TRAVERSE FORMULAS ON FINITE GROUPS

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Abstract. In this paper, we study the right regular representation $R_{\Gamma}$ of a finite group $G$ on the vector space consisting of vector valued functions on $\Gamma \backslash G$ with a subgroup $\Gamma$ of $G$ and give a trace formula using the work of M.-F. Vignéras.

Key words and phrases: representations of a finite group, trace formula.

1. Introduction

Following the suggestion of D. Kazhdan, James Arthur proved the so-called local trace formula for a reductive group $G(F)$ over a non-archimedean local field $F$ investigating the regular representation of $G(F) \times G(F)$ on the Hilbert space $L^2(G(F))$ (cf. [1]-[4]). Motivated by the work of J. Arthur on the local trace formula, M.-F. Vignéras (cf. [10]) gave a trace formula for the regular representation of $G \times G$ in $L^2(G)$ for a finite group $G$. In this paper, motivated by the above mentioned work of M.-F. Vignéras, we study the trace formula of the right regular representation $R_{\Gamma}$ of a finite group $G$ on the vector space of consisting of all vector valued functions on the coset space $\Gamma \backslash G$ for a subgroup $\Gamma$. We derive the trace formula for $R_{\Gamma}(f)$ using the result of M.-F. Vignéras (cf. [10]). This trace formula simplifies the proofs of the well known results on a finite group.

In this paper, we shall study the right regular representation $R$ of $G$ on the vector space $V[\Gamma \backslash G]$ consisting of all vector valued functions on $\Gamma \backslash G$ with values in $V$ and give a trace formula for $R_{\Gamma}(f)$ with a function $f$ on $G$. Using this formula, we derive some well known results.

Notation. We denote by $\mathbb{C}$ the complex number field. For a finite set $A$, we denote by $|A|$ the cardinality of $A$. For a finite group $G$, we denote by $\hat{G}$ the set of all equivalence classes of irreducible representations of $G$. For $\lambda \in \hat{G}$, we let $d_{\lambda}$ be the degree of $\lambda$.

This work was supported by INHA UNIVERSITY Research Grant (INHA-22792).
1991 Mathematics Subject Classification. Primary 20C05, 20C15.
2. Trace Formula

Let $\Gamma$ be a subgroup of a finite group $G$. Let $V$ be a finite dimensional complex vector space. We let $X_\Gamma = \Gamma \backslash G$ and denote by $V_\Gamma$ the vector space consisting of all vector valued functions $\varphi : X_\Gamma \rightarrow V$. We note that $G$ acts on $X_\Gamma$ transitively by right multiplication. We let $R_\Gamma$ be the right regular representation of $G$ on $V_\Gamma$, namely,

$$(R_\Gamma(g)\varphi)(x) = \varphi(xg), \quad g \in G, \quad x \in X_\Gamma.$$ 

For any $g \in G$, we set $X^g_\Gamma = \{ x \in X_\Gamma \mid xg = x \}$. 

**Theorem 1.** Let $G, \Gamma, V_\Gamma, X_\Gamma$ and $X^g_\Gamma$ be as above. We let $\chi_{R_\Gamma}$ be the character of the regular representation $R_\Gamma$ of $G$. For each $\lambda \in \hat{G}$, we let $\chi_\lambda$ be the character of $\lambda$. We assume that $R_\Gamma = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V) \lambda$ is the decomposition of $R_\Gamma$ into irreducibles. Here $m_\lambda(\Gamma, V)$ denotes the multiplicity of $\lambda$ in $R_\Gamma$. Then

1. $$\chi_{R_\Gamma}(g) = \dim_{\mathbb{C}} V \cdot |X^g_\Gamma| \quad \text{for all } g \in G,$$

2. $$m_\lambda(\Gamma, V) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g \in G} |X^g_\Gamma| \chi_\lambda(g^{-1}) \quad \text{for each } \lambda \in \hat{G},$$

3. $$|G|^2 = |\Gamma| \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_\lambda |X^g_\Gamma| \chi_\lambda(g^{-1}).$$

For a function $f \in \mathbb{C}[G]$, we define the endomorphism $R_\Gamma(f)$ of $V_\Gamma$ by

$$R_\Gamma(f) = \sum_{g \in G} f(g) R_\Gamma(g).$$

Then for a function $f \in \mathbb{C}[G],$

4. $$\text{tr } R_\Gamma(f) = \dim_{\mathbb{C}} V \cdot \sum_{g \in G} f(g) |X^g_\Gamma|,$$

and for any $f_1, f_2 \in \mathbb{C}[G],$

5. $$\text{tr } R_{\{1\}}(f_1 * f_2) = \sum_{\lambda \in \hat{G}} d_\lambda \text{tr } (\mathcal{F} f_1(\lambda) \mathcal{F} f_2(\lambda)).$$

Here $f_1 * f_2$ denotes the convolution of $f_1$ and $f_2$ defined by

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h)f_2(h^{-1}g), \quad g \in G,$$
\(d_\lambda\) is the degree of \(\lambda\) and \(\mathcal{F} f(\lambda)\) is the Fourier transform of \(f\) defined by

\[
\mathcal{F} f(\lambda) = \sum_{g \in G} f(g) \lambda(g), \quad \lambda \in \hat{G}.
\]

**Proof.** We let \(V[X]\) be the set of all functions \(\phi : X \to V\) with values in \(V\). We describe a basis for the vector space \(V[X]\) and its dual basis. If \(V = \mathbb{C}\), the vector space \(\mathbb{C}[X]\) has a basis \(\{ \delta_x \mid x \in X \}\), where

\[
\delta_x(y) := \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise}.
\end{cases}
\]

For \(x \in X\) and \(v \in V\), we define the function \(\delta_x \otimes v : X \to V\) by

\[
(\delta_x \otimes v)(y) := \begin{cases} 
v & \text{if } x = y \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(\{ v_1, \ldots, v_n \}\) be a basis for \(V\) with \(\dim_{\mathbb{C}} V = n\). Then it is easy to see that the set \(\{ \delta_x \otimes v_k \mid x \in X, \ 1 \leq k \leq n \}\) forms a basis for \(V[X]\). Let \(V^*\) be the dual space of \(V\). For \(x \in X\) and \(v^* \in V^*\), we define the linear functional \(\delta_x^* \otimes v^* : V[X] \to \mathbb{C}\)

\[
(\delta_x^* \otimes v^*) (\phi) := \langle \phi(x), v^* \rangle, \quad \phi \in V[X].
\]

Suppose \(\{ v_1^*, \ldots, v_n^* \}\) is the dual basis of a basis \(\{ v_1, \ldots, v_n \}\). Then we see easily that the set \(\{ \delta_x^* \otimes v_k^* \mid x \in X, \ 1 \leq k \leq n \}\) forms a basis for the dual space \(V[X]^*\).

We also see that for each \(g \in G\),

\[
< R_\Gamma(g)(\delta_x \otimes v_k), (\delta_x^* \otimes v_k^*) >= \begin{cases} 
1 & \text{if } x g = x \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore \(\chi_{R_\Gamma}(g) = \text{tr } R_\Gamma(g) = n \cdot |X_\Gamma|\) for each \(g \in G\). This proves Formula (1).

We define the hermitian inner product \(\langle , \rangle\) on the group algebra \(\mathbb{C}[G]\) by

\[
\langle f_1, f_2 >= \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}, \quad f_1, f_2 \in \mathbb{C}[G].
\]

Since \(\chi_{R_\Gamma} = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V) \chi_\lambda\), we have

\[
m_\lambda(\Gamma, V) = \langle \chi_{R_\Gamma}, \chi_\lambda \rangle \quad \text{(by Schur orthogonality relation (cf. [6], p. 148))}
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \chi_{R_\Gamma}(g) \chi_\lambda(g^{-1})
\]

\[
= \frac{n}{|G|} \sum_{g \in G} |X_\Gamma^g| \chi_\lambda(g^{-1}). \quad \text{(by (1))}
\]

This proves Formula (2).
We observe that

\[ \dim_{\mathbb{C}} V_{\Gamma} = \frac{|G|}{|\Gamma|} \cdot \dim_{\mathbb{C}} V \]

Since \( R_{\Gamma} = \sum_{\lambda \in \hat{G}} m_{\lambda}(\Gamma, V) \lambda \), we see that

\[ \dim_{\mathbb{C}} V_{\Gamma} = \sum_{\lambda \in \hat{G}} m_{\lambda}(\Gamma, V) \cdot d_{\lambda}, \]

where \( d_{\lambda} \) denotes the degree of \( \lambda \in \hat{G} \). By substituting (2) into (7), we get

\[ \dim_{\mathbb{C}} V_{\Gamma} = \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_{\lambda} |X_{\Gamma}^{g}| \chi_{\lambda}(g^{-1}). \]

Therefore according to (6) and (8), we obtain Formula (3).

Let \( f \in \mathbb{C}[G] \). Then we obtain

\[
\begin{align*}
\text{tr} \ R_{\Gamma}(f) &= \text{tr} \left( \sum_{g \in G} f(g) R_{\Gamma}(g) \right) \\
&= \sum_{g \in G} f(g) \text{tr} \ (R_{\Gamma}(g)) \\
&= \sum_{g \in G} f(g) \chi_{R_{\Gamma}}(g) \\
&= \dim_{\mathbb{C}} V \cdot \sum_{g \in G} f(g) |X_{\Gamma}^{g}|. \quad \text{(by (1))}
\end{align*}
\]

This proves Formula (4).

Finally we shall prove Formula (5). If we take \( \Gamma = \{1\} \), then \( X_{1}^{1} = G \) and \( X_{1}^{g} = \emptyset \) if \( g \neq 1 \). Thus by Formula (4), we get

\[ \text{tr} \ R_{\{1\}}(f) = \dim_{\mathbb{C}} V \cdot |G| f(1). \]

Therefore

\[ f(1) = \frac{\text{tr} \ R_{\{1\}}(f)}{|G| \dim_{\mathbb{C}} V}. \]

We recall the fact (see [6], Corollary 3.4.5) that for any \( f_1, f_2 \in \mathbb{C}[G] \), the following Plancherel formula holds:

\[ (f_1 * f_2)(1) = \frac{\text{tr} \ R(f_1 * f_2)}{|G|} = \frac{1}{|G|} \sum_{\lambda \in \hat{G}} d_{\lambda} \text{tr}_{V_{\lambda}} (\mathcal{F}f_1(\lambda) \mathcal{F}f_2(\lambda)), \]

where \( \text{tr}_{V_{\lambda}}(A) \) denotes the trace of an endomorphism \( A : V_{\lambda} \to V_{\lambda} \) with the representation space of \( \lambda \).
On the other hand, for any \( f_1, f_2 \in \mathbb{C}[G] \), we get

\[(f_1 * f_2)(1) = \frac{\text{tr} R_{\{1\}}(f_1 * f_2)}{|G|} \quad \text{(by (9))}.\]

Hence according to (10) and (11), we obtain Formula (5).

\[\square\]

**Corollary 2.**

(a) \(|G| = \sum_{\lambda \in \hat{G}} d_{\lambda}^2\).

(b) \(|G| = \sum_{g \in G} \sum_{\lambda \in \hat{G}} d_{\lambda} \chi_{\lambda}(g^{-1}).\)

(c) Let \(\lambda_0\) be the trivial representation of \(G\). Then

\[m_{\lambda_0}(\Gamma, V) = \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{g \in G} |X_{\Gamma}^g|\].

In particular, \(m_{\lambda_0}(\{1\}, \mathbb{C}) = 1\).

(d) For any \(f \in \mathbb{C}[G]\),

\[f(1) = \frac{\text{tr} R_{\{1\}}(f)}{|G| \cdot \dim_{\mathbb{C}} V} \]

(e) For a subgroup \(\Gamma\) of \(G\),

\[\sum_{\lambda \in \hat{G}} m_{\lambda}(\Gamma, V)^2 = \frac{(\dim_{\mathbb{C}} V)^2}{|G|} \cdot \sum_{g \in G} |X_{\Gamma}^g|^2 |X_{\Gamma}^{g^{-1}}| \]

(f) For each \(\lambda \in \hat{G}\), \(m_{\lambda}(\{1\}, V) = d_{\lambda} \dim_{\mathbb{C}} V \neq 0\). That is, each \(\lambda \in \hat{G}\) occurs in the regular representation \((R_{\{1\}}, V[G])\) of \(G\) with multiplicity \(d_{\lambda} \cdot \dim_{\mathbb{C}} V\).

(g) For each \(\lambda \in \hat{G}\),

\[\sum_{g \in G} \chi_{\lambda}(g) = \frac{|G|}{\dim_{\mathbb{C}} V} \cdot m_{\lambda}(G, V)\].

**Proof.**

(a) If we take \(\Gamma = \{1\}\), we see easily that \(X_{\{1\}}^1 = G\) and \(X_{\{1\}}^g = \emptyset\) if \(g \neq 1\). Then we get

\[|G|^2 = 1 \cdot \sum_{\lambda \in \hat{G}} d_{\lambda} \cdot |X_{\{1\}}^1| \cdot \chi_{\lambda}(1) \quad \text{(by (3))}\]

\[= |G| \sum_{\lambda \in \hat{G}} d_{\lambda}^2. \quad \text{because } \chi_{\lambda}(1) = d_{\lambda}\]

This proves Formula (a). We recall that another proof of (a) follows from the fact that the group algebra \(\mathbb{C}[G]\) is isomorphic to \(\sum_{\lambda \in \hat{G}} \text{End}(V_{\lambda})\) as algebras, where \(V_{\lambda}\) is the representation space of \(\lambda \in \hat{G}\) (cf. [9]).
(b) We take $\Gamma = G$. It is easy to see that $X_G = \{1\}$ is a point and $X_G^g = X_G$ for all $g \in G$. According to Formula (3), we obtain

$$|G|^2 = |G| \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_{\lambda}(g^{-1}).$$

This proves the statement (b).

(c) It follows from Formula (2).

(d) We take $\Gamma = \{1\}$. Then $X_1^1 = G$ and $X_1^g = \emptyset$ for $g \neq 1$. From Formula (4), we obtain

$$\text{tr} R_{\{1\}}(f) = \dim_{\mathbb{C}} V \cdot |G| f(1).$$

(e) By Schur orthogonality relation, $< \chi_{R_{\Gamma}}, \chi_{R_{\Gamma}} >= \sum_{\lambda \in \hat{G}} m_{\lambda}(\Gamma, V)^2$. On the other hand, according to the formula (1),

$$< \chi_{R_{\Gamma}}, \chi_{R_{\Gamma}} >= \frac{1}{|G|} \sum_{g \in G} \chi_{R_{\Gamma}}(g) \chi_{R_{\Gamma}}(g^{-1}) = (\dim_{\mathbb{C}} V)^2 \cdot \sum_{g \in G} |X_G^g| |X_G^{g^{-1}}|.$$

(f) We take $\Gamma = \{1\}$. Then $X_{\{1\}}^1 = G$, $X_{\{1\}}^g = \emptyset$ for $g \neq 1$, and $V_{\{1\}} = V[G]$. Therefore we obtain the desired result from Formula (2).

(g) We take $\Gamma = G$. We see easily that $|X_G^g| = 1$ for all $g \in G$. According to Formula (2), we get

$$m_{\lambda}(G, V) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g \in G} \chi_{\lambda}(g^{-1}) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g \in G} \chi_{\lambda}(g).$$

Hence this proves the statement (g).

**Theorem 3.** For each $f \in \mathbb{C}[G]$, we have the following trace formula

(12) \[ \text{tr} R_{\Gamma}(f) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g \in G} |X_G^g| |Z_g| f(C_g), \]

where $Z_g$ is the centralizer of $g$ in $G$, $C_g$ is the conjugacy class of $g$ and $f(C_g) = \sum_{h \in C_g} f(h)$.

**Proof.** For $f \in \mathbb{C}[G]$ and $\lambda \in \hat{G}$, we define

$$\lambda(f) := \sum_{g \in G} f(g) \lambda(g).$$
Investigating the spectral decomposition of the regular representation \( R \) of \( G \times G \) on \( \mathbb{C}[G] \) defined by
\[
(R(g_1, g_2)F)(g) = F(g_1^{-1}gg_2), \quad g, g_1, g_2 \in G, \ F \in \mathbb{C}[G].
\]
M.-F. Vigneras (cf. [10], p.284) obtained the following trace formula
\[
|Z_g|F(C_g) = \sum_{\pi \in \hat{G}} \chi_\pi(g^{-1}) \text{ tr} \pi(F)
\]
for any \( g \in G \) and \( F \in \mathbb{C}[G] \).

Let \( R_\Gamma = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V) \lambda \) be the decomposition of \( R_\Gamma \) into irreducibles. If \( f \in \mathbb{C}[G] \),
\[
\text{tr} R_\Gamma(f) = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V) \text{ tr} \lambda(f)
\]
\[
= \frac{\dim_\mathbb{C} V}{|G|} \sum_{\lambda \in \hat{G}} \sum_{g \in G} |X_\Gamma^g| \chi_\lambda(g^{-1}) \text{ tr} \lambda(f) \quad \text{(by (2))}
\]
\[
= \frac{\dim_\mathbb{C} V}{|G|} \sum_{g \in G} |X_\Gamma^g| \left( \sum_{\lambda \in \hat{G}} \chi_\lambda(g^{-1}) \text{ tr} \lambda(f) \right)
\]
\[
= \frac{\dim_\mathbb{C} V}{|G|} \sum_{g \in G} |X_\Gamma^g| |Z_g| f(C_g).
\]
The last equality follows from Formula (13). \( \square \)

**Corollary 4.** Let \( \Gamma \) be a subgroup of \( G \). Then for any \( f \in \mathbb{C}[G] \), we have the following identity
\[
|G| \sum_{g \in G} f(g)|X_\Gamma^g| = \sum_{g \in G} |X_\Gamma^g| |Z_g| f(C_g),
\]
where \( Z_g \) is the centralizer of \( g \) in \( G \), \( C_g \) is the conjugacy class of \( g \) and \( f(C_g) = \sum_{h \in C_g} f(h) \).

**Proof.** The proof follows immediately from Formula (4) and the trace formula (12). \( \square \)

**Remark 5.** The trace formula (12) is similar to the trace formula on the adele group. For the trace formula on the adele group, we refer to [1]-[5], [7] and [8].

**Remark 6.** If \( \Gamma \neq \{1\} \), the multiplicity \( m_\lambda(\Gamma, V) \) of some \( \lambda \in \hat{G} \) may be zero. It is natural to ask when \( m_\lambda(\Gamma, V) \) is not zero. Namely, which \( \lambda \in \hat{G} \) does occur in the regular representation \( (R_\Gamma, V_\Gamma) \) of \( G \)?
REFERENCES

[1] J. Arthur, The trace formula and Hecke operators: Number theory, trace formulas and discrete groups (Oslo, 1987) (1989), Academic Press, 11-27.
[2] ______, Towards a local trace formula: algebraic analysis, geometry and number theory (Baltimore, MD, 1988) (1989), Johns Hopkins Univ. Press, Baltimore, MD, 1-23.
[3] ______, Some problems in local harmonic analysis : harmonic Analysis on reductive groups (Brunswick, ME, 1989) (1991), Prog. Math. Birkhäuser, Boston vol 101, 57-78.
[4] ______, A local trace formula 73 (1991), IHES Publication Math., 5-96.
[5] S. Gelbart, Lectures on Arthur-Selberg Trace Formula, American Math. Soc., Providence, 1996.
[6] R. Goodman and N. Wallach, Representations and Invariants of the Classical Groups, Cambridge University Press, 1998.
[7] A. W. Knapp, Theoretical Aspect of the Trace Formula for GL(2), Proc. Symosia in Pure Math. (1997), American Math. Soc. 61, 355-405.
[8] A. W. Knapp and J. D. Rogawski, Applications of the Trace Formula, Proc. Symosia in Pure Math. (1997), American Math. Soc. 61, 413-431.
[9] J.-P. Serre, Linear representations of finite groups, Springer-Verlag, 1977.
[10] M.-F. Vignéras, An elementary introduction to the local trace formula of J. Arthur. The case of finite groups, Jber. d. Dt. Math.-Verein., Jubiläumstagung 1990, B.G. Teubner Stuttgart (1992), 281-296.

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