Inflationary non-Gaussianity from thermal fluctuations

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Abstract. We calculate the contribution of fluctuations with a thermal origin to the inflationary non-Gaussianity. We find that even a small component of radiation can lead to a large non-Gaussianity. We show that this thermal non-Gaussianity always has positive $f_{NL}$. We illustrate our result in the chain inflation model and the very weakly dissipative warm inflation model. We show that $f_{NL} \sim O(1)$ is general in such models. If we allow a modified equation of state, or some decoupling effects, a large thermal non-Gaussianity of order $f_{NL} > 5$ or even $f_{NL} \sim 100$ can be produced. We also show that the power spectrum of chain inflation should have a thermal origin. In the appendix, we give a clarification on the different conventions used in the literature related to the calculation of $f_{NL}$.

Keywords: cosmological perturbation theory, inflation

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1. Introduction

Inflation has been remarkably successful in solving problems in the standard hot big bang cosmology [1]–[4]. Furthermore inflation shows us that fluctuations of a quantum origin were generated and frozen in to seed the wrinkles in the cosmic microwave background (CMB) [5,6] and today’s large scale structure [7]–[11]. Its prediction of a scale-invariant spectrum which has been confirmed in the experiments over the past decade is remarkable and has been taken to be a great success of the theory.

Since the idea of inflation was proposed, there have been a large number of inflation models. It has become one of the key problems in cosmology to extract more information from experiments in order to distinguish between these inflation models. The key quantities from experiments include the power spectrum of scalar and tensor perturbations, the scalar spectral index and its running, and the non-Gaussianity.

The amount of non-Gaussianity is often estimated using the quantity $f_{\text{NL}}$, which can be written as

$$\zeta = \zeta_g + \frac{3}{5} f_{\text{NL}} \left( \zeta_g^2 - \langle \zeta_g^2 \rangle \right),$$

(1)

where the subscript $g$ denotes the Gaussian part of $\zeta$.

It has been shown that in the simplest single-field slow roll inflation models, the non-Gaussianity estimator $f_{\text{NL}} \sim O(\epsilon, \eta')$ [13,14], where $\epsilon$ and $\eta$ are the slow roll parameters. Such a small non-Gaussianity is not only much smaller than the current observational bound $|f_{\text{NL}}| < 100$, but also well below the sensitivity of the Planck satellite, $f_{\text{NL}} \sim 5$. However, the recent study of inflation models with significantly nonlinear dynamics shows that the non-Gaussianity in them could be large. For example, in the DBI [15], K-inflation [16] and ghost inflation [17] models, $f_{\text{NL}}$ can reach $O(1)$ or larger than $O(1)$ in some parameter regions. Such a large non-Gaussianity is hoped to be observed in future experiments and to shed light on the physics behind these models.

Footnote: Note that, for the definition of $f_{\text{NL}}$, there is a sign difference between the notation of the WMAP group [12] and Maldacena’s calculation [13]. We use the same notation as that of the WMAP group here. The difference is discussed in detail in the appendix.
Recently, Yadav and Wandelt have claimed that, from the WMAP three-year data, $f_{NL}$ is detected at above 99.5% confidence level [18]. They show that, at 95% confidence level, the local shape $f_{NL}$ is in the region

$$26.91 < f_{NL} < 146.71.$$  

(2)

If this result is confirmed by the WMAP five-year data and Planck, a large number of inflation models (without extra mechanisms) will be ruled out, and there is also hope of measuring the shape of $f_{NL}$ and the tri-spectrum $\tau_{NL}$ by Planck.

In this paper, we propose another mechanism to produce potentially large non-Gaussianity. Instead of producing non-Gaussianity from the nonlinear evolution of inflaton, in our mechanism, the large non-Gaussianity stems from the correlation in the initial conditions. We will show that, if the initial condition of the perturbations is prepared in part by thermal fluctuations, there can be strong three-point correlation, inducing large non-Gaussianity.

In many inflation models, the radiation component only takes a very small part in the energy density. But since the non-Gaussianity from the coherent motion of inflaton is highly suppressed, the thermal non-Gaussianity can play a significant part in $f_{NL}$, and in some parameter regions it provides the dominant contribution. This may open a window for us to study the thermal fluctuations in the models.

Indeed, the thermal effects are significantly important in some inflation models. One example is chain inflation. Based on the rapid tunnelling mechanism for the metastable vacua in the string landscape [19, 20], Freese, Spolyar and Liu [21] proposed the so-called chain inflation model in which the metastable vacua during inflation tunnel very rapidly. The density perturbation in chain inflation is calculated by Feldstein and Tweedie in [22] and a simplified version of chain inflation was proposed by Huang in [23].

In the chain inflation models, the average lifetime for a metastable vacuum is much smaller than the Hubble time, so that the vacuum decay via bubble nucleation takes place very rapidly, and there can be many bubbles nucleated within one inflationary horizon. These bubbles eventually collide and the energy stored in the bubble wall decays into radiation. This is very different from the slow roll inflation models in which the decrease in the inflaton energy density is wasted by the cosmic fraction, with very little radiation being left.

Another example with large thermal effect is the warm inflation by Berera and Fang [24] (see also [25] for an introduction and the current status). In the warm inflation model, a fraction of inflaton energy decays into radiation continuously during inflation. The decay from inflaton to radiation can be achieved by an interaction term in the inflaton’s Lagrangian. It is shown in warm inflation that, due to the continuous creation of radiation, the temperature during inflation can be nearly constant [24], so it provides a area for investigating the thermal effects. Previously, the non-Gaussianity of the warm inflation model was studied in [26, 27].

In [27], it is shown that very large non-Gaussianity $f_{NL} \gg 1$ can arise from the nonlinear coupling between the radiation and inflaton on sub-horizon scales. This large non-Gaussianity may be observed by future experiments. However, the starting point of both [26] and [27] is that the thermal fluctuation is exactly Gaussian, which provides a white noise term for the Langevin equation. In this paper, we will consider the non-Gaussianity from thermal fluctuations. So this consideration is complementary to [27].
Moreover, in [27], the strong dissipative regime is considered, while in our paper, we only consider the very weakly dissipative regime.

In such inflation models with thermal radiation, it can be shown that the non-Gaussianity estimator $f_{NL}$ is no longer suppressed by the slow roll parameters. Even when the radiation component is so tiny that it does not qualitatively change the inflationary background, considerable non-Gaussianity $f_{NL} \sim \mathcal{O}(1)$ can be produced.

Furthermore, we suppose, in some cases, a new scale related to the acoustic horizon, or some decoupling scales enters the calculation. In this case, very large non-Gaussianity of the order of $f_{NL} > 5$ or even $f_{NL} \sim 100$ can be produced without fine-tuning.

This paper is organized as follows. In section 2, we develop the general method to calculate the non-Gaussianity of thermal origin. We calculate the two-point and three-point correlation functions of thermal fluctuations. Based on these, we derive the power spectrum and the non-Gaussianity estimator $f_{NL}$. In section 3, we calculate the amount of non-Gaussianity explicitly in the chain inflation model and the thermal inflation model. We conclude in section 4.

2. The thermal correlation functions and non-Gaussianity

In this section, we calculate the correlation functions, power spectrum and non-Gaussianity of thermal fluctuations. We also give a simple estimate of the non-Gaussianity by calculating the back-reaction.

We suppose the energy density takes the form

$$\rho = \rho_0 + \rho_t = \rho_0 + AT^m,$$

where $\rho_0$ is the energy density without thermal origin, for example the effective vacuum energy provided by the inflaton potential. $\rho_t$ is the energy density for the radiation and $A$ is a constant with dimension $[\text{mass}]^{4-m}$. Note that $m = 4$ for usual radiation. While for generality, phenomenologically, we still keep $m$ here.

The correlation functions in thermal equilibrium can be calculated from the partition function of the system:

$$Z = \sum_r e^{-\beta E_r},$$

where $\beta = T^{-1}$.

Let $U \equiv \rho V$ represent the total energy inside a volume $V$. Then the average energy of the system is given by

$$\langle U \rangle = -\frac{d \log Z}{d \beta}.$$

The two-point correlation function for the fluctuations $\delta \rho \equiv \rho - \langle \rho \rangle$ is given by

$$\langle \delta \rho^2 \rangle = \frac{\langle \delta U^2 \rangle}{V^2} = \frac{1}{V^2} \frac{d^2 \log Z}{d \beta^2} = \frac{1}{V^2} \frac{d \langle U \rangle}{d \beta} = \frac{mAT^{m+1}}{V},$$

where in the final equality we have neglected $\langle \rangle$ because the difference is next to the leading order.
Similarly, the three-point correlation function can be expressed as

$$\langle \delta \rho^3 \rangle = \frac{\langle \delta U^3 \rangle}{V^3} = \frac{1}{V^3} \frac{d^3 \log Z}{d \beta^3} = \frac{1}{V^3} \frac{d^2 \langle U \rangle}{d \beta^2} = \frac{m(m+1)AT^{m+2}}{V^2}. \quad (7)$$

Now let us apply the above calculation to inflation. First, we calculate the equation of state $w_r$ for general radiation $\rho_r = AT^m$. For simplicity, we only consider the case that $w_r$ is a positive constant. To do this, we temporarily consider radiation without a source. In the expanding background, consider a comoving volume $V_c$ which is in thermal equilibrium. The conserved radiation entropy within this volume is given by

$$S = \frac{\rho_r + p_r}{T} V_c. \quad (8)$$

The radiation energy density and pressure change with respect to the scale factor $a$ as

$$p_r \sim \rho_r \sim a^{-3(w_r+1)}. \quad (9)$$

Combining (8) and (9), the temperature scales as

$$T \sim (\rho_r + p_r) V_c \sim a^{-3w_r}, \quad (10)$$

so the relation between $\rho_r$ and $T$ can be written as

$$\rho_r \sim T^{(w_r+1)/w_r}. \quad (11)$$

So we have the relation between the sound speed, the equation of state and the parameter $m$ defined in (3) as

$$c_s^2 = w_r = \frac{1}{m-1}. \quad (12)$$

Another important issue is to determine the appropriate size of the thermal system $L$. By determining $L$, we mean that, at length scales smaller than $L$, the fluctuation of the system can be calculated using the thermal dynamics described above, and at scales greater than $L$, the fluctuation is governed by the cosmological perturbation theory. Note that a typical photon in the thermal system has wavelength $T^{-1}$, so there is a lower bound $L \gtrsim T^{-1}$ on $L$. Otherwise, the system is too small to be treated as a thermal system and the above calculation no longer holds. Also, there should be no thermal correlation outside the acoustic horizon $c_s H^{-1}$, so the constraint on $L$ is $T^{-1} \lesssim L \lesssim c_s H^{-1}. \quad 5$

We will argue in the discussion that, from some decoupling mechanism, the explicit value of $L$ may depend on the detailed properties of the thermal system and the dynamics of inflation. In the remainder of the paper, we will hold $L$ as a parameter (sometimes called the ‘thermal horizon’) during the calculation and discuss the most modest limit $L = c_s H^{-1}$ when we come to the final results.

5 In the literature, the length scale $T^{-1}$ is used as $L$ to calculate the thermal fluctuations by some authors [28]. In our calculation of non-Gaussianity, if we choose $L = T^{-1}$, the result turns out to be much more dramatic: we will get very large non-Gaussianity for a much wider class of inflationary models. In our paper, we do not choose to use $L = T^{-1}$, and only take $T^{-1}$ as a lower bound of $L$ here.
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Figure 1. This figure illustrates how the initial condition of perturbation is prepared by thermal fluctuations. The black cycle represents the thermal horizon. A fluctuation \( \delta \rho \) of the system can exit the thermal horizon \( L \) (shown as a shell in the figure) during inflation. This shell outside the thermal horizon cannot return to thermal equilibrium with respect to the original volume. It provides the initial condition of inflationary fluctuations. Fluctuations are created shell by shell.

The fluctuation \( \delta \rho \) can be thought of as an average over all the local fluctuations of the thermal system \( \delta \rho = (a^3/V) \int d^3x \delta \rho(x) \), performing the Fourier transformation, and linking the zero mode of \( k \) to the horizon exit mode \( k = a/L \), as illustrated in figure 1. Then the out-of-thermal-equilibrium initial condition for \( \delta \rho_k \) takes the form

\[
\delta \rho_k = k^{-3/2} \delta \rho.
\]  

(13)

As a check, equation (13) can also be obtained using the window functions.

Since \( L \) is smaller than the inflationary horizon, the relation between the energy density perturbation and the scalar type metric perturbation at the boundary of \( V \) can be calculated using the Poisson equation\(^6\)

\[
\Phi_{kL} = 4\pi G \delta \rho_k L^2,
\]  

(14)

where \( G \) is the Newton constant and \( \Phi_{kL} \) is the Fourier mode of the Newtonian gauge metric perturbation defined as \( ds^2 = a^2(-\Phi d\eta^2 + (1 + 2\Phi) dx^2) \), calculated at \( k = a/L \). Note that \( \Phi \) here is not the Newtonian potential \( \Phi_N \), but rather \( \Phi = -\Phi_N \). We follow the WMAP convention to use \( \Phi \) as the perturbation variable. Further discussion on the conventions can be found in the appendix.

We work in the Newtonian gauge only for simplicity. Since \( L < H^{-1} \), the different choice of gauge is not important inside the thermal horizon (see, for example, \([29]\)). After the inflationary modes leave the thermal horizon, we use \( \Phi \) to describe the mode. It is well known that \( \Phi \) can be made gauge-invariant when considering the more general gauges. So our final result will be independent of gauge choice.

\(^6\) Note that, to be exact, the 00 component of the linearized Einstein is \( -\nabla^2_{\text{ph}} \Phi + 3H \dot{\Phi} + 3H^2 \Phi = 4\pi G \delta \rho \). The sound speed \( c_s \) does not enter this equation. So, as long as \( c_s \) is not too large, even when \( L = c_s H^{-1} \), equation (14) is a very good approximation.
Equation (14) provides a thermal initial condition of $\Phi_k$. After that, the evolution of $\Phi_k$ is governed by cosmological perturbation theory. In the lowest-order slow roll approximation, $\Phi_k$ evolves as

$$\Phi_k \sim \sqrt{1 - k\tau H_1^{(1)}(-k\tau)},$$

(15)

where $\tau$ is the comoving time and $H_1^{(1)}$ is the first-kind Hankel function. It can be shown that $\Phi_k$ oscillates inside the inflationary horizon with nearly constant amplitude, so $|\Phi_k|_{k=aH} \simeq |\Phi_{kL}|$. After horizon crossing, the amplitude of $\Phi_k$ is frozen so that the change of $\Phi_k$ during a few e-folds is negligible. Using (6), (13) and (14), the two-point correlation for $\Phi_k$ at $k = aH$ (and also a few e-folds outside the inflationary horizon) is expressed as

$$\langle \Phi_k^2 \rangle = \frac{4\pi G^2}{2\pi^2} m\rho L k^{-3}.$$

(16)

Note that here $k = |k|$, and in (16) we are actually calculating the correlation between the mode $k$ and $-k$. Similarly, in the three-point correlation function, the quantity we calculate corresponds to the equilateral triangle, satisfying $k_1 + k_2 + k_3 = 0$ and $|k_1| = |k_2| = |k_3| = k$. The phase of $\Phi_k$ in the correlation functions cancels due to momentum conservation.

The power spectrum of $\Phi_k$ from the thermal origin can be written as

$$P_{\Phi} \equiv \frac{k^3}{2\pi^2} \langle \Phi_k^2 \rangle = 8G^2 m\rho L.$$

(17)

Similarly, the three-point function of $\Phi_k$ at $k = aH$ is expressed as

$$\langle \Phi_k^3 \rangle = (4\pi G)^3 m(m+1)\rho T^2 k^{-9/2}.$$

(18)

The non-Gaussianity can be calculated for $\zeta_k$, relating to $\Phi_k$ by $\zeta \simeq \Phi_k/\epsilon$ at a few e-folds outside the inflationary horizon, where $\epsilon \equiv -\dot{H}/H^2$.

Note that this three-point correlation function $\langle \zeta_k^3 \rangle$ is always positive. The positivity is transparent when we choose $L$ as the acoustic horizon $c_s H^{-1}$. In this case, $\zeta$ freezes outside $L$ and does not have a chance to change sign. While, in the case that $L < c_s H^{-1}$, although $\zeta$ has an oscillating solution inside the acoustic horizon, $\langle \zeta_k^3 \rangle$ still cannot change sign. This is because, after the initial condition is prepared, the evolution of $\langle \zeta_k^3 \rangle$ is governed by the interaction Hamiltonian of $\zeta$ as

$$\langle \zeta_k(t)^3 \rangle = -i \int_{t_0}^{t} dt' \langle [\zeta_k(t)^3, H_{\text{int}}(t')] \rangle.$$

(19)

We are assuming the slow roll inflationary scenario, and do not employ other mechanisms to have a large interaction $H_{\text{int}}$. So, once the positive non-Gaussian initial condition is produced, it will stay positive until it is observed in the CMB.

Finally the non-Gaussianity estimator $f_{\text{NL}}$ takes the form

$$f_{\text{NL}} = \frac{5}{18} \frac{k^{-3/2}}{\langle \zeta_k^2 \rangle^2} \frac{\langle \zeta_k^3 \rangle}{\langle \zeta_k^2 \rangle} = \frac{5\epsilon(m+1)}{72\pi G m\rho L^2}.$$

(20)

Note that here we have assumed that the origin of perturbation is completely thermal. A combination of the thermal and quantum origins of the power spectrum and non-Gaussianity will be discussed at the end of this section.
From (20), we see that $f_{NL} \propto L^{-2}$. So the smaller $L$ is, the larger the non-Gaussianity can be. Note that, for modified equation of states, if $|m| \ll 1$, the $m$ in the denominator also enhances the non-Gaussianity.

The non-Gaussianity (20) could also be estimated without calculating the three-point correlation function explicitly. The idea is to calculate the back-reaction. A fluctuation mode which crosses the thermal horizon earlier can change the background for a later fluctuation mode, so leading to non-Gaussian correlation between these two modes.

From the two-point correlation function, or from the standard result in thermodynamics that $\delta T/T \sim \sqrt{1/C_V}$, we have for the second mode

$$\delta_2 \rho \sim \sqrt{mAT^{m+1}/V}.$$  \hspace{1cm} (21)

As the first mode has crossed the thermal horizon by the time the second mode crosses the thermal horizon, the first mode leads to a modification of the background of the thermal system for the second mode. This modification of the background represents the correlation of the two modes, so is the non-Gaussian contribution. At the thermal horizon, this non-Gaussian contribution takes the form

$$\delta_1(\delta_2 \rho) \sim \frac{m+1}{2} \sqrt{mAT^{m-1}/V} \delta_1 T \sim \frac{(m+1)T}{2V}.$$  \hspace{1cm} (22)

As discussed earlier, when the modes reaches the inflationary horizon $k = aH$, we have

$$\zeta - \zeta_g \sim 4\pi G \delta \rho L^2/\epsilon.$$  \hspace{1cm} (23)

So finally the non-Gaussianity $f_{NL}$ is estimated as

$$f_{NL} \sim \frac{5(\zeta - \zeta_g)}{3\zeta_g^2} \sim \frac{5\epsilon(m+1)}{24\pi Gm\rho L^2}.$$  \hspace{1cm} (24)

This differs from (20) only by a factor of 3 and can be considered as in good agreement for a rough estimate. This back-reaction estimate provides a check for the three-point function calculated above, and also explains why the non-Gaussianity can be so large: the thermal horizon is smaller than the inflationary horizon, so a back-reaction calculated at the thermal horizon is larger than the one calculated at the inflationary horizon. This large back-reaction leads to a large non-Gaussianity. Note that, although this estimate cannot give the precise shape of $f_{NL}$, the limit we take is similar to the squeezed limit, which leads to a local shape non-Gaussianity.

Generally, there can also be perturbations from the vacuum fluctuations of the coherent rolling inflaton field. Let us denote this perturbation by $\Phi_k^{\text{vac}}$. Since the vacuum fluctuation and the thermal fluctuation are of different origins, they do not have correlations between each other. So in the two-point and three-point functions, the cross terms such as $\langle \Phi_k \Phi_k^{\text{vac}} \rangle$ vanish. So for the total power spectrum and the non-Gaussianity

$$P_{\Phi}^{\text{tot}} = P_{\Phi}^{\text{vac}} + P_{\Phi}, \quad f_{NL}^{\text{tot}} = \frac{5}{18} k^{-3/2} \frac{\langle \zeta_k^{\text{vac}} \rangle^3 + \langle \zeta_k \rangle^3}{\langle \zeta_k^{\text{vac}} \rangle^2 + \langle \zeta_k \rangle^2},$$  \hspace{1cm} (25)

where $\zeta_k^{\text{vac}}$ and $P_{\Phi}^{\text{vac}}$ are the comoving curvature perturbation and the power spectrum calculated from the vacuum fluctuation of the inflaton field. When $|\Phi_k^{\text{vac}}| \ll |\Phi_k|$, (25) returns to (17) and (20), and when $|\Phi_k^{\text{vac}}| \gg |\Phi_k|$, (25) returns to the power spectrum and the non-Gaussianity with zero temperature.
3. Examples of inflation models with large thermal non-Gaussianity

In the previous section, we have given the general formalism to calculate the thermal perturbations and non-Gaussianity. In this section, we apply the formalism to the chain inflation and the warm inflation.

Although the radiation energy density can be inflated away very easily, as discussed in the introduction, there are mechanisms to continuously produce radiation so that the radiation energy density remains nearly constant. Such mechanisms include the interaction of the radiation with the inflaton, and bubble collision during the chain inflation, which we shall show explicitly.

In the chain inflation model, the vacua tunnel rapidly and the time evolution of the vacuum energy density can be approximated by

$$\rho_0(t) = \rho_0(0) - \alpha t,$$  \hspace{1cm} (26)

where $\alpha$ denotes the averaged decay rate of the vacuum energy. ($\alpha \equiv \sigma/\tau$ in the notation of \[23\]). Suppose that the decreasing energy converts completely into radiation through bubble collisions. Taking into consideration the redshift of the radiation during inflation, the radiation energy density satisfies

$$d\rho_r(t) = \alpha dt - 3H(1 + w_r)\rho_r dt.$$  \hspace{1cm} (27)

By taking the stationary limit $t \gg [(1 + w_r)H]^{-1}$, we have

$$\rho_r = \frac{\alpha}{3(1 + w_r)H} = \frac{2\epsilon \rho}{3(1 + w_r)^2}, \quad \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{4\pi G}{3} \frac{\alpha}{H^3}.$$  \hspace{1cm} (28)

The relation (28) could also be obtained from assuming that the radiation density is produced within one Hubble time, then from (26) and taking $t \sim H^{-1}$, we get directly that $\rho_r$ is of the order of $\alpha/H$.

Note that $\rho_r$ is a slow roll quantity during inflation. As $\rho_r \sim T^m$, when $m$ is not too small, $T$ also changes very slowly during inflation. This verifies the assumption that the radiation density and the temperature during inflation are almost constants.

Several mechanisms have been proposed to calculate the fluctuations in the chain inflation model. In \[22\], the authors showed that the perturbations can come from different paths along which the metastable vacua tunnel. In \[23\], the density perturbation is calculated by applying the standard formalism to the effective scalar field characterizing the tunnelling effect. But up to now the exact mechanism of how the perturbation in chain inflation is generated is still not clear.

In this section, we propose a new mechanism to produce the density perturbation of the chain inflation. We claim that the perturbation can have a thermal origin. It is because, after the bubble collision, the energy contained in the bubble wall becomes radiation. As will be shown later in this section, the radiation density is of the order of $\rho_r \sim \epsilon \rho$, whose thermal fluctuation can exit the horizon and produce a scale-invariant power spectrum. So in a realistic calculation, the fluctuations discussed in \[22\], \[23\] and the thermal fluctuation should be taken into account at the same time.

Using (17), the power spectrum takes the form

$$P_\Phi = \frac{16G^2m\epsilon \rho TL}{3(1 + w_r)}, \quad P_\zeta = \frac{16G^2m\rho TL}{3(1 + w_r)\epsilon}.$$  \hspace{1cm} (29)
Assuming $m$ and $w_r$ are exactly constant, the spectral index takes the form

$$n_s - 1 \equiv \frac{d \ln P_\zeta}{d \ln k} \bigg|_{k=aH} = -5\epsilon + \frac{1}{TL} \frac{d(TL)}{H \, dt}. \quad (30)$$

If $L$ saturates its lower bound $L \sim T^{-1}$, and considering the usual type of radiation $m = 4$, $w_r = 1/3$, then we recover the power spectrum and the spectral index in the simplified chain inflation model [23].

If the thermal perturbation is dominant over other perturbation sources during chain inflation, then the non-Gaussianity can be read off from (20) as

$$f_{NL} = \frac{5(m + 1)(1 + w_r)}{18m(LH)^2}. \quad (31)$$

Note that for the usual type of radiation $m = 4$, $w_r = 1/3$, $L$ should satisfy $L \leq (3H)^{-1}$, so $f_{NL}$ is always larger than $O(1)$. This is very different from the ordinary inflation model in which $f_{NL} \sim O(\epsilon)$.

To go one step further, let us consider the modified radiation $m \neq 4$. We make a modest estimate that $L = c_s H^{-1}$. In this case

$$f_{NL} = \frac{5(m + 1)(1 + w_r)}{18m c_s^2} = \frac{5(m + 1)}{18}. \quad (32)$$

As a phenomenological model, if $m$ is large, one can have large $f_{NL}$.

If the thermal perturbation is not the dominant source in the power spectrum, then we need to compare the thermal and other contributions to estimate the non-Gaussianity. Let the power spectrum from the other origin be $P_{\Phi}^{vac}$, then using (20) and (25), where we have neglected the non-Gaussianity produced by sources other than thermal, we can express $f_{NL}$ as

$$f_{NL} = \frac{5(m + 1)(1 + w_r)}{18m(LH)^2(1 + P_{\Phi}^{vac}/P_{\Phi})^2}. \quad (33)$$

So it is clear that, although the non-Gaussianity is suppressed by the ratio of the spectra, there is no longer the $\epsilon$ suppression. Moreover, if $L \ll H^{-1}$, then the non-Gaussianity can be enhanced by a large amount.

In the case of warm inflation, radiation is continuously produced during slow roll inflation. This process can be modelled by adding a interacting term between inflaton and the radiation component in the Lagrangian. In the slow roll regime, the equation of motion for the inflaton $\varphi$ is

$$3H \dot{\varphi} + \Gamma_{\varphi} \dot{\varphi} + \partial_\varphi V(\varphi) = 0, \quad (34)$$

where $\Gamma_{\varphi}$ is the decay rate for the inflaton to the radiation process. We assume $\Gamma_{\varphi}$ is a constant (or at least a slow roll quantity) here. The equation for the radiation energy density takes the form

$$\dot{\rho}_r + 3H(1 + w_r) \rho_r = \Gamma_{\varphi} \dot{\varphi}^2. \quad (35)$$

A solution for these equations is given by

$$\dot{\rho}_r \simeq 0, \quad \rho_r \simeq \frac{\Gamma_{\varphi}}{3H(1 + w_r)} \dot{\varphi}^2. \quad (36)$$
In [24], it is shown that this solution is an attractor solution and is independent of the initial conditions for the thermal component.

When $\rho_r \lesssim \rho_0$, the universe accelerates. To give a nearly scale-invariant spectrum, we require the universe undergoes a quasi-dS expansion. This requirement can be satisfied when $\rho_r \lesssim \epsilon \rho_0$. This corresponds to the case that $\Gamma_\phi \lesssim H$. When this bound saturates, the power spectrum and the non-Gaussianity coincide with the chain inflation case.

Note that $\rho_r \lesssim \epsilon \rho_0$ is not a necessary condition for quasi-dS expansion. For example, in the strong dissipative regime of warm inflation, this condition can be violated. However, for simplicity, we only consider $\rho_r \lesssim \epsilon \rho_0$ in this paper.

It can also be checked that, when the constraint $\rho_r \lesssim \epsilon \rho_0$ is satisfied, the e-folding number and the slow roll condition are qualitatively the same as the $\rho_r = 0$ case. So the inflationary background does not change very much in this case.

Similarly to the chain inflation case, the thermal power spectrum of warm inflation takes the form
\[ P_\Phi = 8G^2 m \rho_r TL, \quad P_\zeta = 8G^2 m \rho_r TL/\epsilon^2. \]  
(37)

The non-Gaussianity of the warm inflation is
\[ f_{NL} = \frac{5(m+1)\epsilon \rho}{27 m \rho_r (LH)^2 (1 + P_{vac}/P_\Phi)^2}, \]  
(38)

which behaves like the chain inflation model with sub-dominant thermal fluctuation. Note that $\rho_r \lesssim \epsilon \rho_0$, so there is no $\mathcal{O}(\epsilon)$ suppression. The enhancement due to $(LH)^{-2}$ still exists.

To see how $f_{NL}$ behaves at the small $\rho_r$ limit, taking into consideration the observational constraint of the total power spectrum $P_{\zeta}^{\text{tot}} \simeq 2.5 \times 10^{-9}$, $f_{NL}$ can be written as
\[ f_{NL} = \frac{40 m(m+1)G^3 \rho_r T^2}{9\pi P_{\zeta}^{\text{tot}} \epsilon^3}. \]  
(39)

So, when taking the $\rho_r \to 0$ limit, we get $f_{NL} \to 0$. This is reasonable because the $f_{NL}$ we are considering comes from a thermal origin.

It seems surprising at first sight that, in the limit that $\epsilon$ is very small, the radiation component bounded by $\rho_r \lesssim \epsilon \rho$ is tiny, but the thermal non-Gaussianity can still be quite large. The reason for this is that, given the amplitude of the observed CMB power spectrum, when $\epsilon$ is very small the primordial density fluctuation needed to generate the CMB power spectrum also becomes small. So a tiny part of the radiation becomes comparable with the inflaton vacuum fluctuation in the function of generating the fluctuations.

As another special case, let us consider the maximum value of $f_{NL\text{max}}$ for standard radiation, where $\rho_r = AT^4$ and $HL \sim 1$. If the inflaton vacuum is thermalized, $f_{NL\text{max}}$ becomes of the order of 1. However, if the inflaton vacuum is not thermalized, then the maximum is taken when $\rho_r T \sim \epsilon \rho H$, and $f_{NL\text{max}} \sim (P_{\zeta}^{\text{tot}})^{-1/5} \sim 10$. So we conclude that, even for standard radiation, there is a chance that the non-Gaussianity could be observed by the Planck satellite.

Finally, as a clarification, we note that the freezeout for the thermal fluctuation at the horizon crossing is not the same as the freezeout time $t_F$ in [25]. In [25], $t_F$ is defined as the freezeout time for the inflaton field. After $t_F$, the inflaton cannot feel the thermal bath. So the statement that, in the weak dissipative regime $t_F$ should be taken at the horizon crossing, is not in contradiction with our result.
In conclusion, in this paper, we have calculated the non-Gaussianity from thermal effects during inflation. We calculated the two-point and three-point thermal correlation functions, and used these correlation functions to calculate the scalar power spectrum $P_\Phi$ and the non-Gaussianity estimator $f_{\text{NL}}$. We also used an independent method to check the value of $f_{\text{NL}}$.

We have applied our treatment to the chain inflation model. We find that the density perturbation in the chain inflation model may come from thermal fluctuations. This provides a candidate for the origin of the power spectrum of the chain inflation model.

We also calculated the non-Gaussianity of the chain inflation model. We found that, if thermal perturbation is the main source of chain inflation, then the non-Gaussianity $f_{\text{NL}}$ of chain inflation is greater than $O(1)$. Taking into consideration the modified sound speed, the non-Gaussianity can become much larger.

If the thermal perturbation is sub-dominant during chain inflation, then $f_{\text{NL}}$ is suppressed by $P^2_\Phi/(P_\Phi + P^{\text{vac}}_\Phi)^2$. But still, there is no $O(\epsilon)$ suppression and the term $(LH)^{-2}$ can provide a large non-Gaussianity.

As another application, we studied the non-Gaussianity in the warm inflation model. The result of the warm inflation model is similar to the case of chain inflation with a sub-dominant thermal component, and the non-Gaussianity is $O(1)$.

We only studied the $\rho_t \lesssim \epsilon \rho_0$ case in the warm inflation scenario. It is shown that observable non-Gaussianity may already show up in this case. We have not considered in this paper the complementary case $\rho_t > \epsilon \rho_0$. In this latter case, the thermalization of the inflaton vacuum dominates the power spectrum. The thermal part of the non-Gaussianity $f_{\text{NL}}^{\text{thermal}}$ is suppressed in this case. But the thermal non-Gaussianity of the inflaton field should be taken into consideration.

As mentioned in the introduction, in warm inflation models, large non-Gaussianity can also arise from other mechanisms [27]. In this paper, we only considered the non-Gaussianity produced by the thermal fluctuation itself. It would be interesting to combine the mechanism in our paper with that in [27] to get a more complete formula for $f_{\text{NL}}$.

In the warm inflation case, it is clear that the vacuum fluctuation and the thermal fluctuation are two different sources of inflation fluctuations. So it may lead to large isocurvature perturbation. This isocurvature perturbation issue is not discussed in detail in this paper.

In this paper, we calculated the equilateral shape non-Gaussianity. In the backreaction estimate, the shape is something like the local shape. Since the non-Gaussianity from thermal fluctuations can be large, and is hoped to be very likely observed in the near future, it is also important to calculate the more general correlation functions with arbitrary $k$ and obtain the shapes of the non-Gaussianity.

Another important issue is to determine the parameter $L$ given an inflation model, which requires more details on the dynamics of the system. In this paper, we mainly discussed the upper bound of $L$, which is governed by the sound speed $c_s$ of the radiation component. But we note that $L$ may be much smaller than the acoustic horizon. One possible mechanism generating smaller $L$ is the decoupling of the Fourier mode of thermal fluctuations. When the universe expands, the interaction rate $\Gamma$ for the thermal fluctuation
Fourier mode may decrease, so it decouples before reaching the acoustic horizon. We wish to address this issue in the near future.

The generalization of our calculation to other inflation models with radiation is straightforward. For example, our calculation can also be applied to the thermal inflation model [30] or the thermal version of the noncommutative inflation model [28].

A similar analysis can also be performed in the string gas model [31], where the power spectrum also has a thermal origin. The calculation of the non-Gaussianity of the string gas model will be presented in a separate publication [32].

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Appendix. Clarification on conventions

In this appendix, we clarify the conventions we use. This clarification is necessary because a confusion in the conventions (especially for signs) can lead to an extra minus sign in $f_{NL}$, and lead to completely opposite predictions. This is very different from the calculation of the power spectrum, where a confusion of the sign convention usually leads to the same result.

In this paper, we use the WMAP convention. In this convention, the metric perturbation $\Phi$ can be written as (in the Newtonian gauge)

$$ds^2 = a^2 \left( -(1 - 2\Phi) d\eta^2 + (1 + 2\Phi) dx^2 \right).$$

(A.1)

So $\Phi$ is not the Newtonian potential $\Phi_N$, but rather $\Phi = -\Phi_N$. This is the same as the convention used in [33], which is also the same as the convention used in the WMAP group. One can refer to [33] to find the complete definitions. This is of different sign from the convention $\Phi$ used in [29]. (In [29], they use $\phi$ to denote the Newtonian potential and the gauge-invariant quantity $\Phi = \phi$ in the Newtonian gauge.)

For the quantity $\zeta$, there are also different conventions in the literature. In this paper, following the convention of [33], $\zeta$ can be written as

$$\zeta = \Phi - \frac{H}{\partial_t \varphi} \delta \varphi,$$

(A.2)

where $\varphi$ and $\delta \varphi$ are the background value and the perturbations for the inflaton field, respectively. However, in [29], the $\zeta$ parameter they use is of a different sign.

One simple way to check the sign is to relate it to the quantities which have clear physical meaning. There are at least two such quantities: the energy density $\delta \rho$ and the CMB temperature fluctuation $\Delta T/T$. In our convention, the Poisson equation takes the form

$$-\frac{\nabla^2}{a^2} \Phi = 4\pi G \delta \rho.$$

(A.3)
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The CMB temperature fluctuation can be written as

$$\frac{\Delta T}{T} = -\frac{1}{3} \Phi = -\frac{1}{5} \zeta.$$  \hfill (A.4)

Although not related to this paper, we also would like to remind the reader of two more differences in the conventions, which may be used in the calculation of $f_{\text{NL}}$. One is that, in [13], Maldacena uses the same convention of $\zeta$ as the WMAP group, but the equation in footnote 16, $\zeta = -(5/3)\Phi$, does not follow the WMAP convention. So the $f_{\text{NL}}$ defined in [13] is of a different sign from the WMAP convention. The other is that for the so-called 'comoving curvature perturbation' $R$. The $R$ used in [33] (in the comoving gauge) is of a different sign from the $R$ used in [34] outside the horizon.

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