Samaritan’s Dilemma: Classical and quantum strategies in Welfare Game

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Effects of classical/quantum correlations and operations in game theory are analyzed using Samaritan’s Dilemma. We observe that introducing either quantum or classical correlations to the game results in the emergence of a unique or multiple Nash equilibria (NE) which do not exist in the original classical game. It is shown that the strategies creating the NE and the amount of payoffs the players receive at these NE’s depend on the type of the correlation. We also discuss whether the Samaritan can resolve the dilemma acting unilaterally.

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I. INTRODUCTION

Game theory is an interdisciplinary approach that can offer insights into economic, political, and social situations which involve decision makers who have different goals and preferences and who know that their actions affect each other. Since the activities of the decision makers include information processing and a physical system is needed for the implementation of the games, it is no surprise that, currently, game theory is becoming an attractive research topic within the quantum information community.\cite{1,2,3,4,5,6,7,8,9,10,11}

By introducing quantum operations and/or quantum correlations into the game theory, a number of games including Prisoner’s Dilemma (PD)\cite{1}, Battle of Sexes (BoS)\cite{2}, Monty Hall problem (MH)\cite{3}, Chicken Game (CG)\cite{4}, and matching pennies (MP)\cite{5} have been studied. Recently, there has been efforts to form a general theory of quantum games\cite{12,13}. The results of the games when played with the tools of quantum mechanics have been shown to be very different from those of their classical counterparts, e.g. the dilemma of the prisoners have been resolved in the quantum game. Since the field is very new and on its way of development, it is necessary to study quantum versions of various classical games and analyze them within the paradigm of quantum mechanics to understand the new features introduced by quantum mechanics better.

A game Γ can be denoted by \( Γ = \{N, (S_i)_{i \in N}, (u_i)_{i \in N}\}\) where \( N \) is the set of players, \( S_i \) is the strategy set for \( i \)-th player, and \( u_i \) is the payoff function from the set of all possible strategy combinations into the set of real numbers for the \( i \)-th player. Then the payoff for the \( i \)-th player can be denoted as \( u_i(s) \) where \( s \) is the combination of the strategies implemented by all players. A two-player game with each player having the strategy sets \( S_A \) and \( S_B \) can be represented as \( Γ = \{\{\text{Alice, Bob}\}, (S_A, S_B), (u_A, u_B)\} \) from which, for example, Alice’s payoff for the strategy combination of \( (α, β) \) where \( α \in S_A \) and \( β \in S_B \) can be written as \( u_A(α, β) \). In classical game theory, any game is fully described by its payoff matrix. Based on the nature of payoffs, games can be classified in three different ways (i) symmetric, \( u_A(α, β) = u_B(β, α) \), and asymmetric games, \( u_A(α, β) \neq u_B(β, α) \), (ii) zero-sum, \( u_A(α, β) + u_B(β, α) = 0 \) for all \( α \in S_A \) and non-zero sum games, if for all \( α \in S_A \) and \( β \in S_B \), \( u_A(α, β) + u_B(β, α) \neq 0 \), and (iii) coordination if the game has at least one Nash equilibrium (NE), and discoordination games if there is no NE in pure strategies\cite{15,16}.

Nash equilibrium is the most commonly used equilibrium concept for strategic games, and it can be regarded as the steady state of the strategic interaction. The concept of NE is based on the premises that each player acts rationally according to the belief he/she has on the other player, and that the beliefs of each player about the other one is correct. Once the players acts according to NE, no one can take another action to unilaterally deviate from it in order to increase his/her payoff. In an NE, each player’s choice of action is the best response to the actions taken by the other player. Although in pure strategies an NE need not exist, there is always at least one NE in mixed strategies where the players are allowed to randomize among their action. The most common difficulty encountered in the concept of NE is that NE need not be unique. There might be multiple NE’s which avoids making sharp decisions. In such cases, certain NE’s can be isolated as “focal” in that they are clearly better for all players, and they yield a higher payoff to every player than any other NE’s. If there exists only one NE like this, then players will be self-enforced to play it, thus solving...
A two-player-two-strategy game where the difficulty induced by multiple NE’s.

Classical pure strategies. There is no such a situation, therefore the game has no NE in classical pure strategies.

Table II: Payoff matrix for Welfare Game where Alice is the donor and Bob is the beneficiary. Arrows show the best response of a player for a chosen strategy of the other player. Any strategy pointed by two arrows is an NE. In this game, there is no such a situation, therefore the game has no NE in classical pure strategies.

| Alice: Aid (A) | Bob: Work (W) | Bob: Loaf (L) |
|---------------|--------------|--------------|
| (3, 2)        | (−1, 3)      |
|               | ↑            | ↓            |
| Alice: No Aid (N) | (−1, 1)      | (0, 0)       |

The difficulty induced by multiple NE’s.

An example of payoff matrix is given in Table II for a two-player-two-strategy game where \( S_A = S_B = \{ \alpha_1, \beta_1 \} \). The payoffs of players for all possible strategy combinations are represented as \((\ldots, \ldots)\) with the first element being Alice’s payoff and the second Bob’s. This payoff matrix represents a symmetric game if \( d>b \), \( x=w \), and \( d+z=0 \). When \( a>c, d>b, x>y \), or \( c>a, b>d, w>x, z>y \) is satisfied, it represents a discoordination game. On the other hand, for example, if \( a>c, d>b, w>x, z>y \), the game becomes a coordination game with two NE’s in pure strategies.

If we look at the games which have been studied using the quantum mechanical tools, we see that, except the MH and MP games, which are sequential move games, all the games are simultaneous move games. The properties of the simultaneous move games studied so far, then can be listed as: PD&CG: non-zero-sum, symmetric, coordination; and BoS: non-zero-sum, asymmetric, coordination. Although, for these games, it has been shown that pure quantum strategies bring interesting features and, in some cases, resolve the dilemmas of the game, it is not very clear whether quantum strategies can resolve problems in other types of games or payoff matrices. It is, therefore, interesting to study other games belonging to different classes and having different payoff matrix structures. One of such payoff matrices is that of the so-called “Welfare game” or as often referred to as “Samaritan’s dilemma,” which can be classified as a non-zero-sum, asymmetric, and discoordination game. This game is chosen because it appears in a wide range of social, economical, and political issues and the dilemma present in the game is as strong as the dilemmas existing in PD and BoS games.

In this paper, we study the Welfare game whose payoff matrix is shown in Table II. The paper is organized as follows: In Sec. II, we will introduce the classical game and discuss the game using both pure and mixed classical strategies without any shared correlation between the players. Then in Sec. III, we will introduce the quantum version of the game and show the features of the game in different situations: (i) quantum operations with shared quantum correlations, (ii) quantum operations with shared classical correlations, (iii) classical operations with shared classical correlations, and (iv) classical operations with shared quantum correlations. Section IV will include a discussion of whether the players can resolve the dilemma by unilateral actions, and finally, in Sec. V, we will give a brief summary and discussion of our results.

II. WELFARE GAME AND SAMARITAN’S DILEMMA

The Samaritan’s Dilemma arises whenever actual or anticipated “altruistic” behavior of the “Samaritan” (Alice) leads to exploitation on the part of the potential beneficiary (Bob), such that Alice suffers a welfare loss when compared to the situation that would have been obtained if Bob had not acted strategically [14, 15]. Most people have personally experienced this dilemma when confronted with people “in need.” Although there is a desire to help those people who cannot help themselves, there is the recognition that a handout may be harmful to the long-run interests of the recipient. If the condition of the person in need is beyond that person’s control, then there is no dilemma for Samaritan. However, the person in need can influence or create situations which will evoke Samaritan’s help. Then, a dilemma arises because Samaritan wants to help, however, the action of the person in need leads an increase in the amount of help which is not desirable for the Samaritan. Moreover, Samaritan cannot retaliate to minimize or stop this exploitation because doing so, which is a punishment for the people in need, will harm the Samaritan’s own interests in the short run. The Samaritans Dilemma, which was modelled as a two-person strategic game by the Nobel Laureate economist James Buchanan, is widespread in a wide range of distinct issue areas from international politics, government welfare programs, and family issues [14].

The game and the payoff matrix studied in this paper are taken from Ref. [14] where the specific game is named as The Welfare Game. In this game, Alice wishes to aid Bob if he searches for work but not otherwise. On the other hand, Bob searches for work if he cannot get aid from Alice.

Now let us analyze this classical game for pure and mixed strategies. The strategy combinations can be listed as \((A, W)\), \((A, L)\), \((N, W)\), and \((N, L)\). When players use only pure strategies, there is no dominant strategy for neither of the players and, moreover, there is no NE, that is why this is a discoordination game. This can be explained as follows: \((A, W)\) is not an NE because if Alice chooses A, Bob can respond with strategy L where he gets a better payoff (three) as shown with arrow in
Table III (A,L) is not an NE because, in this case, Alice will switch to N. The strategies (N,L) and (N,W) are not NE either, because for the former one Bob will switch to W to get payoff one, whereas for the latter case Alice will switch to A to increase her payoff from $-1$ to 3. Therefore, this game has no NE when played with pure classical strategies.

In mixed classical strategies, we assign the probabilities $p$ and $(1-p)$ to the events that Alice chooses strategies A and N, respectively. In the same way, for Bob’s choices of W and L, we assign the probabilities $q$ and $(1-q)$. Then the payoffs for Alice and Bob become

$$
\begin{align*}
\mathcal{S}_A &= 3pq - p(1-q) - q(1-p), \\
\mathcal{S}_B &= 2pq + 3p(1-q) + q(1-p).
\end{align*}
$$

(1)

It can be easily shown that $p = 0.5$ and $q = 0.2$ correspond to the NE for the game with average payoffs given as $\mathcal{S}_A = -0.2$ and $\mathcal{S}_B = 1.5$. In this case, the payoff of Alice is negative which is not a desirable result for her. (A,L) and (N,L) emerge as the most probable strategies with probabilities 0.4.

Samaritan’s dilemma game is a very good example of how the altruistic and selfish behavior of the players can affect a game and its dynamics. It is different than the other already studied games because it has no NE in pure strategies and it is only Alice who is facing the dilemma. Games with no NE are interesting because they represent situations in which individual players might never settle down to a stable solution. Therefore, the important step in resolving the dilemma is to find an NE on which the players can settle down.

In this work, we look for the strategies utilizing classical or quantum mechanical toolboxes to find solutions to the following problem: Is there a unique NE where the players can settle? If the answer to this question is YES, then we ask how the payoffs of the players compare with each other and to what extent this NE strategy can resolve the dilemma: CASE I: $\mathcal{S}_A < 0$ (insufficient solution), CASE II: $0 \leq \mathcal{S}_A \leq \mathcal{S}_B$ (weak solution), and CASE III: $0 \leq \mathcal{S}_B < \mathcal{S}_A$ (strong solution). Finding a unique NE is the first step in resolving the dilemma. The positive payoffs players receive at a unique NE means that both players are satisfied with the outcome and there is no loss of resources for Alice (CASE II and III). The most desirable solution of the dilemma for Alice is represented in CASE III which can easily be seen from the original payoff matrix shown in Table III.

### III. QUANTUM VERSION OF THE GAME AND EFFECTS OF VARIOUS STRATEGIC SPACES

Since the Welfare Game is a two-player game, quantum strategies can be introduced using the physical model given in Fig. 1. In this physical model, starting from an initial product state $|\psi_0\rangle = |fg\rangle$, the referee prepares the maximally entangled state (MES) $\hat{\rho}_{in} = \hat{J}|\psi_0\rangle\langle\psi_0|\hat{J}^\dagger$ where $\hat{J}$ is the entangling operator defined as

$$
\hat{J}|fg\rangle = \frac{1}{\sqrt{2}}[|fg\rangle + i(-1)^{(f+g)}|(1-f)(1-g)\rangle] \tag{2}
$$

with $f, g = 0, 1$. In this scheme, $\hat{J}$ is chosen such that the original classical game can be reproduced if the classical strategies A and W are defined with the identity operator $\sigma_0$, and N and L are defined with the bit flip operator $i\sigma_y$ when $|\psi_0\rangle = |00\rangle$. Assignment of the operators to the classical strategies denoted by A, W, N, and L is initial state dependent, i.e., when $|\psi_0\rangle = |01\rangle$, A and L should be defined with $\sigma_0$, while N and W with $i\sigma_y$.

Let us assume that the referee prepares $\hat{\rho}_{in}$ and sends one of the qubits of this state to Alice and the other one to Bob. Alice and Bob perform local operations, respectively, denoted by $\hat{U}_A$ and $\hat{U}_B$ on each of their qubits separately. After these operations, the resulting state becomes

$$
\hat{\rho}_{out} = (\hat{U}_A \otimes \hat{U}_B)\hat{\rho}_{in}(\hat{U}_A^\dagger \otimes \hat{U}_B^\dagger). \tag{3}
$$

The referee who receives this final state first performs $\hat{J}^\dagger\hat{\rho}_{out}\hat{J}$ and then makes a projective measurement $\Pi_n = |j\ell\rangle\langle j\ell|_{\{j,\ell=0,1\}}$ with $n = 2j + \ell$ corresponding to the projection onto the orthonormal basis $\{|AW\rangle, |AL\rangle, |NW\rangle, |NL\rangle\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. According to the measurement outcome $n$, the referee assigns to each player the payoff chosen from the payoff matrix given in Table III e.g., if $n = 0$ then the payoffs assigned to Alice and Bob are 3 and 2, respectively. Then the average payoff of the players can be written as

$$
\begin{align*}
\mathcal{S}_A &= \sum_n a_n \operatorname{Tr}(\Pi_n\hat{J}^\dagger\hat{\rho}_{out}\hat{J})_{\hat{J}^\dagger P_{j\ell}} \\
\mathcal{S}_B &= \sum_n b_n \operatorname{Tr}(\Pi_n\hat{J}^\dagger\hat{\rho}_{out}\hat{J})_{\hat{J}^\dagger P_{j\ell}} \tag{4}
\end{align*}
$$

with $a_{\{n=0,1,2,3\}} = \{3, -1, -1, 0\}$ and $b_{\{n=0,1,2,3\}} = \{2, 3, 1, 0\}$ being the payoffs of Alice and Bob chosen from Table III. $P_{j\ell}$ represents the probability of obtaining the measurement outcome $n$. 

![Fig. 1: Schematic configuration of the quantum version of 2 × 2 strategic games.](image)
Players are restricted to choose their operators, $\hat{U}_A$ and $\hat{U}_B$, from two- and one-parameter SU(2) as

$$
\hat{U}_A = \begin{pmatrix}
e^{i\phi_A} \cos \frac{\theta_A}{2} & \sin \frac{\theta_A}{2} \\
-\sin \frac{\theta_A}{2} & e^{-i\phi_A} \cos \frac{\theta_A}{2}
\end{pmatrix},
$$

with $0 \leq \phi_A \leq \pi/2$ and $0 \leq \theta_A \leq \pi$. Bob’s operator, $\hat{U}_B(\theta_B, \phi_B)$, can be obtained in the same way by replacing $A$ with $B$ in Eq. (5). In the following subsections, we investigate the evolution of the Welfare Game with various strategy sets: (A) Quantum operations and quantum correlations, (B) quantum operations and classical correlations, (C) classical operations and classical correlations, and (D) classical operations and quantum correlations.

A. Quantum operations and quantum correlations

Before proceeding further, it is worth noting that if the players use quantum operations with no shared correlated states, then the situation is similar to playing the game with classical mixed strategies which is discussed in Sec. II.

1. One-parameter SU(2) operators

This operator set for Alice and Bob can be obtained from Eq. (4) by setting $\phi_A = 0$ and $\phi_B = 0$. First, let us assume that $|\psi_0\rangle = |00\rangle$, thus the entangled state becomes $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$ with $|\psi_{in}\rangle = (|00\rangle + i|11\rangle)/\sqrt{2}$. Then the payoff for Alice and Bob are found as

$$
\begin{align*}
\$A &= \frac{1}{4}[1 + 3(\cos \theta_A + \cos \theta_B) + 5 \cos \theta_A \cos \theta_B], \\
\$B &= \frac{1}{2}[3 + 2 \cos \theta_A - \cos \theta_A \cos \theta_B].
\end{align*}
$$

It is seen from Eq. (6) that $\$B$ is always positive whereas $\$A$ is either positive or negative depending on the action of Bob. Therefore, Alice cannot always get a positive payoff and hence cannot resolve her dilemma by acting unilaterally and independently of Bob’s strategy. After some straightforward calculations, it can be found that players can achieve an NE if they choose $(\theta_A = \pi/2, \cos \theta_B = -3/5)$ corresponding to the operators $\hat{U}_A = (\sigma_0 + i\sigma_y)/\sqrt{2}$ and $\hat{U}_B = (\sigma_0 + 3i\sigma_y)/\sqrt{5}$ where $\sigma_i$ are the Pauli operators. At this unique NE, the payoffs for the players become $\$A = -0.2$ and $\$B = 1.5.

On the other hand, when the initial state is $|\psi_0\rangle = |01\rangle$, an NE is found at $(\theta_A = \pi/2, \cos \theta_B = 3/5)$ corresponding to $\hat{U}_A = (\sigma_0 + i\sigma_y)/\sqrt{2}$ and $\hat{U}_B = (2\sigma_0 + i\sigma_y)/\sqrt{5}$ with the payoffs $\$A = -0.2$ and $\$B = 1.5.

It is seen that playing the quantum version of the game with one-parameter set of operators and a shared MES between the players reproduces the results of the classical mixed strategy. Both cases have one NE with the same payoffs $\$A = -0.2$ and $\$B = 1.5$. This strategy set resolves the dilemma, but with an insufficient solution for Alice (CASE I).

2. Two-parameter SU(2) operators

This set of quantum operations is defined as given in Eq. (4). For the case when $|\psi_0\rangle = |00\rangle$, the expressions for the payoffs are given as $\$A = 3P_{00} - P_{01} - P_{10}$ and $\$B = 2P_{00} + 3P_{01} + P_{10}$ with

$$
\begin{align*}
P_{00} &= \cos^2(\theta_A/2) \cos^2(\theta_B/2) \cos^2(\phi_A + \phi_B), \\
P_{01} &= |x \sin \phi_B - y \cos \phi_A|^2, \\
P_{10} &= |x \cos \phi_B - y \sin \phi_A|^2.
\end{align*}
$$

where $P_{ij}$ is calculated from Eq. (4) and we set $x = \sin(\theta_A/2) \cos(\theta_B/2)$ and $y = \cos(\theta_A/2) \sin(\theta_B/2)$. A straightforward analysis of these equations reveals that there is a unique NE which appears at $(\theta_A = 0, \phi_A = \pi/2, \theta_B = 0, \phi_B = \pi/2)$ corresponding to $\hat{U}_A = \hat{U}_B = i\sigma_z$ with the payoff $(\$A, \$B)$ given as $(3, 2)$. We see that introducing quantum operations and correlations results in the emergence of this NE point which cannot be seen when the game is played with pure classical strategies. Thus the original discoordination game becomes a coordination game. This unique NE gives Alice the highest payoff she can get in this game. This is the NE point where Alice always wants to achieve because both players benefit from playing the game, moreover, Alice does not lose her resources. Therefore, Alice’s dilemma is resolved in the stronger sense (CASE III). It is also seen that players receive higher payoffs than those obtained if a classical mixed strategy were used.

This new move of the players can be included into the game payoff matrix as a new strategy $\hat{U}$. This, indeed, shows that the quantum strategy introduced into the game transformed the original $(2 \times 2)$ game into a new game which can be described with the new $(3 \times 3)$ payoff matrix. The new payoff matrix of the game is shown in Table III where the new strategy $\hat{U}_A = \hat{U}_B = i\sigma_z$ for the players are depicted as M. This new game payoff matrix includes the original classical game payoff matrix as its subset. Consequently, one can say that the classical game is a subgame of its quantum version. In other words, one can say that in order to reproduce the same results of this quantum game in classical settings, the players, Alice and Bob, should be given the strategy sets $\{A, N, M\}$ and $\{W, L, M\}$, respectively. Then in order to classically communicate their chosen strategy to the referee, each player needs 2 classical bits (c-bits) resulting in a total of 4 c-bits. Note that the same task is completed using four qubits (two prepared by the referee and distributed, and two qubits sent by the players back to the referee after
being operated on) when shared entanglement is used. This corresponds to a communication cost of 2 e-bits (1 e-bit for the referee and 1 e-bit for the players).

On the other hand, when $|\psi_0\rangle = |01\rangle$, we found four NE’s with equal payoffs $(S_A, S_B) = (3, 2)$ when the players choose the following quantum operations: $(\theta_A = \pi, \phi_A = 0, \theta_B = \phi_B = \pi/2), (\theta_A = \pi/2, \phi_A = 0, \theta_B = \phi_B = \pi/2), (\theta_A = 2\pi/3, \phi_A = 0, \theta_B = \pi/3, \phi_B = \pi/2), \text{and } (\theta_A = 3\pi/4, \phi_A = 0, \theta_B = \pi/4, \phi_B = \pi/2)$. These, respectively, correspond to the operators $(U_A, \hat{U}_B)$ as follows $(N, P) = (i\sigma_y, i\sigma_z), (T, Q) = (\sigma_A + i\sigma_B, i\sigma_A + \sigma_B), (Y, R) = \frac{\sigma_A + i\sigma_B}{\sqrt{2}}, \text{and } (Z, S) = (\gamma_0(\sigma_0 + i\sigma_1), i\sigma_0(\sigma_1 + \sigma_1), i\sigma_0(\sigma_1 + \sigma_1))$ where $\gamma_0 = \cos(3\pi/8), \gamma_1 = \tan(3\pi/8), \delta_0 = \cos(\pi/8), \text{and } \delta_1 = \tan(\pi/8)$.

These NE’s have higher payoffs for both players than those obtained when a classical mixed strategy is used. However, in this pure quantum strategy case, the dilemma of the Samaritan (Alice) still continues, because both players cannot decide which NE point to choose. For example, if Alice thinks that Bob will play $i\sigma_y$ then she will play $i\sigma_y$ to reach the first NE. However, since this is a simultaneous move game and there is no classical communication between the players, Bob may play $\frac{i(\sigma_A + \sigma_B)}{\sqrt{2}}$ (because this action will take him to the second NE point) while Alice plays $i\sigma_y$. Such a case will result in the case $S_A < S_B$ and will lower the payoffs of both players. Therefore, still a dilemma exists in the game; however the nature of dilemma has changed.

As we have done for the previous case, these four new strategies which give NE’s can be added to the payoff matrix of the original game. This results in a $(5 \times 6)$ payoff matrix, as depicted in Table III which has the original classical game as a subgame. If the players are given this payoff matrix and restricted to classical communication then a total of 6 e-bits are needed to play this game whereas in quantum strategies what is needed is 2 e-bits.

With these results, it is seen that the dynamics of the game, that is whether the dilemma is resolved or not, the strategies, and the payoffs of the players depend on the initial state and hence on the shared MES. So far, we have considered the case that the referee not only prepares the MES but also informs the players of the initial state. Thus the players knowing the shared MES can choose their best moves to resolve the dilemma. If they do not know which type of the MES they share, the players cannot decide which of the above discussed strategies to apply. Then it becomes interesting to ask the question of what the best strategy is for the players if they do not know the input state or even more interesting is the question of what they can do if the source the referee uses is corrupt.

Assuming that the source prepares the state $|00\rangle$ with probability $p$ and $|01\rangle$ with probability $1 - p$, the state distributed to the players by the referee becomes $\hat{\rho}_n = pJ|00\rangle\langle 00|J + (1 - p)J|01\rangle\langle 01|J$. Hence the payoffs for the players become as $S_A = pS_A + (1 - p)S_B$ and $S_B = pS_B + (1 - p)S_B'$ where the expressions with ‘$\prime$’ and ‘$\prime\prime$’ corresponds to the payoffs the players receive when the input state is $|00\rangle$ and $|01\rangle$, respectively. This modification of the payoffs results in the emergence of new strategies and NE’s for different values of $p$. The problem becomes very unpleasant for the players because the number of NE’s increases preventing the players from resolving the dilemma. To give an idea on the strategies of the players which give rise new NE’s, we listed some of them in Table IV.

### B. Quantum operations and classical correlations

First let us assume that operators are chosen from the two parameter SU(2) set while sharing classical correlations. This can occur, for example, when one or both players induce phase damping on the originally shared MES until the off-diagonal elements of the density operator disappears or the referee distributes such a state to them. If we assume that the initially shared MES is $|\psi_{in}\rangle = (|00\rangle - i|11\rangle)/\sqrt{2}$, then the shared correlation will become the classical correlation $\hat{\rho}_n = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$ after the damping. This new setting of the problem is depicted in Fig. IV from which the payoffs for the players are calculated as

- $S_A = \frac{1}{4} \left[ 1 + 5 \cos \theta_A \cos \theta_B - 3 \sin \theta_A \sin \theta_B \sin(\phi_A + \phi_B) \right]$;
- $S_B = \frac{1}{2} \left[ 3 - \cos \theta_A \cos \theta_B - 2 \sin \theta_A \sin \theta_B \cos \phi_A \sin \phi_B \right]$.

Alice cannot obtain a positive payoff by acting unilaterally and should try for NE. We find that there appears an NE with a payoff $(0.25, 1.5)$ when $\hat{U}_B = (\sigma_0 + i\sigma_1)/\sqrt{2}$ and Alice restricts herself to one parameter SU(2) operators that is $\phi_A = 0$ and $\{ \forall \theta_A : 0 \leq \theta_A \leq \pi \}$. Since neither of the players can make her/his payoff arbitrarily
TABLE IV: New payoff matrix for the Welfare game when players share the quantum correlation \(|01\rangle - i|10\rangle|/\sqrt{2}\), and use two-parameter SU(2) operators. The operators which result in new NE’s are included together with the original payoff matrix (upper left \(2 \times 2\) sub-matrix). The elements in square boxes correspond to NE’s in this game. Gray colored \(2 \times 2\) submatrix corresponds to the original classical game payoff matrix.

| Alice: A | Bob: W | Bob: L | Bob: P | Bob: Q | Bob: R | Bob: S |
|---------|--------|--------|--------|--------|--------|--------|
| (3, 2)  | (3, 2) | (3, 2) | (3, 2) | (3, 2) | (3, 2) | (3, 2) |
| (−1, 1) | (−1, 1) | (−1, 1) | (−1, 1) | (−1, 1) | (−1, 1) | (−1, 1) |
| (0, 0)  | (0, 0)  | (0, 0)  | (0, 0)  | (0, 0)  | (0, 0)  | (0, 0)  |

TABLE V: Strategies of players at the NE points and their corresponding payoffs when the source is corrupted. The players should know the characteristic of the source to decide on their strategy. \(p\) and \((1 - p)\) are, respectively, the probability that \(|00\rangle\langle 00|\) and \(|01\rangle\langle 01|\) are sent from the source.

| \(p\)   | \((\hat{U}_A, \hat{U}_B)\) | \((\hat{S}_A, \hat{S}_B)\) |
|---------|-------------------|-------------------|
| \(1/4\) | \((\sigma_0, \sigma_0)\) | \((0, 11/4)\) |
|         | \((i\sigma_y, i\sigma_y)\) | \((2, 9/4)\) |
| \(1/2\) | \((\sigma_0, \sigma_y)\) | \((1, 5/2)\) |
|         | \((i\sigma_y, \sigma_y)\) | \((1, 5/2)\) |
|         | \((i\sigma_z, \sigma_z)\) | \((1, 5/2)\) |
| \(3/4\) | \((\sigma_0, \sigma_y)\) | \((0, 11/4)\) |
|         | \((i\sigma_z, i\sigma_y)\) | \((2, 9/4)\) |

![Diagram](image)

FIG. 2: Introducing classical correlations into the quantum version of the Welfare game. \(\hat{\rho}_{in}\) is the classical correlations obtained from the MES after it is subjected to noise. The operations inside the dotted boxes are performed by the referee.

as in Eq. (11) with all the \(-\) being replaced by \(+\) and vice versa except the \(+\) in the \(\sin(\cdots)\) term. Consequently, we find an NE when \(\hat{U}_A = (\sigma_0 + i\sigma_y)\sqrt{2}\) and \(\hat{U}_B = i(\sigma_y + \sigma_z)\sqrt{2}\) are chosen which gives the payoffs as \((\hat{S}_A, \hat{S}_B) = (2.5, 1)\). This NE is a self-enforcing point because by the choice of this specific \(\hat{U}_A\), Alice assures a positive payoff for herself and at the same time for Bob which is independent of Bob’s action. When Alice applies this operation then the only thing that one can expect from a rational Bob is to maximize his payoff by applying the above \(\hat{U}_B\). Therefore dilemma is resolved (CASE III).

If the players share a full rank classical correlation that is \(\hat{\rho}_{in} = [\langle 00|00\rangle + |10\rangle\langle 10| + |01\rangle\langle 01| + |11\rangle\langle 11|]/4\), they get constant payoff \((0.25, 1.5)\) independent of their quantum operations, and thus arriving at CASE II.

On the other hand, when the players are restricted to one-parameter SU(2) operators, they get the payoff \((0.25, 1.5)\) at the NE with \(\hat{U}_A = \hat{U}_B = (\sigma_0 + i\sigma_y)\sqrt{2}\) independent of the classical correlation they share.

These results suggest that after the maximally entangled state is distributed by the referee, Alice and/or Bob can induce damping to her/his qubit so that the quantum correlation becomes a classical one. Then they can have unique strategy to go to an NE where both gets positive payoffs. It must be noted that these solutions of the game cannot be achieved by unilateral move of Alice or Bob. That is, the dilemma is resolved only with the condition that players are enforced to play the strategy that will take both of them to the NE.

C. Classical operations and classical correlations

Classical operations are a subset of quantum operations. The players have two operations either the identity operator \(\sigma_0\) or the flip operator \(i\sigma_y\). Here we consider the classical correlations as in the above subsection. Let us assume that the classical correlation is \(\hat{\rho}_{in} = [\langle 00|00\rangle + |11\rangle\langle 11|]/2\). Due to this classical correlation, the diagonal and off-diagonal elements of the payoff matrix of the original game (Table II) are averaged out separately, resulting in a new game with the
| Alice: $\sigma_0$ | Bob: $i\sigma_y$ |
|----------------|----------------|
| $(1.5, 1)$     | $(-1, 2)$      |
| $\uparrow$     | $\downarrow$   |
| Alice: $i\sigma_y$ | $(−1, 2) ← (1.5, 1)$ |

TABLE VI: The new payoff matrix for the Welfare game when players share the classical correlation $\hat{\rho}_n = [\langle 00 | 00 \rangle + | 11 \rangle \langle 11 |]/2$, and use classical operations.

The game becomes more interesting if we now let the players choose a mixed strategy when applying the classical operations, that is Alice and Bob apply $\sigma_0$ and $i\sigma_y$, respectively, with probabilities $p$ and $q$, and $i\sigma_y$, respectively, with probabilities $1-p$ and $1-q$. In this case, it can be easily shown that there is a unique NE when $p = 0.5$ and $q = 0.2$ with payoff $(0.25, 1.5)$. Although, the values of $p$ and $q$ at NE is the same as the case discussed in Section II where only classical mixed strategy for the operators are allowed without any correlated shared state, there is an increase in the payoff of Alice from $−0.2$ to $0.25$ owing to the shared classical correlation. This is interesting because an NE appears where both Alice and Bob have positive payoff, implying that the dilemma is resolved in the weaker sense (CASE II).

### IV. BOB RESTRICTED TO ONLY CLASSICAL OPERATIONS

Since the dilemma in this game is that of Alice, it is natural to ask the question whether she can escape from this dilemma by restricting Bob to only classical operations while she uses quantum operations. We assume that the players share the MES, $[\langle 00 | + i|11 \rangle]/\sqrt{2}$, Alice uses a fixed quantum operator $\hat{U}_A$ chosen from the set of general SU(2) operators given as

$$\hat{U}_A = \begin{pmatrix} e^{i\phi_A \cos \frac{\theta_A}{2}} & e^{i\varphi_A \sin \frac{\theta_A}{2}} \\ -e^{-i\varphi_A \sin \frac{\theta_A}{2}} & e^{-i\phi_A \cos \frac{\theta_A}{2}} \end{pmatrix},$$

with $0 \leq \phi_A, \varphi_A \leq \pi/2$ and $0 \leq \theta_A \leq \pi$, and Bob applies a mixed classical strategy that is he applies $\hat{U}_B^n = \sigma_0$ with probability $p_0$ and $\hat{U}_B^1 = i\sigma_y$ with probability $p_1$ where $p_0 + p_1 = 1$. Then what is the best strategy for Alice to solve her dilemma?

In this setting, the state after Alice and Bob apply their operators becomes

$$\hat{\rho}_{\text{out}} = \frac{1}{2} \sum_{k=0}^1 p_k (\hat{U}_A \otimes \hat{U}_B^k) \hat{\rho}_{\text{in}} (\hat{U}_A^k \otimes \hat{U}_B^k),$$

where $\hat{\rho}_{\text{out}}'$ and $\hat{\rho}_{\text{out}}''$ are the output density operators when Bob applies $\sigma_0$ and $i\sigma_y$, respectively, and we have used $p = p_0$. The average payoffs can be calculated from Eq. (10) as

$$S_A = \sum_n a_n P_{j\ell} = \sum_n a_n (p P_{j\ell} + (1-p) P_{j\ell}'),$$

$$S_B = \sum_n a_n P_{j\ell} = \sum_n b_n (p P_{j\ell} + (1-p) P_{j\ell}'),$$

from which we can write

$$P_{00} = p \cos^2 \frac{\theta_A}{2} \cos^2 \varphi_A + (1-p) \sin^2 \frac{\theta_A}{2} \sin^2 \varphi_A,$$

$$P_{01} = (1-p) \cos^2 \frac{\theta_A}{2} \cos^2 \varphi_A + p \sin^2 \frac{\theta_A}{2} \sin^2 \varphi_A,$$

$$P_{10} = (1-p) \cos^2 \frac{\theta_A}{2} \sin^2 \varphi_A + p \sin^2 \frac{\theta_A}{2} \cos^2 \varphi_A.$$
Classical and quantum correlations are denoted by “Class. Corr.” and “Quant. Corr.”, respectively. Positive (CASE 1). Indeed, when the payoffs are analyzed in detail, it can be seen that Alice can never achieve, for any value of \( \theta_A \) a positive payoff independent of Bob's strategy defined by \( p \).

If Alice chooses her operator from two-parameter SU(2) operators, then it is seen that there is no NE in the game and there is no strategy for Alice to have a positive payoff. For any strategy of Alice, Bob can find a \( p \) value that will minimize Alice’s payoff and vice versa.

When Alice decides to choose her operator as a general SU(2) operator, she will have three parameters to optimize her payoff. In this case, we find that payoffs of both players become independent of Bob’s strategy when Alice’s strategy is either \( \theta = \pi/2, \phi = \varphi = \pi/4 \) resulting in \( \hat{U}_A = [\sigma_0 + i(\sigma_x + \sigma_y + \sigma_z)]/2 \) or \( \theta = \pi/2, \phi = 0, \varphi = \pi/2 \) resulting in \( \hat{U}_A = (\sigma_0 + i\sigma_z)/\sqrt{2} \). The players get the payoffs (0,2.5) and (1,2.5), respectively, for the first and second strategies (CASE II). On the other hand, if Alice applies the operator \( \theta = \pi/2, \phi = 0, \varphi = \pi/4 \) that is \( \hat{U}_A = [\sigma_0 + i(\sigma_x + \sigma_y)/\sqrt{2}] + \sqrt{2} \) or \( \theta = \pi/2, \phi = \pi/4, \varphi = \pi/2 \) that is \( \hat{U}_A = [\sigma_0 + i(\sqrt{2}\sigma_x + \sigma_y)]/2 \), then while Bob’s payoff becomes 2 and is independent of his own strategy, the payoff of Alice is dependent on Bob’s strategy through \( \hat{S}_A = (1 + 3p)/4 \) and \( \hat{S}_A = 1 - 3p/4 \), re-
spectively, for the first and second operators. With these moves of Alice, both players get positive payoffs which resolves the dilemma in the weak sense (CASE II).

A selfish and evil-thinking Bob might try to prevent Alice having a positive payoff. Then the only thing he can do is to induce damping on his qubit until the initial MES becomes a mixed state with no off-diagonal elements (classical correlation). However, in this case, Alice can choose her operator such that $\theta_A = \pi/2$ making both her and Bob’s payoff independent of Bob’s strategy. This results in a positive payoff for both players with $\$A = 0.25$ and $\$B = 1.5$ resolving Alice’s dilemma as in CASE II.

It is understood that by acting unilaterally Alice can resolve her dilemma only when she is allowed to use operators chosen from three-parameter SU(2) set and Bob is restricted to classical operations. Even in this case, only a solution satisfying CASE II is achieved. When Bob is also allowed to use the same type of operator, for each operator of Alice, Bob can always find an operator which will put Alice into dilemma.

### V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have discussed the effects of classical and quantum correlations in the Welfare Game using the quantization scheme proposed by Eisert et al. [1]. This comparative study of the effects of classical/quantum correlations and classical/quantum operations not only confirmed some of the known effects of using the quantum mechanical toolbox in game theory but also revealed that in some circumstances one need not stick to quantum operations or correlations but rather use simple classical correlations and operations. This can be clearly seen in Tables VIII and IX which summarize the results of this study.

The significant effect of using the quantum operations with shared quantum correlation between the players is the emergence of NE’s which turn the game into a coordination game. Together with the results of other games studied using the quantum mechanical toolbox, we can say that there is at least one NE in $2 \times 2$ games with arbitrary payoff matrices when played using one or two parameter SU(2) operators. Sometimes there can emerge more than one NE with either the same or different payoffs based on the structure of the payoff matrix of the original game. When this happens, players are indifferent between the multiple NE. Then it becomes unclear how the players should behave, therefore this NE cannot be taken as a solution for the problem. In such cases, introducing the quantum mechanical toolbox does not make the things easier but rather more complicated.

The surprising result of this study is that the dilemma of Alice can be resolved with the weak (CASE II) and strong (CASE III) solutions even with classical correlations provided that the players are allowed to use quantum operations. Another point to be noted is that the strategies of the players to find an NE and the payoffs they get at this NE depend on the type of the shared quantum correlation or classical correlation. Without a priori information on the distributed or shared correlations, the players cannot make their move.

The power of entanglement for this specific game shows itself in the amount of payoff players get but not whether it resolves the dilemma or not. When MES are allowed with two-parameter SU(2) operators, NE’s with payoffs (3, 2), which has the highest $\$A + \$B$ value achievable in this game, emerge. When the dilemma is resolved with classical correlations, the payoffs the players get are much smaller, $\$A + \$B = 7/2$.

We have to point out that, introducing correlations into a $2 \times 2$ noncooperative game transforms it into a cooperative game which is very different than the original game defined with its own rules. In the original classical game, no communication of any form is allowed between the players; however, in the full quantum strategies, shared entanglement which could be considered as a kind of “spooky communication” between the players are introduced into the game.

In classical mixed strategies, the payoffs become continuous in the mixing probability of the players’ actions, and a compromise becomes possible between the players which assures the existence of an NE. In games played by quantum operations, the inclusion of new strategies and moves into the game is an essential feature. Because, once these new moves are incorporated, both the strategy space and the payoffs become continuous. With this continuity, the players can finely adjust their strategies and come closer to being best responses to each other which leads to one or more NE’s.

|                        | Class. Op. | Quant. Op. |
|------------------------|------------|------------|
| No. Corr.              | I          | I          |
| Class. Corr.           |            |            |
| (|00⟩⟨00| + |11⟩⟨11|)/2 | II         | II         | II         |
| (|01⟩⟨01| + |10⟩⟨10|)/2 | II         | II         | III        |
| Full rank              | II         | II         |
| Quant. Corr.           |            |            |
| (|00⟩ + i|11⟩)/√2 | I          | I          | III        |
| (|01⟩−i|10⟩)/√2 | I          | I          | n.a        |

**TABLE IX**: Solutions to the dilemma in the Welfare game when the players choose classical and quantum strategies. I, II and III, respectively, represent CASE I: $\$A < 0$, CASE II: $0 \leq \$A \leq \$B$, and CASE III: $0 \leq \$B < \$A$. n.a which stands for “not applicable” points out the absence of a unique NE in the game. For the classical operations, only mixed strategies are listed because in pure strategies there is no unique NE in the game. In case of quantum operations, $Q_1$ and $Q_2$ stand for one- and two-parameter SU(2) operators, respectively.
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[16] We restrict the players to one- and two-parameter SU(2) operators because it has already been shown that when the general three-parameter SU(2) set is used in this specific quantization scheme by players sharing an MES, the games do not have an equilibrium [1, 2, 4].
[17] See G. Brassard, e-print quant-ph/0101005. It is well-known that one cannot convey information using only entanglement. However, it is also known that entanglement can be exploited by non-communicating locally separated parties to accomplish certain classes of tasks which are impossible to be completed in the classical world. Games might fall into this class of tasks.