GENUS THETA SERIES, HECKE OPERATORS AND THE BASIS PROBLEM FOR EISENSTEIN SERIES

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Abstract. We derive explicit formulas for the action of the Hecke operator $T(p)$ on the genus theta series of a positive definite integral quadratic form and prove a theorem on the generation of spaces of Eisenstein series by genus theta series. We also discuss connections of our results with Kudla’s matching principle for theta integrals.

1. Introduction

In the theory of theta series of positive definite quadratic forms the problem of giving explicit formulas for the action of Hecke operators on theta series has received some attention [1, 19].

If $p$ is prime to the level $N$ of the quadratic form $q$ of rank $m$ in question, the action of the usual generators $T(p), T_i(p^2)$ of the $p$-part of the Hecke algebra for the group $\Gamma_0^{(n)}(N) \subseteq Sp_n(\mathbb{Z})$ is known [1, 19] except for the case that $n < \frac{m}{2}$ and $\chi(p) = -1$, where $\chi$ is the nebentype character of the degree $n$ theta series of $q$. In this last case it is unknown whether $T(p)$ leaves the space of cusp forms generated by the theta series of positive definite quadratic forms of the same level and rational square class of the discriminant invariant. Some deep results concerning this question have been obtained by Waldspurger [18].

To our surprise, there seem to be no results available even for the question how to describe the action of $T(p)$ on the genus theta series of $q$, i.e., Siegel’s weighted average over the theta series of the quadratic forms $q'$ in the genus of $q$.

The present note intends to fill this gap. It turns out that we have different methods available to express the image of the genus theta series under the operator $T(p)$ in terms of theta series: Using results of Freitag [5], Salvati Manni [14] and Chiera [4] one obtains an expression as a linear combination of theta series of positive definite quadratic forms of level $\text{lcm}(N, 4)$.

We show in Section 5 that this result can be improved to an (explicit) expression as a linear combination of genus theta series of positive definite quadratic forms of level $N$ if $N$ is an odd prime. In fact we prove in that case that any $n + 1$ of the genera of quadratic forms that are rationally equivalent to the given genus and have level dividing $N$ yield a basis of the relevant space of holomorphic Eisenstein series.

This can be generalized to arbitrary square free level under a slightly technical condition on the degree $n$ depending on the $\mathbb{Q}_p$-equivalence class for $p$ dividing $N$
of the given genus of quadratic forms; generalizations to arbitrary level will be the subject of future work.

On the other hand, using the explicit expression for the action of Hecke operators on Fourier coefficients of modular forms given in [1], Siegel’s mass formula and relations between the local densities of quadratic forms we find a much simpler expression: The genus theta series is transformed into a multiple of the genus theta series of a different genus of quadratic forms. If \( \chi(p) = -1 \), the genus involved turns out to be indefinite, and the theta series is the one defined by Siegel \(( n = 1 \) and Maaß [17, 13]. This phenomenon is an instance (with quite explicit data) of the matching principle for Siegel-Weil integrals attached to different quadratic spaces and Maaß [17, 13]. This phenomenon is an instance (with quite explicit data) of the matching principle for Siegel-Weil integrals attached to different quadratic spaces and Maaß [17, 13].

As a consequence of our work we are able to give a positive solution to the basis problem for modular forms in a number of new cases; this will be done in joint work with S. Böcherer.

2. Preliminaries

Let \( L \) be a lattice of full rank on the \( m \)-dimensional vector space \( V \) over \( \mathbb{Q} \), \( q : V \to \mathbb{Q} \) a positive definite quadratic form with \( q(L) \subseteq \mathbb{Z} \), \( B(x, y) = q(x + y) - q(x) - q(y) \) the associated symmetric bilinear form, \( N = N(L) \) the level of \( q \) (i.e., \( N^{-1} \mathbb{Z} = q(L^\#) \mathbb{Z} \), where \( L^\# \) is the dual lattice of \( L \) with respect to \( B \)); we assume \( m = 2k \) to be even.

Let \( R \) be \( \mathbb{Z} \) or \( \mathbb{Z}_p \) for some prime \( p \) and let \( \mathcal{H}_n(R) \) denote the set of half-integral matrices of degree \( n \) over \( R \), that is, \( \mathcal{H}_n(R) \) is the set of symmetric matrices \((a_{ij})\) of degree \( n \) with entries in \( \frac{1}{2}R \) such that \( a_{ii} \) (\( i = 1, \ldots, n \)) and \( 2a_{ij} \) (\( 1 \leq i \neq j \leq n \)) belong to \( R \).

We note that for \( x = (x_1, \ldots, x_n) \in L^n \) the matrix \( q(x) := (\frac{1}{2}B(x_i, x_j)) \) is in the set \( \mathcal{H}_n(\mathbb{Z}) \); we also note that \( \mathcal{H}_n(\mathbb{Z}_p) \) is equal to the set \( M_n^{\text{sym}}(\mathbb{Z}_p) \) of symmetric \( n \times n \) matrices over \( \mathbb{Z}_p \) for \( p \neq 2 \).

For two square matrices \( T_1 \) and \( T_2 \) we write \( T_1 \perp T_2 = (T_1 \ 0 \ T_2) \).

We often write \( a \perp T \) instead of \( (a) \perp T \) if \( (a) \) is a matrix of degree \( 1 \). If \( K = (K, q') \) is a quadratic \( \mathbb{Z}_p \)-lattice with Gram matrix \( T \) with respect to some basis we will freely switch notation between \( T \) and \( K \), so for example if \( K \) is a one-dimensional lattice with basis vector of squared length \( a \) and \( M \) a quadratic lattice with Gram matrix \( T \) we write as above \( a \perp T = (a) \perp T = K \perp T = K \perp M \).

The theta series

\[
\vartheta^{(n)}(L, Z) = \sum_{x=(x_1, \ldots, x_n) \in L^n} \exp(2\pi i \text{tr}(q(x)Z))
\]

of degree \( n \) of \((L, q)\) is well-known to be in the space \( M_k^{(n)}(\Gamma_0^{(n)}(N), \chi) \) of Siegel modular forms of weight \( k = \frac{m}{2} \) and character \( \chi \), where \( \chi \) is the character of \( \Gamma_0^{(n)}(N) \) given by \( \chi \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \tilde{\chi}({\text{det} \, D}) \), \( \tilde{\chi} \) is the Dirichlet character modulo \( N \) given by \( \tilde{\chi}(d) = \left( \frac{-1}{d} \right)^k \text{det} \, L \) for \( d > 0 \) and \( \text{det} \, L \) is the determinant of the Gram matrix of \( L \) with respect to some basis [1].
For definitions and notations concerning modular forms we refer again to [1], we recall that the Hecke operator associated to the double coset
\[
\Gamma_0(N) \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
p & \cdots & p \\
\vdots & \ddots & \vdots \\
p & \cdots & p
\end{pmatrix} \Gamma_0(N)
\]
is as usual denoted by \( T(p) \).

We let \( \{L_1, \ldots, L_h\} \) be a set of representatives of the classes of lattices in the genus of \( L \), put \( w = \sum_{i=1}^{h} \frac{1}{|O(L_i)|} \) (where \( O(L_i) \) is the group of isometries of \( L \) onto itself with respect to \( q \)) and write
\[
\vartheta^{(n)}(\text{gen } L, Z) = \frac{1}{w} \sum_{i=1}^{h} \frac{\vartheta^{(n)}(L_i, Z)}{|O(L_i)|}
\]
for Siegel’s weighted average over the genus.

By Siegel’s theorem (see [10]) the Fourier coefficient \( r(\text{gen } L, A) \) at a positive semi-definite half integral symmetric matrix \( A \) can be expressed as a product of local densities,
\[
r(\text{gen } L, A) = c \cdot (\det A)^{\frac{m-n-1}{2}} (\det L)^{\frac{m}{2}} \prod_{\ell \text{ prime}} \alpha_{\ell}(L, A)
\]
with some constant \( c \).

Here the local density \( \alpha_{\ell}(L, A) \) is given as
\[
\alpha_{\ell}(L, A) = \alpha_{\ell}(S, A) = \ell^j \left( \frac{\chi_{\ell}}{\mathcal{O}_n} \right) \cdot |\# \{ x \in L^n/\ell^j L^n \mid q(x) \equiv A \mod \ell^j \mathcal{O}_n(\mathbb{Z}) \} |
\]
for sufficiently large \( j \) with an additional factor \( \frac{1}{2} \) if \( m = n \) where \( S \) denotes a Gram matrix of \( L \) and where we write
\[
A_j(L, A) = \begin{cases}
\{ x \in L^n/\ell^j L^n \mid q(x) \equiv A \mod \ell^j \mathcal{O}_n(\mathbb{Z}) \} \\
\{ X = (x_{ij}) \in M_{m,n}(\mathbb{Z}_{\ell})/\ell^j M_{m,n}(\mathbb{Z}_{\ell}) \mid A[X] - B \in \ell^j \mathcal{O}_n(\mathbb{Z}_{\ell}) \}
\end{cases}
\]

3. Eisenstein series and theta series

**Proposition 3.1.** Let \( L \) be a lattice of rank \( m = 2k \) with positive definite quadratic form \( q \) of square free level \( N \), let \( n < k - 1 \) and let \( F = \vartheta^{(n)}(\text{gen } L) \) denote the genus theta series of \( L \) of degree \( n \). Then for any prime \( p \nmid N \) the modular form \( F|_k T(p) \) is a linear combination of genus theta series of genera of lattices with positive definite quadratic form of level \( N' = \text{lcm}(N, 4) \).

**Proof.** By [2] \( F|_k T(p) \) is an eigenfunction of infinitely many Hecke operators \( T(\ell) \) for the primes \( \ell \nmid pN \) with \( \chi(\ell) = 1 \) (where \( \chi \) is the nebentyp character for \( \vartheta^{(n)}(L) \)). Proposition 4.3 of [3] implies then that \( G \) is in the space that is generated
by Eisenstein series for the principal congruence subgroup of level $N$; this can also be obtained from Siegel’s main theorem if one uses that this space is Hecke invariant.

We want now to use Theorem 6.9 of [5] (see also [14]) to prove that $G$ is a linear combination of theta series with characteristic for the principal congruence subgroup of level $N' = \text{lcm}(N, 4)$. For this recall that with

$$
\Gamma_1^{(n)}(N') := \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \mod N', \det(A) \equiv 1 \mod N' \}
$$

$$
\Gamma_{sq}^{(n)}(N') := \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \mod N', \det(A) \text{ is congruent to a square } \mod N' \}
$$

$$
\Delta_{sq}^{(n)}(N') := \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}) \mid C \equiv B \equiv 0 \mod N', \det(A) \text{ is congruent to a square } \mod N' \}
$$

we have the cusp function $\Phi_G : \text{Sp}_n(\mathbb{Z})/\Gamma_1^{(n)}(N') \rightarrow \mathbb{C}$ associating to the class $\gamma \Gamma_1^{(n)}(N')$ the value of $G$ in the cusp $\gamma \infty$. The quoted theorem then states that $G$ is a linear combination of theta series as above if and only if $\Phi_G$ is constant on all cosets $\gamma \Gamma_{sq}^{(n)}(N')$. Since $N$ is square free it is known that we can choose a set of representatives $w_i \infty$ of the classes of $\Gamma_{0}^{(n)}(N)$-equivalent cusps where the $w_i$ are certain involutions normalizing the group $\Delta_{sq}^{(n)}(N)$, see e.g. Lemma 8.1]. Since we have $\Gamma_{sq}^{(n)}(N) = \Delta_{sq}^{(n)}(N) \Gamma_1^{(n)}(N)$ and since $G$ transforms under $\Gamma_0^{(n)}(N)$ according to the quadratic character $\chi$ one sees that $\Phi_G$ is constant on the cosets $\gamma \Gamma_{sq}^{(n)}(N)$ and hence also on the cosets $\gamma \Gamma_{sq}^{(n)}(N')$.

Our modular form $G$ is therefore indeed a linear combination of theta series with characteristic for the principal congruence subgroup of level $N' = \text{lcm}(N, 4)$. Since $G$ is in fact a modular form for $\Gamma_0(N')$, Chiera’s Theorem 1 [14] implies that $G$ is a linear combination of theta series $\vartheta^{(n)}(K_j)$ attached to full lattices $K_j$ with quadratic form of level dividing $N'$.

It is well known that the values of the theta series of lattices in the same genus at zero dimensional cusps are the same. From Proposition 3.3 of [5] we can then conclude that $G$ is in fact a linear combination of the $\vartheta^{(n)}(\text{gen}(K_j))$ as asserted.  

4. Action of $T(p)$ and local densities

The action of the Hecke operator $T(p)$ on the Fourier coefficients of a Siegel modular form at nondegenerate matrices $A$ has been described explicitly by Maaß [12] and by Andrianov (Ex. 4. 2. 10 of [11]):

Let $K$ be a $\mathbb{Z}$-lattice with quadratic form of rank $n$ that has Gram matrix $p \cdot A$ with respect to some basis and write $\mathcal{M}_i$ for the set of lattices $M \supset K$ for which $K$ has elementary divisors $(1, \ldots, 1, p, \ldots, p)$ with $(n - i)$ entries $p$.

Then if $F(Z) \in \mathbb{M}_k(\Gamma_0^{(n)}(N), \chi)$,

$$
F(Z) = \sum_{A \geq 0} f(A) \exp(2\pi i \text{tr}(AZ)),
$$

$$
G(Z) = (F | k T(p))(Z) = \sum g_p(A) \exp(2\pi i \text{tr}(AZ)),
$$
one has for non-degenerate $A$:

\[ g_p(A) = \chi(p)^n p^{m-n-1} \sum_{i=0}^{n}(\chi(p)p^{-k})^i p^{i+1} \sum_{M \in M_i} f(M), \]

where by $f(M)$ we denote the Fourier coefficient at an arbitrary Gram matrix of the lattice $M$ (the coefficient $f(A)$ depends only on the integral equivalence class of $A$). Here by convention $f(M)$ is zero if the Gram matrix of $M$ is not half integral.

**Proposition 4.1.** Let

\[ F(Z) := \tilde{g}(\alpha)(\text{gen } L, Z) = \sum_{A \geq 0} f(A) \exp(2\pi i \text{ tr}(AZ)), \]

\[ G(Z) := (F|kT(p))(Z) = \sum_{A \geq 0} g_p(A) \exp(2\pi i \text{ tr}(AZ)). \]

Then $g_p(A) = \lambda_p(L)(c \cdot (\text{det } L)^2 (\text{det } A)^{m-n-1}) \prod_{\ell \text{ prime}} \alpha_p(\tilde{L}_\ell, A)$, where

\[ \lambda_p(L) = p^{n-k-\frac{n(n+1)}{2}} \prod_{j=1}^{n} (1 + \chi(p)p^{-k}) \]

and the $\mathbb{Z}_p$-lattice $\tilde{L}_\ell$ is given by

\[ \tilde{L}_\ell = \begin{cases} L_\ell & \text{if } p = \ell \\ pL_\ell & \text{otherwise.} \end{cases} \]

Here $pL_\ell$ denotes the lattice $L_\ell$ with quadratic form scaled by $p$.

**Proof.** It is (by induction) enough to consider nondegenerate $A$. We write the total factor in front of $f(M)$ for $M \in M_i$ in (4.1) as $\gamma_i$ and rewrite (4.1) in the present situation as

\[ g_p(A) = c \cdot (\text{det } L)^2 \sum_{i=0}^{n} \gamma_i \sum_{M \in M_i} (\text{det } M)^{m-n-1} \prod_{\ell} \alpha_p(L_\ell, M) \]

by inserting the expression for $f(M)$ from (2.1) (Siegel’s theorem).

Since $\text{det } M = p^{2i-n} \text{det } A$ for $M \in M_i$, this becomes

\[ g_p(A) = c \cdot (\text{det } L)^2 \left( \prod_{\ell} \alpha_p(L_\ell, M) \right) \]

\[ \sum_{i=0}^{n} \gamma_i \sum_{M \in M_i} (\text{det } M)^{m-n-1} \prod_{\ell} \alpha_p(L_\ell, M) \]

\[ p^{n-k-\frac{n(n+1)}{2}} = \lambda_p(L) \alpha_p(L, A). \]

Now for $\ell \neq p$ we have $M_\ell = K_\ell$ for all $M$ occurring, hence $\alpha_p(L_\ell, M_i) = \alpha_p(L_\ell, pA) = \alpha_p(pL_\ell, A) = \alpha_p(L_\ell, A)$, for all $\ell \neq p$.

So it remains to prove

\[ p^{n-k-\frac{n(n+1)}{2}} \sum_{i=0}^{n} \gamma_i \sum_{M \in M_i} (\text{det } M)^{m-n-1} \prod_{\ell} \alpha_p(L_\ell, M) = \lambda_p(L) \alpha_p(L, A). \]
We insert \( \gamma_i = \chi(p)^n p^{n-k} \frac{(n+1)}{2} \) to get
\[
p \frac{n(n+1)}{2} \sum_{i=1}^{n} (\chi(p)p^{-k})^{n-i} p^{-i(n+1)} p^{\frac{i(i+1)}{2}} \sum_{M \in \mathcal{M}_i} \alpha_p(L, M) \]
(4.5)
\[
= \prod_{j=1}^{n} (1 + \chi(p)p^{j-k}) \alpha_p(L, A)
\]
as the assertion that we have to prove.

For \( \chi(p) = 1 \) this is proved in [19] (see also [2]), where it is also proved for \( \chi(p) = -1 \) and \( n \geq k \) (in which case the factor \( \lambda_p(L) \) is zero). To prove it for \( \chi(p) = -1 \) notice that \( L_p \) is unimodular even by assumption. By Lemma 3.5 of [15] there exists a polynomial \( G_p(M; X) \) such that \( \alpha_p(\hat{L}_p, M) = G_p(M; \chi_{L_p}(p)p^{-k}) \) is true for all (even) unimodular \( \mathbb{Z}_p \)-lattices \( \hat{L}_p \) of even rank \( 2k \) with \( k \in \mathbb{N} \) and with
\[
\chi_{\hat{L}_p}(p) := \begin{cases} 
1 & \text{if } (-1)^k \det \hat{L}_p \text{ is a square in } \mathbb{Q}_p \\
-1 & \text{otherwise}
\end{cases}
\]
Hence both sides of our assertion (4.5) are polynomials in \( X = \chi(p)p^{-k} \) as \( \hat{L}_p \) varies over (even) unimodular \( \mathbb{Z}_p \)-lattices of (varying) rank \( 2k \). The truth of the assertion for \( \hat{L}_p \) with \( \chi_{\hat{L}_p}(p) = 1 \) and \( k \) arbitrary shows that these polynomials take the same value at infinitely many places, hence must be identical. The assertion is therefore true for all even unimodular \( L_p \) of even rank.

\[\square\]

**Lemma 4.2.** There is a unique isometry class of rational quadratic spaces \( \hat{V} = (\hat{V}, \hat{q}) \) of dimension \( m \), such that
\[
\hat{V}_\ell \cong V'_\ell := \begin{cases} 
\frac{nV_{\ell}}{V_p} & \text{if } p \neq \ell \\
V_p & \text{if } p = \ell
\end{cases}
\]
for finite primes \( \ell \) and \( \hat{V}_\infty = \hat{V} \otimes_{\mathbb{Q}} \mathbb{R} \) is either positive definite or of signature \((m−2, 2)\).

\( \hat{V} \) carries a lattice \( \hat{L} \) such that
\[
\hat{L}_\ell \cong \begin{cases} 
\frac{pL_{\ell}}{L_p} & \text{if } p \neq \ell \\
L_p & \text{if } p = \ell.
\end{cases}
\]
\( \hat{V}_\infty \) is indefinite if and only if \( \chi(p) = -1 \). The same assertion is true if one requires \( \hat{V}_\infty \) to be of signature \((m−2−4j, 2+4j)\) instead of \((m−2, 2)\) for some \( 1 \leq j \leq \frac{m−2}{4} \).

**Proof.** If \( s_\ell V_\ell \) denotes the Hasse symbol of the quadratic space \( V_\ell \) and \( V'_\ell \) is the quadratic \( \mathbb{Q}_\ell \)-space as in (4.9), the discriminant of \( V'_\ell \) is that of \( V_\ell \) and the product of the Hasse symbols \( s_\ell V'_\ell \) over the finite primes \( \ell \) is the Hilbert symbol
\[
(p, (-1)^{\frac{m−2}{2}} \det L)_p \cdot \prod_{\ell \text{ prime}} s_\ell V_\ell
\]
by Hilbert’s reciprocity law, with \((p, (-1)^{\frac{m−2}{2}} \det L)_p = \chi(p)\).

If \( V'_\infty \) is positive definite for \( \chi(p) = 1 \) and of signature \((m−2, 2)\) if \( \chi(p) = -1 \) one sees therefore that \( \text{disc } V'_\ell = \text{disc } V_\ell \) for all \( \ell \) (including \( \infty \)) and \( \prod_{\ell, \infty} s_\ell V'_\ell = 1 \), hence there is a rational quadratic space \( \hat{V} \) such that \( \hat{V}_\ell \cong V'_\ell \) for all \( \ell \) including \( \infty \).
The uniqueness of $\tilde{V}$ is clear from the Hasse-Minkowski theorem, and that $\tilde{L}$ as in (1.7) exists on $\tilde{V}$ is obvious.

We recall that for an integral lattice of positive determinant and even rank Siegel [17] for degree one and Maaß [13] for arbitrary degree defined a holomorphic theta series in the indefinite case whose Fourier coefficients at positive definite $A$ are proportional to the product of the local densities of that lattice, subject to the restriction that the signature $(m_+, m_-)$ satisfies the condition $\min((m_+ + m_- - 3), m_+, m_-) \geq n$. Denote this theta series (if it is defined) for $\tilde{L}$, normalized such that its Fourier coefficient at $A$ is equal to

$$r(\text{gen } \tilde{L}, A) := c \cdot (\det A)^{\frac{m-n-1}{2}}(\det \tilde{L})^{\frac{n}{2}} \prod_{\ell \text{ prime}} \alpha_{\ell}(\tilde{L}, A),$$

by $\vartheta(\tilde{L}, Z)$ or also by $\vartheta(\text{gen } \tilde{L}, Z)$ (notice that this theta series does indeed depend only on the genus of the lattice). The signature condition is in our situation always satisfied if $n = 1$, for bigger $n$ it can be satisfied by choosing $j$ in 4.2 appropriately if $n \leq k - 2$ (with $k = m/2$). If the signature condition is not satisfied, we use the same notation $r(\text{gen } \tilde{L}, A)$ (without knowing a priori whether these numbers are the Fourier coefficients of a modular form).

Then we arrive at the following final result:

**Theorem 4.3.** Let $L$ be as above, $p$ a prime with $p \nmid \det L$, $\tilde{L}$ a quadratic lattice with

$$\tilde{L}_\ell = \begin{cases} pL_\ell & \text{if } p \neq \ell \\ L_p & \text{if } p = \ell. \end{cases}$$

and of signature $(m, 0)$ if $\chi(p) = 1$, of signature $(m - 2, 2)$ if $\chi(p) = -1$. Then

$$\vartheta^{(n)}(\text{gen } L) \mid T(p) = \lambda_p(L)\vartheta^{(n)}(\text{gen } \tilde{L})$$

with

$$\lambda_p(L) = p^{nk - n^2(n+1)} \prod_{j=1}^{n} (1 + \chi(p)p^{j-k}),$$

where $\vartheta^{(n)}(\text{gen } L, Z)$ is a holomorphic modular form of the same level as $L$ whose Fourier coefficient at a positive definite matrix $A$ is equal to $r(\text{gen } \tilde{L}, A)$. The modular form $\vartheta^{(n)}(\text{gen } L, Z)$ is the usual genus theta series if $\tilde{L}$ is positive definite and is equal to the theta series of Siegel and Maaß from above if $\tilde{L}$ is indefinite and this series is defined.

In particular, for all $n < k$ there exists a holomorphic modular form of the same level as $L$ with Fourier coefficients $r(\text{gen } \tilde{L}, A)$ at positive definite matrices $A$.

**Remark.** a) $\lambda_p(L) = 0$ if $n \geq k$ holds with $\chi(p) = -1$, which agrees with Andrianov’s result [2] for this case.

b) In the introduction we mentioned the question whether the space of cusp forms generated by the theta series of positive definite lattices of fixed level and rational square class of the discriminant is invariant under the action of the Hecke operators. In view of our theorem we might reformulate this question by substituting “modular forms” for “cusp forms” and omitting the restriction to positive definite lattices. Since the indefinite theta series of Siegel and Maaß don’t contribute to the space of cusp forms, this doesn’t change the problem with regard to the subspace of cusp forms.
forms.

c) Of course the same result holds true when we take an indefinite lattice $\tilde{L}$ of signature $(m - 2, 2)$ as above as our starting point. The lattices appearing in $\vartheta(\text{gen } \tilde{L}, z) \mid T(p)$ are then positive definite if $\chi(p) = -1$, indefinite if $\chi(p) = +1$.

5. Spaces of genus theta series for odd prime level

We will need some additional notations in this section.

Let $p$ be an odd prime. For a non-zero element $a \in \mathbb{Q}_p$ we put $\chi_p(a) = 1, -1$, or $0$ according as $\mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \mathbb{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbb{Q}_p$. For a non-degenerate half-integral matrix $B$ of even degree $n$, put $\xi_p(B) = \chi_p((-1)^{n/2} \det B)$.

Further for non-negative integers $l, e$ and matrices $A \in \mathcal{H}_m(\mathbb{Z}_p), B \in \mathcal{H}_n(\mathbb{Z}_p)$ define

$$B_e(A, B)^{(l)} = \{X = (x_{ij}) \in A_e(A, B); \ \text{rank}_{\mathbb{Z}_p/pE}(x_{i,j})_{1 \leq i \leq m, 1 \leq j \leq l} = l\}$$

(with $A_e(A, B)$ as in Section 2) and

$$\beta_p(A, B)^{(l)} = \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# B_e(A, B)^{(l)}.$$

We note that

$$\beta_p(A, B)^{(0)} = \alpha_p(A, B).$$

In particular put

$$\beta_p(A, B) = \beta_p(A, B)^{(n)},$$

and call it (as usual) the primitive density. Further for $0 \leq i \leq m$ put

$$\pi_{m,i} = \text{GL}_m(\mathbb{Z}_p)(pE_i \perp E_{m-i}) \text{GL}_m(\mathbb{Z}_p)$$

Furthermore let $H_k = \underbrace{H \perp \ldots \perp H}_{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

Our goal in this section is to prove the following theorem:

**Theorem 5.1.** Let $p$ be an odd prime, $k, n \in \mathbb{N}$ with $n \leq k - 1$ and $p \equiv (-1)^k \mod 4$.

Then the space of modular forms for $\Gamma_0^{(n)}(p)$ spanned by the genus theta series of degree $n$ attached to the genus of positive definite integral quadratic lattices of rank $2k$, level $p$ and discriminant $p^{2r+1}$ for some $0 \leq r < k$ and the space spanned by the genus theta series of degree $n$ (in the sense of Theorem 4.3) attached to the genus of integral quadratic lattices of signature $(2k - 2 - 4j, 2 + 4j)$ (with $1 \leq j \leq \frac{2k-2}{4} $ fixed), level $p$ and discriminant $p^{2r+1}$ for some $0 \leq r < k$ coincide. This space has dimension $n + 1$ and is equal to the space of holomorphic Eisenstein series for the group $\Gamma_0^{(n)}(p)$ of weight $k$ and nontrivial quadratic character.

For each of these signatures the theta series of any $n + 1$ of the $k$ genera of level dividing $p$ and having this signature form a basis of this space of modular forms.

The proof of this theorem will require a few intermediate results which may be of independent interest. A half-integral matrix $S_0$ over $\mathbb{Z}_p$ is called $\mathbb{Z}_p$-maximal if it is the empty matrix or a matrix corresponding to a $\mathbb{Z}_p$-maximal lattice. The main result we need is the following theorem, whose proof again is broken up into several steps:
Theorem 5.2. Let $p$ be an odd prime, let $T \in \mathcal{H}_n(\mathbb{Z}_p)$. Let $k$ be a positive integer, and $S_0$ be a $\mathbb{Z}_p$-maximal half-integral matrix of degree not greater than 2. Then there exist rational numbers $a_i = a_i(k, S_0, T)$ ($i = 0, 1, 2, ..., n$) such that

$$\alpha_p(H_{k-l-1} \perp pH_1 \perp S_0, T) = a_0 + a_1p^l + ... + a_np^{nl}$$

for any $l = 0, 1, ..., k - 1$.

To prove the theorem, first we remark that for $p \neq 2$ a $\mathbb{Z}_p$-maximal matrix $S_0$ of degree not greater than 2 is equivalent over $\mathbb{Z}_p$ to one of the following matrices:

(M-1) $\phi$ (empty matrix),
(M-2) $pu_1$ with $u_1 \in \mathbb{Z}_p^*$,
(M-3) $pu_1$ with $u_1 \in \mathbb{Z}_p$, $\rho_1 = 1$,
(M-4) $u_1 \perp u_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$,
(M-5) $pu_1 \perp pu_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$, $\rho_1 = 1$,
(M-6) $pu_1 \perp pu_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$ such that $-u_1u_2 \notin (\mathbb{Z}_p^*)^2$.

Lemma 5.3. Let $S_0$ be the matrix in Theorem 5.2. For a non-negative integer $l$ put $B_l = B_{S_0,l} = pH_l \perp S_0$ and $\bar{B}_{l,i} = \tilde{B}_{S_0,l,i} = H_l \perp pH_{l-i} \perp S_0$. Let $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$.

1. Let $S_0$ be of type (M-3) or (M-5). Then for any $k \geq n$ we have

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T)$$

$$= \sum_{i=0}^l (-1)^i p^i(i-1/2+i(n-2k+1))C_{2l+1,i} \alpha_p(H_{k+l+1}, -\bar{B}_{l,i} \perp T),$$

where $C_{m,i} = \frac{\prod_{j=1}^{i} (p_{m+2j-1})}{\prod_{j=1}^{i} (p^{m-1})}$ for an odd positive integer $m$ and an integer $i$ such that $i \leq (m-1)/2$.

2. Let $S_0$ be of type (M-1), (M-2), (M-4), or (M-6). Put $\epsilon = \epsilon(S_0) = -1$ or 1 according as $S_0$ is of type (M-6) or not. Then for any $k \geq n$ we have

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T)$$

$$= \sum_{i=0}^l (-1)^i p^i(i-1/2+i(n-2k+1))C_{2l+1,i} \alpha_p(H_{k+l+1}, -\bar{B}_{l,i} \perp T),$$

where $C_{m,i,\epsilon} = \frac{(p^{m/2-\epsilon})^{m/2-1}(p_{m-2}^{-}\epsilon)}{\prod_{j=1}^{i} (p^{m-2j-1})}$ for an even positive integer $m$ and an integer $i$ such that $i \leq m/2$, and $\epsilon = \pm 1$.

3. $\alpha_p(H_{k+l}, -H \perp T) = (1 - p^{-(k+l)})(1 + p^{-(k+l-1)})\alpha_p(H_{k+l-1}, T)$

Proof. By Proposition 2.2 of [8], we have

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T)$$

$$= \sum_{i=0}^{2l+2} (-1)^i p^i(i-1/2+i(n+2l+2)+1-(2k+2l+2))$$

$$\times \sum_{G \in GL_{2l+2}(\mathbb{Q}_p) \setminus \mathbb{Q}_{2l+2}} \alpha_p(H_{k+l+1}, -B_l[G^{-1}] \perp T).$$
We note that \( \alpha_p(H_{k+l+1}, -B_l(G^{-1}) \perp T) = 0 \) if \( G \in \pi_{2l+2,i} \) with \( i \geq l + 1 \). Fix \( i = 0, 1, \ldots, l \). Then by Lemma 2.3 of [7], we have \(-B_l[G^{-1}] \perp T \sim -B_l, i \perp T \) if \( G \in \pi_{i,l} \) and \( B_l[G^{-1}] \in H_{2l+2}(\mathbb{Z}_p) \). Furthermore, by Proposition 2.8 of [7] we have

\[
\#(GL_{2l+2}(\mathbb{Z}_p)) \setminus \{G \in \pi_{2l+2,i} : B_l[G^{-1}] \in H_{2l+2}(\mathbb{Z}_p)\} = \frac{\prod_{j=1}^{i}(p^{2l+2-2j} - 1)}{\prod_{j=1}^{i}(p^j - 1)}.
\]

This proves the assertion (1). Similarly, the assertion (2) can be proved. Now again by Proposition 2.2 of [8] we have

\[
\beta_p(H_{k+l}, H) \alpha_p(H_{k+l-1}, T) = \alpha_p(H_{k+l}, -H \perp T).
\]

On the other hand, we have

\[
\beta_p(H_{k+l}, H) = (1 - p^{-(k+l)})(1 + p^{-(k+l-1)})
\]

(e.g. Lemma 9, [9].) Thus the assertion (3) holds.

Now for a non-degenerate half-integral matrix \( B \) of degree \( n \) over \( \mathbb{Z}_p \) define a polynomial \( \gamma_p(B; X) \) in \( X \) by

\[
\gamma_p(B; X) = \begin{cases} (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i}X^2)(1 - p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases}
\]

For a half-integral matrix \( B \) of degree over \( \mathbb{Z}_p \), let \((\bar{W}, \bar{q})\) denote the quadratic space over \( \mathbb{Z}_p/p\mathbb{Z}_p \) defined by the quadratic form \( \bar{q}(x) = B[x] \mod p \), and define the radical \( R(W) \) of \( \bar{W} \) by

\[
R(W) = \{x \in \bar{W} : \bar{B}(x, y) = 0 \text{ for any } y \in \bar{W}\},
\]

where \( \bar{B} \) denotes the associated symmetric bilinear form of \( \bar{q} \). We then put \( l_p(B) = \text{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p} R(W)^\perp \), where \( R(W)^\perp \) is the orthogonal complement of \( R(W) \) in \( \bar{W} \). Furthermore, in case \( l_p(B) \) is even, put \( \xi_p(B) = 1 \) or \(-1\) according as \( R(W)^\perp \) is hyperbolic or not. Here we make the convention that \( \xi_p(B) = 1 \) if \( l_p(B) = 0 \). We note that \( \xi_p(B) \) is different from \( \xi_p(B) \).

**Lemma 5.4.**

(1) Let \( B \) be a half-integral matrix of degree \( n \) over \( \mathbb{Z}_p \). Put \( l = l_p(B) \). Then we have

\[
\beta_p(H_m, B) = (1 - p^{-m})(1 + \xi_p(B)p^{n-l/2-m}) \prod_{j=0}^{n-l/2-1} (1 - p^{2j-2m})
\]

if \( l \) is even,

\[
\beta_p(H_m, B) = \prod_{j=0}^{n-(l+1)/2} (1 - p^{2j-2m})
\]

if \( l \) is odd.

(2) Let \( T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p) \). Then there exists a polynomial \( F_p(T, X) \) such that \( \alpha_p(H_m, T) = \gamma_p(T; p^{-m})F_p(T, p^{-m}) \).

**Proof.** The assertion (1) follows from Lemma 9, [9]. The assertion (2) is well known (cf. [9]).
Let \((\ , \ , \)_p\) be the Hilbert symbol over \(\mathbb{Q}_p\) and \(h_p\) the Hasse invariant (for the definition of the Hasse invariant, see [10]). Let \(B\) be a non-degenerate symmetric matrix of degree \(n\) with entries in \(\mathbb{Q}_p\). We define

\[
\begin{align*}
\eta_p(B) &= h_p(B)(\det B, (-1)^{(n-1)/2} \det B)_p & \text{if } n \text{ is odd} \\
\xi_p(B) &= \chi_p((-1)^{n/2} \det B) & \text{if } n \text{ is even}.
\end{align*}
\]

From now on we often write \(\xi(B)\) instead of \(\xi_p(B)\) and so on if there is no fear of confusion. For a non-degenerate half-integral matrix \(B\) of degree \(n\) over \(\mathbb{Z}_p\) put \(D(B) = \det B\) and \(d(B) = \text{ord}_p(D(B))\). Further, put

\[
\delta(B) = \begin{cases} 
2[(d(B) + 1)/2] & \text{if } n \text{ is even} \\
(d(B) & \text{if } n \text{ is odd}.
\end{cases}
\]

Let \(\nu(B)\) be the least integer \(l\) such that \(p^l B^{-1} \in \mathcal{H}_n(\mathbb{Z}_p)\). Further put \(\xi'(B) = 1 + \xi(B) - \xi(B)^2\) for a matrix \(B\) of even degree. Then we have

**Proposition 5.5.** Let \(B_1 = (b_1)\) and \(B_2\) be non-degenerate half-integral matrices of degree \(1\) and \(n - 1\), respectively over \(\mathbb{Z}_p\), and put \(B = B_1 \perp B_2\). Assume that \(\text{ord}_p(b_1) \geq \nu(B_2) - 1\).

1. Let \(n\) be even. Then we have

\[
F_p(-(H_i \perp pH_{i-1}) \perp B, p^{-(k+l)})
\]

\[
= \frac{1 - \xi p^{n/2-k}}{1 - p^{n-2k+1}} F_p(- (H_i \perp pH_{i-1}) \perp B_2, p^{(k+l-1)}) + K(B)p^{l-i} F_p( - (H_i \perp pH_{i-1}) \perp B_2, p^{-(k+l)}),
\]

where \(\xi = \xi(B)\), and \(K(B)\) is a rational number depending only on \(B\).

2. Let \(n\) be odd. Then we have

\[
F_p(-(H_i \perp pH_{i-1}) \perp B, p^{-(k+l)})
\]

\[
= \frac{1}{1 - \xi p^{(n+1)/2-k}} F_p(-(H_i \perp pH_{i-1}) \perp B_2, p^{-(k+l-1)}) + K(B)p^{l-i} F_p( - (H_i \perp pH_{i-1}) \perp B_2, p^{-(k+l)}),
\]

where \(\tilde{\xi} = \xi(B_2)\), and \(K(B)\) is a rational number depending only on \(B\). Here we understand that \(B_2\) is the empty matrix and that we have \(\xi = 1\) if \(n = 1\).

**Proof.** (1) Let \(n\) be even. We have \(\text{ord}_p(b_1) \geq \nu(-(H_i \perp pH_{i-1}) \perp B_2) - 1\). Thus by Theorem 4.1 of [8], we have

\[
F_p(-(H_i \perp pH_{i-1}) \perp B, p^{-(k+l)})
\]

\[
= \frac{1 - \xi(l,i)p^{(n+2)/2-(k+l)}}{1 - p^{n+2+1-2(k+l)}} F_p(-(H_i \perp pH_{i-1}) \perp B_2, p^{-(k+l-1)}) + (-1)^{\xi(l,i)+1} \xi(l,i)l \frac{1 - \xi(l,i)p^{(n+2)/2+1-(k+l)}}{1 - p^{n+2+1-2(k+l)}}
\]

\[
\times (p^{(n+2)/2-(k+l)})^2 \delta(l,i) \frac{\delta(l,i) + \xi(l,i)^2 p^{\delta(l,i)}}{2} \times F_p(-(H_i \perp pH_{i-1}) \perp B_2, p^{-(k+l)}).
\]
where $\xi(l, i) = \xi(-(H_i \perp pH_{-i}) \perp B), \xi(l, i)' = \xi(-(H_i \perp pH_{-i}) \perp B), \tilde{\eta}(l, i) = \eta(-(H_i \perp pH_{-i}) \perp B_2), \delta(l, i) = \delta(-(H_i \perp pH_{-i}) \perp B),$ and $\delta(l, i) = \delta(-(H_i \perp pH_{-i}) \perp B_2).$ We note that $\beta(l, i), \xi(l, i),$ and $\eta(l, i)$ are independent of $l$ and $i,$ and they are equal to $\xi, \xi',$ and $\eta(B_2),$ respectively. Furthermore, we have $\delta(l, i) = 2l - 2i + 2[(\text{ord}_p(\text{det} T) + 1)/2]$ and $\tilde{\delta}(l, i) = 2l - 2i + \text{ord}_p(\text{det} \hat{T}).$ Thus the assertion holds. Similarly, the assertion holds in case $n$ is odd.

\[\Box\]

**Proposition 5.6.** Let $S_0$ and the others be as in Lemma 5.3, $T = b_1 \perp b_2 \perp \ldots \perp b_n$ with $\text{ord}_p(b_1) \geq \text{ord}_p(b_2) \geq \ldots \geq \text{ord}_p(b_n).$ Put $\hat{T} = b_2 \perp \ldots b_n.$

1. Assume that $n + \deg S_0$ is even. Put $K(S_0, T) = \frac{1 - p^{n-2k}}{1 - p^{n+2k-\xi}} K(-S_0 \perp T),$ where $\xi = \xi(-S_0 \perp T),$ and $K(-S_0 \perp T)$ is the rational number in Proposition 5.6. Then we have

\[
\alpha_p(H_{k+i+1}, -\tilde{B}_{l,i} \perp T) = \frac{(1 - p^{-(k+i+1)})(1 + p^{-(k+i)})}{1 - p^{n-2k+1}} \alpha_p(H_{k+i}, -B_{l,i} \perp \hat{T}) + p^{l-i} K(S_0, T) \alpha_p(H_{k+i+1}, -\tilde{B}_{l,i} \perp \hat{T}).
\]

2. Assume that $n + \deg S_0$ is odd. Put $K(S_0, T) = (1 - p^{(n-1)/2 - k\xi}) K(-S_0 \perp T),$ where $\tilde{\xi} = \xi(-S_0 \perp \hat{T}),$ and $K(-S_0 \perp T)$ is the rational number in Proposition 5.6. Then we have

\[
\alpha_p(H_{k+i+1}, -\tilde{B}_{l,i} \perp T) = \frac{(1 - p^{-(k+i+1)})(1 + p^{-(k+i)})}{1 - p^{n-2k+1}} \alpha_p(H_{k+i}, -B_{l,i} \perp \hat{T}) + p^{l-i} K(S_0, T) \alpha_p(H_{k+i+1}, -\tilde{B}_{l,i} \perp \hat{T}).
\]

**Proof.** By (1) of Proposition 5.3 and (2) of Lemma 5.4, we have

\[
\alpha_p(H_{k+i+1}, -\tilde{B}_{l,i} \perp T) = \gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+i+1)}) = \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+i+1)}) \times \left[ 1 - \frac{\xi p^{n/2-k}}{1 - p^{n-2k+1}} F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+i)}) + p^{l-i} K(-S_0 \perp T) F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+i+1)}) \right].
\]

We note that

\[
\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+i)}) = \frac{1 - p^{n/2-k\xi}}{(1 - p^{-(k+i+1)})(1 + p^{-(k+i)})} \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+i+1)}),
\]

and

\[
\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+i+1)}) = \frac{1 - p^{n/2-k\xi}}{1 - p^{n-2k}} \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+i+1)}).
\]

Thus the assertion (1) holds.
Now by (2) of Proposition 5.5 and (2) of Lemma 5.4, we have
\[
\alpha_p(H_{k+l+1} - \tilde{B}_l, T) = \gamma_p(-\tilde{B}_l, - T, p^{-(k+l+1)})F_p(-\tilde{B}_l, T, p^{-(k+l+1)})
\]
\[
= \frac{1}{1 - \xi_p(n+1)/2 - k}F_p(-\tilde{B}_l, T, p^{-(k+l)}) + p^{l+1}K(-S_0 \perp T)F_p(-\tilde{B}_l, T, p^{-(k+l+1)}).
\]

We note that
\[
\gamma_p(-\tilde{B}_l, - T, p^{-(k+l)}) = \frac{1 - p^{n + 1 - 2k}}{(1 - p^{-(k+l+1)})(1 + p^{-(k+l)})(1 - \xi_p(n+1)/2 - k)}
\]
and
\[
\gamma_p(-\tilde{B}_l, - T, p^{-(k+l+1)}) = \frac{1}{1 - p^{(n-1)/2 - k\xi_p}}\gamma_p(-\tilde{B}_l, - T, p^{-(k+l+1)}).
\]
Thus the assertion (2) holds.

\[\square\]

**Remark 5.7.** In the above theorem, \(K(S_0, T)\) can be expressed explicitly in terms of the invariants of \(T\).

**Proposition 5.8.** Let \(S_0, T\) and \(\hat{T}\) and the others be as in Proposition 5.6,
\(\alpha_p(H_{k-l-1} \perp B_l, T) = (1 - p^{n - 2k+1})^{-1} \times (1 - p^{-(k+l+1)})^\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) + p^{n-2k+1}(p^{l+1} - 1)\alpha_p(H_{k-l-1} \perp B_l, \hat{T}) + pK(S_0, T)\alpha_p(H_{k-l-1} \perp B_l, \hat{T}),\)
where \(K(S_0, T)\) is the rational number in Proposition 5.6. In particular, if \(n = 1\), for a non-zero element \(T\) of \(\mathbb{Z}_p\), we have
\[
\alpha_p(H_{k-l-1} \perp B_l, T) = 1 + cp^{l},
\]
where \(c = c(S_0, T)\) is the rational number determined by \(T\) and \(S_0\).

(2) Assume that \(S_0\) is of type \((M-1),(M-2),(M-4)\) or \((M-6)\). Put \(l' = l + 1\) or \(l\) according as \(S_0\) is of type \((M-6)\) or not. Let \(\epsilon = \epsilon(S_0)\) be as in Lemma 5.3 and \(\xi = \xi(S_0)\). Put \(\epsilon = -1\) or \(1\) according as \(S_0\) is of type \((M-6)\) or not. Then for non-negative integer \(l \leq k - 1\) we have
\[
\alpha_p(H_{k-l-1} \perp B_l, T) = (1 - p^{n - 2k+1})^{-1} \times (1 - p^{-(k+l' + 1)} \xi)(1 + p^{-k+l' + 1} \xi)\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) + p^{n-2k+1}(p^{l'-1} - \epsilon)(p^{l'+1} + \epsilon)\alpha_p(H_{k-l-1} \perp B_{l-1}, \hat{T}) + K(S_0, T)p^l\alpha_p(H_{k-l-1} \perp B_l, \hat{T}),
\]
where $K(S_0, T)$ is the rational number in Proposition \ref{prop:5.3}. In particular, if $n = 1$, for a non-zero element $T$ of $\mathbb{Z}_p$, we have

$$\alpha_p(H_{k-l-1} \perp B_l, T) = 1 + cp^i,$$

where $c = c(S_0, T)$ is a rational number determined by $T$ and $S_0$. Throughout (1) and (2), we understand $\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) = 1$ if $l = k - 1$.

**Proof.** (1) First let $n + \deg S_0$ be even. Then by (1) of Proposition \ref{prop:6.6} and (1) of Lemma \ref{lem:5.3} we have

$$\beta_p(H_{k+l+1}, -B_l) \alpha_p(H_{k-l-1} \perp B_l, T)$$

$$= \sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k+2)} C_{2l+1, i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})$$

$$\times \left\{ \frac{(1 - p^{-(k+l+1}))(1 + p^{-(k+l)})}{1 - p^{n-2k+1}} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T}) \right\}$$

$$+ \alpha_p(H_{k+l+1}, -\hat{B}_{l,i} \perp \hat{T}) p^{l-i} K(S_0, T) \right\}$$

$$= \frac{(1 - p^{-(k+l+1)})(1 + p^{-(k+l)})}{1 - p^{n-2k+1}}$$

$$\times \sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k+2)} C_{2l+1, i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})$$

$$+ \sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k+2)} (p^{-i} - 1) C_{2l+1, i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})$$

$$+ \sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k+2)} C_{2l+1, i} \alpha_p(H_{k+l+1}, -\hat{B}_{l,i} \perp \hat{T}) p^i K(S_0, T).$$

By (1) of Lemma \ref{lem:5.3} and (1) of Lemma \ref{lem:5.4} we have

$$\frac{(1 - p^{-(k+l+1)})(1 + p^{-(k+l)})}{1 - p^{n-2k+1}}$$

$$\times \sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k+2)} C_{2l+1, i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})$$

$$= (1 - p^{-(k+l+1)})(1 + p^{-(k+l)}) \beta_p(H_{k+l}, -B_l) \alpha_p(H_{k-l-2} \perp B_l, \hat{T})$$

$$= (1 - p^{2l+2-2k}) \beta_p(H_{k+l+1}, -B_l) \alpha_p(H_{k-l-2} \perp B_l, \hat{T})$$

and

$$\sum_{i=0}^l (-1)^i p^{i(i-1)/2 + i(n-2k)} C_{2l+1, i} \alpha_p(H_{k+l+1}, -\hat{B}_{l,i} \perp \hat{T})$$

$$= \beta_p(H_{k+l+1}, -B_l) \alpha_p(H_{k-l-1} \perp B_l, \hat{T}).$$
Furthermore, again by (1) and (3) of Lemma 5.3 and (1) of Lemma 5.4, we have

\[
(1-p^{-(k+l+1)})(1+p^{-(k+l)})(1-p^{-i})\sum_{i=0}^{l}(-1)^{i}p^{-(i-1)/2+i(n-2k+2)}(p^{i} - 1)
\]

\[
\times C_{2l+1,i} \alpha_{p}(H_{k+l}, -\tilde{B}_{l,i} \perp \tilde{T})
\]

\[
= (1-p^{-(k+l+1)})(1+p^{-(k+l)})p^{n-2k+1}(p^{2l} - 1)
\]

\[
\times \sum_{i=1}^{l}(-1)^{i-1}p^{-(i-2)(i-1)/2+(i-1)(n-2k+2)}C_{2l-1,i-1}
\]

\[
\times \alpha_{p}(H_{k+l-1}, -\tilde{B}_{l-1,i-1} \perp H \perp \tilde{T})
\]

\[
= (1-p^{-(k+l+1)})(1+p^{-(k+l)})p^{n-2k+1}(p^{2l} - 1)
\]

\[
\times \sum_{i=1}^{l}(-1)^{i-1}p^{-(i-2)(i-1)/2+(i-1)(n-2k+2)}
\]

\[
\times C_{2l-1,i-1}(1-p^{-(k+l)})(1+p^{-(k+l-1)})
\]

\[
\times \alpha_{p}(H_{k+l-1}, -\tilde{B}_{l-1,i-1} \perp \tilde{T})
\]

\[
= p^{n-2k+1}(p^{2l} - 1)(1-p^{-2(k+l+1)})(1-p^{-2(k+l)})(1+p^{-(k+l-1)})
\]

\[
\times \beta_{p}(H_{k+l-1}, -B_{l-1})\alpha_{p}(H_{k-l-1} \perp B_{l-1}, \tilde{T})
\]

\[
= p^{n-2k+1}(p^{2l} - 1)\beta_{p}(H_{k+l+1}, -B_{l-1})\alpha_{p}(H_{k-l-1} \perp B_{l-1}, \tilde{T}).
\]

This proves the assertion (1) in case \( n + \deg S_{0} \) is odd. Next again by (2) of Proposition 5.6 and (1) of Lemma 5.3, the assertion (1) can be proved in case \( n + \deg S_{0} \) is odd.

(2) First let \( n + \deg S_{0} \) be even. Then by (1) of Proposition 5.6 and (2) of Lemma 5.3, we have

\[
\beta_{p}(H_{k+l+1}, -B_{l})\alpha_{p}(H_{k-l-1} \perp B_{l}, T)
\]

\[
= \sum_{i=0}^{l'}(-1)^{i}p^{-(i-1)/2+i(n-2k+1)}C_{2l',i,\epsilon}
\]

\[
\times \left\{\frac{(1-p^{-(k+l+1)})(1+p^{-(k+l'-1)})}{1-p^{n-2k+1}}\alpha_{p}(H_{k+l}, -\tilde{B}_{l,i} \perp \tilde{T})
\right.
\]

\[
+ \alpha_{p}(H_{k+l+1}, -\tilde{B}_{l,i} \perp \tilde{T})p^{l-i}K(S_{0}, T)\right\}.
\]

We evaluate this further as
Furthermore, again by (1) of Lemma 5.3, and (1) of Lemma 5.4, we have

\[
\frac{(1 - p^{-(k+l'+1)})(1 + p^{-(k+l')})}{1 - p^{n-2k+1}}
\times \sum_{i=0}^{t'} (-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l',i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})
\]

\[+ \sum_{i=0}^{t'} (-1)^i p^{i(i-1)/2+i(n-2k+2)} (p^{i-1} - 1) C_{2l',i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})] \]

\[+ \sum_{i=0}^{t'} (-1)^i p^{i(i-1)/2+i(n-2k)} C_{2l',i} \alpha_p(H_{k+l+1}, -\hat{B}_{l,i} \perp \hat{T}) p^l K(S_0, T).\]

By (1) and (3) of Lemma 5.3 and (1) of Lemma 5.4 we have

\[
(1 - p^{-(k+l'+1)})(1 + p^{-(k+l')})
\times \sum_{i=0}^{t} (-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l',i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})
\]
\[= (1 - p^{-(k+l'+1)})(1 + p^{-(k+l')}) \beta_p(H_{k+l}, -B_l) \alpha_p(H_{k-l-2} \perp B_l, \hat{T}) \]
\[= (1 - \xi p^{l'+1-k})(1 + \xi p^{l'-k}) \beta_p(H_{k+l+1}, -B_l) \alpha_p(H_{k-l-2} \perp B_l, \hat{T}) \]

and

\[
\sum_{i=0}^{l'} (-1)^i p^{i(i-1)/2+i(n-2k)} C_{2l',i} \alpha_p(H_{k+l+1}, -\hat{B}_{l,i} \perp \hat{T})
\]
\[= \beta_p(H_{k+l+1}, -B_l) \alpha_p(H_{k-l-1} \perp B_l, \hat{T}).\]

Furthermore, again by (1) of Lemma 5.3 and (1) of Lemma 5.4 we have

\[
(1-p^{-(k+l'+1)})(1 + p^{-(k+l')}) \sum_{i=0}^{t'} (-1)^i p^{i(i-1)/2+i(n-2k+2)} (p^{i-1} - 1)
\times C_{2l',i} \alpha_p(H_{k+l}, -\hat{B}_{l,i} \perp \hat{T})
\]
\[= (1 - p^{-(k+l'+1)})(1 + p^{-(k+l')}) p^{n-2k+1} (p^{l'} - \epsilon)(p^{l'-1} + \epsilon)
\times \sum_{i=1}^{t'} (-1)^i (-1)^{2i-2+2i-2}(i-1)(n-2k+2)
\times C_{2l'-2,i-1} \alpha_p(H_{k+l}, -\hat{B}_{l-1,i-1} \perp -H \perp \hat{T}),\]
which can be transformed into
\[
(1 - p^{-(k+t+1)}) (1 + p^{-(k+t')}) p^{n-2k+1} (p^{t'} - \epsilon) (p^{t'} - 1 + \epsilon)
\]
\[
\times \sum_{i=1}^{l} (-1)^{i-1} p^{(i-2)(i-1)/2 + (i-1)(n-2k+2)} C_{2t'-2, i-1, e}
\]
\[
\times (1 - p^{-(k+t')}) (1 + p^{-(k+t'-1)}) \alpha_p(H_{k+t-1}, -B_{l-1} \perp \hat{T})
\]
\[
= p^{n-2k+1} (p^{t'} - \epsilon) (p^{t'} - 1 + \epsilon) (1 - p^{-(k+t'+1)}) (1 - p^{-(k+t')})
\]
\[
\times (1 + p^{-(k+t'-1)}) \beta_p(H_{k+t-1}, -B_{l-1}) \alpha_p(H_{k-t-1} \perp B_{l-1}, \hat{T})
\]
\[
= p^{n-2k+1} (p^{t'} - \epsilon) (p^{t'} - 1 + \epsilon)
\]
\[
\times \beta_p(H_{k+t+1}, -B_{l-1}) \alpha_p(H_{k-t-1} \perp B_{l-1}, \hat{T}).
\]
This proves the assertion (2) in case \( n + \deg S_0 \) is odd. Next again by (2) of Proposition 5.6 and Lemma 5.3, the assertion (2) can be proved in case \( n + \deg S_0 \) is odd.

**Proof of Theorem 5.2.** We prove the assertion by induction on \( n \). The assertion for \( n = 1 \) follows from (2) of Proposition 5.3. Let \( n \geq 2 \) and assume that the assertion holds for \( n - 1 \). Then by the induction assumption we have
\[
\alpha_p(H_{s-t-1} \perp B_{l}, \hat{T}) = \sum_{j=0}^{n-1} a_j p^{t_j},
\]
and
\[
\alpha_p(H_{s-t-2} \perp B_{l}, \hat{T}) = \sum_{j=0}^{n-1} a_j' p^{t_j},
\]
where \( a_j = a_j(s, S_0, \hat{T}) \) and \( a_j' = a_j(s - 1, S_0, \hat{T}) \) in Theorem 5.2. We may assume that \( T = b_1 \perp b_2 \perp \ldots \perp b_n \) with \( \ord_p(b_1) \geq \ord_p(b_2) \geq \ldots \geq \ord_p(b_n) \). First assume that \( S_0 \) is of type (M-3) or (M-5). Thus by Proposition 5.3 we have
\[
\alpha_p(H_{k-t-1} \perp B_{l}, T) = \frac{1 - p^{2k+2l+2}}{1 - p^{n-2k+1}} \alpha_p(H_{k-t-2} \perp B_{l}, \hat{T})
\]
\[
+ \frac{p^{n-2k+1}(p^{2l} - 1)}{1 - p^{n-2k+1}} \alpha_p(H_{k-t-1} \perp B_{l-1}, \hat{T})
\]
\[
+ p'K(S_0, T) \alpha_p(H_{k-t-1} \perp B_{l}, \hat{T})
\]
which is equal to
\[
\frac{1 - p^{2k+2l+2}}{1 - p^{n-2k+1}} \sum_{j=0}^{n-1} a_j' p^{t_j}
\]
\[
+ \frac{p^{n-2k+1}(p^{2l} - 1)}{1 - p^{n-2k+1}} \sum_{j=0}^{n-1} a_j' p^{(l-1)j}
\]
\[
+ p'K(S_0, T) \sum_{j=0}^{n-1} a_j p^{t_j}.
\]
For \( 0 \leq j \leq n - 1 \) put
\[
M(j) = \frac{1 - p^{2k+2l+2}}{1 - p^{n-2k+1}} a_j' p^{t_j}
\]
\[
+ \frac{p^{n-2k+1}(p^{2l} - 1)}{1 - p^{n-2k+1}} a_j' p^{(l-1)j} + p'K(S_0, T) a_j p^{t_j}.
\]
Then for \( j \leq n - 2 \), \( M(j) \) is a polynomial in \( p^j \) of degree at most \( n - 1 \). On the other hand,
\[
M(n-1) = \frac{1 - p^{2k+2} \alpha_{n-1} p^{(n-1)} + p^{n-2k} \alpha_{n-1} p^{(n-1)} + a_{n-1} K(S_0, T) p^{(n-1)}}{1 - p^{2k+1}} + a_{n-1} p^{(n-1)} + a_{n-1} K(T) p^n.
\]

Thus \( a_0(H_k - 1 \parallel B_i, T) \) is a polynomial in \( p^j \) of degree at most \( n \). This proves the assertion in case (M-3) or (M-5). Similarly, the assertion can be proved in the remaining case.

**Remark 5.9.** A more careful analysis shows that we have \( a_0(k, S_0, T) = 1 \) in the above theorem.

**Corollary to Theorem 5.2.** Let the notation be as above. For any \( n \)-tuple \((l_1, l_2, \ldots, l_n)\) of complex numbers, put \( \mu(l_1, \ldots, l_n) = \prod_{1 \leq j \leq \leq n} (p^{l_i} - p^{l_j}) \). Then for any integers \( 0 \leq l_1 < \ldots < l_{n+2} \leq k \) and \( T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p) \) we have
\[
\sum_{j=1}^{n+2} (-1)^{j-1} \mu(l_1, ..., l_{j-1}, l_{j+1}, ..., l_{n+2}) a_p(H_{k-l_j-1} \parallel B_{l_j}, T) = 0.
\]

\[\square\]

**Theorem 5.10.** Let \( k \geq n + 1 \). Let \( n + 1 \) integers \( 0 \leq l_1 < \ldots < l_{n+2} \leq k \) be given, let \( \lambda_1, \ldots, \lambda_{n+1} \) be rational numbers such that
\[
\sum_{j=1}^{n+1} \lambda_j a_p(H_{n-l_j+1} \parallel B_{l_j+k-n-2}, T) = 0
\]
for any \( T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p) \). Then we have \( \lambda_1 = \ldots = \lambda_{n+1} = 0 \).

**Proof.** We prove the assertion by induction on \( n \). The assertion for \( n = 1 \) follows from Proposition 5.8. Let \( n \geq 2 \), and assume that the assertion holds for \( n - 1 \). The above relation holds for \( T = p^{2r} \parallel \hat{T} \) with any integer \( r \) and \( \hat{T} \in \mathcal{H}_{n-1}(\mathbb{Z}_p) \cap GL_{n-1}(\mathbb{Q}_p) \). Then by Proposition 5.8,
\[
\sum_{l=1}^{n+1} \lambda_l \{ (1 - p^{2l-2n-2}) a_p(H_{n-l} \parallel B_{l+k-n-2}, \hat{T}) + p^{n-2k+1} (p^{2l+2k-2n-4} - 1) a_p(H_{n-l+1} \parallel B_{l+k-n-3}, \hat{T}) \} + p^{(n-2k+1)r} w(\hat{T}) \sum_{l=1}^{n+1} \lambda_l p^{l+k-n-2} a_p(H_{n-l+1} \parallel B_{l+k-n-2}, \hat{T}) = 0,
\]
where \( w(\hat{T}) \) is a certain rational number depending only on \( \hat{T} \). Thus by taking the limit \( r \to \infty \) we obtain
\[
\sum_{l=1}^{n+1} \lambda_l \{ (1 - p^{2l-2n-2}) a_p(H_{n-l} \parallel B_{l+k-n-2}, \hat{T}) + p^{n-2k+1} (p^{2l+2k-2n-4} - 1) a_p(H_{n-l+1} \parallel B_{l+k-n-3}, \hat{T}) \} = 0 \quad (*)
\]
and
\[ \sum_{l=1}^{n+1} \lambda_l p^{l+k-n-2} \alpha_p(H_{n-l+1} \perp B_{l+k-n-2}, \tilde{T}) = 0 \quad (**) \]

Rewriting (*) we have
\[ \sum_{l=1}^{n} (\lambda_l (1 - p^{2l-2n-2}) + \lambda_{l+1} p^{n-2k+1}(p^{2l+2k-2n-2} - 1)) \times \alpha_p(H_{n-l} \perp B_{l+k-n-2}, \tilde{T}) = 0. \]

Thus by the induction hypothesis, we have
\[ \lambda_l (1 - p^{2l-2n-2}) + \lambda_{l+1} p^{n-2k+1}(p^{2l+2k-2n-2} - 1) = 0 \]
for any \( l = 0, 1, ..., n \). In particular
\[ \lambda_n = -\frac{p^{n-2k+3}(p^{2k-2} - 1)\lambda_{n+1}}{p^2 - 1} \quad (***) \]

On the other hand, by the Corollary to Theorem 5.2 we have
\[ \sum_{l=1}^{n+1} (-1)^{l-1} \mu_l \alpha_p(H_{n-l+1} \perp B_{l+k-n-2}, \tilde{T}) = 0 \quad (***) \]
where \( \mu_l = \mu(k-n-1, ..., l+k-n-3, l+k-n-1, ..., k-1) \). By (***) and (****), and the induction hypothesis, we have
\[ \lambda_l = (-1)^{l-n-1} \frac{\mu_l}{\mu_{n+1}} p^{-l+n+1} \lambda_{n+1}, \]
and in particular
\[ \lambda_n = -\frac{\mu_n}{\mu_{n+1}} p \lambda_{n+1} = -\frac{p^n - 1}{p - 1} \lambda_{n+1}. \quad (***) \]

If \( \lambda_{n+1} \neq 0 \), (****) contradicts (**), since \( n \geq 2 \) and \( k \geq n+1 \). Thus we have \( \lambda_{n+1} = 0 \) and therefore \( \lambda_l = 0 \) for any \( l = 1, ..., n+1 \). This completes the induction.

We can now prove Theorem 5.1. We notice first that the genera of lattices of level \( p \) on the space of the given lattice are represented by lattices \( L^{(i)} \) whose \( p \)-adic completions have a Gram matrix that is \( \mathbb{Z}_p \)-equivalent to \( H_{k-i-1} \perp pH_i \perp S_0 \) with a fixed \( S_0 \) of degree 2 as in Theorem 5.2. Altogether there are \( k \geq n+1 \) such genera.

As a consequence of Siegel’s theorem one sees that the linear independence of any \( n+1 \) of the degree \( n \) theta series of the genera of the \( L^{(i)} \) is implied by the linear independence of the corresponding \( p \)-adic local density functions \( T \mapsto \alpha_p(L^{(i)}, T) \) stated in Theorem 7.10 (notice that the restriction \( k \geq n+1 \) implies that for primes \( \ell \neq p \) the \( \ell \)-adic completion of the lattices \( L^{(i)} \) splits off an orthogonal sum of at least \( n \) unimodular hyperbolic planes so that every even \( T \in M_n^{sym}(\mathbb{Z}) \) is represented at all \( \ell \)-adic completions and the product of the \( \alpha_p(L^{(i)}, T) \) is nonzero). Since all the genus theta series are (by Siegel’s theorem) in the space of Eisenstein series associated to zero-dimensional boundary components (cusps) and since there are \( n+1 \) such cusps in the case of prime level, it is clear that both types of genus theta series generate the full space of Eisenstein series.
Corollary 5.11. Let \( L \) be a lattice on the quadratic space \( V \) over \( \mathbb{Q} \) of level \( p \) as in Theorem 5.1 and put \( F = \vartheta^{(n)}(\text{gen}(L)) \). Then the modular form \( F|_k T(\ell) \) can be expressed as a linear combination of theta series of positive definite lattices of level \( p \) on \( V \) for all primes \( \ell \neq p \).

Proof. This is clear from Theorem 5.10 and Theorem 4.3.

Remark 5.12. a) The result of Theorem 5.1 is more generally true in the case of square free level \( N \), in which case the dimension of the space spanned by the genus theta series becomes \( (n + 1)\omega(N) \) where \( \omega(N) \) is the number of primes dividing \( N \); one has then a basis of genus theta series if one considers \( (n + 1)\omega(N) \) genera of lattices on the same quadratic space \( V \) such that for each \( p \) dividing \( n \) one has \( n + 1 \) local integral equivalence classes. In that case our proof given above requires the restriction that the anisotropic kernel of the quadratic space under consideration has dimension at most 2. Moreover we can not guarantee the holomorphy of the indefinite genus theta series if the character is trivial (i.e., if the underlying quadratic space has square discriminant). One proceeds in the proof as above, adding an induction on the number of primes \( \omega(N) \) dividing \( N \).

b) A different (and much shorter) proof of Theorems 5.2 and 5.10 has been communicated to us by Y. Hironaka and F. Sato [6]. The proof given here gives a little more information (e.g. explicit recursion relations) than theirs. The proof of Hironaka and Sato removes the restriction on the anisotropic kernel mentioned above (if one strengthens the condition on \( n \) to \( n + 1 < k \) in the new cases) and provides also a version for levels that are not square free. The application of that version to the study of the space of Eisenstein series generated by the genus theta series in the case of arbitrary level will be the subject of future work.

6. Connection with Kudla’s matching principle

In Section 4 we have seen that the Hecke operator \( T(p) \) can provide a connection between theta series for lattices in positive definite quadratic space \( (V_1, q_1) \) and in a related indefinite quadratic space \( (V_2, q_2) \). Such a connection has recently been observed in a different setup by Kudla [11]. We sketch his approach briefly in order to study the relation to our construction, for details we refer to [11], Section 4.1.

Let \( (V_1, q_1) \) be a positive definite quadratic space over \( \mathbb{Q} \) of dimension \( m \) and discriminant \( d \), let \( (V_2, q_2) \) be a space of the same dimension \( m \) and discriminant \( d \), but of signature \( (m - 2, 2) \). We fix \( n > 0 \) and an additive character \( \psi \) of \( \mathbb{Q}_\mathbb{A} \). Consider the oscillator representations \( \omega_1 = \omega_{1, \psi} \) of \( \widetilde{Sp}_n(\mathbb{A}) \times O(V_1, q_1)(\mathbb{A}) \) on the Schwartz space \( S((V_1(\mathbb{A}))^n) \) and \( \omega \) of \( \widetilde{Sp}_n(\mathbb{A}) \times O(V_2, q_2)(\mathbb{A}) \) on \( S((V_2(\mathbb{A}))^n) \), where \( \widetilde{Sp}_n(\mathbb{A}) \) denotes the usual metaplectic double cover of the adelic symplectic group \( Sp_n(\mathbb{A}) \).

For \( j = 1, 2 \) we have then for \( \varphi \in S((V_j(\mathbb{A}))^n) \) the theta kernel

\[
\theta(\tilde{g}, h_j; \varphi_j) = \sum_{x \in V_j(\mathbb{Q})} \omega_j(\tilde{g}) \varphi_j(h_j^{-1}x)
\]

\((\tilde{g} \in \widetilde{Sp}_n(\mathbb{A}), h_j \in O(V_j, q_j)(\mathbb{A}))\).
and the theta integral
\[ I(\tilde{g}; \varphi_j) = \int_{O(V_j,s_j)(\mathbb{Q}) \setminus O(V_j,s_j)(\mathbb{A})} \theta(\tilde{g}, h_j, \varphi_j) dh_j \]

which (under our conditions) is absolutely convergent for \( j = 1 \) and for \( j = 2 \) if \( V_2 \) is anisotropic or \( m > n + 2 \).

Let now \( L_j \) be a lattice on \( V_j \) and assume \( \varphi_j \) to be factored as \( \varphi_j = \prod_v \varphi_{j,v} \) over all places \( v \) of \( \mathbb{Q} \), where \( \varphi_{j,p} = 1_{L_{j,p}} \) is the characteristic function of the lattice \( L_{j,p} \) in the \( \mathbb{Q}_p \)-space \( V_{j,p} \) for all finite primes \( p \). Then for \( \varphi_{1,\infty}(x) = \exp(-2\pi \operatorname{tr}(q(x))) \) for \( x \in (V_1 \otimes \mathbb{R})^n \) (the Gaussian vector) the integral \( I(\tilde{g}; \varphi_1) \) is the adelic function corresponding to the Siegel modular form
\[ \psi^{(n)}(\operatorname{gen}(L_j), Z) \]
in the usual way.

For the space \( V_2 \) we consider two different test functions at infinity: If we choose a fixed majorant \( \xi \) of \( q \) and put
\[ \varphi_{2,\infty,\xi}(x) = \exp(-2\pi \operatorname{tr}(\xi(x))) \quad \text{for} \quad x \in (V_2 \otimes \mathbb{R})^n, \]
the value of the theta kernel
\[ \theta(\tilde{g}, 1_{V_2}, \varphi_{2,\infty,\xi} \otimes \prod_{p \neq \infty} \varphi_{2,p}) \]
at \( h_2 = 1_{V_2} \) corresponds to the theta function
\[ \psi^{(n)}(L_2, \xi, Z) = \sum_{x \in L_2^2} \exp(2\pi i \operatorname{tr}(q(x)X)) \exp(-2\pi \operatorname{tr}(\xi(x)Y)) \]
(with \( Z = X + iY \in \mathfrak{H}_n \)) considered by Siegel, and its integral over \( O(V_{2,a})(\mathbb{Q}) \setminus O(V_{2,a})(\mathbb{A}) \) corresponds to the integral of this theta function over the space of majorants \( \xi \); this is a nonholomorphic modular form in the space of Eisenstein series by Siegel’s theorem (or its extension to the Siegel-Weil-Theorem).

Applying a certain differential operator as outlined in [11] to \( \varphi_{2,\infty,\xi} \), we obtain a different test function \( \varphi_{2,\infty,\xi}' \), and the integral of the theta kernel \( \theta(\tilde{g}, h, \varphi_{2,\infty,\xi}' \otimes \prod_{p \neq \infty} \varphi_{2,p}) \) over \( O(V_{2,a})(\mathbb{Q}) \setminus O(V_{2,a})(\mathbb{A}) \) corresponds to the holomorphic theta series of the indefinite lattice \( L_2 \) considered by Siegel in [17] and by Maaß in [13] whenever the latter is defined.

To simplify the discussion, we restrict now (following [11]) to \( n = 1 \). We denote by \( \chi \) the quadratic character of \( \mathbb{Q}_\mathbb{A}/\mathbb{Q}^\times \) defined by
\[ \chi_v(x) = (x, (-1)^{\frac{m-1}{2}}d_v) \]
for all places \( v \), where \((\ , \ )_v \) is the Hilbert symbol. Then associated to \( \varphi_j \) there is a unique standard section \( \Phi_j : \tilde{G}(\mathbb{A}) \times \mathbb{C} \longrightarrow \mathbb{C} \) with \( \Phi_j(\cdot, s) \in I(s, \chi) \), (where \( I(s, \chi) \) is the principal series representation of \( \tilde{G}(\mathbb{A}) \) with parameter \( s \) and character \( \chi \) such that for \( s_0 = \frac{m}{2} - 1 \) one has
\[ \Phi_j(g, s_0) = (\omega_j(\tilde{g})\varphi_j)(0) =: \lambda_j(\varphi_j). \]
With the Eisenstein series

\[ E(\tilde{g}, s; \varphi_j) := \sum_{\gamma \in \mathbb{P}_0 \backslash \mathbb{A}} \Phi_j(\gamma \tilde{g}, s) \]

associated to \( \Phi_j \), the Siegel-Weil theorem asserts that \( E(\tilde{g}, s; \varphi_j) \) is holomorphic at \( s = s_0 \) and that one has the identities

\[ E(\tilde{g}, s_0; \varphi_j) = \kappa \cdot I(\tilde{g}; \varphi_j) \]

where \( \kappa = 2 \) if \( m \leq 2 \) and \( \kappa = 1 \) otherwise.

The above maps \( \lambda_j : S(V(\mathbb{A})) \to I(s_0, \chi) \) factor into a product \( \lambda_j = \prod_v \lambda_{j,v} \) over all places \( v \) of \( \mathbb{Q} \) and Kudla gives the following definition.

**Definition 6.1. (Kudla)**

(a) Let \( v \) be a (finite or infinite) place of \( \mathbb{Q} \), let \( V_{1,v} \) and \( V_{2,v} \) be quadratic spaces over \( \mathbb{Q}_v \) of dimension \( m \) and discriminant \( d \). Then functions \( \varphi_{1,v} \in S(V_{1,v}) \) and \( \varphi_{2,v} \in S(V_{2,v}) \) are said to match if \( \lambda_{1,v}(\varphi_{1,v}) = \lambda_{2,v}(\varphi_{2,v}) \).

(b) Let \( V_1, V_2 \) be quadratic spaces over \( \mathbb{Q} \) of the same dimension \( m \) and discriminant \( d \). Then two test functions \( \varphi_1 \in S(V_1(\mathbb{A})) \) and \( \varphi_2 \in S(V_2(\mathbb{A})) \) match, if \( \lambda_1(\varphi_1) = \lambda_2(\varphi_2) \). Equivalently, two factorisable test functions \( \varphi_1 = \bigotimes_v \varphi_{1,v}, \varphi_2 = \bigotimes_v \varphi_{2,v} \) match if \( \varphi_{1,v} \) and \( \varphi_{2,v} \) match for all places \( v \).

The matching principle observed by Kudla in [11] then states that for matching test functions \( \varphi_1 \in S(V_1(\mathbb{A})), \varphi_2 \in S(V_2(\mathbb{A})) \) one has with \( \Phi(\cdot, s_0) = \lambda_1(\varphi_1) = \lambda_2(\varphi_2) \):

\[ I(\tilde{g}; \varphi_1) = E(\tilde{g}, s_0, \Phi) = I(\tilde{g}; \varphi_2). \]

Although this identity is a trivial corollary of the Siegel-Weil theorem, the matching principle gives highly nontrivial arithmetical identities since the integrals \( I(\tilde{g}, \varphi_1) \) and \( I(\tilde{g}, \varphi_2) \) carry completely different arithmetic information; in [11] the principle is exploited to give identities between degrees of certain special cycles on modular varieties and linear combinations of representation numbers of positive definite quadratic forms. Kudla gives in [11] explicit local matching functions at the infinite place and asserts the existence of local matching functions at the finite places for \( m > 4 \) and for \( m = 4 \) if \( \chi_p \neq 1 \).

We can now state the contribution of our computations from the previous sections to this matching principle:

**Proposition 6.2.** Let \( L,V,q \) be as in the previous sections, let \( n = 1 \) and let \( \varphi_1 = \prod_v \varphi_{1,v} \in S(V(\mathbb{A})) \) be the test function for the positive definite lattice \( L \) as described above. Assume that \( L \) is of square free odd level \( N \) and that all \( p \mid N \) divide the discriminant of \( L \) to an odd power. Let \( \chi \) be the (primitive) quadratic character mod \( N \) with \( \vartheta(L,q) \in M_k(\Gamma_0(N), \chi) \) and let \( p \) be a prime with \( \chi(p) = -1 \). Let \( \vartheta(\text{gen}(L))(\gamma) = \sum c_i \vartheta(\text{gen}(L_i)) \) be the explicit linear combination of theta series of all the positive definite genera of lattices of level \( N \) and discriminant in \( d \cdot (\mathbb{Q}^\times)^2 \) given by the results of Section 5, let \( \psi_i \) be the test function attached to the positive definite lattice \( L_i \) as above. Let \( V_2(q_2) \) be the quadratic space \( V \) of signature \( (m-2,2) \) from Lemma 4.2 in Section 4, let \( L_2 = \tilde{L} \) in the notation of Lemma 4.2 and let

\[ \varphi'_2 = \varphi'_{2,\infty,c} \otimes \prod_{p \neq 0} \varphi_{2,p} \]
be the test function attached to $L_2$ as described above.

Then the test functions
\[ \psi := \sum_i c_i \psi_i \in S(V_1(\mathbb{A})) \]
and
\[ \varphi'_2 \in S(V_2(\mathbb{A})) \]
match and we have
\[ I(\tilde{g}, \psi) = I(\tilde{g}, \varphi'_2). \]

**Proof.** This is clear from the discussion above and Theorem 4.3. \qed

**Remark 6.3.** As already stated in [11] the matching principle can easily be generalized to arbitrary $\tilde{Sp}_n$. In the range of our results in Sections 4 and 5 we have then examples for the matching principle for general $n$ in the same way as described above.
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