Waves in the Skyrme–Faddeev model and integrable reductions

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Abstract
We show that the Skyrme–Faddeev model can be reduced in different ways to completely integrable sectors; the corresponding classes of solutions can be parametrized by specific sets of arbitrary functions. Moreover, using the ansatz of a phase and pseudo-phase reduction, the corresponding ordinary nonlinear wave solutions can be integrated in terms of elliptic functions, leading to periodic solutions. The Whitham averaging method has been exploited in order to describe a slow deformation of periodic wave states, leading to a Hamiltonian system, the integrability of which has been studied.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The main aim of this paper is to obtain the reductions of the so-called Skyrme–Faddeev model, in order to study its integrability properties and determine the existence of periodic solutions. In recent years, the 3D static nonlinear Skyrme–Faddeev $\sigma$-model for the field $\phi : \mathbb{R}^3 \rightarrow S^2$ has attracted special interest, the total free energy of which is given by

$$S_{\text{SF}} = \int d^3x \left[ \frac{1}{4} \rho^2 (\partial_k \phi)^2 + H_k \right], \quad \{H_k = \phi \cdot [\partial_k \phi \times \partial_k \phi], \quad \phi \cdot \phi = 1, \right. \tag{1.1}$$

where $\rho$ is a real positive constant and $H_k$ is an antisymmetric tensor field, the so-called Mermin–Hov vorticity [1–3] in matter physics, expressing the non-irrotational properties of the multi-component superfluid. On the other hand, after a lot of work [4, 5], it was proved that (1.1) can be seen as a special subcase, among many others [6], for the background (classical) field of the quantum pure $SU(2)$ Yang–Mills theory in the infra limit, thus describing a self-consistent ‘mesonic’ field in the context of nuclear physics. Thus, in most studies, the main
interest is to look for localized finite energy solutions, for which one imposes a constant value at spatial infinity, for instance \( \lim_{|x| \to \infty} \phi = (0, 0, 1) \), compactifying to \( S^3 \), the space domain. From the result \( \pi_3(S^2) = \mathbb{Z} \) [7] of the homotopy group theory, one can conclude that all such solutions (hopfions) are separated into sectors labelled by the so-called Hopf index \( Q \). Since in expression (1.1) \( H_{\phi} \) is a closed 2-form on \( S^3 \), derivable from a 1-form \( \alpha_3 \), one has \( Q = \frac{1}{16\pi} \int d^3x \epsilon_{ijk} \partial_i \alpha_3 \partial_j \alpha_3 \) [8–11], expressing the linking number of the pre-images of two independent points taken on the target \( S^2 \). In particular, \( Q = 0 \) corresponds to spherical symmetric solutions, \( Q = 1 \) to toroidal-shaped vortices, but only approximated analytical solutions are known [12]. The existence of many times tangled hopfions has been confirmed by numerical studies [13–17], which have produced a comprehensive analysis of solitons with \( 1 \leq Q \leq 16 \). On the other hand, global analytical considerations [9–11, 18] have shown that the energy of such a knot is bounded from below by \( S_{SF} \geq C \pi^2 \rho |Q|^{3/4} \), where \( C \) is a constant evaluated numerically [17]. The main consequence of the above bound is that hopfions of higher topological charge can be broken only by adding an extra energy contribution for their disentanglement. However, space extended structures were also considered in [19] and [20]. Later it was shown that collections of localized objects may condense in order to form periodic structures in the space [21]. Moreover, as pointed out in [19, 21] the appearance of extended multi-sheeted structures may be energetically more favourable. Thus, the quest for periodic solutions (possibly exact) for the Skyrme–Faddeev model becomes more compelling. On the other hand, in a series of papers [22], it was shown that one may obtain completely integrable sub-systems by adding certain differential constraints to the Lorentz invariant version of (1.1). The integrability corresponds to the existence of infinitely many local conservation laws for the sub-system. We tried different approaches, which led us to identify: (i) the reduction of the system to the well-known d’Alembert–eikonal system [25], assuming the dependence of the field variables only on a single phase and, more generally, a reduced Skyrme–Faddeev system, the solutions of which are parametrized by the arbitrary functions of a single variable (section 2); (ii) explicit periodic solutions in terms of elliptic integrals, limiting the dependence to phase and pseudo-phase variables (section 3); (iii) evolution equations for adiabatically changing parameters of the periodic solutions, by exploiting the averaging Whitham’s method [30] (section 4). Final considerations and an outlook of further researches on the topic are discussed in the conclusions.

2. The Skyrme–Faddeev model and the d’Alembert–eikonal system

First, we consider the four-dimensional relativistic (with the metric tensor \( g_{\mu\nu} = \text{diag}(+, -,-,-) \)) Skyrme–Faddeev model [10] given by the Lagrangian density

\[
\mathcal{L} = \frac{1}{32\pi^2} \left( \partial_\mu \phi \cdot \partial^\mu \phi - \frac{\lambda}{4} (\partial_\mu \phi \times \partial_\nu \phi) \cdot (\partial^\mu \phi \times \partial^\nu \phi) \right) - \kappa (1 - \phi \cdot \phi),
\]

(2.1)

where \( \lambda = \frac{4\rho}{\rho} > 0 \) is a scaling parameter, determining the breaking of the conformal symmetry and \( \kappa \) is a Lagrangian multiplier, implementing the constraint \( \phi \in S^2 \). Thus, the point (and also variational) symmetry group is \( (\mathbb{R}^4 \ltimes SO(3, 1)) \otimes SO(3) \), given by four translations, six boosts/rotations and three gauge transformations [12]. The model (1.1) is obtained by setting \( \partial_0 \phi = 0 \). The geometric constraint \( \phi \cdot \phi = 1 \) can be realized in several ways, but here it seems useful to introduce the polar representation

\[
\phi = (\sin w \cos u, \sin w \sin u, \cos w),
\]

(2.2)
where \( w \) and \( u \) are suitable functions of the variables \((x^0, \ldots, x^3)\) to be determined. The Lagrangian (2.1) becomes
\[
\mathcal{L}_p = \frac{1}{32\pi^2} \left\{ w_{\mu} u^{\mu} + \sin^2 w \left[ u_{\mu} u^{\mu} - \frac{\lambda}{2} (w_{\mu} u^{\mu} u_{\nu} u^{\nu} - w_{\mu} u_{\nu} u^{\mu} u^{\nu}) \right] \right\},
\]
(2.3)
and the Euler–Lagrange equations read
\[
\begin{align*}
\partial_\mu u^{\mu} &= \frac{1}{2} \sin(2w) u_{\nu} u^{\nu} + \frac{\lambda}{2} \sin w u_\nu \partial_\mu [\sin w (u^{\mu} u^{\nu} - u^{\nu} u^{\mu})], \\
w_{\mu} u^{\mu} \sin(2w) + \sin^2 w \left[ \partial_\mu u^{\mu} + \frac{\lambda}{2} w_\nu \partial_\mu (u^{\mu} u^{\nu} - u^{\nu} u^{\mu}) \right] &= 0.
\end{align*}
\]
(2.4)
They contain the field variable \( w \) itself, the first \( w_\nu \) and its second derivatives \( \partial_\mu w_\nu \). Differently, for \( u \) only the first derivatives \( u_\nu \) and the second derivatives \( \partial_\mu u_\nu \) are involved. The first equation can be interpreted as a quadratic constraint among the first and second derivatives of the function \( u \), while the second equation is linear with respect to the first and second derivatives of the function \( u \).

The first observation comes from the assumption \( w = \text{const} \); thus, the above system drastically reduces to the system given by the d’Alembert and the homogeneous eikonal equations:
\[
\partial_\mu u^{\mu} = 0, \quad u_\nu u^{\nu} = 0,
\]
(2.5)
which was investigated in many papers [23–25] and for which the general solution was given in the implicit form
\[
G(u, A_\mu(u)x^{\mu}, B_\mu(u)x^{\mu}) = 0, \quad A_\mu A^{\mu} = B_\mu B^{\mu} = A_\mu B^{\mu} = 0,
\]
(2.6)
with \( G \), \( A_\mu \), and \( B_\mu \) being arbitrary real regular functions. The process to provide an explicit form for \( u \) may lead to multi-valued solutions, the same as for shock waves. The corresponding differentiability singularity in the \( \phi \) field describes a type of a domain wall. Let us note here that the eikonal condition in (2.5) recalls the reducing condition introduced in [22]. Similar considerations can be made in the following reductions we consider.

A different simple reduction can be obtained by assuming \( u = c_v x^{\nu} \) as an ansatz for the system (2.4). This leads to the overdetermined system for \( w \):
\[
\begin{align*}
\left[ 1 - \frac{\lambda}{2} c_v c^v \sin^2 w \right] \partial_\nu w^{\nu} &= \frac{1}{2} \sin(2w) \left[ c_v c^v + \frac{\lambda}{2} (c_v c_\nu w_\mu - c_v w^{\nu} c^{\mu} w_\mu) \right] \\
- \frac{\lambda}{2} \sin^2 w c^\nu c_v \partial_\nu w^{\nu} &= 0.
\end{align*}
\]
(2.7)
\( c^\nu w_\mu \cot w + \frac{\lambda}{4} w_\nu (c^\nu \partial_\mu w^{\nu} - c^\nu \partial_\mu w^{\nu}) = 0. \)

The simplest way to solve the latter system is to restrict ourselves to the special one-phase ansatz [26] \( w = w(\theta) \), where \( \theta = b_v x^{\nu} \) with \( b_v \) being constants, obtaining the ODE
\[
\begin{align*}
b_{\mu} b^{\nu} \left[ 1 - \frac{\lambda}{2} c_v c^v \sin^2 w \right] w''(\theta) &= \frac{c_v c^v}{2} \sin(2w) \left[ 1 + \frac{\lambda}{2} b_\mu b^{\nu} w''(\theta) \right], \\
\end{align*}
\]
(2.7)
Usual methods lead to the solution in terms of elliptic integrals [27]
\[
\begin{align*}
\theta - \theta_0 &= - \frac{\lambda}{2} b^\mu b_\mu \left[ \left( \frac{\lambda}{2} c_v c_\mu - 1 \right) F(\tan^{-1}(a \tan(w))) |m \right. \\
&\left. + i \frac{\lambda}{2} c^\mu c_\mu \Pi \left( n; i \sinh^{-1}(a \tan(w)) \right) - n \left( \frac{\lambda}{2} c_v c^v - 1 \right) \right],
\end{align*}
\]
(2.8)
where $C$ and $\theta_0$ are arbitrary constants and $a = \sqrt{\frac{C^2 + c^2}{C - 1}} + 1 \in \mathbb{R}$, $m = \frac{C^2 + c^2}{C - 1} n = \frac{C - 1}{C + 2e^c n^{-1}}$.

One can weaken the previous constraint on the phases presented above, by imposing

$$w_\mu u^\mu = 0, \quad u_\nu u^\nu = \alpha \quad (\alpha = \text{constant} \in \mathbb{R}).$$

Then, the system (2.4) reduces to the equations

$$\partial_\mu u^\mu = 0, \quad u_\nu u^\nu = \alpha,$$  

$$w_\mu u^\mu = 0, \quad \partial_\mu u^\mu = \frac{\alpha}{2} \sin(2w) \left(1 + \frac{\lambda}{2} w^\mu w_\mu\right),$$

which are highly nonlinear for the $w$ field. The general solution of the d’Alembert–eikonal system (2.9) is given in implicit form by [25]

$$u = A_\mu(\tau) x^\mu + R_1(\tau), \quad B_\mu(\tau) x^\mu + R_2(\tau) = 0,$$  

$$A_\mu A^\mu = \alpha, \quad B_\mu B^\mu = A'_\mu A'^\mu = B_\mu B'^\mu = 0,$$

where the function $\tau$ is implicitly defined by the second relation, the real functions $A_\mu$ and $B_\mu$ satisfy the constraints in the second line and second-order differentiable and the function $R_1$ is differentiable at least up to second order, the same for $R_2$ together with its inverse function. Then, for $\alpha = -\eta^2$, it is useful to reparametrize the solution in the form

$$u = x_\lambda A_\xi(\tau) + A_0(\tau), \quad t = x_\xi B_\lambda(\tau) + B_0(\tau),$$  

$$A_1 = \eta \cos(f(\tau)) \sin(g(\tau)), \quad A_2 = \eta \sin(f(\tau)) \sin(g(\tau)), \quad A_3 = \eta \cos(g(\tau)),$$

with $f(\tau)$ and $g(\tau)$ being arbitrary functions. Consequently, the functions $B_\xi$ are completely determined, because they form a normalized 3-vector simultaneously orthogonal to $A_\lambda$ and $A'_\xi$. The derivatives $B'_\xi$ form a 3-vector parallel to $A'_\xi$.

A different reduction can be obtained by setting the coefficients of all functions of $w$ to zero in equations (2.4), leading to the equations

$$\partial_\mu u^\mu = 0, \quad \partial_\mu u^\mu + \frac{\lambda}{2} w_\nu \partial_\mu (u^\nu w^\mu - u^\mu u^\nu) = 0, \quad u_\nu \partial_\mu (u^\mu u_\nu - u^\nu u^\mu) = 0$$

and constraints

$$w_\mu u^\mu = 0, \quad u_\nu u^\nu \left(1 + \frac{\lambda}{2} w_\mu w^\mu\right) = 0.$$  

(2.13)

The last relationship in (2.13) contains two admissible choices

$$w_\mu u^\mu = 0, \quad u_\nu u^\nu = -\frac{2}{\lambda}. $$  

(2.14)

Since the Skyrme–Faddeev system is characterized by $\lambda \neq 0$, we are mainly interested in the second one. Thus, we will consider now the reduced Skyrme–Faddeev system which is the set of equations

$$\partial_\mu u^\mu = 0, \quad w_\mu w^\mu = -\epsilon^2, \quad u_\mu u^\mu = 0,$$  

$$u_\nu \partial_\mu (u^\mu u_\nu - u^\nu u^\mu) = 0, \quad \epsilon^2 \partial_\mu u^\mu + u_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu) = 0,$$

(2.15)

(2.16)

where $\epsilon^2 = \frac{2}{7}$ for notational clarity. The first two equations in (2.15) are the d’Alembert–eikonal system (2.9), by replacing $u \rightarrow w$ and $\alpha \rightarrow -\epsilon^2$, whose general solution was shown above by (2.11). The third one can be interpreted as an orthogonality condition among the gradients of the two fields. In (2.16), the first equation is a quadratic differential constraint among the derivatives of the function $u$. The second one is a linear PDE for $u$, with variable
coefficients depending on \( w \). However, one can prove that it is just an identity, by using the set of equations (2.15) and the differentiability of \( w \). In conclusion, only the first four equations of (2.15) and (2.16) are independent and form an overdetermined system.

Moreover, let us observe that the first equation in (2.16) can be rewritten as two divergence terms (namely \( \partial_\nu (u_\nu u^\mu u^\nu) \) and \( \partial_\nu (u_\nu w^\mu u^\nu) \)) balanced by a contribution of the form \( (w^\mu u^\nu - w^\nu u^\mu) \partial_\nu w_\mu \). However, the second divergence is vanishing, because of the orthogonality condition in (2.15), while the balancing contribution is the trace of the product of an antisymmetric tensor with a symmetric one, then 0. Finally, recalling that \( w \) satisfies the d’Alembert equation, the equation we considered can be concisely written as

\[
a_{\mu} u_\mu = 0 \quad \text{with} \quad a = u^i u_i, \tag{2.17}
\]

in complete analogy with the last equation in (2.15). Summarizing, the overdetermined system (2.15)–(2.17) describes completely the reduced Skyrme–Faddeev system. Cross differentiation of those equations and systematic substitution of the \( x^\mu\)-derivatives

\[
w_0 = \sqrt{u_m^2 - \epsilon^2}, \quad u_0 = \frac{u_k u_k}{\sqrt{u_m^2 - \epsilon^2}}, \tag{2.18}
\]

lead to a set of compatibility conditions. For the d’Alembert–eikonal equations, it is well known [25] that such a set is finite and any further compatibility condition is satisfied only by the Monge–Ampère equation

\[
\text{Det}[w_{ij}] = 0, \tag{2.19}
\]

and no further relation can be found.

Then, the remaining part of the system involving the field \( u \) variable is given by the linear first-order PDE (2.15-iii) (or (2.18-ii)) and the nonlinear second-order equation in (2.16-i) with variable coefficients depending on \( w \). Their compatibility leads to the two quadratic constraints in \( u \):

\[
\left( u^2 - \epsilon^2 \right) u_m u_k w_{km} + (u_k w_k)^2 u_{mnm} = 2 u_k w_k u_m w_{km}, \tag{2.20}
\]

\[
4u_k w_k u_m w_{pm}(w_m w_{pm} - w_p w_{mm}) + 2(u_k w_k w_{km})^2

\quad + (u_k w_k)^2(w_{mm} w_{pp} - w_{pp}^2) = 2(u^2 - \epsilon^2)(u_k w_{km})^2. \tag{2.21}
\]

Now our aim is to express the above overdetermined system for \( u \) as a first-order linear system of the form

\[
u_0 = Au_1, \quad u_2 = Bu_1, \quad u_3 = Cu_1, \tag{2.22}
\]

where the functions \( A, B \) and \( C \) depend on the first- and second-order derivatives of the field variable \( w \) only. This formulation is analogous to looking for solutions of the Skyrme–Faddeev system (2.4) by the method of hydrodynamic reductions [28, 29], involving a finite number of Riemann invariants. For the sake of clarity, first let us show how such a kind of representation works in two space dimensions, when the relevant equations (2.15-iii) and (2.20) for \( u \) read

\[
u_0 = \frac{w_1}{w_0} u_1 + \frac{w_2}{w_0} u_2, \quad u_1^2 w_{22} + u_2^2 w_{11} = 2u_1 u_2 w_{12}. \tag{2.23}
\]

Then, the first relation above takes the form (2.22-i) if we postulate (2.22-ii). But it is a matter of a direct substitution into (2.23-ii) to determine the \( B \) function by solving a second-degree algebraic equation. Thus, one provides the equations of the form (2.22) by

\[
u_0 = \frac{w_{11} w_{11} + w_{12} w_{12}}{w_{01} w_{11}} u_1, \quad u_2 = \frac{w_{12}}{w_{11}} u_1. \tag{2.24}
\]
Moreover, it is easy to prove also that a solution of the system (2.24) is given by
\[ u = F[w_1, w_2], \]
where \( F \) is an arbitrary real differentiable function of its arguments. Moreover, by direct computation one can prove the vanishing of the quantity \( a \) defined in (2.17) and so the Skyrme–Faddeev system in two dimensions is fulfilled.

In three space dimensions, a similar analysis is much more difficult. In fact, postulating both (2.22-ii) and (2.22-iii) and substituting into (2.20) and (2.21) leads to an algebraic system of fourth degree. Thus, the functions \( A, B \) and \( C \) in (2.22) can be explicitly determined. However, their expressions are too long to be presented here. On the other hand, from the orthogonality condition in (2.15-iii) and by the use of equation (2.18), it is easy to prove that a class of solutions for \( u \) is given by
\[ u = F[w_1, w_2, w_3], \tag{2.25} \]
with \( F \) being an arbitrary differentiable real function. Thus, one has only to check the compatibility of the above result with the equations in (2.16). In the proof, we use the compatibility relations for (2.15), which are sufficient to annihilate all coefficients of the derivatives \( F_{w_i} \), if assumed as independent. This is not the case in 1 + 3 dimensions, where some more information is required. So, one obtains only a set of constraints on the derivatives of \( F \).

A deeper investigation of this case is based on a direct substitution of a general solution for \( w \) in the form (2.11)–(2.12) into the orthogonality condition (2.15-iii). This leads to solving a linear PDE for \( u \) of the form
\[ \left[ X_m \left( B_m'(\tau)A_p(\tau) - A_m'(\tau)B_p(\tau) \right) + B_0'(\tau)A_p(\tau) - A_0'(\tau)B_p(\tau) \right] u_{X_k} = 0, \tag{2.26} \]
having used the transformation
\[ (x_m B_m'(\tau) + B_0'(\tau)) \, d\tau = dt - B_k(\tau) \, dx_k, \quad X_k = x_k, \]
naturally induced by (2.12). Having in mind to solve it via the methods of characteristics, one is led to solve a 3 × 3 ODE system for the unknowns \( X_k(s) \), with \( s \) being an auxiliary variable. Such a system is linear with constant coefficients, depending only on \( \tau \). Moreover, the peculiar symmetry of the constraints (2.11)–(2.12) implies that the coefficients matrix is nilpotent of order 2. Thus, the solution is linear in \( s \), from which one can extract two integrals of motion, in terms of which the solution of (2.26) is expressed. To avoid the complexity of such a consideration in detail, we just mention that a general solution for the complete system of equations is determined by a sole function of a single variable only. One can prove that such a solution can be written without loss of generality just in the form \( u = F(w_i) \), where \( F(a) \) is an arbitrary function. A detailed analysis will be published in a separate paper.

3. Periodic solution

Now, it is well known [12] that the equations of motion given by (1.1) admit harmonic plane wave solutions of the form
\[ \phi = \left( \frac{2A \cos(p_\mu x^\mu)}{A^2 + 1}, \frac{2A \sin(p_\mu x^\mu)}{A^2 + 1}, \frac{1 - A^2}{A^2 + 1} \right), \tag{3.1} \]
where \( A \) parameterizes the amplitude of the third component and remarkably the dispersion law is given by \( p_\mu p^\mu = 0 \). Actually, by using the symmetry group of the model one can find
a 13-parametric family of solutions. In particular, the axis of precession can be arbitrarily fixed by gauge transformations. Nevertheless, they cannot be superimposed, because of the nonlinearity character of the equations of motion. To have a visualization of (3.1) one can think of a periodic circularly polarized unit vector field, whose wave fronts are orthogonal to the direction \( p = (p_1, p_2, p_3) \), maintaining the third projection \( \phi_3 \) constant. The configuration is similar to a cholesteric liquid crystal. Moreover, the energy density of the configuration is constant in the whole space, being equal to \( E = \frac{\Delta F}{4\pi T_1 A T_2} \). Note how solutions with a smaller third component are more energetic.

Looking for the simplest generalization of (3.1), one assumes that

\[
w = \Theta[\theta], \quad u = \Phi[\theta] + \hat{\theta}, \quad \text{where} \quad \theta = \alpha_{\mu} x^\mu, \quad \hat{\theta} = \beta_{\mu} x^\mu,
\]

in which one distinguishes \( \theta \) as the phase from the pseudo-phase \( \hat{\theta} \). From a different point of view, we are looking for invariant solutions under a eight-parametric family of two-dimensional Abelian sub-algebra of the translation symmetry group given by

\[
\left\{ v_x = \sum_{i=1}^{3} \alpha_i t_i - \alpha_0 \sum_{i=1}^{3} t_i, \quad v_\theta = \sum_{i=1}^{3} \beta_i t_i - \beta_0 \sum_{i=1}^{3} t_i \right\}, \tag{3.3}
\]

where \( t_i \) are the generators of the translations. Direct calculation shows that \( \theta \) and \( \hat{\theta} \) in (3.2) are invariants of the sub-algebra above. Actually, by using the adjoint action of the spacetime rotational subgroup, we can conjugate each of the above sub-algebras to exactly one representative sub-algebra of the form (3.3) belonging to a three-parametric sub-family. However, it is easier to deal with all components for notational homogeneity. Thus, the equations of motion reduce to the announced three-parametric family

\[
2B_3 - \frac{\lambda}{4} B \sin^2 \Theta \left[ \Theta_{\theta \theta} = \sin 2\Theta \left( \frac{\lambda}{8} B \Theta^2_\theta + B_3 \Phi^2_\theta + B_2 \Phi_\theta + B_1 \right) \right]
\]

\[
2B_3 \sin^2 \Theta \left[ \Phi_{\theta \theta} + \Theta_{\theta \theta} \sin 2\Theta \left( 2B_3 \Phi_\theta + B_2 \right) = 0 \right],
\]

where \( B_1 = -\beta_{\mu} \beta^\mu, B_2 = -2\alpha_{\mu} \beta^\mu, B_3 = -\alpha_{\mu} \alpha^\mu \) and \( B = B_2^2 - 4B_1 B_3 \).

Of course, these equations provide the above linear solutions (3.1), setting \( B_3 = 0 \) and \( B_2 \neq 0 \), and then \( \Theta = 2 \arctan A, \Phi = \Phi_0 - \frac{B}{B_2} \theta \) and \( p_i = \frac{\beta_i \beta_0 + \alpha_i \alpha_0}{B_2} \). On the other hand, for \( B_3 \neq 0 \), the solution is given by \( \Theta = 2 \arctan A \) and \( \Phi = \Phi_0 - \frac{B_1 + \sqrt{3}}{2B_2} \theta \), with an analogous expression for the \( p_i \).

To deal with the general situation, one uses the expression of the energy–stress tensor \( T^{\mu \nu} = (u^\mu \partial_{\nu} + u^\nu \partial_{\mu}) \mathcal{L}_P - g^{\mu \nu} \mathcal{L}_P \). Substituting the ansatz (2.2) into it, only derivatives with respect to \( \theta \) survive. Thus, the further vanishing divergence is equivalent to take \( \partial_\theta \) over a quantity obtained by contracting \( \mathcal{E}^{\mu} = T^{\mu \nu} a_\nu, \) corresponding to total conserved quantities for the wave, seen as a function of the phase. Their expressions are

\[
E^0 = -\frac{1}{32 \pi^2} \left\{ B_3 \alpha_0 \Theta^2_\theta + \sin^2 \Theta \left[ 2\alpha \cdot \beta \beta_0 + (B_1 - 2\beta^2) \alpha_0 \right. \right.
\]

\[
\left. + B_3 (2\beta_0 + \alpha_0 \Phi_\theta) \Phi_\theta - \frac{\lambda B}{8} \alpha_0 \Theta^2_\theta \right\}, \tag{3.6}
\]

\[
E^i = -\frac{1}{32 \pi^2} \left\{ B_3 \alpha_i \Theta^2_\theta + \sin^2 \Theta \left[ B_2 \beta_i - B_1 \alpha_i + B_3 (2\beta_i + \alpha_i \Phi_\theta) \Phi_\theta - \frac{\lambda B}{8} \alpha_i \Theta^2_\theta \right] \right\}.
\]

\(^5\) When the function \( \Phi[\theta] = 0 \), then one obtains (2.8).
These equations can be used to find expressions for $\Theta_\nu^2$ and $\Phi_\nu$. Precisely, assuming $B_3 \neq 0$, one finds
\begin{equation}
\Theta_\nu^2 = \frac{8B_3(B_1 \sin^2 \Theta + U_3) - 2B_3^2(\sin^2 \Theta + U_2^2 \csc^2 \Theta)}{B_3(8B_3 - \lambda B \sin^2 \Theta)} \quad (3.7)
\end{equation}
\begin{equation}
\Phi_\nu = -\frac{B_3(U_2 \csc^2 (\Theta + 1))}{2B_3} \quad (3.8)
\end{equation}
where the $U_i$ are two constants completely defining the quantities in (3.6) by the expression $E^\mu = U_3\alpha_\mu + \frac{B_i U_i^2}{2}(\beta_i \alpha_\mu - 2\beta_\mu)$. Thus, one has reduced the problem to the quadratures, introducing only two new integration constants besides $[U_2, U_3]$, which determine the amplitudes of the phases. Finally, similar conclusions can be obtained in the case $B_3 = 0$. Despite its involved expression, equation (3.7) can be set in an algebraic form by the transformation
\begin{equation}
\Theta = \arcsin \sqrt{\psi}, \quad (3.9)
\end{equation}
forcing $0 \leq \psi \leq 1$ and satisfying the equation
\begin{equation}
\psi^2_\nu = \frac{64(\psi - 1)(\psi - A_1)(\psi - A_2)}{\lambda^2 \beta \psi_1(\psi_1 - \psi)} \quad (3.10)
\end{equation}
where one has defined the constants $A_{1,2} = \frac{2B_1 U_1 + \sqrt{4B_3^2 \beta^2 - B_1 U_1^2}}{U_1^2}$, related 1–1 to the values of the integrals of motion and $\psi_1 = \frac{8U_2}{2\beta}$. Assuming $A_1$ to be real and setting $0 < A_1 < A_2 < 1$ for instance, by a continuous variation of $\psi_1$, one obtains different behaviour of oscillation amplitudes for $\psi$: $A_2 < \psi < 1$ for $\psi_1 > 0$, $A_1 < \psi < A_2$ for $0 < \psi_1 < A_2$, $A_1 < \psi < 1$ for $\psi_1 = A_2$. So for all choices of $\psi_1$, there is only one oscillating solution, bounded between two of the three zeros of the numerator in (3.10), even if a real or complex unbounded solution may appear (see figure 1).

Analytically, equation (3.10) can be integrated in terms of incomplete elliptic integrals of the third kind [27]. Precisely, by introducing a parametric variable $Z$, one obtains the parametric form
\begin{equation}
\theta(\psi) = \theta_0 + \frac{1}{4} \sqrt{\frac{B_2 \lambda^2}{(A_1 - 1)(A_2 - \psi_1)}} \left[ \frac{(A_1 - A_2)}{\psi_1 - A_2} \right] \left[ \frac{(A_1 - 1)(A_1 - A_2)}{(A_1 - 1)(\psi_1 - A_2)} \right], \quad (3.11)
\end{equation}
which can be expressed in terms of the Weierstrass $P$ function. Furthermore, from (3.8) the function $\Phi$ can be expressed again in terms of incomplete elliptic integrals, namely
\begin{equation}
\Phi = -\frac{B_3 U_2}{2B_3} \left[ \int \frac{d\theta}{\psi(\theta)} + \theta + \Phi_0 \right]
= -\frac{s_1}{2\psi_1} \left[ \sqrt{\frac{2\psi_1 (A_1 - \psi_1)^2 (B_1 \lambda \psi_1 + 2)}{(A_1 - 1)(A_2 - \psi_1)}} \left[ \frac{(A_2 - A_1)}{A_2 - \psi_1} \right] \left[ \frac{(A_1 - 1)(A_1 - A_2)}{(A_1 - 1)(\psi_1 - A_2)} \right] \right]
+ \frac{s_2}{2\psi_1} \left[ \frac{(\psi_1 + s_1)^2}{(A_1 - 1)(A_2 - \psi_1)} \left[ \frac{(A_1 - A_2)\psi_1}{A_1 \psi_1} \right] \left[ \frac{(A_1 - 1)(A_1 - A_2)}{(A_1 - 1)(\psi_1 - A_2)} \right] \right], \quad (3.12)
\end{equation}
where $s_1 = \text{sign } B_2$ and $s_2 = \text{sign } U_2$.

In this parametric form, it is evident that the phases of the spinorial field and the phase $\theta$ do not have the same periodicity (see figures 2 and 5). Thus, the solution is generically
Figure 1. Graphic for the inverse square root of (3.10) for the family of parameters $B = 1, A_1 = 0.1, A_2 = 0.8$ and $-0.45 \leq \psi_1 \leq 1.55$ with steps of 0.1. Colours run from red to violet. Only one bounded periodic solution exists for any set of parameters. The degenerate case $\psi_1 = A_2 = 0.8$ does not coincide, but it corresponds to the confluence of the two yellow curves.

Figure 2. Graphic for $\phi_1$ (green), $\phi_2$ (blue) and $\phi_3$ (red) as a function of $x^1$ for a choice of the parameters $A_1 = 0.2, A_2 = 0.8, \psi_1 = 0.9, B = 1, \lambda = 1, B_1 = 1, \lambda_1 = -1, \lambda_2 = -1$. Accordingly, the wave vectors for the phase and pseudo-phase have been chosen to be $\alpha_\mu = (0, 0, 0, 0.33541)$ and $\beta_\mu = (1.49638, 1, 0, -1.49638)$, respectively.

Quasiperiodic and only for very special choices of the parameters true periodic solutions appear (see figure 4). Then it is convenient to adopt, as we will show in the following section, a method which allows us to describe solutions with a minimal set of parameters, concerning only the periodicity in the phase. Here, note only that the length wave can be made very large when $A_2 \to 1$ and $\psi_1 \to \infty$. In this limit (figure 3), the solution can be expressed in terms of elementary hyperbolic functions. A similar situation occurs when $A_1 = A_2$; then, the elliptic module is 0 and the solutions are given in terms of trigonometric functions. This suggests to us, as we do in the following section, to consider slow deformations of periodic solutions.
Figure 3. Graphic for $\phi_1$ (green), $\phi_2$ (blue) and $\phi_3$ (red) for a choice of the parameters $A_1 = 0.2, A_2 = 0.99, \psi_1 = 20.01, B = 1, \lambda = 1, B_1 = 1, s_1 = -1, s_2 = -1$. The wave vectors are $\alpha_\mu = (0, 0, -1.58153)$ and $\beta_\mu = (-1.04879, 1, 0, 1.04879)$, respectively.

Figure 4. Projection on the plane $(\phi_1, \phi_2)$ of a sample of about 5000 consecutive values of the field $\phi$ along the axes $(0, 0, x^3)$, for the same choice of parameters as in figure 2.

Then, an important observation occurs: one has found a special two-dimensional reduction of the Skyrme–Faddeev model, which is completely integrable, and one may wonder if this is not in the class described in [22]. If one performs the stereographic projection $\phi \to z$ by

$$z = i \tan \left( \frac{u}{2} \right) \exp(-iu), \quad z^* = -i \tan \left( \frac{u}{2} \right) \exp(iu),$$

(3.13)

the constraint imposed by the authors in [22] is expressed by

$$\partial_\mu z \partial^\mu z = 0.$$  

(3.14)
One easily verifies that it is satisfied by the harmonic wave solution (3.1). On the other hand, if one replaces in (3.14) the reduction in (3.2), depending on the phase and pseudo-phase, one obtains the relation

$$\sin^2(\Theta)(\Phi_\theta(B_3\Phi_\theta + B_2) + B_1) - B_3\Theta^3_\theta + i\sin(\Theta)(2B_3\Phi_\theta + B_2)\Theta_\theta = 0$$  \hspace{1cm} (3.15)

from which one sees that the real and imaginary parts have to vanish separately. Replacing condition (3.8) into the last equation, one obtains

$$- B_3\Theta^3_\theta + B_1 \sin^2(\Theta) - iB_2U_2\Theta_\theta \csc(\Theta) - \frac{B_3^2(\sin^2(\Theta) - U_2^2 \csc^2(\Theta))}{4B_3} = 0,$$  \hspace{1cm} (3.16)

saying that, excluding constant $\Theta$ solutions, either $B_2$ or $U_2$ has to be zero. But compatibility with equation (3.7) implies the constancy of $\Theta$ in both cases. So, in conclusion, the solutions found above are out of the subsector described by the constraint (3.14).

In the following section, we average the periodic solutions of the Skyrme–Faddeev model by the Whitham approach.
4. The Whitham averaging method

The Whitham approach was developed for any multi-dimensional system possessing a Lagrangian formulation (see details in [30]). Nevertheless, only some years later, the first averaging on a multi-dimensional example, the well-known three-dimensional Kadomtsev–Petviashvili equation, was provided by Infeld [31]. Now, more than 30 years later, we present the second multi-dimensional example, i.e. we consider the averaging of the four-dimensional Skyrme–Faddeev model. The Whitham approach is nothing but a phenomenological version of the so-called nonlinear WKB method (Wentzel–Kramers–Brillouin) allowing us to investigate slow dynamics of parameters entering into families of periodic solutions. This approach in its linear version is well known in quantum mechanics like separation (in a quasi-classical limit) of motion into fast and slow dynamics. To describe slow dynamics, one needs to average fast dynamics with respect to its periodic solution. In the previous section, such fast dynamics was investigated in detail, finding a multi-parametric family of periodic dispersive wave-trains for the Skyrme–Faddeev system. Now, in the Whitham approach, one can average a family of periodic solutions, depending on a number of arbitrary parameters, which must be sufficient for the correctness of this phenomenological approach. Due to such a nonlinear WKB method, the averaged system allows us to obtain a more general solution than the periodic one (see details in [30]). However, in this paper we restrict our considerations just to the derivation of the averaged system, because such a system has its own interest by virtue of recent development in the theory of integrable multi-dimensional quasilinear systems of first order (see, for instance, [28]). We do not believe that such a quasilinear system is integrable (see the following section). However, we hope to extract an ‘integrable sector’ in further investigations, in particular in the limit the elliptic module goes to 0 or 1.

The lowest order of the modulation approximation is found by substituting the multi-parametric family of periodic solutions (3.10) (see also (3.7) and (3.8)) into the Skyrme–Faddeev Lagrangian (see (2.1) and (2.3)), introducing an averaged Lagrangian, say $L(\gamma, \omega, \beta, \mathbf{k})$, depending on the slowly changing parameters as new dynamic variables $\gamma, \omega, \beta, \mathbf{k}$, which correspond to the derivatives with respect to spacetime variables of the phase $\theta$ and pseudo-phase $\tilde{\theta}$, now not necessarily linear as in (3.2). It means that $\omega = -\theta_k \omega_k, \beta_i = \tilde{\theta}_i \mathbf{X}_i$ and $\gamma = -\theta_k \gamma_k, \beta_i = \tilde{\theta}_i \mathbf{X}_i$, where $X^0, X^1, X^2, X^3$ are the so-called slow variables in comparison with fast variables $x^0, x^1, x^2, x^3$ (see details in [30]).

Thus, one immediately derives the four-dimensional quasilinear system (below we use $\partial_k \equiv \partial_{X_k}$)

$$
\partial_0 L_\omega = \partial_1 L_\omega, \quad \partial_0 L_\gamma = \partial_1 L_{\beta_j},
$$

with the compatibility conditions

$$
\partial_0 k_i^l + \partial_i \omega = 0, \quad \partial_j k_i^l = \partial_i k_j^l, \quad i \neq j, \quad \partial_0 \beta_i^l + \partial_i \gamma = 0, \quad \partial_j \beta_i^l = \partial_i \beta_j^l, \quad i \neq j.
$$

After the substitution of the family of periodic solutions (3.7) and (3.8) into the Lagrangian density (2.3), one obtains

$$
\hat{L}_p = \sin^2(\theta) \left( -\frac{1}{2} \lambda \left( \frac{B_2^2}{4} - B_1 B_3 \right) \Theta_0^2 + B_3 \Phi_0^2 + B_2 \Phi_0 + B_1 \right) + B_3 \Theta_0^2,
$$

which is a function only of $\theta$. Thus, performing an integration over a finite space-like region, contributions from $\theta$-independent coordinates are just time-independent finite multiplicative factors. Then, on a period of the wave, one is led to the Lagrangian

$$
L \equiv \frac{1}{2 \pi} \oint \hat{L}_p d\theta,
$$

12
which, generalizing the standard Whitham approach, is supplemented by the introduction of two natural normalizations (or constraints)
\[ \oint d\theta = 2\pi, \quad \langle \Phi \rangle = \oint \Phi d\theta = 2\pi m, \] (4.3)
where the integer ‘m’ is the number of rotations of the vector \( \phi \) around a value determined by a given pseudo-phase \( \tilde{\theta} \). This situation is very similar to the spin wave configurations called cyclon and extra-cyclon in multiferroic materials ([32]). Then, the corresponding averaged Lagrangian is
\[ L = \left( B_1 - \frac{B_3^2}{4B_3} \right) \left( A_1 + A_2 + W \sqrt{\frac{2}{3}B_3} \right) + \frac{B_2 + 2mB_3}{2B_3} \sqrt{A_1A_2(B_2^2 - 4B_1B_3)}, \] (4.4)
where we introduced the function
\[ W = \frac{1}{2\pi} \oint \frac{(\psi - A_1)(\psi - A_2)(\psi - \psi_1)}{1 - \psi} d\psi. \] (4.5)
One can immediately check that two equations \( L_{A_1} = 0 \) and \( L_{A_2} = 0 \) (see [30]) coincide with normalizations (4.3), while the Euler–Lagrange equations lead to a four-dimensional quasilinear system of the first order (4.2).

The two normalization conditions (4.3) formally allow us to exclude \( A_1 \) and \( A_2 \) from the above construction. This means that solving (4.3), one can derive \( A_1(\gamma, \omega, \beta, k) \) and \( A_2(\gamma, \omega, \beta, k) \). On the other hand, recalling the relation among \( A_i \) and the conserved densities \( E^\mu \), one can use this relation to derive nonlinear dispersion relations for \( \gamma \) and \( \omega \). Thus, the quasilinear system (4.2) contains first-order derivatives with respect to \( x \)’ of eight unknown functions \( \gamma, \omega, \beta, k \) only. However, one can use the five roots \( 0, 1, A_1, A_2 \) and \( \psi_1 \) for the parametrization of the periodic solution (3.10). Nevertheless, the function \( W(\gamma, \omega, \beta, k) \) as well as the averaged Lagrangian cannot be expressed via complete elliptic integrals of the first and second kinds only. For this reason, all partial derivatives \( L_{\omega_2}, L_{\gamma_1}, L_{\gamma_1} \) and \( L_{\psi} \) also contain elliptic integrals of the third kind; thus, all expressions in the quasilinear system (4.2) became too complicated to be presented explicitly in this paper. Indeed, such quasilinear systems are very difficult to write in a compact form due to the complexity of their coefficients. Moreover, it is much more difficult to investigate their integrability properties. Just in the simplest cases, this problem can be solved manually. In all other cases, one needs some symbolic software. Nevertheless, this problem is in principle solvable and we are going to investigate its integrability in the near future.

5. Conclusions

We have found several reductions of the Skyrme–Faddeev model, which can be solved implicitly or via special functions. We have shown the strong connection with the d’Alembert–eikonal system, which allows us to find large classes of solutions. Even in the more complicated system (2.15)–(2.17), we found large classes of solutions in terms of two arbitrary functions. Finally, we have shown that the overdetermined system for the field \( u \) can be formulated as a first-order system of linear PDEs, which can be solved by the ‘method of hydrodynamic reductions’. However, the analysis of the special constraint (2.17) involves several technical complexities and for this only particular solutions are given. Only, we remark that the constraint is a further restriction on the subsector of the solution space described in [22]. In section 3, we have found exact analytic periodic spin waves for the Skyrme–Faddeev model, determined in terms of elliptic integrals of the third kind. Assuming that there exist such periodic dispersive wave-train solutions, we have shown that the Lagrangian can be averaged by the Whitham
method. This provides a Lagrangian for a set of parameters, describing the modulation of periodic waves in terms of a quasilinear system of partial derivatives of the first order. Finally, noting that in general such a system like (4.2) is non-integrable, a deeper analysis is required.

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