Random Walks on Graphs and Approximation of $L^2$-Invariants

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Abstract
In this work, we interpret right multiplication operators $R_w: l^2(G) \rightarrow l^2(G)$, $w \in \mathbb{C}[G]$ as random walk operators on certain labelled graphs we employ that are analogous to Cayley graphs. Applying a generalization of the graph convergence defined by R. I. Grigorchuk and A. Žuk to these graphs gives a new interpretation and proof of a special case of W. Lück’s famous Theorem on the Approximation of $L^2$-Betti numbers for countable residually finite groups by means of exhausting towers of finite-index subgroups. In particular, using this interpretation, the theorem follows naturally from standard theorems in probability theory concerning the weak convergence of probability measures that are characterized by their moments. This paper is mainly a direct adaptation of the ideas of Grigorchuk, Žuk and Lück to this setting. We aim to explain how these ideas are related and give a short exposition of them.

Keywords $L^2$-invariants · Random walks · Spectral density function

Mathematics Subject Classification (2010) 57R19 · 46N99

1 Introduction

A key area of interest in the theory of $L^2$-invariants is the study of the spectral density function $F(R_w): [0, \infty) \rightarrow [0, \infty)$ of a $G$-equivariant bounded operator $R_w: l^2(G) \rightarrow l^2(G)$, given by right multiplication of $w$ (an element of the complex group ring $\mathbb{C}[G]$) on the Hilbert space $l^2(G)$ of square summable formal sums

$$l^2(G) = \left\{ \sum_{g \in G} \lambda_g g : \lambda_g \in \mathbb{C}, \sum_{g \in G} |\lambda_g|^2 < \infty \right\}.$$
The spectral density function $F(R_w)$ is used to define $L^2$-invariants of $R_w$, such as the $L^2$-Betti number, Novikov-Shubin invariant, and Fuglede-Kadison determinant.

In general, $F(R_w)$ is difficult to compute, and there are few instances of actual computations of the $L^2$-invariants of right multiplication operators. For example, for any $w \in \mathbb{C}[G]$ with $G = \mathbb{Z}^d$, recent work by W. Lück [20] provided bounds on the spectral density function of $R_w$, and established that the Novikov-Shubin invariant of $R_w$ is positive. In [5], L. Grabowski exhibited explicit examples of particular groups $G$ and associated operators $A \in M_{a \times b}(\mathbb{C}[G])$ for which the corresponding $L^2$-Betti number is irrational.

This paper studies the spectral density function of $R_w$ using the theory of random walk (or Markov) operators associated to groups. This relies on [7], where R. I. Grigorchuk and A. Żuk defined convergence of sequences of locally finite marked graphs, and showed that the spectral measure associated to the random walk (or Markov) operator of such a graph is continuous with respect to this type of graph convergence. This was subsequently used in [8] to explicitly compute the spectrum of a Markov operator of the lamplighter group, and the approach was further extended in [12] to Markov operators of groups generated by Cayley automata.

An important connection between these two theories is the following correspondence.

1. The random walk operator on the Cayley graph $C_{G,S}$ of $G$ with respect to the generating set $S$. This operator is known as the random walk operator (or Markov operator) associated to the Cayley graph $C_{G,S}$.

2. The $G$-equivariant bounded operator $R_m : l^2(G) \rightarrow l^2(G)$, given by right multiplication of the self-adjoint element $m = \frac{1}{|S|} \sum_{s \in S} s \in \mathbb{C}[G]$.

In particular, computing the spectral measure of the random walk operator on the Cayley graph $C_{G,S}$ gives information on the spectral density function of $R_m$ and the associated $L^2$-invariants. For example, the main result of [8] was used to give a counter-example [10] to the Strong Atiyah Conjecture concerning the range of values of $L^2$-Betti numbers.

The aim of this paper is to extend this correspondence to self-adjoint elements $z = ww^* \in \mathbb{C}[G]$ for an arbitrary element $w$ of $\mathbb{C}[G]$, and use it to study $F(R_w)$. Let $S$ denote a finite symmetric generating set for $G$ (symmetric means if $s \in S$ then $s^{-1} \in S$) such that $z$ can be expressed $z = \sum_{s \in S} c_s s$ for $c_s \in \mathbb{C}$. Our approach is to introduce a certain definition of a labelled graph $\Gamma_{(z,S)}$ that is analogous to a Cayley graph, such that the operator coming from right multiplication by $z$ on $l^2(G)$ is identified with a random walk operator associated to $\Gamma_{(z,S)}$. This operator will be called the Markov-type operator associated to $\Gamma_{(z,S)}$.

In the case that $G$ is a finitely generated residually finite group, it is an important theorem of W. Lück [18, 19] that the spectral density function $F(R_w)$ can be “approximated” by finite-dimensional analogues. We state this theorem as Theorem 1.1 and will introduce the details shortly. Our approach yields a new proof of this theorem that explains the theorem in terms of this theory of random walk operators and shows the connection of these ideas to the work of R. Grigorchuk and A. Żuk [7].

To explain this, we will fix the following notations:

**Setting** Let $G$ be a finitely generated residually finite group. Let $\{K_n\}_{n \in \mathbb{N}}$ be a nested sequence of finite index normal subgroups of $G$ with trivial intersection, i.e.,

$$\{K_n \triangleleft G : [G : K_n] < \infty, \ K_n \supseteq K_{n+1}\}_{n \in \mathbb{N}}.$$
such that $\bigcap_{n \in \mathbb{N}} K_n = \{1\}$. For non-negative integers $a, b \in \mathbb{N}$ and matrix $A \in M_{a \times b}(\mathbb{C}[G])$, let $R_A : (l^2(G))^a \to (l^2(G))^b$ be the operator given by right multiplication by $A$. Set

$$A_n = \pi_n(A) \in M_{a \times b}(\mathbb{C}[G/K_n]),$$

obtained by applying the canonical projection $\pi_n : \mathbb{C}[G] \to \mathbb{C}[G/K_n]$ to the entries of $A$, and let $R_{A_n} : l^2(G/K_n)^a \to l^2(G/K_n)^b$ be the associated right multiplication operator.

The main result of the present paper is a new proof of the following theorem.

**Theorem 1.1** (W. Lück) Let $G$ be a countable residually finite group as described in the Setting, and let $A$ and $A_n$ be given by $w \in \mathbb{C}[G]$ and $w_n = \pi_n(w)$ respectively (i.e., when $a = b = 1$). Let $F(R_w)(\lambda)$ and $F(R_{w_n})(\lambda)$ be the spectral density functions of $R_w$ and $R_{w_n}$ respectively. Then, for all $\lambda \geq 0$ at which $F(R_w)$ is continuous, one has that

$$\lim_{n \to \infty} F(R_{w_n})(\lambda) = F(R_w)(\lambda).$$

This theorem is contained in the proof of Theorem 2.3 of [18], although in that work the theorem is not expressed explicitly in these terms. A presentation of these ideas that contains a discussion from which this version follows can be read in the proof of Theorem 16.3 of Lück’s survey article [19].

The point $\lambda = 0$ is a point at which the spectral density function $F(R_w)$ is continuous. At this point, by definition $F(R_w)(0) = b^{(2)}(R_w)$ and $F(R_{w_n})(0) = b^{(2)}(R_{w_n})$. See [17, Section 2.5] for these definitions. So a corollary of Theorem 1.1 is a special case (for $a = b = 1$) of the following famous theorem of W. Lück concerning the approximation of an $L^2$-Betti number $b^{(2)}(R_A)$ by the sequence of $L^2$-Betti numbers $b^{(2)}(R_{A_n})$ arising from the finite dimensional operators $R_{A_n}$.

**Theorem 1.2** (W. Lück, [18, Theorem 2.3]) Let $G$ be as described in the Setting. For $A \in M_{a \times b}(\mathbb{Q}[G])$, let $b^{(2)}(R_A)$ and $b^{(2)}(R_{A_n})$ be the $L^2$-Betti numbers of $R_A$ and $R_{A_n}$ respectively. Then,

$$b^{(2)}(R_A) = \lim_{n \to \infty} b^{(2)}(R_{A_n}).$$

This statement is a short manipulation away from the main theorem of the article [18, Theorem 0.1], which specifically concerns a proof of what is named there Kazhdan’s equality. Kazhdan’s equality is a limit statement for the Betti numbers of cell complexes $(X_m, A_m)$ appearing in a tower of covering spaces over a cell complex $(X, A)$ arising from a nested sequence of finite index normal subgroups of $\pi_1(X)$ with trivial intersection (as detailed here in the Setting).

The proof of Theorem 1.1 given by W. Lück in [19, Theorem 16.3] employed functional analytic techniques. While the techniques behind the new proof of Theorem 1.1 constitute well-known results, they are nonetheless part of an important and increasingly useful approach of applying tools from geometric group theory to give insights into conjectures involving spectral density functions of right multiplication operators and their associated $L^2$-invariants. Examples of the usefulness of this approach can be seen in [6, 22].

Another motivation for exploring different approaches to spectral density functions arises from some conjectures of T. T. Q. Le that there is a version of Lück’s theorem that relates the growth of the order of torsion to hyperbolic volume in the case that one starts with a hyperbolic 3-manifold [15].
This paper grew out of our attempts to understand the methods to compute the $L^2$-Alexander invariant for knots $K$. Note that the fundamental group $\pi_1(\mathbb{S}^3\setminus K)$ (of the complement of $K$ in the 3-sphere $\mathbb{S}^3$) is a countable residually finite group [11]. The $L^2$-Alexander invariant is an $L^2$-invariant analogue of twisted Alexander polynomials that was introduced by W. Li and W. Zhang in [16]. In particular, 2-bridge knots $K$ have the property that $G = \pi_1(\mathbb{S}^3\setminus K)$ admits a deficiency 1 presentation, and therefore the $L^2$-Alexander invariant of $K$ can, in principle, be computed from an element $w$ of $\mathbb{C}[G]$ (although for most cases, direct computations are difficult to perform). The $L^2$-Alexander invariant is related to an $L^2$-Alexander torsion; $L^2$-Alexander torsions were defined and studied by J. Dubois, S. Friedl, and W. Lück in [2, 3]. It is worth noting that F. Ben Aribi has shown that the $L^2$-Alexander invariant detects the unknot [1].

The paper is organized as follows. Section 2 presents relevant theory on $L^2$-invariants of right multiplication operators and the weak convergence of measures. Section 3 presents the main geometric group theoretic tool of labelled connected marked graphs (denoted lcmg’s) and the corresponding concept of convergence. Section 4 gives the proof of the main result.

2 Preliminaries

2.1 $L^2$-Invariants

For details on the theory of $L^2$-invariants, see [17, Chapters 1, 2, and 13]. Let $G$ be a discrete and countable group. The group ring $\mathbb{C}[G]$ comes with an involution $*$, such that for $\sum_{g \in G} \lambda_g g \in \mathbb{C}[G]$, $\lambda_g \in \mathbb{C}$, $(\sum_{g \in G} \lambda_g g)^* = \sum_{g \in G} \overline{\lambda_g} g^{-1}$, where $\overline{\lambda}$ is the complex conjugate of $\lambda$. One says that $w \in \mathbb{C}[G]$ is self-adjoint if $w^* = w$.

Let $w \in \mathbb{C}[G]$, and let $R_w : l^2(G) \to l^2(G)$ be the associated right multiplication operator. The adjoint of $R_w$ is

$$(R_w)^* = R_{w^*} : l^2(G) \to l^2(G),$$

which is also a right multiplication operator. The operator $R_w$ is self-adjoint (i.e., $(R_w)^* = R_w$) if and only if $w$ is self-adjoint. Furthermore, $R_w$ is an element of the group von Neumann algebra $\mathcal{N}(G)$ consisting of all $G$-equivariant bounded linear operators $l^2(G) \to l^2(G)$. The von Neumann algebra $\mathcal{N}(G)$ has a finite faithful normal trace $\text{tr}_{\mathcal{N}(G)}(T) = \langle T(1), 1 \rangle$, $T \in \mathcal{N}(G)$, where $1 \in l^2(G)$ is the identity element of $G$.

For an arbitrary $T \in \mathcal{N}(G)$, $T^* \circ T$ is self-adjoint, and therefore has a family of spectral projections $\left\{ E_B^{T^* \circ T} \right\}_B$ where

$$E_B^{T^* \circ T} = \chi_B(T^* \circ T),$$

with the collection taken over all Borel subsets $B$ of $[0, \|T\|^2]$, and $\chi_B$ is the characteristic function of $B$.

The spectral density function $F(T)(\lambda)$ is given by

$$F(T)(\lambda) = \text{tr}_{\mathcal{N}(G)}(E_B^{T^* \circ T}_{[0,\lambda^2]}).$$

This is a monotonically increasing right-continuous function. The $L^2$-Betti number of $T$ is defined to be $b^{(2)}(T) := F(T)(0)$. Roughly, $b^{(2)}(T)$ is a measure of the size of the kernel of $T$; one also has that $b^{(2)}(T) = 0$ if and only if $T$ is injective.
2.2 Weak Convergence of Probability Measures

For details, see [13, Chapter 13], [21, Section 11.4], and also [4]. Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence of probability measures on a metric space \(X\) (with Borel \(\sigma\)-algebra \(\mathcal{B}(X)\)). One says that \((\mu_n)_{n \in \mathbb{N}}\) converges weakly to a probability measure \(\mu\) if \(\lim_{n \to \infty} \mu_n(B) = \mu(B)\) for all \(B \in \mathcal{B}(X)\) such that \(\mu(\partial B) = 0\), where \(\partial B\) is the boundary of \(B\). The limit measure of weak convergence can be shown to be unique.

Now for \(X = \mathbb{R}\), let \(\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_{\geq 0}\) be a probability measure on \(\mathbb{R}\). The cumulative distribution function (CDF) \(F_{\mu} : \mathbb{R} \to \mathbb{R}_{\geq 0}\) associated to \(\mu\) is \(F_{\mu}(x) := \mu((-\infty, x])\). Note that the intervals \((-\infty, x]\) have measure zero boundary.

Proposition 2.1 ([13, Theorem 13.23]) For probability measures \((\mu_n)_{n \in \mathbb{N}}\) and \(\mu\) on \(\mathcal{B}(\mathbb{R})\), the following are equivalent:

1. \((\mu_n)_{n \in \mathbb{N}}\) converges weakly to \(\mu\).
2. For all \(x\) on which \(F_{\mu}\) is continuous, \(\lim_{n \to \infty} F_{\mu_n}(x) = F_{\mu}(x)\).

Let \(\mathcal{P}(\mathbb{R})\) be the set of Borel probability measures \(\mu\) on \(\mathbb{R}\) such that all polynomials with real coefficients (denoted \(\mathbb{R}[x]\)) belong to \(L^1(\mu)\); define an equivalence relation on \(\mathcal{P}(\mathbb{R})\): \(\mu_1 \sim \mu_2\) if and only if for every \(p \in \mathbb{R}[x]\), \(\int_{\mathbb{R}} p\, d\mu_1 = \int_{\mathbb{R}} p\, d\mu_2\). For \(\mu \in \mathcal{P}(\mathbb{R})\) and a measurable function \(f\) on \(\mathbb{R}\), the expected value of \(f\) w.r.t. \(\mu\) is \(E_{\mu}(f) = \int_{\mathbb{R}} f(x)\, d\mu(x)\). For \(k \in \mathbb{N}\), the \(k\)th moment of \(\mu\) is \(\mu^{(k)} = E_{\mu}[x^k]\). The measure \(\mu\) is said to be characterized by its moments if the equivalence class of \(\mu\) consists of a single measure. By the Stone-Weierstrass Theorem, any compactly supported measure is characterized by its moments.

The following result is a classical result in the theory of probability measures.

Theorem 2.1 ([21, Theorem 11.4.1]) Let \(\mu \in \mathcal{P}(\mathbb{R})\) and \((\mu_n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R})^\mathbb{N}\) be probability measures such that

1. For every \(p \in \mathbb{R}[x]\), \(\lim_{n \to \infty} \int_{\mathbb{R}} p\, d\mu_n = \int_{\mathbb{R}} p\, d\mu\).
2. \(\mu\) is characterized by its moments.

Then \((\mu_n)_{n \in \mathbb{N}}\) converges weakly to \(\mu\).

3 Labelled Connected Marked Graphs

This section develops the notion of convergence of labelled connected marked graphs (denoted lcmgs) \((X, L, v)\), where \(X\) is a directed graph, \(L\) is a labelling map which labels edges with complex numbers, and \(v\) is a distinguished vertex of \(X\) (details below). This is a slight extension of the convergence of connected marked graphs as described in [7]. The set of all isomorphism classes of lcmgs forms a metric space, which will be denoted by \(\mathcal{LCMG}\). For each lcmg \((X, L, v)\), one obtains an operator \(M_{(X, L, v)}\) that describes a random walk on the lcmg; \(M_{(X, L, v)}\) will be called the Markov-type operator associated to \((X, L, v)\). \(M_{(X, L, v)}\) will be associated a probability measure \(\mu_{v, X, L}\), which will be called the Kesten spectral measure of \((X, L)\) at \(v\). These spectral measures generalize the Kesten spectral measures that occur in [9], which are defined for Markov operators of Cayley graphs; see also [12].

The key result of this section is that the map \((X, L, v) \mapsto M_{(X, L, v)} \mapsto \mu_{v, X, L}\) is weakly continuous with respect to convergence of lcmgs.
A labelled connected marked graph (lcmg) is a connected directed graph \( X = (V(X), E(X)) \) with a distinguished (or marked) vertex \( v \in V(X) \), such that there exist a natural number \( d \) and a real \( b > 0 \) such that

1. \( V(X) \) and \( E(X) \) are both countable.
2. The degree of each vertex is less than \( d \).
3. Edges of \( X \) carry labels in \( \mathbb{C} \), which are determined by a map \( L : E(X) \rightarrow \mathbb{C} \). Edges are allowed to have label 0 \( \in \mathbb{C} \). For each edge \( e \), the label satisfies \( |L(e)| < b \).
4. At most, one self-loop is allowed at each vertex.
5. For any 2 vertices \( v_1, v_2 \in V(X) \), there are no multiple directed edges from \( v_1 \) to \( v_2 \); a directed edge \( v_1 \rightarrow v_2 \) will be unique and will be denoted by \((v_1, v_2)\).

By ignoring the \( \mathbb{C} \)-labels on edges and taking the underlying graph, there is a metric \( d_X \) on \( V(X) \), where \( d_X(v_1, v_2) \) is given by the minimum length of a path in \( X \) from \( v_1 \) to \( v_2 \), where each edge is assigned length 1.

Given lcmg’s \((X_1, L_1, v_1)\) and \((X_2, L_2, v_2)\), say that \((X_1, L_1, v_1)\) is lcmg isomorphic to \((X_2, L_2, v_2)\) if there exists a bijection \( f : V(X_1) \rightarrow V(X_2) \) which is simultaneously

1. Basepoint-preserving: \( f(v_1) = v_2 \).
2. Edge-preserving: \((u, v) \in E(X_1) \) if and only if \((f(u), f(v)) \in E(X_2)\).
3. Label-preserving: \( L(f(u), f(v)) = L(u, v) \).

Let \( \mathcal{LCMG} \) denote the set of all isomorphism classes of lcmg’s. The expression \( [(X, L, v)] \) will refer to the isomorphism class of an lcmg \((X, L, v)\).

For every lcmg \((X, L, v)\), let \( B((X, L, v), r) \) be the ball of radius \( r \geq 0 \) in \( X \) centred at \( v \); this is precisely the full subgraph of \((X, L, v)\) with set of vertices \( \{u \in V(X) : d_X(u, v) \leq r\} \) and with labelling function given by restriction of \( L \). Then, define \( D : \mathcal{LCMG} \times \mathcal{LCMG} \rightarrow \mathbb{R}_{\geq 0} \) by the following, where \([(X_1, L_1, v_1)]\) and \([(X_2, L_2, v_2)]\) denote the isomorphism classes of lcmg’s \((X_1, L_1, v_1)\) and \((X_2, L_2, v_2)\)

\[
D([(X_1, L_1, v_1)], [(X_2, L_2, v_2)]) = \inf S,
\]

where \( S \) denotes the set

\[
\left\{ \frac{1}{n+1} : n \in \mathbb{N} \text{ s.th. } B((X_1, L_1, v_1), n) \text{ and } B((X_2, L_2, v_2), n) \text{ are lcmg iso.} \right\}.
\]

It is then easy to see that \( D \) is a metric on \( \mathcal{LCMG} \). Thus, one says that a sequence of (isomorphism classes of) lcmg’s \( \{[(X_n, L_n, v_n)]\} \) converges to a class \([(X, L, v)]\) if

\[
\lim_{n \to \infty} D([(X_n, L_n, v_n)], [(X, L, v)]) = 0.
\]

The limit class \([(X, L, v)]\) is unique.

The following lemma follows immediately from the definitions.

**Lemma 3.1** Let \( \{[(X_n, L_n, v_n)]\} \) be a sequence of lcmg’s and let \((X, L, v)\) be an lcmg. The following are equivalent:

1. \( \lim_{n \to \infty} D([(X_n, L_n, v_n)], [(X, L, v)]) = 0 \).
2. For every \( r > 0 \), there exists an \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( B((X_n, L_n, v_n), r) \) is lcmg isomorphic to \( B((X, L, v), r) \).

It is also possible to define an involution \( * \) on \( \mathcal{LCMG} \). We define it first on representative lcmg’s and this will obviously give a well-defined operation on isomorphism
classes. For any lmg \((X, L, v)\), define \((X, L, v)^* := (X^*, L^*, v)\) to be a graph \(X^*\) with \(V(X^*) = V(X)\), with the same basepoint \(v\), and for each edge \((v_1, v_2) \in E(X)\) with label \(L((v_1, v_2))\), replace it with an edge \((v_2, v_1)\) with label 
\[L^*((v_2, v_1)) = L((v_1, v_2)).\]

It is clear that \((X^*, L^*, v)\) is also a lmg. \((X, L, v)\) is said to be self-involutive if 
\[(X, L, v)^* = (X, L, v).\]

For any lmg \((X, L, v)\), define the Markov-type operator associated to \((X, L, v)\), 
\[M_{(X, L, v)} : l^2(V(X)) \to l^2(V(X)),\]

as follows 
\[M_{(X, L, v)}(f)(u) = \sum_{(w, u) \in E(X)} L((w, u)) \cdot f(w)\] for all \(f \in l^2(V(X)), u \in V(X)\).

It follows easily from the assumption of a uniform bound on the degrees of vertices and the assumption of a uniform bound on the labels that \(M_{(X, L, v)}\) is a well-defined linear operator 
\[l^2(V(X)) \to l^2(V(X)),\]

and moreover is a bounded linear operator.\(^1\)

The inner product on \(l^2(V(X))\) is given by 
\[
\langle f_1, f_2 \rangle = \sum_{x \in V(X)} f_1(x)\overline{f_2(x)}.
\]

Taking the standard orthonormal basis \(\{\delta_x \in l^2(V(X)) : x \in V(X)\}\), where \(\delta_x(u) = 1\) when \(u = x\) and is 0 otherwise. One has 
\[
\langle M_{(X, L, v)}(\delta_x), \delta_y \rangle = \left\{ \begin{array}{ll} L((x, y)) & \text{if } (x, y) \in E(X), \\
0 & \text{otherwise.} \end{array} \right.
\]

**Proposition 3.1** If an lmg \((X, L, v)\) is self-involutive, then \(M_{(X, L, v)}\) is a self-adjoint bounded linear operator.

**Proof** We have already observed that \(M_{(X, L, v)}\) is a bounded linear operator. So it is enough to check the equation \(\{w_1, M_{(X, L, v)}(w_2)\} = \{M_{(X, L, v)}(w_1), w_2\}\) holds on the basis \(\{\delta_x\}\). Letting \(x, y \in V(X)\) one has 
\[
\langle \delta_x, M_{(X, L, v)}(\delta_y) \rangle = \langle M_{(X, L, v)}(\delta_y), \delta_x \rangle = L((y, x)) = L((x, y)) = \langle M_{(X, L, v)}(\delta_x), \delta_y \rangle.
\]

Here, we have used the assumption that the lmg is self-involutory. \(\square\)

Now, we introduce the spectral measures associated to this operator following \[9\]. Consider a self-involutive lmg \((X, L, v)\), with associated Markov-type operator \(M = M_{(X, L, v)}\). Let \(\{E^M_B\}_B\) be the family of spectral projections determined by \(M\), where \(|M| < \infty\) is the operator norm of \(M\), and \(B\) ranges over Borel subsets of \([-|M|, |M|]\). For \(x, y \in V(X)\), one obtains a collection of measures \(\mu^{X,L}_{x,y}\) on the Borel \(\sigma\)-algebra of \([-|M|, |M|]\]
\[
\mu^{X,L}_{x,y}(B) = E^M_B(\delta_x), \delta_y).
\]
The probability measures 
\[
\{\mu^X_x = \mu^{X,L}_{x,y} : x \in V(X)\}
\]
will be called the Kesten spectral measures of \((X, L)\); individually, \(\mu^X_x\) will be called the Kesten spectral measure of \((X, L)\) at \(x\). For the case of the Markov type operator \(M\), denote

\[^1\text{We would like to thank the referee for pointing out these assumptions are sufficient for this.}\]
The moments of the Kesten spectral measures are given as follows, for a vertex \( a \in V(X) \)
\[
(\mu^{X,L}_a)^{(k)} = \int_{-\infty}^\infty \lambda^k d\mu^{X,L}_a(\lambda) = \int_{-\infty}^\infty \lambda^k d\left( E^M_\lambda(\delta_a), \delta_a \right) = \left( M^k(\delta_a), \delta_a \right) .
\]
The final equality here arises from the spectral representation of the operator \( M \) respecting the inner product. For example, see the identity stated as (2*) of the statement of the spectral theorem for bounded self-adjoint operators appearing as [14, Theorem 9.9-1].

This leads to the following result that is a minor adaptation to this setting (where the edges may have labels) of [7, Lemma 4].

**Theorem 3.1** Let \( \{(X_n, L_n, v_n)\}_{n \in \mathbb{N}} \) be a sequence of lcmg’s such that the corresponding sequence of isomorphism classes converges to a class \([ (X, L, v) ] \). Then, the sequence of Kesten spectral measures \( \{\mu^{X_n,L_n}_{v_n}\}_{n \in \mathbb{N}} \) converges weakly to \( \{\mu^{X,L}_v\} \).

**Proof** By the computation above, the \( k \)th moment of \( \mu^{X,L}_v \) is given by the sum, over all possible directed walks \( \omega \) in \( X \), starting at \( v \) and of metric length \( d_X(x, v) = k \), of the product of the \( \mathbb{C} \)-labels of the edges along each such \( \omega \); by a walk, it is meant that any directed edge may appear more than once in \( \omega \). But since the sequence \( \{ [(X_n, L_n, v_n)] \} \) converges to \( [(X, L, v)] \), then for \( n \) sufficiently large, \( B((X_n, L_n, v_n), k) \) is lcmg isomorphic to \( B((X, L, v), k) \), so that \( (\mu^{X_n,L_n}_{v_n})^{(k)} = \left( \mu^{X,L}_v \right)^{(k)} \). Because the measure \( \mu^{X,L}_v \) is compactly supported, it is characterized by its moments, and hence the convergence of the moments of the measures implies the weak convergence of the measures in this case, by Theorem 2.1. \( \square \)

**4 Cayley Lcmg’s and the Proof of Theorem 1.1**

In this section, the concept of a Cayley lcmg is introduced; this is an adaptation of the concept of a Cayley graph which will correspond to the action of an arbitrary self-adjoint element \( z \in \mathbb{C}[G] \) on \( l^2(G) \) for a finitely generated group \( G \). One may express \( z \) in the form
\[
z = \sum_{s \in S} \lambda_s s,
\]
where \( S \subseteq G \) is a finite symmetric subset (i.e., \( s \in S \implies s^{-1} \in S \)) which we will assume generates \( G \). The coefficients \( \lambda_s \) are allowed to take value 0 for some of the generators \( s \). Note that because \( z = z^* \), \( \lambda_s = \lambda_{s^{-1}} \) for all \( s \in S \); it may also happen that the identity element \( 1 \in G \) belongs to \( S \).

Now, we will build an lcmg \( \Gamma_{(z,S)} \) from the data of the element \( z \in \mathbb{C}[G] \) and the generating set \( S \).

- The vertices \( V(\Gamma_{(z,S)}) \) are the elements of the group \( G \).
- The set of directed edges \( E(\Gamma_{(z,S)}) \) is obtained by taking a directed edge \((g, gs)\) for every element of the group \( g \in G \) and every element \( s \in S \).
- The labelling function is given by \( L((g, gs)) = \lambda_s \) for \( g \in G \) and \( s \in S \).
- The basepoint is chosen to be \( 1 \in G \).
This will be called the \textit{Cayley lcmg} of \( z \) \textit{with respect to} \( S \), denoted \( \Gamma(z,S) \). (For simplicity in this section, we will use the notation \( \Gamma(z,S) \) to refer to the underlying graph as well as the full lcmg \( (\Gamma(z,S), L, 1) \).) Note that the degrees of vertices are bounded (in fact constant in this case) and the labels are also bounded, as is required in the definition of an lcmg. Note that as \( z \) is a self-adjoint element of the group ring, it follows that this is a self-involutive lcmg. Also, note that if \( 1 \in S \), then at each vertex there is a single self-loop labelled \( \lambda_1 \).

Now, suppose that \( G \) is as described in the Setting: \( G \) is countable residually finite, and \( K_n \) denotes a nested sequence of normal finite index subgroups with intersection the identity element. Then, \( \pi_n(S) = \{sK_n : s \in S\} \) is a finite symmetric set of generators of \( G/K_n \). Note that elements of \( \pi_n(S) \) are taken without repetition, i.e., \( \pi_n(S) \) is the collection of pairwise distinct cosets of \( G/K_n \) obtained from the action of \( \pi_n \) on \( S \). Then, \( z_n = \pi_n(z) \) can be expressed in the form

\[
    z_n = \sum_{t \in \pi_n(S)} \lambda_t t.
\]

In a similar way, for each \( n \in \mathbb{N} \), one obtains the Cayley lcmg \( \Gamma(z_n,\pi_n(S)) \) of \( z_n \) with respect to \( \pi_n(S) \), where the basepoint is taken to be the identity element \( 1_n \in G/K_n \).

The following proposition is modelled significantly on [9, Proposition 7.1], with some adaptations for this setting where there are labels present. For the purposes of this elementary exposition, we will give details of the proof here.

\textbf{Proposition 4.1} \textit{The sequence} \( \{[\Gamma(z_n,\pi_n(S))]\}_{n \in \mathbb{N}} \) \textit{of isomorphism classes of lcmg’s converges to the class} \( [\Gamma(z,S)] \).

\textbf{Proof} We will apply Lemma 3.1. Let \( r > 0 \). The assumption \( \bigcap_{n \in \mathbb{N}} K_n = \{1\} \) implies that there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
    V(B(\Gamma(z,S), 2r + 1)) \cap K_n = \{1\}.
\]

We will show that the map

\[
    \alpha_n : V(B(\Gamma(z,S), r)) \to V(B(\Gamma(z_n,\pi_n(S)), r)),
\]

given by \( \alpha_n : x \mapsto \pi_n(x) \) induces an lcmg isomorphism for all \( n \geq N \). According to Lemma 3.1 that proves this proposition.

Fix some \( n \geq N \). First, we make two preliminary observations. Since \( V(B(\Gamma(z,S), 2r + 1)) \cap K_n = \{1\} \) then

1. If \( g \in K_n \) is such that \( g \neq 1 \), then any word \( w \) in the elements of \( S \) that represents \( g \) (recall we are assuming that \( S \) generates \( G \)) must be of length greater than \( 2r + 1 \). Equivalently, any directed path in \( \Gamma(z,S) \) starting at \( 1 \) and ending at \( g \in K_n \) must be of length greater than \( 2r + 1 \).

2. The generating set \( \pi_n(S) \) for \( G/K_n \) contains \( |S| \) pairwise distinct cosets (if \( 1 \in S \), then the identity coset \( 1K_n \) is also included in this collection of cosets).

Now, we carefully check that \( \alpha_n \) is an lcmg isomorphism.

\textbf{The map} \( \alpha_n \) \textbf{is a bijection} That \( \alpha_n \) is surjective holds since \( \pi_n \) is a projection. To prove injectivity, consider vertices \( x, y \in B(\Gamma(z,S), r) \). Then \( x \) and \( y \) can be expressed as words in the elements of \( S \), each of length at most \( r \); each of these words would correspond to a path of length at most \( r \) in \( B(\Gamma(z,S), r) \). If one has \( \pi_n(x) = \pi_n(y) \), then \( x^{-1}y \in K_n \) can be expressed as a word (in the elements of \( S \)) of length at most \( 2r \). This means that there is a...
directed path in $\Gamma_{(z,S)}$ starting at 1 and ending at the element $x^{-1}y$ of $K_n$ of length $2r$. By the choice of $N$, one must have $x^{-1}y = 1$, i.e., $x = y$.

The map $\alpha_n$ is basepoint-preserving This follows from $\pi_n(1) = 1_n$.

The map $\alpha_n$ is edge-preserving Note that in $\Gamma_{(z,S)}$ and $\Gamma_{(zn,\pi_n(S))}$, two vertices $x, y \in G$ (respectively $xK_n$ and $yK_n$) are adjacent by means of a labelled directed edge $x \xrightarrow{\lambda} y$ (respectively $xK_n \xrightarrow{\lambda} yK_n$) if and only if $y = x\lambda$ (respectively $yK_n = xK_n \cdot \lambda K_n = x\lambda K_n$).

Since $\pi_n(xs) = \pi_n(x)\pi_n(s)$, an edge $x \xrightarrow{\lambda} xs$ in $B(\Gamma_{(z,S)}, r)$ gives rise to an edge $xK_n \xrightarrow{\lambda} sK_n$ in $B(\Gamma_{(zn,\pi_n(S))}, r)$. Conversely, suppose vertices $x, y \in B(\Gamma_{(z,S)}, r)$ are such that there is an edge $xK_n \xrightarrow{\lambda} yK_n$ in $\Gamma_{(zn,\pi_n(S))}$. Then, $yK_n = x\lambda K_n \iff y^{-1}xs \in K_n$. Now, express $x$ and $y$ as words of length at most $r$ in the generators $S$. Then, $y^{-1}xs \in K_n$ can be expressed as a word of length at most $2r + 1$ in the generators $S$. This would imply that there is a path of length at most $2r + 1$ in $\Gamma_{(z,S)}$ starting at 1 and ending at an element of $K_n$. By the choice of $N$, one must have that $y^{-1}xs = 1 \iff y = xs$, which shows that there is a labelled directed edge $x \xrightarrow{\lambda} y$ in $\Gamma_{(z,S)}$.

The map $\alpha_n$ is label-preserving This is a direct consequence of the proof that $\alpha_n$ is edge-preserving; each labelled directed edge $x \xrightarrow{\lambda} xs$ in $B(\Gamma_{(z,S)}, r)$ gives rise to one labelled directed edge $xK_n \xrightarrow{\lambda} sK_n$.

One may now prove the main result by just identifying the ingredients of the $L^2$ theory in this Markov operator picture.

**Proof of Theorem 1.1** Let $w \in \mathbb{C}[G]$. By definition the spectral density function of $R_w$ is given by

$$F(R_w)(\lambda) = \text{tr}_{\mathcal{N}(G)}\left( E^{(R_w)^* \circ R_w}_{[0,\lambda^2]} \right) = \text{tr}_{\mathcal{N}(G)}\left( E^{R_{ww^*}}_{[0,\lambda^2]} \right).$$

The second equality follows from the observations that

$$(R_w)^* = R_{w^*} \quad \text{and} \quad Rw_1 \circ Rw_2 = Rw_2w_1$$

for $w_1, w_2 \in \mathbb{C}[G]$.

Therefore, we let $z = ww^*$. For all $n \in \mathbb{N}$, we set $w_n = \pi_n(w) \in \mathbb{C}[G/K_n]$ and $z_n = \pi_n(z) = w_nw_n^*$. Note that both $z$ and $z_n$ are self-adjoint.

Consider the corresponding Markov-type operators $M = M_{\Gamma_{(z,S)}}$ and $M_n = M_{\Gamma_{(zn,\pi_n(S))}}$ associated to the Cayley lcmg’s $\Gamma_{(z,S)}$ and $\Gamma_{(zn,\pi_n(S))}$ respectively, where $n \in \mathbb{N}$.

Observe that under the identifications $V(\Gamma_{(z,S)}) = G$ and $V(\Gamma_{(zn,\pi_n(S))}) = G/K_n$

1. $M = R_z$ and $M_n = R_{z_n}$.
2. $\delta_1 \in l^2(V(\Gamma_{(z,S)}))$ and $\delta_{1_n} \in l^2(V(\Gamma_{(zn,\pi_n(S))}))$ are identified with $1 \in l^2(G)$ and $1_n \in l^2(G/K_n)$ respectively.

By Proposition 4.1, the sequence $\{\Gamma_{(zn,\pi_n(S))}\}_{n \in \mathbb{N}}$ of isomorphism classes of lcmg’s converges to the class $[\Gamma_{(z,S)}]$. Theorem 3.1 then shows that the sequence of Kesten spectral measures $\{\mu_{1_n}^{\Gamma_{(zn,\pi_n(S))}}\}_{n \in \mathbb{N}}$ converges weakly to the Kesten spectral measure $\mu_1^{\Gamma_{(z,S)}}$.

By Proposition 2.1, this implies that for all $\lambda \geq 0$ such that $\mu_1^{\Gamma_{(z,S)}}(\lambda)$ is continuous, the sequence of cumulative distribution functions $\left\{\mu_{1_n}^{\Gamma_{(zn,\pi_n(S))}}(\lambda)\right\}$ converges pointwise to $\mu_1^{\Gamma_{(z,S)}}(\lambda)$. 

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To finish, we identify

\[ \mu^{\Gamma_{1(S),Y}}_1 (\lambda) = \mathbf{EM} \left[ 0, \lambda^2 \right](\delta_1, \delta_1) = \text{tr}_N(G) \left( E_{[0,\lambda^2]}^R \right) = F(R_w)(\lambda). \]

\[ \mu^{\Gamma_{1(\pi_n(S)),Y}}_n (\lambda) = \mathbf{EM}_n \left[ 0, \lambda^2 \right](\delta_1^n, \delta_1^n) = \text{tr}_N(G/K_n) \left( E_{[0,\lambda^2]}^{R_n} \right) = F(R_{w_n})(\lambda). \]

\[ F(R_w)(\lambda) \]

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