The Harmonic GBC Function Map is a Bijection if the Target Domain is Convex

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Abstract

Harmonic generalized barycentric coordinates (GBC) functions have been used for cartoon animation since an early work in 2006[1]. A computational procedure was further developed in [SH] for deformation between any two polygons. The bijectivity of the map based on harmonic GBC functions is still murky in the literature. In this paper we present an elementary proof of the bijection of the harmonic GBC map transforming from one arbitrary polygonal domain $V$ to a convex polygonal domain $W$. This result is further extended to a more general harmonic map from one simply connected domain $V$ to a convex domain $W$ if the harmonic map preserves the orientation of the boundary of the domain $V$. In addition, we shall point out that the harmonic GBC map is also a diffeomorphism over the interior of $V$ to the interior of $W$. Finally, we remark on how to construct a harmonic GBC map from $V$ to $W$ when the number of vertices of $V$ is different from the number of vertices of $W$ and how to construct harmonic GBC functions over a polygonal domain with a hole or holes. We also point out that it is possible to use the harmonic GBC map to deform a nonconvex polygon $V$ to another nonconvex polygon $W$ by a good arrangement of the boundary map between $\partial V$ and $\partial W$. Several numerical deformations based on images are presented to show the effectiveness of the map based on bivariate spline approximation of the harmonic GBC functions.

Key Words: GBC functions, harmonic GBC map, harmonic map, bijection

Mathematics Subject Classification: 31A05, 35J25, 30C60, 53A10

1 Introduction

We are interested in the bijective property of the map based on harmonic GBC functions which transforms from one arbitrary polygonal domain $V \subset \mathbb{R}^2$ to a convex polygonal
domain $W \subset \mathbb{R}^2$. Let us first recall the GBC functions. Given a polygon $V \in \mathbb{R}^2$ of $n$-sides with vertices $v_i, i = 1, \cdots, n$, the functions $\phi_i, i = 1, \cdots, n$ satisfying the following conditions:

$$
\begin{align*}
\sum_{i=1}^{n} \phi_i(x) &= 1 \\
\sum_{i=1}^{n} \phi_i(x)v_i &= x \\
\phi_i(x) &\geq 0, \quad i = 1, \cdots, n,
\end{align*}
$$

for all $x = (x, y) \in V$ and being linear on edges of $V$ with $\phi_i(v_j) = 1$ if $i = j$ and $= 0$ if $i \neq j$. Such functions are called generalized barycentric coordinates (GBC) or GBC functions over $V$.

The study of generalized barycentric coordinates (GBC) started from a seminal work in [W75]. Since then, there are many GBCs which have been constructed. For example, harmonic GBC functions in the 3D setting was explained in [JMDGS06] together with many interesting and useful properties. We refer to a recent survey in [F15] and a book [HS18] edited by leading experts K. Hormann and N. Sukumar. Recently, a general minimization method is proposed to construct GBC functions (cf. [DFL20]). More precisely, consider the following minimization problem:

$$
\min \{ F(\phi_1, \cdots, \phi_n) : \text{subject to } \sum_{i=1}^{n} \phi_i(x) = 1, \sum_{i=1}^{n} \phi_i(x)v_i = x, \phi_i \geq 0, i = 1, \cdots, n \},
$$

where $F$ is a convex function and $V$ has $n$ boundary vertices. For example, one can choose

$$
F(\phi_1, \cdots, \phi_n) = \sum_{i=1}^{n} \int_{V} |\nabla \phi_i(x)|^2 dx
$$

which leads to a new way to compute harmonic GBCs based on bivariate spline functions (cf. [ALW06] and [LS07]). We refer to [DFL20] for a detail explanation. The GBCs have found their applications in geometric design. See, e.g. [JBPS11], [Zetc14], [LL21], and etc.. In addition, they found their applications in numerical solution of partial differential equations (e.g. [MRS14], [RGB14], [FL16], and etc.).

One interesting property of the GBCs functions is that it can form a transform or map from one polygon to another polygon easily. That is, letting $V$ be a polygon with boundary vertices $v_i, i = 1, \cdots, n$ arranged in the counter-clockwise direction and $W$ be another polygon with boundary vertices $w_i, i = 1, \cdots, n$ in the counter-clockwise direction, if $\phi_i, i = 1, \cdots, n$ are GBC functions over $V$ satisfying (1), we can define

$$
F(x, y) = \sum_{i=1}^{n} \phi_i(x, y)w_i, \quad (x, y) \in V.
$$

Then $F$ is a map from $V$ to $W$ when $W$ is a convex polygon. Clearly, the image $\text{Im}(F) \subset W$ due to the properties of (1). One can show that the map is onto which will be proved in the next section. The injectivity of the Wachspress from a convex polygon $V$ to a convex polygon $W$ was established in [FK10]. However, the mean value coordinates may not be injective when a convex polygon $V$ is a pentagon as shown in [FK10]. It is interesting to know if the harmonic GBC map $F$ is injective or not. This paper is also motivated by the work in [SH15], where the researchers proposed an approach to construct a map $F$ from a polygonal domain $V$ to another polygonal domain $W$ by introducing an intermediate convex polygonal domain $\Theta$ such that

$$
F = \Phi_1^{-1} \circ \Phi_0.
$$
Figure 1: An Example of Image Deformation Based on Bijections from Polygons to Rectangle. Clearly the polygon on the right has more than 4 vertices. We added inactive vertices on the left domain to match the number of the vertices of the domain on the right.

where $\Phi_0, \Phi_1$ are two harmonic GBC maps from $V \mapsto \Theta$ and $W \mapsto \Theta$, respectively. The researchers in [SH15] simply used the Radó-Kneser-Choquet theorem to explain that $\Phi_0$ is bijective from $V \mapsto \Theta$ and so is $\Phi_1$. Note that the classic Radó-Kneser-Choquet theorem only explains the bijectivity of the unit ball to a convex domain. See the wiki page of the Radó-Kneser-Choquet theorem on-line. See [AN08] for a recent revisit of the Radó-Kneser-Choquet theorem. If one combines it with a Riemann map from a polygonal domain $V$ to the unit ball $B$, one can establish the bijection between an arbitrary polygon $V$ to a convex function $W$. Unfortunately, the Radó-Kneser-Choquet theorem does not hold for a higher dimensional setting other than $\mathbb{R}^2$. One can not use the Radó-Kneser-Choquet theorem for the bijectivity of 3D harmonic GBC maps. Although the researchers in [SH15] have many nice examples in the $\mathbb{R}^3$ setting, it is necessary to study the bijectivity of harmonic mappings in a general dimensional space. These motivate us to study the bijectivity of the harmonic map based on GBC functions from polygon $V$ to a convex polygon $W$. For simplicity, we consider only the situation in the Euclidean space $\mathbb{R}^2$ in this paper.

The purpose of this paper is to give a proof of the bijective property of $\Phi_i$, $i = 0, 1$ in the $\mathbb{R}^2$ setting. See Theorem [2.1]. In addition, we extend the study to the bijectivity of a general harmonic map under some additional boundary condition. See Theorem [2.2] in the next section. We leave the study of the nontrivial extension of the results in this paper to the multidimensional setting in a future publication. Finally, we point out that polygons $V$ and $W$ of interest may not need to have the same number of vertices. One may add inactive vertices to the boundary of $V$ or $W$ so that the number of vertices of $V$ and $W$ will be the same and the computation of the bijection $F$ can be carried out based on the method in [SH15]. See three numerical examples in Figure 1, Figure 2, which are computed by a similar method to the one in [SH15] with a difference that we employed bivariate spline approximation of harmonic GBC functions discussed in [DFL20].

From these images we can see how the points on the right-hand sides in Figure 1 and Figure 2 are mapped to the points on the left-hand sides by using the harmonic GBC
Figure 2: An Example of Image Deformation Based on Bijections from Polygons to Rectangle. Note that the polygon on the right has more than 4 vertices.

Figure 3: An Example of Image Deformation from an L-shaped Polygon to the H-shaped polygon via a square domain based on the approach in [SH15]
In addition to the proof of Theorems 2.1 and 2.2 we also make a few comments in the end of this paper. One example will be shown that a harmonic GBC map from one nonconvex domain to another nonconvex domain can be bijective. Harmonic GBC functions can be also constructed over a polygonal domain with hole or holes. Does the harmonic GBC map based on these GBC functions have a bijective property? Besides the bijectivity, we will mention that the harmonic GBC map is also a diffeomorphism when \( W \) is convex. Although the harmonic GBC map does not have the property like a conforming map, the harmonic GBC map can be easily computed since harmonic GBC functions over all boundary vertices of \( V \) can be done in parallel and once these functions are found, they can be used to form a bijection to any convex set \( W \). A similar study of the bijectivity of harmonic GBCs in the 3D setting will be commented and difficulties will be pointed out. It seems the extension to the 3D setting is nontrivial.

2 Main Results and Their Proofs

In this section, let us start with harmonic functions over simply connected domains in \( \mathbb{R}^2 \). Let \( V \) and \( W \) be two simply connected domains. Typically, \( V \) and \( W \) are polygons. Let \( u : V \to \mathbb{R} \) is a harmonic function, that is, \( \Delta u = 0 \) over \( \bar{V} \), where \( \Delta = D_x^2 + D_y^2 \) is the standard Laplace operator. For two harmonic functions \( u, v \) on \( V \), let \( F = (u(x, y), v(x, y)) \) which will be called a harmonic map on \( V \) which maps \( V \) to \( \text{Im}(F) \), the image of \( F \) over \( V \).

The harmonic functions have been studied for more than 100 years and many properties are known. Let us mention a few properties without proofs which will be used in the following study. The harmonic function \( u \) will not achieve its maximum nor minimum inside \( V \) except for a constant function \( u \). When \( u \) is a harmonic, so are its derivatives. If a harmonic function \( u \) is a constant over a disk \( D \) (open set) inside \( V \), then \( u \) is a constant over \( V \). Indeed, \( \frac{\partial u}{\partial x} \) over \( V \) will be zero over \( D \) and hence \( \frac{\partial u}{\partial x} \) is zero over \( V \) as a harmonic function is like an analytic function.

Note that when \( V, W \subset \mathbb{R}^2 \) be two separate polygons, \( \partial V \) and \( \partial W \) are piecewise linear boundaries, respectively. By the properties of GBC functions (1) and the definition of \( F \) (3), one can easily check that \( F|_{\partial V} \) is a homeomorphism from \( \partial V \) to \( \partial W \). We can also see that \( F|_{\partial V} \) preserve the orientation of the boundary vertices of \( W \) as both of them are in the counter-clockwise direction.

Our first main result in this paper is to establish the following

**Theorem 2.1.** Let \( V, W \subset \mathbb{R}^2 \) be two polygons. Let \( F = (u(x, y), v(x, y)) : V \mapsto W \) be a harmonic GBC function which maps from \( V \) to \( W \) according to (3). If \( W \) is convex, then \( F \) is a bijection from \( V \) to \( W \).

We remark that the above result can be generalized to the general harmonic map \( F = (u, v) : V \mapsto W \) under the assumptions that

- (1) \( F|_{\partial V} \) is an orientation preserving homeomorphism from the boundary \( \partial V \) of \( V \) to the boundary \( \partial W \) of \( W \).

- (2) \( F \in C^0(V) \cap C^1(\bar{V}) \).

In this general setting, we have
Theorem 2.2. Let \( V, W \subset \mathbb{R}^2 \) be two simply connected domains in \( \mathbb{R}^2 \). Suppose \( u, v \) are harmonic functions over \( V \). Let \( F = (u(x, y), v(x, y)) : V \mapsto W \) be a harmonic map from \( V \) to \( W \). Suppose that \( F \) satisfies (1) and (2) above. Then if \( W \) is convex, \( F \) is a bijection from \( V \) to \( W \).

Note that the map \( F \) is a bijection if \( F \) is surjective and also injective in the sense that \( F \) maps \( V \) onto \( W \) and the mapping is one-to-one. Before proving Theorem 2.1, we first establish the surjectivity.

Lemma 2.3. As the setting in Theorem 2.2, \( F \) maps \( V \) onto \( W \) if \( W \) is convex.

Proof. We will show that the image \( \text{Im}(F) \) of \( F \) over \( V \) is \( W \), i.e. \( \text{Im}(F) = W \).

First of all, if \( F \) is a harmonic GBC, it is easy to see \( \text{Im}(F) \subset W \) by the GBC properties [1]. In general, we claim \( \text{Im}(F) \subset W \) for a general harmonic map \( F \). Otherwise, there exists a point \( p = (u_0, v_0) \in \text{Im}(F) \setminus W \). Then there is a line separate \( W \) and \( p \) because \( W \) is convex. That is, there exist real numbers \( \alpha \) and \( \beta \) such that \( \alpha \cdot u_0 + \beta \cdot v_0 > 0 \), and \( \alpha \cdot u' + \beta \cdot v' < 0 \) for all \((u', v') \in W \). For \( x = (x, y) \in V \), let \( \Phi(x) = \alpha \cdot u(x, y) + \beta \cdot v(x, y) \) be an harmonic function over \( V \) and \( F^{-1}(p) \in V \). Then we have \( \Phi(F^{-1}(p)) > 0 \) and \( \Phi(x) < 0 \) for all \( x \in \partial V \), this leads to a contradiction because it violate the maximum principle of the harmonic function \( \Phi \). Hence, the claim follows.

On the other hand, we want to show that \( W \subseteq \text{Im}(F) \). To do so, we introduce the concept called the contractible curve. If a Jordan curve \( \gamma \) inside \( V \) can be continuously shrunk within \( V \) to a point in \( V \), then \( \gamma \) is called a contractible curve in \( V \). We note that \( V \) is simply connected, \( \partial V \) is a contractible curve in \( V \). Since \( F|_{\partial V} \) is a homeomorphism (continuously bijection) to \( \partial W \), \( \partial W \) must also be contractible in \( \text{Im}(F) \).

Now suppose \( W \not\subseteq \text{Im}(F) \). Then there exists a point \( w \in W \setminus \text{Im}(F) \). Thus, \( \partial W \) is contractible in \( \text{Im}(F) \subset (W \setminus w) \). As \( W \setminus w \) is larger than \( \text{Im}(F) \), \( \partial W \) would be contractible in \( (W \setminus w) \) which is obviously not possible as \( W \setminus w \) is like a punctured disk. Hence, \( W \subseteq \text{Im}(F) \).

Next we shall show that \( F \) is an injection. Recall that \( F = (u, v)^\top \) is homeomorphism at \((x_0, y_0) \in V \) if its Jacobian matrix \( J_0 \) is not singular at the point \((x_0, y_0) \in V \). Suppose that \( F(x, y) \) has a nonzero Jacobian determinant at \( x_0 = (x_0, y_0) \), that is,

\[
\det(J_0) = \det([\nabla u(x_0, y_0); \nabla v(x_0, y_0)]) = \det\left[ \frac{\partial}{\partial x} u(x_0, y_0) \quad \frac{\partial}{\partial y} u(x_0, y_0) \right] \neq 0. \tag{4}
\]

Since \( \det(J_0) \neq 0 \), \( ||J_0 \cdot x|| \geq c \cdot ||x|| \) for some positive constant \( c \) for all \( x \) in a neighborhood of \((x_0, y_0) \). The Taylor expansion at \( x_0 = (x_0, y_0) \) gives us

\[
\|F(x) - F(x_0)\| = \|J_0 \cdot (x - x_0)\| + o(||x - x_0||) \\
\geq c ||x - x_0|| + o(||x - x_0||). \tag{5}
\]

Hence there exist a \( r_0 > 0 \) such that \( F(x) \neq F(x_0) \) for all \( x \in B_{r_0}(x_0) \). Now consider the closed ball \( \overline{B}_r(x_0) \subset \tilde{V} \) if \( r_0 > 0 \) small enough. Apply the same trick to each point on \( \overline{B}_r(x_0) \). We let

\[
r = \min \left\{ r_k \mid x_k \in \overline{B}_{r_k}, B_{r_k}(x_k) \subset \tilde{V}, F(x) \neq F(x_k) \text{ for all } x \in B_{r_k}(x_k) \right\}.
\]

The compactness of \( \overline{B}_{r_0}(x_0) \) implies that \( r > 0 \). Then it is clear that \( F \) restricted on \( B_{r/2}(x_0) \) is a homeomorphism from \( B_{r/2}(x_0) \) to its image \( \text{Im}(F) \) over the ball.

The discussion above leads to the following
Lemma 2.4. Suppose that the Jacobian determinant of \( F \) is not zero at every point of \( V \). Then \( F \) is a local homeomorphism on \( V \), i.e., a continuous, one-to-one and onto map over a neighborhood \( N(x, y) \subset V \) for \((x, y) \in V\).

Unfortunately, the nonzero Jacobian determinant of \( F \) over \( V \) does not guarantee the global injectivity as shown in the following example:

Example 2.5. Let \( F = (u, v) \) with \( u(x, y) = \exp(x/2) \cos(y \exp(-x)) \) and \( v(x, y) = \exp(x/2) \sin(y \exp(-x)) \). Then the Jacobian determinant is equals to 1/2 for all \((x, y) \in \mathbb{R}^2\). However, \( F(0, 0) = F(0, 2\pi) \). i.e. \( F \) is not injective globally.

On the other hand, due to the GBC property, we can see that when \( W \) is a triangle, the map \( F \) is an injection. Indeed, suppose that there are two points \((x_1, y_1)\) and \((x_2, y_2)\) in \( V \) which is also a triangle such that \( F(x_1, y_1) = F(x_2, y_2) \). Then we have

\[
\sum_{i=1}^{3} (\phi_i(x_1, y_1) - \phi_i(x_2, y_2)) w_i = 0.
\]

Without loss of generality, we may assume that \( w_1 = (0, 0) \). As \( W \) is a triangle with nonzero area, it follows that \( (\phi_i(x_1, y_1) - \phi_i(x_2, y_2)) = 0, i = 2, 3 \). That is, \( \phi_i(x_1, y_1) = \phi_i(x_2, y_2), i = 2, 3 \). By the GBC property, \( 1 = \phi_1(x_2, y_2) + \phi_2(x_2, y_2) + \phi_3(x_2, y_2) \) and hence, \( \phi_1(x_1, y_1) = \phi_1(x_2, y_2) \). Now

\[
(x_1, y_1) = \sum_{i=1}^{3} \phi_i(x_1, y_1)v_i = \sum_{i=1}^{3} \phi_i(x_2, y_2)v_i = (x_2, y_2).
\]

We see that \( x_1 = x_2, y_1 = y_2 \) and hence, \( F \) is injective. It is also known that when \( W \) is not a convex polygon, the GBC map may not be an injection. See a counter example in [J13].

In the following, let us prove that the determinant of the Jacobian matrix of \( F \) is nonzero if \( F \) is harmonic and \( W \) is convex. To do so, we need more properties of harmonic functions. Let us begin with the set of critical points of harmonic function.

Definition 2.6. Suppose that \( u \) is harmonic on \( \Omega \subset \mathbb{R}^2 \). The critical set \( C(u) \) of \( u \) is the subset of \( \Omega \) where \( \nabla u \) vanish. More precisely,

\[
C(u) := \{x \in \Omega \mid \nabla u(x) = 0\}.
\]

Lemma 2.7. If \( u \) is non-constant harmonic function, then \( C(u) \) is an empty set or a collection of finitely many isolated points.

Proof. Let \( i = \sqrt{-1} \) and write \( z = x + iy \). Denote by \( u_x, u_y \) the partial derivatives of \( u \). Note that \( f(z) = u_x - iu_y \) is a holomorphic complex function as one can easily check that \( f \) satisfied Cauchy-Riemann equations. A fundamental theory of complex analysis states that the set of zeros of an holomorphic function could only be empty or finitely many isolated point(s). Since \( \nabla u(x, y) = 0 \) if and only if \( f(z) = 0 \), we have \( C(u) = \{x \in \Omega \mid f(z) = 0\} \) is empty or contains finitely many isolated point(s).

Remark. There are few different ways to prove Lemma 2.7. For example, a good way to see is that the critical set \( C(u) \) of a harmonic function \( u : \mathbb{R}^n \to \mathbb{R} \) have Hausdorff dimension at most \( n - 2 \) as discussed in [HHHN]. So when \( n = 2 \), \( C(u) \) is empty or isolated point(s) which are only cases when a set is of Hausdorff dimension 0.
Next we discuss the level curve(s) of harmonic function $u$. Fix $a \in \Omega$. Let $\Gamma_u(a) := \{x \in \Omega | u(x) = u(a)\}$ be the level curve set of $u$ at $u(a)$. The set will have the following properties.

**Lemma 2.8.** Let $u : \Omega \rightarrow \mathbb{R}$ be a non-constant harmonic function. If $a \in C(u)$, then $\Gamma_u(a)$ are curves intersect at $a$. Moreover, the angle between two neighboring curves is $\pi/n$, where $n$ is the number of curves in $\Gamma_u(a)$. Also, any of these curves is not a loop.

**Proof.** WLOG, let $a = 0$. Consider a function $f(z) = u_x - iu_y$. Then $f(0) = 0$ and $f$ is holomorphic and hence, $f$ has a finite multiplicity, say $n$ at $0$. Write $f = nz^{n-1} + O(z^n)$ be the Taylor expansion of $f$ at $0$. So $u = \text{Re}(z^n) + O(z^{n+1})$. Note that $\Gamma_{z^n}(0)$ is $n$ lines intersects at $0$ with angle $\frac{2\pi}{n}$, so $\Gamma_u(0)$ are curves as described above.

Next if there is a level curve which has a loop, say it encloses a nonempty open set $D \subset V$. Because $u$ is harmonic on $V$, $u$ is harmonic on $D$ and hence, $u$ must be a constant on $D$. That is, the critical point set of $u$ contains a nonempty open set $D$ which contradicts to Lemma 2.7. $u$ can have only isolated critical points. Hence, any curve in the level set is not a loop. \hfill $\square$

Next we need the set of extremes.

**Definition 2.9.** Let $\text{Max}_u := \{x \in \partial\Omega | u(x) \geq u(y) \forall y \text{ in a neighborhood of } x\}$ be the set of all local maximal locations. Similarly, let $\text{Min}_u := \{x \in \partial\Omega | u(x) \leq u(y) \forall y \text{ in a neighborhood of } x\}$ be the set of local minimal locations. We denote by $#\text{Max}_u$ the number of connected components in the set $\text{Max}_u$. Similarly, $#\text{Min}_u$ is the number of connected components in the set $\text{Min}_u$.

**Theorem 2.10.** Suppose that $u : \Omega \rightarrow \mathbb{R}^2$ is a harmonic function. If $#\text{Max}_u = $#\text{Min}_u = 1$, then $C(u) = \emptyset$.

**Proof.** If there exist any $a \in C(u)$, by Lemma 2.8, $\Gamma_u(a)$ are curves intersect at $a$. Since $a \in \Omega$, these curves will intersect with $\partial\Omega$ $2n$ times, where $n$ is number of level curves in $\Gamma_u(a)$ and $n \geq 2$. That is, the level curves cut the boundary $\partial\Omega$ into $2n$ pieces with $n \geq 2$. There must have some components of $\text{Max}_u$ or $\text{Min}_u$ between two intersecting points. So $#\text{Max}_u \geq 2$ or $#\text{Min}_u \geq 2$ which is a contradiction. \hfill $\square$

Furthermore, we need more definitions and one more useful lemma. Define a continuous function $f_\alpha = \cos(\alpha)u + \sin(\alpha)v$ over $V$ for each $\alpha \in [0, \pi)$. Let

$$S_\alpha := \{(u,v) \in W | \cos(\alpha)u + \sin(\alpha)v = \max_{(u',v') \in W} \cos(\alpha)u' + \sin(\alpha)v'\}$$

and

$$s_\alpha := \{(u,v) \in W | \cos(\alpha)u + \sin(\alpha)v = \min_{(u',v') \in W} \cos(\alpha)u' + \sin(\alpha)v'\}.$$ 

**Lemma 2.11.** Let $F$, $V$ and $W$ be the same in Theorem 2.2. Then $#\text{Max}_u = $#\text{Min}_u = 1 for any $\alpha \in [0, \pi]$.

**Proof.** We denote by $#S$ the number of connected components in the set $S$ in the proof. Fixed an $\alpha$ in $[0, \pi]$. Note that $f_\alpha$ is the inner product $(\cos\alpha, \sin\alpha) \cdot F$. See an illustration of $f_\alpha$ in Figure 4.

Since $W$ is convex, both $S_\alpha$ and $s_\alpha$ are a set consisting of one connected component. See Figure 4 for the cases when $W$ is convex and is not convex. When $W$ is convex, $S_\alpha$
contains only one component of the image $F$. When $W$ is not convex, $S_\alpha$ may have two or more separate components as in the left of Figure 4 where $\alpha = \pi/4$. Because $f_\alpha$ is harmonic, $f_\alpha$ can only achieve its maximum on the boundary. So $S_\alpha \subset \partial W$. Because $F$ is a continuous bijection on the boundary $\partial V$ to $\partial W$, $F^{-1}(S_\alpha)$ is a set of one component and hence $\#Max_u = \#F^{-1}(S_\alpha) = 1$. Similarly, $\#Min_u = \#F^{-1}(s_\alpha) = 1$. 

Note that even though the non-zero determinant of Jacobian implies local injection, we cannot extend it to $\partial V$ because $F$ is not necessary be differentiable on $\partial V$ as $V$ has corners. However, we can prove the following

**Lemma 2.12.** Suppose that $F : K \to T$, where $K \subset V$ is a compact set and $T \subset W$ is connected. Also, $F \in C^0(K)$ is a local homeomorphism. The pre-image set $F^{-1}(u, v)$ is finite for all $(u, v) \in \text{Im}(F)$.

**Proof.** Suppose that there is a point $(u, v) \in T$ such that the set $F^{-1}(u, v) \subset K$ has an infinitely many points. Then due to the compactness of $K$, there is a subset of points which converge to a cluster point $(x, y) \in K$. Then $F$ is no longer a local homeomorphism at $(x, y)$ which contradicts the result in Lemma 2.4.

Due to the fact that the pre-image set $F^{-1}(u, v)$ consists of finitely many isolated points, we can further show

**Lemma 2.13.** Under the same assumptions in Lemma 2.12, the number of all isolated points in the pre-image set $F^{-1}(u, v)$ is a constant overall $(u, v) \in \hat{W}$.

**Proof.** Suppose $T \subset \hat{W}$ is a connected compact set. Since $F$ is continuous, $K := F^{-1}(T)$ is a compact set. $F|_K$ is a local homeomorphism because $K \subset V$. We first show that $T_i := \{(u, v) \in T | \#F^{-1}(u, v) = i\}$ is an open set for any fixed integer $i \geq 1$, where $\#F^{-1}(u, v)$ stands for the cardinality of the pre-image set $\#F^{-1}(u, v)$ of $(u, v) \in T$.

Let $(u, v) \in T_i$. Consider $F^{-1}(u, v) = \{x_1, x_2, \cdots, x_i\}$, where $x_j$ are isolated points in $V$. Since $F$ is a local homeomorphism, we can choose disjoint neighborhood $N_j$ of $x_j$, $j = 1, 2, \cdots, i$, where $F$ is a homeomorphism from $N_j$ to the image set $F(N_j)$.
R = V \ (\cup_{j=1}^{j=N_j}). Then R is a close set as V is compact and hence, F(R) is a close set. Now we define an open set M = (\cap_{j=1}^{j=N_j}) \setminus F(R) on T. Obviously, M is a neighborhood of (u, v), and F^{-1}(M) is a union of disjoint open sets belongs to \cup_{j=1}^{j=N_j}. By the definition of N_j, F^{-1}(M) \cap N_j is a homeomorphism which means the cardinality #F^{-1}((u', v')) = i for all (u', v') \in M, so M \subset T_{i} and hence T_i is an open set.

Now since T_i is open for any fixed i, it follows that T = \bigcup T_i. As T is connected, it cannot be a disjoint union of more than 2 open set. This implies that T = T_{i} for some i \geq 1. Finally, because this is true for all connected set T \subset W, we can conclude that the number of the pre-image set F^{-1}(u, v) is a constant overall (u, v) \in W. \hfill \Box

We now show that i = 1. Suppose that i \geq 2 on W. We pick 4 points A, B, C, and D in the counterclockwise order as shown on the right of Figure 5. Then we pick a curve called \Gamma_1 connected A and C, and a curve \Gamma_2 connected B and D over W where 2 curves intersect at x = (u, v) on the right of Figure 5. Note that F is a homeomorphism on boundary, but not on the interior. We can see that F^{-1}(\Gamma_1) has i curves jointing F^{-1}(A) and F^{-1}(C) and do not intersects over \hat{V}. Similarly, F^{-1}(\Gamma_2) has i curves jointing F^{-1}(B) and F^{-1}(D) and do not intersects over \hat{V}. By the fact that the intersections of F^{-1}(\Gamma_1) and F^{-1}(\Gamma_2) is F^{-1}(u, v). We conclude that the cardinality #F^{-1}(u, v) = i^2 > i, which is a violation of Lemma 2.13 unless i = 1. Figure 5 shows the case when i = 2.

Let us summarize the discussion above in the following

**Theorem 2.14.** Suppose V, W are two polygons in \mathbb{R}^2. Let F : V \rightarrow W be a harmonic GBC map. Then F is an injection from V to W.

**Proof.** As we have seen that i = 1, i.e. the number of isolated points in the pre-image F^{-1}(u, v) is one for each (u, v) \in W, the map F is injective. \hfill \Box

Combine with Lemma 2.3 we have the following

**Theorem 2.15.** Suppose V, W are two simply connected domains in \mathbb{R}^2. Let F : V \rightarrow W be a harmonic map satisfying F \in C^1(V) \cap C^0(V). Suppose that F|_{\partial V} = \partial W is an orientation preserved homeomorphism and determinant of Jacobian is non-vanish on \hat{V}. Then F is an injection from V to W.

Now we ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** We write (u, v) = F(x, y) and use f_\alpha for \alpha \in [0, \pi). By Lemma 2.11 and Theorem 2.10 we have C(f_\alpha) = \emptyset. This implies the vector \nabla f_\alpha = \nabla F(x, y) \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} \neq 0.
for all \((x,y) \in \tilde{V}\) and for any \(\alpha \in [0, \pi)\). So \(\nabla u\) and \(\nabla v\) are linearly independent in \(V\). Hence, \(\det(\nabla F) = \det([\nabla u; \nabla v]) \neq 0\) for all \((x,y) \in \tilde{V}\), which implies that \(F\) is a local homeomorphism on \(\tilde{V}\) by Lemma 2.4.

Hence, \(F\) is an injection by Theorem 2.14 if \(F\) is a harmonic GBC map and by Theorem 2.15 if \(F\) is a general harmonic map. Together with Lemma 2.3, we complete the proof. \(\square\)

3 Remarks

We have the following remarks in order.

• 1. Note that the convexity condition in Theorem 2.1 is a sufficient condition. It is interesting to see how the map \(F\) from \(V\) to \(W\) behaves when \(W\) may not be a convex domain. We use our bivariate spline harmonic map \(F\) (cf. [DFL20]) to explore the possibility if the map \(F\) is a bijection. It is easy to find many examples that \(F\) does not map into \(W\) if \(W\) is not a convex polygon. However we found a few examples of polygons which can be mapped into an L-shaped domain bijectively using our \(F\) in the numerical sense. See Figure 6 for such an example. We show a J-shaped domain with labeled vertices in \(J\) and in \(L\). We compute the determinants of Jacobi matrix numerically at more than 25,000 points and only found 3 points on the boundary of the L-shaped domain where the determinant of the Jacobi matrix is less than \(1e^{-5}\). These three points locate on the boundary of the L-shaped domain as shown in the red points in Figure 6. The contour lines on the right of Figure 6 show that the mapping is surjective. However we are not able to show such a map is indeed a bijection theoretically.

![Figure 6: Contour lines of \(F\) from a J-shaped domain to an L-shaped domain](image)

In fact, the mapping between the boundary of \(V\) and the boundary \(W\) is important. Many different arrangements of the boundary map between \(V\) and \(W\) lead to the harmonic GBCC map \(F\) which is not bijective in our numerical experiment. We leave the question how to find a good arrangement of the boundaries of \(V\) and \(W\) so that the map is bijective as an open problem for further study.

• 2. As harmonic GBC functions are infinitely many differentiable, we can conclude that \(F\) defined in (3) is not only a bijection, it also a diffeomorphism from \(\tilde{V}\) to \(W\). It is interesting to compare our results, e.g. Theorem 2.1 to the main result, i.e. Theorem 1.3 in [AN08]. For convenience, let us start the main result below. Let
\( B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \) denote the unit disk. Recall the following theorem from \[AN08]\:

**Theorem 3.1** (G. Alexsandriini and V. Nesi, 2008[AN08]). Let \( \Phi : \partial B \mapsto \gamma \subset \mathbb{R}^2 \) be an orientation preserving diffeomorphism of class \( C^1 \) onto a simple closed curve \( \gamma \). Let \( D \) be the bounded domain such that \( \partial D = \gamma \). Let \( U \in C^2(\bar{B}; \mathbb{R}^2) \cap C^1(B; \mathbb{R}^2) \) be the solution to the Dirichlet boundary problem of Laplace equation:

\[
\begin{align*}
\Delta U(x, y) &= 0, \quad (x, y) \in B \\
U(x, y) &= \Phi, \quad (x, y) \in \partial B.
\end{align*}
\]

Then the mapping \( U \) is a diffeomorphism of \( B \) onto \( D \) if and only if

\[
\det(\nabla U) > 0 \text{ everywhere on } \partial B. \tag{6}
\]

Our result uses a convex domain \( W \) instead of the unit \( B \). Also, our \( F \) maps from a polygon \( V \) to \( W \) instead of \( U \) from \( B \) to \( D \) in \[AN08\]. Our \( F \) is an orientation preserving homeomorphism over the piecewise linear boundary \( \partial V \) while the theorem above requires the diffeomorphism \( \Phi \). We do not have a condition similar to (6). One significance of the harmonic GBC map \( F \) is that the diffeomorphism from \( V \) to \( W \) can be easily constructed using the GBC functions.

- **3.** It is also interesting to compare the harmonic GBC map with the Riemann map from any polygon to the disk. The computation of such a Riemann map can be found in book \[GY07\]. Both maps are diffeomorphism from \( V \) to \( W \) when \( W \) is a disk. It is well-known that the Riemann map is a conforming map which preserves the angle and relative locations of the domain \( V \) as shown in the images inside \[GY07\]. However, a harmonic GBC map gives a distorted image called image warping as shown in Figure 3 which may be used for creative artworks such as cartoon images, e.g. in \[JMDGS06\]. One advantage of the harmonic GBC map is that the diffeomorphism from \( V \) to \( W \) can be easily constructed using the GBC functions.

- **4.** Although the harmonic GBC functions are defined over a polygon, they are possible to be defined over a polygonal domain with a hole or holes. We refer to \[HF06\] for mean value coordinates over domains with holes. Indeed, suppose a polygonal domain \( P \) with a hole has \( n \) outer boundary vertices \( v_i, i = 1, \cdots, n \) and \( m \) vertices \( h_j, j = 1, \cdots, m \) on the inner boundary of the hole. One simply defines \( \phi_1, \cdots, \phi_n \) are nonnegative functions which satisfy the standard GBC boundary conditions with additional property: \( \phi_i \)'s are zero at the boundary of the hole. Also, \( \phi_{n+1}, \cdots, \phi_{n+m} \) are nonnegative functions which are zero on the outer boundary of \( P \) and satisfy the standard GBC boundary conditions over the boundary of the hole. All these functions \( \phi_1, \cdots, \phi_n \) and \( \phi_{n+i}, \cdots, \phi_{n+m} \) satisfy the standard GBC properties inside \( P \):

\[
\sum_{i=1}^{n+m} \phi_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \phi_i v_i + \sum_{j=1}^{m} \phi_{n+j} h_j = (x, y),
\]
where \( v_i, i = 1, \cdots, n \) are the vertices of the outer boundary of \( P \) and \( h_j, i = 1, \cdots, m \) are the vertices of the inner boundary of \( P \). See Figs. 7 and 8 for contour plots of some harmonic GBC functions (\( \phi \) is now shown) computed based on the bivariate spline solution to the corresponding Laplace equation over the polygonal domain \( P \) with a hole with an appropriate boundary condition (cf. [ALW06]). It is interesting to know the bijectivity of the harmonic map based on these GBC functions. We leave the question to an open problem.

![Figure 7: Contour plots of GBC functions over a domain with a hole](image1)

![Figure 8: Contour plots of more GBC functions over a domain with a hole](image2)

• 5. Although we can use 3D GBC functions to construct a deforming mapping from a 3D polyhedron to a convex polyhedron, say solid ball as in [SH15], it is easy to see that any extension of the proof of Theorem 2.1 to the 3D setting may not be easy. This is because that the proof uses the zero property of holomorphic functions which has no simple extension in the 3D setting.

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