Three-factor decompositions of $\mathbb{U}_n$ with the three generators in arithmetic progression

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Abstract

Irrespective of whether $n$ is prime, prime power with exponent $> 1$, or composite, the group $\mathbb{U}_n$ of units of $\mathbb{Z}_n$ can sometimes be obtained as $\mathbb{U}_n = \langle x \rangle \times \langle x + k \rangle \times \langle x + 2k \rangle$ where $x, k \in \mathbb{Z}_n$. Indeed, for many values of $n$, many distinct 3-factor decompositions of this type exist. The circumstances in which such decompositions exist are examined. Many decompositions have additional interesting properties. We also look briefly at decompositions of the multiplicative groups of finite fields.

Keywords: generators of groups; units of $\mathbb{Z}_n$.

1 Introduction

An element of $\mathbb{Z}_n$ is a unit of $\mathbb{Z}_n$ if $x$ and $n$ are co-prime. If the prime-power decomposition of $n$ is $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ where $p_1, p_2, \ldots$ are distinct primes, then the number of units is given by Euler’s totient function $\phi_n = |\mathbb{U}_n| = (p_1 - 1)p_1^{\alpha_1 - 1} \cdot (q_1 - 1)q_1^{\beta_1 - 1} \cdot (r_1 - 1)r_1^{\gamma_1 - 1} \cdots$ where $\mathbb{U}_n$ denotes the group of units [6] Chap. 5] We recall the structure of $\mathbb{U}_n$:

- If $n = p_1^{\alpha_1} q_1^{\alpha_2} \cdots$, where $p_1, q_1, r_1, \ldots$ are primes, then

$$\mathbb{U}_n = \mathbb{U}_{p_1^{\alpha_1}} \times \mathbb{U}_{q_1^{\alpha_2}} \times \mathbb{U}_{r_1^{\alpha_3}} \times \cdots$$
• If $p$ is an odd prime, then $\mathbb{U}_p$ is cyclic of order $\phi_p = p^{\alpha-1}(p-1)$;

• $\mathbb{U}_{2\alpha}$ is cyclic of order $2^{\alpha-1}$ if $\alpha \leq 2$, and is isomorphic to $C_2 \times C_{2^{\alpha-2}}$ otherwise.

In particular, although $\mathbb{U}_n$ may be expressible as a direct product of cyclic groups in many different ways, the smallest number of factors is equal to the number $k$ of prime divisors of $n$ if $n$ is odd, or $k - 1$ if $n$ is twice odd, or $k$ if $n$ is four times odd, or $k + 1$ otherwise. The largest number of factors depends on the prime decompositions of $p - 1$ for the prime divisors $p$ of $n$. In particular, if $n$ is prime, then the maximum number of factors is the number of distinct prime divisors of $n - 1$.

We are mainly concerned here with cases where $n$ is odd and $\mathbb{U}_n$ is the product of three cyclic factors. As noted, this forces strong conditions on $n$: it should have at most three prime divisors; and if $n$ is prime, then $n - 1$ should have at least three prime divisors.

If we can write $\mathbb{U}_n = \langle x \rangle \times \langle x + k \rangle \times \langle x + 2k \rangle$ for some $x, k \in \mathbb{Z}_n$, so that the generators are in arithmetic progression (AP). We then have a three-factor AP decomposition of $\mathbb{U}_n$, which we abbreviate to a “3AP decomposition” of $\mathbb{U}_n$. Notable examples are

$$\mathbb{U}_{61} = \langle 9 \rangle_5 \times \langle 11 \rangle_4 \times \langle 13 \rangle_3$$

and

$$\mathbb{U}_{91} = \langle 9 \rangle_3 \times \langle 18 \rangle_{12} \times \langle 27 \rangle_2 = \langle 87 \rangle_6 \times \langle 83 \rangle_4 \times \langle 79 \rangle_3$$

and

$$\mathbb{U}_{65} = \langle 61 \rangle_3 \times \langle 57 \rangle_4 \times \langle 53 \rangle_4$$
where \( x = k \) in each of the 3AP decompositions; and
\[
U_{703} = \langle 700 \rangle_9 \times \langle 701 \rangle_{36} \times \langle 702 \rangle_2,
\]
where the generators are respectively \(-3, -2\) and \(-1 \pmod{703 = 19 \times 37}\).

Our specification (1) may be realised even if the values \( x, x+k \) and \( x+2k \), when reduced \( \pmod{n} \) to lie in the interval \([1, n-1]\), are not in arithmetic progression in \( \mathbb{Z} \). Thus we have
\[
U_{31} = \langle 30 \rangle_2 \times \langle 2 \rangle_5 \times \langle 5 \rangle_3
\]
with \( k \equiv 3 \pmod{31} \). We could equally have written this as
\[
U_{31} = \langle 5 \rangle_3 \times \langle 2 \rangle_5 \times \langle 30 \rangle_2
\]
with \( k \equiv 28 \equiv -3 \pmod{31} \). Faced with such a choice between two equivalent representations, we leave ourselves free to choose whichever seems the more convenient in the context in which it arises. A rule to make the choice with \( 0 < k < (n-1)/2 \) would be unsatisfactory, especially as some decompositions fall into infinite series within which \( k \) lies variously in \((0, (n-1)/2)\) and in \(((n-1)/2, n)\).

Examples where the two outer generators differ by 1 include
\[
U_{275} = \langle 136 \rangle_5 \times \langle -1 \rangle_2 \times \langle 137 \rangle_{20}
\]
and
\[
U_{775} = \langle 386 \rangle_{15} \times \langle -1 \rangle_2 \times \langle 387 \rangle_{20}.
\]

We now note three possibilities:

[A] For some values of \( n \), different 3AP decompositions of \( U_n \) may arise for different factorisations \( \phi_n = a \cdot b \cdot c \). Thus we have
\[
U_{211} = \langle 15 \rangle_6 \times \langle 107 \rangle_5 \times \langle 199 \rangle_7
= \langle 58 \rangle_7 \times \langle 134 \rangle_{15} \times \langle 210 \rangle_2
= \langle 196 \rangle_3 \times \langle 203 \rangle_{35} \times \langle 210 \rangle_2.
\]

[B] For a fixed factorisation \( \phi_n = a \cdot b \cdot c \) for a fixed \( n \), we may have different 3AP decompositions \( \langle x \rangle \times \langle y \rangle \times \langle z \rangle \) of \( U_n \) where the values of \( \text{ord}_n(y) \) are different members of \( \{a, b, c\} \). Thus we have
\[
U_{31} = \langle 30 \rangle_2 \times \langle 2 \rangle_5 \times \langle 5 \rangle_3
= \langle 25 \rangle_3 \times \langle 30 \rangle_2 \times \langle 4 \rangle_5
\]
and
\[
U_{547} = \langle 40 \rangle_3 \times \langle 172 \rangle_{26} \times \langle 304 \rangle_7
= \langle 40 \rangle_3 \times \langle 544 \rangle_7 \times \langle 501 \rangle_{26}
= \langle 520 \rangle_7 \times \langle 40 \rangle_3 \times \langle 107 \rangle_{26}.
\]
We may have different 3AP decompositions of $U_n$ for a fixed ordering of the terms of a fixed factorisation $\phi_n = abcd$ for a fixed $n$. Thus we have

$$U_{191} = \langle 39 \rangle_5 \times \langle 190 \rangle_2 \times \langle 150 \rangle_{19}$$
$$= \langle 184 \rangle_5 \times \langle 190 \rangle_2 \times \langle 5 \rangle_{19}$$

where $184 \equiv 39^2 \pmod{191}$ and $5 \equiv 150^{-2} \pmod{191}$.

**Problem 1** Find a series of primes that behave like 191, with 2 as the order of the middle generator. (Contenders for inclusion in the series are $n = 191, 271$ and 523. A possible series with the order 2 for an outer generator might cover $n = 331, 379, 443$ and 647.)

Clearly, a 3AP decomposition of $U_n$, where $n$ is prime and $n > 4$, cannot exist if the prime-power decomposition of $n - 1$ contains fewer than 3 distinct primes.

Sufficient conditions for the existence of 3AP decompositions of $U_n$ seem to be elusive. Thus only computer search has established that, in the range $n < 300$, a 3AP decomposition of $U_n$ does not exist for any of the values $n = 71, 127, 139, 223$ and 277. (We here exclude the “weak” 3AP decompositions defined in [3 below].

For any $n$ with $n > 4$, there is a *primitive root* of $n$ (an element from $\mathbb{U}_n$ that generates all members of $\mathbb{U}_n$) if and only if $n$ is an odd prime power or twice an odd prime power. In general we write $\lambda_n$ for the maximum order of a member of $\mathbb{U}_n$; if $n$ is odd, with prime power decomposition $n = p^\alpha q^\beta r^\gamma \cdots$, then

$$\lambda_n = \text{lcm}((p - 1)p^{\alpha-1}, (q - 1)q^{\beta-1}, (r - 1)r^{\gamma-1}, \ldots) .$$

We write $\xi_n = \phi_n/\lambda_n$; as shown in [3, §6], $\xi_n$ is even if greater than 1.

**Problem 2** Is there an upper bound on the number of 3AP decompositions of $U_n$ in terms of $\xi(n)$? Conversely, for a given value of $\xi(n) = m$, is it always possible to find $n$ with no 3AP decompositions?

Empirically we have found a tendency for larger values of $\xi(n)$ to be associated with larger numbers of decompositions. The table below, obtained by computer, gives $D$, the maximum number of 3AP decompositions of $U_n$, where $n \leq 1000$ and $\xi(n)$ is prescribed.

| $\xi(n)$ | 1 2 4 6 8 10 12 16 18 20 24 36 |
|----------|------------------|
| $D$      | 10 18 96 182 288 262 496 384 276 204 540 2088 |

### 2 $n$ prime

For $n$ prime, the multiplicative group $\mathbb{U}_n$ is cyclic, and so if it is expressed as a direct product, the factors must have pairwise co-prime orders.
2.1 The case \( n - 1 = 2 \cdot 3 \cdot m \)

We first prove three theorems that apply for prime values \( n \) such that the factors in a 3AP decomposition of \( U_n \) have orders 2, 3 and \( m \), where 2, 3 and \( m \) are pairwise co-prime. The first of these theorems is closely linked to Theorem 2.7 of [1]. We begin with some preliminary remarks.

Our assumption on \( n \) implies that \( n \equiv 7 \) or \( 31 \pmod{36} \), and \( n > 7 \). In particular, since \( n \equiv 3 \pmod{4} \), the quadratic residues have odd order, and the non-residues have even order. In our Theorems, we will be interested in the solutions of the quadratic equation \( x^2 + 3x + 3 = 0 \) in \( \mathbb{Z}_n \). Its discriminant is \(-3\), which (by Quadratic Reciprocity [6, §7.4]) is a square in \( \mathbb{Z}_n \), so the quadratic has two roots in \( \mathbb{Z}_n \). The product of the roots is 3, which is a non-square; so one root (say \( x_1 \)) has odd order, and the other (say \( x_2 \)) has even order. We also note that the values \( y_1 = x_1 + 1 \) and \( y_2 = x_2 + 1 \) satisfy the quadratic equation \( y^2 + y + 1 = 0 \), and so \( \text{ord}_n(y_1) = \text{ord}_n(y_2) = 3 \).

**Theorem 2.1** Let \( n \) be a prime satisfying \( n \equiv 7 \) or \( 31 \pmod{36} \), \( n > 7 \). Suppose that the elements \( x_1 \) and \( x_2 \) from \( U_n \) that satisfy \( x^2 + 3x + 3 \equiv 0 \pmod{n} \) are such that \( \text{ord}_n(x_1) = (n - 1)/6 \). Then \( \text{ord}_n(-(x_1 + 2)) = 3 \) and so

\[
U_n = \langle -x_1 - 2 \rangle_3 \times \langle -1 \rangle_2 \times \langle x_1 \rangle_m
\]

where \( m = (n - 1)/6 \).

**Proof** As noted above, \( 3 = \text{ord}_n(x_2 + 1) \), and \( x_2 + 1 = -x_1 - 2 \), since \( x_1 + x_2 = -3 \). \( \square \)

**Coverage** In the range \( n < 1000 \), Theorem 2.1 covers values as follows:

- \( n = 31 \): \( \langle 25 \rangle_3 \times \langle 30 \rangle_2 \times \langle 4 \rangle_5 \)
- \( n = 43 \): \( \langle 6 \rangle_3 \times \langle 42 \rangle_2 \times \langle 35 \rangle_7 \)
- \( n = 79 \): \( \langle 55 \rangle_3 \times \langle 78 \rangle_2 \times \langle 22 \rangle_{13} \)
- \( n = 211 \): \( \langle 196 \rangle_3 \times \langle 210 \rangle_2 \times \langle 13 \rangle_{35} \)
- \( n = 463 \): \( \langle 21 \rangle_3 \times \langle 462 \rangle_2 \times \langle 440 \rangle_{77} \)
- \( n = 571 \): \( \langle 109 \rangle_3 \times \langle 570 \rangle_2 \times \langle 460 \rangle_{95} \)
- \( n = 751 \): \( \langle 678 \rangle_3 \times \langle 750 \rangle_2 \times \langle 71 \rangle_{125} \)
- \( n = 907 \): \( \langle 522 \rangle_3 \times \langle 906 \rangle_2 \times \langle 383 \rangle_{151} \)

**Theorem 2.2** Let \( n \) be a prime satisfying \( n \equiv 7 \) or \( 31 \pmod{36} \), \( n > 7 \). Suppose that the elements \( x_1 \) and \( x_2 \) from \( U_n \) that satisfy \( x^2 + 3x + 3 \equiv 0 \pmod{n} \) are such that \( \text{ord}_n(x_1) = (n - 1)/2 \) and \( \text{ord}_n(x_2) = (n - 1) \). Then \( \text{ord}_n(x_2 + 1) = 3 \) and \( \text{ord}_n(2x_2 + 3) = (n - 1)/6 \), so that

\[
U_n = \langle 2x_2 + 3 \rangle_m \times \langle x_2 + 1 \rangle_3 \times \langle -1 \rangle_2
\]

where \( m = (n - 1)/6 \).
\textbf{Proof} Since \( U_n = (-1)^2 \times \langle x_1 + 1 \rangle \times C_{(n-1)/6} \), the hypothesis \( \text{ord}_n(x_2) = n - 1 \) shows that \( x_2 = - (x_1 + 1)c \) or \( - (x_2 + 1)c \), where \( \text{ord}_n(c) = (n - 1)/6 \). Hence either \( -(x_1 + 1)x_2 \) or \( - (x_2 + 1)x_2 \) has order \( (n - 1)/6 \). Now \( x_1 + x_2 = -3 \) and \( x_1x_2 = 3 \), so \( -(x_1 + 1)x_2 = -3 - x_2 = x_1 \), which has order \( (n - 1)/2 \), by assumption. So

\[-(x_2 + 1)x_2 = -x_2^2 - x_2 = 2x_2 + 3\]

has order \( (n - 1)/6 \).

\( \square \)

\textbf{Coverage} In the range \( n < 1000 \), Theorem 2.2 covers values as follows:

\begin{align*}
n = 67 & : \langle 59 \rangle_{11} \times \langle 29 \rangle_3 \times \langle 66 \rangle_2 \\
n = 103 & : \langle 10 \rangle_{17} \times \langle 46 \rangle_3 \times \langle 102 \rangle_2 \\
n = 151 & : \langle 86 \rangle_{25} \times \langle 118 \rangle_3 \times \langle 150 \rangle_2 \\
n = 367 & : \langle 200 \rangle_{61} \times \langle 283 \rangle_3 \times \langle 366 \rangle_2 \\
n = 439 & : \langle 343 \rangle_{73} \times \langle 171 \rangle_3 \times \langle 438 \rangle_2 \\
n = 499 & : \langle 279 \rangle_{83} \times \langle 139 \rangle_3 \times \langle 498 \rangle_2 \\
n = 619 & : \langle 505 \rangle_{103} \times \langle 252 \rangle_3 \times \langle 618 \rangle_2 \\
n = 643 & : \langle 355 \rangle_{107} \times \langle 177 \rangle_3 \times \langle 642 \rangle_2 \\
n = 727 & : \langle 563 \rangle_{121} \times \langle 281 \rangle_3 \times \langle 726 \rangle_2 \\
n = 787 & : \langle 28 \rangle_{131} \times \langle 407 \rangle_3 \times \langle 786 \rangle_2 \\
n = 967 & : \langle 682 \rangle_{162} \times \langle 824 \rangle_3 \times \langle 966 \rangle_2
\end{align*}

\textbf{Theorem 2.3} Let \( n \) be a prime satisfying \( n \equiv 7 \) or \( 31 \) (mod 36). Suppose that \( z \) is one of the elements \( x_1 \) and \( x_2 \) that satisfy \( x^2 + 3x + 3 \equiv 0 \) (mod \( n \)) and that \( \text{ord}_n(2^{-1}z) = (n - 1)/6 \). Then

\[ U_n = \langle z + 1 \rangle_3 \times \langle 2^{-1}z \rangle_m \times \langle -1 \rangle_2 \]

where \( m = (n - 1)/6 \).

\textbf{Proof} As for Theorem 2.1. But it depends on \( n \) whether \( z \) is the solution of \( x^2 + 3x + 3 \equiv 0 \) that has the larger or smaller order, and whether \( z \) is \( x_1 \) or \( x_2 \).

\( \square \)

\textbf{Coverage} In the range \( n < 1000 \), Theorem 2.3 covers values as follows:
Note 2.1 In the range $n < 1000$, Theorems 2.1–2.3 exclude $n = 139, 223, 331, 547, 607$ and $859$. All but one of these has $x$-values $x_1$ and $x_2$ with $\text{ord}_n(x_1) = (n-1)/2$ and $\text{ord}_n(x_2) = (n-1)/3$; the exception is $n = 547$, which has $x_1 = 505$ and $x_2 = 39$, with $\text{ord}_n(x_1) = (n-1)/26$ and $\text{ord}_n(x_2) = (n-1)/13$.

Problem 3 It is natural to wonder whether there are infinitely many primes for which the conditions of one of the above theorems are satisfied. Here are some thoughts on this. In all cases we seek primes congruent to 7 or 31 (mod 36); Dirichlet’s Theorem [6, Theorem 2.10] guarantees that infinitely many such primes exist, and indeed they have density $1/6$ among all primes.

Consider Theorem 2.1. We require that an element of order $(n-1)/6$ (necessarily a sixth power) should satisfy $x^2 + 3x + 3 = 0$, so there should be a solution $y$ of the equation $y^{12} + 3y^6 + 3 = 0$. The Chebotarev density theorem [3, 8, section 1.2.2] guarantees that this equation will have a solution in an infinite set (indeed, a set of positive density) of primes. This theorem can further guarantee a set of positive density for which $x_1$ has six distinct sixth roots (so that $n \equiv 1$ (mod 6)) and $x_2$ is a non-square (so that $n \equiv 3$ (mod 4)), but we do not know how to exclude $n \equiv 19$ (mod 36).

A more serious difficulty is that the fact that $x$ is a sixth power guarantees only that its order divides $(n-1)/6$; it does not seem easy to show that the order is precisely this value. Clearly this would be the case if $n = 6q + 1$ with $q$ prime; but it is not even known whether infinitely many primes of this form occur.

Of the 1614 primes less than $10^5$ which are congruent to 7 or 31 (mod 36), there are 494, 476 and 476 that satisfy the conditions of Theorems 2.1–2.3 respectively.

We conclude this subsection with a converse to the preceding theorems.

Theorem 2.4 Any 3AP decomposition of $U_n$ for $n$ prime, in which the generators have orders 2, 3 and $(n-1)/6$, arises as in one of the three preceding theorems.
congruence 85
≡
three cases in turn.

For the two examples for $n$
Proof We already saw that $x$ that $x$
−generator are thus $x$
versa. Let $j$
only element of order 2 is $−$
to prove that $i$
U
in a 3AP decomposition of $n$
has order $\frac{(n-1)}{2}$ and $\frac{(n-1)}{3}$ respectively. Let $j = 3 - i$. From the proof of Theorem 2.2 we see that $2x_i + 3 = -(x_i + 1)x_i$, so that $x_i = -(x_j + 1)(2x_i + 3)$, the product of elements of orders 2, 3 and $\frac{(n-1)}{6}$; so $x_i$ has order $n - 1$. Thus $i = 2$. Now $(x_1 + 1)x_1 = 2x_2 + 3$
has order $\frac{(n-1)}{6}$, so $x_1 = (x_2 + 1)(2x_2 + 3)$ has order $\frac{(n-1)}{2}$.

Finally, the third case obviously gives the situation of Theorem 2.3. □

2.2 The case $n - 1 = 3 \cdot 4 \cdot \mu$
We now prove two theorems that apply for prime values $n$ such that the factors in a 3AP decomposition of $\mathbb{U}_n$ have orders 3, 4 and $\mu$ where 3, 4 and $\mu$ are pairwise co-prime. We give no theorem for the situation where $\mu$ is the order of the middle generator. This case can occur; the smallest example is for $n = 997$.

Theorem 2.5 Let $n$ be a prime satisfying $n \equiv 13, 61, 85$ or $133 \pmod{144}$, $n > 13$. Suppose that there is an element $x$ from $\mathbb{U}_n$ such that $x^2 + 3x + 3 \equiv 0 \pmod{n}$ and such that there is also an element $k$ with $\text{ord}_n(x + 1 + k) = 4$ and $\text{ord}_n(x + 1 + 2k) = \frac{(n-1)}{12}$. Then

$$\mathbb{U}_n = \langle x + 1 \rangle_3 \times \langle x + 1 + k \rangle_4 \times \langle x + 1 + 2k \rangle_{\mu},$$

where $\mu = \frac{(n-1)}{12}$.

Proof As for Theorem 2.1. Note that the condition on $x + 1 + k$ can be written $(x + 1 + k)^2 \equiv -1 \pmod{n}$. □

Coverage In the range $n < 1000$, Theorem 2.5 covers values as follows:

| $n$  | 3AP decomposition of $\mathbb{U}_n$ | $\text{ord}_n(x)$ |
|------|-----------------------------------|-------------------|
| 61   | $\langle 13 \rangle_3 \times \langle 11 \rangle_4 \times \langle 9 \rangle_5$ | $15 = \frac{(n-1)}{4}$ |
| 349  | $\langle 122 \rangle_3 \times \langle 213 \rangle_4 \times \langle 304 \rangle_{29}$ | $58 = \frac{(n-1)}{6}$ |
| 661  | $\{\langle 364 \rangle_3 \times \langle 106 \rangle_4 \times \langle 509 \rangle_{55}\}$ | $66 = \frac{(n-1)}{10}$ |

For the two examples for $n = 661$, the generators of order 55 are related by the congruence $85 \equiv 509^3 \pmod{661}$. 

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Theorem 2.6 Let \( n \) be a prime satisfying \( n \equiv 13, 61, 85 \) or \( 133 \) (mod 144), \( n > 13 \). Suppose that there is an element \( x \) from \( U_n \) such that \( x^2 + 3x + 3 \equiv 0 \) (mod \( n \)) and such that there is also an element \( k \) with \( \text{ord}_n(x + 1 + k) = 4 \) and \( \text{ord}_n(x + 1 - k) = (n - 1)/12 \). Then

\[
U_n = \langle x + 1 - k \rangle_\mu \times \langle x + 1 \rangle_3 \times \langle x + 1 + k \rangle_4
\]

where \( \mu = (n - 1)/12 \).

Proof As for Theorem 2.1. \( \square \)

Coverage In the range \( n < 1000 \), Theorem 2.6 covers values as follows:

| \( n \) | 3AP decomposition of \( U_n \) | \( \text{ord}_n(x) \) |
|-----|--------------------------------|----------------|
| 157 | \( \langle 153 \rangle_{13} \times \langle 12 \rangle_3 \times \langle 28 \rangle_4 \) | 39 = \( (n - 1)/4 \) |
| 229 | \( \langle 161 \rangle_{19} \times \langle 134 \rangle_3 \times \langle 107 \rangle_4 \) | 228 = \( (n - 1) \) |
| 349 | \( \langle 31 \rangle_{29} \times \langle 122 \rangle_3 \times \langle 213 \rangle_4 \) | 58 = \( (n - 1)/6 \) |
| 373 | \( \langle 91 \rangle_{31} \times \langle 284 \rangle_3 \times \langle 104 \rangle_4 \) | 93 = \( (n - 1)/4 \) |
| 997 | \( \langle 226 \rangle_{83} \times \langle 692 \rangle_3 \times \langle 161 \rangle_4 \) | 498 = \( (n - 1)/2 \) |

In the range \( n < 1000 \), Theorems 2.5 and 2.6 fail to provide 3AP decompositions of \( U_n \) for \( n = 277, 421, 709, 733, 853 \) and 877. However, 3AP decompositions for \( n = 421 \) exist for other partitions of \( n - 1 \).

2.3 The case \( n - 1 = 2 \cdot 5 \cdot \nu \)

We now examine what occurs for primes \( n \) such that the factors in a 3AP decomposition of \( U_n \) have orders 2, 5 and \( \nu \), these orders being pairwise co-prime. Amongst primes satisfying \( n \equiv 11, 31, 71 \) and 91 (mod 100), \( n > 11 \), the patterns of occurrence of such 3AP decompositions are very similar to those reported in §2.1 above. For each relevant value of \( n \) there are 4 elements of order 5; their sum is \( -1 \) and their product is \( +1 \). Sometimes more than one of the four can be used.

Type 2.3(a) Analogous to the decompositions obtainable via Theorem 2.1, we now have 3AP decompositions of the form

\[
U_n = \langle -z - 2 \rangle_5 \times \langle -1 \rangle_2 \times \langle z \rangle_\nu .
\]

In the range \( n < 1000 \) they are as follows:

\[
\begin{align*}
n = 31: & \quad \langle 4 \rangle_5 \times \langle 30 \rangle_2 \times \langle 25 \rangle_3 \\
n = 191: & \quad \begin{cases} 
\langle 39 \rangle_5 \times \langle 190 \rangle_2 \times \langle 150 \rangle_{19} \\
\langle 184 \rangle_5 \times \langle 190 \rangle_2 \times \langle 5 \rangle_{19}
\end{cases}
\end{align*}
\]
Type 2.3(b) Analogous to the decompositions obtainable via Theorem 2.2, we have 3AP decompositions of the form

\[ U_n = \langle 2z + 1 \rangle_5 \times \langle z \rangle_5 \times \langle -1 \rangle_2. \]

In the range \( n < 1000 \) they are as follows:

\begin{align*}
n = 271 : & & \langle 10 \rangle_5 \times \langle 270 \rangle_2 \times \langle 259 \rangle_27 \\\n = 431 : & & \langle 405 \rangle_5 \times \langle 430 \rangle_2 \times \langle 24 \rangle_43 \\\n = 691 : & & \langle 89 \rangle_5 \times \langle 690 \rangle_2 \times \langle 600 \rangle_69 \\\n = 991 : & & \langle 799 \rangle_5 \times \langle 990 \rangle_2 \times \langle 190 \rangle_99
\end{align*}

Type 2.3(c) Analogous to the decompositions obtainable via Theorem 2.3, we have 3AP decompositions of the form

\[ U_n = \langle 2z + 1 \rangle_5 \times \langle z \rangle_5 \times \langle -1 \rangle_2. \]

In the range \( n < 1000 \) they are as follows:

\begin{align*}
n = 31 : & & \langle 5 \rangle_3 \times \langle 2 \rangle_5 \times \langle 30 \rangle_2 \\
 = 131 : & & \langle 107 \rangle_{13} \times \langle 53 \rangle_5 \times \langle 130 \rangle_2 \\
 = 311 : & & \langle 13 \rangle_{31} \times \langle 6 \rangle_5 \times \langle 310 \rangle_2 \\
 & & \langle 105 \rangle_{31} \times \langle 52 \rangle_5 \times \langle 310 \rangle_2 \\
 = 491 : & & \langle 203 \rangle_{49} \times \langle 101 \rangle_5 \times \langle 490 \rangle_2 \\
 = 811 : & & \langle 330 \rangle_{81} \times \langle 570 \rangle_5 \times \langle 810 \rangle_2 \\
 = 991 : & & \langle 395 \rangle_{99} \times \langle 197 \rangle_5 \times \langle 990 \rangle_2
\end{align*}

Note 2.2 In the range \( n < 1000 \), no 3AP decomposition of any of the types 2.3(a), 2.3(b) or 2.3(c) exists for \( n = 71, 211, 331, 571, 631 \) or 911.

2.4 Some double-barrelled cases

Two special cases arise for prime \( n \) such that \( U_n \) has more than one 3AP decomposition. These are where we can write either

\[ U_n = \langle k \rangle_a \times \langle k + z \rangle_b \times \langle k + 2z \rangle_c = \langle k - 2z \rangle_b \times \langle k \rangle_a \times \langle k + 2z \rangle_c \]
or

\[ U_n = \langle k \rangle_a \times \langle k + z \rangle_b \times \langle k + 2z \rangle_c = \langle k + z \rangle_b \times \langle k + 2z \rangle_c \times \langle k + 3z \rangle_a . \]

For the first of these we need \( k + z \) and \( k - 2z \) to have the same order (mod \( n \)) and each to be a power of the other (mod \( n \)). For the second we need the same relationship between \( k \) and \( k + 3z \). We have failed to find any theorems to indicate when these cases arise. In the range \( n < 1000 \), the occurrences of the first case are these:

\[
\begin{align*}
U_{67} &= \langle 29 \rangle_3 \times \langle 14 \rangle_{11} \times \langle 66 \rangle_2 = \langle 59 \rangle_{11} \times \langle 29 \rangle_3 \times \langle 66 \rangle_2 \\
U_{211} &= \langle 210 \rangle_2 \times \langle 203 \rangle_{35} \times \langle 196 \rangle_3 = \langle 13 \rangle_{35} \times \langle 210 \rangle_2 \times \langle 196 \rangle_3 \\
U_{271} &= \langle 270 \rangle_2 \times \langle 140 \rangle_{27} \times \langle 10 \rangle_5 = \langle 259 \rangle_{27} \times \langle 270 \rangle_2 \times \langle 10 \rangle_5 \\
U_{331} &= \langle 167 \rangle_{11} \times \langle 83 \rangle_{15} \times \langle 330 \rangle_2 = \langle 4 \rangle_{15} \times \langle 167 \rangle_{11} \times \langle 330 \rangle_2 \\
U_{379} &= \langle 378 \rangle_2 \times \langle 119 \rangle_7 \times \langle 239 \rangle_{27} = \langle 138 \rangle_7 \times \langle 378 \rangle_2 \times \langle 239 \rangle_{27} \\
U_{561} &= \langle 364 \rangle_3 \times \langle 391 \rangle_{20} \times \langle 418 \rangle_{11} = \langle 310 \rangle_{20} \times \langle 364 \rangle_3 \times \langle 418 \rangle_{11} \\
U_{787} &= \langle 407 \rangle_3 \times \langle 203 \rangle_{131} \times \langle 786 \rangle_2 = \langle 28 \rangle_{131} \times \langle 407 \rangle_3 \times \langle 786 \rangle_2 \\
U_{907} &= \langle 906 \rangle_2 \times \langle 714 \rangle_{151} \times \langle 522 \rangle_3 = \langle 383 \rangle_{151} \times \langle 906 \rangle_2 \times \langle 522 \rangle_3
\end{align*}
\]

whereas the occurrences of the second case are these:

\[
\begin{align*}
U_{349} &= \langle 31 \rangle_{29} \times \langle 122 \rangle_3 \times \langle 213 \rangle_4 = \langle 122 \rangle_3 \times \langle 213 \rangle_4 \times \langle 304 \rangle_{29} \\
U_{599} &= \langle 578 \rangle_{23} \times \langle 598 \rangle_2 \times \langle 19 \rangle_{13} = \langle 598 \rangle_2 \times \langle 19 \rangle_{13} \times \langle 39 \rangle_{23}.
\end{align*}
\]

3 Lifts

We now consider how and when an AP decomposition of \( U_n \) can be used to obtain AP decompositions of \( U_{n'} \) where \( n' \) is a power of \( n \) or some other multiple of \( n \).

3.1 Definitions

In this section, we allow weak 3AP decompositions \( U_n = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c \), where one of \( x, y, z \) is allowed to be 1 (so that the corresponding cyclic factor is trivial). For example,

\[ U_7 = \langle 4 \rangle_3 \times \langle 6 \rangle_2 \times \langle 1 \rangle_1 \]  \hspace{1cm} (2)

and

\[ U_{103} = \langle 1 \rangle_1 \times \langle 47 \rangle_6 \times \langle 93 \rangle_{17} \]

are weak 3AP decompositions. Where it aids clarity, we refer to a 3AP decomposition in the original sense as being strong.
If $p$ is a prime satisfying $p \equiv 11 \pmod{12}$ and \( \operatorname{ord}_p(3) = (p - 1)/2 \), then \( U_p = \langle -1 \rangle_2 \times \langle 1 \rangle_1 \times \langle 3 \rangle_{(p-1)/2} \). Likewise if $p$ is a prime satisfying $p \equiv 7 \pmod{12}$ and \( \operatorname{ord}_p(-3) = (p - 1)/2 \), then \( U_p = \langle -3 \rangle_{(p-1)/2} \times \langle -1 \rangle_2 \times \langle 1 \rangle_1 \).

If $n$ divides $n'$, then the map $x \mapsto x \pmod{n}$ is a ring epimorphism from $\mathbb{Z}_{n'}$ to $\mathbb{Z}_n$, and maps $U_{n'}$ onto $U_n$. It preserves the property of forming an arithmetic progression. However, it does not in general map a (weak) 3-AP decomposition of $U_{n'}$ to a (weak) 3-AP decomposition of $U_n$. (It maps a generating set to a generating set, but does not necessarily preserve the direct sum decomposition.) Note in passing that, if $n$ divides $n'$, then $\phi(n)$ divides $\phi(n')$, the quotient being the order of the kernel of the homomorphism from $U_{n'}$ to $U_n$.

Suppose that $n$ divides $n'$, and that the 3AP decompositions

\[ U_n = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c \]

and

\[ U_{n'} = \langle x' \rangle_{a'} \times \langle y' \rangle_{b'} \times \langle z' \rangle_{c'} \]

satisfy $x \equiv x' \pmod{n}$, $y' \equiv y \pmod{n}$ and $z \equiv z' \pmod{n}$. Then we call the second decomposition a lift of the first, with index $n'/n$. Note that we must have $a \mid a'$, $b \mid b'$, $c \mid c'$, and $(a'b'c')/(abc) = \phi(n')/\phi(n)$. For example, the decomposition

\[ U_{49} = \langle 18 \rangle_3 \times \langle 48 \rangle_2 \times \langle 29 \rangle_7 \]

is a lift of the weak 3AP decomposition \( \langle 2 \rangle \) of $U_7$. We further describe $x'$ as being a lift of $x$, and so on.

### 3.2 Lifts from $n$ to $np$, with $p$ an odd prime

We are unable to give necessary and sufficient conditions for lifts to exist. In the remainder of this section, we consider the case where $n' = np$ for an odd prime $p$. (We do not know whether every lift can be obtained by a sequence of lifts where the indices are primes.)

We subdivide the analysis into three cases.

**Case 1: $p^2$ divides $n$.** Let $n = p^k m$ ($k \geq 2$) where $p$ does not divide $m$.

In this case, if an element $x \in U_n$ has \( \operatorname{ord}_{p^k}(x) \) divisible by $p$, then any lift $x'$ of $x$ satisfies \( \operatorname{ord}_{p^k}(x') = p \operatorname{ord}_n(x) \). So at most one of $x$, $y$, $z$ can satisfy this condition if there is a lift which is a 3AP decomposition. Conversely, if, say, $x$ satisfies the condition but $y$ and $z$ do not, then we can choose lifts $y'$ and $z'$ of $y$ and $z$ satisfying $\operatorname{ord}_{p^k}(y') = \operatorname{ord}_n(y)$ and $\operatorname{ord}_{p^k}(z') = \operatorname{ord}_n(z)$, and the unique lift $x'$ of $x$ such that $(x', y', z')$ is an AP in $\mathbb{Z}_{n'}$, to obtain a 3AP decomposition of $U_{n'}$.

For example, below (Case 2, Subcase 2.2) we find a decomposition

\[ U_{275} = \langle 274 \rangle_2 \times \langle 166 \rangle_5 \times \langle 58 \rangle_{20}. \]
This cannot be lifted to a 3AP decomposition of $U_{n'}$ with $n' = 5 \times 275$. On the other hand, the decomposition can be lifted to

$$U_{343} = \langle 18 \rangle_3 \times \langle 324 \rangle_2 \times \langle 323 \rangle_49.$$ 

**Case 2:** $p$ divides $n$ but $p^2$ does not. We further subdivide into four cases according to how many of the orders $a, b, c$ are divisible by $p$. Note that $\phi(np) = \phi(n)p$.

For any $x \in U_n$, with $\text{ord}_n(x) = a$, the lifts of $x$ to $U_{pn}$ belong to an extension of $C_p$ by $C_a$, which is isomorphic to $C_p \times C_a$. Hence, if $p$ does not divide $a$, then one of these lifts (which we call the special lift) has order $a$, and the other $p - 1$ have order $pa$; while if $p$ divides $a$, then all have order $a$.

**Subcase 2.1:** None of $a, b, c$ is divisible by $p$. If a lift is a 3AP decomposition, then two of $x, y, z$ lift to elements of the same orders (and so must be special lifts), while the third lifts to an element with $p$ times the order. Let $x', y', z'$ be the special lifts of $x, y, z$. We call the decomposition unproductive if $(x', y', z')$ is an AP in $\mathbb{Z}_{np}$. In this case, there is no lift to a 3AP decomposition of $U_{np}$. In the contrary productive case, there are three lifts which are 3AP decompositions since we may choose for which two of $x, y, z$ we use special lifts, and the third lift is determined by the AP requirement.

Thus, if we start from the productive strong 3AP decomposition

$$U_{31} = \langle 25 \rangle_3 \times \langle 30 \rangle_2 \times \langle 4 \rangle_5$$

we obtain the three lifts

$$U_{312} = \langle 521 \rangle_3 \times \langle 960 \rangle_2 \times \langle 438 \rangle_{155}$$
$$= \langle 521 \rangle_3 \times \langle 526 \rangle_{62} \times \langle 531 \rangle_5$$
$$= \langle 428 \rangle_{93} \times \langle 960 \rangle_2 \times \langle 531 \rangle_5 .$$

Likewise we can start from the productive strong 3AP decomposition

$$U_{35} = \langle 11 \rangle_3 \times \langle 34 \rangle_2 \times \langle 22 \rangle_4 .$$

to obtain the lifts

$$U_{245} = \langle 116 \rangle_3 \times \langle 244 \rangle_2 \times \langle 127 \rangle_{28}$$
$$= \langle 116 \rangle_3 \times \langle 34 \rangle_{14} \times \langle 197 \rangle_4$$
$$= \langle 46 \rangle_{21} \times \langle 244 \rangle_2 \times \langle 197 \rangle_4$$

and

$$U_{175} = \langle 151 \rangle_3 \times \langle 174 \rangle_2 \times \langle 22 \rangle_{20}$$
$$= \langle 151 \rangle_3 \times \langle 104 \rangle_{10} \times \langle 57 \rangle_4$$
$$= \langle 116 \rangle_{15} \times \langle 174 \rangle_2 \times \langle 57 \rangle_4 .$$
If we start from a productive weak 3AP decomposition, then two of the three lifts are weak but the third (where the identity lifts to an element of order $p$) is strong. This happens in the example (3) of a 3AP decomposition of $\mathbb{U}_{49}$; the two corresponding weak 3AP decompositions lifted from (2) are

$$U_{49} = \langle 18 \rangle_3 \times \langle 34 \rangle_{14} \times \langle 1 \rangle_1 = \langle 46 \rangle_{21} \times \langle 48 \rangle_2 \times \langle 1 \rangle_1.$$ 

Now consider the prime $n = 379$. The strong 3AP decomposition

$$U_{379} = \langle 239 \rangle_{27} \times \langle 378 \rangle_2 \times \langle 138 \rangle_7$$

is unproductive: the special lifts of 239, 378 and 138 are respectively 8956, 143640 and 134683, which happen to be in arithmetic progression (mod 379$^2$). The two weak 3AP decompositions

$$U_{11} = \langle 10 \rangle_2 \times \langle 1 \rangle_1 \times \langle 3 \rangle_5$$

and

$$U_{461} = \langle 1 \rangle_1 \times \langle 48 \rangle_4 \times \langle 95 \rangle_{115}$$

are also unproductive, but these are the only ones with prime modulus less than 1000. It appears that productive decompositions predominate; unproductive ones depend on an accidental coincidence which is comparatively rare.

**Subcase 2.2:** One of $a, b, c$ (say $a$, without loss) is divisible by $p$. Now choose the special lift of either $y$ or $z$, and any non-special lift of the other; the lift of $x$ is determined by the AP requirement. So there are $2(p - 1)$ lifts to 3AP decompositions.

Here is an example. Start from a weak 3AP decompositions of $\mathbb{U}_{55}$:

$$U_{55} = \langle 54 \rangle_2 \times \langle 1 \rangle_1 \times \langle 3 \rangle_{20}. \quad (7)$$

We wish to lift to strong 3AP decompositions of $\mathbb{U}_{275}$. We are in this subcase. All lifts of 3 have order 20, but each of the generators 54 and 1 has one special lift (namely 274 and 1 respectively). So we must use a non-special lift of 1, the special lift of 54, and the lift of 3 which completes the AP:

$$U_{275} = \langle 274 \rangle_2 \times \langle 166 \rangle_5 \times \langle 58 \rangle_{20}$$
$$= \langle 274 \rangle_2 \times \langle 56 \rangle_5 \times \langle 113 \rangle_{20}$$
$$= \langle 274 \rangle_2 \times \langle 221 \rangle_5 \times \langle 168 \rangle_{20}$$
$$= \langle 274 \rangle_2 \times \langle 111 \rangle_5 \times \langle 223 \rangle_{20}$$

The other four lifts (where we use the special lift of 1 and a non-special lift of 54) are weak 3AP decompositions. In the same way, the decomposition

$$U_{55} = \langle 52 \rangle_{20} \times \langle 54 \rangle_2 \times \langle 1 \rangle_1 \quad (8)$$

gives rise to four more strong 3AP decompositions of $\mathbb{U}_{275}$.

Suppose, however, that we consider lifting (7) and (8) from $n = 55$ to $n = 605$. The only lifts of 52, 54, 1 and 3 (mod 605) which have orders 20, 2, 1, and 20 respectively are 602, 604, 1 and 3 (mod 605), which are in AP, so we fail to obtain any 3AP decomposition for $\mathbb{U}_{605}$.
Subcase 2.3: Two of $a, b, c$ (say $a$ and $b$) are divisible by $p$. We must choose a non-special lift of $z$, and any lift of $x$; so there are $p(p - 1)$ lifts to 3AP decompositions of $\mathbb{U}_{n'}$. Suppose, for example, that we take $n = 273 = 3 \times 7 \times 13$ and $p = 3$, to give $n' = 819$. Computer enumeration has shown that there are 108 strong 3AP decompositions of $\mathbb{U}_{273}$. Each perforce has 2 generators whose orders are multiples of 3. The 648 lifts to $\mathbb{U}_{819}$ arise from the strong decompositions.

Subcase 2.4: All three of $a, b, c$ are divisible by $p$. In this case, three distinct prime divisors of $n$ are congruent to 1 (mod $p$), and so $np$ has at least four prime divisors, so no 3AP decomposition of $\mathbb{U}_{np}$ can exist. (Alternatively, note that all lifts of $x, y, z$ have the same orders as the original elements, so the group they generate has the same order as $\langle x, y, z \rangle$.)

Case 3: $p$ does not divide $n$. In this case, $\mathbb{U}_{np} \cong \mathbb{U}_n \times \mathbb{U}_p$. This case is the most difficult and we do not have any general criteria for a lift to exist. However, $\mathbb{U}_{np}$ is a product of at most 3 cyclic groups. From the structure of the group of units, as described in the Introduction, we see that $n = q^\alpha r^\beta$, or $2q^\alpha r^\beta$, or $4q^\alpha$, or $2^\alpha$, for some odd primes $q$ and $r$, and some $\alpha, \beta \geq 0$. In this case, one of the lifts of any generator of a 3AP decomposition of $\mathbb{U}_n$ is a multiple of $p$, and therefore must be disallowed as a spurious lift. The spurious lifts $a', b', c'$ of the three generators $a, b, c$ are in AP. For they form an AP (mod $n$), and a trivial AP (mod $p$). By the Chinese Remainder Theorem, the congruences

\[ b' - a' \equiv c' - b' \pmod{n}, \quad b' - a' \equiv c' - b' \pmod{p}, \]

imply that $b' - a' \equiv c' - b' \pmod{pn}$.

Suppose that we have $n = 31$ and $p = 5$, and we consider lifting

\[ \mathbb{U}_{31} = \langle 25 \rangle_3 \times \langle 30 \rangle_2 \times \langle 4 \rangle_5. \]

The respective spurious lifts are 25, 30 and 35. Two lifts of the 3AP decomposition are available:

\[ \mathbb{U}_{155} = \langle 56 \rangle_3 \times \langle 154 \rangle_2 \times \langle 97 \rangle_{20} \times \langle 87 \rangle_{12} \times \langle 154 \rangle_2 \times \langle 66 \rangle_5. \]

3.3 Lifting to $\mathbb{U}_{p^n}$

In one special case of lifts we can draw a strong conclusion. This case requires a productive 3AP decomposition for $\mathbb{U}_p$, where $p$ is prime; we recall the unproductive examples (1), (5) and (6) given above.

**Theorem 3.1** Let $p$ be an odd prime, and suppose that

\[ \mathbb{U}_p = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c \]

is a productive (possibly weak) 3AP decomposition. Then for any $\alpha \geq 2$, there is a lift of the given decomposition which is a (strong) 3AP decomposition of $\mathbb{U}_{p^n}$.
Proof  We show inductively that there is a lift where two of the lifted elements have the same orders as the originals, and the third has order multiplied by $p^{\alpha-1}$.

For the first step, we are in case 2, subcase 2.1; in this case we saw that any productive decomposition has three lifts, at least one of which is strong.

For the general step, we are in case 1, and we start with a decomposition in which two of the elements have orders coprime to $p$; so the necessary condition for this case is satisfied, and the lift exists. □

4  
$n$ an odd composite integer

4.1  
$n$ a multiple of 3

Theorem 4.1  
A 3AP decomposition of $\mathbb{U}_{3p}$ does not exist for any prime $p$ with $p > 3$.

Proof  Suppose that

$$\mathbb{U}_{3p} = \langle a \rangle \times \langle a + d \rangle \times \langle a + 2d \rangle.$$  

Then $d$ is divisible by 3, since otherwise one of $a$, $a + d$, $a + 2d$ would be a multiple of 3.

If $a \equiv 1 \pmod{3}$ then all three generators are congruent to 1 (mod 3), and so is every element in the group they generate, which is not possible. On the other hand, if $a \equiv 2 \pmod{3}$, then each of the generators has even order (since it has even order in $\mathbb{U}_3$), and so $C_2 \times C_2 \times C_2 \leq \mathbb{U}_{3p} = C_2 \times C_{p-1}$, a contradiction. □

The next result has a similar proof; it is rather special but rules out one particular type of 3AP decomposition.

Theorem 4.2  
There is no 3AP decomposition of $\mathbb{U}_{3m}$ of the form

$$\mathbb{U}_{3m} = \langle a \rangle \times \langle a + m \rangle \times \langle a + 2m \rangle.$$  

Proof  The argument of the preceding theorem shows that $m$ is divisible by 3.

Since all the generators are congruent mod $m$, and projection from $\mathbb{U}_{3m}$ to $\mathbb{U}_m$ is onto, we see that $\mathbb{U}_m$ must be cyclic (and $a$ is a primitive root of $m$), so $m$ is of the form $p_1^t$, or $2p_1^t$ (for some odd prime $p$), or $m = 4$. So necessarily $p = 3$.

But now the order of $a$ (mod $m$) is $2 \cdot 3^{t-1}$, and the same goes for the other generators as well. Their orders (mod 3$m$) are at least as large, so we must have $(2 \cdot 3^{t-1})^3 \leq 2 \cdot 3^t$, which is impossible. □

4.2  
Products of three primes

Theorem 4.1 does not rule out 3AP decompositions of $\mathbb{U}_{3pq}$, where $p$ and $q$ are distinct primes, and indeed these do exist. In this case, a new phenomenon occurs: we can obtain new solutions from old. This works more generally for
the case where $n$ is the product of three odd primes $p, q, r$, and $\xi(n) = 4$ (so that $U_n = C_{\lambda_n} \times C_2 \times C_2$).

Suppose that the abelian group $A$ can be written (adapting our previous notation) as

$$A = \langle x \rangle_{2a} \times \langle y \rangle_2 \times \langle z \rangle_2.$$ 

Then $A$ contains an elementary abelian group $B$ of order 8 generated by $x^a, y, z$. If three elements $x', y', z'$ have the properties that their orders are $2a, 2, 2$ respectively and $\langle (x')^a, y', z' \rangle = B$, then $x', y', z'$ generate cyclic subgroups whose direct product is $A$. If $A = U_n$ for some $n$, then multiplying an arithmetic progression by a fixed unit yields an arithmetic progression; so we look for an element $u$ such that $x' = xu$, $y' = yu$, $z' = zu$ satisfy the above conditions.

We see that $u$ must have order 2, so $u \in B$. If $a$ is even then $(xu)^a = x^a$, while if $a$ is odd then $(xu)^a = x^au$. It is then easy to check that the allowable values of $u$ are as follows:

- $u \in \{x^a, yz, x^a yz\}$ if $a$ is even;
- $u \in \{x^a y, x^a z, yz\}$ if $a$ is odd.

In each case, the possible values of $u$, together with the identity, form a subgroup of $B$; so no further expressions can be obtained by repeating the procedure. Moreover, in each case, $yz$ is an allowed multiplier, and converts $[x, y, z]$ into $[xyz, z, y]$; so the solutions come in pairs, each pair consisting of the first three and the last three terms in the sequence $[x, y, z, xyz]$.

**Theorem 4.3** Suppose that $n$ is the product of three odd primes, and that $U_n = \langle x \rangle_{\lambda} \times \langle y \rangle_2 \times \langle z \rangle_2$ is a 3AP decomposition, where $\lambda = \lambda_n$. Then $U_n = \langle ux \rangle_{\lambda} \times \langle uy \rangle_2 \times \langle uz \rangle_2$ is also a 3AP decomposition, where

- $u \in \{x^{\lambda/2}, yz, x^{\lambda/2} yz\}$ if $\lambda \equiv 0 \pmod{4}$;
- $u \in \{x^{\lambda/2} y, x^{\lambda/2} z, yz\}$ if $\lambda \equiv 2 \pmod{4}$.

In each case, there are two four-term arithmetic progressions whose three-term subprogressions give the stated decompositions.

We call these sets of four decompositions quartets. Here are some examples of quartets, in cases where one of the primes dividing $n$ is 3. We list the values of $n$ and $\lambda$, and the two four-term progressions $[x, y, z, xyz]$; the orders of the terms are $\lambda, 2, 2, \lambda$, and the first and last three give 3AP decompositions.
Case \( \lambda \equiv 0 \pmod{4} \):

- 105; 12; [38, 71, 104, 32], [17, 29, 41, 53]
- 165; 20; [113, 56, 164, 107], [47, 89, 131, 8]
- 285; 36; [98, 191, 284, 92], [212, 134, 56, 263]
- 357; 48; [122, 239, 356, 116], [269, 50, 188, 326]
- 465; 60; [158, 311, 464, 152], [437, 404, 371, 338]

Case \( \lambda \equiv 2 \pmod{4} \):

- 231; 30; [80, 155, 230, 74], [179, 188, 197, 206]
- 483; 66; [164, 323, 482, 158], [95, 461, 344, 227]

In general, there is no requirement that an end-term in a quartet should be the product \( \pmod{n} \) of the other three terms. A counter-example is the following, where the subscript integers are the orders of the terms:

- 315; 12; [8, 4, 131, 6, 254, 6, 62, 4, 62]

4.3 Some results for \( n = pq \) \( (p > 3, q > 3) \)

We now indicate how the role of the value \(-3 \pmod{n}\), as discussed in \( \text{[2]} \) above, carries over to composite values of \( n \).

**Theorem 4.4** Let \( p \) and \( q \) be primes greater than 3, with \( p \equiv 3 \pmod{4} \), and suppose that \( \text{ord}_p(-3) = (p - 1)/2 \) and \( \text{ord}_q(-3) = q - 1 \). (This implies that \( p \equiv 1 \pmod{3} \) and \( q \equiv 2 \pmod{3} \)). Let \( n = pq \), and let \( x \) be the unique element of \( \mathbb{U}_n \) congruent to 1 \( \pmod{p} \) and to \(-3 \pmod{q} \). Then

\[
\mathbb{U}_n = \langle -x - 2 \rangle_{(p-1)/2} \times \langle -1 \rangle_2 \times \langle x \rangle_{q-1},
\]

which is a lift of \( \mathbb{U}_p = \langle -3 \rangle_{(p-1)/2} \times \langle -1 \rangle_2 \times \langle 1 \rangle_1 \).

**Proof** The congruences \( \pmod{3} \) arise by noticing that \(-3\) is a quadratic residue \( \pmod{p} \) and non-residue \( \pmod{q} \), and applying quadratic reciprocity.

We have \( \text{ord}_p(x) = 1 \) and \( \text{ord}_q(x) = q - 1 \), so \( \text{ord}_n(x) = q - 1 \). Also, \(-x - 2\) is congruent to \(-3 \pmod{p}\) and to 1 \( \pmod{q} \), so \( \text{ord}_p(-x - 2) = (p - 1)/2 \) and \( \text{ord}_q(-x - 2) = 1 \), whence \( \text{ord}_n(-x - 2) = (p - 1)/2 \).

Since \( p \equiv 3 \pmod{4} \), we have \( \mathbb{U}_p = \langle -3 \rangle_{(p-1)/2} \times \langle -1 \rangle_2 \), as the orders of the factors are co-prime. So the group \( A = \langle -x - 2 \rangle_{(p-1)/2} \times \langle -1 \rangle_2 \times \langle x \rangle_{q-1} \) projects onto \( \mathbb{U}_p \). Also, \( x \) belongs to the kernel of this projection; since \( x \) is a primitive root of \( q \), the kernel is \( \mathbb{U}_q \). So \( A = \mathbb{U}_n \). \( \square \)
Coverage In the range \( n < 300 \), the coverage of Theorem 4.4 is as follows:

\[
\begin{align*}
35 &= 7 \times 5 : \quad U_{35} = \langle 11 \rangle_3 \times \langle 34 \rangle_2 \times \langle 22 \rangle_4 \\
77 &= 7 \times 11 : \quad U_{77} = \langle 67 \rangle_3 \times \langle 76 \rangle_2 \times \langle 8 \rangle_{10} \\
95 &= 19 \times 5 : \quad U_{95} = \langle 16 \rangle_9 \times \langle 94 \rangle_2 \times \langle 77 \rangle_4 \\
119 &= 7 \times 17 : \quad U_{119} = \langle 18 \rangle_3 \times \langle 118 \rangle_2 \times \langle 99 \rangle_{16} \\
155 &= 31 \times 5 : \quad U_{155} = \langle 121 \rangle_{15} \times \langle 154 \rangle_2 \times \langle 32 \rangle_4 \\
161 &= 7 \times 23 : \quad U_{161} = \langle 116 \rangle_3 \times \langle 160 \rangle_2 \times \langle 43 \rangle_{22} \\
203 &= 7 \times 29 : \quad U_{203} = \langle 88 \rangle_3 \times \langle 202 \rangle_2 \times \langle 113 \rangle_{28} \\
209 &= 19 \times 11 : \quad U_{209} = \langle 111 \rangle_9 \times \langle 208 \rangle_2 \times \langle 96 \rangle_{10} \\
215 &= 43 \times 5 : \quad U_{215} = \langle 126 \rangle_{21} \times \langle 214 \rangle_2 \times \langle 87 \rangle_4
\end{align*}
\]

The case \( 287 = 7 \times 41 \) fails, since \( \text{ord}_{41}(-3) = 8 \). In the range \( q < 300 \), the value \( q = 41 \) is the only prime \( q \) with \( q \equiv 2 \pmod{3} \) and \( \text{ord}_q(-3) \neq q - 1 \). However, in the range \( p < 300 \), there are four primes \( p \) with \( p \equiv 7 \pmod{12} \) and \( \text{ord}_p(-3) \neq (p - 1)/2 \), namely \( p = 67, 103, 151 \) and \( 271 \).

As is hinted in §8 of [3], many special cases arise when we come to consider composite values \( n = pq \) where \( p \) and \( q \) are distinct primes satisfying \( p \equiv q \equiv 1 \pmod{6} \), with \( \gcd(p - 1, q - 1) = 6 \). Accordingly, we do not offer theorems to cover these cases. Instead, for the range \( n < 1000 \), we use Table 1 to list the instances in which we have

\[
U_n = (2x + 3)m \times \langle x + 1 \rangle_3 \times \langle -1 \rangle_2 = \langle -2x - 3 \rangle_m \times \langle -x - 2 \rangle_3 \times \langle -1 \rangle_2
\]

where \( m = \phi(n)/6 \). The following values of \( n \) are not covered: \( 259 = 7 \times 37, 427 = 7 \times 61, 511 = 7 \times 73 \) and \( 973 = 7 \times 139 \).

Now let \( n = pq \) where \( p \) and \( q \) are distinct primes satisfying \( p \equiv q \equiv 5 \pmod{8} \), \( q > 5 \) and \( \gcd(p - 1, q - 1) = 4 \). For the range \( n < 1000 \), Table 2 lists 3AP decompositions of \( U_n \) that are lifts from weak 3AP decompositions of \( U_q \); an asterisk marks a generator lifted from 1 (mod \( q \)). Where the weak 3AP decomposition has a generator of order 4, we classify the lifted 3AP decompositions into three types: if the generator lifted from 1 can be placed first, we have type A when the order of the middle generator is 4, and type C when the order of the last generator is 4, whereas type B has the generator lifted from 1 in the middle.

### 4.4 Lifts from \( n = kp \) to \( n = kp^2 \)

For the range \( n < 1000 \), details of the 3AP decompositions (3APDs) for values of the form \( n = kp^2 \) (\( k \) and \( p \) distinct odd primes, \( k > 3, p > 3 \)) are as in Table 3. With the given restrictions on \( k \) and \( p \), just one value of the form \( n = kp^3 \) lies in the range \( n < 1000 \), namely \( n = 875 \), and it has precisely six 3AP decompositions. Each of these is obtained by further lifting one of the 3AP decompositions for \( n = 175 \). In this further lifting, the orders that are
TABLE 1

Decompositions (9) for $\mathbb{U}_n$ where $n = pq$ as specified in the text

| $n$  | Decomposition |
|------|----------------|
| 91   | $\mathbb{U}_{91} = (33)_2 \times (16)_3 \times (90)_2 = (58)_2 \times (74)_3 \times (90)_2$ |
| 133  | $\mathbb{U}_{133} = (61)_18 \times (30)_3 \times (132)_2 = (72)_18 \times (102)_3 \times (132)_2$ |
| 217  | $\mathbb{U}_{217} = (135)_30 \times (67)_3 \times (216)_2 = (82)_30 \times (149)_3 \times (216)_2$ |
| 247  | $\mathbb{U}_{247} = (137)_36 \times (68)_3 \times (246)_2 = (110)_36 \times (178)_3 \times (246)_2$ |
|      | $= (175)_36 \times (87)_3 \times (246)_2 = (72)_36 \times (159)_3 \times (246)_2$ |
| 301  | $\mathbb{U}_{301} = (271)_42 \times (135)_3 \times (300)_2 = (30)_42 \times (165)_3 \times (300)_2$ |
| 403  | $\mathbb{U}_{403} = (228)_60 \times (315)_3 \times (402)_2 = (175)_60 \times (87)_3 \times (402)_2$ |
| 469  | $\mathbb{U}_{469} = (142)_66 \times (305)_3 \times (468)_2 = (327)_66 \times (163)_3 \times (468)_2$ |
| 553  | $\mathbb{U}_{553} = (205)_78 \times (102)_3 \times (552)_2 = (348)_78 \times (450)_3 \times (552)_2$ |
| 559  | $\mathbb{U}_{559} = (202)_84 \times (380)_3 \times (558)_2 = (357)_84 \times (178)_3 \times (558)_2$ |
| 589  | $\mathbb{U}_{589} = (547)_90 \times (273)_3 \times (588)_2 = (42)_90 \times (315)_3 \times (588)_2$ |
| 679  | $\mathbb{U}_{679} = (26)_96 \times (352)_3 \times (678)_2 = (653)_96 \times (326)_3 \times (678)_2$ |
| 721  | $\mathbb{U}_{721} = (422)_102 \times (571)_3 \times (720)_2 = (299)_102 \times (149)_3 \times (720)_2$ |
| 763  | $\mathbb{U}_{763} = (236)_108 \times (499)_3 \times (762)_2 = (527)_108 \times (263)_3 \times (762)_2$ |
|      | $= (345)_108 \times (172)_3 \times (762)_2 = (418)_108 \times (590)_3 \times (762)_2$ |
| 817  | $\mathbb{U}_{817} = (357)_126 \times (178)_3 \times (816)_2 = (460)_126 \times (638)_3 \times (816)_2$ |
| 871  | $\mathbb{U}_{871} = (59)_132 \times (29)_3 \times (870)_2 = (812)_132 \times (841)_3 \times (870)_2$ |
|      | $= (410)_132 \times (640)_3 \times (870)_2 = (461)_132 \times (230)_3 \times (870)_2$ |
| 889  | $\mathbb{U}_{889} = (674)_126 \times (781)_3 \times (888)_2 = (215)_126 \times (107)_3 \times (888)_2$ |
### TABLE 2
Some lifts from weak 3AP decompositions of $\mathbb{U}_q$, as specified at the end of §4.3

| $n = p \times q$ | 3AP decomposition of $\mathbb{U}_n$ | Type |
|------------------|-------------------------------------|------|
| 65 = 5 × 13     | (27*)₄ × (44)₄ × (61)₃             | A    |
|                  | (53*)₄ × (57)₄ × (61)₃             | A    |
|                  | (53*)₄ × (16)₃ × (44)₄             | C    |
| 145 = 5 × 29     | (88*)₄ × (12)₄ × (81)₇             | A    |
|                  | (117*)₄ × (99)₄ × (81)₇             | A    |
| 185 = 5 × 37     | (43)₄ × (112*)₄ × (181)₉           | B    |
|                  | (38*)₄ × (16)₃ × (179)₄             | C    |
| 265 = 5 × 53     | (213*)₄ × (201)₁₃ × (189)₄         | C    |
| 305 = 5 × 61     | (123*)₄ × (56)₁₅ × (294)₄          | C    |
|                  | (62*)₄ × (24)²₀ × (291)₃           | −    |
|                  | (27)₁₂ × (62*)₄ × (156)₃           | −    |
| 377 = 13 × 29    | (262*)₁₂ × (99)₄ × (313)₇          | A    |
| 377 = 29 × 13    | (287*)₂₈ × (57)₄ × (203)₃          | A    |
|                  | (14*)₂₈ × (146)₃ × (278)₄          | C    |
|                  | (222*)₂₈ × (146)₃ × (70)₄          | C    |
|                  | (235*)₂₈ × (146)₃ × (57)₄          | C    |
| 505 = 5 × 101    | (102*)₄ × (394)₄ × (181)₂₅         | A    |
|                  | (102*)₄ × (414)₄ × (221)₂₅         | A    |
|                  | (203*)₄ × (192)₄ × (181)₂₅         | A    |
|                  | (203*)₄ × (212)₄ × (221)₂₅         | A    |
|                  | (203*)₄ × (56)₂₅ × (414)₄          | C    |
| 545 = 5 × 109    | (33)₄ × (437*)₄ × (296)₂₇          | B    |
|                  | (403)₄ × (437*)₄ × (471)₂₇         | B    |
| 689 = 13 × 53    | (319*)₁₂ × (625)₁₃ × (242)₄         | C    |
| 689 = 53 × 13    | (209*)₅₂ × (317)₄ × (425)₃         | A    |
|                  | (469*)₅₂ × (447)₄ × (425)₃         | A    |
|                  | (456*)₅₂ × (107)₃ × (447)₄         | C    |
|                  | (586*)₅₂ × (107)₃ × (317)₄         | C    |
| 745 = 5 × 149    | (193)₄ × (597)₄ × (256)₃₇          | B    |
|                  | (403)₄ × (597)₄ × (46)₃₇           | B    |
| 785 = 5 × 157    | (158*)₄ × (757)₄ × (571)₃₉         | A    |
|                  | (472*)₄ × (129)₄ × (571)₃₉         | A    |
|                  | (443)₄ × (472*)₄ × (501)₃₉         | B    |
|                  | (158*)₄ × (207)₁₂ × (256)₁₃        | −    |
|                  | (158*)₄ × (326)₃ × (494)₅₂         | −    |
| 865 = 5 × 173    | (693)₄ × (566)₄₃ × (439)₄          | −    |
| 905 = 5 × 181    | (363)₄ × (316)₅ × (269)₃₆         | −    |
| 985 = 5 × 197    | (183)₄ × (592*)₄ × (16)₄₉         | B    |
not multiples of 5 are unchanged, but the orders that are multiples of 5 become multiples of 5$^2$.

As Table 3 indicates, some of the 3AP decompositions for $n = 275$ and $775$ are not lifts, these two $n$-values being distinctive in that they have $p \mid (k - 1)$. How do these exceptional 3AP decompositions arise? One of them is given by

$$U_{275} = \langle 16 \rangle_5 \times \langle 24 \rangle_{10} \times \langle 32 \rangle_4.$$  

This is related to the decomposition

$$U_{55} = \langle 24 \rangle_{10} \times \langle 32 \rangle_4$$

and to the fact that, within $U_{55}$, we have $\langle 16 \rangle_5 \subset \langle 24 \rangle_{10}$. The further 3AP decomposition

$$U_{275} = \langle 181 \rangle_5 \times \langle 244 \rangle_{10} \times \langle 32 \rangle_4$$

arises in the same way, as $181$ is a lift of $16$, and $244$ is a lift of $24$.

Less straightforward situations exist too. Consider, for example, the 3AP decomposition

$$U_{775} = \langle 32 \rangle_4 \times \langle 54 \rangle_{10} \times \langle 76 \rangle_{15}$$

with $n = 31 \cdot 5^2$. If we try lifting to this from $n = 31 \cdot 5 = 155$, we find that $32$, $54$ and $76$ also have orders $4$, $10$ and $15$ (mod $155$), and that

$$U_{155} = \langle 32 \rangle_4 \times \langle 54 \rangle_{10} \times \langle 76 \rangle_{15}$$

Analogous to this, we can rewrite an example from the previous paragraph in the weak form

$$U_{55} = \langle 16 \rangle_5 \times \langle 24 \rangle_{10} \times \langle 32 \rangle_4.$$  

These examples suggest an amusing generalisation of 3AP decompositions to decompositions of the form

$$U_n = \langle x^h \rangle \times \langle (x + k)^i \rangle \times \langle (x + 2k)^j \rangle$$

where $h, i, j \geq 1$, but we do not pursue this idea further here.

5 A class of weak 3AP decompositions

A noteworthy class of weak 3AP decompositions arises for primes $n$ satisfying $n \equiv 1 \pmod{6p}$ where $p$ is an odd prime, $p > 3$. In each of these decompositions, one of the generators has order 6 and another has order $p$. For $n < 300$, such decompositions are as follows:

\begin{align*}
n = 43 & \quad (1)_1 \times (4)_7 \times (7)_6 \\
n = 67 & \quad (1)_1 \times (30)_6 \times (59)_{11} \\
n = 79 & \quad (1)_1 \times (52)_{13} \times (24)_6
\end{align*}
TABLE 3

An enumeration of decompositions for $n = kp^2$

| $n = kp^2$ | # 3APDs | # lifts from $U_{kp}$ | # other |
|------------|---------|-----------------------|---------|
|            |         | from strong 3APDs     | from weak 3APDs |
| $175 = 7 \cdot 5^2$ | 6       | 3                     | 3       | 0       |
| $245 = 5 \cdot 7^2$  | 6       | 3                     | 3       | 0       |
| $275 = 11 \cdot 5^2$ | 68      | 0                     | 8       | 60      |
| $325 = 13 \cdot 5^2$ | 20      | 12                    | 8       | 0       |
| $425 = 17 \cdot 5^2$ | 8       | 0                     | 8       | 0       |
| $475 = 19 \cdot 5^2$ | 6       | 3                     | 3       | 0       |
| $539 = 11 \cdot 7^2$ | 12      | 9                     | 3       | 0       |
| $575 = 23 \cdot 5^2$ | 2       | 0                     | 2       | 0       |
| $605 = 5 \cdot 11^2$ | 0       | 0                     | 0       | 0       |
| $637 = 13 \cdot 7^2$ | 126     | 108                   | 18      | 0       |
| $725 = 29 \cdot 5^2$ | 30      | 18                    | 12      | 0       |
| $775 = 31 \cdot 5^2$ | 188     | 32                    | 24      | 132     |
| $845 = 5 \cdot 13^2$ | 20      | 12                    | 8       | 0       |
| $847 = 7 \cdot 11^2$ | 0       | 0*                    | 0       | 0       |
| $925 = 37 \cdot 5^2$ | 10      | 6                     | 4       | 0       |
| $931 = 19 \cdot 7^2$ | 182     | 156                   | 26      | 0       |

* 3APDs of $U_{kp}$ exist, but the special lifts of the generators are in AP
\[ n = 103 \quad (1)_1 \times (47)_6 \times (93)_{17} \]
\[ n = 139 \quad (97)_6 \times (1)_1 \times (44)_{23} \]
\[ n = 223 \quad \begin{cases} (1)_1 \times (132)_{37} \times (40)_6 \\ (184)_6 \times (1)_1 \times (41)_{37} \end{cases} \]
\[ n = 283 \quad (45)_6 \times (1)_1 \times (240)_{47} \]

Where \( U_n = \langle a \rangle_6 \times \langle 1 \rangle_1 \times \langle c \rangle_p \), we have \( a \equiv (c - 1)^{-1} \pmod{n} \), so that \( U_n = \langle c - 1 \rangle_6 \times \langle c \rangle_p \), a situation discussed in [3, §8.2]. The above weak 3AP decomposition for \( n = 67 \) has the orders of the generators in AP.

### 6 Finite fields

Finite fields of non-prime order can have 3AP decompositions. Clearly this is impossible in fields of characteristic 2, which contain no 3-term arithmetic progressions.

**Example** Using GAP [5], we found the following 3AP decompositions of small finite fields GF\((q)\). In this list, \( \zeta \) denotes the primitive root (denoted by \( Z(q) \) in GAP notation) in the field GF\((q)\). It is a root of the appropriate Conway polynomial [7]; the relevant Conway polynomials are as follows:

\[
\begin{align*}
q = 11^2 : & \quad x^2 + 7x + 2 \\
q = 11^3 : & \quad x^3 + 2x + 9 \\
q = 19^2 : & \quad x^2 - x + 2 \\
q = 19^3 : & \quad x^3 + 4x - 2 \\
q = 23^2 : & \quad x^2 - 2x + 5 \\
q = 29^2 : & \quad x^2 - 5x + 2
\end{align*}
\]

We give only one decomposition for each possible list of orders of the factors:

\[
\begin{align*}
\text{GF}(11^2)^\times & = \langle \zeta^{72} \rangle_5 \times \langle \zeta^{15} \rangle_8 \times \langle \zeta^{80} \rangle_3 \\
\text{GF}(11^3)^\times & = \langle \zeta^{570} \rangle_7 \times \langle \zeta^{532} \rangle_5 \times \langle \zeta^{595} \rangle_{38} \\
& = \langle \zeta^{665} \rangle_2 \times \langle \zeta^{1008} \rangle_{95} \times \langle \zeta^{570} \rangle_7 \\
\text{GF}(19^2)^\times & = \langle \zeta^{144} \rangle_5 \times \langle \zeta^{320} \rangle_9 \times \langle \zeta^{135} \rangle_8 \\
\text{GF}(19^3)^\times & = \langle \zeta^{3429} \rangle_2 \times \langle \zeta^{2970} \rangle_{127} \times \langle \zeta^{5588} \rangle_{27} \\
\text{GF}(23^2)^\times & = \langle \zeta^{176} \rangle_3 \times \langle \zeta^{192} \rangle_{11} \times \langle \zeta^{129} \rangle_{16} \\
\text{GF}(29^2)^\times & = \langle \zeta^{280} \rangle_3 \times \langle \zeta^{720} \rangle_7 \times \langle \zeta^{609} \rangle_{40} \\
& = \langle \zeta^{120} \rangle_7 \times \langle \zeta^{504} \rangle_5 \times \langle \zeta^{385} \rangle_{24}
\end{align*}
\]

Can we have a 3AP decomposition of GF\((q)^\times\) in which two of the generators have orders 2 and 3? As earlier, such a decomposition requires that \((q - 1)/6\)
is co-prime to 6, so that $q \equiv 7$ or $31 \pmod{36}$. But this implies that, if $q = p^n$ with $p$ prime, then $p \equiv 7$ or $31 \pmod{36}$ (since 7 and 31 are primitive $\lambda$-roots of 36 [3], and each is the fifth power of the other). Then elements of orders 2 and 3 lie in the prime subfield, and hence so does the whole AP. So there are no such decompositions other than those of $U_n$ for $n$ prime discussed in [2].

A similar argument shows that a 3AP decomposition of $\mathbb{GF}(11^3)^\times$ into factors of orders 2, 5 and 133 is impossible.

7 Decompositions with more than three factors

As indicated above, we can define 4AP decompositions analogously to 3AP decompositions.

A computer program has shown that no examples of strong 4AP decompositions of $U_n$ exist for prime values of $n$ up to 10000. The smallest composite $n$ for which strong 4AP decompositions of $U_n$ exist is even:

$$U_{104} = \langle 31 \rangle_4 \times \langle 81 \rangle_3 \times \langle 27 \rangle_2 \times \langle 77 \rangle_2$$

$$= \langle 77 \rangle_2 \times \langle 79 \rangle_2 \times \langle 81 \rangle_3 \times \langle 83 \rangle_4.$$ 

The smallest weak 4AP decomposition of $U_n$ with prime $n$ is

$$U_{3613} = \langle 3528 \rangle_4 \times \langle 1148 \rangle_{129} \times \langle 2381 \rangle_7 \times \langle 1 \rangle_1.$$ 

We have no examples with larger numbers of generators that are in AP.

Note The computations reported in this paper were performed using GAP [4], and a package of GAP functions written by the first author for computations in the groups $U_n$, available from [2]. Further documentation of these functions can be found in [3].

References

[1] I. Anderson and D. A. Preece, Obtaining all or half of $U_n$ as $\langle x \rangle \times \langle x + 1 \rangle$. Submitted to ** ** **.

[2] P. J. Cameron, Functions for primitive lambda-roots, available from http://www.maths.qmul.ac.uk/~pjc/csgnotes/PLRfns.txt

[3] P. J. Cameron and D. A. Preece, Notes on Primitive Lambda-roots, http://www.maths.qmul.ac.uk/~pjc/csgnotes/lambda.pdf

[4] N. G. Chebotarev, Determination of the density of the set of prime numbers belonging to a given substitution class [in Russian], Izv. Ross. Akad. Nauk 17 (1923) 205–250.
[5] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4.12; 2008, \(\texttt{http://www.gap-system.org}\).

[6] G. A. Jones and J. M. Jones, *Elementary Number Theory*, London: Springer (1998).

[7] W. Nickel, *Endliche Körper in dem gruppentheoretischen Programmsystem GAP*, Diploma thesis, RWTH Aachen (1988).

[8] Jean-Pierre Serre, *Abelian l-adic representations and elliptic curves* (Revised reprint of the 1968 original ed.), A K Peters, Wellesley, MA, 1998.