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John Erik Fornæss (NTNU)  Dynamics of transcendental Hénon maps  November 2, 2019 1 / 39
Abstract: I will lecture about ongoing joint work with Arosio, Benini and Peters. This mixes the theories of iteration of entire functions in one complex variable and polynomial Henon maps in two complex variables.
Dynamics of transcendental Hénon maps

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Plan of talk

- Polynomials on $\mathbb{C}$
- Transcendental functions on $\mathbb{C}$
- Henon maps on $\mathbb{C}^2$
- Transcendental Henon maps on $\mathbb{C}^2$
This is a joint work with Leandro Arosio, Anna Miriam Benini and Han Peters.

**What is holomorphic dynamics?**

Let $X$ be a complex manifold and let $f : X \to X$ be a holomorphic self-map. Holomorphic dynamics studies the behaviour of the orbits $(z_0, f(z_0), f^2(z_0), \ldots)$, where $z_0 \in X$.

**Example**

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial in one complex variable. Its **Fatou set** is the open set where the family $(f^n)$ is equicontinuous. Its complement is called the **Julia set**.
Polynomial dynamics

There exists a radius $R > 0$ such that $D(0, R)^C$ is mapped into itself and every orbit starting in $D(0, R)^C$ goes to infinity. Hence the escaping set $I_\infty := \{ z : f^n(z) \to \infty \}$ is a Fatou component.

Classification of invariant components [Fatou-Julia]

An invariant Fatou component $\Omega$ different from $I_\infty$ is either

- the basin of attraction of an attracting fixed point $|f'(p)| < 1$ in $\Omega$,
- the basin of attraction of a parabolic fixed point $f'(p) = 1$ in $\partial \Omega$,
- a Siegel disk, biholomorphically equivalent to an irrational rotation on the unit disk $\mathbb{D}$.

There is no wandering Fatou component, that is $\Omega : f^n(\Omega) \neq f^m(\Omega)$ for all $n \neq m$. [Sullivan ‘85]
Transcendental dynamics

If \( f : \mathbb{C} \to \mathbb{C} \) is transcendental (entire with essential singularity at \( \infty \)), there can be

- **escaping** wandering domain [Baker ’76]:
  \[
  f(z) = z + \sin z + 2\pi,
  \]

- **oscillating** wandering domain [Eremenko-Lyubich ’87]

it is an open question whether there can be **orbitally bounded** wandering domains.

**Theorem**

*(Benini-F-Peters (2018))* All entire transcendental functions have infinite entropy.
What about $\mathbb{C}^2$?

A polynomial Hénon map is $F(z, w) = (p(z) - \delta w, z)$, where $p \in \mathbb{C}[z]$ and $\delta \neq 0$ is a constant [Hénon '76]. It is an automorphism of $\mathbb{C}^2$ with constant jacobian $\delta$. 
Let $F$ be a polynomial Hénon map: Oscillating and escaping wandering domains cannot exist. Bounded wandering domains?
Theorem (Astorg-Buff-Dujardin-Peters-Raissy)

(Annals 2016) There is a polynomial map on $\mathbb{C}^2$ with a wandering domain with bounded orbits. (This map is not invertible)

Theorem (Han Peters-David Hahn (2018))

There is an invertible polynomial map on $\mathbb{C}^4$ with a wandering domain with bounded orbits.
Definition

We introduce the family of *transcendental Hénon maps* of the type $F(z, w) = (f(z) - \delta w, z)$, where $f$ is a transcendental function and $\delta \neq 0$ is a constant.

Every such $F$ is an automorphism with constant jacobian $\delta$ and has nontrivial dynamics:

**Theorem (Arosio-Benini-F-Peters (2018), Huu Tai Terje Nguyen (2018))**

*Every transcendental Hénon map $F$ has a periodic point $p$, $F^n(p) = p$.***
We have the existence of an escaping orbit for any transcendental Hénon map. This is known already for entire functions on $\mathbb{C}$.

**Theorem**

Let $F(z, w) = (f(z) - \delta w, z)$ where $f$ is an entire transcendental function. Then there exists an orbit $(z_n, w_n) \to \infty$. 
Theorem

The Julia set of a Henon map is always nonempty.

Proof.

If the Julia set is empty, then there is a subsequence $F^{\circ n_k}$ which converges uniformly on compact sets to a holomorphic map $G : \mathbb{C}^2 \rightarrow \mathbb{P}^2$. Since there is an escaping orbit, $G$ must map at least one point to the line at infinity. The line at infinity is the zero set of a holomorphic function locally. By the Hurwitz theorem it follows that $G$ maps all of $\mathbb{C}^2$ to the line at infinity. However, since $F$ has a periodic point, this is a contradiction.
We explain the main ingredient in the construction of an escaping orbit. It is similar to the proof in one variable. The key ingredient is Wiman Valiron theory.
Let $f(z) = \sum_n a_n z^n$ be an entire transcendental function. For any radius $r$, let $M(r)$ be the maximum value of $|f(z)|, |z| = r$. Note that $a_n r^n \to 0$. Hence there is a power $n = N(r)$ which maximizes $|a_n| r^n$. For a given $r$, pick a point $w_r, |w_r| = r$ for which $|f(w_r)| = M(r)$. Then in a small disc around $w_r$, $f$ is very close to a monomial, $(z/w_r)^{N(r)} f(w_r)$. This shows that the image of this disc maps much closer to infinity and the image will cover a very thick annulus. This makes it possible to repeat and thereby construct an escaping orbit.

More precisely, the main result in Wiman Valiron Theory is the following, but I wont say anything more about it.
Theorem (Wiman-Valiron estimates)

Let $f$ be entire transcendental, $\frac{1}{2} < \alpha < 1$. Let $q$ be a positive integer. Let $r > 0$ and let $w_r$ be a point of maximum modulus for $r$, that is, such that $|w_r| = r$ and $|f(w_r)| = M(r)$. Let $z$ be such that

$$|z - w_r| < \frac{r}{(N(r))^\alpha}, \quad (1)$$

then

$$f(z) = \left(\frac{z}{w_r}\right)^{N(r)} f(w_r)(1 + \epsilon_0), \quad (2)$$

$$f^{(j)}(z) = \frac{N(r)^j}{w_r^j} f(z)(1 + \epsilon_j), \quad (3)$$

for all $1 \leq j \leq q$, where $\epsilon_j$ are functions converging uniformly to 0 in $z$ as $r \to \infty$ provided $r$ stays outside an exceptional set $E$ of finite logarithmic measure.
The disk \( \left\{ |z - w_r| < \frac{r}{(N(r))^\alpha} \right\} \) is called a \textit{Wiman-Valiron disk}. 
We next discuss the theorem mentioned earlier.

**Theorem**

*(Benini-F-Peters (2018)) All entire transcendental functions have infinite entropy.*

This is a first step towards proving that entire Henon maps have infinite entropy. This is still open.
Example

- The map $f = e^{i\theta} \to e^{2i\theta}$ doubles distance. The iterate $f^{\circ n}(e^{i\theta}) \to e^{2^n i\theta}$ multiply distances by $2^n$. The entropy normalizes this to $\frac{\log(2^n)}{n} = \log 2$.

- The map $z \to z^2$ on $\mathbb{C}$ has entropy $\log 2$. This comes from the unit circle. The inside of the circle converges to zero and gives no entropy. The same goes for the outside.

- The map $z \to z^k$ has entropy $\log k$.

- A polynomial $P$ of degree $d$ has entropy $\log d$. A key property is that if $R$ is large enough, then the image $P(\Delta(0, R)) \supset \Delta(0, R)$ and moreover for each $w \in \Delta(0, R)$, there are $d$ preimages $z_1, \ldots, z_d \in \Delta(0, R)$ (counted with multiplicity).
Topological Entropy

Definition (Topological Entropy)

Let $f : X \to X$ be a self-map of a compact metric space $(X, d)$. A set $A \subset X$ is called $(n, \delta)$-separated, for $n \in \mathbb{N}$ and $\delta > 0$, if for any $z \neq w \in A$ there exists $k \leq n - 1$ such that $d(f^k(z), f^k(w)) > \delta$. Let $K(n, \delta)$ be the maximal cardinality of an $(n, \delta)$-separated set. Then the topological entropy is defined as

$$
\text{top}(f) = \sup_{\delta > 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log K(n, \delta) \right\}.
$$
Let $f = \sum_k \epsilon_k z^{n_k}$ for a rapidly increasing sequence $n_k$ and rapidly decreasing sequence $\epsilon_k$. Then $f$ has infinite entropy on $\mathbb{C}$. There will be a sequence $R_k$ so that $f(\Delta(0, R_k)) \supset \Delta(0, R_k)$ and moreover for each $w \in \Delta(0, R_k)$, there are $n_k$ preimages $z_1, \ldots, z_{n_k} \in \Delta(0, R_k)$ (counted with multiplicity).
Topological Entropy

In the case when the space $X$ is not compact, it is not clear how to define entropy. One possibility is to restrict to compact subsets.

**Definition (Topological Entropy in the noncompact case)**

Let $f : X \to X$ be a self-map of a metric space $(X, d)$. Let $Y \subset X$ be a compact subset. A set $A \subset Y$ is called $(n, \delta)$-separated, for $n \in \mathbb{N}$ and $\delta > 0$, if for any $z \neq w \in A$ for which $f^k(z), f^k(w) \in Y$ for all $k \leq n - 1$, there exists $k \leq n - 1$ such that $d(f^k(z), f^k(w)) > \delta$. Let $K(n, \delta, Y)$ be the maximal cardinality of an $(n, \delta)$-separated set. Then the *topological entropy* is defined as

$$
top(Y, f) = \sup_{\delta > 0} \limsup_{n \to \infty} \frac{1}{n} \log K(n, \delta, Y).
$$

$$
top(f) = \sup_{Y \subset X} top(Y, f).
$$
We show that a similar result as for polynomials (see an above example, point 4) also holds for all entire functions:

**Theorem**

Let \( f \) be a transcendental entire function, and let \( n \in \mathbb{N} \). There exists a non-empty bounded open set \( V \subset \mathbb{C} \) so that \( V \subset f(V) \) and such that any point in \( V \) has at least \( n \) preimages for \( f \) in \( V \) counted with multiplicity.
The Kobayashi metric

A key property of the Kobayashi metric is that it is distance decreasing under holomorphic maps.

**Lemma**

The Kobayashi metric on \( \mathbb{C} \setminus \{0, 1\} \) is larger than \( \frac{1}{2|z|\log|z|} \) for all large enough \( |z| \).

This implies that if \( f : \Delta(0, 1) \to \mathbb{C} \setminus \{0, 1\} \), then if \( |f(0)| \) is very large, then \( |f(z)| \) is very large for all \( |z| < 1/2 \). The reason is that the Kobayashi metric is distance decreasing.

More generally, if \( C \subsetneq D \subset \mathbb{C} \) and \( f : D \to \mathbb{C} \setminus \{0, 1\} \) and \( |f(p)| \) is very large for some \( p \in C \), then \( |f(p)| \) is very large for all \( p \in C \).
Note on entire transcendental functions $f$: The max value $M(R)$ for $f$ on the circle of radius $R$ goes to infinity faster then any power $R^j$ of $R$. Another important fact: The Picard theorem says that all values in $\mathbb{C}$ except at most 1 are taken infinitely many times. This has an important consequence:

**Lemma**

There exist for any $j$ arbitrarily large $R$ so that $M(R) > R^j$ and the minimum $m(R)$ on the circle is less than 1.
Corollary

Let $f$ be entire, transcendental. Then there exist arbitrarily large $R$ so that the image of the annulus $A_R = \{ R/2 < |z| < 2R \}$ cannot avoid both 0 and 1.
In fact, we can prove a stronger result: The point 1 can be replaced by any value $\alpha \in A_R$.

**Corollary**

Let $f$ be entire, transcendental. Then there exist arbitrarily large $R$ so that if $f \neq 0$ on $A_R$, then $f(A_R) \supset A_R$.

This suffices to prove that nonvanishing entire transcendental functions have infinite entropy.
Note that if we replace $A_R$ by two halves, $D_R$, midpoints $\theta = \theta_R$, then $f$ will have roots because $D_R$ is simply connected.

**Corollary**

Let $f$ be entire, transcendental. Let $n$ be an integer. Then there exist arbitrarily large $R$ so that if $f \neq 0$ on $A_R$, then $f(A_R) \supset A_R$ and covers $A_R$ at least $n$ times.

We can finally do the same argument, replacing $0$ by any point in $A_R$.

**Theorem**

Let $f$ be a transcendental function. Let $n \in \mathbb{N}$. Then there exist arbitrarily large $R$ and $j$ large and $\theta \in [0, 2\pi]$ so that either $A_R \subset f(D_R)$ or else there exists $\alpha \in A_R \setminus f(D_R)$ so that $\left( A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}}) \right) \subset f(D_R)$.

In the latter case, each $\beta \in \left( A_R \setminus \Delta(\alpha, \frac{1}{R^{j/2}}) \right)$ has at least $n$ distinct and uniformly separated preimages in $D_R$. 
Using this, we prove:

**Theorem**

*(Benini, F, Peters, 2018)* All entire transcendental $f : \mathbb{C} \to \mathbb{C}$ (not a polynomial) have infinite topological entropy.
Theorem (Arosio-Benini-F-Peters)

(Math. Ann. 2019) There are examples of transcendental Hénon maps with

- an escaping wandering domain biholomorphic to $\mathbb{C}^2$,
- an oscillating wandering domain biholomorphic to $\mathbb{C}^2$. 
The oscillating wandering domain

Let $0 < a < 1$. We construct a sequence of maps

$$F_k(z, w) = (f_k(z) + aw, az) \to F$$

with oscillating orbit $(P_n)$ and

$$\text{diam } F^n(B(P_0, 1)) \to 0. \quad (4)$$

We ensure that every $F_k$ has a saddle fixed point at the origin.
Assume that we defined $F_k$ with an orbit $P_1, \ldots, P_{n_k}$.
First step: use the Lambda Lemma to construct a new oscillation $Q_0, \ldots, Q_N$ coming in along the stable manifold of $F_k$ and going out along the unstable manifold of $F_k$. 
Second step: use Runge approximation to obtain $F_{k+1}$ connecting the old orbit $P_0, \ldots, P_{n_k}$ with the new oscillation $(Q_j)$ via a contracting detour $T_0, \ldots, T_M$, long enough to neutralize (possible) expansion on $(Q_j)$. We modify only the 1-dimensional function $f_k$. Finally we send $Q_N$ far away and obtain the point $P_{n_k+1}$. 
Why are the $P_j$’s in different Fatou components?

Let $\Omega_j$ be the Fatou component containing $P_j$. Assume by contradiction that $\Omega_0 = \Omega_m$.

All limit functions on $\Omega_0$ are constant.

Let $K$ be a compact neighborhood of 0 which does not contain any nonzero point of period $m$ of $F$. Then there exists $P_{nj} \to P \neq 0$, $P \in K$.

By normality, $F^{nj} \to P$ on $\Omega_0$, but

$$F^{nj}(P_m) = F^m(F^{nj}(P_0)) \to F^m(P) \neq P.$$
Example

Let $F(z, w) = 2(z, w)$. Then for $(z, w) \neq 0$ the iterates converge to the line of infinity, where the map is the identity. Hence the Fatou set equals $\mathbb{C}^2 \setminus \{(0, 0)\}$. Hence the fixed point $(0, 0)$ is an isolated point in the Julia set.

This is not possible for transcendental Henon maps.

Theorem

(Arosio, Benini, F, Peters) Let $F$ be a transcendental Henon map. Then there can be no fixed point which is an isolated point in the Julia set.
We assume that 0 is an isolated fixed point in the Julia set. (1) First we prove that 0 must be repelling. (2) Secondly we show that this is impossible.
Choose two real numbers $0 < \delta << \epsilon < 1$. Let $A = \{ \delta < \| z \| < \epsilon \}$. Let $U$ be the connected component of the Fatou set which is punctured at the origin. If $\epsilon$ is small enough, $A$ will divide $U$ into three connected components, $A, B, C$ where $B = \{ 0 < \| z \| \leq \delta \}$ and $C = U \setminus (A \cup B)$. If there exists $R$ so that $F^n(A) \subset \mathbb{B}(0, R)$ for all $n$, then by the maximum principle $F^n(B) \subset \mathbb{B}(0, R)$ for all $n$ and then $0$ is in the Fatou set, a contradiction. Hence there must exist a sequence $n_k$ so that $F^{n_k}$ converges uniformly on $A$ to the line at infinity. In particular there is an $n$ so that $f^n(A) \cap \{ \| z \| < \epsilon \} = \emptyset$. We also have that $U = F^n(A) \cup F^n(B) \cup F^n(C)$ which again divides $U$ into three disjoint connected sets. Clearly $F^n(B)$ contains a punctured neighborhood of the origin. It follows that $\{ 0 < \| z \| < \epsilon \} \subset F^n(B)$. This implies that $F^{-n}(\{ \| z \| < \epsilon \}) \subset \{ \| z \| < \delta \}$. Hence both eigenvalues of $(F^{-n})'(0)$ are strictly less than one. Hence the same is true for $(F^{-1})'(0)$ so indeed $0$ is a repelling fixed point for $F$. 

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(2) Suppose that 0 is an isolated repelling fixed point in the Julia set and let $U$ be the Fatou component with a puncture at 0. Since the Jacobian is larger than one, all limits of $F^n$ must be in the line at infinity. Let $V$ be the subset of $\mathbb{C}^2$ consisting of those points for which $F^{-n}(z) \to 0$. This is a Fatou Bieberbach domain. Since $F^{-1}$ has an escaping point, $V$ is not the whole space. So $V$ has a boundary point $p$. Let $A = \{ \delta < \|z\| < \epsilon \}$ for $0 < \delta << \epsilon << 1$. Then the sequence $F^n(A)$ converges uniformly to infinity, and hence cannot cluster at $p$. But there are points $q$ arbitrarily close to $p$ so that $F^{-n}(q) \to 0$. Hence for some $n$, $F^{-n}(q) \in A$. Contradiction.
A stronger result is the following:

**Theorem**

*(Arosio-Benini-F-Peters)*  *There is no isolated point in the Julia set*

and finally:

**Theorem**

*(Arosio-Benini-F-Peters)*  *The Fatou set is pseudoconvex.*
Thank you for listening!