K3 Surfaces, Picard Numbers and Siegel Disks

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Abstract

If a K3 surface admits an automorphism with a Siegel disk, then its Picard number is an even integer between 0 and 18. Conversely, using the method of hypergeometric groups, we are able to construct K3 surface automorphisms with Siegel disks that realize all possible Picard numbers. The constructions involve extensive computer searches for appropriate Salem numbers and computations of algebraic numbers arising from holomorphic Lefschetz-type formulas and related Grothendieck residues.

1 Introduction

Let $X$ be a complex K3 surface, that is, a simply connected compact complex surface with trivial canonical bundle $K_X$. The middle cohomology group $H^2(X, \mathbb{Z})$ equipped with the intersection form is an even unimodular lattice of signature $(3, 19)$. The Hodge decomposition gives an orthogonal direct sum decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

of signatures $(1, 0) \oplus (1, 19) \oplus (1, 0)$. The Picard group (or Néron-Severi group) of $X$ is the lattice $\text{Pic}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, whose rank $\rho(X)$ is called the Picard number of $X$. It is an integer between 0 and 20.

Given a K3 surface automorphism $f : X \to X$, let $\lambda(f)$ be the spectral radius of $f_*|H^{1,1}(X)$. Then $\lambda(f) \geq 1$ and the topological entropy of $f$ is given by $h(f) = \log \lambda(f)$. There exists a constant $\delta(f) \in S^1$ such that $f^*|H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ is the multiplication by $\delta(f)$. If $p \in X$ is a fixed point of $f$ then the holomorphic tangent map $(df)_p : T_p X \to T_p X$ has determinant $\delta(f)$, so the number $\delta(f)$ is called the determinant of $f$ by McMullen [8]. It is referred to as the special eigenvalue of $f$ in our previous paper [5], where $\tau(f) := \delta(f) + \delta(f)^{-1}$ is called the special trace. We remark that $\delta(f)$ is either a root of unity or a conjugate to a Salem number, and if $X$ is projective then $\delta(f)$ must be a root of unity. Here a Salem number is an algebraic integer $\lambda > 1$ which is conjugate to $\lambda^{-1}$ and whose remaining conjugates lie on $S^1$.

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. A map $R : (\mathbb{D}^2, 0) \to (\mathbb{D}^2, 0), (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$ with $\alpha_1, \alpha_2 \in S^1$ is said to be an irrational rotation if $\alpha_1$ and $\alpha_2$ are multiplicatively independent, that is, if $\alpha_1^{m_1} \alpha_2^{m_2} = 1$ with $m_1, m_2 \in \mathbb{Z}$ implies $m_1 = m_2 = 0$. Let $f : X \to X$ be an automorphism of a complex surface $X$. An open subset $U$ of $X$ is said to be a Siegel disk for $f$ centered at $p \in U$ if $f$ preserves $(U, p)$ and $f|_U : U(p) \to U(p)$ is biholomorphically conjugate to an irrational rotation $R : (\mathbb{D}^2, 0) \to (\mathbb{D}^2, 0)$. If $X$ is a K3 surface and $f$ admits a Siegel disk, then $\lambda(f)$ must be a Salem number and $\delta(f)$ must be conjugate to $\lambda(f)$, in particular $X$ must be non-projective and $f$ must have a positive topological entropy (see McMullen [8]).

McMullen [8] synthesized examples of K3 surface automorphisms with a Siegel disk whose underlyng K3 surfaces had Picard number 0. Oguiso [10] found an example of Picard number 8. In [5] we constructed examples of Picard number 12, whose entropy was the logarithm of Lehmer’s number $\Lambda_L \approx 1.17628$, the smallest Salem number ever known, as well as many more examples of Picard number 0. The existence of a Siegel disk imposes a restriction on the Picard number of the underlying K3 surface. In this article we construct K3 surface automorphisms with Siegel disks that realize all possible Picard numbers. Our main result is stated as follows.

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Let $A_u$ (see Gross and McMullen [4, Proposition 3.3]). The minimal polynomial of a Salem number is a polynomial of degree even $n$, when $n = 2$ is proved in a more general context in [3]. If $a$ is an even integer between 0 and 18, Conversely, for any such integer $r$ there exist K3 surface automorphisms $f : X \to X$ with Siegel disks such that $X$ has Picard number $\rho(X) = r$.

The first half of the theorem is just a corollary to [3] Theorem 7.4] and the essential part of the theorem is the second half stating that all Picard numbers $\rho = 0, 2, 4, \ldots, 18$ can be realized by K3 surface automorphisms with Siegel disks. So this article is devoted to establishing the result in the second direction.

The construction of a K3 surface automorphism boils down to a lattice theoretic problem. Let $L$ be an abstract K3 lattice endowed with a Hodge structure $L_C = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ where $L_C := L \otimes \mathbb{C}$. It determines the Picard lattice $Pic := H^{1,1} \cap L$, root system $\Delta := \{ u \in Pic \mid (u, u) = -2 \}$ and Weyl group $W$, the group generated by reflections in root vectors. A positive cone $C^+$ is one of the two connected components of $C := \{ v \in H^2 \mid (v, v) > 0 \}$, where $H^{1,1}_R := H^{1,1} \cap L_R$ with $L_R := L \otimes \mathbb{R}$. We specify a Weyl chamber $K \subset C^+$ as the “Kähler cone”. This is equivalent to dividing $\Delta$ into positive and negative roots $\Delta = \Delta^+ \cup \Delta^-$ in such a manner that $K = \{ v \in C^+ \mid (v, u) > 0 \}$ for any $u \in \Delta^+$. Note that $\Delta^+$ determines a unique set of simple roots, say $\Delta_0$, and vice versa. A Hodge isometry on $L$ is said to be positive if it preserves the connected components $C^\pm$ of $C$. It falls into one of the three types; elliptic, parabolic and hyperbolic. By Torelli theorem and surjectivity of period mapping (see [1] Chap. VIII) any positive Hodge isometry $F : L \to L$ preserving the Kähler cone $K$ lifts to a unique K3 surface automorphism $f : X \to X$ up to isomorphisms. In this article we deal with the case where Pic is negative definite, so that $\Delta$ and $W$ are finite, and $F$ is a positive Hodge isometry of hyperbolic type. Then the resulting lift $f$ is a non-projective K3 surface automorphism of positive entropy.

We realize such structures by the method of hypergeometric groups developed in our article [5]. This method produces a large number of non-projective K3 surface automorphisms of positive entropy with various Picard numbers. From them we look for automorphisms with Siegel disks such that $X$ is a K3 surface automorphism of positive entropy.

The method of Hypergeometric Groups

To review the hypergeometric method, we recall some concepts and terminology on polynomials. In this section all polynomials are monic and defined over $\mathbb{Z}$. Given a polynomial $u(z)$ of degree $n$, its reciprocal is defined by $u^!(z) := z^n u(z^{-1})$. We say that $u(z)$ is palindromic if $u^!(z) = u(z)$ and anti-palindromic if $u^!(z) = -u(z)$. If $u(z)$ is palindromic of even degree $n = 2m$, then there exists a unique polynomial $U(w)$ of degree $m$ such that $u(z) = z^m U(z + z^{-1})$. If $u(z)$ is anti-palindromic of even degree $n = 2m$, then there exists a unique polynomial $U(w)$ of degree $m - 1$ such that $u(z) = (z - 1)(z + 1)z^{m-1} U(z + z^{-1})$. In either case $U(w)$ is referred to as the trace polynomial of $u(z)$. A palindromic polynomial $u(z)$ is said to be unramified if $|u(\pm 1)| = 1$. Such a polynomial is of degree even $n = 2m$, has an even number, say $2t$, of roots outside $S^1$ and satisfies

$$t \equiv m \mod 2, \quad u(1) \cdot u(-1) = (-1)^m$$

(see Gross and McMullen [3] Proposition 3.3]). The minimal polynomial of a Salem number is a Salem polynomial, which is palindromic of even degree and whose trace polynomial is called a Salem trace polynomial. For any unramified Salem polynomial $u(z)$ the congruence in [1] reads $m \equiv t = 1 \mod 2$ and hence

$$\deg u(z) \equiv 2 \mod 4.$$  \hspace{1cm} (2)

Consider a coprime pair of anti-palindromic polynomial $\varphi(z)$ and palindromic polynomial $\psi(z)$ of degree 22. Let $A$ and $B$ be the companion matrices of $\varphi(z)$ and $\psi(z)$ respectively, and let $H := (A, B) \subset \text{GL}(22, \mathbb{Z})$ be
the hypergeometric group generated by $A$ and $B$. Then $C := A^{-1}B$ is a reflection, fixing a hyperplane in $\mathbb{Q}^{22}$ pointwise and sending a nonzero vector $r \in \mathbb{Q}^{22}$ to its negative $-r$. We have a free $\mathbb{Z}$-module of rank 22,

$$L = \langle r, Ar, \ldots, A^{21}r \rangle_\mathbb{Z} = \langle r, Br, \ldots, B^{21}r \rangle_\mathbb{Z},$$

(3)

stable under the action of $H$. We can make $L$ into an $H$-invariant even lattice by providing it with the symmetric bilinear form $(A^{-1}r, A^{-1}r) = \xi_{i-j}$, where $\xi_0 := 2$ and $\{\xi_i\}_{i=1}^\infty$ is defined via the Taylor series expansion

$$\frac{\psi(z)}{\varphi(z)} = 1 + \sum_{i=1}^\infty \xi_i z^{-i} \quad \text{around} \quad z = \infty. \quad (4)$$

The Gram matrix $(B^{\pm 1}r, B^{\pm 1}r)$ for the $B$-basis is given by exchanging $\varphi(z)$ and $\psi(z)$ upside down in formula (4). The lattice $L$ is unimodular if and only if the resultant of $\varphi(z)$ and $\psi(z)$ satisfies

$$\text{Res}(\varphi, \psi) = \pm 1, \quad (5)$$

in which case $\psi(z)$ must be unramified. Indeed, since $\varphi(z)$ is divisible by $(z-1)(z+1)$, the resultant is divisible by $\psi(1) \cdot \psi(-1)$ over $\mathbb{Z}$, hence (5) implies $|\psi(\pm 1)| = 1$. For details we refer to [5] Theorem 2.1.

If the index of $L$ is positive, we replace $L$ by its negative $L(-1)$; otherwise, we keep $L$ as it is. This procedure is referred to as the renormalization of $L$ and the renormalized bilinear form is called the intersection form on $L$. In [5] we give a necessary and sufficient condition for the renormalized lattice $L$ to be a K3 lattice with a Hodge structure such that $A$ is a positive Hodge isometry of hyperbolic type. To review it, let $\Phi(w)$ and $\Psi(w)$ be the trace polynomials of $\varphi(z)$ and $\psi(z)$ respectively, that is,

$$\varphi(z) = (z-1)(z+1)z^{10}\Phi(z + z^{-1}), \quad \psi(z) = z^{11}\Psi(z + z^{-1}).$$

Let $A$ be the multi-set of all complex roots of $\Phi(w)$ counted with multiplicity. Let $A_{\text{on}}$ and $A_{\text{off}}$ be those parts of $A$ which lie on and off the interval $[-2, 2]$ respectively. Define $B$, $B_{\text{on}}$ and $B_{\text{off}}$ in a similar manner for $\Psi(w)$. Then $A_{\text{on}}$ and $B_{\text{on}}$ dissect each other into interlacing components, called trace clusters, such that

$$-2 \leq A_{s+1} < B_s < A_s < \cdots < B_1 < A_1 \leq 2, \quad (6)$$

where one or both of the end clusters $A_1$ and $A_{s+1}$ may be null, while any other cluster must be non-null. Put $A_{>2} := A \cap (2, \infty)$; $|A_{\text{on}}|$ stands for the cardinality of $A_{\text{on}}$ counted with multiplicity; $|A_{\text{on}}| = 0^\nu_0 1^{\nu_1} 2^{\nu_2} 3^{\nu_3}$ means that $A_{\text{on}}$ consists of $\nu_0$ null clusters, $\nu_1$ simple clusters, $\nu_2$ double clusters, $\nu_3$ triple clusters, where $j^{\nu_j}$ is omitted if $\nu_j = 0$. The same rule applies to $B_{\text{on}}$ and other related entities. By “doubles adjacent” we mean the situation in which $A_{\text{on}}$ and $B_{\text{on}}$ contain unique double clusters $A_i$ and $B_j$ respectively, with $A_i$ and $B_j$ being adjacent to each other. If $A_i \cup B_j$ consists of four elements $x_1 < x_2 < x_3 < x_4$, then $x_2$ and $x_3$ are called the inner elements of the adjacent pair (AP). As a part of [5] Theorem 1.2) we have the following.

| case | $s$ | $|A_{\text{on}}|$ | $|B_{\text{on}}|$ | $|A_{>2}|$ | $|B_{\text{off}}|$ | constraints | ST $\tau(A)$ |
|------|----|----------------|----------------|--------|--------------|-------------|-------|
| 1    | 8  | $0^41^33^1$    | $1^8$          | 1      | 3            | $|A_1| = 2$  | middle of TC |
| 2    | 8  | $0^21^63^1$    | $1^73^1$       | 1      | 1            | $|A_1| = 2$  | middle of TC |
| 3    | 8  | $0^11^72^1$    | $1^8$          | 1      | 3            | $|A_1| = 2$  | max $A_1$ |
| 4    | 8  | $0^11^72^1$    | $1^8$          | 1      | 3            | $|A_1| = 2$  | min $A_9$ |
| 5    | 8  | $0^11^72^1$    | $1^73^1$       | 1      | 1            | $|A_1| = 2$  | max $A_1$ |
| 6    | 8  | $0^11^72^1$    | $1^73^1$       | 1      | 1            | $|A_1| = 2$  | min $A_9$ |
| 7    | 9  | $0^21^72^1$    | $1^82^1$       | 1      | 1            | doubles adjacent | inner of AP |
| 8    | 9  | $0^11^9$       | $1^82^1$       | 1      | 1            | $|A_1| = 1, |B_1| = 2$ | element of $A_1$ |
| 9    | 9  | $0^11^9$       | $1^82^1$       | 1      | 1            | $|A_{10}| = 1, |B_1| = 2$ | element of $A_{10}$ |

Table 2.1: Conditions for $A$ to be a positive Hodge isometry of hyperbolic type [5] Table 1.2].
Theorem 2.1 Let \( L = L(\psi, \psi) \) be a unimodular hypergeometric lattice of rank 22. After renormalization, \( L \) is a K3 lattice with a Hodge structure such that \( A \) is a positive Hodge isometry of hyperbolic type, if and only if \( \Phi(\pm 2) \neq 0 \), the roots of \( \Phi(w) \) and \( \Psi(w) \) are all simple and have any one of the configurations in Table 2.1. In this case the special trace \( \tau(A) \) and the Hodge structure up to complex conjugation are uniquely determined by the pair \( (\psi, \psi) \). The location of \( \tau(A) \) is shown in the last column of Table 2.1 where we mean by “middle of TC” that \( \tau(A) \) is the middle element of the unique triple cluster (TC) in \( \mathbf{A}_{\text{on}} \), and by “inner of AP” that \( \tau(A) \) is the inner element in \( \mathbf{A}_{\text{on}} \) of the unique AP of double clusters in \( \mathbf{A}_{\text{on}} \cup \mathbf{B}_{\text{on}} \).

In the situation of Theorem 2.1 \( \varphi(z) \) factors as \( \varphi(z) = \varphi_0(z) \cdot \varphi_1(z) \) where \( \varphi_0(z) \) is a Salem polynomial and \( \varphi_1(z) \) is a product of cyclotomic polynomials. Note that \( \varphi_1(z) \) is divisible by \((z - 1)(z + 1)\). So we write

\[
\varphi_0(z) = S(z), \quad \varphi_1(z) = (z - 1)(z + 1) \cdot C(z).
\]

Let \( \lambda(A) > 1 \) be the Salem number associated with \( S(z) \) and let \( \delta(A)^{\pm 1} \in S^1 \) be the special eigenvalues corresponding to the special trace \( \tau(A) \) in Theorem 2.1, that is, \( \delta(A) + \delta(A)^{-1} = \tau(A) \). Then \( \delta(A) \) is conjugate to \( \lambda(A) \) and the Hodge structure (up to complex conjugation) is given by

\[
L_C = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} = \ell \oplus (\ell + \bar{\ell}) \oplus \bar{\ell},
\]

where \( \ell \) is the eigen-line of \( A \) corresponding to the eigenvalue \( \delta(A) \) and \( \bar{\ell} \) is the complex conjugate to \( \ell \). Specify a positive cone \( C^+ \subset H^{1,1} \) and put \( s := (\lambda/\delta)(r) \) where \( r \) is the vector in \( \mathbf{B} \). Then the intersection form is negative definite on the Picard lattice \( \text{Pic} := H^{1,1} \cap L \), whose rank, i.e. its Picard number is given by

\[
\rho = 22 - \text{deg}(S(z)),
\]

and the vectors \( s, A s, \ldots, A^{\rho-1} s \) form a free basis, the standard basis, of \( \text{Pic} \) (see [5] Theorem 1.5]). The root system \( \Delta := \{ u \in \text{Pic} : (u, u) = -2 \} \) and the Weyl group \( W \) are defined in the usual manner. The lexicographic order on \( \text{Pic} \) with respect to the standard basis leads to a set of positive roots \( \Delta^+ \) and the corresponding Weyl chamber \( \mathcal{K} := \{ v \in C^+ : (v, u) > 0 \text{ for any } u \in \Delta^+ \} \), which we specify as the “\( \text{Kähler} \) cone”.

The matrix \( A \) may not preserve \( \mathcal{K} \), but there is a unique element \( w_A \in W \) such that \( A := w_A \circ A \) preserves \( \mathcal{K} \). We have an algorithm to determine \( \Delta, \Delta^+, \Delta_h, \text{ and } w_A \) explicitly from the initial data \( (\psi, \psi) \), where \( \Delta_h \) is the simple system relative to \( \Delta^+ \) (see [5] Algorithm 7.5]). The Dynkin type of \( \Delta \) can be read off from the intersection relations for the simple roots in \( \Delta_h \). The characteristic polynomial \( \bar{\varphi}(z) \) of \( A \) factors as

\[
\bar{\varphi}(z) = \varphi_0(z) \cdot \bar{\varphi}_1(z),
\]

where \( \varphi_0(z) = S(z) \) is the same Salem polynomial as the one in [7] while \( \bar{\varphi}_1(z) \) is a product of cyclotomic polynomial which, however, may differ from \( \varphi_1(z) \) in [7]. In particular \( A \) and \( \bar{A} \) have the same spectral radius \( \lambda(A) \) and the same special eigenvalue \( \delta(A) \). Preserving the Hodge structure \( [5] \) and the \( \text{Kähler} \) cone \( \mathcal{K} \), the modified matrix \( \bar{A} \) lifts to a K3 surface automorphism \( f : X \to X \) of entropy \( h(f) = \log \lambda(A) \) with special eigenvalue \( \delta(f) = \delta(A) \). Picard lattice \( \text{Pic}(X) \cong \text{Pic} \) and Picard number \( \rho(X) = \rho \) given in [4]. Recall that

\[
\bar{\varphi}_1(z) \quad \text{is the characteristic polynomial of } f^*|\text{Pic}(X).
\]

Moreover, \( \Delta, \Delta^+ \) and \( \Delta_h \) lift to \( \Delta(X), \Delta^+(X) \) and \( \Delta_h(X) \) respectively, where \( \Delta(X) \) is the set of all \((-2)\)-classes in \( \text{Pic}(X) \) with \( \Delta^+(X) \) being its subset of all effective \((-2)\)-classes and \( \Delta_h(X) \) is the set of all \((-2)\)-curves in \( X \). How \( f \) permutes the elements of \( \Delta_h(X) \) is faithfully represented by the action of \( A \) on \( \Delta_h \).

3 Computer Searches

Let \( \mathcal{P} \) be a finite set of polynomials \( \varphi(z) = (z - 1)(z + 1) \cdot S(z) \cdot C(z) \) of degree 22 such that \( S(z) \) is a Salem polynomial and \( C(z) \) is a product of cyclotomic polynomials; see [7]. Similarly let \( \mathcal{Q} \) be a finite set of unramified palindromic polynomials \( \psi(z) \in \mathbb{Z}[z] \) of degree 22, where unramifiedness comes from the remark after [5]. For various choices of \( \mathcal{P} \) and \( \mathcal{Q} \) we make extensive computer searches for those pairs \( (\varphi, \psi) \in \mathcal{P} \times \mathcal{Q} \) which satisfy firstly the unimodularity condition \( [5] \) and secondly all the conditions in Theorem 2.1.

The Salem numbers with any given degree, below any given bound, are finite in their cardinality. Thus we can speak of the \( i \)-th smallest Salem number \( \lambda_i^{(d)} \) of degree \( d \) and its minimal polynomial \( S_i^{(d)}(z) \). The trace polynomial of \( S_i^{(d)}(z) \) is denoted by \( \text{ST}_i^{(d)}(w) \). In his web page [9] Mossinghoff gives a complete list of Salem numbers of small degrees, below certain bounds. A careful inspection of his tables together with the constraint [2] for unramifiedness leads us to the following observation.
Lemma 3.1 Let $d$ be an even integer such that $4 \leq d \leq 22$. Then there exist exactly $N_d$ Salem numbers $\lambda$ of degree $d$ up to bound $\lambda \leq M_d$, where $M_d$ and $N_d$ are given in Table 3.1 with GR := $\phi := (1 + \sqrt{5})/2 \approx 1.61803$ being the golden ratio. Unramified Salem numbers of degree $d$ exist only when $d = 6, 10, 14, 18, 22$, for each of which there are exactly $N'_d$ such numbers up to bound $\lambda \leq M_d$, where $N'_d$ is again given in Table 3.1.

| $d$ | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | total |
|-----|---|---|---|----|----|----|----|----|----|----|-------|
| $M_d$ | 3 | 2.8 | 2.6 | 2.4 | 2.2 | 2 | 1.8 | GR | GR | 1.5 | — |
| $N_d$ | 8 | 34 | 104 | 223 | 314 | 390 | 231 | 141 | 191 | 89 | 1725 |
| $N'_d$ | — | 3 | 29 | — | 67 | — | 42 | — | 30 | 171 | — |

Table 3.1: Salem numbers $\lambda$ of degree $d \leq 22$ up to bound $\lambda \leq M_d$.

The $j$-th cyclotomic polynomial is denoted by $C_j(z)$. In [5, §5.2] we employ unconventional definitions $C_1(z) = (z - 1)^2$ and $C_2(z) = (z + 1)^2$ for $j = 1, 2$, but in this article we take the usual ones $C_1(z) = z - 1$ and $C_2 = z + 1$. For any $j \geq 3$ the polynomial $C_j(z)$ is palindromic of even degree, hence the congruence in [11] with $t = 0$ implies that any unramified cyclotomic polynomial has a degree divisible by 4. By [5, Lemma 5.3] all unramified cyclotomic polynomials $C_1(z)$ of deg $C_1(z) \leq 16$ are exactly those with $l \in L_0$, where

$L_0 := \{12, 15, 20, 21, 24, 28, 30, 36, 40, 42, 48, 60\}.$

(12)

Setting up the “principal” set $\mathcal{P}$ is simple. Put $S(z) = S_1^{(d)}(z)$ for an even integer $d$ with $4 \leq d \leq 20$ and let

$$C(z) = \prod_{j \in J} C_j(z), \quad \text{subject to the degree constraint } d + \sum_{j \in J} \deg C_j(z) = 20,$$

(13)

where $J$ is a finite subset of $\mathbb{Z}_{\geq 3}$ with $J = \emptyset$ for $d = 20$. We remark that $J$ contains neither 1 nor 2 because of $\Phi(\pm 2) \neq 0$ in Theorem 2.1. From (9) the K3 surface to be constructed will have Picard number

$$\rho = 22 - d.$$

(14)

Thus fixing a Picard number $\rho$ is fixing the degree $d$ according to (14) and specifying $\mathcal{P}$ amounts to taking a finite subset of positive integers over which the index $i$ of $S_i^{(d)}(z)$ ranges. The set of $J$’s is determined by $d$ according to (13). In the Appendix we give a list of all Salem polynomials $S_i^{(d)}(w)$ that appear as $S(z)$ explicitly in this article; they are given in terms of their trace polynomials $S_{\mathcal{J}}^{(d)}(w)$.

To set up the “auxiliary” set $\mathcal{Q}$ we observe from Table 2.1 that the trace polynomial $\Psi(w)$ of $\psi(z)$ must have either ten or eight roots in $(-2, 2)$. An instance of the ten-root case is realized by the following setup.

Setup 3.2 Let $\mathcal{Q}$ be the set of all polynomials $\psi(z) = s(z) \cdot c(z)$ of degree 22 such that $s(z) = S_i^{(d)}(z)$ is an unramified Salem polynomial in Lemma 3.1 and $c(z)$ is a product of unramified cyclotomic polynomials,

$$c(z) = \prod_{l \in L} C_l(z), \quad \text{subject to the degree constraint } e + \sum_{l \in L} \deg C_l(z) = 22.$$  

(14)

Here since $e \geq 6$ it follows from (14) that $\deg C_l(z) \leq 16$ for any $l \in L$, hence $L$ must be a subset of $L_0$ in (12).

For Picard numbers $\rho = 2, 4, 6, \ldots, 16$, that is, for $d = 20, 18, 16, \ldots, 6$, Setup 3.2 with some choices of small indices $i$ (or even $i = 1$ only) for $S(z) = S_i^{(d)}(z)$ gives an abundance of solutions $(\varphi, \psi)$ satisfying the conditions in Theorem 2.1. We illustrate this by two computer outputs; one is for $\rho = 6$ ($d = 16$), $i = 1, \ldots, 5$, and the other is for $\rho = 14$ ($d = 8$), $i = 1, \ldots, 16$. In these cases the results are given in Tables 3.2 and 3.3 respectively. In Table 3.3 almost all solutions with $i = 2, \ldots, 15$ are omitted because there are too many of them. We refer to our web page 6 for more extensive outputs that cover all of the cases $\rho = 2, 4, 6, \ldots, 16$.

Remark 3.3 We explain how to look at Tables 3.2, 3.3 and similar tables to be given later. The meanings of the $S(z)$, $C(z)$, $s(z)$, $c(z)$ columns are clear. The ST column indicates the value of the special trace $\tau$, where

$$\tau_0 > \tau_1 > \tau_2 > \cdots > \tau_{d/2-1} \quad \text{with} \quad \tau_0 > 2 > \tau_1$$

(15)
| $S(z)$ | $C(z)$ | $s(z)$ | $c(z)$ | ST | Dynkin | $\tilde{\varphi}_1(z)$ | TrA | SD |
|-------|--------|--------|--------|----|--------|----------------|------|----|
| $S_4^{[16]}$ | C$_8$ | S$_4^{[10]}$ | C$_{36}$ | $\tau_2$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_8$ | S$_5^{[10]}$ | C$_{21}$ | $\tau_2$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_8$ | S$_{4}^{[10]}$ | C$_{28}$ | $\tau_5$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_8$ | S$_{14}^{[10]}$ | C$_{15}$ | $\tau_1$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_8$ | S$_{22}^{[10]}$ | 1 | $\tau_6$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_8$ | S$_{55}^{[22]}$ | 1 | $\tau_7$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{10}^{[10]}$ | C$_{21}$ | $\tau_5$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{15}^{[10]}$ | C$_{28}$ | $\tau_1$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{14}^{[10]}$ | C$_{15}$ | $\tau_2$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{24}^{[10]}$ | C$_{24}$ | $\tau_2$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{15}^{[10]}$ | C$_{15}$ | $\tau_1$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{14}^{[10]}$ | C$_{24}$ | $\tau_1$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{22}^{[10]}$ | 1 | $\tau_2$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{3}C_4$ | S$_{22}^{[10]}$ | 1 | $\tau_5$ | $E_6$ | C$_7^2$ | 3 |
| $S_4^{[16]}$ | C$_{5}$ | S$_{14}^{[18]}$ | C$_{12}$ | $\tau_1$ | $A_4$ | C$_7^2C_3^2$ | 0 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{22}^{[10]}$ | 1 | $\tau_1$ | $A_4$ | C$_7^2C_3^2$ | 0 |
| $S_4^{[16]}$ | C$_{5}$ | S$_{14}^{[18]}$ | C$_{30}$ | $\tau_3$ | $D_6$ | C$_7^2C_2$ | 2 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{22}^{[10]}$ | 1 | $\tau_6$ | $D_6$ | C$_7^2C_2$ | 5 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{15}^{[10]}$ | C$_{28}$ | $\tau_4$ | $D_6$ | C$_7^2C_2$ | 5 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{14}^{[10]}$ | C$_{15}$ | $\tau_4$ | $D_6$ | C$_7^2C_2$ | 5 |
| $S_4^{[16]}$ | C$_{10}$ | S$_{22}^{[10]}$ | 1 | $\tau_3$ | $D_6$ | C$_7^2C_2$ | 5 |
| $S_4^{[16]} | C_{10} | S_{10}^{[10]} | C_{36} | \tau_4 | D_6 | C_7^2C_2 | 5 |
| $S_4^{[16]} | C_{10} | S_{10}^{[10]} | 1 | \tau_3 | D_6 | C_7^2C_2 | 5 |
| $S_4^{[16]} | C_{10} | S_{10}^{[10]} | 1 | \tau_5 | D_6 | C_7^2C_2 | 5 |
| $S_4^{[16]} | C_{12} | S_{24}^{[10]} | C_{42} | \tau_7 | A_1^{\text{II}} \oplus D_4 | C_7^2C_2C_3 | 1 |
| $S_4^{[16]} | C_{12} | S_{10}^{[10]} | C_{42} | \tau_1 | A_1^{\text{II}} \oplus D_4 | C_7^2C_2C_3 | 1 |
| $S_4^{[16]} | C_{12} | S_{14}^{[10]} | C_{30} | \tau_2 | A_1^{\text{II}} \oplus D_4 | C_7^2C_2C_3 | 1 |
| $S_4^{[16]} | C_{12} | S_{14}^{[10]} | C_{15} | \tau_4 | A_1^{\text{II}} \oplus D_4 | C_7^2C_2C_3 | 1 |
| $S_4^{[16]} | C_{12} | S_{14}^{[10]} | C_{20} | \tau_1 | A_1^{\text{II}} \oplus D_4 | C_7^2C_2C_3 | 1 |
| $S_4^{[16]} | C_{3}C_4 | S_{10}^{[10]} | C_{42} | \tau_6 | 0 | C_7^2C_2C_3 | -1 |
| $S_4^{[16]} | C_{3}C_4 | S_{14}^{[10]} | C_{30} | \tau_1 | 0 | C_7^2C_2C_3 | -1 |
| $S_4^{[16]} | C_{3}C_4 | S_{22}^{[10]} | 1 | \tau_3 | 0 | C_7^2C_2C_3 | -1 |
| $S_4^{[16]} | C_{3}C_4 | S_{43}^{[10]} | 1 | \tau_4 | 0 | C_7^2C_2C_3 | -1 |

Table 3.2: Picard number $\rho = 6$ (Setup $\mathbb{2}$)
| $S(z)$ | $C(z)$ | $s(z)$ | $c(z)$ | ST | Dynkin | $\bar{\varphi}_1(t)$ | Tr.Å | SD |
|-------|-------|-------|-------|----|--------|----------------|--------|----|
| $S^{(8)}$ | $C_{28}$ | $S^{(10)}_{14}$ | $C_{36}$ | $\tau_2$ | $A_2$ | $C_1C_2C_{28}$ | 0 |
| $S^{(8)}$ | $C_{28}$ | $S^{(14)}_{14}$ | $C_{30}$ | $\tau_1$ | $A_2$ | $C_1C_2C_{28}$ | 0 |
| $S^{(8)}$ | $C_{28}$ | $S^{(14)}_{234}$ | $C_{30}$ | $\tau_3$ | $A_2$ | $C_1C_2C_{28}$ | 0 |
| $S^{(8)}$ | $C_{28}$ | $S^{(18)}_{65}$ | $C_{12}$ | $\tau_2$ | $A_2$ | $C_1C_2C_{28}$ | 0 |
| $S^{(8)}$ | $C_{28}$ | $S^{(18)}_{109}$ | $C_{12}$ | $\tau_3$ | $A_2$ | $C_1C_2C_{28}$ | 0 |
| $S^{(8)}$ | $C_{4}C_{11}$ | $S^{(6)}_{6}$ | $C_{60}$ | $\tau_1$ | $A_1^{\otimes 2}$ | $C_2^2C_1^{11}$ | $-1$ |
| $S^{(8)}$ | $C_{4}C_{11}$ | $S^{(10)}_{0}$ | $C_{42}$ | $\tau_2$ | $A_1^{\otimes 2}$ | $C_2^2C_1^{11}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(10)}_{14}$ | $C_{42}$ | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(10)}_{10}$ | $C_{12}C_{24}$ | $\tau_2$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(10)}_{12}$ | $C_{12}C_{30}$ | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(14)}_{14}$ | $C_{30}$ | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(18)}_{18}$ | $C_{12}$ | $\tau_2$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(18)}_{22}$ | $C_{12}$ | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(18)}_{18}$ | $C_{60}$ | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{5}C_{20}$ | $S^{(18)}_{22}$ | 1 | $\tau_1$ | $A_1^{\otimes 3}$ | $C_1C_2C_3C_{20}$ | $-1$ |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(10)}_{10}$ | $C_{12}C_{24}$ | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(15)}_{15}$ | $C_{42}$ | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{18}$ | $C_{12}$ | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{22}$ | $C_{12}$ | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{18}$ | $C_{60}$ | $\tau_1$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{22}$ | 1 | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{18}$ | $C_{12}$ | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{15}$ | $S^{(18)}_{22}$ | 1 | $\tau_2$ | $E_6$ | $C_1^4C_2^2C_{15}$ | 3 |
| $S^{(8)}$ | $C_{10}C_{24}$ | $S^{(22)}_{10}$ | 1 | $\tau_2$ | $E_6 \oplus E_8$ | $C_1^2C_2^2$ | 10 |
| $S^{(8)}$ | $C_{3}C_{7}C_{8}$ | $S^{(22)}_{22}$ | 1 | $\tau_2$ | $E_6 \oplus E_8$ | $C_1^2C_2^2$ | 10 |
| $S^{(8)}$ | $C_{3}C_{7}C_{8}$ | $S^{(10)}_{10}$ | $C_{21}$ | $\tau_3$ | $\emptyset$ | $C_1C_2C_{13}$ | 0 |
| $S^{(8)}$ | $C_{13}$ | $S^{(10)}_{13}$ | $C_{36}$ | $\tau_2$ | $\emptyset$ | $C_1C_2C_{13}$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $S^{(8)}$ | $C_{2}C_{9}$ | $S^{(14)}_{14}$ | $C_{15}$ | $\tau_3$ | $E_6$ | $C_1^2C_2C_7$ | 7 |
| $S^{(8)}$ | $C_{4}C_{15}$ | $S^{(10)}_{45}$ | $C_{42}$ | $\tau_1$ | $E_8$ | $C_1^2C_2C_8$ | 10 |
| $S^{(8)}$ | $C_{6}C_{15}$ | $S^{(10)}_{145}$ | $C_{12}C_{24}$ | $\tau_2$ | $E_8$ | $C_1^2C_2C_8$ | 10 |
| $S^{(8)}$ | $C_{13}$ | $S^{(10)}_{13}$ | $C_{42}$ | $\tau_3$ | $A_1 \oplus A_{13}$ | $C_1^{14}$ | 14 |
| $S^{(8)}$ | $C_{3}C_{4}C_{20}$ | $S^{(10)}_{10}$ | $C_{42}$ | $\tau_2$ | $A_1^{\otimes 4} \oplus A_2$ | $C_1^2C_2^2C_3C_{10}$ | 2 |
| $S^{(8)}$ | $C_{3}C_{12}C_{18}$ | $S^{(10)}_{12}$ | $C_{42}$ | $\tau_2$ | $\emptyset$ | $C_1C_2C_3C_{12}C_{18}$ | $-1$ |

Table 3.3: Picard number $\rho = 14$ (Setup $S^{22}$).

| $S(z)$ | $C(z)$ | $s(z)$ | $c(z)$ | ST | Dynkin | $\bar{\varphi}_1(z)$ | Tr.Å | SD |
|-------|-------|-------|-------|----|--------|----------------|--------|----|
| $S^{(8)}$ | $C_{5}C_{18}C_{24}$ | $S^{(14)}_{14}$ | $C_{30}$ | $\tau_1$ | $A_1^{\otimes 4} \oplus A_2^{\otimes 3}$ | $C_1^2C_2^2C_3C_6C_{24}$ | 0 |

Table 3.4: Picard number $\rho = 18$ (Setup $S^{32}$).

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| $S(z)$ | $C(z)$ | $\psi(z)$ | ST | Dynkin | $\bar{\varphi}_1(z)$ | Tr $\bar{A}$ | SD |
|-------|-------|--------|----|--------|----------------|---------|----|
| $S_1^{(4)}$ | $C_{17}$ | 579 | $\tau_1$ | $\emptyset$ | $C_{17}$ | 0 |
| $S_1^{(4)}$ | $C_{32}$ | 289 | $\tau_1$ | $\emptyset$ | $C_{32}$ | 1 |
| $S_1^{(4)}$ | $C_{32}$ | 579 | $\tau_1$ | $\emptyset$ | $C_{32}$ | 1 |
| $S_1^{(4)}$ | $C_{32}$ | 692 | $\tau_1$ | $\emptyset$ | $C_{32}$ | 1 |
| $S_1^{(4)}$ | $C_{40}$ | 40 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{40}$ | 1 |
| $S_1^{(4)}$ | $C_{40}$ | 58 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{40}$ | 1 |
| $S_1^{(4)}$ | $C_{40}$ | 515 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{40}$ | 1 |
| $S_1^{(4)}$ | $C_{40}$ | 692 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{48}$ | 1 |
| $S_1^{(4)}$ | $C_{40}$ | 699 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{48}$ | 1 |
| $S_1^{(4)}$ | $C_{60}$ | 457 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{60}$ | 1 |
| $S_1^{(4)}$ | $C_{60}$ | 699 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{60}$ | 1 |
| $S_1^{(4)}$ | $C_{60}$ | 744 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{60}$ | 1 |
| $S_1^{(4)}$ | $C_{60}$ | 961 | $\tau_1$ | $A_2$ | $C_{1}C_{2}C_{60}$ | 1 |
| $S_1^{(4)}$ | $C_5C_{26}$ | 664 | $\tau_1$ | $A_1^{\oplus 5}$ | $C_1C_2C_5C_{26}$ | 1 |
| $S_1^{(4)}$ | $C_5C_{26}$ | 679 | $\tau_1$ | $A_1^{\oplus 5}$ | $C_1C_2C_5C_{26}$ | 1 |
| $S_1^{(4)}$ | $C_5C_{26}$ | 792 | $\tau_1$ | $A_1^{\oplus 5}$ | $C_1C_2C_5C_{26}$ | 1 |
| $S_1^{(4)}$ | $C_5C_{26}$ | 893 | $\tau_1$ | $A_1^{\oplus 5}$ | $C_1C_2C_5C_{26}$ | 1 |
| $S_1^{(4)}$ | $C_5C_{36}$ | 873 | $\tau_1$ | $A_1^{\oplus 5} \oplus E_6^{\oplus 2}$ | $C_1^2C_2C_5C_{36}$ | 0 |
| $S_1^{(4)}$ | $C_5C_{36}$ | 901 | $\tau_1$ | $A_1^{\oplus 5} \oplus E_6^{\oplus 2}$ | $C_1^2C_2C_5C_{36}$ | 0 |
| $S_1^{(4)}$ | $C_5C_{36}$ | 961 | $\tau_1$ | $A_1^{\oplus 5} \oplus E_6^{\oplus 2}$ | $C_1^2C_2C_5C_{36}$ | 0 |
| $S_1^{(4)}$ | $C_6C_{36}$ | 457 | $\tau_1$ | $E_6^{\oplus 3}$ | $C_1^5C_6C_{36}$ | 3 |
| $S_1^{(4)}$ | $C_6C_{36}$ | 515 | $\tau_1$ | $E_6^{\oplus 3}$ | $C_1^5C_6C_{36}$ | 3 |
| $S_1^{(4)}$ | $C_6C_{36}$ | 699 | $\tau_1$ | $E_6^{\oplus 3}$ | $C_1^5C_6C_{36}$ | 3 |
| $S_1^{(4)}$ | $C_6C_{36}$ | 712 | $\tau_1$ | $E_6^{\oplus 3}$ | $C_1^5C_6C_{36}$ | 3 |
| $S_1^{(4)}$ | $C_3C_4C_{28}$ | 515 | $\tau_1$ | $A_3$ | $C_1^3C_4C_{28}$ | 3 |
| $S_1^{(4)}$ | $C_3C_4C_{28}$ | 870 | $\tau_1$ | $A_3$ | $C_1^3C_4C_{28}$ | 3 |
| $S_1^{(4)}$ | $C_3C_5C_{15}$ | 870 | $\tau_1$ | $\emptyset$ | $C_1C_3C_5C_{15}$ | 1 |
| $S_1^{(4)}$ | $C_3C_5C_{24}$ | 692 | $\tau_1$ | $\emptyset$ | $C_1C_3C_5C_{24}$ | 0 |
| $S_1^{(4)}$ | $C_4C_{10}C_{11}$ | 692 | $\tau_1$ | $D_{11}$ | $C_1^4C_2C_{10}C_{11}$ | 12 |
| $S_1^{(4)}$ | $C_8C_{12}C_{30}$ | 523 | $\tau_1$ | $A_2^{\oplus 2} \oplus E_6 \oplus E_8$ | $C_1^3C_2C_4$ | 11 |
| $S_1^{(4)}$ | $C_9C_{10}C_{18}$ | 711 | $\tau_1$ | $\emptyset$ | $C_9C_{10}C_{18}$ | 2 |
| $S_1^{(4)}$ | $C_{10}C_{12}C_{16}$ | 259 | $\tau_1$ | $A_2 \oplus A_2$ | $C_2^2C_3C_{10}C_{12}C_{16}$ | 2 |
| $S_1^{(4)}$ | $C_3C_5C_{11}$ | 279 | $\tau_1$ | $A_{11}$ | $C_1^3C_2C_3C_{11}$ | 11 |
| $S_1^{(4)}$ | $C_3C_5C_{15}$ | 515 | $\tau_1$ | $A_2^{\oplus 4}$ | $C_1^3C_2^2C_3C_{15}$ | 1 |
| $S_1^{(4)}$ | $C_3C_5C_{15}$ | 870 | $\tau_1$ | $A_2^{\oplus 4}$ | $C_1^3C_2^2C_3C_{15}$ | 1 |
| $S_1^{(4)}$ | $C_3C_5C_{16}$ | 699 | $\tau_1$ | $A_2$ | $C_1^3C_2C_{16}$ | 3 |

Table 3.5: Picard number $\rho = 18$ (Setup 5.2).
are the roots of the trace polynomial $ST(w)$ of $S(z)$. Notice that $\tau_j$ depends on the index $(d, i)$ of $ST(w) = ST^{(d)}(w)$, so should be written $\tau_{ij}^{(d)}$ to be precise, but this dependence is suppressed in notation. The Dynkin column exhibits the Dynkin type of the root system $\Delta$. The $\bar{\phi}_1(z)$ column shows the $\bar{\phi}_1(z)$-component of the characteristic polynomial $\bar{\phi}(z)$ of $\bar{A}$; see [10]. In [7] Lefschetz-type fixed point formulas will be used to look for Siegel disks and the values of $\text{Tr} \bar{A}$ are needed to do so, hence this information is given in the $\text{Tr} \bar{A}$ column. If a solution is marked with $S$ in the SD column, then it can be shown that the automorphism arising from this entry has at least one Siegel disk. (But a blank in this column does not claim non-existence of Siegel disks.)

For $\rho = 18$, i.e. $d = 4$, however, Setup 3.4 with $S(z) = S_1^{(4)}(z)$ leads to only one solution in Table 3.4 for which it is difficult to decide whether the resulting automorphism has Siegel disks or not. Thus we propose an alternative setup which offers a wider variety of candidates for $\psi(z)$.

**Setup 3.4** For Picard number $\rho = 18$, take $S(z) := S_1^{(4)}(z) = z^4 - z^3 - z^2 - z + 1$ to be the minimal polynomial of the Salem number $\lambda_1^{(4)} \approx 1.7220838$, and let $Q$ be the set of all unramified palindromic polynomials,

$$\psi(z) = z^{22} + c_1 z^{21} + \cdots + c_{10} z^{12} + c_{11} z^{11} + c_{10} z^{10} + \cdots + c_1 z + 1 \in \mathbb{Z}[z]$$

such that the following three conditions are satisfied:

(i) $c_j \in \{0, \pm 1, \pm 2\}$ for $j = 1, \ldots, 9$,

(ii) the trace polynomial $\Psi(w)$ of $\psi(z)$ has ten or eight roots on the interval $(-2, 2)$,

(iii) a part of unimodularity condition: the resultant of $S_1^{(4)}(z)$ and $\psi(z)$ is $\pm 1$.

Unramifiedness of $\psi(z)$ implies $\psi(1) = \pm 1$ and $\psi(-1) = \mp 1$, where $\psi(\pm 1)$ must have different signs because the second formula in [11] gives $\psi(1) \cdot \psi(-1) = -1$. Thus $c_{10}$ and $c_{11}$ can be determined from $(c_1, \ldots, c_9)$ by

$$c_{10} = -1 - c_2 - c_4 - c_6 - c_8, \quad c_{11} = c_5^2 := \pm 1 - 2(c_1 + c_3 + c_5 + c_7 + c_9),$$

in a unique manner for $c_{10}$ and in two ways for $c_{11}$. A computer enumeration shows that $Q$ contains a total of 1019 polynomials. They can be identified by the numbering according to the lexicographical order for words $(c_1, \ldots, c_{11})$. All solutions to Setup 3.4 are given in Table 3.5 where $\psi(z)$ is shown by its ID number and $\tau_1 := (1 - \sqrt{13})/2 \approx -1.30278$ is the only root in $(-2, 2)$ of $ST_1^{(4)}(w) = w^2 - w - 3$. For the entry marked with $S$ in the SD column, $\psi(z)$ has ID number 523. Explicitly, this polynomial is given by

$$\psi(z) = z^{22} - z^{21} - 2z^{20} + 2z^{18} + z^{17} - z^{15} - 2z^{14} + z^{12} + z^{11} + z^{10} - 2z^8 - z^7 + z^5 + 2z^4 - 2z^2 - z + 1. \quad (16)$$

It is the minimal polynomial of a Salem number $\lambda \approx 1.72654$ of degree 22, which does not appear in Mossinghoff’s list [9] because $\lambda$ is beyond his bound $M_{22} = 1.5$ in Table 5.4. It is why Setup 3.2 fails to find this solution.

## 4 Fixed Point Formulas

We present two fixed point formulas (FPF’s), originally due to Saito [11], Toledo and Tong [13], which are needed to discuss the existence of Siegel disks. Let $f : X \to X$ be a K3 surface automorphism such that

(C1) $X$ is non-projective and the intersection form on $\text{Pic}(X)$ is negative definite,

(C2) the special eigenvalue $\delta = \delta(f)$ is conjugate to a Salem number.

These conditions are satisfied by all non-projective K3 surface automorphisms produced by the method of hypergeometric groups [5, Theorem 1.5]. Looking for Siegel disks naturally involves questions about fixed points of $f$. We have to control the fixed point set of $f$, which consists of isolated fixed points and possibly occurring fixed curves. Invariant (but not fixed) curves should also be relevant to this issue. By condition (C1) any irreducible curve in $X$ is a $(-2)$-curve [18, Lemma 7.3]. This fact and triviality of the canonical bundle $K_X$ are helpful in discussing questions about fixed curves and invariant curves. We begin by recalling the following.

**Lemma 4.1** If two distinct $(-2)$-curves in $X$ meet then they meet exactly in one point transversally.
Proof. If $C_1$ and $C_2$ are such curves, then $1 \leq C_1 \cdot C_2$ and $(C_1 + C_2)^2 = 2C_1 \cdot C_2 + C_1^2 + C_2^2 = 2(C_1 \cdot C_2 - 2) \leq -2$, since $C_1$ and $C_2$ are $(-2)$-curves, the intersection form on $\text{Pic}(X)$ is even and negative definite, and $C_1 + C_2 \neq 0$ in $\text{Pic}(X)$. Therefore $C_1 \cdot C_2 = 1$ and the assertion follows.

A fixed point $p \in X$ of $f$ is isolated if and only if its multiplicity

$$\mu_p(f) := \dim_{\mathbb{C}} \left( \mathbb{C}[z]/a \right) \quad \text{with} \quad a := (z_1 - f_1(z), z_2 - f_2(z)), \quad (17)$$

is finite, where $(f_1, f_2)$ is the local representation of $f$ in terms of a local chart $z = (z_1, z_2)$ around $p \leftrightarrow z = (0, 0)$, $\mathbb{C}[z]$ is the convergent power series ring in two variables $z = (z_1, z_2)$ and $a$ is its ideal generated by $z_1 - f_1(z)$ and $z_2 - f_2(z)$. Let $\text{Fix}^i(f)$ denote the set of all isolated fixed point of $f$.

**Proposition 4.2** If $N_f$ is the number of $(-2)$-curves fixed pointwise by $f$, then

$$\sum_{p \in \text{Fix}^i(f)} \mu_p(f) = \text{Tr} f^*|H^2(X, \mathbb{C}) + 2(1 - N_f). \quad (18)$$

**Proof.** We use S. Saito’s fixed point formula [11, formula (0.2)] which is stated as

$$L(f) := \sum_{j=0}^4 (-1)^j \text{Tr} f^*|H^j(X, \mathbb{C}) = \sum_{p \in \text{Fix}^i(f)} \mu_p(f) + \sum_{C \in X_0(f)} \chi_C \cdot \mu_C(f) + \sum_{C \in X_1(f)} \tau_C \cdot \mu_C(f),$$

where $X_0(f)$ is the set of all fixed points of $f$ while $X_1(f)$ and $X_2(f)$ are the sets of all irreducible fixed curves of types I and II respectively, $\chi_C$ is the Euler number of the normalization of $C$ and $\tau_C$ is the self-intersection number of $C$. For the definitions of Saito’s indices $\mu_p(f)$ and $\mu_C(f)$ we refer to [2, §3]. For any isolated fixed point $p$, Saito’s index $\mu_p(f)$ coincides with the multiplicity defined in [17], hence the same notation is employed for the two concepts. The formula holds for compact Kähler surfaces [2, Theorem 4.3]. Since $X$ is a K3 surface we have $L(f) = 2 + \text{Tr} f^*|H^2(X)$. Any fixed curve $C$ is a $(-2)$-curve isomorphic to $\mathbb{P}^1$. The differential $df$ acts on the normal bundle $N_C$ to $C$ as multiplication by $\delta \neq 1$. Thus $C \in X_1(f)$ with $\chi_C = 2$ and $X_2(f)$ is empty. An inspection shows that $\mu_C(f) = 1$ and $\mu_p(f) = 0$ at each $p \in C$ (see also [5, §9.3]). Putting all these facts into Saito’s formula we obtain formula [15].

The holomorphic local index of an isolated fixed point $p \in \text{Fix}^i(f)$ is given by the Grothendieck residue

$$\nu_p(f) = \text{Res}_p \omega \quad \text{with} \quad \omega := \frac{dz_1 \wedge dz_2}{(z_1 - f_1(z))(z_2 - f_2(z))}. \quad (19)$$

If $p$ is simple i.e. $\mu_p(f) = 1$ or equivalently if $p$ is transverse to the effect that the tangent map $(df)_p$ does not have eigenvalue $1$, then the index $\nu_p(f)$ admits a simpler representation

$$\nu_p(f) = \frac{1}{\det(I - (df)_p)} = \frac{1}{1 - \text{Tr}(df)_p + \delta}. \quad (20)$$

**Proposition 4.3** If $\delta = \delta(f)$ is the special eigenvalue of $f$ and $N_f$ is the number in Proposition [12] then

$$1 + \delta^{-1} = \sum_{p \in \text{Fix}^i(f)} \nu_p(f) + N_f \frac{1 + \delta}{(1 - \delta)^2}. \quad (21)$$

**Proof.** We use the Toledo-Tong fixed point formula [13, Theorem (4.10)] in 2-dimensional case. If any isolated fixed point $p \in X$ is transverse and if any connected component $C$ of the 1-dimensional fixed point set is also transverse to the effect that $C$ is a smooth curve and the induced differential map $d^0 f$ on the normal line bundle $N_C$ to $C$ has eigenvalue $\lambda_C \neq 1$, then the holomorphic Lefschetz number $L(f)$ is expressed as

$$L(f) := \sum_{j=0}^2 (-1)^j \text{Tr} f^*|H^{0,j}(X) = \sum_{p \in \text{Fix}^i(f)} \nu_p(f) + \sum_C \nu_C(f). \quad (22)$$
Here \( \nu_p(f) \) is given by (20), while if \( TC \) is the tangent bundle to \( C \) and \( \hat{N}_C \) is the dual bundle to \( N_C \), then

\[
\nu_C(f) = \int_C \frac{td(TC)}{1 - \lambda_C \cdot \text{ch}(\hat{N}_C)} = \frac{1}{1 - \lambda_C} \int_C \left\{ \frac{1}{2} c_1(TC) + \frac{\lambda_C \cdot c_1(\hat{N}_C)}{1 - \lambda_C} \right\},
\]

where \( td \) and \( \text{ch} \) stand for Todd class and Chern character respectively. When \( p \in \text{Fix}^c(f) \) is not transverse, \( \nu_p(f) \) can be expressed by the Grothendieck residue in (19) (see Toledo [12, formula (6.3)]).

Currently, we have \( \mathcal{E} = \mathcal{E}(X) \) of all \((-2)\)-curves \( C \) meeting in a point \( p \), then Lemma 5.1 shows that they meets transversally in \( p \), so \((df)_p \) acts on \( T_pX = T_pC_1 \oplus T_pC_2 \) trivially, but this contradicts the fact that \( \text{det}(df)_p = \delta \neq 1 \). Therefore \( C \) is just a single \((-2)\)-curve, which is smooth. Triviality of the canonical bundle \( K_X \) implies that \( \hat{N}_C \) is isomorphic to the tangent bundle \( TC \). Taking \( C \cong \mathbb{P}^1 \), \( \lambda_C = \delta \) and \( \int_C c_1(T^\mathbb{P}^1) = 2 \) into account, we have

\[
\nu_C(f) = \frac{1 + \delta}{(1 - \delta)^2}
\]

for any \((-2)\)-curve \( C \) fixed by \( f \). Thus Toledo-Tong formula (22) leads to the equation (21).

5 Indices on Exceptional Set

The union \( \mathcal{E} = \mathcal{E}(X) \) of all \((-2)\)-curves in \( X \) is referred to as the exceptional set. We are interested in how the isolated fixed points on \( \mathcal{E} \) contribute to the FPF’s (18) and (21). This problem may be considered component-wise for each connected component \( \mathcal{E}' \) of \( \mathcal{E} \) preserved by \( f \). In what follows we denote by \( \mu(f, \mathcal{E}') \) and \( \nu(f, \mathcal{E}') \) the sum of \( \mu_p(f) \) and that of \( \nu_p(f) \) taken over all isolated fixed points \( p \) on \( \mathcal{E} \) respectively.

![Dynkin diagram Γ with a trivalent node E₀.](image)

Figure 5.1: Dynkin diagram \( \Gamma \) with a trivalent node \( E_0 \).

We discuss the case where the dual graph of \( \mathcal{E}' \) is a Dynkin diagram \( \Gamma \) with a trivalent node, that is, of type \( D \) or \( E \) as in Figure 5.1. If \( \Gamma \) is of type \( E_r \) or \( E_8 \) then the automorphism group \( \text{Aut} \Gamma \) is trivial, while if \( \Gamma \) is of type \( D_n \) \((n \geq 5)\) or \( E_6 \) then \( \text{Aut} \Gamma \cong \mathbb{Z}/2\mathbb{Z} \), where the nontrivial automorphism fixes an arm \( E_1, \ldots, E_k \) emanating from the trivalent node \( E_0 \), but permutes the remaining two arms, namely, those containing \( E_{k+} \).

**Lemma 5.1** Let \( \mathcal{E}' \) be a connected component of \( \mathcal{E} \) preserved by \( f \), the dual graph of which is a Dynkin diagram \( \Gamma \) with a trivalent node. Then all isolated fixed points \( p \in \mathcal{E}' \) are simple \( \mu_p(f) = 1 \), that is, transverse.

1. If \( f \) acts on \( \Gamma \) trivially, then \( \mathcal{E}' \) contains exactly one irreducible fixed curve, \( \mu(f, \mathcal{E}') = n - 1 \) and

\[
\nu(f, \mathcal{E}') = -\frac{\delta}{(1 - \delta)^2} \left( \frac{2}{1 + \delta} + \frac{1 + \delta + \cdots + \delta^{n-4}}{1 + \delta + \cdots + \delta^{n-3}} \right) \quad \text{for type } D_n, \ n \geq 5,
\]

\[
\nu(f, \mathcal{E}') = -\frac{\delta}{(1 - \delta)^2} \left( \frac{1}{1 + \delta} + \frac{1 + \delta + \cdots + \delta^{n-5}}{1 + \delta + \cdots + \delta^{n-4}} \right) \quad \text{for type } E_n, \ n = 6, 7, 8.
\]

2. If \( f \) acts on \( \Gamma \) non-trivially, then \( \mathcal{E}' \) contains no irreducible fixed curve and

\[
\mu(f, \mathcal{E}') = n - 1, \quad \nu(f, \mathcal{E}') = \frac{1}{2(1 + \delta)} + \frac{1 + \delta + \cdots + \delta^{n-3}}{2(1 + \delta^{n-2})} \quad \text{for type } D_n, \ n \geq 5,
\]

\[
\mu(f, \mathcal{E}') = 3, \quad \nu(f, \mathcal{E}') = \frac{1}{2(1 + \delta)} + \frac{1 + \delta}{2(1 + \delta^2)} \quad \text{for type } E_6.
\]
Proof. Let $q_j$ be the intersection of $E_j$ and $E_{j+1}$ for $j = 0, \ldots, k-1$. Let $q_\pm$ be the intersection of $E_0$ and $E_k$.

Assertion (1). In this case $E_0$ is a fixed curve of $f$, since the Möbius transformation $f_{E_0}$ fixes the three points $q_0$ and $q_\pm$. Thus one has $(df_{E_0})_{q_0} = 1$, $(df_{E_1})_{q_0} = \delta$, and continues to get $(df_{E_j})_{q_0} = \delta^{-j}$ and $(df_{E_{j+1}})_{q_0} = \delta^{j+1}$ successively for $j = 1, \ldots, k-1$, until arriving at a unique fixed point $q_k \in E_k$ different from $q_{k-1}$, at which $(df)_{q_k}$ has eigenvalues $\delta^{-k}$ and $\delta^{k+1}$ (see Figure 5.2). This argument works smoothly and shows that $q_1, \ldots, q_k$ are transverse fixed points, because $\delta$ is not a root of unity. Moreover, those $k$ points are all of the isolated fixed points on $E_0 \cup \cdots \cup E_k$, and the sum of their $\nu$-indices can be calculated as

$$A_k^+ := \sum_{j=1}^k \nu_{q_j}(f) = \sum_{j=1}^k \frac{1}{(1 - \delta^{-j})(1 + \delta^{j+1})} = \frac{\delta(1 + \delta + \cdots + \delta^{k-1})}{(1 - \delta)(1 + \delta + \cdots + \delta^k).}$$

If $\Gamma$ is of type $D_n$, then the three arms of $\Gamma$ have lengths 1, 1 and $n-3$, so $\mu(f, E') = 1 + 1 + (n-3) = n-1$ and $\nu(f, E') = A^+_1 + A^+_2 + A^+_{n-3}$; this yields (24a). If $\Gamma$ is of type $E_n$ ($n = 6, 7, 8$), then the three arms of $\Gamma$ have lengths 1, 2 and $n-4$, so $\mu(f, E') = 1 + 2 + (n-4) = n-1$ and $\nu(f, E') = A^+_1 + A^+_2 + A^+_{n-3}$; this yields (24b).

![Figure 5.2: An arm $E_1 \cup \cdots \cup E_k$ emanating from the curve $E_0$ of a trivalent node.](image)

Assertion (2). In this case one has $(df_{E_0})_{q_0} = -1$ and $(df_{E_1})_{q_0} = -\delta$, because $f_{E_0}$ fixes $q_0$ and exchanges $q_\pm$. As in the last paragraph one gets $(df_{E_j})_{q_0} = -\delta^{-j}$, $(df_{E_{j+1}})_{q_0} = -\delta^{j+1}$ successively for $j = 1, \ldots, k-1$, until arriving at a unique fixed point $q_k \in E_k$ different from $q_{k-1}$, at which $(df)_{q_k}$ has eigenvalues $-\delta^{-k}$ and $-\delta^{k+1}$ (see Figure 5.2). Do not forget that $f_{E_0}$ has one more fixed point $q'_0 \in E_0$ different from $q_0$, at which $(df)_{q'_0}$ has eigenvalues $-1$ and $-\delta$. The $k + 2$ points $q'_0, q_0, q_1, \ldots, q_k$, which are transverse, are all of the isolated fixed points on $E_0 \cup E_1 \cup \cdots \cup E_k$ and the sum of their $\nu$-indices is given by

$$A_k^- := \nu_{q'_0}(f) + \sum_{j=0}^k \nu_{q_j}(f) = \frac{1}{2(1 + \delta) + \sum_{j=1}^k \frac{1}{(1 - \delta^{-j})(1 + \delta^{j+1})} = \frac{1 + \delta + \cdots + \delta^k}{2(1 + \delta^k).}$$

If $\Gamma$ is of type $D_n$ then $\mu(f, E') = (n - 3) + 2 = n-1$ and $\nu(f, E') = \Lambda^-_{n-3}$, since contributing to $\nu(f, E')$ is only the longest arm of $\Gamma$, with length $n-3$; this yields (24ba). If $\Gamma$ is of type $E_n$ then $\mu(f, E') = 1 + 2 = 3$ and $\nu(f, E') = \Lambda^-_1$, since contributing to $\nu(E')$ is only the shortest arm of $\Gamma$, with length 1; this yields (24ba).

Some calculations related to Lemma 5.1 are made in [5] §9.3 for a couple of examples. Now Lemma 5.1 provides a thorough result of this sort in a unified manner. When $E'$ is a connected component of Dynkin type $A$, things are much subtler due to the possible occurrence of a multiple isolated fixed point on $E'$. In this article no attempt is made to develop a general theory for components of type $A$. Instead, a particular case is discussed in §5 where only one $A_1$-component is present. Even in this case situations are already hard and interesting. The results in the next section will be needed in this context, although they are important in their own light.

6 Grothendieck Residues

Evaluating the residue $\nu_p(f)$ in [10] at a multiple fixed point $p$ is usually difficult, but when $p$ lies on an invariant curve there are cases where this task is more tractable. The aim of this section is to discuss such situations, or more precisely, to do so in a dynamical context, not only with the map $f$ alone but also with its iterates.
Definition 6.1 A fixed point \( p \in X \) of \( f \) is said to be exceptional if \( 2 \leq \mu_p(f) < \infty \) and there exists an \( f \)-invariant curve \( E \) passing through \( p \). For such a curve \( E \), since \( \mu_p(f) \geq 2 \) and \( \text{det}(df)_p = \delta(f) \), one has either (i) \( (df)_E)_p = 1 \); or (ii) \( (df)_E)_p = \delta(f) \), where \( f_E := f|_E \) is the Möbius transformation \( f \) induces on \( E \cong \mathbb{P}^1 \). Let \( \text{Fix}(f) \) be the set of all exceptional fixed points of \( f \). Condition \( \mu_p(f) < \infty \) implies \( \text{Fix}^n(f) \subset \text{Fix}(f) \). We say that \( p \in \text{Fix}^n(f) \) is of type I if \( p \) admits a curve of type (i); and of type II if \( p \) admits no curve of type (i) but a curve of type (ii). Let \( \text{Fix}^n(f) = \text{Fix}_I^n(f) \cup \text{Fix}_II^n(f) \) be the decomposition according to the types.

Lemma 6.2 For any integer \( n \geq 1 \) let \( f^n := f \circ \cdots \circ f \) be the \( n \)-th iterate of \( f \). Then

\[
\text{Fix}^n_I(f) \subset \text{Fix}^n(f), \quad \text{Fix}^n_{II}(f) \subset \text{Fix}^n(f), \quad \text{for any } n \geq 1.
\] (26)

Proof. First let \( p \in \text{Fix}^n_I(f) \) and \( E \) be an \( f \)-invariant but not fixed curve passing through \( p \) with \( (df)_E)_p = 1 \). The Möbius transformation \( f_E \) and its iterates \( f_E^n \) can then be expressed as

\[
f_E(z_1) = \frac{z_1}{1 + z_1} \quad \text{and hence} \quad f_E^n(z_1) = \frac{z_1}{1 + nz_1}, \tag{27}
\]

in terms of a suitable coordinate \( z_1 \) on \( E \) such that \( z_1 = 0 \) at \( p \). Thus \( E \) is not a fixed curve of \( f^n \) for any \( n \geq 1 \).

6.1 Exceptional Fixed Points of Type I

The case of type I is simpler to deal with than that of type II, so we begin with the former.

Theorem 6.3 If \( p \in \text{Fix}^n_I(f) \) then we have \( p \in \text{Fix}^n_{II}(f) \) and

\[
\mu(f^n) = 2, \quad \nu_p(f^n) = \frac{1 + \delta^n}{(1 - \delta^n)^2} \quad \text{for any } n \geq 1.
\] (28)

This theorem will be established after Lemma 6.6. Formulas (28) and (29) tell us that an irreducible fixed curve and an exceptional fixed point of type I have the same holomorphic index.

Suppose \( p \in \text{Fix}^n_{II}(f) \) and let \( E \) be the curve in Definition 6.1. Along with \( (df)_E)_p = 1 \) the tangent map \( (df)_p \) has eigenvalue \( \delta \). Take a local chart \( z = (z_1, z_2) \) around \( p = (0, 0) \) such that \( z_2 \) is in the eigen-direction of eigenvalue \( \delta \), \( E = \{z_2 = 0\} \) and \( z_1 \) is a coordinate on \( E \) such that \( f_E \) is normalized as in (27). Let \( (f_1, f_2) \) be the local representation for \( f \) in this chart. Since \( f \) preserves \( E \), there exist \( g_1(z), g_2(z) \in \mathbb{C}(z) \) such that

\[
f_1(z) = \frac{z_1}{1 + z_1} + z_2 g_1(z), \quad f_2(z) = z_2 \{\delta + g_2(z)\}, \quad g_1(0, 0) = g_2(0, 0) = 0. \tag{29}
\]

Lemma 6.4 For any \( p \in \text{Fix}^n_{II}(f) \) we have \( \alpha = (z_1^2, z_2) \) for the ideal in (17) and hence \( \mu_p(f) = 2 \).

Proof. First we have \( z_2 \in \alpha \), since \( z_2 - f_2(z) = z_2\{1 - \delta - g_2(z)\} \in \alpha \) and \( 1 - \delta - g_2(z) \in \mathbb{C}(z)^\times \). Secondly we have \( z_1^2 \in \alpha \), since \( z_1 - f_1(z) + z_2 g_1(z) = z_1^2/(1 + z_1) \in \alpha \) and \( 1/(1 + z_1) \in \mathbb{C}(z)^\times \). Thus \( (z_1^2, z_2) \subset \alpha \) and the converse inclusion is obvious. As \( 1 \) and \( z_1 \) form a basis of \( \mathbb{C}(z)/\alpha \), we have \( \mu_p(f) = 2 \). \( \square \)

Let \( \varepsilon \approx 0 \) be a small parameter and set \( t := 1/(1 - \varepsilon) \). Consider a perturbation \( f^\varepsilon \) of \( f \) defined by

\[
f^\varepsilon(z) := f(tz_1, t^{-1}z_2), \quad (\varepsilon, z) \approx (0, 0, 0).
\]
Lemma 6.5 For every $\varepsilon \approx 0$ with $\varepsilon \neq 0$ the map $f^\varepsilon$ has exactly two fixed points $p = (0,0)$ and $q^\varepsilon := (\varepsilon,0)$ in a small neighborhood of $(\varepsilon, z_1, z_2) = (0,0,0)$. Moreover, $\delta^\varepsilon := \text{det}(df^\varepsilon)_{q^\varepsilon} = \delta(1 + a\varepsilon^2)$ with $a = O(1)$ as $\varepsilon \to 0$.

Proof. In view of (24) the equation $z_2^* f_2(z) = 0$ reads $z_2 [1 - t\delta - tg_2(tz_1^*, z_1^*)] = 0$, which yields $z_2 = 0$ as $1 - t\delta - tg_2(tz_1^*, z_1^*) \approx 1 - \delta \neq 0$ for $(\varepsilon, z_1, z_2) \approx (0,0,0)$. Again by (25) putting $z_2 = 0$ into $z_1 - f_1^\varepsilon(z) = 0$ gives $z_1(z_1 - \varepsilon)/(1 - \varepsilon) = 0$, i.e., $z_1 = 0, \varepsilon$. Thus the fixed points of $f^\varepsilon$ are exactly $p = (0,0)$ and $q^\varepsilon = (\varepsilon,0)$.

If $\eta$ is a nowhere vanishing holomorphic 2-form on $X$, then equation $f^*\eta = \delta \cdot \eta$ is represented as

$$J_f(z) = \delta \cdot \frac{h(z)}{h(f(z))} \quad \text{with} \quad \eta = h(z) dz_1 \wedge dz_2,$$

where $J_f(z)$ is the Jacobian of $f$. Substituting $z = (t\varepsilon, 0)$ into (29) and using $f(t\varepsilon, 0) = (\varepsilon, 0)$, we have

$$\delta^\varepsilon := \text{det}(df^\varepsilon)_{q^\varepsilon} = \frac{h(t\varepsilon, 0)}{h(f(t\varepsilon, 0))} = \frac{h(0 - \varepsilon, 0)}{h(0, 0)} = \delta \{1 + O(\varepsilon^2)\},$$

since $h(0,0) = c_0 \{1 + c_1 \varepsilon + O(\varepsilon^2)\}$ for some constants $c_0 \in \mathbb{C}^*$ and $c_1 \in \mathbb{C}$.

Lemma 6.6 For any $p \in \text{Fix}_1(f)$ formula (25) holds for $n = 1$.

Proof. It is easy to see that $(df^\varepsilon)_p$ has eigenvalues $(df^\varepsilon)_p \varepsilon = t$ and $\delta^\varepsilon^{-1}$. One then has $(df^\varepsilon)_q \varepsilon = t^{-1}$, since $q^\varepsilon \in E$ is the other fixed point of the Möbius transformation $f^\varepsilon$. So $(df^\varepsilon)_q \varepsilon$ has eigenvalues $t^{-1}$ and $\delta^\varepsilon$. Let $\omega^\varepsilon$ be the 2-form in (19) for $f^\varepsilon$. By continuity principle [3, §5.1] the residue $\nu_p(f)$ is given as the limit of

$$\text{Res}_p \omega^\varepsilon + \text{Res}_q \omega^\varepsilon = \frac{1}{(1-t)(1-t^{-1}\delta)} + \frac{1}{(1-t^{-1})(1-\delta t)} \to \frac{(1-\varepsilon)(1+\delta+a\varepsilon)}{(1-\delta+a\varepsilon)(1-\delta-\varepsilon-a\delta^2)} \to \frac{1+\delta}{(1-\delta)^2} \quad \text{as } \varepsilon \to 0,$$

where formula (20) and Lemma 6.5 are used in the first and second equalities respectively.

Proof of Theorem 6.3. It is an immediate consequence of Lemmas 6.2, 6.4 and 6.6.

6.2 Exceptional Fixed Points of Type II

Let $p \in X$ be a fixed point of $f$ lying on an invariant curve $E$ such that $(df^\varepsilon)_p = \delta$. Note that $f_E$ has a fixed point $q \in E$ different from $p$, at which $(df^\varepsilon)_p = \delta^{-1}$. Along with $\delta$ the tangent map $(df^\varepsilon)_p$ has eigenvalue 1. Take a local chart $z = (z_1, z_2)$ around $p = (0,0)$ such that $z_1$ is in the eigen-direction of eigenvalue 1, $E = \{z_2 = 0\}$, and $z_2$ gives a coordinate on $E \setminus \{q\} \cong \mathbb{C}$ with $q$ located at $z_2 = \infty$, so that $f_E$ is normalized as $f_E(z_2) = \delta z_2$.

Let $(f_1, f_2)$ be the local representation for $f$ in this chart. Since $f$ preserves $E$, we can write

$$f_1(z) = z_1 \{1 + g_1(z)\}, \quad f_2(z) = \delta \{z_2 + z_1 g_2(z)\}, \quad g_1(0,0) = g_2(0,0) = 0,$$

for some $g_1(z), g_2(z) \in \mathbb{C}\{z\}$. Provide $z_1$ and $z_2$ with orders 1 and 2 respectively. Put

$$g_1(z) = \sum_{i,j=0}^\infty a_{ij} z_1^i z_2^j, \quad g_2(z) = \sum_{i,j=0}^\infty b_{ij} z_1^i z_2^j, \quad a_{00} = b_{00} = 0.$$

Lemma 6.7 We have $\mu_p(f) = 2$ if and only if $a_{10} \neq 0$, in which case $a = (z_1^2, z_2)$ for the ideal in (17).

Proof. First we show that $a_{10} \neq 0$ implies $a = (z_1^2, z_2)$ and $\mu_p(f) = 2$. In what follows $u_1(z_1)$ and $v_1(z)$ stand for various elements in $\mathbb{C}\{z_1\}$ and $\mathbb{C}\{z\}$ respectively. Observe that $z_1 - f_1(z) = z_1 \{z_1 u_1(z_1) + z_2 v_1(z)\} \in a$ with $u_1(z_1) \in \mathbb{C}\{z_1\}^\times$. Multiplying it by $u_1(z_1)^{-1}$ yields $v_2(z) := z_1 \{z_1 + z_2 v_2(z)\} \in a$ with $v_3(z) := u_1(z_1)^{-1} v_3(z)$.

One also has $z_2 - f_2(z) = z_2 \{1 - \delta + z_1 v_4(z) + z_1^{k+1} u_2(z_1)\}$ for some $k \geq 0$. So $z_2 - f_2(z) - z_1^2 u_2(z_1) v_2(z) = z_2 v_4(z) \in a$ with $v_3(z) = 1 - \delta + z_1 \{v_3(z) - z_1^{k} u_2(z_1) v_3(z)\} \in \mathbb{C}\{z\}^\times$. Thus $z_2 \in a$ and $z_1^2 = v_2(z) - z_2 z_2 v_3(z) \in a$, hence $a = (z_1^2, z_2)$. Therefore 1 and $z_1$ give a basis of $\mathbb{C}\{z\}/a$, so we have $\mu_p(f) = 2$.

Next we show that $a_{10} = 0$ implies $\mu_p(f) \geq 3$. It suffices to prove the following two claims.
(1) If \( b_{10} \neq 0 \) then 1, \( z_1, z_2 \) are linearly independent in \( \mathbb{C}\{z\}/a \).
(2) If \( b_{10} = 0 \) then 1, \( z_1, z_2^2 \) are linearly independent in \( \mathbb{C}\{z\}/a \).

Note that \( \text{ord}(z_1 - f_1(z)) \geq 3 \) and \( \text{ord}(z_2 - f_2(z)) \geq 2 \). To show claim (1), suppose that \( \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 \in a \) with \( \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \). Since any element of \( a \) has order at least 2, we have \( \alpha_0 = \alpha_1 = 0 \) and \( \alpha_2 z_2 \in a \). Thus \( \alpha_2 z_2 = u(z) \cdot (z_1 - f_1(z)) + v(z) \cdot (z_2 - f_2(z)) \) for some \( u(z) \), \( v(z) \in \mathbb{C}\{z\} \). Its order 2 component yields \( \alpha_2 z_2 = v(0,0)(1 - \delta)z_2 - \delta b_{10} z_2^2 \). As \( b_{10} \neq 0 \) we have \( v(0,0) = 0 \) and hence \( \alpha_2 = 0 \). To show claim (2), suppose that \( \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2^2 \in a \) with \( \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \). Since \( \text{ord}(z_1 - f_1(z))|_{z_2 = 0} \geq 3 \) and \( \text{ord}(z_2 - f_2(z))|_{z_2 = 0} \geq 3 \), any element of \( a_{z_2 = 0} \) has order at least 3. This implies \( \alpha_0 = \alpha_1 = \alpha_2 = 0 \).

**Theorem 6.8** If \( p \in \text{Fix}_0^p(f) \) and \( \mu_p(f) = 2 \) then we have \( p \in \text{Fix}_0^p(f^n) \) and

\[
\mu_p(f^n) = 2, \quad \nu_p(f^n) = \frac{n - 1 + (n+1)\delta^n + (1 + \delta + \cdots + \delta^{n-1})\theta}{n(1 - \delta^n)^2}
\]

for any \( n \geq 1 \), (33)

where in terms of some leading coefficients in \( f_c \) the quantity \( \theta \) is defined by

\[
\theta := \frac{(1 - \delta)a_{20} + \delta a_{01} b_{10}}{(a_{10})^2},
\]

(34)

We establish this theorem by providing four lemmas, where we work with \( f \) in Lemmas 6.9–6.11 and proceed to its iterates \( f^n \) in Lemma 6.12. Hereafter we assume \( (31), (32) \) and \( \mu_p(f) = 2 \) without further comment. Rescaling \( z \rightarrow (\lambda z_1, z_2) \) with \( \lambda \in \mathbb{C}\times \) takes \( f_1(z) \rightarrow \lambda^{-1} f_1(\lambda z_1, z_2) \) and \( f_2(z) \rightarrow f_2(\lambda z_1, z_2) \), hence induces the change of coefficients \( a_{ij} \rightarrow \lambda^i a_{ij} \) and \( b_{ij} \rightarrow \lambda^{i+1} b_{ij} \). We can take \( \lambda = (a_{10})^{-1} \) to get a normalization

\[
a_{10} = 1.
\]

(35)

**Lemma 6.9** We have \( b_{01} = -2 \) under the normalization \( (35) \).

**Proof.** This follows from equation (30). Indeed, substituting \( (31) \) and \( (32) \) into it, we observe

LHS of (30) = \( \delta(1 + (b_{01} + 2)z_1 + O_2) \), \quad RHS of (30) = \( \delta(1 + O_2) \),

where \( O_2 \) stands for various terms of order at least 2. Comparing the first order terms yields \( b_{01} + 2 = 0 \).

Let \( \varepsilon \approx 0 \) be a small parameter and set \( t := 1 - \varepsilon \). Consider a perturbation \( f^\varepsilon \) of \( f \) defined by

\[
f^\varepsilon(z) := f(tz_1, t^{-1} z_2), \quad (\varepsilon, z) \approx (0, 0, 0).
\]

(36)

It is obvious that the origin \( p = (0, 0) \) is a fixed point of \( f^\varepsilon \). Let us find another fixed point.

**Lemma 6.10** For every \( \varepsilon \approx 0 \), \( \varepsilon \neq 0 \), the map \( f^\varepsilon \) has exactly two fixed points \( p = (0, 0) \) and \( q^\varepsilon = (w_1(\varepsilon), w_2(\varepsilon)) \) in a small neighborhood of \( (\varepsilon, z_1, z_2) = (0, 0, 0) \). Under (30) the coordinates of \( q^\varepsilon \) admit an expansion

\[
w_1(\varepsilon) = \varepsilon + A_{20} \varepsilon^2 + O(\varepsilon^3), \quad A_2 = 2 - a_{20} - \frac{\delta a_{01} b_{10}}{1 - \delta},
\]

(36a)

\[
w_2(\varepsilon) = B_2 \varepsilon^2 + O(\varepsilon^3), \quad B_2 = \frac{\delta b_{10}}{1 - \delta}.
\]

(36b)

**Proof.** Put \( F_1^\varepsilon(z) := t g_1(tz_1, t^{-1} z_2) - \varepsilon \) and \( F_2^\varepsilon(z) := (t^{\delta^{-1}} - 1) z_2 + \delta t z_1 t_2 (tz_1, t^{-1} z_2) \). Then \( f^\varepsilon(z) = z \) is equivalent to \( z_1 F_1^\varepsilon(z) = F_2^\varepsilon(z) \) \( = 0 \). If \( z_1 = 0 \), then \( (t^{\delta^{-1}} - 1) z_2 = 0 \) and hence \( z_2 = 0 \), as \( \delta t^{\delta^{-1}} - 1 \approx \delta - 1 \neq 0 \) for \( \varepsilon \approx 0 \). Thus any fixed point other than \( p \) is a solution to the equations \( F_1^\varepsilon(z) = 0 \) and \( F_2^\varepsilon(z) = 0 \). Since

\[
\frac{\partial F_0}{\partial z}(0,0) = \begin{pmatrix} 1 & a_{01} \\ 0 & \delta - 1 \end{pmatrix}, \quad \frac{\partial F_0}{\partial \varepsilon}(0,0) \big|_{\varepsilon = 0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

under (35),

the implicit function theorem implies that near \( (\varepsilon, z_1, z_2) = (0, 0, 0) \) there exists a unique solution \( (z_1, z_2) = (w_1(\varepsilon), w_2(\varepsilon)) \) such that \( w_1(0) = w_2(0) = 0 \). It satisfies \( w_1'(0) = 1, w_2'(0) = 0 \), so admits an expansion as in (30) for some constants \( A_2 \) and \( B_2 \). Using (31), (32), (35) and Lemma 6.8 we have \( F_1^\varepsilon(w_1(\varepsilon), w_2(\varepsilon)) = (a_{20} - 2 + A_2 + a_{01} B_2) \varepsilon^2 + O(\varepsilon^3) \) \( = 0 \) and \( F_2^\varepsilon(w_1(\varepsilon), w_2(\varepsilon)) = (\delta - 1) B_2 + \delta b_{10} \varepsilon^2 + O(\varepsilon^3) \) \( = 0 \), hence \( a_{20} - 2 + A_2 + a_{01} B_2 = 0 = (\delta - 1) B_2 + \delta b_{10} \). This determines \( A_2 \) and \( B_2 \) as in (36a) and (36b) respectively. 

\[\square\]
Lemma 6.11 In terms of \( \theta \) in (34), the holomorphic local index in (19) can be expressed as
\[
\nu_p(f) = \frac{2\delta + \theta}{(1 - \delta)^2},
\]
(37)

Proof. We keep the normalization (35). Since \((df)^p\) has eigenvalues \( t \) and \( \delta t^{-1} \),
\[
\det (I - (df^p)_p) = (1 - t)(1 - \delta t^{-1}) = \frac{\varepsilon(1 - \delta - \varepsilon)}{1 - \varepsilon}.
\]
Calculation of \( \det (I - (df^p)_q^r) \) is much harder, but Lemmas 6.9 and 6.10 together with (31), (32), (35) yield
\[
\det (I - (df^p)_q^r) = \varepsilon [\delta - 1 - \{\delta + (1 - \delta) a_{20} + \delta a_{01} b_{10}\} \varepsilon + O(\varepsilon^2)].
\]
Let \( \omega^c \) be the 2-form in (19) for \( f^c \). By continuity principle [3] §5.1 the residue \( \nu_p(f) \) is given as the limit of
\[
\text{Res}_{p+\varepsilon} \omega^c + \text{Res}_{q} \omega^c = \frac{1}{\det (I - (df^c)_p)} + \frac{1}{\det (I - (df^c)_q^r)} = \frac{2\delta + (1 - \delta) a_{20} + \delta a_{01} b_{10} - \{\delta + (1 - \delta) a_{20} + \delta a_{01} b_{10}\} \varepsilon + O(\varepsilon^2)}{(1 - \delta - \varepsilon)[1 - \delta + \{\delta + (1 - \delta) a_{20} + \delta a_{01} b_{10}\} \varepsilon + O(\varepsilon^2)]} \\
\rightarrow \frac{2\delta + \theta}{(1 - \delta)^2} \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{with} \quad \theta := (1 - \delta) a_{20} + \delta a_{01} b_{10}.
\]

Removing the normalization (35) we obtain formula (37) with \( \theta \) defined in (34).

For any \( n \geq 1 \) the \( n \)-th iterate \( f^n \) can also be represented in the form (31) upon replacing \( \delta \) by \( \delta^n \) in (31) and rewriting the coefficients \( a_{ij} \) and \( b_{ij} \) as \( a_{ij}^{(n)} \) and \( b_{ij}^{(n)} \) in (32), respectively.

Lemma 6.12 For any \( n \geq 1 \) one has \( \mu_p(f^n) = 2 \) and hence \( p \) is an isolated fixed point of \( f^n \). Moreover,
\[
\theta^{(n)} := \frac{(1 - \delta^n) a_{20}^{(n)} + \delta^n a_{01}^{(n)} b_{10}^{(n)}}{(a_{10}^{(n)})^2} = \frac{(1 - \delta^n)((n - 1)(1 - \delta) + \theta)}{n(1 - \delta)} \quad \text{for any} \quad n \geq 1.
\]
(38)

Proof. The obvious composition rule \( f^{n+1} = f \circ f^n \) then leads to a system of recurrence relations
\[
a_{10}^{(n+1)} = a_{10}^{(n)} + 1, \quad a_{01}^{(n+1)} = a_{01}^{(n)} + \delta^n a_{01}, \\
b_{10}^{(n+1)} = b_{10}^{(n)} + \delta^{-n} b_{10}, \quad a_{20}^{(n+1)} = a_{20}^{(n)} + a_{20} + 2a_{10}^{(n)} + \delta^n a_{01} b_{10}^{(n)}.
\]
where the normalization (35) is employed. This system is readily settled as
\[
a_{10}^{(n)} = n \geq 1, \quad a_{01}^{(n)} = \frac{(1 - \delta^n) a_{01}}{1 - \delta}, \\
b_{10}^{(n)} = \frac{(1 - \delta^n) b_{10}^{(n)}}{\delta^{n-1}(1 - \delta)}, \quad a_{20}^{(n)} = n(1 - n) + a_{20} + \frac{\delta}{1 - \delta} \left\{ n - 1 - \frac{(1 - \delta^{n-1})}{1 - \delta} \right\} a_{01} b_{10}^{(n)}.
\]

Lemma 6.7 shows that \( \mu_p(f^n) = 2 < \infty \) and hence \( p \) is an isolated fixed point of \( f^n \) for every \( n \geq 1 \). Substituting the above data into the definition of \( \theta^{(n)} \), we find that a fine cancellation occurs to yield (38).

Proof of Theorem 6.8 It is clear from Lemma 6.12 that \( p \) is an exceptional fixed point of type II relative to \( E \). Lemma 6.11 implies \( \nu_p(f^n) = (2\delta^n + \theta^{(n)})/(1 - \delta^n)^2 \), which combined with (38) yields formula (33).

7 Siegel Disks

Let \( f : X \to X \) be a K3 surface automorphism satisfying the conditions (C1) and (C2) at the beginning of [3]. In [5] Proposition 9.1 we give a criterion for a given fixed point of \( f \) to be the center of a Siegel disk or to be a hyperbolic fixed point. For later use we have to extend it a little bit. Let \( p \in X \) be a fixed point of \( f \). Then the eigenvalues of the tangent map \((df)_p : T_p X \to T_p X \) can be represented as
\[
\alpha_1 := \delta^{\frac{2}{\tau}} \alpha, \quad \alpha_2 := \delta^{\frac{2}{\tau}} \alpha^{-1} \quad \text{for some} \quad \alpha \in \mathbb{C}^\times,
\]
(39)
where the branch of \( \delta^{\frac{2}{\tau}} \) is specified by \( \text{Re}(\delta^{\frac{2}{\tau}}) > 0 \) for the sake of definiteness. Let \( \tau \) be the special trace of \( f \).
Lemma 7.1 Suppose that there exists a rational functions $P(w) \in \mathbb{Q}(w)$ such that $(\alpha + \alpha^{-1})^2 = P(\tau)$.

(1) If $0 \leq P(\tau) \leq 4$ then $p$ is the center of a Siegel disk, provided either (i) $\tau$ admits a conjugate $\tau'$ such that $-2 < \tau' < 2$ and $P(\tau') > 4$; or (ii) $P(\tau)$ is not an algebraic integer.

(2) If $P(\tau) > 4$ then $p$ is a hyperbolic fixed point.

Proof. The cases of (1)-(i) and (2) are proved in [5 Proposition 9.1]. Under the condition that $\alpha_1, \alpha_2 \in \mathbb{Q} \cap S^1$, the fixed point $p$ is the center of a Siegel disk if and only if $\alpha_1$ and $\alpha_2$ are multiplicatively independent [MI] (see [8 Theorem 5.1]). In case (1)-(ii) it follows from $(\alpha + \alpha^{-1})^2 = P(\tau)$ and $0 \leq P(\tau) \leq 4$ that $\alpha_1, \alpha_2 \in \mathbb{Q} \cap S^1$, while assumption (ii) implies that $\tau$ is not an algebraic unit. Suppose that $\alpha_1^n \alpha_2^m = \delta^{(m+n)} \alpha^{m-n} = 1$, that is, $\alpha^{n-m} = \delta^{(m+n)}$ for some $m, n \in \mathbb{Z}$. If $n - m \neq 0$ then $\alpha$ must be an algebraic unit, since so is $\delta$, but this is impossible. Thus one has $n - m = 0$ and $\delta^{(m+n)} = 1$, but the latter equation yields $m + n = 0$, because $\delta$ is not a root of unity as a conjugate to a Salem number. Thus $m = n = 0$ and hence $\alpha_1$ and $\alpha_2$ are MI. \hfill \Box

Remark 7.3 It is sometimes more convenient to express the eigenvalues of $(df)_p$ in the form

$$\beta_1 := \beta, \quad \beta_2 := \delta \beta^{-1}$$

for some $\beta \in \mathbb{C}^\times$. (40)

An obvious variant of Lemma 7.1 in this situation is the following lemma, whose proof is safely omitted.

Lemma 7.2 Suppose that there exists a rational functions $Q(w) \in \mathbb{Q}(w)$ such that $\beta + \beta^{-1} = Q(\tau)$.

(1) If $|Q(\tau)| \leq 2$ then $p$ is the center of a Siegel disk, provided either (i) $\tau$ admits a conjugate $\tau'$ such that $-2 < \tau' < 2$ and $|Q(\tau')| > 2$; or (ii) $Q(\tau)$ is not an algebraic integer.

(2) If $|Q(\tau)| > 2$ then $p$ is a hyperbolic fixed point.

With the help of Lemma 7.1 FPF’s (18) and (21) in Propositions 4.2 and 4.3 often make it possible to determine the rational functions $P(w)$ and $Q(w)$ explicitly. One more piece toward this calculation is to know how the map $f : X \to X$ permutes the $(-2)$-curves in $X$. As is remarked at the end of [9] this can be done by calculating the action of $\tilde{A}$ on the simple system $\Delta_b$ explicitly. Without doing so, however, it is sometimes feasible to get this information by looking at $\tilde{\varphi}_1(z)$ only. Recall from [11] that $\tilde{\varphi}_1(z)$ is the characteristic polynomial of $\tilde{A}|\text{Pic}$, so $\tilde{\varphi}_1(z)$ must be divisible by the characteristic polynomial $\chi(z)$ of $\tilde{A}|\text{Span} \Delta_b$. Thus the shape of $\tilde{\varphi}_1(z)$ constrains that of $\chi(z)$ and hence the way in which $\tilde{A}$ acts on $\Delta_b$ to some extent or fully in some cases. Putting all these ingredients together, we are able to establish, for example, the following result.

| # | $\rho$ | $S(z)$ | $C(z)$ | $\psi(z)$ | ST | Dynkin | $\tilde{\varphi}_1(z)$ | Tr $\tilde{A}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | $S_1^{(12)}$ | 1 | $S_1^{(10)}C_2$ | $\tau_7$ | $A_1$ | $C_1C_2$ | 1 |
| 2 | 4 | $S_8^{(18)}$ | $C_4$ | $S_6^{(6)}C_{48}$ | $\tau_4$ | $A_4^{2}$ | $C_1C_2C_4$ | -1 |
| 3 | 6 | $S_7^{(16)}$ | $C_4C_4$ | $S_3^{(10)}C_{42}$ | $\tau_6$ | $\emptyset$ | $C_1C_2C_4C_4$ | -1 |
| 4 | 8 | $S_1^{(14)}$ | $C_4$ | $S_1^{(14)}C_{24}$ | $\tau_1$ | $E_8$ | $C_4^8$ | 8 |
| 5 | 10 | $S_5^{(12)}$ | $C_{16}$ | $S_3^{(6)}C_{60}$ | $\tau_5$ | $D_9$ | $C_4^8C_2^2$ | 7 |
| 6 | 12 | $S_6^{(10)}$ | $C_{26}$ | $S_1^{(10)}C_{40}$ | $\tau_7$ | $E_6 \oplus E_6$ | $C_4^4C_2^2$ | -1 |
| 7 | 14 | $S_7^{(14)}$ | $C_6C_{12}$ | $S_5^{(10)}C_{42}$ | $\tau_2$ | $\emptyset$ | $C_1C_2C_4C_{12}C_{18}$ | -1 |
| 8 | 16 | $S_5^{(6)}$ | $C_4C_{26}$ | $S_1^{(22)}$ | $\tau_1$ | $D_{16}$ | $C_4^{16}$ | 16 |
| 9 | 18 | $S_1^{(4)}$ | $C_8C_{12}C_{30}$ | $S_1^{(10)}$ | $\tau_1$ | $A_4^{2} \oplus E_6 \oplus E_8$ | $C_1^{12}C_2^2C_4$ | 11 |

Table 7.1: Some pairs $(\varphi, \psi)$ leading to K3 surface automorphisms with Siegel disks.
Theorem 7.4 The pairs $(\varphi, \psi)$ in Table 7.1, which are obtained from Setups 8.2 and 8.3, lead to K3 surface automorphisms with Siegel disks, where $\rho$ is the Picard number and $\varphi(z) = S(z) - C(z)$.

Proof. Let $f : X \to X$ be the K3 surface automorphism lifted from $\hat{A}$ and $\chi(z)$ be the characteristic polynomial of $f^* \text{Span} \Delta_0(X) = A \text{Span} \Delta_0$. Leaving entry #1 in Table 7.1, we deal with the remaining entries.

For entry #2 the map $f$ exchanges the two $A_1$-components of the exceptional set $E = E(X)$. For, otherwise, $\hat{A}$ fixes the two simple roots in $\Delta_0$, having at least two eigenvalues $1$, so $\chi(z)$ and hence $\hat{\varphi}_1(z) = C_1(z)^6, C_2(z)^2$ is divisible by $C_1(z)^2$, a contradiction. For entry #6 a similar reasoning with $\hat{\varphi}_2(z) = C_1(z)^4, C_2(z)^4, C_4(z)^2$ implies that $f$ exchanges the two $E_8$-components of $E$. Thus for these entries $f$ has no fixed points on $E$ and all fixed points of $f$ are isolated, that is, $N_f = 0$ and $\text{Fix}^i(f) = \text{Fix}(f)$ in Table 7.1. This is also the case with entries #3 and #7 for which $E$ is empty. We have $\text{Tr} f^* | H^2(X, \mathbb{C}) = \text{Tr}\hat{A} = -1$ for these four entries. FPF (13) then implies that $f$ admits a unique transverse fixed point $p \in X$. If the eigenvalues of $(df)_p$ are expressed as in (39), then FPF (21) reads
\[
P(w) := \frac{(w + 1)^2}{w + 2}.
\]
For entry #2 we have $0 < P(\tau_1) < 4$ and $P(\tau_8) > 4$; for entry #3 we have $0 < P(\tau_n) < 4$ and $P(\tau_4) > 4$; for entry #6 we have $0 < P(\tau_3) < 4$ and $P(\tau_4) > 4$; for entry #7 we have $0 < P(\tau_2) < 4$ and $P(\tau_3) > 4$. Therefore in these cases $p$ is the center of a Siegel disk by Lemma 7.1(1)-(i).

For entry #4 the exceptional set $E$ itself is the only connected component, which is of type $E_8$. We have $N_f = 1$ and $\mu(f, E) = 7$ from Lemma 5.1(1). So FPF (13) with $\text{Tr} f^* | H^2(X, \mathbb{C}) = \text{Tr}\hat{A} = 8$ shows that $f$ has a unique transverse fixed point $p \in X \setminus E$. If the eigenvalues of $(df)_p$ are expressed as in (39), then FPF (21) reads
\[
1 + \delta^{-1} = \frac{1}{1 - \delta^2(\alpha + \alpha^{-1}) + \delta} - \frac{\delta}{(1 - \delta)^2} \left( \frac{1}{1 + \delta} + \frac{1}{1 + \delta + \delta^2} + \frac{1 + \delta + \delta^2}{1 + \delta + \delta^2 + \delta^3} \right) + \frac{1 + \delta}{(1 - \delta)^2},
\]
where the middle term in the RHS comes from (241) with $n = 8$. This equation gives $(\alpha + \alpha^{-1})^2 = P(\tau)$ with
\[
P(w) := \frac{(w + 2)(w^5 - 5w^3 - w^2 + 5w + 1)^2}{(w^3 - 5w^3 - 5w^3 + 4w + 1)^2}.
\]
We observe $0 < P(\tau_1) < 4$ and $P(\tau_4) > 4$. Hence $p$ is the center of a Siegel disk by Lemma 7.1(1)-(i).

For entry #5 the exceptional set $E$ itself is the only connected component, which is of type $D_9$. The map $f$ acts on the dual graph $\Gamma$ of $E$ non-trivially. For, otherwise, $\hat{A}$ fixes all simple roots in $\Delta_9$, having at least nine eigenvalues $1$, so $\chi(z)$ and hence $\hat{\varphi}_1(z) = C_1(z)^6, C_2(z)^2$ are divisible by $C_1(z)^3$, a contradiction. We have $N_f = 0$ and $\mu(f, E) = 8$ from Lemma 5.1(2). So FPF (13) with $\text{Tr} f^* | H^2(X, \mathbb{C}) = \text{Tr}\hat{A} = 7$ shows that $f$ has a unique transverse fixed point $p \in X \setminus E$. If the eigenvalues of $(df)_p$ are expressed as in (39), then FPF (21) reads
\[
1 + \delta^{-1} = \frac{1}{1 - \delta^2(\alpha + \alpha^{-1}) + \delta} + \frac{1}{2(1 + \delta)} + \frac{1 + \delta + \delta^2 + \delta^3 + \delta^4}{2(1 + \delta^4)},
\]
where the last two terms in the RHS stem from (245) with $n = 9$. This equation gives $(\alpha + \alpha^{-1})^2 = P(\tau)$ with
\[
P(w) := \frac{(w + 2)(w^4 - w^3 - 3w^2 + w + 1)^2}{(w^2 - 2)^2(w^2 + 1)^2(w^2 - w + 1)^2}.
\]
We observe $0 < P(\tau_8) < 4$ and $P(\tau_2) > 4$. Hence $p$ is the center of a Siegel disk by Lemma 7.1(1)-(i).

For entry #6 the exceptional set $E$ itself is the only connected component, which is of type $D_{16}$. The map $f$ acts on the dual graph $\Gamma$ of $E$ trivially, because $\hat{\varphi}_1(z) = \chi(z) = C_1(z)^{16}$. We have $N_f = 1$ and $\mu(f, E) = 15$ from Lemma 5.1(1). So FPF (13) with $\text{Tr} f^* | H^2(X, \mathbb{C}) = \text{Tr}\hat{A} = 16$ shows that $f$ has a unique transverse fixed point $p \in X \setminus E$. If the eigenvalues of $(df)_p$ are expressed as in (39), then FPF (21) reads
\[
1 + \delta^{-1} = \frac{1}{1 - \delta^2(\alpha + \alpha^{-1}) + \delta} - \frac{\delta}{(1 - \delta)^2} \left( \frac{2}{1 + \delta} + \frac{1 + \delta + \cdots + \delta^2}{1 + \delta + \cdots + \delta^2} \right) + \frac{1 + \delta}{(1 - \delta)^2},
\]
where the middle term in the RHS stems from (245) with $n = 16$. This equation gives $(\alpha + \alpha^{-1})^2 = P(\tau)$ with
\[
P(w) := \frac{(w + 2)(w^5 - 2w^7 - 6w^6 + 11w^5 + 11w^4 - 16w^3 - 7w^2 + 5w + 1)^2}{(w^3 - 3w - 1)^2(w^5 - w^3 - 5w^3 + 4w + 5w - 3)^2}.
\]

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We observe $0 < P(\tau_1) < 4$ and $P(\tau_2) > 4$. Hence $p$ is the center of a Siegel disk by Lemma 7.1 (1)-(i).

For entry #9 there are obvious decompositions $E = E(\Delta)_{2}^{2} \cup E(\Delta)_{3}^{1}$, $\Delta_{1} = \Delta_{1}(\Delta_{2})_{4} \cup \Delta_{1}(\Delta_{3})_{2}$, and $\text{Span} \Delta_{5} = \text{Span} \Delta_{5}(\Delta_{2})_{5} \oplus \text{Span} \Delta_{5}(\Delta_{3})_{7} \oplus \text{Span} \Delta_{5}(\Delta_{4})_{8}$, preserved by $f = \tilde{A}$. Let $\chi(z) = \chi_{1}(z) \cdot \chi_{2}(z) \cdot \chi_{3}(z)$ be the corresponding decomposition of characteristic polynomials. As $\deg(\chi(z)) = 2 \times 2 + 6 + 8 = 13 \times 1 + 3 \times 1 + 2 = 18$, we have $\chi(z) = \tilde{\varphi}(z) \cdot (z - \Delta)$. We remark that $\chi_{1}(z) = C_{1}(z)^{13} \cdot C_{2}(z)^{8} \cdot C_{3}(z)$. Since $\tilde{A}$ acts on $\Delta_{5}(\Delta_{3})_{2}$ trivially, we have $\chi_{2}(z) = C_{1}(z)^{5} \cdot C_{2}(z)^{3} \cdot C_{4}(z)$. If $A$ acts on $\Delta_{5}(\Delta_{4})_{8}$ trivially then $\chi_{2}(z) = C_{1}(z)^{8}$, which is absurd. Thus $A$ acts on $\Delta_{5}(\Delta_{4})_{8}$ non-trivially, so that $\chi_{2}(z) = C_{1}(z)^{8} \cdot (z - \Delta)$. This is impossible, so the latter is actually the case. Therefore, $f$ permutes $E(\Delta_{2})_{4}$, acts non-trivially on $\hat{E}(\Delta_{3})_{2}$ and trivially on $\hat{E}(\Delta_{4})_{8}$.

We then have $N_f = 1$ and $\mu(f, E) = \mu(f, E(\Delta_{4})) = 3 + 7 = 10$ from Lemma 5.1 (1)-(2). So FPF (15) with $\text{Tr} f^*|\mathcal{H}^2(X, \mathbb{C}) = \text{Tr} \tilde{A}$ is 11. Thus $f$ has a unique transverse fixed point $p \in X \setminus E$. If the eigenvalues of $(df)_p$ are expressed as $(\delta)$, then FPF (21) can be represented as

$$1 + \delta^{-1} = \frac{1}{1 - \delta^2 (\alpha + \alpha^{-1}) + \delta} + \frac{1}{2(1 + \delta)} + \frac{1 + \delta}{2(1 + \delta^2)}$$

$$- \frac{\delta}{(1 - \delta)^2} \left( \frac{1}{1 + \delta} + \frac{1 + \delta + \delta^2 + \delta^3}{1 + \delta + 2 \delta^2 + 3 \delta^3} + \frac{1 + \delta}{1 - \delta} \right),$$

where the second and third terms in the RHS come from (20b) and (21b) with $n = 8$ respectively. This equation leads to $(\alpha + \alpha^{-1})^2 = P(\tau)$ with the rational function

$$P(w) := \frac{(w + 2)(w^3 - 4w - 2)^2(w^2 - w - 2 + w + 1)^2}{(w^2 - 2)^2(w^3 - 4w^2 - w + 1)^2}.$$

For $\tau = \tau_1$ we observe $0 < P(\tau) < 4$. Using the fact that $\tau$ has minimal polynomial $\text{ST}_{1}^{(4)}(w) = w^3 - w - 3$, we can show that $P(\tau)$ has minimal polynomial $27w^3 - 11w + 1$, which is not monic, so that $P(\tau)$ is not an algebraic integer. Therefore $p$ is the center of a Siegel disk by Lemma 7.1 (1)-(ii).

Remark 7.5 Table (7.4) is just for the sake of illustration, providing only one example for each Picard number $\rho = 2, 4, 6, \ldots, 18$. In fact there are much more pairs $(\varphi, \psi)$ leading to Siegel disks. For example, if $\rho = 4$ we refer to McMullen [8, Table 4]. More examples in this case can be found in Tables 8.2, 8.3, 8.4], which are constructed by the method of hypergeometric groups with $\psi(z)$ being an unramified Salem polynomial of degree 22 and the matrix $B$, in place of $A$, playing the role of a Hodge isometry.

8 Picard Number 2

Let $\lambda_1^{(20)} \approx 1.2326135$ be the smallest Salem number of degree 20, whose minimal polynomial is given by

$$S_1^{(20)}(z) = z^{20} - z^{19} - z^{15} + z^{14} - z^{11} + z^{10} - z^{9} + z^{6} - z^{5} - z + 1.$$

A computer enumeration shows that the solutions to Setup 5.2 with $S(z) = S_1^{(20)}(z)$ are given as in Table 8.1, where the meaning of the last S/H column becomes clear after Theorem 8.1 is stated. Table 8.1 then leads us to consider any K3 surface automorphism $f : X \to X$ such that

- $X$ has Picard number $\rho(X) = 2$ and exceptional set $E(X)$ of Dynkin type $A_1$,
- $f$ has entropy $h(f) = \log \lambda_1^{(20)}$ and special eigenvalue $\delta = \delta(f)$ conjugate to $\lambda_1^{(20)}$,
- $f^{*}|\text{Pic}(X)$ has characteristic polynomial $\tilde{\varphi}_1(z) = C_1(z)^{13} \cdot C_2(z) = (z - 1)(z + 1)$.

We remark that $E(X)$ consists of only one $(-2)$-curve $E \cong \mathbb{P}^1$ and the special trace $\tau := \delta + \delta^{-1}$ is among the roots $\tau_1, \ldots, \tau_9$ of the trace polynomial $\text{ST}_{1}^{(20)}(w)$ such that $2 > \tau_1 > \cdots > \tau_9 > -2$.

Theorem 8.1 The map $f$ has exactly three fixed points in $X$ consisting of a pair $p_{\pm} \in E$ and a single point $p \in X \setminus E$. Each of them is either the center of a Siegel disks ($S$) or a hyperbolic fixed point ($H$), with $p_{\pm}$ being in the same case. How this dichotomy occurs is shown in Table 8.2 for each value of the special trace $\tau$.  

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In case (i) the eigenvalues of \( \varphi \) mean that \( \varphi \) group, having anti-palindromic characteristic polynomial \( \varphi^{(i)} \) = 1

Applying Theorem 8.1 to the entries of Table 8.1, we obtain the S/H column in it, where for example HS and 

Table 8.1: Picard number \( \rho = 20 \) (Setup 32).

| ST | \( \tau_1 \) | \( \tau_2 \) | \( \tau_3 \) | \( \tau_4 \) | \( \tau_5 \) | \( \tau_6 \) | \( \tau_7 \) | \( \tau_8 \) | \( \tau_9 \) |
|----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( p_{\pm} \) | S | S | H | S | S | S | H | H | S |
| \( p \) | S | S | S | S | S | S | S | S | H |

Table 8.2: Center of a Siegel disk (S) or a hyperbolic fixed point (H).

Applying Theorem 8.1 to the entries of Table 8.1 we obtain the S/H column in it, where for example HS and SS mean that \( (p_{\pm}, p) \) is of types (H, S) and (S, S) respectively. Notice that all \( \tau_1, \ldots, \tau_7 \) but \( \tau_2 \) and \( \tau_8 \) appear as special traces. Entry #1 of Table 8.1 is just the first entry of Table 8.1 thus the proof of Theorem 8.1 is completed when Theorem 8.1 is established. The rest of this section is devoted to the proof of Theorem 8.1.

In general if \( F \) is a linear endomorphism with characteristic polynomial \( \varphi(z) \), then by the relation between the generating function for power sums and that for elementary symmetric polynomials we have

\[
\text{Tr}(F^n) = \text{constant} \times (-1)^{n-1} \text{coeff of } z^n \text{ in Maclaurin expansion of } \frac{1}{z} \frac{d}{dz} \log \varphi^\dagger(z) \quad \text{for any } n \geq 1,
\]

where \( \varphi^\dagger(z) \) is the reciprocal to \( \varphi(z) \). Currently, \( F = f^*|H^2(X, \mathbb{Z}) \) is the induced map on middle cohomology group, having anti-palindromic characteristic polynomial \( \varphi(z) = (z - 1)(z + 1)S^{(20)}(z) \). The above formula tells us that \( \text{Tr}(F^n) = \mathbb{1}_3, \mathbb{1}_3, 6, 3, \mathbb{1}_3, \ldots \) for \( n = \mathbb{1}_2, 3, 4, 5, 6, 7, 8, \ldots \) respectively. In particular we notice

\[
\text{Tr}(F) = \text{Tr}(F^3) = \text{Tr}(F^7) = 1.
\]

This observation leads us to consider the map \( f \) together with its third and seventh iterates \( f^3 \) and \( f^7 \).

For the Möbius transformation \( f_E := f|_E \) there are four possibilities:

(i) \( f_E \) has two distinct fixed points \( p_{\pm} \in E \) such that \( (df_E)p_{\pm} = \beta^{\pm 1} \in \mathbb{C}^\times \) with \( \beta \neq 1, \delta^{\pm 1}, \)
(ii) \( f_E \) has a unique fixed point \( p_0 \in E \), in which case \( (df_E)p_0 = 1 \) and \( p_0 \in \text{Fix}_F(f) \),
(iii) \( f_E \) is an identity transformation, that is, \( E \) is a fixed curve of \( f \),
(iv) \( f_E \) has two distinct fixed points \( p_{\pm} \in E \) such that \( (df_E)p_{\pm} = \delta^{\pm 1} \), in which case \( p_{\pm} \in \text{Fix}_F(f) \).

In case (i) the eigenvalues of \( (df)_{p_{\pm}} \) are \( \beta^{\pm 1} \) and \( \delta \beta^{\pm 1} \) as in \( \mathbb{H} \), so \( p_{\pm} \) are transverse fixed points of \( f \).

Lemma 8.2 Case (iv) does not occur.
Proof. FPF (13) is combined with equations (11) and \( N_f = 0 \) to yield
\[
3 = 2 + 1 \leq \mu_{p_+}(f) + \mu_{p_-}(f) + \sum_{p \neq p \pm} \mu_p(f) = 3,
\]
where the sum is taken over all \( p \in \text{Fix}(f) \) such that \( p \neq p \pm \). This shows that \( \mu_{p_+}(f) = 2 \), \( \mu_{p_-}(f) = 1 \) and \( f \) has no other fixed points. The same is true for \( f^3 \). Since \((df)_{p_-} \) has eigenvalues \( \delta^{-1} \) and \( \delta^2 \), FPF (21) and formula (33) in Theorem 6.8 for \( n = 1, 3 \) lead to a system of equations,
\[
1 + \delta^{-1} = \frac{1}{(1 - \delta^{-1})(1 - \delta^2)} + \frac{2\delta + \theta}{(1 - \delta)^2},
\]
\[
1 + \delta^3 = \frac{1}{(1 - \delta^{-3})(1 - \delta^6)} + \frac{2 + 4\delta^3 + (1 + \delta + \delta^2)\theta}{3(1 - \delta^3)^2}.
\]
Eliminating \( \theta \) from it we obtain an algebraic equation \((1 + \delta)(3 + 5\delta^2 - 2\delta^3 + 9\delta^4 - 2\delta^5 + 5\delta^6 + 3\delta^8) = 0 \) for \( \delta \).
This contradicts the fact that the minimal polynomial of \( \delta \) is \( S_{1}^{(20)}(z) \). Thus case (iv) cannot occur. \( \square \)

Put \( \sigma := \delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} \) with branch \( \text{Re}(\delta^{\frac{1}{2}}) > 0 \). Note that \( \sigma = \sqrt{\tau + 2} > 0 \).

Lemma 8.3 In cases (i), (ii) and (iii) the map \( f \) has a unique fixed point \( p \in X \setminus E \), which is transverse. Let \( \delta^{\pm} \in C^\infty \) be the eigenvalues of \((df)_p \) as in (38) and put \( A := e^{-\alpha} + \alpha^{-1} \) and \( B := \beta + \beta^{-1} \), where by convention we understand that \( \beta := 1 \) and \( B := 2 \) in cases (ii) and (iii). Then \( A \) and \( B \) satisfy the equation
\[
\sigma = \frac{\sigma}{\tau - B} + \frac{1}{\sigma - A},
\]
where \( (\tau - A)(\sigma - B) \) does not vanish. In terms of \( B \) the number \( A \) is expressed as
\[
A = \frac{(\tau + 1)B + 2 - \tau^2}{\sigma(B + 1 - \tau)} \quad \text{with} \quad B + 1 - \tau \neq 0.
\]

Proof. In case (i) we have \( \text{Fix}(f) \cap E = \{p \pm \} \), \( \mu_{p_+}(f) = 1 \) and \( N_f = 0 \). In case (ii) we have \( \text{Fix}(f) \cap E = \{p_0\} \), \( \mu_{p_0}(f) = 2 \) and \( N_f = 0 \). In case (iii) we have \( \text{Fix}(f) \cap E = \emptyset \) and \( N_f = 1 \). In any case FPF (13) together with (11) shows that \( f \) has a unique fixed point \( p \in X \setminus E \), which is simple, i.e. transverse.

In case (i), since \( p_\pm \) are transverse fixed point of \( f \), formula (20) gives
\[
\nu_{p_+}(f) + \nu_{p_-}(f) = \frac{1}{(1 - \beta)(1 - \beta^{-1})} + \frac{1}{(1 - \beta^{-1})(1 - \beta)} = \frac{1 + \beta}{1 - \delta B + \delta^2},
\]
so that FPF (21) can be expressed as
\[
1 + \delta^{-1} = \nu_{p_+}(f) + \nu_{p_-}(f) + \nu_{p}(f) = \frac{1 + \delta}{1 - \delta B + \delta^2} + \frac{1}{1 - \delta^2 A + \delta},
\]
which is multiplied by \( \delta^{\frac{1}{2}} \) to yield equation (42). In case (ii) the terms \( \nu_{p_+}(f) + \nu_{p_-}(f) \) in (44) should be replaced by \( \nu_{p_0}(f) \), which is equal to \( (1 + \delta)/(1 - \delta)^2 \) by formula (25) in Theorem 6.3. In case (iii) those terms are not present, but instead a new \((1 + \delta)/(1 - \delta)^2 \) comes in due to the transversality \( N_f = 0 \rightarrow N_f = 1 \). In either case we have (11) and hence (42) with \( B = 2 \). Note that \( \tau - B \neq 0 \) follows from the transversality of \( p_\pm \) in case (i) and from \( B = 2 \) in cases (ii) and (iii), while \( \sigma - A \neq 0 \) follows from the transversality of \( p \).

Equation (42) yields \( \sigma(B + 1 - \tau)A = (\tau + 1)B + 2 - \tau^2 \). If \( B + 1 - \tau = 0 \), that is, \( B = \tau - 1 \) then \( 0 = (\tau + 1)B + 2 - \tau^2 = 1 \), which is impossible. Hence \( B + 1 - \tau \neq 0 \) and \( A \) is expressed as (43). \( \square \)

Lemma 8.4 In case (i) we have \( \beta^n \neq \pm \beta^n \) for \( n = 3, 7 \).

Proof. If \( \beta^3 = \delta \pm 3 \) then \( p_\pm, p \in \text{Fix}(f^3) \) with \( \mu_{p_+}(f^3) \geq 2 \), \( \mu_{p_-}(f^3) = 1 \), \( \mu_p(f^3) \geq 1 \) and \( N_{f^3} = 0 \), so FPF (13) together with (11) leads to a contradiction \( 4 = 2 + 1 + 1 \leq \mu_{p_+}(f^3) + \mu_{p_-}(f^3) + \mu_p(f^3) \leq 3 \). If \( \beta^7 = \delta \pm 7 \) then the same argument with \( f^3 \) replaced by \( f^7 \) yields a similar contradiction. \( \square \)

For \( n \geq 1 \) let \( h_n(w) \in \mathbb{Z}[w] \) be the polynomial such that \( z^n + z^{-n} = h_n(w) \) for \( w = z + z^{-1} \). We have
\[
h_3(w) = w(w^2 - 3), \quad h_7(w) = w(w^6 - 7w^4 + 14w^2 - 7).
\]
Lemma 8.5 In cases (i), (ii) and (iii) the numbers $A$ and $B$ in Lemma 8.3 satisfy two more equations

$$h_n(\sigma) = \frac{h_n(\tau)}{h_n(B) - h_n(A)} + \frac{1}{h_n(\sigma) - h_n(A)}, \quad n = 3, 7,$$

(45)

where all fractions appearing in (45) have nonzero denominators.

Proof. For $n = 3, 7$ we have $\text{Tr}(f^n) = \text{Tr}(f) = 1$ by (11) and

- case (i) for $f$ with $\beta^n \neq 1$ leads to case (i) for $f^n$ by Lemma 8.4
- case (i) for $f$ with $\beta^n = 1$ leads to case (iii) for $f^n$, in which $h_n(B) = \beta^n + \beta^{-n} = 2$,
- case (ii) for $f$ leads to case (ii) for $f^n$, in which $h_n(B) = h_n(2) = 2$,
- case (iii) for $f$ leads to case (iii) for $f^n$, in which $h_n(B) = h_n(2) = 2$.

Thus Lemma 8.3 and its proof apply to $f^n$ in place of $f$. Equations (45) are obtained from (42) by replacing $f$ with $f^n$. This amounts to altering $\delta \mapsto \delta^n, \alpha \mapsto \alpha^n, \beta \mapsto \beta^n$ and so $\xi \mapsto h_n(\xi)$ for $\xi = \tau, \sigma, A, B$ in (42). Here in cases (ii) and (iii) the convention in Lemma 8.3 takes the form $h_n(B) = 2$ for $f^n$, which is fulfilled. \end{proof}

Lemma 8.6 Cases (ii) and (iii) do not occur, hence case (i) actually occurs.

Proof. Recall that we have $B = 2$ in cases (ii) and (iii). Substituting (18) with $B = 2$ into (15) for $n = 3$, we find that $\tau$ satisfies the septic equation $\tau^7 - 3\tau^6 - 9\tau^5 + 17\tau^4 + 39\tau^3 - 7\tau^2 - 50\tau - 20 = 0$. This contradicts the fact that the minimal polynomial of $\tau$ is $S_{11}^{(20)}(w)$. Thus these cases cannot occur altogether. \end{proof}

Lemma 8.7 Let $A$ and $B$ be the numbers in Lemma 8.3. Then we have

$$B = Q(\tau) := -(\tau + 1)(\tau - 2)(\tau^3 - 3\tau + 1), \quad (46a)$$

$$A^2 = P(\tau) := \frac{(\tau^6 - 6\tau^4 - \tau^3 + 10\tau^2 + 3\tau - 4)^2}{(\tau + 2)(\tau^2 - 3)(\tau^3 - \tau^2 - 2\tau + 1)^2}. \quad (46b)$$

Proof. Substituting (18) into (15) we obtain two algebraic equations for $B$, which turn out to factor into

$$\{B - Q(\tau)\} R_3(\tau; B) = 0, \quad \{B - Q(\tau)\} R_5(\tau; B) = 0,$$

over the number field $K := \mathbb{Q}(\tau)$, where $R_3(\tau; x) \in K[x]$ and $R_5(\tau; x) \in K[x]$ are polynomials of degrees 3 and 11 respectively. Moreover $R_3(\tau; x)$ and $R_5(\tau; x)$ have no roots in common (consider their resultant). These facts are verified by Mathematica, which is capable of polynomial calculations over an algebraic number field. Thus we obtain equation (46a). Substituting it into (18) yields

$$A = \frac{\tau^6 - 6\tau^4 - \tau^3 + 10\tau^2 + 3\tau - 4}{\sigma(\tau^2 - 3)(\tau^3 - \tau^2 - 2\tau + 1)},$$

which is squared to give equation (46b), where the relation $\sigma^2 = \tau + 2$ is also used. \end{proof}

Proof of Theorem 8.1 We observe that $|Q(\tau_j)| < 2$ for $j = 1, 2, 4, 5, 6, 9$ and $|Q(\tau_j)| > 2$ for $j = 3, 7, 8$. Thus Lemma 7.2 together with Remark 7.3 implies the second row in Table 8.2. Similarly we observe that $0 < P(\tau_j) < 4$ for $j = 1, \ldots, 8$ and $P(\tau_9) > 4$. Hence Lemma 7.4 yields the third row in Table 8.2. \end{proof}

Remark 8.8 There are examples of K3 surface automorphisms $f : X \to X$ such that $\rho(X) = 12$, the exceptional set $E(X)$ is of type $A_2$, and $f$ has three Siegel disks with centers on $E(X)$ (see [3] Remark 9.7).

A Table of Salem Trace Polynomials

Let $\lambda_i^{(d)}$ be the $i$-th smallest Salem number of degree $d$ and $S_i^{(d)}(z)$ be its minimal polynomial. Here is a list of all Salem polynomials $S_i^{(d)}(z)$ that appear explicitly in this article as the Salem factor $S(z)$ of the polynomial $\varphi(z)$.
They are presented in terms of their trace polynomials \( ST_i^{(d)}(w) \). For each of them numerical computations and symbolic manipulations of the roots \( \tau_0, \tau_1, \ldots, \tau_{d/2-1} \) in (13) can be carried out by using these data.

\[
\begin{align*}
ST_1^{(d)}(w) & = w^2 - w - 3, \\
ST_1^{(3)}(w) & = w^3 - 4w - 1, \\
ST_1^{(8)}(w) & = w^4 - 4w^2 - w + 1, \\
ST_2^{(3)}(w) & = w^4 - 3w^2 + w + 1, \\
ST_1^{(16)}(w) & = w^4 - 2w^3 - 4w^2 + 7w + 1, \\
ST_1^{(10)}(w) & = w^4 - 5w^2 - 2w + 1, \\
ST_1^{(12)}(w) & = w^6 - w^5 - 5w^4 + 4w^3 + 5w^2 - 2w - 1, \\
ST_1^{(14)}(w) & = w^7 - 7w^5 - w^4 + 13w^3 + 4w^2 - 4w - 1, \\
ST_1^{(16)}(w) & = w^8 - w^7 - 8w^6 + 7w^5 + 20w^4 - 14w^3 - 16w^2 + 7w + 1, \\
ST_2^{(16)}(w) & = w^8 + w^7 - 8w^6 - 8w^5 + 19w^4 + 18w^3 - 13w^2 - 10w + 1, \\
ST_3^{(16)}(w) & = w^8 - 8w^6 - w^5 + 20w^4 + 4w^3 - 16w^2 - 3w + 2, \\
ST_4^{(16)}(w) & = w^8 - w^7 - 8w^6 + 7w^5 + 20w^4 - 14w^3 - 17w^2 + 7w + 4, \\
ST_5^{(10)}(w) & = w^9 - 9w^6 - w^5 + 26w^4 + 5w^3 - 25w^2 + 5w + 4, \\
ST_{22}^{(18)}(w) & = w^9 + w^8 - 10w^7 - 11w^6 + 32w^5 + 38w^4 - 33w^3 - 42w^2 + 4w + 7, \\
ST_1^{(20)}(w) & = w^{10} - w^9 - 10w^8 + 9w^7 + 35w^6 - 28w^5 - 49w^4 + 35w^3 + 21w^2 - 15w + 1.
\end{align*}
\]

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