Further robust dissipativity analysis of uncertain stochastic generalized neural networks with Markovian jump parameters

Pharunyou Chanthorn 1, Griengkrai Rajchakit 2,*, Jenjira Thipcha 2, Chanikan Emharuethai 2, Sriraman Ramalingam 3

1 Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
2 Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai, 52290, Thailand.
3 Vel Tech High Tech Dr.Rangarajan Dr.Sakunthala Engineering College, Avadi, Tamil Nadu-600 062, India.
* Correspondence: kreangkri@mju.ac.th; Tel.

Abstract: This paper analyzes the robust dissipativity of uncertain stochastic generalized neural networks (USGNNs) with Markovian jumping parameters and time-varying delays. In practical applications most of the systems refer to uncertainties, hence, the norm-bounded parameter uncertainties and stochastic disturbance are considered. Then, by constructing an appropriate Lyapunov-Krasovskii functional (LKF) and by employing integral inequalities LMI-based sufficient conditions of the considered systems are established. Numerical simulations are given to show the merit of the presented results.

Keywords: Dissipativity analysis; Generalized neural networks; Markovian jump parameters; Stochastic disturbance.

1. Introduction

In the last ten years, neural networks (NNs) have been gaining considerable attention because of their important applications in optimization, signal processing, image processing, associative memories and so forth [1]-[5]. Mainly, generalized neural networks (GNNs) has been considered an energetic research model because of their fitting mathematical model capacity and many powerful results were published concerning the stability of GNNs [15], [16], [35], [36]. On the other hand, time delay is the inherent feature of many physical processes including chemical processes, nuclear reactors, which is major source of instability and poor performance [6]-[12]. Therefore, many efficient approaches and important results have been reported regarding various dynamics of NNs with time-delays [13]-[39].

On the other side, the Markovian jumping neural network (MJNN) has recently been received significant research interest, since MJNN extremely useful model for understanding its dynamics when the NNs incorporate abrupt changes in their structure. To prove this matter, many effective methods have been developed [10]-[16]. On the other hand, the stochastic effects are certainly present in all neural systems. Therefore, the study of stochastic NNs is not only interesting but also essential, because the existence of certain stochastic inputs may affect the behavior of the system [17]-[20]. The stability of stochastic nonlinear systems has recently become an important field of research, and considerable efforts have been devoted to Markovian jumping stochastic NNs, several stability conditions were published recently [21]-[25]. For example, the stability of stochastic static NNs has been investigated in [13] with Markovian switching. On the basis of Lyapunov functional, the exponential stability of stochastic NNs with Markovian jump parameters was studied in [40]. Some other results regarding the proposed problem can be found [22]-[24]. On the other hand, when practical systems are modelled, uncertainties of system parameters are often included. Therefore, many systems refer to uncertainties in practical applications.
From the practical viewpoint, it is important to investigate NNs with uncertain parameters \[18, 19, 23, 24, 28, 40\].

Undoubtedly, dissipative behavior is certainly essential for control and engineering; thus, dissipativity analysis of USGNNs presents a theoretical challenge, which has gained growing attention \[26\]-\[28\]. For instance, in \[29\], three types of neuron activation functions are discussed for global dissipativity of delayed recurrent NNs, i.e. bounded, Lipschitz-continuous and monotonous non-decreasing. In \[30\], the authors analyzed the global dissipativity of NNs with both time-varying delays and unbounded delays. Meanwhile, \((\mathbf{Q}, \mathbf{S}, \mathbf{R})\) dissipativity theory contains multi-dynamic behaviors, which lead to effective known results by changing the weight matrices in their structure and discovered their applications in the field of control and engineering \[27\]-\[32\]. Similar studies on the dissipativity of various NNs can be found in \[28\]-\[33\]. To the best of the authors’ knowledge, the problem of robust dissipativity of USGNNs with norm-bounded uncertainties has not been fully investigated and remains a good challenge.

Motivated by these mentioned shortcomings for the existing results, in this paper we aim to establish robust dissipativity and stability for USGNNs with Markovian jumping parameters. On the basis of Lyapunov functional method, an appropriate LKF is constructed with more delay information and the derivative of LKF has been estimating by new integral inequalities, which has great support to reduce the conservatism of the obtained results. By employing Ito’s formula and some analytic techniques, robust dissipativity and stability conditions are derived in terms of simplified LMI. Numerical simulations are also given to prove the merits of the presented results. The layout of the paper is as follows, the problem is formally defined in the next section. The main results of this paper are presented in section 3. Numerical examples are presented in section 4. Finally, section 5 concludes this paper.

**List of symbols:** Throughout this paper, \(\mathbb{R}^n\) and \(\mathbb{R}^{m \times n}\) denotes the \(n\)-dimensional Euclidean space and the set of \(m \times n\) real matrices, respectively. The superscript \(X^T\) represents the transpose of \(X\). \(\mathcal{P} > 0\) means that \(\mathcal{P}\) is the symmetric positive definite matrix. \(\text{tr}\{\mathcal{D}\}\) denotes the trace of matrix \(\mathcal{D}\), \(\ast\) denotes the elements below the main diagonal of a symmetric block matrix. \((\Omega, \mathcal{F}, \mathcal{P})\) is complete probability space with a natural filtration. \(I_n\) represents the identity matrix with appropriate dimensions. \(\text{diag}\{\cdot\}\) denotes the block diagonal matrix. \(\mathcal{L}_2[0, \infty)\) is the space of an \(n\)-dimensional square integrable function on \([0, \infty)\). \(\mathbb{E}\{\cdot\}\) denote the mathematical expectation.

### 2. Problem description and preliminaries

Let \(\{\epsilon(t), t \geq 0\}\) be a right-continuous Markovian process on \((\Omega, \mathcal{F}, \mathcal{P})\) and taking values in a finite state space \(\mathcal{S} = \{1, 2, \ldots, N\}\) with generator \(\Pi = [\pi_{xy}]_{N \times N}\) given by

\[
\text{Pr}\{\epsilon(t + \Delta t) = y | \epsilon(t) = x\} = \begin{cases} 
\pi_{xy} \Delta t + o(\Delta t), & \text{if } x \neq y, \\
1 + \pi_{xx} \Delta t + o(\Delta t), & \text{if } x = y,
\end{cases}
\]

where \(\Delta t > 0\) and \(\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0\), \(\pi_{xy} \geq 0\) is the transition rate from \(x\) to \(y\) if \(x \neq y\) while \(\pi_{xx} = -\sum_{y=1, y \neq x}^{N} \pi_{xy}\).

We consider the following GNNs with MJP

\[
\begin{cases} 
\dot{p}(t) = -\mathcal{D}(\epsilon(t))p(t) + \mathcal{A}(\epsilon(t))g(\mathcal{W}(\epsilon(t))p(t)) + \mathcal{B}(\epsilon(t))g'(\mathcal{W}(\epsilon(t))p(t - \tau(t))) + u(t) \\
q(t) = g(\mathcal{W}(\epsilon(t))p(t)),
\end{cases}
\]

where \(p(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T \in \mathbb{R}^n\) is the state vector; \(g(\mathcal{W}(\epsilon(t))p(\cdot)) = [g_1(\mathcal{W}(\epsilon(t))p_1(\cdot)), g_2(\mathcal{W}(\epsilon(t))p_2(\cdot)), \ldots, g_n(\mathcal{W}(\epsilon(t))p_n(\cdot))]^T \in \mathbb{R}^n\) is the nonlinear neuron activation function; \(u(t) = [u_1(t), \ldots, u_n(t)]^T \in \mathbb{R}^n\) is the external disturbance which belongs to \(\mathcal{L}_2[0, \infty)\); \(q(t) = [q_1(t), \ldots, q_n(t)]^T \in \mathbb{R}^n\) is the
As mentioned earlier, it is often the case in practice that the NNs is disturbed by environmental noises that affect where
The system (A₁): \( \tau(t) \) is a known time-varying delay of the system which satisfies
\[
0 \leq \tau(t) \leq \tau, \quad \tau(t) \leq \mu. \tag{3}
\]
where \( \tau \) and \( \mu \) are constants.
(A₂): For all \( \zeta_1, \zeta_2 \in \mathbb{R}, \; \zeta_1 \neq \zeta_2 \), the neuron activation function \( g(\cdot) \) is continuous and bounded which satisfies
\[
[g(\zeta_1) - g(\zeta_2) - \Delta_1(\zeta_1 - \zeta_2)]^T [g(\zeta_1) - g(\zeta_2) - \Delta_2(\zeta_1 - \zeta_2)] \leq 0, \tag{4}
\]
where \( \Delta_1 \) and \( \Delta_2 \) are known constant matrices.
The initial condition of (2) is defined by \( p(t) = \phi(t) \) on \(-\tau \leq t \leq 0 \) in \( \phi \in C([-\tau, 0]; \mathbb{R}^n) \).
As mentioned earlier, it is often the case in practice that the NNs is disturbed by environmental noises that affect the stability of the equilibrium. Motivated by this we express a stochastic system whose consequent parts are a set of stochastic Markovian jump GNNs with time-varying delays:
\[
\begin{align*}
\frac{dp(t)}{dt} &= [-\mathcal{D}(e(t))p(t) + \mathcal{A}(e(t))g(W(e(t))p(t)) + \mathcal{B}(e(t))g(W'(e(t))p(t - \tau(t))) + u(t)]dt + \sigma(t, e(t), p(t), p(t - \tau(t)))d\omega(t), \\
q(t) &= g(W(e(t))p(t)).
\end{align*}
\tag{5}
\]
where \( \sigma(t, e(t), p(t), p(t - \tau(t))) \) is the stochastic perturbation. \( \omega(t) = [\omega_1(t), \ldots, \omega_m(t)]^T \in \mathbb{R}^m \) is the Brownian motion \( m \)-space on \( (\Omega, \mathcal{F}, \mathbb{P}) \).
For the purpose of simplicity, let \( e(t) = x \; (x \in S) \). Then \( \mathcal{D}(e(t)) = \mathcal{D}, \; \mathcal{A}(e(t)) = \mathcal{A}, \; \mathcal{B}(e(t)) = \mathcal{B} \) and \( W'(e(t)) = W' \). The system (5) becomes
\[
\begin{align*}
\frac{dp(t)}{dt} &= [-\mathcal{D}p(t) + \mathcal{A}g(Wp(t)) + \mathcal{B}g(W'(p(t - \tau(t)))) + u(t)]dt + \sigma(t, x, p(t), p(t - \tau(t)))d\omega(t), \\
q(t) &= g(Wp(t)).
\end{align*}
\tag{6}
\]
For the sake of convenience, the following abbreviations are adopted in the sequel:
\[
\begin{align*}
\phi(t) &\triangleq -\mathcal{D}p(t) + \mathcal{A}g(Wp(t)) + \mathcal{B}g(W'(p(t - \tau(t)))) + u(t) \\
\sigma(t) &\triangleq \sigma(t, x, p(t), p(t - \tau(t))).
\end{align*}
\tag{7}
\]
The system (6) reads as
\[
\begin{align*}
\frac{dp(t)}{dt} &= \phi(t)dt + \sigma(t)d\omega(t) \\
q(t) &= g(Wp(t)).
\end{align*}
\tag{8}
\]
where

\[ \text{Definition 1.} \quad \text{The NN (6) is said to be mean-square stable if for any } \varepsilon > 0 \text{ there exists a scalar } \omega(\varepsilon) > 0 \text{ such that } \mathbb{E}\{\|p(t)\|^2\} < \varepsilon, \ t > 0, \text{ whenever } \sup_{-\varepsilon \leq t \leq 0} \mathbb{E}\{\|p(t)\|^2\} < \omega(\varepsilon). \text{ In addition, if } \lim_{t \to \infty} \mathbb{E}\{\|p(t)\|^2\} = 0, \text{ for any initial condition, the NN (6) is called mean-square asymptotically stable.}
\]

\[ \text{Definition 2.} \quad \text{The NN (6) is said to be strictly } (Q,S,R) - \gamma \text{ dissipative if, for } \gamma > 0 \text{ and under zero initial condition, the following inequality is satisfied:}
\]

\[ \mathbb{E}\{G(u,q,t_d)\} \geq \mathbb{E}\{\gamma(\nabla u)^2\}, \ \forall t_d \geq 0. \]

\[ \text{Remark 3.} \quad \text{The energy supply function } G(u,q,t_d) \text{ can be express as follows}
\]

\[ G(u,q,t_d) = \langle q, Q \rangle_{t_d} + 2 \langle q, S u \rangle_{t_d} + \langle u, R u \rangle_{t_d}, \ \forall t_d \geq 0,
\]

where \( Q, S, R \in \mathbb{R}^{n \times n} \) with \( Q, R \) are symmetric. The notations \( \langle q, Q \rangle_{t_d}, \langle q, S u \rangle_{t_d} \) and \( \langle u, R u \rangle_{t_d} \) are represents \( \int_0^{t_d} q^T(t) Q q(t) dt, \int_0^{t_d} q^T(t) S u(t) dt \) and \( \int_0^{t_d} u^T(t) R u(t) dt \), respectively.

\[ \text{Hence, the relation (10) can be written in the dissipativity condition:}
\]

\[ \mathbb{J}_{t_d} = \int_0^{t_d} \mathbb{E}\left\{ \begin{bmatrix} q(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ * & R - \gamma I \end{bmatrix} \begin{bmatrix} q(t) \\ u(t) \end{bmatrix} \right\} dt,
\]

\[ \text{Definition 4.} \quad \text{The NN (6) is said to be passive if there exists a scalar } \gamma > 0 \text{ such that for all } t_d \geq 0
\]

\[ 2 \int_0^{t_d} \mathbb{E}\{q^T(t)u(t)\} dt \geq -\gamma \int_0^{t_d} \mathbb{E}\{u^T(t)u(t)\} dt.
\]

\[ \text{holds under all solution with } p(0) = 0.
\]

\[ \text{Lemma 5.} \quad \text{For a given matrix } W = W^T > 0, \text{ given scalars } s_1 \text{ and } s_2 \text{ satisfying } s_1 < s_2, \text{ the following inequality holds for all continuously differentiable function } \Theta \text{ in } [s_1, s_2] \rightarrow \mathbb{R}^d:
\]

\[ \int_{s_1}^{s_2} \Theta^T \left( \begin{bmatrix} s_1 \\ 0 \end{bmatrix} - s_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) W \Theta \left( \begin{bmatrix} s_1 \\ 0 \end{bmatrix} - s_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) ds_1 \geq \frac{1}{(s_2 - s_1)^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Theta^T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \Theta \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[ (\mathcal{A}_3): \text{ There exist matrices } L_{1x} > 0, L_{2x} > 0 \text{ such that for all } x \in S,
\]

\[ \text{tr}\{\sigma^T(t)\sigma(t)\} \leq p^T(t) L_{1x} p(t) + p^T(t) (1-t^2) L_{2x} p(t - t^2).
\]
where

\[ \Theta_1 = \begin{bmatrix} \int_{s_1}^{s_2} \Theta^T(z_1) d z_1 & \int_{s_1}^{s_2} \int_{s_1}^{s_2} \Theta^T(z_2) d z_2 d z_1 & \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \Theta^T(z_3) d z_3 d z_2 d z_1 \end{bmatrix}^T, \]

\[ \Theta_2 = \begin{bmatrix} 9W & -\frac{36}{(s_2-s_1)^2}W & \frac{60}{(s_2-s_1)^2}W \\ \frac{192}{(s_2-s_1)^2}W & -\frac{360}{(s_2-s_1)^2}W & \frac{720}{(s_2-s_1)^2}W \end{bmatrix}. \]

**Lemma 6.** [38] For a given matrix \( R = R^T > 0 \), given scalars \( s_1 \) and \( s_2 \) satisfying \( s_1 < s_2 \), the following inequality holds for all continuously differentiable function \( \vartheta \) in \( [s_1, s_2] \) \( \rightarrow \mathbb{R}^n \):

\[ \int_{s_1}^{s_2} \int_{s_1}^{s_2} \vartheta^T(z_2) R \vartheta(z_2) d z_2 d z_1 \geq \frac{2}{(s_2-s_1)^2} \Theta_2 \Theta_2. \]

where

\[ \Theta_2 = \begin{bmatrix} 6R & -\frac{30}{(s_2-s_1)^2}R & \frac{60}{(s_2-s_1)^2}R \\ \frac{120}{(s_2-s_1)^2}R & -\frac{180}{(s_2-s_1)^2}R & \frac{360}{(s_2-s_1)^2}R \end{bmatrix}. \]

**Lemma 7.** [39] Let \( M \in \mathbb{R}^{n \times n} \) be a positive-definite matrix, vector function \( \vartheta : [s_1, s_2] \rightarrow \mathbb{R}^n \), with scalars \( s_1 < s_2 \), then

\[ \int_{s_1}^{s_2} \vartheta^T(z_1) M \vartheta(z_1) d z_1 \geq \frac{1}{(s_2-s_1)^2} \left[ \int_{s_1}^{s_2} \vartheta(z_1) d z_1 \right]^T M \left[ \int_{s_1}^{s_2} \vartheta(z_1) d z_1 \right]. \]

**Lemma 8.** [40] Let \( \Theta = \Theta^T, j_1 \) and \( j_2 \) be real matrices, \( \mathcal{F}(t) \) satisfies \( \mathcal{F}^T(t) \mathcal{F}(t) \leq I \). Then \( \Theta + (j_1 \mathcal{F}(t) j_2) + (j_1 \mathcal{F}(t) j_2)^T < 0 \), if there exist a scalar \( \varepsilon > 0 \) such that \( \Theta + \varepsilon^{-1} j_1 \mathcal{F}(t) + \varepsilon j_2 \mathcal{F}(t) \) or equivalently

\[ \begin{bmatrix} \Theta & j_1 & \varepsilon j_2 \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0. \]

## 3. Main results

For the notation clearness, the following abbreviations are adopted in the subsequent parts:

\[ p_r \triangleq p(t), \]
\[ p_{r(t)} \triangleq p(t - r(t)), \]
\[ p_c \triangleq p(t - r), \]
\[ q_\varepsilon \triangleq g(\mathcal{W}_t p(t)), \]
\[ q_{c(t)} \triangleq g(\mathcal{W}_t p(t - r(t))), \]
\[ q_r \triangleq q(r), \]
\[ u_r \triangleq u(t), \]
\[ \varphi_r \triangleq \varphi(r). \]
where

\[ \varphi_{r(t)} = \int_{t-r}^{t} \varphi(\sigma_{31}) d\sigma_{31}, \]

\[ \chi_{1} = \int_{t-r}^{t} p(\sigma_{31}) d\sigma_{31}, \]

\[ \chi_{2} = \int_{t-r}^{t} \int_{t-r}^{t} p(\sigma_{32}) d\sigma_{32} d\sigma_{31}, \]

\[ \chi_{3} = \int_{t-r}^{t} \int_{t-r}^{t} \int_{t-r}^{t} p(\sigma_{33}) d\sigma_{33} d\sigma_{32} d\sigma_{31}, \]

\[ \chi_{4} = \int_{t-r}^{t} \int_{t-r}^{t} \int_{t-r}^{t} \int_{t-r}^{t} p(\sigma_{34}) d\sigma_{34} d\sigma_{33} d\sigma_{32} d\sigma_{31}. \]

\[ \xi_{r(t)} = [p_{T}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}, \varphi_{r(t)}^{T}]. \]

### 3.1. Dissipativity analysis

In this subsection, several sufficient conditions for the dissipativity analysis of the considered system model (6) will be established by using LKF and LMI method.

**Theorem 9.** For given scalars \( r \) and \( \mu \), the NNs (6) is \( (\mathbf{Q}, \mathbf{S}, \mathbf{R}) - \gamma \) dissipative, if there exist matrices \( \mathcal{P}(x \in S) > 0, \mathcal{Q}, \mathcal{R} > 0, \mathcal{S} > 0, \mathcal{U} > 0 \), diagonal matrices \( \mathcal{H}_{1} > 0, \mathcal{H}_{2} > 0 \), any matrices \( \mathcal{G}_{1}, \mathcal{G}_{2} \) and scalar \( \gamma > 0 \) such that the following LMIs holds for all \( x \in S) \):

\[ \mathcal{P} \leq \delta_{x} I, \]

\[ \Theta_{1} = (\Theta_{ij})_{12 \times 12} < 0, \]

where

\[ \Theta_{1,1,x} \triangleq \sum_{j=1}^{N} \pi_{E} \mathcal{P}_{y} + \mathcal{Q} + \mathcal{R} + r^{2} \mathcal{U} + \frac{r^{2}}{2} \mathcal{V} + \delta_{x} \mathcal{L}_{1x} - \mathcal{G}_{1} \mathcal{D}_{x} - (\mathcal{G}_{1}^{T} \mathcal{D}_{x})^{T} - \mathcal{H}_{1} \mathcal{W}_{x}^{T} \mathcal{K}_{x} \mathcal{W}_{x}, \]

\[ \Theta_{1,2,x} \triangleq \mathcal{P}_{x} - \mathcal{G}_{1} - (\mathcal{G}_{2}^{T} \mathcal{D}_{x})^{T}, \]

\[ \Theta_{1,5,x} \triangleq \mathcal{G}_{1} \mathcal{A}_{x} + \mathcal{H}_{1} \mathcal{W}_{x}^{T} \mathcal{K}_{x}, \]

\[ \Theta_{1,6,x} \triangleq \mathcal{G}_{1} \mathcal{B}_{x}, \]

\[ \Theta_{1,12,x} \triangleq \mathcal{G}_{1}. \]

**Proof:** Let us consider the LKF candidate for the NNs (6)

\[ \mathcal{V}(t, p_{r}, x) = \sum_{i=1}^{4} \mathcal{V}_{i}(t, p_{r}, x), \]

where

\[ \mathcal{V}_{1}(t, p_{r}, x) = \mathcal{P}_{x}^{T} \mathcal{P}_{x} p_{r}, \]

\[ \mathcal{V}_{2}(t, p_{r}, x) = \int_{t-r(t)}^{t} p^{T}(\sigma_{31}) Q p(\sigma_{31}) d\sigma_{31}, \]

\[ + \int_{t-r(t)}^{t} p^{T}(\sigma_{31}) R p(\sigma_{31}) d\sigma_{31}, \]

\[ + \int_{t-r(t)}^{t} g^{T}(\mathcal{W}_{x}^{2} p(\sigma_{31})) S_{g}(\mathcal{W}_{x}^{2} p(\sigma_{31})) d\sigma_{31}. \]
\[ V_3(t, p, x) = \tau \int_{t-\tau}^{t+\tau} p^{T}(\sigma(t)) \nu p(t, x) \, dt \]

\[ V_4(t, p, x) = \int_{t-\tau}^{t+\tau} \Phi(t) W \nu \Phi(t, x) \, dt \]

Let \( L \) be the weak infinitesimal random process. Then \( V(t, p, x) \) can be calculated by Ito’s formula that

\[ dV(t, p, x) = L V(t, p, x) \, dt + \{ \sigma(t, x, p, p(t)) \, \omega(t) \}, \tag{17} \]

where

\[ L V(t, p, x) = \sum_{i=1}^{4} L V_i(t, p, x). \tag{18} \]

Now we calculate \( L V(t, p, x) \) along the solutions of the NNs (6), one has

\[ L V_1(t, p, x) = 2p^T \Phi \Phi p + \frac{4}{\tau} \left( \sum_{i=1}^{N} \pi_{y_i} \Phi_i \right) p + tr\{ \sigma^T(t) \Phi \sigma(t) \}. \tag{19} \]

\[ L V_2(t, p, x) = \frac{1}{\tau} q p(t) \Phi_{\nu} p(t) + p^T \Phi_{\nu} p(t) - p^T \Phi_{\nu} p(t) \]

\[ + \frac{1}{\tau} \left( \Phi_{\nu}(t) \Phi_{\nu}(t) - \Phi_{\nu}(t) \Phi_{\nu}(t) \right), \]

\[ \leq \frac{1}{\tau} q p(t) \Phi_{\nu} p(t) + p^T \Phi_{\nu} p(t) - p^T \Phi_{\nu} p(t) \]

\[ + \frac{1}{\tau} \left( \Phi_{\nu}(t) \Phi_{\nu}(t) - \Phi_{\nu}(t) \Phi_{\nu}(t) \right). \tag{20} \]

\[ L V_3(t, p, x) = \tau^2 p^T \Phi p(t) - \tau \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt + \frac{\tau^4}{4} p^T \Phi^2 \nu p(t, x) \]

\[ = - \frac{\tau^2}{2} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \tag{21} \]

\[ L V_4(t, p, x) = \tau \Phi \Phi p(t, x) - \int_{t-\tau}^{t+\tau} \Phi^T(\sigma(t)) \Phi p(t, x) \, dt. \tag{22} \]

By using lemma (5), (6) and (7) the integral term in (21)-(22) can be estimate that

\[ - \tau \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \leq \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix}^T \]

\[ \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ = \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ - \frac{\tau^2}{2} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \]

\[ \leq \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix}^T \]

\[ \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ = \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ - \int_{t-\tau}^{t+\tau} \Phi^T(\sigma(t)) \Phi p(t, x) \, dt \leq \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix}^T \]

\[ \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ = \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ - \int_{t-\tau}^{t+\tau} \Phi^T(\sigma(t)) \Phi p(t, x) \, dt \leq \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix}^T \]

\[ \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ = \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ - \int_{t-\tau}^{t+\tau} \Phi^T(\sigma(t)) \Phi p(t, x) \, dt \leq \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix}^T \]

\[ \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]

\[ = \begin{bmatrix} \int_{t-\tau}^{t+\tau} p^T(\sigma(t)) \nu p(t, x) \, dt \end{bmatrix} \]
From (9) and (14), we can get

$$\text{tr}\{\sigma^T(i)P_i\sigma(i)\} \leq \delta_1\text{tr}\{\sigma^T(i)\sigma(i)\} \leq p_i^T\delta_1L_{1x}p_i + p_{i(i)}^T\delta_2L_{2x}p_{i(i)}.$$  

(26)

For any constant matrix $G_1$, $G_2$ with suitable dimension, the subsequent condition holds,

$$2[p_iG_1 + p_{i(i)}G_{2(i)}]^T[-\phi_t - D_xp_t + A_c\phi_t + B_x\phi_{t(i)} + u_t] = 0.$$  

(27)

In addition, from (4) the following inequalities can be obtained

$$(\eta_t - \Delta_1\mathcal{H}_1p_t^*)^T(\eta_t - \Delta_2\mathcal{H}_1p_t) \leq 0,$$  

(28)

$$(\eta_{t(i)} - \Delta_1\mathcal{H}_1p_{t(i)}^*)^T(\eta_{t(i)} - \Delta_2\mathcal{H}_1p_{t(i)}) \leq 0.$$  

(29)

There exist positive diagonal matrices $\mathcal{H}_1, \mathcal{H}_2$, we have

$$0 \leq -\mathcal{H}_1\begin{bmatrix} \mathcal{W}_1p_t^* \\ \mathcal{G}_t \end{bmatrix}^T\begin{bmatrix} \mathcal{K}_1 & -\mathcal{K}_2 \\ * & I \end{bmatrix}\begin{bmatrix} \mathcal{W}_1p_t^* \\ \mathcal{G}_t \end{bmatrix},$$  

(30)

$$0 \leq -\mathcal{H}_2\begin{bmatrix} \mathcal{W}_1p_{t(i)}^* \\ \mathcal{G}_{t(i)} \end{bmatrix}^T\begin{bmatrix} \mathcal{K}_1 & -\mathcal{K}_2 \\ * & I \end{bmatrix}\begin{bmatrix} \mathcal{W}_1p_{t(i)}^* \\ \mathcal{G}_{t(i)} \end{bmatrix},$$  

(31)

where

$$\mathcal{K}_1 = \Delta_1^2 + \Delta_1^2, \mathcal{K}_2 = \Delta_1^2 + \Delta_2^2.$$  

(32)

Combining from (19)-(31), one can obtain

$$\mathbb{E}\{\mathcal{L}(t,p_t,x) - q_t^TQ_tQ_t - 2q_t^TSu_t - u_t^T(R - \gamma I)u_t\} \leq 2p_t^TP_t\phi_t + p_{i(i)}^T\left(\sum_{j=1}^{N}x_j^TP_j\right)p_t + p_{t(i)}^T\delta_1L_{1x}p_t + p_{i(i)}^T\delta_2L_{2x}p_{i(i)} + (1 - \mu)p_{t(i)}^TQ_t\phi_{t(i)} + p_{i(i)}^T\mathcal{K}_1p_{i(i)} + (1 - \mu)p_{i(i)}^TQ_t\phi_{t(i)} + p_{i(i)}^T\mathcal{K}_1p_{i(i)} + (1 - \mu)p_{i(i)}^TQ_t\phi_{t(i)} + p_{i(i)}^T\mathcal{K}_1p_{i(i)}$$

$$-\chi_2^T(\frac{192}{3})V_{x3} + \chi_2^T(\frac{360}{3})V_{x3} - \chi_3^T(\frac{720}{3})V_{x3} + \chi_3^T(\frac{210}{3})V_{x4} + \chi_3^T(\frac{840}{3})V_{x4} + \chi_3^T(\frac{1200}{3})V_{x4}$$

$$+ \chi_2^T(\frac{30}{3})V_{x3} - \chi_2^T(\frac{30}{3})V_{x4} + \chi_3^T(\frac{210}{3})V_{x4} + \chi_3^T(\frac{480}{3})V_{x4} + \chi_3^T(\frac{1200}{3})V_{x4}$$

$$+ v_{t(i)}^T\mathcal{H}_1\phi_{t(i)} + v_{t(i)}^T\mathcal{H}_1\phi_{t(i)} - 2p_{t(i)}^T(G_1B_0)\phi_t - 2p_{t(i)}^T(G_1D_x)p_t + 2p_{t(i)}^T(G_1A_c)\phi_t$$

$$+ 2p_{t(i)}^T(G_2B_0)\phi_{t(i)} + 2p_{t(i)}^T(G_2D_x)p_{t(i)} + 2p_{t(i)}^T(G_2A_c)\phi_{t(i)}$$

$$+ 2p_{t(i)}^T(G_1B_0)\phi_t + 2p_{t(i)}^T(G_2D_x)p_{t(i)} + 2p_{t(i)}^T(G_1A_c)\phi_t$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$+ g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

$$- g_{t(i)}^T(\mathcal{H}_1)\phi_t - p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3\mathcal{W}_1)\phi_{t(i)} + p_{t(i)}^T(\mathcal{W}_2^T\mathcal{H}_2\mathcal{K}_3)\phi_{t(i)}$$

which is equivalent to

$$\mathbb{E}\{\mathcal{L}(t,p_t,x) - q_t^TQ_tQ_t - 2q_t^TSu_t - u_t^T(R - \gamma I)u_t\} \leq \mathbb{E}\{\xi_t^T\phi_{t(i)}\}.$$  

(34)
Therefore, numerous researchers have taken different types of NNs to the investigation of the stability with stochastic inputs. For example, local-field NNs \(H_{\bar{r}}\) has been considered a general form of the system model. Therefore, the results obtained in this literature is more setting the weight matrices in their structure. For example, choose \(\Theta = \Theta_{\bar{r}}\), \(\Theta_{\bar{1},2,3,12,12} < 0\),

\[
\begin{align*}
\Theta_{2} &= (\Theta_{i,j})_{12 \times 12} < 0, \\
\Theta_{1,i} &= \sum_{j=1}^{N} \pi_{ij} P_{j} + Q + R + \varepsilon^{2} \mathcal{U} + \varepsilon^{2} \mathcal{V} + \delta_{x} L_{1x} - G_{1} D_{x} - (G_{1} D_{x})^{T} - H_{1} W_{x}^{T} K_{x} W_{x}, \\
\Theta_{1,2,3} &= P_{x} - G_{1} - (G_{2} D_{x})^{T}, \\
\Theta_{1,5,5} &\equiv G_{1} A_{x} + H_{1} W_{x}^{T} K_{x}, \\
\Theta_{1,6,3} &\equiv G_{1} B_{x}, \\
\Theta_{1,12,3} &\equiv P_{x} + G_{1}, \\
\Theta_{2,2,5} &\equiv r W - G_{2} - G_{1}^{T}, \\
\Theta_{2,2,6} &\equiv G_{2} A_{x}, \\
\Theta_{2,12,2} &\equiv G_{2} B_{x}, \\
\Theta_{3,3,3} &\equiv -(1 - \mu) Q + \delta_{x} L_{2x} - H_{2} W_{x}^{T} K_{x} W_{x}, \\
\Theta_{3,6,3} &\equiv H_{2} W_{x}^{T} K_{x}, \\
\Theta_{4,4,3} &\equiv - S - H_{2}, \\
\Theta_{5,12,3} &\equiv - I, \\
\Theta_{6,6,3} &\equiv -(1 - \mu) S - H_{2}, \\
\Theta_{7,7,3} &\equiv - \frac{1}{\bar{r}} \mathcal{V}, \\
\Theta_{8,8,3} &\equiv - 9 \mathcal{U}, \\
\Theta_{9,9,3} &\equiv - \frac{20}{\bar{r}} \mathcal{U}, \\
\Theta_{10,10,3} &\equiv - \frac{20}{\bar{r}} \mathcal{U} + \delta_{x} W_{x}^{T} K_{x} W_{x}, \\
\Theta_{11,11,3} &\equiv - \frac{120}{\bar{r}} \mathcal{V} - 6 \mathcal{V}, \\
\Theta_{12,12,3} &\equiv - \frac{20}{\bar{r}} \mathcal{V} - 210 \mathcal{V}, \\
\Theta_{13,13,3} &\equiv \frac{480}{\bar{r}} \mathcal{V}, \\
\Theta_{14,14,3} &\equiv - \frac{1200}{\bar{r}} \mathcal{V}, \\
\Theta_{15,15,3} &\equiv - \gamma I.
\end{align*}
\]

**Proof:** Consider the similar LKF (16) and define the following passivity condition for the system (6).

\[
2 \int_{0}^{t_{d}} \mathbb{E} \{q_{u}^{T} u_{t} \} dt \geq - \gamma \int_{0}^{t_{d}} \mathbb{E} \{u_{t}^{T} u_{t} \} dt.
\]

The following proof can be obtained by Theorem (9), we get

\[
\mathbb{E} \{\mathcal{L} \mathcal{V}(t,p_{u},x) - 2q_{u}^{T} u_{t} - u_{t}^{T} \mathcal{V} u_{t} \} \leq \mathbb{E} \{\xi_{i(t)}^{T} \Theta_{i(t)} \}.
\]

Hence \(\Theta < 0\) holds, then (40) implies that

\[
\mathbb{E} \{\mathcal{L} \mathcal{V}(t,p_{u},x) - 2q_{u}^{T} u_{t} - u_{t}^{T} \mathcal{V} u_{t} \} \leq 0.
\]
Integrating (41) from 0 to \(t_d\), under zero initial conditions we obtain

\[
2 \int_0^{t_d} \mathbb{E}\{q_t^T u_t \} dt \geq \mathbb{E}\{V(t_d, p_{(t_d), x}) - V(0, p_{(0), x}) - \int_0^{t_d} u_t^T (\gamma I) u_t dt \} \\
\geq - \int_0^{t_d} \mathbb{E}\{u_t^T (\gamma I) u_t dt \}
\]

for all \(t_d \geq 0\). Therefore, the NNs (6) is passive in the sense of Definition (4). This completes the proof. \(\blacksquare\)

**Remark 13.** When we consider \(u_t = 0\), then NNs (6) turns to

\[
\begin{cases}
    dp(t) = [-D_x p(t) + A_x g(W_t p(t)) + B_x g(W_t p(t - \tau(t)))] dt \\
    q(t) = g(W_t p(t)).
\end{cases}
\]

**Based on Theorem (9), the following Corollary (14) can be obtained.**

**Corollary 14.** For given scalars \(\tau\) and \(\mu\), the NNs (43) is globally asymptotically stable in the mean square, if there exist matrices \(P_t > 0, Q > 0, R > 0, S > 0, U > 0\), and diagonal matrices \(\mathcal{H}_1 > 0, \mathcal{H}_2 > 0\), any matrices \(\mathcal{G}_1, \mathcal{G}_2\) such that the following LMIs holds for all \((x \in S)\):

\[
P_t \leq \delta_x I, \\
\Theta_3 = (\Theta_{i,j})_{11 \times 11} < 0,
\]

where

\[
\begin{align*}
\Theta_{1,1,x} & \triangleq \sum_{i=1}^{N} \pi_i \pi_i P_t + Q + R + r^2 U + \frac{r^4}{4} \Psi' + \delta_x L_t - G_1 D_t - \langle G_1 D_t \rangle^T - \mathcal{H}_1 W_t^T \mathcal{K}_t W_t, \\
\Theta_{1,2,x} & \triangleq P_t - G_t - \langle G_2 D_t \rangle^T, \\
\Theta_{1,5,x} & \triangleq -G_1 A_t + \mathcal{H}_2 W_t^T \mathcal{K}_t, \\
\Theta_{1,6,x} & \triangleq G_1 B_t, \\
\Theta_{1,9,x} & \triangleq -\langle 1 - \mu \rangle Q + \delta_x L_t - \mathcal{H}_2 W_t^T \mathcal{K}_t W_t, \\
\Theta_{3,9,x} & \triangleq \mathcal{H}_2 W_t^T \mathcal{K}_t, \\
\Theta_{1,10,x} & \triangleq -R_t, \\
\Theta_{5,5,x} & \triangleq -S - \mathcal{H}_1, \\
\Theta_{5,6,x} & \triangleq -\langle 1 - \mu \rangle S - \mathcal{H}_2, \\
\Theta_{7,7,x} & \triangleq -\frac{1}{2} W_t, \\
\Theta_{8,8,x} & \triangleq -9 U_t, \\
\Theta_{8,9,x} & \triangleq \frac{460}{r^2} U_t + \frac{108}{r^4} \Psi', \\
\Theta_{9,10,x} & \triangleq \Theta_{10,10,x} = -\frac{720}{r^2} U_t - \frac{210}{r^4} \Psi', \\
\Theta_{10,11,x} & \triangleq \frac{480}{r^3} \Psi', \\
\Theta_{11,11,x} & \triangleq -\frac{120}{r^4} \Psi'.
\end{align*}
\]

**Remark 15.** Suppose there has no stochastic disturbance and \(u_t = 0\), then NNs (6) turns to

\[
\begin{cases}
    dp(t) = [-D_x p(t) + A_x g(W_t p(t)) + B_x g(W_t p(t - \tau(t)))] dt \\
    q(t) = g(W_t p(t)).
\end{cases}
\]

**Based on Theorem (9), the following Corollary (16) can be obtained.**

**Corollary 16.** For given scalars \(\tau\) and \(\mu\), the NNs (46) is globally asymptotically stable, if there exist matrices \(P_t > 0, Q > 0, R > 0, S > 0, U > 0\), and diagonal matrices \(\mathcal{H}_1 > 0, \mathcal{H}_2 > 0\), any matrices \(\mathcal{G}_1, \mathcal{G}_2\) such that the following LMIs holds for all \((x \in S)\):

\[
\Theta_4 = (\Theta_{i,j})_{11 \times 11} < 0,
\]

where

\[
\begin{align*}
\Theta_{1,1,x} & \triangleq \sum_{i=1}^{N} \pi_i \pi_i P_t + Q + R + r^2 U + \frac{r^4}{4} \Psi' + \delta_x L_t - G_1 D_t - \langle G_1 D_t \rangle^T - \mathcal{H}_1 W_t^T \mathcal{K}_t W_t, \\
\Theta_{1,2,x} & \triangleq P_t - G_t - \langle G_2 D_t \rangle^T, \\
\Theta_{1,5,x} & \triangleq -G_1 A_t + \mathcal{H}_2 W_t^T \mathcal{K}_t, \\
\Theta_{1,6,x} & \triangleq G_1 B_t, \\
\Theta_{1,9,x} & \triangleq -\langle 1 - \mu \rangle Q + \delta_x L_t - \mathcal{H}_2 W_t^T \mathcal{K}_t W_t, \\
\Theta_{3,9,x} & \triangleq \mathcal{H}_2 W_t^T \mathcal{K}_t, \\
\Theta_{1,10,x} & \triangleq -R_t, \\
\Theta_{5,5,x} & \triangleq -S - \mathcal{H}_1, \\
\Theta_{5,6,x} & \triangleq -\langle 1 - \mu \rangle S - \mathcal{H}_2, \\
\Theta_{7,7,x} & \triangleq -\frac{1}{2} W_t, \\
\Theta_{8,8,x} & \triangleq -9 U_t, \\
\Theta_{8,9,x} & \triangleq \frac{460}{r^2} U_t + \frac{108}{r^4} \Psi', \\
\Theta_{9,10,x} & \triangleq \Theta_{10,10,x} = -\frac{720}{r^2} U_t - \frac{210}{r^4} \Psi', \\
\Theta_{10,11,x} & \triangleq \frac{480}{r^3} \Psi', \\
\Theta_{11,11,x} & \triangleq -\frac{120}{r^4} \Psi'.
\end{align*}
\]
3.2. Robust dissipativity analysis

In this subsection, we extend the previous dissipativity condition to robust dissipativity analysis of the following uncertain NNs:

\[
\begin{align*}
dp(t) &= \left[ -\left( D_1 + \Delta D_1(t) \right) p(t) + (A_1 + \Delta A_1(t)) g(W_1 p(t)) + (B_1 + \Delta B_1(t)) g(W_2 p(t - \tau(t))) \\
q(t) &= g(W_1 p(t))
\end{align*}
\]

where \( \Delta D_1(t), \Delta D_2(t) \) and \( \Delta A_1(t) \) are the time-varying parameter uncertainties, which are assumed to be of the form

\[
[\Delta D_1(t) \Delta A_1(t) \Delta B_1(t)] = H_t F_t(x)[E_{1x} E_{2x} E_{3x}],
\]

where \( H, E_{1x}, E_{2x}, \) and \( E_{3x} \) are known real constant matrices, and \( F_t(x) \) is an unknown time-varying matrix function satisfying \( F_t(x)^T F_t(x) \leq I \).

Based on Theorem (9) the following Theorem (17) can be derived.

**Theorem 17.** For given scalars \( \tau \) and \( \mu \), the NNs (48) is \( \{Q, S, R\} - \gamma \) dissipative, if there exist matrices \( P_t(x \in S) > 0, Q > 0, R > 0, S > 0, U > 0, \) diagonal matrices \( H_t > 0, H_{t2} > 0, \) any matrices \( G_1, G_2 \) and scalar \( \gamma > 0 \) such that the following LMIs holds for all \( x \in S \):

\[
P_t \leq \bar{\delta}_t I,
\]

\[
\Theta_t = \begin{bmatrix} \Theta & \Gamma_1 & \epsilon \Gamma_2 \\ \Phi & -\epsilon I \\ \Phi & \Phi & -\epsilon I \end{bmatrix} < 0.
\]

where

\[
\begin{align*}
\Theta_{1,1} &\triangleq \sum_{j=1}^{N} \pi_{ij} P_j + Q + R + \frac{\omega}{2} U + \frac{\omega}{2} V' + \delta_t L_1 - G_1 D_1 - (G_1 D_1)^T - H_t W'_t K_t W_t', \\
\Theta_{1,2} &\triangleq P_t - G_1 - (G_1 D_1)^T, \\
\Theta_{1,5} &\triangleq G_1 A_1 + H_t W'_t K_t, \\
\Theta_{1,6} &\triangleq G_1 B_1, \\
\Theta_{1,12} &\triangleq P_t + G_1, \\
\Theta_{2,2} &\triangleq r W_t - G_2 - G_2^T, \\
\Theta_{2,5} &\triangleq G_2 A_1, \\
\Theta_{2,6} &\triangleq G_2 B_1, \\
\Theta_{2,12} &\triangleq G_2, \\
\Theta_{3,3} &\triangleq -(1 - \mu) Q + \delta_t L_2 - H_{t2} W_t W_t', \\
\Theta_{3,5} &\triangleq H_t W'_t K_t, \\
\Theta_{3,6} &\triangleq -S, \\
\Theta_{3,12} &\triangleq -(1 - \mu) S - H_{t2}, \\
\Theta_{7,7} &\triangleq -\frac{1}{T} W_t, \\
\Theta_{8,8} &\triangleq -9 U, \\
\Theta_{8,9} &\triangleq -\frac{9}{4} U, \\
\Theta_{8,10} &\triangleq -\frac{9}{4} U, \\
\Theta_{9,9} &\triangleq -\frac{1}{T} V, \\
\Theta_{9,10} &\triangleq -\frac{1}{T} V', \\
\Theta_{10,10} &\triangleq -\frac{720}{T} U - \frac{210}{T} V', \\
\Theta_{11,11} &\triangleq \frac{180}{T} V', \\
\Theta_{11,12} &\triangleq \frac{1200}{T} V'.
\end{align*}
\]

Proof: Replacing \( D_1, A_1, B_1 \), in LMI (15) with \( (D_1 + \Delta D_1(t)), (A_1 + \Delta A_1(t)), (B_1 + \Delta B_1(t)) \) yields

\[
\hat{\Theta} + (\Gamma_1 F(t) \Gamma_2) + (\Gamma_1 F(t) \Gamma_2)^T < 0.
\]

By using Lemma (8) there exists a scalar \( \epsilon > 0 \), such that

\[
\hat{\Theta} + \epsilon^{-1} \Gamma_1 \Gamma_1^T + \epsilon \Gamma_2 \Gamma_2 < 0.
\]
where
\[
\Gamma_1 = \begin{bmatrix} G_{T_1}^T \mu_{T_1}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,
\]
\[
\Gamma_2 = \begin{bmatrix} E_{T_1}^T \mu_{T_1}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\]

By Schur complement lemma, inequality (53) is equivalent to inequalities (51). It completes the proof.

4. Illustrative examples

In this section, three numerical examples and their simulations are presented to illustrate the effectiveness of the obtained results.

Example 1. Consider the NNs (6) with the following two modes:
\[
D_1 = \begin{bmatrix} 2.3 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & -0.3 \\ 0.4 & 0.2 \end{bmatrix},
\]
\[
W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_{11} = \begin{bmatrix} 0.22 & 0 \\ 0 & 0.22 \end{bmatrix}, \quad L_{21} = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.18 \end{bmatrix},
\]
\[
D_2 = \begin{bmatrix} 1.9 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 0.5 \\ -0.2 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.5 \end{bmatrix},
\]
\[
W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} 0.20 & 0 \\ 0 & 0.20 \end{bmatrix}, \quad L_{22} = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.12 \end{bmatrix}.
\]

Moreover we take,
\[
Q = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}, \quad R = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}.
\]

Let \( \Pi = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \) and \( \tau(t) = 0.2 + 0.1 \sin t \) which satisfies \( \tau = 0.3, \ \mu = 0.2. \) Further, choose \( g_i(p_i(t)) = \tanh(p_i(t)), \ i = 1, 2, \) then we can get \( \Delta_1 = 0, \ \Delta_2 = I, \) from (32) we have \( \mathcal{X}_1 = 0, \ \mathcal{X}_2 = -0.5I. \) LMIs (14)-(15) can be solved by MATLAB. The initial values are chosen as \( p(0) = [0.3, -0.6]^T, \) when taking \( u(t) = 0.01e^{-t} \sin(0.02t) \) \( t > 0, \) the following simulation results can be obtained. Figure 1 shows that time responses of the system (6), Figure 2 depicts the transient response of state of the system (6) and Figure 3 describes the Markovian switching signal.
Figure 1. Time responses of the NNs (6) in Example 1.

Figure 2. Transient response of state variables $p_1(t), p_2(t)$ in Example 1.

Figure 3. Markovian switching signal $e(t)$ in Example 1.
**Example 2.** Consider the GNNs (46) with $x = 1$ and the following parameters:

$$
A = \begin{bmatrix}
-0.0373 & 0.4852 & -0.3351 & 0.2336 \\
-1.6033 & 0.5988 & -0.3224 & 1.2352 \\
0.3394 & -0.0860 & -0.3894 & -0.5785 \\
-0.1311 & 0.3253 & -0.9534 & -0.5015
\end{bmatrix},
B = \begin{bmatrix}
0.8674 & -1.2405 & -0.5325 & 0.0220 \\
0.0474 & -0.9164 & 0.0360 & 0.9816 \\
1.8495 & 2.6117 & -0.3788 & 0.8428 \\
-2.0413 & 0.5179 & 1.1734 & -0.2775
\end{bmatrix},
D = \text{diag}\{1.2769, 0.6231, 0.9230, 0.4480\},
W = \text{diag}\{1, 1, 1, 1\}.
$$

Take $g_i(p_i(t)) = 0.3 \tanh(p_i(t))$, $i = 1, 2, 3, 4$, then we can get $\Delta_1 = 0$, $\Delta_2 = \text{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}$, from (32) we have $\mathcal{K}_1 = 0$, $\mathcal{K}_2 = \text{diag}\{-0.1137^2, -0.1279^2, -0.7994^2, -0.2368^2\}$. LMI (47) can be solved by MATLAB. For various $\mu$, the maximum permissible delay limit $\tau$ is listed in Table 1. From Table 1, it is easy to realize that the obtained result in this paper is less conservative than those results discussed in [34], [35], [36]. For the simulation purpose, take $\tau(t) = 2.6272 + 0.2\sin t$ which satisfies $\tau = 2.8272$. Under the initial conditions $p(0) = [1.5, 1, -1.6, 0.5]^T$ the following simulation can be obtained. According to the Corollary (16) the GNNs (46) is globally asymptotically stable.

| Methods | $\mu$ | 0.1 | 0.5 | 0.9 |
|---------|------|-----|-----|-----|
| [34]    | $\tau$ | 3.8739 | 2.7821 | 2.3279 |
| [35]    | $\tau$ | 4.1903 | 3.0779 | 2.8268 |
| [36]    | $\tau$ | 4.1919 | 3.0790 | 2.8271 |
| Corollary (16) | $\tau$ | 4.1920 | 3.0791 | 2.8272 |

**Figure 4.** Time responses of the GNNs (46) in Example 2.
Example 3. Consider the NNs (48) with the following two modes:

\[
D_1 = \begin{bmatrix} 2.3 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & -0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
L_{11} = \begin{bmatrix} 0.22 & 0 \\ 0 & 0.22 \end{bmatrix}, \quad L_{21} = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.18 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
\]

\[
E_{11} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
D_2 = \begin{bmatrix} 1.9 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 0.5 \\ -0.2 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.5 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
L_{12} = \begin{bmatrix} 0.20 & 0 \\ 0 & 0.20 \end{bmatrix}, \quad L_{22} = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
\]

\[
E_{12} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

Moreover we take,

\[
Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.
\]

Let \( \Pi = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \) and \( \tau(t) = 0.2 + 0.1\sin t \) which satisfies \( \tau = 0.3, \mu = 0.2 \). Further, choose \( g_i(p_i(t)) = \tanh(p_i(t)), i = 1,2 \), then we can get \( \Delta_1 = 0, \Delta_2 = I \), from (32) we have \( \mathcal{K}_1 = 0, \mathcal{K}_2 = -0.5I \). LMI (50)-(51) can be solved by MATLAB. The initial values are chosen as \( p(0) = [0.3, -0.6]^T \), when taking \( u(t) = 0.01e^{-t}\sin(0.02t) t > 0 \), the following simulation results can be obtained. Figure 5 shows that time responses of the system (48), Figure 6 depicts the state of the system (48) under given initial values and and Figure 7 describes the Markovian switching signal.

![Figure 5](image-url)
5. Conclusion

In this article, the problem of dissipativity and stability of USGNNs with Markovian jumping parameters has been investigated. In order to handle this problem easily, an appropriate LKF is constructed and by employing effective integral inequalities, sector bound activation function, Itô’s formula and some analytical techniques, several LMI-based sufficient conditions are derived, whose feasible solution can be verified by MATLAB. Finally, three numerical examples and their simulations are discussed to demonstrate the feasibility and effectiveness of the obtained analytical results. Moreover, it is possible to investigate include the stability and synchronization analysis of various NNs including complex-valued NNs. Thus, in the future work, we will extend the present results to the investigations of stability and synchronization analysis of stochastic complex-valued discrete-time NNs. This will occur in the near future.

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References

1. J. Cao, “Global asymptotic stability of neural networks with transmission delays,” *Int. J. Syst. Sci.*, vol. 31, no. 10, pp. 1313-1316, Nov. 2000.
2. S. Arik, “An analysis of global asymptotic stability of delayed cellular neural networks,” *IEEE Trans. Neural Netw.*, vol. 13, no. 5, pp. 1239-1242, Sep. 2012.
3. Z. Zhao, Q. Song, S. He, “Passivity analysis of stochastic neural networks with time-varying delays and leakage delay,” *Neurocomputing*, vol. 125, pp. 22-27, Feb. 2014.
4. O. M. Kwon, S. M. Lee, J. H. Park, “Improved delay-dependent exponential stability for uncertain stochastic neural networks with time-varying delays,” *Physica Lett. A*, vol. 374, no. 10, pp. 1232-1241, Feb. 2010.
5. H. B. Zeng, J. H. Park, C. F. Zhang, W. Wang, “Stability and dissipativity analysis of static neural networks with interval time-varying delay,” *J. Frankl. Inst.*, vol. 352, no. 3, pp. 1284-1295, Mar. 2015.
6. P. Muthukumar, K. Subramanian, S. Lakshmanan, “Robust finite time stabilization analysis for uncertain neural networks with leakage delay and probabilistic time-varying delays,” *J. Frankl. Inst.*, vol. 353, no. 16, pp. 4091-4113, Nov. 2016.
7. Y. Chen, Z. Wang, Y. Liu, F. E. Alsaadi, “Stochastic stability for distributed delay neural networks via augmented Lyapunov-Krasovskii functionals,” *Appl. Math. Comput.*, vol. 338, pp. 869-881, Dec. 2018.
8. H. Huang, T. Huang, Y. Cao, “Reduced-order filtering of delayed static neural networks with Markovian jumping parameters,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 11, pp. 5606-5618, Nov. 2018.
9. S. Jiao, H. Shen, Y. Wei, X. Huang, Z. Wang, “Further results on dissipativity and stability analysis of Markov jump generalized neural networks with time-varying interval delays,” *Appl. Math. Comput.*, vol. 336, pp. 338-350, Nov. 2018.
10. E. K. Boukas, Z. K. Liu, G. X. Liu, “Delay-dependent robust stability and $H_\infty$ control of jump linear systems with time-delay,” *Int. J. Control*, vol. 74, no. 4, pp. 329-340, Nov. 2010.
11. Y. Y. Cao, J. Lam, L. S. Hu, “Delay-dependent stochastic stability and $H_\infty$ analysis for time-delay systems with Markovian jumping parameters,” *J. Frankl. Inst.*, vol. 340, no. 7, pp. 423-434, Nov. 2003.
12. Q. Zhu, J. Cao, “Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays,” *IEEE Trans. Syst., Man, and Cybern., Part B*, vol. 41, no. 2, pp. 341-353, Apr. 2011.
13. H. Tan, M. Hua, J. Chen, J. Fei, “Stability analysis of stochastic Markovian switching static neural networks with asynchronous mode-dependent delays,” *Neurocomputing*, vol. 151, pp. 864-872, Mar. 2015.
14. S. Zhu, M. Shen, C. C. Lim, “Robust input-to-state stability of neural networks with Markovian switching in presence of random disturbances or time delays,” *Neurocomputing*, vol. 249, pp. 245-252, Aug. 2017.
15. G. Chen, J. Xia, G. Zhuang, “Delay-dependent stochastic stability and dissipativity analysis of generalized neural networks with Markovian jump parameters and two delay components,” *J. Frankl. Inst.*, vol. 353, no. 9, pp. 2137-2158, Jun. 2016.
16. R. Samidurai, R. Manivannan, C. K. Ahn, H. R. Karimi, “New criteria for stability of generalized neural networks including Markov jump parameters and additive time delays,” *IEEE Trans. Syst., Man, and Cybern., Syst.*, vol. 48, no. 4, pp. 485-499, Apr. 2018.
17. H. Wei, R. Li, C. Chen, “State estimation for memristor-based neural networks with time-varying delays,” *Int. J. Mach. Learn. Cybern.*, vol. 6, no. 2, pp. 213-225, Apr. 2015.
18. A. Arunkumar, R. Sakthivel, K. Mathiyalagan and J. H. Park, “Robust stochastic stability of discrete-time fuzzy Markovian jump neural networks,” *ISA Trans.*, vol. 53, no. 4, pp. 1006-1014, Jul. 2014.
19. R. Sakthivel, A. Arunkumar, K. Mathiyalagan, S. Marshal Anthoni, “Robust passivity analysis of fuzzy Cohen-Grossberg BAM neural networks with time-varying delays,” *Appl. Math. Comput.*, vol. 275, no. 7, pp. 213-228, Dec. 2011.
20. S. Blythe, X. Mao, and X. Liao, “Stability of stochastic delay neural networks,” *J. Frankl. Inst.*, vol. 338, no. 4, pp. 481-495, Jul. 2001.
21. Y. Chen, W. Zheng, “Stability analysis of time-delay neural networks subject to stochastic perturbations,” *IEEE Trans. Cybern.*, vol. 43, no. 6, pp. 2122-2134, Dec. 2013.
22. C. Wang, Y. Shen, “Delay-dependent non-fragile robust stabilization and $H_\infty$ control of uncertain stochastic systems with time-varying delay and nonlinearity,” *J. Frankl. Inst.*, vol. 348, no. 8, pp. 2174-2190, Oct. 2011.
23. G. Liu, S. X. Yang, Y. Chai, W. Feng, W. Fu, “Robust stability criteria for uncertain stochastic neural networks of neutral-type with interval time-varying delays,” *Neural Comput. Appl.*, vol. 22, no. 2, pp. 349-359, Feb. 2013.
24. R. Yang, H. Gao, P. Shi, “Novel robust stability criteria for stochastic Hopfield neural networks with time delays,” *IEEE Trans. Syst., Man, and Cybern., Part B*, vol. 39, no. 2, pp. 467-474, Apr. 2009.
25. S. Zhu, Y. Shen, “Passivity analysis of stochastic delayed neural networks with Markovian switching,” *Neurocomputing*, vol. 74, no. 10, pp. 1754-1761, May 2011.
26. J. C. Willems, “Dissipative dynamical systems part I: General theory,” *Arch. Ration. Mech. Anal.*, vol. 45, no. 5, pp. 321-351, Jan. 1972.
27. D. L. Hill, P. J. Moylan, “Dissipative dynamical systems: Basic input-output and state properties,” *J. Frankl. Inst.*, vol. 309, no. 5, pp. 327-357, May 1980.
28. Z. G. Wu, J. H. Park, H. Su, and J. Chu, “Robust dissipativity analysis of neural networks with time-varying delay and randomly occurring uncertainties,” *Nonlinear Dyn.*, vol. 69, no. 3, pp. 1323-1332, Aug. 2012.
29. X. Liao, J. Wang, “Global dissipativity of continuous-time recurrent neural networks with time delay,” *Phys. Rev. E*, 68 (2003) 016118.
30. Q. Song, Z. Zhao, “Global dissipativity of neural networks with both variable and unbounded delays,” *Chaos Solitons Fract.*, 25 (2005) 393-401.
31. Z. Feng and J. Lam, “Stability and dissipativity analysis of distributed delay cellular neural networks,” *IEEE Trans. Neural Netw.*, vol. 22, no. 6, pp. 976-981, Jun. 2011.
32. R. Raja, U. K. Raja, R. Samidurai and A. Leelamani, “Dissipativity of discrete-time BAM stochastic neural networks with Markovian switching and impulses,” *J. Franklin Inst.*, vol. 350, no. 10, pp. 3217-3247, Dec. 2013.
33. J. Cao, K. Yuan, D. W. C. Ho, J. Lam, “Global point dissipativity of neural networks with mixed time-varying delays,” *Chaos*, 16 (2006) 013105.
34. C. K. Zhang, Y. He, L. Jiang, Q. H. Wu, M. Wu, “Delay-dependent stability criteria for generalized neural networks with two delay components,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 7, pp. 1263-1276, Jul. 2014.
35. H. B. Zeng, Y. He, M. Wu, S. Xiao, “Stability analysis of generalized neural networks with time-varying delays via a new integral inequality,” *Neurocomputing*, vol. 161, pp. 148-154, Aug. 2015.
36. B. Wang, J. Yan, J. Cheng, S. Zhong, “New criteria of stability analysis for generalized neural networks subject to time-varying delayed signals,” *Appl. Math. Comput.*, vol. 314, pp. 322-333, Dec. 2017.
37. L. Zhang, L. He, Y. Song, “New results on stability analysis of delayed systems derived from extended Wirtingers integral inequality”, *Neurocomputing*, vol. 283, pp. 98-106, Mar. 2018.
38. R. Samidurai, R. Sriraman, “Non-fragile sampled-data stabilization analysis for linear systems with probabilistic time-varying delays”, *J. Franklin Inst.*, vol. 356, no. 8, pp. 4335-4357, May 2019.
39. K. Gu, V. L. Kharitonov, J. Chen, “Stability of Time-Delay Systems”, Birkhäuser, Boston, MA, 2003.
40. Q. Zhu, J. Cao, “Robust exponential stability of Markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays,” *IEEE Trans. Neural Netw.*, vol. 21, no. 8, pp. 1314-1325, Aug. 2010.