A Bethe Ansatz Solution For The Closed $U_q[sl(2)]$ Temperley-Lieb Quantum Spin Chains

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Abstract

We solve the spectrum of the closed Temperley-Lieb quantum spin chains using the coordinate Bethe ansatz. These models are invariant under the quantum group $U_q[sl(2)]$. 

1 Introduction

Quantum group together with Temperley-Lieb algebra play an important role in the study of integrable spin chains. It may be interesting to study particular Hamiltonians associated with the Temperley-Lieb which are invariant to the quantum group. Taking into account usual toroidal boundary conditions, the Hamiltonians take the form

$$\mathcal{H} = \sum_{n=1}^{N-1} U_{n,n+1} + U_{N1}. \quad (1.1)$$

where $U_{n,n+1}$ operates in a direct product of two $(2s+1)$-dimensional complex spaces $V^{2s+1}$ at positions $n$ and $n + 1$. They are not invariant with respect to $U_q[sl(2)]$ since $U_{N1} \neq U_{1N}$ breaks translational invariance, reflecting the non-cocommutativity of the co-product. Indeed, we know from [1, 2, 3] that very special boundary terms must be considered when we seek these quantum group invariant spin chains. In particular, one possibility to obtain a quantum group invariant Hamiltonian is to consider open boundary conditions, i.e., $U_{N1} = 0$. For the XXZ-Hamiltonian with open boundary conditions one has to apply the Bethe ansatz techniques introduced by Sklyanin[4] using Cherednik’s reflection matrices[5, 6]. By this method the XXZ-Heisenberg model[7], the $spl_q(2,1)$ invariant supersymmetric t-J model[8, 9], the $U_q[sl(n)]$ invariant generalization of the XXZ-chain[10] and the $SU_q(n|m)$ spin chains[3, 11] have been solved for open boundary conditions by this method.

Recently, by means of generalized algebraic nested Bethe ansatz, Karowski and Zapletal[12] presented a class of quantum group invariant $n$-state vertex models with periodic boundary conditions. Also an extension of this method to the case of graded vertex models was analyzed in [13], where a $spl_q(2|1)$ invariant susy t-J model with boundary conditions was presented.

In fact, this type of models were first discussed by Martin[14] from the representations of the Hecke algebra. The study of closed quantum group invariant closed spin chains in the framework of the coordinate Bethe ansatz was presented by Grosse at al. for the $SU_q(2)$ case [15]. In this context it would be interesting to discuss other quantum group invariant closed spin chains. Therefore, it is the purpose of this paper to present and solve, via coordinate Bethe ansatz[16] a closed spin-s Hamiltonian, which in terms of the Temperley-Lieb operators can be written as

$$\mathcal{H} = \sum_{n=1}^{N-1} U_n + U_0 \quad (1.2)$$

where

$$U_0 = GU_{N-1}G^{-1}, \quad G = (Q - U_1)(Q - U_2) \cdots (Q - U_{N-1}) \quad (1.3)$$
satisfying $[\mathcal{H}, G] = 0$ and additionally invariance with respect to the quantum algebra. The operator $G$ shifts the $U_n$ by one unit $GU_nG^{-1} = U_{n+1}$ and maps $U_0$ into $U_1$, which manifest the translational invariance of $\mathcal{H}$.

2 The Temperley-Lieb Hamiltonians

In the basis where $S^z_n$ is diagonal with eigenvectors $|s, n\rangle, |s - 1, n\rangle, \ldots, |-s, n\rangle$ and eigenvalues $s, s - 1, \ldots, -s$, the Hamiltonian densities acting on two neighboring sites are given by

$$
\langle k, l \mid U \mid i, j \rangle = \epsilon(i)\epsilon(k)q^{(i+k)}\delta_{i+1,0}\delta_{k+l,0}
$$

where $\epsilon(i) = (-1)^i$ for $s$ integer and $\epsilon(i) = (-1)^{i+1/2}$ for $s$ semi-integer. Thus $U_n$ denotes the projection on states whose restriction to sites $n$ and $n + 1$ has total spin zero. These Hamiltonians were derived Batchelor and Kuniba from representations of the Temperley-Lieb algebras associated with quantum group $U_q[sl(2)]$. The case $s = 1/2$ was investigated in reference.

In fact, $U_n$ obeys the Temperley-Lieb algebra

$$
U_n^2 = (Q + Q^{-1})U_n, \quad Q + Q^{-1} = [2s + 1]_q
$$

and commutes with the quantum group $U_q[sl(2)]$. The $q$-number notation is $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. This algebra appears in a large class of solvable models and is known to essentially govern their physical properties: $\mathcal{H}$ is an element of a set of infinity quantities conserved which are involutive provided that $U_n$ satisfies the defining relations \((2.2)\).

Having now built common ground for all closed Hamiltonian densities, whose salient feature is that they are spin-zero projectors, we may implement the steps of, where the spectrum of the $A-D$ Temperley-Lieb Hamiltonians with either periodic and free boundary conditions were solved, via a generalization of the coordinate Bethe ansatz.

3 The coordinate Bethe ansatz

Since these Hamiltonians commute with the total spin $S^z_T = \sum_{n=1}^N S^z_n$, the eigenvalues of the operator $r = sN - S^z_T$ can be used to collect the eigenstates of $\mathcal{H}$ in sectors, $\Psi_r$. Due to this $U(1)$ invariance, there always exists a reference state $\Psi_0$ satisfying $\mathcal{H}\Psi_0 = E_0\Psi_0$, with $E_0 = 0$. We take $\Psi_0$ to be $\Psi_0 = \prod_n |s, n\rangle$. This is the only
eigenstate in the sector \( r = 0 \). All other energies will be measured relative to this state.

We will now start to diagonalize \( \mathcal{H} \) in every sector. Nothing interesting happens in sectors with \( r < 2s \). Since \( \mathcal{H} \) is a sum of projectors on spin zero, these states are annihilated by \( \mathcal{H} \).

The first nontrivial sector \( r = 2s \), the correspondent eigenspace is spanned by the states \( |n(-j, j)\rangle = \sum_{n} s \cdots s -j j s \cdots s \rangle \), where \( n = 1, 2, ..., N - 1 \) and \( j = -s, -s + 1, ..., s \). We seek eigenstates of \( \mathcal{H} \) which are linear combinations of these vectors. It is very convenient to consider the linear combination

\[
|\Omega(n)\rangle = \sum_{j=-s}^{s} (-1)^{s+j} q^{s-j} |n(-j, j)\rangle,
\]

which is a highest weight state, \( S^+ |\Omega(n)\rangle = 0 \), and eigenstate of \( U_n \)

\[
U_n |\Omega(n)\rangle = (Q + Q^{-1}) |\Omega(n)\rangle, \quad U_{n\pm1} |\Omega(n)\rangle = \epsilon_s |\Omega(n \pm1)\rangle,
U_n |\Omega(n \pm1)\rangle = \epsilon_s |\Omega(n)\rangle, \quad U_n |\Omega(m)\rangle = 0 \quad \text{for} \quad n \neq \{m \pm1, m\}
\]

where \( \epsilon_s = -1 \) for \( s \) semi-integer and \( \epsilon_s = 1 \) for \( s \) integer. In this basis, all spin-\( s \) Hamiltonians \( \mathcal{H} \) can be treated in a similar way and it affords a considerable simplification in the diagonalization of \( \mathcal{H} \), when one compares with the computations in the usual spin basis\[19\].

### 3.1 One-pseudoparticle eigenstates

Let us consider one free pseudoparticle as a highest weight state which lies in the sector \( r = 2s \)

\[
|\Omega(n)\rangle = \sum_{n=1}^{N-1} A(n) |\Omega(n)\rangle.
\]

Using the eigenvalue equation \( \mathcal{H}|\Omega(n)\rangle = E_{2s} |\Omega(n)\rangle \), one can derive a complete set of equations for the wavefunctions \( A(n) \).

The action of the operator \( G = (Q - U_1) \cdots (Q - U_{N-1}) \) on the states \( |\Omega(n)\rangle \) can be computed using \( (3.2) \). It is simple on the bulk and at the left boundary

\[
G |\Omega(n)\rangle = -\epsilon_s Q^{N-2} |\Omega(n+1)\rangle, \quad 1 \leq n \leq N - 2
\]

but manifests its nonlocality at the right boundary.
\begin{align}
G |\Omega(N-1)\rangle &= \epsilon_s Q^{N-2} \sum_{n=1}^{N-1} (-\epsilon_s Q)^{-n} |\Omega(N-n)\rangle 
\end{align}

Similarly, acting with the operator $G^{-1} = (Q^{-1} - U_{N-1}) \cdots (Q^{-1} - U_1)$, we get

\begin{align}
G^{-1} |\Omega(n)\rangle &= -\epsilon_s Q^{-N+2} |\Omega(n-1)\rangle , \quad 2 \leq n \leq N-1 \\
G^{-1} |\Omega(1)\rangle &= \epsilon_s Q^{-N+2} \sum_{n=1}^{N-1} (-\epsilon_s Q)^{n} |\Omega(n)\rangle 
\end{align}

for the bulk including the right boundary and for the left boundary, respectively.

From these results one can see that the action of $U_0 = GU_{N-1}G^{-1}$ vanishes on the bulk

\begin{align}
U_0 |\Omega(n)\rangle = 0 \quad , \quad 2 \leq n \leq N-2
\end{align}

and is nonlocal at the boundaries

\begin{align}
U_0 |\Omega(1)\rangle &= -\epsilon_s \sum_{n=1}^{N-1} (-\epsilon_s Q)^{n} |\Omega(n)\rangle , \quad U_0 |\Omega(N-1)\rangle = (-\epsilon_s Q)^{-N} U_0 |\Omega(1)\rangle
\end{align}

Next, the action of the operator $U = \sum_{k=1}^{N-1} U_k$ on the states $|\Omega(n)\rangle$ gives the following equations

\begin{align}
U |\Omega(1)\rangle &= (Q + Q^{-1}) |\Omega(1)\rangle + \epsilon_s |\Omega(2)\rangle \\
U |\Omega(n)\rangle &= (Q + Q^{-1}) |\Omega(n)\rangle + \epsilon_s |\Omega(n-1)\rangle + \epsilon_s |\Omega(n+1)\rangle \\
&\quad \text{for } 2 \leq n \leq N-2 \\
U |\Omega(N-1)\rangle &= (Q + Q^{-1}) |\Omega(N-1)\rangle + \epsilon_s |\Omega(N-2)\rangle .
\end{align}

Before we substitute these results into the eigenvalue equation, we will define two new states

\begin{align}
\epsilon_s |\Omega(0)\rangle = U_0 |\Omega(1)\rangle , \quad \epsilon_s |\Omega(N)\rangle = U_0 |\Omega(N-1)\rangle
\end{align}

to include the cases $n = 0$ and $n = N$ into the definition of $\Psi_2$, equation (3.3). Finally, the action of $\mathcal{H} = U + U_0$ on the states $|\Omega(n)\rangle$ is

\begin{align}
\mathcal{H} |\Omega(0)\rangle &= (Q + Q^{-1}) |\Omega(0)\rangle + (-\epsilon_s Q)^N \epsilon_s |\Omega(N-1)\rangle + \epsilon_s |\Omega(1)\rangle \\
\mathcal{H} |\Omega(n)\rangle &= (Q + Q^{-1}) |\Omega(n)\rangle + \epsilon_s |\Omega(n-1)\rangle + \epsilon_s |\Omega(n+1)\rangle
\end{align}
for $1 \leq n \leq N - 2$

$$H |\Omega(N - 1)\rangle = (Q + Q^{-1}) |\Omega(N - 1)\rangle + \epsilon_s |\Omega(N - 2)\rangle$$

$$+ (-\epsilon_s Q)^N \epsilon_s |\Omega(0)\rangle$$

$$H |\Omega(N)\rangle = (Q + Q^{-1}) |\Omega(N)\rangle + \epsilon_s |\Omega(N - 1)\rangle$$

$$+ (-\epsilon_s Q)^N \epsilon_s |\Omega(1)\rangle$$

(3.12)

Substituting these results into the eigenvalue equation $H \Psi_{2s} = E_{2s} \Psi_{2s}$ and using the boundary conditions

$$(-\epsilon_s Q)^N A(x) = A(N + x)$$

(3.13)

we get a complete set of eigenvalue equations for the wavefunction $\Psi_{2s}$

$$E_{2s} A(n) = (Q + Q^{-1}) A(n) + \epsilon_s A(n - 1) + \epsilon_s A(n + 1)$$

for $1 \leq n \leq N - 1$ (3.14)

The plane wave parametrization $A(n) = \xi^n$ solves these eigenvalue equations and the boundary conditions (3.13), provided that:

$$E_{2s} = Q + Q^{-1} + \epsilon_s (\xi + \xi^{-1}) \quad \text{and} \quad \xi^N = (-\epsilon_s Q)^N$$

(3.15)

where $\xi = e^{i\theta}$ and $\theta$ being the momentum.

### 3.2 Two-pseudoparticle eigenstates

Let us now consider the sector $r = 2(2s)$, where we have two interacting pseudoparticles. We seek the corresponding eigenfunction as products of single pseudoparticles eigenfunctions, i.e.

$$\Psi_{4s} = \sum_{x_1 + 1 < x_2} A(x_1, x_2) |\Omega(x_1, x_2)\rangle$$

(3.16)

where

$$|\Omega(x_1, x_2)\rangle = \sum_{i=-s}^{s} \sum_{j=-s}^{s} (-1)^{i+j} q^{2s-i-j} |x_1(-i, i), x_2(-j, j)\rangle$$

(3.17)

To solve the eigenvalue equation $H \Psi_{4s} = E_{4s} \Psi_{4s}$, we recall (3.2) to get the action of $U$ and $U_0$ on the states $|\Omega(x_1, x_2)\rangle$. We have to consider four cases: (i) When the two pseudoparticles are separated in the bulk, the action of $U$ is

$$U |\Omega(x_1, x_2)\rangle = 2(Q + Q^{-1}) |\Omega(x_1, x_2)\rangle + \epsilon_s |\Omega(x_1 - 1, x_2)\rangle + \epsilon_s |\Omega(x_1 + 1, x_2)\rangle$$

$$+ \epsilon_s |\Omega(x_1, x_2 - 1)\rangle + \epsilon_s |\Omega(x_1, x_2 + 1)\rangle$$

(3.18)
\[ i.e., \text{for } x_1 \geq 2 \text{ and } x_1 + 3 \leq x_2 \leq N - 2; \text{ (ii) When the two pseudoparticles are separated but one of them or both are at the boundaries} \]
\[
U|\Omega(1, x_2)\rangle = 2(Q + Q^{-1})|\Omega(1, x_2)\rangle + \epsilon_s|\Omega(2, x_2)\rangle + \epsilon_s|\Omega(1, x_2 - 1)\rangle
+ \epsilon_s|\Omega(1, x_2 + 1)\rangle
\quad (3.19)
\]
\[
U|\Omega(x_1, N - 1)\rangle = 2(Q + Q^{-1})|\Omega(x_1, N - 1)\rangle + \epsilon_s|\Omega(x_1 - 1, N - 1)\rangle
+ \epsilon_s|\Omega(x_1 + 1, N - 1)\rangle + \epsilon_s|\Omega(x_1, N - 2)\rangle
\quad (3.20)
\]
\[
U|\Omega(1, N - 1)\rangle = 2(Q + Q^{-1})|\Omega(1, N - 1)\rangle + \epsilon_s|\Omega(2, N - 1)\rangle + \epsilon_s|\Omega(1, N - 2)\rangle
\quad (3.21)
\]
where \(2 \leq x_1 \leq N - 4\) and \(4 \leq x_2 \leq N - 2\); (iii) When the two pseudoparticles are neighbors in the bulk
\[
U|\Omega(x, x + 2)\rangle = 2(Q + Q^{-1})|\Omega(x, x + 2)\rangle + \epsilon_s|\Omega(x - 1, x + 2)\rangle + \epsilon_s|\Omega(x, x + 3)\rangle
+ U_{x+1}|\Omega(x, x + 2)\rangle
\quad (3.22)
\]
for \(2 \leq x \leq N - 4\) and (iv) When the two pseudoparticles are neighbors and at the boundaries
\[
U|\Omega(1, 3)\rangle = 2(Q + Q^{-1})|\Omega(1, 3)\rangle + \epsilon_s|\Omega(1, 4)\rangle + U_2|\Omega(1, 3)\rangle
\quad (3.23)
\]
\[
U|\Omega(N - 3, N - 1)\rangle = 2(Q + Q^{-1})|\Omega(N - 3, N - 1)\rangle + \epsilon_s|\Omega(N - 4, N - 1)\rangle
+ U_{N-2}|\Omega(N - 3, N - 1)\rangle
\quad (3.24)
\]
Moreover, the action of \(U_0\) does not depend on the pseudoparticles are neither separated nor neighbors. It is vanishes in the bulk
\[
U_0|\Omega(x_1, x_2)\rangle = 0 \quad \text{for } x_1 \neq 1 \text{ and } x_2 \neq N - 1,
\quad (3.25)
\]
and different of zero at the boundaries:
\[
U_0|\Omega(1, x_2)\rangle = -\epsilon_s \sum_{k=1}^{x_2-2} (-\epsilon_s Q)^k |\Omega(k, x_2)\rangle - (-\epsilon_s Q)^{x_2-1}U_{x_2}|\Omega(x_2 - 1, x_2 + 1)\rangle
- \epsilon_s \sum_{k=x_2+2}^{N-1} (-\epsilon_s Q)^{k-2} |\Omega(x_2, k)\rangle
\quad (3.26)
\]
\[
U_0|\Omega(x_1, N - 1)\rangle = (-\epsilon_s Q)^{-N+2} U_0|\Omega(1, x_2)\rangle
\quad (3.27)
\]
where \(2 \leq x_1 \leq N - 3\) and \(3 \leq x_2 \leq N - 2\).
Before we substitute these expressions into the eigenvalue equation, we define new states in order to have consistency between bulk and boundaries terms

\[
U_0 | \Omega(1, x_2) \rangle = \epsilon_s | \Omega(0, x_2) \rangle, \quad U_0 | \Omega(x_{1}, N - 1) \rangle = \epsilon_s | \Omega(x_1, N) \rangle \\
U_0 | \Omega(1, N - 1) \rangle = \epsilon_s | \Omega(0, N - 1) \rangle + \epsilon_s | \Omega(1, N) \rangle \\
U_{x+1} | \Omega(x, x + 2) \rangle = \epsilon_s | \Omega(x + 1, x + 2) \rangle + \epsilon_s | \Omega(x, x + 1) \rangle
\] (3.28)

Acting with \( \mathcal{H} \) on these new states, we get

\[
\mathcal{H} | \Omega(0, x_2) \rangle = 2(Q + Q^{-1}) | \Omega(0, x_2) \rangle + \epsilon_s | \Omega(0, x_2 - 1) \rangle + \epsilon_s | \Omega(0, x_2 + 1) \rangle \\
+ \epsilon_s | \Omega(1, x_2) \rangle + (-\epsilon_s Q)^{N-2} \epsilon_s | \Omega(x_2, N - 1) \rangle
\] (3.29)

\[
\mathcal{H} | \Omega(x_{1}, N) \rangle = 2(Q + Q^{-1}) | \Omega(x_{1}, N) \rangle + \epsilon_s | \Omega(x_{1} - 1, N) \rangle + \epsilon_s | \Omega(x_{1} + 1, N) \rangle \\
+ \epsilon_s | \Omega(x_{1}, N - 1) \rangle + (-\epsilon_s Q)^{-N+2} \epsilon_s | \Omega(1, x_{1}) \rangle
\] (3.30)

\[
\mathcal{H} | \Omega(x, x + 1) \rangle = (Q + Q^{-1}) | \Omega(x, x + 1) \rangle + \epsilon_s | \Omega(x - 1, x + 1) \rangle + \epsilon_s | \Omega(x, x + 2) \rangle
\] (3.31)

Substituting these results into the eigenvalue equation, we get the following equations for wavefunctions corresponding to the separated pseudoparticles.

\[
E_{4s} A(x_{1}, x_{2}) = 2(Q + Q^{-1}) A(x_{1}, x_{2}) + \epsilon_s A(x_{1} - 1, x_{2}) + \epsilon_s A(x_{1} + 1, x_{2}) \\
+ \epsilon_s A(x_{1}, x_{2} - 1) + \epsilon_s A(x_{1}, x_{2} + 1)
\] (3.32)

i.e., for \( x_1 \geq 1 \) and \( x_1 + 3 \leq x_2 \leq N - 1 \). The boundary conditions read now

\[
A(x_{2}, N + x_{1}) = (-\epsilon_s Q)^{N-2} A(x_{1}, x_{2}).
\] (3.33)

The parametrization for the wavefunctions

\[
A(x_{1}, x_{2}) = A_{12} \xi_1^{x_2} \xi_2^{x_1} + A_{21} \xi_1^{x_1} \xi_2^{x_2}
\] (3.34)

solves the equation (3.32) provided that

\[
E_{4s} = 2(Q + Q^{-1}) + \epsilon_s (\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1})
\] (3.35)

and the boundary conditions (3.33) provided that

\[
\xi_2^N = (-\epsilon_s Q)^{N-2} \frac{A_{21}}{A_{12}}, \quad \xi_1^N = (-\epsilon_s Q)^{N-2} \frac{A_{12}}{A_{21}} \Rightarrow \xi^N = (-\epsilon_s Q)^{2(N-2)}
\] (3.36)

where \( \xi = \xi_1 \xi_2 = e^{i(\theta_1 + \theta_2)} \), \( \theta_1 + \theta_2 \) being the total momenta.
Now we include the new states (3.28) into the definition of $\Psi_{4s}$ in order to extend (3.16) to

$$
\Psi_{4s} = \sum_{x_1 < x_2} A(x_1, x_2) |\Omega(x_1, x_2). \tag{3.37}
$$

Here we have used the same notation for separated and neighboring states.

Substituting (3.22) and (3.31) into the eigenvalue equation, we get

$$
E_{4s} A(x, x + 1) = (Q + Q^{-1}) A(x, x + 1) + \epsilon_s A(x - 1, x + 1) + \epsilon_s A(x, x + 2) \tag{3.38}
$$

which gives us the phase shift produced by the interchange of the two interacting pseudoparticles

$$
\frac{A_{21}}{A_{12}} = \frac{1 + \xi + \epsilon_s (Q + Q^{-1}) \xi_2}{1 + \xi + \epsilon_s (Q + Q^{-1}) \xi_1}. \tag{3.39}
$$

We thus arrive to the Bethe ansatz equations which fix the values of $\xi_1$ and $\xi_2$:

$$
\begin{align*}
\xi_2^N &= (-\epsilon_s Q)^{N-2} \left\{ \frac{1 + \xi + \epsilon_s (Q + Q^{-1}) \xi_2}{1 + \xi + \epsilon_s (Q + Q^{-1}) \xi_1} \right\}, \\
\xi_1^N \xi_2^N &= (-\epsilon_s Q)^{2(N-2)} \tag{3.40}
\end{align*}
$$

### 3.3 General eigenstates

The generalization to any $r$ multiple of $2s$ is in principle straightforward. Since the Yang-Baxter equations are satisfied, there is only two-pseudoparticle scattering (if we use the S-matrix language). Therefore neighbor equations, where more the two pseudoparticles become neighbors, are not expected to give any new restrictions. For instance, in the sector $r = 3(2s)$ we have three interacting pseudoparticles with parameters $\xi_1$, $\xi_2$ and $\xi_3$. The corresponding wavefunctions

$$
A(x_1, x_2, x_3) = A_{123} \xi_1^{\epsilon_3} \xi_2^{\epsilon_2} \xi_3^{\epsilon_3} + A_{132} \xi_1^{\epsilon_3} \xi_2^{\epsilon_3} \xi_3^{\epsilon_2} + A_{213} \xi_1^{\epsilon_2} \xi_2^{\epsilon_3} \xi_3^{\epsilon_3} + A_{231} \xi_1^{\epsilon_3} \xi_2^{\epsilon_3} \xi_3^{\epsilon_2} \\
+ A_{312} \xi_1^{\epsilon_3} \xi_2^{\epsilon_2} \xi_3^{\epsilon_3} + A_{321} \xi_1^{\epsilon_3} \xi_2^{\epsilon_3} \xi_3^{\epsilon_2} \tag{3.41}
$$

satisfy the boundary conditions

$$
A(x_2, x_3, N + x_1) = (-\epsilon_s Q)^{N-4} A(x_1, x_2, x_3)
$$

which imply that

$$
\xi_i^N = (-\epsilon_s Q)^{N-4} \frac{A_{ijk}}{A_{jki}} = (-\epsilon_s Q)^{N-4} \frac{A_{ijk}}{A_{kji}}, \quad i \neq j \neq k = 1, 2, 3 \tag{3.42}
$$
These relations show us that the interchange of two pseudoparticles is independent of the position of the third particle. Thus in the sector \( r = p(2s) \), we expect that the \( p \)-pseudoparticle phase shift will be a sum of \( \binom{p}{2} \) two-pseudoparticle phase shifts and the energy is given by

\[
E_{p(2s)} = \sum_{n=1}^{p} \left\{ Q + Q^{-1} + \epsilon_s(\xi_n + \xi_n^{-1}) \right\}
\]  

where

\[
\xi_a^N = (-\epsilon_s Q)^{N-2p+2} \prod_{b \neq a}^{p} \left\{ -\frac{1 + \xi_a \xi_b + \epsilon_s(Q + Q^{-1}) \xi_a}{1 + \xi_a \xi_b + \epsilon_s(Q + Q^{-1}) \xi_b} \right\}, \quad a = 1, \ldots, p
\]

\[
(\xi_1 \xi_2 \cdots \xi_p)^N = (-\epsilon_s Q)^{p(N-2p+2)}
\]

It is not all, in a sector \( r \) we may have \( p \) pseudoparticle and \( N_{s-1}, N_{s-2}, \ldots, N_{-s+1} \) impurities of the type \((s-1), (s-2), \ldots, (-s+1)\), respectively, such that

\[
N_{s-1} + 2N_{s-2} + \cdots + (2s-1)N_{-s+1} = r - 2sp
\]

We call impurity a state \(|a, n\rangle\) flanked by at least two states \(|b, n \pm 1\rangle\) such that \(a+b \neq 0\). Since \( H \) is a sum of projectors on spin zero, these states are annihilated by \( H \). In particular, the do not move under the action of \( H \), which is the reason for their name. Nevertheless, a pseudoparticle can propagate past the isolated impurity, but in so doing causes a shift in its position by two lattice sites. Thus, for a sector \( r \) with \( l \) impurities with parameters \( \xi_1, \ldots, \xi_l \) and \( p \) pseudoparticles with parameters \( \xi_{l+1}, \ldots, \xi_{l+p} \) the energy is given by \( (3.43) \), and the Bethe equations do not depend on impurity type and are given by

\[
\xi_a^N \xi_1^2 \xi_2^2 \cdots \xi_l^2 = (-\epsilon_s Q)^{N-2p+2} \prod_{b=l+1}^{l+p} \left\{ -\frac{1 + \xi_a \xi_b + \epsilon_s(Q + Q^{-1}) \xi_a}{1 + \xi_a \xi_b + \epsilon_s(Q + Q^{-1}) \xi_b} \right\}
\]  

with \( a = l + 1, l + 2, \ldots, l + p \), \( p \geq 1 \), and

\[
\xi^{2p}(\xi_{l+1} \cdots \xi_{l+p})^{N-2p} = (-\epsilon_s Q)^{p(N-2p+2)}
\]

where \( \xi = \xi_1 \xi_2 \cdots \xi_l \xi_{l+1} \cdots \xi_{l+p} \).

4 Conclusion

We have shown that these closed Temperley-Lieb quantum invariant spin chains can be solved by the coordinate Bethe ansatz. A consequence of the nonlocal terms \( U_0 \) is
the arising of boundary conditions depending on the quantum group parameter $q$ via the relation $Q + Q^{-1} = [2s + 1]_q$ and on the number $p$ of pseudoparticles (which is equal to spin sector $r$, when $s = 1/2$).

An interesting extension of this work would be the application of the methods here presented to solve new strongly correlated electronic systems associated with the Temperley-Lieb algebras[20, 21]. This is presently under investigation.

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