Recursive feasibility of continuous-time model predictive control without stabilising constraints*

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Abstract

We consider recursive feasibility of nonlinear continuous-time Model Predictive Control (MPC) without stabilizing terminal costs and constraints. We derive conditions on the horizon length such that the MPC algorithm is recursively feasible assuming local cost controllability, i.e. a controllability property imposed in recent studies on MPC without stabilizing constraints or costs. These conditions also ensure asymptotic stability of the origin w.r.t. the MPC closed loop. For the linear-quadratic case, cost controllability can be replaced by standard assumptions in control. Moreover, we derive results on the relationship between the horizon length and the distance of the initial state from the boundary of the viability kernel.

I. INTRODUCTION

Two key aspects in Model Predictive Control (MPC) are asymptotic stability and recursive feasibility, see, e.g. [1]. The latter property refers to existence of a solution to the optimisation problem recursively invoked in the MPC algorithm. Both properties are often ensured by constructing suitable terminal costs and constraints, which are then added to the optimisation problem to be solved at each time instant. However, recently, researchers have thoroughly analysed so-called “unconstrained” MPC, i.e. MPC schemes without stabilising constraints, w.r.t. their stability behaviour, see, e.g. [2], [3], [4], [5] for references dealing with discrete-time systems and [6], [7] for continuous-time systems and their connection to the former results. See [8] for a study of how these two approaches are linked. The main motivation behind this conceptual shift is simplicity of the approach and its reduced numerical complexity, which explains its pre-dominant use in industry, cp. the discussion in [9, Section 7.4].

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However, a rigorous treatment of recursive feasibility has only been done for discrete-time systems, see [9], [10], [11]. In these references, the authors show that sublevel sets of the finite-horizon value function can be made recursively feasible assuming so-called cost-controllability [12] (the same property as typically invoked for showing asymptotic stability). To this end, a sufficiently long prediction horizon is required, which can be quantified in dependence of the problem data.

In this paper, we derive a continuous-time analogon of these results: We establish a condition that allows one to determine the length of the prediction horizon such that recursive feasibility (and asymptotic stability) of the MPC closed loop is guaranteed. To this end, a thorough investigation of the interplay between open- and closed-loop control is of key importance. In addition, we consider the Linear-Quadratic (LQ) case in detail. Here, we show that the admissible set (also called viability kernel, see [13], [14]) is closely related to the sublevel sets of the finite-horizon value function.

The outline of the paper is as follows: In Section II we present the setting, i.e. nonlinear continuous-time systems with state and input constraints and the MPC algorithm. Section III is devoted to our main result, i.e. a sufficient condition for recursive feasibility (and asymptotic stability). Then, we focus on the LQ case in Section IV before our findings are illustrated by an example in Section V and conclusions are drawn in Section VI.

**Notation:** \( \mathbb{N} \) refers to the set of natural numbers; \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{> 0} \) to the set of nonnegative and positive real numbers resp. For \( x \in \mathbb{R}^n \), \( |x| \) denotes its Euclidean norm while \( \|F\| := \max_{|x|=1} |Fx| \) stands for the induced matrix norm of \( F \in \mathbb{R}^{m \times n} \). For a set \( S \subseteq \mathbb{R}^q \) with \( q \geq n \), \( \text{cl}(S) \), \( \partial S \) and \( \text{proj}_{\mathbb{R}^n}(S) \) refer to its closure, boundary and projection onto \( \mathbb{R}^n \) resp. A continuous function \( \eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) belongs to class \( \mathcal{K}_\infty \) provided that it is strictly increasing, unbounded, and \( \eta(0) = 0 \). The space \( L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \) denotes the set of Lebesgue-measurable functions \( u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m \) that are locally essentially bounded.

**II. Problem setting**

We consider the continuous-time nonlinear system:

\[
\dot{x}(t) = f(x(t), u(t))
\]

with initial condition \( x(0) = x_0 \), where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) are the state and the control at time instant \( t \in \mathbb{R}_{\geq 0} \), respectively. Let the map \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) be continuous and locally Lipschitz continuous with respect to its first argument \( x \) on \( \mathbb{R}^n \setminus \{0\} \). Then, for a given control function \( u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \), there exists a unique solution of the initial value problem, which is denoted by \( x(t) = x(t; x_0, u) \), \( t \in I_{x_0,u} \), where \( I_{x_0,u} \) denotes its maximal interval of existence. Moreover, we employ the
constraint set $\mathcal{E} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ by imposing the constraint
\[
(x(t), u(t)) \in \mathcal{E} \quad \forall t \in [0, \infty).
\] (2)

We introduce the sets
\[
X := \text{proj}_{\mathbb{R}^n}(\mathcal{E}) = \{ x : \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathcal{E} \}
\]
and, for $x \in X$, $U(x) := \{ u \in \mathbb{R}^m : (x, u) \in \mathcal{E} \}$. With $x_0 \in X$ and $T \in \mathbb{R}_{>0}$, we let $\mathcal{U}_T(x_0)$ (resp. $\mathcal{U}_\infty(x_0)$) denote the set of all $u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ such that the solution exists and satisfies the constraint (2) on $[0, T]$ (resp. for all $t \geq 0$). We will refer to the set $\mathcal{U}_T(x_0)$ (resp. $\mathcal{U}_\infty(x_0)$) as the set of admissible control functions with respect to the initial condition $x_0$ and the horizon $T$ (resp. the infinite horizon).

Next, we define the admissible set [13] (also called viability kernel [14]) as the set of states, for which an admissible control exists on $[0, \infty)$.

**Definition 1 (Admissible Set):** The admissible set is defined by
\[
\mathcal{A} := \{ x_0 \in \mathbb{R}^n : \mathcal{U}_\infty(x_0) \neq \emptyset \}.
\]

### A. Continuous-time MPC

Let $x^* \in X$ be a controlled equilibrium, that is, there exists a $u^* \in U(x^*)$ such that $f(x^*, u^*) = 0$. Our goal is to use MPC to asymptotically drive the state to $x^*$ while satisfying the state and control constraints (2). To that end, we define a finite-horizon cost functional
\[
J_T(x_0, u) := \int_0^T \ell(x(s; x_0, u), u(s)) \, ds,
\]
with continuous stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, and consider the finite-horizon optimal control problem
\[
\inf_{u \in \mathcal{U}_T(x_0)} J_T(x_0, u).
\] (3)

with value function $V_T : X \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ defined by $V_T(x_0) := \inf_{u \in \mathcal{U}_T(x_0)} J_T(x_0, u)$. By convention, we will say that $V_T(x_0) = \infty$ when $\mathcal{U}_T(x_0) = \emptyset$. Moreover, we use the abbreviation $V_T^{-1}[0, C] := \{ x \in X \mid V_T(x) \leq C \}$.

We impose the following assumption on $\ell$, which is, e.g., trivially satisfied for quadratic stage cost $\ell(x, u) := x^TQx + u^TRu$ with positive definite weighting matrix $Q \in \mathbb{R}^{n \times n}$.

**A1** Let $\ell(x^*, u^*) = 0$ hold. Moreover, let there be $\underline{\eta}, \overline{\eta} \in K_{\infty}$ satisfying
\[
\underline{\eta}(|x - x^*|) \leq \ell^*(x) \leq \overline{\eta}(|x - x^*|) \quad \forall x \in X
\]
using the abbreviation $\ell^*(x) := \inf_{u \in U(x)} \ell(x, u)$. 

Given a time shift $\delta \in \mathbb{R}_{>0}$, a number $N \in \mathbb{N}$ that specifies the length of the prediction (optimization) horizon, and an initial value $x_0 \in X$, the MPC algorithm is as follows:

1. Set the prediction horizon $T \leftarrow N\delta$ and $p \leftarrow 0$
2. Measure the current state $\hat{x} = x(p\delta; x_0, u_{\text{MPC}})$
3. Find a minimiser $u^* \in \arg \inf_{u \in \mathcal{U}_T(\hat{x})} J_T(\hat{x}, u)$
4. Implement $u_{\text{MPC}}(t) = u^*(t)$, $t \in [p\delta, (p+1)\delta)$
5. Set $p \leftarrow p + 1$ and go to step 2

We assume that a minimizer of problem (3) in step 3 exists if $\mathcal{U}_T(x_0) \neq \emptyset$ holds in order to facilitate exposition of the upcoming analysis, see, e.g., [9] for a detailed discussion of that assumption.

The MPC algorithm iteratively solves a finite-horizon optimal control problem and applies the first $\delta$ time units of the open-loop solution, i.e. $u^*|_{[0,\delta]} \in \mathcal{U}_\delta(\hat{x})$, inducing a time-varying state-feedback law $\mu_{T,\delta} : [0,\delta) \times X \to \mathbb{R}^m$. We denote the resulting closed-loop solution at time $t$ emanating from $x_0$ due to this feedback law by $x_{\mu_{T,\delta}}(t; x_0)$.

The key property to ensure well-posedness of the MPC algorithm assuming so-called initial feasibility, i.e. solvability of the minimization problem (3), is the following.

**Definition 2 (Recursive Feasibility):** A set $S \subset X$ is said to be recursively feasible w.r.t. the feedback law $\mu_{T,\delta}$ if and only if, for each $\hat{x} \in S$, the conditions

- $(x(t; \hat{x}, u^*), u^*(t)) \in \mathcal{E}$ for all $t \in [0,\delta]$ and
- $x(\delta; \hat{x}, u^*) \in S$

hold with $u^*(t) = \mu_{T,\delta}(t; \hat{x})$ depending on $\hat{x}$ and $T$.

**Remark 1:** Note that we do not require $x(t; \hat{x}, u^*) \in S$ on the open interval $t \in (0,\delta)$. Thus, each recursively feasible set is a subset of the admissible set but, in general, not vice versa. In the LQ setting we will show that the interior of the admissible set may be made recursively feasible, under some assumptions and with suitable choices of horizon length and time shift.

**B. Perspectives**

Algorithmically, ensuring admissibility of a control function, i.e. $u \in \mathcal{U}_T(\hat{x})$, is a non-trivial task in the continuous-time setting. Hence, the derivation of sufficient conditions such that only finitely many inequality constraints have to be checked to ensure admissibility are of particular interest, see, e.g. [15, Lemma 1] for an example tailored to the non-holonomic robot. One of our goals for future research is to alleviate this burden by using the characterization of the admissible set’s boundary based on the so-called
theory of barriers [13]. There it is shown that parts of the boundary are made up of integral curves of the system that satisfy a minimum-like principle, which yields conditions on the control. The following example shows that stabilisation is, in general, not possible for all $x_0 \in A$.

**Example 1:** Consider the scalar system $\dot{x}(t) = x(t) + u(t)$ with the constraints $|x(t)| \leq 2$ and $|u(t)| \leq 1$. Then, $\partial A$ is given by the set $\{-1, 1\}$. It can be shown that, for every $x_0 \in \partial A$, there exists a unique input such that the state does not violate the constraints in the future, and that this control renders $\partial A$ invariant. Hence, we get $V_\infty(x_0) = \infty$ for all $x_0 \in \partial A$ if the stage cost satisfies Assumption A1.

However, if the control constraint is relaxed, say $|u(t)| \leq 3$, the viability kernel and the basin of attraction coincide (if the horizon length is sufficiently long).

### III. Recursive Feasibility and Asymptotic Stability

To address stability and recursive feasibility of the MPC closed loop, we impose *cost controllability* [12].

(A2) There exists $\gamma \in \mathbb{R}_{>0}$ and a neighbourhood of $x^*$, labelled $\mathcal{N}$, such that

$$V_\infty(x) \leq \gamma \ell^*(x) \quad \forall x \in \mathcal{N} \cap X.$$ 

Note that we employ the local version following [10], which assumes cost controllability only in a neighbourhood of the controlled equilibrium $x^*$. It is possible to further weaken Assumption A2 by using a growth bound $\gamma$ depending on the length of the prediction horizon, see, e.g. [16] and the references therein for a detailed discussion.

We can extend Assumption A2 to arbitrary sublevel sets of the finite-horizon value function. To this end, note that the constant

$$M := \inf_{x \in X \setminus \mathcal{N}} \ell^*(x) > 0$$

is well-defined in view of Assumption A1.

**Proposition 1:** Let assumptions A1 and A2 hold. Then, for each $C \in \mathbb{R}_{>0}$ such that $\mathcal{N} \cap X \subseteq V_T^{-1}[0, C]$, the inequality

$$V_T(x) \leq \beta \ell^*(x) \quad \forall x \in V_T^{-1}[0, C]$$

holds with $\beta := \max\{\frac{C}{M}, \gamma\}$.

**Proof:** Using Inequality (4), we have $V_T(x) \leq \frac{C}{M} \ell^*(x)$ for all $x \in V_T^{-1}[0, C] \setminus \mathcal{N}$. Moreover, from Assumption A2 we have $V_T(x) \leq \gamma \ell^*(x)$ for all $x \in \mathcal{N} \cap X$. Combining these two inequalities yields the assertion. \hfill \blacksquare
We impose the following assumption:

(A3) For given prediction horizon $T > 0$ and $C > 0$, there exists a constant $\bar{C} \in \mathbb{R}_{>0}$ such that

$$\delta \ell^*(\hat{x}) \leq C V_\delta(\hat{x}) \quad \forall \hat{x} \in V_{\bar{T}}^{-1}[0, C]$$

holds for all $\delta \in (0, T]$

**Remark 2:** The right hand side of Assumption A3 takes the impact of the control on the current state into account, see, e.g., [17, Section 5] for a comparison between the left and the right hand side of Assumption A3. Having $\delta$ also on the left hand side allows one to derive a uniform bound $\bar{C}$; clearly, in MPC (and also in the proof of Theorem [II]), it suffices to have this estimate for one particular $\delta$. In Section [IV], we demonstrate the difference for linear systems.

We now present the main contribution of the paper: the extension of [10, Theorem 4] from discrete to continuous time.

**Theorem 1:** Consider the system (1) and constraint (2). For a given $C > 0$, suppose that Assumptions A1, A2 with $\bar{C}$ hold. Moreover, let $M$ be defined as in (4) and $\beta := \max\{\frac{C}{\bar{M}}, \gamma\}$. Then, for $\delta \in (0, \beta)$, and prediction horizon $N \in \mathbb{N}$ satisfying the condition

$$\max\left\{\frac{C}{M^\delta}, \bar{C}\left(\frac{\beta}{\delta}\right)^2 \right\} : \left(\frac{\beta}{\beta + \delta}\right)^{N-1} < 1,$$

(6)

the following relaxed Lyapunov inequality holds for all $\hat{x} \in V_{\bar{T}}^{-1}[0, C]$ with $\alpha = \alpha_{N, \delta} := \bar{C}(\beta/\delta)^2(\beta/(\beta + \delta))^{N-1}$:

$$V_T(x(\delta; \hat{x}, u_{T, \delta})) \leq V_T(\hat{x}) - (1 - \alpha) \int_0^\delta \ell(s) \, ds.$$  

(7)

Here, we used the abbreviation $\ell(s) := \ell(x(s; \hat{x}, u^*))$ to denote the stage cost evaluated along an arbitrary pair of optimal control $u^* = u^*(\hat{x})$ and solution trajectory $x(s; \hat{x}, u^*)$ of the OCP (3).

**Proof:** Consider $\hat{x} \in V_{\bar{T}}^{-1}[0, C]$, and let $u^* \in \mathcal{U}_T(\hat{x})$ be a solution of the OCP (3), which implies $J_T(\hat{x}, u^*) \leq J_T(\hat{x}, u)$ for all $u \in \mathcal{U}_T(\hat{x})$. Define $\tilde{x} := x(\delta; \hat{x}, u^*)$, and let $\tilde{u}^*$ be the respective solution of (3), i.e. $J_T(\tilde{x}, \tilde{u}^*) = V_T(\tilde{x})$. For any $t \in [0, T]$, we have

$$V_T(\tilde{x}) = \int_0^\delta \tilde{\ell}(s) \, ds + \int_\delta^t \ell(s) \, ds + \int_t^T \ell(s) \, ds$$

(8)

and

$$V_T(\hat{x}) = \int_0^T \tilde{\ell}(s) \, ds \leq \int_0^\delta \ell(s) \, ds + V_{T-\delta} \ell_T(x(t; \hat{x}, u^*))$$

with $\tilde{\ell}(s) := \ell(x(s; \hat{x}, \tilde{u}^*), \tilde{u}^*(s))$. Hence, using (8) to replace $\int_\delta^t \ell(s) \, ds$ in the last inequality yields

$$V_T(\tilde{x}) \leq V_T(\hat{x}) - \int_0^\delta \ell(s) \, ds - \int_t^T \ell(s) \, ds + V_{T-\delta} \ell_T(x(t))$$

(9)

1Note that this equality also holds ($V_T(\hat{x}) = \infty$) if there does not exist an admissible solution of the OCP (3).
with $x(t) := x(t; \hat{x}, u^*)$.

Before we proceed, we make the following preliminary considerations: Since the assumption $V_T(\hat{x}) \leq C$ implies $V_{T-\delta}(x(t; \hat{x}, u^*)) \leq C$ for all $t \in [0, T]$, using Inequality (5) yields $V_{T-\delta}(x(t; \hat{x}, u^*)) \leq \beta \ell^*(x(t; \hat{x}, u^*)) \leq \beta (t)$. Therefore, we have $V_T(\hat{x}) \leq \int_t^T \ell(s) \, ds + \beta (t)$ and, thus, $\int_t^T \ell(s) \, ds \leq \beta (t)$ for all $t \in [0, T]$. For any $\bar{t} \in [0, T - \delta]$, this implies the inequality

$$
\int_{\bar{t}}^{\bar{t} + \delta} \left( \int_{\bar{t}}^{T} \ell(s) \, ds \right) \, dt \leq \beta \int_{\bar{t}}^{\bar{t} + \delta} \ell(t) \, dt,
$$

which is, using $\int_t^T \ell(s) \, ds = \int_{\bar{t}}^{\bar{t} + \delta} \ell(s) \, ds + \int_{\bar{t} + \delta}^{T} \ell(s) \, ds$, equivalent to:

$$
\int_{\bar{t}}^{\bar{t} + \delta} \left( \int_{\bar{t}}^{\bar{t} + \delta} \ell(s) \, ds \right) \, dt + \delta \int_{\bar{t} + \delta}^{T} \ell(s) \, ds \leq \beta \int_{\bar{t}}^{\bar{t} + \delta} \ell(t) \, dt.
$$

Since the first term on the left-hand side is nonnegative, we get $\delta \int_{\bar{t} + \delta}^{T} \ell(s) \, ds \leq \beta \int_{\bar{t}}^{\bar{t} + \delta} \ell(s) \, ds$ for all $\bar{t} \in [0, T - \delta]$. Thus, setting $\bar{t} = p \delta$ for $p \in \{0, 1, \ldots, N - 1\}$, leads to

$$
\frac{\delta}{\beta} \int_{p \delta}^{(p+1) \delta} \ell(s) \, ds \leq \int_{p \delta}^{(p+1) \delta} \ell(s) \, ds.
$$

Adding the term $\int_{(p+1) \delta}^{N \delta} \ell(s) \, ds$ on both sides of this inequality reads as

$$
\frac{\beta + \delta}{\beta} \int_{p \delta}^{N \delta} \ell(s) \, ds \leq \int_{p \delta}^{N \delta} \ell(s) \, ds.
$$

Iteratively applying this inequality until $p = N - 2$ to further estimate the left hand side yields

$$
\int_{p \delta}^{N \delta} \ell(s) \, ds \geq \left( \frac{\beta + \delta}{\beta} \right)^{N - p - 1} \int_{(N - 1) \delta}^{N \delta} \ell(s) \, ds
$$

for $p \in \{0, 1, \ldots, N - 1\}$. Then, first taking $p = 0$ and invoking $\hat{x} \in V_T^{-1}[0, C]$ Assumption A2, and the definition of $\beta$ afterwards, yields

$$
\min \{ \beta \ell^*(\hat{x}), C \} \geq V_T(\hat{x}) \geq \left( \frac{\beta + \delta}{\beta} \right)^{N - 1} \int_{T - \delta}^{T} \ell(s) \, ds.
$$

Using (the first term of) Condition (6), we get $\int_{T - \delta}^{T} \ell(s) \, ds < M \delta$, which implies $\int_{T - \delta}^{T} \ell^*(x(s; \hat{x}, u^*)) \, ds < M \delta$. Since

$$
\delta \left( \inf_{t \in [T - \delta, T]} \ell^*(x(t; \hat{x}, u^*)) \right) \leq \int_{T - \delta}^{T} \ell^*(x(s; \hat{x}, u^*)) \, ds
$$

holds, there exists a time instant $\bar{t} \in [T - \delta, T]$ such that the inequality $\ell^*(x(\bar{t}; \hat{x}, u^*)) < M$ holds. As a consequence, the definition of $M$ ensures $x(\bar{t}; \hat{x}, u^*) \in \mathcal{N}$, cp. (4). Therefore, we have $V_\infty(x(\bar{t}; \hat{x}, u^*)) \leq \beta \ell^*(x(\bar{t}; \hat{x}, u^*))$, which implies

$$
V_{T - \bar{t} + \delta}(x(\bar{t}; \hat{x}, u^*)) \leq \beta \ell^*(x(\bar{t}; \hat{x}, u^*)),
$$

$^2$We like to stress that the following inequality implies finiteness of $V_T(\hat{x})$ and, thus, the existence of an admissible control function $\tilde{u} \in \mathcal{U}_T(\hat{x})$. 

Note that the definition of \( \hat{t} \) and the line of reasoning used to derive Inequality (11) imply
\[
\delta \ell^*(x(\hat{t}; \hat{x}, u^*)) \leq \int_{T-\delta}^T \ell^*(x(s; \hat{x}, u^*)) \, ds \leq \int_{T-\delta}^T \ell(s) \, ds.
\]

Thus, using (9) with \( t = \bar{t} \) and applying the last two inequalities, we get
\[
V_T(\tilde{x}) \leq V_T(\hat{x}) - \int_0^\delta \ell(s) \, ds - \int_{\bar{t}}^T \ell(s) \, ds + \frac{\beta}{\delta} \int_{T-\delta}^T \ell(s) \, ds.
\]

Then, dropping the term \( \int_{\bar{t}}^T \ell(s) \, ds \), \( \hat{t} \in [T-\delta, T] \), and using Inequality (10), we get
\[
V_T(\tilde{x}) \leq V_T(\hat{x}) - \int_0^\delta \ell(s) \, ds + \frac{\beta^2}{\delta} \left( \frac{\beta}{\beta + \delta} \right)^{N-1} \ell^*(\tilde{x}).
\]

Invoking Assumption A3, we get the desired relaxed Lyapunov inequality (7) where \( \alpha_{N,\delta} \) is defined as in (the second term of) Condition (6), i.e. the assertion.

Theorem 1 relates the easily checkable Condition (6) with the Lyapunov-like decrease (7) of the finite-horizon value function, which allows one to conclude asymptotic stability of the equilibrium and can be, thus, considered as a sufficient stability condition, cp. [5]. In addition, it also shows that the sublevel sets of the finite-horizon value function are recursively feasible under the MPC feedback law \( \mu_{T,\delta} \).

For the following statement, we require the set of exceptional points defined by \( \mathcal{O} := \lim_{n \to \infty} V_{\infty}^{-1}[n, \infty[, \) cp. [10]. The set \( \mathcal{O} \) consists of all elements of the state space at which the value function may blow up. Hence, entering the set \( \mathcal{O} \) should be avoided. Using this definition, we can derive the following corollary of Theorem 1, which is the continuous-time analogue of [10, Thm. 6], which involves compact subsets of the infinite-horizon value function’s sublevel sets.

**Corollary 1:** Let the Assumptions A1 - A3 hold and let \( K \subset V_{\infty}^{-1}[0, \infty[ \setminus \mathcal{O} \) be a compact set. Then, for a chosen time shift \( \delta > 0 \) there exists an horizon length \( \bar{T} = \bar{T}(\delta, K) \in (\delta, \infty) \) such that the origin is asymptotically stable w.r.t. the MPC closed loop with domain of attraction containing the set \( K \).

**Proof:** By the same arguments as in [10], for any compact set \( K \subset V_{\infty}^{-1}[0, \infty[ \setminus \mathcal{O} \), there exists a \( C < \infty \) such that \( K \subseteq V_{\infty}^{-1}[0, C] \). Therefore, from Condition (9) and with a chosen time shift \( \delta \), for any initial condition \( x_0 \in K \) the MPC closed loop is stable and recursively feasible if \( N \geq \bar{N} \), i.e. \( \bar{T} := \delta \bar{N} \), where \( \bar{N} = \bar{N}(\delta, K) \in \mathbb{N} \) satisfies
\[
\bar{N} > \frac{\max\{\ln(\delta) + \ln(M) - \ln(C), 2 \ln(\delta/\beta) - \ln(C)\}}{\ln(\beta) - \ln(\beta + \delta)} + 1.
\]

We stress that Corollary 1 implies recursive feasibility for all initial values \( x_0 \) contained in the compact set \( K \).
Remark 3: It may be verified that $\lim_{\delta \to 0^+} \bar{N}(\delta, K) = \infty$ and that we require
\[
\delta < \min \left\{ \frac{C}{\gamma}, \sqrt{C} \right\}
\]
for $\bar{N}(\delta, K) > 1$, where we’ve used the definition of $\beta$ as in Theorem 1.

The importance of considering compact sets $K \subset V_{\infty}^{-1}[0, \infty[\setminus \mathcal{O}$ will be clarified in the next section.

IV. CONSTRAINED LINEAR-QUADRATIC CASE

In Section III we presented conditions under which, for the MPC closed loop, sublevel sets of the finite-horizon value function are recursively feasible and the origin is asymptotically stable. We now consider the LQ case and establish a relationship between the admissible set, see Definition 1 and the level sets of the value function. Then, we specialise Corollary 1 to the LQ case; an important result because there exist algorithms capable of computing compact inner-approximations of viability kernels, see [18], [19], [20]. Most of the results derived for discrete-time systems in [10] carry over to our continuous-time setting in an analogous way. Hence, we only present novel aspects in the current section, and refer the reader to the appendix for the straightforward adaptations.

We focus on linear systems, i.e. system (1) given by
\[
f(x(t), u(t)) := Ax(t) + Bu(t)
\]
with matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Moreover, for ease of notation, we also impose pure control constraints, i.e. $u(t) \in U$, $U \subset \mathbb{R}^m$, for all $t \geq 0$. We impose the following assumptions:

(A4) The pair $(A, B)$ is stabilizable

(A5) $U$ is convex, compact, and contains 0 in its interior

(A6) The set $E$ given by $\{(x, u) \in \mathbb{R}^{n+m} : g(x, u) \leq 0\}$, $p \in \mathbb{N}_0$ and $g \in C^2(\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p)$, is convex, compact, and contains the origin in its interior. Moreover, the mapping $u \mapsto g_i(x, u)$ is convex for all $x \in \mathbb{R}^n$, $i \in \{1, \ldots, p\}$.

Proposition 2: Let the dynamics be given by (14) and Assumptions A5 and A6 hold. Then, the admissible set is compact, convex and contains the origin in its interior.

Proof: The proof that the admissible set is bounded, convex and contains the origin in its interior easily adapts from the discrete-time case as presented in [10], see Propositions 5 and 6 of the appendix.

As detailed in [21, Prop. 3.1] and [22], to have closedness of the admissible set one needs to impose assumptions on the dynamics $f$, in addition to Assumptions A5 and A6. These are that $f$ is at least $C^2$ from $\mathbb{R}^n \times U_1$ to $\mathbb{R}^n$, $U_1$ an open subset containing $U$; that there exists a constant $0 < C < \infty$ such that
sup_{u \in U} |x^T f(x, u)| \leq C(1 + \|x\|^2), \text{ for all } x \in \mathbb{R}^n; \text{ and that the set } f(x, U) \text{ is convex for all } x \in \mathbb{R}^n.

We note that all three of these additional assumptions are satisfied by the linear system (14).

Remark 4: If the functions $g_i$ do not depend on the input, that is $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, the set $A$ is closed without assuming convexity of $g_i(x, \cdot)$, see [13, Prop. 4.1] for details.

Remark 5: Note that in the discrete-time setting with dynamics $x_{k+1} = f(x_k, u_k)$, a set $S$ is said to be control invariant if, for each $x \in S$, there exists $u \in U(x) \text{ such that } f(x, u) \in S$ holds. Hence, establishing closedness of $A$ requires fewer assumptions: Briefly, if $g$ is continuous, closedness can be directly shown by considering the limit of converging sequences $\{x_k\}_{k \in \mathbb{N}} \subseteq A$.

Assumption A4 implies existence of a matrix $F$ such that the state feedback $\mu_F(x) = Fx$ renders the matrix $(A + BF)$ Hurwitz. Thus, see, e.g. [23], there exist constants $\Gamma > 0$ and $\eta > 0$ such that for all $x_0 \in \mathbb{R}^n$:

$$|x(t; x_0, u_F)| \leq \Gamma e^{-\eta(t-t_0)}|x_0| \quad \forall t \in [t_0, \infty].$$

These facts are used to arrive at the following proposition.

**Proposition 3**: Consider the system (14) under assumptions A4 and A6. For all $x \in \lambda A$, with $\lambda \in [0,1]$ there exists a constant $M(\lambda)$ such that $V_\infty(x) \leq M(\lambda)$.

**Proof**: Since the proof essentially uses the same ideas as its discrete-time analogon, we only provide a sketch and refer to Proposition 7 of the appendix for the details. The idea of the proof is to consider a convex combination of two control functions: one that keeps the state-control pair admissible for all time, and the other, given by the linear-quadratic-regulator (LQR) feedback, driving the state to the origin asymptotically. Then, upon reaching a neighbourhood of the origin (which is guaranteed due to the choice of the convex combination) switching to the LQR feedback, which drives the state to the origin while satisfying the constraints.

Proposition 3 states that, for linear systems satisfying Assumptions A4 and A6, the infinite-horizon value function is uniformly bounded on $\lambda A$, $\lambda \in [0,1]$. On the one hand, for given $\delta$, this ensures asymptotic stability of the origin w.r.t. the MPC closed loop with the interior of $A$ contained in the basin of attraction (for a sufficiently large prediction horizon $T = N\delta$). On the other hand, the set $O$ (if present, see Example 1) is restricted to the boundary of $A$.

Next, we state a corollary which combines Corollary 1 with Proposition 3 to provide a result for the LQ case.

**Theorem 2**: Consider the system (14) and suppose that Assumptions A4 - A6 hold with the symmetric,
quadratic, and positive definite stage cost
\[ \ell(x, u) := (x^T, u^T) \begin{pmatrix} Q & N \\ N^T & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}. \]

Let \( K \subset \text{int}(A) \) be compact and \( \delta > 0 \) be given. Then, the origin is asymptotically stable w.r.t. the MPC closed loop with basin of attraction \( S \) containing \( K \) for each prediction horizon \( T = N\delta \) such that \( N \) satisfies Condition (6).

**Proof:** Clearly, there exists \( \lambda \in [0, 1] \) such that \( K \subset \lambda A \). Hence, Proposition 3 implies boundedness of \( V_T \) on \( K \). Consequently, \( K \subseteq V_{\infty}^{-1}[0, \infty[ \backslash \emptyset \) holds. Hence, Theorem 1 and Corollary 1 imply the assertion supposing that Assumptions A1-A3 hold.

Validity of Assumptions A1 and A2 can be shown analogously to the discrete-time case, cp. [10, Thm. 13]: Since the cost is quadratic and positive definite, Assumption A1 holds. In particular, we have \( \ell^*(x) = x^T Q x \leq \sigma_{\max}(Q)|x|^2 \) where \( \sigma_{\max}(Q) \) is the maximal eigenvalue of the matrix \( Q \). Invoking Assumptions A4 and A6 imply that the unique, positive definite solution \( P \) of the algebraic Riccati equation satisfies \( V_{\infty}(x) = x^T P x \) on a neighbourhood \( N \) of the origin. Hence, we have
\[ V_{\infty}(x) \leq \sigma_{\max}(P)|x|^2 \leq \frac{\sigma_{\max}(P)|x|^2}{\sigma_{\min}(Q)} \]
for all \( x \in N \), i.e. Assumption A2. Here, \( \sigma_{\min}(Q) > 0 \) denotes the smallest eigenvalue of \( Q \).

Next, we prove Assumption A3, which trivially holds in the discrete-time setting with \( \bar{C} = 1 \). To this end, we extend the previously presented argumentation based on the algebraic Riccati equation. Taking the constraints into account yields \( V_{\delta}(x) \geq x^T P_{\delta} x \geq \sigma_{\min}(P_{\delta})|x|^2 \). Hence, Assumption A3 holds with \( \bar{C} := T\sigma_{\max}(Q)/\sigma_{\min}(P_{\delta}) \).

Note that, \( T \) may be replaced by \( \delta \) in the verification procedure of Assumption A3 outlined in the proof of Theorem 2 in order to reduce the conservatism in the estimate on the required prediction horizon \( T = N\delta \) based on Condition (6).

Finally, we state the continuous-time analogon of [10, Cor. 15], which relates the sufficient horizon length to the distance of the state to \( A \)'s boundary, see the appendix for a detailed proof.

**Corollary 2:** Consider the system (14) under Assumptions A4 - A6, with the quadratic running cost as in Corollary 2. Given a compact set \( K \subset \text{int}(A) \) there exists a constant \( D \) only depending on the data of the problem, such that:
\[ \sup_{x \in K} V_{\infty}(x) \leq \frac{D}{\text{dist}(K; \partial A)}. \]
Moreover, if for a chosen time shift \( \delta \) satisfying (13), with \( \beta = \max\{ \frac{C}{\text{Dist}(K, \partial A)}, \gamma \} \), \( \bar{N}(K, \delta) \) is chosen satisfying (12), then the origin is asymptotically stable w.r.t. the MPC closed loop with basin of attraction \( \mathcal{S} \) containing the set \( K \).

In the future we intend to further explore conditions under which the horizon length does not blow up as the state approaches the boundary of \( \mathcal{A} \) by using the theory of barriers as mentioned in Subsection II-B.

**V. Example**

We demonstrate the growth of the sufficient horizon length \( N \) for a chosen time shift \( \delta \) as the initial condition, from which the MPC algorithm initiates, approaches the boundary of the admissible set. We consider the double integrator:

\[
\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t),
\]

with \( |u| \leq 1 \) and \( |x_i| \leq 1, \ i = 1, 2 \). As shown in [13], parts of the boundary of the admissible set (called the barrier) are made up of integral curves of the system that, together with a particular control function, satisfy a minimum-like principle and intersect the boundary of the constrained state space in an “ultimate tangentiality condition”. We use this fact to construct the two solid curves that form the barrier, labelled \( \partial \mathcal{A}^- \), shown in Figure 1.

We run the continuous-time MPC algorithm with the running cost \( \ell(x, u) = x_1^2 + x_2^2 + u^2 \) from various initial conditions that approach the boundary of \( \mathcal{A} \), with various time shifts \( \delta \) and horizons \( N \). Table I displays the smallest \( N = N(x_0, \delta) \), for which MPC steers the particular initial state to the origin while maintaining constraint satisfaction. It is interesting to note that the magenta curve, initiating from \( x_0 \in \partial \mathcal{A}^- \), results from a finite horizon length, \( N = 7 \). This emphasises that the infinite-horizon value function may be bounded on the admissible set’s boundary.

| \( x_0 \)    | \( \delta = 0.1 \) | \( \delta = 0.05 \) | \( \delta = 0.03 \) |
|--------------|----------------|----------------|----------------|
| (0.5, 0.5)\( ^T \) | 4    | 7   | 10  |
| (0.6, 0.6)\( ^T \) | 4    | 7   | 11  |
| (0.7, 0.7)\( ^T \) | 5    | 10  | 14  |
VI. CONCLUSION

We analysed the problems of recursive feasibility and asymptotic stability of the MPC algorithm without stabilising terminal costs and constraints in a continuous-time setting. We derived conditions on the prediction horizon and the time shift guaranteeing that arbitrary sublevel sets of the finite-horizon value function enjoy a Lyapunov-like decrease condition along closed-loop trajectories and are, thus, contained in the basin of attraction. For the linear-quadratic case using standard assumption, we argued that the interior of the viability kernel can (essentially) be covered using this technique.

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A. Some facts of the admissible set

We summarise some facts concerning the admissible set for a linear time-invariant system with assumptions A4 and A6. Most of these facts easily adapt from the discrete-time case, as in [10].

Definition 3: A set $S$ is said to be control invariant with respect to the system (1), provided that for all $x_0 \in S \cup_{\infty}(x_0)$ is nonempty and there exists a $u \in \cup_{\infty}(x_0)$ such that $x(t; x_0, u) \in S$ for all $t \in [t_0, \infty[$.

Definition 4: The maximal control invariant set contained in $X$, which we label $C$, is the union of all control invariant sets that are subsets of $X$.

Proposition 4: $A = C$.

Proof: By definition, $C \subseteq A$. Let us show that $A \subseteq C$ by contradiction. Consider a point $x_0 \in A$ at $t_0$ and suppose that $A$ is not control invariant. Then, for all $u \in \cup_{\infty}(x_0)$ there exists a $t_u \in [t_0, \infty[$ such that $x_u := x(t_u; x_0, u) \notin A$, and thus $\cup_{\infty}(x_u) = \emptyset$. Thus $\cup_{\infty}(x_0) = \emptyset$, which contradicts that fact that $x_0 \in A$. Therefore, $A \subseteq C$, and thus $C = A$.

Proposition 5: Consider system (14). If $E$ is convex, then $A$ is convex.

Proof: Consider any initial condition in the convex hull of the two points $x_1, x_2 \in A$, that is, consider $x_3 = \rho x_1 + (1 - \rho)x_2, \rho \in [0, 1]$. Consider the same convex combination of two admissible inputs associated with $x_1$ and $x_2$ at every $t \in [0, \infty[$, that is, consider $u_3(t) = \rho u_1(t) + (1 - \rho)u_2(t)$, with $u_i \in \cup_{\infty}(x_i), i = 1, 2$. By linearity it can be verified that $x(t; x_3, u_3) = \rho x(t; x_1, u_1) + (1 - \rho)x(t; x_2, u_2)$. We have $(x(t; u_i, x_i), u_i(t)) \in E$ for all $t \in [t_0, \infty[, i = 1, 2$. Because $E$ is convex, we have $(x(t; x_3, u_3), u_3(t)) \in E$ for all $t \in [t_0, \infty[$, and thus $x_3 \in A$.

Clearly if $E$ is bounded then $A$ is bounded.

Proposition 6: For the system (14) with assumptions A4 and A6 the origin is in the interior of $A$.

Proof: Clearly $0 \in A$. We need to show that there exists a neighbourhood of 0 contained in $A$. From (15), with a stabilizable pair $(A, B)$, we have $|x(t; x_0, u_F)| \leq \Gamma e^{-\eta(t-t_0)}|x_0|$ for all $t \geq t_0$. Moreover, $|Fx(t; x_0, u_F)| \leq ||F||\Gamma e^{-\eta(t-t_0)}|x_0|$ for all $t \geq t_0$. Thus, for any $\epsilon > 0$ there exists a $\nu := \epsilon / (1 + ||F||)$ such that for all $(x_0, Fx_0) \in \nu B_{n+m}$ we have $(x(t; x_0, u_F), Fx(t; x_0, u_F)) \in \epsilon B_{n+m}$, where $B_{n+m} \subset \mathbb{R}^{n+m}$ is the open unit ball about the origin of $E$. Recall that $0 \in \text{int}(E)$. Thus, we can select $\epsilon > 0$ small enough such that $\epsilon B_{n+m} \subset E$, and so $\text{proj}_{\mathbb{R}^n}(\nu B_{n+m}) \subset A$. 

VII. APPENDIX

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**Proposition 7:** For the system (14) with assumptions A4 and A6 the set \( \lambda \mathcal{A} \) is control invariant for any \( \lambda \in [0, 1] \).

**Proof:** Recall that \( 0 \in \mathcal{A} \) (Proposition 5) and that \( \mathcal{A} \) is control invariant (Proposition 4). Consider a point \( x_0 \in \lambda \mathcal{A} \), then \( x_0/\lambda \in \mathcal{A} \). Thus, there exists a \( u \in U_\infty(x_0/\lambda) \) such that \( x(t; x_0/\lambda, u) \in \mathcal{A} \) for all \( t \geq t_0 \). By linearity it may be verified that \( \lambda x(t; x_0/\lambda, u) = x(t; x_0, \lambda u) \in \lambda \mathcal{A} \) for all \( t \geq t_0 \), with \( \lambda u \in U_\infty(x_0) \) (from A6). Thus, \( \lambda u \) renders \( \lambda \mathcal{A} \) invariant. \( \blacksquare \)

**B. Proof of Proposition 3**

Consider \( x_0 \in \lambda \mathcal{A} \), for which there exists a control, which we label \( u_\lambda \in U_\infty(x_0) \), such that \( x(t; x_0, u_\lambda) \in \lambda \mathcal{A} \) for all \( t \geq t_0 \) (Proposition 7). From (15), because \( (A, B) \) is stabilizable, there exists a \( u_F \in U_\infty(x_0) \) such that \( |x(t; x_0, u_F)| \leq \Gamma e^{-\eta(t-t_0)} |x_0| \). Thus, \( |x(t; x_0, u_F), u_F(t)| \leq \lambda L \min \), where \( L := (1 + \|F\|)d_{max}d_{min}^{-1}, \max \) := \( \sup_{x \in X} |x|, \min \) := \( \inf_{x \in \partial X} |x| \). If \( \lambda L \leq 1 \) then the solution remains in \( \lambda LE \) for all time and converges to the origin, and \( V_\infty(x_0) = \sup_{x \in \lambda \mathcal{A}} J_\infty(x, u_F) \leq \alpha, \alpha \geq 0 \). Otherwise, if \( \lambda L > 1 \), consider \( \mu x(t; x_0, u_\lambda) + (1 - \mu) x(t; x_0, u_F) \), with \( \mu \in [0, 1] \), which is the solution obtained with the control \( \bar{u} := \mu u_\lambda + (1 - \mu) u_F \). It may be verified that: \( |(x(t; x_0, \bar{u}), \bar{u}(t))| \leq [\mu \lambda + (1 - \mu) L \lambda] d_{min} \). Choose \( \mu \) such that \( \mu \lambda + (1 - \mu) L \lambda = 1 \), then \( x(t; x_0, \bar{u}), \bar{u}(t) \) \( \in \mathcal{E} \) for all time. Note that \( \mu > 0 \), (if \( \mu = 0 \) we would have \( u = u_F \) and \( \lambda L \leq 1 \)). With this control, there exists a \( \bar{t} \geq t_0 \) such that \( x(\bar{t}; x_0, u) \in \epsilon \lambda \mathcal{A} \), where we have defined \( \epsilon := 1 - \frac{1 - \lambda}{\lambda} \). From our choice of \( \mu \) we also see that \( \epsilon \epsilon \in [\mu, 1] \). If we consider time \( m \bar{t} \), with \( m \in \mathbb{N}_0 \), we get \( x(m \bar{t}; x_0, u) \in \epsilon^m \lambda \mathcal{A} \). Let \( m(x_0) \) be the smallest integer such that \( \epsilon^m(x_0) \lambda L < 1 \), and let \( \bar{x} := x(m(x_0) \bar{t}; x_0, u) \). If we now switch to the feedback \( u_F \), we get \( |x(t; \bar{x}, u_F)| \leq \epsilon^m(x_0) \lambda L d_{min} \) for all \( t \geq m(x_0) \bar{t} \). Thus, \( x(t; \bar{x}, u_F) \in \lambda L \epsilon^m(x_0) \mathcal{X} \) for all \( t \geq m \bar{t} \), and converges to the origin. We get that:

\[
V_\infty(x_0) = J_{m(x_0) \bar{t}}(x_0, u) + J_\infty(\bar{x}, u_F) \leq m(x_0) \bar{t} \beta + \alpha < \infty.
\]

**C. Proof of Corollary 2**

The proof adapts from [10, Cor. 14]. As in the proof of Proposition 3 consider the constants \( \epsilon \) and \( L \) along with the mapping: \( m(x) := \inf \{ m \in \mathbb{N} : e^m \leq \frac{1}{\epsilon L} \} \). Then it can be shown, after conducting an asymptotic analysis, that \( m(x) \sim \frac{wL \ln(L)}{\partial E(x, \partial \mathcal{A})} \) as \( x \to \partial \mathcal{A} \), where \( w \in [\inf_{x \in \partial \mathcal{A}} |x|, \sup_{x \in \mathcal{A}} |x|] \). The result then follows from the fact, as established in the proof of Proposition 3 that \( V_\infty(x) \leq m(x) \bar{t} \beta + \alpha \).