A NON-COMMUTATIVE HOMOGENEOUS COORDINATE RING
FOR THE THIRD DEL PEZZO SURFACE

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Abstract. Let \( R \) be the free \( \mathbb{C} \)-algebra on \( x \) and \( y \) modulo the relations
\[ x^5 = yxy \quad \text{and} \quad y^2 = xyx \]
edowed with the \( \mathbb{Z} \)-grading \( \deg x = 1 \) and \( \deg y = 2 \). Let \( \mathbb{B}_3 \) denote the blow up of \( \mathbb{P}^2 \) at three non-collinear points. The main result in this paper is that the category of quasi-coherent \( O_{\mathbb{B}_3} \)-modules is equivalent to the quotient of the category of \( \mathbb{Z} \)-graded \( R \)-modules modulo the full subcategory of modules \( M \) such that for each \( m \in M \), \( (x, y)^n m = 0 \) for \( n \gg 0 \). This reduces almost all representation-theoretic questions about \( R \) to algebraic geometric questions about the del Pezzo surface \( \mathbb{B}_3 \). For example, the generic simple \( R \)-module has dimension six. Furthermore, the main result combined with results of Artin, Tate, and Van den Bergh, imply that \( R \) is a noetherian domain of global dimension three.

1. Introduction

We will work over the field of complex numbers. The surface obtained by blowing up \( \mathbb{P}^2 \) at three non-collinear points is, up to isomorphism, independent of the points. It is called the third del Pezzo surface and we will denote it by \( \mathbb{B}_3 \).

Let \( R \) be the free \( \mathbb{C} \)-algebra on \( x \) and \( y \) modulo the relations
\[ x^5 = yxy \quad \text{and} \quad y^2 = xyx. \]

Give \( R \) a \( \mathbb{Z} \)-grading by declaring that
\[ \deg x = 1 \quad \text{and} \quad \deg y = 2. \]

In this paper we prove there is a surprisingly close relationship between the non-commutative algebra \( R \) and the third del Pezzo surface. This relationship can be exploited to obtain a deep understanding of the representation theory of \( R \).

Theorem 1.1. Let \( R \) be the non-commutative algebra \( \mathbb{C}[x, y] \) with relations (1-1). Let \( \text{Gr} R \) be the category of \( \mathbb{Z} \)-graded left \( R \)-modules. Then there is an equivalence of categories
\[ \text{Qcoh} \mathbb{B}_3 \cong \frac{\text{Gr} R}{T} \]
where the left-hand side is the category of quasi-coherent \( O_{\mathbb{B}_3} \)-modules and the right-hand side is the quotient category modulo the full subcategory \( T \) consisting of those modules \( M \) such that for each \( m \in M \), \( (x, y)^n m = 0 \) for \( n \gg 0 \).

Theorem 1.1 is a consequence of the following result.

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Theorem 1.2. Let $R$ be the non-commutative algebra $\mathbb{C}[x, y]$ with relations (1-1). There is an automorphism $\sigma$ of $\mathbb{B}_3$ having order six, and a line bundle $\mathcal{L}$ on $\mathbb{B}_3$ such that $R$ is isomorphic to the twisted homogeneous coordinate ring

$$B(\mathbb{B}_3, \mathcal{L}, \sigma) := \bigoplus_{n \geq 0} H^0(\mathbb{B}_3, \mathcal{L}_n)$$

where

$$\mathcal{L}_n := \mathcal{L} \otimes (\sigma^*) \mathcal{L} \otimes \cdots \otimes (\sigma^*)^{n-1} \mathcal{L}.$$ 

Results of Artin, Tate, and Van den Bergh now imply that $R$ is a 3-dimensional Artin-Schelter regular algebra and therefore has the following properties.

Corollary 1.3. Let $R$ be the algebra $\mathbb{C}[x, y]$ with relations (1-1). Then

1. $R$ is a left and right noetherian domain;
2. $R$ has global homological dimension 3;
3. $R$ is Auslander-Gorenstein and Cohen-Macaulay in the non-commutative sense;
4. the Hilbert series of $R$ is the same as that of the weighted polynomial ring on three variables of weights 1, 2, and 3;
5. $R$ is a finitely generated module over its center [8, Cor. 2.3];
6. $R^{(6)} := \bigoplus_{n=0}^{\infty} R_{6n}$ is isomorphic to $\bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3 \mathcal{O}(-nK_{\mathbb{B}_3}))$;
7. Spec $R^{(6)}$ is the anti-canonical cone over $\mathbb{B}_3$, i.e., the cone obtained by collapsing the zero section of the total space of the anti-canonical bundle over $\mathbb{B}_3$.

This close connection between $R$ and $\mathbb{B}_3$ means that almost all aspects of the representation theory of $R$ can be expressed in terms of the geometry of $\mathbb{B}_3$. We plan to address this question in another paper.

The justification for calling $R$ a non-commutative homogeneous coordinate ring for $\mathbb{B}_3$ is the similarity between the equivalence of categories in Theorem 1.1 and following theorem of Serre:

if $X \subset \mathbb{P}^n$ is the scheme-theoretic zero locus of a graded ideal $I$ in the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ with its standard grading, and $A = S/I$, then there is an equivalence of categories

$$\text{Qcoh} X \cong \frac{\text{Gr} A}{T}$$

where the right-hand side is the quotient category of $\text{Gr} A$, the category of graded $A$-modules, by the full subcategory $T$ of modules supported at the origin.

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2. **THE NON-COMMUTATIVE ALGEBRA** \( R = \mathbb{C}[x, y] \) **WITH** \( x^5 = yxy \) **AND** \( y^2 = xyx \)

The following result is a straightforward calculation. The main point of it is to show that \( R \) has the same Hilbert series as the weighted polynomial ring on three variables of weights 1, 2, and 3.

**Proposition 2.1** (Stephenson). The ring \( R := \mathbb{C}[x, y] \) with defining relations

\[
x^5 = yxy \quad \text{and} \quad y^2 = xyx
\]

is an iterated Ore extension of the polynomial ring \( \mathbb{C}[w] \). Explicitly, if \( \zeta \) is a fixed primitive 6th root of unity, then:

1. \( R = \mathbb{C}[w][z; \sigma][x; \tau, \delta] \) where \( \sigma \in \text{Aut} \mathbb{C}[z] \), \( \tau \in \text{Aut} \mathbb{C}[w, z] \), and \( \delta \) is the \( \tau \)-derivation defined by

\[
\begin{align*}
\sigma(w) &= \zeta w, \\
\tau(w) &= -\zeta^2 w, \quad \tau(z) = \zeta z, \\
\delta(w) &= z, \quad \delta(z) = -w^2;
\end{align*}
\]

2. A set of defining relations of \( R = \mathbb{C}[z, w, x] \) is given by

\[
\begin{align*}
zw &= \zeta wz, \\
xw &= -\zeta^2 wx + z, \\
xz &= \zeta zx - w^2;
\end{align*}
\]

3. \( R \) has basis \( \{w^i z^j x^k \mid i, j, k \geq 0\} \);
4. \( R \) is a noetherian domain;
5. The Hilbert series of \( R \) is \((1 - t)^{-1}(1 - t^2)^{-1}(1 - t^3)^{-1}\).

**Proof.** Define the elements

\[
\begin{align*}
w &:= y - x^2 \\
z &:= xw + \zeta^2 wx \\
&= xy + \zeta^2 yx - \zeta x^3
\end{align*}
\]

of \( R \). Since \( y \) belongs to the ring generated by \( x \) and \( w \), \( \mathbb{C}[x, y] = \mathbb{C}[x, w] = \mathbb{C}[x, w, z] \). It is easy to check that

\[
(2-1) \quad zw = \zeta wz, \quad xw = z - \zeta^2 wx, \quad xz = \zeta zx - w^2.
\]

Let \( R' \) be the free algebra \( \mathbb{C}(w, x, z) \) modulo the relations in \[(2-1)\]. We want to show \( R' \) is isomorphic to \( R \). We already know there is a homomorphism \( R' \rightarrow R \) and we will now exhibit a homomorphism \( R \rightarrow R' \) by showing there are elements \( x \) and \( Y \) in \( R' \) that satisfy the defining relations for \( R \). Define the element \( Y := w + x^2 \) in \( R' \). A straightforward computation in \( R' \) gives

\[
wxw - x^2 w = w^2 + wx^2
\]

so

\[
Y^2 = w^2 + x^2 w + wx^2 + x^4 = xwx + x^4 = xYx.
\]
In the next calculation we make frequent use of the fact that \(1 - \xi + \xi^2 = 0\). Deep breath...

\[
YxY = (w + x^2)zw + wx^3 + z^5
= (w + x^2)(z - \xi^2 wz) + [wx^3 + z^5]
= x^2 z - \xi z x^2 wz + [wz - \xi^2 w^2 x + wx^3 + z^5]
= x(\xi zz - w^2) - \xi z x^2 wz + [wz - \xi^2 w^2 x + wx^3 + z^5]
= (\zeta - \xi^2)(\xi zz - x w^2) x - (z - \xi^2 wz - \zeta (z - \xi^2 wz) x^2
+ [wz - \xi^2 w^2 x + wx^3 + z^5]
= (\zeta^2 - \xi^3)zz^2 - (\zeta - \xi^2)w^2 x - zw + \xi^2 w w z - \xi z x^2 - wz^3
+ [wz - \xi^2 w^2 x + wx^3 + z^5]
= (\zeta^2 - \xi^3)(\xi zz - x w^2) x - (1 - \zeta) wz - \xi w^2 x + z^5]
= \zeta^2 w w z + [(1 - \zeta) wz - \xi w^2 x + z^5]
= \zeta^2 w(z - \xi^2 wz) x + [wz - \xi^2 w^2 x + z^5]
= x^5.
\]

Since \(YxY = x^5\), \(R\) is isomorphic to \(R'\). Hence \(R\) is an iterated Ore extension as claimed. The other parts of the proposition follow easily. \(\square\)

It is an immediate consequence of the relations that \(x^6 = y^3\). Hence \(x^6\) is in the center of \(R\).

3. The del Pezzo surface \(B_3\)

Let \(B_3\) be the surface obtained by blowing up the complex projective plane \(\mathbb{P}^2\) at three non-collinear points. We will write \(E_1\), \(E_2\), and \(E_3\) for the exceptional curves associated to the blowup.

The \(-1\)-curves on \(B_3\) lie in the following configuration (3-1)
where $L_1$, $L_2$, and $L_3$ are the strict transforms of the lines in $\mathbb{P}^2$ spanned by the points that are blown up. (The labeling of the equations for the $-1$-curves will be justified shortly.)

The union of the $-1$-curves is an anti-canonical divisor and is, of course, ample.

3.1. The Picard group of $\mathbb{B}_3$. The Picard group of $\mathbb{B}_3$ is free abelian of rank four. We will identify it with $\mathbb{Z}^4$ by using the ordered basis $H, -E_2, -E_1, -E_3$

where the $E_i$s are the exceptional curves over the points blown up and $H$ is the strict transform of a line in $\mathbb{P}^2$ in general position, i.e., missing the points being blown up. Thus

$$H = (1, 0, 0, 0), \quad E_1 = (0, 0, -1, 0), \quad E_2 = (0, -1, 0, 0), \quad E_3 = (0, 0, 0, -1).$$

The canonical divisor $K$ is $-3H + E_1 + E_2 + E_3$ so the anti-canonical divisor is

$$-K = (3, 1, 1, 1).$$

3.2. Cox’s homogeneous coordinate ring. By definition, Cox’s homogeneous coordinate ring [5] for a complete smooth toric variety is

$$S := \bigoplus_{[L] \in \text{Pic} X} H^0(X, L).$$

For the remainder of this paper $S$ will denote Cox’s homogeneous coordinate ring for $\mathbb{B}_3$.

Let $X, Y, Z, s, t, u$ be coordinate functions on $\mathbb{C}^6$. One can present $\mathbb{B}_3$ as a toric variety by defining it as the orbit space

$$\mathbb{B}_3 := \mathbb{C}^6 - W / (\mathbb{C}^\times)^4$$

where the irrelevant locus, $W$, is the union of nine codimension two subspaces, namely

$$(3-2) \quad \begin{align*}
X &= t = 0 \\
Y &= s = 0 \\
Z &= u = 0 \\
X &= Y = s = t = u = 0 \quad s = s = t = u = 0
\end{align*}$$

and $(\mathbb{C}^\times)^4$ acts with weights

$$\begin{array}{cccc}
X & 1 & 1 & 0 & 1 \\
Y & 1 & 0 & 1 & 1 \\
Z & 1 & 1 & 1 & 0 \\
s & 0 & -1 & 0 & 0 \\
t & 0 & 0 & -1 & 0 \\
u & 0 & 0 & 0 & -1
\end{array}$$

Therefore $S$ is the $\mathbb{Z}^4$-graded polynomial ring

$$S = \mathbb{C}[X, Y, Z, s, t, u]$$

with the degrees of the generators given by their weights under the $(\mathbb{C}^\times)^4$ action. It follows from Cox’s results [5, Sect. 3] that

$$\text{Qcoh}_{\mathbb{B}_3} \equiv \frac{\text{Gr}(S, \mathbb{Z}^4)}{T}$$
where $\text{Gr}(S, \mathbb{Z}^4)$ is the category of $\mathbb{Z}^4$-graded $S$-modules and $\mathcal{T}$ is the full subcategory consisting of all direct limits of modules supported on $W$.

The labelling of the $-1$-curves the diagram (3-1) is explained by the existence of the morphisms

$$\begin{array}{c}
\mathbb{B}_3 \\
\mathbb{P}^2 \\
\mathbb{P}^2 \\
(X, Y, Z, s, t, u) \\
(Xsu, Ytu, Zst) \\
(YZt, ZXs, XYu)
\end{array}$$

that collapse the $-1$-curves.

3.3. **An order six automorphism $\sigma$ of $\mathbb{B}_3$.** The cyclic permutation of the six $-1$-curves on $\mathbb{B}_3$ extends to a global automorphism of $\mathbb{B}_3$ of order six. We now make this explicit.

The category of graded rings consists of pairs $(A, \Gamma)$ consisting of an abelian group $\Gamma$ and a $\Gamma$-graded ring $A$. A morphism $(f, \theta) : (A, \Gamma) \rightarrow (B, \Upsilon)$ consists of a ring homomorphism $f : A \rightarrow B$ and a group homomorphism $\theta : \Gamma \rightarrow \Upsilon$ such that $f(A_i) \subset B_{\theta(i)}$ for all $i \in \Gamma$.

Let $\tau : S \rightarrow S$ be the automorphism induced by the cyclic permutation

$$X \xrightarrow{\tau} Y \xrightarrow{\tau} Z \xrightarrow{\tau} s$$

and let $\theta : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4$ be left multiplication by the matrix

$$\theta = \begin{pmatrix}
2 & -1 & -1 & -1 \\
1 & -1 & -1 & 0 \\
1 & 0 & -1 & -1 \\
1 & -1 & 0 & -1
\end{pmatrix}.$$

Then $(\tau, \theta) : (S, \mathbb{Z}^4) \rightarrow (S, \mathbb{Z}^4)$ is an automorphism in the category of graded rings. Because the irrelevant locus (3-2) is stable under the action of $\tau$, $\tau$ induces an automorphism $\sigma$ of $\mathbb{B}_3$. It follows from the definition of $\tau$ that $\sigma$ cyclically permutes the six $-1$-curves.

Since $(\tau, \theta)^6 = \text{id}_{(S, \mathbb{Z}^4)}$ the order of $\sigma$ divides six. But the action of $\sigma$ on the set of $-1$-curves has order six, so $\sigma$ has order six as an automorphism of $\mathbb{B}_3$.

3.4. Fix a primitive cube root of unity $\omega$. The left action of $\theta$ on $\mathbb{Z}^4 = \text{Pic} \mathbb{B}_3$ has eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ \omega^2 \\ \omega \end{pmatrix},$$

with corresponding eigenvalues $-1, +1, \omega, \omega^2$.

4. **A twisted hcr for $\mathbb{B}_3$**

4.1. **A sequence of line bundles on $\mathbb{B}_3$.** We will blur the distinction between a divisor $D$ and the class of the line bundle $\mathcal{O}(D)$ in $\text{Pic} \mathbb{B}_3$.

We define a sequence of divisors: $D_0$ is zero; $D_1$ is the line $L_1$; for $n \geq 1$

$$D_n := (1 + \theta + \cdots + \theta^{n-1})(D_1).$$
We will write $\mathcal{L}_n := \mathcal{O}(D_n)$. Therefore
$$\mathcal{L}_n = \mathcal{L}_1 \otimes \sigma^* \mathcal{L}_1 \otimes \cdots \otimes (\sigma^*)^{n-1} \mathcal{L}_1.$$ 

For example,
$$\begin{align*}
\mathcal{O}(D_1) &= \mathcal{L}_1 = \mathcal{O}(1, 1, 0, 1) = \mathcal{O}(L_1) \\
\mathcal{O}(D_2) &= \mathcal{L}_2 = \mathcal{O}(1, 1, 0, 0) = \mathcal{O}(L_1 - E_3) \\
\mathcal{O}(D_3) &= \mathcal{L}_3 = \mathcal{O}(2, 1, 1, 1) = \mathcal{O}(L_1 + L_2 - E_3) \\
\mathcal{O}(D_4) &= \mathcal{L}_4 = \mathcal{O}(2, 1, 0, 1) = \mathcal{O}(L_1 + L_2 - E_1 - E_3) \\
\mathcal{O}(D_5) &= \mathcal{L}_5 = \mathcal{O}(3, 2, 1, 1) = \mathcal{O}(L_1 + L_2 + L_3 - E_1 - E_3) \\
\mathcal{O}(D_6) &= \mathcal{L}_6 = \mathcal{O}(3, 1, 1, 1) = \mathcal{O}(L_1 + L_2 + L_3 - E_1 - E_2 - E_3) \\
&= \mathcal{O}(3H - E_1 - E_2 - E_3) \\
&= \mathcal{O}(-K).
\end{align*}$$

**Lemma 4.1.** Suppose that $m \geq 0$ and $0 \leq r \leq 5$. Then
$$D_{6m+r} = D_r - mK.$$

**Proof.** Since $\theta^6 = 1$,
$$\begin{align*}
\sum_{i=0}^{6m+r-1} \theta^i &= (1 + \theta + \cdots + \theta^5) \sum_{j=0}^{m-1} \theta^{kj} + \theta^{km}(1 + \theta + \cdots + \theta^{r-1}) \\
&= (1 + \theta + \cdots + \theta^{r-1}) + m(1 + \theta + \cdots + \theta^5)
\end{align*}$$
where the sum $1 + \theta + \cdots + \theta^{r-1}$ is empty and therefore equal to zero when $r = 0$. Therefore $D_{6m+r} = D_r + mD_6 = D_r - mK$, as claimed. \qed

**4.2. Vanishing results.** For a divisor $D$ on a smooth surface $X$, we write
$$h^i(D) := \dim H^i(X, \mathcal{O}_X(D)).$$

We need to know that $h^1(D) = h^2(D) = 0$ for various divisors $D$ on $\mathbb{P}_3$.

If $D - K$ is ample, then the Kodaira Vanishing Theorem implies that $h^0(K - D) = h^1(K - D) = 0$ and Serre duality then gives $h^2(D) = h^1(D) = 0$.

The notational conventions in section 3.1 identify $\text{Pic} \mathbb{P}_3$ with $\mathbb{Z}^4$ via
$$aH - cE_1 - bE_2 - dE_3 \equiv (a, b, c, d)$$
where $H$ is the strict transform of a line in $\mathbb{P}_2$. The intersection form on $\mathbb{P}_3$ is given by
$$H^2 = 1, \quad E_i.E_j = -\delta_{ij}, \quad H.E_i = 0,$$
so the induced intersection form on $\mathbb{Z}^4$ is
$$(a, b, c, d) \cdot (a', b', c', d') = aa' - bb' - cc' - dd'.$$

**Lemma 4.2.** Let $D = (a, b, c, d) \in \text{Pic} \mathbb{P}_3 \equiv \mathbb{Z}^4$. Suppose that
$$\begin{align*}
(a + 3)^2 &> (b + 1)^2 + (c + 1)^2 + (d + 1)^2 \\
\text{and} \quad b, c, d &> -1, \quad \text{and} \quad a + 1 > b + c, b + d, c + d.
\end{align*}$$

Then $D - K$ is ample, whence $h^1(D) = h^2(D) = 0$. 

\[\]
Proof. The effective cone is generated by \( L_1, L_2, L_3, E_1, E_2, \) and \( E_3 \) so, by the Nakai-Moishezon criterion, \( D - K \) is ample if and only if \((D - K)^2 > 0 \) and \( D.L_i > 0 \) and \( D.E_i > 0 \) for all \( i \). Now \( D - K = (a + 3, b + 1, c + 1, d + 1) \), so \((D - K)^2 > 0 \) if and only if \( 4-1 \) holds and \( (D - K).D' > 0 \) for all effective \( D' \) if and only if \( 4-2 \) holds.

Hence the hypothesis that \( 4-1 \) and \( 4-2 \) hold implies that \( D - K \) is ample. The Kodaira Vanishing Theorem now implies that \( h^0(K - D) = h^1(K - D) = 0 \). Serre duality now implies that \( h^2(D) = h^1(D) = 0 \). \( \square \)

Lemma 4.3. For all \( n \geq 0 \), \( h^1(D_n) = h^2(D_n) = 0 \).

Proof. The value of \( D_n \) for \( 0 \leq n \leq 6 \) is given explicitly in section 4.1. We also note that \( D = D_1 + D_6 = (4, 2, 1, 2) \). It is routine to check that conditions \#1 and \#2 hold for \( D = D_n \) when \( n = 0, 2, 3, 4, 5, 6, 7 \). Hence \( h^1(D_n) = h^2(D_n) = 0 \) when \( n = 0, 2, 3, 4, 5, 6, 7 \).

We now consider \( D_1 \) which is the \(-1\)-curve \( X = 0 \). (Since \((D_1 - K).D_1 = 0, D_1 - K \) is not ample so we can’t use Kodaira Vanishing as we did for the other small values of \( n \).) It follows from the exact sequence \( 0 \rightarrow \mathcal{O}_{\mathbb{B}_3} \rightarrow \mathcal{O}_{\mathbb{B}_3}(D_1) \rightarrow \mathcal{O}_{D_1}(D_1) \rightarrow 0 \) that \( H^p(\mathbb{B}_3, \mathcal{O}_{\mathbb{B}_3}(D_1)) \cong H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1)) \) for \( p = 1, 2 \). However, \( \mathcal{O}_{D_1}(D_1) \) is the normal sheaf on \( D_1 \) and as \( D_1 \) can be contracted to a smooth point on the second del Pezzo surface, \( \mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{D_1}(-1) \). But \( D_1 \cong \mathbb{P}^1 \) so \( H^p(\mathbb{B}_3, \mathcal{O}_{D_1}(D_1)) \cong H^p(\mathbb{P}^1, \mathcal{O}(1)) \) which is zero for \( p = 1, 2 \). It follows that \( h^1(D_1) = h^2(D_1) = 0 \).

Thus \( h^1(D_n) = h^2(D_n) = 0 \) when \( 0 \leq n \leq 7 \). We have also shown that \( D_n - K \) is ample when \( 2 \leq n \leq 7 \). We now argue by induction. Suppose \( n \geq 8 \) and \( D_{n-6} - K \) is ample. Now \( D_n - K = D_{n-6} - K \). Since a sum of ample divisors is ample, \( D_n - K \) is ample. It follows that \( h^1(D_n) = h^2(D_n) = 0 \). \( \square \)

4.3. The twisted homogeneous coordinate ring \( B(\mathbb{B}_3, \mathcal{L}, \sigma) \). We assume the reader is somewhat familiar with the notion of twisted homogeneous coordinate rings. Standard references for that material are [3], [1], [2], and [4].

The notion of a \( \sigma \)-ample line bundle \( \mathbb{B}_3 \) plays a key role in the study of twisted homogeneous coordinate rings. Because \( \mathcal{L}_6 \) is the anti-canonical bundle and therefore ample, \( \mathcal{L}_1 \) is \( \sigma \)-ample. This allows us to use the results of Artin and Van den Bergh in [3] to conclude that the twisted homogeneous coordinate ring

\[
B(\mathbb{B}_3, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} B_n = \bigoplus_{n=0}^{\infty} H^0(\mathbb{B}_3, \mathcal{L}_n)
\]

is such that

\[
\text{Qcoh}_{\mathbb{B}_3} = \frac{\text{Gr} B}{\mathcal{T}}
\]

where \( \mathcal{T} \) is the full subcategory of \( \text{Gr} B \) consisting of those modules \( M \) such that for each \( m \in M, B_n m = 0 \) for \( n \gg 0 \). It then follows that \( B \) has a host of good properties—see [3] for details.

We will now compute the dimensions \( h^0(D_n) \) of the homogeneous \( B_n \) of \( B \). We will show that \( B \) has the same Hilbert series as the non-commutative ring \( \mathcal{R} \), i.e., the same Hilbert series as the weighted polynomial ring with weights 1, 2, and 3. As usual we write \( \chi(D) = h^0(D) - h^1(D) + h^2(D) \). The Riemann-Roch formula is

\[
\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \frac{1}{2} D \cdot (D - K)
\]
where $K$ denotes the canonical divisor. We have $\chi(O_{\mathbb{P}^3}) = 1$ and $K_{\mathbb{P}^3}^2 = 6$.

**Lemma 4.4.** Suppose $0 \leq r \leq 5$. Then

$$h^0(D_{6m+r}) = \begin{cases} (m+1)(3m+r) & \text{if } r \neq 0, \\ 3m^2 + 3m + 1 & \text{if } r = 0 \end{cases}$$

and

$$\sum_{n=0}^{\infty} h^0(D_n) t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

**Proof.** Computations for $1 \leq r \leq 5$ give $D_r^2 = r - 2$ and $D_r \cdot K = -r$. Hence

$$\chi(D_{6m+r}) = 1 + \frac{1}{2}(D_r - mK) \cdot (D_r - (m+1)K)$$

$$= 1 + \frac{1}{2}(D_r^2 - (2m + 1)D_r.K + 6m(m+1)^2)$$

$$= (3m + r)(m + 1)$$

for $m \geq 0$ and $1 \leq r \leq 5$. When $r = 0$, $D_r = 0$ so

$$\chi(D_{6m}) = 3m^2 + 3m + 1.$$

By Lemma 4.3, $\chi(D_n) = h^0(D_n)$ for all $n \geq 0$ so it follows from the formula for $\chi(D_n)$ that

$$h^0(D_{n+6}) - h^0(D_n) = n + 6$$

for all $n \geq 0$.

To complete the proof of the lemma, it suffices to show that $h^0(D_n)$ is the coefficient of $t^n$ in the Taylor series expansion

$$f(t) := \frac{1}{(1-t)(1-t^2)(1-t^3)} = \sum_{n=0}^{\infty} a_n t^n.$$

Because

$$(1 - t^6)f(t) = (1 - t^2)(1 - t)^{-2} = 1 + \sum_{n=1}^{\infty} nt^n,$$

it follows that

$$(1 - t^6)f(t) = a_0 + a_1 t + \cdots + a_5 t^5 + \sum_{n=0}^{\infty} (a_{n+6} - a_n) t^{n+6}$$

$$= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=6}^{\infty} nt^n$$

$$= 1 + t + 2t + \cdots + 5t^5 + \sum_{n=0}^{\infty} (n+6) t^{n+6}.$$

In particular, if $0 \leq r \leq 5$, $a_r = h^0(D_r)$. We now complete the proof by induction. Suppose we have proved that $a_i = h^0(D_i)$ for $i \leq n + 5$. By comparing the expressions in the Taylor series we see that

$$a_{n+6} = a_n + (n+6) = h^0(D_n) + n + 6 = h^0(D_{n+6})$$

where the last equality is given by (4.3). □
4.3.1. Remark. It wasn’t necessary to compute $\chi(D_n)$ in the previous proof. The proof only used the fact that $\chi(D_{n+6}) - \chi(D_n) = n + 6$ which can be proved directly as follows:

$$
\chi(D_{n+6}) - \chi(D_n) = \frac{1}{2} D_{n+6} \cdot (D_{n+6} - K) - \frac{1}{2} D_n \cdot (D_n - K)
$$

$$
= \frac{1}{2} (D_{n+6} - D_n) \cdot (D_{n+6} + D_n - K)
$$

$$
= -K \cdot (D_r - (m+1)K)
$$

$$
= 6(m+1) - K \cdot D_r
$$

$$
= n + 6.
$$

4.4. The isomorphism $R \rightarrow B(\mathbb{B}_3, \mathcal{L}, \sigma)$. The ring $B$ has the following basis elements in the following degrees:

- $O(1, 1, 0, 1) X$
- $O(1, 1, 0, 0) Xu Zt$
- $O(2, 1, 1, 1) XYu YZt XZs$
- $O(2, 1, 0, 1) XYtu YZt^2 XZst X^2su$
- $O(3, 2, 1, 1) XYZtu YZ^2t^2 XZ^2st X^2Zsu X^2Yu^2$
- $O(3, 1, 1, 1) XYZstu YZ^2st^2 XZ^2s^2t X^2Zs^2u X^2Ysu^2 XY^2tu^2 Y^2Zt^2u$

Although $B$ is a graded subspace of Cox’s homogeneous coordinate ring $S$,

the multiplication in $B$ is not that in $S$.

The multiplication in $B$ is Zhang’s twisted multiplication [11] with respect to the automorphism $\tau$ defined in (3-3): the product in $B$ of $a \in B_m$ and $b \in B_n$ is

$$
(4-4) \quad a * B b := a \tau^m(b).
$$

To make it clear whether a product is being calculated in $B$ or $S$ we will write $x$ for $X$ considered as an element of $B$ and $y$ for $Zt$ considered as an element of $B$. Therefore, for example,

$$
x^5 = X \tau(X) \tau^2(X) \tau^3(X) \tau^4(X) \tau^5(X)
$$

$$
= XuYtuZ
$$

$$
= (Zt)Y'(uX)
$$

$$
= Zt\tau^2(X)\tau^3(Zt)
$$

$$
= yxy
$$

and

$$
y^2 = Zt\tau^2(Zt) = Zt(sX) = X(sZ)t = X\tau(zt)\tau^3(X) = xyx.
$$

The following proposition is an immediate consequence of these two calculations.

**Proposition 4.5.** Let $R$ be the free algebra $\mathbb{C}[x, y]$ modulo the relations $x^5 = yxy$ and $y^2 = xyx$. Then there is a $\mathbb{C}$-algebra homomorphism

$$
R = \mathbb{C}[x, y] \rightarrow B(\mathbb{B}_3, \mathcal{L}, \sigma), \quad x \mapsto X, \ y \mapsto Zt.
$$

**Lemma 4.6.** The homomorphism in Proposition 4.5 is an isomorphism in degrees $\leq 6$.

---

1 We will eventually prove that the homomorphism is an isomorphism in all degrees but the low degree cases need to be handled separately.
Proof. By Proposition \[4.7\] \( R \) has Hilbert series \((1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1}\), so the dimension of \( R_n \) in degrees \(1, 2, 3, 4, 5, 6\) is \(1, 2, 3, 4, 5, 7\).

The \( n \)-th row in the following table gives a basis for \( B_n \), \( 1 \leq n \leq 6 \). One proceeds down each column by multiplying on the right by \( x \). There wasn’t enough room on a single line for \( B_6 \) so we put the last two entries for \( B_6 \) on a new line.

\[
\begin{align*}
x &= X \\
x^2 &= Xu \\
x^3 &= XYu \\
x^4 &= XYtu \\
x^5 &= XYZtu \\
x^6 &= XYZstu \\
y &= Zt \\
yx &= YZt \\
yx^2 &= YZt^2 \\
yx^3 &= YZ^2t^2 \\
yx^4 &= YZ^2st^2 \\
yx^5 &= YZ^2st^3 \\
yx^6 &= YZ^2st^2t \\
xy &= XZs \\
xy^2 &= XZstu \\
xy^3 &= XZ^2stu \\
xy^4 &= XZ^2st^2 \\
xy^5 &= XZ^2st^3 \\
xy^6 &= XZ^2st^2t \\

\end{align*}
\]

The products involving \( x \) and \( y \) were made by using the formula \((3-1)\) in the same way that it was used to show that \( x^3 \) = \( yxy \).

**Lemma 4.7.** \( \mathcal{L}_2 \) is generated by its global sections.

Proof. A line bundle \( \mathcal{L} \) on a variety is generated by its global sections if and only if for each point \( p \) on the variety there is a section of \( \mathcal{L} \) that does not vanish at \( p \). In this case, \( H^0(B_3, \mathcal{L}_2) \) is spanned by \( Xu \) and \( Zt \). One can see from the diagram \((3-1)\) that the zero locus of \( Xu \) does not meet the zero locus of \( Zt \), so the common zero locus of \( Xu \) and \( Zt \) is empty. \( \square \)

**Proposition 4.8.** As a \( \mathbb{C} \)-algebra, \( B \) is generated by \( B_1 \) and \( B_2 \).

Proof. It follows from the explicit calculations in Lemma \[4.1\] that the subalgebra of \( B \) generated by \( B_1 \) and \( B_2 \) contains \( B_m \) for all \( m \leq 6 \). It therefore suffices to prove that the twisted multiplication map \( B_2 \otimes B_n \to B_{n+2} \) is surjective for all \( n \geq 5 \).

By definition, \( B_2 = H^0(\mathcal{L}_2) \) and this has dimension two so, by Lemma \[4.7\], there is an exact sequence \( 0 \to N \to B_2 \otimes \mathcal{O}_{B_3} \to \mathcal{L}_2 \to 0 \) for some line bundle \( N \). In fact, \( N \cong \mathcal{L}_2^{-1} \).

By definition, \( \mathcal{L}_{n+2} = \mathcal{L}_2 \otimes \mathcal{M} \) where \( \mathcal{M} \cong \mathcal{O}(D_{n+2} - D_2) \), and the twisted multiplication map \( B_2 \otimes B_n \to B_{n+2} \) is the ordinary multiplication map

\[
B_2 \otimes H^0(\mathcal{M}) = H^0(\mathcal{L}_2) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L}_2 \otimes \mathcal{M}).
\]

Applying \( - \otimes \mathcal{M} \) to the exact sequence \( 0 \to \mathcal{L}_2^{-1} \to B_2 \otimes \mathcal{O}_{B_3} \to \mathcal{L}_2 \to 0 \) and taking cohomology gives an exact sequence

\[
0 \to H^0(\mathcal{L}_2^{-1} \otimes \mathcal{M}) \to B_2 \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L}_2 \otimes \mathcal{M}) \to H^1(\mathcal{L}_2^{-1} \otimes \mathcal{M}).
\]

Hence, if \( h^1(\mathcal{L}_2^{-1} \otimes \mathcal{M}) = 0 \), then \( B_2B_n = B_{n+2} \). But

\[
\mathcal{L}_2^{-1} \otimes \mathcal{M} \cong \mathcal{O}(-D_2 + D_{n+2} - D_2)
\]

so we need to show that \( D_{6m+r} - 2D_2 - K \) is ample whenever \( 6m + r \geq 7 \) and \( 0 \leq r \leq 5 \). By Lemma \[4.1\] \( D_{6m+r} = D_r - mK \). We therefore need to check that conditions \((4.1)\) and \((4.2)\) in Lemma \[4.2\] hold for the divisors \( D \) in the following
\[ D := D_r - 2D_2 - (m + 1)K \]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 r & m \geq 2 & (3m + 1, m - 1, m + 1, m + 1) \\
 r = 0 & m \geq 1 & (3m + 2, m, m + 1, m + 2) \\
 r = 1 & m \geq 1 & (3m + 2, m, m + 1, m + 1) \\
 r = 2 & m \geq 1 & (3m + 3, m + 1, m + 2) \\
 r = 3 & m \geq 1 & (3m + 3, m + 1, m + 2) \\
 r = 4 & m \geq 1 & (3m + 4, 1, m + 1, m + 2) \\
 r = 5 & m \geq 1 & (3m + 4, m + 1, m + 2, m + 2) \\
\end{array}
\]

This is a routine though tedious task. □

**Theorem 4.9.** Let \( R \) be the free algebra \( \mathbb{C}(x, y) \) modulo the relations \( x^5 = yxy \) and \( y^2 = xyx \). Then the \( \mathbb{C} \)-algebra homomorphism

\[ \Phi : R = \mathbb{C}[x, y] \rightarrow B(\mathbb{E}_3, \mathcal{L}, \sigma), \quad x \mapsto X, \quad y \mapsto Zt, \]

is an isomorphism of graded algebras.

**Proof.** By Lemma 4.6, \( B_1 \) and \( B_2 \) are in the image of \( \Phi \). It follows that \( \Phi \) is surjective because \( B \) is generated by \( B_1 \) and \( B_2 \). But \( \Phi(R_n) \subset B_n \), and \( R \) and \( B \) have the same Hilbert series, so \( \Phi \) is also surjective. □

Consider \( R^{(3)} \supset \mathbb{C}[x^3, xy, yx] \). Since \( \dim R_6 = 7 = (\dim R_3)^2 - 2 \) there is a 2-dimensional space of quadratic relations among the elements \( x^3, xy, \) and \( yx \). Hence \( R^{(3)} \) is not a 3-dimensional Artin-Schelter regular algebra. The relations in the degree two component of \( R^{(3)} \) are generated by

\[ (x^3)^2 = (xy)^2 = (yx)^2. \]

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