FINITE RIGID SETS IN CURVE COMPLEXES OF NON-ORIENTABLE SURFACES

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Abstract. A rigid set in a curve complex of a surface is a subcomplex such that every locally injective simplicial map from the set into the curve complex is induced by a homeomorphism of the surface. In this paper, we find finite rigid sets in the curve complexes of connected non-orientable surfaces of genus $g$ with $n$ holes for $g + n \neq 4$.

1. Introduction

The curve complex on a surface, introduced by Harvey [7] in order to study the boundary structure of the Teichmüller space, is an abstract simplicial complex whose vertices are isotopy classes of simple closed curves. The mapping class group of the surface acts on it by simplicial automorphisms. Conversely, except for a few sporadic cases, every simplicial automorphism of the curve complex is induced by a self-homeomorphism of the surface, as shown by Ivanov [12], Korkmaz [13] and Luo [14] in the orientable case, and by Atalan-Korkmaz [3] in the non-orientable case.

In the case the surface is orientable, it was shown by Shackleton [18] that a locally injective simplicial map of the curve complex is induced by a homeomorphism. Aramayona-Leininger [1] introduced a finite subcomplex $X$ in the curve complex such that any locally injective simplicial map from $X$ into the curve complex is also induced from a homeomorphism. They called these sets as finite rigid sets.

The purpose of this paper is to prove the non-orientable version of Aramayona-Leininger result: We prove that if $g + n \neq 4$ then the curve complex of a connected non-orientable surface $N$ of genus $g$ with $n$ boundary components contains a finite rigid set.

Main Theorem. Let $g + n \neq 4$ and let $N$ be a connected non-orientable surface of genus $g$ with $n$ boundary components. There is a finite subcomplex $X_{g,n}$ of the curve complex $\mathcal{C}(N)$ of $N$ such that for any locally injective simplicial map $\phi : X_{g,n} \to \mathcal{C}(N)$ there is a homeomorphism $F : N \to N$ with $\phi(\gamma) = F(\gamma)$ for all $\gamma \in X_{g,n}$.

This theorem is restated and proved as Theorem [20] in Section 5. The proof of it is by induction on the genus $g$. We also show at the end of the paper that in the case $g + n = 4$ the subcomplex $X_{g,n}$ we consider is not rigid.

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After Aramayona-Leininger introduced the finite rigid sets in the curve complex, Maungchang [15] obtained finite rigid subgraphs of the pants graphs of punctured spheres. Aramayona-Leininger in [2] constructed a sequence of finite rigid sets exhausting curve complex of an orientable surface, and Hernández-Hernández [8] obtained an exhaustion of the curve complex via rigid expansion. In particular, they show that for an orientable surface $S$, there exists a sequence of finite rigid sets each has trivial pointwise stabilizer in $\text{Mod}(S)$ so that the union of these sets is the curve complex of $S$.

For non-orientable surfaces, the curve complex was first studied by Scharlemann [17]. After Atalan-Korkmaz [3] proved that the natural map from the mapping class group $\text{Mod}(N)$ of a non-orientable surface $N$ of genus $g$ with $n$ holes to the automorphism group $\text{Aut}(\mathcal{C}(N))$ of the curve complex of $N$ is an isomorphism for $g + n \geq 5$, Irmak generalized this result by showing that superinjective maps and an edge-preserving maps $\mathcal{C}(N) \to \mathcal{C}(N)$ are induced from homeomorphisms $N \to N$ (cf. [10] and [11]).

Here is an outline of the paper. Section 2 contains the relevant definitions and the background. The model of the non-orientable surface and the curves used in the paper are introduced in this section. Section 3 introduces the finite subcomplex $\mathcal{X}_{g,n}$ which will be showed to be a finite rigid set. Section 4 focuses on the topological types of vertices in the finite subcomplex $\mathcal{X}_{g,n}$ and shows that topological types of many elements of $\mathcal{X}_{g,n}$ are preserved under a locally injective simplicial map. Section 5 is devoted to the proof of the Main Theorem. It is shown in the last section, Section 6, that the complex $\mathcal{X}_{g,n}$ is not rigid in the case $g + n = 4$.

The main result of this paper was part of the first author’s Ph.D. thesis [9] at Middle East Technical University. We learned at the ICM 2018 satellite conference ‘Braid Groups, Configuration Spaces and Homotopy Theory’ held in Salvador, Brazil, that Blazej Szepietowski has independently obtained similar results. The authors would like to thank B. Szepietowski for pointing out a mistake in the earlier version. The second author thanks to UMASS Amherst Math. Department, where the writing of this paper is completed.

2. Preliminaries

In this section, we give the basic definitions and facts on the curve complexes. We introduce the model representing a non-orientable surface and present the notation of the simple closed curves on it that will be used throughout the paper.

For two isotopy classes $\alpha$ and $\beta$ of two simple closed curves on a surface, the geometric intersection number of $\alpha$ and $\beta$ is defined as

$$i(\alpha, \beta) = \min\{\text{Card}(a \cap b) \mid a \in \alpha, b \in \beta\},$$

where $\text{Card}(a \cap b)$ denotes the cardinality of $a \cap b$. We say $\alpha$ and $\beta$ are disjoint if $i(\alpha, \beta) = 0$.

Throughout the paper, we will use the terms ‘a hole’ and ‘a boundary component’ to mean the same thing. For simplicity, we will often confuse a curve and its isotopy class. It will be clear from the context which one is meant. Similarly, a homeomorphism and its isotopy class will be confused.
2.1. Abstract simplicial complex.

An abstract simplicial complex on a set (of vertices) $V$ is a subset $K$ of the set of all subsets of $V$ satisfying the followings:

- $K$ does not contain the empty set $\emptyset$,
- If $v \in V$, then $\{v\} \in K$,
- If $\sigma \in K$ and if $\tau \subset \sigma$ with $\tau \neq \emptyset$, then $\tau \in K$.

An element of $K$ is called a simplex of $K$. A face of a simplex $\sigma$ in $K$ is a nonempty subset of $\sigma$. The dimension of a simplex $\sigma$ of $K$ is defined as $|\sigma| - 1$, and the dimension of $K$ is defined as the supremum of the dimensions of all simplices in $K$. An element $v \in V$, identified with the 0-simplex $\{v\}$, is called a vertex of $K$. An edge in $K$ is a 1-simplex. If $\{v_1, v_2\}$ is an edge, we say that $v_1$ and $v_2$ are connected by an edge.

A subcomplex $L$ of a simplicial complex $K$ is a simplicial complex on a subset of $V$ such that if $\sigma \in L$ then $\sigma \in K$. A subcomplex $L$ of $K$ is called a full subcomplex if whenever a set of vertices in $L$ is a simplex in $K$, then it is also a simplex in $L$.

For a vertex $v$ of $K$, the link, $\text{Link}_K(v)$, of $v$ is the full subcomplex of $K$ on the set of vertices connected $v$ by an edge, and the star of $v$ is the full subcomplex containing $v$ and $\text{Link}_K(v)$. Note that $\text{Link}_K(v)$ is a subcomplex of the star of $v$.

2.2. Curve Complex $\mathcal{C}(S)$. The curve complex $\mathcal{C}(S)$ on a compact connected orientable surface $S$, introduced by Harvey [7], is the abstract simplicial complex whose vertices are isotopy classes of nontrivial simple closed curves on $S$, and a set of vertices $\{\alpha_0, \alpha_1, \ldots, \alpha_q\}$ forms a $q$-simplex if $i(\alpha_j, \alpha_k) = 0$ for all $j, k$. A simple closed curve $a$ on $S$ is called nontrivial if it does not bound a disc with at most one hole.

By an easy Euler characteristic argument, it is easy to see that all maximal simplices of $\mathcal{C}(S)$ contains $3g + n - 3$ elements (except for a few sporadic cases), so that the dimension of $\mathcal{C}(S)$ is $3g + n - 4$. Here, $g$ is the genus and $n$ is the number of boundary components of $S$.

2.3. Curve Complex $\mathcal{C}(N_{g,n})$.

Let $N = N_{g,n}$ be a compact connected non-orientable surface of genus $g$ with $n$ holes. The genus of a non-orientable surface is defined as the maximum number of real projective planes in a connected sum decomposition. Equivalently, it is the maximum number of disjoint simple closed curves on $N$ whose complement is connected. Clearly, this definition of the genus works for connected orientable surfaces as well. By convention, we assume that a sphere is a non-orientable surface of genus $0$.

Let $a$ be a simple closed curve on $N$. We denote by $N_a$ the surface obtained by cutting $N$ along $a$. The curve $a$ is called non-separating (respectively separating) if $N_a$ is connected (respectively disconnected). It is called is nontrivial if it bounds neither a disc at most one hole, nor a Möbius band. The curve $a$ is called two-sided (respectively one-sided) if a regular neighbourhood of it is an annulus (respectively a Möbius band). When $a$ is one-sided, $a$ is called essential if either $g = 1$, or $g \geq 2$ and $N_a$ is non-orientable. If $a$ is separating, then we say that it is of type $(p, q)$ if
$N_a$ is the disjoint union of two nonorientable surfaces one of which is of genus $p$ with $q + 1$ holes. In particular, if $N_a$ is the disjoint union of a sphere with $q + 1$ holes and a nonorientable surface of genus $g$ with $n - q + 1$ holes, that is if $a$ bounds a disk with $q$ holes, then we say that $a$ is of type $(0, q)$. Note that a simple closed curve of type $(p, q)$ is also of type $(g - p, n - q)$. We usually assume that $p \leq g - q$.

The curve complex $C(N)$ of the non-orientable surface $N$ is defined similarly to the orientable case; it is an abstract simplicial complex with vertices

$$\{ \alpha \mid \alpha \text{ is the isotopy class of a nontrivial simple closed curve on } N \}.$$ 

A $q$-simplex of $C(N)$ is defined as a set of $(q + 1)$ distinct vertices $\{ \alpha_0, \alpha_1, \ldots, \alpha_q \}$ such that $i(\alpha_j, \alpha_k) = 0$ for all $j \neq k$. We note that $i(\alpha, \alpha) = 1$ if $\alpha$ is one-sided, so that we require $j \neq k$.

We finish this section with the following result stating that the natural simplicial action of $\text{Mod}(N_{g,n})$ on $C(N_{g,n})$ induces an isomorphism:

**Theorem 1.** [3, Theorem 1] For $g + n \geq 5$, the natural map $\text{Mod}(N_{g,n}) \rightarrow \text{Aut}(C(N_{g,n}))$ is an isomorphism.

### 2.4. Top Dimensional Maximal Simplices.

Let $P$ be a set of pairwise disjoint, non-isotopic, nontrivial simple closed curves on $N$. The set $P$ is called a *pair of pants decomposition* of $N$, or simply a *pants decomposition*, if each component of the surface obtained by cutting $N$ along the curves in $P$ is a pair of pants, i.e. a three-holed sphere.

For a connected surface, there is a bijection between maximal simplices of the curve complex and pants decompositions of the surface. In the case the surface is orientable of genus $g$ with $n$ holes, all maximal simplices in the curve complex have the same dimension $3g + n - 4$. But, in the non-orientable case, dimensions of the maximal simplices are not constant.

**Lemma 2.** [3, Proposition 2.3] Let $N$ be a connected non-orientable surface of genus $g \geq 2$ with $n$ holes. Suppose that $(g, n) \neq (2, 0)$. Let $a_r = 3r + n - 2$ and $b_r = 4r + n - 2$ if $g = 2r + 1$, and $a_r = 3r + n - 4$ and $b_r = 4r + n - 4$ if $g = 2r$. Then, there is a maximal simplex of dimension $q$ in $C(N)$ if and only if $a_r \leq q \leq b_r$. In particular, there are precisely $\lfloor g/2 \rfloor$ values which occur as the dimension of a maximal simplex, where $\lfloor g/2 \rfloor$ denotes the smallest integer greater than $g/2$.

A pants decomposition $P$ on a non-orientable surface is called *top dimensional* if its dimension is $b_r$, so that it contains $b_r + 1$ elements. A top dimensional pant decomposition contains exactly $g$ essential one-sided curves (cf. the proof of Lemma 2 above given in [3]). By Lemma 2, we have the following corollary.

**Corollary 3.** Suppose that $g + n \geq 3$ and that $P$ is a top dimensional pants decomposition of $N$. Then $P$ contains $2g + n - 3$ simple closed curves, exactly $g$ of which are essential one-sided.

**Lemma 4.** Let $g \geq 1$, $P$ be a top dimensional pants decomposition of $N$ and let $a \in P$. Then the curve $a$ is either essential one-sided, or of type $(p, q)$ for some $p, q$ with $0 \leq p \leq g, 1 \leq q \leq n - 1$ and $(p, q) \neq (0, 1)$.
Proof. The lemma is trivially true for \( g = 1 \). So suppose that \( g \geq 2 \). Recall that a curve is of type \((p, q)\) if the curve is separating such that both components of its complement are non-orientable and one of these components is of genus \( p \) with \( q + 1 \) holes.

It suffices to prove that the pants decomposition \( P \) can not contain a simple closed curve of type that is not mentioned. The topological types of simple closed curves that are not mentioned in the lemma are as follows:

- A two-sided non-separating simple closed curve whose complement is orientable (in the case \( g \) is even).
- A one-sided simple closed curve whose complement is orientable (in the case \( g \) is odd).
- A separating simple closed curve whose complement is the disjoint union of an orientable surface of genus \( k \) and a nonorientable surface of genus \( g - 2k \) for some \( k \geq 1 \).
- A two-sided non-separating simple closed curve whose complement is nonorientable.

Case 1. Let \( \beta_1 \) be a two-sided non-separating simple closed curve such that the surface \( N_{\beta_1} \) is orientable and let \( P_1 \) be a pants decomposition of \( N \) containing \( \beta_1 \), so that \( Q_1 = P_1 \setminus \{\beta_1\} \) is a pants decomposition of \( N_{\beta_1} \). In this case, \( g \) is even, say \( g = 2r \), \( r \geq 1 \). Since \( N_{\beta_1} \) is an orientable surface of genus \( r - 1 \) with \( n + 2 \) holes, the number of elements of \( Q_1 \) is \(|Q_1| = 3(r - 1) + (n + 2) - 3 \), so that \(|P_1| = 3r + n - 3 \). This number is less than the number \( 2g + n - 3 \). Hence, \( P_1 \) is not top dimensional.

Case 2. Let \( \beta_2 \) be a one-sided simple closed curve such that \( N_{\beta_2} \) is orientable and let \( P_2 \) be a pants decomposition of \( N \) containing \( \beta_2 \). In this case, \( g \) is odd, say \( g = 2r + 1 \), and \( Q_2 \setminus \{\beta_2\} \) is a pants decomposition of \( N_{\beta_2} \), an orientable surface of genus \( r + 1 \) holes. Thus, \(|Q_2| = 3r + (n + 1) - 3 \), so that \(|P_2| = 3r + n - 1 \), which is less than \( 2g + n - 3 \). Hence, \( P_2 \) is not top dimensional.

Case 3. Let \( \beta_3 \) be a separating simple closed curve such that \( N_{\beta_3} \) is the disjoint union of an orientable surface \( S \) of genus \( k \geq 1 \) with \( l + 1 \) holes, and a non-orientable surface \( N' \) of genus \( g - 2k \) with \( n - l + 1 \) holes. Let \( P_3 \) be a pants decomposition of \( N \) containing \( \beta_3 \). Then, the elements of \( P_3 \) lying on \( S \) is a pants decomposition \( Q_3 \) of \( S \), and the elements lying of \( N' \) is a pants decomposition \( Q_3' \) of \( N' \). Then, \( P_3 = Q_3 \cup Q_3' \cup \{\beta_3\} \), \(|Q_3| = 3k + (l + 1) - 3 \), \(|Q_3'| \leq 2(g - 2k) + (n + 1 - l) - 3 \). It follows that \(|P_3| = |Q_3| + |Q_3'| + 1 \leq 2g + n - 3 - k \), which is less than \( 2g + n - 3 \). Thus \( P_3 \) is not top dimensional.

Case 4. Let \( \beta_4 \) be a two-sided non-separating simple closed curve such that \( N_{\beta_4} \) is non-orientable. Let \( P_4 \) be a pants decomposition of \( N \) containing \( \beta_4 \). Since \( N_{\beta_4} \) is a non-orientable surface of genus \( g - 2 \) with \( n + 2 \) holes, the pants decomposition \( Q_4 = P_4 \setminus \{\beta_4\} \) contains at most \( 2(g - 2) + (n + 2) - 3 \) elements, so that \(|P_4| = |Q_4| + 1 \leq 2g + n - 4 \), concluding that \( P_4 \) cannot be top dimensional.

This finishes the proof. \( \square \)

2.5. Our model for \( N_{g,n} \) and curves on it.
Let $R$ be a disc in the plane whose boundary is a \((2g + n)\)-gon such that the edges are labeled as $s_1, e_1, s_2, e_2, \ldots, s_g, e_g, e_{g+1}, e_{g+2}, \ldots, e_{g+n}$ in this order. We denote the edges $s_i$ by boldface semicircles (see Figure 1). For $j = 1, 2, \ldots, n$, let $z_j$ be the corner between the edges $e_{g+k-1}$ and $e_{g-k}$. We glue two copies of $R$ along $e_i$ to get a sphere $S$ with $g$ holes. By identifying the antipodal points on each boundary component of $S$ and by deleting a small open disc about each $z_1, z_2, \ldots, z_n$, we obtain a non-orientable surface $N_{g,n}$ of genus $g$ with $n$ punctures. The disc $R$ will be our model for $N_{g,n}$, so that $N_{g,n}$ is obtained from $R$ by the way we just described.

Figure 1. The disc $R$ on the left represents the surface $N_{g,n}$ on the right, where the interiors of the discs with a cross are to be deleted and the antipodal points on each resulting boundary components are to be identified.

A properly embedded arc on the model $R$ whose endpoints are on the boundary of $R$ (but different from $z_j$) gives a simple closed curve on $N_{g,n}$. Curves we are interested in are represented such arcs on $R$. We fix the following curves on $N_{g,n}$ (cf. Figure 2):

- For $1 \leq i \leq g$, let $\alpha_i$ be the one-sided curve represented by $s_i$.
- For $1 \leq i \leq g$, $1 \leq j \leq g+n$ with $j \neq i, j \neq i-1 \pmod{(g+n)}$, let $\alpha_i^j$ be the one-sided curve represented by a line segment joining the midpoint of $s_i$ and a point of $e_j$.
- For $1 \leq i, j \leq g+n$ and $|i-j| \geq 2$, let $\beta_i^j$ be the two-sided curve represented by a line segment joining a point of $e_i$ to a point of $e_j$.

We take these curves in such a way that they are in minimal position. Note that $\beta_i^j = \beta_j^i$.

2.6. Locally Injective Simplicial Maps. Let $K$ be an abstract simplicial complex and $\gamma$ be a vertex in $K$. Recall that the star of $\gamma$ is the subcomplex of $K$ consisting of all simplices of $K$ containing $\gamma$ as a vertex and all faces of such simplices.

For a subcomplex $L$ of $K$, a simplicial map $\phi : L \to K$ is called locally injective if it is injective on the restriction to the star of every vertex of $L$. 
Remark 5. Since the dimension of a simplex is preserved under a locally injective simplicial map from a subcomplex of $C(N)$ to $C(N)$, the image of a top dimensional pants decomposition under a locally injective simplicial map is again a top dimensional pants decomposition.

2.7. Adjacency Graph of a Pants Decomposition. Let $P$ be a pants decomposition of $N$. Two elements $\alpha$ and $\beta$ of $P$ are called adjacent with respect to $P$ if there exists a connected component, a pair of pants, of $NP$ (the surface obtained by cutting $N$ along all elements of $P$) containing $\alpha$ and $\beta$ as its boundary components. The adjacency graph $A(P)$ of $P$ is the graph whose vertices are the elements of $P$; two vertices $\alpha$ and $\beta$ are connected by an edge if $\alpha$ and $\beta$ are adjacent with respect to $P$ (cf. [4]). We say $P$ is linear if $A(P)$ is linear.

Let $\alpha \in A(P)$ be a vertex of valency one. A vertex $\beta$ is called a $k$th linear successor of $\alpha$ if there is a path $(\alpha = \gamma_0, \gamma_1, \ldots, \gamma_k = \beta)$ from $\alpha$ to $\beta$ such that every $\gamma_i$, $1 \leq i \leq k - 1$, has valency two in $A(P)$. In Figure 3 the vertex
$\beta^n_1$ is a first linear successor, $\beta^n_2$ is a second linear successor and $\beta^n_3$ is a 4th linear successor of $\alpha_1$.

3. Finite Subcomplex $X_{g,n}$ of Curve Complex

A subcomplex $L$ of the curve complex $\mathcal{C}(N)$ of the non-orientable surface $N$ is called \textit{rigid} if every locally injective simplicial map $L \to \mathcal{C}(N)$ is induced from a homeomorphism $N \to N$.

Let $N_{g,n}$ be the compact connected non-orientable surface of genus $g$ with $n$ holes whose model is given in Section 2.5. In this section, we introduce the finite subcomplex $X_{g,n} \subset \mathcal{C}(N_{g,n})$ that will be shown to be rigid in $\mathcal{C}(N_{g,n})$.

3.1. Finite Subcomplex $X_{g,n}$. We use the curves given in the Section 2.5 to introduce the finite subcomplex $X_{g,n}$ of $\mathcal{C}(N_{g,n})$. We define two sets of vertices in $\mathcal{C}(N_{g,n})$ as

\[ X^1_{g,n} = \{ \alpha_i, \alpha^j_i \mid 1 \leq i \leq g, 1 \leq j \leq g+n \text{ and } j \neq i, j \neq i - 1 \mod (g+n) \}, \]
\[ X^2_{g,n} = \{ \beta^i_j \mid 1 \leq i, j \leq g+n, 2 \leq |i-j| \leq g+n-2 \}. \]

We set $X_{g,n} = X^1_{g,n} \cup X^2_{g,n}$ (cf. Figure 4). Although we write $X^1_{g,n}$, $X^2_{g,n}$ and $X_{g,n}$ as sets of vertices, we understand the full subcomplexes of $\mathcal{C}(N)$ containing these sets. Note that the vertices in $X^1_{g,n}$ are one-sided essential, and that the vertices in $X^2_{g,n}$ are separating.

We remark that cutting $N_{g,n}$ along $\alpha_g$ induces an isomorphism
\[ \varphi : \text{Link}_{\mathcal{C}(N_{g,n})}(\alpha_g) \to \mathcal{C}(N_{g-1,n+1}) \]

mapping the link $\text{Link}_{X_{g,n}}(\alpha_g)$ of $\alpha_g$ to $\mathcal{C}(N_{g-1,n+1})$ isomorphically. This allows us to make induction on the genus of the surface in the proof of the main theorem.

We note that for $n \geq 5$, the complex $X_{0,n}$ is the complex $\mathcal{X}$ of Aramayona-Leininger. We may restate Theorem 3.1 in [1] as follows.

\textbf{Theorem 6} (Aramayona-Leininger). \textit{For $n \geq 5$, any locally injective simplicial map $\phi : X_{0,n} \to \mathcal{C}(N_{0,n})$ is induced by a unique orientation preserving homeomorphism $N_{0,n} \to N_{0,n}$}.
3.2. The nonadjacency condition. Let $L$ be a subcomplex of $C(N_{g,n})$ containing a top dimensional pants decomposition $P$ of $N_{g,n}$. We say that $P$ satisfies the nonadjacency condition with respect to $L$ if for every pair $\alpha$ and $\beta$ of nonadjacent vertices in $P$, there exist two disjoint simple closed curves $\delta_\alpha$ and $\delta_\beta$ in $L$ different from $\alpha$ and $\beta$, respectively, such that

$$i(\delta_\alpha, \alpha) \neq 0, i(\delta_\beta, \beta) \neq 0 \text{ and } i(\delta_\alpha, \beta) = 0, i(\delta_\beta, \alpha) = 0$$

and each of them is disjoint from the simple closed curves in $P \setminus \{\alpha, \beta\}$. If $L = X_{g,n}$ we simply say that $P$ satisfies the nonadjacency condition.

The following lemma is the non-orientable version of [13 Lemma 7], which states that for a compact orientable surface $S$ any locally injective simplicial map $C(S) \to C(S)$ preserves the nonadjacency in the adjacency graph of a pants decomposition of $S$. For the sake of completeness, we give a proof of the lemma.

**Lemma 7.** Let $L$ be a subcomplex of $C(N_{g,n})$, $P$ be a top dimensional pants decomposition of $N_{g,n}$ contained in $L$, $A(P)$ be its adjacency graph and let $\phi : L \to C(N_{g,n})$ be a locally injective simplicial map. Suppose that $P$ satisfies the nonadjacency condition with respect to $L$. If $\alpha$ and $\beta$ are nonadjacent in $P$, then $\phi(\alpha)$ is nonadjacent to $\phi(\beta)$ in the pants decomposition $\phi(P)$.

**Proof.** Let $\alpha$ and $\beta$ be two simple closed curves in $P \subset L$ such that they are nonadjacent in $A(P)$. By the assumption, there exist two disjoint closed curves $\delta_\alpha$ and $\delta_\beta$ on $N_{g,n}$ such that $\delta_\alpha \cap \alpha \neq \emptyset, \delta_\beta \cap \beta \neq \emptyset$ and $\delta_\alpha \cap \beta = \emptyset, \delta_\beta \cap \alpha = \emptyset$ and each of them are disjoint from the simple closed curves in $P \setminus \{\alpha, \beta\}$. Hence, $\phi(\delta_\alpha)$ and $\phi(\delta_\beta)$ are disjoint from each other and from all curves in $\phi(P) \setminus \{\phi(\alpha), \phi(\beta)\}$. Also $\phi(\delta_\alpha)$ is disjoint from $\phi(\beta)$ and $\phi(\delta_\beta)$ is disjoint from $\phi(\alpha)$. It is easy to see that $\phi(\delta_\alpha)$ intersects $\phi(\alpha)$ and $\phi(\delta_\beta)$ intersects $\phi(\beta)$.

Assume $\phi(\alpha)$ and $\phi(\beta)$ are adjacent in $A(\phi(P))$. Then there is a subsurface $X$ of $N_{g,n}$ homeomorphic to a pair of pants bounded by $\phi(\alpha)$, $\phi(\beta)$ and another curve (possibly a boundary component), and $\phi(\delta_\alpha)$ and $\phi(\delta_\beta)$ do not intersect the third curve. Since an arc in $\phi(\delta_\alpha) \cap X$ (respectively $\phi(\delta_\beta) \cap X$) connects two points of $\phi(\alpha)$ (respectively $\phi(\beta)$) and since any two such arc on a pair of pants must intersect (cf. [8]), we get a contradiction. \hfill $\square$

4. Topological Types of Vertices in $X_{g,n}$

The purpose of this section is to prove that the topological types of most vertices of $X_{g,n}$ are preserved under locally injective simplicial maps $X_{g,n} \to C(N_{g,n})$. This will be enough for us to prove the main theorem. To this end, let $\phi : X_{g,n} \to C(N_{g,n})$ be a locally injective simplicial map. In this section we assume that $g + n \geq 5$.

4.1. The case $g = 1$. Let $N = N_{1,n}$ denote the non-orientable surface of genus 1 with $n \geq 4$ holes. We show that $\phi$ preserves the topological types of vertices in $X_{1,n}$. Note that in this case $X_{1,n}$ is the subcomplex of $C(N)$ with the vertex set

$$\{\alpha_1, \alpha_1^2, \alpha_1^3, \ldots, \alpha_1^n\} \cup \{\beta_i^j \mid 1 \leq i, j \leq n + 1, 2 \leq |i - j| \leq n - 1\},$$

and, by Corollary [8], a pants decomposition of $N$ contains exactly $n - 1$ curves.
4.1.1. Linear Pants Decompositions. Recall that a pants decomposition is called linear if its adjacency graph is linear.

Lemma 8. For every curve $\beta^j_i \in X_{1,n}$ there is a linear pants decomposition $P^j_i$ in $X_{1,n}$ containing both $\alpha_1$ and $\beta^j_i$.

Proof. Clearly we can assume that $i \leq j - 2$. If $i = 1$

$$P^j_1 = \{\alpha_1, \beta^1_1, \beta^{n-1}_1, \ldots, \beta^1_i, \beta^{j-1}_i, \ldots, \beta^3_i\},$$

and otherwise

$$P^j_i = \{\alpha_1, \beta^{n+1}_2, \beta^2_2, \ldots, \beta^1_2, \beta^{j+1}_i, \beta^{j-1}_i, \ldots, \beta^3_i\}$$

is the desired pants decomposition.

Lemma 9. Let $P$ be a linear pants decomposition of $N$, $\mathcal{A}(P)$ be its adjacency graph and $\gamma \in \mathcal{A}(P)$ be a vertex. The vertex $\gamma$ is of type $(0,k)$ for some $k \geq 3$ if and only if $\gamma$ has valency two in $\mathcal{A}(P)$.

Proof. Suppose first that $\gamma$ is of type $(0,k)$, for some $k \geq 3$. The surface $N_\gamma$ has two connected components, none of which is a pair of pants. Thus, each connected component contains at least an element of $P$ adjacent to $\gamma$, so that $\gamma$ is adjacent to at least two, hence exactly two, vertices in $\mathcal{A}(P)$ since $P$ is linear.

Suppose now that $\gamma$ has valency two in $\mathcal{A}(P)$. Assume $\gamma$ is one-sided. If $N$ is cut along the elements of $P$, one gets $n - 1$ pairs of pants, and there is a unique pair of pants $Y$ such that its boundary components come from $\gamma$ and two other different simple closed curves, say $\beta_1, \beta_2$. But in this case the vertices $\gamma, \beta_1, \beta_2$ form a triangle in the adjacency graph, contradicting the linearity of $\mathcal{A}(P)$. By a similar argument, $\gamma$ can not be of type $(0,2)$ either.

Remark 10. Note that a vertex in the adjacency graph of a linear pants decomposition has exactly two vertices of valency one. These vertices are either one-sided or of type $(0,2)$. In fact, one of them is one-sided and the other is of type $(0,2)$.

Lemma 11. Let $P = \{\gamma_2, \gamma_3, \ldots, \gamma_n\}$ be a linear pants decomposition of $N$, $\mathcal{A}(P)$ be the adjacency graph of $P$ such that $\gamma_i$ is connected to $\gamma_{i+1}$ for $i = 2, \ldots, n - 1$ in $\mathcal{A}(P)$. Then, by changing the label $i$ with $n + 2 - i$ if necessary, $\gamma_n$ is one-sided and $\gamma_i$ is of type $(0,i)$ for $2 \leq i \leq n - 1$.

Proof. By Remark 10 either $\gamma_2$ or $\gamma_n$ is one-sided. By changing the labeling if necessary, we may assume that $\gamma_n$ is one-sided. It remains to prove that $\gamma_k$ is of type $(0,k)$ for $2 \leq k \leq n - 1$.

The proof of this is by induction on $k$. Since the vertex $\gamma_2$ has valency one, it is of type $(0,2)$ by Remark 10. Assume that $\gamma_i$ is of type $(0,i)$ for $i \leq k - 1$. If $N$ is cut along all curves in $P$, one gets $n - 1$ pairs of pants. There is a unique pair of pants $X$ such that two of the holes of $X$ are $\gamma_{k-1}$ and $\gamma_k$ as they are adjacent with respect to $P$. If the third hole of $X$ were a curve $\beta \in P \setminus \{\gamma_{k-1}, \gamma_k\}$, then $\mathcal{A}(P)$ would contain a triangle with vertices $\{\beta, \gamma_{k-1}, \gamma_k\}$, contradicting with the linearity of $P$. Hence, the third hole of $X$ is a hole of $N$ as well. Since $\gamma_{k-1}$ bounds a disc with $k - 1$ holes, it follows that $\gamma_k$ bounds a disc with $k$ holes, i.e. it is of type $(0,k)$.
The following lemma is based on \[1\] Lemma 5.2] proving that if the adjacency graph of a pants decomposition of a torus with \(n \geq 4\) holes does not contain a triangle, then this pants decomposition is either linear or a cyclic pants decomposition.

**Lemma 12.** Let \(P\) be a pants decomposition of \(N\) and let \(\mathcal{A}(P)\) be its adjacency graph. If \(\mathcal{A}(P)\) does not contain any triangle, then \(P\) is linear.

**Proof.** Suppose that \(\mathcal{A}(P)\) does not contain any triangle. The surface obtained by cutting \(N\) along the elements of \(P\) is a disjoint union of pants. Since \(\mathcal{A}(P)\) does not contain any triangle, at least one hole of each such pair of pants is a hole of \(N\). Let \(X\) be such a pair of pants. Then, either

1. exactly two holes of \(X\) are curves in \(P\) one of which is one-sided, or
2. exactly two holes of \(X\) are two-sided curves in \(P\), or
3. only one hole of \(X\) is a curve in \(P\), which must be two-sided.

Since the genus of \(N\) is 1, there is only one pair of pants \(X_1\) of type 1. The pair of pants \(X_2\) glued to \(X_1\) must be of type 2; otherwise (if type 3), the surface would be \(N_{1,3}\). The pair of pant \(X_3\) glued to \(X_1 \cup X_2\) can be of type 2 or 3. If type 3, then \(N = N_{1,4}\). By arguing in this way we conclude that \(N = X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{n-1}\), where \(X_1\) is of type (1), \(X_{n-1}\) is of type 3 and all others are of type 2. It follows that \(\mathcal{A}(P)\) is linear. \(\Box\)

**Lemma 13.** Let \(P \subset \mathcal{X}_{1,n}\) be a linear pants decomposition of \(N\) satisfying the nonadjacency condition with respect to \(\mathcal{X}_{1,n}\). Then \(\phi(P)\) is a linear pants decomposition of \(N\).

**Proof.** Suppose first that \(n = 4\). In this case, the graph \(\mathcal{A}(P)\) contains three vertices and two of them, say \(\alpha, \beta\) are nonadjacent to each other. If the adjacency graph \(\mathcal{A}(\phi(P))\) of \(\phi(P)\) is not linear, then it must be a triangle, i.e. each vertex has valency two. In particular, \(\phi(\alpha)\) is adjacent to \(\phi(\beta)\). Since \(\phi\) preserves nonadjacency with respect to \(P\) by Lemma 7 this is impossible.

Suppose now that \(n \geq 5\). Then \(P\) contains at least four curves. Assume that \(\mathcal{A}(\phi(P))\) contains a triangle. Since \(\mathcal{A}(\phi(P))\) is connected and contains at least four vertices, there exists a vertex \(\phi(\gamma)\) in this triangle which has valency at least three. Thus, there exist at most \(n - 5\) vertices in \(\mathcal{A}(\phi(P))\) nonadjacent to \(\phi(\gamma)\). On the other hand, the number of the vertices in \(\mathcal{A}(P)\) nonadjacent to \(\gamma\) is either \(n - 3\) or \(n - 4\). Since \(\phi\) preserves nonadjacency by Lemma 7 this is a contradiction.

It follows now from Lemma 12 that \(\phi(P)\) is linear. \(\Box\)

**Lemma 14.** Let \(P \subset \mathcal{X}_{1,n}\) be a linear pants decomposition of \(N\) containing \(\alpha_1\). Then \(P\) satisfies the nonadjacency condition.

**Proof.** We may assume, by Lemma 11 that the elements of the pants decomposition \(P\) are \(\gamma_2, \gamma_3, \ldots, \gamma_{n-1}, \alpha_1\), where \(\gamma_k\) is of type \((0, k)\) for \(2 \leq k \leq n-1\). We claim that for every \(\gamma \in P\), there is a vertex \(\delta_\gamma\) in \(\mathcal{X}_{1,n}\) such that \(\delta_\gamma \neq \gamma\) and \((P \setminus \{\gamma\}) \cup \{\delta_\gamma\}\) is also a pants decomposition. Since for any two nonadjacent vertices \(\alpha\) and \(\beta\) there is a separating vertex \(\delta\) such that \(\alpha\) and \(\beta\) lie on different connected components of \(N_\delta\), the lemma follows from this claim,
If \( \gamma = \alpha_1 \) then \( \gamma_{n-1} \in \{ \beta^*_1, \beta^*_{n+1} \} \) since there are only two curves of type \((0, n-1)\) in \( \mathfrak{X}_{1,n} \). If \( \gamma_{n-1} = \beta^*_1 \) then take \( \delta_\gamma = \alpha^*_1 \), and if \( \gamma_{n-1} = \beta^*_{n+1} \) then take \( \delta_\gamma = \alpha^*_2 \).

If \( \gamma = \gamma_k \), then \( \gamma = \beta^j_i \) for some \( i, j \) with \( |j - i| = k \). In this case one of \( \gamma_{k-1} \in \{ \beta^{j-1}_i, \beta^{j+1}_i \} \) and \( \gamma_{k+1} \in \{ \beta^{j+1}_i, \beta^{j-1}_i \} \) exist, and one of the curves \( \beta_{i-1}^{j-1}, \beta_{i-1}^{j+1}, \beta_{i+1}^{j-1}, \beta_{i+1}^{j+1} \) can be taken as \( \delta_\gamma \) (with the convention that \( i - 1 = n + 1 \) if \( i = 1 \)).

\[ \Box \]

Corollary 15. If \( P \subset \mathfrak{X}_{1,n} \) is a linear pants decomposition of \( N \) containing \( \alpha_1 \), then \( \phi(P) \) is linear.

4.1.2. \( \phi \) preserves topological types of vertices. We now prove that the locally injective simplicial map \( \phi : \mathfrak{X}_{1,n} \to C(N) \) preserves the topological types of vertices of \( \mathfrak{X}_{1,n} \).

Proposition 16. For every vertex \( \gamma \in \mathfrak{X}_{1,n} \), the topological types of \( \phi(\gamma) \) and \( \gamma \) are the same.

Proof. Suppose that \( P = \{ \gamma_2, \gamma_3, \ldots, \gamma_{n-1}, \gamma_n \} \) is a linear pants decomposition contained in \( \mathfrak{X}_{1,n} \) with \( \gamma_n = \alpha_1 \) such that \( \gamma_k \) is of type \((0, k)\) and is connected to \( \gamma_{k+1} \) in the adjacency graph \( \mathcal{A}(P) \) for \( k = 2, 3, \ldots, n - 1 \). By Corollary 15 \( \phi(P) \) is a linear pants decomposition of \( N \). Since \( \gamma_2 \) and \( \gamma_n \) have valency one in \( \mathcal{A}(P) \), \( \phi(\gamma_2) \) and \( \phi(\gamma_n) \) have valency one in \( \mathcal{A}(\phi(P)) \). By Remark 11 one of \( \phi(\gamma_2) \) and \( \phi(\gamma_n) \) is one-sided and the other is of type \((0, 2)\).

Now let us consider the linear pants decompositions \( P_1 = \{ \beta^3_1, \beta^4_1, \beta^5_1, \ldots, \beta^n_1, \alpha_1 \} \) and \( P_2 = \{ \beta^1_{n-1}, \beta^2_{n-2}, \ldots, \beta^n_{n-1}, \beta^*_{n+1}, \alpha_1 \} \). It follows from the argument in the previous paragraph that the topological types of \( \phi(\beta^3_1) \) and \( \phi(\beta^*_{n+1}) \) are the same; they are either both one-sided or both of type \((0, 2)\). Since they are disjoint and since any two one-sided curves on \( N \) must intersect, these two curves must be of type \((0, 2)\). Hence, \( \phi(\alpha_1) \) is one-sided.

Let us now consider a curve \( \beta^j_i \) of type \((0, k)\) so that \( k = |j - i| \). By Lemma 8 there is a linear pants decomposition in \( \mathfrak{X}_{1,n} \) containing \( \beta^j_i \) and \( \alpha_1 \). By Corollary 15 \( \phi(P) \) is a linear pants decomposition and by Lemma 10 \( \phi(\beta^j_i) \) is of type \((0, k)\).

Finally, for every \( \alpha^k_1 \), there is a pants decomposition \( P \) containing it. Since \( \phi(\gamma) \) is two-sided for every curve \( \gamma \in P \) different than \( \phi(\alpha^k_1) \), the curve \( \phi(\alpha^k_1) \) must be one-sided.

This finishes the proof of the proposition.

4.2. The case \( g \geq 2 \).

Let \( N \) denote the non-orientable surface \( N_{g,n} \) of genus \( g \geq 2 \) with \( n \) holes given in Section 2.3 where \( g + n \geq 5 \). Recall that by Corollary 3 a top dimensional pants decomposition of \( N \) contains exactly \( 2g + n - 3 \) elements. Let \( \phi : \mathfrak{X}_{g,n} \to C(N) \) be a locally injective simplicial map. Our aim in this subsection is to show that each \( \phi(\alpha_i) \) is one-sided and essential.

4.2.1. Closed Surfaces.

Assume that \( n = 0 \) so that \( g \geq 5 \).
Lemma 17. Let $P$ be a top dimensional pants decomposition of $N$, $\mathcal{A}(P)$ be its adjacency graph and $\alpha$ be a vertex in $\mathcal{A}(P)$. The vertex $\gamma$ is essential one-sided if and only if its valency in $\mathcal{A}(P)$ is two.

Proof. Let $\gamma \in P$ be an element. Since $N$ is closed, by Lemma 14 the curve $\gamma$ is either essential one-sided or of type $(k,0)$ for some $k$, where $2 \leq k \leq g - 2$.

Suppose that $\gamma$ is essential and one-sided. The surface obtained by cutting $N$ along the simple closed curves in $P \setminus \{\gamma\}$ is the disjoint union of $g - 3$ pairs of pants and a subsurface $X$ homeomorphic to the real projective plane with two holes. The curve $\gamma$ lies on $X$. The holes of $X \subset N$ come from two distinct simple closed curves in $P \setminus \{\gamma\}$. Hence, the valency of $\gamma$ in $\mathcal{A}(P)$ is two.

Let $\delta$ be a simple closed curve in $P$ of type $(k,0)$ for some $k$ with $2 \leq k \leq g - 2$. The surface $N_3$ obtained by cutting $N$ along $\delta$ has two connected components $X_1$ and $X_2$, one of them is homeomorphic to $N_{k,1}$, the other is homeomorphic to $N_{g-k,1}$. Each of $X_1$ and $X_2$ contains two of elements in $P$ adjacent to $\delta$ in $\mathcal{A}(P)$. Hence, valency of $\delta$ is four in $\mathcal{A}(P)$.

Since a one-sided curve contained in a top dimensional pants decomposition must be essential, the proof of the lemma is complete. □

Lemma 18. For each $i = 1,2,\ldots,g$, the curve $\phi(\alpha_i)$ is essential one-sided.

Proof. Let $P$ be any pants decomposition in $\mathcal{X}_{g,0}$ containing all $\alpha_i$ such that $P$ satisfies the nonadjacency condition. Note that such a pant decomposition $P$ exists and is of top dimensional. In $\mathcal{A}(P)$, $\alpha_i$ has valency two so that the number of vertices in $\mathcal{A}(P)$ nonadjacent to $\alpha_i$ is $|P| - 3$. Since $P$ satisfies the nonadjacency condition, there are at least $|P| - 3$ vertices in $\mathcal{A}(\phi(P))$ nonadjacent to $\phi(\alpha_i)$. As there is no vertex of valency one in $\mathcal{A}(\phi(P))$, the valency of $\phi(\alpha_i)$ is exactly two. By Lemma 17 $\phi(\alpha_i)$ is one-sided and essential. □

4.2.2. Surfaces with $n \geq 1$ holes. Suppose now that $n \geq 1$ and that $N$ denotes the surface $N_{g,n}$.

Lemma 19. Let $P$ be a top dimensional pants decomposition of $N$, $\mathcal{A}(P)$ be its adjacency graph and let $\gamma$ be a vertex of valency one in $\mathcal{A}(P)$. Then, $\gamma$ is either essential one-sided or of type $(0,2)$.

Proof. Assume that $\gamma$ is neither essential one-sided nor of type $(0,2)$. Since $\gamma$ is contained in the top dimensional pants decomposition $P$, by Lemma 14 $\gamma$ is separating and the surface $N_\gamma$ obtained by cutting $N$ along $\gamma$ is the disjoint union of two subsurfaces of $N$, none of which is a pair of pants. On each of these components, there is a vertex of $P$ adjacent to $\gamma$, so that the valency of $\gamma$ in $\mathcal{A}(P)$ is at least two.

By this contradiction, $\gamma$ is either essential one-sided or of type $(0,2)$. □

Corollary 20. Let $P$ be a top dimensional pants decomposition of $N$ and $\gamma_0 \in P$ have valency one in $\mathcal{A}(P)$. Suppose that a vertex $\gamma_k \in P$ is a $k$th linear successor of $\gamma_0$. If $\gamma_0$ is one-sided, then $\gamma_k$ is of type $(1,k)$, and if $\gamma_0$ is of type $(0,2)$, then $\gamma_k$ is of type $(0,k+2)$. 

**Proof.** Let \((\gamma_0, \gamma_1, \ldots, \gamma_k)\) be a linear path in \(\mathcal{A}(P)\) from \(\gamma_0\) to \(\gamma_k\) such that each \(\gamma_i\) is an \(i\)th linear successor of \(\gamma_0\). The proof then follows from the fact that for \(i = 1, 2, \ldots, k\), the curves \(\gamma_{i-1}\) and \(\gamma_i\) bound a pair of pants. \(\square\)

**Lemma 21.** Let \(P \subset \mathcal{X}_{g,n}\) be a top dimensional pants decomposition satisfying the nonadjacency condition and let \(\gamma_0 \in P\) have valency one in \(\mathcal{A}(P)\). Suppose that \(\gamma_k\) is a \(k\)th linear successor of \(\gamma_0\) in \(\mathcal{A}(P)\). Then \(\phi(\gamma_0)\) has valency one in the adjacency graph \(\mathcal{A}(\phi(P))\) of \(\phi(P)\) and \(\phi(\gamma_k)\) is a \(k\)th linear successor of \(\phi(\gamma_0)\) in \(\mathcal{A}(\phi(P))\).

**Proof.** Let \((\gamma_0, \gamma_1, \ldots, \gamma_k)\) be a linear path such that each \(\gamma_i\) is an \(i\)th linear successor of \(\gamma_0\). Since \(\gamma_1\) is the unique vertex in \(\mathcal{A}(P)\) adjacent to \(\gamma_0\) and since \(\phi\) preserves nonadjacency in \(\mathcal{A}(P)\), \(\phi(\gamma_1)\) is the unique vertex in \(\mathcal{A}(\phi(P))\) adjacent to \(\phi(\gamma_0)\). Hence, \(\phi(\gamma_0)\) has valency one in \(\mathcal{A}(\phi(P))\).

The vertex \(\gamma_i\) is adjacent to \(\gamma_{i-1}\) and \(\gamma_{i+1}\) for \(i = 1, 2, \ldots, k-1\). Since \(\phi\) preserves nonadjacency in \(\mathcal{A}(P)\), the valency of \(\phi(\gamma_i)\) is either one or two in \(\mathcal{A}(\phi(P))\). On the other hand since \(\mathcal{A}(\phi(P))\) is connected, \(\phi(\gamma_i)\) has valency two in \(\mathcal{A}(\phi(P))\). Hence, the vertex \(\phi(\gamma_i)\) must be adjacent to the vertices \(\phi(\gamma_{i-1})\) and \(\phi(\gamma_{i+1})\). Thus, \((\phi(\gamma_0), \phi(\gamma_1), \ldots, \phi(\gamma_k))\) is a linear path, so that \(\phi(\gamma_k)\) is a \(k\)th linear successor of \(\phi(\gamma_0)\). \(\square\)

**Lemma 22.** The curve \(\phi(\alpha_g)\) is one-sided and essential.

**Proof.** For the sake of notational simplicity, let us denote the curve \(\beta_{g-1}^k\) by \(\beta^k\).

Consider the top dimensional pants decomposition

\[ P = \{\alpha_1, \alpha_2, \ldots, \alpha_{g-1}, \alpha_g, \beta_{g+1}^1, \beta_{g+2}^1, \ldots, \beta_{g+n}^1, \beta_2^1, \beta_2^2, \ldots, \beta_2^{g-3}\} \]

in \(\mathcal{X}_{g,n}\), so that the vertex \(\beta_{g+1}^1\) is the unique vertex adjacent to \(\alpha_g\) in the adjacency graph \(\mathcal{A}(P)\) (cf. Figure 5).

Note that \((\alpha_g, \beta_{g+1}^1, \beta_{g+2}^1, \ldots, \beta_{g+n}^1)\) is a linear path in \(\mathcal{A}(P)\) with \(n + 1\) vertices such that for \(1 \leq k \leq n\) the curve \(\beta_{g+k}^1\) is a \(k\)th linear successor of \(\alpha_g\). It is easy to see that the pants decomposition \(P\) satisfies the nonadjacency condition. Hence, the map \(\phi\) preserves the nonadjacency in \(\mathcal{A}(P)\). Since \(\alpha_g\) has valency one in \(\mathcal{A}(P)\), so is \(\phi(\alpha_g)\) in \(\mathcal{A}(\phi(P))\). By Lemma 19, \(\phi(\alpha_g)\) is either essential one-sided or of type \((0, 2)\).

Suppose that \(\phi(\alpha_g)\) is of type \((0, 2)\). By Lemma 21, the vertex \(\phi(\beta_{g+k}^1)\) is a \(k\)th linear successor of \(\phi(\alpha_g)\) in the graph \(\mathcal{A}(\phi(P))\). Since \(\phi(\alpha_g)\) is of type \((0, 2)\), the vertex \(\phi(\beta_{g+n}^1)\) is of type \((0, n + 2)\). But there are no simple closed curve of type \((0, n + 2)\) on \(N\). By this contradiction, \(\phi(\alpha_g)\) must be essential one-sided. \(\square\)

We finish this section with the following corollary.

**Corollary 23.** Let \(g + n \geq 5\). All of the curves \(\phi(\alpha_1), \phi(\alpha_2), \ldots, \phi(\alpha_g)\) are essential one-sided.

**Proof.** The case \(g = 1\) is proved in Proposition 10. So suppose that \(g \geq 2\). By Lemma 22, \(\phi(\alpha_g)\) is essential one-sided. Let \(h : N \to N\) be a homeomorphism with \(h(\phi(\alpha_g)) = \alpha_g\). Now \(\phi = h\phi\) restricts to a locally injective simplicial map \(\text{Link}_{\mathcal{X}_{g,n}}(\alpha_g) \to \text{Link}_{\mathcal{X}_{g,n}}(\alpha_g)\).
The surface $N_{\alpha_g}$ obtained by cutting $N$ along $\alpha_g$ is a non-orientable surface of genus $g-1$ with $n+1$ holes. Consider the injective simplicial map $\varphi : \text{Link}_{C(N)}(\alpha_g) \to \mathcal{C}(N_{\alpha_g})$ induced by cutting $N$ along $\alpha_g$. We then have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Link}_{X_{g,n}}(\alpha_g) & \xrightarrow{\phi} & \text{Link}_{\mathcal{C}(N_{g,n})}(\alpha_g) \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
X_{g-1,n+1} & \xrightarrow{\varphi \tilde{\phi} \varphi^{-1}} & \mathcal{C}(N_{\alpha_g}).
\end{array}
$$

The map $\varphi \tilde{\phi} \varphi^{-1}$ is locally injective simplicial. By Lemma 22, $\varphi \tilde{\phi} \varphi^{-1}$ maps the essential one-sided curve $\varphi(\alpha_{g-1})$ to an essential one-sided curve in $N_{\alpha}$. It follows that $\tilde{\phi}(\alpha_{g-1})$, and hence $\phi(\alpha_{g-1})$, is one-sided and essential.

By continuing in this way, one concludes that $\phi(\alpha_k)$ is one sided essential for all $1 \leq k \leq g$. $\square$

4.2.3. **Curves of type** $(2,0)$. In the proof of the main theorem, we also need that the map $\phi$ preserves the topological types of the curves of type $(2,0)$ in $X_{g,n}$. In the next lemma, we take the subscripts and superscripts modulo $g+n$.

**Lemma 24.** Let $g \geq 2$, $n \geq 1$ and $1 \leq i \leq g-1$. If $\beta = \beta_{i-1}^{i+1} \in X_{g,n}$ so that $\beta$ is of type $(2,0)$, then $\phi(\beta)$ is of type $(2,0)$.

**Proof.** Consider the top dimensional pants decomposition

$$P = \{\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_{i-1}^{i+1}, \beta_{i-1}^{i+2}, \ldots, \beta_{i-1}^{g+n}, \beta_{i-1}^1, \beta_{i-1}^2, \beta_{i-1}^{g-1}, \beta_{i-1}^{g-2}, \ldots, \beta_{i-1}^{g-1}\}.$$

It is easy to see that the pants decomposition $P$ satisfies the nonadjacency condition. By Lemma 22, $\phi$ preserves nonadjacency in the adjacency graph of $P$, i.e. images of nonadjacent vertices in $P$ are nonadjacent in $\phi(P)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{pants_decomposition.png}
\caption{The pants decomposition $P$ containing $\alpha_1$ and its adjacency graph.}
\end{figure}
Figure 6. Possible configurations for $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta_{i-1}^{\pm 1})$ in $\mathcal{A}(\phi(P))$

We note that the vertices $\alpha_i, \alpha_{i+1}$ and $\beta = \beta_{i-1}^{\pm 1}$ form a triangle in $\mathcal{A}(P)$ and $\beta$ is the unique two-sided simple closed curve that is adjacent to both $\alpha_i$ and $\alpha_{i+1}$ in $\mathcal{A}(P)$. Since $\phi(\alpha_j)$ is essential one-sided for each $j = 1, 2, \ldots, g$ and $\beta$ is disjoint from all $\alpha_j$, $\phi(\beta)$ is disjoint from all $\phi(\alpha_j)$, and hence it is a two-sided curve. Since $\phi$ preserves nonadjacency in $\mathcal{A}(P)$, $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$ might only be adjacent to each other and $\phi(\beta)$. If it can be shown that the vertices $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta)$ form a triangle in $\mathcal{A}(\phi(P))$, one can conclude easily that $\phi(\beta)$ is of type (2, 0).

The pants decomposition $\phi(P)$ is top dimensional, so that its adjacency graph $\mathcal{A}(\phi(P))$ is connected. The possible full subgraph of $\mathcal{A}(\phi(P))$ with vertices the vertices $\phi(\alpha_i)$, $\phi(\alpha_{i+1})$ and $\phi(\beta)$ is one of the four graphs given in Figure 6.

It is clear that in a pants decomposition, if two one-sided vertices $\gamma_1, \gamma_2$ are adjacent, then any vertex adjacent to $\gamma_1$ is also adjacent to $\gamma_2$. Since $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$ are essential one-sided simple closed curves, the cases (1) and (2) in Figure 6 is impossible.

We now rule out the case (3). If they form such a graph, then the surface $N$ is a Klein bottle with two holes since $\phi(\alpha_i)$ and $\phi(\alpha_{i+1})$ is not adjacent to any vertex different from $\phi(\beta)$. But we have the assumption $g + n \geq 5$.

Thus we have proved that the only possible configuration is (4) and hence $\phi(\beta)$ bounds a Klein bottle with one hole, i.e., it is of two (2, 0), finishing the proof of the lemma.

\[\square\]

5. PROOF OF THE MAIN RESULT

In this section, we prove the main result of this paper, Theorem 26, also stated as Main Theorem in the introduction. Let $N$ denote the non-orientable surface $N_{g,n}$. 
Lemma 25. Let \( g + n \geq 5 \). Suppose that, for some \( 1 \leq i \leq g \), a locally injective simplicial map \( \phi : \mathcal{X}_{g,n} \to \mathcal{C}(N) \) fixes every vertex of the star of \( \alpha_i \) in \( \mathcal{X}_{g,n} \). Then \( \phi \) fixes every vertex of \( \mathcal{X}_{g,n} \).

Proof. All vertices of \( \mathcal{X}_{g,n} \) but \( \alpha_i^j \) are in the star of \( \alpha_i \). So we need to prove that \( \phi(\alpha_i^j) = \alpha_i^j \) for all \( j \). With the agreement that we take the indices modulo \( g + n \), let us fix some \( j \) with \( j \notin \{i - 1, i\} \) and define the following subsets of \( \mathcal{X}_{g,n} \):

1. \( P_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_g\} \),
2. \( P_2 = \{\beta_{i-1}^j, \beta_{i-1}^{j+1}, \ldots, \beta_{i-1}^{j+n}, \beta_{i-1}^1, \beta_{i-1}^2, \ldots, \beta_{i-1}^{i-3}\} \),
3. \( P_3 = \{\beta_{i+2}^j, \beta_{i+3}^j, \ldots, \beta_i^j\} \).

When we cut \( N \) along the curves in \( P_1 \cup P_2 \cup P_3 \), we get a number of pair of pants and a non-orientable surface \( U \) of genus one with two boundary components. There are only two nontrivial curves on \( U \); \( \alpha_i \) and \( \alpha_i^j \). Since \( \phi(\alpha_i^j) \) lies on \( U \) and is different from \( \alpha_i \), we must have \( \phi(\alpha_i^j) = \alpha_i^j \).

Theorem 26. Let \( g + n \neq 4 \). The finite complex \( \mathcal{X}_{g,n} \) is rigid in \( \mathcal{C}(N) \). That is, any locally injective simplicial map \( \phi : \mathcal{X}_{g,n} \to \mathcal{C}(N) \) is induced from an element \( F \in \text{Mod}(N) \). The mapping class \( F \) is unique up to composition with the involution that interchanges the two faces of the model of \( N \).

Proof. It is easy to see that the sets

\[
\begin{align*}
\mathcal{X}_{1,0} & = \{\alpha_1\} = \mathcal{C}(N_{1,0}), \\
\mathcal{X}_{1,1} & = \{\alpha_1\} = \mathcal{C}(N_{1,1}), \\
\mathcal{X}_{1,2} & = \{\alpha_1, \alpha_2\} = \mathcal{C}(N_{1,2}), \\
\mathcal{X}_{2,0} & = \{\alpha_1, \alpha_2\}, \\
\mathcal{X}_{2,1} & = \{\alpha_1, \alpha_2\}, \\
\mathcal{X}_{3,0} & = \{\alpha_1, \alpha_2, \alpha_3\}
\end{align*}
\]

are rigid. So suppose that \( g + n \geq 5 \).

The proof is by induction on the genus \( g \). Recall that we consider a sphere as a non-orientable surface of genus 0. The base step, the case \( g = 0 \), is proved [11, Theorem 3.1], stated as Theorem 6 above. So assume that \( g \geq 1 \) and that any locally injective simplicial map \( \mathcal{X}_{g-1,m} \to \mathcal{C}(N_{g-1,m}) \) for \( (g - 1) + m \geq 5 \), is induced from a homeomorphism \( N_{g-1,m} \to N_{g-1,m} \).

Let \( \phi : \mathcal{X}_{g,n} \to \mathcal{C}(N) \) be a locally injective simplicial map. By Corollary 23 the curve \( \phi(\alpha_g) \) is a one-sided essential simple closed curve. Thus, by the classification of surfaces, there exists a homeomorphism \( h : N \to N \) such that \( h(\phi(\alpha_g)) = \alpha_g \).

Since \( \phi \) is induced from a homeomorphism of \( N \) if and only if \( h \phi \) is, we may assume, by replacing \( h \phi \) by \( \phi \), that \( \phi(\alpha_g) = \alpha_g \). The map \( \phi \) then induces a locally injective simplicial map

\[
\text{Link}_{\mathcal{X}_{g,n}}(\alpha_g) \to \text{Link}_{\mathcal{C}(N)}(\alpha_g).
\]

The surface \( N_{\alpha_g} \) obtained by cutting \( N \) along \( \alpha_g \) is a non-orientable surface of genus \( g - 1 \) with \( n + 1 \) holes. As in the proof of Corollary 23, consider the injective simplicial map \( \varphi : \text{Link}_{\mathcal{C}(N)}(\alpha_g) \to \mathcal{C}(N_{\alpha_g}) \) induced by cutting \( N \) along \( \alpha_g \) and the
commutative diagram

\[ \begin{array}{ccc}
\text{Link}_{X_{g,n}}(\alpha_g) & \xrightarrow{\phi} & \text{Link}_{C(N)}(\alpha_g) \\
\downarrow & & \downarrow \\
X_{g-1,n+1} & \xrightarrow{\phi} & C(N_{\alpha_g})
\end{array} \]

so that \( \tilde{\phi} = \varphi \phi \varphi^{-1} \) is a locally injective simplicial map. It is easy to see that the map \( \varphi \) on the left-hand side of the diagram is an isomorphism.

By the induction hypothesis, there is a homeomorphism \( f : N_{\alpha_g} \to N_{\alpha_g} \) such that \( f(\gamma') = \tilde{\phi}(\gamma') \) for every curve \( \gamma' \in X_{g-1,n+1} \) on \( N_{\alpha_g} \). That is, \( f(\varphi(\gamma)) = \varphi(\phi(\gamma)) \) for every vertex \( \gamma \in \text{Link}_{X_{g,n}}(\alpha_g) \).

The surface \( N_{\alpha_g} \) has a distinguished boundary component \( \Delta \), coming from \( \alpha_g \). We claim that the homeomorphism \( f \) maps \( \Delta \) to itself.

Case \( g = 1 \): Suppose that \( f(\Delta) \neq \Delta \). Each one of the curves \( \varphi(\beta_1^n) \) and \( \varphi(\beta_2^{n+1}) \) bounds a disc with two holes in \( N_{\alpha_1} \) one of which is \( \Delta \). At least one of the holes \( f(z_1) \) and \( f(z_n) \) is different from \( \Delta \). (Here, \( z_i \) represents the boundary of a small neighborhood of the \( i \)th puncture.) Without loss of generality assume that \( f(z_n) \neq \Delta \). In this case, \( f(\varphi(\beta_1^n)) = \varphi(\phi(\beta_1^n)) \) bounds a disc with two holes none of which is \( \Delta \), so that \( \phi(\beta_1^n) \) is of type \( (0,2) \) in \( C(N_{1,n}) \). But it is of type \( (0,n-1) \). Since \( n \geq 4 \), this is a contradiction, so that \( f(\Delta) = \Delta \). (If \( f(z_1) \neq \Delta \), then use the curve \( \beta_2^{n+1} \).

Case \( g \geq 2 \): If \( n = 0 \), then trivially \( f(\Delta) = \Delta \). Let \( n \geq 1 \).

Suppose that \( f(\Delta) \neq \Delta \), so that \( f(\Delta) \) is a boundary component of \( N_{\alpha_g} \) coming from a boundary component of \( N \). Consider the curve \( \beta_{g-2}^n \) in the link of \( \alpha_g \). Since it is of type \((2,0)\), so is \( \phi(\beta_{g-2}^n) \) by Lemma 24. On the other hand, since the curve \( f(\varphi(\beta_{g-2}^n)) = \varphi(\phi(\beta_{g-2}^n)) \) is either of type \((0,2)\) or of type \((1,1)\), the same holds for \( \phi(\beta_{g-2}^n) \). By this contradiction we must have \( f(\Delta) = \Delta \).

By applying a suitable isotopy on the regular neighborhood of the boundary component \( \Delta \), one may assume that \( f \) sends antipodal points on \( \Delta \) to antipodal points. Hence, \( f \) descends a homeomorphism \( F : N \to N \) with the property that \( F(\alpha_g) = \alpha_g \) and \( F(\gamma) = \phi(\gamma) \) for every vertex \( \gamma \in \text{Link}_{X_{g,n}}(\alpha_g) \).

Now, the automorphism \( F^{-1} \phi \) of \( C(N_{g,n}) \) fixes \( \alpha_g \) and all curves in \( X_{g,n} \) disjoint from \( \alpha_g \). By Lemma 24, \( F^{-1} \phi \) fixes every vertex of \( X_{g,n} \). Thus, \( \phi \) is induces by the homeomorphism \( F : N \to N \).

If \( G \) is another homeomorphism inducing \( \phi \), then \( G^{-1}F \) fixes all curves of \( X_{g,n} \). It can be seen easily that the stabilizer of \( X_{g,n} \) in the mapping class group is a cyclic subgroup of order two generated by the involution interchanging the two copies of \( R \) used as a model on \( N \).

The proof of the theorem is now complete.

\[ \square \]

**Corollary 27.** Let \( g+n \neq 4 \). Every locally injective simplicial map \( X_{g,n} \to C(N) \) is induced by a simplicial automorphism \( C(N) \to C(N) \).
Proof. By Theorem 26, a locally injective simplicial map \( \phi : X_{g,n} \to \mathcal{C}(N) \) is induced by a homeomorphism of \( N \). This homeomorphism induces a simplicial automorphism of \( \mathcal{C}(N) \to \mathcal{C}(N) \) that coincides with \( \phi \) on \( X_{g,n} \).

6. \( X_{g,n} \) FOR \( g + n = 4 \)

In this section, we show that the set \( X_{g,n} \) is not rigid in the case \( g + n = 4 \). Let us define the following subsets \( A_g \) and \( B_g \) of \( X_{g,n} \) such that \( A_g \cup B_g = X_{g,n} \):

\[
A_1 = \{ \alpha_1, \alpha_2, \beta_1^2 \}, \quad B_1 = \{ \alpha_1^3, \beta_1^3 \}, \\
A_2 = \{ \alpha_1, \alpha_2, \alpha_1^2, \beta_1^2 \}, \quad B_2 = \{ \alpha_1^3, \alpha_1^2, \beta_1^3 \}, \\
A_3 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_1^2, \alpha_1^3, \beta_1^3 \}, \quad B_3 = \{ \alpha_1^3, \alpha_1^2, \alpha_1^3, \beta_1^3 \}, \\
A_4 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1^2, \alpha_1^3, \alpha_2^3, \beta_1^3 \}, \quad B_4 = \{ \alpha_1^3, \alpha_1^2, \alpha_1^3, \alpha_2^3, \beta_1^3 \}.
\]

If \( f : N_{g,n} \to N_{g,n} \) is a homeomorphism fixing the one-sided curves \( \alpha_1, \alpha_2, \ldots, \alpha_g \), then the map \( \phi : X_{g,n} \to \mathcal{C}(N_{g,n}) \) defined by

\[
\phi(\gamma) = \begin{cases} 
\gamma & \text{if } \gamma \in A_g \\
f(\gamma) & \text{if } \gamma \in B_g 
\end{cases}
\]

is a locally injective simplicial map. For almost any choice of \( f \) (for instance, \( f \) is the product of the Dehn twist \( t_{\beta_1^2} t_{\beta_1^3} t_{\beta_1^2} \)), the map \( \phi \) is not induced by a homeomorphism of the surface.

Of course, this leaves the question of the existence of a finite rigid set open for the cases \( g + n = 4 \).

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