ON DIFFEOMORPHISMS OF EVEN-DIMENSIONAL DISCS

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Abstract. We determine \( \pi_*(\text{BDiff}_0(D^{2n})) \otimes \mathbb{Q} \) for \( 2n \geq 6 \) completely in degrees \( n \leq 4n - 10 \), far beyond the pseudoisotopy stable range. Furthermore, above these degrees we discover a systematic structure in these homotopy groups: we determine them outside of certain “bands” of degrees.

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1. Introduction

1.1. Context. The topological group Diff_\alpha(D^d) of diffeomorphisms of a disc D^d fixing pointwise a neighbourhood of the boundary is of fundamental interest in geometric topology. As the corresponding group Homeo_\alpha(D^d) of homeomorphisms is contractible, by the Alexander trick, Diff_\alpha(D^d) measures the difference between topological and smooth manifolds.

Before stating our results let us explain this more precisely. For a smooth manifold M of dimension d \neq 4, perhaps with boundary, smoothing theory gives a map

\[
\frac{\text{Homeo}_\alpha(M)}{\text{Diff}_\alpha(M)} \to \Gamma_\alpha \left( \text{Fr}(M) \times_{O(d)} \frac{\text{Top}(d)}{\text{O}(d)} \to M \right)
\]

to the space of sections of the \frac{\text{Top}(d)}{\text{O}(d)}-bundle over M associated to its tangent bundle, trivial over the boundary \partial M, and says that this map is an equivalence onto those path-components which it hits [KS77, Essay IV & V]. For M = D^d the bundle is trivial, and, along with the Alexander trick, this gives an identification [KS77, Theorem V.3.4]

\[
\text{BDiff}_\alpha(D^d) \simeq \Omega^d_\alpha \left( \frac{\text{Top}(d)}{\text{O}(d)} \right).
\]

Thus BDiff_\alpha(D^d) tells us about the smoothing theory of all d-manifolds, and as O(d) is well-understood it also tells us about the group Top(d) = Homeo(\mathbb{R}^d).

In this paper we provide detailed information about \pi_\alpha(BDiff_\alpha(D^{2n})) \otimes \mathbb{Q} for 2n \geq 6; from now on, we will shorten the notation for rational homotopy groups to \pi_\alpha(-)_{\mathbb{Q}}. These were studied by Farrell and Hsiang [FH78] by combining surgery theory and pseudoisotopy theory, giving

\[
\pi_\alpha(\text{BDiff}_\alpha(D^{2n}))_{\mathbb{Q}} = 0 \text{ for } 1 \leq d \leq \min\{\frac{2n-5}{2}, \frac{2n-1}{3}\} \text{ and } 2n \geq 6.
\]

The range comes from the pseudoisotopy stable range, and the formula we gave is the estimate for this range due to Igusa [Igu88, p. 7]. More recently, the second-named author has shown [RW17, Theorem 4.1] that the same result holds in degrees 1 \leq d \leq 2n - 5; here, rather than stabilisation of pseudoisotopies, Morlet’s Lemma of Disjunction [BLR75, Corollary 3.2] is combined with calculations of Berghlund and Madsen [BM20] and of Galatius and the second-named author [GRW14, GRW18].

Recently Weiss [Wei15] has made a remarkable discovery concerning unstable topological Pontrjagin classes. The map BO \to BTop induces an isomorphism on rational cohomology, so there are classes \pi_\alpha(H^q(BO; \mathbb{Q})) which pull back to the classical Pontrjagin classes on BO. By definition, \pi_{n+k} \in H^{4n+4k}(BO(2n); \mathbb{Q}) vanishes for all k > 0, and \pi_\alpha may be written as the square of the (twisted) Euler class \epsilon \in H^2(BO(2n); \mathbb{Q}[\pi_1]). The classes \pi_1 and \epsilon are defined on BTop(2n), respectively by pullback from BTop or as the self-intersection of the zero-section, but it does not follow that the classes \pi_n - \epsilon^2 and \pi_{n+k} for k > 0 vanish here. Weiss’ discovery is that for many n and k they do not vanish, and that furthermore these non-trivial classes transgress to non-trivial cohomology classes on \frac{\text{Top}(2n)}{\text{O}(2n)} which are nontrivial on the image of the Hurwicz map [Wei15, Section 6]. Through (2) they yield non-trivial classes in the rational homotopy groups of BDiff_\alpha(D^{2n}). We will say that such homotopy classes are detected by Pontrjagin classes. Weiss shows that \pi_{2n-1+4k}(\text{BDiff}_\alpha(D^{2n})) contains classes detected by Pontrjagin classes for all large enough n, and for all 0 \leq k \leq \frac{2n}{4} - \text{constant}.

1.2. Statement of results. Our first result is that Pontrjagin classes completely detect the rational homotopy of BDiff_\alpha(D^{2n}) in degrees d \leq 4n - 10.

Theorem A. Let 2n \geq 6. Then in degrees d \leq 4n - 10 we have

\[
\pi_\alpha(\text{BDiff}_\alpha(D^{2n}))_{\mathbb{Q}} = \begin{cases} 
\mathbb{Q} & \text{if } d \geq 2n - 1 \text{ and } d \equiv 2n - 1 \text{ mod } 4, \\
0 & \text{otherwise},
\end{cases}
\]

and the \mathbb{Q}'s are detected by Pontrjagin classes.
Combined with Theorem A of [Kup19], this implies that \( \pi_d(B\text{Diff}_\theta(D^{2n})) \) is finite for \( d < 2n - 1 \) and thus gives an affirmative answer to Question 1(c) of Burghelea in [Kui71] in even dimensions. Theorem A represents the contiguous range of degrees in which we obtain complete information, but we also obtain complete information outside certain “bands” of degrees beyond this. The meaning of this may be most easily appreciated by looking at Figure 1; the formal statement is as follows.

**Theorem B.** Let \( 2n \geq 6 \). Then in degrees \( d \geq 4n - 9 \) we have

\[
\pi_d(B\text{Diff}_\theta(D^{2n}))_\mathbb{Q} = \begin{cases}
\mathbb{Q} & \text{if } d \equiv 2n-1 \mod 4 \text{ and } d \notin \bigcup_{r \geq 2} [2r(n-2) - 1, 2rn - 1], \\
0 & \text{if } d \not\equiv 2n-1 \mod 4 \text{ and } d \notin \bigcup_{r \geq 2} [2r(n-2) - 1, 2rn - 1], \\
? & \text{otherwise},
\end{cases}
\]

and the \( ? \)'s are detected by Pontrjagin classes.

There is a fibration sequence

\[
\Omega^{2n+1}(\text{Top}(\text{Top}(2n))) \longrightarrow \Omega^{2n}(\Omega(\text{Top}(2n))) \longrightarrow \Omega^{2n}(\text{Top}(\text{Top}(2n))),
\]

where the basepoint component of the middle term is identified with \( B\text{Diff}_\theta(D^{2n}) \) by Morlet’s theorem, and where the right-hand term has rational homotopy groups \( \mathbb{Q} \) in degrees \( d \geq 2n - 1 \) such that \( d \equiv 2n - 1 \mod 4 \), and these are (tautologically) detected by Pontrjagin classes. A strengthening of Theorems A and B is then as follows.

**Theorem C.** The rational homotopy groups of \( \Omega^{2n+1}(\text{Top}(\text{Top}(2n))) \) are supported in degrees \( * \in \bigcup_{r \geq 2} [2r(n-2) - 1, 2rn - 2] \).

We refer to the range of degrees \( [2r(n-2) - 1, 2rn - 2] \) as the 2\( r \)-th band. Bands may overlap. There are odd bands as well as a second band, but part of Theorem C is that the rational homotopy groups in these vanish.

To explain the consequences of this calculation for \( B\text{Top}(2n) \) it is convenient, and stronger, to discuss the oriented version \( B\text{Top}(2n). \) This space carries an Euler class \( e \in H^{2n}(B\text{Top}(2n); \mathbb{Z}) \), and has a stabilisation map \( s: B\text{Top}(2n) \rightarrow B\text{Top} \). The following proves Conjectures C1 and C2 of [Bur73], rationally and in even dimensions.

**Corollary D.** For \( 2n \geq 6 \), the map

\[
s \times e: B\text{Top}(2n) \longrightarrow B\text{Top} \times K(\mathbb{Z}, 2n)
\]

is rationally \((6n - 8)-\)connected.

**Remark 1.1.** This may be rephrased in terms of smoothing theory, through (1). Suppose that \( M \) is a 2\( n \)-dimensional closed smooth manifold with \( 2n \geq 6 \), then a topological fibre bundle \( \pi: E \rightarrow S^d \) with fibre \( M \) is smoothable, possibly after pulling back along a map \( S^d \rightarrow S^d \) of non-zero degree, under the following conditions:

(i) when \( d < 2n \) always,

(ii) when \( d = 2n \) if and only if \( \int_E (p_n - c^2)(T_\pi) = 0 \),

(iii) when \( 2n < d < 4n - 10 \) if and only if \( \int_E p_{2n+d}/4(TE) = 0 \) (for \( 2n + d \not\equiv 0 \mod 4 \) this conditions is vacuous).

We can exert some control on how the terms in the fibration sequence (3) interact with each other, by considering the involution on

\[
B\text{Diff}_\theta(D^{2n}) \cong \Omega^{2n}(\text{Top}^{(2n)}(\text{Top}(2n)))
\]

given by conjugating by a reflection of \( D^{2n} \). In Section 6.5 we will describe compatible involutions on the other terms in (3), which on \( \pi_*(\Omega^{2n}(\text{Top}^{(2n)}(\text{Top}(2n))))_\mathbb{Q} \) act by \(-1\)'s, and on \( \pi_*(\Omega^{2n+1}(\text{Top}^{(2n)}(\text{Top}(2n))))_\mathbb{Q} \) by \((-1)^r \) in the \( 2r \)-th band (when such bands overlap one should interpret this as being inconclusive). This is reflected in Figure 1 by colouring the bands: the \((+1)\)-eigenspaces are red, the \((-1)\)-eigenspaces are blue, their overlap grey. This allows us to conclude that elements detected by Pontrjagin classes also
Figure 1. The rational homotopy groups $\pi_*(\text{Diff}_\partial(D^{2n}))_\mathbb{Q}$ for $2n \geq 6$. A dot represents a copy of $\mathbb{Q}$, an empty entry 0, and the shaded area remains unknown. The dotted line shows the Igusa stable range. The colours indicate the eigenspaces of the reflection involution: red is $(+1)$, blue is $(-1)$, and grey may contain either.
exist outside of those bands corresponding to odd \( r \). This in particular implies that 
\( e^2 \neq p_n \in H^{4n}(\text{Top}(2n); \mathbb{Q}) \) for all \( n \geq 3 \), showing that \( \text{Top}(2n)_{\mathbb{Q}} \) is not rationally 
(4n - 1)-connected even for \( 2n = 6,8 \), when the statement of Theorem A is not

**Remark 1.2.** Since \( p_i \in H^*(\text{Top}(2n+2); \mathbb{Q}) \) pulls back to \( p_i \in H^*(\text{Top}(2n); \mathbb{Q}) \)
under the stabilisation map \( \text{Top}(2n) \rightarrow \text{Top}(2n+2) \), copies of \( \mathbb{Q} \) corresponding to 
Pontrjagin classes can be propagated forwards. This was used to complete the pattern
of dots in Figure 1.

In the range of degrees \([2r(n-2) - 1, 2r(n-1)]\) inside the \( 2r \)-th band (corresponding
to the darkly shaded regions in Figure 1) for \( n \) large enough in comparison to \( r \) we
will show that the rational homotopy groups of \( \Omega^{2n+1}_{\mathbb{Q}} \text{Top}(2n) \) may be described as the
homology of a certain chain complex. The terms in this chain complex are independent
of \( n \), though we do not know whether the same is true for the differential (however, we
believe it is). For the fourth band, this chain complex has the form

\[
\mathbb{Q}^2 \leftarrow \mathbb{Q}^4 \leftarrow \mathbb{Q}^{10} \leftarrow \mathbb{Q}^{21} \leftarrow \mathbb{Q}^{15} \leftarrow \mathbb{Q}^3,
\]

supported in degrees \([4n - 9, 4n - 4]\). This complex has Euler characteristic 1, so there
is at least one degree \( d \in [4n - 9, 4n - 5] \) such that

\[
\pi_d \left( \Omega^{2n+1}_{\text{Top}(2n)} \right)_{\mathbb{Q}} \neq 0.
\]

In Corollary 7.15, we show that non-zero classes obtained this way map non-trivially
to \( \pi_d(\text{BDiff}_\partial(D^{2n}))_{\mathbb{Q}} \). This is an instance of a non-zero rational homotopy group of
\( \text{BDiff}_\partial(D^{2n}) \) which is not detected by Pontrjagin classes.

**Remark 1.3.** Watanabe [Wat18] has shown that \( \pi_k(\text{BDiff}_\partial(D^4))_{\mathbb{Q}} \) contains a subspace
\( \mathcal{A}_k \) of certain trivalent graphs with \( 2k \) vertices, by constructing for each such graph
a framed 4-disc bundle such that a corresponding configuration space integral is non-zero. Based on this and his work on diffeomorphism groups of odd-dimensional discs
[Wat09a, Wat09b], we expect his results generalise to show that

\[
\mathcal{A}_k \subset \pi_{(2n-3)k}(\text{BDiff}_\partial(D^{2n}))_{\mathbb{Q}}.
\]

As \( \mathcal{A}_1 = 0 \) and \( \mathcal{A}_2 = \mathbb{Q} \), one first expects to see such a contribution in degree \( 4n - 6 \);
this is indeed compatible with our chain complex above.

### 1.3. Outline of the proof

It is more convenient both for formulating and proving results to work with a variant of \( \text{BDiff}_\partial(D^{2n}) \) which includes framings. The disc has a standard framing \( \ell_0 \), inherited from \( \mathbb{R}^{2n} \), and we are interested in those framings which are equal to \( \ell_0 \) near the boundary. By comparison with the standard framing, such a framing is described by a map \( D^{2n} \rightarrow \text{GL}_{2n}(\mathbb{R}) \) whose value near the boundary is
the identity element. A diffeomorphism acts on a framing through its derivative, so we get an action of \( \text{Diff}_\partial(D^{2n}) \) on \( \Omega^{2n}(\text{GL}_{2n}(\mathbb{R})) \). The **moduli space of framed discs** is by
definition the homotopy quotient

\[
\text{BDiff}_\partial^\ell(D^{2n}) := \Omega^{2n}\text{GL}_{2n}(\mathbb{R}) \sslash \text{Diff}_\partial(D^{2n}).
\]

This fits into a fibre sequence

\[
\Omega^{2n}\text{GL}_{2n}(\mathbb{R}) \rightarrow \text{BDiff}_\partial^\ell(D^{2n}) \rightarrow \text{BDiff}_\partial(D^{2n}),
\]

and we let \( \text{BDiff}_\partial^\ell(D^{2n})_{\ell_0} \) denote the path-component containing the standard framing \( \ell_0 \). The analogue of Morlet’s equivalence (2) with framings is an equivalence
\( \text{BDiff}_\partial^\ell(D^{2n})_{\ell_0} \cong \Omega^{2n}_0\text{Top}(2n) \), so one already sees that adding framings gives a more
direct relation between diffeomorphism groups and \( \text{Top}(2n) \). It has two further advantages: the computations become simpler, and the answer becomes cleaner.
To prove Theorem C we must study the homotopy fibre of the map
\[ \text{BDiff}_{fr}^r(D^{2n})_\ell \cong \Omega_0^{2n} \text{Top}(2n) \longrightarrow \Omega_0^{2n} \text{Top} \cong \prod_{d \geq 0} K(\mathbb{Q}, d). \]

To do so we will exploit a framed version of the Torelli space in terms of its rational cohomology; the excluded ranges of this framed Torelli space in terms of its rational cohomology; the excluded ranges of degrees \([2r(n - 2) - 1, 2r(n - 1)]\) in Theorem C are those in which this calculation is not conclusive; they correspond to the darkly shaded regions above it has a complicated fundamental group, surjecting onto an arithmetic group. In particular, this space is not nilpotent, and one cannot deduce much about its rational homotopy groups of the total space and base space of (4). Results of Galatius and the second-named author [GRW14, GRW18] give a complete description of the cohomology of this space in a range of degrees tending to infinity with \(g\): because of the framing, its rational cohomology is trivial in such a range. However, as mentioned above it has a complicated fundamental group, surjecting onto an arithmetic group. In particular this space is not nilpotent, and one cannot deduce much about its rational homotopy groups from information about its rational cohomology. To get around this we analyse the cohomology of the associated Torelli space, the regular covering space of \(\text{BDiff}_{fr}^r(W_{g,1})\) given by the surjection of its fundamental group to this arithmetic group. In preparation for this, in [KRW20b, KRW19] we have studied the analogous question for the unframed version \(\text{BDiff}_{fr}(W_{g,1})\). In Section 4 we explain how to adapt these arguments to the framed case, and estimate the rational homotopy groups of this framed Torelli space in terms of its rational cohomology; the excluded ranges of degrees \([2r(n - 1) + 1, 2rn - 2]\) in Theorem C are those in which this calculation is not conclusive; they correspond to the lightly shaded regions in Figure 1.

Remark 1.4. Through this method we also obtain information about the rational homotopy groups of the diffeomorphisms of \(W_{g,1}\). In particular, we can determine these
2. Background

This section provides an overview of some well-known material used in this paper. It may be skipped at first reading, and referred back to when necessary.

2.1. Gradings. Let $A$ be a $\mathbb{Q}$-linear abelian symmetric monoidal category, such as the category $\mathbb{Q}\text{-mod}$ of $\mathbb{Q}$-vector spaces $V$ or its subcategory $\mathbb{Q}\text{-mod}^f$ of finite-dimensional $\mathbb{Q}$-vector spaces. We will use $\text{Gr}(A)$ to denote bounded below $\mathbb{Z}$-graded objects $V_*$ in such a category $A$, which has a symmetric monoidal structure with tensor product given by the graded tensor product and symmetry given by the Koszul sign rule. When we define algebraic objects such as graded-commutative algebras or graded Lie algebras in $\text{Gr}(A)$, we use this symmetric monoidal structure.

Notation 2.1. For a graded object $V$, we will denote an $n$-fold grading shift by $V[n]_* := V_{*-n}$. That is, $V[n]$ is $V$ “shifted up by $n$.”

Notation 2.2. We will occasionally use cohomological gradings: $V^* := V_{*-\bullet}$.

2.2. Representation theory of symmetric groups. We write $S_k$ for the group of permutations of the finite set $k := \{1, \ldots, k\}$. Thinking of $S_k$ as a category with a single object, a finite-dimensional rational $S_k$-representation is a functor $S_k \to \mathbb{Q}\text{-mod}^f$. The category of such representations is denoted $\text{Rep}(S_k)$. It is semi-simple because $S_k$ is finite, and the irreducible representations are given by the Specht modules $S^\lambda$. There is one such irreducible representation for each partition $\lambda$ of $k$, and they are all distinct. We will often denote $S^\lambda$ by $(\lambda)$. For example, $S^k = (k)$ is the trivial representation and $S^{k^*} = (1^k)$ is the sign representation.

Many $S_k$-representations come in families indexed by a non-negative integer $k$. Let $\text{FB}$ denote the category whose objects are finite sets and whose morphisms are bijections. Then such families of representations can be assembled to functors $\text{FB} \to \mathbb{Q}\text{-mod}^f$, the category of which we shall denote by $\text{Rep}(\text{FB})$. These are the same as symmetric sequences in finite-dimensional vector spaces.

Replacing $\mathbb{Q}\text{-mod}^f$ by $\text{Gr}(\mathbb{Q}\text{-mod}^f)$, we can define graded representations of $S_k$ and the category $\text{FB}$. We often shorten the notation $\text{Gr}(\text{Rep}(-))$ to $\text{GrRep}(-)$.

2.2.1. Characters and symmetric functions. A reference for the facts in this section is [Mac95]. The representation theory of symmetric groups is captured by the ring of symmetric functions, given by the inverse limit

$$\Lambda := \lim_{\Lambda} \mathbb{Z}[x_1, \ldots, x_r]^{S_r}.$$ 

We will place each variable $x_i$ in weight 1, and write $\Lambda_k$ for the piece of weight $k$. Let $e_k$ denote the $k$th elementary symmetric function, $h_k$ the $k$th complete symmetric function, and $p_k$ the $k$th power sum function. Both the $e_k$ or the $h_k$ provide a polynomial generating set of $\Lambda$, and the $p_k$ provide one of $\Lambda \otimes \mathbb{Q}$. The Schur polynomials $s_\lambda$, for $\lambda$ a partition of $k$, provide a $\mathbb{Z}$-module basis for $\Lambda_k$. There is an involution $\omega: \Lambda \to \Lambda$, uniquely determined by $\omega(e_k) = h_k$, which has the property that $\omega(s_\lambda) = s_{\lambda'}$ with $\lambda'$ the transpose of the partition $\lambda$. 

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Let $R(\mathfrak{S}_k)$ denote the Grothendieck group of the category $\text{Rep}(\mathfrak{S}_k)$ of finite-dimensional rational $\mathfrak{S}_k$-representations. Define the Frobenius character by

$$\text{ch}_k: R(\mathfrak{S}_k) \longrightarrow \Lambda_k$$

$$V \mapsto \text{ch}_k(V) := \sum_{|\lambda|=k} \chi_V(\mathcal{O}_\lambda) \frac{p_{\lambda_1} \cdots p_{\lambda_t}}{1^{\lambda_1} 2^{\lambda_2} \cdots t^{\lambda_t} \lambda_t!},$$

where $\chi_V(\mathcal{O}_\lambda)$ is the value that the character of $V$ takes on the conjugacy class $\mathcal{O}_\lambda$ of cycle type $\lambda$. This is an isomorphism of abelian groups. The Specht module $S^\lambda$ is sent to the Schur polynomial $s_\lambda$; for example, $s_k = h_k$ and $s_{1k} = e_k$. Under this isomorphism the involution $\omega: \Lambda_k \rightarrow \Lambda_k$ corresponds to tensoring with the sign representation.

Let $R(\text{FB})$ denote the Grothendieck group of the category $\text{Rep}(\text{FB})$ of functors $V: \text{FB} \rightarrow \mathbb{Q}\text{-mod}^f$ and let $\hat{\Lambda} = \prod_k \Lambda_k$ be the completion of $\Lambda$ with respect to the filtration given by weight. Then the Frobenius character homomorphisms assemble to an isomorphism

$$\text{ch}: R(\text{FB}) \longrightarrow \hat{\Lambda}$$

$$V \mapsto \prod_k \text{ch}_k(V(\mathfrak{k})).$$

Both the source and target come equipped with additional algebraic structure.

The category $\text{Rep}(\text{FB}) = (\mathbb{Q}\text{-mod}^f)^{\text{FB}}$ of symmetric sequences in finite-dimensional $\mathbb{Q}$-vector spaces has a symmetric monoidal structure given by Day convolution, which makes $R(\text{FB})$ into a commutative ring. Recall that for $F,G \in (\mathbb{Q}\text{-mod}^f)^{\text{FB}}$ the Day convolution $F \otimes G$ is given by the left Kan extension of

$$\text{FB} \times \text{FB} \xrightarrow{F \times G} \mathbb{Q}\text{-mod}^f \times \mathbb{Q}\text{-mod}^f \xrightarrow{\ominus} \mathbb{Q}\text{-mod}^f$$

along $\sqcup: \text{FB} \times \text{FB} \rightarrow \text{FB}$. In terms of the skeleton of $\text{FB}$ given by the finite sets $\mathfrak{k}$,

$$(F \otimes G)(\mathfrak{k}) \cong \bigoplus_{a+b=\mathfrak{k}} \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_\mathfrak{k}} F(a) \otimes G(b).$$

The composition product $\circ$ of symmetric sequences provides an additional, non-commutative, binary operation on $R(\text{FB})$. Under ch, Day convolution corresponds to product of symmetric functions and composition product corresponds to plethysm of symmetric functions.

In particular, the operation $\lambda^k(-) := (1^k) \circ -$ is the $k$th exterior power with respect to Day convolution, and these operations make $R(\text{FB})$ into a $\lambda$-ring [Yau10]. Under ch these operations are sent to $e_k \circ -$ and endow $\hat{\Lambda}$ with a $\lambda$-ring structure, making the Frobenius character into an isomorphism of $\lambda$-rings. Since the $e_k$ are polynomial generators of $\hat{\Lambda}$, we can recover the plethysm operation from the $\lambda$-ring structure.

To add a homological grading, we let $gR(\text{FB})$ denote the Grothendieck group of functors $V_*: \text{FB} \rightarrow \text{Gr}(\mathbb{Q}\text{-mod}^f)$. The Frobenius character gives an isomorphism of $\lambda$-rings

$$\text{ch}: gR(\text{FB}) \longrightarrow \hat{\Lambda}[t][t^{-1}]$$

$$V \mapsto \sum_{n=-\infty}^{\infty} \text{ch}(V_n)t^n,$$

if we use Day convolution and exterior powers formed with respect to Day convolution on the left side, and transfer this $\lambda$-ring structure to the right. Some care is required for explicit formulas on the right hand side, to keep track of the grading as well as the Koszul sign rule, and we will not give details here. We shall refer to ch$(V)$, and variations thereof for graded representations of other groups, as the Hilbert–Poincaré series of $V$.

In a different direction, we can use the usual (internal, or Kronecker) tensor product and exterior powers to make $R(\mathfrak{S}_k)$ into a $\lambda$-ring, and transfer this along the isomorphism $\text{ch}_k: R(\mathfrak{S}_k) \rightarrow \Lambda_k$ to a $\lambda$-ring structure on $\Lambda_k$ (not related to the $\lambda$-ring structure
on \( \hat{\Lambda} \). There is a similar isomorphism \( gR(\mathfrak{S}_k) \to \Lambda_k[[t]][t^{-1}] \) of \( \lambda \)-rings for graded \( \mathfrak{S}_k \)-representations.

2.2.2. Pairs of commuting group actions. Writing \( R(\mathfrak{S}_s \times \mathfrak{S}_k) \) for the Grothendieck group of finite-dimensional rational \( \mathfrak{S}_s \times \mathfrak{S}_k \)-representations, external product gives an isomorphism \( - \otimes - : R(\mathfrak{S}_s) \otimes R(\mathfrak{S}_k) \to R(\mathfrak{S}_s \times \mathfrak{S}_k) \). Thus there is a Frobenius character map

\[
\text{ch}_{s,k} : R(\mathfrak{S}_s \times \mathfrak{S}_k) \to \Lambda_s \otimes \Lambda_k,
\]

which is an isomorphism. The category of finite-dimensional rational \( \mathfrak{S}_s \times \mathfrak{S}_k \)-representations is also semi-simple, and the irreducibles are given by \( S^\lambda \boxtimes S^\mu \) for \( |\lambda| = s \) and \( |\mu| = k \); the Frobenius character sends these to \( s_\lambda \otimes s_\mu \).

The usual (internal, or Kronecker) tensor product and exterior powers make \( R(\mathfrak{S}_s \times \mathfrak{S}_k) \) into a \( \lambda \)-ring. Under \( \text{ch}_{s,k} \) the tensor product goes to the usual multiplication on the tensor product \( \Lambda_s \otimes \Lambda_k \) and the \( \lambda \)-ring structure must satisfy

\[
\lambda^n(a \otimes b) = \lambda^n((a \otimes 1)(1 \otimes b)), \quad \lambda^n(a \otimes 1) = \lambda^n(a) \otimes 1, \quad \lambda^n(1 \otimes b) = 1 \otimes \lambda^n(b).
\]

This uniquely determines a \( \lambda \)-ring structure on \( \Lambda_s \otimes \Lambda_k \), and we shall refer to this construction as the tensor product of \( \lambda \)-rings. With this \( \lambda \)-ring structure on the right side, \( \text{ch}_{s,k} \) is an isomorphism of \( \lambda \)-rings.

Similarly, for \( R(FB \times FB) \), the Grothendieck group of the category \( \text{Rep}(FB \times FB) \) of functors \( V : FB \times FB \to \mathbb{Q}\text{-mod}^I \), the character gives an isomorphism

\[
\text{ch} : R(FB \times FB) \to \hat{\Lambda} \otimes \hat{\Lambda}
\]

\[
V \mapsto \prod_{s,k \geq 0} \text{ch}_{s,k}(V(s,k))
\]

of \( \lambda \)-rings. The \( \lambda \)-ring structure on the left side is given by Day convolution and exterior powers formed with respect to Day convolution, and that on the right side given by the completed tensor product of \( \lambda \)-rings. There is a similar isomorphism for functors valued in graded vector spaces.

A final variant concerns \( R(FB \times F_k) \), the Grothendieck group of the category \( \text{Rep}(FB \times F_k) \) of functors \( V : FB \times F_k \to \mathbb{Q}\text{-mod}^I \). In this case, the character

\[
\text{ch} : R(FB \times F_k) \to \hat{\Lambda} \otimes \Lambda_k
\]

\[
V \mapsto \prod_s \text{ch}_{s,k}(V(s,k))
\]

is an isomorphism of \( \lambda \)-rings. The \( \lambda \)-ring structure on the left term is by interpreting \( FB \times F_k \to \mathbb{Q}\text{-mod}^I \) as a symmetric sequence in \( F_k \)-representations, and using Day convolution and composition product of these. On the right term it is the completed tensor product of \( \lambda \)-rings, using the standard \( \lambda \)-ring structure on \( \hat{\Lambda} \) and the internal one on \( \Lambda_k \). There is a similar isomorphism for functors valued in graded vector spaces.

2.3. Representation theory of arithmetic groups. Let \( \epsilon \in \{-1, 1\} \) and consider \( \mathbb{Z}^{2g} \) equipped with the nonsingular \( \epsilon \)-symmetric pairing \( \lambda : \mathbb{Z}^{2g} \otimes \mathbb{Z}^{2g} \to \mathbb{Z} \) represented by the block-diagonal matrix

\[
\text{diag} \left( \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix} \right)
\]

with respect to the standard basis. We write \( e_1, f_1, e_2, f_2, \ldots, e_g, f_g \) for this ordered basis, called a hyperbolic basis, satisfying

\[
\lambda(e_i, e_j) = 0, \quad \lambda(e_i, f_j) = \delta_{ij}, \quad \lambda(f_i, f_j) = 0.
\]

We write \( H_{\mathbb{Z}}(g) = (\mathbb{Z}^{2g}, \lambda) \) for this bilinear form. The group of automorphisms of \( H_{\mathbb{Z}}(g) \) is denoted by \( O_{g,g}(\mathbb{Z}) \) when \( \epsilon = 1 \) and by \( \text{Sp}_{2g}(\mathbb{Z}) \) when \( \epsilon = -1 \). We will write

\[
H := \mathbb{Q}^{2g} = \mathbb{Z}^{2g} \otimes \mathbb{Q}
\]
when it is considered as a representation of these groups, leaving the \( g \) implicit; it is equipped with the induced \( \epsilon \)-symmetric pairing \( \lambda: H \otimes H \to \mathbb{Q} \), and dually with a form \( \omega \in H^{\otimes 2} \) characterised by \((\lambda \otimes \text{id}) \circ (- \otimes \omega) = \text{id}(-); \) in terms of the hyperbolic basis it is given by

\[
\omega = \sum_{i=1}^{g} f_i \otimes e_i + \epsilon \cdot e_i \otimes f_i.
\]

The groups \( \text{O}_{g,g}(\mathbb{Z}) \) and \( \text{Sp}_{2g}(\mathbb{Z}) \) are the \( \mathbb{Z} \)-points of algebraic groups \( \text{O}_{g,g} \) and \( \text{Sp}_{2g} \) defined over \( \mathbb{Z} \). We also let \( \text{SO}_{g,g} \) denote the identity component of \( \text{O}_{g,g} \). To guarantee that the hypotheses for various results about algebraic groups are satisfied, we adopt the following convention:

**Convention 2.3.** From now on, we will always assume that \( g \geq 2 \).

In this paper, an arithmetic subgroup \( G \) of \( \mathbb{G} \in \{ \text{O}_{g,g}, \text{SO}_{g,g}, \text{Sp}_{2g} \} \) is a finite index subgroup \( G \leq \mathbb{G}(\mathbb{Z}) \) which is Zariski dense in \( \mathbb{G}(\mathbb{Q}) \). This is more restrictive than the usual definition, but will be convenient for us. When \( \mathbb{G} \in \{ \text{SO}_{g,g}, \text{Sp}_{2g} \} \) any finite index subgroup \( G \leq \mathbb{G}(\mathbb{Z}) \) is Zariski dense (recall that we always assume \( g \geq 2 \)); when \( G = \text{O}_{g,g} \) a finite index subgroup \( G \leq \text{O}_{g,g}(\mathbb{Z}) \) if Zariski dense if and only if it is not contained in \( \text{SO}_{g,g}(\mathbb{Z}) \). In particular, in our definition an arithmetic group \( G \) comes with a choice of ambient algebraic group \( \mathbb{G} \).

**Definition 2.4.** A representation \( \phi: G \to \text{GL}(V) \) of an arithmetic group is algebraic if there is a morphism of algebraic groups \( \varphi: \mathbb{G} \to \text{GL}(V) \) which yields \( \phi \) upon taking \( \mathbb{Q} \)-points and restricting to \( G \subset \mathbb{G}(\mathbb{Q}) \). We let \( \text{Rep}(G) \) denote the category of algebraic \( G \)-representations.

When \( \mathbb{G} \in \{ \text{O}_{g,g}, \text{SO}_{g,g}, \text{Sp}_{2g} \} \), a subquotient of an algebraic \( G \)-representation is again algebraic, and every extension of algebraic representations is split [KRW20b, Section 2.1.2]. In particular the category \( \text{Rep}(G) \) of algebraic representations of \( G \) is semi-simple. It is also closed under the formation of tensor products and duals.

By our Zariski density assumption, if \( G' \leq G \) are arithmetic subgroups of the same \( \mathbb{G} \), then an algebraic \( G' \)-representation extends uniquely to an algebraic \( G \)-representation: thus the restriction functor \( \text{Rep}(G) \to \text{Rep}(G') \) is an isomorphism of categories.

**2.3.1. Parity of algebraic representations.** Every algebraic \( G \)-representation extends uniquely to an algebraic \( \mathbb{G}(\mathbb{Z}) \)-representation. The element \(-\text{id} \in \mathbb{G}(\mathbb{Z})\) lies in the centre, so every \( \mathbb{G}(\mathbb{Z}) \)-representation \( V \) canonically decomposes into a \(+1\)-eigenspace \( V_{\text{even}} \) and a \(-1\)-eigenspace \( V_{\text{odd}} \) for the action by this involution; thus every algebraic \( G \)-representation has such a canonical decomposition too (even if \(-\text{id} \) does not lie in \( G \)). We say that an algebraic \( G \)-representation \( V \) is even if \( V = V_{\text{even}} \), and say it is odd if \( V = V_{\text{odd}} \). It is clear from this construction that even and odd representations behave in the expected way under tensor products and linear duals.

**2.3.2. Irreducible algebraic representations and invariants.** For the algebraic groups \( \mathbb{G} \in \{ \text{O}_{g,g}, \text{SO}_{g,g}, \text{Sp}_{2g} \} \) the irreducible algebraic representations of an arithmetic subgroup \( G \) can be classified in terms of the associated complex Lie groups \( \text{O}_{g,g}(\mathbb{C}), \text{SO}_{g,g}(\mathbb{C}) \) and \( \text{Sp}_{2g}(\mathbb{C}) \).

For \( \mathbb{G} \in \{ \text{O}_{g,g}, \text{Sp}_{2g} \} \) we have described this in [KRW20b, Theorem 2.5]: for each partition \( \lambda \) of an integer \( k \geq 0 \) there is a representation \( V_\lambda \) constructed as \( \left[ S^k \otimes H^{[k]} \right]_{\lambda k} \), where \( S^k \) is the Specht module associated to \( \lambda \) and \( H^{[k]} = \ker(\bigoplus H^{\otimes k} \to \bigoplus_{i,j} H^{\otimes k-2}) \) is the common kernel of all contractions using the pairing \( \lambda \). The representation \( V_\lambda \) is always either zero or irreducible: if \( |\lambda| \leq g \) it is irreducible, and all such irreducibles are distinct. Every algebraic representation is a direct sum of \( V_\lambda \)'s.

For \( \mathbb{G} = \text{SO}_{g,g} \) the description of the irreducible representations of \( \text{SO}_{g,g}(\mathbb{C}) \) is more complicated. However, for us only \( G \)-representations which are sums of restrictions of the \( V_\lambda \) from \( \text{O}_{g,g}(\mathbb{Z}) \) will arise, and it will not be important for us to understand whether these remain irreducible or split: when they do split, their summands will
never appear separately. In fact for many purposes we will work “for all large enough $g$”: one may extract from Section 11.6.6 of [Pro07] that the $V_\lambda$ with strictly fewer than $g$ rows are still distinct irreducible $\text{SO}_{g,\mathbb{R}}(\mathbb{Z})$-representations, which will suffice.

As a consequence of this discussion, if $\lambda$ is a nontrivial partition then for all large enough $g$ we have $V_\lambda^G = 0$ for any arithmetic subgroup $G$ of $G \in \{\text{O}_{g,\mathbb{R}}, \text{SO}_{g,\mathbb{R}}, \text{Sp}_{2g}\}$. Also, by Zariski density, on invariants we have

\[ J_{\text{odd}} = V_\lambda^G = 0, \quad \text{and} \quad V_\lambda^G = V_{\text{even}}^G. \]

2.3.3. Schur functors. For each partition $\lambda$ of an integer $k \geq 0$ we have a Schur functor

\[ S_\lambda : \mathbb{Q}\text{-mod} \rightarrow \mathbb{Q}\text{-mod} \]

\[ V \mapsto [S^\lambda \otimes V^{\otimes k}]^G. \]

(More generally, this formula defines Schur functors for graded vector spaces.) By functoriality, $\text{GL}(V)$ act on $S_\lambda(V)$. This yields an algebraic representation of the algebraic group $\text{GL}(V)$. In particular, when $V$ is equal to $H$, this algebraic group is defined over $\mathbb{Z}$ and taking $\mathbb{Z}$-points gives a representation of $\text{GL}_{2g}(\mathbb{Z})$.

For $V = H$ and $G \in \{\text{O}_{g,\mathbb{R}}, \text{SO}_{g,\mathbb{R}}, \text{Sp}_{2g}\}$, there is a canonical morphism $G \rightarrow \text{GL}_{2g}$ of algebraic groups, and thus $S_\lambda(H)$ is an algebraic representation of any arithmetic group $G$ in $G$. It is usually not irreducible as a $G$-representation, and decomposes into a direct sum of $V_\mu$'s. For this branching rule see [KT87, Proposition 2.5.1].

2.3.4. $gr$-algebraic representations. Many of the representations we study are not a priori representations of an arithmetic group $G$ as above, but rather of a larger group $\Gamma$ which fits in an extension

\[ 1 \rightarrow J \rightarrow \Gamma \rightarrow G \rightarrow 1. \]

Definition 2.5. A $\Gamma$-representation is said to be $gr$-algebraic if it admits a finite filtration such that the induced action of $\Gamma$ on the associated graded factors over $G$, and is algebraic as a $G$-representation. We let $\text{Rep}^{gr}(\Gamma)$ denote the category of $gr$-algebraic $\Gamma$-representations.

The class of $gr$-algebraic $G$-representations is closed under subquotients and extensions, as well as tensor products and linear duals [KRW19, Lemma 2.5]. Under certain conditions, $gr$-algebraic representations are automatically algebraic:

Lemma 2.6. If $V$ is a $gr$-algebraic representation of $\Gamma$ on which the subgroup $J$ acts via automorphisms of finite order, then the action of $\Gamma$ on $V$ factors over $G$ and as a $G$-representation $V$ is algebraic.

Proof. Since $V$ is $gr$-algebraic, elements of $J$ must act unipotently. But a unipotent endomorphism of finite order is trivial, so $J$ acts trivially and hence the $\Gamma$-action on $V$ factors through a $G$-action. As algebraic $G$-representations are semi-simple (by our standing assumption that $g \geq 2$), a $gr$-algebraic representation which factors over $G$ is algebraic.

Informally, we describe the conclusion of Lemma 2.6 as $V$ “descends to an algebraic $G$-representation.”

2.3.5. Commuting symmetric group actions. We will also encounter $\mathbb{Q}$-vector spaces with commuting actions of an arithmetic group $G$ as above, and a symmetric group.

Definition 2.7. Let $\text{Rep}(G \times \mathfrak{S}_k)$ denote the category $G \times \mathfrak{S}_k$-representations which are algebraic as a $G$-representation. We will refer to these as algebraic $G \times \mathfrak{S}_k$-representations.

The properties of this category can be understood as a consequence of the previous results. We can write $W \in \text{Rep}(G \times \mathfrak{S}_k)$ canonically as a direct sum of isotypical $\mathfrak{S}_k$-representations $W = \bigoplus W_\lambda$ with $\lambda$ ranging over the partitions of $k$: $W_\lambda$ is the (internal) sum of all $\mathfrak{S}_k$-subrepresentations of $W$ isomorphic to $S^\lambda$. Each $W_\lambda$ is still a
G-representation because the $\mathfrak{G}_k$- and $G$-actions commute. Since subrepresentations of algebraic $G$-representations are again algebraic, $W_\lambda$ is an algebraic $G$-representation. We can now invoke the properties of algebraic $G$-representations to deduce:

**Lemma 2.8.** Suppose that $G$ is an arithmetic group with ambient algebraic group $G \in \{ \text{O}_{g,g} \text{, SO}_{g,g} \text{, Sp}_{2g} \}$ and $g \geq 2$, then $\text{Rep}(G \times \mathfrak{G}_k)$ is closed under subquotients, extensions, tensor products, and linear duals.

Furthermore, we can classify the irreducibles of $\text{Rep}(G \times \mathfrak{G}_k)$: there is a canonical isomorphism

$$\text{Hom}_{\mathfrak{G}_k}(S^\lambda, W) \otimes S^\lambda \rightarrow W_\lambda$$

of $G \times \mathfrak{G}_k$-representations, given by evaluation. By the above considerations, the $G$-representation $\text{Hom}_{\mathfrak{G}_k}(S^\lambda, W)$ is algebraic, so decomposes into a finite direct sum of irreducibles. When $G \in \{ \text{O}_{g,g} \text{, Sp}_{2g} \}$, we conclude $W$ is a direct sum of $G \times \mathfrak{G}_k$-representations of the form $V_\mu \otimes S^\lambda$. These are either irreducible or zero, and irreducible if and only if $V_\mu$ is. When $G = \text{SO}_{g,g}$ the situation is more complicated, but as we explained before, for us only $G$-representations that are direct sums of the restrictions of $V_\lambda$ will arise; these also have such a direct sum decomposition.

2.3.6. **Characters and symmetric functions.** A reference for the facts in this section is [KT87], see also [KRW20b, Section 6.1.3]. Let $G$ be an arithmetic group with ambient algebraic group $G \in \{ \text{O}_{g,g} \text{, SO}_{g,g} \text{, Sp}_{2g} \}$, and $R(G)$ denote the Grothendieck group of algebraic $G$-representations, which has the structure of a $\lambda$-ring using exterior powers.

There is another basis of $\lambda$ of orthogonal, resp. symplectic, Schur polynomials

$$s_{(\lambda)} := \begin{cases} o_\lambda & \text{if } G = \text{O}_{g,g}, \\ sp_\lambda & \text{if } G = \text{Sp}_{2g}. \end{cases}$$

As these are an alternative basis of $\lambda$ there is an automorphism $D: \lambda \rightarrow \lambda$, uniquely determined by $D(s_{(\lambda)}) = s_{(\lambda')}$. The involution satisfies $\omega(sp_\lambda) = o_{\lambda'}$ with $\lambda'$ the transposition of $\lambda$, [KT87, Theorem 2.3.2].

The $s_{(\lambda)}$ are defined so that (for all $g$) the map of $\lambda$-rings

$$\lambda \longrightarrow R(G),$$

uniquely determined by $s_1 \mapsto H$, sends $s_{(\lambda)}$ to the representation $V_\lambda$ (and so is surjective). This map is isomorphism “for large enough $g$”: the components $\lambda_k$ for $k < g$ are mapped isomorphically onto the subgroup spanned by $V_\lambda$ with $|\lambda| < g$.

As for symmetric groups, we can add a homological grading. The result is a surjection of $\lambda$-rings

$$\Lambda[[t]][t^{-1}] \rightarrow gR(G),$$

with $gR(G)$ the Grothendieck group of graded algebraic $G$-representations. In a stable range this has an inverse, which is the Hilbert–Poincaré series of a graded algebraic $G$-representation.

2.4. **The Bousfield–Kan spectral sequence.** We will use spectral sequences which arise from a tower of fibrations of pointed spaces, as discussed in [BK72, Ch. XI. 4.1] and summarised in [KRW19, Section 5.1].

Given a tower of based spaces

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0,$$

we denote by $F_n := \text{hofib}[X_n \rightarrow X_{n-1}]$ the homotopy fibres between adjacent stages, with the convention $X_{-1} = \ast$. The long exact sequences of homotopy groups

$$\pi_2(X_{n-1}) \rightarrow \pi_1(F_n) \rightarrow \pi_1(X_n) \rightarrow \pi_1(X_{n-1}) \xrightarrow{d_2} \pi_0(F_n) \rightarrow \pi_0(X_n) \rightarrow \pi_0(X_{n-1})$$

are extended exact sequences: their rightmost three terms are pointed sets, next three terms groups, and remaining terms abelian groups, and they are exact in an appropriate
sense. These can be assembled to extended exact couple (in the sense of [BK72, §IX.4.1])

\[ D^1 \xrightarrow{i} D^1 \xleftarrow{k} E^1, \]

and its derived couple is again an extended exact couple [BK72, p. 259]. The result is an extended spectral sequence

\[ E^r_{p,q} = \pi_{q-p}(F_p) \Rightarrow \pi_{q-p}(\text{holim}_p X_p), \]

with \( E^r_{p,q} \) is concentrated in the range \( p \geq 0, q - p \geq 0 \). For \( q - p = 0 \) its entries consist of pointed sets, for \( q - p = 1 \) of groups, and \( q - p \geq 2 \) of abelian groups. In the cases which appear in this paper it converges completely in the sense of [BK72, Ch. IX. 5.3], which means that \( \pi_{q-p}(\text{holim}_p X_p) \) is the limit of a tower of epimorphisms with kernels given by entries on the \( E^\infty \)-page.

This spectral sequence is natural in maps of towers of based spaces. It is also natural in the following further sense. The long exact sequence of homotopy groups for \( F_n \to X_n \to X_{n-1} \) comes with a natural \( \pi_1(X_n) \)-action [Spa95, p. 385]. Thus, given a based map from \( Y \) to the tower, we get an action \( \pi_1(Y) \) on the extended couple. The construction of an extended spectral sequence from an extended exact couple is natural in the latter, so we obtain an action of \( \pi_1(Y) \) on the extended spectral sequence. When the spectral sequence converges completely, this action converges to the action of \( \pi_1(Y) \) on \( \pi_* (\text{holim}_p X_p) \). In particular, we can take \( Y = \text{holim}_p X_p \).

2.5. Total homotopy fibres. Let us recall the notion of a total homotopy fibre of a cubical diagram of based spaces [MV15]. A \( k \)-cube is a functor \( X \) from the poset \( P(S) \) of subsets of a finite set \( S \) of cardinality \( k \) to pointed spaces, e.g. for \( S = 2 \) it is a square

\[ X_0 \xrightarrow{\iota} X_1 \xrightarrow{\jmath} X_2 \xrightarrow{\kappa} X_3. \]

We may restrict this to the punctured \( k \)-cube \( P_0(S) \) given by removing the object \( \emptyset \). There is then a natural based map \( X_0 \to \text{holim}_{I \in P_0(S)} X_I \), and its homotopy fibre is the total homotopy fibre

\[ \text{tohofib}_S X := \text{hofib} \left( X_0 \to \text{holim}_{I \in P_0(S)} X_I \right). \]

We will need a general result on the homotopy groups of total homotopy fibres of certain split cubical diagrams.

Definition 2.9. A \( k \)-cube of spaces \( X \) is split up to homotopy if for each pair of disjoint subsets \( I, J \subset \mathbb{Z}_k \) with \( J \) nonempty there are given maps \( s_{I,J} : X_{I \cup J} \to X_I \) such that

(i) when \( I' \) and \( J \) are disjoint subsets and \( I \subset I' \) there are homotopies

\[ s_{I,J} \circ X(I \cup J \to I' \cup J) \simeq X(I \to I') \circ s_{I,J} : X_{I \cup J} \to X_{I'}, \]

(ii) the composition \( X(I \to I \cup J) \circ s_{I,J} \) is homotopic to the identity.

Property (i) says that in the homotopy category the maps \( s \) combine to form a map of \( (k - |J|) \)-cubes from \( \mathbb{Z}_k \setminus J \supset I \) to \( \mathbb{Z}_k \setminus J \supset I \), property (ii) says that this map of cubes splits the map in the opposite direction induced by \( X \).

Lemma 2.10. Suppose that a \( k \)-cube \( \mathbb{Z}_k \supset I \mapsto X_I \) of 1-connected spaces splits up to homotopy. Then the natural homomorphism

\[ \pi_* \left( \text{hofib}(X_I) \right) \to \bigcap_{j \in \mathbb{Z}_k} \ker \left( \pi_* (X_0) \to \pi_* (X_j) \right) \]

is an isomorphism.
Proof. Consider the spectral sequence of [MV15, Theorem 9.6.12], which has the form

$$E^1_{p,q} = \prod_{J \subseteq \mathbb{k}, |J| = p} \pi_q(X_J) \implies \pi_{q-p}(\operatorname{toholib} X_I)$$

with the differential

$$d^1: \prod_{J \subseteq \mathbb{k}, |J| = p} \pi_q(X_J) \longrightarrow \prod_{K \subseteq \mathbb{k}, |K| = p+1} \pi_q(X_K)$$

having $K$th component $\sum_{i \in K} (-1)^{|K|} \chi(K \setminus \{i\} \hookrightarrow K)_*$. Thus, the complex $(E^1_{0,q}, d^1)$ is the totalisation of the $k$-cube of abelian groups $I \mapsto \pi_q(X_I)$. The hypothesis that the $k$-cube consists of 1-connected spaces permits us to neglect basepoints and to ignore that in general, extended spectral sequences have entries which are not abelian groups.

The cube of abelian groups $I \mapsto \pi_q(X_I)$ is split (in the evident sense: replace homotopies in Definition 2.9 by identities), and it is a general property about totalisations of split cubes in an abelian category that

$$\ker \left[ d^1: E^1_{0,q} \rightarrow E^1_{1,q} \right] = \bigcap_{j \in \mathbb{k}} \ker \left[ \pi_q(X_j) \rightarrow \pi_q(X_J) \right]$$

and that the higher homology groups of $(E^1_{0,q}, d^1)$ vanish. This seems to be folklore; we could not find a published proof, so provide the following one.

Let $I \mapsto A_I$ be a split $k$-cube in an abelian category, with $C_p(A) = \prod_{J \subseteq \mathbb{k}, |J| = p} A_I$ and differential $d$ given by the formula above. That $H_0(A) = \bigcap_{j \in \mathbb{k}} \ker[A_{<j} \rightarrow A_{\{j\}}]$ is immediate, and it remains to show that the higher homology vanishes. If $k = 1$ then this chain complex is $d: A_g \rightarrow A_{\{1\}}$ and this differential is split surjective, proving the claim in this case. Suppose then that the claim holds for all split cubes of dimension $< k$. Let $S_p = \prod_{k \in \mathbb{k}, |J| = p} A_I$, which assemble to a subcomplex $S_{<k}$ of $C_*(A)$. Then $S_{<k} = C_{<k}(A')$ with $A' = A_{\{k\}}$ a split $(k-1)$-cube, and $C_*(A)/S_{<k} = C_*(A'')$ with $A'' = A_{\{k\}}$ another split $(k-1)$-cube. This short exact sequence of chain complexes gives

$$\cdots \longrightarrow H_{p-1}(A'') \longrightarrow H_{p-1}(A') \longrightarrow H_p(A) \longrightarrow H_p(A'') \longrightarrow H_p(A') \longrightarrow \cdots$$

By inductive assumption $C_p(A')$ and $C_p(A'')$ only have trivial higher homology, so $H_p(A) = 0$ for $p > 1$, and there is an exact sequence

$$0 \longrightarrow H_0(A) \longrightarrow H_0(A'') \longrightarrow H_0(A') \longrightarrow H_1(A) \longrightarrow 0.$$ 

The argument is completed by observing that the middle map is an epimorphism, as it is induced by the map $A'' \rightarrow A'$ of $(k-1)$-cubes, which is split surjective. □

Recall that $P(\mathbb{k})$ denotes the poset of subsets of $\mathbb{k}$. Let $S(\mathbb{k})$ denote the category whose objects are subsets $I \subseteq \mathbb{k}$, and whose morphisms $\text{mor}_{S(\mathbb{k})}(I, J)$ are bijections $\phi: \mathbb{k} \rightarrow \mathbb{k}$ sending $I$ into $J$. There is an inclusion $P(\mathbb{k}) \rightarrow S(\mathbb{k})$, given by those morphisms with $\phi = \text{id}_{\mathbb{k}}$. In fact, $S(\mathbb{k})$ is equivalent to the Grothendieck construction $P(\mathbb{k}) \times \mathcal{G}_k$ for the evident $\mathcal{G}_k$-action on the poset $P(\mathbb{k})$. Thus any functor $\mathcal{X}: S(\mathbb{k}) \rightarrow \textbf{Top}$ determines, by restriction, a $k$-cube.

Lemma 2.11. A functor $\mathcal{X}: S(\mathbb{k}) \rightarrow \textbf{Top}$ determines a $k$-cube with a $\mathcal{G}_k$-action on its total homotopy fibre $\text{toholib} \mathcal{X}_I$ of $\mathcal{X}_I$. If the values of $\mathcal{X}$ are 1-connected spaces with degreewise finite-dimensional rational homotopy groups, and the $k$-cube is split up to homotopy, then we have an equation

$$\pi_*(\text{toholib} \mathcal{X}_I)_Q = \sum_{j=0}^{k} (-1)^j \text{Ind}_{\mathcal{G}_k \times_q \mathcal{G}_k}^{\mathcal{G}_k} \pi_*(\mathcal{X}_{k-j})_Q \otimes (1^j)$$

in the Grothendieck group $g\mathcal{R}(\mathcal{G}_k)$ of graded $\mathcal{G}_k$-representations.
Proof. The functor $\mathcal{X} : S(k) \to \text{Top}$ provides a $\mathfrak{S}_k$-action on the space $\mathcal{X}_\mathfrak{S}$, as well as on the homotopy limit $\text{holim}_{I \in P_0(k)} \mathcal{X}_I$ of the punctured cube. To see the latter action, we observe that $S_0(k)$ is the Grothendieck construction $P_0(k) \bowtie \mathfrak{S}_k$, so the (homotopy) limit of a $S_0(k)$-diagram over $P_0(k)$ has a residual $\mathfrak{S}_k$-action. There is therefore an induced $\mathfrak{S}_k$-action on the total homotopy fibre.

To obtain the equation, by the proof of Lemma 2.10 the group $\pi_q(\text{tohofib}_{I \in k} \mathcal{X}_I)$ is identified with the Euler characteristic of the complex $(E^{1}_{p,q}, d^1)$, and the homology of this complex is supported in degree $* = 0$, so it is enough to endow the terms $E_{p,q}$ with $\mathfrak{S}_k$-actions such that

(i) $E^{1}_{0,q} = \pi_q(X_\mathfrak{S})$ with its induced $\mathfrak{S}_k$-action,

(ii) $d^1 : E^{1}_{p,q} \to E^{1}_{p+1,q}$ is $\mathfrak{S}_k$-equivariant,

and then take its Euler characteristic as $\mathbb{Z}$-modules with $\mathfrak{S}_k$-action. The identification

$$\text{Ind}^{\mathfrak{S}_k} \pi_q(X_\mathfrak{S}) \boxplus (1^{k-p}) = \prod_{J \subset k \mid |J|=p} \pi_q(X_J) = E^{1}_{p,q}$$

has these properties. \qed

3. Framed diffeomorphisms and self-embeddings

In this section we recall some constructions and definitions from our previous paper [KRW19]. We will need the results concerning Torelli spaces with the tangential structure discussed in Section 8 of [KRW19], but only in the special case of framings. Throughout this section we make the standing assumptions that $n \geq 3$ and $g \geq 2$.

3.1. Framings and diffeomorphisms. We will work with the smooth manifold

$$W_{g,1} := D^{2n} \# (S^n \times S^n)^g,$$

with boundary given by $\partial W_{g,1} = S^{2n-1}$.

**Definition 3.1.** Let $\text{Diff}_\partial(W_{g,1})$ denote the topological group of diffeomorphisms of $W_{g,1}$ fixing a neighbourhood of the boundary pointwise, with the $C^\infty$-topology.

The classifying space $B\text{Diff}_\partial(W_{g,1})$ classifies smooth manifold bundles with fibres $W_{g,1}$ and trivialised boundary. We will also use a variant which classifies such bundles equipped with the tangential structure of a framing of the vertical tangent bundle. Instead of using fibrations over $BO(2n)$ to describe tangential structures, we will use the equivalent model in terms of topological spaces with continuous $GL_{2n}(\mathbb{R})$-actions (see [GRW19, Section 4.5] for a comparison).

Let $\Theta_\delta$ denote the $GL_{2n}(\mathbb{R})$-space given by $GL_{2n}(\mathbb{R})$ with action given by right multiplication. Let $\text{Fr}(TW_{g,1})$ denote the frame bundle associated to the tangent bundle $TW_{g,1}$, which is a principal $GL_{2n}(\mathbb{R})$-bundle. A framing on $W_{g,1}$ is a $GL_{2n}(\mathbb{R})$-equivariant map $\ell : \text{Fr}(TW_{g,1}) \to \Theta_\delta$. There is up to homotopy a unique boundary condition $\ell_0 : \text{Fr}(TW_{g,1})|_{\partial W_{g,1}} \to \Theta_\delta$ which extends to a framing of $W_{g,1}$, as a consequence of [KRW19, Lemma 8.5]. We fix one and for all such a boundary condition.

**Definition 3.2.** Let $\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\delta; \ell_0)$ denote the space of $GL_{2n}(\mathbb{R})$-equivariant maps $\ell : \text{Fr}(TW_{g,1}) \to \Theta_\delta$ extending this boundary condition $\ell_0$.

The derivative of a diffeomorphism of $W_{g,1}$ gives a $GL_{2n}(\mathbb{R})$-equivariant map $\text{Fr}(TW_{g,1}) \to \text{Fr}(TW_{g,1})$ which is the identity near $\partial W_{g,1}$. We denote the topological monoid of such maps by $\text{Bun}_\partial(\text{Fr}(TW_{g,1}))$ and let $\text{Bun}_\partial(\text{Fr}(TW_{g,1}))^*$ denote the submonoid of homotopy-invertible path components. The derivative gives a map of topological monoids $B\text{Diff}_\partial(W_{g,1}) \to \text{Bun}_\partial(\text{Fr}(TW_{g,1}))^*$. The group $\text{Diff}_\partial(W_{g,1})$ acts on $\text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\delta; \ell_0)$ via this map and composition.

**Definition 3.3.** The moduli space of framed manifolds isomorphic to $W_{g,1}$ is the homotopy quotient

$$B\text{Diff}_\partial^W(W_{g,1}) := \text{Bun}_\partial(\text{Fr}(TW_{g,1}), \Theta_\delta; \ell_0) \sslash \text{Diff}_\partial(W_{g,1}).$$
We define the framed Torelli space as $G$. We let $B$. This space is nilpotent by [KRW19, Theorem 8.4], and comes equipped with an action $G$. It comes with an outer action of $G$, for which we write $H$. We have shown in [KRW19, Lemma 8.6] that this action has finitely many orbits.

The path components of $B\text{Diff}^\ell_g(W_{g,1})$ are therefore in bijection with the orbits of the mapping class group $\Gamma_g := \pi_0(\text{Diff}_g(W_{g,1}))$ acting on the set $\text{str}^\ell_g(W_{g,1}) := \pi_0(\text{Bun}_\ell(TW_{g,1}, \Theta; \ell))$ of homotopy classes of framings. We have shown in [KRW19, Lemma 8.6] that this action has finitely many orbits. We denote the path component of $B\text{Diff}^\ell_g(W_{g,1})$ containing the framing $\ell$ by $B\text{Diff}^\ell_g(W_{g,1})_\ell$, and write $\tilde{\Gamma}^\ell_g$ for its fundamental group based at $\ell$.

To compute the homotopy groups of such a path component, we will pass to a covering space with significantly simpler fundamental group. Each diffeomorphism of $W_{g,1}$ fixing a neighbourhood of the boundary induces an automorphism of the middle-dimensional homology group $H_n(W_{g,1}; \mathbb{Z})$. Only those automorphisms preserving the $(-1)^n$-symmetric intersection pairing $\lambda: H_n(W_{g,1}; \mathbb{Z}) \otimes H_n(W_{g,1}; \mathbb{Z}) \to \mathbb{Z}$ can possibly be realised by diffeomorphisms. The spheres $S^n \times \{\ast\}$ and $\{\ast\} \times S^n$ inside $W_{g,1}$ give a hyperbolic basis $e_1, f_1, \ldots, e_g, f_g$ of $H_n(W_{g,1}; \mathbb{Z})$ with respect to which $(H_n(W_{g,1}; \mathbb{Z}), \lambda)$ is identified with $H_2(g)$, and hence the action on homology provides a homomorphism $\alpha_g: \text{Diff}_g(W_{g,1}) \to G_g := \begin{cases} O_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even,} \\ \text{Sp}_{2g}(\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}$

We let $G'_g$ denote the image of $\alpha_g$. It follows from the work of Kreck [Kre79] that $G'_g = G_g$ if $n$ is even or $n = 1, 3, 7$, and that for other odd $n$ the group $G'_g$ is the finite index subgroup $\text{Sp}_{2g}^3(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z})$ of automorphisms preserving the standard quadratic refinement.

The $G'_g$-representation $H = \mathbb{Q}^{2g}$ defines a local coefficient system over $B\text{Diff}_g(W_{g,1})$ for which we write $\mathcal{H}$, and we use the same notation for its pullback to any space over $B\text{Diff}_g(W_{g,1})$.

The kernel of $\alpha_g$ is the called the Torelli group of $W_{g,1}$, and is denoted $\text{Tor}_g(W_{g,1})$. It comes with an outer action of $G'_g$, and hence its classifying space $B\text{Tor}_g(W_{g,1})$ comes with an action of $G'_g$ in the homotopy category. This can also be seen from an alternative definition of $B\text{Tor}_g(W_{g,1})$ as $B\text{Tor}_g(W_{g,1}) = \text{hofib} \left[ B\text{Diff}_g(W_{g,1}) \to BG'_g \right]$.

In general, the composition $B\text{Diff}_g(W_{g,1})_\ell \to B\text{Diff}_g(W_{g,1}) \to B G'_g$ is no longer surjective on $\pi_1$. Instead, by [KRW19, Corollary 8.7] its image is a finite index subgroup of $G'_g$ which we denote $G^\ell_g$. By definition this group fits into a short exact sequence

$$1 \to \tilde{\Gamma}^\ell_g := \ker \left[ \tilde{\Gamma}^\ell_g \to G^\ell_g \right] \to \tilde{\Gamma}^\ell_g \to G^\ell_g \to 1.$$ (7)

We define the framed Torelli space as $B\text{Tor}_g^\ell(W_{g,1}) := \text{hofib} \left[ B\text{Diff}_g(W_{g,1})_\ell \to BG^\ell_g \right]$.

This space is nilpotent by [KRW19, Theorem 8.4], and comes equipped with an action of $G^\ell_g$ in the unbased homotopy category.
Remark 3.5. The groups $G^\ell_g$ and sets $\text{Str}^\ell(W_{g,1})/G^\ell_g$ were determined in [KRW20a], as well as the groups $G^\ell_g(W_{g,1})$ and sets $\text{Str}^\ell(W_{g,1})/\Lambda_g$ which appear in the next section. We shall not use these facts.

3.2. framings and self-embeddings. We have $\partial W_{g,1} = S^{2n-1}$; we fix a disc $D^{2n-1} \subset S^{2n-1}$ and denote it by $\partial W_{g,1}$.

Definition 3.6. Let $\text{Emb}_{\ell/\partial}(W_{g,1})$ denote the topological monoid of embeddings $W_{g,1} \hookrightarrow W_{g,1}$ which are the identity on a neighbourhood of $\partial W_{g,1}$.

Diffeomorphisms are examples of such embeddings, and we let $\text{Emb}_{\ell/\partial}^\infty(W_{g,1}) \subset \text{Emb}_{\ell/\partial}(W_{g,1})$ denote the path components consisting of embeddings which are isotopic to diffeomorphisms.

Remark 3.7. The map $\pi_0(\text{Diff}(W_{g,1})) \to \pi_0(\text{Emb}_{\ell/\partial}(W_{g,1}))$ is in fact surjective when $2n \geq 6$, as its cokernel can be identified with the first inertia group $I(W_g) \subset \Theta_{2n}$ of $W_g$. This vanishes when $n \neq 7$ by [Kos67, Corollary 3.2] and when $n \equiv 3, 5, 6, 7$ (mod 8) by the main result of [Wal62]. We shall not use these facts.

From $\ell_0$ we obtain by restriction a boundary condition $\ell_{1/\partial} := \ell_0|_{\partial W_{g,1}}$ near $\partial W_{g,1}$. Define $\text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_{1/\partial})$ as the space of $GL_{2n}(\mathbb{R})$-equivariant maps $\ell: \text{Fr}(TW_{g,1}) \to \Theta_\ell$, extending $\ell_{1/\partial}$. This has an action of the topological monoid $\text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}))$ of those $GL_{2n}(\mathbb{R})$-equivariant maps $\text{Fr}(TW_{g,1}) \to \text{Fr}(TW_{g,1})$ which are the identity near $\partial W_{g,1}$, and hence of its submonoid $\text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}))^\sim$ of homotopy-invertible path components.

The derivative of a self-embedding $W_{g,1} \hookrightarrow W_{g,1}$ gives a $GL_{2n}(\mathbb{R})$-equivariant map $\text{Fr}(TW_{g,1}) \to \text{Fr}(TW_{g,1})$ which is the identity near $\partial W_{g,1}$. This gives a map of topological monoids $\text{Emb}_{\ell/\partial}^\sim(W_{g,1}) \to \text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}))^\sim$ through which $\text{Emb}_{\ell/\partial}^\sim(W_{g,1})$ acts on $\text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_{1/\partial})$. We define a variant of Definition 3.3 by taking the homotopy quotient.

Definition 3.8. We define

$$B\text{Emb}^\ell_{\ell/\partial}(W_{g,1}) := \text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_{1/\partial}) \sslash \text{Emb}_{\ell/\partial}^\sim(W_{g,1}).$$

Remark 3.9. In accordance to Remark 3.4 this is shortened in comparison with [KRW19], where the notation $B\text{Emb}^\ell_{\ell/\partial}(W_{g,1}; \ell_{1/\partial})$ is used.

The path components of this space are in bijection with the orbits of the group

$$\Lambda_g := \pi_0(\text{Emb}_{\ell/\partial}^\sim(W_{g,1}))$$

acting on the set $\text{Str}_g(W_{g,1}) := \pi_0(\text{Bun}_{\ell/\partial}(\text{Fr}(TW_{g,1}), \Theta_\ell; \ell_{1/\partial}))$ of homotopy classes of framings extending the boundary condition $\ell_{1/\partial}$. There is a natural map

$$B\text{Diff}_g(W_{g,1}) \longrightarrow B\text{Emb}^\ell_{\ell/\partial}(W_{g,1}),$$

which is a surjection on path components by the exact sequence described in [KRW19, Section 8.2.2]

$$0 \longrightarrow \text{Str}^\ell_g(D^{2n}) \longrightarrow \text{Str}^\ell_g(W_{g,1}) \longrightarrow \text{Str}^\ell_g(W_{g,1}) \longrightarrow 0.$$
We define the variant of the framed Torelli space for embeddings as follows:

$$B\text{TorEmb}_{\ell}^{fr, \infty}(W_{g,1}) := \text{hofib} \left[ B\text{Emb}_{\ell}^{fr, \infty}(W_{g,1}) \rightarrow BG_{g}^{fr, \infty}[\ell] \right].$$

This is nilpotent by [KRW19, Proposition 8.19], and comes with an action of $G_{g}^{fr, \infty}[\ell]$ in the unbased homotopy category.

This group does not just act by unbased maps on this space but also acts, after passing to the associated graded of a filtration, on its rational homotopy groups. It is a special case of [KRW19, Proposition 8.11] that for $g \geq 2$ and $i \geq 2$ the action of $\tilde{\Lambda}_{g}^{fr, \ell}$ on the rational homotopy group $\pi_{1}(B\text{Emb}_{\ell}^{fr, \infty}(W_{g,1}), \ell)_{Q}$ is a gr-algebraic representation with respect to the short exact sequence of groups

$$1 \rightarrow L_{g}^{fr, \ell} := \ker \left[ \tilde{\Lambda}_{g}^{fr, \ell} \rightarrow G_{g}^{fr, \infty}[\ell] \right] \rightarrow \tilde{\Lambda}_{g}^{fr, \ell} \rightarrow G_{g}^{fr, \infty}[\ell] \rightarrow 1.$$

**Lemma 3.10.** The group $L_{g}^{fr, \ell}$ is finite.

**Proof.** Following Section 8 of [KRW19], let $\Lambda_{g}^{fr, \infty}[\ell]$ denote the stabiliser of $[[\ell]] \in \text{Str}_{\infty}^{fr}(W_{g,1})$ under $\Lambda_{g}$ and $J_{g}^{fr, \infty}[\ell]$ the kernel of the homomorphism $\Lambda_{g}^{fr, \infty}[\ell] \rightarrow G_{g}^{fr, \infty}[\ell]$. From the long exact sequence of homotopy groups for the fibration sequence

$$\text{Bun}_{\ell}^{fr, \infty}(\text{Fr}(TW_{g,1}), \Theta_{fr}; \ell)_{\partial} \longrightarrow B\text{Emb}_{\ell}^{fr, \infty}(W_{g,1})_{\partial} \longrightarrow B\text{Emb}^{\infty}_{\ell}B(W_{g,1}),$$

and the equivalence $\text{Bun}_{\ell}^{fr, \infty}(\text{Fr}(TW_{g,1}), \Theta_{fr}; \ell)_{\partial} \simeq \text{map}_{\ell}^{fr, \infty}(W_{g,1}, SO(2n)),$ we derive a long exact sequence

$$\cdots \longrightarrow \pi_{1}(\text{Emb}^{\infty}_{\ell}(W_{g,1})) \longrightarrow L_{g}^{fr, \ell}[\ell] \longrightarrow J_{g}^{fr, \infty}[\ell] \longrightarrow \cdots$$

The group $J_{g}^{fr, \infty}[\ell]$ is finite by Lemmas 4.4 and 8.12 of [KRW19]. It therefore suffices to prove that the connecting homomorphism $\partial$ is rationally surjective, as its target is finitely generated. To see this, we use that there is a homomorphism

$$\tau: \text{Hom}(H_{n}(W_{g,1}; Z), \pi_{n+1}(SO(n))) \longrightarrow \pi_{1}(\text{Emb}^{\infty}_{\ell}(W_{g,1}))$$

given as follows. Consider $W_{g,1}$ as a disc $D^{2n}$ with $2g$ $n$-handles attached along maps $\phi_{i}: D^{n} \times \partial D^{n} \hookrightarrow \partial D^{2n}$, the cores of these handles representing homology classes $x_{i} \in H_{n}(W_{g,1}; Z)$. For $\varphi \in \text{Hom}(H_{n}(W_{g,1}; Z), \pi_{n+1}(SO(n)))$ the 1-parameter family of diffeomorphisms $\tau(\varphi)$ is given on the $i$th handle by choosing smooth maps $\varphi(x_{i})$: $(I \times D^{n}, \partial(I \times D^{n})) \rightarrow SO(2n)$ representing the homotopy classes $\varphi(x_{i})$ and considering the maps

$$I \times D^{n} \times D^{n} \longrightarrow D^{n} \times D^{n}$$

$$(t, a, b) \mapsto (\varphi(x_{i}))(t, b) \cdot a, b).$$

These 1-parameter families of diffeomorphisms are the identity on $D^{n} \times \partial D^{n}$, so we extend them by the identity over $D^{2n}$ to a 1-parameter family of diffeomorphism of $W_{g,1}$. They do not fix the boundary, but do fix a small ball in the boundary (disjoint from the handles) and hence indeed represent an element of $\pi_{1}(\text{Emb}^{\infty}_{\ell}(W_{g,1})).$

The composition $\partial \circ \tau$ is the map induced by stabilisation $\pi_{n+1}(SO(n)) \rightarrow \pi_{n+1}(SO(2n))$, and this is rationally surjective for all $n \geq 3$.

Combining this with Lemma 2.6 we get:

**Corollary 3.11.** If $V$ is a $\tilde{\Lambda}_{g}^{fr, \ell}$-representation which is gr-algebraic with respect to (8), then it descends to an algebraic $G_{g}^{fr, \infty}[\ell]$-representation.
3.3. The framed Weiss fibre sequence. The comparison of framed diffeomorphisms and framed self-embeddings proceeds through the Weiss fibre sequence [Wei15, Remark 2.1.2], a delooped version of which is given in [Kup19, Section 4]. The following is [KRW19, Proposition 8.8] specialised to framings. As a boundary condition for framings on $D^{2n}$ we use the restriction of the standard framing.

**Theorem 3.12.** There is a fibration sequence

$$B\Diff_{\partial}^{fr}(W_{g,1}) \to B\Emb_{\ell/\partial}^{fr,\infty}(W_{g,1}) \to B(B\Diff_{\partial}^{fr}(D^{2n})), $$

such that the induced map $B\Diff_{\partial}^{fr}(D^{2n}) \to B\Diff_{\partial}^{fr}(W_{g,1})$ is induced by the inclusion of $D^{2n}$ into $W_{g,1}$.

We may restrict to a path-component of a framing $\ell \in \Str_{\partial}^{fr}(W_{g,1})$, resulting in a fibration sequence of path-connected spaces

$$B\Diff_{\partial}^{fr}(W_{g,1})_\ell \to B\Emb_{\ell/\partial}^{fr,\infty}(W_{g,1})_\ell \to B(B\Diff_{\partial}^{fr}(D^{2n})_{B})$$

for a finite subgroup $B \subset \pi_0(B\Diff_{\partial}^{fr}(D^{2n}))$ [KRW19, Section 8.5.3].

If $F \to E \to B$ is a fibration sequence of based spaces, then the long exact sequence of homotopy groups is one of abelian groups (resp. groups or based sets) with $\pi_1(E,e_0)$-action [Spa95, p. 385]. Thus $\Lambda^{fr,\ell}_g$ acts on the rational homotopy groups for $i \geq 2$, all based at $\ell$,

$$\pi_i(B\Diff_{\partial}^{fr}(W_{g,1})_\ell)_{\mathbb{Q}}, \quad \pi_i(\Emb_{\ell/\partial}^{fr,\infty}(W_{g,1})_\ell)_{\mathbb{Q}}, \quad \pi_i(B(B\Diff_{\partial}^{fr}(D^{2n})_{B}))_{\mathbb{Q}}.$$  

**(10)**

**Lemma 3.13.** For $i \geq 2$, the $\Lambda^{fr,\ell}_g$-action on the groups (10) factors over $G^{\fr,[[\ell]]}_g$ and has the following properties:

(i) $\pi_i(B\Diff_{\partial}^{fr}(W_{g,1})_\ell)_{\mathbb{Q}}$ is an algebraic $G^{\fr,[[\ell]]}_g$-representation.

(ii) $\pi_i(\Emb_{\ell/\partial}^{fr,\infty}(W_{g,1})_\ell)_{\mathbb{Q}}$ is an algebraic $G^{\fr,[[\ell]]}_g$-representation.

(iii) $\pi_i(B(B\Diff_{\partial}^{fr}(D^{2n})_{B}))_{\mathbb{Q}}$ is a trivial representation.

**Proof.** By Corollary 3.11 it suffices to prove $gr$-algebraicity with respect to (8). Assertion (i) follows from (ii) and (iii) using the fact that $gr$-algebraic representations are closed under subrepresentations, quotients, and extensions. Assertion (ii) is a special case of [KRW19, Proposition 8.11]. For assertion (iii), we observe that for a fibration sequence $F \to E \to B$ of based spaces, the $\pi_1(E,e_0)$-action on the higher homotopy groups of $B$ is through the homomorphism $\pi_1(E,e_0) \to \pi_1(B,b_0)$. Since $B(B\Diff_{\partial}^{fr}(D^{2n})_{B})$ is a $(2n-1)$-fold loop space, it is simple. $\square$

4. The homotopy groups of framed diffeomorphisms

In this section we compute the rational homotopy groups of $B\Ton_{\partial}^{fr}(W_{g,1})_\ell$ in the range $* < 4n - 3$ for $g$ sufficiently large, and outside this range excluding certain bands. This uses a framed version of the analysis of the maximal algebraic subrepresentations of the rational cohomology groups of Torelli groups given in Theorem 4.1 of [KRW20b]; by Theorem A of [KRW19] in fact these cohomology groups consist entirely of algebraic representations. We also use Section 8 of [KRW19] to deal with path components. Throughout this section we make the standing assumptions that $n \geq 3$ and $g \geq 2$.

4.1. The cohomology of framed Torelli groups. The work of Galatius and the second-named author [GRW14, GRW18] shows that as long as $n \geq 3$ a certain parametrised Pontrjagin–Thom map

$$\alpha: B\Diff_{\partial}(W_{g,1}) \to \Omega^\infty_0 \MT\theta_{2n}$$

induces an isomorphism on cohomology in a range of degrees tending to infinity with $g$, and in fact is even acyclic in such a stable range. Here $\MT\theta_{2n}$ denotes the Thom spectrum of the virtual vector bundle $-\theta_{2n}^*\gamma$ with $\theta_{2n}: BO(2n)/n \to BO(2n)$ the
generalised Miller–Morita–Mumford classes $\pi: E \to B$ with fibres $W_{g,1}$ and trivialised boundary. The characteristic classes $\kappa_c$ admit a geometric description as generalised Miller–Morita–Mumford classes. Given a smooth manifold bundle $\pi: E \to B$ with fibres $W_{g,1}$ and trivialised boundary, the characteristic class $\kappa_c \in H^{|c| - 2n}(B; \mathbb{Q})$ is given by $\int_{\pi} c(T_\pi(E))$, that is, fibre integration of $c$ applied to the vertical tangent bundle of $E$.

The space $\text{BDiff}_g(W_{g,1})$ classifies smooth manifold bundles with fibre $W_{g,1}$ and trivialised boundary equipped with a framing of the vertical tangent bundle which is standard near the boundary. Thus all characteristic classes $c(T_\pi(E))$ of positive degree vanish, and hence so do all generalised Miller–Morita–Mumford classes. In fact, the analogue of the above result is that as long as $n \geq 3$ the analogous parameterised Pontrjagin–Thom map

$$\alpha^g: \text{BDiff}_g(W_{g,1}) \to \Omega_0^\infty S^{-2n}$$

is acyclic in a stable range, a special case of [GRW17, Corollary 1.8]. As the target has trivial rational cohomology, we have $H^*(\text{BDiff}_g(W_{g,1}); \mathbb{Q}) = 0$ in a stable range.

4.2. Decomposing the framed Torelli groups. The group $G_g$ of Section 3.1 is the automorphism group of the bilinear form $H_Z(g)$. Let $\text{Herm}(\mathbb{Z}, (-1)^n)$ denote the groupoid whose objects are $\{H_Z(g)\}_{g \geq 1}$ and whose morphisms are the isomorphisms of bilinear forms. Direct sum of bilinear forms endows this groupoid with a symmetric monoidality. A symmetric monoidal category yields, as is well-known [Seg74], a connective spectrum: neglecting the notation $\mathbb{Z}$ and $(-1)^n$, which are fixed for our discussion, we write $\text{KH}$ for the spectrum associated to $\text{Herm}(\mathbb{Z}, (-1)^n)$. It is a form of Hermitian $K$-theory of the ring $\mathbb{Z}$.

**Proposition 4.1.** There is a homotopy-commutative square

$$
\begin{array}{ccc}
\text{BDiff}_g(W_{g,1}) & \xrightarrow{\alpha^g} & \Omega_0^\infty S^{-2n} \\
\downarrow & & \downarrow \\
BG_g & \xrightarrow{\gamma} & \Omega_0^\infty \text{KH},
\end{array}
$$

for which the horizontal maps are acyclic in a range of degrees tending to infinity with $g$, and the right vertical map is a loop map.

**Proof.** The infinite loop space $\Omega^\infty \text{KH}$ is equivalent to the group-completion of the monoid $\bigsqcup_{g \geq 0} BG_g = B\text{Herm}(\mathbb{Z}, (-1)^n)$. Let us introduce a further topological monoid.

Fix once and for all a compact submanifold $M_1 \subset [0, 1]^{2n} \times \mathbb{R}^\infty$ which is equal to $[0, 1]^{2n}$ near $(0)[0, 1]^{2n} \times \mathbb{R}^\infty$ and is diffeomorphic to $[0, 1]^{2n} \# S^n \times S^n$ relative to this given identification of the boundary. Let $M_g \subset [0, g] \times [0, 1]^{2n-1} \times \mathbb{R}^\infty$ be obtained as the juxtaposition of $g$ copies of $M_1$. There are then homomorphisms

$$\text{Diff}_\partial(M_g) \times \text{Diff}_\partial(M_{g'}) \to \text{Diff}_\partial(M_{g+g'})$$
given by juxtaposing diffeomorphisms, and this makes $\bigsqcup_{g \geq 0} \text{Diff}_\partial(M_g)$ into an associative monoid (we declare $\text{Diff}_\partial(M_0)$ to be the trivial group). It follows from the
Choosing a diffeomorphism \( M_1 \cong [0, 1]^{2n} \# S^n \times S^n \), we get an identification of \( (H_n(M_1; \mathbb{Z}), \lambda) \) with the form \( H_2(1) \), and hence an identification of \( (H_n(M_g; \mathbb{Z}), \lambda) \) with the form \( H_2(g) = H_2(1)^{2g} \). This gives a map of associative monoids

\[
\bigoplus_{g \geq 0} \text{Diff}_\partial M_g \rightarrow \bigoplus_{g \geq 0} \text{BG}_g
\]

and hence a loop map between their group-completions. Thus there is a loop map \( \Omega_0^\infty \text{MT}_n \rightarrow \Omega_0^\infty \text{KH} \) such that the square

\[
\begin{array}{ccc}
\text{Diff}_\partial(W_{g,1}) & \longrightarrow & \text{Diff}_\partial(X_g) \longrightarrow \Omega_0^\infty \text{MT}_n \\
\downarrow & & \downarrow \\
\text{BG}_g & \longrightarrow & \Omega_0^\infty \text{KH}
\end{array}
\]

commutes up to homotopy. The top horizontal map is acyclic in a range of degrees tending to infinity with \( g \) by [GRW17, Corollary 1.8]. The bottom horizontal map is too, by the refinement of the group-completion theorem to deal with local coefficients, proved in [MP15, RW13], and homological stability for the groups \( G_g \) with abelian local coefficients, which follows from [RWW17, Theorem 5.16].

We extend this to a homotopy-commutative diagram

\[
\begin{array}{ccc}
\text{Diff}_\partial(W_{g,1})_{\ell} & \longrightarrow & \Omega_0^\infty S^{-2n} \\
\downarrow & & \downarrow \\
\text{BG}^\alpha_{g,\ell} & \longrightarrow & \Omega_0^\infty \text{KH}
\end{array}
\]

the outer part of which is the homotopy commutative square in the statement of the proposition. The top right-hand map is induced by the spectrum map \( S^{-2n} \rightarrow \text{MT}_n \), so is an infinite loop map; thus the right-hand composition is a loop map. The top horizontal map is acyclic in a range of degrees by [GRW17, Corollary 1.8] again.

Remark 4.2. The right-hand map in the statement of this proposition presumably admits the structure of an infinite loop map. Using [BBP+17, Theorem 7.14] one does indeed obtain an infinite loop map \( \Omega_0^\infty \text{MT}_n \rightarrow \Omega_0^\infty \text{KH} \) making the square \((11)\) commute up to homotopy, but it is not clear whether these are the same infinite loop space structures as those corresponding to the spectra \( \text{MT}_n \) and \( \text{KH} \). Thus it is unclear whether there is an infinite loop map \( \Omega_0^\infty S^{-2n} \rightarrow \Omega_0^\infty \text{MT}_n \) with respect to this infinite loop structure. The results of [BBP+17] do not apply directly to the framed case as there is no “operadic framing” in the sense of that paper.

As the map \( \text{Diff}_\partial(W_{g,1})_{\ell} \) is acyclic in a stable range, for large enough \( g \) we have

\[
H_1(\text{Diff}_\partial(W_{g,1})_{\ell}; \mathbb{Z}) \cong \pi_{2n+1}(S),
\]

a finite abelian group. We will write

\[
\overline{\text{Diff}}_\partial(W_{g,1})_{\ell} \longrightarrow \text{Diff}_\partial(W_{g,1})_{\ell}
\]

for the corresponding regular \( \pi_{2n+1}(S) \)-cover. The image of the composition

\[
\pi_1(\overline{\text{Diff}}_\partial(W_{g,1})_{\ell}) \longrightarrow \pi_1(\text{Diff}_\partial(W_{g,1})_{\ell}) \longrightarrow C_g^\alpha_{\ell,\ell}
\]
defines a normal subgroup of $G^{fr}[\ell]$. It will play an important role in this paper, warranting its own definition:

**Definition 4.3.** We let $\overline{G}^{fr}[\ell]$ denote the image of $\pi_1(BDiff^f_\ell(W_g,1)) \to G^{fr}[\ell]$.

There is a surjection $\pi_{2n+1}(S) \to G^{fr}[\ell]/\overline{G}^{fr}[\ell]$. By commutativity of the diagram

$$
\xymatrix{
\pi_1(BDiff^f_\ell(W_g,1)) \ar[r] \ar[d] & \pi_1(BDiff^f_\ell(W_g,1)) \ar[r] \ar[d] & \pi_{2n+1}(S) \ar[d] \\
\overline{G}^{fr}[\ell] \ar[r] & G^{fr}[\ell] \ar[r] & \pi_1(KH)
}
$$

the composition along the bottom row is zero, and so the composition

$$
BG^{fr}[\ell] \to BG^{fr}[\ell] \to \Omega_0^\infty KH
$$

lifts to the universal cover $\Omega_0^\infty KH$. Writing $\Omega_0^\infty S^{-2n}$ for the universal cover, we may therefore develop a homotopy-commutative diagram

$$
\xymatrix{
X'_1(g) \ar[r] \ar[d] & B\Tor^f_\ell(W_g,1) \ar[r] \ar[d] & X'_0 \ar[d] \\
A_1(g) \ar[r] & BDiff^f_\ell(W_g,1) \ar[r]^-{\pi^f} & \Omega_0^\infty S^{-2n} \\
A_2(g) \ar[r] & BG^{fr}[\ell] \ar[r] & \Omega_0^\infty KH
}
$$

with rows and columns fibration sequences.

**Remark 4.4.** As mentioned earlier, [KRW20a] describes $G^{fr}[\ell]$. For $n \neq 3$ this group is perfect for large enough $g$, and as $G^{fr}[\ell]/\overline{G}^{fr}[\ell]$ is abelian it follows that $\overline{G}^{fr}[\ell] = G^{fr}[\ell]$. If $n = 3$ the group $G^{fr}[\ell]$ has abelianisation $\mathbb{Z}/4$ for large enough $g$, and $\overline{G}^{fr}[\ell]$ is its commutator subgroup. We shall not use these facts.

**Lemma 4.5.** The covering map $B\Tor^f_\ell(W_g,1) \to B\Tor^f_\ell(W_g,1)$ induces an isomorphism on rational cohomology.

**Proof.** Letting $A := ker(\pi_{2n+1}(S)) \to G^{fr}[\ell]/\overline{G}^{fr}[\ell]$, by construction there is a fibration sequence

$$
\xymatrix{
B\Tor^f_\ell(W_g,1) \ar[r] & B\Tor^f_\ell(W_g,1) \ar[r] & BA.
}
$$

The base is a nilpotent space, and by Theorem 8.4 of [KRW19] so is the total space, so by [BK72, Ch. II 4.5] this is a nilpotent fibration, and hence by [BK72, Ch. II 5.4] $A$ acts nilpotently on each cohomology group $H^n(B\Tor^f_\ell(W_g,1); \mathbb{Q})$. But as $A$ is a finite group it therefore acts trivially, whence the Serre spectral sequence shows that the induced map is an isomorphism on rational cohomology.

The space $A_1(g)$ is the homotopy fibre of the map $\pi^f$ and hence of $\alpha^{fr}$, so it is acyclic in a stable range by Proposition 4.1; $A_2'(g)$ has the same property, rationally.

**Lemma 4.6.** The space $A_2'(g)$ is rationally acyclic in a range of degrees tending to infinity with $g$.

**Proof.** As the universal cover $\Omega_0^\infty KH$ is simply-connected, $A_2'(g)$ is rationally acyclic in a stable range if and only if the map

$$
BG^{fr}[\ell] \to \Omega_0^\infty KH
$$

is an isomorphism on rational cohomology in a stable range. As $BG_g \to \Omega_0^\infty KH$ is acyclic in a stable range by Proposition 4.1, it is enough to show that the maps

$$
BG^{fr}[\ell] \to BG_g \quad \text{and} \quad \Omega_0^\infty KH \to \Omega_0^\infty KH
$$
are rational cohomology isomorphisms in a stable range.

For the first we use that the group $G_{fr}^{g}[r]$ has finite index in $G_g$ by [KRW19, Corollary 8.7]. By construction of the normal subgroup $G_{fr}^{g}[r] < G_g^{fr}[r]$ there is a surjection

$$\pi_{2n+1}(S) \twoheadrightarrow G_{fr}^{g}[r]/G_g^{fr}[r]$$

so $G_{fr}^{g}[r]$ has finite index in $G_g^{fr}[r]$. Thus $G_{fr}^{g}[r]$ is a finite index subgroup of $G_g$. It is therefore an arithmetic group, with ambient algebraic group one of $\{O_{g,g}^r, SO_{g,g}^r, Sp_{2g}^r\}$. The work of Borel [Bor74, Theorem 11.1] calculates the stable rational cohomology of such groups, and the description for $G_g$ and $G_{fr}^{g}[r]$ is the same. (When $n$ is even, although $G_g$ is arithmetic in $O_{g,g}$ it could be the case that $G_{fr}^{g}[r]$ is arithmetic in $SO_{g,g}$. Nonetheless they have the same stable rational cohomology, see the proof of [KRW20b, Theorem 2.3].)

For the second we use the fibration sequence of connected infinite loop spaces

$$\Omega_0^{\infty}KH \rightarrow \Omega_0^{\infty}KH \rightarrow Br_1(\Omega_0^{\infty}KH)$$

and that $\pi_1(\Omega_0^{\infty}KH)$ is finite so has trivial rational cohomology.

The space $X'_0$ is a loop space, and its rational cohomology may therefore be computed from that of the infinite loop space $\Omega_0^{\infty}KH$ as given by Borel [Bor74, Section 11]:

$$H^*(\Omega_0^{\infty}KH; \mathbb{Q}) = \mathbb{Q}[\sigma_{ij}, i > n/2].$$

(Here we take $\sigma_{ij}$ as in Appendix A, cf. Corollary A.3; we will use the same notation for the corresponding classes on spaces rationally equivalent to $\Omega_0^{\infty}KH$.) Namely, because the rational cohomology of $\Omega_0^{\infty}S^{-2n}$ is trivial the Serre spectral sequence for the fibration sequence $X'_0 \rightarrow \Omega_0^{\infty}S^{-2n} \rightarrow \Omega_0^{\infty}KH$ shows that

$$H^*(X'_0; \mathbb{Q}) \cong \mathbb{Q}[\sigma_{ij}, i > n/2]$$

where the class $\sigma_i \in H^i(X'_0; \mathbb{Q})$ transgresses to $\sigma_{i+1} \in H^{i+1}(\Omega_0^{\infty}KH; \mathbb{Q})$.

Because the map $BTor_0^g(W_{g,1}) \rightarrow X'_0$ need not be $\pi_1$-surjective, unfortunately, the space $X'_1(g)$ defined in the diagram (12) need not be path-connected. We will therefore modify the construction slightly to make it so. We do so by replacing $X'_0$ by a certain finite cover, as suggested by the following lemma.

**Lemma 4.7.** The image of $\pi_1(BTor_0^g(W_{g,1})) \rightarrow \pi_1(X'_0)$ has finite index.

**Proof.** As the target is a finitely-generated abelian group, it is enough to show that the map $H_1(BTor_0^g(W_{g,1}); \mathbb{Q}) \rightarrow H_1(X'_0; \mathbb{Q})$ is surjective. By the previous lemma the target is zero if $n$ is even, and there is nothing to show. If $n$ is odd consider the map of Serre spectral sequences for middle and right columns of (12), giving

$$\begin{array}{ccc}
H_2(BG_g^{fr}[r]; \mathbb{Q}) & \xrightarrow{\partial^2} & \cong [H_1(BTor_0^g(W_{g,1}); \mathbb{Q})]\mathbb{Q}[\sigma_2^r] \\
\mathbb{Q}[\sigma_2^r] & \xrightarrow{\partial^2} & H_1(X'_0; \mathbb{Q}) = \mathbb{Q}[\sigma_1^r]
\end{array}$$

As the bottom and left maps are isomorphisms, the right-hand map is surjective.

As $X'_0$ is a loop space, its fundamental group is abelian and hence the cokernel $A$ of $\pi_1(BTor_0^g(W_{g,1})) \rightarrow \pi_1(X'_0)$ is a finite abelian group. Now let

$$q: X'_0 \rightarrow K(A,1)$$

be a map which on $\pi_1$ induces the quotient map; we may think of it as a class $q \in H^1(X'_0; A)$. It is an exercise in elementary homotopy theory to show that homotopy classes, in the category of maps of spaces, from $X \rightarrow Y$ to $* \simeq PK(A,n) \rightarrow K(A,n)$ are in bijection with $H^n(Y,X; A)$. Thus

$$q \in H^0(\Omega_0^{\infty}KH; H^1(X'_0; A)) = H^0(\Omega_0^{\infty}KH; H^2(*, X'_0; A))$$

$$\cong H^2(\Omega_0^{\infty}KH, \Omega_0^{\infty}S^{-2n}; A),$$
where the last identification is via the relative Serre spectral sequence, corresponds to a homotopy class of maps from $\Omega^\infty_0 S^{-2n} \to \Omega^\infty_0 KH$ to $PK(A, 2) \to K(A, 2)$, which we may develop into a homotopy-commutative diagram

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_0' \\
\downarrow & & \downarrow \\
\Omega^\infty_0 S^{-2n} & \longrightarrow & PK(A, 2) \\
\downarrow & & \downarrow \\
\Omega^\infty_0 KH & \longrightarrow & K(A, 2)
\end{array}
$$

(14)

in which rows and columns are homotopy fibre sequences, and the top right-hand horizontal map is homotopic to $q$.

As the composition

$$
B \text{Tor}^k_{fr}(W_g, 1)_{\ell} \longrightarrow X_0' \xrightarrow{q} K(A, 1)
$$

is nullhomotopic, by construction, the map of fibrations from the middle column of (12) to the right column of (12), which is the middle column of (14), lifts to a map to the left column of (14). We can therefore develop a homotopy-commutative diagram

$$
\begin{array}{ccc}
X_1(g) & \longrightarrow & \text{Tor}^k_{fr}(W_g, 1)_{\ell} \\
\downarrow & & \downarrow \\
A_1(g) & \longrightarrow & \Omega^\infty_0 S^{-2n} \\
\downarrow & & \downarrow \\
A_2(g) & \longrightarrow & \text{G}_{fr}[\ell]
\end{array}
$$

(15)

with rows and columns fibrations sequences. As $A$ was a finite abelian group, and the map $q$ may be represented by a loop map, we see that the spaces $A_1(g)$ and $A_2(g)$ are again rationally acyclic in a range tending to infinity with $g$, and the space $X_0$ is a loop space whose rational cohomology is the same as that of $X_0'$, so (13) gives:

**Lemma 4.8.** $H^*(X_0; \mathbb{Q}) \cong \mathbb{Q}[\tau_{4j-2n-1} \mid j > n/2]$, where the class $\tau_i \in H^i(X_0; \mathbb{Q})$ transgresses to $\sigma_{i+1} \in H^{i+1}(\Omega^\infty_0 KH; \mathbb{Q})$.

Since $X_0$ is a loop space its rational cohomology determines its rational homotopy groups: they are given by the dual of the indecomposables of its rational cohomology algebra. Thus the computation of $\pi_*(\text{Tor}^k_{fr}(W_g, 1)_{\ell})_{\mathbb{Q}}$ in a stable range amounts to the computation of $\pi_*(X_1(g))_{\mathbb{Q}}$ in a stable range, as well as the connecting homomorphism.

By construction we have arranged that $\text{Tor}^k_{fr}(W_g, 1)_{\ell} \to X_0$ is $\pi_1$-surjective, and hence that $X_1(g)$ is path-connected. It is reasonable to study the rational homotopy groups of this space via its rational cohomology, as:

**Lemma 4.9.** All spaces in the fibration sequence

$$
X_1(g) \longrightarrow \text{Tor}^k_{fr}(W_g, 1)_{\ell} \longrightarrow X_0
$$

are nilpotent.

**Proof.** The space $\text{Tor}^k_{fr}(W_g, 1)_{\ell}$ is nilpotent by [KRW19, Theorem 8.4], hence so is its the covering space $\text{Tor}^k_{fr}(W_g, 1)_{\ell}$. Then [MP12, Proposition 4.4.1 (i)] implies that $X_1(g)$ is nilpotent. As $X_0$ is a loop space, it is not only nilpotent but simple. \qed
4.3. The rational cohomology of $X_1(g)$. We will compute the rational cohomology of $X_1(g)$ in a stable range by a variant of the method described in [KRW20b], and for the remainder of this section we assume familiarity with the methods of that paper.

In Theorem 4.1 of [KRW20b] we described the rational cohomology of the Torelli spaces $B\text{Tor}_0(W_g, D^{2n}) = B\text{Tor}_0(W_{g,1})$ in a stable range in terms of a functor

$$\mathcal{P}(-, B')_{\geq 0} : d(s)\text{Br} \to \text{Gr}(\mathbb{Q}\text{-mod}),$$

defined in Sections 3.3, 3.4, and 3.5 of [KRW20b]. Here $d(s)\text{Br}$ denotes the downward (signed) Brauer category, described in Section 2.3 of [KRW20b], and $\mathcal{P}(-, B')_{\geq 0}$ is the functor which assigns to a finite set $S$ the vector space with basis the partitions $\{P_i\}_{i \in I}$ of $S$ into (possibly empty) parts $P_i$ which are labelled by a monomial $c_i$ in the Euler class $e$ and the Pontrjagin classes $p_{n-1}, p_{n-2}, \ldots, p_{[(n+1)/4]}$, and such that no parts (1) of size 0 have label of degree $\leq 2n$, (ii) of size 1 have label of degree $< n$, (iii) of size 2 have label of degree 0. This description is given in Remark 3.12 of [KRW20b]. Such a labelled partition is given degree $\sum_{i \in I} n(|P_i| - 2) + |c_i|$. The functoriality on the downward (signed) Brauer category is described in Section 3.5 of [KRW20b].

**Definition 4.10.** Let $\mathcal{P}(-, B')_{\geq 0} : d(s)\text{Br} \to \text{Gr}(\mathbb{Q}\text{-mod})$ be the analogous functor which assigns to a finite set $S$ the vector space with basis the partitions $\{P_i\}_{i \in I}$ of $S$ into parts of size $\geq 3$, with all parts implicitly labelled by the unit 1.\(^1\) Such a labelled partition is given degree $\sum_{i \in I} n(|P_i| - 2)$.

The functoriality on the downward (signed) Brauer category is as follows: to a bijection we assign the natural induced map on labelled partitions, and to a morphism $(d(s)\text{Br})_{\geq 0} : (x, y), \mathcal{D}) : S \to S \setminus \{x, y\}$ we assign the map which gives a partition $\{P_i\}_{i \in I}$ of $S$

(a) if some $P_i$ contains $\{x, y\}$ then returns zero,

(b) if $x$ and $y$ lie in different parts $P_i$ and $P_j$, then returns the partition of $S \setminus \{x, y\}$ given by merging these into a new part $(P_i \setminus \{x\}) \cup (P_j \setminus \{y\})$, and keeping intact all other parts.

A general morphism in $d(s)\text{Br}$ is a composition of such morphisms and bijections, so $\mathcal{P}(-, B')_{\geq 0}$ is determined by these properties.

Our calculation of the $\mathbb{Q}$-cohomology of $X_1(g)$ in a stable range is then as follows, which is the analogue of Theorem 4.1 of [KRW20b] in the framed situation.

**Theorem 4.11.** Let $2n \geq 6$. Then

(i) the monodromy action of $\pi_1(A_2(g))$ on $H^*(X_1(g); \mathbb{Q})$ factors over $\overline{G}_g^{fr}[\ell]$,

(ii) there is a map

$$i^*(K^\vee) \otimes d(s)\text{Br} \left( \mathcal{P}(-, B')_{\geq 0} \otimes \det^{\otimes n} \right) \to H^*(X_1(g); \mathbb{Q})$$

of $\overline{G}_g^{fr}[\ell]$-representations which is an isomorphism in a stable range.

The remainder of this subsection is dedicated to the proof of this theorem.

4.3.1. **Algebraicity.** The monodromy of the left-hand column of (15) gives an action of $\pi_1(A_2(g))$ on $X_1(g)$ in the homotopy category of (unpointed) spaces, and hence an action on the integral or rational cohomology of $X_1(g)$. We will first prove Theorem 4.11 (i), concerning this action. The long exact sequence on homotopy groups for the bottom row of (15) gives a central extension

$$0 \to \pi_2(\Omega_0^{\infty} \overline{K}H) \to \pi_1(A_2(g)) \to \overline{G}_g^{fr}[\ell] \to 0,$$

and we must analyse how the subgroup $\pi_2(\Omega_0^{\infty} \overline{K}H)$ acts on the cohomology of $X_1(g)$.

**Lemma 4.12.** The subgroup $\pi_2(\Omega_0^{\infty} \overline{K}H)$ acts nilpotently on each $H^q(X_1(g); \mathbb{Z})$.\(^2\)

\(^1\)If the only label allowed is the unit 1, which has degree 0, the three conditions of the last paragraph say that there can be no parts of sizes 0, 1, or 2. So the condition that all parts have sizes $\geq 3$ is just a reformulation of these conditions in this case.
Proof. A diagram chase shows that the action of this subgroup is via the map
\[ \partial : \pi_2(\underline{w} S^n \mathbb{KH}) \to \pi_1(X_0) \]
and the monodromy for the top row of (15). We now proceed similarly to the proof of Lemma 4.5. This top row has nilpotent total space and base, so by [BK72, Ch. II 4.5] is a nilpotent fibration, and hence by [BK72, Ch. II 5.4] \( \pi_1(X_0) \) acts nilpotently on each \( H^q(X_1(g); \mathbb{Z}) \). Therefore, \( \pi_2(\underline{w} S^n \mathbb{KH}) \) acts nilpotently on each \( H^q(X_1(g); \mathbb{Z}) \). \( \square \)

Lemma 4.13. For each \( q \), \( H^q(X_1(g); \mathbb{Z}) \) and \( \pi_2(X_1(g)) \) are finitely generated.

Proof. By Lemma 4.9, \( X_1(g) \) is a nilpotent space, so it suffices to prove that its homotopy groups are finitely-generated. The homotopy groups of \( \Omega_{\infty}^n S^{-2n} \) are finitely-generated by a classical theorem of Serre, and those of \( \Omega_{\infty}^n \mathbb{KH} \) are finitely-generated by combining a classical theorem of Borel–Serre [BS73, 11.1(c)] with homological stability [Cha87, §4]. Thus those of \( X_0 \) are finitely-generated too, so by considering the top row of (15) it remains to show that the homotopy groups \( \pi_{t}(B\text{Tor}_{\phi}^f(W_{g,1})) \) are finitely-generated.

To see this we may as well pass down to the finite quotient \( B\text{Tor}_{\phi}^f(W_{g,1}) \) which fits into a fibration sequence
\[ \text{map}_*(W_{g,1}/\partial W_{g,1}, \text{SO}(2n)) \to B\text{Tor}_{\phi}^f(W_{g,1}) \to B\text{Tor}_{\phi}(W_{g,1}), \]
where \( (\dashv) \) denotes those path-components given by the \( \pi_0(\text{Tor}_{\phi}(W_{g,1})) \)-orbit of \( \ell \).

The fibre clearly has finitely-generated homotopy groups, and the base does too by Theorem C (or Corollary 5.5) of [Kup19]. \( \square \)

Lemma 4.14. The \( \pi_1(A_2(g)) \)-action on each \( H^q(X_1(g); \mathbb{Q}) \) is gr-algebraic with respect to the extension (16).

Proof. Consider the fibration given by the top row of (15). First note that all three spaces have rational cohomology of finite type, by Lemma 4.8 for \( X_0 \), Lemma 4.13 for \( X_1(g) \), and Lemma 4.5 and [KRW19, Theorem 8.3] for \( B\text{Tor}_{\phi}^f(W_{g,1}) \).

By the proof of Lemma 4.12 the group \( \pi_1(X_0) \) acts nilpotently on \( H^* (X_1(g); \mathbb{Q}) \), so by [Dwy74] it follows that the associated homology Eilenberg–Moore spectral sequence converges strongly; as all three spaces have rational homology of finite type, it follows by dualising that the cohomology Eilenberg–Moore spectral sequence
\[ E^2_{s,t} = \text{Tor}_{s,t}^H(X_0; \mathbb{Q}, H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q})) \implies H^{t-s}(X_1(g); \mathbb{Q}) \]
also converges strongly. This is a spectral sequence of \( \pi_1(A_2(g)) \)-representations by naturality, and the action on the \( E^2 \)-page is the monodromy action of \( \mathcal{G}_{g}^{\text{gr},[\ell]} \) on \( H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q}) \).

As each \( H^q(X_1(g); \mathbb{Q}) \) is finite-dimensional, by strong convergence of the spectral sequence it has a finite filtration whose filtration quotients are each subquotients of some \( E^2_{s,t} \). Using the fact that gr-algebraic representations are closed under subquotients, extensions, and tensor products, and using the standard bar complex to compute \( E^2_{s,t} \), it thus suffices to show that the \( \mathcal{G}_{g}^{\text{gr},[\ell]} \)-action on \( H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q}) \) is algebraic in each degree.

To see this we use Lemma 4.5 to identify
\[ H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q}) \cong H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q}) \]
via the natural map, which is equivariant with respect to \( \mathcal{G}_{g}^{\text{gr},[\ell]} \to \mathcal{G}_{g}^{\text{gr},[\ell]} \). Now, by Theorem 8.3 of [KRW19], the \( \mathcal{G}_{g}^{\text{gr},[\ell]} \)-representations \( H^*(B\text{Tor}_{\phi}^f(W_{g,1}); \mathbb{Q}) \) are algebraic in each degree, which concludes the argument. \( \square \)

Lemma 4.15. Let \( g \geq 2 \). Any \( \pi_1(A_2(g)) \)-representation \( V \) which is gr-algebraic with respect to (16) descends to an algebraic representation of \( \mathcal{G}_{g}^{\text{gr},[\ell]} \).
Proof. We claim that the subgroup $\pi_2(\Omega_0^\infty \KH) \leq \pi_1(A_2(g))$ acts on $V$ via automorphisms of finite order, whence the result follows from Lemma 2.6. By definition there is a short exact sequence
\[
0 \longrightarrow \pi_2(\Omega_0^\infty \KH) \longrightarrow \pi_2(\Omega_0^\infty \KH) \longrightarrow A \longrightarrow 0
\]
with $A$ a finite abelian group, and we also have $\pi_2(\Omega_0^\infty \KH) = \pi_2(\Omega_0^\infty \KH)$, as $(-)$ denotes a covering space.

When $n$ is even, $\pi_2(\Omega_0^\infty \KH)$ is finite by [Bor74, Section 11], and hence $\pi_2(\Omega_0^\infty \KH)$ is finite too, and so acts on $V$ by automorphisms of finite order.

When $n$ is odd, $\pi_2(\Omega_0^\infty \KH)$, and hence also $\pi_2(\Omega_0^\infty \KH)$, is a finitely-generated abelian group of rank 1 by [Bor74, Section 11]. In fact, $G_{\fr} = \Sp_{2g}(\mathbb{Z})$ which for large enough $g$ is perfect and has $H_2(\Sp_{2g}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}$, so $\pi_2(\Omega_0^\infty \KH) \cong \mathbb{Z}$. Define a space $A_3(g)$ by the homotopy fibre sequence
\[
A_3(g) \longrightarrow BG_{\fr} \longrightarrow \Omega_0^\infty \KH,
\]
so that this space is acyclic in a stable range of degrees. The map from the fibration defining $A_2(g)$, i.e. the bottom row of (15), gives a map of extensions
\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_2(\Omega_0^\infty \KH) & \longrightarrow & \pi_1(A_2(g)) & \longrightarrow & \mathcal{G}_{\fr}^{[t]} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_2(\Omega_0^\infty \KH) & \longrightarrow & \pi_1(A_3(g)) & \longrightarrow & G_{\fr} \longrightarrow 0.
\end{array}
\]
The two outer maps are inclusions of finite index subgroups, so the middle map is too. For $g \geq 3$ the group $G_{\fr} = \Sp_{2g}(\mathbb{Z})$ is perfect, and the lower row is its universal central extension, by construction. Equivalently, it is the pullback to $\Sp_{2g}(\mathbb{Z})$ of the universal cover of $\Sp_{2g}(\mathbb{R})$; by naturality with respect to stabilisation this describes the lower extension for all $g \geq 1$.

At this point we use the theorem of Deligne [Del78, p. 206] on the residual finiteness of the universal central extension $\pi_1(A_3(g))$, viz. that this group is not residually finite as long as $g \geq 2$. Therefore its finite index subgroup $\pi_1(A_2(g))$ is also not residually finite. As the group $\mathcal{G}_{\fr}^{[t]}$ is residually finite, being a subgroup of $\GL_{2g}(\mathbb{Z})$, it follows that there must be a nontrivial element $t \in \pi_2(\Omega_0^\infty \KH)$ which lies in every finite index subgroup of $\pi_1(A_2(g))$. Now $\pi_1(A_2(g))$ is finitely-generated, and by a theorem of Mal’cev [Mal40] any finitely-generated linear group is residually finite. Thus the image of the representation $\rho : \pi_1(A_2(g)) \longrightarrow \GL(V)$ is residually finite, and hence the element $t$ lies in the kernel of $\rho$ and so acts trivially on $V$. But then the subgroup $\pi_2(\Omega_0^\infty \KH) \leq \pi_1(A_2(g))$ acts on $V$ via the finite group $\pi_2(\Omega_0^\infty \KH)/\langle t \rangle$, so acts by automorphisms of finite order as claimed. \hfill $\square$

Combining these lemmas we obtain the following, proving Theorem 4.11 (i):

**Corollary 4.16.** For $g \geq 2$, the subgroup $\pi_2(\Omega_0^\infty \KH)$ acts trivially on the groups $H^s(X_1(g); \mathbb{Q})$, and the induced $\mathcal{G}_{\fr}^{[t]}$-action is algebraic.

4.3.2. From invariants to twisted cohomology. In order to determine the algebraic $\mathcal{G}_{\fr}^{[t]}$-representations $H^s(X_1(g); \mathbb{Q})$, and hence prove Theorem 4.11 (ii), we proceed as in the proof of the analogous Theorem 4.1 of [KRW20b]), using [KRW20b, Proposition 2.16]. Using the notation $H$ of Section 2.3 for the standard representation of $\mathcal{G}_{\fr}^{[t]}$, we need to produce a natural transformation
\[
i_* \mathcal{P}(-)_{\geq 0} \otimes \det \mathcal{O}^n : [H^s(X_1(g); \mathbb{Q}) \otimes H^s(-)] \mathcal{G}_{\fr}^{[t]} : (s) \mathcal{B}_{2g} \longrightarrow \mathcal{G}(\mathbb{Q}-\text{mod})
\]
and show that it is an isomorphism in a stable range.
As a first step towards producing such an isomorphism, we will identify the invariants
\([H^\ast(X_1(g); \Q) \otimes H^{\otimes n}]^{\simeq}_{\simeq}\) with \(H^\ast(B\Diff^\text{fr}_{\partial}(W_{g,1}); \H^{\otimes})\) in a stable range. We will do so by analysing several Serre spectral sequences arising from the diagram (15).

First consider the Serre spectral sequence for the left-hand column of (15), with coefficients in \(\H^{\otimes}\) (which is pulled back from \(B\Gr^{\text{fr},[\ell]}\)). This has the form
\[
1^E_2^{p,q} = H^p(A_2(g); H^q(X_1(g); \Q) \otimes \H^{\otimes}) \implies H^{p+q}(A_1(g); \H^{\otimes}),
\]
where we have used that the coefficient system \(\H\) is trivialised over \([1,1]\). We will now compute the \(E_2\)-page and abutment of this spectral sequence by different means.

By Theorem 4.11 (i) the \(\pi_1(A_2(g))\)-representations \(H^q(X_1(g); \Q)\) factor over \(\Gr^{\text{fr},[\ell]}\). We may therefore consider the Serre spectral sequence
\[
2^E_2^{s,t} = H^s(\Omega^\infty_0 \KH; \Q) \otimes H^t(A_2(g); H^q(X_1(g); \Q) \otimes \H^{\otimes}) \implies H^{s+t}(B\Gr^{\text{fr},[\ell]}; H^q(X_1(g); \Q) \otimes \H^{\otimes})
\]
associated to the fibration given by the bottom row of (15), with coefficients in the \(\Gr^{\text{fr},[\ell]}\)-representation \(H^q(X_1(g); \Q) \otimes \H^{\otimes}\). As this \(\Gr^{\text{fr},[\ell]}\)-representation is algebraic by Corollary 4.16s, by the Borel vanishing theorem, as explained in [KRW20b, Theorem 2.3], we may write the abutment as
\[
H^{s+t}(\Omega^\infty_0 \KH; \Q) \otimes [H^q(X_1(g); \Q) \otimes \H^{\otimes}]^{\Gr^{\text{fr},[\ell]}},
\]
in a stable range of degrees. In this range we therefore have a spectral sequence of \(H^*(\Omega^\infty_0 \KH; \Q)\)-modules which starts with, and abuts to, a free module. By [KRW20b, Lemma 4.3] it must therefore collapse, to give
\[
H^p(A_2(g); H^q(X_1(g); \Q) \otimes \H^{\otimes}) \cong \begin{cases} 0 & \text{if } p > 0, \\ [H^q(X_1(g); \Q) \otimes \H^{\otimes}]^{\Gr^{\text{fr},[\ell]}} & \text{if } p = 0. \end{cases}
\]
It follows that \(1^E_2^{p,q}\) is supported along \(p = 0\) in a stable range, so in this range the spectral sequence \(1^E_2^{p,q}\) collapses, giving an identification
\[
[H^q(X_1(g); \Q) \otimes \H^{\otimes}]^{\Gr^{\text{fr},[\ell]}} \cong H^q(A_1(g); \H^{\otimes}).
\]

We now compute \(H^*(A_1(g); \H^{\otimes})\) by different means. Consider the Serre spectral sequence for the middle row of the diagram (15), with \(\H^{\otimes}\)-coefficients, given by
\[
3^E_2^{a,v} = H^a(\Omega^\infty_0 \s^\sim_0; \Q) \otimes H^v(A_1(g); \H^{\otimes}) \implies H^{a+v}(B\Diff^\text{fr}_{\partial}(W_{g,1}); \H^{\otimes}).
\]
As the simply-connected space \(\Omega^\infty_0 \s^\sim_0\) has finite homotopy groups it also has trivial rational homology, so this spectral sequence is supported on the line \(u = 0\) and hence collapses, to give
\[
H^q(A_1(g); \H^{\otimes}) \cong \K^q(B\Diff^\text{fr}_{\partial}(W_{g,1}); \H^{\otimes}).
\]
Combining this with (17), and observing that all these isomorphisms are the naturally induced ones, shows that the natural map
\[
H^q(B\Diff^\text{fr}_{\partial}(W_{g,1}); \H^{\otimes}) \implies [H^q(X_1(g); \Q) \otimes \H^{\otimes}]^{\Gr^{\text{fr},[\ell]}},
\]
given by restriction, is an isomorphism in a stable range.

As discussed at the beginning of this section, using (18) it now suffices to produce a natural transformation
\[
\overline{\Phi}: i_\ast \P^{\geq 0} \otimes \det^n \implies H^*(B\Diff^\text{fr}_{\partial}(W_{g,1}); \H^{\otimes}); : (s)\Br^\ast \implies \Gr(\Q\text{-mod})
\]
and show that it is an isomorphism in a stable range. This is done in the following sections.
4.3.3. Twisted Miller–Morita–Mumford classes. The space \( \text{BDiff}_0^g(W_{g,1}) \) classifies the data of a smooth \( W_{g,1} \)-bundle \( \pi^* : E^* \to B \) whose boundary \( \partial E^* \) is identified with \( \partial W_{g,1} \times B \), and with a framing of the vertical tangent bundle which agrees with \( \ell_g \) on \( \partial E^* \). We may glue in \( D^{2n} \times B \) along the given identification of the boundaries to obtain a smooth \( W_{g,1} \)-bundle \( \pi : E \to B \) with section \( s : B \to E \) given by the centre of the disc \( D^{2n} \). Using this data, as in Section 3 of [KRW20b] there is defined an \( \epsilon \in H^n(E; H) \) and hence for each finite set \( S \) characteristic classes

\[
\kappa_{\epsilon,S}(\pi) := \pi_!(\epsilon^S) \in H^n(S,-1)(B; H^{\otimes S}).
\]

Universally this gives \( \kappa_{\epsilon,S} \in H^n(S,-1)(\text{BDiff}_0^g(W_{g,1}); H^{\otimes S}) \), which is pulled back from \( \text{BDiff}_0^g(W_{g,1}) \). The interaction of these classes with the maps \( \lambda_{i,j} : H^{\otimes S} \to H^{\otimes S\{i,j\}} \) can be deduced from Proposition 3.10 of [KRW20b], and as follows.

**Proposition 4.17.** For \( k \geq 2 \) we have

\[
\lambda_{1,2}(\pi!(\epsilon^k)) = \begin{cases} (-1)^n2g & \text{if } k = 2, \\ 0 & \text{else.} \end{cases}
\]

For \( a \geq 2 \) and \( b \geq 2 \) we have \( \lambda_{a,a+1}(\pi_!(\epsilon^a) \cdot \pi!(\epsilon^b)) = \pi_!(\epsilon^{1,2,\ldots,a-1} \cdot \epsilon^{a+2,\ldots,a+b}) \).

**Proof.** By Proposition 3.10 of [KRW20b] we have

\[
\lambda_{1,2}(\pi!(\epsilon^k)) = \pi_!(\epsilon^{k-2} \cdot \epsilon(T_\pi E)) + s^*\epsilon(T_\pi E) \cdot \pi!(\epsilon^{k-2}) - \begin{cases} 2 & \text{if } k = 2, \\ 0 & \text{else.} \end{cases}
\]

To analyse the first term, note that as the vertical tangent bundle of the subbundle \( E' \subset E \) is framed, \( e(T_\pi E) \) vanishes when restricted to \( E' \) and so lifts to a class \( e(T_\pi E) \in H^{2n}(E, E'; \mathbb{Q}) \). The relative Serre spectral sequence for the pair of fibrations \( (E, E') \to B \) shows that the map

\[
H^{2n}(E, E'; \mathbb{Q}) \to H^{2n}(W_g, W_{g,1}; \mathbb{Q}) = \mathbb{Q}
\]

given by restricting to a fibre is an isomorphism. As \( e(T_\pi E) \) restricts to \( \chi(W_g) = 2 + (-1)^n2g \) times the generator, and the class \( v \in H^{2n}(E, E'; \mathbb{Q}) \) Poincaré dual to \( s : B \to E \) (constructed in Lemma 3.1 of [KRW20b]) restricts to the generator, we must have the identity \( e(T_\pi E) = \chi(W_g) \cdot v \). But then the first term of the expression above is

\[
\pi_!(\epsilon^{k-2} \cdot e(T_\pi E)) = \chi(W_g)\pi_!(\epsilon^{k-2} \cdot v) = \chi(W_g)s^*\epsilon^{k-2} = \begin{cases} 2 + (-1)^n2g & \text{if } k = 2, \\ 0 & \text{else,} \end{cases}
\]

because \( s^*\epsilon = 0 \) by definition of \( \epsilon \). The second term vanishes, as \( s : B \to E \) lies inside a trivial \( D^{2n} \times B \)-bundle so \( s^*e(T_\pi E) = 0 \) (this did not use the framing). This gives the claimed formula.

By Proposition 3.10 of [KRW20b] we also have

\[
\lambda_{a,a+1}(\pi!(\epsilon^a) \cdot \pi!(\epsilon^b)) = \pi_!(\epsilon^{1,2,\ldots,a-1} \cdot \epsilon^{a+2,\ldots,a+b}) + s^*e(T_\pi E) \cdot \pi!(\epsilon^{a-1} \cdot \epsilon^{b-1}),
\]

but as \( s^*e(T_\pi E) = 0 \) as above the second term vanishes, giving the claimed formula. \( \square \)

Similarly, the effect of the dual forms \( \omega_{i,j} : H^{\otimes S\{i,j\}} \to H^{\otimes S} \) in cohomology are determined by multiplicativity and \( \omega_{1,2}(1) = \pi^!(\epsilon^2) \), as in Proposition 3.10 of [KRW20b].

As in Section 3 of [KRW20b], we define a functor \( \mathcal{P}(-,\otimes_0^g) : (s)\text{Br}_{2g} \to \text{Gr}(\mathbb{Q}\text{-mod}) \) to assign to a finite set \( S \) the vector space with basis the partitions \( \{P_n\} \) of \( S \) with all parts of size \( \geq 2 \). The functoriality on the (signed) Brauer category is as described in Sections 3.4 and 3.5 of [KRW20b], which is arranged so as to give a natural transformation

\[
\Phi : \mathcal{P}(-,\otimes_0^g) \otimes \det^\otimes_0 \to H^*(\text{BDiff}_0^g(W_{g,1}); H(g)^-): (s)\text{Br}_{2g} \to \text{Gr}(\mathbb{Q}\text{-mod})
\]

by sending a partition \( \{P_n\} \) of \( S \) and an orientation \( (s_1 \wedge s_2 \wedge \cdots \wedge s_k)^\otimes_0 \) to \( \pm \prod \kappa_{s_i,\epsilon} \), for a sign as determined in Section 3.3 of [KRW20b]. As at the end of Section 3.5 of [KRW20b] there is an identification of \( \mathcal{P}(-,\otimes_0^g) \) with the Kan extension \( i_! \mathcal{P}(-,\otimes_0^g) \) of the functor \( \mathcal{P}(-,\otimes_0^g) \) on the downwards (signed) Brauer category described in Definition
4.10, so combining the above map with pullback to the cover $B\text{Diff}^\text{fr}_\partial(W_{g,1})_\ell$ gives a natural transformation

$$\overline{\Phi}: i_* \mathcal{P}(-)_{>0} \otimes \det^\otimes \mathcal{H}(g) \otimes \mathcal{H}(g) \otimes \cdots: (s)\text{Br}_{2g} \to \text{Gr}(\mathbb{Q}\text{-mod}).$$

This is the natural transformation promised at the end of Section 4.3.2; it remains to prove that it is an isomorphism in a stable range.

4.3.4. The twisted cohomology calculation. To prove that $\overline{\Phi}$ is an isomorphism in a stable range, we proceed as in the proof of Theorem 3.15 of [KRW20b], with some small changes that we now explain. That is, we suppose that $n$ is odd (the $n$ even case has no significant differences) we let $W$ be a finite dimensional $\mathbb{Q}$-vector space, set $Y := K(W^\vee, n + 1)$, and consider the moduli spaces $B\text{Diff}^\text{fr}_\partial \times Y(W_{g,1})$ classifying framed $W_{g,1}$-bundles equipped with a map to $Y$. We form the diagram

$$\begin{array}{ccc}
\text{map}_*(W_{g,1}/\partial W_{g,1}, Y) & \downarrow & \Omega^\infty_0(S^{-2n} \wedge Y_+), \\
B\text{Diff}^\text{fr}_\partial \times Y(W_{g,1})_\ell & \rightarrow & \Omega^\infty_0(S^{-2n})
\end{array}$$

where the horizontal maps are acyclic in a stable range by [GRW17, Corollary 1.8]. Just as in the proof of Theorem 3.15 of [KRW20b], we have a natural identification

$$H^*(\text{map}_*(W_{g,1}/\partial W_{g,1}, Y); \mathbb{Q}) = \text{Sym}^*(H(g) \otimes W[1]).$$

We may form a new diagram by replacing $\Omega^\infty_0(S^{-2n})$ by its universal cover $\Omega^\infty_0(S^{-2n})$, and taking the associated finite covering spaces of the other spaces in the square: we use $(-)$ to denote these covers too. The spectral sequence for the vertical fibration sequence then takes the form

$$E_2^{p,q} = H^p(B\text{Diff}^\text{fr}_\partial(W_{g,1})_\ell; \Lambda^q(H(g) \otimes W)) \Rightarrow H^{p+q}(B\text{Diff}^\text{fr}_\partial \times Y(W_{g,1})_\ell; \mathbb{Q}).$$

On the other hand the abutment of this spectral sequence agrees in a stable range with

$$H^*(\Omega^\infty_0(S^{-2n} \wedge Y_+); \mathbb{Q}) \cong H^*(\Omega^\infty_0(S^{-2n} \wedge Y_+); \mathbb{Q}) \cong \text{Sym}^*([\text{Sym}^*(W[n + 1])_{>0}])_{>0}.$$

The second identification is just as in the proof of Theorem 3.15 of [KRW20b]. The first identification is because $\Omega^\infty_0(S^{-2n} \wedge Y_+) \rightarrow \Omega^\infty_0(S^{-2n} \wedge Y_+)$ is a finite cover of infinite loop spaces.

At this point there are no further differences from the proof of Theorem 3.15 of [KRW20b], and we follow that argument, concluding that $\overline{\Phi}$ is indeed an isomorphism in a stable range. This finishes the proof of Theorem 4.11.

4.4. The rational homotopy of $X_1(g)$. For the results of this paper it will suffice to make a rather coarse estimate of the rational homotopy groups of $X_1(g)$ from its rational cohomology.

By Theorem 4.11 (i) the action of $\pi_1(A_2(g))$ on $H^*(X_1(g); \mathbb{Q})$ factors over $G^\text{fr}_{g,[\ell]}$. We now give a similar result for the homotopy homotopy groups of $X_1(g)$.

**Lemma 4.18.** For $g$ large enough the nilpotent group $\pi_1(X_1(g))$ is finite and acts trivially on $\pi_q(X_1(g); \mathbb{Q})$ for each $q \geq 2$.

**Proof.** By Theorem 4.11 (ii) for large enough $g$ we have $H^1(X_1(g); \mathbb{Q}) = 0$, as by Definition 4.10 the left-hand side in the statement of Theorem 4.11 (ii) is supported in degrees divisible by $n$. Thus the nilpotent group $\pi_1(X_1(g))$ is finite. By Lemma 4.13 the groups $\pi_q(X_1(g))$ are finitely generated for each $q$, and are nilpotent $\mathbb{Z}[\pi_1(X_1(g))]$-modules by Lemma 4.9. Just as in the proof of Lemma 2.6, elements of the finite group
π₁(X₁(g)) must act trivially on the finite-dimensional vector space πₚ(X₁(g))Q, as they act both nilpotently and with finite order.

It follows from this lemma that for large enough g the action of π₁(A₂(g)) on X₁(g), by unpointed maps, induces a well-defined action on each πₚ(X₁(g))Q. We can prove the analogues of the results of Section 4.3.1 for this action.

**Lemma 4.19.** The π₁(A₂(g))-action on each πₚ(X₁(g))Q is gr-algebraic.

**Proof.** Recall from Lemma 4.9 that X₁(g) is nilpotent and hence may be rationalised. Consider the action in the homotopy category of π₁(A₂(g)) on the rationalisation X₁(g)Q. By Lemma 4.18, X₁(g)Q is simply-connected, so the conclusion follows from Lemma 4.14 using [KRW19, Lemma 2.10] applied to the class of gr-algebraic representations.

Applying Lemma 4.15 gives the following:

**Corollary 4.20.** For g ≥ 2 the subgroup π₂(Ω₀KH) acts trivially on πₚ(X₁(g))Q for each q ≥ 2, and the induced Gₙ⁻[r]-action is algebraic.

Let us now give our coarse estimate of the rational homotopy groups of X₁(g).

**Proposition 4.21.** In a stable range the groups πₚ(X₁(g))Q are supported in degrees n, 2n − 1, and s ∈ Uₙ≥3[r(n − 1) + 1, rn − 2].

Furthermore, in a stable range the Gₙ⁻[r]-invariants [πₚ(X₁(g))Q]²gr are Q in degree 2n − 1, and are otherwise supported in degrees s ∈ U₁≥2[2r(n − 1) + 1, 2rn − 2].

To prepare for the proof of this proposition we must describe some consequences of our description of the cohomology of X₁(g) given in Theorem 4.11, using results from Section 5 of [KRW20b]. Recall from Section 2.3 that we write ω ∈ H⁰⁺ for the form dual to the pairing λ. Here it is convenient to not refer to the hyperbolic basis of H, and rather let {aᵢ}ᵢ=1 g be any basis of H with dual basis {aᵢ}ᵢ=1 g characterized by λ(aᵢ, aⱼ) = δᵢⱼ, where we then have

ω = ∑ᵢ g aᵢ ⊗ aᵢ.

In the notation of that paper, Theorem 4.11 shows that in a stable range we have H⁺(X₁(g); Q) ≅ R for V given by the graded vector space Q[0] with e = 0 ∈ V. In Theorem 5.1 of [KRW20b] we have shown how to obtain a generators and relations description of such rings R⁺, which applied to this case shows that in a stable range the graded-commutative ring H⁺(X₁(g); Q) is generated by the generalised Miller–Mumford classes

κ₁(v₁ ⊗ v₂ ⊗ ⋯ ⊗ v_r) of degree (r - 2)n, for r ≥ 3 and vᵢ ∈ H⁺(W₉,1; Q)

subject to the relations of:

(i) linearity in each vᵢ,

(ii) κ₁(vᵢ(1) ⊗ vᵢ(2) ⊗ ⋯ ⊗ vᵢ(r)) = sign(σ)ⁿ ⋅ κ₁(v₁ ⊗ v₂ ⊗ ⋯ ⊗ v_r),

(iii) ∑ᵢ κ₁(v ⊗ aᵢ) ⋅ κₜ(aᵢ ⊗ w) = κ₁(v ⊗ w), for any tensors v and w,

(iv) ∑ᵢ κ₁(v ⊗ aᵢ ⊗ aₜ) = 0 for any tensor v.

The main consequences of this description that we shall use are that in a stable range the rational cohomology ring of X₁(g) is supported in degrees which are multiples of n, and furthermore has a presentation with generators certain classes κ₁(v₁ ⊗ v₂ ⊗ v₃) of degree n, and all relations in degree 2n.

**Proof of Proposition 4.21.** For a nilpotent based space X of finite type, there is a (dualised) unstable Adams spectral sequence

E²₀,q = Harr_π(H⁺(X; Q))q ⇒ Hom_Z(π_q−p(ΩX), Q)

beginning with the Harrison homology of the augmented ring H⁺(X; Q), and converging to the dual of the rational homotopy groups of X (shifted by 1). In total degree 0,
convergence means that $\bigoplus_{p \geq 0} E_{p,q}^\infty$ is the dual of the rationalisation of the associated graded of the lower central series of the nilpotent group $\pi_0(\Omega X)$. The spectral sequence is natural in based maps. This spectral sequence seems to have been folklore for a long time, but a published reference can be found as [SW11b, Corollary A.2], using [SW11b, Remark 4.10] to identify the $d^1$-differential as the Harrison differential. Using the reduced Harrison complex to compute the $E^2$-page, we see that Harr$_p(H^*(X;\mathbb{Q}))_q$ is a subquotient of $H^{*>0}(X;\mathbb{Q})^\otimes p$, where $H^{*>0}(X;\mathbb{Q})$ denotes the augmentation ideal of $H^*(X;\mathbb{Q})$.

Applying this to $X = X_1(g)$, where we have just discussed $H^*(X_1(g);\mathbb{Q})$ is supported in degrees of which are multiples of $p$, we see that for each $p$ the groups $E^2_{p,q}$ will be supported in $q$-degrees

$$q = mn, (p + 1)n, (p + 2)n, \ldots ,$$

which contribute to total degree

$$q - p = p(n - 1), p(n - 1) + n, p(n - 1) + 2n, \ldots .$$

Thus $\pi_*(\Omega X_1(g);\mathbb{Q})$ is supported in degrees $* \in \bigcup_{n \geq 1}[r(n - 1), rn - 1]$.

We can refine this slightly using the presentation of this ring. As, in a stable range, it has a presentation with generators only in degree $n$ we have Harr$_1(H^*(X_1(g);\mathbb{Q}))_p = 0$ for $q \neq n$; as this presentation has relations only in degree $2n$ we also have Harr$_2(H^*(X_1(g);\mathbb{Q}))_n = 0$ for $q \neq 2n$. It follows that $\pi_*(\Omega X_1(g);\mathbb{Q})$ is supported in degrees $n - 1, 2n - 2, \ldots$, and $* \in \bigcup_{n \geq 1}[r(n - 1), rn - 3]$, which proves the first part.

To prove the claim about the $\overline{\mathcal{C}}^{[r]}_q$-invariants, we study the $\overline{\mathcal{C}}^{[r]}_q$-invariants of these Harrison homology groups in a range, using the presentation of $H^*(X_1(g);\mathbb{Q})$ described above, and a graphical interpretation of it. As we have already mentioned, relation (iii) shows that this ring is generated by the degree $n$ elements $k_1(v_1 \otimes v_2 \otimes v_3)$ for $v_1, v_2, v_3 \in H^\infty(W_{g,1};\mathbb{Q}) = H^\infty \cong H$. The representation $H^{*>1}$ is odd (in the sense of Section 2.3.1), so the representation $H^*(X_1(g);\mathbb{Q})$ has parity $k$ in degree $nk$. Thus $H^{*>0}(X;\mathbb{Q})^\otimes p$ has parity $k$ in degree $nk$ too, so

$$[\text{Harr}_p(H^*(X;\mathbb{Q}))_q]^{\overline{\mathcal{C}}^{[r]}}_q$$

must be supported in bidegrees $(p,q)$ with $q = 2kn$. The same analysis of degrees as in the first part of this proposition shows that $[\pi_*(\Omega X_1(g);\mathbb{Q})]^{\overline{\mathcal{C}}^{[r]}}_q$ is supported in degrees $2n - 2$ and $* \in \bigcup_{n \geq 2}[2k(n - 1), 2kn - 3]$, as claimed.

It remains to compute these invariants in degree $2n - 2$. To do this we will explicitly evaluate $[\text{Harr}_p(H^*(X;\mathbb{Q}))_q]^{\overline{\mathcal{C}}^{[r]}}_q$ for $q < 4n$. As described in Section 5 of [KRW20b] the ring presentation above may be considered graphically, with $k_1(v_1 \otimes v_2 \otimes \cdots \otimes v_n)$ represented by an $r$-valent corolla with legs labelled by the $v_i$, insertion of the form $\omega = \sum a_i \otimes a_i^\#$ represented by inserting an edge between legs, (iii) saying that edges between corollas can be contracted, and (iv) saying that a loop at a corolla makes the entire graph zero. Using this description and the fundamental theorem of invariant theory (see [KRW20b, Section 2.1.4]), we obtain the following description of the invariants

$$[H^{*>0}(X_1(g);\mathbb{Q})^\otimes p]^{\overline{\mathcal{C}}^{[r]}}_q$$

in a stable range. It is the vector space generated by $(\geq 3)$-valent graphs $\Gamma$ equipped with an ordering of the vertices, an ordering of the half-edges at each vertex, an orientation of the edges, and a colouring of each vertex in $\{1, 2, \ldots, p\}$ such that each colour is used at least once. Such graphs are subject to the relations:

(i) reversing the orientation of an edge yields $(-1)^n$ times the graph,

(ii) permuting the half-edges at a vertex, or the set of vertices, introduces a sign as in the proof of Theorem 5.1 of [KRW20b],

(iii) an ordered coloured graph is equivalent to that obtained by contracting an edge from the $k$th to the $(k + 1)$st vertex if they are coloured the same (the merged vertex is given this common colour),
We refer to Section 2.3.6 and Section 6.1 of [KRW20b] for background on symmetric π where the transposition (12) interchanges the two colours. We have

\[ \Theta = \begin{array}{ccc}
& h_3 & \\
v_1 & h_2 & h_5 \\
1 & h_1 & h_4 \\
v_2 & h_6 & \\
2 & \\
\end{array} \in [\mathcal{H}^*(X_1(g); \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]]^* \left[ \mathcal{O}_{n, \omega} \right]. \]

The vertices of this graph have degree \( 0 + n \cdot (\text{val} - 2) = n \). Dividing out shuffles to form the Harrison complex means dividing out by the subspace formed by \( \Theta - (12) \cdot \Theta \), where the transposition (12) interchanges the two colours. We have

\[ (12) \cdot \Theta = \begin{array}{ccc}
& h_3 & \\
v_1 & h_2 & h_5 \\
2 & h_1 & h_4 \\
v_2 & h_6 & \\
1 & \\
\end{array} = (-1)^{3n} \cdot (-1)^n \Theta \]

as to convert this oriented, ordered, coloured graph back to the form of \( \Theta \) we must reverse the orientation of 3 edges, and swap a pair of vertices of degree \( n \) (but we do not need to change the ordering of the half-edges at the vertices). Therefore, after dividing out by shuffles to form the Harrison complex, the class \( \Theta \) remains non-zero and is the only non-zero class in total degree \( q - p \leq 4n - 4 \). It is therefore a permanent cycle, which after dualising gives \( \pi_{2n-2}(\Omega X_1(g))_{\mathbb{Q}}[\pi]^{\mathbb{Q}} \cong \mathbb{Q} \).

We may use the fact that \( \pi_*(X_1(g))_{\mathbb{Q}} \) is supported in degrees \( n, 2n - 1, \) and \( 3n - 2 \) in the range \( * < 4n - 3 \) to determine these groups, as follows.

**Proposition 4.22.** For large enough \( g \) we have

\[ \pi_n(X_1(g))_{\mathbb{Q}} = V_{13} \quad \text{if } n \text{ is odd, } V_3 \text{ if } n \text{ is even,} \]
\[ \pi_{2n-1}(X_1(g))_{\mathbb{Q}} = V_0 + V_{12} + V_{22} \quad \text{if } n \text{ is odd, } V_0 + V_2 + V_{22} \text{ if } n \text{ is even,} \]
\[ \pi_{3n-2}(X_1(g))_{\mathbb{Q}} = V_{2,1} + V_{3,12}. \]

**Proof.** As discussed above, in terms of the notation of Section 5 of [KRW20b] we have \( \mathcal{H}^*(X_1(g); \mathbb{Q}) \cong R^V \) in a stable range, where \( V \) is the graded vector space \( \mathbb{Q}[0] \) with \( e = 0 \in V \). In Section 6 of [KRW20b] we explained how to determine the Hilbert–Poincaré series \( \text{ch}(R^V) \in \Lambda[[t]] \) of the rings \( R^V \) over the representation ring of \( G_{n, \omega} \). We refer to Section 2.3.6 and Section 6.1 of [KRW20b] for background of symmetric functions. Applying the aforementioned recipe to the situation at hand gives \( P_n(t^n) \) as the Hilbert–Poincaré series, where

\[ P_n(x) = D \left( \omega^n \left( \sum_{q=0}^{\infty} h_q \cdot \left( \sum_{p=0}^{\infty} h_p x^{p-2} \right) \right) \right) \in \Lambda[[x]], \]
with \( \omega: \Lambda \to \Lambda \) the involution given by transposing partitions, and \( D: \Lambda \to \Lambda \) the operator sending the Schur polynomial \( s_\lambda \) to \( s_\lambda(\Lambda) \). Explicitly, when \( n \) is odd (resp. \( n \) is even) this is the symplectic (resp. orthogonal) Schur polynomial \( sp_\lambda \) (resp. \( o_\lambda \)). Expanding out up to order \( 3n \), \( P_n(t^n) \) is then \( D(\omega^n(-)) \) applied to
\[
1 + s_3t^n + (s_4 + s_4,1 + s_6)t^{2n} + (s_9 + s_7,2 + s_6,3 + s_6,1 + s_5,2 + s_5,2,2 + s_5 + s_4,3 + s_4,1,1)t^{3n}.
\]

Now by the discussion above, the homology \( H_*(\Omega X_1(g); \mathbb{Q}) \) is supported in degrees \((n-1), 2(n-1), 3(n-1) \) in degrees \(* < 4n-10 \). It follows that in this range it behaves as the Koszul dual of \( H^*(X_1(g); \mathbb{Q}) \), allowing us to formally determine its Hilbert–Poincaré series in this range as \( 1/P_n(-t^{n-1}) \). As \( H_*(\Omega X_1(g); \mathbb{Q}) \) is the enveloping algebra of \( \pi_*(\Omega X_1(g))\mathbb{Q} \), by the Poincaré–Birkhoff–Witt theorem it has a filtration for which there is a natural isomorphism

\[
\text{Sym}^*(\pi_*(\Omega X_1(g))\mathbb{Q}) \cong \text{gr}H_*(\Omega X_1(g); \mathbb{Q})
\]

from the free graded-commutative algebra to the associated graded, and hence an exponential relationship between their Hilbert–Poincaré series. More precisely, if \( Q_n(t) \in \Lambda[[t]] \) denotes the Hilbert–Poincaré series of \( \pi_*(\Omega X_1(g))\mathbb{Q} \), then \( 1/P_n(-t^{n-1}) \) is identified with
\[
\begin{align*}
(n \text{ is odd}) & \quad (1 + h_1 + h_2 + h_3 + \cdots) \circ Q_n(t) \quad \text{if } n \text{ is odd}, \\
(n \text{ is even}) & \quad ((1 + e_1 + e_2 + \cdots) \circ Q_n(t))(1 + h_1 + h_2 + \cdots) \circ Q_n(t) \quad \text{if } n \text{ is even},
\end{align*}
\]

where \( Q_{2n}^\pm(t) = \frac{Q_{2n}(t) \pm Q_{2n}(-t)}{2} \) is the decomposition of \( Q_n(t) \) into even and odd parts. (This is simply implementing the free graded-commutative algebra on the object with Hilbert–Poincaré series \( Q_n(t) \), see Section 7.4 for a related discussion.) Reverting this operation, e.g. using SageMath [Sag20], gives the claimed result. \( \square \)

**Remark 4.23.** The even and odd answers differ precisely by the transposition \( \lambda \mapsto \lambda' \) of partitions, and this is not a coincidence as we now explain. Firstly the series \( P_n(x) \) depends only on the parity of \( n \), and the ring automorphism \( \omega: \Lambda \to \Lambda \) satisfies \( \omega(sp_\lambda) = o_\lambda \), so interchanges the even and odd versions of \( P_n \): we may call them \( P_{\text{even}} \) and \( P_{\text{odd}} \). Second, we have \( \omega(h_n \circ s_\lambda) = (\omega^h h_n) \circ s_\lambda \). This is because \( s_\mu \circ s_\lambda \) is the Frobenius character of the representation \( Ind_{\mathfrak{g}(\mu)}^{\mathfrak{g}(\lambda)} \mathfrak{g}(\lambda|\mu) \otimes (S^\mu \otimes (S^\lambda)|\mu|) \), and applying \( \omega \) corresponds to tensoring with the sign representation of \( \mathfrak{g}(\lambda|\mu) : \) the formula then follows by Frobenius reciprocity. Thus we also have \( \omega(h_n \circ sp_\lambda) = (\omega^h h_n) \circ o_\lambda \).

Let us define \( \overline{Q}_{\text{odd}}(x), \overline{Q}_{\text{even}}(x) \in \Lambda[[x]] \) so that (using the notation \( \overline{Q}_{\text{even}} \) for the even and odd parts)
\[
\frac{1}{P_{\text{odd}}(-x)} = (1 + h_1 + h_2 + \cdots) \circ \overline{Q}_{\text{odd}}(x),
\]
\[
\frac{1}{P_{\text{even}}(-x)} = (1 + e_1 + e_2 + \cdots) \circ \overline{Q}_{\text{even}}(x) \cdot (1 + h_1 + h_2 + \cdots) \circ \overline{Q}_{\text{even}}^+(x).
\]

The discussion above then shows that \( Q_n(t) = \overline{Q}_{\text{odd}}(t^{n-1}) \) (resp. \( \overline{Q}_{\text{even}}(t^{n-1}) \)), when \( n \) is odd (resp. even), in degrees up to \( t^3(n-1) \). If the algebra \( H^*(X_1(g); \mathbb{Q}) \) were Koszul in a stable range then this identity would hold in roughly the same range.

It is easy to see from the definition that the coefficient of \( x^r \) in \( P_{\text{odd}}(-x) \) is a sum of \( sp_\lambda \)'s with \( |\lambda| \equiv r \pmod{2} \), and hence to see from the identity above that \( \overline{Q}_{\text{odd}}(x) \) also has this property. Thus if \( \overline{Q}_{\text{odd}}(x) = \sum r_{\geq 0} c_r x^r \) with \( c_r \in \Lambda \) then
\[
(1 + h_1 + h_2 + \cdots) \circ \overline{Q}_{\text{odd}}(x) = \prod_{r \geq 0} \left( (1 + h_1 + h_2 + \cdots) \circ (c_r x^r) \right) = \prod_{r \geq 0} \left( \sum_{i \geq 0} (h_1 \circ c_r)x^{ri} \right)
\]
so applying $\omega$ gives

$$
\prod_{r \geq 0 \text{ odd}} \left( \sum_{i \geq 0} (c_i \circ \omega(c_r)) x^{r i} \right) \cdot \prod_{r \geq 0 \text{ even}} \left( \sum_{i \geq 0} (h_i \circ \omega(c_r)) x^{r i} \right) = \left( (1 + e_1 + e_2 + \cdots) \circ \omega(\bar{\omega}_{\text{odd}}(x)) \right) \left( (1 + h_1 + h_2 + \cdots) \circ \omega(\bar{\omega}_{\text{odd}}(x)) \right).
$$

This is identified with $\omega(1/P_{\text{odd}}(-x)) = 1/P_{\text{even}}(-x)$, showing that $\omega(\bar{\omega}_{\text{odd}}(x)) = \bar{\omega}_{\text{even}}(x)$ as claimed.

4.5. $X_0$ and Hirzebruch $L$-classes. Finally, we wish to interpret the effect on rational cohomology of the composition

$$
\text{BDiff}^f_{\partial}(D^{2n})_{\ell} \to \text{BTor}^f_{\partial}(W_{g,1})_{\ell} \to X_0,
$$

where $\text{BDiff}^f_{\partial}(D^{2n})_{\ell}$ is the cover corresponding to the image of $H_1(\text{BDiff}^f_{\partial}(D^{2n})_{\ell}) \to \pi_{2n+1}(S)$ and the first map is given by stabilisation. Specifically, in Lemma 4.8 we identified

$$
H^*(X_0; \mathbb{Q}) \cong Q[\sigma_{4j-2n-1} \mid j > n/2]
$$

where $\bar{\sigma}_i$ transgresses to $\sigma_{i+1} \in H_{i+1}(\mathbb{Q}_0^{\text{KH}}; \mathbb{Q}) \cong H^{i+1}(\mathbb{Q}_0^{\text{KH}}; \mathbb{Q})$, and $\sigma_{4j-2n}$ restricts to $\kappa_{\ell,j}$ in the cohomology of $B\text{Homeo}_0(W_{g,1})$. On the other hand, smoothing theory and the Alexander trick gives a map

$$
\text{BDiff}^f_{\partial}(D^{2n}) \to \Omega^{2n+1} \text{Top}(2n) \simeq \Omega^{2n+1} \text{BTop}(2n),
$$

which is a weak equivalence onto the path components that it hits. Thus $\text{BDiff}^f_{\partial}(D^{2n})_{\ell}$ is a covering space of one path component of the right hand side. The topological Hirzebruch $L$-classes $L_j \in H^j(\text{BTop}; \mathbb{Q})$ can be restricted to $\text{BTop}(2n)$, and then looped $(2n+1)$ times to give classes

$$
\Omega^{2n+1}L_j \in H^{4j-2n-1}(\Omega^{2n+1} \text{BTop}(2n); \mathbb{Q}).
$$

In these terms we can then phrase our comparison result, as follows.

**Theorem 4.24.** The composition (19) pulls back $\pi_{4j-2n-1}$ to the class $\Omega^{2n+1}L_j$.

**Proof.** The statement is entirely in rational cohomology, so we can neglect the various principal finite covers denoted $(-)$ and consider the diagram

$$
\text{BDiff}^f_{\partial}(D^{2n})_{\ell} \xrightarrow{j} \text{BTor}^f_{\partial}(W_{g,1})_{\ell} \xrightarrow{i} \text{BDiff}^f_{\partial}(W_{g,1})_{\ell} \xrightarrow{p^*} \text{BG}^f_{\partial}[\ell] \xrightarrow{\{\ast\}} \text{BG}^f_{\partial}[\ell]
$$

for $g$ large enough that the degree $4j-2n-1$ is in the stable range for both $\text{BDiff}^f_{\partial}(W_{g,1})_{\ell}$ and $\text{BG}^f_{\partial}[\ell]$. As $4j - 2n > 0$ we have the class

$$
p^* \sigma_{4j-2n} \in H^{4j-2n}(\text{BDiff}^f_{\partial}(W_{g,1})_{\ell}, \text{BTor}^f_{\partial}(W_{g,1})_{\ell}; \mathbb{Q})
$$

but in the stable range the $Q$-cohomology of $\text{BDiff}^f_{\partial}(W_{g,1})_{\ell}$ is trivial, which identifies this relative cohomology group with $H^{4j-2n-1}(\text{BTor}^f_{\partial}(W_{g,1})_{\ell}; \mathbb{Q})$ and the class $p^*(\sigma_{4j-2n})$ with (the pull back of) $\bar{\sigma}_{4j-2n-1}$. This is simply interpreting the definition of the transgression.

If $M^{4j-2n-1}$ is a stably framed manifold and $f: M \to \text{BTor}^f_{\partial}(W_{g,1})_{\ell}$ is a map, then $\langle \bar{\sigma}_{4j-2n-1}, f_*[M] \rangle$ is evaluated as follows. Choose a nullbordism

$$
F: W \to \text{BDiff}^f_{\partial}(W_{g,1}; \ell_{g})_{\ell}
$$

of the map $i \circ f$ (which is possible after perhaps replacing $[M, f]$ by a multiple in stably framed bordism, again because in the stable range the $Q$-cohomology of $\text{BDiff}^f_{\partial}(W_{g,1})_{\ell}$ is trivial) and then pull back the relative class $p^*(\bar{\sigma}_{4j-2n})$ along

$$(F, f): (W, M) \to (\text{BDiff}^f_{\partial}(W_{g,1})_{\ell}, \text{BTor}^f_{\partial}(W_{g,1})_{\ell})$$
and evaluate against the relative fundamental class of $W$.

On the other hand, if $M^{4j-2n-1}$ is a framed manifold and $g : M \to BDiff^0_0(D^{2n})\eta_0$ is a map classifying a smooth $(D^{2n}, \partial D^{2n})$-bundle

$$\pi : (E, M \times \partial D^{2n}) \to M$$

with a framing $\eta : T_\pi E \to \ell^{2n} \text{ extending } \ell_0$ on the boundary, then $\langle \Omega^{2n+1}L_j, g_*[M] \rangle$ may be evaluated as follows. Use the Alexander trick to obtain a topological $(D^{2n}, \partial D^{2n})$-bundle

$$\tilde{\pi} : (\tilde{E}, M \times [0, 1] \times \partial D^{2n}) \to M \times [0, 1]$$

agreed with $\pi$ over $M \times \{1\}$ and with the trivial bundle $M \times \{0\} \times D^{2n}$ over $M \times \{0\}$. The vertical tangent microbundle $\tau_\pi \tilde{E}$ is then trivialised over

$$\partial \tilde{E} = (M \times \{0\} \times D^{2n}) \cup (M \times [0, 1] \times \partial D^{2n}) \cup (E)$$

so that we have $L_j(\tau_\pi \tilde{E}) \in H^{4j}(\tilde{E}, \partial \tilde{E}; \mathbb{Q})$, and so may form

$$\int L_j(\tau_\pi \tilde{E}) \in H^{4j-2n}(M \times [0, 1], M \times \partial [0, 1]; \mathbb{Q}).$$

Evaluating against the relative fundamental class of $M \times [0, 1]$ gives $\langle \Omega^{2n+1}L_j, g_*[M] \rangle$.

Let us now put these two things together. Let $g : M \to BDiff^0_0(D^{2n})\eta_0$ be given, and consider the topological $(W_{g,1}, \partial W_{g,1})$-bundle

$$\tilde{E}_M \times [0, 1] \to M \times [0, 1]$$

obtained by forming the fibrewise boundary connect sum of $\tilde{E}$ and the trivial $(W_{g,1}, \partial W_{g,1})$-bundle. Over $M \times \{1\}$ this is the smooth bundle $E_2(M \times W_{g,1}) \to M$, and is equipped with a framing of its vertical tangent bundle, which is classified by $f := g \circ j$. As in the second paragraph above, there is a null-bordism $W$ of $M = M \times \{1\}$ and an extension of $E_2(M \times W_{g,1}) \to M$ to a smooth framed $(W_{g,1}, \partial W_{g,1})$-bundle $T \to W$. In total we can consider the topological $(W_{g,1}, \partial W_{g,1})$-bundle

$$\tilde{E} = \tilde{E}_M \times [0, 1] \cup E_2(M \times W_{g,1}) \cup_{E_2(M \times W_{g,1})} T \to (M \times [0, 1]) \cup_{M \times \{1\}} W$$

which is trivialised over $M \times \{0\}$.

The map $p \circ (F, j) : (W, M) \to (BG_{fr,\ell}, \ast)$ classifying the local system of middle cohomology groups extends constantly to a map

$$L : \left((M \times [0, 1]) \cup_{M \times \{1\}} W, M \times \{0\}\right) \to (BG_{fr,\ell}, \ast),$$

because over $M \times \{0, 1\}$ the bundle is obtained from a disc bundle so the local system is canonically trivialised. Thus

$$\langle \sigma_{4j-2n-1}, f_*[M] \rangle = \langle (p \circ (F, j))^* \sigma_{4j-2n}, [W] \rangle = \langle L^* \sigma_{4j-2n}, [(M \times [0, 1]) \cup_{M \times \{1\}} W] \rangle.$$

As the topological $(W_{g,1}, \partial W_{g,1})$-bundle $\tilde{E}$ is trivialised over the boundary $M \times \{0\}$ of the base, by the Family Signature Theorem (Theorem A.2) we have

$$\langle L^* \sigma_{4j-2n}, [(M \times [0, 1]) \cup_{M \times \{1\}} W] \rangle = \left( \int L_j(\tau_{\tilde{E}}), [(M \times [0, 1]) \cup_{M \times \{1\}} W] \right).$$

On the other hand, the vertical tangent bundle of $\tilde{E}$ over $T$ and $M \times [0, 1] \times W_{g,1}$ is trivialised, which as required identifies this with

$$\left( \int L_j(\tau_{\tilde{E}}), [M \times [0, 1]] \right) = \langle \Omega^{2n+1}L_j, g_*[M] \rangle.$$

This computation has the following consequence regarding the rationalisations of the nilpotent spaces in Lemma 4.9.

**Lemma 4.25.** For $g$ sufficiently large, all spaces in the fibration sequence

$$X_1(g) \to B\text{Tor}_g(W_{g,1}) \to X_0$$

are rationally simple.
Proof. By Lemma 4.18, for \( g \) large enough \( X_1(g) \) has finite fundamental group so after rationalisation it is simply-connected. So let us for now take \( g \) sufficiently large. The space \( X_0 \) is a loop space so is simple, even before rationalising. The rationalised fundamental group of \( X_0 \) may be obtained from Lemma 4.8: it is nontrivial if \( n \) is even, and \( \mathbb{Q} \) if \( n \) is odd. Thus if \( n \) is even then \( B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \) is rationally simply-connected.

If \( n \) is odd then we have

\[
\pi_1(B\text{Tor}_\mathcal{O}(W_{g,1})_\ell)_\mathbb{Q} \cong \pi_1(X_0)_\mathbb{Q} \cong \mathbb{Q}.
\]

We consider then the fibre sequence

\[
\mathrm{Diff}^g(D^{2n})_{\ell_0} \to B\text{Diff}^g(D^{2n})_{\ell_0} \to B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \to X_0
\]

obtained from Theorem 3.12 by looping, restricting to certain path components, and taking certain finite covering spaces (the covering space on the right is chosen so that the rightmost map is \( \pi_1 \)-surjective, and hence the fibre is path-connected). To understand the map \( s \), we consider the composition

\[
\Omega^{2n}_0 \mathcal{O}(SO(2n)) \to \Omega^{2n}_0 \text{Top}(2n) \cong B\text{Diff}^g(D^{2n})_{\ell_0} \to B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \to X_0
\]

on \( H^1(\mathbb{Z}; \mathbb{Q}) \). By Theorem 4.24 this pulls back \( \tilde{\sigma}_1 \) to \( \Omega^{2n+1}_0 \mathcal{L}(n+1)/2 \in H^*(\mathcal{O}(SO(2n)); \mathbb{Q}) \) is indecomposable. It follows that the map \( s \) is surjective on \( \pi_1(\mathbb{Z}) \). As the fibration (20) deloops, \( \pi_1(B\text{Diff}^g(D^{2n})_{\ell_0}) \) acts trivially on the higher homotopy groups of \( B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \), so that it is rationally simple. \( \Box \)

Remark 4.26. In fact, the conclusion of Lemma 4.25 holds without the assumption that \( g \) is sufficiently large. As this argument uses Corollary D, we give it separately to make clear that the reasoning is not circular.

Following (7) and (8), we let \( \mathcal{T}_g^{r,\ell} \) and \( \mathcal{T}_g^{fr,\ell} \) denote the fundamental groups of the middle and right terms in (20). The long exact sequence of homotopy groups of that fibration sequence induces an exact sequence

\[
\pi_1(\Omega^{2n}_0 \text{Top}(2n)) \to \mathcal{T}_g^{r,\ell} \to \mathcal{T}_g^{fr,\ell} \to 0.
\]

Now \( \mathcal{T}_g^{fr,\ell} \) is a subgroup of \( \mathcal{L}_g^{fr,\ell} \), so is finite by Lemma 3.10. Thus the map \( s \) is rationally surjective. By Corollary D, the map \( \pi_{2n+1}(\mathcal{O}(SO(2n)))_\mathbb{Q} \to \pi_{2n+1}(\text{Top}(2n))_\mathbb{Q} \) is an isomorphism and hence the latter is 1-dimensional if \( n \) is odd and trivial if \( n \) is even.

If \( n \) is even it follows that \( \mathcal{T}_g^{fr,\ell} \) is finite, so \( B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \) is rationally simply-connected. On the other hand, \( \pi_{2}(X_0)_\mathbb{Q} = 0 \) so we also have \( \pi_1(X_1(g))_\mathbb{Q} = 0 \), so \( X_1(g) \) is rationally simply-connected too.

If \( n \) is odd then as in the proof of Lemma 4.25 the composition

\[
\pi_1(\Omega^{2n}_0 \mathcal{O}(SO(2n)))_\mathbb{Q} \xrightarrow{\cong} \pi_1(\Omega^{2n}_0 \text{Top}(2n))_\mathbb{Q} \xrightarrow{s} (\mathcal{T}_g^{fr,\ell})_\mathbb{Q} \to \pi_1(X_0)_\mathbb{Q}
\]

is an isomorphism, and so the map \( s \) is an isomorphism (as it is both surjective and injective). Using \( \pi_{2}(X_0)_\mathbb{Q} = 0 \) again it follows that \( \pi_1(X_1(g))_\mathbb{Q} = 0 \), so \( X_1(g) \) is again rationally simply-connected. Finally, as the image of \( s \) acts trivially on the higher rational homotopy groups, the space \( B\text{Tor}_\mathcal{O}(W_{g,1})_\ell \) is again rationally simple.

5. The homotopy groups of framed self-embeddings

Theorem 3.12 provided the crucial link between the diffeomorphisms of discs, the diffeomorphisms of \( W_{g,1} \), and the self-embeddings of \( W_{g,1} \). In the previous section we computed \( \pi_1(B\text{Tor}_g^B(W_{g,1})_\ell)_\mathbb{Q} \), and in this section we compute \( \pi_1(B\text{Tor}_\mathcal{O}(g)^{fr}_\ell(W_{g,1})_\ell)_\mathbb{Q} \). Throughout this section we make the standing assumption that \( n \geq 3 \) and \( g \geq 2 \).
5.1. Embedding calculus. In this section we give a minimalist summary of embedding calculus, outsourcing most to Appendix B. Through the device of [KRW19, Section 3.3.3], it applies to self-embeddings of $W_{g,1}$ relative to $1/2\partial W_{g,1}$ and framings. This provides a tower of approximations

\begin{equation}
\cdots \to B\text{Emb}_{1/2\partial}(W_{g,1}) \to BT_k\text{Emb}_{1/2\partial}(W_{g,1}) \times \to BT_{k-1}\text{Emb}_{1/2\partial}(W_{g,1}) \times \to \cdots
\end{equation}

For the top term, we use that every self-embedding of $W_{g,1}$ relative to $1/2\partial W_{g,1}$ is invertible up to isotopy, by the $h$-cobordism theorem. By the convergence results stated in Appendix B, the map

$$B\text{Emb}_{1/2\partial}(W_{g,1}) \to BT_k\text{Emb}_{1/2\partial}(W_{g,1}) \times$$

is $(-n+2)+k(n-2)$-connected and thus we have an equivalence

$$B\text{Emb}_{1/2\partial}(W_{g,1}) \to \text{holim}_{k \to \infty} BT_k\text{Emb}_{1/2\partial}(W_{g,1}) \times.$$

We will approach $B\text{Emb}_{1/2\partial}(W_{g,1})$ by giving a description of the first stage and then, for each $k \geq 2$, a description of the $k$th layer $B\text{Emb}_{1/2\partial}(W_{g,1})^{\times}$.

5.2. The rational homotopy of framed self-embeddings. In the remainder of this section, we use embedding calculus to study the rational homotopy groups

$$\pi_*(B\text{Emb}_{1/2\partial}(W_{g,1})\ell)$$

as representations. (In the cases $* = 0, 1$, rationalisation does not make sense and the path components and fundamental group are understood well enough for our applications.) By the proof of Theorem 8.3 of [KRW19], these are $gr$-algebraic representations of the fundamental group $\hat{\pi}_{*+d}^{gr}$ with respect to (8). By Corollary 3.11, these descend to algebraic representations of $G_{g,[\ell]}$.

Associated to the tower (22) there is a Bousfield–Kan spectral sequence as in Section 2.4. It computes the homotopy groups of $B\text{Emb}_{1/2\partial}(W_{g,1})\ell$ in terms of those of the layers. We postpone discussion of this spectral sequence, and first explain how to compute the rational homotopy groups of the layers.
5.2.1. The first layer. In Lemma 5.1 we identified the first layer with the classifying space of the homotopy automorphisms of \( W_{g,1} \) relative to \( J/\partial W_{g,1} \). In Section 5.3 we identify the rational homotopy groups
\[
\pi_*(\text{BHom}_{J/\partial W_{g,1}}(W_{g,1}))_Q
\]
for \( * > 1 \).
with the derivation Lie algebra of \( L(H[n-1]) \). This implies that
\[
\pi_*(\text{BHom}_{J/\partial W_{g,1}}(W_{g,1}))_Q = 0 \quad \text{unless} \quad * = r(n-1) + 1 \quad \text{for} \quad r > 0.
\]
Each of these is an algebraic \( \underline{\pi}_{g,[r]} \)-representation, and is odd (resp. even) if and only if \( r \) is odd (resp. even). Thus non-trivial invariants can occur only when \( r \) is even. In degrees \( * \leq 4n-10 \), the only non-trivial invariants for \( g \) sufficiently large are given by
\[
\dim \left[ \pi_{2n-1}(\text{BHom}_{J/\partial W_{g,1}}(W_{g,1}))_Q \right]_{\pi_{g,[r]}} = 1.
\]
5.2.2. The higher layers. In Section 5.7 we will give upper bounds on
\[
\pi_*(\text{BL}_k \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q) = \pi_{*-1}(\text{L}_k \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q)_Q
\]
for \( * > 2 \). Using the description of the layers of the embedding calculus tower as section spaces, in [KRW19, Section 5.3] we gave a Federer spectral sequence
\[
\mathcal{E}^2_{p,q} \implies \pi_{q-p}(\text{L}_k \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q).
\]
This is an instance of a strongly convergent Bousfield–Kan spectral sequence. It comes with action of \( \Gamma_g \) by naturality, which factors over \( \Lambda_g \) and restricts to \( \tilde{\Lambda}^g_{r,t} \).
In [KRW19, Lemma 5.4] we described the rationalised \( E^2 \)-page of the Federer spectral sequence as
\[
(F\mathcal{E}^2_{p,q})_Q = \left[ H^q(W^k_{g,1}, \Delta_{J/\partial W_{g,1}})_Q \otimes \pi_q(\text{tohtfib} \text{Emb}(k \setminus I, W_{g,1}))_Q \right]_{\Gamma_g}.
\]
We will explain both terms on the right hand side in due time, for now we note they are algebraic \( \underline{\pi}_{g,[r]} \times \mathfrak{S}_k \)-representations. This spectral sequence converges completely, so for \( q-p > 1 \), \( \pi_{q-p}(\text{L}_k \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q) \) admits a finite filtration whose associated graded consists of subquotients of the terms \( (E^2_{p,q})_Q \) with \( q' - p' = q-p \). Note that we do not “rationalise” the extended spectral sequence, which might not make sense for \( q-p = 0,1 \). Information in these degrees is unnecessary as we understand \( \pi_0 \) and \( \pi_1 \) of \( \text{BEmb}_{J/\partial W_{g,1}}(W_{g,1})_Q \) well enough.
We shall explain and then compute each of the terms in this expression in Sections 5.5 and 5.6 respectively, and take the symmetric group invariants of their tensor products. Let us state our two main results about the higher layers, proven in Section 5.7. We firstly obtain a qualitative result, Proposition 5.32, which we summarise by making a statement true for all \( k \geq 2 \) and \( * \geq 2 \):
\[
\pi_*(\text{BL}_k \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q) = 0 \quad \text{unless} \quad * = r(n-1) - k + 2 \quad \text{for} \quad r \geq k - 2.
\]
These are \( gr \)-algebraic \( \tilde{\Lambda}^g_{r,t} \)-representations with respect to (8), and descend to algebraic \( \underline{\pi}_{g,[r]} \)-representations by Corollary 3.11.
These representations are odd (resp. even) if and only if \( r \) is odd (resp. even), so there can only be non-zero invariants when \( r \) is even. We do computations for the case \( 2r = 2 \): in Computations 5.33, 5.34, and 5.35, for \( g \) sufficiently large we get
\[
\dim \left[ \pi_*(\text{BL}_2 \text{Emb}_{J/\partial W_{g,1}}(W_{g,1})_Q) \right]_{\pi_{g,[r]}} = \begin{cases}
0 & \text{if} \quad * < 4n-4,
1 & \text{if} \quad * = 2n-3,
0 & \text{if} \quad * < 4n-5, \neq 2n-3,
1 & \text{if} \quad * = 2n-4,
0 & \text{if} \quad * < 4n-6, \neq 2n-4.
\end{cases}
\]
In Section 7 we explain a procedure to compute the rational homotopy groups of the layers and in Appendix C we give the results for $* < 5n - 12$.

5.2.3. The Bousfield–Kan spectral sequence. The Bousfield–Kan spectral sequence associated to the tower of fibrations (22) has the form ([BK72, Ch. IX. 4], cf. Section 2.4 or [KRW19, Section 5.1])

\[
\pi_{p,q}(B\text{Emb}^\delta_{l/2\partial}(W_{g,1})) \quad \text{if } p = 0,
\]

\[
\pi_{q-p}(B\text{L}_{p+1}\text{Emb}_{l/2\partial}(W_{g,1})) \quad \text{if } p \geq 1.
\]

As the connectivity of the layers tends to infinity with $n$, this spectral sequence converges completely to $\pi_{q-p}(B\text{Emb}^\delta_{l/2\partial}(W_{g,1}))$. In particular, for $q - p \geq 2$ the rationalisation of this homotopy group has a finite filtration with associated graded given by subquotients of the vector spaces $(BKE^1_{p,q})\mathbb{Q}$. Furthermore, this spectral sequence comes equipped with an action of

\[
\Lambda^r_\ell = \pi_1(B\text{Emb}^\delta_{l/2\partial}(W_{g,1}), \ell)
\]

which converges to the corresponding action of this fundamental group on the higher homotopy groups. It is a spectral sequence of $gr$-algebraic representations of $\Lambda^r_\ell$ with respect to (8), and by Corollary 3.11 these descend to algebraic representations of $\Gamma^r_\ell$. In Sections 5.2.1 and 5.2.2 we described $(BKE^1_{p,q})\mathbb{Q}$ qualitatively, immediately giving:

**Proposition 5.2.** The groups $(BKE^1_{*,*})\mathbb{Q}$ are supported in bidegrees $(p,q)$ with $q = r(n-1) + 1$ and $r \geq p - 1$. Furthermore, the $\Gamma^r_\ell$-invariants $[BKE^1_{*,*}]_{\mathbb{Q}}$ are supported in such bidegrees with $r$ even.

Let us remark on the geometry of this spectral sequence. This proposition says that $(BKE^1_{*,*})\mathbb{Q}$ is supported in bidegrees $(p,q)$ lying in the intervals $[0, r+1] \times \{r(n-1) + 1\}$ with $r \geq 1$. (In Figures 2 and 4 these are displayed as diagonal intervals of slope $-1$, as those charts plot $p$ against $q - p$.) Such intervals contribute to total degrees $[r(n-1), r(n-1) + 1]$, and contribute $\Gamma^r_\ell$-invariants to $(BKE^1_{*,*})\mathbb{Q}$ only if $r$ is even. These provide part of the “bands” discussed in the introduction, as follows.

**Corollary 5.3.** The rational homotopy groups

\[
\pi_*(B\text{Emb}^\delta_{l/2\partial}(W_{g,1}))\mathbb{Q} \quad \text{for } * > 1,
\]

are supported in degrees $* \in \bigcup_{r \geq 1}[r(n-2), r(n-1) + 1]$. 
Furthermore, for \( q \geq 2 \) the non-zero \( \overline{\mathcal{G}}_{r}^{s,[l]} \)-invariants in these groups are supported in degrees \( s \in \bigcup_{r \geq 1} [2r(n - 2), 2r(n - 1) + 1] \).

**Proof.** The discussion above proves the first part, and proves that \([\mathcal{B}(K_{r}, s)]_{Q}^{\overline{\mathcal{G}}_{r}^{s,[l]}}, \mathcal{L}_{s} \) is supported in bidegrees \((p, q)\) with \( p - q \in [2r(n - 2), 2r(n - 1) + 1] \) for some \( r \geq 1 \).

As long as \( g \geq 2 \) extensions of algebraic representations split, as discussed in Section 2.3, and so taking \( \overline{\mathcal{G}}_{r}^{s,[l]} \)-invariants is exact: this gives the desired conclusion about \([\pi_{*}(B\text{Emb}_{1/2,0}(W_{g,1}), \mathcal{L})]_{Q}^{\overline{\mathcal{G}}_{r}^{s,[l]}}, \mathcal{L} \).

As \( n \) increases, more of the intervals \([2r(n - 2), 2r(n - 1) + 1] \) are disjoint. Because the \( d' \)-differentials have bidegree \((r, r - 1) \) (in the indexing of Figures 2 and 4 this corresponds to \((-1, r)\)), in a range where these intervals overlap by at most one degree, the Bousfield–Kan spectral sequence collapses rationally at the \( E^{2} \)-page. In this range the computation of \([\pi_{*}(B\text{Emb}_{1/2,0}(W_{g,1}), \mathcal{L})]_{Q}^{\overline{\mathcal{G}}_{r}^{s,[l]}}, \mathcal{L} \) reduces to computing the homology of certain chain complexes.

The explicit computations done in Sections 5.2.1 and 5.2.2 determine the first such chain complex, corresponding to the second band. The result is displayed in Figure 2 for \( n \) sufficiently large. This is an accurate depiction as long as the fourth band does not overlap: explicitly, \( 2n < 4n - 8 \), or equivalently \( 2n \geq 10 \). We will determine the differential of this chain complex (somewhat indirectly) in Section 6.2. When \( 2n = 6 \) or 8, the range of degrees shown in Figure 2 should also include contributions from the fourth band \([4n - 8, 4n - 3] \). When \( 2n = 6 \) there could even potentially be a non-trivial higher differential in this range, but it can be ruled out by the method of Section 6.5, or using Corollary 5.12 below.

5.3. The rational homotopy of the homotopy automorphisms. The first layer, \( B\text{hAut}_{1/2,0}(W_{g,1}) \) is equivalent to the classifying space of the pointed homotopy automorphisms of \( \vee_{2g} S^{n} \). For use in Section 7.5, we will include information not only for \( 2r = 2 \), but also \( 2r = 4 \). Even though \( H^{\vee} \) can be \( G_{r} \)-equivariantly identified with \( H \), for Section 6.5 it is useful to distinguish the cohomology group \( H^{\vee} = H^{n}(W_{g,1}; \mathbb{Q}) \) from its linear dual \( H = H_{n}(W_{g,1}; \mathbb{Q}) \).

5.3.1. A description in terms of derivation Lie algebras. By the Hilton–Milnor theorem, for \( n \geq 2 \) the rational homotopy Lie algebra of \( \vee_{2g} S^{n} \) is given by

\[
\pi_{*+1}(\vee_{2g} S^{n})_{\mathbb{Q}} \cong L[H[n-1]],
\]

the free Lie algebra on the graded vector space \( H[n - 1] \) with \( H = H_{n}(W_{g,1}; \mathbb{Q}) \).

**Definition 5.4.** For a graded Lie algebra \( \mathcal{L} \) we let \( \text{Der}(\mathcal{L}) \) denote the graded vector space which in grading \( i \) consists of all degree \( i \) derivations (that is \( \mathbb{Q} \)-linear maps \( \phi : \mathcal{L} \to \mathcal{L} \) of degree \( i \) satisfying \( \phi([a, b]) = [\phi(a), b] + (-1)^{|a||b|}[a, \phi(b)] \)). It is equipped with a Lie bracket given by \([\phi, \psi] = \phi \circ \psi - (-1)^{|a||b|} \psi \circ \phi \). We let \( \text{Der}^{*}(\mathcal{L}) \) denote the sub-Lie algebra of those derivations of strictly positive degree.

The Lie algebra \( \text{Der}^{*}(L[H(n - 1)]) \) can be considered as a dg Lie algebra with trivial differential. As such, it captures the rational homotopy type of \( B\text{hAut}^{\text{id}}_{+}(\vee_{2g} S^{n}) \) because it is a Quillen model for the classifying space of the identity component of the pointed homotopy automorphisms. Thus we can use it to compute the rational homotopy groups of \( B\text{hAut}_{1/2,0}(W_{g,1}) \) [BM20, Corollary 3.3]:

**Lemma 5.5.** In positive degrees, we have an isomorphism of graded vector spaces

\[
\pi_{*}(B\text{hAut}_{1/2,0}(W_{g,1}))_{\mathbb{Q}} \cong \text{Der}^{*}(L[H[n-1]])
\]

Since \( L[H[n - 1]] \) is a free Lie algebra, a derivation \( \phi : L[H[n - 1]] \to L[H[n - 1]] \) is completely and uniquely determined by its restriction to the generators \( H[n - 1] \). Thus additively, we have an identification of graded vector spaces

\[
\pi_{*}(B\text{hAut}_{1/2,0}(W_{g,1}))_{\mathbb{Q}} \cong (H[n - 1])^{V} \otimes L[H[n - 1]]
\]
The action of the $\pi_0(\text{hAut}_{1/2}(W_{g,1})) = \text{GL}_2(\mathbb{Z})$ on these higher homotopy groups is in the evident manner, via its action on $H$. In particular, by restriction the rational homotopy groups of $\text{hAut}_{1/2}(W_{g,1})$ are algebraic $\mathcal{G}^{fr,[l]}_g$-representations. Since in degree $r(n-1)$ the derivation Lie algebra of $L(H[n-1])$ is a subquotient of a direct sum of copies of $H^{\otimes r+2}$, it is an odd (resp. even) such representation if and only if $r$ is odd (resp. even). We conclude that:

**Lemma 5.6.** For $s > 1$, we have that

$$\pi_0(\text{BhAut}_s(W_{g,1}))_\mathbb{Q} = 0$$

unless $s = r(n-1) + 1$ for $r > 0$.

Furthermore, these algebraic $\mathcal{G}^{fr,[l]}_g$-representations and can only contain non-zero invariants when $r$ is even.

### 5.3.2. A decomposition into Schur functors

As a $\text{GL}_2(\mathbb{Z})$-representation the free graded Lie algebra is given by

$$L(H[n-1])_{s(n-1)} = \text{Lie}(s) \otimes_{S_s} (H[n-1])^{\otimes 2},$$

where we use the notation $\otimes$ to identify the $S_s$-action on the right term by permutation of the terms in the tensor product (with the Koszul sign rule). Here $\text{Lie}(s)$ denotes the Lie representation of the symmetric group $S_s$. This is the subspace of the free Lie algebra $L[\mathbb{Q}\{x_1, \ldots, x_s\}]$ on generators in degree 0 spanned by those Lie words in which each generator appears exactly once. For $s \leq 6$, they are given in Table 1.

We can incorporate the Koszul sign rule into this representation to give an expression of the free Lie algebra in terms of Schur functors, i.e. write it as a direct sum of functors as in Section 2.3.3:

$$L(H[n-1])_{s(n-1)} = (\text{Lie}(s) \otimes (1^s)^{\otimes n-1}) \otimes_{S_s} H^{\otimes 2}.$$

On restricting to the subgroup $G_2 \leq \text{GL}_2(\mathbb{Z})$ of automorphisms of $H_n(W_{g,1}; \mathbb{Z})$ preserving its intersection form $\lambda$, we may use the duality $x \mapsto \lambda(x, -): H[1-n] \xrightarrow{\sim} (H[-n])^\vee$ to express

$$\text{Der}^+(L(H[n-1])) \cong (H[n-1])^\vee \otimes L(H[n-1]) \cong H[1-n] \otimes L(H[n-1])$$

in terms of Schur functors of $H$, as

$$\text{Der}^+(L(H[n-1]))_{s(n-1)} \cong \text{Der}^n(s + 2) \otimes_{S_{s+2}} H^{\otimes s+2},$$

where the $S_{s+2}$-representation $\text{Der}^n(s + 2)$ depends only on the parity of $n$, and is given by the induced representation

$$\text{Der}^n(s + 2) := \text{Ind}_{S_{s+1}}^{S_{s+2}} (\text{Lie}(s + 1) \otimes (1^s)^{\otimes n-1}) = (\text{Ind}_{S_{s+1}}^{S_{s+2}} \text{Lie}(s + 1)) \otimes (1^s)^{\otimes n-1}.$$

Although both sides of (25) are $\text{GL}_2(\mathbb{Z})$-representations, this isomorphism is only one of $G_2$-representations; it will nonetheless be convenient to make use of the structure of the right-hand side as a $\text{GL}_2(\mathbb{Z})$-representation.

| $s$ | $\text{Lie}(s)$ |
|-----|----------------|
| 1   | $(1)$          |
| 2   | $(1^2)$        |
| 3   | $(2, 1)$       |
| 4   | $(3, 1) + (2, 1^2)$ |
| 5   | $(4, 1) + (3, 2) + (3, 1^2) + (2^2, 1) + (2, 1^3)$ |
| 6   | $(5, 1) + (4, 2) + (2(4, 1^2) + (3^2) + 3(3, 2, 1) + (3, 1^3) + 2(2^2, 1^2) + (2, 1^4)$ |

Table 1. The Lie representations, [Thr42, pages 387–388]. Thrall also gives the cases $7 \leq s \leq 9$ (the case $s = 10$ was corrected in [Bra44]), which we will not reproduce here.
Using this information, we can determine the decomposition into irreducibles of $\text{Der}^+(\mathbb{L}(H[n - 1]))(n-1)$ as an algebraic $G_g$-representation from Table 1.

**Computation 5.7.** For $n$ both odd or even we have

\[
\begin{align*}
\text{Der}^+(\mathbb{L}(H[n - 1]))(n-1) &= S_{1^3} + S_{2,1}, \\
\text{Der}^+(\mathbb{L}(H[n - 1]))_{2(n-1)} &= S_{3,1} + S_{2^2} + S_{2,1^2}, \\
\text{Der}^+(\mathbb{L}(H[n - 1]))_{3(n-1)} &= S_{2,1^3} + S_{2^2,1} + 2S_{3,1,1} + S_{3,2} + S_{4,1}, \\
\text{Der}^+(\mathbb{L}(H[n - 1]))_{4(n-1)} &= S_{2,1^4} + 2S_{2^2,1^2} + S_{2^3} + 3S_{3,1^3} + 3S_{3,2,1} + 2S_{4,1,1} + S_{3^2} + 2S_{4,2} + S_{5,1},
\end{align*}
\]

as $G_g$-representations, expressed in terms of Schur functors of $H$.

We may restrict these to $G_g^{id}$ and decompose them into irreducibles for sufficiently large $g$ (we used SageMath [Sag20]). For $n$ both odd or even, we find

\[
\begin{align*}
\dim \pi_{2n-1}(\text{BH}^*(W_g))_Q &= 1, \\
\dim \pi_{4n-3}(\text{BH}^*(W_g))_Q &= 3.
\end{align*}
\]

5.4. **The rational homotopy of the homotopy automorphisms rel boundary.**

Since every self-embedding of $W_{g,1}$ rel $\partial W_{g,1}$ extends uniquely to a homeomorphism of $W_{g,1}$ rel $\partial W_{g,1}$, by the Alexander trick, there is a factorisation

\[\text{BEmb}_{g,0}(W_{g,1}) \to \text{BH}^0(W_{g,1}) \to \text{BH}^*(W_{g,1}).\]

In particular, any contribution of $\pi_*(\text{BH}^*(W_{g,1}))_Q$ to the homotopy of the framed self-embeddings must be in the image of $\pi_*(\text{BH}^0(W_{g,1}))_Q$. The latter rational homotopy groups also admits a complete description, which we will give following [BM20]. The boundary inclusion $S^{2n-1} \to W_{g,1}$ is represented by $(-1)^n$ times\(^2\) the element

\[\omega = \sum_{i=1}^g [f_i, e_i] \in \pi_{2n-1}(W_{g,1})_Q = \mathbb{L}^2(H[n - 1]) \subset H^\otimes 2.\]

It follows from [BM20, Proposition 5.6] that BH$^0_{g,0}(W_{g,1})$ has as Quillen model the Lie algebra $\text{Der}^+_0(\mathbb{L}(H[n - 1]))$ of those positive degree derivations which annihilate $\omega$, considered as dg Lie algebra with trivial differential.

It is helpful to recall a more concrete description: on [BM20, page 40] the graded vector space $\text{Der}^+_0(\mathbb{L}(H[n - 1]))$ is identified, as a $\pi_0(\text{H}^0(W_{g,1}))$-representation, with the kernel of the surjective linear map given by bracketing:

\[
\ker \left[ H[n - 1] \otimes \mathbb{L}(H[n - 1]) \xrightarrow{[-,-]} \mathbb{L}^2(H[n - 1]) \right] [-2(n - 1)].
\]

5.4.1. **A decomposition into Schur functors.**

As the bracketing map in (26) is surjective, we can extract a decomposition of (26) into Schur functors of $H$ from the discussion in Section 5.3.2. Computation 5.7 and Table 1 give the following.

**Computation 5.8.** For $n$ odd we have

\[
\begin{align*}
\text{Der}^+_1(\mathbb{L}(H[n - 1]))(n-1) &= S_{1^3}, \\
\text{Der}^+_2(\mathbb{L}(H[n - 1]))_{2(n-1)} &= S_{2^2}, \\
\text{Der}^+_3(\mathbb{L}(H[n - 1]))_{3(n-1)} &= S_{3,1^2}, \\
\text{Der}^+_4(\mathbb{L}(H[n - 1]))_{4(n-1)} &= S_{2^3} + S_{3,1^3} + S_{4,2}.
\end{align*}
\]

\(^2\)The conventions of the paper [BM20] have $\omega := \sum_{i=1}^g [e_i, f_i]$. The difference comes from that paper using $x \mapsto \lambda(-, x)$ to identify $H$ with $H^*$, whereas we use $x \mapsto \lambda(x, -)$. It makes no difference to the results we wish to use.
as $G_2$-representations, expressed in terms of Schur functors of $H$. For $n$ even we have the same but with the partitions transposed.

We may restrict these to $\overline{g}^{[i]}$ and decompose them into irreducibles for sufficiently large $g$ (we used SageMath [Sag20]). For $n$ both odd or even, we find

$$\dim [\pi_{2n-1}(\text{BhAut}_0(W_{g,1}))_q]_{\overline{g}^{[i]}} = 1,$$

$$\dim [\pi_{4n-3}(\text{BhAut}_0(W_{g,1}))_q]_{\overline{g}^{[i]}} = 0.$$

**Remark 5.9.** For odd $n$, the Lie algebra $\text{Der}^+_g(L(H[n-1]))$ is, up to regrading, the same as Morita–Sakasai–Suzuki’s $\mathfrak{h}_{g,1}$. In [MSS15], they give a description of the $\text{Sp}_{2g}(\mathbb{Z})$-invariants and decomposition into $V_\lambda$ in a range.

### 5.4.2. Relation to $X_1(g)$

It follows from the previous section that the lowest non-trivial rational homotopy group of $\text{BhAut}_0(W_{g,1})$ is given by

$$\pi_n(\text{BhAut}_0(W_{g,1}))_q \cong \text{Der}^+_g(L(H[n-1]))_{n-1}$$

and by (26), we have an identification of $\text{Der}^+_g(L(H[n-1]))_{n-1}$ as the kernel of $[-, -] : H[n-1] \otimes L^2(H[n-1]) \to L^2(H[n-1])$. By Computation 5.8, this kernel is given by $S_{13}(H[n-1])$, which is isomorphic to $\text{Sym}^n(H)$ if $n$ is even and $\Lambda^3(H)$ when $n$ is odd.

In Section 4.3 we have calculated the rational cohomology of the nilpotent space $X_1(g)$. The lowest non-zero rational cohomology group is the $n$th, and unravelling the description in Theorem 4.11, as we did in the proof of Proposition 4.21, we see that $H^n(X_1(g); \mathbb{Q})$ is given by $S_{13}(H[n-1])$, and so $\pi_n(X_1(g))_q$ is too.

**Proposition 5.10.** For large enough $g$, the composition

$$\pi_n(X_1(g))_q \to \pi_n(B\text{Diff}^0_g(W_{g,1}))_q \to \pi_n(\text{BhAut}_0(W_{g,1}))_q$$

is an isomorphism.

**Proof.** As the two groups are abstractly isomorphic, it suffices to show that the composition is injective. As $\pi_n(X_1(g))_q$ is the lowest rational homotopy group of this space, to show the composition is injective it is enough to show that

$$H^n(\text{BhAut}_0(W_{g,1}); \mathbb{Q}) \to H^n(X_1(g); \mathbb{Q})$$

is surjective. By Theorem 4.11, as unravelled in the proof of Proposition 4.21, the latter group is generated by the characteristic classes $\kappa_i(v_1, v_2, v_3)$ with $v_i \in H^n(W_{g,1}; \mathbb{Q})$.

The definition of the twisted Miller–Morita–Mumford classes $\kappa_i$ given in Section 4.3.3 makes sense for a fibration with fibre $W_g$ and section (see Remark 3.7 of [KRW20b]), and thus they are already defined on the space $\text{BhAut}_0(W_{g,1})$, because gluing in a trivial $D^{2n}$-bundle to the universal fibration gives a $W_g$-fibration with section given by the centre of the disc. Thus the classes $\kappa_i(v_1, v_2, v_3)$ are defined in the cohomology of $\text{BhAut}_0(W_{g,1})$, so the map is surjective as required.

It is interesting to make this more explicit. As in the proof we have characteristic classes $\kappa_1(v_1, v_2, v_3) \in H^n(\text{BhAut}_0(W_{g,1}); \mathbb{Q})$, and we may ask how to evaluate

$$\text{ev} : H^n(W_{g,1}; \mathbb{Q})^{\otimes 3} \otimes \pi_n(\text{BhAut}_0(W_{g,1})) \to \mathbb{Q}$$

$$v_1 \otimes v_2 \otimes v_3 \otimes [W_{g,1} \to E \xrightarrow{\pi} S^n] \longmapsto \int_{S^n} \kappa_1(v_1 \otimes v_2 \otimes v_3)$$

in terms of the description (27). Using $x \mapsto \lambda(x, -) : H \xrightarrow{\sim} H^\vee$, the pairing $\lambda$ on $H$ gives a pairing on $H^\vee$: abusing notation slightly we continue to call it $\lambda$.

**Proposition 5.11.** Let the fibration $W_{g,1} \to E \xrightarrow{\pi} S^n$ correspond to the derivation determined by $\phi : H[n-1] \to L^2(H[n-1]) \subset (H[n-1])^{\otimes 2}$. Then

$$\text{ev}(v_1 \otimes v_2 \otimes v_3 \otimes [W_{g,1} \to E \xrightarrow{\pi} S^n]) = \lambda(\phi(v_1, v_2, v_3)).$$
Proof. If \( W_{g,1} \rightarrow E \xrightarrow{\delta} S^n \) is a fibration, with associated \( W_g \)-bundle \( \tilde{\pi}: \tilde{E} \rightarrow S^n \) and section \( s: S^n \rightarrow \tilde{E} \), then the section provides a splitting of the exact sequence

\[
0 \rightarrow H^n(S^n; \mathbb{Q}) = \mathbb{Q}\{w\} \xrightarrow{s^*} H^n(\tilde{E}; \mathbb{Q}) \rightarrow H^n(W_g; \mathbb{Q}) \rightarrow 0
\]

and hence a map \( \iota: H^n(W_g; \mathbb{Q}) \rightarrow H^n(\tilde{E}; \mathbb{Q}) \). The Poincaré dual of the section \( s \) gives a class \( s(1) = u \in H^{2n}(\tilde{E}; \mathbb{Q}) \) which restricts to a cohomological fundamental class on each fibre. This gives a \( H^*(S^n; \mathbb{Q}) \)-module isomorphism

\[
H^*(\tilde{E}; \mathbb{Q}) = H^*(S^n; \mathbb{Q}) \otimes H^*(W_g; \mathbb{Q}) = \Lambda_\mathbb{Q}(w) \otimes (\mathbb{Q}\{1\} \otimes \iota(H^n(W_g; \mathbb{Q})) \otimes \mathbb{Q}\{u\}).
\]

We have \( u \sim \iota(v) = s(t(s^*(v))) = 0 \), and \( u \sim u = s(t(s(1))) = 0 \) as the section \( s \) may be homotoped off of itself, so the remaining cup products are determined by

\[
\iota(v_1) \sim \iota(v_2) = \lambda(v_1, v_2)(1 \otimes u) + w \otimes \varphi(v_1, v_2)
\]

for a \((-1)^n\)-symmetric map \( \varphi: H^n(W_g; \mathbb{Q}) \otimes H^n(W_g; \mathbb{Q}) \rightarrow H^n(W_g; \mathbb{Q}) \). Then

\[
\int_{S^n} \int_{\tilde{\pi}} \iota(v_1) \sim \iota(v_2) \sim \iota(v_3) = \lambda(\varphi(v_1, v_2), v_3).
\]

The map \( \varphi \) is related to the derivation

\[
\phi \in \text{Der}^+_n((\mathcal{L}(H[n-1]))_{n-1}) \subset \text{Hom}(H, L^2(H[n-1]))
\]

classifying this fibration by

\[
\varphi: H^n(W_g; \mathbb{Q}) \otimes H^n(W_g; \mathbb{Q}) = (H \otimes H)^{\vee} \rightarrow L^2(H[n-1])^{\vee} \xrightarrow{\phi^\vee} H^\vee = H^n(W_g; \mathbb{Q}).
\]

(The fact that the cup product is associative corresponds to \( \phi \) defining a derivation which annihilates \( \omega \).) This gives the claimed formula. \( \square \)

Dualising the isomorphism \( \kappa_1: [H^n(W_{g,1}; \mathbb{Q})]^{\otimes 3} \otimes (1^3)^{\otimes n}_{\mathfrak{S}_3} \rightarrow H^n(X_1(g); \mathbb{Q}) \) gives an isomorphism \( \pi_n(X_1(g)) \rightarrow [H^{\otimes 3} \otimes (1^3)^{\otimes n}_{\mathfrak{S}_3}] \), and for \( w_1, w_2, w_3 \in H \) let us write

\[
t(w_1, w_2, w_3) \in \pi_n(X_1(g)) \mathbb{Q}
\]

for the class corresponding to the (anti)symmetrisation of \( w_1 \otimes w_2 \otimes w_3 \in H^{\otimes 3} \) under this isomorphism. By construction we have

\[
ev(v_1 \otimes v_2 \otimes v_3 \otimes t(w_1, w_2, w_3)) = \sum_{\sigma \in \mathfrak{S}_3} \text{sign}(\sigma)^n \prod_{i=1}^3 v_i(w_\sigma(i)) \in \mathbb{Q},
\]

so the derivation associated to the bundle classified by \( t(w_1, w_2, w_3) \) is determined by the unique \( \lambda: H[n-1] \rightarrow L^2(H[n-1]) \subset (H[n-1])^{\otimes 2} \) with

\[
\lambda(\phi^\vee(v_1, v_2), v_3) = \sum_{\sigma \in \mathfrak{S}_3} \text{sign}(\sigma)^n \prod_{i=1}^3 v_i(w_\sigma(i)).
\]

A brief calculation shows that this is given by

\[
\phi(w) = \sum_{\sigma \in \mathfrak{S}_3} \text{sign}(\sigma)^n \lambda(w_\sigma(3), w)w_{\sigma(1)} \otimes w_{\sigma(2)}.
\]

Using \( x \mapsto \lambda(x, -): H \rightarrow H^{\vee} \) to identify

\[
\text{Der}^+_n([L(H[n-1])])_{n-1} \cong \text{Hom}(H[n-1], L^2(H[n-1])) \cong H[1-n] \otimes L^2(H[n-1])
\]

the element \( \phi \) corresponds to

\[
w_3 \otimes [w_1, w_2] + w_2 \otimes [w_3, w_1] + w_1 \otimes [w_2, w_3].
\]

Recall that in Proposition 4.21 we have shown that \( [\pi_{2n-1}(X_1(g))]_{\mathbb{Q}} \cong \mathbb{Q} \) for all large enough \( g \). The discussion above allows us to analyse the fate of this class in the homotopy groups of \( B\text{Aut}_*(W_{g,1}) \).
Corollary 5.12. The map 
\[ \mathbb{Q} \cong [\pi_{2n-1}(X_1(g))]_\mathbb{Q} \rightarrow [\pi_{2n-1}(B\text{Haut}_*(W_{g,1}))]_\mathbb{Q} \]
is injective.

Proof. In the proof of Proposition 4.21 it was shown that the \(G_g^{[r,d]}\)-invariant class in \(\pi_{2n-1}(X_1(g))_\mathbb{Q}\) was represented by the graph \(\Theta\) in the Harrison complex, so the corresponding homotopy class may be described as the sum of Whitehead products

\[ \Theta := \sum_{i,j,k=1}^g [t(f_i, f_j, f_k), t(e_i, e_j, e_k)] \in \pi_{2n-1}(X_1(g))_\mathbb{Q}. \]

By the discussion above each of \(t(f_i, f_j, f_k)\) and \(t(e_i, e_j, e_k)\) correspond to explicit derivations \(\phi^f_{i,j,k}\) and \(\phi^e_{i,j,k}\) of degree \((n-1)\) in \(\text{Der}^+(L(H[n-1]))\), so we must evaluate the graded commutators \([\phi^f_{i,j,k}, \phi^e_{i,j,k}]\) in this Lie algebra of derivations. Letting \(\epsilon := (-1)^{n-1}2\), in terms of the identification \(\text{Der}^+(L(H[n-1]))_{2(n-1)} = H[1-n] \otimes L^3(H[n-1])\) we calculate

\[ \phi^f_{i,j,k} \circ \phi^e_{i,j,k} = \epsilon e_k \otimes ([f_j, [f_k, e_j]] + [f_k, [f_j, e_k]]) + e_j \otimes ([f_i, [f_k, e_j]] + [f_k, [f_i, e_j]]) + e_i \otimes ([f_j, e_k] + [f_j, [f_k, e_j]]). \]

and

\[ (-1)^n \phi^f_{i,j,k} \circ \phi^f_{i,j,k} = f_k \otimes ([e_j, [f_k, f_j]] + [f_k, [e_j, f_j]]) + f_j \otimes ([e_i, [f_k, f_i]] + [f_k, [e_i, f_i]]) + f_i \otimes ([e_k, f_j] + [e_k, [f_j, e_j]]). \]

The graded commutator is \([\phi^f_{i,j,k}, \phi^e_{i,j,k}] = \phi^f_{i,j,k} \circ \phi^e_{i,j,k} - (-1)^{n-1} \phi^e_{i,j,k} \circ \phi^f_{i,j,k}\), so the class \(\Theta\) maps to

\[ 3 \sum_{i,j=1}^g e_i \otimes ([f_j, [f_i, e_j]] + [e_j, [f_i, f_j]]) + f_i \otimes ([e_j, [f_i, f_j]] + [e_j, [f_i, e_j]]). \]

Using graded commutativity in \(L(H[n-1])\) twice gives

\[ [e_j, [f_i, f_j]] = (-(-1)^{(n-1)(2n-2)})[[f_i, f_j], e_j] \]

\[ = (-(-1)^{(n-1)(2n-2)}(-(-1)^{(n-1)(n-1)}))[[f_j, f_i], e_j] \]

\[ = \epsilon [[f_j, f_i], e_j] \]

which simplifies the expression to

\[ 6 \sum_{i,j=1}^g e_i \otimes [[f_j, f_i], e_j] + f_i \otimes [[e_j, e_i], f_j]. \]

Evaluating this derivation on \(e_\ell\) gives \(6 \sum_{j=1}^g [e_j, e_\ell, f_j] \in L^3(H[n-1])\) which is easily seen to be non-zero in the free Lie algebra: thus the image of the \(G_g^{[r,d]}\)-invariant class \(\Theta\) is non-zero, as required.

\[ \square \]

5.5. The homotopy Lie algebra of total homotopy fibres. We shall now approach the term

\[ \pi_{*+1}(\text{tohofib}_I \text{Emb}(\bar{k} \setminus I, W_{g,1}))_\mathbb{Q} \]
in (23). For each \(x_k \in \text{Emb}(\bar{k}, W_{g,1})\) we have a cubical diagram of ordered configuration spaces

\[ x_k \triangleright I \mapsto \text{Emb}(\bar{k} \setminus I, W_{g,1}). \]

The element \(x_k\) provides basepoints, so as in Section 2.5 we can form the total homotopy fibre tohofib\(_{I \subset x_k} \text{Emb}(x_k \setminus I, W_{g,1})\).
Since the spaces \( \text{Emb}(k, W_{g,1}) \) are simply-connected as long as \( n \geq 2 \), we can canonically identify the rational homotopy groups of this total homotopy fibre with that formed at a different point \( x'_k \in \text{Emb}(k, W_{g,1}) \). More invariantly, each of the groups \( \pi_*(\text{tohofib}_{I \subset x'_k} \text{Emb}(x'_k \setminus I, W_{g,1}))_Q \) is naturally isomorphic to

\[
\colim_{x_k \in \Pi(\text{Emb}(k, W_{g,1}))} \pi_*(\text{tohofib}_{I \subset x_k} \text{Emb}(x_k \setminus I, W_{g,1}))_Q
\]

where \( \Pi(-) \) denotes the fundamental groupoid. We denote this colimit by

\[
\pi_*(\text{tohofib}_{I \subset x_k} \text{Emb}(x_k \setminus I, W_{g,1}))_Q.
\]

Both the symmetric group \( S_k \) and the mapping class group \( \Gamma_g \) act on the cubical diagram and these actions commute, so this is a \( S_k \times \Gamma_g \)-representation. We first study the individual terms \( \text{Emb}(k, W_{g,1}) \) and only then the total homotopy fibre.

### 5.5.1. The homotopy Lie algebra of configuration spaces.

We take \( \text{Emb}(k, \mathbb{R}^d) \) to be based at \( k \ni i \mapsto (i, 0, \ldots, 0) \), and define its rational homotopy groups \( \pi_*(\text{Emb}(k, \mathbb{R}^d))_Q \) with this basepoint. It is not fixed by the \( S_k \)-action which permutes the particles, but under the assumption \( d \geq 3 \) the space \( \text{Emb}(k, \mathbb{R}^d) \) is 1-connected and we nonetheless obtain a well-defined action on its homotopy groups.

These configuration spaces admit additional structure. Firstly, we can forget particles: an injection \( f^*: S \hookrightarrow T \) of finite sets induces by precomposition a map

\[
f^*: \text{Emb}(T, \mathbb{R}^d) \to \text{Emb}(S, \mathbb{R}^d)
\]

and hence a map on rational homotopy groups. Secondly, we can add particles by bringing them in from infinity. To do so, suppose that \( f: S_+ \to T_+ \) is a map of finite pointed sets which is injective on \( S_+ \setminus f^{-1}(t_+) \) (here \( t_+ \) denotes the basepoint element of \( T \)). Letting \( S \subset S_+ \) and \( T \subset T_+ \) denote the subsets of non-basepoint elements, we obtain a map

\[
f^*: \text{Emb}(T, \mathbb{R}^d) \to \text{Emb}(S, \mathbb{R}^d)
\]

as follows: on \( S_+ \setminus f^{-1}(t_+) \) it is given by precomposition with \( f \), and the particles in \( f^{-1}(t_+) \setminus \{s_+\} \) are brought in from infinity. The latter involves a choice, but the homotopy class of map obtained is independent of this choice and gives a well-defined map on rational homotopy groups.

To capture this data, we let \( \text{Fl}_* \) be the category whose objects are finite pointed sets and whose morphisms are basepoint-preserving maps which are injective on those elements not sent to the basepoint. Then the rational homotopy groups assemble to a functor

\[
\pi_{*-1}(\text{Emb}(-, \mathbb{R}^d))_Q: \text{Fl}^{op}_* \to \text{Alg}_{\text{Lie}}(\text{Gr}(\mathbb{Q}\text{-mod}))
\]

\[S_+ \mapsto \pi_{*-1}(\text{Emb}(S, \mathbb{R}^d))_Q\]

whose action on morphisms is described as above.

There is a presentation of \( \pi_{*-1}(\text{Emb}(-, \mathbb{R}^d))_Q \) in terms of the homotopy class \( t_{ij} \in \pi_{d-1}(\text{Emb}(2, \mathbb{R}^d)) \) of the map

\[
\mathbb{S}^{d-1} \to \text{Emb}(2, \mathbb{R}^d)
\]

\[\theta \mapsto (\theta, -\theta).
\]

There is a morphism \( f_{ij}: k_+ \to 2_+ \) in \( \text{Fl} \) given by \( i \mapsto 1, j \mapsto 2 \), and sending all other elements to the basepoint. This induces a map \( f_{ij}^*: \text{Emb}(2, \mathbb{R}^d) \to \text{Emb}(k, \mathbb{R}^d) \) well-defined up to homotopy, and we define \( t_{ij} \in \pi_{*1}(\text{Emb}(k, \mathbb{R}^d)) \) to be \( (f_{ij}^*)_* (t_{12}) \).

A permutation \( \sigma \) of \( k \) gives a map \( \text{Emb}(k, \mathbb{R}^d) \to \text{Emb}(k, \mathbb{R}^d) \) by precomposition, sending \( t_{ij} \) to \( t_{\sigma^{-1}(i), \sigma^{-1}(j)} \). This defines a right action but we shall prefer to consider the associated left action, where \( S_k \) acts through its opposite, so that \( \sigma \cdot t_{ij} = t_{\sigma(i), \sigma(j)} \).

Declaring that the symbol \( t_{ij} \) goes to the element \( t_{ij} \in \pi_{d-1}(\text{Emb}(k, \mathbb{R}^d)) \) we obtain a unique map of graded Lie algebras in \( S_k \)-representations

\[
\text{L}(\mathbb{Q}\{t_{ij} \mid i \neq j \in k\}) \to \pi_{*1}(\text{Emb}(k, \mathbb{R}^d))_Q.
\]
with $|t_{ij}| = d - 2$. This map is surjective and its kernel has been determined. The following definition goes back to [Koh87, Dri89], and also known as Yang–Baxter Lie algebra, or infinitesimal braid Lie algebra.

**Definition 5.13.** Let $S$ be a finite set and $d \geq 0$, then the Drinfel’d–Kohno Lie algebra $t_0(S)$ is the graded Lie algebra given by the quotient of the free graded Lie algebra generated by

1. generators $t_{ij}$ in degree $d - 2$ for each pair $(i, j)$ of distinct elements of $S$,
2. by the ideal generated by the relations
   1. $t_{ij} = (-1)^d t_{ji}$ for $i, j$ distinct,
   2. $[t_{ij}, t_{rs}] = 0$ for $i, j, r, s$ all distinct,
   3. $[t_{ij}, t_{jk} + t_{kj}] = 0$ for $i, j, k$ all distinct.

**Remark 5.14.** Though the notation does not reflect it, $t_0(S)$ depends on an integer $d$.

The Drinfel’d–Kohno Lie algebras assemble to a functor

$$t_0(-): \text{Fl}^p \to \text{Alg}_{\text{Lie}}(\text{Gr}(\mathbb{Q}\text{-mod}))$$

$$S_+ \mapsto t_0(S),$$

whose action on a morphism $f: S \to T_+$ is determined by

$$t_0(f)(t_{ij}) = \begin{cases} tf^{-1}(i)f^{-1}(j) & \text{if } i, j \in f(S), \\ 0 & \text{otherwise.} \end{cases}$$

As before, we consider the left $\mathcal{G}_S$-action on $t_0(S)$ through its opposite, which is determined by $\sigma \cdot t_{ij} = t_{\sigma(i)\sigma(j)}$.

The following result has been proven in varying levels in generality over the years [CG02], [Tam03, Proposition 4.1], [SW11a, Theorem 1], [Ber14, Example 5.5]. The stated version may be deduced from [Pre17, Theorem 14.1.14]:

**Theorem 5.15.** For $d \geq 3$, there is an isomorphism

$$t_0(-) \cong \pi_{*+1}(\text{Emb}(-, \mathbb{R}^d))_\mathbb{Q}$$

of functors $\text{Fl}^p \to \text{Alg}_{\text{Lie}}(\text{Gr}(\mathbb{Q}\text{-mod}))$, uniquely determined by sending the symbol $t_{ij}$ to the element $t_{ij}$.

We will extend this description to the ordered configuration spaces of the manifold $W_{g,1}$ for $2n \geq 4$. That is, we will identify the functor

$$\pi_{*+1}(\text{Emb}(-, W_{g,1}))_\mathbb{Q}: \text{Fl}^p \to \text{Alg}_{\text{Lie}}(\text{GrRep}(\Gamma_g))$$

$$S_+ \mapsto \pi_{*+1}(\text{Emb}(S, W_{g,1}))_\mathbb{Q}$$

whose effect on morphisms is given by relabelling points, forgetting points, or bringing them in by stabilisation. The answer is an extension of the Drinfel’d–Kohno Lie algebra $t_0(\mathcal{X})$ by the $G_g$-representation $H$.

**Definition 5.16.** Let $S$ be a finite set and $n \geq 0$, then the extended Drinfel’d–Kohno Lie algebra $t_n(S)$ is the graded Lie algebra given by the quotient of the free graded Lie algebra generated by

1. elements $t_{ij}$ in degree $2n - 2$ for each pair $(i, j)$ of distinct elements of $S$,
2. a copy $H(r)$ of the $2g$-dimensional vector space $H$ in degree in $n - 1$ for each $r \in S$,

by the ideal generated by the relations

1. $t_{ij} = t_{ji}$ for $i, j$ distinct,
2. $[t_{ij}, t_{rs}] = 0$ for $i, j, r, s$ all distinct,
3. $[t_{ij}, t_{jk} + t_{kj}] = 0$ for $i, j, k$ all distinct,
4. for $a^{(r)} \in H^{(r)}$ and $i, j, r$ all distinct, $[t_{ij}, a^{(r)}] = 0$,
5. for $a^{(i)} \in H^{(i)}$ and $a^{(j)}$ the corresponding vector in $H^{(j)}$, $[t_{ij}, a^{(i)} + a^{(j)}] = 0$,
6. for $a^{(i)} \in H^{(i)}$ and $b \in H^{(j)}$ with $i, j$ distinct, $[a^{(i)}, b^{(j)}] = \lambda(a, b)t_{ij}$. 


Construction of the map \((29)\)

5.17 Remark. Fix a collar \(\mathbb{R}^{2n-1} \times (-\infty, \infty) \hookrightarrow W_{g,1}\) which sends \(\mathbb{R}^{2n-1} \times \{0\}\) to \(\partial W_{g,1}\). Any element of \(\Gamma_g\) can be represented by a diffeomorphism which fixes the collar pointwise. This restricts to an embedding \(e_0: \mathbb{R}^{2n-1} \times \mathbb{R} \hookrightarrow W_{g,1}\). It induces a map

\[\text{Emb}(\mathbb{R}^{2n} , \mathbb{R}^{2n-1} \times \mathbb{R} \hookrightarrow W_{g,1})\]

of configuration spaces which is the first term of \((29)\).

The construction of \(t_g(S)\) is natural in the bilinear form \((H, \lambda)\), so is in particular a Lie algebra object in graded \(G'_g\)-representations: \(g \in G'_g\) acts trivially on \(t_{ij}\) and acts on \(H^{\langle r \rangle}\) by \(a^{\langle r \rangle} \mapsto (ga)^{\langle r \rangle}\). We may consider \(t_g(S)\) as a Lie algebra object in graded \(\Gamma_g\)-representations through the homomorphism \(\Gamma_g \to G'_g\). Doing so, these assemble to a functor

\[t_g(-): \text{Fl}_n^{op} \to \text{Alg}_k(\text{GrRep}(\Gamma_g))\]

whose action on a morphism \(f: S_+ \to T_+\) is determined by

\[
t_g(f)(t_{ij}) = \begin{cases} \bar{t}_{f^{-1}(i)f^{-1}(j)} & \text{if } i, j \in f(S), \\ 0 & \text{otherwise} \end{cases}
\]

In particular, using the opposite of the automorphisms of \(S_+\), \(t_g(S)\) has a left \(\mathcal{S}\)-action which commutes with the action of \(\Gamma_g\); an element \(\sigma \in \mathcal{S}\) acts by \(t_{ij} \mapsto t_{\sigma(i)\sigma(j)}\) and \(a^{\langle i \rangle} \mapsto a^{\langle \sigma(i) \rangle}\).

**Proposition 5.18.** For \(2n \geq 4\) and \(g \geq 0\), there is an isomorphism

\[t_g(-) \cong \pi_{*+1}(\text{Emb}(\cdot, W_{g,1}))_{\mathbb{Q}}\]

of functors \(\text{Fl}_n^{op} \to \text{Alg}_k(\text{GrRep}(\Gamma_g))\).

The proof of this proposition will occupy the next section.

5.5.2. **Proof of Proposition 5.18.** To obtain a natural transformation \(\Phi\) with components

\[\Phi_k: t_g(k) \to \pi_{*+1}(\text{Emb}(k, W_{g,1}))_{\mathbb{Q}}\]

as in Proposition 5.18, we need to construct a map of \(\Gamma_g \times \mathcal{S}\)-representations

\[(28) \quad \bar{t}_g(k) := \mathbb{Q}\{t_{ij} \mid i \neq j \in k\} \oplus H^{\oplus k} \to \pi_{*+1}(\text{Emb}(k, W_{g,1}))_{\mathbb{Q}}.
\]

Using the formula’s after Definition 5.16, the left term gives a functor \(\text{Fl}_n^{op} \to \text{GrRep}(\Gamma_g)\), and we will in fact construct a natural transformation of functors

\[\bar{t}_g(-) := \mathbb{Q}\{t_{ij} \mid i \neq j \in (-)\} \oplus H^{\oplus (-)} \to \pi_{*+1}(\text{Emb}(\cdot, W_{g,1}))_{\mathbb{Q}}.
\]

Since the target is valued in the category \(\text{Alg}_k(\text{GrRep}(\Gamma_g))\) of graded Lie algebras of \(\Gamma_g\)-representations, this extends to a canonical natural transformation with domain

\[L_1(\bar{t}_g(-)): \text{Fl}_n^{op} \to \text{Alg}_k(\text{GrRep}(\Gamma_g)).
\]

To get the desired natural transformation \(\Phi\) with domain \(t_g(-)\), it suffices check that the relations (R1)–(R6) hold in the target for each \(k_+\).

We may replace \(W_{g,1}\) by its interior \(\mathring{W}_{g,1}\), as the inclusions \(\text{Emb}(k, \mathring{W}_{g,1}) \hookrightarrow \text{Emb}(k, W_{g,1})\) are weak equivalences. Then \((28)\) will be obtained from a map

\[(29) \quad \text{Emb}(k, \mathbb{R}^{2n}) \vee \mathring{W}_{g,1} \to \text{Emb}(k, W_{g,1})
\]

which has the desired functoriality up to homotopy; both sides assemble to functors \(\text{Fl}_n^{op} \to \text{Ho}(\text{Top})\) and these maps are natural transformations of such functors.
To obtain the remaining wedge summands of (29) for \(1 \leq i \leq k\), we choose an embedding \(\bar{W}_{g,1} \cup (k \setminus \{i\}) \rightarrow \bar{W}_{g,1}\) as follows. Outside the image of \(e_0\) it is identity and on its image, it is given by \((x, t) \mapsto (x, \lambda(x - i, t))\), with smooth map \(\lambda: \mathbb{R}^{2n-1} \times \mathbb{R} \rightarrow \mathbb{R}\) having the following properties

- It is given by \((x, t) \mapsto t\) if \(\|x\| < 1/4\) or \(t > 3/2\).
- For fixed \(x\), the function \(t \mapsto \lambda(x, t)\) is strictly increasing.
- For fixed \(x\) satisfying \(\|x\| > 3/4\), the image of \(t \mapsto \lambda(x, t)\) is \((1, \infty)\).

This induces a map

\[
\Emb(k, \bar{W}_{g,1} \cup (k \setminus \{i\})) \rightarrow \Emb(k, \bar{W}_{g,1})
\]

of configuration spaces, which we precompose map with the inclusion of the path component

\[
\Emb(i, \bar{W}_{g,1}) \times \prod_{j \neq i} \Emb(j, \ast) \subset \Emb(k, \mathbb{R}^{2n} \cup (k \setminus \{i\}))
\]

which is homeomorphic to \(\bar{W}_{g,1}\).

We consider \(\Emb(k, \bar{W}_{g,1})\) as based at \(k \ni i \mapsto (i, 0, \ldots, 0)\), using the collar coordinates. Similarly, we consider the \(i\)th copy of \(\bar{W}_{g,1}\) as based at \((i, 0, \ldots, 0)\). Then these maps are all basepoint-preserving, and hence induce a map (29).

Having defined (29), we apply its induced map to \(t_{ij} \in \pi_{s+1}(\Emb(k, \mathbb{R}^{2d})_\mathbb{Q})\) to define

\[
t_{ij} \in \pi_{s+1}(\Emb(k, \bar{W}_{g,1}))_\mathbb{Q}.
\]

Similarly, taking the \(i\)th term \(\bar{W}_{g,1}\) and applying the induced map to \(a \in \pi_{s+1}(\bar{W}_{g,1})_\mathbb{Q}\), this defines elements

\[
a^{(i)} \in \pi_{s+1}(\Emb(k, \bar{W}_{g,1}))_\mathbb{Q}.
\]

This completes the definition of the linear map (28).

The construction of (29) is \(\Gamma_{g} \times \mathfrak{S}_{k}\)-equivariant up to homotopy. Thus, to verify it gives a natural transformation on \(\text{Fl}_n\), it suffices to verify naturality with respect to

(i) the standard inclusion \(i_k: k - 1 \rightarrow k\) and
(ii) the standard projection \(p_k: k \rightarrow k - 1\) sending \(k\) to the basepoint.

The map \(i_k^*: \pi_{s+1}(\Emb(-, \bar{W}_{g,1}))_\mathbb{Q}\) is that induced by the map

\[
\pi_k: \Emb(k, \bar{W}_{g,1}) \rightarrow \Emb(k - 1, \bar{W}_{g,1})
\]

which forgets the \(k\)th particle. Similarly, the map \(i_k^*: \bar{I}_g(-)\) is induced by the map

\[
\pi_k: \Emb(k, \mathbb{R}^{2n}) \sqcup \bigvee_{k \setminus \{1\}} \bar{W}_{g,1} 
\rightarrow \Emb(k - 1, \mathbb{R}^{2n}) \sqcup \bigvee_{k - 1 \setminus \{1\}} \bar{W}_{g,1}
\]

forgetting the \(k\)th particle in the first term and the \(k\)th of the wedge of \(\bar{W}_{g,1}\)'s. Now we simply observe that our construction is such that the diagram

\[
\begin{array}{ccc}
\Emb(k, \mathbb{R}^{2n}) \sqcup \bigvee_{k \setminus \{1\}} \bar{W}_{g,1} & \rightarrow & \Emb(k, \bar{W}_{g,1}) \\
\pi_k \downarrow & & \downarrow \pi_k \\
\Emb(k - 1, \mathbb{R}^{2n}) \sqcup \bigvee_{k - 1 \setminus \{1\}} \bar{W}_{g,1} & \rightarrow & \Emb(k - 1, \bar{W}_{g,1})
\end{array}
\]

commutes, proving the case (i).

The map \(p_k^*: \pi_{s+1}(\Emb(-, \bar{W}_{g,1}))_\mathbb{Q}\) is induced by the stabilisation map

\[
s: \Emb(k - 1, \bar{W}_{g,1}) \rightarrow \Emb(k, \bar{W}_{g,1})
\]

bringing in a \(k\)th particle from \((0, \ldots, 0, -\infty)\) with respect to the collar, which is well-defined up to homotopy. Similarly the map \(p_k^*: \bar{I}_g(-)\) is induced by the map

\[
s \vee \iota: \Emb(k - 1, \mathbb{R}^{2n}) \sqcup \bigvee_{k - 1 \setminus \{1\}} \bar{W}_{g,1} 
\rightarrow \Emb(k, \mathbb{R}^{2n}) \sqcup \bigvee_k \bar{W}_{g,1}
\]
given by the stabilisation map $s$ on the first term, and the inclusion $i$ of the first $k - 1$ terms of the wedge of $\Emb(2, W_{g,1})$'s on the second term. These fit in a diagram

\[
\begin{array}{ccc}
\Emb(k - 1, \mathbb{R}^{2n}) \vee \bigvee_{k-1} W_{g,1} & \longrightarrow & \Emb(k - 1, \Emb(2, W_{g,1})) \\
\downarrow{s \vee i} & & \downarrow{s} \\
\Emb(k, \mathbb{R}^{2n}) \vee \bigvee_{k} W_{g,1} & \longrightarrow & \Emb(k, \Emb(2, W_{g,1}))
\end{array}
\]

commuting up to homotopy, proving case (ii).

It now makes sense to ask whether the relations (R1)–(R6) of Definition 5.16 hold in the target. Below we verify that they do:

**Verification of (R1), (R2), and (R3).** By naturality it suffices to verify these relations in the wedge summand $\Emb(k, \mathbb{R}^{2n})$. Here they are a consequence of Theorem 5.15.

**Verification of (R4).** We fix distinct $i, j, r$. There is a map

\[\Emb((i, j), \mathbb{R}^{2n}) \times W_{g,1} \longrightarrow \Emb((i, j, r), W_{g,1})\]

induced by an embedding $\mathbb{R}^{2n} \sqcup W_{g,1} \hookrightarrow W_{g,1}$ constructed similarly to those used to construct (28), and such that the induced map on rational homotopy Lie algebras is surjective onto the subalgebra generated by $t_{ij}$ and $a^{(r)}$s. Now note that relation (R4) holds in the source. The general case follows by naturality with respect to stabilisation.

For relations (R5) and (R6), we compute Whitehead products in the rational homotopy groups of $\Emb(k, \mathbb{R}^{2n})$. A general reference for Whitehead products is [Whi78, Section X.7]. Suppose we have two based maps $f : S^n \to X$ and $g : S^m \to X$. There is a deformation retraction $r : S^n \times S^m \setminus \text{int}(D^{n+m}) \to S^n \vee S^m$ and a boundary inclusion $i : S^{n+m-1} \hookrightarrow S^n \times S^m \setminus \text{int}(D^{n+m})$, then the Whitehead product is given by

\[f \ast g := (f \circ g) \ast r \circ i : S^{n+m-1} \to X.\]

In particular, if the map $(f \vee g) \circ r : S^n \times S^m \setminus \text{int}(D^{n+m}) \to X$ extends over $S^n \times S^m$, then the Whitehead product vanishes.

**Verification of (R6).** By bilinearity of the Whitehead product, it suffices to prove this for $a, b$ representing the cores of $W_{g,1}$. That is, we can take them to be among a hyperbolic basis $e_1, f_1, e_2, f_2, \ldots, e_g, f_g$ of $H$, cf. Section 2.3. By $\Gamma_{g}$-equivariance we may assume $i = 1$ and $j = 2$. By naturality with respect to stabilisation, we may take $k = 2$. Then $[a^{(1)}, b^{(2)}]$ is represented by the map

\[S^n \vee S^n \longrightarrow \Emb(2, \hat{W}_{g,1})\]

given by having a particle travel around a core on the corresponding wedge summand.

If $\omega(a, b) = 0$, these cores can be represented disjointly, and the map extends to $S^n \times S^n$. However, if we are dealing with $a = e_i$ and $b = f_i$, this is not possible. By $\Gamma_{g}$-equivariance we may assume $i = 1$:

**Lemma 5.19.** The relation $[e_1^{(1)}, f_1^{(2)}] = t_{12}$ holds in $\pi_{*+1}(\Emb(2, \hat{W}_{g,1}))/\mathbb{Q}$.

**Proof.** To see that $[e_1^{(1)}, f_1^{(2)}] = t_{12}$, it suffices to construct a map

\[S^n \times S^n \setminus \text{int}(D^{2n}) \longrightarrow \Emb(2, \hat{W}_{g,1})\]

whose restriction to $S^n \vee S^n$ represents $e_1^{(1)}$ and $f_1^{(2)}$, and whose restriction to the boundary represents $t_{12}$. To do so, we observe there is a map $S^n \times S^n \longrightarrow (W_{1,1})^2$ by letting the first particle travel around the first core, and the second particle around the second core. Its composition with the map induced by inclusion $W_{1,1} \hookrightarrow W_{g,1}$ lands in $\Emb(2, \hat{W}_{g,1})$ unless both particles lie the unique intersection point of the core. Removing a small neighbourhood of the form $\text{int}(D^n \times D^n)$ from the inverse image of this point in $S^n \times S^n$ gives us a map

\[S^n \times S^n \setminus \text{int}(D^n \times D^n) \longrightarrow \Emb(2, \hat{W}_{g,1})\]
with desired restriction to $S^n \vee S^n$. Its restriction to boundary is given by the two particles circling around each other once, i.e. $t_{12}$. \hfill \qed

**Verification of (R5).** A similar argument as above deduces this relation from the following lemma:

**Lemma 5.20.** The relation $[t_{12}, e_1(1) + e_1(2)] = 0$ holds in $\pi_+(\text{Emb}(k, W_{g,1})_Q)$. 

**Proof.** It suffices to produce a map $S^{2n-1} \times S^n \to \text{Emb}(2, W_{g,1})$ whose restriction to $S^{2n-1}$ represents $t_{12}$, and whose restriction to $S^n$ represents $e_1(1) + e_1(2)$.

Fix an exponential map $T(S^n \times D^n) \to S^n \times D^n$ which is a fibrewise embedding. As $T(S^n \times D^n)$ is trivial, we may think of this as a map $\mathbb{R}^{2n} \times (S^n \times D^n) \to S^n \times D^n$. Let us restrict to $S^{2n-1} \times S^n \to S^n \times D^n$. This is an embedding when restricted to $S^{2n-1} \times \{t\}$ for any $t \in S^n$. Taking opposite points in these spheres, we obtain a map

$$S^{2n-1} \times S^n \to \text{Emb}(2, S^n \times D^n)$$

The desired map is obtained by composing it with that induced by the embedding $S^n \times D^n \hookrightarrow W_{g,1}$. By construction its restriction to $S^{2n-1} \times \{t\}$ is a representative of $t_{ij}$ and the restriction to $\{x\} \times S^n$ is two particles travelling alongside parallel to the core representing $e_1$. \hfill \qed

At this point we have constructed $\Phi$ with components

$$\Phi_k: t_g(k) \to \pi_+(\text{Emb}(k, W_{g,1}))_Q.$$ 

We will prove by induction over $k$ that $\Phi_k$ is an isomorphism. For $k = 1$, this is a map $\Phi_1: \mathbb{L}(H[n-1]) \to \pi_+(W_{g,1})_Q$ which is an isomorphism by the Hilton–Milnor theorem [Hil55] using $W_{g,1} \simeq \vee_2 S^n$. For the induction step, we use the map

$$\pi_k: \text{Emb}(k, W_{g,1}) \to \text{Emb}(k-1, W_{g,1})$$

which forgets the $k$th particle. On rational homotopy groups it is the map induced by the morphism $i_k: k-1 \to k$ of $F(\ell)$. By construction, the diagram

\[
\begin{array}{ccc}
t_g(k) & \xrightarrow{\Phi_k} & \pi_+(\text{Emb}(k, W_{g,1}))_Q \\
\downarrow \Phi_k & & \downarrow (\pi_k) \\
t_g(k-1) & \xrightarrow{\Phi_{k-1}} & \pi_+(\text{Emb}(k-1, W_{g,1}))_Q \\
\end{array}
\]

commutes. Furthermore, the vertical maps have sections, induced by the morphism $k \to k-1$, of $F(\ell)$ which sends $k$ to the basepoint and is the identity otherwise. Thus the vertical maps are surjective, and we have a map of short exact sequences

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\ker(i_k^*) & \to & \pi_+(W_{g,1, k-1})_Q \\
\downarrow t_g(k) & \xrightarrow{\Phi_k} & \pi_+(\text{Emb}(k, W_{g,1}))_Q \\
\downarrow \Phi_k & & \downarrow (\pi_k) \\
t_g(k-1) & \xrightarrow{\Phi_{k-1}} & \pi_+(\text{Emb}(k-1, W_{g,1}))_Q \\
\downarrow & & \downarrow \\
0 & \to & 0,
\end{array}
\]

where the identification in the bottom-left corner uses that $\pi_k$ is a fibration with fibre $W_{g,1, k-1} := W_{g,1} \setminus \{(k-1) \text{ points}\}$. 


There is an equivalence $W_{g,1,k−1} \simeq W_{g,1} \vee \bigvee_{i=1}^k S^{2n−1}$, so by the Hilton–Milnor theorem the Lie algebra $\pi_{+1}(W_{g,1,k−1})\mathbb{Q}$ is a free graded Lie algebra, generated by $t_{ik}$ for $1 \leq i \leq k − 1$ in degree $2n − 2$ and $H^{(k)}$ in degree $n − 1$. To see that the map

$$\ker(i^*_k) \to \pi_{+1}(W_{g,1,k−1})\mathbb{Q}$$

is surjective, we note that $\ker(i^*_k)$ contains both $t_{ik}$ for $1 \leq i \leq k − 1$ and $H^{(k)}$. It suffices to prove it is generated by these elements: since the target is a free Lie algebra, there are then no relations between the $t_{ik}$ and $H^{(k)}$ in $\ker(i^*_k)$, so the top map is an isomorphism, and by the 5-lemma the middle map is too. We now establish the claim:

**Lemma 5.21.** The Lie algebra $\ker(i^*_k)$ is generated by $t_{ik}$ for $1 \leq i \leq k − 1$ and $a^{(k)}$ for $a \in H$.

**Proof.** We start by noting that $\ker(i^*_k)$ is spanned by the Lie words containing at least one copy of $t_{ik}$ or $a^{(k)}$ for $a \in H$. We must show that these can be rewritten as sums of Lie words which only involve $t_{ik}$ and $a^{(k)}$. Let us say that such a Lie word has type $2r + s$ if it has $r$ entries of the form $t_{ij}$ and $s$ entries of the form $a^{(j)}$; then relations (R1)–(R6) only relate Lie words of the same type (relations (R1)–(R5) further preserve the length of the Lie word, but (R6) does not: type will serve as a proxy for length of Lie words). The proof is by induction over type of the following statement:

$$(I_m) \quad \text{A Lie word in } \ker(i^*_k) \text{ of type } m \text{ is a linear combination of Lie words involving only } t_{ik} \text{ and } a^{(k)}.$$

If a Lie word $z$ in $\ker(i^*_k)$ has type 1 then it must be $a^{(k)}$, so $(I_1)$ holds. If $z$ has type 2 then it must be $t_{ik}$ or $[a^{(j)},b^{(k)}]$. In the first case we are done, and in the second case we are done if $j = k$; if $j \neq k$ then by (R6) we have $[a^{(j)},b^{(k)}] = \lambda(a,b)t_{jk}$, so $(I_2)$ holds.

Let us now consider Lie words of type $m \geq 3$, which in particular have length $\geq 2$. We may therefore write such Lie words as $[x,y]$, and without loss of generality we may suppose that $x$ contains a $t_{ik}$ or $a^{(k)}$. As $x$ has type $< m$, using $(I_{<m})$ we may write it as a linear combination of Lie words involving only $t_{ik}$ and $a^{(k)}$. We therefore proceed by a secondary induction of the following statement:

$$(J_{m,m'}) \quad \text{A Lie word } [x,y] \text{ in } \ker(i^*_k) \text{ of type } m, \text{ with } x \text{ involving only } t_{ik} \text{ or } a^{(k)} \text{ and } (J_{m,m'}) \text{ being of type } m', \text{ is a linear combination of Lie words involving only } t_{ik} \text{ and } a^{(k)}.$$

For each $m$ we will prove this by downwards induction on $m'$: the base case $m' = m$ is vacuous, as there are no such Lie words. We distinguish three cases:

(i) **$x$ and $y$ have a single entry:** Necessarily, we must have $x = t_{ik}$ or $x = a^{(k)}$. There is only something to prove if $y = t_{ij}$ with $i,j \neq k$ or $y = a^{(r)}$ with $r \neq k$. Then the relations of $t_{ij}$ allow us to rewrite $[x,y]$ in the desired form. For example, if $x = t_{ik}$ and $y = a^{(j)}$, use relations (R4) or (R5) depending whether $i = r$ or not.

(ii) **$y$ has at least two entries:** Write $y = [y',y'']$. The Jacobi identity and symmetry gives

$$[x, [y',y'']] = \pm[[y'',x],y'] \pm [[x,y'],y''].$$

Both $[y'',x]$ and $[x,y']$ now contain $t_{ik}$ or $a^{(k)}$ and have type $< m$, so by $(I_{<m})$ they are linear combinations of Lie words containing only $t_{ik}$ and $a^{(k)}$. They also have type $> m'$, as we have appended $y''$ or $y'$ to $x$ which had type $m'$. Thus we have expressed $[x,y]$ as a sum of terms $[u,v]$ with $u$ involving only $t_{ik}$ or $a^{(k)}$ and being of type $> m'$, so by $(J_{m',>m'})$ are done.

(iii) **$x$ has at least two entries:** Write $x = [x',x'']$ with both $x',x''$ having only entries of the form $t_{ik}$ or $a^{(k)}$. The Jacobi identity gives

$$[[x',x''],y] = \pm[[x'',y],x'] \pm [[y,x'],x''].$$
As \([x', y]\) and \([y, x']\) now contain \(t_{ik}\) or \(a(k)\) and have type \(< m\), by \((I \leq m)\) they are linear combinations of Lie words containing only \(t_{ik}\) and \(a(k)\); as \(x'\) and \(x''\) also only contain these elements, we are done. □

This lemma finishes the argument that for each \(k\) the map

\[
\Phi_k : t_j(k) \to \pi_{*+1}(\text{Emb}(k, W_{g,1}))(\mathbb{Q})
\]

is an isomorphism. That this gives an natural isomorphism of functors \(\text{FI}^\text{op} \to \text{Alg}_{\text{Lie}}(\text{GrRep}(\Gamma_g))\) follows from the fact that we established naturality on the generators \(t_j(\mathbb{Z})\) of \(t_j(-)\). Since the \(\Gamma_g\)-action on \(t_j(\mathbb{Z})\) factors over \(G'\), and is algebraic as a representation of this group, we conclude that the same is true for the rational homotopy groups of the configuration spaces \(\text{Emb}(k, W_{g,1})\).

### 5.5.3. The rational homotopy of the total homotopy fibres.

We next express the rational homotopy groups of the total homotopy fibres of cubical diagrams of configuration spaces.

**Lemma 5.22.** Let \(S\) be a finite set and \(n \geq 0\), then the total extended Drinfel’d–Kohno Lie algebra \(f_g(S)\) is given by

\[
f_g(S) := \bigcap_{j \in S} \ker \left[ t_j(S) \to t_j(S \setminus \{j\}) \right].
\]

**Lemma 5.23.** There is a commutative diagram of \(\Gamma_g \times S_k\)-representations

\[
\begin{array}{ccc}
f_g(k) & \xrightarrow{\simeq} & \pi_{*+1}(\text{tohofib} \text{Emb}(k \setminus I, W_{g,1}))\mathbb{Q} \\
\downarrow & & \downarrow \\
t_j(k) & \xrightarrow{\simeq} & \pi_{*+1}(\text{Emb}(k, W_{g,1}))\mathbb{Q}
\end{array}
\]

where both horizontal maps are isomorphisms.

**Proof.** The \(k\)-cube \(I \mapsto \text{Emb}(k \setminus I, W_{g,1})\) is split up to homotopy in the sense of Definition 2.9, by adding configuration points near the boundary of \(W_{g,1}\). It also consists of simply-connected spaces, so by Lemma 2.10 the map from \(\pi_{*+1}(\text{tohofib} \text{Emb}(k \setminus I, W_{g,1}))\mathbb{Q}\) to

\[
\bigcap_{j \in k} \ker \left[ \pi_{*+1}(\text{Emb}(k, W_{g,1}))\mathbb{Q} \to \pi_{*+1}(\text{Emb}(k \setminus \{j\}, W_{g,1}))\mathbb{Q} \right]
\]

is an isomorphism. Combining this with Proposition 5.18 gives the required diagram. □

This implies that the non-zero rational homotopy groups of the total homotopy fibre are concentrated in degrees \(r(n-1)+1\) for \(r > 0\). We will show in Proposition 5.26 that they vanish for \(r\) small in comparison to \(k\). To explicitly compute \(f_g(k)\), an intersection of kernels of several maps, we use that Lemma 2.11 is natural in the cubical diagram to obtain:

**Lemma 5.24.** In the Grothendieck group \(gR(G_g' \times S_k)\) of graded algebraic \(G_g' \times S_k\)-representations, we have an equation

\[
f_g(k) = \sum_{j=0}^{k} (-1)^j \text{Ind}_{S_{k-j} \times S_j}^S t_j(\mathbb{Z}) \otimes (1^j).
\]

To get a qualitative understanding of the graded Lie algebras \(f_g(k)\), we describe a method to compute their restriction to a graded \(G_g' \times S_{k-1}\)-representation. To do so, we introduce for a finite set \(S\), the free graded Lie algebra

\[
L_g(S) := L(\mathbb{Q} \{ t_i \mid i \in S \} \otimes H)
\]
generated by elements $t_i$ in degree $2n - 2$ for all $i \in S$ and a copy of the $2g$-dimensional vector space $H$ in degree $n - 1$. This becomes a graded Lie algebra in $G'_g \times \mathfrak{S}_S$-representations if we let $\sigma \in \mathfrak{S}_S$ act trivially on $H$ and by $t_i \mapsto t_{\sigma(i)}$, and let $g \in G'_g$ act trivially on the $t_i$ and by $a \mapsto ga$. For each $j \in S$, there is a map
$$\mathbb{L}_g(S) \longrightarrow \mathbb{L}_g(S \setminus \{j\}),$$
uniquely determined by sending the generator $t_j$ to 0 and by the identity on the remaining generators.

**Lemma 5.25.** There is an isomorphism of algebraic $G'_g \times \mathfrak{S}_{k-1}$-representations
$$\text{Res}_{\mathfrak{S}_{k-1}}^{G_k} f_g(k) \cong \bigcap_{j \in k-1} \ker \left( \mathbb{L}_g(k-1) \to \mathbb{L}_g(k-1 \setminus \{j\}) \right).$$

**Proof.** Consider cubical diagram $k \supset I \mapsto \text{Emb}(k \setminus I, W_{2,1})$ as a map of $(k-1)$-cubes from $\mathcal{X}: k-1 \supset I \mapsto \text{Emb}(k \setminus I, W_{2,1})$ to $\mathcal{Y}: k-1 \supset I \mapsto \text{Emb}(k-1 \setminus I, W_{2,1})$. The objectwise homotopy fibres are given by
$$\frac{k-1}{I} \mapsto W_{2,1} \vee \bigvee_{k-1 \setminus I} S^{2n-1},$$
so by using [MV15, Corollary 5.4.11, Proposition 5.5.4] we see there is a weak equivalence
$$\text{tohofib} \text{Emb}(k \setminus I, W_{2,1}) \simeq \text{tohofib} \bigg( W_{2,1} \vee \bigvee_{k-1 \setminus I} S^{2n-1} \bigg)$$
equivariant for the $\Gamma_g \times \mathfrak{S}_{k-1}$-action. The right hand side consists of 1-connected spaces, and is split up to homotopy by maps which add wedge summands. Using the Hilton–Milnor theorem [Hil55], we can identify the rational homotopy groups of $W_{2,1} \vee \bigvee_{S} S^{2n-1}$ with $\mathbb{L}(Q(t_i \mid i \in k-1) \oplus H)$ as a $\Gamma_g \times \mathfrak{S}_{k-1}$-representation. The result now follows from Lemma 2.10 and the fact that the action factors over $G'_g$. □

As free graded Lie algebras are easily described in terms of Schur functors, as in Section 2.3.3, from Lemma 5.25 we can compute the decomposition of $f_g(k)$ into irreducibles as a $G'_g \times \mathfrak{S}_{k-1}$-representation for sufficiently large $g$. It is easily deduced that in degree $r(n-1)$ it is an odd (resp. even) $G'_g$-representation if and only if $r$ is odd (resp. even), and using a formula for the dimensions of the homogeneous pieces of free graded Lie algebras [Pet00], we can compute the dimension of $f_g(k)$ in each degree. It also shows:

**Proposition 5.26.** For $n \geq 3$ and $* \geq 1$ we have
$$\pi_{*+1}(\text{tohofib} \text{Emb}(k \setminus I, W_{2,1}))_\mathbb{Q} = 0 \quad \text{unless } * = r(n-1) \text{ for } r \geq 2(k-1).$$

**Proof.** We have already observed that $t_g(k)$ is non-zero only degrees of the form $r(n-1)$ for $r > 0$. To show it also vanishes when $r < 2(k-1)$, observe that an element of $\mathbb{L}_g(k)$ lies in $\bigcap_{j \in k-1} \ker(\mathbb{L}_g(k-1) \to \mathbb{L}_g(k-1 \setminus \{j\}))$ if and only if all generators $t_1, \ldots, t_{k-1}$ appear in it at least once; its degree must thus be at least $2(k-1)(n-1)$. □

We give a general procedure to $f_g(k)$ in Section 7, with results in Appendix C, but here we do the computations by hand. We take $g$ to be sufficiently large ($g \geq 5$ will suffice), and fix a basis $a_1, \ldots, a_{2g}$ of $H$.

**Computation 5.27.** Below we shall compute that
$$f_g(2)(n-1) = \begin{cases} (2) & \text{if } r = 2, \\ H \otimes (1^2) & \text{if } r = 3, \\ H^{\otimes 2} \otimes (1^2)^{\otimes n-1} & \text{if } r = 4. \end{cases}$$
First, as a consequence of Lemma 5.25 in each Lie word in \( f_g(2)_{(n-1)} \) must have at least one entry equal to \( t_{12} \). As this generator lies in degree \( 2(n-1) \), it is one-dimensional spanned by \( t_{12} \). By relation (R1), the non-trivial element of \( \mathfrak{g}_2 \) acts on it trivially. Thus this \( G'_g \times \mathfrak{g}_2 \)-representation is \( (2) \), with \( G'_g \)-acting trivially.

Next, by Lemma 5.25, \( f_g(2)_{(n-1)} \) is \( 2g \)-dimensional and is spanned by Lie words with one entry \( t_{12} \) and one entry \( a_j^{(i)} \) for \( i \in \mathbb{Z}/2 \). Using relation (R5), it has a basis of those Lie words with \( i = 1 \). The non-trivial element of \( \mathfrak{g}_2 \) acts by

\[
[t_{12}, a_j^{(1)}] \mapsto [t_{12}, a_j^{(2)}] = -[t_{12}, a_j^{(1)}]
\]

using (R5), so this is the \( G'_g \times \mathfrak{g}_2 \)-representation \( H \otimes (1^2) \).

Finally, by Lemma 5.25, \( f_g(2)_{(n-1)} \) is \( 2g^2 \)-dimensional and spanned by Lie words with one entry \( t_{12} \) and two entries \( x_i^{(r)} \) with \( i, r \in \mathbb{Z}/2 \). Using the Jacobi relation and anti-symmetry, it is spanned by Lie words of the form

\[
[a_i^{(r)}, [t_{12}, a_j^{(s)}]].
\]

Here we allow \( r = s \), but using relation (R5) we can replace a term \( a_i^{(1)} \) in the inner bracket with \( -a_i^{(2)} \), and vice versa. Using the Jacobi relation, anti-symmetry, and relation (R4), we may assume the superscript \( (2) \) lies in the inner bracket. Thus, it is spanned by the two Lie words \( [a_i^{(1)}, [t_{12}, a_j^{(2)}]] \), which necessarily form a basis. The non-trivial element of \( \mathfrak{g}_2 \) acts by

\[
[a_i^{(1)}, [t_{12}, a_j^{(2)}]] \mapsto [a_i^{(2)}, [t_{12}, a_j^{(1)}]]
\]

\[
= (-1)^n[t_{12}, [a_j^{(1)}, a_i^{(2)}]] + (-1)^n[t_{12}, [a_i^{(1)}, a_j^{(2)}]] \quad \text{Jacobi}
\]

\[
= (-1)^n\lambda(a_j, a_i)[t_{12}, t_{12}] + (-1)^n[a_i^{(1)}, a_i^{(2)}, t_{12}] \quad \text{(R6)}
\]

\[
= (-1)^n-1[a_j^{(1)}, [t_{12}, a_i^{(2)}]] \quad \text{anti-symmetry}.
\]

So it is the \( G'_g \times \mathfrak{g}_2 \)-representation \( H^\otimes 2 \otimes (1^2)^{n-1} \).

**Computation 5.28.** We compute that

\[
f_g(2)_{(n-1)} = \begin{cases} (1^3) & \text{if } r = 4, \\ H \otimes (2,1) & \text{if } r = 5. \end{cases}
\]

First, \( f_g(3)_{(n-1)} \) restricts to \( \text{Lie}(2) = (1^2) \). This is the restriction of a unique \( \mathfrak{g}_3 \)-representation, namely \( (1^3) \).

Next, \( f_g(3)_{(n-1)} \) restricts to the \( G'_g \times \mathfrak{g}_2 \)-representation \( H \otimes ((2) + (1^2)) \), which is \( 2g \)-dimensional. To determine it, we observe that it is spanned by elements of the form

\[
[t_{ij}, [t_{jk}, a_s^{(l)}]].
\]

Such an element is zero unless \( l \in \{i, j, k\} \), by relation (R4). By relation (R5) \([t_{jk}, a_s^{(j)}] = a_s^{(k)} = 0\), so it is spanned by terms \([t_{ij}, [t_{jk}, a_s^{(j)}] = a_s^{(k)}] \). The Jacobi relation gives

\[
-[a_s^{(j)} - a_s^{(k)}, [t_{ij}, t_{jk}]] = [t_{ij}, [t_{jk}, a_s^{(j)}] - a_s^{(k)}] - [t_{ij}, [t_{jk}, a_s^{(j)}] - a_s^{(k)}]
\]

\[
= [t_{ij}, [t_{jk}, a_s^{(j)}] - a_s^{(k)}] - \frac{1}{2}[t_{ij}, [t_{jk}, a_s^{(j)}] - a_s^{(k)}]
\]

Relation (R3) says \([t_{ij}, t_{jk}] = -[t_{ij}, t_{ik}] \), so \([a_s^{(j)} - a_s^{(k)}, [t_{ij}, t_{jk}]]\) is equal to

\[
[a_s^{(j)} - a_s^{(k)}, [t_{ij}, t_{ik}]] = -[t_{ij}, [t_{ik}, a_s^{(j)}] - a_s^{(k)}] + [t_{ki}, [t_{ij}, a_s^{(j)}] - a_s^{(k)}]
\]

\[
= -\frac{1}{2}[t_{ij}, [t_{ik}, a_s^{(j)}] - a_s^{(k)}] - \frac{1}{2}[t_{ki}, [t_{ij}, a_s^{(j)}] - a_s^{(k)}].
\]

For an ordered triple \((i, j, k) \in \mathbb{Z}^3\) of distinct elements, write

\[
z_{i,j,k} := [t_{ij}, [t_{jk}, a_s^{(j)}] - a_s^{(k)}].
\]
Then $f_g(\mathcal{I})_{5(n-1)}$ is the tensor product of $H$ with the quotient of the permutation representation on ordered triples $(i, j, k) \in \mathbb{Z}^3$ of distinct elements. From the above computations we obtain the relation

$$z_{i,j,k} + \frac{1}{2}z_{i,k,j} - \frac{1}{2}z_{j,k,i} = 0.$$  

The quotient of the permutation representation by this relation is $(2, 1)$, and hence there are no further relations. We conclude that $f_g(\mathcal{I})_{5(n-1)} = H \otimes (2, 1)$.

**Computation 5.29.** We compute that

$$f_g(\mathcal{I})_{6(n-1)} = (2^2),$$  

using that it restricts to $\text{Lie}(3) = (2, 1)$. This is the restriction of a unique $\mathfrak{S}_4$-representation, namely $(2^2)$.

### 5.6. The cohomology of products relative to diagonals.

Our next goal is to compute the term $H^*(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q})$. Recall that here $\Delta_{1/2\partial} \subset W^k_{g,1}$ denotes the subspace of those $k$-tuples $(x_1, \ldots, x_k)$ such that $x_i = x_j$ for some $i \neq j$ or $x_i \in 1/2\partial$ for some $i$.

Recall that a partition $\lambda \vdash k$ is a collection of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ with $k = \sum \lambda_i$. Given such a partition, we write $\mu_i = \# \{ i \mid \lambda_i = j \}$ for the number of $\lambda_i$'s which are equal to $j$, so $k = \sum j \cdot \mu_j$. For a group $G$ we write $G/\Sigma_p = G^p \ltimes \Sigma_p$ for the wreath product. Even though $H^*$ can be $G_g$-equivariantly identified with $H$, for Section 6.5 it will be useful to distinguish the cohomology group $H^* = H^n(W^k_{g,1}; \mathbb{Q})$ from its linear dual. We also write $(H^*)^\otimes_k$ for $(H^*)^\otimes_k$ with $\mathfrak{S}_k$ acting by permuting the terms in the tensor product.

**Theorem 5.30.** The $\Gamma_g \times \mathfrak{S}_k$-representations $H^*(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q})$ are isomorphic to

$$\bigoplus_{\lambda \vdash k} \text{Ind}^\mathfrak{S}_k(\mathfrak{S}_1^\otimes \lambda_1 \times \cdots \times (\mathfrak{S}_p^\otimes \lambda_p)) \left( \bigotimes_{s=1}^p (\text{Lie}(s) \otimes (1^s)) \right) \bigotimes (T_H^0)^{n+s-1}),$$

where the summand given by $\lambda$ has degree $\sum \mu_s(n + s - 1)$.

**Proof.** Consider the non-compact manifold $X := W_{g,1} \setminus 1/2\partial W_{g,1}$. The quotient space $W^k_{g,1}/\Delta_{1/2\partial}$ is the homeomorphic to the 1-point compactification $F(k, X)^+$ of the space $F(k, X)$ of ordered configurations in the manifold $X$. Thus we wish to compute $H_*^*(F(k, X); \mathbb{Q})$, as a $\mathfrak{S}_k \times \Gamma_g$-representation.

We will use the work of Petersen [Pet20]. To express his result, we need to introduce some notation. Firstly, for a $V \in \text{Gr}(\mathbb{Q}\text{-mod})^{\text{FB}}$, we write $\Sigma V$ for the functor

$$\mathbb{Z} \longrightarrow V(\mathbb{Z})[s] \otimes (1^s).$$

Secondly, $\otimes_H$ denotes the Hadamard (that is, objectwise) tensor product on objects of $\text{Gr}(\mathbb{Q}\text{-mod})^{\text{FB}}$. Finally, we write $cW$ in $\text{Gr}(\mathbb{Q}\text{-mod})^{\text{FB}}$ for the constant functor with value the graded vector space $W$.

The 1-point compactification $X^+ \simeq \sqrt{2g}S^n$ is formal, so we may take $H^*(X) = H^*[n]$ as a cdga model for $C^*_L(X)$. Then, [Pet20, Corollary 8.8] (adapted to our grading convention and notation) gives an isomorphism in $\text{Gr}(\mathbb{Q}\text{-mod})^{\text{FB}}$

$$\bigoplus_{k \geq 0} H^*_k(F(k, X); \mathbb{Q}) \cong H^*_{CE}(\Sigma cH^*[n] \otimes_H \text{Lie}).$$

Here $\text{Lie}$ denotes the Lie operad, considered as a monoid in $\text{Gr}(\mathbb{Q}\text{-mod})^{\text{FB}}$ with respect to the composition product; this induces the structure of a Lie algebra object on $\Sigma cH^*[n] \otimes_H \text{Lie}$, with respect to which we form Chevalley–Eilenberg homology. As $H^*[n]$ has zero multiplication, the Lie algebra $\Sigma cH^*[n] \otimes_H \text{Lie}$ has trivial bracket, so the appearance of Chevalley–Eilenberg homology simplifies to

$$\bigoplus_{k \geq 0} H^*_k(F(k, X); \mathbb{Q}) \cong S^*(\Sigma cH^*[n] \otimes_H \text{Lie}) \cong S^*(H^*[n] \otimes_H S\text{Lie}),$$
where $\text{SLie}(\mathfrak{k}) = \text{Lie}(\mathfrak{k})[k - 1] \otimes (1^k)$ (using cohomological grading, so this is operadic suspension). In particular, the the term $H^V[n] \otimes \text{SLie}(\mathfrak{k})$ has homological degree $n + k - 1$ and cardinality grading $k$. The expression in the statement of the theorem is obtained by expanding this out, using the definition of “free commutative algebra $S^*$” under Day convolution.

In Table 1, we gave the representations $\text{Lie}(s)$ for $s \leq 6$, which we can combine with Theorem 5.30 to explicitly describe $H^\ast(W^k_{g,1}, \Delta_{1/2}; \mathbb{Q})$. The following proposition collects the properties that we will typically use.

**Proposition 5.31.** $H^\ast(W^k_{g,1}, \Delta_{1/2}; \mathbb{Q})$ has the following properties:

(i) It is concentrated in degrees $* = k + t(n - 1)$ for $1 \leq t \leq k$.

(ii) In degree $* = k + t(n - 1)$ it is odd (resp. even) when $t$ is odd (resp. even).

(iii) In the highest non-zero degree we have $H^{kn}(W^k_{g,1}, \Delta_{1/2}; \mathbb{Q}) \cong (H^V)^{\otimes k} \otimes (1^k)^{\otimes n}$.

(iv) In the next highest non-zero degree we have $H^{(k - 1)n + 1}(W^k_{g,1}, \Delta_{1/2}; \mathbb{Q}) \cong \text{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_k - 2}^\mathfrak{S}_k H^V \boxtimes ((H^V)^{\otimes k - 2} \otimes (1^{k - 2})^{\otimes n})$.

**Proof.** For (i), we use that in Theorem 5.30, a partition $\lambda + k$ contributes to degree $\sum \mu_s (n + s - 1) = k + (\sum \mu_s)(n - 1)$. The sum $\sum \mu_s$ can take any value from 1 to $k$. For (ii), we then observe that in degree $k + (\sum \mu_s)(n - 1)$ the representation $H^V$ arises with tensor power $\sum \mu_s$.

For (iii), the highest degree corresponds to $\sum \mu_s = k$, meaning that $\mu_1 = k$ and all other $\mu_i$ are zero, i.e. $\lambda = (1^k)$. This contributes

$$\text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_k - 1}^\mathfrak{S}_k \mathbb{Q} \wr (H^V)^{\otimes k} \otimes (1^k)^{\otimes n} = (H^V)^{\otimes k} \otimes (1^k)^{\otimes n}.$$  

Finally, for (iv), the second highest degree corresponds to $\sum \mu_s = k - 1$, which can only arise as $\mu_1 = k - 2$, $\mu_2 = 1$, and all other $\mu_i$ are zero, i.e. $\lambda = (2, 1^{k - 2})$. This contributes

$$\text{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_k - 2}^\mathfrak{S}_1 \mathbb{Q} \wr (H^V)^{\otimes k - 2} \otimes (1^{k - 2})^{\otimes n} \boxtimes (\text{Lie}(2) \otimes (1^2)) \wr (H^V) \otimes (1^1)^{\otimes n + 1},$$

but $\text{Lie}(2) = (1^2)$, giving the claimed expression. 

5.7. **The rational homotopy of the layers.** We now prove the two results announced in Section 5.2.2. First is the qualitative result:

**Proposition 5.32.** For $k \geq 2$ and $* \geq 2$, we have that

$$\pi_\ast(BL_q \text{Emb}_{1/2}(W^k_{g,1})_q) = 0 \quad \text{unless } * = r(n - 1) + k - 2 \text{ for } r \geq k - 2.$$  

Furthermore, these are algebraic $\overline{C}_g^{[\ast]}$-representations and can only contain non-zero invariants when $r$ is even.

**Proof.** Recall from Section 5.2.2 that there is a Federer spectral sequence $E^2_{p,q} \Rightarrow \pi_{q-p}(L_k \text{Emb}_{1/2}(W^k_{g,1})_d)$, with rationalised $E^2$-page as in (23):

$$E^2_{p,q} = (H^p(W^k_{g,1}, \Delta_{1/2}; \mathbb{Q}) \otimes \pi_q(\text{toho fb} \text{Emb}(k \setminus I, W^k_{g,1})))^{\otimes n}.$$  

On one hand, Proposition 5.26 says that this vanishes unless $q - 1 = r'(n - 1)$ for $r' \leq 2(k - 1)$. On the other hand 5.31 (i) says this vanishes unless $p = k + t(n - 1)$ for $1 \leq t \leq k$. Hence the total degrees $q - p$ in which it can be non-zero are given by $r(n - 1) - k + 1$ for $r \geq k - 2$. The statement follows once we shift degrees by 1. \(\square\)
It also follows from this argument that the Federer spectral sequence has no differentials, and hence that \( \pi_*(L_{k,\text{Emb}}(W_{g,1})_{\partial})_{\mathbb{Q}} \) has a finite filtration with associated graded given by the terms \((FE^2_{p,q})_{\mathbb{Q}} \) with \( * = q - p \). To justify the claims in Section 5.2.2, it remains to explicitly compute the invariants
\[
[\pi_*(BL_{k,\text{Emb}}(W_{g,1})_{\partial})_{\mathbb{Q}}]^{\mathcal{G}_q[i]}
\]
in degrees \( * \leq 4n - 10 \). By our estimate above, only the layers \( k = 2, 3, 4 \) can have non-zero invariants in this range of degrees. In the following computations we show they are as shown in Figure 2. We assume \( g \) is sufficiently large, cf. Section 2.3.2, and performed our computations in SageMath [Sag20].

**Computation 5.33.** For \( k = 2 \), we get contributions in total degree \( q - p = 2n - 2 \) from two terms:
\[
H^{2n}(W_{g,1}^2, \Delta_{\partial}; \mathbb{Q}) \otimes f_6(2)_{4(n-1)}; \quad H^{n+1}(W_{g,1}^2, \Delta_{\partial}; \mathbb{Q}) \otimes f_6(2)_{3(n-1)}.
\]
We will prove that both have trivial \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \times \mathfrak{S}_2 \)-invariants, using Computation 5.27 and Theorem 5.30.

The first is given by the tensor product of \((H^\vee)^{\otimes 2} \otimes (1^2)^{\otimes n} \) and \( H^{\otimes 2} \otimes (1^2)^{\otimes n-1} \). When we identify \( H^\vee \) with \( H \) and take \( \mathfrak{S}_2 \)-invariants, we get two copies of \( S^2(H) \otimes \Lambda^2(H) \). Decomposing this into a direct sum of irreducible \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \)-representations, we find it contains no \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \)-invariants. The second is given by the tensor product of \( H^\vee \otimes (1^2) \) and \( H \otimes (2) \), which has trivial \( \mathfrak{S}_2 \)-invariants.

**Computation 5.34.** For \( k = 3 \), we get a contribution in total degree \( q - p = 2n - 3 \) from two terms:
\[
H^{3n}(W_{g,1}^3, \Delta_{\partial}; \mathbb{Q}) \otimes f_6(3)_{5(n-1)}; \quad H^{n+2}(W_{g,1}^3, \Delta_{\partial}; \mathbb{Q}) \otimes f_6(3)_{4(n-1)}.
\]
We will prove that the first has trivial \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \times \mathfrak{S}_3 \)-invariants while the latter contains a one-dimensional space of invariants, using Computation 5.28 and Theorem 5.30.

The first is given by the tensor product of \((H^\vee)^{\otimes 2} \otimes (1^3)^{\otimes n} \) and \( H \otimes (2, 1) \). When we identify \( H^\vee \) with \( H \) and take \( \mathfrak{S}_3 \)-invariants, we get \( S_2(H) \otimes H \) and this has one-dimensional \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \)-invariants. The second is given by the tensor product of \( (H^\vee)^{\otimes 2} \otimes ((3) + (2, 1)) \) and \( (1^3) \), which has trivial \( \mathfrak{S}_3 \)-invariants.

**Computation 5.35.** For \( k = 4 \), we get a contribution in degree \( 2n - 4 \) from a single term:
\[
H^{4n}(W_{g,1}^4, \Delta_{\partial}; \mathbb{Q}) \otimes f_6(4)_{6(n-1)}
\]
Using Computation 5.29 and Theorem 5.30, this is given by the tensor product of \((H^\vee)^{\otimes 2} \otimes (1^4)^{\otimes n} \) and \((2, 2) \). When we identify \( H^\vee \) with \( H \) and take \( \mathfrak{S}_4 \)-invariants, we get \( S_2(H) \) and this has a one-dimensional subspace of \( \mathcal{G}^{[\mathfrak{fr},[\ell]}_q \)-invariants.

### 6. Proof of the Main Theorems

We now prove the results announced in the introduction.

#### 6.1. Preparation.

The proofs of our main theorems are based on the diagram
\[
\begin{array}{ccc}
B\text{Diff}^{fr}_{g}(D^{2n})_{\ell_0} & \xrightarrow{X_1(g)} & B\text{Tor}^{fr}_{g}(W_{g,1})_{\ell} \\
\downarrow & & \downarrow \\
B\text{TorEmb}^{fr,\infty}_{1/\partial}(W_{g,1})_{\ell} & \xrightarrow{X_0} & X_0
\end{array}
\]
(30)

in which the row is the top row of (15), \( B\text{TorEmb}^{fr,\infty}_{1/\partial}(W_{g,1})_{\ell} \) is the covering space of \( B\text{TorEmb}^{fr,\infty}_{1/\partial}(W_{g,1})_{\ell} \) making the vertical map to it be \( \pi_1 \)-surjective, and the column is
obtained by looping the Weiss fibre sequence of Theorem 3.12 then restricting to certain path-components and taking certain covering spaces. In particular, the column deloops. Combining Lemma 3.10 and 4.25, we get that all the spaces in (30) are nilpotent and rationally simple. In particular, it makes sense to work with their rationalisations.

**Definition 6.1.** Write $F_n$ for the common homotopy fibre of the dashed compositions
\begin{align}
X_1(g)_\mathbb{Q} &\rightarrow (\mathcal{B}\text{TorEmb}_{/20}(W_{g,1})\ell)\mathbb{Q} \\
(\mathcal{B}\text{Diff}^\ell_2(D^{2n})_0)\mathbb{Q} &\rightarrow (X_0)\mathbb{Q}
\end{align}
formed from (30), identifying the rationalisations of $\mathcal{B}\text{Diff}^\ell_2(D^{2n})_0$ and $\mathcal{B}\text{Diff}^\ell_2(D^{2n})_0$.

By the second description, the space $F_n$ is indeed independent of $g$ as the notation suggests. As $(\mathcal{B}\text{TorEmb}_{/20}(W_{g,1})\ell)\mathbb{Q}$ is simply-connected as a consequence of Lemma 3.10, and $X_1(g)$ is connected, the space $F_n$ is connected and simple. In fact, we have the following identification.

**Proposition 6.2.** There is an equivalence $F_n \simeq (\Omega_O^{2n+1-\text{Top}_n}(\text{Top}_n)^{2n})\mathbb{Q}$, and this space is $(2n - 5)$-connected.

**Proof.** Using the Morlet equivalence $\Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n) \simeq \mathcal{B}\text{Diff}^\ell_2(D^{2n})_0$, we have a map
\[
f : \mathcal{B}\text{Diff}^\ell_2(D^{2n})_0 \rightarrow \Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n)
\]
with homotopy fibre $\mathcal{B}\text{Diff}^\ell_2(D^{2n})_0$. The target of this map has rational cohomology an exterior algebra on the classes $\Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n)^{2n}$ with $4j - 2n - 1 > 0$. By Lemma 4.8 the rational cohomology of $X_0$ is an exterior algebra on classes $\sigma_{4j - 2n - 1}$ with $4j - 2n - 1 < 0$, so after rationalising there are equivalences
\[
(\Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n))_\mathbb{Q} \xrightarrow{\sim} \prod_{d 
mid 2n - 1 \mod 4} K(\mathbb{Q}, d) \xleftarrow{\sim} (X_0)\mathbb{Q},
\]
where the left-hand map is given by the $\Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n)$, and the right-hand map is given by the $\sigma_{4j - 2n - 1}$. By Theorem 4.24 the maps (32) and $f$ are—considered as maps to the product of Eilenberg–Mac Lane spaces—rationally homotopic, and so their homotopy fibres are rationally equivalent.

For the connectivity statement, by [RW17, Corollary 4.2] the space $\mathcal{B}\text{Diff}^\ell_2(D^{2n})_0 \rightarrow \text{Top}_n$ is rationally $(4n - 5)$-connected. As $O \rightarrow \text{Top}_n$ is a rational equivalence, it follows that
\[
(\Omega_0^{2n+1-\text{Top}_n}O)\mathbb{Q} \rightarrow (\Omega_0^{2n+1-\text{Top}_n}(\text{Top}_n))\mathbb{Q} \simeq F_n
\]
is $(2n - 5)$-connected. One calculates that the left-hand side is $(2n + 1)$-connected, so it follows that $F_n$ is $(2n - 5)$-connected. □

We finish our preparation by explaining how the rational homotopy groups of $F_n$ and all spaces appearing in (30) have the structure of algebraic $\overline{\mathcal{G}}_{g,[\ell]}$-representations, in a compatible way.

By the diagram (15), the row of (30) consists of spaces equipped with compatible actions of $\pi_1(A_2(g))$ in the unbased homotopy category. By Corollary 4.20, at the level of rational homotopy groups this action factors through the quotient $\pi_1(A_2(g)) \rightarrow \overline{\mathcal{G}}_{g,[\ell]}$, giving algebraic $\overline{\mathcal{G}}_{g,[\ell]}$-representations.

The column of (30) consists of spaces equipped with compatible actions of $\overline{\mathcal{G}}_{g,[\ell]}$ in the unbased homotopy category, and as these spaces are rationally simple this makes their rational homotopy groups into $\overline{\mathcal{G}}_{g,[\ell]}$-representations, which are algebraic as we have discussed in Section 3.2. The two $\overline{\mathcal{G}}_{g,[\ell]}$-actions on the rational homotopy groups of $\mathcal{B}\text{TorEmb}_{/20}(W_{g,1})\ell$ agree.

As $(\mathcal{B}\text{TorEmb}_{/20}(W_{g,1})\ell)\mathbb{Q}$ is simply-connected, there is an induced unbased $\pi_1(A_2(g))$-action on the homotopy fibre $F_n$, inducing an action on its (rational) homotopy groups
making them gr-algebraic $\pi_1(A_2(g))$-representations: by Lemma 4.15 this action descends to $G_{\ell}^{fr,[d]}$, and makes them algebraic $G_{\ell}^{fr,[d]}$-representations.

6.2. A consequence for embedding calculus. The observations of the last section have a consequence for the differentials in Bousfield–Kan spectral sequence described in Section 5.2.3 in the band $[2n-4,2n-1]$: the spectral sequence in these degrees is shown in Figure 2.

Corollary 6.3. For all large enough $g$ we have

\[
\left[ \pi_{2n-4}(B\text{TorEmb}_{fr,[d]}^{r,n}(W_{g,1})) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] = 0
\]

\[
\left[ \pi_{2n-3}(B\text{TorEmb}_{fr,[d]}^{r,n}(W_{g,1})) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] = 0.
\]

Proof. For the first of these we consider the portion of the $G^{fr,[d]}_{\ell}$-invariant part of the long exact sequence for the fibration (31) given by

\[
\left[ \pi_{2n-4}(X_1(g)) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] \rightarrow \left[ \pi_{2n-4}(B\text{TorEmb}_{fr,[d]}^{r,n}(W_{g,1})) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] \rightarrow \pi_{2n-5}(F_n).
\]

By the second part of Proposition 6.2 the right-hand term vanishes; by the second part of Proposition 4.21 the left-hand term vanishes for all large enough $g$ (as $2n-4 \neq 2n-1$ and $2n-4 < 4n - 3$); thus the middle term vanishes too.

For the second of these we consider the $G^{fr,[d]}_{\ell}$-invariant part of the Bousfield–Kan spectral sequence from Section 5.2.3. The fourth band is the range of degrees $[4n-8,4n-3]$, and $2n - 3 < 4n - 8$ as long as $n \geq 3$ so this band does not interact with degrees $2n-3$ and $2n-4$. Consulting Figure 2, it follows that the differential

\[
d^1: \left[ (BKE_{2,2n-1}^1) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] \rightarrow \left[ (BKE_{2,2n-1}^1) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] = \mathbb{Q}
\]

must be surjective and hence an isomorphism, as this is the only way for the latter term to die in this spectral sequence. As the former term is the only one contributing to total degree $2n-3$, the required vanishing follows. $\square$

6.3. The long exact sequence. Both $\pi_*(BDiff^+_{fr}(D^{2n}))_\ell$ and $\pi_*(X_0)_\ell$ are trivial $G^{fr,[d]}_{\ell}$-representations, so considering $F_n$ to be the homotopy fibre of (32), it follows that the homotopy groups $\pi_*(F_n)$ are also trivial representations of this group.

Now considering $F_n$ as the fibre of (31), we can form the long exact sequence on homotopy groups. Taking $G^{fr,[d]}_{\ell}$-invariants, which is exact for $g \geq 2$, we obtain a long exact sequence

\[
\cdots \rightarrow \pi_*(F_n) \rightarrow \left[ \pi_*(X_1(g))_{\ell} \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] \rightarrow \left[ \pi_*(B\text{TorEmb}_{fr,[d]}^{r,n}(W_{g,1}))_{\ell} \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] \cdots \rightarrow .
\]

Proposition 6.4. $\pi_*(F_n)$ is supported in degrees $* \in \bigcup_{r \geq 2}[2r(n-2) - 1, 2r(n-2)]$.

Proof. As these groups are independent of $g$, we may suppose it is arbitrarily large. In Proposition 4.21 we have computed that

\[
\left[ \pi_{2n-1}(X_1(g)) \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right] = \mathbb{Q}
\]

and that apart from this $\left[ \pi_*(X_1(g))_{\ell} \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right]$ is supported in degrees $* \in \bigcup_{r \geq 2}[2r(n-1) + 1, 2r(n-2)]$.

In Corollary 5.3 we have shown that

\[
\left[ \pi_*(B\text{TorEmb}_{fr,[d]}^{r,n}(W_{g,1}))_{\ell} \right]_{\ell} \left[ G^{fr,[d]}_{\ell} \right]
\]

is supported in degrees $* \in \bigcup_{r \geq 1}[2r(n-2), 2r(n-1) + 1]$, and in fact by Figure 2 and Corollary 6.3 it is supported in degrees $\{2n-1\} \cup \bigcup_{r \geq 2}[2r(n-2), 2r(n-1) + 1]$. 

Remark 6.5 (Miraculous cancellation). This discussion shows that the kernel and cokernel of
\[ \pi_*(X_1(g))\mathbb{Q} \rightarrow \pi_*(B\mathrm{TorEmb}_{g,1}^{\mathbb{Q}}(W_{g,1}))\mathbb{Q} \]
are trivial $G$-representations, and independent of $g$. From the way we have proposed to calculate these groups, in Sections 4 and 5, this is entirely opaque.

By Proposition 4.21 the $r$th band for $\pi_*(X_1(g))\mathbb{Q}$ is $[r(n-1)+1, r(n-2)]$ as long as $r \geq 3$, and is $r(n-1)+1$ for $r \leq 3$. By Corollary 5.3 the $r$th band for $\pi_*(B\mathrm{TorEmb}_{g,1}^{\mathbb{Q}}(W_{g,1}))\mathbb{Q}$ is $[r(n-2), r(n-1)+1]$. In the range where such bands do not overlap, it follows that nontrivial representations may only arise in degrees of the form $r(n-1)+1$.

On one hand, this provides a highly non-trivial verification for the explicit calculations of $\pi_*(B\mathrm{Emb}_{g,1}^{\mathbb{Q}}(W_{g,1}))\mathbb{Q}$ described in Section 7 and Appendix C. On the other hand, in future work we intend to use this miracle to establish Koszulness of the algebra $H^*(X_1(g); \mathbb{Q})$.

6.4. Proof of the main theorems. We will now explain how to deduce our main results from this, starting with the result that the homotopy of $\Omega_{0}^{2n+1}(\frac{\text{Top}(2n)}{\text{Top}(2n)})$ is supported in certain bands.

Proof of Theorem C. This is an immediate consequence of Propositions 6.2 and 6.4. 

The next results concern the rational homotopy of $B\text{Diff}_\beta(D^{2n})$, respectively in degrees $\leq 4n-10$ and outside certain bands.

Proof of Theorems A and B. Using Morlet’s equivalence $\Omega_{0}^{2n}(\frac{\text{Top}(2n)}{\text{Top}(2n)}) \simeq B\text{Diff}_\beta(D^{2n})$, the long exact sequence on rational homotopy groups for the fibration sequence
\[ \Omega^{2n+1}(\frac{\text{Top}(2n)}{\text{Top}(2n)}) \rightarrow \Omega^{2n}(\frac{\text{Top}(2n)}{\text{Top}(2n)}) \rightarrow \Omega^{2n}(\frac{\text{Top}(2n)}{\text{Top}(2n)}), \]
combined with Theorem C and the known rational homotopy groups of $\frac{\text{Top}(2n)}{\text{Top}(2n)}$, gives Theorems A and B.

We end with the comparison between $B\text{STop}(2n)$ and $B\text{STop} \times K(\mathbb{Z}, 2n)$.

Proof of Corollary D. Combining Proposition 6.2 and Theorem C we see that the map
\[ s \times e : B\text{STop}(2n) \rightarrow B\text{STop} \times K(\mathbb{Z}, 2n) \]
given by stabilisation and the Euler class is an epimorphism on rational homotopy groups in degrees $2n+2 \leq * \leq 6n-8$, and an isomorphism in degrees $2n+2 \leq * \leq 6n-9$.

On the other hand by [KS77, p. 246], the map
\[ \frac{\text{STop}(2n)}{\text{SO}(2n)} \rightarrow \frac{\text{Top}(2n)}{\text{O}(2n)} \]
is $(2n+2)$-connected, and the target is rationally contractible. As the analogous map $s \times e : B\text{SO}(2n) \rightarrow B\text{SO} \times K(\mathbb{Z}, 2n)$ is $(4n-1)$-connected it follows that (33) is also an isomorphism on rational homotopy groups in degrees $* \leq 2n+1$. Thus (33) is rationally $(6n-8)$-connected, which proves Corollary D.
6.5. The reflection automorphism. Conjugating by the reflection \( r \) of the disc \( D^{2n} \) in some hyperplane gives an (outer) automorphism of the group \( \text{Diff}_\partial(D^{2n}) \), and hence an automorphism of \( B\text{Diff}_\partial(D^{2n}) \). Under the Morlet equivalence \( B\text{Diff}_\partial(D^{2n}) \cong \Omega_0^\text{Top}(2n)/O(2n) \) this involution is given by sending a map
\[
f: (D^{2n}, \partial D^{2n}) \to \left( \frac{\text{Top}(2n)}{O(2n)} \right),
\]
given by \( f(x) = h(x) O(2n) \) for homeomorphisms \( h(x) \), to the map \( \bar{f}(x) = r \circ h(r(x)) \circ r \circ O(2n) \) (where we also write \( r \) for the corresponding reflection of \( \mathbb{R}^{2n} \)). The analogous formula makes
\[
\Omega_0^\text{Top}(2n)/O(2n) \to \Omega_0^\text{Top}(2n)/O(2n)
\]
a map of spaces with involution, and so induces an involution up to homotopy on the homotopy fibre \( \Omega_0^{2n+1}/\text{Top}(2n) \cong F_n \).

**Theorem 6.6.** These reflection involutions act as follows:

(i) On \( \pi_*\left( \Omega_0^{2n}/\text{Top}(2n) \right) \) it acts by multiplication by \((-1)\).

(ii) On \( \pi_*\left( \Omega_0^{2n+1}/\text{Top}(2n+1) \right) \) it acts by \((-1)^r\).

In (ii) when two such bands overlap then we make no conclusion.

As an application of this, a non-trivial class of \( \pi_*\left( \Omega_0^{2n+1}/\text{Top}(2n+1) \right) \) in the \((-1)^r\)-eigenspace of this involution must have degree \( \geq 6n - 13 \), so we obtain the following extension of Theorem A.

**Corollary 6.7.** Let \( 2n \geq 6 \). Then the map
\[
\pi_*\left( B\text{Diff}_\partial(D^{2n}) \right) \cong \pi_*\left( \Omega_0^{2n}/\text{Top}(2n) \right) = \begin{cases} \mathbb{Q} & \text{if } d \geq 2n - 1 \text{ and } d \equiv 2n - 1 \text{ mod } 4, \\ 0 & \text{otherwise.} \end{cases}
\]
is surjective in degrees \(* \leq 6n - 13\).

**Example 6.8.** In particular, \( \pi_*\left( B\text{Diff}_\partial(D^6) \right) \equiv \mathbb{Q} \) and \( p_3 \neq 0 \in H^{12}(B\text{Top}(6); \mathbb{Q}) \).

**Example 6.9.** Let \( C(D^{2n-1}) \) be the topological group of diffeomorphisms \( f \) of \( D^{2n-1} \times [0, 1] \) which fix \( D^{2n-1} \times \{0\} \cup \partial D^{2n-1} \times [0, 1] \) pointwise. This fits into a fibration sequence studied in pseudoisotopy theory,
\[
\text{Diff}_\partial(D^{2n}) \to C(D^{2n-1}) \to \text{Diff}_\partial(D^{2n-1}).
\]
This is a fibration sequence of spaces with involution, if we use the one described aboe on \( \text{Diff}_\partial(D^{2n}) \) and the “duality” involution on \( C(D^{2n-1}) \), given by reflection in \([0, 1]\) and composition with \( f^{-1}\left|_{D^{2n-1} \times \{1\}} \right. \times \text{id}_{[0, 1]} \). The induced involution on \( \text{Diff}_\partial(D^{2n-1}) \) is inversion, so acts by \(-1\) on all homotopy groups.

Thus, the \((+1)\)-eigenspaces of \( \pi_*\left( B\text{Diff}_\partial(D^{2n}) \right) \) inject into \( \pi_*\left( BC(D^{2n-1}) \right) \). This is in particular the case for non-zero such eigenspaces in the fourth band, which we will discuss in Section 7.5.

To prove Theorem 6.6 we analyse how such reflection involutions act on all our constructions. The manifold \( W_{g,1} \) has an involution \( r_g \) which induces a reflection on \( \partial W_{g,1} = S^{2n-1} \), and with some care one can make the Weiss fibre sequence
\[
B\text{Diff}_\partial(D^{2n}) \to B\text{Diff}_\partial(W_{g,1}) \to B\text{Emb}^{\mathbb{R}}_{g/2g}(W_{g,1})
\]
be one of spaces with involution. However the way we have accessed \( B\text{Diff}_\partial(D^{2n}) \) is by the framed version of this fibre sequence, so we must explain how to obtained reflection automorphisms on the framed versions of these spaces. In fact it will be convenient to work with a variant which is not quite framings, but is close to it.

To construct these we must instead work with the “unoriented version of framings”, which is the tangential structure given by the left \( \text{GL}_{2n}(\mathbb{R}) \)-space \( \text{GL}_{2n}(\mathbb{R})/O(1) \). We write \( \pm fr \) for this tangential structure. The natural map \( \text{GL}_{2n}(\mathbb{R}) \to \text{GL}_{2n}(\mathbb{R})/O(1) \)
associates to any fr-structure $\ell$ a ±fr-structure $\ell^\pm$. It follows from elementary obstruction theory that the maps

$$B\text{Diff}^\pm_\partial(D^{2n})_\ell \rightarrow B\text{Diff}^\pm_\partial(D^{2n})_{\ell^\pm} \quad \text{and} \quad B\text{Diff}^{\pm\ell}_\partial(W_g,1)_\ell \rightarrow B\text{Diff}^{\pm\ell}_\partial(W_g,1)_{\ell^\pm}$$

are weak equivalences.

**Lemma 6.10.** There is a ±fr-structure $\ell^\pm: \text{Fr}(TW_{g,1}) \rightarrow \text{GL}_{2n}(\mathbb{R})/O(1)$ which is invariant under the reflection $r_g$.

**Proof.** Let us first give a useful construction of the reflection $r_g$, see Figure 3. The manifold $W_g = \#^2 S^n \times S^n$ may be obtained by gluing together two copies of the handlebody $V' = \#^2 S^n \times D^n$ along their common boundary $\partial V' = \#^2 S^n \times S^{n-1}$. Choosing a $D^{2n-1} \subset \partial V_g$, we can modify the smooth structure on $V_g$ to make it a manifold with corners, whose boundary strata are $\partial_0 V_g := D^{2n-1}$ and $\partial_1 V_g := \partial V_g \setminus \text{int}(D^{2n-1})$ meeting along their common boundary. Gluing two copies of this modified $V_g$ along $\partial_1 V_g$ gives $W_{g,1}$. Swapping the two copies of $V_g$ therefore gives an involution on $W_{g,1}$: this is our model for the reflection $r_g$.

The fixed set of this involution is $\partial_0 V_g$, and $TW_{g,1}|_{\partial_0 V_g} = e^1 \oplus T\partial_1 V_g$. The manifold $\partial_1 V_g \cong \#^2 S^n \times S^{n-1} \setminus \text{int}(D^{2n-1})$ admits a framing $\ell_1: \text{Fr}(\partial_1 V_g) \rightarrow \text{GL}_{2n-1}(\mathbb{R})$, which induces up to a framing

$$\ell_1: \text{Fr}(TW_{g,1}|_{\partial_1 V_g}) = \text{Fr}(e^1 \oplus T\partial_1 V_g) \rightarrow \text{GL}_{2n}(\mathbb{R}).$$

The ±fr-structure $\ell^\pm_1: \text{Fr}(TW_{g,1}|_{\partial_1 V_g}) \rightarrow \text{GL}_{2n}(\mathbb{R})/O(1)$ is then invariant under pre-composition with the derivative of $r_g$, as this acts by inversion in the first coordinate. There is a single obstruction to solving the extension problem

$$\text{Fr}(TW_{g,1}|_{\partial_1 V_g}) \xrightarrow{\ell^\pm_1} \text{GL}_{2n}(\mathbb{R})/O(1) \xrightarrow{\ell^\pm_2} \text{Fr}(V_g),$$

which lies in $H^n(V_g, \partial_1 V_g; \pi_{n-1}(\text{GL}_{2n-1}(\mathbb{R}))/O(1))$. But this is surjected upon by the choices $H^{n-1}(\partial_1 V_g; \pi_{n-1}(\text{GL}_{2n-1}(\mathbb{R})))$ of the framing $\ell'_1$. Thus after perhaps re-choosing $\ell'_1$, the ±fr-structure $\ell^\pm_1$ extends to a ±fr-structure $\ell^\pm_2$ on $V_g$. We can then extend this to a ±fr-structure $\ell^\pm$ on $W_{g,1}$ by demanding that it be $r_g$-invariant. □

Let us write $\text{Diff}_+ (W_{g,1}) \subset \text{Diff}(W_{g,1})$ for the subgroup of those diffeomorphisms which either fix the boundary or induce the same diffeomorphism as the reflection $r_g$ on the boundary. With the choice of ±fr-structure $\ell^\pm$ given by Lemma 6.10,
the group $\text{Diff}_+(W_{g,1})$ acts on $\text{Bun}^{\pm\text{fr}}(\text{Fr}(\partial D^2); \ell^+)$, and furthermore the reflection $r_g \in \text{Diff}_+(W_{g,1})$ fixes $\ell^\pm$. The reflection therefore gives an involution on the space $B\text{Diff}_\partial^{\pm\text{fr}}(W_{g,1})$, given by conjugating diffeomorphisms by $r_g$ and acting by $r_g$ on $\pm\text{fr}$-structures. The point $\ell^\pm$ is a fixed point of this action.

We similarly get an involution of $B\text{Emb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1})$, making the map

$$B\text{Diff}_\partial^{\pm\text{fr}}(W_{g,1})\ell^\pm \rightarrow B\text{Emb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1})\ell^\pm$$

equivariant. Giving $D^{2\nu}$ the standard $\pm\text{fr}$-structure the reflection equips the space $B\text{Diff}_\partial^{\pm\text{fr}}(D^{2\nu})$ with an involution, which makes the $\pm\text{fr}$ Weiss fibre sequence one of spaces with involution. Furthermore the map

$$B\text{Emb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1})\ell^\pm \rightarrow B\text{Emb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1})\ell^\pm$$

becomes a map of spaces with involution (up to homotopy) if we give $G_{\partial}^{\pm\text{fr},|\ell|}$ the involution given by conjugating, inside $\text{GL}_g(\mathbb{Z})$, by the involution $(r_g)_* = H_n(r_g; \mathbb{Z})$. Using this we obtain in an obvious way compatible involutions up to homotopy of all spaces in the diagram (15).

As usual, we use the abbreviations $H = H_n(W_{g,1}; \mathbb{Q})$ and $H^\vee = H^0(W_{g,1}; \mathbb{Q})$.

**Lemma 6.11.** The $G_{\partial}^{\pm\text{fr},|\ell|}$-invariants of $H^{\otimes p} \otimes (H^\vee)^{\otimes q}$ are trivial if $p + q$ is odd. If $p + q = 2k$ then the action of the involution $(r_g)_*$ on these invariants is as $(-1)^{p+k}$.

*Proof.* The first part follows from the discussion in Sections 2.3.1 and 2.3.2, as $H^{\otimes p} \otimes (H^\vee)^{\otimes q}$ is odd when $p + q$ is odd.

If $p = 2k$ is even and $q = 0$, it follows from the fundamental theorem of invariant theory (see [KRW20b, Section 2.1.4] for a description in our language) that these invariants are obtained from $\omega^{\otimes k}$ by permuting the $2k$ factors. (Recall that $\omega = \sum_{i=1}^{2k} f_i \otimes e_i + (-1)^n e_i \otimes f_i \in H^{\otimes 2}$. The reflection $r_g$ (as constructed in the proof of Lemma 6.10, say) acts as $e_i \mapsto e_i$, $f_i \mapsto -f_i$, so sends $\omega \mapsto -\omega$, and hence $\omega^{\otimes k} \mapsto (-1)^k \omega^{\otimes k}$ as claimed.

If $p + q = 2k$, then we can identify $H^{\otimes p} \otimes (H^\vee)^{\otimes q}$ with $H^{\otimes 2k}$ by identifying $H^\vee$ with $H$ through $\omega$. However, by the previous argument $\omega$ is negated by $(r_g)_*$, so the reflection involution acts on invariants in $H^{\otimes p} \otimes (H^\vee)^{\otimes q}$ by $(-1)^q (-1)^k = (-1)^{p+k}$.

*□*

*Proof of Theorem 6.6.* For part (i), the involution on $\Omega^{2n} \frac{\text{Top}(\partial D^{2n})}{\partial \text{Top}(2n)}$ is given by sending

$$f: (D^{2n}, \partial D^{2n}) \rightarrow \frac{\text{Top}(\partial D^{2n})}{\partial \text{Top}(2n)},$$

of the form $f(x) = h(x) \text{O}(2n)$, to the map $\tilde{f}(x) = r \circ h(r(x)) \circ \text{O}(2n)$. As $\text{Top}$ is an infinite loop space, conjugating by $r$ is homotopic to the identity, so this action is homotopic to $x \mapsto h(r(x)) \text{O}(2n)$, in other words with precomposition by a reflection. This acts by multiplication by $-1$.

For part (ii) we use the fibration sequence

$$F_n \rightarrow X_1(g) \rightarrow B\text{TorEmb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1}),$$

which we have explained above is equipped with compatible involutions up to homotopy.

On the one hand, recall that in the stable range the cohomology of $X_1(g)$ is supported in degrees which are multiples of $n$, and is generated as a ring by its $n$th cohomology, namely the classes $\kappa_j(v_1 \otimes v_2 \otimes v_3)$. In particular $H^{nr}(X_1(g); \mathbb{Q})$ is a quotient of $(H^\vee)^{\otimes 3}$, and similarly the degree $nr$ part of $H^*(X_1(g); \mathbb{Q})^{\otimes p}$ is a quotient of $(H^\vee)^{\otimes 3}$. By following the proof of Proposition 4.21 and using Lemma 6.11, we see that in the band of degrees $[2r(n-1) + 1, 2rn - 2]$ for $r \geq 2$, and in degree $2rn - 1$ for $r = 1$, the reflection acts on the $G_{\partial}^{\pm\text{fr},|\ell|}$-invariants in the rational homotopy of $X_1(g)$ as $(-1)^r$ (as usual, with the understanding that this is inconclusive when two such bands overlap).

On the other hand, recall that the $G_{\partial}^{\pm\text{fr},|\ell|}$-invariants of the rational homotopy of $B\text{TorEmb}^{\pm\text{fr},\text{ze}}_\partial(W_{g,1})$ are concentrated in the bands of degrees $[2r(n-2), 2r(n-1) + 1]$. 

These rational homotopy groups have a filtration whose associated graded is given by subquotients of the $\mathcal{C}_g^{fr,[\ell]}$-invariants of

(i) $\pi_*(\text{BHom}_{j/2 \mathbb{Q}}(W_{g,1})|_{G})$, or  
(ii) $\pi_*(\text{BIL}_{j/2 \mathbb{Q}}(W_{g,1})^{\times}|_{G})$.

For (i), by Lemma 5.5 there is a contribution in degree $2r(n - 1) + 1$ given by a direct sum of quotients of $H^V \oplus H^\otimes 2r + 1$. On its $\mathcal{C}_g^{fr,[\ell]}$-invariants, the reflection acts as $(-1)^{r}$. For (ii), there is a contribution for $* = 2r(n - 1) - k$ when $k \leq 2r + 2$, which is a subquotient of

$\left[ H^k(W_{g,1}, \Delta_{j/2 \mathbb{Q}}; \mathbb{Q}) \otimes \pi_q(\text{tohofib} \text{Emb}(k, I, W_{g,1}))_G \right] \otimes_{k}$

for $q - p + 1 = 2r(n - 1) - k$, $q = (2r + s)(n - 1) + 1$, and $p = s(n - 1) + k$. From the description of the left term in Theorem 5.30 and of the restriction to $\mathcal{C}_g^{fr,[\ell]}|_{G}$ of the right term in Lemma 5.25, we see this is a direct sum of quotients of $(H^V)^{\otimes s} \otimes H^\otimes 2r + s$. On the $\mathcal{C}_g^{fr,[\ell]}$-invariants in these the reflection acts as $(-1)^{r}$.

6.6. The rational homotopy of diffeomorphism groups. We may use the discussion in this section to obtain information about the higher rational homotopy groups of $\text{BDiff}_{\beta}(W_{g,1})$ in stable range. To do so we will combine our discussion of the diagram (30) with the fibration

$$\text{map}_{\beta}(W_{g,1}, \text{SO}(2n)) \to \text{BDiff}_{\beta}(W_{g,1}) \to \text{BDiff}_{\beta}(W_{g,1}).$$

By Proposition 6.4 the space $F_n$ is $(4n - 10)$-connected, and hence the map

$$\pi_*(\text{BDiff}_{\beta}(D^{2n})_G) \to \pi_*(X_0)_G$$

is an isomorphism in the range $* \leq 4n - 10$. This implies that the long exact sequence on rational homotopy groups for the row of (30) splits into short exact sequences in this range of degrees. By Proposition 4.21, in the range $* < 4n - 3$ the rational homotopy groups $\pi_*(X_1(g))_G$ are supported in degrees $n, 2n - 1$, and $3n - 2$, and in Proposition 4.22 we determined these groups.

Remark 6.12. The connectivity of $F_n$ also implies that

$$\pi_*(X_1(g))_G \to \pi_*(\text{BTorEmb}_{j/2 \mathbb{Q}}^{fr,\text{fr}}(W_{g,1}))_G$$

is an isomorphism in the range $* \leq 4n - 10$, and so Proposition 4.21 implies that the rational homotopy groups $\pi_*(\text{BTorEmb}_{j/2 \mathbb{Q}}^{fr,\text{fr}}(W_{g,1}))_G$ are supported in degrees $n, 2n - 1$, and $3n - 2$ in the range $* \leq 4n - 10$, and Proposition 4.22 determines them.

From the point of view of Section 5, knowing a priori that the rational higher homotopy groups of $\text{BEmb}_{j/2 \mathbb{Q}}^{fr,\text{fr}}(W_{g,1})_G$ are supported in degrees $n, 2n - 1$, and $3n - 2$ in this range, and given the calculations of Section 5, one may completely determine the behaviour of the Bousfield–Kan spectral sequence (24) near these degrees and hence determine these groups. We invite the reader to do so using the data in Table 4, and verify that it gives the same as Proposition 4.22.

Corollary 6.13. As long as $g$ is large enough and $n \geq 3$ is odd, in degrees $2 \leq * \leq 4n - 10$ there is a short exact sequence

$$0 \to \bigoplus_{i \geq 0} V_i[2n-1+4i] \oplus V_i[3n] \oplus V_i[2n-1] \oplus V_i[3n][2n-2] \to \pi_*(\text{BDiff}_{\beta}(W_{g,1}))_G \to \bigoplus_{i \geq 0} V_i[3n-5+4i] \oplus V_i[3n-4+4i] \to 0$$

exhibiting $\pi_*(\text{BDiff}_{\beta}(W_{g,1}))_G$ as a graded $\Gamma_g$-representation with respect to

$$1 \to \ker [\Gamma_g \to G'_g] \to \Gamma_g \to G'_g \to 1.$$

If $n$ is even then the same holds with all partitions transposed.
Proof. As discussed above in this range of degrees the long exact sequence of rational homotopy groups for the row of (30) splits into short exact sequences, which combined with Proposition 4.22 and Lemma 4.8 shows that
\[ \pi_*(B\text{Tor}_g(W_{g,1}))_\mathbb{Q} \cong \bigoplus_{i \geq -(2n-1)/4} V_0[2n - 1 + 4i] \oplus (V_5 + V_{x+V_{y2}})[2n-1] \oplus (V_{x2} + V_{y3} + V_{y4})[3n-2] \]
as algebraic $G_g^{fr}[t]$-representations in degrees $* \leq 4n - 10$. In degrees $* \geq 2$ this is the same as the rational homotopy groups of $BDiff_g^f(W_{g,1})$.

There is a fibration sequence
\[ \text{map}_g(W_{g,1}, SO(2n)) \rightarrow \pi_*(W_{g,1}, SO(2n)) \rightarrow \Omega^{2n-1}SO(2n) \]
and (cf. [KRW19, Section 8.2.2]) the rightmost map is zero on homotopy groups as it is given by a sum of Whitehead products and these vanish in the $H$-space $SO(2n)$, giving short exact sequences
\[ 0 \rightarrow \bigoplus_{i = -(2n-1)/4} V_0[2n - 1 + 4i] \rightarrow \pi_*(\text{map}_g(W_{g,1}, SO(2n)))_\mathbb{Q} \rightarrow V_1[n - 1] \oplus \bigoplus_{j=0}^{\lfloor (3n-5)/4 \rfloor} V_1[3n - 5 - 4j] \rightarrow 0 \]

Now consider the long exact sequence of rational homotopy groups for the fibration (34). By comparison with the case $g = 0$, the terms $V_0[2n - 1 + 4i]$ with $i < 0$ are sent isomorphically to those of the same name in the description of $\pi_*(BDiff_g^f(W_{g,1}))_\mathbb{Q}$ above, and the $V_1$ terms are sent to zero as this irreducible representation does not arise in $\pi_*(BDiff_g^f(W_{g,1}))_\mathbb{Q}$. This gives the claimed short exact sequence for $\pi_*(BDiff_g(W_{g,1}))_\mathbb{Q}$, using that irreducible algebraic $G_g^{fr}$-representations are determined by their restrictions to $\overline{G_g^{fr}}[t]$ as long as $g$ is large with respect to the number of rows in the corresponding partition (see the discussion in Section 2.3.2).

For $n$ even we use the even case of Proposition 4.22. □

Remark 6.14. It is natural to ask whether the short exact sequence of Corollary 6.13 is split as $\Gamma_g$-representations, or equivalently whether the Torelli group $\ker [\Gamma_g \rightarrow G_g^{fr}]$ acts trivially on $\pi_1(BDiff_g(W_{g,1}))_\mathbb{Q}$, or equivalently whether the Torelli space $B\text{Tor}_g(W_{g,1})$, which is nilpotent, is rationally simple. If $n \not\equiv 3 \pmod{4}$ then the Torelli group is finite and so this is indeed the case.

If $n \equiv 3 \pmod{4}$ then we believe one can show that it is not the case, using the calculation from [KRW20b, Section 5] of the cohomology ring $H^*(B\text{Tor}_g(W_{g,1}); \mathbb{Q})$ in a stable range, the rational unstable Adams spectral sequence (which is a spectral sequence of graded Lie algebras), and the fact that Hirzebruch $L$-classes contain all possible monomials in Pontrjagin classes with nonzero coefficient [BB18]. We leave the details to the interested reader.

Of course using Proposition 4.21 one can also determine $\pi_*(BDiff_g(W_{g,1}))_\mathbb{Q}$ in higher degrees outside certain bands.

7. The extended Drinfel’d–Kohno Lie algebra via Koszul duality

In Section 5.2 we described the Bousfield–Kan spectral sequence (24) for the embedding calculus Taylor tower (22):
\[ BK_{FP}^{1,p,q} = \begin{cases} \pi_{q-p}(B\text{Aut}_{1/2}(W_{g,1})) & \text{if } p = 0, \\ \pi_{q-p}(BL_{p+1}\text{Emb}_{1/2}(W_{g,1})^\times) & \text{if } p \geq 1, \end{cases} \]
This is a completely convergent extended spectral sequence, and when we rationalise the $E^1$-page in total degree $> 1$ it consists of algebraic $\overline{G_g^{fr}}[t]$-representations. In Section 5.3 we described the rational homotopy groups of the homotopy automorphisms in
with relation (R6) replaced by:

\[ (F^E_{p,q})_Q = \left[ H^p(W^k_{g,1}, \Delta_{i/2a}; \mathbb{Q}) \oplus \pi_q(\text{tohofib Emb}(k \setminus I, W_{g,1})) \right]_{G_k}. \]

This is again a completely convergent extended spectral sequence. Rationally, it collapses at the \( E^2 \)-page and consists of algebraic \( G^R_g \)-representations. Thus, we need to find computable descriptions of both terms

\[ H^*(W^k_{g,1}, \Delta_{i/2a}; \mathbb{Q}) \quad \text{and} \quad \pi_q(\text{tohofib Emb}(k \setminus I, W_{g,1}))_\mathbb{Q} \]

as \( \mathcal{F}^R_g \times \mathfrak{S}_k \)-representations. For the former, a formula was given in Theorem 5.30. For the latter, we gave an algebraic description in terms of a graded Lie algebra \( t_g(k) \), obtained by inclusion-exclusion in terms of extended Drinfel’d–Kohno Lie algebras \( t_g(k) \).

In Section 5.5 we gave a presentation for these graded Lie algebras and we worked out a few cases by hand. But what is missing so far is an algorithmic description of the Frobenius character of \( t_g(k) \).

In this section we will provide exactly that. We will give a filtration of \( t_g(k) \) whose associated graded \( \text{gr} t_g(k) \) is, after forgetting its graded Lie algebra structure, the value at \( H[n-1] \) of a functor

\[ \text{Gr}(\mathbb{Q} \text{-mod}^f) \longrightarrow \text{GrRep}(\mathfrak{S}_k) \]

\[ V \longmapsto \bigoplus_k \text{gr} t(s, k) \otimes \mathfrak{S}_k V^\otimes_2, \]

for certain \( \mathfrak{S}_k \)-representations \( \text{gr} t(\cdot, k) \). It thus suffices to determine these coefficients. By Schur–Weyl duality this task is equivalent to determining \( t_g(k) \) when \( n \) is odd and \( \dim(H) \) is arbitrarily large, as a \( \text{GL}(H) \times \mathfrak{S}_k \)-representation.

To do so, we will show that \( \text{gr} t_g(k) \) is the Koszul dual of the Kriz–Totaro algebra. We then give a description of the latter as a free commutative algebra in the category of symmetric sequences with the Day convolution tensor product. Using symmetric functions, we then give an algorithm to compute the Hilbert–Poincaré series of \( \text{gr} t(\cdot, k) \).

In Appendix C we describe the results of these computations for degrees \( < 5n - 10 \). Finally, in Section 7.5 we give the application of these computations referred to in the introduction: a computation of the fourth “band.”

Throughout this section we will suppose \( n \geq 2 \), so that various constructions yield degreewise finite-dimensional graded vector spaces.

7.1. The associated graded of the extended Drinfel’d–Kohno Lie algebra.

We can lift \( t_g(S) \) and \( f_g(S) \) to Lie algebra objects in filtered objects in \( \text{GrRep}(G'_g \times \mathfrak{S}_S) \), by putting \( t_{ij} \) in filtration degree \(-1\) and \( H^{(r)} \) in filtration degree \( 0 \). Their associated graded are Lie algebra objects in \( \text{GrRep}(G'_g \times \mathfrak{S}_S) \). (They have an additional grading which we shall ignore.) As \( g \geq 2 \), algebraic \( G'_g \times \mathfrak{S}_S \)-representations are semi-simple, so there are non-canonical isomorphisms

\[ t_g(S) \cong \text{gr} t_g(S) \quad \text{and} \quad f_g(S) \cong \text{gr} f_g(S) \]

of graded algebraic \( G'_g \times \mathfrak{S}_S \)-representations. The Lie algebra \( \text{gr} t_g(S) \) has a presentation quite similar to that of \( t_g(S) \).

**Lemma 7.1.** There is a presentation for \( \text{gr} t_g(S) \) equal to that in Definition 5.16 but with relation (R6) replaced by:

\[ (R6') \text{ for } a^{(i)} \in H^{(i)} \text{ and } b \in H^{(j)} \text{ with } i, j \text{ distinct, } [a^{(i)}, b^{(j)}] = 0. \]

**Proof.** We may assume that \( S = k \). It is clear that \( \text{gr} t_g(k) \) is generated by the image of elements \( t_{ij} \) and \( a^{(r)} \), which we shall denote the same way, and that these satisfy
the relations (R1)–(R5) and (R6’). Let us denote by \( \widetilde{\text{gr}} t_g(\mathbb{k}) \) the graded Lie algebra presented by these generators and relations, so that there is a surjective map
\[
\widetilde{\text{gr}} t_g(\mathbb{k}) \twoheadrightarrow \text{gr} t_g(\mathbb{k}),
\]
which we must show is an isomorphism. We will do so by induction over \( k \) in the initial case \( k = 1 \) it is the map \( \text{id} : \mathbb{L}(H[n - 1]) \rightarrow \mathbb{L}(H[n - 1]) \).

For the induction step we will use the description of \( t_g(k) \) as an iterated extension of free graded Lie algebras that appeared in the proof of Proposition 5.18: there is a surjective map \( \text{gr} t_g(k) \twoheadrightarrow \text{gr} t_g(k - 1) \), which is uniquely determined by the fact that it sends generators \( t_{ik} \) and \( a^{(k)} \) to 0 and is the identity on the remaining generators. Using same prescription gives the left vertical map in the commutative diagram
\[
\begin{array}{ccc}
\text{gr} t_g(k) & \twoheadrightarrow & \text{gr} t_g(k) \\
\downarrow j_* & & \downarrow i_* \\
\text{gr} t_g(k - 1) & \twoheadrightarrow & \text{gr} t_g(k - 1).
\end{array}
\]

By the induction hypothesis the bottom horizontal map is an isomorphism, and we also know the top horizontal map is surjective. So to prove that the middle map is also an isomorphism, it suffices prove that \( \dim \ker(j'_*) \leq \dim \ker(i'_*) \) for each degree \( d \). But the argument in Lemma 5.21 goes through with (R6') replacing (R6), and hence the kernel of the left vertical map is generated by \( t_{ik} \) and \( a^{(k)} \). This implies that it is a quotient of \( \mathbb{L}(\{t_{ik} \mid i \in k - 1\} \oplus H[n - 1]) \), which supplies the required inequality. □

In particular, this presentation for \( \text{gr} t_g(S) \) only depends on \( H \) as a graded vector space. We generalise this as follows:

**Definition 7.2.** Let \( S \) be a finite set, \( n \geq 0 \), and \( L \) be a graded Lie algebra, then \( \text{gr} t_L(S) \) is the graded Lie algebra given by the quotient of the free graded Lie algebra generated by

1. elements \( t_{ij} \) in degree \( 2n - 2 \) for each pair \((i, j)\) of distinct elements of \( S\),
2. a copy \( L^{(r)} \) of \( L \) for each \( r \in S\),

by the ideal generated by the relations

(R1) \( t_{ij} = t_{ji} \) for \( i, j \) distinct,

(R2) \( [t_{ij}, t_{rs}] = 0 \) for \( i, j, r, s \) all distinct,

(R3) \( [t_{ij}, t_{ik} + t_{jk}] = 0 \) for \( i, j, k \) all distinct,

(R4) for \( a^{(i)} \in L^{(r)} \) and \( i, j, r \) all distinct, \( [t_{ij}, a^{(r)}] = 0 \),

(R5) for \( a^{(i)} \in L^{(i)} \) and \( a^{(j)} \) the corresponding vector in \( L^{(j)} \), \( [t_{ij}, a^{(i)} + a^{(j)}] = 0 \),

(R6) for \( a^{(i)} \in L^{(i)} \) and \( b \in L^{(j)} \) with \( i, j \) distinct, \( [a^{(i)} b^{(j)}] = 0 \),

(R7) for \( a^{(i)} \) and \( b^{(j)} \) in \( L^{(i)} \) and \( L^{(j)} \), \( [a^{(i)} b^{(j)}] = ([a, b], L) \).

This extends to a functor \( t_L(-) : \text{FI}_n^{\text{op}} \rightarrow \text{Alg}_{\text{Lie}}(\text{Gr}(\mathbb{Q} \text{-mod})) \). In particular, \( t_L(S) \) consists of \( \mathfrak{S}_n \)-representations \( \sigma \in \mathfrak{S}_n \) acting by \( t_{ij} \mapsto t_{\sigma(i) \sigma(j)} \) and \( a^{(i)} \mapsto a^{(\sigma(i))} \). This construction is functorial in the graded Lie algebra \( L \), and we will consider its composition with the free graded Lie algebra functor \( \mathbb{L} : \text{Gr}(\mathbb{Q} \text{-mod}) \rightarrow \text{Alg}_{\text{Lie}}(\text{Gr}(\mathbb{Q} \text{-mod})) \). Evaluating at \( H[n - 1] \) then recovers \( \text{gr} t_g(S) \). Forgetting the Lie algebra structure, we obtain a functor
\[
\text{gr} t_L(-)(\mathbb{k}) : \text{Gr}(\mathbb{Q}\text{-mod}) \rightarrow \text{GrRep}(\mathfrak{S}_n).
\]

(Strictly speaking, we need to assume its input is concentrated in strictly positive or negative degrees for this construction to be degreewise finite-dimensional.)

**Lemma 7.3.** This functor is of the form
\[
V \mapsto \bigoplus_{s \geq 0} \text{gr} t_s(\mathbb{k}) \otimes_{\mathfrak{S}_n} V^{\otimes s}
\]
for graded $\mathcal{G}_s \times \mathcal{G}_k$-representations $\text{gr } t(s,k)$.

Proof. We will prove that the functor $\text{gr } t_L(-)(k)$ is degreewise polynomial in the sense of [Mac95, Appendix 1.A]. From the definition of such functors, it follows they are closed under passing to subquotients and tensor products. As $\text{gr } t_L(-)(k)$ is quadratically presented by a pair of functors $\text{Gr}((\mathbb{Q}\text{-mod}) \to \text{GrRep}(\mathcal{G}_k)$, one for the generators and one of the relations, it suffices to observe that the one for the generators is a degreewise polynomial functor. \hfill \Box

We define

$$\text{gr } f_L(V)(S) := \bigcap_{j \in k} \ker \left[ \text{gr } t_L(V)(S) \to \text{gr } t_L(V)(S \setminus \{j\}) \right].$$

This is a subfunctor of $\text{gr } t_L(-)(k)$, and hence is of the form

$$\text{gr } f_L(-)(k): \text{Gr}((\mathbb{Q}\text{-mod}) \to \text{GrRep}(\mathcal{G}_k) \quad V \mapsto \text{gr } f(s,k) \otimes_{\mathcal{G}_s} V^\otimes \mathbb{Z}$$

for graded $\mathcal{G}_s \times \mathcal{G}_k$-representations $\text{gr } f(s,k)$. These are given by

$$\text{gr } f(s,k) = \bigcap_{j \in k} \ker \left[ \text{gr } t(s,k) \to \text{gr } t(s,k \setminus \{j\}) \right],$$

and can be determined in terms of $\text{gr } t(s,k)$ using the formula in Lemma 5.24. We will collect these for all $s$ and fixed $k$ in a single graded $\text{FB} \times \mathcal{G}_k$-representation

$$\text{gr } f(-,k): S \mapsto \text{gr } f(S,k).$$

7.2. The Kriz–Totaro algebra and its Koszul dual. The Kriz–Totaro algebra arises as the $E^1$-page of the Totaro spectral sequence for $\text{Emb}(S,W_{g,1})$ [Tot96, Theorem 1], the cohomological Leray spectral sequence for the inclusion of the fat diagonal into $W_{g,1}$ (see also [CT78, pp. 117–118] and [Kri94, Theorem 1.1]). We deviate from those references by using a homological grading instead of a cohomological one. As a consequence, the Kriz–Totaro will be non-positively graded. Similarly, the cohomology algebra of $W_{g,1}$ is given by $\mathbb{Q} \otimes H^\vee [-n]$, with $\mathbb{Q}$ in degree $0$ and $H^\vee$ in degree $-n$, and multiplication uniquely determined by requiring that $\mathbb{Q}$ is generated by the unit.

Definition 7.4 (Kriz–Totaro algebra). Let $S$ be a finite set, $n \geq 0$, and $A$ be a graded-commutative algebra. The Kriz–Totaro algebra $\mathcal{T}_A(S)$ is the quotient of the free graded-commutative algebra generated by

- (G1) elements $x_{ij}$ of degree $-(2n-1)$ for each pair $(i,j)$ of distinct elements of $S$,
- (G2) a copy $A^{(r)}$ of $A$ for each $r \in S$,

by the ideal generated by the relations

- (R1) $x_{ij} = x_{ji}$ for all $i,j$ distinct,
- (R2) $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$ for $i,j,k$ all distinct,
- (R3) for $\alpha^{(i)} \in A^{(i)}$ and $\beta^{(j)} \in A^{(j)}$ the corresponding element, $x_{ij}\alpha^{(i)} = x_{ij}\alpha^{(j)},$
- (R4) $1_A^{(i)} = 1$ for all $i$,
- (R5) for any $\alpha^{(i)}, \beta^{(j)} \in A^{(i)}$, $\alpha^{(i)}\beta^{(j)} = (\alpha \cdot_A \beta)^{(i,j)}$.

This algebra admits an action of $\mathcal{G}_S$, with $\sigma \in \mathcal{G}_S$ acting by $x_{ij} \mapsto x_{\sigma(i)\sigma(j)}$ and $\alpha^{(i)} \mapsto \alpha^{(\sigma(i))}$, and is natural in $A$. When we take $A = \mathbb{Q} \otimes H^\vee [-n]$ with $H$ of dimension $2g$, we shall shorten $\mathcal{T}_A(S)$ to $\mathcal{T}_g(S)$.

Remark 7.5. To see this is the same as the algebras described in the above references, use (R4) and (R5) to write $\mathcal{T}_A(S)$ as the quotient of $A^{\otimes S}[x_{ij} \mid i \neq j \in S]$ by the relations (R1)–(R3) (that $x_{ij}^2 = 0$ follows as $x_{ij}$ has odd degree).

Our goal is to show that when $A$ has a PBW-basis in the sense of [LV12, Section 4.3.7] and has Koszul dual $L$, then $\mathcal{T}_A(S)$ is Koszul dual to $\text{gr } t_L(S)$ compatibly with
the actions of $\mathfrak{S}_S$ and naturally in $A$. This allows us to relate the Hilbert–Poincaré series of $T_A(S)$ and $\text{gr} t_L(S)$.

We start with a short overview of Koszul duality of algebras over operads, to fix terminology and notation. References for Koszul duality are [GK94, LV12, Mil12].

Recall a (unital symmetric) operad $P$ in chain complexes is a unital algebra in the category $(\text{Ch}_Q)^{FB}$ of symmetric sequences under composition product. Similarly a cooperad is a counital coalgebra in symmetric sequences. We will interpret graded vector spaces as chain complexes with zero differential; $\text{Gr}(\text{Q-mod}) \subset \text{Ch}_Q$.

An operadic quadratic datum is a graded $\mathfrak{S}_2$-vector space $E$ and a $\mathfrak{S}_2$-invariant subspace $R \subset \text{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_2} E \otimes 2$. We will refer to $E$ as “generators” and to $R$ as “relations.” The quadratic datum $(E, R)$ gives rise to both a quadratic operad and a quadratic cooperad. For the former, we start with the free operad $F(E)$ on $E$ considered as a symmetric sequence. Its 3-ary operations $F(E)(3)$ are given by the $\mathfrak{S}_2$-representation $\text{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_2} E \otimes 2$, so contain $R$. The quadratic operad $O(E, R) \in \text{Op}(\text{Ch}_Q)$ is the initial quotient operad $O$ of $F(E)$ such that $R \to F(E) \to O$ is zero. Similarly, the quadratic cooperad $C(E, R) \in \text{CoOp}(\text{Gr}(\text{Q-mod}))$ is the terminal sub-cooperad $C$ of the cofree cooperad $F^c(E)$ such that $C \to F^c(E) \to F^c(E)(2)/R$ is zero.

We define the quadratic dual operad to $P$ to be $P! := (C[1], R[2])$. If $E$ is finite-dimensional, we can take linear duals to get a cooperad. However, we need to add some signs and grading shifts. Recall the operadic suspension $SP$ of an operad $P$ is the operad with $SP(k) = P(k)[1-k] \otimes (k^k)$, with inverse operation given by the desuspension $S^{-1}P$. Then we define the quadratic dual operad to be

$$P^! := S^{-1}(P!)^\vee.$$  

This has a quadratic presentation as $P^! = O(E^\vee \otimes (1^2), R^\perp)$, with $R^\perp$ the annihilator of $R$ under the pairing $F(E^\vee)(3) \otimes F(E)(3) \to Q$ [LV12, Proposition 7.2.4].Sending $P$ to $P^!$ is a duality, i.e. there is an isomorphism $P \cong (P^!)^!$ natural in operadic quadratic data [LV12, Proposition 7.2.5].

By construction, every quadratic operad has an augmentation $O(E, R) \to \mathbb{1}$ to the monoidal unit for the composition product. Similarly, every quadratic cooperad has a coaugmentation $\mathbb{1} \to C(E, R)$. This is the input for the operadic bar and cobar constructions (see [LV12, Section 6.5] for details)

$$B : \text{Op}_\text{aug}(\text{Ch}_Q) \to \text{CoOp}_\text{coaug}(\text{Ch}_Q)$$

$$\Omega : \text{CoOp}_\text{coaug}(\text{Ch}_Q) \to \text{Op}_\text{aug}(\text{Ch}_Q).$$

A quadratic (co)operad may be endowed with a weight grading, by declaring $E$ has weight 1. Thus, for $P = O(E, R)$, the operadic bar construction $BP$ is naturally bigraded and $H_*(BP)$ splits into a direct sum of subspaces $H_*(BP)^r$ of homogeneous weight $r$. By [LV12, Proposition 7.3.2], there is a natural isomorphism

$$P^! \cong \bigoplus_k H_k(BP)^{(k)}.$$  

A quadratic operad $P$ is Koszul if the inclusion $\bigoplus_k H_k(BP)^{(k)} \to H_*(BP)$ is an isomorphism, i.e. the homology of the bar construction is concentrated on the diagonal. Dually, a quadratic cooperad is Koszul if the homology on the cobar construction is concentrated on the diagonal. If $E$ is finite-dimensional, then the operad $P$ is Koszul if and only if $P^!$ is Koszul [LV12, Proposition 7.4.8].

Example 7.6. The relevant examples for us are the associative operad $\text{Ass}$, the commutative operad $\text{Com}$, and the Lie operad $\text{Lie}$. All three are Koszul, and $\text{Ass}^! \cong \text{Ass}$, while $\text{Com}^! \cong \text{Lie}$ and thus $\text{Lie}^! \cong \text{Com}$.

Given an operad $P \in \text{Op}(\text{Ch}_Q)$, a $P$-algebra in $\text{Ch}_Q$ is a $V \in \text{Ch}_Q$ with the structure of a left $P$-module when regarded as a symmetric sequence in cardinality 0. Similarly for a cooperad $C \in \text{CoOp}(\text{Ch}_Q)$, a $C$-coalgebra is a chain complex with the structure of a left $C$-comodule.
Let \((E,R)\) be an operadic quadratic datum, and \(P = \mathcal{O}(E,R)\) and \(C = \mathcal{C}(E,R)\) the associated quadratic operad and cooperad. An algebraic quadratic datum is a chain complex \(V\) and a subspace \(S \subset E \otimes \mathcal{C}_2 V \otimes \mathcal{C}_2 V^*\). We associate to this a quadratic \(P\)-algebra \(A_P(V,S)\) by taking the initial quotient \(P\)-algebra \(A\) of \(P(V)\) such that \(S \to P(V) \to A\) is zero. Similarly, we associate to it a quadratic \(C\)-coalgebra \(C_C(V,S)\) by taking the terminal sub-\(C\)-algebra \(C\) of \(C(V)\) such that \(C \to C(V) \to \langle E \otimes \mathcal{C}_2 V \otimes \mathcal{C}_2 V^* \rangle / S\) is zero.

If \(E\) and \(V\) are finite-dimensional, we form the quadratic dual \(P\)-algebra

\[
A^! := (C_P(\langle V[-1], S[-2] \rangle))^!.
\]

This admits a quadratic presentation as an \(P\)-algebra by \(A_P(V^*[1], S^*[2])\), with \(S^*\) the annihilator of \(S\) under the pairing

\[
E(V^*) \otimes E(V) \to \mathbb{Q}[2],
\]

where \(E(V)\) stands for \(E \otimes \mathcal{C}_2 V \otimes \mathcal{C}_2 V^*\), and \(E(V^*)\) for \((E^* \otimes (1^2)) \otimes \mathcal{C}_2 (V^*[1]) \otimes \mathcal{C}_2 V^*\). Again this is a duality, i.e. there is a natural isomorphism \(A \xrightarrow{\sim} (A^!)^!\).

The construction of the quadratic \(P\)-algebra \(A_P(V,S)\) is natural in the quadratic datum. In particular, an action of a group on a quadratic datum induces an action on both \(A = A_P(V,S)\) and its quadratic dual \(A^!\), such that \(A \xrightarrow{\sim} (A^!)^!\) is equivariant.

**Example 7.7.** For \(P = \text{Com}\), this prescribes a quadratic commutative algebra \(A_{\text{Com}}(V,S)\) as \(S^*(V)/R\) with \(S \subset S^2(V) = (2) \otimes \mathcal{C}_2 V \otimes \mathcal{C}_2 V\) (this involves a Koszul sign rule). For \(P = \text{Lie}\), this prescribes a quadratic graded Lie algebra \(A_{\text{Lie}}(V,S)\) as \(L(V)/S\) with \(S \subset \Lambda^2(V) = (1^2) \otimes \mathcal{C}_2 V \otimes \mathcal{C}_2 V\). The Koszul dual Lie algebra to \(S^*(V)/R\) is given by \(L(V^*[1])/R^1\), with \(R^1\) the annihilator of \(R\) under the evaluation pairing.

By construction, every quadratic algebra has an augmentation \(A_P(V,S) \to Q\), and every quadratic coalgebra has a coaugmentation \(Q \to C_C(S,V)\). This is the input for the bar and cobar constructions (see [Mil12, Section 2.2] for details)

\[
\begin{align*}
B_P &: \text{Alg}_{P}^{\text{aug}}(\text{Ch}_{\mathbb{Q}}) \to \text{CoAlg}_{P}^{\text{coaug}}(\text{Ch}_{\mathbb{Q}}) \\
\Omega_P &: \text{CoAlg}_{P}^{\text{coaug}}(\text{Ch}_{\mathbb{Q}}) \to \text{Alg}_{P}^{\text{aug}}(\text{Ch}_{\mathbb{Q}}).
\end{align*}
\]

A quadratic (co)algebra may be endowed with a weight grading, by declaring \(E\) has weight 1. Thus, for \(A = A(V,S)\) the bar construction \(B_P A\) is bigraded. By [Mil12, Proposition 3.2.3], there is a natural isomorphism

\[
A \xrightarrow{\sim} \bigoplus_k H_k(B_P A)^{(k)}.
\]

A quadratic \(P\)-algebra \(A\) is Koszul if the inclusion \(\bigoplus_k H_k(B_P A)^{(k)} \to H_*(B_P A)\) is an isomorphism, i.e. the homology of the bar construction is concentrated on the diagonal. Dually, a \(P\)-coalgebra is Koszul if the homology of the cobar construction is concentrated on the diagonal. If \(E\) and \(V\) are finite-dimensional, the \(P\)-algebra \(A\) is Koszul if and only if \(A^!\) is Koszul [Mil12, Theorem 3.3.2].

Let us return to the task at hand: proving that \(T_A(\mathfrak{k})\) is the Koszul dual of \(\text{gr} \mathfrak{t}_L(\mathfrak{k})\). Suppose \(A\) is a quadratic algebra with quadratic datum \((V,R)\) and \(L\) is its Koszul dual. Then we can replace the generators \(A^{(r)}\) and \(L^{(r)}\) by \(V^{(r)}\) and \((V^*[1])^{(r)}\), as well as replace (R7) and (R5) with the quadratic relations from \(R\) and \(R^1[2]\). The resulting alternative presentations of \(\mathfrak{t}_L(S)\) and \(\mathfrak{t}_P(S)\) are not yet quadratic, as the relations (R1) and (R1) are linear. This is of course easily dealt with, by taking \(S = k\) and only using the generators \(t_{ij}\) and \(x_{ij}\) for \(i < j\); this removes relations (R1) and (R1).

The following proposition concerns these modified quadratic presentations. Let us write \(A = A_{\text{Com}}(V,R)\) and \(T_A(S) = A_{\text{Com}}(\widetilde{V},\widetilde{R})\). Here we have

\[
\widetilde{V} = V_X \oplus V_A,
\]

with \(V_X = \mathbb{Q}[x_{ij} \mid i < j]\) and \(V_A = \oplus_{r \in \mathbb{N}} V^{(r)}\). There is a direct sum decomposition \(S^2(\widetilde{V}) = S^2(V_X) \oplus V_X \oplus V_A \oplus S^2(V_A)\), and \(\widetilde{R}\) decomposes accordingly:

\[
\widetilde{R} = R_X \oplus R_{XX} \oplus R_A,
\]
with \( R_X \subseteq S^2(V_X) \) spanned by \((R2)\), \( R_{XA} \subseteq V_X \otimes V_A \) spanned by \((R3)\), and \( R_A \subseteq S^2(V_A) \) spanned by \( k \) copies of \( R \) and \((R5)\). (Note that \((R4)\) does not appear when presenting the Kriz–Totaro algebra quadratically.)

**Proposition 7.8.** Suppose that \( A \) admits a PBW-basis in the sense of [LV12, Section 4.3.7] and has Koszul dual \( L \). Then the commutative algebra \( \mathcal{T}_A(\bar{k}) \) is Koszul, and \((\mathcal{T}_A(\bar{k}))^! \cong \text{gr}_{\ell_L}(\bar{k})\) as objects in \( \text{Alg}_{\mathbb{B}_{\text{fin}}}^{\text{GrRep}(\mathfrak{S}_4))} \).

**Proof.** To prove that \( \mathcal{T}_A(\bar{k}) \) is Koszul we may ignore the group actions. A commutative quadratic algebra is Koszul if and only if its underlying associative algebra is Koszul, cf. [Mil12, Theorem 4.1.6]. For the latter, it suffices to prove that it admits a PBW-basis [LV12, Theorem 4.3.8]. Corollary 2.2 of [Bez94] says that the Kriz–Totaro algebra admits a PBW-basis if \( A \) does.

Now we compute the Koszul dual Lie algebra \( \mathcal{T}_A(\bar{k}) \). By Example 7.7, the graded vector space of generators of this graded Lie algebra is the linear dual of the graded vector space \( \bar{V} \) of generators of \( \mathcal{T}_A(\bar{k}) \), shifted in degree by \(-1\). We fix a basis \( \{\alpha_j\}_{1 \leq j \leq 2g} \) of \( V \), and use the following suggestive notation for a basis of \( \bar{V} \) as objects in \( \text{GrRep}(\mathfrak{S}_4)) \).

- elements \( t_{ij} \) for \( i < j \) in degree \( 2n - 2 \), dual to \( x_{ij} \), and
- elements \( a_{ij}^{(r)} \) for \( 1 \leq j \leq 2g \) in degree \( n - 1 \) for \( r \in S \), dual to \( \alpha_j^{(r)} \).

The relations \( \bar{R} \) are the annihilator of the relations \( \bar{R} \subseteq S^2(\bar{V}) \) of \( \mathcal{T}_A(\bar{k}) \). We have
\[
\bar{R}^\perp = R^{\perp}_{\bar{X}} \oplus R^{\perp}_{\bar{X}A} \oplus R^{\perp}_A,
\]
and below we compute each of these terms.

We claim that \( R^{\perp}_{\bar{X}} \) gives \((R2)\) and \((R3)\): \( \Lambda^2(V_X) \) decomposes as a direct sum
\[
\Lambda^2(V_X) = \Lambda^2(V_X)_{\text{disj}} \oplus \Lambda^2(V_X)_{\text{br}}
\]
with the former spanned by terms \( x_{ij} \land x_{rs} \) for \( i, j, r, s \) all distinct and the latter spanned by terms \( x_{ij} \land x_{jk} \) for \( i, j, k \) all distinct (here we use the convention that for \( i > j \), we have \( x_{ij} = x_{ji} \)). As \( R_X \subseteq \Lambda^2(V_X)_{\text{br}} \), we have \( R^\perp_X \supseteq (\Lambda^2(V_X)_{\text{disj}})^\perp \). This gives \((R2)\).

To compute the annihilator of \( R_X \) in \( (\Lambda^2(V_X)_{\text{br}})^\perp \), we use that \( R_X \) is spanned by terms \( x_{ij} \land x_{jk} + x_{jk} \land x_{ki} + x_{ki} \land x_{ij} \) corresponding to \((R2)\). The span \( R^\perp_{\bar{X}A} \) of the elements \( t_{ij} \land t_{ik} + t_{ij} \land t_{jk} \) thus annihilates \( R_X \), and a dimension count shows it is equal to the annihilator of \( R_X \) in \( (\Lambda^2(V_X)_{\text{br}})^\perp \). This gives \((R3)\).

We claim that \( R^\perp_{\bar{X}A} \) gives \((R4)\) and \((R5)\): \( V_X \otimes V_A \) is a direct sum of two subspaces
\[
V_X \otimes V_A = (V_X \otimes V_A)_{\text{disj}} \oplus (V_X \otimes V_A)_{\text{br}},
\]
the former spanned by terms \( x_{ij} \land \alpha_k^{(i)} \) with \( i, j, r \) distinct, and latter by terms \( x_{ij} \land \alpha_k^{(j)} \).
As \( R_{XA} \subseteq (V_X \otimes V_A)_{\text{br}} \), we have \( R^\perp_{\bar{X}A} \supseteq (V_X \otimes V_A)_{\text{disj}} \). This gives \((R4)\).

To compute the annihilator of \( R_{XA} \) in \( (V_X \otimes V_A)_{\text{br}} \), we use that \( R_{XA} \) is spanned by \( x_{ij} \land \alpha_k^{(i)} - x_{ij} \land \alpha_k^{(j)} \). The span \( R^\perp_{\bar{X}A,br} \) of the elements \( t_{ij} \land \alpha_k^{(i)} + t_{ij} \land \alpha_k^{(j)} \) thus annihilates \( R_{XA} \). Again, a dimension count shows that \( R^\perp_{\bar{X}A,br} \) is the entire annihilator of \( R_{XA} \) in \( (V_X \otimes V_A)_{\text{br}} \). This gives \((R5)\).

We claim that \( R^\perp_A \) gives \((R6)\) and \((R7)\): \( \Lambda^2(V_A) \) decomposes as a direct sum
\[
R_A = \bigoplus_{r \in \bar{k}} R \subset S^2(V_A).
\]

Decomposing \( \Lambda^2(V_X[1]) \) in a similar manner, the annihilator of \( R_A \) consists of the terms \( A^{(r)} \land A^{(s)} \) for \( r < s \in \bar{k} \) as well as \((S^\perp)^{(i)} \) for \( i \in \bar{k} \) which indeed gives \((R6)\) and \((R7)\). Thus we recover the desired quadratic presentation of \( \text{gr}_{\ell_L}(\bar{k}) \).

Finally, we observe that the action of \( \mathfrak{S}_4 \) on \( \mathcal{T}_A(\bar{k}) \) arises through the action on the quadratic datum defining \( \mathcal{T}_A(\bar{k}) \). Thus we get an induced action on \((\mathcal{T}_A(\bar{k}))^!\) through the dual action on its quadratic datum; this is the desired \( \mathfrak{S}_4 \)-action on \( \ell_L(\bar{k}) \).

As \( A = \mathbb{Q} \oplus H[-n] \) clearly admits a PBW-basis, we see that \( \mathcal{T}_A(\bar{k}) \) is Koszul with Koszul dual \((\mathcal{T}_A(\bar{k}))^! = \text{gr}_{\ell_L}(\bar{k}) \).
7.3. The Kriz–Totaro algebra is free. Let us first clarify the title of this section: it is certainly not the case that the commutative algebras $\mathcal{T}_A(S)$ of Definition 7.4 are free. However, taking them all together defines a functor

$$\mathcal{T}_A(-) : FB \to Gr(Q\text{-mod}),$$

and this may be given the structure of a commutative ring object in $Gr(Q\text{-mod})^{FB}$, considered as a symmetric monoidal category via Day convolution, as follows. Take the external product

$$\mathcal{T}_A(S) \otimes \mathcal{T}_A(T) \to \mathcal{T}_A(S \sqcup T)$$

to be the unique map of algebras which send the elements $x_{ij} \text{ or } a^{ij}$ with $i$ and $j$ in either $S$ or $T$ to the elements of the same name with $i$ and $j$ considered as lying in $S \cup T$. This defines a lax monoidality on $\mathcal{T}_A(-)$, and is in fact a lax symmetric monoidality: in other words it is the structure of a commutative ring object in the category of functors. We shall show that $\mathcal{T}_A(-)$ is a free commutative algebra in this category.

Recall from Sections 5.6 and 7.2 the following notations. Firstly, for $V \in Gr(Q\text{-mod})^{FB}$, $SV(\mathfrak{s}) = V(\mathfrak{s})[1 - s] \otimes (1^*)$. Secondly, $\otimes_H$ is the Hadamard tensor product. Finally, we write $cA \in Gr(Q\text{-mod})^{FB}$ for the constant functor with value the graded vector space $A$.

**Proposition 7.9.** There is an isomorphism

$$S^*(cA \otimes_H S^{2n-1}\text{Lie}) \cong \mathcal{T}_A(-),$$

of commutative ring objects in $Gr(Q\text{-mod})^{FB}$, which is natural in the graded-commutative algebra $A$.

This is closely related to the appearance of the Lie representations in Theorem 5.30 via the work of Petersen [Pet20], and apart from some (linear and Poincaré) dualisations is implicit in that paper. We will give a proof in the spirit of [LS86].

**Proof.** For each $k$, the Arnold algebra $G(k)$ is the quotient of the graded-commutative algebra $A[x_{ij} | i \neq j \in \mathbb{K}]$, with $x_{ij}$ in degree $-(2n - 1)$, by the relations (R1) and (R2). (This is the rational cohomology ring of $\text{Emb}(\mathbb{K}, \mathbb{R}^n)$, cf. [Arn69].) To a monomial in the $x_{ij}$’s we may associate a graph with vertices $\mathbb{K}$, by placing an edge from $i$ to $j$ if the term $x_{ij}$ appears. Say the monomial is connected if the corresponding graph is connected. If two monomials are related by (R1) then their associated graphs are equal; if three monomials are related by (R2) then they are either all connected or all disconnected. Thus the linear combinations of monomials with connected graphs gives a well-defined graded subspace $C(k) \leq G(k)$.

If a monomial has an associated graph which is not a tree, we claim it is zero in $G(k)$. To see this, consider a cycle in the associated graph, of length $\ell$. Continuing in this way, we express it as a sum of monomials all having cycles of length 3. But these vanish, using

$$x_{12}x_{23}x_{31} \quad (\text{R2})
\begin{align*}
x_{12}x_{23}x_{31} &= -(x_{23}x_{31} + x_{31}x_{12})x_{31} = 0
\end{align*}
$$
as $x_{31}^2 = 0$. Thus $C(k)$ is spanned by connected monomials whose associated graph is a tree. Such trees have precisely $(k - 1)$ edges, so $C(k)$ is supported in degree $-(2n - 1)(k - 1)$. In fact $G(k)$ is supported in the range of degrees $[-2(2n - 1)(k - 1), 0]$ and $C(k)$ is precisely its homogeneous piece of degree $-(2n - 1)(k - 1)$: in other words, it is the top degree cohomology of $\text{Emb}(\mathbb{K}, \mathbb{R}^{2n})$. It is well-known that this is the representation $\text{Lie}(k) \otimes (1^k)$, see e.g. [Coh95, Theorem 6.1]. It follows that we have $C(k) = \text{Lie}(k) \otimes (1^k)[-2(2n - 1)(k - 1)]$ and so

$$(cA \otimes_H S^{2n-1}\text{Lie})(k) = A \otimes \text{Lie}(k) \otimes (1^k) \otimes \mathbb{K}^{[-2n][-(k - 1)(2n - 1)]} = A \otimes C(k).$$

There is a well-defined map

$$A \otimes C(k) \longrightarrow \mathcal{T}_A(k)$$

$$\alpha \otimes x_{i_1,j_1} \cdots x_{i_r,j_r} \longmapsto \alpha^{(1)} \cdot x_{i_1,j_1} \cdots x_{i_r,j_r},$$

(36)
A monomial in $x_{ij}$’s and $\alpha^{(i)}$’s, representing an element of $T_A(\hat{k})$, yields a graph on the vertices $\hat{k}$ with a labelling of these vertices by elements of $A$. The map (36) is an isomorphism onto the subspace spanned by those monomials whose associated graph is connected: using (R3), as the graph is connected all the labels can be moved to the first vertex, leaving the label $1_A$ at the others.

Using the commutative ring structure of $T_A(-)$, the maps (36) extends to a map of commutative rings

$$S^*(cA \otimes_H S^{2n-1}\text{Lie}) \to T_A(-)$$

as in the statement of the proposition. This is easily checked to be an isomorphism, by interpreting labelled graphs as a disjoint union of connected labelled graphs. 

7.4. The character of the extended Drinfel’d–Kohno Lie algebra. We have described two functors: the associated graded of the extended Drinfel’d–Kohno Lie algebra

$$\mathbb{Q}\text{-mod}^f \to \text{Alg}_{\text{Lie}}(\text{GrRep}(\mathfrak{S}_k))$$

$$H \mapsto \text{gr}\ t(H[n-1])(\hat{k})$$

and the Kriz–Totaro algebra

$$\mathbb{Q}\text{-mod}^f \to \text{Alg}_{\text{Com}}(\text{GrRep}(\mathfrak{S}_k))$$

$$H^\vee \mapsto \mathcal{T}_{\mathbb{Q}\otimes H^\vee[-n]}(\hat{k}).$$

For $V \in \text{GrRep}(FB \times \mathfrak{S}_k)$, we write $\Sigma V$ for the functor $s \mapsto V(s) \otimes ((1') \boxtimes \mathbb{Q})[s]$. The associated graded of the extended Drinfel’d–Kohno Lie algebra is a degreewise polynomial functor of $H$, because by Lemma 7.3 it may be expressed in the form

$$\bigoplus_{s \geq 0} \text{gr}\ t(s, \hat{k}) \otimes_{\mathfrak{S}_s} H[n-1]^\otimes_s = \bigoplus_{s \geq 0} \Sigma^{n-1} \text{gr}\ t(s, \hat{k}) \otimes_{\mathfrak{S}_s} H^\otimes_s$$

for certain graded $\mathfrak{S}_s \times \mathfrak{S}_k$-representations $\text{gr}\ t(s, \hat{k})$. We wish to determine the graded $\mathfrak{S}_s \times \mathfrak{S}_k$-representations $\Sigma^{n-1}\text{gr}\ t(s, \hat{k})$.

Similarly, the Kriz–Totaro algebra is degreewise a polynomial functor of $H^\vee$: taking $A = \mathbb{Q} \oplus H^\vee[-n]$ we read off from Proposition 7.9 that $\mathcal{T}_{\mathbb{Q}\otimes H^\vee[-n]}(\hat{k})$ may be expressed in the form

$$\bigoplus_{s \geq 0} \mathcal{T}(s, \hat{k}) \otimes_{\mathfrak{S}_s} H^\vee[-n]^\otimes_s = \bigoplus_{s \geq 0} \Sigma^{-n}\mathcal{T}(s, \hat{k}) \otimes_{\mathfrak{S}_s} (H^\vee)^\otimes_s,$$

for certain graded $\mathfrak{S}_s \times \mathfrak{S}_k$-representations $\mathcal{T}(s, \hat{k})$.

**Proposition 7.10.**

(i) There is an isomorphism

$$\Sigma^{-n}\mathcal{T}(-, -) \cong S^*\left( ((0) \oplus (1)[-n]) \otimes_H (\mathbb{Q} \boxtimes S^{2n-1}\text{Lie}) \right),$$

of objects in GrRep(FB $\times$ FB), where the free graded-commutative algebra $S^*(-)$ is formed with respect to Day convolution, and is applied to the functor $((0) \oplus (1)[-n]) \otimes_H (\mathbb{Q} \boxtimes S^{2n-1}\text{Lie})$ given by

$$FB \times FB \to \text{Gr}(Q\text{-mod}^f)$$

$$(s, \hat{k}) \mapsto \begin{cases} (0) \otimes S^{2n-1}\text{Lie}(k) & \text{if } s = 0, \\ (1)[-n] \otimes S^{2n-1}\text{Lie}(k) & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For each $k$ the object $\Sigma^{-n}\text{gr}\ t(-, \hat{k}) \in \text{GrRep}(FB \times \mathfrak{S}_k)$ admits the structure of a Lie algebra, the object $\Sigma^{-n}\mathcal{T}(-, \hat{k}) \in \text{GrRep}(FB \times \mathfrak{S}_k)$ admits the structure of a commutative algebra, and these are Koszul dual.
Proof. To prove (i), we must show that
\[ \bigoplus_{s \geq 0} S^s \left( (0) \otimes (1)[-n]) \otimes_H (Q \otimes S^{2n-1}\text{Lie}) \right) \otimes_{\mathfrak{g}_s} (H')^ \otimes \in \text{GrRep}(FB) \]
is isomorphic to \( T_{Q \otimes H'}[-n](-) \), naturally in \( H \). But this is the polynomial expansion of the functor
\[ H \mapsto S^s ( (Q \otimes H)[-n]) \otimes S^{2n-1}\text{Lie}(-) : \text{Q-mod} \rightarrow \text{GrRep}(FB) \]
which is isomorphic to \( T_{Q \otimes H'}[-n](\cdot) \) by Proposition 7.9. Furthermore, by Proposition 7.8 they are Koszul dual.

For (ii) we use Schur–Weyl duality, following Section 2.2 of [SS15]. Let us write \( \text{GrRep}^\text{pol}(\text{Q-mod}) \) for the category of functors \( F : \text{Q-mod} \rightarrow \text{Gr}(\text{Q-mod}) \) which are degreewise polynomial, and consider the Schur–Weyl duality functor
\[ D : \text{GrRep}^\text{pol}(\text{Q-mod}) \rightarrow \text{GrRep}(FB) \]
which is a symmetric monoidal equivalence of categories. Considering objects with \( \mathfrak{g}_s \)-actions it induces a symmetric monoidal equivalence \( D : \text{GrRep}^\text{pol}(\text{Q-mod}) \otimes \mathfrak{g}_s \rightarrow \text{GrRep}(FB \times \mathfrak{g}_s) \) and by construction
\[
D(H) = \text{dim}(V) \rightarrow \colim_{(F)} \left( S \mapsto \Lambda \otimes [t^1, t^{-1}] \right)
\]
the graded version of (6). We will do so in three steps:

(i) Proposition 7.10 (i) gives an expression for \( \text{ch}(\Sigma^{-n}\mathcal{T}(-, -)) \in \Lambda \otimes \Lambda[[t]] \), the graded version of (5). Its \( \Lambda \otimes \Lambda[[t]] \)-component is the Hilbert–Poincaré series of \( \Sigma^{-n}\mathcal{T}(-, \kappa) \).

(ii) Proposition 7.10 (ii) gives that \( \Sigma^{-1}\text{gr t}(\cdot, \kappa) \) is the Koszul dual of \( \Sigma^{-n}\mathcal{T}(-, \kappa) \).

This gives a relationship between their Hilbert–Poincaré series in \( \Lambda \otimes \Lambda[[t]] \), phrased in terms of plethysm.

(iii) Lemma 5.24, upon taking associated graded, gives the Hilbert–Poincaré series of \( \text{gr t}(\cdot, \kappa) \) as an alternating sum of those \( \text{gr t}(\cdot, \kappa') \) for \( k' \leq k \).

We are free to choose \( n \) and will eventually, in Step (iii), take it to be odd.

7.4.1. Step (i). In \( \text{GrRep}(FB \times FB) \), the free graded-commutative algebra \( S^*(-) \) can be described in terms of the composition product. Writing \( A \in \text{GrRep}(FB \times FB) \) as a direct sum \( A_{\text{odd}} \oplus A_{\text{even}} \) of terms concentrated in odd and even degrees, we have \( S^*(A) = \Lambda^*(A_{\text{odd}}) \otimes \text{Sym}^*(A_{\text{even}}) \). Taking the character as in (5), we get
\[
\text{ch}(S^*(A)) = ((1 + e_1 + e_2 + \cdots) \circ \text{ch}(A_{\text{odd}})) \cdot ((1 + h_1 + h_2 + \cdots) \circ \text{ch}(A_{\text{even}})).
\]
Introducing
\[
L_{\text{odd}} := 1 \otimes \text{ch}(S^{2n-1}\text{Lie})_{(\text{odd})} \quad \text{and} \quad L_{\text{even}} := 1 \otimes \text{ch}(S^{2n-1}\text{Lie})_{(\text{even})},
\]
Proposition 7.10 (i) gives the computational instantiation of Step (i):

**Lemma 7.11.** The character of \( \text{ch}(\Sigma^{-n}\mathcal{T}(-, -)) \) is given by
\[
(1 + e_1 + e_2 + \cdots) \circ (L_{\text{odd}} + L_{\text{even}} \cdot (s_1 \otimes 1)t^{-n}) \cdot (1 + h_1 + h_2 + \cdots) \circ (L_{\text{even}} + L_{\text{odd}} \cdot (s_1 \otimes 1)t^{-n})
\]
if \( n \) is odd, and if \( n \) is even, by
\[
(1 + c_1 + c_2 + \cdots) \circ (L_{\text{odd}} \cdot (1 + (s_1 \otimes 1)t^{-n})) \\
\cdot (1 + h_1 + h_2 + \cdots) \circ (L_{\text{even}} \cdot (1 + (s_1 \otimes 1)t^{-n}))
\]

7.4.2. Step (ii). Suppose that \( A = A_{\text{Com}}(V, S) \) is a Koszul quadratic graded-commutative algebra with \( V \) finite-dimensional. Its underlying associative algebra is therefore Koszul, cf. [Mil12, Theorem 4.1.6], so there is also an associative Koszul dual \( A^{!\text{ass}} \), an associative algebra since \( \text{Ass}^! = \text{Ass} \). The relationship between \( A^! \) and \( A^{!\text{ass}} \) is given in [GK94, Theorem 2.3.11]: \( A^{!\text{ass}} \) is the universal enveloping algebra of \( A^! \). By the Poincaré–Birkhoff–Witt theorem, there is an isomorphism \( \text{gr} A^{!\text{ass}} = S^*(A^!) \) for some filtration on \( A^{!\text{ass}} \). Thus we can compute \( \text{ch}(A^! \cdot s) \) by first computing \( \text{ch}(A^{!\text{ass}}) \) and then inverting the operation corresponding to \( S^*(\cdot) \).

It is well-known how to compute the Hilbert–Poincaré series of \( A^{!\text{ass}} \). Let us start with a quadratic datum \((V, S)\) with \( FB \times S_k\)-action. Then we get a quadratic algebra \( A = A(V, S) \) in \( FB \times S_k\)-representations with two gradings: a homological grading and a weight grading, both of which we shall assume are non-negative. We shall need the bigraded Hilbert–Poincaré series \( \text{ch}(A)(t, r) \in \hat{\Lambda} \otimes \Lambda_k[[t, t^{-1}, r]] \), where \( t \) still records the homological grading and \( r \) now records the weight grading. Observing that the proof of [LV12, Theorem 3.5.1] is natural \( A \), we obtain the following generalisation of it, or rather of the inhomogeneously graded version of [Ber14, Corollary 1.6]:

**Lemma 7.12.** For \( A \) a Koszul graded-associative algebra in \( FB \times S_k\)-representations, we have
\[
\text{ch}(A^{!\text{ass}})(t, r) = \frac{1}{\text{ch}(A)(t, -rt^{-1})} \in \hat{\Lambda} \otimes \Lambda_k[[t, t^{-1}, r]].
\]

Next we need to give the inverse to the operation \( S^*(\cdot) \) on the level of symmetric functions. As before \( S^*(\cdot) \) converts direct sums to tensor products, and \( \text{ch} \) sends \( \text{Sym}^*(\cdot) \) to plethysm with \((1+h_1+h_2+\ldots)\) and \( \Lambda^*(\cdot) \) to plethysm with \((1+c_1+c_2+\ldots)\) (here plethysm on \( \hat{\Lambda} \otimes \Lambda[[t]] \) using inner plethysm in the second term). Because our objects are concentrated in even degree when \( n \) is odd, but in both even and odd degrees when \( n \) is even, from now on we take \( n \) to be odd.

Then, to obtain \( 1 + x \) with \( x \) positively graded from \((1+h_1+h_2+\ldots)\circ(1+x)\), we apply the plethystic inverse of \((1+h_1+h_2+\ldots)\), which we denote \( \text{Log}_{\text{even}}(1+) \). This series can be computed iteratively, and a formula can be found in [GG17, Proposition A.4]. Taking the plethystic of this with \( \text{ch}(A^{!\text{ass}})(t, r) \), we get:

**Proposition 7.13.** For \( A \) a Koszul graded-commutative algebra in algebraic \( FB \times S_k\)-representations with \( A^! \) generated in even degrees, we have
\[
\text{ch}(A^!)(t, r) = -\text{Log}_{\text{even}}(1+) \circ (\text{ch}(A)(t, -rt^{-1}) - 1).
\]

When we apply this to the Koszul commutative algebra object \( \Sigma^n T(\cdot, k) \) for \( n \) odd, in which case its Koszul dual is concentrated in even degrees, we get the Hilbert–Poincaré series of \( \Sigma^{n-1} \text{gr } t(\cdot, k) \) in \( \hat{\Lambda} \otimes \Lambda[[t]] \). This completes step (ii).

7.4.3. Step (iii). Implementing the alternating sum is straightforward, as \( \text{ch}_k : \text{Rep}(S_k) \to \Lambda_k \) is given on inductions by
\[
\text{ch}_k \left( \text{Ind}_{S_{k-j} \times e_j}^{S_k} V \boxtimes (1^j) \right) = \text{ch}_{k-j}(V) \cdot s_{1j} \in \Lambda_k.
\]

Thus from Lemma 5.24 we get an equation in \( \Lambda_k \otimes \hat{\Lambda}[[t]] \)
\[
\text{ch(\text{gr f}(s, k))} = \sum_{j=0}^{k} (-1)^j \text{ch(\text{gr t}(s, k-j))} \cdot s_{1j}.
\]
7.5. **The fourth band.** In Appendix C, we give the computation of the invariants on the rationalised $E^1$-page of the Bousfield–Kan spectral sequence (24) for the embedding calculus Taylor tower. In stating those results, we write the cohomology of products relative to diagonals as we did in Lemma 7.3: the expression in Theorem 5.30 is the value on the graded vector space $H^\oplus \llbracket -n \rrbracket$ of the degreewise polynomial functor

$$D_- (k) : \text{Gr}(\mathbb{Q}\text{-mod}) \to \text{GrRep}(\mathfrak{S}_n)$$

for graded $\mathfrak{S}_n \times \mathfrak{S}_k$-representations $D(\underline{s}, \underline{k})$. The formula in 5.30 determines these representations, and Proposition 5.31 makes some cases explicit.

The result is given in Figure 4 for $n$ and $g$ sufficiently large. Using this, we prove:

**Proposition 7.14.** For $n \geq 4$, the Euler characteristic of the $(+1)$-eigenspaces $\pi_* (F_n)^{(+1)}_{\mathbb{Q}}$ over the range of degrees $* \in [4n - 9, 4n - 5]$ is 1.

**Proof.** By the considerations of Section 5.4, the map to the first layer factors as

$$\text{BEmb}_{1/20} (W_{g,1})_t \to \text{Bh\text{-}Emb}(W_{g,1}) \to \text{Bh\text{-}Emb}_{1/20}(W_{g,1}).$$

By Computations 5.7 and 5.8, the right map induces a map on $[\overline{\mathfrak{g}}_{\mathbb{Q}}^{\text{tr}^{[1]}]}$-invariants in the rational homotopy groups, given by

$$0 \equiv [\pi_{4n-3} (\text{Bh\text{-}Emb}(W_{g,1}))_{\mathbb{Q}}] [\overline{\mathfrak{g}}_{\mathbb{Q}}^{\text{tr}^{[1]}}] \to [\pi_{4n-3} (\text{Bh\text{-}Emb}_{1/20}(W_{g,1}))_{\mathbb{Q}}] [\overline{\mathfrak{g}}_{\mathbb{Q}}^{\text{tr}^{[1]}}] \cong \mathbb{Q}^3.$$

In the Bousfield–Kan spectral sequence, there can be no non-zero differentials out of the invariants in the entry $(E^1_{0,4n-3})_{\mathbb{Q}}$ when $n \geq 5$ except $d^1$-differentials. To see this, there can not by any differentials into this entry, and a non-zero $d^r$-differential for

---

**Figure 4.** The dimensions of the invariants in $(E^1_{0,4n-3})_{\mathbb{Q}}$ contributing to degrees $\sim 4n$, for $n$ and $g$ sufficiently large. The indexing and differentials are as in Figure 2.
\( r \geq 2 \) out of this entry can only hit an invariant in a \((+1)\)-eigenspace of the reflection involution from a higher band than the fourth, which are of degree \( \geq 8n - 17 \). This is at least \( 4n - 3 \) when \( n \geq 4 \).

Thus it must be case that the differential

\[
\mathbb{Q}^3 \cong \left[ (E^0_{1,4n-3})_{\mathbb{Q}} \right]^{\mathbb{C}[\xi]} \overset{d^1}{\longrightarrow} \left[ (E^1_{1,4n-3})_{\mathbb{Q}} \right]^{\mathbb{C}[\xi]} \cong \mathbb{Q}^{15}
\]

is injective. This implies that \( \pi_{4n-3}(B\text{TorEmb}_{1,2\mathbb{Q}}(W_{g,1},\ell))_{\mathbb{Q}} = 0 \), and in the remaining degrees in the range \([4n - 8, 4n - 4]\) the Euler characteristic is non-zero by inspection of Figure 4. The result follows from the long exact sequence for (31) and the fact that \( \pi_*(X_1(g))_{\mathbb{Q}} \) is concentrated in degrees \( \geq 4n - 3 \) by the proof of Theorem 6.6. \( \square \)

Since the homotopy groups of \( X_0 \) consists of \((-1)\)-eigenspaces for the reflection involution, we get:

**Corollary 7.15.** For \( n \geq 4 \), the Euler characteristic of \( \pi_*(B\text{Diff}(D^{2n}))_{\mathbb{Q}} \) over the range of degrees \( \ast \in [4n - 9, 4n - 5] \) is 1.

**Remark 7.16.** This is compatible with the suggestion in Remark 1.3 that the cohomology classes arising from configuration space integrals may pair non-trivially against the classes of Corollary 7.15: the reflection involution acts on a cohomology class associated to an element of \( \mathcal{A}_k \) by \((-1)^k\).

Apart from a grading shift Figures 2 and 4 do not depend on whether \( n \) is odd or even. This is a general phenomenon, similar to Remark 4.23.

**Proposition 7.17.** For \( n \) even and odd, the Hilbert–Poincaré series of the \( \mathcal{C}_{\mathbb{Q}} \)-representations \( \pi_*(B\text{Emb}_{1,2}(W_{g,1}))_{\mathbb{Q}} \) differ by an application of the involution \( \omega \) and grading shift only. In particular, the dimensions of the \( \mathcal{C}_{\mathbb{Q}} \)-invariants do not depend on \( n \).

**Proof.** We use the rational collapse of the Federer spectral sequence at the \( E^2 \)-page, as explained following Proposition 5.32, and observe that the entries \( (F \mathcal{E}_n)_{\mathbb{Q}} \) are homogeneous pieces of

\[
[D_{H^{[-n]}_\mathbb{Q}}(k) \otimes f_{L(H^{[n-1]}_\mathbb{Q})}(k)]^{G_k},
\]

using the notation of (35) and (37). We can write the left term as \( \bigoplus_{x \geq 0} D_s(k) \otimes G_x \), \((H^{[-n]})^{\mathbb{Q}} 2 \) and the right term as \( \bigoplus_{x \geq 0} \text{gr} f_{L(k)} \otimes G_x, (H^{[n-1]})^{\mathbb{Q}} 2 \). Thus, these entries are homogeneous pieces of

\[
[\text{gr} f_{S(k)} \otimes D_L(k)]^{G_k} \otimes G_k \times G_k, (H^{\mathbb{Q}} 2 \otimes (1^{s(n-1)} \otimes n-1) \otimes (H^{\mathbb{Q}} 2 \otimes (1^{s(n-1)} \otimes n-1))][s(n-1) - t_n].
\]

Each irreducible \( S^\mu \otimes S^\nu \) which appears in \([\text{gr} f_{S(k)} \otimes D_L(k)]^{G_k}\) contributes a \( \mathcal{C}_{\mathbb{Q}} \)-representation \( S_\mu \otimes S_\nu \) if \( n \) is odd and \( S_\mu \otimes S_\nu \) if \( n \) is even. The character sends these to products \( s_\mu s_\nu \) or \( s_\mu s_\nu \), the result follows by recalling that the involution \( \omega : \hat{\mathbb{A}} \to \hat{\mathbb{A}} \) is a ring homomorphism and satisfies \( \omega(s_\mu) = s_{\mu^t} \). \( \square \)

**Appendix A. The family signature theorem**

An extended version of this appendix is available as [RW19].

A.1. **Twisted signatures.** Let \( H \) be a finitely-generated \( \mathbb{Z} \)-module and \( \lambda : H \otimes \mathbb{R} \to \mathbb{R} \) be an \( \epsilon \)-symmetric nondegenerate bilinear form, with \( \epsilon \in \{1, -1\} \). (The notation differs slightly from the rest of this paper: here \( H \) denotes the rationalisation of such data.)

Let \( O(H, \lambda) \leq GL(H) \) denote the subgroup of those automorphisms of \( H \) which preserve the form \( \lambda \). There is a corresponding local system of abelian groups \( \mathcal{H} \to BO(H, \lambda) \), equipped with an \( \epsilon \)-symmetric nondegenerate bilinear form.

There are cohomology classes

\[
\sigma_i \in H^i(BO(H, \lambda); \mathbb{Q})
\]
in degrees $i \equiv 0 \mod 4$ if $\epsilon = +1$ and $i \equiv 2 \mod 4$ if $\epsilon = -1$ defined as follows. Rational cohomology is dual to rational framed bordism, and if $[f: M \to BO(H, \lambda, \xi)]$ is a framed bordism class of degree $i$ as above then we set

$$\langle \sigma, [f, \xi]\rangle := \text{sign}(H^{i/2}(M; f^*\mathcal{H})).$$

Here the signature of $H^{i/2}(M; f^*\mathcal{H})$ is taken with respect to the bilinear form

$$H^{i/2}(M; f^*\mathcal{H}) \otimes H^{i/2}(M; f^*\mathcal{H}) \xrightarrow{\lambda} H^i(M; f^*\mathcal{H} \otimes \mathcal{H}) \xrightarrow{\lambda} H^i(M; \mathcal{Z}) = \mathbb{Z},$$

which is symmetric because the cup product and $\lambda$ are either both symmetric or both antisymmetric. This expression is well-defined by the usual argument for cobordism-invariance of signatures.

If $(\mathcal{H}', \lambda')$ is another $\epsilon$-symmetric form, then from the description above it is clear that under the map

$$BO(H, \lambda) \times BO(\mathcal{H}', \lambda') \to BO(\mathcal{H} \oplus \mathcal{H}', \lambda \oplus \lambda')$$

the class $\sigma^{\mathcal{H} \oplus \mathcal{H}'}$ pulls back to $\sigma_1 \otimes 1 + 1 \otimes \sigma_1^{\mathcal{H}'}$. As a special case of this, it follows that the classes $\sigma_i$ are compatible under stabilisation of $\epsilon$-symmetric forms.

### A.2. Signatures of Poincaré fibrations

Now suppose that $F^d \to E^{4k} \to B^{4k-d}$ is a fibration with Poincaré base and fibre (and hence Poincaré total space too [Got79, Theorem 1]), with $B$ and $F$ endowed with orientations and such that $\pi_1(B)$ acts trivially on $H^d(F; \mathbb{Z})$. Let

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; \mathbb{Z})) \implies H^{p+q}(E; \mathbb{Z})$$

denote the Serre spectral sequence for this fibration. Meyer [Mey72] has shown that there are identities

$$\text{sign}(E) = \text{sign}(E_2^{*,*}) = \begin{cases} \text{sign}(H^{(4k-d)/2}(B; \mathcal{H}^{d/2}(F; \mathbb{Z}))) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

Here the signature of $E_2^{*,*}$ is taken with respect to the form

$$E_2^{*,*} \otimes E_2^{*,*} \xrightarrow{\lambda} E_2^{*,*} \to \mathbb{Z}[4k - d, d]$$

where the latter map denotes the projection to $E_2^{4k-d,d} = H^{4k-d}(B; \mathcal{H}^{d}(F; \mathbb{Z})) = \mathbb{Z}$.

**Remark A.1.** Meyer assumes various additional hypotheses, most notably that $B$ and $F$ are homology manifolds. This is because he wishes to allow (non locally constant) sheaf coefficients. For locally constant coefficients being Poincaré complexes suffices for his argument to go through.

### A.3. The family signature theorem

The following is the main result we want to record.

**Theorem A.2.** Let $\pi: E \to |K|$ be an oriented topological block bundle with fibre $F^d$. If $d$ is even let $H := H^{d/2}(F; \mathbb{Z})/\text{tors}, \lambda: H \otimes H \to \mathbb{Z}$ denote the cup-product pairing, and let $f: B \to BO(H, \lambda)$ classify the local system $\mathcal{H}^{d/2}(F; \mathbb{Z})/\text{tors}$ over $B$. Then

$$\kappa_{\mathcal{L}_i}(\pi) := \int_{\pi} \mathcal{L}_i(T_\pi E) = \begin{cases} f^*(\sigma_{4i-d}) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

**Proof.** As rational cohomology is dual to rational framed bordism, by naturality it is enough to consider the case where $|K| = B^{4i-d}$ is a stably framed smooth manifold of dimension $4i - d$.

By the discussion in [HLLR17, Section 2] a topological block bundle has a stable vertical tangent microbundle $T_\pi E \to E$, and as $B$ is a topological manifold then by [HLLR17, Lemma 2.5.2] $E$ is a topological manifold and its stable tangent microbundle satisfies

$$TE \cong \pi^*(TB) \oplus T_\pi E \cong \mathbb{R}^{2k-d} \oplus T_\pi E$$
so \( L_i(T_E) = L_i(TE) \) by stability of the \( L \)-classes, and hence
\[
\int_B \int_E L_i(T_E) = \int_E L_i(TE) = \text{sign}(E).
\]

A block bundle determines a fibration with homotopy equivalent fibre, total space, and base, so by Meyer’s result we have
\[
\text{sign}(E) = \begin{cases} 
\text{sign}(H^{(4k-d)/2}(B; H^{d/2}(F; \mathbb{Z}))) & \text{if } d \text{ is even} \\
0 & \text{if } d \text{ is odd}.
\end{cases}
\]
and by definition we have
\[
\text{sign}(H^{(4k-d)/2}(B; H^{d/2}(F; \mathbb{Z}))) = \int_B f^* \sigma_{4k-d}.
\]
Putting these together gives the required identification. \( \square \)

The key ingredient making this argument work is that a topological block bundle over a topological manifold has topological manifold total space, and the Hirzebruch signature theorem holds for topological manifolds (by definition of the topological Hirzebruch \( L \)-classes).

**Corollary A.3.** The natural maps
\[
\begin{align*}
\mathbb{Q}[\sigma_2, \sigma_6, \sigma_{10}, \ldots] & \longrightarrow \lim_{g \rightarrow \infty} H^*(BSp_{2g}(\mathbb{Z}); \mathbb{Q}) \\
\mathbb{Q}[\sigma_4, \sigma_8, \sigma_{12}, \ldots] & \longrightarrow \lim_{g \rightarrow \infty} H^*(BO_{g,9}(\mathbb{Z}); \mathbb{Q})
\end{align*}
\]
are isomorphisms.

**Proof.** By a theorem of Ebert [Ebe11, Theorem A], for any polynomial \( p(x_1, x_2, \ldots, x_r) \) and any even number \( d \), there is a smooth oriented fibre bundle \( \pi: E \rightarrow B \) with \( d \)-dimensional fibres having \( p(\kappa_{\mathcal{L}_{k+1}}, \kappa_{\mathcal{L}_{2+j}}, \ldots, \kappa_{\mathcal{L}_{r+j}}) \neq 0 \in H^*(B; \mathbb{Q}) \), where \( j \) is minimal such that \( 4(1+j) - d > 0 \). With the Family Signature Theorem this implies that the maps in question are injective. On the other hand, by a theorem of Borel [Bor74, Section 11] the right- and left-hand sides are abstractly isomorphic, so the map is in fact an isomorphism. \( \square \)

## Appendix B. Delooping the embedding calculus tower

### B.1. A recollection of embedding calculus.

We recall those features of embedding calculus needed to deal with framed embeddings, referring to [KRW19, Section 3.1] and [Wei99, Wei11, BdBW13] to further information.

Fix a \((d-1)\)-dimensional manifold \( K \), and let \( \text{Mfd}_{d,K} \) denote the category enriched in topological spaces, with objects given by the \( d \)-dimensional manifolds with the boundary identified with \( K \), and morphisms given by spaces of embeddings rel boundary. **Manifold calculus** provides a tower of approximations to presheaves \( F: \text{Mfd}_{d,K}^{op} \rightarrow \text{Top} \), constructed from their restrictions to the subcategory \( \text{Disc}_{\leq k,K} \) of objects diffeomorphic to the disjoint union of \( \leq k \) discs and a collar on \( K \).

It is technically more convenient to work with simplicial sets. We can use the Quillen equivalence \( |-| \simeq \text{Sing}(-) \) to pass between topological spaces and simplicial sets. For the remainder of this appendix, whenever a topological space appears we implicitly apply \( \text{Sing}(-) \). In particular, we will write \( \text{Mfd}_{d,K} \) and \( \text{Disc}_{\leq k,K} \) for the \( \text{sSet} \)-enriched categories obtained by applying \( \text{Sing}(-) \) to the morphism spaces, and consider the \( \text{sSet} \)-enriched categories \( \text{PSh}(-) \) of \( \text{sSet} \)-valued continuous presheaves on these.

These categories of presheaves have a notion of weak equivalence given by object-wise weak equivalences, so we can construct the derived mapping spaces
\[
T_k(F(M)) := \mathbb{R}\text{Map}_{\text{PSh}(\text{Disc}_{\leq k,K})}(\text{Emb}_K(-, M), F(-)),
\]
e.g. using [DK80, 3.1]. These derived mapping spaces can also be computed using model-categorical techniques, [BdBW13, Section 3], and by [DK80, Corollary 4.7] these
constructions are weakly equivalent. By restricting presheaves and evaluating at \( M \), we obtain a tower

\[
\cdots \longrightarrow T_k(F(M)) \longrightarrow T_{k-1}(F(M)) \longrightarrow \cdots.
\]

We obtain embedding calculus when we apply manifold calculus to the representable presheaf \( F(-) = \text{Emb}_K(-,N) \). The result is the embedding calculus Taylor tower \( \text{Emb}_K(M,N) \). This starts at \( T_1(\text{Emb}_K(M,N)) \), and \( T_k(\text{Emb}_K(M,N)) \) is referred to as the \( k \)th stage of the Taylor tower. The embedding calculus tower provides a good approximation to \( \text{Emb}_K(M,N) \) as long as the handle dimension \( h \) of \( M \) relative to \( K \) is sufficiently small with respect to the dimension \( d \) of \( N \): the map

\[
\text{Emb}_K(M,N) \longrightarrow T_k(\text{Emb}_K(M,N))
\]

is \((-h-1+k(d-2-h))\)-connected [GW99, Corollary 2.5]. Thus if \( h<d-2 \) the induced map

\[
\text{Emb}_K(M,N) \longrightarrow \underset{k \to \infty}{\text{holim}} T_k(\text{Emb}_K(M,N))
\]

is a weak equivalence.

This first stage is given by

\[T_1(\text{Emb}_K(M,N)) \simeq \text{Bun}_K(TM,TN).\]

To understand \( \text{Emb}_K(M,N) \) through the embedding calculus Taylor tower, we understand the differences between stages. Fixing an element \( \iota \in \text{Emb}_K(M,N) \) provides a base point in each stage \( T_{k-1}(\text{Emb}_K(M,N)) \) and for \( k \geq 2 \) the \( k \)th layer of the Taylor tower at the identity is the homotopy fibre

\[L_k(\text{Emb}_K(M,N)) := \text{hofib} [T_k(\text{Emb}_K(M,N)) \to T_{k-1}(\text{Emb}_K(M,N))].\]

These layers can be described in terms of section spaces, see [KRW19, Section 3.3.2].

B.2. Delooping the tower. If we specialise to \( M = N \) and \( K = \partial M, \text{Emb}_\partial(M) \) is a simplicial monoid under composition. We will explain that all stages are also simplicial monoids, and the maps between them are maps of simplicial monoids.

In the main body of the paper we are only interested in the higher homotopy groups, so we may restrict to the homotopy-invertible path components: \( T_k \text{Emb}_\partial(M)^\times \) is the derived automorphism space

\[\text{RAut}_{\text{PSh}(\text{Disc} \leq k,K)}(\text{Emb}_\partial(-,M))\]

of the presheaf \( \text{Emb}_\partial(-,M) \). We can then invoke the construction of derived mapping spaces in [DK80, 3.1] to make these into simplicial monoids, or alternatively use model-categorical methods or [DK84, 2.2]; by [DK84, Proposition 2.3] all are weakly equivalent. Either of Dwyer–Kan’s constructions imply that any functor which preserves weak equivalences induces a map on the classifying spaces of derived automorphisms. Delooping the resulting maps, we get the bottom row of the diagram

\[\cdots \longrightarrow \text{BT}_k \text{Emb}_\partial(M)^\times \longrightarrow \text{BT}_{k-1} \text{Emb}_\partial(M)^\times \longrightarrow \cdots\]

It remains to justify the existence of the vertical maps and the commutativity of the diagrams. To do so, we prove in Lemma B.1 that the presheaf \( \text{Emb}_\partial(-,M) \) on \( \text{Mfd}_\partial \) is both cofibrant and fibrant, so that by Yoneda

\[\text{Emb}_\partial(M) \xrightarrow{\simeq} \text{RMap}_{\text{PSh(Mfd)}}(\text{Emb}_\partial(-,M)).\]

We then use Dwyer–Kan’s constructions and pass to homotopy invertible path components to obtain the vertical maps.
Lemma B.1. \( \text{Emb}_\partial(-, M) \in \text{PSh}(\text{Mfd}_\partial) \) is cofibrant and fibrant with respect to the projective model structure.

Proof. Let us recall that the projective model structure on the category \( \text{PSh}(C) \) of simplicial on a \( \text{sSet} \)-enriched category \( C \) has the following properties [BdBW13, Theorem A.2]: (i) weak equivalences and fibrations are objectwise, (ii) cofibrations are retracts which restricts to a presheaf on \( \text{Disc} \). The horizontal homotopy fibres are both \( \text{Bun}_\partial \). Thus the natural map of presheaves

\[
\text{Hom}_C(-, X) \times \partial \Delta^n \longrightarrow \text{Hom}_C(-, X) \times \Delta^n.
\]

This makes clear that \( \text{Emb}_\partial(-, M) \) is cofibrant and fibrant in \( \text{PSh}(\text{Mfd}_\partial) \), as it is representable and objectwise fibrant. \( \square \)

B.3. **Embedding calculus with tangential structures.** We now introduce tangential structures to the embedding calculus tower. A tangential structure is a \( \text{GL}_d(\mathbb{R}) \)-space \( \Theta \). As a boundary condition for \( \theta \)-structures, we fix a \( \text{GL}_d(\mathbb{R}) \)-equivariant map \( \ell_\partial : \text{Fr}(TM) \to \Theta \) near \( \partial M \) and let \( \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \) denote the space of \( \text{GL}_d(\mathbb{R}) \)-equivariant maps \( \text{Fr}(TM) \to \Theta \) extending \( \ell_\partial \). The following generalises Definition 3.8:

\[
\text{BEmb}_\partial^\theta(M)^\times = \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \sslash \text{Emb}_\partial(M)^\times.
\]

Observe that \( \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \) is the value of the presheaf \( \text{Bun}_\partial(-, \Theta; \ell_\partial) \) on \( \text{Mfd}_\partial \), which restricts to a presheaf on \( \text{Disc}_{\leq k, \partial} \). The natural map of presheaves

\[
\text{Bun}_\partial(-, \Theta; \ell_\partial)
\]

is an equivalence because bundle maps form a homotopy sheaf. Using Dwyer–Kan mapping spaces, the simplicial monoid \( \mathbb{R}\text{Aut}_{\text{PSh}(\text{Disc}_{\leq k, \partial})}(\text{Emb}_\partial(-, M), \text{Bun}_\partial(-, \Theta)) \) acts on

\[
\mathbb{R}\text{Map}_{\text{PSh}(\text{Disc}_{\leq k, \partial})}(\text{Emb}_\partial(-, M), \text{Bun}_\partial(-, \Theta))
\]

by first restricting to \( \text{Disc}_{\leq 1, \partial} \) and then precomposing. This allows us to make sense of the homotopy quotient on the right hand side of

\[
\text{BT}_k \text{Emb}_\partial^\theta(M)^\times := \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \sslash \text{T}_k \text{Emb}_\partial(M)^\times.
\]

We caution the reader that the left hand side is a compound symbol, and not obtained by applying embedding calculus to a presheaf or by delooping a simplicial monoid. In particular, using (39) we have:

**Lemma B.2.** \( \text{BT}_1 \text{Emb}_\partial^\theta(M)^\times \simeq \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \sslash \text{Bun}_\partial(\text{Fr}(TM))^\times. \)

Since these constructions are compatible with the restriction from \( \text{Mfd}_\partial \) and between the categories \( \text{Disc}_{\leq k, \partial} \) they give rise to a tower

\[
\text{BEmb}_\partial^\theta(M)^\times \longrightarrow \cdots \longrightarrow \text{BT}_k \text{Emb}_\partial^\theta(M)^\times \longrightarrow \text{BT}_{k-1} \text{Emb}_\partial^\theta(M)^\times \longrightarrow \cdots
\]

**Lemma B.3.** For \( k \geq 2 \) we have a homotopy-cartesian diagram

\[
\text{BT}_k \text{Emb}_\partial^\theta(M)^\times \longrightarrow \text{BT}_k \text{Emb}_\partial(M)^\times
\]

\[
\text{BT}_{k-1} \text{Emb}_\partial^\theta(M)^\times \longrightarrow \text{BT}_{k-1} \text{Emb}_\partial(M)^\times.
\]

Proof. The horizontal homotopy fibres are both \( \text{Bun}_\partial(\text{Fr}(TM), \Theta; \ell_\partial) \). \( \square \)

Thus the higher layers of the ordinary tower are, up to a delooping, the same as those with tangential structures.
Appendix C. Computational results

In this section we give the results of a SageMath [Sag20] implementation of the computational procedure described in Section 7. Our goal is to compute the entries in the chain complexes which compute first four bands, and we include only those intermediate computations necessary to achieve this. The obstruction to further bands is merely computing power and efficiency of the code.

C.1. Extended Drinfel’-Kohno Lie algebras. Section 7 describes a procedure to compute the graded $\mathfrak{g}_s \times \mathfrak{g}_k$-representations $\text{gr}(\mathfrak{g}_s \mathfrak{k})$, from which we may easily deduce $f_g(\mathfrak{k})$. This uses the existing objects and operations, which available in SageMath through its functionality in the package SymmetricFunctions. Firstly, it provides the particular symmetric functions used ($e_k$, $h_k$, $p_k$, $s_\lambda$, $sp_\lambda$, and $o_\lambda$), and readily converts between them. Day convolution of representations is given by the product, plethysm by the command plethysm, inner tensor product by the command kronecker_product, and inner plethysm by the command inner_plethysm. Using these commands, one can perform all the constructions referred to in Section 7, though some work is required to handle pairs of symmetric functions.

We computed $\text{gr}(\mathfrak{g}_s \mathfrak{k})$ and $\text{gr}(\mathfrak{g}_s \mathfrak{k})$ for $g$ large ($g \geq 7$ will suffice) in the following range: $k \leq 6$, $s \leq 5$, and degrees $\leq (r + s)(n - 1)$ with $r + s \leq 10$. This gives enough information to determine $f_g(\mathfrak{k})$ completely for the first four bands.

We record these by fixing an $\mathfrak{g}_s$-representation $S^\lambda$ and taking the power series

$$
\sum_{r=0}^{\infty} \text{ch}_k \left( \text{Hom}_{\mathfrak{g}_s}(S^\lambda, \text{gr}(\mathfrak{g}_s \mathfrak{k})_{r(n-1)} \right) \cdot T^{r+s} \in \Lambda_k[[T]].
$$

The choice of exponent of $T$ is to make it straightforward to read off the degree of the corresponding contribution to $f_g(\mathfrak{k})$: the coefficient of $T^{r+s}$ contributes in degree $(r + s)(n - 1)$.

We give this power series only for $r + s - k \leq 4$; this amounts to restricting our attention to those degrees relevant for computations up to and including the fourth band. Furthermore, in this range it suffices to only record this data for partitions $\lambda$ of numbers $\leq 6 - k$, because $\text{gr}(\mathfrak{g}_s \mathfrak{k})$ is a summand of $\text{gr}(\mathfrak{g}_s \mathfrak{k})$ and it is evident from the presentation of $f_g(\mathfrak{k})$ that only such irreducibles can appear in degrees $\leq (k + 4)(n - 1)$. The results are displayed in Table 2.

C.2. Products relative to diagonals. Theorem 5.30 provides a formula for the cohomology groups $H^*(W^{k \mathfrak{g}_s \mathfrak{k}}_g; \Delta_{r/2}; \mathbb{Q})$ as $\mathcal{H}^0_\mathfrak{g} \times \mathfrak{g}_k$-representation, or equivalently the graded $\mathfrak{g}_s \times \mathfrak{g}_k$-representations $\mathcal{D}(\mathfrak{g}_s \mathfrak{k})$ as in (37). We record these by fixing an $\mathfrak{g}_s$-representation $S^\lambda$ and taking the power series

$$
\sum_{r=0}^{\infty} \text{ch}_k \left( \text{Hom}_{\mathfrak{g}_s}(S^\lambda, \mathcal{D}(\mathfrak{g}_s \mathfrak{k})_{r} \right) \cdot T^{r} \in \Lambda_k[[T]].
$$

The choice of exponent of $T$ is to make it straightforward to read off the degree of the corresponding contribution to $H^*(W^{k \mathfrak{g}_s \mathfrak{k}}_g; \Delta_{r/2}; \mathbb{Q})$: the coefficient of $T^{r}$ is non-zero if and only if $r = s$ and contributes in degree $k + r(n - 1)$.

We give these for $k \leq 4$. The relevant cases for computations up to and including the fourth band are $k \leq 6$ and $r \geq k - 6$, so we can use Proposition 5.31 (iii) and (iv) to deal with the cases $k = 5, 6$. Furthermore, in this range it suffices to only record this data for partitions $\lambda$ of numbers $\leq k$, as it is evident from the formula in Theorem 5.30 that only representations corresponding to such representations can occur. The results are displayed in Table 3.

C.3. The first four layers. The results in the previous two subsections give $f_g(\mathfrak{k})$ and $H^*(W^{k \mathfrak{g}_s \mathfrak{k}}_g; \Delta_{r/2}; \mathbb{Q})$ for $k \leq 6$ in the relevant degrees as algebraic $\mathcal{H}^0_\mathfrak{g} \times \mathfrak{g}_k$-representations. So we can obtain the rationalised entries

$$
(TE_{n,q}^2)_{\mathfrak{g}_k} = [H^*(W^{k \mathfrak{g}_s \mathfrak{k}}_g; \Delta_{r/2}, \mathbb{Q}) \otimes f_g(\mathfrak{k})_{q-1}]_{\mathfrak{g}_k}
$$
for the Federer spectral sequence (23) as $\mathcal{G}_g^{[\ell]}$-representations by taking their tensor product and passing to $\mathcal{G}_g$-invariants. The Federer spectral sequence collapses rationally at the $E^2$-page, and provides the input for computing the rationalised entries on the $E^1$-page $(^{BK}E^1_{s,\ell})_Q$ of Bousfield–Kan spectral sequence.

As we explained in Proposition 5.2, these are supported in bidegrees $(p,q)$ in the intervals $[0, r+1] \times \{ r(n-1)+1 \}$ for $r \geq 1$. In Table 4 we record the entries in these intervals separately for each irreducible $G'_g$-representation $V_\lambda$. (The result shown is for $n$ odd; by Remark 7.17 the result for $n$ even is obtained by transposing the partitions.) The entry in row $\lambda$ and column labelled “$\sim r(n-1)$” is given by the Laurent series

$$\sum_{s=0}^{r+1} \dim \text{Hom}_{\mathcal{G}_g^{[\ell]}}(V_\lambda, (^{BK}E^1_{s,\ell(r(n-1)+1)})_Q) \cdot t^{-s+1} \ell^{s+1} \in \hat{\Lambda}[[t, t^{-1}, \ell]].$$

That is, the variable $\ell$ records the layer and the variable $t$ the amount that its degree deviates from $r(n-1)$. We have once more restricted ourselves to those representations $V_\lambda$ that can actually occur, except that for the sake of brevity we do not display the $V_\lambda$ with $|\lambda| = 6$ for $r = 4$. In particular, the $r$th band consists of odd (resp. even) representations if and only if $r$ is odd (resp. even), and only such partitions are listed.

C.4. Verifications. As with any computer calculation, it is important to verify the results by independent means. Here the checks we have performed:

(i) We computed the cases $k \leq 3$ of Table 2 by hand in Computations 5.27, 5.28 and 5.29, and obtained the same answers.

(ii) Using the Macaulay2 package GradedLieAlgebras one can compute the characters of the homogeneous pieces of quadratically presented graded Lie algebra with $S_k$-action. We used this to verify the computation of $^{gr}f_g^{(0, k)_{(k-1)(2n-2)}}$ for $k \leq 5$.

(iii) Lemma 5.25 says that upon restricting from $S_k$ to $S_{k-1}$, $^{gr}f_g^{(k)}$ can be computed in terms of free graded Lie algebras. These are are obtained by applying Schur functors, so their characters are easy to compute and we verified our answers restrict correctly. For example, from Table 2 it follows that $^{gr}f_g^{(0, 4)_{10(n-1)}} = (2^3) + (3, 1^3) + (4, 2)$ as a $S_6$-representation, which restricts to the $S_5$-representation $(2^2, 1) + (2, 1^3) + (3, 1^2) + (3, 2) + (4, 1)$. This is indeed $\text{Lie}(5)$ as in Table 1.

(iv) The strongest verification of our calculations is that the answers in Table 4 have to satisfy the miraculous cancellation property described in Remark 6.5. In particular, the data in Table 4 must admit, for each non-trivial irreducible representation, a pattern of differentials killing everything below degree $r(n-1)+1$ (i.e. for non-positive powers of $t$). For example, when we consider $V_{3,1}$ for $r = 4$ the pattern is

$$9 \rightarrow 24 \rightarrow 37 \rightarrow 30 \rightarrow 11 \rightarrow 1,$$

for which this is indeed the case. We invite the reader to try other values of $r$ and $\lambda$. Furthermore, the Euler characteristic must match the results of Proposition 4.22 for $r \leq 3$. For example, if we consider $V_{3,12}$ for $r = 3$, it is

$$2 - 2 + 2 - 2 + 1 = 1,$$

as desired.
| $\lambda$ | $S^\lambda$-component of $\text{gr} f(8, 2)_s$ |
|----------|----------------------------------|
| (0)      | $T^2 s_2$                        |
| (1)      | $(T^6 + T^3) s_{12}$             |
| (2)      | $(T^6 + T^4) s_2$                |
| (1, 1)   | $(T^6 + T^4) s_{12} + T^6 s_2$   |
| (3)      | $T^5 s_{12}$                     |
| (2, 1)   | $T^5 s_{12} + T^5 s_2$           |
| (1, 1, 1)| $T^5 s_{12}$                     |
| (4)      | $T^6 s_2$                        |

| $\lambda$ | $S^\lambda$-component of $\text{gr} f(8, 3)_s$ |
|----------|----------------------------------|
| (0)      | $T^4 s_{12} + T^6 s_{21}$         |
| (1)      | $T^7 s_{13} + (2T^7 + T^5) s_{21,1} + T^7 s_{3}$ |
| (2)      | $T^6 s_{13} + T^6 s_{21,1}$      |
| (1, 1)   | $T^6 s_{21,1} + T^6 s_{3}$       |
| (3)      | $T^7 s_{13} + T^7 s_{21,1} + T^7 s_{3}$ |
| (2, 1)   | $T^7 s_{13} + 3T^7 s_{21,1} + T^7 s_{3}$ |
| (1, 1, 1)| $T^7 s_{1,1,1} + T^7 s_{21,1} + T^7 s_{3}$ |

| $\lambda$ | $S^\lambda$-component of $\text{gr} f(8, 4)_s$ |
|----------|----------------------------------|
| (0)      | $T^8 s_{14} + T^8 s_{21,12} + (T^8 + T^6) s_{22} + T^8 s_{3,1}$ |
| (1)      | $T^7 s_{21,12} + T^7 s_{3,1}$    |
| (2)      | $T^8 s_{14} + T^8 s_{21,12} + 2T^8 s_{21,12} + T^8 s_{3,1} + T^8 s_{4}$ |
| (1, 1)   | $2T^8 s_{21,12} + 2T^8 s_{3,1}$ |

| $\lambda$ | $S^\lambda$-component of $\text{gr} f(8, 5)_s$ |
|----------|----------------------------------|
| (0)      | $T^8 s_{3,12}$                    |
| (1)      | $T^9 s_5 + T^9 s_{3,2} + 2T^9 s_{3,12} + T^9 s_{3,2,1} + T^9 s_{15}$ |

| $\lambda$ | $S^\lambda$-component of $\text{gr} f(8, 6)_s$ |
|----------|----------------------------------|
| (0)      | $T^{10} s_{22} + T^{10} s_{3,11} + T^{10} s_{4,2}$ |

Table 2. The power series (41) truncated to powers $T^{r+s}$ with $r + s - k \leq 4$. The coefficient of $T^{r+s}$ in row $(\lambda)$ is the multiplicity of $S^\lambda \otimes S^s$ in the $\mathfrak{g}_s \times \mathfrak{g}_\lambda$-representation $\text{gr} f(8, k)_{(r-\lambda)(n-1)}$, where $s = |\lambda|$. Equivalently, it is the multiplicity of $S^\lambda \otimes S^s$ in the $G_g^{(f)} \times \mathfrak{g}_s$-representation $\text{gr} f(8, k)_{(r+s)(n-1)}$ when $n$ is odd. If $n$ is even, we need to replace $\lambda$ with its transposition $\lambda'$. 

Table 3. The power series (42). The coefficient of $T^\lambda s_\mu$ in row $\lambda$ is the multiplicity of $S^\mu \otimes S^\lambda$ in the $\mathfrak{S}_s \times \mathfrak{S}_k$-representation $D(s,k)$. Equivalently, it is the multiplicity of $S^\mu \otimes S^\lambda$ in the $G_{fr} \times S^k$-representation $H^k(\mathbb{R}^n; \nabla_{A^{1/2}})$ when $n$ is even. If $n$ is odd, we need to replace $\lambda$ with its transposition $\lambda'$.

| $\lambda$ | $S^\lambda$-component of $D(s,2)$ |
|-----------|----------------------------------|
| (0)       | $T_2 s_2$                        |
| (1)       | $(T^5 + T^3)s_{1,1}$             |
| (2)       | $(T^6 + T^4)s_2$                 |
| (1,1)     | $(T^6 + T^4)s_{1,1} + T^6 s_2$   |
| (3)       | $T^5 s_{1,1}$                    |
| (2,1)     | $T^5 s_{1,1} + T^5 s_2$          |
| (1,1,1)   | $T^4 s_{1,1}$                    |
| (4)       | $T^6 s_2$                        |
| (3,1)     | $2T^6 s_{1,1} + T^6 s_2$         |
| (2^2)     | $T^6 s_2$                        |
| (2,1^2)   | $2T^6 s_{1,1} + T^6 s_2$         |
| (1^3)     | $T^6 s_2$                        |

| $\lambda$ | $S^\lambda$-component of $D(s,3)$ |
|-----------|----------------------------------|
| (0)       | 0                                |
| (1)       | $T s_{2,1}$                      |
| (2)       | $T^2 s_{2,1} + T^2 s_3$          |
| (1^2)     | $T^2 s_{2,1} + T^2 s_3$          |
| (3)       | $T^3 s_{1,3}$                    |
| (2,1)     | $T^3 s_{2,1}$                    |
| (1^3)     | $T^3 s_3$                        |

| $\lambda$ | $S^\lambda$-component of $D(s,4)$ |
|-----------|----------------------------------|
| (0)       | 0                                |
| (1)       | $T s_{2,1,2} + T s_{3,1}$        |
| (2)       | $T^2 s_{2,1,2} + T^2 s_{3,2} + 2T^2 s_{3,1}$ |
| (1^2)     | $T^2 s_{2,1,2} + T^2 s_{3,2} + T^2 s_{3,1} + T^2 s_4$ |
| (3)       | $T^3 s_{2,1,2} + T^3 s_{3,1} + T^3 s_4$ |
| (2,1)     | $T^3 s_{2,1,2} + T^3 s_{3,2} + 2T^3 s_{3,1} + T^4 s_4$ |
| (1^3)     | $T^3 s_{2,1,2} + T^3 s_{3,1}$    |
| (4)       | $T^4 s_4$                        |
| (3,1)     | $T^4 s_{3,1}$                    |
| (2^2)     | $T^4 s_{2,2}$                    |
| (2,1^2)   | $T^4 s_{2,1,2}$                  |
| (1^4)     | $T^4 s_{1,4}$                    |
Table 4. The decomposition of the entries \((23)\) on the \(E^1\)-page of the Bousfield–Kan spectral sequence \((24)\) for the embedding calculus Taylor tower, as a \(\tilde{\text{CFk}}_{\lambda'}\)-representation in each of the bands, for \(n\) odd. If \(n\) is even then each \(\lambda\) should be replaced by its transposition \(\lambda'\).

| \(\lambda\) | \(~(n - 1)\) | \(~(2n - 2)\) |
|-------------|--------------|--------------|
| (0)         | \(2t \ell + 2t^2\) | \(t \ell + t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (1)         | \(2t \ell + 2t^2\) | \(2t \ell + 4t^2 + 2t^{-1} \ell^3\) |
| (2)         | \(2t \ell + 2t^2 + 2t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (1^2)       | \(t^2 + t^{-1} \ell^3\) | \(t \ell\) |
| (3)         | \(t \ell + 2t^2 + \ell^3\) | \(t \ell + t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (2, 1)      | \(t \ell + \ell^2\) | \(t \ell + 2t^2 + t^{-1} \ell^3\) |
| (1^3)       | \(t \ell\) | \(t \ell + 2t^2 + t^{-1} \ell^3\) |

| \(\lambda\) | \(~(3n - 3)\) |
|-------------|--------------|
| (1)         | \(2t \ell + 12t^2 + 13t^{-1} \ell^3 + 6t^{-2} \ell^4 + t^{-3} \ell^6\) |
| (2, 1)      | \(2t \ell + 6t^2 + 7t^{-1} \ell^3 + 3t^{-2} \ell^4\) |
| (1^2)       | \(t^2 + t^{-1} \ell^3\) |
| (5)         | \(t \ell + 2t^2 + t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (3, 2)      | \(t \ell + 3t^2 + 3t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (3, 1^2)    | \(t \ell + 2t^2 + 2t^{-1} \ell^3 + 2t^{-2} \ell^4 + t^{-3} \ell^5\) |
| (2, 1^2)    | \(t \ell + 3t^2 + 3t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (2, 1^3)    | \(t \ell + 2t^2 + t^{-1} \ell^3 + t^{-2} \ell^4\) |
| (1^3)       | \(t^2 + t^{-1} \ell^3\) |

| \(\lambda\) | \(~(4n - 4)\) |
|-------------|--------------|
| (0)         | \(3t \ell + 15t^2 + 21t^{-1} \ell^3 + 10t^{-2} \ell^4 + 4t^{-3} \ell^5 + 2t^{-4} \ell^6\) |
| (2)         | \(9t \ell + 26t^2 + 47t^{-1} \ell^3 + 40t^{-2} \ell^4 + 12t^{-3} \ell^5\) |
| (1, 1)      | \(9t \ell + 36t^2 + 48t^{-1} \ell^3 + 29t^{-2} \ell^4 + 11t^{-3} \ell^5 + 3t^{-4} \ell^6\) |
| (4)         | \(3t \ell + 9t^2 + 12t^{-1} \ell^3 + 9t^{-2} \ell^4 + 4t^{-3} \ell^5 + t^{-4} \ell^6\) |
| (3, 1)      | \(9t \ell + 24t^2 + 37t^{-1} \ell^3 + 30t^{-2} \ell^4 + 11t^{-3} \ell^5 + t^{-4} \ell^6\) |
| (2^2)       | \(6t \ell + 20t^2 + 25t^{-1} \ell^3 + 16t^{-2} \ell^4 + 7t^{-3} \ell^5 + 2t^{-4} \ell^6\) |
| (2, 1^2)    | \(9t \ell + 26t^2 + 38t^{-1} \ell^3 + 29t^{-2} \ell^4 + 10t^{-3} \ell^5 + t^{-4} \ell^6\) |
| (1^4)       | \(3t \ell + 11t^2 + 13t^{-1} \ell^3 + 7t^{-2} \ell^4 + 3t^{-3} \ell^5 + t^{-4} \ell^6\) |
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