Kaon Condensation in the Bound-State Approach to the Skyrme Model

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Abstract

We explore kaon condensation using the bound-state approach to the Skyrme model on a 3-sphere. The condensation occurs when the energy required to produce a $K^-$ falls below the electron fermi level. This happens at the baryon number density on the order of 3–4 times nuclear density.

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1 Introduction

Recently it was discovered [1] that kaon condensation in nuclear matter of densities on the order of a few times that of normal nuclear matter may have an impact on the formation of low mass black holes instead of neutron stars for masses on the order of 1.5 solar masses.

Since the proposal of kaon condensation and its possible consequences in 1986 [2], various attempts have been made to determine the critical density at which kaon condensation sets in, by means of a chiral lagrangian [2, 3, 4, 5] and also by means of phenomenological off-shell meson-nucleon interactions [6].

In this paper, we explore kaon condensation within the framework of the bound-state approach to the Skyrme model [7, 8]. In 1985, Callan and Klebanov proposed the bound-state approach to the Skyrme model [7] as a model for exploring the properties of strange baryons. To this day, the bound-state approach to the Skyrme model remains one of the most successful models for hyperons. At around this time, it was suggested that the SU(2) Skyrme model may be used to determine the properties of dense nuclear matter by means of putting skyrmions on a cubic lattice [9]. It was later determined that these SU(2) skyrmions undergo a transition to a smoothed out “dissolved” phase with restored chiral symmetry. [10, 11]. The cubic lattice symmetry was also replaced by other symmetries which resulted in a lower crystal interaction energy without affecting the qualitative phase structure of the crystal [12].

Unfortunately, lattice calculations can be difficult and require involved numerical work. In 1987, Manton proposed an alternative to the lattice. He put a single skyrmion on the surface of a 3-sphere of finite radius [13]. Various densities of nuclear material can be explored simply by varying the radius of the hypersphere. Since there is only one skyrmion, the calculation is much more straightforward. The phase transition may be calculated analytically. While the model seems somewhat contrived, since the nuclear matter density is manifested entirely by the curvature of the hypersphere, it nevertheless produces similar qualitative and quantitative (to within a few percent) results as the lattice calculations at a fraction of the computational cost. Furthermore, a study of two skyrmions on a hypersphere [14] suggests that the simultaneous addition of skyrmions and the increase in hypersphere radius, while keeping density fixed, does not significantly affect the results. Thus, one may be able to continuously connect the lattice approach and the hypersphere approach without substantial change in the observable quantities.

In this paper, we explore kaon condensation by adapting the bound-state approach to the 3-sphere. This approach has already been carried out in [15] using a different version of the Skyrme model, in which vector mesons are included as fundamental fields. It was further reasoned in [15] that the parameter $m_K$, the mass of the kaon in a vacuum, should “run” as a function of the density. At the time, this seemed to be the only means in which kaon condensation could be derived from the Skyrme model. It was assumed that unless the

\footnote{Unfortunately, kinetic energy effects have not been computed with the Skyrme model due to its non-renomalizability, so such comparison of energies is inherently difficult.}
energy of the kaon, $\omega$, reaches zero for a finite value of the 3-sphere radius, kaon condensation would not occur at all, or at least not for reasonably low densities. Without the running kaon mass, $\omega$ reaches zero only at zero radius. However, with more careful consideration we realize that kaon condensation sets in at a point where $\omega$ equals the fermi level of the electrons [3], which may be much higher than its rest mass. This reopened the possibility of using the original version of the Skyrme model, with no running masses, to calculate the onset of kaon condensation. This method has the advantage of requiring fewer input parameters, thus pinning down a more precise (albeit model-dependent) transition density. It is interesting that the final result lies within the range reported in [5] and lies fairly close to the values suggested by [15].

This paper is organized as follows: The bound-state approach is summarized in Section 2 and adapted for the hypersphere in Section 3. The background field is calculated in Section 4. In Section 5, a new discrete symmetry is found and discussed, which leads the way for the calculation of the effective kaon mass, $\omega$, as a function of hypersphere radius in Section 6. Finally, in Section 7, $\omega$ is compared to the electron fermi level and a prediction for the $K^-$ condensation transition density is obtained.

2 The Bound State Approach to the Skyrme Model

A single hyperon may be modeled by the Skyrme model in flat space. The version used by Callan and Klebanov [3] utilizes a simple chiral lagrangian with a commutator stabilizing term

$$
\mathcal{L} = \frac{f_\pi^2}{16} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2 + \frac{f_\pi^2}{8} \text{tr}\mathcal{M}(U + U^\dagger - 2)
$$

$$
= -\frac{f_\pi^2}{16} \text{tr} M_\mu M^\mu + \frac{1}{32e^2} \text{tr}[M_\mu, M_\nu]^2 + \frac{f_\pi^2}{8} \text{tr}\mathcal{M}(U + U^\dagger - 2),
$$

where $U(\vec{x}, t) \in \text{SU}(3)$. $\mathcal{M}$ is proportional to the quark mass matrix and, if we neglect the $u$ and $d$ masses, is given by

$$
\mathcal{M} = \begin{pmatrix}
0 & 0 \\
0 & m_K^2
\end{pmatrix},
$$

where, in general, we write $3 \times 3$ matrices in the partitioned form

$$
\begin{pmatrix}
2 \times 2 & 2 \times 1 \\
1 \times 2 & 1 \times 1
\end{pmatrix}.
$$

$M_\mu$ is defined as

$$
M_\mu := \partial_\mu U U^\dagger.
$$

The standard fit to nucleon and delta masses yields $f_\pi = 129\text{ MeV}$ and $e = 5.45$. The empirical value of the kaon mass is $m_K = 495\text{ MeV}$.
In addition to the local terms, we must also include the Wess-Zumino term, which is written in non-local form as an integral over a five-dimensional disk with spacetime as its boundary:

\[ S_{WZ} = \frac{iN}{240\pi^2} \int_D d^5\vec{x} \varepsilon^{\mu\nu\alpha\beta\gamma} \text{tr}(M_\mu M_\nu M_\alpha M_\beta M_\gamma), \]  

where \( U(\vec{x}, t) \) is continuously extended to the disk (with our convention, \( \varepsilon^{01234} = 1 \)). \( N \) is the number of colors.

In the bound-state approach to the Skyrme model, we use the following anzatz

\[ U(\vec{x}, t) = \sqrt{U_\pi(\vec{x}, t)} K(\vec{x}, t), \]  

\[ U_K(\vec{x}, t) = \exp i \frac{2\sqrt{2}}{f_\pi} \left( \frac{0}{K(\vec{x}, t)^\dagger} \frac{K(\vec{x}, t)}{0} \right), \]

where \( U_\pi \in SU(2) \) and \( K \) is a complex spinor. Substituting this anzatz into the Skyrme action and expanding to second order in the kaon fields, we obtain the bound-state lagrangian

\[ \mathcal{L} = -\frac{f_\pi^2}{4} \text{tr}(A_\mu^2) + \frac{1}{2e^2} \text{tr}([A_\mu, A_\nu]^2) - m_K^2 K^\dagger K + D_\mu K^\dagger D_\mu K \]

\[ + \frac{6}{e^2 f_\pi^2} D_\mu K^\dagger [A_\mu, A_\nu] D_\nu K - \frac{2}{e^2 f_\pi^2} D_\mu K^\dagger D_\nu K \text{tr}(A_\mu A_\nu) \]

\[ + \frac{2}{e^2 f_\pi^2} D_\mu K^\dagger D_\nu K \text{tr}(A_\nu^2) - \frac{2}{e^2 f_\pi^2} K^\dagger K \text{tr}([A_\mu, A_\nu]^2) + \frac{1}{2} K^\dagger K \text{tr}(A_\mu^2) \]

\[ + \frac{iN}{3f_\pi^2 e^2} \varepsilon^{\mu\alpha\beta\gamma} \text{tr}(A_\alpha A_\beta A_\gamma)(D_\mu K^\dagger K - K^\dagger D_\mu K), \]

where

\[ D_\mu K = \partial_\mu K + V_\mu K, \]

\[ A_\mu = \frac{1}{2}(\sqrt{U_\pi}^\dagger \partial_\mu \sqrt{U_\pi} - \sqrt{U_\pi} \partial_\mu \sqrt{U_\pi}^\dagger), \]

\[ V_\mu = \frac{1}{2}(\sqrt{U_\pi}^\dagger \partial_\mu \sqrt{U_\pi} + \sqrt{U_\pi} \partial_\mu \sqrt{U_\pi}^\dagger). \]

The term proportional to \( N \) is the result of integrating the Wess-Zumino term. The lagrangian itself is quadratic in kaon fields and describes non-interacting kaons bound to a background SU(2) skyrmion. The background field is realized by the hedgehog anzatz

\[ U_\pi(\vec{x}, t) = \begin{pmatrix} e^{iF(r)} \hat{r} \cdot \hat{r} & 0 \\ 0 & 1 \end{pmatrix} \]

with the boundary conditions \( F(0) = \pi \) and \( F(\infty) = 0 \). We write the eigenmodes for \( K(\vec{x}, t) \) in terms of its spin and isospin. In particular, we will be interested in S-wave and P-wave kaons, for which the corresponding anzätze are

\[ K_S(\vec{x}, t) = k_S(r) \chi(t), \]

\[ K_P(\vec{x}, t) = ik_P(r) \hat{r} \cdot \hat{r} \chi(t). \]
In each case, $\chi(t)$ are the dynamical variables arranged in a complex spinor.

The field $F(r)$ is determined classically using the non-kaon part of the lagrangian. This baryon field is treated as a background field with which the kaons interact. The energy eigenstates form a Fock space for each kaon mode, where the energy required to create a kaon in a particular mode is given by its classical eigenfrequency (i.e. substitute $\dot{\chi} = -i\omega \chi$ and minimize the action with respect to $\omega$ and $k(r)$). In flat space, the lowest energy modes for P-wave and S-wave are given respectively by $\omega_P = 153$ MeV and $\omega_S = 368$ MeV.

3 The Skyrme Model on a Hypersphere

In [13], Manton suggested that a baryon crystal may be approximated by putting a single Skyrmion on a hypersphere of finite radius. This has the effect of allowing the baryon to interact with itself. We will continue with this idea by considering the bound state between a kaon and a baryon on the hypersphere.

A hypersphere of radius $a$ is described by three angular coordinates $(\rho, \theta, \phi)$ ($0 \leq \rho, \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$) with the following spatial metric

$$ds^2 = a^2(d\rho^2 + \sin^2 \rho(d\theta^2 + \sin^2 \theta d\phi^2)).$$

It is often convenient to replace the $\rho$ coordinate with $x = \cot \rho, -\infty \leq x \leq \infty$. This spreads out the coordinate system around the poles, thereby eliminating coordinate singularities in the lagrangian.

Topologically, the hypersphere is equivalent to $\mathbb{R}^3$ with all the points at infinity mapped to a single point. However, the metric is different, with points near infinity ($r \gg a$) much closer together on the hypersphere than the corresponding points in flat space. For $r \ll a$, this difference is negligible. Since the baryon size in flat space is on the order $1/ef\pi$, it follows that for $a \gg 1/ef\pi$, the results on the hypersphere will not be much different from the results in flat space. However, for $a$ on the order of $1/ef\pi$, the skyrmion begins to interact with itself on the hypersphere, and this can have a significant effect on the results.

We choose to identify the origin $r = 0$ in $\mathbb{R}^3$ with the south pole $\rho = \pi (x = -\infty)$ and the point at infinity with the north pole $\rho = 0 (x = \infty)$. For $r \ll a$, $S^3$ and $\mathbb{R}^3$ are equivalent, both in topology and metric, with corresponding points

$$r \approx a(\pi - \rho) \approx -a/x.$$

There are a few technical remarks in adapting the Skyrme model for the hypersphere. The profile functions $F$ and $k$ on the hypersphere will depend on $\rho (x)$, which replaces $r$ as the radial coordinate. The boundary conditions for $F$ are as follows

|       | at the origin | at infinity |
|-------|--------------|-------------|
| $r$   | 0            | $\infty$    |
| $\rho$ | $\pi$       | 0           |
| $x$   | $-\infty$    | $\infty$    |
| Value of $F$ | $\pi$  | 0           |
In choosing to identify the origin with $\rho = \pi$, we reverse the $\hat{\rho}$ direction compared to the $\hat{r}$ direction in flat space. It is therefore appropriate to reverse the sign of $\varepsilon^{\mu\nu\alpha\beta}$, and hence the Wess-Zumino term.\footnote{If we choose instead to identify the origin with $\rho = 0$, then $F(0) = \pi$ and $F(\pi) = 0$ and the Wess-Zumino term stays the same. The results are equivalent.} In flat space, the matrix $\mathbf{\hat{r}} \cdot \mathbf{\hat{r}}$ depends on the angular coordinates $\theta$ and $\phi$. On the hypersphere, we identify $\mathbf{\hat{r}} \cdot \mathbf{\hat{r}}$ with the same function of $\theta$ and $\phi$, except these coordinates are now the hypersphere coordinates $\theta$ and $\phi$.

Using the anzatz in (3) adapted for the hypersphere, we may compute $A_\mu$ and $V_\mu$. In terms of the normalized coordinate basis for the dual tangent space

$$\omega^c = \mathbf{d}c\sqrt{|g_{cc}|}, \quad c = t, \rho, \theta, \phi,$$

we find $V_\mu$ and $A_\mu$ to given by

$$V_\mu = \frac{1 - \cos F}{2r}i\mathbf{\hat{r}} \cdot (\hat{\phi} \omega_\mu^\theta - \hat{\theta} \omega_\mu^\phi),$$

$$A_\mu = \frac{1}{2a}F'(\rho)i\mathbf{\hat{r}} \cdot \hat{\rho} \omega_\mu^\rho + \frac{\sin F}{2r}i\mathbf{\hat{r}} \cdot (\hat{\theta} \omega_\mu^\theta + \hat{\phi} \omega_\mu^\phi).$$

where $r = r_{\text{phys}} = a \sin \rho$. Using this result and (7), we obtain the (spatially integrated) lagrangian for a P-wave kaon

$$L = \frac{\pi f_\pi}{e} \int_{-\infty}^\infty dx \left\{ \alpha \left( \frac{\sin^2 F}{1 + x^2} + \frac{1}{2}(F'(x))^2 \right) + \frac{1}{\alpha} \left( 2 \sin^4 F + 4(1 + x^2) \sin^2 F(F'(x))^2 \right) \right\}$$

$$+ \chi^\dagger \chi \frac{\pi}{ef_\pi} \int_{-\infty}^\infty dx \left\{ (k'(x))^2 \left( 4\alpha + \frac{8}{\alpha}(1 + x^2) \sin^2 F \right) + \frac{24}{\alpha}(1 + x^2)(1 + \cos F) \sin F F'(x) k(x) k'(x) \right.$$}

$$- \frac{4\omega^2}{e^2 f_\pi^2}(k(x))^2 \left( \frac{\alpha^3}{1 + x^2} + \alpha (F'(x))^2 + \frac{\sin^2 F}{1 + x^2} \right)$$

$$+ \frac{4e\omega N}{f_\pi \pi^2} \sin^2 F F'(x) (k(x))^2 + (k(x))^2 \left( \alpha - (F'(x))^2 \right)$$

$$+ 2 \left( \frac{1 + \cos F}{1 + x^2} - 2 \sin^2 F \right) + \frac{8}{\alpha} \frac{1}{4}(1 + x^2)(1 + \cos F)^2(F'(x))^2$$

$$- 2(1 + x^2) \sin^2 F (F'(x))^2 + (1 + \cos F)^2 \sin^2 F - \sin^4 F$$

$$+ \frac{4m_K^2 \alpha^3}{e^2 f_\pi^2} \frac{1}{(1 + x^2)^2} \right\}, \quad (8)$$

where $\alpha = aef_\pi$ and we use the substitution $x = \cot \rho$ mentioned earlier. One could use (3) instead of (7) to obtain the lagrangian for S-wave kaons. However, as we will see later,
it is possible to obtain this lagrangian from the P-wave lagrangian by substituting $F(x) \mapsto F(x) + \pi$. The ramifications of this result will be explored later.

The remainder of the paper will be spent solving the stationary conditions for $F(x)$ and $k(x)$, and determining $\omega$ as a function of $a$.

4 Finding the Background SU(2) Baryon Field

The SU(2) baryon field is determined by the profile function $F(x)$, which is obtained by solving the stationary conditions for $F$ using the non-kaon part of the lagrangian in (8). Much of this work has been carried out by Manton [13] using different methods.

The differential equation for $F(x)$ is given by

$$0 = \frac{1}{2} F''(x) - \frac{\cos F \sin F}{1 + x^2} + \frac{4}{\alpha^2} \left( (1 + x^2) \sin^2 F F''(x) \right. + (1 + x^2) \cos F \sin F (F'(x))^2 + 2x \sin^2 F F'(x) - \cos F \sin^3 F \right).$$

The boundary conditions for $F$ are given by

$$F(-\infty) = \pi, \quad F(\infty) = 0.$$

The identity profile function $F(x) = \arccot(x)$ ($F(\rho) = \rho$) is a solution of (9) for every $\alpha$. It corresponds to a baryon uniformly distributed on the hypersphere. For $\alpha$ less than the critical value $\alpha_{cr} = 2\sqrt{2}$ it is the only solution and it is stable.

As $\alpha$ crosses the critical value, the identity solution becomes unstable in favor of an asymmetric solution, where the baryon is located at one of the poles. There are actually two equivalent solutions which correspond to each other under the transformation

$$F(x) \mapsto \pi - F(-x),$$

which leaves the differential equation invariant. We always choose the solution with the baryon localized around $x = -\infty$ (where $F = \pi$).

To find the localized solution, we numerically solve the differential equation using the initial conditions provided by the asymptotic solution

$$F(x) \approx \pi + A/x + \ldots \quad x \ll 0$$

and use trial and error with the value of $A$ until we find a solution which approaches zero as $x \to \infty$. For $a \gg 1/ef_\pi$ (i.e. $\alpha \gg 1$), we expect $F(x)$ to approach the solution corresponding to flat space. In particular, $F$ takes its middle value ($\pi/2$) at around $r = a \sin \rho \sim 1/ef_\pi$ which corresponds to $x \sim -\alpha$. Thus $A \sim \alpha$ for large $\alpha$.

3Localizing the baryon around the $F = \pi$ pole, as opposed to the $F = 0$ pole, results in a slightly lower energy for the physically relevant case $m_\pi \neq 0$. 
5 A Discrete Symmetry

In section 4 we found that the non-kaon part of the lagrangian is symmetric under the transformation

\[ F(x) \mapsto \pi - F(-x). \]  

(10)

This transformation maps the baryon from one pole to the other. A simple coordinate transformation corresponding to point inversion, which also maps the baryon from one pole to another, is given by

\[
\begin{align*}
\phi & \mapsto \phi + \pi, \\
\theta & \mapsto \pi - \theta, \\
x & \mapsto -x, \\
F(x) & \mapsto -F(-x),
\end{align*}
\]

(11)

and leaves the original Skyrme lagrangian invariant.\(^4\) \(^{10}\) differs from the coordinate inversion of (11) by the transformation

\[ F(x) \mapsto F(x) + \pi. \]  

(12)

If one examines closely, one realizes that the kaon part of the P-wave lagrangian in (8) is not symmetric under (11) or (12), regardless of the substitution for \(k(x)\), although it is symmetric under (11). However, (12) (and hence (10)) corresponds to a valid symmetry of the bound-state lagrangian. It is no accident that the non-kaon part is symmetric under (10). To determine what this symmetry transformation does to the kaon field, we need to examine the effect of (12) on the Skyrme lagrangian.

The transformation in (12) corresponds to

\[ U_\pi \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U_\pi. \]

The bound-state approach requires the value of \(\sqrt{U_\pi}\). Technically, the square root is ambiguous because \(U_\pi\) is not positive-definite hermitian. We define the square root to be

\[ \sqrt{U_\pi} = \left( e^{i \frac{F(x)}{2} \tilde{\tau} \cdot \hat{r}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \]

Hence (12) corresponds to

\[ \sqrt{U_\pi} \mapsto \left( \begin{pmatrix} i \tilde{\tau} \cdot \hat{r} \\ 0 \end{pmatrix} \right) \sqrt{U_\pi} = \sqrt{U_\pi} \left( \begin{pmatrix} i \tilde{\tau} \cdot \hat{r} \\ 0 \end{pmatrix} \right). \]

\(^4\)The boundary conditions on \(F(x)\) are different.
If we arrange for the kaon field to transform so that

\[ U_K \mapsto W^\dagger U_K W, \quad W := \left( \begin{array}{cc} i\vec{\tau} \cdot \vec{r} & 0 \\ 0^\dagger & 1 \end{array} \right), \]

then \( U(\vec{x}, t) \) itself will transform as

\[ U \mapsto \left( \begin{array}{cc} -1 & 0 \\ 0^\dagger & 1 \end{array} \right) U. \]  

(14)

The Skyrme lagrangian (1) is invariant under (14). The transformation in (13) corresponds to

\[ K \mapsto -i\vec{\tau} \cdot \vec{r} K. \]  

(15)

If we compare the anzätze in (6) and (7) we find that, aside from a trivial change in the sign of \( k(x) \), (15) corresponds to the exchange in the roles of S-wave and P-wave. The crucial conclusion from the discussion above is that the simultaneous transformations (10) and (15) leave the action invariant. This simultaneous transformation does the following:

- It moves the baryon from one pole to the other.
- It does not affect the boundary conditions of \( F(x) \). Therefore, the value of \( F \) at the pole which the baryon is nearest to changes from \( \pi \) to 0 or vice-versa.
- It transforms the kaon from S-wave to P-wave, but does not otherwise move the kaon (\( k(x) \) does not change, except for an irrelevant change in sign).

For \( \alpha > \alpha_{cr} \) the solution for \( F(x) \) does not respect (10). In this phase, the kaon field is not expected to respect (15) either. Thus, there is no symmetry between the S-wave and P-wave kaons. However, for \( \alpha < \alpha_{cr} \), the solution \( F(x) = \arccot(x) \) (\( F(\rho) = \rho \)) does respect (10). In this phase, there is no distinction between S-wave and P-wave kaons. In the next section, we see the value of \( \omega \) for S-wave and P-wave approach each other as the critical value of \( \alpha \) is approached. For \( \alpha < \alpha_{cr} \), there is only one value of \( \omega \).

### 6 Solving for the Kaon Field

The next goal is to solve the kaon part of the lagrangian in (8) for \( \omega \) and the kaon field as a function of \( \alpha \). In the next section, we will compare the electron fermi level to \( \omega \) and determine the value of \( \alpha \) where kaon condensation sets in.

In the non-uniform phase \( \alpha > 2\sqrt{2} \), we substitute the numerical solution to \( F(x) \) into the Euler-Lagrange equations corresponding to the lagrangian and numerically solve for \( k(x) \) and \( \omega \). We do this separately for P-wave and S-wave.

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5This symmetry is broken by introducing a pion mass, which we neglect in this paper.
For P-wave kaons, we use the lagrangian given in (8). The initial conditions are provided by the asymptotic solution
\[ k(x) \approx k_0 + \ldots \quad x \ll 0. \]
Since the differential equation is linear in \( k(x) \), we may fix the value of \( k_0 = 1 \). The value of \( \omega \) is adjusted until we find a solution which approaches zero as \( x \to \infty \). A typical (unnormalized) solution for \( k(x) \) is given in Fig. 1.

For S-wave kaons, we use the S-wave lagrangian obtained from (8) by the substitution \( F(x) \mapsto F(x) + \pi \). This transformation differs from (10) by coordinate inversion, and so the kaon switches poles instead of the baryon. The initial conditions are provided by the asymptotic solution
\[ k(x) \approx -A/x + \ldots \quad x \ll 0. \]
We fix \( A = 1 \) and adjust the value of \( \omega \) until we find a solution which approaches a constant as \( x \to \infty \). A typical (unnormalized) solution for \( k(x) \) is given in Fig. 2.

The solution for \( \omega \) as a function of \( \alpha \) for both S-wave and P-wave is given in Fig. 3.

In the uniform phase \( \alpha < 2\sqrt{2} \), we substitute the analytic solution \( F(x) = \text{arccot}\ x \) (\( F(\rho) = \rho \)) into (8). As explained in the previous section, there is no distinction between S-wave and P-wave kaons in the uniform phase. If we substitute the same analytic form for \( F(x) \) into the lagrangian appropriate for S-wave, we obtain the same differential equation, except for the coordinate inversion \( x \mapsto -x \).

The resulting differential equation is
\[ k''(x) + \frac{k(x)}{(1 + x^2)^2} \left( \frac{1}{4} - x\sqrt{1 + x^2} - x^2 + \lambda \right) = 0, \quad (16) \]
where
\[ \lambda = \frac{3 + \alpha^2}{2 + \alpha^2} \left( \frac{\omega \alpha}{\pi f}\right)^2 + \frac{e^2 N}{\pi^2(2 + \alpha^2) e f} \frac{\omega \alpha}{e f} - \frac{1}{2 + \alpha^2} \left( \frac{m_K}{e f} \right)^2 \alpha^4. \quad (17) \]

The dependence of \( \omega \) on \( \alpha \) is entirely contained in the expression for \( \lambda \). In the introduction, we claimed that \( \omega \to 0 \) as \( \alpha \to 0 \). From (17), it is clear that this is only possible if there exists a solution to (16) for \( \lambda = 0 \). This is not a trivial prediction and there is no reason to expect it. However, if we rewrite (16) in terms of \( \rho \) instead of \( x \), we do in fact find a solution for \( \lambda = 0 \). It is
\[ k(\rho) = \sin \frac{1}{2} \rho. \quad (18) \]
It is this miracle which ultimately is responsible for the fact that the Skyrme model yields kaon condensation at reasonable densities, if at all. Otherwise, \( \omega \sim \lambda/\alpha \), which is the same \( \alpha \)-dependence as the electron fermi energy. It would not be clear for what values of \( \alpha \), if any, that \( \omega \leq \epsilon_F^e \).

\[^6\text{This solution was also found in a different version of the Skyrme model [15].}\]
Solving (17) with $\lambda = 0$ for $\omega$ as a function of $\alpha$, we find

$$\frac{\omega \alpha}{e f_\pi} = \sqrt{\left(\frac{e^2 N}{2\pi^2}\right)^2 + \left(\frac{m_K}{e f_\pi}\right)^2 \alpha^4 (3 + \alpha^2)} - \frac{e^2 N}{2\pi^2}. \tag{19}$$

After plugging in numbers, we get

$$\omega = (3174 \text{MeV}) \sqrt{1 + 0.02433 \alpha^4 (3 + \alpha^2)} - 1. \tag{20}$$

At the critical point, we find $\omega = 332 \text{MeV}$, which is consistent with the numerical solutions for $\alpha > 2\sqrt{2}$. For $\alpha \rightarrow 0$, we find

$$\omega \approx (38.6 \text{MeV}) \alpha^3 (1 + \text{order } \frac{\alpha^4 (3 + \alpha^2)}{40}). \tag{21}$$

It should be noted that $k(\rho)$ given in (18) satisfies

$$k(\rho) = \sin \frac{1}{2} F(\rho).$$

This is the form for $k(\rho)$ which corresponds to a small rigid rotation into the strange directions [7]. Thus the rigid rotator approximation to the bound-state approach given in [16] becomes exact in the uniform phase. Indeed, the formula (19) for $\omega$ can be expressed in the form obtained in [16] using the rigid rotator approximation:

$$\omega = \frac{N}{8\Phi} \left( \sqrt{1 + \left(\frac{m_K}{M_0}\right)^2} - 1 \right), \quad M_0 = \frac{N}{4\sqrt{1}\Phi}. \tag{18}$$

The appropriate values for $\Phi$ and $\Gamma$ are

$$\Phi = \frac{\pi^2 \alpha (3 + \alpha^2)}{4e^3 f_\pi}, \quad \Gamma = \frac{\pi^2 \alpha^3}{e^3 f_\pi}. \tag{19}$$

In light of the fact that $k(\rho)$ as given in (18) is independent of $m_K$, it is no surprise that the form of $k(\rho)$ matches the rigid rotator form. For $m_K = 0$, we expect the SU(3) flavor symmetry to be restored.

The only remaining question is whether or not $\lambda = 0$ actually yields the lowest value of $\omega$ for all $\alpha$ below the critical value. We don’t expect $\lambda \neq 0$ to yield lower energy solutions for any $\alpha < 2\sqrt{2}$ because

- $\omega(\alpha_{\text{cr}})$ is predicted correctly by $\lambda = 0$. Also, the $\lambda = 0$ solution for $k(x)$ matches the numerical solutions we obtain for $k(x)$ for $\alpha > 2\sqrt{2}$ for both S-wave (inverted) and P-wave.

7The other solution to the quadratic equation (corresponding to a minus sign in front of the square root sign) corresponds to $\omega < 0$ which represents the energy of a $K^+$ in nuclear matter. As expected, $|\omega|$ diverges as $N/\alpha$ as $\alpha \rightarrow 0$ due to the Wess-Zumino term.
• $k(\rho)$ given by the $\lambda = 0$ solution contains no nodes, which is what we expect for the ground state solution.

• $k(\rho)$ given by the $\lambda = 0$ solution matches the rigid rotator solution, a situation we expect for $m_K = 0$.

• $\lambda = 0$ must yield the lowest $\omega$ for $\alpha$ sufficiently close to zero. All other values of $\lambda$ result in a divergent $\omega$ as $\alpha \to 0$.

• A sudden jump from $\lambda = 0$ to a different value of $\lambda$ would involve a discontinuous jump in $\omega$ and in $k(\rho)$, neither of which is expected.

7 Determining the Onset of Kaon Condensation

To study the onset of kaon condensation, we need to understand what is going on in the neutron star. In our model, the neutron star consists of a Skyrme lattice of protons and neutrons, as well as enough electrons to balance the charge of the protons. For simplicity, we shall treat these particles as free particles. We assume that the star is in quasi-equilibrium. In particular, the following reaction

$$n \leftrightarrow p^+ + e^- + \bar{\nu}$$

is in equilibrium. This together with conservation of baryon number and charge allows us to solve three equations

$$\varepsilon_F^n = \varepsilon_F^p + \varepsilon_F^e,$$
$$\rho_B = \rho_p + \rho_n,$$
$$\rho_p = \rho_e,$$

for the three variables $\rho_n$, $\rho_p$, $\rho_e$, in terms of $\alpha$. The baryon number density is given by the inverse volume of the hypersphere

$$\rho_B = (2\pi^2 a^3)^{-1}.$$

The fermi energy $\varepsilon_F$ for a spin 1/2 fermion is given in terms of its number density by

$$\varepsilon_F^j = \sqrt{m_f^2 + k_F^2} = \sqrt{m_f^2 + (3\pi^2 \rho_f)^2/3}.$$

This will allow us to calculate $\varepsilon_F^e$ as a function of $\alpha$. Before doing so, we wish to discuss kaon condensation. In free space, the conversion of electrons to kaons via the reaction

$$e^- \leftrightarrow K^- + \nu$$
is unthinkable, because the mass of the kaon is much larger than that of the electron. However, from the previous section we see that the effective mass of the kaon, ω, approaches zero for large baryon densities. Furthermore, for large baryon densities, the fermi level of the electrons (as well as the protons and neutrons) increases. It follows that there exists a value of α for which ε_F^e = ω. Beyond that density, it becomes favorable for electrons to become kaons. Since kaons are bosons, there is no fermi level for the kaons. Every kaon requires the same energy ω to produce. As explained in [1], kaon condensation on the order of a few times normal nuclear matter density may have an impact on the formation of low mass black holes, by reducing the resistance of the neutron star to collapse. We now compute the density of kaon condensation as predicted by our model.

The formula in (22) for the fermi energy is fully relativistic. We shall make the approximation that the fermi-level electrons are ultra-relativistic and the fermi-level nucleons are non-relativistic. Thus the fermi energies are given by

\[ \varepsilon_F^{(p,n)} = m_N + \frac{(3\pi^2 \rho_{p,n})^{2/3}}{2m_N}, \quad \varepsilon_F^e = (3\pi^2 \rho_e)^{1/3}. \]

Conservation of charge and baryon number allows us to express ρ_n, ρ_p and ρ_e in terms of one parameter, \( \gamma = \rho_p/\rho_B \).

\[ \rho_p = \rho_e = \gamma \rho_B, \quad \rho_n = (1 - \gamma) \rho_B. \]

In terms of this parameter, the fermi levels may be reexpressed

\[ \varepsilon_F^n = (3\pi^2 \rho_B)^{1/3} \beta (1 - \gamma)^{2/3} + m_N, \]
\[ \varepsilon_F^p = (3\pi^2 \rho_B)^{1/3} \beta \gamma^{2/3} + m_N, \]
\[ \varepsilon_F^e = (3\pi^2 \rho_B)^{1/3} \gamma^{1/3}, \]
\[ \beta := \frac{(3\pi^2 \rho_B)^{1/3}}{2m_N}. \]

Plugging in numbers yields

\[ (3\pi^2 \rho_B)^{1/3} = (805/\alpha) \text{ MeV}, \]
\[ \beta = 0.429/\alpha. \]

The equilibrium equation then becomes

\[ \beta((1 - \gamma)^{2/3} - \gamma^{2/3}) = \gamma^{1/3}, \]

which we can solve for \( \gamma \) in terms of \( \alpha \).

To get an order of magnitude estimate, we calculate the fermi energies at \( \alpha = 2\sqrt{2} \). This corresponds to \( \beta = 0.152 \). Solving (23) iteratively for \( \beta \) small, we find

\[ \gamma = \beta^3 - 3\beta^5 + \ldots \approx 3.27 \times 10^{-3} \]
from which we compute the fermi levels

\[ \varepsilon^e_F(\alpha_{cr}) = 42.2 \text{ MeV}, \]
\[ \varepsilon^p_F(\alpha_{cr}) - m_N = 0.95 \text{ MeV}, \]
\[ \varepsilon^n_F(\alpha_{cr}) - m_N = 43.2 \text{ MeV}. \]

The relativistic approximations made earlier are justified by these results. Note that \( \omega(\alpha_{cr}) = 332 \text{ MeV} \) is much larger than \( \varepsilon^e_F \) at the phase transition, so that kaon condensation must take place for \( \alpha < 2\sqrt{2} \). However, \( \omega \) decreases quickly from its critical value and \( \varepsilon^e_F \) continues to rise.\(^8\) For \( \alpha = 1, \omega \approx 40 \text{ MeV} \), which is definitely lower than the fermi level at \( \alpha = 1 \). Therefore, kaon condensation occurs somewhere in the regime \( 1 < \alpha < 2\sqrt{2} \).

To find the transition exactly, we note that (23) can be explicitly solved for \( \alpha \) in terms of \( \gamma \):

\[ \alpha = (0.429)(1 - \gamma)^{2/3} - \gamma^{2/3} \gamma^{1/3}. \]

Hence both sides of the equation \( \varepsilon^e_F = \omega \) can be written explicitly in terms of \( \gamma \). Numerical solution yields \( \gamma = 0.016 \).\(^9\) The following values are found at the onset of kaon condensation

\[ \gamma = 0.016, \]
\[ \alpha = 1.58, \]
\[ a = 0.44 \text{ fm}, \]
\[ \omega = \varepsilon^e_F = 129 \text{ MeV}, \]
\[ \varepsilon^p_F - m_N = 8.8 \text{ MeV}, \]
\[ \varepsilon^n_F - m_N = 138 \text{ MeV}, \]
\[ \rho_B = 0.595 \text{ fm}^{-3}. \]

The density for kaon condensation is on the order of a few (3–4) times the density of normal nuclear matter \( \rho_{\text{nucl}} = 0.16 \text{ fm}^{-3} \).

8 Conclusions

We have predicted the onset of kaon condensation to be \( \rho_B = 0.595 \text{ fm}^{-3} = 3.7\rho_{\text{nucl}} \). We offer a brief comparison to the literature in Table 1. Beyond the critical density, an increasing fraction of neutrons is converted to protons and \( K^- \), which softens the star’s resistance to compression. As explained in [1], a critical density on the order of a few times the nuclear

\(^8\)In the uniform phase, the nucleons lose their identity, so the calculation of the fermi level might seem suspicious in this regime. Nevertheless, it is reasonable to expect the fermi level to continue to rise and for (23) to be a reasonable approximation.

\(^9\)This means protons make up 1.6 percent of the nucleons in the neutron star. Most of the protons have already recombined with electrons to form neutrons.
density is sufficient to reduce the upper mass limit beyond which the collapse of a star to a black hole is inevitable. Our results fall within that range.

The key ingredient which allows the bound-state approach to predict a reasonable transition density is using the electron fermi level. Without the fermi level, kaon condensation would not set in until infinite density.

Now that it has been shown that the Skyrme model can produce reasonable results for $K^-$ condensation, several refinements of the calculation are possible. Since the rigid rotator approximation to the bound-state approach becomes exact for $\alpha < 2\sqrt{2}$, we may calculate the complete $1/N$ correction to the $K^-$ energy using the formulae generated in [17]. A more ambitious project is to replace the hypersphere with a real lattice. In addition, a more sophisticated model of the neutron star may be used to determine the electron fermi energy more precisely. None of these refinements is expected to drastically change the results.

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**References**

[1] G. E. Brown and H. A. Bethe, *Astrophys. Jour.* 423 (1994), 659.

[2] D. B. Kaplan and A. E. Nelson, *Phys. Lett.* B175 (1986), 57.

[3] G. E. Brown, K. Kubodera, M. Rho, *Phys. Lett.* B192 (1987), 273; H. D. Politzer, M. B. Wise, *Phys. Lett.* B273 (1991), 156; G. E. Brown, K. Kubodera, M. Rho, V. Thorsson,
Phys. Lett. B291 (1992), 355; V. Thorsson, M. Prakash, J. Lattimer, Nordita-93/29 N, SUNY-NTG-92-33.

[4] G. E. Brown, C.-H. Lee, M. Rho, V. Thorsson, Nucl. Phys. A567 (1994), 937; C.-H. Lee, H. Jung, D.-P. Min, M. Rho, Phys. Lett. B326 (1994), 14; C.-H. Lee, G. E. Brown, M. Rho, SNU/TP-94-28, hep-ph/9403339.

[5] C.-H. Lee, G. E. Brown, D.-P. Min, M. Rho, Preprint hep-ph/9406311 (June 1994).

[6] H. Yabu, S. Nakamura, F. Myhrer, K. Kubodera, Phys. Lett. B315 (1993), 17; M. Lutz, A. Steiner, W. Weise, “Kaons in dense matter”, to be published.

[7] C. Callan and I. Klebanov, Nucl. Phys. B262 (1985), 365; C. Callan, K. Hornbostel and I. Klebanov, Phys. Lett. B202 (1988), 269; I. Klebanov, Strangeness in the Skyrme Model, in Hadrons and Hadronic Matter, Plenum (1990).

[8] N. Scoccola, H. Nadeau, M. Nowak and M. Rho, Phys. Lett. B201 (1988), 425; J. P. Blaizot, M. Rho and N. Scoccola, Phys. Lett. B209 (1988), 27; N. N. Scoccola, D. P. Min, H. Nadeau and M. Rho, Nucl. Phys. A505 (1989), 497; M. Rho, D. O. Riska and N. N. Scoccola, Z. Phys. A341 (1992), 343.

[9] I. Klebanov, Nucl. Phys. B262 (1985), 133.

[10] E. Wust, G. E. Brown, A. D. Jackson, Nucl. Phys. A468 (1987), 450.

[11] A. D. Jackson, J. J. M. Verbaarschot, Nucl. Phys. A484 (1988), 419.

[12] M. Kugler, S. Schtrikman, Phys. Lett. B208 (1988), 491.

[13] N. S. Manton, Comm. Math. Phys. 111 (1987), 469.

[14] A. D. Jackson, A. Wirzba, L. Castillejo, Phys. Lett. B198 (1987), 315.

[15] H. Forkel, A. D. Jackson, M. Rho, N. N. Scoccola, Nucl. Phys. A509 (1990), 673.

[16] D. Kaplan and I. Klebanov, Nucl. Phys. B335 (1990), 45.

[17] K. Westerberg and I. Klebanov, Phys. Rev. D50 (1994), 5834.

Figures

Fig. 1 P-wave kaon wavefunction (unnormalized) for $\alpha = 5$.

Fig. 2 S-wave kaon wavefunction (unnormalized) for $\alpha = 5$.

Fig. 3 Kaon energy (MeV) vs. $\alpha$. 

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Figure 1: P-wave kaon wavefunction for $\alpha = 5$

Figure 2: S-wave kaon wavefunction for $\alpha = 5$

Figure 3: kaon energy (MeV) vs. $\alpha$
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