ALAG - QUANTIZATION

Nik. Tyurin

Abstract. This paper is a continuation of [10]. Here we present the application of ALAG - programme, introduced in [2], [9], to the geometric quantization. The proposed approach is following to "geometrical formulation of quantum mechanics" ([1]). For our ALAG - quantization the correspondence principle is satisfied.

§0. Introduction

The framework of geometric quantization is usually described as follows (see f.e. [5]). Let \((M, \omega)\) is a symplectic manifold which represents a classical mechanical system with finite number of degrees of freedom. Then one understands the geometric quantization as a procedure relating with \((M, \omega)\) a Hilbert space \(\mathcal{H}\) together with a correspondence

\[
Q : \mathcal{C}^\infty(M \to \mathbb{R}) \to O(\mathcal{H})
\]

where \(O(\mathcal{H})\) is the space of self adjoint operators which should satisfy the correspondence principle

\[
Q(\{f, g\}_\omega) = \imath \hbar [Q(f), Q(g)]
\]

as well as the irreducibility condition. Unfortunately such construction couldn’t exist (by the van Hove theorem). Nevertheless one usually uses two basic variants of this procedure: Souriau - Kostant quantization which doesn’t satisfy the irreducibility condition and Berezin quantization which doesn’t satisfy the correspondence principle.

But now the question of quantization can be refined in more invariant fashion: we are looking for a quantum mechanical system which is closely related to the original classical mechanical system. Turning to the language of "geometrical formulation of quantum mechanics" (see [1]) one could reformulate the subject of geometric quantization as follows. For any symplectic manifold \((M, \omega)\) one would like to construct a Kähler manifold \(\mathcal{K}\) (finite or infinite dimensional) endowed with a Kähler triple \((G, I, \Omega)\) where \(\Omega\) is a symplectic form, \(I\) is an integrable complex structure and \(G\) is the corresponding riemannian metric together with a correspondence

\[
\tilde{Q} : \mathcal{C}^\infty(M \to \mathbb{R}) \to \mathcal{N} \subset \mathcal{C}^\infty(\mathcal{K} \to \mathbb{R})
\]

where \(\mathcal{N}\) consists of all functions whose hamiltonian vector fields preserve riemannian metric \(G\):

\[
\mathcal{N} = \{ F \in \mathcal{C}^\infty(\mathcal{K} \to \mathbb{R}) \mid \text{Lie}_{X_F} G = 0 \}.
\]
One requires for this construction that

\[ \tilde{Q}(\{ f, g \}_\omega) = \{ \tilde{Q}(f), \tilde{Q}(g) \}_\Omega \] (\*)

which means that the correspondence principle is satisfied (see [1] for explanation why it’s equivalent to the last one).

Now one can mention that the so called ALAG - programme (see [2], [3], [9]) endows us with a candidate — for any compact smooth orientable symplectic 2n - dimensional manifold \( M \) with integer symplectic form \( \omega \) one can almost canonically construct a set of infinite dimensional Kähler manifolds \( B^{\text{hw},t}_S \) labeled by ”integer” parameter \( S \in H_n(M, \mathbb{Z}) \) and continuous parameter \( t \in \mathbb{R} \). These are called moduli spaces of half weighted Bohr — Sommerfeld lagrangian cycles of fixed volume and topological type.

On the other hand it was pointed out ([9], [10]) that for any such moduli space \( B^{\text{hw},t}_S \) one has a natural correspondence: for every smooth function \( f \) on \( M \) there is a corresponding function \( F_f \) on \( B^{\text{hw},1}_S \); these induced functions we will call special functions. The main goal of the present paper is to prove the following statement.

**Main Theorem.** In the setup of ALAG

1) the correspondence

\[ f \mapsto F_f \]

is an inclusion;

2) for every \( f \in C^\infty(M \to \mathbb{R}) \) the corresponding special function \( F_f \) satisfies the condition mentioned in the framework of the geometrical formulation

\[ \text{Lie}_{X_{F_f}} G = 0; \]

and

3) the correspondence principle in form (\*) is satisfied.

Thus one gets a new approach to the geometric quantization construction via ALAG - programme. At the same time this approach gives us the first application of ALAG - programme in pure geometrical setup. But really the spectrum of validity of ALAG is much more wider and one hopes to see a number of new results in a nearest future.

I would like to cordially thank Korean Institute of Advanced Study for hospitality during this work. This work would be impossible without uninterrupted dialog with the authors of ALAG took place all the time we were staying in Seoul.

§1. Geometrical formulation of quantum mechanics

In this subsection we follow [1] preserving the notations.

Let us briefly recall the main ideas of the geometrical formulation.

In classical mechanical setup one has a symplectic manifold \( M \) which represents the corresponding phase space such that states are represented just by points of \( M \). The set of classical observables is the space of smooth functions \( C^\infty(M \to \mathbb{R}) \) endowed with a Poisson algebra structure which is given by the symplectic form. The measurement aspects are quite simple: one just takes the volume of observables over
a state \( x \in M \). Every smooth function \( f \) generates the corresponding hamiltonian vector field

\[
X_f = \omega^{-1}(df)
\]

which preserves the symplectic form

\[
\text{Lie}_{X_f}\omega = 0
\]

and induces a motion over \( M \) which can be understood as a continuous family of symplectomorphisms of \( M \) and its germ corresponds to \( X_f \).

In quantum mechanics usually one deals with a Hilbert space \( \mathcal{H} \) so a complex vector space (finite or infinite dimensional) endowed with a hermitian inner product \( \langle .,. \rangle \). Quantum states are represented by rays in \( \mathcal{H} \). Quantum observables are self adjoint operators such that for every \( A,B \) one has a skew symmetric operation

\[
\hat{A}, \hat{B} \mapsto i\hbar[\hat{A}, \hat{B}]
\]

which endows the space of self adjoint operators with a Lie algebra structure. The dynamics of the system is described by Schrödinger equation

\[
\psi = -\frac{i}{\hbar} \hat{H} \psi
\]

where \( \hat{H} \) is a preferred observable called the hamiltonian of the system. The corresponding flow is generated by 1 - parameter group \( \exp(i\hat{H}t) \). The measurement aspects are much more complicated: the ideal measurement of an observable \( \hat{A} \) in a state \( \psi \in \mathcal{H} \) yields an eigenvalue of \( \hat{A} \) and immediately after the measurement the state is thrown into the corresponding eigenstate. So roughly speaking for \( \hat{A} \) and \( \psi \) there is a set of projections of the state \( \psi \) to the eigenstates of \( \hat{A} \) which one understands as probabilities (amplitudes)(one can twist every orthonormal eigenbasis such that all coordinates are real nonnegative numbers).

The starting point of discussion is the fact that the space of physical states is not the Hilbert space \( \mathcal{H} \) itself but the space of rays in \( \mathcal{H} \). So instead of \( \mathcal{H} \) one can consider its projectivization

\[
\mathcal{P} = \mathbb{P}(\mathcal{H})
\]

which is a Kähler manifold (infinite or finite dimensional as well as \( \mathcal{H} \)) endowed with

1) symplectic form \( \Omega \)
2) integrable complex structure \( I \)
3) the corresponding riemannian metric \( G \).

One can translate completely all notions of QM to the language of this projective space without any references to the original Hilbert space.

Here it is the vocabulary:

**Quantum states.** Points of \( \mathcal{P} \) represent quantum states.
Quantum observables. Instead of self adjoint operators over $\mathcal{H}$ one takes the corresponding expectation value functions:

$$\hat{A} \mapsto A(\psi) = \frac{1}{2\hbar} G(\psi, \hat{A}\psi).$$

Every such function admits well defined restriction to unit sphere in $\mathcal{H}$ and further to $\mathcal{P}$.

The complement question is: how one can recognize is a smooth function $F$ over $\mathcal{P}$ induced by some self adjoint operator over $\mathcal{H}$? The answer is quite definite: $F$ is induced by a self adjoint operator if and only if the hamiltonian vector $X_F$ preserves the riemannian metric $G$:

$$\text{Lie}_{X_F} G = 0.$$  

Thus one can distinguish such functions over $\mathcal{P}$ and collect these in the set $\mathcal{N}$ of special functions.

Schrödinger vector fields. For such a function $A : \mathcal{P} \to \mathbb{R}$ one gets that the hamiltonian vector field $X_A$ on $\mathcal{P}$ is precisely the same as the corresponding Shroedinger vector field restricted to unit sphere and factorized by the phase rotations.

Commutator. For two observables $\hat{A}, \hat{B}$ one takes the commutator $\hat{C} = \frac{1}{i\hbar} [\hat{A}, \hat{B}]$. Then the induced expectation volume function just equals to the Poisson bracket

$$C = \{A, B\}_\Omega.$$

Schrödinger equation. For a preferred hamiltonian $\hat{H}$ over $\mathcal{H}$ the induced motion on $\mathcal{P}$ is precisely the same as that one which induced by hamiltonian $H$. It means that Schrödinger equation reads as Hamilton equation over $\mathcal{P}$.

Uncertainties. For two special functions $A, B \in \mathcal{N}$ one has riemannian bracket:

$$(A, B)_G = \frac{\hbar}{2} G(X_A, X_B).$$

Then the squared uncertainty of $A$ at the quantum state $p$ equals to

$$(\Delta A)^2(p) = (A, A)_G(p).$$

The uncertainty relation for two quantum observables looks like

$$(\Delta A)(\Delta B) \geq (\frac{\hbar}{2} \{A, B\}_\Omega)^2 + (A, B)_G^2.$$  

If in a point $p$ two hamiltonian vector fields $X_A, X_B$ are $G$ - orthogonal one gets the standard uncertainty relation

$$(\Delta A)(\Delta B) \geq (\frac{\hbar}{2} \{A, B\}_\Omega).$$
**Eigenstates and eigenvalues.** For a self adjoint operator $\hat{A}$ with eigenstates $\psi_i$ and eigenvalues $\lambda_i \in \mathbb{R}$ one gets that the corresponding special function $A$ has these $p_i = \mathbb{P}(\psi_i)$ as critical points with critical values

$$A(p_i) = \lambda_i.$$ 

**Measurement process. Discrete spectrum.** Let one take a state $\psi$ with respect to orthonormal basis $\psi_i$ consists of eigenstates of $\hat{A}$. Then the corresponding amplitudes $\alpha(\psi, \psi_i)$ can be found over $\mathcal{P}$ as follows. For $p = \mathbb{P}(\psi)$ and $p_i = \mathbb{P}(\psi_i)$ consider geodesic distances

$$d(p, p_i)$$

with respect to riemannian metric $G$. Then one has the following equality

$$\alpha(\psi, \psi_i) = \cos^2\left(\frac{d(p, p_i)}{\sqrt{2\hbar}}\right).$$

We already know what are eigenstates and eigenvalues.

**Measurement process. Continuous spectrum.** Almost the same story just instead of points one takes the corresponding critical subsets and measures geodesic distance between point $p$ and such a subset.

Thus one could understand the main difference between classical mechanics and quantum mechanics in the following style (see [1]): together with a symplectic structure one has a compatible riemannian structure on the phase space in the last case.

**Remark.** Here let us note that the projectivization $\mathbb{P}\mathcal{H}$ is a Kähler manifold of special type namely it is an algebraic manifold. This means that one has over $\mathbb{P}\mathcal{H}$ the positive line bundle $\mathcal{O}(1)$ which first Chern class is represented by the Kähler form $\omega$. The space of holomorphic sections $H^0(\mathcal{O}(1))$ is isomorphic to $\mathcal{H}$ and in the next section we discuss how one can reconstruct all data on this Hilbert space.

§2. **Examples**

Now let us consider two well known examples of geometric quantization via this projectivization approach.

First of all turn to Souriau - Kostant method. Here we have a prequantization quadruple $(M, \omega, L, a)$ where $L$ is a hermitian line bundle defined topologically by the first Chern class

$$c_1(L) = [\omega]$$

and $a$ is a hermitian connection defined by

$$F_a = 2\pi i \omega.$$ 

Then the Hilbert space is

$$\mathcal{H} = L^2(M, L)$$

— the completion of the space of all smooth sections of $L \to M$ with respect to the natural norm

$$\|\sigma\|^2 = \int_M <\sigma, \sigma> d\mu,$$
where \( d\mu \) is the Liouville volume form. Then for every function \( f \in C^\infty(M \to \mathbb{R}) \) one takes the operator

\[
Q_f : \mathcal{H} \to \mathcal{H}, \quad Q_f = \nabla_{X_f} + 2\pi if
\]

acting on \( \mathcal{H} \). With well known relationships in differential geometry in hands one has

\[
[Q_f, Q_g] = Q_{\{f, g\}_\omega}.
\]

One can explain this correspondence using some background ideas which could be called as "dynamical correspondence". The point is that one has over \( M, \omega \) the space of symplectomorphism germs represented by global hamiltonian vector fields over \( M \). If one would like to extend a germ generated by \( X_f \) over \( M \) to a one parameter family of automorphisms of the hermitian line bundle \( L \) with fixed hermitian connection \( a \) then there is unique lifting of such type which is described exactly by the vector field corresponds to \( Q_f \). More precisely every hamiltonian vector field \( X_f \) induces a germ of linear transformation of our \( \mathcal{H} \) and this transformation is just the exponential of \( Q_f \). So in style of [1] we will write a vector field on \( \mathcal{H} \) which reads just as

\[
Y(\psi) = Q_f \psi.
\]

If one takes "expectation value" of \( Q_f \) it would be a pure imaginary function since \( Q_f \) is a skew hermitian one. One can rescale this function getting a smooth real function \( \tilde{Q}_f \) on the projective space \( \mathbb{P}(\mathcal{H}) \).

Now every hamiltonian vector field \( X_f \) generates a smooth vector field over \( \mathbb{P} \) which preserves both the symplectic and the riemannian structures over \( \mathbb{P} \). This vector field we will denote as \( \Theta(f) \), emphasizing that it is generated by the smooth "source" function \( f \) on the "source" manifold \( M \). This map

\[
\Theta : C^\infty(M \to \mathbb{R}) \to Vect(\mathbb{P})
\]

we will call the dynamical correspondence. It plays the fundamental role in quantization procedure. Obviously the dynamical correspondence map satisfies the correspondence principle

\[
\Theta(\{f, g\}_\omega) = [\Theta(f), \Theta(g)]
\]

where at the right hand side one has the commutator of two vector fields. It reflects just the fact that \( \Theta \) is defined directly from dynamical properties of the system.

Generalizing this idea let us say that if one has any object over \( M, \omega \) preserved by all symplectomorphisms of \( M \) then there exists the same "dynamical correspondence" between hamiltonian vector fields and infinitesimal automorphisms of this object. We will see further how this idea works in ALAG.

Coming back to Souriau - Kostant quantization one gets the following fact: the induced "dynamical correspondent" vector field \( \Theta(f) \) over \( \mathbb{P} \) is equal to Killing vector field of smooth real function \( \tilde{Q}_f \). This claim is based on the fact that \( Q_f \) is a skew hermitian operator over \( \mathcal{H} \). From the algebraic point of view one can correct the correspondence

\[
f \mapsto Q_f
\]
such that

\[ f \mapsto -iQ_f \]

(we omit the Planck constant every time to clarify the mathematical aspects of the problem), getting self adjoint operators and keeping the correspondence principle but the difficulties which one meets further in this way should be explained with respect to this projection to ”dynamical correspondence” background. Therefore geometrically the well known reducibility of the Souriau - Kostant representation has a root in this ”dynamical discrepancy” when \( \Theta(Q) \) doesn’t coincide with the hamiltonian vector field \( X_{\tilde{Q}_f} \).

Turning to Berezin quantization one should say that it was originally described in the language of projectivization. Here one needs an additional structure - an appropriate complex polarization which can be understood as an integrable complex structure \( I \) compatible with \( \omega \). Thus we come to the framework of the algebraic geometry. Then one has a Hilbert subspace in \( \mathcal{H} \) consists of holomorphic sections of \( L \)

\[ H^0(M_I, L) \subset \mathcal{H}. \]

Let us suppose that our holomorphic line bundle \( L \) is very ample. Then there is smooth holomorphic inclusion

\[ \psi : M_I \to \mathbb{P}H^0(M_I, L)^* \]

defined as usual by the corresponding complete linear system. Combining with hermitian conjugation one gets a smooth antiholomorphic map

\[ \psi_0 : M_I \to \mathbb{P}H^0(M_I, L), \]

and its image is called the space of coherent states. Originally in the Berezin works it was proposed that one takes the space of symbols over the image \( \psi_0(M) \) which are nothing but the expectation functions restricted to \( \psi_0(M) \). The extensions of these symbols over all \( \mathbb{P}H^0(M_I, L) \) can be derived analytically. But we would like to perform this story using so called Berezin - Töplitz operators over \( H^0(M_I, L) \).

Namely since our ambient space \( \mathcal{H} \) is a Hilbert space then one can consider Szöge projector

\[ S : \mathcal{H} \to H^0(M_I, L) \]

which is just the orthogonal projector. Then for every smooth function \( f \) one has the following combination

\[ T_f : H^0(M_I, L) \to H^0(M_I, L), \quad \sigma \mapsto S(f \sigma). \]

These operators are self adjoint. Turning to the geometrical formulation one gets a smooth real function \( \tilde{T}_f \) over \( \mathbb{P}H^0(M_I, L) \) which is a function of special type described in [1]: its hamiltonian vector field should preserve the riemannian metric (and the complex structure) over \( \mathbb{P}H^0(M_I, L) \) as well as the symplectic structure.

One can compute this function using Fourier - Berezin transform. Namely one has a universal function \( u(x, s) \) on the direct product

\[ M \times \mathbb{P}H^0(M_I, L). \]
If we represent $\mathbb{P}H^0(M_I, L)$ by holomorphic sections of unit norms

$$\int_M <s, s> d\mu = 1$$

then

$$u(x, s) = <s(x), s(x)>$$

is a nonnegative smooth function. Then the transform reads as

$$\tilde{T}_f = \int_M f(x)u(x, s)d\mu. \quad (1)$$

On the other hand one couldn’t define an analogy of the dynamical correspondence here. The point is that a generic function $f \in C^\infty(M \to \mathbb{R})$ generates flow which doesn’t preserve the space $H^0(M_I, L)$ and consequently the projective space $\mathbb{P}H^0(M_I, L)$. Thus in the case one should restrict the consideration to the subspace $\mathcal{M} \subset C^\infty(M \to \mathbb{R})$ of functions whose flows preserve the projective space $\mathbb{P}H^0(M_I, L)$. For this case we have a brief and precise description of Berezin quantization contained in [6]. In this description one combines Souriau - Kostant quantization with Berezin approach (see [6]). Then it’s reasonable to define an analogy of the dynamical correspondence which is a map

$$\Theta' : \mathcal{M} \to Vect(\mathbb{P}H^0(M_I, L)).$$

If $f$ belongs to $\mathcal{M}$ then it induces a special function $\tilde{Q}_f$ (just the restriction of the "Souriau - Kostant" - function) such that its Killing vector field coincides with the induced by the dynamical property

$$\Theta'(f) = K(\tilde{Q}_f).$$

And again the dynamical correspondence works establishing that correspondence principle is satisfied in this case in terms of "quantum" Poisson bracket. But this approach is rather special since it allows to quantize only functions of very special type. In Berezin - Töplitz approach one has much more wider situation but of course the integral operator (1) described above has big kernel. At the same time it’s much more complicated problem to define in this approach any kind of the dynamical correspondence. At least we have not now at hands any simple description of that one. Because of this we can not prolong the story just mentioning that in Berezin - Töplitz quantization the correspondence principle holds only asymptotically (see [2]).

§3. DYNAMICAL CORRESPONDENCE IN ALAG

Let $(M, \omega)$ is a compact smooth orientable $2n$ - dimensional symplectic manifold with integer symplectic form $\omega$. Then there exists an infinite dimensional Kähler manifold $\mathcal{B}_S^{huc,1}$ where $S \in H_2(M, \mathbb{Z})$ is a homological class over $M$ and we consider the half weighted cycles of volume 1 just for simplicity (all definitions and constructions are contained in [2], [3] and in [9], where they were firstly introduced).
Moreover this moduli space is an algebraic manifold since the Kähler form represents the first Chern class of Berry bundle. Let us recall that this moduli space consists of pairs \((S, \theta)\) where \(S\) is a Bohr-Sommerfeld cycle in \(M\) of the fixed topological type and \(\theta\) is a half weight on \(S\) such that the corresponding volume

\[
\int_S \theta^2 = 1
\]

— is normalized.

For every smooth function \(f \in C^\infty(M \to \mathbb{R})\) there is an induced smooth function \(F_f\) on the moduli space which is defined by the following formula

\[
F_f(S, \theta) = \tau \int_S f|_S \theta^2
\]  

(2)

where \(\tau\) is a real parameter. This formula (2) gives us a map

\[
C^\infty(M \to \mathbb{R}) \to C^\infty(B^{hw,1}_S \to \mathbb{R}). \quad (3)
\]

On the other hand since the moduli space is defined in terms of symplectic geometry one gets that every symplectomorphism \(\phi\) of \((M, \omega)\) induces an automorphism \(\tilde{\phi}\) of \(B^{hw,1}_S\) which preserves all the structures \(\Omega, I, G\) (see [2], [3]). This gives us a dynamical correspondence

\[
\Theta_{BS} : C^\infty(M \to \mathbb{R}) \to Vect(B^{hw,1}_S)
\]

which satisfies the correspondence principle

\[
\Theta_{BS}\{f, g\}_\omega = [\Theta_{BS}(f), \Theta_{BS}(g)]
\]

just due to the construction. Moreover the image \(Im\Theta_{BS}\) lies in the subspace

\[
Vect_K(B^{hw,1}_S) \subset Vect(B^{hw,1}_S)
\]

consists of the vector fields which preserve the Kähler structure on the moduli space.

Now a natural question arises: for every smooth function \(f\) on \(M\) these two vector fields are proportional

\[
X_{F_f} = 2\tau \Theta_{BS}(f).
\]

We prove this coincidence in the next section by direct computations. Now we would like to remark that the Main Theorem stated in the Introduction is a consequence of this proposition. Indeed:

1) First of all if \(f\) and \(g\) are two distinct functions on \(M\) then the hamiltonian vector fields \(X_{F_f}\) and \(X_{F_g}\) are the same iff \(f = g + c\) where \(c\) is a constant. But it's
clear from (2) that the constants over \( M \) go to constants over \( \mathcal{B}^{h_{w,1}}_S \). Thus the map (3) is an inclusion.

2) Further, we know that \( \Theta_{BS}(f) \) generates the flow which preserves both the symplectic form \( \Omega \) and the riemannian metric \( G \). It means that \( X_{F_f} \) preserves \( G \) to.

3) And due to the fact that the dynamical correspondence satisfies the correspondence principle one gets that

\[
\{ F_f, F_g \} \Omega = 2\tau F_{\{f,g\}} \omega. \tag{4}
\]

We have to mention here that formula (4) has been proved in [10], [11] directly. However now the dynamical correspondence gives us the possibility to reprove this known fact and prove new facts listed above. As well one can see that real parameter \( \tau \) plays the role of a multiple of the Planck constant.

Therefore one can see that the statement of the Main Theorem comes directly from the dynamical correspondence. It reflects just geometrical naturalness of the relationship between infinitesimal symplectic deformations induced by functions \( f \) and \( F_f \) on symplectic manifold \( M \) and Kähler manifold \( \mathcal{B}^{h_{w,1}}_S \) respectively.

At the same time one can see that we have just skipped the discussion for the case of higher level \( k \) (see [2]). It’s easy to see that the construction can be extended to the case of any level. Here one wants to relate our real parameter \( \tau \) appeared in the definition of \( F_f \) and consequently in (4) with \( k \). Since \( k \) is the inverse of the Planck constant (see f.e. [2]) then one could take

\[
\tau = \frac{1}{2k}
\]

going right correspondence during \( BPU \) - map.

\section{4. Computations}

First of all let us recall that the tangent space \( T(S_0, \theta_0) \mathcal{B}^{h_{w,1}}_S \) is modeled by pairs

\[
(\psi_1, \psi_2), \quad \psi_i \in C^\infty(S_0 \to \mathbb{R})
\]

such that

\[
\int_{S_0} \psi_i \theta_0^2 = 0
\]

(see [2], [3]). Now we compute the components of \( \Theta_{BS}(f) \) as follows. The hamiltonian vector field \( X_f \) over \( M \) near \( S_0 \) which is a lagrangian cycle can be decomposed into two parts

\[
X_f = V_f + W_f
\]

where

\[
V_f = \omega^{-1}(d(f|_{S_0}))
\]

and

\[
W_f = X_f - V_f
\]
is parallel to \( S_0 \) (more rigorously \( W_f \) belongs to \( TS_0 \)). As well one can see that

\[ W_f = (\omega^{-1}(df))|_{S_0}. \]

We understand \( V_f \) as "outer" part of \( X_f \) with respect to \( S_0 \) while \( W_f \) is "inner" part whose flow preserves \( S_0 \). Then \( V_f \) gives us the corresponding deformation of \( S_0 \) itself while \( W_f \) deforms \( \theta_0 \) over \( S_0 \). It means that \( \Theta_{BS}(f) \) has the following components

\[
\psi_1(S_0, \theta_0) = f|_{S_0} - c, \\
\psi_2(S_0, \theta_0) = \frac{\text{Lie}_{W_f} \theta_0}{\theta_0}
\]

where \( c \) is the normalized constant

\[ c = \int_{S_0} f|_{S_0} \theta_0^2. \]

The Lie derivative contained in the second equality can be understood as follows. The square \( \theta_0^2 \) gives a volume form \( d\mu_0 \) on \( S_0 \) and one can take the Lie derivative

\[ L_f = \frac{\text{Lie}_{W_f} d\mu_0}{d\mu_0} \]

divided by \( d\mu_0 \). Then

\[ \text{Lie}_{W_f} (\theta_0^2) = 2 \text{Lie}_{W_f} \theta_0 \cdot \theta_0 \]

and consequently

\[ L_f = 2 \frac{\text{Lie}_{W_f} \theta_0}{\theta_0} = 2 \psi_2(S_0, \theta_0). \]

Now let us study the hamiltonian vector field \( X_{F_f} \) over \( B^{huv,1}_S \). The differential \( dF_f \) has the form

\[
dF_f(S_0, \theta_0)(\alpha, \beta) = \tau \int_{S_0} f|_{S_0} 2 \beta \theta_0^2 + \tau \int_{S_0} d\alpha((\omega^{-1}(df))|_{S_0}) \theta_0^2. \tag{5}
\]

Due to simplicity of the symplectic form (see [2], [3]):

\[
\Omega(S_0, \theta_0) < (\alpha, \beta), (\gamma, \delta) >= \int_{S_0} (\alpha \delta - \beta \gamma) \theta_0^2 \tag{6}
\]

one can immediately "convert" the first part

\[ \int_{S_0} f|_{S_0} \beta \theta_0^2 = \Omega < (\psi'_1, 0), (0, \beta) > \]

such that

\[ \psi'_1 = 2\tau f|_{S_0} - c' \]
where $c'$ is the normalized summand as above. Further, the second summand in (5) can be rearrange as follows. First of all $(\omega^{-1}(df))|_{S_0}$ is exactly the "inner" part $W_f$ of the hamiltonian vector field $X_f$. Then one has

\[
\int_{S_0} d\alpha(W_f)\theta_0^2 = \int_{S_0} d\alpha \wedge i_{W_f}d\mu_0 =
\]

\[-\int_{S_0} \alpha d(i_{W_f}d\mu_0) = -\int_{S_0} \alpha \frac{dW_f d\mu_0}{d\mu_0} \theta_0^2
\]

where we use the integration by parts. Substituting to (6) we get

\[
\psi_2' = \tau \frac{\text{Lie}_{W_f}d\mu_0}{d\mu_0} = 2\tau L_f.
\]

Comparing $(\psi_1, \psi_2)$ and $(\psi_1', \psi_2')$ one gets

\[
X_{F_f} = 2\tau \Theta_{BS}(f)
\]

and the proof of the proposition is completed.

§5. ALAG - Quantization

Thus we have seen that the moduli space $B_{S}^{hw,1}$ can be regarded as the quantum phase space for the ALAG - quantization of given symplectic manifold $(M, \omega)$. Over this infinite dimensional Kähler (moreover it’s an algebraic) manifold endowed with symplectic form $\Omega$, integrable complex structure $I$ and the corresponding riemannian metric $G$ one has the space $\mathcal{N}$ of quantum observables consists of all smooth real functions whose hamiltonian vector fields preserve both the symplectic structure and the riemannian metric. Inside the space $\mathcal{N}$ one has subspace of quantized observables

\[
\{F_f\} \subset \mathcal{N}
\]

which is isomorphic as a Lie algebra (up to scaling depending on $\tau$) with $C^\infty(M \to \mathbb{R})$. Thus with the Main Theorem in hands we can say that we’ve performed kinematics and dynamical data of the quantum theory. It remains to define — probabilistic (or measurement) aspects of the theory.

This question includes as well state reduction procedure. And the problem arises in this way is based on the following fact: our infinite dimensional Kähler quantum phase space is noncompact. It implies that some quantum observables have not critical points (= eigenstates) at all in $B_{S}^{hw,1}$. This makes any measurement process impossible for such observables. So one has to construct an appropriate compactification of the moduli space such that for any special function $F_f$ it were a set of critical points. There are a number of usual ways to compactify the space (f.e. the most common is the Gel’fand approach) but here there is some proper way which begins with the considerations of critical points of special functions. Below we outline this way leaving all details to [12]. At the end we discuss two reductions of the method in cases when either real or complex polarization is fixed over $M$.

From formula (5) we have the following simple geometrical fact.
Proposition 2. A point \((S_0, \theta_0) \in \mathcal{B}^{hw,1}_S\) is a critical one for special function \(F_f\) if and only if the hamiltonian vector field \(X_f\) over \(M\) of the original function \(f\) preserves this pair:

\[
f|_{S_0} = \text{const},
\]

\[
\text{Lie}_{X_f} \theta_0 = 0.
\]

Moreover for critical set of any special function \(F_f\) one has very strong geometrical properties. If \(\text{Crit}_i(F_f)\) is a connected smooth component of the critical set of \(F_f\) then

Claim. The submanifold \(\text{Crit}_i(F_f)\) is a complex submanifold of \(\mathcal{B}^{hw,1}_S\).

The proof (contained in [12]) is based on the following description of a tangent vector to \(\text{Crit}_i(F_f)\). A pair \((\psi_1, \psi_2)\) represents the tangent vector iff

\[
\text{Lie}_{X_f} \psi_i = 0.
\]

The symmetry in the condition means that the tangent space to \(\text{Crit}_i(F_f)\) in a smooth point is a complex subspace of \(T\mathcal{B}^{hw,1}_S\).

Now the description of critical points of \(F_f\) in terms intrinsic to the symplectic geometry of the original manifold \(M\) hints how one can complete the moduli space. Roughly speaking together with lagrangian Bohr-Sommerfeld cycles one can consider all isotropic with respect to \(\omega\) subcycles of fixed topological types. Moreover one should require for these subcycles to be intersections of lagrangian Bohr-Sommerfeld cycles with smooth submanifolds of topological types \(D, D^2, \ldots, D^n\) where \(D \in H_{2n-2}(M, \mathbb{Z})\) is Poincare dual to the fixed "symplectic" class \([\omega]\). Let us take a "divisor" \(Y \subset M\) representing \(D\) such that \(Y\) is a symplectic submanifold (one has to impose this condition as we see below). Now let \(\mathcal{B}^{hw,1}_{S,Y}\) consists of all intersections \(S \cap Y\) with some corresponding half weights (one can realize this moduli space in quite natural way: it is the moduli space of half weighted Bohr-Sommerfeld cycles of the fixed volume over symplectic manifold \(Y\) endowed with induced prequantization equipment). Then there exists a smooth function \(f_Y\) over \(M\) such that

1) the corresponding special function \(F_{f_Y}\) has not critical points in \(\mathcal{B}^{hw,1}_S\) at all but

2) has critical points in the compactification component \(\mathcal{B}^{hw,1}_{S,Y}\).

It means that the induced hamiltonian flow contracts the moduli space \(\mathcal{B}^{hw,1}_S\) to the boundary component \(\mathcal{B}^{hw,1}_{S,Y}\). But the last one is not compact itself hence one has to continue the process iterating to the top component which is a "Hilbert scheme" of \(M\).

Really let us take any smooth section \(s \in \Gamma(L)\) whose zero set coincides with the "divisor":

\[
(s)_0 = Y.
\]

Then

\[
f_Y = <s, s>
\]

is a nonnegative smooth function on \(M\). This function is not constant being restricted on every lagrangian cycle and this gives us statement 1) above. And it's
not hard to check that every point of $\mathcal{B}_{S,Y}^{hw,1}$ is stabilized by the induced flow $X_{f_Y}$. Really our function $f_Y$ is identically zero on $Y$ and moreover $Y$ is a component of the critical set $Crit(f_Y)$ of the function. It means that $df_Y$ vanishes at each point of $Y$ hence the same is true for the hamiltonian vector field. Thus the corresponding action on each point $(S', \theta') \in \mathcal{B}_{S,Y}^{hw,1}$ is trivial and this remark gives us statement 2) above.

Now one can repeat the arguments for the boundary component $\mathcal{B}_{S,Y}^{hw,1}$ using a function which doesn’t admit any invariant $n-1$-dimensional isotropic submanifold, coming to isotropic $n-2$-submanifolds etc. etc. and iterating the process one gets a tower of moduli spaces compactifying the original one where the "top" component just corresponds to sets of points of $M$.

A compactification component $\mathcal{B}_{S,Y}^{hw,1} \subset \partial \mathcal{B}_{S,Y}^{hw,1}$ admits natural Kähler structure which is compatible with the original structure on $\mathcal{B}_{S,Y}^{hw,1}$. The same is true for each compactification component. Thus one can expect that the compactified moduli space $\overline{\mathcal{B}}_{S,Y}^{hw,1}$ is an infinite dimensional Kähler manifold. The definition of special function $F_f$ can be easily extended to every compactification component and hence to the compactified moduli space. The dynamical correspondence again ensures that for these extended functions all the statements of the Main Theorem still hold. This means that one gets a real way to quantize classical mechanical systems in the framework of ALAG - programme. And it’s more then reasonable to call this approach as ”ALAG - quantization”.

§6. Polarizations

Here we discuss two examples when ALAG - quantization can be reduced to known ones.

**Real polarization.** Let us suppose that $M, \omega$ is a completely integrable system. It means that there are exist $n$ smooth non constant function $f_i, i = 1, ..., n$ in involution

$$\{f_i, f_j\}_\omega = 0 \quad \forall i, j$$

defining lagrangian fibration

$$\pi : M \to \Delta$$

where $\Delta$ is a convex polytope in $\mathbb{R}^n$ and for any inner point $t \in \Delta \setminus \partial \Delta$ the corresponding fiber $\pi^{-1}(t)$ is a smooth lagrangian cycle. The boundary $\partial \Delta$ represents degenerations: over a point of a hyperplane in $\partial \Delta$ one has isotropic smooth submanifold of dimension $n-1$ etc. (and over vortices of $\Delta$ one has just points). The known quantization procedure has as the Hilbert space $\mathcal{H}$ in this situation the following direct sum. Namely let us take all the lagrangian fibers which are Bohr - Sommerfeld with respect to a prequantization quadruple over $M$. One expects ([7], [8]) that there are finitely many such fibers. Denote these fibers as $S_i, i = 1, ..., l$. Then

$$\mathcal{H} = \sum_{i=1}^{l} \mathbb{C} < S_i >$$

(for details see [7]).
What we get in this situation applying ALAG - programme? Let us construct $\mathcal{B}_{S}^{h_{w},1}$ where $S$ is the class of the fiber in lagrangian fibration. Let us take the induced special functions

$$F_1, ..., F_n \mid F_i = F_{f_i}.$$ 

Let us take the following intersection

$$\mathcal{P} = \text{Crit}(F_1) \cap ... \cap \text{Crit}(F_n) \subset \mathcal{B}_{S}^{h_{w},1}.$$ 

Then this set is the set of mutual eigenstates of quantum observables $F_1, ..., F_n$.

And we have the following

**Proposition 3.** The set $\mathcal{P}$ is a double covering of the set of Bohr - Sommerfeld fibers $\{S_i\}$.

This means that one can reconstruct $\mathcal{H}$ in terms of ALAG. Namely, the set $\mathcal{P}$ is a set of points. If we take the following direct sum

$$\mathcal{H}_{\mathbb{R}} = \sum_i \mathbb{R}p_i$$

then the natural antiholomorphic involution on $\mathcal{B}_{S}^{h_{w},1}$ induces an involution on $\mathcal{P}$ and consequently a complex structure on $\mathcal{H}_{\mathbb{R}}$. This complex space is canonically isomorphic to $\mathcal{H}$.

**Proof.** We have to show that if $S_i$ is a Bohr - Sommerfeld fiber then there exists such half weight $\theta_i$ that pair $(S_i, \pm \theta_i)$ is a critical point of $F_j$ and vice versa. Turning to the Proposition 2 one sees that this fact is quite obvious — for every $j$ our original function $f_j$ is constant along the fiber $S_i$ so it remains to find an appropriate invariant with respect to $W_{f_j}$ half weight $\theta_i$. To do this firstly let us find an invariant volume form $d\mu_i$ over $S_i$. Over $S_i$ we have $n$ nonvanishing pointwise independent hamiltonian vector fields which are parallel to $S_i$:

$$X_{f_1}, ..., X_{f_n}.$$ 

Let us take the corresponding set of the differentials

$$df_1, ..., df_n$$

and perform the top wedge product

$$\eta = df_1 \wedge ... \wedge df_n$$

choosing an appropriate order. This $n$- form is totally zero being restricted to $S_i$ but one can take a $n$ - form $\eta'$ defined by

$$d\mu = \eta \wedge \eta'$$

at each point of $S_i$ where $d\mu$ is the usual symplectic volume form. Of course this form $\eta'$ is not uniquely defined by (7) but its restriction to $S_i$ is unique indeed. One takes

$$d\mu_i = c\eta'|_{S_i},$$
where \( c \) is the normalizing constant. Since all forms which were used in the description are invariant under the Hamiltonian flow induced by each \( f_i \) one gets that the volume form \( d\mu_i \) looks like quite canonical candidate to be the square of the half weight which we want to construct. It remains to mention that there are exactly two half weights over \( S_i \) which give the same form \( d\mu_i \). Thus we get a pair of conjugated points \((S_i, \pm \theta_i) \in B^{hw}_S\) which are critical for every \( F_j \). Moreover there are no other choices of invariant half weights over \( S_i \). One can check that if there exists a half weight \( \theta'_i \) which is invariant under each \( f_i \) then the ratio

\[
\psi = \frac{\theta'_i}{\theta_i} \in C^\infty(S_i \to \mathbb{R})
\]

is a smooth function which should satisfy

\[
L_{W_{f_j}} \psi = 0
\]

for every \( j = 1, \ldots, n \). But again the set of Hamiltonian vector fields \( W_{f_j} \) forms a complete system such that the condition (8) implies that \( \psi \) is a constant. But the normalization condition

\[
\int_{S_i} \theta'_i = 1
\]

means that this constant is equal to \( \pm 1 \).

And in the other direction: if \((S_0, \theta_0)\) is a mutual critical point for every \( F_j \) then removing the half weight part one gets a Bohr-Sommerfeld cycle \( S_0 \) such that for every \( j \) the function \( f_j \) is constant on \( S_0 \). But it means that \( S_0 \) is a fiber and the proof is completed.

**Remark.** One can easily see now why the set of Bohr-Sommerfeld fibers has to be at least discrete. Indeed, if one has a nonisolated Bohr-Sommerfeld fiber \( S_0 \) corresponds to an inner point \( t_0 \) of \( \Delta \)

\[
t_0 \in \Delta \setminus \partial \Delta
\]

then it would be a nonisolated critical point for any special function \( F_j \). This means that there exists a smooth function \( \psi \in C^\infty(S_0 \to \mathbb{R}) \) such that for any original \( f_i \) one has

\[
\text{Lie}_{W_{f_i}} \psi = 0 \quad i = 1, \ldots, n.
\]

Again using the fact that vector fields \( W_{f_i} \) form complete system over \( S_0 \) we get that \( \psi \) has to be constant. These arguments can be refined to the case of the faces of \( \Delta \). The point is that a sequence of smooth Bohr-Sommerfeld fibers can converge a priori to a boundary point representing an isotropic submanifold \( S' \) of dimension less than \( n \). Let us suppose that this limit cycle belongs to a face of \( \partial \Delta \) of dimension \( n - 1 \). Then it has to be a point of our compactification (see Section 5) thus one can deduce that it could not be a limit point for such a sequence. Indeed otherwise there exists a function \( \psi' \) over the submanifold \( S' \) satisfies

\[
\text{Lie}_{W_{f_i}} \psi' = 0, \quad i = 1, \ldots, n - 1
\]
where we change the order for \( f_i \) such that the face containing \( S' \) corresponds to say maximal value of \( f_n \). This function gives us the partial deformation of \( S' \) to the \( n \)-dimensional "resolution". But again this \( \psi \) has to be constant. Therefore one has only two possibility for any sequence of regular Bohr - Sommerfeld fibers: it is either finite or converge to a vortex of \( \partial \Delta \). The last case can be removed in some particular cases.

Thus one can understand the quantization with real polarization as a reduction of ALAG - quantization.

**Complex polarization.** The usual way to quantize a classical system using a complex polarization has been discussed in Section 2. Here we use the approach proposed in [6] combining together the methods of Souriau - Kostant and Berezin (see section 2 above). So let \( M, \omega \) is endowed with a compatible integrable complex structure \( I \) and we take the space of holomorphic section \( H^0(M_I, L) \) as \( \mathcal{H} \). It can be translated (see section 1, 2 above) to the language of projectivization. Then one has

\[
\mathcal{P} = \mathbb{P}H^0(M_I, L)
\]

as the phase space of quantized system.

Again one can relate two phase spaces using so called BPU - map (see [2], [3]):

\[
BPU : \mathcal{B}_S^{hw,1} \to \mathcal{P}.
\]

Thus the first quantum space is fibered over the second one. Now if \( f \) is a quantizable function (see [6]) then one could compare induced quantum observables \( F_f \) and \( \tilde{Q}_f \) over the quantum phase spaces. Postponing any complete computations we just claim that the following fact takes place.

**Proposition 4.** The critical set \( \text{Crit}(F_f) \) is embedded by BPU - map to the critical set \( \text{Crit}(\tilde{Q}_f) \):

\[
BPU(\text{Crit}(F_f)) \subset \text{Crit}(\tilde{Q}_f).
\]

The proof is quite familiar — we again apply the dynamical correspondences for \( \mathcal{B}_S^{hw,1} \) and \( \mathcal{P} \) respectively just noting that BPU - map is invariant under any infinitesimal deformation generating by \( f \) if \( f \) is a quantizable function (recall that it means that \( X_f \) preserves the complex structure and hence acts as an infinitesimal automorphism of \( \mathcal{P} \)). Therefore due to the dynamical correspondence one has that if \( f \) is a quantizable function then

\[
dBPU(X_{F_f}) = K(\tilde{Q}_f)
\]

where \( K \) denotes the Killing vector field. This gives us the statement together with a number of consequent remarks. We’d like just mention here that one which makes the reduction mentioned above. Namely if \( F_f \) has an eigenstate at \((S_0, \theta_0)\) then it implies that \( \tilde{Q}_f \) has an eigenstate at

\[
p_0 = \pi(S_0, \theta_0) \in \mathcal{P}.
\]

So again one can reduce ALAG - quantization to the known setup.
Remark. For a generic function $f$ two critical sets $\text{Crit}(F_f)$ and $\text{Crit}(\tilde{Q}_f)$ don’t coincide modulo $\text{BPU}$ - map. The point is that in this general situation the differential $d\text{BPU}$ ”kills” the hamiltonian vector field $X_{F_f}$ in some points covering critical points ”down stair” (see formula (9)). But if the coincidence doesn’t appear for a smooth function $f$ then one can deform this function such to that for which the coincidence

$$\text{BPU}(\text{Crit}F_f) = \text{Crit}(\tilde{Q}_f)$$

takes place.

On the other hand it’s clear that $\text{BPU}$ - map as it is considered now is a linearization of non linear quantum mechanical system which is our ALAG - quantum mechanical system. At the same time the case with real polarization above can be regarded as a linearization as well. Thus these types of well known quantizations are just two genuinely different linearizations of the ALAG - problem.

Moreover one could try to exploit ALAG - quantization as a link between two different cases of polarized mechanical system. Namely if one has over $M, \omega$ simultaneously real and complex polarizations then $\mathcal{B}_{\mathcal{S}}^{\text{hw},1}$ is an universal object which endows us with a relationship. The constructions of the present section give quite definite way to compare two known quantizations. Namely if over a symplectic manifold $(M, \omega)$ one has both real and complex polarizations then one can relate two Hilbert spaces using our constructions. For this it’s sufficient to find an appropriate smooth function $f_0$ over the original symplectic manifold such that:

1) $f_0$ is an algebraic combination of the integrals $f_1, ..., f_n$ of the system;

2) the induced special function $F_{f_0}$ on the moduli space $\mathcal{B}_{\mathcal{S}}^{\text{hw},1}$ has as the critical set precisely the set of the points which correspond to Bohr - Sommerfield fibers;

3) the corresponding Töplitz operator $T(f_0)$ on $H^0(M_I, L)$ has pairwise different eigenvalues (and of course let $f_0$ preserves the fixed complex structure).

Then applying Propositions 3 and 4 one gets an isomorphism between the corresponding Hilbert spaces. Let us remark that the ambiguity $\pm$ in taking of the half weight part is killed by $\text{BPU}$ - map. At the same time since the critical values of $\tilde{Q}_{f_0}$ are different then the critical points of $\tilde{Q}_{f_0}$ are isolated. It means that one has a preferred basis in $H^0(M_I, L)$, defined by $f_0$. Proposition 4 then gives us that the images of $(S_i, \pm \theta_i)$ belong to this critical set. Now one can formulate the special conditions for this $f_0$ to investigate the isomorphism. If the space of quantizable function is big enough one can deform a given $f_0$, satisfying the conditions 1), 2), 3) above, such that Proposition 4 could be exploited in the opposite direction thus

$$\text{BPU}(\text{Crit}F_f) = \text{Crit}(\tilde{Q}_f)$$

and this would give the isomorphism.

**Conclusion**

As it was pointed out at the end of Section 2 the qualitative difference between classical and quantum dynamics is in presence of an appropriate riemannian metric in the last ones considerations. It plays crucial role in the measurement process which is a distinguished part of the quantum theory. At the same time for any quantum observables considered as a smooth function one has a precise value at
each quantum state which is a point of our projective space. This fact makes some confusion in the application of the geometrical formulation of quantum mechanics. Thus it would be quite reasonable to derive some partial version of what was proposed where some special "things" will play the role of quantum observables. It would be a kind of functions on a super manifold — such functions have not "real" values at points. Below we show that in the framework of ALAG - programme one has some reduction of the ALAG - quantization to a subject including these "things".

Our moduli space \( B^{hw,1}_S \) is fibered over a real smooth manifold \( B_S \) which is the moduli space of original "unweighted" Bohr - Sommerfeld cycles (see [2], [3]):

\[
q : B^{hw,1}_S \to B_S, \quad q(S, \theta) = S.
\]

This fibration is a Lagrangian fibration. Every smooth function \( f \in C^\infty(M \to \mathbb{R}) \) defines a vector field \( Y_f \) on \( B_S \) due to the following arguments. At each \( S \in B_S \) the restriction

\[
\psi = f|_S
\]

gives us a tangent vector

\[
\psi \in T_S B_S = C^\infty(S \to \mathbb{R})/\text{const}.
\]

Generalizing this remark over whole \( B_S \) one gets the vector field \( Y_f \). On the other hand as usual we have a dynamical correspondence

\[
\Theta_S : C^\infty(M \to \mathbb{R}) \to \text{Vect}(B_S)
\]

due to the fact that every symplectomorphism of \( M, \omega \) acts as an automorphism of \( B_S \). It’s a weaker version of Proposition 2 which states that

\[
Y_f = \Theta_S(f).
\]

It implies that map \( \Theta_S \) preserves the Lie algebra structure.

For every function \( f \) one has not any value of this function on a Bohr - Sommerfeld cycle \( S \in B_S \) unless \( f \) is constant alone the submanifold. It means that one has a "measurement" process for \( \tilde{Y}_f \) over \( B_S \) in points where the vector field \( Y_f \) vanishes. Here we denote as \( \tilde{Y}_f \) the "thing" which we would like to define. For these "things" we have a Poisson bracket:

\[
\{\tilde{Y}_f, \tilde{Y}_g\}_B = [\tilde{Y}_f, \tilde{Y}_g].
\]

It’s quite easy to check that this one satisfies all usual conditions. It can be done using the following obvious equality

\[
\{\tilde{Y}_f, \tilde{Y}_g\}_B = \tilde{Y}_{\{f,g\}_\omega}.
\]

Now we can recognize what are these "things" which correspond to smooth real function over \( M \). Namely (see f.e. [4]) each \( \tilde{Y}_f \) looks like a function on an odd
supersymplectic manifold. This manifold is constructed over $\mathcal{B}$ using usual procedure (see [4]) which ”twist” the standard symplectic structure on $T^*\mathcal{B}_S$ getting an example of the supersymplectic structure on

$$\Pi T^*\mathcal{B}_S.$$ Functions in this setup are represented by multivector fields — as well as $\tilde{Y}_f$ has the representation by $Y_f$ over $\mathcal{B}_S$ for every smooth function $f$. Then one can see that the Poisson bracket $\{.,.\}_B$ defined above is nothing but so called Buttin bracket (see [4]). Our ”quantized observables” $\tilde{Y}_f$ are distinguished by the condition that they are represented by vector fields (so they are of degree 1 in odd variables).

We are not ready to go further, leaving this story with super symplectic geometry. We just have to mention that ALAG - programme is in some sense based on super symplectic geometry. Really one could derive that the notion of Bohr - Sommerfeld lagrangian cycle is absolutely equivalent to the notion of lagrangian cycle in some appropriate even super symplectic manifold. This super symplectic manifold is nothing but the principle bundle $P \to M$ associated to our prequantization hermitian line bundle $L \to M$ endowed with our prequantization connection $a$. Over this principle bundle $P$ one has an even super symplectic form $\omega_e$ defined as follows: for every pair of tangent to $P$ vectors $v_1, v_2$ in a point $p \in P$ one can decompose these ones with respect to the fixed connection into horizontal and vertical part. Then $\omega_e$ acts on the horizontal parts as usual $\omega$ (in skew symmetric style) while on the vertical parts as the riemannian pairing (in symmetric style). Now one can say what is a Planckian cycle $\tilde{S} \in P$ (the definition see in [2]): it is just a lagrangian cycle with respect to even super symplectic form $\omega_e$.

Thus one could say that ALAG - programme is based on the super symplectic geometry. It means that one should expect some new results comparing ALAG - constructions with constructions belong to this super symplectic geometry as well as a generalization of ALAG (”non abelian lagrangian algebraic geometry”) in the setup of super symplectic geometry.

References

[1] A. Ashtekar, T. Schilling, Geometrical formulation of quantum mechanics, ArXiv: gr - qc / 9706069.
[2] A. Gorodentsev, A.Tyurin, ALAG, Preprint Max- Planck- Inst.(Bonn), N. 00 - 7.
[3] A. Gorodentsev, A. Tyurin, Abelian lagrangian algebraic geometry, submitted to Izvestya RAN.
[4] O.M. Khudaverdian, Semidensities on odd symplectic super manifolds, arXiv:math.DG/0012256.
[5] N.Hurt, Geometric quantization in action, D.Reidel Publishing Co (1983).
[6] J.Rawnsley, M.Cahen, S.Gutt, Quantization of Kaehler manifolds 1, JGP Vol. 7, N. 1 (1990), 45 - 62.
[7] J. Sniatycki, Geometric quantization and quantum mechanics, Springer (1987).
[8] A. Tyurin, On Bohr - Sommerfeld bases, arXiv:math.AG/9909084.
[9] A. Tyurin, Complexification of Bohr - Sommerfeld condition, preprint Math. Inst. Univ. of Oslo, No. 15 (1999).
[10] N.Tyurin, Hamiltonian dynamics on the moduli space..., preprint MPI (Bonn), 00 - 106.
[11] N.Tyurin, The correspondence principle in abelian lagrangian geometry, submitted to Izvestya RAN.
[12] N. Tyurin, *Compactification of the moduli spaces in ALAG*, in preparation.

MIIT

E-mail address: tyurin@tyurin.mccme.ru ntyurin@newton.kias.re.kr