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Explicit transformation of an intersection of two quadrics to an elliptic curve in Weierstraß form

1. Introduction

This paper is motivated by problems in Diophantine analysis which can be formulated as problems of finding rational points on the intersection of two quadrics. Let us look at two typical examples.

- Example 1: Euler’s problem of concordant forms. Let $M, N \in \mathbb{Z}$ be two different nonzero integers. Euler called $M$ and $N$ concordant or discordant depending on whether or not the system of the equations $X^2 + MY^2 = Z^2$ and $X^2 + NY^2 = W^2$ possesses a nontrivial solution $(X, Y, Z, W) \in \mathbb{Z}^4$ with $Y \neq 0$ (see [3], [7]). This amounts to the question whether or not the intersection $Q_{M,N}$ of the two quadrics $X^2 + MY^2 = Z^2$ and $X^2 + NY^2 = W^2$ possesses a rational point which is nontrivial in the sense that $Y \neq 0$.

- Example 2: Rational squares in arithmetic progression. We can ask when four rational numbers $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$ form a progression $\alpha^2 < \beta^2 < \gamma^2 < \delta^2$ which is part of an arithmetic progression, which is the case if and only if differences between subsequent terms are integral multiples of a (necessarily rational) step size $s$, say

\begin{align*}
\beta^2 - \alpha^2 &= ks, \\
\gamma^2 - \beta^2 &= \ell s, \\
\delta^2 - \gamma^2 &= ms
\end{align*}

where $k, \ell, m \in \mathbb{N}$. If (1) holds for a fixed triplet $(k, \ell, m)$, then $\ell(\beta^2 - \alpha^2) = \ell ks = k\ell s = k(\gamma^2 - \beta^2)$ and $m(\gamma^2 - \beta^2) = m\ell s = \ell ms = \ell(\delta^2 - \gamma^2)$ so that

\begin{align*}
(k + \ell)\beta^2 - k\gamma^2 - \ell\alpha^2 &= 0 \\
-m\beta^2 + (m + \ell)\gamma^2 - \ell\delta^2 &= 0.
\end{align*}

Conversely, if $\alpha, \beta, \gamma, \delta$ are such that (2) holds for a given triplet $(k, \ell, m) \in \mathbb{N}^3$, we can define

\begin{align*}
s := \frac{\beta^2 - \alpha^2}{k} &= \frac{\gamma^2 - \beta^2}{\ell} = \frac{\delta^2 - \gamma^2}{m}
\end{align*}

to get a solution of the original problem (1). Thus given a triplet $(k, \ell, m)$, we ask whether or not the system (2) admits a nontrivial solution $(\alpha, \beta, \gamma, \delta) \neq \lambda \cdot (\pm 1, \pm 1, \pm 1, \pm 1)$ in rational numbers. Now since the equations in (2) are homogeneous, we can interpret $\alpha, \beta, \gamma, \delta$ as projective coordinates of a point in $\mathbb{P}^3(\mathbb{Q})$, and condition (2) can be reformulated by stating that the point $(X_0, X_1, X_2, X_3) := (\beta, \gamma, \alpha, \delta)$ is a rational point in the intersection of the two quadrics

\begin{align*}
Q_1(k, \ell, m) &:= \{(X_0, X_1, X_2, X_3) \in \mathbb{P}^3(\mathbb{R}) \mid (k + \ell)X_0^2 - kX_1^2 - \ellX_2^2 = 0\}, \\
Q_2(k, \ell, m) &:= \{(X_0, X_1, X_2, X_3) \in \mathbb{P}^3(\mathbb{R}) \mid -mX_0^2 + (m + \ell)X_1^2 - \ellX_3^2 = 0\}.
\end{align*}
The purpose of this paper is to present, in detail, the construction of a rationally defined isomorphism (biregular mapping) between a rationally defined smooth intersection of two quadrics in projective three-space $\mathbb{P}^3(K)$ where $K$ is a (not yet specified) field and an elliptic curve in Weierstraß form which maps a distinguished rational point to the point at infinity. The existence of such an isomorphism makes available the elaborate theory of elliptic curves to study number-theoretical problems of the type described before. For example, the construction will yield a rationally defined isomorphism between the intersection $Q_{k,\ell,m} := Q_1(k, \ell, m) \cap Q_2(k, \ell, m)$ and the plane Weierstraß cubic $E_{k,\ell,m}$ given by the affine equation $y^2 = x(x + km)(x + (k+\ell)(\ell+m))$. It turns out that, up to the very last step, the construction involved does not depend on any special characteristics of the equations (2), but works quite generally, and the purpose of this paper is to describe this construction in full generality and in full detail. (The conclusions to be drawn from this construction for the original number-theoretical problem will be discussed elsewhere.)

Most of the ideas and calculations can be found scattered in the literature, albeit in some cases only in a sketchy way, only illustrated by way of example or without considering all special cases which can occur. (References will be given as we progress.) While being only interested in the base-field $\mathbb{Q}$, we note that most of the calculations are valid for an arbitrary base-field $K$, sometimes with the restriction that the characteristic be different from 2 or 3. The calculations consist of several steps, which can be summarized as follows:

- transformation of the intersection of two quadrics to a smooth plane cubic curve;
- transformation of a rationally defined smooth plane cubic curve to an elliptic curve given by a Weierstraß equation;
- in the special situation of $Q_{k,\ell,m}$ transformation of the Weierstraß equation to the equation $y^2 = x(x + km)(x + (k+\ell)(\ell+m))$.

The calculations will be illustrated by a series of diagrams showing the curves that occur during the procedure, both in an affine as well as in a projective coordinate system. The relation between projective coordinates $(X,Y,Z)$ and affine coordinates $(x,y)$ is given by $x = X/Z$ and $y = Y/Z$. The projective coordinates are represented by a triangle whose vertices are the projective points $(0,0,1)$ (leftmost point), $(1,0,0)$ (rightmost point) and $(0,1,0)$ (upper middle point). In each case the drawings for the curves are based on the true equations; however, suitable scaling factors were used in order to make the relevant geometric properties of the curves visible. In addition to the curves themselves, we show those points which are relevant for the actual transformation.

Several times throughout the paper the following result will be invoked:

**Regularity Theorem:** Let $C$ be a nonsingular curve, $V \subseteq \mathbb{P}^n(K)$ a projective variety and $\varphi : C \to V$ a rational mapping. Then $\varphi$ is a morphism, i.e., a mapping which is regular at any point of $C$.

This is a standard result in algebraic geometry which can be found, for example, in [4] (see Chap. I, §6, Prop. 6.8), in [10] (see Chap. II, Sect. 3, Thm. 3, Cor. 1), or in [11] (see Chap. II, §2, Prop. 2.1).
2. Transformation of a quadric intersection to a plane cubic

The transformation of a smooth intersection of two quadrics in projective three-space to a smooth plane cubic curve is briefly sketched in [1], Chap. 8, pp. 36; some special cases are discussed in [2], Sect. 1.4.3, pp. 123-125. The most interesting part of the following calculations is the inverse mapping, which is not to be obtained in a completely trivial way.

The starting point is as follows: We are given two two quadrics $Q_1$ and $Q_2$ in projective three-space whose intersection is a smooth irreducible curve over an algebraic closure of the base-field $K$, and we are given a $K$-rational point $x = (x_0, x_1, x_2, x_3) \in Q_1 \cap Q_2$. (The condition on $Q_1$ and $Q_2$ is satisfied, for example, if $Q_1$ and $Q_2$ generate a separable pencil in the sense of [8], Sect. 2.4.3, p. 74; also see [9], chap. XIII.) The quadrics can be assumed to be given as

$$Q_1 = \{ X \in \mathbb{P}^3(K) \mid X^TAX = 0 \} \quad \text{and} \quad Q_2 = \{ X \in \mathbb{P}^3(K) \mid X^TBX = 0 \}$$

with two symmetric $(4 \times 4)$-matrices $A$ and $B$. (Each quadric can be written in this way over a base-field whose characteristic is different from 2.) Moreover, after re-indexing variables if necessary, we may assume that $x_3 \neq 0$ and hence even $x_3 = 1$. This will be assumed from now on.

To rewrite (5), we consider the coordinate transformation

$$\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -x_1 \\ 0 & 0 & 1 & -x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \text{i.e.,} \quad \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix},$$

which we write for short as $Y = PX$ and $X = QY$ where $Q = P^{-1}$. (Geometrically, the transition from $X = (X_0, X_1, X_2, X_3)$ to $Y = (Y_0, Y_1, Y_2, Y_3)$ is the translation which maps the point $x$ to the point $(0, 0, 0, 1)$.) The equations of the quadrics $Q_1$ and $Q_2$ are then transformed to

$$0 = X^TAX = Y^TQ^TAY = Y^T\begin{bmatrix} a_{00} & a_{01} & a_{02} & u_0 \\ a_{10} & a_{11} & a_{12} & u_1 \\ a_{20} & a_{21} & a_{22} & u_2 \\ u_0 & u_1 & u_2 & 0 \end{bmatrix}Y,$$

$$0 = X^TBX = Y^TQ^TBQY = Y^T\begin{bmatrix} b_{00} & b_{01} & b_{02} & v_0 \\ b_{10} & b_{11} & b_{12} & v_1 \\ b_{20} & b_{21} & b_{22} & v_2 \\ v_0 & v_1 & v_2 & 0 \end{bmatrix}Y,$$

where

$$u_i := \sum_{j=0}^3 a_{ij}x_j \quad \text{and} \quad v_i := \sum_{j=0}^3 b_{ij}x_j \quad \text{for} \quad 0 \leq i \leq 2.$$
The main point of this step is that in the coefficient matrices occurring in (7) the entry at the bottom right vanishes, which means that in the quadrics defined by (7) the variable $Y_3$ occurs only linearly. Thus after the transformation we have obtained the intersection of two quadrics of the form

\begin{align*}
\hat{Q}_1 &= \{(Y_0, Y_1, Y_2, Y_3) \in \mathbb{P}^3(K) \mid q_1(Y_0, Y_1, Y_2) + \ell_1(Y_0, Y_1, Y_2)Y_3 = 0\}, \\
\hat{Q}_2 &= \{(Y_0, Y_1, Y_2, Y_3) \in \mathbb{P}^3(K) \mid q_2(Y_0, Y_1, Y_2) + \ell_2(Y_0, Y_1, Y_2)Y_3 = 0\}
\end{align*}

where $q_1, q_2$ are quadratic and $\ell_1, \ell_2$ are linear. Specifically, we have

\begin{align*}
q_1(Y_0, Y_1, Y_2) &= \sum_{i,j=0}^2 a_{ij}Y_iY_j, \\
q_2(Y_0, Y_1, Y_2) &= \sum_{i,j=0}^2 b_{ij}Y_iY_j, \\
\ell_1(Y_0, Y_1, Y_2) &= 2(u_0Y_0 + u_1Y_1 + u_2Y_2), \\
\ell_2(Y_0, Y_1, Y_2) &= 2(v_0Y_0 + v_1Y_1 + v_2Y_2).
\end{align*}

Moreover, the rational point $x$ is mapped to the rational point $(0, 0, 0, 1)$. Consequently, on $\hat{Q}_1 \cap \hat{Q}_2$ we have $q_1\ell_2 = -\ell_1\ell_2Y_3 = -\ell_2\ell_1Y_3 = q_2\ell_1$. Hence if we define

\begin{equation}
C := \{(Y_0, Y_1, Y_2) \in \mathbb{P}^2(K) \mid q_1(Y_0, Y_1, Y_2)\ell_2(Y_0, Y_1, Y_2) = q_2(Y_0, Y_1, Y_2)\ell_1(Y_0, Y_1, Y_2)\}
\end{equation}

(which is a smooth cubic in the projective plane) then the assignment $(Y_0, Y_1, Y_2, Y_3) \mapsto (Y_0, Y_1, Y_2)$ (which is everywhere defined except at $(0, 0, 0, 1)$) maps $\hat{Q}_1 \cap \hat{Q}_2$ into $C$. As we shall see in a moment, this mapping can be redefined around the point $(0, 0, 0, 1)$ to yield an everywhere defined regular mapping

\begin{equation}
\varphi : \hat{Q}_1 \cap \hat{Q}_2 \to C
\end{equation}

which is then automatically an isomorphism (due to the regularity theorem quoted in the introduction), a fact which in our situation can also be established by explicitly revealing the inverse mapping

\begin{equation}
\psi : C \to \hat{Q}_1 \cap \hat{Q}_2
\end{equation}

rather than by invoking a general principle. Let $(Y_0, Y_1, Y_2) \in C$. If $\ell_1(Y_0, Y_1, Y_2) \neq 0$ then

\begin{equation}
\psi(Y_0, Y_1, Y_2) := \begin{bmatrix}
\ell_1(Y_0, Y_1, Y_2)Y_0 \\
\ell_1(Y_0, Y_1, Y_2)Y_1 \\
-\ell_1(Y_0, Y_1, Y_2)Y_2 \\
-q_1(Y_0, Y_1, Y_2)
\end{bmatrix}
\end{equation}

lies in $\hat{Q}_1 \cap \hat{Q}_2$, and $(\varphi \circ \psi)(Y_0, Y_1, Y_2) = (Y_0, Y_1, Y_2)$. Analogously, if $\ell_2(Y_0, Y_1, Y_2) \neq 0$ then

\begin{equation}
\psi(Y_0, Y_1, Y_2) := \begin{bmatrix}
\ell_2(Y_0, Y_1, Y_2)Y_0 \\
\ell_2(Y_0, Y_1, Y_2)Y_1 \\
\ell_2(Y_0, Y_1, Y_2)Y_2 \\
-q_2(Y_0, Y_1, Y_2)
\end{bmatrix}
\end{equation}
lies in \( \hat{Q}_1 \cap \hat{Q}_2 \), and \((\varphi \circ \psi )(Y_0, Y_1, Y_2) = (Y_0, Y_1, Y_2)\). What if \( \ell_1(Y_0, Y_1, Y_2) = \ell_2(Y_0, Y_1, Y_2) = 0 \) (in which case \((Y_0, Y_1, Y_2)\) clearly also lies in \( C \))? In this case \((Y_0, Y_1, Y_2)\) is the unique point of intersection of the lines \( u_0 Y_0 + u_1 Y_1 + u_2 Y_2 = 0 \) and \( v_0 Y_0 + v_1 Y_1 + v_2 Y_2 = 0 \), which is the (rational) point
\[(u_1 v_2 - u_2 v_1, u_2 v_0 - u_0 v_2, u_0 v_1 - u_1 v_0) =: z.\]

We could again invoke the regularity theorem quoted in the introduction to conclude that \( \varphi \) and \( \psi \) can be extended to become mutually inverse regular mappings in such a way that \( \varphi(0, 0, 0, 1) = z \) and \( \psi(z) = (0, 0, 0, 1) \), but it is also possible to see this in an elementary way by explicitly redefining \( \varphi \) around the point \((0, 0, 0, 1)\). By another elementary argument it can be seen that \( \psi \) does not even need to be redefined around the point \( z \), but that one of the two representations \((14) \) and \((15)\) is necessarily defined at this point. Let us now redefine \( \varphi \) around the point \((0, 0, 0, 1)\). To begin with, we split each of the polynomials defining \( \hat{Q}_1 \) and \( \hat{Q}_2 \) into the part containing \( Y_2 \) and the part not containing \( Y_2 \), thereby writing
\[
q_1 + \ell_1 Y_3 = \alpha_1 Y_2 + \alpha_2, \\
q_2 + \ell_2 Y_3 = \beta_1 Y_2 + \beta_2
\]
where \( \alpha_1, \beta_1 \) are linear polynomials in \((Y_0, Y_1, Y_2, Y_3)\) and where \( \alpha_2, \beta_2 \) are quadratic polynomials in \((Y_0, Y_1, Y_3)\). Specifically, we have
\[
\alpha_1 = 2a_{02} Y_0 + 2a_{12} Y_1 + a_{22} Y_2 + 2u_2 Y_3, \\
\alpha_2 = a_{00} Y_0^2 + a_{11} Y_1^2 + 2a_{01} Y_0 Y_1 + 2u_0 Y_0 Y_3 + 2u_1 Y_1 Y_3, \\
\beta_1 = 2b_{02} Y_0 + 2b_{12} Y_1 + b_{22} Y_2 + 2v_2 Y_3, \\
\beta_2 = b_{00} Y_0^2 + b_{11} Y_1^2 + 2b_{01} Y_0 Y_1 + 2v_0 Y_0 Y_3 + 2v_1 Y_1 Y_3.
\]

Next, we split each of the polynomials \( \alpha_2 \) and \( \beta_2 \) into a part containing \( Y_0 \) and a part containing \( Y_1 \) (where the terms with \( Y_0 Y_1 \) can be arbitrarily assigned to either term), thereby writing
\[
\alpha_2 = \gamma_0 Y_0 + \gamma_1 Y_1, \\
\beta_2 = \delta_0 Y_0 + \delta_1 Y_1
\]
where \( \gamma_0, \gamma_1, \delta_0, \delta_1 \) are all linear in \((Y_0, Y_1, Y_2, Y_3)\). A specific choice is given by
\[
\gamma_0 = a_{00} Y_0 + a_{01} Y_1 + 2u_0 Y_3, \\
\gamma_1 = a_{11} Y_1 + a_{01} Y_0 + 2u_1 Y_3, \\
\delta_0 = b_{00} Y_0 + b_{01} Y_1 + 2v_0 Y_3, \\
\delta_1 = b_{11} Y_1 + b_{01} Y_0 + 2v_1 Y_3.
\]

Now on \( \hat{Q}_1 \cap \hat{Q}_2 \) we have \( q_1 + \ell_1 Y_3 = q_2 + \ell_2 Y_3 = 0 \), hence \( \alpha_1 Y_2 = -\alpha_2 \) and \( \beta_1 Y_2 = -\beta_2 \), hence \( \alpha_1 \beta_2 = -\alpha_1 \beta_1 Y_2 = -\beta_1 \alpha_1 Y_2 = \beta_1 \alpha_2 \) and therefore \( \alpha_1 \delta_0 Y_0 + \alpha_1 \delta_1 Y_1 = \beta_1 \gamma_0 Y_0 + \beta_1 \gamma_1 Y_1 \), i.e.,
\[
(\alpha_1 \delta_1 - \beta_1 \gamma_1) Y_1 = (\beta_1 \gamma_0 - \alpha_1 \delta_0) Y_0.
\]
Thus the polynomial

\[ \lambda := \alpha_1 \beta_1 (\alpha_1 \delta_1 - \beta_1 \gamma_1) \]

satisfies on \( \hat{Q}_1 \cap \hat{Q}_2 \) the equations

\[ \lambda Y_1 = \alpha_1 \beta_1 (\beta_1 \gamma_0 - \alpha_1 \delta_0) Y_0 \]

and

\[
\begin{align*}
\lambda Y_2 &= (\alpha_1 \delta_1 - \beta_1 \gamma_1) \alpha_1 \beta_1 Y_2 \\
&= -(\alpha_1 \delta_1 - \beta_1 \gamma_1) \alpha_1 \beta_2 \\
&= -(\alpha_1 \delta_1 - \beta_1 \gamma_1) \alpha_1 (\delta_0 Y_0 + \delta_1 Y_1) \\
&= -(\alpha_1 \delta_1 - \beta_1 \gamma_1) \alpha_1 \delta_0 Y_0 - \alpha_1 \delta_1 (\beta_1 \gamma_0 - \alpha_1 \delta_0) Y_0 \\
&= \alpha_1 \beta_1 (\gamma_1 \delta_0 - \delta_1 \gamma_0) Y_0
\end{align*}
\]

so that

\[ \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \lambda Y_0 \\ \lambda Y_1 \\ \lambda Y_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \beta_1 (\alpha_1 \delta_1 - \beta_1 \gamma_1) Y_0 \\ \alpha_1 \beta_1 (\beta_1 \gamma_0 - \alpha_1 \delta_0) Y_0 \\ \alpha_1 \beta_1 (\gamma_1 \delta_0 - \delta_1 \gamma_0) Y_0 \end{bmatrix}. \]

Discarding common factors, the mapping \( \varphi \) can thus be rewritten as

\[ \varphi(Y_0, Y_1, Y_2, Y_3) = \begin{bmatrix} \alpha_1 \delta_1 - \beta_1 \gamma_1 \\ \beta_1 \gamma_0 - \alpha_1 \delta_0 \\ \gamma_1 \delta_0 - \delta_1 \gamma_0 \end{bmatrix}. \]

Written in this way, the mapping \( \varphi \) is defined at the point \((0, 0, 0, 1)\); namely, plugging in \((0, 0, 0, 1)\) for \((Y_0, Y_1, Y_2, Y_3)\) in (18) and (20), we find that

\[ \varphi(0, \alpha, 0, 1) = \begin{bmatrix} 2u_2 \cdot 2v_1 - 2v_2 \cdot 2u_1 \\ 2v_2 \cdot 2u_0 - 2u_2 \cdot 2v_0 \\ 2u_1 \cdot 2v_0 - 2v_1 \cdot 2u_0 \end{bmatrix} = \begin{bmatrix} -4(u_1 v_2 - u_2 v_1) \\ -4(u_2 v_0 - u_0 v_2) \\ -4(u_0 v_1 - u_1 v_0) \end{bmatrix} = \begin{bmatrix} u_1 v_2 - u_2 v_1 \\ u_2 v_0 - u_0 v_2 \\ u_0 v_1 - u_1 v_0 \end{bmatrix} \]

which shows that \( \varphi \) maps \((0, 0, 0, 1)\) to \(z\). Thus \( \varphi \) provides an isomorphism from the quadric intersection \( \hat{Q}_1 \cap \hat{Q}_2 \) onto a smooth cubic curve. We want to verify the regularity of the inverse mapping \( \psi \) at the point \(z\) in an elementary way (without invoking the regularity theorem quoted in the introduction). We have the two representations (14) and (15) for \( \psi \), and it is immediately clear that at least one of these is well-defined at any point of \(C\) other than \(z\). However, we claim that at least one of these must be also defined at the point \(z\) (so that no redefinition of \( \psi \) is required about this point). The point \(z\) is uniquely defined by the conditions \( \ell_1(z) = 0 \) and \( \ell_2(z) = 0 \); we need to rule out that, in addition, the conditions \( q_1(z) = 0 \) and \( q_2(z) = 0 \) can simultaneously hold. Now if this were the case
then \((z_1, z_2, z_3, Y_3)\) would be contained in \(\hat{Q}_1 \cap \hat{Q}_2\) for all \(Y_3\), as is clear from equation (9), which means that \(\hat{Q}_1 \cap \hat{Q}_2\) would contain a whole line over an algebraically closed field containing \(K\), contradicting our assumption that \(Q_1 \cap Q_2\) and hence \(\hat{Q}_1 \cap \hat{Q}_2\) is irreducible.

**Example.** We consider the quadrics \(Q_1\) and \(Q_2\) which are defined as the solution set of the equations

\[
X_0^2 + 2X_0X_1 + 2X_1^2 - 6X_1X_2 - 2X_2X_3 + 3X_3^2 = 0
\]

and

\[
-2X_0^2 + X_1^2 + 2X_2^2 - X_3^2 = 0,
\]

respectively. Note that \(Q_1 \cap Q_2\) contains the rational point \((1, 1, 1, 1)\). The isomorphism described before maps \(Q_1 \cap Q_2\) to the smooth cubic \(C\) given by the equation

\[
-2Y_0^3 + 3Y_0^2Y_1 + 6Y_0^2Y_2 + 4Y_0Y_1^2 - 16Y_0Y_1Y_2 + 4Y_0Y_2^2 - 2Y_1^3 - 2Y_1^2Y_2 + 12Y_1Y_2^2 - 8Y_2^3 = 0
\]

so that the coefficients of \(C\) are given by

\[
C_{300} = -2, \quad C_{210} = 3, \quad C_{201} = 6, \quad C_{120} = 4, \quad C_{111} = -16,
\]

\[
C_{102} = 4, \quad C_{030} = -2, \quad C_{021} = -2, \quad C_{012} = 12, \quad C_{003} = -8.
\]

The above isomorphism maps \((1, 1, 1, 1)\) to the rational point \((2, 2, 1)\).
3. Transformation of a smooth plane cubic to Weierstraß form

It is a well-known fact that any smooth cubic curve in the (projective) plane can be transformed into one given by a Weierstraß equation. In nearly any textbook on algebraic geometry or algebraic curves, this is done by choosing an inflection point and transforming this inflection point to the point at infinity (with respect to certain coordinates) in such a way that its tangent is given by the line at infinity. However, if the curve is defined over some field $K$ which is not algebraically closed (in our case $K = \mathbb{Q}$), there may be no inflection point defined over $K$. By a long known but rarely used construction due to Nagell (see [6]) the transformation can be done in such a way that a given $K$-rational point, which is not necessarily an inflection point, is mapped to the point at infinity.

Nagell’s construction seems to be half-forgotten, to the detriment of various approaches found in the literature to translate number-theoretical problems as problems of finding rational points on elliptic curves. (For example, in [5], pp. 6 ff., and in [7], the authors use a mapping of degree 4 instead of an isomorphism, thus losing some information on effects corresponding to torsion points on the Weierstraß curve in question.) We suspect that one reason for Nagell’s construction to sink into oblivion lies in the rather unwieldy calculations to which this construction leads. (This can be seen in the following examples by the fast-growing size of the coefficients showing up in the equations obtained; see also the explicit calculations in Appendix B of [12].)

The following calculations represent a complete and explicit version of Nagell’s construction. However, our calculations follow a geometrically more convenient concept which is briefly sketched in [12], pp. 17 ff (also, see again the explicit example in Appendix B, pp. 311 ff). Our calculations are complete in the sense that non-generic exceptional situations are covered, and they are explicit in the sense that all formulas may directly be realized in any programming language. Let us quickly explain our notations.

Generally, we consider our cubics to be given by homogeneous equations in three variables $X, Y, Z$, i.e., by equations of the form $\sum_{i+j+k=3} \Gamma_{ijk} X^i Y^j Z^k = 0$ where $\Gamma_{ijk} \in K$ denotes the coefficient of the monomial $X^i Y^j Z^k$. Since our cubics are considered to be $(K)$-rationally defined, they contain a fixed $(K)$-rational point, which we denote by $p = (p_x, p_y, p_z) \in K^3$. The transformation from a general cubic $C$ in the plane to a cubic in Weierstraß form is accomplished by a sequence of transformations from one cubic to another, starting with the original cubic $C = C(0)$ given in terms of the variables $(X, Y, Z) = (X_0, Y_0, Z_0)$. The $r$-th step in our algorithm consists of a coordinate transformation (mostly linear, in one instance quadratic) transforming the coordinates $(X_{r-1}, Y_{r-1}, Z_{r-1})$ used for the old curve $C_{(r-1)}$ to the coordinates $(X_r, Y_r, Z_r)$ used for the new curve $C_{(r)}$, thereby mapping the $(K)$-rational point $p_{(r-1)} = (p_{x(r-1)}, p_{y(r-1)}, p_{z(r-1)})$ to the $(K)$-rational point $p_{(r)} = (p_{x(r)}, p_{y(r)}, p_{z(r)})$. At any step in the calculations, all the data which are used for the calculations can be projectively simplified by cancelling common factors, which effects coefficients of cubic equations, coordinates of points in projective space and coordinates defining the slopes of lines. (For example, over the base-field $\mathbb{Q}$ we can always assume the rational point $p_{(r)}$ to have coprime integer coefficients.) We denote by $\Gamma_{ijk}^{(r)}$ the coefficients of the monomial $X_r^i Y_r^j Z_r^k$ in the equation describing the curve $C_{(r)}$;
i.e., the equation for $C_{(r)}$ is written as

\[(29) \quad 0 = \sum_{i+j+k=3} \Gamma^{(r)}_{ijk} X^i Y^j Z^k \]

where $(X, Y, Z) = (X_r, Y_r, Z_r)$ are the coordinates used after the $r$-th transformation. Without loss of generality we may assume that initially $p_x \neq 0$ and hence even that $p_x = 1$. This assumption will be made now. Each of the various coordinate transformations will be explained in geometric terms before being written down explicitly, and subsequently the arithmetical effect of the transformation on the equations of the various cubics involved will be explained. To ease readability, each step is presented on a double-page with diagrams visualizing the transformations used on one page and the associated arithmetical explanations on the other page.
Step 1: We start with an arbitrary cubic $C_{(0)}$ with a distinguished rational point $p^{(0)} = (p_x, p_y, p_z) = (1, p_y, p_z)$. The first transformation is the translation which maps $p^{(0)}$ to $p^{(1)} = (1, 0, 0)$ and hence yields a cubic $C_{(1)}$ with the special distinguished point $p^{(1)} = (1, 0, 0)$.

**Before:** Affine view (left) and projective view (right) of the curve $C_{(0)}$.

**After:** Affine view (left) and projective view (right) of the curve $C_{(1)}$. 
Execution of step 1: The translation which transforms the distinguished point \( p^{(0)} = (p_x, p_y, p_z) = (1, p_y, p_z) \) to the special distinguished point \( p^{(1)} = (1, 0, 0) \) is given by the linear coordinate transformation

\[
\begin{pmatrix}
X_1 \\
Y_1 \\
Z_1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
-p_y & 1 & 0 \\
-p_z & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix}
\]

with inverse

\[
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
p_y & 1 & 0 \\
p_z & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_1 \\
Y_1 \\
Z_1
\end{pmatrix}.
\]

In arithmetical terms, the purpose of this transformation is to make the term \( X^3 \) disappear from the original equation, i.e., to render the coefficient \( \Gamma^{(1)}_{300} \) zero. In fact, plugging (30) into the original equation (29) (with \( r = 0 \)) results in the transformed cubic \( C^{(1)} \) with the equation

\[
\sum_{i+j+k=3} \Gamma^{(1)}_{ijk} X_i^1 Y_j^1 Z_k^1 = 0
\]

where

\[
\begin{align*}
\Gamma^{(1)}_{210} &= \Gamma^{(0)}_{210}p_x^2 + 2\Gamma^{(0)}_{120}p_x p_y + \Gamma^{(0)}_{111}p_z + 3\Gamma^{(0)}_{030}p_y^2 + 2\Gamma^{(0)}_{021}p_y p_z + \Gamma^{(0)}_{012}p_z^2, \\
\Gamma^{(1)}_{201} &= \Gamma^{(0)}_{201}p_x^2 + \Gamma^{(0)}_{111}p_x p_y + 2\Gamma^{(0)}_{102}p_x p_z + \Gamma^{(0)}_{021}p_y^2 + 2\Gamma^{(0)}_{012}p_y p_z + 3\Gamma^{(0)}_{003}p_z^2, \\
\Gamma^{(1)}_{120} &= \Gamma^{(0)}_{120}p_x^2 + 3\Gamma^{(0)}_{030}p_x p_y + \Gamma^{(0)}_{021}p_x p_z, \\
\Gamma^{(1)}_{111} &= \Gamma^{(0)}_{111}p_x^2 + 2\Gamma^{(0)}_{021}p_x p_y + 2\Gamma^{(0)}_{012}p_x p_z, \\
\Gamma^{(1)}_{102} &= \Gamma^{(0)}_{102}p_x^2 + \Gamma^{(0)}_{012}p_x p_y + 3\Gamma^{(0)}_{003}p_x p_z, \\
\Gamma^{(1)}_{030} &= \Gamma^{(0)}_{030}p_x^2, \\
\Gamma^{(1)}_{021} &= \Gamma^{(0)}_{021}p_y^2, \\
\Gamma^{(1)}_{012} &= \Gamma^{(0)}_{012}p_z^2, \\
\Gamma^{(1)}_{003} &= \Gamma^{(0)}_{003}p_z^2.
\end{align*}
\]

(Note that we did not replace \( p_x \) by 1 in order to exhibit the homogeneity of the equations.)

In the above example, we have

\[
\begin{align*}
\Gamma^{(1)}_{300} &= 0, & \Gamma^{(1)}_{210} &= -2, & \Gamma^{(1)}_{201} &= -2, & \Gamma^{(1)}_{120} &= -3, & \Gamma^{(1)}_{111} &= -8, \\
\Gamma^{(1)}_{102} &= 4, & \Gamma^{(1)}_{030} &= -2, & \Gamma^{(1)}_{021} &= -2, & \Gamma^{(1)}_{012} &= 12, & \Gamma^{(1)}_{003} &= -8.
\end{align*}
\]
**Step 2:** In geometrical terms, the purpose of the second transformation is to transform $C_{(1)}$ into a cubic $C_{(2)}$ such that the tangent at the distinguished rational point $(1, 0, 0)$ is given by the equation $Z = 0$.

**Before:** Affine view (left) and projective view (right) of the curve $C_{(1)}$.

**After:** Affine view (left) and projective view (right) of the curve $C_{(2)}$. 

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Execution of step 2: In arithmetical terms, the purpose of this second transformation is to make not only the coefficient of $X^3$, but also the coefficient of $X^2Y$ vanish. Note that the tangent of the cubic $C_{(1)}$ with coefficients (31) at the point $p^{(1)} = (1, 0, 0)$ is given by the equation $0 = \Gamma_{210}^{(1)} Y_1 + \Gamma_{201}^{(1)} Z_1 =: g_y Y_1 + g_z Z_1$. We may assume that $\Gamma_{201}^{(1)} \neq 0$ (since otherwise we could simply exchange the coordinates $Y$ and $Z$ to yield this condition) and hence even that $g_z = 1$. The goal of the transformation is then accomplished by the linear transformation

$$
(32) \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g_y & 1 \end{bmatrix} \quad \text{with inverse} \quad \begin{bmatrix} X_1 \\ Y_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -g_y & 1 \end{bmatrix}.
$$

Plugging (32) into the equation for $C_{(1)}$ results in the transformed cubic $C_{(1)}$ with the equation $\sum_{i+j+k=3} \Gamma_{ijk}^{(2)} X_2^i Y_2^j Z_2^k = 0$ where

$$
\begin{align*}
\Gamma_{201}^{(2)} &= \Gamma_{201}^{(1)} g_z^2, \\
\Gamma_{120}^{(2)} &= \Gamma_{120}^{(1)} g_z^3 - \Gamma_{111}^{(1)} g_y g_z^2 + \Gamma_{102}^{(1)} g_z g_y^2, \\
\Gamma_{111}^{(2)} &= \Gamma_{111}^{(1)} g_z^2 - 2\Gamma_{102}^{(1)} g_y g_z, \\
\Gamma_{102}^{(2)} &= \Gamma_{102}^{(1)} g_z, \\
\Gamma_{030}^{(2)} &= \Gamma_{030}^{(1)} g_z^3 - \Gamma_{021}^{(1)} g_z g_y^2 + \Gamma_{012}^{(1)} g_z g_y^2 - \Gamma_{003}^{(1)} g_y^3, \\
\Gamma_{021}^{(2)} &= \Gamma_{021}^{(1)} g_z^2 - 2\Gamma_{012}^{(1)} g_y g_z + 3\Gamma_{003}^{(1)} g_y^2, \\
\Gamma_{012}^{(2)} &= \Gamma_{012}^{(1)} g_z - 3\Gamma_{003}^{(1)} g_y, \\
\Gamma_{003}^{(2)} &= \Gamma_{003}^{(1)}.
\end{align*}
$$

If $(1, 0, 0)$ happens to be an inflection point of $C_{(2)}$ (i.e., if $\Gamma_{120}^{(2)} = 0$), we can proceed directly to Step 5 (i.e., we let $C_{(5)} := C_{(2)}$ after exchanging the variables $X_2$ and $Y_2$, because after this change of variables we obtain an equation of the form

$$
0 = \Gamma_{300}^{(5)} X_5^3 + \Gamma_{201}^{(5)} X_5^2 Z_5 + \Gamma_{111}^{(5)} X_5 Y_5 Z_5 + \Gamma_{102}^{(5)} X_5 Z_5^2 + \Gamma_{012}^{(5)} Y_5 Z_5^2 + \Gamma_{003}^{(5)} Z_5^3
$$

which is already in Weierstraß form (with one nonzero coefficient more than in the form obtained for $C_{(5)}$ in the other case). Generically, however, the cubic $C_{(2)}$ intersects the tangent $Z_2 = 0$ at the point $(1, 0, 0)$ in a (simple) second point, namely $p^{(2)} = (p_x^{(2)}, p_y^{(2)}, 0) = (\Gamma_{030}^{(2)}, -\Gamma_{120}^{(2)}, 0)$ where $\Gamma_{120}^{(2)} \neq 0$, and this will be assumed to be the starting point for the next step. In our example, we have

$$
\begin{align*}
\Gamma_{300}^{(2)} &= 0, & \Gamma_{210}^{(2)} &= 0, & \Gamma_{201}^{(2)} &= -2, & \Gamma_{120}^{(2)} &= -9, & \Gamma_{111}^{(2)} &= -16, \\
\Gamma_{102}^{(2)} &= -4, & \Gamma_{030}^{(2)} &= -20, & \Gamma_{021}^{(2)} &= -50, & \Gamma_{012}^{(2)} &= -36, & \Gamma_{003}^{(2)} &= -8;
\end{align*}
$$

the distinguished rational point is $p^{(2)} = (-20, 9, 0)$. 

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Step 3: We transform the cubic $C_{(2)}$ into a general Weierstraß cubic in the usual form which, in affine coordinates, is $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. In projective coordinates $(X, Y, Z)$ where $(x, y) = (X/Z, Y/Z)$ this is equivalent to saying that the point $(0, 1, 0)$ lies on the projective curve and that the line $Z = 0$ (the line at infinity) is tangent to $(0, 1, 0)$. Thus the cubic $C_{(3)}$ has the distinguished rational point $p = (1, 0, 0)$, the tangent at $p$ being given by $Z = 0$ and the second intersection point of this tangent with the cubic being given by $q = (0, 1, 0)$.

Before: Affine view (left) and projective view (right) of the curve $C_{(2)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(3)}$. 

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Execution of step 3: If the Weierstraß cubic \( C_{(3)} \) to be constructed is given by the equation \( 0 = \sum_{i+j+k=3} \Gamma_{ijk} X^i Y^j Z^k \), the above conditions are equivalent to \( \Gamma_{030} = \Gamma_{210} = \Gamma_{120} = 0 \). Geometrically, we want to transform \( C_{(2)} \) to a cubic \( C_{(3)} \) such that the second point of intersection of \( C_{(2)} \) with the tangent \( Z = 0 \) at \( p = (1,0,0) \), i.e., the point \( (p_x^{(2)}, p_y^{(2)}, 0) =: (q_x, q_y, 0) \) in the previous step, is transformed into the point \( (0,1,0) \). Since \( q_y \neq 0 \) by assumption, we may as well assume that \( q_y = 1 \). Doing so, we use the coordinate transformation

\[
\begin{bmatrix}
  X_3 \\
  Y_3 \\
  Z_3
\end{bmatrix} = \begin{bmatrix}
  -q_y & q_x & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  X_2 \\
  Y_2 \\
  Z_2
\end{bmatrix}
\text{ with inverse }
\begin{bmatrix}
  X_3 \\
  Y_3 \\
  Z_3
\end{bmatrix} = \begin{bmatrix}
  1 & -q_x & 0 \\
  0 & -q_y & 0 \\
  0 & 0 & -q_y
\end{bmatrix}
\begin{bmatrix}
  X_2 \\
  Y_2 \\
  Z_2
\end{bmatrix}.
\]

Plugging (34) into the equation for \( C_{(2)} \) results in the transformed cubic \( C_{(3)} \) with the equation \( \sum_{i+j+k=3} \Gamma_{ijk}^{(3)} X^i Y^j Z^k = 0 \) where

\[
\begin{align*}
\Gamma_{201}^{(3)} &= \Gamma_{201}^{(2)}, \\
\Gamma_{120}^{(3)} &= -q_y \Gamma_{120}^{(2)}, \\
\Gamma_{111}^{(3)} &= -q_y \Gamma_{111}^{(2)} - 2q_x \Gamma_{201}^{(2)}, \\
\Gamma_{102}^{(3)} &= -q_y \Gamma_{102}^{(2)}, \\
\Gamma_{021}^{(3)} &= q_y^2 \Gamma_{021}^{(2)} + q_x q_y \Gamma_{111}^{(2)} + q_x^2 \Gamma_{201}^{(2)}, \\
\Gamma_{012}^{(3)} &= q_y^2 \Gamma_{012}^{(2)} + q_x q_y \Gamma_{102}^{(2)}, \\
\Gamma_{003}^{(3)} &= q_y^2 \Gamma_{003}^{(2)}.
\end{align*}
\]

The curve \( C_{(3)} \) is characterized by the following properties:

- the point \( (1,0,0) \) lies on \( C_{(3)} \), i.e., \( \Gamma_{300}^{(3)} = 0 \);
- the point \( (0,1,0) \) lies on \( C_{(3)} \), i.e., \( \Gamma_{030}^{(3)} = 0 \);
- the tangent to \( C_{(3)} \) at \( (1,0,0) \) is given by \( Z_3 = 0 \), i.e., \( \Gamma_{210}^{(3)} = 0 \);
- the point \( (1,0,0) \) is not an inflection point of \( C_{(3)} \), i.e., \( \Gamma_{120}^{(3)} \neq 0 \).

In this situation the tangent to \( C_{(3)} \) at the point \( (0,1,0) \) is given by the equation \( h_x X_3 + h_z Z_3 = 0 \) where \( (h_x, h_z) = (\Gamma_{120}^{(3)}, \Gamma_{021}^{(3)}) \) projectively; furthermore, we have \( \Gamma_{120}^{(3)} \neq 0 \), i.e., \( h_x \neq 0 \). In our example, we have

\[
\begin{align*}
\Gamma_{300}^{(3)} &= 0, & \Gamma_{210}^{(3)} &= 0, & \Gamma_{201}^{(3)} &= -2, & \Gamma_{120}^{(3)} &= 81, & \Gamma_{111}^{(3)} &= 64, \\
\Gamma_{102}^{(3)} &= 36, & \Gamma_{030}^{(3)} &= 0, & \Gamma_{021}^{(3)} &= -1970, & \Gamma_{012}^{(3)} &= -2196, & \Gamma_{003}^{(3)} &= -648.
\end{align*}
\]
Step 4: We want to transform $C(3)$ to a cubic $C(4)$ such that the tangent to $C(4)$ at the point $(0, 1, 0)$ is given by the equation $X_4 = 0$.

Before: Affine view (left) and projective view (right) of the curve $C(3)$.

After: Affine view (left) and projective view (right) of the curve $C(4)$. 
Execution of step 4: The goal of this transformation is accomplished by the linear transformation

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} h_x & 0 & h_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} \quad \text{with inverse} \quad \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -h_z \\ 0 & h_x & 0 \\ 0 & 0 & h_x \end{bmatrix} \begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix}.$$  

Plugging (36) into the equation for $C(3)$ results in the transformed cubic $C(4)$ with the equation $\sum_{i+j+k=3} \Gamma^{(4)}_{ijk} X_i^2 Y_j^2 Z_k^2 = 0$ where

$$\begin{align*}
\Gamma_{201}^{(4)} &= \Gamma_{201}^{(3)}, \\
\Gamma_{120}^{(4)} &= h_x \Gamma_{120}^{(3)}, \\
\Gamma_{111}^{(4)} &= h_x \Gamma_{111}^{(3)}, \\
\Gamma_{102}^{(4)} &= h_x \Gamma_{102}^{(3)} - 2h_z \Gamma_{201}^{(3)}, \\
\Gamma_{012}^{(4)} &= h_x^2 \Gamma_{012}^{(3)} - h_x h_z \Gamma_{111}^{(3)}, \\
\Gamma_{003}^{(4)} &= h_x^2 \Gamma_{003}^{(3)} - h_x h_z \Gamma_{102}^{(3)} + h_z^2 \Gamma_{201}^{(3)}. 
\end{align*}$$

The curve $C(4)$ is characterized by the following properties:

- the point $(1, 0, 0)$ lies on $C(4)$, i.e., $\Gamma_{300}^{(4)} = 0$;
- the tangent to $C(4)$ at $(1, 0, 0)$ is given by $Z_4 = 0$, i.e., $\Gamma_{210}^{(4)} = 0$.
- the point $(0, 1, 0)$ lies on $C(4)$, i.e., $\Gamma_{030}^{(4)} = 0$;
- the tangent to $C(4)$ at $(0, 1, 0)$ is given by $X_4 = 0$, i.e., $\Gamma_{021}^{(4)} = 0$.

In our example, we have

$$\begin{align*}
\Gamma_{300}^{(4)} &= 0, & \Gamma_{210}^{(4)} &= 0, & \Gamma_{201}^{(4)} &= -2, & \Gamma_{120}^{(4)} &= 6561, & \Gamma_{111}^{(4)} &= 5184, \\
\Gamma_{102}^{(4)} &= -4964, & \Gamma_{030}^{(4)} &= 0, & \Gamma_{021}^{(4)} &= 0, & \Gamma_{012}^{(4)} &= -4195476, \\
\Gamma_{003}^{(4)} &= -6268808.
\end{align*}$$
Step 5: In this step we will transform $C_{(4)}$ to a Weierstraß cubic $C_{(5)}$ which is characterized by the conditions that the point $(0,1,0)$ is an inflection point of $C_{(5)}$ such that the tangent to $C_{(5)}$ at $(0,1,0)$ is given by the equation $Z = 0$. In arithmetical terms, we want to eliminate the monomial $XY^2$ from the equation of $C_{(4)}$. Once this is done, the terms containing $Y$ are $Y^2Z$, $XYZ$ and $YZ^2$, so that we may split off the factor $Z$ and in the remaining quadratic polynomial then complete the square to get a term $\tilde{Y}^2$. Elimination of $XY^2$ is realized by a quadratic transformation.

Before: Affine view (left) and projective view (right) of the curve $C_{(4)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(5)}$. 
Execution of step 5: The cubic $C_{(4)}$ is given by an equation of the form

$$0 = \Gamma_{201}^{(4)}X_4^2Z_5 + \Gamma_{111}^{(4)}X_5Y_5 + \Gamma_{021}^{(4)}Y_5Z_5 + \Gamma_{003}^{(4)}X_5^2Z_5 + \Gamma_{120}^{(4)}X_4Y_4 + \Gamma_{012}^{(4)}Y_4Z_5^2 + \Gamma_{102}^{(4)}X_4Z_5^2 + \Gamma_{003}^{(4)}Z_5^3.$$  \hfill (38)

Thus letting $C_{(5)}$ be the image of $C_{(4)}$ under the quadratic transformation

$$\begin{bmatrix} X_5 \\ Y_5 \\ Z_5 \end{bmatrix} = \begin{bmatrix} X_4Z_5 \\ X_4Y_4 \\ Z_5^2 \end{bmatrix} =: \rho(X_4, Y_4, Z_4)$$  \hfill (39)

then $\rho : C_{(4)} \to C_{(5)}$ is an isomorphism whose inverse $\psi : C_{(5)} \to C_{(4)}$ is given by

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} X_5^2 \\ Y_5Z_5 \\ X_5Z_5 \end{bmatrix} =: \psi(X_5, Y_5, Z_5).$$  \hfill (40)

Note that $\rho$ is not defined at $(1, 0, 0)$ and $(0, 1, 0)$, whereas $\psi$ is not defined at $(0, 1, 0)$ and $(0, 0, 1)$. This is not a problem because of the regularity theorem formulated at the end of the introduction; however, we do not need to invoke this general result, but can explicitly write down the necessary redefinitions of $\rho$ and $\psi$; see below. Multiplying the equation (38) of the cubic $C_{(4)}$ by $X_4Z_5^2$ and substituting the transformation $\rho$ gives the cubic $C_{(5)}$ with the equation

$$0 = \Gamma_{300}^{(5)}X_5^3 + \Gamma_{201}^{(5)}X_5^2Z_5 + \Gamma_{111}^{(5)}X_5Y_5Z_5 + \Gamma_{021}^{(5)}Y_5Z_5^2 + \Gamma_{003}^{(5)}X_5^3Z_5 + \Gamma_{120}^{(5)}X_4Y_4 + \Gamma_{012}^{(5)}Y_4Z_5^2$$  \hfill (41)

where

$$\Gamma_{300}^{(5)} = \Gamma_{201}^{(4)}, \quad \Gamma_{201}^{(5)} = \Gamma_{102}^{(4)}, \quad \Gamma_{111}^{(5)} = \Gamma_{111}^{(4)}, \quad \Gamma_{102}^{(5)} = \Gamma_{003}^{(4)}, \quad \Gamma_{021}^{(5)} = \Gamma_{120}^{(4)}, \quad \Gamma_{012}^{(5)} = \Gamma_{012}^{(4)}.$$  \hfill (42)

Note that writing (41) in the form

$$0 = \Gamma_{021}^{(5)}Y_5^2Z_5 + \Gamma_{111}^{(5)}X_5Y_5Z_5 + \Gamma_{012}^{(5)}Y_5^2Z_5^2 + \Gamma_{300}^{(5)}X_5^3Z_5 + \Gamma_{201}^{(5)}X_5^2Z_5^2 + \Gamma_{102}^{(5)}X_5Z_5^2$$  \hfill (43)

shows that this is already a (general) Weierstraß equation with the additional property that $(0, 0, 1)$ lies on $C_{(5)}$. Furthermore, we have $\Gamma_{021}^{(5)} \neq 0$, since otherwise the curve would be singular. In our example, we have

$$\Gamma_{300}^{(5)} = -2, \quad \Gamma_{210}^{(5)} = 0, \quad \Gamma_{201}^{(5)} = -4964, \quad \Gamma_{120}^{(5)} = 0, \quad \Gamma_{111}^{(5)} = 5184, \quad \Gamma_{102}^{(5)} = -6268808, \quad \Gamma_{030}^{(5)} = 0, \quad \Gamma_{021}^{(5)} = 6561, \quad \Gamma_{012}^{(5)} = -4195476, \quad \Gamma_{003}^{(5)} = 0.$$  \hfill (43)
• Redefinition of $\rho$ around $(1, 0, 0)$. On $C_{(4)}$ we have

$$(\Gamma^{(4)}_{120} Y_4 + \Gamma^{(4)}_{111} Z_4) X_4 Y_4 = -(\Gamma^{(4)}_{201} X_4^2 + \Gamma^{(4)}_{102} X_4 Z_4 + \Gamma^{(4)}_{012} Y_4 Z_4 + \Gamma^{(4)}_{003} Z_4^2) Z_4.$$ 

Hence letting $\mu(X_4, Y_4, Z_4) := \Gamma^{(4)}_{120} Y_4 + \Gamma^{(4)}_{111} Z_4$, we have

$$\rho_5(X_4, Y_4, Z_4) = \mu(X_4, Y_4, Z_4) \begin{bmatrix} X_4 Z_4 \\ X_4 Y_4 \\ Z_4^2 \end{bmatrix}$$

Then

$$= \begin{bmatrix} \mu(X_4, Y_4, Z_4) & -(\Gamma^{(4)}_{201} X_4^2 + \Gamma^{(4)}_{102} X_4 Z_4 + \Gamma^{(4)}_{012} Y_4 Z_4 + \Gamma^{(4)}_{003} Z_4^2) \end{bmatrix}$$

$$= \mu(X_4, Y_4, Z_4) \begin{bmatrix} X_4 Y_4 \\ Z_4 \end{bmatrix}$$

Since $\mu(1, 0, 0) = 0$, evaluating this representation at $(1, 0, 0)$ yields $\rho(1, 0, 0) = (0, -\Gamma^{(4)}_{201}, 0) = (0, 1, 0)$.

• Redefinition of $\rho$ at $(0, 1, 0)$. On $C_4$ we also have

$$(\Gamma^{(4)}_{102} X_4 + \Gamma^{(4)}_{012} Y_4 + \Gamma^{(4)}_{003} Z_4) Z_4^2 = -(\Gamma^{(4)}_{201} X_4^2 + \Gamma^{(4)}_{102} Y_4^2 + \Gamma^{(4)}_{111} Y_4 Z_4) X_4.$$ 

Hence letting $\lambda(X_4, Y_4, Z_4) := \Gamma^{(4)}_{102} X_4 + \Gamma^{(4)}_{012} Y_4 + \Gamma^{(4)}_{003} Z_4$, we have

$$\rho_5(X_4, Y_4, Z_4) = \lambda(X_4, Y_4, Z_4) \begin{bmatrix} X_4 Z_4 \\ X_4 Y_4 \\ Z_4^2 \end{bmatrix}$$

Then

$$= \begin{bmatrix} \lambda(X_4, Y_4, Z_4) & -(\Gamma^{(4)}_{201} X_4^2 + \Gamma^{(4)}_{102} Y_4^2 + \Gamma^{(4)}_{111} Y_4 Z_4) \end{bmatrix}$$

$$= \mu(X_4, Y_4, Z_4) \begin{bmatrix} X_4 Y_4 \\ Z_4 \end{bmatrix}$$

Since $\lambda(0, 1, 0) = \Gamma^{(4)}_{012}$, this representation yields $\rho(0, 1, 0) = (0, \Gamma^{(4)}_{012}, -\Gamma^{(4)}_{120})$. 20
Redefinition of $\psi$ at $(0, 1, 0)$. On $C_{(5)}$ we have
\[
(\Gamma_{300}^{(5)}X_5 + \Gamma_{201}^{(5)}Z_5)X_5^2 = -(\Gamma_{111}^{(5)}X_5Y_5 + \Gamma_{102}^{(5)}X_5Z_5 + \Gamma_{021}^{(5)}Y_5^2 + \Gamma_{012}^{(5)}Y_5Z_5)Z_5.
\]
Hence letting $\sigma(X_5, Y_5, Z_5) := \Gamma_{300}^{(5)}X_5 + \Gamma_{201}^{(5)}Z_5$, we have
\[
\psi(X_5, Y_5, Z_5) = \sigma(X_5, Y_5, Z_5) \begin{bmatrix} X_5^2 \\ Y_5Z_5 \\ X_5Z_5 \end{bmatrix}
= \begin{bmatrix} -(\Gamma_{111}^{(5)}X_5Y_5 + \Gamma_{102}^{(5)}X_5Z_5 + \Gamma_{021}^{(5)}Y_5^2 + \Gamma_{012}^{(5)}Y_5Z_5)Z_5 \\ \sigma(X_5, Y_5, Z_5)Y_5Z_5 \\ \sigma(X_5, Y_5, Z_5)X_5Z_5 \end{bmatrix}
= Z_5 \begin{bmatrix} -(\Gamma_{111}^{(5)}X_5Y_5 + \Gamma_{102}^{(5)}X_5Z_5 + \Gamma_{021}^{(5)}Y_5^2 + \Gamma_{012}^{(5)}Y_5Z_5) \\ \sigma(X_5, Y_5, Z_5)Y_5 \\ \sigma(X_5, Y_5, Z_5)X_5 \end{bmatrix}.
\]
Since $\sigma(0, 1, 0) = 0$, this representation yields $\psi(0, 1, 0) = (-\Gamma_{021}^{(5)}, 0, 0) = (1, 0, 0)$.

Redefinition of $\psi$ at $(0, 0, 1)$. On $C_{5}$ we also have
\[
(\Gamma_{111}^{(5)}X_5 + \Gamma_{021}^{(5)}Y_5 + \Gamma_{012}^{(5)}Z_5)Y_5Z_5 = -(\Gamma_{300}^{(5)}X_5^2 + \Gamma_{201}^{(5)}X_5Z_5 + \Gamma_{102}^{(5)}Z_5^2)X_5.
\]
Hence letting $\tau(X_5, Y_5, Z_5) := \Gamma_{111}^{(5)}X_5 + \Gamma_{021}^{(5)}Y_5 + \Gamma_{012}^{(5)}Z_5$, we have
\[
\psi(X_5, Y_5, Z_5) = \tau(X_5, Y_5, Z_5) \begin{bmatrix} X_5^2 \\ Y_5Z_5 \\ X_5Z_5 \end{bmatrix}
= \begin{bmatrix} \tau(X_5, Y_5, Z_5)X_5^2 \\ -(\Gamma_{300}^{(5)}X_5^2 + \Gamma_{201}^{(5)}X_5Z_5 + \Gamma_{102}^{(5)}Z_5^2)X_5 \\ \tau(X_5, Y_5, Z_5)X_5 \end{bmatrix}
= X_5 \begin{bmatrix} \tau(X_5, Y_5, Z_5)X_5 \\ -(\Gamma_{300}^{(5)}X_5^2 + \Gamma_{201}^{(5)}X_5Z_5 + \Gamma_{102}^{(5)}Z_5^2) \\ \tau(X_5, Y_5, Z_5)X_5 \end{bmatrix}.
\]
Since $\tau(0, 0, 1) = \Gamma_{012}^{(5)}$, this representation yields $\psi(0, 0, 1) = (0, -\Gamma_{102}^{(5)}, \Gamma_{012}^{(5)})$. 

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Step 6: In this step we transform the generalized Weierstraß cubic $C_{(5)}$ into a Weierstraß cubic $C_{(6)}$, i.e., a cubic for which the only nonzero term containing the variable $Y$ is $Y^2Z$, so that the equation for $C_{(6)}$ has the form $0 = Y^2Z + P(X, Z)$ with a homogeneous polynomial $P(X, Z)$ of degree 3.

Before: Affine view (left) and projective view (right) of the curve $C_{(5)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(6)}$. 
Execution of step 6: The purpose of this step is accomplished by completing the square in the terms containing $Y$. To do so, we multiply the above equation by $4\Gamma_{021}^{(5)}$ and get the complete square $(2\Gamma_{021}^{(5)} Y + \Gamma_{111}^{(5)} X_5 + \Gamma_{012}^{(5)} Z_5)^2 Z_5$ and a polynomial in $X_5$ and $Z_5$ of degree 3. More precisely, we use the linear transformation

$$
\begin{bmatrix}
X_6 \\
Y_6 \\
Z_6
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
\Gamma_{111}^{(5)} & 2\Gamma_{021}^{(5)} & \Gamma_{012}^{(5)} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X_5 \\
Y_5 \\
Z_5
\end{bmatrix},
$$

with inverse

$$
\begin{bmatrix}
X_5 \\
Y_5 \\
Z_5
\end{bmatrix} =
\begin{bmatrix}
2\Gamma_{021}^{(5)} & 0 & 0 \\
-\Gamma_{111}^{(5)} & 1 & -\Gamma_{012}^{(5)} \\
0 & 0 & 2\Gamma_{021}^{(5)}
\end{bmatrix}
\begin{bmatrix}
X_6 \\
Y_6 \\
Z_6
\end{bmatrix}.
$$

Then the transformed cubic is given by $0 = \Gamma_{300}^{(6)} X_6^3 + \Gamma_{201}^{(6)} X_6^2 Z_6 + \Gamma_{102}^{(6)} X_6 Z_6^2 + \Gamma_{021}^{(6)} Y^2 Z_6 + \Gamma_{003}^{(6)} Z_5^3$ with

$$
\begin{align*}
\Gamma_{300}^{(6)} &= 4\Gamma_{021}^{(5)} \Gamma_{300}^{(5)}, \\
\Gamma_{201}^{(6)} &= 4\Gamma_{021}^{(5)} \Gamma_{201}^{(5)} - (\Gamma_{111}^{(5)})^2, \\
\Gamma_{102}^{(6)} &= 4\Gamma_{021}^{(5)} \Gamma_{102}^{(5)} - 2\Gamma_{111}^{(5)} \Gamma_{012}^{(5)}, \\
\Gamma_{021}^{(6)} &= 1, \\
\Gamma_{003}^{(6)} &= 4\Gamma_{021}^{(5)} \Gamma_{003}^{(5)} - (\Gamma_{012}^{(5)})^2.
\end{align*}
$$

The resulting Weierstraß cubic $C_{(6)}$ is characterized by the following properties:

- the point $(0, 1, 0)$ lies on $C_{(6)}$, i.e., $\Gamma_{030}^{(6)} = 0$;
- the tangent to $C_{(6)}$ at $(0, 1, 0)$ is given by $Z_6 = 0$, i.e., $\Gamma_{210}^{(6)} = 0$;
- the point $(0, 1, 0)$ is an inflection point, i.e., $\Gamma_{120}^{(6)} = 0$;
- the only monomial containing $Y$ is $Y^2 Z$, i.e., $\Gamma_{111}^{(6)} = 0$ and $\Gamma_{012}^{(6)} = 0$;
- the coefficient of $Y^2 Z$ is 1, i.e., $\Gamma_{021}^{(6)} = 1$.

In our example, we have

$$
\begin{align*}
\Gamma_{300}^{(6)} &= -52488, & \Gamma_{210}^{(6)} &= 0, & \Gamma_{201}^{(6)} &= -157149072, & \Gamma_{120}^{(6)} &= 0, \\
\Gamma_{111}^{(6)} &= 0, & \Gamma_{102}^{(6)} &= -121019901984, & \Gamma_{030}^{(6)} &= 0, \\
\Gamma_{021}^{(6)} &= 1, & \Gamma_{012}^{(6)} &= 0, & \Gamma_{003}^{(6)} &= -17602018866576.
\end{align*}
$$
Step 7: We transform the Weierstrass cubic $C_{(6)}$ into a Weierstrass cubic $C_{(7)}$ in normal form with the best possible reduction of the coefficients. The goal is to obtain an equation of the form $Y^2Z + P(X,Z) = 0$ such that additionally
- the coefficient $\Gamma_{300}$ of $X^3$ is $-1$;
- the coefficients $\Gamma_{201}, \Gamma_{102}, \Gamma_{003}$ do not contain a factor which can be cancelled by a scaling factor of $X$.

Before: Affine view (left) and projective view (right) of the curve $C_{(6)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(7)}$. 
We then consider the linear transformation

\begin{equation}
\varphi^2 \Gamma_{201}^{(6)}, \quad \varphi^4 \Gamma_{300}^{(6)} \Gamma_{102}^{(6)}, \quad \varphi^6 (\Gamma_{300}^{(6)} \cdot 2 \Gamma_{003}^{(6)}).
\end{equation}

We then consider the linear transformation

\begin{equation}
\begin{bmatrix}
X_7 \\
Y_7 \\
Z_7
\end{bmatrix}
= \begin{bmatrix}
\delta \phi & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & \varphi^3
\end{bmatrix}
\begin{bmatrix}
X_6 \\
Y_6 \\
Z_6
\end{bmatrix}
\quad \text{with inverse}
\begin{bmatrix}
X_6 \\
Y_6 \\
Z_6
\end{bmatrix}
= \begin{bmatrix}
\varphi^2 & 0 & 0 \\
0 & \varphi^3 & 0 \\
0 & 0 & \delta
\end{bmatrix}
\begin{bmatrix}
X_7 \\
Y_7 \\
Z_7
\end{bmatrix}
\end{equation}

Then the transformed cubic \( C(7) \) has the only nonzero coefficients

\begin{equation}
\Gamma_{201}^{(7)} = \Gamma_{201}^{(6)} / \varphi^2, \quad \Gamma_{102}^{(7)} = \delta \Gamma_{102}^{(6)} / \varphi^4, \quad \Gamma_{003}^{(7)} = \delta^2 \Gamma_{003}^{(6)} / \varphi^6.
\end{equation}

Dehomogenization with respect to the variable \( Z = Z_7 \), i.e., letting \( x = X/Z \) and \( y = Y/Z \), yields a classical Weierstraß equation of the form \( y^2 = x^3 + a_2 x^2 + a_4 x + a_6 \) where \( a_2 = -\Gamma_{201}^{(7)}, a_4 = -\Gamma_{102}^{(7)} \) and \( a_6 = -\Gamma_{003}^{(7)} \). In the general situation no further transformations are possible which lead to essential simplifications. Only if the polynomial \( x^3 + a_2 x^2 + a_4 x + a_6 \) has three rational (in fact integral) roots the equation can be further simplified, namely to the form \( y^2 = x(x + A)(x + B) \). In our example, we have

\[ \begin{align*}
\Gamma_{300}^{(7)} &= -1, \quad \Gamma_{210}^{(7)} = 0, \quad \Gamma_{201}^{(7)} = -5988, \quad \Gamma_{120}^{(7)} = 0, \quad \Gamma_{111}^{(7)} = 0, \\
\Gamma_{102}^{(7)} &= -9 222 672, \quad \Gamma_{030}^{(7)} = 0, \quad \Gamma_{021}^{(7)} = 1, \quad \Gamma_{012}^{(7)} = 0, \quad \Gamma_{003}^{(7)} = -2 682 825 616.
\end{align*} \]

This completes the transformation. Let us review what happens to the distinguished rational point on the intersection of quadrics we started with. This point is first mapped to the point \( z \in C(0) \) given by formula (16) above. Next, the point \( z \) is mapped to \((1,0,0) \in C(1)\), and this point is mapped to itself in the transformation from \( C(1) \) to \( C(2) \). If \((1,0,0)\) happens to be an inflection point of \( C(2) \) we apply a coordinate exchange which maps \((1,0,0)\) to \((0,1,0)\); in the generic case, the point \((1,0,0)\) remains fixed both during the transition from \( C(2) \) to \( C(3) \) and during the transition from \( C(3) \) to \( C(4) \) and is then mapped to \((0,1,0) \in C(5) \) by the subsequent quadratic transformation. Thus in all cases, the original distinguished point is mapped to \((0,1,0) \in C(5) \), and this point stays fixed under the remaining transformations.
4. Example: Euler’s concordant forms

As a first example, let us apply the above theory to the intersection \( Q_{M,N} \) of the two quadrics \( X^2 + MY^2 = Z^2 \) and \( X^2 + NY^2 = W^2 \) where \( M \neq N \) are nonzero integers; according to Euler, the numbers \( M \) and \( N \) are called concordant if \( Q_{M,N} \) possesses a rational point which is nontrivial in the sense that \( Y \neq 0 \). As a consequence of the previous discussion, the quadric intersection \( Q_{M,N} \) is isomorphic to a plane elliptic curve \( E_{M,N} \) given by a Weierstraß equation. Note that in [3] and [7] the authors use a mapping of degree 4 instead of a biregular morphism; this is somewhat surprising, since in this way some information on torsion points is lost and since, as we shall see, the biregular morphism is given by a rather simple linear mapping. Also note that in [8] a linear isomorphism from \( Q_{M,N} \) to a smooth plane cubic is given which is rather similar to the first one of our curves, but no transformation of this curve to Weierstraß form is carried out. To apply the theory developed in the previous paragraph we write \((X_0, X_1, X_2, X_3) := (Y, X, Z, W)\) and thus consider the quadrics \( Q_1 \) given by \( MX_0^2 + X_1^2 - X_2^2 = 0 \) and \( Q_2 \) given by \( NX_0^2 + X_1^2 - X_3^2 = 0 \). This corresponds to the equations (5) with \( A = \text{diag}(M, 1, -1, 0) \) and \( B = \text{diag}(N, 1, 0, -1) \). The intersection \( Q_1 \cap Q_2 \) contains the four trivial rational points \((0, 1, \pm 1, \pm 1)\); we choose \( x := (0, 1, 1, 1) \) as the distinguished rational point. We carry out the procedure described in the previous paragraph and visualize the various steps by images generated for the special case \((M, N) = (3, 2)\). The isomorphism used in Section 2 maps \( Q_1 \cap Q_2 \) to the plane cubic whose nonzero coefficients are

\[
\Gamma^{(0)}_{210} = N - M,
\Gamma^{(0)}_{201} = -N,
\Gamma^{(0)}_{021} = -1,
\Gamma^{(0)}_{012} = 1,
\]

and \( \varphi \) maps \( x = (0, 1, 1, 1) \) to \( y = (1, 0, 0) \). In the special case \((M, N) = (3, 2)\) this means that

\[
\Gamma_{210}^{(0)} = -1, \quad \Gamma_{201}^{(0)} = -2, \quad \Gamma_{021}^{(0)} = -1, \quad \Gamma_{012}^{(0)} = 1.
\]
**Left:** Affine view $X_3 = 1$ of the intersection of the quadrics $MX_0^2 + X_1^2 = X_2^2$ and $NX_0^2 + X_1^2 = X_3^2$. **Right:** Affine view of the cubic $C_{(0)}$ to which this quadric intersection is initially transformed.
Step 1: This step is superfluous, because the distinguished point is already $(1,0,0)$. Hence $C_{(1)} = C_{(0)}$.

Step 2: The tangent of the curve $C_{(0)} = C_{(1)}$ at the point $y = (1,0,0)$ is given by $0 = (N - M)Y_1 - NZ_1$. The curve $C_{(2)}$ has the nonzero coefficients $\Gamma_{201}^{(2)} = N^2$, $\Gamma_{030}^{(2)} = M(M - N)$, $\Gamma_{021}^{(2)} = 2M - N$ and $\Gamma_{012}^{(2)} = 1$. In the special case $(M, N) = (3,2)$ this means that $\Gamma_{201}^{(2)} = -4$, $\Gamma_{030}^{(2)} = -3$, $\Gamma_{021}^{(2)} = -4$ and $\Gamma_{012}^{(2)} = -1$.

Before: Affine view (left) and projective view (right) of the curve $C_{(0)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(2)}$. 
**Steps 3, 4 and 5:** These steps are superfluous, because the distinguished point \( p = (1, 0, 0) \) is already an inflection point; simply exchanging \( X_2 \) and \( Y_2 \) already yields a general Weierstraß equation with the nonzero coefficients \( \Gamma^{(5)}_{300} = \Gamma^{(2)}_{030} = M(M - N) \), \( \Gamma^{(5)}_{201} = \Gamma^{(2)}_{021} = 2M - N \), \( \Gamma^{(5)}_{102} = \Gamma^{(2)}_{012} = 1 \) and \( \Gamma^{(5)}_{021} = \Gamma^{(2)}_{201} = N^2 \). In the special case \((M, N) = (3, 2)\) this means that \( \Gamma^{(5)}_{300} = -3 \), \( \Gamma^{(5)}_{201} = -4 \), \( \Gamma^{(5)}_{102} = -1 \) and \( \Gamma^{(5)}_{021} = -4 \).

**Before:** Affine view (left) and projective view (right) of the curve \( C_{(2)} \).

**After:** Affine view (left) and projective view (right) of the curve \( C_{(5)} \).
Step 6: The curve $C_{(6)}$ has the nonzero coefficients $\Gamma_{300}^{(6)} = 4N^2M(M - N)$, $\Gamma_{201}^{(6)} = 4N^2(2M - N)$, $\Gamma_{021}^{(6)} = 1$ and $\Gamma_{102}^{(6)} = 4N^2$. In the special case $(M, N) = (3, 2)$ this means that $\Gamma_{300}^{(6)} = 48$, $\Gamma_{201}^{(6)} = 64$, $\Gamma_{102}^{(6)} = 16$ and $\Gamma_{021}^{(6)} = 1$.

Before: Affine view (left) and projective view (right) of the curve $C_{(5)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(6)}$. 
**Step 7:** We have $\delta = -4N^2M(M - N)$ and $\varphi = 2N$. Hence the curve $C_{(7)}$ has the nonzero coefficients $\Gamma_{300}^{(7)} = -1$, $\Gamma_{201}^{(7)} = 2M - N$, $\Gamma_{102}^{(7)} = -M(M - N)$ and $\Gamma_{021}^{(7)} = 1$ and hence, letting $(X, Y, Z) := (X_7, Y_7, Z_7)$ is given by the equation $Y^2 Z = X^3 - (2M - N)X^2 Z + M(M - N)XZ^2 = X(X - M)(X - (M - N)Z)$. In affine form, this reads $y^2 = x(x - M)(x - (M - N))$. In the special case $(M, N) = (3, 2)$ this means that $\Gamma_{300}^{(7)} = -1$, $\Gamma_{201}^{(7)} = 4$, $\Gamma_{102}^{(7)} = -3$ and $\Gamma_{021}^{(7)} = 1$, and the curve $C_{(7)}$ is in affine form given by $y^2 = x(x - 3)(x - 1)$.

**Before:** Affine view (left) and projective view (right) of the curve $C_{(6)}$.

**After:** Affine view (left) and projective view (right) of the curve $C_{(7)}$. 
We observe that the concatenation of the above coordinate transformations is given by

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} =
\begin{bmatrix}
0 & -M(M-N) & 0 & M(M-N) \\
MN(M-N) & 0 & 0 & 0 \\
0 & -(M-N) & -N & M \\
0 & -(M-N) & -N & M
\end{bmatrix}
\begin{bmatrix}
X_0 \\
X_1 \\
X_2 \\
X_3
\end{bmatrix}.
\]

Thus the isomorphism \( \Phi \) from the quadric intersection \( Q_{M,N} \) to the elliptic curve \( E_{M,N} \) with the affine equation \( y^2 = x(x-M)(x-(M-N)) \) is accomplished by a simple linear map. The original quadric intersection \( Q_{M,N} \) possesses four trivial points with the projective coordinates \((0,1,\pm 1,\pm 1)\). These are mapped by \( \Phi \) to four rational points on the curve \( E_{M,N} \), namely

\[
\begin{align*}
\Phi(0,1,1,1) & = (0,1,0) \quad (\text{point at infinity}), \\
\Phi(0,1,1,-1) & = (M-N, 0,1), \\
\Phi(0,1,-1,1) & = (0, 0, 1), \\
\Phi(0,1,-1,-1) & = (M, 0, 1).
\end{align*}
\]

Not surprisingly, these are exactly the 2-torsion points on \( E_{M,N} \).
5. Example: Rational squares in arithmetic progression

In the special situation of the \((k, \ell, m)\)-problem which was one of the motivations of this paper mentioned in the introduction, we have \(A = \text{diag}(k + \ell, -k, -\ell, 0)\), \(B = \text{diag}(-m, m + \ell, 0, -\ell)\) and \(x = (1, 1, 1, 1)\). The cubic curve \(C(0)\) to which the quadric intersection \(Q_1 \cap Q_2\) is transformed is given by \(\sum_{i+j+k=3} \Gamma_{ijk} X^i Y^j Z^k = 0\) where the only nonzero coefficients \(\Gamma_{ijk} = \Gamma_{ijk}^{(0)}\) are as follows:

\[
\begin{align*}
\Gamma_{210}^{(0)} &= -(k + \ell + m), \\
\Gamma_{201}^{(0)} &= m, \\
\Gamma_{120}^{(0)} &= k + \ell + m, \\
\Gamma_{102}^{(0)} &= -m, \\
\Gamma_{021}^{(0)} &= -(\ell + m), \\
\Gamma_{012}^{(0)} &= \ell + m.
\end{align*}
\]

The distinguished rational point is \(p^{(0)} = (\ell + m, m, k + \ell + m)\). We now follow the general procedure described in the previous paragraph, exemplifying all results for the case \((k, \ell, m) = (2, 3, 5)\) in which we have the coefficients

\[
\begin{align*}
C_{300} &= 0, & C_{210} &= -10, & C_{201} &= 5, & C_{120} &= 10, & C_{111} &= 0, \\
C_{102} &= -5, & C_{030} &= 0, & C_{021} &= -8, & C_{012} &= 8, & C_{003} &= 0
\end{align*}
\]

and the distinguished rational point \((8, 5, 10)\).

Left: Affine view \(X_3 = 1\) of the intersection of the quadrics \((k + \ell)X_0^2 - kX_1^2 - \ell X_2^2 = 0\) and \(-mX_0^2 + (m + \ell)X_1^2 - \ell X_3^2 = 0\), shown for \((k, \ell, m) = (2, 3, 5)\). Right: Affine view of the cubic \(C(0)\) to which this quadric intersection is initially transformed.
Step 1: Translating the distinguished rational point to \((1, 0, 0)\).

Before: Affine view (left) and projective view (right) of the curve \(C_{(0)}\).

After: Affine view (left) and projective view (right) of the curve \(C_{(1)}\).
The curve $C_{(1)}$ has the nonzero coefficients

\[
\begin{align*}
\Gamma_{210}^{(1)} &= k(k + \ell + m), \\
\Gamma_{201}^{(1)} &= \ell m, \\
\Gamma_{111}^{(1)} &= 2(k + \ell)(\ell + m), \\
\Gamma_{021}^{(1)} &= -(\ell + m)^2, \\
\Gamma_{012}^{(1)} &= (\ell + m)^2.
\end{align*}
\]

In the special case $(k, \ell, m) = (2, 3, 5)$ this yields

\[
\begin{align*}
\Gamma_{300}^{(1)} &= 0, & \Gamma_{210}^{(1)} &= 20, & \Gamma_{201}^{(1)} &= 15, & \Gamma_{120}^{(1)} &= 0, & \Gamma_{111}^{(1)} &= 80, \\
\Gamma_{102}^{(1)} &= 0, & \Gamma_{030}^{(1)} &= 0, & \Gamma_{021}^{(1)} &= -64, & \Gamma_{012}^{(1)} &= 64, & \Gamma_{003}^{(1)} &= 0.
\end{align*}
\]

The distinguished rational point after this step is $(1, 0, 0)$. 
Step 2: Adapting the tangent of the distinguished rational point \((1, 0, 0)\).

Before: Affine view (left) and projective view (right) of the curve \(C_{(1)}\).

After: Affine view (left) and projective view (right) of the curve \(C_{(2)}\).
The curve $C_{(2)}$ has the nonzero coefficients

\[
\begin{align*}
\Gamma_{201}^{(2)} &= \ell^2 m^2, \\
\Gamma_{120}^{(2)} &= -2k\ell m(k + \ell)(\ell + m)(k + \ell + m), \\
\Gamma_{111}^{(2)} &= 2\ell m(k + \ell)(\ell + m), \\
\Gamma_{030}^{(2)} &= k(k + \ell)(k + m)(k + \ell + m)(\ell + m)^2, \\
\Gamma_{021}^{(2)} &= -(\ell + m)^2(2k(k + \ell + m) + \ell m), \\
\Gamma_{012}^{(2)} &= (\ell + m)^2.
\end{align*}
\]

The second point of intersection of $C_{(2)}$ with the tangent $Z_2 = 0$ is

\[
p^{(2)} = (k\ell + (k + \ell + m)m, 2\ell m, 0).
\]

In our example we have

\[
\begin{align*}
\Gamma_{300}^{(2)} &= 0, & \Gamma_{210}^{(2)} &= 0, & \Gamma_{201}^{(2)} &= 45, & \Gamma_{120}^{(2)} &= -960, & \Gamma_{111}^{(2)} &= 240, \\
\Gamma_{102}^{(2)} &= 0, & \Gamma_{030}^{(2)} &= 1792, & \Gamma_{021}^{(2)} &= -704, & \Gamma_{012}^{(2)} &= 64, & \Gamma_{003}^{(2)} &= 0
\end{align*}
\]

and $p^{(2)} = (28, 15, 0)$. 
Step 3: Arranging $(0,1,0)$ to be the second intersection point of the tangent at $(1,0,0)$ with the cubic.

Before: Affine view (left) and projective view (right) of the curve $C_{(2)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(3)}$. 
The curve $C_{(3)}$ has the nonzero coefficients

\[
\begin{align*}
\Gamma_{201}^{(3)} &= 1, \\
\Gamma_{120}^{(3)} &= 4k(k + \ell)(\ell + m)(k + \ell + m), \\
\Gamma_{111}^{(3)} &= -2(\ell + m)(3k + 2\ell + m), \\
\Gamma_{021}^{(3)} &= -(\ell + m)^2(3k^2 + 4k\ell + 2km - m^2), \\
\Gamma_{012}^{(3)} &= 4(\ell + m)^2.
\end{align*}
\]

In our example this yields

\[
\begin{align*}
\Gamma_{300}^{(3)} &= 0, & \Gamma_{210}^{(3)} &= 0, & \Gamma_{201}^{(3)} &= 1, & \Gamma_{120}^{(3)} &= 320, & \Gamma_{111}^{(3)} &= -136, \\
\Gamma_{102}^{(3)} &= 0, & \Gamma_{030}^{(3)} &= 0, & \Gamma_{021}^{(3)} &= -496, & \Gamma_{012}^{(3)} &= 320, & \Gamma_{003}^{(3)} &= 0.
\end{align*}
\]
**Step 4:** Adapting the tangent at the point \((0,1,0)\).

*Before:* Affine view (left) and projective view (right) of the curve \(C_{(3)}\).

*After:* Affine view (left) and projective view (right) of the curve \(C_{(4)}\).
The curve $C_{(4)}$ has the nonzero coefficients

\begin{align*}
\Gamma^{(4)}_{201} &= 1, \\
\Gamma^{(4)}_{120} &= 16(\ell + m)k^2(k + \ell)^2(k + \ell + m)^2, \\
\Gamma^{(4)}_{111} &= -8k(k + \ell)(\ell + m)(k + \ell + m)(3k + 2\ell + m), \\
\Gamma^{(4)}_{102} &= 2(\ell + m)(3k^2 + 4k\ell + 2km - m^2), \\
\Gamma^{(4)}_{012} &= -8k(k - m)(k + m)(k + \ell)(k + \ell + m)(k + 2\ell + m)(\ell + m)^2, \\
\Gamma^{(4)}_{003} &= (\ell + m)^2(3k^2 + 4k\ell + 2km - m^2)^2.
\end{align*}

In our example this means

\begin{align*}
\Gamma^{(4)}_{300} &= 0, \quad \Gamma^{(4)}_{210} = 0, \quad \Gamma^{(4)}_{201} = 1, \quad \Gamma^{(4)}_{120} = 6400, \quad \Gamma^{(4)}_{111} = -2720, \\
\Gamma^{(4)}_{102} &= 62, \quad \Gamma^{(4)}_{030} = 0, \quad \Gamma^{(4)}_{021} = 0, \quad \Gamma^{(4)}_{012} = 43680, \quad \Gamma^{(4)}_{003} = 961.
\end{align*}
Step 5: Transformation to a general Weierstraß cubic.

Before: Affine view (left) and projective view (right) of the curve $C_{(4)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(5)}$. 
The curve \( C_{(5)} \) has the nonzero coefficients

\[
\begin{align*}
\Gamma_{300}^{(5)} &= \Gamma_{201}^{(4)} = 1, \\
\Gamma_{201}^{(5)} &= \Gamma_{102}^{(4)} = 2(\ell + m)(3k^2 + 4k\ell + 2km - m^2), \\
\Gamma_{111}^{(5)} &= \Gamma_{111}^{(4)} = -8k(k + \ell)(\ell + m)(k + \ell + m)(3k + 2\ell + m), \\
\Gamma_{102}^{(5)} &= \Gamma_{003}^{(4)} = (\ell + m)^2(3k^2 + 4k\ell + 2km - m^2)^2, \\
\Gamma_{021}^{(5)} &= \Gamma_{120}^{(4)} = 16k^2(k + \ell)^2(k + \ell + m)^2(\ell + m), \\
\Gamma_{012}^{(5)} &= \Gamma_{012}^{(4)} = -8k(k - m)(k + m)(k + \ell)(k + \ell + m)(k + 2\ell + m)(\ell + m)^2.
\end{align*}
\]

In our example this means

\[
\begin{align*}
\Gamma_{300}^{(5)} &= 1, & \Gamma_{210}^{(5)} &= 0, & \Gamma_{201}^{(5)} &= 62, & \Gamma_{120}^{(5)} &= 0, & \Gamma_{111}^{(5)} &= -2720, & \Gamma_{102}^{(5)} &= 961, \\
\Gamma_{030}^{(5)} &= 0, & \Gamma_{021}^{(5)} &= 6400, & \Gamma_{012}^{(5)} &= 43680, & \Gamma_{003}^{(5)} &= 0.
\end{align*}
\]
Step 6: Transformation to a special Weierstraß cubic.

Before: Affine view (left) and projective view (right) of the curve $C_{(5)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(6)}$. 
The curve $C_{(6)}$ has the nonzero coefficients

$$
\Gamma_{300}^{(6)} = 64k^2(k + \ell)^2(k + \ell + m)^2(\ell + m),
\Gamma_{201}^{(6)} = 64k^2(k + \ell)^2(k + \ell + m)^2(\ell + m)^2(3k^2 + 4k\ell + 4\ell^2 + 2km + 4\ell m + 3m^2),
\Gamma_{102}^{(6)} = 64k^2(k + \ell)^2(k + \ell + m)^2(\ell + m)^3 \cdot (3k^4 + 8k^3\ell + 8k^2\ell^2 \cdots \\
\cdots + 4k^3m + 8k^2\ell m + 2k^2m^2 + 8k\ell m^2 + 8\ell^2m^2 + 4km^3 + 8\ell m^3 + 3m^4),
\Gamma_{021}^{(6)} = 1,
\Gamma_{003}^{(6)} = -64k^2(k - m)^2(k + m)^2(k + \ell)^2(k + \ell + m)^2(k + 2\ell + m)^2(\ell + m)^4.
$$

In our example we have

$$
\Gamma_{300}^{(6)} = 25600, \quad \Gamma_{210}^{(6)} = 0, \quad \Gamma_{201}^{(6)} = -5\,811\,200,
\Gamma_{120}^{(6)} = 0, \quad \Gamma_{111}^{(6)} = 0, \quad \Gamma_{102}^{(6)} = 262\,220\,800,
\Gamma_{030}^{(6)} = 0, \quad \Gamma_{021}^{(6)} = 1, \quad \Gamma_{012}^{(6)} = 0, \quad \Gamma_{003}^{(6)} = -1\,907\,942\,400.
$$
Step 7: Transformation to a Weierstraß cubic in normal form.

Before: Affine view (left) and projective view (right) of the curve $C_{(6)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(7)}$. 
In our situation we have $\delta = -64(\ell + m)k^2(k + \ell)^2(k + \ell + m)^2$ and $\varphi = 8(\ell + m)k(k + \ell)(k + \ell + m)$, and the curve $C_{(7)}$ has the nonzero coefficients

\[ \Gamma^{(7)}_{300} = -1, \]
\[ \Gamma^{(7)}_{201} = -(3k^2 + 4k\ell + 4\ell^2 + 2km + 4\ell m + 3m^2), \]
\[ \Gamma^{(7)}_{102} = -(3k^4 + 8k^3\ell + 8k^2\ell^2 + 4k^3m + 8k^2\ell m + 2k^2m^2 + 8k\ell m^2 \cdots + 8\ell^2m^2 + 4km^3 + 8\ell m^3 + 3m^4), \]
\[ \Gamma^{(7)}_{021} = 1, \]
\[ \Gamma^{(7)}_{003} = -(k - m)^2(k + m)^2(k + 2\ell + m)^2. \]

In our example this means

\[ \Gamma^{(7)}_{300} = -1, \quad \Gamma^{(7)}_{210} = 0, \quad \Gamma^{(7)}_{201} = -227, \quad \Gamma^{(7)}_{120} = 0, \quad \Gamma^{(7)}_{111} = 0, \quad \Gamma^{(7)}_{102} = -10243, \]
\[ \Gamma^{(7)}_{030} = 0, \quad \Gamma^{(7)}_{021} = 1, \quad \Gamma^{(7)}_{012} = 0, \quad \Gamma^{(7)}_{003} = -74529. \]
Step 8. In our special situation we can do more than in the general case, since the cubic polynomial in $X_7$ and $Z_7$ splits over the rationals. One easily computes the roots of this polynomial to be $-(k-m)^2$, $-(k+m)^2$ and $-(k+2\ell+m)^2$. The simple substitution $x = \hat{x} - (k-m)^2$ and a further simple substitution to eliminate a common factor 4 in the roots yields the equation

$$\hat{y}^2 = \hat{x}(\hat{x} + km)(\hat{x} + (k + \ell + m)).$$

Before: Affine view (left) and projective view (right) of the curve $C_{(7)}$.

After: Affine view (left) and projective view (right) of the curve $C_{(8)}$.  

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The equation obtained in the previous step has the affine form \( y^2 = p(x) \) where \( p \) has the roots \( -(k-m)^2, -(k+m)^2 \) and \( -(k+2\ell+m)^2 \). The simple substitution \( x = \tilde{x} - (k-m)^2 \) and a further simple substitution to eliminate the common factor 4 in the roots yields the equation \( \tilde{y}^2 = \tilde{x}(\tilde{x}+km)(\tilde{x}+(k+\ell)(\ell+m)) \). Thus we see that the problem of finding rational squares in an arithmetic progression of type \((k, \ell, m)\) is equivalent to finding rational points on the curve \( E_{k,\ell,m} \) given by the Weierstraß equation

\[
(*) \quad y^2 = x(x + km)(x + (k + \ell)(\ell + m)).
\]

In projective form, this curve is given by the homogeneous equation

\[
Y^2Z = X(X + kmZ)(X + (k + \ell)(\ell + m)Z)
\]

where \( x = X/Z \) and \( y = Y/Z \). Thus in our example we end up with the coefficients

\[
\Gamma^{(8)}_{300} = -1, \quad \Gamma^{(8)}_{210} = 0, \quad \Gamma^{(8)}_{201} = -50, \quad \Gamma^{(8)}_{120} = 0, \quad \Gamma^{(8)}_{111} = 0,
\]

\[
\Gamma^{(8)}_{102} = -400, \quad \Gamma^{(8)}_{030} = 0, \quad \Gamma^{(8)}_{021} = 1, \quad \Gamma^{(8)}_{012} = 0, \quad \Gamma^{(8)}_{003} = 0.
\]

From (*) one recognizes the remarkable fact that the torsion group of the resulting curve contains \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) as a subgroup. Moreover, the final coefficients are surprisingly small, due to the cancellation of the terms which was possible in Step 7 (with rather large factors \( \delta \) and \( \varphi \)).
The composition of the above transformations results in an overall transformation $\Phi : \mathbb{P}^3 \to \mathbb{P}^2$, say $\Phi(x_0, x_1, x_2, x_3) = (X, Y, Z)$, which induces a birational (thus a fortiori biregular) mapping of the quadric intersection $Q_{k,\ell,m}$ to the elliptic curve $E_{k,\ell,m}$. This transformation $\Phi$ is represented by a quadratic transformation, defined by

$$X = \sum_{i,j} X_{ij} x_i x_j, \quad Y = \sum_{i,j} Y_{ij} x_i x_j, \quad Z = \sum_{i,j} Z_{ij} x_i x_j$$

where the sums are formed over all indices $0 \leq i, j \leq 3$ such that $i \leq j$ and where the coefficients are given by

$$
\begin{align*}
X_{00} &= -km^2(k + \ell)^2(\ell + m)(k + \ell + m)^2, \\
X_{01} &= km(k + \ell)(\ell + m)(k + \ell + m)^2(k\ell + 2km + \ell m), \\
X_{02} &= -k\ell m^2(k + \ell)(\ell - m)(\ell + m)(k + \ell + m), \\
X_{03} &= -k\ell m(k - m)(k + \ell)^2(\ell + m)(k + \ell + m), \\
X_{11} &= -k^2 m(k + \ell)(\ell + m)^2(k + \ell + m)^2, \\
X_{12} &= k\ell m(k + \ell)(k - m)(\ell + m)^2(k + \ell + m), \\
X_{13} &= k^2 m\ell(k + \ell)(k - m)(\ell + m)(k + \ell + m), \\
X_{22} &= k\ell^2 m^2(k + \ell)(\ell + m)^2, \\
X_{23} &= -k\ell^2 m(k + \ell)(\ell + m)(k^2 + k\ell + \ell m + m^2), \\
X_{33} &= k^2 \ell^2 m(k + \ell)^2(\ell + m), \\
\end{align*}
$$

by

$$
\begin{align*}
Y_{00} &= k\ell m^2(k + \ell)^2(\ell + m)(k + \ell + m)^2, \\
Y_{01} &= k\ell^2 m(k + \ell)(k - m)(\ell + m)(k + \ell + m)^2, \\
Y_{02} &= -k^2 \ell m^2(k + \ell)(\ell + m)(k + \ell + m)(k + 2\ell + m), \\
Y_{03} &= -k\ell m(k + \ell)^2(\ell + m)(\ell + m)^2(k + \ell + m), \\
Y_{11} &= -k^2 \ell m(k + \ell)(\ell + m)^2(k + \ell + m)^2, \\
Y_{12} &= k\ell m(k + \ell)^2(k + m)(\ell + m)^2(k + \ell + m), \\
Y_{13} &= k^2 \ell^2 m(k + \ell)(\ell + m)(k + \ell + m)(k + 2\ell + m), \\
Y_{22} &= -k\ell^2 m^2(k + \ell)(\ell + m)^2(k + \ell + m), \\
Y_{23} &= -k\ell^2 m(k + \ell)(k - m)(\ell + m)(k + \ell + m)^2, \\
Y_{33} &= k^2 \ell^2 m(k + \ell)^2(\ell + m)(k + \ell + m).
\end{align*}
$$
and by

\[
\begin{align*}
Z_{00} &= m^2(k + \ell)^2(k + \ell + m)^2, \\
Z_{01} &= -2km(k + \ell)(\ell + m)(k + \ell + m)^2, \\
Z_{02} &= -2\ell^2m(\ell + m)(k + \ell + m), \\
Z_{03} &= 2k\ell m(k + \ell)^2(k + \ell + m), \\
Z_{11} &= k^2(\ell + m)^2(k + \ell + m)^2, \\
Z_{12} &= 2k\ell m(\ell + m)(k + \ell + m), \\
Z_{13} &= -2k^2\ell(k + \ell)(\ell + m)(k + \ell + m), \\
Z_{22} &= \ell^2m^2(\ell + m)^2, \\
Z_{23} &= -2k\ell^2m(k + \ell)(\ell + m), \\
Z_{33} &= k^2\ell^2(k + \ell)^2.
\end{align*}
\]

The original quadric \(Q_{k,\ell,m}\) possesses eight trivial points with coordinates \((1, \pm 1, \pm 1, \pm 1)\). These are mapped to eight rational points on the elliptic curve \(E_{k,\ell,m}\) via the transformation \(\Phi\). These points are given in projective form \((X, Y, T)\) with integer coordinates as follows:

\[
\begin{align*}
\Phi(1, 1, 1, 1) &= (0, 1, 0) \quad \text{(point at infinity)}, \\
\Phi(1, 1, 1, -1) &= (m(\ell + m), m(\ell + m)(k + \ell + m), 1), \\
\Phi(1, 1, -1, 1) &= (k(k + \ell), -k(k + \ell)(k + \ell + m), 1), \\
\Phi(1, 1, -1, -1) &= (0, 0, 1) \quad \text{(2-torsion point)}, \\
\Phi(1, -1, 1, 1) &= (-m(k + \ell), -m\ell(k + \ell), 1), \\
\Phi(1, -1, 1, -1) &= (-k(\ell + m), 0, 1) \quad \text{(2-torsion point)}, \\
\Phi(1, -1, -1, 1) &= (-km, 0, 1) \quad \text{(2-torsion point)}, \\
\Phi(1, -1, -1, -1) &= (-k(\ell + m), k\ell(\ell + m), 1).
\end{align*}
\]
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