McKay Matrices for Finite-dimensional Hopf Algebras

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Abstract. For a finite-dimensional Hopf algebra $A$, the McKay matrix $M_V$ of an $A$-module $V$ encodes the relations for tensoring the simple $A$-modules with $V$. We prove results about the eigenvalues and the right and left (generalized) eigenvectors of $M_V$ by relating them to characters. We show how the projective McKay matrix $Q_V$ obtained by tensoring the projective indecomposable modules of $A$ with $V$ is related to the McKay matrix of the dual module of $V$. We illustrate these results for the Drinfeld double $D_n$ of the Taft algebra by deriving expressions for the eigenvalues and eigenvectors of $M_V$ and $Q_V$ in terms of several kinds of Chebyshev polynomials. For the matrix $N_V$ that encodes the fusion rules for tensoring $V$ with a basis of projective indecomposable $D_n$-modules for the image of the Cartan map, we show that the eigenvalues and eigenvectors also have such Chebyshev expressions.

1 Introduction

Assume $A$ is a finite-dimensional associative algebra over an algebraically closed field $k$. Let $S_1, S_2, \ldots, S_m$ be the nonisomorphic simple (irreducible) $A$-modules and $P_1, P_2, \ldots, P_m$ be their projective covers, that is, the nonisomorphic indecomposable projective modules such that for each $j = 1, \ldots, m$, the module $P_j$ modulo its radical is isomorphic to $S_j$. The dimensions of these $A$-modules determine two column vectors in $\mathbb{Z}^m$,

$$s = [\dim(S_1) \ \dim(S_2) \ \cdots \ \dim(S_m)]^T$$ and
$$p = [\dim(P_1) \ \dim(P_2) \ \cdots \ \dim(P_m)]^T,$$

where $T$ denotes “transpose.” If $A$ is viewed as a left $A$-module under left multiplication, then

$$A = \bigoplus_{j=1}^m P_j^{\dim(S_j)}$$ and $$p^T s = \sum_{j=1}^m \dim(P_j) \dim(S_j) = \dim(A).$$ (1.2)

When $A$ is semisimple, then $P_j = S_j$ for all $j$, and the second part of (1.2) is the familiar result $\dim(A) = \sum_{j=1}^m (\dim(S_j))^2$.

There are two Grothendieck groups, $G_0(A)$ and $K_0(A)$ associated to $A$:

- $G_0(A)$ is the quotient of the free abelian group on the set of all isomorphism classes $[V]$ of finite-dimensional $A$-modules $V$ subject to the relations $[U] - [V] + [W] = 0$ for each short exact sequence $0 \to U \to V \to W \to 0$ of

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A-modules. By the Jordan-Hölder theorem, this group has a \( \mathbb{Z} \)-module basis consisting of the classes \([S_1], [S_2], \ldots, [S_m]\), and

\[
[A] = \sum_{j=1}^{m} \dim(P_j) [S_j] \quad \text{in} \quad G_0(A).
\]

- \( K_0(A) \) is the quotient of the free abelian group on the set of all isomorphism classes \([V]\) of finite-dimensional projective \( A \)-modules \( V \), subject to the relations \([U] - [V] + [W] = 0\) for each direct sum decomposition \( V = U \oplus W \) of \( A \)-modules. This group has a \( \mathbb{Z} \)-module basis consisting of the classes \([P_1], [P_2], \ldots, [P_m]\) due to the Krull-Remak-Schmidt theorem, and (1.2) says

\[
[A] = \sum_{j=1}^{m} \dim(S_j) [P_j] \quad \text{in} \quad K_0(A).
\]

When \( A \) is a Hopf algebra, the tensor product of two \( A \)-modules is an \( A \)-module with the \( A \)-action given by the coproduct, and the dual vector space of an \( A \)-module is an \( A \)-module via the antipode. Both \( G_0(A) \) and \( K_0(A) \) have products using \( \otimes \), so that \([U][W] = [U \otimes W]\), where \( G_0(A) \) is a ring with a unit element, which is the one-dimensional \( A \)-module \( k \) with action given by the counit, and in general, \( K_0(A) \) is a ring without a unit element.

Let \( V \) be an \( A \)-module, and set \( d = \dim(V) \). The McKay matrix \( M_V \) for tensoring with \( V \) has its \((i,j)\) entry \( M_{ij} = [S_i \otimes V : S_j] \), the multiplicity of \( S_j \) as a composition factor of the \( A \)-module \( S_i \otimes V \), or equivalently, the coefficient of \([S_j]\) when \([S_i \otimes V]\) is expressed as a \( \mathbb{Z} \)-linear combination of the basis elements in \( G_0(A) \). Then

\[
d s_i = \dim(V) \dim(S_i) = \dim(S_i \otimes V)
= \sum_{j=1}^{m} [S_i \otimes V : S_j] \dim(S_j) = \sum_{j=1}^{m} M_{ij} \dim(S_j) = \langle M_V s \rangle_i
\]

shows that the \( i \)th entry of \( M_V s \) is \( d \) times the \( i \)th entry of \( s \) for \( 1 \leq i \leq m \), which implies that \( s \) is a right eigenvector of \( M_V \) for the eigenvalue \( d = \dim(V) \).

One can argue as in the paper [20] by Grinberg, Huang, and Reiner that the dimension vector \( p^\perp = [\dim(P_1) \; \dim(P_2) \; \ldots \; \dim(P_m)] \) is a left eigenvector for \( M_V \) with eigenvalue \( d \). (But note that their McKay matrix and ours are transposes of one another, so for them \( p^\perp \) is a right eigenvector, and \( s \) is a left one.)

The matrix \( M_V \) can be viewed as the adjacency matrix of a quiver (the so-called McKay quiver determined by \( V \)) having nodes labeled by \( 1 \leq i \leq m \) that correspond to the simple modules \( S_i \). There are \([S_i \otimes V : S_j]\) arrows from \( i \) to \( j \). If there is an arrow from \( i \) to \( j \) and one from \( j \) to \( i \), they are replaced by a single undirected edge.

The idea to consider such a quiver and matrix goes back to McKay’s insight [30] that the quivers determined by tensoring with the \( G \)-module \( V = \mathbb{C}^2 \) for a finite subgroup \( G \) of \( SU_2 \) exactly correspond to the affine Dynkin diagrams of types \( A, D, E \). This result, subsequently referred to as the McKay correspondence, has been the inspiration for much work on a host of topics in singularity theory, group theory, orbifolds, and many other subjects.

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In expanding on McKay’s result, Steinberg [35] showed that for the group algebra \( CG \) of any finite group \( G \), the columns of the character table of \( G \) give a complete set of right eigenvectors for the McKay matrix determined by tensoring with any finite-dimensional \( G \)-module \( V \), and the associated eigenvalues are given by the character values \( \chi_j(g_i) \), as \( g_i \) ranges over a set of conjugacy class representatives of \( G \). Group algebras over algebraically closed fields of characteristic \( p > 0 \) were considered in [20], where it was shown that the columns of the Brauer character table of \( G \) are right eigenvectors of \( \mathbb{M}_V \) with corresponding eigenvalues \( \chi_j(g_i) \), as \( g_i \) ranges over a set of representatives for the \( p \)-conjugacy classes of \( G \) (that is, elements whose order is relatively prime to \( p \)). Each such \( g_i \) determines a left eigenvector whose coordinates are the character values of the projective indecomposable modules \( P_j \) evaluated at \( g_i \), and whose eigenvalue is \( \chi_i(g_j) \).

Aside from the result of Grinberg, Huang, and Reiner in [20] mentioned earlier, very little is known about the eigenvalues and eigenvectors of the McKay matrices for arbitrary finite-dimensional Hopf algebras, and the purpose of this paper is to remedy that situation. The specific case of the quantum group \( \mathbb{C}[sl_2] \) at \( q \) an odd root of unity was examined in depth in [3]. In that work, it was shown that the McKay matrix for tensoring with \( V = \mathbb{C}^2 \) has two-dimensional generalized eigenspaces for every eigenvalue different from \( 2 = \dim(V) \), and it is necessary to work with Jordan blocks of size \( 2 \times 2 \) in that example. The main point of [3] is that McKay matrices and quivers determine interesting Markov chains, and in the particular case of \( \mathbb{C}[sl_2] \), the chain exhibits new phenomena due to the existence of such Jordan blocks. The examples in [3] show that the analysis of the rates of convergence of the Markov chains determined by McKay matrices is quite delicate.

Most of the work on characters of Hopf algebras has been in the case that the algebra is semisimple. For example, Witherspoon defined a notion of a character table for finite-dimensional semisimple, almost cocommutative Hopf algebras and showed that the characters provide eigenvectors for the McKay matrix for tensoring with the simple modules (see [39, proof of Thm. 3.2]). In [11], Cohen and Westreich determined Verlinde formulas for semisimple, almost cocommutative Hopf algebras, and in [9, 10], for (non-semisimple) factorizable ribbon Hopf algebras such as the Drinfeld double of the Taft algebra considered here. Such formulas were introduced by Verlinde [38] for diagonalizing fusion relations in 2D rational conformal field theory and have played an important role in physics. They have been considered subsequently in many different contexts.

For a Hopf algebra \( A \) with nonisomorphic simple modules \( S_i \) and corresponding projective covers \( P_i \), the Cartan matrix \( C = (C_{ij}) \) records the multiplicity \( C_{ij} = [P_i : S_j] \) of \( [S_j] \) when \( [P_i] \) is expressed in the \( \mathbb{Z} \)-basis of \( G_0(A) \). Let \( r \) be the rank of \( C \) and select \( r \) of the \( P_i \) so that the corresponding rows of \( C \) are linearly independent. For each simple module \( S_j \) there is an \( r \times r \) matrix \( N^j \), that contains the fusion rules for tensoring those \( r \) projective modules with \( S_j \) and writing the answer \( [P_i \otimes S_j] \) as a linear combination of the \( r \) chosen projectives. It was shown in [9] that the matrices \( N^j \) are diagonalizable and have eigenvectors that can be expressed using the primitive idempotents \( e_i \) corresponding to the projective modules \( P_i \).

In Section 2 of this work, we consider arbitrary finite-dimensional Hopf algebras \( A \) and prove general results about McKay matrices, their eigenvalues, and their (left and right) eigenvectors by using the coproduct and the characters of simple and projective modules (Theorem 2.2 and Proposition 2.4). The tensor product \( P \otimes V \) of a projective
module $P$ with a finite-dimensional module $V$ is projective, and so $[P \otimes V]$ can be written as an integral combination of the classes $[P_j]$ of the projective indecomposable modules. Letting $Q_V = (Q_{ij})$, where $Q_{ij}$ is the multiplicity $[P_i \otimes V : P_j]$ in $[P_i \otimes V]$, we obtain what we term a projective McKay matrix. Theorems 2.11 and 2.13 and Corollary 2.14 show how $Q_V$ and its eigenvectors are related to the McKay matrix of the dual module $V^*$. In the special case that the Hopf algebra is semisimple, the McKay matrix $M_V$ and the projective McKay matrix $Q_V$ are the same, and if the module $V$ is self-dual, then the McKay matrix is orthogonally diagonalizable (Corollary 2.12).

In Section 3, we illustrate the general results of Section 2 by applying them to a family of nonsemisimple Hopf algebras, namely, the Drinfeld double $D_n$ of the Taft algebra $A_n$ for $n$ odd, $n \geq 3$. When $n$ is even, the eigenvalues exhibit different patterns, and that case will not be considered here. The algebras $D_n$ provide a convenient testing ground, as their representation theory has been developed in great detail by Chen and coauthors (see [4, 7, 5, 6, 8, 36]). Unlike the situation for semisimple, almost cocommutative Hopf algebras, the McKay matrices for $D_n$ fail to be diagonalizable. More specifically, we

• determine the eigenvalues, right and left eigenvectors and generalized eigenvectors of the McKay matrix $M_V$ obtained by tensoring the simple $D_n$-modules with one of the two-dimensional simple $D_n$-modules, $V(2,0)$ (Secs. 3.3-3.9);
• express the coordinates of these vectors using Chebyshev polynomials (Secs. 3.4, 3.5, 3.8);
• relate the eigenvectors to character values of the grouplike elements and other special elements of $D_n$ (Secs. 3.6, 3.7, 3.9) and show that the character value of a grouplike element on any simple $D_n$-module can be computed using Chebyshev polynomials (Thm. 3.9 of Sec. 3.6);
• prove that the (generalized) eigenvectors for $M_V$, $V = V(2,0)$, are (generalized) eigenvectors for the McKay matrix of any simple $D_n$-module (Sec. 3.10);
• show that the eigenvalues of the McKay matrix of any simple $D_n$-module can be expressed in terms of Chebyshev polynomials of the second kind (Thm. 3.23 of Sec. 3.11);
• find the eigenvectors and eigenvalues of the projective McKay matrix $Q_V$ by relating them to the McKay matrix of the dual module $V^*$ (Prop. 3.24 of Sec. 3.12);
• determine the structure of the complex Grothendieck algebra $G^*_0(D_n) = \mathbb{C} \otimes_{\mathbb{Z}} G_0(D_n)$ and prove that its Jacobson radical squares to 0; (Thm. 3.31 of Sec. 3.13);
• construct certain idempotents in $G^*_0(D_n)$, and show how they provide an alternate approach to producing the eigenvectors and generalized eigenvectors of $M_V$ (Sec. 3.13);
• compute the eigenvectors and eigenvalues of the matrix $N_V$ that encodes the fusion rules for tensoring a maximal set of independent projective covers in $G_0(D_n)$ with $V = V(2,0)$ (Sec. 3.14).

It has been said that “Chebyshev polynomials are everywhere dense in numerical analysis” (see [28, Sec. 1.1] for a discussion of this quotation). In this paper, Chebyshev polynomials (of the second, third, and fourth kind) are everywhere dense in expressing eigenvalues, eigenvectors, and generalized eigenvectors of McKay matrices and fusion
rules for $D_n$. The characters of the simple $D_n$-modules evaluated on the grouplike elements of $D_n$ also have Chebyshev polynomial expressions.

When $n$ is odd, $D_n$ is a ribbon Hopf algebra [25], and $D_n$ provides an invariant of 3-manifolds [21]. In [2], we determine the unique ribbon element of $D_n$ explicitly. We use the $R$-matrix and ribbon element of the quasitriangular Hopf algebra $D_n$ to obtain an algebra homomorphism from the Temperley-Lieb algebra $TL_k(-q^2 + q^{-2})$ to the centralizer algebra $\text{End}_{D_n}(V^\otimes k)$, when $V$ is any two-dimensional simple $D_n$-module.

In the special case that $V$ is the unique two-dimensional simple module that is self-dual, we show that the homomorphism is injective for all $k \geq 1$ and an isomorphism for $k \leq 2(n - 1)$. This leads to a realization of $\text{End}_{D_n}(V^\otimes k)$ for $V$ self-dual as a diagram algebra.

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2 McKay Matrices for Arbitrary Hopf Algebras

In the classical representation theory of finite groups, the columns of the character table are obtained by evaluating the characters (traces) of the simple modules on one element from each conjugacy class [18, Sec. 2.1]. It is known that these columns are right eigenvectors for any McKay matrix determined by tensoring with a finite-dimensional module of the group (see [35]). In this section, we develop an analog of this result by showing how grouplike elements and generalizations of skew primitive elements in an arbitrary Hopf algebra can be used to construct eigenvectors for McKay matrices. In the special case that the algebra is semisimple, more detailed results are possible.

Throughout Section 2, $A$ denotes a finite-dimensional Hopf algebra over an algebraically closed field $k$. All $A$-modules are assumed to be finite-dimensional, and all tensor products are over $k$. We adopt Sweedler’s notation for the coproduct $\Delta$ applied to an element $x \in A$,

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$ 

2.1 Right eigenvectors from traces of simple modules

We will use the coproduct of $A$ to obtain right (generalized) eigenvectors of McKay matrices by taking the trace of certain elements of $A$ on simple modules. To accomplish
this, we apply the following well-known results on traces. Parts (a) and (b) hold for any finite-dimensional algebra \( A \).

**Lemma 2.1** (a) Assume \( S, T, U \) are finite-dimensional modules over the algebra \( A \) such that \( U \cong S/T \). Then for any \( x \in A \), the trace \( \text{tr}_U(x) \) of \( x \) on \( U \) satisfies
\[
\text{tr}_U(x) = \text{tr}_S(x) + \text{tr}_T(x).
\]
(b) If \( U_1, \ldots, U_s \) are the composition factors of a finite-dimensional \( A \)-module \( U \), where \( U_i \) occurs with multiplicity \( c_i \), then for any \( x \in A \),
\[
\text{tr}_U(x) = c_1\text{tr}_{U_1}(x) + \cdots + c_s\text{tr}_{U_s}(x).
\]
(c) (See [27, Proposition 10.21 (b)].) For any finite-dimensional \( A \)-modules \( U \) and \( W \), and any \( x \in A \),
\[
\text{tr}_{U \otimes W}(x) = \sum_{i} \text{tr}_U(x_{(i)}) \text{tr}_W(x_{(2)}).
\]

Let \( S_1, S_2, \ldots, S_m \) be a \( \mathbb{Z} \)-basis of simple modules for the Grothendieck ring \( G_0(A) \) of the Hopf algebra \( A \), and for any \( x \in A \) set
\[
\text{Tr}_S(x) := [\text{tr}_{S_1}(x) \text{tr}_{S_2}(x) \ldots \text{tr}_{S_m}(x)]^T.
\]
Observe that \( \text{Tr}_S(1) = [\dim(S_1) \dim(S_2) \ldots \dim(S_m)]^T = \mathbf{s} \), the vector of dimensions of the simple \( A \)-modules. We know that \( \mathbf{s} \) gives a right eigenvector for any McKay matrix determined by tensoring with a finite-dimensional \( A \)-module. Next we explore some other elements of the Hopf algebra \( A \) that give such right eigenvectors.

**Theorem 2.2** Assume \( M_V = (M_{ij}) \), where \( M_{ij} = [S_i \otimes V : S_j] \), is the McKay matrix associated to tensoring with an \( A \)-module \( V \). Then the trace of \( x \in A \) on \( S_i \otimes V \) for any \( 1 \leq i \leq m \) is given by
\[
\sum_{x} \text{tr}_{S_i}(x_{(1)})\text{tr}_V(x_{(2)}) = \sum_{j=1}^{m} M_{ij}\text{tr}_{S_j}(x), \quad \text{and} \quad \sum_{x} M_V \text{Tr}_S(x) = \sum_{x} \text{Tr}_V(x_{(2)}) \text{Tr}_S(x_{(1)}).
\]

**Proof** This follows from Lemma 2.1 (c) and the fact that \( M_{ij} \) is the multiplicity of \( S_j \) as a composition factor of \( S_i \otimes V \). Equation (2.3) gives the \( i \)th coordinate of the matrix equation in (2.4).

**Corollary 2.3** For the McKay matrix \( M_V \) associated to tensoring the simple \( A \)-modules with \( V \), the following hold:

(a) When \( g \in A \) is grouplike, then \( \Delta(g) = g \otimes g \), and (2.4) says
\[
M_V \text{Tr}_S(g) = \text{tr}_V(g) \text{Tr}_S(g).
\]
Consequently, for every grouplike element of \( A \), \( \text{Tr}_S(g) \) is a right eigenvector for \( M_V \) with eigenvalue \( \text{tr}_V(g) \). When \( g = 1 \), this reverts to the result that the dimension vector \( \mathbf{s} \) is a right eigenvector with eigenvalue \( \text{tr}_V(1) = \dim(V) \).
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(b) When \( x \in A \) has the property that \( \Delta(x) = x \otimes y + z \otimes x \) for some nonzero \( y, z \in A \), then (2.4) says

\[
M_V \text{Tr}_S(x) = \text{tr}_V(y) \text{Tr}_S(x) + \text{tr}_V(x) \text{Tr}_S(z). \tag{2.6}
\]

The next result draws some useful conclusions from this equation. In part (2)(b) of Proposition 2.4, the phrase generalized right eigenvector of \( M_V \) with eigenvalue \( \lambda \) refers to a vector \( u \) such that \((M_V - \lambda I)u\) is a nonzero right eigenvector of \( M_V \) for \( \lambda \). Similarly, in Proposition 2.7 (2)(b) below, a generalized left eigenvector for the matrix \( Q_V^T \) with eigenvalue \( \lambda \) is a vector \( w \) such that \( w(Q_V^T - \lambda I) \) is a nonzero left eigenvector for \( Q_V^T \) with eigenvalue \( \lambda \).

**Proposition 2.4** Assume (2.6) holds for some \( x \in A \), and \( \text{Tr}_S(x) \neq 0 \).

1. If \( \text{tr}_V(x) = 0 \), then \( \text{Tr}_S(x) \) is a right eigenvector of \( M_V \) with eigenvalue \( \text{tr}_V(y) \).
2. If \( \text{tr}_V(x) \neq 0 \), and \( \text{Tr}_S(z) \) is a right eigenvector of \( M_V \) with eigenvalue \( \lambda \), then
   - \( \text{tr}_V(y) \neq \lambda \) implies that \( (\text{tr}_V(y) - \lambda) \text{Tr}_S(x) + \text{tr}_V(x) \text{Tr}_S(z) \) is a right eigenvector for \( M_V \) with eigenvalue \( \text{tr}_V(y) \);
   - \( \text{tr}_V(y) = \lambda \) and \( \text{Tr}_S(z) \neq \lambda \) imply that \( \text{Tr}_S(x) \) is a generalized right eigenvector for \( M_V \) with eigenvalue \( \lambda \); \( M_V \text{Tr}_S(x) = \lambda \text{Tr}_S(x) + \text{tr}_V(x) \text{Tr}_S(z) \).

**Remark 2.5** The case \( \text{tr}_V(y) = \lambda \) and \( \text{Tr}_S(x) = \delta \text{Tr}_S(z) \) for some \( 0 \neq \delta \in \mathbb{k} \) cannot happen when \( \text{Tr}_S(z) \) is assumed to be a right eigenvector for \( M_V \) with eigenvalue \( \lambda \) in (2), as this would imply that \( \text{Tr}_S(x) \) is a right eigenvector for \( M_V \) with both eigenvalue \( \lambda \) and eigenvalue \( \lambda + \delta^{-1} \text{tr}_V(x) \neq \lambda \).

### 2.2 Left eigenvectors from traces of projective modules

This section discusses using projective modules to produce left eigenvectors. Suppose \( P_1, P_2, \ldots, P_m \) are the nonisomorphic indecomposable projective modules for the Hopf algebra \( A \). The tensor product of any \( A \)-module \( V \) with a projective \( A \)-module is projective, hence the corresponding isomorphism class is a \( \mathbb{Z} \)-linear combination of the \([P_j] \) in the Grothendieck group \( K_0(A) \) (see [26, Sec. 3.1]). We use \([P_i \otimes V : P_j]\) to denote the multiplicity of \([P_j]\) in \([P_i \otimes V]\) and define

\[
Q_V = (Q_{ij}), \quad \text{where} \quad Q_{ij} = [P_i \otimes V : P_j]. \tag{2.7}
\]

We refer to the matrix \( Q_V \) as the projective McKay matrix and relate \( Q_V \) to a McKay matrix in Theorem 2.11 below. If \( x \in A \) and \( \Delta(x) = \sum x_{i(1)} \otimes x_{i(2)} \) then

\[
\sum_x \text{tr}_{P_i}(x_{i(1)}) \text{tr}_V(x_{i(2)}) = \text{tr}_{P_i \otimes V}(x) = \sum_j Q_{ij} \text{tr}_{P_j}(x) = \sum_j \text{tr}_{P_j}(x) (Q_V^T)_{ji}. \tag{2.8}
\]

This is the \( i \)th component of the matrix equation

\[
\text{Tr}_P(x) Q_V = \sum_x \text{tr}_V(x_{i(2)}) \text{Tr}_P(x_{i(1)}), \quad \text{where} \tag{2.9}
\]
We have the following analogs of (2.5) and (2.6):

\[ \text{Tr}_P(x) := \begin{bmatrix} \text{tr}_{P_1}(x) & \text{tr}_{P_2}(x) & \ldots & \text{tr}_{P_m}(x) \end{bmatrix}. \quad (2.10) \]

**Corollary 2.6** Let \( Q^A_V \) be the McKay matrix associated to tensoring the projective indecomposable \( A \)-modules with \( V \). Then

(a) When \( g \in A \) is grouplike,

\[ \text{Tr}_P(g)Q^A_V = \text{tr}_V(g) \text{Tr}_P(g). \quad (2.11) \]

Consequently, for every grouplike element of \( A \), \( \text{Tr}_P(g) \) is a left eigenvector for \( Q^A_V \) with eigenvalue \( \text{tr}_V(g) \). When \( g = 1 \), the eigenvalue is \( \text{tr}_V(1) = \dim(V) \), and the eigenvector \( \text{Tr}(1) \) is just the dimension vector \( \mathbf{p} = [\dim(P_1) \ \dim(P_2) \ \ldots \ \dim(P_m)] \).

(b) When \( x \in A \) has the property that \( \Delta(x) = x \otimes y + z \otimes x \) for some nonzero \( y, z \in A \), then (2.9) says

\[ \text{Tr}_P(x)Q^A_V = \text{tr}_V(y) \text{Tr}_P(x) + \text{tr}_V(x) \text{Tr}_P(z). \quad (2.12) \]

The next result is the projective version of Proposition 2.4.

**Proposition 2.7** Assume (2.12) holds for \( x \in A \) and \( \text{Tr}_P(x) \neq 0 \).

1. If \( \text{tr}_V(x) = 0 \), then \( \text{Tr}_P(x) \) is a left eigenvector for \( Q^A_V \) with eigenvalue \( \text{tr}_V(y) \).
2. If \( \text{tr}_V(x) \neq 0 \), and \( \text{Tr}_P(z) \) is a left eigenvector for \( Q^A_V \) with eigenvalue \( \lambda \), then

   (a) \( \text{tr}_V(y) \neq \lambda \) implies that \( (\text{tr}_V(y) - \lambda) \text{Tr}_P(x) + \text{tr}_V(x) \text{Tr}_P(z) \) is a left eigenvector for \( Q^A_V \) with eigenvalue \( \text{tr}_V(y) \);

   (b) \( \text{tr}_V(y) = \lambda \) and \( \text{Tr}_P(x) \neq k \text{Tr}_P(z) \) imply that \( \text{Tr}_P(x) \) is a generalized left eigenvector for \( Q^A_V \) with eigenvalue \( \lambda \): \( \text{Tr}_P(x)Q^A_V = \lambda \text{Tr}_P(x) + \text{tr}_V(x) \text{Tr}_P(z) \).

**Remark 2.8** As in Remark 2.5, the case \( \text{tr}_V(y) = \lambda \) and \( \text{Tr}_P(x) = \delta \text{Tr}_P(z) \) for some \( 0 \neq \delta \in k \) cannot happen when \( \text{Tr}_P(z) \) is assumed to be a left eigenvector for \( Q^A_V \) with eigenvalue \( \lambda \).

### 2.3 The Cartan map and Cartan matrix

We will use the Cartan map to relate McKay matrices and projective McKay matrices for any finite-dimensional Hopf algebra \( A \).

The two Grothendieck groups mentioned in the Introduction are related by the Cartan map \( c : K_0(A) \rightarrow G_0(A) \), \([P] \mapsto [P]\). The Cartan matrix is the integral matrix \( C = (C_{ij}) \) representing \( c \) in the bases \( \{[P_j] \mid 1 \leq j \leq m\} \) and \( \{[S_j] \mid 1 \leq j \leq m\} \) of \( K_0(A) \) and \( G_0(A) \) respectively. It has as its \( (i,j) \) entry \( C_{ij} = [P_i : S_j] \), the multiplicity of \( [S_j] \) when \( [P_i] \) is written as a \( \mathbb{Z} \)-combination of the classes \( [S_j] \), which is also the multiplicity of \( S_j \) in a composition series for \( P_i \). When \( A \) is semisimple, \( C \) is the identity matrix, as \( P_j = S_j \) for all \( j \). In general, the Cartan matrix is not invertible.

Here is a tiny example exhibiting a non-invertible \( C \): The Sweedler Hopf algebra has a basis \( 1, a, b, ab \), where \( a^2 = 0, b^2 = 1, ba = -ab \) (it is the baby Taft algebra \( A_2 \) - compare Section 3.1 with \( q = -1 \)). It has two one-dimensional simple modules \( S_a \) and
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If \( Q(a) \) and\( Q(b) \) are two-dimensional and have both simple modules as composition factors. Thus, \( C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). In fact, the Cartan matrix of any Taft algebra \( A_n \) defined using a primitive root of unity of order \( n \geq 2 \) is the \( n \times n \) matrix with all entries equal to 1 (see [27, Exer. 10.2.4]).

Now suppose that \( Q_V = (Q_{ij}) \) is the projective McKay matrix for tensoring with \( V \) as in (2.7). Then the relation between \( Q_V \) and the McKay matrix \( M_V \) for tensoring simple modules with \( V \) can be described as follows:

**Proposition 2.9** For any \( A \)-module \( V \),

(a) \( Q_VC = CM_V \), where \( C = (C_{ij}) \) is the Cartan matrix.

(b) If \( v \) is a right eigenvector for \( M_V \) with eigenvalue \( \lambda \), then \( Cv \) is a right eigenvector for \( Q_V \) with eigenvalue \( \lambda \).

(c) If \( w \) is a left eigenvector for \( Q_V \) with eigenvalue \( \mu \), then \( wcs \) is a left eigenvector for \( M_V \) with eigenvalue \( \mu \).

**Proof** (a) With notation as in (2.7), we have on one hand

\[
[P_i \otimes V] = \sum_{r=1}^{m} Q_{ir}[P_r] \overset{c}{\longrightarrow} \sum_{r,t=1}^{m} Q_{ir}t_s[S_t] = \sum_{r,t=1}^{m} Q_{ir}C_{rt}[S_t],
\]

and on the other,

\[
[P_i \otimes V] \overset{c}{\longrightarrow} [P_i \otimes V] = [P_i][V] = \sum_{r,t=1}^{m} C_{rt}[S_t][V] = \sum_{r,t=1}^{m} C_{rt}M_{rt}[S_t].
\]

Thus, \( Q_VC = CM_V \).

(b) If \( v \) is a right eigenvector for \( M_V \) with eigenvalue \( \lambda \), then \( Q_VCv = CM_Vv = \lambda Cv \) so that \( Cv \) is a right eigenvector for \( Q_V \) with eigenvalue \( \lambda \).

(c) Similarly, if \( w \) is a left eigenvector for \( Q_V \) with eigenvalue \( \mu \), then \( wcs \) is a left eigenvector for \( M_V \) with eigenvalue \( \mu \). \( \blacksquare \)

For simplicity, we usually omit the brackets on the isomorphism class representatives of \( K_0(A) \) and \( G_0(A) \) in what follows unless they are needed for clarity. We will use the next result, which can be found for example in [16, Prop. 9.2.3].

**Proposition 2.10** Let \( P_i \) be the projective cover of the simple module \( S_i \). Then for any \( A \)-module \( N \),

\[
\dim_k \text{Hom}_A(P_i, N) = [N : S_i],
\]

the multiplicity of \( S_i \) in a Jordan-Hölder series of \( N \).

For every \( P \in K_0(A) \) and \( V, W \in G_0(A) \), the following holds (see [26, Sec. 3.1])

\[
\dim_k \text{Hom}_A(P \otimes V, W) = \dim_k \text{Hom}_A(P, W \otimes V^*). \quad (2.13)
\]
This will enable us to relate the projective McKay matrix to the McKay matrix of the dual module.

Assume $M_V = (M_{ij})$ is the McKay matrix for tensoring with $V \in G_0(A)$, and $M_{V^*} = (M^*_{ij})$ is the McKay matrix for tensoring with the dual module $V^*$. Let $M^T_{V^*} = (M^*_{ji})$ be the transpose of $M_{V^*}$, and let $Q_V = (Q_{ij})$ be the projective McKay matrix for $V$. Then we have the following consequence of Proposition 2.10 and (2.13). Variations of this result have appeared in several different contexts such as [1, Lem. 8], [29, Lem. 10], and it can be regarded as a special case of the tensor category result [15, Prop. 6.1.2].

**Theorem 2.11** Assume $V$ is a module for a finite-dimensional Hopf algebra $A$. Then

$$Q_{ij} = \dim_k \text{Hom}_A(P_i \otimes V, S_j) = \dim_k \text{Hom}_A(P_i, S_j \otimes V^*) = [S_j \otimes V^* : S_i] = M^*_{ji}.$$ 

Therefore, $Q_V = M^T_{V^*}$.

**Corollary 2.12** For any module $V$ over a finite-dimensional Hopf algebra $A$,

(a) $M^T_{V^*} C = C M_V$, where $C$ is the Cartan matrix.

(b) If $v$ is a right eigenvector of $M_V$ with eigenvalue $\lambda$, then $Cv$ is a right eigenvector for $M^T_{V^*}$, with eigenvalue $\lambda$. Similarly, if $w$ is a left eigenvector of $M^T_{V^*}$, with eigenvalue $\mu$, then $wC$ is a left eigenvector for $M_V$ with eigenvalue $\mu$.

(c) If $C$ is invertible, then $M^T_{V^*} = CM_V C^{-1}$. Therefore, $M_V$ and $M^T_{V^*}$ (and also $M_{V^*}$) have the same eigenvalues when $C$ is invertible.

(d) If $A$ is semisimple, the Cartan matrix is the identity matrix, and consequently, $M^T_{V^*} = M_V$. Moreover, if $V$ is self-dual, then $M_V$ is symmetric, hence orthogonally diagonalizable.

### 2.4 Eigenvectors from projective modules

We combine results from the previous sections to obtain left eigenvectors from traces of projective modules. Theorem 2.11 implies that $Q^T_{V} = M^*_{V^*}$ holds for any finite-dimensional Hopf algebra $A$, where $V^*$ is the dual module to $V$, and $Q_V$ is the projective McKay matrix. As a consequence of (2.9), we have

**Theorem 2.13** For any $A$-module $V$ and all $x \in A$,

(a) $\text{Tr}_P(x) M_{V^*} = \sum_x \text{tr}_V(x_{(2)}) \text{Tr}_P(x_{(1)})$, where 

$$\text{Tr}_P(x) = [\text{tr}_{P_1}(x) \ \text{tr}_{P_2}(x) \ \ldots \ \text{tr}_{P_m}(x)].$$ 

(b) If $(V^*)^* \cong V$, then $\text{Tr}_P(x) M_V = \sum_x \text{tr}_V(x_{(2)}) \text{Tr}_P(x_{(1)})$.

**Corollary 2.14** Under the assumption that $(V^*)^* \cong V$ for the $A$-module $V$, the following results hold (compare these to the corresponding results for simple modules (2.5) and (2.6)):

(a) When $g \in A$ is grouplike,

$$\text{Tr}_P(g) M_V = \text{tr}_{V^*}(g) \text{Tr}_P(g).$$  

(2.14)
Hence, for every grouplike element of $A$, $\text{Tr}_P(g)$ is a left eigenvector of $M_V$ of eigenvalue $\text{tr}_V(g)$. The vector $\text{Tr}_P(1)$ is just the dimension vector 
$p = [\dim(P_1) \ \dim(P_2) \ \ldots \ \dim(P_m)]$, and the eigenvalue is 
$\text{tr}_V(1) = \dim(V^*) = \dim(V) = \text{tr}_V(1)$.

(b) When $x \in A$ has the property that $\Delta(x) = x \otimes y + z \otimes x$ for some nonzero $y, z \in A$, then (2.9) says 
$$\text{Tr}_P(x)M_V = \text{tr}_V(y) \text{Tr}_P(x) + \text{tr}_V(x) \text{Tr}_P(z).$$

### 2.5 Eigenvectors for McKay matrices from the Grothendieck algebra $G(A)$

Next we describe a way to produce left eigenvectors and generalized left eigenvectors for McKay matrices using the Grothendieck algebra $G(A) = \mathbb{C} \otimes_{\mathbb{Z}} G_0(A)$ of any Hopf algebra $A$. The classes $[S_1], [S_2], \ldots, [S_m]$ of the nonisomorphic simple modules give a $\mathbb{C}$-basis for $G(A)$.

**Proposition 2.15** (a) Let $V$ be an $A$-module, and assume 
$[X] = c_1[S_1] + \cdots + c_m[S_m] \in G(A)$, $c_j \in \mathbb{C}$ for all $j$, is an eigenvector for the right multiplication operator $R_{[V]}$ of $G(A)$ with eigenvalue $\lambda$. Then $[c_1, c_2, \ldots, c_m]$ is a left eigenvector for $M_V$ with eigenvalue $\lambda$.
(b) Suppose that $[Y] = d_1[S_1] + \cdots + d_m[S_m] \in G(A)$ has the property that 
$(R_{[V]} - \lambda I)^{\ell}[Y] = 0$. Then for the McKay matrix $M_V$, we have 
$[d_1, d_2, \ldots, d_m](M_V - \lambda I)^{\ell} = 0$.

**Proof** (a) We are assuming that $R_{[V]}([X]) = \lambda [X]$, or more specifically, 
$\lambda \sum_{i=1}^m c_i[S_i] = \lambda [X]$.
Since multiplication in $G(A)$ is given by tensoring, this implies that 
$$\sum_{i=1}^m c_i[S_i \otimes V : S_j] = \lambda c_j$$
holds for each $j$, where $[S_i \otimes V : S_j]$ is the multiplicity of $[S_j]$ in $[S_i \otimes V]$. However, 
$[S_i \otimes V : S_j] = M_{ij}$, the $(i, j)$ entry of $M_V$. Thus, 
$\sum_{i=1}^m c_i M_{ij} = \lambda c_j$ for all $j$, which says 
$[c_1, c_2, \ldots, c_m]M_V = \lambda [c_1, c_2, \ldots, c_m]$.
Part (b) follows by induction, with part (a) providing the $\ell = 1$ case. 

**Remark 2.16** Grothendieck algebras in general do not have to be commutative, so we do need to specify the ‘right’ multiplication above.

### 2.6 Eigenvectors for McKay matrices from $K(A)$

To determine additional information about the McKay matrix $M_V$, we consider the operator $R_{[V]}$ of right multiplication by $[V^*]$ on the finite-dimensional complex vector space $K(A) = \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$. Suppose $[Y]$ is an eigenvector for $R_{[V]}$ with eigenvalue $\xi$. Then $[Y][V^*] = \xi[Y]$, where $[Y] = \sum_{i=1}^m y_i[P_i]$, a $\mathbb{C}$-linear combination of the
projective covers. The matrix $Q_{V^*}$ has $(i,j)$ entry equal to the multiplicity of $[P_j]$ in $[P_i \otimes V^*] = [P_i][V^*]$, which implies

$$[y_1 \ y_2 \ \ldots \ y_m]Q_{V^*} = \xi[y_1 \ y_2 \ \ldots \ y_m].$$

Using the fact that $Q_{V^*}^T = M_{\xi_{V^*}}$ from Theorem 2.11 and (2.16), we have

**Proposition 2.17** Assume that the $A$-module $V$ satisfies $(V^*)^* \cong V$. Let $[Y]$ be a left eigenvector for $R_{V^*}$ with eigenvalue $\xi$ in $K_0^G(A)$, and let $[y_1 \ y_2 \ \ldots \ y_m]$ be its coordinate vector relative to the basis $\{[P_i]\}_{i=1}^m$ of $K_0^G(A)$ of projective covers. Then

$$M_V[y_1 \ y_2 \ \ldots \ y_m]^T = \xi[y_1 \ y_2 \ \ldots \ y_m]^T,$$

so that $[y_1 \ y_2 \ \ldots \ y_m]^T$ is a right eigenvector for $M_V$ with eigenvalue $\xi$.

**Remark 2.18** When the antipode $S$ of $A$ has the property that $S^2$ is an inner automorphism of $A$, then $(V^*)^* \cong V$ holds for all $A$-modules $V$ (see [27, Lem. 10.2(a)]). Any semisimple Hopf algebra will have $(V^*)^* \cong V$ for all $V$, as will any symmetric algebra [32]. Drinfeld doubles are always symmetric (i.e. have a nondegenerate, symmetric, associative bilinear form). In particular, the Drinfeld double $D_n$, which we investigate in detail in Section 3, has the property $(V^*)^* \cong V$ for all $V$, and $D_n$ is not semisimple. Also, since the quantum group $u_q(sl_2)$ for $q$ a root of unity has a unique simple module of each dimension, $(V^*)^* \cong V$ holds for all $u_q(sl_2)$-modules $V$, and $u_q(sl_2)$ is not semisimple.

### 2.7 Eigenvectors for McKay matrices of semisimple Hopf algebras

In this section, we assume that the Hopf algebra $A$ is semisimple. The assumption of semisimplicity enables us to say more about the eigenvectors and characters of $A$.

Following [26] and [39, p. 886], we define

$$\langle X, Y \rangle = \dim_k \hom_A(X^*, Y),$$

for any two $A$-modules $X$ and $Y$ in $G_0^A(A)$, where $X^*$ denotes the dual module of $X$. This generates a nondegenerate, symmetric associative $\mathbb{C}$-bilinear form on $G_0^A(A)$. Therefore, $G_0^A(A)$ is a symmetric algebra with dual bases $\{S_1, S_2, \ldots, S_m\}$ and $\{S_1^*, S_2^*, \ldots, S_m^*\}$, where the $S_i$ are representatives of the isomorphism classes of simple $A$-modules.

Let $\xi_1, \xi_2, \ldots, \xi_s$ be the $G_0^A(A)$-characters afforded by the simple $G_0^A(A)$-modules. Since $A$ is semisimple, so is the Grothendieck algebra $G_0^A(A)$ (see, for example, [41] and [39]). Therefore, by [13, Sec. 9B] we know that the characters of $G_0^A(A)$ are linearly independent over $\mathbb{C}$. Moreover, if $E_1, E_2, \ldots, E_s$ are the primitive central idempotents of $G_0^A(A)$, then $\xi_i(E_j) = \xi_i(1)$, and $\xi_i(E_j) = 0$ for $j \neq i$. The idempotents $E_1, E_2, \ldots, E_s$ form a $\mathbb{C}$-basis for the center of $G_0^A(A)$.

By Proposition 9.17 (i) of [13], the duality of the bases $\{S_i^*\}$, $\{S_j\}$ relative to the symmetric bilinear form above can be used to define

$$D_i = \sum_{j=1}^m \xi_i(S_j^*)S_j = \sum_{j=1}^m \xi_i(S_j)S_j^*, \quad \text{for } 1 \leq i \leq s, $$

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and to prove that $\zeta_i(\mathcal{D}_i) \neq 0$. Then applying [13, Prop. 9.17 (iii)], we have

**Proposition 2.19** Assume that the Hopf algebra $A$ is semisimple, and let $\zeta_1, \zeta_2, \ldots, \zeta_s$ be the characters of the simple $G^0(A)$-modules. Then for all $1 \leq i \leq s$, the primitive central idempotent $E_i$ of $G^0(A)$ corresponding to the character $\zeta_i$ is given by

$$E_i = \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\mathcal{D}_i,$$

where $\zeta_i(\mathcal{D}_i) = \sum_{j=1}^m \zeta_i(S_j(\mathcal{D}_i)) = \zeta_i \left( \bigoplus_{j=1}^m S_j \otimes S_j \right),$.

and $S_j, 1 \leq j \leq m$, are the simple $A$-modules.

**Remarks 2.20** (a) In [39], Witherspoon investigated $G^0(A)$ when $A$ is a semisimple, almost cocommutative Hopf algebra. Under these assumptions, $G^0(A)$ is both semisimple and commutative, so the simple modules for $G^0(A)$ are one-dimensional. The expression for the primitive central idempotents coming from [39, (3.4)] is basically the same as the one given in Proposition 2.19; however, since $\zeta_i(1) = 1$ for all $i$ when $A$ is semisimple and almost cocommutative, that factor does not appear in [39].

(b) The primitive central idempotents $E_i$ form a basis for $G^0(A)$ when $A$ is semisimple and almost cocommutative, and in fact, the simple $G^0(A)$-modules are exactly the $A_0\mathcal{E}_i$. Therefore, $s = m$ in the situation considered in [39]. This is not true when $A$ is an arbitrary semisimple Hopf algebra. Each $E_i$ is the identity element of a matrix block of the semisimple algebra $G^0(A)$, but these blocks do not have to be one-dimensional. The Hopf algebra that is (14) in Kashina’s classification [24] of 16-dimensional semisimple Hopf algebras is an example of this phenomenon. The Grothendieck algebra $G^0(A)$ has a $2 \times 2$ matrix block in that case.

Assume $A$ is semisimple and $V$ is an $A$-module that is central in $G^0(A)$. Then $V = \sum_{i=1}^s \lambda_i E_i$, which implies that $E_i V = \lambda_i E_i$, that is, $E_i$ is an eigenvector for right multiplication by $V$ in $G^0(A)$. Therefore, by Propositions 2.15 and 2.19, we have for the McKay matrix $M_V = (M_{ij})$,

$$\lambda_i E_i = \lambda_i \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^m \zeta_i(S_j)S_j = E_i V = \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^m \zeta_i(S_j)S_j V$$

$$= \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^m \zeta_i(S_j) \left( \sum_{\ell=1}^m [S_j \otimes V : S_{\ell}]S_\ell \right)$$

$$= \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{\ell=1}^m \left( \sum_{j=1}^m \zeta_i(S_j)M_{\ell j} \right) S_\ell.$$

In other words, $\lambda_i \zeta_i(S_j) = \sum_{\ell=1}^m \zeta_i(S_j)M_{\ell j}$. This says that $[\zeta_i(S_1) \ldots \zeta_i(S_m)]$ for $1 \leq i \leq s$, is a left eigenvector for $M_V$ with eigenvalue $\lambda_i$ when $V = \sum_{i=1}^s \lambda_i E_i$ is a central element of $G^0(A)$. Hence, we have shown the following
The Drinfeld Double of the Taft Algebra and its Modules

3.1 Preliminaries

The Drinfeld double $D_n$ of the Taft algebra $A_n$ has a presentation as the Hopf algebra over $k$ with generators $a, b, c, d$ that satisfy the following relations:

\[
ba = qab, \quad db = qbd, \\
ca = qac, \quad dc = qcd, \\
bc = cb, \quad da - qad = 1 - bc, \\
a^n = 0 = d^n, \quad b^n = 1 = e^n.
\]

The coproduct, counit, and antipode of $D_n$ are given by

\[
\Delta(a) = a \otimes b + 1 \otimes a, \quad \Delta(d) = d \otimes c + 1 \otimes d, \\
\Delta(b) = b \otimes b, \quad \Delta(c) = c \otimes c, \\
\varepsilon(a) = 0 = \varepsilon(d), \quad \varepsilon(b) = 1 = \varepsilon(c), \\
S(a) = -ab^{-1}, \quad S(b) = b^{-1}, \quad S(c) = c^{-1}, \quad S(d) = -dc^{-1}.
\]

The Taft algebra $A_n$ is the Hopf subalgebra generated by $a$ and $b$. It follows from (3.2) and the fact that $\Delta$ is an algebra homomorphism that the elements $b^ic^k$ for $0 \leq i, k \leq n - 1$ are grouplike.
3.1.2 The Cartan map for $D_n$

The simple $D_n$-modules $V(\ell, r)$ are indexed by a pair $(\ell, r)$ where $1 \leq \ell \leq n$ and $r \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (the integers modulo $n$). Then $V(\ell, r)$ is a $k$-vector space of dimension $\ell$ with basis $v_1, v_2, \ldots, v_\ell$ and with $D_n$-action given by

\begin{align*}
  a. v_j &= v_{j+1}, \quad 1 \leq j < \ell, & a. v_\ell &= 0, \\
  b. v_j &= q^{\ell-j-1} v_j, & c. v_j &= q^{j-(r+\ell)} v_j, \quad 1 \leq j \leq \ell, \\
  d. v_j &= \alpha_{\ell-1}(\ell) v_{j-1}, \quad 1 \leq j \leq \ell, & d. v_1 &= 0,
\end{align*}

where

$$
\alpha_i(\ell) = \frac{(q^i - 1)(1 - q^{i-\ell})}{q - 1} \quad \text{for } 1 \leq i \leq n - 1. \tag{3.4}
$$

From Chen et al. [5, 8], the following hold:

1. $V(1, 0)$ is the trivial $D_n$-module with action given by the counit $\varepsilon$.
2. $V(\ell, r) \otimes V(1, s) \cong V(\ell, r + s)$.
3. $V(\ell, r) \otimes V(\ell', s)$ is completely reducible if and only if $\ell + \ell' \leq n + 1$. In this case, if $m = \min(\ell, \ell')$, then

$$
V(\ell, r) \otimes V(\ell', s) \cong \bigoplus_{j=1}^m V(\ell + \ell' + 1 - 2j, r + s + j - 1). \tag{3.5}
$$

Let $P(\ell, r)$ be the projective cover of the simple $D_n$-module $V(\ell, r)$. Chen [6] has shown that any indecomposable projective left $D_n$-module is isomorphic to one of the modules $P(\ell, r)$ for $1 \leq \ell < n$ or to $V(n, r)$ for some $r \in \mathbb{Z}_n$, and the module $P(\ell, r)$ for $1 \leq \ell < n$ has the following structure. There is a chain of submodules $P(\ell, r) \supset \text{soc}^2(P(\ell, r)) \supset \text{soc}(P(\ell, r)) \supset (0)$ such that

1. $\text{soc}(P(\ell, r))$ is the socle of $P(\ell, r)$ (the sum of all the simple submodules), and $\text{soc}(P(\ell, r)) \cong V(\ell, r)$;
2. $\text{soc}^2(P(\ell, r))/\text{soc}(P(\ell, r)) \cong V(n - \ell, r + \ell) \oplus V(n - \ell, r + \ell)$;
3. $P(\ell, r)/\text{soc}^2(P(\ell, r)) \cong V(\ell, r)$.

Therefore, $[P(\ell, r)] = 2[V(\ell, r)] + 2[V(n - \ell, r + \ell)]$ in the Grothendieck group $G_0(D_n)$, and the dimension of $P(\ell, r)$ is $2n$ for $1 \leq \ell < n$. Hence, it follows that $[P(\ell, r)] = [P(n - \ell, r + \ell)]$ holds in $G_0(D_n)$ for all $1 \leq \ell < n$ and all $r \in \mathbb{Z}_n$. The modules $V(n, r)$ for all $r \in \mathbb{Z}_n$ are the only $D_n$-modules that are both simple and projective.

3.1.2 The Cartan map for $D_n$

We consider an extension of the Cartan map of $D_n$ to a $\mathbb{C}$-linear map (also denoted $c$), $c : K^G_0(D_n) \rightarrow G^C_0(D_n)$, $[P] \mapsto [P]$. Then $c([P(\ell, r)]) = [P(\ell, r)] = 2[V(\ell, r)] + 2[V(n - \ell, \ell + r)]$ for all $1 \leq \ell \leq \frac{n+1}{2}$, and $r \in \mathbb{Z}_n$, and $c([V(n, r)]) = [V(n, r)]$ for all $r \in \mathbb{Z}_n$. Therefore, the images under $c$ of the basis elements of $K^G_0(D_n)$ are $\mathbb{C}$-linearly independent elements of $G^C_0(D_n)$, and it follows that $\dim(\text{im}(c)) = \frac{n(n+1)}{2}$. Since the elements $[P(\ell, r)] - [P(n - \ell, \ell + r)]$, $1 \leq \ell \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, are linearly independent elements of the kernel of $c$ and $\dim(\ker(c)) = \frac{n(n+1)}{2} - \frac{n-1}{2}$.
\[ n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}, \] they form a basis for the kernel. To summarize, we have the following result.

**Proposition 3.1**

(a) The elements \(2[V(\ell, r)] + 2[V(n-\ell, \ell + r)]\) for all \(1 \leq \ell \leq \frac{n-1}{2}, \ r \in \mathbb{Z}_n\), and \([V(n, r)], r \in \mathbb{Z}_n\), form a basis for the image \(\text{im}(c)\) of the Cartan map \(c : K_G^0(D_n) \rightarrow G_G^0(D_n)\). Therefore, \(\dim(\text{im}(c)) = \frac{n(n+1)}{2}\).

(b) The rank of the Cartan matrix \(C\) of \(D_n\) is \(\frac{n(n+1)}{2}\).

(c) The elements \([P(\ell, r)] - [P(n-\ell, \ell + r)], 1 \leq \ell \leq \frac{n-1}{2}, r \in \mathbb{Z}_n\), form a basis for the kernel of \(c\), and \(\dim(\ker(c)) = \frac{n(n-1)}{2}\).

### 3.2 The McKay matrix for tensoring with the \(D_n\)-module \(V(2, 0)\)

Throughout this section, we assume that \(V\) is the two-dimensional simple \(D_n\)-module \(V(2,0)\) with basis \(\{v_1, v_2\}\). Now it follows from (3.5) and [8, Prop. 3.1, Thms. 3.3 and 3.5] that for \(V = V(2,0)\),

1. \(V(1, r) \otimes V = V(2, r)\);
2. \(V(\ell, r) \otimes V = V(\ell + 1, r) \oplus V(\ell - 1, r + 1)\) for \(2 \leq \ell < n\);
3. \(V(n, r) \otimes V \cong P(n-1, r + 1)\);
4. \(P(1, r) \otimes V \cong P(2, r) \oplus 2V(n, r + 1)\);
5. \(P(\ell, r) \otimes V \cong P(\ell + 1, r) \oplus P(\ell - 1, r + 1)\) for \(2 \leq \ell < n - 1\);
6. \(P(n-1, r) \otimes V \cong P(n-2, r + 1) \oplus 2V(n, r)\).

The McKay matrix for tensoring the simple \(D_n\)-modules \(V(\ell, r)\) with \(V \coloneqq V(2,0)\) is the \(n^2 \times n^2\) matrix \(M_V = [M_{(\ell, r), (\ell', s)}]\), whose entry \(M_{(\ell, r), (\ell', s)}\) is given by the composition series multiplicity

\[ M_{(\ell, r), (\ell', s)} = [V(\ell, r) \otimes V : V(\ell', s)]. \]

We assume that the numbering of the rows and columns of \(M_V\) is first by \(\ell = 1\), then by \(\ell = 2\), etc., and for each \(\ell\) the numbering is \(r = 0, 1, \ldots, n-1\); that is, we are numbering by lexicographic order, and we will often simply write \(M\) for \(M_V\) when the choice of \(V\) is unambiguous. Using the decomposition formulas above, we observe that the McKay matrix \(M\) then can be displayed using \(n \times n\) blocks, where \(I = I_n\) is the \(n \times n\) identity matrix, and \(Z\) is the \(n \times n\) cyclic permutation matrix as presented below

\[
M = \\
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
Z & 0 & 1 & \cdots & 0 & 0 \\
0 & Z & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & Z & 0 \\
2I & 0 & 0 & \cdots & Z & 0 \\
\end{pmatrix},
\]

\[
Z = \\
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}. \quad (3.6)
\]

There are identity matrices on the superdiagonal of \(M\), and the matrix \(Z\) is on the subdiagonal of \(M\) except for the last row (corresponding to the modules \(V(n, r)\) for \(r \in \mathbb{Z}_n\)).
where the nonzero entries are $2I$ and $2Z$, due to the fact that $V(n, r) \otimes V(2, 0) = P(n-1, r+1)$, which has composition factors $V(n-1, r+1)$ (twice) and $V(1, r)$ (twice). Assume $Y = \text{diag}(X, X, \ldots, X)$, where the $n \times n$ matrix $X$ diagonalizes $Z$,

$$XZX^{-1} = D := \text{diag}(1, q, \ldots, q^{n-1}),$$

and $q$ is as before, a primitive $n$th root of unity in $k$ for $n$ odd and $\geq 3$. Then

$$M' = YMY^{-1} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ D & 0 & I & \cdots & 0 & 0 \\ 0 & D & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D & 0 \\ 2I & 0 & 0 & \cdots & 2D & 0 \end{pmatrix}.$$  \hspace{1cm} (3.7)

3.3 Characteristic polynomial and characteristic roots of $M_V, V = V(2, 0)$

We want to determine the characteristic roots of $M = M_V$, which we can do by computing the characteristic roots of $M'$, as they are the same. The advantage to working with $M'$ is that its entries lie in the commutative ring of diagonal matrices, and so usual matrix operations apply.

Consider the characteristic polynomial of $M'$, which can be found by computing

$$\det(tI_n^2 - M') = \det \begin{pmatrix} tI & -I & 0 & \cdots & 0 & 0 \\ -D & tI & -I & \cdots & 0 & 0 \\ 0 & -D & tI & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -D & tI \\ -2I & 0 & 0 & \cdots & -2D & tI \end{pmatrix}.$$  \hspace{1cm} (3.8)

Define polynomials $U_k(t, D)$ recursively by

$$U_0(t, D) = 1, \quad U_1(t, D) = tI, \quad U_k(t, D) = tU_{k-1}(t, D) - DU_{k-2}(t, D), \quad k \geq 2.$$  \hspace{1cm} (3.8)

These polynomials are related to Chebyshev polynomials of the second kind, as we explain in the next section. In computing the determinant of $tI_n^2 - M'$, we will abbreviate $U_k(t, D)$ as $U_k$. We perform row operations on $tI_n^2 - M'$ using the matrices $-I$ on the superdiagonal to eliminate the entries beneath them. Therefore, after using the $-I$
in the first row and then the \(-1\) in the second row, we have
\[
\begin{pmatrix}
-1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & -1 & 0 \\
0 & 0 & 0 & \cdots & -D \ tI \\
-2I & 0 & 0 & \cdots & -2D \ tI
\end{pmatrix}
\]
\[
\begin{pmatrix}
-1 & 0 & \cdots & 0 & 0 \\
0 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & \cdots & -D \ tI \\
-2I & 0 & 0 & \cdots & -2D \ tI
\end{pmatrix}
\]
respectively. Continuing, we obtain
\[
\begin{pmatrix}
U_1 & -1 & 0 & \cdots & 0 & 0 \\
U_2 & 0 & -1 & \cdots & 0 & 0 \\
U_3 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 \\
U_{n-2} & \vdots & \ddots & -1 & 0 \\
U_{n-1} & 0 & 0 & \cdots & 0 & -1 \\
-2D U_{n-2} & 2I & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
so that after the final step, the result of using the bottommost \(-1\) on the superdiagonal is
\[
\begin{pmatrix}
U_1 & -1 & 0 & \cdots & 0 & 0 \\
U_2 & 0 & -1 & \cdots & 0 & 0 \\
U_3 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 \\
U_{n-2} & \vdots & \ddots & -1 & 0 \\
U_{n-1} & 0 & 0 & \cdots & 0 & -1 \\
-2D U_{n-1} & 2I & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
Therefore, setting
\[
p_n(t, D) := rU_{n-1}(t, D) - 2DU_{n-2}(t, D) - 2I = U_n(t, D) - DU_{n-2}(t, D) - 2I
\]
we have \(\det(rU_{n-2} - M') = p_n(t, D)\), where \(M'\) is as in (3.7), and the characteristic roots of \(M'\), hence also of \(M\), are the roots of \(p_n(t, D)\).
Here are the first few polynomials \(p_n(t, D)\):
\[
\begin{align*}
n = 3: \quad & t^3 - 3Dt - 2I \\
n = 5: \quad & t^5 - 5Dt^3 + 5D^2t - 2I \\
n = 7: \quad & t^7 - 7Dt^5 + 14D^2t^3 - 7D^3t - 2I \\
n = 9: \quad & t^9 - 9Dt^7 + 27D^2t^5 - 30D^3t^3 + 9D^4t - 2I \\
n = 11: \quad & t^{11} - 11Dt^9 + 44D^2t^7 - 77D^3t^5 + 55D^4t^3 - 11D^5t - 2I \\
n = 13: \quad & t^{13} - 13Dt^{11} + 65D^2t^9 - 156D^3t^7 + 182D^4t^5 - 91D^5t^3 + 13D^6t - 2I.
\end{align*}
\]
These are polynomials with coefficients that are $n \times n$ diagonal matrices. Each diagonal entry $q^k$ of $D$ for $k \in \mathbb{Z}_n$ determines a polynomial $p_n(t, q^k)$ in $q$ and $t$ with coefficients in $\mathbb{Z}$. The characteristic roots of $M$ are obtained by setting those $n$ polynomials equal to 0. For example, when $n = 7$, the polynomials are $p_n(t, q^k) = t^7 - 7q^k t^6 + 14q^{2k} t^5 - 7q^{3k} t^4 + q^{4k} t^3$, and the characteristic roots of $M$ are the roots of those 7 polynomials.

### 3.4 Right eigenvectors for $M_V$, $V = V(2, 0)$, and Chebyshev polynomials

The polynomials $p_n(t, D)$ are related to Chebyshev polynomials $U_k(t)$ of the second kind, which are defined recursively by the formulas

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_k(t) = 2t U_{k-1}(t) - U_{k-2}(t) \quad \text{for} \quad k \geq 2. \quad (3.11)$$

Setting $\mathcal{U}_k(t) = U_k(\frac{t}{2})$, we have

$$\mathcal{U}_0(t) = 1, \quad \mathcal{U}_1(t) = t, \quad \mathcal{U}_k(t) = t \mathcal{U}_{k-1}(t) - \mathcal{U}_{k-2}(t) \quad \text{for} \quad k \geq 2. \quad (3.12)$$

There are a number of closed-form formulas for Chebyshev polynomials of the second kind. Replacing $t$ by $\frac{x}{2}$ in one such formula (see for example, [31, 18.5.10 with $\lambda = 1$] or [14, (23), p. 185]) gives the following expression for $\mathcal{U}_k(t)$:

$$\mathcal{U}_k(t) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} t^{k-2j}. \quad (3.13)$$

The polynomials $\mathcal{U}_k(t, D)$, which were defined in the previous section, satisfy a similar recursion (3.8), and as a result,

$$\mathcal{U}_k(t, D) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} t^{k-2j} D^j. \quad (3.14)$$

The polynomial $\mathcal{U}_k(t)$ is $U_k(t, D)$ with $D$ and the $n \times n$ identity matrix $I$ replaced by 1. Thus, when $D$ and $I$ are replaced by 1 in $p_n(t, D)$, we obtain

$$p_n(t) := p_n(t, 1) = r \mathcal{U}_{n-1}(t) - 2 \mathcal{U}_{n-2}(t) - 2 = U_n(t) - U_{n-2}(t) - 2. \quad (3.15)$$

Assume now that $n = 2h + 1$ for $h \geq 1$. We aim to show

$$p_n(t) = p_{2h+1}(t) = (t - 2) W_h^2(t). \quad (3.16)$$

where $W_h(t) = U_h(t) + U_{h-1}(t)$ for all $h \geq 1$, by appealing to results on Chebyshev polynomials of the fourth kind.

The sum of two consecutive Chebyshev polynomials of the second kind is a Chebyshev polynomial of the fourth kind. These polynomials are defined recursively by the following formulas (see [28, Secs. 1.2.3, 1.2.4]):

$$W_0(t) = 1, \quad W_1(t) = 2t + 1, \quad W_k(t) = 2t W_{k-1}(t) - W_{k-2}(t) \quad \text{for} \quad k \geq 2. \quad (3.17)$$

Thus, they satisfy the same recursion as the polynomials $U_k(t)$, except $W_1(t) = 2t + 1$, while $U_1(t) = 2t$. In particular, $W_k(t) = U_k(t) + U_{k-1}(t)$ for all $k \geq 1$ by [28, (1.54)].
Chebyshev polynomials have integer coefficients and complex roots. Suppose \( x = e^{i\theta} \in \mathbb{C} \). Then \( z := \frac{x + x^{-1}}{2} = \cos(\theta) \in \mathbb{C} \), and it follows from \([28, (1.54)]\) that the relation

\[
U_k(z) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}}
\]

(3.18)

holds. Moreover, by \([28, (1.57)]\),

\[
W_k(z) = \frac{x^{2k+1} - x^{-1}}{x^2 - 1} = x^k + x^{k-1} + \cdots + x^{-(k-1)} + x^{-k} = x^{-k} \frac{x^{2k+1} - 1}{x - 1}.
\]

(3.19)

Hence, for \( U_k(x + x^{-1}) = U_k\left(\frac{x + x^{-1}}{2}\right) \) and \( W_k(x + x^{-1}) := W_k\left(\frac{x + x^{-1}}{2}\right) \), we can conclude the following:

**Proposition 3.2** Assume \( U_k(t) \) is defined as in (3.12). Set \( W_0(t) = 1 = U_0(t) \), and let \( W_k(t) = U_k(t) + U_{k-1}(t) \) for \( k \geq 1 \). Let \( x = e^{i\theta} \in \mathbb{C} \) be chosen so that \( t = x + x^{-1} \). Then for all \( k \geq 1 \), the following hold:

(a) \( U_k(t) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}} \);

(b) \( W_k(t) = x^k + x^{k-1} + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = x^{-k} \frac{x^{2k+1} - 1}{x - 1} \);

(c) \( p_n(t) = U_n(t) - U_{n-2}(t) - 2 = (t - 2)W_n^{2}(t) \), for \( n = 2h + 1, h \geq 1 \).

**Proof** Only the last equality in (c) needs to be verified, and we proceed to show that the two sides of (c) are equal by computing both by induction on \( h \) and comparing them. When \( h = 1 \),

\[
W_1^2(t) = x^2 + 2x + 3 + 2x^{-1} + x^{-2}
\]

\[
= (x^2 + 1 + x^{-2}) + 2(x + x^{-1}) + 2 = U_2(t) + 2U_1(t) + 2U_0(t).
\]

Assuming the statement

\[
U_{2h}(t) + 2U_{2h+1}(t) + \cdots + 2U_1(t) + 2U_0(t) = W_h^2(t)
\]

(3.20)

for \( h \geq 1 \), we have

\[
U_{2h+2}(t) + 2U_{2h+1}(t) + 2U_{2h}(t) + \cdots + 2U_1(t) + 2U_0(t)
\]

\[
= \left( x^{2h+2} + x^{2h} + \cdots + x^{-2h} + x^{-2h-2} \right) + \left( 2x^{2h+1} + 2x^{2h-1} + \cdots + 2x^{-2h+1} + 2x^{-2h-1} \right)
\]

\[
+ \left( x^{2h} + x^{2h-2} + \cdots + x^{-2h+2} + x^{-2h} \right) + U_{2h}(t) + 2U_{2h-1}(t) + \cdots + 2U_1(t) + 2U_0(t)
\]

\[
= \left( x^{2h+2} + 2x^{2h+1} + 2x^{2h} + \cdots + 2 + \cdots + 2x^{-2h} + 2x^{-2h-1} + x^{-2h-2} \right)
\]

\[
+ \left( x^{2h} + 2x^{2h-1} + 3x^{2h-2} + \cdots + (2h + 1) + \cdots + 3x^{-2h+2} + 2x^{-2h+1} + x^{-2h} \right)
\]

\[
= x^{2h+2} + 2x^{2h+1} + 3x^{2h} + \cdots + (2h + 3) + \cdots + 3x^{-2h} + 2x^{-2h-1} + x^{-2h-2}
\]

\[
= W_{h+1}^2(t).
\]

This completes the induction step and proves (3.20).
Now on the other hand, we claim that when $n = 2h + 1 \geq 3$, 

$$p_n(t) = (t - 2) \left[ 2U_{n-1}(t) + 2U_{n-2}(t) + \cdots + 2U_0(t) \right]. \tag{3.21}$$

We argue this by induction on $n$. The base case when $n = 3$ follows from a direct calculation. Suppose the statement is true for $n$. Then by (3.15), for $n + 1$ we have 

$$p_{n+1}(t) = \left[ 2U_{n+1}(t) - 2U_{n-1}(t) \right] = (t - 2) \left[ 2U_{n}(t) + 2U_{n-1}(t) - 2U_{n-1}(t) \right] = (t - 2) \left[ 2U_{n}(t) + 2U_{n-1}(t) - 2U_{n-1}(t) \right] = (t - 2) \left[ 2U_{n}(t) + 2U_{n-1}(t) + \cdots + 2U_0(t) \right].$$

Therefore, the assertion $p_n(t) = (t - 2)W_n^2(t)$ follows by comparing (3.20) and (3.21). 

Applying (3.19) with $k = h$, we deduce

**Corollary 3.3** For $h \geq 1$, the roots of $W_n(t)$ as a polynomial in $x$ are all $x \neq 1$ which are roots of unity in $\mathbb{C}$ of order $2h + 1$. As a polynomial in $t$, the roots of $W_n(t)$ are all $t = x + x^{-1}$, where $x$ is a root of unity of order $2h + 1$ in $\mathbb{C}$ and $x \neq 1$.

**Remark 3.4** We are assuming that $D_n$ is defined over an algebraically closed field $k$ of characteristic $0$, and $q$ is a primitive $n$th root of unity in $k$ for $n = 2h + 1$, $h \geq 1$. Since the subfield of $k$ generated by 1 and $q$ is isomorphic to the subfield of $\mathbb{C}$ generated by 1 and $x$, where $x \neq 1$ is a root of unity of order $2h + 1$ as in Corollary 3.3, we will identify $x$ with $q$ in what follows.

**Corollary 3.5** Assume $n \geq 3$, $n$ odd. The characteristic roots of the McKay matrix $M$ in (3.6) are $\lambda_{j,r} = q^r(q^j + q^{-j}) = q^rU_j(q^j + q^{-j})$, where $r \in \mathbb{Z}_n$, $0 \leq j \leq \frac{n-1}{2}$, and $q$ is a primitive $n$th root of unity. Each root $\lambda_{0,r} = 2q^r$ has multiplicity 1, and each root $\lambda_{j,r}$ for $j \neq 0$ has multiplicity 2.

The next result gives an expression for right eigenvectors of $M = MV$, $V = V(2,0)$, in terms of the Chebyshev polynomials.

**Proposition 3.6** Assume $r \in \mathbb{Z}_n$ and $0 \leq j \leq \frac{n-1}{2}$, and let $v_0$ be the right eigenvector of the matrix $Z$ in (3.6) corresponding to the eigenvalue $q^{2r}$ given by $v_0 = \begin{bmatrix} q^r & \cdots & q^{(n-1)r} \end{bmatrix}^T$. For $1 \leq \ell \leq n - 1$, set $v_\ell := q^{\ell r}U_\ell(q^j + q^{-j})v_0$, where $U_\ell$ is as in (3.12). Then $v_{j,r} = [v_0 \ v_1 \ \cdots \ v_{n-1}]^T$ is a right eigenvector of $M$ corresponding to the eigenvalue $\lambda_{j,r} = q^r(q^j + q^{-j})$. 

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Proof. We check directly that the given vector $v_{j,r}$ is a right eigenvector of $M$, that is, we verify

$$Mv_{j,r} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \lambda_{j,r} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

(3.22)

holds by comparing both sides of (3.22). We assume that $v_0$ is as in the statement of the proposition, and argue this forces $\nu_{r} := q^{rr}U_{r}(q^{\ell} + q^{-\ell})v_{0}$ to hold for $1 \leq \ell \leq n - 1$, where $U_{r}$ is as in (3.12). The comparison involves checking

$$v_{0} = \lambda_{j,r}v_{r-1} - Zv_{r-2}$$

(3.23)

for all $0 \leq r \leq n - 1$, and $Z = q^{rr}U_{r}(q^{\ell} + q^{-\ell})v_{0}$.

Remark 3.7 In the expression for the right eigenvector $v_{j,r}$ of $M$ in (3.22) corresponding to the eigenvalue $\lambda_{j,r} = q^{\ell}(q^{1} + q^{-1})$, the last vector component is

$$v_{n-1} = q^{n-1r}U_{n-1}(q^{1} + q^{-1})v_{0} = q^{n-1r}U_{n-1}(q^{1} + q^{-1})v_{0} = 0$$

(3.24)

when $j \neq 0$ by Proposition 3.2 (a).

3.5 Generalized right eigenvectors for $M$

Using the Chebyshev polynomials $U_{r}(q^{1} + q^{-1})$, we now describe generalized right eigenvectors for $M$.

Theorem 3.8 Assume $r \in \mathbb{Z}_{n}$ and fix a choice of $j \in \{1, 2, \ldots, n-1\}$. Let $\nu = \begin{bmatrix} 1 & q^{2} & \cdots & q^{n-1} \end{bmatrix}^T$ be the right eigenvector of the matrix $Z$ corresponding to the eigenvalue $q^{2r}$ as in Proposition 3.6, and set $x_{0} = \nu$. For any $1 \leq k \leq n - 1$, assume

$$x_{k} := q^{kr}U_{k}\nu + q^{(k-1)r} \sum_{s=0}^{k-1} (k-2s)U_{k-1-2s}\nu,$$

(3.24)
where $\mathcal{U}_k$ is shorthand for $\mathcal{U}_k(q^j + q^{-j})$. Then $x_{j,t} = [x_0 \ x_1 \ \ldots \ x_{n-2} \ x_{n-1}]^T$ is a generalized right eigenvector of $M_V$ corresponding to the eigenvalue $\lambda_{j,t} = q^t (q^j + q^{-j})$, and $M_V x_{j,t} = \lambda_{j,t} x_{j,t} + v_{j,t}$, where $v_{j,t}$ is the right eigenvector for $M_V$ in Proposition 3.6.

Proof. The proof amounts to showing that the matrix equation below holds

$$M_V x_{j,t} = \lambda_{j,t} x_{j,t} + v_{j,t},$$

(3.25)

when $x_{j,t}$ has vector components given by (3.24).

Row 0 of $M_V x_{j,t}$ says that $x_1 = \lambda_{j,t} x_0 + v = q^t \mathcal{U}_1 x_0 + v$, which is true for $x_1$ in (3.24). Next we compute rows $k = 2, \ldots, n - 1$ of $M_V x_{j,t}$ proceeding by induction to show that $x_k = \lambda_{j,t} x_{k-1} - Z x_{k-2} + q^{(k-1)r} \mathcal{U}_{k-1} v$ must hold for $2 \leq k \leq n - 1$. To facilitate this, we write

$$x_k = q^{kr} \mathcal{U}_k v + q^{(k-1)r} \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2s) \mathcal{U}_{k-1-2s} v = q^{kr} \mathcal{U}_k v + q^{(k-1)r} \Sigma(k),$$

(3.26)

where $\Sigma(k) = \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2s) \mathcal{U}_{k-1-2s} v$.

(3.27)

Now for row $k$ of $M_V x_{j,t}$ verifying that (3.26) holds involves using the Chebyshev recursion $\mathcal{U}_{k+1} = (q^j + q^{-j}) \mathcal{U}_k - \mathcal{U}_{k-1}$ for $1 \leq k \leq n - 1$ (which we write $\mathcal{U}_{k+1} = \mathcal{U}_1 \mathcal{U}_k - \mathcal{U}_{k-1}$ here for the sake of brevity) and showing that

$$x_{k+1} = \lambda_{j,t} x_k - Z x_{k-1} + q^{kr} \mathcal{U}_k v$$

$$= q^{(k+1)r} \mathcal{U}_1 \mathcal{U}_k v + q^{kr} \mathcal{U}_1 \Sigma(k) - q^{2r} x_{k-1} + q^{kr} \mathcal{U}_k v$$

$$= q^{(k+1)r} \left( \mathcal{U}_{k+1} v + \mathcal{U}_{k-1} v \right) + q^{kr} \left( \mathcal{U}_1 \Sigma(k) + \mathcal{U}_k v \right)$$

$$- q^{2r} \left( q^{(k-1)r} \mathcal{U}_{k-1} v + q^{(k-2)r} \Sigma(k-1) \right)$$

$$= q^{(k+1)r} \mathcal{U}_{k+1} v + q^{kr} \left( \mathcal{U}_1 \Sigma(k) - \Sigma(k-1) + \mathcal{U}_k v \right).$$

(3.28)

We see that (3.26) will hold for $k + 1$, if we can show that

$$\Sigma(k+1) = \mathcal{U}_1 \Sigma(k) - \Sigma(k-1) + \mathcal{U}_k v, \quad (\mathcal{U}_1 = q^j + q^{-j}).$$

(3.29)

Case $k$ odd, $k = 2t + 1$ for $t \geq 1$. We start from the right-hand side and use the Chebyshev recursion relation. When we encounter a term $\mathcal{U}_t v$ with $\ell < 0$, we assume it is 0 and drop it from the equation. We use the fact that $\lfloor \frac{k-1}{2} \rfloor = t = \lfloor \frac{k}{2} \rfloor$ and $\lfloor \frac{k-2}{2} \rfloor = t - 1$. Then
\[ \mathcal{U}_1 \Sigma(k) - \Sigma(k-1) + \mathcal{U}_k \nu \]
\[ = \mathcal{U}_1 \sum_{s=0}^{t} (k-2s) \mathcal{U}_{k-2-2s} \nu - \sum_{s=0}^{t-1} (k-1-2s) \mathcal{U}_{k-2-2s} \nu + \mathcal{U}_k \nu \]
\[ = \sum_{s=0}^{t} (k-2s) \mathcal{U}_{k-2s} \nu + \sum_{s=0}^{t-1} (k-2s) \mathcal{U}_{k-2-2s} \nu - \sum_{s=0}^{t-1} (k-1-2s) \mathcal{U}_{k-2-2s} \nu + \mathcal{U}_k \nu \]
\[ = \sum_{s=0}^{t} (k-2s) \mathcal{U}_{k-2s} \nu + \sum_{s=0}^{t-1} \mathcal{U}_{k-2-2s} \nu + \mathcal{U}_k \nu \]
\[ = \sum_{s=0}^{t} (k+1-2s) \mathcal{U}_{k-2s} \nu - \sum_{s=0}^{t} \mathcal{U}_{k-2s} \nu + \sum_{s=0}^{t-1} \mathcal{U}_{k-2-2s} \nu + \mathcal{U}_k \nu \]
\[ = \sum_{s=0}^{t} (k+1-2s) \mathcal{U}_{k-2s} \nu = \sum_{s=0}^{t} (k+1-2s) \mathcal{U}_{k-2s} \nu = \Sigma(k+1). \]

**Case k even.** Since the argument just requires minor adjustments to the one above when \( k \) is even, we omit the proof.

What remains to be done is to compute the last row of \( M \nu \mathbf{x}_{j,r} \), and to show that \( 2 \mathbf{x}_0 + 2Z \mathbf{x}_{n-2} = \lambda_j \nu \mathbf{x}_{n-1} + q^{(n-1)r} \mathcal{U}_{n-1} \nu \). We will use the relations \( Z \mathbf{x}_k = q^{2r} \mathbf{x}_k \) for all \( 0 \leq k \leq n-1 \), and \( \mathcal{U}_{n-1} \nu = 0 \) and \( \mathcal{U}_n \nu = \nu \) which come from \( \mathcal{U}_k \nu = q^{(k+1)r - q^{(k+1)r}} \nu \) when \( k = n-1, n \). Since \( \mathcal{U}_{n-1} \nu = 0 \), what we end up showing is that \( 2 \mathbf{x}_0 + 2Z \mathbf{x}_{n-2} = \lambda_j \nu \mathbf{x}_{n-1} \) for all \( n \geq 1 \). Now the computation in (3.28) for \( k = n-1 \) with some rearranging and with \( \mathcal{U}_1 (n-1) - \Sigma(n-2) + \mathcal{U}_{n-1} \nu \) replaced with \( \Sigma(n) = \sum_{s=0}^{n-1} (n-2s) \mathcal{U}_{n-1-2s} \nu \) implies

\[ Z \mathbf{x}_{n-2} - \lambda_j \nu \mathbf{x}_{n-1} = q^{(n-1)r} \mathcal{U}_{n-1} \nu - q^{n} \mathcal{U}_n \nu - q^{(n-1)r} \sum_{s=0}^{n-1} (n-2s) \mathcal{U}_{n-1-2s} \nu \]
\[ = -\nu - q^{(n-1)r} \sum_{s=1}^{n-1} (n-2s) \mathcal{U}_{n-1-2s} \nu \quad \text{(using } \mathcal{U}_{n-1} \nu = 0 \text{ and } \mathcal{U}_n \nu = \nu). \]

Therefore, for the last row we have

\[ 2 \mathbf{x}_0 + 2Z \mathbf{x}_{n-2} - \lambda_j \nu \mathbf{x}_{n-1} = 2 \mathbf{v} + q^{2r} \mathbf{x}_{n-2} - \nu - q^{(n-1)r} \sum_{s=1}^{n-1} (n-2s) \mathcal{U}_{n-1-2s} \nu \]
\[ = \mathbf{v} + q^{2r} \left( q^{(n-2)r} \mathcal{U}_{n-2} \nu + \sum_{s=0}^{n-1} (n-2s) \mathcal{U}_{n-3-2s} \nu \right) \]
\[ - q^{(n-1)r} \sum_{s=1}^{n-1} (n-2s) \mathcal{U}_{n-1-2s} \nu \]
3.6 Right eigenvectors from grouplike elements of \( D_n \)

In this section, we focus on the grouplike elements \( b^i c^k \), \( 0 \leq i, k \leq n - 1 \), of \( D_n \) and compute the trace vectors \( \text{Tr}_S(b^i c^k) \) explicitly. Relation (2.5) with \( g = b^i c^k \) says \( \text{MV}_S(b^i c^k) = \text{tr}_V(b^i c^k) \text{Tr}_S(b^i c^k) \), so that \( \text{Tr}_S(b^i c^k) \) is a right eigenvector of eigenvalue \( \text{tr}_V(b^i c^k) \) for the McKay matrix \( \text{MV}_V \), \( V = V(2, 0) \). We identify the eigenvalue \( \text{tr}_V(b^i c^k) \) with \( \lambda_{i,r} = q^j + q^{-j} \) for certain values of \( j \) and \( r \) that depend on \( i \) and \( k \). Since there is a unique right eigenvector of \( \text{MV}_V \) corresponding to the eigenvalue \( \lambda_{i,r} \) up to scalar multiples by Theorem 3.8, by comparing the vectors \( \text{Tr}_S(b^i c^k) \) with the vectors \( v_{j,r} \) in Section 3.4, we obtain an expression for the characters \( \eta_{\ell,s} \) of the simple \( D_n \)-modules \( V(\ell, s) \) evaluated on the grouplike elements \( b^i c^k \) of \( D_n \) in terms of Chebyshev polynomials of the second kind. The trace vectors \( \text{Tr}_S(b^i c^k) = \text{Tr}_S(b^i c^k) \) for \( i, k \in \mathbb{Z}_n \) are shown to give a complete set of right eigenvectors for \( \text{MV}_V \).

The simple \( D_n \)-module \( V(\ell, s) \), \( 1 \leq \ell \leq n \) and \( s \in \mathbb{Z}_n \), has a basis \( \{v_1, v_2, \ldots, v_\ell\} \) with \( D_n \)-action prescribed by (3.3), which implies that \( b^i c^k \) has the following character value on \( V(\ell, s) \):

\[
\eta_{\ell,s}(b^j c^k) := \text{tr}_V(b^j c^k) = \sum_{\tau=1}^{\ell} q^{(s+\tau-1)j+(\tau-(\ell+s))k} = q^{(s-1)i-(\ell-s)k} \sum_{\tau=1}^{\ell} q^{(\tau+i-k)k}. \tag{3.30}
\]

We assume that the simple modules of \( D_n \) are ordered so that for a fixed grouplike element \( g \),

\[
\text{Tr}_S(g) = [u_1 \ u_2 \ \ldots \ u_{n-1} \ u_n]^T, \quad \text{where} \quad u_\ell = [\eta_{\ell,0}(g) \ \eta_{\ell,1}(g) \ \ldots \ \eta_{\ell,n-1}(g)] \quad \text{for} \ 1 \leq \ell \leq n. \tag{3.31}
\]

**Theorem 3.9** Assume \( i, k \in \mathbb{Z}_n \), and \( V = V(2, 0) \). Then

(a) \( \text{Tr}_S(b^i c^k) \) is a right eigenvector for \( \text{MV}_V \) of eigenvalue \( \lambda_{i,r} = \eta_{2,0}(b^i c^k) = q^j + q^{-j} \) if

\[
j = \pm \frac{i + k}{2} \pmod{n} \quad \text{and} \quad \ell = \frac{i - k}{2} \pmod{n}, \tag{3.32}
\]

or equivalently, if for \( 0 \leq j \leq \frac{n+1}{2} \) and \( r \in \mathbb{Z}_n \),

\[
i = j + r \pmod{n}, \quad k = j - r \pmod{n} \quad \text{or} \quad i = -j + r \pmod{n}, \quad k = -j - r \pmod{n}. \tag{3.33}
\]

(b) \( \text{Tr}_S(b^i c^k) = v_{j,r} \), and for all \( 1 \leq \ell \leq n \) and \( s \in \mathbb{Z}_n \),

\[
\eta_{\ell,s}(b^i c^k) = q^{(s+\tau-1)j} \text{U}_{\tau-1}(q^j + q^{-j}), \tag{3.34}
\]

where \( j \) and \( r \) are as in (3.32), and \( v_{j,r} \) is as in Proposition 3.6.
(c) \( \text{Tr}_S(b^j e^k) = \text{Tr}_S(b^{-k} e^{-i}) \) for all \( i, k \in \mathbb{Z}_n \). Therefore, the character \( \eta_{\ell,s} \) of \( V(\ell, s) \) satisfies

\[
\eta_{\ell,s}(b^j e^k) = \eta_{\ell,s}(b^{-k} e^{-i})
\]

(3.35)

for all \( 1 \leq \ell \leq n \) and \( s \in \mathbb{Z}_n \).

(d) The vectors \( \text{Tr}_S(b^j e^k) = \text{Tr}_S(b^{-k} e^{-i}) \) for \( i, k \in \mathbb{Z}_n \) give a complete set of right eigenvectors for the McKay matrix \( M_V \) for tensoring with the \( D_n \)-module \( V = V(2, 0) \).

**Proof** We already know that \( \text{Tr}_S(b^j e^k) \) is a right eigenvector of \( M_V \) of eigenvalue \( \eta_{2s,0}(b^i e^k) \) for \( i, k \in \mathbb{Z}_n \), (see (1) of Corollary 2.3).

(a) By (3.30) with \( \ell = 2 \) and \( s = 0 \),

\[
\eta_{2s,0}(b^j e^k) = q^j + q^{-k} = q^{-j} (q^i + q^{-i}) = q^j (q^i + q^{-i}) = \lambda_{j,r},
\]

(3.36)

when \( j \) and \( r \) are as in (3.32). Conversely, given \( r \in \mathbb{Z}_n \) and \( 0 \leq j \leq \frac{n-1}{r} \), it is easy to verify that \( \eta_{2s,0}(b^j e^k) = \lambda_{j,r} \) for the specified values of \( i \) and \( k \) in (3.33).

(b) Let \( \text{Tr}_S(g) \) and \( u_r \) be as in (3.31) for \( g = b^j e^k \), and assume \( 2r = i - k \) and \( 2j = i + k \mod n \). We argue first that \( u_1 \) is the right eigenvector \[ 1 q^{2r} q^{4r} ... q^{2(n-1)r} \]

for the cyclic \( n \times n \) matrix \( Z \) with corresponding eigenvalue \( q^{2r} \). For this, observe that by (3.30),

\[
\eta_{1s}(b^j e^k) = q^{(s-1)i-(s+1)k} q^{j+k} = q^{si-sk}.
\]

(3.37)

Thus \( \frac{\eta_{1s}(b^j e^k)}{\eta_{1,s-1}(b^j e^k)} = q^{i-k} = q^{2r} \) for all \( s \in \mathbb{Z}_n \). Since \( \eta_{1,0}(b^j e^k) = 1 \), we have

\[
u_1 = \left[ 1 q^{2r} q^{4r} ... q^{2(n-1)r} \right]^T,
\]

which is an eigenvector for \( Z \) of eigenvalue \( q^{2r} \), and \( \eta_{1s}(b^j e^k) = q^{2sr} \), for all \( s \in \mathbb{Z}_n \).

Now the first vector components \( u_1 \) and \( v_0 \) of \( \text{Tr}_S(g) \) and \( v_{j,r} \) are identical, and the subsequent vector components \( u_r \) of \( \text{Tr}_S(g) \) must satisfy the same relations as the vector components \( v_{\ell-1} \) in the proof of Proposition 3.6 for \( \ell \geq 1 \). Thus, \( \text{Tr}_S(b^j e^k) = v_{j,r} \), where \( j \) and \( r \) are as in (3.32) and

\[
u_r = v_{\ell-1} = q^{(\ell-1)r} U_{\ell-1}(q^j + q^{-i}) \mathbf{w}_0 = q^{(\ell-1)r} U_{\ell-1}(q^j + q^{-i}) u_1.
\]

(3.38)

Equating component \( s \) on both sides for \( 0 \leq s \leq n-1 \) gives the assertion in (3.34) – the character value \( \eta_{\ell,s}(b^j e^k) \) of \( b^j e^k \) on \( V(\ell, s) \) is given by \( \eta_{\ell,s}(b^j e^k) = q^{(\ell-1)s} U_{\ell-1}(q^j + q^{-i}) \), where \( j \) and \( r \) are as in (3.32).

(c) We have seen in the proof of (a) that \( \eta_{2s,0}(b^j e^k) = q^j + q^{-k} \) for any \( i, k \in \mathbb{Z}_n \). Therefore, \( \eta_{2s,0}(b^{-k} e^{-i}) = q^{-k} + q^i = \eta_{2s,0}(b^j e^k) \). Since \( \text{Tr}_S(b^j e^k) \) and \( \text{Tr}_S(b^{-k} e^{-i}) \) are two right eigenvectors of \( M_V \) with the same eigenvalue, and since \( \eta_{1s}(b^j e^k) = q^{si-sk} = \eta_{1s}(b^{-k} e^{-i}) \) for all \( s \in \mathbb{Z}_n \), follows from (3.37), we obtain as in the proof of (b) that \( \text{Tr}_S(b^j e^k) = \text{Tr}_S(b^{-k} e^{-i}) \). As a result, \( b^j e^k \) and \( b^{-k} e^{-i} \) have the same character value on any simple \( D_n \)-module \( V(\ell, s) \), that is \( \eta_{\ell,s}(b^j e^k) = \eta_{\ell,s}(b^{-k} e^{-i}) \) for all \( 1 \leq \ell \leq n \), and \( s \in \mathbb{Z}_n \).
3.7 Generalized right eigenvectors as trace vectors

The goal of this section is to show that generalized right eigenvectors for the McKay matrix $M_V$ for $V = V(2, 0)$ can also be realized as trace vectors on simple modules, but for traces of non-grouplike elements. Generalized eigenvectors occur only for the eigenvalues $\lambda_{j, r}$ with $j \neq 0$, and the corresponding (generalized) eigenspace is two-dimensional in that case. We will use the coproduct expression for the trace in (2.4) and will require quantum integers $[\ell] = 1 + q + \cdots + q^{\ell-1}$, the quantum factorial $[\ell]! = [\ell][\ell-1] \cdots [1]$, and the quantum binomial coefficient

$$\binom{\ell}{i} = \frac{[\ell]!}{[i]![\ell-i]!}$$

for $\ell, i \in \mathbb{Z}_{\geq 0}$, $\ell \geq i$, where $[0] = [0]! = 1$ is understood.

Chen has studied a family of Hopf algebras $H(p, q)$, where $p$ and $q$ are arbitrary scalars, and Lemma 2.7 of [4] gives an expression for coproduct for the algebra $H(p, 1)$. When $q$ is an $n$th root of unity, $H(p, q)$ modulo a certain Hopf ideal is a finite-dimensional quasi-triangular Hopf algebra $H_n(p, q)$, and the algebra $H_n(1, q)$ is isomorphic to $D_n$. Chen’s coproduct formula can be modified by replacing binomial coefficients with their $q$ analogues to give a coproduct formula for specific powers of the generators of $D_n$. We will use these specializations in the next result.

Lemma 3.10 For $\ell \in \mathbb{Z}_{\geq 0}$,

$$\Delta(d^\ell a^r) = \sum_{i=0}^{\ell} \binom{\ell}{i} d^i a^r \otimes b^i \epsilon^{\ell-i} a^{\ell-i} + \text{nilpotent terms}$$

$$= d^\ell a^r \otimes b^\ell \epsilon^\ell + [\ell]^2 d^{\ell-1} a^{\ell-1} \otimes b^{\ell-1} \epsilon^{\ell-1} da + \text{nilpotent terms.}$$

By nilpotent terms, we mean terms $x \otimes y$ such that $y$ acts as a nilpotent endomorphism on $V$.

Proof From [4, Lemma 2.7], we deduce

$$\Delta(a^r) = \sum_{i=0}^{\ell} \binom{\ell}{i} d^i \otimes a^{\ell-i} b^i, \quad \Delta(d^\ell) = \sum_{i=0}^{\ell} \binom{\ell}{i} d^i \otimes \epsilon^i d^{\ell-i}.$$

Recall that $ba = qab$, $db = qbd$ in $D_n$. Therefore,
Therefore, \(s = M\) Theorem 3.11 \(M\) and the action of the generators of 

Now from (3.36) we know that 

The term \(b^\ell c^\ell d^{-\ell} a^{\ell-1}\) is nilpotent on \(V\) only if \(\ell - t = 0\) or \(\ell - t = 1\), i.e. \(t = \ell\) or 

The following result gives a formulation of the generalized right eigenvectors for 

Theorem 3.11 \(\) Given \(0 \leq i, k \leq n - 1\) with \(k \neq -i (\text{mod } n)\), choose \(1 \leq s \leq n - 1\) such that 

\[ \gamma_\ell = \frac{[\ell + 1]^2 q^{-1-\ell+s}}{[\ell][s-\ell](q-1)} \gamma_{\ell+1} \]

for \(\ell = s-1, s-2, \ldots, 1\). Then \(\sum_{\ell=1}^s \gamma_\ell \text{Tr}_V(b^i c^k d^\ell a^t)\) is in the generalized eigenspace of 

M = \(M_V\), \(V = V(2,0)\), as trace vectors on the simple modules. Recall that \(\{\text{Tr}_S(b^i c^k)\}\) gives a list of right eigenvectors for \(M_V\), with repetitions described by Theorem 3.9 (c).

Proof \(\) From the coproduct expression in (3.39) and Lemma 2.1 (c), we have 

\[ M \text{Tr}_V(d^t a^\ell) = \text{tr}_V(b^i c^\ell) \text{Tr}_V(d^t a^\ell) + [\ell]^2 \text{tr}_V(b^{\ell-1} c^{\ell-1} d a) \text{Tr}_V(d^t a^{\ell-1}). \]

And similarly, 

\[ M \text{Tr}_S(b^i c^k d^\ell a^t) = \text{tr}_V(b^{\ell+i} c^{\ell+k}) \text{Tr}_S(b^i c^k d^\ell a^t) \]

\[ + \text{tr}_V(b^{\ell+i-1} c^{\ell+k-1} d a) [\ell]^2 \text{Tr}_S(b^i c^k d^{\ell-1} a^{\ell-1}). \]

Now from (3.36) we know that \(\text{tr}_V(b^i c^k) = q^i + q^{-k}\) for any \(i, k \in \mathbb{Z}_n\). Using that fact and the action of the generators of \(D_n\) on the basis \(\{v_1, v_2\}\) of \(V\) in (3.3), we have 

\[ \text{tr}_V(b^{\ell+i} c^{\ell+k}) = q^{\ell+i} + q^{-\ell-k}, \]

\[ da.v_1 = a_1(2)v_1 = (1-q^{-1})v_1, \quad b^{\ell+i-1} c^{\ell+k-1}.v_1 = q^{-\ell-k+1}v_1, \]

\[ \text{tr}_V(b^{\ell+i-1} c^{\ell+k-1} d a) = (1-q^{-1})q^{-\ell-k+1}. \]

Therefore, 

\[ M \text{Tr}_S(b^i c^k d^\ell a^t) = (q^{\ell+i} + q^{-\ell-k}) \text{Tr}_S(b^i c^k d^\ell a^t) \]

\[ + [\ell]^2 (1-q^{-1})q^{-\ell-k+1} \text{Tr}_S(b^i c^k d^{\ell-1} a^{\ell-1}). \] (3.40)
By a change of the index of summation on the second summand,

\[
\mathbf{M} \left( \sum_{\ell=1}^{s} \gamma_{\ell} \text{Tr}_{S}(b^{i}c^{k}d^{\ell}a^{j}) \right) \\
= \sum_{\ell=1}^{s} (q^{\ell+i} + q^{-\ell-k}) \text{Tr}_{S}(b^{i}c^{k}d^{\ell}a^{j}) \gamma_{\ell} \\
+ \sum_{t=0}^{s-1} (t+1)^{2}(1-q^{-1})q^{-\ell-k} \text{Tr}_{S}(b^{i}c^{k}d^{\ell}a^{j}) \gamma_{\ell+1} \\
= (1 - q^{-1})q^{-k} \text{Tr}_{S}(b^{i}c^{k}) \gamma_{1} + (q^{s+i} + q^{-s-k}) \gamma_{s} \text{Tr}_{S}(b^{i}c^{k}d^{s}a^{s}) \\
+ \sum_{t=1}^{s-1} \left( (q^{\ell+i} + q^{-\ell-k}) \gamma_{\ell} + [(t+1)^{2}(1-q^{-1})q^{-\ell-k}] \gamma_{\ell+1} \right) \text{Tr}_{S}(b^{i}c^{k}d^{\ell}a^{j}).
\]

We claim this is equal to

\[
\lambda \left( \sum_{\ell=1}^{s} \gamma_{\ell} \text{Tr}_{S}(b^{i}c^{k}d^{\ell}a^{j}) \right) + \text{constant} \cdot \text{Tr}_{S}(b^{i}c^{k}),
\]

for \( \lambda = q^{i} + q^{-k} = \text{tr}_{V}(b^{i}c^{k}) = \lambda_{j,r} \), where \( j \) and \( r \) are as in (3.32).

First, the terms agree at \( \ell = s \), where we recall that \( s = -(i+k) \mod n \) so that

\[
\lambda = q^{s+i} + q^{-s-k} = q^{-k} + q^{i}.
\]

When \( 1 \leq \ell \leq s-1 \), the following is true

\[
(q^{\ell+i} + q^{-\ell-k}) \gamma_{\ell} + [(t+1)^{2}(1-q^{-1})q^{-\ell-k}] \gamma_{\ell+1} = \lambda \gamma_{\ell} = \gamma_{\ell}(q^{\ell} + q^{-k}),
\]

if and only if the following is true:

\[
\gamma_{\ell} = \frac{[(t+1)^{2}(1-q^{-1})q^{-\ell-k}] \gamma_{\ell+1}}{q^{i} + q^{-k} - q^{\ell+i} - q^{-\ell-k}} = \frac{[(t+1)^{2}(q-1)q^{-1-\ell-k}] \gamma_{\ell+1}}{q^{1-\ell-k} - q^{-\ell-k}(1-q^{-1})} \\
= \frac{[(t+1)^{2}q^{-1-\ell-k}]}{[(t+1)^{2}q^{-1-\ell-k}] \gamma_{\ell+1}} = \frac{[(t+1)^{2}q^{-1-k}]}{[(t+1)^{2}q^{-1-k}] \gamma_{\ell+1}} = \frac{[(t+1)^{2}q^{-\ell}s]}{[(t+1)^{2}q^{-\ell}s] \gamma_{\ell+1}}.
\]

3.8 Left eigenvectors and generalized left eigenvectors of \( M_{V}, V = V(2,0) \)

In this section, we compute left (generalized) eigenvectors of the McKay matrix \( \mathbf{M} = M_{V} \) for \( V = V(2,0) \) using modified Chebyshev polynomials \( \mathcal{L}_{j}(t) \) that are defined recursively by

\[
\mathcal{L}_{0}(t) = 2, \quad \mathcal{L}_{1}(t) = t, \quad \mathcal{L}_{k}(t) = t \mathcal{L}_{k-1}(t) - \mathcal{L}_{k-2}(t) \text{ for } k \geq 2.
\]  

\[\text{(3.41)}\]

**Proposition 3.12** Assume \( x \neq 1 \), and let \( t = x + x^{-1} \) as in Proposition 3.2. Then

(a) For \( k \geq 2 \), \( \mathcal{L}_{k}(t) = \mathcal{U}_{k}(t) - \mathcal{U}_{k-2}(t) \).

(b) For all \( k \geq 0 \), \( \mathcal{L}_{k}(t) = x^{k} + x^{-k} \).
The leftmost column involves comparing $w$ showing that Corollary 3.14 of Proposition 3.12 (b).

Proof (a) We have $L_2(t) = tL_1(t) - L_0(t) = t^2 - 2 = t^2 - 1 - 1 - U_2(t) - U_0(t)$, and for $k > 2$ the proof of (a) is an easy inductive argument starting from this base case.

(b) The relation $L_k(t) = x^k + x^{-k}$ clearly holds for $k = 0, 1$. Proposition 3.2 (a) says that $U_k(t) = x^k + x^{-k-2} + \cdots + x^{-(k-2)} + x^{-k}$ for all $k \geq 2$. Part (b) follows readily from that and part (a).

\[ \begin{aligned}
\text{Proposition 3.13} & \quad \text{For } r \in \mathbb{Z}_n, \text{ let } w_0 \neq 0 \text{ be a left eigenvector of } Z \text{ corresponding to the eigenvalue } q^{2r}. \text{ Assume } 0 \leq j \leq \frac{n-1}{2}, \text{ and set } w_k = q^{kr} L_k(q^j + q^{-j})w_0 = q^{kr}(q^j + q^{-j})w_0 \\
& \quad \text{for } 1 \leq k \leq n-1, \text{ where } L_k \text{ is the modified Chebyshev polynomial defined in (3.41). Then } w_{j,r} = [w_{n-1} \ w_{n-2} \ \ldots \ \ w_1 \ w_0] \text{ is a left eigenvector of } M \text{ corresponding to eigenvalue } \lambda_{j,r} = q^r(q^j + q^{-j}).
\end{aligned} \]

Proof The proof amounts to comparing the left and right sides of $w_{j,r} M = \lambda_{j,r} w_{j,r}$ and showing that equality holds when $w_k = q^{kr} L_k(q^j + q^{-j})w_0 = q^{kr}(q^j + q^{-j})w_0$, where the last "=" results from $L_k(q^j + q^{-j}) = q^k + q^{-k}$, which is a direct consequence of Proposition 3.12 (b).

We expand the left-hand side of $w_{j,r} M$ starting from the rightmost column and ask if the two sides of $w_{j,r} M = \lambda_{j,r} w_{j,r}$ are equal:

$w_1 \overset{?}{=} \lambda_{j,r} w_0 = q^r(q^j + q^{-j})w_0 = q^r L_1(q^j + q^{-j})w_0$

$w_2 \overset{?}{=} \lambda_{j,r} w_1 - w_0(2Z) = w_0(\lambda_{j,r}^2 - 2Z) = w_0(\lambda_{j,r}^2 - 2q^{2r})$

$= w_0 q^{2r}((q^j + q^{-j})^2 - 2) = q^{2r} L_2(q^j + q^{-j})w_0$

$w_k \overset{?}{=} \lambda_{j,r} w_{k-1} - w_{k-2}Z = \lambda_{j,r} q^{(k-1)r} L_{k-1}(q^j + q^{-j})w_0 - q^{(k-2)r} L_{k-2}(q^j + q^{-j})w_0 Z$

$= q^{kr}(q^j + q^{-j})L_{k-1}(q^j + q^{-j}) - L_{k-2}(q^j + q^{-j}) \overset{?}{=} w_0$

$= q^{kr} L_k(q^j + q^{-j})w_0, (3 \leq k \leq n-1)$. The leftmost column involves comparing $w_{n-2}Z + 2w_0$ with $\lambda_{j,r} w_{n-1}$, which we do by showing that

$w_{n-2}Z + 2w_0 - \lambda_{j,r} w_{n-1}$

$= q^{(n-2)r} L_{n-2}(q^j + q^{-j})w_0 Z + 2w_0 - \lambda_{j,r} q^{(n-1)r} L_{n-1}(q^j + q^{-j})w_0$

$= q^{nr} L_{n-2}(q^j + q^{-j})w_0 + 2w_0 - q^{nr}(q^j + q^{-j}) L_{n-1}(q^j + q^{-j})w_0$

$= (q^{j(n-2)} + q^{-j(n-2)})w_0 + 2w_0 - (q^j + q^{-j})(q^{j(n-1)} + q^{-j(n-1)})w_0$

$= (q^{j(n-2)} + q^{-j(n-2)} + 2 - q^j - q^{-j}) \overset{?}{=} 0$. The next corollary relates the above eigenvector results to the dimension vectors.

Corollary 3.14 The dimension vector $s = [\dim(S_1) \ \dim(S_2) \ \ldots \ \dim(S_n)]^T$ of the simple $D_n$-modules is a right eigenvector corresponding to the eigenvalue $\lambda_{0,0} = 2 = \dim(V(2,0))$. The dimension vector $p^T = [\dim(P_1) \ \dim(P_2) \ \ldots \ \dim(P_n)]$ of the projective indecomposable $D_n$-modules is a left eigenvector corresponding to $\lambda_{0,0}$. 

\[ \begin{aligned}
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\end{aligned} \]
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Proof Observe that \( v_0 = [1 1 \cdots 1 1]^T \) and \( w_0 = v_{0}^{T} \) are right and left (resp.) eigenvectors of the matrix \( Z \) corresponding to the eigenvalue 1, and \( U_0(2) = k + 1 \) and \( L_1(2) = 2 \) for all values of \( k \geq 0 \). Therefore, by Propositions 3.6 and 3.13, \( v_k = (k+1)v_0 \) and \( w_k = 2w_0 \) for all \( 1 \leq k \leq n - 1 \). Consequently, \( v_{0,0} = s \) and \( w_{0,0} = \frac{1}{n}p^2 \), and the corresponding eigenvalue is \( \lambda_{0,0} = 2 = \dim(V(2,0)) \).

Remark 3.15 Corollary 3.14 confirms the result in [20, Sec. 3] mentioned in the Introduction in the specific case of the Drinfeld double \( D_n \) of the Taft algebra and the McKay matrix for tensoring with \( V(2,0) \).

Proposition 3.16 Let \( r \in \mathbb{Z}_n \) and assume \( w = w_0 \), where \( w_0 Z = q^2 q \) as in Proposition 3.13. Fix a choice of \( j \in \{1, \ldots, \frac{n+1}{2} \} \) and set \( U_k = U_k(q^k + q^{-k}) \) for all \( k \geq 0 \). Assume \( y_0 = w, \ y_1 = q^{r} U_1 w + w, \) and \( y_k = q^{k r} (U_k - U_{k-2}) w + k q^{(k-1) r} U_{k-1} w \) for \( 2 \leq k \leq n - 1 \). Then \( y_{j,r} := [y_{n-1} \ y_{n-2} \ \ldots \ y_1 \ y_0] \) is a generalized left eigenvector for \( M_V \) corresponding to the eigenvalue \( \lambda_{j,r} = q^{r} (q^j + q^{-j}) \), and \( y_{j,r} \) satisfies

\[
\begin{aligned}
 y_{j,r} \cdot M &= [y_{n-1} \ y_{n-2} \ \ldots \ y_1 \ y_0] \\
 &= \lambda_{j,r} [y_{n-1} \ y_{n-2} \ \ldots \ y_1 \ y_0] + [q^{(n-1)r} L_{n-1} w \ q^{(n-2)r} L_{n-2} w \ \ldots \ q^{r} L_1 w \ w] \\
 &= \lambda_{j,r} y_{j,r} + w_{j,r},
\end{aligned}
\]

where \( w_{j,r} \) is as in Proposition 3.13, and \( L_k = U_k(q^k + q^{-k}) \) for \( k \geq 1 \).

Proof The proof entails comparing the entries on both sides of (3.42) starting with column 0 (the rightmost) and proceeding to column \( n - 1 \) (the leftmost). This involves noting that

\[
\begin{align*}
y_1 &= \lambda_{j,r} y_0 + w = q^{r} U_1 w + w, \\
y_2 &= \lambda_{j,r} y_1 - 2y_0 Z + q^{r} L_1 w = q^{r} (q^j + q^{-j})(q^{r} U_1 w + w) - 2q^{r} w + q^{r} L_1 w \\
&= q^{2r} (U_2 - U_0) w + 2U_1 w,
\end{align*}
\]

and then arguing inductively for \( k \geq 3 \), using the relation \( L_k = U_k - U_{k-2} \) to show that

\[
\begin{align*}
y_{k+1} &= \lambda_{j,r} y_k - y_{k-1} Z + q^{k r} L_k w \\
&= q^{r} (q^j + q^{-j}) [q^{k r} (U_k - U_{k-2}) + k q^{(k-1) r} U_{k-1}] w \\
&\quad - q^{(k+1)r} (U_{k-1} - U_{k-3}) w - (k - 1) q^{k r} U_{k-2} w + q^{k r} (U_k - U_{k-2}) w
\end{align*}
\]
It remains to check the leftmost column, which involves verifying that

\[ y_{n-2}Z + 2y_0 = \lambda_{j,r} y_{n-1} - q^{(n-1)r} \mathcal{L}_{n-1} \mathbf{w} = 0. \]

Now by the right-hand side of the above calculation for \( y_{k+1} \) with \( k = n - 1 \), we have

\[
\begin{align*}
  y_{n-2}Z + 2y_0 & = \lambda_{j,r} y_{n-1} - q^{(n-1)r} \mathcal{L}_{n-1} \mathbf{w} \\
  &= -q^{nr}(\mathcal{U}_n - \mathcal{U}_{n-2}) \mathbf{w} - nq^{(n-1)r} \mathcal{U}_{n-1} \mathbf{w} + 2\mathbf{w} \\
  &= -\mathcal{L}_n \mathbf{w} - 0 + 2\mathbf{w} = -(nq^r + q^{-n})\mathbf{w} + 2\mathbf{w} = 0,
\end{align*}
\]

since \( \mathcal{U}_{n-1} \mathbf{w} = \frac{q^{n-1} - q^{-n}}{q^r - q^{-r}} \mathbf{w} = 0 \) by Proposition 3.2 (a).

### 3.9 Left eigenvectors from grouplike elements of \( D_n \)

Next we determine the vector \( \text{Tr} \mathcal{P}(g) \) explicitly for \( g = b^j c^k \), a grouplike element of \( D_n \).

In contrast to the situation for right eigenvectors, only the \( n \) vectors \( \text{Tr} \mathcal{P}(b^j c^{k-i}) \), \( i \in \mathbb{Z}_n \), are nonzero.

We assume an ordering \( P_1, P_2, \ldots, P_{n^2} \) of the projective indecomposable \( D_n \)-modules \( \mathcal{P}(\ell, r) \), \( 1 \leq \ell \leq n - 1 \), \( V(n, r) \), \( r \in \mathbb{Z}_n \), first by \( \ell \) and then by \( r \). Since the dual of the simple module \( \mathcal{V}(\ell, r) \) is \( \mathcal{V}(\ell, 1 - r - \ell) \) by [22, Thm. 4.3], we see \( (\mathcal{V}(\ell, r))^* \cong \mathcal{V}(\ell, r) \), so that \( (\mathcal{V}^*)^* \cong \mathcal{V} \) holds for any finite-dimensional \( D_n \)-module. Using this fact, we have from (2.14) that for every grouplike element of \( D_n \) and every \( D_n \)-module \( V \), \( \text{Tr} \mathcal{P}(g) M_V = \text{tr}_V(g) \mathcal{P}(g) \), where \( \text{Tr}_V(g) = [\text{tr}_P(g) \text{ tr}_{P_2}(g) \ldots \text{ tr}_{P_{n^2}}(g)] \). Hence for every grouplike element of \( D_n \), \( \text{Tr} \mathcal{P}(g) \) is a left eigenvector of \( M_V \) of eigenvalue \( \text{tr}_V(g) \).

To compute the vector \( \text{Tr} \mathcal{P}(g) \) for \( g = b^j c^k \), we use the explicit description of the projective modules \( \mathcal{P}(\ell, r) \) given in [7, Lem. 2.1], showing for \( 1 \leq \ell < n \) that \( \mathcal{P}(\ell, r) \) has a basis \( P_1, P_2, \ldots, P_{2n} \) such that the actions of \( b \) and \( c \) are diagonal:

\[
(b^i c^j) P_r = \begin{cases} 
q^{(r-1)i+j+\ell-r} P_r & \text{for } 1 \leq t \leq n, \\
q^{(r-1)i+j+\ell-r} P_r & \text{for } n+1 \leq t \leq 2n.
\end{cases}
\]

Thus, for \( \ell < n \),

\[
\text{tr}_{\mathcal{P}(\ell, r)}(b^i c^j) = q^{(r-1)i+j+\ell-r} \sum_{i=1}^{n} q^{i+k} + q^{(r-1)i+j+\ell-r} \sum_{i=n+1}^{2n} q^{i+k}.
\]

As shown in (3.30), we have for \( \mathcal{V}(n, r) \),

\[
\text{tr}_{\mathcal{V}(n, r)}(b^i c^j) = q^{(r-1)i+j+\ell-r} \sum_{i=1}^{n} q^{i+k}.
\]
Observe that when $i + k \neq 0 \pmod{n}$, the trace in these expressions is 0; consequently, $\text{Tr}_i(b^i c^{-i}) = 0$, when $i + k \neq 0 \pmod{n}$. Hence, we may assume that $k = -i \pmod{n}$ and obtain

$$
\text{tr}_{(t, i)}(b^i c^{-i}) = \left( q^{(r-1)i+(r+\ell)i} + q^{(r-1+\ell)i+i} \right) n = 2nq^{(2r+\ell-1)i} \quad \text{for } \ell < n,
$$

$$
\text{tr}_{(n, r)}(b^i c^{-i}) = q^{(r-1)i+i} \sum_{i=1}^{n} q^{(i+i)} = nq^{(2r-1)i}.
$$

(3.43)

Except for the extra factor of 2 that occurs when $\ell < n$, the second equation in (3.43) is the same as the first one with $\ell = n$. This motivates us to define for a fixed value of $i$, a vector with the following $n$ components:

$$
t_{\ell, i} = \begin{cases} 
2nq^{(r+\ell)i} & 2nq^{(r-1)i} \quad 2nq^{(r+3)i} \quad \ldots \quad 2nq^{(r+n-3)i} \\
nq^{(r-1)i} & nq^{ri} \quad nq^{2ri} \quad \ldots \quad nq^{(n-3)ri} 
\end{cases} \quad \ell \neq n, 
\begin{cases} 
nq^{ri} & nq^{2ri} \quad \ldots \quad nq^{(n-3)ri} 
\end{cases} \quad \ell = n.
$$

(3.44)

From the calculations above, we can conclude that the following holds.

**Proposition 3.17** Assume $i \in \mathbb{Z}_n$, and let $t_{\ell, i}$ be as in (3.44) for $1 \leq \ell \leq n$. Then

$$
\text{Tr}_i(b^i c^{-i}) = [t_{1,i} \ t_{2,i} \ \ldots \ \ t_{n,i}]
$$

is a left eigenvector of the McKay matrix $M_V$ corresponding to the eigenvalue $\text{tr}_{V^i}(b^i c^{-i})$ for any finite-dimensional $D_n$-module $V$.

**Remark 3.18** In comparison to the right eigenvector situation in Theorem 3.9 (d), no nonzero left eigenvectors corresponding to the eigenvalues $q^j (q^i + q^{-i})$ for $j \neq 0$ arise from evaluating projective characters on grouplike elements of $D_n$.

**Example 3.19** When $V = V(2, 0)$, we have $V^* = V(2, -1)$, and (3.3) tells us that the matrix of $b^i c^{-i}$ on $V^*$ relative to the basis $\{v_1, v_2\}$ is

$$
\begin{pmatrix} 
q^{-i} & 0 \\
0 & q^{-i}
\end{pmatrix} = \begin{pmatrix} 
q^{-i} & 0 \\
0 & q^{-i}
\end{pmatrix}.
$$

Hence, $\text{tr}_{V^i}(b^i c^{-i}) = 2q^{-i} = \lambda_{0,(n-1)i}$. Since these eigenvalues are distinct for $i \in \mathbb{Z}_n$, the left eigenvectors $\text{Tr}_i(b^i c^{-i})$ for $M_V$ are linearly independent. When $n = 3$, the vectors $\text{Tr}_i(b^i c^{-i})$ and their corresponding eigenvalues are

$$
\begin{align*}
\text{Tr}_i(1) &= [6 \ 6 \ 6 \ 6 \ 6 \ 6 \ 3 \ 3 \ 3] \quad \lambda_{0,0} = 2, \\
\text{Tr}_i(bc^{-1}) &= [6 \ 6q \ 6q^2 \ 6q^2 \ 6q \ 3q^2 \ 3] \quad \lambda_{0,1} = \lambda_{0,-2} = 2q, \\
\text{Tr}_i(b^2 c^{-2}) &= [6 \ 6q^2 \ 6q \ 6q^2 \ 6q \ 3q^2 \ 3] \quad \lambda_{0,2} = \lambda_{0,-1} = 2q^2.
\end{align*}
$$

(3.45)

**3.10 Tensoring with $V(\ell, s)$**

Let $g = [V(1, 1)]$ and $x = [V(2, 0)]$ in the Grothendieck ring $G_0(D_n)$. Then $g^n = 1 = [V(1, 0)]$ and $x^k = [V(2, 0)]^k = [V^{nk}] = [V]^k$ in $G_0(D_n)$. The following results are
In addition, if \( f \) and \( g \) are \( [V(\ell, s)] \) for all \( s \in \mathbb{Z}_m \), it follows that \( G_0(D_n) \) is generated by \( g \) and \( x \). Moreover,

\[
[V(\ell, 0)] = \sum_{i=0}^{\lfloor \frac{\ell-1}{i} \rfloor} (-1)^i \binom{\ell-1}{i} g^i x^{\ell-2i} \quad \text{for all } 1 \leq \ell \leq n. \tag{3.47}
\]

Thus, this defines a sequence of elements of \( G_0(D_n) \) given by

\[
f_0(x, g) = 1 = [V(1, 0)], \quad f_1(x, g) = x = [V(2, 0)]
\]

and satisfying

\[
f_\ell(x, g) = x f_{\ell-1}(x, g) - g f_{\ell-2}(x, g) = \sum_{i=0}^{\lfloor \frac{\ell-1}{i} \rfloor} (-1)^i \binom{\ell-1}{i} g^i x^{\ell-2i}, \quad 2 \leq \ell \leq n. \tag{3.48}
\]

In addition, if \( f(x, g) := f_n(x, g) - g f_{n-2}(x, g) - 2 \), then by Lemma 3.3 of [40] we have

\[
f(x, g) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{i} g^i x^{n-2i} - 2 = 0. \tag{3.49}
\]

If \( R \) is the group algebra \( \mathbb{Z}G \), where \( G \) is the cyclic group generated by \( g \), then \( G_0(D_n) \cong R[x]/(f(x, x)) \) is a commutative \( \mathbb{Z} \)-algebra of dimension \( n^2 \) with a basis given by \( \{x^i| i \in \mathbb{Z}_m, 0 \leq k \leq n - 1 \} \). Under the correspondence \( 1 \leftrightarrow I = I_n, \ I l \leftrightarrow x, \ D \leftrightarrow g \), we have that \( U_k(t, D) \leftrightarrow f_k(x, g) \) for \( 0 \leq k \leq n \), since the recursion relations are the same. Moreover, \( p_n(t, D) \leftrightarrow f(x, g) \) implies that

\[
p_n(t, D) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{i} g^i x^{n-2i} - 2 = 0. \tag{3.50}
\]

These considerations give the following result (compare [40, Thm. 3.4]).

**Proposition 3.20** The Grothendieck ring \( G_0(D_n) \cong \mathbb{Z}[g, x]/(g^n-1, f(x, g)) \).

**Example 3.21** When \( n = 11, \) (3.49) says

\[
f(x, g) = x^{11} - \frac{11}{10} \binom{10}{9} g x^9 + \frac{11}{2} \binom{9}{8} g^2 x^7 - \frac{11}{8} \binom{8}{7} g^3 x^5 + \frac{11}{7} \binom{7}{6} g^4 x^3 - \frac{11}{6} \binom{6}{5} g^5 x^2 - 2
\]

\[
= x^{11} - 11g x^9 + 44g^2 x^7 - 77g^3 x^5 + 55g^4 x^3 - 11g^5 x^2 - 2
\]

and (3.50) says

\[
p_n(t, D) = t^{11} - 11t D^9 + 44t^2 D^7 - 77t^3 D^5 + 55t^4 D^3 - 11t^5 D - 2I = 0, \quad \text{(compare (3.10)).}
\]
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Now suppose that $M_{(s)}$ is the McKay matrix for tensoring with $V(\ell, s)$. In computing matrices here, we order the rows and columns as usual, first by $\ell$ and then by $s$, so that the order is $(1, 0), (1, 1), \ldots, (1, n-1), (2, 1), \ldots, (2, n-1), \ldots, (n, 0), \ldots, (n, n-1)$ as before. Let $Z^{(s)} = \text{diag}(Z^{s}, Z^{s}, \ldots, Z^{s})$ ($n$ copies), where $Z$ is the cyclic permutation matrix in (3.6), and let $I_{n}^{s}$ be the $n^{2} \times n^{2}$ identity matrix. Assume $M = M_{(2,0)}$, the McKay matrix for tensoring with $V = V(2, 0)$. Then (3.47) implies that

$$M_{(s)} = \sum_{i=0}^{n+1} (-1)^{i} \left( \begin{array}{c} \ell - 1 - i \\ i \end{array} \right) Z^{(\ell+s)} M_{(\ell, s)}^{\ell-1-2i} \quad \text{for all } 1 \leq \ell \leq n, s \in \mathbb{Z}_{n}.$$  \hspace{1cm} (3.5)  

Here are a few special cases:

$$M_{(1,0)} = I_{n}^{2}, \quad M_{(2,0)} = M, \quad M_{(3,0)} = M^{2} - Z^{(1)}, \quad M_{(4,0)} = M^{3} - 2Z^{(1)}M, \quad M_{(5,0)} = M^{4} - 3Z^{(1)}M^{2} + Z^{(2)}.$$  

**Corollary 3.22** The left and right (generalized) eigenvectors are the same for all the matrices $M_{(\ell, s)}$, $1 \leq \ell \leq n, s \in \mathbb{Z}_{n}$.

**Proof** Since $M$ and $Z^{(1)}$ commute, we can simultaneously upper-triangularize them and find a basis of common right (generalized) eigenvectors for them. Similarly, we can simultaneously lower-triangularize $M$ and $Z^{(1)}$ and find a basis of common left (generalized) eigenvectors. These vectors will be common left and right (generalized) eigenvectors for all powers of $M$ and $Z^{(1)}$, hence for all the matrices $M_{(\ell, s)} = Z^{(s)}M_{(\ell, 0)}$. \hfill \Box

### 3.11 Eigenvalues for the McKay matrix of any simple $D_{n}$-module.

In this section, we use the results above to determine the eigenvalues of the McKay matrix $M_{(\ell, s)}$ for all $1 \leq \ell \leq n$ and $s \in \mathbb{Z}_{n}$. Recall from Proposition 3.6 that $v_{j,r} = [v_{0}^{r} \quad v_{1}^{r} \quad \ldots \quad v_{n-1}^{r}]^{T}$ is a right eigenvector for $M = M_{(2,0)}$ with eigenvalue $\lambda_{j,r} = q^{j}(q^{j} + q^{-j})$ when $\lambda_{0}$ is a right eigenvector for $Z$ with eigenvalue $q^{2j}$, and $v_{k} = Z^{(s)}U_{k}(q^{1} + q^{-j})v_{0}$ for $1 \leq k \leq n - 1$. It follows that $Z^{(s)}v_{j,r} = q^{2jr}v_{j,r}$ for all $s \in \mathbb{Z}_{n}$. As a consequence, we have the next result by combining (3.51) with (3.13).

**Theorem 3.23** (a) Assume $v_{j,r} = [v_{0} \quad v_{1} \quad \ldots \quad v_{n-1}]^{T}$ is a right eigenvector for $M = M_{(2,0)}$ with eigenvalue $\lambda_{j,r} = q^{j}(q^{j} + q^{-j})$ as in Proposition 3.6. Then

$$M_{(s)}v_{j,r} = \sum_{i=0}^{n+1} (-1)^{i} \left( \begin{array}{c} \ell - 1 - i \\ i \end{array} \right) q^{2jr} q^{(\ell-1-2i)r} (q^{j} + q^{-j})^{\ell-1-2i} v_{j,r},$$

$$= q^{(\ell-1)r} \sum_{i=0}^{n+1} (-1)^{i} \left( \begin{array}{c} \ell - 1 - i \\ i \end{array} \right) (q^{j} + q^{-j})^{\ell-1-2i} v_{j,r},$$

$$= q^{(\ell-1)r} U_{\ell-1}(q^{j} + q^{-j}) v_{j,r}.$$  \hspace{1cm} (3.52)
Hence, for all \(1 \leq \ell \leq n\) and for all \(s \in \mathbb{Z}_n\), \(v_{j,r}\) is a right eigenvector for 
\[M_{(\ell,s)} = Z^{(s)}M_{(\ell,0)}\] 
with eigenvalue 
\[q^{(\ell-1+2s)r} U_{\ell-1}(q^j) \] 
and with eigenvalue 
\[q^{(\ell-1+2s)r} \ell \] when \(j = 0\).

(b) Assume \(w_{j,r} = [w_{n-1} \ w_{n-2} \ldots \ w_1 \ w_0]\) is a left eigenvector for \(M\) with eigenvalue 
\[q^r (q^j + q^{-j})\] as in Proposition 3.13. Then \(w_{j,r}\) is a left eigenvector for 
\[M_{(\ell,s)} = Z^{(s)}M_{(\ell,0)}\] 
with eigenvalue 
\[q^{(\ell-1+2s)r} (q^j + q^{-j})\] for all \(1 \leq \ell \leq n\) and for all \(s \in \mathbb{Z}_n\).

Proof: Relation (3.23) follows directly from (3.52) and the fact that \(v_{j,r}\) is a right eigenvector for 
\(Z^{(s)}\) with eigenvalue \(q^{2sr}\). The last equality in (3.52) comes from Proposition 3.2 (a). Part (b) is similar.

3.12 Eigenvectors for the projective McKay matrices of \(D_n\).

Recall from Section 2.2 that the projective McKay matrix for tensoring with a finite-dimensional module \(V\) is given by \(Q_V = (Q_{ij})\), where \(Q_{ij} = [P_i \otimes V : P_j]\). We have shown in Theorem 2.11 that \(Q_V = M_V^T\), where \(M_V\) is the McKay matrix for tensoring with the dual module \(V^*\). In particular, since for the simple \(D_n\)-module \(V(\ell, s)\), we have 
\[V(\ell, s)^* = V(\ell, 1 - \ell - s),\] 
we can conclude the following about \(Q_{(\ell,s)}\) using Theorem 3.23.

Proposition 3.24 Assume \(v_{j,r}\) and \(w_{j,r}\) are as in Theorem 3.23. Then for all \(1 \leq \ell \leq n\) and for all \(s \in \mathbb{Z}_n\),

(a) \(v_{j,r}^\top = [v_0 \ v_1 \ldots \ v_{n-1}]\) is a left eigenvector for \(Q_{(\ell,s)}\) with eigenvalue 
\[q^{(\ell-1-2s)s} U_{\ell-1}(q^j) \] 
and with eigenvalue 
\[q^{(\ell-1-2s)s} \ell \] when \(j = 0\).

(b) \(w_{j,r}^\top = [w_{n-1} \ w_{n-2} \ldots \ w_1 \ w_0]\) is a right eigenvector for \(Q_{(\ell,s)}\) with eigenvalue as in (a) for all \(1 \leq \ell \leq n\) and for all \(s \in \mathbb{Z}_n\).

3.13 Multiplication operators in Grothendieck algebras and idempotents.

In Section 2.7, we have described a method for constructing a left eigenvector for the 
McKay matrix \(M_V\) using an eigenvector for the right multiplication operator \(R_V\) in the 
Grothendieck algebra of an arbitrary Hopf algebra. The next result has a similar flavor 
but is stated in greater generality for an arbitrary finite-dimensional algebra \(A\).

Proposition 3.25 Suppose \(A\) is an algebra with basis \(\{b_1, b_2, \ldots, b_k\}\), \(k \in \mathbb{Z}_{\geq 1}\), such that 
\[b_i b_j = \sum_{k=1}^{k} a_{ij}^{(k)} b_k\] for all \(1 \leq i, j \leq k\). Let \(M_j = (a_{ij}^{(k)})\) be the McKay matrix for 
multiplying by \(b_j\) (on the right). Assume \(u = [u_1 \ u_2 \ldots \ u_k]\) is a common left eigenvector for
Moreover, we show in part (a) of that theorem, that there are \( G \) elements that result from tensoring with \( R \) corresponding to distinct eigenvalues of the McKay matrix. In Proposition 3.13, we have described the McKay Matrices for Finite-dimensional Hopf Algebras.

**Proof** It follows from the relations \( uM_j = \beta_j u \) that \( \sum_{t=1}^k u_t u_{i,t} = \beta_j u_t \) for all \( 1 \leq j, t \leq k \). Then for \( e_u = \sum_{i=1}^k u_i b_i \), we have

\[
e_u^2 = \left( \sum_{i=1}^k u_i b_i \right)^2 = \left( \sum_{i=1}^k u_i b_i \right) \left( \sum_{j=1}^k u_j b_j \right) = \sum_{i,j=1}^k u_i u_j b_i b_j = \sum_{i,j=1}^k u_i u_j b_i b_j = \sum_{j,t=1}^k u_j \beta_j u_t b_t = \sum_{j,t=1}^k \beta_j u_t b_t = \sum_{j=1}^k \beta_j u_j \left( \sum_{i=1}^k u_i b_i \right) = e_u e_u.
\]

**Corollary 3.26** Assume \( A = G_0^\mathbb{C}(D_n) = \mathbb{C} \otimes G_0(D_n) \), and let \( b_1, b_2, \ldots, b_{n^2} \) be an ordering of the simple modules for the Drinfeld double \( D_n \), first by \( \ell = 1, \ldots, n \), and then for a given value of \( \ell \) by \( r = 0, 1, \ldots, n - 1 \), so that \( b_1, b_2, \ldots, b_{n^2} \) is a basis for \( A \), and \( b_1 = V(1,0) \) is the unit element of \( A \). Let \( M_j \) be the McKay matrix for tensoring with \( b_j \), and let \( u = [u_1 \ u_2 \ldots u_n] \) be a nonzero common left eigenvector for matrices \( M_j \) (such exist by Corollary 3.22). Assume \( u \) has eigenvalue \( \beta_j \) relative \( M_j \) for all \( j \). Then \( e_u = \sum_{i=1}^n u_i b_i \in A \) satisfies \( e_u^2 = c_u e_u \) where \( c_u = \sum_{j=1}^n \beta_j u_j \), and when \( c_u = \sum_{j=1}^n \beta_j u_j \neq 0 \), then \( e_u = c_u^{-1} e_u \) is an idempotent in \( A \).

**Remark 3.27** In Proposition 3.13, we have described \( \frac{n(n+1)}{2} \) left eigenvectors corresponding to distinct eigenvalues of the McKay matrix \( M \) for tensoring with \( V(2,0) \). They correspond to eigenvectors for the right multiplication operator \( R_x = [V(2,0)] \) of \( G_0^\mathbb{C}(D_n) \) as in Section 3.10. They are common left eigenvectors for the \( n^2 \) McKay matrices that result from tensoring with any simple module \( V(\ell, s) \). Each such left eigenvector \( u = [u_1 \ u_2 \ldots u_n] \) with \( c_u \neq 0 \) gives an idempotent \( e_u = c_u^{-1} \sum_{i=1}^n u_i b_i \) in \( G_0^\mathbb{C}(D_n) \). Moreover, such idempotents are distinct, \( u \neq v \implies e_u \neq e_v \), because the \( b_i \) determine a basis for \( G_0^\mathbb{C}(D_n) \). This suggests that we should be able to locate \( \frac{n(n+1)}{2} \) linearly independent idempotents in \( G_0^\mathbb{C}(D_n) \). We accomplish this in Theorem 3.31 (b) below. Moreover, we show in part (a) of that theorem, that there are \( \frac{n(n+1)}{2} \) linearly independent elements that square to 0 and form a basis for the Jacobson radical of \( G_0^\mathbb{C}(D_n) \).

Our aim here is to identify a \( \mathbb{C} \)-basis of \( G_0^\mathbb{C}(D_n) \) consisting of elements that square to 0 and idempotents. Our calculations will be based on the following well-known result (see for example, [18, Sec. 2.2]). If \( CG \) is the group algebra of a finite group and \( S \) is a simple \( G \)-module, then

\[
e_S := \frac{\dim(S)}{|G|} \sum_{g \in G} \chi_S(g^{-1}) g
\]
is a central idempotent in $\mathbb{C}G$, and $\varepsilon_S$ projects any finite-dimensional $G$-module onto the $S$-isotypic component. Moreover, $\varepsilon_S \varepsilon_T = 0$ whenever $S$ and $T$ are simple, nonisomorphic modules.

Recall that $G_0^G(D_n) \cong \mathbb{C}G[x]/(f(x, g))$, where $G = \langle g \rangle$,

$$f(x, g) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \frac{n-i}{n-i} x^{n-2i} - 2,$$

and $g = [V(1, 1)], \chi = [V(2, 0)]$. Therefore, it follows for $u \in \mathbb{Z}_n$ that $E_u = \frac{1}{n} \sum_{v=0}^{n-1} q^{uv} g^v$ is an idempotent in $\mathbb{C}G \subset G_0^G(D_n)$ corresponding to the character $\chi_u(g) = q^{n^2}$. In the notation of Corollary 3.5, we can write $\varepsilon_u$ so that $\varepsilon_u = g\varepsilon_u = \delta_{au'} \varepsilon_u$. Moreover, $g\varepsilon_u = \delta_{au'} \varepsilon_u$. Note that

$$g\varepsilon_u = \frac{1}{n} \sum_{v=0}^{n-1} q^{uv} g^{v+1} = \frac{q^n}{n} \sum_{v=0}^{n-1} q^{-2(v+1)} g^{v+1} = q^n \varepsilon_u,$$

(3.54)

so that $\mathbb{C}G = \bigoplus_{u \in \mathbb{Z}_n} \varepsilon_u$ is a decomposition of the group algebra $\mathbb{C}G$ into simple $G$-modules, where $\varepsilon_u$ is the one-dimensional $G$-module with corresponding character $\chi_u$.

As a consequence of (3.54), we know that

$$f(x, g)\varepsilon_u = f(x, q^u)\varepsilon_u = p_n(x, q^u)\varepsilon_u$$

(3.55)

in the notation of Corollary 3.5. We can write $u = 2r$ for some $r$, since $n$ is odd and 2 is invertible modulo $n$, and then

$$f(x, q^{2r}) = p_n(x, q^{2r}) = (x - A_0, r) \prod_{j=1}^{n-1} (x - A_j, r)^2.$$

(3.56)

**Definition 3.1** For $r \in \mathbb{Z}_n$, let

$$F_{j, r} := \frac{f(x, q^{2r})}{x - A_j, r} \varepsilon_{2r}, \quad 0 \leq j \leq \frac{n-1}{2},$$

$$G_{j, r} := \frac{f(x, q^{2r})}{(x - A_{j, r})^2} \varepsilon_{2r}, \quad 1 \leq j \leq \frac{n-1}{2}.$$

(3.57)

**Proposition 3.28** The elements defined in (3.57) satisfy the relations

(a) $x F_{j, r} = A_{j, r} F_{j, r}$ and $g F_{j, r} = q^{2r} F_{j, r};$

(b) $x G_{j, r} = A_{j, r} G_{j, r} + F_{j, r}$ and $g G_{j, r} = q^{2r} G_{j, r}.$

**Proof** The fact that $g F_{j, r} = q^{2r} F_{j, r}$ and $g G_{j, r} = q^{2r} G_{j, r}$ hold is a consequence of $g \varepsilon_{2r} = q^{2r} \varepsilon_{2r}$. The relations involving $x$ follow easily from

$$(x - A_{j, r}) F_{j, r} = (x - A_{j, r}) \frac{f(x, q^{2r})}{x - A_{j, r}} \varepsilon_{2r} = f(x, q^{2r}) \varepsilon_{2r} = f(x, g) \varepsilon_{2r} = 0,$$

$$(x - A_{j, r}) G_{j, r} = (x - A_{j, r}) \frac{f(x, q^{2r})}{(x - A_{j, r})^2} \varepsilon_{2r} = F_{j, r}.$$
The ideal $F$ is idempotent using the elements $G$. The following products are 0 whenever $F \neq s$:

(a) $F_j F_k = F_k F_j = G_j G_k = G_k G_j = 0$ when $j \neq k$.

(b) The following products are 0 whenever $j \neq k$:

$F_j F_k, F_j G_k, G_j G_k$.

(c) $F_{j,r}^2 = 0$ when $1 \leq j \leq \frac{n-1}{2}$.

(d) The ideal $F = \text{span}_k \{F_{j,r} \mid 1 \leq j \leq \frac{n-1}{2}, r \in \mathbb{Z}_n \}$ satisfies $F^2 = 0$.

Proof (a) Since $E_{2s} E_{2s} = 0$ whenever $r \neq s$, it is apparent that $F_j F_k = F_k F_j = G_j G_k = G_k G_j = 0$ for $r \neq s$ and any choice of $j$ and $k$. (b) Suppose $m_j = 1$ when $j = 0$, and $m_j = 1$ or 2 when $1 \leq j \leq \frac{n-1}{2}$. Then when $j \neq k$, we have

$$f(x, q^{2r}) E_{2s} = f(x, q^{2r}) E_{2s} = f(x, q^{2r}) E_{2s} = f(x, q^{2r}) E_{2s} = 0.$$

(3.58)

Part (b) is now clear from the calculation in (3.58). For part (c), observe that equation (3.58) holds when $k = j \neq 0, m_j = m_k = 1$. Part (d) follows from (b) and (c).

Proposition 3.30 The elements $\{\xi_r^{-1} F_{0,r} = \xi_r^{-1} f(x, q^{2r}) E_{2s} \mid r \in \mathbb{Z}_n \}$ are (nonzero) orthogonal idempotents, where $\xi_r = \prod_{j=1}^{\frac{n+1}{2}} \left(2q^r - q^r(q^j + q^{-j})\right)^2 \neq 0$.

Proof Orthogonality is a consequence of Proposition 3.29 (a), and the remaining assertions follow from (3.56) and the calculation

$$F_{0,r}^2 = \frac{f(x, q^{2r}) E_{2s}}{x - A_{0,r}} E_{2s} = \frac{f(x, q^{2r}) E_{2s}}{x - A_{0,r}} = \frac{f(x, q^{2r}) E_{2s}}{x - A_{0,r}} = \xi_r F_{0,r},$$

$$= \left(\prod_{j=1}^{\frac{n+1}{2}} (x - A_{j,r})^2\right) F_{0,r} = \left(\prod_{j=1}^{\frac{n+1}{2}} (A_{k,r} - A_{j,r})^2\right) F_{0,r} = \xi_r F_{0,r},$$

where $\xi_r = \left(\prod_{j=1}^{\frac{n+1}{2}} (2q^r - q^r(q^j + q^{-j}))\right)^2 \neq 0$, which implies $\xi_r^{-1} F_{0,r}$ is an idempotent.

So far we have identified $n$ idempotents $\xi_r^{-1} F_{0,r}, r \in \mathbb{Z}_n$, in $G_0^1(D_n)$ and $\frac{n(n-1)}{2}$ elements $F_{j,r}, 1 \leq j \leq \frac{n+1}{2}, r \in \mathbb{Z}_n$, that square to 0. Next we will find some more idempotents using the elements $G_{j,r}$. In Sec. 3.13.2, we will examine the elements $F_{j,r}$ and $G_{j,r}$ explicitly for $n = 3$. 
3.13.1 Idempotents from the elements $G_{j,r}$
We begin by computing $G_{j,r}F_{j,r}$ and $G_{j,r}^2$ for $1 \leq j \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_u$. Now

$$G_{j,r}F_{j,r} = \left( \frac{f(x, q^{2r})}{(x - A_{j,r})^2} \right) F_{j,r} = (x - A_{0,r}) \left( \prod_{k=1, k \neq j}^{n-1} (x - A_{k,r}) \right) F_{j,r}$$

$$= (A_{j,r} - A_{0,r}) \left( \prod_{k=1, k \neq j}^{n-1} (A_{j,r} - A_{k,r}) \right) F_{j,r} = \vartheta_{j,r} F_{j,r},$$

where

$$\vartheta_{j,r} = (A_{j,r} - A_{0,r}) \left( \prod_{k=1, k \neq j}^{n-1} (A_{j,r} - A_{k,r}) \right). \tag{3.59}$$

This shows that $G_{j,r}F_{j,r}$ is a nonzero multiple of $F_{j,r}$. We also have

$$G_{j,r}^2 = \left( \frac{f(x, q^{2r})}{(x - A_{j,r})^2} \right)^2 F_{j,r} = \left( \frac{f(x, q^{2r})}{(x - A_{0,r})^2} \right) F_{j,r} = (x - A_{0,r}) \left( \prod_{k=1, k \neq j}^{n-1} (x - A_{k,r}) \right) G_{j,r}.$$

For $1 \leq k \leq \frac{n-1}{2}$, $k \neq j$,

$$(x - A_{k,r})G_{j,r} = (A_{j,r} - A_{k,r})G_{j,r} + F_{j,r}$$

$$(x - A_{k,r})^2 G_{j,r} = (A_{j,r} - A_{k,r})(x - A_{k,r})G_{j,r} + (x - A_{k,r})F_{j,r}$$

$$= (A_{j,r} - A_{k,r})^2 G_{j,r} + 2(A_{j,r} - A_{k,r})F_{j,r}$$

$$(x - A_{0,r})G_{j,r} = (A_{j,r} - A_{0,r})G_{j,r} + F_{j,r}.$$

These computations and (3.60) imply that

$$G_{j,r}^2 = \vartheta_{j,r} G_{j,r} + v_{j,r} F_{j,r},$$

for some scalar $v_{j,r}$, where $\vartheta_{j,r} \neq 0$ is as in (3.59). Then

$$(G_{j,r} - \frac{v_{j,r}}{\vartheta_{j,r}} F_{j,r})^2 = G_{j,r}^2 - 2 \frac{v_{j,r}}{\vartheta_{j,r}} G_{j,r} F_{j,r}$$

$$= \vartheta_{j,r} G_{j,r} + v_{j,r} F_{j,r} - 2 \frac{v_{j,r}}{\vartheta_{j,r}} \vartheta_{j,r} F_{j,r}$$

$$= \vartheta_{j,r} G_{j,r} - v_{j,r} F_{j,r} = \vartheta_{j,r} \left( G_{j,r} - \frac{v_{j,r}}{\vartheta_{j,r}} F_{j,r} \right),$$

which shows that

$$G_{j,r}^r := \vartheta_{j,r} \left( G_{j,r} - \frac{v_{j,r}}{\vartheta_{j,r}} F_{j,r} \right) \tag{3.61}$$

is an idempotent in $G_0^u(D_n)$ for $\vartheta_{j,r}$ as in (3.59) and some $v_{j,r} \in \mathbb{C}$ for each $1 \leq j \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_u$.

In summary, we have the following
Theorem 3.31 (a) The $\frac{n(n-1)}{2}$ elements $F_{j,r}$, $1 \leq j \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, determine a $\mathbb{C}$-basis for the Jacobson radical $\mathcal{J}$ of the Grothendieck algebra $G^0_\mathcal{C}(D_n)$ and $\mathcal{J}^2 = (0)$.

(b) The elements $\xi_{\ell_0}^{-1}F_{0,r}$ and $\mathcal{G}_{j,r}'$ for $r \in \mathbb{Z}_n$ and $1 \leq j \leq \frac{n-1}{2}$ are orthogonal idempotents, and they form a basis for $G^0_\mathcal{C}(D_n)$ modulo $\mathcal{J}$.

(c) Suppose $S_1, S_2, \ldots, S_n$ is an ordering of the nonisomorphic simple $D_n$-modules, first by $\ell = 1, \ldots, n$ and then by $r = 0, 1, \ldots, n - 1$. If $F_{j,r} = c_{1,j}^{r}S_1 + c_{2,j}^{r}S_2 + \cdots + c_{n,j}^{r}S_n$, then $f_{j,r} := [c_{1,j}^{r} \ c_{2,j}^{r} \ \cdots \ c_{n,j}^{r}]$ is a left eigenvector for the McKay matrix $M_V$, $V = V(2,0)$, for all $0 \leq j \leq \frac{n-1}{2}$ and $r \in \mathbb{Z}_n$ corresponding to the eigenvalue $\lambda_{j,r} = q^j(q^r + q^{-r})$.

(d) If $\mathcal{G}_{j,r} = d_{1,j}^{r}S_1 + d_{2,j}^{r}S_2 + \cdots + d_{n,j}^{r}S_n$, then $g_{j,r} := [d_{1,j}^{r} \ d_{2,j}^{r} \ \cdots \ d_{n,j}^{r}]$ is a generalized left eigenvector for the McKay matrix $M_V$ such that for all $1 \leq j \leq \frac{n-1}{2}$ and $r \in \mathbb{Z}_n$, $g_{j,r}M_V = \lambda_{j,r}g_{j,r} + f_{j,r}$.

(e) The vectors $f_{j,r}$, $0 \leq j \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, give a complete set of left eigenvectors, and the vectors $g_{j,r}$, $1 \leq j \leq \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, give a complete set of generalized left eigenvectors for the McKay matrix $M_V$, $V = V(2,0)$, hence, for any McKay matrix $M_{\xi(s)}$ by Corollary 3.22.

Proof From Proposition 3.28 (a) we know that the elements $F_{j,r}$ are eigenvectors for the multiplication operator $R_x$, $x = [V(2,0)]$, of $G^0_\mathcal{C}(D_n)$, so the corresponding coordinate vectors $f_{j,r}$ relative to the basis of nonisomorphic simple modules will be left eigenvectors for $M = M_V$ by Proposition 2.15. Part (b) of Proposition 3.28 shows that $xg_{j,r} = \lambda_{j,r}g_{j,r} + F_{j,r}$. Therefore, the coordinate vector $g_{j,r}$ of $\mathcal{G}_{j,r}$ will be a generalized left eigenvector for $M_V$ corresponding to the eigenvalue $\lambda_{j,r}$ by Proposition 2.15 (b).

3.13.2 Computations for $D_3$

When $n = 3$, we have $f(x, q^{2r}) = x^3 - 3q^{2r}x - 2$, and

$$f(x, q^{2r}) = x^2 + 2q^{r}x + q^{2r} = (x + q^{r})^2, \quad \text{and}$$

$$\xi_{r} = (2q^{r} - q^{r}(q + q^{-1}))^2 = (3q^{r})^2 = 9q^{2r}.$$ 

Thus, $\xi_{r}^{-1}F_{0,r} = (9q^{2r})^{-1}(x^2 + 2q^{r}x + q^{2r})E_{2r} = \frac{1}{3}(q^{r}x^2 + 2q^{2r}x + 1)E_{2r}$ is an idempotent for $r = 0, 1, 2$. Now

$$F_{1,r} = \frac{f(x, q^{2r})}{x - A_{1,r}}E_{2r} = (x - A_{0,r})(x - A_{1,r})E_{2r},$$

(3.62)

so that

$$F_{1,r}^2 = (x - A_{0,r})^2(x - A_{1,r})^2E_{2r} = (x - A_{0,r})f(x, q^{2r})E_{2r} = (x - A_{0,r})f(x, q^{r})E_{2r} = 0.$$

Hence, the elements $F_{1,r}, r \in \mathbb{Z}_3$, square to 0 in agreement with Proposition 3.29 (c).

Finally,

$$\mathcal{G}_{1,r} = \frac{f(x, q^{2r})}{(x - A_{1,r})^2}E_{2r} = (x - A_{0,r})E_{2r}$$

and
\(\mathcal{G}_{1,r}^2 = (x - \lambda_{0,r})\mathcal{G}_{1,r} = (\lambda_{1,r} - \lambda_{0,r})\mathcal{G}_{1,r} + F_{1,r}.\)

This tells us that by taking \(v_{1,r} = 1\) and \(\vartheta_{1,r} = \lambda_{1,r} - \lambda_{0,r} = q'(q + q^{-1}) - 2q' = -3q',\)

\[
\left(\mathcal{G}_{1,r} + \frac{1}{3q'}F_{1,r}\right)^2 = -3q'\mathcal{G}_{1,r} + F_{1,r} - 2F_{1,r} = -3q'\left(\mathcal{G}_{1,r} + \frac{1}{3q'}F_{1,r}\right),
\]

and therefore by setting \(\mathcal{G}_{1,r}' = -\frac{1}{3q'}(\mathcal{G}_{1,r} + \frac{1}{3q'}F_{1,r}),\) we get an idempotent for \(r \in \mathbb{Z}_3.\)

Writing \((x,1)\) for \(V(\ell, r)\) and recalling that \(x = [V(2, 0)]\) and \(g = [V(1, 1)]\) gives

\[
\xi_r^{-1}F_{0,r} = (9q'^2)^{-1}(x^2 + 2q'x + q'^2)E_{2r}
\]

\[
= \frac{1}{27}(q'^2x^2 + 2q'^2x + 1)(q'^{-2r}g^2 + q'^{-2r} + 1)
\]

\[
= \frac{1}{27}\left(\xi_r(3,0) + \xi_r(1,1) + 2q'^2(2,0) + (1,0)\right)\left(q'^{2r}(1,2) + \xi_r(1,1) + (1,0)\right)
\]

\[
= \frac{1}{27}\left(3,2 + q'^2(3,1) + q'(3,0) + 2q'^2(2,2)
\right.
\]

\[
+ 2(2,1) + 2q'^2(2,0) + 2q'^2(1,2) + 2q'^2(1,1) + 2(1,0)\right).
\]

Ordering the summands from \((1,0)\) to \((3,2),\) ignoring the factor of \(\frac{1}{27},\) and recording the coefficients, we have

\[
f_{0,r} = [2 - 2q' 2q'^2 - 2q'^2 2 - 2q' q' q'^2 1].
\]

Multiplying \(f_{0,r}\) by 3 and then setting \(r = 0, 1, 2,\) we obtain the left eigenvectors of \(M_V\) for \(V = V(2, 0)\) in (3.45) exactly:

\[
\text{Trp}(1) = [6 6 6 6 6 3 3 3] \quad \lambda_{0,0} = 2,
\]

\[
\text{Trp}(bc^{-1}) = [6 6q 6q^2 6q^2 6 6q 3q 3q^2 3] \quad \lambda_{0,1} = 2q,
\]

\[
\text{Trp}(bc^{-2}) = [6 6q^2 6q 6 6q^2 3q^2 3 3] \quad \lambda_{0,2} = 2q^2.
\]

Now

\[
\mathcal{G}_{1,r} = \frac{f(x, q'^2)}{(x - \lambda_{0,r})}E_{2r} = (x - \lambda_{0,r})E_{2r} = (x - 2q')(\frac{1}{3}(q'^2 + q'^2g + 1))
\]

\[
= \frac{1}{3}(1 - 2q'(1,0) - 2q'^2(1,1) - 2(1,2) + (2,0) + q'(2,1) + q'^2(2,2)).
\]

Therefore, \(g_{1, r} = \frac{1}{3}[-2q'^2, -2q'^2, -2, 1, q', q'^2, 0, 0, 0],\) and \(\lambda_{1,r} = q'(q + q^{-1}) = -q'.\)

From (3.62), we can deduce that

\[
f_{1,r} = \frac{1}{3}[-q'^2, -1, -q', -q'^2, -q'^2, -1, 1, q', q'^2].\]

Then

\[
g_{1,r}M_V = \frac{1}{3}[-2q', -2q'^2, -2, 1, q', q'^2, 0, 0, 0] \begin{pmatrix} 0 & 1 & 0 \\ Z & 0 & 1 \\ 2I & 2Z & 0 \end{pmatrix}
\]

\[
= \frac{1}{3}[q'^2, 1, q', -q'^2, -q'^2, -2, 1, q', q'^2]
\]

\[
= \frac{-q'}{3}[-2q', -2q'^2, -2, 1, q', q'^2, 0, 0, 0] + \frac{1}{3}[-q'^2, -1, -q', -q'^2, -1, 1, q', q'^2]
\]

\[= \lambda_{1,r}g_{1,r} + f_{1,r}.\]
so that $g_{1,r}$ is a generalized left eigenvector for $M_V$ with eigenvalue $\lambda_{1,r}$ for $r \in \mathbb{Z}_3$.

### 3.14 Fusion rules for tensoring a maximal set of independent projective modules in $G_0(D_n)$ with $V$

We have seen in Proposition 3.1 that the Cartan map $c$ for $D_n$ has rank $\frac{n(n+1)}{2}$, and that the modules $P(\ell, r) - P(n - \ell, \ell + r)$ lie in the kernel of $c$ for $1 \leq \ell \leq \frac{n(n-1)}{2}$. Following [9], we let $N_V$ be the matrix that records tensoring a projective module $P$ with $V = V(2,0)$ and writing the answer $[P \otimes V]$ as a $\mathbb{Z}$-combination of isomorphism classes of projectives whose images form a $\mathbb{Z}$-basis for $c(K_0(D_n)) \subseteq G_0(D_n)$. Since the Cartan map has rank $\frac{n(n+1)}{2}$, and we use only the modules $V(n,r), P(1,r), \ldots, P(\frac{n-1}{2},r)$ in forming $N_V$. We assume that ordering and take all values of $r$ for each type, first for $V(n,r)$, then for $P(1,r)$ etc. From the tensor rules (3.2), we have that the resulting matrix $N_V$ is $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ for any $n = 2h + 1$ with $h \geq 1$, and

$$N_V = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
2Z & 0 & I & \cdots & 0 & 0 \\
0 & Z & 0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & Z & 0 & I \\
0 & 0 & 0 & \cdots & Z & Z^{h+1}
\end{pmatrix}, \quad (3.63)
$$

where $I$ is the $n \times n$ identity matrix, and $Z$ is the $n \times n$ cyclic matrix in (3.6). In this section, we show that the matrix $N_V$ has right eigenvectors whose components involve the modified Chebyshev polynomials $L_k(t)$ of Section 3.8, and left eigenvectors whose components involve the Chebyshev polynomials $V_k(t)$ of the third kind [28, Sec. 1.2.3], which are defined by

$$V_0(t) = 1, \quad V_1(t) = t - 1, \quad V_k(t) = tV_{k-1}(t) + V_{k-2}(t), \quad k \geq 2. \quad (3.64)$$

They are expressible in terms of other Chebyshev polynomials via the relations

$$V_k(t) = U_k(t) - U_{k-1}(t), \quad \text{and} \quad L_k(t) = V_k(t) + V_{k-1}(t) \text{ for all } k \geq 1. \quad (3.65)$$

The first identity can be found in [28, 1.17], and the second comes from $L_k(t) = U_k(t) - U_{k-2}(t)$ and the first. More specifically, we show the following for all $n = 2h + 1 \geq 3$:

**Theorem 3.32** (a) The matrix $N_V$ in (3.63) has eigenvalues $\lambda_{j,r} = q^j(q^r + q^{-r})$ for $0 \leq j \leq \frac{n-1}{2}, r \in \mathbb{Z}_n$, (each with multiplicity one), so the matrix $N_V$ is diagonalizable (as expected from [9]).

(b) Let $v \neq 0$ satisfy $Zv = q^{2r}v$, and assume $L_k$ stands for $L_k(q^j + q^{-j})$. Then

$$[v \quad q^r L_1 v \quad \ldots \quad q^{hr} L_k v]$$

is a right eigenvector for $N_V$ of eigenvalue $\lambda_{j,r}$ for $0 \leq j \leq \frac{n-1}{2}, r \in \mathbb{Z}_n$.

(c) Let $w \neq 0$ satisfy $Zw = q^{2r}w$, and assume $V_k$ stands for $V_k(q^j + q^{-j})$. Then

$$[q^{hr} V_h w \quad \ldots \quad q^j V_1 w \quad w]$$
is a left eigenvector for $N_V$ with eigenvalue $\lambda_{j,r}$ for $0 \leq j \leq \frac{n-1}{2}, r \in \mathbb{Z}_n$.

**Proof**  We will argue that (b) and (c) hold, and part (a) will follow.

(b) We compare both sides of this equation and verify they are indeed equal:

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
2Z & 0 & 1 & \ldots & 0 & 0 \\
0 & Z & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & Z & 0 \\
0 & 0 & 0 & \ldots & 0 & Z^{h+1}
\end{pmatrix}
\begin{pmatrix}
v \\
q^r \mathcal{L}_1 v \\
q^{2r} \mathcal{L}_2 v \\
q^{3r} \mathcal{L}_3 v \\
\vdots \\
q^{hr} \mathcal{L}_h v \\
q^{(h+1)r} \mathcal{L}_{h+1} v
\end{pmatrix}
= \lambda_{j,r}
\begin{pmatrix}
v \\
q^r \mathcal{L}_1 v \\
q^{2r} \mathcal{L}_2 v \\
q^{3r} \mathcal{L}_3 v \\
\vdots \\
q^{hr} \mathcal{L}_h v \\
q^{(h+1)r} \mathcal{L}_{h+1} v
\end{pmatrix}.
\]

Row 0 on the left is $q^r \mathcal{L}_1 v = q^r (q^j + q^{-j}) v$, which equals $\lambda_{j,r} v$, the entry in row 0 on the right. Row 1 says $2q^{2r} \mathcal{L}_2 v = q^{2r} (L_0 + L_2) v = q^{2r} (q^j + q^{-j}) \mathcal{L}_1 v = \lambda_{j,r} q^r \mathcal{L}_1 v$. Now for rows $2 \leq s \leq h - 1$, we have

\[
Zq^{(s-2)r} \mathcal{L}_{s-2} v + q^{sr} \mathcal{L}_s v = q^{sr} (L_{s-2} + \mathcal{L}_s) v = q^{sr} (q^j + q^{-j}) \mathcal{L}_{s-1} v = \lambda_{j,r} q^{(s-1)r} \mathcal{L}_{s-1} v.
\]

Finally, for the last row, recall that $h = \frac{n-1}{2}$ so that $h+1 = \frac{n+1}{2}$. Then on the left we have

\[
Zq^{(h-1)r} \mathcal{L}_{h-1} v + Z^{h+1} q^{hr} \mathcal{L}_h v = q^{(h+1)r} (\mathcal{L}_{h-1} + q^{2(h+1)+h} \mathcal{L}_h) v = q^{(h+1)r} (\mathcal{L}_{h-1} + \mathcal{L}_h) v.
\]

On the right, the last entry is

\[
\lambda_{j,r} q^{hr} \mathcal{L}_h v = q^{(h+1)r} (q^j + q^{-j}) \mathcal{L}_h = q^{(h+1)r} (\mathcal{L}_{h+1} + \mathcal{L}_{h-1}) v.
\]

So comparing the left and right sides, we see that the argument boils down to whether $\mathcal{L}_h v = \mathcal{L}_{h+1} v$. But since

\[
\mathcal{L}_h (q^j + q^{-j}) = q^{hj} + q^{-hj} = q^{(h+1)j} + q^{(h+1)j} = \mathcal{L}_{h+1} (q^j + q^{-j}),
\]

the left and right sides are indeed equal, so (b) holds.

(c) The connection with the Chebyshev polynomials $U_k(t)$ in (3.65) is the one that will be most helpful in proving part (c). Recall we know by Proposition 3.2(a) that for $t = x + x^{-1}$ and all $k \geq 1$,

\[
U_k(t) = x^k + x^{k-2} + \ldots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}}.
\]

Therefore,

\[
V_k(t) = U_k(t) - U_{k-1}(t) = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}} - \frac{x^k - x^{-k}}{x - x^{-1}}
\]
3.15 Further Questions

In this paper, we have proven results on McKay matrices of arbitrary finite-dimensional Hopf algebras and have illustrated them for the Drinfeld double $D_{n}$ of the Taft algebra, but there remain many interesting open questions, even for semisimple Hopf algebras.

(1) When is the McKay matrix symmetric or normal, hence orthogonally diagonalizable?

It is shown in [39] that the McKay matrix corresponding to any simple module is orthogonally diagonalizable when $A$ is semisimple and almost cocommutative, and...
we have shown in Corollary 2.12 that if $A$ is semisimple and $V$ is self-dual, then $M_V$ is symmetric. The McKay matrix $M_V$, $V = V(2, 0)$, for the nonsemisimple Drinfeld double $D_n$ is not symmetric, and for the algebra $A$ that is (14) in Kashina’s classification [24] of 16-dimensional semisimple Hopf algebras there is a module $V \not\cong V^*$ such that $M_V$ is not symmetric.

(2) For which Hopf algebras do the (generalized) right eigenvectors of McKay matrices correspond to columns in something that can be regarded as a character table?

(3) When can all the (right) eigenvectors of the McKay matrix $M_V$ be obtained from traces of grouplike elements? We have shown this is possible for $D_n$ in Sec. 3.6. It is possible for Radford’s Hopf algebra $A(n, m)$ which is also not semisimple [33, Exer. 10.5.9], but fails to be true for the Kac-Palyutkin algebra which is semisimple [23]. We have seen in Sec. 3.9 that for $D_n$, only $n$ of the $\frac{n(n+1)}{2}$ linearly independent left eigenvectors can be realized as trace vectors of grouplike elements on projective covers.

(4) Under what assumptions can the (generalized) eigenvectors of McKay matrices be related to central elements of the Hopf algebra $A$ or cocommutative elements in $A^*$?

(5) When are the eigenvalues of the fusion matrix $N_V$ obtained by tensoring a maximal independent set of indecomposable projective modules with $V$ the same as the eigenvalues for the McKay matrix $M_V$? They are for Drinfeld double $D_n$ and $V = V(2, 0)$.

(6) What can be said about the (generalized) eigenvectors of matrices that encode the fusion relations in the more general context of tensor or fusion categories (see e.g. [19], [34])? In [17], Etingof, Nikshych, and Ostrik introduced the Frobenius-Perron dimension of a fusion category as the spectral radius of a matrix representing the fusion relations.

(7) When $n$ is even, do the (generalized) eigenvectors of the McKay matrices for tensoring $D_n$-modules have expressions in terms of Chebyshev polynomials, and what can be said about the multiplicities of the eigenvalues?

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