Breached pair superfluidity: a brief review

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Interior gap superfluidity was introduced together with Frank Wilczek. Later on together with our collaborators, we generalized this new possibility of superfluidity to a broader concept, breached pair superfluidity. In the occasion to celebrate Professor Frank Wilczek’s seventieth birthday and his productive career in several major areas in physics, I dedicate this note to recall the exciting times of developing this idea, the main aspects of the proposed phase, and the discussion on its stability condition.

1. Introduction

Modern history of physics has proudly recorded a group of great luminaries having broadly impacted on the areas of high energy, statistical, and condensed matter physics. Among those who are still live and active in today’s physics, many would immediately point out Frank Wilczek in this class. I was lucky to have had the privilege of working with him—during my postdoctoral times at MIT—on one of the topics of great interest in such an interdisciplinary area. It was about exotic superfluidity arising from the effect of mismatched Fermi surfaces.

The phenomenon of mismatched Fermi surfaces occurs in a number of physical systems. It happens in electronic superconductors when an in-plane magnetic field is applied or when superconductivity and ferromagnetism coexist. It should happen in dense quantum chromodynamics (QCD) matter—arguably the inside of neutron stars or a quark-gluon plasma realized in the heavy ion collision experiments at CERN. It can be artificially tuned to occur, not as a problem but as a new parameter regime for possible new effects, in cold atomic gases through the mixtures of different atomic species whose populations are separately controlled and conserved.

Understanding the Zeeman effect in the electronic superconductors was a key motivation in the early proposal by Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) to consider the instability of Bardeen-Cooper-Schrieffer (BCS) phase towards alternative energetically favored states to accommodate the Fermi level difference due to spin polarization. This class of states are now known as FFLO phases, in which each Cooper pair carries a finite center-of-mass momentum whose scale is set by the Fermi surface difference, i.e. $Q \sim 1.25p_F$ by mean field theory.

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Frank Wilczek and his collaborators are among the early pioneers to initiate the field of color superconductivity. The condensed matter concept of FFLO phase was extended to the understanding of the phase diagram of dense QCD matter, for which crystalline color superconductivity was proposed. The study of quark matter at finite density turns into a new condensed matter physics problem. A fundamental change in comparison is the role of the usual Coulomb interaction between electrons mediated by U(1) electromagnetic gauge bosons (photons) being replaced by the strong interaction mediated by SU(3) color gauge bosons known as gluons. And there are more degrees of freedom and higher symmetry involved, such as a variety of Cooper pairs with several flavors and colors present at the appropriate density level (for that matter, also the energy scale).

In high density QCD, the phenomenon of mismatched Fermi surfaces is guaranteed to exist by two facts combined: six quark flavors carry two different fractions of electrical charge and each flavor has its own unique mass, hence different from each other. Let us illustrate this by considering an intermediate high density quark matter with average chemical potential $\mu \sim 400 \text{MeV}$, which probably corresponds to the nucleon density in the core of neutron stars. At such an energy scale, the charm, top and bottom flavors are not present due to their individual masses being significantly high above. The QCD matter then is made of (approximately) massless up and down quarks and massive strange quarks. The strange quark has a mass $M_s \sim 300 \text{MeV}$, comparable to the chemical potential scale under consideration. It plays a crucial role, as we see below, to cause a mismatch in flavor chemical potentials. Electric neutrality requires a nonzero density of electrons present, estimated to be $\mu_e \sim 50 \text{MeV}$. In many body physics, $\mu$ and $\mu_e$ are treated as Lagrangian multiplier to enforce the conserved quantum numbers of quark flavor and electric charge, respectively. A simple algebra shows that they are related to the individual flavor chemical potentials and Fermi momenta as follows:

$$\begin{align*}
\mu_u &= \mu - \frac{2}{3} \mu_e = 367 \text{MeV}, \\
\mu_d &= \mu + \frac{1}{3} \mu_e = 417 \text{MeV}, \\
\mu_s &= \mu + \frac{1}{3} \mu_e = 417 \text{MeV}, \\
\mu_F &= \sqrt{\mu^2_s - M_s^2} = 289 \text{MeV}.
\end{align*}$$

The above crude estimate shows that the three flavors would have three different Fermi surfaces. An immediate consequence is that the color-flavor-locking superconductivity would be unstable towards crystalline color superconductivity, the manifestation of FFLO in high density QCD. This was analyzed in the context of color superconductivity by Alford, Bowers, and Rajagopal (for reviews, see). The BCS equivalent in this context is the color-flavor-locking (CFL) superconductivity with equal population pairing, proposed for the extreme high density limit where all flavors would enjoy approximately the same Fermi surfaces with chemical potential difference negligible at such a high-density scale. It was discussed in 1970s and 1980s (see Bailin and Love) that dense quark matter might become superfluid.
Frank, together with his collaborators, pioneered in revitalizing the idea of color superconductivity. He also knew well of the FFLO idea from his close collaborators and former students who extended the FFLO concept to the intermediate density regime of quark matter to introduce crystalline color superconductivity. I was fortunate to be exposed to this interesting interface between condensed matter and high energy physics, immediately after I went to MIT as a postdoctoral fellow with Frank in 2001. I quickly learned a great deal of color superconductivity and QCD matter phase diagram from him, and we realized that some of the ideas could be quantum simulated by ultra-cold atomic gases.

Progressive developments in ultra-cold gases have revitalized interest in some basic qualitative questions of quantum many-body theory, because they promise to make a wide variety of conceptually interesting parameter regimes, which might previously have seemed academic or excessively special, experimentally accessible. Advances in the spatiotemporal control and readout of ultra-cold quantum gases are rapidly expanding the scientific range of current and near-future experiments in this growing field. Emergent tools of quantum control in AMO physics represent an important bridge to the exploration of complex many-body problems based on fully understood few-particle subsystems. One of the most remarkable is the unprecedented flexibility to directly control the interaction between the selected atomic internal states—which play the role of “spins” or “flavors”—through the technique known as Feshbach resonance. It has been demonstrated that interaction between fermionic atoms can be precisely tuned, by dialing an external magnetic field, from attractive to repulsive, from weak to infinitely strong. Trapping cold atoms and molecules in optical lattices brings a whole new set of quantum manipulations. Among them, the band masses and interaction between atoms on the same or neighboring sites are all becoming experimentally controlled with unprecedented flexibilities. Novel forms in geometry or (dynamic) Floquet engineering have been demonstrated and used to explore a wide range of interesting quantum states of matter, from simulating topological insulators and fractional quantum Hall effects to exploring antiferromagnetism and orbital superfluidity.

2. Aspects of breached pair superfluidity

The FFLO actually represents a class of superconducting states, because the FFLO order parameter in principle can be a superposition of Cooper pairs condensed at a set of different finite momenta,

\[ \Delta(r) \equiv \langle \psi_\sigma(r) \psi_{\sigma'}(r) \epsilon_{\sigma\sigma'} \rangle \sim \sum_{\{Q_\alpha\}} \alpha e^{iQ_\alpha \cdot r} \Delta_\alpha \]

where I assume a fermion model of two spins \( \sigma = \uparrow, \downarrow \), \( \epsilon_{\sigma\sigma'} \) is the antisymmetric tensor, and the summation over \( \alpha \) represents a set of Cooper pair momenta under choice to minimize the postulated ground state energy. Hence, it yields a rich phase diagram of different ordering crystalline structures. In other words, generic
FFLO phases break both spatial translational and rotational symmetries in addition to the phase U(1) rotation symmetry. A FFLO pairing order parameter with a single momentum component is an exception as it would be still translationally invariant. The rich phase diagram however poses an experimental challenge to realize, observe and identify the nature and symmetry of the actual FFLO phase being realized, in part due to an abundance of competing FFLO phases which differ in crystalline symmetry. The energy difference between similar but symmetrically different FFLO phases often is very small, from variational calculations. The overall condensation energy saved in FFLO scenario is also exponentially smaller than that of the corresponding BCS phase if the condition of Fermi surface match restores while everything else being kept the same. The reason is quite easy to understand. Unlike the BCS case, FFLO pairing only takes place in certain spots of the Fermi surfaces, so the density of states of fermionic particle states participating in pairing and condensation is fractional, not a whole shell along the Fermi surface in a 3D setting. The energy gap and critical temperature both depend on the density of states of paired fermions exponentially, if the BCS mean field theory is taken as guidance.

The new experimental regime in cold atomic gases motivated Frank and me to think whether it was possible to have some homogeneous phase — on this regard like the BCS but not like the FFLO—that can be energetically better than the non-homogeneous FFLO. Our initial proposal, as a phenomenological trial wavefunction for the superfluid ground state was interior gap superfluidity \(^\text{28}\). Later on, we realize this represents just one special limit of a more general possibility \(^\text{29-31}\). In the following, I will first review the general state and then point the interior gap as a special situation of the general scenario.

Let us use an example of model Hamiltonian to introduce the breached pair superfluid (BP) state \(^\text{29,31}\)

\[
H = \sum_{p\sigma} \left[ \frac{p^2}{2m_{\sigma}} - \mu_{\sigma} \right] \psi_{p\sigma}^\dagger \psi_{p\sigma} + \sum_{pp'q} V_{pp'}(q) \psi_{q+p}^\dagger \psi_{q}^\dagger \psi_{q-p} \psi_{q-p'},
\]

where \(\sigma = \{\uparrow, \downarrow\}\) (or \{A, B\}) is the spin indices, \(m_{\sigma}\) are the masses of fermions, and \(\mu_{\sigma}\) are the chemical potentials to enforce the condition of spin population imbalance, say \(N_{\uparrow} < N_{\downarrow}\). Unlike previous studies, a key additional feature that I believe was first explored by Frank and me \(^\text{28}\) is the introduction of mass imbalance \(m_{\uparrow} \neq m_{\downarrow}\). It was found that the larger the mass ratio is, the better the BP state is favored. As a concrete example, assume that the spin down species is much heavier than the spin up species \((m_{\uparrow} \gg m_{\downarrow})\) without loss of generality. In order for the BP state to be energetically favorable, the interaction \(V_{pp'}(q)\) has to be strongly momentum dependent. (A contact \(\delta\)-like interaction in the real space corresponds to a \(q\)-independent constant in momentum space by Fourier transformation, which does not work.) Two types of interaction, each of special interest, were found to
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work\textsuperscript{31} One is of the (modified) BCS type of interaction

\[ V_{pp'}^{I}(q) = \begin{cases} g \text{ (constant)} & \text{if } p, p' \in [p_F^\sigma - \lambda/2, p_F^\sigma + \lambda/2], \\ 0. \end{cases} \tag{4} \]

where \( p_F^\sigma \) are the Fermi momentum for the two spins (\( \sigma = \uparrow, \downarrow \)) and the non-vanishing condition in the above is for momenta sitting within the stripes of width \( \lambda \) around each of two Fermi surfaces. Here is a subtle point—\( \lambda \) behaves like the ultraviolet cutoff, in the sense \( \lambda \gg \Delta/v_F \) (where \( \Delta \) is the energy gap due to Cooper pairing), but should be kept smaller than the Fermi surface mismatch, \( \lambda < \delta p_F = p_F^\uparrow - p_F^\downarrow \). This type of model was used in the calculation of our 2003 PRL\textsuperscript{28} when first introducing the concept of interior gap superfluidity. Such a condition was briefly indicated in a sentence put in parentheses under Eq. (1) in Ref.\textsuperscript{28} The work by Wu and Yip\textsuperscript{32} was on a qualitatively different model— their interaction is point-like in the position space, which transforms into a uniform constant in momentum space with a momentum cut-off scale. In other words, the type of interaction considered in Wu-Yip paper is not what used in our model calculation. This was the source of early debate and confusion. It was clarified in our later 2005 PRL work with Forbes et al.\textsuperscript{31}

Another type is of momentum structure falling off as a Gaussian\textsuperscript{31}

\[ V_{pp'}^{II}(q) = ge^{-q^2/\lambda^2}, \quad \forall p, p'. \tag{5} \]

Similar to type I interaction in a qualitative sense, for the BP phase to prevail, the falling off momentum scale \( \lambda \) needs to be within the Fermi momentum difference between the two spins.

The BP state is similar to the BCS in terms of symmetry of the superfluid order parameter—Cooper pairs condensed at zero momentum and the state is uniform in position space. Then, how does the state accommodate the spin population difference \( N_\uparrow < N_\downarrow \)? This question can be answered precisely by contrasting the many-body wavefunctions of the well-known BCS state

\[ |\Psi_{BCS}\rangle = \prod_p (u_p \psi^\uparrow_p \psi^\downarrow_p + v_p \psi^\downarrow_p \psi^\uparrow_p) |0\rangle, \tag{6} \]

and the new BP state,

\[ |\Psi_{BP}\rangle = \prod_{p : p < p_\Delta^-} (u_p + v_p \psi^\uparrow_{p-p_\Delta^-} \psi_{p_\Delta^-}^{\downarrow}) \prod_{p : p \in [p_\Delta^-, p_\Delta^+]} \psi^\downarrow_p \prod_{p : p > p_\Delta^+} (u_p + v_p \psi^\uparrow_{p-p_\Delta^+} \psi_{p_\Delta^+}^{\downarrow}) |0\rangle. \tag{7} \]

Momentum-space phase separation. The BP state accomplishes the population imbalance by a mechanism that I would refer to as momentum-space phase separation of superfluid and normal components. The momentum region of \( p : p \in [p_\Delta^-, p_\Delta^+] \) is a normal component breach: it is filled by only one fermion species, namely, the majority species (spin \( \downarrow \) in our example here), hence there is no pairing. Superfluid components exist, in a way similar to what happens in BCS state, in the momentum regions outside this “normal” breach.
The BP variational wavefunction consists of the usual coherence variables \((u_p, v_p)\) as well as additional variational parameters \(p^\pm_\Delta\), to be determined by variationally minimizing the ground state energy with fixed chemical potentials and solving the energy gap equation self-consistently. The following summarizes the results obtained in this manner,

\[
\left\{ \begin{array}{l}
|u_p|^2 \\
|v_p|^2
\end{array} \right\} = \frac{1}{2} \left( 1 \pm \frac{\epsilon_p^+}{\sqrt{\epsilon_p^{+2} + |\Delta|^2}} \right), \quad \epsilon_p^\pm = \frac{\epsilon_\uparrow(p) \pm \epsilon_\downarrow(p)}{2}
\]

with \(\epsilon_\sigma(p) = p^2/2m_\sigma - \mu_\sigma\). The Bogoliubov quasi-particle spectrum in the BP superconducting state takes a different form than in the BCS state, shown as follows:

\[
E_p^\pm = \epsilon_p \pm \sqrt{\epsilon_p^{+2} + |\Delta|^2}.
\]

Taking the usual notation, \(\Delta\) denotes the excitation energy gap appearing in the superconducting state. What is special in the BP excitation spectrum is the presence of gapless Fermi surface while being superconducting (or superfluid for charge neutral fermions). The gapless surfaces are the zero mode solution of the quasi-particle excitation in momentum space,

\[
E_p^+ E_p^- = 0 \quad \Rightarrow \quad p = p^\pm_\Delta.
\]

The peculiar features of the BP state are illustrated in Fig. [1]. The momentum dependence of the fermion occupation numbers, \(n_\sigma(p)\), tells two important points. First, it shows where the pairing spectral weight is taking place most. That is the region where \(n_\sigma(p)\) deviates from 1 most due to fermion pairing. Note this is the ground state property \((T = 0)\) under discussion. In the example we adopt here \((m_\uparrow \ll m_\downarrow\text{ and } p_{F\uparrow}^\downarrow < p_{F\downarrow}^\uparrow)\), Cooper pairing occurs mostly around the smaller Fermi surface, hence initially called “interior gap” superfluidity\(^{28}\). If one had considered the opposite case (still \(m_\uparrow \ll m_\downarrow\text{ but } p_{F\uparrow}^\downarrow > p_{F\downarrow}^\uparrow)\), one would find fermion pairing should occur most along the larger Fermi surface, and then one would see something like “exterior gap”\(^{29}\). Second, the occupation numbers tell where the breach (a region of no Cooper pairing) is precisely located. It is the place in momentum where the minority species is 0 and the majority is 1. In both “interior” and “exterior” gap case, the breach is the hallmark of the new superfluid\(^{29,31}\).

Let us summarize the characteristic features of the BP phase:

(a) It realizes a momentum-space phase separation to accommodate the population imbalance between two spins.
(b) It exhibits coexisting superfluid and normal Fermi liquid components in the quantum ground state through such a momentum-space phase separation.
(c) It is a superfluid with a full surface of gapless quasiparticle excitations. In the case illustrated in Fig. [1] the BP phase has 2 gapless surfaces (BP2). At certain critical point, one gapless surface is found possible (BP1)\(^{29,33}\).
(d) Unlike the fully gapped BCS phase, the superfluid density is not equal to the total fermion density, but to the density of only those fermions filled outside the breach.
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Fig. 1. Comparison of BP and BCS states revealed from excitation spectra and fermion occupation numbers. In this illustration of example, we assume the condition $m_\uparrow \ll m_\downarrow$ and $N_\uparrow < N_\downarrow$ with two species fermions denoted as $A, B$ (or $\uparrow, \downarrow$ exchangeably). The region bounded between $p_\Delta^\pm$ is a breach in the momentum space, where only the majority species is present and no pairing takes place. Within the breach, the occupation numbers are 1 and 0 for the majority and minority fermion, respectively (right panel).

(e) Unlike the FFLO phase, it does not spontaneously break the translational and rotational symmetries.

3. Remarks

Since the initial proposal, the concept of interior gap and BP phase has received much attention and debates regarding its stability and competition with other viable phases. The confusion has something to do with that the BP phase we proposed is related to one of the solutions to the superconducting gap equation studied by Sarma in early years, which was correctly realized as an unstable solution at his analysis. (It is sometime jointly called Sarma or BP phase in literature). The key point is that we introduced the BP phase, despite some similarity to Sarma’s state in the structure of wavefunction, with a large mass ratio. This is one of the key ingredients to have a physically stable state, to be highlighted next.

Based on studies and discussions so far, our general conclusion is that the BP phase requires two crucial conditions (a) a relatively large mass ratio between the two fermion species and (b) strong momentum dependence of a two-body attractive
s-wave interaction. The latter is equivalent to requiring strong spatial dependence of the interaction in position space, by Fourier transformation. The examples we have found include the Gaussian type of long range interaction in position space or the type of interaction restricted to a narrow momentum strip as in the BCS model. The BP phase has been found as a competing phase in the phase diagram by other studies. An effective field theory approach by Son and Stephanov\[33\] showed a universal phase diagram of a homogenous gapless superfluid phase which corresponds to the BP phase. The quantum critical theory based on the renormalization group analysis by Yang and Sachdev\[34,35\] showed the gapless BP phase is stable in a 2D superfluid. Dynamical mean field theory by Dao et al\[40\] found that the attractive Hubbard model yields a polarized phase closely connected to the physics of the Sarma or BP phase (with two fermi surfaces) down to very low temperatures. Furthermore, Dukelsky et al obtained the phase diagram of an exact solvable model by using the algebraic Richardson-Gaudin techniques, which shows not only a breached pair (BP) phase but also another exotic phase that they dubbed “breached” LOFF phase.\[41\] This model generalizes what was introduced by me with Frank.\[28\]

The presence of gapless Fermi surfaces was found to manifest itself by some remarkable, unconventional aspects such as inducing a spatially oscillating potential between superfluid vortices, akin to the RKKY indirect-exchange interaction in non-magnetic metals.\[52\] It was remarked that the “interior gap”/BP superfluidity might be relevant to explain the experimental observation of unpaired electrons present in the heavy-fermion superconductor CeCoIn\[5\].\[43\] In the context of high density QCD, gapless color superconductivity—a phase related to the Sarma/BP—was proposed as a natural candidate for quark matter in the cores of compact stars at zero and finite temperature by Huang and Shovkovy through a detailed analysis\[44\] and their earlier paper.\[45\] For trapped cold ensembles of atoms, the effect of mass and population imbalance in gapless superfluidity was analyzed by Bedaque, Caldas, and Rupak\[46,47\] (where the authors did not consider momentum-dependent interaction) suggesting real-space phase separation of normal and superfluid components and by Yi and Duan and later by Lin with them\[48,49\] putting forward experimental signatures detecting breached pair phases with 1 or 2 Fermi surfaces (BP1 and BP2).

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