Generalized parabolic structures over smooth curves with many components and principal bundles over reducible nodal curves

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Abstract

Let \( Y \) be a possibly non-connected smooth projective curve, \( y_1, y_1', y_2, \ldots, y_{2\nu}, y_{2\nu}' \) different points of \( Y \), \( r \in \mathbb{N} \), \( d \in \mathbb{Z} \), \( \delta \in \mathbb{Q}_{>0} \), \( \kappa = (\kappa_1, \ldots, \kappa_\nu) \in \mathbb{Q}_{\geq 0}^\nu \) and \( e = (e_1, \ldots, e_\nu) \in \mathbb{Z}_{\geq 0}^\nu \) with \( e_i \leq r \). We construct a projective moduli space of \((\kappa, \delta)\)-(semi)stable singular principal \( G \)-bundles of rank \( r \), degree \( d \), with generalized parabolic structure of type \( e \) supported on the divisors \( D_1 = y_1 + y_1', \ldots, D_\nu = y_{2\nu} + y_{2\nu}' \). In case \( Y \) is the normalization of a connected and reducible projective nodal curve \( X \), there exists a closed subscheme coarsely representing the subfunctor corresponding to those bundles that descend to \( X \). We prove that the descent operation gives a bijection between the set of isomorphism classes of singular principal \( G \)-bundles of type \( e \) on \( X \) and the set of isomorphism classes of descending singular principal \( G \)-bundles with generalized parabolic structures of type \( e \) satisfying certain condition on \( Y \). If the stable locus is dense inside the moduli space of descending singular principal \( G \)-bundles, the descent operation induces a birational, surjective and proper morphism onto the schematic closure of the space of \( \delta \)-stable singular principal \( G \)-bundles of type \( e \). This generalizes the known results over irreducible curves.

Keywords Principal bundles · Generalized parabolic structures · Reducible nodal curves · Compactification

Mathematics Subject Classification 14D22 · 14E20 · 14H60 · 14L24

1 Introduction

Let \( X \) be a smooth projective curve over the field of complex numbers \( \mathbb{C} \), \( \mathcal{E} \) a locally free sheaf on \( X \) and \( x \in X \) a closed point. A parabolic structure on \( \mathcal{E} \) at \( x \) is a flag of vector spaces \((0) \subset E_1 \subset \cdots \subset E_s \subset \mathcal{E}(x)\) together with weights \( 0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_s < 1 \)

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(weighted flags for short). The study of parabolic locally free sheaves began with the seminal work of V. B. Mehta and C. S. Seshadri [18]. They defined a (natural) semistability condition for such objects and proved the existence of a coarse projective moduli space for semistable parabolic locally free sheaves. Furthermore, they proved that the isomorphism classes of parabolic locally free sheaves that are stable coincide with the set of equivalence classes of irreducible unitary representations of the topological fundamental group of $X$ (see [18, Theorem 4.1]).

The concept of parabolic locally free sheaf can be generalized by considering weighted flags supported on divisors of the smooth projective curve $X$. These objects are called generalized parabolic locally free sheaves, and they were introduced by U. Bhosle in [2]. The importance of generalized parabolic locally free sheaves is not only the possible link to the space of representations of the topological fundamental groups but also the link to the geometry of the moduli spaces of torsion-free sheaves on nodal curves. To be more precise, if $\pi: Y \to X$ is the normalization map of a reducible projective nodal curve, then there exists a coarse projective moduli space for generalized parabolic locally free sheaves (the parabolic structure being supported on $q_1 + q_2 = \pi^{-1}(p)$) on $Y$ together with a morphism to the moduli space of torsion-free sheaves on $X$ of rank $r$ and degree $d$ making the former moduli space a desingularization of the later provided $(r, d) = 1$ (see [3]).

The above ideas have been useful to approach the problem of compactifying the moduli space of principal $G$-bundles over an irreducible nodal curve. In [24], A. Schmitt realized that, once a faithful representation $\rho: G \to \text{SL}(V)$ is fixed, every principal $G$-bundle can be seen as a pair $(\mathcal{E}, \tau)$ formed by a locally free sheaf $\mathcal{E}$ and a non-trivial morphism of algebras $\tau: S^\bullet (V \otimes \mathcal{E})^G \to \mathcal{O}_X$. These objects are called singular principal $G$-bundles and they carry a semistability condition, which depends (a priori) on a positive rational parameter $\delta \in \mathbb{Q}_{>0}$. Then, the main result is that there exists a coarse projective moduli space for $\delta$-(semi)stable singular principal $G$-bundles and it coincides with the classical moduli space [22], provided that $\delta$ is large enough. This motivated the works [6, 26, 27], where U. Bhosle generalized the definition of singular principal $G$-bundles, as well as the $\delta$-(semi)stability condition, over an irreducible nodal curve in a natural way and proved the existence of a projective moduli space for them, while A. Schmitt studied the asymptotic behavior of the $\delta$-(semi)stability condition obtaining a similar result as that of the smooth case. The study of the asymptotic behavior of the $\delta$-(semi)stability condition becomes harder when the curve has singularities, and it was carried out in [26, 27] by considering singular principal $G$-bundles on $X$ as singular principal $G$-bundles with generalized parabolic structures on the normalization $Y$. Therefore, the moduli spaces of singular principal $G$-bundles with generalized parabolic structures over a smooth projective curve play an important role in this problem.

Likewise, the use of generalized parabolic structures allows to find interesting results regarding the geometry of the moduli space of Hitchin pairs over a reducible curve. In [7], U. Bhosle constructs a morphism between the moduli space of Hitchin pairs with generalized parabolic structure over the normalization $Y$ and the moduli space of Hitchin pairs over the reduced curve $X$, showing that under certain condition this is a birational morphism and its image contains all stable Higgs bundles. See [17] for an overview of the subject.

On the other hand, singular principal $G$-bundles with generalized parabolic structures turn out to be useful also in the problem of compactifying the moduli space of principal Higgs $G$-bundles over an irreducible nodal curve. In [16], A. Lo Giudice and A. Pustetto enlarge the category of principal Higgs $G$-bundles on the nodal curve to the category of singular principal $G$-bundles together with a Higgs field, which can be seen as singular principal $G$-bundles with generalized parabolic structure together with a Higgs field on the normalization of the
nodal curve. Again, the moduli space of the last objects plays an important role in the study of the moduli space of the first objects.

1.1 Goal of the paper

The goal of this work is to study the relationship between singular principal $G$-bundles on a reducible nodal projective curve and singular principal $G$-bundles with generalized parabolic structures on the normalization. We work over an algebraically closed field $\mathbb{C}$ of characteristic zero.

Let $X$ be a projective nodal curve with nodes $x_1, \ldots, x_v$ and $l$ irreducible components, and $\pi : Y = \bigsqcup_{i=1}^l Y_i \to X$ its normalization. We fix an ample invertible sheaf $\mathcal{O}_X(1)$ on $X$ and we denote by $\mathcal{O}_Y(1)$ the ample invertible sheaf obtained by pulling $\mathcal{O}_X(1)$ back to $Y$. We denote by $y_{1i}, y_{2i}$ the points in the preimage of the $i$-th nodal point $x_i$, $D_i = y_{1i} + y_{2i}$ the corresponding divisor on $Y$ and $D = \sum D_i$ the total divisor. Let $G$ be a semisimple linear algebraic group, $\rho : G \hookrightarrow \text{SL}(V)$ a faithful representation of dimension $r \in \mathbb{N}$, $\delta \in \mathbb{Q}_{>0}$ and $d \in \mathbb{Z}$. Let $\text{SPB}(\rho)^{\delta-(s)s}_{r,d}$ be the moduli space of $\delta$-(semi)stable singular principal $G$-bundles of rank $d$ and degree $d$ over $X$ (see [19]) and $J(r)$ the set $\{(e_1, \ldots, e_v) \in \mathbb{Z}^v \; | \; 0 \leq e_i \leq r\}$. Then, there is a stratification, $\text{SPB}(\rho)^{\delta-(s)s}_{r,d} = \bigcup_{\xi \in J(r)} \text{SPB}(\rho)^{\delta-(s)s}_{r,d,\xi}$, where $\text{SPB}(\rho)^{\delta-(s)s}_{r,d,\xi}$ parametrizes singular principal bundles, $(\mathcal{F}, \tau)$, with $\mathcal{F}_{x_i} \simeq \mathcal{O}_{X,x_i}^{e_i} \oplus m_{x_i}^{r-e_i}.$

Given $\kappa = (\kappa_1, \ldots, \kappa_2) \in \mathbb{Q}_{\geq 0}^2$ with $\kappa_1 < 1$, we prove that there exists a coarse projective moduli space, $D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta)$ for $(\kappa, \delta)$-(semi)stable descending singular principal $G$-bundles with generalized parabolic structures over $Y = \bigsqcup_{i=1}^l Y_i$ of rank $d$, degree $\delta(\kappa, r) = d - \sum_{i=1}^v (r - e_i)$ and type $\xi$ supported on the divisors $D_i$ (see Theorem 5) together with a morphism (see Eq. 25)

$$\Theta : D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta) \longrightarrow \text{SPB}(\rho)^{\delta-(s)s}_{r,d}. $$

We show that the restriction $\Theta_\xi : D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta) \longrightarrow \bigcup_{\xi \leq \xi} \text{SPB}(\rho)^{\delta-(s)s}_{r,d,\xi}$ to each component induces an isomorphism between a (functorially well defined) dense open subscheme of the stable locus $W_\xi \subset D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta)$ and $\text{SPB}(\rho)^{\delta-(s)s}_{r,d,\xi}$ (see Theorem 7). Therefore, $\Theta_\xi$ induces a birational surjective and proper morphism

$$D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta) \longrightarrow \overline{\text{SPB}(\rho)^{\delta-(s)s}_{r,d,\xi}}$$

when the stable locus is dense inside $D(\rho)^{(\kappa, \delta)-(s)s}_{r,d}(\kappa, \delta)$. In case the density property does not hold, this result reduces the study of specializations of $\delta$-stable singular principal $G$-bundles on $X$ to the study of specializations of $(\kappa, \delta)$-stable descending singular principal $G$-bundles with generalized parabolic structure on $Y$ which is, a priori, more tractable.

1.2 Outline of the paper

In Sect. 2, we introduce the definitions of generalized parabolic swamps and generalized parabolic singular $G$-bundles of a given type, as well as the semistability conditions. In Sect. 3, we prove the existence of a coarse projective moduli space for generalized parabolic $(\kappa, \delta)$-(semi)stable swamps of a given type. The main difficulty here is to find the linearized projective embedding that makes the semistability condition to coincide with the Hilbert–
Mumford semistability. In Sect. 4, we prove the existence of a coarse projective moduli space for \((\kappa, \delta)-(\text{semi})\text{stable singular principal} \ G\text{-bundles}. By [19, Theorem 5.5], this is a direct consequence of the results proved in Sect. 3. In Sect. 5, we construct the moduli space for descending singular principal bundles over the normalization of a projective nodal curve, as well as the morphism \(\Theta\) that relates it with the closure of the stable locus of the moduli space of singular principal bundles over the nodal curve, and we prove the main result.

2 Preliminaries

We denote by \(\mathbb{N}_0\) the set \(\mathbb{N} \cup \{0\}\). Given a real number \(x \in \mathbb{R}\), we define \([x]_+ = \max\{x, 0\}\).

For any scheme \(X\), \(X^*\) denotes its functor of points. Given a coherent sheaf \(\mathcal{E}\) over a scheme \(X\), we denote by \(\text{Proj}(\mathcal{E})\) the projective spectrum of its symmetric algebra, \(\text{Proj}(S^*(\mathcal{E}))\).

Let \(Y = \bigsqcup_{i=1}^q Y_i\) be a disjoint union of smooth projective and irreducible curves, \(j_i : Y_i \to Y\) the natural embedding of the \(i\)-th component, \(\mathcal{O}_Y(1)\) an ample invertible sheaf and \(\mathcal{O}_{Y_i}(1) = \mathcal{O}_Y(1)|_{Y_i} = j_i^*\mathcal{O}_Y(1)\) the restriction of \(\mathcal{O}_Y(1)\) to the component \(Y_i\). The degrees \(\deg(\mathcal{O}_Y(1))\) and \(\deg(\mathcal{O}_{Y_i}(1))\) are denoted by \(h\) and \(h_i\), respectively.

Given a coherent sheaf, \(\mathcal{E}\), on \(Y\), it holds that \(\mathcal{E} = \bigoplus_{i=1}^q j_i^*(\mathcal{E}_i)\), where \(\mathcal{E}_i\) is the restriction of \(\mathcal{E}\) to \(Y_i\). The multirank of \(\mathcal{E}\) is defined as the tuple \(\alpha = (r_1, \ldots, r_q)\), where \(r_i = \text{rk}(\mathcal{E}_i) \in \mathbb{N}_0\), while the multidegree is defined as \(d = (d_1, \ldots, d_q)\), where \(d_i = \deg(\mathcal{E}_i) \in \mathbb{Z}\). The multiplicity of \(\mathcal{E}\), \(\alpha(\mathcal{E})\), is defined as the leading coefficient of its Hilbert polynomial and its degree, \(\deg(\mathcal{E})\), is defined as \(\chi(\mathcal{E}) - (\alpha/h)\chi(\mathcal{O}_Y)\). Since the Euler–Poincaré characteristic is additive, the degree can be expressed in terms of \(d_1, \ldots, d_q, a, h_1, \ldots, h_q\). If \(r \in \mathbb{N}\) and \(\text{rk}(\mathcal{E}_i) = r\) for all \(i\) (we will say that \(\mathcal{E}\) has rank \(r\)), then \(P_{\mathcal{E}}(n) = an + r \chi(Y) + d\), and it holds \(\alpha = hr\) and \(d = \sum_{i=1}^q d_i\).

Given a coherent sheaf, \(\mathcal{E}\), on \(Y\), and a point \(y \in Y\), we denote by \(\mathcal{E}(y)\) the fiber of \(\mathcal{E}\) at \(y\). Let \(y_1^1, y_2^1, \ldots, y_1^v, y_2^v\) be \(2v\) different points of \(Y\). Denote by \(D_i\) the divisor \(y_1^i + y_2^i\). Given a coherent sheaf \(\mathcal{E}\) over \(Y\), we use the notation \(r(i) := \dim(\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))\), \(i = 1, \ldots, v\).

2.1 Generalized parabolic structures

Definition 1 Let \(\mathcal{E}\) be a locally free sheaf. A generalized parabolic structure on \(\mathcal{E}\) is a tuple \(q = (q_1, \ldots, q_v)\) where \(q_i\) is a quotient, \(\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) \to R_i \to 0\), of vector spaces. We say that the tuple \((\mathcal{E}, q)\) is a generalized parabolic locally free sheaf. The tuple of dimensions \(\underline{e} = (e_1 = \dim(R_1), \ldots, e_v = \dim(R_v))\) is called the type of \((\mathcal{E}, q)\). The multirank, multidegree and degree of \((\mathcal{E}, q)\) are defined as the multirank, multidegree and degree of \(\mathcal{E}\).

Remark 1 1. Let \(R := \bigoplus R_i\) be the total vector space. Since the supports of the divisors \(D_i\) are disjoint, we have \(\Gamma(D, \mathcal{E}|_D) = \bigoplus \Gamma(D_i, \mathcal{E}|_{D_i}) = \bigoplus \mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i)\). Then, we can form the quotient \(q := \bigoplus q_i : \Gamma(D, \mathcal{E}|_D) \to R \to 0\).
2. Let \((\mathcal{E}, q)\) be a generalized parabolic locally free sheaf. Every subsheaf \(\mathcal{F} \subset \mathcal{E}\) inherits a generalized parabolic structure \(q' = (q'_1, \ldots, q'_v)\), \(q'_i\) defined as the projection onto its image of the linear map \(\mathcal{F}(y_1^i) \oplus \mathcal{F}(y_2^i) \to \mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) \to R_i\).

Definition 2 Let \((\mathcal{E}, q)\) and \((\mathcal{E}', q')\) be generalized parabolic locally free sheaves on \(Y\). A morphism between them is a tuple \((f, u_1, \ldots, u_v)\) where \(f : \mathcal{E} \to \mathcal{E}'\) is a morphism of \(\mathcal{O}_Y\)-modules and \(u_i : R_i \to R'_i\) is a linear map such that \(q'_i \circ (f(y_1^i) \oplus f(y_2^i)) = u_i \circ q_i\), where \(f(y)\) denotes the induced linear map between the fibers at \(y \in Y\).
**Notation 1** Given a tuple \( e = (e_1, \ldots, e_v) \in \mathbb{N}_0^v \), we will denote by \( I(e) \) the set \( \{i \in \{1, \ldots, v\} \text{ such that } e_i \neq 0 \} \) and by \( v'(e) \) the number of elements in \( I(e) \). We will drop the dependency of \( e \) in \( v'(e) \) if there is no risk of confusion.

**Definition 3** Let us pick \( \kappa := (\kappa_1, \ldots, \kappa_v) \in \mathbb{Q}_{\geq 0}^v \). Given a generalized parabolic locally free sheaf, \( (\mathcal{E}, q) \), of multirank \( r = (r_1, \ldots, r_n) \), degree \( d \) and type \( e := (e_1, \ldots, e_v) \in \mathbb{N}_0^v \) with \( e_i \leq r(i) \), we define its \( \kappa \)-parabolic degree as

\[
\kappa \text{-pardeg}(\mathcal{E}) := d - \sum_{i \in I(e)} \kappa_i e_i.
\]

**Remark 2** 1. Let \( \kappa, e, r, d \) and \( (\mathcal{E}, q) \) be as above. We will use the following notation:

\[
\kappa \text{-par} \mu(\mathcal{E}) := \frac{\kappa \text{-pardeg}(\mathcal{E})}{\alpha},
\]

\[
\kappa \text{-par} \chi(\mathcal{E}(n)) := \chi(\mathcal{E}(n)) - \sum_{i \in I(e)} \kappa_i e_i,
\]

\[
\kappa \text{-par} h^0(\mathcal{E}(n)) := h^0(Y, \mathcal{E}(n)) - \sum_{i \in I(e)} \kappa_i e_i.
\]

2. Let \( \kappa, e, r, d \) and \( (\mathcal{E}, q) \) be as above and assume \( r = (r, \ldots, r) \). Let \( \mathcal{F} \subseteq \mathcal{E} \) be a subsheaf and consider the generalized parabolic structure on \( \mathcal{F} \) inherited from \( (\mathcal{E}, q) \). Then,

\[
\kappa \text{-pardeg}(\mathcal{F}) = \text{deg}(\mathcal{F}) - \sum_{i \in I(q)} \kappa_i \text{dim}(q_i(\mathcal{F}(y_1^i) \oplus \mathcal{F}(y_2^i))).
\]

3. If there is no confusion, we will drop the dependency on \( \kappa \) from the notations introduced so far.

From now on, we will focus on the case \( r = (r, \ldots, r) \) and \( e_i \leq r \) for \( i = 1, \ldots, v \).

### 2.2 Swamps with generalized parabolic structures

Let us fix the following data: \( r \in \mathbb{N}, d \in \mathbb{Z}, e := (e_1, \ldots, e_v) \in \mathbb{N}_0^v \) with \( e_i \leq r, \kappa := (\kappa_1, \ldots, \kappa_v) \in \mathbb{Q}_{\geq 0}^v \), \( a, b, c \in \mathbb{N}_0 \) and an invertible sheaf \( \mathcal{L} \) on \( Y \).

**Definition 4** A swamp with generalized parabolic structure of type \( (a, b, c, \mathcal{L}, e) \), rank \( r \) and degree \( d \) is a triple \( (\mathcal{E}, q, \phi) \) where \( (\mathcal{E}, q) \) is a generalized parabolic locally free sheaf of rank \( r \), degree \( d \) and type \( e \), and \( \phi : (\mathcal{E} \otimes a)^\oplus b \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L} \) is a nonzero morphism of \( \mathcal{O}_Y \)-modules.

**Definition 5** Let \( (\mathcal{E}, q, \phi) \) and \( (\mathcal{F}, p, \theta) \) be swamps with generalized parabolic structure on \( Y \). A morphism between them is a morphism of \( \mathcal{O}_Y \)-modules \( f : \mathcal{F} \to \mathcal{G} \) compatible with both structures.

**Notation 2** To shorten, we will denote the tuple \( (a, b, c, \mathcal{L}, e) \) that defines the type of a generalized parabolic swamp by the symbol \( \text{tp} \).

Let \( \phi : (\mathcal{E} \otimes a)^\oplus b \to \det(\mathcal{E})^{\otimes c} \otimes \mathcal{L} \) be a swamp on \( Y \), and \( (\mathcal{E}_\bullet, m) \) a weighted filtration. For each \( \mathcal{E}_i \in \mathcal{E}_\bullet \), denote by \( \alpha_i \) its multiplicity and by \( \alpha \) the multiplicity of \( \mathcal{E} \). Let us set...
\( \kappa := (\kappa_1, \ldots, \kappa_t) \in \mathbb{Q}_{\geq 0}^t \) and \( \Gamma := \sum \lambda_i \Gamma_i(\alpha_i) \), where \( \Gamma_i(l) = (l - \alpha, \ldots, l - \alpha, l, \alpha, \ldots, \alpha) \). Let us denote by \( J \) the set [multi-indices \( I = (i_1, \ldots, i_{\lambda_i}) | i_j \in \{1, \ldots, t + 1\} \)]. We define

\[
\mu(\mathcal{E}_i, m, \phi) := -\min_{I \in J} \left\{ \Gamma_{a_i} + \cdots + \Gamma_{a_{\lambda_i}} | \phi|_{(\mathcal{E}_i \otimes \cdots \otimes \mathcal{E}_{a_{\lambda_i}})^{a_{\lambda_i}}} \neq 0 \right\},
\]

\[
P_k(\mathcal{E}, m) := \sum_{i=1}^s m_i (\kappa \text{-pardeg}(\mathcal{E}) a_i - \kappa \text{-pardeg}(\mathcal{E}_i) a_i),
\]

where \( \Gamma_{a_j} \) means the the \( a_j \)-th component of \( \Gamma \).

**Definition 6** A generalized parabolic swamp \( (\mathcal{E}, q, \phi) \) of rank \( r \) and type \( tp = (a, b, c, \mathcal{L}, e) \) is \((\kappa, \delta)\)-(semi)stable if for every weighted filtration \( (\mathcal{E}_i, m) \) of \( \mathcal{E} \), the inequality \( P_k(\mathcal{E}_i, m) + \delta \mu(\mathcal{E}_i, m, \phi)(\geq 0) \) holds.

**Remark 3** 1. Let \( \phi: (\mathcal{E}^{a_i} \otimes b) \rightarrow \det(\mathcal{E}) \otimes \mathcal{L} \) be a swamp on \( Y \), and let \( (\mathcal{E}, m) \) be a weighted filtration. Given a multi-index \( I = (i_1, \ldots, i_{\lambda_i}) \), we define

\[
\zeta_i(\mathcal{E}_i, I) := \# \{ k \in I | (i_1, \ldots, i_{\lambda_i}) | \alpha_k = \alpha_i \},
\]

\( \alpha_i \) (resp. \( \alpha_k \)) being the multiplicity of \( \mathcal{E}_i \) (resp. \( \mathcal{E}_k \)), and denote by \( \zeta_i(\mathcal{E}_i, I) \) the number \( \zeta_i(\mathcal{E}_i, I) \) where \( I \) is a multi-index giving the minimum in \( \mu(\mathcal{E}_i, m, \phi) \). Then, we have (see [28, Eq. (2.9)])

\[
\mu(\mathcal{E}_i, m, \phi) = \sum_{i=1}^t m_i (\alpha_i a - \zeta_i(\mathcal{E}_i, \alpha_i)).
\]

2. In order to check the above semistability condition, it suffices to consider saturated filtrations, that is, those that satisfy that \( \mathcal{E}/\mathcal{E}_i \) is locally free for every \( i \).

3. Observe that there is a positive integer \( A \), depending only on the numerical input data, \( r, a, b, c \) and \( \mathcal{L} \), such that it is enough to check the \( \delta \)-semistability condition for weighted filtrations with \( m_i < A \). This follows from [12, Lemma 1.4] changing ranks by multiplicities.

Let \( S \) be a scheme and \( \pi_{S_S} : S \times D_i \rightarrow S \) be the projection onto the first factor. Let us set \( S_D_i := S \times D_i \subset S \times Y \). A family of generalized parabolic locally free sheaves of rank \( r \), degree \( d \), and type \( e \) parametrized by \( S \) is a tuple \( (\mathcal{E}_S, q_S) \) where \( \mathcal{E}_S \) is a family of locally free sheaves on \( Y \) parametrized by \( S \) of rank \( r \) and degree \( d \), and \( q_S = (q_{S1}, \ldots, q_{S\lambda}) \),

\[
q_S : \pi_{S_S}(\mathcal{E}_S|_{S_D_i}) \rightarrow R_i \rightarrow 0 \text{ being a locally free quotient sheaf of rank } e_i \text{ on } S.
\]

A family of generalized parabolic swamps of rank \( r \), degree \( d \), and type \( tp = (a, b, c, \mathcal{L}, e) \) is a quadruple \( (\mathcal{E}_S, q_S, N_S, \phi_S) \) where \( (\mathcal{E}_S, q_S) \) is a family of generalized parabolic locally free sheaves of rank \( r \), degree \( d \), and type \( e \), \( N_S \) is an invertible sheaf on \( S \), and \( \phi_S : (\mathcal{E}_S^{a_i} \otimes b) \rightarrow \det(\mathcal{E}_S)^{\otimes c} \otimes \pi_Y^*: \mathcal{L} \otimes \pi_S^*: N_S \) is a morphism of locally free sheaves on \( S \times Y \) such that \( \phi_S|_{s \times Y} \) is nonzero for all \( s \in S \). Finally, \((\kappa, \delta)\)-(semi)stable families are families which are \((\kappa, \delta)\)-(semi)stable fiberwise. Then, one can introduce the moduli problem defined by the functor

\[
SGPS_{r, d, \mathcal{L}, \mathcal{E}}^{(\kappa, \delta), (s)}(S) = \left\{ \begin{array}{l}
\text{isomorphism classes of families of} \\
\text{(semi)stable generalized parabolic} \\
\text{swamps } (\mathcal{E}_S, q_S, N_S, \phi_S) \text{ parametrized} \\
\text{by } S \text{ with rank } r, \text{ degree } d \text{ and type } tp
\end{array} \right\}.
\]
2.3 Singular principal $G$-bundles with generalized parabolic structures

Let us pick $r \in \mathbb{N}$, $d \in \mathbb{Z}$, $\mathfrak{e} := (e_1, \ldots, e_r) \in \mathbb{N}_0^r$ with $e_i \leq r$ and $\kappa := (\kappa_1, \ldots, \kappa_r) \in \mathbb{Q}_{\geq 0}^r$. Let $G$ be a semisimple linear algebraic group, and let $\rho : G \hookrightarrow \text{SL}(V)$ be a faithful representation.

**Definition 7** A singular principal $G$-bundle over $Y$ of rank $r$ and degree $d$ is a pair $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a locally free sheaf of rank $r$ and degree $d$, and $\tau : S^*(V \otimes \mathcal{E})^G \rightarrow \mathcal{O}_Y$ is a non-trivial morphism of $\mathcal{O}_Y$-algebras.

**Definition 8** A singular principal $G$-bundle with a generalized parabolic structure over $Y$ of rank $r$, degree $d$ and type $\mathfrak{e}$ is a triple $(\mathcal{E}, \tau, q)$ where $(\mathcal{E}, q)$ is a generalized parabolic locally free sheaf of rank $r$, degree $d$ and type $\mathfrak{e}$, and $(\mathcal{E}, \tau)$ is a singular principal $G$-bundle.

**Definition 9** Let $(\mathcal{E}, \tau, q)$ and $(\mathcal{G}, \lambda, p)$ be singular principal $G$-bundles with generalized parabolic structure on $Y$. A morphism between them is a morphism of $\mathcal{O}_Y$-modules $f : \mathcal{F} \rightarrow \mathcal{G}$ compatible with both structures.

Following [19, Theorem 5.5], we can assign to any singular principal $G$-bundle a swamp of type $(a, b, 0, \mathcal{O}_Y)$ for certain natural numbers $a$, $b$ that depend only on the numerical input data,

\[
\begin{align*}
\{ \text{isomorphism classes of singular principal } G\text{-bundles} \} & \rightarrow \{ \text{isomorphism classes of swamps of type } (a, b, 0, \mathcal{O}_Y) \}, \\
(\mathcal{E}, \tau, q) & \mapsto (V \otimes \mathcal{E}, \phi_\tau, q)
\end{align*}
\]

this map being injective. Thus, we can define, for any weighted filtration $(\mathcal{E}_\bullet, m)$, the semistability function $\mu(\mathcal{E}_\bullet, m, \tau)$ as $\mu(\mathcal{E}_\bullet, m, \phi_\tau)$ (see [19, Definition 6.1]).

**Definition 10** A generalized parabolic singular principal $G$-bundle of rank $r$ degree $d$ and type $\mathfrak{e}$, $(\mathcal{E}, \delta)$, is $(\kappa, \delta)$-(semi)stable if for every weighted filtration $(\mathcal{E}_\bullet, m)$ of $\mathcal{E}$, the inequality $P_\kappa(\mathcal{E}_\bullet, m) + \delta \mu(\mathcal{E}_\bullet, m, \tau)(\geq)0$ holds.

Then, one can define a family as in the case of swamps and introduce the moduli problem defined by the functor

\[
\text{SPBGPS}(\rho)_{r, d, \mathfrak{e}}(S) = \left\{ \text{isomorphism classes of families of } (\kappa, \delta)\text{-}(semi)stable singular principal } G\text{-bundles with generalized parabolic structure on } Y \text{ parametrized by } S \text{ with rank } r \text{ degree } d \text{ and type } \mathfrak{e} \right\}.
\]

2.4 Some calculations in geometric invariant theory

Let $U$ be a vector space of dimension $p$. Recall that a basis $u := \{u_1, \ldots, u_p\}$ of the vector space $U$, together with a vector $\mathcal{U} = (\gamma_1, \ldots, \gamma_p) \in \mathbb{Z}_p$ such that $\gamma_1 \leq \ldots \leq \gamma_p$ and $\sum_{i=1}^p \gamma_i = 0$, defines a one-parameter subgroup $\lambda(u, \mathcal{U}) : G_m \rightarrow \text{SL}(U)$. Conversely, every one-parameter subgroup of $\text{SL}(U)$ arises in this way (see [28, Example 1.5.1.12]). Furthermore, every one-parameter subgroup of $\text{SL}(U)$ determines a weighted flag $(U_\bullet, m)$.
of $U$ and every weighted flag arises in this way as well. It turns out that the Hilbert–Mumford function, $\mu(\cdot, \lambda)$ depends only on the associated weighted flag of $\lambda$ and not on $\lambda$ itself (see [28, Proposition 1.5.1.35, Example 1.5.1.36]).

We derive the explicit expression of the Hilbert–Mumford criterion (see [21, Theorem 2.1, Proposition 2.3]) in some situations that will be important for our purposes. Similar calculations can be found along [28], so we will skip some details.

2.4.1 Example 1

Let $p, r$ be integers such that $1 \leq e \leq p$. Let $\mathcal{G}_r := \text{Grass}_e(U^\oplus 2)$ be the Grassmannian of $e$-dimensional quotients of $U^\oplus 2$, $U$ being a $p$-dimensional vector space, and let $N$ be a positive integer. The Grassmannian can be embedded into the projective space through the Plücker embedding $\mathcal{G}_r \hookrightarrow \mathbb{P}(\wedge^e U^\oplus 2)$. The group $\text{SL}(U)$ acts on both spaces through the diagonal $\delta : \text{SL}(U) \hookrightarrow \text{SL}(U^\oplus 2)$ in the obvious way, and $i$ is $\text{SL}(U)$-equivariant. If $\mathcal{O}(1)$ is the tautological invertible sheaf on $\mathbb{P}(\wedge^e U^\oplus 2)$, then $\mathcal{L} := i^* \mathcal{O}(1)$ is an $\text{SL}(U)$-linearized very ample invertible sheaf.

Let $\lambda : \mathbb{G}_m \rightarrow \text{SL}(U)$ be a one parameter subgroup, $u = \{u_1, \ldots, u_p\}$ a basis of $U$, $\gamma_1 \leq \ldots \leq \gamma_p$ integers such that $\lambda = \lambda(u, \gamma)$, and $[\tau : U^\oplus 2 \rightarrow E \rightarrow 0] \in \mathcal{G}_r$ a rational point in the Grassmannian. Then, we have

$$\mu_{\mathcal{L}}([\tau], \lambda(u, \gamma)) = \sum_{i=1}^{s} i \dim(\text{Ker}(\tau)) - p \dim(\text{Ker}(\tau) \cap (U_i \oplus U_i))$$

$$= \sum_{i=1}^{s} p \dim_\tau(U_i \oplus U_i) - ie,$$

where $(U_*, m)$ is the weighted filtration associated with $\lambda$.

2.4.2 Example 2

Let $Y_1, \ldots, Y_l$ be smooth projective connected curves, and consider their disjoint union, $Y := \bigsqcup Y_i$. Let $\mathcal{N}_1, \ldots, \mathcal{N}_l$ be invertible sheaves on $Y_1, \ldots, Y_l$, respectively, and denote by $\mathcal{N} := \bigoplus \mathcal{N}_i$ the corresponding invertible sheaf on $Y$. Let us choose $r, n \in \mathbb{N}$, and let $U$ be a vector space of dimension $p > r$. For each $i$, we define

$$\mathcal{G}_1 := \mathbb{P}(\text{Hom}\left(\bigwedge^r U, H^0(Y_i, \mathcal{N}_i(N))\right)).$$

Let us consider the product $\mathcal{G}_1 \times \cdots \times \mathcal{G}_l$, where $\mathcal{G}_1 := \mathbb{P}(\text{Hom}\left(\bigwedge^r U, H^0(Y_1, \mathcal{N}_1(N))\right))$. We denote by $\pi_i$ the projection of $\mathcal{G}_1 \times \cdots \times \mathcal{G}_l$ onto $\mathcal{G}_i \times \cdots \times \mathcal{G}_l$. Given natural numbers $b_1, \ldots, b_l \in \mathbb{N}$, we may consider the very ample invertible sheaf $\mathcal{O}_{\mathcal{G}_1}(b_1) \otimes \cdots \otimes \mathcal{O}_{\mathcal{G}_l}(b_l)$ with the obvious $\text{SL}(U)$-linearization. For the sake of simplicity, we will use the symbol $\mathcal{L}$ to denote the invertible sheaf $\mathcal{O}_{\mathcal{G}_1}(1)$. Clearly, $\mu_{\mathcal{L}}([g], \lambda) = \sum_{i=1}^{l} b_i \mu_{\mathcal{L}}^{\mathcal{G}_i}([g], \lambda)$, $[g]$ being the $i$-th component of $[g] \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_l$. Therefore, the calculation of the semistability function of points of $\mathcal{G}_1 \times \cdots \times \mathcal{G}_l$ with respect to $\mathcal{L}$ is reduced to the calculation of the semistability function of points of $\mathcal{G}_1 \times \cdots \times \mathcal{G}_l$ with respect to $\mathcal{L}_1$. Let $\mathcal{E}$ be a locally free quotient sheaf of rank $r$, and $q : U \otimes \mathcal{O}_Y(-n) \rightarrow \mathcal{E} \rightarrow 0$ a quotient whose determinant is isomorphic to $\mathcal{N}$. Restricting to the $i$-th component, twisting by $n$, taking the
r-th exterior power and global sections, we obtain a morphism $H^0(L(q_i(n))) : \wedge^r U \to H^0(Y, \mathcal{M}_i(rn))$, whose equivalence class defines a point $[H^0(L(q_i(n)))] \in \mathbb{G}_{1,N}$. Now, a short calculation shows that

$$
\mu_{\mathscr{L}, j}([H^0(L(q_j(n))]), \lambda) = \sum_{i=1}^s m_i (\text{rk}(\mathcal{E}_i|_{Y_j}) p - r \dim(U_i)),
$$

$(U_i, m_i)$ being the i-th term of the weighted filtration associated with $\lambda$ and $\mathcal{E}_i|_{Y_j}$, the restriction to $Y_j$ of the saturated subsheaf generated by $U_i$.

### 2.4.3 Example 3

Let us consider the same situation as in Example 2. Let $\mathscr{L}$ be an invertible sheaf on $Y$, $U$ a $p$-dimensional vector space and set $a, b, c, n \in \mathbb{N}$. Given an invertible sheaf $\mathcal{N}$ on $Y$, we define the projective space

$$
\mathbb{G}_{2,N} := \text{P}(\text{Hom}(U_{a,b}, H^0(Y, \mathcal{N} \otimes \mathcal{L}(na))))^\vee,
$$

where $U_{a,b} := (U \otimes a) \oplus b$. Let $(q, \phi)$ be a pair given by a locally free quotient sheaf of rank $r$, $q : U \otimes \mathcal{E}_Y(-n) \to \mathcal{E}$, whose determinant is isomorphic to $\mathcal{N}$ and a nonzero morphism $\phi : (\mathcal{E} \otimes a) \oplus b \to \mathcal{N} \otimes \mathcal{L}$. Let $\Delta : U_{a,b} \to U_{a,b}$ be the diagonal linear map, and consider the morphism $H^0((q(n) \otimes a) \oplus b) \circ \Delta : U_{a,b} \to H^0(Y, (\mathcal{E} \otimes a) \oplus b \otimes \mathcal{L}(na))$. Twisting $\phi$ by $\mathcal{E}_Y(na)$, we get $H^0(\phi(na)) : H^0(Y, (\mathcal{E} \otimes a) \oplus b \otimes \mathcal{L}(na)) \to H^0(Y, \mathcal{N} \otimes \mathcal{L}(na))$. The composition of both morphisms gives rise to a rational point in $\mathbb{G}_{2,N}$,

$$
[H^0(\phi(na)) \circ H^0((q(n) \otimes a) \oplus b) \circ \Delta : U_{a,b} \to H^0(Y, \mathcal{N} \otimes \mathcal{L}(na))] \in \mathbb{G}_{2,N}.
$$

Let $\mathbf{u} = \{u_1, \ldots, u_p\}$ be a basis of $U$. For any multi-index $I = (i_1, \ldots, i_a)$ with $i_j \in \{1, \ldots, p\}$, we define $u_I := u_{i_1} \otimes \cdots \otimes u_{i_a}$ and $u_I := (0, \ldots, 0, u_I, 0, \ldots, 0)$ appearing in the $k$-th position. Then, the elements $u_I^k$ form a basis of $U_{a,b}$ and the group $\text{SL}(U)$ acts on $\mathbb{G}_{2,N}$ in the obvious way. We want to compute the semistability function for any point $T \in \mathbb{G}_{2,N}$ of form (3) with respect to the natural $\text{SL}(U)$-linearization of $\mathcal{E}_{\mathbb{G}_{2,N}}$ (1). Let $\lambda : \mathbb{G}_m \to \text{SL}(U)$ be a one parameter subgroup. Then, there exists a basis $\mathbf{u} = \{u_1, \ldots, u_p\}$ of $U$ and integers $\gamma_1 \leq \ldots \leq \gamma_p$ with $\sum \gamma_i = 0$ such that $\lambda(z)u_i = z^{\gamma_i}u_i, \forall z \in \mathbb{C}_m$.

For any multi-index $I = (i_1, \ldots, i_a)$ we take $u_I$ and define $\gamma_I = \gamma_{i_1} + \cdots + \gamma_{i_a}$. The obvious operation, $g \cdot u_I^k$, $\forall g \in \text{SL}(U)$, makes $\mathbb{G}_m$ to act naturally on $U_{a,b}$ (therefore, on $\mathbb{G}_{2,N}$) through $\lambda : \mathbb{G}_m \to \text{SL}(U)$, and we have $\mu([T], \lambda) = -\min[\gamma_I|T(u_I^k) \neq 0]$.

Given a multi-index $I = (i_1, \ldots, i_a)$ we want to compute $\gamma_I = \gamma_{i_1} + \cdots + \gamma_{i_a}$ for $\gamma = (i - p, \ldots, i - p, i, \ldots, i)$. Let us set $\xi(I,i) := \#\{j | i_j \leq i\}$. Then, $1, \ldots, \xi(I,i) \leq i$ and $i_{\xi(I,i)+1}, \ldots, i_a > i$, so $\gamma_I = (i-p)\xi(I,i) + (a - \xi(I,i)) = ia - \xi(I,i)p$. A short calculation shows

$$
\mu([T], \lambda) = \sum_{i=1}^s m_i (\xi(I_1, dimU_I) p - dimU_i a),
$$

$(U_\bullet, m)$ being the weighted flag associated with $\lambda$ and $I = (i_1, \ldots, i_a)$ is the multi-index giving the minimum of the semistability function.
3 Moduli space for generalized parabolic swamps

We assume that the following data are fixed along this section: \( r \in \mathbb{N}, d \in \mathbb{Z}, \varepsilon := (e_1, \ldots, e_v) \in \mathbb{N}_0^v \) with \( e_i \leq r, \kappa := (\kappa_1, \ldots, \kappa_v) \in \mathbb{Q}^+_{\geq 0}, a, b, c \in \mathbb{N}_0 \) and an invertible sheaf \( \mathcal{L} \) on \( Y \). Recall that \( h := \text{deg}(\mathcal{O}_Y(1)) \) and \( h_i := \text{deg}(\mathcal{O}_Y(1)) \).

The main result of this section is Theorem 3 which demonstrates the existence of a coarse projective moduli space for \((\kappa, \delta)\)-(semi)stable swamps with generalized parabolic structure of given type \( \text{tp} = (a, b, c, \mathcal{L}, \varepsilon) \) and with rank and degree equal to \( r \) and \( d \), respectively.

3.1 Boundedness for generalized parabolic swamps

Let us denote by \( E_{d,r} \) the set of isomorphism classes of locally free sheaves on \( Y \) of rank \( r \) and degree \( d \). Recall that a family of isomorphism classes of sheaves \( E \subset E_{d,r} \) on \( Y \) is bounded if and only if there is a natural number \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and every locally free sheaf \( \mathcal{E} \in E, h^1(Y, \mathcal{E}(n)) = 0 \) and \( \mathcal{E}(n) \) is globally generated. Boundedness for locally free sheaves appearing in \((\kappa, \delta)\)-(semi)stable swamps with generalized parabolic structures (Proposition 2) will follow from the next observation.

Let \( \mathcal{E} \) be a locally free sheaf over \( Y \) and \((\mathcal{E}_q, m)\) a weighted filtration, with \( \mathcal{E}_q \equiv (0) \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s \subset \mathcal{E} \). Let us consider a partition of the multi-index \( I := (1, 2, \ldots, s) \), \( I = I_1 \cup I_2 \); let us say \( I_1 = (i_1, \ldots, i_t) \) and \( I_2 = (k_1, \ldots, k_{s-t}) \). Then, a short calculation (see [12, Lemma 1.6] for the connected case) shows that

\[
\left( \sum_{i=1}^{s} m_i \right) a(\alpha - 1) \geq \mu(\mathcal{E}_q, m, \phi) \geq -\left( \sum_{i=1}^{s} m_i \right) a(\alpha - 1),
\]

where \( \mathcal{E}_q^I = \mathcal{E}_I \). The following results are important direct consequences of Eq. (4).

**Proposition 1** A generalized parabolic swamp \((\mathcal{E}, q, \phi)\) is \((\kappa, \delta)\)-(semi)stable if and only if for any weighted filtration \((\mathcal{E}_q, m)\), such that \( \text{par} \mu(\mathcal{E}_q) \geq \text{par} \mu(\mathcal{E}) - C_1 \), where \( C_1 = a\delta + r \nu \), the inequality \( P_\kappa(\mathcal{E}_q, m) + \delta \mu(\mathcal{E}_q, m, \phi) \geq 0 \) holds.

**Proof** We just need to show the inverse implication. Let \((\mathcal{E}_q, m)\) be a weighted filtration such that \( \text{par} \mu(\mathcal{E}_q) < \text{par} \mu(\mathcal{E}) - C_1 \) for all \( i \). Since \( \kappa\text{-pardeg}(\mathcal{E}_q)\alpha - \kappa\text{-pardeg}(\mathcal{E})\alpha_i < -C_1\alpha\alpha_i \), Eq. (4) implies that \( P_\kappa(\mathcal{E}_q, m) + \delta \mu(\mathcal{E}_q, m, \phi) \geq \sum_{i=1}^{s} m_i (a\alpha\alpha_i - 1 + r \nu \alpha), \) is bounded by a constant depending only on \( a, \delta, r, h, \nu, d \). This, in particular, means that for any locally free sheaf \( \mathcal{E} \) of rank \( r \) and degree \( d \) appearing in a \((\kappa, \delta)\)-(semi)stable swamp
with a generalized parabolic structure of type \((a, - , - , - )\) \(^1\), we have that \(\deg(\mathcal{E}|_Y)\) is bounded from below and above by constants depending only on \(a, \delta, \alpha, \nu, d\) which we will denote by \(A_-(a, \delta, r, h, v, d)\) and \(A_+(a, \delta, r, h, v, d)\), or just by \(A_-\) and \(A_+\) if there is no confusion.

### 3.2 The Gieseker space and map

#### 3.2.1 The parameter space

Let \(H\) be an effective divisor of degree \(h\) in \(Y\) such that \(\mathcal{O}_Y(H) \simeq \mathcal{O}_Y(1)\). By Proposition \(2\), there exists a natural number \(n_0 \in \mathbb{N}\) such that for every \(n \geq n_0\) and every \((\kappa, \delta)\)-(semi)stable generalized parabolic structure of type \(\mathbf{tp} = (a, b, c, \mathcal{L}, \mathcal{E})\) of rank \(r\) and degree \(d\) we have

\[
\begin{align*}
H^1(Y, \mathcal{E}(n)) &= H^1(Y, \det(\mathcal{E}(rn))) = H^1(Y, \det(\mathcal{E})^\mathcal{O}_c \otimes \mathcal{L} \otimes \mathcal{O}_Y(\alpha n)) = 0, \\
\mathcal{E}(n), \det(\mathcal{E}(rn)), \det(\mathcal{E})^\mathcal{O}_c \otimes \mathcal{L} \otimes \mathcal{O}_Y(\alpha n) &\text{ are globally generated.}
\end{align*}
\]

(5)

Let us pick \(n \geq n_0\), and \(d = (d_1, \ldots, d_l) \in \mathbb{Z}^l\) with \(d = \sum_{i=1}^l d_i\). Let \(U\) be the vector space \(\mathbb{C}^\oplus p\) where \(p := r(h(\mathcal{O}_Y)) + d + an\) (recall \(a = hr\)). Let us denote by \(Q^0\) the quasi-projective scheme parametrizing equivalence classes of quotients \(q: U \otimes \pi^*_Y \mathcal{O}_Y(-n) \rightarrow \mathcal{E}\), where \(\mathcal{E}\) is a locally free sheaf of rank \(r\) and multidegree \(d = (d_1, \ldots, d_l)\) on \(Y\), such that the induced map \(U \rightarrow H^0(Y, \mathcal{E}(n))\) is an isomorphism. On \(Q^0 \times Y\), we have the universal quotient \(q_{Q^0}: U \otimes \pi^*_Y \mathcal{O}_Y(-n) \rightarrow \mathcal{E}_{Q^0}\). Since \(n > n_0\), the sheaf \(H := \mathcal{H}\mathcal{O}_{Q^0}(U_{a,b} \otimes \mathcal{O}_{Q^0}, \pi_{Q^0*}(\det(\mathcal{E}_{Q^0})^\mathcal{O}_c \otimes \pi^*_Y \mathcal{L} \otimes \pi^*_Y \mathcal{O}_Y(na)))\) is locally free. Let \(\pi^*\): \(\mathfrak{h} = \text{P}(H^\vee) \rightarrow Q^0\) be the corresponding projective bundle and \(q^*_h: U \otimes \pi^*_Y \mathcal{O}_Y(-n) \rightarrow \mathcal{E}_{Q^0}\) the pullback of the universal quotient to \(\mathfrak{h} \times Y\). Now, the tautological invertible quotient on \(\mathfrak{h}\), \(\mathfrak{p}^*H^\vee \rightarrow \mathfrak{p}_h(1) \rightarrow 0\), induces a morphism on \(\mathfrak{h} \times Y\), \(s_h: U_{a,b} \otimes \mathcal{O}_{\mathfrak{h}} \rightarrow \det(\mathcal{E}_{Q^0})^\mathcal{O}_c \otimes \pi^*_Y \mathcal{L} \otimes \pi^*_Y \mathcal{O}_Y(na) \otimes \pi^*_Y \mathcal{O}_Y(1)\). From the universal quotient, we obtain a surjective morphism \((q^*_{Q^0})^\oplus b: U_{a,b} \otimes \pi^*_Y \mathcal{O}_Y(-na) \rightarrow (\mathcal{E}_{Q^0})^\oplus b\). Denoting by \(\mathcal{K}\) its kernel, we get a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
0 & \rightarrow & U_{a,b} \otimes \pi^*_Y \mathcal{O}_Y(-na) & \rightarrow & (\mathcal{E}_{Q^0})^\oplus b & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \mathcal{E}_{Q^0} = \frac{\mathfrak{h} \otimes \pi^*_Y \mathcal{O}_Y(1)}{\mathfrak{h} \otimes \pi^*_Y \mathcal{O}_Y(-na)} & \rightarrow \\
\det(\mathcal{E}_{Q^0})^\mathcal{O}_c \otimes \pi^*_Y \mathcal{L} \otimes \pi^*_Y \mathcal{O}_Y(1)
\end{array}
\]

From [12, Lemma 3.1], it follows that there is a closed subscheme \(\mathfrak{G} \subset \mathfrak{h}\) over which \(s_h \otimes \pi^*_Y \mathcal{O}_Y(-na)\) factorizes through a morphism \(\phi_{\mathfrak{G}}: (\mathcal{E}_{Q^0})^\oplus b \rightarrow \det(\mathcal{E}_{Q^0})^\mathcal{O}_c \otimes \pi^*_Y \mathcal{L} \otimes \pi^*_Y \mathcal{O}_Y\), \(\pi^*_Y \mathcal{O}_Y\) being the pullback of the restriction of \(\mathcal{O}_Y(1)\) to \(\mathfrak{G}\). Then, on the scheme \(\mathfrak{G} \times Y\) we have a family of swamps \((\mathcal{E}_{\mathcal{G}}, \mathcal{M}_\mathcal{G}, \phi_{\mathcal{G}})\) parametrized by \(\mathfrak{G}\). In order to include the parabolic structure, we need to consider the Grassmannian \(\mathcal{G}_{r_i} := \text{Grass}_{e_i}(U^\oplus 2)\) of \(e_i\) dimensional quotients of \(U^\oplus 2\). Let us define

\[
Z := \mathfrak{G} \times \prod_i \mathcal{G}_{r_i}
\]

(6)

and denote by \(c_i: Z \rightarrow \mathcal{G}_{r_i}\) the \(i\)-th projection. Let \(q^i_z: U^\oplus 2 \otimes \mathcal{O}_Z \rightarrow R_{Z,i}\) be the pullback of the universal quotient of the Grassmannian \(\mathcal{G}_{r_i}\) by the projection \(c_i\) (if \(i \notin I(\mathcal{E})\) then \(R_{Z,i}\)

\(^1\) This means that the first component is fixed and equal to \(a\) but the others are left to be free.
is the zero sheaf and \( \mathcal{G}_t \) is just a point) and \( q_Z : U^{\oplus 2v} \otimes O_Z \to \bigoplus_1^v R_{Z,i} \) the direct sum. Let \( \mathcal{G} \times Y \to \mathcal{G} \), \( Z \times Y \to Z \) be the two projections, \( \mathcal{N}_Z \) the pullback of \( \mathcal{N}_{\mathcal{G}} \) to \( Z \), and \( q_Z, \mathcal{E}_Z \) and \( q_Z \) the pullbacks of the corresponding objects over \( \mathcal{G} \times Y \to Z \times Y \). Let us consider the morphisms \( \pi^i : Z \times D_i \to Z \). For each \( i \), there are quotients \( f_i : U^{\oplus 2} \otimes \mathcal{O}_Z \to \pi^i_a(\mathcal{E}_Z|_{D_i}) \) and we can form \( f : = \bigoplus f_i : U^{\oplus 2v} \otimes \mathcal{O}_Z \to \bigoplus \pi^i_a(\mathcal{E}_Z|_{D_i}) \). The morphisms \( q_Z \) and \( f \) give rise to the diagram

\[
0 \to \text{Ker}(f) \to U^{\oplus 2v} \times \mathcal{O}_Z \overset{f}{\to} \bigoplus \pi^i_a(\mathcal{E}_Z|_{D_i}) \to 0
\]

Let us denote by \( \mathcal{J}_d \subset Z \) the closed subscheme given by the zero locus of the morphism \( q' \) (see [12, lemma 3.1] again). Then, the restriction of \( q_Z \) to \( \mathcal{J}_d \) factorizes through

\[
q'_{\mathcal{J}_d} : \bigoplus \pi^i_a(\mathcal{E}_Z|_{D_i})|_{\mathcal{J}_d} = \bigoplus \pi^i_{3,d}(\mathcal{E}_{\mathcal{J}_d}|_{D_i}) \to \bigoplus R_{Z,i}|_{\mathcal{J}_d} = \bigoplus R_{3,d,i}.
\]

Since \( f \) and \( q_Z \) are diagonal morphisms, we deduce that \( q'_{\mathcal{J}_d} \) is also diagonal. Therefore, \( q'_{\mathcal{J}_d} \) is determined by \( v \) morphisms \( q'_{3,d} : \pi^i_{3,d} : (\mathcal{E}_{\mathcal{J}_d}|_{D_i}) \to R_{3,d,i} \). Denote by \( (q'_{3,d}, \mathcal{E}_{\mathcal{J}_d}, \mathcal{N}_{\mathcal{J}_d}, \phi_{\mathcal{J}_d}) \) the restriction of \( (q_{3,d}, \mathcal{E}_Z, \mathcal{N}_Z, \phi_Z) \) to \( \mathcal{J}_d \). Then, we have a universal family of generalized parabolic swamps, \( (\mathcal{E}_{\mathcal{J}_d}, q'_{3,d}, \mathcal{N}_{\mathcal{J}_d}, \phi_{\mathcal{J}_d}) \), with rank \( r \), multidegree \( (d_1, \ldots, d_l) \) and type \( tp = (a, b, c, \mathcal{L}, \mathcal{E}) \). Let us define

\[
I(r, d, \kappa, \delta, tp) := \left\{ (d_1, \ldots, d_l) \in \mathbb{Z}^l \mid \text{satisfying } d_1 + \ldots + d_l = d \text{ and such that there exists a } (\kappa, \delta) \text{-semistable with generalized parabolic structure of rank } r, \text{ multidegree}(d_1, \ldots, d_l) \text{ and type } tp = (a, b, c, \mathcal{L}, \mathcal{E}) \right\}
\]

From Remark 4, it follows that for every multi-index \( (d_1, \ldots, d_l) \in I(r, d, \kappa, \delta, tp) \) we have \( A_- \leq d_i \leq A_+ \), \( i = 1, \ldots, l \). Thus, \( I(r, d, \kappa, \delta, tp) \) is a finite set. Then, we define

\[
\mathcal{J} := \bigsqcup_{d \in I(r, d, \kappa, \delta, tp)} \mathcal{J}_d.
\]

By Eq. 5, it follows that every \( (\kappa, \delta) \)-semistable generalized parabolic swamp of rank \( r \), degree \( d \) and type \( tp \) determines a point in \( \mathcal{J} \).

### 3.2.2 The Gieseker space and map

We will show that there is a natural injective map from the parameter space \( \mathcal{J}_d \) into certain projective scheme, this map being \( \text{SL}(U) \)-equivariant. Let \( n_0 \in \mathbb{N} \) be as in Sect. 3.2.1.

Let us fix a Poincaré invertible sheaf \( \mathcal{P}_c \) on \( Y_1 \times \text{Pic}^d(Y_1) \) and let \( n > n_0 \) be a natural number. Define the sheaf \( \mathcal{G}'_1 = \mathcal{H}om_{\mathcal{O}_{\text{Pic}^d(Y_1)}}(\bigwedge^n U \otimes \mathcal{O}_{\text{Pic}^d(Y_1)}, \mathcal{P}(\text{Pic}^d(Y_1), n)) \). Since \( n > n_0 \), the above sheaf is locally free, and we can consider the corresponding projective bundle on \( \text{Pic}^d(Y_1) \), \( \mathcal{G}'_1 = \mathcal{P}(\mathcal{G}'_1) \). Note that the determinant map \( \mathcal{E}_{\mathcal{J}_d} \mapsto \bigwedge \mathcal{E}_{\mathcal{J}_d}|_{Y_1} = \bigwedge (\mathcal{E}_{\mathcal{J}_d}|_{Y_1}) \) defines a morphism \( \partial_i : \mathcal{J}_d \to \text{Pic}^d(Y_1) \). Consider now on \( \mathcal{J}_d \times Y \) the universal
quotient \( \eta_\mathcal{J}_d : U \otimes \pi_\mathcal{Y}_Y^* \mathcal{O}_Y(-n) \to \mathcal{E}_\mathcal{J}_d \). Restricting to the \( i \)-th component, twisting by \( n \) and taking determinants, we get \( \bigwedge^i q_\mathcal{J}_d \mathcal{J}_d(n) : \bigwedge^i U \otimes \mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y} \to \bigwedge^i \mathcal{E}_\mathcal{J}_d|_{\mathcal{Y}_Y} \otimes \pi_\mathcal{Y}_Y^* \mathcal{O}_Y(nr) \). Let \( \mathcal{N}_i \) be an invertible locally free sheaf on \( \mathcal{J}_d \) such that \( \bigwedge^i \mathcal{E}_\mathcal{J}_d|_{\mathcal{Y}_Y} = (q_i \times id_{\mathcal{Y}_Y})^* \mathcal{P}_i \otimes \pi_\mathcal{J}_d^* \mathcal{N}_i \). Then, we have a point \( \pi_\mathcal{J}_d^* (\bigwedge^i q_\mathcal{J}_d \mathcal{J}_d(n)) \in \mathbb{G}_1^*(\mathcal{J}_d) \) for each \( i \).

Let us define now \( \mathcal{G}_2 := \mathcal{H}om_{\mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y}}(U_{a,b} \otimes \mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y}, \pi_{\mathcal{J}_d \times \mathcal{Y}_Y}^* \mathcal{P} \otimes \pi_\mathcal{Y}_Y^* \mathcal{L} \otimes \pi_\mathcal{Y}_Y^* \mathcal{O}_Y(na)) \). For \( n > n_0 \), \( \mathcal{G}_2 \) is also locally free and we can consider the corresponding projective bundle on \( \mathcal{P}ic(\mathcal{Y}_Y) \), \( \mathcal{G}_2 = \mathcal{P}(\mathcal{G}_2^*) \). Consider now the universal quotient \( q_\mathcal{J}_d : U \otimes \mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y} \to \mathcal{E}_\mathcal{J}_d \) and the universal swamp \( \phi_\mathcal{J}_d : (\mathcal{E}_\mathcal{J}_d^*(\mathcal{G}_2))^{\otimes b} \to \det(\mathcal{E}_\mathcal{J}_d)^{\otimes c} \otimes \pi_\mathcal{Y}_Y^* \mathcal{L} \otimes \pi_\mathcal{Y}_Y^* \mathcal{O}_Y(na)) \). Let \( \mathcal{N} \) be an invertible sheaf on \( \mathcal{J}_d \) such that \( \det(\mathcal{E}_\mathcal{J}_d) = (\mathcal{O}_n \times \mathcal{D}(\mathcal{Y}_Y)) \otimes \pi_\mathcal{Y}_Y^* \mathcal{N} \) and note that \( U_{a,b} \otimes \mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y} \simeq \pi_\mathcal{J}_d^* (U_{a,b} \otimes \mathcal{O}_{\mathcal{J}_d \times \mathcal{Y}_Y}) \). Composing \( (q_\mathcal{J}_d \mathcal{J}_d(n))^{\otimes a} \otimes b \) with \( \phi_\mathcal{J}_d \), we obtain a point \( \psi \circ (\pi_\mathcal{J}_d \mathcal{J}_d(n))^{(\otimes a)}^{\otimes b} \in \mathcal{G}_2^*(\mathcal{J}_d) \). Altogether, with the obvious morphism to the Grassmannians, \( \mathcal{J}_d \to \prod_{i \in I(\mathcal{G})} \mathcal{G}_1 \), give us the so-called Gieseker morphism

\[
\text{Gies} : \mathcal{J}_d \longrightarrow \left( \left( \mathbb{G}_1^1 \times \ldots \times \mathbb{G}_1^1 \right) \times \mathbb{P}(\mathcal{G}_2) \right) \times \left( \prod_{i \in I(\mathcal{G})} \mathcal{G}_1 \right) =: \mathbb{G}. \tag{8}
\]

**Proposition 3** \( \text{Gies} : \mathcal{J}_d \to \mathbb{G} \) is injective and \( SL(U) \)-equivariant.

**Proof** Follows as in the connected case (see for instance [11, Lemma 4.3]). \( \square \)

### 3.3 Semistability

We will see that making \( n > n_0 \) even larger, \( \mathcal{J}_d \) contains all \((k, \delta)\)-(semi)stable generalized parabolic swamps of the given type, rank and degree. Let \( \mathcal{J}_d^{(k, \delta)\text{-ss}} \subset \mathcal{J}_d \) be the subscheme consisting of \((k, \delta)\)-(semi)stable generalized parabolic swamps. In order to show that the quotient \( \mathcal{J}_d^{(k, \delta)\text{-ss}} / SL(U) \) exists and is projective and contains \( \mathcal{J}_d^{(k, \delta)\text{-ss}} / SL(U) \) as an open subscheme, we first find a linearized invertible sheaf on \( \mathbb{G} \) for which \( \text{Gies}^{-1}(\mathcal{G}^{\text{ss}}) = \mathcal{J}_d^{(k, \delta)\text{-ss}} \) and then we show that \( \text{Gies}|_{\mathcal{J}_d^{(k, \delta)\text{-ss}}} \) is a proper morphism. The main auxiliary result is given in §3.3.2 (see Theorem 1) regarding the sectional semistability condition.

#### 3.3.1 Semistability in the Gieseker space

Let \( i_1, \ldots, i_{\nu} \) be the indices in \( I(\mathcal{G}) \). Let \( b_1, \ldots, b_1, c, k_{i_1}, \ldots, k_{i_{\nu}} \) be integers and consider the ample invertible sheaf on \( \mathcal{G}, \mathcal{O}_\mathcal{G}(b_1, \ldots, b_1, c, k_{i_1}, \ldots, k_{i_{\nu}}) \). Consider the obvious linearization on it and let \( \mathcal{G}^{\text{ss}} \) be the set of points which are (semi)stable with respect to the given linearization. Consider a weighted flag \((U^*, m)\), where \( U^* : (0) \subset U_1 \subset \ldots \subset U_\nu \subset U \), and \( m = (m_1, \ldots, m_\nu) \). Let \( \lambda : \mathcal{O}_m \to SL(U) \) be a one-parameter subgroup whose associated weighted flag is \((U^*, m)\). Let \( \lambda \) be a rational point of \( \mathcal{J}_d \) and \( \text{Gies}(\lambda) = (t_1, \ldots, t_{\nu}, i_{\nu-1}, i_{\nu}) \) its image in \( \mathbb{G} \). Let \( \mathcal{Q} : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E} \) be the locally free quotient sheaf corresponding to \( \lambda \). The weighted filtration \((U^*, m)\) induces a filtration of \( \mathcal{E} \) defined by \( \mathcal{E}_u := q(U_u \otimes \mathcal{O}_Y(-n)) \subset \mathcal{E} \). If we define \( \mathcal{E}_u^i := \mathcal{E}_u|_{Y_u}, \alpha_u := \alpha(\mathcal{E}_u) \) and \( l_u := \dim(U_u) \), assuming that \( h^1(Y, \mathcal{E}_u|_{Y_u}) = 0 \) and \( h^1(Y, \mathcal{E}_u(n)) \), the semistability
function is given by (see Sect. 2.4)

\[
\mu(\lambda, \text{Gies}(t)) = \sum_{i=1}^{l} b_i \sum_{u=1}^{s} m_u (\text{rk}(\mathcal{E}^i_u)p - rh^0(Y, \mathcal{E}_u(n))) \\
+ c \sum_{u=1}^{s} m_u (\zeta(l_0, l_u)p - ah^0(Y, \mathcal{E}_u(n))) \\
+ \sum_{i \in I(\mathcal{E})} k_i \sum_{u=1}^{s} m_u (p \dim(q_i(\mathcal{E}^i_u(y^i_1) \oplus \mathcal{E}^i_u(y^i_2))) - e_i h^0(Y, \mathcal{E}_u(n))).
\]

(9)

Let us fix now a particular polarization (recall \( h_i = \deg(\mathcal{E}^i_{Y_i}(1)) \)),

\[
\begin{align*}
&b_i := bh_i, b := p - b', b' := \sum_{i=1}^{l} \kappa_i e_j, \\
&c := \delta rh = \sum_{i=1}^{l} \delta rh_i, \\
&k_i := \kappa_i \alpha.
\end{align*}
\]

(10)

Then, Eq. (9) becomes,

\[
\mu(\lambda, \text{Gies}(t)) = \sum_{u=1}^{s} m_u \left\{ b \alpha_u p - h^0(Y, \mathcal{E}_u(n)) \alpha p \\
+ c \zeta(l_0, l_u)p + \sum_{i \in I(\mathcal{E})} \alpha \kappa_i p \dim(q_i(\mathcal{E}^i_u(y^i_1) \oplus \mathcal{E}^i_u(y^i_2))) \right\}.
\]

Again, since \( b = p - b' - b'_1 = a \delta \) and \( \alpha_u = \sum_{i=1}^{l} h_i \text{rk}(\mathcal{E}^i_u) \), we get

\[
\frac{\mu(\lambda, \text{Gies}(t))}{p} = \sum_{u=1}^{s} m_u \left\{ p \alpha_u - \alpha h^0(Y, \mathcal{E}_u(n)) + \sum_{i=1}^{l} h_i (r \zeta(l_0, l_u) - \text{rk}(\mathcal{E}^i_u)) \\
+ \sum_{i \in I(\mathcal{E})} \alpha \kappa_i \dim(q_i(\mathcal{E}^i_u(y^i_1) \oplus \mathcal{E}^i_u(y^i_2))) - b'_2 \alpha_u \right\}.
\]

Since \( h^1(Y, \mathcal{E}(n)) = h^1(Y, \mathcal{E}_u(n)) = 0 \), we have

\[
p \alpha_u - \alpha h^0(\mathcal{E}_u(n)) = \alpha_u P_{\mathcal{E}}(n) - \alpha P_{\mathcal{E}_u}(n) = \alpha_u \deg(\mathcal{E}) - \alpha \deg(\mathcal{E}_u).
\]

We also know that \( \kappa^- \text{-pardeg}(\mathcal{E}_u) = \deg(\mathcal{E}_u) - \sum_{i \in I(\mathcal{E})} \kappa_i \dim(q_i(\mathcal{E}^i_u(y^i_1) \oplus \mathcal{E}^i_u(y^i_2))) \) (see Remark 2) and \( \kappa^- \text{-pardeg}(\mathcal{E}) = \deg(\mathcal{E}) - \sum_{i \in I(\mathcal{E})} \kappa_i e_i \). Then, we finally get

\[
\frac{\mu(\lambda, \text{Gies}(t))}{p} = \sum_{u=1}^{s} m_u \left\{ \alpha_u \kappa^- \text{-pardeg}(\mathcal{E}) - \alpha \kappa^- \text{-pardeg}(\mathcal{E}_u) + \delta (\alpha \zeta(l_0, l_u) - a \alpha_u) \right\}.
\]

### 3.3.2 Sectional semistability

In the next theorem, we adapt the result [25, Theorem 2.12] to our case.
Theorem 1 There exists $n_1 \in \mathbb{N}$ such that for very $n > n_1$ and every $(\kappa, \delta)$-(semi)stable generalized parabolic swamp, $(\mathcal{E}, q, \phi)$, the inequality

$$\sum_{i=1}^{s} m_i (\text{par} \chi(\mathcal{E}(n)) \alpha_i - \text{par} h^0(\mathcal{E}_i(n)) \alpha + \delta \mu(\mathcal{E}_*, m, \phi)(\geq 0)$$

holds for every weighted filtration $(\mathcal{E}_*, m)$.

Proof Let $C_1$ be the constant given in Proposition 2. For any other constant, $C_2$, we denote by $E(C_1, C_2, d, \alpha)$ the family of isomorphism classes of locally free sheaves $\mathcal{E}'$ satisfying

a) $\mu(\mathcal{E}') \geq \frac{d}{\alpha} - C_2$, b) $1 \leq \alpha' \leq \alpha - 1$ and c) $\mu_{\max}(\mathcal{E}') \leq \frac{d}{\alpha} + C_1$. This is a bounded family, so there is a natural number $n(C_1, C_2, d, \alpha) \in \mathbb{N}$ such that $\mathcal{E}(n)$ is globally generated and $h^1(Y, \mathcal{E}(n)) = 0$ for each $\mathcal{E} \in E(C_1, C_2, d, \alpha)$ and for every $n \geq n(C_1, C_2, d, \alpha)$.

Let us assume that another constant $C_2$ is fixed. Let $\mathcal{E}$ be a locally free sheaf appearing in a $(\kappa, \delta)$-(semi)stable swamp of rank $r$ and degree $d$, and let $\mathcal{E}' \subset \mathcal{E}$ be a (locally free) subsheaf.

We distinguish two cases:

Case 1: We assume that $\mathcal{E}'$ belongs to $E(C_1, C_2, d, \alpha)$. Then, if $n \geq n(C_1, C_2, d, \alpha)$, we have that $\mathcal{E}'(n)$ is globally generated and $h^1(Y, \mathcal{E}'(n)) = 0$. Thus, we deduce $\text{par} \chi(\mathcal{E}(n)) \alpha' - \text{par} h^0(\mathcal{E}'(n)) \alpha = \kappa$-pardeg $(\mathcal{E}) \alpha' - \kappa$-pardeg $(\mathcal{E}') \alpha$.

Case 2: Assume that $\mathcal{E}'$ does not belong to the above family. Applying Le Potier–Simpson estimate (see [15, Corollary 3.3.8]), we get

$$h^0(Y, \mathcal{E}'(n)) \leq \alpha' \left( \frac{\alpha' - 1}{\alpha'} \left[ \frac{d}{\alpha} + C_1 + n + B \right] + \frac{1}{\alpha'} \left[ \frac{d}{\alpha} - C_2 + n + B \right] \right),$$

where $B := -1 + \alpha(\alpha + 1)/2$. Let $n'(C_1, C_2, d, \alpha) \in \mathbb{N}$ be the smallest natural number making $\frac{d}{\alpha} + C_1 + n + B$ and $\frac{d}{\alpha} - C_2 + n + B$ to be positive. Then, a short calculation shows that $h^0(Y, \mathcal{E}'(n)) \leq \alpha'(\frac{d}{\alpha} + n + B - C_2^2 + C_1(\alpha - 1))$ for every $n \geq n'(C_1, C_2, d, \alpha)$. From this, we deduce that

$$\chi(\mathcal{E}(n)) \alpha' - h^0(Y, \mathcal{E}'(n)) \alpha \geq \left( \frac{\alpha}{h} (1 - g) + d + \alpha n \right) \alpha'$$

$$- \alpha \left( \frac{d}{\alpha} + n + B' - C_2 \alpha + C_1(\alpha - 1) \right) \alpha'$$

$$= \alpha \alpha' \left( \frac{1}{h} - B \right) - \frac{\alpha' \alpha}{h} g + \alpha' C_2 - C_1 \alpha \alpha' (\alpha - 1)$$

$$\geq \alpha \alpha' (-B) + C_2 - C_1 \alpha (\alpha - 1)^2$$

$$\geq \alpha^2 (-B) + C_2 - C_1 \alpha (\alpha - 1)^2,$$

where the last inequality follows from the fact that $B$ is positive. Then,

$$\text{par} \chi(\mathcal{E}(n)) \alpha_i - \text{par} h^0(\mathcal{E}_i(n)) \alpha$$

$$= \chi(\mathcal{E}(n)) \alpha_i - h^0(Y, \mathcal{E}_i(n)) \alpha$$

$$\geq \alpha_i \sum_{j \in I(\mathcal{E})} \kappa_j e_j - \alpha \sum_{j \in I(\mathcal{E})} \kappa_j \dim(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j)))$$

$$\geq \alpha^2 (-B) + C_2 - C_1 \alpha (\alpha - 1)^2$$

$$\geq \alpha_i \sum_{j \in I(\mathcal{E})} \kappa_j e_j - \alpha \sum_{j \in I(\mathcal{E})} \kappa_j \dim(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j)))$$
\[
\geq \alpha^2(-B) + C_2 - C_1\alpha(\alpha - 1)^2 - r\alpha \left( \sum_{j \in I(q)} \kappa_j \right).
\]

Let us set \( K := -[B]_+\alpha^2 + C_2 - C_1\alpha(\alpha - 1)^2 - r\alpha(\sum_{j \in I(q)} \kappa_j) \). Then, \( \text{par}\chi(\mathcal{E}(n))\alpha_i - \text{parh}^0(\mathcal{E}_i(n))\alpha \geq K \). Let us assume now that \( C_2 \) is large enough so that \( K > \delta\alpha(\alpha - 1) \) and let

\[ n > n_1 := \max \{ n_0, n(C_1, C_2, d, \alpha), n'(C_1, C_2, d, \alpha) \}, \]

\( n_0 \) being that natural number considered in § 3.2.1. Let \( (\mathcal{E}, m) \) be a weighted filtration with \( \mathcal{E}_r(\mathcal{E}) \equiv (0) \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s \subset \mathcal{E} \) and \( m = (m_1, \ldots, m_s) \). We make a partition of this filtration as follows. Let \( j_1, \ldots, j_t \) be the indices such that \( \mu(\mathcal{E}_{j_i}) \geq \frac{d}{\alpha} - C_2, \mathcal{E}_{j_i}(n) \) is globally generated and \( h^1(Y, \mathcal{E}_{j_i}(n)) = 0 \) for \( i = 1, \ldots, t \). Let \( l_1, \ldots, l_{s-t} \) be the set of indices \( \{1, 2, \ldots, s\} \setminus \{j_1, \ldots, j_t\} \) in an increasing order. Let us consider the weighted filtrations \( (\mathcal{E}_{\bullet, 1}, m_1) \) and \( (\mathcal{E}_{\bullet, 2}, m_2) \) defined as

\[
\mathcal{E}_{\bullet, 1} \equiv (0) \subset \mathcal{E}_{j_1} \subset \ldots \subset \mathcal{E}_{j_t} \subset \mathcal{E}, \quad m_1 = (m_{j_1}, \ldots, m_{j_t}),
\]

\[
\mathcal{E}_{\bullet, 2} \equiv (0) \subset \mathcal{E}_{l_1} \subset \ldots \subset \mathcal{E}_{l_{s-t}} \subset \mathcal{E}, \quad m_2 = (m_{l_1}, \ldots, m_{l_{s-t}}).
\]

From Eq. (4), we find that \( \mu(\mathcal{E}_{\bullet, \mathbf{0}, \phi}) \geq \mu(\mathcal{E}_{\bullet, 2, \mathbf{0}, \phi}) - (\sum_{q=1}^t m_{j_q})\alpha(\alpha - 1) \). Thus

\[
\sum_{i=1}^s m_i(\text{par}\chi(\mathcal{E}(n))\alpha_i - \text{parh}^0(\mathcal{E}_i(n))\alpha) + \delta\mu(\mathcal{E}_{\bullet, \mathbf{0}, \phi}) \geq \sum_{q=1}^t m_{j_q}(\text{par}\chi(\mathcal{E}(n))\alpha_{j_q} - \text{parh}^0(\mathcal{E}_{j_q}(n))\alpha) + \delta\mu(\mathcal{E}_{\bullet, 1, m_1, \phi})
\]

\[
+ \left( \sum_{q=1}^{t-s} m_{l_q} \right) K - \delta \left( \sum_{q=1}^{s-t} m_{l_q} \right) \alpha(\alpha - 1) \geq 0,
\]

which proves the theorem. \( \square \)

3.3.3 \((\kappa, \mathfrak{d})\)-semistability and Hilbert–Mumford semistability

The goal now is to prove Theorem 2, which shows that \((\kappa, \mathfrak{d})\)-(semi)stability is equivalent to GIT (semi)stability in the Gieseker space under some conditions.

Let \( B := -1 + \alpha(\alpha + 1)/2 \) be the constant given in the proof of Theorem 1 and let \( K' \) be a constant such that \( d + K' > 0 \) and with the property

\[
\alpha K' > \max \left\{ d(w - \alpha) + a\gamma + a\delta(\alpha - 1) + B(\alpha - 1)w = 1 \ldots \alpha - 1 \right\}, \tag{11}
\]

Proposition 4 Let us assume \( \kappa_1 \leq 1 \). There exists \( n_2 \in \mathbb{N} \) and a constant \( C_3 \) such that for every \( n \geq n_2 \) and for any triple \( t = (q : U \otimes \phi_Y(-n) \to \mathcal{E}, q, \phi) \) of degree \( d \) and multiplicity \( \alpha \) whose induced map \( U \to H^0(Y, \mathcal{E}(n)) \) is injective and determines a semistable point in the Gieseker space, \( \mathbb{G}(\mathfrak{d}) \), it holds \( \mu_{\max}(\mathcal{E}) \leq \mu(\mathcal{E}) + C_3 \).

Remark 5 Although the triple \( t \) is not assumed to be in \( \mathfrak{I} \), we still denote by \( \text{Gies}(t) \) the point in \( \mathfrak{G} \) defined by \( t \).
Let us consider the locally free sheaf $\hat{E}$ for every subsheaf $E$, since in such case we would have

$$\mu(\hat{E}'') \leq \mu(\hat{E}) < \frac{d + K'}{\alpha(\hat{E}')} \leq d + K' \leq \mu(\hat{E}) + C_3$$

for every subsheaf $\hat{E}'' \subset \hat{E}$, where $C_3 := \frac{d}{\alpha} (\alpha - 1) + K'$.

Let $Q := \hat{E}/\hat{E}'$ be the (semistable) locally free quotient sheaf. Let us use the notation $\alpha' := \alpha(\hat{E}')$, $\alpha'' := \alpha(Q)$, $d' := \deg(\hat{E}')$, $d'' := \deg(Q)$, $\mu' := \mu(\hat{E}')$ and $\mu'' := \mu(Q)$. We will prove the statement by contradiction. To do so, let us assume that $d' \geq d + K'$. We have $h^0(Y, Q(n)) \leq \alpha''[\mu'' + n + B]_+$ for all $n \in \mathbb{N}$ (see [15, Corollary 3.3.8]). Then, we have to study two different cases.

**Case 1:** $h^0(Y, Q(n)) \leq \alpha''(\mu'' + n + B)$. Let us set $U' := H^0(Y, \hat{E}'(n)) \cap U$. Then,

$$\dim(U') \geq p - h^0(Y, Q(n))\alpha\left(\frac{1 - g}{h}\right) + d + \alpha n - \alpha''(\mu'' + n + B)$$

$$\geq \alpha\left(\frac{1 - g}{h} + n\right) + d - d'' - \alpha''\left(\frac{1 - g}{h} + n\right) - \alpha'' B$$

$$\geq \alpha'\left(\frac{1 - g}{h} + n\right) + d + K' - B(\alpha - 1).$$

Let us consider the locally free sheaf $\hat{E} := \text{Im}(U' \otimes \mathcal{O}_Y(-n) \to \hat{E})$. We have $U' \subset H^0(Y, \hat{E}(n)) \cap U$ (see [12, Lemma 3.3.3], which also holds in our case), $\text{rk}(\hat{E}|_{Y_i}) \leq \text{rk}(\hat{E}'|_{Y_i})$ and $\hat{E}$ is generically generated by global sections. Let $\{u_1, \ldots, u_p\}$ be a basis of $U'$ and complete it to a basis $\mathcal{B} = \{u_1, \ldots, u_p\}$ of $U$. Let $\lambda = \lambda(u, \gamma_p(i))$ be the associated one-parameter subgroup. Then, we obtain

$$\mu_G(\lambda, t_1, i) = \text{prk}(\hat{E}|_{Y_i}) - r \dim(U') \leq \text{prk}(\hat{E}'|_{Y_i}) - r \dim(U').$$

Since $\xi(I, i) \leq a$, we also have $\mu_G(\lambda, t_2) \leq a(p - \dim(U'))$. Therefore,

$$\mu_G(\lambda, \text{Gies}(t)) = \sum_{i=1}^{l} b_i \mu_{G_1}(\lambda, t_1, i) + c \mu_{G_2}(\lambda, t_2)$$

$$+ \sum_{i \in I(\mathcal{G})} k_i(p \dim(q_i(\hat{E}|_{Y_1} \oplus \hat{E}|_{Y_2})) - e_i \dim(U'))$$

$$\leq \sum_{i=1}^{l} h_i(p - a\delta - \sum_{i \in I(\mathcal{G})} k_i e_i)(\text{prk}(\hat{E}'|_{Y_i}) - r \dim(U'))$$

$$+ \sum_{i=1}^{l} h_i \delta r a(p - \dim(U'))$$

$$+ \sum_{i \in I(\mathcal{G})} k_i a(p \dim(q_i(\hat{E}'|_{Y_1} \oplus \hat{E}'|_{Y_2})) - e_i \dim(U')).$$
A short calculation gives us

\[
\frac{\mu_G(\lambda, \text{Gies}(t))}{p} \leq \alpha' \left( p - \left( \sum_{i \in I(\mathcal{G})} \kappa_i e_i \right) \right) - \alpha \left( \dim(U') \right) - \sum_{i \in I(\mathcal{G})} \kappa_i \alpha \left( \dim(q_i(\mathcal{E}'(y'_1) \oplus \mathcal{E}'(y'_2))) \right) + a\delta(\alpha - \alpha').
\] (12)

Since \( p = \alpha(n + \frac{1-g}{h}) + d \) and \( \dim(U') \geq d + K' + \alpha'(n + \frac{1+g}{h}) - B(\alpha - 1) \), we deduce that,

\[
\frac{\mu_G(\lambda, \text{Gies}(t))}{p} \leq a\delta(\alpha - \alpha') - \alpha K' + B\alpha(\alpha - 1) - \alpha' \left( \sum_{i \in I(\mathcal{G})} \kappa_i e_i \right)
+ \alpha \left( \sum_{i \in I(\mathcal{G})} \kappa_i \dim(q_i(\mathcal{E}'(y'_1) \oplus \mathcal{E}'(y'_2))) \right) + d(\alpha' - \alpha).
\]

Since \( \alpha' \left( \sum_{i \in I(\mathcal{G})} \kappa_i e_i \right) > 0, \alpha \sum_{i \in I(\mathcal{G})} \kappa_i \dim(q_i(\mathcal{E}'(y'_1) \oplus \mathcal{E}'(y'_2))) \leq \alpha vv r \) (because \( \kappa_i \leq 1 \)) and \( \alpha - \alpha' < \alpha - 1 \), we get \( \mu_G(\lambda, \text{Gies}(t)) < 0 \). However, \( \text{Gies}(t) \) is semistable so we obtain a contradiction.

**Case 2:** \( h^0(\mathcal{Y}, \mathcal{D}(n)) = 0 \). Let \( n'_2 \) be the smallest natural number \( n \in \mathbb{N} \) such that \( n > \frac{g-1}{h} \).

Let us assume \( n \geq n'_2 \), then, \( \dim(U') = p \). The same calculation as before (see Eq. (12)) gives rise to the inequality

\[
\frac{\mu_G(\lambda, \text{Gies}(t))}{p} \leq \alpha' \left( p - \sum_{i \in I(\mathcal{G})} \kappa_i e_i \right) - \alpha \left( \dim(U') \right)
- \sum_{i \in I(\mathcal{G})} \kappa_i \dim(q_i(\mathcal{E}'(y'_1) \oplus \mathcal{E}'(y'_2))) + a\delta(\alpha - \alpha')
\leq (\alpha' - \alpha)(p - a\delta) + a\nu r.
\]

Let \( n''_2 \) be the smallest natural number \( n \in \mathbb{N} \) such that \( p - a\delta > \frac{-a\nu r}{\alpha - \alpha} \) (recall that \( p = r_X(\mathcal{O}_Y) + d + an \)). If we choose \( n \geq n_2 := \max\{n'_2, n''_2\} \), then \( \mu_G(\lambda, \text{Gies}(t)) < 0 \) and we get again a contradiction. □

**Theorem 2** Let us assume \( \kappa_i \leq 1 \). There exists \( n_3 \in \mathbb{N} \) such that for every natural number \( n \geq n_3, t = (q : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}, q, \phi) \in \text{Gies}^{-1}(\mathbb{G}(s)^3) \) if and only if \( (\mathcal{E}, q, \tau) \) is \( (\kappa, \delta) \)-semistable.

**Proof** Let \( C_3 \) be the constant given in Proposition 4. Let us set \( \overline{C}_1 = \frac{1}{h} + C_1 \). Let \( E(\overline{C}_1, C_3, d, \alpha) \), be the family of locally free sheaves satisfying a) \( \mu(\mathcal{E}') \geq \frac{d}{\alpha} - \overline{C}_1 \), b) \( 1 \leq \alpha' \leq \alpha - 1 \) and c) \( \mu_{\max}(\mathcal{E}') \leq \frac{d}{\alpha} + C_3 \). This family is bounded, so there is a natural number, \( n'_3 \in \mathbb{N} \), large enough such that \( \mathcal{E}'(n) \) is globally generated and \( h^1(\mathcal{Y}, \mathcal{E}'(n)) = 0 \) for any \( \mathcal{E}' \) in this family and for every \( n \geq n'_3 \). Let us define \( n_3 := \max\{n_0, n_1, n_2, n'_3\} \) and fix \( n \geq n_3 \).

1) From the construction of the parameter space, we know that \( q \) induces an isomorphism \( U \cong H^0(\mathcal{Y}, \mathcal{E}(n)) \). Then, by Proposition 4, \( \text{Gies}(t) \in \mathbb{G}(s)^3 \) implies \( \mu_{\max}(\mathcal{E}) \leq \frac{d}{\alpha} + C_3 \). We
also know, by Proposition 1, that $(\mathcal{E}, q, \phi)$ is \((\kappa, \delta)\)-(semi)stable if and only if $P_\kappa(\mathcal{E}, m) + \delta \mu(\mathcal{E}, m, \phi) \geq 0$ for every $(\mathcal{E}, m)$ with $\mu(\mathcal{E}), \mu(\mathcal{E}_j) \geq \mu(\mathcal{E}) - C_1$. Observe that, in this case, $\mu(\mathcal{E}) \geq \mu(\mathcal{E}_j) \geq \mu(\mathcal{E}) - C_1$. Then $\frac{\mu(\mathcal{E})}{\mu(\mathcal{E}_j)} \geq \frac{\mu(\mathcal{E})}{\mu(\mathcal{E}) - C_1}$. Now, let us take a weighted filtration $(\mathcal{E}, m)$ of $\mathcal{E}$ with $\mu(\mathcal{E}_j) \geq \mu(\mathcal{E}) - C_1$, so each $\mathcal{E}_j$ belongs to $E(C_1, C_3, \alpha, \alpha)$. Let $u = \{u_1, \ldots, u_p\}$ be a basis of $U$, such that there are indices $l_1, \ldots, l_s$ with $U^{(l_j)} := \langle u_1, \ldots, u_{l_j} \rangle \cong H^0(Y, \mathcal{E}(n))$ for each $j$. Let us define $\nu = \sum_{j=1}^s \alpha_j \nu_{p(l_j)}$ and consider the one-parameter subgroup $\lambda(u, \nu_{p(l_j)})$. Looking at the calculations at the beginning of Sect. 3.3.1, we have
\[
0 \leq \mu_G(\lambda(u, \nu), \text{Gies}(t)) \leq \sum_{u=1}^s m_u \left( (\alpha_u \nu_{\mathcal{E}(\mathcal{E})} - \alpha \nu_{\mathcal{E}(\mathcal{E})} + \delta \alpha \zeta(I_0, I_u) - a \alpha_u) \right)
\leq \sum_{u=1}^s m_u \left( (\alpha_u \nu_{\mathcal{E}(\mathcal{E})} - \alpha \nu_{\mathcal{E}(\mathcal{E})} + \delta \alpha \zeta(I_0, I_u) - a \alpha_u) \right)
= P_\kappa(\mathcal{E}, m) + \delta \mu(\mathcal{E}, m, \tau).
\]
$\hat{\mathcal{E}}_u$ being the saturated subsheaf generated by $\mathcal{E}_u, \alpha_u$ being the multiplicity of $\mathcal{E}_u$, and $I_0$ being a multi-index giving the minimum in $\mu_G(\lambda(u, \nu), \text{Gies}(t))$. The last inequality follows from the fact that $\alpha_u := \alpha(\hat{\mathcal{E}}_u) = \alpha_u$ and $\nu_{\mathcal{E}(\mathcal{E})} \geq \nu_{\mathcal{E}(\mathcal{E})}$. We conclude that the swamp is \((\kappa, \delta)\)-(semi)stable.

2) By Theorem 1, it follows that
\[
\sum_{i=1}^s m_i (\text{par}_G(\mathcal{E}(\mathcal{E})(n)) \alpha_i - \text{par} h^0(\mathcal{E}_i(n)) \alpha_i) + \delta \mu(\mathcal{E}, m, \phi) \geq 0
\]
for any weighted filtration $(\mathcal{E}, m)$ of $\mathcal{E}$. Let $\lambda$ be a one-parameter subgroup and $(U, m')$ a weighted filtration such that $\lambda = \lambda(U, m')$. This filtration, together with the quotient $q_t : U \otimes \Theta_Y(-n) \rightarrow \mathcal{E}$, induces a chain
\[
(0) \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_s \subseteq \mathcal{E}
\]
and, therefore, a filtration $\mathcal{E} = (0) \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_s \subseteq \mathcal{E}$, formed by the different subsheaves collected in the above chain. Let $J = (i_1, \ldots, i_s)$ be the multi-index defined by the following condition: $i_j \in \{1, \ldots, s\}$ is the maximum among those $k \in \{1, \ldots, s\}$ such that $\mathcal{E}_j = \mathcal{E}_k$. Let us denote by $m_j$ the sum of the numbers $m^j_k$ corresponding to those sheaves in the chain (14) which are equal to $\mathcal{E}_i$, i.e., $m_j = m_k + m_{k+1} + \ldots + m_{i_j}, (k, k+1, \ldots, i_j)$ being the indices such that $\mathcal{E}_k = \mathcal{E}_{k+1} = \ldots = \mathcal{E}_{i_j} = \mathcal{E}_j$. We obtain in this way a weighted filtration $(\mathcal{E}, m)$. Multiplying by $p$ in Eq. (14), we get
\[
0 \leq \sum_{i=1}^s m_i \left( p^2 \alpha_i - p h^0(Y, \mathcal{E}_i(n)) \alpha + \alpha (\mathcal{E}_i(n) - a \alpha_i) \right)
+ p \sum_{j \in I(\mathcal{E})} \kappa_j \dim(q_j(\mathcal{E}_i(y_1^j) + \mathcal{E}_i(y_2^j))) \alpha_i - p \left( \sum_{j \in I(\mathcal{E})} \kappa_j e_j \right) \alpha_i.
\]
The inverse calculation presented in Sect. 3.3.1 gives

\[ 0 \leq \sum_{u=1}^{l} b_u \sum_{i=1}^{s} m_i(r(\mathcal{E}_u^i)p - r h^0(Y, \mathcal{E}_u(n))) \]
\[ + c \sum_{i=1}^{s} m_i(\xi_i(\mathcal{E}_u^i)p - a h^0(Y, \mathcal{E}_u(n))) \]
\[ + \sum_{j \in I(\mathfrak{q})} k_j \sum_{i=1}^{s} m_i(p \dim(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) - e_j h^0(Y, \mathcal{E}_i(n))). \] (15)

Let \( I_0 \) be a multi-index giving the minimum in \( \mu(\mathcal{E}_u, m, \phi) \). Taking into account Remark 3 1) and the fact \( l_i := \dim U_i \leq h^0(Y, \mathcal{E}_u^i(n)) \), Eq. (15) leads to

\[ 0 \leq \sum_{u=1}^{l} b_u \sum_{i=1}^{s'} m_i'(r(\mathcal{E}_u^i)p - r l_i) \]
\[ + c \sum_{i=1}^{s'} m_i'(\xi_i(\mathcal{E}_u^i) p - a l_i) \]
\[ + \sum_{j \in I(\mathfrak{q})} k_j \sum_{i=1}^{s'} m_i'(p \dim(q_j(\mathcal{E}_i(y_1^j) \oplus \mathcal{E}_i(y_2^j))) - e_j l_i) \]
\[ \leq \mu_G(\lambda(U_\bullet, m'), \text{Gies}(t)), \]

and the result is proved. \( \square \)

3.4 The moduli space

Let \( \mathcal{J}_d^{(\kappa, \delta)-(s)s} \subset \mathcal{J}_d \) be the open subscheme representing \((\kappa, \delta)-(semi)stable\) objects. The last step before proving the existence of the moduli space consists in showing that the restriction of the Gieseker map to the \((\kappa, \delta)-(semistable)\) locus is proper.

**Proposition 5** Let us assume \( \kappa_i < 1 \) for \( i \in I(\mathfrak{q}) \). There exists \( n_4 \in \mathbb{N} \) with \( n_4 \geq n_3 \) such that the Gieseker morphism, \( \text{Gies}: \mathcal{J}_d^{(\kappa, \delta)-(s)s} \rightarrow G^{(s)s} \), is proper for any \( d \in I(r, d, \kappa, \delta, \mathfrak{t}p) \) and for every \( n > n_4 \).

**Proof** We use the valuative criterion for properness. Let \( (\mathcal{O}, m, k) \) be a DVR, \( K \) being its field of fractions and assume we have a commutative diagram

\[ \text{Spec}(K) \xrightarrow{h_K} \mathcal{J}_d^{(\kappa, \delta)-(s)s} \]
\[ \downarrow \]
\[ \{0, \eta\} = S : \xrightarrow{h} \mathcal{O} \rightarrow G^{(s)s}, \]

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where 0 and \( \eta \) are the closed point and the generic point of \( S \), respectively. The morphism \( h_K \) is given by a family \( (q_K, \phi_K) \) over \( Y_K := Y \times \text{Spec}(K) \), where

\[
q_K : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}_K \to 0
\]

\[
\phi_K : (\mathcal{E}_K^{\otimes a})^{\oplus b} \to \det(\mathcal{O}_K)^{\otimes c} \otimes \mathcal{L}_K
\]

(16)

Let us show that \( h_K \) can be extended to a family, \( \hat{h} = (q_S, \phi_S, q_S) \), over \( Y \times S \). The quotient \( q_K \) defines a point in the Quot scheme of quotients of \( U \otimes \mathcal{O}_Y(-n) \) with Hilbert polynomial \( P(n) \). Therefore, there exists a (unique) flat extension

\[
q_S : U \otimes \mathcal{O}_Y(-n) \to \mathcal{E}_S \to 0
\]

over \( Y \times S \). We define the sheaves \( \mathcal{M} := \pi_S^*(\det(\mathcal{O}_S)^{\otimes c} \otimes \mathcal{O}_S^{\otimes 1} \mathcal{O}_Y(\mathcal{L})) \) and \( \mathcal{G} = \pi_S^*((U^{(a)} \otimes \mathcal{O}_Y)) \). Both sheaves are locally free, so we can form the projective space over \( S \), \( \text{pr}_S : \mathbb{P} := \text{Proj} \left( \text{Hom}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{G}) \right) \to S \), which carries a tautological morphism over \( \mathbb{P} \times Y \), \( \mathcal{G}' \to \mathcal{M}' \), where

\[
\mathcal{G}' = \text{pr}_S^*(\text{pr}_{\mathbb{P}}^*U^{(a)} \otimes \mathcal{O}_Y)
\]

\[
\mathcal{M}' = (\text{id}_Y \times \text{pr}_S)^*(\det(\mathcal{O}_S)^{\otimes c} \otimes \mathcal{O}_Y(\mathcal{L})) \otimes \mathcal{O}_Y \mathcal{L} \otimes \text{pr}_S^*\mathcal{O}_{\mathbb{P}}(1)
\]

The canonical morphism \( \Delta : \mathcal{G}' \to (U^{(a)} \otimes \mathcal{O}_Y) \) induces a diagram

\[
\begin{array}{ccc}
\mathcal{G}' & \longrightarrow & \mathcal{G}' \\
\downarrow & & \downarrow \\
\mathcal{M}' & \longrightarrow & (\text{id}_Y \times \text{pr}_S)^*(\mathcal{E}_S(n)^{\otimes a})^{\oplus b}
\end{array}
\]

\( \mathcal{K} \) being the kernel of the horizontal arrow. Let \( S \subset \mathbb{P} \) be the closed subscheme over which \( g \) is the zero morphism, i.e., over which the tautological morphism factorizes through \( (\text{id}_Y \times \text{pr}_S)^*(\mathcal{E}_S(n)^{\otimes a})^{\oplus b} \). Thus, we have a morphism

\[
(\text{id}_Y \times \text{pr}_S)^*(\mathcal{E}_S^{\otimes a})^{\oplus b} \to (\text{id}_Y \times \text{pr}_S)^*\det(\mathcal{E}_S)^{\otimes c} \otimes \mathcal{O}_Y \mathcal{L} \otimes \text{pr}_S^*\mathcal{O}_{\mathbb{P}}(1)
\]

over \( S \times Y \). Note now that the morphism \( \phi_K : (\mathcal{E}_K^{\otimes a})^{\oplus b} \to \det(\mathcal{O}_K)^{\otimes c} \otimes \mathcal{L}_K \) defines a point \( \text{Spec}(K) \to S \). Since \( S \) is projective this point extends (uniquely) to a point \( \text{Spec}(\mathcal{O}) \to S \), i.e., to a morphism

\[
\phi_S : (\mathcal{E}_S^{\otimes a})^{\oplus b} \to \det(\mathcal{O}_S)^{\otimes c} \otimes \mathcal{O}_Y \mathcal{L} \otimes N.
\]

(18)

Let us extend now the parabolic structure. Since \( \mathcal{E}_S, \eta \simeq \mathcal{E}_K \), we have an isomorphism \( \pi_{K*}(\mathcal{E}_S, \eta|_{D_i}) \simeq \pi_{K*}(\mathcal{E}_K|_{D_i}) \). Thus, composing with \( \pi_{K*}(\mathcal{E}_K|_{D_i}) \to R_K \to 0 \), we obtain a surjection \( \pi_{K*}(\mathcal{E}_S, \eta|_{D_i}) \to R_K \to 0 \). Observe that the morphism \( \pi_S : D_i \times S \to S \) is finite, thus affine and proper. By flat base change, we know that \( \pi_{K*}(\mathcal{E}_S, \eta|_{D_i}) = j_*\pi_{S*}(\mathcal{E}_S|_{D_i}) \). \( j \) being the open embedding \( j : \eta \hookrightarrow S \). Taking the push-forward and composing with the canonical map \( \pi_{S*}(\mathcal{E}_S|_{D_i}) \to j_*j_*\pi_{S*}(\mathcal{E}_S|_{D_i}) \), give rise to a morphism \( \pi_{S*}(\mathcal{E}_S|_{D_i}) \to j_*R_K \). Let \( R_S \subset j_*R_K \) be its image. Then, by [14, Proposition 2.8.1], \( R_S \) is \( \mathcal{O} \)-flat (thus a free \( \mathcal{O} \)-module) and the quotient

\[
q_{iS} : \pi_{S*}(\mathcal{E}_S|_{D_i}) \to R_S \to 0
\]

extends \( q_i : \pi_K^*(\mathcal{E}_K|_{D_i}) \to R_K \to 0 \) (thus \( \text{rk}(R_S) = e_i \)). Then, the family given in (17), (18), (19), \( \hat{h} = (q_S, \phi_S, q_S) \), extends the family given in Eq. (16) to \( S \). Clearly, the family
(q, \phi, q_{\hat{\lambda}}) defines an S-valued point \( t : S \to G \) in the Gieseker space. Since \( t(\eta) = h(\eta) \), we deduce that \( t(0) = h(0) \); thus, it defines a semistable point in \( G \). Let us show that \( q(0) \) induces an isomorphism \( U \simeq H^0(Y, \mathcal{E}_0(n)) \). To show that it is injective, we consider the linear map \( U \) induces an isomorphism \((1, 2, 3) A. L. \) Muñoz Castañeda

we deduce that \( \omega \) does not. Then, by Serre duality, there is a non-trivial morphism \( \mathcal{O} \). Let us demonstrate that \( E \) defines an

\[ \mu_G(\lambda, t(0)) = \sum_{i=1}^l b_i \mu_{G_1}(\lambda, t_{1,i}(0)) + c \mu_{G_2}(\lambda, t_{2}(0)) \]

\[ + \sum_{i \in I(G)} k_i \mu_{\mathbb{G}_m}(\lambda, t_{3,i}(0)) \]

\[ = \sum_{i=1}^l b_i(-r \dim(H)) + c \alpha(-\dim(H)) \]

\[ + \sum_{i \in I(G)} k_i(p \dim(t_{10}(H \oplus H) - e_i \dim(H)) \]

\[ = \sum_{i=1}^l h_i(p - a \delta - \sum_{j \in I(G)} \kappa j e_j)(-r \dim(H)) + \sum_{i=1}^l h_i \delta r a(-\dim(H)) \]

\[ + \sum_{i \in I(G)} \kappa i \alpha(-e_i \dim(H)) = -\alpha p \dim(H) \geq 0 \]

so we must have \( \dim(H) = 0 \), i.e., \( U \to H^0(Y, \mathcal{E}_0(n)) \) is injective. Let us show that it is in fact an isomorphism. For that, we just need to show that \( h^1(Y, \mathcal{E}_0(n)) = 0 \). Suppose it does not. Then, by Serre duality, there is a non-trivial morphism \( \mathcal{O} \). Let \( \mathcal{G} \) be its image, and consider the linear map \( \Omega : U \leftarrow H^0(Y, \mathcal{E}_0(n)) \to H^0(Y, \mathcal{G}) \). Let \( H \subset U \) be the kernel of \( \Omega \), \( \lambda \) the corresponding one-parameter subgroup and \( \mathcal{F} \subset \mathcal{E}_0(n) \) the subsheaf generated by \( H \). Since \( t(0) \) is semistable, we obtain:

\[ 0 \leq \frac{\mu(\lambda, \text{Gies}(t))}{p} = p \alpha_{\mathcal{G}} - \alpha \dim(H) + \delta \sum_{i=1}^l h_i(r \xi(I_0, \dim(H)) - \text{ark}(\mathcal{F})) \]

\[ + \sum_{i \in I(G)} \alpha \kappa_i \dim(q_i(\mathcal{F}(y^1_i) \oplus \mathcal{F}(y^2_i))) - b_2 \alpha_{\mathcal{G}}. \]

where \( \alpha_{\mathcal{G}} := \alpha(\mathcal{G}) \). Since \( h_0^0(Y, \mathcal{G}) \geq p - \dim(H) \), we get

\[ 0 \leq -p \alpha_{\mathcal{G}} + a h^0(Y, \mathcal{G}) + \delta \sum_{i=1}^l h_i(r \xi(I_0, \dim(H)) - \text{ark}(\mathcal{F})) \]

\[ + \sum_{i \in I(G)} \alpha \kappa_i \dim(q_i(\mathcal{F}(y^1_i) \oplus \mathcal{F}(y^2_i))) - b_2 \alpha_{\mathcal{G}} \]

and, therefore, \( h_0^0(Y, \mathcal{G}) \geq \frac{p}{a} + M \), with \( M = -\frac{\delta}{a} h(a - 1) + r v \). Note that \( p = a n + d + r \chi(\mathcal{O}_Y) \). Let us define \( n_5 \in \mathbb{N} \) as the smallest natural number such that \( h^0(Y, \omega_Y) < h^0(Y, \mathcal{G}) \). Then, if \( n \geq n_5 \) we get a contradiction since \( h^0(Y, \omega_Y) \geq h^0(Y, \mathcal{G}) \) always holds, so \( h^1(Y, \mathcal{E}_0(n)) = 0 \).

Let us demonstrate that \( \mathcal{E}_0(n) \) has no torsion by contradiction. Let us assume that it has torsion, \( T \subset \mathcal{E}_0(n) \), supported on the divisors \( D_i \), set \( T := H^0(Y, T) \) and \( H := \)
Since $\kappa_i < 1$, we must have $\dim(T_{D_i}) = 0$, that is $T = 0$, so $\mathcal{E}_{(0)}$ has no torsion supported on the divisors $D_i$. Furthermore, from the last calculation it is clear that there cannot be any torsion subsheaf supported outside the divisors $D_i$; therefore, $\mathcal{E}_{(0)}$ is locally free. Thus, the extended family defines a point in $\mathcal{G}_d$. Since the corresponding point in $\mathcal{G}$ lies in the semistable locus, we deduce that the extended family lies in the semistable locus, $\mathcal{G}^{ss}(s)$, as well and by Theorem 2 we are done. \hfill \Box

Let $d \in I(r, d, \kappa, \delta, \mathcal{E})$ be as in §3.2.1, Eq. (7), and let $\mathcal{G}_d$ be the parameter space constructed in Sect. 3.2.1. Over $Y \times \mathcal{G}_d$ there is a universal family satisfying the local universal property (follows as in [27, Proposition 2.8]). Note also that the natural $\text{SL}(U)$ action on $\mathcal{G}^{0}$, $\mathfrak{h}$ and $\mathcal{G}_r$ determines an action on the space $\mathcal{G}_d$, $\Gamma : \text{SL}(U) \times \mathcal{G}_d \rightarrow \mathcal{G}_d$, and that the universal family satisfies the glueing property as well (again it follows as in [27, Proposition 2.10]).

**Theorem 3** Let us fix $\mathcal{E} = (a, b, c, \mathcal{L}, \mathcal{E})$ and assume $\kappa_i < 1$ for $i \in I(\mathcal{E})$. There exist a projective scheme $\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss}$ and an open subscheme $\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),s} \subset \text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss}$ together with a natural transformation

$$\alpha_{\mathcal{E}}^{(s)} : \text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss} \rightarrow h_{\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss}}$$

with the following properties:

1. For every scheme $S$ and every natural transformation

$$\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss} \rightarrow h_S,$$

there exists a unique morphism $\varphi : \text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss} \rightarrow S$ with $\alpha' = h(\varphi) \circ \alpha_{\mathcal{E}}^{(s)}$.

2. The scheme $\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),s}$ is a coarse moduli space for $\text{SGPS}_{r,d,\mathcal{E}}^{(\kappa,\delta),ss}$.

**Proof** Consider the Gieseker map $\text{Gies} : \mathcal{G}_d \rightarrow \mathcal{G}$, which is injective and $\text{SL}(U)$-equivariant (see Proposition 3). Let us consider on $\mathcal{G}$ the polarization given in Sect. 3.3.1, and set $\mathcal{L} := \text{Gies}^* \mathcal{O}(b_1, \ldots, b_l, c, k_1, \ldots, k_r)$. From ([21, Chap. 2, §1]), we know that $\text{Gies}^{-1}(\mathcal{G}^{(s)}(s)) = \mathcal{G}_d^{ss}$ and, therefore, Theorem 2 implies that $\mathcal{G}_d^{ss} = \mathcal{G}_d^{(\kappa,\delta),ss}$. By Proposition 5, we deduce...

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that the restriction of the Gieseker map to the semistable locus is an \( \text{SL}(U) \)-equivariant injective and proper morphism. Thus

a. the good quotient \( \text{SGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-\text{ss}} := \frac{\mathcal{O}_r^{(k,\delta)-\text{ss}}}{\text{SL}(U)} \) exists and is projective,

b. the geometric quotient \( \text{SGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-s} := \frac{\mathcal{O}_d^{(k,\delta)-s}}{\text{SL}(U)} \) exists and is an open subscheme of \( \text{SGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-\text{ss}} \).

Define \( \text{SGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-s}(\rho) := \bigsqcup_{d \in I(r,d,\mathfrak{g},\mathfrak{d},\mathfrak{t},\mathfrak{p})} \text{SGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-s}(\rho) \). Now, 1) and 2) follow from this construction, the local universal property and the glueing property. \( \square \)

4 Moduli space for generalized parabolic singular principal bundles

The construction of a projective moduli space for the moduli functor given in (2) is now straightforward due to Eq. (1) (see [6, 19, 24, 27] for the connected case). Thus, we will only give a sketch of the construction in order to avoid repetition.

Let us pick \( r \in \mathbb{N}, d \in \mathbb{Z}, \mathfrak{g} := (e_1, \ldots, e_v) \in \mathbb{N}_0^v \) with \( e_i \leq r \) and \( \kappa := (\kappa_1, \ldots, \kappa_v) \in \mathbb{Q}_0^v \). Let \( G \) be a semisimple linear algebraic group and \( \rho : G \hookrightarrow \text{SL}(V) \) a faithful representation.

**Theorem 4** If \( k_i < 1 \) for each \( i \in I(\mathfrak{g}) \), there exists a projective scheme \( \text{SPBGPS}(\rho)^{(k,\delta)-\text{ss}}_{r,d,\mathfrak{g}} \) and an open subscheme \( \text{SPBGPS}(\rho)^{(k,\delta)-s}_{r,d,\mathfrak{g}} \subset \text{SPBGPS}(\rho)^{(k,\delta)-\text{ss}}_{r,d,\mathfrak{g}} \) together with a natural transformation

\[
\alpha^{(s)s} : \text{SPBGPS}(\rho)^{(k,\delta)-s}_{r,d,\mathfrak{g}} \to h_{\text{SPBGPS}(\rho)^{(k,\delta)-s}_{r,d,\mathfrak{g}}}
\]

with the following properties:

1. For every scheme \( S \) and every natural transformation

\[
\alpha' : \text{SPBGPS}(\rho)^{(k,\delta)-s}_{r,d,\mathfrak{g}} \to h_S,
\]

there exists a unique morphism \( \varphi : \text{SPBGPS}(\rho)^{(k,\delta)-s}_{r,d,\mathfrak{g}}(\rho) \to S \) with \( \alpha' = h(\varphi) \circ \alpha^{(s)s} \).

2. The scheme \( \text{SPBGPS}^{(k,\delta)-s}_{r,d,\mathfrak{g}}(\rho) \) is a coarse moduli space for the moduli functor \( \text{SPBGPS}_{r,d,\mathfrak{g}}^{(k,\delta)-s}(\rho) \).

**Proof** Let us fix \( n \in \mathbb{N} \) and \( d = (d_1, \ldots, d_l) \in \mathbb{Z}^l \) with \( d = \sum_{i=1}^{l} d_i \). Let \( U \) be the vector space \( \mathbb{C}^p \) where \( p := r \chi(\mathcal{E}_Y) + d + \alpha n \) (recall \( \alpha = hr \)). Denote by \( Q_0^0 \) the quasi-projective scheme parametrizing equivalence classes of quotients \( q : U \otimes \pi_Y^* \mathcal{E}_Y(-n) \to \mathcal{E} \) where \( \mathcal{E} \) is a locally free sheaf of rank \( r \) and multidegree \( (d_1, \ldots, d_l) \) on \( Y \), and such that the induced map \( U \to H^0(Y, \mathcal{E}(n)) \) is an isomorphism. On \( Q_0^0 \times Y \), we have the morphism, \( h : S^*(V \otimes U \otimes \pi_Y^* \mathcal{E}_Y(-n)) \to S^*(V \otimes \mathcal{E}_Q^0) \). Let \( s \in \mathbb{N} \) be as in [19, Theorem 4.2, Remark 4.3]. Then, \( h(\bigoplus_{i=1}^{s} S^1(V \otimes U \otimes \pi_Y \mathcal{E}_Y(-n))) \) contains a set of generators of \( S^*(V \otimes \mathcal{E}_Q^0)^G \). Observe that every morphism \( k : \bigoplus_{i=1}^{s} S^1(V \otimes U \otimes \mathcal{E}_Y(-n)) \to \mathcal{E}_Y \) breaks into a family of morphisms \( k_i : S^1(V \otimes U) \to \mathcal{E}_Y(-n) \simeq S^1(V \otimes U \otimes \mathcal{E}_Y(-n)) \to \mathcal{E}_Y \). We denote by the same symbols the induced linear maps \( k_i : S^1(V \otimes U) \to H^0(Y, \mathcal{E}_Y(in)) \). From this point onwards, we can proceed as in [19, §6.1] and we end up with a closed subscheme \( \mathbb{D} \subset Q^* \) together with a universal family (\( \mathfrak{q}_D, \mathfrak{e}_D, \mathfrak{t}_D \)) of singular principal \( G \)-bundles of rank \( r \) and multidegree \( (d_1, \ldots, d_l) \). In order to include the parabolic structure as well we
need to consider the Grassmannians $\mathcal{G}_{r_1} := \text{Grass}_{e_i}(U^\oplus 2)$ of $e_i$ dimensional quotients of $U^\oplus 2$. Define $Z := \mathbb{D} \times \prod_i \mathcal{G}_{r_1}$, and denote by $c_i : Z \to \mathcal{G}_{r_1}$ the projection onto the $i$-th Grassmannian. Consider the pullback of the universal quotient of the $i$-th Grassmannian to $Z$, $q'_Z : U^\oplus 2 \otimes \mathcal{O}_Z \to R_{Z,i}$, and take the direct sum $q_Z : U^\oplus 2 \otimes \mathcal{O}_Z \to \bigoplus_i^\nu R_{Z,i}$. Denote by $q_Z$, $\mathcal{E}_Z$ and $\tau_Z$ the pullbacks to $Z \times \mathcal{Y}$ of the corresponding objects over $\mathbb{D}$. Consider the morphism $\pi^i : Z \times \{y_1^i, y_2^i\} \to Z$. For each $i$, there are quotients $f_i : U^\oplus 2 \times \mathcal{O}_Z \to \pi^i_*(\mathcal{E}_Z|D_i)$ and we can form $f := \oplus(f_i) : U^\oplus 2 \times \mathcal{O}_Z \to \bigoplus \pi^i_*(\mathcal{E}_Z|D_i)$. Consider the following diagram,

$$
0 \longrightarrow \text{Ker}(f) \longrightarrow U^\oplus 2 \times \mathcal{O}_Z \overset{f}{\longrightarrow} \bigoplus \pi^i_*(\mathcal{E}_Z|D_i) \longrightarrow 0
$$

Denote by $\mathcal{M}_d(G) \subset Z$ the closed subscheme given by the zero locus of the morphism $q'$ (see [12, lemma 3.1]). Then, the restriction of $q_Z$ to $\mathcal{M}_d(G)$ factorizes

$$
\bigoplus \pi^i_*(\mathcal{E}_Z|D_i)|_{\mathcal{M}_d(G)} \longrightarrow \bigoplus R_{\mathcal{M}_d(G),i}
$$

Since $f$ and $q_Z$ are diagonal morphisms, we deduce that $q_{\mathcal{M}_d(G)}$ is also diagonal. Therefore, $q_{\mathcal{M}_d(G)}$ is determined by $\nu$ morphisms $q^{\mathcal{M}_d}_{\mathcal{M}_d(G)} : \pi^i_{\mathcal{M}_d(G)}(\mathcal{E}_{\mathcal{M}_d(G)}|D_i) \to R_{\mathcal{M}_d(G),i}$. Denote by $(q_{\mathcal{M}_d(G)}, \mathcal{E}_{\mathcal{M}_d(G)}, \tau_{\mathcal{M}_d(G)})$ the restriction of $(q_Z, \mathcal{E}_Z, \tau_Z)$ to $\mathcal{M}_d(G)$. Then, $(\mathcal{E}_{\mathcal{M}_d(G)}, q_{\mathcal{M}_d(G)}, \tau_{\mathcal{M}_d(G)})$ is a universal family of singular principal $G$-bundles with generalized parabolic structure. Note that there is a natural $\text{GL}(U)$ action on the space $\mathcal{M}_d(G)$, $\Gamma : \text{GL}(U) \times \mathcal{M}_d(G) \to \mathcal{M}_d(G)$. We can view this $\text{GL}(U)$-action as a $(\mathbb{C}^* \times \text{SL}(U))$-action, and we can construct the quotient of $\mathcal{M}_d(G)$ by $\text{GL}(U)$ in two steps, considering the actions of $\mathbb{C}^*$ and $\text{SL}(U)$ separately. From Eq. (1) and the results of Sect. 3, it follows that if $n \in \mathbb{N}$ is large enough, the quotient $\mathcal{M}^{(\kappa, \delta)-(s)s}_{d}(G)\/(\mathbb{C}^* \times \text{SL}(U))$ exists, $\mathcal{M}^{(\kappa, \delta)-(s)s}_{d}(G)$ being the locus parametrizing $(\kappa, \delta)$-(semi)stable singular principal $G$-bundles with generalized parabolic structure. From Remark 4 and Definition 10, it follows that the set $I(r, d, \epsilon, \kappa, \delta)$ of those multidegrees $d$ for which there exist $(\kappa, \delta)$-semistable singular principal $G$-bundles with generalized parabolic structure of type $\epsilon$ is finite. Then,

$$
\text{SPBGP}(\rho)(\kappa, \delta)-(s)s := \bigcup_{d \in I(r, d, \epsilon, \kappa, \delta)} \mathcal{M}^{(\kappa, \delta)-(s)s}_{d}(G)\/(\mathbb{C}^* \times \text{SL}(U))
$$

satisfies the conditions of the statement.

\hfill $\Box$

### 5 Application to principal bundles on reducible nodal curves

Let $X$ be a projective nodal curve with nodes $x_1, \ldots, x_\nu$. Let $X_1, \ldots, X_\ell$ be the irreducible components of $X$ and $\pi : Y = \coprod_{i=1}^\ell Y_i \to X$ its normalization. Let $\mathcal{O}_X(1)$ be an ample invertible sheaf on $X$ and denote by $\mathcal{O}_Y(1)$ the ample invertible sheaf obtained by pulling $\mathcal{O}_X(1)$ back to $Y$. As usual, $h$ is the degree of $\mathcal{O}_Y(1)$, $y_1^i, y_2^i$ are the points in the preimage...
of the $i$-th nodal point $x_i$, $D_i = y_1^i + y_2^i$ is the corresponding divisor on $Y$ and $D = \sum D_i$ is the total divisor.

### 5.1 Torsion-free sheaves over a reducible nodal curve

Let $\mathcal{F}$ be a torsion-free sheaf on $X$ of rank $r$. C. S. Seshadri showed (see [29, Chapter 8]) that for each nodal point $x$ (regardless of how many components this point lies on), there is a natural number $0 \leq l \leq r$ such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{l_i} \oplus m_x^{r-l}$. Then, it is said that a torsion-free sheaf of rank $r$ is of type $l_i$.

If $\mathcal{F}$ is a torsion-free sheaf on $X$, then the canonical map $\alpha : \mathcal{F} \to \pi_*\pi^*(\mathcal{F})$ is injective, and $\mathcal{F} := \text{Coker}(\alpha)$ is a torsion sheaf supported on the nodes. If $\mathcal{F}$ is of rank $r$ and of type $l_i$, a short calculation shows that

$$\deg(\pi^*\mathcal{F}) = \deg(\mathcal{F}) + rv - \sum l_i,$$

$$\deg(T(\mathcal{F})) = 2(rv - \sum l_i),$$

(20)

$T(\mathcal{F})$ being the torsion subsheaf of $\pi^*\mathcal{F}$ (see [1] for the irreducible case).

**Proposition 6** If $\mathcal{F}$ is a torsion-free sheaf of rank $r$ and type $l = (l_1, \ldots, l_v)$ on $X$, then the natural morphism $\beta : \mathcal{F} \to \pi_*\pi^*(\mathcal{F})$, where $\pi_* := \pi^*(\mathcal{F})/T(\mathcal{F})$, is injective and $\text{length}(\text{Coker}(\beta)) = l := \sum l_i$. Furthermore, $\text{Coker}(\beta) = \bigoplus_{i=1}^v \mathbb{C}D_i$.

**Proof** Let $\mathcal{F}$ be a torsion-free sheaf on the nodal curve $X$ and let $T(\mathcal{F})$ be the torsion subsheaf of $\pi^*\mathcal{F}$. The natural morphism $\beta : \mathcal{F} \to \pi_*\pi^*(\mathcal{F})/T(\mathcal{F})$ is injective at every smooth point so it is injective since $\mathcal{F}$ is torsion-free. Let us consider now the exact sequence

$$0 \to \mathcal{F} \to \pi_*\pi^*(\mathcal{F})/T(\mathcal{F}) \to \text{Coker}(\beta) \to 0.$$  

(21)

Then, we have $\chi(\pi^*(\mathcal{F})/T(\mathcal{F})) = \chi(\mathcal{F}) + \text{length}(\text{Coker}(\beta))$ and, therefore, $r\chi(\mathcal{F}) + \deg(\pi^*(\mathcal{F})/T(\mathcal{F})) = r\chi(\mathcal{F}) + \deg(\mathcal{F}) + \text{length}(\text{Coker}(\beta))$. However, $\chi(\mathcal{F}) = \chi(\mathcal{F}) = v$, so $\text{length}(\text{Coker}(\beta)) = rv + \deg(\pi^*(\mathcal{F})/T(\mathcal{F})) - \deg(\mathcal{F})$ and applying Eq. (20) we obtain the result.

**Corollary 1** Let $\mathcal{F}$ be a torsion-free sheaf of rank $r$ and type $l = (l_1, \ldots, l_v)$ on $X$. Suppose there exists a locally free sheaf $\mathcal{E}$ on $Y$ of the same rank and an injection $i : \mathcal{F} \hookrightarrow \pi_*\mathcal{E}$. Then $\text{length}(\text{Coker}(i)) = e$ if and only if $\text{length}(\text{Coker}(\pi_*\mathcal{E})) = e - l$, where $l = \sum l_i$.

**Proof** Let $\mathcal{F}$ be a torsion-free sheaf of rank $r$ on $X$ and suppose there exists a locally free sheaf of rank $r$, $\mathcal{E}$, on the normalization and an injection $i : \mathcal{F} \hookrightarrow \pi_*\mathcal{E}$. Then, there is an injection $\lambda : \mathcal{E}_0 \hookrightarrow \mathcal{E}$ such that $\pi_*\mathcal{E}_0 \circ \beta = i$. From the above observation, it follows that $\text{Coker}(i)/\text{Coker}(\beta) \cong \text{Coker}(\pi_*\mathcal{E}_0)$. Hence, we deduce that $\text{length}(\text{Coker}(\pi_*\mathcal{E})) = \text{length}(\text{Coker}(\mathcal{F}) - \text{length}(\text{Coker}(\beta)))$. Since $\text{length}(\text{Coker}(i)) = e$ and $\text{length}(\text{Coker}(\beta)) = l$, we can conclude using Proposition 6.

### 5.2 Descending singular principal $G$-bundles

We recall the descending operation introduced in [26, §2].

Let us fix $r \in \mathbb{N}$, $d \in \mathbb{Z}$ and $e := (e_1, \ldots, e_v) \in \mathbb{N}_0^v$ with $e_i \leq r$. Let $(\mathcal{E}, g, \tau)$ be a singular principal $G$-bundle with generalized parabolic structure on $Y$ with rank $r$, degree...
Let us consider the natural surjection $\text{ev}_D = \oplus \text{ev}_i : \mathcal{E} \to \mathcal{E}|_D = \bigoplus \mathcal{E}|_{D_i}$ and take the push-forward, $\pi_*(\mathcal{E}(D)) : \pi_*(\mathcal{E}(D)) \to R_\mathcal{E}$. Since $\pi_*(\mathcal{E}(D))$ is precisely the vector space $\bigoplus (\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i))$ supported on the nodes, we can consider $R = \bigoplus R_i$ as a torsion sheaf supported on the nodes and compose $\pi_*(\mathcal{E}(D))$ with $q$ to get the morphism $q \circ \pi_*(\mathcal{E}(D)) : \pi_*(\mathcal{E}) \to R \to 0$. Defining $\mathcal{F} = \text{Ker}(q \circ \pi_*(\mathcal{E}(D)))$, we get an exact sequence

$$0 \to \mathcal{F} \hookrightarrow \pi_*(\mathcal{E}) \xrightarrow{\mu} R \to 0,$$

where $\mathcal{F}$ is a torsion-free sheaf of rank $r$ and degree $d + \sum_{i=1}^v (r - e_i)$, and $R$ has length $\text{length}(R) := e_1 + \cdots + e_v$.

**Remark 6** Let $(\mathcal{E}, q)$ be a generalized parabolic locally free sheaf of rank $r$, degree $d$ and type $\mathcal{E}' = (e'_1, \ldots, e'_v)$. For each $i = 1, \ldots, v$, denote by $K_i$ the kernel of the $i$-th parabolic structure $\mathcal{E}(y_1^i) \oplus \mathcal{E}(y_2^i) \to R_i$ and by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the kernel of the induced linear map $K_i \to \mathcal{E}(y_1^i)$ (resp. $K_i \to \mathcal{E}(y_2^i)$). From [3, Proposition 3.7], it follows that the associated torsion-free sheaf $\mathcal{F}$ satisfies $\mathcal{F}_{xi} \simeq \mathcal{O}_{\mathbb{X}}(\mathcal{E}_{xi}) \oplus_{\mathbb{C}} m_{e_i}^{-e_i}$, where $e_i = 2r - e'_i - \text{dim}(C_1) - \text{dim}(C_2)$.

It remains to construct $\tau' : \text{Spec}(\mathcal{F} \otimes V)^G \to \mathcal{O}_{\mathbb{X}}$ from the data $(\mathcal{E}, q, \tau)$. Let us consider the canonical isomorphism $\pi^* (\text{Spec}(\mathcal{F} \otimes V)^G) \simeq \text{Spec}(\pi^* (\mathcal{F}) \otimes V)^G$. The identity map $\pi^* \mathcal{E} \to \pi^* \mathcal{E}$ induces a morphism $\pi^* (\mathcal{F}) \to \pi^* (\mathcal{E})$ by adjunction and therefore a morphism of algebras $\pi^* (\mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \otimes \mathcal{E}$ which, in turn, induces a morphism of algebras $\mathcal{F} \otimes \mathcal{E} \to \mathcal{F} \otimes \mathcal{E}$.

This induces a diagram

$$\begin{array}{ccc}
S^* (V \otimes \mathcal{F})^G & \xrightarrow{\tau'} & S^* (V \otimes \pi_* \mathcal{E})^G \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{X}} & \xrightarrow{\pi_* \mathcal{E}} & \mathcal{O}_Y \\
\oplus \bigoplus_{i=1}^v \mathcal{C}_{xi} & \to & 0
\end{array}$$

**Definition 11** Given $r \in \mathbb{N}$, $d \in \mathbb{Z}$ and $\mathcal{E} := (e_1, \ldots, e_v) \in \mathbb{N}_0^v$ with $e_i \leq r$, a descending $G$-bundle of rank $r$, degree $d$ and type $\mathcal{E}$ on $Y$ is a singular principal $G$-bundle with generalized parabolic structure of rank $r$, degree $d$ and type $\mathcal{E}$, $(\mathcal{E}, q, \tau)$, such that $\tau'$ takes values in $\mathcal{O}_{\mathbb{X}} \subset \pi_*(\mathcal{E})^G$.

**Definition 12** Let us fix $r \in \mathbb{N}$, $d \in \mathbb{Z}$ and $\mathcal{E} := (e_1, \ldots, e_v) \in \mathbb{N}_0^v$ with $e_i \leq r$, and $\delta \in \mathbb{Q}_{\geq 0}$. Given $\kappa := (\kappa_1, \ldots, \kappa_v) \in \mathbb{Q}_{\geq 0}^v$, a descending $G$-bundle of rank $r$, degree $d$ and type $\mathcal{E}$ is $(\kappa, \delta)$-(semi)stable if the underlying singular principal $G$-bundle with generalized parabolic structure is $(\kappa, \delta)$-(semi)stable.

A family of descending $G$-bundles parametrized by a scheme $S$ is defined in the obvious way, and we can consider the moduli functor,

$$\text{D}((\kappa, \delta))_{\text{ss}} (S) = \text{isomorphism classes of families of $(\kappa, \delta)$-(semi)stable descending $G$-bundles on $Y$ parametrized by $S$ with rank $r$ degree $d$ and type $\mathcal{E}$}.$$

One can demonstrate the next theorem following a similar argument as given for proving Theorem 4 and [27, Main Theorem].

**Theorem 5** If $\kappa_i < 1$ for $i \in I(\mathcal{E})$ for each $i$, there exist a projective scheme $\text{D}((\kappa, \delta))_{\text{ss}}$ and an open subscheme $\text{D}((\kappa, \delta))_{\text{ss}} \subset \text{D}((\kappa, \delta))_{\text{ss}}$ together with a natural transformation

$$\alpha^{(s)} : \text{D}((\kappa, \delta))_{\text{ss}} \to h_{\text{D}((\kappa, \delta))_{\text{ss}}}.$$
with the following properties:

1. For any scheme $S$ and any natural transformation 
   \[ \alpha' : D(\rho)_{r,d,e}^{(\kappa,\delta,\eta)-s} \to h_S, \]
   there exists a unique morphism $\varphi : D(\rho)_{r,d,e}^{(\kappa,\delta,\eta)-s} \to S$ with $\alpha' = h(\varphi) \circ \alpha^{(s)s}$.

2. The scheme $D(\rho)_{r,d,e}^{(\kappa,\delta,\eta)-s}$ is a coarse moduli space for the moduli functor $D(\rho)_{r,d,e}^{(\kappa,\delta,\eta)-s}$.

### 5.3 Relation to the moduli space of principal $G$-bundles over a reducible nodal curve. Specializations

Let us fix $r \in \mathbb{N}$, $d \in \mathbb{Z}$ and $e := (e_1, \ldots, e_v) \in \mathbb{N}_0^v$ with $e_i \leq r$. Let $(\mathcal{E}, q, \tau)$ be a descending $G$-bundle of rank $r_d$, degree $d$ and type $e$, and $(\mathcal{F}, \tau')$ the induced singular principal $G$-bundle. Recall that both sheaves, $\mathcal{E}$ and $\mathcal{F}$, are related through the exact sequence given in Eq. (22) where the morphism $w$ factorizes over the surjection $q : \pi_*(\mathcal{E}|_D) \to R$. For any subsheaf $\mathcal{I} \subset \mathcal{E}$, the image of $w$ restricted to $\pi_*(\mathcal{I}) \subset \pi_*(\mathcal{E})$ is precisely $\bigoplus_{i=1}^v q_i(\mathcal{I}(y^i_1) \oplus \mathcal{I}(y^i_2))$. Therefore, we can construct the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \pi_*(\mathcal{E}) & \stackrel{w}{\longrightarrow} & R & \longrightarrow & 0 \\
&& \downarrow \alpha && \downarrow \pi && \downarrow && \\
0 & \longrightarrow & \text{Ker}(p') & \longrightarrow & \pi_*(\mathcal{I}) & \bigoplus_{i=1}^v q_i(\mathcal{I}(y^i_1) \oplus \mathcal{I}(y^i_2)) & \longrightarrow & 0
\end{array}
\]

and we define $S(\mathcal{I}) := \text{Ker}(p')$. If $\mathcal{I}$ is saturated, then $S(\mathcal{I})$ is clearly saturated. This construction allows us to attach to any weighted filtration $(\mathcal{E}_s, m)$ of $\mathcal{E}$ formed by saturated subsheaves a weighted filtration $(S(\mathcal{E}_s), m)$ of $\mathcal{F}$ formed by saturated subsheaves. Moreover, any saturated subsheaf of $\mathcal{F}$ can be constructed from a saturated subsheaf of $\mathcal{E}$ (follows as in the connected case [27]).

In what follows, we will use the notation $1$ for $(1, \ldots, 1) \in \mathbb{Q}^v_{\geq 0}$.

**Proposition 7** Let $(\mathcal{E}, q, \tau)$ be a descending $G$-bundle of rank $r_d$ and type $e$, and $(\mathcal{F}, \tau')$ the induced singular principal $G$-bundle on $X$. Then, $(\mathcal{F}, \tau')$ is $(\kappa, \delta)$-(semi)stable if and only if $(\mathcal{E}, q, \tau)$ is a $(1, \delta)$-(semi)stable $G$-bundle with a generalized parabolic structure.

**Proof** This is proved as in the irreducible case [27, Proposition 5.2.2] \(\square\)

**Proposition 8** Let $r \in \mathbb{N}_0$, $d \in \mathbb{Z}$ and $e := (e_1, \ldots, e_v) \in \mathbb{N}_0^v$ with $e_i \leq r$. There exists $\epsilon = \epsilon(e) \in \mathbb{R} \cap (0, 1)$, such that for any $\kappa \in \mathbb{Q}_{\geq 0}^v$ with $1 - \epsilon < k_i < 1$, any integral parameter $\delta$, and any singular principal $G$-bundle $(\mathcal{E}, q, \tau)$ with a generalized parabolic structure of rank $r_d$, degree $d$ and type $e$, we have:

1. if $(\mathcal{E}, q, \tau)$ is $(\kappa, \delta)$-semistable, then it is $(1, \delta)$-semistable,
2. if $(\mathcal{E}, q, \tau)$ is $(1, \delta)$-stable, then it is $(\kappa, \delta)$-stable.

**Proof** Recall that the $(\kappa, \delta)$-(semi)stability condition for a singular principal $G$-bundle with a generalized parabolic structure has to be checked just for the weighted filtrations $(\mathcal{E}_s, m)$ of $\mathcal{E}$ for which $m_i < A$ for a suitable constant $A$ depending only on the numerical input data (see Remark 3). This implies that we can find a natural number $n$ such that $P_1(\mathcal{E}_s, m) + \delta \mu(\mathcal{E}_s, m, \tau) \in \mathbb{Z}[\frac{1}{n}]$ for all such weighted filtrations. A short calculation shows that for

\[ \square \] Springer
every generalized parabolic bundle \((\mathcal{E}, \varphi)\) and every weighted filtration \((\mathcal{E}_\bullet, m)\) we have \(P_1(\mathcal{E}^\bullet, m) - P_k(\mathcal{E}^\bullet, m) \leq v r e A a^2.\) In fact, we can also show that \(P_1(\mathcal{E}^\bullet, m) - P_k(\mathcal{E}^\bullet, m) \geq -v r e A a^2.\) Take \(e\) so that the inequality \(v r e A a^2 < \frac{1}{n}\) holds. Now, 1) and 2) follow by a similar to the one given in [27, Proposition 5.2.3.].

\[\square\]

Let \(r \in \mathbb{N}, d \in \mathbb{Z}\) and \(\varepsilon = (e_1, \ldots, e_v) \in \mathbb{N}_0^v.\) Denote by \(\mathfrak{D}_{r,d(\varepsilon),\varepsilon}\) the set of isomorphism classes of descending \(G\)-bundles over \(Y\) with rank \(r\) type \(\varepsilon\) and degree \(d(\varepsilon, r) = \deg - \sum_{i=1}^v (r - e_i)\), and by \(\mathfrak{G}P\mathfrak{B}_{r,d,\varepsilon}\) the set of isomorphism classes of singular principal \(G\)-bundles over \(Y\) of rank \(r\) degree \(d\) and type \(\varepsilon\). From Corollary 1, it follows that there is a map \(\Theta_\varepsilon : \mathfrak{D}_{r,d(\varepsilon),\varepsilon} \rightarrow \bigcup_{\varepsilon' \leq \varepsilon} \mathfrak{G}P\mathfrak{B}_{r,d,\varepsilon'}\)

**Theorem 6** \(\Theta_\varepsilon\) induces a bijection \(\Theta^{-1}_\varepsilon : \mathfrak{G}P\mathfrak{B}_{r,d,\varepsilon} \rightarrow \mathfrak{D}_{r,d(\varepsilon),\varepsilon}\)

**Remark 7** From Remark 6, it follows that \(\Theta^{-1}_\varepsilon : \mathfrak{G}P\mathfrak{B}_{r,d,\varepsilon}\) consists exactly of those descending singular principal \(G\)-bundles \((\mathcal{E}, \varphi, \tau)\) \(\in \mathfrak{D}_{r,d(\varepsilon),\varepsilon}\) satisfying \(\text{dim}(C_i^1) + \text{dim}(C_i^2) = 2(r - e_i)\) for \(i = 1, \ldots, v.\)

**Proof** Let \((\mathcal{F}, \tau)\) be a singular principal \(G\)-bundle of rank \(r\), degree \(d\) and type \(\varepsilon\), and consider the exact sequence

\[
0 \rightarrow T(\mathcal{F}) \rightarrow \pi^*(\mathcal{F}) \rightarrow \mathcal{E}_0 = \pi^* \mathcal{F}/T(\mathcal{F}) \rightarrow 0.
\]  

(24)

Since \(S^*(V \otimes \pi^* \mathcal{F})^G \rightarrow S^*(V \otimes \mathcal{E}_0)^G\) is still surjective we find a closed immersion \(\text{Spec}(S^*(V \otimes \mathcal{E}_0)^G) \leftarrow \text{Spec}(S^*(V \otimes \pi^* \mathcal{F})^G).\) We have the following diagram

\[
\text{Spec}(S^*(V \otimes \mathcal{E}_0)^G) \leftarrow \text{Spec}(S^*(V \otimes \pi^* \mathcal{F})^G) \rightarrow \text{Spec}(S^*(V \otimes \mathcal{F})^G).
\]

The morphism \(\pi^*(\tau) : \pi^*(S^*(V \otimes \mathcal{F})^G) = S^*(V \otimes \pi^* \mathcal{F})^G \rightarrow \pi^* \mathcal{E}_0 = \mathcal{E}_0\) is the one that we obtain by adjunction when we take the composition of \(S^*(V \otimes \mathcal{F})^G\) with the natural inclusion of rings \(\mathcal{E}_0 \subset \pi^* \mathcal{E}_0\). Let us denote by \(W\) the open subset \(Y \setminus \pi^{-1}(\text{Sing}(X))\).

Restricting the exact sequence (24) to this open subset, we obtain \(\pi^* \mathcal{F}|_W = \mathcal{E}_0|_W\) so \(\text{Spec}(S^*(V \otimes \mathcal{E}_0|_W)^G) = \text{Spec}(S^*(V \otimes \pi^* \mathcal{F}|_W)^G)\) which means that the restriction \(\pi^*(\tau|_W)\) takes values in \(\text{Spec}(S^*(V \otimes \mathcal{E}_0|_W)^G)\). From the chain of immersions

\[
\text{Spec}(S^*(V \otimes \mathcal{E}_0|_W)^G) \leftarrow \text{Spec}(S^*(V \otimes \mathcal{E}_0)^G) \rightarrow \text{Spec}(S^*(V \otimes \pi^* \mathcal{F})^G)
\]

it follows that \(\pi^*(\tau)\) must then take values in \(\text{Spec}(S^*(V \otimes \mathcal{E}_0)^G)\), that is, the morphism \(S^*(V \otimes \pi^* \mathcal{F})^G \rightarrow \mathcal{E}_0\) factorizes through the surjection

\[
S^*(V \otimes \pi^* \mathcal{F})^G \rightarrow S^*(V \otimes \mathcal{E}_0)^G \rightarrow 0
\]

and we denote by \(\tau_0\) the morphism of algebras \(S^*(V \otimes \mathcal{E}_0)^G \rightarrow \mathcal{E}_0.\) On the other hand, given a node \(x \in X\), we have \(\pi_*(\mathcal{E}_0)_x \otimes \mathcal{O}_{X,x}/m_x \simeq \mathcal{E}_0(y_1) \oplus \mathcal{E}_0(y_2).\) Therefore, the surjection \(\pi_*(\mathcal{E}_0) \rightarrow \text{Coker}(\beta)\) defined in Proposition 6 induces a surjection \(q_i^0 : \mathcal{E}_0(y_i^1) \oplus \mathcal{E}_0(y_i^2) \rightarrow \text{Coker}(\beta)_x\) of dimension \(e_i\) for each \(i = 1, \ldots, v,\) which, in turn, induces a generalized parabolic structure of type \(\varepsilon = (e_1, \ldots, e_v).\) From this construction, it follows that the singular principal \(G\)-bundle with generalized parabolic structure \((\mathcal{E}_0, \tau_0, q^0)\) of rank \(r\), degree...
Let us pick \( r \in \mathbb{N} \), \( d \in \mathbb{Z} \), \( \delta \in \mathbb{Z}_{>0} \) and define \( J(r) := \{ \underline{e} = (e_1, \ldots, e_v) \in \mathbb{N}_0^v | e_i \leq r \} \). For each \( \underline{e} \in J(r) \), we choose \( \epsilon(\underline{e}) \) as in Proposition 8 and define \( \epsilon := \min_{\underline{e} \in J(r)} \epsilon(\underline{e}) \).

Let \( \kappa \) be such that \( 1 - \epsilon < \kappa < 1 \). Let \( \text{SPB}(\rho)_{r,d}^{\delta,(\text{s})} \) be the moduli space of \( \delta \)-(semi)stable singular principal \( G \)-bundles of rank \( r \) and degree \( d \) on the nodal curve \( X \) (see [19]). Then, Proposition 7 and Proposition 8 imply that, for each \( \underline{e} \in J(r) \), there is a proper morphism

\[
\Theta : D(\rho)_{r,d,\underline{e}}^{(\underline{e},\delta),(\text{s})} := \prod_{\underline{e} \in J(r)} D(\rho)_{r,d,(\underline{e},r),\underline{e}}^{(\underline{e},\delta),(\text{s})} \rightarrow \text{SPB}(\rho)_{r,d}^{\delta,(\text{s})}
\]

between the moduli spaces, where \( d(\underline{e}, r) = d - \sum_{i=1}^v (r - e_i) \). Given an element \( \underline{e} \in J(r) \), we denote by \( \text{SPB}(\rho)_{r,d,\underline{e}}^{\delta,(\text{s})} \) the subscheme that parametrizes singular principal \( G \)-bundles, \((\mathcal{F}, \tau)\), with \( \mathcal{F} \) a torsion-free sheaf of type \( \underline{e} \). Then, by Corollary 1, \( \Theta \) induces a proper morphism

\[
\Theta_{\underline{e}} : D(\rho)_{r,d,(\underline{e},r),\underline{e}}^{(\underline{e},\delta),(\text{s})} \rightarrow \bigcup_{\underline{e}' \leq \underline{e}} \text{SPB}(\rho)_{r,d,\underline{e}'}^{\delta,(\text{s})}.
\]

Let us denote by \( \overline{\text{SPB}(\rho)_{r,d,\underline{e}}^{\delta,(\text{s})}} \) the schematic closure in \( \text{SPB}(\rho)_{r,d}^{\delta,(\text{s})} \), which lies in the closed subscheme \( \bigcup_{\underline{e}' \leq \underline{e}} \text{SPB}(\rho)_{r,d,\underline{e}'}^{\delta,(\text{s})} \). Obviously, \( \Theta_{\underline{e}} \) maps \( \Theta_{\underline{e}}^{-1}(\text{SPB}(\rho)_{r,d,\underline{e}}^{\delta,(\text{s})}) \) to \( \text{SPB}(\rho)_{r,d,\underline{e}}^{\delta,(\text{s})} \).
Theorem 7 If the open subscheme $D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon} \subset D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon}$ is dense, then $\Theta_\varepsilon$ induces a birational, proper and surjective morphism

$$\Theta_\varepsilon : D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon} \longrightarrow \overline{\text{SPB}(\rho)^{b, s}_{r, d, \varepsilon}}.$$ 

Proof From Proposition 8 and Theorem 6, we deduce that $\Theta_\varepsilon$ induces an isomorphism $\Theta_\varepsilon^{-1}(\text{SPB}(\rho)^{\delta, s}_{r, d, \varepsilon}) \cong \text{SPB}(\rho)^{\delta, s}_{r, d, \varepsilon}$. Let us denote by $\mathcal{W}_\varepsilon$ the dense open subscheme of $D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon}$ parametrizing descending principal bundles with generalized parabolic structure such that $\dim(C_i^1) + \dim(C_i^2) = 2(r - e_i)$ for $i = 1, \ldots, v$ (see Remark 6). From Proposition 7 and Remark 7, it follows that $\Theta_\varepsilon^{-1}(\text{SPB}(\rho)^{\delta, s}_{r, d, \varepsilon}) = \mathcal{W}_\varepsilon \cap D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon}$. Therefore, it is a dense open subscheme. Finally, since $\Theta_\varepsilon$ is proper, the isomorphism $\Theta_\varepsilon^{-1}(\text{SPB}(\rho)^{\delta, s}_{r, d, \varepsilon}) \cong \text{SPB}(\rho)^{\delta, s}_{r, d, \varepsilon}$ extends to a surjective and proper morphism $\Theta_\varepsilon : D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon} \longrightarrow \overline{\text{SPB}(\rho)^{b, s}_{r, d, \varepsilon}}$. \hfill $\square$

6 Further comments

There are some questions and some remarks that deserve to be pointed out.

The condition imposed in Theorem 7 holds if there exists a $(\varepsilon, \delta)$-stable descending $G$-bundle in each component of $D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon}$. This problem is still open even in case $G = \text{GL}(n)$ up to the knowledge of the author (see [3, Remark 3.10]). Regarding the case considered in this article, it would be interesting to know to what extent this problem depends on the representation we have fixed. Likewise, it would be important to understand the geometry of $D(\rho)^{(\varepsilon, \delta)-\text{ss}}_{r, d(\varepsilon, r), \varepsilon}$. For instance, which configurations of the parameters make the moduli space to be normal?

Although most of the article deals with the existence of the moduli space of descending singular principal $G$-bundles, it is worth pointing out Theorem 6. The restriction of this correspondence to principal $G$-bundles has been useful to study the Picard group of the stack of principal $G$-bundles on $X$ [4]. The correspondence established in this article might be helpful to compute the Picard group of the moduli space of singular principal $G$-bundles following the same strategy as in [4, 5] and, eventually, to determine the Picard group of the universal moduli space of singular principal $G$-bundles over $\overline{M}_g$ [20]. See [9, 10] for related questions.

Finally, it would be interesting to generalize [16] to reducible curves. This would allow to study Hitchin maps for generalized parabolic $G$-Hitchin pairs [8] over stable curves and to face certain interesting open questions in the field (see [23]).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.
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