INVESTMENT AND CONSUMPTION IN REGIME-SWITCHING MODELS WITH PROPORTIONAL TRANSACTION COSTS AND LOG UTILITY

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(Communicated by Qing Zhang)

Abstract. A continuous-time and infinite-horizon optimal investment and consumption model with proportional transaction costs and regime-switching was considered in Liu [4]. A power utility function was specifically studied in [4]. This paper considers the case of log utility. Using a combination of viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation and convex analysis of the value function, we are able to derive the characterizations of the buy, sell and no-transaction regions that are regime-dependent. The results generalize Shreve and Soner [6] that deals with the same problem but without regime-switching.

1. Introduction. The optimal investment and consumption problems with transaction costs have attracted considerable attentions in the fields of financial mathematics and financial engineering. A continuous-time model with proportional transaction costs and infinite horizon is studied by Davis and Norman [3], and Shreve and Soner [6], among others, where the risky asset (a stock) is assumed to follow a constant geometric Brownian motion (GBM). It is shown that the solvency region can be partitioned into three non-overlapped cones, corresponding to the buy, sell and no-transaction regions, respectively. The optimal investment policy is to keep the system state within the no-transaction region for all the time. Specifically, if the investor’s initial position is outside the no-transaction region, he/she should sell or buy stocks immediately to bring his/her position into the no-transaction region. The investor should then trade only when his/her position is on the boundary of the no-transaction region, and buy or sell only the amount necessary to keep the position from leaving the boundary.

Along another line, regime-switching models have been well accepted in recent years as improved models for asset prices. In this setting, asset prices are dictated by a number of GBMs coupled by a finite-state Markov chain, that represents

2010 Mathematics Subject Classification. Primary: 91G80, 93E20; Secondary: 60H30.

Key words and phrases. Investment and consumption model, proportional transaction costs, regime-switching model, Hamilton-Jacobi-Bellman equation, viscosity solution, log utility.

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various randomly changing market conditions. The regime-switching models are capable of capturing the dynamical change of the asset prices across different stages of business cycles. Motivated by the strong modeling ability of the regime-switching models, we studied in Liu [4] an infinite-horizon problem of optimal investment and consumption with proportional transaction costs in a continuous-time regime-switching model. In particular, [4] considers a power utility function $U(c) = c^\gamma / \gamma$ with $0 < \gamma < 1$. We adopt the methodology of [6] and extend the fundamental results of [6] to the regime-switching model for the power utility case. Note that inclusion of regime-switching in the problem formulation significantly complicates the analysis and solution due to the fact that the associated Hamilton-Jacobi-Bellman (HJB) equation to the optimal control problem is given by a system of coupled variational inequalities (in contrast with a single one without regime-switching as in [6]). We have shown that, with regime-switching, the three regions (buy, sell and no-transaction) become regime-dependent, that is, they can be different for different state of the Markov chain and, when a regime-switching occurs, those three regions may also change. These results are very useful in application since they can help the investors make the right decisions regarding investment and consumption with varying market regimes.

In this paper we continue the study with a different utility function, namely, the log utility function given by $U(c) = \ln c$. Note that when the log utility is used in the optimization problem, the value function takes $-\infty$ on the boundary of the problem solvency region, which is different than the power utility as we treated in [4]. In Section 2 we present the formulation of the investment and consumption problem, and the associated HJB system. In Section 3 we presents the fundamental properties of the value function, in particular, the viscosity solution property of the value function to the HJB system. In Section 4 we use the viscosity solution property and the convexity property of the value function to characterize the (regime-dependent) buy, sell and no-transaction regions associated with the optimal control. We provide further remarks and conclude the paper in Section 5. In addition, we include an appendix in Section 6 that provides a proof of the viscosity solution property and a convex analysis of the value function.

2. Problem formulation. In this work we consider the problem that an investor distributes his/her wealth between a bond and a stock and consumes from the bond account in such a way that a total utility from consumption over an infinite time horizon be maximized. The bond price and the stock price are assumed to depend on the market regime, which is modeled by a continuous-time Markov chain $\alpha_t$ that takes value in a finite set $\mathcal{M} = \{1, \ldots, m_0\}$ where $m_0 > 0$ denotes the total number of regimes considered for the market. Specifically, we assume

$$dB(t) = r(\alpha_t)B(t)dt$$

(2.1)

for the bond price $B(t)$ and

$$dS(t) = S(t)[\eta(\alpha_t)dt + \sigma(\alpha_t)dW(t)]$$

(2.2)

for the stock price $S(t)$, where $W(t)$ is an one-dimensional standard Brownian motion, $r : \mathcal{M} \to \mathbb{R}$ is the interest rate, $\eta : \mathcal{M} \to \mathbb{R}$ is the drift of the stock, and $\sigma : \mathcal{M} \to \mathbb{R}$ is the volatility of the stock. We assume that $r(i) > 0$ and $\sigma(i) > 0$ for each $i \in \mathcal{M}$. Both $W(t)$ and $\alpha_t$ are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions.
of right continuity and completeness. Moreover, \( \alpha_t \) is observable and independent of \( W(t) \).

Let \( x(t) \) and \( y(t) \) denote the dollar amounts at time \( t \) in the bond and stock accounts, respectively. Then the equations governing the system state \( (x(t), y(t)) \) are given by:

\[
\begin{align*}
\dot{x}(t) &= [r(\alpha_t)x(t) - C(t)]dt - (1 + \lambda)dM(t) + (1 - \mu)dN(t), \\
\dot{y}(t) &= y(t)[\eta(\alpha_t)dt + \sigma(\alpha_t)dW(t)] + dM(t) - dN(t),
\end{align*}
\]

with given initial values \( x(0^-) = x, \ y(0^-) = y, \) and \( \alpha_0 = i \), where \( C(t) \) denotes the consumption rate at time \( t \), \( M(t) \) and \( N(t) \) denote the cumulative purchases and sales of the stock at time \( t \), respectively. The triple \( (C, M, N) = \{(C(t), M(t), N(t)), t \geq 0\} \) is a control policy for the optimal investment and consumption problem. The constants \( \lambda \) and \( \mu \) stand for the proportional transaction costs associated with, respectively the purchases and sales of the stock. We assume that \( 0 \leq \mu < 1 \) and \( 0 \leq \lambda < 1 \).

The open subset \( \Pi \) of \( \mathbb{R}^2 \), defined by

\[
\Pi = \{(x, y) \in \mathbb{R}^2 : x + (1 + \lambda)y > 0, \ x + (1 - \mu)y > 0\}
\]

is the solvency region of the problem. Let

\[
\partial_1 \Pi = \{(x, y) \in \mathbb{R}^2 : y \geq 0, \ x + (1 - \mu)y = 0\},
\]
\[
\partial_2 \Pi = \{(x, y) \in \mathbb{R}^2 : y \leq 0, \ x + (1 + \lambda)y = 0\}.
\]

Then the boundary of \( \Pi \) can be written as \( \partial \Pi = \partial_1 \Pi \cup \partial_2 \Pi \). Also note that \( \Pi = \Pi \cup \partial \Pi \).

**Definition 2.1.** Given \( (x, y) \in \Pi \) and \( i \in \mathcal{M} \), a control policy \( (C, M, N) \) is called admissible with respect to the initial state \( (x, y, i) \) if the following conditions hold:

1. \( C, M, N \) are \( \mathcal{F} \)-adapted processes.
2. The process \( C \) satisfies:

\[
C(t) \geq 0, \ \text{a.s.,} \ \int_0^T C(t)dt < \infty, \ \text{a.s.,} \ \forall T \geq 0.
\]

3. The processes \( M \) and \( N \) are right-continuous with left limits (RCLL), non-negative and nondecreasing.
4. \( (x(t), y(t)) \in \Pi, \ \forall t \geq 0, \) where \( (x(t), y(t)) \) is the solution of (2.3) when the control \( (C, M, N) \) is being used.

Let \( \mathcal{A}(x, y, i) \) denote the collection of admissible controls for the initial state \( (x, y, i) \).

Let \( U : [0, \infty) \to \mathbb{R} \) be the utility function. In this paper we consider the log utility function given by \( U(x) = \ln x \). Given an initial state \( (x, y, i) \) with \( (x, y) \in \Pi, \ i \in \mathcal{M}, \) and an admissible control \( (C, M, N) \in \mathcal{A}(x, y, i) \), The total discounted utility \( J \) from consumption is defined by:

\[
J(x, y, i, C, M, N) = \mathbb{E} \left[ \int_0^\infty e^{-\beta t}U(C(t))dt \right] = \mathbb{E} \left[ \int_0^\infty e^{-\beta t}\ln(C(t))dt \right],
\]

where \( \beta > 0 \) is a discount factor. The value function \( V \) is then given by

\[
V(x, y, i) = \sup_{(C, M, N) \in \mathcal{A}(x, y, i)} J(x, y, i, C, M, N).
\]
Remark 2.1. Note that if \((x, y) \in \partial \Pi\), then for any \(i \in \mathcal{M}\), the only admissible control policy is the one that jumps immediately to \((0, 0)\) and remains there and has \(C \equiv 0\). Consequently \(V(x, y, i) = -\infty\) on the boundary \(\partial \Pi\).

For notational convenience, in what follows we let \(a_i = a(i), i \in \mathcal{M}\) for a function \(a : \mathcal{M} \rightarrow \mathbb{R}\), and \(w_i(x, y) = w(x, y, i)\) for a function \(w : \Pi \times \mathcal{M} \rightarrow \mathbb{R}\).

For each \(i \in \mathcal{M}\), let the operator \(L_i\) be defined by

\[
L_i \phi(x, y) = \beta \phi(x, y) - \frac{1}{2} \sigma_i^2 y^2 \phi_{yy}(x, y) - \eta_i y \phi_y(x, y) - r_i x \phi_x(x, y),
\]

where \(\phi\) is a function in \(C^2(\Pi)\). With regime-switching present in the model, the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal control problem is the following system of \(m_0\) coupled variational inequalities:

\[
\min \left\{ L_i \phi(x, y, i) - \tilde{U}(w(x, y, i)) - Q \phi(x, y, \cdot)(i),
\right. \\
(1 + \lambda) w_x(x, y, i) - w_y(x, y, i), \\
\left. \quad -(1 - \mu) w_x(x, y, i) + w_y(x, y, i) \right\} = 0, \quad (x, y) \in \Pi, \quad i \in \mathcal{M},
\]

where \(\tilde{U}(x) = \sup_{c \geq 0} \{ U(c) - cx \}\) is the convex conjugate of \(U\) and

\[
Q \phi(x, y, \cdot)(i) = \sum_{j \neq i} q_{ij} \left[ w(x, y, j) - w(x, y, i) \right] = \sum_{j=1}^{m_0} q_{ij} \left[ w(x, y, j) \right]
\]

where \((q_{ij})_{m_0 \times m_0}\) is the intensity matrix of the Markov chain \(\alpha_t\). Note that \(q_{ij}\)s satisfy: \(q_{ij} \geq 0\) if \(i \neq j\), \(q_{ii} \leq 0\) and \(q_{ii} = - \sum_{j \neq i} q_{ij}\) for each \(i = 1, \ldots, m_0\).

For the log utility function \(U(c) = \ln c\), we have

\[
\tilde{U}(x) = \sup_{c \geq 0} \{ \ln c - cx \} = - \ln x - 1.
\]

3. Properties of the value function. In this section we study the properties of the value function \(V\) defined by (2.8) for the log utility. First of all, The value function \(V\) has the homotheticity property.

Proposition 3.1.

\[
V(\theta x, \theta y, i) = \frac{1}{\beta} \ln \theta + V(x, y, i),
\]

for all \(\theta > 0, (x, y) \in \Pi\) and for each \(i \in \mathcal{M}\).

Next, in view of the concavity of the utility function \(U\) and the linearity of the state equation (2.3), as detailed in [6, Proposition 3.1], we have

Proposition 3.2. The value function \(V\) is jointly concave in \(x\) and \(y\) for each \(i \in \mathcal{M}\).

Corollary 3.1. The value function \(V\) is continuous in \(\Pi\) for each \(i \in \mathcal{M}\).
To show that \( V \) is also continuous on the boundary \( \partial \Pi \), i.e., \( V(x, y, i) \to -\infty \) as \( (x, y) \to \partial \Pi \), we will establish an upper bound for \( V \) that equals \(-\infty\) at \( \partial \Pi \). This is done by considering a power utility function \( U(c) = c^\gamma / \gamma \), \( 0 < \gamma < 1 \).

Let \( V_\gamma \) denote the value function of the optimization problem for the power utility function \( U(c) = c^\gamma / \gamma \). That is, \( V_\gamma(x, y, i) \) is defined by (2.8) where the power utility function is used in (2.7). Consider the function

\[
h(\gamma) = \frac{\beta - r^* \gamma}{\gamma} - \frac{\theta^*}{2(1 - \gamma)}, \quad \gamma \in (0, 1),
\]

(3.2)

where \( r^* = \max_{i \in \mathcal{M}} r_i \) and \( \theta^* = \max_{i \in \mathcal{M}} \frac{(\eta_i - r_i)^2}{\sigma_i^2} \). Then it is readily seen that \( h \) has a unique root \( \gamma_0 \in (0, 1) \) such that \( h(\gamma) > 0 \), \( \forall \gamma \in (0, \gamma_0) \). Consequently, let \( A(\gamma) = \frac{\gamma - h(\gamma)}{\gamma(1 - h(\gamma))} \), then \( A(\gamma) > 0 \), \( \forall \gamma \in (0, \gamma_0) \). Define a function \( \varphi : \Pi \to \mathbb{R} \) by

\[
\varphi(x, y) = \frac{1}{\gamma} A^{-1}(\gamma)(x + \kappa y)^\gamma, \quad (x, y) \in \Pi,
\]

(3.3)

where \( \kappa \) is a constant satisfying \( 1 - \mu \leq \kappa \leq 1 + \lambda \). Using the operator \( L_i \) defined by (2.9) for each \( i \in \mathcal{M} \), and the power utility function in (2.11), we have

\[
L_i \varphi(x, y) - \tilde{U}(\varphi(x, y)) = A^{-1}(\gamma)[x + \kappa y]^\gamma \times
\]

\[
\left[ \frac{\beta - \gamma r_i}{\gamma} - \frac{(\eta_i - r_i)^2}{2(1 - \gamma)\sigma_i^2} - \frac{1 - \gamma}{\gamma} A(\gamma) + \frac{1}{2(1 - \gamma)} \left( (1 - \gamma)\sigma_i x \kappa y - \eta_i - r_i \right)^2 \right] \geq A^{-1}(\gamma)[x + \kappa y]^\gamma \left[ h(\gamma) - \frac{1 - \gamma}{\gamma} A(\gamma) \right] = 0.
\]

(3.4)

By following the proof of [6, Proposition 5.1], we can show

**Proposition 3.3.** Let \( \gamma \in (0, \gamma_0) \). Then the value function \( V_\gamma \) for the power utility function \( U(c) = c^\gamma / \gamma \) has the following upper bound:

\[
V_\gamma(x, y, i) \leq \frac{1}{\gamma} A^{-1}(\gamma)(x + \kappa y)^\gamma, \quad \forall (x, y) \in \Pi, \forall i \in \mathcal{M}.
\]

(3.5)

Using Proposition 3.3, we establish the following upper bound for \( V \), the value function for the log utility function \( U(c) = \ln c \) considered in this paper.

**Proposition 3.4.**

\[
V(x, y, i) \leq \frac{1}{\beta} \ln(x + \kappa y) + \frac{1}{\beta} \ln \beta + \frac{r^* - \beta}{\beta^2} + \frac{\theta^*}{2\beta^2}, \quad \forall (x, y) \in \Pi, \forall i \in \mathcal{M}.
\]

(3.6)

**Proof.** Since \( \ln c \leq \frac{c^\gamma}{\gamma} - 1 / \gamma \), \( \forall c \geq 0 \), \( \forall \gamma \in (0, 1) \), in view of (2.8), we have \( V(x, y, i) \leq V_\gamma(x, y, i) - \frac{1}{\beta} \). It then follows by using (3.5) that

\[
V(x, y, i) \leq \lim_{\gamma \to 0^+} \frac{A^{-1}(\gamma)(x + \kappa y)^\gamma - 1}{\gamma}
\]

\[
= \frac{1}{\beta} \ln(x + \kappa y) + \frac{1}{\beta} \ln \beta + \frac{r^* - \beta}{\beta^2} + \frac{\theta^*}{2\beta^2}.
\]

\( \square \)

By taking \( \kappa = 1 - \mu \) and \( \kappa = 1 + \lambda \), respectively in (3.6), we can easily see that \( V(x, y, i) \) has limit \(-\infty\) as \( (x, y) \) approaches the boundary \( \partial \Pi \). This result, combined with Corollary 3.1, produces the continuity of the value function \( V \) on \( \Pi \).
Theorem 3.1. The value function $V$ is continuous on $\overline{\Pi}$ for each $i \in M$.

The next proposition establishes a lower bound for the value function $V$. This result generalizes [6, Proposition 3.4] to the regime-switching model for the case of log utility.

Proposition 3.5.

$$V(x, y, i) \geq \begin{cases} \frac{1}{\beta} \ln[x + (1 - \mu)y] + \frac{1}{\beta} \ln \beta + \frac{r_* - \beta}{\beta^2}, & \forall i \in M, \forall (x, y) \in \overline{\Pi}, y \geq 0, \\ \frac{1}{\beta} \ln[x + (1 + \lambda)y] + \frac{1}{\beta} \ln \beta + \frac{r_* - \beta}{\beta^2}, & \forall i \in M, \forall (x, y) \in \overline{\Pi}, y < 0, \end{cases}$$

where $r_* = \min_{i \in M} r_i$.

Proof. First we note that in view of Remark 2.1, $V(x, y, i) = -\infty$ on the boundary $\partial \Pi = \partial_1 \Pi \cup \partial_2 \Pi$, which agrees with the lower bounds given in (3.7).

We now show that the lower bounds (3.7) hold in $\Pi$. To this end, given $i \in M$ and $(x, y) \in \Pi$. We consider the admissible control $(C, M, N)$ in $A(x, y, i)$ that takes one initial jump to a point on the $x$-axis (if $y \neq 0$), makes no further transfers between the bond and the stock afterwards, and consumes at a rate proportional to the bond account. Specifically, if $y > 0$, we set $M(0) = 0$, $N(0) = y$, resulting in $x(0) = x + (1 - \mu)y$, $y(0) = 0$; if $y < 0$, we set $M(0) = -y$, $N(0) = 0$, resulting in $x(0) = x + (1 + \lambda)y$, $y(0) = 0$. For both cases, we set $C(t) = \theta x(t)$, $t \geq 0$, where $\theta > 0$ is a positive constant. Applying the policy $(C, M, N)$ to (2.3), we have $y(t) = 0$, $t \geq 0$ and

$$dx(t) = [r(\alpha_t)x(t) - \theta x(t)]dt,$$

whose solution is

$$x(t) = x(0)e^{\int_0^t [r(\alpha_s) - \theta]ds}$$

with

$$x(0) = \begin{cases} x + (1 - \mu)y, & \text{if } y \geq 0, \\ x + (1 + \lambda)y, & \text{if } y < 0. \end{cases}$$

It follows that

$$V(x, y, i) \geq \mathbb{E} \left[ \int_0^\infty e^{-\beta t} U(C(t)) \, dt \right]$$

$$= \mathbb{E} \left[ \int_0^\infty e^{-\beta t} \ln(\theta x(t)) \, dt \right]$$

$$= \frac{1}{\beta} \ln(\theta x(0)) + \mathbb{E} \left[ \int_0^\infty e^{-\beta t} \int_0^t [r(\alpha_s) - \theta]ds \, dt \right]$$

$$\geq \frac{1}{\beta} \ln(\theta + \frac{1}{\beta} \ln|x(0)| + \frac{r_* - \theta}{\beta^2} t)$$

$$\geq \frac{1}{\beta} \ln \beta + \frac{1}{\beta} \ln|x(0)| + \frac{r_* - \theta}{\beta^2},$$

noting that the maximum value of $\frac{1}{\beta} \ln \theta + \frac{r_* - \theta}{\beta^2}$ is attained at $\theta = \beta$.

This completes the proof. □

The next result, whose proof is similar to [4, Proposition 3.2], generalizes [8, Proposition 2.1] to the regime-switching model and log utility function treated in
this paper. For completeness, we include the proof in the appendix (see 6.1 Proof of Proposition 3.6).

**Proposition 3.6.** The value function $V$ is strictly increasing in $x$ and increasing in $y$ for each $i \in \mathcal{M}$.

Now we present the viscosity solution property of the value function $V$ to the HJB system (2.10) for which the proof is based on the following dynamic programming (DP) equation. Let $\tau$ be an almost surely finite stopping time. Then

$$V(x, y, i) = \sup_{(C, M, N) \in \mathcal{A}(x, y, i)} E \left[ \int_0^\tau e^{-\beta t} \ln(C(t)) dt + e^{-\beta \tau} V(x(\tau), y(\tau), \alpha_\tau) \right].$$

(3.9)

Moreover, if $(C^*, M^*, N^*)$ is an optimal control, then

$$V(x, y, i) = E \left[ \int_0^\tau e^{-\beta t} \ln(C^*(t)) dt + e^{-\beta \tau} V(x^*(\tau), y^*(\tau), \alpha_\tau) \right],$$

(3.10)

where $(x^*(t), y^*(t)), t \geq 0$ denotes the solution to (2.3) when $(C^*, M^*, N^*)$ is being used.

The notion of viscosity solution to the HJB system (2.10) for the regime-switching model considered in this paper is a natural modification of that for the second-order nonlinear partial differential equations introduced in Crandall, Ishii and Lions [2]. For completeness and later use, we include the definition in this section.

**Definition 3.2.** Consider a function $v : \Pi \times \mathcal{M} \to \mathbb{R}$ with $v_i(x, y) = v(x, y, i) \in \mathbb{R}, (x, y) \in \Pi, i \in \mathcal{M}$. Suppose that $v_i$ is continuous in $(x, y)$ on $\Pi$ for each $i \in \mathcal{M}$. $v$ is a viscosity solution of the HJB system (2.10) if,

1. $v$ is a viscosity subsolution of (2.10) in $\Pi$, that is, for each $i \in \mathcal{M}$,

$$\min \left\{ L_i \varphi(x_0, y_0) - Qv(x_0, y_0, \cdot)(i) - \tilde{U}(\varphi(x_0, y_0)), \right.$$

$$(1 + \lambda)\varphi_x(x_0, y_0) - \varphi_y(x_0, y_0),$$

$$-(1 - \mu)\varphi_x(x_0, y_0) + \varphi_y(x_0, y_0) \right\} \leq 0,$$

whenever $\varphi \in C^2(\Pi)$ such that $v_i - \varphi$ has a maxima at $(x_0, y_0) \in \Pi$ and $v_i(x_0, y_0) = \varphi(x_0, y_0)$, and

2. $v$ is a viscosity supersolution of (2.10) in $\Pi$, that is, for each $i \in \mathcal{M}$,

$$\min \left\{ L_i \psi(x_0, y_0) - Qv(x_0, y_0, \cdot)(i) - \tilde{U}(\psi(x_0, y_0)), \right.$$

$$(1 + \lambda)\psi_x(x_0, y_0) - \psi_y(x_0, y_0),$$

$$-(1 - \mu)\psi_x(x_0, y_0) + \psi_y(x_0, y_0) \right\} \geq 0,$$

whenever $\psi \in C^2(\Pi)$ such that $v_i - \psi$ has a minima at $(x_0, y_0) \in \Pi$ and $v_i(x_0, y_0) = \psi(x_0, y_0)$.

In view of Definition 3.2, the following corollary is immediate.

**Corollary 3.2.** Let the function $v : \Pi \times \mathcal{M} \to \mathbb{R}$ be a viscosity solution of the HJB system (2.10). If for some $i \in \mathcal{M}$, $v_i$ is twice differentiable at a point $(x_0, y_0) \in \Pi$, 

then
\[
\min \left\{ L_i v(x_0, y_0, i) - Q v(x_0, y_0, \cdot)(i) - U(v_x(x_0, y_0, i)), \right. \\
(1 + \lambda) v_x(x_0, y_0, i) - v_y(x_0, y_0, i), \\
\left. - (1 - \mu) v_x(x_0, y_0, i) + v_y(x_0, y_0, i) \right\} = 0,
\]
that is, the function \( v_i \) satisfies the equation (3.13) in the classical sense at the point \((x_0, y_0)\).

The following property will be used in the rest of the paper. Given \( i \in \mathcal{M} \) and two points \((x_1, y_1), (x_2, y_2) \in \Pi\). \((x_2, y_2) \) can be reached from \((x_1, y_1) \) by a transaction if there exist two constants \( m, n \geq 0 \) such that
\[
x_2 = x_1 - (1 + \lambda)m + (1 - \mu)n, \quad y_2 = y_1 + m - n.
\]
In this case, for each admissible control \((C, M, N) \in \mathcal{A}(x_2, y_2, i)\), an admissible control in \( \mathcal{A}(x_1, y_1, i) \) can be constructed by simply adding this transaction to \((C, M, N) \) at time \( t = 0 \). This results in the following proposition.

**Proposition 3.7.** If \((x_2, y_2) \in \Pi \) can be reached from \((x_1, y_1) \in \Pi \) by a transaction, then we have \( V(x_2, y_2, i) \leq V(x_1, y_1, i) \) for any \( i \in \mathcal{M} \).

**Theorem 3.3.** The value function \( V \) defined by (2.8) is a viscosity solution of the HJB system (2.10) in \( \Pi \).

We conclude this section with a property of the value function that will be used in the next section. The proofs are similar to [6, Proposition 3.6, Corollary 3.7].

**Proposition 3.8.** Give \( i \in \mathcal{M} \) and \((x, y) \in \Pi\). Then for all \( h \geq 0 \), we have
\[
V(x + h, y, i) \geq \begin{cases} 
\frac{1}{2} \ln \left[ 1 + \frac{h}{x + (1 + \lambda)y} \right] + V(x, y, i), & \text{if } y \geq 0, \\
\frac{1}{2} \ln \left[ 1 + \frac{h}{x + (1 - \mu)y} \right] + V(x, y, i), & \text{if } y < 0.
\end{cases}
\]

**Corollary 3.3.** Give \( i \in \mathcal{M} \) and \((x_0, y_0) \in \Pi\). Let \( \phi : \Pi \to \mathbb{R} \) be a differentiable function satisfying \( \phi(x_0, y_0) = V(x_0, y_0, i) \). If \( \phi \geq V_i \) or \( \phi \leq V_i \) on \( \Pi \), then,
\[
\phi_x(x_0, y_0) \geq \begin{cases} 
\frac{1}{\beta \|x_0 + (1 + \lambda)y_0\|}, & \text{if } y_0 \geq 0, \\
\frac{1}{\beta \|x_0 + (1 - \mu)y_0\|}, & \text{if } y_0 < 0.
\end{cases}
\]

4. **Further analysis of the value function.** We continue the analysis of the value function \( V \) for the log utility function \( U(c) = \ln c \). It can be shown that the solvency region \( \Pi \) can be partitioned into three cones corresponding to the three arguments of the minimum operator in (2.10). However, due to the introduction of regime-switching, the three cones become regime-dependent, namely, they can be different for different regimes. Nevertheless, the proof is very similar to [6, Section 6] as each of the \( m_0 \) functions \( V_i, 1 \leq i \leq m_0 \) can be dealt with separately. Hence coupling due to regime-switching does not complicate the analysis. For the sake of completeness and easy reference, we include the analysis in the appendix (see 6.3 Partition of the Solvency Region).
Similar to [6, Section 8], in view of the homotheticity property (3.1) of the value function, each $V_i$ is transformed to a single variable function by making a proper change of variable. Let $\mathcal{I} = \left( -\frac{1}{\mu}, \frac{1}{\mu} \right)$ and

$$u_i(z) = u(z, i) = V(1 - z, z, i), \ \forall z \in \mathcal{I}, \ \forall i \in \mathcal{M}.$$ (4.1)

Using (3.1), we have

$$V(x, y, i) = \frac{1}{\beta} \ln(x + y) + V \left( \frac{x}{x + y}, \frac{y}{x + y}, i \right)$$

$$= \frac{1}{\beta} \ln(x + y) + u \left( \frac{y}{x + y}, i \right), \ \forall (x, y) \in \Pi \setminus \{(0, 0)\}, \ \forall i \in \mathcal{M}. \quad (4.2)$$

Let $z = \frac{y}{x+y}$. With the connection $w_i(x, y) = \frac{1}{\beta} \ln(x + y) + \varpi_i(z)$, direct computations show that the system of HJB equations (2.10) for $u_i$ is transformed into the following system of HJB equations for $\varpi_i$, $1 \leq i \leq m_0$:

$$\min \left\{ \left( \beta - q_{ii} \right) \varpi_i(z) - \frac{1}{\beta} d_{i1}(z) - d_{i2}(z) \varpi_i'(z) - d_{i3}(z) \varpi_i''(z) + \sum_{j \neq i} q_{ij} \varphi_j(z) \right\}$$

$$- \tilde{U} \left( \frac{1}{\beta} - z \varphi_i'(z) \right), \ \frac{1}{\beta} - \frac{1}{\beta} d_{i4}(z) \varphi_i'(z), \ \frac{1}{\beta} - d_{i5}(z) \varphi_i'(z) \right\} = 0, \ z \in \mathcal{I}, \ i \in \mathcal{M},$$ (4.3)

where

$$d_{i1}(z) = r_i + (\eta_i - r_i)z - \frac{1}{2} \sigma_i^2 z^2,$$

$$d_{i2}(z) = (\eta_i - r_i)z(1 - z) - \sigma_i^2 z^2(1 - z),$$

$$d_{i3}(z) = \frac{1}{2} \sigma_i^2 z^2(1 - z)^2,$$

for $i \in \mathcal{M}$ and

$$d_{i4}(z) = \frac{1 - \mu z}{\mu}, \ d_{i5}(z) = \frac{1 + \lambda z}{\lambda}.$$ 

Note that $d_4$ and $d_5$ do not depend on the regime $i$.

**Proposition 4.1.** The function $u$ defined in (4.1) is a viscosity solution of the HJB system (4.3) in $\mathcal{I}$.

**Proof.** We only show the subsolution property since the supersolution property can be shown analogously. Fix an $i \in \mathcal{M}$. Let $\phi \in C^2(\mathcal{I})$ and $z_0 \in \mathcal{I}$ be such that $u_i - \phi$ has a maxima at $z_0$ and $u_i(z_0) = \phi(z_0)$. Define

$$\varphi(x, y) = \frac{1}{\beta} \ln(x + y) + \phi \left( \frac{y}{x + y} \right), \ \forall (x, y) \in \Pi.$$ 

Then we have $\varphi \in C^2(\Pi)$,

$$\varphi(x, y) \geq \frac{1}{\beta} \ln(x + y) + u_i \left( \frac{y}{x + y} \right) = V(x, y, i), \ \forall (x, y) \in \Pi,$$

and

$$\varphi(1 - z_0, z_0) = \phi(z_0) = u_i(z_0) = V(1 - z_0, z_0, i),$$

$$\forall (x, y) \in \Pi. \quad (4.4)$$

Thus $\varphi$ is a subsolution of the HJB system (4.3). Q.E.D.
that is, $V_i - \varphi$ has a maxima at $(1 - z_0, z_0) \in \Pi$. Using the viscosity subsolution property of $V_i$ to (2.10), we have

$$
\min \left\{ L_i \varphi(1 - z_0, z_0) - QV(1 - z_0, z_0, \cdot)(i) - \tilde{U}(\varphi_x(1 - z_0, z_0)), \\
(1 + \lambda)\varphi_x(1 - z_0, z_0) - \varphi_y(1 - z_0, z_0), \\
- (1 - \mu)\varphi_x(1 - z_0, z_0) + \varphi_y(1 - z_0, z_0) \right\} \leq 0,
$$

which becomes

$$
\min \left\{ \left(\beta - q_{ii}\right)\phi(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)\phi'(z_0) - d_{i3}(z_0)\phi''(z_0) - \sum_{j \neq i} q_{ij} u_j(z_0) \\
- \tilde{U} \left( \frac{1}{\beta} - z_0 \phi'(z_0) \right), \frac{1}{\beta} + d_1(z_0)\phi'(z_0), \frac{1}{\beta} - d_5(z_0)\phi'(z_0) \right\} \leq 0.
$$

(4.4)

It follows from Proposition 3.2 that $u_i$ is a concave function of $z$ in $\mathcal{I}$ for each $i \in \mathcal{M}$ as $u_i$ is obtained by evaluating $V_i$ along a line segment of $x + y = 1 \in \tilde{\Pi}$. By applying the arguments [6, P648-649] to each function $u_i$, $i \in \mathcal{M}$, we have the following results. For any given $z \in \mathcal{I}$, the difference quotient $\frac{u_i(z + h) - u_i(z)}{h}$ is a nonincreasing function of $h$, so the right derivative $D^+ u_i(z)$ and left derivative $D^- u_i(z)$ exist and are finite, where

$$
D^\pm u_i(z) = \lim_{h \to 0^\pm} \frac{u_i(z + h) - u_i(z)}{h}.
$$

Moreover, $D^+$ is right continuous and $D^-$ is left continuous; they are both nonincreasing and agree except on a countable subset $\mathcal{N}_i$ of $\mathcal{I}$. Consequently, $u_i$ is differentiable on $\mathcal{I} \setminus \mathcal{N}_i$ and $\partial u_i(z) = [D^+ u_i(z), D^- u_i(z)]$ for all $z \in \mathcal{I}$, where $\partial u_i$ denotes the sub-differential of $u_i$ which can be defined similar to (6.34). Since $u''_i$ is defined almost everywhere in $\mathcal{I}$ and is nonincreasing, its pointwise derivative $u''_i$ is also defined almost everywhere in $\mathcal{I}$ and is locally integrable. Similar to Corollary 3.2, it can be shown that $u_i$ satisfies (4.3) almost everywhere in the classical sense.

Presented in the next proposition, the $C^1$ property of the function $u_i$ generalizes [6, Proposition 8.2] to the case with regime-switching and log utility.

**Proposition 4.2.** For each $i \in \mathcal{M}$, the function $u_i$ is $C^1$ on $\mathcal{I} \setminus \{0\}$. If for some $i \in \mathcal{M}$, $u_i$ is not continuously differentiable at the point 0, then we have

$$
V(x, 0, i) = \frac{1}{\beta} \ln x + u_i(0), \forall x > 0,
$$

(4.5)

where $u_i(0)$ satisfies the equation

$$
(\beta - q_{ii})u_i(0) - \sum_{j \neq i} q_{ij} u_j(0) - \frac{r_i}{\beta} + 1 - \ln \beta = 0.
$$

(4.6)

Moreover, the left and right derivatives of $u_i$ at 0 exist and are limits of the derivatives of $u_i$ from the left and right sides of 0, respectively.
Proof. Given \( i \in \mathcal{M} \). Let \( \mathcal{I}_i \) denote the subset of \( \mathcal{I} \) on which \( u_i \) is twice differentiable. Then \( \mathcal{I}_i \) has full measure according to the discussion above. Take \( z_0 \in \mathcal{I} \setminus \mathcal{I}_i \). Let \( \{ z_n^+ \}, \{ z_n^- \}, n \geq 1 \) be two sequences in \( \mathcal{I}_i \) such that \( z_n^+ \uparrow z_0 \) and \( z_n^- \downarrow z_0 \) as \( n \to \infty \). Then
\[
\partial u_i(z_0) = [D^+ u_i(z_0), D^- u_i(z_0)] = [\lim_{n \to \infty} u'_i(z_n^+), \lim_{n \to \infty} u'_i(z_n^-)],
\]
since \( D^+ \) is right continuous and \( D^- \) is left continuous.

Because \( u_i \) satisfies (4.3) in the classical sense at each point \( z_n^\pm \), it follows that
\[
\begin{cases}
\frac{1}{\beta} + d_4(z_n^+)u'_i(z_n^+) \geq 0, \\
\frac{1}{\beta} - d_5(z_n^-)u'_i(z_n^-) \geq 0.
\end{cases}
\]

Sending \( n \to \infty \) yields
\[
\begin{cases}
\frac{1}{\beta} + d_4(z_0)D^+ u_i(z_0) \geq 0, \\
\frac{1}{\beta} - d_5(z_0)D^- u_i(z_0) \geq 0.
\end{cases}
\]

Note that \( d_4(z) > 0 \) and \( d_5(z) > 0, \forall z \in \mathcal{I} \). Then we have,
\[
\begin{cases}
\frac{1}{\beta} + d_4(z_0)\theta > 0, \\
\frac{1}{\beta} - d_5(z_0)\theta > 0,
\end{cases} \quad \forall \theta \in (D^+ u_i(z_0), D^- u_i(z_0)). \tag{4.7}
\]

Now suppose \( D^+ u_i(z_0) < D^- u_i(z_0) \) so that such an \( \theta \) does exist. For any \( \varepsilon > 0 \), define:
\[
\phi_\varepsilon(z) = u_i(z_0) + \theta(z - z_0) - \frac{(z - z_0)^2}{\varepsilon}, \quad z \in \mathcal{I}.
\]

Then \( \phi_\varepsilon(z_0) = u_i(z_0), \phi_\varepsilon \in C^2(\mathcal{I}) \) and \( \phi_\varepsilon \) dominates \( u_i \) locally at \( z_0 \). Due to the viscosity subsolution property of \( u_i \) to (4.3), \( \phi_\varepsilon \) satisfies (4.4) at \( z_0 \). In view of (4.7) we indeed obtain
\[
(\beta - q_{ii})\phi_\varepsilon(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)\theta + d_{i3}(z_0)\frac{2}{\varepsilon} - \sum_{j \neq i} q_{ij}u_j(z_0) - \tilde{U}\left(\frac{1}{\beta} - z_0\theta\right) \leq 0,
\]
or equivalently,
\[
(\beta - q_{ii})u_i(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)\theta + d_{i3}(z_0)\frac{2}{\varepsilon} - \sum_{j \neq i} q_{ij}u_j(z_0) - \tilde{U}\left(\frac{1}{\beta} - z_0\theta\right) \leq 0. \tag{4.8}
\]

If \( z_0 \neq 0,1 \), then \( d_{i3}(z_0) > 0 \) and (4.8) can not hold for all \( \varepsilon > 0 \). Consequently, such \( \theta \) does not exist. Therefore \( D^+ u_i(z_0) = D^- u_i(z_0), \) i.e., \( \partial u_i(z_0) \) is a singleton whenever \( z_0 \neq 0,1 \). It follows that \( u_i \) is \( C^1 \) on \( \mathcal{I} \setminus \{0,1\} \) in view of Proposition 6.1.

Now consider the point \( z_0 = 1 \). Again, that \( u_i \) satisfies (4.3) in the classical sense at each point \( z_n^\pm \) implies that
\[
(\beta - q_{ii})u_i(z_n^\pm) - \frac{1}{\beta}d_{i1}(z_n^\pm) - d_{i2}(z_n^\pm)u_i'(z_n^\pm) - d_{i3}(z_n^\pm)u_i''(z_n^\pm) - \sum_{j \neq i} q_{ij}u_j(z_n^\pm) - \tilde{U}\left(\frac{1}{\beta} - z_n^\pm u_i'(z_n^\pm)\right) \geq 0. \tag{4.9}
\]
The local integrability of \( u_i'' \) implies that we can choose the sequences \( \{ z_n^\pm \} \), \( n \geq 1 \) such that
\[
\lim_{n \to \infty} |d_{i3}(z_n^\pm) u_i''(z_n^\pm)| = 0.
\]

Sending \( n \to \infty \) in (4.9) yields
\[
(\beta - q_{ii})u_i(1) - \frac{1}{\beta} d_{i1}(1) - \sum_{j \neq i} q_{ij} u_j(1) - \tilde{U} \left( \frac{1}{\beta} - D^\pm u_i(1) \right) \geq 0. \tag{4.10}
\]

On the other hand, if \( D^+ u_i(1) < D^- u_i(1) \) and \( \theta \in (D^+ u_i(1), D^- u_i(1)) \), then (4.8) becomes
\[
(\beta - q_{ii})u_i(1) - \frac{1}{\beta} d_{i1}(1) - \sum_{j \neq i} q_{ij} u_j(1) - \tilde{U} \left( \frac{1}{\beta} - \theta \right) \leq 0. \tag{4.11}
\]

Sending \( \theta \downarrow D^+ u_i(1) \) and \( \theta \uparrow D^- u_i(1) \), respectively in (4.11) yields the reverse direction of the inequality (4.10). Therefore we obtain equality in (4.10). It follows that \( \tilde{U}(\frac{1}{\beta} - D^+ u_i(1)) = \tilde{U}(\frac{1}{\beta} - D^- u_i(1)) \). In view of (2.13), we have
\[
\ln(\frac{1}{\beta} - D^+ u_i(1)) = \ln(\frac{1}{\beta} - D^- u_i(1)).
\]
Hence \( D^+ u_i(1) = D^- u_i(1) \) and \( \partial u_i(1) \) is a singleton. Consequently, \( u_i \) is \( C^1 \) at \( z_0 = 1 \).

Finally, we consider the point \( z_0 = 0 \). Similar to (4.10), we have
\[
(\beta - q_{ii})u_i(0) - \frac{1}{\beta} d_{i1}(0) - \sum_{j \neq i} q_{ij} u_j(0) - \tilde{U} \left( \frac{1}{\beta} \right) \geq 0.
\]
Since \( d_{i1}(0) = r_i \) and \( \tilde{U}(\frac{1}{\beta}) = -1 + \ln \beta \), it follows that,
\[
(\beta - q_{ii})u_i(0) - \frac{r_i}{\beta} - \sum_{j \neq i} q_{ij} u_j(0) + 1 - \ln \beta \geq 0. \tag{4.12}
\]
If \( D^+ u_i(0) < D^- u_i(0) \) and \( \theta \in (D^+ u_i(0), D^- u_i(0)) \), then (4.8) becomes
\[
(\beta - q_{ii})u_i(0) - \frac{r_i}{\beta} - \sum_{j \neq i} q_{ij} u_j(0) + 1 - \ln \beta \leq 0. \tag{4.13}
\]
Combining (4.12) and (4.13), we have (4.6).

**Corollary 4.1.** The value function \( V_i(x, y) \) is \( C^1 \) on \( \Pi \setminus \{(x, 0) : x > 0\} \) for each \( i \in \mathcal{M} \). If \( V_i \) is not continuously differentiable on \( \{(x, 0) : x > 0\} \) for some \( i \in \mathcal{M} \), then \( V_i(x, 0) \) is given by (4.5) for this \( i \). Moreover, even if \( V_i \) is not continuously differentiable on \( \{(x, 0) : x > 0\} \) for some \( i \in \mathcal{M} \), the partial derivative \( V_x(x, 0, i) \) is defined and continuous there, and the two one-sided partial derivatives:
\[
V_y(x, 0 \pm, i) = \lim_{h \to 0 \pm} \frac{V(x, h, i) - V(x, 0, i)}{h}, \quad x > 0
\]
between are defined and satisfy the one-sided continuity conditions:
\[
V_y(x, 0 \pm, i) = \lim_{(\tilde{x}, \tilde{y}) \to (x, 0 \pm)} V_y(\tilde{x}, \tilde{y}, i), \quad x > 0.
\]

**Proof.** In view of (4.2) and Proposition 4.2, we see that \( V_i(x, y) \) is \( C^1 \) on \( \Pi \setminus \{(x, 0) : x > 0\} \) for each \( i \in \mathcal{M} \). Since \( V(x, 0, i) = \frac{1}{\beta} \ln x + u_i(0), \quad x > 0 \), \( V_x(x, 0, i) = \frac{1}{\beta x} \).
is defined and continuous on \( \{(x, 0) : x > 0\} \). Assume that \( V_i \) is not continuously differentiable on \( \{(x, 0) : x > 0\} \) for some \( i \in \mathcal{M} \). We have

\[
V_y(x, 0 \pm, i) = \lim_{h \to 0 \pm} \frac{1}{h} \ln(x + h) + u \left( \frac{h}{x + h}, i \right) - \frac{1}{h} \ln x - u(0, i)
\]

\[
= \frac{1}{\beta x} + \frac{D^\pm u_i(0)}{x}.
\]  \hspace{1cm} (4.14)

On the other hand, for \((\tilde{x}, \tilde{y}) \neq (0, 0)\),

\[
V_y(\tilde{x}, \tilde{y}, i) = \frac{1}{\beta (\tilde{x} + \tilde{y})} + u'\left( \frac{\tilde{y}}{\tilde{x} + \tilde{y}}, i \right) \frac{\tilde{x}}{(\tilde{x} + \tilde{y})^2},
\]

and

\[
\lim_{(\tilde{x}, \tilde{y}) \to (x, 0 \pm)} V_y(\tilde{x}, \tilde{y}, i) = \frac{1}{\beta x} + \frac{D^\pm u_i(0)}{x}.
\]

\( \square \)

From Section 6.3 of the Appendix we have, for each \( i \in \mathcal{M} \), \( \Pi \) can be partitioned into three open and nonintersecting cones (possibly empty) \( SA^i \), \( BU^i \) and \( NT^i \). Let \( \Pi^* = \Pi \setminus \{(x, 0) : x > 0\} \). Then from Corollary 4.1, \( V_i \) is \( C^1 \) on \( \Pi^* \) for each \( i \in \mathcal{M} \). In view of Corollary 6.2 and Proposition 6.4, we can characterize those cones by:

\[
SA^i = \{(x, y) \in \Pi^* : -(1 - \mu)V_x(x, y, i) + V_y(x, y, i) = 0\}, \tag{4.15}
\]

\[
BU^i = \{(x, y) \in \Pi^* : (1 + \lambda)V_x(x, y, i) - V_y(x, y, i) = 0\}, \tag{4.16}
\]

and

\[
NT^i \setminus \{(x, 0) : x > 0\} = \{(x, y) \in \Pi^* : -(1 - \mu)V_x(x, y, i) + V_y(x, y, i) > 0,
\]

\[
(1 + \lambda)V_x(x, y, i) - V_y(x, y, i) > 0\}. \tag{4.17}
\]

Note that in view of (6.51), we must have \( SA^i \neq \Pi \). Otherwise, \( V_i \) would not be equal to \(-\infty\) on the boundary \( \partial_2 \Pi \). Similarly, we must have \( BU^i \neq \Pi \). This means that both \( SA^i \) and \( BU^i \) are proper subsets of \( \Pi \).

**Proposition 4.3.** If \( \eta_i > r_i - q_i \), then \( NT^i \neq \emptyset \).

**Proof.** Corollary 6.1 tells us that \( SA^i \cap BU^i = \emptyset \) for each \( i \in \mathcal{M} \). Suppose \( \eta_i > r_i - q_i \) for some \( i \in \mathcal{M} \) and \( NT^i = \emptyset \). Then the two cones \( SA^i \) and \( BU^i \) must be nonempty and would share a half-line boundary \( H^i = SA^i \cap BU^i \setminus \{(0, 0)\} \) in \( \Pi \).

We divide the discussion of \( H^i \) into two cases.

**Case 1.** If \( V_i \) is \( C^1 \) on \( H^i \), then (4.15) and (4.16) imply

\[
\begin{align*}
-(1 - \mu)V_x(x, y, i) + V_y(x, y, i) &= 0, \\
(1 + \lambda)V_x(x, y, i) - V_y(x, y, i) &= 0,
\end{align*}
\]

due to the continuity of the partial derivatives of \( V \). This leads to \( V_x(x, y, i) = V_y(x, y, i) = 0, \ (x, y) \in H^i \), a contradiction with (6.37).

**Case 2.** If \( V_i \) is not \( C^1 \) on \( H^i \), then we must have \( H^i = \{(x, 0) : x > 0\} \). In view of (4.5), (6.51) and the continuity of \( V_i \), we must have

\[
A_i = B_i = u_i(0).
\]  \hspace{1cm} (4.18)
If $x > 0$ and $y > 0$ (note that the proof for $x > 0$ and $y < 0$ is completely analogous), then $(x, y) \in SA^i$ and $V_i(x, y) = \frac{1}{\beta} \ln[x + (1 - \mu)y] + u_i(0)$ in view of (6.51) and (4.18). Clearly, $V_i$ is twice differentiable at $(x, y)$. In view of Corollary 3.2, $V_i$ satisfies (3.13) in the classical sense at the point $(x, y)$. Then we have
\[
\mathcal{L}_i V(x, y, i) - QV(x, y, \cdot)(i) - \tilde{U}(V, x, y, i) \geq 0. \tag{4.19}
\]
On the other hand, direct calculation yields:
\[
\begin{align*}
\mathcal{L}_i V(x, y, i) - QV(x, y, \cdot)(i) - \tilde{U}(V, x, y, i) &= \ln[x + (1 - \mu)y] + \beta u_i(0) \\
&\quad + \frac{\sigma_i^2 y^2 (1 - \mu)^2}{2\beta^2 [x + (1 - \mu)y]^2} - \frac{\eta_i y (1 - \mu)}{\beta^2 [x + (1 - \mu)y]^2} - \frac{r_i x}{\beta^2 [x + (1 - \mu)y]^2} \\
&\quad - \sum_{j \neq i} q_{ij} \left( V(x, y, j) - \frac{1}{\beta} \ln[x + (1 - \mu)y] - u_i(0) \right) + 1 + \ln \left( \frac{1}{\beta^2 [x + (1 - \mu)y]^2} \right) \\
&= (\beta - q_i) u_i(0) - \sum_{j \neq i} q_{ij} u_j(0) + 1 - \ln \beta - \frac{r_i}{\beta} + \frac{y (1 - \mu) (r_i - \eta_i)}{\beta [x + (1 - \mu)y]} \\
&\quad + \frac{\sigma_i^2 y^2 (1 - \mu)^2}{2\beta^2 [x + (1 - \mu)y]^2} + \sum_{j \neq i} q_{ij} \left( u_j(0) - V(x, y, j) + \frac{1}{\beta} \ln[x + (1 - \mu)y] \right) \\
&= y \left\{ \frac{\sigma_i^2 y (1 - \mu)^2}{2\beta^2 [x + (1 - \mu)y]^2} + \frac{(1 - \mu) (r_i - \eta_i)}{\beta [x + (1 - \mu)y]} \\
&\quad + \sum_{j \neq i} q_{ij} \frac{u_j(0) - V(x, y, j) + \frac{1}{\beta} \ln[x + (1 - \mu)y]}{y} \right\} 
\end{align*}
\tag{4.20}
\]
where the last equality uses (4.6). Using (4.2), we have
\[
\begin{align*}
\lim_{y \to 0^+} \frac{u_j(0) - V(x, y, j) + \frac{1}{\beta} \ln[x + (1 - \mu)y]}{y} &= \lim_{y \to 0^+} \frac{u_j(0) - u_j \left( \frac{y}{x+y} \right) - \frac{1}{\beta} \ln(x + y) + \frac{1}{\beta} \ln(x + (1 - \mu)y)}{y} \\
&= - \frac{D^+ u_j(0)}{x} - \frac{\mu}{\beta x} = \frac{1 - \mu}{\beta x} - V_y(x, 0+, j) \\
\end{align*}
\]
where (4.14) (with $i$ being replaced by $j$) is used for the last equality. It follows that
\[
\begin{align*}
\lim_{y \to 0^+} \left\{ \frac{\sigma_i^2 y (1 - \mu)^2}{2\beta^2 [x + (1 - \mu)y]^2} + \frac{(1 - \mu) (r_i - \eta_i)}{\beta [x + (1 - \mu)y]} \\
&\quad + \sum_{j \neq i} q_{ij} \frac{u_j(0) - V(x, y, j) + \frac{1}{\beta} \ln[x + (1 - \mu)y]}{y} \right\} \\
&= \frac{(1 - \mu) (r_i - \eta_i)}{\beta x} + \sum_{j \neq i} q_{ij} \left( \frac{1 - \mu}{\beta x} - V_y(x, 0+, j) \right) \\
&= \frac{(1 - \mu) (r_i - \eta_i - q_{ii})}{\beta x} - \sum_{j \neq i} q_{ij} V_y(x, 0+, j) \leq \frac{(1 - \mu) (r_i - \eta_i - q_{ii})}{\beta x} < 0
\end{align*}
\]
since \( V_y(x,0+,j) \geq 0 \) and \( r_i - \eta_i - q_{ii} < 0 \). Therefore for \( y \) sufficiently close to \( 0+ \), we have
\[
\mathcal{L}_iy(x,y,i) - Qy(x,y,\cdot)(i) - \bar{U}(V_x(x,y,i)) < 0,
\]
which contradicts (4.19).

**Remark 4.1.** Note that if \( \eta_i \leq r_i - q_{ii} \) for some regime \( i \in \mathcal{M} \), then it may be possible to have \( NT^i = \emptyset \) and \( H^i = \{(x,0) : x > 0\} \). In this case, the optimal investment policy is to clean up positions in risky asset immediately at \( t = 0 \) (sell if \( y(0-) > 0 \) and buy if \( y(0-) < 0 \)) and never hold the risky asset, as long as the regime stays at \( i \).

If \( NT^i \neq \emptyset \) for \( i \in \mathcal{M} \), then there exist two constants \( \theta^i_1, \theta^i_2 \) in \( \mathcal{I} \) with \( \theta^i_1 < \theta^i_2 \) such that
\[
NT^i = \left\{(x,y) \in \mathcal{I} : \frac{y}{x+y} < \theta^i_2 \right\}.
\]
We have the following property.

**Proposition 4.4.** If \( NT^i \neq \emptyset \) for \( i \in \mathcal{M} \), then the function \( u_i \) defined by (4.1) is \( C^2 \) on \( (\theta^i_1, \theta^i_2) \setminus \{0,1\} \). Moreover, \( u_i \) satisfies the following equation in the classical sense on this set:
\[
(\beta - q_{ii})u_i(z) - \frac{1}{\beta}d_{i1}(z) - d_{i2}(z)u'_i(z) - d_{i3}(z)u''_i(z) - \sum_{j \neq i} q_{ij}u_j(z) = 0, \quad z \in (\theta^i_1, \theta^i_2) \setminus \{0,1\}.
\]

**Proof.** From (4.17), we have
\[
\begin{align*}
- (1 - \mu)V_x(1 - z, z, i) + V_y(1 - z, z, i) & > 0, \\
(1 + \lambda)V_x(1 - z, z, i) - V_y(1 - z, z, i) & > 0,
\end{align*}
\]
for all \( z \in (\theta^i_1, \theta^i_2) \setminus \{0\} \).

Since \( V_x(1 - z, z, i) = \frac{1}{\beta} - z u'_i(z) \), \( V_y(1 - z, z, i) = \frac{1}{\beta} + (1 - z) u'_i(z) \), it follows that
\[
\begin{align*}
\frac{1}{\beta} + d_4(z)u'_i(z) & > 0, \\
\frac{1}{\beta} - d_5(z)u'_i(z) & > 0,
\end{align*}
\]
for all \( z \in (\theta^i_1, \theta^i_2) \setminus \{0\} \).

Next, we show that \( u_i \) is a viscosity solution of (4.22) on \( (\theta^i_1, \theta^i_2) \setminus \{0\} \). To this end, let \( \phi \in C^2((\theta^i_1, \theta^i_2) \setminus \{0\}) \) and \( z_0 \in (\theta^i_1, \theta^i_2) \setminus \{0\} \) be such that \( u_i - \phi \) has a maxima at \( z_0 \) and \( u_i(z_0) = \phi(z_0) \). Since it has been shown in Proposition 4.2 that \( u_i \) is \( C^1 \) on \( \mathcal{I} \setminus \{0\} \), we have \( u_i'(z_0) = \phi'(z_0) \). Using this equality and the fact (Proposition 4.1) that \( u \) is a viscosity solution of HJB system (4.3) in \( \mathcal{I} \), we have:
\[
\begin{align*}
\min \left\{(\beta - q_{ii})\phi(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)\phi'(z_0) - d_{i3}(z_0)\phi''(z_0) - \sum_{j \neq i} q_{ij}u_j(z_0) \right. \\
&\quad \left. - \bar{U}\left(\frac{1}{\beta} - z_0\phi'(z_0)\right), \quad \frac{1}{\beta} + d_4(z_0)u'_i(z_0), \quad \frac{1}{\beta} - d_5(z_0)u'_i(z_0)\right\} \leq 0, \quad z \in \mathcal{I}, \quad i \in \mathcal{M},
\end{align*}
\]
In view of (4.24), we have

\[
(\beta - q_{ii})\phi(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)\phi'(z_0) - d_{i3}(z_0)\phi''(z_0) - \sum_{j \neq i} q_{ij}u_j(z_0) - \bar{U}\left(\frac{1}{\beta} - z\phi'(z_0)\right) \leq 0.
\]

This shows that \(u_i\) is a viscosity subsolution of (4.22). The establishment of the supersolution property of \(u_i\) to (4.22) is straightforward and omitted.

We now define a continuous function \(\chi_i : (\theta_1^i, \theta_2^i) \setminus \{0\} \rightarrow \mathbb{R}\) by

\[
\chi_i(z) = (\beta - q_{ii})u_i(z) - \frac{1}{\beta}d_{i1}(z) - d_{i2}(z)u'_i(z) - \sum_{j \neq i} q_{ij}u_j(z) - \bar{U}\left(\frac{1}{\beta} - zu'_i(z)\right)
\]

and consider the following ordinary differential equation:

\[
-d_{i3}(z)w''_{ie}(z) + \chi_i(z) = \varepsilon,
\]

where \(\varepsilon\) is a real number (not necessarily positive). Let \([a, b] \subset (\theta_1^i, \theta_2^i) \setminus \{0, 1\}\) be a non-degenerate closed interval on which \(d_{i3} \neq 0\). For \(z \in [a, b]\), let

\[
w_{ie}(z) = u_i(a) + \frac{(u_i(b) - u_i(a) - g_{ie}(b))(z - a)}{b - a} + g_{ie}(z),
\]

where

\[
g_{ie}(z) = \int_a^z \frac{\chi_i(\nu) - \varepsilon}{d_{i3}(\nu)} \, d\nu d\zeta.
\]

Then it is easy to check that \(w_{ie}\) is a \(C^2\) solution of (4.26) with \(w_{ie}(a) = u_i(a)\) and \(w_{ie}(b) = u_i(b)\).

Suppose \(\varepsilon > 0\) (\(\varepsilon < 0\)) and \(u_i - w_{ie}\) achieves its maxima (minima) on \([a, b]\) at an interior point \(z_0 \in (a, b)\). Then we can use \(w_{ie}\) as a testing function for the subsolution (supersolution) property of \(u_i\) to (4.22). It then follows that

\[
(\beta - q_{ii})u_i(z_0) - \frac{1}{\beta}d_{i1}(z_0) - d_{i2}(z_0)w'_i(z_0) - d_{i3}(z_0)w''_i(z_0)
\]

\[
- \sum_{j \neq i} q_{ij}u_j(z_0) - \bar{U}\left(\frac{1}{\beta} - zw'_i(z_0)\right) \leq 0 (\geq 0).
\]

Note that \(w''_{ie}(z_0) = u''_i(z_0)\). We then have

\[-d_{i3}(z_0)w''_{ie}(z_0) + \chi_i(z_0) \leq 0 (\geq 0),\]

a contradiction to (4.26). Hence \(u_i - w_{ie}\) can achieve its maxima (minima) on \([a, b]\) only at the endpoints \(a\) or \(b\), which is 0. Then \(u_i \leq w_{ie}\) (\(u_i \geq w_{ie}\)) on \([a, b]\). Let \(\varepsilon \downarrow 0\) (\(\varepsilon \uparrow 0\)), we conclude that \(u_i \leq w_{i0}\) (\(u_i \geq w_{i0}\)), i.e., \(u_i = w_{i0}\). This shows that \(u_i\) is a \(C^2\) function on \((a, b)\).

\[\square\]

**Corollary 4.2.** If \(NT^i \neq \emptyset\) for \(i \in \mathcal{M}\), then in the set \(NT^i \setminus \{(x, y) : x = 0 \text{ or } y = 0\}\), the function \(V_i\) is \(C^2\) and satisfies the equation

\[
\mathcal{L}_iV(x, y, i) - QV(x, y, i)(i) - \bar{U}(V_z(x, y, i)) = 0
\]

in the classical sense.

**Remark 4.2.** Shreve and Soner [6, Corollary 8.8] show that the buy region always contains the cone \(\{(x, y) \in \Pi, y < 0\}\) for the model without regime-switching. For the regime-switching model considered in this paper, we are unable to reach the
5. Concluding remarks. In this paper we studied an infinite-horizon problem of optimal investment and consumption with proportional transaction costs in a continuous-time regime-switching model. We considered the case of log utility and extended the fundamental results of Shreve and Soner [6] to the regime-switching model.

We would like to note that the uniqueness property of the viscosity solution is important, in particular to the numerical solutions of the HJB equation. The approach for establishing the uniqueness of viscosity solution in Tao et al. [7] will be helpful for the HJB system considered in our work. While the present work and Liu [4] are focused on the qualitative analysis of the investment and consumption problem with proportional transaction costs and regime-switching, developing efficient numerical methods are very interesting projects for future research. The new numerical schemes will enable us to find the optimal control, i.e., the buy, sell, and no-transaction regions via numerical examples that can help us gain a deeper insight into the complicated behavior of the optimal dynamics of the investment and consumption problems across different market regimes. Another interesting topic is to study the investment and consumption in the regime-switching models with exponential utility function.

6. Appendix.

6.1. Proof of Proposition 3.6. First, note that \( A(x_1, y, i) \subset A(x_2, y, i) \) if \( x_1 \leq x_2 \).

It then follows that \( V(x_1, y, i) \leq V(x_2, y, i) \), i.e., \( V \) is increasing in \( x \) for each \( i \in M \).

Similarly we can show that \( V \) is increasing in \( y \) for each \( i \in M \).

Now suppose there exist an \( i_0 \) and a \( y_0 \) such that \( V(x, y_0, i_0) \) is not strictly increasing in \( x \). Consequently, assume that there exist \( x_1 < x_2 \) such that \( V(x_1, y_0, i_0) = V(x_2, y_0, i_0) \). It follows that \( V(x, y_0, i_0) = V(x_1, y_0, i_0) \) for all \( x \in [x_1, x_2] \) since \( V(x, y_0, i_0) \) is increasing in \( x \). Note that \( V(x, y_0, i_0) \) is concave in \( x \), hence the interval \([x_1, x_2]\) can not be finite. Therefore there exists \( x_0 > 0 \) such that

\[
V(x, y_0, i_0) = V(x_0, y_0, i_0), \quad \forall x \in [x_0, \infty). \tag{6.1}
\]

Consider the triple \((x_0, y_0, i_0)\) and let \((C^\varepsilon, M^\varepsilon, N^\varepsilon) \in A(x_0, y_0, i_0)\) be an \( \varepsilon \)-optimal control for given \( \varepsilon > 0 \). Then we have

\[
V(x_0, y_0, i_0) \leq E \left[ \int_0^\infty e^{-\beta t} U(C^\varepsilon(t)) dt \right] + \varepsilon, \tag{6.2}
\]

where \( U(x) = \ln x \). Depending on the sign of \( y_0 \), we continue the analysis in two cases.

Case 1. \( y_0 \geq 0 \).

Pick \( x_1 \) such that

\[
x_1 > \max \left( x_0, U^{-1} \left[ \beta \left( E \left[ \int_0^\infty e^{-\beta t} U(C^\varepsilon(t)) dt \right] + \varepsilon \right) \right] / r_* \right), \tag{6.3}
\]

where \( U^{-1}(x) = e^x \) for the log utility function. We claim that the policy \((r_*x_1, 0, 0) \in A(x_1, y_0, i_0)\). To prove that, applying the policy \((r_*x_1, 0, 0)\) to the system equation
(2.3), we have:
\[
\begin{align*}
\begin{cases}
    dx(t) = [r(\alpha_t) x(t) - r_* x_1] dt, \\
    dy(t) = [\eta(\alpha_t) dt + \sigma(\alpha_t) dW(t)], \\
    x(0) = x_1, y(0) = y_0, \alpha_0 = \iota_0.
\end{cases}
\end{align*}
\]
Recall that \( r_* = \min_{i \in \mathcal{M}} r_i \). It’s readily seen that \( x(t) \geq x_1 \) and \( y(t) \geq 0 \) for all \( t \geq 0 \). Hence \( (x(t), y(t)) \in \Pi \) for all \( t \geq 0 \).

We obtain from (6.2) and (6.3) that
\[
V(x_0, y_0, i_0) \leq E \left[ \int_0^{\infty} e^{-\beta t} U(C^*(t)) dt \right] + \varepsilon
\]
\[
< \frac{1}{\beta} U(r_* x_1) = \int_0^{\infty} e^{-\beta t} U(r_* x_1) dt \leq V(x_1, y_0, i_0),
\]
which is a contradiction to (6.1).

**Case 2.** \( y_0 < 0 \).

Pick \( x_1 \) such that
\[
x_1 + (1 + \lambda) y_0 > \max \left( x_0, U^{-1} \left[ \beta \left( E \left[ \int_0^{\infty} e^{-\beta t} U(C^*(t)) dt \right] + \varepsilon \right) \right] / r_* \right).
\]
In this case, the policy \( (r_*(x_1 + (1 + \lambda) y_0), -y_0, 0) \in \mathcal{A}(x_1, y_0, i_0) \). Indeed, applying this policy, we have:
\[
\begin{align*}
\begin{cases}
    dx(t) = [r(\alpha_t) x(t) - r_* (x_1 + (1 + \lambda) y_0)] dt, \\
    dy(t) = y(t)[\eta(\alpha_t) dt + \sigma(\alpha_t) dW(t)], \\
    x(0) = x_1 + (1 + \lambda) y_0, y(0) = 0, \alpha_0 = \iota_0.
\end{cases}
\end{align*}
\]
Then \( x(t) \geq x_1 + (1 + \lambda) y_0 > 0 \) and \( y(t) = 0 \) for all \( t \geq 0 \). Hence \( (x(t), y(t)) \in \Pi \) for all \( t \geq 0 \).

It follows from (6.2) and (6.4) that
\[
V(x_0, y_0, i_0) \leq E \left[ \int_0^{\infty} e^{-\beta t} U(C^*(t)) dt \right] + \varepsilon
\]
\[
< \frac{1}{\beta} U(r_*(x_1 + (1 + \lambda) y_0)) = \int_0^{\infty} e^{-\beta t} U(r_*(x_1 + (1 + \lambda) y_0)) dt
\]
\[
\leq V(x_1, y_0, i_0),
\]
which again is a contradiction to (6.1).

**6.2. Proof of Theorem 3.3.** The following formula will be used in the proof of Theorem 3.3 and throughout the paper. Let \( v : \Pi \times \mathcal{M} \to \mathbb{R} \) be such that \( v_i \in C^2(\Pi) \) for each \( i \in \mathcal{M} \). Let \( (x_0, y_0) \in \Pi, i \in \mathcal{M} \). Let \( (C, M, N) \in \mathcal{A}(x_0, y_0, i) \) and \( (x(t), y(t)), t \geq 0 \) be the corresponding solution of (2.3). Let \( \tau \) be an almost surely finite stopping time. By applying the generalized Itô’s formula for RCLL semi-martingales to the Markovian-modulated process \( e^{-\beta t} v(x(t), y(t), \alpha_t) \) (see for instance, [1] and [5]), we have:
\[
e^{-\beta \tau} v(x(\tau), y(\tau), \alpha_\tau) = v(x_0, y_0, i)
\]
\[
- \int_0^\tau e^{-\beta t} [L_{\alpha_t} v(x(t), y(t), \alpha_t) + C(t) v_x(x(t), y(t), \alpha_t)] dt
\]
where

\[ C \]

argument by contradiction. Assume that there exist an

\[ \varepsilon > 0 \]

\[ \exists x, y \in \mathbb{R} \]

\[ v(x(t), y(t), \alpha_i) \]

\[ v(x(t), y(t), \alpha_i) \]

\[ \sum_{0 \leq t \leq T} e^{-\beta t} [v(x(t), y(t), \alpha_i) - v(x(t), y(t), \alpha_i^{-})] \]

\[ + \int_0^T e^{-\beta t} Qv(x(t), y(t), \cdot)(\alpha_i)dt \]

\[ + \int_0^T e^{-\beta t} Q\sigma(\alpha_i)y(v(x(t), y(t), \alpha_i)dW(t) + \int_0^T dM_v(t) \]

\[ = v(x_0, y_0, i) - \int_0^T e^{-\beta t} [\mathcal{L}_\alpha v(x(t), y(t), \alpha_i) + C(t)v_x(x(t), y(t), \alpha_i)] dt \]

\[ + \int_0^T e^{-\beta t} [-(1 + \lambda)v_x(x(t), y(t), \alpha_i) + v_y(x(t), y(t), \alpha_i)] dM(t) \]

\[ + \int_0^T e^{-\beta t} [(1 - \mu)v_x(x(t), y(t), \alpha_i) - v_y(x(t), y(t), \alpha_i)] dN(t) \]

\[ + \int_0^T e^{-\beta t} Qv(x(t), y(t), \cdot)(\alpha_i)dt \]

\[ + \int_0^T e^{-\beta t} \sigma(\alpha_i)y(v(x(t), y(t), \alpha_i)dW(t) + \int_0^T dM_v(t), \]

where

\[ M^c(t) = M(t) - \sum_{0 \leq s \leq t} (M(s) - M(s-)), \]

\[ N^c(t) = N(t) - \sum_{0 \leq s \leq t} (N(s) - N(s-)) \]

\[ (1 + \lambda)\varphi_x(x_0, y_0) - \varphi_y(x_0, y_0) > 0, \quad (6.7) \]

\[ -(1 - \mu)\varphi_x(x_0, y_0) + \varphi_y(x_0, y_0) > 0, \quad (6.8) \]

\[ \mathcal{L}_{i_0} \varphi(x_0, y_0) - QV(x_0, y_0, \cdot)(i_0) - \tilde{U}(\varphi_x(x_0, y_0)) > \varepsilon \quad (6.9) \]

for some number \( \varepsilon > 0 \), where \( \tilde{U} \) is given by (2.13) for the log utility.

Since \( \varphi \) is smooth, \( V \) and \( \tilde{U} \) are continuous, there exists an \( \delta \)-neighborhood of \( (x_0, y_0) \): \( B_{\delta}(x_0, y_0) = \{(x, y) : |x, y| - (x_0, y_0)| < \delta \} \subset \Pi \) for some \( \delta > 0 \) such that

\[ (1 + \lambda)\varphi_x(x, y) - \varphi_y(x, y) > 0, \quad (6.10) \]

\[ -(1 - \mu)\varphi_x(x, y) + \varphi_y(x, y) > 0, \quad (6.11) \]

and

\[ \mathcal{L}_{i_0} \varphi(x, y) - QV(x, y, \cdot)(i_0) - \tilde{U}(\varphi_x(x, y)) > \varepsilon \quad (6.12) \]
whenever \((x, y) \in B_\delta(x_0, y_0)\).

Consider an optimal control \((C^*, M^*, N^*)\) for the initial state \((x_0, y_0, i_0)\) and let \((x^*(t), y^*(t)), t \geq 0\) denote the corresponding solution of the state equation (2.3) with \((x^*(0), y^*(0)) = (x_0, y_0)\) and \(\alpha_0 = i_0\). We now show that almost surely \((x^*(t), y^*(t))\) does not have jumps at \(t = 0\), i.e., \(M^*(0) = N^*(0) = 0\), a.s. Indeed, since
\[
\lim_{h \to 0} \frac{\varphi(x_0 - (1 + \lambda)h, y_0 + h) - \varphi(x_0, y_0)}{h} = -(1 + \lambda)\varphi_x(x_0, y_0) + \varphi_y(x_0, y_0) < 0,
\]
thus, for sufficiently small \(h > 0\), we have
\[
\varphi(x_0 - (1 + \lambda)h, y_0 + h) - \varphi(x_0, y_0) < 0. \quad (6.13)
\]
Suppose that \((x^*(t), y^*(t))\) had a jump of size \(h\) at \(t = 0\) in the direction \((-1+\lambda, 1)\). We can assume that \(h\) is sufficiently small for which (6.13) holds (otherwise, we can complete the purchase of stock at time \(t = 0\) in multiple transactions; each with an amount small enough). Also note that \(\alpha_t\) has no jumps almost surely at \(t = 0\).

From the dynamic principle equation (3.10), we have
\[
V(x_0, y_0, i_0) = V(x_0 - (1 + \lambda)h, y_0 + h, i_0), \quad (6.14)
\]
from which we deduce
\[
\varphi(x_0, y_0) \leq \varphi(x_0 - (1 + \lambda)h, y_0 + h), \quad (6.15)
\]
a contradiction to (6.13). Hence \(M^*(0) = 0\). Similarly we can show that almost surely \((x^*(t), y^*(t))\) has no jumps at \(t = 0\) in the direction \((1-\mu, -1)\). So \(N^*(0) = 0\).

Let \(\tau_1 = \inf\{t \geq 0 : (x^*(t), y^*(t)) \notin B_\delta(x_0, y_0)\}\) be the first exit time of the optimal state trajectory from the neighborhood \(B_\delta(x_0, y_0)\). Then by what we have just proved, \(\tau_1 > 0\) a.s. Let \(\tau_\alpha\) be the first jump time of \(\alpha_t\). Let \(\tau = \tau_\alpha \land \tau_1\). Then \(\tau > 0\) a.s.

Define
\[
\Phi(x, y, i) = \begin{cases} \varphi(x, y), & \text{if } i = i_0, \\ V(x, y, i), & \text{if } i \neq i_0. \end{cases} \quad (6.16)
\]
Consider \(e^{-\beta\tau}\Phi(x^*(\tau), y^*(\tau), \alpha_\tau), 0 \leq t < \tau\). Using (6.5) and noting that \(\alpha_t = i_0\) for \(0 \leq t < \tau\), we have:
\[
e^{-\beta\tau}\Phi(x^*(\tau), y^*(\tau), \alpha_\tau) = \varphi(x_0, y_0)
\]
\[
- \int_0^\tau e^{-\beta t} \mathcal{L}\varphi(x^*(t), y^*(t)) + C^*(t)\varphi_x(x^*(t), y^*(t)) dt
\]
\[
+ \int_0^\tau e^{-\beta t} [-(1 + \lambda)\varphi_x(x^*(t), y^*(t)) + \varphi_y(x^*(t), y^*(t)) + \varphi_x(x^*(t), y^*(t))] dM^*(t)
\]
\[
+ \int_0^\tau e^{-\beta t} [(1 - \mu)\varphi_x(x^*(t), y^*(t)) - \varphi_y(x^*(t), y^*(t))] dN^*(t)
\]
\[
+ \int_0^\tau e^{-\beta t} Q\Phi(x^*(t), y^*(t), \cdot)(i_0) dt
\]
\[
+ \int_0^\tau e^{-\beta t} \sigma(\alpha_t)\varphi_y(x^*(t), y^*(t)) dW(t) + \int_0^\tau dM_\varphi(t). \quad (6.17)
\]
Note that both \( \varphi(x^*(t), y^*(t)) \) and \( y^*(t)\varphi_y(x^*(t), y^*(t)) \) are bounded on \([0, \tau)\). It follows that
\[
E \left[ \int_0^\tau e^{-\beta t} \sigma(\alpha_t) y^*(t) \varphi_y(x^*(t), y^*(t)) dW(t) + \int_0^\tau dM_\varphi(t) \right] = 0. \tag{6.18}
\]
Since
\[
\Phi(x^*(\tau), y^*(\tau), \alpha_\tau) = \begin{cases} 
\varphi(x^*(\tau), y^*(\tau)) \geq V(x^*(\tau), y^*(\tau), i_0) \text{ if } \tau = \tau_1 < \tau_\alpha, \\
V(x^*(\tau), y^*(\tau), \alpha_\tau) \text{ if } \tau = \tau_\alpha,
\end{cases}
\]
for both cases, we have
\[
\Phi(x^*(\tau), y^*(\tau), \alpha_\tau) \geq V(x^*(\tau), y^*(\tau), \alpha_\tau). \tag{6.19}
\]
Also note that
\[
Q\Phi(x^*(t), y^*(t), \cdot)_{(i_0)} = \sum_{i \neq i_0} q_{i,i_0} [\Phi(x^*(t), y^*(t), i) - \Phi(x^*(t), y^*(t), i_0)]
\]
\[
= \sum_{i \neq i_0} q_{i,i_0} [V(x^*(t), y^*(t), i) - \varphi(x^*(t), y^*(t))]
\]
\[
\leq \sum_{i \neq i_0} q_{i,i_0} [V(x^*(t), y^*(t), i) - V(x^*(t), y^*(t), i_0)]
\]
\[
= QV(x^*(t), y^*(t), \cdot)_{(i_0)}.
\tag{6.20}
\]
In view of (2.11), we have
\[
\tilde{U}(\varphi_x(x^*(t), y^*(t))) \geq U(C^*(t)) - C^*(t)\varphi_x(x^*(t), y^*(t)),
\]
i.e.,
\[
C^*(t)\varphi_x(x^*(t), y^*(t)) \geq U(C^*(t)) - \tilde{U}(\varphi_x(x^*(t), y^*(t))). \tag{6.21}
\]
Taking expectation in (6.17) and Using (6.10), (6.11), (6.19), (6.20), and (6.21) yield
\[
E \left[ e^{-\beta \tau} V(x^*(\tau), y^*(\tau), \alpha_\tau) \right] \leq \varphi(x_0, y_0) - E \left[ \int_0^\tau e^{-\beta t} \mathcal{L}_{i_0} \varphi(x^*(t), y^*(t)) \right]
\]
\[
- \tilde{U}(\varphi(x^*(t), y^*(t))) - QV(x^*(t), y^*(t), \cdot)_{(i_0)} dt - \int_0^\tau e^{-\beta t} U(C^*(t)) dt.
\]
In view of (6.12) and since \( \varphi(x_0, y_0) = V(x_0, y_0, i_0) \), we obtain
\[
V(x_0, y_0, i_0) \geq E \left[ \int_0^\tau e^{-\beta t} U(C^*(t)) dt + e^{-\beta t} V(x^*(\tau), y^*(\tau), \alpha_\tau) \right] + \varepsilon E \left[ \int_0^\tau e^{-\beta t} \right]
\]
\[
= E \left[ \int_0^\tau e^{-\beta t} U(C^*(t)) dt + e^{-\beta \tau} V(x^*(\tau), y^*(\tau), \alpha_\tau) \right] + \varepsilon \beta (1 - E \left[ e^{-\beta \tau} \right]). \tag{6.22}
\]
Since \( \tau > 0 \) a.s., \( E \left[ e^{-\beta \tau} \right] < 1 \). It follows that
\[
V(x_0, y_0, i_0) > E \left[ \int_0^\tau e^{-\beta t} U(C^*(t)) dt + e^{-\beta \tau} V(x^*(\tau), y^*(\tau), \alpha_\tau) \right], \tag{6.23}
\]
a contradiction to the dynamic programming equation (3.10). Therefore \( V \) is a viscosity subsolution of (2.10) in \( \Pi \).
Step 2. \( V \) is a viscosity supersolution of (2.10) in \( \Pi \). We need to show that for each \( i \in \mathcal{M} \), each function \( \psi \in C^2(\Pi) \) and each \((x_0, y_0) \in \Pi \) such that \( V_i(x_0, y_0) = \psi(x_0, y_0) \), \( V_i(x, y) \geq \psi(x, y) \), \( \forall (x, y) \in \Pi \), we have

\[
\min \left\{ \mathcal{L}_i \psi(x_0, y_0) - QV(x_0, y_0 \cdot)(i) - \tilde{U}(\psi_x(x_0, y_0)), \right.
\]
\[
(1 + \lambda)\psi_x(x_0, y_0) - \psi_y(x_0, y_0), \\
\left. -(1 - \mu)\psi_x(x_0, y_0) + \psi_y(x_0, y_0) \right\} \geq 0. \quad (6.24)
\]

To this end, we show that each term in (6.24) is nonnegative.

Pick \( M_0 > 0 \) such that \((x_0 - (1 + \lambda)M_0, y_0 + M_0) \in \Pi \). Consider the investment policy \( M(t) = M_0 \) and \( N(t) = 0 \) for all \( t \geq 0 \), i.e., purchasing an amount \( M_0 \) of the stock at time \( t = 0 \) and making no trading afterwards. By Proposition 3.7, we have

\[
V(x_0, y_0, i) \geq V(x_0 - (1 + \lambda)M_0, y_0 + M_0, i).
\]

It follows that

\[
\psi(x_0 - (1 + \lambda)M_0, y_0 + M_0) - \psi(x_0, y_0) \leq 0.
\]

Dividing by \( M_0 \) and sending \( M_0 \to 0 \), we obtain

\[
-(1 + \lambda)\psi_x(x_0, y_0) + \psi_y(x_0, y_0) \leq 0. \quad (6.25)
\]

Similarly, by using the policy \( M(t) = 0 \) and \( N(t) = N_0 \) for all \( t \geq 0 \), where \( N_0 > 0 \) is chosen in such a way that \((x_0 + (1 - \mu)N_0, y_0 - N_0) \in \Pi \), we can show that

\[
-(1 - \mu)\psi_x(x_0, y_0) - \psi_y(x_0, y_0) \leq 0. \quad (6.26)
\]

Finally we consider the control \((C, M, N) \in \mathcal{A}(x_0, y_0, i)\) given by \( M(t) = N(t) = 0 \) for all \( t \geq 0 \), \( C(t) = c > 0 \) for \( 0 \leq t \leq \tau \), where the stopping time \( \tau \) is defined next. Since \((x_0, y_0) \in \Pi \), there exists a number \( \delta > 0 \) such that \( B_\delta(x_0, y_0) = \{(x, y) : |(x, y) - (x_0, y_0)| < \delta \} \subset \Pi \). Let \( \tau_\delta = \inf \{t \geq 0 : (x(t), y(t)) \notin B_\delta(x_0, y_0)\} \) be the first exit time of the state trajectory from the neighborhood \( B_\delta(x_0, y_0) \), where \((x(t), y(t))\) is the solution of (2.3) when the policy \((C, M, N) = (c, 0, 0)\) is being used. Let \( \tau = \tau_\delta \wedge \tau_\alpha \) where \( \tau_\alpha \) is the first jump time of \( \alpha_i \). Then \( \tau > 0 \) a.s. Using the dynamic programming equation (3.9), we have

\[
\psi(x_0, y_0) = V(x_0, y_0, i) \geq E \left[ \int_0^\tau e^{-\beta t} U(c) dt + e^{-\beta \tau} V(x(\tau), y(\tau), \alpha_{\tau}) \right]. \quad (6.27)
\]

Define

\[
\Psi(x, y, j) = \left\{ \begin{array}{ll}
\psi(x, y), & \text{if } j = i, \\
V(x, y, j), & \text{if } j \neq i.
\end{array} \right. \quad (6.28)
\]

Using (6.5) we obtain

\[
E \left[ e^{-\beta \tau} \Psi(x(\tau), y(\tau), \alpha_{\tau}) \right] = \psi(x_0, y_0)
\]
\[
- E \left[ \int_0^\tau e^{-\beta t} [\mathcal{L}_i \psi(x(t), y(t)) + c \psi_x(x(t), y(t))] dt \right]
\]
\[
+ E \left[ \int_0^\tau e^{-\beta t} Q \Psi(x(t), y(t), \cdot)(i) dt \right]. \quad (6.29)
\]

Since

\[
\Psi(x(\tau), y(\tau), \alpha_{\tau}) = \left\{ \begin{array}{ll}
\psi(x(\tau), y(\tau)), & \text{if } \tau = \tau_\delta < \tau_\alpha, \text{ and} \\
V(x(\tau), y(\tau), \alpha_{\tau}), & \text{if } \tau = \tau_\alpha,
\end{array} \right.
\]

...
we have
\[
\Psi(x(\tau), y(\tau), \alpha_\tau) \leq V(x(\tau), y(\tau), \alpha_\tau).
\] (6.30)
Note that
\[
Q \Psi(x(t), y(t), \cdot)(i) = \sum_{j \neq i} q_{ij} \left[ \Psi(x(t), y(t), j) - \Psi(x(t), y(t), i) \right]
\]
\[
= \sum_{j \neq i} q_{ij} \left[ V(x(t), y(t), j) - \psi(x(t), y(t)) \right]
\]
\[
\geq \sum_{j \neq i} q_{ij} \left[ V(x(t), y(t), j) - V(x(t), y(t), i) \right]
\]
\[
= QV(x(t), y(t), \cdot)(i).
\] (6.31)
Using (6.30) and (6.31) in (6.29), we have
\[
\begin{align*}
E \left[ e^{-\beta t} V(x(\tau), y(\tau), \alpha_\tau) \right] & \geq \psi(x_0, y_0) \\
- E \left[ \int_0^\tau e^{-\beta t} \left[ \mathcal{L}_t \psi(x(t), y(t)) + cv_x(x(t), y(t)) \right] dt \right] \\
+ E \left[ \int_0^\tau e^{-\beta t} QV(x(t), y(t), \cdot)(i) dt \right].
\end{align*}
\] (6.32)
Combining (6.32) with (6.27) yields
\[
E \left[ \int_0^\tau e^{-\beta t} \left[ \mathcal{L}_t \psi(x(t), y(t)) + cv_x(x(t), y(t)) - U(c) - QV(x(t), y(t), \cdot)(i) \right] dt \right] \geq 0.
\] (6.33)
Note that \( \tau \to 0 \) a.s. as \( \delta \to 0^+ \). Dividing both sides of (6.33) by \( E[\tau] \), sending \( \delta \to 0^+ \), we get
\[
\mathcal{L}_t \psi(x_0, y_0) + cv_x(x_0, y_0) - U(c) - QV(x_0, y_0, \cdot)(i) \geq 0,
\]
or equivalently,
\[
\mathcal{L}_t \psi(x_0, y_0) - QV(x_0, y_0, \cdot)(i) \geq U(c) - cv_x(x_0, y_0),
\]
Taking maximum of the right hand side over \( c > 0 \), then moving to left, we obtain
\[
\mathcal{L}_t \psi(x_0, y_0) - QV(x_0, y_0, \cdot)(i) - \bar{U}(\psi_x(x_0, y_0)) \geq 0,
\]
which is the required inequality.
This completes the proof of Theorem 3.3.

6.3. Partition of the solvency region. Given \( i \in \mathcal{M} \) and \((x, y) \in \Pi\), the sub-differential of \( V \) is defined by
\[
\partial V(x, y, i) = \{ (\delta^x_x, \delta^y_y) \in \mathbb{R}^2 : V(\bar{x}, \bar{y}, i) \leq V(x, y, i) + \delta^x_x(\bar{x} - x) + \delta^y_y(\bar{y} - y), \ \forall(\bar{x}, \bar{y}) \in \Pi \}.
\] (6.34)
Note that the value function \( V_i(x, y) \) is concave and finite in \( \Pi \). It follows that \( \partial V(x, y, i) \) is a nonempty, compact and convex subset of \( \mathbb{R}^2 \). Also \( V_0(x, y) \) is differentiable at a point \((x, y) \in \Pi\) if and only if \( \partial V(x, y, i) \) is a singleton. In this case we have \( \partial V(x, y, i) = \{ (V_x(x, y, i), V_y(x, y, i)) \} \).
Applying [6, Lemma 6.1 and Proposition 6.2] to each function \( V_i \) for \( i \in \mathcal{M} \), we have:
Lemma 6.1. Given $i \in \mathcal{M}$. Let $\{(x_n, y_n), n \geq 1\}$ be a sequence of points in $\Pi$ with limit $(x_0, y_0) \in \Pi$. If $(\delta_x^n, \delta_y^n) \in \partial V(x_n, y_n, i)$ for every $n \geq 1$, then the sequence $\{(\delta_x^n, \delta_y^n), n \geq 1\}$ is bounded and every limit point of the sequence is in $\partial V(x_0, y_0, i)$.

Proposition 6.1. Let $O$ be an open subset of $\Pi$. Then $V_i$ is of $C^1$ in $O$ if and only if $\partial V(x, y, i)$ is a singleton for every $(x, y) \in O$.

Next, for given $i \in \mathcal{M}$, $(x, y) \in \Pi$, and $(\delta_x^i, \delta_y^i) \in \partial V(x, y, i)$, define

$$
\phi(\tilde{x}, \tilde{y}, i) = V(x, y, i) + \delta_x^i(\tilde{x} - x) + \delta_y^i(\tilde{y} - y), \forall (\tilde{x}, \tilde{y}) \in \Pi.
$$

In view of (6.34), we have $\phi(\tilde{x}, \tilde{y}, i) \geq V(\tilde{x}, \tilde{y}, i), \forall (\tilde{x}, \tilde{y}) \in \Pi$. For each $\mu > 0$ satisfying $(x - (1 - \mu)h, y + h) \in \Pi$ from which $(x, y)$ can be reached by a transaction, Proposition 3.7 implies that

$$
\phi(x, y, i) = V(x, y, i) \leq V(x - (1 - \mu)h, y + h, i)
$$

and

$$
\leq \phi(x - (1 - \mu)h, y + h, i) + \phi(x, y, i) + h[1 - (1 - \mu)\delta_x^i + \delta_y^i] = \phi(x, y, i) + h[1 - \delta_x^i + \delta_y^i].
$$

It then follows that

$$
-(1 - \mu)\delta_x^i + \delta_y^i \geq 0, \forall (\delta_x^i, \delta_y^i) \in \partial V(x, y, i), \forall (x, y) \in \Pi.
$$

Similarly, we can show

$$
(1 + \mu)\delta_x^i - \delta_y^i \geq 0, \forall (\delta_x^i, \delta_y^i) \in \partial V(x, y, i), \forall (x, y) \in \Pi.
$$

Since for $(x, y) \in \Pi$, $x + (1 - \mu)y > 0$ and $x + (1 + \mu)y > 0$, it follows from Corollary 3.3 that $\delta_x^i = \phi_x(x, y, i) > 0$, and then from (6.35) that $\delta_y^i > 0$. To summarize, we have

$$
\delta_x^i > 0, \delta_y^i > 0, \forall (\delta_x^i, \delta_y^i) \in \partial V(x, y, i), \forall (x, y) \in \Pi.
$$

For $\theta$ sufficiently close to 1, using the homotheticity property (3.1), we have

$$
\frac{1}{\beta} \ln \theta = V(\theta x, \theta y, i) - V(x, y, i)
$$

and

$$
\leq \phi(\theta x, \theta y, i) - V(x, y, i) = (\theta - 1)(x\delta_x^i + y\delta_y^i).
$$

Dividing both sides of (6.38) by $\theta - 1$, setting $\theta \to 1+$ and $\theta \to 1-$, respectively, we obtain

$$
x\delta_x^i + y\delta_y^i = \frac{1}{\beta} \ln \theta, \forall (\delta_x^i, \delta_y^i) \in \partial V(x, y, i), \forall (x, y) \in \Pi.
$$

Now, for $(\tilde{x}, \tilde{y}) \in \Pi$, $\theta > 0$, we have

$$
V(\tilde{x}, \tilde{y}, i) = V\left(\theta \frac{\tilde{x}}{\theta}, \theta \frac{\tilde{y}}{\theta}, i\right) = \frac{1}{\beta} \ln \theta + V\left(\frac{\tilde{x}}{\theta}, \frac{\tilde{y}}{\theta}, i\right) \leq \frac{1}{\beta} \ln \theta + \phi\left(\frac{\tilde{x}}{\theta}, \frac{\tilde{y}}{\theta}, i\right)
$$

and

$$
= \frac{1}{\beta} \ln \theta + V(x, y, i) + \delta_x^i\left(\frac{\tilde{x}}{\theta} - x\right) + \delta_y^i\left(\frac{\tilde{y}}{\theta} - y\right)
$$

and

$$
= V(\theta x, \theta y, i) + \frac{\delta_x^i}{\theta}(\tilde{x} - x \theta) + \frac{\delta_y^i}{\theta}(\tilde{y} - y \theta).
$$

It follows that $(\theta^{-1}\delta_x^i, \theta^{-1}\delta_y^i) \in \partial V(\theta x, \theta y, i)$. Thus $\theta^{-1}\partial V(x, y, i) \subset \partial V(\theta x, \theta y, i)$. The reverse set containment can be established by simply replacing $\theta$ by $\frac{1}{\theta}$, $x$ by $\theta x$ and $y$ by $\theta y$. Therefore we have

$$
\theta^{-1}\partial V(x, y, i) = \partial V(\theta x, \theta y, i), \forall (x, y) \in \Pi, \forall \theta > 0, \forall i \in \mathcal{M}.
$$
Using the sub-differentials, the solvency region $\Pi$ can be partitioned into three convex cones. First, for $i \in \mathcal{M}$, $(x, y) \in \Pi$, $(\bar{x}, \bar{y}) \in \Pi$, $(\delta^i_x, \delta^i_y) \in \partial V(x, y, i)$ and $(\delta^j_x, \delta^j_y) \in \partial V(\bar{x}, \bar{y}, i)$. In view of (6.34), we have

$$V(\bar{x}, \bar{y}, i) \leq V(x, y, i) + \delta^i_x(\bar{x} - x) + \delta^i_y(\bar{y} - y)$$

$$\leq V(\bar{x}, \bar{y}, i) + \delta^j_x(x - \bar{x}) + \delta^j_y(y - \bar{y}) + \delta^i_x(\bar{x} - x) + \delta^i_y(\bar{y} - y).$$

It follows that

$$(\delta^i_x - \delta^j_x)(x - \bar{x}) + (\delta^i_y - \delta^j_y)(y - \bar{y}) \leq 0. \quad (6.41)$$

For $i \in \mathcal{M}$, $(x, y) \in \Pi$, define

$$\vartheta^+(x, y, i) = \max\{-1(1 - \mu)\delta^i_x + \delta^j_y : (\delta^i_x, \delta^j_y) \in \partial V(x, y, i)\},$$

$$\vartheta^-(x, y, i) = \min\{-1(1 - \mu)\delta^i_x + \delta^j_y : (\delta^i_x, \delta^j_y) \in \partial V(x, y, i)\}. \quad (6.42)$$

Note that the maxima and minima in (6.42) are attained since $\partial V(x, y, i)$ is compact. In addition, $\vartheta^+(x, y, i) \geq \vartheta^-(x, y, i) \geq 0$ due to (6.35).

Consider the half line $L_1$ originating at the point $(1 + \lambda, -1)$ on $\partial_2 \Pi$ and parallel to $\partial_1 \Pi$, given by the parametric equations:

$$L_1 : \begin{cases} x(\rho) = 1 + \lambda - (1 - \mu)\rho, \\ y(\rho) = -1 + \rho, \end{cases} \quad \forall \rho \geq 0. \quad (6.43)$$

Let

$$\rho^*_i := \inf\{\rho > 0 : \vartheta^-(x(\rho), y(\rho), i) = 0\}, \quad (6.44)$$

where $\rho^*_0 = \infty$ if the above set is empty.

**Lemma 6.2.** Given $i \in \mathcal{M}$. For $0 < \rho < \bar{\rho} < \infty$, we have

$$\vartheta^+(x(\bar{\rho}), y(\bar{\rho}), i) \leq \vartheta^-(x(\rho), y(\rho), i). \quad (6.45)$$

If $0 < \rho^*_0 < \infty$, then $\vartheta^-(x(\rho^*_0), y(\rho^*_0), i) = 0$ and

$$\vartheta^+(x(\rho), y(\rho), i) = 0, \quad \forall \rho > \rho^*_0. \quad (6.46)$$

Let $i \in \mathcal{M}$ be fixed. We partition $\Pi$ into two open and convex cones (possibly empty) as following:

$$SA^i = \{(x, y) \in \Pi : (\theta x, \theta y) = (x(\rho), y(\rho)) \text{ for some } \theta > 0 \text{ and some } \rho \text{ with } \rho^*_0 < \rho < \infty\},$$

$$\Pi \setminus SA^i = \{(x, y) \in \Pi : (\theta x, \theta y) = (x(\rho), y(\rho)) \text{ for some } \theta > 0 \text{ and some } \rho \text{ with } 0 < \rho < \rho^*_0\}. \quad (6.47)$$

**Proposition 6.2.** Given $i \in \mathcal{M}$. We have

$$-(1 - \mu)\delta^i_x + \delta^j_y = 0, \quad \forall (\delta^i_x, \delta^j_y) \in \partial V(x, y, i), \quad \forall (x, y) \in SA^i. \quad (6.47)$$

In a similar way, we consider the half line $L_2$ originating at the point $(-(1 - \mu), 1)$ on $\partial_1 \Pi$ and parallel to $\partial_2 \Pi$, given by the parametric equations:

$$L_2 : \begin{cases} \bar{x}(\rho) = -(1 - \mu) + (1 + \lambda)\rho, \\ \bar{y}(\rho) = 1 - \rho, \end{cases} \quad \forall \rho \geq 0. \quad (6.48)$$

Let

$$\bar{\rho}^*_0 := \inf\{\rho > 0 : (1 + \lambda)\delta^i_x - \delta^j_y = 0 \text{ for some } (\delta^i_x, \delta^j_y) \in \partial V(\bar{x}(\rho), \bar{y}(\rho), i)\}. \quad (6.49)$$
Let \( i \in \mathcal{M} \) be fixed. We partition \( \Pi \) into two open and convex cones (possibly empty) as following:
\[
BU^i = \{(x, y) \in \Pi : (\theta x, \theta y) = (\tilde{x}(\rho), \tilde{y}(\rho)) \text{ for some } \theta > 0 \text{ and some } \rho \text{ with } \\
\tilde{\rho}_0 < \rho < \tilde{\rho}_0 \},
\]
\[
\Pi \setminus BU^i = \{(x, y) \in \Pi : (\theta x, \theta y) = (\tilde{x}(\rho), \tilde{y}(\rho)) \text{ for some } \theta > 0 \text{ and some } \rho \text{ with } \\
0 < \rho < \tilde{\rho}_0 \}.
\]

**Proposition 6.3.** Given \( i \in \mathcal{M} \). We have
\[
(1 + \lambda)\delta^i_x - \delta^i_y = 0, \forall (\delta^i_x, \delta^i_y) \in \partial V(x, y, i), \forall (x, y) \in BU^i. \quad (6.50)
\]

**Corollary 6.1.** \( SA^i \cap BU^i = \emptyset, \forall i \in \mathcal{M} \).

**Corollary 6.2.** For each \( i \in \mathcal{M} \), the value function \( V_i \) is \( C^1 \) in \( SA^i \cup BU^i \).

Note that Corollary 6.2 implies that \(-(1 - \mu) V_x(x, y, i) + V_y(x, y, i) = 0\) for \((x, y) \in SA^i \) and \((1 + \lambda) V_x(x, y, i) - V_y(x, y, i) = 0\) for \((x, y) \in BU^i \). In view of (6.39), we also have \( xV_x(x, y, i) + yV_y(x, y, i) = \frac{1}{2} \). This leads to the following result:
\[
V(x, y, i) = \left\{ \begin{array}{l}
\frac{1}{2} \ln[x + (1 - \mu)y] + A_i, \forall (x, y) \in SA^i, \\
\frac{1}{2} \ln[x + (1 + \lambda)y] + B_i, \forall (x, y) \in BU^i,
\end{array} \right.
\]
for some regime-dependent numbers \( A_i, B_i, i \in \mathcal{M} \).

For each \( i \in \mathcal{M} \), let \( NT^i = \Pi \setminus SA^i \cup BU^i \). Then we have:

**Proposition 6.4.** Given \( i \in \mathcal{M} \). we have
\[
\begin{cases}
-(1 - \mu)\delta^i_x + \delta^i_y > 0, \\
(1 + \lambda)\delta^i_x - \delta^i_y > 0,
\end{cases} \forall (\delta^i_x, \delta^i_y) \in \partial V(x, y, i), \forall (x, y) \in NT^i. \quad (6.52)
\]

**Acknowledgments.** We would like to thank the two anonymous referees and the editors for their valuable comments, which helped to improve the exposition of this paper. This paper acknowledges the financial support from the National Natural Science Foundation of China (71473235), Zhejiang Provincial Natural Science Foundation of China (LY16G030023), Key Humanities and Social Science Projects in Zhejiang Province University (2016GH019), Zhejiang Provincial Key Research Base of Management Science and Engineering, and Zhejiang Industrial Development Policy Key Research Center of Philosophy and Social Science of Zhejiang Province.

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Received August 2016; revised October 2016.

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