Density-dependent NN-interaction from subleading chiral 3N-forces: short-range terms and relativistic corrections

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Abstract

We derive from the subleading contributions to the chiral three-nucleon force (short-range terms and relativistic corrections, published in Phys. Rev. C84, 054001 (2011)) a density-dependent two-nucleon interaction $V_{\text{med}}$ in isospin-symmetric nuclear matter. The momentum and $k_f$-dependent potentials associated with the isospin operators ($1$ and $\vec{\tau}_1 \cdot \vec{\tau}_2$) and five independent spin-structures are expressed in terms of loop functions, which are either given in closed analytical form or require at most one numerical integration. Our results for $V_{\text{med}}$ are most helpful to implement subleading chiral 3N-forces into nuclear many-body calculations.

1 Introduction and summary

Three-nucleon forces are an indispensable ingredient in accurate few-nucleon and nuclear structure calculations. Nowadays, chiral effective field theory is the appropriate tool to construct systematically the nuclear interactions [1]. Precise two-nucleon potentials have been developed at next-to-next-to-next-to-leading order (N$^3$LO) in the chiral (small momentum) expansion [1, 2, 3]. The extension of the NN-potential to N$^4$LO, with various higher-order two- and three-pion exchange contributions, has been accomplished in refs. [4, 5]. Three-nucleon forces appear first at N$^2$LO, where they consist of a zero-range contact-term (parameter $c_E$), a mid-range $1\pi$-exchange component (parameter $c_D$) and a long-range $2\pi$-exchange component (parameters $c_{1,3,4}$). The calculation of the subleading chiral three-nucleon forces, built up by many pion-loop diagrams, has been performed for the long-range contributions in ref. [6] and completed with the short-range terms and relativistic corrections in ref. [7]. Moreover, the extension of the $2\pi$-exchange component of the chiral three-nucleon force to N$^4$LO has been accomplished in ref. [8] and the corresponding (ten independent) 3N-contact terms quadratic in momenta have been derived in ref. [9].

For the variety of existing many-body methods that are commonly employed in calculations of nuclear matter or medium mass and heavy nuclei it is technically very challenging to include chiral three-nucleon forces directly. In ref. [10] a decomposition of the chiral three-nucleon forces at N$^2$LO and N$^3$LO in a momentum-space partial wave basis has been proposed, which should make ab initio studies of few-nucleon scattering, nuclei and nuclear matter computationally much more efficient. In that endeavor one is working with large file sizes for the 3-body matrix elements in each individual spin-isospin channel, while the values of the low-energy constants and the form of the regulator function can be chosen freely. An alternative and simpler approach is to employ instead a density-dependent two-nucleon interaction $V_{\text{med}}$ that reflects the underlying three-nucleon force. The analytical calculation of $V_{\text{med}}$ from the leading chiral 3N-force at N$^2$LO (involving the parameters $c_{1,3,4}$, $c_D$ and $c_E$) has been presented in ref. [11]. When restricting to on-shell scattering of two-nucleons in isospin-symmetric spin-saturated nuclear matter, the resulting in-medium NN-interaction $V_{\text{med}}$ has the same isospin- and spin-structure as the free NN-potential. The subsequent decomposition into partial wave matrix elements has provided a good illustration of the (repulsive or attractive) effects of the various components of $V_{\text{med}}$ in different spin-isospin channels. Together

\footnote{This work has been supported in part by DFG and NSFC (CRC110).}
with extensions to isospin-asymmetric nuclear matter or spin-polarized neutron matter, the in-medium interaction $V_{\text{med}}$ derived from the leading chiral 3N-force has found many applications in recent years \[12,13,14\]. Furthermore, the approach has been generalized to the strangeness sector by deriving in-medium hyperon-nucleon interactions \[15\] from the leading order chiral YNN-forces (where $Y=\Lambda, \Sigma$).

The purpose of the present paper is to continue the construction of the in-medium NN-interaction $V_{\text{med}}$ to N$^3$LO by treating the subleading contributions to the chiral 3N-force. We focus here on the short-range terms and relativistic corrections, which have been derived in ref. \[7\] by using the method of unitary transformations. The calculation of $V_{\text{med}}$ from the remaining long-range 3N-forces (divided into diagram classes of two-pion-exchange topology, two-pion-one-pion-exchange topology, and ring topology in ref. \[6\]) is yet more demanding and relegated to a future publication. In section 2, we outline the diagrammatic calculation of $V_{\text{med}}$ from a generic 3N-interaction $V_{3\text{N}}$ by closing one nucleon-line to an in-medium loop. This way one encounters four different topologies of in-medium NN-scattering diagrams: self closings, short-range vertex corrections, pionic vertex corrections, and double exchanges. The basis of five independent spin-operators, into which the in-medium interaction $V_{\text{med}}$ is decomposed, is also given. In section 3 the one-pion-exchange-contact part of the 3N-force is treated and shown to produce an effectively vanishing in-medium NN-interaction $V_{\text{med}}^{(1S_0)} = 0$. Section 4 deals with the two-pion-exchange-contact topology and analytical expressions are given for the obtained contributions to $V_{\text{med}}$. In section 5 the leading relativistic corrections to the chiral 3N-force, divided into $1\pi$-exchange-contact and $2\pi$-exchange topologies, are treated. The corresponding contributions to $V_{\text{med}}$ are expressed in terms of loop-functions $\Gamma_\nu(p,k_f)$, $\gamma_\nu(p,k_f)$, $G_\nu(p,q,k_f)$ and $K_\nu(p,q,k_f)$ which are listed in their explicit form in the appendix. Moreover, in section 6 the subleading 3N-contact potential of ref. \[9\] is converted into an in-medium NN-interaction $V_{\text{med}}$ linear in density $\rho = 2k_f^3/3\pi^2$ and quadratic in momenta.

In summary, after eventual partial-wave projection (see herefore e.g. eqs.(38)-(41) in ref.\[11\]) our results for $V_{\text{med}}$ are suitable for easy implementention of subleading chiral 3N-forces into nuclear many-body calculations.

2 In-medium NN-interaction from one-loop diagrams

![Figure 1: Generic form of the 3N-interaction. The dashed line symbolizes pion-exchange and the wiggly line a short-range interaction.](image)

First, we outline how the in-medium interaction $V_{\text{med}}$ is constructed and calculated diagramatically. The generic form of a chiral 3N-interaction $V_{3\text{N}}$ is shown in Fig. 1. The dashed line symbolizes (one or two) pion-exchange and the wiggly line a short-range interaction (without a momentum-dependent propagator). In order to obtain from $V_{3\text{N}}$ a density-dependent NN-interaction, one
has to close one nucleon-line and integrate the resulting loop, \((2\pi)^{-4}\int d^4l\), over the medium part 
\(-2\pi\delta(l_0)\theta(k_f - |\vec{l}|)\) of the heavy nucleon particle-hole propagator. The process of closing one of the 
three identical fermionic nucleon-lines in \(V_{3N}\) leads to four different topologies for NN-scattering 
diagrams: self closings, short-range vertex corrections, pionic vertex corrections, and double exchanges, which are separately shown in Figs. 2, 3, 4 and 5. The respective mirror graphs, resulting 
from the interchange of both nucleons \(N_1 \leftrightarrow N_2\), have of course to be added. With reference to 
these four topologies, the contributions to the in-medium NN-interaction are distinguished and denoted as \(V_{\text{med}}^{(0)}, V_{\text{med}}^{(1)}, V_{\text{med}}^{(2)}\) and \(V_{\text{med}}^{(3)}\). In the case of a \(2\pi\)-exchange 3N-force the two types of 
pionic vertex corrections will be combined into just one expression for \(V_{\text{med}}^{(1)}\). This reduced specifi-
cation is employed in subsections 5.2 and 5.3. Individual contributions to \(V_{\text{med}}\) are represented in 
terms of loop functions, that are defined by Fermi sphere integrals \((2\pi)^{-1}\int d^3l \theta(k_f - |\vec{l}|)\) over pion 
propagators.

Figure 2: Self closings of a nucleon line generating the contribution \(V_{\text{med}}^{(0)}\) to the in-medium NN-
interaction. Mirror graphs \(N_1 \leftrightarrow N_2\) have to be supplemented.

Figure 3: Vertex corrections by the short-range interaction generating the contribution \(V_{\text{med}}^{(1)}\). Mir-
ror graphs \(N_1 \leftrightarrow N_2\) have to be supplemented.

Figure 4: Vertex corrections by pion-exchange generating the contribution \(V_{\text{med}}^{(2)}\). Mirror graphs 
\(N_1 \leftrightarrow N_2\) have to be supplemented.

In chiral effective field theory, 3N-interactions depend on the spins \(\vec{\sigma}_j\), isospins \(\vec{\tau}_j\), and momenta 
of the three involved nucleons. Denoting the in-going and out-going nucleon momenta by \(\vec{p}_j\) and \(\vec{p}_j'\),
Figure 5: Double exchanges generating the contribution $V_{\text{med}}^{(3)}$ to the in-medium NN-interaction. Mirror graphs $N_1 \leftrightarrow N_2$ have to be supplemented.

($j = 1, 2, 3$), the associated in-medium momentum-transfers $\vec{q}_j = \vec{p}_j' - \vec{p}_j$ satisfy the relation $\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$. In order to determine the in-medium NN-interaction $V_{\text{med}}$, we consider the on-shell scattering process $N_1(\vec{p}) + N_2(-\vec{p}) \rightarrow N_1(\vec{p}') + N_2(-\vec{p}'')$ in the center-of-mass frame, such that $|\vec{p}| = |\vec{p}'| = p$. The in-going and outgoing momenta are $\pm \vec{p}$ and $\pm \vec{p}'$, and $\vec{q} = \vec{p}' - \vec{p}$ is the momentum transfer with $0 \leq q \leq 2p$. For NN-scattering in isospin-symmetric spin-saturated nuclear matter of density $\rho = 2k_f^3/3\pi^3$, the isospin operators appearing in $V_{\text{med}}$ are the same as in free space, namely $1$ and $\vec{\tau}_1 \cdot \vec{\tau}_2$. Although Galilei invariance is broken by the nuclear medium, this similarity applies for the chosen on-shell kinematics ($|\vec{p}| = |\vec{p}'|$) also to the spin-momentum operators. Therefore one can expand $V_{\text{med}}$ in terms of the following five independent spin operators:

$$1, \quad \vec{\sigma}_1 \cdot \vec{\sigma}_2, \quad \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}, \quad i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) , \quad \vec{\sigma}_1 \cdot \vec{p}\vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p}' \vec{\sigma}_2 \cdot \vec{p}''. \quad (1)$$

The quadratic spin-orbit operator will be used occasionally for notational convenience and it possesses the following decomposition:

$$\vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}) = q^2 \left( p^2 - \frac{q^2}{4} \right) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \left( \frac{q^2}{2} - p^2 \right) \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} - \frac{q^2}{2} (\vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p}' \vec{\sigma}_2 \cdot \vec{p}''). \quad (2)$$

As a side remark, we note that the sign-convention for the NN-potential is chosen such that ordinary $1\pi$-exchange gives $V_{1\pi} = -(g_A/2f_\pi)^2(m_\pi^2 + q^2)^{-1}\vec{\tau}_1 \cdot \vec{\tau}_2 \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}$. The occurring parameters are $g_A = 1.3, f_\pi = 92.2$ MeV and $m_\pi = 138$ MeV. After these preparations, we turn now to the enumeration of the contributions to the in-medium NN-potential $V_{\text{med}}$, following the presentation of subleading chiral $3N$-forces $V_{3N}$ in ref. [7].

3 One-pion-exchange-contact topology

For the $1\pi$-exchange-contact topology two nonzero contributions to $V_{3N}$ have been derived in section II of ref. [7], which we take from eq.(2.1):

$$V_{3N} = -\frac{g_A^4 C_T m_\pi}{16\pi f_\pi^4} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{m_\pi^2 + q_1^2} \vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_3 \cdot \vec{q}_1, \quad (3)$$

and eq.(2.3):

$$V_{3N} = \frac{g_A^4 C_T m_\pi}{32\pi f_\pi^4} \frac{\vec{\sigma}_1 \cdot \vec{q}_1}{m_\pi^2 + q_1^2} \left[ 2\vec{\tau}_1 \cdot \vec{\tau}_3 \vec{\sigma}_3 \cdot \vec{q}_1 - \vec{\tau}_1 \cdot (\vec{\tau}_2 \times \vec{\tau}_3) \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3) \cdot \vec{q}_1 \right]. \quad (4)$$

\footnote{Going off-shell ($|\vec{p}| \neq |\vec{p}'|$) introduced some spin-momentum operators that are unfamiliar from the NN-potential. In practice the off-shell extrapolation [11] of $V_{\text{med}}$ is performed by the substitution $p^2 \rightarrow (p^2 + p'^2)/2$ (or $pp'$).}
We remind that the parameter $C_T$ belongs to the leading order NN-contact potential $V_{\text{ct}} = C_S + C_T \vec{s}_1 \cdot \vec{s}_2$ and the factor $m_\pi/8\pi$ in $V_{3N}$ stems from pion-loop integrals evaluated in dimensional regularization. Computing all the in-medium diagrams with self closings, vertex corrections and double exchanges (shown in Figs. 2, 3, 4, 5) for both 3N-interactions in eqs.(3,4) and summing up the individual contributions to $V_{\text{med}}$, one obtains the following total result:

$$V_{\text{med}} = (1 + 2 - 3) \frac{g_A^4 C_T m_\pi k_f^3}{24\pi^3 f_\pi^4} \frac{\vec{r}_1 \cdot \vec{r}_2}{m_\pi + q^2} \vec{s}_1 \cdot \vec{q} \vec{s}_2 \cdot \vec{q} + \frac{q_A^4 C_T m_\pi}{48\pi^3 f_\pi^4} (2k_f^3 - 3m_\pi^2 \Gamma_0) [\vec{r}_1 \cdot \vec{r}_2 (1 - 2\vec{s}_1 \cdot \vec{s}_2) - 3]$$

$$+ \frac{g_A^4 C_T m_\pi}{32\pi^3 f_\pi^4} (3 + \vec{r}_1 \cdot \vec{r}_2) [(\Gamma_0 + 2\Gamma_1 + \Gamma_3)(\vec{s}_1 \cdot \vec{p} \vec{s}_2 \cdot \vec{p} + \vec{s}_1 \cdot \vec{p}' \vec{s}_2 \cdot \vec{p}') + 2\Gamma_2 \vec{s}_1 \cdot \vec{s}_2].$$

The first line is obviously zero. From the known eigenvalues $\vec{s}_1 \cdot \vec{s}_2 \rightarrow 4S - 3$ and $\vec{r}_1 \cdot \vec{r}_2 \rightarrow 4I - 3$, with $S = 0, 1$ the total spin and $I = 0, 1$ the total isospin of the two-nucleon system, one concludes that the second line contributes only in the $^1S_0$ partial wave. The term in the third line is operative only for total isospin $I = 1$ and partial wave projection shows furthermore that it contributes only to the $s$-wave. Therefore, the in-medium potential remaining in the $^1S_0$ partial wave is:

$$V_{\text{med}}^{(^1S_0)} = \frac{g_A^4 C_T m_\pi}{4\pi^3 f_\pi^4} \left\{ \frac{2k_f^2}{3} - m_\pi^2 \Gamma_0 - p^2(\Gamma_0 + 2\Gamma_1 + \Gamma_3) - 3\Gamma_2 \right\} = 0,$$

and it vanishes by the identity:

$$\int \frac{d^4l}{2\pi} \frac{m_\pi^2 + (\vec{l} + \vec{p})^2}{m_\pi^2 + (\vec{l} + \vec{p})^2} = \frac{2k_f^3}{3} = m_\pi^2 \Gamma_0 + 3\Gamma_2 + p^2(\Gamma_3 + 2\Gamma_1 + \Gamma_0).$$

The last part in eq.(7) follows by expanding the numerator and expressing the Fermi sphere integrals in terms of the loop functions $\Gamma_\nu(p, k_f)$, which are given in explicit form in the appendix. The effective vanishing of $V_{\text{med}}$ from the $1\pi$-exchange-contact topology is an important consistency check, since the underlying $V_{3N}$ vanishes after complete antisymmetrization as demonstrated at the end of section II in ref. [7].

4 Two-pion-exchange-contact topology

Next, we come to the $2\pi$-exchange-contact topology, for which ref.[7] has derived two contributions to $3N$-interaction $V_{3N}$. The structurally simpler one written in eq.(3.5) has the form:

$$V_{3N} = -\frac{g_A^2 C_T}{24\pi^3 f_\pi^4} \vec{r}_1 \cdot \vec{r}_2 \vec{s}_2 \cdot \vec{s}_3 [m_\pi + (2m_\pi^2 + q_1^2) A(q_1)],$$

with the pion-loop function

$$A(q_1) = \frac{1}{2q_1} \arctan \frac{q_1}{2m_\pi}.$$ 

The corresponding self-closing term $V_{\text{med}}^{(0)} = 0$ due to vanishing spin- or isospin-traces and the short-range vertex corrections introduce merely a factor $-3p$, leading to the result:

$$V_{\text{med}}^{(1)} = \frac{g_A^2 C_T k_f^3}{12\pi^3 f_\pi^4} \vec{r}_1 \cdot \vec{r}_2 [m_\pi + (2m_\pi^2 + q^2) A(q)].$$

5
Pionic vertex corrections and double exchanges differ only with respect to the isospin operator and therefore can be combined to one single expression:

$$V_{\text{med}}^{(2)} + V_{\text{med}}^{(3)} = \frac{g_A^4 C_T}{720 \pi^3 f_\pi^4} \sigma_1 \cdot \sigma_2 (3 + \bar{\tau}_1 \cdot \bar{\tau}_2) \left\{ k_f m_\pi (p^2 + 11 k_f^2 - 4 m^2_\pi) ight.$$ 

$$+ (p + k_f)^2 \left[ \frac{k_f}{p} (k_f^2 + 10 m^2_\pi) - 5 m^2_\pi + \frac{7 k_f^2}{4} + \frac{k_f p}{2} - \frac{p^2}{4} \right] \arctan \frac{p + k_f}{2m_\pi}$$ 

$$+ (p - k_f)^2 \left[ \frac{k_f}{p} (k_f^2 + 10 m^2_\pi) + 5 m^2_\pi - \frac{7 k_f^2}{4} + \frac{k_f p}{2} + \frac{p^2}{4} \right] \arctan \frac{p - k_f}{2m_\pi}$$ 

$$+ \frac{m_\pi^2}{p} (5 p^2 - 5 k_f^2 + 4 m^2_\pi) \ln \left( \frac{4 m^2_\pi + (p + k_f)^2}{4 m^2_\pi + (p - k_f)^2} \right) \right\}.$$ (11)

Note that this term acts only in the $^1 S_0$ partial wave with $\sigma_1 \cdot \sigma_2 (3 + \bar{\tau}_1 \cdot \bar{\tau}_2) \rightarrow -12$.

The other 3N-interaction provided by the 2$\pi$-exchange-contact topology reads according to eq.(3.1) in ref. [7]:

$$V_{3N} = \frac{g_A^4 C_T}{48 \pi f_\pi^4} \left\{ 2 \bar{\tau}_1 \cdot \bar{\tau}_2 \bar{\sigma}_2 \cdot \bar{\sigma}_3 \left[ 3 m_\pi - \frac{m_\pi^3}{4 m^2_\pi + q^2} + 2 (2 m^2_\pi + q^2) A(q_1) \right] \right.$$ 

$$+ 9 \left[ \bar{\sigma}_1 \cdot \tilde{q}_1 \bar{\sigma}_2 \cdot \tilde{q}_1 - q_1^2 \bar{\sigma}_1 \cdot \bar{\sigma}_2 \right] A(q_1) \right\},$$ (12)

where the first part is structurally equivalent to eq.(8). It is advantageous to combine self closings and short-range vertex corrections:

$$V_{\text{med}}^{(0)} + V_{\text{med}}^{(1)} = \frac{g_A^4 C_T}{12 \pi^3 f_\pi^4} \left\{ \frac{3}{2} (\bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q} - q^2 \bar{\sigma}_1 \cdot \bar{\sigma}_2) A(q) \right.$$ 

$$+ \bar{\tau}_1 \cdot \bar{\tau}_2 \left[ \frac{m_\pi^3}{4 m^2_\pi + q^2} - 3 m_\pi - 2 (2 m^2_\pi + q^2) A(q) \right] \right\},$$ (13)

since for these the medium effect amounts to merely a factor of density $\rho = 2 k_f^3 / 3 \pi^2$. The contributions from pionic vertex corrections and double exchanges are also preferably combined into one expression, which reads:

$$V_{\text{med}}^{(2)} + V_{\text{med}}^{(3)} = \frac{g_A^4 C_T}{40 \pi^3 f_\pi^4} \left[ \frac{H_1}{2} - \frac{H_0}{9} (3 + \bar{\tau}_1 \cdot \bar{\tau}_2) \bar{\sigma}_1 \cdot \bar{\sigma}_2 \right]$$ 

$$+ \frac{g_A^4 C_T}{64 \pi^3 f_\pi^4} \left[ H_2 \bar{\sigma}_1 \cdot \bar{\sigma}_2 + H_3 (\bar{\sigma}_1 \cdot \tilde{p} \bar{\sigma}_2 \cdot \tilde{p} = \bar{\sigma}_1 \cdot \tilde{p} \bar{\sigma}_2 \cdot \tilde{p}) \right].$$ (14)

The occurring $(p, k_f)$-dependent functions $H_{0,1,2,3}$ have the following analytical forms:

$$H_0 = k_f m_\pi (p^2 + 21 k_f^2 - 19 m^2_\pi) + m_\pi^3 (35 p^2 - 35 k_f^2 - 44 m^2_\pi) \ln \left( \frac{4 m^2_\pi + (p + k_f)^2}{4 m^2_\pi + (p - k_f)^2} \right)$$ 

$$+ \left[ \frac{k_f^3}{p} (k_f^2 + 10 m^2_\pi) - 5 p^2 m^2_\pi + 5 k_f p + \frac{5}{2} p^2 k_f^2 \right.$$

$$+ 15 m^2_\pi (k_f^2 + 2 m^2_\pi) + \frac{1}{4} (15 k_f^4 - p^4) \right] \arctan \frac{p + k_f}{2 m_\pi}$$ 

$$+ \left[ \frac{k_f^3}{p} (k_f^2 + 10 m^2_\pi) + 5 p^2 m^2_\pi - 5 k_f p - \frac{5}{2} p^2 k_f^2 \right.$$

$$- 15 m^2_\pi (k_f^2 + 2 m^2_\pi) + \frac{1}{4} (p^4 - 15 k_f^4) \right] \arctan \frac{p - k_f}{2 m_\pi},$$ (15)
\[ H_1 = k_f m_\pi (p^2 - 9k_f^2 + 36m_\pi^2) + \frac{2m_\pi^3}{p}(5k_f^2 + 12m_\pi^2 - 5p^2) \ln \frac{4m_\pi^2 + (p + k_f)^2}{4m_\pi^2 + (p - k_f)^2} \]
\[ + \left[ \frac{k_f^5}{p} - 60m_\pi^4 + 5k_f^3 p + \frac{5}{2} p^2 k_f^2 + \frac{1}{4} (15k_f^4 - p^4) \right] \arctan \frac{p + k_f}{2m_\pi} \]
\[ + \left[ \frac{k_f^5}{p} + 60m_\pi^4 + 5k_f^3 p - \frac{5}{2} p^2 k_f^2 + \frac{1}{4} (p^4 - 15k_f^4) \right] \arctan \frac{p - k_f}{2m_\pi}, \quad (16) \]

\[ H_2 = \frac{k_f m_\pi}{7p^2} \left[ \frac{19k_f^4}{10} + \frac{16k_f^2}{15}(6m_\pi^2 - 19p^2) + 8m_\pi^4 + \frac{68p^2 m_\pi^2}{2} + \frac{p^4}{2} \right] + \frac{m_\pi}{8p^3} \left[ -\frac{64}{7} m_\pi^6 \right] \ln \frac{4m_\pi^2 + (p + k_f)^2}{4m_\pi^2 + (p - k_f)^2} \]
\[ + \left[ \frac{2k_f^7}{35p^3} + 2k_f^3 p + \frac{6}{5} p^2 k_f^2 + k_f^4 - \frac{p^4}{7} - 16m_\pi^4 \right] \arctan \frac{p + k_f}{2m_\pi} \]
\[ + \left[ \frac{2k_f^7}{35p^3} + 2k_f^3 p - \frac{6}{5} p^2 k_f^2 - k_f^4 + \frac{p^4}{7} + 16m_\pi^4 \right] \arctan \frac{p - k_f}{2m_\pi}, \quad (17) \]

\[ H_3 = \frac{k_f m_\pi}{140p^4} \left[ 41p^4 + 8p^2(13k_f^2 - 3m_\pi^2) - 3(19k_f^4 + 64k_f^2 m_\pi^2 + 80m_\pi^4) \right] \]
\[ + \frac{m_\pi}{p^5} \left[ \frac{m_\pi^2}{4} (3k_f^4 - 2k_f p^2 - p^4) + \frac{3}{5} m_\pi^4 (3k_f^2 + p^2) + \frac{12}{7} m_\pi^6 + \frac{3}{16} (k_f^2 - p^2)^3 \right] \]
\[ \times \ln \frac{4m_\pi^2 + (p + k_f)^2}{4m_\pi^2 + (p - k_f)^2} + \frac{(p + k_f)^4}{35p^3} \left[ (4p^3 - 16k_f^2 p^2 + 12k_f^2 p - 3k_f^2) \arctan \frac{p + k_f}{2m_\pi} \right] \]
\[ - \frac{(p - k_f)^4}{35p^3} \left[ (4p^3 + 16k_f^2 p^2 + 12k_f^2 p + 3k_f^2) \arctan \frac{p - k_f}{2m_\pi} \right]. \quad (18) \]

These results have been obtained by using the following reduction formulas for Fermi sphere integrals over even functions \( F(s) = F(-s) \):

\[
\int \frac{d^3l}{2\pi} F(|\vec{l} + \vec{p}|) \{l_i, l_i l_j \} = \int_{p-k_f}^{p+k_f} ds \frac{s}{2p} \left[ k_f^2 - (p - s)^2 \right] F(s) \{1, \chi_1 p_i, \chi_2 \delta_{ij} + \chi_3 p_i p_j \}, \quad (19) \]

with the polynomial weighting-functions:

\[
\chi_1 = \frac{1}{4p^2} (s^2 + 2sp - 3p^2 - k_f^2), \quad (20) \\
\chi_2 = \frac{1}{24p^2} [k_f^2 - (p - s)^2] (s^2 + 4sp + p^2 - k_f^2), \quad (21) \\
\chi_3 = \frac{1}{8p^4} [k_f^4 + 2k_f^2 (p^2 - sp - s^2) + (p - s)^2 (s^2 + 4sp + 5p^2)]. \quad (22) 
\]

Note that the analytical result in eq.\((11)\) was obtained with the first of the three reduction formulas.
5 Leading relativistic corrections

Next, we treat the relativistic $1/M$-corrections to the chiral 3N-interaction, which can be sub-divided into diagrams of $1\pi$-exchange-contact topology (with parameter combination $g_A^2 C_{S,T}/f_\pi^2$) and diagrams of $2\pi$-exchange topology (proportional to $g_A^2/f_\pi^4$). The corresponding expressions for $V_{3N}$ depend on constants $\tilde{\beta}_{8,9}$ which parametrize a unitary ambiguity of these 3N-potentials. In order to be consistent with the underlying NN-potential, we follow ref. [3] and choose the values $\tilde{\beta}_8 = 1/4$ and $\tilde{\beta}_9 = -1/4$. One should note here that the misprint $\tilde{\beta}_9 = 0$ in eq.(4.14) of ref.[7] has been corrected in ref. [5].

5.1 $1\pi$-exchange-contact topology

The $1/M$-correction to the $\pi N$-coupling combined with the 4N-contact vertex ($\sim C_{S,T}$) leads to the 3N-interaction written in eq.(4.6) of ref.[7], and setting $\tilde{\beta}_9 = -1/4$ it reads:

$$V_{3N} = -\frac{g_A^2}{16Mf_\pi^2} \frac{\vec{r}_1 \cdot \vec{r}_2}{m_\pi^2 + q^2} \left\{ C_T \left[ i \vec{s}_1 \cdot (\vec{p}_1 + \vec{p}_1') (\vec{s}_2 \times \vec{s}_3) \cdot \vec{q}_1 + 3\vec{s}_1 \cdot \vec{q}_1 \vec{s}_2 \cdot \vec{q}_3 + 3i\vec{s}_1 \cdot \vec{q}_1 (\vec{s}_2 \times \vec{s}_3) (\vec{p}_2 + \vec{p}_2') \right] + 3C_S \vec{s}_1 \cdot \vec{q}_1 \vec{s}_2 \cdot \vec{q}_3 \right\}.$$  \hspace{1cm} (23)

While self closings vanish (tr $\vec{r}_{1,2} = 0$, tr $\vec{s}_3 = 0$ or $\vec{q}_3 = 0$), the short-range vertex corrections produce a $1\pi$-exchange NN-potential modified by a factor linear in density $\rho$:

$$V_{\text{med}}^{(1)} = \frac{g_A^2 k_f^2}{16\pi^2 Mf_\pi^2} (C_T - C_S) \frac{\vec{r}_1 \cdot \vec{r}_2}{m_\pi^2 + q^2} \vec{s}_1 \cdot \vec{q} \vec{s}_2 \cdot \vec{q}.$$  \hspace{1cm} (24)

Furthermore, one gets from pionic vertex corrections the contribution:

$$V_{\text{med}}^{(2)} = \frac{3g_A^2 C_T}{8\pi^2 Mf_\pi^2} \left\{ 2p^2 \left( \Gamma_3 - \Gamma_0 \right) + \frac{3g_A^2}{4} \left( \Gamma_0 + \Gamma_1 + 4\Gamma_2 \right) \vec{s}_1 \cdot \vec{s}_2 ight. \\
- \left. \frac{3}{2} \left( \Gamma_0 + \Gamma_1 \right) \vec{s}_1 \cdot \vec{q} \vec{s}_2 \cdot \vec{q} + \left( \Gamma_0 - \Gamma_3 \right) \left( \vec{s}_1 \cdot \vec{p} \vec{s}_2 \cdot \vec{p} + \vec{s}_1 \cdot \vec{p}' \vec{s}_2 \cdot \vec{p}' \right) \right\} \\
+ \frac{9g_A^2}{32\pi^2 Mf_\pi^2} \left( \Gamma_0 + \Gamma_1 \right) \left[ \left( C_S - C_T \right) i \left( \vec{s}_1 + \vec{s}_2 \right) \cdot \left( \vec{q} \times \vec{p} \right) - C_S q^2 \right],$$  \hspace{1cm} (25)

and from double exchanges the contribution:

$$V_{\text{med}}^{(3)} = \frac{g_A^2 \vec{r}_1 \cdot \vec{r}_2}{16\pi^2 Mf_\pi^2} \vec{s}_1 \cdot \vec{s}_2 \left\{ C_T \left[ \frac{3g_A^2}{2} \left( \Gamma_0 + \Gamma_1 \right) - p^2 \left( 5\Gamma_0 + 6\Gamma_1 + \Gamma_3 \right) \right] - \left( 2C_T + 3C_S \right) \Gamma_2 \right\} \\
+ \frac{g_A^2 \vec{r}_1 \cdot \vec{r}_2}{32\pi^2 Mf_\pi^2} \left\{ 3 \left( C_S - C_T \right) \left( \Gamma_0 + \Gamma_1 \right) \vec{s}_1 \cdot \vec{q} \vec{s}_2 \cdot \vec{q} \\
+ \left[ C_T \left( 5\Gamma_0 + 6\Gamma_1 + \Gamma_3 \right) - 3C_S \left( \Gamma_0 + 2\Gamma_1 + \Gamma_3 \right) \right] \left( \vec{s}_1 \cdot \vec{p} \vec{s}_2 \cdot \vec{p} + \vec{s}_1 \cdot \vec{p}' \vec{s}_2 \cdot \vec{p}' \right) \right\} \\
+ \frac{3g_A^2 \vec{r}_1 \cdot \vec{r}_2}{16\pi^2 Mf_\pi^2} C_T \left\{ \left( \Gamma_0 + \Gamma_1 \right) \left[ \frac{q^2}{2} - i \left( \vec{s}_1 + \vec{s}_2 \right) \cdot \left( \vec{q} \times \vec{p} \right) \right] - 3\Gamma_2 - p^2 \left( \Gamma_0 + 2\Gamma_1 + \Gamma_3 \right) \right\},$$  \hspace{1cm} (26)

which both include a spin-orbit term $\sim i \left( \vec{s}_1 + \vec{s}_2 \right) \cdot \left( \vec{q} \times \vec{p} \right)$.

The retardation correction to the $1\pi$-exchange-contact diagram leads to the 3N-interaction written in eq.(4.3) of ref.[7], and setting $\tilde{\beta}_8 = 1/4$ it reads:

$$V_{3N} = \frac{g_A^2 \vec{s}_1 \cdot \vec{q}_1 \vec{r}_1 \cdot \vec{r}_2}{16Mf_\pi^2 \left( m_\pi^2 + q_1^2 \right)^2} \left\{ \vec{q}_1 \cdot \vec{q}_3 \left( C_S \vec{s}_2 \cdot \vec{q}_1 + C_T \vec{s}_3 \cdot \vec{q}_1 \right) \\
+ i C_T \left( \vec{s}_2 \times \vec{s}_3 \right) \cdot \vec{q}_1 \left( 3\vec{p}_1 + 3\vec{p}_1' + \vec{p}_2 + \vec{p}_2' \right) \cdot \vec{q}_1 \right\}.$$  \hspace{1cm} (27)
Again, $V_{\text{med}}^{(0)} = 0$ and for the contributions from vertex corrections and double exchanges one finds the results:

$$V_{\text{med}}^{(1)} = \frac{g_A^2 k_f^3}{48\pi^2 M f_\pi^2} (C_S - C_T) q \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{(m_\pi^2 + q^2)^2} \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q},$$

(28)

$$V_{\text{med}}^{(2)} = \frac{3g_A^2 C_S q^2}{32\pi^2 M f_\pi^2} \left[ \Gamma_0 + \Gamma_1 - m_\pi^2 (\gamma_0 + \gamma_1) \right]$$

$$+ \frac{3g_A^2 C_T}{8\pi^2 M f_\pi^2} \left\{ (\gamma_2 + \gamma_4) \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} + \left[ 2\Gamma_2 - \frac{4k_f^3}{3} + 2m_\pi^2 (2\Gamma_0 - m_\pi^2 \gamma_0 - \gamma_2) + \frac{q^2}{4} (\gamma_2 + \gamma_4) + \left( \frac{q^2}{4} - 4p^2 \right) \left( m_\pi^2 \gamma_0 + m_\pi^2 \gamma_1 + \gamma_2 + \gamma_4 - \Gamma_0 - \Gamma_1 \right) \right] \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$$+ \left[ \Gamma_0 + 2\Gamma_1 + \Gamma_3 - m_\pi^2 (\gamma_0 + 2\gamma_1 + \gamma_3) - 4(\gamma_2 + \gamma_4) + \left( \frac{q^2}{8} - 2p^2 \right) (\gamma_0 + 3\gamma_1 + 3\gamma_3 + \gamma_5) \right] \left( \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} \right) \right\},$$

(29)

$$V_{\text{med}}^{(3)} = \frac{g_A^2 \vec{\tau}_1 \cdot \vec{\tau}_2}{16\pi^2 M f_\pi^2} (C_T - C_S) (\gamma_2 + \gamma_4) \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}$$

$$+ \frac{g_A^2 C_S}{32\pi^2 M f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \left\{ 2\Gamma_2 - (2m_\pi^2 + q^2) \gamma_2 - q^2 \gamma_4 \right\} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \left[ \Gamma_0 + 2\Gamma_1 + \Gamma_3 - m_\pi^2 (\gamma_0 + 2\gamma_1 + \gamma_3) - \frac{q^2}{2} (\gamma_0 + 3\gamma_1 + 3\gamma_3 + \gamma_5) \right] \left( \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} \right) \right\}$$

$$+ \frac{g_A^2 C_T}{16\pi^2 M f_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \left\{ \frac{2k_f^3}{3} + m_\pi^2 (2\Gamma_0 - \Gamma_1) + \frac{q^2}{2} \right\} \left[ m_\pi^2 (\gamma_0 + \gamma_1) - \Gamma_1 \right]$$

$$+ \left[ \left( 4p^2 + \frac{q^2}{2} \right) \left( m_\pi^2 (\gamma_0 + \gamma_1) + \gamma_2 + \gamma_4 - \Gamma_0 - \Gamma_1 \right) \right] \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$$+ \left[ \frac{3}{2} \left( m_\pi^2 (\gamma_0 + 2\gamma_1 + \gamma_3) - \Gamma_0 - 2\Gamma_1 - \Gamma_3 \right) + 4(\gamma_2 + \gamma_4) \right] \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} \right\} \right\}.$$  (30)

The new loop functions $\gamma_\nu (p, k_f)$ are defined by Fermi sphere integrals over a squared pion propagator and their analytical expressions (involving arctangents and logarithms) are given in the appendix.

5.2 2$\pi$-exchange topology

The $1/M$-correction to the isovector Weinberg-Tomozawa $\pi\pi$NN-vertex combined with two ordinary $\pi$N-couplings gives rise to the 3N-interaction written in eq.(4.8) of ref. [7]:

$$V_{3N} = - \frac{g_A^2}{16M f_\pi^4} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_3 \cdot \vec{q}_3}{(m_\pi^2 + q_1^2)(m_\pi^2 + q_3^2)} \vec{\tau}_1 \cdot (\vec{\tau}_2 \times \vec{\tau}_3) \left[ \vec{\sigma}_2 \cdot (\vec{q}_1 \times \vec{q}_3) + i \frac{1}{2}(\vec{p}_2 \times \vec{p}_3) \cdot (\vec{q}_3 - \vec{q}_1) \right].$$  (31)

Here, we have multiplied $V_{3N}$ with a factor 2, since it is symmetric under $1 \leftrightarrow 3$ to account already for this permutation. Alternatively, one could distinguish the equivalent couplings to nucleon 1 and
nucleon 3 and the factor 2 would emerge at the end of the calculation. Obviously, the self closings vanish \((\operatorname{tr} \bar{\tau}_{1,2,3} = 0)\), while one gets the following contributions from pionic vertex corrections:

\[
V_{\text{med}}^{(1)} = \frac{g_A^2}{32\pi^2 M f_\pi} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{m_\pi^2 + q^2} \left[ 13 \Gamma_2 - m_\pi^2 \Gamma_1 + 3p^2(\Gamma_0 + 2\Gamma_1 + \Gamma_3) \right] \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q},
\]

and double exchanges:

\[
V_{\text{med}}^{(3)} = \frac{g_A^2}{16\pi^2 M f_\pi} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{(m_\pi^2 + q_1^2)(m_\pi^2 + q_3^2)} \left\{ \left( 16 \Gamma_2 + 2p^2(\Gamma_1 + \Gamma_3) + (2m_\pi^2 + q^2)(p^2(G_0 - G_{0*})) \right) + \left[ \left( \frac{m_\pi^2 + q^2}{2} - p^2 \right) G_0 + \left( \frac{m_\pi^2 + q^2}{2} + 4p^2 \right) G_1 - G_2 \right] \right\}. \]

The loop functions \(G_\nu(p, q, k_f)\) are defined by Fermi sphere integrals over the product of two different pion propagators and given in the appendix.

The retardation correction to the above 2\(\pi\)-exchange mechanism leads to the 3N-interaction written in eq.(4.2) of ref.\[7\]:

\[
V_{3N} = \frac{ig_A^2}{32M f_\pi^2} \frac{\bar{\sigma}_1 \cdot \bar{q}_1 \bar{\sigma}_3 \cdot \bar{q}_3}{(m_\pi^2 + q_1^2)(m_\pi^2 + q_3^2)} \bar{\tau}_1 \cdot (\bar{\tau}_3 \times \bar{\tau}_3) \left[ \bar{q}_3 \cdot (\bar{p}_3 + \bar{p}_3') - \bar{q}_1 \cdot (\bar{p}_1 + \bar{p}_1') \right],
\]

where we have again multiplied \(V_{3N}\) with a factor 2, due to its symmetry under \(1 \leftrightarrow 3\). One finds the following contributions from pionic vertex corrections:

\[
V_{\text{med}}^{(1)} = \frac{g_A^2}{32\pi^2 M f_\pi} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{m_\pi^2 + q^2} \left[ m_\pi^2 \Gamma_1 - \Gamma_2 + p^2(\Gamma_0 + 2\Gamma_1 + \Gamma_3) \right] \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q},
\]

and double exchanges:

\[
V_{\text{med}}^{(3)} = \frac{g_A^2}{32\pi^2 M f_\pi} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{m_\pi^2 + q^2} \left\{ 6\Gamma_2 + 2p^2(\Gamma_1 + \Gamma_3) + (2m_\pi^2 + q^2)(p^2(G_0 - G_{0*})) \right\}. \]

The star in the index of a loop function \(G_{\nu*}\) symbolizes an extra factor \(\bar{l}\) in the respective Fermi sphere integral \((2\pi)^{-1} \int d^3 l \theta(k_f - l)\).

The \(1/M\)-correction to the \(\pi\)N-coupling leads via the mechanism of two consecutive pion-exchanges to the 3N-interaction written in eq.(4.5) of ref.\[7\], and setting \(\beta_0 = -1/4\) it reads:

\[
V_{3N} = \frac{g_A^4}{64M f_\pi^2} \frac{\bar{\sigma}_1 \cdot \bar{q}_1}{(m_\pi^2 + q_1^2)(m_\pi^2 + q_3^2)} \left\{ \bar{\tau}_1 \cdot \bar{\tau}_3 \left[ 3\bar{\sigma}_3 \cdot \bar{q}_3 (i\bar{\sigma}_2 \cdot (\bar{q}_1 \times (\bar{p}_2 + \bar{p}_2')) + q_1^2) \right] + i\bar{\sigma}_3 \cdot (\bar{p}_3 + \bar{p}_3') \bar{\sigma}_2 \cdot (\bar{q}_1 \times \bar{q}_3) \right\}
\]

The contribution from self closings

\[
V_{\text{med}}^{(0)} = -\frac{g_A^4}{16\pi^2 M f_\pi^4} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{(m_\pi^2 + q^2)^2} \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q},
\]
comes exclusively from the $q^2_1$-term in the first line of eq.(37). Moreover, one obtains the following contributions from pionic vertex corrections:

$$V_{\text{med}}^{(1)} = \frac{g_A^4}{128\pi^2 M f^4 \pi} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{m^2_\pi + q^2} \left\{ p^2 (11\Gamma_0 + 6\Gamma_1 - 5\Gamma_3) - q^2 (2\Gamma_0 + 5\Gamma_1 + 3\Gamma_3) + 3m^2_\pi \Gamma_1 - 41\Gamma_2 \right\} \left\{ \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} - q^2 (\Gamma_0 + 2\Gamma_1 + \Gamma_3)(\vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p'} \vec{\sigma}_2 \cdot \vec{p'}) \right\}, (39)$$

and double exchanges:

$$V_{\text{med}}^{(3)} = \frac{3g_A^4}{128\pi^2 M f^4 \pi} \left\{ 4k_3^2 - 3q^2 (\Gamma_0 + \Gamma_1) + 3m^2_\pi [(2m^2_\pi + q^2)G_0 - 4\Gamma_0] + \left[ 2G_2 - \Gamma_0 + 2\Gamma_1 + (m^2_\pi + 2p^2 + q^2)G_0 + 2(4p^2 - q^2)(G_1 + G_3) - 4m^2_\pi G_1 \right] i(\vec{\sigma}_1 + \vec{\sigma}_2)(\vec{q} \times \vec{p}) + 4(G_0 + 2G_1) \vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}) \right\} + \frac{g_A^4}{64\pi^2 M f^4 \pi} \left\{ (8p^2 - 3q^2)(\Gamma_0 + \Gamma_1) + \frac{4}{3}k_3^2 - 4m^2_\pi \Gamma_0 + (2m^2_\pi + q^2)[(q^2 - 4p^2)(G_0 + 2G_1) + m^2_\pi G_0] + 6G_2 - \Gamma_0 + (m^2_\pi + 6p^2 - q^2)G_0 + 6(4p^2 - q^2)(G_1 + G_3) \right\} i(\vec{\sigma}_1 + \vec{\sigma}_2)(\vec{q} \times \vec{p}). (40)$$

Note that the isoscalar part of $V_{\text{med}}^{(3)}$ includes a quadratic spin-orbit term $\sim \vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p})$. For the choice $\vec{\beta}_8 = 0$, this term would be absent and furthermore $V_{\text{med}}^{(3)}$ would carry the overall isospin factor $3 + 2\vec{\tau}_1 \cdot \vec{\tau}_2$.

Finally, there are the retardation corrections to the (consecutive) $2\pi$-exchange. These generate the $3\text{N}$-interaction written in eq.(4.1) of ref. [7], and setting $\vec{\beta}_8 = 1/4$ it reads:

$$V_{3\text{N}} = \frac{g_A^4}{64 M f^4 \pi} \frac{\vec{\sigma}_1 \cdot \vec{q_1} \vec{\sigma}_3 \cdot \vec{q_3}}{(m^2_\pi + q^2)^2} \left\{ - \vec{q_1} \cdot \vec{q_3} [\vec{\tau}_1 \cdot \vec{\tau}_3 \vec{q_1} \cdot \vec{q_3} + \vec{\tau}_1 \cdot (\vec{\tau}_2 \times \vec{\tau}_3) \vec{\sigma}_2 \cdot (\vec{q_1} \times \vec{q_3})] + i [\vec{\tau}_1 \cdot \vec{\tau}_3 \vec{\sigma}_2 \cdot (\vec{q_1} \times \vec{q_3}) - \vec{\tau}_1 \cdot (\vec{\tau}_2 \times \vec{\tau}_3) \vec{q_1} \cdot \vec{q_3}] \vec{q_1} \cdot (3\vec{p_1}' + 3\vec{p_1} + \vec{p}_2 + \vec{p}_2')] \right\}. (41)$$

The contribution from self closings:

$$V_{\text{med}}^{(0)} = \frac{g_A^4 k_3^2}{48\pi^2 M f^4 \pi} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{(m^2_\pi + q^2)^3} \vec{q} \vec{\sigma}_2 \cdot \vec{q}, (42)$$

stems from the first term $\sim (\vec{q_1} \cdot \vec{q_3})^2$ in eq.(41). The full result from the pionic vertex corrections takes the form:

$$V_{\text{med}}^{(1)} = \frac{g_A^4}{64\pi^2 M f^4 \pi} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{m^2_\pi + q^2} \left\{ \frac{q^2}{m^2_\pi + q^2} \left[ 3(\Gamma_2 + \Gamma_4) - \frac{k_3^2}{3} + \frac{q^2}{4}(\Gamma_0 + 3\Gamma_1 + 3\Gamma_3 + \Gamma_5) \right] + \frac{8k_3^2}{3} + 4\Gamma_2 + 4m^2_\pi (\gamma_0 - \gamma_2 - 2\Gamma_0) + q^2 \left[ \Gamma_3 + \frac{\Gamma_1 - \Gamma_0}{2} - 2\gamma_2 - 2\gamma_4 + \frac{m^2_\pi}{2} (\gamma_0 - \gamma_1 - 2\gamma_3) \right] + (8p^2 - q^2) \left[ m^2_\pi (\gamma_0 + \gamma_1) - \Gamma_0 - \Gamma_1 - \gamma_2 - \gamma_4 - \frac{q^2}{4} (\gamma_0 + 3\gamma_1 + 3\gamma_3 + \gamma_5) \right] \right\}. (43)$$

For reasons of clarity, we split the contribution $V_{\text{med}}^{(3)}$ from double exchanges into two pieces. Eval-
and the terms in second line of eq.(41) yield the additional contribution:

\[
V_{\text{med}}^{(3)} = \frac{g_4^4}{128\pi^2 M f_\pi^4} \left\{ \begin{array}{l}
\bar{q}^2 \left[ q^2 (3\Gamma_1 - 2\gamma_2 - q^2 \gamma_3 + 10\gamma_4 + 2p^2\gamma_5) - \frac{8K_0^3}{3} - 8(3\Gamma_2 + p^2\Gamma_3) \\
+ (3m_\pi^2 + 3q^2 + 2p^2)(4\Gamma_0 - q^2\gamma_1) - (2m_\pi^2 + q^2)(3m_\pi^2 + 3q^2 + 4p^2)(\gamma_0 + G_0) \\
+ 4(2m_\pi^2 + q^2)(3\gamma_2 + p^2\gamma_3 + G_{0*}) + (2m_\pi^2 + q^2)^2 \left( (2m_\pi^2 + q^2 + 4p^2) \frac{K_0}{4} - K_{0*} \right) \end{array} \right. \\
\times \left[ (2m_\pi^2 + q^2 + 4p^2) \frac{K_0}{4} - K_{0*} \right] \left[ \bar{q}^2 \bar{q}^2 - 2K_{0*} - \gamma_2 - G_2 \right] (q^2 \bar{q}^2 \bar{q}^2 - \bar{q}^2 \bar{q}^2 \bar{q}^2) \\
+ 3 \left[ (2m_\pi^2 + q^2 + 4p^2) \frac{K_0}{2} + 2K_1 + 2K_3 \right] - \gamma_0 - 2\gamma_1 - \gamma_3 \\
- G_0 - 4G_1 - 4G_3 - 2(K_{0*} + 4K_{1*} + 4K_{3*}) \right\}.
\]

The new loop functions \( K_{\nu}(p, q, k_f) \) are defined and given in explicit form in the appendix.

### 5.3 1/M-correction to medium insertion

One also needs to consider relativistic corrections to the in-medium N-propagator (or particle-hole propagator). Taking into account the (off-shell) kinetic energy \( \vec{l}^2/2M \), the medium-insertion reads:

\[
-2\pi \delta \left( l_0 - \frac{\vec{l}^2}{2M} \right) \theta(k_f - |\vec{l}|) = 2\pi \left[ -\delta(l_0) + \frac{\vec{l}^2}{2M} \delta'(l_0) + \ldots \right] \theta(k_f - |\vec{l}|),
\]

where we have expanded in \( 1/M \). The \( l_0 \)-integral is evaluated by the formula \( \int dl_0 \delta'(l_0)\mathcal{I}(l_0) = -\mathcal{I}'(0) \). When neglecting consistently the external kinetic energies \( p_0 = p'_0 = p^2/2M \), the pion propagators as well as the isoscalar \( c_{2,3} \)-vertices \([6]\) give rise to an integrand \( \mathcal{I}(l_0) \) that depends quadratically on \( l_0 \), thus \( \mathcal{I}'(0) = 0 \). A linear dependence on \( l_0 \) is provided at leading order by the isovector Weinberg-Tomozawa \( \pi\pi\text{NN}-\text{vertex} \). When combined with two ordinary \( \pi\pi \)-couplings, the
pertinent diagram resembling a $2\pi$-exchange NN-interaction generates (after closing one nucleon line to a loop) the following contributions to the in-medium NN-potential $V_{\text{med}}$:

$$V_{\text{med}}^{(1)} = -\frac{g_\pi^2}{32\pi^2 M f_\pi^4} \frac{\bar{\tau}_1 \cdot \bar{\tau}_2}{m_n^2 + q^2} \left[3\Gamma_2 + 5\Gamma_4 + p^2(\Gamma_3 + \Gamma_5)\right] \bar{s}_1 \cdot \bar{q} \bar{s}_2 \cdot \bar{q},$$

$$V_{\text{med}}^{(3)} = \frac{g_\pi^2 \bar{\tau}_1 \cdot \bar{\tau}_2}{32\pi^2 M f_\pi^4} \left\{6\Gamma_2 + 2p^2\Gamma_3 - (2m_n^2 + q^2)G_{0*} + (G_{0*} + 2G_{1*}) i(\bar{s}_1 + \bar{s}_2) \cdot (\bar{q} \times \bar{p})\right\}.$$  \hspace{1cm} (47)

(48)

Note that these contributions $V_{\text{med}}^{(1,3)}$ differ from those in eqs. (32,33), which had their origin in the $1/M$-correction to the Weinberg-Tomozawa vertex.

\section{Subleading three-nucleon contact potential}

The subleading three-nucleon contact potential (appearing at $N^4$LO in the chiral counting) has been analyzed in ref. \cite{9}. Taking into account Poincare symmetry and Fierz constraints the initial list of 116 operators could be reduced to 10 independent ones. The resulting 3N-contact potential $V_{3N}$, written in eq. (15) of ref. \cite{9}, depends quadratically on the nucleon momenta and it involves ten parameters, called $E_1, \ldots, E_{10}$. Working out for each term $\sim E_i$ the closing of one nucleon line to a loop, one obtains from the subleading 3N-contact potential the following in-medium NN-interaction:

$$\frac{V_{\text{med}}}{\rho} = E_1 \left(\frac{6}{5}k_f^2 + 2p^2 - 3q^2\right) + E_2 \left[(\bar{s}_1 \cdot \bar{s}_2 + 3) \left(\frac{3}{5}k_f^2 + p^2\right) - \bar{s}_1 \cdot \bar{s}_2 q^2\right]$$

$$+ E_3 \left[(\bar{s}_1 \cdot \bar{s}_2 + 3) \left(\frac{2}{5}k_f^2 + p^2\right) - \bar{s}_1 \cdot \bar{s}_2 q^2\right] + E_4 \left[(\bar{s}_1 \cdot \bar{s}_2 \bar{s}_1 \cdot \bar{s}_2 + 9) \left(\frac{3}{5}k_f^2 + p^2\right) - \bar{s}_1 \cdot \bar{s}_2 \bar{s}_1 \cdot \bar{s}_2 q^2\right]$$

$$+ 3E_5 \left[\bar{s}_1 \cdot \bar{p} \bar{s}_2 \cdot \bar{p} + \bar{s}_1 \cdot \bar{p}' \bar{s}_2 \cdot \bar{p}' + \frac{2}{5}k_f^2(\bar{s}_1 \cdot \bar{s}_2 - 1) - \frac{2}{3}p^2 + q^2 - \bar{s}_1 \cdot \bar{q} \bar{s}_2 \cdot \bar{q}\right]$$

$$+ 3E_6 \left[(\bar{s}_1 \cdot \bar{s}_2 + 3) \left[\frac{1}{2}(\bar{s}_1 \cdot \bar{p} \bar{s}_2 \cdot \bar{p} + \bar{s}_1 \cdot \bar{p}' \bar{s}_2 \cdot \bar{p}') - \frac{p^2}{3} + \frac{k_f^2}{5}(\bar{s}_1 \cdot \bar{s}_2 - 1) + \bar{s}_1 \cdot \bar{s}_2 \left(\frac{q^2}{3} - \bar{s}_1 \cdot \bar{q} \bar{s}_2 \cdot \bar{q}\right)\right]$$

$$+(9E_8 - 7E_7) \frac{i}{2} (\bar{s}_1 + \bar{s}_2) \cdot (\bar{q} \times \bar{p}) + E_9 \left[\bar{s}_1 \cdot \bar{q} \bar{s}_2 \cdot \bar{q} + \frac{q^2}{2} - p^2 - \frac{3}{5}k_f^2 - \frac{i}{2}(\bar{s}_1 + \bar{s}_2) \cdot (\bar{q} \times \bar{p})\right]$$

$$+ E_{10} \left[\bar{s}_1 \cdot \bar{s}_2 \bar{s}_1 \cdot \bar{q} \bar{s}_2 \cdot \bar{q} + \frac{3}{2}q^2 - 3p^2 - \frac{9}{5}k_f^2 - \frac{3i}{2}(\bar{s}_1 + \bar{s}_2) \cdot (\bar{q} \times \bar{p})\right],$$

which depends linearly on the density $\rho = 2k_f^3/3\pi^2$ and quadratically on the momenta $\bar{p}, \bar{q}, k_f$. The above expression for $V_{\text{med}}$ contributes only to s- and p-wave matrix elements and to $^3S_1-^3D_1$ mixing.

\section*{Appendix: Loop-functions}

In this appendix we specify all the loop functions which have been used in section 5 to express the contributions to the in-medium NN-interaction $V_{\text{med}}$.

The functions $\Gamma_\nu(p,k_f)$ with $\nu = 0, \ldots, 5$ arise from Fermi sphere integrals $(2\pi)^{-1} \int d^3l \theta(k_f - |\vec{l}|)$ over a pion propagator $[m_\pi^2 + (\vec{l} + \vec{p})^2]^{-1}$, supplemented by tensorial factors $1 (\nu = 0), l_i (\nu = 1), l_ii_j (\nu = 2, 3)$ or $l_ii_jjk (\nu = 4, 5)$. Their respective analytical expressions read:

$$\Gamma_0 = k_f - m_\pi \left[\arctan \frac{k_f + p}{m_\pi} + \arctan \frac{k_f - p}{m_\pi}\right] + \frac{m_\pi^2 + k_f^2 - p^2}{4p} \ln \frac{m_\pi^2 + (k_f + p)^2}{m_\pi^2 + (k_f - p)^2}.$$  \hspace{1cm} (50)
\[
\Gamma_1 = \frac{k_f}{4p^2}(m^2_\pi + k^2_f + p^2) - \Gamma_0 - \frac{1}{16p^3} [m^2_\pi + (k_f + p)^2] [m^2_\pi + (k_f - p)^2] \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (51)
\]

\[
\Gamma_2 = \frac{k^3_f}{9} + \frac{1}{6}(k^2_f - m^2_\pi - p^2) \Gamma_0 + \frac{1}{6}(m^2_\pi + k^2_f - p^2) \Gamma_1, \quad (52)
\]

\[
\Gamma_3 = \frac{k^3_f}{3p^2} - \frac{m^2_\pi + k^2_f + p^2}{2p^2} \Gamma_0 - \frac{m^2_\pi + k^2_f + 3p^2}{2p^2} \Gamma_1, \quad (53)
\]

\[
\Gamma_4 = \frac{m^2_\pi}{3} \Gamma_0 + \frac{k_f}{64} \left[ \frac{5p^2}{3} - 3m^2_\pi - \frac{31k^2_f}{9} + \frac{1}{3p^2}(3k^4_f - 14k^2_f m^2_\pi - 17m^4_\pi) - \frac{(k^2_f + m^2_\pi)^3}{p^2} \right]
\]
\[+ \frac{1}{768p^5} [m^2_\pi + (k_f + p)^2] [m^2_\pi + (k_f - p)^2] \left[ 3(k^2_f + m^2_\pi)^2 + 2p^2(k^2_f + 7m^2_\pi) - 5p^4 \right]
\times \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (54)
\]

\[
\Gamma_5 = -\Gamma_0 + \frac{k_f}{64} \left[ 29 + \frac{5}{p^6}(k^2_f + m^2_\pi)^3 + \frac{25k^2_f + 141m^2_\pi}{3p^2} + \frac{1}{3p^4}(17k^4_f + 86k^2_f m^2_\pi + 69m^4_\pi) \right]
\]
\[\frac{1}{256p^5} [m^2_\pi + (k_f + p)^2] [m^2_\pi + (k_f - p)^2] \left[ 5(k^2_f + m^2_\pi)^2 + 2p^2(7k^2_f + 9m^2_\pi) + 29p^4 \right]
\times \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (55)
\]

The functions \(\gamma_\nu(p, k_f)\) with \(\nu = 0, \ldots, 5\) arise from Fermi sphere integrals over a squared pion propagator \([m^2_\pi + (\vec{p} + \vec{l})^2]^{-2}\), supplemented by tensorial factors \(1 (\nu = 0), l_i (\nu = 1), l_i l_j (\nu = 2, 3)\) or \(l_i l_j l_k (\nu = 4, 5)\). They satisfy the relation \(\gamma_\nu = -\partial \Gamma_\nu / \partial m^2_\pi\) and their analytical expressions read:

\[
\gamma_0 = \frac{1}{2m_\pi} \left[ \arctan \frac{k_f + p}{m_\pi} + \arctan \frac{k_f - p}{m_\pi} \right] - \frac{1}{4p} \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (56)
\]

\[
\gamma_1 = -\gamma_0 - \frac{k_f}{2p^2} + \frac{p^2 + k^2_f + m^2_\pi}{8p^3} \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (57)
\]

\[
\gamma_2 = \frac{k_f}{8p^2}(3p^2 - k^2_f - m^2_\pi) - m^2_\pi \gamma_0 + \frac{1}{32p^6} \left[ (p^2 + k^2_f + m^2_\pi)^2 - 4p^2 (p^2 + m^2_\pi) \right] \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (58)
\]

\[
\gamma_3 = \gamma_0 + \frac{k_f}{8p^4} \left[ 7p^2 + 3k^2_f + 3m^2_\pi \right] - \frac{1}{32p^5} \left[ 3(k^2_f + m^2_\pi)^2 + 2p^2(3k^2_f + 5m^2_\pi) + 7p^4 \right] \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (59)
\]

\[
\gamma_4 = m^2_\pi \gamma_0 + \frac{k_f}{16p^4} \left[ (k^2_f + m^2_\pi)^2 + \frac{4p^2}{3}(k^2_f + 3m^2_\pi) - 5p^4 \right]
\]
\[+ \frac{p^2 - k^2_f - m^2_\pi}{64p^5} \left[ 5p^4 + 2p^2(3k^2_f + 3m^2_\pi) + (k^2_f + m^2_\pi)^2 \right] \ln \frac{m^2_\pi + (k_f + p)^2}{m^2_\pi + (k_f - p)^2}, \quad (60)
\]
\[ \gamma_5 = -\gamma_0 - \frac{k_f}{p^2} \left[ \frac{5}{16p^4}(k_f^2 + m_\pi^2)^2 + \frac{19}{16} + \frac{2k_f^2 + 3m_\pi^2}{3p^2} \right] + \frac{1}{64p^4} \left[ 5p^4(3k_f^2 + 7m_\pi^2) + 19p^6 + 3p^2(k_f^2 + m_\pi^2)(3k_f^2 + 7m_\pi^2) + 5(k_f^2 + m_\pi^2)^3 \right] \ln \frac{m_\pi^2 + (k_f + p)^2}{m_\pi^2 + (k_f - p)^2}. \] (61)

The functions \( G_\nu(p, q, k_f) \) with \( \nu = 0, \ldots, 3 \) arise from Fermi sphere integrals over two different pion propagators \([m_\pi^2 + (l+p)^2]^{-1}[m_\pi^2 + (l+p')^2]^{-1}\), multiplied by tensorial factors \(1(\nu = 0), l_i(\nu = 1)\) or \(l_i l_j(\nu = 2, 3)\). In order to construct this set of functions one starts with the one-parameter (radial) integrals:

\[ G_{0,0,**} = \frac{2}{q} \int_0^{k_f} dl \frac{\{l, l^3, l^5\}}{B + q^2l^2} \ln \frac{ql + \sqrt{B + q^2l^2}}{\sqrt{B}}, \] (62)

with the abbreviation \( B = [m_\pi^2 + (l + p)^2][m_\pi^2 + (l - p)^2] \) and solves systems of linear equations:

\[ G_1 = \frac{1}{4p^2 - q^2} \left[ \Gamma_0 - (m_\pi^2 + p^2)G_0 - G_{0*} \right], \] (63)

\[ G_{1*} = \frac{1}{4p^2 - q^2} \left[ 3\Gamma_2 + p^2\Gamma_3 - (m_\pi^2 + p^2)G_{0*} - G_{**} \right], \] (64)

\[ G_2 = (m_\pi^2 + p^2)G_1 + G_{0*} + G_{1*}, \] (65)

\[ G_3 = \frac{1}{4p^2 - q^2} \left[ \frac{\Gamma_1}{2} - 2(m_\pi^2 + p^2)G_1 - G_{0*} - 2G_{1*} \right]. \] (66)

The functions \( K_{\nu,**}(p, q, k_f) \) with \( \nu = 0, \ldots, 3 \) arise from Fermi sphere integrals over the symmetrized combination \([m_\pi^2 + (l + p)^2]^{-2}[m_\pi^2 + (l + p')^2]^{-1} + [m_\pi^2 + (l + p)^2]^{-1}[m_\pi^2 + (l + p')^2]^{-2}\) of pion propagators, supplemented by tensorial factors \(1(\nu = 0), l_i(\nu = 1)\) or \(l_i l_j(\nu = 2, 3)\). By definition the relation \( K_\nu = -\partial G_\nu/\partial m_\pi^2 \) holds. Again, one starts with four functions represented by one-parameter integrals:

\[ K_{0,0,**,**} = 2 \int_0^{k_f} dl \{l, l^3, l^5, l^7\} \frac{m_\pi^2 + l^2}{B + q^2l^2} \left[ \frac{l}{B} + \frac{1}{q\sqrt{B + q^2l^2}} \right] \ln \frac{ql + \sqrt{B + q^2l^2}}{\sqrt{B}}, \] (67)

and solves linear equations for the others:

\[ K_1 = \frac{1}{4p^2 - q^2} \left[ \gamma_0 + G_0 - (m_\pi^2 + p^2)K_0 - K_{0*} \right], \] (68)

\[ K_{1*} = \frac{1}{4p^2 - q^2} \left[ 3\gamma_2 + p^2\gamma_3 + G_{0*} - (m_\pi^2 + p^2)K_{0*} - K_{**} \right], \] (69)

\[ K_2 = (m_\pi^2 + p^2)K_1 - G_1 + K_{0*} + K_{1*}, \] (70)

\[ K_3 = \frac{1}{4p^2 - q^2} \left[ \frac{\gamma_1}{2} - 2(m_\pi^2 + p^2)K_1 + 2G_1 - K_{0*} - 2K_{1*} \right], \] (71)

\[ K_{1**} = \frac{1}{4p^2 - q^2} \left[ G_{**} - (m_\pi^2 + p^2)K_{**} - K_{***} + \gamma_{**} \right], \] (72)

\[ K_{2*} = (m_\pi^2 + p^2)K_{1*} - G_{1*} + K_{**} + K_{1**}, \] (73)

\[ K_{3*} = \frac{1}{4p^2 - q^2} \left[ \frac{1}{2}(5\gamma_4 + p^2\gamma_5) - 2(m_\pi^2 + p^2)K_{1*} + 2G_{1*} - K_{**} - 2K_{1**} \right]. \] (74)

We remind that a * indicates an additional power of \( l^2 \) in the integrand. The new auxiliary function \( \gamma_{**} \) in eq.(72) is given by:

\[ \gamma_{**} = (p^4 + 5m_\pi^4 - 10p^2m_\pi^2)\gamma_0 + 4k_f \left( p^2 + \frac{k_f^2}{6} - m_\pi^2 \right) + \frac{m_\pi^4 - p^4}{p} \ln \frac{m_\pi^2 + (k_f + p)^2}{m_\pi^2 + (k_f - p)^2}. \] (75)
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