On the nature of the new group LB1

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The new local group LB1 introduced previously will be studied in detail, depicting its unique nature that makes it a new group in fundamental physics. It will be made clear that even though most of its elements are Lorentz transformations, one unique discrete transformation not present in the Lorentz groups, is making this group into a new group. In fact there will be two kinds of discrete transformations, one a Lorentz transformation, the other not. It is these discrete transformations that allow for an isomorphism between the group SO(2) and LB1. An isomorphism between the compact group SO(2) and the non-compact group SO(1, 1) plus two kinds of discrete transformations that make up the new group LB1.
I. INTRODUCTION

The purpose of this manuscript is to make clear the nature of the group LB1 through a detailed study of its elements. To these end we will consider just the fundamental elements and steps associated to tetrad construction in a four-dimensional Lorentzian curved spacetime. The whole analysis as given in manuscript \textsuperscript{1}, will be considered already understood. Therefore, we will start this work introducing the tetrads found in the aforementioned paper\textsuperscript{1} that locally and covariantly diagonalize the electromagnetic stress-energy tensor. This local tetrad electromagnetic gauge transformations reflect the existence of an isomorphism between the local internal group $U(1)$ and the local group of spacetime transformations LB1. By means of these tetrad vectors we will proceed to the study of their local gauge transformation properties and analyze one by one all the group elements in section \textsuperscript{II}. We use a metric with sign conventions $-++$. If $F_{\mu\nu}$ is the electromagnetic field then $f_{\mu\nu} = (G^{1/2}/c^2) F_{\mu\nu}$ is the geometrized electromagnetic field.

II. LOCAL GAUGE TRANSFORMATIONS ON BLADE ONE

In geometrodynamics, the Maxwell equations, $f_{\mu'\nu'} = 0$ and $*f_{\mu'\nu'} = 0$ are telling us that two potential vector fields $A_\nu$ and $*A_\nu$ exist\textsuperscript{2}. Note that the star in $*A_\nu$ is just a name. The symbol $\cdot$ stands for covariant derivative with respect to the metric tensor $g_{\mu\nu}$. When we make the transformation,

$$A_\alpha \rightarrow A_\alpha + \Lambda_\alpha, \quad (1)$$

$f_{\mu\nu} = A_\nu;\mu - A_\mu;\nu$ remains invariant, and the transformation,

$$*A_\alpha \rightarrow *A_\alpha + *\Lambda_\alpha, \quad (2)$$

leaves $*f_{\mu\nu} = *A_\nu;\mu - *A_\mu;\nu$ invariant, as long as the functions $\Lambda$ and $*\Lambda$ are scalars. These two potentials are not independent from each other, but necessary in the construction of the new local tetrad that we will introduce shortly. We briefly remind ourselves that the original expression for the electromagnetic stress-energy tensor $T_{\mu\nu} = f_{\mu\lambda} f_{\nu}^\lambda + *f_{\mu\lambda} *f_{\nu}^\lambda$ was given in terms of the electromagnetic tensor $f_{\mu\nu}$ and its dual $*f_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} f^{\sigma\tau}$.
After a local duality transformation,

\[ f_{\mu\nu} = \xi_{\mu\nu} \cos \alpha + *\xi_{\mu\nu} \sin \alpha , \]  

(3)

where the local scalar \(\alpha\) is the complexion, we are able to write the stress-energy in terms of the extremal field \(\xi_{\mu\nu}\) and its dual. We can express the extremal field as,

\[ \xi_{\mu\nu} = e^{-*\alpha} f_{\mu\nu} = \cos \alpha f_{\mu\nu} - \sin \alpha * f_{\mu\nu}. \]  

(4)

Extremal fields satisfy the equation

\[ \xi_{\mu\nu} * \xi^{\mu\nu} = 0 . \]  

(5)

This a condition imposed on extremal fields in order to find a local scalar named the complexion \(\alpha\). It can be proved that condition (5) and through the use of the general identity,

\[ A_{\mu\nu} B^{\nu\alpha} - *B_{\mu\alpha} * A^{\nu\alpha} = \frac{1}{2} \delta_{\mu}^{\nu} A_{\alpha\beta} B^{\alpha\beta} , \]  

(6)

which is valid for every pair of antisymmetric tensors in a four-dimensional Lorentzian spacetime, when applied to the case \(A_{\mu\alpha} = \xi_{\mu\alpha}\) and \(B^{\nu\alpha} = *\xi^{\nu\alpha}\) yields the equivalent condition,

\[ \xi_{\alpha\mu} * \xi^{\mu\nu} = 0 . \]  

(7)

With all these elements we proceed to introduce without further delay the new orthonormal tetrad,

\[ U^\alpha = \xi^{\alpha\lambda} \xi_{\rho\lambda} A^\rho / (\sqrt{-Q/2} \sqrt{A_{\mu} \xi^{\mu\sigma} \xi_{\nu\sigma} A^{\nu}}) \]  

(8)

\[ V^\alpha = \xi^{\alpha\lambda} A_{\lambda} / (\sqrt{A_{\mu} \xi^{\mu\sigma} \xi_{\nu\sigma} A^{\nu}}) \]  

(9)

\[ Z^\alpha = *\xi^{\alpha\lambda} * A_{\lambda} / (\sqrt{*A_{\mu} * \xi^{\mu\sigma} * \xi_{\nu\sigma} * A^{\nu}}) \]  

(10)

\[ W^\alpha = *\xi^{\alpha\lambda} * \xi_{\rho\lambda} A^\rho / (\sqrt{-Q/2} \sqrt{*A_{\mu} * \xi^{\mu\sigma} * \xi_{\nu\sigma} * A^{\nu}}) . \]  

(11)
where \( Q = \xi_{\mu\nu} \xi^{\mu\nu} = -\sqrt{T_{\mu\nu}T^{\mu\nu}} \) according to equations (39) in \(^2\). \( Q \) is assumed not to be zero, because we are dealing with non-null electromagnetic fields. The first two eigenvectors of the stress-energy tensor \(^9\) with eigenvalue \( Q/2 \), the last two \(^10\) with eigenvalue \(-Q/2\). Schouten defined what he called a two-bladed structure in a spacetime\(^4\). Tetrad vectors \(^9\) define the local blade one and vectors \(^10\) define the local blade two.

Let us introduce some names. Setting aside normalization factors, the tetrad vectors have two essential components, see reference\(^1\). For instance in vector \( U_{\text{not-normalized}}^{\alpha} = \xi^{\alpha\lambda} \xi_{\rho\lambda} X^\rho \) there are two main structures. First, the skeleton, in this case \( \xi^{\alpha\lambda} \xi_{\rho\lambda} \), and second, the gauge vector \( X^\rho \). In vector \( Z_{\text{not-normalized}}^{\alpha} = \ast \xi^{\alpha\lambda} Y_\lambda \), the skeleton is \( \ast \xi^{\alpha\lambda} \), and the gauge vector is \( Y^\rho \). The gauge vectors it was proved in manuscript\(^1\) could be anything that does not make the tetrad vectors trivial. That is, the tetrad \(^9\) diagonalizes the stress-energy tensor for any non-trivial gauge vectors \( X^\mu \) and \( Y^\mu \). It was therefore proved that we can make different choices for \( X^\mu \) and \( Y^\mu \). We made the choices \( X^\mu = A^\mu \) and \( Y^\mu = \ast A^\mu \). For the purpose of making the notation compatible with that of manuscript\(^1\), let us introduce the non-normalized local tetrad that covariantly diagonalizes the electromagnetic stress-energy tensor,

\[
\begin{align*}
V^{\alpha}_{(1)} &= \xi^{\alpha\lambda} \xi_{\rho\lambda} A^\rho \\
V^{\alpha}_{(2)} &= \sqrt{-Q/2} \xi^{\alpha\lambda} A_\lambda \\
V^{\alpha}_{(3)} &= \sqrt{-Q/2} \ast \xi^{\alpha\lambda} \ast A_\lambda \\
V^{\alpha}_{(4)} &= \ast \xi^{\alpha\lambda} \ast \xi_{\rho\lambda} \ast A^\rho ,
\end{align*}
\]

We are going to use this particular version in order to study the tetrad local electromagnetic gauge transformations. Vector \( V_{(1)}^{\alpha} \) is assumed to be timelike, see reference\(^1\). We reiterate that without repeating the analysis in manuscript\(^1\) we proceed to study the different possible local tetrad transformation cases on blade one resorting to the same notation employed in this previous work. In order to simplify the notation we are going to write \( \Lambda_{\alpha} = \Lambda_\alpha \).

The purpose of this work is to study the different cases that arise when we consider the change in \(^12\) under \(^1\),
\[ \tilde{V}_\alpha^{(1)} = V_\alpha^{(1)} + \xi^{\alpha\lambda} \xi_{\rho\lambda} \Lambda^\rho, \]  
\[ \tilde{V}_\alpha^{(2)} = V_\alpha^{(2)} + \sqrt{-Q/2} \xi^{\alpha\lambda} \Lambda_\lambda, \]

Since the second terms in (16-17) and according to equation (7) belong in the local plane or blade one, we proceed then to write equations (16-17) as,

\[ \tilde{V}_\alpha^{(1)} = V_\alpha^{(1)} + C V_\alpha^{(1)} + D V_\alpha^{(2)}, \]  
\[ \tilde{V}_\alpha^{(2)} = V_\alpha^{(2)} + E V_\alpha^{(1)} + F V_\alpha^{(2)}. \]

After some algebraic work it was found in reference 1 that have the following relations between coefficients,

\[ E = D \]  
\[ F = C. \]

It was also found that,

\[ C = (-Q/2) V_\alpha^{(1)} \Lambda^\sigma / (V_\beta^{(2)} V_\beta^{(2)}), \]  
\[ D = (-Q/2) V_\alpha^{(2)} \Lambda^\sigma / (V_\alpha^{(1)} V_\alpha^{(1)}). \]

After all this algebraic work we would like to calculate the norm of the transformed vectors \( \tilde{V}_\alpha^{(1)} \) and \( \tilde{V}_\alpha^{(2)} \),

\[ \tilde{V}_\alpha^{(1)} V_{(1)\alpha} = [(1 + C)^2 - D^2] V_\alpha^{(1)} V_{(1)\alpha} \]  
\[ \tilde{V}_\alpha^{(2)} V_{(2)\alpha} = [(1 + C)^2 - D^2] V_\alpha^{(2)} V_{(2)\alpha}, \]

where the relation \( V_\alpha^{(1)} V_{(1)\alpha} = -V_\alpha^{(2)} V_{(2)\alpha} \) has been used. It is evident from equations (24-25) that two situations might arise. Either \( [(1 + C)^2 - D^2] \) is positive, or negative. Equality to zero will be analyzed at the end. The condition for these transformations to keep the
timelike or spacelike character of $V^{\alpha}_{(1)}$ and $V^{\alpha}_{(2)}$ is $[(1 + C)^2 - D^2] > 0$. If this condition is satisfied, then we can normalize the transformed vectors $\tilde{V}^{\alpha}_{(1)}$ and $\tilde{V}^{\alpha}_{(2)}$ in expressions (18-19) as follows,

\[
\frac{\tilde{V}^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} = \frac{(1 + C)}{\sqrt{(1 + C)^2 - D^2}} \frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} + \frac{D}{\sqrt{(1 + C)^2 - D^2}} \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} \quad (26)
\]

\[
\frac{\tilde{V}^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} = \frac{D}{\sqrt{(1 + C)^2 - D^2}} \frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} + \frac{(1 + C)}{\sqrt{(1 + C)^2 - D^2}} \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} . \quad (27)
\]

The condition $[(1 + C)^2 - D^2] > 0$ enables two possible situations, $1 + C > 0$ or $1 + C < 0$. When $1 + C > 0$, the transformations (26,27) are manifesting that an electromagnetic gauge transformation on the vector field $A^\alpha$, that leaves invariant the electromagnetic field $f_{\mu\nu}$, generates a boost transformation on the normalized tetrad vector fields \( \left( \frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}}, \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} \right) \). When the case $1 + C < 0$ is fulfilled, equations (26,27) can be rewritten,

\[
\frac{\tilde{V}^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} = \frac{[-(1 + C)]}{\sqrt{(1 + C)^2 - D^2}} \frac{(-V^{\alpha}_{(1)})}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} + \frac{[-D]}{\sqrt{(1 + C)^2 - D^2}} \frac{(-V^{\alpha}_{(2)})}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} \quad (28)
\]

\[
\frac{\tilde{V}^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} = \frac{[-D]}{\sqrt{(1 + C)^2 - D^2}} \frac{(-V^{\alpha}_{(1)})}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}} + \frac{[-(1 + C)]}{\sqrt{(1 + C)^2 - D^2}} \frac{(-V^{\alpha}_{(2)})}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} . \quad (29)
\]

Equations (28,29) represent the composition of two transformations. An inversion of the normalized tetrad vector fields \( \left( \frac{V^{\alpha}_{(1)}}{\sqrt{-V^{\beta}_{(1)} V_{(1)\beta}}}, \frac{V^{\alpha}_{(2)}}{\sqrt{V^{\beta}_{(2)} V_{(2)\beta}}} \right) \), and a boost.

If the case $[(1 + C)^2 - D^2] < 0$ is satisfied, the vectors $V^{\alpha}_{(1)}$ and $V^{\alpha}_{(2)}$ will change their timelike or spacelike nature,

\[
\tilde{V}^{\alpha}_{(1)} V_{(1)\alpha} = [-(1 + C)^2 + D^2] (-V^{\alpha}_{(1)} V_{(1)\alpha}) \quad (30)
\]

\[
(-\tilde{V}^{\alpha}_{(2)} V_{(2)\alpha}) = [-(1 + C)^2 + D^2] V^{\alpha}_{(2)} V_{(2)\alpha} . \quad (31)
\]

These are improper transformations on blade one. The normalized tetrad vectors $V^{\alpha}_{(1)}$ and $V^{\alpha}_{(2)}$ transform as,
In manuscript transformations \([32,33]\) have been described as “For \(D > 0\) and \(1+C > 0\) these transformations \([32,33]\) represent improper space inversions on blade one. If \(D > 0\) and \(1+C < 0\), equations \([32,33]\) are improper time reversal transformations on blade one.” This description is inaccurate. Let us see why. We can rewrite transformations \([32,33]\) for \(D > 0\) as the composition of two different kinds of transformations. First, a local boost given by \(\Lambda_o^2 = \frac{D}{\sqrt{-(1+C)^2+D^2}}, \Lambda_o^1 = \frac{(1+C)}{\sqrt{-(1+C)^2+D^2}}\), \(\Lambda_1^1 = \frac{D_1}{\sqrt{-(1+C)^2+D^2}}\). Second, a discrete transformation given by \(\Lambda_o^0 = 0, \Lambda_o^1 = 1, \Lambda_1^0 = 1, \Lambda_1^1 = 0\). We notice that this discrete transformation is not a Lorentz transformation. If the case is that \(D < 0\), we can proceed to analyze in analogy to \([28,29]\). Then, the normalized tetrad vectors transform as,

\[
\frac{\tilde{V}_{\alpha}^{(1)}}{\sqrt{V_{\beta}^{(1)} V_{\beta}^{(1)}}} = \frac{(1+C)}{\sqrt{-(1+C)^2+D^2}} + \frac{D}{\sqrt{-(1+C)^2+D^2}} \frac{V_{\alpha}^{(1)}}{\sqrt{V_{\beta}^{(1)} V_{\beta}^{(1)}}} \]

\[
\frac{\tilde{V}_{\alpha}^{(2)}}{\sqrt{V_{\beta}^{(2)} V_{\beta}^{(2)}}} = \frac{-D}{\sqrt{-(1+C)^2+D^2}} + \frac{(1+C)}{\sqrt{-(1+C)^2+D^2}} \frac{V_{\alpha}^{(2)}}{\sqrt{V_{\beta}^{(2)} V_{\beta}^{(2)}}} \]

Analogously to the previous case we wrote in manuscript \([34,35]\) “For \(D < 0\) and \(1+C < 0\) these transformations \([34,35]\) represent the composition of inversions, and improper space inversions on blade one. If \(D < 0\) and \(1+C > 0\), equations \([34,35]\) are inversions composed with improper time reversal transformations on blade one.” Once again this description is inaccurate. We can rewrite transformations \([34,35]\) for \(D < 0\) as the composition of three different kinds of transformations. First, a local boost given by \(\Lambda_o^0 = \frac{-D}{\sqrt{-(1+C)^2+D^2}}, \Lambda_o^1 = \frac{-(1+C)}{\sqrt{-(1+C)^2+D^2}}, \Lambda_1^1 = \frac{-D}{\sqrt{-(1+C)^2+D^2}}\). Second, a discrete transformation given by \(\Lambda_o^0 = 0, \Lambda_o^1 = 1, \Lambda_1^0 = 1, \Lambda_1^1 = 0\), which again is not a Lorentz transformation. Third, a full inversion. For the equality \(D = 1+C\) we can see using equations \([18,19]\) and \([20,21]\) that,
\[ \tilde{V}_\alpha^{(1)} = (1 + C) V_\alpha^{(1)} + (1 + C) V_\alpha^{(2)} \]  
(36)  
\[ \tilde{V}_\alpha^{(2)} = (1 + C) V_\alpha^{(2)} + (1 + C) V_\alpha^{(1)} . \]  
(37)

Equations (36-37) show that any vector on blade one transforms as,

\[ A V_\alpha^{(1)} + B V_\alpha^{(2)} \rightarrow A \tilde{V}_\alpha^{(1)} + B \tilde{V}_\alpha^{(2)} = (1 + C) (A + B) (V_\alpha^{(1)} + V_\alpha^{(2)}) . \]  
(38)

This is clearly a non-injective transformation. At the same time we know that there is an inverse transformation,

\[ \tilde{V}_\alpha^{(1)} = V_\alpha^{(1)} - \xi^{\alpha\lambda} \xi_{\rho\lambda} \Lambda^\rho = V_\alpha^{(1)} \]  
(39)  
\[ \tilde{V}_\alpha^{(2)} = V_\alpha^{(2)} - \sqrt{-Q/2} \xi^{\alpha\lambda} \Lambda^\lambda = V_\alpha^{(2)} . \]  
(40)

Then, the conclusion must be, that there could not exist a scalar function that satisfies the initial assumption \( D = 1 + C \). Analogous for \( D = -(1 + C) \).

III. CONCLUSIONS

There is an isomorphism between the “internal” group of local electromagnetic gauge transformations and the local group LB1 of spacetime transformations. This “connection” is established through new tetrads. These new tetrads setting aside normalization factors consist of two main elements. The skeleton and the gauge vector. The gauge dependence occurs through the two local gauge vectors involved in the construction of the four tetrad vectors that allow for the study of its local gauge transformations. When the elements of the group LB1 are analyzed in detail we found the following. It is composed by the group \( SO(1,1) \), the boosts, a discrete full inversion; so far all Lorentz transformations, and a new discrete transformation given by \( \Lambda^o_o = 0, \Lambda^o_1 = 1, \Lambda^1_o = 1, \Lambda^1_1 = 0 \), which is clearly not a Lorentz transformation\(^6\). We know from reference\(^1\) that the local group of electromagnetic gauge transformations is independently isomorphic to the local group of spatial tetrad rotations on blade two, that is \( SO(2) \). We notice immediately by transitivity that the group \( SO(2) \) is isomorphic to the group LB1. A compact group is isomorphic to
a non-compact group $SO(1,1)$ plus two different kinds of discrete transformations. This happens because of the discrete full inversion and the discrete transformation $\Lambda^o_o = 0$, $\Lambda^o_1 = 1$, $\Lambda^1_o = 1$, $\Lambda^1_1 = 0$, that we would like to call a “switch” and evidently is not a Lorentz transformation. Nonetheless we called this new group LB1, that is, Lorentz blade one, because all but one of its elements are Lorentz transformations. This new group LB1 allows for a “link” between the “internal” transformations and “spacetime” transformations. We quote from\textsuperscript{10} “...A second problem which at present is the subject of lively interest is the identity between the gravitational field and the electromagnetic field. The mind striving after unification of the theory cannot be satisfied that two fields should exist which, by their nature, are quite independent. A mathematically unified field theory is sought in which the gravitational field and the electromagnetic field are interpreted only as different components or manifestations of the same uniform field, the field equations where possible no longer consisting of logically mutually independent summands”.

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