Dilations of semigroups on von Neumann algebras and noncommutative $L^p$-spaces

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Abstract

We prove that any $w^*$-continuous semigroup of factorizable Markov maps acting on a von Neumann algebra $M$ equipped with a state can be dilated by a group of Markov $*$-automorphisms in a manner analogous to the discrete case of one factorizable Markov operator. We also give a version of this result for strongly continuous semigroups of operators acting on noncommutative $L^p$-spaces, examples of semigroups to which the results of this paper can be applied and applications of these results to functional calculus of the generators of these semigroups.

1 Introduction

The study of dilations of operators is of central importance in operator theory and has a long tradition in functional analysis. Suppose $1 < p < \infty$. A classical result from seventies essentially due to Akcoglu [AKS] (see also [Pel]) says that a positive contraction $T: L^p(\Omega) \to L^p(\Omega)$ on an $L^p$-space $L^p(\Omega)$ admits a positive isometric dilation $U$ on a bigger $L^p$-space than the initial $L^p$-space, i.e. there exists another measure space $\Omega'$, two positive contractions $J: L^p(\Omega) \to L^p(\Omega')$ and $P: L^p(\Omega') \to L^p(\Omega)$ and a positive invertible isometry $U: L^p(\Omega') \to L^p(\Omega')$ such that $T^k = PU^kJ$ for any integer $k \geq 0$. Note that in this situation, $J$ is an isometric embedding whereas $JP$ is a contractive projection.

Later, Fendler [Fen1] proved a continuous version of this result for any strongly continuous semigroup $(T_t)_{t \geq 0}$ of positive contractions on an $L^p$-space $L^p(\Omega)$. More precisely, this theorem says that there exists a measure space $\Omega'$, two positive contractions $J: L^p(\Omega) \to L^p(\Omega')$ and $P: L^p(\Omega') \to L^p(\Omega)$ and a strongly continuous group of positive invertible isometries $(U_t)_{t \in \mathbb{R}}$ on $L^p(\Omega')$ such that $T_t = PU_tJ$ for any $t \geq 0$, see also [Fen2].

In the noncommutative setting, measure spaces and $L^p$-spaces are replaced by von Neumann algebras and noncommutative $L^p$-spaces and positive maps by completely positive maps. In their remarkable paper [JLM], Junge and Le Merdy showed that there exists no “reasonable” analog of Akcoglu result for completely positive contractions acting on noncommutative $L^p$-spaces. It is a striking difference with the world of classical (=commutative) $L^p$-spaces of measure spaces.

Independently, Kümmerer, Maassen, Haagerup and Musat introduced and studied dilations of well-behaved completely positive unital operators on noncommutative probability spaces (=von Neumann algebras equipped with states), the so-called Markov operators [Kum1] [Kum2].

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These dilations induce dilations on the associated noncommutative $L^p$-spaces. The following definition of these operators is considered in [AnD], [HaM] and [Ric].

Definition 1.1 Let $(M, \phi)$ and $(N, \psi)$ be von Neumann algebras equipped with normal faithful states $\phi$ and $\psi$, respectively. A linear map $T : M \to N$ is called a $(\phi, \psi)$-Markov map if:

1. $T$ is completely positive,
2. $T$ is unital,
3. $\psi \circ T = \phi$,
4. $T \circ \sigma^\phi_t = \sigma^\psi_t \circ T$, for all $t \in \mathbb{R}$, where $\sigma^\phi_t$ and $\sigma^\psi_t$ denote the automorphism groups of the states $\phi$ and $\psi$, respectively.

In particular, when $(M, \phi) = (N, \psi)$, we say that $T$ is a $\phi$-Markov map. Such an operator $T$ induces a contraction $T : L^p(M) \to L^p(M)$ on the associated noncommutative $L^p$-space $L^p(M)$ for any $1 \leq p < \infty$, see for example [AnD] lemma 2.4.

The following definition is essentially due to Kümmner (see [Kum2] Definition 2.1.1). We refer to [Kum1] [Kum2] [Kum3] [KuM] for physical interpretations of this notion.

Definition 1.2 Let $M$ be a von Neumann algebra with a normal faithful finite state $\phi$ and let $T : M \to M$ be a $\phi$-Markov map. We say that $T$ is dilatable if there exists a von Neumann algebra $N$ with a normal faithful state $\psi$, a $*$-automorphism $U$ of $N$ leaving $\psi$ invariant and a $(\phi, \psi)$-Markov $*$-monomorphism $J : M \to N$ satisfying

$$T^k = EU^kJ, \quad k \geq 0.$$ 

where $E = J^* : N \to M$ is the canonical faithful normal conditional expectation preserving the states associated with $J$.

Note that Haagerup and Musat [HaM] Theorem 4.4] have succeeded in characterizing dilatable Markov maps. Indeed, they proved that a $\phi$-Markov map $T$ is dilatable if and only if $T$ is factorizable in the sense of [AnD], i.e. there exists a von Neumann algebra $N$ equipped with a faithful normal state $\psi$ and $*$-automorphisms $J_0 : M \to N$ and $J_1 : M \to N$ such that $J_0$ is $(\phi, \psi)$-Markov and $J_1$ is $(\phi, \psi)$-Markov, satisfying, moreover, $T = J_0^* \circ J_1$.

Now, we introduce the continuous version of this definition from [Arh2] Definition 1.3] inspired by Fendler result, see also [KumM] Definition page 4].

Definition 1.3 Let $M$ be a von Neumann algebra equipped with a normal faithful state $\phi$. Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of $\phi$-Markov maps on $M$. We say that the semigroup is dilatable if there exist a von Neumann algebra $N$ equipped with a normal faithful state $\psi$, a $w^*$-continuous group $(U_t)_{t \in \mathbb{R}}$ of $*$-automorphisms of $N$, a $*$-monomorphism $J : M \to N$ such that each $U_t$ is $\psi$-Markov and $J$ is $(\phi, \psi)$-Markov satisfying

$$(1.1) \quad T_t = EU_tJ, \quad t \geq 0,$$

where $E = J^* : N \to M$ is the canonical faithful normal conditional expectation preserving the states associated with $J$.

Note that such a dilation induces an isometric dilation similar to the one of Fendler theorem for the strongly continuous semigroup induced by the semigroup $(T_t)_{t \geq 0}$ on the associated noncommutative $L^p$-space $L^p(M)$ for any $1 \leq p < \infty$.

Our main result is the following theorem:
Theorem 1.4  let $M$ be a von Neumann algebra equipped with a normal faithful state $\phi$. Let $(T_t)_{t \geq 0}$ be a $w^*$-semigroup of factorizable $\phi$-Markov maps on $M$. Then the semigroup $(T_t)_{t \geq 0}$ is dilatable.

In particular, this result implies that all $w^*$-semigroups of selfadjoint Markov Fourier multipliers are dilatable, see Corollary 5.1. We also prove Theorem 4.4 which is a variant of this result for noncommutative $L^p$-spaces useful even for non-$\sigma$-finite von Neumann algebras. Finally, we refer to the paper in preparation [JRS] for related results.

In the last section, we also give applications to $H^\infty$ functional calculus which is a very useful and important tool in various areas: harmonic analysis of semigroups, multiplier theory, Kato’s square root problem, maximal regularity in parabolic equations, control theory, etc. For detailed information we refer the reader to [Han], [JMX] and [KW] and to the references therein.

The paper is organized as follows. The first section gives background Section 3 gives a proof of Theorem 1.4. In the following section 4 we describe and prove a noncommutative $L^p$ analog of this result. In section 5 we give examples of dilatable semigroups. Finally, we conclude in section 6 with the applications of our results to functional calculus.

2 Preliminaries

Noncommutative $L^p$-spaces We use Haagerup noncommutative $L^p$-spaces. We refer to the survey [PiX] and to the papers [Ray2] and [Pis1] for more information.

Markov operators Note that a linear map $T: M \to N$ satisfying conditions (1) – (3) of Definition 1.1 is automatically normal. If, moreover, condition (4) is satisfied, then it was proved in [AcC] (see also [AnD], Lemma 2.5) that there exists a unique completely positive unital map $T^*: N \to M$ such that

$$\phi(T^*(y)x) = \psi(yT(x)), \quad x \in M, y \in N.$$  

(2.1)

It is easy to show that $T^*$ is a $(\psi, \phi)$-Markov map. Moreover, we say that a $\phi$-Markov map $T: M \to M$ is selfadjoint if $T = T^*$. Indeed, it is not difficult to prove that a $(\phi, \psi)$-Markov $*$-homomorphism is always injective. Suppose $T(x) = 0$. We have $\phi(x) = \psi(T(x)) = 0$. Hence $x = 0$ by the positivity of $T$ and the faithfulness of $\phi$. Now if $y \in M$ satisfies $T(y) = 0$. We have $T(y)^*T(y) = 0$. Since $T$ is a $*$-homomorphism, we infer that $T(y^*y) = 0$. We deduce that $y^*y = 0$ and therefore that $y = 0$.

Ultraproducts of Banach spaces Let $(X_n)_{n \geq 1}$ be a sequence of Banach spaces, and let $\ell^\infty(N, X_n)$ be the Banach space of all sequences $(x_n)_{n \geq 1} \in \prod_{n=1}^\infty X_n$ with $\sup_{n \geq 1} \|x_n\|_{X_n} < \infty$ with the norm $\|(x_n)_{n \geq 1}\| = \sup_{n \geq 1} \|x_n\|_{X_n}$. Let $U$ be a free ultrafilter on $N$. The Banach space ultraproduct $(X_n)^U$ is defined as the quotient $\ell^\infty(N, X_n)/J_U$, where $J_U$ is the closed subspace of all $(x_n)_{n \geq 1} \in \ell^\infty(N, X_n)$ which satisfies $\lim_{n \to U} \|x_n\|_{X_n} = 0$. An element of $(X_n)^U$ represented by $(x_n)_{n \geq 1} \in \ell^\infty(N, E)$ is written as $(x_n)^U$. For any $(x_n)^U \in (X_n)^U$, one has $\|(x_n)^U\| = \lim_{n \to U} \|x_n\|_{X_n}$. If $(T_n: X_n \to Y_n)_{n \geq 1}$ is a bounded sequence of bounded linear operators, we can define the ultraproduct map $T: (X_n)^U \to (Y_n)^U$, $(x_n)^U \mapsto (T_n(x_n))^U$. We refer to [DJM], section 8 for more information.

If $1 < p < \infty$, an ultraproduct of noncommutative $L^p$-spaces is a noncommutative $L^p$-space, see [Ray1]. However, the Banach space ultraproduct of von Neumann algebras is not a von Neumann algebra in general.
Ultraproducts of von Neumann algebras  If \( \phi \) is a normal faithful state on a von Neumann algebra \( M \), we define \( \| \cdot \|_\phi^1 \) by
\[
\|x\|_\phi^1 = \phi(x^*x + xx^*)^\frac{1}{2}, \quad x \in M.
\]
Let us now define the (Ocneanu) ultraproduct \((M_n, \phi_n)^U\) of a sequence \((M_n, \phi_n)_{n \geq 1}\) of σ-finite von Neumann algebras equipped with normal faithful states \( \phi_n \) with respect to a free ultrafilter \( \mathcal{U} \) over \( \mathbb{N} \). Define \( \ell^\infty(N, M_n) \) the C*-algebra of sequences \((x_n)_{n \geq 1} \in \prod_{n=1}^{\infty} M_n \) such that \( \sup_{n \geq 1} \|x_n\|_{M_n} < +\infty \) endowed with the norm \( \| (x_n) \|_{\ell^\infty(N, M_n)} = \sup_{n \geq 1} \|x_n\|_{M_n} \). Let \( \mathcal{U} \) be free ultrafilter on \( \mathcal{N} \). We let
\[
\mathcal{I}_U(M_n, \phi_n) := \left\{ (x_n)_{n \geq 1} \in \ell^\infty(N, M_n) : \|x_n\|_{\phi_n}^1 \underset{n \to \mathcal{U}}{\longrightarrow} 0 \right\},
\]
and also, with the abbreviated notation \( \mathcal{I}_U \) for \( \mathcal{I}_U(M_n, \phi_n) \), let
\[
\mathcal{M}^U(M_n, \phi_n) := \{ (x_n)_{n \geq 1} \in \ell^\infty(N, M_n) : (x_n)_{n} \mathcal{I}_U \subset \mathcal{I}_U, \text{ and } \mathcal{I}_U(x_n)_{n} \subset \mathcal{I}_U \}.
\]
It is then apparent that \( \mathcal{M}^U(M_n, \phi_n) \) is a C*-algebra (with pointwise operations and supremum norm) in which \( \mathcal{I}_U(M_n, \phi_n) \) is a closed ideal. We then define
\[
(M_n, \phi_n)^U := \mathcal{M}^U(M_n, \phi_n)/\mathcal{I}_U(M_n, \phi_n)
\]
(the quotient C*-algebra). Moreover, \( (M_n, \phi_n)^U \) is a W*-algebra. We denote the image of \((x_n)_{n \geq 1} \in \mathcal{M}^U(M_n, \phi_n) \) in \( (M_n, \phi_n)^U \) as \((x_n)^U\). Finally, the following defines a normal faithful state \((\phi_n)^U\) on \( (M_n, \phi_n)^U \):
\[
(\phi_n)^U((x_n)^U) := \lim_{n \to \mathcal{U}} \phi_n(x_n), \quad (x_n)^U \in (M_n, \phi_n)^U.
\]
See [AHW], [AnH] and [Pis2, Section 9.10] for more information.

Similarly to [Ued], page 352, if \((J_n: M_n \to N_n)_{n \geq 1}\) is a sequence of \((\phi_n, \psi_n)\)-Markov *-homomorphism then it is not difficult to prove that we can define the ultraproduct map \((J_n)^U: (M_n)^U \to (N_n)^U\), \((x_n)^U \mapsto (J_n(x_n))^U\) which is a \(((\phi_n)^U, (\psi_n)^U)\)-Markov *-monomorphism. Indeed the \(J_n\)'s induce a *-homomorphism \( J = \oplus J_n: \ell^\infty(N, M_n) \to \ell^\infty(N, N_n) \). Moreover it is easy to check that if \((x_n)_{n \geq 1} \in \mathcal{M}^U(M_n, \phi_n) \) then \((J_n(x_n))_{n \geq 1} \in \mathcal{M}^U(N_n, \psi_n) \) and if \((x_n)_{n \geq 1} \in \mathcal{I}_U(M_n, \phi_n) \) then \((J_n(x_n))_{n \geq 1} \in \mathcal{I}_U(N_n, \psi_n) \). The restriction of \( \oplus J_n \) gives a map \( \mathcal{M}^U(M_n, \phi_n) \to \mathcal{M}^U(N_n, \psi_n) \). The quotient map is the ultraproduct map \((J_n)^U\). Moreover, if \( \mathbb{E}_n: N_n \to M_n \) is the canonical faithful normal conditional expectation preserving the states associated with \( J_n \) then the equation
\[
(\mathbb{E}_n)^U((x_n)^U) = (\mathbb{E}_n(x_n))^U
\]
gives rise to a well-defined normal faithful conditional expectation \((\mathbb{E}_n)^U: (N_n)^U \to (M_n)^U\) such that \((\phi_n)^U \circ (\mathbb{E}_n)^U = (\psi_n)^U\).

Convexity  A normed linear space \( X \) is locally uniformly convex if for any \( \varepsilon > 0 \) and any \( x \in X \) with \( \|x\| = 1 \) there exists \( \delta(\varepsilon, x) > 0 \) such that \( \|y\| = 1 \) and \( \frac{\|y+x\|}{2} > 1 - \delta(\varepsilon, x) \) imply \( \|y - x\| \leq \varepsilon \). It is clear from the definition that uniform convexity implies local uniform convexity.
Semi-groups of operators  Let $X$ be a Banach space. Recall that a semigroup $(T_t)_{t \geq 0}$ of operators on $X$ is strongly continuous if the map $t \mapsto T_t x$ is continuous from $\mathbb{R}^+$ into $X$ for any $x \in X$.

Let $X$ be a dual Banach space with predual $X_*$. We say that a semigroup $(T_t)_{t \geq 0}$ of operators on $X$ is $w^*$-continuous if the map $t \mapsto \langle y, T_t x \rangle_{X_*, X}$ is continuous on $\mathbb{R}^+$ for any $x \in X$ and any $y \in X_*$. In passing, note that the weak* topology on the Banach space $B(X)$ is the topology of pointwise convergence on $X$ endowed with the $\sigma(X, X_*)$-topology.$^1$

Representations of groups and kernels  Let $X$ be a Banach space. Let $\pi : G \to B(X)$ be a representation of a group $G$ on $X$. Then we say that $\pi$ is bounded when $\sup \{ \| \pi_i \| : t \in G \} < \infty$.

We need some notions and results of the papers [DLG1] and [DLG2]. Recall that a non-empty subset $D$ of an algebraic semigroup $\mathcal{S}$ is called a two-sided ideal if $\mathcal{S} D \subseteq D$ and if $D \mathcal{S} \subseteq D$. If $\mathcal{S}$ is a semigroup, the intersection of all the two-sided ideals of $\mathcal{S}$ is called the kernel of $\mathcal{S}$. If $\mathcal{S}$ be a compact (Hausdorff) semitopological semigroup, that is a semigroup with separately continuous semigroup operations, then it is known [DLG1] Theorem 2.3] that its kernel is non-empty.

Let $\pi : G \to B(X)$ be a (non-continuous) bounded representation of a topological group $G$ on a reflexive Banach space $X$. Then we denote by

$$X_c = \{ x \in X : t \mapsto \pi_t x \text{ is continuous from } G \to X \}$$

the subspace of continuously translating elements of $X$ for the representation $\pi$. Let $V(e)$ be the set of all neighbourhoods $V$ of the identity $e$ of $G$. We then set $S^c(\pi)$ be the closure in the weak operator topology of the convex hull of $\bigcap_{V \in V(e)} \{ \pi_t : t \in V \}$, endowed with the weak operator topology and called the convex semigroup of $\pi$ over the identity $e$. Then it is known [DLG2] Lemma 2.3] that $S^c(\pi)$ is a compact semitopological semigroup. A consequence of [DLG1] Theorem 7.2] is that the kernel $K(\pi)$ of $S^c(\pi)$ consists entirely of projections. The result [DLG2] Lemma 2.4] (and the remarks before) gives the following result.

**Theorem 2.1** Let $X$ be a reflexive Banach space and $\pi : G \to B(X)$ be a (non-continuous) bounded representation of a commutative topological group $G$. Then the kernel $K(\pi)$ of the convex semigroup $S^c(\pi)$ of $\pi$ contains a unique idemotent $Q$ and $Q$ is a bounded projection of $X$ on $X_c$ with $Q \pi_t = \pi_t Q$ for any $t \in G$.

Accumulation points  Let $(x_i)_{i \in I}$ be a net in a topological space $X$. An accumulation point of the net $(x_i)_{i \in I}$ is an element of the intersection $\bigcap_{F \in \mathcal{F}} F$ where

$$\mathcal{F} = \{ F \subseteq X : \text{there exists } i_0 \in I \text{ such that } \{ x_i : i \geq i_0 \} \subseteq F \}$$

or equivalently a limit of some subnet of $(x_i)_{i \in I}$.

3 Dilations of semigroups on von Neumann algebras

Suppose that $X$ is a dual Banach space $X$ with predual $X_*$. It is well-known [BM] Exercise 1.12 page 251] that the space $B_{w^*}(X)$ of weak* continuous operator of $B(X)$ is a semitopological semigroup with respect to the weak* topology and that the mapping

$$B(X_*) \xrightarrow{T} B_{w^*}(X)$$

1. A net $(T_i)$ in $B(X)$ converges to a point $T \in B(X)$ in the weak* topology if and only if for any $x \in X$ and any $y \in X_*$, we have $\langle y, T_i x \rangle_{X_*, X} \to \langle y, T x \rangle_{X_*, X}$.
is a weak operator-weak* homeomorphism. Moreover, it is easy to prove that $B_{w^*}(X)$ is a closed subspace of $B(X)$.

Let $G$ be a topological group and let $\pi : G \to B_{w^*}(X)$ be a (non-continuous) bounded representation on a dual Banach space $X$ by weak* continuous operators. Then, we define

$$X_{w^*} = \left\{ x \in X : t \mapsto \langle y, \pi_t x \rangle_{X, \ast} \text{ is continuous from } G \text{ to } \mathbb{C} \text{ for any } y \in X_\ast \right\}$$

called the subspace of weak* continuously translating elements of $X$. Let $\mathcal{V}(e)$ be the set of all neighbourhoods $V$ of the identity $e$ of $G$. We then set $S^{w^*}(\pi)$ be the closure in the weak* topology of $B(X)$ of the convex hull of $\bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$, endowed with the weak* operator topology where the closure (or equivalently in $B_{w^*}(X)$). Note that each set $\overline{\{ \pi_t : t \in V \}}^{w^*}$ is a subset of $B_{w^*}(X)$.

The following three propositions are weak* analogs of the results of [DLG2] lemma 2.3 and [DLG2] lemma 2.4.

**Lemma 3.1** Let $\pi : G \to B_{w^*}(X)$ be a bounded (non-continuous) representation of a topological group $G$ on a dual Banach space $X$ such that $\pi_t$ is $w^*$-continuous for any $t \in G$. The sets $\bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$ and $S^{w^*}(\pi)$ are compact semitopological semigroups.

**Proof** : The subset $\{ \pi_t : t \in V \}$ of the dual Banach space $B(M)$ is norm-bounded, hence the set $\overline{\{ \pi_t : t \in V \}}^{w^*}$ is compact for the weak* topology by Alaoglu’s theorem. We deduce that the intersection $\bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$ is compact. Using [DS] Exercise 3, page 511] and the homeomorphism, we conclude that its closed convex hull $S^{w^*}(\pi)$ is also compact.

Let $T$ and $R$ be elements of $\bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$. Let $U$ be a neighbourhood of $R$ for the weak* topology. For any neighbourhood $V$ of $e$, we have $W \cap \{ \pi_t : t \in V \} \neq \emptyset$. Hence there exists $t_{V,U} \in V$ such that $\pi_{t_{V,U}} \in U$. The net $(t_{V,U})^\infty$ converge to $e$ in $G$ and the net $(\pi_{t_{V,U}})$ converge to $R$ in the weak* topology.

Let $V$ be an element of $\mathcal{V}(e)$. Choose $W$ in $\mathcal{V}(e)$ such that $W^2 \subset V$. Note that $T \in \overline{\{ \pi_t : t \in W \}}^{w^*}$. We have

$$\overline{\{ \pi_t : t \in W \}} \cdot \overline{\{ \pi_t : t \in W \}} \subset \overline{\{ \pi_t : t \in V \}} \subset \bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}.$$ 

If $V \subset W$, i.e. if $(W,U) \preceq (V,U)$, we have $t_{V,U} \in W$ and

$$T \cdot \pi_{t_{V,U}} \in \overline{\{ \pi_t : t \in W \}}^{w^*} \cdot \overline{\{ \pi_t : t \in V \}}^{w^*} \subset \bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}.$$ 

Passing to the limit, we deduce that $TR \in \bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$ for any $V \in \mathcal{V}(e)$. Hence $TR$ belongs to the set $\bigcap_{V \in \mathcal{V}(e)} \overline{\{ \pi_t : t \in V \}}^{w^*}$, i.e. this latter set is a semigroup. Finally its closed convex hull $S^{w^*}(\pi)$ is clearly also a semigroup.  

**Proposition 3.2** Let $\pi : G \to B_{w^*}(X)$ be a bounded (non-continuous) representation of a commutative topological group $G$ on a dual Banach space $X$ such that $\pi_t$ is $w^*$-continuous for any $t \in G$. The kernel of the compact semitopological semigroup $S^{w^*}(\pi)$ contains a unique projection $Q$ such that $Q \pi_t = \pi_t Q$ for any $t \in G$.

2. Declare that $(V_1, U_1) \preceq (V_2, U_2)$ if $V_2 \subset V_1$ and $U_2 \subset U_1$.  

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Proof: For any \( t \in G \), we have
\[
\pi_t \left( \bigcap_{V \in V(e)} \{ \pi_s : s \in V \}^{w^*} \right) \pi_{t^{-1}} = \bigcap_{V \in V(e)} \pi_t \{ \pi_t : t \in V \}^{w^*} \pi_{t^{-1}} = \bigcap_{V \in V(e)} \{ \pi_s : s \in V \}^{w^*}.
\]
So \( \sigma \mapsto \pi_t \sigma \pi_{t^{-1}} \) is an automorphism of the semigroup \( S^{w^*}(\pi) \). But any automorphism of \( S^{w^*}(\pi) \) preserves the least ideal (the kernel). In particular, by uniqueness of the projection \( Q \), we deduce that \( \pi_t Q \pi_{t^{-1}} = Q \) for any \( t \in G \).

**Proposition 3.3** Let \( \pi: G \to B(X) \) be a bounded (non-continuous) representation of a topological group \( G \) on a dual Banach space \( X \). The set \( X_{w^*} \) consists of precisely those \( x \in X \) which are fixed under all \( T \) in \( \bigcap_{V \in V(e)} \{ \pi_t : t \in V \}^{w^*} \):
\[
X_{w^*} = \left\{ x \in X : \forall T \in \bigcap_{V \in V(e)} \{ \pi_t : t \in V \}^{w^*}, T(x) = x \right\}.
\]

Proof: Consider \( x \in X_{w^*} \). If \( y \in X_* \) then for any \( \varepsilon > 0 \), using the continuity of \( t \mapsto \langle y, \pi_t x \rangle_{X_*} \) at \( e \), we see that there exists a neighbourhood \( V_{\varepsilon,x,y} \in V(e) \) such that for any \( t \in V_{\varepsilon,x,y} \)
\[
|\langle y, \pi_t x \rangle_{X_*} - \langle y, x \rangle_{X_*}| < \varepsilon.
\]
Let \( \sigma \) be an element of the closure \( \{ \pi_t : t \in V_{\varepsilon,x,y} \}^{w^*} \). There exists a net \( (\pi_{t_i})_{i \in I} \) with \( t_i \in V_{\varepsilon,x,y} \) converging to \( \sigma \) in the weak* topology. For any \( i \in I \), we have
\[
|\langle y, \pi_{t_i} x \rangle_{X_*} - \langle y, x \rangle_{X_*}| < \varepsilon.
\]
Passing to the limit, we obtain
\[
|\langle y, \sigma(x) - x \rangle_{X_*}| = |\langle y, \sigma(x) \rangle_{X_*} - \langle y, x \rangle_{X_*}| < \varepsilon.
\]
Now, if \( \sigma_0 \in \bigcap_{V \in V(e)} \{ \pi_t : t \in V \}^{w^*} \) then for any \( \varepsilon > 0 \) and any \( y \in X_* \) the element \( \sigma_0 \) belongs to \( \{ \pi_t : t \in V_{\varepsilon,x,y} \}^{w^*} \). For any \( y \in X_* \), we deduce that
\[
|\langle y, \sigma_0(x) - x \rangle_{X_*}| = 0
\]
We conclude that \( \sigma_0(x) = x \).

For the reverse inclusion, let \( x \in X \) fixed by all elements of \( \bigcap_{V \in V(e)} \{ \pi_t : t \in V \}^{w^*} \), i.e. suppose that for any \( \sigma \in \bigcap_{V \in V(e)} \{ \pi_t : t \in V \}^{w^*} \) we have \( \sigma(x) = x \). Consider a net \( (t_i)_{i \in I} \) in \( G \) converging to the identity \( e \). Since the representation \( \pi \) is bounded, the set \( \{ \pi_t : t \in G \}^{w^*} \) is weak* compact. Using the continuous map \( B(X) \to X, T \mapsto Tx \), where the spaces are equipped with the weak* topology, we see that the subset \( \{ \pi_t : t \in G \}^{w^*} \cdot x \) of \( X \) is compact for the weak* topology.
Note that an accumulation point of the net \((\pi_i)_{i \in I}\) is an element of \(\bigcap_{F \in \mathcal{F}} F^w\) where
\[
\mathcal{F} = \{ F \subset B(X) : \text{there exists } i_0 \in I \text{ such that } \{ \pi_i : i \geq i_0 \} \subset F \}.
\]
For any neighbourhood \(V \in \mathcal{V}(e)\) there exists \(i_V\) such that \(i \geq i_V\) imply \(t_i \in V\) and thus \(\pi_{t_i} \in \pi(V)\). Thus the set \(\{ \pi_i : i \geq i_V \}\) is included in \(\{ \pi_i : t \in V \}\). Then the set \(\{ \pi_i : t \in V \}\) belongs to \(\mathcal{F}\). We deduce that
\[
\bigcap_{F \in \mathcal{F}} F^w \subset \bigcap_{V \in \mathcal{V}(e)} \{ \pi_i : t \in V \}\bigwedge^w.
\]
We conclude that the net \((\pi_i)_{i \in I}\) can have accumulation points only in the intersection \(\bigcap_{V \in \mathcal{V}(e)} \{ \pi_i : t \in V \}\bigwedge^w\). Now, it is not difficult to see that the net \((\pi_i, x)_{i \in I}\) of \(X\) can only have accumulation points in \(\bigcap_{V \in \mathcal{V}(e)} \{ \pi_i : t \in V \}\bigwedge^w \cdot x = \{ x \}\). Lying in the weak* compact subset \(\{ \pi_i : t \in G \}\bigwedge^w \cdot x\) of \(X\), we infer that it converges weak* to \(x\). Consequently, the map \(t \mapsto \pi_i x\) is weak* continuous at \(t = e\), hence everywhere, completing the proof.

The following result is a particular case of the combination of [BGKS Theorem 1.2], [BGKS Proposition 5.5], [BGKS Remark 5.6] and [BGKS Corollary 4.3] (and its proof), see also [KuN, Theorem 2.4]. Here, we use the fact that a unital completely positive map \(T : M \to M\) on a von Neumann algebra \(M\) is a Schwarz map [Pan, Proposition 3.3], i.e:
\[
T(x^*)T(x) \leq T(x^*x), \quad x \in M.
\]

**Theorem 3.4** Let \(M\) be a von Neumann algebra equipped with a normal faithful state \(\phi\). Let \(\mathcal{S}\) be a semigroup of normal unital completely positive maps \(T : M \to M\) leaving \(\phi\) invariant. The closure \(\overline{\mathcal{S}}^w\) of the convex hull \(\text{co}(\mathcal{S})\) of \(\mathcal{S}\) in the weak* topology of \(B(M)\) is a compact semitopological semigroup and its kernel is a singleton \(\{ E \}\) where \(E\) is a faithful normal conditional expectation \(E : M \to M\) leaving \(\phi\) invariant satisfying
\[
\text{Ran } E = \{ x \in M : T(x) = x \text{ for any } T \in \mathcal{S} \}.
\]

The following lemma is a generalization of [Fen1, Lemma 3]. Thanks the uniformly convexity of noncommutative \(L^p\)-spaces [PX Corollary 5.2], this lemma can be applied to noncommutative \(L^p\)-spaces.

**Lemma 3.5** Let \(X\) be a Banach space and let \(Y\) be a locally uniformly convex Banach space. Let \((T_t)_{t \geq 0}\) be a strongly continuous semigroup of contractions on \(X\). Let \((U_t)_{t \in \mathbb{Q}}\) be a (non continuous) group of isometries on \(X\) and \(J : X \to Y\) and \(P : Y \to X\) two contractions such that \(T_t = PU_tJ\) for any \(t \in \mathbb{Q}^+\). If \(x \in X\) then the map
\[
\mathbb{Q} \ni t \mapsto U_tJx
\]
is continuous from \(\mathbb{Q}\) to \(Y\) with its norm topology.

**Proof** : Let \(x \in X\) with \(\|x\| = 1\). Note that \(\|Jx\|_Y = \|x\|_X = 1\). By the locally uniform convexity of \(Y\), if \(\varepsilon > 0\) there exists \(\delta(\varepsilon, x) > 0\) such that if \(y \in Y\) satisfies \(\|y\|_Y = 1\) and \(\frac{\|y - Jx\|_Y}{2} \geq 1 - \delta(\varepsilon, x)\) we have \(\|y - Jx\|_Y \leq \varepsilon\). Since \((U_t)_{t \in \mathbb{Q}}\) is a group of isometries, it suffices
to show, for $x \in X$, the continuity from the right of the map $s \mapsto U_t J x$ at $t = 0$. Given $\varepsilon > 0$, by the strong continuity of $(T_t)_{t \in \mathbb{Q}}$ there exists $\delta > 0$ such that $0 \leq t \leq \delta$ imply $\|T_t x - x\| \leq \delta(\varepsilon, x)$. Hence, for any $t \in \mathbb{Q} \cap [0, \delta]$ we have

$$
\|U_t J x + J x\|_Y \geq \|P U_t J x + P J x\|_X = \|T_t x + x\|_X \\
= \|2x - (x - T_t x)\|_X \geq \|2x\|_X - \|T_t x - x\|_X \geq 2 - 2\delta(\varepsilon, x).
$$

Hence $\frac{\|U_t J x + J x\|_Y}{2} \geq 1 - \delta(\varepsilon, x)$. Since $\|U_t J x\|_Y = \|J x\|_Y = \|x\|_X = 1$, we infer $\|U_t J x - J x\|_Y \leq \varepsilon$.

The following lemma is a variant of the above Lemma.

**Lemma 3.6** Let $M$ and $N$ be von Neumann algebras equipped with normal faithful states $\phi$ and $\psi$. Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of $\phi$-Markov maps on $M$. Let $(U_t)_{t \in \mathbb{Q}}$ be a group of $\ast$-automorphisms of $N$ leaving $\psi$ invariant and $J : M \to N$ a $(\phi, \psi)$-Markov $\ast$-monomorphism such that $T_t = \mathbb{E} U_t J$ for any $t \in \mathbb{Q}^+$ where $\mathbb{E} : N \to M$ is the canonical faithful normal conditional expectation preserving the states associated with $J$. For any $x \in M$ and any $y \in L^1(N)$, the map

$$Q \mapsto \langle y, U_t J(x) \rangle_{L^1(N), N}
$$

is continuous.

**Proof**: We fix $1 < p < \infty$. The semigroup $(T_t)_{t \geq 0}$ induces a strongly continuous semigroup of contractions on $L^p(M)$ and the semigroup $(U_t)_{t \in \mathbb{Q}}$ induces a group of isometries on $L^p(N)$. Moreover $J$ induces an isometric embedding of $L^p(M)$ into $L^p(N)$ and $\mathbb{E}$ a contractive map from $L^p(N)$ onto $L^p(M)$. For any $x \in L^p(M)$, by Lemma 3.5, the map $t \mapsto U_t J x$ is continuous from $\mathbb{Q}$ to $L^p(N)$ with its norm topology. Let $t_0 \in \mathbb{Q}$ and let $D \in L^1(N)$ the density operator of $\psi$. Note that $D \frac{\varepsilon}{\varepsilon} \in L^{2p}(N)$. For any $z \in L^{2p}(N)$ and any $x \in L^p(M)$, we have

$$
\langle D \frac{\varepsilon}{\varepsilon} z D \frac{\varepsilon}{\varepsilon}, U_t J x \rangle_{L^1(N), N} = \langle z, D \frac{\varepsilon}{\varepsilon} U_t J x D \frac{\varepsilon}{\varepsilon} \rangle_{L^{2p}(N), L^p(N)} \\
\xrightarrow{t \to t_0} \langle z, D \frac{\varepsilon}{\varepsilon} U_{t_0} J x D \frac{\varepsilon}{\varepsilon} \rangle_{L^{2p}(N), L^p(N)} = \langle D \frac{\varepsilon}{\varepsilon} z D \frac{\varepsilon}{\varepsilon}, U_{t_0} J x \rangle_{L^1(N), N}.
$$

Recall that $D \frac{\varepsilon}{\varepsilon} L^{2p}(N) D \frac{\varepsilon}{\varepsilon}$ is norm dense in $L^1(N)$. Now, with a $\frac{\delta}{\varepsilon}$-argument, it is not difficult to complete this proof.

Now we can prove our first main result. We use a similar strategy to the one of Fendler [Fen1]. However, the method of [Fen1] does not apply identically to our context. We adapt instead some trick of the proof of [AFM, Corollary 6.2] using some results from the papers [DLG1] and [DLG2].

**Theorem 3.7** Let $M$ be a von Neumann algebra equipped with a normal faithful state $\phi$. Let $(T_t)_{t \geq 0}$ be a $w^*$-semigroup of factorizable $\phi$-Markov map on $M$. Then the semigroup $(T_t)_{t \geq 0}$ is dilatible.

**Proof**: For a finite set $B \subset \mathbb{Q}$ let $U_B = \{n \in \mathbb{N} : nt \in \mathbb{Z} \text{ for any } t \in B\}$. Then the set of all sets $\{U_B : B \subset \mathbb{Q}, B \text{ finite}\}$ is closed under finite intersections and thus constitutes the basis of some filter $\mathcal{F}$ which is contained in some ultrafilter $\mathcal{U}$.

Using [Ham, Theorem 4.4], for any integer $n \geq 0$, we note that the operator $T_{n} : M \to M$ is dilatible. This means that there exist a von Neumann algebra $N_{\frac{n}{n}}$ equipped a normal
faithful state $\psi_\frac{1}{\mathcal{A}}$, a $*$-automorphism $S_{\frac{1}{\mathcal{A}}}$ of $N_n$ leaving $\psi_\frac{1}{\mathcal{A}}$ invariant and a $(\phi, \psi_\frac{1}{\mathcal{A}})$-Markov $*$-monomorphism $J: M \to N_{\frac{1}{\mathcal{A}}}$ such that

$$(T_{\frac{1}{\mathcal{A}}})^k = E_{\frac{1}{\mathcal{A}}}(S_{\frac{1}{\mathcal{A}}})^k J_{\frac{1}{\mathcal{A}}}, \quad k \geq 0,$$

where $E_{\frac{1}{\mathcal{A}}}: N_{\frac{1}{\mathcal{A}}} \to M$ is the canonical $\psi_{\frac{1}{\mathcal{A}}}$-preserving normal faithful conditional expectation associated with $J_{\frac{1}{\mathcal{A}}}$. For $t \in \mathbb{Q}$, we define the operator $S_{\frac{1}{\mathcal{A}},t}: N_{\frac{1}{\mathcal{A}}} \to N_{\frac{1}{\mathcal{A}}}$ by

$$S_{\frac{1}{\mathcal{A}},t} = \begin{cases} (S_{\frac{1}{\mathcal{A}}})^{nt} & \text{if } nt \in \mathbb{Z} \\ \text{Id}_{N_{\frac{1}{\mathcal{A}}}} & \text{if } nt \notin \mathbb{Z}. \end{cases}$$

If $B = \{t_1, \ldots, t_k\} \subset \mathbb{Q}^+$ is a finite subset, then for $t \in B$ and $n \in U_B$ we have $nt \in \mathbb{Z}$ and thus

$$(3.2) \quad T_t = (T_{\frac{1}{\mathcal{A}}})^{nt} = E_{\frac{1}{\mathcal{A}}}(S_{\frac{1}{\mathcal{A}}})^{nt} J_{\frac{1}{\mathcal{A}}} = E_{\frac{1}{\mathcal{A}}}S_{\frac{1}{\mathcal{A}},t}J_{\frac{1}{\mathcal{A}}},$$

i.e. the following diagram commutes.

$$\begin{array}{ccc}
N_{\frac{1}{\mathcal{A}}} & \xrightarrow{S_{\frac{1}{\mathcal{A}},t}} & N_{\frac{1}{\mathcal{A}}} \\
J_{\frac{1}{\mathcal{A}}} \downarrow & & \downarrow E_{\frac{1}{\mathcal{A}}} \\
M & \xrightarrow{T_t} & M
\end{array}$$

We consider the following ultraproducts of von Neumann algebras

$$M^I = (M, \phi)^I \quad \text{and} \quad \tilde{N} = (N_{\frac{1}{\mathcal{A}}}, \psi_{\frac{1}{\mathcal{A}}})^I.$$ 

We equip $\tilde{N}$ with the normal faithful state $\psi = (\psi_{\frac{1}{\mathcal{A}}})^I$. Let $\mathcal{I}$ be the canonical inclusion $\mathcal{I}: M \to M^I x \mapsto (x, x, \ldots)^I$ and let $E: M^I \to M$, $(x_n)^I \mapsto \lim_{n \to \mathcal{I}} x_n$ be the conditional expectation associated with the canonical inclusion $\mathcal{I}: M \to M^I$. We introduce the operators

$$J = (J_{\frac{1}{\mathcal{A}}})^I \mathcal{I}, \quad \tilde{S}_t = (S_{\frac{1}{\mathcal{A}},t})^I, \quad t \in \mathbb{Q}.$$ 

Observe that the map $\tilde{J}: M \to \tilde{N}$ is a $(\phi, \psi)$-Markov $*$-monomorphism. For any $t \in \mathbb{Q}$, note also that the map $\tilde{S}_t: \tilde{N} \to \tilde{N}$ is a $*$-automorphism of $\tilde{N}$ (hence $w^*$-continuous) leaving $\psi$ invariant.

Let $\tilde{E}: \tilde{N} \to M$ be the canonical $\psi$-preserving faithful normal conditional expectation associated with $\tilde{J}$. We have $\tilde{E} = E \circ (E_{\frac{1}{\mathcal{A}}})^I$.

Let us check that

$$\tilde{S}: \quad \mathbb{Q} \quad \rightarrow \quad B(\tilde{N})$$

$$t \quad \mapsto \quad \tilde{S}_t$$

is a representation and that it defines a dilation of the semigroup $(T_t)_{t \in \mathbb{Q}^+}$. If $t, t' \in \mathbb{Q}$, $x = (x_n)^I \in \tilde{N}$ an if $n \in U_{(t,t')}$ (i.e. for $n$ sufficiently large) then we have $nt, nt' \in \mathbb{Z}$ and $n(t+t') = nt + nt' \in \mathbb{Z}$. Then we obtain

$$S_{\frac{1}{\mathcal{A}},t+t'}(x_n) = (S_{\frac{1}{\mathcal{A}}})^{nt+nt'}(x_n) = (S_{\frac{1}{\mathcal{A}}})^{nt}(S_{\frac{1}{\mathcal{A}}})^{nt'}(x_n) = S_{\frac{1}{\mathcal{A}},t}(S_{\frac{1}{\mathcal{A}},t'}(x_n)).$$
We have \((S_{n,t}\cdot x_n)^U = (S_{n,t} \cdot (x_n)^U, \) and thus
\[
\tilde{S}_{t+t'}((x_n)^U) = \tilde{S}_t \tilde{S}_{t'}((x_n)^U).
\]
Moreover, for \(t \in \mathbb{Q}^+ \) and \(x \in M, \) we have
\[
\mathbb{E}\tilde{S}_{t} J x = \mathbb{E}(E_+)^U (S_+^t)^U (J^+)^U T x = \mathbb{E}(E_+)^U (S_+^t)^U (J^+)^U T x = \lim_{n \to \infty} \mathbb{E}(E_+^n S_+^n t J^+ x)^U.
\]
By (3.2), if \(n \in U(1), \) we have \(E_+^n S_+^t J^+ x = T_t x. \) We deduce that
\[
\mathbb{E}\tilde{S}_{t} J = T_t, \quad t \in \mathbb{Q}^+.
\]
We define \( \mathcal{S} \) to be the semigroup \( \bigcap_{V \in V(0)} \{ S \in \mathbb{Q}^+ \}. \) From Theorem 3.4, we deduce that the kernel of the weak* closure \( co(\mathcal{S}) = \mathcal{S}'^w(\tilde{S}_t)_{t \in \mathbb{Q}^+} \) of the convex hull \( co(\mathcal{S}) \) of \( \mathcal{S} \) is a singleton \( \{ \mathcal{E}' \} \) where \( \mathcal{E}' : \widetilde{N} \to \widetilde{N} \) a faithful normal conditional expectation preserving \( \varphi_U \) satisfying
\[
\text{Ran} \mathcal{E}' = \{ x \in \widetilde{N} : T(x) = x \text{ for any } T \in \mathcal{S} \}.
\]
By Proposition 3.3, the subspace \( \widetilde{N}_{w^*} \) of weak* continuously translating elements of \( \widetilde{M} \) of the representation \( Q \to B(\widetilde{N}), \) \( t \mapsto \tilde{S}_t \) is equal to the fixed point subspace of \( \mathcal{S} \):
\[
\widetilde{N}_{w^*} = \{ x \in \widetilde{N} : T(x) = x \text{ for any } T \in \mathcal{S} \}.
\]
Hence the von Neumann algebra \( \text{Ran} \mathcal{E}' \) is equal to \( \widetilde{N}_{w^*} \) and is invariant under the operator \( \tilde{S}_t \) for any \( t \in \mathbb{Q} \) by Proposition 3.2. By Proposition 3.3, the range \( \text{Ran}(\tilde{J}) \) of the map \( J : M \to \widetilde{N} \) is contained in the subspace \( \widetilde{N}_{w^*} \) of continuously translating elements of \( \widetilde{N} \) of the representation \( Q \to B(\widetilde{N}), \) \( t \mapsto \tilde{S}_t. \) Now, it is easy to obtain (1.1) by letting \( N = \widetilde{N}_{w^*} \) and
\[
U_t = \mathcal{E}\tilde{S}_t \widetilde{N}_{w^*}, \quad \text{for all } t \in \mathbb{Q}.
\]
where we consider \( \mathcal{E}' \) as an operator from \( N \) on \( \widetilde{N}_{w^*}. \) Finally, we let \( J : M \to \widetilde{N}_{w^*} \) the canonical *-monomorphism and \( \mathcal{E} : \widetilde{N}_{w^*} \to M \) the associated conditional expectation. We conclude that
\[
T_t = E U_t J, \quad t \geq 0.
\]

**Remark 3.8** We refer to [AHW], [AnH], [CL] and [Oza] for QWEP von Neumann algebras. We say [Arh2, Definition 1.2] that \( T \) is QWEP-factorizable if the definition of factorizability of the introduction is satisfied with a QWEP von Neumann algebra \( N. \) Similarly, we say [Arh2, Definition 1.3] that \((T_t)_{t \geq 0}\) is QWEP-dilatable if the definition 1.3 is satisfied with a QWEP von Neumann algebra \( N. \) It is easy to see that if each \( T_t \) is QWEP-factorizable then the semigroup \((T_t)_{t \geq 0}\) is QWEP-dilatable. Indeed, the proof [HaM, Theorem 4.4] of gives a QWEP von Neumann algebra (use [Oza, Proposition 4.1 (ii)b and (iii)]) and note that if each \( N_\pm \) has QWEP, then \( N \) has QWEP (use the proof of [AHW, Lemma 4.3] and [Oza, Proposition 4.1 (ii)].
4 Dilations of semigroups on noncommutative $L^p$-spaces

The goal is to prove Theorem 4.4 below which is a noncommutative $L^p$ variant of Theorem 3.7. Suppose $1 < p < \infty$. Recall the definition of [JLM] page 239 which says that a contraction $T: L^p(M) \to L^p(M)$ on a noncommutative $L^p$-space is dilatable if there exist a noncommutative $L^p$-space $L^p(N)$, two contractions $J: L^p(M) \to L^p(N)$ and $P: L^p(N) \to L^p(M)$ and an isometry $U: L^p(N) \to L^p(N)$ such that $T^k = PU^kJ$ for any $k \geq 0$. Now, we introduce a variant.

**Definition 4.1** Suppose $1 \leq p < \infty$. We say that a completely positive contraction $T: L^p(M) \to L^p(M)$ on a noncommutative $L^p$-space is completely positively dilatable if there exist a noncommutative $L^p$-space $L^p(N)$, two completely positive contractions $J: L^p(M) \to L^p(N)$ and $P: L^p(N) \to L^p(M)$ and a completely positive invertible isometry $U: L^p(N) \to L^p(N)$ such that

$$T^k = PU^kJ, \quad k \geq 0.$$  

**Remark 4.2** Note that a dilatable $\phi$-Markov $T: M \to M$ on a von Neumann algebra $M$ equipped with a state $\phi$ induces a completely positively dilatable contraction on the associated noncommutative $L^p$-space $L^p(M)$.

In this section, we use Banach ultraproducts. The same method that the beginning of the proof of Theorem 3.7 with the stability of the class of noncommutative $L^p$-spaces under Banach ultraproducts [Ray1] gives the following result.

**Lemma 4.3** Suppose that $(T_t)_{t \geq 0}$ is a (not necessarily strongly continuous) semigroup of contractions on $L^p(M)$ such that each operator $T_t$ is completely positively dilatable. Then there exists a noncommutative $L^p$-space $L^p(\tilde{N})$, a group $(U_t)_{t \in \mathbb{Q}}$ of completely positive invertible isometries of $L^p(\tilde{N})$ and two completely positive contractions $\tilde{J}: L^p(M) \to L^p(\tilde{N})$ and $\tilde{P}: L^p(\tilde{N}) \to L^p(M)$ such that

$$T_t = \tilde{P}U_t\tilde{J}, \quad t \in \mathbb{Q}^+.$$  

Moreover, if $M$ has QWEP, then $\tilde{N}$ has QWEP.

One more time, if the semigroup $(T_t)_{t \geq 0}$ is strongly continuous, the above ultraproduct construction yields a too big space $L^p(\tilde{N})$ such that one can expect the representation $\tilde{U}: t \mapsto \tilde{U}_t$ of $\mathbb{Q}$ to be continuous on $L^p(\tilde{N})$. However, it is still possible to restrict $t \mapsto \tilde{U}_t$ to a smaller subspace on which the desired continuity holds. Again, the method of [Fen1] does not apply to our context.

**Theorem 4.4** Suppose $1 < p < \infty$. Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup of completely positive contractions on a noncommutative $L^p$-space $L^p(M)$ such that each $T_t: L^p(M) \to L^p(M)$ is completely positively dilatable. Then there exists a noncommutative $L^p$-space $L^p(N)$, a strongly continuous group of completely positive isometries $U_t: L^p(N) \to L^p(N)$ and two completely positive contractions $J: L^p(M) \to L^p(N)$ and $P: L^p(N) \to L^p(M)$ such that

$$T_t = PU_tJ, \quad t \geq 0.$$  

Moreover, if $M$ has QWEP, then $N$ has QWEP.

**Proof:** By Lemma 4.3, we obtain a representation $\tilde{U}: \mathbb{Q} \to B(L^p(\tilde{N}))$ by completely positive isometric operators. We have

$$T_t = \tilde{P}U_t\tilde{J}, \quad t \in \mathbb{Q}^+.$$  

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By Theorem 4.1 we deduce that the kernel $K(\hat{U})$ of $S^p(\hat{U})$ contains a unique element $Q_c$ and that $Q_c: L^p(N) \rightarrow L^p(\hat{N})$ is the projection from the Banach space $L^p(\hat{N})_c$ of continuously translating elements. Furthermore, we have

$$\hat{U}_t Q_c = Q_c \hat{U}_t, \quad t \in \mathbb{Q}.$$  

Thus the range $L^p(\hat{N})_c$ of the projection $Q_c$ is invariant under the operator $\hat{U}_t$ for any $t \in \mathbb{Q}$. Moreover, we infer from Lemma 4.3 that the range Ran($\hat{J}$) of the map $\hat{J}$ given by Lemma 4.3 is contained in the subspace $L^p(\hat{N})_c$ of continuously translating elements of $L^p(\hat{N})$ of the representation $\hat{U}$. Furthermore, since each operator $\hat{U}_t$ is isometric and completely positive, hence contractive, we see that the convex semigroup $S^p(\hat{U})$ of $\hat{U}$ over the identity consists of completely positive contractions only. It follows that $Q_c$ is also contractive and completely positive and consequently that the subspace $L^p(\hat{N})_c$ is completely positive complemented in $L^p(\hat{N})$, hence a noncommutative $L^p$-space $L^p(N)$ by the main result of [ArR]. Now, we define

$$U_t = Q_c \hat{U}_t|_{L^p(\hat{N})_c} \quad \text{for all } t \in \mathbb{Q}$$

where we consider $Q_c$ as an operator from $L^p(\hat{N})$ on $L^p(\hat{N})_c$. Finally, we let $J: L^p(M) \rightarrow L^p(\hat{N})_c$ be the canonical embedding of $L^p(M)$ into $L^p(\hat{N})_c$ and $P = \tilde{F}_p|_{L^p(\hat{N})_c}$. We conclude that

$$T_t = P \hat{U}_t J, \quad t \geq 0.$$  

It follows from the complete positivity of the projection $Q_c$ and from (4.2) that the induced isometry $U_t: L^p(N) \rightarrow L^p(\hat{N})$ is also completely positive.

\[\hspace{-0.4cm}\]

**Remark 4.5** Note that there exists some completely positive contractive map $T: S^p \rightarrow S^p$ which does not admit an isometric dilation on a noncommutative $L^p$-space, see [JLM]. See also [Arh1] and [ALM] for more information on dilations on noncommutative $L^p$-spaces.

5 **Semigroups of selfadjoint Fourier multipliers**

As we said in the introduction, Haagerup and Musat [HaM, Theorem 4.4] have characterised dilatable Markov maps. Indeed, they proved that if $T: M \rightarrow M$ is a $\phi$-Markov map on a von Neumann algebra $M$ equipped with a state $\phi$ then $T$ is dilatable if and only if $T$ is factorizable in the sense of [AnD]. This result allows us to give concrete examples of dilatable semigroups.

Suppose that $G$ is a discrete group. We denote by $\varepsilon_G$ the neutral element of $G$. We denote by $\lambda_g: \ell^2_G \rightarrow \ell^2_G$ the unitary operator of left translation by $g$ and $\text{VN}(G)$ the von Neumann algebra of $G$ spanned by the $\lambda_g$’s where $g \in G$. It is an infinite algebra with its canonical faithful normal finite trace given by

$$\tau_G(x) = \langle \varepsilon_G, x(\varepsilon_G) \rangle_{\ell^2_G}$$

where $(\varepsilon_g)_{g \in G}$ is the canonical basis of $\ell^2_G$ and $x \in \text{VN}(G)$. A Fourier multiplier is a normal linear map $T: \text{VN}(G) \rightarrow \text{VN}(G)$ such that there exists a complex function $t: G \rightarrow \mathbb{C}$ such that $T(\lambda_g) = t_g \lambda_g$ for any $g \in G$. In this case, we denote $T$ by $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$. It is well-known that a Fourier multiplier $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$ is completely positive if and only if the function $t$ is positive definite. It is easy to see that a $\tau_G$-Markov Fourier multiplier $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$ is selfadjoint if and only if $t: G \rightarrow \mathbb{C}$ is a real function.

Using the factorisability of selfadjoint $\tau_G$-Markov Fourier multipliers of [Ric], we deduce the following result:
Corollary 5.1 Let $G$ be a discrete group. Let $(T_t)_{t \geq 0}$ be a $w^*$-semigroup of selfadjoint $\tau_G$-Markov Fourier multipliers on the von Neumann algebra $VN(G)$. Then the semigroup $(T_t)_{t \geq 0}$ is dilatable.

6 Applications to $H^\infty$ functional calculus

We start with a little background on sectoriality and $H^\infty$ functional calculus. We refer to [Haas, [KW], [JMX] and [Arh2] for details and complements. Let $X$ be a Banach space. A closed densely defined linear operator $A : D(A) \subseteq X \to X$ is called sectorial of type $\omega$ if its spectrum $\sigma(A)$ is included in the closed sector $\Sigma_{\omega}$, and for any angle $\omega < \theta < \pi$, there is a positive constant $K_{\theta}$ such that

$$\|(\lambda - A)^{-1}\|_{X \to X} \leq \frac{K_{\theta}}{|\lambda|}, \quad \lambda \in \mathbb{C} - \Sigma_{\theta}.$$ 

If $-A$ is the negative generator of a bounded strongly continuous semigroup on a $X$ then $A$ is sectorial of type $\frac{\pi}{2}$. We also recall that sectorial operators of type $\frac{\pi}{2}$ coincide with negative generators of bounded analytic semigroups.

For any $0 < \theta < \pi$, let $H^\infty(\Sigma_{\theta})$ be the algebra of all bounded analytic functions $f : \Sigma_{\theta} \to \mathbb{C}$, equipped with the supremum norm $\|f\|_{H^\infty(\Sigma_{\theta})} = \sup\{|f(z)| : z \in \Sigma_{\theta}\}$. Let $H^\infty(\Sigma_{\theta}) \subset H^\infty(\Sigma_{\theta})$ be the subalgebra of bounded analytic functions $f : \Sigma_{\theta} \to \mathbb{C}$ for which there exist $s, c > 0$ such that $|f(z)| \leq c|z|^s(1 + |z|)^{-2s}$ for any $z \in \Sigma_{\theta}$.

Given a sectorial operator $A$ of type $0 < \omega < \pi$, a bigger angle $\omega < \theta < \pi$, and a function $f \in H^\infty(\Sigma_{\theta})$, one may define a bounded operator $f(A)$ by means of a Cauchy integral (see e.g. [Haas Section 2.3] or [KW Section 9]). The resulting mapping $H^\infty(\Sigma_{\theta}) \to B(X)$ taking $f$ to $f(A)$ is an algebra homomorphism. By definition, $A$ has a bounded $H^\infty(\Sigma_{\theta})$ functional calculus provided that this homomorphism is bounded, that is, there exists a positive constant $C$ such that $\|f(A)\|_{X \to X} \leq C\|f\|_{H^\infty(\Sigma_{\theta})}$ for any $f \in H^\infty(\Sigma_{\theta})$. In the case when $A$ has a dense range, the latter boundedness condition allows a natural extension of $f \mapsto f(A)$ to the full algebra $H^\infty(\Sigma_{\theta})$.

Note a noncommutative $L^p$-space is a UMD Banach space [Pix Corollary 7.7]. Using the connection between the existence of dilations in UMD spaces and $H^\infty$ functional calculus [KW Corollary 10.9], Theorem 14.1 and the angle reduction principle of [JMX Proposition 5.8] we obtain:

Theorem 6.1 Let $M$ be a von Neumann algebra equipped with a normal faithful state $\phi$. Let $(T_t)_{t \geq 0}$ be a $w^*$-semigroup of factorizable $\phi$-Markov maps on $M$. Suppose $1 < p < \infty$. We let $-A_p$ be the generator of the induced strongly continuous semigroup $(T_t)_{t \geq 0}$ on the Banach space $L^p(M)$. Then for any $\theta > \pi|\frac{1}{p} - \frac{1}{2}|$, the operator $A_p$ has a bounded $H^\infty(\Sigma_{\theta})$ functional calculus.

This result give a partial answer to the question of [JMX page 57]. This theorem is applicable to any $w^*$-semigroup $(T_t)_{t \geq 0}$ of selfadjoint $\tau_G$-Markov Fourier multipliers on the von Neumann algebra $VN(G)$ of a discrete group $G$.

Using Remark 3.8 we can also simplify the vector-valued variant of [Arh2 Theorem 1.4]. Here, we use the notations from [Arh2].

Theorem 6.2 Let $M$ be a von Neumann algebra with QWEP equipped with a normal faithful state. Let $(T_t)_{t \geq 0}$ be a $w^*$-continuous semigroup of QWEP-factorizable $\phi$-Markov on $M$. Suppose $1 < p, q < \infty$ and $0 < \alpha < 1$. Let $E$ be an operator space such that $E = (OH(I), F)_{\alpha}$.
for some index set $I$ and for some OUMD$_q$ operator space $F$ with $\frac{1}{p} = \frac{1-\alpha}{2} + \frac{\alpha}{q}$. We let $-A_p$ be the generator of the strongly continuous semigroup $(T_t \otimes \text{Id}_E)_{t \geq 0}$ on the vector valued noncommutative $L^p$-space $L^p(M, E)$. Then for some $0 < \theta < \frac{\pi}{2}$, the operator $A_p$ has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.

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References

[AcC] L. Accardi and C. Cecchini. Conditional expectations in von Neumann algebras and a theorem of Takesaki. J. Funct. Anal. 45 (1982), 245–273.

[AnD] C. Anantharaman-Delaroche. On ergodic theorems for free group actions on noncommutative spaces. Probab. Theory Related Fields 135 (2006), no. 4, 520–546.

[AnH] H. Ando and U. Haagerup. Ultraproducts of von Neumann algebras. J. Funct. Anal. 266 (2014), no. 12, 6842–6913.

[AHW] H. Ando, U. Haagerup and C. Winsløw. Ultraproducts, QWEP von Neumann algebras, and the Effros–Maréchal topology. J. Reine Angew. Math. 715 (2016), 231–250.

[AsK] M. Akcoglu and L. Sucheston. Dilations of positive contractions on $L_p$ spaces. Canad. Math. Bull. 20 (1977), no. 3, 285–292.

[Arh1] C. Arhancet. On Matsaev’s conjecture for contractions on noncommutative $L^p$-spaces. Journal of Operator Theory 69 (2013), no. 2, 387–421.

[Arh2] C. Arhancet. Analytic semigroups on vector valued noncommutative $L^p$-spaces. Studia Math. 216 (2013), no. 3, 271–290.

[ALM] C. Arhancet and C. Le Merdy. Dilation of Ritt operators on $L^p$-spaces. Israel J. Math. 201 (2014), no. 1, 373–414.

[AFM] C. Arhancet, S. Fackler and C. Le Merdy. Isometric dilations and $H^\infty$ calculus for bounded analytic semigroups and Ritt operators. To appear in Transactions of the American Mathematical Society. Preprint, arXiv:1504.00471.

[ArR] C. Arhancet and Y. Raynaud. Completely positive contractive projections on noncommutative $L^p$-spaces. Preprint.

[BGKS] A. Bátkai, U. Groh, D. Kunzenti-Kovács and M. Schreiber. Decomposition of operator semigroups on $W^*$-algebras. Semigroup Forum 84 (2012), no. 1, 8–24.

[BJM] J. F. Berglund, H. D. Junghenn and P. Milnes. Analysis on semigroups. Function spaces, compactifications, representations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

[CL] V. Capraro and M. Lupini. Introduction to Sofic and hyperlinear groups and Connes’embedding conjecture. With an appendix by Vladimir Pestov. Lecture Notes in Mathematics, 2136. Springer, Cham, 2015.

[DS] N. Dunford and J. T. Schwartz. Linear operators. Part I. General theory. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.
[Ric] É. Ricard. A Markov dilation for self-adjoint Schur multipliers. Proc. Amer. Math. Soc. 136 (2008), no. 12, 4365–4372.

[Ued] Y. Ueda. Fullness, Connes’χ-groups, and ultra-products of amalgamated free products over Cartan subalgebras. Trans. Amer. Math. Soc. 355 (2003), no. 1, 349–37.

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