FORMS OF DIFFERENTIAL LIE ALGEBRAS OVER $\mathbb{C}(t)$
ASSOCIATED WITH
COMPLEX SIMPLE LIE ALGEBRAS

AKIRA MASUOKA AND YUTA SHIMADA

Abstract. Discussed here is descent theory in the differential context that everything is equipped with a differential operator. To answer a question personally posed by Prof. Pianzola, we determine all forms of the differential Lie algebras over $\mathbb{C}(t)$ associated with complex simple Lie algebras. Hopf-Galois Theory, a ring-theoretic counterpart of theory of torsors for group schemes, plays a role to grasp the forms mentioned above from torsors.

Key Words: differential Lie algebra, Hopf algebra, affine group scheme, descent theory, Amitsur cohomology

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1. Introduction—Problem and Answer

Throughout in this paper, rings, algebras and Hopf algebras are assumed to be commutative, unless otherwise stated.

We let $\delta$ mean “differential” and use the symbol $\delta$ to indicate differential operators in general. A $\delta$-ring is thus a (commutative) ring $R$ equipped with an additive operator $\delta : R \to R$ satisfying the Leibniz rule $\delta(xy) = (\delta x)y + x(\delta y)$, $x, y \in R$. It is called a $\delta$-field if the ring is a field. The rational function field $\mathbb{C}(t)$ in one variable is regarded as a $\delta$-field with respect to the standard operator $\delta t = 1$, $\delta c = 0$, $c \in \mathbb{C}$. A $\delta$-$\mathbb{C}(t)$-Lie algebra $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}(t)$ which is equipped with an additive operator $\delta : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta(aX) = (\delta a)X + a(\delta X), \quad [\delta X, Y] = [\delta X, Y] + [X, \delta Y]$$

where $a \in K$ and $X, Y \in \mathfrak{g}$. In the same way a $\delta$-$R$-Lie algebra is defined for any $\delta$-ring $R$. Let $\mathfrak{g}$ be a $\delta$-$\mathbb{C}(t)$-Lie algebra. Given a $\delta$-ring map $\mathbb{C}(t) \to R$ (that is, a ring map preserving the $\delta$-operator), the base extension $\mathfrak{g} \otimes_{\mathbb{C}(t)} R$ of $\mathfrak{g}$ is naturally a $\delta$-$R$-Lie algebra. A form of $\mathfrak{g}$ is a $\delta$-$\mathbb{C}(t)$-Lie algebra $\mathfrak{f}$ such that

$$\mathfrak{g} \otimes_{\mathbb{C}(t)} R \simeq \mathfrak{f} \otimes_{\mathbb{C}(t)} R \text{ as } \delta$-R-Lie algebras$$

for some $\mathbb{C}(t) \to R \neq 0$. In this case we say that $\mathfrak{f}$ splits by $R$.

Let $n \geq 2$. We can and do regard the $\mathbb{C}(t)$-Lie algebra $\mathfrak{sl}_n(\mathbb{C}(t))$ which consists of all traceless matrices $X = (x_{ij})$ with $x_{ij} \in \mathbb{C}(t)$, as a $\delta$-$\mathbb{C}(t)$-Lie algebra with respect to the entry-wise $\delta$-operator $\delta(x_{ij}) := (\delta x_{ij})$. Professor Pianzola posed the following problem personally to the first-named author.
Problem 1.1. Describe all forms of the $\delta$-$\mathcal{C}(t)$-Lie algebra $\mathfrak{sl}_n(\mathbb{C}(t))$.

Apparently, to generalize the problem, one can replace $\mathfrak{sl}_n(\mathbb{C})$ with a complex simple Lie algebra $\mathfrak{g}_0$ (of finite dimension), and regard $\mathbb{C}(t)$-Lie algebra
\begin{equation}
\mathfrak{g}_0(\mathbb{C}(t)) = \mathfrak{g}_0 \otimes_{\mathbb{C}} \mathbb{C}(t)
\end{equation}
as a $\delta$-$\mathcal{C}(t)$-Lie algebra with the $\delta$ operating on the tensor factor $\mathbb{C}(t)$. The notation \[\text{(1.1)}\] is used since $\mathfrak{g}_0$ is seen to give the functor $R \mapsto \mathfrak{g}_0 \otimes_{\mathbb{C}} R$, and $\mathfrak{g}_0 \otimes_{\mathbb{C}} \mathbb{C}(t)$ is then its value. We aim at solving the following generalized problem.

Problem 1.2. Given a complex simple Lie algebra $\mathfrak{g}_0$, describe all forms of $\mathfrak{g}_0(\mathbb{C}(t))$.

Recall that the complex simple Lie algebras over $\mathbb{C}$ are classified, labeled as $A_\ell$ ($\ell \geq 1$), $B_\ell$ ($\ell \geq 2$), $C_\ell$ ($\ell \geq 3$), $D_\ell$ ($\ell \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. Let $\mathfrak{g}_0$ be a complex simple Lie algebra, and let $\Gamma$ denote the automorphism group of the associated Dynkin diagram. Explicitly, the group is given by
\begin{equation}
\Gamma = \begin{cases}
\{1\} & \text{type } A_2, B_\ell \ (\ell \geq 2), \ C_\ell \ (\ell \geq 3), \ E_7, \ E_8, \ F_4 \text{ or } G_2; \\
\mathbb{Z}_2 & \text{type } A_\ell \ (\ell \geq 2), \ D_\ell \ (\ell \geq 5) \text{ or } E_6; \\
\mathfrak{S}_3 & \text{type } D_4
\end{cases}
\end{equation}
according to the type of $\mathfrak{g}_0$; see [9, Table 3 on Page 298]. Here and in what follows $\mathbb{Z}_n$ denotes the cyclic group of order $n$. The action by $\Gamma$ naturally gives rise to automorphisms of $\mathfrak{g}_0$, which forms a group naturally identified with the outer-automorphism group $\text{Out}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$.

Roughly speaking, our answer, Theorem [13], to the problem tells that all non-trivial forms are obtained by the Galois Descent for which $\Gamma$ (and its subgroups for type $D_4$) act as Galois group(s). To make a precise statement we introduce below the notion of being quasi-isomorphic.

Lemma 1.3. If $\mathfrak{g} = (\mathfrak{g}, \delta)$ is a $\delta$-$R$-Lie algebra, then for any element $D \in \mathfrak{g}$,
\[\delta + \text{ad} \ D : \mathfrak{g} \to \mathfrak{g}, \ X \mapsto \delta X + [D, X]\]
is a $\delta$-operator with which $\mathfrak{g}$ is again a $\delta$-$R$-Lie algebra.

Indeed, one sees, more generally, that for any $R$-linear derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$, $(\mathfrak{g}, \delta + \mathcal{D})$ is a $\delta$-$R$-Lie algebra. Note that the inner derivation $\text{ad} \ D$ above is $R$-linear.

Definition 1.4. Let $R$ be as above. We say that two $\delta$-$R$-Lie algebras $\mathfrak{g}_i = (\mathfrak{g}_i, \delta_i), \ i = 1, 2$, are quasi-isomorphic, if there is an element $D_1 \in \mathfrak{g}_1$ such that
\[(\mathfrak{g}_1, \delta_1 + \text{ad} \ D_1) \simeq (\mathfrak{g}_2, \delta_2) \text{ as } \delta$-$R$-Lie algebras.

The condition is equivalent to saying that there is an element $D_2 \in \mathfrak{g}_2$ such that $(\mathfrak{g}_1, \delta_1) \simeq (\mathfrak{g}_2, \delta_2 + \text{ad} \ D_2)$, as is easily seen. It follows that the quasi-isomorphism gives an equivalence relation among all $\delta$-$R$-Lie algebras.

Theorem 1.5. A $\delta$-$\mathcal{C}(t)$-Lie algebra is a form of $\mathfrak{g}_0(\mathbb{C}(t))$ if and only if it is quasi-isomorphic to one of those listed below, according to the case $\Gamma = \{1\}, \mathbb{Z}_2$ or $\mathfrak{S}_3$; see [1,2].
(1) Case $\Gamma = \{1\}$. $g_0(\mathbb{C}(t))$.
(2) Case $\Gamma = \mathbb{Z}_2$. (i) $g_0(\mathbb{C}(t))$;
    (ii) $g_0(L)^T$, where $L/\mathbb{C}(t)$ is a quadratic field extension.
(3) Case $\Gamma = S_3$. (i) $g_0(\mathbb{C}(t))$;
    (ii) $g_0(L)^{S_2}$, where $L/\mathbb{C}(t)$ is a quadratic field extension;
    (iii) $g_0(L)^{S_3}$, where $L/\mathbb{C}(t)$ is a cubic Galois extension;
    (iv) $g_0(L)^F$, where $L/\mathbb{C}(t)$ is a Galois extension of fields with Galois
    group $\Gamma (= S_3)$.

We should immediately add some explanations about the statement above.
First, any finite field extension $L/\mathbb{C}(t)$ uniquely turns into an extension
of $\delta$-fields, whence $g_0(L)$ turns into a $\delta$-$L$-Lie algebra. Second, in (ii) of
(2) and (iv) of (3) above, the group $\Gamma$ is supposed to act diagonally on
$g_0(L) = g_0 \otimes_{\mathbb{C}} L$, as outer-automorphisms on $g_0$, and as the Galois
group on $L$. In addition, $g_0(L)^T$ denotes the $\Gamma$-invariants in $g_0(L)$, which is in fact a
$\delta$-$\mathbb{C}(t)$-Lie algebra by Galois Descent; see Section 4.3. Third, in (ii) of (3),
we choose arbitrarily an order 2 subgroup $\mathbb{Z}_2$ of $\Gamma (= S_3)$, and let it act on
$g_0$ by restriction. The $\delta$-$\mathbb{C}(t)$-Lie algebra $g_0(L)^{S_2}$ which results in the same
way as above does not depend (up to isomorphism) on the choice since the
order 2 subgroups are conjugate to each other; on the other hand it may
depend on $L$. Finally, in (iii) of (3), we suppose that $\mathbb{Z}_3$ is the unique order
3 subgroup of $\Gamma (= S_3)$, and let it act on $g_0$ by restriction, again.

The theorem will be proved in the final Section 4 which contains explicit
descriptions (see Section 4.0 of the non-trivial $\delta$-$\mathbb{C}(t)$-Lie algebras listed in
(ii) of (2) and (ii)-(iv) of (3). The preceding two sections provide preliminaries,
some of which are beyond what will be needed, but are of interest
by themselves. Section 2 presents descent theory in the differential context;
Section 3 prepares technical tools mainly from Hopf-Galois Theory, which
is a ring-theoretic counterpart of theory of torsors for group schemes. In
particular, bi-Galois Theory [10] will play a role in two stages (see Sections
3.5 and 4.3), when we grasp the forms in question from $\delta$-torsors.

2. DESCENT THEORY IN DIFFERENTIAL CONTEXT

2.1. $\delta$-$R$-Objects. Let $R$ be a $\delta$-ring. A $\delta$-$R$-module is an $R$-module $M$
equipped with an additive operator $\delta : M \to M$ satisfying $\delta(xm) = (\delta x)m\ + x(\delta m)$, $x \in R$, $m \in M$. All $\delta$-$R$-modules form a symmetric tensor category
$\delta$-$R$-Modules with respect to the tensor product $M_1 \otimes_R M_2$, the unit object
$R$ and the obvious symmetry $M_1 \otimes_R M_2 \to M_2 \otimes_R M_1$, $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$;
the $\delta$-operator on $M_1 \otimes_R M_2$ is given by $\delta(m_1 \otimes m_2) = \delta m_1 \otimes m_2 + m_1 \otimes \delta m_2$.

The $\delta$-$R$-Lie algebra defined in the previous section is precisely the Lie
algebra in the category $\delta$-$R$-Modules. In general, any linear object, such
as algebra or Hopf algebra, in $\delta$-$R$-Modules is called a $\delta$-$R$-object, so as $\delta$-$R$-algebra or $\delta$-$R$-Hopf algebra; important is the fact that the structure is
defined by morphisms of $\delta$-$R$-Modules between tensor powers of the object.
Given a $\delta$-$R$-algebra $S$, we have the base-extension functor

$$\otimes_R S : \delta$-$R$-Modules \to \delta$-$S$-Modules,$$

which induces base-extension functors for linear objects such as above.
We are concerned with descent theory (see [13], for example) in differential context. To make this clearer, let us fix a \( \delta \)-\( R \)-object \( A \). A \( \delta \)-\( R \)-object \( B \) is called a form of \( A \), if there exists a \( \delta \)-\( R \)-algebra \( S \) such that (i) \( S \) is faithfully flat as an \( R \)-algebra, and (ii) \( A \otimes R S \simeq B \otimes R S \) as \( \delta \)-\( S \)-objects. In this case \( B \) is said to be split by \( S \). The \( \delta \)-automorphism-group functor of \( A \) is the functor

\[
\text{Aut}_\delta(A) : \text{\( \delta \)-\( R \)-Algebras} \to \text{Groups}, \ T \mapsto \text{Aut}_{\delta,T}(A \otimes_R T)
\]

(2.1) from the category \( \delta \)-\( R \)-Algebras of \( \delta \)-\( R \)-algebras to the category Groups of groups, which associates to each \( \delta \)-\( R \)-algebra \( T \) the automorphism group \( \text{Aut}_{\delta,T}(A \otimes_R T) \) of the \( \delta \)-\( T \)-object \( A \otimes_R T \). When constructing the 1st Amitsur cohomology (pointed) set as in [13] Section 17.6], replace faithfully flat homomorphisms of rings and automorphism-group functors with our \( \text{Aut}_{\delta} \) groups, which associates to each \( \delta \) from the category \( \text{\( \mathcal{A} \)ffine schemes} \) a functor which is representable, and turns out, indeed, to be an automorphism-group functor.

We remark that for any functor \( G : \text{\( \delta \)-\( R \)-Algebras} \to \text{Groups} \), the cohomology set \( H^1(S/R, G) \) is defined just as the one in (2.2). The set will appear in what follows (see (3.6)) only when \( G \) is representable, and turns out, indeed, to be an automorphism-group functor.

Remark 2.1. We have used so far the base-on-right notation \( A \otimes_R T \) which denotes the extended base on the right, which seemingly looks nicer than the base-on-left notation \( T \otimes_R A \). In fact, we may and do (when it is natural) use the latter notation.

3. AFFINE \( \delta \)-\( K \)-GROUPS, THEIR LIE ALGEBRAS AND TORSORS

In this section \( K \) denotes a \( \delta \)-field. We assume that the characteristic char \( K \) of \( K \) is zero.

3.1. AFFINE \( \delta \)-\( K \)-GROUPS AND THEIR LIE ALGEBRAS. An affine \( \delta \)-\( K \)-group scheme \( G \) is by definition a representable functor \( G : \text{\( \delta \)-\( K \)-Algebras} \to \text{Groups} \) (see (2.1)); this will be called an affine \( \delta \)-\( K \)-group for short. Such a functor \( G \) is uniquely represented by a \( \delta \)-\( K \)-Hopf algebra, say \( H \), and is presented so as \( G = \text{Spec}_K(H) \) or \( \text{Spec}_{\delta,K}(H) \). We say that \( G \) is algebraic, or it is an affine algebraic \( \delta \)-\( K \)-group, if \( H \) is finitely generated as a \( K \)-algebra. If one forgets \( \delta \), then \( G = \text{Spec}(H) \) is an affine \( K \)-group, which has the Lie algebra

\[
\text{Lie}(G) = \text{Der}_K(H, K).
\]

(3.1) Recall that this consists of all \( K \)-linear maps \( D : H \to K \) that satisfy

\[ D(ab) = D(a)\epsilon(b) + \epsilon(a)D(b), \ a, b \in H, \] where \( \epsilon : H \to K \) is the counit of
$H$. This is in fact a $\delta$-$K$-Lie algebra with respect to the operator defined by

$$(\delta D)(a) := \delta(Da) - D(\delta a), \quad D \in \text{Lie}(G), \ a \in H.$$  

Note that the canonical pairing $H \otimes_K \text{Lie}(G) \to K$ is a morphism in $\delta$-$K$-\textit{Modules}. We have $\dim_K(\text{Lie}(G)) < \infty$, if $G$ is algebraic.

**Remark 3.1.** The notion of “algebraic” defined above would be rather restricted for those who would like to work intensively in differential algebra. It should be distinguished from the more natural (for those above) notion of “$\delta$-algebraic”, which will be discussed briefly in Section 3.3 being less crucial for our purpose though.

3.2. $\delta$-$K$-\textit{Torsors} and \textit{Galois} $\delta$-$K$-\textit{algebras}. An affine $\delta$-$K$-scheme is by definition a representable set-valued functor $X : \delta$-$K$-\textit{Algebras} $\to$ \textit{Sets}. It is uniquely represented by a $\delta$-$K$-algebra, say $A$, being presented so as $X = \text{Spec}_A(A)$; it is said to be algebraic if $A$ is finitely generated as a $K$-algebra. The category of affine $\delta$-$K$-schemes, whose morphisms are natural transformations, has direct products. The direct product $X_1 \times X_2$ of two affine $\delta$-$K$-schemes $X_i = \text{Spec}_A(A_i), i = 1, 2,$ is represented by $A_1 \otimes_K A_2$. The notion of group object of the category is naturally defined, and such an object is precisely an affine $\delta$-$K$-group. Given an affine $\delta$-$K$-group $G = \text{Spec}_A(H)$, the notion of right (or left) $G$-equivariant objects is defined, as well. Such an object is called a right (or left) $G$-equivariant $\delta$-$K$-scheme. Giving such a $\delta$-$K$-scheme $X = \text{Spec}_A(A)$ is the same as giving a right (or left) $H$-comodule $\delta$-$K$-algebra; it is an object $A$ in $\delta$-$K$-\textit{Algebras} equipped with a morphism $A \to A \otimes_K H$ (or $A \to H \otimes_K A$) in the category which satisfy the co-associativity and the counit property. Obviously, $G$ itself is $G$-equivariant on both sides.

Let $R$ be a $\delta$-$K$-algebra. An affine $\delta$-$K$-group or (equivariant or ordinary) $\delta$-$K$-scheme $X = \text{Spec}_A(A)$ has the base change $X_R = \text{Spec}_{\delta R}(A \otimes_K R)$; it is by definition the functor $T \mapsto X(T)$ defined on $\delta$-$R$-\textit{Algebras}, where each $T \in \delta$-$R$-\textit{Algebras} is regarded naturally as a $\delta$-$K$-algebra. We can discuss forms of $X$; it is the same as discussing forms of $A$.

Let $G = \text{Spec}_A(H)$ be an affine $\delta$-$K$-group. A form of the right $G$-equivariant $\delta$-$K$-scheme $G$ is called a right $\delta$-$K$-torsor for $G$. To be explicit it is a right $G$-equivariant $\delta$-$K$-scheme $X$ such that $X_R \simeq G_R$ as right $G$-equivariant $\delta$-$R$-schemes for some non-zero $\delta$-$K$-algebra $R$. Such an $X$ is uniquely represented by a right $H$-comodule $\delta$-$K$-algebra $B$ which is a form of $H$. Such a form $B$ is characterized as a right $H$-\textit{Galois} $\delta$-$K$-algebra [8, Section 8.1]; it is by definition a non-zero right $H$-comodule $\delta$-$K$-algebra $B$ such that the $\delta$-$K$-algebra map

$$(3.2) \quad \tilde{\rho} : B \otimes_K B \to B \otimes_K H, \ \tilde{\rho}(b \otimes c) = bc$$

is an isomorphism. Here and in what follows, $\rho : B \to B \otimes_K H$ denotes the structure map. Note that $\tilde{\rho}$ is a $\delta$-$B$-algebra isomorphism (with the base-on-left notation, see Remark 2.1), and $B$ splits by $B$ itself.

The analogous notions of left $\delta$-$K$-\textit{torsors} for $G$ and of left $H$-\textit{Galois} $\delta$-$K$-\textit{algebras} are defined in the obvious manner, and those two are in one-to-one correspondence.
3.3. Affine $\delta$-algebraic $\delta$-$K$-groups. An $\delta$-$K$-algebra $A$ is said to be $\delta$-finitely generated if it is generated as a $K$-algebra by finitely many elements $a_1, \ldots, a_n$, together with their iterated differentials $\delta^r a_1, \ldots, \delta^r a_n$, $r > 0$. An extension $L/K$ of $\delta$-fields said to be $\delta$-finitely generated, if $L$ is the quotient field of some $\delta$-$K$-finitely generated $\delta$-$K$-subalgebra of $L$.

An affine $\delta$-group $G = \text{Spec}_\delta(H)$ is said to be $\delta$-algebraic if the $\delta$-$K$-Hopf algebra $H$ is $\delta$-finitely generated as a $\delta$-$K$-algebra; see Remark 3.1. Obviously, “algebraic” implies “$\delta$-algebraic”.

Lemma 3.2. Every right (or left) $\delta$-$K$-torsor for an affine $\delta$-$K$-group $G$ splits by some $\delta$-$K$-field. It splits by a $\delta$-finitely generated extension $L/K$ of $\delta$-fields, if $G$ is $\delta$-algebraic.

Proof. Suppose that $B$ is a right $H$-Galois $\delta$-$K$-algebra, as above. Choose arbitrarily a maximal $\delta$-stable ideal $m$ of $B$, and construct $R = B/m$, a simple $\delta$-$K$-ring. Since $\text{char} K = 0$, $R$ is an integral domain by [12] Lemma 1.17. The quotient field $L = Q(R)$ of $R$ uniquely turns into a $\delta$-$K$-field. By applying $L \otimes H \to \hat{\rho}$, it follows that $B$ splits by $L$, proving the first assertion. If $H$ is $\delta$-finitely generated, then $B$ and $R$ are so. It follows that the $L/K$ above is $\delta$-finitely generated, proving the second.

Proposition 3.3. Suppose that $A$ is $\delta$-$K$-object of finite $K$-dimension. Then $\text{Aut}_\delta(A)$ is an affine $\delta$-algebraic $\delta$-$K$-group, and $A$ splits by some $\delta$-finitely generated extension $L/K$ of $\delta$-fields.

Proof. We have only to prove that $\text{Aut}_\delta(A)$ is an affine $\delta$-algebraic $\delta$-$K$-group, since the rest then follows from the preceding Lemma. Choose a $K$-basis $v_1, \ldots, v_n$ of $A$. Let

$$F = K[x_{ij}, x_{ij}', x_{ij}'' , \ldots]$$

denote the free $\delta$-$K$-algebra in indeterminates $x_{ij}$, where $1 \leq i, j \leq n$. Let

$$G = F_d ( = F[1/d])$$

denote the localization by the determinant $d = \det X$ of the $n \times n$ matrix $X = (x_{ij})_{i,j}$, which has the indeterminates above as entries. This $G$ has the $\delta$-operator uniquely extending the one $\delta x_{ij}^{(r)} = x_{ij}^{(r+1)}$, $r \geq 0$, on $F$. We have a $G$-linear bijection $\phi : A \otimes_K G \to A \otimes_K \hat{G}$ determined by

$$\phi(v_j \otimes 1) = \sum_{i=1}^n v_i \otimes x_{ij}, \quad 1 \leq j \leq n.$$ 

This last is alternatively expressed as $\phi(v_1 \otimes 1, \ldots, v_n \otimes 1) = (v_1, \ldots, v_n) \otimes X$ by matrix presentation; such presentation will be used in [3.40], [3.31] and [3.5], as well.

Let $$H = G/a,$$

where $a$ is the smallest $\delta$-stable ideal of $G$ such that the base extension $\phi_H : A \otimes_K H \to A \otimes_K H$ of $\phi$ along $G \to G/a = H$ is an endomorphism of the $\delta$-$H$-object $A \otimes_K H$; obviously, it is necessarily an automorphism. This $a$ is, in fact, given by the relations which ensure that $\phi_H$ commutes with
the structure maps of $A$ (see [13 Section 7.6]), and with the $\delta$-operator. Explicitly, the latter relation for commuting with $\delta$-operator is

$$XD = DX + \delta X,$$

where $D \in M_n(K)$ is the matrix determined by

$$\delta(v_1, \ldots, v_n) = (v_1, \ldots, v_n)D.$$

We see that $H$ represents the functor $\text{Aut}_\delta(A)$ regarded to be set-valued. In fact, for every $R \in \delta$-$K$-Algebras, we have the natural bijection

$$\text{Spec}_\delta(H)(R) \to \text{Aut}_\delta(R \otimes_K R), \; f \mapsto \text{the base extension of } \phi_H \text{ along } f.$$  

By Yoneda's Lemma, $H$ uniquely turns into a $\delta$-$K$-Hopf algebra with respect to the familiar Hopf-algebra structure

$$\Delta X = X \otimes X, \quad \epsilon X = I, \quad SX = X^{-1},$$

where $\Delta$, $\epsilon$ and $S$ denote the coproduct, the counit and the antipode, respectively, and it represents the group-valued functor $\text{Aut}_\delta(A)$. Since $H$ is obviously $\delta$-finitely generated, the desired result follows. \hfill $\square$

For $K$ as above, we choose and fix an extension $U/K$ of $\delta$-fields into which every $\delta$-finitely generated extension $L/K$ of $\delta$-fields can be embedded. There exists such an extension; a universal extension [6 Chapter III, Section 7] over $K$ is an example.

For an affine $\delta$-algebraic $\delta$-$K$-group $G$, we define $H^1_\delta(K, G)$ by

$$H^1_\delta(K, G) := H^1(U/K, G).$$

The $\delta$-automorphism-group functor $\text{Aut}_\delta(G) : T \mapsto \text{Aut}_\delta(T(G_T))$ of the right $G$-equivariant $\delta$-$K$-scheme $G$ is naturally isomorphic to $G$ itself; $\text{Aut}_\delta(T(G_T))$ consists of the natural automorphisms of the functor $G_T : \delta$-$T$-Algebras $\to$ Groups. This fact, combined with Lemma 5.2 shows that $H^1_\delta(K, G)$ classifies all right $\delta$-$K$-torsors for $G$.

For a $\delta$-$K$-object $A$ of finite $K$-dimension, we define

$$H^1_\delta(K, \text{Aut}_\delta(A)) := H^1(U/K, \text{Aut}_\delta(A)).$$

This classifies all forms of $A$, as is seen from Proposition 8.3.

3.4. $\delta$-$K$-Bi-torsors and bi-Galois $\delta$-$K$-algebras. Let $G = \text{Spec}_\delta(H)$ be an affine $\delta$-$K$-group. Suppose that $X = \text{Spec}_\delta(B)$ is a right $\delta$-$K$-torsor for $G$, or in other words, $B = (B, \rho)$ is a right $H$-Galois $\delta$-$K$-algebra. Tracing the argument of [10] modified into our differential situation, we see that there exists uniquely (up to isomorphism) a pair $(H', \lambda)$ of a $\delta$-$K$-Hopf algebra $H'$ and a left $H'$-comodule $\delta$-$K$-algebra structure $\lambda : B \to H' \otimes_K B$ such that (i) $(B, \lambda)$ is a left $H'$-Galois $\delta$-$K$-algebra, and (ii) $\lambda$ and $\rho$ commute in the sense that

$$(\lambda \otimes \text{id}_H) \circ \rho = (\text{id}_{H'} \otimes \rho) \circ \lambda.$$

We say that $B$ is an $(H', H)$-bi-Galois $\delta$-$K$-algebra. Accordingly, we have uniquely a pair of an affine $\delta$-$K$-group $G'$ and its action on $X$ from the left, such that (i) $X$ is a left $\delta$-$K$-torsor for $G'$, and (ii) the actions on $X$ by $G'$
and by $G$ commute with each other. We say that $X$ is a $δ$-$K$-bi-torsor. We write

\begin{equation}
H^B, \ G^X
\end{equation}

for $H'$, $G'$, respectively. If $B$ (or equivalently, $X$) is trivial, or namely if $B = H$ (or $X = G$), then $H^B = H$ and $G^X = G$. This, applied after some base extension, shows the following:

**Proposition 3.4.** $H^B$ and $G^X$ are forms of $H$ and of $G$, respectively.

With $K$ replaced by a non-zero $δ$-ring $R$, the results above remain true if the relevant $δ$-$R$-Hopf algebra is flat over $R$. We remark that $δ$-$R$-torsors are then required, in addition to the $\tilde{ρ}$ being isomorphic, to be faithfully flat over $R$.

### 3.5. Interpretation of $H^1_δ(K, G)$

Let $G = \text{Spec}_δ(H)$ be an affine algebraic $δ$-$K$-group, and set $g := \text{Lie}(G)$. Then $g$ is a $δ$-$K$-Lie algebra of finite $K$-dimension, whence the $δ$-automorphism-group functor $\text{Aut}_δ(g)$ is an affine $δ$-algebraic $δ$-$K$-group by Proposition 3.3. We see that the left adjoint action by $G$ on $g$ gives rise to a morphism of affine $δ$-algebraic $δ$-$K$-groups

$$\text{Ad} : G \to \text{Aut}_δ(g),$$

which induces naturally a map between the cohomology sets

$$\text{Ad}_δ : H^1_δ(K, G) \to H^1_δ(K, \text{Aut}_δ(g)).$$

Given a right $δ$-$K$-torsor $X$ for $G$, we define

\begin{equation}
\delta X := \text{Lie}(G^X).
\end{equation}

This is a form of $g = \text{Lie}(G)$, since $G^X$ is a form of $G$; see Proposition 3.4.

**Proposition 3.5.** $\text{Ad}_δ$ is interpreted in terms of forms so as

\begin{equation}
[a \text{ a right $δ$-$K$-torsor } X \text{ for } G] \mapsto [\delta X],
\end{equation}

where $[\ ]$ indicates isomorphism classes.

**Proof.** In this proof we write $⊗$ for $⊗_K$, and use the base-on-left notation for base extensions; see Remark 2.1. Suppose that $X = \text{Spec}_δ(B)$ is a right $δ$-$K$-torsor for $G$, or in other words, $B = (B, ρ)$ is a right $H$-Galois $δ$-$K$-algebra.

Let $γ ∈ G(B ⊗ B)$. This gives the automorphism $ℓ_γ : (B ⊗ B) ⊗ H \to (B ⊗ B) ⊗ H$ of the right ($(B ⊗ B) ⊗ H$)-Galois $δ$-$(B ⊗ B)$-algebra $(B ⊗ B) ⊗ H$ given by

$$ℓ_γ((b ⊗ c) ⊗ h) = (b ⊗ c)(h(1)) ⊗ h(2), \quad b, c ∈ B, \ h ∈ H.$$

Here and in what follows, we let

$$\Delta(h) = h(1) ⊗ h(2). \quad (Δ \circ \text{id}) \circ \Delta(h) = h(1) ⊗ h(2) ⊗ h(3)$$

denote the coproduct on $H$. The right co-adjoint action $\text{Coad}_γ : (B ⊗ B) ⊗ H \to (B ⊗ B) ⊗ H$ by $γ$ is defined by

$$(\text{Coad}_γ)((b ⊗ c) ⊗ h) = (b ⊗ c)γ(h(1))γ^{-1}(h(3)) ⊗ h(2).$$
This is an automorphism of the $\delta-(B \otimes B)$-Hopf algebra $(B \otimes B) \otimes H$. Note that $\ell_\gamma$ turns into an isomorphism of left $((B \otimes B) \otimes H)$-Galois $\delta-(B \otimes B)$-algebras, if one twists through $\text{Coad} \gamma$ the obvious co-action by $(B \otimes B) \otimes H$ on the domain. Explicitly, this means that

$$
(3.11) \quad \text{Coad} \gamma \otimes_{B \otimes B} \ell_\gamma \circ \Delta_{(B \otimes B) \otimes H} = \Delta_{(B \otimes B) \otimes H} \circ \ell_\gamma \text{ on } (B \otimes B) \otimes H,
$$

where $\Delta_{(B \otimes B) \otimes H}$ denotes the coproduct on $(B \otimes B) \otimes H$.

Suppose that the $\gamma$ above is a 1-cocycle for computing $H^1_K(B/K, G)$ which gives the form $B$ through $\bar{\rho}$. This means that one has the commutative diagram

$$
\begin{array}{ccc}
(B \otimes B) \otimes B & \xrightarrow{d_1\bar{\rho}} & (B \otimes B) \otimes H \\
\downarrow & & \downarrow \ell_\gamma \\
(B \otimes B) \otimes H & \xrightarrow{d_2\bar{\rho}} & (B \otimes B) \otimes H
\end{array}
$$

of right $((B \otimes B) \otimes H)$-Galois $\delta-(B \otimes B)$-algebras, where $d_i, i = 1, 2$, denote the base extensions along

$$
B \to B \otimes B, \ b \mapsto 1 \otimes b, \ b \otimes 1.
$$

Recall that $B$ is an $(H^B, H)$-bi-Galois $\delta$-$K$-algebra. By [10, Theorem 3.5], the Hopf algebra $H^B$ consists of the elements $\sum_i b_i \otimes c_i$ in $B \otimes B$ such that

$$
(3.12) \quad \sum_i (b_i)(0) \otimes (c_i)(0) \otimes (b_i)(1)(c_i)(1) = \sum_i b_i \otimes c_i \otimes 1 \text{ in } (B \otimes B) \otimes H,
$$

where $\rho(b) = b(0) \otimes b(1)$. Moreover,

$$
(3.13) \quad \mu : B \otimes H^B \to B \otimes B, \ \mu(b \otimes z) = bz
$$

is an isomorphism of left $(B \otimes H^B)$-Galois $\delta$-$B$-algebras. Define

$$
\nu := \bar{\rho} \circ \mu : B \otimes H^B \to B \otimes H,
$$

and recall from Section 3.4 uniqueness of the pair $(H', \lambda)$, and apply it first over $B$, and next over $B \otimes B$. Then one sees the following. First, there uniquely exists an isomorphism $\theta : B \otimes H^B \to B \otimes H$ of $\delta$-$B$-Hopf algebras such that

$$
(\theta \otimes \nu) \circ \Delta_{B \otimes H^B} = \Delta_{B \otimes H} \circ \nu,
$$

where $\Delta_{B \otimes H^B}$ and $\Delta_{B \otimes H}$ denote the coproducts on the $\delta$-$B$-Hopf algebras. In fact, this $\theta$ is the unique isomorphism between the two $\delta$-$B$-Hopf algebras that is compatible with their co-actions on $B \otimes B$. Next, the last commutative diagram, with $((B \otimes B) \otimes H)^{(-)}$ applied (see [38]), induces the commutative diagram

$$
\begin{array}{ccc}
(B \otimes B) \otimes H^B & \xrightarrow{d_1\theta} & (B \otimes B) \otimes H \\
\downarrow & & \downarrow \text{Coad} \gamma \\
(B \otimes B) \otimes H & \xrightarrow{d_2\theta} & (B \otimes B) \otimes H
\end{array}
$$

of $\delta-(B \otimes B)$-Hopf algebras; notice from (3.11) that $\ell_\gamma$ induces $\text{Coad} \gamma$.  


Notice from (3.1) that \( g^X = \text{Der}_r(H^B, K) \). Then one sees that \( \theta \) induces an isomorphism
\[
\theta^* : B \otimes g \xrightarrow{\sim} B \otimes g^X
\]
of \( \delta \)-\( B \)-Lie algebras. Moreover, the last commutative diagram induces by duality the commutative diagram
\[
(B \otimes B) \otimes g^X \xrightarrow{d_1(\theta^*)} (B \otimes B) \otimes g \xleftarrow{d_2(\theta^*)} (B \otimes B) \otimes g
\]
of \( \delta \)-\((B \otimes B)\)-Lie algebras, where the horizontal arrow indicates the left adjoint action by \( \gamma^{-1} \). We may reverse the direction of the arrow, changing the label into the left adjoint action \( \text{Ad} \gamma \) by \( \gamma \). The result shows that \( \text{Ad} \gamma \), regarded as a 1-cocycle in \( \text{Aut}_s(g)(B \otimes B) \), gives the form \( g^X \) of \( g \). This proves that \( \text{Ad}_r : H^1(B/K, G) \to H^1(B/K, \text{Aut}_s(g)) \) is interpreted by the same formula as (3.10).

Given a \( \delta \)-\( K \)-algebra map \( B \to U \), \( H^1(B/K, G) \) and \( H^1(B/K, \text{Aut}_s(g)) \) are naturally embedded into \( H^1(K, G) \) and \( H^1(K, \text{Aut}_s(g)) \), respectively. By the result shown above the interpretation is true, restricted to the image of \( H^1(B/K, G) \). The proposition now follows, since \( H^1(K, G) \) is the union of all those images, where \( B \) ranges over all right \( H \)-Galois \( \delta \)-\( K \)-algebras; see Lemma 3.2. \( \square \)

Let \( G = \text{Spec}_\delta(H) \) be an affine algebraic \( \delta \)-\( K \)-group with \( g = \text{Lie}(G) \), as above. Recall from (3.1) that \( g = \text{Der}_r(H, K) \).

**Proposition 3.6.** Regard \( H \) merely as the trivial right \( H \)-Galois \( K \)-algebra, forgetting \( \delta \) on it.

1. Given an element \( D \in g \), define
\[
(3.14) \quad \delta_D : H \to H, \quad \delta_D(h) = \delta h + D(h^{(1)})h^{(2)},
\]
where \( \delta \) denotes the original operator on \( H \). Then this is a \( \delta \)-operator with which \( H \) is made into a right \( H \)-Galois \( \delta \)-\( K \)-algebra. Conversely, such a \( \delta \)-operator uniquely arises in this way.

2. Given an element \( D \in g \), let \( X_D \) denote the right \( \delta \)-\( K \)-torsor for \( G \) which is represented by the right \( H \)-Galois \( \delta \)-\( K \)-algebra \( (H, \delta_D) \) obtained above. Then the form \( g^{X_D} \) of \( g \) is the \( K \)-Lie algebra \( g \) equipped with the new \( \delta \)-operator
\[
\delta + \text{ad} D : g \to g, \quad z \mapsto \delta z + [D, z],
\]
where \( \delta \) denotes the original operator on \( g \). Thus \( g^{X_D} \) is quasi-isomorphic to the original \( g \); see Definition 1.4.

**Proof.** (1) Suppose that \( \delta_1 \) is a desired operator, or namely, \((H, \delta_1)\) is a right \( H \)-Galois \( \delta \)-\( K \)-algebra. Then one sees that \( \delta_1 - \delta : H \to H \) is a \( K \)-linear derivation and is at the same time a right \( H \)-comodule map. It follows that \( \delta_1 \) is necessarily of the form \( \delta_D \) with \( D \in g \) uniquely determined. Such \( \delta_D \) is seen to be a desired operator for any \( D \), indeed.
For every form of $g$ we see that the present $H'$ is the $K$-Hopf algebra $H$ equipped with the $\delta$-operator

$$H \to H, \ h \mapsto \delta h + D(h_{(1)})h_{(2)} - h_{(1)}D(h_{(2)}) \Leftrightarrow$$

This implies the desired result. \hfill $\Box$

A simple consequence of the proposition above is the following.

**Corollary 3.7.** Let $g$ be a $\delta$-Lie algebra of finite $K$-dimension. Once the Lie algebra $\text{Lie}(G)$ of some affine algebraic $\delta$-group $G$ is shown to be a form of $g$, then every $\delta$-Lie algebra quasi-isomorphic to $\text{Lie}(G)$ is a form of $g$, as well.

3.6. **Differential $\delta$-$K$-objects arising from $C$-linear objects.** Let $K$ be a $\delta$-field of characteristic zero. Let

$$C = C_K \ (= \{ x \in K \mid \delta x = 0 \})$$

denote the field of constants in $K$, which is necessarily of characteristic zero. In this subsection we let $\otimes$ denote the tensor product $\otimes_C$ over $C$.

Let $A_0$ be a $C$-linear object. We can and do regard the base extension $A_0 \otimes K$ as a $\delta$-$K$-object with respect to the operator $\delta_0$ defined by

$$\delta_0 : A_0 \otimes K \to A_0 \otimes K, \ a \otimes x \mapsto a \otimes \delta x.$$ 

For every $\delta$-algebra $R$, $A_0 \otimes R$ is similarly a $\delta$-$R$-object, and is a base extension of the $\delta$-$K$-object $A_0 \otimes K$ above.

**Proposition 3.8.** If the automorphism-group functor $\text{Aut}(A_0)$ of $A_0$ happens to be an affine $C$-group, represented by a $C$-Hopf algebra $H_0$, then the $\delta$-automorphism-group functor $\text{Aut}_\delta(A_0 \otimes K)$ of the $\delta$-$K$-object $(A_0 \otimes K, \delta_0)$ is an affine $\delta$-$K$-group, represented by the $\delta$-$K$-Hopf algebra $H_0 \otimes K$.

**Proof.** Let $R \in \delta$-$K$-Algebras. One sees that every automorphism of the $\delta$-$R$-object $A_0 \otimes R$ restricts to an automorphism of $A_0 \otimes C_R$ over the $C$-algebra $C_R$ of constants in $R$, and so it is uniquely presented as the base extension of the restriction. This shows $\text{Aut}_{\delta, R}(A_0 \otimes R) = \text{Aut}_{C_R}(A_0 \otimes C_R)$; this last is naturally isomorphic to $\text{Spec}_{C_R}(H_0)(C_R) = \text{Spec}_{\delta, K}(H_0 \otimes K)(R)$. This proves the proposition. \hfill $\Box$

We remark that the proposition follows from the proof of Proposition 3.3 if $(\text{dim}_K A_0 \otimes K) = \text{dim}_C A_0 < \infty$. For the relation 3.5 turns into $\delta X = 0$ since $D = O$.

**Corollary 3.9.** If $\text{dim}_C A_0 < \infty$, then $\text{Aut}_\delta(A_0 \otimes K)$ is an affine algebraic $\delta$-$K$-group.

**Proof.** This follows from the proposition above, since $\text{Aut}(A_0)$ is an affine algebraic $C$-group under the assumption; see [13, Section 7.6]. \hfill $\Box$

The following result would be worth presenting, though it is not essentially used in this paper.

**Proposition 3.10.** Assume $\text{dim}_C A_0 < \infty$, and that the field $C$ is algebraically closed. Then every form of the $\delta$-$K$-object $(A_0 \otimes K, \delta_0)$ splits by some (finitely generated) Picard-Vessiot extension $L$ over $K$. 

---

**Note:** The above text is a continuation of the previous discussion on differential algebras, focusing on the properties and structures related to $\delta$-$K$-objects and their associated Lie algebras and automorphism groups. The document emphasizes the importance of characteristic zero fields and their role in defining and understanding these algebraic structures. The propositions and corollaries provided offer insights into the behavior of these objects under various algebraic operations and transformations.
Proof. By the preceding results the first assumption implies $\text{Aut}(A_0) = \text{Spec}_C(H_0)$ and $\text{Aut}_\delta(A_0 \otimes K) = \text{Spec}_{\delta, K}(H_0 \otimes K)$, where $H_0$ is a finitely generated $C$-Hopf algebra.

Let $B$ be a right $(H_0 \otimes K)$-Galois $\delta$-$K$-algebra, and regard it as a form of the right $(H_0 \otimes K)$-comodule $\delta$-$K$-algebra $H_0 \otimes K$. We should prove that this form $B$ splits by some $L/K$ as above. It suffices to prove that there exists a $\delta$-$K$-algebra map from $B$ to such an $L$, since $B$ splits by $B$, itself.

We have the $\delta$-$B$-algebra isomorphism $\tilde{\rho} : B \otimes_K B \cong B \otimes H_0$ as in (3.2). Choose a simple quotient $\delta$-$K$-algebra $R$ of $B$, as in the proof of Lemma 3.2. Then $R$ is an integral domain by [12, Lemma 1.17], as before. This is finitely generated as a $K$-algebra since $B$ is such. The quotient field $L = \mathbb{Q}(R)$ of $R$ uniquely turns into a $\delta$-field, which is necessarily a finitely generated extension over $K$. The second assumption above, combined with [1, Lemma 4.2], implies that the field $C_L$ of constants in $L$ equals $C$. This $L/K$ will be proved to be a desired Picard-Vessiot extension by [1, Definition 1.8 and Theorem 3.11], if one sees that the canonical $\delta$-$R$-algebra map $\tilde{\rho}^{-1} : H_0 \to B \otimes_K B$ with the natural surjection $B \otimes_K B \to R \otimes_K R$, which apparently takes values in $C_{R \otimes_K R}$. Indeed, this is seen from the commutative diagram

$$
\begin{array}{ccc}
B \otimes H_0 & \overset{\tilde{\rho}^{-1}}{\longrightarrow} & B \otimes_K B \\
\downarrow & & \downarrow \\
R \otimes C_{R \otimes_K R} & \longrightarrow & R \otimes_K R.
\end{array}
$$

Here the vertical arrow on the left-hand side naturally arises from the composite of $\tilde{\rho}^{-1}|_{H_0} : H_0 \to B \otimes_K B$ with the natural surjection $B \otimes_K B \to R \otimes_K R$, which apparently takes values in $C_{R \otimes_K R}$. □

4. PROOF OF THE THEOREM AND COMPUTATIONS

Throughout in this section we let $K := \mathbb{C}(t)$, and write $\otimes$ for $\otimes_K$.

Suppose that we are in the situation of Section [1]. Let $\mathfrak{g}_0$ be a complex simple Lie algebra, and let $\mathfrak{g} = \mathfrak{g}_0(K)$ denote the $\delta$-$K$-Lie algebra as in (1.1).

4.1. Two key facts. One key fact for us is the following description of the automorphism-group scheme $\text{Aut}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$. Recall that the finite group $\Gamma = \text{Out}(\mathfrak{g}_0)$ of outer-automorphisms of $\mathfrak{g}_0$ is explicitly given by (1.2); this $\Gamma$ will be identified with the associated, finite constant group scheme. Let $G_0^\circ$ be the adjoint simple $\mathbb{C}$-group associated with $\mathfrak{g}_0$. A natural action by $\Gamma$ on $G_0^\circ$ constitutes an affine algebraic $\mathbb{C}$-group

$$
G_0 = G_0^\circ \rtimes \Gamma
$$

of semi-direct product, so that

$$
\text{Lie}(G_0) = \text{Lie}(G_0^\circ) = \mathfrak{g}_0,
$$

and the adjoint action by $G_0$ on $\mathfrak{g}_0$ gives an isomorphism

$$
(4.1) \quad \text{Ad} : G_0 \xrightarrow{\sim} \text{Aut}(\mathfrak{g}_0)
$$
of affine algebraic $C$-groups. By restriction this $Ad$ induces the identity $\Gamma = \text{Out}(g_0)$. Note that $G_0^0$ is the connected component of $G_0$ containing the identity element. See [3 Chapter 4, Section 4, 1°].

Suppose $G_0 = \text{Spec}_C(H_0)$, and define

$$G = \text{Spec}_{\delta-C(t)}(H_0 \otimes_C C(t)).$$

Then one sees $g = \text{Lie}(G)$. Moreover, it follows from (4.1) and Proposition 3.8 that the adjoint action by $G$ on $g$ gives an isomorphism

$$Ad : G \rightarrow \text{Aut}_{\delta}(g)$$

of affine algebraic $\delta-C(t)$-groups. This together with Proposition 3.5 prove the following.

**Proposition 4.1.** Every form of the $\delta$-$K$-Lie algebra $g$ uniquely arises, as described by (3.10), from a right $\delta$-$K$-torsor for $G$.

Another key fact is the cohomology vanishing of the (non-differential) Amitsur 1st cohomology due to Steinberg [11, III, 2.3, Theorem 1′] (see also [7, Section 25]),

$$H^1(K, F) = 0,$$

where $F$ is a connected affine algebraic $K$-group. This is proved more generally when $K$ is replaced by a perfect field, say $K'$, of dimension $\leq 1$ [11, Definition on Page 78], and in addition, $F$ is assumed to be smooth if $\text{char} K' > 0$; note that every affine algebraic $K$-group is necessarily smooth since $\text{char} K = 0$. One sees that $K (= C(t))$ is a $(C_1)$-field by Tsen’s Theorem, whence $K$ is of dimension $\leq 1$ by [11, Corollary on Page 80].

4.2. **Proof of Theorem 1.5, Part 1: Case $\Gamma = \{1\}$.** In this case, $G$, regarded as an affine $K$-group, is connected. By (4.2) applied to this $G$, it follows that every right $\delta$-$K$-torsor for $G$, regarded as a right $K$-torsor, is trivial. Propositions 3.6 and 4.1 conclude the proof.

4.3. **Galois Descent.** To proceed to Parts 2 and 3, suppose that we are now in Case $\Gamma \neq \{1\}$.

Note that $\Gamma$, regarded as a finite constant $C$-group scheme, is represented by the dual $(C\Gamma)^*$ of the group algebra $C\Gamma$; this $(C\Gamma)^*$ is the separable part $\pi_0(H_0)$ of $H_0$ [13 Page 49], that is, the largest separable subalgebra (in fact, Hopf subalgebra) of the $C$-Hopf algebra $H_0$. Suppose $G_0^0 = \text{Spec}_C(J_0)$, and define

$$H := H_0 \otimes_C K, \quad J := J_0 \otimes_C K, \quad Z := (C\Gamma)^* \otimes_C K (= (K\Gamma)^*),$$

which are naturally $\delta$-$K$-Hopf algebras, such that $G = \text{Spec}_\delta(H)$, in particular. One sees that $Z \subset H$ is a $\delta$-$K$-Hopf subalgebra, and

$$J = H/(Z^+),$$

where $(Z^+)$ is the ideal (in fact, $\delta$-stable Hopf ideal) generated by the augmentation ideal $Z^+ = \text{Ker}(\epsilon : Z \rightarrow K)$ of $Z$, that is, the kernel of the counit. Since $\Gamma$ acts innerly on $G_0^0 (\subset G_0)$ from the right, it acts from the left on $J$ as $\delta$-$K$-Hopf-algebra automorphisms. The action gives rise by adjoint to the co-action $J \rightarrow J \otimes Z$ by $Z = (K\Gamma)^*$, so that the associated smash coproduct

13
Choose arbitrarily a right $\delta$-$K$-torsor $X = \text{Spec}_\delta(B)$ for $G = \text{Spec}_\delta(H)$. In view of Proposition 4.1 we wish to describe the $\delta$-$K$-Lie algebra $\mathfrak{g}^X (= \text{Lie}(G^X))$. Let $H' := H^B$, or in other words, suppose $G^X = \text{Spec}_\delta(H')$, so that $B$ is an $(H', H)$-bi-Galois $\delta$-$K$-algebra. We are going to prove the following.

**Proposition 4.2.** There is a finite-dimensional $\delta$-$K$-Hopf subalgebra $Z'$ of $H'$ such that

1. $Z'$ is separable as a $K$-algebra;
2. the associated quotient $\delta$-$K$-Hopf algebra

\begin{equation}
J' := H'/(Z' +) \quad (\text{cf. (4.3)})
\end{equation}

has the trivial separable part, $\pi_0(J') = K$, or in other words, it includes no non-trivial separable $K$-subalgebra.

This implies that the affine $K$-group $\text{Spec}(H')$ includes $\text{Spec}(J')$ as the connected component containing the identity element, and thereby concludes

\begin{equation}
\mathfrak{g}^J = \text{the Lie algebra of the affine } \delta-K\text{-group } \text{Spec}_\delta(J')
\end{equation}

as $\delta$-$K$-Lie algebras. Therefore, we aim first to prove the proposition above, and then to describe the $\mathfrak{g}^X$ above.

Let $\rho : B \to B \otimes H$ denote the structure map on $B$, and define

$$R := \rho^{-1}(B \otimes Z).$$

Then this $R$ is a right $\delta$-$K$-Galois algebra for $Z$, or in other words, $\text{Spec}_\delta(R)$ is a right $\delta$-$K$-torsor for the finite constant $\delta$-$K$-group scheme $\Gamma_K$ given by $\Gamma$; it arises from the the right $\delta$-$K$-torsor $X = \text{Spec}_\delta(B)$ for $G$ through the restriction map $H^1_\delta(K, G) \to H^1_\delta(K, \Gamma_K)$ which is defined in the differential situation, as well, just as in the ordinary situation. Note that $R$ is naturally a $\delta$-$K$-algebra of finite $K$-dimension, and is a Galois $K$-algebra with Galois group $\Gamma$ in the classical sense that the $K$-algebra map $R \times K \to \text{End}_K(R)$ which arises from the natural module-action on $R$ by the semi-direct product $R \rtimes \Gamma$ is an isomorphism. Note that $R \rtimes \Gamma$ is naturally a non-commutative $\delta$-$K$-algebra with $\Gamma (= \{1\} \times \Gamma)$ included in constants. A $\delta$-$(R \rtimes \Gamma)$-module is thus an $R$-module $M$ equipped with an additive operator $\delta$ and a $\Gamma$-action of $K$-linear automorphisms, such that

$$\begin{align*}
\delta(\gamma m) &= \gamma(\delta m), \\
\delta(am) &= (\delta a)m + a(\delta m), \\
\gamma(am) &= (\gamma a)(\gamma m),
\end{align*}$$

where $\gamma \in \Gamma$, $a \in R$ and $m \in M$. We call this a $(\delta, \Gamma)$-$R$-module, to treat $\delta$ and $\Gamma$ on an equality, and let $(\delta, \Gamma)$-$R$-Modules denote the category of those modules. The classical Galois Descent Theorem tell us that the functor $M \mapsto M^\Gamma$, $\Gamma$-invariants in $M$, gives the category equivalence

$$(\delta, \Gamma)$-$R$-Modules \xrightarrow{\sim} \delta$-$K$-Modules,$$

whose quasi-inverse is given by the base-extension functor $\otimes_K R$. In fact, this is a symmetric tensor equivalence, so that there is induced the category equivalence between their (commutative-)algebra objects, or between any other kind of linear objects. The category on the left-hand side has the

$Z \hookrightarrow J$ (see [8, Definition 10.6.1]) coincides with $H$. Here one should recall $G_0 = \Gamma \ltimes G_0^0 (= G_0^0 \rtimes \Gamma)$.
tensor product $\otimes_R$, the unit object $R$ and the obvious symmetry, while the category on the right-hand side has the tensor product $\otimes_K$, the unit object $K$ and the obvious symmetry. A commutative algebra in $(\delta, \Gamma)$-$R$-Modules will be called a $(\delta, \Gamma)$-$R$-algebra; it descends to a $\delta$-$K$-algebra by the category equivalence above. Similarly, a $(\delta, \Gamma)$-$R$-Hopf or Lie algebra is defined, and it descends to a $\delta$-$K$-object.

We have the commutative diagram

$$
\begin{array}{cccc}
B \otimes B & \rightarrow & B \otimes H \\
\downarrow & & \downarrow \\
R \otimes R & \rightarrow & R \otimes Z \\
\downarrow \text{mult} & & \downarrow \text{id}_R \otimes \epsilon \\
R & & \\
\end{array}
$$

of $\delta$-$K$-algebras, where the upper horizontal arrow indicates the isomorphism $\tilde{\rho}$ (see (3.2)) associated with the structure map $\rho : B \rightarrow B \otimes H$ on $B$, and the lower one is the analogous isomorphism for the right $Z$-Galois $\delta$-$K$-algebra $R$. In addition, $\text{mult} : R \otimes R \rightarrow R$ indicates the multiplication $x \otimes y \mapsto xy$. By the base extensions along the two diagonal arrows $\text{mult} : R \otimes R \rightarrow R$ and $\text{id}_R \otimes \epsilon : R \otimes H \rightarrow R$, the $\tilde{\rho}$ induces the isomorphism

$$(4.6) \quad B \otimes_R B \rightarrow B \otimes J = B \otimes_R (J \otimes R).$$

Recall that $\Gamma$ acts on $J$ as $\delta$-$K$-Hopf algebra automorphisms. Then one sees that $J \otimes R$ is a $(\delta, \Gamma)$-$R$-Hopf algebra, and hence descends to a $\delta$-$K$-Hopf algebra

$$J := (J \otimes R)^\Gamma.$$

The composite $B \rightarrow B \otimes H \rightarrow B \otimes J = B \otimes_R (J \otimes R)$ of the structure map on $B$ with the natural surjection onto $B \otimes_R (J \otimes R)$ is a $(\delta, \Gamma)$-$R$-algebra map, and hence descends to a $\delta$-$K$-algebra map $B^\Gamma \rightarrow B^\Gamma \otimes J$, which we call $\varrho$.

**Lemma 4.3.** $B^\Gamma$ is a right $J$-Galois $\delta$-$K$-algebra by the $\varrho$ above.

**Proof.** One sees that $\varrho$ satisfies the co-associativity and the counit property since the last composite does. One sees that $(4.6)$ is an isomorphism of $(\delta, \Gamma)$-$R$-algebras, and descends to $\tilde{\varrho} : B^\Gamma \otimes B^\Gamma \rightarrow B^\Gamma \otimes J$, which is, therefore, an isomorphism. □

Recall $H' = H^B$, and define $Z' := Z^R$, so that $R$ is a $(Z', Z)$-bi-Galois $\delta$-$K$-algebra.

**Lemma 4.4.** $Z'$ is a finite-dimensional $\delta$-$K$-Hopf subalgebra of $H'$ which has the property (i) of Proposition 4.2, that is, $Z'$ is separable as a $K$-algebra.

**Proof.** By (3.12) we have $Z' \subset H'$. This inclusion is compatible with the Hopf-algebra structure maps, as is seen from the construction of $H^B$ given in [10, Theorem 3.5]. To verify this here only for the coproduct, recall from (3.13) that $H' \subset B \otimes B$ gives rise to a left $B$-linear isomorphism $B \otimes H' = B \otimes B$. Therefore, we have $H' \otimes H' \subset B \otimes H' \otimes H' = B \otimes B \otimes H' =$
$B \otimes B \otimes B$. The construction cited above tells us that the coproduct on $H'$ is the restriction of $B \otimes B \to B \otimes B \otimes B$, $b \otimes c \mapsto b \otimes 1 \otimes c$. This, combined with the analogous restriction of $R \otimes R \to R \otimes R \otimes R$ to the coproduct $Z' \to Z \otimes Z'$, shows the desired compatibility, as is verified by a commutative diagram in cube.

The $K$-algebras $Z$, $R$ and $Z'$ turn to be mutually isomorphic after base extension to some algebraically closed field. It follows that $Z'$ is finite-dimensional separable, since $Z$ is.

Define $J' := H'/((Z')^+)$, as in (4.4). The proof of Proposition 4.2 completes by proving the next lemma. The following proposition describes the $\delta$-$K$-Lie algebra $g^X$, see (3.9).

**Lemma 4.5.** $B^\Gamma$ is a $(J', J)$-bi-Galois $\delta$-$K$-algebra, and $J'$ has the property (ii) of Proposition 4.2.

Proof. The same argument as proving Lemma 4.3 shows that $B^\Gamma$ is a left $J'$-Galois $\delta$-$K$-algebra. Here one should notice that $\Gamma$ acts (or $Z$ co-acts) trivially on $H'$, and hence on $J'$. Indeed, $B^\Gamma$ is bi-Galois, since the structure maps $H' \otimes B \leftarrow B \to B \otimes H$ on $B$ commute with each other (see (3.7)), and hence those on $B^\Gamma$ do.

Note that $\pi_0(J) = \pi_0(J_0) \otimes_K K$ equals $K$. This is equivalent to saying that the $K$-algebra $J$ contains no non-trivial idempotent even after base extension to some (or any) algebraically closed field. It follows that $J$ and $J'$ have the same property, since $J$ and $J$, as well as $J$ and $J'$, are mutually isomorphic after base extension such as above.

Since $g_0(R) = g_0 \otimes_K R$, on which $\Gamma$ acts diagonally, is a $(\delta, \Gamma)$-R-Lie algebra, it descends to $g_0(R)^\Gamma$, a $\delta$-$K$-Lie algebra. Our aim of this subsection is achieved by the following.

**Proposition 4.6.** The $\delta$-$K$-Lie algebra $g^X$ is quasi-isomorphic to $g_0(R)^\Gamma$.

Proof. Recall (4.5) and the result of Proposition 4.2 that $B^\Gamma$ is a $(J', J)$-bi-Galois $\delta$-$K$-algebra. By Steinberg’s Cohomology-Vanishing (4.2) applied to the connected affine $K$-group Spec$(J)$, we see that the right $J$-Galois $K$-algebra $B^\Gamma$ is isomorphic to $J$. This together with Proposition 3.6 prove the desired result.

We add an important consequence.

**Corollary 4.7.** The forms of $g_0(K)$ are precisely the $\delta$-$K$-Lie algebras quasi-isomorphic to $g_0(R)^\Gamma$, where $R$ ranges over all right $(K \Gamma)^*$-Galois $\delta$-$K$-algebras.

Proof. By Propositions 4.4 and 4.6 every form is quasi-isomorphic to some $g_0(R)^\Gamma$. Conversely, any $g_0(R)^\Gamma$ is apparently a form, whence any one that is quasi-isomorphic to $g_0(R)^\Gamma$ is, as well, by (4.5) and Corollary 3.7.

4.4. **Proof of Theorem 1.5.** Part 2: Case $\Gamma = \mathbb{Z}_2$. In this case, the right $(K \Gamma)^*$-Galois $\delta$-$K$-algebras $R$ are precisely (i) the trivial one $(K \Gamma)^*$ (equipped with the obvious $\delta$-operator), and (ii) the quadratic field extensions over $K$ (equipped with the $\delta$-operator uniquely extending the one on $K$). By Corollary 1.7 it remains to show that for $R = (K \Gamma)^*$ in (i), we
have $g_0(R)^\Gamma \simeq g_0(K)$. Let $\text{Map}(\Gamma, g_0(K))$ denote the $\Gamma$-set of all maps $\Gamma \to g_0(K)$, equipped with the action

$$\gamma f : \gamma' \mapsto f(\gamma'\gamma),$$

where $\gamma, \gamma' \in \Gamma$ and $f \in \text{Map}(\Gamma, g_0(K))$. Regard this naturally as the direct product of $\#\Gamma$-copies of the $\delta$-$K$-Lie algebra $g_0(K)$. Then we see that associating to $x \otimes a \in g_0 \otimes_C (KT)^*$, the map $\gamma \mapsto \gamma x \otimes a(\gamma)$ gives a $\Gamma$-equivariant isomorphism

$$g_0(R) \cong \text{Map}(\Gamma, g_0(K))$$

of $\delta$-$K$-Lie algebras, whose restriction to the $\Gamma$-invariants is the desired $g_0(R)^\Gamma \simeq g_0(K)$. This completes the proof.

4.5. **Proof of Theorem 1.5, Part 3:** Case $\Gamma = S_3$. In this case, let $R$ be a right $(KT)^*$-Galois $\delta$-$K$-algebra. In view of Corollary 4.7 we wish to show that $g_0(R)^\Gamma$ is such as in Part 3 of the theorem. This is obvious when $R$ is either trivial or a Galois field extension $L/K$ with $\Gamma = \text{Gal}(L/K)$; notice from the preceding case that $g_0(R)^\Gamma = g_0(K)$ if $R$ is trivial. We may thus exclude these two cases.

To describe $R$, note that $R$ is artinian as a ring, and $\Gamma$-simple in the sense that it does not include any non-trivial $\Gamma$-stable ideal. Since the action by $\Gamma$ on $R$ commutes with the $\delta$-operator, $R$ is a module algebra over the $\mathbb{C}$-Hopf algebra $\mathbb{C}\Gamma \otimes_C \mathbb{C}[\delta]$, which is *artinian simple* or AS in the sense of [1, Definition 11.6]; see the original [2, Definition 2.6] as an alternate. This $\mathbb{C}$-Hopf algebra is the group algebra $\mathbb{C}\Gamma$ tensored with the polynomial algebra $\mathbb{C}[\delta]$ in which $\delta$ is primitive. Choose arbitrary a maximal (or equally, minimal) ideal $m$ of $R$, and let $\Gamma'$ be the subgroup of $\Gamma$ consisting of all elements that stabilize $m$. By [1, Proposition 11.5] we have (a) $\Gamma' \simeq \mathbb{Z}_2$ or (b) $\Gamma' \simeq \mathbb{Z}_3$, with the extremal cases being excluded. Moreover, there exists a $\delta$-$K$-field $L$ such that $R$ is naturally isomorphic to the $(\delta, \Gamma)$-$K$-algebra $\text{Map}(\Gamma' \setminus \Gamma, L)$ consisting of all maps from the set of right cosets $\Gamma' \setminus \Gamma$ to $L$. This $\text{Map}(\Gamma' \setminus \Gamma, L)$ is naturally isomorphic to the direct product of $[\Gamma : \Gamma']$-copies of $L$, as $\delta$-$K$-algebra, and possesses the $\Gamma$-action presented below.

Suppose that $\mathbb{Z}_2$ is an arbitrarily chosen subgroup of $\Gamma$ of order 2, and $\mathbb{Z}_3$ is the unique subgroup of $\Gamma$ of order 3, so that we have $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_2$.

Case (a). We may suppose $\Gamma' = \mathbb{Z}_2$ (see [1, Proposition 11.5 (1)]) and $\Gamma' \setminus \Gamma = \mathbb{Z}_3$. If $\gamma \in \Gamma$, $\gamma' \in \mathbb{Z}_3$, and $f \in \text{Map}(\mathbb{Z}_3, L)$, then we have

$$\gamma f : \gamma' \mapsto \begin{cases} f(\gamma \gamma'), & \text{if } \gamma \in \mathbb{Z}_3 \\ \gamma f(\gamma'^{-1}), & \text{if } 0 \neq \gamma \in \mathbb{Z}_2. \end{cases}$$

Case (b). We have $\Gamma' = \mathbb{Z}_3$, and we may suppose $\Gamma' \setminus \Gamma = \mathbb{Z}_2$. If $\gamma \in \Gamma$, $\gamma' \in \mathbb{Z}_2$, and $f \in \text{Map}(\mathbb{Z}_2, L)$, then we have

$$\gamma f : \gamma' \mapsto \begin{cases} f(\gamma \gamma'), & \text{if } \gamma \in \mathbb{Z}_2; \\ \gamma f(\gamma'), & \text{if } \gamma \in \mathbb{Z}_3, \gamma' = 0 \text{ in } \mathbb{Z}_2; \\ \gamma^{-1}f(\gamma'), & \text{if } \gamma \in \mathbb{Z}_3, \gamma' \neq 0 \text{ in } \mathbb{Z}_2. \end{cases}$$

In either case, since $R$ is right $(KT)^*$-Galois, $\Gamma'$ must act non-trivially on $L$, so that $L/K$ is a Galois field extension with $\Gamma' = \text{Gal}(L/K)$. Conversely,
if $L/K$ is such, then $R$ is seen to be a right $(K\Gamma)^\ast$-Galois $\delta$-$K$-algebra, splitting by $\otimes_K L$. Moreover, $g_0(R)$ is naturally isomorphic to the $(\delta, \Gamma)$-$R$-Lie algebra $\text{Map}(\Gamma' \setminus \Gamma, g_0(L))$ equipped with the obviously induced structure. We see

$$g_0(R)^\Gamma \simeq \text{Map}(\Gamma' \setminus \Gamma, g_0(L))^\Gamma = \left\{ \{\text{all constant maps } \mathbb{Z}_3 \to g_0(L)\}^{\mathbb{Z}_2} \right\} \text{ in Case (a)}$$

$$\text{Map}(\mathbb{Z}_2, g_0(L)^{\mathbb{Z}_2})^{\mathbb{Z}_2} \text{ in Case (b)}$$

which completes the proof.

### 4.6. Explicit non-trivial forms

Let us describe explicitly (up to quasi-isomorphism) the non-trivial forms of $g_0(K)$ listed in (ii) of Part 2 and (ii)–(iv) of Part 3 of the theorem, separately for four types. For all those, quadratic field extensions are needed. Such an extension $L/K$ is of the form

$$L = K(\sqrt{\alpha}) = \{a + b\sqrt{\alpha} \mid a, b \in K\},$$

where $\alpha \in K^\times \setminus (K^\times)^2$. The generator of $\text{Gal}(L/K) (= \mathbb{Z}_2)$ sends each element $x = a + b\sqrt{\alpha}$ to

$$x = a - b\sqrt{\alpha}. \quad (4.7)$$

We will use this symbol $\overline{x}$, regardless of $\alpha$.

#### 4.6.1. Type $A_\ell$ ($\ell \geq 2$)

We have $g_0 = \mathfrak{sl}_n(\mathbb{C})$, where $n = \ell + 1 \geq 3$. The order 2 outer-automorphism is conjugate to $X \mapsto -X$. For a quadratic extension field $L = K(\sqrt{\alpha})$ over $K$ as above, the generator of $\Gamma (= \mathbb{Z}_2)$ may supposed to act on $g_0(L)$ by $X = (x_{ij})_{i,j} \mapsto -X = (\overline{x}_{ji})_{i,j}$; see [4, Chapter IX, Theorem 5]. We see

$$g_0(L)^\Gamma = \mathfrak{o}_n(K) \oplus \sqrt{\alpha}(\text{Sym}_n(K) \cap \mathfrak{sl}_n(K)),$$

where $\text{Sym}_n(K)$ (resp., $\mathfrak{o}_n(K)$) denotes the $K$-subspace of $\mathfrak{gl}_n(L)$ consisting of all matrices $X$ with entries in $K$ that are symmetric (resp., skew-symmetric, $^t X = -X$).

#### 4.6.2. Type $D_\ell$ ($\ell \geq 5$)

Let $m = 2\ell$. We have $g_0 = \mathfrak{o}_m(\mathbb{C})$, which consists of all skew-symmetric $m \times m$ complex matrices. The order 2 outer-automorphism is conjugate to $X \mapsto DXD$, where $D = \text{diag}(-1, 1, \ldots, 1)$; see [4, Chapter IX, Theorem 6]. For a quadratic extension field $L = K(\sqrt{\alpha})$ over $K$ as above, we see

$$g_0(L)^\Gamma = \left\{ \begin{pmatrix} 0 & -\sqrt{\alpha}^t X \\ \sqrt{\alpha} X & Y \end{pmatrix} \right| X \in K^{m-1}, Y \in \mathfrak{o}_{m-1}(K) \right\}, \quad (4.8)$$

where by writing $X \in K^{m-1}$, we mean that $X$ is an $(m-1)$-columned vector with entries in $K$. 
4.6.3. **Type** E\(_6\). Here we follow [5, Section 7] for the construction. Let \( \mathfrak{J} \) be the exceptional central simple Jordan algebra over \( \mathbb{C} \), and let \( \mathfrak{J}^+ \) denote the subspace of \( \mathfrak{J} \) which consists of the elements \( a \) with trace zero, \( T(a) = 0 \). We have the general linear complex Lie algebra \( \mathfrak{gl}(\mathfrak{J}) \) on the \( \mathbb{C} \)-vector space \( \mathfrak{J} \). Given an element \( a \in \mathfrak{J}^+ \), we have an element \( R_a \in \mathfrak{gl}(\mathfrak{J}) \) given by \( R_a(x) = xa = ax \), \( x \in \mathfrak{J} \). Let \( \mathfrak{R}_J \) be the subspace of \( \mathfrak{gl}(\mathfrak{J}) \) which consists of all \( R_a, a \in \mathfrak{J}^+ \). The complex simple Lie algebra \( \mathfrak{g}_0 \) of type \( E_6 \) is the Lie subalgebra of \( \mathfrak{gl}(\mathfrak{J}) \) generated by \( \mathfrak{R}_J \). We have \( \mathfrak{g}_0 = \mathfrak{R}_J \oplus f_0 \), where we set \( f_0 := [\mathfrak{R}_J, \mathfrak{R}_J] \); this is a Lie subalgebra of \( \mathfrak{g}_0 \) such that \( [\mathfrak{R}_J, f_0] = \mathfrak{R}_J \), and is in fact the complex simple Lie algebra of type \( F_4 \). The order 2 outer-automorphism of \( \mathfrak{g}_0 \) is conjugate to \( X \mapsto -X^* \), where \( X^* \) denotes the operator adjoint to \( X \) with respect to the trace form \( (a, b) \mapsto T(ab) \). More explicitly this is given by

\[
X \mapsto \begin{cases} -X & \text{if } X \in \mathfrak{R}_J^+; \\ X & \text{if } X \in f_0. \end{cases}
\]

Therefore, we have

\[
\mathfrak{g}_0(L)^\Gamma = (\mathfrak{R}_J \otimes \mathbb{C} \sqrt[3]{\alpha}) \oplus f_0(K).
\]

4.6.4. **Type** D\(_4\). The complex Lie algebra \( \mathfrak{g}_0 \) is the Lie algebra \( \mathfrak{o}_8(\mathbb{C}) \) of skew-symmetric \( 8 \times 8 \) complex matrices. We follow E. Cartan [3] for the explicit description of outer-automorphisms. We discuss for each group action, separately as in Part 3 of the theorem.

(ii) **Action by** \( \mathbb{Z}_2 \). The argument above for D\(_\ell\) (\( \ell \geq 5 \)) works for \( \ell = 4 \), as well, so that \( \mathfrak{g}_0(L)^{\mathbb{Z}_2} \) is given by the right-hand side of (1.8) with \( m = 8 \).

(iii) **Action by** \( \mathbb{Z}_3 \). Choose a generator \( \sigma \) of the group. The relevant Galois extension is a cubic one, and it is of the form \( L = K(\sqrt[3]{\beta}) \), where \( \beta \in K^\times \backslash (K^\times)^3 \). The generator \( \sigma \) acts on \( L \) so that \( \sqrt[3]{\beta} \mapsto \omega \sqrt[3]{\beta} \), where \( \omega \) is a primitive 3rd root of 1.

We suppose that the rows and the columns of matrices in \( \mathfrak{g}_0 (= \mathfrak{o}_8(\mathbb{C})) \) are indexed by the elements 0, 1, \( \cdots \), 7 of \( \mathbb{Z}_8 \) in this order. Every matrix \( X = (x_{ij})_{i,j \in \mathbb{Z}_8} \) in \( \mathfrak{g}_0 \) is uniquely determined by the seven vectors

\[
(4.9) \quad t \langle x_{0,i}, x_{i+1,i+5}, x_{i+4,i+5}, x_{i+2,i+3} \rangle, \quad i \in \mathbb{Z}_8 \backslash \{0\}
\]

in \( \mathbb{C}^4 \). The action by \( \mathbb{Z}_3 \) on \( \mathfrak{g}_0 \) is (up to conjugation) such that \( \sigma \) acts on the vectors above as the \( \mathbb{C} \)-linear automorphisms given by the matrix

\[
(4.10) \quad S = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}
\]
which is seen to have 1, 1, \omega and \omega^2 as eigen-values; see [3 Section 4]. Set 
\sqrt{-3} := 1 + 2\omega, a square root of -3. Then we have the eigen-vectors
\begin{equation}
(4.11) \quad v_1^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_1^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_\omega = \begin{pmatrix} \sqrt{-3} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\omega^2} = \begin{pmatrix} -\sqrt{-3} \\ 1 \\ 1 \\ 1 \end{pmatrix}
\end{equation}
of \mathcal{S} which are associated with 1, 1, \omega and \omega^2, respectively; these form a basis of \mathbb{C}^4. Let \mathcal{L} denote the \mathcal{L}-vector space of all 4-collumded vectors with entries in \mathcal{L}. Define a 4-dimensional \mathcal{L}-subspace of \mathcal{L}^4 by
\begin{equation}
(4.12) \quad \Xi_{\mathcal{L}/\mathcal{K}} = K(v_1^{(1)} + Kv_1^{(2)} + K\sqrt{-3}v_{\omega^2} + K(\sqrt{-3})^2v_{\omega}).
\end{equation}
We see now easily
\[ g_0(\mathcal{L})^\Gamma = \left\{ X = (x_{ij})_{i,j\in \mathbb{Z}_8} \in g_0(\mathcal{L}) \mid \begin{pmatrix} x_{0,i} \\ x_{i+1,i+5} \\ x_{i+4,i+5} \\ x_{i+2,i+3} \end{pmatrix} \in \Xi_{\mathcal{L}/\mathcal{K}}, \quad i \in \mathbb{Z}_8 \setminus \{0\} \right\}. \]
(iv) **Action by** \( \Gamma(= \mathfrak{S}_3) \). The relevant Galois extension is described by the following.

**Lemma 4.8.** A Galois extension field over \( \mathcal{K} \) with Galois group \( \Gamma(= \mathfrak{S}_3) \) is the same as a field \( \mathcal{L} \) of the form \( \mathcal{L} = K(\sqrt{-3}, \sqrt{-3}) \), where
\begin{enumerate}[(a)]
\item \( \alpha \in K^\times \setminus (K^\times)^2 \), so that \( K(\sqrt{-3})/\mathcal{K} \) is a quadratic field extension,
\item \( \beta \in M^\times \setminus (M^\times)^3 \), where \( M = K(\sqrt{-3}) \), and
\item \( \beta \omega \in (K^\times)^3 \), where \( \omega \) is such as given by (4.7).
\end{enumerate}
For such an \( \mathcal{L} \), we have
\begin{enumerate}[(x)]
\item an order 3 element \( \sigma \) and an order 2 element \( \tau \) of \( \Gamma \), which necessarily generate \( \Gamma \), satisfying \( \sigma \tau = \tau \sigma^2 \),
\item a primitive 3rd root \( \omega \) of 1, and
\item an element \( \gamma \) of \( K^\times \) such that \( \gamma^3 = \beta \omega \) (see (c) above),
\end{enumerate}
with which the action by \( \Gamma \) is presented as
\[ \sigma : \sqrt{-3} \mapsto \sqrt{-3}, \quad \frac{\beta}{\sqrt{-3}} \mapsto \omega \sqrt{-3}; \quad \tau : \sqrt{-3} \mapsto -\sqrt{-3}(= \sqrt{-3}), \quad \frac{\beta}{\sqrt{-3}} \mapsto \frac{\gamma}{\sqrt{-3}}. \]
**Proof.** Given \( \beta \) such as in (b), we have a cubic extension \( M(\sqrt{-3})/M \). One sees that \( \beta \omega \in (M^\times)^3 \) if and only if \( M(\sqrt{-3}) = M(\sqrt{-3}) \). If \( \gamma \in M^\times \) and \( \gamma^3 = \beta \omega \), then \( \gamma/\sqrt{-3} \) is a 3rd root of \( \beta \). A point is only to see that \( \sqrt{-3} \mapsto \gamma/\sqrt{-3} \) gives an involution which extends \( M \to M, x \mapsto -x \) if and only if \( \gamma \in K^\times \).

**Example 4.9.** Recall \( K = \mathbb{C}(t) \). One can prove that \( \alpha = 1 - t^3 \) and \( \beta = 1 + \sqrt{1 - t^3} \) satisfy the conditions above. Indeed, a point is to prove \( \beta \notin (M^\times)^3 \), reducing it to show directly that there is no triple of polynomials \( a, b, c \) in \( \mathbb{C}[t] \) with \( c \) monic, such that \( a^3 + 3ab^2c = c^3, 3a^2b + b^3c = c^3 \). The result shows that there exists a Galois extension \( L/\mathcal{K} \) with Gal(\( L/\mathcal{K} \)) = \( \mathfrak{S}_3 \).

Let \( L = K(\sqrt{-3}, \sqrt{-3}), M = K(\sqrt{-3}), \sigma, \tau, \omega \) and \( \gamma \) be as in Lemma 4.8. Recall from the proof of the lemma that \( \gamma/\sqrt{3} \) is a 3rd root of \( \beta \), and denote it by \( \sqrt{3} \); so that one has \( \tau(\sqrt{3}) = \sqrt{3} \) and \( \tau(\sqrt{3}) = \sqrt{3} \).
The action by $\Gamma$ on $\mathfrak{g}_0(=\mathfrak{o}_8(\mathbb{C}))$ is (up to conjugation) such that the generators $\sigma$ and $\tau$ act on the seven vectors in (4.9) as the $K$-linear automorphisms given by the matrix $S$ in (4.10) and the diagonal matrix $D = \text{diag}(-1, 1, 1, 1)$, respectively. The latter action by $\tau$ on $\mathfrak{g}_0$ coincides with the above mentioned outer-automorphism $X \mapsto DXD$ for type $D_\ell$, when $\ell = 4$.

Note $L = M(\sqrt[3]{\beta})$, and apply the previous result for the action by $\langle \sigma \rangle$ (on $L/M$). Then by using the $M$-subspace $\Xi_{L/M}$ of $L^4$ defined by (4.12) (modified into the present situation) we have

$$
\mathfrak{g}_0(L)^{\langle \sigma \rangle} = \left\{ X = (x_{ij})_{i,j \in \mathbb{Z}_8} \in \mathfrak{g}_0(L) \left| \begin{pmatrix} x_{0,i} \\
x_{i+1,i+5} \\
x_{i+4,i+5} \\
x_{i+2,i+3} \
\end{pmatrix} \in \Xi_{L/M}, \ i \in \mathbb{Z}_8 \setminus \{0\} \right. \right\}.
$$

By using the vectors given in (4.11) we define a 4-dimensional $K$-subspace of $L^4$ by

$$
\Theta_{L/K} = Kv_1^{(1)} + K\sqrt[3]{\beta}v_2 + K\left( \sqrt[3]{\beta}v_\omega^2 + 3\sqrt[3]{\alpha}v_\omega \right)
+ K\sqrt[3]{\alpha}\left( \sqrt[3]{\beta}v_\omega^2 - 3\sqrt[3]{\alpha}v_\omega \right).
$$

We see now easily

$$
\mathfrak{g}_0(L)^\Gamma = (\mathfrak{g}_0(L)^{\langle \sigma \rangle})^{\langle \tau \rangle} = \\
\left\{ X = (x_{ij})_{i,j \in \mathbb{Z}_8} \in \mathfrak{g}_0(L) \left| \begin{pmatrix} x_{0,i} \\
x_{i+1,i+5} \\
x_{i+4,i+5} \\
x_{i+2,i+3} \
\end{pmatrix} \in \Theta_{L/K}, \ i \in \mathbb{Z}_8 \setminus \{0\} \right. \right\}.
$$

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Akira Masuoka, Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail address: akira@math.tsukuba.ac.jp

Yuta Shimada, Graduate School of Pure and Applied Sciences, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail address: shimada@math.tsukuba.ac.jp