Research Article

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On Poincaré duality for pairs (G,W)

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Abstract: Let $G$ be a group and $W$ a $G$-set. In this work we prove a result that describes geometrically, for a Poincaré duality pair $(G, W)$, the set of representatives for the $G$-orbits in $W$ and the family of isotropy subgroups. We also prove, through a cohomological invariant, a necessary condition for a pair $(G, W)$ to be a Poincaré duality pair when $W$ is infinite.

Keywords: Poincaré duality pairs, Cohomology of groups, Cohomological invariants

MSC: 20J05, 55P20, 55U30

1 Introduction

Bieri and Eckmann [4] introduced the concept of relative cohomology $H^*(G, S; M)$ for a group pair $(G, S)$, where $G$ is a group, $S$ is a family of subgroups of $G$ and $M$ is a $\mathbb{Z}_2 G$-module. Dicks and Dunwoody [6] worked with that concept from another point of view. Instead of dealing with $S$ a family of subgroups, they worked with a $G$-set $W$, defining the groups $H^*(G, W; M)$. In Section 2 we recall some definitions and results about relative cohomology of groups and we describe, in details, the equivalence between the theories of Dicks-Dunwoody and Bieri-Eckmann.

By using the relative cohomology theory, Bieri and Eckmann introduced the concept of Poincaré duality pair $(G, S)$ and gave a topological interpretation for those pairs. Dicks and Dunwoody, with their notation for relative cohomology, also gave a topological interpretation for Poincaré duality pairs $(G, W)$.

In Section 3 we present some concepts about Poincaré duality pairs and a result that describes topologically, for a $PD^n$-pair $(G, W)$, the set $E$ of orbit representatives and the family $S$ of isotropy subgroups.

In Section 4, based in [1] and [2] and by using the notation from Dicks and Dunwoody, we present a characterization of the types of Poincaré duality pairs and, through of a generalized invariant “end”, a cohomological criterion for a pair $(G, W)$ to be a Poincaré duality pair.

2 The equivalence between the theories of Dicks-Dunwoody and Bieri-Eckmann

To begin with we recall some definitions and results which will be useful in this work. For details see [4] and [6].

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Definition 2.1. Let $G$ be a group. A $\mathbb{Z}_2G$-projective resolution of a $\mathbb{Z}_2G$-module $M$ is an exact sequence of $\mathbb{Z}_2G$-modules: $\cdots \rightarrow F_0 \xrightarrow{\delta_0} F_1 \xrightarrow{\delta_1} \cdots \rightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\epsilon} M \rightarrow 0$ in which each $F_i$ is projective. The map $F_0 \xrightarrow{\epsilon} M$ is called augmentation map and we denote the projective resolution by $F \rightarrow M$.

Definition 2.2. Let $G$ be a group, $M$ a $\mathbb{Z}_2G$-module and $F \rightarrow \mathbb{Z}_2$ a projective resolution of $\mathbb{Z}_2$ over $\mathbb{Z}_2G$. The homology groups of $G$ with coefficients in $M$ are, for all $n \in \mathbb{Z}$, defined by $H_n(G; M) = H_n(F \otimes_{\mathbb{Z}_2} G)$. The cohomology groups of $G$ with coefficients in $M$ are, for all $n \in \mathbb{Z}$, defined by $H^n(G; M) = H^n(Hom_{\mathbb{Z}_2}(F, Hom_G(M, F)))$.

Let $G$ be a group and let $S = \{S_i, i \in I\}$ be a family of subgroups of $G$. The pair $(G, S)$ is called a group pair. Consider $\mathbb{Z}_2(G/S)$ the free $\mathbb{Z}_2$-module generated by the cosets $gS_i$, which $G$ acts by left multiplication. The map $\varepsilon : \mathbb{Z}_2(G/S) \rightarrow \mathbb{Z}_2$ defined on the generators by $\varepsilon(gS_i) = 1$ is called usual augmentation map and we denote by $\Delta$ the kernel of $\varepsilon$.

Now we recall the concept of relative cohomology of groups due to Bieri and Eckmann.

Definition 2.3. Let $(G, S)$ be a group pair, $S = \{S_i, i \in I\}$, $M$ a $\mathbb{Z}_2G$-module and $F \rightarrow \mathbb{Z}_2$ a $\mathbb{Z}_2G$-projective resolution of the trivial $\mathbb{Z}_2G$-module $\mathbb{Z}_2$. The relative (co)homology groups for $(G, S)$, with coefficients in $M$ are, for all $k \in \mathbb{Z}$, defined by

- $H_k(G, S; M) = H_{k-1}(F \otimes_{\mathbb{Z}_2} (\Delta \otimes_{\mathbb{Z}_2} M))$
- $H^k(G, S; M) = H^{k-1}(Hom_{\mathbb{Z}_2}(G, Hom_{\mathbb{Z}_2}(\Delta, M)))$

where the $G$-actions in $Hom_{\mathbb{Z}_2}(\Delta, M)$ and $\Delta \otimes_{\mathbb{Z}_2} M$ are given, respectively, by $g f(x) = g f(g^{-1} x)$ and $g(x \otimes m) = g x \otimes g m$ (diagonal actions).

Remark 2.4.

(i) If $S = \emptyset$, the relative cohomology is simply the ordinary cohomology of groups, i.e., $H^k(G, \emptyset; M) = H^k(G; M)$ and $H_k(G, \emptyset; M) = H_k(G; M)$.

(ii) It is convenient to write, for any family $S = \{S_i, i \in I\}$ of subgroups of $G$,

$$H_k(S; M) = \bigoplus_{i \in I} H_k(S_i; M) \text{ and } H^k(S; M) = \prod_{i \in I} H^k(S_i; M),$$

where $M$ is a $\mathbb{Z}_2S$-module, i.e., a $\mathbb{Z}_2S_i$-module for all $i$. If $M$ is a $\mathbb{Z}_2G$-module then $M$ is a $\mathbb{Z}_2S$-module by restrictions.

We will see now the definition of relative cohomology due to Dicks and Dunwoody.

Definition 2.5. Consider $W$ a $G$-set and $\mathbb{Z}_2W$ the free $\mathbb{Z}_2$-module generated by $W$. Let $\eta : \mathbb{Z}_2W \rightarrow \mathbb{Z}$ be the augmentation map, $\Delta' = \ker \eta$, $M$ a $\mathbb{Z}_2G$-module and $P \rightarrow \Delta'$ a $\mathbb{Z}_2G$ projective resolution of $\Delta'$. The groups of relative cohomology of the pair $(G, W)$, with coefficients in $M$ are, for all $k \in \mathbb{Z}$, defined by

$$H_k(G, W; M) = H_{k-1}(P \otimes_{\mathbb{Z}_2} G M) \text{ and } H^k(G, W; M) = H^{k-1}(Hom_{\mathbb{Z}_2}(P, Hom_{\mathbb{Z}_2}(G, M))).$$

Now we present the equivalence between the Definitions 2.3 and 2.5.

Theorem 2.6. Let $G$ be a group and $M$ a $\mathbb{Z}_2G$-module. Suppose either

(i) $W$ is a $G$-set, $E = \{w_i, i \in I\}$ is a set of representatives for the $G$-orbits in $W$ and $S = \{G w_i | i \in I\}$ is the family of isotropy subgroups; or

(ii) $S = \{S_i | i \in I\}$ is a family of subgroups of $G$, $W = \bigcup_{i \in I} G/S_i$ and $G$ acts in $W$ by left translation in $G/S_i$, $i \in I$.

Then the relative (co)homology groups of the pair $(G, W)$ with coefficients in $M$ (given by Definition 2.5) and the relative (co)homology groups of the pair $(G, S)$ with coefficients in $M$ (given by Definition 2.3) are isomorphic, i.e., for all $k \in \mathbb{Z}$, we have

$$H_k(G, W; M) \simeq H_k(G, S; M) \text{ and } H^k(G, W; M) \simeq H^k(G, S; M).$$
Proof. (i) Firstly, consider a pair \((G, W)\) and \(E = \{w_i \mid i \in I\}\) a set of orbit representatives in \(W\). We have
\[
W = \bigcup_{w_i \in E} G(w_i),
\]
where \(G(w_i) = \{g w_i \mid g \in G\}\) is the orbit of the element \(w_i \in E\). Consider, for each \(w_i \in I\), the isotropy subgroup \(G_{w_i} = \{g \in G \mid g w_i = w_i\}\). Using the one-to-one correspondence \(G(w_i) \leftrightarrow G/G_{w_i}\) we have
\[
\mathbb{Z}_2 W = \mathbb{Z}_2 \left[ \bigcup_{w_i \in E} G(w_i) \right] = \bigoplus_{w_i \in E} \mathbb{Z}_2 [G(w_i)] = \bigoplus_{w_i \in E} \mathbb{Z}_2 [G/G_{w_i}].
\]
Let \(S = \{G_{w_i} \mid i \in I\}\) be the family of isotropy subgroups. We have \(\mathbb{Z}_2 W = \mathbb{Z}_2 (G/S)\). It follows from Definition 2.3 that \(\mathbb{Z}_2 W\) coincides with the augmentation \(\eta: \mathbb{Z}_2 W \to \mathbb{Z}_2\) from Definition 2.3 coincides with the augmentation \(\eta: \mathbb{Z}_2 W \to \mathbb{Z}_2\) from Definition 2.5. Moreover \(\Delta = \ker \epsilon = \ker \eta\). Consider a \(\mathbb{Z}_2 G\)-projective resolution of the module \(\mathbb{Z}_2\):
\[
F: \cdots \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]
It follows from Definition 2.3 that \(H_k(G, S; M) = H_{k-1}(F \otimes_{\mathbb{Z}_2} G, \Delta \otimes_{\mathbb{Z}_2} M)\). Now, for all \(k \in \mathbb{Z}\), \(P_k = F_k \otimes_{\mathbb{Z}_2} \Delta\) is a \(\mathbb{Z}_2 G\)-module with diagonal \(G\)-action. Since \(\Delta\) is \(\mathbb{Z}_2\)-free (see [5]), it follows that \(P_k\) is \(\mathbb{Z}_2 G\)-projective and the sequence
\[
P: \cdots \rightarrow \delta_4 \rightarrow \delta_3 \rightarrow \delta_2 \rightarrow \delta_1 \rightarrow \delta_0 \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]
is exact. Thus \(P \rightarrow \Delta\) is a \(\mathbb{Z}_2 G\)-projective resolution of \(\Delta\). By Definition 2.5 we have \(H_k(G, W; M) = H_{k-1}(P \otimes_{\mathbb{Z}_2} G, M)\). Observe that, since \(\Delta\) is \(\mathbb{Z}_2\)-free, we have \(\Delta = \otimes \mathbb{Z}_2\) and thus
\[
P_1 \otimes_{\mathbb{Z}_2} G M = (F_1 \otimes_{\mathbb{Z}_2} \Delta) \otimes_{\mathbb{Z}_2} G M = \oplus (F_i \otimes_{\mathbb{Z}_2} G M).
\]
On the other hand,
\[
F_1 \otimes_{\mathbb{Z}_2} G (\Delta \otimes_{\mathbb{Z}_2} M) = F_1 \otimes_{\mathbb{Z}_2} G (\oplus \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} M) = \oplus (F_i \otimes_{\mathbb{Z}_2} G M).
\]
Therefore \(P \otimes_{\mathbb{Z}_2} G M = F \otimes_{\mathbb{Z}_2} G (\Delta \otimes_{\mathbb{Z}_2} M)\) and we conclude that \(H_k(G, S; M) \simeq H_k(G, W; M)\).

To calculate the cohomology groups due to Bieri-Eckmann we consider the projective resolution (1) and, by Definition 2.3, \(H^k(G, S; M) = H^{k-1}(Hom_{\mathbb{Z}_2 G}(F, Hom_{\mathbb{Z}_2 G}(\Delta, M)))\). Considering now the projective resolution (2) and Definition 2.5, we have \(H^k(G, W; M) = H^k(Hom_{\mathbb{Z}_2 G}(P, M))\). Through the projective resolution (2) we can form the following cochain complex:
\[
Hom_{\mathbb{Z}_2 G}(P, M) : 0 \rightarrow \widetilde{\delta}_0 Hom_{\mathbb{Z}_2 G}(P_0, M) \rightarrow \widetilde{\delta}_1 Hom_{\mathbb{Z}_2 G}(P_1, M) \rightarrow \cdots
\]
where \(\widetilde{\delta}_i(f) = f \circ \delta_{i-1}, i > 0\) and \(\widetilde{\delta}_0 = 0\). Observe that \(Hom_{\mathbb{Z}_2 G}(P, M) = Hom_{\mathbb{Z}_2 G}(P \otimes \Delta, M)\). Thus we have \(H^k(G, S; M) \simeq H^k(G, W; M)\).

(ii) Now, consider a group pair \((G, S)\) where \(S = \{S_i \mid i \in I\}\) is a family of subgroups of \(G\) and let \(W = \bigcup_{i \in I} G/S_i\) be. For all \(w = hS_i \in W\) we have \(g w = ghS_i, \forall g \in G\). This defines a \(G\)-action in \(W\). For \(w \in W\) we have the \(G\)-orbit
\[
G(w) = \{g w \mid g \in G\} = \{g hS_i \mid g \in G\} = G/S_i.
\]
The set \(E = \{w_i = 1S_i \mid i \in I\}\) is a set of orbit representatives and
\[
G_{w_i} = \{g \in G \mid g w_i = w_i\} = \{g \in G \mid g \in S_i\} = S_i
\]
is the isotropy subgroup for \(w_i\). With this notation, and by using the same idea of the first part of the proof, we have \(H_k(G, S; M) \simeq H_k((G, W); M)\) and \(H^k(G, S; M) \simeq H^k((G, W); M)\).
3 Poincaré duality pairs

In this section, before proving the main result, we recall some definitions and results about duality groups and pairs due to Bieri and Eckmann [4] and Dicks and Dunwoody [6].

**Definition 3.1.** A group $G$ is called a duality group of dimension $n$, or simply a $D^n$-group, if there exist a $\mathbb{Z}_2 G$-module $C$, called the dualizing module of $G$, and natural isomorphisms

$$H^k(G; M) \cong H_{n-k}(G; C \otimes M)$$

for all integers $k$ and all $\mathbb{Z}_2 G$-modules $M$. In the special case where $C = \mathbb{Z}_2$, we say that $G$ is a Poincaré duality group of dimension $n$, or simply a $PD^n$-group.

**Definition 3.2.** A duality pair of dimension $n$, or simply a $D^n$-pair, consists of a group pair $(G, S)$ and a $\mathbb{Z}_2 G$-module $C$, where $S = \{S_i, i \in I\}$ is a finite family of $D^{n-1}$-subgroups of $G$ with dualizing module $C$ and natural isomorphisms

$$H^k(G; M) \cong H_{n-k}(G; S_i \otimes M), \quad (1)$$

$$H^k(G, S; M) \cong H_{n-k}(G; C \otimes M), \quad (2)$$

for all $\mathbb{Z}_2 G$-modules $M$ and all $k \in \mathbb{Z}$. $C$ is called the dualizing module of the $D^n$-pair $(G, S)$. If $C = \mathbb{Z}_2$ the duality pair $(G, S)$ is called a Poincaré duality pair, or simply a $PD^n$-pair.

**Remark 3.3.** If $(G, S)$ is a $PD^n$-pair we can show that the isomorphisms (1) and (2) in Definition 3.2 are equivalent ([3]). Hence, to show that a group pair $(G, S)$ is a $PD^n$-pair it is enough to prove only one of the isomorphisms given in Definition 3.2.

In view of Remark 3.3 we can define Poincaré duality pairs in a simpler way.

**Definition 3.4.** A group pair $(G, S)$ is called of Poincaré duality pair of dimension $n$ ($PD^n$-pair), if there exists a natural isomorphism

$$H^k(G; M) = H_{n-k}(G, S; M) \quad (3)$$

for all $\mathbb{Z}_2 G$-module $M$ and all $k \in \mathbb{Z}$.

**Definition 3.5.** A group pair $(G, S)$, with $S = \{S_i, i \in I\}$, is realised by a pair of CW-complexes $(X, Y)$, if $X$ is a $K(G, 1)$-complex and $Y$ is a subcomplex of $X$ whose components $Y_i, i \in I$, are $K(S_i, 1)$ complexes, so that the maps $i_\#: \pi_1(Y_i) \rightarrow \pi_1(X)$ induced by the inclusions $i : Y_i \hookrightarrow X$ are injective and map $\pi_1(Y_i)$ on $S_i \subset G$, after a convenient choice of paths connecting base points. The pair $(X, Y)$ is called an Eilenberg-MacLane pair and is denoted by $K(G, S, 1)$.

The next result provides a topological interpretation for $PD^n$-pairs (see [4]).

**Theorem 3.6.** If $(G, S)$ admits an Eilenberg-MacLane pair $(X, Y)$, where $X$ is a compact manifold of dimension $n$ and $Y = \partial X$, then $(G, S)$ is a $PD^n$-pair.

In the following we present the definition of $PD^n$-pair and its topological interpretation given by Dicks and Dunwoody in [6].

**Definition 3.7.** A pair $(G, W)$ is a Poincaré duality pair of dimension $n$, or simply a $PD^n$-pair, if there exists a natural isomorphism

$$H^k(G; M) \cong H_{n-k}(G, W; M)$$

for all $\mathbb{Z}_2 G$-modules $M$ and all $k \in \mathbb{Z}$.
Remark 3.8. If $(G, W)$ is a $PD^n$-pair and $S = \{G_{w_i} \mid i \in I\}$ is the family of isotropy subgroups of a set of orbits representatives $E = \{w_i, \ i \in I\}$ in $W$ then, it follows from Definitions 3.4 and 3.7 and Theorem 2.6, that

$$H^k(G; M) = H_{n-k}(G, W; M) = H_k(G, S; M)$$

Hence $(G, S)$ is a $PD^n$-pair according to Bieri and Eckmann. By using [4, Theorem 4.2], we have, for a $PD^n$-pair $(G, W)$, the following results:

(i) $W$ falls into finitely many $G$-orbits.
(ii) For each $w \in W$, the isotropy subgroup $G_w$ is a $PD^{n-1}$-group.

Theorem 3.9. Let $X$ be a compact $n$-manifold, $\tilde{X}$ its universal covering space, and suppose that $\partial \tilde{X}$ and the components of the boundary $\partial X$ are all contractible; let $G = \pi_1(X)$ and let $W$ be the $G$-set of components of $\partial \tilde{X}$. Then $(G, W)$ is a Poincaré duality pair of dimension $n$.

The next theorem provides a relation between the topological interpretations for $PD^n$-pairs given by the Theorem 3.6 due to Bieri Eckmann and Theorem 3.9 due to Dicks-Dunwoody, describing the set of orbit representatives in $W$ and the family of isotropy subgroups.

Theorem 3.10. Let $X$ be a compact $n$-manifold, which is also a CW-complex, with boundary $\partial X = \bigcup_{i \in I} X_i$, where $X_i, \ i \in I$, are the components of $\partial X$. Consider $\tilde{X}$ the universal covering of $X$ and suppose that $\tilde{X}$ and the components of the boundary $\partial \tilde{X}$ are all contractible. Let $G = \pi_1(X)$ and $W$ the $G$-set of components of $\partial \tilde{X}$. Then, (i) $E = \{\tilde{X}_i \mid i \in I\}$ is a set of orbit representatives in $W$, where, for each $i \in I$, $\tilde{X}_i$ is a copy of the universal covering of $X_i$. (ii) For each $\tilde{X}_i \in E$, we have $G_{\tilde{X}_i} = \pi_1(X_i)$. (iii) If $S = \{G_{\tilde{X}_i} \mid i \in I\}$ is the family of isotropy subgroups given by $E$ then $(X, \partial X)$ is an Eilenberg-MacLane pair realising the group pair $(G, S)$. Hence, $(G, W)$ is a $PD^n$-pair according to Dicks-Dunwoody and $(G, S)$ is a $PD^n$-pair according to Bieri-Eckmann.

Proof. (i) Consider the covering map $p : \tilde{X} \to X$. Since $p$ is a local homeomorphism we have $\partial \tilde{X} = p^{-1}(\partial X)$ and so $W$ is the set of path components of $p^{-1}(\partial X)$. On the other hand, $\tilde{X}_i = p^{-1}(X_i)$ consists of copies of the universal covering of $X_i$. Thus, for each $i \in I$, we have a set $J_i$ and a family of path connected sets $\tilde{X}_{ij}$, $j \in J_i$, such that $\tilde{X}_i = \bigcup_{j \in J_i} \tilde{X}_{ij}$. Hence

$$\partial \tilde{X} = \bigcup_{i \in I} \tilde{X}_i = \bigcup_{i \in I} \bigcup_{j \in J_i} \tilde{X}_{ij}.$$

It follows that

$$W = \{\tilde{X}_{ij} \mid i \in I, j \in J_i\} = \bigcup_{i \in I} \{\tilde{X}_{ij} \mid j \in J_i\}.$$
It follows that $g \in A(\overline{X}_i, p_i) = \pi_1(X_i)$ and so, $G\overline{X}_i \subset \pi_1(X_i)$. Therefore $S_i = G\overline{X}_i = \pi_1(X_i), \forall i \in I$.

(iii) It follows from the hypotheses that $X$ is a $K(G, 1)$ and $X_i = K(S_i, 1)$-subcomplex of $X$ for all $i \in I$. Therefore, $(X, \partial X)$ is an Eilenberg-MacLane-pair realising $(G, S)$. Finally, by the hypotheses of the theorem it follows that $(G, W)$ is a $PD^n$-pair according to Dicks-Dunwoody and from $(iii)$ it follows that $(G, S)$ is a $PD^n$-pair according to Bieri-Eckmann.

\[\text{4 A cohomological criterion for Poincaré duality pairs}\]

Let $(G, S)$ a group pair with $S = \{S_i \mid i \in I\}$ a family of subgroups with infinite index in $G$ and consider the $\mathbb{Z}_2G$-module $\mathbb{Z}_2(G/S) = \bigoplus_{i \in I} \mathbb{Z}_2(G/S_i)$. Based in [1], we have defined the algebraic invariant

$$E(G, S, \mathbb{Z}_2(G/S)) = 1 + \dim \ker \text{res}^G_S,$$

where $\text{res}^G_S : H^1(G; \mathbb{Z}_2(G/S)) \to \prod_{i \in I} H^1(S_i; \mathbb{Z}_2(G/S))$ is the restriction map induced by the inclusions $S_i \hookrightarrow G$ for $i \in I$. Denote $E(G, S, \mathbb{Z}_2(G/S))$ by $E(G, S)$.

The following result, based in [2, Theorem 1], provides a necessary condition for a group pair $(G, S)$ to be a $PD^n$-pair.

**Theorem 4.1.** If $(G, S)$ is a $PD^n$-pair, with $[G : S] = \infty$ for all $S \in S$, then $E(G, S) = 1$.

This cohomological criterion can be revised in the notation of Dicks-Dunwoody. Before proving the result we have to make some remarks.

Let $G$ be a group, $S$ a subgroup of $G$, $M$ a $\mathbb{Z}_2G$-module and $i : S \to G$ the inclusion map. Consider the restriction map

$$\text{res}^G_S : H^1(G; M) \to H^1(S; M).$$

**Lemma 4.2.** Let $(G, W)$ be a pair where $G$ is a group and $W$ is a $G$-set. Consider $w, u \in W$ representatives for the same $G$-orbit in $W$. If $G_u$ and $G_w$ are the correspondent isotropy subgroups then $\ker \text{res}^G_{G_u} = \ker \text{res}^G_{G_w}$.

**Proof.** Consider $w, u \in W$ representatives for the same $G$-orbit in $W$. Then there exists $\sigma \in G$ such that $u = \sigma w$ and it is easy to see that $G_u = \sigma G_w \sigma^{-1}$. By [8, Corollary 2-3-2], the inner automorphism $\varphi_\sigma : G \to G$ given by $\varphi(x) = \sigma x \sigma^{-1}$ induces the identity in cohomology, i.e., $id = \varphi_\sigma : H^1(G; M) \to H^1(G; M)$. Besides, $\varphi(G_u) = G_w$ and the induced homomorphism $\varphi_\sigma^* : H^1(G_u; M) \to H^1(G_w; M)$ is an isomorphism. Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
H^1(G; M) & \xrightarrow{\text{res}^G_{G_u}} & H^1(G_u; M) \\
\downarrow{id} & & \downarrow{\varphi_\sigma^*} \\
H^1(G; M) & \xrightarrow{\text{res}^G_{G_w}} & H^1(G_w; M)
\end{array}
$$

Then $\text{res}^G_{G_w} = \varphi_\sigma^* \circ \text{res}^G_{G_u}$ and so, $\ker \text{res}^G_{G_u} = \ker \text{res}^G_{G_w}$.

\[\square\]
Consider now two sets, \( E \) and \( E' \), of orbit representatives for the \( G \)-orbits in \( W \). We have the restriction maps:

\[
\text{res}^G_W : H^1(G, M) \to \prod_{u \in E} H^1(G_u, M) \quad \text{and} \quad \text{res}'^G_W : H^1(G, M) \to \prod_{u \in E'} H^1(G_u, M)
\]

In view of the previous result and by \([2, \S 1, (1.1)]\), it follows that

\[
\ker \text{res}^G_W \cong \bigcap_{w \in E} \ker \text{res}^G_{G_w} = \bigcap_{u \in E'} \ker \text{res}'^G_{G_u} = \ker \text{res}'^G_W.
\]

Hence we have the following result.

**Lemma 4.3.** In view of the previous considerations, \( \dim \ker \text{res}^G_W \) is independent of the choice of the set \( E \) of orbit representatives.

By using this Lemma, we can adapt the definition of the invariant \( E(G, S) \) to the notation from Dicks and Dunwoody.

**Definition 4.4.** Let \((G, W)\) be a pair where \( G \) is a group and \( W \) is a \( G \)-set such that \([G : G_w] = \infty \) for all \( w \in E \), where \( E \) is a set of orbit representatives for the \( G \)-orbits of \( W \), and whose morphisms are maps \( \psi : (G, W) \to (G', W') \) consisting of

- A homomorphism \( \alpha : G \to G' \);
- A map of \( G \)-sets \( \phi : W \to W' \), with \( \phi(E) \subseteq E' \), (where \( E \) and \( E' \) are sets of orbit representatives for the \( G \)-orbits of \( W \) and the \( G' \)-orbits of \( W' \), respectively) such that \( \alpha(G_w) \subseteq G_{\phi(w)} \), \( \forall w \in E \).
- A homomorphism \( f : \mathbb{Z}^I W \to \mathbb{Z}^I W' \) satisfying \( f(gw) = \alpha(g)f(w) \) \( \forall g \in G, w \in W \).

Now, we can rephrase Theorem 4.1 providing a necessary condition for a pair \((G, W)\) to be a Poincaré duality pair.

**Theorem 4.6.** Let \((G, W)\) be a Poincaré duality pair of dimension \( n \). Consider \( E \) a set of representatives to the \( G \)-orbits in \( W \) and suppose \([G : G_w] = \infty \) for all \( w \in E \). Then \( E(G, W) = 1 \).

**Proof.** Since \((G, W)\) is a \( PD^n \)-pair it follows that \( E \) is finite. The proof is similar to that of \([2, \text{Theorem 1}]\), replacing the group pair \((G, S)\) by the pair \((G, W)\) and putting \( S = \{G_w \mid w \in E\} \). The proof consists in to calculating the groups that appears in the exact sequence

\[
0 \to H^0(G; \mathbb{Z}_2 W) \to \bigoplus_{w \in E} H^0(G_w; \mathbb{Z}_2 W) \xrightarrow{\delta} H^1(G, \mathbb{Z}_2 W) \xrightarrow{J} H^1(G; \mathbb{Z}_2 W) \to \cdots
\]

given in \([4, \text{Proposition 1.1}]\). We have \( H^0(G; \mathbb{Z}_2 W) = \bigoplus_{w \in E} \mathbb{Z}_2(G/G_w)^G = 0 \) and so, \( \delta \) is a monomorphism. By using Shapiro’s Lemma and duality, we have \( H^1(G, \mathbb{Z}_2 W) \cong \bigoplus_{w \in E} \mathbb{Z}_2(G/G_w) \). Hence, \( \dim H^1(G, \mathbb{Z}_2 W) < \infty \).

On the other hand,

\[
\bigoplus_{w \in E} H^0(G_w; \mathbb{Z}_2 W) = \bigoplus_{w \in E} H^0(G_w; \mathbb{Z}_2(G/G_w)) = [\bigoplus_{w \in E} (\mathbb{Z}_2(G/G_w))^{G_w} \oplus \bigoplus_{u \neq w \in E} (\mathbb{Z}_2(G/G_u))^{G_u}].
\]

Since \( \mathbb{Z}_2 \cong \mathbb{Z}_2(G/G_u))^{G_u} \), it follows that

\[
\bigoplus_{w \in E} \mathbb{Z}_2 \cong \bigoplus_{u \neq w \in E} (\mathbb{Z}_2(G/G_u))^{G_u} \cong \left[ \bigoplus_{w \in E} (\mathbb{Z}_2(G/G_w))^{G_w} \oplus \bigoplus_{u \neq w \in E} (\mathbb{Z}_2(G/G_u))^{G_u} \right] \cong \bigoplus_{w \in E} \mathbb{Z}_2. \]

Thus, \( \mathbb{Z}_2(G/G_u))^{G_u} \cong \mathbb{Z}_2 \) and \( \mathbb{Z}_2(G/G_u))^{G_u} = 0 \) for all \( w, u \in E \) with \( u \neq w \). It follows that \( \delta \) is an isomorphism and so, \( J \) is trivial which provides \( \ker \text{res}^G_W = 0 \). Then we have \( \dim \ker \text{res}^G_W = 0 \) and \( E(G, W) = 1 \).

\( \square \)
The next result was proved in [7].

**Theorem 4.7.** Let \((G, S)\) be a \(PD^n\)-pair. Then, only one of the statements is true:

(i) \(S\) consists of only one subgroup \(S\) with \([G : S] = 2\).

(ii) \(S\) consists of two copies of \(G\).

(iii) \([G : S] = \infty\) for all \(S \in S\).

By using this result and Theorem 2.6 we can characterize the types of \(PD^n\)-pairs with the notation of Dicks-Dunwoody.

**Theorem 4.8.** Let \((G, W)\) be a \(PD^n\)-pair. Then, only one of the statements is true:

(i) \(W\) consists of exactly two elements and the \(G\)-action in \(W\) is transitive.

(ii) \(W\) consists of exactly two elements and the \(G\)-action in \(W\) is trivial.

(iii) \([G : G_w] = \infty\) for all \(w \in W\), and \(W\) is infinite.

**Proof.** We have that \(W = \bigcup_{w_j \in E} G(w_j) = \bigcup_{w_j \in E} G/G_{w_j}\), where \(E = \{w_j \mid i \in I\}\) is a set of orbit representatives in \(W\). Let \(S = \{S_i = G_{w_j} \mid i \in I\}\) be the family of isotropy subgroups. Since \((G, W)\) is a \(PD^n\)-pair we have that \((G, S)\) is a Poincaré duality pair (according to Bieri-Eckmann). By Theorem 4.7 we have three types of \(PD^n\)-pairs \((G, S)\).

In case (i), we have \(S = \{S\} = \{G_w\}\), with \([G : S] = 2\). Hence \(W = G/G_w\) has two elements and the \(G\)-action in \(W\) is transitive.

In case (ii), we have \(S = \{S_1, S_2\} = \{G_{w_1}, G_{w_2}\} = \{G, G\}\) and, since \(W = \bigcup_{i=1}^{i=2} G/G_{w_i}\), it follows that \(G/G_{w_i} = G/G = \{1\}\) and so, \(G/G_{w_i}\) has one element for \(i = 1, 2\). Therefore \(W\) has exactly two elements. Since \(G_{w_i} = G\) for \(i = 1, 2\) the \(G\)-action in \(W\) is trivial.

Finally, in case (iii), we have that \([G : S_i] = [G : G_{w_i}] = \infty\). Since, for \(w \in W\), there exist \(g \in G\) and \(w_k \in E\) such that \(G_w = gG_{w_k}g^{-1}\) we have \([G : G_w] = \infty\), for all \(w \in W\). Hence, we conclude that \(W\) is infinite, because \(W = \bigcup_{w_j \in E} G/G_{w_j}\). \(\square\)

**Example 4.9.** The only connected one-manifold-with-boundary is a line segment \(X = \overline{AB}\), with boundary consisting of the two endpoints \(A\) and \(B\) on \(X\) is its own universal covering. We have \(G = \pi_1(X) = \{1\}\) and \(W = \{A, B\}\). The \(G\)-action in \(W\) is trivial and \((G, W)\) is a \(PD^1\)-pair of type (ii). Now, if \(X\) is an annulus, i.e., a region bounded by two concentric circles or a Möbius band, then the universal covering \(\overline{X}\) of \(X\) is an infinity band delimited by two lines \(r\) and \(s\) which are the components of the boundary of \(\overline{X}\). We have \(G = \pi_1(X) = \mathbb{Z}\), \(W = \{s, r\}\). In the first case the \(G\)-action in \(W\) is trivial and \((G, W)\) is a \(PD^2\)-pair of type (ii). In the case of the Möbius band, the \(G\)-action in \(W\) is transitive and \((G, W)\) is a \(PD^2\)-pair of type (i).

**Example 4.10.** Consider a torus with one hole. We have \(G = \pi_1(X) = \langle t \rangle \ast \langle s \rangle\). If \(W\) consists of the components of boundary of the universal covering of \(X\), by Theorem 3.10, the \(G\)-action in \(W\) is transitive and for all \(w \in W\) we have \(W = G/G_w\), where \(G_w = \langle t s t^{-1} r^{-1} \rangle\). In this case \((G, W)\) is a \(PD^2\)-pair of type (iii), because \(W\) is infinite, and by Theorem 4.6, \(E(G, W) = 1\). But, if \(S = \langle s \rangle\) and \(W = G/S\) then, by considering [1, Example 2.2(c)] and the proof of Theorem 2.6 above, it follows that \(E(G, W) = \infty\). Hence, by Theorem 4.6, in this case, \((G, W)\) is not a \(PD^n\)-pair.

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