Laser-Ion Lens and Accelerator

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Generation of highly collimated monoenergetic relativistic ion beams is one of the most challenging and promising areas in ultra-intense laser-matter interactions because of the numerous scientific and technological applications that require such beams. We address this challenge by introducing the concept of laser-ion lensing and acceleration (LILA). Using a simple analogy with a gradient-index lens, we demonstrate that simultaneous focusing and acceleration of ions is accomplished by illuminating a shaped solid-density target by an intense laser pulse at $\sim 10^{23}$ W/cm$^2$ intensity, and using radiation pressure of the laser to deform/shape the target into a cubic micron spot. We show that the LILA process can be approximated using a simple deforming mirror model, and then validate it using three-dimensional particle-in-cell simulations of a two-species plasma target comprised of electrons and ions. Extensive scans of the laser and target parameters identify the stable propagation regime where the Rayleigh-Taylor (RT)-like instability is suppressed. Stable focusing is found at different laser powers (from few- to multi-petawatt). Focused ion beams with the focused density of order $10^{23}$ cm$^{-3}$, energies in access of 750 MeV, and energy density up to $2 \times 10^{18}$ J/cm$^3$ at the focal point are predicted for future multi-petawatt laser systems.

A focusing optical lens is one of the oldest and best-known scientific instruments. The operating principle of a lens can be easily understood in either wave or corpuscular description of light: by causing a photon impinging on its central portion travel longer distance than by a photon impinging onto its periphery, we can ensure that both photons reach the focal point at the same time. Thus, focusing is ensured by the judicious variation of the lens thickness: thicker at the center, thinner at the edge. While the speed of the photons is piecewise constant inside and outside the lens, this is not a necessary condition for light focusing. For example, in a gradient index (GRIN) lens\textsuperscript{1} the light speed continuously varies across the lens, thus ensuring that all photons arrive at the focal point at the same time, regardless of their entry point. Motivated by the concept of a GRIN lens focusing light using non-uniform matter, we pose the following question: is it possible to focus matter using light?

The key to developing such a “matter lens” is the realization that, just as matter can change the velocity/direction of a photon, an intense flux of photons can do the same for the matter. This can be accomplished using the concept of radiation pressure acceleration (RPA)\textsuperscript{2,3} developed in the context of laser-ion acceleration of thin targets. The idea is schematically illustrated in Fig. 1, where the target is shaped in such a way that its outer (thinner) regions are accelerated to higher velocities than its central (thicker) region. We analytically demonstrate that, for a judicious choice of target density distribution, the resulting continuous velocity variation across the target enables its focusing into an infinitesimally small spot. The important feature of RPA-based focusing of the matter is that the target is not only focused, but also accelerated; hence, we refer to this scheme as a Laser-Ion Lens and Accelerator (LILA).

Just as the wave nature of light prevents its focusing to a geometric point by the GRIN lens, several fundamental plasma effects impose limits on the minimal focal spot of a realistic laser-propelled target. Those effects include Coulomb explosion\textsuperscript{4} and Rayleigh-Taylor (RT)-like instability\textsuperscript{7,8} that are known to break up constant-thickness targets, as well as plasma heating by the laser pulse. Under a simplifying assumption about the target as an initially cold two-species (electrons and single-charge ions) plasma, we describe the results of our fully-kinetic particle-in-cell (PIC) simulations and demonstrate that the RT-like instability and Coulomb explosion are effectively suppressed in a converging flow of the plasma. The result is a tightly-focused, quasi-monoenergetic, and nearly-neutral relativistic beam produced by the LILA. Scientific and industrial applications of such beams include are extremely wide-ranging: proton radiography\textsuperscript{9}, fast ignition of fusion targets\textsuperscript{10,11}, production of warm dense matter\textsuperscript{12,13}, hadron cancer therapy\textsuperscript{14,15}, and particle nuclear physics\textsuperscript{17,18}.

The LILA concept owes its feasibility to recent advances in solid-state laser technology that have enabled the generation of ultra-short laser pulses with intensities well above $I_{\text{rel}} > 10^{18}$ W/cm$^2$\textsuperscript{19} corresponding to the normalized vector potential $a_0 \equiv \omega_0/c = 1 \mu$m. Because of the wide range of their applications, laser-driven ion accelerators represent one of the most exciting areas of plasma physics at high energy density. In addition to RPA, where an over-dense thin target is propelled by the radiation pressure $P = 2I/c$ of a circular polarized laser with ultra-high intensity $I > 10^{23}$ W/cm$^2$\textsuperscript{17,20,22}, several ion accelerating scenarios are under investigation. Those include target normal sheath acceleration (TNSA)\textsuperscript{23,24}, shock wave\textsuperscript{25,26}, and laser break-out afterburner (BOA)\textsuperscript{27} acceleration.

Because the emphasis of this work is on simultaneous
acceleration and focusing of the target to a wavelength-scale focal spot, we concentrate on the rest of the manuscript on the RPA approach. While it is possible that under some conditions other ion acceleration technique could also provide focusing, their investigation is beyond the scope of this manuscript. As the starting point, we develop a simple theory describing the dynamics of a laser-propelled deformed thin target under a simplifying assumption that the target acts on the incident laser light as a "perfect" reflecting mirror. The resulting model is used to derive the optimal target shape that enables ideal focusing of the target into a focal point.

I. RESULTS

A. Deformable mirror model of LILA

Interaction of the circularly polarized planar laser wave with a thin dense target, whose thickness \(d(r_0)\) decreases from the target center \((r_0 = 0)\) toward its edge \((r_0 < R_0)\) can be simplified by describing the target as an ideal mirror which is deformed during its motion by the slowly changing radiation pressure \(P\) applied normally to the target surface. Because of the variation of the areal mass \(dm/\delta S = m_\text{i}n_\text{i}d_\text{t}\) (where \(m_\text{i}\) is the ion mass and \(n_\text{i}\) is the target density), different parts of the target experience different accelerations. The initially flat target bends because of the higher velocity of its periphery and eventually focuses to a small area by the applied radiation pressure. The evolving shapes of the target at different moments in time are schematically shown in Fig. [4].

Despite the simplicity of the deformable mirror (DM) model, which neglects many plasma phenomena such as laser heating of the target [28, 29] and spatial separation between light electrons and heavy ions [7], it is found to be useful for predicting the optimal thickness profile and deriving scaling laws of target’s focusing and acceleration.

Assuming that an initially planar target starts out, and remains, axially-symmetric, we use the Lagrangian coordinates [30] to describe the motion of ring-shaped elements of the target. The two coordinates, \(x(r_0, t)\) and \(r(r_0, t)\), are functions of the time \(t\) and the element’s initial radial position \(r_0\): \(x(r_0, t = 0) = 0\) and \(r(r_0, t = 0) = r_0\). The number of the ions \(\delta N\) contained in a ring element of the width \(\delta r_0\) and radius \(r_0\) is conserved during its motion: \(\delta N = 2\pi n_\text{i}d_\text{t}r_0\delta r_0\). The element’s area \(\delta S(r_0, t)\) and the unit vector \(\vec{n}(r_0, t)\) normal to the element’s surface are changing with time according to

\[
\delta S = 2\pi r(r_0, t)[r'(r_0, t)^2 + x'(r_0, t)^2]^{1/2}\delta r_0, \quad \vec{n} = \frac{r'(r_0, t)e_\gamma - x'(r_0, t)e_\rho}{[r'(r_0, t)^2 + x'(r_0, t)^2]^{1/2}},
\]

where ‘ stands for a derivative with respect to \(r_0\), and \((e_\rho, e_\gamma)\) are the unit vectors in the propagation and radial directions, respectively.

When photons are reflected from a reflecting target moving with velocity \(\vec{v}\), the reflection angle \(\alpha_r\) differs from the incidence angle \(\alpha_i\). However, the change of the photon momentum \(\hbar\vec{k}\) is always directed along the surface normal because of the accompanying red-shifting of the photon frequency [31, 32]. After cumbersome but straightforward calculations, we find that

\[
\frac{\Delta k}{k_0} = 2\frac{(\beta \cos \phi - \cos \alpha_i)}{(1 - \beta^2 \cos^2 \phi)}\vec{n}, \quad (3)
\]

Using momentum conservation, we obtain the equation of motion for the target element:

\[
\delta N \frac{\partial \vec{v}}{\partial t} = -\left(\frac{E^2}{4\pi} \delta S \cos \alpha_i \kappa \right) \frac{\Delta k}{|k_0|}, \quad (4)
\]

where \(\vec{v} = \vec{v}/\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}\) is the dimensionless relativistic ion momentum.

Using a geometric optics analogy with an aberration-free parabolic lens [1], we consider a parabolically shaped
target with finite radius $R_0$ and variable thickness given by $d(r_0) = d_0 \ast (1 - r_0^2/2R_0^2)$, where $d_0$ is the target thickness at the center and $R_c$ is the radius of curvature, see Fig. 1. Normalizing the coordinates according to $x \to x/R_c$, $r \to r/R_c$, $r_0 \to r_0/R_c$, $t \to ct/R_c$ by $R_c$, and $d \to d/d_0$, the target’s equations of motion are expressed as

$$\frac{\partial \vec{p}}{\partial t} = \frac{gR_c}{c^2} (\cos \alpha_i - \beta \cos \phi) \frac{r}{d(r_0)} (r' e_x - x' e_r),$$

(5)

$$\frac{\partial \vec{r}}{\partial t} = \frac{\vec{v}}{c} = \frac{\vec{p}}{1 + |\vec{p}|^2}.$$

(6)

where $g = E^2/2\pi d_0 n_e n_0$ is the initial acceleration of the central point of the target, and $d = (1 - r_0^2/2)$ is the normalized target thickness. The trigonometric functions in Eq. (5) can be expressed as $\cos \phi = \vec{n} \cdot \vec{v}/v$ and $\cos \alpha_i = \vec{n} \cdot \vec{e}_r$, where $\vec{n}$ is given by Eq. (2). Assuming an initially stationary target ($\vec{p}(r_0, t = 0) = 0$) for all values of $r_0$ in the $r_0 < R_0/R_c$ range, we observe that the target dynamics is determined by only two dimensionless parameters: the normalized target radius $R_0/R_c$ and acceleration $\Gamma \equiv gR_c/c^2$. The final target energy becomes relativistic for $\Gamma > 1$.

Figure 2 shows several snapshots of the target shape for two laser amplitudes ($a_0 = 10$: sub-relativistic acceleration, and $a_0 = 100$: relativistic acceleration) and two initial target radii ($R_0 = R_c$: large target, and $R_0 = 2/3 R_c$: small target). The calculations were performed using Eqs. (6)(6), i.e. within the DM model approximation. In all four cases, the parabolically shaped target is focused to a very small spot at the focusing distance $x = L_f$ and no aberration is observed. In the sub-relativistic case ($\Gamma = 0.021$), the target undergoes significant bending and the final energy of protons reaches $10$MeV at the focal point $L_f \approx 3 R_c$. In the relativistic case ($\Gamma = 2.1$), the bending is less significant and the final proton energy is $750$MeV at $L_f \approx 6 R_c$. In fact, it can be analytically demonstrated that $L_f \approx 2.8 R_c$ in the limit of $\Gamma \to 0$. On the other hand, $L_f$ grows without a limit as $\Gamma \to \infty$. Another important observation from Fig. 2 is that the focusing length is essentially independent on the initial target radius: the small and the large targets focus at the same point. Therefore, within the limits of the DM model, the target dynamics is parametrized by $\Gamma$ alone.

In reality, the applicability of the DM model is limited by the complex dynamics of the multi-species plasmas that includes plasma heating (which cannot be completely eliminated even for circular polarization because of the non-planar nature of the bending target), charge separation between electrons and ions, the Coulomb explosion that follows from such separation, and the RT-like instability. Below we demonstrate that, despite the complexity of relativistic laser-plasma interactions, the conclusions of the DM model largely hold, and that simultaneous focusing/acceleration by the LILA mechanism is feasible under a variety of laser powers.

**B. 3D Particle-in-Cell Simulations of LILA**

We provide proof-of-principle of the LILA concept by performing three-dimensional (3D) simulations using a first-principles PIC code VLPL [25]. As our first example, we assume a fully-ionized two-species (electrons and protons) thin parabolically shaped plasma target, with the radii $R_0 = 8 \mu m$ and $R_c = 7 \mu m$, and electron/proton density $n = 100 \text{sc}$, irradiated by a circularly-polarized planar wave with wavelength $\lambda_0 = 1 \mu m$, intensity $I = 1.75 \times 10^{22} \text{W/cm}^2$ (dimensionless amplitude $a_0 = 80$), and the estimated power over target area $P = 35 \text{PW}$. Note that such laser powers will be accessible within a few years at several user facilities throughout the world, including ELI [34] in Europe, OMEGA-EP-OPAL in North America [25] and Cekko-EXA [35] in Asia. The target thickness $d_0 = 300 \text{nm}$ at its center is chosen to be slightly larger than the optimal thickness: $d_{\text{opt}} = (\lambda/\pi)(n_e/n) a_0 \approx 250 \text{nm}$ [37].

**FIG. 3.** A 3D PIC simulation of LILA. (a) Snapshots of ion densities. Black-dashed lines: target position from the DM model. The focal spot (peak plasma density) is achieved at $t_f = 133.3 fs$ (b) Proton energy spectrum and energy density distribution (in the inset) at $t = t_f$. (c) Proton phase space $(E_k, \theta)$ distribution and normalized emittance $\varepsilon_a$ (dotted line) vs energy $E_k$ at $t = t_f$. Target composition: electrons and protons ($m_e/m_\text{sc} = 1836$). Target parameters: $R_0 = 8 \mu m$, $R_c = 7 \mu m$, $n_0 = 100 n_e$, and $d_0 = 300 \text{nm}$. Laser parameters: $\lambda_0 = 1 \mu m$ and $I = 1.75 \times 10^{22} \text{W/cm}^2$ ($a_0 = 80$).

As seen from Fig. 3(a), the bending and focusing of the target are captured quite well by the DM model. Positions of the target obtained from the VLPL simulation and from the model at different time moments are very close to each other. But in contrast to the model, in the more realistic simulations, the target deteriorates from the edges where its thickness significantly smaller than $d_{\text{opt}}$. Also, we observe that the focused target signifi-
cantly stretches in the longitudinal (x-) dimension. Thus, unlike an ideal mirror, a realistic plasma target cannot be focused into a point. However, as we observe from Fig. 3 (a), a significant fraction of the ion energy is focused into a focal spot measuring less than 4µm in every dimension. Nevertheless, the density of the focused ions is ≈1.5 times larger than the initial target density.

That some of the ions are found outside of the focal point is not unexpected. In fact, it is known that a fraction of the ions is left in the tail of the target, and that only some of the ions gain large energy through the RPA mechanism. For the parameters of the simulation presented in Fig. 3 approximately one-half of all ions are accelerated and focused into a hot spot. Another deviation from the simplified DM model is that the focal length \( L_f^{(PIC)} \approx 4 R_c \approx 28 \mu m \) found from the PIC simulations is shorter than \( L_f^{(DM)} \approx 5 R_c = 35 \mu m \) predicted by the DM model.

Because \( \Gamma \approx 1.6 \) for the simulation parameters of Fig. 3 the target ions at the focal spot acquire relativistic energies: 250MeV < \( E_k \) < 1500 MeV as confirmed by the ion energy spectrum plotted as a solid line in Fig. 3 (b) that peaks at \( E_k^{(peak)} \approx 750 \text{MeV} \). To quantify the degree of directionality of the LILA ions we have plotted in Fig. 3 (b) the normalized emittance \( \epsilon_n(\Gamma) \equiv \langle |p|/mc \rangle \sqrt{\langle z^2 \rangle \langle \zeta^2 \rangle} - \langle z \zeta \rangle^2 \rangle \) as a function of ion energy. Here \( \zeta = p_z/p_e \), and the brackets \( \langle \cdots \rangle \) denote averaging over all particles with energies close to \( E_k \). Remarkably, \( \epsilon_n(\Gamma) \) has a minimum around \( E_k \approx E_k^{(peak)} \), indicating that the accelerated beam is not only focused and quasi-monoenergetic, but also highly directional.

Indeed, the proton beam distribution plotted in Fig. 3 (c) in the \( (E_k, \theta) \) phase space (where \( \theta \) is the angle between ion velocity and the x-axis) confirms that the angular spread of the peak energy ions at the focal spot is very small: \( \Delta \theta_{max} \approx 5^\circ \). This corresponds to the remarkably low emittance of the focal spot quasi-monoenergetic ions: \( \epsilon_{min} \approx 0.035(\pi \cdot \text{mm} \cdot \text{mrads}) \). The resulting concentration of high-energy ions in such a small focal volume produces extremely high energy density \( u_k \) plotted in the inset of Fig. 3 (b), with its peak reaching \( u_k^{(max)} \approx 2 \times 10^{13} \text{J} \cdot \text{cm}^{-3} \).

To understand why the DM model remains quite accurate in this regime of ultra-intense laser-matter interaction, we calculated the electron spectrum at \( t = t_f \), as well as the electron energy density distribution in space. As one can see from Fig. 3 (a), the electrons at the focal spot remain significantly colder than ions: their energy spectrum peaks at \( E_{el}^{(max)} \approx 100 \text{MeV} \), and the peak energy density reaches only \( u_{el}^{(max)} \approx 3.3 \times 10^{12} \text{J} \cdot \text{cm}^{-3} \). Therefore, the two-species plasma target behaves as quasi-neutral, with moving slightly ahead of the ions to provide the charge separation needed to generate accelerating electric fields.

We note that not only the laser pulse deforms and focuses the target, but the target itself strongly modiﬁes the initially planar laser wavefront. The wavefront co-evolves with the target shape during its acceleration/focusing, and by the time \( t = t_f \) it acquires a bowl-like shape shown in Fig. 3 (b). Note that only a small part of the planar laser wave contained within the cylinder with the volume \( 14 \lambda_c \times R_0^2 \) interacts with the target until the focal point is reached with power. The laser energy contained in such a volume is \( U_L \approx 1,600 \text{J} \), and \( \eta \approx 16\% \) of \( U_L \) is transferred to the ions at the hot focal spot.

C. LILA Scaling and Stability with Respect to RT-like Instability

With the DM model validated by 3D PIC simulation for at least some laser/target parameters, we now turn to

![Figure 4](image-url) (a) The electron spectrum and energy density distribution (inset) at the focal spot. (b) Distribution of the laser intensity \( I \) at \( t = t_f \). Black electron density contours indicate the electron beam’s location at \( t = t_f \).

![Figure 5](image-url) (a) Stable (crosses, bullets) and unstable (triangles) target focusing in the \( (\Gamma, R_c) \) parameter space. (b) Focal length \( L_f \) and (c) momentum \( p_z \) as functions of \( \sqrt{\Gamma} \) from the DM model (solid line) and 3D PIC simulations (crosses, bullets). (d) Example of the unstable acceleration/focusing of the target, corresponds to \( \Gamma = 1.61 \) and \( R_c/\lambda_L = 7 \). (e) Ions energy spectrum of the unstable target in (d) at moment \( t = 133fs \).
obtain simple scalings of the energy gain and focal distance of the converging ions. As demonstrated earlier in Sec. [A], the dynamics of the target focusing and acceleration within the DM model is determined by a single dimensionless parameter $\Gamma$. In particular, the ion momentum $p_x$ at the focal point and the focusing length $L_f$ in this model can be approximated by

$$p_x/m_e c \approx \Gamma^{1/2}, \quad L_f/R_e \approx 2\Gamma^{1/2} + 2.8. \quad (7)$$

In reality, approaching these scalings requires that the target does not succumb to RT-like instability. Therefore, we have carried out a series of VLPL simulations to examine the influence of the RT instability on the target focusing, and to verify the scalings given by Eq. (7). The results of these simulation corresponding to the normalized quantities $\Gamma$ and $R_c/\lambda_L$ listed in Table I are shown in Fig. 5. Three simulations with different values of $R_c/\lambda_L$ are performed for each value of $\Gamma$. In all simulations, the target is assumed to be irradiated by a planar circularly polarized laser wave, and its following initial parameters are used: the radius $R_0 = 1.14 R_c$, maximum thickness $d_0 = 1.2 d_{opt}$, and plasma density $n_0 = 100 n_c$. For a more detailed choice of parameters, please see Methods.

| $\Gamma^{1/2}$ (Petawatt) | 0.67(2.8) | 0.87(8) | 1.07(18) | 1.27(35) |
|--------------------------|----------|---------|----------|----------|
| $\times R_c/\lambda_L$   | 1.9      | 2.6     | 3.4      | 4.1      |
| $\bullet R_c/\lambda_L$  | 3.2      | 4.5     | 5.8      | 7.0      |
| $\Delta R_c/\lambda_L$  | 6.2      | 7.5     | 8.8      | 10       |

The identification of the stable LILA regimes was done by analyzing the transverse size of the hot spot at the focal point, as well as the particle/energy densities within it. For example, the simulation results shown in Figure 5 correspond to $\Gamma = 1.61$ and $R_c/\lambda_L = 7$ exemplify a stable focusing case. Strong convergence of the target appears to suppress the instability. In fact, one of the characteristic signatures of the RT instability is the breakup of the target into multiple high-density plasma clumps. Such instability onset is indeed observed at $t \approx 26.6fs$. However, at the later times, the clumps converge towards the axis and merge with each other, thereby effectively suppressing the instability. Figure 5(d) shows the example of the unstable target focusing which corresponds to $\Gamma = 1.61$ and $R_c/\lambda_L = 10$. RT instability breaks target into large clumps and the radiation pressures fail to provide any focusing because the whole target is transparent to the laser after $t = 66 fs$. At $t = 133 fs$, the whole target is already dispersed. Compare with the focused target at Figure 3, its ion density is smaller by one order of magnitude and its spectrum is not mono-energetic.

One immediate observation from Figure 5(a) is that the target focusing is stabilized at small values of $R_c$, but is disrupted for the larger targets. Qualitatively, this can be understood by observing that larger $R_c$ corresponds to longer focusing time, thus supporting more e-foldings for the developing RT-like instability. Figure 5(a) further implies that, for given target size, its focusing/acceleration is stabilized for large values of $\Gamma$. This result is consistent with earlier calculations \[38\]: higher laser power accelerates ions to higher velocities and, therefore, provides less time for the instability to grow. Not surprisingly, for the stable acceleration/focusing regime, one can rely on the predictions of the DM model for the focal length $f$ and the ion momentum $p_x$. Indeed, the results obtained with VLPL simulations are in agreement with Eq.(7) as shown in Figures 3(b,c). Here the ion momentum $p_x$ is obtained at the maximum of ion distribution function; it is found to be close to the momentum average over all ions inside the hot spot.

II. CONCLUSION AND DISCUSSION

In the main text, we have illustrated the concept of simultaneously focusing/accelerating a parabolic target irradiated by a planar laser wave. In more realistic situations, shaping the target according to the transverse profile of the laser is necessary. For example: if a Gaussian-shaped laser pulses with a finite spot size $\sigma_L$ are used, i.e. $I = I_0 \exp(-r^2/\sigma_L^2)$. The DM model predicts that one must correct the parabolic shape of the target correspondingly to account for the finite laser spot size: $d(r_0) \rightarrow d(r_0) \exp(-r^2/\sigma_L^2)$, and excellent performance of LILA is confirmed in 3D PIC simulation.

In conclusion, by shaping the target in a specific way, one can simultaneously accelerate and focus the target by the circular-polarized laser pulse. A tightly-focused ultra-short ions beam with high particle density and high energy density can be obtained at the focal point. Scaling laws for the target focal length and final energy are obtained from a deformable mirror model and found to be consistent with full 3D PIC simulations. These scaling laws are useful in designing target geometry and choosing the required laser power and duration. Depending on those parameters, a wide range of ion kinetic energies – from 200MeV to 700MeV can be obtained, with future applications such as proton therapies \[39\], spallation physics \[40\], and many others.

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IV. METHODS

A. Dimensional analysis of DM equations.

When the target acceleration is small, $\Gamma \ll 1$, target undergoes significant bending and moves with sub-relativistic speed. Neglecting $\beta$ in Eq. (5) and $|\vec{v}|^2$ in Eq. (6) and introducing renormalized time $\tilde{t} = t/\Gamma^{1/2}$, one can exclude the acceleration parameter $\Gamma$ from DM equations:

$$\frac{\partial^2 \vec{r}}{\partial \tilde{t}^2} = \frac{\cos^2 \alpha_x}{d(r_0)} \frac{r}{r_0} (r' \vec{e}_x - x' \vec{e}_x),$$

(8)

Solving this equation, we find that in the sub-relativistic limit the focal length is fixed $L_f = 2.95 R_i$ and proton momentum at the focal point $p_x \approx 1.6 \Gamma^{1/2}$.

In the opposite case when the target acceleration is large, $\Gamma >> 1$, the target bends insignificantly $\alpha_x \approx 0$, $\phi \approx 0$, and $\alpha'' \approx |d''| = 1, \phi'' \approx |d''| = 1$. The DM equations (5) and (6) take the form

$$\dot{\vec{p}}_y = \frac{\Gamma (1 - \beta)}{d(1 + \beta)} \vec{r} - \vec{p}_x \frac{\Gamma (1 - \beta)}{d(1 + \beta)} r',$$

$$\dot{\vec{p}}_x = \frac{\Gamma \chi^2}{d(4p_x^2)} \vec{r},$$

(9)

where ‘over-dot’ denotes the time derivative. Since $p_x \gg 1$, we can approximate $\gamma \approx p_x$, $\eta \approx p_x \gamma$ and $(1 - \beta)/(1 + \beta) \approx 1/4p_x^2$. For targets with small radius $r_0 \ll 1$, we also can approximate contraction of the target area by introducing a transverse contraction coefficient $\chi(t)$ such that $r(t) = \chi(t)r_0$ and then transform the Eqs. (9) to:

$$\frac{\partial}{\partial \tilde{t}} (p_x \chi)r_0 = -\frac{\Gamma \chi}{d(4p_x^2)} \vec{x}',$$

$$\dot{\chi} = \frac{\Gamma \chi^2}{d(4p_x^2)}$$

(10)

(11)

Dropping the terms $\propto r_0^2$, we derive from Eq. (10) the following equation for $\chi$:

$$\frac{\partial}{\partial \tilde{t}} \left[ \frac{4p_x^2}{\chi} \frac{\partial}{\partial \tilde{t}} (p_x \chi) \right] r_0 = -\Gamma \chi'.$$

(12)

Bending of the target is caused by the variation of longitudinal velocity $v_x$ along the target radius which is resulted from the variation of the target thickness. Taking derivative of Eq. (11) with respect to $r_0$ and using the relationship $\vec{x}' \approx p_x / p_x'$, we find after simple transformation that

$$\frac{\partial}{\partial \tilde{t}} (4p_x^5 v_x') = r_0 \Gamma \chi^2.$$

(13)

Equations (12) and (13) are supplemented by the equation of motion of the target as a whole:

$$4p_x^2 \frac{\partial}{\partial \tilde{t}} p_x = \Gamma \chi^2.$$

(14)

It follows from Eqs. (13) and (14) that $v_x = r_0 / 3p_x^2$. This reduces Eq. (12) to

$$\frac{\partial}{\partial \tilde{t}} \left[ \frac{4p_x^2}{\chi} \frac{\partial}{\partial \tilde{t}} (p_x \chi) \right] = -\frac{\Gamma}{3p_x^2}.$$

(15)

Our derivation shows that the target dynamics can be described in paraxial approximation by two ordinary differential Eqs. (14) and (15) for longitudinal momentum $p_x$ and transverse contraction coefficient $\chi$. One can exclude $\Gamma$ from these equations by introducing new variables $\bar{p}_x = p_x / \Gamma^{1/2}$ and $\tilde{t} = t / \Gamma^{1/2}$. Therefore, $p_x \propto \Gamma^{1/2}$ and $\eta \propto \Gamma^{1/2}$. Rigorous analysis of Eqs. (14) leads to formulas $L_f \approx 2\Gamma^{1/2} + 2.95$ and $p_x \approx \Gamma^{1/2}$.

We conclude from this consideration that at least in paraxial approximation (when bending is small) the target is focused in the small area when its thickness changes by parabolic law.

B. Choice of Parameters

In all simulations listed in Table I, we have chosen the target thickness at its center point $d_0 = \eta_{1,2} \lambda_{opt}$, and $\eta_1$ is a number of order unity. Because the target thickness decreases at the periphery of the target, we have additionally imposed a condition of $\eta_1 > 1$. The radius of the target was chosen to be comparable to its radius of curvature $R_i$: $R_0 = \eta_2 R_i$, where $\eta_2$ is a number of order unity. This restriction was made from practical consideration: if $\eta_2 \approx \sqrt{2}$ for the parabolic target thickness profile, then the target becomes very thin at its edge. On the other hand, if $\eta_2 \ll 1$, then the target is essentially planar, and the new physics associated with LILA cannot be captured. Therefore, for all simulations we have chosen $\eta_1 = 1.2$ and $\eta_2 = 1.14$. For these numerical values of $\eta_{1,2}$, the wave amplitude is estimated from the expression for the acceleration in terms of $a_0$ given by $g = 2\pi a_0 (c^2 / \lambda_L) (d_{opt} / d_0) (m_e / m_i)$, and from the definition of $\Gamma$ that can be expressed as $g = \Gamma c^2 / R_i$. Combining these expressions and taking the value of $\eta_1$ into account results in the following expression for $a_0$: $a_0 = (\Gamma c / R_i) (m_e / m_i) (d_0 / d_{opt}) / (2\pi) \approx 350 \Gamma \lambda_L / R_i$. Inserting this expression of $a_0$ into a formula for the laser power incident upon a circle with the radius $R_0$, and taking into account the value of $\eta_2$, the following expression for the laser power $P_L$ is obtained: $P_L \approx \Gamma L \times 2.8 PW$. By varying $\Gamma$, we have considered the targets with the radii $R_0$ as small as $2\lambda_L$ (close to the diffraction limit) and as large as $10\lambda_L$. For the range of $\Gamma$’s tested in our simulations, the equivalent laser power varied between $P_L \approx 2.8 PW$ for $\Gamma \approx 0.45$ to $P_L \approx 35 PW$ for $\Gamma = 1.61$ as listed in Table I.

C. Numerical Simulation

All 3D simulations presented in the paper are conducted by the 3D PIC Code: Virtual Laser Plasma Lab (VLPL) [34]. All simulations used the same spatial resolution of $\Delta x \times \Delta y \times \Delta z = \lambda_L / 100 \times \lambda_L / 12 \times \lambda_L / 12$, time resolution of $\Delta t = 9 \times 10^{-9} \lambda_L / c$, and $N_{macro} = 80$ macro-particles per cell.
