On the generalization of classical Zernike system

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Received 26 January 2023; revised 20 November 2023
Accepted for publication 5 January 2024
Published 18 January 2024

Recommended by Professor Beatrice Pelloni

Abstract

We generalize the results obtained recently (Blasco et al 2023 Nonlinearity 36 1143) by providing a very simple proof of the superintegrability of the Hamiltonian

\[ H = \vec{p}^2 + F(\vec{q} \cdot \vec{p}), \quad \vec{q}, \vec{p} \in \mathbb{R}^2, \]

for any analytic function \( F \). The additional integral of motion is constructed explicitly and shown to reduce to a polynomial in canonical variables for polynomial \( F \). The generalization to the case \( \vec{q}, \vec{p} \in \mathbb{R}^n \) is sketched.

Keywords: Zernike system, superintegrability, integrals of motion, dilatation

Mathematics Subject Classification numbers: 37J06

1. Introduction

In the recent paper [1] the natural generalization of the classical Zernike system [2–4] (for the quantum version see [3–9]) has been proposed. It is defined by the Hamiltonian

\[ H = \vec{p}^2 + \sum_{n=1}^{N} \gamma_n (\vec{q} \cdot \vec{p})^n \]

with \( \vec{q} = (q_1, q_2), \vec{p} = (p_1, p_2) \) being canonical variables.

It has been shown in [1] that (1) is maximally superintegrable and admits, apart from energy and angular momentum, an additional integral of motion which is polynomial of degree \( N \) in momenta.

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In the present paper we discuss a number of new results concerning the Hamiltonians generalizing that considered in [1]. More precisely:

- we show that the more general Hamiltonians of the form
  \[
  H = \vec{p}^2 + F(\vec{q} \cdot \vec{p}) ,
  \]
  with \( F(\cdot) \) being an arbitrary real analytic (or even smooth) function, are superintegrable;
- what is even more important, we explain that the superintegrability property is an almost immediate consequence of the structure of the Hamiltonian. In fact, one could consider the arbitrary Hamiltonian \( H = H(\vec{p}^2, \vec{q} \cdot \vec{p}) \). Since \( H \) is autonomous and rotation invariant, \( H \) and
  \[
  L \equiv \varepsilon_{ij} q_i p_j
  \]
  are two commuting integrals of motion. Moreover, since \( \vec{q} \cdot \vec{p} \) is the generator (in the sense of canonical transformations) of dilatations, any function \( C(\vec{p}) \) homogeneous of degree zero, is also an integral of motion. Note, that out of three integrals of motion, \( H, L \) and \( C \), only \( H \) depends on the choice of \( F(\cdot) \); therefore, for generic \( F \), these integrals are functionally independent implying superintegrability. The only obstacle is that the additional integral \( C(\vec{p}) \) is singular somewhere in momentum space. We show that for the particular Hamiltonians (2) one can replace \( C(\vec{p}) \), by using a very simple identity (cf equations (8) and (A2) below), with nonsingular combination of \( C, H \) and \( L \). We suspect that similar procedure is possible for more general Hamiltonians \( H(\vec{p}^2, \vec{q} \cdot \vec{p}) \) as well;
- having the explicit form of the superintegral for the Hamiltonian (2) we reproduce the result of [1]: the superintegral is a polynomial in momenta provided \( F \) is such a polynomial;
- we sketch the argument that (2) continues to be superintegrable if the Euclidean plane is replaced by the surface of constant curvature;
- we extend our results to the case of configuration space of arbitrary dimensions.

2. Superintegrability

In order to prove the superintegrability of the Hamiltonian (2) let us note that \( \vec{q} \cdot \vec{p} \) generates dilatations. Therefore, it Poisson-commutes with any homogeneous function of degree zero. Any such function is a constant of motion for the Hamiltonian (2) provided it is a constant of motion for the free Hamiltonian \( H_0 = \vec{p}^2 \). For example, one can take \( Q_k(p)/R_k(p) \) with \( Q_k \) and \( R_k \) being homogeneous polynomials of degree \( k \). The simplest choice seems to be \( \frac{p_1^2}{p_2^2} \) or \( \frac{p_2^2}{p_1^2} \). However, such integrals are singular functions of the momenta while we would like to find the analytic ones.

To this end note that \( F(\vec{q} \cdot \vec{p}) \), being analytic, can be rewritten as
  \[
  F(\vec{q} \cdot \vec{p}) = A \left( (\vec{q} \cdot \vec{p})^2 \right) + (\vec{q} \cdot \vec{p}) B \left( (\vec{q} \cdot \vec{p})^2 \right)
  \]
  with
  \[
  A \left( (\vec{q} \cdot \vec{p})^2 \right) \equiv \frac{1}{2} \left( F(\vec{q} \cdot \vec{p}) + F(-\vec{q} \cdot \vec{p}) \right)
  \]
\[
B \left( (\vec{q} \cdot \vec{p})^2 \right) \equiv \frac{1}{2 \vec{q} \cdot \vec{p}} (F(\vec{q} \cdot \vec{p}) - F(-\vec{q} \cdot \vec{p})) \tag{6}
\]

being analytic as well. Due to \((\vec{q} \cdot \vec{p})^2 = \vec{q}^2 \cdot \vec{p}^2 - L^2\) one has
\[
H = \vec{p}^2 + A \left( \vec{q}^2 \cdot \vec{p}^2 - L^2 \right) + (\vec{q} \cdot \vec{p}) B \left( \vec{q}^2 \cdot \vec{p}^2 - L^2 \right). \tag{7}
\]

Finally, note a simple identity
\[
\frac{p_2^2}{p^2} \cdot H - \frac{p_2^2}{p^2} A (-L^2) - \frac{p_2^2}{p^2} B (-L^2) = p_2^2 + \frac{p_2^2}{p^2} \left( A \left( \vec{q}^2 \cdot \vec{p}^2 - L^2 \right) - A \left( -L^2 \right) \right) + \frac{p_2^2}{p^2} \cdot (\vec{q} \cdot \vec{p}) \left( B \left( \vec{q}^2 \cdot \vec{p}^2 - L^2 \right) - B \left( -L^2 \right) \right) + q_1 p_2 B (-L^2). \tag{8}
\]

The left hand side of equation (8) obviously provides an integral of motion functionally independent of \(H\) and \(L\) while the right hand side is explicitly analytic. If \(F\) is a polynomial the integral (8) is a polynomial of the same degree. Similar identity holds under the replacement \(1 \leftrightarrow 2\).

3. Polar coordinates

Due to the rotational invariance it is interesting to study the Hamiltonian (2) in polar coordinates. To this end we make the canonical transformation to polar variables [1]:

\[
q_1 = r \cos \varphi \quad \quad p_1 = p_r \cos \varphi - \frac{p_\varphi}{r} \sin \varphi \\
q_2 = r \sin \varphi \quad \quad p_2 = p_r \sin \varphi + \frac{p_\varphi}{r} \cos \varphi \tag{9}
\]

where \(p_\varphi = L\) is the integral of motion. The Hamiltonian acquires the form
\[
H = p_r^2 + \frac{p_\varphi^2}{r^2} + F(r \cdot p_r) \tag{10}
\]

while the canonical equations of motion read

\[
\dot{r} = \frac{\partial H}{\partial p_r} = 2p_r + F'(r \cdot p_r) r \\
\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{2p_\varphi^2}{r^2} - F'(r \cdot p_r) p_r \\
\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{2p_\varphi}{r^2} \\
\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0. \tag{11}
\]
It is now easy to see that our system is superintegrable. Namely, there exists third, functionally independent, integral of motion:

\[ C = \text{any periodic function of } \left( \arctan \left( \frac{r \cdot p_r}{p_\varphi} \right) - \varphi \right). \]  

(12)

For example, for any natural \( n \) one can take

\[ C_n \equiv \sin \left( 2n \left( \arctan \left( \frac{r \cdot p_r}{p_\varphi} \right) - \varphi \right) \right). \]  

(13)

An elementary computation yields

\[ C_n = \left( \frac{z^{2n} - \bar{z}^{2n}}{2|z|^{2n}} \right) \cos 2n\varphi - \left( \frac{z^{2n} + \bar{z}^{2n}}{2|z|^{2n}} \right) \sin 2n\varphi \]  

(14)

with

\[ z \equiv p_\varphi + ir \cdot p_r. \]  

(15)

Obviously, there is only one functionally independent integral \( C_n \). It is convenient to choose the simplest one,

\[ C_1 = \left( \frac{z^2 - \bar{z}^2}{2|z|^2} \right) \cos 2\varphi - \left( \frac{z^2 + \bar{z}^2}{2|z|^2} \right) \sin 2\varphi \]  

(16)

\( H(E) \) and \( p_\varphi(L) \) can be expressed in terms of \( z, \bar{z} \) and \( r \) as follows

\[ E = H = \frac{|z|^2}{r^2} + F \left( \frac{z - \bar{z}}{2i} \right) \]  

(17)

\[ L = p_\varphi = \frac{z + \bar{z}}{2}. \]  

(18)

Assume now that \( F(\cdot) \) is a polynomial; actually, we shall consider here only the case of \( F(\cdot) \) being a monomial and leave the general case to the appendix. So we put

\[ F(r \cdot p_r) = \gamma (r \cdot p_r)^N, \quad \gamma \in \mathbb{R}. \]  

(19)

We show that the third integral of motion can be chosen as a polynomial in the momenta. The starting point is our integral \( C_1 \), equation (16). We consider two cases:

(i) \( N = 2k \)

Let us define

\[ G \equiv H - (-1)^k \gamma p_\varphi^{2k}. \]  

(20)

Obviously, \( G \) is an integral of motion; in terms of \( z, \bar{z} \) variables it reads

\[ G = \frac{|z|^2}{r^2} + \gamma \frac{(-1)^k}{2^{2k}} \left( (z - \bar{z})^{2k} - (z + \bar{z})^{2k} \right). \]  

(21)
The second term on the right hand side is proportional to \(|z|^2\) (the terms \(z^{2k}\) and \(\bar{z}^{2k}\) cancel). Therefore, the integral of motion
\[
\tilde{C} \equiv G \cdot C_1 \tag{22}
\]
is a polynomial in \(z, \bar{z}\) of degree \(N\) which implies that \(\tilde{C}\) is a polynomial in \(p_r\) and \(p_\phi\) (consequently, in \(p_1\) and \(p_2\)) of the same degree.

(ii) \(N = 2k+1\)

In this case the previous argument does not work unless \(\gamma\) is purely imaginary. However, one can proceed as follows. Note that, given any \(\chi = \chi(E, L)\), \(G = G(E, L)\), the following function is an integral of motion
\[
\tilde{C}_\chi \equiv G \left( \frac{z^2 - \bar{z}^2}{2|z|^2} \right) \cos (2\varphi + \chi) - G \left( \frac{z^2 + \bar{z}^2}{2|z|^2} \right) \sin (2\varphi + \chi) \tag{23}
\]
which can be rewritten as
\[
\tilde{C}_\chi = R \cos 2\varphi - S \sin 2\varphi \tag{24}
\]
\[
R \equiv G \left( \frac{z^2 - \bar{z}^2}{2|z|^2} \right) \cos \chi - G \left( \frac{z^2 + \bar{z}^2}{2|z|^2} \right) \sin \chi \tag{25}
\]
\[
S \equiv G \left( \frac{z^2 - \bar{z}^2}{2|z|^2} \right) \sin \chi + G \left( \frac{z^2 + \bar{z}^2}{2|z|^2} \right) \cos \chi. \tag{26}
\]

Now, in order to \(\tilde{C}_\chi\) become a polynomial in \(z\) and \(\bar{z}\), \(G\) and \(\chi\) should be chosen in such a way that both
\[
\frac{(z^2 - \bar{z}^2)}{i} G \cos \chi - (z^2 + \bar{z}^2) G \sin \chi \tag{27}
\]
are proportional to \(|z|^2\). Taking the appropriate linear combinations of the above expressions we finally conclude that
\[
G \cos \chi - i G \sin \chi \tag{28}
\]
should be proportional to \(\bar{z}\). Now,
\[
E = \frac{|z|^2}{r^2} + i \gamma (-1)^{k+1} \frac{(z - \bar{z})^{2k+1}}{2^{2k+1}}. \tag{29}
\]
Therefore, one can take
\[
G \sin \chi = E, \quad G \cos \chi = (-1)^k \gamma p_\phi^{2k+1}. \tag{30}
\]
With this choice \(\tilde{C}_\chi\) becomes a polynomial in \(z\) and \(\bar{z}\) of degree \(N = 2k+1\).
For general $F(\cdot)$ (in particular, for $F(\cdot)$ being a general polynomial) the relevant superintegral is derived in appendix. It reads

$$
\tilde{C} = C_1 \left( H - A \left( \frac{(z + \bar{z})^2}{4} \right) \right) + B \left( \frac{(z + \bar{z})^2}{4} \right) + \left( \frac{z + \bar{z}}{4|z|^2} \right) \left( (z^2 + \bar{z}^2) \cos 2\varphi + \frac{z^2 - \bar{z}^2}{i} \sin 2\varphi \right)
$$

(31)

where $C_1$, $A$ and $B$ are given by (5), (6) and (16), respectively.

4. Simple examples

Consider the classical Zernike system,

$$
H = p_r^2 + \frac{p_\varphi^2}{r^2} + \gamma_1 (r \cdot p_r) + \gamma_2 (r \cdot p_r)^2
$$

(32)
or

$$
H = \frac{|z|^2}{r^2} + \frac{\gamma_1}{2l} (z - \bar{z}) - \frac{\gamma_2}{4} (z - \bar{z})^2;
$$

(33)

here $F(\cdot)$ is no longer a monomial. However, one easily checks that the choice

$$
G \cos \chi = \gamma_1 p_\varphi
$$

$$
G \sin \chi = E + \gamma_2 p_\varphi^2
$$

(34)
yields, through equations (19)–(21), the integral $I' - I$ from [1] (cf equation (2.2) therein).

Let us now consider the monomial of third order, $N = 3 \ (k = 1)$. Then

$$
E = \frac{|z|^2}{r^2} + i \frac{\gamma (z - \bar{z})^3}{2r^3}.
$$

(35)

Using equation (30) one can check that our integral $\tilde{C}_x$ takes the form

$$
\tilde{C}_x = \left( p_r^2 - \frac{p_\varphi^2}{r^2} \right) + \gamma \left( r^3 p_\varphi^3 - 2r \cdot p_r p_\varphi^2 \right) \cos 2\varphi + \left( \frac{2r p_\varphi}{r} + \gamma \left( 2r^2 p_\varphi^2 - p_\varphi^3 \right) \right) \sin 2\varphi
$$

(36)

which agrees with the result following from table 2 of [1].

Let us note that obviously one can choose the additional integral of motion as an element of arbitrary fixed representation of the $SO(2)$ rotation group generated by $p_\varphi$. Equation (16) tells us that this can be always real irreducible sum of two complex representations corresponding to the characters $\pm 2$.

Using the algorithm described in previous sections one can easily produce numerous examples of superintegrable Zernike-like systems, together with corresponding superintegrals. For example, consider the following Hamiltonian

$$
H = \tilde{p}^2 + \lambda e^{i\tilde{p}}.
$$

(37)
Using equations (5)–(8) one finds immediately

$$\tilde{C} = \frac{p^2}{\beta^2} H - \frac{\lambda p^2}{\beta^2} \cos L - \frac{\lambda p_1 p_2}{\beta^2} \sin L.$$  \hfill (38)

Note that \( \tilde{C} \) is no longer a polynomial in momenta. Each term on the right hand side is separately an integral of motion and they are combined in such a way as to cancel the \( \frac{1}{p^2} \) singularity.

5. Conclusions

We have shown that the Hamiltonian (2) is maximally superintegrable for any choice of the function \( F(\cdot) \). For \( F \) being a polynomial it is easy to show that the additional integral of motion can be also chosen to be a polynomial in momenta of the same degree. Moreover, it is an element of the representation of \( SO(2) \) group being the direct sum (in complex domain) of representations described by the characters \( \pm 2 \). In other words, when the complicated polynomial in Cartesian coordinates, representing the relevant integral, is expressed in terms of polar ones, all trigonometric functions cancel except \( \sin^2 \phi \) and \( \cos^2 \phi \).

We considered only the Euclidean case. However, it is immediate to see that one can proceed analogously if the Euclidean plane is replaced by sphere or hyperbolic space. The only difference is that now the additional integral of motion is any periodic function of

$$\arctan \left( \frac{\hat{T}_k(\varrho \cdot p_\varphi)}{p_\varphi} \right) - \psi$$  \hfill (39)

with the notation adopted from [1]. Finally, let us note that the Hamiltonian (2) continues to be maximally superintegrable in any dimension, \((\vec{q}, \vec{p}) \in \mathbb{R}^{2n}\). As an example consider \( n = 3 \). Passing to the spherical coordinates we find

$$H = p_r^2 + \frac{\vec{L}^2}{r^2} + F (r \cdot p_r).$$  \hfill (40)

Now, the motion on \( S^2 \) as configuration space, defined by the Hamiltonian \( \vec{L}^2 \), is maximally superintegrable. Let \((I_1, I_2, \psi_1, \psi_2)\) be the corresponding action-angle variables; one can take \( I_1 = L_3 \equiv p_\varphi \), \( I_2 = |\vec{L}| \). Then \( I_1, I_2 \) and any periodic function of \( \psi_1 \) are independent integrals of motion. The fourth one is provided by the Hamiltonian (40). The latter can be rewritten as

$$H = p_r^2 + \frac{p_\varphi^2}{r^2} + F (r \cdot p_r).$$  \hfill (41)

Repeating the reasoning presented in section III we find the fifth integral

$$C = \text{any periodic function of } \left( \arctan \left( \frac{r \cdot p_\varphi}{I_2} \right) - \psi_2 \right).$$  \hfill (42)

For arbitrary \( n \) we note that the geodesic motion on \( S^{n-1} \) is again maximally superintegrable yielding \( 2(n - 1) - 1 \) integrals of motion. Together with total energy we have \( 2n - 2 \) integrals. The remaining one can be constructed along the lines sketched above.

Data availability statement

No new data were created or analysed in this study.
Acknowledgment

We are grateful to Prof. Krzysztof Andrzejewski and Paweł Maślanka for helpful discussions and useful suggestions. This paper was supported by the IDUB grant, Decision No 54/2021.

Appendix

In order to find the form of the regular integral in polar coordinates let us note that the integral $C_1$, equation (16), when expressed in Cartesian coordinates, reads

$$C_1 = \frac{2p_1 p_2}{\vec{p}^2}.$$  \hspace{1cm} (A1)

It can be used to construct the regular integral in analogy with equation (8)

$$\tilde{C} = C (H - A (-L^2)) + \frac{2Lp^2}{\vec{p}^2} B (-L^2).$$  \hspace{1cm} (A2)

Expressing (A2) back in polar coordinates one obtains (31).

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