Katugampola-type fractional differential equations with delay and impulses

Abstract

Our aim in this note is to study the existence of solutions of a Katugampola-type fractional impulsive differential equation with delay. We use successive approximation method to show the existence of solutions. In the end, an example is given to verify the hypothetical results.

Keywords: katugampola fractional derivative, impulsive equations, time delay

Introduction

Because of its wide applicability in biology, medicine and in more and more fields, the theory of fractional differential equations (FDEs) has recently been attracting increasing interest, see for instance and references therein. Impulsive differential equations have played an important role in modelling phenomena, especially in describing dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulse differential systems to describe the model since the last century. For the basic theory on impulsive differential equations, the reader can refer to the books Motivated by the papers, the aim of this note is to discuss the existence and uniqueness of solutions of Katugampola-type FDEs with delay and impulses.

Consider the Katugampola-type FDEs with delay and impulse of the form,

\[
\begin{align*}
\frac{^pD_t^\alpha}{\omega} \mathcal{Z}(t) &= \mathcal{Y}(t, \mathcal{Z}_t), \quad t \in T, \quad \mathcal{Z} \in \mathbb{C}([0, T]); \\
\Delta\mathcal{Z}(t_k) &= I^k_{\omega, t} \mathcal{Z}(t_k), \quad k = 1, 2, \ldots, m; \\
\mathcal{Z}(0) &= 0,
\end{align*}
\]

where \( \frac{^pD_t^\alpha}{\omega} \) is the generalized fractional derivative in Caputo sense, \( \omega \in \mathbb{R}^+ \), \( \omega > 0 \), \( 0 = t_0 < t_1 < \cdots < t_m = T \), \( \omega \in \mathbb{C}(\mathbb{R} \times \mathbb{R}) \) and \( I^k_{\omega, t} \) are given functions satisfying some assumptions that will be specified later. \( \Delta\mathcal{Z}(t_k) = \mathcal{Z}(t_k) - \mathcal{Z}(t_{k-1}) \) and \( \mathcal{Z}(t_k) \) represent the right and left limits of \( \mathcal{Z}(t) \) at \( t = t_k \) respectively, and they satisfy that \( \mathcal{Z}(t_k) = \mathcal{Z}(t_{k-1}) \). If \( \mathcal{Z}(t_k) \in \mathcal{C}([-\mu, T], \mathbb{R}) \), then for any \( t \in [0, T] \), define \( \mathcal{I}_\omega(t) = \mathcal{Z}(t + \theta) \) for \( \theta \in [-\mu, 0] \), where \( \mathcal{I}_\omega(t) \) represents the history of the state from time \( t - \mu \) to present time \( t \), \( \omega \in \mathbb{C}([\mu, 0], \mathbb{R}) \) and \( \psi(0) = 0 \).

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preliminary results. In Section 3, the existence and uniqueness of the problem(1) are obtained by successive approximation method. In Section 4, an example is given to demonstrate the effectiveness of the main results.

Preliminaries

In this section, we recollect several definitions of fractional derivatives and integrals from the papers.

**Definition 4.1** The fractional (arbitrary) order integral of the function \( \mathcal{Y} \in L^1([a, b], \mathbb{R}) \) of order \( \omega \in \mathbb{R}^+ \) is defined by

\[
\mathcal{I}_\omega^\mathcal{Y}(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t - s)^{\omega-1} \mathcal{Y}(s) \, ds,
\]

where \( \Gamma \) is the gamma function.

**Definition 4.2** For a function \( \mathcal{Y} \) given on the interval \([a, b]\), the Caputo fractional order derivative of \( \mathcal{Y} \), is defined by

\[
\mathcal{D}_\omega^\mathcal{Y}(t) = \frac{\Gamma(\omega)}{\Gamma(\omega - 1)} \int_a^t (t - s)^{\omega-1} \mathcal{Y}(s) \, ds,
\]

where \( \omega \in \mathbb{R}^+ \).

**Definition 4.3** The generalized left-sided fractional integral \( \mathcal{I}_s^\omega \mathcal{Y}(t) \) of order \( \omega \in \mathbb{C}(\mathbb{R} \times \mathbb{R}) \) is defined by

\[
\mathcal{I}_s^\omega \mathcal{Y}(t) = \frac{1}{\Gamma(\omega)} \int_t^b (s - t)^{\omega-1} \mathcal{Y}(s) \, ds,
\]

for \( t > \alpha \), if the integral exists.

**Definition 4.4** The generalized fractional derivative, corresponding to the generalized fractional integral (4), is defined for \( 0 < \alpha < t \), by

\[
\mathcal{D}_s^\omega \mathcal{Y}(t) = \frac{\Gamma(\omega)}{\Gamma(\omega - 1)} \int_t^b (s - t)^{\omega-1} \mathcal{Y}(s) \, ds,
\]
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\[(\rho^a D^\alpha_z 5\beta)(t) = \int_0^{\alpha^a} \left( \frac{d}{d(\alpha^a_t)} \right)^n (t'\omega-a)^n \left( \rho^a - x^a \right)^{n-a-1} s^{\rho^a-1} \delta(s) ds, \quad \text{if \ the \ integral \ exists.} \]

**Definition 4.5** The Caputo-type generalized fractional derivative, \( \rho^a D^\alpha_z \delta(t) \) is defined via the above generalized fractional derivative (5) as follows

\[(\rho^a D^\alpha_z \delta)(t) = \int t^a \frac{d}{dt} \left( \frac{d}{dt} \right)^{n-1} (t'\omega-a)^{n-a} \delta(s) ds, \quad \text{for each, } a \in \mathbb{R}. \]

It can be decomposed as

\[\rho^a D^\alpha_z \delta(t) = \int t^a \frac{d}{dt} \left( \frac{d}{dt} \right)^{n-1} (t'\omega-a)^{n-a} \delta(s) ds. \]

**Remark 4.7** In Caputo sense, the Katugampola fractional derivative operator \( \rho^a D^\alpha_z \) is a left inverse of the integral operator \( \rho^a I^\alpha_z \) but in general is not a right inverse,

\[\rho^a D^\alpha_z \left( \rho^a I^\alpha_z \delta(t) \right) = \delta(t) \]

and the following holds

\[\rho^a D^\alpha_z \left( \rho^a I^\alpha_z \delta(t) \right) = \delta(t) - \sum_{k=0}^{n-a} \frac{1}{k!} (t'\omega-a)^{k} \delta(s) ds. \]

**Lemma 4.8** Assume that \( \delta \in \mathcal{C} \), \( T > 0 \). A function \( \delta \in \mathcal{C} \) is a solution of the initial value problem

\[
\begin{align*}
\rho^a D^\alpha_z \delta(t) & = 5(t) \quad \text{at } t = 0, \quad t \in \mathbb{R};
\Delta \delta(t_i) & = I_i(\delta(t_i)), \quad k = 1, 2, \ldots, m; \\
\delta(t) & = \psi(t), \quad t \in [-\mu, 0],
\end{align*}
\]

if and only if \( \delta \) satisfies the following integral equation

\[
\begin{align*}
\psi(t) & = \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds + \sum_{j=1}^{k+1} I_j(\delta(t_j)), \\
& + \sum_{j=1}^{k} \left( t'\omega-a \right)^{j} 5(s) ds + \sum_{j=1}^{k} I_j(\delta(t_j)).
\end{align*}
\]

**Proof.** Assume that \( \delta \) satisfies (9). One can see, from Remark 2.7

and \( \psi(0) = 0 \), that

\[
3(t) = \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds, \quad \text{for } t \in [0, t_1].
\]

In view of \( \left[ t' \right] - \left[ t' \right] = I(\delta(t)), \) we get that

\[
3(t) = I(\delta(t)) + \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds.
\]

It follows that, for \( t \in [t_1, t_2], \)

\[
3(t) = \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds + \sum_{j=1}^{k} I_j(\delta(t_j)).
\]

In consequence, we can see, by means of \( 3(t) = I(\delta(t)) + I_2(\delta(t)), \) that

\[
3(t) = \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds + \sum_{j=1}^{k} I_j(\delta(t_j)).
\]

Conversely, if \( \delta \) is a solution of (10), one can obtain by a direct computation, that \( \rho^a D^\alpha_z \delta(t) = 5(t), \) \( t \neq t_1, \ t \in [0, T], \) and

\[
\Delta \delta(t_i) = I_i(\delta(t_i)), \quad k = 1, 2, \ldots, m.
\]

This completes the proof.

**Existence and uniqueness results**

Initially, set \( C_0 = [v \in \mathcal{C} : \mathbb{R}], \) \( \nu(0) = 0 \). For each, \( v \in C_0, \) we denote by \( \nu \) the function defined by

\[
\nu(t) = v(t), \quad 0 \leq t \leq T \quad \text{and } \nu(0) = 0.
\]

If \( \delta \) is a solution of (1), then \( \delta(t) \) can be decomposed as

\[
\begin{align*}
& 3(t) = \rho^{a-a-\alpha} (t, 0) + \sum_{j=1}^{k+1} \left( t'\omega-a \right)^{j} 5(s) ds + \sum_{j=1}^{k} I_j(\delta(t_j)) \\
& + \sum_{j=1}^{k} I_j(\delta(t_j)).
\end{align*}
\]
\[3(\tau)=\tau(t)+\phi(t)\text{ for } -\mu \leq t \leq T,\] which implies that \(\exists \tau_0, \phi\) for \(0 \leq t \leq T,\) where
\[
\phi(t)=0, \text{ for } 0 \leq t \leq T, \text{ and } \phi(t)=\psi(t), -\mu \leq t \leq 0. \quad (12)
\]

Therefore, the problem (1) can be transformed into the following fixed point problem of the operator \(N:\mathbb{C}_0 \rightarrow \mathbb{R},\)
\[
Nv(t)=\frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \left( (t^\alpha-s^\alpha)^{\alpha-1}s^\alpha \right) \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} f(\tau(t_j)), t_e(t_j, \tau_j), k=0,1,2,\ldots,m. \quad (13)
\]

Now, let us present our main results.

**Theorem 5.1** For the functions \(\varphi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})\) and \(L \in \mathbb{R},\) assume the following conditions hold

- There exists a continuous function \(\alpha \in [0, T] \rightarrow \mathbb{R}^+\) satisfying
  \[\delta(t, p_j) = \delta(t, a_j) \leq (\alpha(\varphi(t)))^{-1}\rho(s, q(s)), \text{ for } s \in [0, T];\]
- There exists a constant \(L > 0\) such that
  \[\sum_{j=1}^{\alpha(\varphi(t))} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
  \[> 0, \text{ for } \alpha > 0, \text{ and } \rho \in \mathbb{R}, t_e(t_j, \tau_j); \]
- There exists a constant \(M > 0\) such that
  \[\|N(t, \phi)\| \leq M, \text{ where } \|\phi\| \text{ is defined in (12)}.
\]

**Proof**

To complete the proof, we shall use the method of successive approximations. Define a sequence of functions \(v_n \in [0, T] \rightarrow \mathbb{R},\)
\[
v_0(t) = 0, v_n(t) = Nv_{n-1}(t). \quad (14)
\]

Since \(v_0(t) = 0,\) it is easy to see from (11) that \(v_n(t) = 0\) for \(s \in [0, T].\)

Thus, we have,
\[v(t) - v_n(t) \leq \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds + \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} f(\tau(t_j)), t_e(t_j, \tau_j), k=0,1,2,\ldots,m. \quad (13)
\]

It follows from the above inequalities with \(N_1 < 1\) that \(v_n(t) \rightarrow v(t)\) uniformly convergent to \(v(t)\) with respect to \(t.\)

In what follows, we shall show that \(v(t)\) is a solution of the equation (1). Observe that
\[v(t) = \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds + \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} \frac{\rho_{\alpha}(t)}{\Gamma(\alpha)} \int_{\mathbb{R}^+} f(s, \tau_0, \phi) ds
\]
\[+ \sum_{j=1}^{k-1} f(\tau(t_j)), t_e(t_j, \tau_j), k=0,1,2,\ldots,m. \quad (13)
\]

Therefore,

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\[ \frac{D^\rho_{\alpha(T)}}{\Gamma(\alpha)}(t^\nu\phi(t)) + \alpha(t) \sum_{j=1}^{m} a_j(t) \phi(t) \] (17)

where \( a_j(t) \) are non-negative.

In consequence, we can see that for a sufficiently large number \( n \geq n_0 \),

\[ |v(t) - Nv(t)| \leq |v(t) - v_n(t)| + |v_n(t) - Nv(t)| \]

Let us take, \( \rho = \frac{1}{2}, \Gamma(\alpha + 1) > \frac{4}{10} \) and \( \mu \) is a non-negative constant, \( \mu(\theta) = \frac{1}{\Gamma(\alpha + 1/2)} \) for \( -\mu \leq \theta \leq 0 \) and \( 0 \leq \theta \leq 1 \).

Finally, we prove the uniqueness of the solution. Assume that \( v_1, v_2 : [0, T] \rightarrow \mathbb{R} \) are two solutions of (1). Note that

\[ |v_1(t) - v_2(t)| \leq \frac{1}{2} \frac{D^\rho_{\alpha(T)}}{\Gamma(\alpha)}(t^{\nu+1}) [\alpha(t)_{\sup}](x(t) - T\nu(x)) \] (18)

According to the conditions \( (A_1) \), the uniqueness of the problem (1) follows immediately, which completes the proof.

An illustrative example

Consider the following Katugampola-type fractional impulsive differential equation with delay of the form

\[ \hat{v}(t) = \frac{\hat{v}(t)}{(9 + e^t)(1 + \frac{1}{3})} \]

Now, we can see that

\[ |\hat{v}(t, \alpha) - \hat{v}(t, \alpha)| \leq \frac{e^{-t}}{(9 + e^t)(1 + \frac{1}{3})} \leq \frac{e^{-t}}{(9 + e^t)} \]

where \( \alpha(t) = e^{-t} \) and \( \alpha = \alpha_{\sup}(t) = \frac{1}{10} \), so the condition \( (A_1) \) is satisfied.

On the other hand, we get that

\[ |\hat{v}(p, \alpha)| \leq \frac{3}{2} e^{-t} \leq \frac{3}{2} |\hat{v}(p, \alpha)|. \]

which satisfies the condition \( (A_2) \) of Theorem 3.1 with \( L = \frac{1}{3} \).

By a direct computation, we obtain that

\[ \lim_{t \to 0^+} \hat{v}(p, \alpha) = \frac{1}{3} \frac{1}{2} \frac{1}{10} \]

and

\[ |\hat{v}(p, \alpha)| \leq \frac{e^{-t}}{(9 + e^t)} \leq \frac{e^{-t}}{10} \]

As a result, the equations in (20) satisfy all the hypotheses in Theorem 3.1 which guarantees that (20) has a unique solution.

Conclusion

In this note, the existences of solutions of a Katugampola-type fractional impulse differential equation with delay were investigated. The successive approximation method was employed to show the existence of solutions. The example reflects the applicability of the proposed method.
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Conflict of interest
Author declares that there is no conflict of interest.

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