Solar system tests of the equivalence principle and constraints on higher-dimensional gravity

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In most studies of equivalence principle violation by solar system bodies, it is assumed that the ratio of gravitational to inertial mass for a given body deviates from unity by a parameter \( \Delta \) which is proportional to its gravitational self-energy. Here we inquire what experimental constraints can be set on \( \Delta \) for various solar system objects when this assumption is relaxed. Extending an analysis originally due to Nordtvedt, we obtain upper limits on linearly independent combinations of \( \Delta \) for two or more bodies from Kepler’s third law, the position of Lagrange libration points, and the phenomenon of orbital polarization. Combining our results, we extract numerical upper bounds on \( \Delta \) for the Sun, Moon, Earth and Jupiter, using observational data on their orbits as well as those of the Trojan asteroids. These are applied as a test case to the theory of higher-dimensional (Kaluza-Klein) gravity. The results are three to six orders of magnitude stronger than previous constraints on the theory, confirming earlier suggestions that extra dimensions play a negligible role in solar system dynamics and reinforcing the value of equivalence principle tests as a probe of nonstandard gravitational theories.

I. INTRODUCTION

We investigate the consequences of possible violations of the equivalence principle (EP) for solar system bodies whose ratio of gravitational mass \( m_g \) to inertial mass \( m_i \) is given by

\[
m_g/m_i = 1 + \Delta .
\]

The study of this problem has a long history stretching back to Newton \([\text{[1]}]\). In modern times it has been most closely associated with Nordtvedt \([\text{[2]}]\), who investigated the possibility that \( \Delta \) is proportional to the body’s relative gravitational self-energy \( U \)

\[
\Delta = \eta U ,
\]

\[
U = -\frac{G}{c^2} \int \rho(x) \rho(x') \frac{d^3x d^3x'}{|x - x'|} / \int \rho(x) d^3x ,
\]

where \( \eta \) is a universal constant made up of parametrized post-Newtonian (PPN) parameters, \( \eta \approx 4\beta - \gamma - 3 - 10\zeta_3/3 - \alpha_1 + 2\alpha_2/3 - 2\zeta_1/3 - \zeta_2/3 \). In the standard PPN gauge, \( \gamma \) is related to the amount of space curvature per unit mass, \( \beta \) characterizes the nonlinearity of the theory, \( \xi \) and \( \alpha_1 \) refer to possible preferred-location and preferred-frame effects, and the \( \zeta_k \) allow for possible violations of momentum conservation (see \([\text{[3]}]\) for discussion). One has \( \eta = 0 \) in standard general relativity, where \( \gamma = \beta = 1 \). In 4D scalar-tensor theories, by contrast, \( \gamma = (1 + \omega)/(2 + \omega) \) and \( \beta = 1 + \omega'/(2(2 + \omega)(3 + 2\omega)^2) \), where \( \omega = \omega(\phi) \) is the generalized Brans-Dicke parameter and \( \omega' = d\omega/d\phi \).

The relative gravitational self-energy \( U \) can be calculated for most objects in the solar system, subject to uncertainties in their mass density profiles. Thus, for example, \( U_S \approx -10^{-5} \) for the Sun, while Jupiter has \( U_J \approx -10^{-9} \). More precise estimates are available for the Earth and Moon, \( U_E = -4.64 \times 10^{-10} \) and \( U_M = -1.9 \times 10^{-11} \) respectively \([\text{[4]}]\). With \( U \) known, constraints on EP violation (in theories of this kind) take the form of upper limits on the PPN parameter \( \eta \). The latest such bounds (from lunar laser ranging, or LLR) are of order \( |\eta| \leq 10^{-3} \) \([\text{[5]}]\).

There exist, however, theories of gravity in which EP violations are not necessarily related to gravitational self-energy. In such theories, analysis of solar system data can lead only to upper limits on \( \Delta \) for the individual bodies involved. These limits will in general differ from object to object. They will, however, typically be orders of magnitude stronger than the above-mentioned experimental bound on \( \eta \) (since they are not diluted by the small factor \( U \)). In what follows, we obtain numerical constraints on \( \Delta \) for the Sun, Moon, Earth and Jupiter, and apply these as an example to higher-dimensional (Kaluza-Klein) gravity, in which EP violations are related to the curvature of the extra part of the spacetime manifold.

II. EQUIVALENCE PRINCIPLE VIOLATIONS IN KALUZA-KLEIN GRAVITY

Theories of gravity in more than four dimensions are calculated to be older than general relativity \([\text{[6]}]\), and most often associated with the names of Kaluza \([\text{[7]}]\) and Klein \([\text{[8]}]\) (for recent reviews, see \([\text{[9]}]\)). The extra dimensions have traditionally been assumed to be compact, in order to explain their nonappearance in low-energy physics. The past few years, however, have witnessed an explosion of new interest in non-compactified theories of higher-dimensional gravity \([\text{[10]}]\). In such theories the dimensionality of spacetime may in principle manifest itself at
exponentially accessible energies. We examine here the theory of Kaluza-Klein gravity \cite{10}, and focus on the prototypical five-dimensional (5D) case, although the extension to higher dimensions is straightforward in principle.

In 5D Kaluza-Klein gravity, objects such as stars and planets are modelled by a static, spherically-symmetric analog of the 4D Schwarzschild solution known as the soliton metric. This may be written (following \cite{21}), but switching to nonisotropic form, and defining \( a \equiv 1/\alpha, b \equiv \beta/\alpha \) and \( M_\ast \equiv 2m \)
\[
dS^2 = A^a dt^2 - A^{-a-b} dr^2 - A^{1-a-b} \times \nonumber \\
\quad r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - A^b dy^2 ,
\]
where \( y \) is the fifth coordinate, \( A(r) \equiv 1 - 2M_\ast/r \), \( M_\ast \) is a parameter related to the mass of the object at the center of the geometry, and the constants \( a, b \) satisfy a consistency relation \( a^2 + ab + b^2 = 1 \), or
\[
a = -b/2 \pm (1 - 3b^2/4)^{1/2} ,
\]
which follows from Einstein's field equations. Equation \( \text{(2.3)} \) reduces to the 4D Schwarzschild solution on hypersurfaces \( y = \text{const} \) as \( b \rightarrow 0 \) and \( a \rightarrow +1 \).

Solitons differ in interesting ways from 4D black holes, as discussed by many authors \cite{16,17,21,23}. Of most relevance for our purposes is the fact that a soliton's gravitational mass (as identified from the asymptotic behaviour of \( g_\infty \)) is given by \( m_g = aM_\ast \), while its inertial mass (as obtained using the Landau-Lifshitz energy-momentum pseudotensor) turns out to be \( m_i = (a+b/2)M_\ast \). Using equations (1.1) and (2.2), it follows that
\[
\Delta = 1/2 \pm 3b^2/4 \right)^{-1/2} .
\]

It is important to recognize that \( b \) is not a universal constant like \( \eta \), but depends in principle on local physics \cite{23} and may vary from soliton to soliton. (Birkhoff's theorem in its usual form does not hold in higher-dimensional general relativity \cite{24}.) To constrain the theory, one therefore hopes to apply as many tests as possible to a given astrophysical system.

The classical tests of general relativity have been worked out for the soliton metric. Data from long-baseline radio interferometry, ranging to the Mars Viking lander, and the perihelion precession of Mercury imply that \( |b| \leq 0.07 \) in the solar system \cite{25,26}. In what follows, therefore, we keep only terms of second order or lower in \( b \). We also drop the negative roots in equation \( \text{(2.2)} \) [ie, the positive roots in equation \( \text{(2.3)} \)], in order to eliminate the possibility of negative gravitational and/or inertial mass, and to ensure that the 4D Schwarzschild solution is recovered (for \( y = \text{const} \)) as \( b \rightarrow 0 \) \cite{10}. This leaves us with
\[
\Delta = -b/2 ,
\]
which may be compared with equation \( \text{(1.2)} \) for PPN-type theories. In Kaluza-Klein gravity, experimental constraints on EP violation translate directly into upper bounds on the metric parameter \( b \), which characterizes the departure from flatness of the fifth dimension in the vicinity of the central mass.

III. THE MODIFIED THIRD LAW OF KEPLER

Let us now proceed to see what experimental constraints can be placed on the parameter \( \Delta \) for solar system bodies. Given any inverse-square law central force \( F = -\kappa/r^2 \), it may be shown that the two-body problem admits as solutions elliptical orbits satisfying
\[
\tau = 2\pi a^{3/2} \sqrt{\mu/\kappa} ,
\]
where \( \tau \) is the orbital period, \( a \) the semi-major axis, and \( \mu = m_1 m_2 / (m_1 + m_2) \) the reduced mass of the system \cite{27}.

We are concerned in this paper with the case in which \( \kappa = G m_g,1 m_g,2 \), where the gravitational masses \( m_{g,i} \) are related to the inertial ones \( m_{i,k} \) by equation \( \text{(1.1)} \). Re-arranging equation \( \text{(2.2)} \), one then obtains the following modified form of Kepler's third law
\[
G(m_1 + m_2 + m_2 \Delta_1 + m_1 \Delta_2) = \omega^2 a^3 ,
\]
where \( \omega = 2\pi/\tau \) is the mean orbital angular frequency, and \( m_k \) refers (here and elsewhere in the remainder of this paper) to the gravitational mass of the \( k \)th body.

Equation \( \text{(3.2)} \) was first obtained by Nordtvedt using a somewhat different argument \cite{3}.

If all bodies have the same value of \( \Delta \), we will not see violations of the EP. The common part of \( \Delta_1 \) and \( \Delta_2 \) can be absorbed into a rescaled gravitational constant. One way to see this explicitly is to rewrite equation \( \text{(3.2)} \) in the form
\[
G(1 + \Delta_1)(m_1 + m_2) + Gm_1(\Delta_2 - \Delta_1) = \omega^2 a^3 .
\]

There are two modifications to Kepler's third law here: a rescaling of the value of \( G \) in the first term, and a completely new second term, which depends only on the difference \( \Delta_2 - \Delta_1 \). This latter term is a clear manifestation of EP violation in the system.

Nevertheless, we can put experimental constraints on the values of \( \Delta_1 \) and \( \Delta_2 \), even in cases where they are equal. To see this, divide equation \( \text{(3.3)} \) through by \( Gm_1 \) and rearrange to obtain
\[
\frac{\omega^2 a^3}{Gm_1} - \left( 1 + \frac{m_2}{m_1} \right) = (\Delta_2 - \Delta_1) + \Delta_1 \left( 1 + \frac{m_2}{m_1} \right) .
\]

To within experimental uncertainty, of course, the left-hand side of this equation vanishes for planetary two-body systems; this is a statement of Kepler's third law. (More accurately, we may say that in standard theory,
where $\Delta_1$ and $\Delta_2$ are assumed to vanish, a statistical best-fit value is chosen for $Gm_1$ in such a way as to force
the left-hand side as close to zero as possible for all sys-
tems observed.) Even when $(\Delta_2 - \Delta_1) = 0$, therefore,
observation imposes an upper bound on the possible size of
the last term on the right-hand side. This reflects the fact
that we have excellent experimental data, not only
on possible violations of the EP, but also on the value of
$Gm_1$ (as well as $\omega$, $a$ and $m_2/m_1$) for
most solar system
objects. (The situation is less satisfactory for $G$ and $m_1$
considered individually \cite{28}, but these quantities are not
needed here.)

We can use this extra knowledge to set limits on the
sum, rather than the difference of $\Delta_1$ and $\Delta_2$. In combi-
nation with more purely EP-based constraints from the
three-body problem, this will allow us to extract limits
on the individual terms $\Delta_1$ and $\Delta_2$. We therefore collect
terms in $\Delta_1$ and $\Delta_2$ and recast equation (3.4) in a form
more suitable for contact with observation

$$\frac{m_2}{m_1} \Delta_1 + \Delta_2 = \left( \frac{m_s}{m_1} \right) \left( \frac{\omega}{k} \right)^2 \left( \frac{a}{A} \right)^3 - \left( 1 + \frac{m_2}{m_1} \right),$$

where $k$ is the Gaussian constant, $a$ the semi-major
axis of the orbit, and $A$ the length of the astronomical
unit (AU). The quantities $k$ and $A$ are related via
$k^2 A^3 \equiv Gm_s$ to the gravitational mass of the Sun
(which is actually inferred in practice from a statistical
fit to experimental data on $A$). The length of the AU
($A = 1.5 \times 10^{11}$ m) is currently known from planetary
 ranging data to better than $\pm 20$ m \cite{29}.

To within experimental accuracy, the right-hand side
of equation (3.5) vanishes (as remarked above; Kepler’s
third law). Our upper limits on the left-hand side are then
just the sum of the relevant uncertainties in $m_s/m_1$, $\omega$, $a$, $A$ and $m_2/m_1$ ($k$ is an exact quantity).

For the Sun-Jupiter system ($m_1 = m_5$ and $m_2 = m_1$),
$\omega(= 0.53$ rad/yr) has an uncertainty which may be
conservatively estimated at $1$ arcsec/250 yr \cite{30}. The mean
Sun-Jupiter distance ($= 5.2$ AU) is likewise uncertain to
about 200 km, based on post-fit residuals to all observa-
tional data \cite{29} (not individual ranging measurements to
Jupiter, which are good to as little as 3 km \cite{31}). And
the mass ratio $m_5/m_1 (= 1047)$ is known to $\pm 0.0005$ \cite{29}.
Combining these numbers, we find that

$$| \frac{\Delta_1}{1047} + \Delta_2 | \leq 8 \times 10^{-7}.$$  

(3.6)

The bulk of this is due to the uncertainty in $a$, with a small
remainder coming from uncertainty in $\omega$. The un-
certainties in $A$ (that is, in $Gm_s$) and $m_5/m_1$ are unim-
portant by comparison.

For the Earth-Moon system ($m_1 = m_m$, $m_2 = m_m$),
$\omega(= 0.23$ rad/day) has an uncertainty of less than
0.03 arcsec/century \cite{12}. The mass ratio $m_m/(m_m + m_m)(= 3.3 \times 10^5)$ is known to better than $\pm 0.002$ \cite{33},
while $m_e/m_m (= 81)$ is good to $\pm 0.00002$ \cite{14}. The mean
Earth-Moon distance ($= 3.8 \times 10^5$ km) has been estab-
lished to within 0.8 m by LLR data \cite{33}. Let us take the
uncertainty in the length of the semi-major axis $a$ to be
less than twice this figure, or 1.6 m. We then obtain

$$\frac{\Delta_k}{81} + \Delta_m \leq 1 \times 10^{-8}.$$  

(3.7)

In this case, while uncertainty in $a$ is still responsible
for about three-quarters of the total, there are also sub-
stantial contributions from uncertainties in $m_e/m_s$ and
$m_m/m_e$. The decrease in importance of error in $a$ (re-
lative to the Sun-Jupiter case) may be attributed to the
high quality of the LLR data.

### IV. MIGRATION OF THE STABLE LAGRANGE
POINTS AND THE TROJAN ASTEROIDS

We turn next to the effects of EP violation on three-
bodied motion in the solar system, beginning with the
problem in which a test body is located at one of the
stable Lagrange points ($L = L_4$ or $L_5$) in the orbit of
one massive body $M_2$ (a planet or moon, say) around
another ($M_1$). Nordtvedt \cite{3} has shown that the location
of $L$ is sensitive to $\Delta_1 \neq 0$ for the largest of the three
masses. We follow his notation, which is based on the
assumption of circular orbits, but extend the calculation
to cases in which all three objects can in principle display
EP violations of the form (1.1).

The point $L$ is located at points where radial acceler-
tion toward the $M_1$-$M_2$ center of mass ($C$, say) is
precisely balanced by centrifugal acceleration, and where
there is no net acceleration perpendicular to this direc-
tion. The situation is illustrated in Fig. 1, where $r_1$ and
$r_2$ denote the distances from $L$ to $M_1$ and $M_2$ respec-
tively, while $r$, $R_1$ and $R_2$ are the distances of $L$, $M_1$ and
$M_2$ from $C$. It is straightforward to show \cite{3} that

$$(r_1/r_2)^3 = (1 + \Delta_1)/(1 + \Delta_2).$$  

(4.1)

The effect of the $\Delta$-terms, in other words, is to “de-
equilateralize” the triangle made up by $L$, $M_1$ and $M_2$.

It is further possible to solve explicitly for $r_1$ and $r_2$;
one finds that

$$\frac{r_1/(R_1 + R_2)^3}{(R_1 + R_2)/3} = (1 + \Delta_1)/(1 + \Delta_2)$$

$$\frac{r_2/(R_1 + R_2)^3}{(R_1 + R_2)/3} = (1 + \Delta_1)/(1 + \Delta_1).$$  

(4.2)

Denoting the unperturbed values of $r_1$ and $r_2$ by $R$ ($\equiv R_1 + R_2$), and assuming that the $\Delta$-terms are much less
than unity, we see that they lead to small movements of
$L$ toward (or away from) $M_1$ and $M_2$, as follows

$$\delta r_1 \approx -(\Delta_2 - \Delta_1)(R_1 + R_2)/3$$

$$\delta r_2 \approx -(\Delta_1 - \Delta_1)(R_1 + R_2)/3. $$  

(4.3)

In the limits $\Delta_1 \to 0$ and $\Delta_2 \to 0$, these results reduce
to those given in \cite{3}.
Experimentally, one could use these results to obtain constraints on $(\Delta_k - \Delta_1)$ by ranging to a satellite at $L$ and looking for nonzero values of $\delta r_k$ ($k = 1$ or 2). In the case of the Earth-Moon system ($M_1 = \text{Earth, } M_2 = \text{Moon}$) this would require an artificial satellite such as the SOHO spacecraft (which is however located at $L_1$ rather than $L_4$ or $L_5$). In the case of the Sun-Jupiter system ($M_1 = \text{Sun, } M_2 = \text{Jupiter}$) one could instead make use of the larger Trojan asteroids, of which 413 have now been identified [35], and some $2 \times 10^9$ estimated to exist with radii over 1 km [36]. Such a procedure would however be hindered by low signal-to-noise levels and uncertainties involving factors such as surface topography.

An observational quantity offering greater promise is the angular position $\theta$ of the stable Lagrange point relative to the line through $M_1$ and $M_2$ (see Fig. 1). For different values of $r_1$ and $r_2$, one will measure different angles $\delta \theta$. From the law of sines

$$\tan(\theta + \delta \theta) = \frac{(R - \delta r_1) \sin \beta}{R_1 - (R - \delta r_1) \cos \beta}, \quad (4.4)$$

where $\beta$ may be determined from the law of cosines

$$R_1 - (R - \delta r_1) \cos \beta = \frac{(R - \delta r_2)^2 - R^2 + (R - \delta r_1)^2 - 2R(R - \delta r_1) \cos \beta}{\sqrt{3(R_1^2 + R_1 R_2 + R_2^2)}}. \quad (4.5)$$

Differentiating, and linearizing in the small parameters $\delta r_1$ and $\delta r_2$, one can show that

$$\delta \theta = \frac{(2R_1 + R_2) \delta r_1 - (R_1 + 2R_2) \delta r_2}{3\sqrt{3}(R_1^2 + R_1 R_2 + R_2^2)}. \quad (4.6)$$

Inserting equations (4.3) into this expression, we obtain the general result

$$\delta \theta = \frac{(R_1 + R_2)}{3\sqrt{3}(R_1^2 + R_1 R_2 + R_2^2)} \left[(R_1 + 2R_2)(\Delta_1 - \Delta_L) - (2R_1 + R_2)(\Delta_2 - \Delta_L)\right]. \quad (4.7)$$

That this is a pure EP-violating effect can be seen from the fact that only differences of $\Delta$-terms appear. In cases where $\Delta_1 = \Delta_2 = \Delta_1$, there is no migration of the Lagrange point and $\delta \theta = 0$.

Let us now specialize to cases in which $M_1$ is much more massive than $M_2$, so that $R_2 \gg R_1$. (This is not a serious limitation as it describes all situations of interest in the solar system.) We also follow Nordtvedt [3] in assuming that the test object at $L$ differs from the much larger masses $M_1$ and $M_2$ in such a way that $\Delta_L \ll \Delta_1$ and $\Delta_L \ll \Delta_2$. This leaves us with

$$\delta \theta = \frac{1}{3\sqrt{3}}(2\Delta_1 - \Delta_2), \quad (4.8)$$

a result that can be used to set constraints on $(2\Delta_1 - \Delta_2)$ if we have experimental data on any possible angular shift of the Lagrange points from their expected locations 60 degrees behind and ahead of Jupiter.

It happens that we do have good observations of a number of Trojan asteroids, the first of which was discovered over ninety years ago. To locate their mean angular position with sufficient precision for EP tests is however a significant challenge because these objects undergo librations about the Lagrange points, with periods that are typically a good deal longer — of order 150 yr [21] — than the timescale over which they have been observed. Only twelve Trojans (588 Achilles, 911 Agamenon, 1404 Ajax, 1173 Anchises, 1172 Aneas, 1437 Dionysus, 624 Hector, 659 Nestor, 1143 Odysseus, 617 Patroclus, 884 Priamus and 1208 Troilus) have been observed for sixty years or more. So one has to fit orbits to an incomplete arc of observations. Moreover the older observational data will be subject to larger random scatter than more recent measurements. There are also several potential sources of systematic error, including observational selection effects and the nonuniform distribution of asteroids, which may not necessarily cancel themselves out as the libration centers follow Jupiter around the sky.

The problem has nevertheless been seriously tackled in a pair of papers by Orellana and Vucetich [37,38]. (The authors set out to obtain constraints on the PPN parameter $\eta$, but we may relate this simply to $\delta \theta$ by a constant factor, given as $\delta \theta = 1.26\eta$ arcsec in [38].) A formal least-squares statistical fit (including a model for systematic errors) produces the rather surprising result $\delta \theta = -0.21 \pm 0.09$ arcsec, which could perhaps be taken as a potential EP-violating signal. A more conservative procedure, however, leads to a final value of $\delta \theta = -0.18 \pm 0.15$ arcsec, which is consistent with no signal. Based on this latter limit we adopt the upper bound $|\delta \theta| \leq 0.33$ arcsec. This compares reasonably with planetary angular measurements, which are typically accurate to between 0.1 and 1 arcsec [29]. In combination with equation (4.8) this bound leads to the constraint

$$|2\Delta_0 - \Delta_1| \leq 8 \times 10^{-6}. \quad (4.9)$$

This is an order of magnitude weaker than the limit (3.6) on the sum $\Delta_0/1047 + \Delta_1$ for the Sun-Jupiter system from Kepler’s third law, and almost three orders of magnitude weaker than the corresponding limit (3.7) for the Earth-Moon system. Nevertheless equation (4.9) will prove useful in constraining the value of $\Delta_1$.

V. ORBITAL POLARIZATION AND THE LUNAR NORDTVEDT EFFECT

We turn next to the best-known probe of EP violations in the solar system, the polarization of a two-body orbit in the field of a third body, also known variously as the Nordtvedt Effect and the Lunar Eötvös experiment. This was investigated by Nordtvedt in the context of the Earth-Moon-Sun system [33,34] and the Sun-Earth-Jupiter system [35], and has been the subject of numerous studies since [12,13,14,15]. We follow the approach of Will [1], but
extend his notation to general three-body systems. The situation is illustrated in Fig. 2.

Including their mutual attraction (but neglecting any quadrupole or higher moments), the two massive bodies \(M_1\) and \(M_2\) described by equation (1) will fall toward \(M_3\) with different accelerations given by

\[
a_1 = -(1 + \Delta_1)(m_3 x_1/r_1^3 - m_2 x/r^3),
\]

\[
a_2 = -(1 + \Delta_2)(m_3 x_2/r_2^3 + m_1 x/r^3),
\]

where \(x_1\) and \(x_2\) are the 3-vectors connecting \(M_3\) to \(M_1\) and \(M_2\) respectively, and \(x\) is the 3-vector from \(M_1\) to \(M_2\). (The lengths of these vectors are \(r_1\), \(r_2\) and \(r\) respectively.) The relative acceleration \(a = a_2 - a_1\) between \(M_1\) and \(M_2\) may then be written

\[
a = -m_\ast \frac{x}{r^2} + (\Delta_1 - \Delta_2) m_3 \frac{x_1}{r_1^2} + (1 + \Delta_2) m_3 \frac{x_2}{r_2^2},
\]

where \(m_\ast \equiv m_1(1 + \Delta_2) + m_2(1 + \Delta_1)\). The first term is the standard Newtonian acceleration, which gives rise to the circular (unperturbed) orbit of \(M_2\) around \(M_1\), assumed here to be coplanar with \(M_3\). The second, “Nordtvedt term” will be treated as a perturbation of this orbit. (We follow Will [4] in neglecting the third, tidal term at this stage.) The acceleration \(a = d^2 x/dt^2\) and angular momentum \(h = x \times (dx/dt)\) satisfy

\[
d^2 r/dt^2 = x \cdot a/r + h^2/r^3,
\]

\[
dh/dt = x \times a.
\]

Defining \(\delta a \equiv (\Delta_1 - \Delta_2)m_3/r_1^2\), linearizing about the unperturbed orbit (so that \(r \equiv r_0 + \delta r\) and \(h \equiv h_0 + \delta h\)) and solving the resulting pair of differential equations, we find the following periodic oscillation in distance between \(M_1\) and \(M_2\)

\[
\delta r = (\Delta_1 - \Delta_2) A_{EP} \cos D,
\]

where \(A_{EP}\), the EP-violating amplitude, is given by

\[
A_{EP} \equiv \left[ \frac{1 + 2\omega_2/(\omega_2 - \omega_1)}{2(\omega_2/\omega_1) - 1} \right] r_1,
\]

and \(D \equiv (\omega_2 - \omega_1)t\) is the synodic phase. Here \(\omega_1\) is the orbital angular frequency of \(M_1\) about \(M_3\), \(\omega_2\) is the orbital angular frequency of \(M_2\) about \(M_1\), and we have made use of the relations \(m_\ast r_0^2 = \omega_1^2 r_0\), \(m_3 r_1^2 = \omega_2^2 r_1\) and \(h_0/r_0^2 = \omega_0^2\).

Equation (5.4) represents a polarization, or alignment of the orbit of \(M_2\) about \(M_1\) along the direction either toward (if \(\Delta_1 > \Delta_2\)) or away from \(M_3\) (if \(\Delta_1 < \Delta_2\)). For the Earth-Moon system (\(M_1 = \text{Earth}, M_2 = \text{Moon}, M_3 = \text{Sun}\)), we use \(\omega_1 = 1.991 \times 10^{-7}\) rad/s, \(\omega_2 = 2.662 \times 10^{-6}\) rad/s and \(r_1 = 1.496 \times 10^{11}\) m to obtain \(A_{EP} = 1.84 \times 10^{10}\) m. Due to amplification by tidal effects, however, this figure is thought to be a considerable underestimate [8]. More refined calculations, both analytical and numerical, now indicate a final value of \(A_{EP} = 2.9 \times 10^{10}\) m [1].

All these amplitudes are, of course, far larger than any observed anomalous fluctuations in the Earth-Moon distance at this frequency. (The largest known term at the synodic frequency has an amplitude of approximately 100 km, but can be modelled so accurately that it does not contribute to the uncertainty in \(\delta r\).) Indeed, analysis of LLR data now constrains any such fluctuations to be less than 1.3 cm in size [3]. Using the latest above-mentioned value for \(A_{EP}\), we infer an upper bound

\[
|\Delta_E - \Delta_M| \leq 4.4 \times 10^{-13}.
\]

For PPN-type theories, in which \(\Delta_E = \eta U_E\) and \(\Delta_M = \eta U_M\) with the values of \(U_E\) and \(U_M\) given in § 5, this leads to the constraint \(|\eta| \leq 1.0 \times 10^{-3}\), which is currently the strongest limit on this parameter. (It has been weakened only slightly, to \(|\eta| \leq 1.3 \times 10^{-3}\), by a recent study of possible masking by composition-dependent effects [12].)

VI. ORBITAL POLARIZATION AND THE SOLAR NORDTVEDT EFFECT

One can also apply equation (5.4) to the Sun-Earth system with Jupiter as the third mass (\(M_1 = \text{Sun}, M_2 = \text{Earth}, M_3 = \text{Jupiter}\)), as suggested in [8]. This might be dubbed a solar Eötvös experiment since it is the difference in accelerations of the Earth and Sun, and the size of anomalous variations in the solar-terrestrial distance, that are of interest. For this case we have \(\omega_1 = 1.679 \times 10^{-8}\) rad/s, \(\omega_2 = 1.991 \times 10^{-7}\) rad/s and \(r_1 = 7.783 \times 10^{11}\) m, resulting in an uncorrected Nordtvedt factor of \(A_{EP} = 1.09 \times 10^{11}\) m.

A realistic upper limit on anomalous fluctuations in the Sun-Earth distance at the relevant frequency (corresponding to a period of 1.09 yr) might be several times the uncertainty in the length of the AU (±20 m); we take here a value of 100 m. (There are ordinary classical perturbations of amplitude ~ 2000 km at this frequency [5], but these can be compensated for very accurately, since the ratio of Jovian to solar mass is known to better than a part in 10^6 [28].) Using equations (5.4) and (5.5), we then obtain

\[
|\Delta_E - \Delta_M| \leq 9 \times 10^{-10},
\]

which is some four orders of magnitude stronger than the constraint on \(\Delta_E\) and \(\Delta_M\) obtained from the Trojan asteroids in equation (4.3). This is a reflection of two things: the uncertainties discussed in § 5 in connection with the motion of these bodies, and the fact that our measurements of the distance to the Sun, like that to the Moon, rest ultimately on the great precision of ranging data (in this case, from the Viking lander rather than the Apollo retroreflectors).
To complete the set of constraints, we must finally apply the modified Kepler’s third law, equation (3.3), to our last pair of bodies. For the Sun-Earth system \((m_1 = m_\odot\) and \(m_2 = m_\oplus\)), uncertainty in \(\omega (= 2\pi\ \text{rad/yr})\) will be insignificant. For definiteness one could perhaps use the total drift error in the Earth’s inertial mean longitude, which is 0.003 arcsec/century \([29]\). As in the Earth-Moon case, let us take the uncertainty in semi-major axis \(a\) (here 1 AU) to be 40 m, or twice that in the length of the AU itself. Uncertainty in the mass ratio \(m_\oplus/m_\odot(= 1/334,000)\) may be obtained from the limits 
\[
\delta(m_\oplus/m_\odot) \leq 0.002 \quad \text{and} \quad \delta(m_\oplus/m_\odot) \leq 0.00002
\]
giving a figure of \(\delta(m_\oplus/m_\odot) = 2 \times 10^{-14}\). Combining these numbers, we find that 
\[
\left| \frac{\Delta_k}{334,000} + \Delta_\oplus \right| \leq 7 \times 10^{-9}. \tag{6.2}
\]
This is due almost entirely to the uncertainty in \(m_\oplus/m_\odot\) — small though that is — with the tiny remainder coming from uncertainty in the values of \(a\) and \(A\).

**VII. INDEPENDENT LIMITS ON \(\Delta_k\)**

We now proceed to extract individual constraints on the parameters \(\Delta_k\) by combining equations (3.6) and (4.9), (5.6) and (6.2), respectively. All possible pairs of inequalities may be written in the form 
\[
\begin{align*}
|\Delta_1 + c_1 \Delta_2| &\leq \epsilon_1, \\
|\Delta_1 - c_2 \Delta_2| &\leq \epsilon_2, \tag{7.1}
\end{align*}
\]
where \(c_1, c_2, \epsilon_1\) and \(\epsilon_2\) are positive. We wish to obtain upper limits on the quantities \(\Delta_k\).

Squaring equations (7.1), taking their sum and difference, and solving the resulting pair of quadratic equations, we find that
\[
\begin{align*}
|\Delta_1| &\leq \frac{|(c_1 + c_2)(\epsilon_1 + \epsilon_2) - (c_1 - c_2)(\epsilon_1 - \epsilon_2)|}{2(c_1 + c_2)}, \\
|\Delta_2| &\leq \frac{\epsilon_1 \pm \epsilon_2}{c_1 + c_2}, \tag{7.2}
\end{align*}
\]
where the signs must be evaluated together (ie, if the upper sign is selected for \(\Delta_1\), then it must be selected for \(\Delta_2\) as well).

We apply these results to the Sun-Jupiter system \((1=8.2=1)\) by substituting \(c_1 = 1047\) and \(c_1 = 8 \times 10^{-4}\) from equation (4.4) (Kepler’s modified third law), and \(c_2 = 0.5\) and \(c_2 = 4 \times 10^{-6}\) from equation (4.9) (migration of the stable Lagrange point) to obtain
\[
\begin{align*}
|\Delta_1| &\leq 5 \times 10^{-6}, \tag{7.3} \\
|\Delta_2| &\leq 8 \times 10^{-7}. \tag{7.4}
\end{align*}
\]
Similarly, for the Earth-Moon system \((1=2=1)\), equation (3.7) gives \(c_1 = 81\) and \(c_1 = 8 \times 10^{-7}\) (Kepler’s modified third law), while equation (5.6) gives \(c_2 = 1\) and \(c_2 = 4 \times 10^{-13}\) (lunar Nordtvedt effect), so that
\[
\begin{align*}
|\Delta_1| &\leq 1 \times 10^{-8}, \tag{7.5} \\
|\Delta_2| &\leq 1 \times 10^{-8}. \tag{7.6}
\end{align*}
\]

Finally, for the Sun-Earth system \((1=8.2=1)\), we take \(c_1 = 334,000\) and \(c_1 = 2 \times 10^{-3}\) from equation (4.4) (Kepler’s law), while \(c_2 = 1\) and \(c_2 = 9 \times 10^{-10}\) from equation (5.6) (solar Nordtvedt effect), so that
\[
\begin{align*}
|\Delta_1| &\leq 8 \times 10^{-9}, \tag{7.7} \\
|\Delta_2| &\leq 7 \times 10^{-9}. \tag{7.8}
\end{align*}
\]

The limit (7.7) obtained for the Sun from the solar Nordtvedt effect is nearly three orders of magnitude stronger than that derived from the Trojan asteroids, equation (7.3). This is perhaps not surprising, given the discussion in §VI above. But the fact that the lunar and solar Nordtvedt effects lead to equally strong constraints on \(\Delta_2\) for the Earth — equations (7.7) and (7.8) respectively — is somewhat unexpected, in view of the widespread belief that LLR data provides by far the strongest probe of EP violations in the solar system. This may be partly explained by the fact that the strength of both Nordtvedt effect-based limits ultimately derives from ranging measurements, as discussed in §VI.

Statistical effects may also play a role here. As uncertainties in our orbital and other parameters \((\chi, \eta)\) say, we have in each case used residuals from published fits to a fixed number of solution parameters. These fits, however, do not generally incorporate all the \(\chi\) (they are typically sensitive to at most a single EP-violating parameter, \(\eta\)). We have, in other words, relied on more degrees of freedom than are actually present in the solutions. This is not necessarily a problem, but will tend to underestimate our uncertainties. The results least affected will be those based on the lunar Nordtvedt effect — equation (5.6) — for which EP-violating terms have been included in the solution sets. Our other results — equations (4.4), (4.7), (4.9), (6.1) and (6.2) — may be less robust in comparison. A fully rigorous future treatment would rely on new statistical fits incorporating an independent parameter \(\Delta_k\) for each body, rather than parametrizing EP violations with a single universal constant.

**VIII. APPLICATION TO KALUZA-KLEIN GRAVITY**

Let us now see what equations (7.3) – (7.8) imply for 5D Kaluza-Klein gravity, in which as we recall from equation (2.4), \(\Delta \approx -b/2\), where \(b\) is a free parameter which characterizes the departure from flatness of the fifth dimension in the vicinity of the massive body. Our constraints on this parameter follow immediately from equations (7.4), (7.6), (7.7) and (7.8), and may be summarized together as follows
These upper bounds on $b$ are far more stringent than any obtained so far by other means. In particular, the result (8.3) for the Earth is more than three orders of magnitude stronger than that which may be expected using data on geodetic precession from the upcoming Gravity Probe B satellite [25,26,39]. Equation (8.2) for Jupiter is more stringent by five orders of magnitude than the limits which may be obtained using light deflection by the giant planet [20,40]. And the upper limit (8.4) for the Sun is more than six orders of magnitude tighter than those set so far from the classical tests of general relativity (light deflection using long-baseline radio interferometry, Mercury’s perihelion precession, and time delay using the Mars Viking lander [25,26]).

IX. CONCLUSIONS

We have looked for the constraints imposed by solar system data on theories in which the ratio of gravitational to inertial mass differs from unity by some factor $\Delta$ which may in principle differ from body to body. For two objects characterized by $\Delta_1$ and $\Delta_2$, upper bounds on the sum $|\Delta_1 + c_1 \Delta_2|$ have been found from Kepler’s third law, while the difference $|\Delta_1 - c_2 \Delta_2|$ can be constrained by data on the position of Lagrange libration points, and orbital polarization in the field of a third body (the Nordtvedt effect). (Here $c_1$ and $c_2$ are known constants.) Combining these results, we have extracted independent upper limits on $\Delta$ for the Sun, Moon, Earth and Jupiter, using experimental data on their orbits as well as those of the Trojan asteroids. In particular, we find that $\Delta \leq 10^{-8}$ for the Sun, Earth and Moon, and $10^{-6}$ for Jupiter.

As a test case, we have applied these results to five-dimensional Kaluza-Klein gravity, in which $\Delta$ can be shown to depend on a parameter $b$ of the five-dimensional metric which characterizes the curvature of the extra dimension near the central mass. Our upper bounds on this parameter are three to six orders of magnitude stronger than existing limits on $b$ from the classical (and geodetic precession) tests of general relativity, and confirm earlier conclusions [10,20] that a fifth dimension, if any, plays no significant role in the dynamics of the solar system.

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\[ |b_3| \leq 2 \times 10^{-8}, \]
\[ |b_5| \leq 2 \times 10^{-6}, \]
\[ |b_6| \leq 2 \times 10^{-8}, \]
\[ |b_8| \leq 2 \times 10^{-10}. \]

IX. CONCLUSIONS

We have looked for the constraints imposed by solar system data on theories in which the ratio of gravitational to inertial mass differs from unity by some factor $\Delta$ which may in principle differ from body to body. For two objects characterized by $\Delta_1$ and $\Delta_2$, upper bounds on the sum $|\Delta_1 + c_1 \Delta_2|$ have been found from Kepler’s third law, while the difference $|\Delta_1 - c_2 \Delta_2|$ can be constrained by data on the position of Lagrange libration points, and orbital polarization in the field of a third body (the Nordtvedt effect). (Here $c_1$ and $c_2$ are known constants.) Combining these results, we have extracted independent upper limits on $\Delta$ for the Sun, Moon, Earth and Jupiter, using experimental data on their orbits as well as those of the Trojan asteroids. In particular, we find that $\Delta \leq 10^{-8}$ for the Sun, Earth and Moon, and $10^{-6}$ for Jupiter.

As a test case, we have applied these results to five-dimensional Kaluza-Klein gravity, in which $\Delta$ can be shown to depend on a parameter $b$ of the five-dimensional metric which characterizes the curvature of the extra dimension near the central mass. Our upper bounds on this parameter are three to six orders of magnitude stronger than existing limits on $b$ from the classical (and geodetic precession) tests of general relativity, and confirm earlier conclusions [10,20] that a fifth dimension, if any, plays no significant role in the dynamics of the solar system.

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\[ |b_3| \leq 2 \times 10^{-8}, \]
\[ |b_5| \leq 2 \times 10^{-6}, \]
\[ |b_6| \leq 2 \times 10^{-8}, \]
\[ |b_8| \leq 2 \times 10^{-10}. \]

These upper bounds on $b$ are far more stringent than any obtained so far by other means. In particular, the result (8.3) for the Earth is more than three orders of magnitude stronger than that which may be expected using data on geodetic precession from the upcoming Gravity Probe B satellite [25,26,39]. Equation (8.2) for Jupiter is more stringent by five orders of magnitude than the limits which may be obtained using light deflection by the giant planet [20,40]. And the upper limit (8.4) for the Sun is more than six orders of magnitude tighter than those set so far from the classical tests of general relativity (light deflection using long-baseline radio interferometry, Mercury’s perihelion precession, and time delay using the Mars Viking lander [25,26]).
FIGURE CAPTIONS

FIG. 1. Effect of EP violations on the stable three-body configuration of Lagrange.

FIG. 2. The Nordtvedt effect: two bodies $M_1$ and $M_2$ fall toward $M_3$ with different accelerations.
