A note on characterizations of $G$-normal distribution

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Abstract

In this paper, we show that the $G$-normality of $X$ and $Y$ can be characterized according to the form of $f$ such that the distribution of $\lambda + f(\lambda)Y$ does not depend on $\lambda$, where $Y$ is an independent copy of $X$ and $\lambda$ is in the domain of $f$. Without the condition that $Y$ is identically distributed with $X$, we still have a similar argument.

Keywords: $G$-normal distribution, Characterization.

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1 Introduction

In the classical framework, let $X$ and $Y$ be two independent random vectors and $f$ be a function defined on an interval of $\mathbb{R}$. Nguyen and Sampson ([2]) obtained that the distribution of $X$ and $Y$ can be characterized according to the form of $f$ such that the distribution of the random vector $\lambda X + f(\lambda)Y$ does not depend on $\lambda$, where $\lambda$ takes on values in the domain of $f$. These results complement previously obtained characterizations where $X$ and $Y$ are required to be identically distributed and for one value $\lambda^*$, $\lambda^* + f(\lambda^*)Y$ has the same distribution as $X$ which are discussed in Kagan et al. ([1]). All these results are related to the Marcinkiewicz theorem (see [1]), which simply says that under suitable conditions if $X$ and $Y$ are independent and identically distributed, and $\lambda_1X + \tau_1Y$ and $\lambda_2X + \tau_2Y$ have the same distribution, then $X$ and $Y$ have a normal distribution. Note that, in fact, $\tau_i$, $i = 1,2$, can not be arbitrary constants, but must satisfy $\tau_i = \sqrt{1 - \lambda_i^2}$, $i = 1,2$.

Recently, Peng systemically established a time-consistent fully nonlinear expectation theory (see [3], [4] and [5]).

As a typical and important case, Peng (2006) introduced the $G$-expectation theory (see [6] and the references therein). In the $G$-expectation framework ($G$-framework for short), the notion of independence, identically distributed and $G$-normal distribution were established.

Motivated by their works, we obtain several characterizations of $G$-normal distribution. The paper is organized as follow: In section 2, we recall some notations and results that we will use in this paper. In section 3, we obtain our main results. In section 4, we give some comments.

2 Preliminaries

We present some preliminaries in the theory of sublinear expectation, $G$-normal distribution under $G$-framework. More details can be found in Peng [6].

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Definition 2.1 Let $\Omega$ be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on $\Omega$, namely $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}: \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$;

(b) Constant preservation: $\hat{E}[c] = c$;

(c) Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;

(d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for each $\lambda \geq 0$. $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2 Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{b, \text{Lip}}(\mathbb{R}^n)$, where $C_{b, \text{Lip}}(\mathbb{R}^n)$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^n$.

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}[]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b, \text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\varphi(x, Y)|_{x = X}]$.

Definition 2.4 (G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normally distributed if for each $a, b \geq 0$ we have

$$aX + b\hat{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of $X$, i.e., $\hat{X} \overset{d}{=} X$ and $\hat{X} \perp X$. Here the letter $G$ denotes the function

$$G(A) := \frac{1}{2} \hat{E}[\langle AX, X \rangle]: S_d \to \mathbb{R},$$

where $S_d$ denotes the collection of $d \times d$ symmetric matrices.

Peng [6] showed that $X = (X_1, \cdots, X_d)$ is G-normally distributed if and only if for each $\varphi \in C_{b, \text{Lip}}(\mathbb{R}^d)$, $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following $G$-heat equation:

$$\partial_t u - G(D^2_x u) = 0, \ u(0, x) = \varphi(x).$$

The function $G(\cdot): S_d \to \mathbb{R}$ is a monotonic, sublinear mapping on $S_d$ and $G(A) = \frac{1}{2} \hat{E}[\langle AX, X \rangle] \leq \frac{1}{2} |A|\hat{E}[|X|^2]$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}([\gamma A]),$$

where $S_d^+$ denotes the collection of nonnegative elements in $S_d$. 

2
3 characterizations of $G$-normal distribution

We only consider non-degenerate random variable $X$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, i.e. $\hat{E}[X^2] \geq -\hat{E}[-X^2] \geq 0$. From the definition of $G$-normal distribution, a equivalent characterization of $G$-normal distribution is that, for any $a, b > 0$

$$\frac{a}{\sqrt{a^2 + b^2}}X + \frac{b}{\sqrt{a^2 + b^2}}Y \overset{d}{=} X,$$

where $Y$ is an independent copy of $X$. Denote $\lambda = \frac{a}{\sqrt{a^2 + b^2}}$, then

$$\hat{\lambda}X + \sqrt{1 - \lambda^2}Y \overset{d}{=} X.$$

We are interested in the case that $\sqrt{1 - \lambda^2}$ is replaced by $f(\lambda)$ which is a nonnegative function of $\lambda$. Actually we have the following theorem

**Theorem 3.1** Let $f$ be a nonnegative function defined on some interval of $\mathbb{R}$ and $X$ be a non-degenerate random variable on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, for any $\lambda$ which is in the domain of $f$,

$$\lambda X + f(\lambda)Y \overset{d}{=} X$$

where $Y$ is an independent copy of $X$, then:

(i) $X$ is $G$-normal distributed;

(ii) $f(\lambda) = \sqrt{1 - \lambda^2}$.

**Proof.** Denote $\hat{E}[X] = \overline{\mu}$, $-\hat{E}[-X] = \underline{\mu}$, $\hat{E}[X^2] = \sigma^2$ and $-\hat{E}[-X^2] = \overline{\sigma}^2$, then

$$\hat{E}[\lambda X + f(\lambda)X] = f(\lambda)\overline{\mu} + \lambda'\overline{\mu} - \lambda'\underline{\mu} = \overline{\mu}.$$  \hspace{1cm} (1)

$$-\hat{E}[-\lambda X - f(\lambda)X] = f(\lambda)\underline{\mu} - \lambda'\overline{\mu} + \lambda'\underline{\mu} = \underline{\mu}.$$  \hspace{1cm} (2)

$$\hat{E}[(\lambda X + f(\lambda)X)^2] = \lambda^2\sigma^2 + f(\lambda)^2\sigma^2 + 2f(\lambda)(\overline{\mu}\hat{E}[(\lambda X)^+]) - \underline{\mu}\hat{E}[(\lambda X)^-]) = \sigma^2.$$  \hspace{1cm} (3)

$$-\hat{E}[-(\lambda X + f(\lambda)X)^2] = \lambda^2\overline{\sigma}^2 + f(\lambda)^2\overline{\sigma}^2 + 2f(\lambda)(\underline{\mu}\hat{E}[(\lambda X)^+]) - \overline{\mu}\hat{E}[(\lambda X)^-]) = \overline{\sigma}^2.$$  \hspace{1cm} (4)

Without loss of generalization, we assume that there exist $b > a > 0$ such that the interval $(a, b)$ is a subset of the domain of $f$.

We shall prove that $\overline{\mu} = 0$. Otherwise suppose $\overline{\mu} \neq 0$. Thus, for $\lambda \in (a, b)$, from (1)

$$f(\lambda) = 1 - \lambda.$$

Substitute in (3) and (4),

$$\sigma^2 = \lambda^2(2\sigma^2 - \overline{\mu}\hat{E}[X^+] + \underline{\mu}\hat{E}[X^-]) - 2\lambda(\sigma^2 - \overline{\mu}\hat{E}[X^+] + \underline{\mu}\hat{E}[X^-]) + \sigma^2.$$

$$\overline{\sigma}^2 = \lambda^2(2\overline{\sigma}^2 + \overline{\mu}\hat{E}[X^+] - \underline{\mu}\hat{E}[X^-]) - 2\lambda(\overline{\sigma}^2 + \overline{\mu}\hat{E}[X^+] - \underline{\mu}\hat{E}[X^-]) + \overline{\sigma}^2.$$
This contradiction yields that \( \mu = 0 \). Similarly, \( \mu = 0 \). Then
\[
\sigma^2 = \lambda^2 \sigma^2 + f(\lambda) \sigma^2.
\]

Hence, \( f(\lambda) = \sqrt{1 - \lambda^2} \). Thus \( Y \) is \( G \)-normal distributed. \( \square \)

Moreover we can still have the following theorem without the condition that \( Y \) and \( X \) are identically distributed.

**Theorem 3.2** Let \( X, Y \) be two non-degenerate random variables on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) and \( f \) be a given non-negative function defined on some interval of \( \mathbb{R} \). Assuming that \( Y \) is independent with \( X \), and \( \lambda X + f(\lambda) Y \) is a non-degenerate random variable whose distribution does not depend on \( \lambda \) for all \( \lambda \) in the domain of \( f \), then:

(i) \( f(\lambda) = \sqrt{a - b \lambda^2} \) for some \( a, b > 0 \).

(ii) \( X \) and \( Y \) are \( G \)-normal distributed with \( \sigma_X^2 = b \sigma_Y^2 \), \( \sigma_X^2 = b \sigma_Y^2 \).

**Proof.** Denote \( \hat{y} = \mathbb{E}[X] = \mathbb{E}_X, -\mathbb{E}[-X] = \mathbb{E}_X \) and \( -\mathbb{E}[-X^2] = \sigma_X^2, \mathbb{E}[Y] = \mathbb{E}_Y, -\mathbb{E}[-Y] = \mathbb{E}_Y, -\mathbb{E}[-Y^2] = \mathbb{E}_Y^2 \), we have
\[
\hat{E}[\lambda X + f(\lambda) Y] = f(\lambda) \mathbb{E}_Y + \lambda^+ \mathbb{E}_X + \lambda^- \mathbb{E}_X = \mathbb{n}. \tag{5}
\]
\[
-\hat{E}[-(\lambda X + f(\lambda) Y)] = f(\lambda) \mathbb{E}_Y + \lambda^+ \mathbb{E}_X - \lambda^- \mathbb{E}_X = \mathbb{n}. \tag{6}
\]
\[
\hat{E}[(\lambda X + f(\lambda) Y)^2] = \lambda^2 \sigma_X^2 + f(\lambda)^2 \sigma_Y^2 + 2 f(\lambda) (\mathbb{E}_Y \hat{E}[(\lambda X)^+] - \mathbb{E}_Y \hat{E}[(\lambda X)^-]) = \sigma^2. \tag{7}
\]
\[
-\hat{E}[-(\lambda X + f(\lambda) Y)^2] = \lambda^2 \sigma_X^2 + f(\lambda)^2 \sigma_Y^2 + 2 f(\lambda) (\mathbb{E}_Y \hat{E}[(\lambda X)^+] - \mathbb{E}_Y \hat{E}[(\lambda X)^-]) = \sigma^2. \tag{8}
\]
Since the distribution of \( \lambda X + f(\lambda) Y \) does not depend on \( \lambda, \mathbb{n}, \mathbb{h}, \sigma^2 \) and \( \sigma^2 \) do not depend on \( \lambda \).

Without loss of generalization, we assume that there exist \( b > a > 0 \) such that the interval \((a, b)\) is a subset of the domain of \( f \).

We shall prove that \( \rho_Y = \rho_X = 0 \). Otherwise suppose \( \rho_Y \neq 0 \) (and then \( \rho_X \neq 0 \) for \( \lambda \in (a, b) \) from (7)). Thus
\[
f(\lambda) = \frac{\mathbb{h}}{\rho_Y} - \lambda \frac{\mathbb{h}}{\rho_X}. \tag{9}
\]
Substitute in (7) and (8),
\[
\sigma^2 = \lambda^2 (\sigma_X^2 + \sigma_Y^2 \frac{\rho_X}{\rho_Y}) - \rho_X \hat{E}[X^+] + \frac{\rho_X}{\rho_Y} \mathbb{E}_X \hat{E}[X^-] - 2 \lambda \mathbb{h} (\sigma_Y^2 \frac{\rho_X}{\rho_Y} - \hat{E}[X^+] + \frac{\rho_X}{\rho_Y} \hat{E}[X^-]) + \frac{h^2}{\rho_Y^2} \sigma_Y^2.
\]
\[
\sigma^2 = \lambda^2 (\sigma_X^2 + \sigma_Y^2 \frac{\rho_X}{\rho_Y}) + \rho_X \hat{E}[X^-] - \frac{\rho_X}{\rho_Y} \mathbb{E}_X \hat{E}[X^+] - 2 \lambda \mathbb{h} (\sigma_Y^2 \frac{\rho_X}{\rho_Y} + \hat{E}[X^-] - \frac{\rho_X}{\rho_Y} \hat{E}[X^+] + \frac{h^2}{\rho_Y^2} \sigma_Y^2). \tag{10}
\]
depend on $\lambda$.

This contraction yields that $\overline{\mu}_X = \overline{\mu}_Y = 0$. Similarly, $\mu_X = \mu_Y = 0$. Hence $\overline{h} = h = 0$ and

$\overline{\sigma}^2 = \lambda^2 \overline{\sigma}_X^2 + f(\lambda) \overline{\sigma}_Y^2$.

$\sigma^2 = \lambda^2 \sigma_X^2 + f(\lambda) \sigma_Y^2$.

We have $f(\lambda) = \sqrt{\frac{a}{\sigma_Y^2} - \frac{b}{\sigma_Y^2} \lambda^2} = \sqrt{a - b \lambda^2}$ where $a = \overline{\sigma}_X^2$, $b = \overline{\sigma}_Y^2$ and $\sigma_X^2 = b \sigma_Y^2$, $\sigma_X^2 = b \sigma_Y^2$.

The domain of $f(\lambda)$ is the interval $[-\sqrt{\frac{b}{a}}, \sqrt{\frac{b}{a}}]$.

For $\lambda = 0$ or $\lambda = \sqrt{\frac{b}{a}}$, the two random variables $\sqrt{a} Y$ and $\sqrt{b} X$ are identically distributed according to

$$\lambda (\sqrt{\frac{b}{a}} X + \sqrt{\frac{a}{b}} Y)$$

By Theorem 3.1, $\sqrt{\frac{b}{a}} X$ and $\sqrt{\frac{a}{b}} Y$ are $G$-normal distributed. Hence $X$ and $Y$ are $G$-normal distributed. □

4 Comments

Note that the distribution uncertainty of a $G$-normal distribution is not just the parameter of the classical normal distributions, it is difficult to obtain characterization results of $G$-normal distributions. Let $g_\lambda(u, v) = \lambda u + f(\lambda)v$ be a family of functions from $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. With the nondependence of the distribution of $g_\lambda(X, Y)$ upon $\lambda$, our results concern which possible forms of $g_\lambda(u, v)$ provide $G$-normality of $X$ and $Y$. In conclusion, our results complement the characterizations of $G$-normal distribution.

Acknowledgment

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