A toy model for slowly growing wormholes as effective topology changes

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We present a toy model for growing wormholes as a model of effective low-energy topology changes. We study the propagation of quantum fields on a 1 + 1 spacetime analogous to the trouser-leg topology change. A low-energy effective topology change is produced by a physical model which corresponds to a barrier smoothly changing the tunneling probability between two spatial regions.

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In 1957 Wheeler argued that the topology of space must fluctuate at what he called the “Planck scale” of distances [1]. He concluded that in a linearized theory of gravity the Riemann tensor would exhibit fluctuations violent enough to practically pinch off portions of space or produce multiply connected “wormholes.” Even though the scales involved meant that the linearized theory would fail, Wheeler suggested that these phenomena would, nonetheless, occur.

Interest in the possibility of wormholes was greatly increased by the Morris-Thorne wormhole solution [2] of classical general relativity. They found that if a wormhole can be formed or found by accident then it can in principle be stretched to allow ‘faster-than-light’ travel or time-travel [3]. Indeed, one of the most bizarre possibilities of wormholes would be to escape from the gravitational pull of a black-hole from well within its Schwarzschild radius [4].

The possibility of creating wormhole structures faces severe problems: Classically, the weak energy condition must be violated, both to supply suitable matter for the construction of a Morris-Thorne wormhole and to circumvent classical singularity theorems [5]. However, quantum mechanics allows for violation of the weak energy condition [6]. Unfortunately, quantum fields propagating on a time varying topology introduce new problems: Anderson and DeWitt [7] performed the first model calculations with a quantum field propagating on a classical spacetime manifold which instantaneously changed its topological connectedness. They showed that this sort of ideal topology change leads to the production of infinitely bright flashes of energy [8,9].

How might we evade an instantaneous topological change? We argue that an effective low-energy theory can behave exactly like a gradual topology change. The fundamental idea is that the propagating fields should see the change occur only at the amplitude level: Quantum fields propagating on the spacetime experience a gradually increasing tunneling probability from the original to the new ‘topological’ configuration. For wormhole ‘creation’ this could be equivalent to slowly stretching a microscopic wormhole to a large size. See Fig. 1. For such a gradual stretching the low-energy dynamics of fields on this background spacetime would evolve smoothly from those on a spacetime without a wormhole to those with one. Initially, only modes with energies above the Planck-scale can tunnel between the upper and lower sheets, however, this path becomes more and more accessible to low-energy modes as the wormhole is gradually enlarged. The growing wormhole is acting as a time-varying waveguide. This procedure delays the problem discovered by Anderson and DeWitt until the Planck-scale where it is no longer applicable. As far as the low-energy effective theory is concerned this would be an effective topology change, whether or not it was fundamentally so. The important point is that the fields see smoothly changing tunneling probabilities instead of a discontinuous change in the underlying topology. In fact, any mechanism which yields this behavior might mediate a low-energy effective topology change, in this sense our proposal is more general than merely slowly growing a classical wormhole.

Calculations along the lines of those described in Fig. 1 involving an actual growing wormhole are too hard, so instead we consider a 1 + 1 dimensional toy model with Lagrangian density as

$$\mathcal{L} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial x} \right)^2 - 2V(t) \delta(x) \phi^2 \right], \quad (1)$$

where $2V(t)$ represents the ‘height’ of the tunneling barrier between the left and right half-spaces. As the barrier height is slowly shrunk from being huge to being zero the initially effectively disconnected half-spaces re-connect. In some sense this is a good model for the
physics we are interested in for a growing wormhole in that the field modes see a gradually increasing tunneling probability between the half-spaces, and that only the high frequency modes can effectively tunnel the barrier which otherwise strongly reflects the low frequency modes. However, in other respects this model is not so similar to the wormhole scenario, and instead resembles a sort of opened-string pants diagram which we might draw as Fig. 3. The crotch has been drawn as sort of hazy to remind us that the geometric optics limit is failing and wavepackets are dispersively reflected and transmitted during the time-varying dynamics of the barrier. Finally, we note that our toy model is closely analogous to the trousers spacetime originally considered by Anderson and DeWitt in their study of the behavior of quantum fields propagating on a time-varying topology.

Let us consider an adiabatic expansion of $\phi(0, t)$ for an incoming harmonic wave: e.g., $U_{in}(t) = e^{-i\omega t}$ and $V_{in}(t) = 0$. To first order in the time-derivative of the barrier height we find:

$$\phi(0, t) \simeq \frac{i\omega}{i\omega - V(t)} e^{-i\omega t} \left(1 + \frac{V'(t)}{[i\omega - V(t)]^2}\right)$$

which may be checked by direct substitution into the differential equation (4). For the model barrier time-dependence

$$V(t) = \frac{V_0 e^{-\omega_0 t} + V_1 e^{\omega_0 t}}{e^{-\omega_0 t} + e^{\omega_0 t}},$$

we find that the first-order derivative terms in (7) are small for all $t$ and $\omega$ when

$$\omega_0 \ll \frac{\min(V_0, V_1)}{|V_1 - V_0|},$$

i.e., for a sufficiently slow change in barrier height we have obtained a uniformly accurate approximation to the general solution of Eq. (3).

Our Lagrangian density (8) has an associated time-independent inner product

$$(u_1, u_2) \equiv i \int_{-\infty}^{\infty} dx \overline{\pi_1} \partial_\tau u_2 - \partial_\tau \overline{\pi_1} u_2,$$

where the overline represents complex conjugation. Using this and the result that

$$\int_0^\infty dx e^{\pm i\omega x} = \pi \delta(\omega) \pm i \mathcal{P} \frac{1}{\omega},$$

where $\mathcal{P}$ represents the Cauchy principle value function, we may determine the in-coming traveling wave mode solutions for a static barrier. The solutions are

\[
\begin{align*}
U_{in}^\omega &= \frac{1}{\sqrt{2\pi\omega}} \begin{cases} 
e^{-i\omega u} + \rho e^{-i\omega v}, & x < 0, \\
\tau e^{-i\omega u}, & x \geq 0,
\end{cases} \\
v_{in}^\omega &= \frac{1}{\sqrt{2\pi\omega}} \begin{cases} \tau e^{-i\omega v}, & x < 0, \\
e^{-i\omega u} + \rho e^{-i\omega v}, & x \geq 0,
\end{cases}
\end{align*}
\]

so the static barrier has reflection and transmission coefficients

$$\rho = \tau - 1 \quad \text{and} \quad \tau = \frac{i\omega}{i\omega - V},$$

respectively (thus, $|\rho|^2 + |\tau|^2 = 1$ so energy is conserved in this case). The right-headed incoming mode $U_{in}^\omega$ is illustrated in Fig. 3. These modes form a complete orthonormal set according to the inner product (10) with

\[
\begin{align*}
(u^\omega_{in}, u^\omega_{in}) &= (v^\omega_{in}, v^\omega_{in}) = \delta(\omega - \omega'), \\
(u^\omega_{in}, v^\omega_{in}) &= (u^\omega_{in}, \overline{v^\omega_{in}}) = (u^\omega_{in}, \overline{v^\omega_{in}}) = 0.
\end{align*}
\]

FIG. 2. Spacetime diagram around the ‘crotch’ (dashed line) of our toy ‘wormhole.’ Time is running up. As the barrier height $2V'(t)$ is gradually lowered from being huge to being zero the disconnected half-spaces slowly reconnect.
annihilation and creation operators by decomposing the unity. This language may be translated to that of mode
commutation relations for \( \hat{\psi} \), respectively. By virtue of the equal-time equations of motion for right- and left-headed traveling-
waves, we have

\[
\hat{\psi}_+ = \frac{1}{\sqrt{2\pi\omega}} \left[ \frac{\tau e^{i\omega x}}{e^{i\omega} + \tau e^{i\omega}}, \quad x < 0, \right.
\]
\[
\hat{\psi}_- = \frac{1}{\sqrt{2\pi\omega}} \left[ \frac{\tau e^{-i\omega x}}{e^{-i\omega} + \tau e^{-i\omega}}, \quad x > 0, \right.
\]

Canonical quantization of the field proceeds by enforcing the standard equal-time commutation relations with
conjugate momentum \( \hat{p}(x, t) \equiv \partial\hat{\phi}(x, t)/\partial t \) via

\[
\left[ \hat{\phi}(x, t), \hat{p}(x', t) \right] = i \delta(x - x'),
\]

where for convenience we choose Planck’s constant as unity. This language may be translated to that of mode
annihilation and creation operators by decomposing the field into any complete orthonormal set of modes with
respect to the inner product \( \langle \rangle \), for instance

\[
\hat{\phi}(x, t) = \frac{1}{\sqrt{2 \pi}} \int_0^\infty d\omega \left[ u_\omega(x, t) \hat{a}_\omega + v_\omega(x, t) \hat{b}_\omega + \text{h.c.} \right],
\]

with \( u_\omega \) and \( v_\omega \) denoting full solutions to the classical equations of motion for right- and left-headed traveling-
wave modes, respectively. By virtue of the equal-time commutation relations for \( \hat{\phi} \) and \( \hat{p} \) the annihilation
and creation operators satisfy the canonical commutation relations

\[
[\hat{a}_\omega, \hat{a}^\dagger_\omega] = [\hat{b}_\omega, \hat{b}^\dagger_\omega] = \delta(\omega - \omega'), \quad [\hat{a}_\omega, \hat{b}_\omega] = 0. \tag{17}
\]

This completes the canonical quantization for our one-dimensional field.

The Bogoliubov transformations may be derived in the following way: Using the time-evolved incoming modes
with \( V = V(t) \) and the static outgoing modes [Eq. (14)] with \( V \) replaced by the final value \( V_1 \) we obtain an expression for the annihilation operator for the outgoing
modes as

\[
\hat{a}_\omega^\text{out} = \int_0^\infty d\omega' \left[ (u_\omega^\text{out}, u_\omega^\text{in}) \right]_{t_1} \hat{a}^\text{in} + (u_\omega^\text{out}, v_\omega^\text{in}) \right]_{t_1} \hat{b}^\text{in} \]
\[
+ (u_\omega^\text{out}, \tau_\omega^\text{in}) \right]_{t_1} \hat{a}^\text{in} + (u_\omega^\text{out}, \tau_\omega^\text{in}) \right]_{t_1} \hat{b}^\text{in} \right]. \tag{18}
\]

In fact, it is sufficient for the purposes of determining the number of quanta generated to calculate the latter two
terms alone. We may determine \( \hat{b}^\text{out} \) by symmetry. Consider the time-evolved mode \( u_\omega^\text{in} \): It’s Bogoliubov coefficient with the static mode \( u_\omega^\text{out} \) is given by

\[
\langle \hat{n}_\omega(t) \rangle = \langle \hat{a}^\dagger_\omega(t) \hat{a}_\omega(t) \rangle = \frac{1}{2\pi\sqrt{\omega'd}} \left( (V_1 - V_0) \tau_0 \overline{\tau}_1 e^{-i(\omega + \omega')t_1} \right.
\]
\[
- \pi \tau_1 e^{-i\omega t_1} [\phi(0, t_1) - \phi(0, t_1)] + \omega' \int_{t_1}^t dt' e^{-i\omega't'} [\phi(0, t') - \phi(0, t')] \right),
\]

where \( \phi(0, t) \equiv \tau_0 e^{-i\omega t} \) and subscripts \( j = 0, 1 \) refer to using \( V = V_j \). The Bogoliubov coefficient for the time-

evolved mode \( u_\omega^\text{in} \) with static \( u_\omega^\text{out} \) is identical.

How many quanta are produced and with what spectrum? The power spectrum of right-going quanta is given by

\[
\langle \hat{n}_\omega(R) \rangle = \langle \hat{a}^\dagger_\omega \hat{a}_\omega \rangle = 2 \int_0^\infty d\omega' \left| \langle u_\omega^\text{out}, v_\omega^\text{in} \rangle \right|^2 \bigg|_{t = t_1} . \tag{20}
\]

By symmetry the left- and right-going quanta have the same power spectra.

Consider the model time-dependence for the barrier given by Eq. (8). Taking the lowest order adiabatic expansion as

\[
\phi(0, t) \simeq \frac{i\omega}{\omega - V(t)} e^{-i\omega t}, \tag{21}
\]

and the limit \( t_0 \to -\infty \) we have

\[
\left\langle u_\omega^\text{in}(t), u_\omega^\text{out} \right\rangle \simeq \frac{\sqrt{\omega'd}}{2\pi(i\omega - V_0)} \left( \frac{e^{-i(\omega + \omega')t_1}}{(\omega + \omega')(i\omega - V_1)} + i \int_{-\infty}^{t_1} dt' e^{-i(\omega + \omega')t'} \right. \]
\[
\left. \left. \left. \frac{e^{-i(\omega + \omega')t'}}{(i\omega - V_0)e^{-2\omega t'} + (i\omega - V_1)} \right). \tag{22}
\]

This integral may be performed with a suitably chosen contour (see Fig. 3) yielding for \( t_1 \) asymptotically large:

\[
\frac{i}{\Omega A_1} \int_{-\infty}^{t_1} dt \frac{e^{-i\omega t}}{A_0 e^{-2\omega t} + A_1} \tag{23}
\]
\[
= \frac{-e^{-i\omega t_1}}{\Omega A_1} + \frac{\pi}{2\omega A_1 \sinh(\pi\Omega/2\omega_0)} \left( \frac{A_1}{A_0} \right) \frac{\sinh(\pi\Omega/2\omega_0)}{\sinh(\pi\omega_0/2)}. \]

The inner-product then reduces to
\[
\left( \hat{\Omega}_{\omega}^{\text{in}(t)}, \hat{\Omega}_{\omega'}^{\text{out}} \right) \simeq \frac{\sqrt{\omega'} (V_1 - V_0)}{4\omega_0 (i\omega - V_0) (i\omega - V_1)} \frac{1}{\sinh[\pi (\omega + \omega')/2\omega_0]} \left( \frac{i\omega - V_1}{i\omega - V_0} \right)^{i(\omega + \omega')/2\omega_0} \]

which as expected is independent of \( t_1 \).

The overlap between the distant-past incoming and distant-future outgoing field modes is then

\[
\left| \left( \hat{\Omega}_{\omega}^{\text{in}(t)}, \hat{\Omega}_{\omega'}^{\text{out}} \right) \right|^2 \simeq \frac{\omega' (V_1 - V_0)^2}{16\omega_0^2 (\omega^2 + V_0^2)(\omega^2 + V_1^2)} \exp \left\{ \frac{\omega + \omega'}{\omega_0} [\arctan(\omega'/V_1) - \arctan(\omega'/V_0)] \right\} \frac{\sinh^2[\pi (\omega + \omega')/2\omega_0]}{\sinh^2[\pi (\omega + \omega')/2\omega_0]} . \tag{25}
\]

The integral of this expression yields the power spectrum \( (\omega_0) \) and may be well approximated by noting that the only significant contributions come from \( \omega' \ll \omega_0, V_1 \):

\[
\langle \hat{n}_\omega (R) \rangle \simeq -\frac{\omega (V_1 - V_0)^2}{2\pi^2 V_0^2 V_1^2} \ln \left( 1 - e^{-\pi \omega/\omega_0} \right) \tag{26}
\]

\[
\simeq \frac{(V_1 - V_0)^2}{2\pi^2 V_0^2 V_1^2} \times \begin{cases} -\omega \ln(\pi \omega/\omega_0), & \omega \ll \omega_0, \\ \omega e^{-\pi \omega/\omega_0}, & \omega \gg \omega_0 . \end{cases}
\]

The total energy generated is (recalling \( \hbar = 1 \))

\[
\mathcal{E} = \int_0^\infty d\omega \omega [\langle \hat{n}_\omega (L) \rangle + \langle \hat{n}_\omega (R) \rangle]
\]

\[
\simeq \frac{\omega_0^3 (V_1 - V_0)^2}{\pi^3 V_0^2 V_1^2} \int_0^\infty dx x^2 \ln \left( 1 - e^{-x} \right)
\]

\[
= \frac{\omega_0^3 (V_1 - V_0)^2}{45\pi V_0^2 V_1^2} \ll (\hbar) \omega_0 , \tag{27}
\]

so the adiabatic response is very weak even as \( V_0 \to \infty \) and \( V_1 \to 0 \). We note that both of these limits can only be taken within our calculations as a double limit with \( \omega_0 \to 0 \) thus ensuring the adiabaticity condition \( (3) \) continues to be satisfied.

We have studied a relativistic quantum field theory with tunneling through a time-varying barrier. The smoothly changing tunneling probability leads to an open-string pants spacetime with a hazy ‘crotch.’ Our toy model calculations suggest that slowly growing a microscopic wormhole should appear as an effective low-energy topology change. Finally, we have shown that quantum fields propagating on such an effective topology change need not have the non-renormalizable difficulties associated with instantaneous topology changes of the underlying manifold.

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FIG. 4. of “A toy model for slowly growing wormholes as effective topology changes,” by S. L. Braunstein, submitted to Physical Review D.

FIG. 4. Contour used to perform integral (23). The crosses denote poles.