Reconstructions for Analytic Signals Based on Greedy Algorithm

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Abstract. In this paper, we utilize greedy algorithm to approximate analytic signals and transfer functions sparsely. Based on reproducing kernels of Hardy space, a redundant dictionary is constructed. The approximating strategy relies on matching pursuit which is an implementation of greedy algorithm. A numerical example and Python codes are provided to illustrate the idea.

1. Introduction
Orthogonal basis has been widely used in many differential fields such as signal processing and system identification. A reason for utilizing orthogonal basis is that stable linear system can be described by orthogonal functions properly. The first investigations were FIR, Laguerre and Kautz models which were built in Fourier, Laguerre and Kautz bases respectively. Since then, a huge amount of research articles on rational orthogonal bases were published [1,2].

Let $\mathbb{D}$ be the unit disc of the complex plain. Define the generalized orthogonal bases in $\mathbb{D}$ as follows:

$$B_n(z) = \frac{1 - |\xi_n|^2}{1 - \xi_n z} \prod_{k=1}^{n-1} \frac{z - \xi_k}{1 - \xi_k z}$$

where $\xi_k \in \mathbb{D}$. Under certain condition, the functions defined as above can be considered as complete basis in Hardy spaces $H_p(\mathbb{D})$ or Disk Algebra $A(\mathbb{D})$. In general, the parameters $\xi_k$ are also called poles of the bases functions. This orthogonal basis is usually used to construct transfer functions, that is, for a given system function $T(z)$, it can be reconstructed by

$$\tilde{T}(z) = \sum_{k=1}^{n} c_k B_k(z)$$

where $c_k$ is the coefficient of the basis which is to be estimated. In practice, the structure of basis function is fixed and the coefficients would be computed. The advantage of this structure lies in its orthogonality and variability. In other words, the poles $\xi_k$ need to be chosen properly to derive better approximations. The set of $B_n(z)$ is in fact a dictionary of $H_2(\mathbb{D})$ and hence selecting poles for the transfer function can be considered as the best rational approximation problem which is still an open problem. Besides rational orthogonal basis, the transfer function can be also represented by shifted Cauchy kernels:

$$e_{\xi}(z) = \frac{1}{1 - \xi z}$$
This Cauchy kernel is also called rational wavelet. The rational function can be derived by Gram-Schmidt orthogonalization algorithm using \( e_k(z) \)[3]. When one utilizes \( e_k(z) \), the transfer function can be written as

\[
\hat{T}(z) = \sum_{k=1}^{n} l_k e_{x_k}(z)
\]

In order to get solution of \( l_k \), one can use \( l_2 \) norm least-square algorithm. In this paper, we utilize greedy algorithm to get better reconstruction for system function which is a known rational function. The rest of this paper is organized as following. Section 2 is dedicated to greedy algorithm for its principle and practice. In Section 3, we handle the discrete model by sampling the continuous signal on the boundary. In the last Section, a numerical example will be shown to illustrate the idea.

2. Greedy Algorithm

For a linear system whose solution is known to be sparse, one is interested in solving this set of linear equations as follows:

\[
y = Ax
\]

where \( y \in R^m \), \( A \in R^{m \times n} \), \( x \in R^n \) and \( m < n \). It is well known that such under determined system has infinite solutions and these solutions has a generic formulation in free variables [4]. So called sparse solution is a kind of special solution which is sparse. Suppose the solution is \( k \) sparse, in other words, it has only \( k \) nonzero elements and \( n - k \) zeros. A \( k \) sparse solution means it has \( k \) degrees of freedom, in other words, the solution has \( k \) positions which correspond to several values. In most cases, one would known positions with nonzero values. Assume \( \Theta \) be the set of such nonzero positions, one can choose the columns corresponding to these positions and denote it as \( A_\Theta \). Suppose that the number of equations \( m \) is larger than \( k \), one must solve the concise version of over-determined problem as the following equation:

\[
y = A_\Theta x_\Theta
\]

where \( x_\Theta \) denotes some special solution whose support set is \( \Theta \). Hence one can derive that [5]:

\[
x_\Theta = (A_\Theta^T A_\Theta)^{-1} A_\Theta^T y.
\]

If the unknown values are obtained, the rest elements of the solution will be filled with zeros.

However, the position with nonzero elements in \( x \) will not be given in advance. In this situation, there are \( c_n^k \) possibilities for the solution \( x \) which means one must choose \( k \) elements in \( n \) positions. In mathematical words, the problem is rewritten as

\[
\min_{x} \|x\|_0 \leq s \quad s. t. \quad y = Ax
\]

where \( \|x\|_0 \) denotes the quasi-norm of \( x \) which is not a norm but only counts the number of nonzero elements in \( x \). It can be seen this is a combinatorial problem, in other words, an NP-hard problem. Until now there is no efficient methods to handle NP-hard problem [6,7].

Consider a simpler problem. In this case suppose the solution has only one nonzero which means the solution \( x \) is a 1 sparse vector. Then one can easily reconstruct \( x \) as follows:

\[
A^T y = A^T A x
\]

In general cases, there exists two categories of techniques to solve it. One is employing approximate methods and the other is using convex strategies. In this paper, we adopt the first technique. Instead of examining exhaustively for the best candidate, an approximation alternative used by greedy algorithm is to search for a single best column at each iteration. The greedy algorithm strategy is outlined as follows [8-10]:

1. Start with a support set which is empty initially. This set is used to hold the nonzero elements.
2. Choose the best atom element in the dictionary at each iteration and add this element to the support set.
3. Update the solution step by step and compute the objective function which is also called residual.
4. Check if the solution satisfies the stopping criterion. If yes then exit, otherwise return to step 2.
3. The Model Statement

Basically, the model for the transfer functions or analytic signals reads as follows:

\[ T(z) = \sum_{k=0}^{\infty} c_k z^k \]

\( T(z) \) is called an analytic signal which is assumed to be a rational function. In addition, we impose assumptions as follows:

1. The system \( T(z) \) is linear time-invariant, stable and causal with single input and single output.
2. The system \( T(z) \) belongs to Hardy space, namely \( H_2(\mathbb{D}) \).
3. The responses of the system \( \{E_k\} \) are sampled from the boundary of \( T(z) \).

Under these assumptions, \( T(z) \) can be approximated by

\[ T(z) \approx \hat{T}(z) = \sum_{k=1}^{n} l_k e^{\xi_k(z)} \]

and \( E_k = T(e^{iw_k}) \) where \( w_k = 2\pi(k-1)/n \). Hence one can derive a discrete version of the equation:

\[ E = Ax \]

where the matrix \( A \) stands for the sampling matrix [11]

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 - \xi_1 e^{iw_1} & 1 - \xi_2 e^{iw_1} & \cdots & 1 - \xi_n e^{iw_1} \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 - \xi_1 e^{iw_n} & 1 - \xi_2 e^{iw_n} & \cdots & 1 - \xi_n e^{iw_n}
\end{bmatrix}
\]

and \( x = [x_1, x_2, \ldots, x_n]^T \). If the parameters \( \xi_1, \xi_2, \ldots, \xi_n \) are determined, one can compute the coefficients \( x \) to derive an approximation.

4. Numerical Example

In this part, an example will be given to illustrate the idea based on greedy algorithm. The analytic signal reads:

\[ f(z) = 10 - 5.44 z (1 - 0.29 z)(1 - 0.71 z) \]

We utilize greedy strategy to approximate the original signal \( f(z) \). Let \( R_0(z) = f(z) \), then \( R_0(z) = (R_0, g_1)g_1(z) + R_1(z) \). And \( R_1(z) \) can be considered as another signal, one derives \( R_1(z) = (R_1, g_2)g_2(z) + R_2(z) \). The process continues until the approximation is close enough to the original signal. The Python code is as follows.

```python
import numpy as np
from scipy.linalg import norm

n_a_r = 30
n_a_theta = 60
a_r = np.linspace(0,1,n_a_r+2)[1:-1]
a_theta = np.linspace(0,2*np.pi,n_a_theta+1)[0:-1]
a_set = np.kron(a_r,np.exp(a_theta*1j))
a_set = np.append(a_set,0+0j)
n_sample = 128
t = np.linspace(0,2*np.pi,n_sample+1)[-1]
z = np.exp(t*1j)
design_matrix = np.zeros(shape=(t.size,a_set.size),dtype=np.complex)
for k in range(a_set.size):
a = a_set[k]
```
design_matrix[:,k] = np.sqrt(1-np.abs(a)**2)/(1-np.conj(a)*z)
design_matrix[:,k] = design_matrix[:,k]/norm(design_matrix[:,k])
f = (10-5.44*z)/(1-0.29*z)*(1-0.71*z))
sparse_K = 6
resid = f
index_set = []
f_approx = 0
print(norm(resid))
for k in range(sparse_K):
    x = design_matrix.conj().T @ resid
    ind = np.argmax(np.abs(x))
    resid = resid - x[ind]*design_matrix[:,ind]
    f_approx = f_approx + x[ind]*design_matrix[:,ind]
    index_set.append(ind)
    print(norm(resid))
plt.figure(figsize=(10,4))
plt.subplot(1,2,1)
plt.title('the real part of the signal')
plt.plot(f_approx.real,'r',f.real)
plt.subplot(1,2,2)
plt.title('the imaginary part of the signal')
plt.plot(f_approx.imag,'r',f.imag)
plt.savefig('aaa.png',dpi = 1300)

Figure 1. The Real Part and Imaginary Part of the Original Signal

Table 1 shows the values of the parameters. In this case, we take the six largest atoms to construct the approximation.

| Table 1. the values of the parameters |
|--------------------------------------|
| $a_1 = 0.51612903 + 0. j$           | $a_2 = 0.93548387 + 0. j$ |
| $a_3 = 0.67853038 + 0.49298118j$   | $a_4 = -0.06406583 - 0.60954568j$ |
| $a_5 = 0.85901879 - 0.27911212j$   | $a_6 = -0.45547586 + 0.41011231j$ |

Table 2 shows the norms of residues for each iteration. The original discrete signal is a vector with norm 129.8038523861962. We list the norms after each step of greedy approximation.
Table 2. the norms of residues for each iteration

| Iteration | Norm          |
|-----------|---------------|
| 1         | $10.581980461085044$ |
| 2         | $9.567514821900433$  |
| 3         | $8.092492181223287$  |
| 4         | $6.33274286426407$   |
| 5         | $4.598360880292408$  |
| 6         | $3.964608580565063$  |

5. Conclusions
In this article, we adopt greedy algorithm (i.e. Matching Pursuit) to rebuild an analytic signal or a transfer function of an LTI system. Based on reproducing kernels of Hardy space, a redundant dictionary is constructed. The redundant dictionary $e_k(z)$ enable us deriving a sparse reconstruction. The approximating strategy relies on matching pursuit which is an implementation of greedy algorithm. A numerical example and Python codes are provided to illustrate the idea. We conclude that Hardy space $H_2(D)$ and Cauchy kernels are suitable for reconstructions of analytic signals and rational transfer functions.

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