Application of the Dupin cyclide in temple architecture

N A Salkov
Moscow State Academic Art Institute named after V. I. Surikov, Moscow, Russia

Abstract. In recent years, the construction of churches and temples has been rapidly developed, so the need for design and support for such constructions arose. A new department called “Temple architecture” even appeared at MArchI (Moscow Architectural Institute). The article considers a method for geometric construction of the surface of such objects as the dome of temples and churches from the parts of Dupin cyclides. As a result, two variants of domes made from the compartments of the Dupin cyclide were proposed. Based on the theoretical research, it becomes clear that there can be more design options, and if we use a special case of cyclides for construction - compartments of rotating cones - then many times more.

Keywords: Dupin cyclide, temple architecture, surface design, descriptive geometry, modeling.

1. Introduction
In recent years, interest has been arisen [1–5] in such a remarkable surface as Dupin cyclide [6, 7]. On the one hand, these surfaces are the only only cyclic, whose focal surfaces degenerate into two curves — conics, and therefore (and not only) they are interesting from a geometrical point of view.

On the other hand, Dupin cyclides can be used in various spheres: in the design of pipelines of different diameters and directions [8], in architecture and construction in the form of covering from compartments of Dupin cyclides [8], in computer modeling, the formation of conic surfaces [7]: methods for construction of conics proposed in works [9–10] along with V.A. Korotkiy’s works [11; 12] can be used in computer simulation.

The use of special cases of a Dupin cyclide in the construction of coverings in the educational process is also shown in the work [8]. The use of the Dupin cyclide for solving Apollonius’ and Fermat’s problems [8] is shown in the work [3]. This can solve dense laying problems in an automated mode. In the work [13], based on the properties of the Dupin cyclide, all possible contact methods were proposed to achieve first-order smoothness when joining circumferences of different diameters from zero to infinity. The listed possibilities of using Dupin cyclides are not limited to this.

Since tori, conical and cylindrical surfaces of rotation (figure. 1) are particular cases of the Dupin cyclide, the possibilities of using the surface increase many times over, and in this work it is impossible to list them all.

But there is another direction in the application of Dupin cyclides - its use in temple architecture.

Currently, the construction of churches and temples takes on an impressive scale: they are being built everywhere. With the blessing of His Holiness Kirill, Patriarch of Moscow and All Russia, on September 1, 2016, the Department of Temple Architecture was opened at the Moscow Architectural Institute (MArchI).
a) torus  b) rotating cone  c) rotating cylinder

**Figure 1.** Special cases of Dupin cyclides

Figure 2 shows the Hazrat Sultan Mosque: its image (figure 2, a) and a computer model (figure 2, c). The dome consists of a spherical belt and a compartment of a rotating cone (figure 2, b).

**Figure 2.** Kazakhstan, Astana, The Hazrat Sultan Mosque

**Figure 3.** The dome of St. Peter's Basilica

**Figure 4.** The Basilica of the Sacre Coeur, Paris

**Figure 5.** The compartment of the torus

Figure 3 shows the dome of St. Peter's Basilica, figure 4 – the Basilica of the Sacre Coeur in Paris. Domes on the cathedrals are made in the form of torus's compartments (figure 5).

All three images show a combination of two different surfaces: figure 6 is a sphere with a torus, figure 7 and figure 8 – adjoining tori.
Figure 6. The onion dome of the Alekseevsky monastery (Sokolniki)

Figure 7. The main dome of the temple of the Alekseevsky monastery

Figure 8. The Cathedral of Christ the Savior

The figure 9 shows the geometric construction of the conjugation of a sphere with a torus: graphical (figure 9, a) and the computer one (figure 9, b).

The figure 10 shows the conjugation of two tori: the graphical construction (figure 10, a) and the computer one (figure 10, b).

Figure 9.

Figure 10.

Figure 11. Rotating sphere-cone

Figure 11 shows another option for constructing onion domes – a combination of conjugated spheres with a rotating cone: figure 11, a gives a graphic image, figure 11, b – computer interpretation.

As you can see, in the architecture of churches and temples, that are built and under construction in the Russian Federation, various configurations of domes, called poppy heads or onions, are used. And all of them are somehow connected with the Dupin cyclide.
Figure 12. Figure 13.

Figure 12 presents one of the onions of the Protection of the Holy Virgin Orthodox Cathedral (St. Basil's Cathedral), which also includes the compartments of Dupin cyclides. On figure 13 these compartments are twisted along a helical line.

2. Formulation of the problem

Such onion domes for temples have been made artificially so far, one can say manually, without applying theoretical knowledge of geometric modeling, which is now applied in any direction of construction and production in the form of engineering geometry. Based on the foregoing, there is a practical need to develop the possibility of using Dupin cyclides in the design of the onions of domes of churches and temples.

3. Theory

We will develop a variant of joining the compartments of the Dupin cyclide for the construction of onion domes of the temples like those presented on figure 12 and figure 13.

3.1. Assignment of Dupin Cyclides with three lines and a sphere

Statement. If we take two lines \( l \) and \( t \) intersecting at the point \( O \), a sphere \( \Omega \) of radius \( R \) centered at the point \( O \), and the line \( a \) intersecting the lines \( l \) and \( t \), then these geometric figures define a Dupin cyclide. Consider this statement as an example.

In the works [7; 14] it is shown that the Dupin cyclide is uniquely defined by its outlines.

Let the lines \( l \) and \( t \) intersecting in the point \( O \) be given (figure 14), a sphere \( \Omega \) of radius \( R \) centered at the point \( O \), and the line \( a \) intersecting the lines \( l \) and \( t \) at the points \( O_1 \) and \( O_2 \), respectively.

The lines \( l \), \( t \), and a make up the plane \( \Sigma \). This sphere of radius \( R \) intersects the lines \( l \) and \( t \) at four points \( M \), \( N \), \( I \) and \( 2 \). Take two points \( M \) and \( N \), and draw another line \( d \) through them (figure 15). The straight line \( d \), intersecting the straight line \( a \), gives the point \( j \), which we will consider the axis \( j \) (figure 15), perpendicular to the plane \( \Sigma \). The axis \( j \) and the line \( d \) make up a plane intersecting a given sphere along the circle \( m \).

Next, draw a circumference: the \( m_1 \) with the center \( O_1 \) of radius \( R_1 = O_1M \) and the circumference \( m_2 \) with the center \( O_2 \) of radius \( R_2 = O_2N \).

The circles \( m_1 \) and \( m_2 \) will be the outlines of the Dupin cyclide, and the segment \( MN \) is the projection of the tangent circle \( m \) of this sphere with the surface of the Dupin cyclide. The straight line \( a \) in this case is the common axis of symmetry of the Dupin cyclide.

Indeed, a Dupin cyclide generally has 10 parameters. We calculate the parameters of these geometric conditions. The plane \( \Sigma \) contains 3 parameters, three straight lines in this plane – 6 more parameters, finally the sphere \( \Omega \), since its center is already defined, has only the remaining parameter – radius \( R \). Total – 10 parameters. Thus, the Dupin cyclide is a given surface.
It is possible to count without the plane Σ. One of the lines has 4 parameters in \( \mathbb{R}^3 \). The second straight line, since it intersects with the first, and therefore one parameter is already fixed, has 3 parameters. The third straight line, in connection with the intersection of the first two and therefore two parameters are already fixed, has 2 parameters free. And, finally, the sphere whose center is defined has only 1 parameter free - the radius \( R \). Total – 10 parameters.

Figure 14. Figure 15.

Thus, it is proved that three mutually intersecting lines and the sphere uniquely define \( \infty^0 \) Dupin cyclides. In the general case, four. In this example, it is the only one since one pair of points is taken: \( M \) and \( N \).

Figure 16.
3.2. Dupin cyclides enveloping the same sphere

Let’s consider figure 16. In accordance with the previously identified properties of the Dupin cyclide [15], the apex \( K \) of the rotating cone in contact with this sphere \( \Omega \) along the circle \( m \) (figure 16, segment \( M_2N_2 \)), which, in turn, touches the Dupin cyclide along the same circle, will belong to the axis \( i^1 \), passing in the plane \( \Sigma \) and perpendicular to the axes \( j^1 \) and \( a^1 \).

Let’s draw another straight line of the form \( a \) (the main axes of Dupin cyclides) — the straight line \( a^2 \) (figure 16). We construct the second Dupin cyclide according to the constructions already indicated above. As you can see, both cyclides are touching along the circle \( m \) (figure 16, segment \( M_2N_2 \)), their axes \( i^1 \) and \( i^2 \) intersect at the point \( K \). The line of intersection of order 16 splits into a circle and a curve of order 16.

Based on the construction, we have a smooth transition of the first order of smoothness between two Dupin cyclides. We use this property to create surfaces from the compartments of the Dupin cyclide, as shown in [15].

3.3. Modeling of the onion dome of the temple from the compartments of Dupin Cyclides

To model the onion dome of the temple from the compartments of Dupin cyclides, similar to how one of the domes of St. Basil’s Cathedral (figure 12, dark inserts), it is necessary that the axis \( i \) be vertical (figure 17). In this case, the upper part of the dome will be the closest to the traditional aesthetic perception of the dome. The lower part can have both a vertical axis \( i \) and an inclined one.

Figure 18 shows one of the options for joining the compartments of the Dupin cyclide to create a so-called poppy head. Figure 19 shows the variants of the designed poppy head's surface: from the outer compartments of the Dupin cyclide (figure 19, a) and from the inner compartments (figure 19, b).

Both axes of the Dupin cyclide (both the \( i^1 \) axis and the \( i^2 \) axis) are vertical, and in this example, are the same. Therefore, the point \( K \) — the apex in contact with the sphere common to both cyclides — cannot be determined.

The main axis of symmetry of the both cyclides \( a^1 \) and \( a^2 \) will be parallel, and not be at an acute angle, as in the figure 16.

The lower cyclide is symmetric about the \( i^1 \) axis; for the upper cyclide, the \( i^2 \) axis is tangent to both outline generatrices.

The selected compartments of the Dupin cyclide are shown by the thick lines, so even at this stage you can imagine a poppy head.
4. Discussion of results
The results were presented in the form of a report at the annual All-Russian Scientific Conference with International Participation “Problems of Engineering Geometry”, held at the Russian Technological University (MIREA) on October 21st, 2019, and received a positive assessment.

5. Conclusion
There have been no works related to the temple architecture of the poppy heads of domes, consisting of the Dupin cyclides so far. This work shows the possibility of geometric modeling of the surfaces of poppy heads from the compartments of Dupin cyclides with the possibility of applying different variants. The proposed options are not limited to the possibility of geometric modeling of poppy heads. There can be much more options, especially since the combinations of cyclides and rotating cones are not included in this work.

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