HETEROGENEOUS DISCRETE KINETIC MODEL AND ITS DIFFUSION LIMIT

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Abstract. A revertible discrete velocity kinetic model is introduced when the environment is spatially heterogeneous. It is proved that the parabolic scale singular limit of the model exists and satisfies a new heterogeneous diffusion equation that depends on the diffusivity and the turning frequency together. An energy functional is introduced which takes into account spatial heterogeneity in the velocity field. The monotonicity of the energy functional is the key to obtain uniform estimates needed for the weak convergence proof. The Div-Curl lemma completes the strong convergence proof.

1. Introduction. There has been a lot of controversy about the correct diffusion equation when the diffusivity $D = D(x)$ is not constant. We claim in the paper that the reason for the long-standing controversy is that diffusivity alone is not enough to explain diffusion phenomena in heterogeneous environments, and additional information such as turning frequency is needed. The purpose of the paper is to introduce a spatially heterogeneous discrete velocity kinetic model and show rigorously that its diffusion limit (or parabolic scale limit) exists and satisfies a heterogeneous diffusion equation,

$$u_t = \nabla \cdot \left( \sqrt{\mu^{-1} D} \nabla (\sqrt{\mu D} u) \right),$$

where $\mu = \mu(x)$ is the turning frequency of the kinetic model. Discrete kinetic models in the literature usually include spatial heterogeneity only in the turning frequency. We introduce a fully heterogeneous discrete model that includes the heterogeneity even in the velocity field. It seems that isotropic diffusion equations are always in this form if they are derived from a revertible random walk system (see Section 2). Note that this is the same diffusion equation which has passed the thought experiment test in [12].

The kinetic theory is the foundation of statistical thermodynamics which provides a molecular-level explanation of the classical thermodynamics. In particular, various discrete velocity kinetic equations are used as mathematical models to show such molecular-level dynamics rigorously (see Platkowski and Illner [18] for discussions).

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Broadwell and Carleman models [1, 2] are well-known examples. In one space dimension, the Carleman model is written as

\begin{align}
    u_t^+ + vu_x^+ &= k(u^+, u^-, x)(u^- - u^+), \\
    u_t^- - vu_x^- &= k(u^+, u^-, x)(u^+ - u^-),
\end{align}

where $u^+$ is the density of particles that move to right with a constant speed $v > 0$ and $u^-$ is the one that move to left. The turning frequency $k(u^+, u^-, x)$ is proportional to the total population, $u := u^+ + u^-$, in the Carleman model, which models binary collisions between two molecules. Taylor [25], Goldstein [7], and Kac [11] took a constant turning frequency and derived telegrapher’s equation. The Carleman model (2) has been generalized by taking $k = u^\alpha$ to derive nonlinear diffusion equations, where $\alpha = 1$ gives the Carleman model and $\alpha = 0$ the Goldstein-Taylor model. Pulvirenti and Toscani [19] considered the parabolic limit for $0 \leq \alpha < 1$ and showed the convergence to the fast diffusion equation. Lions and Toscani [13] extended it to the case of all $\alpha < 1$, which now includes the slow diffusion when $\alpha < -1$. See [20, 21, 22] for other subsequent results.

However, the focus of the paper is on spatial heterogeneity, not nonlinearity. The reason is that Brownian particles collide with background molecules but not with each other. Therefore, binary collisions of the Carleman model are suitable for modeling thermodynamics rather than diffusion phenomena and hence we consider solitary collisions with background particles. The key is to include the effect of background molecules when the environment is heterogeneous. The temperature and the pressure are two main macroscopic level quantities which decide the speed and the density of molecules. The speed and the turning frequency of the discrete kinetic model depend on these quantities. There are recently proposed continuum velocity kinetic models that include spatial heterogeneity in the speed and the turning frequency (see [9, 10]). However, as far as authors know, there is no discrete kinetic equation in the literature that is designed to study a spatially heterogeneous diffusion. The issues and difficulties in constructing a spatially heterogeneous discrete and continuum velocity kinetic equation are discussed in Section 2.

Three diffusion laws are often taken when the diffusivity is heterogeneous, which are

\begin{align}
    u_t &= \nabla \cdot (D\nabla u), \\
    u_t &= \nabla \cdot (\sqrt{D}(\nabla \sqrt{D} u)), \\
    u_t &= \Delta(Du).
\end{align}

These three laws are called Fick [6], Wereide [26], and Chapman [3], respectively. If $\mu$ is constant, (1) becomes Wereide’s diffusion law (4) and if $D$ is also constant, the four diffusion laws are all identical. There has been a lot of controversy over which one is the correct diffusion equation. See Milligen et al. [14] for a comparison of diffusion laws with experimental data, where none of them gives a satisfying result. The three diffusion laws are based on an assumption that the diffusion phenomenon is decided by the diffusivity $D$ only even in a heterogeneous environment. From this viewpoint, the new diffusion (1) is against the belief that the diffusivity alone can decide the diffusion phenomenon.

We consider an $n$-dimensional heterogeneous discrete velocity kinetic model that consists of $2n$ equations for $k = 1, \cdots, n$, 

of the system (6)–(7) comes from the conservation law, \( \mu \) (see [13, (1.7)]), which is equivalent with (10) since \( \epsilon > 0 \) appears after a change of time and space variables in the Carleman model (2) corresponds to \( \frac{k^2}{2} \) for the turning frequency \( \mu \).

We consider the problem with initial values,

\[ u^i(x, 0) = u_0^i(x), \quad i = 1+, 1-, \cdots, n+, n-, \]  

and the periodic boundary conditions,

\[ u^{k_±}(x, t) = u^{k_±}(y, t) \quad \text{if} \quad \text{mod} (x - y, 2) = 0. \]  

The periodic boundary condition (9) implies that the space domain is actually an \( n \)-dimensional torus without boundary and we can forget boundary effects for simplicity. The speed \( v = v(x) \) and the turning frequency \( \mu = \mu(x) \) are scalar-valued periodic functions of modulo 2 for the compatibility with the boundary condition.

The diffusivity \( D \) of a discrete velocity kinetic equation is given by

\[ D = \frac{v^2}{n\mu} \]  

(see Remark 1). This relationship holds for both linear and nonlinear models, as well as homogeneous and heterogeneous models. For example, for the generalized Carleman model with \( k = u^a, v = 1, \alpha < -1, \) and \( n = 1, \) the diffusivity is \( D = \frac{1}{2\alpha} \) (see [13, (1.7)]), which is equivalent with (10) since \( \mu = 2k. \) Note that the left side of the system (6)–(7) comes from the conservation law,

\[ u^k_{t} \pm \frac{1}{\epsilon} \nabla \cdot (v \epsilon_k u^{k_±}) = u^{k_±} \pm \frac{1}{\epsilon} (v u^{k_±})_{x_k}. \]

We denote the population density of the whole species by

\[ u(x, t) = \sum_{j=1}^{n} u^j(x, t). \]

The small parameter \( \epsilon > 0 \) appears after a change of time and space variables in a parabolic scale. The solution of the system depends on the small parameter and we will denote the solution by \( u^i \) when the dependency on \( \epsilon \) is needed explicitly.

The main result of the paper is the following theorem.
Theorem 1.1 (Strong convergence to a weak solution). Let $u_i^0 \in L^4(\Omega)$ for $i = 1, \ldots, n$, $v(x)$ and $\mu(x)$ be bounded and bounded away from zero, and $\nabla v$ be bounded (see (15)–(17)). Let $u^\epsilon$ be the solution of (6)–(9) and $u^* = \sum_i u_i^\epsilon$ the corresponding total population. Then, $u^\epsilon$ converges in $L^2(\Omega \times (0, T))$ for any $T > 0$ and its limit $u$ is a weak solution of (1).

Note that the classical energy functional is not applicable for a heterogeneous problem. The key step toward the proof is the construction of the energy functional given in Definition 4.1, which is designed to count the spatial heterogeneity in the speed $v(x)$. Then, the monotonicity of the energy functional gives the subsequential convergence in a weak sense. Finally, the Div-Curl lemma of compensated compactness theory completes the passage to the limit in the strong sense.

The paper is organized as follows. In Section 2, we discuss the meaning and the property of the discrete velocity kinetic model (6)–(7) by comparing it with continuum velocity kinetic models. In particular, the revertibility is introduced as a key requirement to be a meaningful heterogeneous random walk system. In Section 3, notation of the paper is introduced. Since we tackle the $n$-dimensional case directly, the use of well-organized notation helps analysis. The existence of the problem is borrowed from classical semigroup theory. In Section 4, the energy functional is introduced and $L^2$ uniform estimates are obtained by using the monotonicity of the functions. In Section 5, the convergence is obtained using the uniform $L^2$ estimates and the Div-Curl lemma. A discussion on the correct diffusion law is given in Section 6 with possible applications to biological dispersal problems.

2. Revertible velocity jump process. There are a variety of random walk systems from position jumps to velocity jumps, from discrete models to continuum models. If different models give you different diffusion equations when the environment is heterogeneous, how can you find the correct diffusion equation? In this section we discuss the effect of heterogeneity in a random walk system and introduce revertible random walk system. We claim that the revertibility is the key to obtain the correct diffusion equation and the same diffusion law as (1) is obtained as long as the system is isotropic and revertible.

Let $X_\ell$ be the position of a particle after $\ell$ number of random walks. If the random walk system is homogeneous, the expectation is the same as its initial position, i.e., $E(X_\ell) = E(X_0)$. This is the property that makes the probability density function converge to a constant eventually. However, the equality fails in general if the random walk system is heterogeneous. The expectation always moves toward a position with shorter walk length and longer jumping time. We call a random walk system revertible if $E(X_2) = E(X_0)$ when the second walk is in the opposite direction of the first one. Then, we can avoid undesired drift phenomena if the random walk system us revertible (see Kim and Seo [12]).

In a homogeneous case, the kinetic equation is written as

$$p_t + \frac{1}{\epsilon} v \cdot \nabla p = \frac{\mu}{\epsilon^2} \int_V \left( q(v)p(v', x, t) - q(v)p(v, x, t) \right) dv',$$

where $p = p(v, x, t)$ is the density (or probability) of particles with velocity $v \in V$ at $(x, t) \in Q_\infty$, $V \subset \mathbb{R}^n$ is the collection of velocities that a particle may take, the constant coefficient $\mu$ is the turning frequency, and $q(v)$ is the probability for a particle to take the velocity $v$ after a collision. Othmer et al. [15] took the kinetic equation as a velocity jump process. Hillen and Othemer [9, 16] derived that the
singular limit of the total population density, \( u(t, t) := \int_V p(v, x, t) dv \), satisfies
\[
\frac{du}{dt} = \nabla \cdot (D \nabla u),
\]
where \( D \) is the diffusivity tensor,
\[
D = \frac{1}{\mu} \int_V (v \otimes v) q(v) dv.
\]
The discrete velocity kinetic model of the paper is obtained by taking
\[
V = \{ \pm v_0 e_i : i = 1, \cdots, n \}, \quad q(\pm e_k) = q_k, \quad \sum_{k=1}^{n} 2q_k = 1.
\]
If the probability \( q_k \) is independent of \( k \), i.e., \( q_k = \frac{1}{2n} \), the random walk system becomes isotropic and one obtains the same equations as in (6)–(7) after replacing the nonconstant speed \( v(x) \) with the constant one \( v_0 \). Since there is no directional difference in \( V \) and \( q \), we expect an isotropic diffusion. The convergence of the homogeneous discrete kinetic equation to the diffusion equation,
\[
\frac{du}{dt} = \Delta u, \quad D(x) := \frac{v_0^2}{n\mu},
\]
is classical.

The spatial heterogeneity would be included in \( \mu \) and \( q(v) \). For example, if the temperature increases, the speed of background molecules increases, and hence Brown particles are more likely to take a higher speed. The turning frequency of Brownian particles also depends on the molecules’ speed and density. To include such spatial variations, we may take a heterogeneous kinetic equation,
\[
\frac{dp}{dt} + \frac{1}{\epsilon} v \cdot \nabla p = \frac{\mu(x)}{\epsilon^2} \int_V \left( q(v, x)p(v', x, t) - q(v', x)p(v, x, t) \right) dv'. \quad (11)
\]
In this equation, the rate of collision \( \mu(x) \) varies spatially and the probability \( q(v, x) \) also varies spatially. In this model, the velocity is not changed until the next collision, which is the fundamental hypothesis of the kinetic theory. The diffusion equation derived formally from the heterogeneous kinetic equation is
\[
\frac{du}{dt} = \nabla \cdot \frac{1}{\mu} \nabla (\mu Du), \quad D(x) := \frac{1}{\mu} \int_V (v \otimes v) q(v, x) dv. \quad (12)
\]
See Hillen and Painter [10] for a derivation when \( \mu \) constant and Kim and Seo [12, Section 2] for a derivation when \( \mu \) is not constant.

There are two weaknesses of these continuum models. First, the derivations are only formal and there is no rigorous convergence proof yet. An alternative way is to construct discrete kinetic models related to these continuum models. In such models, a particle may pick up a velocity of different speeds at a different position and keeps it until the next collision. Working with such a discrete kinetic equation would provide needed analytical results. However, constructing such a discrete model is very tricky, and as far as the authors know, there is no such discrete system in the literature.

The second weakness is that the velocity jump process (11) is not revertible. One of the reasons for the lack of revertibility is that the setting \( \mu \) only as a function of \( x \) is an oversimplification. It should account for the speed and density of background
molecules and we don’t know how to set it up correctly. For example, the mean collision time $\Delta t$ is given by a relation
\[
\int_0^{\Delta t} \mu(x(t))dt = 2,
\]
where $x(t)$ is the position of the particle at time $t$. Let the second walk be in the opposite direction of the first one. Since the integral is independent of the particle speed, $X_2 \neq X_0$ in general if the mean speed of the second walk is not identical to the first one. Therefore, the heterogeneous kinetic equation (11) is not revertible.

Kim and Seo [12] introduced a new idea that a particle takes a vector field after a collision and moves along an integral curve of the chosen vector field. The spatial heterogeneity can be included in the vector fields $v_\alpha$ as well as in the probability $q$. Note that this setting violates the fundamental hypothesis of kinetic theory that the particle velocity persists until the next collision. However, we obtain the revertibility instead. The corresponding kinetic equation is written as
\[
p_t + \frac{1}{\epsilon} \nabla \cdot (v_\alpha \mu) = \frac{\mu(x)}{\epsilon^2} \int_A \left( q(v_\alpha, x) p(\alpha', x, t) - q(v_\alpha, x) p(\alpha, x, t) \right) d\alpha',
\]
where $A$ is the index set of velocity vector fields, i.e., $V = \{v_\alpha; \alpha \in A\}$, and $q(v_\alpha, x)$ is the probability to take the velocity vector field $v_\alpha$ after a collision at the position $x$. The corresponding diffusion equation is
\[
u_t = \nabla \cdot \left( \frac{1}{\mu} \nabla \cdot (\mu Du) \right) - \nabla \cdot \left( \frac{1}{\mu} \mathbb{N} u \right), \quad \mathbb{N} = \int_A (\nabla v_\alpha) v_\alpha q(v_\alpha, x) d\alpha.
\]
The correction term $\mathbb{N}$ appears since $v_\alpha$ is a vector field and hence $v_\alpha \nabla p \neq \nabla \cdot (v_\alpha p)$.

We can easily construct a discrete kinetic equation corresponding to (13). For example, we may take
\[
V = \{v_{k\pm}(x) := \pm v(x)e_k : k = 1, \cdots, n\}, \quad \text{and} \quad q(v_{k\pm}, x) = \frac{1}{2n},
\]
where there are $2n$ discrete vector fields $v_i$, $i = 1, \cdots, n\pm$, with a speed $v(x) > 0$. If $u^i$ is the density of particles that move along the vector field $v_i$, the revertible kinetic equation (13) is written as
\[
u^i_t + \frac{1}{\epsilon} \nabla \cdot (u^i v_i) = \frac{\mu(x)}{2n\epsilon^2} \sum_{j=1\pm}^{n\pm} (u^j - u^i), \quad i = 1, \cdots, n\pm.
\]
These are the discrete kinetic equations of the paper in (6)–(7). In other words, we have a discrete kinetic model for the modified revertible kinetic equation (13), but not for the classical one (11).

**Remark 1.** Under the choice of discrete system (14), the anisotropic diffusivity tensor in (12) is written as
\[
D(x) = \frac{\nu^2}{2n\mu} \sum_{k=1}^{n} (e_k \otimes e_k + e_{-k} \otimes e_{-k}) = \frac{\nu^2}{2n\mu} (I + I) = \frac{\nu^2}{n\mu} I.
\]
Hence, it gives the isotropic diffusivity $D = \frac{\nu^2}{n\mu}$ in (10).

The spatial heterogeneity in $q(v, x)$ is taken from the position at the collision moment. We may call the velocity jump process based on (11) an Ito type since the spatial heterogeneity is taken at the starting position of a new random walk.
The corresponding diffusion equation is Chapman’s law (5). If a velocity jump process follows the modified kinetic equation (13), the spatial heterogeneity in the velocity vector field \( v_\alpha \) is involved continuously along the path of a particle. If a particle moves backward in the second walk, i.e., if it takes \( v_{-\alpha} \), it returns to the exactly same position after the second walk. This behavior makes the random walk revertible and gives a behavior of the Stratonovich integral which takes the information from the middle point between the starting and the arriving points.

3. Notation and existence. We introduce the notations for the discrete kinetic model of the paper. We denote the space and time domain as \( \Omega_T := \Omega \times (0, T) \). Vectors are denoted in bold characters. The special feature of the model is in the spatial heterogeneity in the particle speed \( v = v(x) \) and the turning frequency \( \mu = \mu(x) \). For the compatibility with the periodic boundary condition, the two coefficients are assumed to satisfy the same periodic boundary condition,

\[
v(x) = v(y), \quad \mu(x) = \mu(y) \quad \text{if} \quad \text{mod}(x - y, 2) = 0.
\] (15)

We also assume that \( v(x) \) and \( \mu(x) \) are bounded and bounded away from zero. To reduce the number of parameters, we assume that there exists \( M > 0 \) such that

\[
M^{-1} < v(x) < M \quad \text{and} \quad M^{-1} < \mu(x) < M.
\] (16)

In addition, we assume that the partial derivatives of \( v \) are bounded, i.e.,

\[
\left| \frac{\partial v(x)}{\partial x_k} \right| < M, \quad k = 1, \ldots, n.
\] (17)

We are interested in the singular limit of solutions of (6)–(8) as \( \epsilon \to 0 \). In these limiting process, the spatial heterogeneity in \( v \) and \( \mu \) are treated as macroscopic-scale distributions. We take the following notations:

\[
\mathbf{u} = (u_1^+, u_1^-, \ldots, u_n^+, u_n^-) \in \mathbb{R}^{2n},
\]

\[
J_{i,j} = \frac{v(x)}{\epsilon} (u_i^j - u_j^i), \quad i,j = 1,\ldots,n,\pm
\]

\[
u^k = u^{k+} + u^{k-}, \quad k = 1,\ldots,n,
\]

\[
J_k = J_{k+,k-}, \quad k = 1,\ldots,n,
\]

\[
\mathbf{J} = (J_1,\ldots,J_n) \in \mathbb{R}^n.
\]

Solutions of (6)–(8) depend on the parameter \( \epsilon \). If necessary, we explicitly denote the dependency on \( \epsilon \) as follows:

\[
u^\epsilon, \; u^{i,\epsilon}, \; J_{i,j}^\epsilon, \; J_k^\epsilon \quad \text{and} \quad \mathbf{J}^\epsilon.
\]

However, for simplicity, we denote them without \( \epsilon \) when the parameter \( \epsilon \) is fixed. If necessary, we also denote the dependency of the flux on \( u \) as follows:

\[
\mathbf{J}^u, \; J_{i,j}^u, \; J_k^u \quad \text{and} \quad \mathbf{J}^u^\epsilon.
\]

We reserve \( i \) and \( j \) to denote one of the \( 2n \) directions, i.e., \( i,j \in \{1,\ldots,n,\pm\} \), and \( k \) to denote one of \( n \) coordinates, i.e., \( k \in \{1,\ldots,n\} \).

Remark 2. If one wants to know the dependency of upper and lower bounds of \( v \) and \( \mu \), one may take \( 0 < m_1 < v(x) < M_1 \) and \( 0 < m_2 < \mu(x) < M_2 \). In this paper we look for simpler expressions using only one parameter \( M > 0 \).
The existence and the uniqueness of the weak solution of (6)–(8) come from classical semigroup theory (see [4, Section 3], [8, Section 3], and [17, Sections 1, 3, and 4]). For example, we may write (6)–(7) in an operator form,

\[
\frac{\partial}{\partial t} U = GU + BU,
\]

where

\[
U := \begin{pmatrix} u_1^{1+} \\ u_1^{-} \\ \vdots \\ u_n^{1+} \\ u_n^{-} \end{pmatrix}, \quad GU := \frac{v(x)}{\epsilon} \begin{pmatrix} -\partial_{x_i} u_1^{1+} \\ \vdots \\ -\partial_{x_i} u_n^{1+} \end{pmatrix},
\]

and

\[
BU := \frac{1}{\epsilon} \begin{pmatrix} -u_1^+ \partial_{x_i} v(x) \\ \vdots \\ -u_n^+ \partial_{x_i} v(x) \\ -u_1^- \partial_{x_i} v(x) \\ \vdots \\ -u_n^- \partial_{x_i} v(x) \end{pmatrix} + \frac{\mu(x)}{2n\epsilon^2} \begin{pmatrix} \sum_{j=1}^{n^\pm} (u_j^+ - u_1^+) \\ \vdots \\ \sum_{j=1}^{n^\pm} (u_j^+ - u_n^+) \\ \sum_{j=1}^{n^\pm} (u_j^- - u_1^-) \\ \vdots \\ \sum_{j=1}^{n^\pm} (u_j^- - u_n^-) \end{pmatrix}.
\]

Since \(v(x)\) is bounded and the above differential operator \(G\) is a contraction (see [8, Section 3]), we can verify that \(G : D(G) \to [L^p(\Omega_T)]^{2n}\) is a continuous semigroup on \(U\), where we are interested in the case with \(p \geq 2\). The other operator \(B\) is bounded and linear, which can be considered as a perturbation [17, Section 3].

The domain of the linear operator \(G\) is

\[
D(G) = \{(u_1^{1+}, \ldots, u_n^{1-}) \in [L^p(\Omega)]^{2n} \mid \partial_k u_k^{x \pm} \in L^p(\Omega) \text{ and periodic} \}.
\]

Now, we may apply Theorem 1.3 in [17, Section 4] and obtain a unique solution \(u'(x,t) \in C([0,T], [L^p(\Omega)]^{2n})\) of (6)–(7) for an initial value \(u_0^e \in [L^p(\Omega)]^{2n}\).

4. Energy functional and its monotonicity. In this section, we obtain uniform \(L^2\)-estimates of \(u'\) and \(J_{u'e}\) which depend only on the initial value. First, we introduce an energy functional.

**Definition 4.1** (Energy functional). Let \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) be a nondecreasing function. For a given nonconstant speed \(v : \Omega \to \mathbb{R}_+\), the energy of a fractional population distribution \(u^j : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+\) at time \(t > 0\) is defined by

\[
\mathcal{E}_v(u^j)(t) := \int_{\Omega} \Phi_v(u^j(x,t), x)dx \quad \text{with} \quad \Phi_v(u^j, x) = \int_0^{u^j} \psi(v(x)\tau)d\tau.
\]

For given \(2n\) fractional population distributions, \(u = (u_1^{1+}, \ldots, u_n^{1-}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+^{2n}\), the total energy of the solution is defined by

\[
\mathcal{E}_v(u)(t) := \sum_{j=1}^{n^\pm} \mathcal{E}_v(u^j)(t).
\]

Note that since \(\psi\) is a nondecreasing function, \(\Phi\) is a convex function with respect to variable \(u\). Let \(\Psi\) be the antiderivative of \(\psi\) with \(\Psi(0) = 0\). Then, \(\Phi_v\) is given by

\[
\Phi_v(u, x) = \frac{1}{v(x)}\Psi(v(x)u) \quad \text{or} \quad \Psi(v(x)u) = v(x)\Phi_v(u, x).
\]

We take \(\psi(s) = s^{p-1}\) in this paper which is a nondecreasing function when \(p \geq 2\).
Theorem 4.2. Let $v$ and $\mu$ satisfy (15)–(17) and $u^i$, $i = 1\pm, \cdots, n\pm$, be smooth solutions of (6)–(8). Then,

1. The total energy $E_v(u)(t)$ decreases in time and hence

$$E_v(u)(t) \leq E_v(u(0)), \quad t > 0.$$ 

2. For all $T > 0$,

$$\sum_{i,j=1}^{n\pm} \|J_{i,j}^{\mu}\|^2_{L^2(\Omega_T)} \leq 2nM^3 \sum_{j=1}^{n\pm} \|u^j_0\|^2_{L^2(\Omega)}.$$ 

Proof. Multiply $\psi(v(x)u^{k\pm})$ to (6)–(7) and obtain

$$\psi(v(x)u^{k\pm})u_t^{k\pm} = \frac{\psi(v(x)u^{k\pm})}{\epsilon} (v(x)u^{k\pm})x_k = \frac{\psi(v(x)u^{k\pm})\mu(x)}{2n\epsilon^2} \sum_{j=1}^{n\pm} (u^j - u^{k\pm}).$$

We will integrate this equation over $\Omega$ and add them for $i = 1\pm, \cdots, n\pm$. First, the summation of the time derivative terms gives

$$\sum_{i=1}^{n\pm} \int_{\Omega} \psi(v(x)u^i)u^i_1 dx = \sum_{i=1}^{n\pm} \int_{\Omega} \left( \frac{d}{dt} \int_{0}^{t} \psi(v(x)\tau) d\tau \right) dx = \frac{d}{dt} E_v(u).$$

The integration of the second term, is zero due to the periodic boundary condition, i.e.,

$$\int_{\Omega} \psi(v(x)u^{k\pm})(v(x)u^{k\pm})x_k dx = \int_{\Omega} \frac{\partial}{\partial x_k} \psi(v(x)u^{k\pm}) dx = 0.$$ 

Therefore, the summation of the flux terms is zero. The summation of the third term, the turning term, give

$$\sum_{i=1}^{n\pm} \int_{\Omega} \left( \frac{\psi(v(x)u^i)\mu(x)}{2n\epsilon^2} \sum_{j=1}^{n\pm} (u^j - u^i) \right) dx$$

$$= \frac{1}{2n\epsilon^2} \int_{\Omega} \mu(x) \sum_{i,j=1}^{n\pm} \psi(v(x)u^i)(u^j - u^i) dx$$

$$= -\frac{1}{4n\epsilon^2} \int_{\Omega} \mu(x) \sum_{i,j=1}^{n\pm} (\psi(v(x)u^i) - \psi(v(x)u^j))(u^i - u^j) dx$$

$$= -\frac{1}{4n\epsilon^2} \int_{\Omega} \frac{\mu(x)}{v(x)} \sum_{i,j=1}^{n\pm} (\psi(v(x)u^i) - \psi(v(x)u^j))(v(x)u^i - v(x)u^j) dx$$

$$\leq -\frac{1}{4n\epsilon^2 M^2} \int_{\Omega} \sum_{i,j=1}^{n\pm} (\psi(v(x)u^i) - \psi(v(x)u^j))(v(x)u^i - v(x)u^j) dx \leq 0,$$

where the first inequality comes from the estimate $\frac{\mu(x)}{v(x)} \geq M^{-2}$ given by uniform bounds in (16), and the second inequality comes from the monotonicity of the function $\psi$. By combining above inequalities, we have

$$\frac{d}{dt} E_v(u) \leq 0,$$

and the first assertion of the theorem is completed.
To show the second assertion we take \( \psi(s) = s \). Then, the above inequality is written by

\[
\frac{\partial}{\partial t} \mathcal{E}_v(u) \leq -\frac{1}{4nM^2} \int_{\Omega} \sum_{i,j=1}^{n+} |J_{i,j}^{u,\epsilon}|^2 \, dx.
\] (18)

The integration of (18) over \((0,T)\) gives

\[
\frac{1}{4nM^2} \int_{0}^{T} \int_{\Omega} \sum_{i,j=1}^{n+} |J_{i,j}^{u,\epsilon}|^2 \, dx \, dt \leq \mathcal{E}_v(u_0) - \mathcal{E}_v(u(x,T)) \leq \mathcal{E}_v(u_0).
\]

Therefore, the \( L^2 \)-norm of \( J_{i,j}^{u,\epsilon} \) is uniformly bounded with respect to \( \epsilon \) and \( T > 0 \) by

\[
\sum_{i,j=1}^{n+} \|J_{i,j}^{u,\epsilon}\|_{L^2(\Omega_{T})} \leq 4nM^2\mathcal{E}_v(u_0).
\]

For \( \psi(s) = s \),

\[
\mathcal{E}_v(u_0) = \int_{\Omega} \frac{v(x)}{2} \sum_{j=1}^{n+} (u_0^j)^2 \, dx \leq \frac{M}{2} \sum_{j=1}^{n+} \|u_0^j\|^2_{L^2(\Omega)}.
\]

Therefore, \( 2nM^3 \sum_{j=1}^{n+} \|u_0^j\|^2_{L^2(\Omega)} \) is an upper bound.

The case of our main interest is with \( \psi(s) = s^{p-1} \) for \( p = 2 \) or 4. In such cases, we can easily show that \( L^p \) norm of the solution is nonincreasing because the weight function \( v(x) \) is bounded by (16).

**Corollary 1.** Assume further that \( p \geq 2 \) and \( u_0 \in [L^p(\Omega)]^{2n} \). Then, we have \( u \in C([0,T];[L^p(\Omega)]^{2n}) \) and there exists a constant \( C = C(M,p) \) such that

\[
\|u(t)\|_{[L^p(\Omega)]^{2n}} \leq C\|u_0\|_{[L^p(\Omega)]^{2n}}, \quad t > 0.
\] (19)

**Proof.** Let \( \psi(s) = s^{p-1} \). Then,

\[
\Phi_v(u, x) = \int_{0}^{u} \psi(v(x) \tau) \, d\tau = \int_{0}^{u} v(x)^{p-1} \tau^{p-1} \, d\tau = v(x)^{p-1} \frac{u^p}{p}.
\]

Since the speed \( v(x) \) is bounded by (16), we obtain

\[
\frac{1}{M^{p-1}} \frac{u^p}{p} \leq \Phi_v(u, x) \leq M^{p-1} \frac{u^p}{p}, \quad u \geq 0.
\]

Replace \( u \) with the solution \( u^j \) and integrate it over \( \Omega \). Then, for \( t \geq 0 \),

\[
\frac{1}{pM^{p-1}} \|u^j(t)\|^p_{L^p(\Omega)} \leq \mathcal{E}_v(u^j)(t) \leq \frac{M^{p-1}}{p} \|u^j(t)\|^p_{L^p(\Omega)}.
\]

The monotonicity of the total energy (Theorem 4.2) implies that

\[
\|u(t)\|_{[L^p(\Omega)]^{2n}} = \sum_{j=1}^{n+} \|u^j(t)\|^p_{L^p(\Omega)}
\]

\[
\leq pM^{p-1} \mathcal{E}_v(u(x, t)) \leq pM^{p-1} \mathcal{E}_v(u_0(x)) \leq M^{2(p-1)} \sum_{j=1}^{n+} \|u_0^j\|^p_{L^p(\Omega)} = M^{2(p-1)} \|u_0\|^p_{[L^p(\Omega)]^{2n}}.
\]
Therefore, (19) holds with $C = M \frac{2(n-1)}{p}$.

Theorem 4.2 and Corollary 1 are for smooth solutions. We can extend it to weak solutions using standard approximation techniques.

**Theorem 4.3.** Let $u \in C([0,T]; [L^p(\Omega)]^{2n})$ be a weak solution of (6)–(8) with an initial value $u_0 \in [L^2(\Omega)]^{2n}$. Then,

1. The total energy $E_v(u)(t)$ decreases in time and hence $E_v(u)(t) \leq E_v(u_0(x)), \quad t > 0$.

2. There exists a constant $C$ such that, for all $t > 0$,

$$
\|u(t)\|_{L^p(\Omega)^{2n}} \leq C\|u_0\|_{L^p(\Omega)^{2n}}.
$$

**Proof.** Denote $u = \{u^j\}_{j=1}^{\pm n}$ and the initial value $u_0 = (u_0^1, u_0^0, \ldots, u_0^n) \in [L^2(\Omega)]^{2n}$. Let $u_\delta \in [L^2(\Omega)]^{2n}$ be a sequence of smooth functions that converge to $u_0$ as $\delta \to 0$ and $u_\delta$ be a smooth solution with these smooth initial values. Since the relation $u_0 \to u$ is a Lipschitz continuous mapping from $[L^p(\Omega)]^{2n}$ to $C([0,T]; [L^p(\Omega)]^{2n})$ and the problem is linear, we have

$$
\|u^\delta_j - u^\delta_k\|_{C([0,T]; [L^p(\Omega)]^{2n})} \leq C\|u_0^\delta_j - u_0^\delta_k\|_{L^p(\Omega)^{2n}}
$$

for a constant $C > 0$. Therefore, the sequence of smooth solutions $\{u^\delta\}$ converge to the weak solution $u$. Since a smooth solution $u^\delta$ satisfies Theorem 4.2 and Corollary 1, we can deduce that $u$ satisfies the conclusions of the theorem corresponding to Theorem 4.2(1) and Corollary 1 by the continuity of the norms.

5. **Convergence and Div-Curl lemma.** The main theoretical part of the paper is in obtaining the singular limit as $\varepsilon \to 0$. First, by adding the $2n$ equations in (6)–(7), we obtain a conservation law for the total population,

$$
u_t + \nabla \cdot J = 0. \quad (20)
$$

By subtracting (7) from (6), we obtain $n$ equations for each components of the flux,

$$
\frac{\varepsilon^2}{v(x)} \frac{\partial J_k}{\partial t} + (v(x)u^k)_{x_k} = -\frac{\mu(x)}{v(x)} J_k, \quad k = 1, \ldots, n. \quad (21)
$$

To show the convergence of the singular limit as $\varepsilon \to 0$, we need to show the convergence of the following two sequences,

$$(v(x)u^{k,\varepsilon})_{x_k} \to (v(x)\frac{u}{\varepsilon})_{x_k} \quad \text{and} \quad \frac{\varepsilon^2}{v} \frac{\partial J_k}{\partial t} \to 0. \quad (21)
$$

The main part of this section is to show the convergence of these two. If they are done, (21) implies

$$
J_k^\varepsilon \to -\frac{v(x)}{\varepsilon \mu(x)} (v(x)u)_{x_k}.
$$

After substituting them into (20), we obtain

$$
u_t = \nabla \cdot \left( \frac{v(x)}{\varepsilon \mu(x)} \nabla (v(x)u) \right) \quad \text{for} \quad (x,t) \in \Omega \times (0, \infty). \quad (22)
$$

If the diffusivity $D$ is given as in (10), we may rewrite (22) in the form of our diffusion law (1) and completes the proof of Theorem 1.1. In this section, we show that the solutions of (6)–(9) converge to a weak solution of (22) as $\varepsilon \to 0$. 


Lemma 5.1. Let \( u^{i,\epsilon}, j = 1, \ldots, n, \) be weak solutions of (6)–(9) with initial values \( u^j_0 \in L^2(\Omega) . \) Then, for any given \( T > 0, \) there is a weakly convergent subsequence \( u^{i,\epsilon} \) such that, as \( \epsilon \to 0, \)
\[
 u^{i,\epsilon} \to u^i \quad \text{weakly in } L^2(\Omega_T),
\]
\[
 J^{u,\epsilon}_{i,j} \to J^u_{i,j} \quad \text{weakly in } L^2(\Omega_T).
\]

Proof. We have already shown that \( u^{i,\epsilon} \) and \( J^{u,\epsilon} \) are uniformly bounded in \( L^2(\Omega_T) . \)
Therefore, there exist weakly convergent subsequences \( \{u^{i,\epsilon} \} \) and their limits \( u^i \in L^2(\Omega_T), \) i.e.,
\[
 u^{i,\epsilon} \to u^i \quad \text{weakly in } L^2(\Omega_T).
\]
By taking further subsequence, we may assume \( \epsilon^{1+} \) is a subsequence of \( \epsilon^{-} \), \( \epsilon^{2-} \) is a subsequence of \( \epsilon^{1+} \), and so on. In other words, \( \epsilon^{j} \) is a subsequence of \( \epsilon^{j'} \) for \( j' < j. \)

By denoting \( \epsilon^{\ell} := \epsilon^{n+} \), we have
\[
 u^{i,\epsilon} \to u^i \quad \text{weakly in } L^2(\Omega_T), \quad j = 1, \ldots, n \pm .
\]
Since \( J^{u,\epsilon}_{i,j} \) is again a uniformly bounded sequence in \( L^2(\Omega_T) \) for \( i, j = 1, \ldots, n, \)
we may repeat the process and obtain a subsequence of \( \epsilon^{\ell} \), which is denoted by \( \epsilon^{\ell} \)
again, such that
\[
 J^{u,\epsilon}_{i,j} \to J^u_{i,j} \quad \text{weakly in } L^2(\Omega_T)
\]
for all \( i, j = 1, \ldots, n \pm . \)

Next, we show that the obtained subsequence satisfies
\[
 (v(x)u^{k,\epsilon})_{x_k} \to (v(x)u_{x_k}) \quad \text{in } H^{-1}(\Omega_T), \tag{23}
\]
\[
 \frac{\epsilon^2}{v} \frac{\partial J^{\epsilon}_{k}}{\partial t} \to 0 \quad \text{in } H^{-1}(\Omega_T). \tag{24}
\]
This part is needed to make the formal derivation of the diffusion equation (22) to be rigorous. In the rest of this section, we obtain the convergence and complete the convergence of the singular limit to the unique solution of (1).

Lemma 5.2. Let \( u^{i,\epsilon} \) be the subsequence obtained in Lemma 5.1. Then, the convergences in (23) and (24) hold as \( \ell \to \infty. \)

Proof. For \( i, j = 1, \ldots, n \pm, \) we already know that \( L^2 \) norm of \( J^{\epsilon}_{i,j} \) is uniformly bounded independent to \( \epsilon \) from Theorem 4.2(2). Using this fact,
\[
 \|u^{i,\epsilon} - u^{k,\epsilon}\|_{L^2(\Omega_T)} = \frac{\epsilon^2}{v(x)} \|J^{\epsilon}_{i,j}\|_{L^2(\Omega_T)} \leq \epsilon^2 M \|J^\epsilon_{i,j}\|_{L^2(\Omega_T)} \to 0 \quad \text{as } \ell \to \infty.
\]
Equivalently, \( u^{i,\epsilon} \to u^i \) weakly in \( L^2(\Omega_T) \) and hence
\[
 (v(x)u^{k,\epsilon})_{x_k} \to (v(x)u_{x_k}) \quad \text{in } H^{-1}(\Omega_T).
\]
Let \( K := \{ \phi \in H^1_0(\Omega_T) : \|\phi\|_{H^1_0(\Omega_T)} \leq 1 \}. \) Then, since \( J^{\epsilon}_{i,j} \) is bounded in \( L^2(\Omega_T), \)
\[
 \|\frac{\partial J^{\epsilon}_{k}}{\partial t}\|_{H^{-1}(\Omega_T)} = \sup_{\phi \in K} \|\partial_t J^{\epsilon}_{k,-k}\| \sup_{\phi \in K} \|\phi\|_{L^2(\Omega_T)} \|J^{\epsilon}_{k,-k}\|_{L^2(\Omega_T)} \leq \|J^{\epsilon}_{k,-k}\|_{L^2(\Omega_T)} < \infty.
\]
Therefore, we have
\[
\| \epsilon^2 \frac{\partial J_k^{\epsilon \ell}}{\partial t} \|_{H^{-1}(\Omega_T)} \leq M \epsilon^2 \| \frac{\partial J_k^{\epsilon \ell}}{\partial t} \|_{H^{-1}(\Omega_T)} \to 0 \quad \text{in } H^{-1}(\Omega_T).
\]
The convergence in (23) and (24) are obtained.

Now, we are going to prove the strong convergence when the initial values \(u_0^\ell\) are placed in \(L^4(\Omega)\). The key ingredient is the Div-Curl lemma.

**Lemma 5.3** (Div-Curl Lemma). Suppose that \(A \subset \mathbb{R}^{n+1}\) is open and two sequences of functions \(\mathbf{w}^\ell, \mathbf{z}^\ell : A \to \mathbb{R}^{n+1}\) are given. Suppose further that
\[
\begin{align*}
\mathbf{w}^\ell &\rightharpoonup \mathbf{w} \text{ weakly in } [L^2(A)]^{n+1}, \\
\mathbf{z}^\ell &\rightharpoonup \mathbf{z} \text{ weakly in } [L^2(A)]^{n+1}, \\
\nabla \cdot \mathbf{w}^\ell &\text{ is bounded in } L^2(A), \\
\text{curl}(\mathbf{z}^\ell) &\text{ is bounded in } [L^2(A)]^{(n+1)^2},
\end{align*}
\]

Then,
\[
\langle \mathbf{w}^\ell, \mathbf{z}^\ell \rangle \to \langle \mathbf{w}, \mathbf{z} \rangle \text{ in the distribution sense,}
\]
where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(\mathbb{R}^{n+1}\).

To apply the Div-Curl Lemma, we first arrange solutions and fluxes in a form that fits to the lemma.

**Lemma 5.4.** Let \(u_0^\ell \in L^2(\Omega)\), \(\mathbf{w}^\ell = (\mathbf{J}^\ell, u^\epsilon)\), and \(\mathbf{z}^\ell = (0, v(x)u^\epsilon)\). Then, both sequences \(\mathbf{w}^\ell\) and \(\mathbf{z}^\ell\) are in \([L^2_x(A_{\Omega_T})]^{n+1}\) and \(\langle \mathbf{w}^\ell, \mathbf{z}^\ell \rangle \to \langle \mathbf{w}, \mathbf{z} \rangle\) in the distribution sense.

**Proof.** It is enough to show that \(\mathbf{w}^\ell\) and \(\mathbf{z}^\ell\) satisfy the four conditions in (25)–(28).

By Lemma 5.1, \(\mathbf{w}^\ell \rightharpoonup \mathbf{w}\) and \(\mathbf{z}^\ell \rightharpoonup \mathbf{z}\) weakly in \([L^2(\Omega_T)]^{n+1}\) and hence the first two conditions (25) and (26) are satisfied. Equation (20) implies (27), i.e.,
\[
\nabla_{x,t} \cdot \mathbf{w}^\ell = u_0^\ell + \nabla \cdot \mathbf{J}^\ell = 0.
\]

It is left to show the compactness condition (28) on \(\text{curl}(\mathbf{z}^\ell)\) in \([H^{-1}(\Omega_T)]^{(n+1)^2}\).

Since \(u^\epsilon = \sum_{k=1}^n u_k^\epsilon\),
\[
u^\epsilon = nu_x^\epsilon + \sum_{k'=1}^n (u_{k'}^\epsilon - u_k^\epsilon),
\]
and
\[
\partial_{x_k}(v(x)u^\epsilon) = n \partial_{x_k}(v(x)u_k^\epsilon) + \sum_{k'=1}^n \partial_{x_k}[v(x)(u_{k'}^\epsilon - u_k^\epsilon)].
\]

By (21), we have
\[
\partial_{x_k}(v(x)u_k^\epsilon) = -\frac{\mu(x)}{v(x)} J_k^\epsilon - \frac{\epsilon^2}{v(x)} \frac{\partial J_k^\epsilon}{\partial t}.
\]

The compactness of \(\frac{\epsilon^2}{v(x)} \frac{\partial J_k^\epsilon}{\partial t}\) has been obtained in Lemma 5.2. In addition, \(J_k^\epsilon = \frac{1}{\epsilon^2} v(x)(u_k^\epsilon - u_k^{-\epsilon})\) is bounded in \(L^2(\Omega_T)\) by Theorem 4.2(2) and is a compact operator in \(H^{-1}(\Omega_T)\). Thus, we can conclude that \((v(x)u_k^\epsilon)_{x_k}\) is compact in \(H^{-1}(\Omega_T)\). We also have
\[
\|v(x)(u_{k'}^\epsilon - u_k^\epsilon)\|_{L^2(\Omega_T)} = \|\epsilon^2(J_{k'}^\epsilon - J_k^\epsilon)\|_{L^2} \leq 2\epsilon^2 \sqrt{U_0},
\]
which converges to 0 as $\ell \to 0$. Thus, $\partial_{x_k}[v(x)(u_k^{\ell,\epsilon} - u_k^{\ell,\epsilon})]$ are compact in $H^{-1}(\Omega_T)$. By (29), $\partial_{x_k}(v(x)u^{\epsilon})$ is also compact in $H^{-1}(\Omega_T)$ for all $k$ and it completes the proof.

Lemma 5.5. If $u_0 \in L^4(\Omega)$ for all $i = 1, \cdots, n \pm$ (or $u_0 \in [L^4(\Omega)]^{2n}$), there is a sequence $\epsilon_\ell \to 0$ as $\ell \to \infty$ such that, for all $T > 0$,

$$ u^{\ell,\epsilon} \to u \quad \text{strongly in } L^2(\Omega_T). $$

Proof. Lemma 5.1 implies that, for all $i = 1, \cdots, n \pm$,

$$ u^{i,\epsilon} \to u^i, \quad u^{\epsilon} \to u \quad \text{weakly in } L^2(\Omega_T). $$

From Lemma 5.2, we have $u^i = \frac{1}{n} u$ for some $u \in L^2(\Omega_T)$. We denote $u^{k,\epsilon} := u^{k+,\epsilon} + u^{k-,\epsilon}$. Then, the Div-Curl lemma implies that

$$ v(x)(u^{k,\epsilon})^2 \to v(x)(u^k)^2 = \frac{v(x)}{n^2} u^2 $$

in the distribution sense. The uniform boundedness of $(u^{k,\epsilon})^2$ and the fact that $(\frac{1}{n} u)^2$ is in $L^2$ comes from Corollary 1 with $p = 4$. In addition, $v(x)$ is well-defined and bounded away from zero. Therefore,

$$ (u^{k,\epsilon})^2 \to \frac{1}{n^2} u^2 \quad \text{weakly in } L^2(\Omega_T). $$

Since $u^{\epsilon} = \sum_{k=1}^n u^{k,\epsilon}$, we obtain the weak convergence of $(u^{\epsilon})^2 \to u^2$. The strong $L^2$ convergence comes from [22, Lemma 7], which states that $u^{\epsilon} \to u$ strongly in $L^2(\Omega_T)$ if $|\Omega_T| < \infty$, $u^{\epsilon} \to u$, and $(u^{\epsilon})^2 \to u^2$ weakly in $L^2(\Omega_T)$.

Now we finish the proof of the main theorem.

Proof of Theorem 1.1. Note that $u^{\epsilon} := \sum_{j=1}^{n} u^{i,j,\epsilon} \to u$ strongly in $L^2(\Omega_T)$. In addition, for $i, j = 1, \cdots, n$, $u^{i,j,\epsilon} \to u^i$, $\nabla \cdot J^{\epsilon} \to 0$ as $\ell \to \infty$.

Therefore, the sequence of solutions of the system (20)–(21),

$$ u_t^{\epsilon} + \nabla \cdot J^{\epsilon} = 0, $$

$$ \frac{\epsilon^2}{v} \frac{\partial J_k^{\epsilon}}{\partial t} + (v(x)u^{k,\epsilon})_{x_k} = -\frac{\mu(x)}{v(x)} J_k^{\epsilon}, $$

converge to a solution of

$$ u_t + \nabla \cdot J = 0, $$

$$ \left( \frac{v(x)}{n} u \right)_{x_k} = -\frac{\mu(x)}{v(x)} J_k $$

in the distribution sense. After the substitution of (31) into (30), we can see that the limit $u$ is the weak solution of the diffusion equation

$$ u_t = \nabla \cdot \left( \frac{v(x)}{n\mu(x)} \nabla (v(x)u) \right) $$

with the periodic boundary condition and the initial condition. It is classical that the weak solution of (32) is unique. This implies that the subsequential convergence
as $\epsilon \ell \to 0$ is actually the convergence as $\epsilon \to 0$. Since the diffusivity $D$ is given by (10), (32) is written as (1) and the proof is completed.

**Remark 3.** In one space dimension ($n = 1$), there is a simpler way to obtain the diffusion equation (32) as a singular limit of solutions to discrete kinetic equations. Consider a new space variable given by

$$y = \int_{x_0}^{x} \frac{1}{v(s)} ds. \quad (33)$$

This change of variable stretches or shrinks the space according to the given speed $v(x)$ and makes the nonconstant speed in $x$ variable a constant one $\tilde{v}(y) = 1$ in $y$ variable. The particle density in a new space variable becomes $w(y,t) = v(x)u(x,t)$.

However, the turning frequency $\tilde{\mu}$ in the new variable $y$ is not changed, i.e., $\tilde{\mu}(y) = \mu(x)$.

Since $dy = \frac{1}{v(x)} dx$, we have

$$w_t = vu_t = v \left( \frac{v(x)}{\mu(x)} (vu)_{x} \right)_x = v \frac{dy}{dx} \left( \frac{v(x)}{\mu(x)} \frac{dy}{dx} w_y \right)_y = \left( \frac{1}{\tilde{\mu}(y)} w_y \right)_y,$$

which is Fick’s law (3) with $D(y) = \frac{1}{\tilde{\mu}(y)}$. It is classical that the singular limit of a discrete kinetic model with a constant speed and a heterogeneous turning frequency,

$$w^+_t + \frac{1}{\epsilon} w^+_y = \frac{\tilde{\mu}(y)}{\epsilon^2} (w^- - w^+),$$

$$w^-_t - \frac{1}{\epsilon} w^-_y = \frac{\tilde{\mu}(y)}{\epsilon^2} (w^+ - w^-),$$

converges to Fick’s law. Therefore, after changing the space variable $y$ back to the original one $x$, we may obtain the diffusion equation (32) for $n = 1$. However, this technique works for one space dimension only. This one dimensional argument has been used in several examples in a different context (see [5, Section 2]).

6. **Discussion: What is the correct diffusion law?** It is left to answer the question whether (1) is the correct diffusion law. There are subtle issues to consider before answering this important question. In the paper, we have proved that (1) is the diffusion equation satisfied by the singular limit of solutions to the discrete kinetic model (6)–(9). However, that is not enough to say (1) is the correct one since other diffusion laws in (3)–(5) can be satisfied by the limits of other kinetic or random walk models. Hence, the main focus should be on constructing kinetic and random walk models correctly.

The revertibility is an important requirement when we build up a random walk model. It seems that, as long as a random walk system is revertible, the obtained diffusion equation is in the form of (1). Another issue is the choice of parameters of heterogeneity. In (6)–(7), we have the spatial heterogeneity in the frequency $\mu$ and the speed $v$. If one considers different discrete kinetic equations with more heterogeneous components, a different diffusion law may emerge. On the other hand, if we reduce the heterogeneity by taking a constant turning frequency $\mu$, for example, (1) is equivalent to Wereide’s diffusion law (4). Therefore, we cannot argue that (4) is wrong since it is different from (1).

The best way to validate a diffusion law is to compare the solution with physical experiments. However, experiments did not help much so far (see [14]). The main reason is that the diffusivity $D$ alone is not enough to determine the phenomenon
if the environment is heterogeneous. Since \( \mu \) and \( D \) are two independent functions, we need two parameters and (1) can be written as

\[
u_t = \nabla \cdot (K \nabla (\mu u)), \quad D = KM \text{ and } \mu = M/K, \tag{34}
\]

where \( K \) is the diffusion conductivity and \( M \) is the motility. We need to measure the two quantities, \( K \) and \( M \), to explain the diffusion phenomenon.

Choosing a different diffusion model may lead to completely opposite conclusions and the right choice of a diffusion law is essential to build up a correct theory in a heterogeneous environment. We end the discussion with such an example taken from [23]. Shigesada et al. [24] took Fick’s diffusion law (3) and proposed a biological invasion model,

\[
u_t = \nabla \cdot (D(x)\nabla u) + (r(x) - u)u, \quad x \in \mathbb{R},
\]

where the diffusivity \( D \) and the growth rate \( r \) are periodic functions given by

\[
(D(x), r(x)) = \begin{cases} 
(1, 1), & mL < x \leq mL + L_a, \\
(D_b, -r_b), & mL + L_a < x \leq mL + L, \quad m \in \mathbb{Z}.
\end{cases}
\]

In this model the growth rate in undesirable patches is negative \( r = -r_b < 0 \) and in desired ones is positive \( r = 1 \).

The question is whether it is beneficial for survival to increase the dispersal rate in undesirable patches. The answer of Shigesada et al. [24] is against it. They showed from the solution analysis that the chance for survival increases if the dispersal rate decreases in bad patches, i.e., if \( D_b < 1 \). This paradox arises from the use of a wrong diffusion model that does not fit to the situation. Seo and Kim [23] replaced Fick’s law with the diffusion model (1) and considered

\[
u_t = \nabla \cdot (\sqrt{\mu - 1}D \nabla (\sqrt{\mu}Du)) + (r(x) - u)u.
\]

For a fair comparison, \( \mu \) is assumed to be constant which actually gives Wereide’s law (4). They showed that the chance for survival increases if the dispersal rate increases in a harsh environment, i.e., if \( D_b > 1 \). This is exactly the opposite conclusion of Shigesada et al. and there is no paradox anymore.

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