Fermionic Casimir effect in de Sitter spacetime

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Abstract. The Casimir densities are investigated for a massive spinor field in de Sitter spacetime with an arbitrary number of toroidally compactified spatial dimensions. The vacuum expectation value of the energy-momentum tensor is presented in the form of the sum of corresponding quantity in the uncompactified de Sitter spacetime and the part induced by the non-trivial topology. The latter is finite and the renormalization is needed for the first part only. The asymptotic behavior of the topological term is investigated in the early and late stages of the cosmological expansion. When the comoving lengths of the compactified dimensions are much smaller than the de Sitter curvature radius, to the leading order the topological part coincides with the corresponding quantity for a massless fermionic field and is conformally related to the corresponding flat spacetime result with the same topology. In this limit the topological term dominates the uncompactified de Sitter part and the back-reaction effects should be taken into account. In the opposite limit, for a massive field the asymptotic behavior of the topological part is damping oscillatory.

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1. Introduction
Many of high energy theories of fundamental physics are formulated in higher dimensional spacetimes and it is commonly assumed that the extra dimensions are compactified. In particular, the idea of compactified dimensions has been extensively used in supergravity and superstring theories. From an inflationary point of view universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [1]. These models may play an important role by providing proper initial conditions for inflation (for physical motivations of considering compact universes see also [2]). As it was argued in [3], there are many reasons to expect that in string theory the most natural topology for the universe is that of a flat compact three-manifold. The quantum creation of the universe having toroidal spatial topology is discussed in [4] and in references [5] within the framework of various supergravity theories. In addition to the theoretical work, there has been a large activity to search for signatures of non-trivial topology by identifying ghost images of galaxies, clusters or quasars. Recent progress in observations of the cosmic microwave background provides an alternative way to observe the topology of the universe [6]. If the scale of periodicity is close to the particle horizon scale then the changed appearance of the microwave background sky pattern offers a sensitive probe of the topology.

The compactification of spatial dimensions leads to a number of interesting quantum effects which include instabilities in interacting field theories [7], topological mass generation [8, 9, 10]
and symmetry breaking [10][11]. In the case of non-trivial topology the boundary conditions imposed on fields give rise to the modification of the spectrum for vacuum fluctuations and, as a result, to the Casimir-type contributions in the vacuum expectation values of physical observables (for the topological Casimir effect and its role in cosmology see [12, 13]). A characteristic feature of models with compactified dimensions is the presence of moduli fields which parametrize the size and the shape of the extra dimensions and the Casimir effect has been used for the generation of the effective potential for these moduli. The Casimir energy can also serve as a model for dark energy needed for the explanation of the present accelerated expansion of the universe (see [14, 15, 16] and references therein).

Quantum field theory in de Sitter (dS) spacetime has been extensively studied during the past two decades. Much of early interest was motivated by the questions related to the quantization of fields on curved backgrounds. dS spacetime has a high degree of symmetry and numerous physical problems are exactly solvable on this background. The importance of this theoretical work increased by the appearance of the inflationary cosmology scenario [17]. In most inflationary models an approximately dS spacetime is employed to solve a number of problems in standard cosmology. During an inflationary epoch quantum fluctuations in the inflaton field introduce inhomogeneities and may affect the transition toward the true vacuum. These fluctuations play a central role in the generation of cosmic structures from inflation. More recently astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background [18] indicate that at the present epoch the Universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the Universe would accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. Hence, the investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future.

One-loop quantum effects for various spin fields on the background of dS spacetime have been discussed by several authors (see, for instance, [19]-[34] and references therein). The effects of the toroidal compactification of spatial dimensions in dS spacetime on the properties of quantum vacuum for a scalar field with general curvature coupling parameter are investigated in [35, 36, 37] (the quantum effects in braneworld models with dS spaces and in higher-dimensional brane models with compact internal spaces were discussed in [38, 39]). In the present talk, based on [40, 41] we present the results on the Casimir effect for a fermionic field on background of dS spacetime with spatial topology \( \mathbb{R}^p \times (S^1)^q \).

The paper is organized as follows. In the next section the plane wave eigenspinors in \((D+1)\)-dimensional dS spacetime with an arbitrary number of toroidally compactified dimensions are constructed. In section 3 these eigenspinors are used for the evaluation of the vacuum expectation value of the energy-momentum tensor in both cases of the fields with periodicity and antiperiodicity conditions along compactified dimensions. The behavior of these quantities is investigated in asymptotic regions of the parameters. The special case of topology \( \mathbb{R}^{D-1} \times S^1 \) with the corresponding numerical results is discussed in section 4. The main results are summarized in section 5.

### 2. Plane wave eigenspinors in de Sitter spacetime with compactified dimensions

We consider a quantum fermionic field \( \psi \) on background of \((D+1)\)-dimensional de Sitter spacetime, dS\(_{D+1}\), described by the line element

\[
 ds^2 = dt^2 - e^{2t/\alpha} \sum_{l=1}^{D} (dz^l)^2 ,
\]

where the parameter \( \alpha \) in the expression for the scale factor is related to the corresponding cosmological constant \( \Lambda \) by the formula \( \alpha^2 = D(D - 1)/(2\Lambda) \). We assume that the spatial
coordinates \( z^l, l = p + 1, \ldots, D \), are compactified to \( S^1 \) of the length \( L_l \): \( 0 \leq z^l \leq L_l \), and for the other coordinates we have \( -\infty < z^l < +\infty, l = 1, \ldots, p \). Hence, we consider the spatial topology \( \mathbb{R}^p \times (S^1)^q \), where \( q = D - p \).

The dynamics of the field is governed by the covariant Dirac equation

\[
i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu,
\]

where \( \gamma^\mu = \epsilon^\mu_{(a)} \gamma^{(a)} \) are the generalized Dirac matrices and \( \Gamma_\mu \) is the spin connection. The latter is given in terms of the flat-space Dirac matrices \( \gamma^{(a)} \) by

\[
\Gamma_\mu = \frac{1}{4} \epsilon^{(a)}_{(b)} \gamma^{(b)} \epsilon^\nu_{(a)} \epsilon_{(b)\nu;\mu}.
\]

In these relations \( \epsilon^\mu_{(a)} \) are the tetrad components defined by \( \epsilon^\mu_{(a)} \epsilon^\nu_{(b)} \eta^{ab} = g^{\mu\nu} \), with \( \eta^{ab} \) being the Minkowski spacetime metric tensor. In the \((D + 1)\)-dimensional flat spacetime the Dirac matrices are \( N \times N \) matrices with \( N = 2^{[(D+1)/2]} \), where the square brackets mean the integer part of the enclosed expression. We will assume that these matrices are given in the chiral representation:

\[
\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a^+ & 0 \end{pmatrix}, \quad a = 1, 2, \ldots, D.
\]

From the anticommutation relations for the Dirac matrices one has \( \sigma_a \sigma_b^+ + \sigma_b \sigma_a^+ = 2\delta_{ab} \).

In the discussion below we will denote the position vectors along the uncompactified and compactified dimensions by \( z_p = (z^1, \ldots, z^p) \) and \( z_q = (z^{p+1}, \ldots, z^D) \). One of the characteristic features of the field theory on backgrounds with non-trivial topology is the appearance of inequivalent types of fields with the same spin [42]. For fermion fields the boundary conditions along the compactified dimensions can be either periodic (untwisted field) or antiperiodic (twisted field). First we consider the field with periodicity conditions (no summation over \( l \)):

\[
\psi(t, z_p, z_q + L_l \epsilon_l) = \psi(t, z_p, z_q),
\]

where \( l = p + 1, \ldots, D \) and \( \epsilon_l \) is the unit vector along the direction of the coordinate \( z^l \).

The case of a fermionic field with antiperiodicity conditions will be discussed below. We are interested in the effects of non-trivial topology on the vacuum expectation value (VEV) of the energy-momentum tensor of the fermionic field assuming that the field is prepared in the Bunch-Davies vacuum state (also called Euclidean vacuum). In order to evaluate this VEV, we will use the direct mode-summation procedure. In this approach the knowledge of the complete set of properly normalized eigenspinors, \( \{ \psi_\beta^{(+)}, \psi_\beta^{(-)} \} \), is needed. Here the collective index \( \beta \) presents a set of quantum numbers specifying the solutions. By virtue of spatial translation invariance the spatial part of the eigenfunctions \( \psi_\beta^{(l)} \) can be taken in the standard plane wave form \( e^{\pm i k z} \).

We will decompose the wave vector into the components along the uncompactified and compactified dimensions, \( k = (k_p, k_q) \) with

\[
k = \sqrt{k_p^2 + k_q^2}.
\]

For a spinor field with periodicity conditions along the compactified dimensions the corresponding wave vector has the components

\[
k_q = (2\pi n_{p+1}/L_{p+1}, \ldots, 2\pi n_D/L_D),
\]

where \( n_{p+1}, \ldots, n_D = 0, \pm 1, \pm 2, \ldots \).
Choosing the basis tetrad in the form
\[ e_\mu^{(0)} = \delta_\mu^0, \quad e_\mu^{(a)} = e^{l/\alpha} \delta_\mu^a, \quad a = 1, 2, \ldots, D, \] (8)
for the positive frequency plane wave eigenspinors corresponding to the Bunch-Davies vacuum and satisfying the periodicity conditions along the compactified dimensions we have
\[ \psi_\beta^{(+)} = A_\beta \eta^{(D+1)/2} e^{i k \cdot r} \left( \begin{array}{c} H^{(1)}_{1/2-i \alpha m}(k \eta) w_\sigma^{(+)} \\ -i(n \cdot \sigma) H^{(1)}_{-1/2-i \alpha m}(k \eta) w_\sigma^{(+)} \end{array} \right), \]
where \( \beta = (k, \sigma), \quad n = k/k, \quad w_\sigma^{(+)} = \delta_{l\sigma}. \) In (9), \( H^{(1)}_l(x) \) is the Hankel function, \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_D), \) and we are using the notation
\[ \eta = \alpha e^{-l/\alpha}, \quad 0 \leq \eta < \infty. \] (10)
Note that \( \tau = -\eta \) is the conformal time coordinate, in terms of which the dS line element (1) takes the conformally flat form. Similarly, the negative frequency eigenspinors have the form
\[ \psi_\beta^{(-)} = A_\beta \eta^{(D+1)/2} e^{-i k \cdot r} \left( \begin{array}{c} i(n \cdot \sigma) H^{(2)}_{1/2+i \alpha m}(k \eta) w_\sigma^{(-)} \\ H^{(2)}_{-1/2+i \alpha m}(k \eta) w_\sigma^{(-)} \end{array} \right), \]
with \( w_\sigma^{(-)} = iw_\sigma^{(+)}. \)

The coefficients \( A_\beta \) in the expressions for the eigenspinors are determined from the orthonormalization condition
\[ \int d^D x \sqrt{\gamma} \psi_\beta^{(\pm)} \psi_\beta^{(\pm)\dagger} = \delta_{\beta\beta'}, \] (12)
with \( \gamma \) being the determinant of the spatial metric. On the right-hand side of this condition the symbol \( \delta_{\beta\beta'} \) is understood as the Dirac delta function for the continuous indices and the Kronecker delta for the discrete ones. By making use of the Wronskian relation for the Hankel functions, we can see that
\[ A^2_\beta = \frac{k e^{\pi \alpha m}}{2^{2\pi + 2\pi p - 1} V_q \alpha D}, \] (13)
where \( V_q = L_{p+1} \cdots L_D \) is the volume of the compactified subspace. For a massless fermionic field the Hankel functions in (9) and (11) are expressed in terms of exponentials and we have the standard conformal relation \( \psi_\beta^{(\pm)} = (\eta/\alpha)^{(D+1)/2} \psi_\beta^{(M)(\pm)} \) between eigenspinors defining the Bunch-Davies vacuum in dS spacetime and the corresponding eigenspinors \( \psi_\beta^{(M)(\pm)} \) for the Minkowski spacetime with spatial topology \( \mathbb{R}^p \times (S^1)^q. \)

The plane wave eigenspinors for a twisted spinor field are constructed in a similar way. For this field we have antiperiodicity conditions along the compactified dimensions:
\[ \psi(t, z_p, z_q + L_t e_t) = -\psi(t, z_p, z_q). \] (14)
The corresponding eigenspinors are given by expressions (9), (11), where now the components of the wave vector along the compactified dimensions are given by the formula
\[ k_q = (\pi(2n_{p+1} + 1)/L_{p+1}, \ldots, \pi(2n_D + 1)/L_D), \] (15)
with \( n_{p+1}, \ldots, n_D = 0, \pm 1, \pm 2, \ldots, \) and
\[ k^2 = k_p^2 + \sum_{t=p+1}^{D} [\pi(2n_t + 1)/L_t]^2. \] (16)
Note that the physical wave vector is given by the combination \( k\eta/\alpha. \)
3. Vacuum expectation value of the energy-momentum tensor

The compactification of the spatial dimensions leads to the modification of the spectrum for zero-point fluctuations of fields and as a result of this the VEVs of physical observables are changed. Among the most important quantities characterizing the properties of the vacuum is the expectation value of the energy-momentum tensor. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as a source of gravity in the Einstein equations and plays an important role in modelling a self-consistent structure of the quantum field at a given point, the energy-momentum tensor acts as a source in the VEV of the energy-momentum tensor we will use the direct mode summation approach. The corresponding mode-sum has the form

\[ \langle 0 | T_{\mu\nu} | 0 \rangle = \frac{i}{2} \int d\mathbf{k}_p \sum_{\mathbf{k}_{q,\sigma}} [\bar{\psi}_{\mathbf{k},\sigma}^{(-)} \gamma_\mu \psi_{\mathbf{k},\sigma}^{(-)}] - (\nabla_\mu \bar{\psi}_{\mathbf{k},\sigma}^{(-)} \gamma_\nu \psi_{\mathbf{k},\sigma}^{(-)}) + \text{h.c.}, \]  

(17)

where the eigenspinors \( \psi_\beta^{(-)} = \psi_{\mathbf{k},\sigma}^{(-)} \) are given by \( \| \) and \( \bar{\psi}_{\mathbf{k},\sigma}^{(-)} = \psi_{\mathbf{k},\sigma}^{(-)} + \gamma^0 \) is the Dirac adjoint. For a fermionic field with periodic boundary conditions, substituting the eigenspinors \( \| \) into this formula, for the energy density and the vacuum stresses we find (no summation over \( l \))

\[ \langle 0 | T_0^0 | 0 \rangle = \frac{2^{-p} \eta^{D+1} \alpha^{-D-1}}{\pi^{p/2} \Gamma(p/2) V_q} \int_0^\infty dk_p k_p^{p-1} \sum_{n_{p+1}=-\infty}^{+\infty} \sum_{n_p=-\infty}^{+\infty} k^2 \times \text{Re} \left[ H^{(1)}_{1/2+i\alpha m}(k) H^{(2)}_{-1/2+i\alpha m}(k) - H^{(1)}_{-1/2+i\alpha m}(k) H^{(2)}_{1/2+i\alpha m}(k) \right], \]  

(18)

\[ \langle 0 | T_1^l | 0 \rangle = \frac{2^{-p} \eta^{D+1} \alpha^{-D-1}}{\pi^{p/2} \Gamma(p/2) V_q} \int_0^\infty dk_p k_p^{p-1} \sum_{n_{p+1}=-\infty}^{+\infty} \sum_{n_p=-\infty}^{+\infty} k^2 \times \text{Re} \left[ H^{(1)}_{1/2+i\alpha m}(k) H^{(2)}_{1/2+i\alpha m}(k) \right], \]  

(19)

with \( l = 1, 2, \ldots, D \), and the off-diagonal components vanish. The VEVs given by \( \| \) and \( \| \) are divergent and need some regularization procedure. To make them finite we can introduce a cut-off function \( \varphi_\lambda(k) \) with the cut-off parameter \( \lambda \), which decreases sufficiently fast with increasing \( k \) and satisfies the condition \( \varphi_\lambda(k) \to 1 \), for \( \lambda \to 0 \).

As the next step, we apply to the series over \( n_{p+1} \) in \( \| \) and \( \| \) the Abel-Plana summation formula \( \| \) \( \| \)

\[ \sum_{n=0}^{\infty} f(n) = \int_0^\infty dx f(x) + i \int_0^\infty dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}, \]  

(20)

where the prime means that the term \( n = 0 \) should be halved. The term in the VEV with the first integral in the right-hand side of this formula corresponds to the energy-momentum tensor in dS spacetime with topology \( \mathbb{R}^{p+1} \times S^1 \) of index \( q-1 \). As a result, the VEV of the energy-momentum tensor is presented in the decomposed form

\[ \langle T_1^l \rangle_{p,q} = \langle T_1^l \rangle_{p+1,q-1} + \Delta_{p+1}(T_1^l)_{p,q}. \]  

(21)

Here \( \langle T_1^l \rangle_{p+1,q-1} \) is the VEV of the energy-momentum tensor for the topology \( \mathbb{R}^{p+1} \times S^1 \) of index \( q-1 \) and the part (no summation over \( l \))

\[ \Delta_{p+1}(T_1^l)_{p,q} = \frac{N \eta^{D+2} L_{p+1}}{(2\pi)^{(p+1)/2} V_q \alpha^{D+1}} \sum_{n=1}^{\infty} \sum_{n_{p+2}=-\infty}^{+\infty} \sum_{n_D=-\infty}^{+\infty} \int_0^\infty dx x \times \frac{\text{Re}[I_{1/2-i\alpha m}(\eta x) - I_{1/2+i\alpha m}(\eta x)]}{\cosh(\alpha m \pi)(L_{p+1} n)^{p+1}} f_p(n L_{p+1} \sqrt{x^2 + k^2_{n_{q-1}}}), \]  

(22)
is due to the compactness of the \((p+1)\)th dimension. In \([22]\), \(I_\nu(x)\) and \(K_\nu(x)\) are the modified Bessel functions and we have introduced the notations

\[
f_\nu(x) \equiv x^\nu K_\nu(x), \quad k_{n_{-1}}^2 = \sum_{l=p+2}^D (2\pi n_l/L_l)^2. \tag{23}
\]

For the separate components the functions \(f_p^{(l)}(y)\) in formula \([22]\) have the form

\[
\begin{align*}
    f_p^{(l)}(y) &= f_{(p+1)/2}(y), \quad l = 0, 1, \ldots, p, \\
    f_p^{(p+1)}(y) &= -p f_{(p+1)/2}(y) - y^2 f_{(p-1)/2}(y), \\
    f_p^{(l)}(y) &= k_l^2 (nL_{p+1})^2 f_{(p-1)/2}(y), \quad l = p + 2, \ldots, D,
\end{align*}
\tag{24}
\]

where \(k_l = 2\pi n_l/L_l\). From \([22]\) we see that the topological part depends on the variable \(\eta\) and the length scales \(L_l\) in the combinations \(L_l/\eta\). Noting that \(a(\eta)L_l\) is the comoving length with \(a(\eta) = \alpha/\eta\) being the scale factor, we conclude that the topological part of the energy-momentum tensor is a function of comoving lengths of the compactified dimensions.

The topological term \([22]\) is finite and by the recurrence relation \([21]\) the renormalization procedure for the VEV of the energy-momentum tensor is reduced to the renormalization of the corresponding VEV in uncompactified dS spacetime. It can be seen that the topological part is traceless for a massless field and is covariantly conserved: \(\Delta_{p+1}(T_{p,q}^k)_{\nu} = 0\). From the first relation in \([21]\) we see that the vacuum stresses along the uncompactified dimensions are equal to the energy density,

\[
\Delta_{p+1}(T_{p,q}^0)_{\nu} = \Delta_{p+1}(T_{1}^1)_{p,q} = \cdots = \Delta_{p+1}(T_{p}^p)_{p,q}, \tag{25}
\]

and, hence, in the uncompactified subspace the equation of state for the topological part of the energy-momentum tensor is of the cosmological constant type. Note that the topological parts are time-dependent and they break the dS symmetry.

For a massless fermionic field, the modified Bessel functions in \([22]\) are expressed in terms of elementary functions and after the integration over \(x\) one finds (no summation over \(l\))

\[
\Delta_{p+1}(T_{l}^l)_{p,q} = \frac{2N(\eta/\alpha)^{D+1}}{(2\pi)^{p/2+1}V_q L_{p+1}^{D+1}} \sum_{n=1}^{+\infty} \cdots \sum_{n_{D}=-\infty}^{+\infty} \frac{g_p^{(l)}(nL_{p+1}k_{n_{-1}})}{n_{p+2}^{D+1}}, \tag{26}
\]

with the notations

\[
\begin{align*}
    g_p^{(l)}(y) &= f_{p/2+1}(y), \quad l = 0, 1, \ldots, p, \\
    g_p^{(p+1)}(y) &= - (p + 1) f_{p/2+1}(y) - y^2 f_{p/2}(y), \\
    g_p^{(l)}(y) &= (nL_{p+1}k_l)^2 f_{p/2}(y), \quad l = p + 2, \ldots, D.
\end{align*}
\tag{27}
\]

The massless fermionic field is conformally invariant in any dimension and in this case the problem under consideration is conformally related to the corresponding problem in the Minkowski spacetime with spatial topology \(R^p \times (S^1)^q\). Formula \([26]\) could also be obtained from the relation \(\Delta_{p+1}(T_{l}^l)_{p,q} = a^{-D-1}(\eta) \Delta_{p+1}(T_{l}^{(M)})_{p,q}\), with \(a(\eta)\) being the scale factor. Comparing expression \([26]\) with the corresponding formula from \([36]\) for a conformally coupled massless scalar field, we see that the following relation takes place: \(\Delta_{p+1}(T_{l}^{(scalar)})_{p,q} = -N \Delta_{p+1}(T_{l}^{(M)})_{p,q}\).
After the repetitive application of the recurrence formula (21), the VEV of the energy-momentum tensor for the topology \( \mathbb{R}^p \times (S^1)^q \) is presented in the form

\[
\langle T^k_l \rangle_{p,q} = \langle T^k_l \rangle_{dS,ren} + \langle T^k_l \rangle_c, \quad \langle T^k_l \rangle_c = \sum_{l=1}^{q} \Delta_{D+1-l} \langle T^k_l \rangle_{D-l,l},
\]

where \( \langle T^k_l \rangle_{dS,ren} \) is the renormalized VEV in the uncompactified dS spacetime and \( \langle T^k_l \rangle_c \) is the topological part. For \( D = 3 \) the renormalized VEV of the energy-momentum tensor for fermionic field in uncompactified dS spacetime is investigated in [24] (see also [12]). The corresponding expression has the form

\[
\langle T^k_l \rangle_{dS,ren} = \frac{\delta^k_l}{16 \pi^2 \alpha^4} \left\{ 2m^2 \alpha^2 (m^2 \alpha^2 + 1) [\ln(m \alpha) - \Re \psi(i m \alpha)] + m^2 \alpha^2/6 + 11/60 \right\},
\]

where \( \psi(x) \) is the logarithmic derivative of the gamma-function. In the limit of zero mass, only the last term in curly brackets contributes and the corresponding energy density is positive. For large values of the mass we have:

\[
\langle T^k_l \rangle_{dS,ren} \approx -\frac{\delta^k_l}{960 \pi^2 \alpha^6 m^2}, \quad m \alpha \gg 1,
\]

and the energy density is negative.

In higher dimensions a closed expression can be obtained for even values of \( D \). In this case for the renormalized VEV of the energy-momentum tensor in uncompactified dS spacetime we have the formula [41]

\[
\langle T^k_l \rangle_{dS,ren} = \frac{2 N m^{D+1} \delta^k_l}{(4 \pi)^{(D+1)/2} (D+1)} \frac{\Gamma ((1 - D)/2)}{e^{2 \pi m \alpha} + 1} \prod_{j=0}^{D/2-1} \left[ 1 + \frac{(j + 1/2)^2}{m^2 \alpha^2} \right].
\]

For a massless fermionic field this tensor vanishes. We could expect this result, since in odd-dimensional spacetimes the trace anomaly is absent. For large values of the mass, \( m \alpha \gg 1 \), the VEV for the energy-momentum tensor is exponentially suppressed. This is in contrast with the case of even dimensional spacetimes, where the suppression in the large mass limit is power-law. In particular, for 5-dimensional de Sitter spacetime formula (31) leads to the result

\[
\langle T^k_l \rangle_{dS,ren} = \frac{m^5 \delta^k_l}{15 \pi^2 e^{2 \pi m \alpha} + 1} \left( 1 + \frac{5}{2 m^2 \alpha^2} + \frac{9}{16 m^4 \alpha^4} \right), \quad D = 4.
\]

The corresponding energy density is positive. In figure 1 we have plotted the vacuum energy density \( \langle T^0_0 \rangle_{dS,ren} \) as a function of the parameter \( m \alpha \) for spatial dimensions \( D = 3, 4, 6 \).

In uncompactified dS spacetime the energy-momentum tensor \( \langle T^k_l \rangle_{dS,ren} \) is time-independent and corresponds to a gravitational source of the cosmological constant type. These properties directly follow from the dS invariance of the Bunch-Davies vacuum state. Combining with the initial cosmological constant \( \Lambda \), one-loop effects in uncompactified dS spacetime lead to the effective cosmological constant

\[
\Lambda_{\text{eff}} = \Lambda + 8 \pi G_{D+1} \langle T^0_0 \rangle_{dS,ren},
\]

where \( G_{D+1} \) is the gravitational constant in \( (D + 1) \)-dimensional spacetime. In the discussion above we have assumed that the quantum state of a fermionic field is the Bunch-Davies vacuum state. In [44] it was shown that for a scalar field with a wide range of mass and
curvature coupling parameter the expectation values of the energy-momentum tensor in arbitrary physically admissible states approaches the expectation value in the Bunch-Davies vacuum at late times.

Now we return to the consideration of the topological part in the VEV of the energy-momentum tensor and discuss its behavior in the asymptotic regions of the parameters. For small values of the comoving length $a(\eta)L_{p+1}$ with respect to the dS curvature radius, $a(\eta)L_{p+1} \ll \alpha$, to the leading order the topological part in the VEV of the energy-momentum tensor coincides with that for a massless field given by formula (26). In particular, the topological part of the vacuum energy density is positive. This limit corresponds to the early stages of the cosmological evolution, $t \to -\infty$, and at these stages the total VEV is dominated by the topological part. In this limit the back-reaction effects of the topological terms are important and these effects can change the dynamics essentially (for a discussion of back-reaction effects from vacuum fluctuations on the dynamics of dS spacetime see, for instance, [45] and references therein).

For large values of the comoving length, $a(\eta)L_{p+1} \gg \alpha$, the leading term in the topological part is given by the expression (no summation over $l$):

$$\Delta_{p+1}\langle T^l_l \rangle_{p,q} \approx \frac{NB_l \cos[2mt - 2\alpha m \ln(\alpha/L_{p+1}) - \phi_l]}{2^{p/2} \pi^{(p+1)/2} V_q L_{p+1}^p \cosh(\alpha m \pi) e^{(D+1)t/\alpha}}.$$  \hspace{1cm} (34)

Here the coefficients $B_l$ and the phases $\phi_l$ are defined by the formula

$$B_l e^{i\phi_l} = \frac{2^{-i\alpha m}}{\Gamma(1/2 + i\alpha m)} \sum_{n=1}^{\infty} \sum_{n_{p+2}=\infty}^{+\infty} \cdots \sum_{n_D=-\infty}^{+\infty} h_p^{(l)}(nL_{p+1}k_{n-q-1}) h_{p+1}^{(p+1)}(nL_{p+2}k_{n-q-1}) h_{p+2}^{(p+2)}(nL_{p+3}k_{n-q-1}) \cdots,$$  \hspace{1cm} (35)

with the notations

$$h_p^{(l)}(x) = f_{p/2+1+i\alpha m}(x), \quad l = 0, 1, \ldots, p,$$
$$h_{p+1}^{(p+1)}(x) = -(p + 1 + 2i\alpha m)f_{p/2+1+i\alpha m}(x) - x^2 f_{p/2+i\alpha m}(x),$$
$$h_{p+2}^{(p+2)}(x) = (L_{p+1}k_m)^2 f_{p/2+i\alpha m}(x), \quad l = p + 2, \ldots, D.$$  \hspace{1cm} (36)
Relation (34) describes the asymptotic behavior of the topological part in the late stages of cosmological evolution corresponding to the limit $t \to +\infty$. In this limit the behavior of the topological part for a massive spinor field is damping oscillatory. The damping factor in the amplitude and the oscillation frequency are the same for all terms in the sum of (28) and the total topological term behaves as

$$\langle T^k_i \rangle_c \propto \delta^k_i e^{-(D+1)t/\alpha} \cos \left(2mt + \phi^i_c\right), \quad t \to +\infty.$$  

(37)

As the vacuum energy-momentum tensor for uncompactified dS spacetime is time-independent, we have similar damping oscillations in the total energy-momentum tensor $\langle T^k_i \rangle_{dS,\text{ren}} + \langle T^k_i \rangle_c$. This type of oscillations are absent in the case of a massless field when the topological parts decay monotonically as $e^{-(D+1)t/\alpha}$. Note that in the case of a scalar field the vanishing of the topological part is monotonic or oscillatory in dependence of the mass and the curvature coupling parameter of the field [35, 36].

For a fermionic field with antiperiodicity conditions (14), the VEV of the energy-momentum tensor is found in a way similar to that for the periodicity conditions. The corresponding formulae for the topological parts are obtained from those for the field with periodicity conditions inserting the factor $(-1)^n$ in the summation over $n$ and replacing the definition for $k^2_{nq-1}$ by

$$k^2_{nq-1} = \sum_{l=p+2}^D \left[\pi(2n_l + 1)/L_l\right]^2.$$  

(38)

In situations where the main contribution comes from the term with $n = 1$, the topological parts in the VEV of the energy-momentum tensor for fields with periodicity and antiperiodicity conditions have opposite signs.

4. Special case

In the special case of a single compactified dimension with the length $L$ (topology $R^{D-1} \times S^1$), for the topological part in the VEV of the energy-momentum tensor for an untwisted field we have (no summation over $l$)

$$\langle T^k_i \rangle_c = \frac{N(\eta/L)^D}{(2\pi)^{D/2}\alpha^{D+1}\cosh(\alpha m\pi)} \sum_{n=1}^{\infty} \frac{1}{nD} \int_0^\infty dx \int_0^\infty f^{(l)}_{D-1}(nxL/\eta) \Re\left[I^2_{1/2-i\lambda m}(x) - I^2_{1/2+i\lambda m}(x)\right],$$

(39)

For a massless fermionic field this formula reduces to (no summation over $l$)

$$\langle T^k_i \rangle_c = \frac{N\zeta(D+1)}{\pi^{(D+1)/2}} \left(\eta/\alpha L\right)^{D+1} \Gamma\left(D/2 + 1\right), \quad \langle T^D_i \rangle_c = -D\langle T^0_0 \rangle_c,$$

(40)

with $l = 0, 1, \ldots, D - 1$ and $\zeta(x)$ being the Riemann zeta function. In accordance with the asymptotic analysis given before, the expressions (40) describe the asymptotic behavior of the topological part in the VEV of the energy-momentum tensor for a massive field at early stages of the cosmological expansion. The asymptotics at late stages are obtained from general formula (41) and have the form (no summation over $l$)

$$\langle T^k_i \rangle_c \approx NC_i \cos\left[2mt - 2\alpha m \ln(\alpha/L) - \phi^i_c\right] \frac{\cos[2mt - 2\alpha m \ln(\alpha/L) - \phi^i_c]}{\pi^{D/2}\cosh(\alpha m\pi)}(Le^{\ell/\alpha})^{D+1},$$

(41)
where now $C_l$ and $\phi_l$ are defined by the relations

$$C_l e^{i\phi_l} = \frac{\Gamma((D + 1)/2 + i\alpha m)}{\Gamma(1/2 + i\alpha m)} \zeta(D + 1 + 2i\alpha m), \ l = 0, 1, \ldots, D - 1,$$

$$C_D e^{i\phi_D} = -(D + 2i\alpha m)C_0 e^{i\phi_0}. \ \ \ (42)$$

For a massless field, (41) is reduced to the exact result (40). The corresponding formulae for a twisted spinor field are obtained by adding the coefficient $(2 - D - 1)$ in the first formula of (40) and the coefficient $(2 - D - 2i\alpha m)$ in the first formula of (42).

For $D = 3$ the functions $f_D^{(l)}(x)$ in (39) are expressed in terms of exponentials and after the summation over $n$ we find (no summation over $l$)

$$\langle T_l^1 \rangle_c = \frac{(\eta/L)^3}{\pi\alpha^4 \cosh(\alpha m \pi)} \int_0^\infty dx x G_1(Lx/\eta) \text{Re}[I_{-1/2-i\alpha m}^2(y) - I_{1/2+i\alpha m}^2(y)], \ \ \ (43)$$

where the following notations are introduced

$$G_0(y) = y^2 \ln(1 - e^{-y}), \ G_3(y) = G_0(y) - 2G_1(y), \ G_1(y) = G_2(y) = yL_2(e^{-y}) + L_3(e^{-y}), \ \ \ (44)$$

with $L_n(z) = \sum_{k=1}^\infty z^k/k^n$ being the polylogarithm function. In particular, for a massless fermionic field we have

$$\langle T_1^1 \rangle_c = \frac{2\pi^2}{45} \left( \frac{\eta}{\alpha L} \right)^4 \text{diag}(1, 1, 1, -3). \ \ \ (45)$$

In the case of a twisted field and for the topology $R^2 \times S^1$ one has (no summation over $l$)

$$\langle T_1^1 \rangle_c = \frac{(\eta/L)^3}{\pi\alpha^4 \cosh(\alpha m \pi)} \int_0^\infty dx x G_1^{(tw)}(Lx/\eta) \text{Re}[I_{-1/2-i\alpha m}^2(y) - I_{1/2+i\alpha m}^2(y)], \ \ \ (46)$$

with the notations

$$G_0^{(tw)}(y) = y^2 \ln(1 + e^{-y}), \ G_3^{(tw)}(y) = G_0^{(tw)}(y) - 2G_1^{(tw)}(y),$$

$$G_1^{(tw)}(y) = yL_2(e^{-y}) + L_3(e^{-y}), \ \ l = 1, 2. \ \ \ (47)$$

For a massless twisted field one has the result

$$\langle T_1^1 \rangle_c = \frac{-7\pi^2}{180} \left( \frac{\eta}{\alpha L} \right)^4 \text{diag}(1, 1, 1, -3). \ \ \ (48)$$

Note that for a massless field in dS$_3$ having spatial topology $R^3 \times S^1$ the topological part in the VEV of the energy-momentum tensor has the form

$$\langle T_1^1 \rangle_{c,m=0} = \frac{b\zeta(5)}{\pi^2} \left( \frac{\eta}{\alpha L} \right)^5 \text{diag}(1, 1, 1, -4), \ \ \ (49)$$

where $b = 3$ for the field with periodicity condition and $b = -45/16$ for the field with antiperiodicity condition. This topology corresponds to the original Kaluza-Klein model.

In figure 2 we have plotted the topological parts in the VEVs of the energy density and the vacuum stress along the compactified dimension for untwisted (full curves) and twisted (dashed curves) spinor fields as functions of the ratio $L/\eta$ for the value of the parameter $\alpha m = 1$. The numbers near the curves indicate the component of the energy-momentum tensor ($l = 0$ for the energy density and $l = D$ for the stress). The left panel is for dS$_4$ with topology $R^2 \times S^1$ and
Figure 2. The topological parts in the VEVs of the energy density \((l = 0)\) and the vacuum stress along the compactified dimension \((l = D)\) in units of the dS curvature scale, \(\alpha^{D+1}(T^l_t)\), as functions of the ratio \(L/\eta\) for the value of the parameter \(\alpha m = 1\) and in the special case of spatial topology \(R^{D-1} \times S^1\). Full/dashed curves correspond to spinor fields with periodicity/antiperiodicity conditions along the compactified dimension. The left (right) panel corresponds to \(D = 3\) \((D = 4)\).

the right panel is for dS\(_5\) having spatial topology \(R^3 \times S^1\). Note that the ratio \(L/\eta = a(\eta) L/\alpha\) is the comoving length of the compactified dimension in units of the dS curvature radius. For a massless fermionic field in dS spacetime the corresponding VEVs are given by formulae (45), (48) and (49). As we have explained before, in the limit \(L/\eta \ll 1\), these expressions are the leading terms in the corresponding asymptotic expansion for the VEV of the energy-momentum tensor of the massive field.

From the asymptotic formula (34) it follows that for large values of the ratio \(L/\eta\) the topological part oscillates. As in the scale of figure 2 the oscillations are not well seen, we illustrate this oscillatory behavior in figure 3, where the topological parts in the energy density are plotted for untwisted (full curves) and twisted (dashed curves) fermionic fields \(L/\eta\) for \(\alpha m = 4\). As in figure 2 the left panel is for dS\(_4\) with topology \(R^2 \times S^1\) and the right panel is for dS\(_5\) with spatial topology \(R^3 \times S^1\). The first zero with respect to \(L/\eta\) and the distance between neighbor zeros decrease with increasing values of the parameter \(\alpha m\).

5. Conclusion
We have investigated the Casimir densities for a massive spinor field in \((D + 1)\)-dimensional dS spacetime with an arbitrary number of toroidally compactified spatial dimensions assuming that the field is prepared in the Bunch-Davies vacuum state. For the evaluation of the vacuum expectation value of the energy-momentum tensor the mode-summation procedure is employed. In this approach we need the corresponding eigenspinors satisfying appropriate boundary conditions along the compactified dimensions. These eigenspinors are constructed in section 2 for fields with both periodicity and antiperiodicity conditions. By using the eigenspinors and applying to the mode-sum the Abel-Plana formula, the VEV for the spatial topology \(R^p \times (S^1)^q\) is presented in the form of the sum of the corresponding quantity in the topology \(R^{p+1} \times (S^1)^{q-1}\) and of the part which is induced by the compactness of \((p + 1)\)th dimension. For the field with periodicity conditions the topological part is given by formula (22). The corresponding formula for the field with antiperiodicity conditions is obtained from that for the field with periodicity
conditions inserting the factor \((-1)^n\) in the summation over \(n\) and replacing the definition for \(k_{n+1}^2\) by \((38)\).

The topological part is finite and the renormalization procedure for the VEV of the energy-momentum tensor is reduced to that for the uncompactified dS spacetime. This part is time-dependent and breaks the dS symmetry. The corresponding vacuum stresses along the uncompactified dimensions coincide with the energy density and, hence, in the uncompactified subspace the equation of state for the topological part of the energy-momentum tensor is of the cosmological constant type. For a massless fermionic field the problem under consideration is conformally related to the corresponding problem in the Minkowski spacetime with spatial topology \(R^D \times (S^1)^q\) and we have the standard relation

\[
\langle T_{l}^{k} \rangle_{c} = a^{-(D+1)}(\eta)\langle T_{l}^{k} \rangle^{(M)}
\]

between the topological terms.

For a massive fermionic field, in the limit when the comoving length of a compactified dimension is much smaller than the dS curvature radius, the topological part in the VEV of the energy-momentum tensor coincides with the corresponding quantity for a massless field and is conformally related to the VEV in toroidally compactified Minkowski spacetime. In particular, the topological part in the vacuum energy density is positive for an untwisted fermionic field. This limit corresponds to the early stages of the cosmological evolution and the topological part dominates over the uncompactified dS part. At these stages the back-reaction effects of the topological term are important and these effects can essentially change the dynamics of the model. In the opposite limit, when the comoving lengths of the compactified dimensions are large with respect to the dS curvature radius, in the case of a massive field the asymptotic behavior of the topological part is damping oscillatory and to the leading order is given by formula \((34)\). These formula describes the behavior of the topological part in the late stages of the cosmological expansion. As the uncompactified dS part is time-independent, we have similar oscillations in the total VEV as well. Note that this type of oscillatory behavior is absent for a massless fermionic field.

**Figure 3.** The topological part in the VEV of the energy density as a function of the ratio \(L/\eta\) for the value of the parameter \(\alpha m = 4\) and in the special case of spatial topology \(R^{D-1} \times S^1\). Full/dashed curves correspond to spinor fields with periodicity/antiperiodicity conditions along the compactified dimension. The left (right) panel corresponds to \(D = 3\) (\(D = 4\)).
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