesting observation from Figure 4 is that, in the undersmoothed regime,\(^5\)
the regret is maximized around $\delta = 4$. We note that, in a randomized trial with $\pi_t = 0.5$ throughout, $\delta = 4$ corresponds to an effect size that is twice the standard deviation of its difference-in-means estimator. In other words, $\delta = 4$ is an effect size that can be reliably detected using a randomized trial run on all samples, but that would be difficult to detect using just a fraction of the data. The fact that regret is maximized around $\delta = 4$ is consistent with an intuition that the hardest problems for bandit algorithms are those with effects we can detect—but just barely.

We also again find that Thompson sampling in general has unstable behavior—even when it is operating in a regime where it achieves low regret. In Figure 5, we display realizations of Thompson sampling with $\delta = \mu_1 - \mu_2 = 4$ and $c = 1/4^2$ corresponding to $\nu_n = \delta_n$ in (39). As seen in Figure 4, this is not the regret-optimal choice of $c$ (and $c = 0$ would be better), but may correspond to something an analyst hoping to achieve stable performance would use. Figure 5, however, dispels any illusions of stability. Although Thompson sampling usually identifies the first arm as the better one and then de-emphasizes pulling the bad arm, in some realizations it ends up convinced the second arm is better and spends almost all its draws on the bad arm.

Figure 5: Distribution of the final state of two-arm Thompson sampling, with $\delta = \mu_1 - \mu_2 = 4$, $\sigma^2 = 1$ and $c = 1/4^2$. Here, the error on each arm means $S_{k,t}/Q_{k,1} - \mu_k$, and the error on the arm difference means is $S_{1,t}/Q_{1,1} - S_{2,t}/Q_{2,1} - (\mu_1 - \mu_2)$.

Thompson sampling blows up in the super-diffusive regime. With small values of $c > 0$, the bump near $\delta = 4$ in Figure 4 is just a local maximum, and the curve will diverge when $\delta$ gets very large.

6Here, Thompson sampling did a reasonable job learning which arm is better: It only finished with $S_{1,t}/Q_{1,1} - S_{2,t}/Q_{2,1} < 0$ in 2.9% of simulations, whereas a randomized trial with $\pi_t = 0.5$ throughout would get the sign wrong with probability $\Phi(-\delta/2) = \Phi(-2) = 2.3%$. So even though Thompson sampling is often completely wrong on the magnitude of the effect, it’s rarely wrong on the sign here.
6 Discussion

An immediate open question is whether the driving Brownian motions, $W_k$, in the time-change theorem (Theorem 3) are mutually independent. If this can be established, it would make the analysis of general $K$-arm bandits substantially more tractable. For instance, we expect that it may allow us extend the properties of undersmoothed Thompson sampling (Theorem 9) to general $K$-arm bandits. More precisely, suppose that in a general $K$-arm bandit problem the gap between the mean rewards of the top two arms are $\Delta/\sqrt{n}$. Then, in the super-diffusive limit of $\Delta \to \infty$, one may expect that the scaled regret $R \prec 1/\Delta$ with an undersmoothed version of Thompson sampling whereas $R \to \infty$ under smoothed Thompson sampling.

A second open question is whether we can show that a solution to the PDE in (22) exists, when $L$ is explicitly given in (22). Classical theory on this type of PDE, known as the Cauchy problem, typically requires that the second-order operator $\eta\eta^\top$ be uniformly positive semi-definite, also known as the ellipticity condition [Karatzas and Shreve, 2005]. This is clearly violated in our setting, because there is no diffusion along the $Q_t$ coordinates, and therefore $\eta\eta^\top$ is zero in the lower diagonal entries. It would be interesting to see whether such limitations can be overcome by exploiting additional structures of the problem.

Finally, another interesting follow-up question is whether the approach used here can be used to build confidence intervals using data from sequential experiments, thus adding to the line of work pursued by Hadad et al. [2019], Howard et al. [2018], and others.

7 Proofs of Main Results

We will use the following elementary lemma repeatedly; the proof of is given in Appendix A.1.

**Lemma 11** (Gaussian Tail Bounds). For all $x < -\sqrt{2\pi/(9 - 2\pi)}$:

$$
\Phi(x) \leq \frac{1}{|x|} \exp(-x^2/2), \quad \Phi(x) \geq \frac{1}{3|x|} \exp(-x^2/2).
$$

(41)

This immediately implies that for all $x > \sqrt{2\pi/(9 - 2\pi)}$:

$$
\Phi(x) \leq 1 - \frac{1}{3x} \exp(-x^2/2), \quad \Phi(x) \geq 1 - \frac{1}{x} \exp(-x^2/2).
$$

(42)

7.1 Proof of Theorem 2

**Proof.** The proof is based on the martingale framework of Stroock and Varadhan [2007], which hinges on showing that an appropriately scaled generator of the discrete-time Markov process converges to the infinitesimal generator of the diffusion process. We begin with a review of the relevant results of the Stroock and Varadhan framework. Fix $d \in \mathbb{N}$. Let $(Z^n_t)_{t \in \mathbb{N}}$ be a sequence of time-homogeneous Markov chains taking values in $\mathbb{R}^d$, indexed by $n \in \mathbb{N}$. Denote by $\Pi^n$ the transition kernel of $Z^n$:

$$
\Pi^n(z, A) = \mathbb{P}(Z^n_{t+1} \in A \mid Z^n_t = z), \quad z \in \mathbb{R}^d, A \subseteq \mathbb{R}^d.
$$

(43)

Let $\tilde{Z}^n_t$ be the piece-wise linear interpolation of $Z^n_t$:

$$
\tilde{Z}^n_t = (1 - tn + \lfloor tn \rfloor)Z^n_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)Z^n_{\lfloor tn \rfloor + 1}, \quad t \in [0, 1].
$$

(44)
Define $K^n(z, A)$ to be the scaled transition kernel:

$$K^n(z, A) = n\Pi^n(z, A). \tag{45}$$

Finally, define the functions

$$a^n_{k,l}(z) = \int_{x : |x - z| \leq 1} (x_k - z_k)(x_l - z_l)K^n(z, dx),$$

$$b^n_k(z) = \int_{x : |x - z| \leq 1} (x_k - z_k)K^n(z, dx),$$

$$\Delta^n_k(z) = K^n(z, \{x : |x - z| > \epsilon\}).$$

We will use the following result. A proof of the theorem can be found in Stroock and Varadhan [2007, Chapter 11] or Durrett [1996, Chapter 8]. For conditions that ensure the uniqueness and existence of the Itô diffusion (49), see Karatzas and Shreve [2005, Chapter 5, Theorem 2.9].

**Theorem 12.** Fix $d$. Let $\{a_{k,l}\}_{1 \leq k, l \leq d}$ and $\{b_k\}_{1 \leq k \leq d}$ be bounded Lipschitz-continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. Suppose that for all $k, l \in \{1, \ldots, d\}$ and $\epsilon, R > 0$

$$\lim_{n \to \infty} \sup_{z : |z| < R} |a^n_{k,l}(z) - a_{k,l}(z)| = 0, \tag{46}$$

$$\lim_{n \to \infty} \sup_{z : |z| < R} |b^n_k(z) - b_k(z)| = 0, \tag{47}$$

$$\lim_{n \to \infty} \sup_{z : |z| < R} \Delta^n_k(z) = 0. \tag{48}$$

If $Z^n_0 \to z_0$ as $n \to \infty$, then $\{Z^n_t\}_{t \in [0,1]}$ converges weakly in $\mathcal{C}$ to the unique solution to the stochastic differential equation

$$dZ_t = b(Z_t)dt + \eta(Z_t)dB_t, \tag{49}$$

where $Z_0 = z_0$ and $\{\eta_{k,l}\}_{1 \leq k, l \leq d}$ are dispersion functions such that $a(z) = \eta(z)\eta^\top(z).$

We are now ready to prove Theorem 2. We will use the compact representation of the SDE given in (20), with $Z^n = (Z^n, S^n)$. $Z_t = (Q_t, S_t)$ and $b$ and $\eta$ defined as in (18) and (19), respectively. To prove the convergence of $\tilde{Z}^n$ to the suitable diffusion limit, we will evoke Theorem 12 (here $d \leftrightarrow 2K$). It suffices to verify the convergence of the corresponding generators in (46) through (47). To start, the following technical lemma [Durrett, 1996, Section 8.8] will simplify the task of proving convergence by removing the need of truncation in the integral; the proof is given in Appendix A.2. Define

$$m^n_p(z) = \int |x - z|^p K^n(z, dx), \quad a^n_{k,l}(z) = \int (x_k - z_k)(x_l - z_l)K^n(z, dx),$$

$$b^n_k(z) = \int (x_k - z_k)K^n(z, dx).$$

The decomposition from $a$ to $\eta$ is unique only up to rotation. However, the resulting stochastic differential equation is uniquely defined by $a$. This is because the distribution of the standard Brownian motion is invariant under rotation, and hence any valid decomposition would lead to the same stochastic differential equation.
Lemma 13. Fix $p \geq 2$ and suppose that for all $R < \infty$,
\[
\lim_{n \to \infty} \sup_{\|z\| < R} m_n^p(z) = 0. \tag{50}
\]
\[
\lim_{n \to \infty} \sup_{\|z\| < R} |\tilde{a}_{k,l}^n(z) - a_{k,l}(z)| = 0, \tag{51}
\]
\[
\lim_{n \to \infty} \sup_{\|z\| < R} |\tilde{b}_{k}^n(z) - b_k(z)| = 0, \tag{52}
\]

Then, the convergence in (46) through (48) holds.

In what follows, we will use $z = (q, s)$ to denote a specific state of the Markov chain $Z^n$. The transition kernel of the pre-limit chain $Z^n$ can be written as
\[
\Pi^n((q, s), (q + e_k/n, s + e_k ds/\sqrt{n})) = \bar{\psi}_k^n(q, s)P_k^n(ds), \quad k = 1, \ldots, K, \tag{53}
\]
and zero elsewhere, where $e_k \in \{0, 1\}^K$ is the unit vector where the $k$th entry is equal to 1 and all other entries are 0, and $\{P_k^n\}_{k=1}^K$ are the reward probability measures. Define $K^n(z, A) = n\Pi^n(z, A)$.

We next define the limiting functions $a$ and $b$. The function $b$ is defined as in (18):
\[
b_k(z) = \begin{cases} 
\psi_k(z), & k \in \mathcal{I}_Q, \\
\psi_k(z)\mu_k, & k \in \mathcal{I}_S,
\end{cases} \tag{54}
\]
and we let
\[
a_{ij}(z) = (\eta \eta^\top)_{k,l}(z) \tag{55}
\]
where $\eta$ is defined in (19). That is,
\[
a_{k,l}(z) = \begin{cases} 
\psi_k(z)\sigma_k^2, & \text{if } k = l \in \mathcal{I}_S, \\
0, & \text{otherwise}.
\end{cases} \tag{56}
\]

Fix $R > 0$. We show that the corresponding $a^n$ and $b^n$ converge to the functions $a$ and $b$ defined above, uniformly over the compact set $\{z : \|z\| \leq R\}$. In light of Lemma 13, it suffices to verify the convergence in (50) through (52) for $p = 4$. Starting with (50), we have that
\[
m_n^4(z) = \int \|z' - z\|^4 n\Pi^n(z, dz')
\]
\[
= \sum_{k=1}^K n\bar{\psi}_k^n(z) \int_{w \in \mathbb{R}} \left(\frac{1}{n^2} + \frac{w^2}{n}\right)^2 P_k^n(dw)
\]
\[
\leq \sum_{k=1}^K n\bar{\psi}_k^n(z) \left(\frac{2}{n^4} + \frac{1}{n^2} \int_{w \in \mathbb{R}} w^4 P_k^n(dw)\right)
\]
\[
= \frac{2}{n} + \frac{1}{n} \mathbb{E}_{Z \sim P_k^n}[Z^4]
\]
\[
n \to 0, \tag{57}
\]
as $n \to 0$, uniformly over all $z$, where the last step follows from the assumption that the reward distributions admit bounded fourth moments. This shows (50).
For the drift term $b$, we consider the following two cases; together, they prove (52).

**Case 1**, $k \in \mathcal{I}_Q$. For all $k \in \mathcal{I}_Q$, and $n \in \mathbb{N}$,

$$
\tilde{b}_k^n(z) = \int (q'_k - q_k) K^n(z, dz') = \frac{1}{n} (n \tilde{\psi}_k^n(z)) \xrightarrow{n \to \infty} \psi_k(z).
$$

**Case 2**, $k \in \mathcal{I}_S$. For all $k \in \mathcal{I}_S$,

$$
\tilde{b}_k^n(z) := \int (s'_k - s_k) K^n(z, dz') = \tilde{\psi}_k^n(z) n \int w \sqrt{n} P_k^n(dw) = \tilde{\psi}_k^n(z) \mu_k \xrightarrow{n \to \infty} \psi_k(z).
$$

For the variance term $a$, we consider the following three cases:

**Case 1**, $k, l \in \mathcal{I}_Q$. Note that under the multi-arm bandit model, only one arm can be chosen at each time step. This means that only one coordinate of $\mathcal{Q}^n$ can be updated at time, immediately implying that for all $n$ and $k, l \in \mathcal{I}_Q, k \neq l$,

$$
\tilde{a}_{k,l}^n(z) = \int (q'_k - q_k)(q'_l - q_l) K^n(z, dz') = 0.
$$

For the case $k = l$, we note that for all $k \in \mathcal{I}_Q$, and all sufficiently large $n$

$$
\tilde{a}_{k,k}^n(z) = \frac{1}{n^2} n \tilde{\psi}_k^n(z) \xrightarrow{n \to \infty} 0.
$$

**Case 2**: $k \in \mathcal{I}_Q, l \in \mathcal{I}_S$, or $k \in \mathcal{I}_S$ and $l \in \mathcal{I}_Q$.

$$
\tilde{a}_{k,l}^n(z) = \int (q'_k - q_k)(q'_l - q_l) K^n(z, dz') = \tilde{\psi}_k^n(z) \mathbb{E}_{Z \sim P_k^n}[Z] = \tilde{\psi}_k^n(z) \mu_k / \sqrt{n} \xrightarrow{n \to \infty} 0.
$$

**Case 3**: $k, l \in \mathcal{I}_S$. This case divides into two further sub-cases. Suppose that $k \neq l$. Similar to the logic in Case 1, because only one coordinate of $\mathcal{Q}^n$ can be updated at a given time step, we have

$$
\tilde{a}_{k,l}^n(z) = 0, \quad k \neq l.
$$
Suppose now that \( k = l \). We have
\[
\tilde{a}_{n,k,l}(z) = \int (q_{k'} - q_k)^2 K^n(z, dz')
\]
\[
= \tilde{\psi}_k(z) \int w^2(nP^n_k(\sqrt{1}dw))
\]
\[
= \tilde{\psi}_k(z)E_{Z \sim P^n_k} [Z^2]
\]
\[
\xrightarrow{n \to \infty} \psi_k(z) \sigma_k^2
\]
\[= a_{k,l}(z). \quad (64)\]

We note that due to Assumption 1, the convergence of \( \tilde{b}_n \), \( \tilde{a}_n \) and \( m_{n,p} \) to their respective limits holds uniformly over compact sets. We have thus verified the conditions in Lemma 13, further implying (46) through (47). Note that because \( \psi_k \) is bounded and Lipschitz-continuous, so are \( a \) and \( b \). This proves the convergence of \( Z^n \) to the diffusion limit in \( C \).

Finally, to prove the convergence of \( E[f(Z^n_1)] \) to \( E[f(Z_t)] \), note that the weak convergence of \( Z^n \) in \( C \) implies that the marginal distribution, \( Z^n_t \), converges weakly to \( Z_t \), as \( n \to \infty \). The result then follows immediately from the continuous mapping theorem and the bounded convergence theorem. This completes the proof of Theorem 2.

7.2 Proof of Theorem 3

Proof. The proof is based on an application of random time change in Itô diffusion. Fix \( k \in \{1, \ldots, K\} \). From (20), we have that
\[
S_{k,t} = Q_{k,t} \mu_k + \sigma_k \int_0^t \sqrt{\psi_k(Z_s)} dB_{k,t}.
\]
It thus suffices to show
\[
\int_0^t \sqrt{\psi_k(Z_t)} dB_{k,t} = \tilde{W} Q_{k,t}
\]
in distribution, where \( \tilde{W} \) is a one-dimensional standard Brownian motion. To this end, let all processes be defined on a common probability space, and denote by \( \mathcal{F}_t \) the natural filtration associated with \( B_t \). Define:
\[
c(t) = \psi_k(Z_t) = \frac{d}{dt}Q_{k,t},
\]
\[
\beta(t) = \int_0^t c(s) ds = Q_{k,t}, \quad \beta_{\infty} = \sup_{t \geq 0} \beta(t),
\]
\[
\alpha(q) = \inf\{t : \beta(t) \geq q\}, \quad q \in [0, \beta_{\infty}).
\]

Note that both \( \beta \) and \( c \) are \( \mathcal{F}_t \)-adapted, and \( \beta(t) = \int_0^t c(s) ds \). Furthermore, since \( Q_{k,t} \) is strictly increasing in \( t \) almost surely, we have that, almost surely, \( \alpha(\cdot) \) is invertible, and
\[
\alpha(\beta(t)) = t. \quad (67)
\]

Under these conditions, we have the following random time change formula from Oksendal [2013, Corollary 8.5.5]:
Lemma 14. Define \( Y_t = \int_0^t \sqrt{c(s)} dB_s \), where \( B_s \) is an \( n \)-dimensional standard Brownian motion. Then, \( Y_{\alpha(t)} \) is also an \( n \)-dimensional standard Brownian motion.

The lemma thus implies that there exists a Brownian motion \( \tilde{W} \) defined on the same probability space such that \( Y_{\alpha(t)} = \tilde{W}_t \) almost surely. Using (67), we have that

\[
\tilde{W}_{Q_{k,t}} = Y_{\alpha(\beta(t))} = Y_t = \int_0^t \sqrt{\psi_k(Z_s)} dB_{k,t}, \quad \text{a.s.}
\]

This shows (66) and completes the proof.

7.3 Proof of Theorem 7

Proof. We will use the characterization of the diffusion process in (30). Although our main focus is on the process \( Z_t \) restricted to the \([0,1]\) interval, the diffusion process itself is in fact well defined on \( t \in [0, \infty) \). A useful observation we will make here is that, for any \( c > 0 \) and \( \mu \), we have that almost surely

\[
Q_t \to \infty, \quad \text{as} \quad t \to \infty.
\]

This fact can be verified by noting that if \( Q_t \) were bounded over \([0, \infty)\), then its drift \( \Pi(c,Q_t) \) would have been bounded from below by a strictly positive constant, leading to a contradiction. Define \( \tau^c(q) \) to be the first time \( Q_t \) reaches \( q \), which is well defined by the above reasoning for all \( q > 0 \):

\[
\tau^c(q) = \inf\{t : Q_t \geq q\}.
\]

Here, we make the dependence on \( c \) explicit. Note that for all \( c > 0 \), \( Q_t \) and \( \tau^c \) are increasing and continuous and \( Q_t = \tau^{-1}(t) \). From (30), and with the change of variables \( u = s^{-1} \), we have that for \( q > 0 \):

\[
\tau^c(q) = \int_0^q \Phi\left( \frac{s\mu + W_s}{\sigma \sqrt{s + \sigma^2 c}} \right) ds
= \int_{1/q}^{\infty} u^{-2} \Phi\left( \frac{\mu + uW_{1/u}}{\sigma \sqrt{u + u^2 \sigma^2 c}} \right) du
= \int_{1/q}^{\infty} u^{-2} \Phi\left( \frac{\mu + \tilde{W}_u}{\sigma \sqrt{u + u^2 \sigma^2 c}} \right) du
= \int_{1/q}^{\infty} h^c(u) du
\]

where \( \tilde{W}_t = tW_{1/t} \), and

\[
h^c(u) = u^{-2} \Phi\left( \frac{\mu + \tilde{W}_u}{\sigma \sqrt{u + u^2 \sigma^2 c}} \right).
\]

It is well known that if \( W \) is a standard Brownian motion, then so is \( \tilde{W}_t \). By the law of iterated logarithm of Brownian motion, we have that

\[
\limsup_{t \to \infty} \frac{\sqrt{t \log \log t}}{|\tilde{W}_t|} = 1, \quad \text{a.s.}
\]
Fix a sample path of $W$ such that the above is satisfied. Then, there exist $M \in (1/q, \infty)$ and $D > 0$, such that

$$W_t \geq -D \sqrt{t \log \log t}, \quad \forall t \geq M. \quad (74)$$

We now consider two cases depending on the sign of $\mu$. First, suppose that $\mu \geq 0$. Define

$$g(u) = \begin{cases} 
    u^{-2}/\Phi \left( \frac{u - |\bar{W}_u|}{\sigma \sqrt{u}} \right), & 0 \leq u < M, \\
    u^{-2}/\Phi \left( -\frac{D}{\sigma} \sqrt{\log \log u} \right), & u \geq M. 
\end{cases} \quad (75)$$

It follows that

$$h^c(u) \leq g(u), \quad \forall c, u \geq 0. \quad (76)$$

We now show that $g$ is integrable, i.e.,

$$\int_M^\infty g(u) du < \infty \quad (77)$$

for all $M > 1$. Recall the following lower bound on the cdf of standard normal from Lemma 11: for all sufficiently large $u$

$$\Phi(x) \geq \frac{-x}{\sqrt{2\pi}} \frac{1}{1 + x^2} \exp(-x^2/2), \quad x < 0. \quad (78)$$

We thus have that, for all sufficiently large $u$,

$$g(u) = u^{-2}/\Phi \left( \frac{D}{\sigma} \sqrt{\log \log u} \right) \leq u^{-2} \sqrt{2\pi} \frac{1}{1 + D^2 \log \log u} \exp \left( \frac{D^2}{2\sigma^2} \log \log u \right) \leq 2u^{-2} \sqrt{2\pi} \frac{D}{\sigma} \sqrt{\log \log u (\log u)^{D^2/2\sigma^2}} \leq b_1 u^{-2} (\log u)^{b_2}, \quad (79)$$

where $b_1$ and $b_2$ are positive constants. Noting that

$$(\log u)^a \ll \sqrt{u} \quad (80)$$

as $n \to \infty$ for any constant $a > 0$, we have that $b_1 u^{-2} (\log u)^{b_2}$ is integrable over $(a, \infty)$ for any $a \in (0, 1)$. This proves the integrability of $g$ in (77).

Using (76), (77) and the dominated convergence theorem, we thus conclude that, for all $q > 0$,

$$\lim_{c \downarrow 0} \tau^c(q) = \tau^0(q) = \int_0^q \Phi \left( \frac{s\mu + W_s}{\sigma \sqrt{s}} \right) ds, \quad a.s. \quad (81)$$

Recall that $Q_t = (\tau^c)^{-1}(t)$, the above thus implies that, for all $t \in [0, 1]$,

$$Q_t \overset{c \downarrow 0}{\longrightarrow} \tilde{Q}_t := (\tau^0)^{-1}(t), \quad a.s. \quad (82)$$

---

8Here and henceforth the notation $f \ll g$ denotes the asymptotic relation $f(x)/g(x) \to 0$ as $x$ tends to an appropriate limit; $f \gg g$ is defined analogously, with $f(x)/g(x) \to \infty$. 

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Finally, the point-wise convergence implies uniform convergence over the compact interval \([0, 1]\) since \(Q_t\) is 1-Lipschitz. This proves our claim in the case where \(\mu \geq 0\). The case for \(\mu < 0\) follows an essentially identical set of arguments after adjusting the constants \(M\) and \(D\) in (75), recognizing that the behavior of \(\mu + \tilde{W}^u\) is largely dominated by that of \(\tilde{W}^u\) when \(u\) is large, which can be in turn bounded by the law of iterated logarithm. This completes the proof of Theorem 7.

7.4 Proof of Theorem 9

Proof. Define the limit cumulative regret \(R_t\) as:

\[
R_t = (\mu)_+ t - \mu Q_t, \quad t \in [0, 1],
\]

where \(Q_t\) is the diffusion limit associated with \(Q^*_n\). Note that \(R_1\) corresponds to the scaled cumulative regret \(R\) in (32). We first prove the statements in the almost sure sense. The proof is divided into four cases.

Case 1: \(c > 0, \mu \to -\infty\). When \(c > 0\), the drift of \(Q_t\) is given by

\[
\Pi(c, Q_t) = \Phi \left( \frac{Q_t \mu + W_{Q_t}}{\sigma \sqrt{Q_t + \sigma^2 c}} \right).
\]

For the sake of contradiction, suppose there exists a constant \(C > 0\) such that for all sufficiently large \(\mu\):

\[
\sup_{t \in [0, 1]} Q_t \leq C/|\mu|.
\]

This would imply that there exists \(B > 0\) such that for all sufficiently large \(\mu\),

\[
\frac{Q_t \mu + W_{Q_t}}{\sigma \sqrt{Q_t + \sigma^2 c}} \geq -C + B \quad \forall t \in [0, 1].
\]

That is, the drift of \(Q_t\) is positive and bounded away from zero for all \(t\), independently of \(\mu\). This implies that

\[
\liminf_{\mu \to -\infty} Q_1 > 0,
\]

leading to a contradiction with (85). We conclude that \(Q_1 \gg 1/|\mu|\) as \(\mu \to -\infty\) and hence

\[
\lim_{\mu \to -\infty} R_1 = \lim_{\mu \to -\infty} |\mu| Q_1 = \infty,
\]

as claimed.

Case 2: \(c > 0, \mu \to \infty\). Fix \(\alpha \in (1, 2)\). Define

\[
\epsilon_{\mu} = \mu^{-\alpha}.
\]

Decompose regret as:

\[
R_1 = R_{\epsilon_{\mu}} + (R_1 - R_{\epsilon_{\mu}}).
\]

Clearly, we have that

\[
R_{\epsilon_{\mu}} \leq \mu \epsilon_{\mu} = \mu^{-(\alpha - 1)}.
\]

For the second term in (90), note that almost surely there exist constants \(b_1, b_2 > 0\) such that

\[
\frac{Q_t \mu + W_{Q_t}}{\sigma \sqrt{Q_t + \sigma^2 c}} \geq \frac{Q_t \mu - b_1}{b_2}, \quad \forall t \in [0, 1].
\]
This shows that, there exists constant $c_1 > 0$ such that the drift of $Q_t$ is greater than $c_1$ for all $t \in (0, \epsilon_\mu)$. As a result, we have that for all sufficiently large $\mu$,\[ Q_t \geq c_1 \mu^\alpha, \quad \forall t \geq \epsilon_\mu. \quad (93) \]

Using Lemma 11, (92) and (93) together imply that for all large $\mu$\[ R_1 - R_{\epsilon_\mu} = \mu \int_{\epsilon_\mu}^1 (1 - \Pi(0, Q_t))dt \leq \frac{1}{c_1 \mu^{\alpha-1}} \exp(-c_2^2 \mu^{2\alpha}) \quad (94) \]

Combining together (91) and (94), we have that for all large $\mu$,
\[ R_1 \leq \mu^{-(\alpha-1)} + \frac{1}{c_1 \mu^{\alpha-1}} \exp(-c_2^2 \mu^{2\alpha}) \xrightarrow{\mu \to \infty} 0, \quad \text{a.s.} \quad (95) \]

**Case 3: $c = 0, \mu \to -\infty$.** We follow a similar set of steps as in Case 2. The main difference here is that $\mu$ is now negative, and as such the arguments here need to be adjusted accordingly. Fix $\alpha \in (1,2)$. Define the stopping time\[ S_{\mu} = \inf\{ t : Q_t > |\mu|^{-\alpha} \}, \quad (96) \]
with $S_{\mu} = 1$ if $Q_1 \leq |\mu|^{-\alpha}$. Decompose $R_1$ with respect to $S_{\mu}$ as:\[ R_1 = R_{S_{\mu}} + (R_1 - R_{S_{\mu}}). \quad (97) \]

We next bound the two terms on the right-hand side of the equation above separately. Since $Q_t$ is non-decreasing, we have\[ R_{S_{\mu}} \leq |\mu| \cdot Q_{S_{\mu}} = |\mu|^{-(\alpha+1)}. \quad (98) \]

For the second term, the intuition is that by the time $Q_t$ reaches $|\mu|^{-3/2}$, the drift in $Q_t$ will have already become overwhelmingly small for the rest of the time horizon. To make this rigorous, note the following facts:

1. By LIL of Brownian motion, almost surely, there exists constant $C$ such that\[ \limsup_{q \downarrow 0} \sup_{x \in [q, 1]} \frac{|W_x|}{\sqrt{x}} \leq C \sqrt{\log \log(1/q)}. \quad (99) \]

2. $\mu \sqrt{Q_{S_{\mu}}} = |\mu|^{-1-\alpha/2}$, and therefore $\mu \sqrt{Q_{S_{\mu}}} \gg \sqrt{\log \log(1/Q_{S_{\mu}})}$ a.s. as $\mu \to -\infty$.

Combining these facts along with the normal cdf tail bounds from Lemma 11 we have that almost surely, there exists constant $b > 0$, such that for all sufficiently small $\mu$,
\[ R_1 - R_{S_{\mu}} = |\mu| \int_{S_{\mu}}^1 \Pi(0, Q_t)dt \leq |\mu| \left( \sup_{t \in [S_{\mu}, 1]} \frac{\mu \sqrt{Q_t}}{\sigma} + \frac{W_{Q_t}}{\sigma \sqrt{Q_t}} \right) \leq |\mu| |\mu|^b. \quad (100) \]
Putting together (98) and (100) shows that
\[ R_1 \leq \mu^{-(\alpha-1)} + |\mu| \exp(-|\mu|b) \mu^{-\infty} 0, \quad a.s. \] (101)

**Case 4:** \( c = 0, \mu \to \infty \). In this case, we would like to argue that \( Q_t \) will increase rapidly as \( \mu \) grows, and to do so, it is important to be able to characterize the drift of \( Q_t \) near \( t = 0 \). To this end, it will be more convenient for us to work with the following re-parameterization of the diffusion process, which we have already encountered in the proof of Theorem 7, (71). Let \( \eta \) be a function such that
\[ \eta(x) \prec 1/x^2, \quad \text{as} \quad x \to \infty. \] (102)
Define
\[ \tau_{\mu} = \inf\{t : Q_t \geq 1 - \eta(\mu)\}. \] (103)
It follows from the definition that, if we can show that almost surely
\[ \tau_{\mu} < 1, \] (104)
for all sufficiently large \( \mu \), then it follows that for all large \( \mu \)
\[ R_1 = \mu(1 - Q_1) \leq \mu \eta(\mu), \quad a.s. \] (105)
Using a change of variable of \( u = s^{-1} \), identical to that in (71), we have that
\[ \tau_{\mu} = \int_{(1-\eta(\mu))^{-1}}^{\infty} u^{-2} \left( \Phi \left( \frac{\mu}{\sigma \sqrt{u}} + \tilde{W}_u \right) \right)^{-1} du \]
\[ = \int_{1/(1-\mu^{-2})}^{\infty} u^{-2} \xi(\mu, u) du \] (106)
where \( \tilde{W}_t = tW_{1/t} \) is a standard Brownian motion, and
\[ \xi(\mu, u) \triangleq \left( \Phi \left( \frac{\mu}{\sigma \sqrt{u}} + \tilde{W}_u \right) \right)^{-1}. \] (107)

We now bound the above integral using a truncation argument. For \( K > (1 - \eta(\mu))^{-1} \), we write
\[ \tau_{\mu} = \int_{(1-\eta(\mu))^{-1}}^{\infty} u^{-2} \xi(\mu, u) du \]
\[ = \int_{(1-\eta(\mu))^{-1}}^{K} u^{-2} \xi(\mu, u) du + \int_{K}^{\infty} u^{-2} \xi(\mu, u) du \]
\[ \leq \left( \sup_{u \in [(1-\eta(\mu))^{-1}, K]} \xi(\mu, u) \right) \int_{(1-\eta(\mu))^{-1}}^{\infty} u^{-2} du + \int_{K}^{\infty} u^{-2} \xi(\mu, u) du \]
\[ = \left( \sup_{u \in [(1-\eta(\mu))^{-1}, K]} \xi(\mu, u) \right) (1 - \eta(\mu)) + \int_{K}^{\infty} u^{-2} \xi(\mu, u) du. \] (108)

The following lemma bounds the second term in the above equation; the proof is given in Appendix A.3.
Lemma 15. For any $\delta \in (0, 1)$, there exists $C > 0$ such that for all large $\mu$ and $K$:

$$
\int_0^\infty u^{-2}\xi(\mu, u)\,du \leq CK^{-(1-\delta)}, \ a.s. \tag{109}
$$

Bounding the first term is more delicate, and will involve taking $\mu$ to infinity in a manner that depends on $K$. Fix $\gamma \in (0, 1)$ and consider the sequence of $\mu$:

$$
\mu_n = n^{\frac{1}{2} + \gamma}, \ n \in \mathbb{N}. \tag{110}
$$

By LIL of Brownian motion, we have that there exists $C > 0$ such that for all sufficiently large $K$

$$
\inf_{u \in [(1-\eta(\mu))^{-1}, K]} \frac{\tilde{W}_u}{\sqrt{u}} \geq -C\sqrt{\log \log K}, \ a.s. \tag{111}
$$

Combining this with the lower bound on the normal cdf (Lemma 11), we have

$$
\sup_{u \in [(1-\eta(\mu))^{-1}, K]} \xi(\mu, u) \leq 1 + \exp\left(-\frac{-(\mu_K/\sqrt{K} - C\sqrt{\log \log K})^2}{\mu_K/\sqrt{K} - C\sqrt{\log \log K}}\right) \leq 1 + \exp(-K\gamma) \tag{112}
$$

for all large $K$.

Fix $\nu \in (0, 1)$, and $\delta, \gamma \in (0, 1/4)$ such that

$$
2 > \frac{1 - \delta}{1/2 + \gamma} > 2 - \nu. \tag{113}
$$

Note that such $\delta$ and $\gamma$ exist for any $\nu$, so long as we ensure that both $\delta$ and $\gamma$ are sufficiently close to 0. Combining (108), (112) and Lemma 15, we have that there exist $c_1, c_2 > 0$ such that for all large $K$:

$$
\tau_{\mu_K} \leq 1 - \eta(K^{\frac{1}{2} + \gamma}) + \exp(-K\gamma) + c_1 K^{-(1-\delta)}
\leq 1 - \eta(K^{\frac{1}{2} + \gamma}) + c_2 K^{-(1-\delta)}, \ a.s. \tag{114}
$$

Fix $\alpha$ such that

$$
2 - \nu < \alpha < \frac{1 - \delta}{(1/2 + \gamma)} < 2. \tag{115}
$$

This choice of $\alpha$ exists because of (113). We now set $\eta$ to be

$$
\eta(x) = x^{-\alpha}. \tag{116}
$$

We have that for all sufficiently large $K$

$$
\tau_{\mu_K} \leq 1 - K^{-\alpha(1/2 + \gamma)} + c_2 K^{-(1-\delta)} < 1, \ a.s., \tag{117}
$$

where the last inequality follows from (115). Combining the above equation, (104), (105) and the fact that $\nu$ can be arbitrarily close to 0, we have thus shown that for all $\alpha \in (1, 2)$,

$$
R_1 \leq \mu\eta(\mu) = \mu^{-(\alpha-1)}, \ a.s., \tag{118}
$$

for all large $\mu$. This proves our main claim in this case, that is, almost surely

$$
R_1 \asymp 1/\mu, \quad \text{as } \mu \to \infty. \tag{119}
$$

\hfill \Box
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A Additional Proofs

A.1 Proof of Lemma 11

*Proof.* For the lower bound, we have that for all $x < 0$:

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-s^2/2)ds \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \frac{-x}{s} \exp(-s^2/2)ds < \frac{1}{|x|} \exp(x^2/2),
$$

where (a) follows from the fact that $-x/s \leq 1$ for all $s \geq -x$. For the upper bound, define $f(x) = \frac{1}{\sqrt{2\pi(x^2+1)}} \exp(-x^2/2) - \Phi(-x)$. We have that $f(0) = -\Phi(0) < 0$, $\lim_{x \to -\infty} = 0$, and

$$
f'(x) = \frac{1}{\sqrt{2\pi(1+x^2)^2}} \exp(-x^2/2) > 0, \quad \forall x > 0.
$$

This implies that $f(x) < 0$ for all $x > 0$, which further implies that

$$
\Phi(-x) \geq \frac{|x|}{\sqrt{2\pi(x^2+1)}} \exp(-x^2/2), \quad \forall x < 0.
$$

The claim follows by noting that

$$
\frac{|x|}{\sqrt{2\pi(x^2+1)}} \geq \frac{1}{3|x|} \quad \text{whenever} \quad |x| \geq \sqrt{2\pi/(9 - 2\pi)}.
$$

A.2 Proof of Lemma 13

*Proof.* The proof is based on Durrett [1996], and we include it here for completeness. For $\Delta^n_\in$, note that for all $\epsilon > 0$,

$$
\Delta^n_\in(z) \leq \frac{1}{\epsilon^p} m^p_\in(z).
$$

The convergence of (50) thus implies that of (48). For $b^n_\epsilon$, note that

$$
|\hat{b}^n_\epsilon(z) - b^n_\epsilon(z)| = \int_{x:|x-z| > 1} |x-z| K^n(z, dx) \leq m^n_\in(z),
$$

which completes the proof.

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where the last step follows from the assumption $p \geq 2$. We have thus proven that (50) and (52) together imply (47). Finally, for $a_{k,l}^n$, we have

$$|\bar{a}_{k,l}(z) - a_{k,l}^n(z)| \leq \int_{x:|x-z|>1} |(x_k - z_k)(x_l - z_l)| K^n(z, dx)$$

$$\leq \int_{x:|x-z|>1} |x - z|^2 K^n(z, dx) \leq m^n_p(z), \quad (124)$$

whenever $p \geq 2$, where the first step follows from the Cauchy-Schwarz inequality, and the second step from the observation that $(x_k - z_k)^2 \leq |x - z|^2$ for all $k$. This shows (50) and (51) together imply (46), completing our proof.

A.3 Proof of Lemma 15

We will follow a similar set of arguments as those in the proof of Theorem 7, (79), exploiting the LIL of Brownian motion along with tail bounds on the normal cdf. By LIL, we have that there exists constant $C > 0$ such that for all large $u$

$$\tilde{W}_u / \sqrt{u} \leq -C \sqrt{\log \log u}, \ a.s. \quad (125)$$

Using the lower bound on the normal cdf:

$$\Phi(x) \geq \frac{1}{\sqrt{2\pi}} \frac{-x}{1 + x^2} \exp(-x^2/2), \quad x < 0,$$

We have that for all large $u$, almost surely:

$$\xi(\mu, u) = 1/\Phi \left( \frac{\mu}{\sigma \sqrt{u}} + \frac{\tilde{W}_u}{\sigma \sqrt{u}} \right) \leq 1/\Phi \left( \frac{\tilde{W}_u}{\sigma \sqrt{u}} \right) \leq b_1 (\log u)^{b_2}, \quad (126)$$

where $b_1$ and $b_2$ are positive constants that do not depend on $u$, which further implies that for any $\alpha \in (0, 1)$

$$\int_{K}^\infty u^{-2} \xi(\mu, u) \leq 1/K^{1-\alpha}, \quad (127)$$

for all large $K$. 32