No-Go for Quantum Seals

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Authors note: After posting this manuscript, we were made aware of a number of previous works along very similar lines, such as Refs. [1–5], amongst others. We had formulated this concept and arrived at our results independently, and regret that as a result we did not properly cite these works and failed to put our work in a proper context. We are grateful to the individuals who brought this to our attention, and we are gratified to find that our ideas, while not as new as we had believed, are at least of interest to the community. We are currently working on assessing how our results relate to these prior works. In the mean time, we remain proud of our work, and we present it below in its original form.

Abstract

We introduce the concept of a quantum “seal” and investigate its feasibility. We define a seal as a state provided by Alice to Bob along with a set of instructions which Bob can follow in order to “break the seal” and read the classical message stored inside. We define two success criteria for a seal: the probability Bob can successfully read the message if he follows Alice’s instructions without any further input must be high, and if Alice asks for the state back from Bob, the probability Alice can tell if Bob broke the seal without permission must be high. We prove that within the constraints of quantum mechanics these two criteria are mutually exclusive. Finally, we derive upper and lower bounds on the achievability of a probabilistic seal for various message lengths, and show that for a 1-bit message our bound is tight.

1 Introduction

We investigate whether it is possible to both communicate classical information using a quantum state, but also verify whether that information has been extracted. We phrase this problem using the scenario of a quantum “seal,” in analogy to the impressions made in wax or clay that have been used to ensure the integrity of letters and packages for thousands of years [6]. Alice wishes to give Bob a message that is only to be opened and read by Bob at a later date when an agreed upon set of conditions has been met. For example, the conditions might be “Do not open the message until the third night of Hannukah,” or “Only open the message if instructed to do so by Alice.” To this end, Alice gives Bob a quantum state and a classical description of a quantum measurement. If Bob makes the measurement described by Alice, with high probability he will extract the classical value of the message. Bob should be able to make this measurement without further information from Alice. However, at any time before the agreed upon conditions have been met, Alice can request the state back, and she would like to detect with high probability whether Bob has cheated and read the message, thereby “breaking the seal” prematurely. A seal therefore has two success criteria: 1) the probability Bob can successfully read the message if he follows Alice’s instructions must be high, and 2) the probability Alice can tell if Bob broke the seal without permission must be high.
A quantum seal would represent a novel quantum information technology with a number of potential applications. For example, the message Alice gives to Bob could be a one-time cipher pad protected by the seal, so that when required Bob could use the pad without any additional communication over a secure channel, but prior to that Alice could periodically verify that copies of the pad had not been made. By similar logic, a quantum seal could protect launch codes or single-use passwords. Quantum seals might even be used to balance civil liberties with security. For example, a government might need access to telephone records in the event of an imminent threat, but otherwise promises that the records will remain private and will not be used for other purposes. By providing the records under a quantum seal, the telephone company could both guarantee access while also periodically checking that the records were not tampered with.

Sadly, we prove a no-go result for quantum seals: if Bob can read the message with high probability, Alice will be unlikely to detect his measurement. This may seem counterintuitive given results from quantum key distribution (QKD), which tell us that measurements made by an eavesdropper can be detected. However, QKD schemes involve distributing a random string to be used as a key, which can then be used to securely transfer classical information. The robustness of QKD relies not on the necessary disturbance induced by measurement but on the impossibility of measuring an unknown state in two non-orthonormal bases simultaneously [7, 8]. Here we require that Bob be able to access the classical information stored behind the seal with no additional input from Alice. Viewed in this context, it is perhaps not so surprising that any intuition from QKD fails.

Our scenario also differs from previous studies of cryptographic measurement disturbance in several respects. Previous works assume Bob (or an eavesdropper) has a description of the possible states that Alice might send [7–11]. In our case, we want to give Bob the minimum amount of information he needs to read the message, so we do not give him a description of the state, but rather a description of the measurement he should make. Second, in the context of quantum cryptography, it is most important to give lower bounds on the size of the change in the state due to Bob’s measurement [7, 12–17]. In our case, our goal is to upper bound the size of the change in the state due to Bob’s measurement.

There are two results that are particularly relevant to this work. The first is the Gentle Measurement Lemma [18, 19], which states that if a measurement outcome occurs with high probability, and if that outcome is measured, then the state after the measurement is difficult to distinguish from the original state. Part of our results amount to extending the Gentle Measurement Lemma to the case when a measurement outcome should occur with high probability, but whether that outcome actually occurred when the measurement was performed is unknown.

The second relevant concept is that of one-time memories [20]. A one-time memory is one of several protocols (such as bit-commitment and oblivious transfer), for which there are quantum no-go results [21–24]. However, were they to exist, one-time memories could implement the functionality of a seal straightforwardly and simply. A one-time memory is a device that contains two messages $s, t \in \{0, 1\}^n$. Once one message is read, the other is destroyed. To create a seal from a one-time memory, Alice would set $s$ to be the message, and set $t$ to be a random string. Bob could therefore learn $s$ perfectly without further input from Alice. However, if he were to break the seal early, $t$ would become inaccessible. He could try to make a new one-time memory to give back to Alice as a fake, but he would not know what string to store as $t$ and his guess would be inaccurate with high probability. Thus, Alice could ask for the box back, try to learn $t$, and if it was inaccessible or not the string she stored, she would know Bob had cheated. Since seals are weaker than one-time memories, our no-go for quantum seals does not necessarily follow from the no-go for quantum one-time memories. However, we note that under certain physically realistic assumptions (such as no entangling operations), quantum one-time memories can exist [25], which implies the existence of quantum seals under similar restricted scenarios.
2 Preliminaries

2.1 Notation and Quantum Measurement

We use $\mathcal{H}$ to denote a Hilbert space, and $\mathcal{D}(\mathcal{H})$ to denote the set of positive linear operators acting on $\mathcal{H}$ with trace one; $\mathcal{D}(\mathcal{H})_1$ is the set of density matrices on $\mathcal{H}$. For $N \in \mathbb{N}$, we let $[N] = \{1, \ldots, N\}$. $I_A$ denotes the identity operator on $\mathcal{H}_A$, but we drop the subscript if clear from context.

A quantum measurement is described by a positive operator value measure (POVM). A POVM is a set of operators $\{E_i\}_{i \in [N]}$ acting on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^{N} E_i = I$ and $E_i \succeq 0$. Given a state $\rho \in \mathcal{D}(\mathcal{H})$, the probability of measuring outcome $i$ is $\text{tr}(E_i \rho)$.

There are an infinite number of ways to implement a given POVM [26, 27] (see [28] for a nice description of methods to implement a POVM) and the implementation affects the state the system is left in after the measurement. Here, we think of the implementation as a two step process. In the first step, we apply what we call the standard implementation: if $\rho$ is measured and outcome $i$ is obtained, the system is transformed as $\rho \rightarrow \sqrt{E_i} \rho \sqrt{E_i} / \text{tr}(E_i \rho)$. (Averaging over the possible outcomes, $\rho$ is transformed as $\rho \rightarrow \sum_i \sqrt{E_i} \rho \sqrt{E_i}$) In the second step, a completely positive trace preserving (CPTP) map is applied to the resultant state, and this map can depend on the outcome $i$ of the first step.

The trace distance between quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ is $\frac{1}{2} \| \rho - \sigma \|_1$, where $\|A\|_1 = \text{tr}(\sqrt{A^* A})$. The trace distance has an important operational meaning: given a state promised to be either $\rho$ or $\sigma$ with equal probability, the maximum probability of correctly guessing which is the case is $\frac{1}{2} + \frac{1}{4} \| \rho - \sigma \|_1$.

The Gentle Measurement Lemma [18, 19] says that if a state has high overlap with a POVM operator, and that outcome is measured, then the post-measurement state will not differ considerably from the pre-measurement state. More precisely, given a POVM operator $E_i$ on $\mathcal{H}$ and a state $\rho \in \mathcal{D}(\mathcal{H})$, where $\text{tr}(E_i \rho) \geq q$, then

$$\| \rho - \sqrt{E_i} \rho \sqrt{E_i} \|_1 \leq 2\sqrt{1 - q}. \quad (1)$$

2.2 Set-Up

We assume the message protected by the seal takes a value $m \in [M]$ for $M \geq 2$. Alice encodes the message into a pure state $|\psi_m\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ refers to System $A$, the part of the system that Alice retains, while $\mathcal{H}_B$ refers to System $B$, the part of the system Alice gives to Bob as the sealed message. (If Alice instead would prefer to encode into a mixed state, we can always purify the state without loss of generality. See e.g. [29].) Thus Alice sends to Bob the state $\rho_m \in \mathcal{D}(\mathcal{H}_B)$, where $\rho_m$ is the reduced density matrix of $|\psi_m\rangle$ on $\mathcal{H}_B$.

In addition to the state $\rho_m$, Alice gives Bob a classical description of a POVM $\hat{E} = \{E_{ij}\}_{i \in [M], j \in [M]}$. She promises Bob that if he performs the POVM $\hat{E}$ on $\rho_m$, he will get an outcome $(i, j)$ such that $i = m$ with probability at least $p$. This promise implies

$$\sum_{j \in [M_m]} \text{tr} \left( E_{mj} \rho_m \right) \geq p. \quad (2)$$

After giving System $B$ to Bob, Alice may ask for it to be returned to her at any point. Bob’s goal in our scenario is to make a measurement on $\rho_m$ that allows him to determine $m$, but when he returns the system to Alice, she cannot detect his measurement. Alice’s goal is to design a state $\rho_m$ and a POVM $\hat{E}$, with the properties described above, such that Bob can not learn $m$ without
significantly altering $\rho_m$, so that when Alice asks for System $B$ to be returned, she can reliably determine whether Bob has cheated.

We use two metrics to judge Alice’s success in detecting Bob’s potential breaking of the seal. First, suppose Alice knows that with probability $1/2$, Bob measured in such a way as to learn the message $m$ with probability at least $p$, and with probability $1/2$, he did nothing. We call $p_{\text{guess}}$ her maximum success probability in guessing whether or not Bob measured. Second, suppose Alice would like to detect if Bob made a measurement that allowed him to learn the message with probability at least $p$, but never wants to falsely accuse Bob of cheating if he didn’t actually read the message. We call $p_{\text{NFP}}$ (NFP for “no false positives”) her maximum success probability in this task. If Alice chooses her message state $|\psi_m\rangle$ from some distribution of states, we take $p_{\text{guess}}$ and $p_{\text{NFP}}$ to be her success probabilities averaged over her choice of state.

For the quantum seal to behave as desired, we would like to have a protocol in which $p$ is large, and $p_{\text{guess}}$ or $p_{\text{NFP}}$ are large, so that Bob can read the message correctly with high probability, but also Alice can detect if he read it or not. However, we show that if $p$ is large, Bob can implement a POVM on $\rho_m$ in such a way that $p_{\text{guess}}$ and $p_{\text{NFP}}$ will be small.

## 3 A Naive Approach that Fails

In this section, we present a straightforward strategy that seems promising, yet ultimately fails.

Let $M = 2$, so Alice wants to encode a binary message. Suppose Alice creates a state on $3q$ qubits that she plans to send entirely to Bob; that is, she sets $\mathcal{H}_A = 1$, and $\mathcal{H}_B = \mathbb{C}^{2^q}$. She chooses $\sigma, \tau \in S_{3q}$ uniformly at random, where $S_n$ is the symmetric group of degree $n$. Then Alice gives one of the following states to Bob, depending on whether she wants the message to be “1” or “2”:

$$|\psi_1\rangle = U_\sigma |0\rangle^{\otimes 2q} |+\rangle^{\otimes q},$$

$$|\psi_2\rangle = U_\tau |1\rangle^{\otimes 2q} |+\rangle^{\otimes q},$$

(3)

where $U_\sigma$ (respectively $U_\tau$) is a unitary that acts on a Hilbert space of $3q$ qubits, and permutes the qubit registers according to the permutation $\sigma$ (resp. $\tau$).

Alice tells Bob to measure each qubit using the POVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ (i.e. the standard basis projective measurement), and if the number of 0 outcomes is at least $3q/2$, he should decide $m = 1$, and otherwise, he should decide $m = 2$. If Bob uses the standard implementation of Alice’s POVM, he will perform a projective measurement, and he will be able to read the message perfectly, since the the number of 0 outcomes will be at most $q$ when $m = 1$, and at least $2q$ when $m = 2$.

After the standard implementation, Bob is left with a standard basis state, which will be nearly orthogonal to the original state. Thus if Alice asked for the state back, she would with high probability be able to detect Bob’s measurement.

Bob could try to disguise his measurement by applying a CPTP map to alter his state after measurement. Let’s assume, without loss of generality, that $m = 1$, and also that Bob knows that Alice originally sent a state of the form $U_\sigma |0\rangle^{\otimes 2q} |+\rangle^{\otimes q}$ for some $\sigma \in S_{3q}$. (This extra information can only help Bob.) Bob can replace qubits in registers where he got outcome 1 with states $|+\rangle$, to try to make his state closer to Alice’s original state. However with extremely high probability in the limit of large $q$ (using e.g. Hoeffding’s inequality [30]), he will measure 0’s in about half of the registers that originally contained the state $|+\rangle$. For large $q$, Bob has a vanishingly small probability of correctly guessing where these “false 0” registers are, and if he guesses incorrectly, it will make his overlap worse. Thus, there is very little Bob can do to recover from the measurement; the seal has been broken, and Alice will detect his measurement.
So why does this protocol fail? While Alice told Bob that he should measure in the standard basis, Bob can instead use Alice’s instructions to make a different but related measurement. He measures using the standard implementation of the POVM \( \{ \Pi_1, \Pi_2 \} \) where \( \Pi_1 \) is the projector onto standard basis states whose strings have more than \( 3q/2 \) zeros, and \( \Pi_2 \) is the projector onto the remaining standard basis states. Bob has thereby combined all of the measurement operators that correspond to a given outcome into a single measurement operator. For any choice of \( \sigma, \tau \in S_{3q}, \) \( \Pi_1|\psi_1\rangle = |\psi_1\rangle \) and \( \Pi_2|\psi_2\rangle = |\psi_2\rangle \). Thus Bob can deterministically distinguish the value of the message without disturbing the state and breaking the seal, and Alice will be completely unaware of his measurement.

In the next section, we show that there is always a way for Bob to cheat in a manner similar to this, as long as Alice wants Bob to be able to read the message with high probability.

4 No-Go For Quantum Seals

We will show that a good strategy for Bob is to apply the standard implementation of the POVM \( \{ F_i \}_{i \in [M]} \), for

\[
F_i = \sum_{j \in [M]} E_{i,j} \tag{4}
\]

where \( E_{i,j} \) are the elements of Alice’s recommended POVM. If outcome \( F_i \) occurs, Bob decides the message is \( i \). Averaged over Bob’s outcome, the full state on \( \mathcal{H}_A \otimes \mathcal{H}_B \) after measurement is (see Section 2.1)

\[
\sum_{i=1}^{M} \mathcal{I}_A \otimes \sqrt{F_i} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{F_i}. \tag{5}
\]

Now if Alice asks for Bob to return his system, and he did not make a measurement, she will have the state \( |\psi_m\rangle \langle \psi_m| \). If he did make the measurement using the POVM in Eq. (4), the state will be that in Eq. (5).

We first bound \( p_{\text{guess}} \). Alice’s goal is to determine which state she possesses. We assume Alice knows that if Bob made a measurement, he measured using the standard implementation of the POVM in Eq. (4), as this information can only help her. Then using the properties of the trace distance (see Section 2.1) and Eq. (5), the probability that she correctly guesses whether Bob has read the message is:

\[
p_{\text{guess}} \leq \frac{1}{2} + \frac{1}{4} \left\| |\psi_m\rangle \langle \psi_m| - \sum_{i=1}^{M} \mathcal{I}_A \otimes \sqrt{F_i} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{F_i} \right\|_1
\]

\[
\leq \frac{1}{2} + \frac{1}{4} \left\| |\psi_m\rangle \langle \psi_m| - \mathcal{I}_A \otimes \sqrt{F_m} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{F_m} \right\|_1
\]

\[
+ \frac{1}{4} \sum_{i \in [M] \setminus m} \left\| \mathcal{I}_A \otimes \sqrt{F_i} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{F_i} \right\|_1, \tag{6}
\]

where we’ve used the triangle inequality. By the Gentle Measurement Lemma (Eq. (1)) [18, 19] and Alice’s promise (Eq. (2)), we have the first norm in the second line of Eq. (6) is bounded as

\[
\left\| |\psi_m\rangle \langle \psi_m| - \mathcal{I}_A \otimes \sqrt{F_m} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{F_m} \right\|_1 \leq 2 \sqrt{1 - \text{tr}(F_m \rho_m)}. \tag{7}
\]
Figure 1: An upper bound on the probability Alice can correctly guess if Bob has cheated, $p_{\text{guess}}$, is plotted as a function of the probability $p$ that Bob can successfully read the sealed message if he follows Alice’s instructions (solid red line). This bound holds for any message length ($M \geq 2$). A lower bound on the achievable $p_{\text{guess}}$ in the case of a single bit message ($M = 2$), using the protocol described in Sec. 5, is also shown (dashed red line). The shaded red portion in between the two lines represents what may be achievable for $M = 2$ using a different protocol.

We can rewrite the sum of norms in Eq. (6) using the fact that $\mathcal{I}_A \otimes \sqrt{\mathcal{F}_i} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{\mathcal{F}_i}$ is positive semidefinite, and that $\sum_{i \in [M]} \mathcal{F}_i = \mathbb{I}_B$:

$$\sum_{i \in [M] \setminus m} \left\| \mathcal{I}_A \otimes \sqrt{\mathcal{F}_i} |\psi_m\rangle \langle \psi_m| \mathcal{I}_A \otimes \sqrt{\mathcal{F}_i} \right\|_1 = \sum_{i \in [M] \setminus m} \text{tr} \left( \mathcal{I}_A \otimes \mathcal{F}_i |\psi_m\rangle \langle \psi_m| \right) = 1 - \text{tr} (\mathcal{F}_m \rho_m). \quad (8)$$

Combining Eqs. (2), (4) and (6) to (8), we have

$$p_{\text{guess}} \leq \frac{1}{2} + \frac{1}{4} \left( 2 \sqrt{1 - \text{tr}(\mathcal{F}_m \rho_m)} + 1 - \text{tr}(\mathcal{F}_m \rho_m) \right) \leq \frac{1}{2} + \frac{1}{4} \left( 2 \sqrt{1 - p} + 1 - p \right), \quad (9)$$

which we plot in Fig. 1.

Next we bound $p_{\text{NFP}}$. If Bob is honest, when he returns System $B$ to Alice, she will have the state $|\psi_m\rangle$. Therefore, Alice needs a two-outcome POVM, such that one outcome will never occur if Bob is honest. The POVM that achieves this is $\{ \mathcal{I}_{AB} - |\psi_m\rangle \langle \psi_m|, |\psi_m\rangle \langle \psi_m| \}$, where the first outcome
Figure 2: Upper bounds on the probability Alice can know with certainty that Bob has cheated, $p_{NFP}$, are plotted as a function of the probability $p$ that Bob can successfully read the will if he follows Alice’s instructions, for different ranges of possible message lengths $M$. For the case of a single bit message, $M = 2$, the plotted function is achievable (solid green line), as proven in Sec. 5, but for the larger values of $M$ the plot only represents upper bounds on $p_{NFP}$.

will only be observed if Bob is dishonest. Thus

$$p_{NFP} = \text{tr} \left( (I_{AB} - |\psi_m\rangle \langle \psi_m|) \left( \sum_{i \in [W]} (I_A \otimes \sqrt{F_i}|\psi_m\rangle \langle \psi_m|I_A \otimes \sqrt{F_i}) \right) \right)$$

$$= 1 - \sum_{i \in [M]} |\langle \psi_m|I_A \otimes \sqrt{F_i}|\psi_m\rangle|^2$$

(10)

Now

$$\langle \psi_m|I_A \otimes \sqrt{F_i}|\psi_m\rangle \geq \langle \psi_m|I_A \otimes F_i|\psi_m\rangle = \text{tr}(F_i \rho_m)$$

(11)

because the eigenvalues of $F_i$ are between 0 and 1. Plugging in and using Cauchy-Schwarz, we have

$$p_{NFP} \leq 1 - \text{tr}(F_i \rho_m)^2 - \frac{(\sum_{i \in [M]} \text{tr}(F_i \rho_m))^2}{M - 1} = 1 - \text{tr}(F_m \rho_m)^2 - \frac{(1 - \text{tr}(F_m \rho_m))^2}{M - 1}$$

(12)

When $M \gg 1$, using Eq. (2), we have $p_{NFP} \leq 1 - p^2$, while in the case of $M = 2$, we have $p_{NFP} \leq (1 - p)^2$. Bounds on $p_{NFP}$ for several values of $M$ are shown in Fig. 2.
By either metric ($p_{\text{guess}}$ or $p_{\text{NFP}}$), we see there is a trade off. If Alice wants Bob to be able to read the message with probability close to 1, then she will not be able to detect with high probability whether he has broken the seal.

5 Achievability

In this section, we investigate whether Alice can achieve the bounds of Section 4. We design a specific strategy for Alice and determine the probability with which she can detect Bob’s cheating. This allows us to put lower bounds on $p_{\text{guess}}$ and $p_{\text{NFP}}$ since, of course, the specific strategy we choose might not be optimal.

Alice decides to send Bob a single qubit with a binary message encoded, and she sends Bob the full state; that is, $\mathcal{H}_A = 1$, $\mathcal{H}_B = \mathbb{C}^2$, and $M = 2$. Alice chooses $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2\}$ where

$$\mathcal{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and tells Bob that his maximum probability of success is $p > 1/2$. In fact, Alice even gives Bob the additional information that $|\psi_1\rangle$ and $|\psi_2\rangle$ have the form

$$|\psi_1\rangle = \sqrt{p}|0\rangle + e^{i\phi}\sqrt{1-p}|1\rangle,$$

$$|\psi_2\rangle = \sqrt{1-p}|0\rangle + e^{i\phi}\sqrt{p}|1\rangle,$$

for $\phi$ that Alice has chosen uniformly at random from $[0, 2\pi]$. Since this additional information can only help Bob, it will still allow us to put lower bounds on $p_{\text{guess}}$ and $p_{\text{NFP}}$.

Alice’s success probabilities $p_{\text{guess}}$ and $p_{\text{NFP}}$ are always analyzed in the case that Bob makes a measurement that obtains the correct outcome with probability at least $p$. But note that the unique optimal POVM for distinguishing $|\psi_1\rangle$ and $|\psi_2\rangle$ is the POVM $\mathcal{E}_1$ and $\mathcal{E}_2$ from Eq. (13) [31], and this measurement only succeeds with probability $p$. This implies that Bob has no choice but to measure using an implementation of this POVM.

The standard implementation applies a projective measurement into the $\{|0\rangle, |1\rangle\}$ basis. Next, Bob could apply a CPTP map to shift his state away from either $|0\rangle$ or $|1\rangle$. However, note that Bob has lost all knowledge of the initial state $|\psi_m\rangle$.

Bob wants to minimize $p_{\text{guess}}$ on average over Alice’s choice of $\phi$. Using the fact that the trace norm is equal to the Euclidean norm on the Bloch sphere [29], and using the fact that the average distance from a point to the edge of a circle is minimized when the point is in the center of the circle, we have that Bob’s optimal strategy is to return a state of the form:

$$\rho(x) = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}.$$ (16)

A straightforward calculation then shows that Bob will minimize the trace distance to Alice’s state on average by returning $\rho(p)$ if $m = 1$, and $\rho(1-p)$ if $m = 2$. Luckily for Bob, if he simply returns the the result of his standard implementation (either the state $|0\rangle$ or $|1\rangle$) to Alice, on average over his measurement outcomes, he will return precisely $\rho(p)$ in the case that $m = 1$, and $\rho(1-p)$ when $m = 2$. This gives us Bob’s optimal strategy if he is trying to minimize $p_{\text{guess}}$.

We next show that Bob should use the same strategy if he instead would like to minimize $p_{\text{NFP}}$. We can calculate $\text{tr}((I - |\psi_m\rangle\langle\psi_m|)\rho)$ for any single qubit state $\rho$ that Bob might return to Alice. Averaging this quantity over $\phi$, we find that Bob minimizes his probability of being detected if he
sends Alice $\rho(1)$ when $m = 1$ and $\rho(0)$ when $m = 2$. However, because Bob doesn’t know whether $m = 1$ or $m = 2$, the best he can do is to guess that $m = 1$ when he gets outcome $\mathcal{E}_1$, and send $\rho(1)$ in that case. Likewise he should send $\rho(0)$ when he gets outcome $\mathcal{E}_2$. This is the same strategy as in the case of $p_{\text{guess}}$.

We can now bound $p_{\text{guess}}$ (assuming Alice knows Bob implements the optimal strategy, since there is no reason for Bob to use any other strategy):

$$p_{\text{guess}} \geq \frac{1}{2} + \frac{1}{2} \|\rho(p) - |\psi_1\rangle\langle\psi_1|\|_1 = \frac{1}{2} + \frac{\sqrt{2}p(1-p)}{4}. \quad (17)$$

Likewise, we can bound $p_{\text{NFP}}$ as

$$p_{\text{NFP}} \geq \text{tr}(\left((I - |\psi_1\rangle\langle\psi_1|)\rho(p)\right)) = (1-p)2p. \quad (18)$$

Comparing with Eq. (12), we see that our bound on $p_{\text{NFP}}$ is tight, at least in the case of a single bit message!

Thus, while the upper bounds of Section 4 tell us that we can not achieve arbitrary large values for $p$, $p_{\text{guess}}$, and $p_{\text{NFP}}$, in this section, we find that Alice can strike a reasonable balance, at least in the case of $M = 2$. For example, if Alice can tolerate Bob being able to read the message with probability 0.8, Alice can detect if he measured with probability 0.32, so she will catch Bob approximately a third of the time.

6 Conclusion

In this work, we have proved that the transfer of classical information using a quantum state is mutually exclusive to being able to detect with high probability that the information was extracted by measurement. We proved this by introducing the concept of a “seal,” which is a quantum state that Alice gives Bob, along with instructions for how that state is to be measured in order to properly “read” it while breaking the seal. Furthermore, we would like the seal to satisfy the two requirements that (1) Bob has a high probability of reading the message correctly if he follows Alice’s instructions, and (2) Alice is able to detect with high probability if Bob has read it and “broken the seal.” We presented a naive approach that fails, and then proceeded to place upper bounds on the success probabilities for the criteria using any quantum states, proving that the dual requirements of a “seal” are in direct competition with each other. We also derived lower bounds on the achievability of these success criteria, showing that while a truly deterministic seal is not possible within the realm of quantum mechanics, a probabilistic seal that works only some of the time is.

We leave open several questions: is the standard implementation of the POVM $\{F_i\}_{i \in [M]}$ always the optimal one for Bob to use? What is a good strategy for Bob if he would like to read the message with less certainty, but have an even smaller probability of breaking the seal? We showed that our upper bound on $p_{\text{NFP}}$ is tight for the case of $M = 2$; is our bound tight for larger values of $M$?

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