‘Thermal’ ambience and fluctuations in classical field theory

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Abstract

A plane monochromatic wave will not appear monochromatic to a non-inertial observer. We show that this feature leads to a ‘thermal’ ambience in an accelerated frame even in classical field theory. When a real, monochromatic, mode of a scalar field is Fourier analyzed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum consists of three terms: (i) a factor \(1/2\) that is typical of the ground state energy of a quantum oscillator, (ii) a Planckian distribution \(N(\Omega)\) and—most importantly—(iii) a term \(\sqrt{N(N+1)}\), which is the root mean square fluctuations about the Planckian distribution. It is the appearance of the root mean square fluctuations that motivates us to attribute a ‘thermal’ nature to the power spectrum. This result shows that some of the ‘purely’ quantum mechanical results might have a classical analogue. The ‘thermal’ ambience that we report here also proves to be a feature of observers stationed at a constant radius in the Schwarzschild and de-Sitter spacetimes.

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1 Introduction

It is well known that quantization of a field in Minkowski and Rindler coordinates are not equivalent [1]. It is also known that the response of a uniformly accelerating detector in the Minkowski vacuum is a thermal spectrum [2, 3]. In both these situations, one obtains the thermal spectrum in the strict sense of the word: Not only that the mean occupation number in any mode is Planckian, but the fluctuations around the mean is also characterized by the standard thermal noise. These results suggest that the quantum fluctuations in the vacuum appear as thermal fluctuations in the uniformly accelerated frame. Similar results also arise in Schwarzschild and de-Sitter spacetimes [4, 5].

In contrast to quantum theory, classical field theory does not admit any intrinsic fluctuations. The absence of concepts such as vacuum and fluctuations in classical field theory may lead us to believe that non-trivial phenomena as the one mentioned above will not have any classical analogue. We shall show, however, that such is not the case. In this paper, we discuss a fairly non-trivial and interesting effect that arises purely in the context of classical field theory and which probably has serious implications for such phenomena as black hole entropy. We find that, when a real monochromatic wave mode of a classical field is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum has a ‘thermal’ nature—both as regards the mean occupation number as well as the fluctuations around the mean. Similar results arise in the Schwarzschild and de Sitter spacetimes as well.

This paper is organized as follows. In sections 2 and 3 we show that a ‘thermal’ ambience is a feature of uniformly accelerated observers in Minkowski spacetime. In section 4 we explain as to how such an effect can arise in Schwarzschild and de-Sitter spacetimes too. Finally, in section 5 we discuss the possible implications of our analysis.

2 ‘Thermal’ ambience in an accelerated frame

Consider a massless, minimally coupled, scalar field which satisfies the Klein-Gordon equation

$$\Box \Phi \equiv \Phi^\mu_{\mu} = 0. \quad (1)$$

In flat spacetime, the basis solutions to the above Klein-Gordon equation in the Minkowski coordinates \((t, x)\) can be taken to be plane waves labeled by the wave number \(k\).
vector $k$:

$$\Phi(t, x) = \cos(\omega t - k \cdot x), \quad (2)$$

where $\omega = |k|$. We now ask: Consider an observer who is moving on an arbitrary trajectory $(t(\tau), x(\tau))$, parametrized by the proper time $\tau$. How will this observer view the above Minkowski plane wave mode?

The moving observer will see the scalar field varying with respect to his (her) proper time in a manner determined by the function $\Phi [t(\tau), x(\tau)]$. If the observer is in inertial motion then the monochromatic wave will appear to be another monochromatic wave with a Doppler shifted frequency. But in general, for noninertial trajectories, the wave will not appear to be monochromatic for the moving observer but will prove to be a superposition of waves with different frequencies. To determine the exact decomposition of the wave, we should Fourier analyze the Minkowski mode in the frame of the observer. The Fourier transform of the Minkowski plane wave with respect to the proper time $\tau$ of the observer in motion is described by the integral

$$\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \; \Phi [t(\tau), x(\tau)] \; e^{-i \Omega \tau}. \quad (3)$$

This expression gives the amplitude of a component with frequency $\Omega$ (as defined by the moving observer) present in the original monochromatic wave. Given a particular plane wave, we can always align the coordinates such that the wave is traveling along the $x$ axis, i.e. the wave vector is given by $k = (k, 0, 0)$. Then the plane wave mode (2) reduces to

$$\Phi(t, x) = \cos(\omega t - kx) \quad (4)$$

and its Fourier transform is given by the integral

$$\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \; \cos [\omega t(\tau) - kx(\tau)] \; e^{-i \Omega \tau}. \quad (5)$$

We shall now specialize to the case of an observer who is accelerating uniformly with respect to the Minkowski coordinates with a proper acceleration $g$. We shall also assume that the observer is accelerating along the $x$-axis. The world line of such an observer in the Minkowski coordinates $(t, x, y, z)$ is given by the relations $[6]$

$$t = t_0 + g^{-1} \sinh(g \tau) \quad ; \quad x = x_0 + g^{-1} \cosh(g \tau) \quad ; \quad y = y \quad \text{and} \quad z = z, \quad (6)$$

where $t_0$ and $x_0$ are constants and $\tau$ is the proper time as measured by the clock in the frame of the uniformly accelerated observer. The world line of such a
uniformly accelerating observer is a hyperbola in the \((t, x)\) plane parametrized by the two constants \(t_0\) and \(x_0\). The asymptotes of this hyperbola are the past and the future light cones that intersect at the point \((t_0, x_0)\). To see how the plane wave \(\Phi\) will be viewed by such an observer, we substitute the coordinate transformations \(6\) in the Fourier integral \(5\), and obtain that 

\[
\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \cos \left( \omega [t_0 - x_0 + g^{-1} \sinh(\tau) - g^{-1} \cosh(\tau)] \right) e^{-i\Omega \tau} \\
= \int_{-\infty}^{\infty} d\tau \cos \left( \omega g^{-1} e^{-\tau} - \beta \right) e^{-i\Omega \tau} \\
= \left( \frac{1}{2g} \right) e^{-i\phi} \left( e^{-(\Omega/4\Omega_0)} e^{-i\beta} + e^{(\Omega/4\Omega_0)} e^{i\beta} \right) \Gamma \left( i\Omega g^{-1} \right), \quad (7)
\]

where

\[
\phi = \Omega g^{-1} \ln(\omega g^{-1}); \quad \Omega_0 = g/2\pi \quad \text{and} \quad \beta = \omega(t_0 - x_0). \quad (8)
\]

In the above integral we have assumed that the plane wave is traveling to the right so that \(k = \omega\). The resulting power spectrum per logarithmic interval in frequency is given by \(P(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2\) and can be written in a remarkable form:

\[
P(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2 = \left( \frac{\pi}{2g} \right) \left( \coth \left( \Omega/2\Omega_0 \right) + \csch \left( \Omega/2\Omega_0 \right) \cos(2\beta) \right) \\
= \left( \frac{\pi}{g} \right) \left\{ \frac{1}{2} + N + \sqrt{N(N+1)} \cos(2\beta) \right\}, \quad (9)
\]

where

\[
N(\Omega) = \left( \frac{1}{\exp(\Omega/\Omega_0) - 1} \right). \quad (10)
\]

We shall now consider various features of this result.

To begin with we note that this result is a purely classical one and hence \(\hbar\) does not appear anywhere. In ordinary units, \(\Omega_0 = (g/2\pi c)\) has the correct dimensions, viz. per second, for a frequency. The quantity \(N(\Omega)\) is a Planckian in terms of frequencies and is again independent of \(\hbar\). Usually, one tries to express the Planckian distribution in terms of energies of the ‘quanta’ labeled by frequency \(\Omega\) and in such a case we need to write frequencies as, say \(\Omega = (E/\hbar)\), thereby artificially introducing \(\hbar\); but the result, stated as a power spectrum in frequency space, makes perfect conceptual sense as it stands. (For example, radio astronomers measure the power spectrum in frequency space and may not think in terms of photons.) Of course, to obtain a quantity with the dimension of
temperature we again need to introduce a $h$ into the quantity $\Omega_0$. (For this reason the word thermal has been appearing within quotes throughout this paper.)

The analysis done above could have been carried out even in the days before quantum theory—it uses only classical relativity. But it is our knowledge of quantum theory that allows a suggestive interpretation of the three terms in the power spectrum: The first term—viz. the factor $(1/2)$—is typical of the ground state energy of a quantum oscillator. The second term $N$ is a Planckian distribution in $\Omega$, as already mentioned. *Note that these two terms are totally independent of the original frequency $\omega$ of the plane wave!*

The third term is still more remarkable. When we vary the constants $t_0$ and $x_0$ this term fluctuates between $-\sqrt{N(N+1)}$ and $+\sqrt{N(N+1)}$. The magnitude of this fluctuation (which is the root mean square deviation about the mean value) is exactly what one would have obtained for a strictly thermal distribution of massless bosonic quanta in quantum field theory. In fact it is this fluctuation that motivates us to attribute a ‘thermal’ nature to the power spectrum.

While this result is very suggestive, it must be noted that $\beta$ is related to $t_0$ and $x_0$ by equation (8). If the original plane wave had an extra phase $\delta$, then the argument of the cosine term will pick up $2\delta$ additionally. So by choosing the constants $\delta, t_0$ and $x_0$ suitably, it possible to kill the fluctuations in the power spectrum. (It is also easy to verify that one cannot choose the constants to cancel the first two terms as well.) The implications of this result are not clear.

It may be noted that the existence of the three terms is a direct consequence of our choosing a *real* plane wave. If the same analysis is repeated for a complex mode for the scalar field, say $\Phi(t, x) = \exp(-i(\omega t - kx))$, then the resultant power spectrum per logarithmic frequency interval is given by

$$P(\Omega) = \left(\frac{2\pi}{g}\right) N,$$

where $N$ is given by (10). We do not get the zero-point term or the fluctuations. Of course, in classical field theory, one must use *real* modes and that is exactly what we have done here.

Finally, let us consider the limit of $\omega \to 0$. In this limit, the field in the inertial frame reduces to an unimportant constant—which could be thought of as closest to the concept of a ‘vacuum’ in the classical theory. The Fourier integral as well as the phase $\phi$ in equation (8) diverges when $\omega \to 0$; but the power
spectrum—which is the squared modulus of the amplitude—is well defined:

\[ P(\Omega) \bigg|_{\omega \to 0} = \left( \frac{\pi}{g} \right) \left\{ \frac{1}{2} + N + \sqrt{N(N + 1)} \right\}. \tag{12} \]

However, as long as \( \omega \) is treated as a ‘regulator’ one can say that the accelerated observer will see these terms even in the limit of \( \omega \to 0 \). This is very reminiscent of the inertial vacuum appearing as a Planckian spectrum to the accelerated observer in a manner which is completely independent of the original wave mode.

## 3 Generalization to other field configurations

In the last section we have carried out our analysis for real Minkowski waves that were traveling to the right. It is straightforward to verify that one obtains the same power spectrum for left moving waves, \( i.e \) when \( k = -\omega \).

A more general case is as follows. Consider a function of \( \Phi(t - x) \) that satisfies the Klein-Gordon equation and is either odd or even in \( t - x \). Such a function \( \Phi(t - x) \), which will represent a wave packet that is traveling along the \( x \) axis, can be Fourier decomposed into the following form

\[ \Phi(t - x) = \int_{-\infty}^{\infty} d\alpha f(\alpha) \exp i\alpha(t - x). \tag{13} \]

The function \( f(\alpha) \) will prove to be odd or even depending on whether \( \Phi(t - x) \) is odd or even. Substituting the transformation equations (6) in (13) and Fourier transforming as before with respect to the proper time of the Rindler observer, we obtain that

\[ \tilde{\Phi}(\Omega) = g^{-1} \Gamma(i\Omega g^{-1}) \left( e^{(\Omega/4\Omega_0)} F_1(\Omega) \pm e^{-(\Omega/4\Omega_0)} F_2(\Omega) \right), \tag{14} \]

where the plus sign is to be chosen if \( \Phi(t - x) \) is an even function and the minus sign if \( \Phi(x - t) \) is an odd function. The distributions \( F_1(\Omega) \) and \( F_2(\Omega) \) are described by the integrals

\[ F_1(\Omega) = \int_0^{\infty} d\alpha f(\alpha) \exp - \left( i\Omega g^{-1} \ln(g^{-1}\alpha) \right) e^{i\alpha(t_0 - x_0)} \tag{15} \]

and

\[ F_2(\Omega) = \int_0^{\infty} d\alpha f(\alpha) \exp - \left( i\Omega g^{-1} \ln(g^{-1}\alpha) \right) e^{-i\alpha(t_0 - x_0)}, \tag{16} \]
where $\Omega_0$ is given by (8). Now we obtain that

$$\mathcal{P}(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2 = \left( \frac{\pi}{g \sinh(\Omega/2\Omega_0)} \right) \left\{ e^{(\Omega/2\Omega_0)} |F_1(\Omega)|^2 + e^{-(\Omega/2\Omega_0)} |F_2(\Omega)|^2 \right. \left. \pm \left( F_1^*(\Omega)F_2(\Omega) + F_1(\Omega)F_2^*(\Omega) \right) \right\}. \quad (17)$$

This spectrum, of course, does not have a thermal nature since it depends explicitly on the form of $f(\alpha)$.

But a simplification occurs if we treat $f(\alpha)$ as a stochastic variable so that when averaged over an ensemble of realizations, it satisfies the relation

$$\langle f(\alpha) f^*(\alpha') \rangle = P(\alpha) \delta(\alpha - \alpha'), \quad (18)$$

with some power spectrum $P(\alpha)$, such that $\int_{-\infty}^{\infty} d\omega \, P(\omega) = 2C$. In such a case, when $|F_1(\Omega)|^2$ and $|F_2(\Omega)|^2$ are averaged over the stochastic variable $f(\alpha)$, both reduce to a constant independent of $\Omega$, i.e.

$$\langle |F_1(\Omega)|^2 \rangle = \langle |F_2(\Omega)|^2 \rangle = \int_0^\infty d\alpha \, P(\alpha) = C. \quad (19)$$

The power spectrum (17) when averaged over the stochastic variable $f(\alpha)$ is given by

$$\langle \mathcal{P}(\Omega) \rangle = \left( \frac{4\pi C}{g} \right) \left\{ \frac{1}{2} + N \pm \sqrt{N(N+1)} \cos(2\beta') \right\}, \quad (20)$$

where $\beta'$ is a function of $(t_0 - x_0)$ and is defined by the relation

$$\cos(2\beta') = \left( \frac{1}{2C} \right) \left\langle F_1^*(\Omega)F_2(\Omega) + F_1(\Omega)F_2^*(\Omega) \right\rangle = \left( \frac{1}{C} \right) \int_0^\infty d\alpha \, P(\alpha) \cos[2\alpha(t_0 - x_0)]. \quad (21)$$

So a stochastic wave field in the Minkowski frame will also reproduce all the three terms in the power spectrum obtained earlier.

The wave field described above did not have explicit random phases. It is possible to define a random wave field in a different way. Consider the following random superposition of real modes for the scalar field

$$\Phi(t, x) = \int_{-\infty}^{\infty} d\omega \, A(\omega) \cos[\omega(t - x) + \theta(\omega)], \quad (22)$$

where $A(\omega)$ and $\theta(\omega)$ are stochastic variables satisfying the relations

$$\langle A(\omega) A(\omega') \rangle = \tilde{P}(\omega) \delta(\omega - \omega') \quad ; \quad \langle \theta(\omega) \rangle = 0 \quad (23)$$
and \( \bar{P}(\omega) \) is an arbitrary function of \( \omega \) such that \( \bar{C} = \int_{-\infty}^{\infty} d\omega \bar{P}(\omega) \) is a finite constant. We can now set \( t_0 = x_0 = 0 \) in (8) without any loss of generality. Substituting the coordinate transformations (8) in the scalar field configuration given by (22) and Fourier transforming the same with respect to the proper time of the uniformly accelerated observer, we obtain

\[
\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega A(\omega) \cos \left( \omega \left[ t(\tau) - x(\tau) \right] + \theta(\omega) \right) e^{-i\Omega \tau} \\
= \int_{-\infty}^{\infty} d\omega A(\omega) \int_{-\infty}^{\infty} d\tau \cos \left( \omega g^{-1} e^{-g\tau} - \theta(\omega) \right) e^{-i\Omega \tau} \\
= \left( \frac{1}{2g} \right) \Gamma(i\Omega g^{-1}) \int_{-\infty}^{\infty} d\omega A(\omega) e^{-i\phi} \\
\times \left( e^{-(\Omega/4\Omega_0)} e^{-i\theta(\omega)} + e^{(\Omega/4\Omega_0)} e^{i\theta(\omega)} \right),
\]

(24)

where \( \phi \) and \( \Omega_0 \) are given by (8). The power spectrum per logarithmic frequency interval, \( \text{viz.} \) the quantity \( \left( \Omega |\tilde{\Phi}(\Omega)|^2 \right) \) when averaged over the stochastic variables \( A(\omega) \) and \( \theta(\omega) \) then reduces to

\[
\langle P(\Omega) \rangle = \left( \frac{\pi \bar{C}}{g} \right) \left\{ \frac{1}{2} + N \right\}.
\]

(25)

In this case, the random phases have averaged out the fluctuation term, \( \text{viz.} \) the factor \( \sqrt{N(N+1)} \) that had appeared in the power spectrum (8). A somewhat similar result was obtained earlier by Boyer [8]. He modeled the zero-point fluctuations as due to random superposition of Minkowski plane wave modes, and used it as a basis for investigating the ‘spectrum’ observed by a uniformly accelerating observer. He showed that the correlation function of an accelerating observer ‘in a random classical scalar zero-point radiation’ exactly matches the correlation function of an inertial observer in a thermal background. Our analysis here shows that the effect reported by Boyer arises when a random superposition of Minkowski real modes are simply Fourier analyzed in the frame of a uniformly accelerating observer (cf. equation (23)). \( \text{But notice that, such an approach has killed a very interesting} \ \sqrt{N(N+1)} \text{term which was originally present.} \)

Finally, we discuss a case in which the observer is moving in a direction perpendicular to the wave vector. Consider an observer who is uniformly accelerating along the \( y \) axis, \( \text{i.e.} \) in a direction perpendicular to which the plane wave is traveling (which we always take to be the \( x \)-axis). If the proper acceleration of the observer is \( g \), then the coordinate transformations to the uniformly accelerated frame are given by

\[
t = t_0 + g^{-1} \sinh(g\tau) \quad ; \quad x = x \quad ; \quad y = y_0 + g^{-1} \cosh(g\tau) \quad \text{and} \quad z = z.
\]

(26)
Substituting these transformations in the Fourier transform (5), we obtain
\[
\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \cos \left( \omega (t_0 + g^{-1} \sinh(g\tau)) - kx \right) e^{-i\Omega\tau} = \left( g^{-1} K_{(i\Omega/g)} \left( \omega g^{-1} \right) \left( e^{-i(\Omega/4\Omega_0)} e^{-i(\omega_0-kx)} + e^{i(\Omega/4\Omega_0)} e^{i(\omega_0-kx)} \right) \right),
\]
where \( K_{(i\Omega/g)} \) is the Bessel function of imaginary order. The resulting power spectrum
\[
P(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2 = 2\Omega g^{-2} \left| K_{(i\Omega/g)} \left( \omega g^{-1} \right) \right|^2 \left\{ \cosh(\Omega/2\Omega_0) + \cos[2(\omega_0-kx)] \right\}
\]
\[
= 4\Omega g^{-2} \sinh(\Omega/2\Omega_0) \left| K_{(i\Omega/g)} \left( \omega g^{-1} \right) \right|^2 \times \left\{ \frac{1}{2} + N + \sqrt{N(N+1)} \cos[2(\omega_0-kx)] \right\},
\]
does not have a thermal nature because of the coefficients multiplying the expression in the curly brackets. Therefore, thermal ambience arises only for observers whose acceleration is along the same axis as the direction of propagation of the wave.

It is however interesting to ask: What happens to the power spectrum (28) in the limit of \( \omega \to 0 \)? In this limit, the original wave field is a constant and any direction of motion for the observer should be equivalent. Hence we expect to see the ‘thermal’ ambience in this limit even for this observer. This is indeed the case: In the limit of \( \omega \to 0 \)
\[
K_{(i\Omega_0 g^{-1})} \left( \omega g^{-1} \right) \approx 2^{(i\Omega_0 g^{-1} - 1)} (\omega g^{-1})^{-1} \Gamma(i\Omega_0 g^{-1}).
\]
Substituting the above approximation for \( K_{(i\Omega_0 g^{-1})} \left( \omega g^{-1} \right) \) in (28) one recovers the result given in (9) with \( \beta \) set to zero. This result also holds for a wave propagating in an arbitrary direction, as is to be expected.

4 ‘Thermal’ ambience in Schwarzschild and de-Sitter spacetimes

In this section, we shall briefly comment on the generalization of the above results to Schwarzschild and de-Sitter spacetimes. The solution to the Klein-Gordon equation in these spacetimes cannot be expressed in terms of simple functions in (3+1) dimensions and hence we will work in (1+1) dimensions.
In \((1+1)\) dimensions, the Schwarzschild spacetime is described by the line-element
\[
d s^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2.
\] (30)

In terms of the Regge-Wheeler coordinates \((t, r^*)\) \[9\], where
\[
r^* = r + 2M \ln \left(\frac{r}{2M} - 1\right),
\] (31)
the Schwarzschild line-element turns out to be conformal to the flat space metric, \(i.e.\)
\[
d s^2 = \left(1 - \frac{2M}{r}\right) (dt^2 - dr^*^2).
\] (32)

And, in terms of the Kruskal-Szekeres coordinates \((v, u)\) \[9\], which are related to the Regge-Wheeler coordinates \((t, r^*)\) by the transformations
\[
u = u_0 + e^{r^*/4M} \cosh(t/4M) \quad \text{and} \quad v = v_0 + e^{r^*/4M} \sinh(t/4M),
\] (33)
(where \(u_0\) and \(v_0\) are arbitrary constants) the Schwarzschild line-element reduces to
\[
ds^2 = \left(\frac{32M^3}{r}\right) e^{-(r/2M)} (dv^2 - du^2).
\] (34)

The proper time \(\tau\) of an observer stationed at a constant \(r\) is then related to the Schwarzschild time coordinate \(t\) by the equation
\[
\tau = \lambda(r) t \quad \text{where} \quad \lambda(r) = \left(1 - \frac{2M}{r}\right)^{1/2}.
\] (35)

Just as the trajectory of a uniformly accelerating observer is a hyperbola in the plane of the Minkowski coordinates, the world line of an observer stationed at a constant \(r\) is a hyperbola in the \((v, u)\) plane. And, the asymptotes of this hyperbola are the past and the future horizons of the Schwarzschild spacetime that intersect at the point \((v_0, u_0)\).

As is well known the action for a minimally coupled scalar field is conformally invariant in \((1+1)\) dimensions. Hence the normal modes of a massless, minimally coupled scalar field in conformally flat metrics are just plane waves. So the normal mode solutions of the Schwarzschild spacetime in the Kruskal-Szekeres coordinates \((v, u)\) are just plane waves. Consider a single real mode described by the equation
\[
\Phi(v, u) = \cos (\omega v - ku).
\] (36)
We would like to know how an observer located at constant (Schwarzschild) radial coordinate \( r \) will describe this mode. Assuming that the plane wave is traveling to the right, \( i.e. \ k = \omega \) and Fourier tranforming the monochromatic wave given in equation (36) with respect to the proper time \( \tau \) of an observer stationed at a constant \( r \), we obtain that

\[
\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau \, \Phi[v(\tau), u(\tau)] \, e^{-i\Omega \tau} = \lambda \int_{-\infty}^{\infty} dt \, \cos \left( \omega e^{(r^* - t)/4M} - \beta \right) e^{-i\Omega t} = 2M\lambda \, e^{-i\mu} \left( e^{-2\pi\Omega M\lambda} e^{-i\beta} + e^{2\pi\Omega M\lambda} e^{i\beta} \right) \Gamma (4i\Omega M\lambda), \tag{37}
\]

where

\[
\mu = 4\Omega M\lambda \ln \left( \omega e^{r^*/4M} \right) \quad \text{and} \quad \beta = \omega (v_0 - u_0). \tag{38}
\]

The resulting power spectrum per logarithmic frequency interval is then

\[
P(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2 = (4\pi M\lambda) \left\{ \frac{1}{2} + N + \sqrt{N(N+1)} \cos(2\beta) \right\}, \tag{39}
\]

where

\[
N(\Omega) = \left( \frac{1}{\exp(8\pi M\Omega\lambda) - 1} \right). \tag{40}
\]

We once again obtain the three terms discussed before.

The analysis for the de-Sitter spacetime is similar. The line-element that describes the de-Sitter spacetime is

\[
ds^2 = (1 - H^2 r^2) \, dt^2 - (1 - H^2 r^2)^{-1} \, dr^2. \tag{41}
\]

In terms of the ‘Regge-Wheeler’ coordinates \((t, r^*)\) corresponding to the de-Sitter spacetime, where

\[
r^* = H^{-1} \arctanh(Hr), \tag{42}
\]

the de-Sitter line-element turns out to be

\[
ds^2 = (1 - H^2 r^2) \left( dt^2 - dr^{*2} \right). \tag{43}
\]

The ‘Kruskal-Szekeres’ coordinates \((v, u)\) corresponding to the de-Sitter spacetime are related to the coordinates ‘Regge-Wheeler’ coordinates \((t, r^*)\) by the equations

\[
u = u_0 + e^{Hr^*} \cosh(Ht) \quad \text{and} \quad v = v_0 + e^{Hr^*} \sinh(Ht). \tag{44}
\]
The de-Sitter line-element in terms of the coordinates \((v, u)\) then reduces to
\[
\begin{equation}
    ds^2 = H^{-2} (1 - Hr)^2 (dv^2 - du^2).
\end{equation}
\]

Consider an observer who is stationed at a constant \(r\) in de-Sitter spacetime. The world line of such an observer, just as in the Schwarzschild case, is a hyperbola in the \((v, u)\) plane whose asymptotes are the past and the future horizons of the de-Sitter spacetime that intersect at the point \((v_0, u_0)\). The proper time \(\tau\) of this observer is related to the de-Sitter time coordinate \(t\) as follows
\[
    \tau = \lambda t, \quad \text{where now} \quad \lambda = (1 - H^2 r^2)^{1/2}.
\]

For the case of a real wave as given in (36), where the coordinates \(v\) and \(u\) are now related to de-Sitter coordinates \(t\) and \(r\) by the equations (44) and (42), the power spectrum per logarithmic frequency interval as seen by the observer stationed at a constant \(r\) is
\[
    \mathcal{P}(\Omega) \equiv \Omega |\tilde{\Phi}(\Omega)|^2 = (\pi H^{-1}\lambda) \left\{ \frac{1}{2} + N + \sqrt{N(N+1)} \cos(2\beta) \right\},
\]
where
\[
    N(\Omega) = \left( \frac{1}{\exp(2\pi\Omega H^{-1}\lambda) - 1} \right).
\]

In evaluating the powers spectrum above, it has been assumed that \(k = \omega\), so that \(\beta = \omega(v_0 - u_0)\). The similarity to the previous results are obvious.

### 5 Conclusions

In conclusion, we would like to stress those aspects of our results which are unexpected and contrast them with those which could have been anticipated with some hindsight.

To begin with, the following fact is well-known: In quantum field theory, the amplitude for transition of an Unruh-DeWitt detector, up to the first order in perturbation theory, is described by an integral that is similar in form to (3) [2, 3]. When the scalar field is decomposed in terms of the Minkowski modes, the transition probability, per unit proper time, of a uniformly accelerating Unruh-DeWitt detector turns out to be a thermal spectrum (see for instance [10]). It might, therefore, seem that when a traveling wave is Fourier transformed with respect to the proper time of a uniformly accelerated observer, the resulting power spectrum will have a thermal nature.
However, there are some subtleties involved. To begin with, the modes of the quantum field are complex while here we are dealing with real plane wave modes. This makes the vital difference. As we have mentioned before, while a complex mode like \( e^{\omega t - ikx} \) will give a Planckian distribution it will not yield the two other terms we have obtained in our analysis. In this sense, the real wave is quite different from the complex one. We stress the fact that, when a real Minkowski mode is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum not only contains a Planckian distribution but also contains the root mean square fluctuations about the Planckian. As mentioned earlier, it is the appearance of these fluctuations that motivates us to attribute a ‘thermal’ nature to the power spectrum. We know of no simple way to guess at this answer.

Secondly, note the effect survives in the power spectrum even in the limit of \( \omega \to 0 \). This is the closest to what one can call a ‘classical’ vacuum—and our result shows that such a mode, with infinitesimal frequency, leads to a thermal ambience in the accelerated frame which is totally independent of the properties of the original wave. This result suggest that there is a deep connection between plane waves, accelerated frames and thermal fluctuations even at the classical level. This connection could be worth exploring.

A somewhat similar analysis, *viz.* Fourier analyzing the Minkowski modes in the frame of an uniformly accelerated observer was carried out earlier by Gerlach [11]. He had constructed a linear superposition of Minkowski modes in (3+1) dimensions such that the modulus square of the amplitude of these modes (which represents the total classical energy of these modes) to be equivalent to that of the ground state energy of a quantum oscillator. Fourier analyzing such a field configuration with respect to the proper time of a uniformly accelerating observer, Gerlach had obtained a power spectrum (in a particular semiclassical limit) similar in form to equation (9). He had presented his result as a ‘heuristic derivation of the thermal spectrum’ that arises in quantum field theory due to the inequivalent quantization in Minkowski and Rindler coordinates. Our results and emphasis are different in several ways. To begin with, the effect we are reporting here is a feature of classical field theory and no quantum processes are involved. It is physically motivated in a clear and simple manner and we do not have to resort to any superposition of modes. Secondly, our results are *exact* for a real, monochromatic plane wave while Gerlach needed to resort to some approximations because of the particular superposition of modes he had chosen. Thirdly, we would like to draw attention to the zero-frequency limit of the wave,
when it takes a life of its own in the accelerated frame. This result, as far as we
know, has not been noted in the literature before. Finally, Gerlach had offered
no explanation for the appearance of the factor $\cos(2\beta)$ as the coefficient of the
fluctuation term. Our analysis clearly shows that it arises due to the shift in the
origin of the Minkowski coordinates.

In section 4 we have shown that a thermal ambience is a feature of black hole
spacetimes too. There has been an earlier attempt by Frolov [12] in which he had
modeled the black hole as a black body cavity and obtained a thermal spectrum
for the radiation leaking out of a cavity But Frolov had invoked quantum theory
to obtain his results. Now, knowing that a ‘thermal’ ambience can be a feature
of black hole spacetimes too, we are presently investigating the possibility of
interpreting the origin of black hole entropy purely classically.

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