Estimates for the differences of positive linear operators and their derivatives

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Abstract
The present paper deals with the estimate of the differences of certain positive linear operators and their derivatives. Our approach involves operators defined on bounded intervals, as Bernstein operators, Kantorovich operators, genuine Bernstein-Durrmeyer operators, and Durrmeyer operators with Jacobi weights. The estimates in quantitative form are given in terms of the first modulus of continuity. In order to analyze the theoretical results in the last section, we consider some numerical examples.

Keywords Positive linear operators · Differences of operators · Bernstein operators · Durrmeyer operators · Kantorovich operators

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1 Introduction

The de la Vallée Poussin operators of a $2\pi$-periodic integrable function $f$ are defined as

$$L_n(f; x) = \frac{1}{2\pi} \frac{(n!)^2}{(2n)!} 2^{2n} \int_{-\pi}^{\pi} f(u) \left( \cos \frac{x - u}{2} \right)^{2n} du.$$
These operators are trigonometric analogues of the Bernstein operators. It is well-known that de la Vallée-Poussin operator commutes with the derivative. Indeed, for $f \in C^1_{2\pi}[-\pi, \pi]$,

$$L_n(f; x) = \frac{1}{2\pi} \left( \frac{n!}{(2n)!} \right)^2 \frac{(2n)!}{2^n} \int_{-\pi}^{\pi} f(x + t) \left( \cos \frac{t}{2} \right)^{2n} dt$$

and we get

$$(L_n(f; x))' = \frac{1}{2\pi} \left( \frac{n!}{(2n)!} \right)^2 \frac{(2n)!}{2^n} \left\{ \int_{-\pi}^{\pi} f'(x + t) \left( \cos \frac{t}{2} \right)^{2n} dt - f(\pi) \left( \cos \frac{\pi-x}{2} \right)^{2n} \right\} = L_n(f'; x).$$

Thus, $(L_n f)^{(k)} = L_n(f^{(k)})$, for $f \in C^k_{2\pi}[-\pi, \pi]$. Certainly, this property is not available for the Bernstein operators $B_n$. The polynomials $(B_n f)^{(k)}$ and $B_n^{-k}(f^{(k)})$ have degree $n - k$ and converge to $f^{(k)}$. This remark motivated us to estimate in terms of moduli of continuity the differences $(L_n f)^{(k)} - L_n^{-k}(f^{(k)})$ for certain positive linear operators, as Bernstein, Kantorovich, genuine Bernstein-Durrmeyer, and Durrmeyer operators with Jacobi weights.

The study of differences of certain positive and linear operators has as starting point the problem proposed by Lupas [18], namely the question raised by him was to give an estimate for $B_n \circ \overline{B}_n - \overline{B}_n \circ B_n$, where $B_n$ and $\overline{B}_n$ are Bernstein operators and Beta operators, respectively. A solution for the problem proposed by Lupas was given for a more general case in [13]. Some interesting results on this topic were established by Gonska et al. in [12] and [14]. New estimates of the differences of certain operators are provided in a recent paper of Acu et al. [2]. These estimates improve some results concerning the differences of the $U^\rho_n$ operators studied in [20, 21]. Very recently, Aral et al. [6] obtained some quantitative results in terms of weighted modulus of continuity for differences of certain positive linear operators defined on unbounded intervals. Also, some estimates for the Chebyshev functional of these operators were provided.

Through the paper $\|\cdot\|$ denotes the supremum norm and $\omega(f, \cdot)$ is the modulus of continuity of the function $f$. Let $\langle x \rangle_n$ and $(x)_n$, $x \in \mathbb{R}$ be the rising and falling factorials respectively, given by $\langle x \rangle_n = \prod_{v=0}^{n-1} (x + v)$, $(x)_n = \prod_{v=0}^{n-1} (x - v)$ for $n \in \mathbb{N}$, $\langle x \rangle_0 = (x)_0 = 1$.

2 The Bernstein operators

Bernstein operators are one of the most important sequences of positive linear operators. These operators were introduced by Bernstein [9] and were intensively studied. For $f \in C [0, 1]$, the Bernstein operators are defined by

$$B_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) \ f \left( \frac{k}{n} \right), \ x \in [0, 1].$$
where \( p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \ k = 0, 1, \ldots, n. \)

**Theorem 2.1** For Bernstein operators, the following property holds:

\[
\left\| (B_n f)^{(r)} - B_{n-r} \left( f^{(r)} \right) \right\| \leq \frac{(r-1)r}{2n} \| f^{(r)} \| + \omega \left( f^{(r)}, \frac{r}{n} \right), \ f \in C^r[0, 1], \ r = 0, 1, \ldots, n. \tag{1}
\]

**Proof** The above differences can be written as

\[
(B_n f; x)^{(r)} - B_{n-r} (f^{(r)}(x)) = (n)_r \sum_{i=0}^{n-r} p_{n-r,i}(x) \Delta_i^n f \left( \frac{i}{n} \right) - \sum_{i=0}^{n-r} p_{n-r,i}(x) f^{(r)} \left( \frac{i}{n-r} \right)
\]

\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \left\{ (n)_r \Delta_i^n f \left( \frac{i}{n} \right) - f^{(r)} \left( \frac{i}{n-r} \right) \right\}
\]

\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \left\{ \frac{(n)_r}{n^r} r! \left[ \frac{i}{n}, \ldots, \frac{i+r}{n}; f \right] - f^{(r)} \left( \frac{i}{n-r} \right) \right\}
\]

\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \left\{ \frac{(n)_r}{n^r} f^{(r)}(\xi_i) - f^{(r)} \left( \frac{i}{n-r} \right) \right\}
\]

\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \left\{ \left( \frac{(n)_r}{n^r} - 1 \right) f^{(r)}(\xi_i) + f^{(r)}(\xi_i) - f^{(r)} \left( \frac{i}{n-r} \right) \right\},
\]

where \( \frac{i}{n} \leq \xi_i \leq \frac{i+r}{n} \).

We have

\[
0 \leq 1 - \frac{(n)_r}{n^r} \leq \frac{r(r-1)}{2n} \quad \text{and} \quad \frac{i}{n} \leq \frac{i+r}{n-r} \leq \frac{i}{n-r}, \quad \text{for} \ 0 \leq i \leq n-r.
\]

Therefore,

\[
\left\| (B_n f)^{(r)} - B_{n-r} \left( f^{(r)} \right) \right\| \leq \frac{(r-1)r}{2n} \| f^{(r)} \| + \omega \left( f^{(r)}, \frac{r}{n} \right).
\]

\( \square \)

Following the suggestion of a referee, we conclude this section with a second proof and an improved form of the inequality (1).

Let \( W_r = U_1 + \cdots + U_r \), where \( (U_j)_{j \geq 1} \) is a sequence of independent identically distributed random variables having the uniform distribution on [0, 1].

Let \( n > r \) and \( 0 \leq i \leq n-r \). We have \( 0 \leq W_r \leq r \) almost surely, and consequently

\[
\left| f^{(r)} \left( \frac{i}{n} + \frac{1}{n} W_r \right) - f^{(r)} \left( \frac{i}{n-r} \right) \right| \leq \omega \left( f^{(r)}, \frac{r}{n} \right) \text{ almost surely.} \tag{2}
\]
According to [3, (36)] (see also [4, 5]), we have

$$\Delta r^r f(x) = \alpha r^r \mathbb{E} f^{(r)}(x + \alpha W_r), \ x, \alpha \in \mathbb{R},$$

where $\mathbb{E}$ stands for mathematical expectation.

Let $C_{n,r} := \frac{(n)_r}{n^r}$. Using (3) and (2) we get

$$\left| B_n(f; x)^{(r)} - C_{n,r} B_{n-r}(f^{(r)}; x) \right| \leq C_{n,r} \sum_{i=0}^{n-r} p_{n-r,i}(x) \left( \mathbb{E} f^{(r)} \left( \frac{i}{n} + \frac{1}{n} W_r \right) - f^{(r)} \left( \frac{i}{n-r} \right) \right) \leq C_{n,r} \sum_{i=0}^{n-r} p_{n-r,i}(x) \mathbb{E} f^{(r)} \left( \frac{i}{n} + \frac{1}{n} W_r \right) - f^{(r)} \left( \frac{i}{n-r} \right) \leq C_{n,r} \sum_{i=0}^{n-r} p_{n-r,i}(x) \omega \left( f^{(r)}, \frac{r}{n} \right).$$

Therefore,

$$\left| B_n(f; x)^{(r)} - C_{n,r} B_{n-r}(f^{(r)}; x) \right| \leq C_{n,r} \omega \left( f^{(r)}, \frac{r}{n} \right). \quad (4)$$

Now,

$$\left| B_n(f; x)^{(r)} - B_{n-r}(f^{(r)}; x) \right| \leq \left| B_n(f; x)^{(r)} - C_{n,r} B_{n-r}(f^{(r)}; x) \right| + (1 - C_{n,r}) \left| B_{n-r}(f^{(r)}; x) \right|,$$

and (4) shows that

$$\left\| B_n(f)^{(r)} - B_{n-r}(f^{(r)}) \right\| \leq \frac{(r - 1)r}{2n} \left\| f^{(r)} \right\| + C_{n,r} \omega \left( f^{(r)}, \frac{r}{n} \right), \quad (5)$$

which improves the inequality (1).

Similar improvements can be considered for the corresponding inequalities in the next sections, but they will be deferred to a forthcoming paper.

### 3 The Kantorovich operators

These operators are the integral modification of Bernstein operators and were introduced by Kantorovich [16] as follows:

$$K_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \ f \in L_1[0, 1]. \quad (6)$$

The Kantorovich operators are related to the Bernstein polynomials by

$$K_n(f; x) = \left[ B_{n+1}(F; x) \right]', \text{ where } F(x) = \int_0^x f(t)dt.$$
Theorem 3.1 The Kantorovich operators verify
\[
\| (K_n f)^{(r)} - K_{n-r} \left( f^{(r)} \right) \| \leq \frac{(r+1)r}{2(n+1)} \| f^{(r)} \| + \omega \left( f^{(r)}, \frac{r+1}{n+1} \right), \quad f \in C^r[0, 1], \ r = 0, 1, \ldots n.
\]

Proof The \( r \)th derivative of Kantorovich polynomials can be written as follows:
\[
(K_n(f; x))^{(r)} = (B_{n+1} F(x))^{(r+1)}(x) = \sum_{i=0}^{n-r} p_{n-r,i}(x) \left( (n+1)_{r+1} \Delta_{\frac{1}{n+1}}^{r+1} F \left( \frac{i}{n+1} \right) \right)
\]
\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) (n+1)_{r+1} (r+1)! \left[ \frac{i}{n+1}, \ldots, \frac{i+r+1}{n+1}; F \right]
\]
\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) (n+1)_{r+1} F^{(r+1)}(\xi_i) = \sum_{i=0}^{n-r} p_{n-r,i}(x) (n+1)_{r+1} f^{(r)}(\xi_i).
\]

For the differences of Kantorovich operators, we obtain
\[
(K_n(f; x))^{(r)} - K_{n-r} \left( f^{(r)}(x) \right)
\]
\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \frac{(n)_r}{(n+1)^r} f^{(r)}(\xi_i) - \sum_{i=0}^{n-r} (n-r+1) p_{n-r,i}(x) \int_{\frac{i}{n-r+1}}^{\frac{i+1}{n-r+1}} f^{(r)}(t) dt
\]
\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \frac{(n)_r}{(n+1)^r} f^{(r)}(\xi_i) - \sum_{i=0}^{n-r} p_{n-r,i}(x) f^{(r)}(\eta_i)
\]
\[
= \sum_{i=0}^{n-r} p_{n-r,i}(x) \left\{ \left( \frac{(n)_r}{(n+1)^r} - 1 \right) f^{(r)}(\xi_i) + f^{(r)}(\xi_i) - f^{(r)}(\eta_i) \right\},
\]
where \( \frac{i}{n+1} \leq \xi_i \leq \frac{i+r+1}{n+1} \) and \( \frac{i}{n-r+1} \leq \eta_i \leq \frac{i+1}{n-r+1} \).

Let us remark that
\[
0 \leq 1 - \frac{(n)_r}{(n+1)^r} \leq \frac{r(r+1)}{2(n+1)}.
\]
Therefore,
\[
\| (K_n f)^{(r)} - K_{n-r} \left( f^{(r)} \right) \| \leq \frac{(r+1)r}{2(n+1)} \| f^{(r)} \| + \omega \left( f^{(r)}, \frac{r+1}{n+1} \right).
\]

In order to extend the above result, we will define the operator
\[
Q_k^n f := \frac{n^k(n-k)!}{n!} \left( B_n(f^{(-k)}) \right)^{(k)}, \quad f \in C[0, 1],
\]
where \( f^{(-k)} \) is an antiderivative of order \( k \) for the function \( f \).
Theorem 3.2 For the operators $Q_k^n$, the following property holds:

$$\left\| \left( Q_k^n f \right)^{(r)} - Q_{n-r}^k \left( f^{(r)} \right) \right\| \leq \frac{(2k + r - 1)r}{2n} \left\| f^{(r)} \right\| + \omega \left( f^{(r)} , \frac{k + r}{n} \right), \quad f \in C'[0, 1], \quad r = 0, 1, \ldots, n.$$ 

Proof The above inequality follows from

$$\left\| \left( Q_k^n f \right)^{(r)} - Q_{n-r}^k \left( f^{(r)} \right) \right\| = \frac{n^k (n - k)!}{n!} \left( B_n \left( f^{(-k)} \right) \right)^{(k+r)} (n - r) f^{(r)} + \omega \left( f^{(r)} , \frac{k + r}{n} \right).$$

4 The Durrmeyer operators with Jacobi weights

The classical Durrmeyer operators are the integral modification of Bernstein operators so as to approximate Lebesgue integrable functions defined on the interval $[0, 1]$. These operators were introduced by Durrmeyer [11] and, independently, by Lupasch [17] and are defined as

$$M_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{f(t)} p_{n,k}(t) f(t) \, dt, \quad x \in [0, 1].$$
Let \( w(\alpha, \beta)(x) = x^\alpha(1 - x)^\beta \), \( \alpha, \beta > -1 \) be a Jacobi weight function on the interval \((0, 1)\) and \( L_p^{(\alpha, \beta)}[0, 1] \) be the space of Lebesgue-measurable functions \( f \) on \([0, 1]\) for which the weighted \( L_p \)-norm is finite. The Durrmeyer operators can be generalized as follows:

\[
M_n^{(\alpha, \beta)}(f; x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{1}{c_{n,k}^{(\alpha, \beta)}} \int_0^1 p_{n,k}(t) w^{(\alpha, \beta)}(t) f(t) dt,
\]

where \( c_{n,k}^{(\alpha, \beta)} = \int_0^1 p_{n,k}(t) w^{(\alpha, \beta)}(t) dt \) and \( f \in L_1^{(\alpha, \beta)}[0, 1] \). See [7] and [19].

The classical Durrmeyer operators \( M_n \) are obtained for \( \alpha = \beta = 0 \).

In order to give the estimate for the difference of the Durrmeyer operators, we need the following result (see, e.g., [2]):

**Lemma 4.1** Let \( F : C[0, 1] \to \mathbb{R} \) be a positive linear functional with \( F(e_0) = 1 \) and \( F(e_1) = b \). Then \( b \in [0, 1] \), and for each \( \varphi \in C^2[0, 1] \),

\[
|F(\varphi) - \varphi(b)| \leq \frac{1}{2} \| \varphi'' \| (F(e_2) - e_2(b)),
\]

where \( e_r(x) = x^r \), \( r = 0, 1, \ldots \).

Let \( \varphi \in C^2[0, 1] \). With fixed \( 0 \leq r \leq n \) and \( 0 \leq k \leq n - r \), consider the functional

\[
A_{n,k}^{(\alpha, \beta)}(\varphi) := (n + \alpha + \beta + r + 1) \int_0^1 p_{n+\beta+\alpha+r,k+\alpha+r}(t) \varphi(t) dt
- (n + \alpha + \beta - r + 1) \int_0^1 p_{n-r+\alpha+\beta,k+\alpha}(t) \varphi(t) dt
= B_{n,k}^{(\alpha, \beta)}(\varphi) - C_{n,k}^{(\alpha, \beta)}(\varphi),
\]

where

\[
B_{n,k}^{(\alpha, \beta)}(\varphi) = (n + \alpha + \beta + r + 1) \int_0^1 p_{n+\beta+\alpha+r,k+\alpha+r}(t) \varphi(t) dt,
C_{n,k}^{(\alpha, \beta)}(\varphi) = (n + \alpha + \beta - r + 1) \int_0^1 p_{n-r+\alpha+\beta,k+\alpha}(t) \varphi(t) dt.
\]

**Lemma 4.2** The functional \( A_{n,k}^{(\alpha, \beta)} \) verifies

\[
|A_{n,k}^{(\alpha, \beta)}(\varphi)| \leq \frac{1}{4} \| \varphi'' \| \frac{n + \alpha + \beta + 3}{(n + \alpha + \beta + 3)^2 - r^2} + \omega \left( \varphi, \frac{r(n - r + |\beta - \alpha|)}{(n + 2 + \alpha + \beta)^2 - r^2} \right),
\]

where \( \omega \) is the first order modulus of continuity.
Proof By simple calculations, we get
\[
B_{n,k}^{(\alpha,\beta)}(e_0) = 1, \quad B_{n,k}^{(\alpha,\beta)}(e_1) = \frac{k + \alpha + r + 1}{n + r + \alpha + \beta + 2},
\]
\[
B_{n,k}^{(\alpha,\beta)}(e_2) = \frac{(k + \alpha + r + 1)(k + \alpha + r + 2)}{(n + r + \alpha + \beta + 2)(n + r + \alpha + \beta + 3)},
\]
\[
C_{n,k}^{(\alpha,\beta)}(e_0) = 1, \quad C_{n,k}^{(\alpha,\beta)}(e_1) = \frac{k + \alpha + 1}{n - r + \alpha + \beta + 2},
\]
\[
C_{n,k}^{(\alpha,\beta)}(e_2) = \frac{(k + \alpha + 1)(k + \alpha + 2)}{(n - r + \alpha + \beta + 2)(n - r + \alpha + \beta + 3)}.
\]
Therefore, using Lemma 4.1,
\[
|A_{n,k}^{(\alpha,\beta)}(\varphi)| = |B_{n,k}^{(\alpha,\beta)}(\varphi) - C_{n,k}^{(\alpha,\beta)}(\varphi)| \leq |B_{n,k}^{(\alpha,\beta)}(\varphi) - \varphi\left(\frac{k + \alpha + r + 1}{n + r + \alpha + \beta + 2}\right)|
\]
\[
+ |C_{n,k}^{(\alpha,\beta)}(\varphi) - \varphi\left(\frac{k + \alpha + 1}{n - r + \alpha + \beta + 2}\right)| + |\varphi\left(\frac{k + \alpha + r + 1}{n + r + \alpha + \beta + 2}\right) - \varphi\left(\frac{k + \alpha + 1}{n - r + \alpha + \beta + 2}\right) |
\]
\[
\leq \frac{1}{2}\|\varphi''\|(k + \alpha + r + 1)(\beta + 1 + n - k) + (k + \alpha + 1)(\beta + 1 + n - k)
\]
\[
+ \omega\left(\varphi, \frac{r(n - r - |\beta - \alpha|)}{(2 + \alpha + \beta + n + r)(2 + \alpha + \beta + n + r - k)}\right)
\]
\[
\leq \frac{1}{4}\|\varphi''\|\frac{n + \alpha + \beta + 3}{(n + \alpha + \beta + 3)^2 - r^2} + \omega\left(\varphi, \frac{r(n - r - |\beta - \alpha|)}{(n + 2 + \alpha + \beta)^2 - r^2}\right).
\]

Theorem 4.1 For Durrmeyer operators with Jacobi weights, the following property holds:
\[
\left\|\frac{\Gamma(n + \alpha + \beta + r + 1)\Gamma(n - r + 1)}{\Gamma(n + \alpha + \beta + 2)\Gamma(n + 1)}\left(M_n^{(\alpha,\beta,f)}(\varphi^{(r)}) - M_{n-r}^{(\alpha,\beta,f)}(\varphi^{(r)})\right)\right\|
\]
\[
\leq \frac{1}{4}\|\varphi^{(r+2)}\|\frac{n + \alpha + \beta + 3}{(n + \alpha + \beta + 3)^2 - r^2} + \omega\left(\varphi^{(r)}, \frac{r(n - r - |\beta - \alpha|)}{(n + 2 + \alpha + \beta)^2 - r^2}\right),
\]
where \(f \in C^{r+2}[0, 1]\), \(r = 1, \ldots, n\).

Proof In [1], Abel et al. proved the following identity for the derivatives of \(M_n^{(\alpha,\beta,f)}\):
\[
\left(M_n^{(\alpha,\beta,f)}(\varphi^{(r)})\right) = \frac{(n)_r}{(n + \alpha + \beta + 2)_r} M_{n-r}^{(\alpha+r,\beta+r,f)}(\varphi^{(r)}), \tag{8}
\]
where \(f^{(r)} \in L^w_{1}[0, 1]\) and \(r \leq n\).

By simple calculations, it can be shown that
\[
\left(M_n^{(\alpha,\beta,f)}(\varphi^{(r)})\right) = \frac{\Gamma(n + \alpha + \beta + 2)\Gamma(n + 1)}{\Gamma(n + \alpha + \beta + r + 1)\Gamma(n - r + 1)} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+\alpha+r,k+\alpha+r}(t) f^{(r)}(t) dt.
\]
We can write
\[
\frac{\Gamma(n + \alpha + \beta + r + 2) \Gamma(n - r + 1)}{\Gamma(n + \alpha + \beta + 2) \Gamma(n + 1)} \left( M_n^{(\alpha, \beta)}(f; x) \right)^{(r)} - M_{n-r}^{(\alpha, \beta)}(f^{(r)}; x) = (n + \alpha + \beta + r + 1) \sum_{k=0}^{n-r} p_n-r,k(x) \int_0^1 p_{n+\beta+\alpha+r,k+k+\alpha+r}(t) f^{(r)}(t) \, dt

- \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 \frac{1}{t^\alpha(1-t)^\beta} \int_0^1 p_{n-r,k}(t) t^\alpha(1-t)^\beta f^{(r)}(t) \, dt

= \sum_{k=0}^{n-r} p_{n-r,k}(x) \left\{ (n + \alpha + \beta + r + 1) \int_0^1 p_{n+\beta+\alpha+r+k+\alpha+r}(t) f^{(r)}(t) \, dt

- (n + \alpha + \beta - r + 1) \int_0^1 p_{n-r+\alpha+\beta+k+\alpha}(t) f^{(r)}(t) \, dt \right\} = \sum_{k=0}^{n-r} p_{n-r,k}(x) A_{n,k}^{(\alpha, \beta)}(f^{(r)}).
\]

Using Lemma 4.2 for \( f \in C^{r+2}[0, 1] \), the proof is complete.
The differences of Durrmeyer operators with Jacobi weights can be written as

\[
\| (M_n^{(\alpha, \beta)} f)^{(r)} - M_{n-r}^{(\alpha, \beta)} (f^{(r)}) \| \\
\leq \left| \frac{\Gamma(n-r+1)\Gamma(n+\alpha+\beta+r+2)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+2)} - 1 \right| \| (M_n^{(\alpha, \beta)} f)^{(r)} \| + \theta(f; n, r)
\]

\[
\leq \left( 1 - \frac{(n)_r}{\langle n+\alpha+\beta+2 \rangle_r} \right) \| f^{(r)} \| + \theta(f; n, r)
\]

\[
\leq \left( 1 - \frac{n-r+1}{n+\alpha+\beta+2} \right) r \| f^{(r)} \| + \theta(f; n, r)
\]

\[
= \frac{r(r+\alpha+\beta+1)}{n+\alpha+\beta+2} \| f^{(r)} \| + \theta(f; n, r).
\]

Using Theorem 4.1 the proof is complete. \qed

**Corollary 4.1** For Durrmeyer operators, the following property holds:

\[
\left\| \frac{(n+r+1)(n-r)!}{(n+1)!n!} (M_n f)^{(r)} - M_{n-r} (f^{(r)}) \right\| \\
\leq \frac{1}{4} \| f^{(r+2)} \| \left( \frac{n+3}{(n+3)^2-r^2} + \omega \left( f^{(r)}, \frac{r(n-r)}{(n+2)^2-r^2} \right) \right),
\]

where \( f \in C^{r+2}[0, 1], \) \( r = 1, \ldots, n. \)

**Corollary 4.2** Let \( f \in C^{r+2}[0, 1], \) \( r = 0, 1, \ldots, n. \) The Durrmeyer operators verify

\[
\| (M_n f)^{(r)} - M_{n-r} (f^{(r)}) \| \leq \frac{r(r+1)}{n+2} \| f^{(r)} \| + \frac{n+3}{4[(n+3)^2-r^2]} \| f^{(r+2)} \|
\]

\[
+ \omega \left( f^{(r)}, \frac{r(n-r)}{(n+2)^2-r^2} \right).
\]

## 5 The genuine Bernstein-Durrmeyer operators

The genuine Bernstein-Durrmeyer operators (see [10, 15]) are defined as follows:

\[
U_n(f; x) = (1-x)^n f(0) + x^n f(1) + (n-1) \sum_{k=1}^{n-1} \left( \int_0^1 f(t) p_{n-k-1}(t) dt \right) p_{n,k}(x), \quad f \in C[0, 1].
\]

These operators are limits of the Bernstein-Durrmeyer operators with Jacobi weights (see [7, 8, 19]), namely

\[
U_n f = \lim_{\alpha \to -1, \beta \to -1} M_n^{(\alpha, \beta)} f.
\]
Theorem 5.1 For the genuine Bernstein-Durrmeyer operators, the following property holds:

\[
\left\| \frac{(n + r - 1)!(n - r)!}{(n - 1)!n!} (U_n f)^{(r)}(x) - U_{n-r} f^{(r)} \right\| 
\leq \frac{1}{4} \frac{n + 1}{(n + 1)^2 - r^2} \left\| f^{(r+2)} \right\| + \frac{r}{n + r} \left\| f^{(r+1)} \right\| 
+ \omega \left( f^{(r)}, \frac{r(n - 2 - r)}{n^2 - r^2} \right),
\]

where \( f \in C^{r+2}[0, 1] \), \( r = 1, \ldots, n - 2 \).

Proof First,

\[
\frac{(n + r - 1)!(n - r)!}{(n - 1)!n!} (U_n f(x))^{(r)} - U_{n-r} \left( f^{(r)}(x) \right) = \frac{(n + r - 1)!(n - r)!}{(n - 1)!n!} (n)_r \frac{1}{(n - r - 1)!} 
\times \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+r-2,k+r-1}(t) f^{(r)}(t) dt - U_{n-r} \left( f^{(r)}(x) \right)
\]

\[
= (n + r - 1) \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+r-2,k+r-1}(t) f^{(r)}(t) dt 
- p_{n-r,0}(x) f^{(r)}(0) - p_{n-r,n-r}(x) f^{(r)}(1) 
- (n - r - 1) \sum_{k=1}^{n-r-1} p_{n-r,k}(x) \int_0^1 p_{n+r-2,k-1}(t) f^{(r)}(t) dt 
\]

\[
= (n + r - 1) p_{n-r,0}(x) \int_0^1 p_{n+r-2,r-1}(t) f^{(r)}(t) dt 
+ (n + r - 1) p_{n-r,n-r}(x) \int_0^1 p_{n+r-2,n-1}(t) f^{(r)}(t) dt 
\]

\[
- p_{n-r,0}(x) f^{(r)}(0) - p_{n-r,n-r}(x) f^{(r)}(1) + (n + r - 1) 
\times \sum_{k=1}^{n-r-1} p_{n-r,k}(x) \int_0^1 p_{n+r-2,k+r-1}(t) f^{(r)}(t) dt 
- (n - r - 1) \sum_{k=1}^{n-r-1} p_{n-r,k}(x) \int_0^1 p_{n+r-2,k-1}(t) f^{(r)}(t) dt 
\]

\[
= p_{n-r,0}(x) \int_0^1 (n + r - 1) p_{n+r-2,r-1}(t) \left( f^{(r)}(t) - f^{(r)}(0) \right) dt 
+ p_{n-r,n-r}(x) \int_0^1 (n + r - 1) p_{n+r-2,n-1}(t) \left( f^{(r)}(t) - f^{(r)}(1) \right) dt 
\times \sum_{k=1}^{n-r-1} p_{n-r,k}(x) \int_0^1 \left[ (n + r - 1) p_{n+r-2,k+r-1}(t) - (n - r - 1) p_{n-r-2,k-1}(t) \right] f^{(r)}(t) dt.
\]
It follows that
\[
\left| \int_0^1 (n + r - 1)p_{n+r-2,r-1}(t) \left( f^{(r)}(t) - f^{(r)}(0) \right) dt \right|
\leq \int_0^1 (n + r - 1)p_{n+r-2,r-1}(t) \| f^{(r+1)} \| t dt = \| f^{(r+1)} \| \frac{r}{n + r},
\]
\[
\left| \int_0^1 (n + r - 1)p_{n+r-2,n-1}(t) \left( f^{(r)}(t) - f^{(r)}(1) \right) dt \right| \leq \| f^{(r+1)} \| \frac{r}{n + r}.
\]
Using Lemma 4.2, we get
\[
\int_0^1 \left[ (n + r - 1)p_{n+r-2,k+r-1}(t) - (n - r - 1)p_{n-r-2,k-1}(t) \right] f^{(r)}(t) dt
= A_{n-2,k-1}(f^{(r)}) \leq \frac{1}{4} \| f^{(r+2)} \| \frac{n + 1}{(n + 1)^2 - r^2} + \omega \left( f^{(r)}, \frac{r(n - 2 - r)}{n^2 - r^2} \right).
\]
Since \( \sum_{k=1}^{n-r-1} p_{n-r,k}(x) \leq 1, p_{n-r,0}(x) + p_{n-r,n-r}(x) \leq 1 \), the proof is complete. \( \square \)

6 Numerical results

In this section, we will give some numerical examples in order to show the relevance of the theoretical results.

Example 1 Let \( f(x) = \frac{1}{32\pi} \{ 4\pi x \cos(2\pi x) - \pi \cos(2\pi x) - 6 \sin(2\pi x) \} \), \( r = 3 \) and \( E_{n,r}(f; x) = \left| (B_n(f; x))^{(r)} - B_{n-r}(f^{(r)}(x)) \right| \). In Fig. 1, the graphs of the functions \( f^{(r)}, B_{n-r}(f^{(r)}), \) and \( (B_n f)^{(r)} \) for \( n = 50 \) and \( r = 3 \) are given. Also, for \( n \in \{50, 100, 150\} \), the absolute values of the differences are illustrated in Fig. 2.

Example 2 Let \( f(x) = -\frac{1}{4\pi^2} \sin(2\pi x) - \frac{32}{\pi^2} \sin \left( \frac{1}{4} \pi x \right) \), \( r = 2 \) and \( E_{n,r}(f; x) = \left| (K_n(f; x))^{(r)} - K_{n-r}(f^{(r)}(x)) \right| \). In Fig. 3, the graphs of the functions \( f^{(r)}, K_{n-r}(f^{(r)}), \) and \( (K_n f)^{(r)} \) for \( n = 50 \) and \( r = 2 \) are given. Also, for \( n \in \{50, 100, 150\} \), the absolute values of the differences are illustrated in Fig. 4.

Example 3 Let \( f(x) = \frac{1}{20} x^5 - \frac{3}{32} x^4 + \frac{13}{192} x^3 - \frac{3}{128} x^2 \), \( r = 2 \) and \( E_{n,r}(f; x) = \left| (M_n(f; x))^{(r)} - M_{n-r}(f^{(r)}(x)) \right| \). In Fig. 5, the graphs of the functions \( f^{(r)}, M_{n-r}(f^{(r)}), \) and \( (M_n f)^{(r)} \) for \( n = 50 \) and \( r = 2 \) are given. Also, for \( n \in \{50, 100, 150\} \), the absolute values of the differences are illustrated in Fig. 6.
Fig. 1 Approximation process by $B_{n-r}(f^{(r)})$ and $(B_n f)^{(r)}$

Fig. 2 Error $E_{n,r}(f; x)$, for $n \in \{50, 100, 150\}$
Fig. 3  Approximation process by $K_{n-r}(f^{(r)})$ and $(K_n f)^{(r)}$

Fig. 4  Error $E_{n,r}(f; x)$, for $n \in \{50, 100, 150\}$
Fig. 5  Approximation process by $M_{n-r}(f^{(r)})$ and $(M_n f)^{(r)}$

Fig. 6  Error $E_{n,r}(f; x)$, for $n \in \{50, 100, 150\}$
Fig. 7 Approximation process by $U_n r(f^{(r)})$ and $(U_n f)^{(r)}$

Fig. 8 Error $E_{n,r}(f; x)$, for $n \in \{30, 40, 50\}$
Example 4 Let \( f(x) = \frac{1}{20}x^5 - \frac{17}{144}x^4 + \frac{7}{72}x^3 - \frac{1}{32}x^2 \), \( r = 2 \) and \( E_{n,r}(f; x) = \left| (U_n(f; x))^{(r)} - U_{n-r}(f^{(r)}(x)) \right| \). In Fig. 7 are given the graphs of the functions \( f^{(r)} \), \( U_{n-r}(f^{(r)}) \) and \( (U_n f)^{(r)} \) for \( n = 50 \) and \( r = 2 \). Also, for \( n \in \{30, 40, 50\} \), the absolute values of the differences are illustrated in Fig. 8.

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