Gravitational interaction for light-like motion in classical and quantum theory

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Abstract

On the basis of an exact vacuum solution of Einstein’s equations, *vis.* the pencil-of-light field, we study the light-like motion of test and non-test objects. We also consider the quantum theoretical interaction of massless scalar particles through virtual gravitons. The dragging phenomenon is manifested and its agreement with astronomical observations established.

This paper submitted to arXiv is a somewhat reedited copy of my article dedicated to Dr. Ivar Piir in a volume published on the occasion of his 60th birthday in 1989 in Tartu by the Estonian Academy of Sciences.

1 Introduction

The interplay of classical and quantum physics is very profound. On the one hand, the classical Poisson brackets are often heuristically replaced by commutators, or similar procedures are performed. On the other hand, the Ehrenfest theorem helps to “deduce” the classical theory from the quantum one; there also exist other forms of the correspondence principle. One of these was noted by I.Piir [3]: it related the cross-section of scattering of a photon on the Schwarzschild centre with the classical light bending effect in the Schwarzschild field. This analysis, published as early as in 1957, retains its fundamental importance even today. Continuing the study, I extended these results to include this scattering of other particles, as well as to make use of only “one half” of the correspondence principle limit, *i.e.* when one
of the two mutually scattered particles became a classical centre, while the other continues to be a quantum theoretical object (see [2, 3]).

Another trend of research in classical General Relativity at all times was to study dragging effects, beginning with Einstein’s admission of Mach’s principle [4], and the approximate results of Thirring and Lense [5, 6]. Usually, the dragging effects are related to the rotation of the gravitational field source (this reflects the idea of Mach’s principle); however, similar and not less striking phenomena are also predicted for motion in the NUT field [7] and in the pencil-of-light field [8]. Neither of these fields is associated with the rotation, but they, and also the Kerr field, have exact electromagnetic analogues, *vis.* the characteristic magnetic or quasi-magnetic fields. Therefore one could treat the dragging phenomenon as a totality of quasi-magnetic or -electric effects in the gravitation theory.

However, the quantum theoretical manifestations of dragging are practically unknown (not mentioning the spin-orbital type effects for the general relativistic Dirac field, namely effects in the primarily quantized theory; see [9, 3]). I dare to try to fill this gap and give here new examples of the classical and quantum theoretical parallels in this field, taking the opportunity of the 60th birthday of my dear friend and highly respected colleague Ivar Piir, in the hope that he would enjoy reading here some lines on the subjects of quantum and classical theory of gravitation, the theory so close and exciting for both of us.

An approximate expression of the gravitational field of a light-like linearly extended object was found in 1931 (my birth-year) by Tolman, Ehrenfest, and Podolsky [10]. An exact solution for this field of a pencil of light [8] turned out to be a special case of the well-known Peres’ wave [11]. This is in fact not a wave, the field is atationary, though it belongs to the Petrov type N. In our discussion of the classical dragging phenomena we shall consider a further generalization of this field to a case when the “pencil” possesses angular momentum oriented along it (which is the spatial direction of its light-like motion) [12]:

\[
\begin{align*}
\frac{ds^2}{\kappa} &= 2dv(du + \kappa \ln \sigma \rho dv + gd\varphi) - \rho^2 d\phi^2 - d\rho^2.
\end{align*}
\]

(1)

Here \(\kappa = 8\gamma\varepsilon\), \(\gamma\) being the Newtonian gravitational constant and \(\varepsilon\), the linear energy density along the \(z\) axis; the function \(g(v) \left( \frac{dv}{dt} \neq 0 \right)\) describes the angular momentum of the source. In this vacuum solution \(\kappa\) may depend on \(v\), but here we assume it to be constant for the sake of simplicity (but for constant \(g\) the angular momentum vanishes).
2 A classical case of the light-like motion

Now we take \( ds^2 = 0 \) and denote the affine parameter along geodesics by \( \lambda \). The motion of a test particle is described by the geodesic equations,

\[
\frac{d}{d\lambda} \left( g_{\alpha\beta} \frac{dx^\beta}{d\lambda} \right) = \frac{1}{2} g_{\mu\nu,\alpha} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda},
\]

which are readily integrated for \( \alpha \) labelling the variables \( v \) and \( \varphi \):

\[
v = A\lambda + B, \quad A = \text{const., } B = \text{const}
\]

(the second integral of motion) and

\[
\frac{d\varphi}{d\lambda} = \frac{Ag + C}{\rho^2},
\]

where \( C = \text{const} \) (the first integral). The equations which correspond to the variables \( u \) and \( \rho \), are

\[
\frac{d^2 u}{d\lambda^2} = -g \frac{d^2 \varphi}{d\lambda^2} - \frac{2\kappa A}{\rho} \frac{dp}{d\lambda},
\]

and

\[
\frac{d^2 \rho}{d\lambda^2} = -\frac{\kappa A^2}{\rho} + \frac{(Ag + C)^2}{\rho^3}.
\]

\( ds^2 = 0 \) is itself a first integral of motion; it reads

\[
2A \frac{du}{d\lambda} = -2\kappa A^2 \ln \sigma \rho - \frac{A^2 g^2}{\rho^2} + \frac{C^2}{\rho^2} + \left( \frac{dp}{d\lambda} \right)^2.
\]

One may replace by it one of the above equations, or use it in order to check the results on which the equation (7) imposes a certain constraint (the light-like motion case).

We consider first the simplest cases:

1. Let \( \varphi = \text{const.} \); then equation (4) yields \( A = 0 = C \). The equation (7) itself reduces to \( \frac{dp}{d\lambda} = 0 \), i.e. \( \rho = \text{const.} \), and from (5) we get \( u = K\lambda + L \), \( K \) & \( L \) being constants.

2. Let \( \rho = \text{const.} \). Then from (6) it follows that \( \frac{d\varphi}{d\lambda} = \sqrt{\kappa A} \rho = \text{const.} \), which can be realized only if \( A = 0 \) (since we admit \( \frac{dg}{dv} \neq 0 \)). The problem is thus reduced to the first case.
This shows absolutely no influence of the spinning pencil of light upon a test photon moving parallel (not antiparallel!) to it. The picture becomes more obvious if we put $\sqrt{2}u = t + z$ and $\sqrt{2}v = t - z$ (cf. [8, 12]).

The other cases describe photons moving non-parallel or (instantaneously) antiparallel to the pencil of light. The most interesting case among these would be when $\frac{d\phi}{d\lambda}$ and $\frac{d\rho}{d\lambda}$ both vanish at (say) $\lambda = 0$, $A \neq 0$:

3°. Let us choose fairly arbitrarily $\rho = f(\lambda)$ with $\left(\frac{df}{d\lambda}\right)|_{\lambda=0}$. Now a substitution of $\rho$ into equation (6) determines the function $g(v)$ (with the help of equation (13)). The equation (14) then yields $\phi(\lambda)$ via a straightforward integration, and the equation (5) yields $u(\lambda)$ as well. Constraints on the constants which enter the integrals emerge after substitution of them into (7). A further adjustment of the function $f(\lambda)$ should now be made in order to meet the condition with $\left(\frac{df}{d\lambda}\right)|_{\lambda=0}$. As an example of an adequate choice of the function $f$ we can produce here:

$$\rho = \rho_0 \left[1 + \frac{\kappa}{2} \left(\frac{A\lambda}{\rho_0}\right)^2\right]^{-1}.$$ 

Then

$$\frac{d\phi}{d\lambda} = \lambda \rho_0 A^2 C^3 \left[1 + \kappa \left(\frac{A\lambda}{\rho_0}\right)^2\right]^{1/2} \times \left[2 + \kappa \left(\frac{A\lambda}{\rho_0}\right)^2 + \frac{\kappa^2}{4} \left(\frac{A\lambda}{\rho_0}\right)^4\right]^{1/2}.$$ 

This solution describes both the gravitational attraction and angular dragging in the field of a spinning pencil of light for the case of non-parallel motion of a test photon. No interaction in the case of parallel motion (1° & 2°) gives a manifestation of dragging too, though a peculiar one; this case seems to be very important since it corresponds to exact non-interaction in a self-consistent problem (see [8, 12]). This means that a superposition of fields of two parallel pencils of light is also an exact vacuum Einstein field, and it generally admits only one (null) Killing vector, $\partial_u$.

3 Light-like motion in quantum theory

Within the scope of the quantum theory, we shall consider here interaction of two massless particles through virtual gravitons. Examples of such calculations can be found in [3], Section 7.3, though one has to set the rest masses
equal to zero in the respective formulae, so that I would have no need to consider in [3] the case of a parallel motion of incoming particles. If only two particles are present, we may transform the picture (via a Lorentz transformation) so that the particles will move either parallel or antiparallel to each other. The case of the parallel motion is a very special one, since it remains invariant under all transformations, while antiparallel, on the contrary, is a more general one. The latter case admits a further transformation to the centre-of-mass frame which was considered in [3]. However, the laboratory frame of Ref. 3 becomes now futile, since one of the particles there has to be supposed to be at rest which cannot hold when its rest-mass vanishes. Hence the parallel motion should be considered in a general frame, and the antiparallel motion too, because we are interested in a final transition to the scattering by a classical field.

We suppose the interacting particles to be the scalar ones, but the scalar fields to be distinguishable; this will simplify the calculations. The standard two-vertex diagram of Fig. 7 in [3] describes the process under consideration where \( r \) and \( s \) are the four-momenta of the incoming particles, and \( p \) and \( q \) those of the outgoing ones.

We shall use the expression (7.3.4) for the differential cross-section

\[
dσ = (2\pi)^2 p_0^2 |F|^2 \left[ \frac{\vec{r}}{r_0} - \frac{\vec{s}}{s_0} \right] \times \left[ 1 + \frac{\partial q_0}{\partial p_0} \right]^{-1},
\]

where the matrix element is (7.3.3),

\[
F = \frac{i\kappa^2 [(p \cdot r)(q \cdot s) - (p \cdot q)(r \cdot s)]}{4(2\pi)^2(p_0q_0r_0s_0)^{1/2}(p - r)^2},
\]

\( \kappa = \sqrt{2\kappa} \), \( \kappa \) being Einstein’s gravitational constant, and the conservation law \( p_\nu u + q_\nu = r_\nu + s_\nu \) is to be taken into account, as well as relations like \( p_0^2 = \vec{p}^2 \). It is easy to check that \( p \cdot r = q \cdot s, p \cdot s = q \cdot r, p \cdot q = r \cdot s \). Then

\[
(p \cdot r)(q \cdot s) - (p \cdot q)(r \cdot s) - (p \cdot s)(q \cdot r) = -2(p \cdot s)(p \cdot q).
\]

In the case of antiparallel motion the conservation law gives \( q_0^2 = \vec{q}^2 = (r_0 - s_0)^2 - p_0^2 + 2p_0q_0 \), so that \( \frac{\partial q_0}{\partial p_0} = 1 \), since \( \frac{\partial r_0}{\partial p_0} = \frac{\partial s_0}{\partial p_0} = 0 \) by definition. Moreover, \( |\vec{r}/r_0 - \vec{s}/s_0| = 2 \). If we denote \( \vec{r} \cdot \vec{p}/r_0p_0 = \cos \theta \), then \( r \cdot p = 2r_0p_0 \sin^2(\theta/2), s \cdot p = 2s_0p_0 \cos^2(\theta/2) \) and \( F = i\kappa^2 s_0 [2(2\pi)^2 r_0]^{-1} \cot^2(\theta/2) \). Hence,

\[
dσ = \frac{\kappa^4 p_0^2 s_0}{(8\pi)^2 r_0} \cot^4(\theta/2) d\Omega,
\]

5
which gives in the center-of-mass frame exactly the same expression which the equation (7.3.12) will yield if $m = M = 0$:

$$d\sigma_C = \kappa^4 p_0^2 (8\pi)^{-2} \cot^4 (\theta/2) d\Omega.$$  

On the other hand, in the case of parallel motion both the numerator and denominator in (9), as well as $|\vec{r}/r_0 - \vec{s}/s_0|$, vanish. One can cure this indeterminacy by starting with non-zero rest masses and passing to the limit of zero ones. Then $d\sigma_p = 0$.

One can now pass to scattering on a classical point-like particle moving with the velocity of light by taking $q_0 = s_0 \gg p_0 = r_0$. The cross-sections then become

$$d\sigma_{\perp} = \frac{\kappa^4 s_0^2}{(8\pi)^2} \cot^4 (\theta/2) d\Omega, \quad d\sigma_{\parallel} = 0,$$

which mean that the interaction between two light-like objects (with zero rest-masses) in a head-on collision ($\perp$) is four times greater than between a beam of light and a Schwarzschild “point” mass at rest (see [1, 2, 3]),

$$d\sigma_{\text{Schw}} = \frac{\kappa^4 m^2}{(16\pi)^2} \cot^4 (\theta/2) d\Omega,$$

whereas two classical objects, two quantum theoretical ones, or a pair consisting of one classical and one quantum object, moving in one and the same direction ($\parallel$) with the velocity of light (thus having no rest-masses), do not interact at all. If both objects are non-relativistic, we have to make use of the equation (7.3.14) in [3] with $p_0 = m$; the corresponding cross-section becomes another four times smaller than that of (12) (there enters also a difference in the angular dependence, but it is insignificant for a scattering in small angles).

### 4 Concluding remarks

In our simple calculations above we have shown the existence of a far-reaching harmony between quantum and classical physics also in the ultra-relativistic region, where a peculiar property of absolute non-interaction between any number of light-like particles moving in parallels, is revealed. This property is very natural from the standpoint of relativistic causality, and it leads to
a remarkable fact of exact superimposition ability of corresponding gravita-
tional fields in general relativity. See also a different approach developed by
Bonnor [13].

There exists even a very sensitive test which indisputably proves the theo-
etrical prediction of non-interaction between light-like objects moving in
parallels. This is the astronomical observation of optical images of distant
galaxies, which appears completely undisturbed, however far the emitters
of radiation were located (except of the influence of the interstellar non-
relativistic matter).

It is necessary to stress again (see [3], p. 249) the invalidity of the assertion
that the limit of a classical centre is achieved when the field intensity tends to
infinity (see [13], p. 191 in the Russian edition ⇒ pp. 260—262 in the English
edition); in reality, it is the property of Newtonian inertia (energies and/or
rest masses) of quanta of the corresponding field (not its intensity), that
should grow ‘infinitely’ in order that it would be possible to pass to scattering
of particles on the corresponding classical “centre”.

Acknowledgement

I am grateful to Dr. Ivar Piir for the inspirations he has always incited in me
and which have always led to fascinating studies of the enigmas of Nature.
Palju, palju önne, edu ja kordaminekuid!

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1This book has a very typical history for the Soviet era: When I had defended my
second doctoral Tesis (equivalent to the German “habil” Dissertation) in Moscow in 1970
having used [3] as the very Thesis, I received an invitation to publish its English translation
by the main scientific Hungarian publishing house. But when I stepped publicly forward
at the Peoples’ Friendship University in Moscow against the terrorists’ act in the Munich
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Olympic games (1972), I promptly got a letter from the publishing house in Budapest on the withdrawal of this offer, since they “have information that the English version of this book is in the process of being published elsewhere”. Naturally, there was no such an “elsewhere” at all.