ON GRADED BIALGEBRA DEFORMATIONS

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ABSTRACT. We introduce the graded bialgebra deformations, which explain Andruskiewitsch-Schneider’s liftings method. We also relate this graded bialgebra deformation with the corresponding graded bialgebra cohomology groups, which is the graded version of the one due to Gerstenhaber-Schack.

1. Introduction

The classification of finite-dimensional pointed Hopf algebras is a basic problem in the theory of Hopf algebras. It is well-known that any pointed Hopf algebra $H$ has a coradical filtration, with respect to which one associates a coradically-graded Hopf algebra $\text{gr}H$. Following Andruskiewitsch and Schneider, the classification problem can be divided into two parts. One is the classification of all coradically-graded pointed Hopf algebras. The other is to find all possible pointed Hopf algebras $H$ with $\text{gr}H$ isomorphic to a given coradically-graded pointed Hopf algebra. The second part is just the lifting method in [1] and [2]. One of our motivations is to relate the lifting method with certain bialgebra deformation theory.

The deformation theory for algebras is initiated by Gerstenhaber in [4], and its analogue for bialgebras appeared first in [5] (also see [6] and [10]). Inspired by the graded algebra deformation theory in [11] and [8], we develop in this paper the theory of graded bialgebra deformations and their corresponding cohomology groups. Moreover this deformation theory can be used to explain Andruskiewitsch-Schneider’s lifting method.

Key words and phrases. Graded Bialgebras, Liftings, Deformations.

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The paper is organized as follows. In section 2, first we recall the notion of liftings and introduce the graded bialgebra deformations, and we show that the lifting is just the same as the graded bialgebra deformation in the sense of Theorem 2.2. The graded-rigid bialgebras are also studied, see Corollary 2.3 and Corollary 2.4. In section 3, we introduce the notion of graded “hat” bialgebra cohomology groups for graded bialgebras, which controls the graded bialgebra deformations, see Theorem 3.3.

2. LIFTINGS AND GRADED BIALGEBRA DEFORMATIONS

We will work on a base field $\mathbb{K}$. All unadorned tensors are over $\mathbb{K}$. We refer the notion of graded bialgebras and filtered bialgebras to [13], the notion of graded linear maps to [9] and [7].

2.1. Let us recall Andruskiewitsch-Schneider’s liftings method, for more details, see [2]. Note that the lifting defined here is a slight generalization.

Throughout, $B = \oplus_{i \geq 0} B_{(i)}$ will be a graded bialgebra over $\mathbb{K}$, with identity element $1_B$, multiplication map $m$, counit $\varepsilon$, and comultiplication $\Delta$. Then $B$ has a natural bialgebra filtration

$$B_0 \subseteq B_1 \subseteq \cdots \subseteq B_i \subseteq \cdots,$$

where $B_i = \oplus_{j \leq i} B_{(j)}$ for any $n \geq 0$.

A lifting of the graded bialgebra $B$ is a filtered bialgebra structure, denoted by $\hat{U}$, on the underlying filtered vector space $B$ with the above filtration such that

$$\text{gr} \hat{U} = B$$

as graded bialgebras, where $\text{gr} \hat{U}$ is the graded bialgebra associated to the filtered bialgebra $\hat{U}$ ([13], p.226). (By $\text{gr} \hat{U} = B$, we use the natural identification of the underlying space $\text{gr} \hat{U}$ with $B$, that is $\text{gr} \hat{U}_{(i)} = B_i / B_{i-1} \cong B_{(i)}$ for each $i \geq 0$.)

For any lifting $U$ of the graded bialgebra $B$, it follows from the definition that $U$ and $B$ have the same identity element and the counit. Therefore, to give a lifting $U$, we just need to define the multiplication $m_U$ and comultiplication $\Delta_U$.

Two liftings $U, V$ of the graded bialgebra $B$ are said to be equivalent, if there is filtered bialgebra isomorphism $\theta : U \rightarrow V$ such that $\text{gr} \theta = \text{Id}_B$, where $\text{gr} \theta$ is the graded morphism associated to $\theta$, and here again we use the identifications $\text{gr} U = B$ and $\text{gr} V = B$ (as graded bialgebras).

Denote by

$$\text{Lift}(B)$$

the set of equivalent classes of all the liftings of the graded bialgebra $B$. 
2.2. In this subsection, we will study graded bialgebra deformations of the graded bialgebra $B = \oplus_{i \geq 0} B(i)$.

Let $l \in \mathbb{N} \cup \{+\infty\}$. Consider the space $B[t]/(t^{l+1})$, which is viewed as a free module over $\mathbb{K}[t]/(t^{l+1})$, and also a graded $\mathbb{K}$-space with $\deg t = 1$ and $\deg b = n$, if $b \in B(n)$. If $l = +\infty$, then $B[t]/(t^{l+1})$ means $B[t]$ and $\mathbb{K}[t]/(t^{l+1})$ means $\mathbb{K}[t]$.

An $l$-th level graded bialgebra deformation of $B$ consists of

$$m^l_i : (B \otimes B)[t]/(t^{l+1}) \rightarrow B[t]/(t^{l+1})$$

and

$$\Delta^l_i : B[t]/(t^{l+1}) \rightarrow (B \otimes B)[t]/(t^{l+1}) \cong B[t]/(t^{l+1}) \otimes_{\mathbb{K}[t]/(t^{l+1})} B[t]/(t^{l+1})$$

which are $\mathbb{K}[t]/(t^{l+1})$-linear and homogeneous maps of degree zero such that

(i) $B[t]/(t^{l+1})$ is a bialgebra over $\mathbb{K}[t]/(t^{l+1})$ with identity element $1_B$, multiplication $m^l_i$, counit $\varepsilon^l_i$ and comultiplication $\Delta^l_i$, where the counit $\varepsilon^l_i : B[t]/(t^{l+1}) \rightarrow \mathbb{K}[t]/(t^{l+1})$ is given by $\varepsilon^l_i(bt^j) = \varepsilon(b)t^j$, $b \in B$, $0 \leq j \leq l$;

(ii) $m^l_i \equiv m \otimes \text{Id}_{\mathbb{K}[t]/(t^{l+1})}$ and $\Delta^l_i \equiv \Delta \otimes \text{Id}_{\mathbb{K}[t]/(t^{l+1})} \mod(t)$, where $m$ and $\Delta$ are the multiplication and comultiplication of $B$, respectively.

Denote by $(B[t]/(t^{l+1}), m^l_i, \Delta^l_i)$ the above $l$-th level graded bialgebra deformation.

From now on, we will abbreviate $l$-th level graded bialgebra deformations as $l$-deformations, and $+\infty$-deformations will be referred simply as deformations. Denote by $\mathcal{E}^l(B)$ the set of all $l$-deformations of the graded bialgebra $B$, and $\mathcal{E}^{+\infty}(B)$ is written as $\mathcal{E}(B)$. Elements of $\mathcal{E}(B)$ will be written as $(B[t], m_t, \Delta_t)$.

Two $l$-deformations $(B[t]/(t^{l+1}), m^l_i, \Delta^l_i)$ and $(B[t]/(t^{l+1}), m'^l_i, \Delta'^l_i)$ are said to be isomorphic, if there exists an isomorphism of $\mathbb{K}[t]/(t^{l+1})$-bialgebras

$$\phi : (B[t]/(t^{l+1}), m^l_i, \Delta^l_i) \rightarrow (B[t]/(t^{l+1}), m'^l_i, \Delta'^l_i)$$

such that $\phi$ is homogeneous of degree zero and

$$\phi \equiv \text{Id}_B \otimes \text{Id}_{\mathbb{K}[t]/(t^{l+1})} \mod(t).$$

Denote by

$$iso\mathcal{E}^l(B) \; (\text{resp.} \; iso\mathcal{E}(B))$$

the set of isoclasses of $l$-deformations (resp. deformations) of the graded bialgebra $B$, for $l \in \mathbb{N}$. 
2.3. Use the notation as above. Consider an element \((B[t]/(t^{l+1}), m'_l, \Delta'_l)\) of \(\mathcal{E}^l(B)\). By definition, we can write

\[
m'_l(a \otimes b) = \sum_{0 \leq s \leq l} m_s(a \otimes b)t^s,
\]

and

\[
\Delta'_l(c) = \sum_{0 \leq s \leq l} \Delta_s(c)t^s,
\]

where \(a, b, c \in B\), and \(m_s : B \otimes B \rightarrow B\) and \(\Delta_s : B \rightarrow B \otimes B\) are homogeneous of degree \(-s\). Note that \(m_0 = m\) and \(\Delta_0 = \Delta\).

It is easy to check that the associativity of \(m'_l\), the compatibility of \(m'_l\) and \(\Delta'_l\), and the coassociativity of \(\Delta'_l\) are equivalent to the following identities, respectively, for each \(1 \leq n \leq l\),

\[
\begin{align*}
\text{(2.3)} & \quad am_n(b \otimes c) - m_n(ab \otimes c) + m_n(a \otimes bc) - m_n(a \otimes b)c = \sum_{1 \leq s \leq n-1} m_s(m_{n-s}(a \otimes b) \otimes c) - m_s(a \otimes m_{n-s}(b \otimes c)), \\
\text{(2.4)} & \quad m_n(a_1 \otimes b_1(1) \otimes a_2 b_2(2) - \Delta(m_n(a \otimes b)) + a_1 b_1(1) \otimes m_n(a_2 \otimes b_2(2)) \\
& \quad \quad + a_1(1) b_l \otimes a_2(2) b_r - \Delta_n(ab) + a_1 b_1(1) \otimes a_r b(2) \\
& \quad = -\sum_{0 \leq s, r, s', r' \leq n-1, s+s'+r+r'=n} (m_r \otimes m_{r'}) \circ \tau_{23} \circ (\Delta_s \otimes \Delta_{s'})(a \otimes b) \\
& \quad \quad + \sum_{1 \leq s \leq n-1} \Delta_s(m_{n-s}(a \otimes b)),
\end{align*}
\]

and

\[
\begin{align*}
\text{(2.5)} & \quad c_1(1) \otimes \Delta_n(c_2(2)) - (\Delta \otimes \text{Id}) \circ \Delta_n(c) + (\text{Id} \otimes \Delta) \circ \Delta_n(c) - \Delta_n(c_1(1)) \otimes c_2(2) \\
& \quad = \sum_{1 \leq s \leq n-1} (\Delta_{n-s} \otimes \text{Id}) \circ \Delta_s(c) - (\text{Id} \otimes \Delta_{n-s}) \circ \Delta_s(c),
\end{align*}
\]

where we use Sweedler’s notation \(\Delta(a) = a_1(1) \otimes a_{(2)}\), \(a \in B\), and in the second identity we use the notation \(\Delta_n(a) = a_l \otimes a_r\) and \(\Delta_n(b) = b_l \otimes b_r\), and the map \(\tau_{23}\) is the canonical flip map at the second and third positions.

Let \((B[t]/(t^{l+1}), m'_l, \Delta'_l)\) and \((B[t]/(t^{l+1}), m'_l, \Delta'_l)\) be two \(l\)-deformations with the maps \(m_s, \Delta_s\) and \(m'_s, \Delta'_s\) as in (2.1) and (2.2). An isomorphism \(\phi\) between these deformations is given by

\[
\phi(a) = \sum_{0 \leq s \leq l} \phi_s(a)t^s, \quad a \in B,
\]
where \( \phi_s : B \rightarrow B \) is a homogeneous map of degree \(-s\). Note that \( \phi_0 = \text{Id}_B \). The fact that \( \phi \) is a morphism of \( \mathbb{K}[t]/(t^{l+1})\)-bialgebras implies \( \phi \) preserves the identity element \( 1_B \) and the counit \( \varepsilon_t \), and it satisfies, for each \( 1 \leq n \leq l \),

\[
(2.7) \quad (m_n - m'_n)(a \otimes b) = a\phi_n(b) - \phi_n(ab) + \phi_n(a)b 
\]

\[
+ \sum_{0 < s < n} \{ \phi_s(a)\phi_{n-s}(b) - \phi_s(m_{n-s}(a \otimes b)) + \sum_{r + r' = n-s} m'_s(\phi_r(a) \otimes \phi_{r'}(b)) \} 
\]

and

\[
(2.8) \quad (\Delta_n - \Delta'_n)(c) = \Delta(\phi_n(c)) - c_{(1)} \otimes \phi_n(c_{(2)}) - \phi_n(c_{(1)}) \otimes c_{(2)} 
\]

\[
+ \sum_{0 < s < n} \{ \Delta'_s(\phi_{n-s}(c)) - (\phi_s \otimes \phi_{n-s})(\Delta(c)) - \sum_{r + r' = n-s} (\phi_r \otimes \phi_{r'})(\Delta_s(c)) \},
\]

for all \( a, b, c \in B \). Note that above discussion works for all \( l \in \mathbb{N} \cup \{+\infty\} \).

The analogue of the following lemma is well-known in classical deformation theory.

**Lemma 2.1.** There exist restriction maps \( r_{l',l} : \mathcal{E}^l(B) \rightarrow \mathcal{E}^{l'}(B) \) for every \( l > l' \in \mathbb{N} \), and maps \( r_l : \mathcal{E}(B) \rightarrow \mathcal{E}^l(B) \) such that

\[
\mathcal{E}(B) = \varprojlim_{l \in \mathbb{N}} \mathcal{E}^l(B).
\]

**Proof.** The restriction map \( r_{l',l} \) is given as follows: given \( (B[t]/(t^{l+1}), m'_l, \Delta'_l) \) in \( \mathcal{E}^{l'}(B) \) with the maps \( m_s \) and \( \Delta_s \) defined in (2.1) and (2.2), just define \( m'_l \) and \( \Delta'_s \) as follows: \( m'_l := \sum_{0 \leq s \leq l'} m_s t^s \) and \( \Delta'_s := \sum_{0 \leq s \leq l'} \Delta_s t^s \); it is direct to check that \( (B[t]/(t^{l+1}), m'_l, \Delta'_l) \) is the desired element in \( \mathcal{E}^{l'}(B) \). The map \( r_l \) is defined in a similar way, and then the result is obvious. \( \square \)

A graded bialgebra \( B = \oplus_{i \geq 0} B_{(i)} \) is called graded-rigid if the set \( \text{iso} \mathcal{E}(B) \) has only one element, i.e., any deformation of \( B \) is isomorphic to the trivial one.

2.4. We have the following observation, which says that the graded bialgebra deformations coincide with the liftings.

**Theorem 2.2.** Let \( B = \oplus_{i \geq 0} B_{(i)} \) be a graded bialgebra. There exists a natural bijection

\[
\text{Lift}(B) \simeq \text{iso} \mathcal{E}(B).
\]

**Proof.** We will construct a map \( F : \text{Lift}(B) \rightarrow \text{iso} \mathcal{E}(B) \). Given a lifting \( U \) of \( B \). Denote by \( m_U \) and \( \Delta_U \) the multiplication and comultiplication maps...
of $U$. Since $U$ is a filtered bialgebra, we have
\[ m_U : B_i \otimes B_j \longrightarrow B_{i+j} \quad \text{and} \quad \Delta_U : B_n \longrightarrow \sum_{i+j=n} B_i \otimes B_j. \]

Therefore, for any $s \geq 0$, there uniquely exist homogeneous maps of degree $-s$, say $m_s : B \otimes B \longrightarrow B$ and $\Delta_s : B \longrightarrow B \otimes B$, such that
\[ m_U(a \otimes b) = \sum_{s \geq 0} m_s(a \otimes b) \quad \text{and} \quad \Delta_U(c) = \sum_{s \geq 0} \Delta_s(c). \]

By gr$U = B$ as graded bialgebras, we have $m_0 = m$ and $\Delta_0 = \Delta$.

Now Define $F(U) = (B[t], m_t, \Delta_t)$ as follows
\[ m_t(a \otimes b) := \sum_{s \geq 0} m_s(a \otimes b)t^s \quad \text{and} \quad \Delta_t(c) := \sum_{s \geq 0} \Delta_s(c)t^s. \]

It is direct to check that $F(U)$ is a deformation. $F$ is well-defined, i.e., it maps equivalent liftings to isomorphic deformations. In fact, for given liftings $U$ and $V$, an equivalence $\theta$ of $U$ and $V$ is a filtered isomorphism, hence for any $s \geq 0$, there determines a unique homogeneous map $\phi_s : B \longrightarrow B$ of degree $-s$ such that
\[ \theta(a) = \sum_{s \geq 0} \phi_s(a), \quad a \in B. \]

Then define a $\mathbb{K}[t]$-linear map $\phi : B[t] \longrightarrow B[t]$ such that $\phi(a) = \sum_{s \geq 0} \phi_s(a)t^s$. Hence $\phi$ is an isomorphism between the deformations $F(U)$ and $F(V)$.

On the other hand, by seeing (2.1) and (2.2), one obtains that $F$ is a bijection. This completes the proof.

An immediate consequence of Theorem 2.2 is

**Corollary 2.3.** Let $B = \bigoplus_{i \geq 0} B_i$ be a graded bialgebra. Then $B$ is graded-rigid implies that, for any filtered bialgebra $U$ such that gr$U \simeq B$ as graded bialgebras, we have $U \simeq B$ as bialgebras.

If we assume that $B$ is coradically-graded, the converse is also true.

**Proof.** By Theorem 2.2, $B$ is graded-rigid if and only if Lift($B$) is a single element set, i.e., every lifting of $B$ is trivial.

For the first statement, such a filtered bialgebra $U$ with gr$U \simeq B$ gives rise to a lifting on $B$, denoted by $U'$, such that $U \simeq U'$ (as bialgebras). Since $B$ is graded-rigid, we get $U' \simeq B$, thus we are done.

For the second one, assume $B$ is coradically-graded. Let $U$ be a lifting of $B$. Thus by the assumption, there exists an isomorphism $\theta : U \simeq B$. Note that $\theta$ preserves the coradical filtration, thus gr$\theta$ can be viewed as a graded automorphism of $B$. Thus take $\theta' = (\text{gr}\theta)^{-1} \circ \theta : U \simeq B$. So $\theta'$ realizes an equivalence between the lifting $U$ and the trivial lifting. This proves that $B$ is graded-rigid.

\[ \blacksquare \]
2.5. In this subsection, we assume that the base field $\mathbb{K}$ is algebraically closed of characteristic zero. One can define the variety $\text{Bialg}_n$ of the bialgebra structures on $n$-dimensional spaces, which carries a natural $GL_n(\mathbb{K})$-action by base changes, see [12] and [8]. Recall that a bialgebra $B$ is called rigid if $GL_n(\mathbb{K})$-orbit of $\text{Bialg}_n$ containing $B$ is Zariski open. In fact, we have

**Corollary 2.4.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $B = \bigoplus_{i \geq 0} B_{(i)}$ a finite dimensional graded bialgebra over $\mathbb{K}$. If $B$ is rigid and coindcally-graded, then $B$ is graded-rigid in the sense of 2.3.

**Proof.** By Corollary 2.3, we only need to show that every filtered bialgebra $U$ with $\text{gr}U \simeq B$ is isomorphic to $B$. Assume the dimension of $B$ is $n$. By Theorem 3.4 in [8], $B$ is a degeneration of $U$, i.e., lies the closure of the orbit of $U$ (in the variety $\text{Bialg}_n$). However the $GL_n(\mathbb{K})$-orbit of $B$ is open, we obtain that $B$ and $U$ belong to the same $GL_n(\mathbb{K})$-orbit, i.e., $B \simeq U$ as bialgebras, finishing the proof. ■

3. Graded bialgebra cohomology

In this section we will relate the graded bialgebra deformations with corresponding cohomology groups, which will be a graded (and normalized) version of “hat” bialgebra cohomology groups introduced in [5] (also see [10]).

3.1. Let $(B, m, e, \Delta, \varepsilon)$ be a bialgebra. Again we will use Sweedler’s notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, $a \in B$.

Let us recall the bicomplex in [5] or [10], p.619. For this end, we need the following maps, where $p, q \geq 1$ and all $b$’s are in $B$, $\lambda^p : B^{\otimes p+1} \to B^{\otimes p}$ and $\rho^p : B^{\otimes p+1} \to B^{\otimes p}$ are given by

$$\lambda^p(b^1 \otimes \cdots \otimes b^{p+1}) = b^{(1)}_1 b^2 \otimes \cdots \otimes b^{(p)}_p b^{p+1},$$

$$\rho^p(b^1 \otimes \cdots \otimes b^{p+1}) = b^{(1)}_1 b^{p+1}_1 \otimes \cdots \otimes b^p b^{p+1}_p.$$  

Dually, the maps $\sigma^q : B^{\otimes q} \to B^{\otimes q+1}$ and $\tau^q : B^{\otimes q} \to B^{\otimes q+1}$ are given by

$$\sigma^q(b^1 \otimes \cdots \otimes b^q) = (b^{(1)}_1 \cdots b^{(1)}_q) \otimes b^{(2)}_1 \otimes \cdots \otimes b^{(2)}_q,$$

$$\tau^q(b^1 \otimes \cdots \otimes b^q) = b^{(1)}_1 \otimes \cdots \otimes b^{(1)}_q \otimes (b^{(2)}_1 \cdots b^{(2)}_q).$$

In addition, we need $\Delta^p_i : B^{\otimes p} \to B^{\otimes p+1}$ and $\mu^q_j : B^{\otimes q+1} \to B^{\otimes q}$, $1 \leq i \leq p$ and $1 \leq j \leq q$, which are given by

$$\Delta^p_i(b^1 \otimes \cdots \otimes b^p) = b^1 \otimes \cdots \otimes b^{i(1)}_1 \otimes b^{i(2)}_2 \otimes \cdots \otimes b^p,$$

$$\mu^q_j(b^1 \otimes \cdots \otimes b^{q+1}) = b^1 \otimes \cdots \otimes b^j b^{j+1} \otimes \cdots \otimes b^{q+1}.$$ 

Let $C^{p,q} = \text{Hom}_\mathbb{K}(B^{\otimes q}, B^{\otimes p})$, $p, q \geq 1$. Define

$$\delta^p_{\lambda,q} : C^{p,q} \to C^{p+1,q} \quad \text{and} \quad \delta^p_{\sigma,q} : C^{p,q} \to C^{p+1,q}$$

Using these maps, the graded cohomology can be defined.
which are given by
\[
\delta_h^{p,q}(f) = \lambda^p \circ (\text{Id} \otimes f) + \sum_{i=1}^{q} (-1)^i f \circ \mu_i^q + (-1)^{q+1} \rho^p \circ (f \otimes \text{Id})
\]
\[
\delta_c^{p,q}(f) = (\text{Id} \otimes f) \circ \sigma^q + \sum_{j=1}^{p} (-1)^j \Delta_j^p \circ f + (-1)^{p+1} (f \otimes \text{Id}) \circ \tau^q
\]
for \( f \in C^{p,q} \), where \text{Id} denotes the identity map of \( B \).

It is direct to check that \((C^{p,q}, \delta_h^{p,q}, \delta_c^{p,q})\) is a bicomplex (see [10], p.619), i.e.,
\[
\delta_h^{p+1,q} \circ \delta_h^{p,q} = 0, \quad \delta_c^{p+1,q} \circ \delta_c^{p,q} = 0.
\]

We will introduce a sub-bicomplex of the above bicomplex. Let \( m = \text{Ker} \varepsilon \). Denote by \( i : m \to B \) the inclusion map, and \( \pi : B \to m \) is given by \( \pi(b) = b - \varepsilon(b)1_B \), \( b \in B \). Set \( D^{p,q} = \text{Hom}_X(m^{\otimes q}, m^{\otimes p}) \), \( p, q \geq 1 \). Note that we have a natural embedding \( D^{p,q} \hookrightarrow C^{p,q} \) by identifying \( f \in D^{p,q} \) with \( i^\otimes p \circ f \circ \pi^\otimes q \in C^{p,q} \).

We have the following observation

**Lemma 3.1.** Use the above notation. Then \( \delta_h^{p,q}(D^{p,q}) \subseteq D^{p,q+1} \) and \( \delta_c^{p,q}(D^{p,q}) \subseteq D^{p+1,q} \).

**Proof.** Just note that \( f \in C^{p,q} \) lies in \( D^{p,q} \) if and only if
\[
(\text{Id}^{\otimes j-1} \otimes \varepsilon \otimes \text{Id}^{\otimes p-j}) \circ f = 0
\]
and
\[
f(b_1 \otimes \cdots \otimes b^{i-1} \otimes 1 \otimes b^{i+1} \otimes \cdots \otimes b^q) = 0,
\]
for any \( 1 \leq i \leq q, 1 \leq j \leq p \) and any \( b^i \in B \). Then the lemma follows from the definition of \( \delta_h^{p,q} \) and \( \delta_c^{p,q} \) immediately.

### 3.2

From now on \( B = \bigoplus_{i \geq 0} B_{(i)} \) will be a graded bialgebra. In this case \( m \subseteq B \) is a graded subspace. Consider \( D^{p,q}_{(l)} := \text{Hom}_X(m^{\otimes q}, m^{\otimes p}_{(l)}) \), \( l \in \mathbb{Z} \), whose elements are homogeneous maps from \( m^{\otimes q} \) to \( m^{\otimes p} \) of degree \( l \). Note that \( D^{p,q}_{(l)} \subseteq D^{p,q} \hookrightarrow C^{p,q} \). We have the following

**Lemma 3.2.** \( \delta_h^{p,q}(D^{p,q}_{(l)}) \subseteq D^{p,q+1}_{(l)} \) and \( \delta_c^{p,q}(D^{p,q}_{(l)}) \subseteq D^{p+1,q}_{(l)} \) for each \( l \in \mathbb{Z} \), \( p, q \geq 1 \).

**Proof.** Set \( C^{p,q}_{(l)} = \text{Hom}_X(B^{\otimes q}, B^{\otimes p}_{(l)}) \). Clearly, \( D^{p,q}_{(l)} = D^{p,q} \cap C^{p,q}_{(l)} \). From the definition of \( \delta_h^{p,q} \) and \( \delta_c^{p,q} \), one sees that they preserve the degrees, i.e., \( \delta_h^{p,q}(C^{p,q}_{(l)}) \subseteq C^{p,q+1}_{(l)} \) and \( \delta_c^{p,q}(C^{p,q}_{(l)}) \subseteq C^{p+1,q}_{(l)} \). Now the result follows from Lemma 3.1.
Denote by $\delta_{h,l}^{p,q}$ (resp. $\delta_{c,l}^{p,q}$) the restriction of the maps $\delta_{h}^{p,q}$ (resp. $\delta_{c}^{p,q}$) to the subspace $D_{l}^{p,q}$. Thus by Lemma 3.2, we get a bicomplex $(D_{l}^{p,q}, \delta_{h,l}^{p,q}, \delta_{c,l}^{p,q})$ for each $l \in \mathbb{Z}$.

There is a canonical way to construct a complex from a given bicomplex. Set

$$\hat{D}_{l}^{n} = \bigoplus_{p+q=n+1, p,q \geq 1} D_{l}^{p,q}, \quad n \geq 1;$$

define $\partial_{l}^{n} : \hat{D}_{l}^{n} \longrightarrow \hat{D}_{l}^{n+1}$ by

$$\partial_{l}^{n}|_{D_{l}^{n+1-p,q}} := \delta_{h,l}^{p,q} + (-1)^{q}\delta_{c,l}^{p,q}, \quad 1 \leq q \leq n.$$

Hence, for each $l \in \mathbb{Z}$, we get a complex

$$0 \longrightarrow \hat{D}_{l}^{1} \longrightarrow \hat{D}_{l}^{2} \longrightarrow \hat{D}_{l}^{3} \longrightarrow \hat{D}_{l}^{4} \longrightarrow \cdots$$

We define the $n$-th cohomology group of the above complex to be the $n$-th graded “hat” bialgebra cohomology of degree $l$ of the graded bialgebra $B$, which will be denoted by $\hat{h}_{l}^{n}(B)_{(l)}$, $n \geq 1$, $l \in \mathbb{Z}$.

It is very useful to write out $\hat{h}_{l}^{2}(B)_{(l)}$ and $\hat{h}_{l}^{3}(B)_{(l)}$ explicitly from the definition. In what follows, we will use the maps $\delta_{h,l}^{p,q}$ and $\delta_{c,l}^{p,q}$, instead of $\delta_{h}^{p,q}$ and $\delta_{c}^{p,q}$ for simplicity. We have the following facts.

1. The cohomology group $\hat{h}_{l}^{2}(B)_{(l)}$ consists of all pairs $(f, g)$, where $f : m \otimes m \longrightarrow m$ and $g : m \longrightarrow m \otimes m$ are homogeneous maps of degree $l$, satisfying the following relations:

$$\delta_{h}^{1,2}(f) = 0, \quad \delta_{c}^{1,2}(f) + \delta_{h}^{2,1}(g) = 0, \quad \delta_{c}^{2,1}(g) = 0,$$

i.e., for any $a, b, c \in m$, we have

$$af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0,$$

$$f(a_{(1)} \otimes b_{(1)}) \otimes a_{(2)}b_{(2)} - \Delta(f(a \otimes b)) + a_{(1)}b_{(1)} \otimes f(a_{(2)} \otimes b_{(2)}) + a_{(1)}g(b)_{l} \otimes a_{(2)}g(b)_{r} - g(ab) + g(a)b_{(1)} \otimes g(a)b_{(2)} = 0,$$

$$c_{(1)} \otimes g(c_{(2)}) - (\Delta \otimes \mathrm{Id})(g(c)) + (\mathrm{Id} \otimes \Delta)(g(c)) - g(c_{(1)}) \otimes c_{(2)} = 0,$$

where we write $g(b) = g(b)_{l} \otimes g(b)_{r}$, $b \in B$.

Two pairs $(f, g) = (f', g')$ in $\hat{h}_{l}^{2}(B)_{(l)}$ if and only if there exists a homogeneous map $\theta : m \longrightarrow m$ of degree $l$ such that, for any $a, b, c \in m$,

$$(f - f')(a \otimes b) = a\theta(b) - \theta(ab) + \theta(a)b,$$

$$(g - g')(c) = \Delta(\theta(c)) - c_{(1)} \otimes \theta(c_{(2)}) - \theta(c_{(1)}) \otimes c_{(2)}.$$

2. The group $\hat{h}_{l}^{3}(B)_{(l)}$ consists of all triples $(F, H, G)$, where

$$F : m \otimes m \otimes m \longrightarrow m, \quad H : m \otimes m \longrightarrow m \otimes m, \quad G : m \longrightarrow m \otimes m \otimes m$$
are homogeneous maps of degree \( l \), subject to the relations:
\[
\delta_h^{1,3}(F) = 0, \quad \delta_c^{2,2}(F) = \delta_c^{1,3}(H), \quad \delta_c^{2,2}(H) = -\delta_h^{1,3}(G), \quad \delta_c^{1,1}(G) = 0.
\]
Note that \((F, H, G) = 0\) in \( \hat{h}_b^3(B)_1 \) if and only if there exists \((f, g) \in \hat{D}_b^2\) such that
\[
(F, H, G) = \partial_{(1)}^2((f, g)),
\]
which can be written out explicitly by the definition of \( \partial_{(1)}^2 \).

3.3. Now we are at the position to present our main observations, which relate the graded bialgebra deformations of the graded bialgebra \( B \) with the cohomology groups \( \hat{h}_b^3(B)_l \) and \( \hat{h}_b^3(B)_{l(1)} \) (compare [5], Section 5).

**Theorem 3.3.** Let \( B = \bigoplus_{i \geq 0} B_{(i)} \) be a graded bialgebra. Use the notation as above. Then

1. There is a bijection between iso\( \mathcal{E}^1(B) \) and \( \hat{h}_b^2(B)_{(-1)} \).
2. If \( \hat{h}_b^2(B)_{(-l)} = 0 \) for each \( l \geq 1 \), then the graded bialgebra \( B \) is graded-rigid.
3. The obstruction to extend an element of \( \mathcal{E}^l(B) \) to \( \mathcal{E}^{l+1}(B) \) lies in \( \hat{h}_b^3(B)_{(-l-1)}, l \geq 1 \). In particular, if \( \hat{h}_b^3(B)_{(-l)} = 0 \), one can extend any element of \( \mathcal{E}^l(B) \) to \( \mathcal{E}^{l+1}(B) \).

**Proof.** (1). Recall from 2.2 that an element in \( \mathcal{E}^1(B) \) is just given by \((B[t]/(t^2), m_1^1, \Delta_1^1)\). As in 2.3, write
\[
m_1^1(a \otimes b) = ab + f(a \otimes b)t, \quad \Delta_1^1(c) = \Delta(c) + \epsilon(c)t,
\]
where \( f : B \otimes B \to B \) and \( g : B \to B \otimes B \) are homogeneous of degree \(-1\). Note that \( 1_B \) is the identity element of \( B[t]/(t^2) \), hence \( f(1_B \otimes b) = f(b \otimes 1_B) = 0 \) for all \( b \in B \). Moreover, for \( a, b \in m, \epsilon_1^1(m_1^1(a \otimes b)) = 0 \) implies that \( \epsilon_1^1(ab + f(a \otimes b)t) = 0 \), i.e., \( f(a \otimes b) \in m \). Thus we may view \( f \) belongs to \( D_{(-1)}^{1,2} \). Dually one can show that \( g \in D_{(-1)}^{2,1} \).

Note that \( m_1^1 \) is an associative multiplication on \( B[t]/(t^2) \), thus we get
\[
f(a \otimes b)c - f(a \otimes bc) + f(ab \otimes c) - af(b \otimes c) = 0, \quad \forall a, b, c \in B.
\]
Therefore we get equation (3.1). Similarly, the fact that \( \Delta_1^1 \) is an algebra morphism (resp. that \( \Delta_1^1 \) is an coassociative comultiplication) gives us equation (3.2) (resp. equation (3.3)), i.e., \((f, g)\) can be viewed as an element in \( \hat{h}_b^2(B)_{(-1)} \).

Suppose that \((B[t]/(t^2), m_1^1, \Delta_1^1)\) and \((B[t]/(t^2), m_1^1, \Delta_1^1, \Delta_1^1)\) are two isomorphic deformations, with \((f, g)\) and \((f', g')\) defined as above, respectively. Let \( \phi \) (see also 2.3) be the isomorphism. We may write
\[
\phi(a) = a + \theta(a)t, \quad \forall a \in B,
\]
for some homogeneous map $\theta : B \rightarrow B$ of degree $-1$ (note that the map $\theta$ may be viewed as a map from $m$ to $m$). Now it is direct to check that $\theta$ realizes an equivalence of $(f, g)$ and $(f', g')$ in $\hat{h}_B^2(B)_{(-1)}$. Now we have obtained a map from $E^1(B)$ to $\hat{h}_B^2(B)_{(-1)}$, sending $(B[t]/(t^2), m_1, \Delta_1)$ to $(f, g)$. One can easily see that the correspondence is bijective, as required.

(2). To prove that $B$ is graded-rigid, we just need to show that $isoE(B)$ is a single-element set.

Let $(B[t], m_t, \Delta_t)$ be an element in $E(B)$. As before, write

$$m_t(a \otimes b) = \sum_{s=0}^{\infty} m_s(a \otimes b)t^s \quad \text{and} \quad \Delta_t(c) = \sum_{s=0}^{\infty} \Delta_s(c)t^s.$$  

Note that $m_0 = m$ and $\Delta_0 = \Delta$, and $m_s$ and $\Delta_s$ are homogeneous maps of degree $-s$. By a similar argument as (1), we may view $m_s \in D^{1,2}_{(-s)}$ and $\Delta_s \in D^{2,1}_{(-s)}$. Moreover, from (1), we see that $(m_1, \Delta_1)$ can be viewed as an element in $\hat{h}_B^2(B)_{(-1)}$. Now by the assumption, there exists a homogeneous map $\theta_1 : m \rightarrow m$ of degree $-1$, such that (see (3.4) and (3.5))

$$m_1(a \otimes b) = a\theta_1(b) - \theta_1(ab) + \theta_1(a)b,$$

$$\Delta_1(c) = \Delta(\theta_1(c)) - c_{(1)} \otimes \theta_1(c_{(2)}) - \theta_1(c_{(1)}) \otimes c_{(2)}.$$  

Take $\phi_1 : B[t] \rightarrow B[t]$ to be a $\mathbb{K}[t]$-linear map such that

$$\phi_1(a) = a + \theta_1(a)t, \quad a \in B.$$  

Note that $\phi_1$ is a bijective map preserving the identity $1_B$ and the counit $\varepsilon_t$. Consider the deformation

$$(B[t], m'_t = \phi_1 \circ m_t \circ (\phi_1^{-1} \otimes \phi_1^{-1}), \Delta'_t = (\phi_1 \otimes \phi_1) \circ \Delta_t \circ \phi_1^{-1}).$$  

We have

$$m'_t(a \otimes b) = ab + m'_2(a \otimes b)t^2 + m'_3(a \otimes b)t^3 + \cdots,$$

$$\Delta'_t(c) = \Delta(c) + \Delta'_2(c)t^2 + \Delta'_3(c)t^3 + \cdots$$  

where $m'_s$ and $\Delta'_s$ are homogeneous maps of degree $-s$, $s \geq 2$. Now by comparing (2.3-5) and (3.1-3), we see that $(m_2', \Delta_2')$ can be viewed as an element in $\hat{h}_B^2(B)_{(-2)}$. Hence there exists a homogeneous map $\theta_2 : m \rightarrow m$ of degree $-2$, such that (again see (3.4) and (3.5))

$$m_2'(a \otimes b) = a\theta_2(b) - \theta_2(ab) + \theta_2(a)b,$$

$$\Delta'_2(c) = \Delta(\theta_2(c)) - c_{(1)} \otimes \theta_2(c_{(2)}) - \theta_2(c_{(1)}) \otimes c_{(2)}.$$  

Take $\phi_2 : B[t] \rightarrow B[t]$ to be a $\mathbb{K}[t]$-linear map such that

$$\phi_2(a) = a + \theta_2(a)t^2, \quad \forall a \in B.$$  

Now consider the following deformation

$$(B[t], m''_t = \phi_2 \circ m'_t \circ (\phi_2^{-1} \otimes \phi_2^{-1}), \Delta''_t = (\phi_2 \otimes \phi_2) \circ \Delta'_t \circ \phi_2^{-1}).$$
Thus we get the following deformation
\[ m_t''(a \otimes b) = ab + m_3''(a \otimes b)t^3 + m_3''(a \otimes b)t^4 + \cdots, \]
\[ \Delta_t''(c) = \Delta(c) + \Delta_3''(c)t^3 + \Delta_2''(c)t^4 + \cdots \]
Similarly, we may view that \((m_3'', \Delta_3'')\) lies in \(h^2_B(B_{(-3)})\). By assumption and comparing (3.1-3), we have a homogeneous map \(\theta_3 : m \rightarrow m\) such that
\[ m_3''(a \otimes b) = a\theta_3(b) - \theta_3(ab) + \theta_3(a)b, \]
\[ \Delta_3''(c) = \Delta(\theta_3(c)) - c(1) \otimes \theta_3(c(2)) - \theta_3(c(1)) \otimes c(2). \]
Now define \(\phi_3 : B[t] \rightarrow B[t]\) to be a \(\mathbb{K}[t]\)-linear map such that
\[ \phi_3(a) = a + \theta_3(a)t^3, \quad \forall a \in B. \]
Thus we get the following deformation
\[ (B[t], m^{''''}_t) = \phi_3 \circ m^{''}_t \circ \phi_3^{-1}, \]
whose coefficients of \(t, t^2\) and \(t^3\) vanish. Now one can define \(\theta_4\) and \(\phi_4\), and so on.

Finally, define the infinite composition \(\cdots \phi_3 \circ \phi_2 \circ \phi_1\) to be \(\phi\). Note that the \(\mathbb{K}[t]\)-linear isomorphism \(\phi : B[t] \rightarrow B[t]\) is well-defined on every \(a \in B\), which preserves the identity \(1_B\) and the counit \(\varepsilon_t\). (In fact, \(\phi_s(a) = a + \theta_s(a)t^s\) where \(\theta_s : m \rightarrow m\) is homogeneous of degree \(\neg s\), hence, for each fixed \(a \in B_{(i)}\), \(\phi_s(a) = a\) for \(s \geq i\). Consequently, \(\phi(a)\) has only nonzero coefficients of \(t^i\) for \(0 \leq s \leq i\).) By the construction of each map \(\phi_s\), we obtain that the deformation \((B[t], \phi \circ m_4 \circ (\phi^{-1} \otimes \phi^{-1}), (\phi \otimes \phi) \circ \Delta_t \circ \phi^{-1})\) is trivial, which is also equivalent to the given deformation. Thus we have proved (2).

(3). Let \((B[t]/(t^{l+1}), m_l^I, \Delta_l^I)\) be an element in \(\mathcal{E}^l(B)\). Write
\[ m_l^I(a \otimes b) = \sum_{0 \leq s \leq l} m_s(a \otimes b)t^s \quad \text{and} \quad \Delta_l^I(c) = \sum_{0 \leq s \leq l} \Delta_s(c)t^s, \]
where \(m_s\) and \(\Delta_s\) are homogeneous maps of degree \(\neg s\). By the same argument as above, one can show that \(m_s\) (resp. \(\Delta_s\)) can be viewed as maps from \(m \otimes m\) to \(m\) (resp. from \(m\) to \(m \otimes m\)).

To extend \((B[t]/(t^{l+1}), m_l^I, \Delta_l^I)\) to some element in \(\mathcal{E}^{l+1}(B)\), we just need to find some homogeneous maps \(f : m \otimes m \rightarrow m\) and \(g : m \rightarrow m \otimes m\) of degree \(\neg (l+1)\) such that \((B[t]/(t^{l+2}), m_{l+1}^{I+1} + t^{l+1}f, \Delta_l^I + t^{l+1}g)\) is an bialgebra over \(\mathbb{K}[t]/(t^{l+2})\).

The associativity of \(m_{l+1}^I + t^{l+1}f\) is equivalent to
\[ (m_{l+1}^I + t^{l+1}f)((m_{l+1}^I + t^{l+1}f)(a \otimes b)) \otimes c) = (m_{l+1}^I + t^{l+1}f)(a \otimes ((m_{l+1}^I + t^{l+1}f)(b \otimes c))), \]
for all \(a, b, c \in B\). Since \(m_l^I\) is associative, then the above identity holds if and only if the two-sides have the same coefficients of the term \(t^{l+1}\). Thus

\[ m_{l+1}^I + t^{l+1}f \]
by direct computation, we get

\[ F(a \otimes b \otimes c) := \sum_{s=1}^{l} m_s(m_{t+1-s}(a \otimes b) \otimes c) - m_s(a \otimes m_{t+1-s}(b \otimes c)) \]
\[ = af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c \]
\[ = \delta^{1,2}_h(f)(a \otimes b \otimes c). \]

Similarly, one obtains that the compatibility of the multiplication \( m^l_t + t^{l+1}f \) and comultiplication \( \Delta^l_t + t^{l+1}g \), and the coassociativity of \( \Delta^l_t + t^{l+1}g \) are equivalent to the following two identities, respectively,

\[ H(a \otimes b) := \sum_{s=1}^{l} \Delta_s(m_{t+1-s}(a \otimes b)) - \sum_{s+r+s'+r'=l+1} (m_{s'} \otimes m_{r'}) \circ \tau_{23}(\Delta_s(a) \otimes \Delta_r(b)) \]
\[ = (\delta^{1,2}_c(f) + \delta^{2,1}_c(g))(a \otimes b), \]

and

\[ G(c) := \sum_{s=1}^{l} (\Delta_s \otimes \text{Id}) \circ \Delta_{t+1-s}(c) - (\text{Id} \otimes \Delta_s) \circ \Delta_{t+1-s}(c) \]
\[ = c_{(1)} \otimes g(c_{(2)}) - (\Delta \otimes \text{Id})(g(c)) + (\text{Id} \otimes \Delta)(g(c)) - g(c_{(1)}) \otimes c_{(2)} \]
\[ = \delta^{2,1}_c(g)(c), \]

where \( a, b, c \in m \), and \( \tau_{23} \) is the flip map with respect to the second and third positions.

Now it is direct to check that the element \((F, H, G) \in \hat{D}_{(-l-1)} \) is a cocycle (exactly as \( H \) in the case of algebras and \( H \) in the case of non-graded bialgebras), i.e., it lies in the kernel of the differential \( \beta^2_{(-l-1)} \) from (2.3-5), therefore, it can be viewed as an element in the cohomology group \( \hat{H}^3_{(-l-1)}(B) \). Now by comparing the above three identities with (3.6), we obtain that if \( \hat{h}^3_{b}(B)_{(-l-1)} = 0 \), then such maps \( f, g \) always exist, i.e., we can extend \((B[t]/(t^{l+1}), m^l_t, \Delta^l_t) \) to \((B[t]/(t^{l+2}), m^l_t + t^{l+1}f, \Delta^l_t + t^{l+1}g) \). Note that by the above three equivalences, one deduces that \((B[t]/(t^{l+2}), m^l_t + t^{l+1}f, \Delta^l_t + t^{l+1}g) \) belongs to \( \mathcal{E}^{l+1}(B) \). This completes the proof.

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