AN ANALYTIC STRUCTURE EMERGING IN PRESENCE OF INFINITELY MANY ODD COORDINATES

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Abstract. We show that the spectrum of the locally convex nonstandard hull of an infinite dimensional Grassmann algebra contains a nontrivial analytic part.

§1. Introduction

The present note contributes to the program of featuring even geometry as a “collective effect in infinite-dimensional odd geometry,” as suggested by Manin [14]. In our earlier work [19, 22] we proposed to apply the construction of nonstandard hull [12] to a normed Grassmann algebra $\bigwedge(\nu)$ with an infinitely large number of odd generators, $\nu$, in order to construct the superspace $X$ limit of finite dimensional purely odd superspaces as a locally ringed superspace associated to the nonstandard hull algebra $\hat{\bigwedge}(\nu)$ in a universal way. We have demonstrated that the resulting superspace $X$ is infinite-dimensional both in odd sector and, remarkably, in even sector as well, in the sense that the spectrum of the hull algebra $\hat{\bigwedge}(\nu)$, which serves as the underlying compact space for $X$, is infinite-dimensional.

It seems, however, that the nonstandard hull of the normed Grassmann algebra admits practically no derivations other than nilpotent, which makes it hardly possible to develop a far-reaching version of “even” analysis and geometry over it. In this note we propose to consider instead a nonstandard hull formed with respect to a certain natural non-normable locally convex topology on an infinite-dimensional Grassmann algebra. Namely, we start with the (completed) exterior algebra $\bigwedge l_1$ over a countably infinite set $\Xi$ of odd generators which has the strongest locally convex algebra topology making $\Xi$ into a bounded set. The spectrum, $\Sigma(\widehat{\bigwedge l_1})$, of the resulting locally convex nonstandard hull $\widehat{\bigwedge l_1}$ carries a nontrivial analytic structure, namely, there exists a homeomorphism $\Phi$ from the unit disc $D \subset \mathbb{C}$ to $\Sigma(\widehat{\bigwedge l_1})$ such that for every element $a \in \widehat{\bigwedge l_1}$ the composition $a \circ \Phi$ is a holomorphic function from $D$ to $\mathbb{C}$. The presence of such analytic parts in the spectrum of a locally convex algebra is known to be uncommon (cf. [29]). The result indicates

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that infinite dimensional purely odd superspaces may in a sense “generate” the complex analyticity.

For an elaborate discussion of the present circle of ideas the reader is referred to our paper [22]. We follow the “classical” Robinson’s approach to nonstandard analysis [26, 30], because apparently no other known formalization of nonstandard analysis (such as Internal Set Theory) provides adequate means to deal with nonstandard hulls. Either of the books [2, 15] serves as a self-contained introduction to supergeometry in the form we need. Terminology and notation from theory of topological algebras, locally convex spaces, infinite-dimensional holomorphy, and Banach-Lie theory follow [29], [27], [5], and [3], respectively. Throughout the paper, \( \mathbb{C} \) is the basic field.

§2. FREE LOCALLY CONVEX GRADED COMMUTATIVE ALGEBRAS

A rich collection of “grassmannian” algebras can be obtained by invoking the categorial concept of a universal arrow to forgetful functor [13]. Here we will present just one of them, and we refer the reader for further examples of the kind to [9, 20, 21, 23, 24]. The present particular construct seems to be new, although numerous similar algebras, known for quite a while in the folklore, are now surfacing in mathematical physics every now and then.¹

Recall that a \( \mathbb{Z}_2 \)-graded algebra \( \Lambda \) (we will term such algebras just graded) is graded commutative if for all \( x, y \in \Lambda \) one has \( xy = (-1)^{\bar{x}\bar{y}}yx \), \( x, y \in \Lambda^0 \cup \Lambda^1 \), where \( \bar{x} \in \mathbb{Z}_2 \) is the parity of \( x \), such that \( x \in \Lambda^{\bar{x}} \).

2.1. Theorem. Let \( E \) be a locally convex space. There exist a complete locally convex graded commutative unital algebra \( \bigwedge E \) and a continuous linear operator \( i : E \to \bigwedge E^1 \) such that every continuous linear operator \( f \) from \( E \) to the odd part \( \Lambda^1 \) of a complete locally convex graded commutative unital algebra \( \Lambda \) gives rise in a unique way to a continuous unital graded algebra homomorphism \( \tilde{f} : \bigwedge E \to \Lambda \) with \( f = \tilde{f} \circ i \).

The proof employs certain pretty standard devices (cf. [20, 21, 23, 24]). What really matters, is the following result which details the structure of the free locally convex graded commutative algebra \( \bigwedge E \).

2.2. Theorem. Let \( E \) be a locally convex space. The following are true.

(i) The mapping \( i : E \to \bigwedge E^1 \) is an embedding of locally convex spaces;
(ii) the algebra generated by \( i(E) \) in \( \bigwedge E \) is the exterior algebra over \( E \), and it is dense in \( \bigwedge E \);
(iii) the completed \( n \)-th symmetric power of \( i(E) \) in \( \bigwedge E, \bigwedge^n(E) \), is the locally convex quotient space of the \( n \)-th projective tensor power \( E^\otimes n \) under the antisymmetrization map;
(iv) the algebra \( \bigwedge E \) is the locally convex direct sum of subspaces \( \bigwedge^n(E), n \in \mathbb{N} \).

Proof. The only nontrivial point is the verification of the continuity of multiplication in an algebra constructed as the locally convex direct sum of subspaces \( \bigwedge^n(E), n \in \mathbb{N} \). This is done with the help of the fact that on \( \bigwedge E \) three topologies coincide: the topology of the locally convex direct sum, the topology of the direct limit of the

¹ The author has benefited from discussing this topic with J. Baez, U. Bruzzo, Z. Jaskolski, T. Loring, and N.C. Phillips.
spaces $\bigoplus_{i=0}^{n} \bigwedge^{i}(E)$, $i \in \mathbb{N}$ in the category of Tychonoff spaces and the box product topology (see [27]). □

We will mention just two examples. If $E = \mathbb{C}^{n}$ is finite dimensional then the algebra $\bigwedge E$ is nothing else but the familiar Grassmann algebra with $n$ generators. The case where $E = \mathbb{C}^{\infty} \equiv \varinjlim \mathbb{C}^{n}$ was actually studied in [10]. For other related examples, see [24].

2.3. Theorem. Let $\Gamma$ be a set. Then there exists a locally convex unital graded commutative algebra $\Lambda$ topologically generated by its bounded set $\Gamma$ of odd elements, such that every mapping sending $\Gamma$ to a bounded subset of the odd part of a locally convex unital graded commutative algebra $M$ extends to a continuous unital graded algebra homomorphism $\Lambda \to M$. The algebra $\Lambda$ is nothing more than $\bigwedge l_{1}(\Gamma)$, the free locally convex graded commutative unital algebra over the Banach space $l_{1}(\Gamma)$.

Proof. Reduces to verification of the universality property of the kind described in the Theorem for the algebra $\bigwedge l_{1}(\Gamma)$, which, in turn, follows from Theorem 2.1 and the following statement known in functional analysis: the topology on the space $l_{1}(\Gamma)$ is the strongest complete locally convex topology making $\Gamma$ into a bounded subset. □

We will be interested in the algebra $\bigwedge l_{1}$ over the set $\Gamma = \mathbb{N}$ of countably many free odd generators. In a sense, it is one of the most natural “grassmannian” algebras.

Denote by $M_{n}$ the totality of all $n$-element subsets of $\mathbb{N}$ inheriting from $\mathbb{N}$ the natural order, and let $M = \cup_{n \in \mathbb{N}} M_{n}$. For any $\mu = (\mu_{1}, \mu_{2}, \ldots, \mu_{k}) \in M$ put $l(\mu) = k$ and $\xi^{\mu} := \xi_{\mu_{1}} \xi_{\mu_{2}} \cdots \xi_{\mu_{k}}$. Then every element $x \in \bigwedge^{n} l_{1}$ is uniquely represented as

$$x = \sum_{\mu \in M_{n}} a_{\mu} \xi^{\mu}$$

with $a_{\mu} \in \mathbb{C}$ and

$$\sum_{\mu \in M_{n}} |a_{\mu}| < +\infty$$

An arbitrary element $x \in \bigwedge l_{1}$ can be represented as a finite sum $x = x_{1} + \cdots + x_{n}$, where $x_{i} \in \bigwedge^{n_{i}} l_{1}$ for some $n_{i} \in \mathbb{N}$. Every subspace $\bigwedge^{n} l_{1}$ is a complete normable locally convex space, although it does not carry any distinguished norm. Actually, the algebra $\bigwedge l_{1}$ can be “manufactured” from a well-known Banach-Grassmann algebra $B_{\infty}$ [7] by decomposing the latter algebra into the $l_{1}$ type sum of its normed subspaces isomorphic to $\bigwedge^{n} l_{1}$ each, and then rearranging the parts, this time as a locally convex direct sum. Recall that $B_{\infty}$ is the completion of the Grassmann algebra on an infinite set $\xi_{1}, \ldots, \xi_{n}, \ldots$ of odd generators, endowed with the norm

$$\| x \| \equiv \left\| \sum_{\mu \in M} a_{\mu} \xi^{\mu} \right\| \overset{def}{=} \sum_{\mu \in M} |a_{\mu}| C,$$

and an element of $B_{\infty}$ is any sum $\sum_{\mu \in M} a_{\mu} \xi^{\mu}$ such that the above norm is finite.

The algebra $\bigwedge l_{1}$ is, as a locally convex space, an $(LB)$-space (the direct locally convex limit of the Banach spaces $\bigoplus_{i=0}^{n-1} \bigwedge^{n} l_{1}$).

Remark that there is a canonical continuous algebra monomorphism $j : \bigwedge l_{1} \to B_{\infty}$. Its restriction to every subspace $\bigoplus_{i=0}^{n-1} \bigwedge^{n} l_{1}$ is an embedding of locally convex spaces.
§3. Superspaces

A locally ringed superspace \([15]\) is defined by an analogy with a locally ringed (or geometric) space \([4]\). It is a pair \(M = (M_0, S_M)\) consisting of a topological space \(M_0\) and a sheaf \(S_M\) of graded commutative rings on it such that every stalk \(S_{M,x}, x \in M_0\) is a local ring. Denote by \(m_x\) the radical of the ring \(S_{M,x}\). A morphism between two superspaces \(M\) and \(N\) is a pair \(\phi = (\phi_0, \phi^\#)\) where the mapping \(\phi_0: M_0 \to N_0\) is continuous and \(\phi^\#: S_N \to \phi_0^* S_M\) is a sheaf morphism such that for every \(x \in M_0\), \(\phi^\# m_{\phi_0 x} \subset m_x\).

This concept is crucial for supergeometry, because a supermanifold is nothing more than a locally ringed superspace obtained by patching together the model superspaces of a special form, the superdomains. However, what we need here, is the following particular concept: a purely odd supermanifold \(C_{0,q}\) of dimension \((0, q)\) is a superspace whose underlying topological space is one-pointed and the algebra of global sections of the (constant) structure sheaf is the Grassmann algebra \(\bigwedge^{l_1}\).

We are interested in the behaviour of the purely odd superspace \(C_{0,q}\) as \(q \to \infty\). To do that, we need a superspace of infinite purely odd dimension. There are several different approaches to infinite-dimensional supermanifold theory \([18, 28]\), which is far from being in its final form yet. Happily enough, it would be consistent with all of them, to assume that a superspace whose underlying topological space is one-pointed and the algebra of global sections of the constant structure sheaf is \(\bigwedge l_1\), is an example of a supermanifold of dimension \((0, \infty)\). We will denote this superspace by \(C_{0,\infty}\).

Our main idea is to consider, within a nonstandard model of analysis, the superspace \(C_{0,\infty}\), and to assume that the global function algebra on an associated superspace \(X\) arising as the limit of superspaces \(C_{0,q}\), \(q \to \infty\), or the “collective effect” in presence of infinitely many odd coordinates \(\xi_1, \ldots, \xi_n, \ldots\) on \(C_{0,\infty}\), is the nonstandard hull of the global function algebra \(\bigwedge l_1\) on \(C_{0,\infty}\), that is, the quotient space of the algebra of all finite elements of \(\bigwedge l_1\) by the ideal of all infinitesimals. In a sense, the nonstandard hull is the “observable part” of the algebra \(\bigwedge l_1\), or (in the spirit of the French school \([11]\)) its “shadow” in the standard world.

§4. Nonstandard hull of the grassmannian algebra \(\bigwedge l_1\)

Let \(\mathcal{M}\) be a higher order set-theoretic structure, and \(*\mathcal{M}\) be a higher order nonstandard model of analysis enlargement of \(\mathcal{M}\) \([26, 30]\) which is at least \(\aleph_1\)-saturated. Let \(E \in \mathcal{M}\) be a standard locally convex space. Denote

\[
fin E \overset{def}{=} \{ x \in^* E : p(x) \text{ is a finite element of } ^*\mathbb{R} \}
\]

for every standard continuous prenorm \(p\), and

\[
\mu_E(0) \overset{def}{=} \{ x \in^* E : p(x) \approx 0 \text{ for every standard continuous prenorm } p \}.
\]

The quotient linear space \(\hat{E}\) of the principal galaxy of \(E\), \(fin E\), by the monad of zero, \(\mu_E(0)\), becomes a standard locally convex space if being endowed with a family of prenorms \(\bar{p}\), where \(p\) runs over the collection of all continuous prenorms on \(E\), by letting \(\bar{p}(x) \overset{def}{=} p(x)\) for any element \(x \in \hat{E}\) of the form \(x = \pi_\alpha x\), \(x \in fin E\), \(\alpha \in \text{fin } E\).
where $\pi_E : fin E \to \hat{E}$ is the quotient linear mapping. The locally convex space $\hat{E}$ is termed the nonstandard hull of $E$. The LCS $E$ canonically embeds into $\hat{E}$ as a locally convex subspace. If $\mathfrak{M}$ is saturated enough, then $\hat{E}$ is complete. For more on this, including nonstandard hulls of internal rather than standard spaces, see [12, 30, 6].

If $A$ is a standard locally convex topological algebra then the principal galaxy $fin A$ is a (generally speaking, external) subalgebra of $A$, the monad of zero $\mu_A(0)$ is an ideal of $fin A$, and the nonstandard hull $\hat{A}$ is a locally convex topological algebra belonging to a variety of algebras generated by $A$. The algebra $A$ itself is isomorphic to a topological subalgebra of $\hat{A}$ in a canonical way.

As a consequence, the nonstandard hull $\widehat{\Lambda}l_1$ of the grassmannian algebra $\Lambda l_1$ is a complete locally convex graded commutative algebra, and the nonstandard hull $\hat{B}_\infty$ is a graded commutative Banach algebra. Because of the functorial properties of the operation of forming nonstandard hull, the mapping $j : \Lambda l_1 \to B_\infty$ gives rise to a continuous algebra homomorphism $\hat{j} : \widehat{\Lambda}l_1 \to \hat{B}_\infty$.

4.1. Theorem. The algebra $\widehat{\Lambda}l_1$ is a locally convex direct sum of its complete normable locally convex subspaces $\widehat{\Lambda}^n l_1$; in particular, $\widehat{\Lambda}l_1$ is an (LB)-space. The mapping $\hat{j} : \widehat{\Lambda}l_1 \to B_\infty$ is a continuous algebra monomorphism, and for every $n \in \mathbb{N}$, the restriction of $\hat{j}$ to the nonstandard hull $\widehat{\Lambda}^n l_1$ is an isomorphism of locally convex spaces.

Proof. The algebra $\Lambda l_1$ can be represented as the box product of spaces $\Lambda^n l_1$, and it is easy to show that the nonstandard hull of the box product of a standard countable family of LCS's is canonically isomorphic to the box product of their nonstandard hulls. The remaining properties are straightforward. □

Remark that the nonstandard hull $\hat{B}_\infty$ is bigger than just the $l_1$-type sum of its Banach subspaces $\widehat{\Lambda}^n l_1$ [22].

In our paper [22], we investigated the structure of the nonstandard hull $\widehat{\Lambda}(\nu$) of the (internal) $\ast$finite dimensional Grassmann algebra with $\nu$ odd generators, endowed with an $l_1$-type norm. Although this latter algebra is only a normed subalgebra of $B_\infty$, and therefore the hull $\widehat{\Lambda}(\nu$) is isometric to a proper closed subalgebra of the hull $\hat{B}_\infty$, it is easy to see that $\widehat{\Lambda}(\nu$) is a retract of $\hat{B}_\infty$, and therefore virtually all results can be extended from the case of $\widehat{\Lambda}(\nu$) to $\hat{B}_\infty$. In particular, the following holds.

4.2. Theorem (cf. [22]). The spectrum $\Sigma(\hat{B}_\infty)$ of the nonstandard hull $\hat{B}_\infty$ is an inseparable compact space. The number of connected components of $\Sigma(\hat{B}_\infty)$ is uncountable; however, $\Sigma(\hat{B}_\infty)$ contains a connected inseparable subspace. The space $\Sigma(\hat{B}_\infty)$ contains a topological copy of the cube $I^n$ for each natural number $n$, therefore the topological dimension of $\Sigma(\hat{B}_\infty)$ is infinite in any sense. □

The Gelfand spectrum $\Sigma(A)$ of a topological algebra $A$ is defined as in case of Banach algebras, but now one should require that $\Sigma(A)$ be formed by continuous characters of the algebra $A$ only. The mapping $\hat{j} : \Lambda l_1 \to B_\infty$ leads to the dual continuous mapping between Gelfand spectra of both algebras, $j^* : \Sigma(B_\infty) \to \Sigma(\widehat{\Lambda}l_1)$. One can show that it is neither one-to-one nor onto. However, the constructions in
4.3. Theorem. The spectrum \( \Sigma(\widehat{l}_1) \) of the nonstandard hull \( \widehat{l}_1 \) is an inseparable compact space. The number of connected components of \( \Sigma(\widehat{l}_1) \) is uncountable; however, \( \Sigma(\widehat{l}_1) \) contains a connected inseparable subspace. The space \( \Sigma(\widehat{l}_1) \) contains a topological copy of the cube \( I^n \) for each natural number \( n \), therefore the topological dimension of \( \Sigma(\widehat{l}_1) \) is infinite in any sense. \( \square \)

§5. Automorphisms of the algebra \( \widehat{l}_1 \)

Let \( u \in GL(l_1) \) be an automorphism of the Banach space \( l_1 \). In view of existence of canonical embedding \( l_1 \rightarrow \widehat{l}_1 \), the operator \( u \) becomes a linear mapping \( u : l_1 \rightarrow \widehat{l}_1 \). Since the set \( u(\Xi) \) is bounded, the operator \( u \) can be extended to a standard automorphism \( \hat{u} \) of \( \widehat{l}_1 \), which in turn is lifted to an automorphism, \( \hat{u} \), of the hull algebra \( \widehat{l}_1 \). This way the general linear group \( GL(l_1) \) acts on the algebra \( \widehat{l}_1 \) by automorphisms. We denote this action by \( \tau \) and endow the group \( GL(l_1) \) with the uniform operator topology.

5.1. Theorem. The action \( \tau \) is analytic as a mapping \( GL(l_1) \times \widehat{l}_1 \rightarrow \widehat{l}_1 \).

Proof. Since every subspace \( \widehat{\bigwedge}^{n} l_1 \) of \( \widehat{l}_1 \) is closed under the action \( \tau \), the action itself decomposes into the locally convex direct sum of actions \( \tau_n \) of \( GL(l_1) \) on \( \widehat{\bigwedge}^{n} l_1 \), and it is sufficient to prove the analyticity of each mapping \( \tau_n : GL(l_1) \times \bigwedge^n l_1 \rightarrow \bigwedge^n l_1 \), as \( n \in \mathbb{N} \). Since \( GL(l_1) \) is a Banach-Lie group and the locally convex space \( \bigwedge^n l_1 \) is complete normable, the analyticity of the action would follow at once from its continuity as a homomorphism \( GL(l_1) \rightarrow GL(\bigwedge^n l_1) \) with respect to the uniform operator topology on the latter group. It is enough to check the continuity of this homomorphism at the identity element only. Let us endow, for the purpose of our proof, the locally convex space \( \bigwedge^n * l_1 \) with a norm induced from the Banach algebra \( *B_\infty \) (a particular choice of the norm does not affect the uniform operator topology). Let \( \epsilon > 0 \) be arbitrary standard with \( \epsilon < 1 \); we will find a neighbourhood of identity, \( U \), in \( GL(l_1) \) such that for every \( g \in U \) and \( x \in \bigwedge^n * l_1 \) with \( \| x \| \leq 1 \) one has \( \| \tau_g(x) - x \| \leq \epsilon \). Choose, using the uniform continuity of the action of \( GL(l_1) \) on \( l_1 \), a neighbourhood of identity, \( U \subset GL(l_1) \), such that for every \( g \in U \) and \( x \in l_1 \) with \( \| x \| \leq 1 \) one has \( \| \tau_g(x) - x \| \leq \epsilon/2^{n-1} \). Since the unit ball in \( \bigwedge^n * l_1 \) is the convex-circled envelope of the set of monomials \( f^u, u \in *, M_n \), one can
6.1. Theorem. There exists a homeomorphism $\Phi$ from the unit disc $D \subset \mathbb{C}$ into $\Sigma(\widehat{l_1})$ such that for every element $a \in \widehat{l_1}$ the composition $\hat{a} \circ \Phi$ is a holomorphic function from $D$ to $\mathbb{C}$.

Proof. For every $z \in D$, the dilatation of $l_1$ by the element $e^z$, i.e., the mapping $x \mapsto e^z x$, belongs to $GL(l_1)$; we will denote this dilatation simply by $e^z$. Now fix a continuous character $\chi$ (that is, an element of $\Sigma(\widehat{l_1})$) sending the element $\theta$ (constructed in §4) to 1. (Such a character exists in the spectrum of the Banach algebra $\hat{B}_\infty$, as it is well known, and it is enough to take its restriction to $\widehat{l_1}$ as $\chi$.) For each $z \in D$ define a new character $\chi_z$ by letting $\chi_z(a) = \chi(\tau e^z a)$ for all $a \in \widehat{l_1}$. The arising mapping $\Phi: z \mapsto \chi_z$ from $D$ to $\Sigma(\widehat{l_1})$ is one-to-one (because $\chi_z(\theta) = e^{2z}$ for all $z \in D$) and is continuous. Since the topology on $\Sigma(\widehat{l_1})$ is that of pointwise convergence on elements of $\widehat{l_1}$, then in order to prove the continuity of $\Phi$, it is enough to show that all mappings $z \mapsto \chi_z(a)$ from $D$ to $\mathbb{C}$ are continuous as $a \in \widehat{l_1}$. This follows from a stronger result: for every $a \in \widehat{l_1}$, the mapping $z \mapsto \chi_z(a)$ (in other form, $\hat{a} \circ \Phi$) is analytic, as the composition of a chain of analytic maps between complete locally convex spaces:
§7. Conclusion: further developments

Since we are working in an analytic rather than purely algebraic context, it seems reasonable to modify the definition of a locally ringed (= geometric) superspace by letting algebras of superfunctions (= sections of the structure sheaf) carry a topology, which, for a number of reasons, should be complete and locally $m$-convex in the sense of [1, 16]. The restriction homomorphisms must be, of course, continuous. One can show that, like in the case of algebraic prime spectra [4], there exists a universal solution to the problem of associating to any graded commutative Banach algebra $A$ a superspace in the above sense; the underlying topological space of the universal superspace is the Gelfand spectrum $\Sigma(A)$. (A related construction — in the “purely even” case — can be found in [25].) The superspace $X$ which we consider as the “collective effect” arising in $\mathbb{R}^{0,\infty}$ is just the solution to a universal problem of such kind for the hull algebra $\hat{\mathfrak{l}}_1$. (A superspace denoted by the same letter $X$ in [19, 22] is some proper subsuperspace.) The reduced subsuperspace, $X_{\text{red}}$, of $X$ [15] is obtained by taking quotient of the structure sheaf of $X$ by the ideal of nilpotents (or, in our topologized case, quasinilpotents).

It turns out that the non-nilpotent derivations of the hull algebra $\hat{\mathfrak{l}}_1$ are abundant, and they give birth to vector fields on the geometric space $X_{\text{red}}$. (One can obtain those derivations as elements of the nonstandard hull of the well-known [8] Lie superalgebra of derivations of a Grassmann algebra.) Thereby, the space $X_{\text{red}}$ admits a substantial differential geometry of its own.

Another possibility to treat the problem is to invoke the synthetic approach, based on topos theory, and the first step may be to give an appropriate locally convex nonstandard hull of an infinite-dimensional Grassmann algebra the structure of a smooth algebra in the sense of [17].

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