Estimates of multipolar coefficients for searching for cosmic ray anisotropies with non-uniform or partial sky coverage

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Abstract. We study the possibility of extracting the multipolar moments of an underlying distribution from a set of cosmic rays observed with non-uniform or even partial sky coverage. We show that if the degree is assumed to be upper bounded by \( L \), each multipolar moment can be recovered whatever the coverage, but with a variance increasing exponentially with the bound \( L \) if the coverage is zero somewhere. Despite this limitation, we show the possibility of testing predictions of a model without any assumption on \( L \) by building an estimate of the covariance matrix seen through the exposure function.

Keywords: cosmic ray detectors, ultra high energy cosmic rays
1. Introduction

Anisotropy in the arrival directions of cosmic rays is a major observable in attempts to understand their origin. Magnetic fields bend their trajectories in such a way that transport of cosmic rays is mainly diffusive up to high energies: this makes their angular distribution isotropic. Nevertheless, above the so-called knee of cosmic rays up to the ankle, there are predictions for anisotropies that are small but increasing with energy, predictions which of course depend on the regular and the turbulent components of the assumed galactic magnetic field, as well as the assumed distribution of sources and composition of cosmic rays [1, 2]. Further, at ultrahigh energies, cosmic ray arrival directions are expected to be less and less smeared out by galactic and extragalactic magnetic fields, leading to a possible extraction of information about the positions of the sources [7]–[11]. Hence, it is clear that any evidence for an anisotropy, or any limit on anisotropies in the cosmic ray locations observed by experiments, provides some of the most important constraints upon models.

The multipole expansion up to a given order $L$ is a powerful tool for studying the structures standing out from the noise down to an angular scale of $\approx \pi/L$, whatever the shape of the underlying celestial pattern. In practice, the number of significant coefficients is limited by the angular resolution of the detector and, on the other hand, by the available...
statistics of observation. However, ground based experiments cover a limited range in declination, so it is impossible to apply ‘off the shelf’ the formalism of multipole moments: any one of the coefficients may be modified in an unpredictable way by the unseen part of the sky. Methods have been developed for studying the CMB with an incomplete coverage [3–6], but here we are faced with a different problem: we cannot suppose a priori that the distribution of cosmic rays is described by a power spectrum, because we want to detect possible non-isotropic structures, a priori unknown. In other words, the information carried by the $a_{\ell m}$ cannot be reduced to only the knowledge of the $C_{\ell}$.

One purpose of this paper is to study the possibility of estimating the multipole moments of a distribution of points over a sphere in the case of a non-uniform or even a partial coverage of the sky, together with the limitations of such an approach. The estimation of dipoles and quadrupoles was studied in [12]–[14]. Here, we use the moments of the observed distribution on a set of orthogonal functions: either the spherical harmonics themselves, or a set of functions tailored to the coverage function. With these two different methods, we show that the interference between the modes induced by the non-uniformity or the hole of the coverage can be removed assuming a bounded expansion in the conjugate space, allowing us to recover the underlying multipole moments. However, in accordance with the simple intuition that it is impossible to describe the unseen part of the sky, we point out that the uncertainty on the recovered coefficients increases with the assumed bound $L$ of the expansion. We show that the larger the hole in the coverage of the sky, the faster the increase of uncertainty with $L$. After some general considerations about the description of point processes on a sphere, sections 2–4 are dedicated to these methods whereas section 5 illustrates them with some examples.

Because of the incomplete knowledge of the distribution of cosmic ray sources, and the stochastic nature of the propagation through magnetic fields, the anisotropies that we want to characterize are not reducible to explicit models: they may be interpreted as a particular realization of a random process. This means that some model predictions are better expressed as average values of the coefficients, with their covariance matrix. This matrix is not necessarily diagonal for describing the physics that we are interested in, contrary to the case for a power spectrum. We show in section 6 that under reasonable assumptions, an estimation can be performed with a partial sky coverage, evading the problem of setting a bound on the expansion.

2. Generalities about point processes on a sphere

The number of cosmic rays $n(\theta, \varphi)$ observed as a function of $\Omega = (\theta, \varphi)$ is a random process that we can model using the following quantity:

$$n(\Omega) = \frac{1}{N} \sum_{i=1}^{N} \delta(\Omega, \Omega_i)$$

where $\delta$ is the Dirac function on the surface of the unit sphere, and $\Omega_i$ the position of the $i$th cosmic ray. This distribution follows a Poisson law with an averaged density that we will denote by $\mu(\Omega)$:

$$\mu(\Omega) = \omega(\Omega) \lambda(\Omega).$$
Here, $\lambda$ is the density of the distribution of cosmic rays and $\omega$ is the exposure function of the experiment. The multipole coefficients of the function $\lambda(\theta, \varphi)$ defined on the unit sphere express its expansion in spherical harmonics:

$$\lambda(\theta, \varphi) = \sum_{\ell,m} a_{\ell m} Y^m_\ell (\theta, \varphi) \quad (\ell \geq 0, -\ell \leq m \leq \ell).$$

In this paper, we choose to normalize the spherical harmonics in such a way that

$$\int d\Omega Y^m_\ell(\Omega) Y^{m'}_{\ell'}(\Omega) = 4\pi \delta_{\ell\ell'} \delta_{mm'}.$$ Together with the normalization $\int d\Omega \lambda(\Omega) = 4\pi$, our convention leads to $a_{00} = 1$ which is, in the context of this study, a natural system of units. For convenience, we will use hereafter the notation $\sum_{\ell,m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}$.

With a uniform sky coverage, it is easy to obtain an unbiased evaluation of these coefficients from a sample of $N$ points $(\theta_i, \varphi_i)$ distributed independently according to the density $\lambda$:

$$\overline{a}_{\ell m} = \frac{1}{N} \sum_{i=1}^{N} Y^m_\ell (\theta_i, \varphi_i).$$

If the distribution is roughly uniform (that is, $|a_{\ell m}| \ll 1$ for all $(\ell, m) \neq (0,0)$), these estimators are quasi-optimal, weakly correlated and their variances are close to $1/N$; otherwise the variances can be approximated from the quadratic moments:

$$\text{var}(\overline{a}_{\ell m}) = \frac{1}{N} \sum_{i} \left( (Y^m_\ell (\theta_i, \varphi_i))^2 - (\overline{a}_{\ell m})^2 \right).$$

These properties are due to the orthogonality of the spherical harmonics, and cannot be used directly if the coverage of the sphere is not uniform, that is, if the distribution actually observed is $\lambda(\theta, \varphi) \omega(\theta, \varphi)$, where $\omega$ is a non-uniform function eventually vanishing in some regions.

However, if we suppose that the expansion of $\lambda$ in spherical harmonics is bounded to degree $L$ (at least to a good approximation), we are going to see that it is possible to recover—within limitations that we will discuss in detail—the multipolar coefficients even in the case of partial sky coverage.

Throughout the paper, we will consider by default an exposure function not covering the whole sky in a realistic way since we use the function calculated by Sommers [12] describing the coverage of the sky of the Southern site of the Pierre Auger observatory as long as the acceptance of the detector is saturated up to a local zenith angle $\theta_{\text{max}}$. This function is shown in figure 1 with $\theta_{\text{max}} = 60^\circ$, which guarantees in a realistic way this ideal function to be meaningful [17].

3. Estimation through the deconvolution of the exposure function

3.1. The estimate

In this section, we describe an estimate of the $a_{\ell m}$ coefficients based on the interpretation of the estimate

$$\overline{b}_{\ell m} = \frac{1}{N} \sum_{i=1}^{N} Y^m_\ell (\theta_i, \varphi_i).$$

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in terms of a convolution between the underlying $a_{\ell m}$ coefficients of the density $\lambda(\theta, \varphi)$ and a kernel which depends on the $\omega(\theta, \varphi)$ function. To some extent, this approach is the equivalent of the MASTER one within the CMB framework [15], except that we are interested here in building a linear estimate of the $a_{\ell m}$ coefficients rather than a quadratic estimate of the $C_{\ell}$ ones. As the cosmic rays are observed through the exposure function $\omega$, the estimate $b_{\ell m}$ is not an estimate of the multipolar coefficients of the density $\lambda$, but an estimate of the multipolar coefficients of $\omega \lambda$. The $a_{\ell m}$ coefficients are thus related to the $b_{\ell m}$ ones through the following convolution:

$$b_{\ell m} = \sum_{\ell', m'} [K]_{\ell m'}^{\ell m} a_{\ell' m'}.$$ 

The kernel $K$ is entirely determined by the specific exposure function. Indeed, on using the completeness relation of the spherical harmonics, the elements of the kernel $[K]_{\ell m'}^{\ell m}$ read

$$[K]_{\ell m'}^{\ell m} = \int_{4\pi} d\Omega Y_{\ell m}^*(\Omega)\omega(\Omega)Y_{\ell m'}^{*}(\Omega).$$

This relation was referred to as the convolution theorem in [16], as this is the analog on the sphere of the convolution theorem for a Fourier transform. Then, on using direct numerical results for $K$ and $K^{-1}$ for the specific exposure function $\omega$, the underlying $a_{\ell m}$ coefficients can be formally recovered through the following estimate:

$$a_{\ell m} = \sum_{\ell', m'} [K^{-1}]_{\ell m'}^{\ell m} b_{\ell' m'}.$$
3.2. Statistical properties of the estimate

The observed $N$ points are sampled according to a Poissonian process on the sphere. Averaging over a large number of realizations of $N$ events distributed according to $\mu(\theta, \phi)$, it is elementary to compute the first and the second moment of $n(\Omega)$:

$$\langle n(\Omega) \rangle_P = \mu(\Omega)$$

$$\langle n(\Omega) n(\Omega') \rangle_P = \mu(\Omega) \mu(\Omega') + \mu(\Omega) \delta(\Omega, \Omega')$$

where the subscript $P$ stands for Poisson. The average of the $b_{\ell m}$ estimate then reads

$$\langle b_{\ell m} \rangle_P = \left( \int d\Omega \mu(\Omega) Y_{\ell m}^*(\Omega) \right)_P$$

leading to the following averaged $a_{\ell m}$ estimate:

$$\langle a_{\ell m} \rangle_P = \sum_{\ell_1, m_1} [K^{-1}]_{\ell \ell_1}^{m_1 m} \sum_{\ell_2, m_2} K_{\ell_1 \ell_2}^{m_1 m_2} a_{\ell_2 m_2}$$

Thus, it is clear that we have built an unbiased estimate. Turning to the covariance, we get in the same way

$$\text{cov}(\bar{b}_{\ell m}, \bar{b}_{\ell' m'}) = \int d\Omega d\Omega' \mu(\Omega) \mu(\Omega') Y_{\ell}^{m}(\Omega) Y_{\ell'}^{m'}(\Omega')$$

$$+ \int d\Omega d\Omega' \mu(\Omega) \delta(\Omega, \Omega') Y_{\ell}^{m}(\Omega) Y_{\ell'}^{m'}(\Omega') - \langle \bar{b}_{\ell m} \rangle_P \langle \bar{b}_{\ell' m'} \rangle_P.$$
In the case of partial coverage, the spherical harmonics are no longer orthogonal, and behave in such a way that the coefficients of $K^{-1}$ only satisfy the expression

$$\sum_{\ell_1, m_1}^{L} [O]^{mm}_{\ell\ell_1} [K^{-1}]^{m_1 m_'}_{\ell_1 \ell'} = \int_{\Delta\Omega} d\Omega \frac{1}{\omega(\Omega)} Y^m_\ell(\Omega) Y^{m'}_{\ell'}(\Omega)$$

where $\Delta\Omega$ is the non-zero region of $\omega$, and

$$[O]^{mm'}_{\ell\ell'} = \int_{\Delta\Omega} d\Omega Y^m_\ell(\Omega) Y^{m'}_{\ell'}(\Omega).$$

It is then obvious, in this latter case, that $K^{-1}$ is invertible only if $L$ is finite, and that the coefficients of $K^{-1}$ strongly depend on the assumed bound $L$, leading to an indeterminacy of each coefficient as $L$ is increasing. This indeterminacy is nothing but the mathematical traduction from it being impossible to know the distribution of cosmic rays in the uncovered region of the sky.

4. Estimation through dedicated orthogonal functions

In this section, we describe another way, more intuitive, to recover the underlying $a_{\ell m}$ coefficients, by applying the Gram–Schmidt procedure to the $\omega(\theta, \varphi) Y^m_\ell(\theta, \varphi)$ with $\ell \leq L$, which allows us to build orthogonal functions from the coverage function. Then, by applying the formalism of moments to these functions, the $a_{\ell m}$ are obtained with linear combinations of these moments.

4.1. Applying the Gram–Schmidt procedure

The scalar product being defined as

$$\langle f | g \rangle = \frac{1}{4\pi} \int f^*(\theta, \varphi) g(\theta, \varphi) d\Omega,$$

the normalized spherical harmonics may be written as

$$Y^m_\ell(\theta, \varphi) = P^m_\ell(\cos \theta) e^{im\varphi}$$

where the $P^m_\ell$ are the associated Legendre functions supposed here to be normalized:

$$\frac{1}{2} \int_{-1}^{1} P^m_\ell(x)^2 dx = 1.$$

In practical computations we use the real functions $Y^0_\ell(\theta, \varphi)$ for $m = 0$, and \{\(\sqrt{2} P^m_\ell(\theta, \varphi) \cos(m\varphi), \sqrt{2} P^m_\ell(\theta, \varphi) \sin(m\varphi)\)\} for $1 \leq m \leq \ell$. For convenience we keep the notation with the $Y^m_\ell$ hereafter.

Let us suppose first that $\omega$ is a function of $\theta$ only (for example, if the coverage is uniform in right ascension). Then $\omega Y^m_\ell$ and $\omega Y^{m'}_{\ell'}$ are orthogonal if $m \neq m'$, and the orthogonalization may be performed separately for each value of $m$, combining the $\omega Y^m_\ell$ with $m \leq \ell \leq L$. If $N(f)$ represents the function $f$ after normalization, we just need to set, for a given $m$,

$$Q^m_{[m]} = N(\omega P^m_{[m]})$$

$$Q^m_{[m]+1} = N(\omega P^m_{[m]+1} - \langle Q^m_{[m]} | \omega P^m_{[m]+1} \rangle) Q^m_{[m]}$$
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\[ Q_{|m|+2}^m = N(\omega P_{|m|+2}^m - \langle Q_{|m|}^m | \omega P_{|m|+2}^m \rangle Q_{|m|}^m - \langle Q_{|m|+1}^m | \omega P_{|m|+2}^m \rangle Q_{|m|+1}^m) \]
\[ \ldots \]
\[ Q_{|m|+p}^m = N(\omega P_{|m|+p}^m - \sum_{k=0}^{p-1} \langle Q_{|m|+k}^m | \omega P_{|m|+p}^m \rangle Q_{|m|+k}^m) \]
\[ \ldots \].

Then the normalized functions \( Z_{\ell}^m \) defined on the sphere by

\[ Z_{\ell}^m(\theta, \varphi) = Q_{\ell}^m(\cos \theta) e^{im\varphi} \]

are orthogonal to each other, and the subset of \( Z_{\ell}^m \) with \(|m| \leq \ell \leq L\) generates the same subspace as the \( \omega Y_{\ell}^m \) with \(|m| \leq \ell \leq L\). We can express them through a set of coefficients \( C_{\ell \ell'}^m \):

\[ Z_{\ell}^m(\theta, \varphi) = \sum_{\ell' = m}^{\ell} C_{\ell \ell'}^m \omega(\theta) Y_{\ell'}^m(\theta, \varphi). \]

If \( \omega \) depends on both \( \theta \) and \( \varphi \), the same procedure can be applied, but the orthogonal functions are mixtures of different values of \( m \), and there is no canonical way to obtain them; anyway it is possible to build a basis preserving the subset generated by \( 0 \leq \ell \leq L \) whatever \( L \) is. For simplicity, we do not develop such a formalism here. In particular, as only a small dependence on \( \varphi \) is expected in the case that we are interested in, it is possible to weight the events to account for this variation of the exposure as a function of the right ascension, and hence the formalism applied here can be applied 'off the shelf'.

To illustrate the method, figure 2 displays the shape of the \( \omega Y_{\ell}^m \) and the \( Z_{\ell}^m \), and the triangular matrix of the transformation \( C_{\ell \ell'}^m \) for \( L = 15, m = 0 \) (on a logarithmic scale), in the case of the coverage function displayed in figure 1. One can see that the off-diagonal terms (in absolute value) grow rapidly well above 1 with \( \ell \) and dominate over the diagonal ones (the coefficients for other values of \( m \) have a quite similar pattern). This strong ‘mismatch’ between the \( \omega Y_{\ell}^m \) and the \( Z_{\ell}^m \) is suggested by the shapes of the functions and may be understood qualitatively in the following way: for large values of \( \ell - |m| \), the \( \theta \) dependence of \( Y_{\ell}^m \) has \( \ell - |m| \) oscillations over the full interval \([0, \pi]\), while \( Z_{\ell}^m \) has the same number of oscillations over the covered interval; this makes difficult a matching of \( Z_{\ell}^m \) to functions like \( \omega Y_{\ell}^m \) which have less oscillations over this interval.

4.2. Estimating the multipole coefficients

Points being distributed according to the density \( \lambda(\theta, \varphi) \) (to be evaluated), and detected with a probability \( \omega(\theta) \) (supposed to be known), the observed points are distributed according to \( \omega \lambda \); this function may be expanded over the \( Z_{\ell}^m \) defined from \( \omega \) as explained above:

\[ \omega \lambda = \sum_{\ell, m} \alpha_{\ell m} Z_{\ell}^m \]

and an unbiased estimator of the \( \alpha_{\ell m} \) is obtained from the points:

\[ \overline{\alpha}_{\ell m} = \frac{1}{N} \sum_i Z_{\ell}^m(\theta_i, \varphi_i). \]
Figure 2. Transformation between the $\omega Y^0_\ell$ and the $Z^0_\ell$ with the coverage function of figure 1. Top: shape (as a function of the declination) for $\ell \leq 4$ ($\omega Y^0_\ell$ on the left side, $Z^0_\ell$ on the right side). Bottom: coefficients for $\ell \leq 15$; left: the $C^0_{\ell\ell'}$, on a logarithmic scale: the area (in the units of the axes) is $\ln|C^0_{\ell\ell'}|/10$ (that is: a point represents 1, a unit square represents $2.2 \times 10^4$); the off-diagonal coefficients are in green if negative, in red if positive; right: the $D^0_{\ell\ell'}$ (inverse matrix) on a linear scale, 1:1, with the same sign convention.

If $\lambda$ is quasi-uniform, $\omega \lambda$ is almost proportional to $Z^0_0$: the coefficient $\alpha_{00}$ is greatly dominant. Then these estimators are quasi-optimal; if $\omega$ is not constant, the $\alpha_{\ell m}$, for a given value of $m$, may be correlated. If $N$ is large, their covariance matrix is approximately given by quadratic moments:

$$\text{cov}(\alpha_{\ell m}, \alpha_{\ell' m}) \simeq \frac{1}{N} \sum_i Z^m_\ell(\theta_i, \varphi_i) Z^m_{\ell'}(\theta_i, \varphi_i) - \overline{\alpha}_{\ell m} \overline{\alpha}_{\ell' m}.$$
It is now easy to obtain estimators of the multipole coefficients of $\lambda$ at a given order $L$ by substituting the expressions for the $Z^m_{\ell}$:

$$\omega \lambda \simeq \sum_{\ell,m} \overline{\alpha}_{\ell m} \sum_{\ell' = m}^L C^m_{\ell \ell'} \omega Y^m_{\ell'} ,$$

that is,

$$\lambda \simeq \sum_{\ell,m} \overline{\alpha}_{\ell m} Y^m_{\ell} \quad \text{with} \quad \overline{\alpha}_{\ell m} = \sum_{\ell' = 1}^L C^m_{\ell' \ell} \overline{\alpha}_{\ell' m} .$$

The $\overline{\alpha}_{\ell m}$ with different values of $m$ are not correlated, and the covariance matrix of the $\overline{\alpha}_{\ell m}$ is given by

$$\text{cov}(\overline{\alpha}_{\ell_1 m}, \overline{\alpha}_{\ell_2 m}) = \sum_{\ell', \ell''} C^m_{\ell_1 \ell'} C^m_{\ell_2 \ell'} \text{cov}(\overline{\alpha}_{\ell' m}, \overline{\alpha}_{\ell'' m}).$$

5. Illustrations

To illustrate the statistical properties of the estimates, we show here some simple applications of the methods in the case of the exposure shown in figure 1. For the sake of clarity, we will refer to the method presented in section 3 as method 1, and to the method presented in section 4 as method 2.

5.1. Behavior of variances with $L$

For illustrations, we use here method 1. At first, we restrict the bound $L$ to 1, so we are interested here in research into a dipolar component only. We show in figure 3 the reconstruction of the coefficient $a_{10}$ in the case of an indeed dipolar distribution, whose excess of events points towards equatorial North with a magnitude $a_{10} = 0.1$. The red histogram drawn shows the occurrence number of each reconstructed value of $\overline{\alpha}_{10}$ in the case of $N = 10^5$ events generated by a Monte Carlo method according to

$$\mu(\theta, \varphi) = \omega(\theta, \varphi)[1 + a_{10} Y^0_0(\theta, \varphi)].$$

Over the histogram is plotted a Gaussian curve whose average and standard deviation parameters are the ones determined in section 3.2. This curve matches the histogram in such a way that the statistics previously determined by calculation do indeed describe the properties of the estimators. Let us note that under the assumption of a purely dipolar distribution (i.e. $L = 1$) the reconstruction of the multipolar coefficients is obtained in a very reasonable way.

Let us continue to illustrate the method by looking at the same multipolar coefficients, still in the case of a purely dipolar distribution, but increasing the bound $L$ to 2 and 3. Still in figure 3, the blue and the green histograms and Gaussian curves plot the same quantities as the red ones but for $L = 2$ and 3 respectively, and illustrate the extremely fast degradation of the accuracy of the reconstruction of $a_{10}$ by more than a factor 2 for each additional order.

This tendency towards widening of the laws is largely confirmed when one looks at the reconstruction of any coefficients $a_{\ell m}$ as a function of $L$. We show this property in figure 4.
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for the $\{a_{\ell m}\}_{\ell \leq 2}$ set of coefficients, which illustrates clearly that it becomes increasingly difficult to give a meaning to the reconstructed values of the coefficients as soon as the maximum order of development is greater than 3.

5.2. Comparison of the two methods

Two samples of points were simulated according to a slightly anisotropic distribution ($a_{10}/a_{00} = 0.05, a_{11} = 0$, i.e. dipole moment along $z$), multiplied by the coverage function drawn in figure 1; the $a_{\ell m}$ were estimated by both methods with the bound $L$ going from 1 to 5. Figure 5 shows that they give comparable results, and that the difference between them is generally smaller that the intrinsic difference between the samples (statistical fluctuations). Once again, one can see the divergence of the variances with increasing $L$.

5.3. Highly non-uniform coverage of the whole sky

In contrast, with a complete coverage (even highly non-uniform), the size of the variance is stabilized at large $L$. This is illustrated in figure 6, comparing a partial coverage (cf figure 1) to this coverage completed by a small fraction of the same function in the opposite hemisphere, in such a way that there is no fully unseen region. Even a relatively small relative exposure in the Northern part of the sky allows us to recover the coefficients with almost the same precision as if the exposure was uniform over the whole sky. Note however that if the exposure in the opposite hemisphere tends to zero, even if the phenomenon of
stabilization at large $L$ remains, the variance at any $\ell$ increases, tending towards a plateau determined roughly by $1/\sqrt{N'}$ where $N'$ is in that case the total number of events which would be observed on the full sky through a uniform window but with a low absolute coverage, in such a way that $N'$ is small. Of course, the larger the size of the relative exposure tending to 0, the faster the increase of the variance towards this plateau.

### 5.4. Angular distribution in the covered region

We have shown that using a large value of $L$ in the case of a partial coverage of the sky forbids giving for any $a_{\ell m}$ coefficient an interpretation of an individual multipolar moment. Nevertheless, one may wonder about the significance of the full set of coefficients $\{a_{\ell m}\}$. As a toy example, we generated a distribution of points according to the exposure function of figure 1 times the function shown in figure 7 (top left) which is a combination of $Y_1^1$, $Y_2^2$ and $Y_3^3$. Top right in figure 7, we show the reconstructed sky assuming $L$ to be equal to 3, which illustrates that the reconstructed sky matches the injected one in the covered region even if the variance on each reconstructed multipolar coefficient is already large (as shown in preceding subsections) for $L = 3$. Increasing the value of $L$ to 5 (bottom left) or 10 (bottom right) do not change this property of the expansion, as only additional statistical fluctuations appear due to the finite number of points. On these plots, we hide the unseen part of the sky, where the reconstructed expansion is meaningless.

### 5.5. Hypothesis test

Any sky observed through an exposure function $\omega$ can thus be described precisely in the observed part of the sky by increasing $L$ to a sufficient value. However, the interpretation
Figure 5. Comparison of reconstructed coefficients $a_{\ell m}$ ($\ell = 0, 1$) for two simulated dipolar samples (red, blue) with the two methods (solid circles: method 1 presented in section 3; open circles: method 2 presented in section 4), with a bound $L$ from 1 to 5. Top left: $a_{00}$; top right: $a_{10}$; bottom: $a_{11}$ and $a_{1-1}$.

of each multipolar moment is problematic, because it depends strongly on the cut $L$. We want now to build a statistical test to obtain a reasonable value of $L$ from the data themselves.

Starting from a hypothesis on $L$ and the corresponding reconstructed $\{a_{\ell m}\}$ coefficients, the likelihood function $\mathcal{L}_L$ built from the realization is

$$\mathcal{L}_L = \prod_{i=1}^{N} \left( \omega(\Omega_i) \sum_{\ell m} a_{\ell m} Y_{\ell}^{m}(\Omega_i) \right).$$

For any particular realization, from this likelihood (which depends on $L$), we apply the method of the likelihood ratio to accept or to reject (within some chosen threshold) a null hypothesis $H_0(\mathcal{L}_{L_0})$ with respect to another hypothesis $H_1(\mathcal{L}_{L_1})$ by computing

$$\lambda = \frac{\mathcal{L}_{L_0}}{\mathcal{L}_{L_1}}.$$
Figure 6. Dependence on the bound $L$ of the variances of estimated $a_{\ell m}$ with a partial or a complete (but not uniform) coverage. Black circles: coverage function of figure 1; red squares: the same function plus 0.1 times the symmetric one with respect to the equatorial plane; blue triangles: the same plus 0.2 times the symmetric one. The solid symbols correspond to a simulated dipolar distribution along the $z$ axis; the open ones correspond to the $a$ axis in the equatorial plane.

Figure 7. Top left: toy injected sky (combination of $Y_1^1$, $Y_2^2$ and $Y_3^1$), in equatorial coordinates. Then recovered skies using different assumptions for $L$, and using the exposure function of figure 1. Top right: recovered sky assuming $L = 3$. Bottom left: recovered sky assuming $L = 5$. Bottom right: recovered sky assuming $L = 10$. The unseen part of the sky is hidden.
Asymptotically, for a sample obeying the hypothesis $H_0$, $-2 \ln \lambda$ is distributed according to a $\chi^2$ with a number of degrees of freedom equal to the number of extra parameters in the $H_1$ hypothesis with respect to $H_0$. The value of $\lambda$ for any particular realization can thus be used to validate (or to reject) an assumption on $L$.

As an example, let us assume that the cosmic rays follow a symmetrical quadrupolar distribution $1 + 0.1 \sin^2 \theta - 0.2 \cos^2 \theta$, and let us use once again the exposure function shown in figure 1. By restricting the reconstruction to a dipolar distribution, one then finds an artifact amplitude of about 5%. To test the relevance of the hypothesis of a purely dipolar sky, one can thus—starting from this sky—estimate whether it is necessary or not to increase the degree of the expansion by calculating the ratio of the likelihood between the null hypothesis $L = 1$ and another hypothesis on $L$, $L = 2$ for instance. To show the behavior of the test, we generated 1000 different realizations of the quadrupolar pattern with 100 000 points each, then we reconstructed the parameters of the expansion within the two hypotheses, and finally computed the ratio of likelihoods. In this case, the hypothesis $L = 2$ introduces five more parameters $\{a_{2m}\}$, and the expected values of $-2 \ln \lambda$ are asymptotically distributed as a $\chi^2$ with five degrees of freedom. We plot the result at the top of figure 8: by choosing the threshold of the test to be 5% (vertical line at $-2 \ln \lambda = 11.07$), only eight realizations over 1000 are accidentally accepted (red histogram). In contrast, repeating the same procedure for the $L = 2$ null hypothesis with respect to the $L = 3$ hypothesis, we show at the bottom of figure 8 that the distribution obtained perfectly matches the asymptotical expected one (a $\chi^2$ with seven degrees of freedom in that case). With the same partial coverage, a similar test on samples of 1000
points gives a poor discrimination between different hypotheses, and with only 100 points the test is completely irrelevant.

This procedure may be used to define a ‘likely minimum value’ $L_{\text{min}}$ of $L$, and to prevent a wrong interpretation of multipolar coefficients obtained with a lower value, which are then biased (as the artifact dipole obtained above from a symmetric quadrupole). Of course, a given sample cannot provide by itself an absolute maximum for $L$, and in the presence of a hole the multipolar coefficients remain undefined without an external assumption; however let us point out that in many cases, the values of the coefficients at a given order have no intrinsic physical meaning if the distribution is of higher order.

6. Testing model predictions

Let us consider a distribution $\lambda(\theta, \varphi)$ with coefficients $a_{\ell m}$ on the $Y_{\ell m}$; the observed distribution $\omega(\theta)\lambda(\theta, \varphi)$ has coefficients $\alpha_{\ell m}$ on the $Y_{\ell m}$, and the relation

$$a_{\ell m} = \sum_{\ell' = \ell}^{\infty} C_{\ell' \ell}^{m} \alpha_{\ell' m}$$

may be inverted, because for each value of $m$ the matrix $C_{\ell' \ell}$ is triangular, and the coefficients of the inverse relation may be computed exactly for any value of $\ell$:

$$\alpha_{\ell m} = \sum_{\ell' = \ell}^{\infty} D_{\ell' \ell}^{m} a_{\ell' m}.$$ 

The values of the $D_{\ell' \ell}^{m}$ are displayed in figure 2 (right) with the same example as in the left plot, but on a linear scale: contrary to the $C_{\ell' \ell}^{m}$, they remain below 1 in absolute value, and practically negligible far away from the diagonal\(^3\).

As a consequence, if a model gives predictions about the $a_{\ell m}$, it will be possible, in some cases, to deduce predictions on the $\alpha_{\ell m}$, which can be tested without any assumption on $L$. In that sense, the compatibility of a model may be checked with observations over an incomplete sky with a precision depending on the available statistics (but, of course, it can never prove that this model is the only possible one).

If a model makes a deterministic prediction, comparing the $\alpha_{\ell m}$ to the predicted values may be a convenient way to test this model up to a given order of multipolarity, that is, down to a given angular scale. The method is potentially more interesting if the predictions are probabilistic. As we emphasized in the introduction, this is a relevant framework for describing high energy cosmic ray physics. Indeed, even in a situation with a well-defined and structured configuration of sources, propagation of cosmic rays unavoidably leads to a probabilistic nature of the observable sky, that is to say, a probabilistic nature of the multipolar moments. Each class of models has intrinsically a natural variance encrypted in the $a_{\ell m}$ covariance matrix. Further, some models do not try to build a well-defined configuration of sources, but pick up randomly cosmic rays at sources according to some distributions, making it even more impossible to circumvent the characterization of a particular data set through a relevant statistical tool.

\(^{3}\) However, in practice, through the matrix inversion, the numerical divergence of the $C_{\ell' \ell}^{m}$ limits the expansion to $L \simeq 15$ for this kind of coverage function; this is sufficient for most studies on sky anisotropies.
Consequently, the discrimination of models through an exploratory search in a data set is potentially extremely powerful in looking for the distance of the full covariance matrix from the expected one. The simplest example is a model predicting random $a_{\ell m}$ following independent Gaussian laws with variances $\sigma^2_\ell$. In that case, the covariance matrix for the $\alpha_{\ell m}$ reads

$$\text{cov}(\alpha_{\ell_1 m}, \alpha_{\ell_2 m}) = \sum_{\ell' = m}^{\infty} D_{\ell_1 \ell_2}^m D_{\ell_1 \ell_2}^m \sigma^2_{\ell'}$$

provided, of course, that this series converges: this is true if the series of $\sigma^2_\ell$ converges (as it should do for physical models), and if the $D_{\ell_1 \ell_2}^m$ have the behavior suggested by figure 2.

7. Conclusions

To cope with a partial sky coverage, a formalism using the computation of moments on orthogonal functions was developed for recovering the angular distribution of the incident flux from a sample of $N$ observed points. If the multipolar expansion is assumed to be upper bounded by $\ell \leq L$, the coefficients $a_{\ell m}$ may be estimated with a variance proportional to $1/N$ as usual, with a penalty factor increasing exponentially with $L$ if there is a hole in the coverage (but stabilizing rapidly if the coverage is nowhere vanishing, even highly non-uniform). Two methods were tested, giving similar results, and practically the same variances.

Statistical tests based on likelihood ratios may be built to check a hypothesis on the distribution, for example a given bound $\ell \leq L$. In any case, it is possible to express predictions of a model in terms of coefficients which can be computed without any assumption on $L$, and tested against the moments found with a sample of observed points.

The methods presented in this paper may be applied to any cosmic ray data set, provided that the arrival directions and the coverage of the sky are known with a reasonable precision.

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