FUNDAMENTAL SOLUTIONS TO SOME ELLIPTIC EQUATIONS WITH DISCONTINUOUS SENIOR COEFFICIENTS AND AN INEQUALITY FOR THESE SOLUTIONS.

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Abstract. Let \( Lu := \nabla \cdot (a(x)\nabla u) = -\delta(x-y) \) in \( \mathbb{R}^3 \), \( 0 < c_1 \leq a(x) \leq c_2 \), \( a(x) \) is a piecewise-smooth function with the discontinuity surface \( S \) which is smooth. It is proved that in a neighborhood of \( S \) the behavior of the function \( u \) is given by the formula:

\[
u(x,y) = \begin{cases} (4\pi a_+)^{-1}[r_{xy}^{-1}bR^{-1}], & y_3 > 0, \\ (4\pi a_-)^{-1}[r_{xy}^{-1}-bR^{-1}], & y_3 < 0. \end{cases} \tag{\star}
\]

Here the local coordinate system is chosen in which the origin lies on \( S \), the plane \( x_3 = 0 \) is tangent to \( S \), \( a_+(a_-) \) is the limiting value of \( a(x) \) on \( S \) from the half-space \( x_3 > 0, \ (x_3 < 0) \), \( r_{xy} := |x-y|, \ R := \sqrt{\rho^2 + (|x_3| + |y_3|)^2}, \ \rho := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}, \ b := (a_+-a_-)/(a_++a_-). \) If \( S \) is the plane \( x_3 = 0 \) and \( a(x) = a_+ \) in \( x_3 > 0, \ a(x) = a_- \) in \( x_3 < 0 \), then (\star) is the global formula for \( u \) in \( \mathbb{R}^3 \). Inequality for the fundamental solution for small and large \( |x-y| \) follows from formula (\star).

1. INTRODUCTION.

There are many papers on the behavior, as \( x \to y \), of the fundamental solutions to the elliptic equations of the form

\[
Lu := \sum_{i,j=1}^{n} \partial_{ij} [a_{ij}(x)u_{ij}(x,y)] = -\delta(x-y) \text{ in } \mathbb{R}^n, \ u_{ij} := \frac{\partial u}{\partial x_j} = \partial_j u \tag{1.1}
\]

for smooth coefficients \( a_{ij} \). Methods of pseudo-differential operators theory give expansion in smoothness of the solution to (1). In [LSW] existence of the unique solution to (1) with the properties

\[
0 < c_1 r^{2-n} \leq u(x,y) \leq c_2 r^{2-n}, \ u \in H^1_{\text{loc}}(\mathbb{R}^n), \ r := |x-y|, \tag{1.2}
\]

is obtained under the assumption that \( a_{ij} \) are bounded real-valued measurable functions such that

\[
a_1 \sum_{i=1}^{n} t_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)t_i t_j \leq a_2 \sum_{i=1}^{n} t_i^2, \ a_1, a_2 = \text{const} > 0. \tag{1.3}
\]

Our purpose is to give an analytical formula for the fundamental solution of the basic model operator (1.1), namely the operator with

\[
a_{ij}(x) = \delta_{ij} a(x), \quad d(x) = \begin{cases} a_+, & x_3 > 0, \\ a_-, & x_3 < 0. \end{cases} \tag{1.4}
\]
Here $a_+$ and $a_-$ are positive constants,

$$
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j,
\end{cases}
$$

and $u(x, y)$ is the unique solution of the problem:

$$
Lu := \sum_{i=1}^{n} \partial_i (a(x)u_i) = -\delta(x-y) \text{ in } \mathbb{R}^n, 
$$

$$
[u]_S = 0, \quad [a(x)u_N]_S = 0, 
$$

the symbol $[u]_S$ denotes the jump of $u$ across $S$, that is,

$$
[u] = u_+ - u_-, \quad u_{\pm} := \lim_{\varepsilon \to 0} u(s \pm \varepsilon N),
$$

$s$ is a point on $S$, $N$ is the unit normal to $S$ directed along $x_3$, $u_N$ is the normal derivative on $S$, $S$ is the plane $x_3 = 0$, $[au_N] := a_+u_{N}^+ - a_-u_{N}^-$. 

Problem (1.5)-(1.6) is important in many applications and is called a transmission problem. The solution to (1.5)-(1.6) is sought in the class $H^1_{\text{loc}}(\mathbb{R}^n \setminus y) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, where $H^1 := W^{1,2}$ and $W^{\ell,p}_{\text{loc}}$ is the Sobolev space of functions whose distributional derivatives up to the order $\ell$ belong to $L^p_{\text{loc}}$. If

$$
a_{ij}(x) = \begin{cases} 
a_{ij}^+, & x_3 > 0, \\
a_{ij}^-, & x_3 < 0,
\end{cases} 
$$

and the constant matrices $a_{ij}^\pm$ are positive definite, then there exists an orthogonal coordinate transformation which reduces $a_{ij}^+$ to $\delta_{ij}$ and $a_{ij}^-$ to $\lambda_j \delta_{ij}$, $\lambda_j > 0$. We do not give the formula for $u(x, y)$ in this more general case.

Finally note that for discontinuous coefficients equation (1.5) is understood in the weak sense, namely as the identity:

$$
\int_{\mathbb{R}^n} a(x)u_i(x,y)\phi_i(x) \, dx = \phi(y),
$$

The identity (6) for $u \in H^1_{\text{loc}}(\mathbb{R}^n \setminus y) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ implies conditions (1.6).

Let $n = 3$. The formula for the solution to problem (1.4)-(1.6), or the equivalent problem (1.4), (1.7) is given in Theorem 1.1.

**Theorem 1.1.** The unique solution to problem (1.4)-(1.6) is:

$$
u(x, y) = \begin{cases} 
\frac{1}{4\pi a_+} \left[ \frac{b}{r} + \frac{b}{R} \right], & y_3 > 0, \\
\frac{1}{4\pi a_-} \left[ \frac{b}{r} - \frac{b}{R} \right], & y_3 < 0
\end{cases}
$$

where $R := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (|x_3| + |y_3|)^2}$. 

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Corollary 1.1. The following inequality holds for \( r \to 0 \):

\[
|u(x, y)| < c|x - y|^{-1},
\]

where the constant \( c > 0 \) does not depend on \( x \) and \( y \).

Thus, the fundamental solution of the equation (1.1) with discontinuous senior coefficients has a different representation than the fundamental equation for the similar operator with continuous coefficients, but satisfies similar inequality for small \(|x - y|\).

A formula, similar to (1.8) can be derived by the same method for \( n > 3 \) as well. Formula (1.8) allows one to get asymptotics of \( u(x, y) \) and of \( \nabla_x u(x, y) \) as \(|x - y| \to 0\). Such asymptotics are useful in the study of inverse problems for discontinuous media [3].

In section 2 we prove Theorem 1.1. In section 3 various generalizations and applications are discussed.

2. PROOF OF THEOREM 1.

The proof is given for \( n = 3 \), but it holds with obvious small changes for \( n > 3 \).

The idea of the proof is to take the Fourier transform of equation (1.5) with respect to the variables \( \hat{x} := (x_1, x_2) \), to solve the resulting problem for an ordinary differential equation analytically, and then to Fourier-invert the solution of this problem.

Let us go through the steps.

Step 1. Let \( y_1 = y_2 = 0 \) without loss of generality (since \( u \) is translation-invariant in the plane \((x_1, x_2)\)). Denote

\[
w(\xi, x_3, y_3) := \int_{\mathbb{R}^2} e^{i\xi \cdot \hat{x}} u(\hat{x}, x_3, y) d\hat{x}; \quad u = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} w e^{-i\xi \cdot x} d\xi.
\]

\( \xi := (\xi_1, \xi_2) \), \( \xi^2 = |\xi|^2 = \xi^2_1 + \xi^2_2 \). Denote \( w' := \frac{\partial w}{\partial x_3} \).

Let us Fourier-transform equation (1.5), with \( a(x) \) given in (1.4), and get

\[
w''(\xi, x_3, y_3) - \xi^2 w(\xi, x_3, y_3) = \begin{cases} -\frac{1}{a_+} \delta(x_3 - y_3), & x_3 > 0, \\ -\frac{1}{a_-} \delta(x_3 - y_3), & x_3 < 0, \end{cases}
\]

\[
w(\xi, +0, y_3) - w(\xi, -0, y_3) = 0, \quad a_+ w'(\xi, +0, y_3) - a_- w'(\xi, -0, y_3) = 0.
\]

In what follows we omit \( \xi \) in the variables of \( w \), and write \( w(x_3, y_3) \) for brevity. Thus, \( w \) solves problem (2.2)-(2.3) and satisfies the condition

\[
w(\pm \infty, y_3) = 0.
\]
Assume that \( y_3 \neq 0 \). Then problem (2.2)-(2.4) has a solution and this solution is unique. A lengthy but straightforward calculation yields the formula for \( w \):

\[
    w = \begin{cases} 
        \frac{\exp(-|\xi|x_3-y_3)}{2|\xi|a_+} + b \frac{\exp(-|\xi|(x_3+|y_3|))}{2|\xi|a_+}, & y_3 > 0 \\
        \frac{\exp(-|\xi||x_3-y_3|)}{2|\xi|a_-} - \frac{\exp(-|\xi|(x_3+|y_3|))}{2|\xi|a_-}, & y_3 < 0
    \end{cases}
\]

where \( b := \frac{a_+ - a_-}{a_+ + a_-} \), \( a_-, a_+ > 0 \). (2.5)

Step 2. The function \( u(x, y) \) is obtained from \( w \) by the second formula (2.1). Let us denote \( |\xi| := v, \rho := |\hat{x}| = \sqrt{x_1^2 + x_2^2} \), and remember that \( y_1 = y_2 = 0 \). Since \( w \) depends on \( |\xi| \) and does not depend on the angular variable, \( |\xi| := v \), we have

\[
    u = \frac{1}{(2\pi)^2} \int_0^\infty d\nu \int_0^{2\pi} e^{-iv\rho \cos \phi} w d\phi = \frac{1}{2\pi} \int_0^\infty ds \nu w J_0(\nu\rho)
\]

where \( J_0(x) \) is the Bessel function and we have used the known formula:

\[
    \frac{1}{2\pi} \int_0^{2\pi} e^{iv\rho \cos \phi} d\phi = J_0(v\rho).
\]

We need another well-known formula:

\[
    \int_0^\infty e^{-\nu t} J_0(\nu\rho) d\nu = \frac{1}{\sqrt{\rho^2 + t^2}}, \quad t > 0
\]

From (2.5), (2.7) and (2.8) we get (1.8) with \( y_1 = y_2 = 0 \). Therefore, recalling the translation invariance of \( u \) in the horizontal directions, we get (1.8).

Theorem 1.1 is proved. □

Remark 2.1 Note that the limits of \( u(x, y) \) as \( y_3 \rightarrow \pm 0 \) exist and are equal:

\[
    u(x, \hat{y}, +0) = u(x, \hat{y}, -0) = \frac{1}{2\pi\tau(a_+ + a_-)}.
\]

A result similar to (2.9) is mentioned in [K, p.318], however the argument [K] is not clear: the differentiation is done in the classical sense but the functions involved have no classical derivatives: they have a jump.

3. GENERALIZATIONS, APPLICATIONS.

This section contains some remarks.
Remark 3.1 First, note that if \( a(x) \) is a piecewise-smooth function with a smooth discontinuity surface \( S \), \( s \in S \), \( a_\pm = \lim_{s \to 0} a(s \pm \varepsilon N) \), where \( N \) is the exterior normal to \( S \) at the point \( s \), then the main term of the asymptotics of the fundamental solution \( u(x,y) \) in a neighborhood \( U_s \) of the point \( s \in S \) is given by formula (1.8) in which \( x, y \in U_s \). This follows from the fact that the main term in smoothness of the solution to an elliptic equation in \( U_s \) is the same as to the equation with constant coefficients which are limits of \( a(x) \) as \( x \to s \).

In our case, this “frozen-coefficients” model problem is given by equations (1.4)-(1.6). This argument shows that the same conclusion holds if the coefficient \( a(x) \) in \( \mathbb{R}^3 \) and in \( \mathbb{R}^3 \) is not smooth but just Lipschitz-continuous.

Remark 3.2 In principle, our method for calculation of \( u(x,y) \) for the model problem (1.4)-(1.6) is applicable for the model problem (1.4) with anisotropic matrix.

Remark 3.3 We only mention that our result concerning asymptotics of \( u(x,y) \) as \( |x-y| \to 0 \) for piecewise-smooth coefficients is applicable to inverse problems of geophysics and inverse scattering problems for acoustic and electromagnetic scattering by layered bodies.

For example, if the governing equation is [R, p.14]:

\[
\nabla \cdot [a(x)\nabla u] + k^2 q(x)u = \delta(x-y) \quad \text{in } \mathbb{R}^3,
\]

\(k = \text{const} > 0\), say \( k = 1\), \( q(x) = 1 + p(x)\), where \( p(x) \) is a compactly supported real-valued function, \( p(x) \in L^2_{\text{loc}}(\mathbb{R}^3)\), \( \text{supp}(p(x)) \subset \mathbb{R}^3 := \{ x : x_3 < 0 \} \), \( a(x) = 1 + A(x)\), where \( A(x) \) is compactly supported piecewise-smooth function with finitely many closed compact smooth surfaces \( S_j \subset \mathbb{R}^3 \) of discontinuity. Across these surfaces the transmission conditions (1.6) hold, and at infinity \( u \) satisfies the radiation condition. Then \( u(x,y) \) is uniquely determined.

An inverse problem is: given \( g(x) \) and \( u(x,y) \) for all \( x, y \in S := \{ x : x_3 = 0 \} \) and a fixed \( k = 1\), can one uniquely determine \( a(x)\), in particular, the discontinuity surfaces \( S_j \)?

To explain how Theorem 1.1 can be used in this inverse problem, note that if two systems of surfaces \( S_j^{(1)} \) and \( S_j^{(2)} \) and two functions \( a_1 \) and \( a_2 \) produce the same surface data on \( S \) for all \( x, y \in S \), then an orthogonality relation [R, pp. 65, 86] holds:

\[
\int v(x)\nabla u_1(x,y)\nabla u_2(x,z) \, dx = 0, \quad \forall y, z \in D_{12}', \quad (3.1)
\]

where \( v(x) = a_1 - a_2\), \( u_m(x,y)\), \( m = 1, 2\), are the fundamental solutions corresponding to the obstacle \( D_m \) (i.e., to \( a_m \) and \( S_j^{(m)}m\), \( D_{12}' := \mathbb{R}^3 \setminus D_{12}, D_{12} := D_1 \cup D_2\).

Let us prove, e.g., that \( \partial D_1 = \partial D_2\), using (3.1). If there is a part of \( \partial D_1 \) which lies outside \( D_2\), and \( s \) is a point at this part, then, assuming (for simplicity only) that \( v(x) \) is piecewise-constant and using for \( \nabla u_1 \) and \( \nabla u_2 \) formulas, which follow from (1.8) as \( y = z \to s\), we conclude that the left-hand side of (3.1) is an integral which contains a part, unbounded as \( y = z \to s \in \partial D_1\): \( c \int |x-y|^{-1} \, dx\), \( c = \text{const} \neq 0\). This contradicts to (3.1). Therefore there is no part of \( \partial D_1 \) which lies outside \( D_2\). Likewise, there is not part of \( \partial D_2 \) which lies outside \( \partial D_1\). Thus, \( \partial D_1 = \partial D_2\). Similarly one proves that \( S_j^{(1)} = S_j^{(2)} \) for all \( j\), provided (3.1) holds.

A detailed presentation of such an argument is given in the paper by C. Athanasiadis, A.G. Ramm and I. Stratis, Inverse acoustic scattering by layered obstacle (in preparation).
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