ROLLER BOUNDARIES FOR MEDIAN SPACES AND ALGEBRAS

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ABSTRACT. We construct compactifications for median spaces with compact intervals, generalising Roller boundaries of CAT(0) cube complexes. Examples of median spaces with compact intervals include all finite rank median spaces and all proper median spaces of infinite rank. Our methods also apply to general median algebras, where we recover the zero-completions of [BM93]. Along the way, we prove various properties of halfspaces in finite rank median spaces and a duality result for locally convex median spaces.

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1. Introduction.

The aim of this paper is to construct a compactification with good median properties for certain classes of median algebras and median spaces. Median algebras were originally introduced in order theory as a common generalisation of dendrites and lattices; they have been extensively studied in relation to semi-lattices (see e.g. [Sho52, Isb80, BH83]) and, more recently, in more geometrical terms because of their connections to CAT(0) cube complexes and median spaces (for instance in [Rol98, Bow13, Bow14]).
A metric space $X$ is said to be median if, for any three points $x_1, x_2, x_3$ of $X$, there exists a unique median, i.e., a unique point $m = m(x_1, x_2, x_3) \in X$ such that $d(x_i, x_j) = d(x_i, m) + d(m, x_j)$ for all $1 \leq i < j \leq 3$. In this case, we refer to the induced map $m: X^3 \to X$ as the median map. The 0-skeleton of any CAT(0) cube complex becomes a median metric space if we endow it with the restriction of the intrinsic path metric of the 1-skeleton. More generally, every real tree and every ultralimit of median spaces is median. The latter include, in particular, all asymptotic cones of cube complexes.

Further examples of median spaces arise from Guirardel cores of pairs of actions on real trees [Gui05] and from asymptotic cones of coarse median groups of finite rank [Bow13, Zei16]. Examples of the latter are provided by mapping class groups, cubulated groups and most irreducible 3-manifold groups. Finally, we remark that $L^1(X, \mu)$ is a median space for any measure space $(X, \mu)$. For additional literature on the subject, see for instance [vdV93, Ver93, Nic08, CDH10, Bow16] and references therein.

The theory of median spaces has two essentially distinct flavours. On the one hand, the study of infinite dimensional median spaces is strongly related to functional analysis. For instance, a locally compact, second countable group admits a (metrically) proper action on a median space if and only if it has the Haagerup property. Similarly, Kazhdan groups are precisely those that can act on median spaces only with bounded orbits [CMV04, CDH10].

On the other hand, the study of finite dimensional median spaces (finite rank in our terminology) strongly resembles that of CAT(0) cube complexes, with the additional pathologies typical of real trees.

A key feature of cube complexes is that they come equipped with a collection of hyperplanes. These give each CAT(0) cube complex a canonical structure of space with walls [HP98]. Conversely, every space with walls can be canonically embedded into a CAT(0) cube complex [Sag95, Nic04, CN05]. In fact, the relationship between spaces with walls and cube complexes can be viewed as a form of duality; see Corollary 4.10 in [Nic04]. A similar phenomenon arises for more general median spaces, as we now describe.

Spaces with measured walls were introduced in [CMV04]; they provide useful characterisations for the Haagerup and Kazhdan properties [CMV04, DCTV08, CDH10]. Each space with measured walls $Z$ can be canonically embedded into a median space $M(Z)$ called its medianisation. Conversely, to every median space $X$ there corresponds a canonical set of walls, namely its convex walls; this induces a structure of space with measured walls on $X$ such that $X \hookrightarrow M(X)$ [CDH10].

We prove the following analogue of the duality result available for cube complexes (Corollary 4.10 in [Nic04]). In particular, this applies to all complete, finite rank median spaces:

\footnote{For connected median spaces, the \textit{rank} can be defined as the supremum of the topological dimensions of locally compact subsets; see Section \ref{sec:rank}.}
**Theorem A.** For every complete, locally convex median space $X$, the inclusion $X \hookrightarrow M(X)$ is a surjective isometry.

As in cube complexes, walls split every median space into halfspaces. In general, the behaviour of halfspaces can be extremely complicated. For instance, if $(X, \mu)$ is a standard probability space with no atoms, every halfspace of $L^1(X, \mu)$ is dense (Example 2.24). This however does not happen in finite rank median spaces:

**Proposition B.** In a complete, finite rank median space every halfspace is either open or closed. If $h_1 \supseteq ... \supseteq h_k$ is a chain of halfspaces with $k > 2 \cdot \text{rank}(X)$, the closure $\overline{h_k}$ is contained in the interior of $h_1$.

Many of the analogies between median spaces and CAT(0) cube complexes resemble those between real trees and simplicial trees. It is thus natural to wonder (see e.g. Question 1.11 in [CDH10]) whether a group acting on a median space with unbounded orbits (resp. properly) must have an action on a CAT(0) cube complex with unbounded orbits (resp. proper).

Both questions have a negative answer. For instance, irreducible lattices in $O(4,1;\mathbb{R}) \times O(3,2;\mathbb{R})$ do not have property (T), but all their actions on CAT(0) cube complexes fix a point; see Theorem 6.14 in [Cor15]. Furthermore, Baumslag-Solitar groups $BS(m,n)$ with $m \neq n$ have the Haagerup property, but do not act properly on any CAT(0) cube complex [Hag07].

We are however unable to answer the previous questions under the additional assumption that the median space be of finite rank. Even for groups acting on real trees, it is a delicate matter [Min16]. See [CD17] for a discussion of similar problems.

In the present paper, we bring the analogies between finite dimensional CAT(0) cube complexes and finite rank median spaces one step further, by extending to median spaces the construction of the Roller compactification.

Roller boundaries of CAT(0) cube complexes are implicit in [Rol98], although the definition that is most commonly used today probably first appeared in [BCG+09]. They have been profitably used to obtain various interesting results, for instance, without attempting to be exhaustive, in [BCG+09, NS13, CFI16, Fer18, FLM18].

It is well-known that Roller boundaries of cube complexes can be given the following two equivalent characterisations. We denote by $\mathcal{H}$ the set of halfspaces of the cube complex $X$ and do not distinguish between $X$ and its 0-skeleton.

1. We can embed $X$ into $2^\mathcal{H}$ by mapping each vertex $v$ to the set $\sigma_v := \{ h \in \mathcal{H} \mid v \in h \}$. The space $2^\mathcal{H}$ is compact with the product topology. Thus, the closure $\overline{X}$ of $X$ inside $2^\mathcal{H}$ is compact; we refer to it as the Roller compactification. The Roller boundary is the set $\partial X := \overline{X} \setminus X$.
2. An ultrafilter on $\mathcal{H}$ is a maximal subset $\sigma \subseteq \mathcal{H}$ such that any two halfspaces in $\sigma$ intersect. The Roller compactification $\overline{X}$ coincides
with the subset of $2^\mathcal{H}$ consisting of all ultrafilters. Boundary points correspond to ultrafilters that contain infinite descending chains of halfspaces.

It should be noted that a different definition of the Roller boundary appears in \cite{Gur05}; see \cite{Gen16} for a discussion of this alternative notion.

For a general median space $X$, we will give four equivalent definitions of the Roller compactification. We sketch them here to illustrate the issues that arise when leaving the discrete world of cube complexes.

(1) As in cube complexes, we denote the set of halfspaces by $\mathcal{H}$. In principle, one could try to define a compactification as we did above, namely by taking the closure of the image of $X \hookrightarrow 2^\mathcal{H}$. However, if $X$ is not discrete, this results in a space that is too large and carries little geometrical meaning: the double dual $X^{\diamond\diamond}$ \cite{Rol98}. See Remark 2.18 and Example 4.5 for the pathologies that may arise; here we simply remark that the inclusion $X \hookrightarrow X^{\diamond\diamond}$ needs not be continuous. Instead, given $x, y \in X$, we denote by $I(x, y)$ the union of all medians $m(x, y, z)$ with $z \in X$; we refer to $I(x, y)$ as the interval between $x$ and $y$. In many interesting cases, intervals are compact. We define the Roller compactification $X$ as the closure of the image of the map:

$$X \hookrightarrow \prod_{x,y \in X} I(x, y)$$

$$z \mapsto (m(x, y, z))_{x,y},$$

and we set $\partial X := X \setminus X$. For CAT(0) cube complexes, this provides a new characterisation of the customary Roller boundary. A similar construction was considered in \cite{War58} for dendrites.

(2) As in cube complexes, we could try to consider the set of ultrafilters on $\mathcal{H}$; this would however result again in the double dual $X^{\diamond\diamond}$. Instead, we endow $\mathcal{H}$ with a $\sigma$-algebra of subsets and a measure; we need a finer $\sigma$-algebra than the one in \cite{CDH10}, see Section 3. We then only consider measurable ultrafilters, identifying sets with null symmetric difference. This is an alternative description of the Roller compactification $\overline{X}$ (see Theorem 4.15), although no natural topology arises via this construction.

(3) Given a general median algebra $M$, the zero-completion of $M$ was introduced in \cite{BM93}. We will recover the same object from a more geometrical perspective in Section 4.1. When $M$ is the median algebra arising from the median space $X$, the zero-completion of $M$ is identified with the Roller compactification $\overline{X}$. This shows that $\overline{X}$ has itself a natural structure of median algebra.

(4) Finally, $\overline{X}$ can be naturally identified with the horofunction compactification of $X$. For 0-skeleta of CAT(0) cube complexes, this
is an unpublished result of U. Bader and D. Guralnik; also see [CL11, FLM18].

Each of the four definitions is better suited to the study of a particular aspect of $X$. Their interplay yields:

**Theorem C.** Let $X$ be a complete, locally convex median space with compact intervals. The Roller compactification $\overline{X}$ is a compact, locally convex, topological median algebra. The inclusion $x \mapsto x$ is a continuous morphism of median algebras, whose image is convex and dense. It is an embedding if, in addition, $X$ is connected and locally compact.

The class of locally convex median spaces with compact intervals encompasses both complete, finite rank median spaces and (possibly infinite dimensional) CAT(0) cube complexes. For the latter, we recover the usual Roller compactification. We also remark that every complete, connected, locally compact median space is proper and thus has compact intervals; examples of such spaces that are locally convex and infinite-rank appear e.g. in [CD17].

As for CAT(0) cube complexes, Roller compactifications of median spaces are endowed with an extended metric $d: \overline{X} \times \overline{X} \to [0, +\infty]$. Thus, they are partitioned into *components*, namely maximal subsets of points, any two of which are at finite distance. The space $X$ always forms an entire component; moreover:

**Theorem D.** Let $X$ be complete and finite rank. Every component of $\partial X$ is a complete median space of strictly lower rank.

In subsequent work, Theorem D will allow us to prove a number of results by induction on the rank. In [Fio18], we use Roller boundaries to extend to finite rank median spaces the machinery developed in [CS11] and part of Hagen’s theory of unidirectional boundary sets (UBS) [Hag13]. As a consequence, we obtain in [Fio18] a version of the Tits alternative for groups acting freely on finite rank median spaces.

In [Fio19], we generalise to finite rank median spaces the superrigidity result of [CFI16]. As a consequence, if $\Gamma$ is an irreducible lattice in a higher rank semisimple Lie group, every action of $\Gamma$ on a complete, connected, finite rank median space must fix a point.

This is in sharp contrast to the behaviour of actions on infinite rank median spaces. Indeed, it was shown in [CD17] that, for $k, n \geq 2$, all uniform lattices in $(\text{PSL}_k \mathbb{R})^n$ admit proper, cocompact actions on complete, connected, infinite-rank median spaces. Note that this phenomenon is specific to non-discrete median spaces, as every cocompact cube complex is finite dimensional.

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2See Lemma 4.6 in [Bow16] and Proposition I.3.7 in [BH99].

3To be precise, [CD17] only proves that the action is *cobounded*, but it is possible to show that the target median space is proper. The authors have informed me that this stronger result (implying cocompactness) will appear in the next version of their preprint.
Structure of the paper. In Section 2 we give definitions and basic results. We study convexity, intervals and halfspaces; we prove Proposition B. In Section 3 we construct a $\sigma$-algebra and a measure on the set of halfspaces of a median space; we also prove Theorem A. In Section 4.1 we study zero-completions of median algebras; our perspective is different from the one in [BM93], but we show that our notions are equivalent. Section 4.2 is devoted to Roller compactifications of median spaces; we prove Theorem C there. Finally, we analyse components in Section 4.3 and prove Theorem D.

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2. Preliminaries.

2.1. Median algebras and median spaces. For definitions and various results on median algebras from a geometric perspective, we refer the reader to [CDH10, Bow13, Bow16, Rol98]. In the following discussion we consider a median algebra $M$ with median map $m$.

Given $x, y \in M$ the interval $I(x, y)$ is the set of points $z \in M$ satisfying $m(x, y, z) = z$. A finite or infinite sequence of points $x_k \in M$ is a discrete geodesic if $x_k \in I(x_m, x_n)$ whenever $m < k < n$. A subset $C \subseteq M$ is said to be convex if $m(x, y, z) \in C$ whenever $x, y \in C$ and $z \in M$; equivalently $I(x, y) \subseteq C$ for all $x, y \in C$. Any collection of pairwise-intersecting convex subsets has the finite intersection property; this fact is usually known as Helly’s Theorem, see e.g. Theorem 2.2 in [Rol98].

A convex halfspace is a nonempty convex subset $\mathfrak{h} \subseteq M$ whose complement $\mathfrak{h}^* := M \setminus \mathfrak{h}$ is also convex and nonempty. A convex wall is an unordered pair $\mathfrak{w} := \{\mathfrak{h}, \mathfrak{h}^*\}$, where $\mathfrak{h}$ is a convex halfspace; we will generally simply speak of halfspaces and walls, unless we need to avoid confusion with the notions in Section 2.2. The wall $\mathfrak{w}$ separates subsets $A, B$ of $M$ if $A \subseteq \mathfrak{h}$ and $B \subseteq \mathfrak{h}^*$ or vice versa. Any two disjoint convex subsets can be separated by a wall, see e.g. Theorem 2.7 in [Rol98].
We will denote the collections of all walls and all halfspaces of $M$ by $\mathcal{W}(M)$ and $\mathcal{H}(M)$, respectively, or simply by $\mathcal{W}$ and $\mathcal{H}$ when the context is clear; there is a natural two-to-one projection $\pi : \mathcal{H} \to \mathcal{W}$. Given subsets $A, B \subseteq M$, we write $\mathcal{W}(A|B)$ for the set of walls separating them and set

$$\mathcal{H}(A|B) := \{ h \in \mathcal{H} \mid B \subseteq h, \ A \subseteq h^* \}. $$

We will simply write $\sigma_A$ for $\mathcal{H}(\emptyset|A)$ and confuse the singleton $\{ x \}$ with the point $x$. We will refer to sets of the form $\mathcal{W}(x|y)$ and $\mathcal{H}(x|y)$ as wall-intervals and halfspace-intervals, respectively.

A pocset $(P, \preceq, *)$ consists of a poset $(P, \preceq)$ equipped with an order-reversing involution $*$, such that every element $a \in P$ is incomparable with $a^*$. Elements $a, b$ of a pocset are transverse if any two elements of the set $\{a,a^*,b,b^*\}$ are incomparable. Considering the median algebra $M$, the triple $(\mathcal{H}, \subseteq, *)$ is a pocset. Halfspaces $h, t$ are transverse if and only if all the intersections $h \cap t, h \cap t^*, h^* \cap t, h^* \cap t^*$ are nonempty. We say that two walls are transverse if they correspond to transverse halfspaces.

A partial filter is a subset $\sigma \subseteq P$ such that there do not exist $a, b \in \sigma$ with $a \preceq b^*$. If moreover for every $a \in P$ we either have $a \in \sigma$ or $a^* \in \sigma$, we say that $\sigma$ is an ultrafilter. A filter is a partial filter $\sigma$ such that $b \in \sigma$ whenever $a \preceq b$ and $a \in \sigma$. Some care is needed when comparing our terminology to that of [CFI16], as their notion of “partially defined ultrafilter” coincides with our notion of “filter”.

Every partial filter $\sigma \subseteq P$ is contained in a filter; the smallest such filter is the set of all $b \in P$ such that $a \preceq b$ for some $a \in \sigma$. Every filter is contained in an ultrafilter; in fact, ultrafilters are precisely filters that are maximal under inclusion. Given any subset $A \subseteq M$, the set $\sigma_A \subseteq \mathcal{H}$ is a filter; it is an ultrafilter if and only if $A$ consists of a single point.

A subset $\sigma \subseteq P$ is said to be inseparable if, whenever $a \preceq b \preceq c$ and $a, c \in \sigma$, we also have $b \in \sigma$. Given a subset $\sigma \subseteq P$, its inseparable closure is the smallest inseparable subset of $P$ containing $\sigma$. It consists precisely of all those $b \in P$ such that there exist $a, c \in \sigma$ with $a \preceq b \preceq c$. Note that the inseparable closure of a partial filter is again a partial filter; all filters are inseparable.

The set $\{-1,1\}$ has a unique structure of median algebra. If $k \in \mathbb{N}$, a $k$-hypercube is the median algebra $\{-1,1\}^k$, given by considering the median map of $\{-1,1\}$ separately in all coordinates. The rank of the median algebra $M$ is the maximal $k \in \mathbb{N}$ such that we can embed a $k$-hypercube into $M$; if $M$ has at least two points, we have rank($M$) $\in [1, +\infty]$. The rank of $M$ coincides with the maximal cardinality of a set of pairwise-transverse halfspaces, see Proposition 6.2 in [Bow13].

Given a subset $C \subseteq M$ and $x \in M$, $y \in C$, we say that $y$ is a gate for $(x, C)$ if $y \in I(x, z)$ for every $z \in C$. We say that $C \subseteq M$ is gate-convex if a gate for $(x, C)$ exists for every $x \in M$. If $C$ is gate-convex, there is a unique gate for $(x, C)$ for every $x \in M$; thus we can define a gate-projection $\pi_C : M \to C$. Gate-convex subsets are always convex, but the converse is
not true in general; see Lemma 2.6 for an obstruction. The interval $I(x, y)$ is always gate-convex with gate-projection given by $\pi(z) = m(x, y, z)$; on the other hand, if $C \subseteq I(x, y)$ is gate-convex, we have $C = I(\pi_C(x), \pi_C(y))$.

**Proposition 2.1.** A map $\phi: M \to M$ is a gate-projection to its image if and only if, for all $x, y, z \in M$, we have $\phi(m(x, y, z)) = m(\phi(x), \phi(y), \phi(z))$. In this case, we also have $\phi(m(x, y, z)) = m(\phi(x), \phi(y), \phi(z))$; in particular, gate-projections map intervals to intervals.

**Proof.** See Proposition 5.1 in [BH83] and 5.8 in [Isb80]. Note that “retract” and “Čebyšev ideals” are alternative terminology for “gate-convex subset”; Isbell works in the more general context of isotropic media. □

**Lemma 2.2.**
(1) If $C_1 \subseteq M$ is convex and $C_2 \subseteq M$ is gate-convex, the projection $\pi_{C_2}(C_1)$ is convex. If moreover $C_1 \cap C_2 \neq \emptyset$, we have $\pi_{C_2}(C_1) = C_1 \cap C_2$.

(2) If $C_1, C_2 \subseteq M$ are gate-convex, the sets $\pi_{C_1}(C_2)$ and $\pi_{C_2}(C_1)$ are gate-convex with gate-projections $\pi_{C_1} \circ \pi_{C_2}$ and $\pi_{C_2} \circ \pi_{C_1}$, respectively.

(3) If $C_1, C_2 \subseteq M$ are gate-convex and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is gate-convex with gate-projection $\pi_{C_1} \circ \pi_{C_2} = \pi_{C_2} \circ \pi_{C_1}$. In particular, if $C_2 \subseteq C_1$, we have $\pi_{C_2} = \pi_{C_2} \circ \pi_{C_1}$.

(4) If $C_1, C_2 \subseteq M$ are gate-convex, we have $\pi_{C_1} \circ \pi_{C_2} \circ \pi_{C_1} = \pi_{C_1} \circ \pi_{C_2}$.

**Proof.** For part 1, see 1.8 and the corollary to 2.5 in [Isb80]. Part 2 follows from Proposition 2.1 above and part 3 is an immediate consequence. Part 4 follows from part 3 and the observation that $\pi_{C_1} \circ \pi_{C_2}$ is the gate-projection to $\pi_{C_1}(C_2) \subseteq C_1$. □

Each convex subset $C \subseteq M$ is also a median subalgebra. In particular, we can consider the collection $\mathcal{H}(C)$ of all halfspaces of the median algebra $C$.

**Proposition 2.3.** If $C \subseteq M$ is gate-convex, there is a one-to-one correspondence

$$
\{h \in \mathcal{H}(M) \mid h \cap C \neq \emptyset, \ h^* \cap C \neq \emptyset\} \longleftrightarrow \mathcal{H}(C)
$$

$$
h \longleftrightarrow h \cap C
$$

$$
\pi^{-1}_C(h) \longleftrightarrow \mathcal{H}(C)
$$

Moreover, if $h \cap C, \mathcal{H} \subseteq C \in \mathcal{H}(C)$, then $h \subseteq \mathcal{H}$ if and only if $h \cap C \subseteq \mathcal{H} \cap C$.

**Proof.** If $h \in \mathcal{H}(M)$ and $h \cap C, h^* \cap C$ are both nonempty, they are halfspaces of $C$. All halfspaces of $C$ arise this way: given a partition $C = C_1 \cup C_2$ where $C_1$ are both convex, we obtain a partition $M = \pi^{-1}_C(C_1) \cup \pi^{-1}_C(C_2)$ and Proposition 2.1 ensures that the $\pi^{-1}_C(C_i)$ are also convex. Finally, part 1 of Lemma 2.2 implies that, if $h \cap C \in \mathcal{H}(C)$, then $\pi_C(h) \subseteq h \cap C$; equivalently, $h \subseteq \pi^{-1}_C(h \cap C)$. Similarly, $h^* \subseteq \pi^{-1}_C(h^* \cap C)$, thus $h = \pi^{-1}_C(h \cap C)$. □

In particular, for all $x, y \in M$, we can identify $\mathcal{H}(x, y) \simeq \mathcal{H}(I(x, y))$ canonically. We will also (slightly improperly) consider the sets $\mathcal{H}(C)$ as
subsets of $\mathcal{H}(M)$ from now on. Given subsets $C_1, C_2 \subseteq M$, we say that $(x_1, x_2)$ is a pair of gates for $(C_1, C_2)$ if $x_1$ is a gate for $(x_2, C_1)$ and $x_2$ is a gate for $(x_1, C_2)$.

**Lemma 2.4.** If $C_1, C_2 \subseteq M$ are gate-convex, for every $y_1 \in C_1$ and $y_2 \in C_2$ there exists a pair of gates $(x_1, x_2)$ for $(C_1, C_2)$ such that $y_1 x_1 x_2 y_2$ is a discrete geodesic. Moreover, $\mathcal{H}(C_1 | C_2) = \mathcal{H}(x_1 | x_2)$.

**Proof.** Set $x_2 := \pi_{C_2}(y_1)$ and $x_1 := \pi_{C_1}(x_2)$; by part 2 of Lemma 2.2, we have $\pi_{C_2}(x_1) = \pi_{C_2} \pi_{C_1}(x_2) = x_2$. Hence $(x_1, x_2)$ is a pair of gates and the fact that $y_1 x_1 x_2 y_2$ is a discrete geodesic follows from the gate property. Another consequence of $x_1$ and $x_2$ being gates is that the sets $\mathcal{H}(x_1 | x_2)$, $\mathcal{H}(x_1 | C_2)$ and $\mathcal{H}(C_1 | x_2)$ coincide; this yields the last part of the lemma. \[\square\]

**Lemma 2.5.** Let $\mathcal{H} \subseteq \mathcal{X}$ be a subset with $\mathcal{H} \cap \mathcal{X}(x | y) \neq \emptyset$ for every $x, y \in M$. The rank of $M$ coincides with the maximal cardinality of a set of pairwise-transverse halfspaces in $\mathcal{H}$.

**Proof.** It suffices to prove that, if $h_1, ..., h_{k-1}, h \in \mathcal{H}$ are pairwise transverse, there exists $h' \in \mathcal{H}$ such that $h_1, ..., h_{k-1}, h'$ are pairwise transverse. Pick points $x \in h_1 \cap ... \cap h_{k-1} \cap h^*$, $y \in h_1 \cap ... \cap h_{k-1} \cap h^*$, $u \in h_1 \cap ... \cap h_{k-1} \cap h$ and $v \in h_1 \cap ... \cap h_{k-1} \cap h$; these exist by Helly’s Theorem. The intervals $I := I(x, y)$ and $J := I(u, v)$ are disjoint since $I \subseteq h^*$ and $J \subseteq h$; thus there exists $h' \in \mathcal{H}(I | J) \cap \mathcal{H}$, by Lemma 2.4. It is immediate to check that $h_1, ..., h_{k-1}, h'$ are pairwise transverse. \[\square\]

A median algebra $M$ is a **topological median algebra** if it is endowed with a Hausdorff topology so that the median map $m$ is continuous. We speak of a **locally convex** median algebra if, in addition, every point has a basis of convex neighbourhoods. A topological median algebra is said to have **compact intervals** if, for every $x, y \in M$, the interval $I(x, y)$ is compact.

**Lemma 2.6.** Let $M$ be a topological median algebra $M$ with compact intervals. A convex subset $C \subseteq M$ is gate-convex if and only if it is closed.

**Proof.** The fact that gate-convex subsets are closed holds in any topological median algebra. Indeed, if $C$ is gate-convex with projection $\pi$ and $x \notin C$, the points $x$ and $\pi(x)$ are distinct. Setting $I := I(x, \pi(x))$, part 1 of Lemma 2.2 gives $\pi_I(C) = I \cap C = \{ \pi(x) \}$. Since the median map $m$ is continuous by definition, the projection $\pi_I$ is also continuous and $\pi_I^{-1}(I \setminus \{ \pi(x) \})$ is an open neighbourhood of $x$ disjoint from $C$. Hence $C$ is closed.

Now suppose that $M$ has compact intervals and that $C \subseteq M$ is closed and convex. Given $x \in M$, we consider the family $\mathcal{G} := \{ I(x, y) \cap C \mid y \in C \}$; by Helly’s Theorem, any two elements of $\mathcal{G}$ intersect. Another application of Helly’s Theorem shows that $\mathcal{G}$ has the finite intersection property. By compactness, the intersection of all elements of $\mathcal{G}$ is nonempty. Any point in this intersection is a gate for $(x, C)$; this proves that $C$ is gate-convex. \[\square\]

**Lemma 2.7.** Let $M$ be a compact topological median algebra.
(1) Projections to gate-convex sets are continuous.
(2) If \( C_1, C_2 \subseteq M \) are convex and compact, the convex hull of \( C_1 \cup C_2 \) is compact.

Proof. Suppose that the projection \( \pi \) to a gate-convex subset \( C \subseteq M \) is not continuous. There exist \( y \in M \) and a net \( (y_j)_{j \in J} \) converging to \( y \), such that \( \pi(y_j) \) does not converge to \( \pi(y) \). By compactness, there exists a subnet \( (z_k)_{k \in K} \) such that \( (\pi(z_k))_{z \in K} \) converges to a point \( z \neq \pi(y) \); since \( C \) is closed by Lemma 2.6, we have \( z \in C \). Thus, \( \pi(z_k) = m(z_k, \pi(y), \pi(z_k)) \) converges to \( m(y, \pi(y), z) = \pi(y) \) for \( k \in K \); this implies \( z = \pi(y) \), a contradiction.

For part 2, the map \( f : M \to M \) given by \( f(x) := m(x, \pi C_1(x), \pi C_2(x)) \) is continuous by part 1 and Lemma 2.6. The hull of \( C_1 \cup C_2 \) is precisely the fixed-point set of \( f \); this easily follows from Proposition 2.3 in [Rol98] and the gate-property. We conclude that the hull is closed, hence compact. \( \square \)

Given a metric space \( X \) and \( x, y \in X \), we denote by \( I(x, y) \) the interval between \( x \) and \( y \), i.e. the set of points \( z \in X \) such that \( d(x, y) = d(x, z) + d(z, y) \). We say that \( X \) is a median space if, for all \( x, y, z \in X \), the intersection \( I(x, y) \cap I(y, z) \cap I(z, x) \) consists of a single point, which we denote by \( m(x, y, z) \). This defines a median-algebra structure on \( X \) with the same notion of interval; in particular, we can define rank, convexity and gate-convexity for subsets of \( X \).

A complete median space is geodesic if and only if it is connected, see Lemma 4.6 in [Bow16]. In this case, the interval \( I(x, y) \) is simply the union of all geodesics with endpoints \( x \) and \( y \).

If \( X \) is a geodesic median space, its rank coincides with the supremum of the topological dimensions of its locally compact subsets — even when either of the two quantities is infinite; see Theorem 2.2 and Lemma 7.6 in [Bow13] for one inequality and Proposition 5.6 in [Bow16] for the other. We prefer to speak of rank, rather than dimension, as the metric spaces that we will be interested in could well be disconnected, e.g. 0-skeleta of CAT(0) cube complexes.

Example 2.8. Let \((\Omega, \mathcal{B}, \mu)\) be a measure space.

(1) The space \( L^1(\Omega, \mu) \) is median when endowed with the metric induced by its norm. The median map is determined by the property that \( m(f, g, h)(x) \) is the middle value of \( \{ f(x), g(x), h(x) \} \), for almost every \( x \) and all \( f, g, h \in L^1(\Omega, \mu) \).

(2) Given any \( E \subseteq \Omega \), the collection \( \mathcal{M}_E \) of all \( F \subseteq \Omega \) such that \( E \triangle F \) is measurable and finite-measure can be given the pseudometric

\[
d(A, B) = \mu(A \triangle B).
\]

This makes sense as \( A \triangle B = (A \triangle E) \triangle (B \triangle E) \). Identifying sets at distance zero, the space \( \mathcal{M}_E \) can be isometrically embedded into \( L^1(\Omega, \mu) \) by mapping \( F \mapsto 1_{F \triangle E} \) and it inherits a median metric. A point lies in the set \( m(A, B, C) \) if and only if it lies in at least two of
the sets $A, B, C \subseteq \Omega$; the interval $I(A,B)$ can be recognised as the collection of sets $Z$ satisfying $A \cap B \subseteq Z \subseteq A \cup B$.

Let $X$ be a median space throughout the rest of this section. The median map $m$ is 1-Lipschitz if we endow $X^3$ with the $\ell^1$ metric. If $C \subseteq X$ is convex and $x \in X$, a point $z \in C$ is a gate for $(x,C)$ if and only if $d(x, C) = d(x,z)$; gate-projections are 1-Lipschitz. Gate-convex sets are closed and convex; the converse holds in complete median spaces. See [CDH10] for further details and examples.

In complete median spaces we can complement Lemma 2.4 above.

**Lemma 2.9.** If $X$ is complete and $C_1, C_2 \subseteq X$ are closed and convex, the points $z_1 \in C_1$, $z_2 \in C_2$ form a pair of gates for $(C_1, C_2)$ if and only if $d(z_1, z_2) = d(C_1, C_2)$. In particular, disjoint closed convex sets always have positive distance.

**Proof.** If $d(z_1, z_2) = d(C_1, C_2)$, it is immediate that $(z_1, z_2)$ is a pair of gates. Conversely, given a pair of gates $(z_1, z_2)$, we set $I := I(z_1, z_2)$; if $z'_i \in C_i$, we have $\pi_I(z'_i) = z_i$ by part 1 of Lemma 2.2 and the observation that $C_i \cap I = \{z_i\}$. Since $\pi_I$ is 1-Lipschitz, we have $d(z_1, z_2) \leq d(z'_1, z'_2)$; hence $d(z_1, z_2) = d(C_1, C_2)$ by the arbitrariness of $z'_i$.

**Lemma 2.10.** If $X$ has finite rank, it is locally convex.

**Proof.** Given $x \in X$, $\epsilon > 0$ and $y, z \in B(x, \epsilon)$, we have $I(y, z) \subseteq B(x, 2\epsilon)$. Thus $X$ is “weakly locally convex” in the sense of [Bow13] and we conclude by Lemma 7.1 in [Bow13]. □

The class of locally convex median spaces encompasses both finite rank median spaces and infinite dimensional CAT(0) cube complexes. Instead, the median space $L^1([0,1])$ is not locally convex; as we shall see in Example 2.24 the convex hull of any nonempty open subset is the entire $L^1([0,1])$.

**Lemma 2.11.** Suppose $X$ is complete and locally convex and let $\{C_i\}_{i \in I}$ be a collection of convex subsets of $X$ with nonempty intersection $K$. The intersection of $\{\overline{C_i}\}_{i \in I}$ is $\overline{K}$.

**Proof.** We only need to prove that, if $x \in \overline{C_i}$ for all $i \in I$, then $x \in \overline{K}$. Given $\epsilon > 0$, let $N_\epsilon \subseteq B(x, \epsilon)$ be a convex neighbourhood of $x$; denote by $\pi_\epsilon$ the gate-projection to $\overline{N}_\epsilon$. Since $C_i \cap N_\epsilon \neq \emptyset$ for all $i \in I$, part 1 of Lemma 2.2 implies that $\pi_\epsilon(C_i) = C_i \cap \overline{N}_\epsilon$ and

$$\pi_\epsilon(K) \subseteq \bigcap_i \pi_\epsilon(C_i) \subseteq \left( \bigcap_i C_i \right) \cap \overline{N}_\epsilon = K \cap \overline{N}_\epsilon.$$ 

Hence $K$ intersects $B(x, 2\epsilon)$ for all $\epsilon > 0$ and, by the arbitrariness of $\epsilon$, we conclude that $x \in \overline{K}$. □
2.2. SMW’s, PMP’s and SMH’s. Let \( X \) be a set. A wall is an unordered pair \( \{ h, h^* \} \) corresponding to any partition \( X = h \cup h^* \). The wall \( h \cup h^* \) separates subsets \( A, B \subseteq X \) if \( A \subseteq h \) and \( B \subseteq h^* \) or vice versa. As in Section 2.1, we use the notation \( \mathcal{W}(A \mid B) \) to refer to walls separating \( A \) and \( B \).

**Definition 2.12.** We say that the 4-tuple \((X, \mathcal{W}, \mathcal{D}, \mu)\) is a space with measured walls (SMW) if \( \mathcal{W} \) is a collection of walls of \( X \) and the measure \( \mu \), defined on the \( \sigma \)-algebra \( \mathcal{D} \subseteq 2^X \), satisfies \( \mu(\mathcal{W}(x \mid y)) < +\infty \) for all \( x, y \in X \).

If \((X, \mathcal{W}, \mathcal{D}, \mu)\) is a space with measured walls, the associated collection of halfspaces is the set \( \mathcal{H} \) of those \( h \subseteq X \) such that \( \{ h, h^* \} \in \mathcal{W} \). It is endowed with a two-to-one projection \( \pi: \mathcal{H} \to \mathcal{W} \) given by \( \pi(h) = \{ h, h^* \} \). We can define a pseudo-metric on \( X \) by setting \( \text{dist}_\mu(x, y) := \mu(\mathcal{W}(x \mid y)) \). When this is a genuine metric, we speak of a faithful SMW.

Let \((X', \mathcal{W}', \mathcal{D}', \mu')\) be another SMW. Any map \( f: X \to X' \) such that \( \{ f^{-1}(h), f^{-1}(h^*) \} \in \mathcal{W} \) whenever \( \{ h, h^* \} \in \mathcal{W}' \) induces a map \( \mathcal{W} \to \mathcal{W}' \).

**Definition 2.13.** A pointed measured pocset (PMP) is a 4-tuple \((\mathcal{P}, \mathcal{D}, \eta, \sigma)\), where \( \mathcal{P} \) is a pocset, \( \mathcal{D} \) is a \( \sigma \)-algebra of subsets of \( \mathcal{P} \), the measure \( \eta \) is defined on \( \mathcal{D} \) and \( \sigma \subseteq \mathcal{P} \) is a (not necessarily measurable) ultrafilter.

In analogy to the terminology of \([\text{CDH10}]\), we say that an ultrafilter \( \sigma' \subseteq \mathcal{P} \) is admissible if \( \sigma' \Delta \sigma \in \mathcal{D} \) and \( \eta(\sigma' \Delta \sigma) < +\infty \). We will denote by \( \mathcal{M}(\mathcal{P}, \mathcal{D}, \eta, \sigma) \) (or simply \( \mathcal{M} \)) the set of admissible ultrafilters associated to the pointed measured pocset \((\mathcal{P}, \mathcal{D}, \eta, \sigma)\). As in Example 2.3, we can equip \( \mathcal{M} \) with the median pseudo-metric \( d(\sigma_1, \sigma_2) := \eta(\sigma_1 \Delta \sigma_2) \). We identify admissible ultrafilters at zero distance, so that \( \mathcal{M} \) becomes a median space.

Two PMP’s \((\mathcal{P}, \mathcal{D}, \eta, \sigma)\) and \((\mathcal{P}', \mathcal{D}', \eta', \sigma')\) are isomorphic if there exists an isomorphism of pocsets \( f: \mathcal{P} \to \mathcal{P}' \) such that \( f \) and \( f^{-1} \) are measurable, \( f_\ast \eta = \eta' \) and \( \eta(f^{-1}(\sigma') \Delta \sigma) < +\infty \). Any isomorphism of the two PMP’s induces an isometry of the corresponding median spaces \( \mathcal{M} \) and \( \mathcal{M}' \).

Given a SMW \((X, \mathcal{W}, \mathcal{D}, \mu)\), we always obtain a PMP \((\mathcal{H}, \pi^\ast \mathcal{D}, \pi^\ast \mu, \sigma_x)\), where \( \sigma_x \subseteq \mathcal{H} \) is the set of halfspaces containing \( x \). Here \( \pi^\ast \mathcal{D} \) denotes the \( \sigma \)-algebra \( \{ \pi^{-1}(E) \mid E \in \mathcal{D} \} \) and \( \pi^\ast \mu(\pi^{-1}(E)) = \mu(E) \). The choice of \( x \in X \) does not affect the isomorphism type of the PMP.

We simply denote by \( \mathcal{M}(X) \) the associated median space of admissible ultrafilters; unlike in \([\text{CDH10, CD17}]\), for us this is a genuine metric space. We have a pseudo-distance-preserving map \( X \to \mathcal{M}(X) \) given by \( y \mapsto \sigma_y \). If \((X, \mathcal{W}, \mathcal{D}, \mu)\) is faithful, this is an isometric embedding. We have recovered the following:
Proposition 2.14 ([CDH10]). Any faithful space with measured walls can be isometrically embedded into a median space.

The embedding is canonical in that every automorphism of $X$ as SMW extends to an isometry of $M(X)$. Note however that the restriction of the metric to $X$ does not have to be median, as $X$ might not be a median subalgebra. The following is a partial converse to the previous proposition.

Theorem 2.15 ([CDH10]). Let $Y$ be a median space, $\mathcal{W}$ its set of convex walls and $\mathcal{B}$ be the $\sigma$-algebra generated by wall-intervals. There exists a measure $\nu$ on $\mathcal{W}$ such that $\nu(\mathcal{W}(x|y)) = d(x, y)$ for all $x, y \in Y$. In particular, $(Y, \mathcal{W}, \mathcal{B}, \mu)$ is a faithful space with measured walls and we have isometric embeddings

$$Y \hookrightarrow M(Y) \hookrightarrow L^1(\mathcal{H}, \pi^* \mu).$$

Summing up, we can associate a median space $M(X)$ to every faithful SMW $(X, \mathcal{W}, \mathcal{B}, \mu)$ and a faithful SMW to every median space. One can wonder whether the compositions

$$\text{SMW} \hookrightarrow \text{median space} \hookrightarrow \text{SMW},$$

$$\text{median space} \hookrightarrow \text{SMW} \hookrightarrow \text{median space},$$

are the identity. While this has no hope of being true in the first case (we could have taken a set of non-convex walls of a median space), we will show in Corollary 3.11 that $Y \simeq M(Y)$ for all locally convex median spaces $Y$.

We conclude by introducing the following variation on the notion of SMW, which will be more useful to us in the following treatment.

Definition 2.16. The 4-tuple $(X, \mathcal{H}, \mathcal{B}, \nu)$ is a space with measured halfspaces (SMH) if $\mathcal{H} \subseteq 2^X$ is a collection of subsets of $X$ closed under taking complements, $\mathcal{B} \subseteq 2^\mathcal{H}$ is a $\sigma$-algebra and $\nu$ is a measure defined on $\mathcal{B}$ satisfying $\nu(\mathcal{H}(x|y)) = \nu(\mathcal{H}(y|x)) < +\infty$ for all $x, y \in X$.

If $h \in \mathcal{H}$, we set $h^* := X \setminus h$ and, if $E \subseteq \mathcal{H}$, we define $E^* := \{h^* \mid h \in E\}$. We borrow the notation $\mathcal{H}(A|B)$ and $\sigma_A$ from Section 2.1. Note that, if $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a SMW with associated collection of halfspaces $\mathcal{H}$ and projection $\pi: \mathcal{H} \to \mathcal{W}$, the 4-tuple $(X, \mathcal{H}, \pi^* \mathcal{B}, \pi^* \mu)$ is not a space with measured halfspaces; indeed, $\mathcal{H}(x|y) \notin \pi^* \mathcal{B}$ if $x \neq y$.

A pseudo-metric on $X$ and a notion of homomorphism of SMH’s can be defined exactly as we did for SMW’s. Given a space with measured halfspaces $(X, \mathcal{H}, \mathcal{B}, \nu)$ and a point $x \in X$, we obtain a PMP $(\mathcal{H}, \mathcal{B}, \nu, \sigma_x)$; we write $M(X)$ instead of $M(\mathcal{H}, \mathcal{B}, \nu, \sigma_x)$. The discussion in Section 5 of [CDH10] works identically if we replace the symbol $\mathcal{W}$ with $\mathcal{H}$ everywhere.

In particular, we have:

Theorem 2.17. Let $Y$ be a median space, $\mathcal{H}$ the set of convex halfspaces and $\mathcal{B}$ be the $\sigma$-algebra generated by halfspace-intervals. There exists a measure $\nu$ on $\mathcal{H}$ such that $\nu(\mathcal{H}(x|y)) = d(x, y)$ for all $x, y \in Y$. In particular,
(Y, ℱ, ℬ, ν) is a faithful space with measured halfspaces and we have isometric embeddings

\[ Y \hookrightarrow M(Y) \hookrightarrow L^1(ℱ, ν). \]

Note that \( * : ℱ \to ℱ \) is a measure-preserving involution. We will always denote the \( σ \)-algebras in Theorems 2.15 and 2.17 by the same symbol as there is no chance of confusion.

Remark 2.18. Embedding \( Y \hookrightarrow M(Y) \) as in Theorem 2.17, each point \( y \in Y \) is represented by all measurable ultrafilters \( σ \subseteq ℱ \) that have null symmetric difference with \( σ_y \). Some of these ultrafilters can be “bad”: the intersection of all halfspaces in \( σ \) can be empty, rather than \( \{ y \} \).

As an example, consider \( Y = ℝ \) with its standard metric and \( y = 0 \). The natural ultrafilter \( σ_0 \) representing 0 consists of the halfspaces \((−∞, a)\) for \( a > 0 \), \((-∞, a] \) for \( a \geq 0 \), \((a, +∞)\) for \( a < 0 \) and \([a, +∞)\) for \( a \leq 0 \). However, \( σ_0 := (σ_0 \setminus \{(−∞, 0]\}) \cup \{(0, +∞)\} \) also is an admissible ultrafilter representing 0. Here, measurability and nullity of \( σ_0 \triangle σ_0 \) follow from the observation that \( \{(−∞, 0]\} \) coincides with the intersection of the countable family of halfspace-intervals \( ℱ(\frac{1}{n}|0) \), \( n \geq 1 \).

We remark that the measures in Theorems 2.15 and 2.17 can be extended to complete \( σ \)-algebras \( ℬ_0 \supseteq ℬ \), i.e. with the property that any subset of a null set is measurable; see e.g. Theorem 1.36 in [Rud87].

2.3. Intervals and halfspaces. Let \( X \) be a median space throughout this section. It is not hard to use the arguments of [Bow14] to prove the following generalisation of Theorem 1.14 in [BCG+09]; still, we provide a proof below for the convenience of the reader.

Proposition 2.19. Let \( X \) be complete of rank \( r < +∞ \). For every \( x, y \in X \), there exists an isometric embedding \( I(x, y) \hookrightarrow ℓ^1_r \), where \( ℓ^1_r \) is endowed with the \( ℓ^1 \) metric.

In particular, we obtain the following useful fact (for connected median spaces, also see Corollary 1.3 in [Bow14]).

Corollary 2.20. In a complete, finite rank median space intervals are compact.

To prove Proposition 2.19, observe that, if \( M \) is a median algebra and \( σ_1, σ_2 \subseteq ℱ(M) \) are ultrafilters, antichains in the poset \( σ_1 \setminus σ_2 \) correspond to sets of pairwise-transverse halfspaces and thus have cardinality bounded above by \( \text{rank}(M) \). Hence, Dilworth’s Theorem [Dil50] yields the following:

Lemma 2.21. Let \( M \) be a median algebra with \( \text{rank}(M) = r < +∞ \) and let \( σ_1, σ_2 \) be ultrafilters on \( ℱ \). We can decompose \( σ_1 \setminus σ_2 = C_1 \sqcup \ldots \sqcup C_k \), where \( k \leq r \) and each \( C_i \) is nonempty and totally ordered by inclusion.

Note that, in general, we have no guarantee that the chains provided by the previous lemma are measurable.
Proof of Proposition 2.19. Assume that $X = I(x, y)$ for simplicity. We will produce maps $f_1, \ldots, f_r : X \to [0, d(x, y)]$ such that for every $u, v \in X$ we have $d(u, v) = \sum |f_i(u) - f_i(v)|$.

We first do this under the assumption that $X$ be finite. By Lemma 2.21, we can decompose $\mathcal{H}(x|y) = C_1 \sqcup \ldots \sqcup C_r$, where each $C_i$ is a finite set that is totally ordered by inclusion. Let $\nu$ be the measure on $\mathcal{H}$ that is provided by Theorem 2.17. Since $X$ is finite, the singletons of $\mathcal{H}$ are halfspace-intervals (see e.g. Section 3 in [Bow13]), hence measurable; thus, each $C_i$ is measurable. For $z \in X$ and $1 \leq i \leq r$, we set $f_i(z) := \nu(C_i \cap \mathcal{H}(x|z))$. It is straightforward to check that these define the required embedding.

In the general case, let $\mathcal{M}$ be the set of finite subalgebras of $X$ containing $\{x, y\}$. Every finite subset of $X$ is contained in an element of $\mathcal{M}$ by Lemma 4.2 in [Bow13]; in particular, $(\mathcal{M}, \subseteq)$ is a directed set. Every $M \in \mathcal{M}$ is a finite interval with endpoints $x, y$, so the previous discussion yields maps $f_1^M, \ldots, f_r^M : M \to [0, d(x, y)]$ defining an isometric embedding $M \hookrightarrow \mathbb{R}^r$. We can extend each $f_i^M$ to a function $\tilde{f}_i^M : X \to [0, d(x, y)]$ that takes $X \setminus M$ to zero. This defines nets $P_i : \mathcal{M} \to [0, d(x, y)]^X$. The space $[0, d(x, y)]^X$ is compact by Tychonoff’s Theorem, thus a subnet of $P_i$ converges. Its limit is a function $f_i : X \to [0, d(x, y)]$ and it is immediate to check that $f_1, \ldots, f_r$ yield the required embedding $X \hookrightarrow \mathbb{R}^r$.

We now proceed to examine various properties of the halfspaces of $X$.

Proposition 2.22. If $X$ has finite rank and $h \in \mathcal{H}$, either $\partial h := \overline{h} \cap \overline{h^*}$ is empty or it is a closed, convex subset with $\text{rank}(\partial h) \leq \text{rank}(X) - 1$.

Proof. Follows from Lemma 2.10 above and Lemma 7.5 in [Bow13].

We remark that, in a median space, closures of convex sets are convex, but interiors of convex sets need not be; in particular, the closure of a halfspace needs not be a halfspace. For instance, consider a real tree $T$, a branch point $x \in T$ and a connected component $\mathfrak{h}$ of $T \setminus \{x\}$. This is a halfspace of $T$, as both $\mathfrak{h}$ and $\mathfrak{h}^*$ are convex. The interior of $\mathfrak{h}^*$, however, is not convex, as it coincides with $\overline{\mathfrak{h}}^* \setminus \{x\}$. In particular, $\overline{\mathfrak{h}} = \mathfrak{h} \cup \{x\}$ is not a halfspace.

Nevertheless, we have the following:

Corollary 2.23. In a complete, finite rank median space, each halfspace is either open or closed (possibly both).

Proof. We proceed by induction on $\text{rank}(X)$; if the rank is zero, $\mathcal{H} = \emptyset$ and there is nothing to prove. Now assume the result for all median spaces of rank at most $\text{rank}(X) - 1$ and suppose $\mathfrak{h} \in \mathcal{H}(X)$ is neither open nor closed. Then $\partial \mathfrak{h}$ is nonempty and we have a partition $\partial \mathfrak{h} = (\partial \mathfrak{h} \cap \mathfrak{h}) \sqcup (\partial \mathfrak{h} \cap \mathfrak{h}^*)$. By Helly’s Theorem, the convex set $\partial \mathfrak{h} \cap \mathfrak{h} = \overline{\mathfrak{h} \cap \mathfrak{h}^*} \cap \mathfrak{h}$ is nonempty; the same argument yields $\partial \mathfrak{h} \cap \mathfrak{h}^* \neq \emptyset$. The above partition of $\partial \mathfrak{h}$ must then arise from a halfspace of $\partial \mathfrak{h}$ and the inductive hypothesis guarantees that $\partial \mathfrak{h} \cap \mathfrak{h}$ is either open or closed. Note that $\mathfrak{h} = \pi_{\partial \mathfrak{h}}^{-1}(\partial \mathfrak{h} \cap \mathfrak{h})$, by Proposition 2.3. Since $\pi_{\partial \mathfrak{h}}$ is continuous, $\mathfrak{h}$ is either open or closed, a contradiction. \qed
The situation can be completely different in infinite rank median spaces:

**Example 2.24.** The space $X = L^1([0,1])$ is complete, median and all its halfspaces are dense. In order to see the latter, let us write $B_R$ for the $R$-ball around the origin and let us denote characteristic functions by $\chi$. Given a function $f \in L^1([0,1])$ with $\|f\|_1 < 2R$, there exists a measurable partition $[0,1] = P \cup Q$ such that $\|f \cdot \chi_P\|_1 < R$ and $\|f \cdot \chi_Q\|_1 < R$. By part 1 of Example 2.8 the interval between $f \cdot \chi_P$ and $f \cdot \chi_Q$ contains $f$. This shows that $B_{2R}$ is contained in the convex hull of $B_R$ for every $R > 0$; in particular, the hull of $B_R$ is the entire $X$. Since any nonempty open subset of $X$ can be translated so that it contains a ball around the origin, we conclude that the hull of any nonempty open set is the entire $X$. Thus, every proper convex subset of $X$ must have empty interior and all halfspaces are dense.

**Example 2.24.** Let us consider the compact topological median algebra $M = \{0,1\}^\mathbb{N}$ with the product topology; we identify $M$ with the power set $2^\mathbb{N}$ of $\mathbb{N}$. There is a one-to-one correspondence between walls of $M$ and ultrafilters $\mathcal{U} \subseteq 2^\mathbb{N}$. Indeed, observe that, given $A, B \subseteq \mathbb{N}$, we have $I(A, B) = \{C \subseteq \mathbb{N} : A \cap B \subseteq C \subseteq A \cup B\}$. Since $M = I(\emptyset, \mathbb{N})$, every wall of $M$ has a side $h \subseteq 2^\mathbb{N}$ containing $\mathbb{N}$ and a side $h^* \subseteq 2^\mathbb{N}$ containing $\emptyset$. Since $A \cap B \in I(A, B)$, the collection $h \subseteq 2^\mathbb{N}$ is closed under taking intersections. If $A \subseteq B$ and $A \in h$, we have $B \in I(A, \mathbb{N}) \subseteq h$. Finally, for every $A \subseteq \mathbb{N}$, the collection $h$ contains exactly one among $A$ and $\mathbb{N} \setminus A$, as $M = I(A, \mathbb{N} \setminus A)$. This shows that $h \subseteq 2^\mathbb{N}$ is an ultrafilter. Conversely, it is easy to check that every ultrafilter $\mathcal{U} \subseteq 2^\mathbb{N}$ is a halfspace of $M$.

Now, $M$ has some obvious walls coming from its product structure. The associated halfspaces can be described explicitly by setting one coordinate of $\{0,1\}^\mathbb{N}$ to 0 or 1; note that, in a finite product $\{0,1\}^n$, all halfspaces would be of this form. In terms of the correspondence established above, these walls are exactly principal ultrafilters (see e.g. Definition 10.15 in [DK17]). However, it is a well-known consequence of the axiom of choice that there exist also non-principal ultrafilters. These will correspond to additional walls of $M$, which — unlike the previous ones — yield dense halfspaces of $M$.

Endowing the product $X = \prod\{0, \frac{1}{n}\}$ with its $\ell^1$ metric, we obtain a compact median metric space. As a median algebra, this is isomorphic to $M$ above, which yields a natural correspondence between walls of $X$ and walls of $M$. The space $X$ is totally disconnected, but the walls of the geodesic median space $Y = \prod\{0, \frac{1}{n}\}$ are not much better behaved: by Lemma 6.5 in [Bow13], every halfspace of $X$ is of the form $h \cap Y$ for some $h \in \mathcal{H}(Y)$. We remark that — unlike $L^1([0,1])$ — $M$, $X$ and $Y$ are also locally convex.

Even in finite rank median spaces, walls can display more complicated behaviours than hyperplanes in CAT(0) cube complexes. Consider for instance the rank-two median space pictured in Figure 1 it is obtained by glueing

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4Here we consider the classical set-theoretical notion of ultrafilter, not the one introduced in Section 2.1. See e.g. Definition 10.12 in [DK17].
together three halfplanes, each endowed with the $\ell^1$ metric. The pictured halfspaces (if is open and $k$ is closed) satisfy $h \not\subseteq k$, but $d(h, k^*) = 0$; indeed, $h$ and $k$ share a portion of their frontier isometric to a ray.

Another pathology appears in the space $I$ in Figure 2 below, which we view as a subset of $\mathbb{R}^2$ with the restriction of the $\ell^1$ metric. The halfspaces $h$, $k$ are open and $h \not\subseteq k$.

These issues can easily be circumvented, at least in finite rank spaces; this is the content of Proposition 2.26 below.

**Lemma 2.25.** Let $X = I(x, y)$ be complete and finite rank. If, for $h$, $k \in \mathcal{H}$, we have $y \in h \subseteq k$ and $d(x, h) = d(x, k)$, then $h = k$.

**Proof.** Observe that the gate-projections of $x$ to $k$ and $h$ coincide by part 3 of Lemma 2.2; we denote them by $z$. If $w \in k$, the sequence $xzwy$ is a discrete geodesic; thus, $w \in h$. This proves that $k \subseteq h$, while the other inclusion is obvious. \qed

The hypotheses of Lemma 2.25 do not imply that $h = k$ or $h = k$; see for instance $h \subseteq k$ in Figure 2.

**Proposition 2.26.** Let $X$ be complete of rank $r < +\infty$ and let $h_1 \supseteq \ldots \supseteq h_k$ be a chain of halfspaces.

1. If $d(h_1^*, h_k) = 0$ and each $h_i$ is open, then $k \leq r$.
2. In general, if $d(h_1^*, h_k) = 0$ we have $k \leq 2r$.
3. If there exists $x \in h_1^*$ such that $d(x, h_1) = d(x, h_k)$, then $k \leq r + 1$.

**Proof.** If $d(h_1^*, h_k) = 0$, no $h_i$ can simultaneously be open and closed or we would have $h_1^* \cap h_k \subseteq h_i^* \cap h_i = h_i^* \cap h_i = \emptyset$ and $d(h_1^*, h_k) > 0$ by Lemma 2.9.
We prove part 1 by induction on $r$; the case $r = 0$ is trivial. If $r \geq 1$, observe that $C := \partial h_k$ is closed, convex and nonempty, since $h_k$ is not closed. If $i \leq k - 1$, we have $h_i \subseteq h_k$ and $h_i \cap h^*_k \neq \emptyset$, hence $h_i \cap C \neq \emptyset$ by Helly’s Theorem. Similarly $h^*_i \cap C \neq \emptyset$, since $h^*_i \subseteq h^*_k$ and since by assumption we have $h^*_i \cap h_k \supseteq h^*_1 \cap h_k = \emptyset$.

Proposition 2.3 implies that $h_i \cap C$ are a chain of distinct halfspaces of $C$, for $i \leq k - 1$ and, by Proposition 2.22, the rank of $C$ is at most $r - 1$. By Helly’s Theorem and Lemma 2.9, we have $h^*_i \cap h^*_k \cap C \neq \emptyset$, since $h^*_i \subseteq h^*_k$ and since by assumption we have $h^*_i \cap h_k \supseteq h^*_1 \cap h_k = \emptyset$.

Each of the bounds in Proposition 2.26 is sharp. An example with $r = 2$ is given by the median space in Figure 2. For part 1 one can consider the chain $h \subseteq \mathfrak{k}$ and $h \subseteq \mathfrak{k} \subseteq \mathfrak{F}$ for part 3; for part 2 one needs to add $(\tau h)^*$, where $\tau$ is the natural involution of $I$.

**Corollary 2.27.** Let $X$ be complete and finite rank. Every totally ordered subset $C \subseteq \mathcal{H}$ has a countable subset $C_0 \subseteq C$ that is cofinal in $C$.

**Proof.** Pick a point $x \in X$ and define $\delta_x : C \to \mathbb{R}$ by $\delta_x(h) := d(x, h^*)$ if $x \in h$ and $\delta_x(h) := -d(x, h)$ if $x \in h^*$. The map $\delta_x$ is monotone and, by part 3 of Proposition 2.26, it has finite fibres. The image of $\delta_x$ is a separable metric space; let $A \subseteq \delta_x(C)$ be a countable dense subset. If $\delta_x(C)$ has a maximum or a minimum, add them to $A$ and set $C_0 := \delta_x^{-1}(A)$. It is immediate to check that this is cofinal in $C$ (upwards and downwards).
3. Measure theory on the poset of halfspaces.

3.1. A finer $\sigma$-algebra on $H$. Let $X$ be a median space. The $\sigma$-algebras introduced in Theorems 2.15 and 2.17 are often too restrictive to work comfortably with. As an example, the ultrafilters $\sigma_x$, $x \in X$, need not be measurable, for instance when working with non-separable median spaces. The latter might look like pathological examples, but we remark that, on the contrary, an interesting class of finite rank median spaces arises from asymptotic cones of coarse median groups and these are rarely separable.

Since the measures constructed in Theorems 2.15 and 2.17 originate from Carathéodory’s construction, one might wish to consider instead the $\sigma$-algebras of additive sets for the outer measures $\mu^*$ and $\nu^*$. Unfortunately, even in simple examples, one might obtain measure spaces that are not semifinite. We choose a different path.

We say that a subset $E \subseteq H$ is morally measurable if $E \cap H(x \mid y)$ lies in $B_0$ for all $x, y \in X$; here $B_0$ is the completion of $B$, see the end of Section 2.2. Morally measurable sets form a $\sigma$-algebra $\hat{B} \supseteq B$. For every $z \in X$, the ultrafilter $\sigma_z \subseteq H$ is morally measurable since $\sigma_z \cap H(x \mid y) = H(x \mid m)$, where $m = m(x, y, z)$.

We say that a morally measurable subset $E \subseteq H$ is morally null if $\nu(E \cap H(x \mid y)) = 0$ for all $x, y \in X$. In particular, subsets of morally null sets are morally null. Given a morally measurable set $E$, we define

$$\hat{\nu}(E) := \sup \left\{ \sum_{i \in I} \nu(E \cap H(x_i \mid y_i)) \mid \bigcup_{i \in I} H(x_i \mid y_i) \subseteq H \right\}.$$  

We allow $I$ to be of any cardinality, although restricting to finite or countable index sets would not affect the value of $\hat{\nu}$. A morally measurable set $E$ is morally null if and only if $\hat{\nu}(E) = 0$.

We recall that a measure space $(\Omega, \mu)$ is semifinite if every measurable subset $E \subseteq \Omega$ with $\mu(E) = +\infty$ contains a measurable subset $F \subseteq E$ with $0 < \mu(F) < +\infty$.

Proposition 3.1. The triple $(H, \hat{B}, \hat{\nu})$ is a semifinite measure space.

We prove Proposition 3.1 below; the nontrivial part is showing that $\hat{\nu}$ is a measure. Since differences of halfspace-intervals are finite disjoint unions of halfspace-intervals (see [CDH10]), the following is an easy observation.

Lemma 3.2. Any finite (countable) union of halfspace-intervals is a finite (countable) disjoint union of halfspace-intervals.

Proof of Proposition 3.1. It suffices to prove that $\hat{\nu}$ is additive and $\sigma$-subadditive. Let $\{E_n\}_{n \geq 0}$ be a collection of pairwise-disjoint, morally measurable sets with union $E$. Given pairwise-disjoint $H(x_1 \mid y_1), ..., H(x_k \mid y_k)$, we have

$$\sum_{i=1}^k \nu(E \cap H(x_i \mid y_i)) = \sum_{n \geq 0} \sum_{i=1}^k \nu(E_n \cap H(x_i \mid y_i)) \leq \sum_{n \geq 0} \hat{\nu}(E_n)$$
and this proves the inequality \( \nu(E) \leq \sum \nu(E_n) \). Now, if \( E, F \) are disjoint morally measurable sets and \( \epsilon > 0 \), we can find finitely many points \( x_i, y_i \in X \) such that \( \nu(E) - \epsilon \leq \sum \nu(E \cap \mathcal{H}(x_i|y_i)) \) and, similarly, points \( u_j, v_j \in X \) satisfying an analogous inequality for \( F \). By Lemma 3.2, the union of all \( \mathcal{H}(x_i|y_i) \) and \( \mathcal{H}(u_j|v_j) \) can be decomposed as a finite disjoint union of halfspace-intervals \( \mathcal{H}(w_k|z_k) \). Thus,

\[
\nu(E) + \nu(F) - 2\epsilon \leq \sum_i \nu(E \cap \mathcal{H}(x_i|y_i)) + \sum_j \nu(F \cap \mathcal{H}(u_j|v_j))
\leq \sum_k \nu((E \cup F) \cap \mathcal{H}(w_k|z_k)) \leq \nu(E \cup F)
\]

and, by the arbitrariness of \( \epsilon \), we obtain \( \nu(E) + \nu(F) \leq \nu(E \cup F) \). We have already shown subadditivity, so \( \nu(E) + \nu(F) = \nu(E \cup F) \). \( \Box \)

**Lemma 3.3** (Properties of \( \nu \)).

1. For every \( E \in \mathcal{B} \) there exist pairwise-disjoint \( \mathcal{H}(x_n|y_n), n \geq 0 \), such that \( \nu(E) = \sum \nu(E \cap \mathcal{H}(x_n|y_n)) \).
2. \( \nu(E) \leq \nu(E) \) for all \( E \in \mathcal{B} \); in particular, \( \nu \ll \nu \).
3. \( \nu(E) = \nu(E) \) if \( E \in \mathcal{B} \) and \( \nu(E) < +\infty \); in particular, if \( E \) is morally null, \( \nu(E) \) is either 0 or +\( \infty \).

**Proof.** Part 1 follows from Lemma 3.2 and part 2 is direct from definitions. The ring used to define the measure \( \nu \) consists of unions of halfspace-intervals (see [CDH10]); thus, if \( \nu(E) < +\infty \), the set \( E \) is contained in a countable union of halfspace-intervals, hence in a disjoint one, by Lemma 3.2. Now part 3 is straightforward. \( \Box \)

### 3.2. Properties of the \( \sigma \)-algebra \( \mathcal{B} \).

Let \( X \) be a complete median space throughout this section.

**Lemma 3.4.** Singletons in \( \mathcal{H} \) are morally measurable and the following are equivalent for \( h \in \mathcal{H} \):

- \( h \) is an atom for \( \nu \);
- \( h \) is clopen;
- \( d(h, h^*) > 0 \).

**Proof.** If one side of the wall \( \{h, h^*\} \) is not closed, say \( h \), we can find points \( x_n \in h \) converging to some \( y \in h^* \). Thus, \( \{h\} \) lies in the intersection of the sets \( \mathcal{H}(y|x_n) \) and it is morally null (hence morally measurable, since the \( \sigma \)-algebra \( \mathcal{B}_0 \) is complete). Otherwise, \( h \) is clopen; by Lemma 2.9 this is equivalent to \( d(h, h^*) > 0 \). Lemma 2.4 provides a pair \( (x, y) \) of gates for \( (h, h^*) \), hence the set \( \{h\} = \mathcal{H}(h^*|h) = \mathcal{H}(y|x) \) lies in \( \mathcal{B} \) and has positive measure. Conversely, if \( h \) is an atom, it is easy to see that \( d(u, v) \geq \nu(\{h\}) \) for all \( u \in h, v \in h^* \). \( \Box \)

**Lemma 3.5.** A complete median space \( X \) is connected if and only if the measure space \( (\mathcal{H}(X), \nu) \) has no atoms.
Chapter I.1 of [BH99]). Let \( z \in I \) and no atoms, we prove that, for every \( d(x, z) = d(z, y) \). This implies that \( X \) is geodesic (see e.g. Remark 1.4(1) in Chapter I.1 of [BH99]). Let \( F_x \subseteq I(x, y) \) be the subset of points \( z \) such that \( d(x, z) \leq d(z, y) \); we endow it with a structure of poset by declaring that \( z_1 \preceq z_2 \) whenever \( z_2 \in I(z_1, y) \). Chains in \( F_x \) correspond to Cauchy nets in \( I(x, y) \) and these converge; thus, \( F_x \) is inductive and Zorn’s Lemma yields a maximal element \( \overline{z} \). Exchanging the roles of \( x \) and \( y \), the same construction provides a point \( \overline{w} \). Suppose for the sake of contradiction that we have \( d(x, \overline{z}) < \frac{1}{2}d(x, y, \overline{w}) \). By maximality of \( \overline{z} \) and \( \overline{w} \), the interval \( I(\overline{z}, \overline{w}) \) consists of the sole points \( \overline{z} \) and \( \overline{w} \). By Proposition 2.3 we conclude that \( H(\overline{z}, \overline{w}) \) consists of a single halfspace, hence an atom, contradiction.

Given a point \( x \in X \) and a convex subset \( C \subseteq X \) we define their adjacencies:

\[ \text{Adj}_x := \{ h \in H \mid x \not\in h, \ x \in \overline{h} \}, \]
\[ \text{Adj}(C) := \{ h \in H \mid h \cap C = \emptyset, \ h \cap C^c \neq \emptyset \}. \]

Note that in general \( \text{Adj}_x \neq \text{Adj}(\{x\}) = \emptyset \).

**Lemma 3.6.** If \( X \) is locally convex, adjacencies are morally null. In particular, \( \sigma_C \) and \( H(C) \) are morally measurable for every convex subset \( C \subseteq X \).

**Proof.** Given \( x, y \in X \), let \( K \) be the intersection of all halfspaces in \( \text{Adj}_x \cap \sigma_y \); by Lemma 2.11 we have \( x \in \overline{K} \) and we can find points \( x_n \in K \) converging to \( x \). Thus \( \text{Adj}_x \cap \sigma_y \) is contained in the intersection of the sets \( H(x | x_n) \) and it is morally null. By the arbitrariness of \( y \), we conclude that \( \text{Adj}_x \) is morally null. For every convex subset \( C \subseteq X \) and points \( u, v \in X \), we have

\[ \text{Adj}(C)^* \cap H(u | v) \subseteq \text{Adj}(C)^* \sigma_u \subseteq H(\overline{u}, C), \]

where \( \overline{u} \) is the gate-projection of \( u \) to \( C^c \); the last inclusion follows from part 1 of Lemma 2.2. The set \( H(\overline{u}, C) \subseteq \text{Adj}(C) \) is morally null, hence \( \text{Adj}(C) \) is morally null. Moreover, \( \sigma_C \setminus \sigma_u = H(u | C) = H(u | \overline{u}) \cup H(\overline{u}, C) \), where \( H(\overline{u}, C) \) is morally null; as a consequence, \( \sigma_C \) is morally measurable. The same holds for \( H(C) = H(\sigma_C \cup \sigma_C^c) \).

We will denote by \( H^0 \) the collection of nowhere-dense halfspaces and set \( H^\infty := H^0 \cup (H^0)^c \). If \( h \in H \setminus H^\infty \) we say that \( h \) is thick.

**Corollary 3.7.** If \( X \) is locally convex, the set \( H^\infty \) is morally null.

**Proof.** This follows from Lemma 3.6 and the observation that, for every \( x, y \in X \), we have \( H^0 \cap H(x | y) \subseteq \text{Adj}^*_y \).

**Proposition 3.8.** Let \( X \) be locally convex. If \( X \) is separable, the measure space \( (H, \hat{\nu}) \) is \( \sigma \)-finite. The converse holds if \( X \) has finite rank.
Proof. If \( \{x_n\}_{n \geq 0} \) is a countable dense subset of \( X \), all halfspace-intervals \( \mathcal{H}(x_n|x_m) \) have finite measure and their union contains \( \mathcal{H} \setminus \mathcal{H}^\times \); thus \( (\mathcal{H}, \hat{\nu}) \) is \( \sigma \)-finite by Corollary 3.7.

Conversely, if \( (\mathcal{H}, \hat{\nu}) \) is \( \sigma \)-finite, part 1 of Lemma 3.3 implies that there exists \( \{x_n\}_{n \geq 0} \subseteq X \) such that the sets \( \mathcal{H}(x_n|x_m) \) cover \( \mathcal{H} \) up to a morally null set. By Lemma 6.4 in [Bow13] and Corollary 2.20 above, hulls of separable subsets of \( X \) are separable; thus, the hull \( C \) of \( \{x_n\}_{n \geq 0} \) is separable. If there existed a point \( z \notin C \), the gate \( \tau \) for \( (z, C) \) would produce a positive-measure set \( \mathcal{H}(z|\tau) \) disjoint from the union of the \( \mathcal{H}(x_n|x_m) \), contradiction. Thus \( C \) is dense in \( X \) and \( X \) is separable.

Recall that a subset \( C \subseteq \mathcal{H} \) is said to be inseparable if it contains every halfspace \( j \in \mathcal{H} \) such that there exist \( h, t \in C \) with \( h \subseteq j \subseteq t \).

**Lemma 3.9.** If \( X \) has finite rank, any inseparable subset \( C \subseteq \mathcal{H} \) is morally measurable. In particular, every filter \( \sigma \subseteq \mathcal{H} \) is morally measurable.

**Proof.** It suffices to prove the lemma under the additional assumption that \( C \subseteq \mathcal{H}(x|y) \), for points \( x, y \in X \). By Lemma 2.21 and Corollary 2.27, \( C \) is a countable union of subsets of the form \( \mathcal{H}(t^*|h) \), with \( h, t \in C \). Each of these is morally measurable by Lemma 3.6.\( \square \)

We can extend the notion of admissibility in Section 2.2 and [CDH10] as follows. We say that a partial filter \( \sigma \subseteq \mathcal{H} \) is tangible if it is morally measurable and \( \hat{\nu}(\sigma \setminus \sigma_x) < +\infty \) for some (equivalently, all) \( x \in X \). For a morally measurable ultrafilter \( \sigma \), tangibility is equivalent to having \( \hat{\nu}(\sigma \Delta \sigma_x) < +\infty \), since \( (\sigma_x \setminus \sigma) = (\sigma \setminus \sigma_x)^* \). For instance, all admissible ultrafilters \( \sigma \) on the PMP \( (\mathcal{H}, \mathcal{B}, \nu, \sigma_x) \) are tangible. Indeed, they are morally measurable since \( \sigma = \sigma_x \sqcup [(\sigma \Delta \sigma_x) \setminus \sigma_x] \), where \( \sigma_x \) is morally measurable and \( \sigma \Delta \sigma_x \in \mathcal{B} \).

We denote by \( \mathcal{M}(X) \) the set of tangible ultrafilters, identifying ultrafilters with \( \hat{\nu} \)-null symmetric difference. The analogy with the notation \( \mathcal{M}(X) \) of Section 2.2 is justified by Corollary 3.11 below; for now, we simply observe that there are isometric embeddings \( X \hookrightarrow \mathcal{M}(X) \hookrightarrow \mathcal{H}(X) \) as a consequence of parts 2 and 3 of Lemma 3.3. The following is a key result.

**Lemma 3.10.** Let \( X \) be locally convex. For every tangible filter \( \sigma \subseteq \mathcal{H} \), there exists \( x \in X \) such that \( \hat{\nu}(\sigma \setminus \sigma_x) = 0 \).

**Proof.** If there exists \( x_0 \in X \) such that \( \hat{\nu}(\sigma \cap \mathcal{H}(x_0|y)) = 0 \) for all \( y \in X \), then \( \hat{\nu}(\sigma \setminus \sigma_x) = 0 \). Indeed, given any \( u, v \in X \), we can set \( m := m(x_0, u, v) \) and \( (\sigma \setminus \sigma_{x_0}) \cap \mathcal{H}(u|v) = \sigma \cap \mathcal{H}(m|v) \subseteq \sigma \cap \mathcal{H}(x_0|v) \), the latter being morally null.

If instead \( \hat{\nu}(\sigma \cap \mathcal{H}(x_0|y)) > 0 \) for some \( x_0, y \in X \), then there exists \( z \in I(x_0,y) \) such that \( z \neq x_0 \) and \( \mathcal{H}(x_0|z) \subseteq \sigma \). Indeed, by Lemma 3.6 there exists \( h \in \sigma \cap \mathcal{H}(x_0|y) \) such that \( d(x_0, h) > 0 \). Letting \( z \) be the gate for \( (x_0, h) \), if \( t \in \mathcal{H}(x_0|z) \) we have \( h \subseteq \overline{h} \subseteq t \); hence, the fact that \( \sigma \) is a filter and \( h \in \sigma \) implies that \( t \in \sigma \).
Now, we construct a countable ordinal \( \eta \) and an injective net \((x_{\alpha})_{\alpha \leq \eta}\) such that all the following are satisfied:

1. \((\sigma \setminus x_{\alpha+1}) \cup (\sigma_{x_{\alpha+1}} \setminus \sigma_{x_{\alpha}}) = (\sigma \setminus \sigma_{x_{\alpha}})\) for each \( \alpha \leq \eta \);
2. if \( \alpha \) is a limit ordinal, \( \bigcap_{\beta<\alpha} \sigma \setminus \sigma_{x_{\beta}} = \sigma \setminus \sigma_{x_{\alpha}} \) up to a morally null set;
3. \( \hat{\nu} (\sigma \setminus \sigma_{x_{\alpha}}) = 0 \).

The net is constructed by transfinite induction, starting with an arbitrary choice of \( x_0 \). Suppose that \( x_\beta \) has been defined for all \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, the inductive hypothesis implies that the disjoint union of the sets \( \sigma_{x_{\alpha+1}} \setminus \sigma_{x_{\alpha}} \), for \( \beta < \alpha \), is contained in \( \sigma \setminus \sigma_{x_0} \), up to a morally null set. Hence \( \sum_{\beta<\alpha} \|d(x_\beta, x_{\beta+1})\| \leq \nu(\sigma \setminus \sigma_{x_0}) < +\infty \) and \((x_\beta)_{\beta<\alpha}\) is a Cauchy net. We define \( x_\alpha \) to be its limit; we only need to check condition 2 and it follows from Lemma 3.6.

If \( \alpha = \beta + 1 \), we look at \( \sigma \setminus \sigma_{x_\beta} \); if this is morally null, we stop and set \( \eta = \beta \). Otherwise, we can find \( y \in X \) so that \( \hat{\nu}(\sigma \cap H(x_\beta|y)) > 0 \) and we have already shown that there exists \( z \in X \) with \( H(x_\beta|z) \subseteq \sigma \); we set \( x_\alpha := z \). By construction \( \sigma_{x_{\alpha}} \setminus \sigma_{x_{\beta}} \subseteq \sigma \setminus \sigma_{x_{\beta}} \) and, since \( \sigma \) is a filter, we have \( \sigma \setminus (\sigma_{x_{\beta}} \setminus \sigma_{x_{\alpha}}) = \emptyset \); in particular, \( \sigma \setminus \sigma_{x_{\alpha}} \subseteq \sigma \setminus \sigma_{x_{\beta}} \). From this we immediately get condition 1, i.e. \( \sigma \setminus \sigma_{x_\beta} = (\sigma \setminus \sigma_{x_{\alpha}}) \cup (\sigma_{x_{\alpha}} \setminus \sigma_{x_{\beta}}) \).

We conclude by remarking that, since \( d(x_{\alpha+1}, x_\alpha) > 0 \) for all ordinals \( \alpha \) and \( \sum_{\beta<\alpha} \|d(x_\beta, x_{\beta+1})\| \leq \nu(\sigma \setminus \sigma_{x_0}) < +\infty \), the process must terminate for some countable ordinal \( \eta \).

**Corollary 3.11.**

1. If \( X \) is locally convex and \( \sigma \subseteq H \) is a tangible ultrafilter, there exists \( x \in X \) such that \( \hat{\nu}(\sigma \Delta \sigma_x) = 0 \).
2. If \( X \) has finite rank and \( \sigma \subseteq H \) is a tangible filter, there exists a convex subset \( C \subseteq X \) such that \( \hat{\nu}(\sigma \Delta \sigma_C) = 0 \).

**Proof.** Let \( \sigma \) be a tangible filter and let \( x \in X \) be such that \( \hat{\nu}(\sigma \setminus \sigma_x) = 0 \), as provided by Lemma 3.10. If \( \sigma \) is an ultrafilter, we immediately obtain \( \hat{\nu}(\sigma \Delta \sigma_x) = 0 \). Otherwise, suppose that \( X \) has finite rank and let \( C \) be the intersection of all \( H \setminus \sigma \cap \sigma_x \); since \( x \in C \), this is a nonempty convex subset and the filter \( \sigma_C \) contains \( \sigma \cap \sigma_x \). In particular, we have \( \hat{\nu}(\sigma \setminus \sigma_C) = 0 \).

Suppose \( \xi \in \sigma_C \setminus \sigma \) and consider the gate-projection \( \pi: X \to F^* \). Observe than no \( \xi \in \sigma \cap \sigma_x \) can be contained in \( \xi \) or we would have \( \xi \in \sigma \); hence every \( \xi \in \sigma \cap \sigma_x \) intersects \( \xi^* \). Part 1 of Lemma 2.2 then implies that, given any \( y \in C \), the projection \( \pi(y) \) still lies in \( C \); in particular, \( C \cap F^* \neq \emptyset \). If \( \xi \) were open, we would have \( F^* = \xi^* \) and the previous statement would contradict the fact that \( C \subseteq \xi \); thus, \( \xi \) is closed by Corollary 2.22.

If \( u \in X \) and \( \pi \) is its gate-projection to \( C \), we have \( \pi \in F^* \) for every \( \xi \in \sigma_C \setminus (\sigma \∪ \sigma_u) \), since \( C \cap F^* \neq \emptyset \). Since every such \( \xi \) is closed, we have \( \pi \in F^* \cap \xi \) and \( \sigma_C \setminus (\sigma \∪ \sigma_u) \subseteq \text{Adj}_\pi \). The arbitrariness of \( u \) and Lemma 3.6 imply that \( \hat{\nu}(\sigma_C \setminus \sigma) = 0 \).

An immediate consequence of part 1 of Corollary 3.11 is:
Corollary 3.12. For every complete, locally convex median space $X$, the isometric embedding $X \hookrightarrow M(X)$ is surjective. In particular, the spaces $X$, $M(X)$ and $\mathcal{M}(X)$ are isometric.

An interesting consequence of Corollary 3.12 is the following.

Corollary 3.13. If $X$ is locally convex with corresponding SMH-structure $(X, \mathcal{H}, \mathcal{B}, \nu)$. Then $\text{Aut}(X, \mathcal{H}, \mathcal{B}, \nu) = \text{Isom}(X)$.

An analogous result holds for the SMW-structure.

4. Compactifying median spaces and algebras.

4.1. The zero-completion of a median algebra. Let $M$ be a median algebra. We denote by $\mathcal{J}(M)$ (or simply by $\mathcal{J}$) the poset of all intervals $I(x,y)$ with $x, y \in M$, ordered by inclusion; singletons are allowed. Note that the poset $(\mathcal{J}, \subseteq)$ is not a directed set. Still, whenever $I, I', I'' \in \mathcal{J}$ and $I \subseteq I' \subseteq I''$, we have $\pi_I|_{I''} = \pi_I|_{I'} \circ \pi_{I'}|_{I''}$ and it makes perfect sense to consider the inverse limit

$$\lim_{\leftarrow} I := \left\{ (x_I)_I \in \prod_{I \in \mathcal{J}} I \mid \pi_{I \cap J}(x_I) = \pi_{I \cap J}(x_J), \forall I, J \in \mathcal{J}, I \cap J \neq \emptyset \right\}.$$

Note that $I \cap J \in \mathcal{J}$ whenever $I, J \in \mathcal{J}$; indeed, we showed in Section 2.1 that every gate-convex subset of $I$ is itself an interval.

The product of all $I \in \mathcal{J}$ has a structure of median algebra given by considering median maps component by component; the inverse limit $\lim_{\leftarrow} I$ also inherits a structure of median algebra and we have a monomorphism $i: M \hookrightarrow \lim_{\leftarrow} I$ given by mapping $x \mapsto (\pi_I(x))_I$.

Definition 4.1. We denote the median algebra $\lim_{\leftarrow} I$ simply by $\overline{M}$ and refer to it as the zero-completion of $M$.

This terminology comes from [BM93] where the same notion is defined from a different perspective; more on this in Remark 4.7.

We also have a monomorphism $i: M \hookrightarrow 2^\mathcal{H}$ given by mapping $x \mapsto \sigma_x$; the space $2^\mathcal{H}$ can be endowed with the product median algebra structure and the product topology. Here the set $2 = \{0,1\}$ is equipped with the discrete topology and its unique structure of median algebra.

Definition 4.2. The closure of $i(M)$ in $2^\mathcal{H}$ with the induced median-algebra structure will be denoted by $M^{\infty}$ and we will refer to it as the double dual of $M$ (compare [Rol98]).

Lemma 4.3. (1) The median algebra $M^{\infty}$ coincides with the subset of $2^\mathcal{H}$ consisting of ultrafilters on $\mathcal{H}$.

(2) There is a monomorphism $j: \overline{M} \hookrightarrow M^{\infty}$ such that $i = j \circ i$. 
Proof. Ultrafilters on $\mathcal{H}$ form a closed subset of $2^\mathcal{H}$ with the product topology; thus every element of $M^\infty$ is an ultrafilter. Every neighbourhood of an ultrafilter $\sigma \subseteq \mathcal{H}$ is of the form $\{ A \subseteq \mathcal{H} \mid h_1, ..., h_k \in A, \ h_{k+1}, ..., h_n \notin A \}$, where $h_1, ..., h_n$ lie in $\sigma$ and hence intersect pairwise. Any such neighbourhood intersects $i(M)$ as it contains $i(x)$ for any $x \in h_1 \cap ... \cap h_n$; the latter is nonempty by Helly’s Theorem. Hence, every ultrafilter on $\mathcal{H}$ lies in $M^\infty$.

Regarding part 2, given $(x_I) \in \varprojlim I$ we will construct an ultrafilter $\sigma \subseteq \mathcal{H}$ such that $\sigma \cap \mathcal{H}(I) = \sigma_x \cap \mathcal{H}(I)$ for all $I \in \mathcal{J}$. This ultrafilter is unique and the corresponding map $j: \overline{M} \to M^\infty$ is easily seen to be a monomorphism. It suffices to show that the sets

$$\Omega_1 := \{ h \in \mathcal{H} \mid \exists I \text{ s.t. } x_I \in h \in \mathcal{H}(I) \},$$

$$\Omega_2 := \{ h \in \mathcal{H} \mid x_I \in h, \forall I \text{ s.t. } h \in \mathcal{H}(I) \},$$

coincide and set $\sigma := \Omega_1 = \Omega_2$. This is indeed an ultrafilter: $\Omega_1$ contains at least one side of every wall of $M$, while any two halfspaces in $\Omega_2$ intersect.

The inclusion $\Omega_2 \subseteq \Omega_1$ is immediate; proving the other amounts to showing that $x_j \in h \in \mathcal{H}(J)$ and $x_I \notin h \in \mathcal{H}(I)$ cannot happen at the same time. We argue by contradiction.

Observe that $m_I := \pi_I(x_I) \in h$ and $m_J := \pi_J(x_I) \in h^*$ by part 1 of Lemma 2.2. Let $I' := I(x_I, m_I)$ and $J' := I(x_I, m_J)$; since $x_I \in I' \subseteq I$, we have $x_I = \pi_I(x_I) = x_I$ and similarly $x_I = x_I$. Set $K := I(x_I, x_I)$.

Since $I' \subseteq K$ and $J' \subseteq K$, we have $\pi_I(x_K) = x_I$ and $\pi_J(x_K) = x_I$. However, $\{h, h^*\} \in \mathcal{H}(I') \cap \mathcal{H}(J')$, so the previous equalities imply that $x_K \in h^*$ and $x_K \in h$, respectively, a contradiction.

\begin{corollary}
\begin{enumerate}
\item The embedding $i: M \to \overline{M}$ has convex image. Thus, given $x, y \in M$, the notion of $I(x, y)$ is the same in $M$ and $\overline{M}$.
\item For every $J \in \mathcal{J}$, the projection $p_J: \overline{M} \to J$ that $\overline{M}$ inherits from $\prod I$ is precisely the gate-projection $\pi_J: \overline{M} \to J$.
\end{enumerate}
\end{corollary}

\begin{proof}
Consider points $x, y \in M$ and set $J := I(x, y) \in \mathcal{J}$. Given $z \in \overline{M}$, we write $\overline{z}$ instead of $p_I(z)$. Assuming $m(i(x), i(y), z) = z$, we will show that $z = i(\overline{z})$. This yields $z \in i(M)$ and proves part 1.

If we had $z \neq i(\overline{z})$, there would exist $I \in \mathcal{J}$ such that $z_I \neq p_I(z_I)$; note that $p_I(z_I) = \pi_I(z_I)$, where $\pi_I$ denotes the gate-projection $M \to I$. Let $h \in \mathcal{H}(M)$ be a halfspace lying in $\mathcal{H}(\pi_I(z_I)|z_I)$; by Proposition 2.3 we have $h \in \mathcal{H}(z_I|z_I)$. The equality $\Omega_1 = \Omega_2$ in the proof of Lemma 4.3 now shows that $h \notin \mathcal{H}(J)$. Since $z_I \in h^*$, we have $J \subseteq h^*$; in particular, $\pi_I(x)$ and $\pi_I(y)$ lie in $h^*$. However, since $m(i(x), i(y), z) = z$, we have $m(\pi_I(x), \pi_I(y), z) = p_I m(i(x), i(y), z) = z_I \notin h$, a contradiction.

For part 2, note that $\pi_J(z) = m(x, y, z)$ lies in $i(M)$ by part 1. Thus,

$$m(x, y, z) = p_J m(x, y, z) = m(p_J(x), p_J(y), p_J(z)) = m(x, y, z) = z_I.$$
The median algebras $\overline{M}$ and $M^{\infty}$ can coincide; for instance, this is the case for 0-skeleta of a CAT(0) cube complexes. However, the following example shows that $\overline{M}$ and $M^{\infty}$ differ in general.

**Example 4.5.** Consider the median algebras $N = N$ and $M = N \cup \{+\infty\}$; in both cases $m(x, y, z) = y$ if $x \leq y \leq z$. For every $k \in \mathbb{N}$, both $M$ and $N$ have a wall $w_k$ separating $k$ and $k + 1$; in $M$, there is an additional wall $w_{\infty}$ separating $\mathbb{N}$ and $+\infty$. Observe that $M = \overline{N} = N^{\infty}$ and, since $M = I(0, +\infty)$, we also have $M = \overline{M}$.

However, $M^{\infty} = M \cup \{-\infty\}$, where $-\infty$ is represented by the ultrafilter that picks the side containing $-\infty$ for every wall $w_k$ and the side containing $\mathbb{N}$ for the wall $w_{\infty}$. The point $-\infty$ is “bigger than any natural number”, but still “smaller than $+\infty$”; also compare Remark 2.18.

Given $a \in M$, we say that a convex subset $C \subseteq M$ is a-directed if $a \in C$ and, for every $x, y \in C$, there exists $z \in C$ such that $x, y \in I(a, z)$.

**Lemma 4.6.** Fix $a \in M$. There is a one-to-one correspondence between points of $\overline{M}$ and gate-convex, a-directed subsets $C \subseteq M$. Points $b \in M \subseteq \overline{M}$ correspond to intervals $I(a, b)$.

**Proof.** If $C \subseteq M$ is gate-convex and a-directed, the projection $C_I := \pi_I(C)$ is a gate-convex, $\pi_I(a)$-directed subset of $I$ by part 2 of Lemma 2.2. It follows that there exists a (unique) point $x_I \in C_I$ such that $C_I = I(\pi_I(a), x_I)$. By Proposition 2.1, we obtain a point $\xi_C := (x_I)_I \in \lim I$.

Conversely, given $\xi \in \overline{M}$, we can consider the interval $I(a, \xi) \subseteq \overline{M}$ and set $C_\xi := I(a, \xi) \cap M$. Since $M$ is convex in $\overline{M}$, the map $u \mapsto m(a, \xi, u)$ takes $M$ into itself and it is a gate-projection $M \to C_\xi$ by Proposition 2.1.

If $x, y \in C_\xi$, we have $x, y \in I(a, z)$ with $z := m(x, y, \xi) \in C_\xi$. Thus $C_\xi$ is gate-convex and a-directed.

Observe that, setting $z := m(a, \xi, \pi_I(\xi))$, we have $\pi_I(z) = \pi_I(\xi)$ and $z \in I(a, \xi) \cap M$; hence, $\pi_I(I(a, \xi) \cap M) = I(\pi_I(a), \pi_I(\xi))$ for all $I \in \mathcal{J}$, i.e. $\xi = \xi_{C_\xi}$ for every $\xi \in \overline{M}$. We conclude by observing that $C_1 \neq C_2$ implies $\xi_{C_1} \neq \xi_{C_2}$; indeed, if $x \in C_1 \setminus C_2$ and $J := I(a, x)$, we have $\pi_J(\xi_{C_1}) = x$ and $\pi_J(\xi_{C_2}) \neq x$, since $\pi_J(C_1) = J$ and $\pi_J(C_2) = C_2 \cap J \neq J$. $\square$

**Remark 4.7.** When zero-completions were originally defined in [BM93], points of $\overline{M}$ were compatible meet-semilattice operations, see Theorem 1 in *op. cit.* The terminology “zero-completion” is due to the fact that all semilattice operations have a zero in $\overline{M}$, while they need not have one in $M$. By Lemma 4.6 above and Theorem 5.5 in [BH83], our definition of $\overline{M}$ yields the same object. Zero-completions do not seem to have been studied outside of [BM93]; we will develop their theory further in the next sections, especially in the case when $M$ arises from a median space.

**Lemma 4.8.** If $C \subseteq M$ is gate-convex, the zero-completion $\overline{C}$ canonically embeds into $\overline{M}$ as a gate-convex subset with $\overline{C} \cap M = C$. 
Proof. If \( \xi = (x_I)_I \in \lim I \), we set \( \pi(\xi) := (\pi_I \pi_C(x_I))_I \in \prod I \). This is an element of \( \lim I \) as, if \( J \subseteq K \) are intervals of \( M \), we have
\[
\pi_J(\pi_K \pi_C(x_K)) = \pi_J \pi_C(\pi_K(x_K)) = \pi_J \pi_C(\pi_K(\xi)) = \pi_J \pi_C(\pi_J(\xi)) = \pi_J \pi_C(x_J),
\]
by Corollary 4.4 and part 4 of Lemma 2.2. We obtain a map \( \pi: \overline{M} \to \overline{M} \), which is a gate-projection onto its image, by Proposition 2.1. The image of \( \pi \) is the set of those \( (x_I)_I \) with \( \pi_I \pi_C(x_I) = x_I \), i.e. \( x_I \in \pi_I(C) \), for all \( I \in J \).
Restricting the index set to \( \mathcal{J}(C) \), we obtain a morphism \( f: \text{im} \pi \to \overline{C} \).
Embedding \( \overline{M} \hookrightarrow \overline{M}^{\infty} \) as in Lemma 4.3, all points of \( \text{im} \pi \) are represented by ultrafilters containing \( \sigma_C \). In terms of ultrafilters, \( f \) is the restriction of the map that takes each ultrafilter on \( \mathcal{H}(M) \) to its intersection with the subset \( \mathcal{H}(C) \subseteq \mathcal{H}(M) \); in particular, \( f \) is injective.

We now show that \( f \) is surjective; given \( \xi \in \overline{C} \), let \( \sigma \subseteq \mathcal{H}(C) \) be the ultrafilter representing \( \xi \), as provided by Lemma 4.3. If \( I \in \mathcal{J}(M) \), there exist points \( u, v \in C \) such that \( \pi_I(\sigma) = I(\pi_I(u), \pi_I(v)) \); we set \( J := I(u, v) \).
If \( \xi_J \) is the coordinate of \( \xi \) corresponding to \( J \in \mathcal{J}(C) \), we set \( x_I := \pi_I(\xi_J) \).
Note that \( x_I \) is represented by the ultrafilter \( (\sigma_C \cap \mathcal{H}(I)) \cup (\sigma \cap \mathcal{H}(I)) \) on \( \mathcal{H}(I) \).
In particular, our definition of \( x_I \) does not depend on the choice of the points \( u, v \) and \( (x_I)_I \) satisfies the compatibility condition necessary to define a point \( \eta \in \overline{M} \). It is clear that \( \eta \in \text{im} \pi \) and \( f(\eta) = \xi \). Thus, we have identified \( \overline{C} \simeq \text{im} \pi \); the fact that \( \overline{C} \cap M = C \) is a trivial observation. \( \square \)

Given \( \mathfrak{h} \in \mathcal{H}(M) \), there exists a unique halfspace \( \tilde{\mathfrak{h}} \in \mathcal{H}(\overline{M}) \) satisfying \( \tilde{\mathfrak{h}} \cap M = \mathfrak{h} \). This can be observed by applying Proposition 2.3 to an interval \( I(x, y) \) with \( x \in \mathfrak{h} \) and \( y \in \mathfrak{h}^* \).

The halfspace \( \tilde{\mathfrak{h}} \) can be further characterised by \( \xi \in \tilde{\mathfrak{h}} \Leftrightarrow \mathfrak{h} \in j(\xi) \), where \( j \) is as in Lemma 4.3. Indeed, Corollary 4.4 implies that, for every interval \( I \subseteq M \), we have \( j(\xi) \cap \mathcal{H}(I) = \sigma_{\pi_I(\xi)} \cap \mathcal{H}(I) \).
If \( \mathfrak{h} \in \mathcal{H}(I) \), Proposition 2.3 then yields
\[
\xi \in \tilde{\mathfrak{h}} \Leftrightarrow \pi_I(\xi) \in \mathfrak{h} \Leftrightarrow \mathfrak{h} \in j(\xi).
\]

Note that not all halfspaces of \( \overline{M} \) are of the form \( \tilde{\mathfrak{h}} \); indeed, by convexity of \( M \), we have \( \mathcal{H}(M, \xi) \neq \emptyset \) for every \( \xi \in \overline{M} \setminus M \). An explicit example of such a wall was given in Example 4.5 for the median algebra \( N \) (in the notation of the example, we have \( \mathfrak{m}_\infty \in \mathcal{H}(N) \setminus +\infty \) and \( +\infty \in \partial N \)).

Nevertheless, we can identify the set of halfspaces of \( M \) with a subset \( \mathcal{H}(M) \subseteq \mathcal{H}(\overline{M}) \) and any two points of \( \overline{M} \) are separated by an element of \( \mathcal{H}(M) \). Indeed, if \( \xi, \eta \in \overline{M} \) are distinct, then \( \pi_I(\xi) \neq \pi_I(\eta) \) for some interval \( I \subseteq M \) and, for any \( \mathfrak{h} \in \mathcal{H}(\pi_I(\xi)|\pi_I(\eta)) \), we have \( \mathfrak{h} \in \mathcal{H}(\xi|\eta) \).

**Lemma 4.9.** We have \( \text{rank}(\overline{M}) = \text{rank}(M) \).

**Proof.** Immediate from the discussion above and Lemma 2.5 \( \square \)

**Lemma 4.10.** If there exists a topology on \( M \) for which it is a compact topological median algebra, then \( M = \overline{M} \).
Proof. Given $\xi = (x_I)_I \in \lim^{-} I$, consider the projections $\pi_I: M \to I$ and set $C_I := \pi_I^{-1}(x_I) \subseteq M$. These are convex sets by Proposition 2.1 and they pairwise intersect as they all contain $\xi$; moreover, they all intersect $M$, which is convex in $\overline{M}$. Helly’s Theorem implies that $\{C_I \cap M \mid I \in \mathcal{I}\}$ has the finite intersection property.

Now endow $M$ with its compact topology. Each $C_I \cap M = (\pi_I|_M)^{-1}(x_I)$ is closed in $M$, hence compact. Thus, the intersection of all the $C_I \cap M$ is nonempty; any $x$ in this intersection satisfies $\pi_I(x) = x_I$ for every interval $I$, i.e. $\xi = \iota(x)$.

In the rest of the section, we suppose that $M$ is a topological median algebra. Endowing the product of all intervals with the product topology, the zero-completion $\overline{M}$ inherits a topology. The inclusion $\iota: M \hookrightarrow \overline{M}$ is always continuous, but in general not a topological embedding (cf. Proposition 4.20). Classical examples of this phenomenon are provided by locally infinite trees, since in that case $\overline{M}$ coincides with the usual Roller compactification.

For instance, if $T$ is a geodesically complete (real or simplicial) tree and $x \in T$ is a point of infinite degree, $\iota(x)$ does not lie in the interior of $\iota(T)$; in particular, $\iota(T)$ is not open in $\overline{T}$. Another interesting case is when $T$ is a bounded tree and $T \setminus \{x\}$ contains infinitely many connected components of diameter at least 1; here the map $\iota: T \to \overline{T}$ is a continuous bijection, but still not a homeomorphism.

The following is an easy observation.

Lemma 4.11. If $M$ is locally convex, $\overline{M}$ is as well.

Lemma 4.12. If $M$ has compact intervals, $\overline{M}$ is compact and $M$ is dense in $\overline{M}$.

Proof. Compactness is immediate from the observation that $\overline{M}$ is a closed subset of $\prod I$. We prove density by showing that for every $\xi \in \overline{M}$ and for every finite collection of intervals $I_1, ..., I_k \in \mathcal{I}(M)$, there exists $x \in M$ such that $\pi_{I_i}(x) = \pi_{I_i}(\xi)$ for all $i$.

Let $C$ be the convex hull of $I_1 \cup ... \cup I_k$ in $\overline{M}$; it is compact by Lemma 2.7 and $C \subseteq M$ since $M$ is convex in $\overline{M}$. Note that $C$ is gate-convex in $\overline{M}$ by Lemma 2.6 let $\pi: \overline{M} \to C$ be the corresponding gate-projection. By part 3 of Lemma 2.2, we can set $x := \pi(\xi) \in C \subseteq M$.

If $M$ has compact intervals and $C \subseteq M$ is gate-convex, Lemma 4.12 implies that the subset of $\overline{M}$ that we identified with the zero-completion $C$ in Lemma 4.8 coincides with the closure of $C$ in the topology of $\overline{M}$.

4.2. The Roller compactification of a median space. Let $X$ be a complete, locally convex median space with compact intervals throughout this section. This encompasses all finite rank median spaces (see Corollary 2.20), all (possibly infinite dimensional) $\text{CAT}(0)$ cube complexes and all (possibly infinite rank) complete, connected, locally compact, locally convex median spaces (see [CD17] for examples).
**Definition 4.13.** In this context, we will refer to the zero-completion $\overline{X}$ as *Roller compactification* and to $\partial X := \overline{X} \setminus X$ as *Roller boundary*.

The renaming is justified by the fact that the median metric on $X$ induces an additional structure on $\overline{X}$, which we shall study in this and the following section. There is a strong analogy with Roller boundaries of CAT(0) cube complexes and, indeed, if $X$ is the 0-skeleton of a CAT(0) cube complex, our notion of Roller boundary coincides with the usual one. The following proposition sums up what we already know about $X$; it roughly corresponds to the first and third definitions of $\overline{X}$ that we gave in the introduction.

**Theorem 4.14.** (1) The Roller compactification $\overline{X}$ is a locally convex, compact, topological median algebra.

(2) The inclusion $i: X \hookrightarrow \overline{X}$ is a continuous morphism with convex, dense image.

(3) For every closed convex subset $C \subseteq X$, the closure of $C$ in $\overline{X}$ is gate-convex and naturally identified with the Roller compactification of $C$.

(4) If $X$ is separable, the topology of $\overline{X}$ is separable and metrisable.

We remark that $\partial X$ need not be closed, compare Proposition 4.20 below. In analogy with the space $\mathcal{M}(X)$ and Definition 4.2, we introduce

$$\overline{\mathcal{M}}(X) := \left\{ \sigma \subseteq \mathcal{H} \mid \sigma \in \hat{B}, \sigma \text{ ultrafilter} \right\} / \sim,$$

where $\sigma_1 \sim \sigma_2$ if $\hat{\nu}(\sigma_1 \triangle \sigma_2) = 0$. We can give $\overline{\mathcal{M}}(X)$ a median-algebra structure by defining the median map as in part 2 of Example 2.8. If $X$ has finite rank, all ultrafilters are morally measurable by Lemma 3.9 and $\overline{\mathcal{M}}(X)$ is simply a quotient of $X^\infty$. By Corollary 3.12, we have a monomorphism $X \simeq \mathcal{M}(X) \hookrightarrow \overline{\mathcal{M}}(X)$. The following result corresponds to the second definition of the Roller compactification that we gave in the introduction.

**Theorem 4.15.** The map $j: \overline{X} \hookrightarrow X^\infty$ introduced in Lemma 4.3 takes values in the set of morally measurable ultrafilters and it descends to an isomorphism $\overline{j}: \overline{X} \overset{\sim}{\rightarrow} \overline{\mathcal{M}}(X)$ extending $X \hookrightarrow \mathcal{M}(X)$.

**Proof.** If $\xi = (x_I)_I \in \varprojlim I$, we have $j(\xi) \cap \mathcal{H}(I) = \sigma_{x_I} \cap \mathcal{H}(I)$ for every interval $I$, hence $j(\xi)$ is morally measurable. We get a morphism $\overline{j}: \overline{X} \rightarrow \overline{\mathcal{M}}(X)$ extending $X \hookrightarrow \overline{\mathcal{M}}(X)$. If $\eta = (y_I)_I$ satisfies $\hat{\nu}(j(\xi) \triangle j(\eta)) = 0$, we have $\hat{\nu}((\sigma_{x_I} \triangle \sigma_{y_I}) \cap \mathcal{H}(I)) = 0$, i.e. $x_I = y_I$, for all $I \in \mathcal{I}$. Thus, $\overline{j}$ is injective.

If $\sigma \subseteq \mathcal{H}$ is a morally measurable ultrafilter, each $\sigma \cap \mathcal{H}(I)$ is a morally measurable ultrafilter on $\mathcal{H}(I)$ and it is tangible since $\hat{\nu}(\mathcal{H}(I)) < +\infty$. Corollary 3.12 provides $z_I \in I$ with $\hat{\nu}((\sigma_{z_I} \cap \mathcal{H}(I)) = 0$. We obtain $\zeta := (z_I)_I \in \varprojlim I$ with $\hat{\nu}(j(\zeta) \cap \mathcal{H}(I)) = 0$; hence, $\overline{j}$ is also surjective. \qed

In general, given $\xi \in \overline{X}$ and a morally measurable ultrafilter $\sigma \subseteq \mathcal{H}$ representing $\xi$, we could have $h \in \sigma$ even if $\xi \notin h$; see e.g. Remark 2.18. However, we have already observed that this does not happen for $\sigma = j(\xi)$.\[\]
Since $j(x) = \sigma_x$ for every $x \in X$, we will also denote $j(\xi)$ by $\sigma_\xi$ from now on. This should be viewed as a canonical choice of an ultrafilter representing $\xi$.

**Lemma 4.16.** A sequence $(\xi_n)_{n \geq 0}$ in $\overline{X}$ converges to $\xi \in \overline{X}$ if and only if
\[
\nu(\limsup_{n \to +\infty} \sigma_\xi \triangle \sigma_{\xi_n}) = 0.
\]

**Proof.** Given $I \in \mathcal{J}$, Lemma 3.6 implies that $\pi_I(\xi_n) \to \pi_I(\xi)$ if and only if
\[
0 = \nu\left(\limsup_{n \to +\infty} \pi_I(\xi_n) \triangle \pi_I(\xi)\right) = \nu\left(\limsup_{n \to +\infty} (\sigma_\xi \triangle \sigma_{\xi_n}) \cap \mathcal{H}(I)\right).
\]
Since $\xi_n \to \xi$ if and only if $\pi_I(\xi_n) \to \pi_I(\xi)$ for all $I \in \mathcal{J}$, convergence corresponds precisely to $\limsup \sigma_\xi \triangle \sigma_{\xi_n}$ being morally null. \[\square\]

We can endow $\overline{X} \simeq \overline{\mathcal{H}}(X)$ with an extended metric
\[
d(\sigma_1, \sigma_2) := \frac{1}{2} \cdot \nu(\sigma_1 \triangle \sigma_2) \in [0, +\infty);
\]
the restriction to $\mathcal{H}(X) \simeq X$ coincides with the usual metric on $X$.

**Lemma 4.17.**
1. The median map $m: \overline{X}^3 \to \overline{X}$ is 1-Lipschitz with respect to the extended metric $d$.
2. If $C \subseteq \overline{X}$ is closed and convex, the gate-projection $\pi: \overline{X} \to C$ is 1-Lipschitz with respect to the extended metric $d$.

**Proof.** Part 1 follows from part 2 applied to intervals. Denote by $\sigma_C$ the set of $h \in \mathcal{H}$ such that $C \subseteq h$ and set $\mathcal{H}(C) := \mathcal{H} \setminus (\sigma_C \cup \sigma_C^*)$. By part 1 of Lemma 4.2, we have $\sigma_\pi(\xi) = (\sigma_\xi \cap (\mathcal{H}(C) \cup \sigma_C))$, for all $\xi \in \overline{X}$. Thus, for all $\xi, \eta \in \overline{X}$,
\[
2 \cdot d(\pi(\xi), \pi(\eta)) = \nu((\sigma_\xi \triangle \sigma_\eta) \cap \mathcal{H}(C)) \leq \nu(\sigma_\xi \triangle \sigma_\eta) = 2 \cdot d(\xi, \eta).
\]
\[\square\]

If $\xi \in \overline{X}$ and $(\xi_n)_{n \geq 0}$ is a sequence in $\overline{X}$ with $d(\xi_n, \xi) \to 0$, part 2 of Lemma 4.17 shows that $\pi_I(\xi_n) \to \pi_I(\xi)$ for every interval $I \subseteq X$; hence $\xi_n \to \xi$ in $\overline{X}$. The converse does not hold: consider a sequence $(x_n)_{n \geq 0}$ in $X \subseteq \overline{X}$ that converges to a point $\xi \in \partial X$ (these exist by Lemma 4.12). Corollary 3.12 shows that $d(\xi_n, \xi) = +\infty$ for all $n \geq 0$, so $d(x_n, \xi) \not\to 0$.

The following notion is due to Gur [Gur05] for CAT(0) cube complexes.

**Definition 4.18.** A **component** of $\overline{X}$ is a $\sim$-equivalence class of morally measurable ultrafilters for the equivalence relation
\[
\sigma_1 \approx \sigma_2 \iff d(\sigma_1, \sigma_2) = +\infty.
\]

Note that the subset $X \subseteq \overline{X}$ always forms a single component. Part 2 of Example 2.8 implies the following.

**Proposition 4.19.** The restriction of the metric $d$ to any component of $\overline{X}$ gives it a structure of median space. Each component is a convex in $\overline{X}$.

The study of components of $\partial X$ will be the subject of Section 4.3.
Proposition 4.20. If $X$ is connected and locally compact, the inclusion $\iota : X \rightarrow \overline{X}$ is a topological embedding.

Proof. Given $x_0 \in X$, choose $0 < \delta < +\infty$ and let $F$ be a finite, $\delta$-dense subset of $B_{2\delta}(x_0)$; this exists since $X$ is proper by Proposition 3.7 in Chapter I.3 of [BH99]. Let $U$ be the set of points $\xi \in \overline{X}$ such that $d(\pi_I(\xi), x_0) < \delta$ for all intervals $I = I(x_0, y)$, with $y \in F$; this is a neighbourhood of $x_0$ in $\overline{X}$. If $\eta \in \overline{X}$ and $d(\eta, x_0) \geq 2\delta$, there exists $z \in I(x_0, \eta)$ with $d(x_0, z) = 2\delta$, since $X$ is geodesic. Choosing $y \in F$ with $d(y, z) < \delta$ we have

$$d(m(\eta, x_0, y), x_0) > d(m(\eta, x_0, z), x_0) - \delta = d(z, x_0) - \delta = \delta.$$ 

Thus $U \subseteq B_{2\delta}(x_0) \subseteq X$ and $x_0 \in U$. Since $x_0$ and $\delta$ were arbitrary, this shows that the map $\iota$ is open. \qed

Note that connectedness cannot be dropped from the statement of Proposition 4.20. For instance, let $T$ be the tree obtained by glueing countably many rays $\{r_n \mid n \in \mathbb{N}\}$ by their origins. Consider the closed subset $X \subseteq T$ obtained by removing the open interval $(0, n) \subseteq (0, +\infty)$ from the ray $r_n$, for each $n \geq 0$. The median space $X$ is proper, but the inclusion $\iota : X \rightarrow \overline{X}$ does not have open image; indeed, in $\overline{X}$ the endpoints at infinity of the rays $r_n$ converge to their common origin.

We conclude this section by presenting one more characterisation of $\overline{X}$. Fixing $x_0 \in X$, we denote by $C_{\text{Lip}}(X)_{x_0}$ the set of 1-Lipschitz functions $X \rightarrow \mathbb{R}$ taking $x_0$ to 0; we endow this space with the topology of pointwise convergence, which is compact and coincides with the topology of uniform convergence on compact subsets. The map

$$B_{x_0} : X \hookrightarrow C_{\text{Lip}}(X)_{x_0} \quad x \mapsto d(x, \cdot) - d(x, x_0);$$

is continuous and it is customary to refer to $B_{x_0}(X)$ as the horofunction compactification (or Busemann compactification) of $X$; indeed, it does not depend on the basepoint $x_0$. The following is an extension of an unpublished result of U. Bader and D. Guralnik in the case of CAT(0) cube complexes (see e.g. the appendix to [CL11]).

Proposition 4.21. The identity map of $X$ extends to a homeomorphism between its Roller and Busemann compactifications.

Proof. Since $B_{x_0}(x)[z] = d(z, m(z, x, x_0)) - d(x_0, m(z, x, x_0))$, we can construct an extension of $B_{x_0}$ taking values in the space $\mathbb{R}^X_{x_0}$ of functions $X \rightarrow \mathbb{R}$ taking $x_0$ to 0:

$$\tilde{B}_{x_0} : \overline{X} \rightarrow \mathbb{R}^X_{x_0} \quad \xi \mapsto d(\cdot, m(\cdot, \xi, x_0)) - d(x_0, m(\cdot, \xi, x_0)).$$

This is well-defined due to the convexity of $X \subseteq \overline{X}$. If $\xi, \eta \in \overline{X}$ and $\xi \neq \eta$, there exists $h \in \mathcal{H}$ with $\tilde{h} \in \mathcal{H}(\xi|\eta)$. Without loss of generality,
we can assume that $x_0 \in \mathcal{h}^*$. Pick a point $x \in \mathcal{h}$ and set $u := m(x_0, x, \eta)$, $v := m(x_0, u, \xi)$; since $\mathcal{h} \in \mathcal{M}(v|u)$, we have $u \neq v$. In particular,
$$\tilde{B}_{x_0}(\xi)[u] = d(u, v) - d(x_0, v) > -d(x_0, v) = \tilde{B}_{x_0}(\eta)[u].$$
This shows that $\tilde{B}_{x_0}$ is injective; we now prove that it is continuous for the topology of pointwise convergence. Given $\xi \in X$, $\x \in X$ and $\epsilon > 0$, there exists a neighbourhood $U$ of $\xi$ such that, for every $\eta \in U$, the projections of $\xi$ and $\eta$ to $I(x_0, \x)$ are at distance smaller than $\epsilon/2$. In particular, for $\eta \in U$,
$$|\tilde{B}_{x_0}(\xi)[x] - \tilde{B}_{x_0}(\eta)[x]| \leq 2 \cdot d(m(x, \xi, x_0), m(x, \eta, x_0)) < \epsilon.$$ 
Continuity of $\tilde{B}_{x_0}$ and Lemma \ref{lemma:injectivity} imply that $\tilde{B}_{x_0}(X)$ coincides with the Busemann compactification. Finally, since $X$ is compact, $\tilde{B}_{x_0}$ is a closed map, hence a homeomorphism.

As a consequence of the proof of Proposition \ref{proposition:injectivity}, we can define 1-Lipschitz Busemann functions for points in the Roller boundary.

Corollary 4.22. For every $\xi \in X$ and $x_0 \in X$, the function $X \to \mathbb{R}$ defined by
$$z \mapsto d(z, m(z, \xi, x_0)) - d(x_0, m(z, \xi, x_0))$$
is 1-Lipschitz.

4.3. Components of the Roller boundary. Let $X$ be a complete, locally convex median space with compact intervals. In this section, we study the structure of the median spaces arising as components of $\partial X$. Our first goal is to obtain the following.

Proposition 4.23. Components of $\overline{X}$ are complete.

To do so, we need to relate the extended metric on $\overline{X}$ to its restriction to the intervals of $X$.

Proposition 4.24. For every $\xi, \eta \in \overline{X}$, we have
$$d(\xi, \eta) = \sup_{I \in \mathcal{I}(X)} d(\pi_I(\xi), \pi_I(\eta)).$$

Proof. The inequality $\geq$ follows from Lemma \ref{lemma:extension}. Given $\epsilon > 0$, we will produce an interval $I \subseteq X$ with $d(\pi_I(\xi), \pi_I(\eta)) \geq d(\xi, \eta) - \epsilon$. By the definition of $\tilde{\nu}$, there exist points $x_1, ..., x_n, y_1, ..., y_n \subseteq X$ such that $\mathcal{M}(x_i|y_i)$ are pairwise-disjoint and
$$\sum_{k=1}^{n} \tilde{\nu}((\sigma_{\xi} \setminus \sigma_{\eta}) \cap \mathcal{M}(x_i|y_i)) \geq d(\xi, \eta) - \epsilon.$$
Suppose that $n$ is minimal among the integers for which such an inequality holds; we will show that $n = 1$, which will conclude the proof. Suppose for the sake of contradiction that $n \geq 2$ and set $u := m(\eta, x_1, x_2), \ldots, x_k).$
$v := m(\xi, y_1, y_2)$. Observe that

$$(\sigma_\xi \setminus \sigma_\eta) \cap (\mathcal{H}(x_1|y_1) \cup \mathcal{H}(x_2|y_2)) \subseteq (\sigma_\xi \setminus \sigma_\eta) \cap \mathcal{H}(u|v),$$

which, applying $\tilde{v}$, violates the minimality of $n$. \qed

**Proof of Proposition 4.23** Let $(\xi_n)_{n \geq 0}$ be a Cauchy sequence in a component of the Roller boundary. By Lemma 4.17, the sequence $\pi_I(\xi_n)$ is also Cauchy for every $I \in \mathcal{F}$ and it has a limit $\xi_I \in I$. These points define a point $\xi := (\xi_I)_I \in \lim I$. By Proposition 4.24,

$$d(\xi, \xi_n) = \sup_{I \in \mathcal{F}} d(\xi_I, \pi_I(\xi_n)) = \sup_{I \in \mathcal{F}} \lim_{m \to +\infty} d(\pi_I(\xi_m), \pi_I(\xi_n)) \leq \sup_{I \in \mathcal{F}} \lim_{m \to +\infty} \sup_{n \to } d(\xi_m, \xi_n) = \lim_{m \to +\infty} d(\xi_m, \xi_n)$$

and the latter converges to zero as $n$ goes to infinity. \qed

**Proposition 4.25.** Components of $\overline{X}$ have compact intervals.

**Proof.** Given points $\xi, \eta$ in the component $Z \subseteq \overline{X}$, let $\pi_n$ be the gate-projection to an interval $I_n \subseteq X$ with $\pi_n((\sigma_\xi \setminus \sigma_\eta) \cap \mathcal{H}(I_n))) \leq \frac{1}{n}$; these exist for every $n \geq 1$, by Proposition 4.24. Since $X$ has compact intervals, every sequence in $I(\xi, \eta)$ has a subsequence $(\xi_k)_{k \geq 0}$ with the property that $(\pi_n(\xi_k))_{k \geq 0}$ converges for every $n \geq 1$. For all $k, h, n \geq 1$, we have $d(\pi_n(\xi_k), \pi_n(\xi_h)) \geq d(\xi_k, \xi_h) - \frac{1}{n}$; thus, $(\xi_k)_{k \geq 0}$ is Cauchy and compactness of $I(\xi, \eta)$ follows from Proposition 4.23. \qed

The following should better justify the terminology introduced in Definition 4.18.

**Proposition 4.26.** If $X$ is connected, each component $Z \subseteq \overline{X}$ is connected.

Note however that $Z$ is not a connected component of $\overline{X}$ as the latter is connected, being the closure of $X$. 

**Proof.** By Lemma 3.5 and Proposition 4.23, it suffices to prove that no halfspace of $Z$ is an atom. Suppose for the sake of contradiction that there exists $t \in \mathcal{H}(Z)$ with $d(t, t^*) > 0$; let $(\xi, \eta)$ be a pair of gates for $(t, t^*)$, as provided by Lemma 2.4. Since $Z$ is convex in $\overline{X}$, the interval between $\xi$ and $\eta$ in $\overline{X}$ consists of the sole points $\xi$ and $\eta$. Let $\pi : \overline{X} \to I(\xi, \eta) = \{\xi, \eta\}$ be the corresponding gate-projection; since $\pi$ is 1-Lipschitz and $X$ is connected, we must have either $\pi(X) = \{\xi\}$ or $\pi(X) = \{\eta\}$. However, since $\xi \neq \eta$, there exists $h \in \mathcal{H}$ such that $\tilde{h} \in \mathcal{H}(\xi|\eta)$; hence $\pi(h^*) = \{\xi\}$ and $\pi(h) = \{\eta\}$, a contradiction. \qed

Observe that, if $Z$ is a component of the Roller boundary and $h \in \mathcal{H}$ is such that $\tilde{h} \cap Z$ and $\tilde{h}^* \cap Z$ are both nonempty, then they are halfspaces for the median-space structure of $Z$. The corresponding walls of $Z$ are enough to separate points in $Z$. However, not all walls of the median space $Z$ arise this way, essentially due to the fact that not every wall of the median algebra...
\(\mathcal{X}\) arises from a wall of \(X\). Still, almost every wall of \(Z\) comes from the above construction; see Proposition \ref{prop:wall-from-walls} below.

**Lemma 4.27.** Let \(Z\) be a component of the Roller boundary. The sets \(\sigma_Z := \{ h \in \mathcal{H} \mid Z \subseteq h \}\) and \(\mathcal{H}_Z := \mathcal{H} \setminus (\sigma_Z \cup \sigma_Z^*)\) are morally measurable.

**Proof.** By part 1 of Lemma \ref{lem:gamma}, we have \(\sigma_Z \cap \mathcal{H}(I) = \sigma_{\pi_I(Z)} \cap \mathcal{H}(I)\) for every interval \(I \subseteq X\) and \(\pi_I(Z)\) is convex. The statement now follows from Lemma \ref{lem:convexity}.

Lemma \ref{lem:gate-projection} allows us to define gate-projections to boundary components. Namely, if \(Z\) is a component of the Roller boundary, we have a morally measurable decomposition \(\mathcal{H} = \mathcal{H}_Z \cup \sigma_Z \cup \sigma_Z^*\) and we can consider the map \(\text{res}_Z : 2^\mathcal{H} \to 2^\mathcal{H}\) that takes \(E \subseteq \mathcal{H}\) to \((E \cap \mathcal{H}_Z) \cup \sigma_Z\). Using Lemma \ref{lem:gate-projection}, it is immediate to observe that \(\text{res}_Z\) sends morally measurable ultrafilters to morally measurable ultrafilters and hence induces a map \(\pi_Z : \mathcal{X} \to \mathcal{X}\).

**Proposition 4.28.** The map \(\pi_Z\) is the gate-projection to the closure of \(Z\) in \(\mathcal{X}\); this is canonically identified with the Roller compactification of \(Z\) and will be denoted unambiguously by \(Z \subseteq \mathcal{X}\).

**Proof.** By Proposition \ref{prop:gate-projection} the map \(\pi_Z\) is a gate-projection to some gate-convex set \(C \subseteq \mathcal{X}\). To avoid confusion, we denote by \(W\) the closure of \(Z\) in \(\mathcal{X}\). Given \(\xi \in \mathcal{X}\), we have \(\pi_Z(\xi) = \xi\) if and only if \(\tilde{\nu}(\sigma_Z \setminus \sigma_\xi) = 0\); hence, \(\pi_Z\) is the identity on \(Z\) and, by part 1 of Lemma \ref{lem:gamma}, it follows that \(W \subseteq C\).

Suppose for the sake of contradiction that there exists \(\xi \in C \setminus W\). By Lemma \ref{lem:gate-convex}, \(W\) is gate-convex, so there exists a gate \(\eta\) for \((\xi, W)\). Since \(\xi \neq \eta\), we have \(\pi_I(\xi) \neq \pi_I(\eta)\) for some interval \(I \subseteq X\); every \(h \in \mathcal{H}\) such that \(h \in \mathcal{H}(\pi_I(\xi)) \cap \pi_I(\eta)\) satisfies \(h \in \mathcal{H}(\xi \mid \eta) = \mathcal{H}(\xi \mid W)\), hence \(h \in \sigma_Z \setminus \sigma_\xi\). This implies that \(\tilde{\nu}(\sigma_Z \setminus \sigma_\xi) > 0\), contradicting the fact that \(\xi \in C\).

We are left to identify \(W\) with the Roller compactification \(Z\). We have a continuous morphism \(f : X \to Z\) mapping each \(\xi \in \mathcal{X}\) to \((\pi_J(\xi))_{J \in \mathcal{J}(Z)}\). Since \(f(Z) = Z\), Lemma \ref{lem:closed-compact} and Proposition \ref{prop:surjective} imply that \(f\) is surjective. Moreover, \(f(\xi) = f(\eta)\) if and only if no \(h \in \mathcal{H}_Z\) satisfies \(h \in \mathcal{H}(\xi | \eta)\). If \(\xi, \eta\) are distinct and lie in \(W\), we have \(0 < d(\xi, \eta) = \tilde{\nu}((\sigma_\xi \setminus \sigma_\eta) \cap \mathcal{H}_Z)\). Thus, the restriction of \(f\) to \(W\) is an isomorphism.

In the rest of the section we will have to assume in addition that \(X\) has finite rank; the necessity of this will be discussed below.

**Proposition 4.29.** Suppose \(X\) is a complete, finite rank median space and let \(Z\) be a component of \(\partial X\). Then:

1. we have \(\pi_Z(X) \subseteq Z\);
2. every thick halfspace of \(Z\) is of the form \(\tilde{h} \cap Z\) for a unique \(h \in \mathcal{H}\);
3. we have \(\text{rank}(Z) \leq \text{rank}(X) - 1\).
Proof. Given \( x \in X \) and \( \xi \in Z \), let \( \{ h_1, \ldots, h_k \} \) be a maximal set of pairwise-transverse halfspaces in \((\sigma_\xi \setminus \sigma_x) \cap \mathcal{H}_Z\). We have:

\[
\begin{align*}
    d(\xi, \pi_Z(x)) &= \tilde{\nu}((\sigma_\xi \setminus \sigma_x) \cap \mathcal{H}_Z) \\
    &\leq \sum_{i=1}^{k} \tilde{\nu}(\mathcal{H}(x|h_i)) + \sum_{i=1}^{k} \tilde{\nu}(\mathcal{H}(h_i^*|\xi)) \\
    &\leq \sum_{i=1}^{k} d(x, h_i) + \sum_{i=1}^{k} d(\xi, h_i^* \cap Z) < +\infty.
\end{align*}
\]

We now prove part 2. Given the partition \( Z = \xi \cup \xi^* \) associated to a halfspace of \( Z \), we obtain a partition of \( X \) into the convex subsets \( \pi_Z^{-1}(\xi) \) and \( \pi_Z^{-1}(\xi^*) \). These are halfspaces of \( X \), unless \( \pi_Z(X) \subseteq \xi \) or \( \pi_Z(X) \subseteq \xi^* \). We show that this cannot happen if \( \xi \) is thick. Pick a point \( \xi \in \xi^* \) with \( d(\xi, \xi) > 0 \); let \( \eta \) be the gate for \((\xi, \xi^*)\). Since \( \xi \neq \eta \), there exists a halfspace \( h \in \mathcal{H} \) with \( h \in \mathcal{H}(\xi|\eta) \subseteq \mathcal{H}(\xi|\xi^*) \). Thus, \( \pi_Z(x) \in \xi^* \) for every \( x \in h^* \); in particular, \( \pi_Z(X) \not\subseteq \xi \). A symmetric argument shows that \( \pi_Z(X) \not\subseteq \xi^* \).

Finally, we prove part 3. Suppose for the sake of contradiction that \( \text{rank}(Z) \geq r = \text{rank}(X) \). We have already observed that the halfspaces \( h \cap Z \) with \( h \in \mathcal{H}_Z \) are enough to separate points of \( Z \). Lemma \ref{lemma:median} then shows that there exist \( h_1, \ldots, h_r \in \mathcal{H} \) such that \( \xi_i := h_i \cap Z \in \mathcal{H}(Z) \) are pairwise transverse; in particular, \( h_1, \ldots, h_r \) must be pairwise transverse. Pick \( x \in h_1^* \cap \ldots \cap h_r^* \) and \( \xi \in \xi_1 \cap \ldots \cap \xi_r \). Given two halfspaces \( h, \xi \in \sigma_\xi \setminus \sigma_x \subseteq \mathcal{H} \), either \( h \subseteq \xi \) or \( \xi \subseteq h \) or they are transverse; observe that \( \{ h_1, \ldots, h_r \} \subseteq \sigma_\xi \setminus \sigma_x \).

If \( h \in (\sigma_\xi \setminus \sigma_x) \cap \sigma_Z \), we cannot have \( h \subseteq h_i \) for any \( i \) since \( \sigma_Z \) is a filter and \( h_i \in \mathcal{H}_Z \). Moreover, since \((\sigma_\xi \setminus \sigma_x) \cap \mathcal{H}_Z \) has finite measure by part 1, the set \((\sigma_\xi \setminus \sigma_x) \cap \sigma_Z \) has infinite measure. Finally, the halfspaces \( h \in (\sigma_\xi \setminus \sigma_x) \cap \sigma_Z \) such that \( h \subseteq h_i \) for some \( i \) form a subset of finite measure, bounded above by the sum of the distances from \( x \) to each \( h_i \); we conclude that there exists \( h \in \sigma_\xi \setminus \sigma_x \) that is transverse to \( h_1, \ldots, h_r \), a contradiction. \( \square \)

Part 1 of Proposition \ref{prop:median} can fail without the finite rank assumption. Let \( X \) be the 0-skeleton of the \( \text{CAT}(0) \) cube complex whose vertex set is the restricted product \( \{0,1\}^{(\mathbb{N})} \) and whose edges join sequences with exactly one differing coordinate. Hyperplanes are in one-to-one correspondence with natural numbers and the Roller compactification \( \overline{X} \) can be identified with the unrestricted product \( \{0,1\}^\mathbb{N} \). Let \( \xi \in \overline{X} \) be the point whose coordinates are all \( 1 \); its component \( Z \) consists of sequences with only finitely many zeroes. It is immediate to observe that \( Z = X \); in particular, \( \pi_Z \) is the identity on all of \( \overline{X} \).

In general, even in finite rank, non-thick halfspaces of a component \( Z \subseteq \partial X \) need not be of the form \( h \) for some \( h \in \mathcal{H} \). Consider the median space \( X \) in Figure \ref{fig:median}. It is an infinite descending staircase with steps of constant height and exponentially-decreasing width; we consider \( X \) as a subset of \( \mathbb{R}^2 \) with the restriction of the \( \ell^1 \) metric. It is a complete median space of rank two.
Let $\xi \in \overline{X}$ be the point "at the bottom" of the staircase $X$ and let $Z \subseteq \overline{X}$ be its component of the Roller boundary. It is easy to notice that $\{\xi\}$ is a halfspace of $Z$, while, for every $h \in H_Z$, the set $\overline{h} \cap Z$ either does not contain $\xi$ or contains a neighbourhood of $\xi$ in $Z$.

**Proposition 4.30.** Let $X$ be a complete finite rank median space with distinct components $Z_1, Z_2 \subseteq \partial X$ satisfying $\text{rank}(Z_1) = \text{rank}(Z_2) = k$. There exists a component $W \subseteq X$ such that $\text{rank}(W) \geq k + 1$ and $W \cap I(\eta_1, \eta_2) \neq \emptyset$ for every $\eta_1 \in Z_1$ and $\eta_2 \in Z_2$.

**Proof.** As in the proof of Proposition 4.29, Lemma 2.5 yields pairwise-transverse halfspaces $h_1, \ldots, h_k \in \mathcal{H}_{Z_1}$. Suppose that $h_i \in \mathcal{H}_{Z_2}$ if and only if $i \leq s$, for some $0 \leq s \leq k$. Similarly, let $\xi_1, \ldots, \xi_k \in \mathcal{H}_{Z_2}$ be pairwise transverse, with $\xi_j \in \mathcal{H}_{Z_1}$ if and only if $j \leq t$, for some $0 \leq t \leq k$.

Up to replacing some of these halfspaces with their complements, we can assume that $h_i \cap \xi_j \neq \emptyset$ and $h_i^* \cap \xi_j^* \neq \emptyset$ for all $1 \leq i, j \leq k$ and, in addition, $\xi_j^* \cap Z_1 \neq \emptyset$ and $h_i \cap Z_2 \neq \emptyset$. This can be achieved as follows. We start by ensuring that $h_i \in \sigma_{Z_2}$ and $\xi_j \in \sigma_{Z_1}$ if $i > s$ and $j > t$. If $i \leq s$ and there exists $1 \leq j \leq k$ such that $h_i$ and $\xi_j$ are not transverse, we pick the side of the wall $\{h_i, h_i^*\}$ that intersects both $\xi_j$ and $\xi_j^*$; since the $h_i$ are pairwise transverse, they all determine the same side of $\{h_i, h_i^*\}$. Finally, we pick sides for the walls $\{\xi_j, \xi_j^*\}$ with $j \leq t$ in a similar way. Now, Helly's Theorem...
implies that there exist points
\[ \xi_1 \in \tilde{h}_1 \cap ... \cap \tilde{h}_k \cap \tilde{\xi}_1 \cap \ldots \cap \tilde{\xi}_k \cap Z_1, \]
\[ \xi_2 \in \tilde{h}_1 \cap ... \cap \tilde{h}_k \cap \tilde{\xi}_1 \cap \ldots \cap \tilde{\xi}_k \cap Z_2. \]

The set \( \sigma_{\xi_2} \setminus \sigma_{\xi_1} \) has infinite measure, since \( Z_1 \) and \( Z_2 \) are distinct. On the other hand, the sets \( \sigma_{\eta_i} \cap \sigma_{\xi_1}^*, \sigma_{\eta_i} \cap \sigma_{\xi_2}^* \) and \( \sigma_{\xi_2} \cap \sigma_{\xi_1}^* \) all have finite measure; indeed, \( d(\xi_1, h_i \cap Z_1) \), \( d(\tilde{h}_i, \tilde{\xi}_1 \cap Z_1) \) and \( d(\tilde{\xi}_2, \tilde{h}_j \cap Z_2) \) are finite. We conclude that there exists \( h \in \sigma_{\xi_2} \setminus \sigma_{\xi_1} \) not lying in any of these sets; in particular, \( h \) is either transverse to all the \( h_i \) or it is transverse to all the \( \tilde{h}_j \). Without loss of generality, let us assume that we are in the former case. By Helly’s Theorem, we can choose points \( x_1 \in h_1 \cap ... \cap h_k \cap h^* \) and \( x_2 \in h_1 \cap ... \cap h_k \cap h \); we set \( \xi'_1 := m(\xi_1, \xi_2, x_1) \), \( \xi'_2 := m(\xi_1, \xi_2, x_2) \). Observe that \( \xi'_1, \xi'_2 \) belong to the interval \( I(\xi_1, \xi_2) \) and, by Lemma 4.17, we have \( d(\xi'_1, \xi'_2) \leq d(x_1, x_2) < +\infty \).

In particular, \( \xi'_1 \) and \( \xi'_2 \) lie in the same component of \( X \), which we denote by \( W \). Since \( h_1, ..., h_k \) all separate \( x_1 \) and \( x_2 \) and they intersect \( I(\xi_1, \xi_2) \) nontrivially, they also separate \( \xi'_1 \) and \( \xi'_2 \). Hence, \( h_1, ..., h_k \) all lie in \( \mathcal{H}_W \) and \( \text{rank}(W) \geq k + 1 \).

Finally, \( I(\eta_1, \eta_2) \) intersects \( W \) for all \( \eta_i \in Z_i \). Indeed, projecting \( \xi'_1 \) to \( I(\eta_1, \eta_2) \) we only move it by a finite amount:
\[
d(\xi'_1, m(\eta_1, \eta_2, \xi'_1)) = d(m(\xi_1, \xi_2, \xi'_1), m(\eta_1, \eta_2, \xi'_1)) \leq d(\xi_1, \eta_1) + d(\xi_2, \eta_2) < +\infty.
\]

\[ \square \]

**Corollary 4.31.** Every convex subset \( C \subseteq X \) intersects a unique component of \( X \) of maximal rank.

**Proof.** Suppose \( C \) intersects two distinct components \( Z_1, Z_2 \) of \( X \) of maximal rank. Given \( \eta_1 \in C \cap Z_1 \) and \( \eta_2 \in C \cap Z_2 \), we have \( I(\eta_1, \eta_2) \subseteq C \) and this interval intersects a component of \( X \) of strictly higher rank by Proposition 4.30; a contradiction. \[ \square \]

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