Holography in Gödel–type spacetimes and their generalizations

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Abstract. In the contribution we review some facts concerning holography in the Gödel-type universes, adopting the approach presented in the paper Boyda at all [1]. Then the analysis is extended to an inhomogeneous cylindrically symmetric solution, found in framework of Einstein–Maxwell–dilaton gravity, which violates the causality. In particular, the question is raised, whether holographic screens shield the closed timelike curves from an observer, thus providing a chronology protection mechanism.

1. Introduction
Holographic principle plays an important role in contemporary string theory, but it affects also considerations coupled with the classical general relativity. The classical relativity theory is widely known for a possibility of causality violation. Perhaps one of the earliest and the most famous is Gödel solution. In attempt to resolve the problem of chronology violation, a number of ways has been propounded [2, 3]. The holographic principle, although not yet completely understood, provides an approach how to find a chronology enforcement mechanism. In the paper [1], by using the holographic entropy bound, a possible reasoning was put forward, which respects the causality in Gödel universe.

We start with a very brief exposition of Gödel–type solutions [4]. They form stationary cylindrically symmetric and spacetime homogeneous spacetimes whose underlying manifolds have topology \( \mathbb{R}^4 \). The metric tensor can be in the usual cylindrical coordinates written down as

\[
ds^2 = ds_3^2 + dz^2,
\]

where the non–trivial part \( G_3 \) has the metric

\[
ds^2 = dt^2 + \frac{8\Omega}{m^2} \text{sh}^2 \left( \frac{mr}{2} \right) dt d\varphi + \frac{4}{m^2} \text{sh}^2 \left( \frac{mr}{2} \right) \left[ \left( \frac{4\Omega^2}{m^2} - 1 \right) \text{sh}^2 \left( \frac{mr}{2} \right) - 1 \right] d\varphi^2,
\]

with constants \( m \) and \( \Omega \).

The Killing vector \( \partial_\varphi \), which is tangent to circles \( t, r, z = \text{const} \), becomes null for \( r_C = \frac{2}{m} \text{argsh} \left( \frac{4\Omega^2}{m^2} - 1 \right)^{-1/2} \) and timelike for \( r > r_C \), implying that closed timelike curves (CTC) appear in Gödel–type spacetimes once \( 4\Omega^2 > m^2 \). In the original Gödel solution [5] it is \( 2\Omega^2 = m^2 \). Because of the homogeneity for every point of the spacetime a closed timelike circle can be found that passes through it, but every closed timelike curve must intersect outer region \( r > r_C \). Consider a stationary observer located on the time axis, who sends out a congruence
of lightrays. Null geodesics emitted from the origin follow a spiral trajectory. In a finite affine parameter they reach velocity of light surface at \( r = r_C \). Circles at \( r_C \) are closed null circles. Then the lightrays begin to refocus and in the same affine parameter they reach the origin (see the Figure 1 bellow).

2. A review of the covariant entropy bound

We sketch briefly the basic concepts of the holographic principle [6]. Consider a 2–dimensional spacelike surface \( S \) in a spacetime. There are 4 null congruences orthogonal to it, two of them are outgoing and two of them are incoming. Exactly two of them will be non–expanding. In a generic spacetime the expansion \( \theta \) of a surface of codimension 2 is defined by

\[
\theta = h_{\mu\nu} D_\mu \zeta_\nu ,
\]

with \( \zeta_\mu \) being a null vector orthogonal to the surface and \( h_{\mu\nu} \) and \( D_\mu \) being an induced metric and a covariant derivative on it. We say that the non–expanding congruences form light sheet \( L(S) \) of the surface \( S \) so long as their expansion is non–positive. Let a geodesic observer sends out lightrays in all directions from the origin at a fixed time. The holographic screen \( \mathcal{H} \) is reached when an area of a 2–dimensional surface generated by the wave front of the lightrays is maximal. Alternatively one can follow the incoming lightrays into past until they reach the surface when the geodesics no longer expand. This surface again defines the holographic screen. It is obvious that a holographic screen satisfies the property \( \theta = 0 \).

Take a 2–dimensional spacelike surface \( S \) with an area \( A(S) \) and define its light sheet \( L(S) \). According to the covariant entropy bound conjecture an entropy \( S \) on \( L(S) \) satisfies

\[
S(L(S)) \leq \frac{A(S)}{4} .
\]

There is no complete proof of (2), but it is generally accepted. So far, there is no known counterexample of the covariant entropy bound [6].

3. Preferred holographic screens in the Gödel spacetime

Preferred screens in \( \mathcal{G}_3 \). Consider again the observer sending the lightrays into all directions. Because of the rotational invariance all the geodesics will reach the same radial distance in the same affine parameter \( \lambda \) and in the same time \( t \). Holographic screen is obtained by maximizing the area of constant \( t \) and \( r \)

\[
A(r) = \frac{4\pi}{m} \text{sh} \left( \frac{mr}{2} \right) \sqrt{\frac{4\Omega^2}{m^2} - 1} \text{sh}^2 \left( \frac{mr}{2} \right) - 1 .
\]

This yields that the location of the holographic screen \( \mathcal{H} \) is at \( r_\mathcal{H} = \frac{2}{m} \text{argsh} \left[ \frac{1}{\sqrt{2}} \left( \frac{4\Omega^2}{m^2} - 1 \right)^{-1/2} \right] \). It is less than the critical radius \( r_C \) for closed timelike curves, so that the CTC are either shielded from the observer by the screen or they are separated into two causal pieces. The last statement provides us with an idea of the chronology protection mechanism – shielding of CTC by an holographic screen. It is depicted in the following figure, where \( \gamma_1 \) and \( \gamma_2 \) are CTC, first of which is shielded while the other is broken into two pieces.

Preferred screens in Gödel spacetime. It is easy to obtain straightforwardly holographic screens in the 4–dimensional Gödel spacetime by taking the direct product of \( \mathcal{H} \) and \( \mathbb{R} \). This screen is non–compact and corresponds to so called observer “delocalized” on the rotation axis [1]. In addition, there are also compact holographic screens [1]. From the homogeneity of Gödel solutions it follows that other observers see different, but isomorphic screens.
4. Two classes of solutions in Einstein–Maxwell–dilaton gravity

In two subsequent paragraphs two classes of stationary cylindrically symmetric solutions are discussed, that were obtained within framework of Einstein–Maxwell–dilaton (EMD) plus fluid (Euler) gravity. Thus two classes of solutions will be exhibited, which describe a charged, generally rotating, perfect fluid, which is approved to carry a dilatonic charge. In cylindrical coordinates \((t, \phi, z, r)\) both classes of solutions follow from a general stationary cylindrically symmetric line element which takes the form

\[
ds^2 = e^{2\alpha} (dt + df)^2 - l^2 d\phi^2 - e^{2\gamma} dz^2 - e^{2\delta} dr^2 ,
\]

where the unknown functions \(\alpha, \gamma, \delta, f\) and \(l\) depend on \(r\) only. We require the metric (3) to be a solution of \(\varepsilon\)-order EMD(Euler) gravity with the action

\[
S = \int_M \left[ *R + 16\pi * \mu - d\phi \wedge *d\phi + 2 \varepsilon e^{\phi} F \wedge *F \right] ,
\]

where \(\varepsilon\) is a real parameter. Because to tackle the problem exactly is impossible generically, we demand the metric (3) to fulfill the equations of motion following from action (4) up to first order in \(\varepsilon\). Furthermore, \(p, \mu\) and \(\phi\) are a fluid pressure, an energy density and a dilatonic field respectively.

The Maxwell field is supposed to be purely magnetic, so that in the fluid comoving coordinate system the Lorentz force acting on the fluid particles vanishes. It is convenient to define an orthogonal basis of 1–forms as follows

\[
\Theta^0 = e^\alpha (dt + fd\phi) , \quad \Theta^1 = ld\phi , \quad \Theta^2 = e^\gamma dz , \quad \Theta^3 = e^\delta dr .
\]

Excluding the spacetimes with electric currents parallel to the symmetry axis, the electromagnetic 2–form gets the form

\[
F = B_r \Theta^1 \wedge \Theta^2 + B_z \Theta^3 \wedge \Theta^1 ,
\]
and it should satisfy the generalized Maxwell equations

\[-* d * (e^\phi F) = \frac{4\pi}{e} j,\]

(7)

\(j\) being the current density.

**First class.** This is a case with \(\alpha\) being constant, without loss of generality equal to zero. Then we obtain the solution

\[ds^2 = \left(dt + \frac{2\Omega}{C}\gamma d\varphi\right)^2 - l^2 d\varphi^2 - e^{2\gamma} dz^2 - C^{-2}l^{-2}(d\tau^2),\]

\[l^2 = \frac{8\pi \rho}{C^2} e^{2\gamma} - \frac{4\Omega^2 + \phi_1^2 - 4\varepsilon B^2}{2C^2} \gamma + \nu,\]

(8)

with \(\Omega, C, B, \nu\) and \(\phi_1\) being constants whose physical meaning will be now clarified. Note that in an arbitrary (non-constant \(C^2\)) function appears in (8) reflecting the radial rescaling possibility. A simple calculation shows that the fluid particles move along geodesic worldlines. Further the Lorentz force vanishes, so that \(u \wedge * F = 0\) and the Bianchi identity

\[u \wedge * \left[\mu du + d(p u) + \varepsilon \rho e^\phi F\right] = 0,\]

(9)

where \(\rho\) is a charge density, implies that the pressure is constant. The vorticity 1-form, the dilaton and the Maxwell field become

\[\omega = \Omega dz,\]

\[\phi = \phi_0 + \phi_1 z, \quad \phi_0 = \text{const}\]

(10)

\[F = B \exp\left(-\frac{1}{2}\phi_0 - \frac{1}{2}\phi_1 z - \gamma\right) \Theta^3 \wedge \Theta^1, \quad B = \text{const}.\]

(11)

From (10) it is clear that \(\Omega\) physically corresponds to a rotation of the fluid. The constant \(\phi_1\) is the longitudinal part of gradient of the dilaton and enters the stress–energy tensor. Equation (12) gives us an interpretation of \(B\), it measures the longitudinal magnetic field. It can be shown that \(C\) indicates how much the spacetime with metric (3) fails to be spacetime homogeneous. Finally, \(\nu\) is an integration constant, whose value should be determined in order that (3) satisfies axial symmetry condition. Much more detailed reading about properties of the solution (3) as well as other solutions can be found in [7].

**Second class.** In the second class the function \(\gamma\) is set to zero. The resulting metric in this case become

\[ds^2 = e^{2\alpha} (dt + f d\varphi)^2 - l^2 d\varphi^2 - dz^2 - C^{-2}l^{-2}(d\alpha^2),\]

(13)

with \(l\) and \(f\) being given by

\[f = -\frac{\Omega}{C} e^{-2\alpha} + F,\]

\[C^2 l^2 = \Omega^2 e^{-2\alpha} + \frac{1}{4}\phi_1^2 e^{2\alpha} - 2\varepsilon B^2 \alpha^2 + D\alpha + E.\]

(14)

Again an arbitrary function \(\alpha\) appears in (13) corresponding to the possibility of radial rescaling. There are new integration constants \(D, E, F\) which are to be determined in order that (13) represents an axially symmetric spacetime. The dilaton continues to be linearly dependent on the longitudinal direction,

\[\phi = \phi_0 + \phi_1 z,\]

(15)
while the pressure becomes radially dependent

\[ 16\pi p = \left[D - 2\varepsilon B^2 (1 + 2\alpha)\right] e^{-2\alpha} + \phi_1^2. \]  

(16)

Because the pressure is inhomogeneous and the Lorentz force, being vanishing, does not compensate the pressure gradient, the fluid particles do not follow geodesic worldlines. The fluid acceleration 1–form is \( \dot{u} = -d\alpha \) and the vorticity is still given by equation (10). The solution (13) and (14) has a number of interesting properties [8]. Moreover, it turns out that it survives also if the string–inspired theory of gravity is taken into account, in which case the action (4) is enriched by contribution of higher derivatives terms.

5. Holography in the first class

In this paragraph we apply the same approach as we did in the case of Gödel solution. To be concrete we will consider a particular solution found in [9], which reduces to a solution found in [4] in the limit when \( C \to 0 \). It is given by the following appropriate specialization of the integration constants

\[ 16\pi p = -C^2 \nu = m^2, \quad \gamma = \frac{2C}{m^2} \sh^2 \left( \frac{m r}{m} \right), \quad 4\Omega^2 = 4\varepsilon B^2 - \phi_1^2 + 2m^2(1 - C). \]  

(17)

This particular solution is important, since it is axially symmetric and fulfils the elementary flatness condition, and it reduces to a Gödel–type solution (1) if \( C \to 0 \) [9]. Note that it satisfies the energy conditions provided that \( C \leq 0 \). The last inequality implies the chronology violation, which is global, since there are no chronology horizons. We decompose (8), provided that the choice of the integration constants (17) has been done, into the non–trivial part \( G_3 \) and the flat piece along the longitudinal direction \( z \). Consider an observer located at the origin sending out lightrays in all the directions. Because of the symmetry, the lightrays will reach the same radial distance within the same affine parameter. So one can employ again the idea in [1] and investigate a location of an holographic screen by looking for a radial coordinate \( r_H \), which maximizes \( g_{\varphi\varphi} \). It can be shown that the metric function \( g_{\varphi\varphi} \) is maximized for \( r_H \), such that \( r_H < r_C \), where \( r_C \) is the radial coordinate for which \( g_{\varphi\varphi} = 0 \), i.e. it corresponds to the velocity of light surface. More precisely, it can be shown that

\[ r_H \leq \frac{2}{m} \argsh \left( \frac{m}{2\sqrt{2}\Omega} \right), \]

\[ r_C \geq \frac{2}{m} \argsh \left( \frac{m}{2\Omega} \right). \]  

(18)

A location of the screen in the entire spacetime is obtained by cartesian product of 2–dimensional surface \( r_H = \text{const in } G_3 \) times \( \mathbb{R} \). Thus, because every CTC must intersect the region \( r > r_C \), the chronology is protected for an observer delocalized along the rotation axis. A question arises whether there is a compact holographic screen and where it is located, but it will be studied elsewhere [10]. A brief remark on this subject can be found in the next paragraph.

6. Towards the holography in the second class

In the case of the second class (13) it is not possible to look for the holographic screen by using the above decomposition. So thus it is inevitable to search for null geodesic congruence which is generated by a moving observer. The situation is complicated by the fact that the solution is no longer spacetime homogeneous, so that different screens are not isomorphic in general. Further disadvantage consists of a fact that the geodesics cannot be determined explicitly except in
special cases. Nevertheless one can introduce a null tetrad \((l, n, m, \bar{m})\) by

\[
\begin{align*}
    l &= \frac{A}{\sqrt{2}} \left(\Theta^0 + \Theta^1\right), \\
    n &= \frac{1}{A\sqrt{2}} \left(\Theta^0 - \Theta^1\right), \\
    m &= \frac{1}{\sqrt{2}} \left(\Theta^2 + i\Theta^3\right), \\
    \bar{m} &= \frac{1}{\sqrt{2}} \left(\Theta^2 - i\Theta^3\right),
\end{align*}
\]  

(19)

where \(A\) is an arbitrary function. Although neither the null congruence \(l\) nor \(n\) is geodesic, by a sequence of three independent Lorentz rotations can be the tetrad (19) brought in a form, in which a new null vector field \(l\) generates a geodesic congruence [10] (composition of a rotation with \(l\) unchanged, then a rotation with \(n\) left, and finally rotation in \(m, \bar{m}\) plane). Holographic screen then can be determined from vanishing of the expansion scalar. This holds both for the first and the second class of solutions and it motivates our effort to continue the analysis of holography in this type of spacetimes.

7. Conclusion
After a brief review of basic concepts concerning the Gödel-type solutions and the holographic principle, we introduced two classes of solution in EMD(Euler) gravity, which are stationary and exhibit the cylindrical symmetry. Then we treated the holographic screens in the first class of solutions, and we outlined how to deal with the task more generally. We have found that at this level, the causality is protected.

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