Quantum Entanglement transfer between spin-pairs

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(Dated: October 7, 2011)

We investigate the transfer of entanglement from source particles (SP) to target particles (TP) in the Heisenberg interaction \( H = \vec{s}_1 \cdot \vec{s}_2 \). In our research, TP are two qubits and SP are two qubits or qutrits. When TP are two qubits, we find that no matter what state the TP is initially prepared in, at the specific time \( t = \pi \), the entanglement of TP can attain to 1 after interaction with SP which stay on the maximally entangled state. For the TP are two qutrits, we find that the maximal entanglement of TP after interaction is relative to the initial state of TP and always cannot attain to 1 to almost all of initial states of TP. But we discuss an iterated operation which can make the TP to the maximal entangled state.

PACS numbers: 03.67.-a, 03.65.Ud, 42.50.Dv
Keywords: entanglement transfer; Maximally entangled state; qubit; qutrit

INTRODUCTION

Entanglement has been considered as a physical resource of quantum information processing. It is profoundly important in quantum teleportation [1], quantum computing [2], cryptography [3], and quantum games [4,5]. Using Jayness-Cumming-type interaction between atoms and cavity fields [6], entanglement transfer from two-mode squeezed vacuum state to two separable atoms which are prepared in pure state has been studied in Ref. [7]. Recently, Zou et al have discussed the entanglement transfer from some entangled two-mode fields to mixed qubits [8] and Zhang et al have discussed the entanglement transfer from photons to atoms [9].

In the paper [9], they conclude that the maximally entangled state of photons can lead the atoms to the maximal state of entanglement by nonlinear interaction. That is, the entanglement among photons transfers to the atoms by the interaction. Similarly, the transfer process between atoms and atoms stimulated our interests and then we research it. Fortunately, we find some interesting things in the cases of two pairs of qutrits. In the process of study, we consider the Heisenberg interaction \( H = \vec{s}_1 \cdot \vec{s}_2 \), and mainly focus on the entanglement transfer from source particles (SP) which are two qubits or qutrits to the target particles (TP) which contain two qubits. When SP are two qubits, the process of the transfer is clear and easy to understand, because the quantity of entanglement between two qubits is well-defined. But when SP are two qutrits, we have not a good definition of entanglement degree and only have two invariants \( I_1 \) and \( I_2 \) [10]. In the second section, we introduce the model and the study method. In the third section, the transfer between two pairs of qubits is studied. And in the Fourth section, we study the cases of transfer from two qutrits to two qubits. The Final section is the conclusion.

THE MODEL AND THE RESEARCH METHOD

As shown in Fig.1, there are four particles in this model, and two of them 1 and 2 are called target particles (TP) which are two qubits and initially stay on the pure state \( |\psi_{12}\rangle \)

\[
|\psi_{12}\rangle = \cos \theta_1 |00\rangle + \sin \theta_1 |11\rangle \tag{1}
\]

The other two particles 3 and 4 are called source particles (SP) which are two qubits or two qutrits and initially stay on the pure state \( |\psi_{34}\rangle \). The initial density operator of the whole system is

\[
\rho(0) = \rho_{12}(0) \otimes \rho_{34}(0) = |\psi_{12}\rangle \langle \psi_{12}| \otimes |\psi_{34}\rangle \langle \psi_{34}| \tag{2}
\]

where \( \rho_{12}(0) \) is the initial density operator of TP and \( \rho_{34}(0) \) is similar. Then, particles 1 and 3 interact under the interaction Hamiltonian \( H = \vec{s}_1 \cdot \vec{s}_2 \) and particles 2 and 4 are also acted with \( H \).
The time evolution operator of the whole system can be derived \( U(t) = u_{13}(t) \otimes u_{24}(t) \), where \( u_{13}(t) \) is the time operator between particles 1 and 3, then \( u_{24}(t) \) is similar. Now we can obtain the density operator \( \rho(t) \) of the system at time \( t \)

\[
\rho(t) = U(t)\rho(0)U^\dagger(t) = u_{13}(t) \otimes u_{24}(t)\rho_{12}(0) \otimes \rho_{34}(0) u_{13}^\dagger(t) \otimes u_{24}^\dagger(t) \]

To get the entanglement between two target particles, one needs to trace over the SP 3 and 4 to obtain the reduced density operator for the TP \( \rho_{12}(t) \)

\[
\rho_{12}(t) = \text{Tr}_{3,4}\rho(t) \tag{4}
\]

From (1), we can calculate the initial quantity of entanglement \( E_{12}(0) \) of TP

\[
E_{12}(0) = 2|\sin \theta_1 \cos \theta_1| \tag{5}
\]

where we use the Negativity [11,12] to denote the quantity of entanglement. For a bipartite system described by the density matrix, the negativity criterion for entanglement of the two subsystems is given by the following formula: 
\[
\varepsilon = -2 \sum_i \lambda_i^2 \quad \text{where the sum is taken over the negative eigenvalues } \lambda_i \text{ of the partial transposition of the density matrix } \rho.
\]

The value \( \varepsilon = 0 \) indicates that the two subsystems are separable. This function varies between 0 and 1, and monotonically increases as the entanglement grows. Then from (4), we can calculate the Negativity of TP at the time \( t \)—\( E_{12}(t) \). We can know the detailed process of the entanglement transfer from SP to TP by studying the \( E_{12}(t) \).

**ENTANGLEMENT TRANSFER FROM TWO QUBITS TO TWO QUBITS**

The initial state of SP \( |\psi_{34}\rangle \) is

\[
|\psi_{34}\rangle = \cos \theta_2 |00\rangle + \sin \theta_2 |11\rangle \tag{6}
\]

And the interaction Hamiltonian between 1 and 3 is

\[
H = \hat{s}_1 \cdot \hat{s}_3 = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \tag{7}
\]

The time evolution operator between particle 1 and 3 \( u_{13}(t) \) is

\[
u_{13}(t) = e^{-\frac{it}{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{1}{2}(1 + e^{it}) & -\frac{1}{2}(-1 + e^{it}) & 0 \\
0 & \frac{1}{2}(-1 + e^{it}) & \frac{1}{2}(1 + e^{it}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \tag{8}
\]

The interaction Hamiltonian and the time evolution operator between particle 2 and 4 are similar with particle 1 and 3.

The initial quantity of entanglement of SP \( E_{34}(0) \) is

\[
E_{34}(0) = 2|\sin \theta_2 \cos \theta_2| \tag{9}
\]

After some calculations, the reduced density operator \( \rho_{12}(t) \) for TP is derived

\[
\rho_{12}(t) = \begin{pmatrix}
A(t) & 0 & 0 & F(t) \\
0 & B(t) & 0 & 0 \\
0 & 0 & C(t) & 0 \\
F^*(t) & 0 & 0 & D(t)
\end{pmatrix} \tag{10}
\]

where the matrix basis is chosen as \( \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \} \). The coefficients in Eq. (10) are functions of time and given by

\[
A(t) = (\cos \theta_2 \sin \theta_1 \sin^2 \frac{t}{2} - \sin \theta_2 \cos \theta_1 \cos^2 \frac{t}{2})^2 + \cos^2 \theta_1 \cos^2 \theta_2 \\
B(t) = \frac{1}{4} \sin^2 t \sin^2 (\theta_1 + \theta_2) \\
C(t) = B(t) \\
D(t) = (\cos \theta_2 \sin \theta_1 \cos^2 \frac{t}{2} - \sin \theta_2 \cos \theta_1 \sin^2 \frac{t}{2})^2 + \sin^2 \theta_1 \sin^2 \theta_2 \\
F(t) = e^{-it} [(\cos \theta_2 \sin \theta_2 \sin^2 \frac{t}{2} + \cos^2 \theta_1 e^{2it} \sin^2 \theta_1) + \cos \theta_1 \sin \theta_1 \cos^2 \frac{t}{2} (\cos^2 \theta_2 + e^{2it} \sin^2 \theta_2)] \tag{11}
\]

And the analytic expression of the quantity of entanglement of TP at the time \( t \), \( E_{12}(t) \) is very complicated. And one who wants to know its real form, can calculate it using the following formula—\( E_{12}(t) = \text{Max} \{ 0, \sqrt{(B-C)^2 + 4|F|^2 - (B+C)} \} \). From now on, we use the numerical method to study the problem and give some numerical results.

First, we find that \( E_{12}(t) \) has a period that is \( 2\pi \), that is to say

\[
E_{12}(2k\pi) = E_{12}(0) = 2|\sin \theta_1 \cos \theta_1| \tag{12}
\]

where \( k \) is the natural number.

Second, when \( t = \pi \),

\[
E_{12}(\pi) = E_{34}(0) = 2|\sin \theta_2 \cos \theta_2| \tag{13}
\]

That is to say, the entanglement of TP can attain to the entanglement of SP. Assuming that the entanglement of SP is 1 and initially the entanglement of TP is very small, we can enhance the entanglement of TP and make them stay on the maximal entangled state under the interaction \( H \).
TP increases from 0 to 1 when \( t \) of TP contains five sections. (i). the entanglement from 1 to 0. To the picture (b), (c) and (d), the initial state of SP are two qutrits and so we search the transfer process from two qutrits to two qubits. When \( t = \pi \), \( \theta = \frac{\pi}{2} \), \( x = \frac{\pi}{2} \), \( y = \frac{\pi}{2} \), \( z = \frac{\pi}{2} \) and then decreases from \( \pi \) to zero; (ii). \( E_{12} \) stays on the zero during a period of time; (iii). \( E_{12} \) increases from zero to \( E_{34}(0) \) when \( t = \pi \) and then decreases from \( E_{34}(0) \) to zero; (iv). \( E_{12} \) also stays on the zero during a period of time; (v). \( E_{12} \) increases from zero to \( E_{12}(0) \) when \( t = 2\pi \).

**ENTANGLEMENT TRANSFER FROM TWO QUTRITS TO TWO QUBITS**

In this section, the SP are two qutrits and so we research the transfer process from two qutrits to two qubits. The initial state of SP \(|\psi_{34}\rangle\) is

\[
|\psi_{34}\rangle = k_0 |00\rangle + k_1 |11\rangle + k_2 |22\rangle
\]  

(14)

where \( k_0, k_1, k_2 \) satisfy the normalizing condition \(|k_0|^2 + |k_1|^2 + |k_2|^2 = 1\).

And the interaction Hamiltonian between 1 and 3 is

\[
H = \vec{s}_1 \cdot \vec{s}_3 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(15)

The time evolution operator between particle 1 and 3 \( u_{13}(t) \) is

\[
u_{13}(t) = \begin{pmatrix}
x_1 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & x_3 & 0 \\
0 & 0 & x_4 & 0 & x_3 \\
0 & 0 & x_3 & 0 & x_2 \\
0 & 0 & 0 & 0 & x_1
\end{pmatrix}
\]

(16)

where \( x_1 = e^{-it}, x_2 = \frac{e^{it} + 2e^{-it}}{3}, x_3 = -\sqrt{2}e^{-it} + e^{-it}, x_4 = 2e^{it} + e^{-it} \).

Just as the last section, particle 2 and 4 do the same operation with particle 1 and 3 in the interaction Hamiltonian and time evolution operator. The two invariants of \(|\psi_{34}\rangle\) is [10]

\[
I_1 = |k_0|^4 + |k_1|^4 + |k_2|^4 \\
I_2 = |k_0|^6 + |k_1|^6 + |k_2|^6
\]

(17)

After some calculations, the reduced density operator \( \rho_{12}(t) \) for TP is derived

\[
\rho_{12}(t) = \begin{pmatrix}
a(t) & 0 & 0 & c(t) \\
0 & b(t) & 0 & 0 \\
0 & 0 & c(t) & 0 \\
e^{it} & 0 & 0 & d(t)
\end{pmatrix}
\]

(18)

where the matrix basis is chosen as \{\( |00\rangle, |01\rangle, |10\rangle, |11\rangle \}\). The coefficients in Eq. (18) are functions of time and given by

![Diagram](image)
$$a(t) = \frac{k_2^2}{81} \cos^2 \theta_1 + \frac{e^{-3it}}{81} [(2 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$[(1 + 2 e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$b(t) = -\frac{2e^{-3it}}{81} [(1 + 2 e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$[(2 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + (1 + 2 e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$c(t) = b(t)$$

$$d(t) = \frac{k_2^2}{81} \sin^2 \theta_1 + \frac{e^{-3it}}{81} [2(-1 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$[(2 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + (1 + 2 e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$e(t) = \frac{e^{-3it}}{9} 2(-1 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1$$

$$+ \frac{k_2^2}{9} \sin \theta_1 [(1 + 2 e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$+ \frac{e^{-3it}}{81} [(2 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + 2(-1 + e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

$$[(2 + e^{\frac{3it}{2}})^2 k_1 \cos \theta_1 + (1 + 2 e^{\frac{3it}{2}})^2 k_0 \sin \theta_1]$$

Compared with the last section, the analytic expression of the quantity of entanglement of TP at the time \( t \)— \( E_{12}(t) \) is more complicated. We still use the numerical method to study the problem and give some numerical results from now on.

First, we find that \( E_{12}(t) \) has a period that is \( \frac{4\pi}{3} \)

$$E_{12} \left( \frac{4k\pi}{3} \right) = E_{12}(0) = 2|\sin \theta_1 \cos \theta_1| \quad (20)$$

where \( k \) is the natural number.

Second, being similar to the case that SP are two qubits, when \( t = \pi/2 \) i.e. \( t = \pi/3 \), SP can influence the entanglement of TP to a great extent. So we can calculate the analytic expression of the reduced density operator \( \rho_{12}(t) \) and the entanglement at the time \( t = \frac{\pi}{3} \), denoted as \( \rho_{12}(\frac{\pi}{3}) \), \( E_{12}(\frac{\pi}{3}) \). Obviously, \( E_{12}(\frac{\pi}{3}) \) is a function of \( \{\theta_1, k_0, k_1, k_2\} \) and by some transformations the parameters can become \( \{\theta_1, I_1, I_2\} \). For a given value of \( \theta_1 \), there is a pair of \( I_1, I_2 \) that makes the entanglement \( E_{12}(\frac{\pi}{3}) \) to the maximal quantity. Considering the complexity of analytical form of \( E_{12}(\frac{\pi}{3}) \), we have to use the numerical method to find the maximal quantity of \( E_{12}(\frac{\pi}{3}) \), denoted as \( E_{12}\text{Max}(\frac{\pi}{3}) \), and \( I_1, I_2 \) matched with \( E_{12}\text{Max}(\frac{\pi}{3}) \) to a special \( \theta_1 \).

Now, we introduce how to find the \( E_{12}\text{Max}(\frac{\pi}{3}) \). Firstly, the area to satisfy the physical condition is very small in the picture of number pair \( I_1 - I_2 \). To show it clearly, we need to make a transformation

\[
I_1' = I_1 \\
I_2' = I_2 - \frac{3}{2}I_1
\]

\[
(19)
\]

From Fig. 3 one can see the 9 maximal points are almost on the line AB. The real equation of frontier AB is very complicated, so we give an approximate one. That
And the error is small or equal to 1%. So we can infer that to a special \( \theta_1 \), the point \( \{I_1, I_2\} \) that makes \( E_{12}(\frac{2\pi}{3}) \) attain to the maximal quantity is approximately on the line AB. Now, you may ask that how \( \theta_1 \) is matched with \( I_1, I_2 \). By some calculations, we can get a better approximate result

\[
I_1 = -0.08756 \sin^2 2\theta_1 - 0.07911 \sin 2\theta_1 + 0.5 \quad (23)
\]

From the above discussion, we find that there are some differences between case 1 and case 2. Firstly, the relation between the maximal entanglement of TP after interaction with SP which stay on the state A initially; and the case that SP are two qubits whose entanglement is 1.

Case 1: When SP are two qubits, we can enhance the entanglement of TP to 1 by interacting with SP whose entanglement is 1.

Case 2: When SP are two qutrits, generally, we cannot enhance the entanglement of TP to 1 by the interaction with SP; even if SP is on the maximally entangled state. More interesting thing is that to a special initial entanglement of TP, there exists a state in all states of qutrits which satisfy the physical condition. Compared with other states of qutrits, this state can make a largest enhancement to the entanglement of TP after interaction. Generally, this state is not the maximally entangled state, but it can make TP to a larger entangled state than the maximally entangled state did.

Considering this thing carefully, we find that in the case 1, the maximal entanglement of TP after interaction has no relation with the initial state of TP and always can attain to 1 to all initial states of TP, as long as SP stay on the maximally entangled state initially. In the case 2, the maximal entanglement of TP after interaction is relative to the initial state of TP and always cannot attain to 1 to all states of TP except one condition that the initial state of TP and SP is the maximally entangled states. But this case is useless. Because the initial state of TP is already the maximally entangled state, it is not necessary to interact with SP. Therefore, we only have interest on the initial state of TP which is not maximally entangled state.

From the above discussion, we find that there are some differences between case 1 and case 2. Firstly, the relation between the maximal entanglement of TP after

\[
I_2' = -\frac{2}{3} I_1' - \frac{1}{6} \quad (22)
\]
interaction and the entanglement of TP initially is different. Secondly, the behavior and function of maximally entangled state is different in the entanglement transfer process. Our next work is to give the reason that brings about those differences exactly.

Now, we consider the reason from two aspects. On the one hand, the difference derives from the form of interaction. So we will use more general form of interaction, for example

\[ H = J_1 s_1^x s_2^x + J_2 s_1^y s_2^y + J_3 s_1^z s_2^z + B_1 s_1^z + B_2 s_2^z. \]

And we will adjust the quantity of those parameters and see whether the difference will vanish or not. On the other hand, we conjecture that the difference of entanglement between qubits and qutrits lead to those differences. As we all know, there does not exist a well concept to describe the entanglement of two qutrits. We expect to find a proper and general definition of entanglement of two qutrits based on the entanglement transfer to obtain the essential reason.

This work is supported in part by NSF of China Grant Nos. 10575053 and 10605013, Program for New Century Excellent Talents in University, and the Project-sponsored by SRF for ROCS, SEM.

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