Singularities in the metric of the classical solutions to the Einstein equations (Schwarzschild, Kerr, Reissner–Nordström and Kerr–Newman solutions) lead to appearance of generalized functions in the Einstein tensor that are not usually taken into consideration. The generalized functions can be of a more complex nature than the Dirac $\delta$-function. To study them, a technique has been used based on a limiting solution sequence. The solutions are shown to satisfy the Einstein equations everywhere, if the energy-momentum tensor has a relevant singular addition of non-electromagnetic origin. When the addition is included, the total energy proves finite and equal to $mc^2$, while for the Kerr and Kerr–Newman solutions the angular momentum is $mc^2$, whereas for the Reissner–Nordström and Kerr–Newman solutions correspond to the point charge in the classical electrodynamics, the result obtained allows us to view the point charge self-energy divergence problem in a new fashion.

**Keywords**: Self-energy; classical electron; point charge.

**Introduction**

A major general relativity principle is the equality of inert and gravitating masses. However, the classical solutions to the Einstein equations (Schwarzschild, Kerr, Reissner–Nordström and Kerr–Newman solutions) do not satisfy that principle at first sight. For the Schwarzschild and Kerr solutions the energy-momentum tensor and, hence, self-energy are zero, for the Reissner–Nordström and Kerr–Newman solutions the self-energy is infinite, whereas the gravitating mass is finite for all these solutions. A reason for this unconformity can be that the above solutions satisfy the Einstein equations not in the entire space. A property common to all of the solutions is existence of singularity of form $\sim 1/r$, $\sim 1/r^2$ in the metric. This suggests that the Einstein tensor containing second derivatives of the metric can contain the generalized functions, which are lost in the direct differentiation and, therefore, are...
not included in the energy-momentum tensor. The paper demonstrates that the Einstein tensor for the aforementioned solutions in fact contains the generalized functions, which can be of a more complex nature than the Dirac $\delta$-function. If we require validity of the Einstein equations in the entire space, including $r = 0$, then an appropriate singular term must be added to the energy-momentum tensor.

1. The Analogy with Electrostatics

It is simplest to elucidate the method determining if the generalized function appears in the singular function differentiation by the example of electrostatics. The point charge potential

$$\varphi = \frac{e}{r} \tag{1}$$

is singular at point $r = 0$ and satisfies Poisson equation:

$$\Delta \varphi = -4\pi \rho, \tag{2}$$

where $\rho = e\delta(r)$. A method to ascertain this is the following. Replace potential (1) by the nonsingular function of form

$$\tilde{\varphi} = \frac{e}{r} \theta(r - r_0) + \left(\frac{3e}{2r_0} - \frac{e r^2}{2 r_0^3}\right) \theta(r_0 - r), \tag{3}$$

where $\theta(x)$ is Heaviside function ($\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$). Having substituted this potential into (2), we find that $\tilde{\varphi}$ will be a solution to the Poisson equation for charge density

$$\rho = \frac{3e}{4\pi r_0^3} \theta(r_0 - r). \tag{4}$$

The integral of (4) over volume is independent of $r_0$ and equal to $e$. In the limit $r_0 \to 0$, $\tilde{\varphi} \to e/r$, $\rho \to e \delta(r)$, i.e. the limit of solution (3) corresponds to the presence of a point source with charge $e$ in the origin of coordinates and is a solution to equation (2). It is easy to show that this result is independent of the choice of the potential in range $r < r_0$, with the smooth behavior of the potential at point $r = r_0$ being not necessary. The result is always single: in the limit $r_0 \to 0$, the potential is $\varphi = \frac{e}{r}$ and the charge density is $\rho = e \delta(r)$. Below we apply a similar procedure to the classical solutions of the Einstein equations.

2. Self-Energy

What should be meant by the self-energy in the general relativity is not a trivial question. This question is typically solved using the energy-momentum pseudotensor (see, e.g., in Ref. 1 and references thereof). A demerit of the approach is that the

\[\text{For example, in electrostatics for the point charge potential we have } \Delta \varphi = -4\pi e \delta(r), \text{ while the direct differentiation yields } \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) = 0.\]
Point Charge Self-Energy in the General Relativity

A self-energy definition can be suggested based on the energy-momentum tensor of fields and material only. For stationary and static solutions there is Killing vector $\xi_0 = \partial/\partial t$ generating conserved current $J^\mu = T^\mu_\nu \xi^\nu_0$, where $\xi^\nu_0 = (1, 0)$ are the contravariant vector components. As $\nabla^\mu J^\mu = 0$, conservation law

$$\frac{d}{dt} \int d^3x \sqrt{-g} J^0 = \int dS_k \sqrt{-g} J^k_0.$$  \hspace{1cm} (5)

is satisfied. If the energy density is defined as a zero component of the current, then total energy

$$E = \int d^3x \sqrt{-g} J^0 = \int d^3x \sqrt{-g} T^0_0$$  \hspace{1cm} (6)

will be independent of a choice of the system of coordinates.

3. Reissner–Nordström Metric

The Reissner–Nordström solution is of the form

$$ds^2 = \frac{\Phi}{r^2} dt^2 - \frac{r^2 dr^2}{\Phi} - r^2 \left( \sin^2 \theta d\phi^2 + d\theta^2 \right),$$  \hspace{1cm} (7)

where $\Phi = r^2 - 2 m r + Q^2$ ($m$ and $Q$ are the mass and charge, respectively). This solution satisfies Einstein equations

$$G^\mu_\nu = 8\pi T^\mu_\nu,$$  \hspace{1cm} (8)

where $T^\mu_\nu = \frac{1}{4\pi} \left( F^\mu_\alpha F^\alpha_\nu + \frac{1}{4} g^\mu_\nu g^{\alpha\beta} F_\alpha^\alpha F_\beta^\beta \right)$ is the electromagnetic field energy-momentum tensor everywhere, except for point $r = 0$, at which the solution is singular. The singularity structure of the tensor $G^\mu_\nu$ and nature of the appearing generalized function can be found out using a procedure similar to that described in Section 1.

Consider the metric of form (7), having substituted the following function for $\Phi$ in it:

$$\tilde{\Phi} = (r^2 - 2 m r + Q^2) \theta(r - r_0) + \frac{r^2}{r_0^2} \left( r_0^2 - 2 m r_0 + Q^2 \right) \theta(r_0 - r).$$  \hspace{1cm} (9)

In so doing the metric becomes non-singular and in the limit $r_0 \to 0$ transfers to metric (7). The energy-momentum tensor corresponding to the metric can be derived from the Einstein equations. The $(0,0)$ component of the tensor is

$$T^0_0 = \frac{1}{8\pi} G^0_0 = \frac{Q^2}{8\pi r^4} \theta(r - r_0) + \left( -\frac{Q^2}{8\pi r^2 r_0^2} + \frac{m}{4\pi r^2 r_0} \right) \theta(r_0 - r).$$  \hspace{1cm} (10)

Those units are used, in which the gravitational constant and light speed are equal to 1.
In this expression the first term is the electrostatic field energy confined in range $r > r_0$. The second term appearing from the metric smoothing does not disappear in the limit $r_0 \to 0$. The self-energy in the solution constructed is

$$E = \int d^3x \sqrt{-g} T_0^0 = \frac{Q^2}{2r_0} + \left( \frac{Q^2}{2r_0} + m \right) = m.$$  \hfill (11)

Result (11) can be shown to be independent of the metric smoothing method. In the limit $r_0 \to 0$ relation (10) can be written as

$$T_0^0 = \frac{1}{\sqrt{-g}} \left( m \delta(r) + \frac{1}{2} Q^2 \varpi(r) \right).$$  \hfill (12)

Here $\varpi(r)$ is the generalized function determined by the following integration rule:

$$\int f(r) \varpi(r) d^3x = \int \frac{f(r) - f(0)}{r^4} d^3x,$$

where $f(r)$ is a bounded smooth function.

For the Schwarzschild metric ($Q = 0$ in (7)) the term $m \delta(r)$ in $T_0^0$ that corresponds to a point source can be obtained straightforwardly when the presence of term $\sim \Delta(1/r)$ in $G_0^0$ is considered. A more complicated generalized function $\varpi(r)$ appears as a source when $Q \neq 0$. It owes its origin to the presence of term $\sim \Delta(1/r^2)$ in $G_0^0$. Thus, the Schwarzschild and Reissner–Nordström solutions can be extended to the entire space, if the point source is added to the energy-momentum tensor.

4. Kerr-Newman Metric

Apply the same procedure to the Kerr–Newman metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \Psi \left( dt - \frac{(r^2 x_k + r(\mathbf{x} \times \mathbf{a})_k + a_k \mathbf{a} x^k)}{r(r^2 + a^2)} dx^k \right)^2.$$  \hfill (13)

Here

$$\Psi = -2 m r + \frac{Q^2}{r^2 + \frac{a^2}{r^2}},$$  \hfill (14)

where $\mathbf{a}$ is a space vector, $a$ is its module, $r$ is determined by equation

$$r^4 - r^2(x^2 - a^2) - (\mathbf{a} \mathbf{x})^2 = 0.$$  \hfill (15)

It is easy to show that this is an ordinary Kerr-Schild representation, if vector $\mathbf{a}$ is directed along axis $z$. The surfaces of constant $r$ are ellipsoids of revolution, whose axis coincides with the direction of the vector $\mathbf{a}$. With $r = 0$ the ellipsoid degenerates to a disk of radius $a$. The metric is continuous on the disk, but the components of the metric and electromagnetic field 4-potential and strength undergo a kink, while the electromagnetic field strength a discontinuity (see Fig. 1). This means that on the disk there is a singular distribution of mass, charge, and currents, which is not embodied in the electromagnetic field energy-momentum tensor.
Construct a solution similar to solution (9). To do this, in (13) replace $\Psi$ by function

$$\tilde{\Psi} = -\frac{2m r + Q^2}{r^2 + \frac{ax}{r}^2} \theta(r - r_0) + \frac{2m r_0 + Q^2}{r_0^2 + \frac{ax}{r_0}^2} \theta(r_0 - r).$$

The constructed solution is continuous everywhere, but has a derivative discontinuity at $r = r_0$. To satisfy the Einstein equations, the $(0,0)$-component of the energy-momentum tensor should be of the following form,

$$T_{00}^0 = \frac{1}{8\pi} G_0^0 = \frac{Q^2}{4\pi} \left( \frac{\rho^2}{r^2} - \frac{r^2}{\rho^2} - \frac{1}{2} \frac{1}{\rho^2} \right) \theta(r - r_0) +$$

$$r_0^2 \left( 2M r_0 - Q^2 \right) \frac{\left( r_0^4 - 3ax^2 \right)}{8\pi r_0^2 \left( r_0^4 + ax^2 \right)} \frac{\left( r_0^4 - ax^2 + r^6 a^2 + r^2 ax^2 a^2 \right)}{\left( r_0^4 + ax^2 \right)^3} \theta(r_0 - r) +$$

$$\delta(r - r_0) \left( M r_0 \left( r_0^4 - 3ax^2 + Q^2 \left( -r_0^4 + ax^2 \right) \right) \left( -ax^2 + r_0^2 a^2 \right) \right) \frac{\left( -ax^2 + r_0^2 a^2 \right)}{8\pi \left( r_0^4 + ax^2 \right)^3},$$

where $\rho = r^2 + \frac{ax^2}{r^2}$. The contribution to the self-energy made by each of three parts of $T_{00}^0$ with $r_0 \to 0$ constitutes diverging quantities, but in the aggregate the divergencies are surprisingly compensated:

$$E = \frac{Q^2}{4r_0} + \frac{Q^2 \left( r_0^2 + a^2 \right) \arctan \left( \frac{a}{r_0} \right)}{4 r_0^2 a} +$$

$$\left( 2m r_0 - Q^2 \right) \frac{-3}{4 r_0} + \frac{\left( 5 r_0^2 + a^2 \right)}{4 r_0^2 a} \arctan \left( \frac{a}{r_0} \right) -$$

$$\frac{\left( 2Q^2 - 5m r_0 \right)}{2 r_0} + \frac{\left( 2Q^2 r_0 - m \left( 5 r_0^2 + a^2 \right) \right) \arctan \left( \frac{a}{r_0} \right)}{2 r_0 a} = m.$$

The first, second and third lines in this relation are the contributions of ranges $r > r_0$, $r < r_0$ and surface $r = r_0$, respectively. It can be shown that, like for the
Reissner-Nordström metric, the result is independent of the smoothing method. The result obtained also applies to the Kerr metric (it will suffice to set \( Q = 0 \) in (13)).

The energy-momentum tensor also allows us to find the total moment of the system. The contributions to the total moment made by ranges \( r > r_0, r < r_0 \) and surface \( r = r_0 \) diverge in the limit \( r_0 \to 0 \), but the divergencies are compensated and the total moment proves equal to \( mca \).

The difficulties that appeared in the derivation of the relations in this paper have been overcome thanks to code Mathematica 5, Wolfram Research, Inc.

5. Conclusion

As shown in this paper, for the Schwarzschild, Kerr, Reissner–Nordström and Kerr–Newman solutions to satisfy the Einstein equations in the entire space, including \( r = 0 \), singular terms containing generalized functions should be added to the energy-momentum tensor. In so doing the total energy proves finite and equal to \( mc^2 \) for any solution. For the nonzero-charge solutions the addition plays the role of Poincare tensions, i.e. infinite negative mass is located at the center.

The existence of negative mass for the nonzero-charge solutions is also evidenced by the fact that for trial particles the gravitational attraction transfers to repulsion even at classical radius \( \frac{Q^2}{mc^2} \). One can determine this having analyzed the equations of motion of trial particles.

The presence of the singular terms in the energy-momentum tensor follows from the formal requirement of the solution validity in the entire space. The physical interpretation of the complete energy-momentum tensor may require involvement of other physical fields or matter.

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