Integrability in 3 + 1 Dimensions: Relaxing a Tetrahedron Relation

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Abstract

I propose a scheme of constructing classical integrable models in 3 + 1 discrete dimensions, based on a relaxed version of the problem of factorizing a matrix into the product of four matrices of a special form.

It is well known that in the 2 + 1-dimensional case the integrability of classical models is closely connected with the problem of re-factorizing the product of three special block matrices in a similar product, but taken in the reverse order $[1, 3, 4]$. It seems that all the known classical integrable systems can be obtained from there as some particular and/or limit cases $[2, 3]$. Moreover, the quantization of at least some systems arising from this re-factorizing problem is amazingly straightforward $[4]$.

The similar problem for the product of four matrices (“tetrahedron relation”, see below) cannot, generally, be solved, because of lack of parameters. However, it turns out that this problem can be relaxed in such a way that it becomes solvable, and many solutions of the “evolution equations” of the corresponding dynamical system can be written out in algebraic-geometrical form, namely employing an algebraic curve and divisors in it.

By the “tetrahedron relation” I mean in this paper the following decomposition of a $6 \times 6$ matrix $K$ of complex numbers:

$$K = A_{123}B_{145}C_{246}D_{356},$$

(1)

where the subscripts show in which lines and columns a matrix can have nontrivial elements, while all other elements are zeroes if they are non-diagonal and unities if they are

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diagonal. For example,

\[
A_{123} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

We will also write \( A \) instead of \( A_{123} \) and so on.

The matrix \( K \) has 36 parameters, while the product \( ABCD \) has only 30 of them (4 matrices each having 9 parameters, but 6 of those must be subtracted because of the obvious “gauge” freedom in \( A, B, C \) and \( D \) not changing \( K \), compare Chapter 2 of [1]). So let us relax the relation (1) in the following way. Let \( \Phi_k \) and \( \Psi_k \), where \( k = 1, 2, 3, 4, 5 \), be five given six-dimensional vectors, and let us search for such matrices \( A, B, C \) and \( D \) that

\[
ABCD \Phi_k = \Psi_k
\]

for all \( k \). The conditions (2) impose exactly 30 restrictions on \( A, B, C \) and \( D \), as needed.

We can depict each of the matrices \( A, B, C \) and \( D \) as a vertex with three “incoming” and three “outgoing” edges. Then, the product \( ABCD \) will be represented as a tetrahedron with six “inner” edges and twelve “outer” edges. Six of the latter are incoming, and to them the components of vectors \( \Phi_k \) are attached, and six others are outgoing, and to them the components of vectors \( \Psi_k \) are attached. Now, the fact that we can find \( A, B, C \) and \( D \) from (2) means that the numbers attached to the inner edges are also determined from \( \Phi_k \) and \( \Psi_k \) (up to the gauge freedom: we can attach to each edge an arbitrary constant and multiply all the edge’s five numbers by that constant).

This suggests to introduce the following dynamical system, in the spirit of [1]. Let there be several planes in the three-dimensional space. Let them be permitted to move arbitrarily, but each remaining parallel to its initial position. We will call edges of such figure the (open) intervals of lines where two planes intersect (we will attach some orientation to each of those lines) between points (vertices) where three planes intersect. To each edge five complex numbers (“dynamical variables”) will correspond, and all those numbers must be consistent in the sense that there must exist \( 3 \times 3 \) matrices in each vertex transforming five incoming 3-vectors into five outgoing ones. As already explained, for just one tetrahedron this consistency means that the variables on inner edges are expressed through those on outer edges, and the reader can invent more examples where also the variables can be chosen freely for some edges and then all the other variables get fixed.

The elementary move in the system can be described as passing a plane through the intersection point of three other planes, or “turning a tetrahedron inside out“. In such move, the numbers corresponding to the tetrahedron’s outer edges remain intact, while those corresponding to the inner edges change according to the change of “relaxed decomposition” (3), e.g., instead of (3) we get

\[
D'C'B'A' \Phi_k = \Psi_k.
\]
An algebro-geometric solution for this system reads as follows. Let us take an algebraic curve $\Gamma$, and attach to each plane, say plane number $j$, two divisors $D_j^{(1)}$ and $D_j^{(2)}$ in $\Gamma$ of degree 2. For simplicity, we can assume that such a divisor is just two arbitrary points of $\Gamma$. To be exact, we will attach $D_j^{(1)}$ to one open half-space in which plane number $j$ divides the space, and $D_j^{(2)}$—to the other open half-space, and nothing to the plane itself.

Let us also take some divisor $D$ whose degree will be specified later, and put in correspondence to each point $x$ of the space the divisor

$$D(x) = D - \sum_j D_j^{(\alpha_j)},$$

where the sum is taken over all such $j$ that $x$ does not belong to the plane number $j$, and $\alpha_j = 1$ or 2 according to which half-space the point $x$ belongs.

We will be interested, as in [4], in divisors corresponding this way to the edges. Let us choose the degree of $D$ so that the degree of $D(x)$, if $x$ belongs to an edge, be $g$—the genus of $\Gamma$. If each $D_j^{(\alpha_j)}$ is just two points, this means that the space of meromorphic functions with poles in the points of $D$ and zeroes in points of all $D_j^{(\alpha_j)}$ is one-dimensional, that is, to each edge corresponds a meromorphic function $f$, determined up to a constant multiplier. Now let us choose five more points $z_1, z_2, z_3, z_4, z_5 \in \Gamma$ (the same for all edges), and put in correspondence to the edge the numbers $f(z_1), f(z_2), f(z_3), f(z_4)$ and $f(z_5)$. The same reasoning as in [1, 4] shows that this provides a solution for our dynamical system.

References

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