Refined Asymptotics for Rate-Distortion using Gaussian Codebooks for Arbitrary Sources

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Abstract

The rate-distortion saddle-point problem considered by Lapidoth (1997) consists in finding the minimum rate to compress an arbitrary ergodic source when one is constrained to use a random Gaussian codebook and minimum (Euclidean) distance encoding is employed. We extend Lapidoth’s analysis in several directions in this paper. Firstly, we consider refined asymptotics. In particular, when the source is stationary and memoryless, we establish the second-order, moderate, and large deviation asymptotics of the problem. Secondly, by “random Gaussian codebook”, Lapidoth refers to a collection of random codewords, each of which is drawn independently and uniformly from the surface of an $n$-dimensional sphere. To be more precise, we term this as a spherical Gaussian codebook. We also consider i.i.d. Gaussian codebooks in which each random codeword is drawn independently from a product Gaussian distribution. We derive the second-order, moderate, and large deviation asymptotics when i.i.d. Gaussian codebooks are employed. Interestingly, in contrast to the recent work on the channel coding counterpart by Scarlett, Tan and Durisi (2017), the dispersions for spherical and i.i.d. Gaussian codebooks are identical. Our bounds on the optimal error exponent for the spherical case coincide on a non-empty interval of rates above the rate-distortion function. The optimal error exponent for the i.i.d. case is established for all rates.

Index Terms

Lossy data compression, Rate-distortion, Gaussian codebook, Mismatched encoding, Minimum distance encoding, Ensemble tightness, Second-order asymptotics, Dispersion, Moderate deviations, Large deviations

I. INTRODUCTION

In the traditional lossy data compression problem [1, Section 3.6], one seeks to find the minimum rate of compression of a source while allowing it to be reconstructed within a distortion $D$ at the output of the decompressor. Shannon [2] established the rate-distortion function for stationary and memoryless sources. However, practical considerations on the system design often necessitate a particular encoding strategy. This then constitutes a mismatch problem in which the codebook is optimized for a source with one distribution but used to compress a source of a different distribution. For example, one might be interested to use a Gaussian codebook—a codebook that is optimal for a memoryless Gaussian source—to compress a source that is arbitrary. For all ergodic sources with second moment $\sigma^2$, Lapidoth [3, Theorem 3] established that the (ensemble) rate-distortion function is

$$R_{\sigma^2}(D) = \frac{1}{2} \log \max \left\{ 1, \frac{\sigma^2}{D} \right\}. \tag{1}$$

The term “Gaussian codebook” requires some qualifications; Lapidoth [3] uses this term to mean a collection of random codewords each of which is drawn independently and uniformly from the surface of a sphere in $n$-dimensions. In this work, we term this random codebook as a spherical Gaussian codebook and, for the sake of comparison, we also consider i.i.d. Gaussian codebooks in which each component of each codeword is drawn independently from a (univariate) Gaussian distribution. In the spirit of recent emphases on refined asymptotics that bring to light the tradeoff between the coding rate, the blocklength, and the probability of excess-distortion, in this paper, we establish ensemble-tight second-order coding rates and moderate deviations constants. We also establish bounds on the ensemble error exponent and show that these bounds are tight for a non-empty set of rates above $R_{\sigma^2}(D)$ in (1).

A. Main Contributions and Related Works

Our main contributions are as follows:

(i) We conduct a second-order asymptotic analysis [4]–[6] for the rate-distortion saddle-point problem for stationary and memoryless sources that satisfy mild technical conditions. Here, the probability of excess-distortion is allowed to be non-vanishing and the spotlight is shone on the additional rate, above the rate-distortion function, required at finite blocklengths to compress the source to within the prescribed probability of excess-distortion. This work complements that of Kostina and Verdú [7] and Ingber and Kochman [8] who established the second-order asymptotics (or dispersion) for compressing (discrete and Gaussian) memoryless sources when the encoder is unconstrained. Our main result here is somewhat surprising. We show that the mismatched dispersions (and the first-order coding rates or rate-distortion...

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functions) for spherical and i.i.d. Gaussian codebooks are identical; this implies that there is no performance loss in terms of the backoff from the first-order fundamental limit regardless of which type of Gaussian codebook one uses. This is in stark contrast to the recent work by Scarlett, Tan, and Durisi [9] on the channel coding counterpart of this problem [10]. It was shown in [9] that the dispersions for both types of codebooks are different. For the lossy source coding case, the dispersions are common and depend only on the second and fourth moments of the source through a simple formula. We provide intuition for why this is the case after the statement of Theorem 1. We recover the dispersion of lossy compression of Gaussian memoryless sources (GMSes) [7], [8] by particularizing the arbitrary source to be a Gaussian.

(ii) Next, we conduct moderate deviations analysis [11], [12] for the same problem under an additional, albeit mild, assumption on the source. Here, the rate of the codebook approaches the rate-distortion function at a speed slower than the reciprocal of the square root of the blocklength. One then seeks the subexponential rate of decay of the probability of excess-distortion. This analysis complements that of Tan [13] who considered the unconstrained encoding case for (discrete and Gaussian) memoryless sources. This was generalized to the successive refinement problem by the present authors [14]. We again show that the moderate deviations constants are identical and that for GMSes can be easily recovered.

(iii) Finally, we consider the large deviations regime [15], [16] in which the rate of the codebook is constrained to be strictly above the rate-distortion function and one seeks to establish the exponential rate of decay of the probability of excess-distortion. Our analysis complements that of Itakura and Kubo [17] who used ideas from Marton [18] to find the excess-distortion exponent for compressing a GMS. We establish the optimal error exponent for i.i.d. Gaussian codebooks. In addition, we derive bounds on the optimal error exponent for spherical Gaussian codebooks and show that they coincide for a non-empty interval of rates directly above \( R^*_n(D) \) in (1). We recover the excess-distortion exponent of lossy compression of GMSes [17] by particularizing the source to be a Gaussian.

B. Organization of the Rest of the Paper

The rest of the paper is organized as follows. We set up the notation, formulate our problem precisely, and present existing results in Section II. In Section III, we present our main results. These include results concerning second-order, moderate, and large deviation asymptotics. Sections IV to VI are devoted to the proofs for each of these asymptotic results respectively.

II. THE RATE-DISTORTION SADDLE-POINT PROBLEM

A. Notation

Random variables and their realizations are in upper (e.g., \( X \)) and lower case (e.g., \( x \)) respectively. All sets are denoted in calligraphic font (e.g., \( \mathcal{X} \)). We use \( \mathbb{R}, \mathbb{R}_+, \) and \( \mathbb{N} \) to denote the set of real numbers, non-negative real numbers, and natural numbers respectively. For any two natural numbers \( a \) and \( b \) we use \( [a : b] \) to denote the set of all natural numbers between \( a \) and \( b \) (inclusive). We use \( \exp(x) \) to denote \( e^x \). All logarithms are base \( e \). We use \( \text{Q}(\cdot) \) to denote the standard Gaussian complementary cumulative distribution function (cdf) and \( \text{Q}^{-1}(\cdot) \) its inverse. For any random variable \( X \), we use \( \Lambda_X(\theta) \) to denote the cumulant generating function \( \log \mathbb{E}[\exp(\theta X)] \) (where \( \theta \in \mathbb{R} \)). We use \( \Lambda_X^{(t)}(t) \) (where \( t \in \mathbb{R} \)) to denote the Fenchel-Legendre transform (convex conjugate) of the cumulant generating function, i.e., \( \sup_{\theta_0 \geq 0} \{ \theta t - \Lambda_X(\theta) \} \). Let \( X^n := (X_1, \ldots, X_n) \) be a random vector of length \( n \) and \( x^n = (x_1, \ldots, x_n) \) is a particular realization. We use \( \|x^n\| = \sqrt{\sum_1^n x_i^2} \) to denote the \( \ell_2 \) norm of a vector \( x^n \in \mathbb{R}^n \). Given two sequences \( x^n \) and \( y^n \), the quadratic distortion measure (squared Euclidean norm) is defined as \( d(x^n, y^n) := \frac{1}{n} \|x^n - y^n\|^2 = \frac{1}{n} \sum_{i=1}^n (x_i - y_i)^2 \). For any two sequences \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \), we write \( a_n \sim b_n \) to mean \( \lim_{n \to \infty} a_n / b_n = 1 \).

B. System Model

Consider arbitrary source \( X \) with distribution (probability mass function or probability density function) \( f_X \) satisfying

\[
\mathbb{E}[X^2] = \sigma^2, \quad \zeta := \mathbb{E}[X^4] < \infty, \quad \mathbb{E}[X^6] < \infty. \tag{2}
\]

In this paper, we consider memoryless sources and thus \( X^n \) is an i.i.d. sequence where each component is generated according to \( f_X \). We consider the rate-distortion saddle-point problem [3, Theorem 3] with an admissible distortion level \( 0 < D < \sigma^2 \). This is the lossy source coding problem [1, Section 3.6] where one is constrained to use random Gaussian codebooks (spherical or i.i.d.) and an encoding strategy which chooses the codeword that minimizes the quadratic distortion measure.

Definition 1. An \((n, M)\)-code for the rate-distortion saddle-point problem consists of

- A set of \( M \) codewords \( \{Y^n(i)\}_{i=1}^M \) known by both the encoder and decoder;
- An encoder \( f \) which maps the source sequence \( X^n \) into the index of the codeword that minimizes the quadratic distortion with respect to the source sequence \( X^n \), i.e.,

\[
f(X^n) := \arg\min_{i \in [1:M]} d(X^n, Y^n(i)). \tag{3}
\]
• A decoder $\phi$ which declares the reproduced sequence as the codeword with index $f(X^n)$, i.e.,

$$\phi(f(X^n)) = Y^n(f(X^n)).$$

(4)

Throughout the paper, we consider random Gaussian codebooks. To be specific, we consider two types of Gaussian codebooks.

• First, we consider the spherical Gaussian codebook where each codeword $Y^n$ is generated independently and uniformly over a sphere with radius $\sqrt{n(\sigma^2 - D)}$, i.e.,

$$Y^n \sim f^n_{sp}(y^n) = \frac{\delta(\|y^n\|^2 - n(\sigma^2 - D))}{S_n(\sqrt{n(\sigma^2 - D)})},$$

(5)

where $\delta(\cdot)$ is the Dirac delta function, $S_n(r) = n\pi^{n/2}r^{n-1}/\Gamma(n/2)$ is the surface area of an $n$-dimensional sphere with radius $r$, and $\Gamma(\cdot)$ is the Gamma function. For GMSes, the spherical codebook is second-order optimal (cf. [7, Theorem 40]).

• Second, we consider the i.i.d. Gaussian codebook where each codeword $Y^n$ is generated independently according to the following product Gaussian distribution with variance $\sigma^2 - D$, i.e.,

$$Y^n \sim f^n_{iid}(y^n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(\sigma^2 - D)}} \exp\left\{-\frac{y_i^2}{2(\sigma^2 - D)}\right\}.$$  

(6)

The i.i.d. Gaussian codebook is also second-order optimal (cf. [7, Theorem 12]) for a GMS.

Here, we mention in passing that the techniques herein may also be applicable to the truncated i.i.d. Gaussian codebook [16, Eqns. (7.3.12)-(7.3.15)] where the codewords are generated from an i.i.d. Gaussian distribution as in (6) but the distribution is then truncated to the annulus $\{y^n \in \mathbb{R}^n : \sigma^2 - D - \eta n < \|y^n\|^2 \leq \sigma^2 - D\}$ where $\eta$ is a fixed positive number. This distribution is useful for deriving error exponent and second-order results. See [19], [20] for discussions.

The (ensemble) excess-distortion probability is defined as

$$P_{e,n} := \Pr\{d(X^n, \phi(f(X^n))) > D\}$$

(7)

$$= \mathbb{E}_{f^n_X} \left[ (1 - \Pr\{d(X^n, Y^n) \leq D\})^M \right| X^n,$$

(8)

where (8) follows from [7, Theorem 9] and the inner probability (over $Y^n$ which is independent of $X^n$) is calculated either with respect to the right hand side of (5) if we use a spherical Gaussian codebook or the right hand side of (6) if we use an i.i.d. Gaussian codebook. Note that the probability in (7) is averaged over the source distribution as well as the random codebook. This is in contrast to the traditional lossy source coding analysis [7], [18] where the excess-distortion probability is averaged over the source distribution only. The additional average over the codebook allows us to pose questions concerning ensemble tightness in the spirit of [9], [21].

C. Existing Results and Definitions

Let $M_{sp}^n(n, \varepsilon, \sigma^2, D)$ be the minimum number of codewords required to compress a length-$n$ source sequence so that the excess-distortion probability with respect to distortion level $D$ is no larger than $\varepsilon \in (0, 1)$ when a spherical Gaussian codebook is used. Similarly, let $M_{iid}^n(n, \varepsilon, \sigma^2, D)$ be the corresponding quantity when an i.i.d. Gaussian codebook is used. Lapidoth [3, Theorem 3] showed that for any ergodic source with finite second moment $\sigma^2$ and any $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} \frac{1}{n} \log M_{sp}^n(n, \varepsilon, \sigma^2, D) = \frac{1}{2} \log \frac{\sigma^2}{D} \text{ nats per source symbol.}$$

(9)

As we will show via a by-product of Theorem 1, for any source satisfying (2) and any $\varepsilon \in (0, 1)$, we also have

$$\lim_{n \to \infty} \frac{1}{n} \log M_{iid}^n(n, \varepsilon, \sigma^2, D) = \frac{1}{2} \log \frac{\sigma^2}{D} \text{ nats per source symbol.}$$

(10)

In this paper, we are interested in second-order, large, and moderate deviations analyses. These analyses provide a refined understanding of the tradeoff between the rate, the blocklength and the excess-distortion probability. In the study of second-order asymptotics, a non-vanishing excess-distortion probability is allowed and we aim to find the back-off from the first-order coding rate (the rate-distortion function) $\frac{1}{2} \log \frac{\sigma^2}{D}$.

**Definition 2.** Fix any $\varepsilon \in [0, 1)$. A number $L$ is said to be a spherical-achievable second-order coding rate if there exists a sequence of $(n, M)$-codes using spherical Gaussian codebooks such that

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left( \log M - \frac{n}{2} \log \frac{\sigma^2}{D} \right) \leq L,$$

(11)

$$\limsup_{n \to \infty} P_{e,n} \leq \varepsilon.$$  

(12)
The infimum of all spherical-achievable second-order coding rates is the optimal spherical second-order coding rate and is denoted as \( L_{sp}^*(\varepsilon) \). In a similar manner, we define i.i.d.-achievable second-order coding rates and denote their infimum as the optimal i.i.d. second-order coding rate \( L_{iid}^*(\varepsilon) \).

The moderate deviations regime interpolates between the large deviations (cf. Definition 4 to follow) and the second-order regimes. In this regime, we are interested in a sequence of \((n, M)\)-codes whose rates approach the first-order coding rate \( \frac{1}{2} \log \frac{\sigma^2}{\mathbb{D}} \) and whose excess-distortion probabilities vanish simultaneously.

**Definition 3.** Consider any sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) such that as \( n \to \infty \)
\[
\xi_n \to 0 \quad \text{and} \quad \sqrt{\frac{n}{\log n}} \xi_n \to \infty.
\] (13)

A non-negative number \( \nu \) is said to be a spherical-achievable moderate deviations constant if there exists a sequence of \((n, M)\)-codes using spherical Gaussian codebooks such that
\[
\limsup_{n \to \infty} \frac{1}{n\xi_n} \left( \log M - \frac{n}{2} \log \frac{\sigma^2}{\mathbb{D}} \right) \leq 1,
\] (14)
\[
\liminf_{n \to \infty} -\frac{1}{n\xi_n^2} \log P_{e,n} \geq \nu.
\] (15)

The supremum of all spherical-achievable moderate deviations constants is the optimal spherical moderate deviations constant and is denoted as \( \nu_{sp}^* \). Similarly, we can define i.i.d.-achievable moderate deviations constants and their supremum is the optimal i.i.d. moderate deviations constant \( \nu_{iid}^* \).

In the large deviations regime, we characterize the speed of the exponential decay of the excess-distortion probability for codes with a rate upper bounded by \( R \).

**Definition 4.** A non-negative number \( E \) is said to be a rate-\( R \) spherical-achievable error exponent if there exists a sequence of \((n, M)\)-codes using spherical Gaussian codebooks such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log M \leq R,
\] (16)
\[
\liminf_{n \to \infty} -\frac{1}{n} \log P_{e,n} \geq E.
\] (17)

The supremum of all rate-\( R \) spherical-achievable error exponents is denoted as the optimal spherical error exponent \( E_{sp}^*(R) \). Similarly, we can define the optimal i.i.d. error exponent \( E_{iid}^*(R) \).

### III. Main Results

#### A. Second-Order Asymptotics

Our first result pertains to the second-order coding rate. Recall the definitions of \( \sigma^2 \) and \( \zeta \) in (2). Let the mismatched dispersion be defined as
\[
V(\sigma^2, \zeta) := \frac{\zeta - \sigma^4}{4 \sigma^4} = \frac{\text{Var}[X^2]}{4 (\text{E}[X^2])^2}.
\] (18)

**Theorem 1.** Consider an arbitrary memoryless source \( X \) satisfying (2). For any \( \varepsilon \in [0, 1) \),
\[
L_{sp}^*(\varepsilon) = L_{iid}^*(\varepsilon) = \sqrt{V(\sigma^2, \zeta)} Q^{-1}(\varepsilon).
\] (19)

The proof of Theorem 1 is provided in Section IV. In the proof, we show that for any \( \varepsilon \in (0, 1) \),
\[
\log M_{sp}(n, \varepsilon, \sigma^2, D) = \log M_{iid}(n, \varepsilon, \sigma^2, D) = \frac{n}{2} \log \frac{\sigma^2}{\mathbb{D}} + \sqrt{n V(\sigma^2, \zeta) Q^{-1}(\varepsilon)} + O(\log n).
\] (20)

A few remarks are in order.

First, in contrast to Scarlett, Tan, and Durisi [9] where spherical and i.i.d. Gaussian codebooks achieve different second-order coding rates for the channel coding saddle-point problem (where the codebook is Gaussian and the channel is additive and its noise is non-Gaussian) [10], we observe no performance gap in second-order asymptotics between two kinds of Gaussian codebooks in the rate-distortion saddle-point counterpart. We provide some intuition why this is so. For the channel coding problem [9], [10], the performance gap between the dispersions appears because the probability of dominant error event depends strongly on the choice of the random codebook since the mutual information random variable that appears in the random coding

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1In Definition 2, we allow \( \varepsilon = 0 \) (signifying that \( P_{e,n} \) is required to vanish) but in the definitions of \( M_{sp}^*(n, \varepsilon, \sigma^2, D) \) and \( M_{iid}^*(n, \varepsilon, \sigma^2, D) \) we enforce that \( \varepsilon \in (0, 1) \).
union (RCU) bound [12] is a strong function of the code distribution. Indeed, it is advantageous to use a spherical code as this ensures that the codewords have exact power $P$. For an i.i.d. code, each of the codewords has power $P$ averaged over the $n$ time slots. Due to this additional randomness (in addition to the channel noise), the dispersion or variability for the i.i.d. case is worse (higher) than that for the spherical case. This is analogous to the well-known fact that for discrete memoryless channels, fixed composition codes [15] generally perform better than i.i.d. codes [16] in the error exponents regime [19, Chapter 2] [22, Section 5]. In contrast, for the rate-distortion saddle-point problem we consider in this paper, there is no “power constraint” on the codewords. Thus, as can be gleaned in the proof in Section IV, the dominant error event in the present rate-distortion case concerns the atypicality of the source sequence $X^n$. The Gaussian codebook plays a diminished role in the contribution to the overall randomness or dispersion of the system.

Second, for a GMS (i.e., $X \sim \mathcal{N}(0, \sigma^2)$), the dispersion is known to be $V(\sigma^2, \zeta) = \frac{1}{\theta}$ independent of the distortion level or the source variance [7], [8]. Hence, when specialized to GMSes, our results are consistent with existing results, namely that both spherical and i.i.d. Gaussian codebooks achieve the optimal second-order coding rate for the rate-distortion problem [7, Theorems 13 and 40]. Furthermore, our results strengthen those of Lapidoth [3, Theorem 3] in the sense that, using i.i.d. Gaussian codebooks, it is true that for an arbitrary memoryless source satisfying (2) and for any $\varepsilon \in (0, 1)$, the first-order result in (10) holds.

Third, we remark that as (20) indicates, the larger the mismatched dispersion for a source, the larger the back-off from the first-order coding rate given a fixed non-vanishing excess-distortion probability $\varepsilon$ and a blocklength $n$. This mismatched dispersion is a function of the fourth moment of the source. Indeed, $V(\sigma^2, \zeta)$ is related to the kurtosis of the source.

Finally, we mention in passing that our proof strategy differs significantly from Lapidoth’s [3] who, for the direct and ensemble converse parts, invoked a theorem due to Wyner [23] concerning packings and coverings of $n$-spheres. The analysis used to prove this theorem and the subsequent ones naturally require more refined estimates on various probabilities.

**B. Moderate Deviation Asymptotics**

**Theorem 2.** Consider an arbitrary memoryless source $X$ satisfying (2) and $\Lambda_X(\theta)$ is finite for some positive number $\theta$. If $V(\sigma^2, \zeta)$ is positive,

$$\nu^*_{sp} = \nu^*_{iid} = \frac{1}{2V(\sigma^2, \zeta)}. \quad (21)$$

The proof of Theorem 2 is provided in Section V. Several remarks are in order.

First, we observe that similarly to second-order asymptotics (Theorem 1), the dispersion $V(\sigma^2, \zeta)$ is a fundamental quantity that governs the speed of convergence of the excess-distortion probability to zero.

Second, we remark the constraint in (13) is mild. In fact, $\xi_n = n^{-t}$ for any $t \in (0, 0.5)$ satisfies (13). We believe that the second condition in (13) may be relaxed by the more common moderate deviations condition [11], [12], namely $\sqrt{n}\xi_n \to \infty$, if a more careful analysis is employed.

Third, we remark that the condition on the source where $\Lambda_X(\theta)$ is finite for some $\theta > 0$ is also mild. It is used to assert that the probability that an average of a sum of i.i.d. random variables deviates from its mean by a constant amount decays exponentially. See [24, Theorem B.4.1] and Lemma 5 to follow. Note that this condition is satisfied if $X$ is distributed according to many common distributions such as discrete, Gaussian, Maxwell, or Rayleigh. However, this condition is violated if $X$ has a Cauchy distribution, which does not even have a finite mean.

**C. Large Deviation Asymptotics**

We present several definitions before stating our main result. Given $s \in \mathbb{R}$ and any non-negative number $z$, define

$$c(z) := \min\{z, 2\sigma^2 - |\sigma^2 - 2D|\}, \quad (22)$$

$$R_{sp}(z) := \frac{1}{2} \log \left(1 - \frac{(z + \sigma^2 - 2D)^2}{4z(\sigma^2 - D)}\right), \quad (23)$$

$$R_{iid}(s, z) := \frac{1}{2} \log(1 + 2s) + \frac{sz}{(1 + 2s)(\sigma^2 - D)} - \frac{sD}{\sigma^2 - D}, \quad \text{and} \quad (24)$$

$$s^*(z) := \max \left\{0, \frac{\sigma^2 - 3D + \sqrt{(\sigma^2 - D)^2 + 4zD}}{4D} \right\}. \quad (25)$$

We remark that $R_{sp}(z)$ is the rate of the exponential decay of the non-excess-distortion probability for any source sequence $x^n$ whose power is $z = \frac{1}{n}\|x^n\|^2$ when its reproduction sequence is generated according to $f_{Y^n}^{sp}$ in (5), i.e.,

$$\lim_{n \to \infty} -\frac{1}{n} \log \Pr\{d(x^n, Y^n) \leq D\} = R_{sp}(z), \quad \text{where} \quad Y^n \sim f_{Y^n}^{sp}. \quad (26)$$

Similarly, $R_{iid}(s^*(z), z)$ is the exponent of the non-excess-distortion probability for any source sequence $x^n$ with power $z$ when $Y^n \sim f_{Y^n}^{iid}$. 


The Fenchel-Legendre transform of the cumulant generating function of \( X^2 \) (cf. Section II-A) is

\[
\Lambda_{X^2}(t) := \sup_{\theta \geq 0} \{ \theta t - \Lambda_{X^2}(\theta) \}, \quad t \in \mathbb{R}.
\] (27)

This is also known as the large deviations rate function [25] of \( X^2 \). For brevity in presentation of the following theorem, let

\[
r_2 := \sqrt{\sigma^2 - D + \sqrt{D}}.
\] (28)

**Theorem 3.** The following bounds on the optimal error exponents hold.

- If \( R < \frac{1}{2} \log \frac{\sigma^2}{D} \), then for both spherical and i.i.d. Gaussian codebooks,

\[
E^*_\text{sp}(R) = E^*_\text{iid}(R) = 0.
\] (29)

- If \( R \geq \frac{1}{2} \log \frac{\sigma^2}{2} \),
  - For the spherical Gaussian codebook, the optimal error exponent satisfies

\[
\Lambda_{X^2}(c(\alpha)) \leq E^*_\text{sp}(R) \leq \Lambda^*_\text{sp}(\alpha),
\] (30)

where \( \alpha \in [\sigma^2, r_2^2] \) is implicitly determined by \( R \) through the equation

\[
R = R^*_\text{sp}(\alpha).
\] (31)

  - For the i.i.d. Gaussian codebook, the optimal error exponent is

\[
E^*_\text{iid}(R) = \Lambda^*_\text{sp}(\alpha),
\] (32)

where \( \alpha \geq \sigma^2 \) is implicitly determined by \( R \) through the equation

\[
R = R^*_\text{iid}(s^*(\alpha), \alpha).
\] (33)

The proof of Theorem 3 is provided in Section VI. Several remarks are in order.

First, as can be gleaned in the proof of Theorem 3 (cf. Lemma 6), \( R^*_\text{sp}(z) \) and \( R^*_\text{iid}(s^*(z), z) \) are both increasing functions of \( z \) if \( z \geq \sigma^2 \) and \( R^*_\text{sp}(\sigma^2) = R^*_\text{iid}(s^*(\sigma^2), \sigma^2) = \frac{1}{2} \log \frac{\sigma^2}{2} \). Further, \( \Lambda_{X^2}(t) > 0 \) for \( t > \sigma^2 \) and \( \Lambda_{X^2}(t) = 0 \) otherwise. Combining these two facts, we can conclude that \( E^*_\text{sp}(R) \) and \( E^*_\text{iid}(R) \) are both positive for rates \( R > \frac{1}{2} \log \frac{\sigma^2}{2} \). That \( E^*_\text{sp}(R) > 0 \) for \( R > \frac{1}{2} \log \frac{\sigma^2}{2} \) recovers the achievability part of [3, Theorem 3] without recourse to Wyner’s theorem [23].

Second, the lower and upper bounds in (30) on the optimal error exponent for the spherical Gaussian codebook match if and only if \( c(\alpha) = \alpha \), i.e., \( \alpha \leq 2\sigma^2 - |\sigma^2 - 2D| \). Recalling that \( R^*_\text{sp}(z) \) is increasing in \( z \) if \( z \geq \sigma^2 \), we conclude that the bounds in (30) match for all rates \( R \leq R^*_\text{sp}(\sigma^2 + 2D) \) if \( \sigma^2 \geq 2D \) and for all rates \( R \leq R^*_\text{sp}(3\sigma^2 - 2D) \) if \( D < \sigma^2 < 2D \). We illustrate this point in in Figure 1 by choosing \( \sigma^2 = 12 \) and \( D = 4 \) and plotting the lower and upper bounds in (30) for a discrete and a Rayleigh distribution.
Third, for the i.i.d. Gaussian codebook, if we consider that each codeword is generated according to (6) with \( \sigma^2 \) replaced by \( \alpha \in \mathbb{R}_+ \), then the right hand side of (33) is replaced by \( \frac{1}{2} \log \frac{\alpha}{D} \). Under this scenario, by particularizing the result to a Gaussian codebook, we can recover the achievability result of Ihara and Kubo [17].

IV. PROOF OF SECOND-ORDER ASYMPTOTICS (THEOREM 1)

A. Preliminaries for the Spherical Codebook

In this subsection, we present some definitions and preliminary results for spherical Gaussian codebooks. For simplicity, let the variance or power of \( Y \) be \( P_Y := \sigma^2 - D \). Further, for any \( \varepsilon \in (0, 1) \), let

\[
V := \text{Var}[X^2] = \zeta - \sigma^4, \tag{34}
\]

\[
a_n := \sqrt{\frac{V}{n}}, \tag{35}
\]

\[
b_n := \sqrt{\frac{V}{n}Q^{-1}(\varepsilon)}, \tag{36}
\]

where the second equality in (34) follows from the definition in (2). Note that for any \( x^n \), \( \Pr\{d(x^n, Y^n) \leq D\} \) depends on \( x^n \) only through its norm \( \|x^n\| \). For any \( x^n \) such that \( \frac{1}{n}\|x^n\|^2 = z > 0 \), let

\[
\Psi(n, z) := \Pr\{d(x^n, Y^n) \leq D\} \tag{37}
\]

where \( Y_1 \) is the first element of sequence \( Y^n = (Y_1, \ldots, Y_n) \) and (42) follows because \( Y^n \) is spherically symmetric so we may take \( x^n = \left(\sqrt{n}z, 0, \ldots, 0\right) \) (cf. (9)).

Let \( Z := \frac{1}{n}\|X^n\|^2 \) be the random variable representing the average power of a source sequence \( X^n \). Further, let \( f_Z \) be the corresponding probability distribution function (pdf) of \( Z \). Recall that \( r_2 = \sqrt{\sigma^2 - D} + \sqrt{D} = \sqrt{P_Y} + \sqrt{D} \) (cf. (28)) and let

\[
r_1 := \sqrt{\sqrt{P_Y} - \sqrt{D}}. \tag{43}
\]

Kostina and Verdú [7, Theorem 37] showed that for any \( z < r_1^2 \) or \( z > r_2^2 \),

\[
\Psi(n, z) = 1, \tag{44}
\]

and otherwise

\[
\Psi(n, z) \geq \frac{\Gamma\left(\frac{n+2}{2}\right)}{\sqrt{\pi n \Gamma\left(\frac{n}{2}\right)}} \left(1 - \frac{(z + P_Y - D)^2}{4z P_Y}\right)^{-\frac{n-1}{2}} =: g(n, z), \tag{45}
\]

where \( \Gamma(\cdot) \) is the Gamma function. Hence, from (8) and (44), we conclude that the excess-distortion probability for the spherical Gaussian codebook equals to

\[
P_{e,n} = \Pr\{Z < r_1^2\} + \Pr\{Z > r_2^2\} + \int_{r_1^2}^{r_2^2} (1 - \Psi(n, z))^M f_Z(z) \, dz. \tag{46}
\]

B. Achievability Proof for the Spherical Codebook

Using the definition of \( g(\cdot) \) in (45), we conclude that \( g(n, z) \) is a decreasing function of \( z \) if \( z \leq |P_Y - D| \). Invoking the definitions of \( b_n \) in (36), \( r_1 \) in (43) and \( r_2 \) in (28), we conclude that \( r_1^2 \leq |P_Y - D| \) and \( r_2^2 \geq \sigma^2 + b_n \) for \( n \) large enough. Thus, combining (43), (46) and noting that \( \Psi(n, z) \leq 1 \), for sufficiently large \( n \), we can upper the excess-distortion probability as follows:

\[
P_{e,n} \leq \Pr\{Z < |P_Y - D|\} + \int_{|P_Y - D|}^{\sigma^2 + b_n} (1 - g(n, z))^M f_Z(z) \, dz + \Pr\{Z > \sigma^2 + b_n\} \tag{47}
\]

\[
\leq \Pr\{Z < |P_Y - D|\} + \int_{|P_Y - D|}^{\sigma^2 + b_n} \exp\{-Mg(n, z)\} f_Z(z) \, dz + \Pr\{Z > \sigma^2 + b_n\}, \tag{48}
\]

\[
\leq \Pr\{Z < |P_Y - D|\} + \exp\{-M\overline{g}(n, \sigma^2 + b_n)\} + \Pr\{Z > \sigma^2 + b_n\} \tag{49}
\]

\(^{2}\)Note that we use \( \alpha \) while Ihara and Kubo [17] use \( \alpha^2 \) to mean the same quantity.
where (48) follows since \((1 - a)^M \leq \exp\{-Ma\}\) for any \(a \in [0, 1)\); and (49) follows since \(g(n, z)\) is decreasing in \(z\) for \(z \geq |P_Y - D|\). Let the third central moment of \(X^2\) be defined as

\[
T := E[|X^2 - \sigma^2|^3].
\]  

Using the definitions of \(V\) in (34), \(T\) in (50) and the Berry-Esseen theorem, we conclude that

\[
\Pr\{Z < |P_Y - D|\} = \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 < \sigma^2 - 2D \right\}
\leq \frac{6T}{\sqrt{n} V^{3/2}} + \exp \left\{ - \frac{2n(\sigma^2 - |\sigma^2 - 2D|)^2}{V} \right\}
\leq \frac{6T}{\sqrt{n} V^{3/2}} + \exp \left\{ - \frac{2n(\sigma^2 - |\sigma^2 - 2D|)^2}{V} \right\}
= O\left( \frac{1}{\sqrt{n}} \right),
\]  

where (53) follows since \(Q(a) \leq \exp\{-\frac{a^2}{2}\}\) while (54) follows since \(T\) (cf. (50)) is finite for sources satisfying (2) and \(\sigma^2 - |\sigma^2 - 2D| > 0\) due to the fact that \(\sigma^2 > D\). Similarly, using the definition of \(b_n\) in (36) and the Berry-Esseen theorem, we have

\[
\Pr\{Z > \sigma^2 + b_n\} = \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > \sigma^2 + b_n \right\}
\leq \varepsilon + \frac{6T}{\sqrt{n} V^{3/2}}
\leq \varepsilon + O\left( \frac{1}{\sqrt{n}} \right).
\]  

Choose \(M\) such that

\[
\log M = - \log g(n, \sigma^2 + b_n) + \log \left( \frac{1}{2} \log n \right)
= n \left( \frac{1}{2} \log \frac{\sigma^2}{\sigma} + \frac{b_n}{2} \sigma^2 + O\left( \frac{\log n}{n} \right) \right)
= n \left( \frac{1}{2} \log \frac{\sigma^2}{\sigma} + \sqrt{n V(\sigma^2, \varepsilon)} Q^{-1}(\varepsilon) + O(\log n) \right),
\]  

where (59) follows from the Taylor expansion of \(g(n, \sigma^2 + b_n)\) (cf. (45)) and noting that \(\Gamma(\frac{n+1}{2}) / \Gamma(\frac{n}{2}) = \Theta(\sqrt{n})\); and (60) follows from the definition of \(b_n\) (cf. (36)) and \(\sqrt{n V(\sigma^2, \varepsilon)}\) (cf. (18)). Thus, with the choice of \(M\) in (58), we conclude that

\[
\exp\{-M g(n, \sigma^2 + b_n)\} = \frac{1}{\sqrt{n}}.
\]  

Hence, combining (49), (54), (57) and (61), we have shown that there exists a sequence of \((n, M)\)-codes satisfying (60) and

\[
\limsup_{n \to \infty} P_{e, n} \leq \varepsilon.
\]  

C. Ensemble Converse for the Spherical Gaussian Codebook

We now show that the result in (20) is ensemble tight. From Stam’s paper [26, Eq. (4)], the distribution of \(Y_1\) is

\[
f_{Y_1}(y) = \frac{1}{\sqrt{\pi n P_Y}} \frac{\Gamma(n/2)}{\Gamma(n/2)} \left( 1 - \frac{y^2}{nP_Y} \right)^{\frac{n-3}{2}} 1\{y^2 \leq n P_Y \}.
\]  

Recall the definitions of \(a_n\) in (35) and \(b_n\) in (36). Define the sets

\[
\mathcal{P} := \{ r \in \mathbb{R} : b_n < r - \sigma^2 \leq a_n \},
\]  

\[
\mathcal{Q} := \{ r \in \mathbb{R} : r + P_Y - D \geq 0 \}.
\]
Then, for any $z \in \mathcal{P} \cap \mathcal{Q}$ satisfying $\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}} \leq \sqrt{nP_Y}$, using the definition of $\Psi(\cdot)$ in (42), we obtain that

$$\Psi(n, z) = \Pr \left\{ Y_1 \geq \sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}} \right\}$$

(66)

$$= \int_{\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}}}^{\sqrt{nP_Y}} \frac{1}{\sqrt{\pi n P_Y}} \Gamma \left( \frac{n}{2} \right) \left( 1 - \frac{y^2}{nP_Y} \right)^{\frac{n-3}{2}} dy$$

(67)

$$\leq \int_{\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}}}^{\sqrt{nP_Y}} \frac{1}{\sqrt{\pi n P_Y}} \Gamma \left( \frac{n}{2} \right) \left( 1 - \frac{(z + P_Y - D)^2}{4zP_Y} \right)^{\frac{n-3}{2}} dy$$

(68)

$$\leq \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{n-1}{2} \right) \left( 1 - \frac{(z + P_Y - D)^2}{4zP_Y} \right)^{\frac{n-3}{2}}$$

(69)

$$= \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{n-1}{2} \right) \exp \left\{ \frac{n-3}{2} \log \left( 1 - \frac{(z + P_Y - D)^2}{4zP_Y} \right) \right\} =: \mathcal{g}(n, z),$$

(70)

where (67) follows from the definition in (63) and the condition that $z \in \mathcal{Q}$ (cf. (65)) which implies $\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}} \geq 0 > -\sqrt{nP_Y}$; (68) follows since $(1 - \frac{y^2}{nP_Y})$ is decreasing in $y$ for positive $y$; and (69) follows by enlarging the integration region (recall that $\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}} \geq 0$). Note that $\mathcal{g}(n, z)$ is decreasing in $z$ for $z \geq |P_Y - D|$ and $\mathcal{g}(n, z) \geq 0$ for all $z \in \mathcal{P}$. Hence, for any $z \in \mathcal{P} \cap \mathcal{Q}$ such that $\sqrt{\frac{n(z + P_Y - D)}{2\sqrt{z}}} > \sqrt{nP_Y}$, we still have $\mathcal{g}(n, z) \geq \Psi(n, z)$.

Recall that $Z = \frac{1}{\sqrt{n}} \|X^n\|^2$ and $f_{\mathcal{g}}$ is the corresponding pdf of $Z$. Thus, according to (8), for $n$ sufficiently large, we have

$$P_{e,n} = \mathbb{E}_{X^n} \left[ (1 - \Pr \{ d(X^n, Y^n) \leq D \})^M | X^n \right]$$

(71)

$$= \int_0^\infty (1 - \Psi(n, z))^M f_\mathcal{g}(z) \, dz$$

(72)

$$\geq \int_0^\infty (1 - \mathcal{g}(n, z))^M 1 \{ z \in \mathcal{P} \cap \mathcal{Q} \} f_\mathcal{g}(z) \, dz$$

(73)

$$\geq \int_{z \in \mathcal{P} \cap \mathcal{Q}} (1 - \mathcal{g}(n, \sigma^2 + b_n))^M f_\mathcal{g}(z) \, dz$$

(74)

$$\geq \int_{z \in \mathcal{P} \cap \mathcal{Q}} \exp \left\{ - M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \right\} f_\mathcal{g}(z) \, dz$$

(75)

$$\geq \int_{z \in \mathcal{P} \cap \mathcal{Q}} \exp \left\{ - M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \right\} \left\{ M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \leq \frac{1}{\sqrt{n}} \right\} f_\mathcal{g}(z) \, dz$$

(76)

$$\geq \left( 1 - \frac{1}{\sqrt{n}} \right) \int_{z \in \mathcal{P} \cap \mathcal{Q}} \left\{ M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \leq \frac{1}{\sqrt{n}} \right\} f_\mathcal{g}(z) \, dz$$

(77)

$$= \left( 1 - \frac{1}{\sqrt{n}} \right) \Pr \left\{ Z \in \mathcal{P} \cap \mathcal{Q}, M \leq \frac{1 - \mathcal{g}(n, \sigma^2 + b_n)}{\mathcal{g}(n, \sigma^2 + b_n)} \frac{1}{\sqrt{n}} \right\}$$

(78)

$$= \left( 1 - \frac{1}{\sqrt{n}} \right) \Pr \left\{ Z \in \mathcal{P} \cap \mathcal{Q}, \log M \leq \log \left( 1 - \mathcal{g}(n, \sigma^2 + b_n) \right) - \log \mathcal{g}(n, \sigma^2 + b_n) - \frac{1}{2} \log n \right\}$$

(79)

$$\geq \left( 1 - \frac{1}{\sqrt{n}} \right) \Pr \left\{ Z \in \mathcal{P} \cap \mathcal{Q}, \log M \leq - \log 2 - \log \mathcal{g}(n, \sigma^2 + b_n) - \frac{1}{2} \log n \right\}$$

(80)

where (72) follows from the definition of $\Psi(n, z)$ in (42); (73) follows by restricting $z \in \mathcal{P} \cap \mathcal{Q}$ and using the definition of $\mathcal{g}(\cdot)$ in (70); (74) follows since $\mathcal{g}(n, z)$ is decreasing in $z$ for $z \in \mathcal{P} \cap \mathcal{Q}$; (75) follows since $(1 - a)^M \geq \exp \{ - M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \}$ for any $a \in [0, 1)$; (77) follows since $M \frac{\mathcal{g}(n, \sigma^2 + b_n)}{1 - \mathcal{g}(n, \sigma^2 + b_n)} \leq \frac{1}{\sqrt{n}}$, $\exp \{ - a \}$ is decreasing in $a$, and $\exp \{ - a \} \geq 1 - a$ for $a \geq 0$; and (80) follows since $\mathcal{g}(n, z) \leq \frac{1}{2}$ for $n$ large enough if $z > \sigma^2$.

Combining (70), (80) and applying a Taylor expansion of $\mathcal{g}(n, \sigma^2 + b_n)$ similarly to (59), we conclude that for any $(n, M)$-code such that

$$\log M \leq - \log 2 - \frac{1}{2} \log n - \log \mathcal{g}(n, \sigma^2 + b_n)$$

(81)

$$= n \left( \frac{1}{2} \log \frac{\sigma^2 + b_n}{D} + \frac{b_n}{2\sigma^2} + O \left( \frac{\log n}{n} \right) \right),$$

(82)

we have

$$P_{e,n} \geq \left( 1 - \frac{1}{\sqrt{n}} \right) \Pr \{ Z \in \mathcal{P} \cap \mathcal{Q} \}. $$

(83)
The following lemma is essential to complete the converse proof.

**Lemma 4.** Consider any source $X$ such that (2) are satisfied and $\sigma^2 < \infty$. Then, we have

$$\Pr\{Z \in P \cap Q\} \geq \varepsilon + O\left(\frac{1}{\sqrt{n}}\right).$$  \hspace{1cm} (84)

The proof of Lemma 4 is deferred to Appendix A.

Using the definition of $V(\sigma^2, \zeta)$ in (18), the definition of $b_n$ in (36), the bounds in (82), (83), and Lemma 4, we conclude that for any sequence of $(n, M)$-codes such that

$$\log M \leq \frac{n}{2} \log \frac{\sigma^2}{D} + \sqrt{nV(\sigma^2, \zeta)}Q^{-1}(\varepsilon) + O(\log n),$$  \hspace{1cm} (85)

we have

$$\limsup_{n \to \infty} P_{e,n} \geq \liminf_{n \to \infty} P_{e,n} \geq \varepsilon$$  \hspace{1cm} (86)

as desired.

D. Preliminaries for the I.I.D. Gaussian Codebook

Now we consider the i.i.d. Gaussian codebook (cf. (6)). Note that $\Pr\{d(x^n, Y^n) \leq D\}$ depends on $x^n$ only through its norm $\|x^n\|$ (cf. [17]). Given any sequence $x^n$ such that $\frac{1}{n}\|x^n\|^2 = z$, define

$$\Upsilon(n, z) := \Pr\{d(x^n, Y^n) \leq D\}.$$  \hspace{1cm} (87)

From (6), we obtain that

$$f_{Y^n}(y^n) = \frac{1}{(2\pi(\sigma^2 - D))^{n/2}} \exp\left\{-\frac{\|y^n\|^2}{2(\sigma^2 - D)}\right\}.$$  \hspace{1cm} (88)

Since $f_{Y^n}(y^n)$ is decreasing in $\|y^n\|$, we conclude that $\Upsilon(n, z)$ is a decreasing function of $z$ (cf. [17]). Using the definition of $\Upsilon(\cdot)$ in (87), we have

$$\Upsilon(n, z) = \Pr\{\|x^n - Y^n\|^2 \leq nD\} \hspace{1cm} (89)$$

$$= \Pr\left\{\sum_{i=1}^{n}(Y_i - \sqrt{z})^2 \leq nD\right\} \hspace{1cm} (90)$$

$$= \Pr\left\{-\frac{1}{nP_Y}\sum_{i=1}^{n}(Y_i - \sqrt{z})^2 \geq -\frac{D}{P_Y}\right\}. \hspace{1cm} (91)$$

where (90) follows since the probability depends on $x^n$ only through its power and thus we can choose $x^n$ such that $x_i = \sqrt{z}$ for all $i \in [1 : n]$ (cf. [9, Eq. (94)]). For the i.i.d. Gaussian codebook, each $Y_i \sim N(0, P_Y)$ and hence $\frac{1}{nP_Y}(Y_i - \sqrt{z})^2$ is distributed according to a non-central $\chi^2$ distribution with one degree of freedom.

Given $z$ and $s$, let

$$\kappa(s) := \frac{(P_Y(1 + 2s) + 2z)^2}{P_Y(1 + 2s)^3}.$$  \hspace{1cm} (92)

Using the result of [27, Section 2.2.12] concerning the cumulant generating function of a non-central $\chi^2$ distribution, the definition of $R_{\text{iid}}(\cdot)$ in (24), the definition of $s^*(\cdot)$ in (25), and the Bahadur-Ranga Rao (strong large deviations) theorem for non-lattice random variables [25, Theorem 3.7.4], we obtain

$$\Upsilon(n, z) \sim \frac{\exp\{-nR_{\text{iid}}(s^*(z), z)\}}{s^*(z)\sqrt{\kappa(s^*(z))}2\pi n}, \hspace{1cm} n \to \infty.$$  \hspace{1cm} (93)
E. Achievability Proof for the I.I.D. Gaussian Codebook

According to (8), the excess-distortion probability under the i.i.d. Gaussian codebook can be upper bounded as follows:

\[
P_{e,n} = \mathbb{E}\left[\left(1 - \Pr\{d(X^n, Y^n) \leq D\}\right)^M \mid X^n\right]
\]

\[
= \int_0^\infty (1 - \Upsilon(n, z))^M f_Z(z) \, dz
\]

\[
\leq \int_{\sigma^{-a_n}}^{\sigma^{+b_n}} f_Z(z) \, dz + \int_{\sigma^{-a_n}}^{\sigma^{+b_n}} (1 - \Upsilon(n, z))^M f_Z(z) \, dz + \int_{\sigma^{+b_n}}^\infty f_Z(z) \, dz
\]

\[
\leq \exp\{-M\Upsilon(n, z)\} f_Z(z) \, dz + \Pr\{Z < \sigma^2 - a_n\} + \Pr\{Z > \sigma^2 + b_n\}
\]

\[
\leq \exp\{-M\Upsilon(n, \sigma^2 + b_n)\} + \Pr\{Z < \sigma^2 - a_n\} + \Pr\{Z > \sigma^2 + b_n\},
\]

where (96) follows since \(\Upsilon(n, z) \geq 0\); (97) follows since \((1 - a)^M \leq \exp\{-Ma\}\); and (98) follows since \(\Upsilon(n, z)\) is decreasing in \(z\) and \(\Pr\{\sigma^2 - a_n \leq Z \leq \sigma^2 + b_n\} \leq 1\).

Using the definitions of \(R_{\text{iid}}(\cdot)\) in (24) and \(s^*(\cdot)\) in (25), we have

\[
R_{\text{ iid}}(s^*(\sigma^2 + b_n), \sigma^2 + b_n) = \frac{1}{2} \log \frac{P_Y + \sqrt{P_Y^2 + 4(\sigma^2 + b_n)D}}{2D} + \frac{z(P_Y - 2D + \sqrt{P_Y^2 + 4(\sigma^2 + b_n)D})}{2P_Y} - \frac{P_Y - 2D + \sqrt{P_Y^2 + 4(\sigma^2 + b_n)D}}{4P_Y}
\]

\[
= \frac{1}{2} \log \frac{\sigma^2}{D} + \frac{b_n}{2\sigma^2} + O(b_n^2),
\]

\[
= \frac{1}{2} \log \frac{\sigma^2}{D} + \sqrt{\frac{V(\sigma^2, \zeta)}{n}} Q^{-1}(\varepsilon) + O\left(\frac{1}{n}\right),
\]

where (100) follows from a Taylor expansion at \(z = \sigma^2\) and recalling that \(P_Y = \sigma^2 - D\); and (101) follows from the definitions of \(V(\sigma^2, \zeta)\) in (18) and \(b_n\) in (36).

Choose \(M\) such that

\[
\log M \geq -\log \Upsilon(n, \sigma^2 + b_n) + \log \left(\frac{1}{2} \log n\right).
\]

Then, we have

\[
\exp\{-M\Upsilon(n, \sigma^2 + b_n)\} \leq \frac{1}{\sqrt{n}}.
\]

Further, using the result in (93) and (101), we obtain

\[
\log M \geq \frac{n}{2} \log \frac{\sigma^2}{d} + \sqrt{nV(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n).
\]

Similarly as the proof of Lemma 4, using the Berry-Esseen theorem and the definition of \(a_n\) in (35), we obtain

\[
\Pr\{Z < \sigma^2 - a_n\} = \Pr\left\{\frac{1}{n} \sum_{i=1}^n (X_i^2 - \sigma^2) < \sqrt{\frac{\log n}{n}}\right\}
\]

\[
\leq Q\left(\sqrt{\frac{\log n}{n}}\right) + \frac{6T}{\sqrt{n}} V_{3/2}
\]

\[
= O\left(\frac{1}{\sqrt{n}}\right).
\]

Hence, combining (57), (98), (103) and (107), we conclude that there exists a sequence of \((n, M)\)-codes using i.i.d. Gaussian codebooks with \(M\) satisfying (104) and

\[
\limsup_{n \to \infty} P_{e,n} \leq \varepsilon.
\]

F. Ensemble Converse for the I.I.D. Gaussian Codebook

The ensemble converse proof for the i.i.d. Gaussian codebook is omitted since it is similar to the ensemble converse proof for the spherical Gaussian codebook in Section IV-C starting from (71) except for the following two points: i) replace \(\Upsilon(n, z)\) with \(\Upsilon(n, z)\); ii) replace \(\mathcal{P} \cap \mathcal{Q}\) with \(\mathcal{P}\).
V. PROOF OF MODERATE DEVIATION ASYMPOTICS (THEOREM 2)

A. Preliminaries

The following slight strengthening of the Chernoff bound [24, Theorem B.4.1] is crucial in our proof.

**Lemma 5.** Given an i.i.d. sequence $X^n$, suppose that the cumulant generating function $\Lambda_{|X|}(\theta)$ is finite for some positive number $\theta$. Then for any $t > \mathbb{E}[X]$, 
\[
\Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i > t \right\} \leq \exp\{-n\Lambda_X^*(t)\}.
\] (109)

and for any $t < \mathbb{E}[X]$, 
\[
\Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i < t \right\} \leq \exp\{-n\Lambda_X^*(t)\}.
\] (110)

In both cases, $\Lambda_X^*(t) > 0$.

In other words, if the threshold $t$ deviates from the mean by a constant amount, the probability in question decays exponentially fast.

B. Achievability Proof for the Spherical Gaussian Codebook

Let 
\[
c_n := 2\sigma^2\xi_n.
\] (111)

The proof of the achievability follows from Section IV-B up till (49) with $c_n$ taking the role of $b_n$. Invoking Lemma 5, we conclude that under the conditions in Theorem 2, we have
\[
\Pr\{Z < |P_Y - D|\} = \Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 < |P_Y - D| \right\} \leq \exp\{-nt_1\}
\] (113)

for some $t_1 > 0$ since $|P_Y - D| = |\sigma^2 - 2D| < \sigma^2$ due to the fact that $\sigma^2 > D$. Invoking the moderate deviations theorem [25, Theorem 3.7.1] and the definition of $V$ in (34), we conclude that
\[
\Pr\{Z > \sigma^2 + c_n\} = \Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - \sigma^2) > c_n \right\}
\] (114)

\[
= \exp\left\{ -\frac{nc_n^2}{2V} + o(nc_n^2) \right\}
\] (115)

\[
= \exp\left\{ -\frac{nc_n^2}{2V(\sigma^2, \xi_n)} + o(n\xi_n^2) \right\},
\] (116)

where (116) follows from the definitions of $V(\sigma^2, \xi_n)$ in (18), $V$ in (34) and $c_n$ in (111).

Recall the definition of $g(\cdot)$ in (45). Choose $M$ such that
\[
\log M = -\log g(n, \sigma^2 + c_n) + \log n
\] (117)

\[
= n\left( \frac{1}{2} \log \frac{\sigma^2}{D} + \xi_n + o(\xi_n) \right),
\] (118)

where (118) follows from a Taylor expansion similar to (59), the definition of $c_n$ in (111), and the conditions on $\xi_n$ in (13). With this choice of $M$, we have
\[
\exp\{-Mg(n, \sigma^2 + c_n)\} = \frac{1}{n}.
\] (119)

Combining the results in (49), (113), (116), (118) and (119), we conclude that there exists a sequence of $(n, M)$-codes such that
\[
\lim sup_{n \to \infty} \frac{1}{n\xi_n} \left( \log M - \frac{n}{2} \log \frac{\sigma^2}{D} \right) \leq 1,
\] (120)

\[
\lim inf_{n \to \infty} \frac{-1}{n\xi_n^2} \log P_{e,n} \geq \frac{1}{2V(\sigma^2, \xi_n)}.
\] (121)
C. Ensemble Converse Proof for the Spherical Gaussian Codebook

The converse proof follows from Section IV-C up till (83). However, we use \( \mathcal{P}', c_n \) in place of \( \mathcal{P}, b_n \) where

\[
\mathcal{P}' := \{ r \in \mathbb{R} : \xi_n < r - \sigma^2 < 2\xi_n \}.
\]  

(122)

Now recall the definition of \( \overline{g}(\cdot) \) in (70). Applying a Taylor expansion similar to (118), invoking Lemma 5, and using the moderate deviations theorem in [25, Theorem 3.7.1], we conclude that for any \( (n, M) \)-code such that

\[
\log M \leq -\log 2 - \frac{1}{2} \log n - \log \overline{g}(n, \sigma^2 + c_n)
\]

\[
= n \left( \frac{1}{2} \log \frac{\sigma^2}{D} + \xi_n + o(\xi_n) \right),
\]

(123)

(124)

we have

\[
\frac{P_{e,n}}{1 - \frac{\sigma^2}{D}} \geq \exp \left\{ - \frac{n\xi_n^2}{2V(\sigma^2, \zeta)} + o(n\xi_n^2) \right\} - \exp \left\{ - \frac{4n\xi_n^2}{2V(\sigma^2, \zeta)} + o(n\xi_n^2) \right\} - \exp \{-nt_2\},
\]

(125)

for some \( t_2 > 0 \). Note that the first term on the right hand side of (125) dominates as \( n \to \infty \). From the results in (124) and (125), we conclude that for any sequence of \( (n, M) \)-codes such that

\[
\limsup_{n \to \infty} \frac{1}{n\xi_n} \left( \log M - \frac{n}{2} \log \frac{\sigma^2}{D} \right) \leq 1,
\]

(126)

we have

\[
\liminf_{n \to \infty} -\frac{1}{n\xi_n} \log P_{e,n} \leq \limsup_{n \to \infty} -\frac{1}{n\xi_n} \log P_{e,n} \leq \frac{1}{2V(\sigma^2, \zeta)}.
\]

(127)

D. Proof for the I.I.D. Gaussian Codebook

The proof for i.i.d. Gaussian codebooks is similar to the proof in Section IV-E and IV-F with the use of Lemma 5 and [25, Theorem 3.7.1] as in Sections V-B and V-C and is thus omitted.

VI. PROOF FOR LARGE DEVIATION ASYMPTOTICS (THEOREM 3)

A. Preliminaries

The following properties of the quantities \( R_{sp}(z) \) in (23), \( R_{iid}(s^\ast(z), z) \) (cf. (24) and (25)) and \( \Lambda_{X^2}(t) \) in (27) are useful in the proof of Theorem 3.

Lemma 6. The following claims hold.

(i) Concerning \( R_{sp}(z) \),
   
   (a) \( R_{sp}(z) \) is increasing in \( z \) if \( z \geq |\sigma^2 - 2D| \);
   
   (b) \( R_{sp}(\sigma^2) = \frac{1}{2} \log \frac{\sigma^2}{2\sigma^2} \), \( \lim_{z \to -\sigma^2} R_{sp}(z) = \infty \).

(ii) Concerning \( R_{iid}(s^\ast(z), z) \),
   
   (a) \( s^\ast(z) > 0 \) if and only if \( z > \max(0, 2D - 2\sigma^2) \);
   
   (b) \( R_{iid}(s^\ast(z), z) = 0 \) if \( z = \max(0, 2D - 2\sigma^2) \); and \( R_{iid}(s^\ast(\sigma^2), \sigma^2) = \frac{1}{2} \log \frac{\sigma^2}{2\sigma^2} \);
   
   (c) \( R_{iid}(s^\ast(z), z) = \sup_{\lambda \geq 0} R_{iid}(s^\ast(z), z) \) and thus \( R_{iid}(s^\ast(z), z) \) is increasing in \( z \) for \( z \geq \max(0, 2D - 2\sigma^2) \).

(iii) Concerning \( \Lambda_{X^2}(t) \) (cf. [25, Chapter 3]),
   
   (a) \( \Lambda_{X^2}(t) \) is convex and non-decreasing in \( t \) for \( t \geq 0 \);
   
   (b) \( \Lambda_{X^2}(t) = 0 \) if \( t \leq \sigma^2 \);
   
   (c) \( \Lambda_{X^2}(t) \) is increasing in \( t \) for \( t \geq \sigma^2 \).

The proof of Lemma 6 is omitted since it follows either from simple algebra or from [25, Chapter 3].
B. Achievability Proof for the Spherical Gaussian Codebook

Recall that $P_Y = \sigma^2 - D$. Invoking the definitions of $r_1$ in (43) and $g(\cdot)$ in (45), we conclude that $r_1^2 \leq |\sigma^2 - 2D| < \sigma^2$ and $g(n, z)$ is decreasing in $z$ if $z \in ([|\sigma^2 - 2D|], r_1^2)$.

Using the expression for the excess-distortion probability in (46), given any $\alpha$ such that $\alpha \in [\sigma^2, r_1^2)$, we can upper bound $P_{e,n}$ as follows

$$P_{e,n} = \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 < r_1^2 \right\} + \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > r_1^2 \right\} + \int_{r_1^2}^{\sigma^2} (1 - \Psi(n, z))^2 f_Z(z) dz \quad (128)$$

$$\leq \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 < |\sigma^2 - 2D| \right\} + \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > \alpha \right\} + \int_{|\sigma^2 - 2D|}^{\sigma^2} (1 - g(n, z))^2 f_Z(z) dz \quad (129)$$

$$= \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - \sigma^2) < |\sigma^2 - 2D| - \sigma^2 \right\} + \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > \alpha \right\} + (1 - g(n, \alpha))^2 M \quad (130)$$

$$= \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - \sigma^2) > |\sigma^2 - 2D| - \sigma^2 \right\} + \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > \alpha \right\} + (1 - g(n, \alpha))^2 M \quad (131)$$

$$\leq 2 \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > \min\{\alpha, 2\sigma^2 - |\sigma^2 - 2D|\} \right\} + \exp\{-Mg(n, \alpha)\} \quad (132)$$

$$= 2 \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > c(\alpha, D) \right\} + \exp\{-Mg(n, \alpha)\}, \quad (133)$$

where (130) follows since $g(n, z)$ is decreasing for $z \geq |\sigma^2 - 2D|$ and $\alpha \geq \sigma^2 > |\sigma^2 - 2D|$; (132) follows since $(1 - a)^M \leq \exp\{-Ma\}$ for any $a \in [0, 1)$; and (133) follows from the definition of $c(\cdot)$ in (22). Now, given any positive $\delta \in (0, 1)$, recalling the definition of $R_{\text{sp}}(\cdot)$ in (23), we choose $M$ such that

$$\log M = (1 + \delta)(n - 1)R_{\text{sp}}(\alpha) + \log \frac{\sqrt{\pi n \Gamma(n+1)}}{\Gamma(\frac{n+2}{2})}. \quad (134)$$

Using the definitions of $R_{\text{sp}}(\cdot)$ in (23) and $g(\cdot, z)$ in (45), we obtain that

$$\exp\{-Mg(n, z)\} = \exp\{-\exp\{(n - 1)\delta R_{\text{sp}}(\alpha)\}\}, \quad (135)$$

which vanishes doubly exponentially fast for $\alpha \geq \sigma^2$. Invoking Cramér’s Theorem [25, Theorem 2.2.3] and the definition of $\Lambda^*_{X^2}(\cdot)$ in (27), we obtain that

$$\Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 > c(\alpha, D) \right\} \leq \exp\{-n\Lambda^*_{X^2}(c(\alpha, D))\}. \quad (136)$$

Therefore, using (133), (134), (135), (136) and noting that $\sqrt{\pi n \Gamma(n+1)} / \Gamma(\frac{n+2}{2}) = \Theta(1/\sqrt{n})$, we conclude that for all $\alpha \in [\sigma^2, r_1^2)$, there exists a sequence of $(n, M)$-codes such that

$$\lim_{n \to \infty} \frac{1}{n} \log M = (1 + \delta)R_{\text{sp}}(\alpha), \quad (137)$$

$$\liminf_{n \to \infty} \frac{1}{n} \log P_{e,n} \geq \Lambda^*_{X^2}(c(\alpha, D)). \quad (138)$$

Now, note that $\sigma^2 > |\sigma^2 - 2D|$ due to the fact that $\sigma^2 > D$. Recall that $R_{\text{sp}}(z)$ is increasing in $z$ for $z \geq \sigma^2$ (cf. Claim (i)(a) in Lemma 6) and $R_{\text{sp}}(\sigma^2) = \frac{1}{2} \log \frac{2D}{\sigma^2}$ (cf. Claim (i)(b) in Lemma 6). Hence, by letting $\delta \downarrow 0$, (137) and (138) imply that for all $R \geq \frac{1}{2} \log \frac{2D}{\sigma^2}$, there exists a sequence of $(n, M)$-codes such that

$$\lim_{n \to \infty} \frac{1}{n} \log M = R, \quad (139)$$

and (138) both hold, where $\alpha$ is determined from $R = R_{\text{sp}}(\alpha)$ (cf. (23)).

The proof for $R \in [0, \frac{1}{2} \log \frac{2D}{\sigma^2})$ follows trivially by noting that any $(n, M)$-code satisfies that $P_{e,n} \leq 1$. 


C. Ensemble Converse Proof for the Spherical Gaussian Codebook

Fix any \( \alpha \) such that \( \alpha \in [\sigma^2, r_2^2] \) (cf. (28) for the definition of \( r_2 \)). Let
\[
\bar{\mathcal{P}} := \{ r \in \mathbb{R} : \alpha \leq r < r_2^2 \}, \\
\hat{\mathcal{Q}} := \{ r \in \mathbb{R} : r - |\sigma^2 - 2D| \geq 0 \}.
\]

Note that \( r \in \hat{\mathcal{Q}} \) implies that \( r + (\sigma^2 - 2D) \geq 0 \).

Using the result in (8) and the definition of \( \bar{g}(\cdot) \) in (70), we conclude that for sufficiently large \( n \),
\[
P_{e,n} \geq \int_{\alpha}^{r_2^2} (1 - \bar{g}(n, z)) M 1\{ z \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \} f_Z(z) \, dz
\]
\[
\geq (1 - \bar{g}(n, \alpha))^M \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}
\]
\[
\geq \exp \left\{ -M \frac{\bar{g}(n, \alpha)}{1 - \bar{g}(n, \alpha)} \right\} \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}
\]
\[
\geq \exp \left\{ -2M \bar{g}(n, \alpha) \right\} \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}.
\]

where (143) follows since \( \bar{g}(n, z) \) is decreasing in \( z \) for \( z \geq |\sigma^2 - 2D| \) and \( \alpha \geq \sigma^2 \geq |\sigma^2 - 2D| \); (144) follows since \( (1-a)^M \geq \exp\{ -M \frac{a}{1-a} \} \) for any \( a \in [0,1) \); and (145) follows since \( \bar{g}(n, \alpha) \leq \frac{1}{2} \) for \( n \) sufficiently large.

For any \( M \) such that
\[
\log M \leq -\log \bar{g}(n, \alpha) - \log 2 - \frac{1}{2} \log n,
\]
using (145) and the inequality that \( \exp\{-a\} \geq 1 - a \), we have that for sufficiently large \( n \),
\[
P_{e,n} \geq \left(1 - \frac{1}{\sqrt{n}}\right) \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}.
\]

Note that
\[
\Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}
\]
\[
= \Pr \left\{ \max\{\alpha, |\sigma^2 - 2D|\} \leq \frac{1}{n} \sum_{i=1}^{n} X_i^2 < r_2^2 \right\}
\]
\[
= \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \geq \alpha \right\} - \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \geq r_2^2 \right\},
\]

where the final equality holds because \( \alpha^2 \) is chosen to be in \( [\sigma^2, r_2^2] \) so \( \alpha^2 \geq \sigma^2 > |\sigma^2 - 2D| \). Invoking Cramér’s theorem [25, Theorem 2.2.3], we obtain that for sufficiently large \( n \) and any positive number \( \delta \in (0,1) \),
\[
\Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \in \bar{\mathcal{P}} \cap \hat{\mathcal{Q}} \right\}
\]
\[
\geq \exp \left\{ -n(1 + \delta) \Lambda_{\chi^2}(\alpha) \right\} - \exp \left\{ -n(1 + \delta) \Lambda_{\chi^2}(r_2^2) \right\}
\]
\[
\geq \frac{1}{2} \exp \left\{ -n(1 + \delta) \Lambda_{\chi^2}(\alpha) \right\},
\]

where (151) holds since \( \alpha^2 \leq \alpha < r_2^2 \) and \( \Lambda_{\chi^2}(t) \) (cf. (27)) is increasing in \( t \) for all \( t \geq \sigma^2 \) (cf. Claim (iii)(c) in Lemma 6).

Invoking the definition of \( \bar{g}(\cdot) \) in (70), the definition of \( R_{\text{sp}}(\cdot) \) in (23), and the bounds in (146), (147), (151), we conclude that for any sequence of \( (n, M) \)-codes and any \( \alpha \in [\sigma^2, r_2^2] \) if
\[
\limsup_{n \to \infty} \frac{1}{n} \log M \leq R_{\text{sp}}(\alpha),
\]
then
\[
\liminf_{n \to \infty} -\frac{1}{n} \log P_{e,n} \leq (1 + \delta) \Lambda_{\chi^2}(\alpha).
\]
Note that $\sigma^2 > |\sigma^2 - 2D|$ because $\sigma^2 > D$. Recall that $R_{sp}(z)$ is increasing in $z$ for $z \geq \sigma^2$ (cf. Claim (i)(a) in Lemma 6) and $R_{sp}(\sigma^2) = \frac{1}{2} \log \frac{2}{\pi}$ (cf. Claim (i)(b) in Lemma 6). Hence, the converse proof is complete by letting $\delta \downarrow 0$ and noting that $\Lambda_{X^2}(t) = 0$ if $t \leq \sigma^2$ (cf. Claim (iii)(b) in Lemma 6).

**D. Achievability Proof for the I.I.D. Gaussian Codebook**

Fix $\alpha$ such that $\alpha > \max\{0, 2D - \sigma^2\}$. Invoking the conclusion in (93), for any $x^n$ such that $\frac{1}{n}\|x^n\|^2 \leq \alpha$, we have that for sufficiently large $n$ and any positive $\delta$,

$$\Upsilon\left(n, \frac{1}{n}\|x^n\|^2\right) \geq \Upsilon(n, \alpha) \geq \exp\{-n(1+\delta)R_{iid}(s^*(\alpha), \alpha)\}. \tag{154}$$

Invoking (8), the excess-distortion probability can be upper bounded as follows

$$P_{e,n} = E_{X^n}\left[(1 - \Pr\{d(X^n, Y^n) \leq D\})^M|X^n\right]$$

$$= \int_0^\infty (1 - \Upsilon(n, z))^M f_Z(z) \, dz \tag{156}$$

$$= \int_0^\alpha (1 - \Upsilon(n, z))^M f_Z(z) \, dz + \Pr\left\{\frac{1}{n}\sum_{i=1}^n X_i^2 > \alpha\right\} \tag{157}$$

$$\leq \int_0^\alpha (1 - \exp\{-n(1+\delta)R_{iid}(s^*(\alpha), \alpha)\})^M f_Z(z) \, dz + \Pr\left\{\frac{1}{n}\sum_{i=1}^n X_i^2 > \alpha\right\} \tag{158}$$

$$\leq \exp\{-M\exp\{-n(1+\delta)R_{iid}(s^*(\alpha), \alpha)\}\} + \Pr\left\{\frac{1}{n}\sum_{i=1}^n X_i^2 > \alpha\right\}. \tag{159}$$

where (157) follows from the definition of $\Upsilon(\cdot)$ in (87); (158) follows since $\Upsilon(n, z) \leq 1$; (159) follows from (155); and (160) follows since $(1 - a)^{\alpha} \leq \exp\{-Ma\}$ for any $a \in [0, 1]$.

Recall the definitions of $s^*(\cdot)$ in (25) and $R_{iid}(\cdot)$ in (24). Choose $M$ such that

$$\log M = n(1 + 2\delta)R_{iid}(s^*(\alpha), \alpha). \tag{160}$$

Invoking the definition of $\Lambda_{X^2}(\cdot)$ in (27), the conclusion in (160) and Cramér’s Theorem [25, Theorem 2.2.3], we conclude that for sufficiently large $n$ and arbitrary positive $\delta$,

$$P_{e,n} \leq \exp\{-\exp\{n\delta R_{iid}(s^*(\alpha), \alpha)\}\} + \exp\{-n\Lambda_{X^2}(\alpha)\}. \tag{161}$$

Recall that $R_{iid}(s^*(\sigma^2), \sigma^2) = \frac{1}{2} \log \frac{2}{\pi}$ and $R_{iid}(s^*(\cdot), \cdot)$ is positive and increasing for $\alpha > \max\{0, 2D - \sigma^2\}$ (cf. Claim (ii) in Lemma 6. Thus, the first term in (162) vanishes doubly exponentially fast since $\alpha > \max\{0, 2D - \sigma^2\}$. From (161) and (162), we conclude that for any $\alpha > \max\{0, 2D - \sigma^2\}$, there exists a sequence of $(n, M)$-codes such that

$$\lim_{n \to \infty} \frac{1}{n}\log M = (1 + 2\delta)R_{iid}(s^*(\alpha), \alpha). \tag{163}$$

$$\liminf_{n \to \infty} \frac{1}{n}\log P_{e,n} \geq \Lambda_{X^2}(\alpha). \tag{164}$$

The proof of the achievability part is done by letting $\delta \downarrow 0$ and using the fact that $\Lambda_{X^2}(t) = 0$ if $t \leq \sigma^2$ (cf. Claim (iii)(b) in Lemma 6).

**E. Ensemble Converse Proof for the I.I.D. Gaussian Codebook**

Fix any $\alpha$ such that $\alpha > \max\{0, 2D - \sigma^2\}$. Using the strong large deviations result in (93), we conclude that for $n$ large enough and any positive number $\delta \in (0, 1)$, given any $x^n$ such that $\frac{1}{n}\|x^n\|^2 \geq \alpha$,

$$\Upsilon\left(n, \frac{1}{n}\|x^n\|^2\right) \leq \Upsilon(n, \alpha) \leq \exp\{-n(1-\delta)R_{iid}(s^*(\alpha), \alpha)\}. \tag{165}$$
From (8) and (166), we conclude that for sufficiently large $n$,

$$P_{e,n} = \int_0^\infty (1 - \Upsilon(n, z))^M f_Z(z) \, dz$$

(167)

$$\geq \int_\alpha^\infty (1 - \Upsilon(n, z))^M f_Z(z) \, dz$$

(168)

$$\geq \int_\alpha^\infty (1 - \exp \{- n(1 - \delta)R_{iid}(s^*(\alpha), \alpha)\})^M f_Z(z) \, dz$$

(169)

$$\geq \exp \{- 2M \exp \{- n(1 - \delta)R_{iid}(s^*(\alpha), \alpha)\} \Pr \left\{ \frac{1}{n} \sum_{i=1}^n X_i^2 > \alpha \right\},$$

(170)

where (170) follows because i) $(1 - a)^M \geq \exp \{- M \frac{1}{1-a} \}$ and ii) for $n$ sufficiently large, $\exp \{- n(1 - \delta)R_{iid}(s^*(\alpha), \alpha)\} \leq \frac{1}{e}$. Using the bound in (170) and Cramér’s theorem [25, Theorem 2.2.3], we conclude that for any $\alpha > \max(0, 2D - \sigma^2)$, if $M$ is chosen such that

$$\log M \leq n(1 - \delta)R_{iid}(s^*(\alpha), \alpha) - \log n - \log 2,$$

(171)

then for sufficiently large $n$, using the inequality $\exp(-a) \geq 1 - a$ for $a \in [0, 1)$, we obtain

$$P_{e,n} \geq \left(1 - \frac{1}{n}\right) \exp \{- n(1 - \delta)\Lambda_{X^2}^*(\alpha)\}.$$  

(172)

Hence, given any $\alpha > \max(0, 2D - \sigma^2)$, we have shown that for any sequence of $(n, M)$-codes such that

$$\limsup_{n \to \infty} \frac{1}{n} \log M \leq (1 - \delta)R_{iid}(s^*(\alpha), \alpha),$$

(173)

we have

$$\liminf_{n \to \infty} \frac{1}{n} \log P_{e,n} \leq (1 - \delta)\Lambda_{X^2}^*(\alpha).$$

(174)

Finally, recall that $R_{iid}(s^*(\sigma^2), \sigma^2) = \frac{1}{2} \log \frac{\sigma^2}{D}$ and $R_{iid}(s^*(\alpha), \alpha)$ is positive and increasing for $z > \max(0, 2D - \sigma^2)$ (cf. Claim (ii) in Lemma 6). Thus, the converse proof is complete by letting $\delta \downarrow 0$ and using the fact that $\Lambda_{X^2}^*(t) = 0$ if $t \leq \sigma^2$ (cf. Claim (iii)(b) in Lemma 6).

APPENDIX

A. Proof of Lemma 4

Note that $T$ (cf. (50)) is finite since $E[X^6]$ is finite (cf. (2)). Using the definitions of $a_n$ in (35), $b_n$ in (36), $P$ in (64), and the Berry-Esseen theorem, we obtain

$$\Pr\{Z \in P\} = \Pr\left\{ \frac{1}{n} \sum_{i=1}^n X_i^2 > \sigma^2 + b_n \right\} - \Pr\left\{ \frac{1}{n} \sum_{i=1}^n X_i^2 > \sigma^2 + a_n \right\}$$

(175)

$$= \Pr\left\{ \frac{1}{n} \sum_{i=1}^n (X_i^2 - \sigma^2) > \sqrt{\frac{V}{n}} Q^{-1}(\varepsilon) \right\} - \Pr\left\{ \frac{1}{n} \sum_{i=1}^n (X_i^2 - \sigma^2) > \sqrt{\frac{\log n}{n}} \right\}$$

(176)

$$\geq \varepsilon - \frac{6T}{\sqrt{n} V^{3/2}} - \left( Q\left(\sqrt{\log n}\right) + \frac{6T}{\sqrt{n} V^{3/2}} \right)$$

(177)

$$\geq \varepsilon + O\left( \frac{1}{\sqrt{n}} \right),$$

(178)
where (178) follows since \( Q(x) \leq \exp\left(\frac{-x^2}{2}\right) \) and \( T \) is finite. Similarly, using the definition of \( Q \) in (65) and the Berry-Esseen theorem, we obtain

\[
\Pr\{Z \notin Q\} = \Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 + P_Y - D \leq 0 \right\} \\
= \Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \leq \sigma^2 + (D - P_Y - \sigma^2) \right\} \\
= \Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \leq \sigma^2 - 2P_Y \right\} \\
\leq Q\left( \frac{2 \sqrt{\frac{n}{V}} P_Y + 6T}{\sqrt{nV^{3/2}}} \right) \\
\leq \exp\left\{ -\frac{2nP_Y}{V} \right\} + \frac{6T}{\sqrt{nV^{3/2}}} \\
= O\left( \frac{1}{\sqrt{n}} \right),
\]  

(179)

where (184) follows since the first term in (183) vanishes exponentially fast and is thus dominated by the second term. Combining (178) and (184), we have

\[
\Pr\{Z \in \mathcal{P} \cap \mathcal{Q}\} \geq \Pr\{Z \in \mathcal{P}\} - \Pr\{Z \notin \mathcal{Q}\} \\
\geq \varepsilon + O\left( \frac{1}{\sqrt{n}} \right).
\]  

(185)

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