One more theorem on norm equivalence in the Lebesgue space

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Abstract

In this paper we consider a norm based on the infinitesimal generator of the shift semigroup in a direction. The relevance of such a focus is guaranteed by an abstract representation of a fractional integro-differential operator by means of a composition of the corresponding infinitesimal generator. The main result of the paper is a theorem establishing equivalence of norms in functional spaces. Even without mentioning the relevance of this result for the constructed theory, we claim it deserves to be considered itself.

Keywords: Equivalence of norms; compact embedding of spaces; infinitesimal generator; m-accretive operator; uniformly elliptic operator.

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1 Introduction

The facts that have motivated us to write this paper lie in the fractional calculus theory. Basically, an event that a differential operator with a fractional derivative in final terms underwent a careful study [14], [8] have played an important role in our research. The main feature is that there exists various approaches to study the operator and one of them is based on an opportunity to represent it in a sum of a senior term and an a lower term, here we should note that this method works if the senior term is selfadjoint or normal. Thus, in the case corresponding to a selfadjoint senior term, we can partially solve the problem having applied the results of the perturbation theory, within the framework of which the following papers are well-known [4], [10], [11], [12], [13], [17]. Note that to apply the last paper results we must have the mentioned above representation. In other cases we can use methods of the papers [7], which are relevant if we deal with non-selfadjoint operators and allow us to study spectral properties of operators. In the paper [9] we explore a special operator class for which a number of spectral theory theorems can be applied. Further, we construct an abstract model of a differential operator in terms of m-accretive operators and call it an m-accretive operator transform, we find such conditions that being imposed guaranty that the transform belongs to the class. One of them is a compact embedding of a space generated by
an m-accretive operator (infinitesimal generator) into the initial Hilbert space. Note that in the case corresponding to the second order operator with the Kiprianov operator in final terms we have obtained the embedding mentioned above in the one-dimensional case only. In this paper we try to reveal this problem and the main result is a theorem establishing equivalence of norms in function spaces in consequence of which we have a compact embedding of a space generated by the infinitesimal generator of the shift semigroup in a direction into the Lebesgue space. We should note that this result do not give us a useful concrete application in the built theory for it is more of an abstract generalization. However this result, by virtue of popularity and well known applicability of the Lebesgue spaces theory, deserves to be considered itself.

2 Preliminaries

Let $C_i, \ i \in \mathbb{N}_0$ be real constants. We assume that a value of $C$ is positive and can be different in various formulas but values of $C_i$ are certain. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the set of linear bounded operators on $\mathcal{H}$. Denote by $\mathcal{L}$ the closure of an operator $L$. Denote by $\mathcal{D}(L), \mathcal{R}(L), \mathcal{N}(L)$ the domain of definition, the range, and the kernel or null space of an operator $L$ respectively. Consider a pair of complex Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the notation $\mathcal{H}_2 \subset \mathcal{H}_1$ means that $\mathcal{H}_2$ is dense in $\mathcal{H}_1$ as a set of elements and we have a bounded embedding provided by the inequality $\|f\|_{\mathcal{H}_2} \leq C_0 \|f\|_{\mathcal{H}_1}, \ C_0 > 0, \ f \in \mathcal{H}_2$, moreover any bounded set with respect to the norm $\mathcal{H}_2$ is compact with respect to the norm $\mathcal{H}_1$. An operator $L$ is called bounded from below if the following relation holds $\text{Re}(Lf, f)_{\mathcal{H}_1} \geq \gamma_L \|f\|_{\mathcal{H}_1}^2$, $f \in \mathcal{D}(L), \ \gamma_L \in \mathbb{R}$, where $\gamma_L$ is called a lower bound of $L$. An operator $L$ is called accretive if $\gamma_L = 0$. An operator $L$ is called strictly accretive if $\gamma_L > 0$. An operator $L$ is called m-accretive if the next relation holds $(A + \zeta)^{-1} \in \mathcal{B}(\mathcal{H})$, $\|(A + \zeta)^{-1}\| \leq (\text{Re}\zeta)^{-1}, \ \text{Re}\zeta > 0$. Assume that $T_t, \ (0 \leq t < \infty)$ is a semigroup of bounded linear operators on $\mathcal{H}$, by definition put

$$A f = - \lim_{t \to +0} \left( \frac{T_t - I}{t} \right) f,$$

where $\mathcal{D}(A)$ is a set of elements for which the last limit exists in the sense of the norm $\mathcal{H}$. In accordance with definition \[13\ p.1, 13\] the operator $-A$ is called the infinitesimal generator of the semigroup $T_t$. Using notations of the paper \[5\] we assume that $\Omega$ is a convex domain of the $n$-dimensional Euclidean space $\mathbb{E}^n$, $P$ is a fixed point of the boundary $\partial \Omega$, $Q(r, e)$ is an arbitrary point of $\Omega$; we denote by $e$ a unit vector having a direction from $P$ to $Q$, denote by $r = |P - Q|$ the Euclidean distance between the points $P, Q$, and use the shorthand notation $T := P + e, t \in \mathbb{R}$. We consider the Lebesgue classes $L_p(\Omega), \ 1 \leq p < \infty$ of complex valued functions. For the function $f \in L_p(\Omega)$, we have

$$\int_{\Omega} |f(Q)|^p dQ = \int_{\omega} d\chi \int_0^{d(e)} |f(Q)|^p r^{n-1} dr < \infty, \ (1)$$

where $d\chi$ is an element of solid angle of the unit sphere surface (the unit sphere belongs to $\mathbb{E}^n$) and $\omega$ is a surface of this sphere, $d := d(e)$ is the length of the segment of the ray going from the point $P$ in the direction $e$ within the domain $\Omega$. We use a shorthand notation $P \cdot Q = P_iQ_i = \sum_{i=1}^n P_iQ_i$
for the inner product of the points \( P = (P_1, P_2, ..., P_n) \), \( Q = (Q_1, Q_2, ..., Q_n) \) which belong to \( \mathbb{E}^n \). Denote by \( D(f) \) a weak partial derivative of the function \( f \) with respect to a coordinate variable with index \( 1 \leq i \leq n \). We assume that all functions have a zero extension outside of \( \Omega \). Everywhere further, unless otherwise stated, we use notations of the papers [3], [5], [6].

**Lemma 1.** Assume that \( A \) is a closed densely defined operator, the following condition holds

\[
\|(A + t)^{-1}\|_{\mathcal{B} \to \mathcal{B}} \leq \frac{1}{t}, \quad t > 0, \tag{2}
\]

where a notation \( \mathcal{B} := \mathcal{B}(A + t) \) is used. Then the operator \( A \) is \( m \)-accretive.

*Proof.* Using (2) consider

\[
\|f\|_{\mathcal{B}}^2 \leq \frac{1}{t^2} \|(A + t)f\|_{\mathcal{B}}^2; \quad \|f\|_{\mathcal{B}}^2 \leq \frac{1}{t^2} \{\|Af\|_{\mathcal{B}}^2 + 2t \text{Re}(Af, f)_{\mathcal{B}} + t^2 \|f\|_{\mathcal{B}}^2\};
\]

\[
t^{-1} \|Af\|_{\mathcal{B}}^2 + 2 \text{Re}(Af, f)_{\mathcal{B}} \geq 0, \quad f \in D(A).
\]

Let \( t \) be tended to infinity, then we obtain

\[
\text{Re}(Af, f)_{\mathcal{B}} \geq 0, \quad f \in D(A). \tag{3}
\]

It means that the operator \( A \) is accretive. Due to (3), we have \( \{\lambda \in \mathbb{C} : \text{Re}\lambda < 0\} \subset \Delta(A) \), where \( \Delta(A) = \mathbb{C} \setminus \overline{\sigma(A)} \). Applying Theorem 3.2 [3] p.268, we obtain that \( A - \lambda \) has a closed range and \( \text{nul}(A - \lambda) = 0 \), \( \text{def}(A - \lambda) = \text{const}, \forall \lambda \in \Delta(A) \). Let \( \lambda_0 \in \Delta(A) \), \( \text{Re}\lambda_0 < 0 \). Note that in consequence of inequality (3), we have

\[
\text{Re}(f, (A - \lambda)f)_{\mathcal{B}} \geq -\text{Re}\lambda \|f\|_{\mathcal{B}}^2, \quad f \in D(A). \tag{4}
\]

Since the operator \( A - \lambda_0 \) has a closed range, then

\[
\mathfrak{H} = \mathcal{R}(A - \lambda_0) \oplus \mathcal{R}(A - \lambda_0)^\perp.
\]

We remark that the intersection of the sets \( D(A) \) and \( \mathcal{R}(A - \lambda_0)^\perp \) is zero, because if we assume the contrary, then applying inequality (4), for arbitrary element \( f \in D(A) \cap \mathcal{R}(A - \lambda_0)^\perp \) we get

\[
-\text{Re}\lambda_0 \|f\|_{\mathcal{B}}^2 \leq \text{Re}(f, [A - \lambda_0]f)_{\mathcal{B}} = 0,
\]

hence \( f = 0 \). It implies that

\[
(f, g)_{\mathcal{B}} = 0, \quad \forall f \in \mathcal{R}(A - \lambda_0)^\perp, \quad \forall g \in D(A).
\]

Since \( D(A) \) is a dense set in \( \mathfrak{H} \), then \( \mathcal{R}(A - \lambda_0)^\perp = 0 \). It implies that \( \text{def}(A - \lambda_0) = 0 \) and if we take into account Theorem 3.2 [3] p.268, then we come to the conclusion that \( \text{def}(A - \lambda) = 0, \forall \lambda \in \Delta(A) \), hence the operator \( A \) is \( m \)-accretive. The proof is complete. \( \square \)

Assume that \( \Omega \subset \mathbb{E}^n \) is a convex domain, with a sufficient smooth boundary (\( C^3 \) class) of \( n \)-dimensional Euclidian space. For the sake of the simplicity we consider that \( \Omega \) is bounded. Consider the shift semigroup in a direction acting on \( L_2(\Omega) \) and defined as follows \( T_t f(Q) = f(Q + et) \), where \( Q \in \Omega, Q = P + er \). The following lemma establishes a property of the infinitesimal generator \( -A \) of the semigroup \( T_t \).
**Lemma 2.** We claim that $A = \tilde{A}_0$, $N(A) = 0$, where $A_0$ is a restriction of $A$ on the set $C_0^\infty(\Omega)$.

**Proof.** Let us show that $T_t$ is a strongly continuous semigroup ($C_0$ semigroup). It can be easily established due to the continuous in average property. Using the Minkowskii inequality, we have

$$
\left\{ \int_{\Omega} |f(Q + et) - f(Q)|^2 dQ \right\}^{\frac{1}{2}} \leq \left\{ \int_{\Omega} |f(Q + et) - f_m(Q + et)|^2 dQ \right\}^{\frac{1}{2}} + \left\{ \int_{\Omega} |f(Q) - f_m(Q)|^2 dQ \right\}^{\frac{1}{2}} + \left\{ \int_{\Omega} |f_m(Q) - f_m(Q + et)|^2 dQ \right\}^{\frac{1}{2}} = I_1 + I_2 + I_3 < \varepsilon,
$$

where $f \in L_2(\Omega)$, $\{f_n\}_1^\infty \subset C_0^\infty(\Omega)$; $m$ is chosen so that $I_1, I_2 < \varepsilon/3$ and $t$ is chosen so that $I_3 < \varepsilon/3$. Thus, there exists such a positive number $t_0$ so that

$$
\|T_t f - f\|_{L_2} < \varepsilon, \ t < t_0,
$$

for arbitrary small $\varepsilon > 0$. Hence in accordance with the definition $T_t$ is a $C_0$ semigroup. Using the assumption that all functions have the zero extension outside $\bar{\Omega}$, we have $\|T_t\| \leq 1$. Hence we conclude that $T_t$ is a $C_0$ semigroup of contractions (see [15]). Hence by virtue of Corollary 3.6 [15, p.11], we have

$$
\| (\lambda + A)^{-1} \| \leq \frac{1}{\text{Re} \lambda}, \ \text{Re} \lambda > 0. \quad (5)
$$

Inequality (5) implies that $A$ is m-accretive. It is the well-known fact that an infinitesimal generator $-A$ is a closed operator, hence $A_0$ is closeable. It is not hard to prove that $\tilde{A}_0$ is an m-accretive operator. For this purpose let us rewrite relation (5) in the form

$$
\| (\lambda + \tilde{A}_0)^{-1} \|_{R \rightarrow \delta} \leq \frac{1}{\text{Re} \lambda}, \ \text{Re} \lambda > 0,
$$

applying Lemma 1, we obtain that $\tilde{A}_0$ is an m-accretive operator. Note that there does not exist an accretive extension of an m-accretive operator (see [3]). On the other hand it is clear that $\tilde{A}_0 \subset A$. Thus we conclude that $\tilde{A}_0 = A$. Consider an operator

$$
Bf(Q) = \int_0^t f(P + e[r - t]) dt, \ f \in L_2(\Omega).
$$

It is not hard to prove that $B \in B(L_2)$, applying the generalized Minkowskii inequality, we get

$$
\|Bf\|_{L_2} \leq \int_0^{\text{diam} \Omega} dt \left( \int_{\Omega} |f(P + e[r - t])| dQ \right)^{1/2} \leq C \|f\|_{L_2}.
$$

Note that the fact $A_0^{-1} \subset B$, follows from the properties of the one-dimensional integral defined on smooth functions. Using Theorem 2 [16, p.555], the proved above fact $\tilde{A}_0 = A$, we deduce that $A^{-1} = \tilde{A}_0^{-1}$, hence $A^{-1} \subset B$. The proof is complete. \qed
3 Main results

Consider a linear space $L^2_2(\Omega) := \{ f = (f_1, f_2, \ldots, f_n), f_i \in L_2(\Omega) \}$, endowed with the inner product

$$(f, g)_{L^2_2} = \int_\Omega (f, g)_{E^n} dQ, \ f, g \in L^2_2(\Omega).$$

It is clear that this pair forms a Hilbert space and let us use the same notation $L^n_2(\Omega)$ for it. Consider a semilinear form

$$t(f, g) := \sum_{i=1}^n \int_\Omega (f, e_i)_{E^n} \overline{(g, e_i)}_{E^n} dQ, \ f, g \in L^2_2(\Omega),$$

where $e_i$ corresponds to $P_i \in \partial \Omega$, $i = 1, 2, \ldots, n$.

**Lemma 3.** The points $P_i \in \partial \Omega$, $i = 1, 2, \ldots, n$ can be chosen so that the form $t$ generates an inner product.

**Proof.** It is clear that we should only establish an implication $t(f, f) = 0 \Rightarrow f = 0$. Since $\Omega \in \mathbb{E}^n$, then without lose of generality we can assume that there exists $P_i \in \partial \Omega$, $i = 1, 2, \ldots, n$, such that

$$\Delta = \begin{vmatrix} P_{11} & P_{12} & \ldots & P_{1n} \\
P_{21} & P_{22} & \ldots & P_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} & P_{n2} & \ldots & P_{nn} \end{vmatrix} \neq 0, \quad (6)$$

where $P_i = (P_{i1}, P_{i2}, \ldots, P_{in})$. It becomes clear if we remind that in the contrary case, for arbitrary set of points $P_i \in \partial \Omega$, $i = 1, 2, \ldots, n$, we have

$$P_n = \sum_{k=1}^{n-1} c_k P_k, \ c_k = \text{const},$$

from what follows that we can consider $\Omega$ at least as a subset of $\mathbb{E}^{n-1}$. Continuing this line of reasonings we can find such a dimension $p$ that a corresponding $\Delta \neq 0$ and further assume that $\Omega \in \mathbb{E}^p$. Consider a relation

$$\sum_{i=1}^n \int_\Omega |(\psi, e_i)_{E^n}|^2 dQ = 0, \ \psi \in L^n_2(\Omega).$$

It follows that $(\psi(Q), e_i)_{E^n} = 0$ a.e. $i = 1, 2, \ldots, n$. Note that every $P_i$ corresponds to the set $\vartheta_i := \{ Q \subset \vartheta_i : (\psi(Q), e_i)_{E^n} \neq 0 \}$. Consider $\Omega' = \Omega \setminus \bigcup_{i=1}^n \vartheta_i$, it is clear that $\text{mes} \left( \bigcup_{i=1}^n \vartheta_i \right) = 0$.

Note that due to the made construction, we can reformulate the obtained above relation in the coordinate form

$$\begin{cases} (P_{11} - Q_1) \psi_1(Q) + (P_{12} - Q_2) \psi_2(Q) + \ldots + (P_{1n} - Q_n) \psi_n(Q) = 0 \\
(P_{21} - Q_1) \psi_1(Q) + (P_{22} - Q_2) \psi_2(Q) + \ldots + (P_{2n} - Q_n) \psi_n(Q) = 0 \\
\vdots \\
(P_{n1} - Q_1) \psi_1(Q) + (P_{n2} - Q_2) \psi_2(Q) + \ldots + (P_{nn} - Q_n) \psi_n(Q) = 0 \end{cases}.$$
where $\psi = (\psi_1,\psi_2,\ldots,\psi_n)$, $Q = (Q_1,Q_2,\ldots,Q_n)$, $Q \in \Omega'$. Therefore, if we prove that

$$\Lambda(Q) = \begin{vmatrix}
P_{11} - Q_1 & P_{12} - Q_2 & \cdots & P_{1n} - Q_n \\
P_{21} - Q_1 & P_{22} - Q_2 & \cdots & P_{2n} - Q_n \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} - Q_1 & P_{n2} - Q_2 & \cdots & P_{nn} - Q_n
\end{vmatrix} \neq 0 \text{ a.e.},$$

then we obtain $\psi = 0 \text{ a.e.}$ Assume the contrary i.e. that there exists such a set $\Upsilon \subset \Omega$, mess $\Upsilon \neq 0$, so that $\Lambda(Q) = 0$, $Q \in \Upsilon$. We have

$$\begin{vmatrix}
P_{11} - Q_1 & P_{12} - Q_2 & \cdots & P_{1n} - Q_n \\
P_{21} - Q_1 & P_{22} - Q_2 & \cdots & P_{2n} - Q_n \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} - Q_1 & P_{n2} - Q_2 & \cdots & P_{nn} - Q_n
\end{vmatrix} - \begin{vmatrix}
P_1 - Q_1 & Q_2 & \cdots & Q_n \\
P_2 - Q_1 & P_{22} - Q_2 & \cdots & P_{2n} - Q_n \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} - Q_1 & P_{n2} - Q_2 & \cdots & P_{nn} - Q_n
\end{vmatrix} = \begin{vmatrix}
P_{11} & P_{12} & \cdots & P_{1n} \\
P_{21} & P_{22} & \cdots & P_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} & P_{n2} & \cdots & P_{nn}
\end{vmatrix} - \sum_{j=1}^{n} \Delta_j = 0,
$$

where

$$\Delta_j = \begin{vmatrix}
P_{11} & P_{12} & \cdots & P_{1n} \\
P_{21} & P_{22} & \cdots & P_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{j-11} & P_{j-12} & \cdots & P_{j-1n} \\
Q_1 & Q_2 & \cdots & Q_n \\
P_{j+11} & P_{j+12} & \cdots & P_{j+1n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} & P_{n2} & \cdots & P_{nn}
\end{vmatrix}.$$
Now, let us prove that \( \Upsilon \) belongs to a hyperplane in \( \mathbb{E}^n \), we have

\[
\left| \begin{array}{cccc}
P_{11} - Q_1 & P_{12} - Q_2 & \cdots & P_{1n} - Q_n \\
P_{21} - P_{11} & P_{22} - P_{12} & \cdots & P_{2n} - P_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1} - P_{n-11} & P_{n2} - P_{n-12} & \cdots & P_{nn} - P_{n-1n}
\end{array} \right| = 0.
\]

Hence \( \Upsilon \) belongs to a hyperplane generated by the points \( P_i, i = 1, 2, \ldots, n \). Therefore \( \text{mes} \mathcal{S} = 0 \), and we obtain \( \psi = 0 \) a.e. The proof is complete. \( \Box \)

Consider a pre Hilbert space \( L_2^2(\Omega) := \{ f : f \in L_2^2(\Omega) \} \) endowed with the inner product

\[
(f, g)_{L_2^2} := \sum_{i=1}^n \int_{\Omega} (f, e_i)_{L_2}(g, e_i)_{L_2} dQ, \quad f, g \in L_2^2(\Omega),
\]

where \( e_i \) corresponds to \( P_i \in \partial \Omega, i = 1, 2, \ldots, n \), condition \( [\text{II}] \) holds. The following theorem establishes a norm equivalence.

**Theorem 1.** The norms \( \| \cdot \|_{L_2^2} \) and \( \| \cdot \|_{L_2^2} \) are equivalent.

**Proof.** Consider the space \( L_2^2(\Omega) \) and a functional \( \varphi(f) := \|f\|_{L_2^2}, f \in L_2^2(\Omega) \). Let us prove that \( \varphi(f) \geq C, f \in U \), where \( U := \{ f \in L_2^2(\Omega), \|f\|_{L_2^2} = 1 \} \). Assume the contrary, then there exists such a sequence \( \{\psi_k\}_1^\infty \subset U \), so that \( \varphi(\psi_k) \to 0, k \to \infty \). Since the sequence \( \{\psi_k\}_1^\infty \) is bounded,
then we can extract a weekly convergent subsequence \( \{ \psi_{k_j} \} \) and claim that the week limit \( \psi \) of the sequence \( \{ \psi_{k_j} \} \) belongs to \( U \). Consider a functional

\[
L_g(f) := \sum_{i=1}^{n} \int_{\Omega} (f, e_i)_{L_2} (g, e_i)_{L_2} dQ, \quad f, g \in L_2^2(\Omega).
\]

Due to the following obvious chain of the inequalities

\[
|L_g(f)| \leq \sum_{i=1}^{n} \left\{ \int_{\Omega} \| (f, e_i)_{L_2} \|^2 dQ \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \| (g, e_i)_{L_2} \|^2 dQ \right\}^{\frac{1}{2}} \leq n \| f \|_{L_2^2} \| g \|_{L_2^2}, \quad f, g \in L_2^2(\Omega),
\]

we see that \( L_g \) is a linear bounded functional on \( L_2^2(\Omega) \). Therefore, by virtue of the weak convergence of the sequence \( \{ \psi_{k_j} \} \), we have \( L_g(\psi_{k_j}) \to L_g(\psi), \quad k_j \to \infty \). On the other hand, recall that since it was supposed that \( \varphi(\psi_k) \to 0, \quad k \to \infty \), then we have \( \varphi(\psi_{k_j}) \to 0, \quad k \to \infty \). Hence applying (4), we conclude that \( L_g(\psi_{k_j}) \to 0, \quad k_j \to \infty \). Combining the given above results we obtain

\[
L_g(\psi) = \sum_{i=1}^{n} \int_{\Omega} (\psi, e_i)_{L_2} (g, e_i)_{L_2} dQ = 0, \quad \forall g \in L_2^2(\Omega).
\]

Taking into account (8) and using the ordinary properties of Hilbert space, we obtain

\[
\sum_{i=1}^{n} \int_{\Omega} |(\psi, e_i)_{L_2}|^2 dQ = 0.
\]

Hence in accordance with Lemma 3, we get \( \psi = 0 \) a.e. Notice that by virtue of this fact we come to the contradiction with the fact \( \| \psi \|_{L_2^2} = 1 \). Hence the following estimate is true \( \varphi(f) \geq C, \quad f \in U \). Having applied the Cauchy Schwartz inequality to the Euclidian inner product, we can also easily obtain \( \varphi(f) \leq \sqrt{n} \| f \|_{L_2^2}, \quad f \in L_2^2(\Omega) \). Combining the above inequalities, we can rewrite these two estimates as follows \( C_0 \leq \varphi(f) \leq C_1, \quad f \in U \). To make the issue clear, we can rewrite the previous inequality in the form

\[
C_0 \| f \|_{L_2^2} \leq \varphi(f) \leq C_1 \| f \|_{L_2^2}, \quad f \in L_2^2(\Omega), \quad C_0, C_1 > 0.
\]

The proof is complete. \( \square \)

Consider a pre Hilbert space

\[
\tilde{S}_\lambda := \{ f, g \in C_0^\infty(\Omega), \quad (f, g)_{\tilde{S}_\lambda} = \sum_{i=1}^{n} (A_i f, A_i g)_{L_2} \},
\]

where \( -A_i \) is an infinitesimal generator corresponding to the point \( P_i \). Here, we should point out that the form \( (\cdot, \cdot)_{\tilde{S}_\lambda} \) generates an inner product due to the fact \( N(A_i) = 0, \quad i = 1, 2, ..., n \) proved in Lemma 2. Let us denote a corresponding Hilbert space by \( S_\lambda \).
Corollary 1. The norms $\| \cdot \|_{\mathfrak{H}_A^n}$ and $\| \cdot \|_{H_0^1}^n$ are equivalent, we have a bounded compact embedding 
$$\mathfrak{H}_A^n \subset \subset L_2(\Omega).$$

Proof. Let us prove that 
$$Af = - (\nabla f, e)_{\mathfrak{H}_C^n}, f \in C_0^\infty(\Omega).$$

Using the Lagrange mean value theorem, we have 
$$\int_{\Omega} \left| \left( \frac{T_i - t}{t} \right) f(Q) - (\nabla f, e)_{\mathfrak{H}_C^n}(Q) \right|^2 dQ = \int_{\Omega} \left| (\nabla f, e)_{\mathfrak{H}_C^n}(Q_\xi) - (\nabla f, e)_{\mathfrak{H}_C^n}(Q) \right|^2 dQ,$$

where $Q_\xi = Q + e\xi, 0 < \xi < t$. Since the function $(\nabla f, e)_{\mathfrak{H}_C^n}$ is continuous on $\Omega$, then it is uniformly continuous on $\Omega$. Thus, for arbitrary $\varepsilon > 0$, a positive number $\delta > 0$ can be chosen so that 
$$\int_{\Omega} \left| (\nabla f, e)_{\mathfrak{H}_C^n}(Q_\xi) - (\nabla f, e)_{\mathfrak{H}_C^n}(Q) \right|^2 dQ < \varepsilon, t < \delta,$$

from what follows the desired result. Taking it into account, we obtain 
$$\|Af\|_{L_2} = \left\{ \int_{\Omega} \| (\nabla f, e)_{\mathfrak{H}_C^n} \|^2 dQ \right\}^{1/2} \leq \left\{ \int_{\Omega} \| f \|_{\mathfrak{H}_C^n}^2 \sum_{i=1}^n |D_i f|^2 dQ \right\}^{1/2} = \| f \|_{H_0^1}, f \in C_0^\infty(\Omega).$$

Using this estimate, we easily obtain $\| f \|_{\mathfrak{H}_C^n} \leq C \| f \|_{H_0^1}, f \in C_0^\infty(\Omega)$. On the other hand, as a particular case of formula (9), we obtain $C_0 \| f \|_{H_0^1} \leq \| f \|_{\mathfrak{H}_C^n}, f \in C_0^\infty(\Omega)$. Thus, we can combine the previous inequalities and rewrite them as follows $C_0 \| f \|_{H_0^1} \leq \| f \|_{\mathfrak{H}_C^n} \leq C \| f \|_{H_0^1}, f \in C_0^\infty(\Omega)$. Passing to the limit at the left-hand and right-hand side of the last inequality, we get 
$$C_0 \| f \|_{H_0^1} \leq \| f \|_{\mathfrak{H}_C^n} \leq C \| f \|_{H_0^1}, f \in H_0^1(\Omega).$$

Combining the fact $H_0^1(\Omega) \subset \subset L_2(\Omega)$, (Rellich-Kondrashov theorem) with the lower estimate in the previous inequality, we complete the proof.

Uniformly elliptic operator in the divergent form

Consider a uniformly elliptic operator 
$$-\mathcal{T} := -D_j(a^{ij} D_{x^j}), a^{ij}(Q) \in C^2(\Omega), a^{ij} \xi_i \xi_j \geq \gamma_a |\xi|^2, \gamma_a > 0, i, j = 1, 2, \ldots, n,$$

$$\text{D}(\mathcal{T}) = H^2(\Omega) \cap H_0^1(\Omega).$$

The following theorem gives us a key to apply results of the paper [9] in accordance with which a number of spectral theorems can be applied to the operator $-\mathcal{T}$. Moreover the conditions established bellow are formulated in terms of the operator $A$, what reveals a mathematical nature of the operator $-\mathcal{T}$. 

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Theorem 2. We claim that
\[ -\mathcal{T} = \frac{1}{n} \sum_{i=0}^{n} A_i^* G_i A_i, \] (10)
the following relations hold
\[ -\text{Re}(\mathcal{T} f, f)_{L^2} \geq C\|f\|_{\mathcal{H}_0^2}; \quad |(\mathcal{T} f, g)_{L^2}| \leq C\|f\|_{\mathcal{H}_0^2}\|g\|_{\mathcal{H}_0^2}, \quad f, g \in C_0^\infty(\Omega), \]
where $G_i$ are some operators corresponding to the operators $A_i$.

Proof. It is easy to prove that
\[ \|A_if\|_{L^2} \leq C\|f\|_{\mathcal{H}_0^1}, \quad f \in H_0^1(\Omega), \] (11)
for this purpose we should use a representation $A_if(Q) = - (\nabla f, e_i)_{\mathbb{E}^n}$, $f \in C_0^\infty(\Omega)$. Applying the Cauchy Schwarz inequality, we get
\[ \|A_if\|_{L^2} \leq \left\{ \int_{\Omega} |(\nabla f, e_i)_{\mathbb{E}^n}|^2 dQ \right\}^{1/2} \leq \left\{ \int_{\Omega} \|\nabla f\|_{\mathbb{E}^n}^2 \|e_i\|_{\mathbb{E}^n}^2 dQ \right\}^{1/2} = \|f\|_{\mathcal{H}_0^1}, \quad f \in C_0^\infty(\Omega). \]

Passing to the limit at the left-hand and right-hand side, we obtain \ref{eq:11}. Thus, we get $H_0^1(\Omega) \subset D(A_i)$. Let us find a representation for the operator $G_i$. Consider the operators
\[ B_if(Q) = \int_0^r f(P_i + e[r - t])dt, \quad f \in L_2(\Omega), \quad i = 1, 2, ... n. \]

It is obvious that
\[ \int_{\Omega} A_i(B_i f \cdot g) dQ = \int_{\Omega} A_i B_i f \cdot g dQ + \int_{\Omega} B_i f \cdot A_i g dQ, \quad f \in C^2(\Omega), \quad g \in C_0^\infty(\Omega). \] (12)

Using the divergence theorem, we get
\[ \int_{\Omega} A_i(B_i f \cdot g) dQ = \int_{\sigma} (e_i, n)_{\mathbb{E}^n}(B_i f \cdot g)(\sigma)d\sigma, \] (13)
where $S$ is the surface of $\Omega$. Taking into account that $g(S) = 0$ and combining \ref{eq:12}, \ref{eq:13}, we get
\[ -\int_{\Omega} A_i B_i f \cdot g dQ = \int_{\Omega} B_i f \cdot \overline{A_i g} dQ, \quad f \in C^2(\Omega), \quad g \in C_0^\infty(\Omega). \] (14)

Suppose that $f \in H^2(\Omega)$, then there exists a sequence $\{f_n\}_1^\infty \subset C^2(\bar{\Omega})$ such that $f_n \to f$ (see \cite[p.346]{16}). Using this fact, it is not hard to prove that $\mathcal{T} f_n \to \mathcal{T} f$. Therefore $A_i B_i f_n \to A_i B_i f$, since $B_i$ is continuous (see proof of Lemma 2). Using these facts, we can extend relation \ref{eq:14} to the following
\[ -\int_{\Omega} \mathcal{T} f \cdot \bar{g} dQ = \int_{\Omega} B_i f \overline{A_i g} dQ, \quad f \in D(\mathcal{T}), \quad g \in C_0^\infty(\Omega). \] (15)
Note, that it was previously proved that $A_i^{-1} \subset B_i$ (see the proof of Lemma 2), $H^1_0(\Omega) \subset D(A_i)$. Hence $G_i A_i f = B_i T f, f \in D(T)$, where $G_i := B_i T B_i$. Using this fact we can rewrite relation (15) in a form
\[
- \int_\Omega T f \cdot \tilde{g} dQ = \int_\Omega G_i A_i f \overline{A_i g} dQ, \, f \in D(T), \, g \in C_0^\infty(\Omega). \tag{16}
\]

Note that in accordance with Lemma 2 we have
\[
\forall g \in D(A_i), \exists \{g_n\}^\infty_n \subset C_0^\infty(\Omega), g_n \rightarrow g.
\]

Therefore, we can extend relation (16) to the following
\[
- \int_\Omega T f \cdot \tilde{g} dQ = \int_\Omega G_i A_i f \overline{A_i g} dQ, \, f \in D(T), \, g \in D(A_i). \tag{17}
\]

Relation (17) indicates that $G_i A_i f \in D(A_i^*)$ and it is clear that $-T \subset A_i^* G_i A_i$. On the other hand, in accordance with Chapter VI, Theorem 1.2 [2], we have that $-T$ is a closed operator. Using the divergence theorem we get
\[
- \int_\Omega D_j(a^{ij} D_i f) \tilde{g} dQ = \int_\Omega a^{ij} D_i f \overline{D_j g} dQ, \, f \in C^2(\Omega), \, g \in C_0^\infty(\Omega).
\]

Passing to the limit at the left-hand and right-hand side of the last inequality, we can extend it to the following
\[
- \int_\Omega D_j(a^{ij} D_i f) \tilde{g} dQ = \int_\Omega a^{ij} D_i f \overline{D_j g} dQ, \, f \in H^2(\Omega), \, g \in H_0^1(\Omega).
\]

Therefore, using the uniformly elliptic property of the operator $-T$, we get
\[
- \text{Re} \left( T f, f \right)_{L_2} \geq \gamma_a \int_\Omega \sum_{i=1}^n |D_i f|^2 dQ = \gamma_a \| f \|^2_{L_2}, \, f \in D(T). \tag{18}
\]

Using the Poincaré-Friedrichs inequality, we get $- \text{Re} \left( T f, f \right)_{L_2} \geq C \| f \|^2_{L_2}, \, f \in D(T)$. Applying the Cauchy-Schwarz inequality to the left-hand side, we can easily deduce that the conditions of Lemma 1 are satisfied. Thus, the operator $-T$ is $m$-accretive. In particular, it means that there does not exist an accretive extension of the operator $-T$. Let us prove that $A_i^* G_i A_i$ is accretive, for this purpose combining (16), (18), we get $(G_i A_i f, A_i f)_{L_2} \geq 0, f \in C_0^\infty(\Omega)$. Due to the relation $\tilde{A}_0 = A$, proved in Lemma 2 the previous inequality can be easily extended to $(G_i A_i f, A_i f)_{L_2} \geq 0, f \in D(G_i A_i)$. In its own turn, it implies that $(A_i^* G_i A_i f, f)_{L_2} \geq 0, f \in D(A_i^* G_i A_i)$, thus we have obtained the desired result. Therefore, taking into account the facts given above, we deduce that $-T = A_i^* G_i A_i, i = 1, 2, \ldots, n$ and obtain (10). Applying the Cauchy-Schwarz inequality to the inner sums, then using Corollary 1, we obtain
\[
\left| \int_\Omega T f \cdot \tilde{g} dQ \right| = \left| \int_\Omega a^{ij} D_i f \overline{D_j g} dQ \right| \leq a_1 \int_\Omega \| \nabla f \|^2_{L^2} \| \nabla g \|^2_{L^2} dQ \leq
\]
\[ \leq a_1 \|f\|_{H^1_0} \|g\|_{H^1_0} \leq C \|f\|_{H^n_0} \|g\|_{H^n_0}, \quad f, g \in C^\infty_0(\Omega), \]

where

\[ a_1 = \sup_{Q \in \overline{\Omega}} \sqrt{\sum_{i,j=1}^{n} |a_{ij}(Q)|^2}. \]

On the other hand, applying (11), (18) we get

\[ -\text{Re}(\mathcal{T} f, f) \geq C \|f\|_{H^n_0}^2, \quad f \in C^\infty_0(\Omega). \]

The proof is complete. \( \square \)

Thus, by virtue of Corollary 1 and Theorem 2, we are able to claim that theorems (A) - (C) can be applied to the operator \(-\mathcal{T}\).

### 4 Conclusions

In this paper we have established a norm equivalence in the Lebesgue space, it gives us an opportunity to reveal more fully a true mathematical nature of a differential operator. As a consequence of the mentioned equivalence we have a compact embedding of a space generated by the infinitesimal generator of the shift semigroup in a direction into the Lebesgue space. The considered particular case corresponds to a uniformly elliptic operator which is not selfadjoint under minimal assumptions regarding its coefficients. Thus, the opportunity to apply spectral theorems in the natural way becomes relevant, since there are not many results devoted to the spectral properties of non-selfadjoint operators. Along with all these, by virtue of popularity and the well-known applicability of the Lebesgue spaces theory, the result related to the norm equivalence deserves to be considered itself.

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