1-RATIONAL SINGULARITIES AND QUOTIENTS BY REDUCTIVE GROUPS

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ABSTRACT. We prove that good quotients of algebraic varieties with 1-rational singularities also have 1-rational singularities. This refines a result of Boutot on rational singularities of good quotients.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Generalising the Hochster-Roberts theorem [HR74], Jean-François Boutot proved that the class of varieties with rational singularities is stable under taking good quotients by reductive groups, see [Bou87].

In this short note we study varieties with 1-rational singularities. This is the natural class of singular varieties to which projectivity results for Kähler Moishezon manifolds generalise, cf. [Nam02]. Our main result is:

**Theorem 3.1.** Let $G$ be a complex reductive Lie group and let $X$ be an algebraic $G$-variety such that the good quotient $\pi : X \to X//G$ exists. If $X$ has 1-rational singularities, then $X//G$ has 1-rational singularities.

In our proof, which forms a part of the author’s thesis [Gre08a], we follow [Bou87] and we check that in Boutot’s arguments it is possible to separate the different cohomology degrees.

**Theorem 3.1** has been used in [Gre08b] to prove projectivity of compact momentum map quotients of algebraic varieties.

**Acknowledgements.** The author wants to thank Miles Reid for kindly answering his questions via e-mail. The author gratefully acknowledges the financial support of the Mathematical Sciences Research Institute, Berkeley, by means of a postdoctoral fellowship during the 2009 "Algebraic Geometry" program.

2. PRELIMINARIES

2.1. **Singularities and resolution of singularities.** We work over the field $\mathbb{C}$ of complex numbers. If $X$ is an algebraic variety we denote by $\text{tdim}_x X$ the dimension of the Zariski tangent space at $x \in X$. I.e., if $m_x$ denotes the maximal ideal in $\mathcal{O}_{X,x}$, then $\text{tdim}_x X := \dim_{\mathbb{C}} m_x/m_x^2$. A point $x$ in an algebraic variety $X$ is called *singular* if $\text{tdim}_x X > \dim X$. We denote the singular set of an algebraic variety $X$ by $X_{\text{sing}}$. A variety $X$ is called *smooth* if $X_{\text{sing}} = \emptyset$. 

*Date:* 22nd January 2009.

*Mathematical Subject Classification:* 14L30, 14L24, 14B05.

*Keywords:* group actions on algebraic varieties, good quotient, singularities.
Definition 2.1. Let \( X \) be an algebraic variety. A resolution of \( X \) is a proper birational surjective morphism \( f : Y \to X \) from a smooth algebraic variety \( Y \) to \( X \).

If \( X \) is an algebraic variety, by a theorem of Hironaka [Hir64] there exists a resolution \( f : Y \to X \) by a projective morphism \( f \) such that the restriction \( f : f^{-1}(X \setminus X_{\text{sing}}) \to X \setminus X_{\text{sing}} \) is an isomorphism. See also [BM91], [EH02], and [Kol07] for later improvements and simplifications of Hironaka’s proof.

2.2. 1-rational singularities. In this section we introduce the class of singularities studied in this note.

Definition 2.2. An algebraic variety \( X \) is said to have 1-rational singularities, if the following two conditions are fulfilled:

1. \( X \) is normal,
2. for every resolution \( f : \tilde{X} \to X \) of \( \tilde{X} \), we have \( R^1 f_* \mathcal{O}_{\tilde{X}} = 0 \).

Proposition 2.3. Let \( X \) be a normal algebraic variety. If there exists one resolution \( f_0 : X_0 \to X \) such that \( R^1 (f_0)_* \mathcal{O}_{X_0} = 0 \), then \( X \) has 1-rational singularities.

Proof. Let \( f_1 : X_1 \to X \) be a second resolution of \( X \). By [Hir64], there exits a smooth algebraic variety \( Z \) and resolutions \( g_0 : Z \to X_0 \) and \( g_1 : Z \to X_1 \) such that the following diagram commutes

\[
\begin{array}{ccc}
Z & \xrightarrow{g_0} & \ \ \ X_0 \\
\downarrow & & \downarrow \ f_0 \\
X_1 & \xrightarrow{g_1} & \ \ \ X
\end{array}
\]

For \( j = 0, 1 \) there exists a spectral sequence (see [Wei94]) with lower terms

\[
0 \to R^1 (f_j)_* (g_j)_* \mathcal{O}_Z \to R^1 (f_j \circ g_j)_* \mathcal{O}_Z \to (f_j)_* (R^1 (g_j)_* \mathcal{O}_Z) \to \cdots .
\]

Since \( g_0 \) and \( g_1 \) are resolutions of smooth algebraic varieties, we have \( R^1 (g_j)_* \mathcal{O}_Z = 0 \) and \( (g_j)_* \mathcal{O}_Z = \mathcal{O}_{X_j} \) for \( j = 0, 1 \) (see [Hir64] and [Uen75]). It follows that

\[
0 = R^1 (f_0)_* \mathcal{O}_{X_0} \cong R^1 (f_0 \circ g_0)_* \mathcal{O}_Z \cong R^1 (f_1 \circ g_0)_* \mathcal{O}_Z \cong R^1 (f_1)_* \mathcal{O}_{X_1} .
\]

Remark 2.4. The proof shows that, if \( f_1 : X_1 \to X \) and \( f_2 : X_2 \to X \) are two resolutions of \( X \), there is an isomorphism \( R^1 (f_1)_* \mathcal{O}_{X_1} \cong R^1 (f_2)_* \mathcal{O}_{X_2} \). From this, it follows that having 1-rational singularities is a local property. Furthermore, if \( X \) is an algebraic \( G \)-variety for an algebraic group \( G \), and \( f : \tilde{X} \to X \) is a resolution of \( X \), then the support of \( R^1 f_* \mathcal{O}_X \) is a \( G \)-invariant subvariety of \( X \).

2.2.1. Rational singularities. In this section we shortly discuss the relation of the notion “1-rational singularity” to the more commonly used notion of “rational singularity”.

Definition 2.5. An algebraic variety \( X \) is said to have rational singularities, if the following two conditions are fulfilled:

1. \( X \) is normal,
(2) for every resolution $f : \tilde{X} \to X$ of $X$ we have $R^j f_* O_{\tilde{X}} = 0$ for all $j = 1, \ldots, \dim X$.

**Remark 2.6.** Again, the vanishing of the higher direct image sheaves is independent of the chosen resolution.

Due to a result of Malgrange [Mal57, p. 236] asserting that $R^{\dim X} f_* O_X = 0$ for every resolution $f : \tilde{X} \to X$ of an irreducible variety $X$, an algebraic surface has 1-rational singularities if and only if it has rational singularities. These notions differ in higher dimensions as is illustrated by the following example.

**Example 2.7.** Let $Z$ be a smooth quartic hypersurface in $\mathbb{P}^3$ and let $X$ be the affine cone over $Z$ in $\mathbb{C}^4$. The variety $X$ has an isolated singularity at the origin. Let $L$ be the total space of the line bundle $O_Z(-1)$, i.e., the restriction of the dual of the hyperplane bundle of $\mathbb{P}^3$ to $Z$. Then, blowing down the zero section $Z_L \subset L$ and setting $\tilde{X} := L$, we obtain a map $f : \tilde{X} \to X$ which is a resolution of singularities, an isomorphism outside of the origin $0 \in X$ with $f^{-1}(0) = Z_L \cong Z$. We claim that $0 \in X$ is a 1-rational singularity which is not rational.

To see that the origin is a normal point of $X$ it suffices to note that it is obtained as the blow-down of the maximal compact subvariety $Z_L$ of the smooth variety $L$.

To compute $(R^j f_* O_{\tilde{X}})_0$, we use the Leray spectral sequence

$\cdots \to H^j(X, O_X) \to H^j(\tilde{X}, O_{\tilde{X}}) \to H^0(X, R^j f_* O_{\tilde{X}}) \to H^{j+1}(X, O_X) \to \cdots$

and the fact that $X$ is affine to show that $H^j(\tilde{X}, O_{\tilde{X}}) \cong H^0(X, R^j f_* O_{\tilde{X}}) = (R^j f_* O_{\tilde{X}})_0$. Expanding cohomology classes into Taylor series along fibres of $L$, we get that $H^j(\tilde{X}, O_{\tilde{X}}) \cong \bigoplus_{k \geq 0} H^j(Z, O_Z(k))$. Hence, we have

$$
(1) \quad (R^j f_* O_{\tilde{X}})_0 \cong \bigoplus_{k \geq 0} H^j(Z, O_Z(k)) \quad \text{for all } j \geq 1.
$$

It follows from [1] and [Har77, Chap III, Ex 5.5] that $(R^1 f_* O_{\tilde{X}})_0 = 0$, and hence that $0 \in X$ is a 1-rational singularity. Since the canonical bundle $K_Z$ of $Z$ is trivial, it follows from Serre duality that $H^2(Z, O_Z(k)) \cong H^0(Z, O_Z(-k))$. As a consequence, we get

$$(2) \quad H^2(Z, O_Z(k)) = \begin{cases} 
\mathbb{C} & \text{for } k = 0, \\
0 & \text{otherwise}.
\end{cases}$$

Together with (1) this implies that $(R^2 f_* O_{\tilde{X}})_0 = \mathbb{C}$. Consequently, the singular point $0 \in X$ is not a rational singularity.

### 2.3. Good quotients.

**Definition 2.8.** Let $G$ be a complex reductive Lie group acting algebraically on an algebraic variety $X$. An algebraic variety $Y$ together with a morphism $\pi : X \to Y$ is called **good quotient** of $X$ by the action of $G$, if

1. $\pi$ is $G$-invariant, surjective, and affine,
2. $(\pi, O_X)G = O_Y$.

**Example 2.9.** Let $X$ be an affine $G$-variety. Then the algebra of invariants $\mathbb{C}[X]^G$ is finitely generated, and its inclusion into $\mathbb{C}[X]$ induces a regular map $\pi$ from $X$ to $Y := \text{Spec}(\mathbb{C}[X]^G)$ which fulfills (1) and (2) above, i.e., $\pi$ is a good quotient.
3. Singularities of Good Quotients: Proof of the Main Result

Let $G$ be a complex reductive Lie group and let $X$ be an algebraic $G$-variety such that the good quotient $X//G$ exists. We study the singularities of $X//G$ relative to the singularities of $X$.

More precisely, we prove the following theorem, which is the main result of this note.

**Theorem 3.1.** Let $G$ be a complex reductive Lie group and let $X$ be an algebraic $G$-variety such that the good quotient $\pi : X \rightarrow X//G$ exists. If $X$ has 1-rational singularities, then $X//G$ has 1-rational singularities.

Before proving the theorem we explain two technical lemmata. The first one discusses the relation between cohomology modules of $X$ and of $X//G$.

**Lemma 3.2.** Let $X$ be an algebraic $G$-variety with good quotient $\pi : X \rightarrow X//G$. Then the natural map $\pi^* : H^1(X//G, \mathcal{O}_{X//G}) \rightarrow H^1(X, \mathcal{O}_X)$ is injective.

**Proof.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of $X//G$. We can compute the cohomology module $H^1(X//G, \mathcal{O}_{X//G})$ via Čech cohomology with respect to the covering $\mathcal{U}$. Since $\pi$ is an affine map, $\pi^{-1}(\mathcal{U}) := \{\pi^{-1}(U_i)\}_{i \in I}$ is an affine open covering of $X$ and we can compute the cohomology module $H^1(X, \mathcal{O}_X)$ via Čech cohomology with respect to the covering $\pi^{-1}(\mathcal{U})$.

Let $\eta = (\eta_{ij}) \in C^1(\mathcal{U}, \mathcal{O}_{X//G})$ be a Čech cocycle such that the pullback of the associated cohomology class $[\eta] \in H^1(X//G, \mathcal{O}_{X//G})$ fulfills $\pi^*([\eta]) = 0 \in H^1(X, \mathcal{O}_X)$. Then, there exists a cocycle $\nu = (\nu_i) \in C^0(\pi^{-1}(\mathcal{U}), \mathcal{O}_X)$ such that

$$\pi^*(\eta_{ij}) = \nu_i|_{\pi^{-1}(U_i)} - \nu_j|_{\pi^{-1}(U_j)} \in \mathcal{O}_X(\pi^{-1}(U_{ij})).$$

Averaging $\nu_i \in \mathcal{O}_X(\pi^{-1}(U_i))$ over a maximal compact subgroup $K$ of $G$ we obtain invariant functions $\bar{\nu}_i \in \mathcal{O}_X(\pi^{-1}(U_i))^G$. Since $\pi^*(\eta_{ij}) \in \mathcal{O}_X(\pi^{-1}(U_{ij}))^G$, the cocycle $\bar{\nu} = (\bar{\nu}_i) \in C^0(\pi^{-1}(\mathcal{U}), \mathcal{O}_X)$ fulfills

$$\pi^*(\eta_{ij}) = \bar{\nu}_i|_{\pi^{-1}(U_i)} - \bar{\nu}_j|_{\pi^{-1}(U_j)} \in \mathcal{O}_X(\pi^{-1}(U_{ij})).$$

For all $i$, there exist a uniquely determined function $\tilde{\nu}_i \in \mathcal{O}_{X//G}(U_i)$ with $\pi^*(\tilde{\nu}_i) = \bar{\nu}_i$. Consequently, we have

$$\eta_{ij} = \tilde{\nu}_i|_{U_{ij}} - \tilde{\nu}_j|_{U_{ij}} \in \mathcal{O}_{X//G}(U_{ij}).$$

Therefore, $[\eta] = 0 \in H^1(X//G, \mathcal{O}_{X//G})$ and $\pi^*$ is injective, as claimed. \( \square \)

The second lemma will be used to obtain information about the singularities of an algebraic variety $X$ from information about the singularities of a general hyperplane section $H$ of $X$ and vice versa.

**Lemma 3.3.** Let $X$ be a normal affine variety with $\dim X \geq 2$ and let $f : \bar{X} \rightarrow X$ be a resolution of singularities. Let $\mathcal{L} \subset \mathcal{O}_X(X)$ be a finite-dimensional subspace, such that the associated linear system is base-point free, and such that the image of $X$ under the associated map $\varphi_{\mathcal{L}} : X \rightarrow \mathbb{P}_H$ fulfills $\dim(\varphi_{\mathcal{L}}(X)) \geq 2$. If $h \in \mathcal{L}$ is a general element, then the following holds for the corresponding hyperplane section $H \subset X$:

1. The preimage $\bar{H} := f^{-1}(H)$ is smooth and $f|_{\bar{H}} : \bar{H} \rightarrow H$ is a resolution of $H$.
2. We have $R^if_*\mathcal{O}_H \cong R^if_*\mathcal{O}_{\bar{H}} \otimes \mathcal{O}_H$ for $j = 0, 1$. 

\( \square \)
Proof. 1) This follows from Bertini’s theorem (see [Har77, Chap III, Cor 10.9]).

2) In the exact sequence

\[ 0 \to \mathcal{O}_X(-\tilde{H}) \overset{m_0}{\to} \mathcal{O}_X \to \mathcal{O}_{\tilde{H}} \to 0, \]

the map \( m \) is given by multiplication with the equation \( h \in \mathcal{O}_X(X) \) defining \( H \) and \( \tilde{H} \). Pushing forward the short exact sequence \((3)\) by \( f_* \) yields the long exact sequence \((4)\)

\[ 0 \to f_* \mathcal{O}_X(-\tilde{H}) \overset{m_0}{\to} f_* \mathcal{O}_X \to f_* \mathcal{O}_{\tilde{H}} \to \]

\[ \to R^1f_* \mathcal{O}_X(-\tilde{H}) \overset{m_1}{\to} R^1f_* \mathcal{O}_X \to R^1f_* \mathcal{O}_{\tilde{H}} \to \]

\[ \to R^2f_* \mathcal{O}_X(-\tilde{H}) \overset{m_2}{\to} R^2f_* \mathcal{O}_X \to R^2f_* \mathcal{O}_{\tilde{H}} \to \]

\[ \to \ldots. \]

Since \( X \) is an affine variety, the exact sequence above is completely determined by the following sequence of finite \( \mathcal{O}_X(X) \)-modules:

\[ 0 \to \Gamma(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \overset{m_0}{\to} \Gamma(\tilde{X}, \mathcal{O}_\tilde{X}) \to \Gamma(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \]

\[ \to H^1(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \overset{m_1}{\to} H^1(\tilde{X}, \mathcal{O}_\tilde{X}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \]

\[ \to H^2(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \overset{m_2}{\to} H^2(\tilde{X}, \mathcal{O}_\tilde{X}) \to H^2(\tilde{X}, \mathcal{O}_{\tilde{H}}) \to \]

\[ \to \ldots. \]

The maps \( m_j, j = 0, 1, 2 \) are given by multiplication with the element \( h \in \mathcal{O}_X(X) \). We claim that we can choose \( h \in \mathcal{L} \) in such a way that \( m_1 \) and \( m_2 \) are injective. Indeed, if \( R \) is a commutative Noetherian ring with unity, \( M \) is a finite \( R \)-module, and \( Z_R(M) \) denotes the set of zero-divisors for \( M \) in \( R \), we have

\[ Z_R(M) = \bigcup_{P \in \text{Ass } M} P, \]

where \( \text{Ass } M \) is the finite set of assassins (or associated primes) of \( M \). Hence, the set of zero-divisors for \( H^1(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \) and \( H^2(\tilde{X}, \mathcal{O}_\tilde{X}(-\tilde{H})) \) is a union of finitely many prime ideals \( P \) of \( \mathcal{O}_X(X) \). Since the linear system associated to \( \mathcal{L} \) is base-point free, the general element \( h \) of \( \mathcal{L} \) lies in \( \mathcal{L} \setminus \bigcup P \). For such a general \( h \in \mathcal{L} \setminus \bigcup P \), the maps \( m_1 \) and \( m_2 \) in the sequences \((5)\) and \((4)\) are injective.

Since \( \mathcal{O}_\tilde{X}(-\tilde{H}) \cong f^*(\mathcal{O}_X(-H)) \), the projection formula for locally free sheaves (see [Har77, Chap III, Ex 8.3]) yields

\[ R^if_* \mathcal{O}_\tilde{X}(-\tilde{H}) \cong R^if_* \mathcal{O}_\tilde{X} \otimes \mathcal{O}_X(-H). \]

Furthermore, for \( j = 0, 1 \) the image of \( m_j \) coincides with the image \( \mathcal{B}_j \) of the natural map \( R^if_*, \mathcal{O}_\tilde{X} \otimes \mathcal{O}_X(-H) \to R^if_*, \mathcal{O}_\tilde{X} \). Since \( m_1 \) and \( m_2 \) are injective by the choice of \( H \), it follows that

\[ R^if_*, \mathcal{O}_\tilde{H} \cong R^if_*, \mathcal{O}_\tilde{X} / \mathcal{B}_j \quad \text{for } j = 0, 1. \]

Tensoring with \( R^if_*, \mathcal{O}_\tilde{X} \) is right-exact, and hence the exact sequence

\[ 0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0 \]

yields \( R^if_*, \mathcal{O}_\tilde{H} \cong R^if_*, \mathcal{O}_\tilde{X} \otimes \mathcal{O}_H \), as claimed. \( \square \)
Proof of Theorem 3.1. Since the claim is local and \( \pi \) is an affine map, we may assume that \( X//G \) and \( X \) are affine.

First, we prove that normality of \( X \) implies normality of \( X//G \). We have to show that \( C[X//G] \cong C[X]^G \) is a normal ring. So let \( \alpha \in \text{Quot}(C[X]^G) \subset C(X)^G \) be an element of the quotient field of \( C[X]^G \) and assume that \( \alpha \) fulfills a monic equation

\[
a^n + c_1 a^{n-1} + \cdots + c_n = 0
\]

with coefficients \( c_j \in C[X]^G \subset C[X] \). Since \( C[X] \) is normal by assumption, it follows that \( \alpha \in C(X)^G \cap C[X] = C[X]^G \). Hence, \( C[X]^G \) is normal. As a consequence, we can assume in the following that \( X \) is \( G \)-irreducible.

We prove the claim by induction on \( \dim X//G \). For \( \dim X//G = 0 \) there is nothing to show. For \( \dim X//G = 1 \) we notice that \( X//G \) is smooth. So, let \( \dim X//G \geq 2 \). Let \( \pi : X \to X//G \) denote the quotient map and let \( p_X : \tilde{X} \to X \) be a resolution of \( X \). First, we prove that a general hyperplane section \( H \subset X//G \) has 1-rational singularities. If \( H \) is a general hyperplane section in \( X//G \), Lemma 3.3 applied to \( \pi^{-1}(H) \) yields that \( p_X|_{\tilde{H}} : \tilde{H} \to \pi^{-1}(H) \) is a resolution, where \( \tilde{H} = p_X^{-1}(\pi^{-1}(H)) \), and that

\[
R^j(p_X)_\ast \mathcal{O}_{\tilde{X}} \otimes \mathcal{O}_{\pi^{-1}(H)} = R^j(p_X)_\ast \mathcal{O}_{\tilde{H}} \quad \text{for } j = 0, 1.
\]

Since \( f_\ast \mathcal{O}_{\tilde{X}} = \mathcal{O}_X \) by Zariski’s main theorem, it follows from the case \( j = 0 \) that \( \pi^{-1}(H) \) is normal. Alternatively, one could invoke Seidenberg’s Theorem (see e.g. [BS95, Thm. 1.7.1]). Together with the case \( j = 1 \), this implies that \( \pi^{-1}(H) \) has 1-rational singularities. By induction, it follows that \( H = \pi^{-1}(H)//G \) has 1-rational singularities.

Let \( p : Z \to X//G \) be a resolution of \( X//G \). As we have seen above, a general hyperplane section \( H \) of \( X//G \) has 1-rational singularities and the restriction of \( p \) to \( \tilde{H} := p^{-1}(H) \) is a resolution of \( H \). It follows that \( \mathcal{O}_H \otimes R^1 p_\ast \mathcal{O}_Z = R^1 p_\ast \mathcal{O}_{\tilde{H}} = 0 \). Consequently, the support of \( R^1 p_\ast \mathcal{O}_Z \) does not intersect \( H \) and hence, \( \text{supp}(R^1 p_\ast \mathcal{O}_Z) \) consists of isolated points.

Since the claim is local, we can assume in the following that \( R^1 p_\ast \mathcal{O}_Z \) is supported at a single point \( x_0 \in X//G \).

The group \( G \) acts on the fibre product \( Z \times_{X//G} X \) such that the map \( p_X : Z \times_{X//G} X \to X \) is equivariant. One of the \( G \)-irreducible components \( \tilde{X} \) of \( Z \times_{X//G} X \) is birational to \( X \), and, by passing to a resolution of \( \tilde{X} \) if necessary, we can assume that \( p_X : \tilde{X} \to X \) is a resolution of \( X \).

We obtain the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p_X} & \tilde{X} \\
\pi \downarrow & & \downarrow p_Z \\
X//G & \leftarrow & Z.
\end{array}
\]

Since \( R^1 p_\ast \mathcal{O}_Z \) is supported only at \( x_0 \), we have \( (R^1 p_\ast \mathcal{O}_Z)_{x_0} = H^0(X//G, R^1 p_\ast \mathcal{O}_Z) \). Recall that \( X//G \) is affine, hence, the Leray spectral sequence

\[
0 \to H^1(X//G, \mathcal{O}_{X//G}) \to H^1(Z, \mathcal{O}_Z) \to H^0(X//G, R^1 p_\ast \mathcal{O}_Z) \to H^2(X//G, \mathcal{O}_{X//G}) \to \cdots
\]

implies that it suffices to show that \( H^1(Z, \mathcal{O}_Z) = 0 \). 

\[\square\]
Since $X$ is affine and has 1-rational singularities, it follows from the Leray spectral sequence
\[ 0 \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^0(X, R^1f_*\mathcal{O}_{\tilde{X}}) \to \cdots \]
that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(X, \mathcal{O}_X) = 0$. Consequently, it suffices to show that there exists an injective map $H^1(Z, \mathcal{O}_Z) \hookrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

We introduce the following notation: $U = (X//G) \setminus \{x_0\}$, $U' = \pi^{-1}(U) \subset X$, $\tilde{U} = p_{\tilde{X}}^{-1}(U') \subset \tilde{X}$. Let $\mathcal{O}_U$ be a resolution. Let $w \in W$ and $Y = p^{-1}(w)$. Then $H^i_Y(Z, \mathcal{O}_Z) = 0$ for all $i < \dim W$.

\textbf{Sketch of proof.} Set $n := \dim W$. As a first step, we compactify $p$. There exist projective completions $\overline{Z}$ and $\overline{W}$ of $Z$ and $W$, respectively, and a resolution $\overline{p} : \overline{Z} \to \overline{W}$ such that $\overline{p}^{-1}(W) = Z$ and $\overline{p}|_Z = p$. Since $Z$ is an open neighbourhood of $Y$ in $\overline{Z}$, the Excision Theorem of Local Cohomology (see [Har77], Chap III, Ex 2.3) implies that $H^1_Y(\overline{Z}, \mathcal{O}_{\overline{Z}}) = H^1_Y(Z, \mathcal{O}_Z)$. Let $\mathcal{K}_{\overline{Z}}$ be the locally free sheaf associated to the canonical bundle of $\overline{Z}$. The Formal Duality Theorem (see [Har70]) implies that the dual $H^j_Y(\overline{Z}, \mathcal{O}_{\overline{Z}})^* \cong H^j_Y(\overline{Z}, \mathcal{O}_{\overline{Z}})$ is isomorphic to $(R^{n-j}/\overline{p}_*\mathcal{K}_{\overline{Z}})_w$, where $\mathcal{K}_{\overline{Z}}$ denotes completion with respect to the maximal ideal of $\mathcal{O}_{\overline{Z},w} = \mathcal{O}_{\overline{W},w}$. In summary, we have obtained an isomorphism
\[ H^j_Y(Z, \mathcal{O}_Z)^* \cong (R^{n-j}/\overline{p}_*\mathcal{K}_{\overline{Z}})_w \quad \text{for all } j = 0, \ldots, n. \]

By Grauert-Riemenschneider vanishing (see e.g. [Laz04] Chap 4.3.B)), the term on the right hand side equals zero for $n-j \geq 1$. This proves the claim. 

Since $\dim X//G \geq 2$, Proposition 3.4 yields $H^1_Y(Z, \mathcal{O}_Z) = 0$. As a consequence of (7), the map $h_{Z,V}$ is injective.

The restriction of $p$ to $V = p^{-1}(U)$ is a resolution of $U$. Since the support of $R^1p_*\mathcal{O}_Z$ is concentrated at $x_0$, the variety $U$ has 1-rational singularities, and the Leray spectral sequence
\[ 0 \to H^1(U, \mathcal{O}_U) \xrightarrow{h_{U,V}} H^1(V, \mathcal{O}_V) \to H^0(U, R^1p_*\mathcal{O}_U) \to \cdots \]
yields that $h_{U,V}$ is bijective. Similar arguments show that $h_{U',\tilde{U}}$ is bijective. Furthermore, Lemma 3.2 implies that $h_{U',U'}$ is injective.
By the considerations above, the map
\[ h_{Z,\tilde{U}} := h_{U',\tilde{U}} \circ h_{U,\tilde{U}} \circ h_{U,V}^{-1} \circ h_{Z,V} \]
is injective. Diagram (6) implies \( h_{Z,\tilde{U}} = h_{\tilde{X},\tilde{U}} \circ h_{Z,\tilde{X}} \), and therefore \( h_{Z,\tilde{X}} \) is injective. Consequently, we have \( H^1(Z, \mathcal{O}_Z) = 0 \). This concludes the proof of Theorem 3.1. \( \square \)

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