Localization of the first eigenfunction of a convex domain

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Abstract

We study the first Dirichlet eigenfunction of the Laplacian in a n-dimensional convex domain. For domains of a fixed inner radius, estimates of Chiti [5], [6], imply that the ratio of the $L^2$-norm and $L^\infty$-norm of the eigenfunction is minimized when the domain is a ball. However, when the eccentricity of the domain is large the eigenfunction should spread out at a certain scale and this ratio should increase. We make this precise by obtaining a lower bound on the $L^2$-norm of the eigenfunction and show that the eigenfunction cannot localize to too small a subset of the domain. As a consequence, we settle a conjecture of van den Berg, [17], in the general n-dimensional case.

The main feature of the proof is to obtain sufficiently sharp estimates on the first eigenvalue in order to estimate the first derivatives of the eigenfunction.

1 Introduction and statement of results

Let $\Omega \subset \mathbb{R}^n$ be a convex domain and let $\lambda$ be the first eigenvalue of the Dirichlet Laplacian on $\Omega$. We denote the corresponding eigenfunction by $u$ so that

$$
\begin{cases}
(\Delta + \lambda)u = 0 \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega.
\end{cases}
$$

This first eigenfunction is of one sign, and we choose it so that $u(x) > 0$ in $\Omega$. Our starting point for studying the behaviour of $u$ and its level sets is that the convexity of $\Omega$ ensures that $u$ is log-concave, [4]. In particular the superlevel sets

$$\{x \in \Omega : u(x) > c\}$$

are convex subsets of $\Omega$. It is natural to study the shape of the level sets of $u$ and how they depend on the geometry of $\Omega$ and the level under consideration. The quantity $|u(x)|^2$ can be interpreted as an (unnormalized) density for a free quantum particle in the domain $\Omega$. The shape and location of the superlevel sets where $u$ is comparable to its maximum value therefore correspond to the parts of $\Omega$ where the particle is most likely to be found. In this paper, we will obtain a lower bound on the $L^2(\Omega)$-norm of $u$ in terms of its $L^\infty(\Omega)$-norm and length scales coming from the shape of $\Omega$ (see Theorem 1.1 below).

Where Laplace eigenfunctions localize, that is the region of $\Omega$ where they are of large magnitude relative to the rest of the domain, has received recent attention. For example, the torsion function has been used as a landscape function for predicting where Laplace eigenfunctions will localize, [2], [1], [16]. While in general the first eigenfunction can localize to a small subset of $\Omega$, relative to $\Omega$ itself, our result will place a restriction on how small this region can be.

In [2], [3], Chiti provides a lower bound on the $L^2(\Omega)$-norm of $u$ of the form

$$
\|u\|_{L^2(\Omega)} \geq c_n^* \text{ inrad}(\Omega)^{n/2} \|u\|_{L^\infty(\Omega)}
$$

(1)

Here $\text{inrad}(\Omega)$ is the inner radius of $\Omega$. The constant $c_n^* > 0$ depends only on the dimension, and is explicitly given in terms of Bessel functions (and their zeros). (In fact, this bound holds for any bounded, connected domain $\Omega$.) Moreover, the constant $c_n^*$ cannot be improved since equality in (1) holds when $\Omega$ is a ball. However, for $\Omega$ convex and when the diameter of $\Omega$ is large compared to its inner
radius, one expects the eigenfunction to *spread out* along the diameter of $\Omega$, and for the $L^2(\Omega)$-norm to increase relative to the $L^\infty(\Omega)$-norm. In terms of the estimate in (1), the question is then whether an estimate of the form
\[
\| u \|_{L^2(\Omega)} \geq c_n (\text{diam}(\Omega)/\text{inrad}(\Omega))^n \text{inrad}(\Omega)^{n/2} \| u \|_{L^\infty(\Omega)}
\]
holds for all convex $\Omega$, and some uniform $\alpha > 0$. Repeated applications of the Harnack inequality in overlapping balls is not sufficient to establish $\| u \|_{L^2(\Omega)}$ for any $\alpha > 0$, and so any improvement of (1) must use the fact that $u$ is an eigenfunction in a fundamental way. Kröger, [14], in two dimensions, and van den Berg, [17], in higher dimensions studied the first eigenfunction of a thin sector. Via a separation of variables in polar coordinates, and the properties of the resulting Bessel function in the radial variable, this example of the sector ensures that the maximal value of $\alpha$ for which (2) could hold is $\alpha = \frac{1}{6}$. Based on the intuition that the sector should be the convex domain for which the eigenfunction spreads out the least, van den Berg made the following conjecture:

**Conjecture 1** ([17]) There exists a constant $c_n > 0$, depending only on the dimension $n$, such that
\[
\| u \|_{L^2(\Omega)} \geq c_n (\text{diam}(\Omega)/\text{inrad}(\Omega))^{1/6} \text{inrad}(\Omega)^{n/2} \| u \|_{L^\infty(\Omega)}.
\]

The two dimensional case of this conjecture has been established in [8]. Their proof uses an eigenvalue bound for the first eigenvalue of a class of one dimensional Schrödinger operators, and the work of Grieser and Jerison, [12], [11], on the first eigenfunction of a convex, planar domain.

In this paper, we bound $\| u \|_{L^2(\Omega)}$ from below in the general $n$-dimensional case. We call $K$ a John ellipsoid associated to $\Omega \subset \mathbb{R}^n$ if $K$ is contained within $\Omega$ and the dilation of $K$ about its centre with scaling factor $n$ contains $\Omega$. John’s lemma [13] ensures that such an ellipsoid $K$ exists. We now fix a John ellipsoid $K$ and define $N_j$ to be the lengths of the axes of $K$ with
\[
N_1 \geq N_2 \geq \cdots \geq N_n.
\]
Our main theorem provides a lower bound on the scale at which the eigenfunction can localize by establishing a lower bound on the $L^2(\Omega)$-norm of $u$ in terms of its $L^\infty(\Omega)$-norm, and the length scales $N_j$.

**Theorem 1.1** There exists a constant $c_n > 0$, depending only on the dimension $n$, such that
\[
\| u \|_{L^2(\Omega)} \geq c_n N_n^{n/2} \prod_{j=1}^{n-1} \left( \frac{N_j}{N_n} \right)^{1/6} \| u \|_{L^\infty(\Omega)}.
\]

In particular, $\prod_{j=1}^{n-1} \left( \frac{N_j}{N_n} \right)^{1/6} \geq \left( \frac{N_1}{N_n} \right)^{1/6}$, and $N_1, N_n$ are comparable to the diameter, inner radius of $\Omega$, up to a factor depending only on $n$. Therefore, Theorem 1.1 settles Conjecture 1.

**Remark 1.1** Let $M_1 \geq M_2 \geq \cdots \geq M_n$ be the lengths of the axes of a John ellipsoid for the superlevel set $\{ x \in \Omega : u(x) > \frac{1}{2} \max_{\Omega} u \}$. In the course of proving Theorem 1.1 we will show that $M_j \geq c_n N_j^{1/3}$ for some constant $c_n > 0$. In terms of localization, this shows that the eigenfunction does not localize in a subset of $\Omega$ smaller than this.

To prove Theorem 1.1 we first obtain an upper bound on the directional derivatives of $u$ in terms of the length scales $N_j$. After a rotation we will assume that the axes of $K$ lie along the coordinate axes.

**Theorem 1.2** There exists a constant $C_n$, depending only on the dimension $n$, such that for each $j$, $1 \leq j \leq n$, the derivative $\partial_{x_j} u(x)$ satisfies
\[
\| \partial_{x_j} u \|_{L^2(\Omega)} \leq C_n N_n^{-1} \left( \frac{N_j}{N_n} \right)^{-1/3} \| u \|_{L^2(\Omega)}.
\]

**Remark 1.2** If we denote $u_m$ to be the $m$-th Dirichlet eigenfunction of $\Omega$, then the estimate in Theorem 1.2 continues to hold, with a constant $C_n$ replaced by a constant $C_{m,n}$ depending only on $m$ and $n$. 

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2. To prove Theorem 1.1, we will also use in a crucial way the log concavity of the eigenfunction. In particular, this will allow us to reduce estimating the $L^1$ and we will also use the variational formulation of the first eigenvalue, $\lambda_1$, on the first Dirichlet eigenvalue of $(\Omega;u)$. Via a dilation we can also assume that $N_n = 1$ when proving these theorems, and by taking a constant multiple of $u$, we also assume that $\max_\Omega u = 1$. To prove Theorem 1.2 we will begin by using the eigenfunction equation to write

$$\int_\Omega |\nabla u|^2 \, dx = \lambda \int_\Omega |u|^2 \, dx,$$

and we will also use the variational formulation of the first eigenvalue,

$$\lambda = \inf \left\{ \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx} : v \in H^1_0(\Omega), v \neq 0 \right\}.$$

These can be combined to reduce the proof of Theorem 1.2 to obtaining sufficiently sharp upper bounds on the eigenvalue $\lambda$ in terms of the eigenvalues of $(n-j)$-dimensional cross-sections of $\Omega$ (see Proposition 2.2). We prove the desired eigenvalue bounds by induction on $j$, and will carry out the proof in Section 2. To prove Theorem 1.1 we will also use in a crucial way the log concavity of the eigenfunction $u$, $\|u\|_{L^2(\Omega)}$. In particular, this will allow us to reduce estimating the $L^2(\Omega)$-norm of $u$ to estimating the lengths of the axes of a John ellipsoid associated to the superlevel set

$$\Omega_{1/2} = \{ x \in \Omega : u(x) > \frac{1}{2} \}.$$

The desired estimate follows from using the derivative bounds in Theorem 1.2 and we will prove Theorem 1.1 in Section 2. Finally, in Section 3 we discuss known estimates in the two dimensional case, and future directions in higher dimensions. In [12], Jerison introduces a length scale $L$ depending on the geometry of the convex, planar domain, and together with Grieser uses it to study the shape of the first (and second) eigenfunction, $\|u\|_{L^2(\Omega)}$. In particular, their results imply comparable upper and lower bounds on $\|u\|_{L^2(\Omega)}$ in terms of this length scale $L$. It is natural to ask how to construct analogous length scales controlling the shape of the first eigenfunction in higher dimensions, and in Section 4 we discuss this in more detail.

Remark 1.3 Throughout, constants which we will denote by $C, C_1, c_1$ etc, are constants which depend only on the dimension. We also say that two quantities are comparable (and write as $\sim$) if they can be bounded in terms of each other up to a constant depending only on $n$.

2 Gradient bounds for the eigenfunction

In this section we prove Theorem 1.2. The key step in the proof is to obtain appropriate upper bounds on the eigenvalue $\lambda$. In fact, we will carry out an inductive step, which will require estimates on the first Dirichlet eigenvalue of $(n-k)$-dimensional cross-sections of $\Omega$ for $0 \leq k \leq n - 1$. To write down the eigenvalue bounds that we will establish, we first introduce the following notation: Given $i$, with $1 \leq i \leq n - k - 1$, and a point $x \in \mathbb{R}^{n-k}$, we write

$$x = (x_1, x_2, \ldots, x_{n-k}) = (X_i, X'_{n-k-i}) \in \mathbb{R}^{n-k},$$

with $X_i \in \mathbb{R}^i$, $X'_{n-k-i} \in \mathbb{R}^{n-k-i}$. Now let $W$ be a $(n-k)$-dimensional convex domain. For each $Y_i \in \mathbb{R}^i$, we denote the $(n-k-i)$-dimensional cross-sections of $W$ by

$$W(Y_i) = \{ x = (X_i, X'_{n-k-i}) \in W : X_i = Y_i \}.$$

For us, $W$ will either be the original convex domain $\Omega$ (with $k = 0$) or a $(n-k)$-dimensional cross-section of $\Omega$, for some $1 \leq k \leq n - 1$.

Definition 2.1 For a $(n-k)$-dimensional convex domain $W$, let $\lambda(W)$ be its first Dirichlet eigenvalue. For $i$, with $1 \leq i \leq n - k - 1$, and $Y_i \in \mathbb{R}^i$, let $\mu(Y_i;W)$ be the first Dirichlet eigenvalue of $W(Y_i)$, and define $\mu_i^*(W)$ by

$$\mu_i^*(W) = \min_{Y_i} \mu(Y_i;W).$$
We also formally define $\mu_{n-k}^*(W) = 0$, and then for $1 \leq i \leq n - k$ set

$$\delta_i(W) = \lambda(W) - \mu_i^*(W).$$

We can obtain gradient bounds on the first Dirichlet eigenfunction of $W$ in terms of $\delta_i(W)$ via the following proposition.

**Proposition 2.2** Let $u_W(x)$ be the first Dirichlet eigenfunction of $W$. Then, for each $1 \leq i \leq n - k$, with $\delta_i(W)$ as in Definition 2.1, the gradient bounds

$$\sum_{\ell=1}^{i} \int_W |\partial_{x_{\ell}} u_W(x)|^2 \, dx \leq \delta_i(W) \int_W |u_W(x)|^2 \, dx$$

hold. In particular, $\delta_i(W) \geq 0$ for all $i$.

**Proof of Proposition 2.2** Since $u_W$ is a Dirichlet eigenfunction with eigenvalue $\lambda(W)$ we have

$$\int_W |\nabla u_W(x)|^2 \, dx = \lambda(W) \int_W |u_W(x)|^2 \, dx.$$  \hspace{1cm} (4)

For $i = n - k$, we have $\delta_{n-k}(W) = \lambda(W)$ and then the estimate holds (with equality) immediately. We now fix $i$ with $1 \leq i < n - k$. For each $X_i \in \mathbb{R}^d$ such that $W(X_i)$ is non-empty, the function $u_W(X_i, \cdot)$ is an admissible test function for the first eigenvalue on $W(X_i)$. Therefore,

$$\sum_{\ell=i+1}^{n-k} \int_{W(X_i)} |\partial_{x_{\ell}} u_W(X_i, X'_{n-k-i})|^2 \, dX'_{n-k-i} \geq \mu_i(X_i; W) \int_{W(X_i)} |u_W(X_i, X'_{n-k-i})|^2 \, dX'_{n-k-i} \geq \mu_i^*(W) \int_{W(X_i)} |u_W(X_i, X'_{n-k-i})|^2 \, dX'_{n-k-i}.$$

Since this holds for each $X_i$, we integrate in $X_i$ and then use it in (4) to get

$$\mu_i^*(W) \int_W |u_W(x)|^2 \, dx + \sum_{\ell=1}^{i} \int_W |\partial_{x_{\ell}} u_W(x)|^2 \, dx \leq \lambda(W) \int_W |u_W(x)|^2 \, dx.$$

The estimate in the proposition then follows from the definition of $\delta_i(W)$. \hspace{1cm} \Box

As before, we set $u(x) = u_{\Omega}(x)$, $\lambda = \lambda(\Omega)$, and for ease of notation, we write $\mu(Y; \Omega) = \mu(Y_i)$, $\mu_i^* = \mu_i^*(\Omega)$. Using Proposition 2.2 in order to prove Theorem 1.2 it is sufficient to establish the following eigenvalue bounds.

**Proposition 2.3** There exists a constant $C_n$ such that for all $j$, $1 \leq j \leq n$,

$$\mu_j^* \leq \lambda \leq \mu_j^* + C_n N_j^{-2/3}.$$  

From Proposition 2.2 we have $\lambda - \mu_j^* \geq 0$, and so we only need to prove the upper bound. Since $\mu_n^* = 0$, and $\Omega$ has inner radius comparable to $N_n = 1$, the estimate in the proposition certainly holds for $j = n$. We will prove Proposition 2.3 by induction on $j$ (starting with $j = n$ as the base case, and then decreasing $j$). To establish the inductive step we will use the variational formulation of the first Dirichlet eigenvalue. We will construct an appropriate test function involving the eigenfunctions corresponding to the minimal eigenvalue $\mu_j^*$ of the $j$-dimensional cross-sections of $\Omega$. To demonstrate the method let us first use it to prove the proposition in the two dimensional case. (In two dimensions, the estimate in Proposition 2.3 is also contained in the work of Jerison [12] and Grieser-Jerison [11].)
Proof of Proposition 2.3 in two dimensions: In the two dimensional case, we just need to consider \( j = 1 \). After a translation along the \( x_1 \)-axis, we may assume that the minimal value \( \mu_1^* = \mu_1(Y_1) \) is attained at \( Y_1 = 0 \). (Note that this point is at the height of the domain \( \Omega \) in the \( x_2 \)-direction is largest.) Let \( \psi(x_2) \) be the corresponding \( L^2(\Omega(0)) \)-normalized first Dirichlet eigenfunction of the interval \( \Omega(0) \), extended to be zero outside of \( \Omega(0) \). By the properties of the John ellipsoid of \( \Omega \), we can find a point \( x = (x_1, x_2) \in \Omega \) with \( |x_1| = N_1 \), and so without loss of generality, we assume that \( x^* = (N_1, x_2^*) \in \Omega \) for some \( x_2^* \). By translating in the \( x_2 \)-direction we may assume that \( x_2^* = 0 \), and after this translation there still exists a constant \( C \) such that \( |x_2| \leq C \) on the support of \( \psi(x_2) \).

We now define a test function that we can use in the variational formulation of the first eigenvalue \( \lambda \): We set \( v(x_1, x_2) \) to be the function

\[
v(x_1, x_2) = \chi(x_1) \psi(x_2 N_1/(N_1 - x_1)) .
\]

Here \( \chi(x_1) \geq 0 \) is a smooth cut-off function, such that

\[
\begin{align*}
\chi(x_1) &= 1 \text{ for } \frac{2}{3} N_1^{1/3} \leq x_1 \leq N_1^{1/3}, \\
\chi(x_1) &= 0 \text{ for } x_1 \geq 2 N_1^{1/3}, \quad x_1 \leq \frac{2}{3} N_1^{1/3}.
\end{align*}
\]

The function \( \chi(x_1) \) can in particular be chosen so that \( |\chi'(x_1)| \leq CN_1^{-1/3} \). The domain \( \Omega \) contains the interval \( \Omega(0) \) and the point \( x^* = (N_1, 0) \), and so also contains the convex hull of these two sets. Therefore, for each \( x_1 \in [0, N_1] \), the cross-section \( \Omega(x_1) \) contains the interval \( \frac{N_1 - x_1}{N_1} \Omega(0) \). In particular, this ensures that \( v(x_1, x_2) \) is equal to zero on the complement of \( \Omega \), and we can use it in the variational formulation of the first eigenvalue \( \lambda \). That is,

\[
\lambda \leq \frac{\int_{\Omega} |\nabla v(x)|^2 \, dx}{\int_{\Omega} |v(x)|^2 \, dx} .
\]

We can write the right hand side of (5) as

\[
\begin{align*}
\int_{\Omega} \chi(x_1)^2 \frac{N_2}{(N_1 - x_1)^2} |\psi'(x_2 N_1/(N_1 - x_1))|^2 \, dx &+ \int_{\Omega} |\partial_{x_1} v(x)|^2 \, dx \\
\int_{\Omega} \chi(x_1)^2 |\psi(x_2 N_1/(N_1 - x_1))|^2 \, dx &+ \int_{\Omega} |\partial_{x_1} v(x)|^2 \, dx,
\end{align*}
\]

and on the support of \( \chi(x_1) \) we have

\[
\left| \frac{N_1}{N_1 - x_1} - 1 \right| \leq CN_1^{-2/3}.
\]

Therefore, since \( \psi(x_2) \) is an eigenfunction on \( \Omega(0) \) with eigenvalue \( \mu_1^* \), we have

\[
\lambda \leq \mu_1^* + CN_1^{-2/3} + \frac{\int_{\Omega} |\partial_{x_1} v(x)|^2 \, dx}{\int_{\Omega} |v(x)|^2 \, dx} .
\]

The \( x_1 \)-derivative of \( v \) is given by

\[
\partial_{x_1} v(x_1, x_2) = \chi'(x_1) \psi(x_2 N_1/(N_1 - x_1)) - \chi(x_1) \frac{x_2 N_1}{(N_1 - x_1)^2} \psi'(x_2 N_1/(N_1 - x_1)) .
\]

We have \( |\chi'(x_1)| \leq CN_1^{-1/3} \), \( |\psi'(x_2 N_1/(N_1 - x_1))| \leq C \), and \( |x_2| \leq C \) on the support of \( \psi \). Combining this with the estimate \( N_1/(N_1 - x_1)^2 \leq CN_1^{-1} \) on the support of \( \chi(x_1) \), from (7) we obtain

\[
\lambda \leq \mu_1^* + CN_1^{-2/3} ,
\]

as required.

We now prove the general case.
Proof of Proposition 2.2. We first recall that the estimate in the proposition holds for \( j = n \), and that the lower bound holds for all \( j \). We will prove the upper bound by induction on \( j \), using \( j = n \) as the base case. Our inductive hypothesis is that there exists constants \( C_j \) such that

\[
\lambda \leq \mu_j^* + C_j N_j^{-2/3}
\]

for \( k + 1 \leq j \leq n \), and we will prove that there exists a constant \( C_k \) such that (8) holds for \( j = k \). Analogously to the two dimensional case, we will prove this estimate by using an appropriate test function in the variational formulation for \( \lambda \). The minimal value \( \mu_k^* \) is given by \( \mu(Y_k) \) for some \( Y_k \in \mathbb{R}^k \), and we let \( \psi(X'_{n-k}) \) be the \( L^2(\Omega(Y_k)) \)-normalized first Dirichlet eigenfunction of the \((n - k)\)-dimensional cross-section \( \Omega(Y_k) \), and extended to be zero outside \( \Omega(Y_k) \). (We recall that in our notation \( X'_{n-k} = (x_{k+1}, x_{k+2}, \ldots, x_n) \).) Our test function will involve this eigenfunction, and we first use Proposition 2.2 to establish bounds on the components of the gradient of \( \psi(X'_{n-k}) \), under the inductive hypothesis.

Lemma 2.4. Assuming that the estimate in (8) holds for \( j \) satisfying \( k + 1 \leq j \leq n \), there exists a constant \( C \) (depending on the constants \( C_j \)) so that for each such \( j \) in this range,

\[
\int_{\Omega(Y_k)} |\partial_{x_j} \psi(X'_{n-k})|^2 \, dX'_{n-k} \leq C N_j^{-2/3} \int_{\Omega(Y_k)} |\psi(X'_{n-k})|^2 \, dX'_{n-k} = C N_j^{-2/3}.
\]

Proof of Lemma 2.4. The eigenfunction \( \psi(X'_{n-k}) \) on \( \Omega(Y_k) \) has eigenvalue \( \mu_k^* \), and analogously to Definition 2.1 for \( k + 1 \leq j \leq n \), we define \( \mu_k^* \) to be the minimum eigenvalue over all \((n-j)\)-dimensional cross-sections of \( \Omega(Y_k) \) in the \( X_{n-j} \) variables. Since \( \Omega(Y_k) \subset \Omega \), by the definitions of the minima \( \mu_{k,j}^* \) and \( \mu_j^* \) we automatically have

\[
\mu_j^* \leq \mu_{k,j}^*.
\]

Combining this with the inductive hypothesis in (8), for each \( k + 1 \leq j \leq n \) we obtain

\[
\mu_k^* \leq \lambda \leq \mu_j^* + C_j N_j^{-2/3} \leq \mu_{k,j}^* + C_j N_j^{-2/3}.
\]

Therefore, setting \( W \) to be the \((n-k)\)-dimensional convex domain \( \Omega(Y_k) \), and using the notation from Definition 2.1 we have

\[
\delta_i(W) = \lambda(W) - \mu_i^*(W) = \mu_k^* - \mu_{k,i+k}^* \leq C_{i+k} N_{i+k}^{-2/3}.
\]

for \( 1 \leq i \leq n-k \). The gradient bounds in the statement of the lemma then immediately follow from Proposition 2.2 using that \( \psi(X'_{n-k}) \) is \( L^2(\Omega(Y_k)) \)-normalized.

We now define the test function that we will use to bound \( \lambda \). We first translate the domain \( \Omega \) in the \( X_k \)-variables so that the point \( Y_k \) with \( \mu(Y_k) = \mu_k^* \) is at the origin, which we denote by \( 0_k \). Then, using the above notation, \( \psi(X'_{n-k}) \) is the first Dirichlet eigenfunction of the \((n-k)\)-dimensional cross-section \( \Omega(0_k) \). By the properties of the John ellipsoid of \( \Omega \), there exists a \( k \)-dimensional parallelepiped \( P \) of dimensions comparable to \( N_1 \times N_2 \times \cdots \times N_k \) contained in the intersection of \( \Omega \) with a \( k \)-dimensional plane \( \{X'_{n-k} = \text{constant} \} \). By translating \( \Omega \) in the \( X_{n-k}' \) variables we will assume that this \( k \)-dimensional plane is \( \{X'_{n-k} = 0'_{n-k} \} \). Note that after this translation, there exists a constant \( C \) such that

\[
\text{proj}_j(\Omega(0_k)) \subset \{ |x_j| \leq C N_j \}
\]

for \( k + 1 \leq j \leq n \). Here \( \text{proj}_j(\Omega(0_k)) \) is the projection of \( \Omega(0_k) \) onto the \( x_j \)-axis. Since \( \Omega \) contains the above parallelepiped \( P \), there exists a \((k-1)\)-dimensional sphere contained in \( \{X'_{n-k} = 0'_{n-k} \} \), centred at the origin \( 0_k \) in the \( X_k \)-variables, of radius \( R_1 \) with \( R_1 \sim N_1 \), and with the following property: There exists a direction \( e \) in the \( X_k \)-variables and number \( \theta_k \), with \( \theta_k \sim N_k/N_1 \), such that the subset, \( S_k \), of the sphere making an angle at most \( \theta_k \) with \( e \), is contained within \( \Omega \). (Note that in the case of \( k = 1 \), the sphere is 0-dimensional, and the above reduces to the existence of a point in \( \Omega \) at a distance comparable to \( N_1 \) from the \((n-1)\)-dimensional cross-section \( \Omega(0_1) \).)
We now let $\Gamma_k$ be the $k$-dimensional cone in the $X_k$-variables generated by the set $S_k$, with vertex at the origin $0_k$. This cone $\Gamma_k$ contains a $k$-dimensional cube of side length comparable to $N_k^{1/3}$, at a distance comparable to $N_k^{-2/3}$ from the origin. We can therefore define a cut-off function $\chi(X_k)$ adapted to this cube (so that $\chi(X_k) = 1$ in the middle half of the cube, and 0 outside the cube), with $|\nabla \chi(X_k)| \leq C N_k^{-1/3}$. Our test function is then

$$w(x) = w(X_k, X_{n-k}') = \chi(X_k) \psi \left( X_{n-k}^{'} R_1 / (R_1 - r_k) \right).$$

(11)

Here $r_k = (x_1^2 + x_2^2 + \cdots + x_k^2)^{1/2}$ is the distance to the origin $0_k$ in the $X_k$-plane. Since $\Omega$ is convex, it contains the convex hull of the $(n - k)$-dimensional cross-section $\Omega(0_k)$ and the set $S_k$. Therefore, given $X_k \in S_k$, $s \in [0,1]$, the $(n-k)$-dimensional cross-section of $\Omega$ at $s X_k \in \Gamma_k$ contains the set

$$\left( \frac{R_1 - |sX_k|}{R_1} \right) \Omega(0_k) = (1-s)\Omega(0_k).$$

Thus, the test function $w(x)$ vanishes outside of $\Omega$, and so can be used to obtain an upper bound on $\lambda$. We therefore have

$$\lambda \leq \frac{\int_{\Omega} |\nabla_X w(x)|^2 \, dx}{\int_{\Omega} |w(x)|^2 \, dx} + \frac{\int_{\Omega} |\nabla_{X_{n-k}'} w(x)|^2 \, dx}{\int_{\Omega} |w(x)|^2 \, dx},$$

(12)

and we deal with each term separately. We can write the second term in (12) as

$$\frac{\int_{\Omega} \frac{R_k^2}{(R_1 - r_k)^2} |\nabla \chi(X_k)|^2 \left( |\nabla_{X_{n-k}'} \psi \right) (X_{n-k}^{'} R_1 / (R_1 - r_k))|^2 \, dx}{\int_{\Omega} |\nabla \chi(X_k)|^2 |\psi \left( X_{n-k}^{'} R_1 / (R_1 - r_k) \right)|^2 \, dx},$$

(13)

and on the support of $\chi(X_k)$ we have

$$\left| \frac{R_1}{R_1 - r_k} - 1 \right| \leq CN_k^{-2/3}.$$  

(14)

Therefore, since $\psi(X_{n-k}')$ has eigenvalue $\mu_k^*$ on $\Omega(0)$, we can bound the quantity in (13) by $\mu_k^* + CN_k^{-2/3}$. We now turn to the first term in (12). We can bound the magnitude of $\nabla_X w(x)$ by

$$\left| (\nabla \chi(X_k) \psi \left( X_{n-k}^{'} R_1 / (R_1 - r_k) \right) \right| + \left| \chi(X_k) \frac{R_1}{R_1 - r_k}X_{n-k}^{'} \left( \nabla_{X_{n-k}'} \psi \right) \left( X_{n-k}^{'} R_1 / (R_1 - r_k) \right) \right|. \quad (15)$$

Since $|\nabla \chi(X_k)| \leq C N_k^{-1/3}$, the contribution from the first term in (15) leads to a contribution of size $CN_k^{-2/3}$ to (12). Using $|R_1 - r_k| \geq c N_1$, together with the lengths of the projections of $\Omega(0)$ onto each axis from (10), we can bound the second term in (15) by

$$CN_1^{-1} \sum_{j=k+1}^{n} N_j \left| (\partial_{x_j} \psi) \left( X_{n-k}^{'} R_1 / (R_1 - r_k) \right) \right|. \quad (15)$$

Therefore, by Lemma 2.3, we can bound the contribution to (12) from the second term in (15) by

$$CN_1^{-2} \sum_{j=k+1}^{n} N_j^2 N_j^{-2/3} = CN_1^{-2} \sum_{j=k+1}^{n} N_j^{4/3}. \quad (15)$$

Since $N_1 \geq N_2 \geq \cdots \geq N_n$, this can be bounded by $CN_1^{-2} N_{k+1}^{4/3} \leq CN_k^{-2/3}$. Putting everything together, we obtain

$$\lambda \leq \mu_k^* + CN_k^{-2/3}.$$  

This is precisely the inductive step, and so completes the proof of the proposition. $\square$
Remark 2.1 Denoting \( \lambda_m \) to be the \( m \)-th Dirichlet eigenvalue of \( \Omega \), a small modification of the proof of Proposition 2.3 ensures the existence of a constant \( C_{m,n} \) such that
\[
\mu_j^* \leq \lambda_m \leq \mu_j^* + C_{m,n}N_j^{-2/3}.
\] (16)
The only change is that in place of \( \chi(X_k) \), we require \( m \) functions \( \chi_m(X_k) \), with \( |\nabla \chi_m(X_k)| \leq C_m N_j^{-1/3} \), chosen such that
\[
w_m(x) = \chi_m(X_k) \psi(X'_m_{-k}R_k/(R_1 - r_k))
\]
are orthogonal. The estimate in (16) in particular ensures that if \( u_m \) is the corresponding \( m \)-th eigenfunction, then it also satisfies the derivative estimates in Theorem 1.2 with a constant \( C_{m,n} \).

3 A lower bound on the \( L^2(\Omega) \)-norm of the eigenfunction

In this section we prove Theorem 1.1 by combining the derivative estimates from Theorem 1.2 with the log concavity of the eigenfunction. Since \( u \) is log concave, the superlevel set \( \Omega_{1/2} \) is a convex subset of \( \Omega \). In particular, we can associate a John ellipsoid \( E \) onto \( \Omega \). In particular, we can associate a John ellipsoid \( E \) onto \( \Omega \). The first step is to show that \( \Omega \) is certainly comparable to 1. Given \( \Omega \) we use the log concavity of \( u \), the superlevel set \( \Omega \) is maximized when \( t \) as \( t \) varies give the \( (n-1) \)-dimensional slices of \( \Omega \) which are orthogonal to \( w_k \). For each \( t \), we can consider the \( L^2(\Omega_w(t)) \)-norm squared of \( u \),
\[
\int_{\Omega_w(t)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k),
\] (17)
where \( d\sigma_{n-1}(x; w_k) \) is the flat \( (n-1) \)-dimensional surface measure on \( \Omega_w(t) \). Suppose that the expression in (17) is maximized when \( t = t^* \), and set
\[
B_k^* = \int_{\Omega_w(t^*)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k).
\]
We can now use Theorem 1.2 to obtain a lower bound on the \( L^2 \)-norm of \( u \) in terms of \( B_k^* \).
Lemma 3.2 There exists a constant $c_2 > 0$ such that for each $k$, $1 \leq k \leq n-1$, and any such direction $w_k$,

$$\int_{\Omega} |u(x)|^2 \, dx \geq c_2 B_k^* N_k^{1/3}.$$  

Proof of Lemma 3.2: Fix a point $x_{t^*} \in \Omega_{w_k}(t^*)$ and for each $s$ choose $x_s$ such that $(x_{t^*} - x_s) \cdot w_k = t^* - s$ and $|x_{t^*} - x_s| = |t^* - s|$. Then, extending $u$ by zero outside $\Omega$, for any $t$ we can write

$$u(x_t) = u(x_{t^*}) + \int_{t^*}^{t} \partial_{w_k} u(x_s) \, ds,$$

where $\partial_{w_k} u$ is the directional derivative $w_k \cdot \nabla u$. This implies that

$$|u(x_t)|^2 \geq \frac{1}{2} |u(x_{t^*})|^2 - \left( \int_{t^*}^{t} \partial_{w_k} u(x_s) \, ds \right)^2 \geq \frac{1}{2} |u(x_{t^*})|^2 - |t - t^*| \left| \int_{t^*}^{t} |\partial_{w_k} u(x_s)|^2 \, ds \right|.$$  

We now integrate over the $(n-1)$ variables orthogonal to $w_k$. Since $w_k$ lies in the projection of $\mathbb{R}^n$ onto the first $k$ coordinates, we can use Theorem 1.2 with $j \leq k$ to bound $\partial_{w_k} u$. We therefore have

$$\int_{\Omega_{w_k}(t)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k) \geq \frac{1}{2} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k) - C |t - t^*| N_k^{-2/3} \int_{\Omega} |u(x)|^2 \, dx. \quad (18)$$

In particular, for

$$|t - t^*| \leq \frac{1}{4} C^{-1} N_k^{2/3} \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{-1} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k),$$

the estimate in (18) implies that

$$\int_{\Omega_{w_k}(t)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k) \geq \frac{1}{4} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, d\sigma_{n-1}(x; w_k) = \frac{1}{4} B_{k}^*.$$  

Therefore,

$$\int_{\Omega} |u(x)|^2 \, dx \geq \frac{1}{16} C^{-1} N_k^{2/3} \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{-1} \left( B_{k}^* \right)^2,$$

and rearranging implies the estimate in the lemma. \qed

The final step is to show that for each $k$, $1 \leq k \leq n-1$, we can choose such a unit direction $w_k$ lying in $k$-dimensional space spanned by $e_1, e_2, \ldots, e_k$, such that

$$B_{k}^* \geq c_3 \prod_{j=1, j \neq k}^{n} M_j.$$ \quad (19)

Inserting this in Lemma 3.2 and using the upper bound in Lemma 3.1 implies that $M_k$ is bounded from below by a multiple of $N_k^{1/3}$. The lower bound in Lemma 3.1 then gives the estimate in Theorem 1.1.

We are left to prove (19), and we first consider $k = 1$: Consider the $(n-1)$-dimensional cross-sections of $\Omega_{1/2}$ perpendicular to $w_1 = e_1$. Since $\Omega_{1/2}$ has volume comparable to $\prod_{j=1}^{n} M_j$ and diameter comparable to $M_1$, the volume of one of these cross-sections must be at least comparable to $\prod_{j=2}^{n} M_j$. In particular, this ensures that $B_1^* \geq \frac{1}{4} c \prod_{j=2}^{n} M_j.$

For $k \geq 2$, we first choose a unit direction $w_k$ in the intersection of the $k$-dimensional plane spanned
by $e_1, e_2, \ldots, e_k$ and the $(n-k+1)$-dimensional plane spanned by $v_k, v_{k+1}, \ldots, v_n$. Taking the $(n-1)$-dimensional cross-sections of $\Omega_{1/2}$ perpendicular to $w_k$, the volume of one of these cross-sections must be at least $c \prod_{j=1, j \neq k}^{n} M_j$. To see this, we first note that there is a $(n-k+1)$-dimensional cross-section of $\Omega_{1/2}$ which is perpendicular to $v_1, v_2, \ldots, v_{k-1}$ and contains a $(n-k+1)$-dimensional ellipsoid $E$ with axes of lengths $M_k, M_{k+1}, \ldots, M_n$. In particular, the volume of one of the $(n-k)$-dimensional cross-sections of $E$ which is perpendicular to $w_k$ must be at least $c \prod_{j=k+1}^{n} M_j$. But $w_k$ is also perpendicular to $v_1, v_2, \ldots, v_{k-1}$, and the projection of $\Omega_{1/2}$ onto the $v_j$-direction is comparable to $M_j$. Therefore, there exists a $(n-k) + (k-1) = (n-1)$-dimensional cross-section of $\Omega_{1/2}$ perpendicular to $w_k$ of volume at least $c \left( \prod_{j=1}^{k-1} M_j \right) \left( \prod_{j=k+1}^{n} M_j \right)$. This ensures that $B_k^* \geq \frac{c}{L} \prod_{j=1, j \neq k}^{n} M_j$, and (19) holds.

4 The two-dimensional case

Theorem [11] provides a lower bound on the $L^2(\Omega)$-norm of $u$. In two dimensions, Jerison and Grieser have given a precise characterization of the shape of $u$ in terms of the geometry of $\Omega$. To state this, we first rotate so that the projection of the planar domain onto the $x_2$-axis is the smallest and dilate so that this projection is of length 1. Then, we can write $\Omega$ as

$$\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 : a \leq x_1 \leq b, f_1(x_1) \leq x_2 \leq f_2(x_1) \}.$$ 

Here $b - a$ is comparable to $N_1, f_1, f_2$ are convex, concave functions respectively, and $0 \leq h(x) = f_2(x_1) - f_1(x_1)$ is a concave function, attaining a maximum of 1.

**Definition 4.1 ([12])** Define $L$ to be the largest value such that $1 - L^{-2} \leq h(x_1) \leq 1$ on an interval $I$ of length $L$.

Since $h(x_1)$ is concave, the value of $L$ satisfies $cN_1^{1/3} \leq L \leq CN_1$, and $L \sim N_1, L \sim N_1^{1/3}$ is attained when $\Omega$ is a rectangle, circular sector respectively. Any intermediate value of $L$ can be obtained by, for example, forming the trapezoid of a rectangle of diameter $L$ attached to a right angled triangle. In [12], [10], [11], Grieser and Jerison obtain estimates on the first and second Dirichlet eigenfunction in terms of this length scale $L$. Their approach is to perform an approximate separation of variables in $\Omega$. Since the cross-section of $\Omega$ at $x_1$ has eigenvalue $\frac{\pi^2}{h(x_1)^2}$, a separation of variables leads to the ordinary differential operator

$$L = \frac{d^2}{dx_1^2} + \frac{\pi^2}{h(x_1)^2}$$

on the interval $[a, b]$. Grieser and Jerison approximate $\lambda$ and $u$ in terms of the first eigenvalue and eigenfunction of $L$, and the approximation becomes stronger as the diameter of $\Omega$ increases. As a consequence of their work, the following $L^2(\Omega)$-bound holds in this planar case.

**Theorem 4.2 (Grieser-Jerison, [11])** There exists an absolute constant $C$ such that the superlevel set \{ $u > \frac{1}{2} \max_{\Omega} u$ \} has diameter bounded between $C^{-1}L$ and $CL$, and

$$C^{-1}L^{1/2} \| u \|_{L^\infty(\Omega)} \leq \| u \|_{L^2(\Omega)} \leq CL^{1/2} \| u \|_{L^\infty(\Omega)}.$$ 

Using the definition of $L$ from Definition 4.1 to compare the estimate in Theorem 4.2 with the lower bound in Theorem 1.1 in two dimensions, we note the following. When $L$ is comparable to $N_1^{1/3}$, such as for a circular sector or right angled triangle, the bounds in the two theorems agree and in particular the lower bound in Theorem 1.1 is sharp. However, for $L \gg N_1^{1/3}$ Theorem 4.2 says that the eigenfunction $u$ has spread out by more than $N_1^{1/3}$ in the $x_1$-direction and so the $L^2(\Omega)$-norm of $u$ is larger than that given in Theorem 1.1.

In higher dimensions, we can begin an analogous discussion. Consider the thin sector in $\mathbb{R}^n$ of the form

$$\{ (r, \theta) : 0 < r < N_1, \theta \in D^{n-1} \},$$
where $D^{n-1}$ is a geodesic disc of radius 1 in $S^{n-1}$. As shown in [17], for this domain, the lower bound given in Theorem 1.1 is sharp. If the domain $\Omega$ is instead a parallelepiped, then the superlevel set \( \{ u > \frac{1}{2} \max_{\Omega} u \} \) takes up a uniform portion of the whole domain. For a parallelepiped, this leads to the estimate

\[
\|u\|_{L^2(\Omega)} \sim \text{Volume}(\Omega)^{1/2} \|u\|_{L^\infty(\Omega)} \sim \prod_{j=1}^n N_j^{1/2} \|u\|_{L^\infty(\Omega)}.
\]

Therefore, in dimensions higher than two it is natural to ask whether one can define analogous length scales to that of $L$ from Definition 4.1 which govern the shape of the first eigenfunction.

**Question 4.3** Fix $c$, with $0 < c < 1$. Can we use the geometry of $\Omega$ to determine $n$ length scales $M_1 \geq M_2 \geq \cdots \geq M_n$, and $n$ directions $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ such that the John ellipsoid of

\[
\{ x \in \Omega : u(x) > c \max_{\Omega} u \}
\]

has axes along the directions $v_j$ and of lengths comparable to $M_j$?

This question is open in any dimension higher than two. Let us normalize $\Omega \subset \mathbb{R}^n$ so that it has inner radius equal to 1, and its projection onto the $x_n$-axis is of length comparable to 1. Then, we can certainly choose $v_n$ to point in the $x_n$-direction and take $M_n \sim 1$. The question is then to determine the remaining $n - 1$ length scales and orientation. The results of this paper show that the lengths $M_j$ must satisfy the lower bound $M_j \geq cN_j^{1/3}$. In [2], another preliminary step towards answering this question has been carried out: Consider the operator

\[
-\Delta_{x_1, x_2} + \frac{\pi^2}{h(x_1, x_2)^2}, \tag{20}
\]

with Dirichlet boundary conditions on a two dimensional convex domain $D$. Here $h(x_1, x_2)$ is a concave function on $D$, attaining a minimum of 1. The first eigenfunction of this operator still has convex superlevel sets and in [2], length scales $L_1, L_2$ and an orientation of the domain $D$ are found in terms of $D$ and $h$, which govern the intermediate level sets of this first eigenfunction. In particular, the $L^2(D)$-norm is comparable to $L_1^{1/2} L_2^{1/2}$ multiplied by the $L^\infty(D)$-norm of the eigenfunction.

The operator in (20) can be used to make progress of answering the question in the three dimensional case. For three dimensional domains of the form

\[
\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in D, 0 \leq x_3 \leq h(x_1, x_2) \},
\]

an approximate separation of variables into $(x_1, x_2)$ and $x_3$-variables leads to the operator in (20). It is shown in [3] that when $L_1$ and $L_2$ are sufficiently close in size ($L_1 \leq L_2^{3/2}$), this separation of variables provides a good approximation to the first eigenfunction of $\Omega$. In particular, referring back to Question 4.3 in this case we can set $M_1 = L_1$, $M_2 = L_2$, $M_3 = 1$, and the orientation of $D$ also governs the behaviour of the first Dirichlet eigenfunction of $\Omega$. To make further progress towards fully answering Question 4.3 even in the three dimensional case, a key step is to determine the orientation of the superlevel sets of $u$, as in general it will not be the same as that of $\Omega$ itself. Especially as the dimension of $\Omega$ increases, it is unclear how to determine this orientation.

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