PALINDROMES IN TWO-DIMENSIONAL WORDS

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Abstract. A two-dimensional (2D) word is a 2D palindrome if it is equal to its reverse and it is an HV-palindrome if all its columns and rows are 1D palindromes. We study some combinatorial and structural properties of HV-palindromes and its comparison with 2D palindromes. We investigate the maximum number of distinct non-empty HV-palindromic sub-arrays in any finite 2D word, thus, proving the conjecture given by Anisiua et al. We also find the least number of HV-palindromes in an infinite 2D word over a finite alphabet size $q$.

1. Introduction

Palindromes are extensively studied in 1-dimensional words by several authors [1, 4, 11, 12]. There is an increasing interest in the combinatorial properties of palindromes in mathematics, theoretical computer science, and biology. Due to its symmetrical properties, this concept was generalized to two-dimension. Such a construction has significance in detecting bilateral symmetry of an image and face recognition technologies [6, 13].

Identifying palindromes in arrays dates back to 1994, when authors in [15], described an array to be a palindrome if all rows and columns are 1D palindromes. These structures are referred to as HV-palindromes in [3], where H and V stand for horizontal and vertical respectively. It was much later in 2001 when Berthé et al., [5] formally defined a 2D palindrome to be an array which is equal to its reverse. It can be easily observed that an array whose all rows and columns are 1D palindromes is equal to its reverse. Hence, HV-palindromes is a sub-class of 2D palindromes. Recently, there has been a rise in research that deals with the concept of 2D palindromes. A relation between 2D palindromes and 2D primitive words was studied in [17]. An algorithm for finding the maximal 2D palindromes was given in [9]. The maximum and the least number of 2D palindromes in an array was studied in [3, 14] and [20] respectively.

The main idea of this paper is to study the structure of a special type of 2D word called an HV-palindrome. The motivation of studying this structure came from the conjecture mentioned in [3] which speculates about the maximum number of HV-palindromes in a 2D word of size $(2, n)$ for a given $n$. We settle this conjecture in affirmative and generalize the result to words of larger sizes.

The paper is organized as follows. Section 3 deals with the characterization of a word to be an HV-palindrome. Further, the 2D words whose all 2D palindromes are HV-palindromes are also characterized. In Section 4, we count the number of possible 2D palindromes and HV-palindromes for a given array size.

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and investigate the number of 2D palindromic and HV-palindromic conjugates of a 2D word. Lastly, in Section 5, we find the maximum number of HV-palindromes in a finite 2D word and the least number of HV-palindromes in an infinite 2D word for a given alphabet size $q$. We end the paper with some concluding remarks.

2. Basic definitions and notations

An alphabet $\Sigma$ is a finite non-empty set of symbols. A 1D word is defined to be a sequence of letters. $\Sigma^*$ denotes the set of all words over $\Sigma$ including the empty word $\lambda$. $\Sigma^+ = \Sigma \setminus \lambda$. The length of a word $w \in \Sigma^*$ is the number of symbols in a word and is denoted by $|w|$. The reversal of $w = a_1a_2 \cdots a_n$ is defined to be a string $w^R = [a_{n+1-i}]_{1 \leq i \leq n} = a_n \cdots a_2a_1$ where $a_i \in \Sigma$. $\text{Alph}(w)$ is the set of all factors of $w$ of length 1. A word $w$ is said to be a palindrome if $w = w^R$. The concepts of prefix, suffix, primitivity, and conjugates are as usual. For all other concepts in formal language theory and combinatorics on words, the reader is referred to [14, 18].

2.1. Two-dimensional arrays. A two-dimensional word $w = [w_{i,j}]_{1 \leq i \leq m, 1 \leq j \leq n}$ over $\Sigma$ of size $(m, n)$ is defined to be a two-dimensional rectangular array of letters. If both $m$ and $n$ are infinite, then $w$ is an infinite 2D word. A factor of $w$ is a sub-array of $w$. In the case of 2D words, an empty word is a word of size $(0, 0)$, and we use the notation $\lambda$ to denote such a word. The set of all 2D words including the empty word $\lambda$ over $\Sigma$ is denoted by $\Sigma^{**}$ whereas, $\Sigma^{++}$ is the set of all non-empty 2D words over $\Sigma$. Note that, the words of size $(m, 0)$ and $(0, m)$ for $m > 0$ are not defined.

**Definition 2.1.** Let $u = [u_{i,j}]$ and $v = [v_{i,j}]$ be two 2D words over $\Sigma$ of size $(m_1, n_1)$ and $(m_2, n_2)$, respectively.

1. The column concatenation of $u$ and $v$ (denoted by $\odot$) is a partial operation, defined if $m_1 = m_2 = m$, and it is given by

$$
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
u_{1,1} & \cdots & u_{1,n_1} & v_{1,1} & \cdots & v_{1,n_2} & \\
& & \vdots & \ddots & \vdots & \cdots & \vdots \\
& & u_{m_1} & \cdots & u_{m_1,n_1} & v_{m_1} & \cdots & v_{m_1,n_2} \\
\end{array}
$$

The column closure of $u$ (denoted by $u^{\odot}$) is defined as $u^{\odot} = \bigcup_{i \geq 0} u^{i\odot}$ where $u^{0\odot} = \lambda, u^{1\odot} = u, u^{n\odot} = u \odot u^{(n-1)\odot}$.
(2) The row concatenation of $u$ and $v$ (denoted by $\odot$) is a partial operation defined if $n_1 = n_2 = n$, and it is given by

$$
\begin{array}{cccc}
  u_{1,1} & \cdots & u_{1,n} \\
  \vdots & \ddots & \vdots \\
  u_{m_1,1} & \cdots & u_{m_1,n} \\
  v_{1,1} & \cdots & v_{1,n} \\
  \vdots & \ddots & \vdots \\
  v_{m_2,1} & \cdots & v_{m_2,n}
\end{array}
$$

The row closure of $u$ (denoted by $u^{\odot}$) is defined as $u^{\odot} = \bigcup_{i \geq 0} u_i^{\odot}$ where $u_0^{\odot} = \lambda, u_1^{\odot} = u, u_n^{\odot} = u \odot u^{(n-1)^{\odot}}$.

In [2], a prefix of a 2D word $w$ is defined to be a rectangular sub-block that contains one corner of $w$, whereas suffix of $w$ is defined to be a rectangular sub-array that contains the diagonally opposite corner of $w$. However, throughout this paper, we consider prefix/suffix of a 2D word $w$ to be a rectangular sub-array that contains the top-left/bottom-right corner of $w$. This was formally defined in [17] as follows.

**Definition 2.2.** Given $u \in \Sigma^{**}, v \in \Sigma^{**}$ is said to be a prefix of $u$ (respectively, suffix of $u$), denoted by $v \leq_p u$ (respectively $v \leq_s u$) if $u = (v \odot x) \sqcup y$ or $u = (v \sqcup x) \odot y$ (respectively, $u = y \odot (x \odot v)$ or $u = y \odot (x \sqcup v)$) for $x, y \in \Sigma^{**}$.

If a sub-array $v$ occurs as both the prefix and suffix of a 2D word $w$, then it is called a *border* of $w$.

**Definition 2.3.** Let $w = [w_{i,j}]_{1 \leq i \leq m, 1 \leq j \leq n}$ be a 2D word. The reverse and transpose of $w$ denoted by $w^R$ and $w^T$ respectively are defined as

$$
\begin{array}{cccc}
w_{m,n} & w_{m,n-1} & \cdots & w_{m,1} \\
\vdots & \ddots & \ddots & \vdots \\
w_{m-1,n} & w_{m-1,n-1} & \cdots & w_{m-1,1} \\
\vdots & \ddots & \ddots & \vdots \\
w_{1,n} & w_{1,n-1} & \cdots & w_{1,1}
\end{array}
$$

$$
\begin{array}{cccc}
w_{1,1} & w_{2,1} & \cdots & w_{m,1} \\
\vdots & \ddots & \ddots & \vdots \\
w_{1,2} & w_{2,2} & \cdots & w_{m,2} \\
\vdots & \ddots & \ddots & \vdots \\
w_{1,n} & w_{2,n} & \cdots & w_{m,n}
\end{array}
$$

If $w = w^R$, then $w$ is said to be a *two-dimensional (2D) palindrome* ([5, 9]).

**Example 2.4.** Let $\Sigma = \{a, b, c\}$ and let

$$
\begin{array}{ccc}
a & b & c \\
\hline
b & c & c \\
a & c & b \\
a & c & b
\end{array}
$$

Note that $w = w^R$ and hence $w$ is a 2D palindrome.

Note that the rows and columns of a 2D palindrome are not always 1D palindromes. We observe that if all columns and rows of a finite 2D word are palindromes, then the word itself is a palindrome. Such palindromes are referred to as *HV-palindromes* in [3].
Example 2.5. Let \( \Sigma = \{a, b\} \) and let
\[
\begin{array}{ccc}
a & b & a \\
\hline
u = b & c & b \\
a & b & a
\end{array}
\]
Note that every row and every column of \( u \) is a 1D palindrome. Thus, \( u \) is an HV-palindrome.

We recall the notion of horizontal and vertical palindromes from [19].

Definition 2.6. Let \( w \) be a 2D word. The horizontal palindromes of \( w \) are the palindromic factors of \( w \) of size \((1, j)\) where \( j \geq 1 \) and vertical palindromes of \( w \) are the palindromic factors of \( w \) of size \((i, 1)\) where \( i \geq 2 \).

The palindromes of size \((1, 1)\) are trivial. Note that, all horizontal and vertical palindromes are HV-palindromes. We now recall the notion of the center of a 2D word defined in [9].

Definition 2.7. Center is the position that results in an equal number of columns to the left and right, as well as an equal number of rows above and below i.e. in a word of size \((m, n)\), if \( m \) and \( n \) are odd, then the center is at location \( \left( \left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \right) \). If \( m \) or/and \( n \) is even, the center is in between rows or/and columns respectively.

Example 2.8. The center of \( w \) in Example 2.4 is in between the rows and columns of the sub-array \( cc \ominus cc \). However, for the word \( u \) from Example 2.5, the center of \( u \) is the sub-word \( c \).

For more information pertaining to two-dimensional word concepts, we refer the reader to [8, 10, 5, 9].

3. Structure of an HV-palindrome

By definition, if a 2D word \( w \) of size \((m, n)\) is a palindrome, then \( w = w^R \) and \( w \) accepts the following structures.

- If \( m = n \), then \( w \) admits four symmetries namely identity, two diagonal reflections and 180° rotation.
- If \( m \neq n \), then \( w \) admits two symmetries namely identity and 180° rotation.

Hence, given a 2D palindrome \( w = [w_{i,j}] \) of size \((m, n)\), let \( p_1 \) be the prefix of size \( \left( \left\lfloor \frac{m}{2} \right\rfloor, n \right) \) and \( p_2 = w_{\lfloor \frac{m}{2} \rfloor, 1} w_{\lfloor \frac{m}{2} \rfloor, 2} \cdots w_{\lfloor \frac{m}{2} \rfloor, n} \). Then, \( w \) is of the form

\[
\begin{align*}
(1) & \quad p_1 \ominus p_1^R, \text{ if } m \text{ is even.} \\
(2) & \quad p_1 \ominus p_2 \ominus p_1^R, \text{ if } m \text{ is odd.}
\end{align*}
\]

In addition to the symmetries in a 2D palindrome, an HV-palindrome is preserved under reflections about horizontal and vertical axis containing the center of the word. Due to such symmetrical properties, we give the exact structure of an HV-palindrome.

Theorem 3.1. (Structure theorem of HV-palindromes)

Given an HV-palindrome \( w = [w_{i,j}] \) of size \((m, n)\), let \( u = u_1 \ominus u_2 \ominus u_{\lfloor \frac{m}{2} \rfloor} \) be the prefix of \( w \) of size
(⌈\(m/2\)⌉, ⌈\(n/2\)⌉), \(v = u_1^R \oplus u_2^R \oplus u_{\frac{m}{2}}^R\), \(p_1 = w_1[w_2 \cdots w_{\frac{m}{2}} \cdots w_{\frac{n}{2}}] \) and \(p_2 = w[w_{\frac{m}{2}} \cdots w_{\frac{n}{2}}]
\). Then, \(w\) is of the form

| \(m\) | \(n\) | Structure of \(w\) | \(m\) | \(n\) | Structure of \(w\) |
|------|------|----------------|------|------|----------------|
| even | even | \(u \ v \ u^R \) | odd  | odd  | \(u \ p_1^T \ v \) |
| odd  | even | \(u \ v \)     | odd  | odd  | \(u \ p_1^T \ v \) |
|       |      | \(p_2 \ p_2^R\) |       |       | \(p_2 \ x \ p_2^R\) |
|       |      | \(v^R \ u^R\)   |       |       | \(v^R \ (p_1^T)^R \ u^R\) |

where \(x = w_{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil}\).

**Proof.** We give the proof of case when \(m\) and \(n\) are both even and the rest of the cases follow similarly. Let \(w = [w_{i,j}]\) be an HV-palindrome of size \((m, n)\) and \(u = u_1 \oplus u_2 \oplus u_{\frac{m}{2}}\) be the prefix of \(w\) of size \((\frac{m}{2}, \frac{n}{2})\). Now, as every row of \(w\) is a palindrome, \(u_1 u_1^R \oplus u_2 u_2^R \oplus u_{\frac{m}{2}} u_{\frac{m}{2}}^R = u \oplus v\) is the prefix of \(w\) of size \((\frac{m}{2}, n)\). Also, as every column of \(w\) is a palindrome, then \(w = u \ v \ u^R\).

We observe that similar to the construction in case of a 2D palindrome in [17], an HV-palindrome of size \((m, n)\) can be constructed from an HV-palindrome of size \((m + 1, n + 1)\) by removing the \(\lceil \frac{m}{2} \rceil^{th}\) row and \(\lceil \frac{n}{2} \rceil^{th}\) column of \(w\). We also observe the following result.

**Lemma 3.2.** If \(w\) is an HV-palindrome of size \((m, n)\), then the word obtained by removal of first and last \(k\) rows of \(w\), for \(1 \leq k \leq \frac{\lceil m \rceil}{2}\) and first and last \(r\) columns of \(w\), for \(1 \leq r \leq \frac{\lceil n \rceil}{2}\) is an HV-palindrome of size \((m - 2k, n - 2r)\). This result is also true in the case of 2D palindromes.

3.1. **Characterization of an HV-palindrome.** In this section, we first give two necessary and sufficient conditions for a 2D word to be an HV-palindrome and then give a characterization of 2D words such that all of its palindromic sub-words are HV-palindromes i.e., words with no non-HV-palindromes. We begin the section by stating a necessary and sufficient condition for a 2D word to be an HV-palindrome.

**Proposition 3.3.** Let \(w = w_1 \oplus w_2 \cdots \oplus w_m = u_1 \oplus u_2 \cdots \oplus u_n\) be a 2D word of size \((m, n)\), where \(w_i\) and \(u_j\) be the \(i^{th}\) row and \(j^{th}\) column of \(w\) respectively. Then, \(w\) is an HV-palindrome if and only if \(w_i = w_{n-i+1}\) for \(1 \leq i \leq \frac{\lceil m \rceil}{2}\) and \(u_j = u_{n-j+1}\) for \(1 \leq j \leq \frac{\lceil n \rceil}{2}\).

**Proof.** Let \(w = w_1 \oplus w_2 \cdots \oplus w_m = u_1 \oplus u_2 \cdots \oplus u_n\) be an HV-palindrome. This implies each \(w_i\) and \(u_j\) is a palindrome. Every row of \(w\) is a palindrome if and only if \(u_j = u_{n-j+1}\) for \(1 \leq j \leq \frac{\lceil n \rceil}{2}\). Every column is a palindrome if and only if \(w_i = w_{n-i+1}\) for \(1 \leq i \leq \frac{\lceil m \rceil}{2}\). \(\square\)
One can easily observe that the above result does not hold true in the case of 2D palindromes that are not HV-palindromes. For example, the word \(w\) given in Example 2.4 does not satisfy the conditions given in Proposition 3.3.

It is well known that a 1D word \(u\) is a 1D-palindrome if and only if \(u = (xy)^i x\) for some 1D palindromes \(x, y \in \Sigma^*\) and \(i \geq 1\).

It was proved in [17], that for 2D palindromic words the condition is only sufficient. We now show that the condition is both necessary and sufficient for an HV-palindrome.

**Proposition 3.4.** Let \(x, y\) be HV-palindromes. Then, \(u = (x \oplus y)^i \ominus x\) or \(u = (x \ominus y)^i \ominus x, i \geq 1\) if and only if \(u\) is an HV-palindrome.

**Proof.** Let \(u = (x \oplus y)^i \ominus x\), then

\[
u^R = [(x \oplus y)^i \ominus x]^R = x^R \ominus (y^R \ominus x^R)^i = (x^R \ominus y^R)^i \ominus x^R = (x \oplus y)^i \ominus x
\]

Hence, \(u \in P_{2d}\). A similar proof follows for \(u = (x \ominus y)^i \ominus x\). Now as \(x\) and \(y\) are HV-palindromes, then \(x \oplus y \ominus x\) is an HV-palindrome. Hence, \(u\) is an HV-palindrome for \(i \geq 1\).

Conversely, let \(u\) be an HV-palindrome of size \((r, s)\), then \(u = u_1 \oplus u_2 \cdots \oplus u_r\) and by Proposition 3.3, \(u_i = u_{n-i+1}\) for \(1 \leq i \leq \lceil \frac{r}{2} \rceil\) and every row of \(u\) is a palindrome. Let \(x = u_1\) and \(y = u_2 \ominus u_3 \cdots \ominus u_{r-1}\). By Lemma 3.2, \(y\) is an HV-palindrome. Thus, \(u = (x \ominus y) \ominus x\).

Theorem 3.5. All 2D palindromic sub-words of a 2D word \(w\) are HV-palindromes if and only if \(w\) has no sub-word of the form

\[
\begin{array}{ccc}
x & u & y \\
v & p & v^R \\
y & u^R & x \\
\end{array}
\]

where \(x, y \in \Sigma\) such that \(x \neq y\), \(u\) and \(v^T\) are 1D words and \(p\) is a 2D palindrome (may be empty).
Proof. Let \( w \) be a 2D word with no sub-word of the form

\[
\begin{array}{ccc}
x & u & y \\
v & p & v^R \\
y & u^R & x
\end{array}
\]

where \( x \neq y \) such that \( x, y \in \Sigma \), \( u \) and \( v^T \) are 1D words and \( p \) is a 2D palindrome. We show that all of its 2D palindromic sub-words are HV-palindromes. Let \( p' \) be a 2D palindromic sub-word of \( w \) of size \((r, s)\), with \( r, s \geq 2 \), then it must be of the form

\[
\begin{array}{ccc}
x_1 & u_1 & x_1 \\
v_1 & p_1 & v_1^R \\
x_1 & u_1^R & x_1
\end{array}
\]

where \( x_1 \in \Sigma \), \( u_1 \) and \( v_1 \) are 1D words and \( p_1 \) is a 2D palindrome. We show that \( p' \) is an HV-palindrome. Consider the first and last row of \( p' \). We show that they are the same. If not, consider the first position where they are different, say it is the \( i \)th position. Let the \( i \)th position of the first and last row of \( p' \) be \( a \) and \( b \) respectively, where \( a \neq b \in \Sigma \). As \( p' \) is a 2D palindrome, then the \((s - i + 1)\)th position of the first and last row of \( p' \) are \( b \) and \( a \) respectively. Then \( p' \) has a sub-word of the form

\[
\begin{array}{ccc}
\vdots & \cdots & \vdots \\
p_1 & v_1^R \\
\vdots & \cdots & \vdots
\end{array}
\]

where \( a \neq b \), \( u_1 \) and \( v_1^T \) are 1D words and \( p_1 \) is a 2D palindrome which is a contradiction. Hence, the first row and the last row of \( p' \) are same. Similarly, we can show that the \( j \)th and \((r - j + 1)\)th row of \( p' \) are same for \( 1 \leq j \leq \lfloor \frac{r}{2} \rfloor \). Now, consider the word \((p')^T\) which is a palindrome of size \((s, r)\). Apply the same procedure on \((p')^T\) to show that the \( j \)th and \( s - j + 1 \)th row of \((p')^T\) are same for \( 1 \leq j \leq \lfloor \frac{s}{2} \rfloor \). This implies \( j \)th and \( s - j + 1 \)th column of \( p' \) is same for \( 1 \leq j \leq \lfloor \frac{s}{2} \rfloor \). Thus, by Proposition 3.3 \( p_1 \) is an HV-palindrome. Conversely, if all palindromes of \( w \) are HV-palindromes, then any sub-word of the form

\[
\begin{array}{ccc}
x & u & y \\
v & p & v^R \\
y & u^R & x
\end{array}
\]

where \( x, y \in \Sigma \) are distinct, \( u \) and \( v^T \) are 1D words and \( p \) is a 2D palindrome is itself a non-HV-palindrome which is a contradiction. \( \square \)

3.2. Borders in an HV-palindrome. We now count the number of borders in an HV-palindrome. We first observe a general result for 2D palindromes.

**Proposition 3.6.** Every border of a 2D palindrome is a 2D palindrome.
Proof. Let \( w = [w_{i,j}] \) be a 2D palindrome of size \((m, n)\). Let \( v \) be a border of size \((i, j)\), then

\[
\begin{align*}
v &= \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,j} & w_{m-i+1,n-j+1} & w_{m-i+1,n-j+2} & \cdots & w_{m-i+1,n} \\
       w_{2,1} & w_{2,2} & \cdots & w_{2,j} & w_{m-i+2,n-j+1} & w_{m-i+2,n-j+2} & \cdots & w_{m-i+2,n} \\
       \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
       w_{i,1} & w_{i,2} & \cdots & w_{i,j} & w_{m,n-j+1} & w_{m,n-j+2} & \cdots & w_{m,n} \end{bmatrix} \\
\end{align*}
\]

Now, as \( w \) is a palindrome, then for \( 0 \leq k \leq i - 1 \)

\[
(2) \quad w_{1+k,1}w_{1+k,2}\cdots w_{1+k,j} = (w_{m-k,n-j+1}w_{m-k,n-j+2}\cdots w_{m-k,n})^R 
\]

Hence, by Equations 1 and 2, \( v \) is a 2D palindrome. \(\square\)

Corollary 3.7. Every border of an HV-palindrome is a 2D palindrome.

Remark 3.8. Border of an HV-palindrome need not be an HV palindrome. In Example 2.5, \( ab \odot ba \) is a border but is not an HV-palindrome.

We now count the number of borders in a 2D palindrome and an HV-palindrome. It is clear that the maximum number of borders in a word of size \((m, n)\) is \(mn\) and is achieved when \(|Alph(w)| = 1\). Let \( BOR(w) \) be the set of all borders of \( w \), then we have the following result.

Lemma 3.9. Let \( w \) be a word of size \((m, n)\).

(1) If \( w \) is a 2D palindrome, then \( 1 \leq |BOR(w)| \leq mn \).

(2) If \( w \) is an HV-palindrome, then \( 3 \leq |BOR(w)| \leq mn \).

Proof. Let \( w \) be a word of size \((m, n)\), \( m, n \geq 2 \).

(1) If \( w \) is a 2D palindrome, then the prefix of size \((1, 1)\) is a border of \( w \). It can be observed in Example 2.5 that the lower bound is tight.

(2) If \( w \) is an HV-palindrome, then the prefixes of size \((1, n)\), \((m, 1)\) and \((1, 1)\) are borders of \( w \). We show the existence of words that achieve the lower bound. Let

\[
\begin{align*}
  ababa \\
  w = bbabb \\
  ababa
\end{align*}
\]

Here, \( w \) is an HV-palindrome with only 3 borders. \(\square\)

4. Counting Palindromes

It can be easily observed that the maximum number of distinct 1D palindromes of length \( n \) over an alphabet size \( k \) is \( k^{\lceil n/2 \rceil} \). In this section, we first count the maximum number of 2D palindromes (HV-palindromes) that can be obtained over a given alphabet \( \Sigma \). We then determine the maximum and the
minimum number of 2D palindromes (HV-palindromes) that can appear in the conjugacy class of a given 2D word $w$.

**Theorem 4.1.** Let $\Sigma$ be a finite alphabet such that $|\Sigma| = k$. Then,

1. the maximum number of distinct 2D palindromes of size $(m, n)$ is $k^i$ where $i = \lceil \frac{mn}{2} \rceil$.
2. the maximum number of distinct HV-palindromes of size $(m, n)$ is $k^j$ where $j = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$.

**Proof.** Let $w$ be a palindrome of size $(m, n)$ such that $w = w_1 \oplus w_2 \oplus \cdots \oplus w_m$, where $w_i$ is the $i$th row of $w$. Since $w = w^R$, the 1D word $u = w_1 w_2 \cdots w_m$ is a 1D palindrome. Let $v$ be the prefix of $u$ of length $\lceil \frac{mn}{2} \rceil$. Then, $v^R$ is a suffix of $u$. Thus, we have $\lceil \frac{mn}{2} \rceil$ distinct choices of letters from $\Sigma$ and therefore, there are $k^{\lceil \frac{mn}{2} \rceil}$ distinct 2D palindromes of size $(m, n)$ over $\Sigma$. If $w$ is an HV-palindrome of size $(m, n)$, then by Proposition 3.3, we have $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ distinct choices of letters in the prefix of $w$ of size $(\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil)$. Hence, there are $k^{\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil}$ distinct HV-palindromes of size $(m, n)$ over $\Sigma$.

We illustrate with the following example.

**Example 4.2.** Consider the binary alphabet $\Sigma = \{a, b\}$. The set of all 2D-palindromes of size $(2, 2)$ is given by:

$$\begin{align*}
\{ & aa, ab, ba, bb \\
& aa', ba', ab', bb' \}
\end{align*}$$

which has exactly 4 2D palindromes and 2 HV-palindromes of size $(2, 2)$.

### 4.1. Palindromes in a conjugacy class.

It was proved in [12] that a conjugacy class of a 1D word contains at most two 1D palindromes. We now count the number of 2D palindromes and HV-palindromes in a conjugacy class of a 2D word. We recall the definition of conjugates in an array from [10].

**Definition 4.3.** Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be respectively the $m$ rows and the $n$ columns of a word $w$ of size $(m, n)$. The cyclic rotation of $k$ columns, for $1 \leq k \leq n$ denoted by $\otimes_k^{\text{Col}}$ is defined as the word

$$\otimes_k^{\text{Col}} = v_{n-k+1} \oplus \cdots \oplus v_n \oplus v_1 \oplus v_2 \cdots \oplus v_{n-k-1} \oplus v_{n-k}$$

Similarly, the cyclic rotation of $k$ rows, for $1 \leq k \leq m$ denoted by $\otimes_k^{\text{Row}}$ is defined as the word

$$\otimes_k^{\text{Row}} = u_{m-k+1} \oplus \cdots \oplus u_m \oplus u_1 \oplus u_2 \cdots \oplus u_{m-k-1} \oplus u_{m-k}$$

Then, the conjugacy class of $u$ denoted by $\text{Conj}(u)$ is defined as

$$\text{Conj}(u) = \{ \otimes_i^{\text{Col}} \otimes_j^{\text{Row}} u, 1 \leq i \leq n, 1 \leq j \leq m \}$$

Note that, given any 2D word $w$ of size $(m, n)$, the number of elements in its conjugacy class can be at most $mn$. We illustrate with the following example.
Example 4.4. Consider the 2D word $w = abc \oplus cbb \oplus bbc \oplus cba$ of size $(4, 3)$. Then, the conjugacy class of $w$ is given by:

$$Conj(w) = \begin{cases} abc~cbb~bbc~cba~bbc~bca~bcb~acb~cbb~bcb \\ cbb~bcb~cba~bbc~bca~bcb~bcb~bcb~acb~cbb \\ bbb~cba~abc~cba~bcb~bca~bca~cbb~acb~acb \\ cba~abc~cbb~abc~bca~bcb~bca~acb~acb~cbb \\ \end{cases}$$

Remark 4.5. For a 2D word $w$ of size $(m, n)$, if $|Alph(w)| = 1$, $|Conj(w)| = 1$ and if $|Alph(w)| = mn$, $|Conj(w)| = mn$. However, the converse need not be true as illustrated in Example 4.4. If the 2D word is a 2D palindrome and if $|Alph(w)| = \lceil \frac{mn}{2} \rceil$, then $|Conj(w)| = mn$. If $w$ is an HV-palindrome and if $|Alph(w)| = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$, then $|Conj(w)| = mn$.

We now count the maximum number of 2D palindromes (HV-palindromes) in the conjugacy class of a 2D word. We call such conjugates as palindromic (HV-palindromic) conjugates and denote them by $PALConj(w)$ ($HVPALConj(w)$). Note that, for the $w$ given in Example 4.4 we have,

$$PALConj(w) = \begin{cases} abc~bcb \\ cbb~cba \\ bbc~abc \\ cba~cbb \end{cases}, \quad HVPALConj(w) = \emptyset$$

We give another example of a 2D word $v = abba \oplus aaaa \oplus aaaa \oplus abba$ of size $(4, 4)$ where $|HVPALConj(v)| = |PALConj(v)| = 4$.

$$PALConj(v) = HVPALConj(v) = \begin{cases} abba~baab~aaaa~aaaa \\ aaaa~aaaa~abba~baab \\ aaaa~aaaa~abba~baab \\ abba~baab~aaaa~aaaa \end{cases}$$

We first recall the following result for 1D words from [12].

**Theorem 4.6.** A conjugacy class of a 1D word contains at most two palindromes and it has exactly two if and only if it contains a word of the form $(wu)^i$, where $wu$ is a primitive word and $i \geq 1$.

We have the following result.

**Theorem 4.7.** Let $w$ be a 2D word of size $(m, n)$, then

$$0 \leq |PALConj(w)| \leq \begin{cases} 4, \quad \text{if } m, n \text{ are even,} \\ 1, \quad \text{if } m, n \text{ are odd,} \\ 2, \quad \text{otherwise.} \end{cases}$$
Proof. It is clear that there exist words (for example, \( w = aa \oplus ab \)) with no palindromic conjugates. Also, note that for a 2D word \( w \) of size \((m, n)\) such that \(|\text{Alph}(w)| = 1, |PALConj(w)| = 1\) and for \( w \), such that \(|\text{Alph}(w)| = mn, |PALConj(w)| = 0\).

We now find the maximum number of palindromic conjugates that \( w \) can have. If \( w \) has no palindromic conjugates, then we are done. Otherwise, assume that \( v \in PALConj(w) \). Note that, \( PALConj(w) = PALConj(v) \). Now, \( v = v_1 \oplus v_2 \oplus \cdots \oplus v_n = u_1 \oplus u_2 \oplus \cdots \oplus u_m \), where \( v_i \) is the \( i^{th} \) column of \( v \) and \( u_j \) is the \( j^{th} \) row of \( v \). As \( v \) is a palindrome, then it is a 1D palindrome over the alphabet of columns i.e., the set \( \{v_1, v_2, \ldots v_n\} \). It is also a 1D palindrome over the alphabet of rows i.e., the set \( \{u_1, u_2, \ldots u_m\} \). Also, note that \( v' \) is a 2D palindrome iff it is a 1D palindrome over its alphabet of columns and its rows. We have the following cases.

Case 1: If \( m \) and \( n \) are both even, then by Theorem 4.6, as \( n \) is even, there can be at most 2 1D palindromic conjugates of \( v \) say \( v \) and \( v' \) over its alphabet of columns. Each of them can be expressed as alphabet of rows say the set \( B \). Again, as \( m \) is even, by Theorem 4.6, there are at most 2 palindromic conjugates of \( v \) and \( v' \) each as a 1D word over \( B \). In total, there are at most 4 palindromic conjugates in the conjugacy class of \( w \). Hence, for some \( i_1, i_2, j_1 \) and \( j_2 \),

\[
PALConj(w) = \{ \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w, \circlearrowright_{i_2}^{\text{Col}} \circlearrowleft_{j_2}^{\text{Row}} w, \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_2}^{\text{Row}} w, \circlearrowright_{i_2}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w \}
\]

Case 2: If \( m \) is even and \( n \) is odd, then \( v \) can be expressed as a 1D word over its alphabet of rows. As \( m \) is odd, by Theorem 4.6 there is no palindromic conjugate of \( v \), other than \( v \). Again \( v \) can be expressed as a 1D word over the alphabet of columns say the set \( A \) to obtain at most 2 palindromic conjugates over \( A \). In total, there are at most 2 palindromic conjugates in the conjugacy class of \( w \). Hence, for some \( i_1, j_1 \) and \( j_2 \),

\[
PALConj(w) = \{ \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w, \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_2}^{\text{Row}} w \}
\]

Similar is the case when \( m \) is odd and \( n \) is even. Here, for some \( i_1, i_2 \) and \( j_1 \),

\[
PALConj(w) = \{ \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w, \circlearrowright_{i_2}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w \}
\]

Case 3: If \( m, n \) are both odd, by Theorem 4.6 there is no palindromic conjugate of \( v \), other than \( v \). Hence, for some \( i_1 \) and \( j_1 \),

\[
PALConj(w) = \{ \circlearrowright_{i_1}^{\text{Col}} \circlearrowleft_{j_1}^{\text{Row}} w \}
\]

□

Theorem 4.7 also holds in case of HV-palindromes. We have the following result.
Corollary 4.8. Let $w$ be a 2D word of size $(m, n)$, then

$$0 \leq |HV\text{PALConj}(w)| \leq \begin{cases} 4, & \text{if } m, n \text{ are even,} \\ 1, & \text{if } m, n \text{ are odd,} \\ 2, & \text{otherwise.} \end{cases}$$

Remark 4.9. We now find the structure of the words that achieve the above upper bound in case of HV-palindromes. Let $v \in HV\text{PALConj}(w)$, then by the structure of $v$, $i_1 = 0, j_1 = 0, i_2 = \frac{n}{2}$ and $j_2 = \frac{m}{2}$.

Case 1: If $m$ and $n$ are even, then to have 4 distinct palindromic conjugates, we have the following:

(a) $v \neq \circ_{\text{Col}} \frac{m}{2} v$ and $\circ_{\text{Row}} \frac{n}{2} v \neq \circ_{\text{Col}} \circ_{\text{Row}} \frac{m}{2} \frac{n}{2} v$. This implies that the prefix of size $(m, \frac{n}{2})$ is not an HV-palindrome.

(b) $v \neq \circ_{\text{Row}} \frac{n}{2} v$ and $\circ_{\text{Col}} \frac{m}{2} v \neq \circ_{\text{Col}} \circ_{\text{Row}} \frac{m}{2} \frac{n}{2} v$. This implies that the prefix of size $(\frac{m}{2}, n)$ is not an HV-palindrome.

(c) $w \neq \circ_{\text{Col}} \circ_{\text{Row}} \frac{m}{2} \frac{n}{2} v$. This implies that the prefix of size $(\frac{m}{2}, \frac{n}{2})$ is not a 2D palindrome.

(d) $\circ_{\text{Col}} \frac{m}{2} v \neq \circ_{\text{Row}} \frac{n}{2} v$. This implies that the prefix of size $(\frac{m}{2}, \frac{n}{2})$ is not a 2D palindrome.

Hence, in this case $|PAL\text{Conj}(v)| = 4$ iff the prefix of size $(\frac{m}{2}, \frac{n}{2})$ of $v$ is not a 2D palindrome.

Case 2: For $m$ even and $n$ odd, $|PAL\text{Conj}(v)| = 2$ i.e. $v \neq \circ_{\text{Row}} \frac{n}{2} v$ iff the prefix of size $(\frac{m}{2}, n)$ is not an HV-palindrome.

Case 3: For $m$ odd and $n$ even, $|PAL\text{Conj}(v)| = 2$ i.e. $v \neq \circ_{\text{Col}} \frac{m}{2} v$ iff the prefix of size $(m, \frac{n}{2})$ is not an HV-palindrome.

5. Bounds on the number of palindromes

In this section, we find the maximum and the least number of HV-palindromes in any 2D finite and infinite word respectively. We also compare these results with the existing results on 2D palindromes.

5.1. On the maximum number of HV-palindromes. In this section, we find the maximum number of non-empty distinct HV-palindromes in a 2D word $w$ of size $(m, n)$. It was proved in [1] and [19] that there are at most $n$ palindromes in a 1D word of length $n$ and at most $2n + \lfloor \frac{n}{2} \rfloor - 1$ palindromes in a two-row array of size $(2, n)$ respectively. Further, it was conjectured in [3] that the number of HV-palindromes in any 2D word of size $(2, n)$ is less than or equal to $2n$. We give a proof of this conjecture. We also extend the result to a word of size $(m, n)$ and find an upper bound.

Theorem 5.1. The maximum number of HV-palindromes in any 2D word of size $(2, n)$ is $2n$, $n \geq 1$.

Proof. Let $w$ be a 2D word of size $(2, n)$. The number of HV-palindromes in $w$ is the sum of horizontal palindromes in $w$ and the HV-palindrome of size $(2, t)$, $t \geq 1$ in $w$. It can be observed that the HV-palindromes of size $(2, t)$, $t \geq 1$ are of the form $p \ominus p$, where $p$ is a 1D palindrome. Let the number of the HV-palindromes of size $(2, t)$, $t \geq 1$ in $w$ be $k$. Then, there are $k$ palindromes of the form $p_i \ominus p_i$ where $1 \leq i \leq k$. This implies there are $k$ common horizontal palindromes $p_i$ in both the rows of $w$ and hence,
these $k$ horizontal palindromes should be counted only once. Thus, as only one horizontal palindrome can be created on the concatenation of a letter, then the maximum number of horizontal palindromes in $w$ is $\leq 2n - k$. The number of HV-palindromes in $w \leq 2n - k + k = 2n$. \qed

We now give an example of a word of size $(2, n)$ that achieves this bound. Let $w = (a \oplus a)^n \oplus$. It has $n$ horizontal palindromes: $a^i$ for $1 \leq i \leq n$ and $n$ HV-palindromes: $(a \oplus a)^j \oplus$ of size $(2, j)$ for $1 \leq j \leq n$. We recall the following from [19].

**Lemma 5.2.** Let $w$ be a $2D$ word of size $(m, n)$ for $m, n \geq 2$. Then, the column concatenation of $w$ and $(x_1 \oplus x_2 \cdots \oplus x_m)$, where $x_i \in \Sigma$ for each $i$ creates at most one distinct palindrome of size $(m, t)$, $t \geq 1$.

**Remark 5.3.** Note that this also holds true in the case of HV-palindromes.

**Theorem 5.4.** The maximum number of HV-palindromes in any $2D$ word of size $(m, n)$ for $m, n \geq 2$ is

$$a_k = \begin{cases} 
\frac{(k+1)}{2}^2 n, & \text{if } k \text{ is odd} \\
\frac{(k+1)}{2}^2 - \frac{1}{4} n, & \text{if } k \text{ is even}. 
\end{cases}$$

**Proof.** We prove by induction on $m$. The base case for $m = 2$ is clear from Theorem 5.1. Assume the result to be true for $m = k - 1$ and let the number of HV-palindromes in a word of size $(k - 1, n)$ be less than or equal to $a_{k-1}$. Let $w$ be a word of size $(k, n)$. Then by induction, the number of HV-palindromes in the prefix of size $(k - 1, n)$ is bounded above by $a_{k-1}$. On concatenation of the last row, new palindromes of size $(i, t)$, $t \geq 1$ for $1 \leq i \leq k$ can be created. We observe that an HV-palindrome of size $(l, t)$, $t \geq 1$ and an HV-palindrome of size $(k - l + 1, t)$ cannot be both newly created for $l \geq \lceil \frac{k}{2} \rceil$ by the addition of the same symbol in the last row. Also, by Lemma 5.2, concatenation of last row leads to the creation of at most $n$ HV-palindromes of size $(i, t)$, $t \geq 1$ for a fixed $i$. Hence, there can be at most $\lceil \frac{k}{2} \rceil n$ new palindromes on concatenation of last row. Hence, the maximum number of HV-palindromes in any 2D word of size $(k, n)$ is

$$a_{k-1} + \frac{k}{2} n = a_k.$$

Solving the recurrence for even and odd values of $m$ and with initial condition $a_2 = 2n$, we have

$$a_k = \begin{cases} 
\frac{(k+1)}{2}^2 n, & \text{if } k \text{ is odd} \\
\frac{(k+1)}{2}^2 - \frac{1}{4} n, & \text{if } k \text{ is even}. 
\end{cases}$$

Hence, the result holds for $m = k + 1$ and thus, the result follows by induction for all $m \geq 2$. \qed

The following table depicts the maximum number of distinct non-empty HV-palindromic sub-arrays in any binary word in $\Sigma^{m \times n}$ for larger values of $m$ and $n$ obtained by a computer program.

| $m \times n$ | Max (HV) | $m \times n$ | Max (HV) |
|--------------|----------|--------------|----------|
| 3 $\times$ 2 | 6        | 3 $\times$ 6 | 20       |
| 3 $\times$ 3 | 10       | 4 $\times$ 2 | 8        |
| 3 $\times$ 4 | 13       | 4 $\times$ 3 | 13       |
| 3 $\times$ 5 | 17       | 4 $\times$ 4 | 19       |
5.2. **On the least number of HV-palindromes.** In this section, we find the least number of non-empty distinct palindromes in a 2D infinite word $w$ with $|\text{Alph}(w)| = q$ for a given alphabet size $q$.

A 1D infinite word is an infinite sequence of symbols. It was proved in [7] that there are at least 8 palindromes in an infinite 1D word. A 2D infinite word is an infinite array of symbols i.e. an array with infinite rows and columns. We recall the following result on the least number of 2D palindromes in an infinite 2D word from [20].

**Theorem 5.5.** The least number of 2D palindromes in an infinite 2D word is

\[
\begin{align*}
\infty, & \quad \text{if } |\text{Alph}(w)| = 1 \\
20, & \quad \text{if } |\text{Alph}(w)| = 2 \\
q, & \quad \text{if } |\text{Alph}(w)| = q, \text{ where } q \geq 3
\end{align*}
\]

We have the following result for HV-palindromes.

**Theorem 5.6.** The least number of HV-palindromes in an infinite 2D word is

\[
\begin{align*}
\infty, & \quad \text{if } |\text{Alph}(w)| = 1 \\
14, & \quad \text{if } |\text{Alph}(w)| = 2 \\
q, & \quad \text{if } |\text{Alph}(w)| = q, \text{ where } q \geq 3
\end{align*}
\]

**Proof.** Let $w$ be a 2D infinite word with $|\text{Alph}(w)| = q$. We have the following cases.

- **Case 1:** If $|\text{Alph}(w)| = 1$, then $w = (x^\infty)^\otimes$, where $x \in \Sigma$. This word has infinite HV-palindromes: $(x^i)^\otimes$ for $i, j \geq 1$.

- **Case 2:** If $|\text{Alph}(w)| = 2$, then let $w$ be a binary word on $\Sigma = \{a, b\}$. It was shown in [7], that any finite 1D binary word of length greater than 8, has at least 8 palindromic factors. All these palindromes are HV-palindromes. Since every infinite 1D word must have at least 8 HV-palindromes, every row and column of $w$ has at least 8 HV-palindromes. The only palindromes that can be common to both are the trivial palindromes i.e. $a$ and $b$. Thus, $w$ has at least 6 horizontal and 6 vertical non-trivial HV-palindromes. Thus, any 2D infinite binary word $w$ has at least $8 + 8 - 2 = 14$ HV-palindromes. We give an example of the word that achieves the bound.

Let $u_1 = ababba$ and $u_{i+1}$ be the 1-cyclic shift of $u_i$ for $1 \leq i \leq 5$ and $v = u_1 \ominus u_2 \cdots \ominus u_6$ Note that, for the given $v$, $(v^{2\otimes})^{2\otimes}$ has exactly 14 HV-palindromes: $\{p, p^T : p \in A\}$ where

\[
A = \{a, b, aa, bb, aba, bab, abba, baab\}
\]

As there is no palindrome of size $(i, 1)$ or $(1, j)$ for $i, j \geq 5$ in $(v^{2\otimes})^{2\otimes}$, thus, there are only 14 HV-palindromes in $(v^{\infty\otimes})^{\infty\otimes}$ which is the required word.
• Case 3: If $|\text{Alph}(w)| = q$, where $q \geq 3$, then there are at least $q$ trivial HV-palindromes. We give a word with exactly $q$ HV-palindromes. Let $\text{Alph}(w) = \{a_1, a_2, \cdots, a_q\}$. We show the existence of an infinite 2D word $w$ that achieves this bound. Let

$$u = a_1 \ a_2 \ \cdots \ a_{q-1} \ a_q$$

$$u = \begin{array}{cccc}
a_1 & a_2 & \cdots & a_{q-1} & a_q \\
a_2 & a_3 & \cdots & a_q & a_1 \\
\vdots & \vdots & & \vdots & \vdots \\
a_q & a_1 & \cdots & a_{q-2} & a_{q-1}
\end{array}$$

Then, the word $w = (u^{\infty})^{\infty}$ has $q$ HV-palindromes: $\{a_1, a_2, \cdots, a_q\}$.

As the palindromes in the case of $q \geq 3$ in the above theorem are all trivial. We find the least number of HV-palindromes in an infinite 2D word $w$ with at least one HV-palindrome such that $|\text{Alph}(w)| \geq 3$.

**Theorem 5.7.** The least number of HV-palindromes in an infinite 2D word $w$ with $|\text{Alph}(w)| = q$ that has at least one non-trivial HV-palindrome is

$$\left\{ \begin{array}{ll} 5, & \text{if } q = 3, \\
q + 1, & \text{if } q > 3. 
\end{array} \right.$$  

**Proof.** Let $w$ be a 2D infinite word with $|\text{Alph}(w)| = q$. We have the following cases.

• **Case 1:** If $|\text{Alph}(w)| = 3$, then consider an infinite 2D word with exactly one non-trivial HV-palindrome. We observe that this non-trivial HV-palindrome should occur as a sub-word and is of one of the forms $xx, xyx$ or their transpose where $x, y \in \Sigma$. If $|\text{Alph}(w)| = 3$ i.e $\Sigma = \{a, b, c\}$, then we cannot construct a sub-word of size $(3, 3)$ with the above as prefixes with only one non-trivial HV-palindrome. Hence, there is no infinite 2D word with exactly one non-trivial HV-palindrome when $|\text{Alph}(w)| = 3$. Thus, we construct an infinite 2D word $w$ with exactly two non-trivial HV-palindromes such that $|\text{Alph}(w)| = 3$. Let $w = (a(abc)^{\infty})^{\infty} \ominus (v^{\infty})^{\infty}$ for $v = abc$. This word has only two non-trivial HV-palindromes $aa$ and $a \ominus b \ominus a$. Thus, the least number of HV-palindromes in an infinite 2D word $w$ with $|\text{Alph}(w)| = 3$ that has at least one non-trivial HV-palindrome is 5.

• **Case 2:** If $|\text{Alph}(w)| = q$, for $q \geq 4$, then the least number of HV-palindromes in an infinite 2D word $w$ with at least one non-trivial HV-palindrome should be greater than or equal to $q + 1$. We
give the existence of such a word with exactly $q + 1$ HV-palindromes. Let $w = (u^{\infty \ominus})^{\infty \ominus}$ for

$$
\begin{array}{ccccccc}
   a_1 & a_2 & a_3 & \cdots & a_{q-2} & a_{q-1} \\
   a_1 & a_3 & a_4 & \cdots & a_{q-1} & a_q \\
   a_2 & a_4 & a_5 & \cdots & a_q & a_1 \\
   \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
   a_{q-2} & a_q & a_1 & \cdots & a_{q-4} & a_{q-3} \\
   a_{q-1} & a_1 & a_2 & \cdots & a_{q-3} & a_{q-2}
\end{array}
$$

$w$ has one non-trivial HV-palindrome $a_1 \ominus u_1$ along with trivial palindromes.

\[\square\]

6. Conclusions

HV-palindromes is a special class of 2D palindromes in which every row and column is a 1D palindrome. We witnessed certain important combinatorial properties by investigating the structure of an HV-palindrome. We have an affirmative answer to the conjecture proposed in [3] for the maximum number of HV-palindromes in a word of size $(2, n)$ and a generalization for a word of size $(m, n)$, $m \geq 2$. We also analyzed the least number of HV-palindromes in an infinite 2D word. In future, it will be interesting to study and compare the properties of 1D palindromes and HV-palindromes.

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