Infinitesimally thin static scalar shells surrounding charged Gauss-Bonnet black holes

Shahar Hod

The Ruppin Academic Center,
Emeq Hefer 40250, Israel
The Hadassah Academic College,
Jerusalem 91010, Israel
E-mail: shaharhod@gmail.com

ABSTRACT: We reveal the existence of a new form of spontaneously scalarized black-hole configurations. In particular, it is proved that Reissner-Nordström black holes in the highly charged regime \( Q/M > (Q/M)_{\text{crit}} = \sqrt{21}/5 \) can support thin matter shells that are made of massive scalar fields with a non-minimal coupling to the Gauss-Bonnet invariant of the curved spacetime. These static scalar shells, which become infinitesimally thin in the dimensionless large-mass \( M\mu \gg 1 \) regime, hover a finite proper distance above the black-hole horizon [here \( \{ M, Q \} \) are respectively the mass and electric charge of the central supporting black hole, and \( \mu \) is the proper mass of the supported scalar field]. In addition, we derive a remarkably compact analytical formula for the discrete resonance spectrum \( \{ \eta(Q/M, M\mu; n) \}_{n=0}^{\infty} \) of the non-trivial coupling parameter which characterizes the bound-state charged-black-hole-thin-massive-scalar-shell cloudy configurations of the composed Einstein-Maxwell-scalar field theory.

KEYWORDS: Black Holes, Classical Theories of Gravity

ArXiv ePrint: 2201.03503
1 Introduction

Wheeler’s no-hair conjecture [1, 2] has asserted that black-hole solutions of the coupled Einstein-matter field equations should describe bald spacetimes in which spatially regular static scalar field configurations cannot be supported. In accord with this influential conjecture, various no-hair theorems [3–9] have ruled out the existence of scalarized black-hole spacetimes that are made of scalar fields with a minimal (and also with a non-trivial) coupling to the Ricci curvature scalar.

Intriguingly, recent explorations [10–24] of the coupled Einstein-matter field equations have revealed the existence of hairy (scalarized) black-hole spacetimes in which the externally supported hair is non-trivially (non-minimally) coupled to the spatially-dependent Gauss-Bonnet curvature invariant \(G\) of the spacetime [25–30].

This physically interesting phenomenon, known as black-hole spontaneous scalarization, is closely related to the fact that, in composed Einstein-Gauss-Bonnet-scalar field theories, the Klein-Gordon equation which determines the spatial behavior of the scalar field \(\phi\) in the curved spacetime contains an effective mass term (a Gauss-Bonnet-scalar-field direct interaction term) of the linearized form \(-\bar{\eta}\phi G\). This spatially-dependent effective mass term leads, for large enough values of the non-trivial coupling parameter \(\bar{\eta}\) [see eq. (3.15) below], to the formation of a negative (attractive) black-hole-field binding potential well in the vicinity of the black-hole horizon.

---

1See [25–30] for the interesting case of spatially regular asymptotically flat static scalar field configurations with a non-minimal coupling to the Maxwell electromagnetic invariant of a central supporting charged black hole.
Recently, the physically important phenomenon of black-hole spontaneous scalarization has been explored, using numerical techniques, in the context of the charged black-hole solutions of the composed Einstein-Maxwell-Gauss-Bonnet-scalar field theory \[23, 24\]. Interestingly, it has been demonstrated in \[23, 24\] that the sharp boundary between bald black holes and hairy (scalarized) black-hole spacetimes is marked by the presence of charged Reissner-Nordström black holes that support spatially regular asymptotically flat bound-state configurations of the non-minimally coupled linearized scalar fields.

The physical significance of the bound-state charged-black-hole-linearized-scalar-field cloudy configurations (the term scalar ‘clouds’ is usually used in the physics literature to describe linearized field configurations that are supported by central black holes with spatially regular horizons \[31–34\]) stems from the fact that these composed configurations determine the charge-dependent critical existence-line $\tilde{\eta} = \tilde{\eta}(Q/M)^2$ of the non-trivial Einstein-Maxwell-Gauss-Bonnet-scalar field theory.

The main goal of the present paper is to explore the physical and mathematical properties of non-minimally coupled linearized massive scalar field configurations (massive scalar clouds) that are supported by charged Reissner-Nordström black holes with spatially regular horizons. In particular, using analytical techniques, we shall reveal the physically intriguing fact that the addition of a mass term to the supported non-minimally coupled scalar fields [see the action (2.1), which characterizes the composed Einstein-Maxwell-massive-scalar field theory] allows the existence of infinitesimally thin static scalar shells that hover a finite proper distance above the horizons of highly charged Gauss-Bonnet black holes.

In addition, below we shall derive a remarkably compact analytical resonance formula that describes, in the large-mass $M\mu \gg 1$ regime, the functional dependence $\tilde{\eta} = \tilde{\eta}(Q/M, M\mu)$ of the critical existence line, which characterizes the composed charged-blackhole-thin-massive-scalar-shell bound-state cloudy configurations, on the electric charge $Q$ of the central supporting black hole and on the dimensionless proper mass $M\mu^3$ of the non-minimally coupled scalar field.

2 Description of the system

We study the physical and mathematical properties of ‘cloudy’ black-hole configurations which are made of central charged Reissner-Nordström black holes that support spatially regular bound-state static configurations of linearized massive scalar fields. The composed Einstein-Maxwell-Gauss-Bonnet-nonminimally-coupled-massive-scalar field theory is characterized by the action \[23, 24\]^4

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{4} R - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} \nabla_{\alpha} \phi \nabla^{\alpha} \phi - \frac{1}{2} \mu^2 \phi^2 + f(\phi) G \right], \quad (2.1)$$

\(^2\)One can assume, without loss of generality, that the electric charge of the central supporting black hole is characterized by the relation $Q \geq 0$.

\(^3\)Here the physical field parameter $\mu$ stands for $\mu/\hbar$. Hence, the mass parameter of the non-minimally coupled scalar field has the dimensions of length$^{-1}$.

\(^4\)We shall use gravitational units in which $8\pi G = c = \hbar = 1$. 

- 2 -
where $\mu$ is the mass of the non-minimally coupled scalar field. As we shall explicitly prove below, the supported matter configurations owe their existence to the presence, in the action (2.1), of a direct (non-minimal) coupling between the massive scalar field $\phi$ and the Gauss-Bonnet invariant

$$\mathcal{G} \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$

that characterizes the curved black-hole spacetime.

As discussed in [13, 14, 25], the leading-order functional behavior

$$f(\phi) = \frac{1}{2} \eta \phi^2$$

of the scalar coupling function guarantees that the bald Reissner-Nordström black-hole spacetime is a valid solution of the composed Einstein-Maxwell-scalar field equations in the weak-field $\phi \to 0$ limit.\(^5\) The strength of the non-minimal coupling between the supported massive scalar field and the Gauss-Bonnet curvature invariant (2.2) is controlled by the physical parameter $\eta$.\(^6\)

The curved line element \[^7\]

$$ds^2 = -h(r) dt^2 + \frac{1}{h(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

with

$$h(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

characterizes the supporting Reissner-Nordström black hole of mass $M$ and electric charge $Q$. The roots

$$r_\pm = M \pm (M^2 - Q^2)^{1/2}$$

of the metric function (2.5) determine the radii of the (outer and inner) black-hole horizons.

The action (2.1) of the composed Einstein-Maxwell-massive-scalar field theory yields the Klein-Gordon differential equation \[^24\]

$$\nabla^\nu \nabla_\nu \phi = \mu^2_{\text{eff}} \phi$$

for the eigenfunctions of the supported scalar field configurations, where the effective mass term

$$\mu^2_{\text{eff}}(r; M, Q) = \mu^2 - \eta \mathcal{G} ,$$

which depends on the Gauss-Bonnet curvature invariant

$$\mathcal{G}_{RN}(r; M, Q) = \frac{8}{r^8} (6M^2 r^{-2} - 12MQ^2 r + 5Q^4)$$

\[^5\]It is interesting to stress the fact that the charge-dependent critical existence-line of the physical system, which marks the boundary between bald and hairy black-hole solutions of the field equations, is universal in the sense that it characterizes different Einstein-Maxwell-scalar field theories whose weak scalar field behavior is described by the leading-order functional expansion $f(\phi) \propto \phi^2 \cdot [1 + O(\phi^2)]$ [23, 24].

\[^6\]The physical parameter $\eta$, which couples the supported massive scalar field $\phi$ to the Gauss-Bonnet curvature invariant $\mathcal{G}$, has the dimensions of length $^2$.

\[^7\]We use here the Schwarzschild spacetime coordinates $(t, r, \theta, \phi)$.
of the Reissner-Nordström black-hole spacetime (2.4), reflects the non-trivial massive-scalar-field-Gauss-Bonnet coupling in the composed field theory (2.1).

Intriguingly, one finds that, depending on the relative magnitudes of the physical parameters \( \{\eta, \mu\} \) of the composed field theory (2.1), the radially-dependent effective mass term (2.8) may become negative in the vicinity of the black-hole outer horizon. Below we shall use analytical techniques in order to prove that this property of the effective scalar-field-Gauss-Bonnet mass term (2.8) may allow the existence of infinitesimally thin non-minimally coupled massive scalar shells (thin massive scalar clouds) that hover a finite proper distance above the horizons of highly charged [see eq. (3.10) below] Reissner-Nordström black holes.

Substituting the functional decomposition

\[
\phi(r, \theta, \varphi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \varphi) \tag{2.10}
\]

for the static non-minimally coupled massive scalar field into eq. (2.7) [here \( Y_{lm}(\theta, \varphi) \) with \( l \geq |m| \) are the familiar spherical harmonic functions] and using the curved black-hole line element (2.4), one obtains the radial differential equation [24]

\[
\frac{d}{dr} \left[ r^2 h(r) \frac{dR}{dr} \right] - \left[ \mu^2 r^2 + l(l+1) \right] R + \eta \left( \frac{48 M^2}{r^4} - \frac{96 M Q^2}{r^5} + \frac{40 Q^4}{r^6} \right) R = 0 , \tag{2.11}
\]

which determines the spatial behavior of the supported massive scalar clouds in the curved black-hole spacetime (2.4).

The radial differential equation (2.11), supplemented by the physically motivated boundary conditions of spatially regular (bounded) functional behavior of the scalar field at the black-hole outer horizon [13, 14, 37],

\[
\psi(r = r_H) < \infty , \tag{2.12}
\]

and an asymptotic exponential decay of the massive scalar eigenfunction at spatial infinity [13, 14, 37],

\[
\psi(r \to \infty) \sim r^{-1} e^{-\mu r} \to 0 , \tag{2.13}
\]

determine the discrete resonance spectrum \( \{M^{-2} \eta(M, Q, \mu; n)\}_{n=0}^{n=\infty} \) of the dimensionless coupling parameter that characterizes the bound-state Reissner-Nordström-black-hole-nonminimally-coupled-massive-scalar-field cloudy configurations of the composed Einstein-Maxwell-Gauss-Bonnet-scalar field theory (2.1). In particular, the critical existence-line of the physical system, which marks the boundary between bald Reissner-Nordström black holes and hairy (scalarized) black-hole solutions, is determined by the fundamental \( (n = 0) \) resonant mode \( \eta_0 = \eta_0(M, Q, \mu) \) [see eq. (6.3) below].

\*\*For brevity, we shall henceforth omit the angular harmonic indexes \( \{l, m\} \) of the supported massive scalar fields.
3 The discrete resonance spectrum of the composed black-hole

nonminimally-coupled-linearized-massive-scalar-field configurations

In the present section we shall prove the intriguing existence, in the composed Einstein-

Maxwell-Gauss-Bonnet-massive-scalar field theory (2.1), of infinitesimally thin scalar shells

with positive ($\eta > 0$) values of the non-minimal Gauss-Bonnet-scalar-field coupling param-

eter that are supported in highly-charged black-hole spacetimes a finite proper distance

above the black-hole horizons. In addition, we shall use analytical techniques in the di-
mensionless large-mass regime

$M\mu \gg 1$ \hspace{1cm} (3.1)

in order to determine the discrete resonance spectrum \( \{\eta(M,Q,\mu;n)\}_{n=0}^{\infty} \) of the com-

posed Reissner-Nordström-black-hole-nonminimally-coupled-massive-scalar-field bound-

state cloudy configurations.

In particular, we shall explicitly prove that the composed Reissner-Nordström-black-

hole-nonminimally-coupled-massive-scalar-field system is amenable to an analytical treat-

ment in the dimensionless large-mass regime (3.1), which corresponds to the dimensionless

large-coupling regime

$\bar{\eta} \equiv \frac{\eta}{M^2} \gg 1.$ \hspace{1cm} (3.2)

To this end, it is convenient to define the radial scalar eigenfunction

$\psi \equiv rR,$ \hspace{1cm} (3.3)

in terms of which the radial equation (2.11) can be expressed in the mathematically compact

form

$$\frac{d^2 \psi}{dy^2} - V\psi = 0,$$ \hspace{1cm} (3.4)

where the differential relation$^9$

$$dy = \frac{dr}{h(r)}$$ \hspace{1cm} (3.5)

determines the new radial coordinate $y(r)$. The effective potential in the Schrödinger-like

radial differential equation (3.4), which characterizes the composed black-hole-massive-scalar-field system, is given by the (rather cumbersome) functional expression

$$V(r; M, Q, \mu, l, \bar{\eta}) = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left[ \mu^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} - \bar{\eta} \cdot V_{GB} \right].$$ \hspace{1cm} (3.6)

The presence of the Gauss-Bonnet term

$$V_{GB}(r; M, Q) = \frac{48M^4}{r^6} - \frac{96M^3Q^2}{r^7} + \frac{40M^2Q^4}{r^8}$$ \hspace{1cm} (3.7)

in the effective interaction potential (3.6) is a direct consequence of the non-trivial (non-

minimal) coupling between the Gauss-Bonnet curvature invariant (2.9) of the charged black-hole spacetime (2.4) and the supported massive scalar field.

$^9$Note that the new radial coordinate belongs to the infinite regime $y \in [-\infty, \infty]$. 
We shall henceforth consider composed black-hole-massive-scalar-field cloudy configurations in the dimensionless large-mass regime (3.1) [or equivalently, in the dimensionless large-coupling regime (3.2)], in which case the effective black-hole-massive-field interaction potential (3.6) can be written in the form\(^{10}\)

\[
V(r; M, Q, \mu, \bar{\eta}) = h(r) \left[ \mu^2 - \bar{\eta} \left( \frac{48M^4}{r^6} - \frac{96M^3Q^2}{r^7} + \frac{40M^2Q^4}{r^8} \right) \right] \cdot \left( 1 + O[(M\mu)^{-2}] \right). \quad (3.8)
\]

The Gauss-Bonnet term (3.7) has a peak whose charge-dependent radius is given by the simple functional relation

\[
r_{\text{peak}}(M, Q) = \frac{5Q^2}{3M}. \quad (3.9)
\]

Interestingly, one finds that, in the dimensionless charge-to-mass ratio regime

\[
\frac{Q}{M} \geq \left( \frac{Q}{M} \right)_{\text{crit}} = \frac{\sqrt{21}}{5}, \quad (3.10)
\]

the radial peak (3.9) is located outside the black-hole outer horizon [that is, \(r_{\text{peak}} \geq r_+\) in the regime (3.10)]. Below we shall prove that this intriguing physical property of the non-trivial Gauss-Bonnet term (3.7) allows the existence of infinitesimally thin massive scalar shells that hover a finite proper distance above the outer horizons of central supporting Reissner-Nordström black holes in the highly charged regime (3.10).\(^{11,12}\)

Substituting (3.9) into eq. (3.7), one finds the functional relations\(^{13}\)

\[
\max_r \{V_{\text{GB}}(r \geq r_+; M, Q)\} \quad (3.11)
\]

\[
= \begin{cases} 
\frac{17496M^{10}}{78125Q^{12}} & \text{for } Q/M \geq (Q/M)_{\text{crit}} \\
\frac{8[5M^2Q^4 + 12M^3(M^2 - Q^2)^{3/2} - 18M^4Q^2 + 12M^6]}{[M + (M^2 - Q^2)^{1/2}]^8} & \text{for } Q/M \leq (Q/M)_{\text{crit}}
\end{cases}
\]

\(^{10}\)Note that the functional expression (3.8) is valid in the \(r = O(M)\) regime [see eqs. (3.1), (3.2), and (3.9)].

\(^{11}\)The main goal of the present paper is to prove the existence of infinitesimally thin matter shells that are supported in Gauss-Bonnet black-hole spacetimes a finite proper distance above the black-hole horizons. We therefore focus in this paper on non-minimally coupled massive scalar fields with positive \((\eta > 0)\) coupling parameters that are supported by highly charged \((Q > Q_{\text{crit}})\) black holes. It is worth noting that thin matter shells with \(\eta > 0\) can also be supported by Gauss-Bonnet black holes in the complementary \(Q \leq Q_{\text{crit}}\) regime. However, these scalar configurations are effectively connected to the black-hole outer horizon (that is, in this case the classically allowed region of the composed black-hole-massive-scalar-field binding potential extends all the way to black-hole horizon). Likewise, thin scalar shells in the negative \((\eta < 0)\) coupling regime can also be supported by charged Gauss-Bonnet black holes. However, these shells are also connected to the black-hole outer horizon (that is, in this case the classically allowed region of the binding potential starts at the black-hole horizon).

\(^{12}\)It is important to emphasize that we shall analyze in the present paper scalar field configurations which are not strictly zero at the black-hole horizon. In particular, we shall consider supported massive scalar fields which, in the large-mass regime (3.1), are exponentially suppressed outside a classically allowed narrow radial interval [see eqs. (4.1) and (4.4) below] with a penetration depth into a classically forbidden region that scales as \(1/\mu\) and therefore becomes infinitesimally small in the large-mass regime \(M\mu \gg 1\).

\(^{13}\)Here we have used the fact that, in the dimensionless sub-critical regime \(Q/M \leq (Q/M)_{\text{crit}}\), the Gauss-Bonnet term (3.7) is a monotonically decreasing function in the exterior \(r \geq r_+\) region of the black-hole spacetime, which implies the relation \(V_{\text{GB}}(r \geq r_+; M, Q) \leq V_{\text{GB}}(r_+; M, Q) = \frac{4M^2}{r_+^2} - \frac{96M^3Q^2}{r_+^3} + \frac{40M^2Q^4}{r_+^4}\) in the regime \(Q/M \leq \sqrt{21}/5\).
for the maximal value of the non-trivial Gauss-Bonnet term (3.7) in the exterior regions of charged Reissner-Nordström black holes.

### 3.1 Upper bound on the proper masses of non-minimally coupled scalar clouds

In the present subsection we shall derive a charge-dependent upper bound on the allowed proper masses of the non-minimally coupled scalar fields that can be supported by the central charged Reissner-Nordström black holes. To this end, we point out that the presence of a binding (attractive) potential well outside the black-hole outer horizon provides a necessary condition for the existence of static bound-state scalar field configurations (scalar clouds) that are supported in the curved black-hole spacetime.

In particular, the requirement

\[ V(r_t^- \leq r \leq r_t^+) \leq 0 \quad \text{with} \quad r_t^- \geq r_+ \]  

(3.12)

[here \( \{r_t^-, r_t^+\} \) with \( r_t^- \geq r_+ \) are the characteristic classical turning points of the effective curvature potential (3.6)] yields the series of inequalities\(^{14}\)

\[ \mu^2 - \bar{\eta} \cdot \max_r \{V_{GB}(r)\} \leq \mu^2 - \bar{\eta} \cdot V_{GB}(r) \]  

\[ \leq \mu^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} - \bar{\eta} \cdot V_{GB}(r) \leq 0 \quad \text{for} \quad \bar{\eta} > 0 . \]

Taking cognizance of eqs. (3.11) and (3.13), one finds the charge-dependent upper bound

\[ M\mu \leq \sqrt{\bar{\eta}} \cdot \begin{cases} \frac{54\sqrt{30}}{625} \cdot \left( \frac{M}{Q} \right)^6 & \text{for} \quad \frac{Q}{M} \geq \frac{\sqrt{21}}{5} \\ \sqrt{8[5M^4Q^4 + 12M^5(M^2 - Q^2)^{3/2} - 18M^6Q^2 + 12M^8]} & \text{for} \quad \frac{Q}{M} \leq \frac{\sqrt{21}}{5} \end{cases} \]

(3.14)

on the allowed proper masses of the supported non-minimally coupled scalar fields.

Interestingly, the upper bound (3.14) can be expressed as the lower bound

\[ \bar{\eta} \geq (M\mu)^2 \cdot \begin{cases} \frac{78125}{17496} \cdot \left( \frac{Q}{M} \right)^{12} & \text{for} \quad \frac{Q}{M} \geq \frac{\sqrt{21}}{5} \\ \frac{[M + (M^2 - Q^2)^{1/2}]^8}{8[5M^4Q^4 + 12M^5(M^2 - Q^2)^{3/2} - 18M^6Q^2 + 12M^8]} & \text{for} \quad \frac{Q}{M} \leq \frac{\sqrt{21}}{5} \end{cases} \]

(3.15)

on the dimensionless value of the non-minimal coupling parameter which characterizes the composed charged-black-hole-nonminimally-coupled-linearized-massive-scalar-field bound-state configurations.

\(^{14}\)Here we have used the inequality \(2M/r^3 - 2Q^2/r^4 \geq 0\) which characterizes the exterior \( r \geq r_+ \) region of the black-hole spacetime (2.4).
3.2 The resonance spectrum of the composed black-hole-non-minimally-coupled-massive-scalar-field cloudy configurations

In the present subsection we shall focus our attention on the dimensionless super-critical charge regime [see eq. (3.10)]

\[ \frac{Q}{M} \geq \frac{\sqrt{21}}{5} \]  

(3.16)

which, as we shall explicitly prove below, is characterized by the presence of arbitrarily thin scalar shells that hover a finite proper distance above the central charged black holes [see eq. (4.4) below].

Interestingly, we shall prove that the discrete resonance spectrum \( \{ \bar{\eta}(M, Q, \mu; n) \}_{n=0}^{\infty} \) of the dimensionless physical parameter \( \bar{\eta} \), which characterizes the non-minimally coupled Einstein-Maxwell-massive-scalar field theory (2.1), can be determined analytically in the eikonal large-mass regime (3.1) [which corresponds to the large-coupling regime (3.2), see eq. (3.15)]. In particular, the radial differential equation of the supported non-minimally coupled massive scalar fields with the familiar Schrödinger-like form (3.4) is characterized by the well known second-order WKB quantization condition [38–40]

\[ \int_{y_t^+}^{y_t^-} dy \sqrt{-V(y; M, Q, M \mu, \bar{\eta})} = \left( n + \frac{1}{2} \right) \pi \quad ; \quad n = 0, 1, 2, \ldots \]  

(3.17)

where the classical turning points of the effective black-hole-field binding potential (3.8) [with \( V(y_t^-) = V(y_t^+) = 0 \)] determine the integration limits in (3.17) and \( n \in \{ 0, 1, 2, \ldots \} \) is the discrete resonance parameter of the physical system. The WKB resonance condition (3.17) can be expressed, using the differential relation (3.5), in the mathematically more convenient form

\[ \int_{r_t^-}^{r_t^+} dr \sqrt{-\frac{V(r; M, Q, M \mu, \bar{\eta})}{h(r)^2}} = \left( n + \frac{1}{2} \right) \pi \quad ; \quad n = 0, 1, 2, \ldots \]  

(3.18)

Using the dimensionless relations [see eqs. (3.9) and (3.14)]

\[ M \mu = \sqrt{\bar{\eta}} \cdot \frac{54\sqrt{30}}{625} \cdot \left( \frac{M}{Q} \right)^6 \cdot (1 - \epsilon) \quad ; \quad \epsilon \geq 0 \]  

(3.19)

and

\[ r \equiv r_{\text{peak}} \cdot (1 + x) \]  

(3.20)

which define the auxiliary physical variables \( \{ \epsilon, x \} \), one can write the effective radial potential (3.8) of the composed black-hole-non-minimally-coupled-massive-scalar-field cloudy configurations in the dimensionless form\(^\text{15}\)

\[ M^2 \frac{V(r)}{[h(r)]^2} = \bar{\eta} \cdot \frac{34992Q^2}{3125(25Q^2 - 21M^2)} \left( \frac{M}{Q} \right)^{12} \left( -\epsilon + 9 \cdot x^2 \right) \]

\[ \cdot \left\{ 1 + O \left[ \frac{x}{25(Q/M)^2 - 21}; \frac{x^3}{[25(Q/M)^2 - 21]^2} \right] \right\} \]  

\[(3.21)\]

\(^\text{15}\)Here we have used the assumptions \( |x| \ll 25(Q/M)^2 - 25 \) and \( \epsilon \ll [25(Q/M)^2 - 25]^2 \). Taking cognizance of eqs. (3.23), (3.24), and (3.25), one finds that these assumptions correspond to the large-mass requirement \( M\mu \gg [25(Q/M)^2 - 25]^{3/2} \).
where we have used here the functional relation [see eqs. (2.5), (3.9), and (3.20)]

\[ h(r) = \frac{25Q^2 - 21M^2}{25Q^2} \cdot \left\{ 1 + O\left[ \frac{x}{25(Q/M)^2 - 21} \right] \right\}. \quad (3.22) \]

Substituting eqs. (3.9), (3.20), and (3.21) into eq. (3.18) and defining

\[ z = \frac{3}{\sqrt{\epsilon}} \cdot x, \quad (3.23) \]

one obtains the characteristic WKB resonance condition

\[ \epsilon \cdot \frac{12 \sqrt{15}}{25} \left( \frac{M}{Q} \right)^4 \sqrt[4]{\frac{\bar{\eta}Q^2}{25Q^2 - 21M^2}} \int_{-1}^{1} dz \sqrt{1 - z^2} = \left( n + \frac{1}{2} \right) \cdot \pi \quad ; \quad n = 0, 1, 2, \ldots \quad (3.24) \]

for the composed black-hole-massive-field system, which yields the discrete resonance spectrum\(^{16}\)

\[ \epsilon(Q/M, \bar{\eta}) = \frac{1}{\sqrt{\bar{\eta}}} \cdot \frac{25}{6\sqrt{15}} \left( \frac{Q}{M} \right)^4 \sqrt[4]{\frac{25Q^2 - 21M^2}{Q^2}} \cdot \left( n + \frac{1}{2} \right) \quad ; \quad n = 0, 1, 2, \ldots \quad (3.25) \]

Interestingly, and most importantly for our analysis, one finds from (3.25) the characteristic relation \( \epsilon \ll 1 \) in the large-coupling \( \bar{\eta} \gg 1 \) regime.

Substituting the analytically derived relation (3.25) into (3.19), one obtains the large-mass (large-coupling) resonance formula

\[ \sqrt{\bar{\eta}(Q/M, M\mu; n)} = \frac{625}{54\sqrt{30}} \left( \frac{Q}{M} \right)^6 \cdot M\mu \quad (3.26) \]

\[ + \frac{25}{6\sqrt{15}} \left( \frac{Q}{M} \right)^4 \sqrt[4]{\frac{25Q^2 - 21M^2}{Q^2}} \cdot \left( n + \frac{1}{2} \right) \quad ; \quad n = 0, 1, 2, \ldots \]

for the composed Reissner-Nordström-black-hole-nonminimally-coupled-massive-scalar-field cloudy configurations.

4 The effective radial widths of the supported scalar clouds

In the present section we shall determine the effective widths of supported massive scalar field configurations in the charged Reissner-Nordström black-hole spacetime (2.4). In particular, we shall explicitly prove that, in the dimensionless large-mass \( M\mu \gg 1 \) regime, the supported scalar clouds can be made arbitrarily thin.

The effective widths of the supported matter configurations in the charged black-hole spacetime are determined by the classically allowed radial region

\[ \Delta r(Q/M, M\mu) \equiv r_{t+} - r_{t-} \quad (4.1) \]

\(^{16}\)Here we have used the simple integral relation \( \int_{-1}^{1} dz \sqrt{1 - z^2} = \pi/2. \)
of the effective binding potential (3.8), which characterizes the composed black-hole-
massive-scalar-field system (2.1). Taking cognizance of eqs. (3.9), (3.20), and (3.23) with
\[ \Delta z = 1 - (-1) = 2 \] [see eq. (3.24)], one finds the remarkably simple functional expression
\[
\frac{\Delta r(Q/M, M \mu)}{M} = \frac{10}{9} \left( \frac{Q}{M} \right)^2 \cdot \sqrt{\epsilon}
\] (4.2)
for the effective widths of the scalar clouds in the charged Reissner-Nordström black-hole spacetime (2.4).

Substituting the analytically derived functional relation (3.25) into eq. (4.2), one finds
the dimensionless expression
\[
\frac{\Delta r(Q/M, M \mu)}{M} = \frac{1}{\bar{\eta}^{\frac{1}{4}}} \cdot \frac{50}{6\sqrt{15}} \left( \frac{Q}{M} \right)^4 \left( \frac{25Q^2 - 21M^2}{Q^2} \right)^{\frac{1}{2}} \cdot \sqrt{n + \frac{1}{2}}
\] (4.3)
for the effective widths of the supported massive scalar field configurations, which in the
large-mass regime (3.1) can be expressed in the form [see eq. (3.26)]
\[
\frac{\Delta r(Q/M, M \mu)}{M} = \frac{Q}{M} \cdot \frac{2^\frac{3}{4}}{3} \left( \frac{25Q^2 - 21M^2}{Q^2} \right)^{\frac{1}{4}} \cdot \sqrt{n + \frac{1}{2}} \cdot \frac{1}{\sqrt{M \mu}}.
\] (4.4)

From eq. (4.4) one learns that, in the dimensionless large-mass \( M \mu \gg 1 \) regime, the sup-
ported massive scalar clouds in the charged Reissner-Nordström black-hole spacetime (2.4)
can be made arbitrarily thin.\(^{17}\)

5 Penetration depth of the massive scalar field into the classically for-
bidden region

In the present section we shall analyze the spatial behavior of the non-minimally coupled
massive scalar field configurations outside the narrow radial interval (4.1) [see also (4.4)]
of the classically allowed region.

It is important to stress the fact that the scalar field is not strictly zero outside the
classically allowed region (4.1). However, as we shall now demonstrate explicitly, the
effective penetration depth of the scalar eigenfunction into the classically forbidden region
becomes infinitesimally small in the large-mass \( M \mu \gg 1 \) regime (3.1).

In particular, according to the standard WKB analysis [38–40], the radial function
that characterizes the thin massive scalar field configurations is exponentially suppressed
\[
\psi_{WKB}(r; M, Q, \mu, \bar{\eta}) \sim e^{-\int \sqrt{V(y)} dy},
\] (5.1)
in the classically forbidden region outside the narrow radial interval (4.1). Interestingly,
and most importantly for our analysis, one finds the relation [see eqs. (3.8) and (3.26)]
\[
\sqrt{V} \propto M \mu \gg 1
\] (5.2)
\(^{17}\)It should be emphasized that the massive scalar field is exponentially suppressed in the classically
forbidden region outside the narrow radial interval (4.4).
in the large-mass regime (3.1). Thus, the effective penetration depth of the WKB scalar eigenfunction into the classically forbidden region [that is, into the region outside the classically allowed narrow radial interval (4.1)] scales as $1/\mu$ and therefore becomes infinitesimally small in the large mass $M\mu \gg 1$ regime (3.1) that we consider in the present paper.

For example, for a Reissner-Nordström black hole with $Q/M = 0.99$ supporting a non-minimally coupled massive scalar field with $M\mu = 100$ one finds from eqs. (3.5), (3.8), (3.26), and (5.1) the extremely small ratio $\psi(r = r_+)/\psi(r = r_{\text{peak}}) \simeq 1.2 \times 10^{-219}$ [It is important to note that this dimensionless ratio becomes even smaller for larger field masses. For example, for a supported massive scalar field with $M\mu = 1000$ one finds the characteristic small ratio $\psi(r = r_+)/\psi(r = r_{\text{peak}}) \simeq 8.7 \times 10^{-2190}$].

6 Summary and discussion

Recent studies of the composed Einstein-Gauss-Bonnet-scalar field theory have revealed the physically intriguing fact that asymptotically flat black holes with spatially regular horizons can be spontaneously scalarized [10–24].

Motivated by this highly interesting observation, we have studied, using analytical techniques, the physical and mathematical properties of static matter configurations (linearized scalar clouds) which are made of massive scalar fields with a direct non-minimal coupling to the Gauss-Bonnet curvature invariant $G$ of a central supporting charged Reissner-Nordström black hole.

The main results derived in this paper and their physical implications are as follows:

1. We have derived the charge-dependent upper bound (3.14) on the allowed proper masses of the non-minimally coupled scalar fields that can be supported in the curved spacetimes of charged Reissner-Nordström black holes.

2. It has been explicitly proved that, in the highly charged $Q/M \geq (Q/M)_{\text{crit}} = \sqrt{21}/5$ regime [see eq. (3.10)], the addition of a mass term to the supported scalar fields allows the existence of thin static scalar shells that are non-minimally coupled to the Gauss-Bonnet curvature invariant and hover a finite proper distance above the horizon of the central charged black hole. In particular, the supported thin shells are characterized by the dimensionless relation [see eq. (3.9)]

$$\frac{r_{\text{peak}}}{M} = 5\left(\frac{Q}{M}\right)^2 > \frac{r_+}{M} \quad \text{for} \quad \frac{Q}{M} > \left(\frac{Q}{M}\right)_{\text{crit}} = \frac{\sqrt{21}}{5}. \quad (6.1)$$

3. It has been shown that the supported scalar clouds are characterized by the effective dimensionless widths\(^{18}\)

$$\frac{\Delta r(Q/M, M\mu)}{M} = \frac{Q}{M} \cdot \frac{2\frac{3}{2}}{3} \left(\frac{25Q^2}{Q^2} - 21M^2\right)^{\frac{1}{4}} \cdot \frac{1}{\sqrt{M\mu}}. \quad (6.2)$$

\(^{18}\)Here we have substituted $n = 0$ in eq. (4.4) for the fundamental mode of the composed black-hole-massive-scalar-field system.
Intriguingly, the analytically derived functional expression (6.2) implies that the supported scalar configurations, which are made of non-minimally coupled massive scalar fields, can be made arbitrarily thin in the dimensionless large-mass $M\mu \gg 1$ regime.

4. Using a WKB analysis, we have derived the remarkably compact analytical resonance formula

$$\sqrt{\eta}(Q/M,M\mu) = \frac{625}{54\sqrt{30}} \left(\frac{Q}{M}\right)^6 \cdot M\mu + \frac{25}{12\sqrt{15}} \left(\frac{Q}{M}\right)^4 \sqrt{\frac{25Q^2 - 21M^2}{Q^2}}$$

(6.3)

for the critical existence-line that characterizes the composed Reissner-Nordström-black-hole-nonminimally-coupled-linearized-massive-scalar-field cloudy configurations in the dimensionless large-mass (large-coupling) $M\mu \gg 1$ regime.

Finally, it is worth emphasizing the fact that the analytically derived critical existence-line (6.3) marks, in the dimensionless large-mass regime (3.1), the sharp boundary between bald black-hole solutions of the Einstein-Maxwell-Gauss-Bonnet-massive-scalar-field theory (2.1) and hairy (scalarized) black-hole spacetimes that characterize the composed physical system.

Acknowledgments

This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] R. Ruffini and J.A. Wheeler, *Introducing the black hole*, Phys. Today 24 (1971) 30.

[2] J.D. Bekenstein, *Black hole hair: 25 years after*, in the 2nd International Sakharov Conference on Physics, (1996), [gr-qc/9605059](https://arxiv.org/abs/gr-qc/9605059) [insPIRE].

[3] J.D. Bekenstein, *Nonexistence of baryon number for static black holes*, Phys. Rev. D 5 (1972) 1239 [insPIRE].

[4] T.P. Sotiriou, *Black holes and scalar fields*, Class. Quant. Grav. 32 (2015) 214002 [arXiv:1505.02248] [insPIRE].

[5] C.A.R. Herdeiro and E. Radu, *Asymptotically flat black holes with scalar hair: a review*, Int. J. Mod. Phys. D 24 (2015) 1542014 [arXiv:1504.08209] [insPIRE].

[6] T.P. Sotiriou and V. Faraoni, *Black holes in scalar-tensor gravity*, Phys. Rev. Lett. 108 (2012) 081103 [arXiv:1109.6324] [insPIRE].

---

$^{19}$Here we have substituted $n = 0$ in the analytically derived resonance spectrum (3.26) for the fundamental mode of the composed black-hole-massive-scalar-field system.
[7] A.E. Mayo and J.D. Bekenstein, No hair for spherical black holes: Charged and nonminimally coupled scalar field with selfinteraction, Phys. Rev. D 54 (1996) 5059 [gr-qc/9602057] [inSPIRE].

[8] S. Hod, No nonminimally coupled massless scalar hair for spherically symmetric neutral black holes, Phys. Lett. B 771 (2017) 521 [arXiv:1911.08371] [inSPIRE].

[9] S. Hod, No hair for spherically symmetric neutral black holes: Nonminimally coupled massive scalar fields, Phys. Rev. D 96 (2017) 124037 [arXiv:2002.05903] [inSPIRE].

[10] T.P. Sotiriou and S.-Y. Zhou, Black hole hair in generalized scalar-tensor gravity, Phys. Rev. Lett. 112 (2014) 251102 [arXiv:1312.3622] [inSPIRE].

[11] T.P. Sotiriou and S.-Y. Zhou, Black hole hair in generalized scalar-tensor gravity: An explicit example, Phys. Rev. D 90 (2014) 124063 [arXiv:1408.1698] [inSPIRE].

[12] T.P. Sotiriou, Gravity and scalar fields, Lect. Notes Phys. 892 (2015) 3 [arXiv:1404.2955] [inSPIRE].

[13] D.D. Doneva and S.S. Yazadjiev, New Gauss-Bonnet black holes with curvature-induced scalarization in extended scalar-tensor theories, Phys. Rev. Lett. 120 (2018) 131103 [arXiv:1711.01187] [inSPIRE].

[14] H.O. Silva, J. Sakstein, L. Gualtieri, T.P. Sotiriou and E. Berti, Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling, Phys. Rev. Lett. 120 (2018) 131104 [arXiv:1711.02080] [inSPIRE].

[15] P.V.P. Cunha, C.A.R. Herdeiro and E. Radu, Spontaneously scalarized Kerr black holes in extended scalar-tensor-Gauss-Bonnet gravity, Phys. Rev. Lett. 123 (2019) 011101 [arXiv:1904.09997] [inSPIRE].

[16] A. Dima, E. Barausse, N. Franchini and T.P. Sotiriou, Spin-induced black hole spontaneous scalarization, Phys. Rev. Lett. 125 (2020) 231101 [arXiv:2006.03095] [inSPIRE].

[17] S. Hod, Spontaneous scalarization of Gauss-Bonnet black holes: analytic treatment in the linearized regime, Phys. Rev. D 100 (2019) 064039 [arXiv:1912.07630] [inSPIRE].

[18] S. Hod, Gauss-Bonnet black holes supporting massive scalar field configurations: the large-mass regime, Eur. Phys. J. C 79 (2019) 966 [arXiv:2101.02219] [inSPIRE].

[19] S. Hod, Onset of spontaneous scalarization in spinning Gauss-Bonnet black holes, Phys. Rev. D 102 (2020) 084060 [arXiv:2006.09399] [inSPIRE].

[20] D.D. Doneva, L.G. Collodel, C.J. Krüger and S.S. Yazadjiev, Black hole scalarization induced by the spin: 2+1 time evolution, Phys. Rev. D 102 (2020) 104027 [arXiv:2008.07391] [inSPIRE].

[21] C.A.R. Herdeiro, E. Radu, H.O. Silva, T.P. Sotiriou and N. Yunes, Spin-induced scalarized black holes, Phys. Rev. Lett. 126 (2021) 011103 [arXiv:2009.03904] [inSPIRE].

[22] E. Berti, L.G. Collodel, B. Kleihaus and J. Kunz, Spin-induced black-hole scalarization in Einstein-scalar-Gauss-Bonnet theory, Phys. Rev. Lett. 126 (2021) 011104 [arXiv:2009.03905] [inSPIRE].

[23] Y. Brihaye and B. Hartmann, Spontaneous scalarization of charged black holes at the approach to extremality, Phys. Lett. B 792 (2019) 244 [arXiv:1902.05760] [inSPIRE].

[24] C.A.R. Herdeiro, A.M. Pombo and E. Radu, Aspects of Gauss-Bonnet scalarisation of charged black holes, Universe 7 (2021) 483 [arXiv:2111.06442] [inSPIRE].
[25] C.A.R. Herdeiro, E. Radu, N. Sanchis-Gual and J.A. Font, *Spontaneous scalarization of charged black holes*, Phys. Rev. Lett. **121** (2018) 101102 [arXiv:1806.05190] [inSPIRE].

[26] P.G.S. Fernandes, C.A.R. Herdeiro, A.M. Pombo, E. Radu and N. Sanchis-Gual, *Spontaneous scalarisation of charged black holes: coupling dependence and dynamical features*, Class. Quant. Grav. **36** (2019) 134002 [Erratum ibid. **37** (2020) 049501] [arXiv:1902.05079] [inSPIRE].

[27] S. Hod, *Spontaneous scalarization of charged Reissner-Nordström black holes: analytic treatment along the existence line*, Phys. Lett. B **798** (2019) 135025 [arXiv:2002.01948] [inSPIRE].

[28] S. Hod, *Reissner-Nordström black holes supporting nonminimally coupled massive scalar field configurations*, Phys. Rev. D **101** (2020) 104025 [arXiv:2005.10268] [inSPIRE].

[29] M. Khodadi, A. Allahyari, S. Vagnozzi and D.F. Mota, *Black holes with scalar hair in light of the Event Horizon Telescope*, JCAP **09** (2020) 026 [arXiv:2005.05992] [inSPIRE].

[30] S. Hod, *Analytic treatment of near-extremal charged black holes supporting non-minimally coupled massless scalar clouds*, Eur. Phys. J. C **80** (2020) 1150 [inSPIRE].

[31] S. Hod, *Stationary scalar clouds around rotating black holes*, Phys. Rev. D **86** (2012) 104026 [Erratum ibid. **86** (2012) 129902] [arXiv:1211.3202] [inSPIRE].

[32] S. Hod, *Stationary resonances of rapidly-rotating Kerr black holes*, Eur. Phys. J. C **73** (2013) 2378 [arXiv:1311.5298] [inSPIRE].

[33] S. Hod, *Kerr-Newman black holes with stationary charged scalar clouds*, Phys. Rev. D **90** (2014) 024051 [arXiv:1406.1179] [inSPIRE].

[34] C.A.R. Herdeiro and E. Radu, *Kerr black holes with scalar hair*, Phys. Rev. Lett. **112** (2014) 221101 [arXiv:1403.2757] [inSPIRE].

[35] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman, San Francisco U.S.A. (1973).

[36] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford University Press, Oxford U.K. (1983).

[37] C.F.B. Macedo, J. Sakstein, L. Gualtieri, H.O. Silva and T.P. Sotiriou, *Self-interactions and Spontaneous Black Hole Scalarization*, Phys. Rev. D **99** (2019) 104041 [arXiv:1903.06784] [inSPIRE].

[38] L.D. Landau and E.M. Lifshitz, *Quantum mechanics*, 3rd edition, Pergamon, U.K. (1977), see chapter VII.

[39] J. Heading, *An introduction to phase integral methods*, Wiley, New York U.S.A. (1962).

[40] C.M. Bender and S.A. Orszag, *Advanced mathematical methods for scientists and engineers*, McGraw-Hill, New York U.S.A. (1978).