ISOMETRIC SHIFTS AND METRIC SPACES

JESÚS ARAUJO AND JUAN J. FONT

Abstract. Let $M$ be a complete metric space. If $C^*(M)$ admits an isometric shift, then $M$ is separable.

1. Introduction

Shift operators play an important role in many disciplines such as Perturbation Theory, Engineering Mathematics, Scattering Theory, Stochastic Processes, . . . (see [11]). R.M. Crownover ([3]) was the first to extend the definition of shift operator from separable Hilbert spaces to arbitrary Banach spaces without using basis. Namely, if $\mathcal{K}$ is a Banach space, then $T : \mathcal{K} \to \mathcal{K}$ is said to be an (isometric) shift operator if

1. $T$ is a linear isometry,
2. The codimension of $T(\mathcal{K})$ in $\mathcal{K}$ is 1,
3. $\bigcap_{n=1}^{\infty} T^n(\mathcal{K}) = \{0\}$.

If Condition 3 is removed, then we have a codimension 1 linear isometry.

In [3], Gutek, Hart, Jamison and Rajagopalan extended many of the results obtained by J.R. Holub in [4] concerning isometric shift operators on the Banach space $C(X)$ ($X$ compact Hausdorff). First, they classified codimension 1 linear isometries on $C(X)$ using the following result: let $T : C(X) \to C(X)$ be a codimension 1 linear isometry. Then there exists a closed subset $X_0$ of $X$ such that either

(i) $X_0 = X \setminus \{p\}$

where $p$ is an isolated point of $X$, or

(ii) $X_0 = X$

and such that there exists a continuous map $h$ of $X_0$ onto $X$ and a function $a \in C(X_0)$, $|a| \equiv 1$, such that

$$(T f)(x) = a(x) \cdot f(h(x))$$

2000 Mathematics Subject Classification. Primary 47B38; Secondary 54D65, 46J10.

Research of the first author was partially supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT, PB98-1102).

Research of the second author was partially supported by Fundació Caixa Castelló.
for all $x \in X_0$.

The proof of this result is based on a well known theorem of Hol-
hsztynski (8). Those isometries that satisfy Condition (i) are said to
be of type I. Those satisfying Condition (ii) are said to be of type II.
These two classes are not disjoint. Farid and Varadarajan (4) de-
voted part of their paper to clarify the above classification. Finally, in
4 the author proposes an alternative (disjoint) classification based on
the separation properties of the range of $T$. Thus, $T$ is of type II if
and only if $T(C(X))$ separates all the points of $X$ except two and is of
type I which is not of type II if and only if $T(C(X))$ separates all the
points of $X$.

Codimension 1 linear isometries on arbitrary function algebras have
also been studied and classified in 2 by using the results in 1. Re-
cently, K. Izuchi (10) has characterized Douglas algebras which admit
codimension 1 linear isometries, thus solving the conjecture settled in
2.

Another question which has also been addressed in the context of iso-
metric shifts is the characterization of those compact Hausdorff spaces
$X$ which admit such operators, that is, the existence of isometric shifts
on $C(X)$. In 3, the authors proved that nonseparable spaces without
isolated points do not admit isometric shifts and even that there is no
nonseparable space which admits isometric shifts of type II. They left
open the question of the existence of spaces without isolated points
which admit isometric shifts. This question was answered in the pos-
tive by R. Haydon (7). He proved the existence of isometric shifts
of type II when $X$ is either connected or the Cantor set. However it
is still an open question whether there exists a nonseparable compact
space $X$ which admits an isometric shift. In this paper we show that no
nonseparable metric (noncompact) space admits isometric shifts. We
also provide an example of an isometric shift (of type I) with several
interesting features.

2. Preliminaries

Let $\mathbb{K}$ denote the field of real or complex numbers. If $X$ is a com-
pact (respectively locally compact) Hausdorff space, then $C(X)$ (re-
spectively $C_0(X)$) stands for the Banach space of all $\mathbb{K}$-valued con-
tinuous functions defined on $X$ (respectively which vanish at infinity),
equipped with its usual supremum norm. If $M$ is a metric space, then
we shall write $C^*(M)$ to denote the normed space of all bounded $\mathbb{K}$-
valued continuous functions defined on $M$. As usual, $\beta M$ stands for
the Stone-Cech compactification of $M$. Given $f \in C(X)$, we shall
consider that $c(f)$ is its cozero set.
If $U$ is a subset of $X$, then $\text{cl}_X(U)$ and $\text{int}_X(U)$ denote its closure and its interior in $X$, respectively.

3. ISOMETRIC SHIFTS ON $C^*(M)$ AND SEPARABILITY

Let $M$ be a complete metric space and let $T : C^*(M) \longrightarrow C^*(M)$ be an isometric shift. Then $T$ induces an isometric shift (which we continue to denote by $T$) on $C(\beta M)$.

**Theorem 3.1.** Let $M$ be a complete metric space. If $C^*(M)$ admits an isometric shift $T$, then $M$ is separable.

**Proof.** Let us first assume $T$ to be of type II which is not of type I. According to [3, Lemma 2.2], the map $h : \beta M \longrightarrow \beta M$ is a surjective continuous map such that there exists $x_0 \in \beta M$ in such a way that $h^{-1}\{x\}$ consists of just one point for every $x \in \beta M \setminus \{x_0\}$ and $h^{-1}\{x_0\}$ consists of two points of $\beta M$, say $x_1, x_2$. Also, since $T$ is not of type I, then the points $x_1$ and $x_2$ are not isolated. Furthermore, in [3, Theorem 2.5], it is proven that the set

$$D := \bigcup_{k=-\infty}^{\infty} h^k(\{x_0\})$$

is a countable dense subset in $\beta M$. We are going to see that this set is contained in $M$ and in this way we give an explicit countable dense subset in $M$.

First we have, by [3, Theorem 2.6], that if $\beta M/R$ is the quotient space for the equivalence relation defined as $xRy$ whenever $h(x) = h(y)$, then the map $h^R : \beta M/R \longrightarrow \beta M$ sending each class $x^R$ into the image $h(x)$ of any $x \in x^R$ is a surjective homeomorphism. This implies in particular that the image of a $G_\delta$-point in $\beta M/R$ is a $G_\delta$-point in $\beta M$ and vice versa. Let us recall that the only points in $\beta M$ which are $G_\delta$ are those in $M$.

Let us check which the $G_\delta$-points in $\beta M/R$ are. Suppose that $x^R \in \beta M/R$ satisfies that there exists $x \in M$ with $x \in x^R$. Clearly, if $x^R$ is the singleton $\{x\}$, then $x^R$ is $G_\delta$. Otherwise, as we remark above, $x^R$ consists of two points, $x_1, x_2$, and is the only point in $\beta M/R$ which is not a singleton. Then it is apparent that $x^R$ is $G_\delta$ if and only if both $x_1, x_2 \in M$.

Suppose next that $x^R = \{x_1, x_2\}$, and that $x_1 \in M$. Since $T$ is not of type I, then the points $x_1$ and $x_2$ are clearly not isolated. Thus, there exists a sequence $(y_n)$ in $M \setminus \{x_1, x_2\}$ converging to $x_1$. Also each $y_n$ is a $G_\delta$-point, and consequently so is $h^R(y_n) = h(y_n)$, that is, the sequence $(h(y_n))$ is contained in $M$, and converges to $x_0$. But this
implies in particular that \( x_0 \in M \) ([14, Theorem 8.3.2]). Conversely, if we assume that \( x_0 \in M \), then \( x_0 \) is a \( G_\delta \)-point of \( \beta M \) and, consequently, so is \( h^{-1}(\{x_0\}) = \{x_1, x_2\} \). This implies, as stated above, that both \( x_1 \) and \( x_2 \) belong to \( M \). Summing up, we have proven that \( x_0 \in M \) if and only if \( x_1 \in M \) or \( x_2 \in M \), and that this fact yields \( x_1, x_2 \in M \).

Let us now assume that \( x_0 \notin M \), which is to say that \( x_1, x_2 \notin M \). Then it is easy to check that the restriction of the map \( h \) to \( M \), \( h : M \to M \), is bijective and continuous, and its inverse \( h^{-1} : M \to M \) is also continuous. Consequently, the map \( T : C^*(M) \to C^*(M) \) sending each \( f \) into \( a \cdot f \circ h \), \( |a| \equiv 1 \), is clearly a surjective linear isometry, that is, it is not a codimension 1 isometry, against our hypothesis. We deduce that \( x_0 \) must belong to \( M \), and consequently \( x_1, x_2 \) belong to \( M \). Hence, \( h^{-1}(\{x_0\}) \subset M \).

A similar reasoning leads to the fact that \( h^k(\{x_0\}) \subset M \) for every integer \( k \). That is, \( D \subset M \), as was to be proved.

Let us now assume that \( T : C(\beta M) \to C(\beta M) \) is of type I. Thus, there exist an isolated point \( p \in \beta M \) and a homeomorphism ([14, Lemma 2.2]) \( h \) of \( \beta M \setminus \{p\} \) onto \( \beta M \) and a function \( a \in C(\beta M \setminus \{p\}) \), \( |a| \equiv 1 \), such that

\[
(Tf)(x) = a(x) \cdot f(h(x))
\]

for all \( x \in \beta M \setminus \{p\} \). Consider the set \( A = \{p, h^{-1}(p), h^{-2}(p), \ldots\} \). Then \( Y := \beta M \setminus \overline{\beta M}(A) \) is a locally compact space and \( h : Y \to Y \) is a surjective homeomorphism. Hence we have a surjective isometry \( S : C_0(Y) \to C_0(Y) \) defined to be

\[
(Sf)(x) = \hat{a}(x) \cdot f(h(x)),
\]

where \( \hat{a} \) is the restriction to \( Y \) of \( a \).

For any \( f \in C_0(Y) \), we can define a function \( \hat{f} \in C(\beta M) \) such that \( \hat{f} = f \) on \( Y \) and 0 on \( \beta M \setminus Y \). As a consequence, a linear continuous functional \( \mu \) (indeed a regular complex measure) can be defined on \( C_0(Y) \) to be \( \mu(f) := (T\hat{f})(p) \).

**Claim 1.** Assume that there is \( f \in C_0(Y) \) such that \( \mu(f) = 0 \) and \( (\mu \circ S^{-n})(f) = 0 \) for all \( n \in \mathbb{N} \). Then \( f \equiv 0 \).

Let us suppose, contrary to what we claim, that there is \( f \in C_0(Y) \), \( f \neq 0 \), such that \( \mu(f) = 0 \) and \( (\mu \circ S^{-n})(f) = 0 \) for all \( n \in \mathbb{N} \). Let us check that \( \hat{f} \in \mathcal{R}(T^n) \) for all \( n \in \mathbb{N} \).

Since \( S : C_0(Y) \to C_0(Y) \) is a surjective isometry, there is \( g \in C_0(Y) \) such that \( S(g) = f \). If \( x \in Y \), then

\[
(T\hat{g})(x) = a(x) \cdot \hat{g}(h(x)) = \hat{a}(x) \cdot g(h(x)) = (Sg)(x) = f(x) = \hat{f}(x).
\]
That is, $T\hat{g} = \hat{f}$ on $Y$.

On the other hand, $(T\hat{g})(p) := \mu(g) = \mu(S^{-1}f) = (\mu \circ S^{-1})(f)$. By assumption, $(\mu \circ S^{-1})(f) = 0$. Hence, $(T\hat{g})(p) = 0 = \hat{f}(p)$.

Next, from the representation of the isometric shift $T$, we know that $(T\hat{g})(h^{-n}(p)) = a(h^{-n}(p)) \cdot \hat{g}(h^{-n+1}(p))$, but $h^{-n+1}(p) \in \beta M \setminus Y$, which is to say that $\hat{g}(h^{-n+1}(p)) = 0$.

Finally, it is apparent, from the above two paragraphs and from density, that $T\hat{g} \equiv 0$ on $\beta M \setminus Y$. Hence, gathering the information above, we infer that $T\hat{g} = \hat{f}$, i.e., $\hat{f} \in R(T)$.

Let us now check that $\hat{f} \in R(T^{n})$. To see this, it suffices to prove that $\hat{g} \in R(T)$. Since $S$ is surjective, there is $g_{1} \in C_{0}(Y)$ such that $S(g_{1}) = g$. Furthermore $(T\hat{g}_{1})(p) := \mu(g_{1}) = \mu(S^{-2}f) = (\mu \circ S^{-2})(f) = 0 = \hat{g}(p)$. Hence, as above, we deduce that $T\hat{g}_{1} = \hat{g}$. In like manner, we can obtain $g_{2}, g_{3}, \ldots, g_{n}, \ldots$ to show that $\hat{f} \in R(T^{n})$ for all $n \in \mathbb{N}$. This fact contradicts the definition of isometric shift and the proof of Claim 1 is complete.

It is well-known that every regular complex measure $\theta$ can be written as $\theta = (\theta_{1} - \theta_{2}) + i(\theta_{3} - \theta_{4})$, where $\theta_{i}, i = 1, 2, 3, 4$, are regular positive measures. Hence each of the regular complex measures $\mu, \mu \circ S^{-1}, \mu \circ S^{-2}, \ldots, \mu \circ S^{-n}, \ldots$ can be divided into four regular positive measures. As a consequence, we get a new sequence of regular positive measures, which we shall denote by $\{\mu_{n}\}_{n \in \mathbb{N}}$. With no loss of generality, we can assume that all these measures are normalized.

Since the space of regular measures on a locally compact space is a Banach space, we can define a regular positive measure as follows:

$$\eta := \sum_{n=1}^{\infty} \frac{\mu_{n}}{2^{n}}$$

**Claim 2.** For every nonempty open subset $U$ of $Y$, $\eta(U) > 0$.

Let us suppose that there exists a nonempty open subset $U$ of $Y$ such that $\eta(U) = 0$. Hence we can find $f \in C_{0}(Y), f \neq 0$, such that $c(f) \subset U$. Consequently,

$$\mu_{n}(f) = \int_{Y} f d\mu_{n} = 0$$

for all $n \in \mathbb{N}$. Finally, Claim 1 yields $f \equiv 0$, a contradiction.

Let us now define a (open) subset $N := M \setminus \text{cl}_{\beta M}(\{p, h^{-1}(p), h^{-2}(p), \ldots\})$ of $M$. Next we consider the family, say $\mathcal{F}_{1}$, of all subsets $B$ of $N$ which satisfy the following property: if $x, y \in B$, then $d(x, y) \geq 1$ or
\[ d(x, y) = 0, \] where \( d \) denotes the metric in \( N \) induced from \( M \). Let us choose a chain \((A_n)\) of elements of \( \mathcal{F}_1 \) ordered by inclusion. Since \( \bigcup_{\alpha} A_\alpha \in \mathcal{F}_1 \), Zorn’s lemma yields a maximal element, say \( M_1 \).

**Claim 3.** \( M_1 \) is a countable set.

Assume the contrary. Then there exists an uncountable family \( \Delta \) of indexes such that \( M_1 = \{x_\alpha : \alpha \in \Delta\} \).

Since \( N \) is an open subset of \( M \), there is, for each \( \alpha \in \Delta \), a constant \( M_\alpha > 0 \) such that the open ball \( B(x_\alpha, M_\alpha) \subset N \).

Next, for each \( \alpha \in \Delta \), take \( m_\alpha := \inf \left\{ \frac{1}{3} M_\alpha \right\} \)
and consider the open ball \( B(x_\alpha, m_\alpha) \). It is clear, from the definition of \( \mathcal{F}_1 \), that if \( \alpha, \beta \in \Delta \), \( \alpha \neq \beta \), then
\[ B(x_\alpha, m_\alpha) \cap B(x_\beta, m_\beta) = \emptyset. \]

Now, for every \( \alpha \in \Delta \), we can define the set
\[ V_\alpha := \text{int}_{\beta M}(\text{cl}_{\beta M}(B(x_\alpha, m_\alpha))). \]
It is apparent that \( V_\alpha \cap M = B(x_\alpha, m_\alpha) \) for each \( \alpha \in \Delta \) and that \( V_\alpha \cap V_\beta = \emptyset \) if \( \alpha \neq \beta \). Furthermore, each \( V_\alpha \) is contained in \( Y \) since \( Y \) is open.

Summarizing, we have found an uncountable pairwise disjoint family of open subsets \( \{V_\alpha : \alpha \in \Delta\} \) in \( Y \).

We know, by Claim 2, that \( \eta(V_\alpha) > 0 \) for all \( \alpha \in \Delta \). Hence, there is \( n_0 \in \mathbb{N} \) such that the set
\[ \gamma := \left\{ \alpha \in \Delta : \eta(V_\alpha) > \frac{1}{n_0} \right\} \]
is not countable since neither is \( \Delta \). Let us choose a countable subset \( \{\alpha_1, \alpha_2, ..., \alpha_n, ...\} \) of indexes in \( \gamma \). Then,
\[ \eta\left( \bigcup_{n=1}^{\infty} V_{\alpha_n} \right) = \sum_{n=1}^{\infty} \eta(V_{\alpha_n}) = +\infty. \]
This contradiction completes the proof of Claim 3.

As in the paragraph before Claim 3, we can define, for every \( n \in \mathbb{N} \), the family \( \mathcal{F}_n \) of all subsets \( B \) of \( N \) which satisfy the following property: if \( x, y \in B \), then \( d(x, y) \geq 1/n \) or \( d(x, y) = 0 \). In like manner, we
obtain, for every $n \in \mathbb{N}$, a maximal element $M_n$ of $F_n$ which turns out to be countable.

Let us now see that the countable set

$$D := \bigcup_{n=1}^{\infty} M_n$$

is dense in $N$. To this end, choose $x \in N \setminus D$ and $\epsilon > 0$. Then there exists $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \epsilon$. Since $x \notin D$, then $x \notin M_{m_0}$. This fact implies the existence of $y \in M_{m_0}$ such that $d(x, y) < 1/m_0$. That is, there is an element $y$ of $D$ in the open ball $B(x, \epsilon)$ and the density of $D$ in $N$ follows.

Finally, it is clear that the countable set

$$D \cup \{p, h^{-1}(p), h^{-2}(p), \ldots\}$$

is dense in $M$ and we are done.

4. Example

In [4], Haydon showed a method to provide isometric shifts of type II. However, it is remarkable the scarcity of examples of isometric shifts of type I. In this final section we provide an example of an isometric shift of type I, which is not of type II, with several additional features. Indeed, in [3], the authors raised the question whether, for an isometric shift of type I, the set $D := \{p, h^{-1}(p), h^{-2}(p), \ldots\}$ was always dense in $X$. The question was answered in the negative by Farid and Varadarajan ([4]) by providing an example of an isometric shift of type I such that $X \setminus \text{cl}_X(D)$ was a (finite) nonempty subset. Our example shows somehow that $D$ can be far from being dense in $X$ in the sense that $X \setminus \text{cl}_X(D)$ is uncountable. Our $X$ also has, contrary to what Holub conjectured in [9], an infinite connected component (see also [3, Corollary 2.1]).

Example. Let $\partial D$ denote the unit circle in $\mathbb{C}$, and let

$$X = \partial D \cup \left\{ \frac{1}{n} : n \in \mathbb{N}, n \geq 2 \right\} \cup \{0\}.$$ 

It is clear that $X$ is a compact metric space. Let us show that $X$ admits an isometric shift of type I by constructing it explicitly.

Let $T : C(X) \to C(X)$ be the following operator. Take any $f \in C(X)$ and define, for each $e^{i\theta} \in \partial D$,

$$(Tf)(e^{i\theta}) := f(e^{i(\theta+\sqrt{2})}).$$

It is clear that, given any $e^{i\theta} \in \partial D$, the sequences $(e^{i(\theta+2n\sqrt{2})})$ and $(e^{i(\theta+(2n-1)\sqrt{2})})$ are dense in $\partial D$. Then we take in $\partial D$ the point $1 = e^{i0}$. 
Clearly the evaluation map \( \delta_1 \) is continuous in \( C(X) \) and its norm is equal to 1. So, for \( f \in C(X) \), we define

\[
(Tf)(1/2) := -(\delta_1 + \delta_{e^{-i\sqrt{2}}})(f)/2 = -f(1)/2 - f(e^{-i\sqrt{2}})/2.
\]

Next, for \( n \geq 3 \), we define

\[
(Tf)(1/n) := -f(1/n - 1),
\]

and

\[
(Tf)(0) := -f(0).
\]

It is clear that \( Tf \in C(X) \) and that \( T \) is an isometry. In fact \( T \) is a codimension 1 linear isometry of type I (being \( p = 1/2 \)), which is not of type II since the range of \( T \) separates all the points of \( X \) (see [3]). Let us see that it is also a shift operator.

Suppose that \( g \in C(X) \) satisfies \( g \in \bigcap_{n=1}^{\infty} R(T^n) \). We have to prove that \( g = 0 \). First we have that \( (g(1/n)) \) must be a convergent sequence, and it converges to the value \( g(0) \). Also, we have that

\[
g(1/2) = -(T^{-1}g)(1/2) - (T^{-1}g)(e^{-i\sqrt{2}})/2,
\]

by construction. In the same way \( g(1/3) = -(T^{-1}g)(1/2) = (T^{-2}g)(1/2) + (T^2g)(e^{-i\sqrt{2}})/2 = (T^{-1}g)(e^{-i\sqrt{2}})/2 + (T^{-1}g)(e^{-i2\sqrt{2}})/2 \) and, in general, for \( n \geq 2, \ n \in \mathbb{N} \),

\[
g(1/n) = (-1)^{n+1}((T^{-1}g)(e^{-(n-1)i\sqrt{2}})/2 + (T^{-1}g)(e^{-(n-2)i\sqrt{2}})/2).
\]

In particular, we have that the sequence

\[
\left( -1 \right)^{n+1} \frac{(T^{-1}g)(e^{-(n-1)i\sqrt{2}}) + (T^{-1}g)(e^{-(n-2)i\sqrt{2}})}{2}
\]

must converge to \( g(0) \), because \( g \) is continuous.

On the other hand, by the density of points of the form \( e^{i2n\sqrt{2}}, \ n \in \mathbb{N} \), we have that given any point \( z_0 \in \partial D \), there exists a sequence \( (n_k) \) of even numbers such that \( (e^{-in_k\sqrt{2}}) \) converges to \( z_0 \) as \( k \) tends to infinity. Also \( e^{i(n_k-1)\sqrt{2}} \) converges to \( z_0 e^{-i\sqrt{2}} \). Since \( T^{-1}g \) is continuous, this implies that \( ((T^{-1}g)(e^{-in_k\sqrt{2}})) \) goes to \( (T^{-1}g)(z_0) \), and that \( ((T^{-1}g)(e^{i(n_k-1)\sqrt{2}})) \) goes to \( (T^{-1}g)(z_0 e^{-i\sqrt{2}}) \). We deduce that

\[
\left( \frac{(T^{-1}g)(e^{-in_k\sqrt{2}}) + (T^{-1}g)(e^{-in_k\sqrt{2}})}{2} \right)
\]

converges to

\[
\frac{(T^{-1}g)(z_0) + (T^{-1}g)(z_0 e^{-i\sqrt{2}})}{2}.
\]
On the other hand, we know that the above sequence converges to $g(0)$. But a similar approach can be taken for a sequence of odd natural numbers $(m_k)$ instead of $(n_k)$. In this case we will obtain that
\[
\frac{\left((T^{-1}g)(e^{-i(m_k-1)\sqrt{2}}) + (T^{-1}g)(e^{-im_k\sqrt{2}})\right)}{2}
\]
converges to
\[
\frac{(T^{-1}g)(z_0) + (T^{-1}g)(z_0e^{-i\sqrt{2}})}{2},
\]
and on the other hand, it must converge to $-g(0)$. As a consequence, we deduce that $g(0) = -g(0) = 0$, and that, for every $z_0 \in \partial D$,
\[
(T^{-1}g)(z_0) + (T^{-1}g)(z_0e^{-i\sqrt{2}}) = 0.
\]
In particular, this implies that for every $z_0 \in \partial D$, $(T^{-1}g)(z_0e^{-i\sqrt{2}}) = (T^{-1}g)(z_0e^{i\sqrt{2}})$. Consequently, the sequence
\[
\left((T^{-1}g)(e^{i2n\sqrt{2}})\right)
\]
is constant. By the density of points $e^{i2n\sqrt{2}}$, $n \in \mathbb{N}$, we conclude that $T^{-1}g$ is constant on $\partial D$. In particular, this implies that the sequence $(|g(1/n)|)$ is constant. Since it converges to $|g(0)| = 0$, we conclude that $g(1/n) = 0$ for every $n \in \mathbb{N}$. As a consequence it is easy to see that $T^{-1}g \equiv 0$ on $\partial D$. But this clearly implies that $g = 0$, as we wanted to prove.

REFERENCES

[1] J. Araujo and J.J. Font, *Linear isometries between subspaces of continuous functions*. Trans. Amer. Math. Soc. **349** (1997), 413-428.
[2] J. Araujo and J.J. Font, *Codimension 1 linear isometries on function algebras*. Proc. Amer. Math. Soc. **127** (1999), 2273-2281.
[3] R.M. Crownover, *Commutants of shifts on Banach spaces*. Michigan Math. J. **19** (1972), 233-247.
[4] F.O. Farid and K. Varadajaran, *Isometric shift operators on $C(X)$*. Can. J. Math. **46** (3) (1994), 532-542.
[5] J.J. Font, *Isometries on function algebras with finite codimensional range*. Manuscripta Math. **100** (1999), 13-21.
[6] A. Gutek, D. Hart, J. Jamison and M. Rajagopalan, *Shift Operators on Banach Spaces*. J. Funct. Anal. **101** (1991), 97-119.
[7] R. Haydon, *Isometric Shifts on $C(K)$*. J. Funct. Anal. **135** (1996), 157-162.
[8] H. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*. Studia Math. **26** (1966), 133-136.
[9] J.R. Holub, *On Shift Operators*. Canad. Math. Bull. **31** (1988), 85-94.
[10] K. Izuchi, *Douglas algebras which admit codimension 1 linear isometries*. Proc. Amer. Math. Soc. (To appear).
[11] N. K. Nikol’skii, *Treatise on the shift operator*. Springer-Verlag, (1986).
[12] M. Rajagopalan and K. Sundaresan, *Backward shifts on Banach spaces $C(X)$*. J. Math. Anal. Appl. 202 (1996), 485-491.
[13] M. Rajagopalan and K. Sundaresan, *Generalized backward and forward shifts on function spaces*. J. Analysis 7 (1999), 78-81.
[14] A. Wilansky, *Topology for analysis*. Waltham, Mass.-Toronto, Ont.-London, (1970).

Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Facultad de Ciencias, Avda. de los Castros, s. n., E-39071 Santander, Spain
E-mail address: araujoj@unican.es

Departamento de Matemáticas, Universitat Jaume I, Campus Penyeta Roja, E-12071 Castellón, Spain
E-mail address: font@mat.uji.es