Non-Perturbative Renormalization Group Analysis of the Ohmic Quantum Dissipation

Ken-Ichi Aoki∗ and Atsushi Horikoshi †

Institute for Theoretical Physics, Kanazawa University, Kakuma-machi Kanazawa 920-1192, Japan

November 13, 2018

Abstract

We analyze quantum tunneling with the Ohmic dissipation by the non-perturbative renormalization group method. We calculate the localization susceptibility to evaluate the critical dissipation for the quantum-classical transition, and find considerably larger critical dissipation compared to the previous semi-classical arguments.

PACS numbers: 11.10.Hi, 03.65.-w, 73.40.Gk

1 Introduction

Caldeira and Leggett proposed a method to derive the quantum dissipative behavior from a microscopic theory, which consists of the target system and the environment [1]. The action of their model is written as follows,

\[
S[q, \{x_\alpha\}] = \int dt \left\{ \frac{1}{2} \dot{M}q^2 - V_0(q) + \sum_\alpha \left[ \frac{1}{2} m_\alpha \dot{x}_\alpha^2 - \frac{1}{2} m_\alpha \omega_\alpha^2 x_\alpha^2 \right] - q \sum_\alpha C_\alpha x_\alpha \right\},
\]

where \(q(t)\) is the variable of the target system in a potential \(V_0(q)\), and \(x_\alpha(t)\) is the harmonic oscillators representing the environment. The target system is coupled linearly to each oscillator with strength \(C_\alpha\). If we set the parameters \(m_\alpha, \omega_\alpha (> 0), C_\alpha\) in a suitable way, the dissipative term (for the Ohmic dissipation, \(\dot{q}\)) arises in the effective classical equation of motion of \(q(t)\) after elimination of the environmental variables with the proper boundary condition.

∗e-mail : aoki@hep.s.kanazawa-u.ac.jp
†Present address: Japan Science and Technology Corporation, and Department of Chemistry, Faculty of Science, Nara Women’s University, Nara 630-8506, Japan; e-mail : horikosi@cc.nara-wu.ac.jp
In this letter, we study the quantum mechanics of this system by the Euclidean path integral over \( q(\tau) \) and \( x_\alpha(\tau) \). We integrate the variable \( x_\alpha \) to define the effective action for the target system,

\[
Z = \frac{1}{N} \int Dq \prod_\alpha \int Dx_\alpha e^{-\frac{i}{\hbar} S_E} = \frac{1}{N_q} \int Dq e^{-\frac{i}{\hbar} \int d\tau \left[ \frac{1}{2} Mq'^2 + V_0(q) \right] - \frac{i}{\hbar} \Delta S[q]},
\]

where \( S_E \) is the Euclidean action

\[
S_E[q, \{ x_\alpha \}] = \int d\tau \left\{ \frac{1}{2} Mq'^2 + V_0(q) + \sum_\alpha \left[ \frac{1}{2} m_\alpha \dot{x}_\alpha^2 + \frac{1}{2} m_\alpha \omega_\alpha^2 x_\alpha^2 \right] + q \sum_\alpha C_\alpha x_\alpha \right\},
\]

\( \mathcal{N} \) and \( \mathcal{N}_q \) are normalization constants and \( \Delta S[q] \) is a term generated by the quantum effects of environment \( x_\alpha \). Caldeira and Leggett studied the influence of \( \Delta S \) on the quantum tunneling of the target system \( q \) and they found that the quantum tunneling is suppressed by the Ohmic dissipation [1]. Following their analysis, many theoretical works have been done and the localization (quantum-classical) transition has been suggested to occur [2, 3, 4, 5, 6, 7, 8]. The validity of their results have been suggested by elaborate experiments [9]. However, the theoretical works so far (instanton, perturbation, etc.) depend on some expansions with respect to small parameters which are not always valid.

For example, for evaluation of the tunneling rate through the double well barrier,

\[
V_0(q) = -\frac{1}{2} M\omega_0^2 q^2 + \lambda_0 q^4,
\]

the dilute gas instanton calculation is valid only in \( \lambda_0 \rightarrow 0 \) region [10].

In this letter we adopt the non-perturbative renormalization group (NPRG) method [11, 12, 13] to analyze the dissipative quantum tunneling in the Caldeira-Leggett model. Since the NPRG is formulated without any series expansion, it is a powerful tool to analyze the non-perturbative features of quantum field theory, such as the phase structure of the theory [14]. For the quantum tunneling problem, the NPRG method can reproduce the exact tunneling rate through the double well barrier (Eq. (4)) in the wide parameter \( \lambda_0 \) region [10, 15, 16]. Therefore the NPRG is expected to be effective in the non-perturbative analysis of the dissipative quantum tunneling. We can discuss the possibility of the localization (quantum-classical) transition by investigating the “phase structure” of the dissipative quantum mechanics. The NPRG also helps us to interpret the effects of \( \Delta S \) as effective infrared or ultraviolet cutoffs for the quantum fluctuation, and we can readily understand the dissipation effects as suppression or enhancement of the quantum features of the effective target system [17].

## 2 Dissipative effective action

The effective interaction \( \Delta S \) defined in Eq. (2) takes the following form,

\[
\Delta S[q] = - \int d\tau \int ds \; q(\tau) \; \alpha(\tau - s) \; q(s).
\]

The non-local coupling coefficient is given by

\[
\alpha(\tau - s) = \int \frac{d\omega}{2\pi} \sum_\alpha \frac{C_\alpha^2}{2m_\alpha \omega^2 + \omega_\alpha^2} \; e^{i\omega(\tau - s)}
\]

\[
= \int_0^\infty \frac{d\omega}{2\pi} \; J(\omega) \; e^{-\omega|\tau - s|},
\]

where

\[
J(\omega) = \frac{\omega_\alpha^2}{2m_\alpha \omega^2 + \omega_\alpha^2} \; e^{i\omega(\tau - s)}
\]

and

\[
I(\omega) = \int_0^\infty \frac{d\omega'}{2\pi} \; J(\omega') \; e^{-\omega'|\tau - s'|},
\]

are the non-local coupling coefficients. The integral equations for \( I(\omega) \) and \( J(\omega) \) can be solved numerically.
where we have introduced the spectral density function $J(\omega)$ characterizing the environment,

$$J(\omega) = \sum \frac{C_\alpha^2}{4m_\alpha \omega_\alpha} (2\pi) \delta(\omega - \omega_\alpha).$$

If we set $J(\omega) = \eta \omega$, the Ohmic dissipation term $-\eta \dot{q}$ arises in the effective classical equation of motion of $q(t)$ with the proper boundary condition of the environmental variables [5, 7].

Generally $\Delta S$ consists of the local component $\Delta S_L$ and the non-local component $\Delta S_{NL}$. We identify the component $\Delta S_{NL}$ as the dissipation term, since it corresponds to the odd power term in the Fourier transform and then it is related to the time reversal symmetry breaking. On the other hand, the component $\Delta S_L$ consists of the even power term in the Fourier transform and then it does not contribute the time reversal symmetry breaking in the effective classical equation of motion, that is, it is irrelevant to the dissipation. Therefore, we introduce a suitable counterterm in the Euclidean action $S_E$ to cancel the component $\Delta S_L$ [1].

For the Ohmic dissipation $J(\omega) = \eta \omega$, we identify the dissipative part as follows:

$$\Delta S_{NL}[q] = \frac{\eta}{4\pi} \int d\tau \int ds \frac{(q(\tau) - q(s))^2}{|\tau - s|^2}. \tag{8}$$

It should be noted that the dissipation term, $\Delta S_{NL}[q]$, never breaks the time reversal symmetry in the action level. The actual dissipative effect $(-\eta \dot{q})$ arises only in the classical equation of motion level [5, 7]. Using the Fourier transform of $q(\tau)$, $\Delta S_{NL}$ is also represented as [7]

$$\Delta S_{NL}[q] = \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{\eta |\omega|}{\omega} \tilde{q}(\omega) \tilde{q}(-\omega). \tag{9}$$

Note that $\eta$ is a dimensionful parameter, $|\eta| = [M \omega] = [\text{mass}] \times [\text{time}]^{-1}$. We study the influence of the dissipation term $\Delta S_{NL}$ on the quantum behaviors of $q(\tau)$.

### 3 NPRG equation with the Ohmic dissipation

Now let us proceed to the NPRG analysis of the Caldeira-Leggett model [18]. Originally the NPRG method has been formulated and used mainly in the statistical mechanics or the quantum field theory to analyze the critical phenomena [11, 12, 13, 14]. Recently, the NPRG method has been found even effective in the quantum mechanics particularly for the non-perturbative analysis [10, 15, 16].

We derive the NPRG equation for the quantum mechanics with the dissipation term. In the NPRG method, the theory is defined as an effective theory with a high frequency cutoff $\Lambda$, and is described by the Wilsonian effective action $S_\Lambda[q]$, which changes with respect to the scale $\Lambda$. We now employ the local potential approximation (LPA),

$$S_\Lambda[q] = \int d\tau \left[ \frac{1}{2} M q^2 + \frac{\eta}{4\pi} \int ds \frac{(q(\tau) - q(s))^2}{|\tau - s|^2} + V_\Lambda(q) \right], \tag{10}$$

3
where the effective action is limited to have only local potential term, the Wilsonian effective potential \( V_\Lambda(q) \), in addition to the fixed kinetic term and the fixed dissipation term. Both the particle mass \( M \) and the dissipation strength \( \eta \) are constant and only the Wilsonian effective potential \( V_\Lambda(q) \) is changed as the scale \( \Lambda \) is lowered. Note that the LPA is the leading order of the derivative expansion of \( S_\Lambda[q] \) and has nothing to do with the prescription that we ignore the local part \( \Delta S_L \) in the component \( \Delta S \). The LPA means that any quantum correction to the derivative coupling is ignored, but does not mean that any quantum correction from the derivative coupling is ignored. We carry out the path integration over the highest frequency degrees of freedom of \( \tilde{q}(\omega) \), \( \Lambda - \Delta \Lambda < |\omega| \leq \Lambda \), up to the one-loop order,

\[
V_{\Lambda - \Delta \Lambda}(q) = V_\Lambda(q) + \hbar \int_{\Lambda - \Delta \Lambda}^{\Lambda} \frac{d\omega}{2\pi} \log \left( M\omega^2 + \eta |\omega| + \frac{\partial^2 V_\Lambda}{\partial q^2} \right),
\]

(11)

where we have omitted a proper constant term which should make the argument of the logarithm to be dimensionless, since it contributes only to the \( q \)-independent part of the effective action. Taking the limit \( \Delta \Lambda \to 0 \) in Eq. (11), we have the NPRG equation

\[
\Lambda \frac{\partial V_\Lambda}{\partial \Lambda} = -\frac{\hbar}{2\pi} \log \left( 1 + \frac{\eta}{\Lambda M} + \frac{1}{\Lambda^2 M} \frac{\partial^2 V_\Lambda}{\partial q^2} \right),
\]

(12)

which should be called the LPA Wegner-Houghton equation with dissipation. This is because in the \( \eta \to 0 \) limit Eq. (12) becomes the ordinary LPA Wegner-Houghton equation [12, 13], which describes the change of the Wilsonian effective potential \( V_\Lambda(q) \) with respect to the scale \( \Lambda \). The extra term \( \eta/(\Lambda M) \) originates from the dissipation term \( \Delta S_{NL} \). The NPRG equation (Eq. (12)) is obtained with the approximation LPA; the leading order of the derivative expansion of \( S_\Lambda[q] \). The derivative expansion does not rely on any small parameter which controls the series expansion, and therefore the effectiveness of the NPRG equation is expected to be free from the smallness of the parameters in the system.

4 Analysis of Dissipative Quantum Tunneling

We regard the system variable \( q \) as a macroscopic collective coordinate and discuss how the quantum mechanical behavior of the macroscopic coordinate \( q \) is affected by the environmental effects. As a typical phenomenon, the quantum tunneling of \( q \) has been analyzed by many authors [1, 2, 3, 5]. We set the bare potential of \( q \) as the double well type (Eq. (4)). In this system with vanishing \( \eta \), the coordinate \( q \) tunnels through the potential barrier and it oscillates between two wells. The main question is whether it also oscillates or not even in the \( \eta \neq 0 \) quantum dissipative dynamics. Such an oscillating behavior is particularly called the macroscopic quantum coherence and the inclination of ceasing the oscillation corresponds to decoherence [6].

Now we briefly summarize the previous results obtained for the macroscopic quantum coherence in double well potential systems with the Ohmic dissipation. Usually the first energy gap \( \Delta E = E_1 - E_0 \) is calculated as the physical quantity because it corresponds to the tunneling amplitude between two wells. Caldeira-Leggett evaluated it by using the semi-classical (instanton) approximation which is valid for \( \lambda_0 \to 0 \) and the perturbation with respect to \( \eta \), and then they found the dissipation term \( \Delta S_{NL} \) suppresses the quantum
Renormalization group analyses have been done for the instanton gas system within the dilute gas approximation which is valid also for $\lambda_0 \to 0$ and $\eta \to 0$ region [2, 3]. They predict a remarkable phenomenon that $\Delta E$ vanishes at a critical value $\eta_c = 2\pi \hbar \lambda_0/(M\omega_0^2)$, where the decoherence occurs. This phenomenon is often called the quantum-classical transition because the system looks classical in a limited sense that the quantum tunneling does not occur [19]. This is nothing but the spontaneous $Z_2$ symmetry breaking. In these works the energy gap $\Delta E$ was treated as the order parameter of the transition.

These results are obtained by using the semi-classical approximation and/or the perturbation theory with respect to $\eta$. Their reliability depend on the smallness of the couplings $\lambda_0$ and $\eta$. Therefore, to get more general and reliable results free of such limitation, we must employ an analyzing tool which does not need the series expansion with respect to any couplings. We expect our method using the LPA Wegner-Houghton equation with dissipation (Eq. (12)) will work best for the large coupling region, because the employed approximation, LPA, is the leading order of the derivative expansion and does not depend any small parameter. In fact, it has been found that the NPRG analysis of quantum tunneling based on the LPA Wegner-Houghton equation works very well in the wide parameter ($\lambda_0$) region [10].

We analyze the dissipative quantum tunneling by solving the NPRG equation (Eq. (12)) numerically. This equation is a two-dimensional partial differential equation for the Wilsonian effective potential $V_\Lambda(q)$ with respect to $\Lambda$ and $q$. The initial condition of the potential is the bare potential $V_{\Lambda_0}(q) = -\frac{1}{2}M\omega_0^2q^2 + \lambda_0q^4$ at the initial cutoff $\Lambda_0$. We solve the differential equation toward the infrared limit $\Lambda \to 0$ and finally obtain the physical effective potential $V_{\text{eff}}(q) = \frac{1}{2}M\omega_{\text{eff}}^2q^2 + \lambda_{\text{eff}}q^4 + \cdots$. We can exploit the physical information of the quantum system from the effective potential $V_{\text{eff}}(q)$. First of all, the frequency squared starting with the negative value $\omega_{\Lambda_0}^2 = -\omega_0^2$ finally reaches a positive value (for small $\eta$), $\omega_{\text{eff}}^2 > 0$, that is, the effective potential $V_{\text{eff}}(q)$ becomes a single well form. It means that due to the quantum tunneling, the $Z_2$ symmetry (parity) does not break spontaneously, and the expectation value of $q$ is vanishing. The tunneling effects are automatically incorporated when we integrate (solve) the NPRG equation toward the infrared.

It should be noted that for non-vanishing $\eta$, the simple equivalence between the energy gap $\Delta E$ and the effective frequency $\omega_{\text{eff}}$, $\Delta E = \hbar \omega_{\text{eff}}$ [10], does not hold. The long range ($\tau \to \infty$) behavior of the two point function

$$\lim_{\tau \to \infty} \langle \Omega| T\hat{q}(\tau)\hat{q}(0) |\Omega\rangle = \lim_{\tau \to \infty} \int \frac{d\omega}{2\pi} e^{i\omega \tau} \frac{\hbar}{M\omega^2 + \eta|\omega| + M\omega_{\text{eff}}^2},$$

is actually independent of $\omega_{\text{eff}}$ and is determined by $\eta$ with power damp behavior even in case of infinitesimal $\eta$. This singular behavior comes from the nature of the environmental degrees of freedom where the harmonic oscillator frequencies are assumed to distribute continuously up to zero ($\omega_0 = 0$). Even if there is an infrared cutoff, it does not change the situation much and the lowest frequency environment dominates the long range correlator of the target system. In this sense the semi-classical arguments calculating the first energy gap $\Delta E = E_1 - E_0$ as the order parameter of the quantum-classical transition in the quantum dissipative systems seems somewhat doubtful, because the correspondence between the tunneling amplitude and the first energy gap $\Delta E$ comes from the long range behavior of the transition amplitude.
Here, instead of analyzing the long range correlator, we adopt another physical quantity to describe the possible quantum-classical transition, the localization susceptibility. We add a source term

\[ J \tilde{q}_T(0) \equiv J \int_{-T/2}^{T/2} d\tau \, q(\tau) \tag{14} \]

to the Euclidean action \( S_E \) (Eq. (3)). Then the localization susceptibility \( \chi \) is defined by

\[ \chi \equiv \left. \frac{\langle \langle q(\tau) \rangle \rangle_J}{dJ} \right|_{J=0} = \lim_{T \to \infty} \frac{1}{T} \left. \frac{\langle \langle q_T(0) \rangle \rangle_J}{dJ} \right|_{J=0} = \lim_{T \to \infty} \frac{1}{T} \left. \frac{d^2 \log Z}{dJ^2} \right|_{J=0}, \tag{15} \]

and it is exactly given by the effective potential as follows:

\[ \chi = \left( \left. \frac{d^2 V_{\text{eff}}(q)}{dq^2} \right|_{q=q_{\text{min}}} \right)^{-1} = \frac{1}{M \omega_{\text{eff}}^2}, \tag{16} \]

where \( q_{\text{min}} \) is a minimum of \( V_{\text{eff}}(q) \) and it is actually vanishing in the sub-critical region. These definitions are valid also for non-vanishing \( \eta \). If we find a divergent and scaling behavior of \( \chi \) toward \( \eta = \eta_c \), we may conclude that the localization transition occurs at \( \eta_c \) and it is the second order phase transition.

![Figure 1: The effective frequency \( \omega_{\text{eff}} \) with initial cutoff \( \Lambda_0 = 10^4 [h \omega_0] \).](image)

![Figure 2: Critical scaling fit of localization susceptibility \( \chi \) for \( \lambda_0 = 1.0 [M^2 \omega_0^2 / \hbar] \).](image)

The results are shown in Fig.1 for several values of \( \lambda_0 \). In previous works for \( \eta = 0 \) system [10, 15, 16], the NPRG analyses have been found to work excellently in these \( \lambda_0 \) region \( (\lambda_0 \gtrsim 0.1 [M^2 \omega_0^2 / \hbar]) \) while the dilute gas instanton approximation does not work at all there. Therefore, we expect the NPRG analysis also works well for these \( \lambda_0 \) values even in \( \eta \neq 0 \) systems. We find that for every value of \( \lambda_0 \), \( \omega_{\text{eff}} \) decreases as \( \eta \) becomes large. This \( \eta \) and \( \lambda_0 \) dependence of \( \omega_{\text{eff}} \) is naturally understandable by considering the dissipation effect in Eq. (12). The quantum fluctuation of the dynamical variable \( q \) is suppressed below the effective infrared cutoff \( \omega_{\text{IR}} = \eta / M \). Therefore, the larger \( \eta \)
causes the stronger cutoff effect and results in the smaller $\omega_{\text{eff}}$. For larger $\lambda_0$, the effective frequency of the system is larger, and therefore larger $\eta$ is required to effectively cutoff the quantum fluctuation. These behaviors are qualitatively consistent with those of $\Delta E$ obtained by the instanton approximation.

Then, what happens for large $\eta$ case? The expected phenomenon is the quantum-classical transition characterized by complete disappearance of the tunneling. The NPRG method may not work precisely in the $\omega_{\text{eff}}/\omega_0 \to 0$ region because Eq. (12) becomes singular there and the numerical error increases. Therefore, according to the standard technique of analyzing the critical phenomena, we evaluate the critical dissipation $\eta_c$ from the diverging behavior of the localization susceptibility. We fit $\chi(\eta)$ with the critical exponent form

$$\chi = C|\eta - \eta_c|^{-\gamma}. \quad (17)$$

We show an example of fit for $\lambda_0 = 1.0[M^2\omega_0^3/\hbar]$ in Fig.2. We conclude that the localization susceptibility exhibits a divergent behavior with a power scaling, which indicates the second order phase transition of localization. We obtain the critical dissipation $\eta_c$ and the critical exponent $\gamma$ for these $\lambda_0$, which are plotted in Fig.3. The previous analysis using the dilute gas instanton gives a simple relation $\eta_c = 2\pi\hbar\lambda_0/(M\omega_0^2)$ which is also shown in Fig.3 for comparison. The NPRG results for $\eta_c$ are systematically larger compared to those of the instanton. As for the critical exponent $\gamma$, we observe the universality property except for the smallest $\lambda_0 = 0.1[M^2\omega_0^3/\hbar]$, where NPRG results may not be reliable.

The NPRG works very well in the large $\lambda_0$ region [10, 15, 16]. Strictly speaking, tunneling phenomena are no longer characteristic in such parameter region. It can be easily seen by evaluating a dimensionless quantity $r = 8\sqrt{2}\hbar\lambda_0/(M^2\omega_0^3)$, which is the ratio of the zero-point energy of the particle at each well with $\eta = 0$ to the height of the interwell barrier. Tunneling phenomena are dominant for $r \ll 1$ case. For $\lambda_0 = 0.1[M^2\omega_0^3/\hbar]$, 

![Figure 3: $\lambda_0$ dependence of critical dissipation $\eta_c$ and critical exponent $\gamma$.](image-url)
$r \sim 1.1$ is obtained. Therefore, our analysis in the large $\lambda_0$ region ($\lambda_0 \gtrsim 0.1 [M^2 \omega_0^2 / \hbar]$) rather claims suppression of the quantum oscillation over the small inter-well barrier. The critical phenomenon that we observed here is induced by a destruction of the quantum coherence and can be interpreted as spontaneous $Z_2$ symmetry breaking-like phenomenon in quantum mechanics.

5 Summary

The Caldeira-Leggett model for dissipative quantum mechanics was analyzed by means of the NPRG method. We derived the LPA Wegner-Houghton equation with dissipation, which is the local potential approximated NPRG equation with the Ohmic dissipation term. We applied it to analysis of dissipative quantum tunneling (or dissipative quantum oscillation) and investigated whether the quantum-classical transition occurs or not. Since the first energy gap $\Delta E = E_1 - E_0$ is unsuitable as the order parameter of the dissipative phase transition, we calculated the localization susceptibility $\chi$ from the effective potential $V_{\text{eff}}(q)$. The observed $\chi$ diverges toward $\eta = \eta_c(\lambda_0)$ with the critical scaling, and therefore we concluded that $\eta_c(\lambda_0)$ is the critical dissipation for the quantum-classical transition and this transition seems the second order phase transition. The values of the critical dissipation $\eta_c(\lambda_0)$ are rather larger than the values obtained by the semi-classical analyses, where the first energy gap $\Delta E$ is used as the order parameter of the quantum-classical transition. As for the non-Ohmic dissipations, the analogous systematic study using the NPRG equation will be reported elsewhere [17]. There must remain, however, many fundamental and subtle problems to be studied about the notion of quantum-classical transition and the critical dissipation. The NPRG method will be effective also for those problems.

Acknowledgement

We would like to thank I. Sawada for fruitful and encouraging discussions and suggestions. K.-I. Aoki is partially supported by the Grant-in Aid for Scientific Research (#12874029) from the Ministry of Education, Science and Culture.

References

[1] A.O. Caldeira and A.J. Leggett, Phys. Rev. Lett. 46 (1981) 211; Ann. Phys. 149 (1983) 374.

[2] S. Chakravarty, Phys. Rev. Lett. 49 (1982) 681.

[3] A.J. Bray and M.A. Moore, Phys. Rev. Lett. 49 (1982) 1545.

[4] M.P.A. Fisher and W. Zwerger, Phys. Rev. B32 (1985) 6190.

[5] K. Fujikawa, S. Iso, M. Sasaki and H. Suzuki, Phys. Rev. Lett. 68 (1992) 1093; Phys. Rev. B46 (1992) 10295.
[6] U. Weiss, *Quantum Dissipative Systems*, 2nd Edition, World Scientific, Singapore, 1999.

[7] S.-Y. Lee, H. Kim, D.K. Park, C.S. Park and J.K. Kim, Phys. Rev. B60 (1999) 308.

[8] B. Zhou, J.-Q. Liang and F.-C. Pu, Phys. Lett. B496 (2000) 218.

[9] For a recent review, see R. Fazio and H. van der Zant, Phys. Rep. 355(4) (2001) 235.

[10] K.-I. Aoki, A. Horikoshi, M. Taniguchi, and H. Terao, in Proceedings of the Workshop on The Exact Renormalization Group, Faro, Portugal, September 1998, World Scientific, Singapore, 1999, p. 194; Prog. Theor. Phys. 108 (2002) 571.

[11] K.G. Wilson and J.B. Kogut, Phys. Rep. 12 (1974) 75.

[12] F. Wegner and A. Houghton, Phys. Rev. A8 (1973) 401.

[13] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B270 (1986) 687.

[14] K.-I. Aoki, Int. J. Mod. Phys. B14 (2000) 1249.

[15] A.S. Kapoyannis and N. Tetradis, Phys. Lett. A276 (2000) 225.

[16] D. Zappala, Phys. Lett. A290 (2001) 35.

[17] K.-I. Aoki and A. Horikoshi, Phys. Rev. A66 (2002) 042105.

[18] The perturbative renormalization group analysis has been done for the dissipative quantum mechanics with periodic potential and interesting localization-delocalization transition has been discovered [4].

[19] The quantum-classical transition has been also discussed as the decay-rate phase transition, which means that the decay rate of a metastable state changes from the quantum tunneling dominated one to the thermal activation dominated one at critical temperature $T_c$ [7, 8].