Construct holomorphic invariants in Čech cohomology by a combinatorial formula

Hanlong Fang

Abstract. Let \( E \) be a holomorphic vector bundle over a complex manifold \( X \). We introduce \( T \) invariants of \( E \) in Čech cohomology groups via a combinatorial formula. When \( X \) is compact, up to certain normalized factors, the \( T \) invariants of \( E \) coincide with the power sums of the Chern roots of \( E \). We refine the first \( T \) invariant and show that it detects the degeneration of the Fröhlicher spectral sequence and characterizes the flatness of holomorphic line bundles.

1. Introduction

Since the pioneer works of Stiefel, Whitney, Pontrjagin, Steenrod, Chern, Weil, Grothendieck, etc., to name a few, a great number of invariants of fiber bundles have been discovered. Among these invariants, Chern classes, which were originally constructed in [12] by integrating invariant polynomials of curvature forms, play a central role.

Chern classes have been refined in various settings through different methods, in order to obtain the properties of manifolds (or even singular varieties). Aeppli [1] introduced the cohomology groups that bears his name, when investigating the function theory on Stein manifolds. In developing the Nevanlinna theory of holomorphic sections of ample holomorphic vector bundles, Bott and Chern [8] introduced Bott–Chern cohomology, which now serves as a powerful tool in the study of non-Kähler manifolds (see [2–4] and references therein) and is also related to arithmetic characteristic classes (see [20]).

Bott [6,7] initiated the study of constructing Chern classes by transition functions through his operators for iterated differences. Lehmann [26], Lehman-Suwa [27], Corrêa-Suwa [14], etc., to name a few, represented characteristic classes in Čech-de Rham or Čech-Bott–Chern cohomology, when investigating the localization problem of characteristic classes. Sharygin [30] developed an algorithm to represent Chern classes of smooth principle \( GL_n \)-bundles by Čech cocycles in the Čech-de Rham complex.

We continue the investigation along the above works by focusing more on the holomorphic aspect. By restricting to the holomorphic category, we produce Čech
cocycles in the usual Čech cohomology group instead of the double complex via an explicit formula. Since our approach is purely algebraic, it is immediately carried over to locally free sheaves on schemes over general fields. Moreover, motivated by [15], we construct new holomorphic invariants refining the first Chern classes, the refined first $T$ invariant, and investigate its geometric meaning.

We first introduce the following key combinatorial formula. Let $X$ be a complex manifold and $E$ a holomorphic vector bundle over $X$. Let $U := \{U_i\}_{i \in I}$ be an open cover of $X$, where $I$ is an ordered set, and $g := \{g_{ij}\}$ a system of transition functions associated with a certain trivialization of $E$ with respect to $U$. Denote by $\Omega^k$ the sheaf of germs of holomorphic differential $k$-forms. For a positive integer $k$ and indices $i_1, i_2, \ldots, i_{k+1} \in I$, define $t_{i_1 \cdots i_{k+1}} \in \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k)$ by

$$t_{i_1 \cdots i_{k+1}} := \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot tr\left(g_{i_{\sigma(1)}i_{\sigma(k+1)}}^{-1} d g_{i_{\sigma(1)}i_{\sigma(k+1)}} \right) \cdot d g_{i_{\sigma(2)}i_{\sigma(k+1)}} \cdot d g_{i_{\sigma(3)}i_{\sigma(k+1)}} \cdots d g_{i_{\sigma(k)}i_{\sigma(k+1)}};$$

(1)

define a Čech $k$-cocycle $\hat{f}_{k, U}(E)$ by

$$\hat{f}_{k, U}(E) := \bigoplus_{i_1 < \cdots < i_{k+1}} t_{i_1 \cdots i_{k+1}} \in \bigoplus_{i_1 < \cdots < i_{k+1}} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k).$$

(2)

Here $S_{k+1}$ is the permutation group of $\{1, 2, \ldots, k+1\}$; $sgn(\sigma) = 1$ for $\sigma$ an even permutation and $sgn(\sigma) = -1$ for $\sigma$ an odd permutation; $tr$ is the trace operator of differential form-valued matrices, where the product between differential forms is understood as the wedge product.

**Theorem 1.1.** (The combinatorial formula) $\hat{f}_{k, U}(E)$ is a Čech $k$-cocycle and defines an element $f_{k, U}(E) \in \hat{H}^k(U, \Omega^k)$ independent of the choice of $g$.

Since for every refinement $V$ of $U$ the above construction is compatible with the natural restriction $I_{U \mid V} : \hat{H}^k(U, \Omega^k) \to \hat{H}^k(V, \Omega^k)$, we have

**Theorem 1.2.** The image of $f_{k, U}(E)$ in the Čech cohomology group $\hat{H}^k(X, \Omega^k)$ under the canonical homomorphism is independent of the choice of $U$. Denote the image by $f_k(E)$ and call it the $k$-th $T$ invariant of $E$.

When $X$ is compact, we can view $f_k(E)$ as an element of $H^k(X, \Omega^k)$ naturally. Then,

**Theorem 1.3.** Let $X$ be a compact complex manifold and $E$ a holomorphic vector bundle over $X$. Let $ch(E)$ be the Chern character of $E$ in the ring $\bigoplus_{k=0}^{\infty} H^k(X, \Omega^k)$ (see 4). Then,

$$\text{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k! \cdot (2\pi \sqrt{-1})^k} \cdot f_k(E)$$

(3)

in the ring $\bigoplus_{k=0}^{\infty} H^k(X, \Omega^k)$. Here $f_0(E) = 1$ by convention.
Next we can lift the first $T$ invariant from the Čech cohomology group $\check{H}^1(X, d\Omega^0)$ to $\check{H}^1(X, \Omega^1)$ (see 2 for the definitions) as follows.

Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of a complex manifold $X$ and $g := \{g_{ij}\}$ a system of transition functions associated with a certain trivialization of a holomorphic vector bundle $E$ with respect to $\mathcal{U}$. For indices $i_1, i_2 \in I$, define an element $t_{i_1 i_2} \in \Gamma(U_{i_1 i_2}, d\Omega^0)$ by

$$t_{i_1 i_2} := \sum_{\sigma \in S_2} \frac{\text{sgn}(\sigma)}{2!} \cdot \text{tr}(g_{i_1}^{-1}(i_{\sigma(1)}i_{\sigma(2)}) d g_{i_1}(i_{\sigma(1)}i_{\sigma(2)})) = \text{tr}(g_{i_1}^{-1} d g_{i_1 i_2});$$

(4)

define a Čech 1-cochain $\hat{f}^r_{1, \mathcal{U}}(E)$ by

$$\hat{f}^r_{1, \mathcal{U}}(E) := \bigoplus_{i_1 < i_2} t_{i_1 i_2} \in \bigoplus_{i_1 < i_2} \Gamma(U_{i_1 i_2}, d\Omega^0).$$

(5)

**Theorem 1.4.** $\hat{f}^r_{1, \mathcal{U}}(E)$ is a Čech 1-cocycle and defines an element $f^r_{1, \mathcal{U}}(E) \in \check{H}^1(\mathcal{U}, d\Omega^0)$ independent of the choice of $g$. The image of $f^r_{1, \mathcal{U}}(E)$ in the Čech cohomology group $\check{H}^1(X, d\Omega^0)$ under the canonical homomorphism is independent of the choice of $\mathcal{U}$. Denote the image by $f^r_1(E)$ and call it the refined first $T$ invariant.

We call a holomorphic line bundle $E$ flat if its transition functions can be taken as constant functions; call $E$ $\mathcal{Q}$-flat if $mE$ is flat for a certain positive integer $m$.

We have

**Theorem 1.5.** Let $X$ be a compact complex manifold and $\pi : E \to X$ a holomorphic line bundle. $E$ is $\mathcal{Q}$-flat if and only if the refined first $T$ invariant of $E$ is trivial in $H^1(X, d\Omega^0)$.

When $X$ is a compact Kähler manifold, the refined first $T$ invariants coincide with the first $T$ invariants. To determine whether a manifold has a line bundle whose refined first $T$ invariant is strictly finer than its first $T$ invariant, we prove that

**Theorem 1.6.** Let $X$ be a compact complex manifold. There is a line bundle of $X$ with trivial first $T$ invariant but non-trivial refined first $T$ invariant if only if in the Frölicher spectral sequence of $X$,

$$E^{0,1}_2 \neq E^{0,1}_3.$$  

(6)

**Remark 1.7.** By Example 3.11 in [28], the refined first $T$ invariant can be strictly finer than the first $T$ invariant.

We now briefly describe the organization of the paper and the basic ideas for the proofs. The most technical part of the paper is the proof of Theorem 1.1. But as long as one has a correct guess of the formula, the remaining is a routine algebraic computation combining the permutation technique, the properties of the trace operator and the use of the cocycle condition of the transition functions. To compare the $T$ invariants and the Chern characters in the Dolbeault cohomology, we use the
Hirsch lemma and the splitting principle. Noticing that the first $T$ invariants are in a smaller sheaf $d\Omega^0$, we can lift the first $T$ invariant from $H^1(X, \Omega^1)$ to $H^1(X, d\Omega^0)$. Since the combinatorical formula is proved by an algebraic computation, we can generalize the previous discussion easily to the locally free sheaves of schemes over general fields.

The organization of the paper is as follows: In 2.1, we introduce some notations and basic facts of holomorphic vector bundles and Čech cohomology of various sheaves. In 2.2, we prove Theorem 1.1. In 3.1, we prove Theorem 1.2 and Theorem 1.3. In 3.2, we lift the first $T$ invariant from $\check{H}^1(X, \Omega^1)$ to $\check{H}^1(X, d\Omega^0)$ and prove Theorems 1.4 and 1.5. In 3.3, we prove Theorem 1.6, the strict refinement criterion. In 3.4, we generalize the above discussion to locally free sheaves of schemes over general fields. In 3.5, we discuss the basic properties of the cohomological ring in which the refined $T$ invariants live. In Appendix A, we include a detailed proof of Lemma 2.5 which is used in the proof of Theorem 1.1; it shows that the cocycles constructed in Theorem 1.1 is independent of the choice of the transition functions associated with a cover.

2. The combinatorial formula

2.1. Preliminary

In this subsection, we recall some notations and basic facts of holomorphic vector bundles and Čech cohomology which will be used throughout the paper.

Suppose that $X$ is a complex manifold and $E$ is a holomorphic vector bundle over $X$ of rank $M$. Then, there is an open cover $U := \{U_i\}_{i \in I}$ of $X$ and a holomorphic trivialization $(E, \{U_i\}, \{\phi_i\})$ of $E$, where $I$ is an ordered set, as follows

$$\phi_i : E\mid_{U_i} \cong U_i \times \mathbb{C}^M, \quad i \in I. \quad (7)$$

(For example, we can choose $U$ to be any Stein open cover of $X$, for holomorphic vector bundles are holomorphically trivial over Stein sets.) The maps

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^M \to (U_i \cap U_j) \times \mathbb{C}^M \quad (8)$$

are vector-space automorphisms of $\mathbb{C}^M$ in each fiber and hence give rise to maps

$$g_{ij} : (U_i \cap U_j) \to GL(M, \mathbb{C})$$

$$g_{ij}(z) = \phi_i \circ \phi_j^{-1}\big|_{z \times \mathbb{C}^M} \quad (9)$$

We call such $g := \{g_{ij}\}$ the system of (matrix-valued) transition functions associated with trivialization $(E, \{U_i\}, \{\phi_i\})$.

Remark 2.1. In the following, we usually refer to a system of transition functions without mentioning the trivialization it is associated with, for we will only use the information of the transition functions in this paper.
Notice that there is a non-abelian Čech cohomology interpretation of the holomorphic vector bundles as follows (see [18] or [10] for more details). Since \( g_{ij} \) is a \( GL(M, \mathbb{C}) \)-valued holomorphic function and \( g_{ij}g_{jk}g_{ki} \equiv 1 \) on \( U_{ijk} := U_i \cap U_j \cap U_k \), one can associate \( g \) with the following Čech 1-cocycle

\[
\tilde{g} := \bigoplus_{i_1 < i_2} g_{i_1i_2} \bigg|_{U_{i_1i_2}} \in \bigoplus_{i_1 < i_2} \Gamma_{\text{hol}}(U_{i_1i_2}, GL(M, \mathbb{C})),
\]

where \( \Gamma_{\text{hol}}(U_{i_1i_2}, GL(M, \mathbb{C})) \) is the (nonabelian) group consisting of \( GL(M, \mathbb{C}) \)-valued holomorphic functions over \( U_{i_1i_2} \). If \((E, \{U_i\}, \{\phi_i\})\) is another trivialization of \( E \) with respect to the same cover, there exists a Čech 0-cochain \( h := \bigoplus_{i_1} h_{i_1} \in \bigoplus_{i_1} \Gamma_{\text{hol}}(U_{i_1}, GL(M, \mathbb{C})) \)

\[
h := \bigoplus_{i_1} h_{i_1} \in \bigoplus_{i_1} \Gamma_{\text{hol}}(U_{i_1}, GL(M, \mathbb{C}))
\]

such that \( \tilde{g}_{ij} = h^{-1}_{i_1} g_{ij} h_{j} \) for \( i, j \in I \). Denote by \( \tilde{H}^1_{\text{hol}}(U, GL(M, \mathbb{C})) \) the quotient group under the above relation. It is easy to verify that each holomorphic vector bundle of rank \( M \) determines a well-defined element in the Čech cohomology group

\[
\tilde{H}^1_{\text{hol}}(X, GL(M, \mathbb{C})) := \lim_{\mathcal{U}} \tilde{H}^1_{\text{hol}}(\mathcal{U}, GL(M, \mathbb{C}))
\]

where the direct limit runs over all the open covers.

Denote by \( \Omega^k \) be the sheaf of germs of holomorphic \( k \)-forms for \( k \geq 0 \); by convention, \( \Omega^0 = \mathcal{O}_X \). Recall that \( \tilde{H}^k(\mathcal{U}, \Omega^k) \) is the \( k \)-th Čech cohomology group of the sheaf \( \Omega^k \) with respect to \( \mathcal{U} \). Let \( I_{\mathcal{U}_V} \) be the natural induced map from \( \tilde{H}^k(\mathcal{U}, \Omega^k) \) to \( \tilde{H}^k(\mathcal{V}, \Omega^k) \) for open covers \( \mathcal{U} \) and \( \mathcal{V} \) such that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \); denote by \( \{\tilde{H}^k(\mathcal{U}, \Omega^k), I_{\mathcal{U}_V}\} \) the direct system indexed by the direct set formed by the open covers under refinement. \( \tilde{H}^k(X, \Omega^k) \), the \( k \)-th Čech cohomology group of \( \Omega^k \), is the direct limit of \( \{\tilde{H}^k(\mathcal{U}, \Omega^k), I_{\mathcal{U}_V}\} \)

\[
\tilde{H}^k(X, \Omega^k) := \lim_{\mathcal{U}} \tilde{H}^k(\mathcal{U}, \Omega^k); \tag{13}
\]

there is a canonical homomorphism for each open cover \( \mathcal{U} \),

\[
L_{\mathcal{U}} : \tilde{H}^k(\mathcal{U}, \Omega^k) \rightarrow \tilde{H}^k(X, \Omega^k). \tag{14}
\]

Also, there is a natural homomorphism from Čech cohomology to sheaf cohomology,

\[
L : \tilde{H}^k(X, \Omega^k) \rightarrow H^k(X, \Omega^k). \tag{15}
\]

By Leray’s theorem, it is clear that if each \( U_i \) in \( \mathcal{U} \) is Stein, then \( H^k(X, \Omega^k) \cong \tilde{H}^k(\mathcal{U}, \Omega^k) \), and if each open cover has a Stein cover as its refinement, then \( L \) is an isomorphism (see Chapter VI of [19] for reference). In particular, if \( X \) is a compact complex manifold, \( L \) is an isomorphism.

Following [22], for integer \( q \geq 0 \), we denote by \( d\Omega^q \) the sheaf of germs of closed holomorphic \((q + 1)\)-forms. We call an open cover \( \mathcal{U} := \{U_i\}_{i \in I} \) a Stein cover if each \( U_i \) is Stein, and a good cover if each \( U_i \) is Stein and each nonempty
intersections $U_{i_1\ldots i_p}$ is contractible. Recall that for each good cover $\mathcal{U}$ and each $q > 0$, $p > 0$ and $k \geq 0$, $H^q(U_{i_1\ldots i_p}, d\Omega^k) = 0$ (see 3.3 of [15]), and hence
\[ H^{q+1}(X, d\Omega^q) \cong \check{H}^{q+1}(\mathcal{U}, d\Omega^q) \quad \text{for } q \geq 0. \] (16)

Moreover, if each open cover has a good cover as its refinement, then the following natural map from Čech cohomology to sheaf cohomology is an isomorphism,
\[ L : \check{H}^{q+1}(X, d\Omega^q) \cong H^{q+1}(X, d\Omega^q) \quad \text{for } q \geq 0; \] (17)

In particular, $L$ is an isomorphism for compact complex manifolds (see the good cover lemma in Appendix I of [15] for instance). Moreover, the natural inclusion $J_{q+1} : d\Omega^q \rightarrow \Omega^{q+1}$ of $\mathbb{C}$-sheaves induces a natural homomorphism between cohomology groups
\[ j_{q+1} : H^{q+1}(X, d\Omega^q) \rightarrow H^{q+1}(X, \Omega^{q+1}) \quad \text{for } q \geq 0. \] (18)

2.2. Proof of Theorem 1.1

Let $X$ be a complex manifold and $E$ a holomorphic vector bundle over $X$; denote the rank of $E$ by $M$. In this subsection, we prove Theorem 1.1 by the following lemmas.

**Lemma 2.2.** Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of $X$, where $I$ is an ordered set, and $g := \{g_{ij}\}$ be a system of transition functions of $E$ with respect to $\mathcal{U}$. For each integer $k$ and $i_1, \ldots, i_{k+1} \in I$, define $t_{i_1...i_{k+1}} \in \Gamma(U_{i_1...i_{k+1}}, \Omega^{\leq})$ by
\[ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \text{tr}(g^{-1}_{i_{\sigma(1)}i_{\sigma(k+1)}} g^{-1}_{i_{\sigma(2)}i_{\sigma(k+1)}} g^{-1}_{i_{\sigma(3)}i_{\sigma(k+1)}} \cdots g^{-1}_{i_{\sigma(k)}i_{\sigma(k+1)}}). \] (19)

Then $t_{i_1...i_{j+1}...i_{k+1}} = -t_{i_1...i_{j+1}...i_{k+1}}$. Hence, $t_{i_1...i_{k+1}} = 0$ if there is a repeated index in $\{i_1, \ldots, i_{k+1}\}$.

**Remark 2.3.** We use the following convention in Lemma 2.2. $d g_{ij}$ is the matrix derived by differentiating matrix $g_{ij}$ entry by entry. The product $\cdots d g_{\sigma(1)\sigma(k)}^{-1} g_{\sigma(2)\sigma(k)}^{-1} d g_{\sigma(2)\sigma(k)}^{-1} \cdots$ is the matrix product of matrices with differential form-valued entries. For example, if
\[ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \] (20)
then
\[ dg = \begin{pmatrix} dg_{11} & dg_{12} \\ dg_{21} & dg_{22} \end{pmatrix}, \quad dh = \begin{pmatrix} dh_{11} & dh_{12} \\ dh_{21} & dh_{22} \end{pmatrix}, \] (21)
and
\[ dgdh = \begin{pmatrix} dg_{11} \wedge dh_{11} + dg_{12} \wedge dh_{21} & dg_{11} \wedge dh_{12} + dg_{12} \wedge dh_{22} \\ dg_{21} \wedge dh_{11} + dg_{22} \wedge dh_{21} & dg_{21} \wedge dh_{12} + dg_{22} \wedge dh_{22} \end{pmatrix}. \] (22)
Proof of Lemma 2.2. Denote the permutation \((j, j + 1) \in S_{k+1}\) by \(\tau\); namely, \(\tau(j) = j + 1, \tau(j + 1) = j\) and \(\tau(q) = q\) for each \(q, 1 \leq q \leq k + 2\) and \(q \neq j, j + 1\).

Recall that \(t_1 \cdots t_{k+1}\) is given by (19). Then,

\[
t_{1 \cdots i+j+1 \cdots i+k+1} = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot tr
\]

\[
(\text{23})
\]

\[
\bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) = sgn(\tau) \cdot \sum_{\sigma \in S_{k+1}} \frac{sgn(\tau \circ \sigma)}{(k+1)!} \cdot tr
\]

\[
\bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) \bigg( g_{(24)}^{-1} d g_{(24)}^{1} \bigg) = -t_{1 \cdots i+j+1 \cdots i+k+1}.
\]

Clearly, \(t_{1 \cdots i+k+1} = 0\) if there is a repeated index in \(\{i_1, \ldots, i_{k+1}\}\) by equality (23).

We complete the proof of Lemma 2.2.

We define a Čech \(k\)-cochain \(\hat{f}_k(E, g)\) by

\[
\hat{f}_k(E, g) := \bigoplus_{i_1 < \cdots < i_{k+1}} t_{1 \cdots i+k+1} \in \bigoplus_{i_1 < \cdots < i_{k+1}} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k).
\]

According to (19) and the proof of Lemma 2.2, it is clear that \(t_{j_1 \cdots j_{k+1}} = sgn(\sigma) \cdot t_{j_{\sigma(1)} \cdots j_{\sigma(k+1)}}\).

Hence, for convenience, we can extend the components of the Čech \(k\)-cochain \(\hat{f}_k(E, g)\) in (24) to all \((k+1)\)-tuples of elements in \(I\).

Next, we will prove that \(\hat{f}_k(E, g)\) is closed.

Lemma 2.4. The Čech \(k\)-cochain \(\hat{f}_k(E, g)\) defined by (24) is a Čech \(k\)-cocycle.

Proof of Lemma 2.4. It suffices to prove that for any \(i_1, \ldots, i_{k+2} \in I\) the following equality holds:

\[
\sum_{j=1}^{k+2} (-1)^{j+1} t_{i_{1 \cdots i+j+1 \cdots i+k+2}} = 0,
\]

where \(\hat{i}_j\) is the usual notation for omitting the \(j\)-th index \(i_j\).

Let \(\sigma \in S_{k+1}\); for \(j = 1, \ldots, k + 2\), define the \(j\)-th lift \(\hat{\sigma}^j \in S_{k+2}\) of \(\sigma\) as follows.

\[
\hat{\sigma}^j(q) = \begin{cases} 
\sigma(q) & \text{if } \sigma(q) < j \text{ and } 1 \leq q \leq k + 1; \\
\sigma(q) + 1 & \text{if } \sigma(q) \geq j \text{ and } 1 \leq q \leq k + 1; \\
j & \text{if } q = k + 2.
\end{cases}
\]

(26)
Denote by $S^j_{k+2}$ the set $\{\tau \in S_{k+2} | \tau(k+2) = j\}$. Notice that
\begin{equation}
\text{sgn}(\hat{\sigma}^j) = \text{sgn}(\sigma) \cdot (-1)^{k+2-j},
\end{equation}
and
\begin{equation}
S_{k+2}^j = \bigcup_{j=1}^{k+2} S^j_{k+2} \text{ where } S^j_{k+2} \cap S^j_{k+2} = \emptyset \text{ for } i \neq j.
\end{equation}

We then have the following formula under the above notation for $j = 1, \ldots, k+2$.

\begin{equation}
(-1)^{j+1} t_1 \cdot t_2 \cdots t_{k+2}
= \sum_{\sigma \in S_{k+2}} (-1)^{j+1} \frac{\text{sgn}(\sigma)}{(k+1)!} \cdot \text{tr}(g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}}^{-1}) \cdot d g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}} g_{\hat{\sigma}(2)i_{\hat{\sigma}(k+1)}}^{-1} \cdots g_{\hat{\sigma}(k+1)i_{\hat{\sigma}(k+1)}}^{-1}
= \sum_{\sigma \in S_{k+2}} (-1)^{k+1} \frac{\text{sgn}(\hat{\sigma})}{(k+1)!} \cdot \text{tr}(g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}}^{-1}) \cdot d g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}} g_{\hat{\sigma}(2)i_{\hat{\sigma}(k+1)}}^{-1} \cdots g_{\hat{\sigma}(k+1)i_{\hat{\sigma}(k+1)}}^{-1}
= \sum_{\sigma \in S_{k+2}} (-1)^{k+1} \frac{\text{sgn}(\hat{\sigma})}{(k+1)!} \cdot \text{tr}(g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}}^{-1}) \cdot d g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}} g_{\hat{\sigma}(2)i_{\hat{\sigma}(k+1)}}^{-1} \cdots g_{\hat{\sigma}(k+1)i_{\hat{\sigma}(k+1)}}^{-1}.
\end{equation}

Substituting into (25), we derive that
\begin{equation}
\sum_{j=1}^{k+2} (-1)^{j+1} t_1 \cdots t_{k+2}
= \sum_{j=1}^{k+2} \sum_{\sigma \in S^j_{k+2}} (-1)^{k+1} \frac{\text{sgn}(\hat{\sigma})}{(k+1)!} \cdot \text{tr}(g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}}^{-1}) \cdot d g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}} g_{\hat{\sigma}(2)i_{\hat{\sigma}(k+1)}}^{-1} \cdots g_{\hat{\sigma}(k+1)i_{\hat{\sigma}(k+1)}}^{-1}
= \sum_{\sigma \in S_{k+2}} (-1)^{k+1} \frac{\text{sgn}(\hat{\sigma})}{(k+1)!} \cdot \text{tr}(g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}}^{-1}) \cdot d g_{\hat{\sigma}(1)i_{\hat{\sigma}(k+1)}} g_{\hat{\sigma}(2)i_{\hat{\sigma}(k+1)}}^{-1} \cdots g_{\hat{\sigma}(k+1)i_{\hat{\sigma}(k+1)}}^{-1}.
\end{equation}

Notice that $g_{\alpha \beta} = g_{\alpha \gamma} \cdot g_{\gamma \beta}$ and hence $dg_{\alpha \beta} = dg_{\alpha \gamma} \cdot g_{\gamma \beta} + g_{\alpha \gamma} \cdot dg_{\gamma \beta}$. Then, we have
\begin{equation}
g_{\alpha \beta}^{-1} dg_{\alpha \beta} = (g_{\gamma \beta}^{-1} dg_{\alpha \gamma} g_{\gamma \beta} + g_{\alpha \gamma}^{-1} dg_{\gamma \beta}) = g_{\gamma \beta}^{-1} (dg_{\alpha \gamma} + (-1)g_{\alpha \gamma}^{-1} dg_{\gamma \beta}) g_{\gamma \beta}.
\end{equation}
Substituting (31) into (29) by taking $\gamma = i_{\hat{\sigma}(k+2)}$, $\beta = i_{\hat{\sigma}(k+1)}$ and $\alpha = i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, i_{\hat{\sigma}(3)}, \ldots, i_{\hat{\sigma}(k)}$, we get
$$(-1)^{j+1} t_{i_1, \ldots, i_j} = \sum_{\hat{\sigma} \in S_{k+2}} (-1)^{k+1} \cdot \text{sgn}(\hat{\sigma}) \cdot \left\{ \prod_{l=1}^{k} g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \cdot g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \right\}$$

\begin{equation}
+ (-1)^{j+1} t_{i_1, \ldots, i_j} = \sum_{\hat{\sigma} \in S_{k+2}} (-1)^{k+1} \cdot \text{sgn}(\hat{\sigma}) \cdot \left\{ \prod_{l=1}^{k} g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \cdot g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \right\}.
\end{equation}

(32)

Notice that in the last step we use the fact that $tr(ABC) = tr(BCA)$.

To simplify the notation, for elements $p_1, \ldots, p_{k+1} \in I$ (not necessarily distinct), we define

$$\Delta(p_1, \ldots, p_{k+1}) := tr(g_{p_1 p_{k+1}}^{-1} \cdot g_{p_2 p_{k+1}}^{-1} \cdot g_{p_3 p_{k+1}}^{-1} \cdot g_{p_4 p_{k+1}}^{-1} \cdot \ldots \cdot g_{p_{k+1} p_{k+1}}^{-1});$$

(33)

for positive integer $j$ and $s_1, \ldots, s_j$ such that $1 \leq j \leq k$ and $1 \leq s_1 < s_2 < \cdots < s_j \leq k$, and an element $p_{k+2} \in I$, define

$$\Delta_{s_1 \ldots s_j, p_{k+2}}(p_1, \ldots, p_{k+1})$$

$$:= \Delta(p_1, \ldots, p_{k+1}),$$

(34)

namely, one replaces the indices $p_{s_1}, \ldots, p_{s_j}$ by $p_{k+2}$ in $\Delta(p_1, \ldots, p_{k+1})$.

Under the above notation, we can expand (32) and derive that

$$(-1)^{j+1} t_{i_1, \ldots, i_j} = \sum_{\hat{\sigma} \in S_{k+2}} (-1)^{k+1} \cdot \text{sgn}(\hat{\sigma}) \cdot \left\{ \prod_{l=1}^{k} g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \cdot g_{\hat{\sigma}(l) / \hat{\sigma}(l+2)} \right\}$$

$$- \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k+1)}, i_{\hat{\sigma}(k+2)}) = \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, i_{\hat{\sigma}(3)}, \ldots, i_{\hat{\sigma}(k+1)});$$

$$- \cdots - \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, i_{\hat{\sigma}(3)}, \ldots, i_{\hat{\sigma}(k+1)}, i_{\hat{\sigma}(k+2)})$$

(35)

Claim I. For $1 \leq j \leq k + 2$, $2 \leq m \leq k$ and $1 \leq s_1 < \cdots < s_m \leq k$, we have that

$$\sum_{\hat{\sigma} \in S_{k+2}} \text{sgn}(\hat{\sigma}) \cdot \Delta_{s_1 \ldots s_m, i_{\hat{\sigma}(k+1)}}(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) = 0.$$

(36)

Proof of Claim I. Denote by $\tau$ the permutation $(s_1, s_2) \in S_{k+2}$. Since $1 \leq s_1 < s_2 \leq k$, $\hat{\sigma} \circ \tau \in S_{k+2}$ for each $\hat{\sigma} \in S_{k+2}$. It is easy to verify that the following composition map is a bijection for $1 \leq j \leq k + 2$,

$$T_\tau : S_{k+2} \xrightarrow{\cong} S_{k+2},$$

$$\hat{\sigma} \mapsto \hat{\sigma} \circ \tau.$$

(37)
Moreover, we have that

\[ \Delta_{s_1 \cdots s_m, i(\sigma k+1)} (i(1), i(2), \ldots, i(k), i(k+2)) = \Delta_{s_1 \cdots s_m, i(\sigma \tau k+1)} (i(1), i(2), \ldots, i(k), i(k+2)). \]  

(38)

Combining bijection map (37) and equality (38), we derive that

\[
\sum_{\hat{\sigma} \in S_{k+2}} sgn(\hat{\sigma}) \cdot \Delta_{s_1 \cdots s_m, i(\hat{\sigma} k+1)} (i(1), i(2), \ldots, i(k), i(k+2)) = \sum_{\hat{\sigma} \circ \tau q \in S_{k+2}} sgn(\hat{\sigma} \circ \tau q) \cdot \Delta_{s_1 \cdots s_m, i(\hat{\sigma} \circ \tau q k+1)} (i(1), i(2), \ldots, i(k), i(k+2)) \]

(39)

Therefore, we have

\[
\sum_{\hat{\sigma} \in S_{k+2}} sgn(\hat{\sigma}) \cdot \Delta_{s_1 \cdots s_m, i(\hat{\sigma} k+1)} (i(1), i(2), \ldots, i(k), i(k+2)) = 0 .
\]  

(40)

We complete the proof of Claim I. \( \square \)

Denote the permutation \((q, k + 1) \in S_{k+2}\) by \( \tau_q \) for \( 1 \leq q \leq k + 2 \). Then, the following map is a bijection for \( 1 \leq q \leq k + 2 \),

\[
T_{\tau_q} : S_{k+2} \xrightarrow{\cong} S_{k+2}, \quad \hat{\sigma} \mapsto \hat{\sigma} \circ \tau_q
\]

(41)

\( sgn(\tau_q) = -1 \) for \( q \neq k + 1 \) and \( sgn(\tau_q) = 1 \) for \( q = k + 1 \).
By Claim I and (35), we have that
\[
\sum_{j=1}^{k+2} (-1)^{j+1} t_{i_1 \ldots i_j \ldots i_{k+2}} = \frac{(-1)^{k+1}}{(k + 1)!} \sum_{\hat{\sigma} \in S_{k+2}} sgn(\hat{\sigma})
\]

\[
\cdot \left\{ \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) - \Delta(i_{\hat{\sigma}(k+1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) - \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(k+1)}, i_{\hat{\sigma}(3)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) - \ldots - \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, i_{\hat{\sigma}(3)}, \ldots, i_{\hat{\sigma}(k-1)}, i_{\hat{\sigma}(k+1)}, i_{\hat{\sigma}(k+2)}) \right\}
\]

\[
= \frac{(-1)^{k+1}}{(k + 1)!} \sum_{\hat{\sigma} \in S_{k+2}} sgn(\hat{\sigma}) \cdot \left\{ (k + 1) \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) \right\}
\]

Together with (30), we have that
\[
\sum_{j=1}^{k+2} (-1)^{j+1} t_{i_1 \ldots i_j \ldots i_{k+2}} = \frac{(-1)^{k+1}}{(k + 1)!} \sum_{\hat{\sigma} \in S_{k+2}} sgn(\hat{\sigma}) \cdot \left\{ (k + 1) \Delta(i_{\hat{\sigma}(1)}, i_{\hat{\sigma}(2)}, \ldots, i_{\hat{\sigma}(k)}, i_{\hat{\sigma}(k+2)}) \right\}
\]

\[
= \frac{(-1)^{k+1}(k + 1) \cdot sgn(\tau_{k+2})}{(k + 1)!} \sum_{\hat{\sigma} \circ \tau_{k+2} \in S_{k+2}} sgn(\hat{\sigma} \circ \tau_{k+2}) \cdot \Delta(i_{\hat{\sigma} \circ \tau_{k+2}(1)}, \ldots, i_{\hat{\sigma} \circ \tau_{k+2}(k)}, i_{\hat{\sigma} \circ \tau_{k+2}(k+1)})
\]

\[
= -(k + 1) \cdot \sum_{j=1}^{k+2} (-1)^{j+1} t_{i_1 \ldots i_j \ldots i_{k+2}}.
\]

Therefore, we complete the proof of Lemma 2.4. \hfill \square

By Lemma 2.4, \( \tilde{f}_k(E, g) \) determines an element in \( \tilde{H}^k(\mathcal{U}, \Omega^k) \); we denote this element by \( f_k(E, g) \). Next, we will prove that \( f_k(E, g) \) is independent of the choice of \( g \).

**Lemma 2.5.** The element \( f_k(E, g) \in \tilde{H}^k(\mathcal{U}, \Omega^k) \) is independent of the choice of the system of transition functions of \( E \) with respect to the open cover \( \mathcal{U} \); namely, for \( g \) and \( \tilde{g} \) systems of transition functions with respect to \( \mathcal{U} \), \( f_k(E, g) = f_k(E, \tilde{g}) \) in \( \tilde{H}^k(\mathcal{U}, \Omega^k) \).

**Proof of Lemma 2.5.** See Appendix A. \hfill \square
Proof of Theorem 1.1. Theorem 1.1 follows from Lemmas 2.4 and 2.5. □

Remark 2.6. The above construction holds for the sheaf $A^k$ consisting of germs of smooth differential $k$-forms. However, the resulting invariant cocycle is trivial for $H^k(X, A^k) = 0, k ≥ 1$. That seems to be a reason of using Čech-de Rham complex as in [30] to define Chern classes of smooth bundles.

3. T invariants in $H^k(X, Ω^k)$ and $H^1(X, dΩ^0)$

In this section, we will define the $T$ invariants and the refined first $T$ invariant of holomorphic vector bundles.

3.1. $T$ invariants in the Dolbeault cohomology

Proof of Theorem 1.2. Let $U := \{U_i\}_{i ∈ I}$ be an open cover of $X$, where $I$ is an ordered set, and $g := \{g_{ij}\}$ be a system of transition functions associated with a certain trivialization of $E$ with respect to $U$. By Theorem 1.1, $f_k, U(E)$ is an well-defined element in $\hat{H}^k(U, Ω^k)$.

Suppose $V := \{V_j\}_{j ∈ J}$ is a refinement of $U$. Choose a map $c : J → I$ such that $V_j ⊂ U_{c(j)}$ for all $j ∈ J$. For each $j_1, j_2 ∈ J$, define a matrix-valued function $\tilde{g}_{j_1,j_2}$ as follows.

$$\tilde{g}_{j_1,j_2} : V_{j_1,j_2} → GL(M, C)$$

$$x \mapsto g_{c(j_1)c(j_2)}(x);$$

then $\tilde{g} := \{\tilde{g}_{j_1,j_2}\}$ is a system of transition functions associated with a certain trivialization of $E$ with respect to $V$. On the other hand, recall that the natural restriction $I_{UV} : \hat{H}^k(U, Ω^k) → \hat{H}^k(V, Ω^k)$ is induced by the following map between Čech complexes

$$\Gamma_{UV} : \tilde{C}^\bullet(U, Ω^k) → \tilde{C}^\bullet(V, Ω^k)$$

$$(\tilde{ξ}_{i_1⋯i_p}) \mapsto (\tilde{ξ}_{c(j_1)⋯c(j_p)})_{\mid V_{j_1⋯j_p}}.$$ (45)

In the following, we will show that $I_{UV}(f_k, U(E)) = f_k, V(E)$. Let $\tilde{f}_k, U(E) = (t_{i_1⋯i_{k+1}})$ be the Čech cocycle constructed in Theorem 1.1 corresponding to $f_k, U(E)$, where $t_{i_1⋯i_{k+1}}$ is given by (19). Then by (45), $\Gamma_{UV}(E)$ maps $f_k, U(E)$ to a Čech cocycle $\Gamma_{UV}(f_k, U(E)) := (\tilde{t}_{i_1⋯i_{k+1}})$, where

$$\tilde{t}_{i_1⋯i_{k+1}} = \sum_{σ ∈ S_{k+1}} sgn(σ) \cdot tr(g_{c(j_σ(1))c(j_σ(k+1))}dgc(j_σ(1))c(j_σ(k+1))gc(j_σ(2))c(j_σ(k+1)))$$

$$dgc(j_σ(2))c(j_σ(k+1)) \cdot gc(j_σ(3))c(j_σ(k+1))dgc(j_σ(3))c(j_σ(k+1)) \cdots gc(j_σ(k))c(j_σ(k+1))dgc(j_σ(k))c(j_σ(k+1)).$$ (46)
Noticing (44), we can rewrite (46) as follows.

\[ \tilde{f}_{j_1 \ldots j_{k+1}} = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \text{tr} \left( \tilde{g}_{j_{\sigma(1)}j_{\sigma(k+1)}}^{-1} d\tilde{g}_{j_{\sigma(1)}j_{\sigma(k+1)}} \tilde{g}_{j_{\sigma(2)}j_{\sigma(k+1)}}^{-1} \cdot \tilde{g}_{j_{\sigma(3)}j_{\sigma(k+1)}}^{-1} \cdots \tilde{g}_{j_{\sigma(k)}j_{\sigma(k+1)}}^{-1} d\tilde{g}_{j_{\sigma(k)}j_{\sigma(k+1)}} \right). \]  

(47)

Then \( I_{\mathcal{U} \mathcal{V}}(f_k, \mathcal{U}(E)) = f_k, \mathcal{V}(E) \).

Since any two open covers \( \mathcal{U}^1 \) and \( \mathcal{U}^2 \) have a common refinement \( \mathcal{V} \),

\[ [f_k, \mathcal{U}^1(E)] = [f_k, \mathcal{U}^2(E)] = [f_k, \mathcal{V}(E)], \]

(48)

where \([f_k, \bullet(E)]\) is the equivalence class of \( f_k, \bullet(E) \) in \( \check{H}^k(X, \Omega^k) \). Then \( f_k(E) \) is independent of the choice of the cover \( \mathcal{U} \). We complete the proof of Theorem 1.2.

\[ \square \]

**Corollary 3.1.** Let \( X \) be a compact complex manifold and \( \pi : E \to X \) a holomorphic vector bundle. Then \( \check{H}^k(X, \Omega^k) \cong H^k(X, \Omega^k) \). Moreover, there is a finite Stein cover \( \mathcal{U} \) of \( X \) such that \( f_k(E) = \tilde{f}_k, \mathcal{U}(E) \) under the natural isomorphisms \( \check{H}^k(\mathcal{U}, \Omega^k) \cong \check{H}^k(X, \Omega^k) \cong H^k(X, \Omega^k) \), where \( \tilde{f}_k, \mathcal{U}(E) \) is the Čech k-cocycle defined by (1) and (2).

**Proof of Corollary 3.1.** Since \( X \) is compact, each open cover of \( X \) has a finite Stein cover as its refinement. In particular, there is a finite Stein cover \( \mathcal{U} \) of \( X \).

Recall the following natural homomorphisms,

\[ \check{H}^k(\mathcal{U}, \Omega^k) \xrightarrow{L_\mathcal{U}} \check{H}^k(X, \Omega^k) \xrightarrow{L} H^k(X, \Omega^k). \]

(49)

Notice that \( \check{H}^k(\mathcal{U}, \Omega^k) \cong H^k(X, \Omega^k) \), for coherent sheaves are acyclic over Stein sets. Then, \( L \) is surjective.

Suppose \( \mathcal{V} \) is an open cover of \( X \) and \( a \in \check{H}^k(\mathcal{V}, \Omega^k) \) such that \( L([a]) = 0 \), where \([a]\) is the equivalence class of \( a \) in \( \check{H}^k(X, \Omega^k) \). Then, there is a finite Stein open cover \( \mathcal{W} \) of \( X \) as a refinement of \( \mathcal{V} \). Since \( (L \circ L_{\mathcal{W}})([L_{\mathcal{V}}(a)]) = L([a]) = 0 \) and \( L \circ L_{\mathcal{W}} \) is an isomorphism, \([a] = [L_{\mathcal{W}}(a)] = 0 \). Therefore, \( H^k(X, \Omega^k) \cong \check{H}^k(X, \Omega^k) \). Then \( f_k(E) = \tilde{f}_k, \mathcal{U}(E) \) under the natural isomorphisms \( \check{H}^k(\mathcal{U}, \Omega^k) \cong \check{H}^k(X, \Omega^k) \cong H^k(X, \Omega^k) \).

\[ \square \]

Before proving Theorem 1.3, we recall the notion of the Chern character of a vector bundle (see [24] for reference). Let \( E \) be a vector bundle over \( X \) and \( \gamma_1, \ldots, \gamma_M \) be the Chern roots of \( E \). Then, \( \text{ch}(E) \) the Chern character of \( E \) is defined as follows.

\[ \text{ch}(E) := \sum_{i=1}^{M} \exp \gamma_i = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (\gamma_1^k + \cdots + \gamma_M^k). \]

(50)

Notice that \( \text{ch}(E) \) can be written as a polynomial whose variables are the Chern classes of \( E \) by Newton’s identities. Moreover, when \( E \) is a holomorphic vector bundle, one can view \( \text{ch}(E) \) as an element in the ring \( \bigoplus_{k=0}^{\infty} H^k(X, \Omega^k) \).

We first prove Theorem 1.3 in the case when \( E \) is a line bundle.
**Lemma 3.2.** Let $X$ be a compact complex manifold. Let $\pi : E \to X$ be a holomorphic line bundle of $X$. $f_k(E)$ is defined as in Theorem 1.2. Then,

$$
\text{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k! \cdot (2\pi \sqrt{-1})^k} \cdot f_k(E).
$$

**Proof of Lemma 3.2.** Let $U := \{U_i\}_{i=1}^l$ be a Stein open cover of $X$ and $g := \{g_{i_1i_2}\}$ be a system of transition functions associated with a certain trivialization of $E$ with respect to $U$.

When $k = 1$, (1) reads

$$
t_{i_1i_2} = \frac{1}{2} tr(g_{i_1i_2}^{-1} d g_{i_1i_2}) - \frac{1}{2} tr(g_{i_2i_1}^{-1} d g_{i_2i_1}).
$$

Since $E$ is a line bundle and hence $g_{i_1i_2}$ is a function, we have that

$$
d g_{i_2i_1} = d g_{i_1i_2}^{-1} = -g_{i_1i_2}^{-1} d g_{i_2i_1} g_{i_1i_2}^{-1}.
$$

Substituting (53) into (52), we derive that

$$
t_{i_1i_2} = \frac{1}{2} g_{i_1i_2}^{-1} d g_{i_1i_2} - \frac{1}{2} g_{i_2i_1}^{-1} (-g_{i_1i_2}^{-1} d g_{i_2i_1} g_{i_1i_2}^{-1}) = d \log(g_{i_1i_2}).
$$

Similarly to the proof of Proposition 1 in 1.1 of [21], we conclude that $f_1(E) = (2\pi \sqrt{-1}) \cdot c_1(E)$ in $H^1(X, \Omega^1)$. Note that since $g_{ij}(z) \neq 0$ and $U_{ij}$ are simply connected, one can define a branch of log, and hence $d \log(g_{i_1i_2})$ is well defined.

Similarly, when $k \geq 2$,

$$
t_{i_1\cdots i_{k+1}} = \frac{1}{k+1} \sum_{\alpha=1}^{k+1} \sum_{1 \leq u_1^\alpha < \cdots < u_{k+1}^\alpha \leq k+1, \ u_1^\alpha, \ldots, u_{k+1}^\alpha, \alpha \text{ are distinct}} \text{sgn} \left( \begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ u_1^\alpha & u_2^\alpha & \cdots & u_k^\alpha & \alpha \end{pmatrix} \right) \cdot \left( d \log g_{i_1^\alpha i_1} \wedge d \log g_{i_2^\alpha i_2} \wedge \cdots \wedge d \log g_{i_{k+1}^\alpha i_{k+1}} \right).
$$

Since $\log g_{ik} = \log g_{ij} + \log g_{jk}$, we have that

$$
t_{i_1\cdots i_{k+1}} = \frac{1}{k+1} \sum_{\alpha=1}^{k+1} \sum_{1 \leq u_1^\alpha < \cdots < u_{k+1}^\alpha \leq k+1, \ u_1^\alpha, \ldots, u_{k+1}^\alpha, \alpha \text{ are distinct}} \text{sgn} \left( \begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ u_1^\alpha & u_2^\alpha & \cdots & u_k^\alpha & \alpha \end{pmatrix} \right) \cdot \left( d \log g_{i_1^\alpha i_{(k+1)}} + d \log g_{i_{(k+1)}i_1} \wedge \cdots \wedge (d \log g_{i_{(k+1)}i_{(k+1)}} + d \log g_{i_{(k+1)}i_{(k+1)}}) \right)$$

$$
= d \log g_{i_1i_{(k+1)}} \wedge d \log g_{i_2i_{(k+1)}} \wedge \cdots \wedge d \log g_{i_{(k+1)}i_{(k+1)}}$$

$$
= d \log g_{i_1i_2} \wedge d \log g_{i_2i_3} \wedge \cdots \wedge d \log g_{i_{k}i_{(k+1)}}.
$$

By formula (14.24) on Page 174 of [11], $f_k(E)$ is the $k$-fold cup product of $f_1(E)$ with itself, that is,

$$
f_k(E) = f_1(E) \cup f_1(E) \cup \cdots \cup f_1(E) \in \tilde{H}^k(U, \Omega^k).
$$
Then,
\[
\text{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot c_k^1(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left( \frac{f_1(E)}{2\pi \sqrt{-1}} \right)^k
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (2\pi \sqrt{-1})^k \cdot f_k(E).
\]

We complete the proof of Lemma 3.2. \(\square\)

Next we prove Theorem 1.3 for the vector bundle with a full flag structure.

**Lemma 3.3.** Let \(X\) be a compact complex manifold. Let \(\pi : E \to X\) be a holomorphic vector bundle of \(X\) with a filtration by holomorphic subbundles
\[
E = E_M \supset E_{M-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 = 0
\]
with line bundle quotients \(L_i = E_i/E_{i-1}\). \(f_k(E)\) is defined as in Theorem 1.2. Then,
\[
\text{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (2\pi \sqrt{-1})^k \cdot f_k(E).
\]

**Proof of Lemma 3.3.** Since \(E\) has a full flag structure, there is a Stein open cover \(U : = \{U_i\}_{i=1}^l\) of \(X\) and a system of transition functions \(g : = \{g_{ij}\}\) associated with a certain trivialization of \(E\) with respect to \(U\) such that \(g_{ij}\) is an upper triangular matrix as follows.
\[
g_{ij} = \begin{pmatrix}
g_{ij}^1 & * & \cdots & * & * \\
0 & g_{ij}^2 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & g_{ij}^{M-1} & * \\
0 & 0 & \cdots & 0 & g_{ij}^M
\end{pmatrix}
\text{ for } 1 \leq i, j \leq l.
\]

Similarly to (55) and (56), computation yields that
\[
t_{i_1 \cdots i_{k+1}} = \sum_{j=1}^{M} \sum_{\alpha=1}^{k+1} \sum_{1 \leq u_1^\alpha < \cdots < u_k^\alpha \leq k+1} \left| u_{1}^{\alpha} \right| \left| u_{2}^{\alpha} \right| \cdots \left| u_{k}^{\alpha} \right| \cdot \left( \frac{1}{u_1^\alpha} \frac{2}{u_2^\alpha} \cdots \frac{k+1}{u_k^\alpha} \right) \cdot (d \log g_{i_{1}\alpha}^j \wedge d \log g_{i_{2}\alpha}^j \wedge d \log g_{i_{3}\alpha}^j \wedge \cdots \wedge d \log g_{i_{k}\alpha}^j)
\]
\[
= \sum_{j=1}^{M} d \log g_{i_{1}i_{(k+1)}}^j \wedge d \log g_{i_{2}i_{(k+1)}}^j \wedge \cdots \wedge d \log g_{i_{k}i_{(k+1)}}^j
\]
\[
= \sum_{j=1}^{M} d \log g_{i_{1}i_{2}}^j \wedge d \log g_{i_{1}i_{3}}^j \wedge \cdots \wedge d \log g_{i_{k}i_{(k+1)}}^j
\]
\[
=: t_{i_1 \cdots i_{k+1}}^1 + t_{i_1 \cdots i_{k+1}}^2 + \cdots + t_{i_1 \cdots i_{k+1}}^M.
\]
Notice that by Lemma 3.2, \( f_k(L_j), j = 1, \ldots, M, \) is given by
\[
f_k(L_j) = \bigoplus_{1 \leq i_1 < \cdots < i_{k+1} \leq l} t^j_{i_1 \cdots i_{k+1}} \in \bigoplus_{1 \leq i_1 < \cdots < i_{k+1} \leq l} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k);
\]

hence,
\[
f_k(E) = \sum_{j=1}^{M} f_k(L_j) = \sum_{j=1}^{M} (f_1(L_j))^k.
\]

Therefore,
\[
\text{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=1}^{M} c^j_k(L_j) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=1}^{M} \left( \frac{f_1(L_j)}{2\pi \sqrt{-1}} \right)^k
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{k!} (2\pi \sqrt{-1})^k \sum_{j=1}^{M} f^j_k(L_j) = \sum_{k=0}^{\infty} \frac{1}{k!} (2\pi \sqrt{-1})^k \cdot f_k(E).
\]

We complete the proof of Lemma 3.3. \( \square \)

**Proof of Theorem 1.3.** The idea is to use the splitting principle. As in [17], we construct a split manifold for \((X, E)\) as follows. Let \( P(E) \) be the projective bundle of \( E \); denote the projection map by \( \sigma : P(E) \to X \) and denote the pullback of \( E \) to \( P(E) \) by \( \sigma^{-1}(E) \). Repeating this procedure, we can find a compact complex manifold \( X' \) and a holomorphic map \( \pi : X' \to X \) such that the pullback of \( E \) to \( X' \) has a filtration of holomorphic subbundles:
\[
\pi^{-1}(E) = E_M \supset E_{M-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 = 0
\]

with line bundle quotients \( L_i = E_i / E_{i-1} \).

Let \( \mathcal{U} := \{ U_i \}_{i=1}^{l} \) be a Stein cover of \( X \); let \( \mathcal{V} \) be a Stein cover of \( X' \) such that \( \mathcal{V} \) is a refinement of the cover \( \mathcal{U}^* := \pi^* \mathcal{U} = \{ \pi^{-1}(U_i) \}_{i=1}^{l} \) of \( X' \). Then, we have the following commutative diagram for each positive integer \( k \):
\[
\begin{array}{ccc}
\tilde{H}^k(\mathcal{U}, \Omega^k_X) & \xrightarrow{T_k} & \tilde{H}^k(\mathcal{V}, \Omega^k_X) \\
\downarrow{L_{\mathcal{U}}} & & \downarrow{L_{\mathcal{V}}} \\
\tilde{H}^k(X, \Omega^k_X) & \xrightarrow{\pi^*_k} & \tilde{H}^k(X', \Omega^k_{X'}) \\
\downarrow{L_X} & & \downarrow{L_{X'}} \\
H^k(X, \Omega^k_X) & \xrightarrow{\tilde{\pi}^*_k} & H^k(X', \Omega^k_{X'}).
\end{array}
\]

Here \( T_k = L_{\mathcal{U}^* \mathcal{V}} \circ \tilde{\pi}_k^* \), where \( \tilde{\pi}_k^* \) is the natural pullback homomorphism from \( \tilde{H}^k(\mathcal{U}, \Omega^k_X) \) to \( \tilde{H}^k(\mathcal{U}^*, \Omega^k_{X'}) \); \( \pi^*_k \) is the induced homomorphism between the direct limits; \( \pi^*_k \) is the natural pullback map from \( H^k(X, \Omega^k_X) \) to \( H^k(X', \Omega^k_{X'}) \). Moreover, \( L_{\mathcal{U}}, L_{\mathcal{V}}, L_X \) and \( L_{X'} \) are all isomorphisms.
Let $\hat{f}_k, U(E)$ be the Čech cocycle representing $f_k(E) \in H^k(X, \Omega^k)$ and let $\hat{f}_k, \nu(\pi^{-1}(E))$ be the Čech cocycle representing $f_k(\pi^{-1}(E)) \in H^k(X', \Omega^k_{X'})$. Recall that the pullback of a system of transition functions of $E$ with respect to $U$ gives a system of transition functions of $\pi^{-1}(E)$ with respect to $U^*$. In the same way as the proof of Theorem 1.2 (see (46) and (47)), we can prove the following identity,

$$(\Pi_k^* \circ L_{U})(\hat{f}_k, U(E)) = (L_{\nu})(\hat{f}_k, \nu(\pi^{-1}(E))),$$

and hence

$$(\pi_k^*)(f_k(E)) = f_k(\pi^{-1}(E)).$$

On the other hand, by representing the Chern classes by the integrations of invariant polynomials of the curvature forms of a Hermitian metric, it is clear that

$$(\pi_k^*)(ch(E)) = ch(\pi^{-1}(E)) \in \bigoplus_{i=0}^{\infty} H^k(X', \Omega^k_{X'}).$$

By the Hirsch lemma (see Lemma 18 of [13] or Lemma 3.3 of [29]), the following natural homomorphism is an embedding

$$\pi^*: \bigoplus_{i=0}^{\infty} H^k(X, \Omega^k_X) \hookrightarrow \bigoplus_{i=0}^{\infty} H^k(X', \Omega^k_{X'});$$

in particular, $\pi_k^*$ is injective. Therefore, in order to prove Theorem 1.3, it suffices to prove that

$$ch(\pi^{-1}(E)) = \sum_{k=0}^{\infty} \frac{1}{k! \cdot (2\pi \sqrt{-1})^k} \cdot f_k(\pi^{-1}(E)).$$

By Lemma 3.3, we draw the conclusion. \qed

### 3.2. Refinement of the first Chern class

In the following, we will show that the first $T$ invariant of a holomorphic vector bundle can be lifted to the group $\hat{H}^1(X, d\Omega^0)$. Recall that by Theorem 1.3, the first $T$ invariant coincides with the first Chern class, up to a certain normalized factor, in the Dolbeault cohomology group $H^1(X, \Omega^1)$. Hence, the lifted first $T$ invariant can be viewed as a refinement of the first Chern class in $H^1(X, \Omega^1)$.

**Proof of Theorem 1.4.** The proof is similar to that of Theorem 1.2. We only need to show that the coefficients of the Čech cochains take values in the sheaf $d\Omega^0$.

Notice that

$$d\left(tr(g_{i_1i_2}^{-1} dg_{i_1i_2}) \right) = tr\left(- g_{i_1i_2}^{-1} dg_{i_1i_2} g_{i_1i_2}^{-1} dg_{i_1i_2} \right) = 0,$$

since $tr(AA) = 0$ for any differential 1-form-valued $M \times M$ matrix $A$. Therefore, $t_{i_1i_2} \in \Gamma(U_{i_1i_2}, d\Omega^0)$. 

Suppose that \( \{ \tilde{g}_{ij} \} \) is another system of transition functions associated with a certain trivialization of \( E \) with respect to \( \mathcal{U} \). Then, there exists a Čech 0-cochain

\[
h := \bigoplus_{i_1} h_{i_1} \in \bigoplus_{i_1} \Gamma_{hol}(U_{i_1}, GL(M, \mathbb{C})),
\]

(74)
such that \( \tilde{g}_{ij} = h_{i_1}^{-1} g_{ij} h_{j} \) for \( i, j \in I \). As in the proof of Lemma 2.5, we can define a Čech 0-cochain

\[
h_0(E, g, \tilde{g}) := \bigoplus_{i_1} s_{i_1} \in \bigoplus_{i_1} \Gamma(U_{i_1}, \Omega^1),
\]

(75)
such that, for any elements \( i_1, i_2 \in I \),

\[
\tilde{t}_{i_1 i_2} - t_{i_1 i_2} = (s_{i_1} - s_{i_2})\big|_{U_{i_1 i_2}}.
\]

(76)

Recalling (168), computation yields that

\[
s_{i_1} = -\text{tr} \left( h_{i_1}^{-1} dh_{i_1} \right).
\]

(77)

In order to show that \( \hat{f}_1(E) \) is a well-defined element of \( \tilde{H}^1(\mathcal{U}, d\Omega^0) \), which is independent of the choice of the system of the transition functions, it suffices to to prove that, in addition to (75) and (76),

\[
s_{i_1} \in \Gamma(U_{i_1}, d\Omega^0) \text{ for } i_1 \in I.
\]

(78)

By (77), (78) holds in the same way as (73).

The remaining of the proof is similar to the proof of Theorem 1.2 which we omit here to avoid repetition. We complete the proof of Theorem 1.4.

\[\square\]

**Corollary 3.4.** Let \( X \) be a compact complex manifold. Let \( \pi : E \to X \) be a holomorphic vector bundle on \( X \). Then \( \tilde{H}^1(X, d\Omega^0) \cong H^1(X, d\Omega^0) \). Moreover, there is a finite good cover \( \mathcal{U} \) of \( X \) such that \( \tilde{f}_1^r(E) = \hat{f}_1^r(\mathcal{U}) \) under the natural isomorphisms \( \tilde{H}^1(\mathcal{U}, d\Omega^0) \cong H^1(X, d\Omega^0) \cong \tilde{H}^1(X, d\Omega^0) \cong H^1(X, d\Omega^0) \), where \( \hat{f}_1^r(\mathcal{U}) \) is the Čech 1-cocycle defined by (4) and (5).

**Proof of Corollary 3.4.** Since \( X \) is a compact complex manifold, each open cover of \( X \) has a finite good cover as its refinement (see Appendix I in [15] for instance). In particular, there is a finite good cover \( \mathcal{U} \) of \( X \). Recall the following natural homomorphisms,

\[
\tilde{H}^1(\mathcal{U}, d\Omega^0) \xrightarrow{L_{\mathcal{U}}} \tilde{H}^1(X, d\Omega^0) \xrightarrow{L} H^1(X, d\Omega^0),
\]

(79)

Then \( \tilde{H}^1(\mathcal{U}, d\Omega^0) \cong H^1(X, d\Omega^0) \) (see 3.3 of [15]). The remaining of the proof is similar to that of Corollary 3.1, which we omit it to avoid repetition. \[\square\]
Proof of Theorem 1.5. If $E$ is $\mathcal{Q}$-flat, then for a certain positive integer $m$ the line bundle $mE$ has a system of transition functions which are all constant. $(4)$ yields that $f_1^*(mE) = 0$. Since $H^1(X, d\Omega^0)$ is a vector space over $\mathbb{C}$, $f_1^*(mE) = \frac{1}{m} \cdot f_1^*(mE) = 0$.

Next assume $f_1^*(E) = 0$ in $H^1(X, d\Omega^0)$. Recall the following short exact sequence of $\mathbb{C}$-sheaves (see [22] or [15]),

$$0 \to \mathbb{C} \to \Omega^0 \to d\Omega^0 \to 0. \quad (80)$$

Combined with the exponential sheaf sequence $0 \to \mathbb{Z} \xrightarrow{2\pi \sqrt{-1}} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$ (see [21]), we have the following commutative diagram of sheaves

$$
\begin{array}{ccc}
0 & \to & \mathbb{Z} \\
\downarrow j & & \downarrow id \\
0 & \to & \mathbb{C} \to \Omega^0 \to d\Omega^0 \\
\end{array}
$$

Here $j(n) = 2\pi \sqrt{-1} \cdot n, n \in \mathbb{Z}$; by convention, $\mathcal{O}_X = \Omega^0$ and $id$ is the identity map; $G(g) = d\log(g)$ for each germ $g$ of sheaf $\mathcal{O}_X^*$.

Taking the corresponding long exact sequences of cohomology groups, we derive that

$$
\begin{align*}
&\to H^0(X, \Omega^0) \to H^0(X, d\Omega^0) \to H^1(X, \mathbb{C}) \\
&\to H^1(X, \Omega^0) \to H^1(X, d\Omega^0) \to H^2(X, \mathbb{C}), \\
&\to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \\
&\to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).
\end{align*}
$$

Notice that the group structure of the sheaf $\mathcal{O}_X^*$ is given by multiplication instead of addition, and $H^*(X, \mathcal{O}_X^*)$ should be understood as Čech cohomology groups (see [21]). On the other hand, since the other groups in $(83)$ are isomorphic to their corresponding Čech cohomology groups, we will view all groups in $(83)$ as Čech cohomology groups in the following.

Let $\mathcal{U} := \{U_i\}_{i=1}^l$ be a good cover of $X$. We have that (see 3.3 of [15] for instance),

$$
\begin{align*}
H^p(X, \Omega^0) &\cong \tilde{H}^p(\mathcal{U}, \Omega^0), & H^p(X, d\Omega^0) &\cong \tilde{H}^p(\mathcal{U}, d\Omega^0), & H^p(X, \mathbb{C}) &\cong \tilde{H}^p(\mathcal{U}, \mathbb{C}), \\
H^p(X, \mathcal{O}_X^0) &\cong \tilde{H}^p(\mathcal{U}, \mathcal{O}_X^0), & H^p(X, \mathbb{Z}) &\cong \tilde{H}^p(\mathcal{U}, \mathbb{Z}) & \text{for } p \geq 0.
\end{align*}
$$

Moreover, we can derive the following commutative diagram from $(81)$,

$$
\begin{array}{ccc}
\tilde{H}^1(\mathcal{U}, \mathbb{Z}) & \to & \tilde{H}^1(\mathcal{U}, \mathcal{O}_X) \\
\downarrow j_* & & \downarrow id \\
\tilde{H}^1(\mathcal{U}, \mathcal{O}_X) & \xrightarrow{h} & \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^0) \\
\downarrow \tilde{f}_{1, id} & & \downarrow \tilde{j}_* \\
\tilde{H}^2(\mathcal{U}, \mathbb{Z}) & \xrightarrow{c_1} & \tilde{H}^2(\mathcal{U}, \mathbb{Z}) \\
\end{array}
$$

Moreover, we can derive the following commutative diagram from $(81)$,
Here $c_1$ is the first Chern class map with the image in $\tilde{H}^2(\mathcal{U}, \mathbb{Z})$ and $Id$ is the identity map; since each element of $\tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ can be one to one identified with a holomorphic line bundle over $X$ up to biholomorphisms, the map $\tilde{f}_{1,\mathcal{U}}^*$ is induced by the refined first $T$ invariant map.

By a slight abuse of notation, we still denote by $E$ the element in $\tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ corresponding to the holomorphic line bundle $E$. Since $\tilde{H}^1(\mathcal{U}, d\Omega^0) \cong \tilde{H}^1(\mathcal{U}, d\Omega^0) \cong H^1(X, d\Omega^0)$, $\tilde{f}_{1,\mathcal{U}}^*(E) = 0$ in $\tilde{H}^1(\mathcal{U}, d\Omega^0)$; hence, $c_1^1(f_{1,\mathcal{U}}^*(E)) = 0$ in $\tilde{H}^2(\mathcal{U}, \mathbb{C})$. Then, $c_1^1(E)$ is a torsion element in $\tilde{H}^2(\mathcal{U}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. Take a positive integer $m$ such that $c_1^1(mE) = 0$. Since the horizontal lines of (85) are exact, there is an element $e \in \tilde{H}^1(\mathcal{U}, \mathcal{O}_X)$ such that $h(e) = mE$. Since $D(Id(e)) = \tilde{f}_{1,\mathcal{U}}^*(mE) = 0$, there is an element $\xi \in \tilde{H}^1(\mathcal{X}, \mathbb{C})$ such that $B(\xi) = Id(e)$. Note that $\xi$ is a Čech 1-cocycle with coefficients in $\mathbb{C}$, and hence it can be represented as follows

$$
\xi = \bigoplus_{1 \leq j_1 < j_2 \leq l} \xi_{j_1,j_2} \in \bigoplus_{1 \leq j_1 < j_2 \leq l} \Gamma(U_{j_1,j_2}, \mathbb{C}). \tag{86}
$$

By chasing diagram (85) in the reverse order, we have a system of transition functions associated with a certain trivialization of $mE$ with respect to $\mathcal{U}$ as follows:

$$
g_{j_1,j_2} = \exp(\xi_{j_1,j_2}) \in \Gamma(U_{j_1,j_2}, \mathbb{C}) \text{ for } 1 \leq j_1 < j_2 \leq l. \tag{87}
$$

We complete the proof of Lemma 1.5. \hfill \Box

To indicate the geometric meaning of the refined first $T$ invariants, we introduce the following notation whose consistency is ensured by Theorem 1.5.

**Definition 3.5.** Let $X$ be a compact complex manifold and $E$ be a holomorphic line bundle over $X$. Let $f_{1}^\mathcal{U} \in H^1(X, d\Omega^0)$ be the refined first $T$ invariant defined as Theorem 1.4. Then we call $\frac{1}{2\pi\sqrt{-1}} f_{1}^\mathcal{U}$ the $\mathbb{Q}$-flat class map of $E$. Moreover, the map associating each line bundle with its $\mathbb{Q}$-flat class is called the $\mathbb{Q}$-flat class map and is denoted by $F$.

**Remark 3.6.** It is clear that $F$ is a homomorphism from $H^1(X, \mathcal{O}_X^*)$ to $H^1(X, d\Omega^0)$ such that $F(\mathcal{O}_X) = 0$ and $F(E_1 \otimes E_2) = F(E_1) + F(E_2)$ for line bundles $E_1$ and $E_2$.

Denote by $i_2$ the natural homomorphism from $H^1(X, \Omega^1)$ to $H^2(X, \mathbb{C})$. We then have

**Proposition 3.7.** Let $X$ be a compact complex manifold. The first Chern class maps factor through the $\mathbb{Q}$-flat class map $F$ as follows:

$$
H^1(X, \mathcal{O}_X^*) \xrightarrow{F} H^1(X, d\Omega^0) \xrightarrow{j_1} H^1(X, \Omega^1) \xrightarrow{i_2} H^2(X, \mathbb{C}). \tag{88}
$$

That is, for each holomorphic line bundle $E$ of $X$, $(j_1 \circ F)(E)$ is the first Chern class of $E$ in the Dolbeault cohomology group $H^1(X, \Omega^1)$; $(i_2 \circ j_1 \circ F)(E)$ is the first Chern class of $E$ in the De Rham cohomology group $H^2(X, \mathbb{C})$.

**Proof of Proposition 3.7.** Let $\mathcal{U} := \{U_i\}_{i=1}^\mathcal{U}$ be a good cover of $X$ and view all the above groups as Čech cohomology groups $H^\bullet(\mathcal{U}, \cdot)$. Then, Proposition 3.7 is an easy consequence of Theorem 1.3 and Proposition 1 in 1.1 of [21]. \hfill \Box
3.3. Strict refinement criterion

It is well known that when $X$ is a non-Kähler manifold the homomorphism $i_2$ in (88) may not be injective, so that the first Chern class in the Dolbeault cohomology may be finer than the first Chern class in the De Rham cohomology (see Example 3.10).

In the following, we will give an criterion for when the refined $T$ invariant (or equivalently, the $\mathcal{Q}$-flat class) is strictly finer than the first $T$ invariant (or equivalently, the first Chern class in the Dolbeault cohomology).

First, let us recall some notation of Frölicher spectral sequence following [16]. For $p, q \geq 0$, denote by $pT_r^n$ the vector space consisting of differential forms locally given by

$$\sum_{r \geq p; r+s=n} \phi_{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s} dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_r} \wedge d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_s}, \quad (89)$$

where $z'_{\nu}$s are local holomorphic coordinates. For $r \geq 0$, define $pT_r^n$, $pB_r^n$, and $E_{r}^{p,q}$ by

$$pT_r^n := \{ t | t \in pT^n; dt \in pT^{n+1} \}, \quad (90)$$

$$pB_r^n := \{ t | t \in pT^n; t = ds; s \in pT^{n-1} \}, \quad (91)$$

$$E_{r}^{p,q} := \frac{pT_r^{p+q}}{pT_r^{p+q} + pB_r^{p+q}}. \quad (92)$$

Notice that the exterior differentiation induces the following cochain complex

$$\cdots \rightarrow E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \rightarrow \cdots. \quad (93)$$

Moreover, if we denote by $K_r^{p,q} \subset E_r^{p,q}$ the $d_r$-kernel, then

$$E_{r+1}^{p,q} = \frac{K_{r}^{p,q}}{d_r(E_r^{p+r,q-r+1})}. \quad (94)$$

We also collect some complexes for small $p, q, r$ as follows:

$$E_1^{0,0} \rightarrow E_1^{1,0} \rightarrow E_1^{2,0} \rightarrow E_1^{3,0},$$

$$0 \rightarrow E_2^{1,0} \rightarrow E_2^{2,0} \rightarrow E_2^{3,0} \rightarrow 0 \rightarrow \cdots; \quad (95)$$

$$0 \rightarrow E_3^{1,0} \rightarrow E_3^{2,0} \rightarrow E_3^{3,0} \rightarrow 0 \rightarrow \cdots;$$

$$0 \rightarrow E_4^{1,0} \rightarrow E_4^{2,0} \rightarrow E_4^{3,0} \rightarrow 0 \rightarrow \cdots.$$
Theorem 3.8. Let $X$ be a compact complex manifold. The first Chern classes factor through the $\mathcal{Q}$-flat class map $F$ as follows:

$$H^1(X, \mathcal{O}^*) \xrightarrow{F} H^1(X, d\Omega^0) \xrightarrow{j_1} H^1(X, \Omega^1) \xrightarrow{i_2} H^2(X, \mathbb{C}).$$  \hspace{1cm} (96)

Denote by $j_1|_{\text{Im}(F)}$ the restriction of $j_1$ to the image of $F$. Then $j_1|_{\text{Im}(F)}$ is injective if and only if

$$E^0_{2,1} = E^0_{3,1}. \hspace{1cm} (97)$$

Proof of Theorem 3.8. We first prove that $j_1|_{\text{Im}(F)}$ is injective if and only if $j_1$ is. If $j_1$ is injective, then it is clear that $j_1|_{\text{Im}(F)}$ is injective. Now suppose $j_1|_{\text{Im}(F)}$ is injective. Recalling diagram (85), we have the following commutative diagram:

$$\begin{array}{cccc}
H^1(X, \mathcal{O}_X) & \xrightarrow{h} & H^1(X, \mathcal{O}_X^*) \\
\downarrow{Id} & & \downarrow{F} \\
H^1(X, \Omega^0) & \xrightarrow{D} & H^1(X, d\Omega^0) & \xrightarrow{i_2} H^2(X, \mathbb{C}) \\
\downarrow{j_1} & & \downarrow{i_2} \\
H^1(X, \Omega^1) & & & .
\end{array} \hspace{1cm} (98)$$

Assume that there is an element $\xi \in H^1(X, d\Omega^0)$ such that $j_1(\xi) = 0 \in H^1(X, \Omega^1)$. Then, $(i_2 \circ j_1)(\xi) = 0 \in H^2(X, \mathbb{C})$ and hence there is an element $\tau \in H^1(X, \Omega^0)$ such that $D(\tau) = \xi$. Thus, $(h \circ (Id)^{-1})(\tau)$ gives a line bundle $E \in H^1(X, \mathcal{O}_X^*)$. Since the diagram is commutative, we have that $F(E) = \xi$ and $(j_1 \circ F)(E) = j_1(\xi) = 0$. Because $j_1|_{\text{Im}(F)}$ is injective, we conclude that $\xi = F(E) = 0$. Therefore, $j_1$ is injective.

Next we will apply the Frölicher spectral sequence to prove that $j_1$ is injective if and only if (97) holds. Combining diagrams (85) and (98), we have

$$0 \longrightarrow H^0(X, d\Omega^0) \longrightarrow H^1(X, \mathcal{O}) \longrightarrow H^1(X, \Omega^0) \xrightarrow{D} H^1(X, d\Omega^0) \xrightarrow{i_2} H^2(X, \mathbb{C}) \xrightarrow{j_1} H^1(X, \Omega^1). \hspace{1cm} (99)$$

It is easy to see that

$$\text{ker } j_1 \subset \text{Im } D, \hspace{1cm} (100)$$

and hence $\text{ker } j_1 = 0$ if and only if

$$\text{dim } \text{Im } (j_1 \circ D) = \text{dim } \text{Im } D. \hspace{1cm} (101)$$

By (94), we have

$$E^0_{2,1} = \text{ker } (E^0_{1,1} \xrightarrow{d_1} E^1_{1,1}), \quad E^0_{3,1} = E^0_{4,1} = \cdots = \text{ker } (E^0_{2,1} \xrightarrow{d_2} E^2_{2,0}) = \cdots = \text{ker } (E^0_{1,0} \xrightarrow{d_1} E^1_{1,0}), \hspace{1cm} (102)$$

$$E^1_{2,0} = E^1_{3,0} = \cdots = \frac{\text{ker } (E^1_{1,0} \xrightarrow{d_1} E^2_{1,0})}{\text{Im } (E^0_{0,0} \xrightarrow{d_1} E^1_{1,0})}. \hspace{1cm} (103)$$
and hence (see Theorem 3 in [16])

$$\dim H^1(X, \mathbb{C}) = \dim E^{0,1}_3 + \dim E^{1,0}_3.$$  \hfill (104)

Since (see Lemma 1 in [16])

$$E^{p,q}_1 \cong H^q(X, \Omega^p) \cong H^{p,q}(X),$$  \hfill (105)

it is easy to see that $H^0(X, d\Omega^0) = E^{1,0}_2 = E^{1,0}_3$.

We will introduce the double complex interpretation of $E^{p,q}_r$. Recall the following double complex in 14 of [11]:

$$
\begin{array}{c}
0 \\
\delta \uparrow \\
Z^{0,1} \xrightarrow{\delta} Z^{1,1} \\
\delta \uparrow \\
Z^{1,0} \xrightarrow{\delta} Z^{2,0} \\
\delta \uparrow \\
0
\end{array}
$$

where $Z^{p,q}$ are differential forms on $X$ of bidegree $(p, q)$ ($p$-th wedge of $dz^\nu$ and $q$-th wedge of $d\bar{z}^\mu$). Notice that an element in $E^{0,1}_r$ can be represented by an element $b \in Z^{0,1}$ such that $\bar{\partial}b = 0$; an element of $E^{0,1}_2$ can be represented by an element $b \in Z^{0,1}$ such that $\bar{\partial}b = 0$ and there is an element $c \in Z^{1,0}$ with $\bar{\partial}b = \partial c$; an element of $E^{0,1}_3$ can be represented by an element $b \in Z^{0,1}$ such that $\bar{\partial}b = 0$, and there is an element $c \in Z^{1,0}$ with $\bar{\partial}b = \delta c$ and $\partial c = 0$. Therefore, $H^1(X, \Omega^0)$ has a filtration

$$H^1(X, \Omega^0) = E^{0,1}_1 \supset E^{0,1}_2 \supset E^{0,1}_3;$$  \hfill (107)

by (104), $\text{Im}(D) \cong E^{0,1}_1 / E^{0,1}_2$.

Now in order to prove Theorem 3.8, it remains to show that $\text{Im}(j_1 \circ D) \cong E^{0,1}_1 / E^{0,1}_2$. Notice that the map $j_1 \circ D$ is induced by the following differential map between sheaves

$$d : \Omega^0 \rightarrow \Omega^1,$$  \hfill (108)

and $H^q(X, \Omega^p) \cong H^{p,q}(X)$. We derive that

$$(j_1 \circ D)(b) = \bar{\partial}(b),$$  \hfill (109)

for each $b \in Z^{0,1}$ such that $\bar{\partial}b = 0$. Noticing that $\bar{\partial}b = 0 \in H^1(X, \Omega^1)$ if and only if there is an element $c \in Z^{1,0}$ such that $\bar{\partial}c = \bar{\partial}b = 0$, we conclude that $\text{Im}(j_1 \circ D) \cong E^{0,1}_1 / E^{0,1}_2$.

Therefore, we complete the proof of Theorem 3.8. \qed

**Proof of Theorem 1.6.** Theorem 1.6 follows from Theorem 3.8. \qed
**Remark 3.9.** Recall that a complex manifold $X$ is said to have Property (H) if $X$ is compact and the following equality holds (see [15]).

$$
\dim H^1(X, \mathbb{C}) = \dim H^0(X, d\Omega^0) + \dim H^1(X, \mathcal{O}_X). \tag{110}
$$

If $X$ has Property (H), then the restriction of $j_1$ to the image of $F$ and the restriction of $i_2 \circ j_1$ to the image of $F$ are both injective. In particular, this is always true for Kähler manifolds and manifolds of Fujiki class $\mathcal{C}$.

**Example 3.10.** (See [9]) Let $X$ be a Calabi-Eckmann manifold which is diffeomorphic to the product manifold $S^{2u+1} \times S^{2v+1}$ $(u,v \geq 1)$. Then, $\dim H^1(X, d\Omega^0) = \dim H^1(X, \Omega^1) = 1$ and $\dim H^2(X, \mathbb{C}) = 0$. Therefore, the first Chern class in the Dolbeault cohomology is strictly finer than the first Chern class in the De Rham cohomology for a certain line bundle. However, the refined first $T$-invariant in $H^1(X, d\Omega^0)$ is isomorphic to its image in $H^1(X, \Omega^1)$ for every holomorphic vector bundle of $X$.

**Example 3.11.** (See [28]) Consider the real, nilpotent subgroup of $GL(6, \mathbb{C})$

$$
G_2 := \left\{ \begin{pmatrix}
1 & 0 & 0 & -\bar{y}_1 & w_1 \\
1 & -x_1 & 0 & w_2 \\
1 & 0 & 0 & y_1 \\
1 & 0 & y_2 \\
1 & 0 & \cdots & 1 \\
0 & \cdots & 1
\end{pmatrix} : x_1, y_1, y_2, z_1, w_1, w_2 \in \mathbb{C} \right\}. \tag{111}
$$

Regarding $x_1, y_1, y_2, z_1, w_1, w_2$ as complex coordinates we can identify $G_2$ with $\mathbb{C}^6$ and then multiplication on the left with a fixed element is holomorphic. Taking the quotient with respect to the discrete subgroup $\Gamma := G_2 \cap GL(6, \mathbb{Z}[i])$ acting on the left yields a (compact) nilmanifold with left-invariant complex structure $X_2 := \Gamma \backslash G_2$.

If we call the matrix in (111) $A$, then the space of left-invariant differential $(1, 0)$-forms is spanned by the components of $A^{-1}dA$ which yields the following:

$$
U := \{ dx_1, dy_1, dy_2, dz_1, \omega_1, \omega_2 \} \tag{112}
$$

where

$$
\omega_1 = dw_1 - \bar{y}_1 dz_1, \\
\omega_2 = dw_2 - \bar{z}_1 dy_1 + x_1 dy_2. \tag{113}
$$

The differential form $\bar{\omega}_1$ defines a class $[\bar{\omega}_1]_2$ in $E_2^{0,1}$ and

$$
d_2([\bar{\omega}_1]_2) = [dx_1 \wedge dy_2]_2 \neq 0 \text{ in } E_2^{2,0}. \tag{114}
$$

Therefore, $E_2^{0,1} \neq E_3^{0,1}$. By Theorem 3.8, we have a line bundle whose $Q$-flat class is not zero but its first Chern class in the Dolbeault cohomology is zero.
Remark 3.12. ([5]) Let $X$ be a compact complex manifold and $E$ be a vector bundle over $X$. Then, $E$ has a holomorphic connection if and only if the Atiyah class $a(E)$ of $E$ is trivial, where

$$a(E) \in H^1(X, \Omega^1 \otimes \text{End}(E)).$$

Notice that when $E$ is a line bundle $a(E)$ is the first Chern class in the Dolbeault cohomology. Therefore, when $E_2^{0,1} = E_3^{0,1}$ in the Frölicher spectral sequence of $X$ a line bundle has a holomorphic connection if and only if it is flat up to a positive multiple.

3.4. $T$ invariants for locally free sheaves of schemes over general fields

In this subsection, we will discuss the analogues of the $T$ invariants for the locally free sheaves of schemes over a general field. Let $X$ be a scheme over a field $K$. Denote by $\Omega_{X/\text{Spec} K}$ the sheaf of Kähler differentials of $X$ over $\text{Spec} K$. Denote by $\Omega^k$ the $k$-th exterior power sheaf of $\Omega_{X/\text{Spec} K}$ for $k \geq 1$. Let $E$ be a locally free sheaf over $X$. Let $U := \{U_i\}_{i \in I}$ be an open cover of $X$ (in the Zariski topology) and $g := \{g_{ij}\}$ be a system of transition functions associated with a certain trivialization of $E$ with respect to $U$ (similarly to (7), (8) and (9) in 2).

Theorem 3.13. Suppose that $k \geq 1$ and $(k+2)!$ is not divisible by the characteristic of $K$. For any indices $i_1, i_2, \ldots, i_{k+1} \in I$, define $t_{i_1 \cdots i_{k+1}} \in \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k)$ by

$$t_{i_1 \cdots i_{k+1}} := \sum_{\sigma \in S_{k+1}} \text{sgn}(\sigma) \cdot tr(g_{i_{\sigma(1)}i_{\sigma(k+1)}}^{-1} dg_{i_{\sigma(1)}i_{\sigma(k+1)}}^{-1} dg_{i_{\sigma(2)}i_{\sigma(k+1)}}^{-1} \cdots g_{i_{\sigma(k)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(k)}i_{\sigma(k+1)}}^{-1} d g_{i_{\sigma(k)}i_{\sigma(k+1)}}^{-1}) \cdot$$

$$\cdot g_{i_{\sigma(1)}i_{\sigma(k+1)}}^{-1} dg_{i_{\sigma(2)}i_{\sigma(k+1)}}^{-1} \cdots g_{i_{\sigma(k)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(k)}i_{\sigma(k+1)}}^{-1});$$

define a Čech $k$-cochain $\hat{f}_{k, U}(E)$ by

$$\hat{f}_{k, U}(E) := \bigoplus_{i_1 < \cdots < i_{k+1}} t_{i_1 \cdots i_{k+1}} \in \bigoplus_{i_1 < \cdots < i_{k+1}} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k).$$

Then $\hat{f}_{k, U}(E)$ is a Čech $k$-cocycle and defines an element $f_{k, U}(E) \in \check{H}^k(U, \Omega^k)$ independent of the choice of $g$. Denote by $f_k(E)$ the image of $f_{k, U}(E)$ in the Čech cohomology group $\check{H}^k(X, \Omega^k)$ under the canonical homomorphism and call it the $k$-th $T$ invariant in $\check{H}^k(X, \Omega^k)$.

Proof of Theorem 3.13. The proof is the same as the proof of Theorem 1.2. It suffices to verify that when we go through the proof of Theorem 1.2 the equalities therein are non-trivial even if the characteristic of $K$ char($K$) is non-zero.

We can verify that based on the assumption that $(k+2)!$ is not divisible by char($K$). For instance, (116) is well-defined since $(k+1)!$ is not divisible by char($K$); (23), (39), (158), and (167) hold non-trivially since char($K$) $\neq 2$; (43) holds non-trivially since $(k+2)$ is not divisible by char($K$).

We thus complete the proof of Theorem 3.13. □
Since $\check{H}^k(X, \Omega^k) \cong H^k(X, \Omega^k)$ for separated schemes, we can compute the $T$ invariants in sheaf cohomology for separated scheme, and further define the cohomological Chern character for separated schemes as follows.

**Definition 3.14.** Let $X$ be a separated scheme over a field $K$. Let $E$ be a local free sheaf over $X$. Suppose that the characteristic of $K$ is large enough and that $X$ has a finite affine open cover. Then the cohomological Chern character is defined by

$$
\text{ch}_{coh}(E) := \sum_{k=0}^{\infty} \frac{1}{k! \cdot (2\pi \sqrt{-1})^k} \cdot f_k(E).
$$

Here we make the convention that when $f_k(E) = 0 \in H^k(X, \Omega^k)$, $\frac{1}{k! \cdot (2\pi \sqrt{-1})^k} \cdot f_k(E) = 0$, even if $k$ is divisible by the characteristic of $K$.

### 3.5. Cohomological groups $H^k(X, d\Omega^{k-1}), k \geq 2.$

In this subsection, we will introduce the cohomological ring $\Phi_X := \bigoplus_{i=0}^{\infty} H^i(X, d\Omega^{i-1})$ where the products of refined first $T$ invariants live. Notice that in general cases, the $k$-th $T$ invariant, $k \geq 2$, cannot be refined to an element of $H^k(X, d\Omega^{k-1})$, for the coefficients of the cocycle defined by (1) are not $d$-closed.

Let $X$ be a compact complex manifold and let $\Phi_X$ be the vector space defined by

$$
\Phi_X := \bigoplus_{i=0}^{\infty} H^i(X, d\Omega^{i-1}),
$$

where $H^0(X, d\Omega^{-1}) = \mathbb{C}$ by convention. Recall the following short exact sequences

$$
0 \to \mathbb{C} \to \Omega^0 \to d\Omega^0 \to 0 \\
0 \to d\Omega^0 \to \Omega^1 \to d\Omega^1 \to 0 \\
\ldots \\
0 \to d\Omega^q \to \Omega^{q+1} \to d\Omega^{q+1} \to 0 \\
\ldots;
$$

and the associated long exact sequence

$$
\to H^k(X, \mathbb{C}) \to H^k(X, \Omega^0) \to H^k(X, d\Omega^0) \to H^{k+1}(X, \mathbb{C}) \to, \\
\to H^k(X, \Omega^1) \to H^k(X, d\Omega^1) \to H^{k+1}(X, d\Omega^0) \to H^{k+1}(X, \Omega^1) \to, \\
\ldots \ldots \\
\to H^k(X, \Omega^{q+1}) \to H^k(X, d\Omega^{q+1}) \to H^{k+1}(X, d\Omega^q) \to H^{k+1}(X, \Omega^{q+1}) \to, \\
\ldots \ldots.
$$

Then, we can prove that
Lemma 3.15. Denote the complex dimension of \( X \) by \( N \). Then,
\[
H^p(X, d\Omega^q) = 0 \quad \text{for} \ p \geq N + 1 \quad \text{and} \ q \geq 0.
\]  \hspace{1cm} (122)

Proof of Lemma 3.15. Notice that \( H^p(X, \mathbb{C}) = H^p(X, \Omega^q) = 0 \) for \( p \geq N + 1 \) and \( q \geq 0 \). Then, the lemma follows from the long exact sequences (121). \( \square \)

Next we will give a ring structure for the vector space \( \Phi_X \) as follows. Let \( \mathcal{U} := \{ U_i \}_{i=1}^d \) of \( X \) be a finite, good open cover in the sense that each \( U_i \) is Stein and each nonempty intersection \( U_{i_1 \ldots i_p} \) is contractible. Then,
\[
H^{q+1}(X, d\Omega^q) \cong \hat{H}^{q+1}(\mathcal{U}, d\Omega^q) \quad \text{for} \ q \geq 0,
\]  \hspace{1cm} (123)
where \( \hat{H}^{q+1}(\mathcal{U}, d\Omega^q) \) is the \((q + 1)\)-th \( \check{C}ech \) cohomology group of the sheaf \( d\Omega^q \) with respect to \( \mathcal{U} \). In the following we will identify the sheaf cohomology groups and their corresponding \( \check{C}ech \) cohomology groups.

Let \( \xi \in \hat{H}^r(\mathcal{U}, d\Omega^{r-1}) \) and \( \eta \in \hat{H}^s(\mathcal{U}, d\Omega^{s-1}) \) be \( \check{C}ech \) cocycles as follows:
\[
\xi = \bigoplus_{1 \leq i_1 < \ldots < i_{r+1} \leq l} u_{i_1 \ldots i_{r+1}} \in \bigoplus_{1 \leq i_1 < \ldots < i_{r+1} \leq l} \Gamma(U_{i_1 \ldots i_{r+1}}, d\Omega^{r-1}),
\]
\[
\eta = \bigoplus_{1 \leq i_1 < \ldots < i_{s+1} \leq l} v_{i_1 \ldots i_{s+1}} \in \bigoplus_{1 \leq i_1 < \ldots < i_{s+1} \leq l} \Gamma(U_{i_1 \ldots i_{s+1}}, d\Omega^{s-1}),
\]  \hspace{1cm} (124)
where \( 0 \leq r, s \leq N \). We can define the cup product \( \xi \cup \eta \in \hat{H}^{r+s}(\mathcal{U}, d\Omega^{r+s-1}) \) by (see [11])
\[
\xi \cup \eta := \bigoplus_{1 \leq i_1 < \ldots < i_{r+s+1} \leq l} w_{i_1 \ldots i_{r+s+1}} \in \bigoplus_{1 \leq i_1 < \ldots < i_{r+s+1} \leq l} \Gamma(U_{i_1 \ldots i_{r+s+1}}, d\Omega^{r+s-1}),
\]  \hspace{1cm} (125)
where
\[
w_{i_1 \ldots i_{r+s+1}} = u_{i_1 \ldots i_{r+1}} \wedge v_{i_{r+1} \ldots i_{r+s+1}}.
\]  \hspace{1cm} (126)

Since \( dw_{i_1 \ldots i_{r+s+1}} = 0 \),
\[
w_{i_1 \ldots i_{r+s+1}} \in \Gamma(U_{i_1 \ldots i_{r+s+1}}, d\Omega^{r+s-1}) \quad \text{for} \ 1 \leq i_1 < \ldots < i_{r+s+1} \leq l.
\]  \hspace{1cm} (127)
Moreover, it is easy to verify that \( \xi \cup \eta = \eta \cup \xi \) and hence \( \Phi_X \) is a commutative ring.

By the natural inclusion we have the following proposition.

Proposition 3.16. Denote by \( \Psi_X \) the cohomology ring
\[
\Psi_X := \bigoplus_{i=1}^N H^i(X, \Omega^i).
\]  \hspace{1cm} (128)
Then, there is a ring homomorphism
\[
j : \Phi_X \rightarrow \Psi_X
\]  \hspace{1cm} (129)
induced by the natural group homomorphisms
\[
j_p : H^p(X, d\Omega^{p-1}) \rightarrow H^p(X, \Omega^p) \quad \text{for} \ p \geq 0.
\]  \hspace{1cm} (130)
Although the homomorphism (129) is neither injective nor surjective in general even if \( X \) is a projective manifold, it is true that \( j \) is an isomorphism for \( X = \mathbb{C}P^n \).

**Proposition 3.17.** The ring homomorphism (129) is an isomorphism for \( X = \mathbb{C}P^n \).

**Proof of Proposition 3.17.** We will prove by induction on \( k = 1, 2, \ldots \) that the following properties hold:

1. \( H^p(\mathbb{C}P^n, d\Omega^{k-1}) = 0 \) for \( 0 \leq p \leq k - 1 \);
2. \( H^p(\mathbb{C}P^n, d\Omega^{k-1}) \cong H^{p+1}(\mathbb{C}P^n, d\Omega^{k-2}) \cong \cdots \cong H^{p+k}(\mathbb{C}P^n, \mathcal{O}) \) for \( p \geq k \);
3. the natural homomorphism \( H^k(\mathbb{C}P^n, d\Omega^{k-1}) \rightarrow H^k(\mathbb{C}P^n, \Omega) \) is an isomorphism.

(131)

For \( k = 1 \), we first consider the following short exact sequence of sheaves (see (120)):

\[
0 \rightarrow \mathcal{O} \rightarrow \Omega^0 \rightarrow d\Omega^0 \rightarrow 0,
\]
and its associated long exact sequence of cohomology groups (see (121))

\[
0 \rightarrow H^0(\mathbb{C}P^n, \mathcal{O}) \rightarrow H^0(\mathbb{C}P^n, \Omega^0) \rightarrow H^0(\mathbb{C}P^n, d\Omega^0) \rightarrow H^1(\mathbb{C}P^n, \mathcal{O}) \rightarrow \cdots
\]

(133)

Noticing that \( H^0(\mathbb{C}P^n, \mathcal{O}) = H^0(\mathbb{C}P^n, \Omega^0) \) and \( H^p(\mathbb{C}P^n, \Omega^0) = 0 \) for \( p \geq 1 \), Properties (1) and (2) hold.

Moreover, by diagram chasing in the Čech-algebraic De Rham double complex, we have the following commutative diagram

\[
\begin{array}{ccc}
H^1(\mathbb{C}P^n, d\Omega^0) & \xrightarrow{\cong} & H^2(\mathbb{C}P^n, \mathcal{O}) \\
\downarrow & & \downarrow \\
& H^1(\mathbb{C}P^n, \Omega^1). & \\
\end{array}
\]

(134)

Therefore, Property (3) holds for \( k = 1 \).

Now suppose that Properties (1), (2) and (3) hold for all \( k \) such that \( 1 \leq k \leq l \). We are going to prove that Properties (1), (2) and (3) hold for \( k = l + 1 \). Similarly, considering the following long exact sequence

\[
0 \rightarrow H^0(\mathbb{C}P^n, d\Omega^{l-1}) \rightarrow H^0(\mathbb{C}P^n, \Omega^l) \rightarrow H^0(\mathbb{C}P^n, d\Omega^l) \rightarrow H^1(\mathbb{C}P^n, d\Omega^{l-1}) \rightarrow \\
\cdots \rightarrow H^p(\mathbb{C}P^n, d\Omega^{l-1}) \rightarrow H^p(\mathbb{C}P^n, \Omega^l) \rightarrow H^p(\mathbb{C}P^n, d\Omega^l) \rightarrow H^{p+1}(\mathbb{C}P^n, d\Omega^{l-1}) \rightarrow \\
\rightarrow H^{p+1}(\mathbb{C}P^n, d\Omega^{l-1}) \rightarrow \cdots
\]

(135)

For an integer \( p \) such that \( 0 \leq p \leq l - 2 \), noticing that \( H^p(\mathbb{X}, \Omega^l) = H^{p+1}(\mathbb{X}, \Omega^l) = 0 \), we have \( H^p(\mathbb{C}P^n, d\Omega^l) \cong H^{p+1}(\mathbb{C}P^n, d\Omega^{l-1}) = 0 \). For an integer \( p \) such that \( p \geq l + 1 \), noticing that \( H^p(\mathbb{X}, \Omega^l) = H^{p+1}(\mathbb{X}, \Omega^l) = 0 \),
we have $H^p(\mathbb{C}P^n, d\Omega^l) \cong H^{p+1}(\mathbb{C}P^n, d\Omega^{l-1}) \cong \cdots \cong H^{p+l+1}(\mathbb{C}P^n, \mathbb{C})$. Moreover, we have the following long exact sequences,

\[ H^{l-1}(\mathbb{C}P^n, \Omega^l) \to H^{l-1}(\mathbb{C}P^n, d\Omega^l) \to H^l(\mathbb{C}P^n, d\Omega^{l-1}) \to H^l(\mathbb{C}P^n, \Omega^l) \to \]
\[ \to H^l(\mathbb{C}P^n, d\Omega^l) \to H^{l+1}(\mathbb{C}P^n, d\Omega^{l-1}), \]

(136)

and

\[ H^{l-2}(\mathbb{C}P^n, \Omega^{l+1}) \to H^{l-2}(\mathbb{C}P^n, d\Omega^{l+1}) \to H^{l-1}(\mathbb{C}P^n, d\Omega^l) \to H^{l-1}(\mathbb{C}P^n, \Omega^{l+1}), \]

(137)

\[ H^{l-3}(\mathbb{C}P^n, \Omega^{l+2}) \to H^{l-3}(\mathbb{C}P^n, d\Omega^{l+2}) \to H^{l-2}(\mathbb{C}P^n, d\Omega^{l+1}) \to H^{l-2}(\mathbb{C}P^n, \Omega^{l+2}), \]

\[ \cdots \]

(138)

\[ H^0(\mathbb{C}P^n, \Omega^2) \to H^0(\mathbb{C}P^n, d\Omega^2) \to H^1(\mathbb{C}P^n, d\Omega^{2-1}) \to H^1(\mathbb{C}P^n, \Omega^2), \]

(139)

By a backward induction based on (137), (138), (139) and (140), we conclude that

\[ H^{l-1}(\mathbb{C}P^n, d\Omega^l) = H^{l-2}(\mathbb{C}P^n, d\Omega^{l+1}) = \cdots = H^0(\mathbb{C}P^n, d\Omega^{2l-1}) = 0 \]

(141)

Substituting (141) into (136), we have that

\[ 0 \to H^l(\mathbb{C}P^n, d\Omega^{l-1}) \to H^l(\mathbb{C}P^n, \Omega^l) \to H^l(\mathbb{C}P^n, d\Omega^l) \to 0, \]

(142)

Since $\dim H^l(\mathbb{C}P^n, d\Omega^{l-1}) = \dim H^l(\mathbb{C}P^n, \Omega^l) = \dim H^l(\mathbb{C}P^n, \Omega^l)$, we have that $H^l(\mathbb{C}P^n, d\Omega^l) = 0$.

Notice that Properties (1) and (2) in (131) hold for $k = l + 1$ as above. To prove Property (3) $k = l + 1$, we recall the following natural homomorphisms

\[ H^{l+1}(\mathbb{C}P^n, d\Omega^l) \to H^{l+2}(\mathbb{C}P^n, d\Omega^{l-1}) \to \cdots \to H^{2l+2}(\mathbb{C}P^n, \mathbb{C}) \]

(143)

Moreover, by diagram chasing in the Čech-algebraic De Rham double complex, we have the following commutative diagram

\[
\begin{array}{ccc}
H^{l+1}(\mathbb{C}P^n, d\Omega^l) & \mathbb{H} \to & H^{l+2}(\mathbb{C}P^n, d\Omega^{l-1}) \\
& \mathbb{H} \to & \cdots \\
& \mathbb{H} \to & H^{2l+2}(\mathbb{C}P^n, \mathbb{C}) \\
& \mathbb{H} \to & H^{l+1}(\mathbb{C}P^n, \Omega^{l+1}) \\
\end{array}
\]

(144)

Therefore, Property (3) holds for $k = l + 1$.

We conclude Proposition 3.17 by Property (3). \qed

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability The datasets supporting the conclusions of this article are included within the article and its additional files.

A. Proof of Lemma 2.5

Proof. Since \( \{g_{ij}\} \) and \( \{\tilde{g}_{ij}\} \) are two systems of transition functions of \( E \) with respect to \( \mathcal{U} \), there exists a Čech 0-cochain

\[ h := \bigoplus_{i_1} h_{i_1} \in \bigoplus_{i_1} \Gamma_{\text{hol}}(U_{i_1}, GL(M, \mathbb{C})), \]  

such that \( \tilde{g}_{ij} = h_{i_1}^{-1} g_{ij} h_{j} \) for \( i, j \in I \). Let \( \hat{f}_k(E, g) \) and \( \hat{f}_k(E, \tilde{g}) \) be the Čech \( k \)-cocycles associated with \( f_k(E, g) \), \( f_k(E, g) \), respectively, as follows.

\[ \hat{f}_k(E, g) = \bigoplus_{i_1 < \cdots < i_{k+1}} t_{i_1 \cdots i_{k+1}} \in \bigoplus_{i_1 < \cdots < i_{k+1}} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k), \]  

\[ \hat{f}_k(E, \tilde{g}) = \bigoplus_{i_1 < \cdots < i_{k+1}} \tilde{t}_{i_1 \cdots i_{k+1}} \in \bigoplus_{i_1 < \cdots < i_{k+1}} \Gamma(U_{i_1 \cdots i_{k+1}}, \Omega^k). \]

Here \( t_{i_1 \cdots i_{k+1}} \) and \( \tilde{t}_{i_1 \cdots i_{k+1}} \) are defined by \((24)\) with respect to \( g \) and \( \tilde{g} \), respectively. To prove Lemma 2.5, it suffices to prove that there is a Čech \((k - 1)\)-cochain

\[ h_{k-1}(E, g, \tilde{g}) = \bigoplus_{j_1 < \cdots < j_k} s_{j_1 \cdots j_k} \in \bigoplus_{j_1 < \cdots < j_k} \Gamma(U_{j_1 \cdots j_k}, \Omega^k), \]  

such that for any \( i_1, \ldots, i_{k+1} \in I \), we have

\[ \tilde{t}_{i_1 \cdots i_{k+1}} - t_{i_1 \cdots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j-1} s_{i_1 \cdots i_{j-1} j \cdots i_{k+1}}|_{U_{i_1 \cdots i_{k+1}}}. \]

Notice that

\[ \tilde{g}_{\alpha\beta}^{-1} d\tilde{g}_{\alpha\beta} = h_{\beta}^{-1} (g_{\alpha\beta}^{-1} d g_{\alpha\beta} + g_{\alpha\beta}^{-1} h_{\alpha} dh_{\alpha}^{-1} g_{\alpha\beta} + (-1) \cdot h_{\beta} dh_{\beta}^{-1}) h_{\beta}. \]
Construct holomorphic invariants in Čech cohomology

Applying (150) for \( \beta = i_{\sigma(k+1)} \) and \( \alpha = i_{\sigma(1)} \cdot i_{\sigma(2)} \cdot i_{\sigma(3)} \cdot \ldots \cdot i_{\sigma(k)} \), we get

\[
\tilde{t}_{i_1 \ldots i_{k+1}} = \sum_{\sigma \in S_{k+1}} \frac{\text{sgn}(\sigma)}{(k + 1)!} \cdot \text{tr}\left( \tilde{g}^{-1}_{i_{\sigma(1)}i_{\sigma(k+1)}} d\tilde{g}_{i_{\sigma(2)}i_{\sigma(k+1)}} \tilde{g}^{-1}_{i_{\sigma(3)}i_{\sigma(k+1)}} d\tilde{g}_{i_{\sigma(4)}i_{\sigma(k+1)}} \ldots \tilde{g}^{-1}_{i_{\sigma(k)}i_{\sigma(k+1)}} \right)
\]

\[
= \sum_{\sigma \in S_{k+1}} \frac{\text{sgn}(\sigma)}{(k + 1)!} \cdot \text{tr}\left\{ \prod_{l=1}^{k} h_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} d\tilde{g}_{i_{\sigma(l)}i_{\sigma(k+1)}} \right\} + \frac{g^{-1}_{i_{\sigma(l)}i_{\sigma(k+1)}} h_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1}}{(k + 1)!} \cdot \text{tr}\left\{ \prod_{l=1}^{k} \left( h_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} \right) \right\}.
\]

In order to compute the monomials appearing in the expansion of (151) by distribution law, we introduce the following notation. For integer \( t \geq 0 \) and integers \( a_1, \ldots, a_t \), we define the counting function \( \delta_{a_1 \ldots a_t} \) from \( \mathbb{Z} \) to the set \( \{0, 1\} \) as follows.

\[
\delta_{a_1 \ldots a_t}(l) = \begin{cases} 
1 & \text{if } l \in \{a_1, \ldots, a_t\}; \\
0 & \text{if } l \notin \{a_1, \ldots, a_t\}.
\end{cases}
\]

Notice that when \( t = 0 \), \( \delta_{a_1 \ldots a_t} \equiv 0 \); we denote it by \( \delta_\emptyset \).

Fix a permutation \( \sigma \in S_{k+1} \). For integers \( j, n \) such that \( j \geq 0, n \geq 0 \) and \( j + n \leq k \), and distinct integers \( u_1, \ldots, u_j, v_1, \ldots, v_n \in \{1, \ldots, k\} \) (by convention if \( j = 0 \), there is no \( u \); if \( n = 0 \), there is no \( v \)), we define

\[
\Delta^\sigma_{u_1 \ldots u_j, v_1 \ldots v_n} (\tilde{t}_{i_1 \ldots i_{k+1}}) := \text{tr}\left[ \prod_{l=1}^{k} \left( g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} d\tilde{g}_{i_{\sigma(l)}i_{\sigma(k+1)}} \right) \right]^{-\delta_{u_1 \ldots u_j}(l) - \delta_{v_1 \ldots v_n}(l)} \cdot \left( g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} h_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} \right) \right].
\]

Note that here we use the convention that

\[
\left( g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} \right)^{0} = \left( g_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} h_{i_{\sigma(l)}i_{\sigma(k+1)}}^{-1} \right)^{0} = I_{M \times M}.
\]

where \( I_{M \times M} \) is the identity matrix of rank \( M \). We also denote \( \Delta^\sigma_{u_1 \ldots u_j, v_1 \ldots v_n} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) by \( \Delta^\sigma_{\emptyset, v_1 \ldots v_n} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) when \( j = 0 \) and \( n \geq 1 \); denote \( \Delta^\sigma_{u_1 \ldots u_j, v_1 \ldots v_n} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) by \( \Delta^\sigma_{u_1 \ldots u_j, \emptyset} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) when \( n = 0 \) and \( j \geq 1 \); denote \( \Delta^\sigma_{u_1 \ldots u_j, v_1 \ldots v_n} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) by \( \Delta^\sigma_{\emptyset, \emptyset} (\tilde{t}_{i_1 \ldots i_{k+1}}) \) when \( j = n = 0 \).
Applying distribution law to (151), we have
\[
\tilde{t}_{i_1 \cdots i_{k+1}} = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{n \geq 2, 0 \leq j \leq k-n; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ 1 \leq v_1 < v_2 < \cdots < v_n \leq k; \ u_1, \ldots, u_j, v_1, \ldots, v_n \text{ are distinct}} \Delta_{u_1 \cdots u_j, v_1 \cdots v_n}^\sigma \tilde{t}_{i_1 \cdots i_{k+1}} 
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{n=1, 1 \leq j \leq k-1; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ u_1, \ldots, u_j, v \text{ are distinct}} \Delta_{u_1 \cdots u_j, v}^\sigma \tilde{t}_{i_1 \cdots i_{k+1}} 
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{n=0, 1 \leq j \leq k; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ u_1, \ldots, u_j, v \text{ are distinct}} \Delta_{u_1 \cdots u_j, v}^\sigma \tilde{t}_{i_1 \cdots i_{k+1}} 
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq v_1 \leq k; \ u_1, \ldots, u_j, v \text{ are distinct}} \Delta_{u_1 \cdots u_j, v}^\sigma \tilde{t}_{i_1 \cdots i_{k+1}} 
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]

We first compute \( I_1 \) as follows.

**Claim I.** For integers \( j, n \) such that \( n \geq 2 \) and \( 0 \leq j \leq k-n \), and distinct integers \( u_1, \ldots, u_j, v_1, \ldots, v_n \in \{1, \ldots, k\} \), the following identity holds,
\[
\sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \Delta_{u_1 \cdots u_j, v_1 \cdots v_n}^\sigma \tilde{t}_{i_1 \cdots i_{k+1}} = 0.
\]

Therefore, \( I_1 \equiv 0 \).

**Proof of Claim I.** The proof is similar to the proof of Claim I of Lemma 2.4. Denote by \( \tau \) the permutation \((v_1, v_2) \in S_{k+1} \). Notice that
\[
\begin{align*}
\Delta_{u_1 \cdots u_j, v_1 \cdots v_n}^{\sigma \tau} \tilde{t}_{i_1 \cdots i_{k+1}} &= tr \left\{ \prod_{l=1}^{k} \left( g_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{l-1} dg_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))} \right)^{1-b_{k-1-u_j}} \cdot dh_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{1-b_{k-1-v_n}} 
\right. \\
& \quad \cdot \left( -h_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))} dh_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{1-b_{k-1-v_n}} \right) \biggr) \\
&= tr \left\{ \prod_{l=1}^{k} \left( g_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{l-1} dg_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))} \right)^{1-b_{k-1-u_j}} \cdot dh_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{1-b_{k-1-v_n}} 
\right. \\
& \quad \cdot \left( -h_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))} dh_{\sigma(\tau (1)) (\sigma (\tau (2)) (k+1))}^{1-b_{k-1-v_n}} \right) \biggr) \\
&= \Delta_{u_1 \cdots u_j, v_1 v_2 \cdots v_n}^{\sigma} \tilde{t}_{i_1 \cdots i_{k+1}}.
\end{align*}
\]
Hence,

\[
\sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \Delta_{u_1 \cdots u_j, v_1 v_2 \cdots v_n}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}) = \sum_{\sigma \sigma \tau \in S_{k+1}} \frac{sgn(\sigma \circ \tau)}{(k+1)!} \cdot \Delta_{u_1 \cdots u_j, v_1 v_2 \cdots v_n}^{\sigma \circ \tau}(\tilde{t}_{i_1 \cdots i_{k+1}}) = sgn(\tau) \cdot \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \Delta_{u_1 \cdots u_j, v_1 v_2 \cdots v_n}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}) = - \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \Delta_{u_1 \cdots u_j, v_1 v_2 \cdots v_n}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}).
\]

(158)

We complete the proof of Claim I.

\[\Box\]

In order to compute \(I_4\), we introduce the following notation.

Fix \(\sigma \in S_{k+1}\). For integers \(j, l\) such that \(j \geq 1\) and \(0 \leq l \leq k - j\), and distinct integers \(u_1, \ldots, u_j, x_1, \ldots, x_l \in \{1, \ldots, k\}\) (by convention when \(l = 0\), there is no \(x\)), we define

\[
\Delta_{(u_1 \cdots u_j; x_1 \cdots x_l)}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}) = \text{tr}\left\{ \prod_{l=1}^{k} \left( g_{i_\sigma(l)i_\sigma(u_1)}^{-1} d g_{i_\sigma(l)i_\sigma(u_1)} \right)^{1-\delta_{x_1 \cdots x_l}(l)-\delta_{u_1 \cdots u_j}(l)} \cdot \left( -g_{i_\sigma(x_1)}^{-1} d g_{i_\sigma(x_1)} \right)^{\delta_{x_1 \cdots x_l}(l)} \cdot \left( g_{i_\sigma(l)i_\sigma(x_1)}^{-1} h_{i_\sigma(l)i_\sigma(x_1)} d h_{i_\sigma(l)i_\sigma(x_1)} \right)^{\delta_{u_1 \cdots u_j}(l)} \right\}. \tag{159}
\]

We also denote \(\Delta_{(u_1 \cdots u_j; x_1 \cdots x_l)}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}})\) by \(\Delta_{(u_1 \cdots u_j; y)}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}})\) when \(l = 0\).

Claim II.

\[
I_4 = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k; \quad 1 \leq u_1 < u_2 < \cdots < u_j \leq k;} \Delta_{(u_1 \cdots u_j; y)}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}) + \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k-1; \quad 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \quad 1 \leq x_1 \leq k, x_1 \notin \{u_1, \ldots, u_j\}} \Delta_{(u_1 \cdots u_j; x_1)}^{\sigma}(\tilde{t}_{i_1 \cdots i_{k+1}}). \tag{160}
\]

Proof of Claim II. Recall the following identities

\[
g_{y_\beta}^{-1} h_{y_\gamma}^{-1} d h_{y_\gamma} g_{y_\beta} = g_{a_\beta}^{-1} s y_\alpha^{-1} h_{y_\gamma}^{-1} d h_{y_\gamma} g_{y_\alpha} g_{a_\beta} \quad \text{and} \quad g_{b_\beta}^{-1} d g_{b_\beta} = g_{c_\alpha}^{-1} g_{d_\alpha}^{-1} d g_{d_\alpha} g_{c_\alpha} + g_{a_\beta}^{-1} d g_{a_\beta}. \tag{161}
\]
Applying (161), we have

\[
\Delta_{u_1 \cdots u_j}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}}) = \text{tr} \left\{ \prod_{l=1}^{k} \left( g_{i_\sigma(l)}^{-1} i_{\sigma(k+1)} g_{i_\sigma(l)} g_{i_\sigma(l)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l) \right\} \\
\cdot \left( g_{i_\sigma(l)} h_{i_\sigma(l)} d h_{i_\sigma(l)}^{-1} g_{i_\sigma(l)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l)
\]

\[
= \text{tr} \left\{ \prod_{l=1}^{k} \left( g_{i_\sigma(u_1)} g_{i_\sigma(u_1)} i_{\sigma(k+1)} g_{i_\sigma(u_1)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l) \right\}
\]

\[
+ g_{i_\sigma(u_1)} g_{i_\sigma(u_1)} i_{\sigma(k+1)} \delta_{u_1 \cdots u_j}(l)
\]

\[
\cdot \left( g_{i_\sigma(u_1)} h_{i_\sigma(u_1)} d h_{i_\sigma(u_1)}^{-1} g_{i_\sigma(u_1)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l)
\]

\[
= \text{tr} \left\{ \prod_{l=1}^{k} \left( g_{i_\sigma(l)} g_{i_\sigma(l)} i_{\sigma(k+1)} + (-1) \cdot g_{i_\sigma(u_1)} g_{i_\sigma(u_1)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l) \right\}
\]

\[
\cdot \left( g_{i_\sigma(u_1)} h_{i_\sigma(u_1)} d h_{i_\sigma(u_1)}^{-1} g_{i_\sigma(u_1)} i_{\sigma(k+1)} \right) \delta_{u_1 \cdots u_j}(l)
\]

\[
= \sum_{0 \leq l \leq k-j; \ 1 \leq x_1 < \cdots < x_l \leq k; \ u_1, \ldots, u_j, x_1, \ldots, x_l \text{ are distinct}} \Delta_{(u_1 \cdots u_j; x_1 \cdots x_l)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}}),
\]

Then,

\[
I_k = \sum_{\sigma \in S_k+1} \frac{\text{sgn}(\sigma)}{(k+1)!} \sum_{j \geq 1; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ u_1, \ldots, u_j, x_1, \ldots, x_l \text{ are distinct}} \Delta_{(u_1 \cdots u_j; x_1 \cdots x_l)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}})
\]

\[
= \sum_{\sigma \in S_k+1} \frac{\text{sgn}(\sigma)}{(k+1)!} \sum_{1 \leq i \leq k} \Delta_{(u_1 \cdots u_j; x_i)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}})
\]

\[
+ \sum_{\sigma \in S_k+1} \frac{\text{sgn}(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ 1 \leq x_1 \leq k; \ x_1 \notin [u_1, \ldots, u_j]} \Delta_{(u_1 \cdots u_j; x_1 \cdots x_l)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}})
\]

\[
+ \sum_{\sigma \in S_k+1} \frac{\text{sgn}(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k-2; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ 2 \leq i \leq k-j; \ 1 \leq x_1 < \cdots < x_i \leq k; \ u_1, \ldots, u_j, x_1, \ldots, x_i \text{ are distinct}} \Delta_{(u_1 \cdots u_j; x_1 \cdots x_i)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}})
\]

\[
= J_1 + J_2 + J_3.
\]

Interchanging the order of summation, we have that

\[
J_1 = \sum_{1 \leq j \leq k-2; \ 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \ u_1, \ldots, u_j, x_1, \ldots, x_i \text{ are distinct}} \sum_{\sigma \in S_{k+1}} \frac{\text{sgn}(\sigma)}{(k+1)!} \cdot \Delta_{(u_1 \cdots u_j; x_1 \cdots x_i)}^\sigma (\tilde{g}_{i_1 \cdots i_{k+1}})
\]

(164)
Hence, it suffices to prove that for integers \( j, l \) such that \( 1 \leq j \leq k - j \), and distinct integers \( u_1, \ldots, u_j, x_1, \ldots, x_l \in \{1, \ldots, k\} \), the following equality holds,

\[
\sum_{\sigma \in S_{k+1}} sgn(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}) \equiv 0. \quad (165)
\]

We shall prove equality (165) in the same way as the proof of Claim I of Lemma 2.4. Denote by \( \tau \) the permutation \((x_1, x_2) \in S_{k+1}\). Then,

\[
\begin{align*}
\Delta^\sigma\tau_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}) &= tr \left\{ \prod_{l=1}^{k} (g^{-1}_{l(\sigma(\sigma_0)l)}g_{l(\sigma(\sigma_0)l)}^{1-\delta_{1-x_1(l)}-\delta_{u_1-u_j(l)}}) \\
&\quad \cdot (-g_i_{l(\sigma(\sigma_0)l)}dg_{l(\sigma(\sigma_0)l)}) \delta_{1-x_1(l)} \\
&\quad \cdot (g_i_{l(\sigma(\sigma_0)l)}dg_{l(\sigma(\sigma_0)l)}) \delta_{u_1-u_j(l)} \right\} \\
&= tr \left\{ \prod_{l=1}^{k} (g^{-1}_{l(\sigma(\sigma_0)l)}g_{l(\sigma(\sigma_0)l)}) \delta_{1-x_1(l)}-\delta_{u_1-u_j(l)} \right\} \\
&\quad \cdot (-g_i_{l(\sigma(\sigma_0)l)}dg_{l(\sigma(\sigma_0)l)}) \delta_{1-x_1(l)}(g_i_{l(\sigma(\sigma_0)l)}dg_{l(\sigma(\sigma_0)l)}) \delta_{u_1-u_j(l)} \\
&= \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}) + \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}).
\end{align*}
\]

Therefore, we have that

\[
\sum_{\sigma \in S_{k+1}} sgn(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}) = \sum_{\sigma \in S_{k+1}} sgn(\sigma) \cdot \Delta^\sigma\tau_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}})
\]

\[
= sgn(\tau) \cdot \sum_{\sigma \in S_{k+1}} sgn(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}})
\]

\[
= - \sum_{\sigma \in S_{k+1}} sgn(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j; x_1 \cdots x_l)}(\tilde{t}_{1 \cdots i_{k+1}}).
\]

We complete the proof of Claim II. \( \square \)

We now construct a Čech \((k - 1)\)-cochain \( s \) as follows. For \( i_1, i_2, \ldots, i_{k+1} \in I \) and \( 1 \leq \alpha \leq k + 1 \), define

\[
S_{i_1 \cdots i_{k+1}; i_1 \cdots i_{k+1}} := (-1)^{\alpha-1} \frac{1}{(k+1)!} \sum_{1 \leq j \leq k-1; 1 \leq u_1 < \cdots < u_j \leq k; \sigma(v_1) = \alpha \atop 1 \leq v_1 < k; u_1, \ldots, u_j, v_1 \text{ are distinct}} sgn(\sigma) \cdot \Delta_{u_1 \cdots u_j; v_1}(\tilde{t}_{1 \cdots i_{k+1}})
\]

\[
+ (-1)^{\alpha-1} \frac{1}{(k+1)!} \sum_{1 \leq u_1 \leq k} \sum_{\sigma \in S_{k+1}, \sigma(v_1) = \alpha} sgn(\sigma) \cdot \Delta_{\emptyset; v_1}(\tilde{t}_{1 \cdots i_{k+1}})
\]

\[
+ (-1)^{\alpha-1} \frac{1}{(k+1)!} \sum_{1 \leq j \leq k; 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \sigma(k+1) = \alpha \atop \sigma(v_1) = \alpha} sgn(\sigma) \cdot \Delta_{(u_1 \cdots u_j; \emptyset)}(\tilde{t}_{1 \cdots i_{k+1}})
\]
\[ + \frac{(-1)^{\alpha-1}}{(k+1)!} \sum_{1 \leq j \leq k-1; \sigma \in S_{k+1}, \sigma(x_1)=\alpha} \sum_{1 \leq u_1 < u_2 < \cdots < u_j \leq k; \sigma(v_1)=\alpha} \sum_{1 \leq v_1 \leq k; u_1, \ldots, u_j, v_1 \text{ are distinct}} sgn(\sigma) \cdot \Delta^\sigma_{u_1 \cdots u_j; v_1} (\hat{t}_{i_1 \cdots i_k+1}). \]

(168)

**Claim III.** The above definition depends on \((i_1, \ldots, i_{\alpha}, i_{\beta}, \ldots, i_{k+1})\) but not on \((i_1, \ldots, i_{k+1})\); namely, if \((i_1, \ldots, i_{\alpha}, i_{\beta}, \ldots, i_{k+1}) = (j_1, \ldots, j_{\beta}, \ldots, j_{k+1})\),

\[ s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+1} = s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+1}}. \]

**Proof of Claim III.** For any integers \(\alpha, \beta\) and \(i_1, \ldots, i_{k+2}\) such that \(1 \leq \alpha < \beta \leq k + 2\) and \(i_1, i_2, \ldots, i_{k+1} \in I\), we can define \(s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+2}}\) and \(s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+2}}\) by (168). Firstly, we have

\[ s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+2}} \]

\[ = \frac{(-1)^{\alpha-1}}{(k+1)!} \sum_{j=0; \sigma \in S_{k+1}, \sigma(v_1)=\alpha \leq \beta} \sum_{1 \leq v_1 \leq k; u_1, \ldots, u_j, v_1 \text{ are distinct}} sgn(\sigma) \cdot \Delta^\sigma_{u_1 \cdots u_j; v_1} (\hat{t}_{i_1 \cdots i_{\beta} \cdots i_{k+2}}) \]

(169)

\[ + \frac{(-1)^{\alpha-1}}{(k+1)!} \sum_{1 \leq j \leq k; \sigma \in S_{k+1}, \sigma(v_1)=\alpha \leq \beta} \sum_{1 \leq u_1 < u_2 < \cdots < u_j \leq k; \sigma(k+1)=\alpha} sgn(\sigma) \cdot \Delta^\sigma_{u_1 \cdots u_j; v_1} (\hat{t}_{i_1 \cdots i_{\beta} \cdots i_{k+2}}) \]

\[ + \frac{(-1)^{\alpha-1}}{(k+1)!} \sum_{1 \leq j \leq k-1; \sigma \in S_{k+1}, \sigma(v_1)=\alpha \leq \beta} \sum_{1 \leq u_1 < u_2 < \cdots < u_j \leq k; \sigma(x_1)=\alpha} sgn(\sigma) \cdot \Delta^\sigma_{u_1 \cdots u_j; x_1} (\hat{t}_{i_1 \cdots i_{\beta} \cdots i_{k+2}}) \]

\[ =: \kappa^\alpha_1 + \kappa^\alpha_2 + \kappa^\alpha_3 + \kappa^\alpha_4. \]

Notice that in the definition of \(s_{i_1 \cdots i_{\alpha} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+1}}\), the number \(\alpha\) on the right hand side is referring to the position of the omitted index on the left hand side. Therefore, \(s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+2}}\) takes the following form,

\[ s_{i_1 \cdots i_{\alpha} \cdots i_{\beta} \cdots i_{k+1}; 1 \cdots i_{\beta} \cdots i_{k+2}} \]

\[ = \frac{(-1)^{\beta-2}}{(k+1)!} \sum_{1 \leq j \leq k-1; \tau \in S_{k+1}, \tau(v_1)=\beta-1} \sum_{1 \leq u_1 < u_2 < \cdots < u_j \leq k; \tau(x_1)=\alpha} \sum_{1 \leq v_1 \leq k; u_1, \ldots, u_j, v_1 \text{ are distinct}} sgn(\tau) \cdot \Delta^\tau_{u_1 \cdots u_j; v_1} (\hat{t}_{i_1 \cdots i_{\beta} \cdots i_{k+2}}) \]

\[ + \frac{(-1)^{\beta-2}}{(k+1)!} \sum_{j=0; \tau \in S_{k+1}, \tau(v_1)=\beta-1} \sum_{1 \leq v_1 \leq k; u_1, \ldots, u_j, v_1 \text{ are distinct}} sgn(\tau) \cdot \Delta^\tau_{v_1} (\hat{t}_{i_1 \cdots i_{\beta} \cdots i_{k+2}}) \]
Moreover, to compare the signature, we define a bijection by Claim IV.

Next we will prove Claim IV used above.

Construct holomorphic invariants in Čech cohomology

In order to prove Claim III, it suffices to prove that

$$s_{i_1 \ldots i_a \ldots i_\beta \ldots i_{k+2}} = s_{i_1 \ldots i_a \ldots i_\beta \ldots i_{k+2}}$$

(171)

By Claim IV to be proved in the following, formula (171) holds. We complete the proof of Claim III.

Next we will prove Claim IV used above.

Claim IV.

$$\kappa_i^\alpha = (-1)^{\alpha-\beta+1} \cdot \kappa_i^\beta \quad \text{for} \quad i = 1, 2, 3, 4.$$  (172)

Proof of Claim IV. We will prove Claim IV based on a case by case argument.

Case I ($\kappa_i^\beta = (-1)^{\alpha-\beta+1} \cdot \kappa_i^\beta$). It suffices to prove that for fixed integer $j$ such that $1 \leq j \leq k - 1$, and distinct integers $u_1, \ldots, u_j, \nu_1 \in \{1, \ldots, k\}$, the following equality holds,

$$\sum_{\sigma \in S_{k+1}, \sigma(v_1) = \alpha} sgn(\sigma) \cdot \Delta_{u_1 \ldots u_j, \nu_1}^{\sigma}(\tilde{t}_{i_1 \ldots i_\beta \ldots i_{k+2}}) = (-1)^{\alpha-\beta+1} \sum_{\tau \in S_{k+1}, \tau(v_1) = \beta-1} sgn(\tau) \cdot \Delta_{u_1 \ldots u_j, \nu_1}^{\tau}(\tilde{t}_{i_1 \ldots i_\beta \ldots i_{k+2}}).$$  (173)

In order to compare the terms on both sides, we lift each element of $S_{k+1}$ to an element of $S_{k+2}$ as follows. Denote by $S_{k+1}^\beta$ the set consisting of all bijections from $\{1, 2, \ldots, k+1\}$ to $\{1, \ldots, \beta, \ldots, k+2\}$; denote by $S_{k+1}^\alpha$ the set consisting of all bijections from $\{1, 2, \ldots, k+1\}$ to $\{1, \ldots, \alpha, \ldots, k+2\}$. Define a bijection

$$\hat{\sigma} : S_{k+1} \longrightarrow S_{k+1}^\beta$$

by

$$\hat{\sigma}(l) = \begin{cases} 
\sigma(l) & \text{if} \quad 1 \leq l \leq k + 1 \quad \text{and} \quad \sigma(l) < \beta, \\
\sigma(l) + 1 & \text{if} \quad 1 \leq l \leq k + 1 \quad \text{and} \quad \sigma(l) \geq \beta. 
\end{cases}$$  (175)

Moreover, to compare the signature, we define a bijection

$$\beta : S_{k+1}^\beta \longrightarrow \{\eta | \eta \in S_{k+2}, \eta(k+2) = \beta\}$$

$$\hat{\sigma} \mapsto [\hat{\sigma}]_\beta$$  (176)
by

$$[	ilde{\sigma}]_\beta(l) = \begin{cases} \tilde{\sigma}(l) & \text{if } 1 \leq l \leq k + 1, \\ \beta & \text{if } l = k + 2; \end{cases}$$  \tag{177}

then, the following equality holds,

$$\text{sgn}(\sigma) = (-1)^{k + 2 - \beta} \cdot \text{sgn}(\tilde{\sigma})_{\beta}.$$  \tag{178}

Similarly, we can define a bijection

$$\tilde{\bullet} : S_{k + 1} \longrightarrow S_{k + 1}^{\alpha} \tau \longmapsto [\tau]_{\alpha}$$  \tag{179}

by

$$[\tau]_{\alpha}(l) = \begin{cases} \tau(l) & \text{if } 1 \leq l \leq k + 1 \text{ and } \tau(l) < \alpha, \\ \tau(l) + 1 & \text{if } 1 \leq l \leq k + 1 \text{ and } \tau(l) \geq \alpha; \end{cases}$$  \tag{180}

define a bijection

$$\alpha : S_{k + 1}^{\alpha} \longrightarrow \{ \eta | \eta \in S_{k + 2}, \eta(k + 2) = \alpha \}$$  \tag{181}

by

$$[\tau]_{\alpha}(l) = \begin{cases} \tau(l) & \text{if } 1 \leq l \leq k + 1, \\ \alpha & \text{if } l = k + 2, \end{cases}$$  \tag{182}

the following equality holds,

$$\text{sgn}(\tau) = (-1)^{k + 2 - \alpha} \cdot \text{sgn}([\tau]_{\alpha}).$$  \tag{183}

Rewriting (153) under the above notation, we have

$$\Delta^\sigma_{u_1 \cdots u_j, v_1} (\tilde{t}_{i_1} \cdots \tilde{t}_{i_{k+2}}) = tr \left\{ \prod_{l=1}^{k} (g_{i_{\tilde{t}_l}(l)}^{-1} g_{i_{\tilde{t}_l}(l)} d g_{i_{\tilde{t}_l}(l)} i_{\tilde{t}_l}(k+1))^{1-\delta_{u_1 \cdots u_j}(l) - \delta_{v_1}(l)} \cdot (g_{i_{\tilde{t}_l}(l)}^{-1} h_{i_{\tilde{t}_l}(l)} d h_{i_{\tilde{t}_l}(l)} i_{\tilde{t}_l}(k+1))^{\delta_{u_1 \cdots u_j}(l)} \cdot (-h_{i_{\tilde{t}_l}(l)} d h_{i_{\tilde{t}_l}(l)}^{-1} i_{\tilde{t}_l}(k+1))^{\delta_{v_1}(l)} \right\}$$  \tag{184}

and

$$\Delta^\tau_{u_1 \cdots u_j, v_1} (\tilde{t}_{i_1} \cdots \tilde{t}_{i_{k+2}}) = tr \left\{ \prod_{l=1}^{k} (g_{i_{\tilde{\tau}_l}(l)}^{-1} g_{i_{\tilde{\tau}_l}(l)} d g_{i_{\tilde{\tau}_l}(l)} i_{\tilde{\tau}_l}(k+1))^{1-\delta_{u_1 \cdots u_j}(l) - \delta_{v_1}(l)} \cdot (g_{i_{\tilde{\tau}_l}(l)}^{-1} h_{i_{\tilde{\tau}_l}(l)} d h_{i_{\tilde{\tau}_l}(l)} i_{\tilde{\tau}_l}(k+1))^{\delta_{u_1 \cdots u_j}(l)} \cdot (-h_{i_{\tilde{\tau}_l}(l)} d h_{i_{\tilde{\tau}_l}(l)}^{-1} i_{\tilde{\tau}_l}(k+1))^{\delta_{v_1}(l)} \right\}.$$  \tag{185}
In order to prove equality (173), it suffices to construct a bijection
\[ P_{\alpha, \beta, u_1, \ldots, u_j, v_1} : \{ \sigma \in S_{k+1} | \sigma(v_1) = \alpha \} \rightarrow \{ \tau \in S_{k+1} | \tau(v_1) = \beta - 1 \}, \]
(186)
such that
\[ sgn(\sigma) \cdot \Delta_{u_1 \ldots u_j, v_1}^\sigma (\tilde{t}_1 \cdots \tilde{t}_{\beta-1} i_{\alpha} \cdots i_{k+2}) = (191) \]
\[ (193) \]
\[ (-1)^{\alpha - \beta + 1} \cdot sgn(P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma)) \cdot \Delta_{u_1 \ldots u_j, v_1}^\sigma (\tilde{t}_1 \cdots \tilde{t}_{\beta-1} i_{\alpha} \cdots i_{k+2}). \]
(187)

For each \( \sigma \in S_{k+1} \) such that \( \sigma(v_1) = \alpha \), we define \( P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma) \) as follows. Take \( \tilde{\sigma} \) as in (175) and define \( \eta \in S_{k+1}^\alpha \) by
\[ \eta(l) = \begin{cases} \tilde{\sigma}(l) & \text{if } 1 \leq l \leq v_1 - 1, \\ \beta & \text{if } l = v_1, \\ \tilde{\sigma}(l) & \text{if } v_1 + 1 \leq l \leq k + 1; \end{cases} \]
(188)

let \( P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma) \) be the inverse image of \( \eta \) under the bijection \( \bar{\cdot} \), that is, \( P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma) = \eta \). It is easy to verify that \( P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma) \in \{ \tau \in S_{k+1} | \tau(v_1) = \beta - 1 \} \).

Then, by (184) and (185), we have
\[ \Delta_{u_1 \ldots u_j, v_1}^\sigma (\tilde{t}_1 \cdots \tilde{t}_{\beta-1} i_{\alpha} \cdots i_{k+2}) = \text{tr} \left\{ \prod_{l=1}^k (g_{1\Theta(l)\Theta(l+1)} d g_{1\Theta(l)\Theta(l+1)})^{1 - h_{\Theta-1} - \Theta(l)} h_{\Theta(l)} \right\} \]
(189)
\[ - h_{1\Theta(l+1)} d h_{1\Theta(l+1)}^{1 - h_{\Theta-1} - \Theta(l)} g_{1\Theta(l)\Theta(l+1)}^{1 - h_{\Theta-1} - \Theta(l)} g_{1\Theta(l)\Theta(l+1)}^{1 - h_{\Theta-1} - \Theta(l)} \]
\[ = \Delta_{\tilde{u}_1 \cdots \tilde{u}_j, \tilde{v}_1}^\sigma (\tilde{t}_1 \cdots \tilde{t}_{\beta-1} i_{\alpha} \cdots i_{k+2}). \]

To complete the proof of Claim IV in this case, it suffices to prove the following equality,
\[ sgn(P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma)) = (-1)^{\alpha - \beta + 1} \cdot sgn(\sigma). \]
(190)

Since
\[ sgn(P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma)) = (-1)^{k+2-\alpha} \cdot sgn([P_{\alpha, \beta, u_1, \ldots, u_j, v_1}(\sigma)])_{\alpha} \]
(191)
and
\[ (-1)^{\alpha - \beta + 1} \cdot sgn(\sigma) = (-1)^{\alpha - \beta + 1} \cdot (-1)^{k+2-\beta} \cdot sgn([\sigma])_{\beta}, \]
(192)
it suffices to prove that
\[ sgn([\eta])_{\alpha} = - sgn([\sigma])_{\beta}. \]
(193)

Since \([\eta]_{\alpha} = \iota \circ [\sigma]_{\beta} \in S_{k+2} \) where \( \iota \in S_{k+2} \) is the permutation \((\alpha, \beta)\), equality (193) holds.
Therefore, \( \kappa_1^\alpha = (-1)^{\alpha - \beta + 1} \cdot \kappa_1^\beta \).

**Case II** \( (\kappa_2^\alpha = (-1)^{\alpha - \beta + 1} \cdot \kappa_2^\beta) \). Similarly, it suffices to prove that for fixed integer \( v_1 \) such that \( 1 \leq v_1 \leq k \) the following equality holds:

\[
\sum_{\sigma \in S_{k+1}, \sigma(v_1) = \alpha} \text{sgn}(\sigma) \cdot \Delta^{\alpha}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}) = (-1)^{\alpha - \beta + 1} \cdot \sum_{\tau \in S_{k+1}, \tau(v_1) = \beta - 1} \text{sgn}(\tau) \cdot \Delta^{\beta}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}).
\]

(194)

Define \( \tilde{\sigma}, [\tilde{\sigma}]_\beta, \bar{\tau} \) and \([\bar{\tau}]_\alpha\) in the same way as (175), (177), (180) and (182). Rewriting (184) under the above notation, we have

\[
\Delta^{\alpha}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}) = \text{tr} \left\{ \prod_{l=1}^{k} (g_{\eta(l)}^{-1} d g_{\eta(l)}(h_{\eta(l)})^{-1})^{1-\delta_{v_1}(l)} \cdot (h_{\eta(l)}^{-1} d h_{\eta(l)})^{\delta_{v_1}(l)} \right\}.
\]

(195)

and

\[
\Delta^{\beta}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}) = \text{tr} \left\{ \prod_{l=1}^{k} (g_{\eta(l)}^{-1} d g_{\eta(l)}(h_{\eta(l)})^{-1})^{1-\delta_{v_1}(l)} \cdot (h_{\eta(l)}^{-1} d h_{\eta(l)})^{\delta_{v_1}(l)} \right\}.
\]

(196)

Notice that equality (194) holds if there exists a bijection

\[ P_{\alpha,\beta,v_1} : \{ \sigma \in S_{k+1} | \sigma(v_1) = \alpha \} \to \{ \tau \in S_{k+1} | \tau(v_1) = \beta - 1 \}, \]

such that

\[ \text{sgn}(\sigma) \cdot \Delta^{\alpha}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}) = (-1)^{\alpha - \beta + 1} \cdot \text{sgn}(P_{\alpha,\beta,v_1}(\sigma)) \cdot \Delta^{\beta}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}). \]

(198)

For each \( \sigma \in S_{k+1} \) such that \( \sigma(v_1) = \alpha \), we define an element \( P_{\alpha,\beta,v_1}(\sigma) \in S_{k+1} \) as follows. Take \( \tilde{\sigma} \) as in (175) and define \( \eta \in S_{k+1}^\alpha \) by

\[
\eta(l) = \begin{cases} \tilde{\sigma}(l) & \text{if } 1 \leq l \leq v_1 - 1, \\ \beta & \text{if } l = v_1, \\ \tilde{\sigma}(l) & \text{if } v_1 + 1 \leq l \leq k + 1; \end{cases}
\]

(199)

let \( P_{\alpha,\beta,v_1}(\sigma) \) be the inverse image of \( \eta \) under the map \( \bar{\sigma} \), that is, \( \bar{P}_{\alpha,\beta,v_1}(\sigma) = \eta \). It is easy to verify that \( P_{\alpha,\beta,v_1}(\sigma) \in \{ \tau \in S_{k+1} | \tau(v_1) = \beta - 1 \} \). Then, by (184) and (185), we have that

\[
\Delta^{P_{\alpha,\beta,v_1}(\sigma)}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}) = \text{tr} \left\{ \prod_{l=1}^{k} (g_{\eta(l)}^{-1} d g_{\eta(l)}(h_{\eta(l)})^{-1})^{1-\delta_{v_1}(l)} \cdot (h_{\eta(l)}^{-1} d h_{\eta(l)})^{\delta_{v_1}(l)} \right\}
\]

(200)

\[
= \Delta^{\sigma}_{\vartheta,v_1}(\tilde{t}_1 \ldots \tilde{t}_\beta \ldots \tilde{t}_{k+2}).
\]
Notice that
\[
\text{sgn}(P_{\alpha,\beta,v_1}(\sigma)) = (-1)^{k+2-\alpha} \cdot \text{sgn}([P_{\alpha,\beta,v_1}(\sigma)]_{\alpha}) = (-1)^{k+2-\alpha} \cdot \text{sgn}([\eta]_{\alpha}).
\]
(201)
\[
(-1)^{\alpha-\beta+1} \cdot \text{sgn}(\sigma) = (-1)^{\alpha-\beta+1} \cdot (-1)^{k+2-\beta} \cdot \text{sgn}([\widehat{\sigma}]_{\beta}).
\]
(202)
Since $[\eta]_{\alpha} = \iota \circ [\widehat{\sigma}]_{\beta} \in S_{k+2}$ where $\iota \in S_{k+2}$ is the permutation $(\alpha, \beta), \text{sgn}([\sigma]_{\beta}) = -\text{sgn}([\eta]_{\alpha})$.
Then, $\kappa_2^\alpha = (-1)^{\alpha-\beta+1} \cdot \kappa_3^\beta$.

**Case III ($\kappa_2^\beta = (-1)^{\alpha-\beta+1} \cdot \kappa_3^\beta$).** Similarly we will show that for fixed integers $j, u_1, \ldots, u_j$ such that $1 \leq j \leq k$ and $1 \leq u_1 < \cdots < u_j \leq k$, the following equality holds:
\[
\sum_{\sigma \in S_{k+1}, \sigma(k+1)=\alpha} \text{sgn}(\sigma) \cdot \Delta^\sigma_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}) = (-1)^{\alpha-\beta+1}
\]
\[
\cdot \sum_{\tau \in S_{k+1}, \tau(k+1)=\beta-1} \text{sgn}(\tau) \cdot \Delta^\tau_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}).
\]
(203)
Define $\widehat{\sigma}$, $[\widehat{\sigma}]_{\beta}$, $\overline{\sigma}$ and $[\overline{\sigma}]_{\alpha}$ in the same way as (175), (177), (180) and (182). Rewriting (159) under the new notation, we have
\[
\Delta^\widehat{\sigma}_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}) = \text{tr} \left\{ \prod_{l=1}^{k} (g_{\overline{t}_{i_l}(i_l \overline{t}_{i_l})}^{-1} d\overline{g}_{\overline{t}_{i_l}(i_l \overline{t}_{i_l})})^{1-\delta_{u_1 \ldots u_j}} \right\},
\]
(204)
\[
\Delta^\overline{\sigma}_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}) = \text{tr} \left\{ \prod_{l=1}^{k} (g_{\overline{t}_{i_l}(i_l \overline{t}_{i_l})}^{-1} d\overline{g}_{\overline{t}_{i_l}(i_l \overline{t}_{i_l})})^{1-\delta_{u_1 \ldots u_j}} \right\}.
\]
(205)
Notice that equality (203) holds if there exists a bijection
\[
P_{\alpha,\beta} : \{ \sigma \in S_{k+1} \mid \sigma(k+1) = \alpha \} \rightarrow \{ \tau \in S_{k+1} \mid \tau(k+1) = \beta-1 \},
\]
(206)
such that
\[
\text{sgn}(\sigma) \cdot \Delta^\sigma_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}) = (-1)^{\alpha-\beta+1} \cdot \text{sgn}(P_{\alpha,\beta}(\sigma)) \cdot \Delta^{P_{\alpha,\beta}(\sigma)}_{(u_1 \ldots u_j; \emptyset)}(\overline{t}_{i_1 \ldots \hat{i}_j \cdots i_{k+2}}).
\]
(207)
For each $\sigma \in S_{k+1}$ such that $\sigma(k+1) = \alpha$, we define an element $P_{\alpha,\beta}(\sigma) \in S_{k+1}$ as follows. Take $\widehat{\sigma}$ as in (175) and define $\eta \in S_{k+1}^{\alpha}$ by
\[
\eta(l) = \begin{cases} 
\widehat{\sigma}(l) & \text{if } 1 \leq l \leq k, \\
\beta & \text{if } l = k+1;
\end{cases}
\]
(208)
let $P_{\alpha, \beta}(\sigma)$ be the inverse image of $\eta$ under the map $\overline{\ast} \cdot \sigma$, that is, $\overline{P_{\alpha, \beta}(\sigma)} = \eta$. It is easy to verify that $P_{\alpha, \beta}(\sigma) \in \{ \tau \in S_{k+1} | \tau(k + 1) = \beta - 1 \}$. Then, by (159), we have that

$$\Delta_{P_{\alpha, \beta}(\sigma)(u_1 \ldots u_j; \emptyset)}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}) = tr \{ \prod_{l=1}^{k} (g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} d g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})})^{1-\delta_{u_1 \ldots u_j(l)}} \cdot (g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} h_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} \delta_{u_1 - u_j(l)}) \} = tr \{ \prod_{l=1}^{k} (g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} d g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})})^{1-\delta_{u_1 \ldots u_j(l)}} \cdot (g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} h_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} g_{ij(\tilde{t}_a \ldots \tilde{t}_{k+1})}^{-1} \delta_{u_1 - u_j(l)}) \} = \Delta_{\sigma(u_1 \ldots u_j; \emptyset)}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}).$$

Since

$$sgn(P_{\alpha, \beta}(\sigma)) = (-1)^{k+2-\alpha} \cdot sgn([\overline{P_{\alpha, \beta}(\sigma)}]_{\sigma}) = (-1)^{k+2-\alpha} \cdot sgn([\eta]_{\alpha})$$

and

$$(-1)^{\alpha-\beta+1} \cdot sgn(\sigma) = (-1)^{\alpha-\beta+1} \cdot (-1)^{k+2-\beta} \cdot sgn([\hat{\sigma}]_{\beta}),$$

it suffices to prove that

$$sgn([\eta]_{\alpha}) = -sgn([\hat{\sigma}]_{\beta}).$$

Noticing that $[\eta]_{\alpha} = \iota \circ [\hat{\sigma}]_{\beta} \in S_{k+2}$ where $\iota \in S_{k+2}$ is the permutation $(\alpha, \beta)$, (212) holds.

Therefore, $\kappa^a_{3} = (-1)^{\alpha-\beta+1} \cdot \kappa^b_{3}$.

**Case IV** ($\kappa^a_{4} = (-1)^{\alpha-\beta+1} \cdot \kappa^b_{4}$). Finally we will prove that for fixed integer $j$ such that $1 \leq j \leq k - 1$, and distinct integers $u_1, \ldots, u_j, x_1 \in \{ 1, \ldots, k \}$, the following equality holds:

$$\sum_{\sigma \in S_{k+1}, \sigma(x_1) = \alpha} sgn(\sigma) \cdot \Delta_{\sigma(u_1 \ldots u_j; x_1)}^{\tau}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}) = (-1)^{\alpha-\beta+1} \cdot \sum_{\tau \in S_{k+1}, \tau(x_1) = \beta-1} sgn(\tau) \cdot \Delta_{\tau(u_1 \ldots u_j; x_1)}^{\tau}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}).$$

It suffices to prove that there exists a bijection

$$P_{\alpha, \beta, u_1 \ldots u_j, x_1} : \{ \sigma \in S_{k+1} \mid \sigma(x_1) = \alpha \} \rightarrow \{ \tau \in S_{k+1} \mid \tau(x_1) = \beta - 1 \},$$

such that

$$sgn(\sigma) \cdot \Delta_{\sigma(u_1 \ldots u_j; x_1)}^{\tau}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}) = (-1)^{\alpha-\beta+1} \cdot sgn(P_{\alpha, \beta, u_1 \ldots u_j, x_1}(\sigma)) \cdot \Delta_{P_{\alpha, \beta, u_1 \ldots u_j, x_1}(\sigma)}^{\tau}(\tilde{t}_1 \ldots \tilde{t}_a \ldots \tilde{t}_{k+2}).$$
For each $\sigma \in S_{k+1}$ such that $\sigma(x_1) = \alpha$, we define an element $P_{\alpha, \beta, u_1 \ldots u_j, x_1}(\sigma) \in S_{k+1}$ as follows. Take $\widehat{\sigma}$ as in (175) and define $\eta \in S_{k+1}$ by

$$\eta(l) = \begin{cases} \widehat{\sigma}(l) & \text{if } 1 \leq l \leq x_1 - 1, \\ \beta & \text{if } l = x_1, \\ \widehat{\sigma}(l) & \text{if } x_1 + 1 \leq l \leq k + 1; \end{cases} \quad (216)$$

let $P_{\alpha, \beta, u_1 \ldots u_j, x_1}$ be the inverse image of $\eta$ under the map $\bullet$, that is, $P_{\alpha, \beta, u_1 \ldots u_j, x_1} = \eta$. It is easy to verify that $P_{\alpha, \beta, u_1 \ldots u_j, x_1} \in \{ \tau \in S_{k+1} \mid \tau(x_1) = \beta - 1 \}$. Then by (159), we have5

$$\Delta_{(u_1 \ldots u_j; x_1)}(\widehat{i}_1 \ldots \widehat{i}_\beta \ldots \widehat{i}_{k+2}) = \text{tr} \left\{ \prod_{l=1}^{k} \left( g^{-1}_{l\eta(l)\eta(u_1)}) d g_{l\eta(l)\eta(u_1)} \right)^{-(\delta_{x_1}(l) - \delta_{u_1 \ldots u_j}(l))} \cdot \left( -g_{l\eta(u_1)} d g_{l\eta(u_1)} \right)^{\delta_{x_1}(l)} \cdot \left( g^{-1}_{l\eta(l)\eta(u_1)} d g_{l\eta(l)\eta(u_1)} \right)^{-\delta_{u_1 \ldots u_j}(l)} \right\}, \quad (217)$$

and

$$\Delta_{(u_1 \ldots u_j; x_1)}(\widehat{i}_1 \ldots \widehat{i}_\alpha \ldots \widehat{i}_{k+2}) = \text{tr} \left\{ \prod_{l=1}^{k} \left( g^{-1}_{l\eta(l)\eta(u_1)} d g_{l\eta(l)\eta(u_1)} \right)^{-(\delta_{x_1}(l) - \delta_{u_1 \ldots u_j}(l))} \cdot \left( -g_{l\eta(u_1)} d g_{l\eta(u_1)} \right)^{\delta_{x_1}(l)} \cdot \left( g^{-1}_{l\eta(l)\eta(u_1)} d g_{l\eta(l)\eta(u_1)} \right)^{-\delta_{u_1 \ldots u_j}(l)} \right\}. \quad (218)$$

Hence,

$$\Delta_{(u_1 \ldots u_j; x_1)}(\widehat{i}_1 \ldots \widehat{i}_\sigma \ldots \widehat{i}_{k+2}) = \Delta_{(u_1 \ldots u_j; x_1)}(\widehat{i}_1 \ldots \widehat{i}_\alpha \ldots \widehat{i}_{k+2}). \quad (219)$$

Next we are going to prove that

$$\text{sgn}(P_{\alpha, \beta, u_1 \ldots u_j, x_1}(\sigma)) = (-1)^{\alpha - \beta + 1} \cdot \text{sgn}(\sigma). \quad (220)$$

Similarly, it suffices to prove that

$$\text{sgn}([\eta]_{\alpha}) = -\text{sgn}([\sigma]_{\beta}). \quad (221)$$

Since $[\eta]_{\alpha} = \iota \circ [\widehat{\sigma}]_{\beta} \in S_{k+2}$ where $\iota \in S_{k+2}$ is the permutation $(\alpha, \beta)$, (221) holds.

Therefore, $\kappa_{4}^{\alpha} = (-1)^{\alpha - \beta + 1} \cdot \kappa_{4}^{\beta}$.

We complete the proof of Claim IV.

By Claim III, for integer $k \geq 1$ and elements $j_1, \ldots, j_k \in I$, we can define

$$s_{j_1 \ldots j_k} := s_{\widehat{j}_1 \ldots \widehat{j}_k; \beta; j_1 \ldots j_k}, \quad (222)$$

where $\beta$ is any element in $I$. Noticing that only the transition functions $g_{\gamma \delta}$ where $\gamma, \delta \in \{ j_1, \ldots, j_k \}$ appearing in (168), $s_{j_1 \ldots j_k} \in \Gamma(U_{j_1 \ldots j_k}, \Omega^k)$. Thus, we can define a Čech $(k - 1)$-cochain $h_{k-1}(E, g, \widehat{g})$ by

$$h_{k-1}(E, g, \widehat{g}) := \bigoplus_{j_1 < \cdots < j_k} s_{j_1 \ldots j_k} \in \bigoplus_{j_1 < \cdots < j_k} \Gamma(U_{j_1 \ldots j_k}, \Omega^k). \quad (223)$$
Similarly, we can extend the components of $h_{\xi-1}(E, g, \tilde{g})$ to all $k$-tuples of elements in $I$. Then in the same way as Lemma 2.2, we can show that (168) and (222) are compatible with this extension.

Next, we prove that $\partial h_{\xi-1}(E, g, \tilde{g})$ is cohomologous to $\hat{f}_k(E, g)$ by $h_{\xi-1}(E, g, \tilde{g})$.

**Claim V.** $\partial h_{\xi-1}(E, g, \tilde{g}) = \hat{f}_k(E, \tilde{g}) - \hat{f}_k(E, g)$. That is, for any elements $i_1, \ldots, i_{k+1} \in I$,

$$
\tilde{t}_{i_1 \ldots i_{k+1}} - t_{i_1 \ldots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^j s_{i_1 \ldots \hat{i}_j \ldots i_{k+1}} \bigg| _{U_{i_1 \ldots i_{k+1}}} .
$$

(224)

**Proof of Claim V.** Recall that

$$
\sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \cdot \Delta_{\emptyset, \emptyset}^{\sigma}(\tilde{t}_{i_1 \ldots i_{k+1}}) = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} .
$$

(225)

Hence by Claims I and II, and formulas (155) and (225), we have that

$$
\tilde{t}_{i_1 \ldots i_{k+1}} - t_{i_1 \ldots i_{k+1}} = \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k-1; 1 \leq u_1 \leq \ldots \leq u_j \leq k; u_1, \ldots, u_j, v_1 \text{ are distinct}} \Delta_{u_1 \ldots u_j, v_1}^{\sigma} (\tilde{t}_{i_1 \ldots i_{k+1}})
$$

$$
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k; 1 \leq u_1 < \ldots < u_j \leq k} \Delta_{u_1 \ldots u_j, \emptyset}^{\sigma} (\tilde{t}_{i_1 \ldots i_{k+1}})
$$

$$
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k; 1 \leq u_1 < \ldots < u_j \leq k} \Delta_{\emptyset, v_1}^{\sigma} (\tilde{t}_{i_1 \ldots i_{k+1}})
$$

$$
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k; 1 \leq u_1 < u_2 < \ldots < u_j \leq k} \Delta_{(u_1 \ldots u_j; x)}^{\sigma} (\tilde{t}_{i_1 \ldots i_{k+1}})
$$

$$
+ \sum_{\sigma \in S_{k+1}} \frac{sgn(\sigma)}{(k+1)!} \sum_{1 \leq j \leq k-1; 1 \leq u_1 < u_2 < \ldots < u_j \leq k; 1 \leq x_1 \leq k, x_1 \notin \{ u_1, \ldots, u_j \} } \Delta_{(u_1 \ldots u_j; x_1)}^{\sigma} (\tilde{t}_{i_1 \ldots i_{k+1}}).
$$

(226)

Notice that for each fixed $\xi$ such that $1 \leq \xi \leq k+1$, $S_{k+1}$ is the disjoint union of the following sets,

$$
\{ \sigma \in S_{k+1} | \sigma(\xi) = \alpha \} \text{ where } \alpha = 1, \ldots, k+1.
$$

(227)
Then, by interchanging the order of summation in (225), we have

\[
\tilde{t}_{i_1 \cdots i_{k+1}} - t_{i_1 \cdots i_{k+1}} = \sum_{\alpha=1}^{k+1} \frac{1}{(k+1)!} \sum_{1 \leq j \leq k-1; \atop 1 \leq u_1 < \cdots < u_j \leq k; \atop 1 \leq v_1 \leq k; \atop u_1, \ldots, u_j, v_1 \text{ are distinct}} \text{sgn}(\sigma) \cdot \Delta^\sigma_{u_1 \cdots u_j, v_1} \left( \tilde{t}_{i_1 \cdots i_{k+1}} \right)
\]

\[
+ \sum_{\alpha=1}^{k+1} \frac{1}{(k+1)!} \sum_{j=0; \atop 1 \leq v_1 \leq k; \atop \sigma(v_1) = \alpha} \text{sgn}(\sigma) \cdot \Delta^\sigma_{\emptyset, v_1} \left( \tilde{t}_{i_1 \cdots i_{k+1}} \right)
\]

\[
+ \sum_{\alpha=1}^{k+1} \frac{1}{(k+1)!} \sum_{1 \leq j \leq k; \atop 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \atop \sigma(k+1) = \alpha} \text{sgn}(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j); \emptyset} \left( \tilde{t}_{i_1 \cdots i_{k+1}} \right)
\]

\[
+ \sum_{\alpha=1}^{k+1} \frac{1}{(k+1)!} \sum_{1 \leq j \leq k-1; \atop 1 \leq u_1 < u_2 < \cdots < u_j \leq k; \atop \sigma(x_1) = \alpha \atop 1 \leq x_1 \leq k, x_1 \neq [u_1, \ldots, u_j]} \text{sgn}(\sigma) \cdot \Delta^\sigma_{(u_1 \cdots u_j); x_1} \left( \tilde{t}_{i_1 \cdots i_{k+1}} \right).
\]

(228)

Recalling (168), we conclude that

\[
\tilde{t}_{i_1 \cdots i_{k+1}} - t_{i_1 \cdots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha-1} s_{i_1 \cdots i_{\alpha-1} \cdots i_{k+1}} \left| U_{i_1 \cdots i_{k+1}} \right|.
\]

(229)

We complete the proof of Claim V. \( \square \)

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