An adjacency labeling scheme based on a tree-decomposition

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Abstract

In this paper we look at the problem of adjacency labeling of graphs. Given a family of undirected graphs the problem is to determine an encoding-decoding scheme for each member of the family such that we can decode the adjacency information of any pair of vertices only from their encoded labels. Further, we want the length of each label to be short (logarithmic in \( n \), the number of vertices) and the encoding-decoding scheme to be computationally efficient. We proposed a simple tree-decomposition based encoding scheme and used it give an adjacency labeling of size \( O(k \log k \log n) \)-bits. Here \( k \) is the clique-width of the graph family. We also extend the result to a certain family of \( k \)-probe graphs.

Keywords: Clique-widths Hereditary Classes Implicit representation.

1 Introduction

Adjacency labeling is a method to store adjacency information implicitly within vertex labels such that we can determine the adjacency between two vertices just from their labels. To be useful in practice we want these labels to be compact and easy to encode-decode. This is a powerful technique for lossless compression of graphs. Since the decoding is fully local, it makes these schemes particularly useful for storing graphs on distributed systems. It is an active area of research to determine adjacency labeling schemes for various graph families of practical importance.

1.1 Preliminaries

Let \( G = (V, E) \) be an undirected graph with vertex set \( V (|V| = n) \) and edge set \( E (|E| = m) \). We assume \( G \) has no self-loops or parallel edges. Let \( \mathcal{F}_n \) be a family of graphs on the vertex set \( V \) of size \( n \). For any \( u, v \in V \), we define \( \text{adj}(u, v) = 1 \) if \( \{u, v\} \in E \) and 0 otherwise.

Definition 1 (modified from [5]). An \( L \)-bit adjacency labeling scheme of a graph family \( \mathcal{F}_n \) is a pair of functions \( \text{enc} : \mathcal{F}_n \to (V \to \{0, 1\}^L) \) and \( \text{dec} : \{0, 1\}^L \times \{0, 1\}^L \to \{0, 1\} \) such that for all \( G = (V, E) \in \mathcal{F}_n \) and for all \( u, v \in V \),

\[
\text{adj}(u, v) = \text{dec}(\text{enc}(G)(u), \text{enc}(G)(v)).
\]
We say there is an $L$-bit adjacency labeling for $F_n$.

We write $\text{enc}(G)(u) = \text{enc}(u)$ when the graph $G$ is clear from the context. According to above definition a labeling scheme is local; as it determines the adjacency only based on the vertex labels. For a labeling scheme $\langle \text{enc}, \text{dec} \rangle$ to be useful in practice we want both functions, $\text{enc}$ and $\text{dec}$, to be efficiently computable. Here we use the qualifier “adjacency” labeling to distinguish it from other types labeling schemes (see below). However, in their seminal paper, authors in $[20]$ referred to such a scheme simply as an $L$-labeling of $G$. In general the $\langle \text{enc, dec} \rangle$ pair may be used as an efficient storage-retrieval scheme for $F_n$ with respect to some predicate $P$. For example $P$ could be the predicate that a triple of three vertices forms a triangle in $G$. Another example is the distance labeling problem $[4]$ where given a pair of vertex labels the decoder outputs the shortest path distance between them.

In this paper we are only concerned with adjacency labeling. There is a simple yet beautiful connection between adjacency labeling and *induced universal* graphs of a hereditary graph family.

**Definition 2.** A graph property $\mathcal{P}$ is said to be *hereditary* if it is closed under taking induced subgraphs.

**Definition 3.** $[2, 3, 1]$ A graph $G_\mathcal{P}$ of size $f(n)$ (for some time-constructible$^1$ function $f : \mathbb{N} \to \mathbb{N}$) is called universal for $\mathcal{P}$ if every graph $G \in \mathcal{P}$ with at most $n$ vertices is an induced subgraph of $G_\mathcal{P}$.

An adjacency labeling for a hereditary family is said to be *efficient* if $k = O(\log n)$. It is an easy exercise to note that having an efficient adjacency labeling for a hereditary family implies that there is a induced universal graph $G_\mathcal{P}$ with $O(n^{O(1)})$ vertices. In this paper we give an adjacency labeling for graphs parameterized over its *clique-width*. Upto a constant factor, this scheme is efficient for graphs of bounded clique-width.

**Definition 4.** $[14]$ The *clique-width* (denoted by $cw(G)$) of a graph $G$ is the minimum number of labels (of vertices) to construct $G$ using the following four operations:

1. Create a vertex in $v$ with label $i$ (denoted by $(v, i)$)
2. Disjoint union $G_1 \oplus G_2$ $^2$ of two labeled graphs $G_1$ and $G_2$
3. Join operation $\eta_{i,j}$ : adds edges between every pair of vertices one with label $i$ and another with label $j$ ($i \neq j$)
4. Relabel operation $\rho_{i \rightarrow j}$ relabels vertices having label $i$ with label $j$

A construction of $G$ using the above operations is known as a $k$-expression where $cw(G) = k$. A $k$-expression can be equivalently represented as a rooted binary tree$^3$ $T$ (called a union tree $[19]$) as follows. Leaves of $T$ corresponds to the labeled (with their initial labels) vertices $(v, i)$’s of $G$. Each internal node correspond to a union operation. Lastly, each internal node is decorated with a (possibly empty) sequence of join and relabel operations. We use the notation $d_z$ to denote the decorator for the node $z$.

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$^1$f(n) can be computed in time $O(f(n))$.

$^2$The vertex set of $V(G_1 \oplus G_2)$ of $G_1 \oplus G_2$ is $V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$.

$^3$It is a tree and not a DAG as the same graph does not take part in two separate union operation.
Figure 1: $T$ is a union tree of the graph $G$. However, $T$ is not a proper union tree. The induced subgraphs $G_x = G[\{a, b, c\}]$ has edges $ac$ and $bc$ but $H_x$, the graph corresponding to the subtree $T_x$ rooted at $x$, has no such edges.

We say $k$ is the width of $T$. For some internal vertex $x$ of $T$ let $T_x$ be the subtree rooted at $x$. Let $G_x$ be the induced subgraph of $G$ determined by the leaves of $T_x$. Then $T_x$ (including any join or relabel operations in $d_x$) is a union tree for some spanning subgraph $H_x$ of $G_x$. Borrowing the terminology from [19] we say $T$ is a proper union tree of $G$ if for every internal vertex $x \in T$, $H_x = G_x$ (see example in Fig. 1). It is an easy exercise (see lemma 1 in [19]) to show that we can transform any union tree in linear time to a proper one of the same width representing the same graph. Henceforth we shall assume without loss of generality that we are working with proper union trees.

In this paper, we also look at a generalization of $k$-expressions and study adjacency labeling of the corresponding graph family. Recently, a new width parameter was proposed [18, 10].

Figure 2: Top right graph $H$ corresponds to the $k$-expression $t = \eta_{1,3}(t_1 \oplus t_2)$ where $t_1 = \rho_{1 \rightarrow 2}\eta_{2,3}(\eta_{1,2}(\eta_{1,2}((a, 1) \oplus (b, 2)) \oplus (c, 3)))$ and $t_2 = \rho_{3 \rightarrow 2}\eta_{1,2}\eta_{2,3}(\eta_{1,2}((d, 1) \oplus (e, 2)) \oplus (\eta_{2,3}((f, 3) \oplus (g, 2)))). G$ can be embedded into $H$ using two independent sets $N_1, N_2$ as illustrated by the union tree $T$ of $G$.

Definition 5. (from [12]) Let $\mathcal{F}$ be a family of graphs. The $\mathcal{F}$-width of a graph $G$ is the minimum number $k$ of independent sets $N_1, \ldots, N_k$ in $G = (V, E)$ such that there exists $H$ is a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.
\( H = (V, E') \in \mathcal{F} \) where the following holds: 1) \( G \) is a spanning subgraph of \( H \) and 2) for every edge \((u, v) \in E' \setminus E\) there exists an \( i \in [k] \) with \( u, v \in N_i \).

A graph which has an \( \mathcal{F} \)-width of \( k \) is known as a \( k \)-probe \( \mathcal{F} \)-graph\(^6\). In this paper we consider the adjacency labeling of \( w_k \)-probe \( C_k \)-graphs. Here \( C_k \) is the family of graphs with clique-width \( \leq k \). We can represent a tree decomposition of \( w_k \)-probe \( C_k \)-graphs via a minor modification to the proper union tree structure (Fig. 2). The labels of each leaf now has an additional \( w_k \)-length binary vector. Specifically, each leaf corresponds to a tuple \((u, i, M_u)\) where \( M_u[j] = 1 \iff u \in N_j \); and \( i \) is \( u \)'s initial label in the \( k \)-expression (as before). Adjacency is determined as follows (using definition 4 and 5). Let \( z = lca(u, v) \). Then \( u, v \) are adjacent if and only if: 1) there is a join operation in \( d_z \) between the current labels of \( u \) and \( v \) and 2) \( M_u \) and \( M_v \) do not have a common 1. It should be noted that a \( w_k \)-probe \( C_k \)-graph has a clique-width \( \leq k 2^{w_k} \).

The motivation for studying adjacency labelling of \( k \)-probe \( \mathcal{F} \)-graphs are threefold. Firstly, they are a generalization of probe-graphs \(^{12}\), which can model some natural problems. For example a type of DNA mapping problem can be formulated as a recognition problem for probe-graphs of intervals \(^{11}\). Secondly, this family of graphs do not have a bounded genus. This may make finding an compact adjacency labeling challenging; especially if \( \mathcal{F} \) does not posses tree decomposition (like a union tree). Existing approaches such as those developed in \(^{16}\) does not extended to graph families whose genus is not bounded. We leave this as an open problem. Finally, depending on \( \mathcal{F} \), many computationally hard problems exhibit efficient algorithms when the \( \mathcal{F} \)-width is bounded. For example, the recognition problem for graphs of bounded \( B \)-width is fixed parameter tractable \(^{12}\). Here \( B \) is the class of block graphs \(^{6}\). A graph is a block graph if it is chordal and has no induced subgraph isomorphic to the diamond graph \((K_4 - e)\).

1.2 Summary of Our Results

Our main result is as follows.

**Theorem 1.** Suppose \( C_{k,n} \) is a family of graphs with \( n \)-vertices having a clique-width at most \( k \). Then \( C_{k,n} \) has an adjacency labeling scheme of size \( O(k \log k \log n) \). If \( k \) is bounded the above result is optimal upto a constant factor.

Briefly, we apply a recursive transformation on the union tree to obtain a tree of \( O(\log n) \) depth. This transformation preserves the lowest common ancestor relations between the leaves and allows us to encode the adjacency information contained within the internal nodes and the leaves with \( O(k \log k) \) bits. We also get the following generalization as a corollary.

**Corollary 1.** There is an \( O(k \log k \log n + w_k) \)-bits labeling scheme for \( w_k \)-probe \( C_k \)-graphs of size \( n \).

**Proof.** This immediately follows from Theorem 1 and the fact that we need an additional \( w_k \)-bits to encode the vectors \( M_u \). \(\square\)

\(^5\)Here \([n] = \{1, \ldots, n\}\).

\(^6\)Some authors call them probe-\( k \) \( \mathcal{F} \)-graph\(^{12}\)
1.3 Previous and Related Work

Adjacency labeling schemes studied in this paper closely follows the paradigm introduced in [20, 23]. However, the study of adjacency labeling schemes goes back more than half a century [8, 7]. Since then many results have been discovered for a wide variety of graph classes. A comprehensive overview and some interesting open problems can be found in [26, 24] and the references therein. So we restrict our discussion to results which are closely related to ours.

A folklore result is that cographs have $O(\log n)$-bit adjacency labeling. This follows from the fact that a cograph is a permutation graphs and for which an adjacency labeling follows trivially (for each vertex store $(i, \pi(i))$) [26]. In [17] authors gave a $(\log n + O(k \log \log \frac{n}{k}))$-bits adjacency labeling scheme for graphs of tree-width $k$.

Only a handful of results are known with respect to the clique-width parameter. There is a parallel line of research based on ordered binary decision diagrams (OBDD). OBDD’s are a generalization of union trees in the setting of boolean functions. In [21] authors gave a $O(n^{\frac{k^2}{\log k}})$-sized, $O(\log n)$-depth OBDD with an encoding size of $O(\log k \log n)$-bits. This scheme is based on a bottom up tree decomposition approach originally introduced in [22]. In contrast our decomposition scheme is top-down. An improvement was proposed based on a tree-decomposition approach similar to ours [19]. Here the author gave a $O(kn)$-sized data structure that supported $O(1)$-time adjacency quires. A more recent result on OBDD type storage scheme for small clique-width graph can be found in [9]. However, these representations are not local and the adjacency queries are performed with the help of a global data structure (the OBDD or something similar). The result closest to ours can be found in [15, 26]. The first paper uses the language of monodic second-order logic. There, authors gave an adjacency labeling scheme, which in the language of this paper, translates to a label of size $O(f(k) \log n)$ bits. In their paper authors did not give an explicit expression for $f(k)$. In [26] (chapter 11) the author hinted at an $O(\log n)$ adjacency labeling for graphs with bounded clique-width. The proposal uses a recursive decomposition by successively finding balanced $k$-modules for any graph with clique-width $k$. Although explicit bounds were not provided with respect to $k$, we expect that working out the details can give a bound similar to ours.

2 A caterpillar-type balanced decomposition

In this section we give a simple balanced decomposition (discussed shortly) of a rooted tree (not necessarily binary). Here we work with a generic rooted tree $T$ having $n$ leaves (hence $\leq n - 1$ internal nodes). Later in section 3 we will use this result to prove theorem 1.

It is clear that every tree can be constructed starting from $K_1$ by repeatedly adding pendant edges. However, it may require $O(n)$ iterations to construct a tree with $n$ vertices. We show that each tree on $n$ leaves can be constructed within $O(\log n)$ iterations using a slightly more relaxed operation. We assume each non-root vertex $v$ has either no children or at least two children (that is, $v$ does not have exactly one child). Let $L_n$ denote the set of all trees with at most $n$ leaves. See Fig. 3 for all trees in $L_3$, where the root is colored red.

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7We thank an anonymous reviewer for pointing this out.
Definition 6. A caterpillar (see Fig. 4) is a tree for which there exists a root-leaf path $P$ such that all vertices outside $P$ are leaves.

Figure 4: A rooted caterpillar tree.

Let $T_0, T_1, ..., T_k$ be disjoint trees. The operation of adding $T_1, ..., T_k$ to $T_0$ creates a tree obtained by identifying roots of $T_1, ..., T_k$ with $k$ distinct leaves of $T_0$, respectively (see Fig. 5). Let $C_0$ denote the class of all caterpillars. For each positive integer $p$, let $C_p$ consist of trees obtained by adding trees from $C_{p-1}$ to trees from $C_0$. Note that $C_{p-1} \subseteq C_p$ since $K_1 \in C_0$.

A path $P$ of a tree $T$ is called an $r$-path if $r$ is an end of $P$. Let $P$ be an $r$-path of a tree $T$. Suppose $T \neq P$. Let $T \setminus P$ denote the graph obtained by deleting $V(P)$ from $T$. Then each component of $T \setminus P$ must be one of the following two types: those that have no edges, which we call trivial, and those that have at least one edge, which we call nontrivial. Let $T_0$ consist of all edges that are incident with at least one vertex of $P$. Then $T_0$ is a caterpillar (with root $r$). Suppose $T \neq T_0$. Then at least one component of $T \setminus P$ is nontrivial. Let $T_1, ..., T_k$ be all such components. Then,

(i) the root of $T_i$ is the vertex of $T_i$ that is closest to $r$ (in $T$).
(ii) $E(T_0), E(T_1), ..., E(T_k)$ form a partition of $E(T)$
(iii) $T$ can be obtained by adding $T_1, ..., T_k$ to $T_0$.

The following theorem gives a structural relationship between $L_n$ and $C_p$.

Theorem 2. For every integer $n \geq 1$, we have $L_n \subseteq C_p$, where $p = \lfloor \log_2 n \rfloor$.

To prove the theorem we use the following lemma.
Lemma 7. Let $T$ be a tree with $n \geq 1$ leaves. Then $T$ has an $r$-path $P$ such that each component of $G \setminus P$ has at most $n/2$ leaves.

Proof. If $n = 1$ then the path with only one vertex $r$ satisfies the requirement. So we assume $n \geq 2$. Under this assumption, for any $r$-path $P$, $T \setminus P$ must have at least one component. This allows us to define for any $r$-path $P$:

$$h(P) = \max\{t \mid T \setminus P \text{ has component with } t \text{ leaves}\}$$

Let $P$ be an $r$-path that minimizes $h(P)$. We prove that $P$ satisfies the lemma.

![Figure 6: The tree used in the proof of Lemma 7](image)

Suppose on the contrary that $P$ does not satisfy the lemma. That is, $T \setminus P$ has a component $T_1$ with $n_1 > n/2$ leaves. Let the ends of $P$ be $r$ and $w$ and let the root of $T_1$ be $z$, as illustrated in the Fig. 6 above. Let $Q$ be the unique path of $T$ between $r$ and $z$. We prove that $h(Q) < n_1 \leq h(P)$, which will be a desired contradiction.

To estimate $h(Q)$ we observe that $T \setminus Q$ has two types of components: those that are disjoint from $T_1$ and those that are contained in $T_1$. For the ones that are disjoint from $T_1$, the number of leaves each of them may have is bounded by $n - n_1$, which is smaller than $n_1$. Next, we consider a component $T'$ of $T \setminus Q$ with $T' \subseteq T_1$. Since $T_1$ has $n_1 > n/2 \geq 1$ leaves, $z$ is not a leaf of $T$. By the assumption we made in the beginning of Section 2, $z$ has at least two children. It follows that $T'$ does not contain all leaves of $T_1$, which implies that $T'$ has fewer than $n_1$ leaves. Therefore, we have shown that every component of $T \setminus Q$ has fewer than $n_1$ leaves. Consequently, $h(Q) < n_1$, contradicting the choice of $P$. This contradiction proves the lemma.

Proof of Theorem 2.

Proof. As we observed earlier, every tree in $\mathcal{L}_3$ is a caterpillar, so we have $\mathcal{L}_n \subseteq \mathcal{C}_0$, for $n = 1, 2, 3$, and thus the theorem holds for $n = 1, 2, 3$. Suppose the theorem holds for $n - 1$, where $n \geq 4$. We prove that the theorem holds for $n$, and this would prove the theorem.

Let $T$ be a tree with $n \geq 4$ leaves. We need to show $T \in \mathcal{C}_{[\log_2 n]}$. We may assume that $T$ is not a caterpillar because otherwise $T \in \mathcal{C}_0 \subseteq \mathcal{C}_{[\log_2 n]}$. By Lemma, $T$ has an $r$-path $P$ such that each component of $T \setminus P$ has at most $n/2$ leaves. Since $T$ is not a caterpillar, $T \setminus P$ has at least one nontrivial component. Let $T_1, ..., T_k$ be all such components. By our induction hypothesis, each $T_i$ belongs to $\mathcal{C}_{[\log_2 (n/2)]} = \mathcal{C}_{[\log_2 n]} - 1$. It follows that $T \in \mathcal{C}_{[\log_2 n]}$ since $T$ is obtained by adding $T_1, ..., T_k$ to $T_0$. This completes our induction and it proves the theorem.
Remark 8. The decomposition in the above theorem can be computed in linear time as follows. First we apply depth first search to compute for each node \( u \) in \( T \) the number of leaves in the subtree \( T_u \). Then we apply a slightly modified heavy-light decomposition (see for example [25]) to obtain a decomposition of \( T \) into disjoint paths \( P \). If there is a \( r \)-path in \( P \) (there can be at most one) then use it as \( P \). Otherwise pick any child \( u \) of \( r \) and take \((r, u)\) as \( P \). We do not need to recompute the heavy-light decomposition for the recursive case and rather use the one computed for \( T \). The heavy-light decomposition can be computed in linear time and hence also the caterpillar decomposition.

3 An Adjacency Labeling Scheme

Recall that a proper union tree \( T \) is a rooted binary tree with root \( r \) and has \( n \) leaves. The initial label of a leaf \( u \) will be denoted as \( c_0(u) \). For an internal node \( x \in T \) let \( L_x \) be the set of leaves in the subtree \( T_x \). We begin with a lemma.

\[ \text{Figure 7: We want to determine the information needed to compute the adjacency between } u \text{ and } v \text{ given we already know their lowest common ancestor } x. \]

Lemma 9. Suppose \( G \) is a graph of clique-width \( k \) and \( T \) be a proper union tree of \( G \). We consider a node \( x \) of \( T \) as shown in Fig. 7. Let \( x = \text{lca}(u, v) \), where \( u \) and \( v \) are two leaf nodes. Given \( x, c_0(u) \) and \( c_0(v) \) we can determine \( \text{adj}(u, v) \) for all \( v \in T_z \) with an additional \( O(k \log k) \)-bits of information stored locally at \( u \).

This \( O(k \log k) \)-bits of information will serve to perform adjacency queries between \( u \) and \( v \) given we already know their lowest common ancestor \( x \).

Proof. We consider the situation shown in Fig. 7. Let \( B_x \) be the set of unique labels assigned to the leaves of the subtree rooted at \( x \) after applying \( d_x \). In order to determine \( \text{adj}(u, v) \) it is sufficient to know; 1) the labels of \( u \) and \( v \) after applying the decorators \( d_y \) and \( d_z \) respectively and 2) the decorator \( d_x \). However, we do not need to know the entirety of \( d_x \) but only whether \( c_x(v) \in C_x(u) \), which is defined next. Suppose \( c_x(u) \) \( (c_x(v)) \) are the labels of \( u \) (resp. \( v \)) before applying \( d_x \). From \( d_x \) we can easily determine the set of labels \( C_x(u) \subseteq B_y \cup B_z \) such that,

\[ \forall i \in C_x(u) \exists \eta_{i \rightarrow c_x(u)} \text{ or } \eta_{c_x(u) \rightarrow i} \in d_x \]
It is important to note that when defining the set $C_x(u)$ we consider the labels from the set $B_y \cup B_z$ before any re-labeling due to $d_x$.\footnote{Alternatively, we may assume that all relabeling operations in $d_x$ proceeds all join operations\cite{13}} As an example, suppose $B_y = \{1, 2\}$, $B_z = \{3, 5\}$, $c_x(u) = 1$ and
\[ d_x = \rho_3 \rightarrow 2 \eta_1 \rightarrow 2 \rho_2 \rightarrow 5 \eta_5 \rightarrow 1 \]
then $C_x = \{2, 3, 5\}$ and not simply $\{2, 5\}$. Clearly $|C_x(u)| \leq k - 1$ and it takes $O(k \log k)$-bits to store $|C_x(u)|$. Next, we need to retrieve $c_x(v)$ for any $v \in L_z$. This can be done by storing an additional $O(k \log k)$-bits at $u$. This follows from the fact that, given an initial labeling of $L_z$, the sub-$k$-expression induced by $T_z$ (including applying the decorator $d_z$) is just a re-labeling. This re-labeling can be stored as a list $(F_x(u))$ of size $k$ where each value is between 1 and $k$. The $c_0(v)^{th}$ entry of this list gives $c_x(v)$. Finally, we use $O(\log k)$ bits to store $c_x(u)$ at $u$. \hfill $\square$

Going forward, we will describe an encoding of each leaf as an alternating sequence of labels of two types. One containing path information and the other containing adjacency information. For the latter, we will use $C_x(u), F_x(u)$ and $c_x(u)$. We let $A_x(u) = (C_x(u), F_x(u), c_x(u))$.

**Theorem 1.** Suppose $C_{k,n}$ is a family of graphs with $n$-vertices having a clique-width at most $k$. Then $C_{k,n}$ has an adjacency labeling scheme of size $O(k \log k \log n)$. If $k$ is bounded the above result is optimal upto a constant factor.

![Figure 8](image)

Figure 8: The figure shows the tree obtained after contracting the smaller bushes. To make $T_s$ a proper union tree, the node $x$ is removed and $y$ is made a child of $z$.

**Proof.** First we start from the $r$-path decomposition of $T$ as described in the previous section. Let $P$ be a $r$-path. From Lemma 7 we know that the subtrees attached to $P$ have $\leq n/2$ leaves. Let $T_{\text{large}}$ be a possibly empty collection of subtrees which has between $n/4$ and $n/2$
leaves. These subtrees are identified with light blue color in Fig 8-a. Note that $0 \leq |T_{\text{large}}| \leq 4$. Consider the sequence(s) of smaller subtrees (we will call them bushes) between the trees in $T_{\text{large}}$. These bushes are highlighted with orange regions in the figure. There may be no such bushes between two large trees. Now we collate the bushes between two successive large subtrees (while descending along $P$) to create larger bushes until the total number of leaves among them is $\geq n/4$ but $\leq n/2$. At this point we call it a super-bush and restart the gathering process on the remaining bushes until we get another super-bush or we reach the end of the bushes. In the latter case we create a super-bush with whatever we have gathered upto that point. That is, we group the bushes (if any) between two successive large subtrees into $\geq n/4$ sized super-bushes (but no larger than $n/2$) except may be a constant number of groups which can have $< n/4$ leaves. We identify each super-bush with a tree, the root of which is the node closest to $r$. Further, we attached the tree to $P$ using the node of the super-bush which was closest to $r$. For example, in Fig. 8-a for the super-bush starting from the node $s$ we create a tree $T_s$ with $s$ as the root. We attach $T_s$ to $P$ where the node $s$ was previously located. From our construction, the number of such attachments will also be a constant. The decorators remain with the original vertices and the new (orange vertices in Fig. 8-b) vertices on $P$ does not contain any decorators. The resulting tree, denoted by $T'$, is not necessarily a valid union tree. However, we ensure that each subtree attached to $P$ is a proper union tree (Fig. 8-b).

First we informally describe the decoding scheme; this will give us an idea of what information to encode within the labels. Let $u, v$ be a pair of leaves in $T$ (Fig. 8). Let $x = \text{lca}(u, v)$. From lemma 9 we see that labels of size $O(k \log k)$-bits are sufficient to determine $\text{adj}(u, v)$ given $x, c_0(u)$ and $c_0(v)$. It remains to be determined the number of such labels we need to determine adjacency between $u$ and any other vertex in $G$. Trivially, we can maintain one such label for each node on the root-leaf path (in $T$) terminating in $u$. Since a path (in the caterpillar-decomposition) can have arbitrary length we will need $\Omega(\log n)$-bits to encode the path information in the final label. To reduce the encoding size and get our claimed bound we make the following crucial observation. It is not necessary to determine the $\text{lca}(u, v)$ explicitly. It suffices to know $A_x(u), c_0(u)$ and $c_0(v)$ to determine $\text{adj}(u, v)$. By taking a recursive approach, we show that we only need to remember $O(\log n)$ number of adjacency-type labels per vertex $u$. Since, each adjacency information requires $O(k \log k)$ bits labels, we get the bound claimed in the theorem. This recursive encoding scheme is determined based on $T'$. Let $l_1(u)$ (resp. $l_1(v)$) be the index of the subtree $u$ (resp. $v$) is a leaf of on the path $P$ in $T'$. For example, in Fig. 8 we have $l_1(u) = 3$ and $l_1(v) = 5$. There are two cases:

(i) $l_1(u) = l_1(v)$ Then $u, v \in L_s$ for some node $s \in P$ (see Fig. 8-b). To determine $\text{adj}(u, v)$, we recurse on the subtree $T_s$. According to our construction $T_s$ is a proper union tree corresponding to the induced subgraph $G[L_s]$. Then we determine an adjacency labeling scheme for $G[L_s]$ using $T_s$. This is used to determine $\text{adj}(u, v)$. This recursive construction is possible, since any induced subgraph of $G$ has a clique-width $\leq k$ and $T$ is a proper union tree.

(ii) $l_1(u) \neq l_1(v)$ We assume without loss of generality that $l_1(u) < l_1(v)$ (the case $l_1(u) > l_1(v)$ is symmetric). In this case we use $A_x(u), c_0(u)$ and $c_0(v)$ to determine $\text{adj}(u, v)$. 

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This completes the informal description of the decoding. From this, an encoding scheme emerges naturally.

**Encoder:** Generate $T'$ from $T$ and for each $u \in V$ we compute $(l_1(u), A_{P(u)}(u))$. Here, $P(u)$ is the lowest ancestor of $u$ on the path $P$ (in Fig. 8-b $P(u) = x$). For notational simplicity we denote $A_{P(u)}(u) = A_1(u)$. Then, perform the encoding recursively on each induced subgraphs of $G$ corresponding to the subtrees attached to $P$. For the vertex $u$ this process gives a sequence of labels $((l_1(u), A_1(u)))$'s. Appending to this sequence its initial label $c_0(u)$ gives the final encoding:

$$\text{enc}(u) = (c_0(u), (l_1(u), A_1(u)), \ldots, (l_p(u), A_p(u))),$$

where $p = O(\log n)$ (from Theorem 2). It is clear from the construction that $\text{enc}$ uses $O(k \log k \log n)$-bits.

**Decoder:** Given two strings $\text{enc}(u)$ and $\text{enc}(v)$ first we check the labels $(l_1(u), A_1(u))$ and $(l_1(v), A_1(v))$. If $l_1(u) < l_1(v)$ then we use $A_1(u), c_0(u)$ and $c_0(v)$ to determine $\text{adj}(u, v)$. The case $l_1(u) > l_1(v)$ is symmetric. Otherwise, $l_1(u) = l_1(v)$. In this case we proceed to check the next pair of labels $(l_2(u), A_2(u))$ and $(l_2(v), A_2(v))$ and so on. In general, let $i$ be the smallest number such that $l_i(u) \neq l_i(v)$. Then, using either $A_i(u)$ or $A_i(v)$ and $c_0(u), c_0(v)$ we can determine $\text{adj}(u, v)$. By our construction there is always such an $i \leq p$ such that $l_i(u) \neq l_i(v)$.

**Correctness:** We induct on the depth of the recursive construction. For the base case, we take $p = 0$ and the correctness follows trivially. Assume the encoder-decoder works correctly whenever the decomposition has depth $\leq p - 1$. This takes care of the case $l_1(u) = l_1(v)$. For the remaining case assume $l_1(u) < l_1(v)$. Then correctness follows from lemma 9.

**Remark 10.** Recall from Remark 1, given a union tree $T$ we can determine the recursive decomposition in $O(|T|)$ time. Additionally, $O(k \log k \log n)$ time is spent processing each leaf of $T$. Thus $\text{enc}$ can be computed in $O(kn \log k \log n)$ time. Decoding can be done in linear time in the size of the labels (i.e., in $O(k \log k \log n)$ time).

## Acknowledgement

The author would like to thank Guoli Ding for many discussions and considerable advice. In particular for the proof of caterpillar decomposition. We also thank anonymous reviewers for their helpful comments.

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