A MULTIPARAMETRIC QUANTUM SUPERSPACE AND ITS LOGARITHMIC EXTENSION

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ABSTRACT. We introduce a multiparametric quantum superspace with \( m \) even generators and \( n \) odd generators whose commutation relations are in the sense of Manin such that the corresponding algebra has a Hopf superalgebra. By using its Hopf superalgebra structure, we give a bicovariant differential calculus and some related structures such as Maurer-Cartan forms and the corresponding vector fields. It is also shown that there exists a quantum supergroup related with these vector fields. Moreover, we introduce the logarithmic extension of this quantum superspace in the sense that we extend this space by the series expansion of the logarithm of the grouplike generator, and we define new elements with nonhomogeneous commutation relations. It is clearly seen that this logarithmic extension is a generalization of the \( \kappa \)-Minkowski superspace. We give the bicovariant differential calculus and the related algebraic structures on this extension. All noncommutative results are found to reduce to those of the standard superalgebra when the deformation parameters of the quantum \((m+n)\)-superspace are set to one.

1. INTRODUCTION

Noncommutative geometry has gained more attention of researchers as research domain in the fields of mathematics and mathematical physics since noncommutative differential geometry was broadly introduced by Connes [1] in 1986. In particular, quantum groups (Refs. [11, 13, 14, 25, 31, 32, 33]) and quantum spaces (Refs. [22, 24, 32, 33]) are explicit realizations of noncommutative spaces and play a fundamental role in the theory of the integrable models, conformal field theory [7, 10] and the classification of knots and links [8, 20]. The quantum (super)spaces have been envisioned by many as a paradigm for the general programme of quantum deformed physics [27]. Thus, based on the fact that the study of differential calculus is a main mathematical tool in the quantum deformed physics, many efforts have been accomplished in order to develop noncommutative differential structures on quantum superspaces(groups) (Refs. [2, 3, 4, 5, 6, 12, 17, 18, 19, 21, 23, 26, 28, 29, 30, 31, 35, 36]). In particular, as a fundamental work, the study of differential calculus on noncommutative
space of quantum groups was initiated by Woronowicz \cite{26}. In Woronowicz’s approach, the differential calculus on quantum groups is inferred by Hopf algebra structure of quantum groups and this calculus is extended to graded differential Hopf algebras. Later, Wess and Zumino introduced the differential calculus on the quantum (hyper-)plane which is covariant with respect to the quantum group \cite{9}.

In this paper, we give a multiparametric quantum \((m + n)\)-superspace on which we define a Hopf superalgebra, and its bicovariant differential calculus is given by using its Hopf superalgebra structure. Then we define new generators with nonhomegeneous commutation relations via the series expansion of logarithm of the grouplike generator. First we recall some definitions and statements which shall be used throughout the paper.

An associative algebra is a vector space \(\mathcal{A}\) over a field \(K\) together with a bilinear mapping, namely, the multiplication \(\mu: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\) satisfying
\[
\mu \circ (\text{id} \otimes \mu) = \mu \circ (\mu \otimes \text{id})
\]
for all \(a, b, c \in \mathcal{A}\). Moreover, if there exists a mapping \(\eta: K \to \mathcal{A}\) such that
\[
\mu \circ (\eta \otimes \text{id}) = \text{id} = \mu \circ (\text{id} \otimes \eta),
\]
where \(\text{id}\) stands for the identity mapping, then \(\mathcal{A}\) is a unital algebra. A coalgebra is a \(K\)-vector space \(\mathcal{A}\), together with two linear mappings, \(\Delta_A: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\) and \(\varepsilon_A: \mathcal{A} \to K\) (the coproduct and the counit, respectively) which satisfy
\[
(\Delta_A \otimes \text{id}) \circ \Delta_A = (\text{id} \otimes \Delta_A) \circ \Delta_A
\]
\[
(\varepsilon_A \otimes \text{id}) \circ \Delta_A = \text{id} = (\text{id} \otimes \varepsilon_A) \circ \Delta_A.
\]
A bialgebra is both a unital associative algebra and a coalgebra, with the compatibility conditions that \(\Delta_A\) and \(\varepsilon_A\) are both algebra homomorphisms with \(\Delta(1_A) = 1_A \otimes 1_A\) and \(\varepsilon_A(1_A) = 1_K\). A Hopf algebra is a bialgebra together with a linear mapping \(S_A: \mathcal{A} \to \mathcal{A}\), the antipode, which satisfies
\[
\mu \circ (S_A \otimes \text{id}) \circ \Delta_A = \eta \circ \varepsilon_A = \mu \circ (\text{id} \otimes S_A) \circ \Delta_A.
\]

Let \(\Omega\) be a bimodule over any Hopf algebra \(\mathcal{A}\) and \(\Delta_R: \Omega \to \Omega \otimes \mathcal{A}\) be a linear homomorphism. One says that \((\Omega, \Delta_R)\) is a right-covariant bimodule if
\[
\Delta_R(ab + ap'a') = \Delta_A(a)\Delta_R(p) + \Delta_R(p')\Delta_A(a')
\]
for all \(a, a' \in \mathcal{A}\) and \(p, p' \in \Omega\) and
\[
(\Delta_R \otimes \text{id}) \circ \Delta_R = (\text{id} \otimes \Delta_A) \circ \Delta_R,
\]
\[
\mu \circ (\text{id} \otimes \varepsilon_A) \circ \Delta_R = \text{id}.
\]
Let \(\Delta_L: \Omega \to \mathcal{A} \otimes \Omega\) be a linear homomorphism. One says that \((\Omega, \Delta_L)\) is a left-covariant bimodule if
\[
\Delta_L(ab + p'a') = \Delta_A(a)\Delta_L(p) + \Delta_L(p')\Delta_A(a')
\]
for all $a, a' \in \mathcal{A}$ and $p, p' \in \Omega$ and

$$
(id \otimes \Delta_L) \circ \Delta_L = (\Delta_L \otimes id) \circ \Delta_L,
$$

$$
\mu \circ (\varepsilon_L \otimes id) \circ \Delta_L = id.
$$

A bicovariant bimodule over $\mathcal{A}$ is a bimodule $\Omega$ with linear mappings $\Delta_R, \Delta_L$ such that $(\Omega, \Delta_L)$ is the left covariant bimodule, $(\Omega, \Delta_R)$ is the right covariant bimodule, and

$$
(id \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes id) \circ \Delta_R.
$$

Let $\mathcal{A}$ be endowed with a linear mapping $d : \mathcal{A} \to \Omega$ satisfying the Leibniz rule $d(ab) = d(a)b + ad(b)$, and $\Omega$ is the linear span of elements $\lambda b$ with $a, b \in \mathcal{A}$. Then the tuple $(\Omega, d)$ is called the first order differential calculus over $\mathcal{A}$. The algebra of higher order differential forms (or differential graded algebra) is a $\mathbb{N}_0$-graded algebra $\Omega^\Delta = \bigoplus_{n \geq 0} \Omega^n$, for all $i, j \in \mathbb{Z}$. Consider a unital associative superalgebra generated by even generators $a_i, i = 1, 2, ..., m$ and odd ones $a_i, i = m + 1, ..., m + n$ satisfying the commutation relations as follows:

$$
a_i a_j = (-1)^{ij} p_i^z p_j^{-z} a_j a_i,
$$

for $i, j = 1, 2, ..., m + n$. Note that $\hat{i} \in \mathbb{Z}_2$ stands for the parity of the generator $a_i$, and it follows from (10) that $a_i^2 = 0$ for $\hat{i} = 1$. Throughout this paper, we denote by $\mathcal{A}$ this quantum superalgebra. At this position, we remark that $\mathcal{A}$ is a generalization of the quantum superalgebra considered in Ref. [15]. It is clear that this superalgebra has a Hopf superalgebra structure with the following mappings:

$$
\Delta(a_i) = a_i^\hat{i} \otimes a_i + a_i \otimes a_i^\hat{i}, \quad \Delta(a_1) = a_1 \otimes a_1
$$

$$
\varepsilon(a_i) = 0, \quad \varepsilon(a_1) = 1
$$

$$
S(a_i) = -a_i^{-z} a_i a_i^{-z}, \quad S(a_1) = a_1^{-1},
$$

where $i = 2, 3, ..., m + n$. Note that the tensor product in $\mathcal{A} \otimes \mathcal{A}$ is given as follows:

$$
(f \otimes u)(v \otimes g) = (-1)^{\hat{a}\hat{g}} f v \otimes u g, \quad f, g, u, v \in \mathcal{A}.
$$

Remark 1.1. By a brief survey on more general algebras such as $\Gamma$-graded and color(Lie) algebras [37] and Lie $\tau$-algebras [38, 39, 40], one can see that the commutation relation given  by (10) can be represented by a bicharacter $\alpha : \Gamma \times \Gamma \to k^*$ where $\Gamma$ is an abelian group and $\alpha$ holds the conditions $\alpha(f, g, h) = \alpha(g, h, f) = \alpha(h, f, g)$.
\[ \alpha(f, h)\alpha(g, h) = \alpha(f, g)\alpha(f, h). \]
Furthermore, one can define the multiplication in the tensor product \( \mathcal{A} \otimes \mathcal{A} \) by
\[
(a_i \otimes a_j)(a_k \otimes a_l) = \alpha(g_j, g_k)a_ia_k \otimes a_ja_l
\]
where \( \alpha(g_i, g_j) = (-1)^{ij} p_j^{z_j}p_i^{-z_i} \) and \( g_i \)'s are the elements in \( \Gamma \) which as degree correspond to the elements \( a_i \)'s. Thus this basic multiplication can be extended to the all homogeneous elements (ordered monomials) by using the properties of \( \alpha \). However, when we take into account the coproduct defined in (11) together with the multiplication (13), we easily see that the coproduct does not preserve the commutation relation (11). This discrepancy is caused by the existence of \( a_1^{z_1} \) in the coproduct (11) and the fact that the generator \( a_1 \) is a grouplike element. Thus we can overcome from this inconsistency by setting all generators \( a_i \)'s as the primitive elements, that is, \( \Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i \).

The differential algebra \( \Omega^\wedge \) of all differential forms over \( \mathcal{A} \) can be given by the following relations of the generators \( a_i \)'s with their differentials \( d(a_i) \)'s:
\[
a_i d(a_j) = (-1)^{\hat{i}j+1}p_j^{z_j}p_i^{-z_i}d(a_j)a_i,
\]
and the relations among differentials
\[
d(a_i) \wedge d(a_j) = (-1)^{\hat{i}(\hat{j}+1)+1}p_j^{z_j}p_i^{-z_i}d(a_j) \wedge d(a_i),
\]
where \( d : \Omega^\wedge \to \Omega^\wedge \) is the exterior differential operator satisfying
\[
d^2 = 0,
\]
and the graded Leibniz rule
\[
d(u \wedge v) = (du) \wedge v + (-1)^{\hat{u}v}u \wedge (dv).
\]
Note that the parity of \( d(w) \) is given as \( \hat{w} + 1 \) for \( w \in \Omega^n \ (n = 0, 1, 2, ...) \), that is, \( d \) is of degree one. Moreover, one can give the right covariant bimodule structure on the space of 1-forms via \( \Delta_R : \Omega^1 \to \Omega^1 \otimes \mathcal{A} \), defined by \( \Delta_R(d(f)) = ((d \otimes \text{id}) \circ \Delta)(f) \), and the left covariant bimodule one by \( \Delta_L : \Omega^1 \to \Omega^1 \otimes \mathcal{A} \), defined by \( \Delta_L(d(f)) = ((\text{id} \otimes d) \circ \Delta)(f) \) for \( f \in \mathcal{A} \), where we use the graded tensor product of mappings. This bimodule structure is also extended to the space of all higher-order forms \( \Omega^\wedge \). Therefore, if we set \( d(f) = \sum_{k=1}^{m+n} d(a_k)\partial_{a_k}(f) \), from the differential calculus above, it follows that the Weyl superalgebra corresponding to \( \mathcal{A} \) is given by the relations (11), the following relations of the derivative operators with the generators with \( a_i \)'s:
\[
\partial_{a_i}a_j = \delta_{ij} + (-1)^{\hat{i}j}p_j^{z_j}p_i^{-z_i}a_j\partial_{a_i}, \quad i, j = 1, 2, ..., m + n
\]
and the relations among the derivative operators
\[
\partial_{a_i}\partial_{a_j} = (-1)^{\hat{i}j}p_j^{z_j}p_i^{-z_i}\partial_{a_j}\partial_{a_i},
\]
where \( \partial_{a_i} : A \rightarrow A \) is a linear operator acting on a monomial ordered of the form 
\[ f = a_1^{k_1} a_2^{k_2} \ldots a_{m+n}^{k_{m+n}} \]
as follows:

\[
(20) \quad \partial_{a_i}(a_1^{k_1} a_2^{k_2} \ldots a_{m+n}^{k_{m+n}}) = (-1)^{i+j} a_i^{-1} k_i \sum_{r=0}^{i-1} s r k_r \prod_{r=1}^{i-1} p_r^{-k_r} a_1^{k_1} a_2^{k_2} \ldots a_i^{k_i-1} \ldots a_{m+n}^{k_{m+n}}
\]

where \( f_i = a_1^{k_1} a_2^{k_2} \ldots a_{i-1}^{k_{i-1}} \).

**Remark 1.2.** It is clearly seen from (19) that the partial derivatives yield a representation of the superalgebra \( A \).

Now we shall construct a quantum supergroup of the vector fields corresponding to Maurer-Cartan forms on \( A \). First, let us start by the right-invariant Maurer-Cartan formula for any \( f \in A \) [20]:

\[ w_f := \mu((d \otimes S_A) \Delta(f)), \]

where \( \mu \) stands for the multiplication. Thus we have

\[
(22) \quad \omega_{a_1} = da_1 a_1^{-1}
\]
\[
\omega_{a_i} = da_i a_i^{-z_i} - z_i da_1 a_1^{-1} a_i a_i^{-z_i}, \quad i = 2, 3, \ldots, m+n.
\]

We also need the commutation relations of the Maurer-Cartan forms \( \omega_{a_i} \)'s with the generators \( x_i \)'s:

\[
(23) \quad a_i \omega_{a_j} = (-1)^{i+j+1} p_j^{z_i} \omega_{a_j}, \quad i, j = 1, 2, \ldots, m+n.
\]

Thus, taking into account the fact

\[ d := \omega_{a_1} T_{a_1} + \omega_{a_2} T_{a_2} + \ldots + \omega_{a_{m+n}} T_{a_{m+n}} = da_1 \partial_{a_1} + da_2 \partial_{a_2} + \ldots + da_{m+n} \partial_{a_{m+n}} \]

where \( T_{a_i} \)'s are the vector fields corresponding to the Maurer-Cartan forms, we can write the vector fields in terms of the partial derivatives \( \partial_{a_i} \)'s:

\[ T_{a_1} = \sum_{k=1}^{m+n} z_k a_i \partial_{a_1}, \quad T_{a_i} = a_i \partial_{a_i}, \quad i = 2, 3, \ldots, m+n. \]

To give the algebra of the vector fields, we compute the following super commutative algebra relations by using (11), (12) and (13) as follows:

\[ T_{a_i} T_{a_j} = (-1)^{i+j} T_{a_j} T_{a_i}, \quad i, j = 1, 2, \ldots, m+n. \]

To construct Hopf superalgebra structure, it is sufficient to give the Leibniz rules related with the vector fields. For this, consider any monomial \( f = a_1^{k_1} a_2^{k_2} \ldots a_{m+n}^{k_{m+n}} \) and any element \( g \) of \( A \). In what follows we shall use the relation of \( f \) with the Maurer-Cartan form \( \omega_{a_i} \):

\[
(24) \quad f \omega_{a_1} = (-1)^{i} \omega_{a_1} f,
\]
\[
f \omega_{a_i} = (-1)^{i} f \omega_{a_i} p_j^{z_i} \omega_{a_j}, \quad i = 2, 3, \ldots, m+n.
\]
Now we consider the action of $d$ on $f \cdot g$:
\[
d(f \cdot g) = (\omega_{a_1}T_{a_1} + \omega_{a_2}T_{a_2} + \ldots + \omega_{a_{m+n}}T_{a_{m+n}})(f)g + (-1)^{f}(\omega_{a_1}T_{a_1} + \omega_{a_2}T_{a_2} + \ldots + \omega_{a_{m+n}}T_{a_{m+n}})(g).
\]
To collect with respect to $\omega_i$’s, we use the relations given by (24):
\[
(\omega_{a_1}T_{a_1} + \omega_{a_2}T_{a_2} + \ldots + \omega_{a_{m+n}}T_{a_{m+n}})(f \cdot g) = \omega_{a_1}(T_{a_1}(f) \cdot g + f \cdot T_{a_1}(g)) + \omega_{a_i} \left( T_{a_i}(f) \cdot g + (-1)^{f}p_1^{i-1}z_i^k f \cdot T_{a_i}(g) \right).
\]
This last equation yields the following Leibniz rules:
\[
T_{a_1}(f \cdot g) = T_{a_1}(f) \cdot g + f \cdot T_{a_1}(g)
\]
\[
T_{a_i}(f \cdot g) = T_{a_i}(f) \cdot g + (-1)^{f}p_1^{i-1}z_i^k f \cdot T_{a_i}(g), \quad i = 2, 3, \ldots, m + n.
\]
Thus using the above relation with the fact that the graded tensor product is $(X \otimes Y)(f \otimes g) = (-1)^f X(f) \otimes Y(g)$ and $\mu(\Delta(X)(f \otimes g)) := X(f \cdot g)$ for any two elements $X$ and $Y$ with degrees $\hat{X}$ and $\hat{Y}$, respectively, we have the following deformed coproducts for the quantum Lie superalgebra:
\[
\Delta(T_{a_1}) = T_{a_1} \otimes 1 + 1 \otimes T_{a_1}
\]
\[
\Delta(T_{a_i}) = T_{a_i} \otimes 1 + p_1^{a_{a_1}} \otimes T_{a_i}, \quad i = 2, 3, \ldots, m + n.
\]
Note that we also use $T_{a_1}(f) = (\sum_{i=1}^{m+n} z_i^k) f$ obtained from (20). From the Hopf algebra axioms, we also obtain the counit and the antipode as follows:
\[
\varepsilon(T_{a_i}) = 0, \quad i = 1, 2, \ldots, m + n
\]
\[
S(T_{a_1}) = -T_{a_1}, \quad S(T_{a_i}) = -p_1^{a_{a_1}} T_{a_i}, \quad i = 2, 3, \ldots, m + n.
\]

2. Nonhomegeneous commutation relations derived from $A$

Let us generalize $A$ to a new algebra obtained by considering formal series in the grouplike element $a_1$ such that
\[
(\sum c_k a_1^k) a_j = a_j \sum c_k(p_j a_1)^k, \quad c_k \in \mathbb{C},
\]
and the multiplication of two power series in $a_1$ is defined through the usual Cauchy product. Now, set
\[
x_1 := ln(a_1), \quad e^{x_1} := a_1
\]
\[
x_i := a_1^{-1} a_i, \quad i = 2, 3, \ldots, m + n
\]
\[
h_i := ln(p_i), \quad i = 1, 2, \ldots, m + n
\]
such that for $i, j = 2, 3, \ldots, m + n$
\[
[x_1, x_i] = h_i x_i, \quad x_i x_j = (-1)^{i} p_1^{1-z_i} p_j^{1-z_j} x_j x_i,
\]

\[
[107x214]and the multiplication of two power series in a
\[
(26)
\]
\[
\sum_{i=1}^{m+n} z_i^k f \cdot T_{a_i}(g)
\]
\[
(25)
\]
\[
(27)
\]
\[
(24)
\]
\[
(23)
\]
\[
(22)
\]
\[
(21)
\]
\[
(20)
\]
where \( [u, v] = uv - vu \). Let \( \mathcal{M} \) denote a new algebra generated by \( x_i \)'s. From (27), the following noncommutative relations are obtained:

\[
\begin{align*}
(28) & \quad x_k^i x_j^l = x_j^l (x_1 + lh_j)^k, \quad x_i^k x_j^l = \left((-1)^{ij}p_i^{1-z_j}p_j^{z_i} - 1\right)^{kl} x_j^l x_i^k
\end{align*}
\]

for \( k, l \in \mathbb{N} \) and \( i, j = 2, 3, \ldots, m + n \).

Using Hopf superalgebra structure of the algebra \( \mathcal{A} \), one can easily see that the coproduct for the algebra \( \mathcal{M} \) appears as:

\[
(29) \quad \Delta_{\mathcal{M}}(x_i) = e^{(z_i - 1)x_1} \otimes x_i + x_i \otimes e^{(z_i - 1)x_1}, \quad i = 1, 2, \ldots, m + n
\]

the counit and the antipode are given as follows:

\[
(30) \quad \varepsilon_{\mathcal{M}}(x_i) = 0, \quad S_{\mathcal{M}}(x_i) = -e^{(1-z_i)x_1} x_i e^{(1-z_i)x_1}, \quad i = 1, 2, \ldots, m + n.
\]

Remark 2.1. In fact, it is clear that the algebra \( \mathcal{M} \) with commutation relations (27) is a generalization of \( \kappa \)-deformed superspace as the superspace extension of the \( \kappa \)-deformed Minkowski space \([41, 42]\).

Since \( \mathcal{M} \) has a Hopf superalgebra structure, it is well known that there exists a bicovariant differential calculus over \( \mathcal{M} \). In order to see this differential calculus explicitly, we want to see commutation parameters of the relevant relations in terms of the parameters \( h_i \)'s. To achieve this, one can use the following mappings having the properties of the right(left) bicovariant structure:

\[
(31) \quad \Delta_R(d(f)) = ((d \otimes \text{id}) \circ \Delta_{\mathcal{M}})(f), \quad \Delta_L(d(f)) = ((\text{id} \otimes d) \circ \Delta_{\mathcal{M}})(f)
\]

for \( f \in \mathcal{M} \). However, because of nonhomegeneous commutation relations (27), we need to perform too much computation to realize the approach mentioned above. For this approach, the interested reader is referred to the bicovariant differential calculus on the \( \kappa \)-Minkowski space \([13]\). Instead the approach used in \([14]\), we use an approach that requires a simple computation, based on the following lemma \([16]\):

**Lemma 2.2.** The following series is a representation of \( \ln(a_1) \)

\[
(32) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (a_1 - 1)^k,
\]

which means that it is consistent with relations (27). Moreover, based on the action of operator \( \partial_{a_1} \) (see \([27]\)), this representation yields the fact that \( \partial_{a_1}(\ln a_1) = a_1^{-1} \).

**Theorem 2.3.** We have the following relations of the generators \( a, b \) and \( \beta \) with their differentials, and these relations yield a bicovariant differential calculus over \( \mathcal{M} \)

\[
(33) \quad [x_1, dx_1] = 0, \quad [x_1, dx_i] = h_i dx_i, \quad [x_i, dx_j]_{\eta(i,j)} = 0,
\]
where $i = 2, 3, ..., m + n$, $j = 1, 2, ..., m + n$, and $[,]$ and $[,]_h$ are $\mathbb{Z}_2$-graded commutator and $q$-commutator defined by $[a, b] = ab - (-1)^{ab}ba$, $[a, b]_q = ab - (-1)^{ab}ba$, respectively, and $\eta(i, j) = p_i^{1 - z_i} p_j^{z_j - 1}$.

**Proof.** We first differentiate $ln(a_i)$, $a_i^{-1}a_i$, by using the relations (14), which results in

\begin{equation}
\begin{aligned}
    dx_i = da_i a_i^{-1}, \\
    dx_i = h_i^{-1}da_i a_i^{-1} - h_i^{-1}da_1 a_1^{-1} a_i a_1^{-1},
\end{aligned}
\end{equation}

for $i = 2, 3, ..., m + n$. Using the series (32), (34) and (14) in $x_i dx_i$ implies $x_i dx_i = (-1)^{(j+1)}dx_i x_1 + h_i dx_i$. The other noncommutative relations are obtained in similar way. Now, we should prove that the relations (33) yield a bicovariant differential calculus with (31). Indeed, we first show that the mappings $\Delta_R$ and $\Delta_L$, acting on $\mathcal{M}$ as $\Delta_M$ does, and the differentials as follows, preserve the relations (33):

\begin{equation}
\begin{aligned}
    \Delta_R(dx_i) = (z_i - 1)e^{(z_i - 1)x_i}dx_i \otimes x_i + dx_i \otimes e^{(z_i - 1)x_i}, \\
    \Delta_L(dx_i) = e^{(z_i - 1)x_i} \otimes dx_i + (-1)^i x_i \otimes (z_i - 1)e^{(z_i - 1)x_i} dx_i.
\end{aligned}
\end{equation}

For this, for example, we show that

\begin{equation}
\begin{aligned}
    \Delta_L(x_j dx_j) = & (1 \otimes x_1 + x_1 \otimes 1) \left( e^{(z_j - 1)x_j} \otimes dx_j + (-1)^{(j+1)}x_j \otimes (z_j - 1)e^{(z_j - 1)x_j} dx_j \right) \\
    = & e^{(z_j - 1)x_j} \otimes (h_j dx_j + dx_j x_1) + (-1)^{(j+1)}x_j \otimes (z_j - 1)x_1 e^{(z_j - 1)x_j} dx_j + x_1 e^{(z_j - 1)x_j} \otimes dx_j \\
    & + (-1)^{(j+1)}(h_j x_j + x_j x_1) \otimes (z_j - 1)e^{(z_j - 1)x_j} dx_j \\
    = & \left( e^{(z_j - 1)x_j} \otimes dx_j + (-1)^{(j+1)}x_j \otimes (z_j - 1)e^{(z_j - 1)x_j} dx_j \right) (1 \otimes x_1 + x_1 \otimes 1) \\
    & + h_j \left( e^{(z_j - 1)x_j} \otimes dx_j + (-1)^{(j+1)}x_j \otimes (z_j - 1)e^{(z_j - 1)x_j} dx_j \right) \\
    = & \Delta_L(dx_j) \Delta_L(x_1) + h_j \Delta_L(dx_j).
\end{aligned}
\end{equation}

That is, $\Delta_L$ leaves invariant the commutation relation $[x_i, dx_j] = h_j dx_j$, $j = 2, 3, ..., m + n$. It is also readily seen that the mappings given by (35) hold the conditions of the bicovariant bimodule. Moreover, if we apply the exterior differential operator $d$ to each relation in (33), we have the following relations among the differentials

\begin{equation}
\begin{aligned}
    dx_i \wedge dx_j = (-1)^{(i+1)(j+1)}\eta(i, j)dx_j \wedge dx_i, \quad i, j = 1, 2, ..., m + n
\end{aligned}
\end{equation}

which is preserved under the mappings $\Delta_R$, $\Delta_L$.

**Corollary 2.4.** *The differential algebra with the relations (27), (33) and (36) has a graded Hopf algebra structure induced by $\hat{\Delta} = \Delta_R + \Delta_L$.***
One can also easily obtain deformation relations between the operators and the generators of $\mathcal{M}$ using the Leibniz rule and the differential calculus of $\mathcal{M}$ as follows

$$
[\partial_{x_1}, x_1] = 1, \quad [\partial_{x_i}, x_1] = h_i \partial_{x_i}, \quad i = 2, 3, ..., m + n,
$$

$$
[\partial_{x_i}, x_j, \eta(j,i)] = \delta_{ij}, \quad i = 1, 2, ..., m + n, \quad j = 2, 3, ..., m + n.
$$

Using the nilpotency rule $d^2 = 0$, one gets commutation relations

$$
[\partial_{x_i}, \partial_{x_j}] \eta(i,j) = 0, \quad i, j = 1, 2, ..., m + n.
$$

Finally, the deformed Weyl superalgebra $\mathbb{C} \langle x_1, x_2, ..., x_{m+n}, \partial_{x_1}, \partial_{x_2}, ..., \partial_{x_{m+n}} \rangle$ is given by the defining relations (27), (37) and (38). This deformed Weyl superalgebra becomes the usual Weyl superalgebra when the all parameters $p_1, p_2, ..., p_{m+n} \to 0$.

3. Maurer-Cartan 1-forms on $\mathcal{M}$

The right-invariant Maurer-Cartan form corresponding to any $f \in \mathcal{M}$ can be given by the following formula [26]:

$$
w_f := m((d \otimes S_M) \Delta_M(f)),
$$

where $m$ stands for the multiplication. Thus we have

$$
\omega_{x_i} = [dx_i + (1 - z_i)dx_1 x_i] e^{(1-z_i) x_1}, \quad i = 1, 2, ..., m + n.
$$

Denote the algebra generated by $\omega_{x_1}, \omega_{x_2}, ..., \omega_{x_{m+n}}$ by $\Theta$. First determine all commutation relations about the Maurer-Cartan forms. The generators of $\Theta$ and the generators of $\mathcal{M}$ satisfy the following rules

$$
[x_1, \omega_{x_1}] = 0, \quad [x_1, \omega_{x_i}] = h_i \omega_{x_1}, \quad \omega_{x_i}, \omega_{x_j} = (-1)^{i(j+1)} \partial_{x_j} \omega_{x_i} - \partial_{x_i} \omega_{x_j},
$$

where $i = 2, 3, ..., m + n, j = 1, 2, ..., m + n$. The commutation rules of the generators of $\Theta$ are of the following form:

$$
[\omega_{x_i}, \omega_{x_j}] = 0, \quad i, j = 1, 2, ..., m + n.
$$

The algebra $\Theta$ is a graded Hopf algebra with the following comappings: for $i = 1, 2, ..., m + n$, the coproduct $\Delta_\Theta : \Theta \to \Theta \otimes \Theta$ is defined by

$$
\Delta_\Theta(\omega_{x_i}) = \omega_{x_i} \otimes 1 + 1 \otimes \omega_{x_i},
$$

The counit $\varepsilon_\Theta : \Theta \to \mathbb{C}$ is given by

$$
\varepsilon_\Theta(\omega_{x_i}) = 0,
$$

and the antipode $S_\Theta : \Theta \to \Theta$ is defined by

$$
S_\Theta(\omega_{x_i}) = -\omega_{x_i}.
4. Quantum Lie superalgebra of vector fields

In this section we give vector fields corresponding to the Maurer-Cartan 1-forms on \( M \) and Lie superalgebra of the vector fields. First we rewrite the Maurer-Cartan forms as follows:

\[
d x_i = \omega_{x_i} e^{(z_i-1)x_1} + (z_i - 1)dx_1 x_i, \quad i = 1, 2, ..., m + n
\]

and consider the exterior differential \( d \) in the following form

\[
d = \omega_{x_1} T_{x_1} + \omega_{x_2} T_{x_2} + \cdots + \omega_{x_{m+n}} T_{x_{m+n}},
\]

where \( T_{x_i} \)’s are vector fields corresponding to the Maurer-Cartan forms. We can determine the vector fields in terms of the partial derivative operators holding relations given in (37) and (38). By inserting (46) into the expression

\[
d = dx_1 \partial_{x_1} + dx_2 \partial_{x_2} + \cdots + dx_{m+n} \partial_{x_{m+n}},
\]

we can find the (quantum) Lie superalgebra generators expressed in terms of the operators:

\[
T_{x_1} \equiv \partial_{x_1} + \sum_{i=2}^{m+n} (z_i - 1)x_i \partial_{x_i},
\]

\[
T_{x_i} \equiv e^{(z_i-1)x_1} \partial_{x_i}, \quad i = 2, 3, ..., m + n.
\]

Now, we can obtain the commutation relations of these generators, as follows, using (27), (37) and (38):

\[
[T_{x_i}, T_{x_j}] = 0, \quad i, j = 1, 2, ..., m + n,
\]

where the parity of \( T_{x_i} \) is the same with one of \( x_i \). The commutation relations in (50) should be compatible with monomials in \( M \). To realize this, it is sufficient to get commutation relations between the generators of Lie superalgebra and the coordinates of \( M \). They are derived by (27) and (37) as

\[
[T_{x_1}, x_1] = 1, \quad [T_{x_i}, x_1] = h_i T_{x_i}, \quad i = 2, 3, ..., m + n,
\]

\[
[T_{x_i}, x_j]_{\eta(x_j)} e^{(x_j-1)h_j} = e^{(x_i-1)x_1} \delta_{ij}, \quad i = 2, 3, ..., m + n, \quad j = 2, 3, ..., m + n.
\]

5. Conclusion

We introduced a multiparametric quantum \((m+n)\)-superspace which as algebra is noncommutative in the sense of Manin superplane, and as Hopf superalgebra is cocommutative. Then we see that a bi-covariant differential calculus on this quantum superspace results in the partial derivatives which represent the superalgebra on this quantum superspace. Moreover, we constructed a noncomutative Hopf superalgebra (quantum supergroup) related with the vector fields corresponding to the Maurer-Cartan forms on this quantum \((m+n)\)-superspace. We also define new elements with nonhomogeneous relations by the logarithm of the grouplike element...
in the quantum (m+n)-superspace with homogeneous relations. The most interesting part of this algebra induced from the quantum (m+n)-superspace is that it reduces to the κ-deformed Minkowski superspace by some convenient constrains on the deformation parameters $p_i$'s and arbitrary integers $z_i$'s. Finally, a bicovariant differential calculus and the relevant results over the logarithmic extension of the quantum (m+n)-superspace are given.

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