OPTIMAL HARDY INEQUALITY FOR THE FRACTIONAL LAPLACIAN ON $L^p$

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Abstract. For the fractional Laplacian we give Hardy inequality which is optimal in $L^p$ for $1 < p < \infty$. As an application, we explicitly characterize the contractivity of the corresponding Feynman-Kac semigroups on $L^p$.

1. Introduction

Hardy inequalities are of paramount importance in harmonic analysis, functional analysis, partial differential equations, potential theory and probability. They are applied to embedding theorems, Gagliardo–Nirenberg interpolation inequalities and in real interpolation theory, see Chua [21], Kalajdzievski and Pietruska-Paluba [40]. They yield contractivity of operator semigroups, a priori estimates, existence and regularity results for solutions of PDEs, plus their asymptotics and qualitative properties, see, e.g., Maz’ya [51], Arendt, Goldstein and Goldstein [2], Barras and Goldstein [4], and Vazquez and Zuazua [61]. The important connection between Hardy-type inequalities and superharmonic functions was exploited, e.g., by Ancona [1], Fitzsimmons [29], Bogdan and Dyda [10], Dyda [26], Devyver, Fraas and Pinchover [25], Bogdan, Dyda and Kim [11], Bogdan, Jakubowski, Grzywny and Pilarczyk [13]. In particular, the inequalities are connected to sharp estimates of the heat kernel of $\Delta^{\alpha/2} + \kappa|x|^{-\alpha}$ [13], see also Calvaruso, Metafune, Negro and Spina [18].

For an account of the history of Hardy-type inequalities we refer to Opic and Kufner [52]. The subject was initiated in 1920, when Hardy [34] discovered that

$$\int_0^\infty [u'(x)]^2 \, dx \geq \frac{1}{4} \int_0^\infty \frac{u(x)^2}{x^2} \, dx,$$

for absolutely continuous functions $u$ such that $u(0) = 0$ and $u' \in L^2(0, \infty)$. The classical Hardy inequality in $\mathbb{R}^d$ for $d \geq 2$ is

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} \, dx, \quad u \in L^2(\mathbb{R}^d).$$

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Here the left-hand side of (2) is considered infinite if the distributional gradient of $u$ is not a square-integrable function, see, e.g., [11, (30) and (32)] for this formulation.

In 2000 Fitzsimmons [29] proved an abstract analogue of (2), in which the Dirichlet integral appearing on the left-hand side of (2) is replaced by a general symmetric Dirichlet form $E$ in the sense of Fukushima, Oshima, Takeda [33]. The rule stemming from [29] is the following: If $\mathcal{L}$ is the generator of the form and function $h$ is superharmonic, i.e., $h \geq 0$ and $\mathcal{L}h \leq 0$, then $E(u, u) \geq -\int u^2 \mathcal{L}h / h$.

The paper [11] gives similar results in the setting of symmetric transition densities, with explicit construction of the function $h$, Riesz’ charge $-Lh$, and the Hardy weight, or Fitzsimmons’ ratio, $-\mathcal{L}h / h$. The resulting Hardy inequalities, in fact, Hardy identities, in some cases are optimal in the sense of large weight and large functional domain, desirably the whole of $L^2$.

The present work extends part of the results of [11] to the setting of $L^p$ spaces with arbitrary $p \in (1, \infty)$. We focus on integral forms related to the semigroup of fractional Laplacian and give optimal inequalities in this case. We also show that the inequalities lead to optimal contractivity results for related operator semigroups on $L^p$. From our presentation it should also be evident that the approach applies to more general sub-Markovian semigroups.

1.1. Sobolev-Bregman forms. Let $d \in \mathbb{N}$ and $0 < \alpha < 2$. We consider the fractional Laplacian,

$$\Delta^{\alpha/2}u(x) := -(-\Delta)^{\alpha/2}u(x) := \lim_{\epsilon \to 0^+} \int_{|y-x| > \epsilon} (u(y) - u(x)) \nu(x, y) \, dy, \quad x \in \mathbb{R}^d.$$  

Here, $u \in C^2_c(\mathbb{R}^d)$, 

$$\nu(x, y) = A_{d,-\alpha} |y-x|^{-d-\alpha}, \quad x,y \in \mathbb{R}^d,$$

and $A_{d,-\alpha} = 2^\alpha \Gamma((d + \alpha)/2) \pi^{-d/2}/|\Gamma(-\alpha/2)|$. We further consider

$$E[u] := E(u, u) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \nu(x, y) \, dy \, dx,$$

defined for every (Borel measurable) $u : \mathbb{R}^d \to \mathbb{R}$. The natural domain $\mathcal{D}(E)$ of the form consists of those functions $u \in L^2(\mathbb{R}^d)$ for which $E[u] < \infty$. By [11, Proposition 5], for all $0 < \alpha < d \wedge 2$, $0 \leq \beta \leq d - \alpha$, and $u \in L^2(\mathbb{R}^d)$, we have the Hardy-type identity

$$E[u] = \kappa_\beta \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha} \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(x)h(y)\nu(x, y) \, dy \, dx,$$

where $h(x) := |x|^{-\beta}$, and

$$\kappa_\beta = \frac{2^\alpha \Gamma(\frac{\beta+\alpha}{2}) \Gamma(\frac{d-\beta}{2})}{\Gamma(\frac{\beta}{2}) \Gamma(\frac{d-\beta-\alpha}{2})}.$$  

Before [11], the identity (4) was given by Frank, Lieb and Seiringer in [30, (4.3)] for functions $u \in C^\infty_0(\mathbb{R}^d \setminus \{0\})$ and $\beta \in [0, (d - \alpha)/2]$. We note that the coefficient $\kappa_\beta$
is increasing on $[0,(d - \alpha)/2]$, symmetric with respect to $(d - \alpha)/2$ and takes on the maximal value at $\beta = (d - \alpha)/2$, which is

$$\kappa(d-\alpha)/2 = 2^\alpha \Gamma\left(\frac{d + \alpha}{4}\right)^2 \Gamma\left(\frac{d - \alpha}{4}\right)^{-2},$$

see [30, Lemma 3.2] or [11, p. 237]. Correspondingly, the following optimal fractional Hardy inequality holds

$$E[u] \geq \kappa(d-\alpha)/2 \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha} \, dx \quad \text{for all} \quad u \in L^2(\mathbb{R}^d). \quad (6)$$

The inequality is also known as Hardy-Rellich inequality and was proved by Herbst [35, (2.6)], Beckner [6, Theorem 2] and Yafaev [62]. We will propose an optimal analogue of the inequality appropriate for $L^p(\mathbb{R}^d)$. To this end for $p \in (1,\infty)$ and (Borel measurable) $u : \mathbb{R}^d \to \mathbb{R}$ we define the Sobolev-Bregman form, or the $p$-form,

$$E_p[u] := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(u(x)^{p-1} - u(y)^{p-1}) \nu(x,y) \, dy \, dx. \quad (7)$$

Here and below we use the notation

$$a^{(k)} := |a|^k \operatorname{sgn} a, \quad a, k \in \mathbb{R},$$

where $0^k = 0$. The $p$-form is well defined since the integrand in (7) is nonnegative for every $u$. In fact, to compare $E_p$ with $E = E_2$, we recall that

$$4(p-1)p^{-2}(b^{(p/2)} - a^{(p/2)})^2 \leq (b - a)(b^{(p-1)} - a^{(p-1)}) \leq 2(b^{(p/2)} - a^{(p/2)})^2. \quad (8)$$

The inequality holds true for all $p \in (1,\infty)$ and $a, b \in \mathbb{R}$, see, e.g., Liskevich, Perelmuter and Semenov [46, Lemma 2.1]. By (6) and (8),

$$E_p[u] \geq 4(p-1)p^{-2} \kappa(d-\alpha)/2 \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} \, dx, \quad u \in L^p(\mathbb{R}^d). \quad (9)$$

We remark that the inequality (9) is given in Cialdea and Maz’ya [22, p. 231]. Our goal is to improve the constant. To this end, inspired by Bogdan, Dyda and Luks [12, (9)], we consider the Bregman divergence:

$$F_p(a, b) := |b|^p - |a|^p - p a^{(p-1)}(b - a), \quad a, b \in \mathbb{R}. \quad (10)$$

It is the second-order Taylor remainder of the convex function $\mathbb{R} \ni x \mapsto |x|^p$, so $F_p(a, b) \geq 0$. For instance, $F_2(a, b) = (b - a)^2$. The function $F_p$ may be used to quantify the regularity of functions in integral forms generalizing (3). Indeed, the symmetrization of $F_p$ is

$$\frac{1}{2}(F_p(a, b) + F_p(b, a)) = \frac{p}{2}(b - a)(b^{(p-1)} - a^{(p-1)}), \quad (11)$$

which is the expression in the definition of $E_p$, up to the factor of $p$. In passing we refer the reader to Bogdan, Grzywny, Pietruska-Pałuba and Rutkowski [15] for references to applications of Bregman divergence in analysis, statistical learning and optimization, and to Bogdan and Wiącek [17] for a martingale connection.
1.2. Main results. For $\beta \in \mathbb{R}$ we denote $h_\beta(x) := |x|^{-\beta}$, $x \in \mathbb{R}^d$. Of course, for $a \in \mathbb{R}$,
\begin{equation}
 h_\beta(x)^a = h_{a\beta}(x), \quad x \in \mathbb{R}^d.
\end{equation}
We propose the following Hardy-type identity for the Sobolev-Bregman forms.

**Theorem 1.** If $0 < \alpha < d \wedge 2$, $0 \leq \beta \leq (d - \alpha) \wedge (d - \alpha)/(p - 1)$, $h = h_\beta$ and $u \in L^p(\mathbb{R}^d)$, then
\begin{equation}
 \mathcal{E}_p[u] = \kappa_{(\alpha-1)/p} + \frac{(p-1)\kappa_\beta}{p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^{\alpha}} dx
 + \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p \left( u(x), u(y) \right) h(x)^{p-1} h(y)^{p-1} h(x,y) \nu(x,y) dy dx.
\end{equation}
In particular, for $\beta = (d - \alpha)/p$ we obtain
\begin{equation}
 \mathcal{E}_p[u] = \kappa_{(\alpha-1)/p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^{\alpha}} dx
 + \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p \left( u(x), u(y) \right) h(x)^{p-1} h(y)^{p-1} h(x,y) \nu(x,y) dy dx,
\end{equation}
and, of course,
\begin{equation}
 \mathcal{E}_p[u] \geq \kappa_{(\alpha-1)/p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^{\alpha}} dx \quad \text{for all} \quad u \in L^p(\mathbb{R}^d).
\end{equation}
The results extend the Hardy identities from \cite{10} and the ground-state representations of Frank, Lieb and Seiringer in \cite[Proposition 4.1]{30}, see also Frank and Seiringer \cite{31} and Beckner \cite{7}. The proofs of (13) and (14) are given in Section 3. In Lemma 8 of Section 3 we also show that (15) improves (9), namely for $p \neq 2$ we have
\begin{equation}
 \kappa_{(\alpha-1)/p} > \frac{4(p-1)}{p^2} \frac{2^\alpha \Gamma \left( \frac{d+\alpha}{4} \right)^2}{\Gamma \left( \frac{d+\alpha}{4} \right)^2}.
\end{equation}
Figure 1 compares both sides of (16), by showing $\kappa_{(\alpha-1)/p}$ of (15) (solid line) and $4(p-1)p^{-2}\kappa_{(\alpha-1)/2}$ of (9) (dashed line) as functions of $1/p \in (0,1]$ for $d = 3$ and $\alpha = 1$. Here is a statement deeper than (16), which we prove in Section 4.

**Theorem 2.** The constant in (15) is sharp.

The result was previously known only for $p = 2$; we again refer to \cite[(2.6)]{35}, \cite[proof of Theorem 2]{6} or \cite{62}.

For the sake of comparison we recall the fractional Hardy-Sobolev inequality from \cite[Theorem 1.1]{31}:
\begin{equation}
 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^p |x|^{d+ps} dx dy \geq C_{d,s,p} \int_{\mathbb{R}^d} |u(x)|^p |x|^{ps} dx.
\end{equation}
The constant $C_{d,s,p}$ is known and sharp, too. Of course, the forms in (7) and (17) are different generalizations of the quadratic form in (3). Correspondingly, the inequalities (15) and (17) are different optimal generalizations of the fractional Hardy inequality (6). In passing we refer the reader to \cite[Section 6]{15} for an additional insight into the
comparison of (15) and (17) and to Frank and Seiringer [32] for a version of (17) for half-spaces.

Not unexpectedly, our main application of Theorem 1 and 2 is to the $L^p$ contractivity of the Feynman-Kac semigroup $\tilde{P}_t$ generated by $\Delta^{\alpha/2} + \kappa_{\delta} |x|^{-\alpha}$ with $\kappa_{\delta}$ given by (5).

**Theorem 3.** Let $0 < \alpha < 2 \wedge d$, $\delta \in [0, (d - \alpha)/2]$, $1 < p < \infty$ and $0 < t < \infty$. The operator $\tilde{P}_t$ is a contraction on $L^p(\mathbb{R}^d)$ if and only if $\kappa_{\delta} \leq \kappa_{(d-\alpha)/p}$.

The proof of this characterization is given in Section 5, see also Remark 1 there. The result is analogous to the classical case ($\alpha = 2$), where the operator $\Delta + \kappa |x|^{-2}$ generates a contraction semigroup on $L^p(\mathbb{R}^d)$ if and only if $\kappa \leq \kappa_{(d-2)/p} = (d-2)(p-1)p^{-2}$, see Kovalenko, Perelmuter and Semenov [43], see also Liskevich and Semenov [48, Theorems 1 and 2] and [2, Corollary 1.2]. Theorem 3 is illustrated by Figure 2. Notably, if $\alpha = 2$ then $\kappa_{\beta} = \beta(d-2-\beta)$ and (16) becomes equality, so we have improved Hardy inequality for the nonlocal operator $\Delta^{\alpha/2}$ but not for the local operator $\Delta$.

Here is the last theorem of the paper.

**Theorem 4.** Let $0 < \alpha < 2 \wedge d$, $\delta \in [0, (d - \alpha)/2]$, $1 < p < \infty$ and $0 < t < \infty$. The operator $\tilde{P}_t$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\delta < d/p^*$, where $p^* = \max\{p, p/(p-1)\}$.

The result is proved in Section 5. As we shall see, the result is a consequence of the estimates of $\tilde{P}_t$ given in [13]. An illustration of Theorem 4 is given in Figure 3.

The structure of the paper is as follows. In Section 2 we give preliminaries and ponder definitions. The proof of Theorem 1 is given in Section 3. The proof of Theorem 2 is in Section 4. In Section 5 we prove Theorem 3 and Theorem 4. In Section 1.3 we point out to related results in the literature and broader perspectives.
Figure 2. The shaded area shows the values of $1/p$, for which, given \( \delta \in [0, (d - \alpha)/2] \) and \( \Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}, \tilde{P}_t \) are contractive on \( L^p(\mathbb{R}^d) \).

Figure 3. The shaded area shows the values of $1/p$, for which, given \( \delta \in [0, (d - \alpha)/2] \) and \( \Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}, \tilde{P}_t \) are bounded on \( L^p(\mathbb{R}^d) \).

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1.3. Further discussion. The main results of the paper are the optimal Hardy identity (14) and inequality (15). In this section, however, we try to give a broader perspective for the considered Sobolev-Bregman forms and point out to connections and inspirations. It is well known that the ramification of the potential theory of symmetric Markovian processes and semigroups by means of the theory of Dirichlet forms on $L^2$ spaces turned out extremely successful [33], Ma and Röckner [49]. Therefore many authors worked to extend it to the case of the $L^p$ spaces. The development is related to the fact that $\langle -Af,f^{p-1} \rangle \geq 0$ for $f$ in the domain of the generator, see (40) below. We recall that for $p \in (1, \infty)$ the dual space of $L^p$ is, of course, $L^{p/(p-1)}$ and for $u \in L^p$ we have $u^{(p-1)} \in L^{p/(p-1)}$, and $\|u\|_p^p = \int |u|^p = \int |u^{(p-1)}|^{p/(p-1)} = \int u^{(p-1)}u$. Therefore $\|u\|_p^{2-p}u^{(p-1)}$ yields a linear functional on $L^p$ appropriate for testing the dissipativity of generators in the Lumer-Phillips theorem, see, e.g., Pazy [53, Section 1.4] or [22, p. 3]. For the semigroups generated by local operators we refer to Langer and Maz’ya [45], Sobol and Vogt [58, Theorem 1.1] and the monograph [22], see also the historical comments in Section 4.7 therein.

As we shall experience first-hand in Section 5, the forms $E_p$ capture the evolution of $L^p$ norms of functions upon the action of operator semigroups. The connection is well known, since at least the paper of Varopoulos [60, (1,1)]. It was then used, e.g., to study perturbations of semigroups on $L^p$ in [46, Theorem 3.2] and [48]. For nonlocal operators we refer the reader to Farkas, Jacob and Schilling [27, (2.4)] and to the monograph of Jacob [37, (4.294)]. As a rule the authors use the intermediary $L^2$ setting, relying on inequalities similar to (8) and to (39) below. Therefore the resulting inequalities do not allow for optimal constants, certainly in the case of nonlocal generators, cf. (9) and (15); see also [22, p. 231]. Suboptimal constants may lead to suboptimal qualitative results, as shall be evident from the proof of Theorem 3. We avoid the inaccuracy of (8) by systematically using the Bregman divergence – we work intrinsically in $L^p$ and therefore obtain the optimal constants and the optimal range of exponents in Theorems 2 and 3.

We remark that (39) and (40) can be traced back to the papers [60] and Liskevich, Semenov [47]. In our notation it reads there as $\langle -Af,f^{p/(p-1)} \rangle \approx \langle -A^{1/2}f^{p/(p/2)},A^{1/2}f^{p/(p/2)} \rangle$, see [47, Theorem 1], see also Kinzebulatov and Semenov [42, Proposition 8] for recent developments and [41] for applications to symmetric stable processes with drift, understood as solutions of stochastic differential equations.

Finally, it may be hard to point out the first occurrence of (8). Our best guess is [47, Lemma 1] and [46, Lemma 2.1], see also [60, p. 246], Bakry [3, p. 39], Stroock [59, Lemma 9.9 and p. 134] and Carlen, Kusuoka and Stroock [19, p. 269] for formulations with nonnegative arguments or one-sided comparison. We also point out to the calculations with forms and powers in Davies [24, Chapter 2 and 3], to the calculations in [22, Section 7.6] and [15, (2.19)], and to Lemma 5 below.

2. Preliminaries

We use “:=” to indicate definitions, e.g., $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$, and $a_+ := a \vee 0$. All the functions considered below are Borel measurable either by construction or assumptions. If $\mathcal{F}$ is a family of functions, then we let $\mathcal{F}_+ = \{f \in \mathcal{F} :$
\[ f \geq 0 \}. \text{ For nonnegative functions } f \text{ and } g \text{ we write } f(x) \approx g(x) \text{ to indicate that there are numbers } 0 < c < C < \infty \text{ such that } cf(x) \leq g(x) \leq Cf(x) \text{ for all the considered arguments } x. \text{ We call such comparisons two-sided or } \textit{sharp}. \text{ For an open subset } D \text{ of the } d\text{-dimensional Euclidean space } \mathbb{R}^d, \text{ we let } C_c^\infty(D) \text{ be the space of smooth functions with compact support in } D. \\

Note that \(|x|^p)' = px^{(p-1)} \) on \( \mathbb{R} \) for \( p \in (1, \infty) \). We shall focus on related inequalities.

**Lemma 5.** For \( p \in (1, \infty) \) there are constants \( C, C' > 0 \) such that for \( a, b \in \mathbb{R} \),

\[
(18) \quad 0 \leq |b|^p - |a|^p - pa^{(p-1)}(b - a) \leq C|b - a|^\lambda(|b| + |a|)^{p-\lambda}, \quad \lambda \in [0, 2],
\]

\[
(19) \quad |b|^p - |a|^p \leq (p + C)|b - a|(|b| + |a|)^{p-1},
\]

\[
(20) \quad |b^{(p-1)} - a^{(p-1)}| \leq C'|b - a|(|b| + |a|)^{p-1-\lambda}, \quad \lambda \in [0, 1].
\]

**Proof.** The inequality (18) with \( \lambda = 2 \) follows from [12, Lemma 6]. In fact, we have

\[
(21) \quad |b|^p - |a|^p - pa^{(p-1)}(b - a) \approx (b - a)^2(|b| + |a|)^{p-2}, \quad a, b \in \mathbb{R},
\]

see [12, (11)]. Here is, however, another proof: if \( a \neq 0 \), then we let \( x = b/a \) and arrive at the following equivalent statement of (18):

\[
0 \leq |x|^p - 1 - p(x - 1) \leq C|x - 1|^\lambda(|x| + 1)^{p-\lambda}, \quad x \in \mathbb{R}.
\]

Since \( x \mapsto |x|^p \) is strictly convex, both sides are continuous and strictly positive on \( \mathbb{R} \setminus \{1\} \). Let first \( \lambda = 2 \). By a compactness argument it is enough to notice that

\[
\lim_{x \to \pm \infty} \frac{|x|^p - 1 - p(x - 1)}{(x - 1)^2(|x| + 1)^{p-2}} = 1,
\]

and, by L'Hôpital's rule,

\[
\lim_{x \to 1} \frac{|x|^p - 1 - p(x - 1)}{(x - 1)^2} = \frac{p(p - 1)}{2}.
\]

When \( \lambda \in [0, 2] \), we have that \( |b - a|^2(|b| + |a|)^{p-2} \leq |b - a|^\lambda(|b| + |a|)^{p-\lambda} \), which gives (18). (19) follows from (18) with \( \lambda = 1 \), and (20) can be proved like (18). \qed

2.1. Functions and kernels. Let \( g \) denote the Gaussian kernel

\[
g_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)} \quad t > 0, \quad x \in \mathbb{R}^d.
\]

Let \( \eta_t(s) \geq 0 \) be the density function of the \( \alpha/2 \)-stable subordinator at time \( t \), see, e.g., Bliedtner and Hansen [8, Section V.3], Schilling, Song and Vondraček [57, Chapter 13] or Sato [55, Section 6.30]. In particular, \( \eta_t(s) = 0 \) for \( s \leq 0 \), and we have the following formula for the Laplace transform of \( \eta_t \):

\[
\int_0^\infty e^{-us}\eta_t(s) \, ds = e^{-tu^{\alpha/2}}, \quad u \geq 0, \quad t > 0.
\]

We define, by \textit{subordination}, the convolution semigroup of functions (the \( \alpha \)-stable semigroup):

\[
p_t(x) = \int_0^\infty g_s(x)\eta_t(s) \, ds, \quad t > 0, \quad x \in \mathbb{R}^d.
\]
It is well known that the functions $p_t$ satisfy the following scaling property:

\[(24) \quad p_t(x) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}} x), \quad t > 0, \quad x \in \mathbb{R}^d.\]

We consider the following transition probability density

\[(25) \quad p_t(x, y) = p_t(y - x), \quad t > 0, \quad x, y \in \mathbb{R}^d.\]

Clearly, $p_t(x, y)$ is symmetric in the space variables: $p_t(x, y) = p_t(y, x)$. It is well known that

\[(26) \quad p_t(x, y) \leq c \min \left( t^{-\frac{d}{\alpha}}, t|x - y|^{-\frac{d}{\alpha}} \right), \quad t > 0, \quad x, y \in \mathbb{R}^d,
\]

and so we can record the following convenient estimate

\[(27) \quad p_t(x, y)/t \leq c \nu(x, y), \quad t > 0, \quad x, y \in \mathbb{R}^d,
\]

see, e.g., Bogdan, Grzywny and Ryznar [16, Theorem 21]. Also,

\[(28) \quad p_t(x, y)/t \to \nu(x, y) \text{ as } t \to 0^+,
\]

see, e.g., Cygan, Grzywny and Trojan [23, Proof of Theorem 6]. We will use (27) and (28) to support arguments based on the following general fact.

**Lemma 6.** If nonnegative functions satisfy $f_n \leq cf$ for $n \in \mathbb{N}$, and $f = \lim f_n$, then $\lim \int f_n d\mu = \int f d\mu$ for any measure $\mu$.

**Proof.** If the integral $\int f d\mu$ is finite, then the dominated convergence theorem applies. Otherwise, by Fatou’s lemma, $\int f d\mu = \infty = \lim \inf \int f_n d\mu = \lim \int f_n d\mu$, too. \(\square\)

For $\alpha < d$ and $\beta \in (0, d)$, we let

\[f_\beta(t) = c t^{(d - \alpha - \beta)/\alpha}, \quad t \in \mathbb{R}.\]

Here $c \in (0, \infty)$ is a normalizing constant so chosen that

\[(29) \quad \int_0^\infty f_\beta(t) p_t(x) dt = |x|^{-\beta} = h_\beta(x), \quad x \in \mathbb{R}^d.
\]

For $\beta \in (0, d - \alpha)$ we let

\[(30) \quad q_\beta(x) := \frac{1}{h_\beta(x)} \int_0^\infty f_\beta(t) p_t(x) dt, \quad x \in \mathbb{R}^d.
\]

We note that the choice of $c$ does not affect $q_\beta$. By [11, Section 4],

\[(31) \quad q_\beta(x) = \kappa_\beta |x|^{-\alpha}.
\]

By [11, the proof of Proposition 5], the function $\beta \mapsto \kappa_\beta$ is increasing on $(0, (d - \alpha)/2]$ and decreasing on $[(d - \alpha)/2, d - \alpha)$. Furthermore, $\kappa_\beta = \kappa_{d - \alpha - \beta}$.

We denote, as usual, $P_t u(x) = \int_{\mathbb{R}^d} u(y)p_t(x, y) dy$ for $u : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, if the integral is well defined.

Since $\int_{\mathbb{R}^d} p_t(x, y) dy = \int_{\mathbb{R}^d} p_t(x, y) dx = 1$, by Schur’s test, for every $p \in [1, \infty]$,

\[(32) \quad \|P_t f\|_p \leq \|f\|_p, \quad f \in L^p(\mathbb{R}^d).
\]

In fact, since $p_t(x, y) \leq p_t(0, 0) = t^{-d/\alpha} p_1(0, 0) < \infty$, by Young inequality,

\[(33) \quad \|P_t f\|_\infty \leq c_t \|f\|_p, \quad f \in L^p(\mathbb{R}^d).
\]
By [11, Eq. 7], for $0 \leq \beta \leq d - \alpha$,

$$P_t h_{\beta} \leq h_{\beta}. \tag{34}$$

In this sense, $h_{\beta}$ is supermedian when $0 \leq \beta \leq d - \alpha$.

2.2. The domain of the form $\mathcal{E}_p$. Recall that $p \in (1, \infty)$. As usual, $L^p(\mathbb{R}^d)$ is the collection of all the real-valued functions on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |f(x)|^p dx < \infty$; we identify functions $u, v \in L^p(\mathbb{R}^d)$ if $u = v$ a.e. on $\mathbb{R}^d$. The dual space of $L^p(\mathbb{R}^d)$ is, of course, $L^q(\mathbb{R}^d)$, where $q = p/(p - 1)$. We also consider the canonical pairing $\langle u, v \rangle = \langle v, u \rangle = \int_{\mathbb{R}^d} u(x)v(x)dx$, $u \in L^p(\mathbb{R}^d)$, $v \in L^q(\mathbb{R}^d)$.

For arbitrary Borel measurable $u : \mathbb{R}^d \to \mathbb{R}$ we have $(b - a)(b^{(p-1)} - a^{(p-1)}) \geq 0$, the nonlinear form

$$\mathcal{E}_p[u] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(u^{(p-1)}(x) - u^{(p-1)}(y))v(x,y)dy \, dx \tag{35}$$

is well-defined, perhaps infinite, for every $u : \mathbb{R}^d \to \mathbb{R}$. We define its natural domain:

$$\mathcal{D}(\mathcal{E}_p) = \{ u \in L^p(\mathbb{R}^d) : \mathcal{E}_p[u] < \infty \}.$$ 

For $p = 2$, as usual, we let

$$\mathcal{E}[u] = \lim_{t \to 0} \frac{1}{t} \langle u - P_t u, u \rangle,$$

where $u \in L^2(\mathbb{R}^d)$ is arbitrary and the expression under the limit is decreasing in $t$ (see Hille and Phillips [36, Section 22.3] and [33, Lemma 1.3.4]). We also let

$$\mathcal{D}(\mathcal{E}) = \{ u \in L^2(\mathbb{R}^d) : \mathcal{E}[u] < \infty \}.$$ 

Note that the quantity suggested by the definition of $\mathcal{E}[u]$,

$$\mathcal{E}^{(t)}(u,v) := \frac{1}{t} \langle u - P_t u, v \rangle, \quad t > 0,$$

makes sense for all $u \in L^p(\mathbb{R}^d)$ and $v \in L^q(\mathbb{R}^d)$ because $P_t L^p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. In fact, $(P_t)_{t \geq 0}$ form a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^d)$, see, e.g., Kwaśnicki [44], and we denote by $\mathcal{D}_p(\Delta^{\alpha/2})$ the domain of $\Delta^{\alpha/2}$ considered as the generator of this semigroup. We can now characterize $\mathcal{D}(\mathcal{E}_p)$. We note that (39) is a variant of [46, Theorem 3.1], see also [46, Remark (3)].

**Lemma 7.** Let $p > 1$. Then for every $u \in L^p(\mathbb{R}^d)$ we have

$$\mathcal{E}_p[u] = \lim_{t \to 0} \mathcal{E}^{(t)}(u,u^{(p-1)}). \tag{36}$$

Furthermore,

$$\mathcal{D}(\mathcal{E}_p) = \{ u \in L^p(\mathbb{R}^d) : \sup_{t > 0} \mathcal{E}^{(t)}(u,u^{(p-1)}) < \infty \} \tag{37}$$

$$= \{ u \in L^p(\mathbb{R}^d) : \text{finite } \lim_{t \to 0} \mathcal{E}^{(t)}(u,u^{(p-1)}) \text{ exists} \}.$$ 

For arbitrary Borel measurable $u : \mathbb{R}^d \to \mathbb{R}$ we also have

$$\frac{4(p-1)}{p^2} \mathcal{E}[u^{(p/2)}] \leq \mathcal{E}_p[u] \leq 2\mathcal{E}[u^{(p/2)}]. \tag{39}$$
and $\mathcal{D}(\mathcal{E}_p) = \mathcal{D}(\mathcal{E})^{(2/p)} := \{v^{(2/p)} : v \in \mathcal{D}(\mathcal{E})\}$. Finally, $\mathcal{D}_p(\Delta^{\alpha/2}) \subset \mathcal{D}(\mathcal{E}_p)$ and

$$
(40) \quad \mathcal{E}_p[u] = -\langle \Delta^{\alpha/2} u, u^{(p-1)} \rangle, \quad u \in \mathcal{D}_p(\Delta^{\alpha/2}).
$$

**Proof.** Since $\int_{\mathbb{R}^d} p_t(x,y)dy = 1$ for all $t > 0$, $x \in \mathbb{R}^d$, the symmetry of $p_t(x,y)$ yields

$$
\mathcal{E}^{(t)}(u,u^{(p-1)}) = \frac{1}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(x,y)(u(x) - u(y))dy \, u(x)^{p-1} \, dx
$$

$$
= \frac{1}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(y,x)(u(y) - u(x))dx \, u(y)^{p-1} \, dy
$$

$$
= \frac{1}{2t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(x,y)(u(x) - u(y))(u(x)^{p-1} - u(y)^{p-1}) \, dy \, dx
$$

for every $u \in L^p(\mathbb{R}^d)$. By Lemma 6, (27) and (28) we get (36). Since the limit in (36) exists for all $u \in L^p(\mathbb{R}^d)$, we obtain (37) and (38), using dominated convergence. From (8) we obtain (39). Of course, if $u \in \mathcal{D}(\mathcal{E})$ and $u = u^{(2/p)}$, then by (39), $\mathcal{E}_p[u] \leq 2\mathcal{E}[(u^{(2/p)})^{(p/2)}] = 2\mathcal{E}[v] < \infty$, so $u \in \mathcal{D}(\mathcal{E}_p)$. Conversely, if $u \in \mathcal{D}(\mathcal{E}_p)$ and we define $v = u^{(p/2)}$, then by (39), $\mathcal{E}[v] \leq p^2(4(p-1))^{-1}\mathcal{E}_p[u] < \infty$, so $v \in \mathcal{D}(\mathcal{E})$ and clearly $u = v^{(2/p)}$.

Let $u \in \mathcal{D}_p(\Delta^{\alpha/2})$. Consider again $\frac{1}{t}(u - P_t u, u^{(p-1)})$ as $t \to 0^+$. By the definition of the semigroup generator [44], we see that $(P_t u - u)/t$ converges to $\Delta^{\alpha/2} u$ in $L^p(\mathbb{R}^d)$. Since $u^{(p-1)} \in L^{p/(p-1)}(\mathbb{R}^d)$, the expression $\mathcal{E}^{(t)}(u,u^{(p-1)})$ tends to (finite) $-\langle \Delta^{\alpha/2} u, u^{(p-1)} \rangle$. On the other hand, it converges to $\mathcal{E}_p[u]$ by (36).

### 3. Hardy identity and inequality

**Proof of Theorem 1.** For $\beta$ as in the assumptions of the theorem, we write $h = h_\beta$, $f = f_\beta$, etc. Take $v \in L^p(\mathbb{R}^d)$ and let $u = vh$, so that $v h = u$ a.e. The factorization $v h = u$ is the essence of Doob’s conditioning, which inspired the calculations in [11, Theorem 2] and in what follows. Let $t > 0$. Of course, $v h \in L^p(\mathbb{R}^d)$, and by (34), $v P_t h \in L^p(\mathbb{R}^d)$. Consider (36). We have

$$
\mathcal{E}^{(t)}(vh,vh^{(p-1)}) = \frac{1}{t}(vh,vh^{(p-1)}) - \frac{1}{t}(P_t(vh),(vh)^{p-1})
$$

$$
= \frac{p-1}{p} \langle v \frac{h-P_th}{t}, vh^{(p-1)} \rangle + \frac{p-1}{p} \langle v \frac{P_th}{t}, vh^{(p-1)} \rangle
$$

$$
+ \frac{1}{p} \langle v^{(p-1)} \frac{h^{p-1}-P_th^{p-1}}{t}, vh \rangle + \frac{1}{p} \langle v^{(p-1)} \frac{P_th^{p-1}}{t}, vh \rangle
$$

$$
- \langle \frac{P_t(vh)}{t}, (vh)^{(p-1)} \rangle = I^{(1)}_t + I^{(2)}_t + J_t,
$$
where

\[
I^{(1)}_t := \frac{p - 1}{p} \langle v h - P_t h, (vh)^{(p-1)} \rangle = \frac{p - 1}{p} \int_{\mathbb{R}^d} |v(x)|^p h(x)^{p-1} \frac{(h - P_t h)(x)}{t} \, dx,
\]
\[
I^{(2)}_t := \frac{1}{p} \langle v^{(p-1)} h^{p-1} - P_t h^{p-1}, vh \rangle = \frac{1}{p} \int_{\mathbb{R}^d} |v(x)|^p h(x)^{p-1} \frac{(h^{p-1} - P_t h^{p-1})(x)}{t} \, dx,
\]
\[
J_t := \frac{1}{p} \langle v^{(p-1)} P_t h^{p-1}, vh \rangle + \frac{p - 1}{p} \langle P_t h, (vh)^{(p-1)} \rangle - \langle \frac{P_t (vh)}{t}, (vh)^{(p-1)} \rangle.
\]

By the symmetry of \(p_t(x, y)\) and the definition of \(F_p\),
\[
J_t = \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v(x)|^p h(y)^{p-1} h(x) \frac{p_t(x, y)}{t} \, dy \, dx
\]
\[
+ \frac{p - 1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v(x)|^p h(x)^{p-1} h(y) \frac{p_t(x, y)}{t} \, dy \, dx
\]
\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)^{(p-1)} v(y) h(x)^{p-1} h(y) \frac{p_t(x, y)}{t} \, dy \, dx
\]
\[
= \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ |v(y)|^p - |v(x)|^p - pv(y)^{(p-1)} (v(y) - v(x)) \right] h(x)^{p-1} h(y) \frac{p_t(x, y)}{t} \, dy \, dx
\]
\[
= \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p (v(x), v(y)) h(x)^{p-1} h(y) \frac{p_t(x, y)}{t} \, dy \, dx.
\]

By Lemma 6, (27) and (28),
\[
\lim_{t \to 0} J_t = \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p (v(x), v(y)) h(x)^{p-1} h(y) \nu(x, y) \, dy \, dx
\]
\[
= \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p \left( \frac{u(x)}{h(x)}, \frac{u(y)}{h(y)} \right) h(x)^{p-1} h(y) \nu(x, y) \, dy \, dx.
\]

We next consider \(I^{(1)}_t\) and recall that \(h(x) = \int_0^\infty f(s)p_s(x) \, ds\), so for \(x \neq 0\),
\[
(h - P_t h)(x) = \int_0^\infty f(s)p_s(x) \, ds - \int_0^\infty f(s)p_{s+t}(x) \, ds
\]
\[
= \int_0^\infty [f(s) - f(s-t)] p_s(x) \, ds.
\]

Thus,
\[
I^{(1)}_t = \frac{p - 1}{p} \int_{\mathbb{R}^d} |v(x)|^p h(x)^{p-1} \int_0^\infty \frac{1}{t} [f(s) - f(s-t)] p_s(x) \, ds \, dx.
\]

We have \(0 \leq f(s) - f(s-t) \leq Ctf'(s)\) for \(s, t > 0\). By Lemma 6 and (31) we get
\[
\lim_{t \to 0} I^{(1)}_t = \frac{p - 1}{p} \int_{\mathbb{R}^d} |v(x)|^p h(x)^{p-1} \int_0^\infty f'(s)p_s(x) \, ds \, dx
\]
\[
= \frac{p - 1}{p} \int_{\mathbb{R}^d} |v(x)|^p h(x)^p q(x) \, dx = \frac{(p - 1)\kappa_\beta}{p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^{\alpha}} \, dx.
\]
We can treat $I_t^{(2)}$ analogously. The analogy has several layers. We let $p' = p/(p - 1)$. We have $p = p'/(p' - 1)$ and $(p' - 1)(p - 1) = 1$. By considering (12) and (29) we get

$$I_t^{(2)} = \frac{p'}{p'} - 1 \int_{\mathbb{R}^d} \left| v(x) \right|^{(p' - 1)} |v' h_{(p-1)\beta}(x) h_{(p-1)\beta}(x) - P_{t h_{(p-1)\beta}(x)} | \frac{dx}{t}.$$  

We note that $0 \leq (p - 1)\beta \leq d - \alpha$ and by the case of $I_t^{(1)}$ we get

$$\lim_{t \to 0} I_t^{(2)} = \frac{\kappa(p-1)\beta}{p} \int_{\mathbb{R}^d} \left| u(x) \right|^p \frac{dx}{|x|^\alpha}.$$  

By (36), (41), (42) and (43) we get (13). □

For $\beta = (d - \alpha)/p$, we have $\kappa(p-1)\beta = \kappa_{d-\alpha-\beta} = \kappa_\beta$, which yields (14). As aforementioned in Introduction, our constant is better than that in (9).

**Lemma 8.** If $p \neq 2$, then $\kappa_{(d-\alpha)/p} > 4(p-1)p^{-2}\kappa_{(d-\alpha)/2}$.

**Proof.** Let $p > 2$. Denote

$$r(t) = \kappa_{2t} = \frac{2^\alpha \Gamma(\frac{d}{2} + t) \Gamma(\frac{d}{2} - t)}{\Gamma(t) \Gamma((d - \alpha)/2 - t)}, \quad t \in (0, \frac{d-\alpha}{4}).$$

Put $t = \frac{d-\alpha}{2p}$. Then $p = \frac{d-\alpha}{2t}$ and

$$\frac{4(p-1)}{p^2} = \frac{16t^2}{(d - \alpha)^2} \left( \frac{d - \alpha}{2t} - 1 \right) = \frac{16}{(d - \alpha)^2} t ((d - \alpha)/2 - t) =: s(t).$$

We only need to verify that

$$r(t) > \kappa_{(d-\alpha)/2}s(t), \quad t \in (0, \frac{d-\alpha}{4}).$$

Notice that $r(0^+) = 0$ and $s(0^+) = 0$ and $r(\frac{d-\alpha}{4}) = \kappa_{(d-\alpha)/2} = \kappa_{(d-\alpha)/2}s(\frac{d-\alpha}{4})$. Let

$$F(t) = \ln r(t) - \ln \kappa_{(d-\alpha)/2}s(t).$$

It suffices to show that $F(t) > 0$ for $t \in (0, \frac{d-\alpha}{4})$. By [11, Proof of Proposition 5],

$$F'(t) = \sum_{k=0}^{\infty} \left( \frac{1}{\frac{d}{2} + k - t} - \frac{1}{t + \frac{d}{2} + k} - \frac{1}{(d - \alpha)/2 + k - t} + \frac{1}{t + k} \right) - \left( \frac{1}{t} - \frac{1}{(d - \alpha)/2 - t} \right),$$

hence

$$F'(t) = \sum_{k=0}^{\infty} \left( \frac{1}{\frac{d}{2} + k - t} - \frac{1}{t + \frac{d}{2} + k} - \frac{1}{(d - \alpha)/2 + k - t} + \frac{1}{t + k} \right)$$

$$= \sum_{k=0}^{\infty} \left( \frac{\frac{d}{2} - 1}{(t + 1 + k)(t + \frac{d}{2} + k)} - \frac{\frac{d}{2} - 1}{(\frac{d}{2} + k - t)(\frac{d}{2} + k - t + \frac{d}{2} + k - t)} \right)$$

$$= \alpha - \frac{2}{2} \sum_{k=0}^{\infty} \frac{k(d - \alpha) + (d - \alpha)(d + 2)/4 - t(d + 2 + 4k)}{(t + 1 + k)(t + \frac{d}{2} + k)(\frac{d}{2} + k - t)(\frac{d}{2} + k - t + \frac{d}{2} + k - t)}.$$

The numerator of every term in the last series is decreasing in $t$ and equal to 0 for $t = \frac{d-\alpha}{4}$, so $F'(t) < 0$ for $t \in (0, \frac{d-\alpha}{4})$. Since $F(\frac{d-\alpha}{4}) = 0$, $F(t) > 0$ for $t \in (0, \frac{d-\alpha}{4})$. We
now consider the case $p \in (1, 2)$. Let $q = \frac{p}{p-1}$. Of course, $q \in (2, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have $\kappa/(d-\alpha)/q = \kappa/(d-\alpha)/p$, since $(d-\alpha)/q + (d-\alpha)/p = d-\alpha$ and $\kappa/\beta$ is symmetric with respect to $(d-\alpha)/2$. Furthermore, $4(p-1)p^{-2} = 4/(pq) = 4(q-1)q^{-2}$. □

4. Optimality

In this section we prove Theorem 2. The argument is rather technical due to integrability issues with the intended test function for (15). We start with three auxiliary lemmas. The following decomposition of $E_p[u]$ is different and simpler than (13) but we should note that the second term needs not be nonnegative or finite.

**Lemma 9.** Under the assumptions of Theorem 1 we have

$$E_p[u] = \kappa_{\beta} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} \, dx$$

$$+ \lim_{t \to 0^+} \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{h(x) - h(y)} \left( \frac{u(x)^{(p-1)} h(x) - u(y)^{(p-1)} h(y)}{h(x) - h(y)} \right) \frac{h(x) h(y) p_t(x, y)}{t} \, dy \, dx,$$

if additionally $u \in L^p(\mathbb{R}^d, |x|^{-\alpha} \, dx)$.

**Proof.** Thus, $u \in L^p(\mathbb{R}^d, (1 + |x|^{-\alpha}) \, dx)$. Let $v = u/h$. By (36),

$${E_p[u]} \equiv \lim_{t \to 0^+} E^{(t)}(vh, (vh)^{(p-1)}).$$

By the proof of Theorem 1, $vP_t h \in L^p(\mathbb{R}^d)$. We have

$$E^{(t)}(vh, (vh)^{(p-1)}) = \langle v \frac{h - P_t h}{t}, (vh)^{(p-1)} \rangle + \langle \frac{vP_t h - P_t(vh)}{t}, (vh)^{(p-1)} \rangle =: I_t + J_t.$$

By (42) and the assumption $u \in L^p(\mathbb{R}^d, |x|^{-\alpha} \, dx)$

$$\lim_{t \to 0^+} I_t = \kappa_{\beta} \int_{\mathbb{R}^d} \frac{|v(x) h(x)|^p}{|x|^\alpha} \, dx < \infty.$$

Symmetrizing the integrand in $J_t$ we get

$$J_t = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x) - v(y)) h(y) v(x)^{(p-1)} h(x) p_t(x, y) \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x) - v(y)) \left( (v(x)^{(p-1)} h(x)^{(p-1)} h(y) - v(y)^{(p-1)} h(y)^{(p-1)} h(x) \right) \frac{p_t(x, y)}{2t} \, dy \, dx.$$

Since $u = vh$, the result follows. □

Here is an informal idea of the proof of Theorem 2. Let

$$(44) \quad u(x) = |x|^{-\beta/(p-1) \wedge |x|^{-\beta}}, \quad x \in \mathbb{R}^d.$$  

Since $h = h_\beta$, the integrand under the limit in Lemma 9 is zero if $|x|, |y| \leq 1$ and if $|x|, |y| \geq 1$, and it has negative sign otherwise. This reverses the Hardy inequality, but, unfortunately we need to face the aforementioned integrability issues. The next result gives a preparation. For $r > 0$ we let $B(0, r) = \{x \in \mathbb{R}^d : |x| < r \}$. Also, we let $B_1 = B(0, 1)$ and $B_1^c = \mathbb{R}^d \setminus B_1$.  

Lemma 10. If $p > 2$ and $\mu > \beta \vee (d - \alpha)/p$, then
\[ \int_{B_1^c} \int_{B_1^c} |x|^{\beta - \mu} - |y|^{\beta - \mu} \left| |x|^{\beta - (p - 1)\mu} - |y|^{\beta - (p - 1)\mu} \right| |x|^{-\beta} |y|^{-\beta} \nu(x, y) \, dy \, dx < \infty. \]

Proof. For $x \in B_1^c$ we let $D_1(x) = B_1^c \cap B(0, |x|/2) = \{y : 1 \leq |y| < |x|/2\}$, $D_2(x) = B_1^c \cap B(0, 2|x|)^c = B(0, 2|x|)^c = \{y : 2|x| \leq |y|\}$, and $D_3(x) = B_1^c \setminus (D_1(x) \cup D_2(x)) = \{y : |x|/2 \leq |y| < 2|x|\}$. The sets form a partition of $B_1^c$. Indeed, the integrand is symmetric in $x, y$, on the left we integrate over $\{x, y \in B_1^c : 2|y| \leq |x|\}$, and on the right over $\{x, y \in B_1^c : 2|x| \leq |y|\}$. Then for $y \in D_1(x)$ we have $|y| \leq |x|$ and $|x - y| \geq |x|/2$, hence
\[ \int_{B_1^c} \int_{D_1(x)} |x|^{-d - \alpha} |y|^{-\mu} \, dy \, dx \leq 2^{d + \alpha} \int_{B_1^c} \int_{D_1(x)} |x|^{-d - \alpha} |y|^{-\mu} \, dy \, dx = 2^{d + \alpha} \int_{B_1^c} \int_{B(0, 2|x|)^c} |x|^{-d - \alpha} |y|^{-\mu} \, dy \, dx = c \int_{B_1^c} |y|^{-\mu - \alpha} \, dy < \infty. \]

For $y \in D_2(x)$ we have $|y| \geq |x|$ and $|x - y| \geq |y|/2$, hence
\[ \int_{B_1^c} \int_{D_2(x)} |x|^{-d - \alpha} |y|^{-\mu} \, dy \, dx \leq 2^{d + \alpha} \int_{B_1^c} \int_{D_2(x)} |x|^{-d - \alpha} |y|^{-\mu} \, dy \, dx = c \int_{B_1^c} |x|^{-\mu - \alpha} \, dx < \infty. \]

We next note that for $a, b > 0$ and $\gamma > 0$,
\begin{equation}
|a^{-\gamma} - b^{-\gamma}| \leq \gamma |a - b| (a \wedge b)^{-\gamma - 1}.
\end{equation}

Indeed, if, say, $a \leq b$, then
\[ a^{-\gamma} - b^{-\gamma} = \gamma \int_a^b s^{-\gamma - 1} \, ds \leq \gamma (b - a) a^{-\gamma - 1}. \]

For $y \in D_3(x)$ we have $|x|/2 < |y| < 2|x|$, therefore,
\[ \int_{B_1^c} \int_{D_3(x)} \leq c_1 \int_{B_1^c} \int_{B(0, 3|x|)} |x|^{-\mu - 2} |y|^{-d - \alpha + 2} \, dy \, dx \leq c_1 \int_{B_1^c} \int_{B(0, 3|x|)} |x|^{-\mu - 2} |z|^{-d - \alpha + 2} \, dz \, dx = c_2 \int_{B_1^c} |x|^{-\mu - \alpha} \, dx < \infty. \]

\[ \square \]

Here is another auxiliary estimate.

Lemma 11. If $p > 2$, $0 < \beta < \mu$, $R > 1$ and $A_R = \{x \in \mathbb{R}^d : 1/R < |x| < 1\}$, then
\[ \int_{A_R} \int_{B_1^c} \left| 1 - |y|^{\beta - \mu} \right| \left| |x|^{(p - 2)\beta} - |y|^{\beta - (p - 1)\mu} \right| |x|^{-\beta} |y|^{-\beta} \nu(x, y) \, dy \, dx \leq c (1 \vee R^{(p - 1)\beta - d} \vee \log R). \]

Proof. We split the integral as follows:
\[ \int_{A_R} \int_{B_1^c} = \int_{D_1} \int_{B_1^c} + \int_{D_2} \int_{D_3} + \int_{D_2} \int_{D_4} =: I_1 + I_2 + I_3, \]
To estimate $\int_{D_1} I_1 \leq c \int_{D_1} \int_{B_1^c} |x|^{-(p-1)\beta} |y|^{-d-\alpha-\beta} dy dx = c_2 \int_{D_1} |x|^{-(p-1)\beta} dx \\
\leq c(1 \vee R^{(p-1)\beta-d} \vee \log R).$

If $x \in D_2$, $y \in D_3$, then $1 - |y|^{\beta-\mu} < |x|^{\beta-\mu} - |y|^{\beta-\mu}$ and $|x|^{-(p-2)\beta} - |y|^{-(p-2)\mu} < |x|^{\beta-(p-1)\mu} - |y|^{\beta-(p-1)\mu}$, so by (45),

$$|I_2| \leq c \int_{D_2} \int_{D_3} |x|^{-(p-1)\beta} |x-y|^{-d-\alpha+2} dy dx \leq c \int_{D_2} \int_{D_3} |x-y|^{-d-\alpha+2} dy dx \leq c_1 \int_{D_2} \int_{B(0,3)} |z|^{-d-\alpha+2} dz dx < \infty.$$  

To estimate $I_3$, we note that for $x \in D_2$, $y \in D_4$ we have $|x-y| \geq \frac{1}{2} |y|$, so

$$I_3 \leq c \int_{D_2} \int_{D_4} |x-y|^{-d-\alpha} |y|^{-\beta} dy dx \leq c_2 \int_{D_4} |y|^{-d-\beta-\alpha} dy dx < \infty.$$

Proof of Theorem 2. By Theorem 1 and (14), $\mathcal{E}_p[u] \geq \kappa_{(d-\alpha)/p} \int_{\mathbb{R}^d} |u(x)|^p |x|^{-\alpha} dx$ for $u \in L^p(dx)$. Let $\kappa > \kappa_{(d-\alpha)/p}$. It suffices to verify that

$$\mathcal{E}_p[u] < \kappa \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx < \infty$$

for some $u \in L^p(\mathbb{R}^d, (1 \vee |x|^{-\alpha}) dx)$.

We first consider the case $p > 2$. Let $(d-\alpha)/p < \beta < (p-1)(d-\alpha)/p$. We note that $\kappa_{\beta} > \kappa_{(d-\alpha)/p}$. As usual, let $h(x) = h_{\beta}(x) = |x|^{-\beta}$. Let $\mu > \beta \vee d/p$, $R > 1$ and

$$(48) \quad u_R(x) = |x|^{-\beta/(p-1)} \wedge |x|^{-\beta} \wedge R^{\mu-\beta} |x|^{-\mu}, \quad x \in \mathbb{R}^d,$$

comp. (44). This will be our test function for the Hardy inequality (15). We have $u_R \in L^p(\mathbb{R}^d, (1 + |x|^{-\alpha}) dx)$. Therefore not only the Hardy inequality holds true for $u_R$, but also the representation from Lemma 9 is valid:

$$\mathcal{E}_p[u_R] = \kappa_{\beta} \int_{\mathbb{R}^d} \frac{|u_R(x)|^p}{|x|^\alpha} dx \\
+ \lim_{t \to 0} \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{u_R(x)}{h(x)} - \frac{u_R(y)}{h(y)} \right) \left( \frac{u_R(x)^{(p-1)}}{h(x)} - \frac{u_R(y)^{(p-1)}}{h(y)} \right) h(x)h(y) \frac{p_h(x,y)}{t} dy dx,$$

and the first integral on the right-hand side is finite. We next show that the limit of the above double integral can be made negative by choosing a sufficiently large $R$. Let $B_1 = B(0,1)$, $B_2 = B_2^R = B(0,R) \setminus B(0,1)$, $B_3 = B_3^R = B(0,R)^c$. By symmetry, we
can split the integral as follows:
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} = \int_{B_1} \int_{B_1} + \int_{B_2} \int_{B_2} + 2 \int_{B_1} \int_{B_2} + \int_{B_1} \int_{B_3} + 2 \int_{B_1} \int_{B_3} + 2 \int_{B_2} \int_{B_3} =: I_1 + I_2 + 2I_3^R + I_4^R + 2I_5^R + 2I_6^R.
\]
Note that \( u_R(x) = |x|^{-\beta/(p-1)} \) on \( B_1 \), \( u_R(x) = |x|^{-\beta} \) on \( B_2 \) and \( u_R(x) = R^{\mu-\beta}|x|^{-\mu} \) on \( B_3 \). We see that \( I_1 = I_2 = 0 \) and the the integrand in \( I_3^R \) is negative. Moreover, observe that \( I_3^R \) decreases when \( R \) increases, so there is a constant \( A_0 > 0 \) such that
\[
\limsup_{t \to 0} \int_{B_1} \int_{B_2} \int_{B_3} \left( \frac{u_R(x)}{h(x)} - \frac{u_R(y)}{h(y)} \right) h(x)h(y)\frac{p(x,y)}{t} \ dy \ dx < -A_0
\]
for all \( R > 1 \). We also have
\[
|I_4^R| \leq C \int_{B_1} \int_{B_2} \left| x \right|^\beta/\beta/(p-1) - R^{\mu-\beta} \left| y \right|^\beta/\beta/(p-1) \left| 1 - R^{(p-1)-(\mu-\beta)} \right| \left| y \right|^{\beta-(p-1)/\mu} \left| x \right|^{-\beta} \left| y \right|^{-\beta} \nu(x,y) \ dy \ dx
\]
\[
= CR^{d-\alpha-p} \int_{B_1} \int_{B_2} \left| x \right|^\beta/\beta/(p-1) - \left| y \right|^{-\beta} \nu(x,y) \ dy \ dx + C \int_{B_1} \int_{B_3} \left| x \right|^{-\beta} \left| y \right|^{\beta-\mu} R^{\mu-\beta} \nu(x,y) \ dy \ dx
\]
\[
\leq c \int_{B_1} \int_{B_3} \left| y \right|^{-\beta} \ dy + cR^{\mu/(\mu-\beta)} \int_{B_3} \left| y \right|^{\beta-\mu} \ dy = c_1 \left( R^{-\beta-\alpha} + R^{-(p-1)-(\beta-\alpha)} \right) \to 0 \text{ as } R \to \infty.
\]
Finally,
\[
|I_5^R| \leq C \int_{B_2} \int_{B_3} \left( 1 - R^{\mu-\beta} \left| y \right|^\beta/\beta/(p-1) \left| x \right|^{(p-2)-\beta} - R^{(p-1)-(\mu-\beta)} \left| y \right|^{\beta-(p-1)/\mu} \left| x \right|^{-\beta} \left| y \right|^{-\beta} \nu(x,y) \ dy \ dx
\]
\[
= CR^{d-\alpha-p} \int_{B_2} \int_{B_3} \left( 1 - \left| y \right|^{\beta/\beta/(p-1)} \right| x \right|^{(p-2)-\beta} - \left| y \right|^{\beta-(p-1)/\mu} \left| x \right|^{-\beta} \left| y \right|^{-\beta} \nu(x,y) \ dy \ dx
\]
\[
0 \text{ as } R \to \infty,
\]
see Lemma 11. Hence, for \( R \) sufficiently large, \( \mathcal{E}_p[u_R] < \kappa \beta \int_{\mathbb{R}^d} |u_R(x)|^p |x|^{-\alpha} \ dx \).

Since \( \beta \mapsto \kappa \beta \) is continuous, symmetric about \((d-\alpha)/2\) and increasing on \((0,(d-\alpha)/2)\), there is \( \beta \in ((d-\alpha)/p,(p-1)(d-\alpha)/p) \) such that \( \kappa(d-\alpha)/p < \kappa \beta < \kappa \).

Then \( \mathcal{E}_p[u_R] < \kappa \int_{\mathbb{R}^d} |u_R(x)|^p |x|^{-\alpha} \ dx \), which proves (47), and the theorem, for \( p > 2 \).

We now consider \( p \in (1,2) \). Let \( p' < \frac{p}{p-1} \) be its H"older conjugate. Of course, \( p' \in (2,\infty) \) and \((p-1)(p'-1) = 1\), or \((p-1)p' = p\). For \( u \in L^p(\mathbb{R}^d) \) we let \( v = u^{(p-1)} \),
i.e., $v^{(p'/2-1)} = u$. We have $|u|^p = |v|^{p'}$, so $\int_{\mathbb{R}^d} |v|^{p'}\,dx = \int_{\mathbb{R}^d} |u|^p\,dx$ and $v \in L^{p'}(\mathbb{R}^d)$. Also,

$$\mathcal{E}_p[v] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x)^{p'-1} - v(y)^{p'-1})(v(x) - v(y))\nu(x,y)\,dy\,dx = \mathcal{E}_p[u]$$

and

$$\int_{\mathbb{R}^d} \frac{|v(x)|^{p'}}{|x|^\alpha}\,dx = \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha}\,dx.$$ 

It follows from the first part of the proof that the Hardy inequality holds for the exponents $p$ and $p'$ with the same constants. In particular, the constant $\kappa_{(d-\alpha)/p}$ is optimal on $L^{p'}(\mathbb{R}^d)$. This conforms with our claim, since $\kappa_{(d-\alpha)/p'} = \kappa_{(d-\alpha)/p}$, as noted in the proof of Lemma 8.

The following technical result will be useful in Section 5. Its proof may be of independent interest, since it sheds some light on the structure of $\mathcal{D}(\mathcal{E}_p)$.

Lemma 12. If $p > 2$ then the inequality (47) holds for some function in $C_c(\mathbb{R}^d)_+$. 

Proof. Assume that $\kappa > \kappa_{(d-\alpha)/p'}$. Let $u$ be the function defined by (48) with suitable $R$ and such that (47) holds. The function is radially decreasing, meaning that $u(x) \leq u(y)$ if $|x| \geq |y|$. For $\epsilon > 0$ we let $\phi_\epsilon(a) = (a - \epsilon) \vee 0$, $a \in \mathbb{R}$. Of course,

$$|\phi_\epsilon(a)| \leq |a| \quad \text{and} \quad |\phi_\epsilon(b) - \phi_\epsilon(a)| \leq |b - a|, \quad a,b \in \mathbb{R}.$$ 

Consequently, by (21),

$$F_p(\phi_\epsilon(u(x)), \phi_\epsilon(u(y))) \approx (\phi_\epsilon(u(y)) - \phi_\epsilon(u(x)))^2(|\phi_\epsilon(u(y))| + |\phi_\epsilon(u(x))|)^{p-2} \leq (u(y) - u(x))^2 (|u(y)| + |u(x)|)^{p-2} \approx F_p(u(x), u(y)).$$

Since $\phi_\epsilon(u) \to u$ as $\epsilon \to 0$, from the dominated convergence theorem we get

$$\mathcal{E}_p[\phi_\epsilon(u)] \to \mathcal{E}_p[u].$$ 

By Fatou’s lemma

$$\liminf_{\epsilon \to 0} \int \frac{\phi_\epsilon(u)^p}{|x|^\alpha}\,dx \geq \int \frac{|u|^p}{|x|^\alpha}\,dx,$$

therefore taking a sufficiently small $\epsilon$ we get that (47) holds with $\phi_\epsilon(u)$. Slightly abusing the notation, below we write $u$ for the latter function, so $u := \phi_\epsilon(u)$. The function is radially decreasing and compactly supported and we have $u^{(p'/2)} \in \mathcal{D}(\mathcal{E})$, which follows from (8). Let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be smooth, compactly supported, radially decreasing and such that $\int \varphi = 1$. For $\eta > 0$ denote $\varphi_\eta(x) = \eta^{-d} \varphi(x/\eta)$ and let

$$v_\eta = u^{(p'/2)} \ast \varphi_\eta.$$ 

Each $v_\eta$ is smooth, nonnegative and of compact support. It is also radially decreasing, as a convolution of two such functions, see Beckner [5, page 171]. It is evident that $v_\eta \to u^{(p'/2)}$ (pointwise) as $\eta \to 0$. By the arguments from [14, the proof Lemma A.5] we see that $\mathcal{E}[v_\eta - u^{(p'/2)}] \to 0$, hence by Vitali’s theorem, the functions

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto (v_\eta(x) - v_\eta(y))^2$$

...
are uniformly integrable with respect to $\nu(x,y)dx dy$, see Schilling [56, Theorem 22.7].

Let

$$u_\eta = (v_\eta)^{(2/p)} = \left(u^{(p/2)} * \varphi_\eta\right)^{(2/p)}.$$  

Since $v_\eta \geq 0$ is smooth and radially decreasing, $u_\eta$ is smooth, despite the fractional exponent in its definition. It is clear that $u_\eta \to u$ pointwise. Since

$$F_p(u_\eta(x), u_\eta(y)) \approx (v_\eta(x) - v_\eta(y))^2,$$

the left-hand side is uniformly integrable with respect to $\nu(x,y)dx dy$. By Vitali’s theorem again, $\mathcal{E}_p[u_\eta] \to \mathcal{E}_p[u]$ as $\eta \to 0$. By Fatou’s lemma, taking sufficiently small $\eta$ we see that (47) holds with $u := u_\eta$. The proof is complete. \qed

5. Application to parabolic equation

In this section we prove for $p \in (1, \infty)$ that the Feynman-Kac semigroup generated by the Schrödinger operator $\Delta \alpha/2 + \kappa_{(d-\alpha)/p}|x|^{-\alpha}$ is a contraction on $L^p(\mathbb{R}^d)$, and the threshold $\kappa_{(d-\alpha)/p}$ is sharp. We shall also see that the semigroup generated by $\Delta \alpha/2 + \kappa|x|^{-\alpha}$ is bounded on $L^p(\mathbb{R}^d)$ if and only if either $d/p^* \geq (d-\alpha)/2$ and $\kappa \leq \kappa_{(d-\alpha)/2}$ or $d/p^* < (d-\alpha)/2$ and $\kappa \leq \kappa_{d/p^*}$. Here $p^* \equiv \max\{p, p/(p-1)\}$.

As usual, let $\alpha \in (0,2)$ and $\alpha < d \in \mathbb{N}$. For $\delta \in [0, (d-\alpha)/2]$, $\kappa = \kappa_\delta$ and $q = q_\delta$ as in (31), we define (cf. [13]) the Schrödinger perturbation of $p_t$ from (25) by $q = q_\delta$:

$$\tilde{p}_t = \sum_{n=0}^{\infty} p_t^{(n)}.$$  

Here for $t > 0$ and $x, y \in \mathbb{R}^d$ we let $p_t^{(0)}(x,y) = p_t(x,y)$ and

$$p_t^{(n)}(x,y) = \int_0^t \int_{\mathbb{R}^d} p_s(x,z)q(z)p_{t-s}^{(n-1)}(z,y) \, dz \, ds$$

$$= \int_0^t \int_{\mathbb{R}^d} p_s^{(n-1)}(x,z)q(z)p_{t-s}(z,y) \, dz \, ds, \quad n \geq 1.$$  

From the general theory [11], $\tilde{p}_t$ is a symmetric transition density, i.e., the following Chapman-Kolmogorov equation holds:

$$\int_{\mathbb{R}^d} \tilde{p}_s(x,z)\tilde{p}_t(z,y) \, dz = \tilde{p}_{t+s}(x,y).$$

The scaling of $\tilde{p}_t(x,y)$ is the same as that of $p_t(x,y)$ [13, Lemma 2.2]:

$$\tilde{p}_t(x,y) = t^{-d/2} \tilde{p}_1(xt^{-1/2}, yt^{-1/2}), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$  

For $t > 0$ and $x \neq 0$ we define the Feynman-Kac semigroup

$$\tilde{P}_t f(x) = \int_{\mathbb{R}^d} \tilde{p}_t(x,y)f(y) \, dy.$$
The integral certainly makes sense if $f$ is nonnegative, but the following result in fact shows that $\tilde{P}_t$ may be contractive on $L^p(\mathbb{R}^d)$. Before we proceed, we note that for nonnegative functions $f, g$ on $\mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{p}_t(x,y) f(y) dy \ g(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{p}_t(x,y) g(y) dy \ f(x) dx. \tag{57}$$

By Hölder inequality, for each $p \in (1, \infty)$ the operator norm of $\tilde{P}_t$ on $L^p$ is the same as on $L^{p/(p-1)}$ - below this fact will be referred to as the duality argument. Furthermore, by (56) we have

$$\|\tilde{P}_t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \|\tilde{P}_t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)}, \quad t > 0. \tag{58}$$

Remark 1. For clarity we note that the heat kernel $\tilde{p}_t$ of $\Delta^{\alpha/2} + \kappa|x|^{-\alpha}$ can be defined also for $\kappa \in (-\infty, 0)$, see Jakubowski and Wang [39], see also Cho, Kim, Song and Vondraček [20]. Then $0 \leq \tilde{p}_t \leq p_t$, so $\|\tilde{P}_t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq \|P_t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = 1$ for every $p \geq 1$. Further, for $\kappa > \kappa_{(d-\alpha)/2}$ we have $\tilde{p}_t(x,y) \equiv \infty$ by [13, Corollary 4.11], so $\|\tilde{P}_t\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \infty$ in this case. The remaining cases are resolved in Theorem 3 and 4, by considering $\kappa = \kappa_\delta$ with $\delta \in [0, (d-\alpha)/2]$. We recall that analogous perturbations of local elliptic operators have been widely investigated. We refer, e.g., to Kovalenko, Perelmuter and Semenov [43]. In particular, the approach covers Hardy-type perturbations of the classical Laplacian. See also [18].

The relation of the Hardy inequality to the contractivity of the corresponding Feynman-Kac semigroups is also the subject of [2]. In [2, Corollary 1.2] the authors use the Hardy inequality to prove that the semigroup corresponding to the operator $-\Delta + \kappa|x|^{-2}$ is contractive on $L^p(\mathbb{R}^d)$ for $d \geq 2$ if and only if $\kappa \leq (d-2)^2(p-1)/p^2$. This accords well with Theorem 3 because $(d-2)^2(p-1)/p^2 = \kappa_{(d-\alpha)/p}$, if we let $\alpha = 2$.

Let us first present an informal idea of the proof of Theorem 3. Consider

$$u_t = \Delta^{\alpha/2} u + \kappa|x|^{-\alpha} u, \quad t > 0, \ x \in \mathbb{R}^d, \tag{59}$$

$$u(0, x) = f(x), \quad x \in \mathbb{R}^d.$$ 

The semigroup solution of this Cauchy problem is $u(t, x) = \tilde{P}_t f(x)$, at least for $f$ in the domain of the generator. We multiply both sides of (59) by $u^{(p-1)}$ and integrate

$$\int_{\mathbb{R}^d} u_t u^{(p-1)} dx = \int_{\mathbb{R}^d} u^{(p-1)} \Delta^{\alpha/2} u dx + \kappa \int_{\mathbb{R}^d} |u|^p |x|^{-\alpha} dx. \tag{60}$$

By calculus and (15),

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{|u|^p}{p} dx \leq - \kappa_{(d-\alpha)/p} + \kappa \int_{\mathbb{R}^d} |u|^p |x|^{-\alpha} dx \leq 0,$$

hence $\|u(t, \cdot)\|^p_p$ is decreasing, and so $\|\tilde{P}_t f\|^p_p \leq \|f\|^p_p$. On the other hand, for $\kappa > \kappa_{(d-\alpha)/p}$ we have the opposite inequality for the function from Lemma 12. We now present rigorous arguments (some technical results used in the proof are still deferred to Section 6).
Proof of Theorem 3. In the case of $p = 2$, the condition $\delta \leq (d - \alpha)/2$ is clearly met, while the contractivity of $\tilde{P}_t$ is proved in [13, Proposition 2.4], using Schur’s test. Let $p \in (2, \infty)$. For $M \in (0, \infty)$ let $q^{(M)}(x) = q(x) \wedge M$, where, recall, $q = q_\alpha$. In a similar manner as above, we define the Schrödinger perturbation of $p_t$ by $q^{(M)}$, which we denote $\tilde{p}^{(M)}_t$. Since $q^{(M)}$ is bounded, by Phillips’ perturbation theorem the domain of the generator $\Delta^{\alpha/2}$ of the strongly continuous semigroup $P_t$ on $L^p(\mathbb{R}^d)$ is the same as the domain of the generator, $\Delta^{\alpha/2} + q^{(M)}$, of the strongly continuous semigroup

$$\tilde{P}^{(M)}_t f(x) := \int_{\mathbb{R}^d} \tilde{p}^{(M)}_t(x, y)f(y)dy.$$  

We refer to Phillips [54, Theorem 3.2] and [44, Lemma 4.2] for details. Let $f$ be in the domain of $\Delta^{\alpha/2}$ on $L^p(\mathbb{R}^d)$. Let $u^{(M)}(t, x) = \tilde{P}^{(M)}_t f(x)$. Denote $u^{(M)}(t) = u^{(M)}(t, \cdot)$. From the general semigroup theory, the mapping $[0, \infty) \ni t \mapsto u^{(M)}(t) \in L^p(\mathbb{R}^d)$ is continuously differentiable. We then verify that $t \mapsto |u^{(M)}(t)|^p$ and $t \mapsto u^{(M)}(t)^{p-1}$ are continuous in $L^1(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, respectively, see Lemma 14. Then it follows from Lemma 16 that $|u^{(M)}(t)|^p$ is continuously differentiable in $L^1(\mathbb{R}^d)$ and

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u^{(M)}(t)|^p dx = \int_{\mathbb{R}^d} \frac{d}{dt} |u^{(M)}(t)|^p dx = \int_{\mathbb{R}^d} pu^{(M)}(t)^{(p-1)} \frac{d}{dt} u^{(M)}(t) dx$$

$$= p \int_{\mathbb{R}^d} u^{(M)}(t)^{(p-1)}(\Delta^{\alpha/2} u^{(M)}(t) + q^{(M)} u^{(M)}(t)) dx$$

$$\leq p \left(-\mathcal{E}_p[u^{(M)}(t)] + \int_{\mathbb{R}^d} q^{(M)} |u^{(M)}(t)|^p dx \right) \leq 0,$$

provided $\kappa \in [0, \kappa_{(d-\alpha)/p}]$, see (60). We then get $\int_{\mathbb{R}^d} |\tilde{P}^{(M)}_t f|^p dx \leq \int_{\mathbb{R}^d} |f|^p dx$. This extends to all $f \in L^p(\mathbb{R}^d)$ by the density of the domain of the generator. We then let $M \to \infty$. By monotone convergence, $\tilde{p}^{(M)}_t \uparrow \tilde{p}_t$ and for every nonnegative $f \in L^p(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\tilde{P}_t f)^p dx \leq \int_{\mathbb{R}^d} (\tilde{P}^{(M)}_t f)^p dx \leq \int_{\mathbb{R}^d} f^p dx.$$  

The estimate for general $f \in L^p(\mathbb{R}^d)$ follows by considering $f = f_+ - f_-$, see (67).

The case of $p \in (1, 2)$ results from duality with the same range of $\kappa$ as for the Hölder conjugate of $p$, that is the contractivity of $\tilde{P}_t$ holds for $\kappa \in [0, \kappa_{(d-\alpha)/p}]$ with $p' = p/(p-1)$. But $\kappa_{(d-\alpha)/p'} = \kappa_{(d-\alpha)-(d-\alpha)/p} = \kappa_{(d-\alpha)/p}$. Thus, the contractivity of $\tilde{P}_t$ on $L^p(\mathbb{R}^d)$ is proved if $p \in (1, \infty)$ and $\kappa \leq \kappa_{(d-\alpha)/p}$.

We next assume $\kappa > \kappa_{(d-\alpha)/p}$ and prove that the contractivity fails. To this end we first consider $p \in (2, \infty)$. By Lemma 12 there is a nonnegative $f \in C^\infty_c(\mathbb{R}^d)$ such that

$$\mathcal{E}_p[f] < \kappa \int_{\mathbb{R}^d} |f(x)|^p |x|^\alpha dx < \infty.$$  

It is well known that $C^\infty_c(\mathbb{R}^d)$ is a subset of the domain of the generator of the semigroup $\{P_t, t \geq 0\}$ acting on $L^p$, see, e.g., Jacob [38, Theorem 3.3.11] or Farkas, Jacob and Schilling [28, Theorem 1.4.2 or Proposition 2.1.1]; or use [55, Theorem 31.5], (27) and
Therefore $f$ is in the domain of $\Delta^{\alpha/2}$ on $L^p(\mathbb{R}^d)$. For $M \in (0, \infty)$ we write $u^{(M)}(t,x) = \tilde{P}_t^{(M)} f(x)$ or $u^{(M)}(t) = \tilde{P}_t^{(M)} f$. By (61),

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u^{(M)}(t)|^p dx = p \int_{\mathbb{R}^d} u^{(M)}(t)^{(p-1)} \Delta^{\alpha/2} u^{(M)}(t) + q^{(M)} u^{(M)}(t) \, dx$$

$$= p \left( -\mathcal{E}_p[u^{(M)}(t)] + \int_{\mathbb{R}^d} q^{(M)} |u^{(M)}(t)|^p dx \right).$$

In particular,

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u^{(M)}(t)|^p dx \bigg|_{t=0} = p \left( -\mathcal{E}_p[f] + \int_{\mathbb{R}^d} q^{(M)} |f|^p dx \right)$$

(63)

$$= p \left( \int_{\mathbb{R}^d} \left( q^{(M)}(x) - \frac{\kappa}{|x|^\alpha} \right) |f|^p dx \right) + p \left( \kappa \int_{\mathbb{R}^d} \frac{|f|^p}{|x|^\alpha} dx - \mathcal{E}_p[f] \right).$$

The last term is strictly positive by (62). Since $q^{(M)}(x) \to \kappa |x|^{-\alpha}$, for sufficiently large $M$ the expression in (63) is strictly positive, and so for such $M$ we get,

$$\left. \frac{d}{dt} \|u^{(M)}(t)\|_p \right|_{t=0} > 0.$$

Therefore for small $t > 0$ we have

$$\|f\|_p^p = \|u^{(M)}_0\|_p^p < \|u^{(M)}(t)\|_p^p \leq \|\tilde{P}_t f\|_p^p.$$

The case $p \in (1,2)$ follows by the duality argument. \hfill \qed

To complement Theorem 3 we note that for $x,y \in \mathbb{R}^d$ and $t > 0$,

$$\tilde{p}_t(x,y) \approx \left( 1 + e^{\delta/\alpha} |x|^{-\delta} \right) \left( 1 + e^{\delta/\alpha} |y|^{-\delta} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).$$

The result is given in [13, Theorem 1] (for $x=0$ or $y=0$ we have $p_t^{(1)}(x,y) = \infty$; see [39, Lemma 3.7 and the comment before Corollary 3.9], so $\tilde{p}_t(x,y) = \infty$). Denote $H(x) = |x|^{-\delta} \vee 1$, $x \in \mathbb{R}^d$. Clearly,

$$\tilde{p}_1(x,y) \approx H(x) H(y) p_1(x,y), \quad x,y \in \mathbb{R}^d.$$

Proof of Theorem 4. By (58) it is enough to consider $t = 1$. Let $B = B(0,1) \subset \mathbb{R}^d$. Denote $q = p/(p-1)$. If $\delta \geq d/q$, then $\int_B H^q = \infty$, so there is $f \in L^p_+$ such that $\int_B H(y)f(y) = \infty$, see, e.g., [9, Corollary 4.4.5]. By (65), $\tilde{P}_f = \infty$ on $\mathbb{R}^d$. Also, if $\delta \geq d/p$, then $\tilde{P}_f 1_B \geq cH 1_B$, so $f := 1_B \in L^p$ but $\int \tilde{P}_f|^p \geq c \int_B H^p = \infty$. If $0 \leq \delta < d/p$, then we let $f \in L^p_+$, $g \in L^q_+$ and consider

$$I(f,g) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x) H(x) H(y) p_1(x,y) f(y) dy dx$$

$$= I(f 1_B, g 1_B) + I(f 1_B, g 1_{B^c}) + I(f 1_{B^c}, g 1_B) + I(f 1_{B^c}, g 1_{B^c}).$$

By (32) and Hölder inequality,

$$I(f 1_{B^c}, g 1_{B^c}) \leq \|f\|_p \|g\|_q.$$
By Hölder inequality, \( \int_B gH \leq c\|g\|_q \), where \( c = (\int_B |x|^{-\delta q})^{1/p} \). Similarly, \( \int_B fH \leq c\|f\|_p \), where \( c = (\int_B |x|^{-\delta p})^{1/q} \). Furthermore, since \( \delta \) is bounded, \( I(f1_B, g1_B) \leq \int_B gH \int_B fH \leq c\|g\|_q\|f\|_p \). By (33), \( I(f1_Bc, g1_B) \leq \int_B gH\|p1f\|_\infty \leq c\|g\|_q\|f\|_p \). Similarly, \( I(f1_B, g1_B^c) \leq c\|g\|_q\|f\|_p \). By (65), \( \tilde{P}_1 \) is bounded on \( L^p \).

6. Appendix

For the convenience of the reader, we add some details to the arguments used in the proof of Theorem 3. In what follows, we consider a measure space \((X, \mu)\) and the spaces \( L^p(X, d\mu) \) for \( p \geq 1 \). For simplicity, we write \( L^p \) for \( L^p(X, d\mu) \). Recall that \( p/(p-1) \) is the Hölder conjugate exponent of \( p \in (1, \infty) \). (i.e. we have \( p^{-1} + (p/(p-1))^{-1} = 1 \).

Lemma 13 (Continuity). Let \( p \in (1, \infty) \) and \( r \in \left[ \frac{p}{p-1}, \infty \right) \). If \( f \in L^p \), \( g \in L^r \), then \( \|fg\|_{\frac{pr}{p+r}} \leq \|f\|_p\|g\|_r \). If \( fn \to f \) in \( L^p \) and \( gn \to g \) in \( L^r \), then \( fn gn \to fg \) in \( L_{\frac{pr}{p+r}} \).

Proof. Of course, \( r > 1 \). Also, \( \frac{p+r}{pr} = \frac{1}{p} + \frac{1}{r} = 1 \), thus \( \frac{pr}{p+r} \in [1, \infty) \). For \( f \in L^p \), \( g \in L^r \), by Hölder inequality with exponents \( \frac{pr}{r} \) and \( \frac{p+r}{p} \),

\[
\int |fg|^{\frac{pr}{p+r}} \leq \left( \int |f|^p \right)^{\frac{r}{p+r}} \left( \int |g|^r \right)^{\frac{p}{p+r}} < \infty,
\]

and we get the first statement. The second statement is verified as follows,

\[
\|fn gn - fg\|_{\frac{pr}{p+r}} = \|fn gn - fn g + fn g - fg\|_{\frac{pr}{p+r}} \leq \|fn\|_p\|gn - g\|_r + \|fn - f\|_p\|g\|_r \to 0,
\]

as the sequence \( fn \) is bounded in \( L^p \). \( \square \)

Let \( p \in (1, \infty) \). Assume that \( [0, \infty) \ni t \mapsto u(t) \in L^p \). We will relate the continuity and differentiability properties of \( u \) in \( L^p \) to those of \( |u|^p \) in \( L^1 \). We denote

\[
\Delta_h u(t) = u(t+h) - u(t), \quad \text{if } t, t+h \geq 0.
\]

We say that \( u \) is continuous in \( L^p \) at \( t \geq 0 \), if \( \Delta_h u(t) \to 0 \) in \( L^p \) as \( h \to 0 \), and we say \( u \) is continuously differentiable at \( t \geq 0 \) if \( u'(t) := \lim_{h \to 0} \frac{1}{h} \Delta_h u(t) \) in \( L^p \) with continuous \( u' \). Of course, \( u'(0) \) is the right-hand side derivative in the above setting.

Lemma 14. If \( u \) is continuous in \( L^p \), then \( |u|^p \) and \( u^{(p-1)} \) are continuous in \( L^1 \) and \( L_{\frac{p}{p-1}} \), respectively.

Proof. By (19), \( |\Delta_h u(t)|^p \leq (p+C)|\Delta_h u(t)||u(t+h) + |u(t)||^{p-1} \). Similarly, by (20), \( |\Delta_h u(t)|^{p-1} \leq c'|\Delta_h u(t)|^{\lambda}(|u(t+h) + |u(t)||)^{p-1-\lambda} \), and we can pick \( \lambda > 0 \) such that \( p-1-\lambda > 0 \). The results follow from Lemma 13. \( \square \)

Lemma 15 (Differentiability). If \( [0, \infty) \ni t \mapsto u(t) \) is continuously differentiable in \( L^p \), then \( |u|^p \) is continuously differentiable in \( L^1 \) and \( (|u|^p)' = pu^{(p-1)}u' \).

For comparison, Marinelli, Röckner [50, p. 4] assert that for \( p \geq 2 \) the function \( \phi : H \ni x \mapsto \|x\|^p \) is weakly differentiable for every Hilbert space \( H \), with the Fréchet derivative \( D\phi(x) : y \mapsto p\|x\|^{p-2}(x, y) \).
Proof of Lemma 15. By Lemma 13, \( u^{(p-1)} u' \) is continuous in \( L^1 \). By (18), for \( \lambda \in [0,2] \),
\[
\frac{1}{h} \left( \Delta_h |u|^p(t) - pu(t)^{(p-1)} \Delta_h u(t) \right) \leq C h^{\lambda-1} \frac{1}{h} \Delta_h u(t)^\lambda (|u(t) + h| + |u(t)|)^{p-\lambda}.
\]
We pick \( \lambda > 1 \) such that \( p - \lambda > 0 \) and the result follows. \( \square \)

Recall that \((P_t, t \geq 0)\) is a strongly continuous operator semigroup on \( L^p \). Let \( f \) be in the domain of its generator \( A \). Let \( u(t) = P_t f \). Then \( u'(t) = P_tA f = AP_t f = Au(t) \).

By Lemma 15 we obtain the following result.

Lemma 16. \( |u(t)|^p \) is differentiable in \( L^1 \) with the derivative
\[
( |u(t)|^p )' = pu(t)^{(p-1)} u'(t) = pu(t)^{(p-1)} P_t A f, \quad t \geq 0.
\]

Finally, recall that in the proof of Theorem 3 we only proved the contractivity of the semigroup for \( \kappa \leq \kappa_{d-\alpha}/p \) and nonnegative \( f \in L^p \). This suffices because for general \( f \in L^p \) we may write \( f = f_+ - f_- \) and we have \( (P_t f)_+ \leq (P_t f)_+ \) and \( (P_t f)_- \leq (P_t f)_- \), hence \( |P_t f|^p \leq (P_t f_+)^p + (P_t f_-)^p \). Therefore,
\[
\int |P_t f(x)|^p dx \leq \int P_t f_+(x)^p dx + \int P_t f_-(x)^p dx \leq \int f_+(x)^p dx + \int f_-(x)^p dx = \int |f(x)|^p dx.
\]

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