A New Quadratic Bound for the Manickam-Miklós-Singhi Conjecture

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Abstract
More than twenty-five years ago, Manickam, Miklós, and Singhi conjectured that for positive integers \( n, k \) with \( n \geq 4k \), every set of \( n \) real numbers with nonnegative sum has at least \( \binom{n-1}{k-1} \) \( k \)-element subsets whose sum is also nonnegative. We verify this conjecture when \( n \geq 8k^2 \), which simultaneously improves and simplifies a bound of Alon, Huang, and Sudakov and also a bound of Pokrovskiy when \( k < 10^{45} \).

1 Introduction
Manickam, Miklós, and Singhi [17, 18] conjectured in 1988 that

\textbf{Conjecture 1.1} For positive integers \( n, k \) with \( n \geq 4k \), every set of \( n \) real numbers with nonnegative sum has at least \( \binom{n-1}{k-1} \) \( k \)-element subsets whose sum is also nonnegative.

Conjecture 1.1 was motivated by studies of the first distribution invariant in certain association schemes, and may also be considered an analogue of the famous Erdős-Ko-Rado theorem [10]. The Erdős-Ko-Rado theorem states that if \( n > 2k \), then any family of \( k \)-element subsets of an \( n \)-element set with the property that any two subsets have nonempty intersection has size at most \( \binom{n-1}{k-1} \); moreover the unique extremal family is a star, the family of \( k \)-element subsets containing a fixed element.

Conjecture 1.1 is similar to the Erdős-Ko-Rado theorem, not only in the appearance of the binomial coefficient \( \binom{n-1}{k-1} \), but also because the family of \( k \)-element subsets with nonnegative sum attains this lower bound and forms a star when one of the \( n \) real numbers equals \( n - 1 \) and the remaining \( n - 1 \) numbers equal \(-1\). As in the Erdős-Ko-Rado theorem, \( n \) must be large enough with respect to \( k \), otherwise there exist \( n \) real numbers with nonnegative sum and fewer than \( \binom{n-1}{k-1} \) \( k \)-element subsets with nonnegative sum.

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Such examples can be easily constructed when \( n = 3k + r \) and \( 1 \leq r \leq k/7 \). Although Conjecture 1.1 and the Erdős-Ko-Rado theorem share the same bound and extremal example, there is no obvious way to translate one question into the other.

Conjecture 1.1 has attracted a lot of attention due to its connections with the Erdős-Ko-Rado theorem [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 21], but still remains open. For more than two decades, Conjecture 1.1 was known to hold only when \( k | n \) [18] or when \( n \) is at least an exponential function of \( k \) [4, 6, 17, 21]. In their recent breakthrough paper, Alon, Huang, and Sudakov [2] obtained the first polynomial bound \( n \geq \min\{33k^2, 2k^3\} \) on Conjecture 1.1. Later, Aydinian and Blinovsky [3] and Frankl [11] gave different proofs of Conjecture 1.1 for a cubic range. Recently, a linear bound \( n > 10^{46}k \) has been obtained by Pokrovskiy [20]. Finally, there are also several works that verify Conjecture 1.1 for small \( k \) [8, 13, 16, 19].

The main result of this paper verifies Conjecture 1.1 when \( n \geq 8k^2 \). In particular, Theorem 1.2 simultaneously improves and simplifies the bound \( n \geq \min\{33k^2, 2k^3\} \) of Alon, Huang, and Sudakov [2] and also the bound \( n \geq 10^{46}k \) of Pokrovskiy [20] when \( k < 10^{45} \). Note that there is no loss of generality in assuming that the \( n \) real numbers in Conjecture 1.1 are listed in decreasing order and sum to zero.

**Theorem 1.2** Let \( X = \{x_1, \ldots, x_n\} \subset \mathbb{R} \) be a set of \( n \) real numbers whose sum is zero, and assume \( x_i \geq x_j \) if \( i \leq j \). If \( n \geq 8k^2 \), then at least \( (n-1)k(n-k-1) \) \( k \)-element subsets of \( X \) have nonnegative sum. Moreover, if equality holds, the family of \( k \)-element subsets with nonnegative sum is a star on \( x_1 \), \( \{S \in \binom{X}{k} : x_1 \in S\} \).

The proof of Theorem 1.2 is similar to that of Theorem 1.3 in [9], where we tackle the Manickam-Miklós-Singhi conjectures for sets and vector spaces simultaneously. For the reader’s convenience, we present the calculations for the case of sets in full detail in this unpublished manuscript.

## 2 Bose-Mesner Matrices

We will need a lemma involving inclusion matrices \( W_{jk} \) and Kneser matrices \( \overline{W}_{jk} \). Let \( W_{jk} \) (respectively \( \overline{W}_{jk} \)) denote the \( \binom{n}{j} \times \binom{n}{k} \) matrix whose rows are indexed by the \( j \)-element subsets of \( X \), whose columns are indexed by the \( k \)-element subsets of \( X \), and where the entry in row \( Y \) and column \( S \) is 1 if \( Y \subset S \) (respectively if \( Y \cap S = \emptyset \)) and is 0 otherwise.

Define \( \vec{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) to be a vector that lists the \( n \) real numbers in Theorem 1.2. Without loss of generality, we may assume that \( \vec{x} \neq \vec{0} \), and observe that \( \vec{x} \) is orthogonal to \( \vec{1} \). We will show that \( W_{1k}^T \vec{x} \) has at least \( \binom{n-1}{k-1} \) nonnegative entries when \( n \geq 8k^2 \).

An important observation by Wilson [23] is that \( W_{1k}^T \vec{f} \) is an eigenvector of the Bose-Mesner matrix

\[
B_j = \overline{W}_{jk}^T W_{jk} \tag{2.1}
\]

for \( 0 \leq j \leq k \) with eigenvalue \( -(\binom{k-1}{j-1}) \binom{n-j-1}{k-1} \). We include a proof for completeness.
Lemma 2.1 (Wilson, [23]) For $0 \leq j \leq k$, we have $W_{1k}^T \vec{x}$ is an eigenvector of the Bose-Mesner matrix $B_j$ with eigenvalue

$$ -(k-1) \binom{n-j-1}{j-1}.$$  

Proof. Since $W_{1k}^T$ has full column rank [12, 15, 22] and $\vec{x} \neq \vec{0}$, we have that $W_{1k}^T \vec{x} \neq \vec{0}$. For $S \in \binom{X}{j}$ and $T \in \binom{X}{1}$, observe that $W_{jk}W_{1k}^T(S,T)$ counts the number of $k$-element subsets of $X$ that contain $S \cup T$. Hence,

$$ W_{jk}W_{1k}^T(S,T) = \begin{cases} \binom{n-j}{k-j-1} & \text{if } T \subset S \\ \binom{n-j-1}{k-j} & \text{if } T \not\subset S. \end{cases} $$

Since $J\vec{x} = \vec{0}$, we have

$$ W_{jk}W_{1k}^T = \left( \begin{array}{c} n-j-1 \\ k-j \end{array} \right) W_{1j}^T + \left( \begin{array}{c} n-j-1 \\ k-j-1 \end{array} \right) J \vec{x} = \left( \begin{array}{c} n-j-1 \\ k-j \end{array} \right) W_{1j}^T \vec{x}. \quad (2.2) $$

For $A \in \binom{X}{j}$ and $B \in \binom{X}{1}$, observe that $\overline{W}_{jk}^T W_{1j}^T(A,B)$ counts the number of $j$-element subsets of $X$ that are disjoint from $A$ and that contain $B$. Hence,

$$ \overline{W}_{jk}^T W_{1j}^T = \left( \begin{array}{c} n-j-1 \\ j-1 \end{array} \right) \overline{W}_{1k}^T = \left( \begin{array}{c} n-j-1 \\ j-1 \end{array} \right) (J-W_{1k}^T) \vec{x} = -\left( \begin{array}{c} n-j-1 \\ j-1 \end{array} \right) W_{1k}^T \vec{x}, \quad (2.3) $$

since $J\vec{x} = \vec{0}$. Multiplying (2.2) on the left by $\overline{W}_{jk}^T$ and applying (2.3) yields

$$ B_j W_{1k}^T = -\left( \begin{array}{c} n-j-1 \\ k-j \end{array} \right) \left( \begin{array}{c} n-j-1 \\ j-1 \end{array} \right) W_{1k}^T \vec{x} = -\left( \begin{array}{c} k-1 \\ j-1 \end{array} \right) \left( \begin{array}{c} n-j-1 \\ k-1 \end{array} \right) \overline{W}_{1k}^T \vec{x}, \quad (2.4) $$

which proves that $W_{1k}^T \vec{x}$ is an eigenvector of the Bose-Mesner matrix $B_j$ with eigenvalue $-\left( \begin{array}{c} k-1 \\ j-1 \end{array} \right)(n-j-1)/(k-1)$.

We will use Lemma 2.1 to obtain lower bounds on the number of nonnegative $k$-element subsets that intersect $\{x_1, \ldots, x_k\}$ and that contain $x_1$ respectively.

## 3 Bounds from Eigenvalues

The main result of this section is Lemma 3.3, which shows that if $n \geq Ck^2$ and there are at most $\binom{n-1}{k-1}$ nonnegative $k$-element subsets of $X$, then at least $\left( 1 - \frac{6}{C} \right) \binom{n-1}{k-1}$ $k$-element subsets on $x_1$ have nonnegative sum.

Henceforth, we write $W_{1k}^T \vec{x} = \vec{b}$, and index the entries of $\vec{b}$ with subsets $S \in \binom{X}{k}$. Let $A = \{x_1, \ldots, x_k\}$ and note that $A$ is the $k$-element subset of $X$ with largest sum. Using Lemma 2.1, we give a lower bound on the number of nonnegative $k$-element subsets of $X$ that intersect $A$. 


Lemma 3.1  There are greater than \( \binom{n-k-1}{k-1} \) nonnegative \( k \)-element subsets of \( X \) that intersect \( A = \{ x_1, \ldots, x_k \} \).

**Proof.** Observe that \( b_A \) is a largest entry of \( \vec{b} \) and that \( b_A > 0 \) since \( \vec{b} \neq \vec{0} \) and \( \vec{b} \) is orthogonal to \( \vec{1} \).

For \( S, T \in \binom{X}{k} \), observe that \( B_j(S, T) \) counts the number of \( j \)-element subsets of \( X \) that lie in \( T \) and are disjoint from \( S \). Hence,

\[
B_j(S, T) = \binom{k - |S \cap T|}{j}.
\]

(3.5)

By Lemma 2.1, the dot product of the row of \( B_k \) corresponding to \( A \) and \( \vec{b} \) equals \( -(n-k-1) b_A \). Hence, (3.5) with \( j = k \) yields

\[
\sum_{S \cap A = \emptyset} b_S = -\binom{n - k - 1}{k - 1} b_A.
\]

(3.6)

Since \( \vec{b} \) is orthogonal to \( \vec{1} \), we see that

\[
\sum_{S \cap A \neq \emptyset} b_S = \binom{n - k - 1}{k - 1} b_A.
\]

(3.7)

Since \( b_A \) is a largest entry of \( \vec{b} \), there are greater than \( \binom{n-k-1}{k-1} \) nonnegative \( k \)-element subsets of \( X \) that intersect \( A \). \( \square \)

Recall that \( A = \{ x_1, \ldots, x_k \} \) is the \( k \)-element subset of \( X \) with largest sum. Let \( C = \{ x_1, x_{k+1}, \ldots, x_{2k-1} \} \) and note that \( C \) is the \( k \)-element subset of \( X \) with largest sum such that \( |A \cap C| = 1 \). Using Lemma 2.1 we give a lower bound on \( b_C \), the sum of \( C \), under the assumptions that \( n \geq k^2 \) and that there are at most \( \binom{n-1}{k-1} \) \( k \)-element subsets with nonnegative sum in \( X \).

**Lemma 3.2** Let \( A = \{ x_1, \ldots, x_k \} \) and let \( C = \{ x_1, x_{k+1}, \ldots, x_{2k-1} \} \). If \( n \geq k^2 \) and there are at most \( \binom{n-1}{k-1} \) nonnegative \( k \)-element subsets of \( X \) then \( b_C \), the sum of \( C \), satisfies

\[
b_C \geq \left( 1 - \frac{(2k-1)(k-1)}{n - 2k + 1} \right) b_A.
\]

(3.8)

**Proof.** By Lemma 2.1 with \( j = k - 1 \), the dot product of the row of \( B_{k-1} \) corresponding to \( A \) and \( \vec{b} \) equals \( -(k - 1) \binom{n-k}{k-1} b_A \). Hence, by (3.5),

\[
k \sum_{S \cap A = \emptyset} b_S + \sum_{|S \cap A| = 1} b_S = -(k - 1) \binom{n-k}{k-1} b_A.
\]

(3.9)

Hence,

\[
\sum_{|S \cap A| = 1} b_S = \left( \binom{n-k-1}{k-1} - (k - 1) \binom{n-k-1}{k-2} \right) b_A,
\]

(3.10)
which is nonnegative if and only if \( n \geq k^2 \). Observe that for any \( S \in \binom{X}{k} \) such that \(|S \cap A| = 1\), we have \( b_C \geq b_S \). We claim that

\[
 b_C \geq \left( \frac{(n-k-1)-(k-1)(n-k-1)}{n-k-1} \right) b_A, \tag{3.11}
\]

otherwise as \( b_A \) is a largest entry of \( \bar{b} \), (3.10) implies there are at least \( \binom{n-1}{k-1} \) nonnegative entries \( b_S \) where \(|S \cap A| = 1\). Since \( A \) has nonnegative sum, we would get greater than \( \binom{n-1}{k-1} \) nonnegative \( k \)-element subsets of \( X \) if (3.11) does not hold.

Now, we show that the fraction on the right hand of (3.11) is at least the fraction on the right hand side of (3.8). We have

\[
\binom{n-k-1}{k-1} - (k-1)\binom{n-k-1}{k-2} = \left( 1 - \frac{(k-1)^2}{n-2k+1} \right) \binom{n-k-1}{k-1}. \tag{3.12}
\]

We also have that

\[
\frac{(n-k-1)}{(n-1)} = \frac{(n-k-1) \cdots (n-2k+1)}{(n-1) \cdots (n-k+1)} > \left( 1 - \frac{k}{n-k+1} \right)^{k-1} > 1 - \frac{k(k-1)}{n-k+1}. \tag{3.13}
\]

Putting (3.12) and (3.13) together yields (3.8).

Now we give a lower bound on the number of nonnegative \( k \)-element subsets that contain \( x_1 \) under the assumptions that \( n \geq k^2 \) and that there are at most \( \binom{n-1}{k-1} \) \( k \)-element subsets of \( X \) with nonnegative sum.

**Lemma 3.3** If \( n \geq k^2 \) and there are at most \( \binom{n-1}{k-1} \) nonnegative \( k \)-element subsets of \( X \), then the number of nonnegative \( k \)-element subsets that contain \( x_1 \) is at least

\[
\left( 1 - \frac{(6k-3)(k-1)}{n-2k+1} \right) \binom{n-1}{k-1}. \tag{3.14}
\]

**Proof.** Recall that \( A = \{x_1, \ldots, x_k\} \) and that \( C = \{x_1, x_{k+1}, \ldots, x_{2k-1}\} \). By Lemma 2.1, the dot product of the row of \( B_k \) corresponding to \( C \) and \( b \) equals \(-\binom{n-k-1}{k-1}b_C\). Hence,

\[
\sum_{S \cap C \neq \emptyset, S \cap A \neq \emptyset} b_S + \sum_{S \cap C \neq \emptyset, S \cap A = \emptyset} b_S = \sum_{S \cap C \neq \emptyset} b_S = \binom{n-k-1}{k-1}b_C. \tag{3.15}
\]

We claim that

\[
\sum_{S \cap C \neq \emptyset, S \cap A = \emptyset} b_S \leq \left( \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \right) b_A. \tag{3.16}
\]

Otherwise, there would be at least \( \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \) nonnegative entries \( b_S \) such that \( S \cap C \neq \emptyset \) and \( S \cap A = \emptyset \) as \( b_A \) is a largest entry. By Lemma 3.1, this would yield
greater than \((n-1)\) nonnegative \(k\)-element subsets. Hence, (3.16) holds, which implies by Lemma 3.2, (3.13), (3.15), and (3.16) that

\[
\sum_{\substack{S \cap C \neq \emptyset, \\
x \in X \setminus \emptyset}} b_S \geq \left(1 - \frac{k(k - 1)}{n - k + 1}\right) \left(2 - \frac{(2k - 1)(k - 1)}{n - 2k + 1}\right) - 1 \left(\frac{n - 1}{k - 1}\right) b_A
\]

\[
\geq \left(1 - \frac{(4k - 1)(k - 1)}{n - 2k + 1}\right) \left(\frac{n - 1}{k - 1}\right) b_A.
\]

(3.17)

Let \(\mathcal{F}_i\) be the family of \(k\)-element subsets of \(X\) that contain \(x_i\) but not \(x_1\) and intersect \(A\) and \(C\). We have

\[
\sum_{x_1 \in S} b_S = \sum_{\substack{S \cap C \neq \emptyset, \\
x_1 \notin S}} b_S - \sum_{\substack{S \cap C \neq \emptyset, \\
x_1 \notin S}} b_S = \sum_{\substack{S \cap C \neq \emptyset, \\
x_1 \notin S}} b_S - \sum_{i=2}^{n} |\mathcal{F}_i|x_i.
\]

(3.18)

We first show that if \(i \in \{2, \ldots, 2k - 1\}\) then

\[
|\mathcal{F}_i| = \left(\frac{n - 2}{k - 1}\right) - \left(\frac{n - k - 1}{k - 1}\right).
\]

(3.19)

Without loss of generality suppose that \(x_i \in A \setminus \{x_1\}\). There are \(\binom{n-2}{k-1}\) \(k\)-element sets of \(X\) that contain \(x_i\) but not \(x_1\). From these, we subtract the \(\binom{n-k-1}{k-1}\) \(k\)-element subsets of \(X\) that contain \(x_i\) but do not intersect \(C\).

Now we determine \(|\mathcal{F}_i|\) when \(i \in \{2k, \ldots, n\}\). Let \(\mathcal{G}_i\) (respectively \(\mathcal{H}_i\)) be the family of \(k\)-element subsets of \(X\) that contain \(x_i\) but not \(x_1\) and intersect \(A\) (respectively \(C\)). We have \(\mathcal{F}_i = \mathcal{G}_i \cap \mathcal{H}_i\) so by inclusion-exclusion,

\[
|\mathcal{F}_i| = |\mathcal{G}_i \cap \mathcal{H}_i| = |\mathcal{G}_i| + |\mathcal{H}_i| - |\mathcal{G}_i \cup \mathcal{H}_i|
\]

\[
= 2 \left(\left(\frac{n - 2}{k - 1}\right) - \left(\frac{n - k - 1}{k - 1}\right)\right) - \left(\left(\frac{n - 2}{k - 1}\right) - \left(\frac{n - 2k}{k - 1}\right)\right)
\]

\[
= \left(\frac{n - 2}{k - 1}\right) - 2 \left(\frac{n - k - 1}{k - 1}\right) + \left(\frac{n - 2k}{k - 1}\right).
\]

(3.20)

By (3.18), (3.19), and (3.20),

\[
\sum_{\substack{S \cap C \neq \emptyset, \\
x_1 \notin S \setminus \emptyset}} b_S = |\mathcal{F}_2| \sum_{i=2}^{2k-1} x_i + |\mathcal{F}_2k| \sum_{i=2k}^{n} x_i = |\mathcal{F}_2|(b_A + b_C - 2x_1) + |\mathcal{F}_2k|(x_1 - b_A - b_C)
\]

\[
= (2|\mathcal{F}_2| - |\mathcal{F}_2k|)(-x_1) + (|\mathcal{F}_2| - |\mathcal{F}_2k|)(b_A + b_C)
\]

\[
< 2(|\mathcal{F}_2| - |\mathcal{F}_2k|)b_A = 2 \sum_{j=k+2}^{2k} \left(\frac{n - j}{k - 2}\right) b_A
\]

\[
< 2(k - 1) \left(\frac{n - k - 2}{k - 2}\right) b_A < \frac{2(k - 1)^2}{n - 1} \left(\frac{n - 1}{k - 1}\right) b_A.
\]
By (3.17), (3.18), and (3.21), we have
\[
\sum_{x_1 \in S} b_S \geq \left( 1 - \frac{(6k - 3)(k - 1)}{n - 2k + 1} \right) \binom{n - 1}{k - 1} b_A.
\] (3.22)

Hence, a lower bound on the number of nonnegative \( k \)-element subsets that contain \( x_1 \) is given by (3.14).

4 Bounds from Averaging

The main result of this section is Lemma 4.1, which shows that if \( T \in \binom{X}{k} \) is a \( k \)-element subset with negative sum, then there are at least \( \binom{n-2k}{k-1} \) nonnegative \( k \)-element subsets of \( X \) that are disjoint from \( T \). The proof of Lemma 4.1 is similar to Manickam’s and Singhi’s proof of Conjecture 1.1 when \( k \mid n \) [18] and has also been observed by others [2, 11, 21].

**Lemma 4.1**  If \( T \in \binom{X}{k} \) has negative sum, then there are at least
\[
\binom{n - 2k}{k - 1} \geq \left( 1 - \frac{(2k - 1)(k - 1)}{n - 2k + 1} \right) \binom{n - 1}{k - 1}
\] (4.23)
nonnegative \( k \)-element subsets of \( X \) that are disjoint from \( T \).

**Proof.** Write \( n = mk + r \) where \( 0 \leq r \leq k - 1 \). Since \( T \in \binom{X}{k} \) has negative sum, adding the smallest \( r \) elements of \( X \setminus T \) to \( T \) yields a \( (k + r) \)-element subset \( U \in \binom{X}{k+r} \) with negative sum. Let
\[
\mathcal{F} := \left\{ S \in \binom{X \setminus U}{k} : b_S \geq 0 \right\}
\] (4.24)
be the family of \( k \)-element sets disjoint from \( U \) that have nonnegative sum.

Consider a random permutation \( \pi \in S_X \) that fixes \( U \). Partition the \( (m-1)k \) elements of \( X \setminus U \) into \( k \)-element sets \( S = \{S_1, \ldots, S_{m-1}\} \), and define the indicator random variable \( Z_i \) to be 1 if \( \pi(S_i) \) has nonnegative sum and 0 otherwise. Let \( Z = \sum_{i=1}^{m-1} Z_i \) and note that \( Z \geq 1 \) because some \( k \)-element subset in \( \pi(S) \) must have nonnegative sum as the sum of the elements of \( X \setminus U \) is positive. On the other hand, \( \mathbb{E}(Z_i) \) is the probability that a randomly chosen \( k \)-element subset that is disjoint from \( U \) has nonnegative sum. Hence,
\[
\mathbb{E}(Z_i) = \frac{|\mathcal{F}|}{(n-k-r)k}.
\] (4.25)

By linearity of expectation,
\[
1 \geq \mathbb{E}(Z) = (m-1)\mathbb{E}(Z_i) = \frac{(n-k-r)|\mathcal{F}|}{k(n-k-r)}.
\] (4.26)

Since \( 0 \leq r \leq k - 1 \), we have
\[
|\mathcal{F}| \geq \binom{n - k - r - 1}{k - 1} \geq \binom{n - 2k}{k - 1} \geq \left( 1 - \frac{(2k - 1)(k - 1)}{n - 2k + 1} \right) \binom{n - 1}{k - 1}.
\] (4.27)

Since \( U \in \binom{X}{k+r} \) contains \( T \in \binom{X}{k} \), each \( k \)-element subset in \( \mathcal{F} \) is also disjoint from \( T \). \( \square \)
5 Proof of Theorem 1.2

Finally, we prove Theorem 1.2.

**Proof of Theorem 1.2** If all $k$-element subsets containing $x_1$ have nonnegative sum, then there are at least $\binom{n-1}{k-1}$ $k$-element subsets of $X$ with nonnegative sum.

Otherwise, some $k$-element subset $T \in \binom{X}{k}$ containing $x_1$ has negative sum. Suppose, for a contradiction, that there are at most $\binom{n-1}{k-1}$ nonnegative $k$-element subsets in this case. By Lemma 3.3, there are at least

$$\left(1 - \frac{(6k-3)(k-1)}{n-2k+1}\right) \binom{n-1}{k-1} \quad (5.28)$$

nonnegative $k$-element subsets containing $x_1$ since $n \geq 8k^2$.

Since $T \in \binom{X}{k}$ has negative sum, by Lemma 4.1, there are at least

$$\left(1 - \frac{(2k-1)(k-1)}{n-2k+1}\right) \binom{n-1}{k-1} \quad (5.29)$$

nonnegative $k$-element subsets of $X$ that have trivial intersection with $T$.

Since $T$ contains $x_1$, none of the $k$-element subsets counted in (5.29) contain $x_1$. Summing (5.28) and (5.29), there are at least

$$\left(2 - \frac{(8k-4)(k-1)}{n-2k+1}\right) \binom{n-1}{k-1} \quad (5.30)$$

nonnegative $k$-element subsets in $X$. For $n \geq 8k^2$, however, the expression in (5.30) is greater than $\binom{n-1}{k-1}$, which contradicts our assumption. □

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References

[1] N. Alon, H. Aydinian, and H. Huang. Maximizing the number of nonnegative subsets. arXiv preprint arXiv:1312.0248, 2013.

[2] N. Alon, H. Huang, and B. Sudakov. Nonnegative $k$-sums, fractional covers, and probability of small deviations. *J. Combin. Theory Ser. B*, 102(3):784–796, 2012.

[3] H. Aydinian and V.M. Blinovsky. A remark on the problem of nonnegative $k$-subset sums. *Probl. Inf. Transm*, 48(4):347–351, 2012.

[4] A. Bhattacharyya. On a conjecture of Manickam and Singhi. *Discrete Math.*, 272(2-3):259–261, 2003.
[5] T. Bier. A distribution invariant for association schemes and strongly regular graphs. *Linear Algebra Appl.*, 57:105–113, 1984.

[6] T. Bier and N. Manickam. The first distribution invariant of the Johnson-scheme. *Southeast Asian Bull. Math.*, 11(1):61–68, 1987.

[7] V. Blinovskiy. Minimal number of edges in hypergraph guaranteeing perfect fractional matching and MMS conjecture. arXiv preprint arxiv:1310.0989, 2013.

[8] A. Chowdhury. A note on the Manickam-Miklós-Singhi conjecture. *European J. Combin.*, 35:131–140, 2014.

[9] A. Chowdhury, G. Sarkis, and S. Shahriari. The Manickam-Miklós-Singhi conjectures for sets and vector spaces. arXiv preprint arXiv:1309.2212, 2013.

[10] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[11] P. Frankl. On the number of nonnegative sums. *J. Combin. Theory Ser. B*, 103(5):647–649, 2013.

[12] J. E. Graver and W. B. Jurkat. The module structure of integral designs. *J. Combinatorial Theory Ser. A*, 15:75–90, 1973.

[13] S. Hartke and D. Stolee. A linear programming approach to the Manickam-Miklós-Singhi conjecture. *European J. Combin.*, 36:53–70, 2014.

[14] H. Huang and B. Sudakov. The minimum number of nonnegative edges in hypergraphs. arXiv preprint arXiv:1309.2549, 2013.

[15] W.M. Kantor. On incidence matrices of finite projective and affine spaces. *Math. Z.*, 124:315–318, 1972.

[16] N. Manickam. *On the distribution invariants of the association scheme*. PhD thesis, Ohio State University, 1986.

[17] N. Manickam and D. Miklós. On the number of nonnegative partial sums of a nonnegative sum. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 385–392. North-Holland, Amsterdam, 1988.

[18] N. Manickam and N. M. Singhi. First distribution invariants and EKR theorems. *J. Combin. Theory Ser. A*, 48(1):91–103, 1988.

[19] G. Marino and G. Chiaselotti. A method to count the positive 3-subsets in a set of real numbers with non-negative sum. *European J. Combin.*, 23(5):619–629, 2002.

[20] A. Pokrovskiy. A linear bound on the Manickam, Miklós, and Singhi conjecture. arXiv preprint arxiv:1308.2176, 2013.
[21] M. Tyomkyn. An improved bound for the Manickam-Miklós-Singhi conjecture. *European J. Combin.*, 33(1):27–32, 2012.

[22] R.M. Wilson. The necessary conditions for $t$-designs are sufficient for something. *Utilitas Math.*, 4:207–215, 1973.

[23] R.M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984.