Distributed Average Tracking for Lipschitz-Type Nonlinear Dynamical Systems

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Abstract

In this paper, a distributed average tracking problem is studied for Lipschitz-type nonlinear dynamical systems. The objective is to design distributed average tracking algorithms for locally interactive agents to track the average of multiple reference signals. Here, in both the agents’ and the reference signals’ dynamics, there is a nonlinear term satisfying the Lipschitz-type condition. Three types of distributed average tracking algorithms are designed. First, based on state-dependent-gain designing approaches, a robust distributed average tracking algorithm is developed to solve distributed average tracking problems without requiring the same initial condition. Second, by using a gain adaption scheme, an adaptive distributed average tracking algorithm is proposed in this paper to remove the requirement that the Lipschitz constant is known for agents. Third, to reduce chattering and make the algorithms easier to implement, a continuous distributed average tracking algorithm based on a time-varying boundary layer is further designed as a continuous approximation of the previous discontinuous distributed average tracking algorithms.

Key words: Distributed average tracking, nonlinear dynamics, adaptive algorithm, continuous algorithm.

1 Introduction

In the past two decades, there have been lots of interests in the distributed cooperative control [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], and [13], for multi-agent systems due to its potential applications in formation flying, path planning and so forth. Distributed average tracking, as a generalization of consensus and cooperative tracking problems, has received increasing attentions and been applied in many different perspectives, such as distributed sensor networks [14], [15] and distributed coordination [16], [17]. For practical applications, distributed average tracking should be investigated for signals modeled by more and more complex dynamical systems.

The objective of distributed average tracking problems is to design a distributed algorithm for multi-agent systems to track the average of multiple reference signals. The motivation of this problem comes from the coordinated tracking for multiple camera systems. Spurred by the pioneering works in [18], and [19] on the distributed average tracking via linear algorithms, real applications of related results can be found in distributed sensor fusion [14], [15], and formation control [16]. In [20], distributed average tracking problems were investigated by considering the robustness to initial errors in algorithms. The above-mentioned results are important for scientific researchers to build up a general framework to investigate this topic. However, a common assumption in the above works is that the multiple reference signals are constants [19] or achieving to values [18]. In practical applications, reference signals may be produced by more general dynamics. For this reason, a class of nonlinear algorithms were designed in [21] to track multiple reference signals with bounded deviations. Then, based on non-smooth control approaches, a couple of distributed algorithms were proposed in [22] and [23] for agents to track arbitrary time-varying reference signals with bounded deviations and bounded second deviations, respectively. Using discontinuous algorithms, further, [24] studied the distributed average tracking problems for multiple signals generated by linear dynamics.

Motivated by the above mentioned observations, this paper is devoted to solving the distributed average tracking problem for Lipschitz-type nonlinear dynamical systems. Three DAT algorithms are proposed in this paper. First of all, based on relative states of neighboring agents, a class of distributed discontinuous DAT algorithms are proposed with robustness to initial condi-
tions. Then, a novel class of distributed algorithms with adaptive coupling strengths are designed by utilizing an adaptive control technique. Different from [22], [23] and [24], the proposed algorithms are based on node adaptive laws. Further, a class of continuous algorithms are given to reduce chattering. Compared with the above existing results, the contributions of this paper are threefold. First, main results of this paper extend the dynamics of the reference signals and agents from linear systems [22] and [23] to nonlinear systems, which can describe more complex dynamics. Second, by using adaptive control approaches, the requirements of all global information are removed, which greatly reduce the computational complexity for large-scale networks. Third, compared with existing results in [24], new continuous algorithms are redesigned via the boundary layer concept to reduce the chattering phenomenon. Continuous algorithms in this paper is more appropriate for real engineering applications.

Notations: Let \( R^n \) and \( R^{n \times n} \) be sets of real numbers and real matrices, respectively. Denote by \( 1 \) a column vector with all entries equal to one. The matrix inequality \( A > (\geq) B \) means that \( A - B \) is positive (semi-) definite. Denote by \( A \otimes B \) the Kronecker product of matrices \( A \) and \( B \). For a vector \( x = (x_1, x_2, \cdots, x_n)^T \in R^n \), let \( \|x\| \) denote the 2-norm of \( x \), \( h(x) = \|x\| \), \( h(x) = \|x\| + \epsilon \). For a set \( V \), \( |V| \) represents the number of elements in \( V \).

2 Preliminaries

2.1 Graph Theory

An undirected (simple) graph \( G \) is specified by a vertex set \( V \) and an edge set \( E \) whose elements characterize the incidence relations between distinct pairs of \( V \). The notation \( i \sim j \) is used to denote that node \( i \) is connected to node \( j \), or equivalently, \( (i, j) \in E \). We make use of the \( |V| \times |E| \) incidence matrix, \( D(G) \), for a graph with an arbitrary orientation, i.e., a graph whose edges have a head (a terminal node) and a tail (an initial node). The columns of \( D(G) \) are then indexed by the edge set, and the \( i \)th row entry takes the value 1 if it is the initial node of the corresponding edge, \(-1 \) if it is the terminal node, and zero otherwise. The diagonal matrix \( \Delta(G) \) of the graph contains the degree of each vertex on its diagonal. The adjacency matrix, \( A(G) \), is the \( |V| \times |V| \) symmetric matrix with zero in the diagonal and one in the \( (i, j) \)th position if node \( i \) is adjacent to node \( j \). The graph Laplacian [25] of \( G \), \( L := \frac{1}{2} D(G)D(G)^T = \Delta(G) - A(G) \), is a rank deficient positive semi-definite matrix.

An undirected path between node \( i_1 \) and node \( i_s \) on undirected graph means a sequence of ordered undirected edges with the form \( (i_k; i_{k+1}) \), \( k = 1, \cdots, s - 1 \). A graph \( G \) is said to be connected if there exists a path between each pair of distinct nodes.

Assumption 1 Graph \( G \) is undirected and connected.

Lemma 1 [25] Under Assumption 1, zero is a simple eigenvalue of \( L \) with \( 1 \) as an eigenvector and all the other eigenvalues are positive. Moreover, the smallest nonzero eigenvalue \( \lambda_2 \) of \( L \) satisfies \( \lambda_2 = \min_{x \neq 0} \frac{x^TLx}{x^Tx} \).

3 Main results

3.1 Robust distributed average tracking algorithms design

Consider a multi-agent system consisting of \( N \) physical agents described by the following nonlinear dynamics

\[ \dot{x}_i(t) = Ax_i(t) + Bf(x_i, t) + Bu_i, \]

where \( A \in R^{n \times n} \) and \( B \in R^{n \times p} \) both are constant matrices with compatible dimensions, \( x_i(t) \in R^n \) and \( u_i(t) \in R^p \) is the state and control input of the \( i \)th agent, respectively, and \( f : R^n \times R^+ \rightarrow R^p \) is a nonlinear function. Suppose that there is a time-varying reference signal, \( r_i(t) \in R^n, i = 1, 2, \cdots, N \), which generated by the following Lipschitz-type nonlinear dynamical systems:

\[ \dot{r}_i(t) = Ar_i(t) + Bf(r_i, t), \]

where \( r_i(t) \in R^n \) is the state of the \( i \)th reference signal.

It is assumed that agent \( i \) has access to \( r_i(t) \), and agent \( i \) can obtain the relative information from its neighbors denoted by \( N_i \).

Assumption 2 \( (A, B) \) is stabilizable.

The main objective of this paper is to design a class of distributed controller \( u_i(t) \) for physical agent \( i \) in (1) to track the average of multiple reference signals \( r_i(t) \) generated by the general nonlinear dynamics (2), i.e.,

\[ \lim_{t \to \infty} \left( x_i(t) - \frac{1}{N} \sum_{i=1}^{N} r_i(t) \right) = 0, \]

where each agent has only local interaction with its neighbors.

Assumption 3 For \( \forall \theta_i(t) \in R^n, i = 1, 2 \) and \( \forall t > 0 \), the nonlinear function \( f : R^n \times R^+ \rightarrow R^p \) satisfies a Lipschitz-type condition: \( \|f(\theta_1, t) - f(\theta_2, t)\| \leq \gamma \|\theta_1 - \theta_2\| \), where \( \gamma \in R^+ \) and \( f(0, t) = 0 \).

As it was mentioned, there are many applications that the physical agents should track a time varying trajectory, where each agent has an incomplete copy of this
trajectory. While, the physical agents and reference trajectory might be described by more complicated dynamics rather than the linear dynamics in real applications. Therefore, we consider a more general group of physical agents, where the nonlinear function \( f(\cdot, t) \) in their dynamics satisfies the Lipschitz-type condition.

Therefore, a distributed average tracking controller algorithm is designed as

\[
u(t) = K_1(p_i(t) - r_i(t)) + K_2 \dot{x}_i(t) + \mu \phi_i h(K_2 \dot{x}_i(t)) + \alpha \theta_i \Phi h(K_2 \dot{x}_N(t))
\]

with a distributed average tracking filter algorithm as proposed as follows:

\[
u_i(t) = s_i(t) + r_i(t), \quad \dot{s}_i(t) = A s_i(t) + BK_1(p_i(t) - r_i(t)) + \alpha \theta_i \Phi h(K_2 \dot{x}_i(t)) + \alpha \theta_i \Phi h(K_2 \dot{x}_N(t))
\]

where \( K_1 \) and \( K_2 \) control gain matrices, respectively, to be determined.

Then, using the controller (3) for (1), one gets the tracking error system

\[
\dot{x}_i(t) = (A + BK_2) \dot{x}_i(t) + B(f(x_i, t) - f(r_i, t)) + \mu \phi_i h(K_2 \dot{x}_i(t)) + \alpha \theta_i \Phi h(K_2 \dot{x}_N(t))
\]

Following from (2) and (4), one gets

\[
\dot{p}_i(t) = (A + BK_1)p_i(t) - BK_1r_i(t) + BF(r_i, t) + \alpha \theta_i \Phi h(K_2 \dot{x}_i(t)) + \alpha \theta_i \Phi h(K_2 \dot{x}_N(t))
\]

Define \( M = I_N - \frac{1}{N} 11^T \). Then \( M \) satisfies following properties: Firstly, it is easy to see that 0 is a simple eigenvalue of \( M \) with 1 as the corresponding right eigenvector and 1 is the other eigenvalue with multiplicity \( N - 1 \), i.e., \( M 1 = 1^T 1 = 0 \). Secondly, since \( L^T = L \), one has \( LM = L(I_N - \frac{1}{N} 11^T) = L - \frac{1}{N} L11^T = L - \frac{1}{N} L11^T \). Finally, \( M^2 = M(I_N - \frac{1}{N} 11^T) = M - \frac{1}{N} M11^T = M \).

Define \( \xi(t) = (M \otimes 1) p(t) \), where \( \xi(t) = (\xi_1^T(t), \xi_2^T(t), \ldots, \xi_N^T(t))^T \). Then, it follows that \( \xi(t) = 0 \) if and only if \( p_1(t) = p_2(t) = \cdots = p_N(t) \). Therefore, the consensus problem of (6) is solved if and only if \( \xi(t) \) asymptotically converges to zero. Hereafter, we refer to \( \xi(t) \) as the consensus error. By noting that \( LM = L + ML \), it is not difficult to obtain from (8) that the consensus error \( \xi(t) \) satisfies

\[
\dot{\xi}(t) = (M \otimes (A + BK_1)) \xi(t) - (M \otimes BK_1) r(t) + \alpha (M \otimes B) H(L \otimes K_1) \xi(t) + (M \otimes B) F(r, t),
\]

**Algorithm 1:** Under Assumptions 1 and 2, for multiple reference signals in (2), the distributed average tracking algorithms (4) and (3) can be constructed as follows

1. Solve the following algebraic Riccati equations (AREs):

\[
P_iA + A^T P_i - P_iBB^T P_i + Q_i = 0,
\]

with \( Q_i \) to obtain matrices \( P_i \), where \( i = 1 \). Then, choose \( K_i = -B^T P_i \). Choose the parameters \( \alpha \geq \gamma + \beta P_i \), \( \beta > 0 \), \( \mu \geq \gamma \) and \( \nu > 0 \).

**Theorem 1** Under Assumptions 1-3, by using the distributed average tracking controller algorithm (3) with the distributed average tracking filter algorithm (4), the
state $x_i(t)$ in (1) will track the average of multiple reference signals $r_i(t)$, $i = 1, 2, \cdots, N$, generated by the Lipschitz-type nonlinear dynamical systems (2) if the parameters $\alpha$, $\beta$, $\mu$, $\nu$ and the feedback gains $K_i, i = 1, 2$, are designed by Algorithm 1.

**Proof:** The proof contains three steps. First, it is proved that for the $i$th agent, \( \lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{k=1}^{N} p_k(t) \right) = 0 \). Consider the Lyapunov function candidate

\[
V_1(t) = \xi^T(L \otimes P_1)\xi.
\]  

(11)

By the definition of $\xi(t)$, it is easy to see that $(1^T \otimes L)\xi = 0$. For the connected graph $\mathcal{G}$, it then follows from Lemma 1 that

\[
V_1(t) \geq \lambda_2 \lambda_{\text{min}}(P_1) \|\xi\|^2,
\]  

(12)

where $\lambda_{\text{min}}(P_1)$ is the smallest eigenvalue of the positive matrix $P_1$. The time derivative of $V_1$ along (9) can be obtained as follows

\[
\dot{V}_1 = \xi^T(L \otimes P_1)\dot{\xi} + \xi^T(L \otimes P_1)\dot{\xi} = \xi^T(M \otimes (A + BK_1))^T(L \otimes P_1)\xi + \xi^T(L \otimes P_1)(M \otimes (A + BK_1))\xi - 2\xi^T(L \otimes P_1)(M \otimes BK_1)r(t) + 2\alpha \xi^T(L \otimes P_1)(M \Theta \otimes B)H(L \otimes K_1)\xi(t) + 2\xi^T(L \otimes P_1)(M \otimes B)F(r(t), t).
\]  

(13)

Substituting $K_1 = -BB^T P_1$ into (13), it follows from the fact $LM = ML = L$ and Assumption 3 that

\[
\dot{V}_1 = \xi^T[L \otimes (A^T P_1 + P_1 A) - 2(L \otimes P_1 BB^T P_1)]\xi + 2\xi^T(L \otimes P_1 BB^T P_1)r(t) - 2\alpha \xi^T(L \otimes P_1 BB^T P_1)r(t) + 2\xi^T(L \otimes P_1)^T(B^T P_1)\xi
\]

\[
= \xi^T[L \otimes (A^T P_1 + P_1 A) - 2(L \otimes P_1 BB^T P_1)]\xi + 2\sum_{i=1}^{N} \left( \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \right)^T B^T P_1 r_i
\]

\[
- 2\alpha \sum_{i=1}^{N} \partial_i \left( \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \right)^T + 2\alpha \sum_{i=1}^{N} \partial_i \left( \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \right)^T [f(r_i(t), t) - f(0, t)]
\]

\[
\leq \xi^T[L \otimes (A^T P_1 + P_1 A) - 2(L \otimes P_1 BB^T P_1)]\xi + 2\alpha \sum_{i=1}^{N} \partial_i \left( \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \right)^T [f(r_i(t), t) - f(0, t)]
\]

Since $\alpha > \gamma + \|B^T P_1\|, \beta > 0$, one has

\[
\dot{V}_1 \leq \xi^T(L \otimes (A^T P_1 + P_1 A) - 2(L \otimes P_1 BB^T P_1))\xi
\]

\[
\leq \lambda_2 \lambda_{\text{min}}(P_1) \|\xi\|^2.
\]  

(15)

It follows from (10) that $P_1 A + A^T P_1 - P_1 BB^T P_1 \leq -Q_1$. Therefore, we have

\[
\dot{V}_1 < -\eta_1 V_1
\]  

(16)

where $\eta_1 = \frac{\lambda_{\text{min}}(Q_1)}{\lambda_{\text{max}}(P_1)}$. Thus, one has

\[
\lim_{t \to \infty} \xi_i(t) = \lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{k=1}^{N} p_k(t) \right) = 0.
\]

Second, it is proved that $\lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{k=1}^{N} r_k(t) \right) = 0$. Let $r^*(t) = \frac{1}{N} \sum_{i=1}^{N} r_i(t)$. It follows from (2) that

\[
\dot{r}^*(t) = A r^*(t) + \frac{1}{N} B \sum_{i=1}^{N} f(r_i(t), t).
\]  

(17)

Let $p^*(t) = \frac{1}{N} \sum_{i=1}^{N} p_i(t)$. It follows from (2) that

\[
\dot{p}^*(t) = (A + BK_1)p^*(t) - BK_1 r^*(t) + \frac{1}{N} B \sum_{i=1}^{N} f(r_i(t), t)
\]

\[
+ \alpha \sum_{i=1}^{N} \partial_i \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right).
\]  

(18)
Denote \( \zeta(t) = p^*(t) - r^*(t) \), one has

\[
\dot{\zeta}(t) = \dot{p}^*(t) - \dot{r}^*(t) = (A + BK_1)p^*(t) - BK_1r^*(t) - Ar^*(t) + \alpha \sum_{i=1}^N \partial_i h \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right)
= \sum_{i=1}^N \partial_i h \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right) = (A + BK_1)\zeta(t) + \omega(t),
\]

where \( \omega(t) = \alpha \sum_{i=1}^N \partial_i h \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right) \).

We then use input-to-state stability to analyze the system (19) by treating the term \( \omega(t) \) as the input and \( \zeta(t) \) as the states. Since (10) with \( K_1 = -B^TP_1 \), one has \( A + BK_1 \) is Hurwitz. Thus, the system (19) with zero input is exponentially stable and hence input-to-state stable. Since \( \lim_{t \to \infty} (p_i(t) - \frac{1}{N} \sum_{k=1}^N p_k(t)) = 0 \). One has \( \lim_{t \to \infty} \omega(t) = 0 \). Thus, it follows that \( \lim_{t \to \infty} \zeta(t) = 0 \), which implies that \( \lim_{t \to \infty} \left( \frac{1}{N} \sum_{i=1}^N p_i(t) - \frac{1}{N} \sum_{i=1}^N r_i(t) \right) = 0 \). Therefore, one obtains \( \lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{i=1}^N p_i(t) \right) = 0 \).

\[
\left( p_i(t) - \frac{1}{N} \sum_{i=1}^N p_i(t) \right) + \lim_{t \to \infty} \left( \frac{1}{N} \sum_{i=1}^N p_i(t) - \frac{1}{N} \sum_{i=1}^N r_i(t) \right) = 0.
\]

Third, it is proofed that \( \lim_{t \to \infty} \left( x_i(t) - \frac{1}{N} \sum_{i=1}^N r_i(t) \right) = 0 \). Consider the candidate Lyapunov function

\[
V_2 = \ddot{x}^T (I \otimes P_2) \ddot{x},
\]

with \( P_2 > 0 \). By taking the derivative of \( V_2 \) along (7), one gets

\[
\dot{V}_2 = \ddot{x}^T (I \otimes ((A + BK_2)^TP_2 + P_2(A + BK_2))) \ddot{x} + 2\ddot{x}^T (I \otimes P_2B) (F(x, t) - F(r, t)) + 2\mu (\Phi \otimes P_2B) H((I \otimes K_2) \ddot{x}(t)) = 0.
\]

Using \( K_2 = -B^TP_2 \), one has

\[
\dot{V}_2 = \ddot{x}^T (I \otimes ((A^TP_2 + P_2A - 2P_2BB^TP_2))) \ddot{x} + 2\ddot{x}^T (I \otimes P_2B) (F(x, t) - F(r, t)) - 2\mu (\Phi \otimes P_2B) H((I \otimes B^TP_2) \ddot{x}(t)) = 0.
\]

Since \( \mu \geq \gamma \) and \( \nu > 0 \), one has

\[
\dot{V}_2 \leq \ddot{x}^T (I \otimes ((A^TP_2 + P_2A - 2P_2BB^TP_2))) \ddot{x} + 2\ddot{x}^T (I \otimes P_2B) (F(x, t) - F(r, t)) + 2\mu (\Phi \otimes P_2B) H((I \otimes B^TP_2) \ddot{x}(t)) - 2\mu \sum_{i=1}^N \phi_i(B^TP_2 \ddot{x}(t)) h(B^TP_2 \ddot{x}_i).
\]

Thus, the system (22) can be solved. This completes the proof.

### 3.2 Adaptive distributed average tracking algorithms design

Note that, in above subsection, the proposed distributed average tracking algorithms (3) and (4) require that the parameters \( \alpha \) and \( \mu \) satisfy the conditions \( \alpha \geq \gamma + \|B^TP_1\| \) and \( \mu \geq \gamma \), which depend the Lipschitz constant \( \gamma \). Since the \( \gamma \) is a global information, for a local agent, it becomes difficult to obtain \( \gamma \). Therefore, to overcome the global information restriction, we design an adaptive distributed average tracking controller algorithm

\[
u_i(t) = K_1(p_i(t) - r_i(t)) + K_2 \bar{x}_i(t) + \mu_i(t) \phi_i h(K_2 \bar{x}_i(t)) + \alpha_i(t) \partial_i h \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right)
\]

and an adaptive distributed average tracking filter algorithm

\[
p_i(t) = s_i(t) + r_i(t),
\]

\[
p_i(t) = s_i(t) + r_i(t),
\]
\[
\dot{s}_i(t) = A s_i(t) + BK_1(p_i(t) - r_i(t)) \\
+ \alpha_i(t) \partial_i \mathbf{B} \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right),
\]

with two time-varying parameters \( \mu_i(t) \) and \( \alpha_i(t) \) satisfying the following adaptive update strategies:

\[
\dot{\mu}_i(t) = \kappa_i \dot{\phi}_i \|K_2 \tilde{x}_i(t)\|,
\]

and

\[
\dot{\alpha}_i(t) = \chi_i \partial_i \left( \sum_{j \in N_i} K_1(\xi_i(t) - \xi_j(t)) \right),
\]

respectively, where \( \kappa_i, \chi_i \) are adaptive parameters to be determined.

By substituting adaptive controller (25) into (1), one obtains

\[
\dot{x}_i(t) = (A + BK_2) \tilde{x}_i(t) + B(f(x_i, t) - f(r_i, t)) \\
+ \mu_i(t) \Phi_i \mathbf{B}h[K_2 \tilde{x}_i(t)],
\]

where \( \mu_i(t) \) is given by (27). According to (2) and (26), one has

\[
\dot{p}_i(t) = (A + BK_1)p_i(t) - BK_1 r_i(t) + Bf(r_i, t) \\
+ \alpha_i(t) \partial_i \mathbf{B} \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right),
\]

where \( \alpha_i(t) \) is given by (28).

Then, the closed-loop systems in matrix form are obtained:

\[
\dot{x}(t) = (I \otimes (A + BK_2)) \tilde{x}(t) + (I \otimes B)(f(x, t) - f(r, t)) \\
+ (\mu(t) \Phi \otimes \mathbf{B})h[(I \otimes K_1) \tilde{x}(t)],
\]

with

\[
\dot{\xi}(t) = (I \otimes (A + BK_1))\xi(t) - (M \otimes BK_1) r(t) \\
+ (M \otimes B)f(r, t) + (M \alpha(t) \Theta \otimes \mathbf{B})h((L \otimes K_1) \xi(t))
\]

where \( \mu(t) = \text{diag}(\mu_1(t), \mu_2(t), \ldots, \mu_N(t)) \), and \( \alpha(t) = \text{diag}(\alpha_1(t), \alpha_2(t), \ldots, \alpha_N(t)) \), respectively.

**Assumption 4** It is assumed that \( r_i \) is bounded.

**Algorithm 2**: Under Assumptions 1-4, for multiple reference signals in (2), the adaptive distributed average tracking algorithms (25)-(28) is designed by the following two steps:

1. Solve the AREs (10) to obtain \( K_i, i = 1, 2 \).
2. Choose the parameters \( \kappa > 0, \chi > 0, \beta > 0, \) and \( \nu > 0 \).

**Theorem 2** Under Assumptions 1-4, the adaptive distributed average tracking algorithms (25)-(28) solve the distributed average tracking problem of the multi-agent system (1) with the reference dynamical system (2) if the parameters are given by Algorithm 2.

**Proof**: First, consider the following Lyapunov candidate,

\[
\dot{V}_3 = \xi^T (L \otimes P_1) \xi + \sum_{i=1}^N \tilde{\alpha}_i(t)^2 / \chi_i,
\]

where \( \tilde{\alpha}_i(t) = \alpha_i(t) - \alpha \). As proved in Theorem 1, the derivation of (33) along (32) and (28) is given by

\[
\dot{V}_3 \leq \xi^T (L \otimes (A^T P_1 + P_1 A) - 2L \otimes P_1 BB^T P_1) \xi \\
- 2 \sum_{i=1}^N ((\alpha_i(t) - \gamma - \|B^T P_1\|) \|r_i\| + \alpha_i(t)\beta) \\
\bigg\| \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \bigg\| \\
+ 2 \sum_{i=1}^N \tilde{\alpha}_i(t) \partial_i \left( \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \right) \\
= \xi^T (L \otimes (A^T P_1 + P_1 A) - 2L \otimes P_1 BB^T P_1) \xi \\
- 2 \sum_{i=1}^N ((\alpha - \gamma - \|B^T P_1\|) \|r_i\| + \alpha \beta) \\
\bigg\| \sum_{j \in N_i} [B^T P_1(\xi_i(t) - \xi_j(t))] \bigg\|.
\]

Adaptively updating \( \alpha > \gamma + \|B^T P_1\| > 0 \), and choosing \( \beta > 0 \), one has

\[
\dot{V}_3 \leq -\xi^T (L \otimes Q_1) \xi \triangleq -U(t) \leq 0,
\]

which implies that \( V_3(t) \) is non-increasing. Then, according to (33), it follows that \( \xi, \alpha_i(t) \) are bounded. It is following from Assumption 4 that \( r \) is bounded. One has \( \|F(r, t)\| = \|F_r(t) - F(0, t)\| \leq \gamma \|r\| \) which implies that \( F(r, t) \) is bounded. Therefore, \( \xi \) is bounded, which implies that \( \lim_{t \to \infty} V_3(t) \) exists and is finite. Since (35), one has one has \( \int_0^t U(t) dt \) exists and is finite. By noting that \( \dot{U}(t) \) is also bounded. Therefore, \( U(t) \) is uniform continuity. By utilizing Barbalat’s Lemma, it guarantees \( \lim_{t \to \infty} u(t) = 0 \). Thus, one has \( \lim_{t \to \infty} \xi(t) = 0 \). Noting that \( \chi > 0, \beta > 0 \), one has \( \alpha_i(t) \) is monotonically increasing and bounded. Thus, \( \alpha_i(t) \) converges to some finite constants. Thus, it follows that \( \lim_{t \to \infty} \xi_i(t) = \lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{k=1}^N p_k(t) \right) = 0 \).
Second, similar to the proof in Theorem 1, one has
\[
\dot{\zeta}(t) = (A + BK_1)\zeta(t) + \varpi(t),
\]  
(36)
where \(\varpi(t) = \sum_{i=1}^{N} \alpha_i(t)\dot{\theta}_i(h_i)\sum_{j \in N_i} K_1 (p_i(t) - p_j(t))\). Note that \(\alpha_i(t)\) converges to some finite constants. By leveraging input-to-state stability to analyze the system (36), one has \(\lim_{t \to \infty} \zeta(t) = 0\). Then, one has \(\lim_{t \to \infty} \left( p_i(t) - \frac{1}{N} \sum_{k=1}^{N} r_k(t) \right) = 0 \). Third, consider the following Lyapunov candidate
\[
V_4 = \tilde{x}^T (I \otimes P_2) \tilde{x} + \sum_{i=1}^{N} \tilde{\mu}_i(t)^2\frac{\kappa_i}{\kappa_i'},
\]  
(37)
where \(\tilde{\mu}_i(t) = \mu_i(t) - \mu\). As the proof given by Theorem 1, one has the derivation of (37) along (31) and (27),
\[
\dot{V}_4 \leq \tilde{x}^T (I \otimes (A P_2 + P_2 A - 2P_2BB^T P_2)) \tilde{x}
- 2\sum_{i=1}^{N} (\mu_i(t) - \gamma)\|x_i - r_i\| + \mu_i(t)\nu\|BPT_2 \tilde{x}_i(t)\| \\
+ 2\sum_{i=1}^{N} \tilde{\mu}_i(t)\tilde{\phi}_i(BP_2 T_2 \tilde{x}_i(t)) \\
\leq \tilde{x}^T (I \otimes (A P_2 + P_2 A - 2P_2BB^T P_2)) \tilde{x}
- 2\sum_{i=1}^{N} (\mu - \gamma)\|x_i - r_i\| + \mu \nu\|BPT_2 \tilde{x}_i(t)\|.  
\]  
(38)
Adaptively updating \(\mu \geq \gamma\) and choosing \(\nu > 0\), one has
\[
\dot{V}_4 \leq -\tilde{x}^T (I \otimes Q_2) \tilde{x} \triangleq -W(t) \leq 0,
\]  
(39)
which implies that \(V_4(t)\) is non-increasing. Then, according to (37), it follows that \(\tilde{x}, \mu_i(t)\) are bounded. It is following from Assumption 4 and (30) that \(r\) and \(p\) are bounded. One has \(\|F(x, t) - F(r, t)\| \leq \gamma\|x - r\| \leq \gamma(\|\tilde{x}\| + \|p\| + \|r\|)\), which implies that \(F(x, t) \to F(r, t)\) is bounded. Therefore, from (31), one has \(\tilde{x}\) is bounded, which implies that \(\lim_{t \to \infty} V_4(t)\) exists and is finite. Thus, \(\int_{0}^{\infty} W(t) dt\) exists and is finite. By noting that \(W(t)\) is also bounded, \(W(t)\) is uniform continuity. By utilizing Barbalat’s Lemma, it guarantees \(\lim_{t \to \infty} W(t) = 0\). Thus, one has \(\lim_{t \to \infty} \tilde{x}(t) = 0\). Noting that \(\kappa_i > 0, \nu > 0\), one has \(\mu_i(t)\) is monotonically increasing and bounded. Thus, \(\mu_i(t)\) converges to some finite constants. It follows that \(\lim_{t \to \infty} \bar{x}_i(t) = 0\), which implies \(\lim_{t \to \infty} \bar{x}_i(t) = 0\). The proof is completed.

**Remark 1** Differing from the robust distributed average tracking algorithms (3) and (4) in above subsection, the adaptive algorithms (25)-(28) are local fashion without knowing the global information \(\gamma\).

### 3.3 Continuous distributed average tracking algorithms design

In the above subsections, the distributed average tracking algorithms are designed based on the discontinuous function \(h(x)\), which may generate chattering phenomenon. In order to reduce the chattering in real applications and make the controller easier to implement, based on the boundary layer concept, we replace the discontinuous function \(h(x)\) by a continuous approximation \(h_{\varepsilon}(x)\), and propose a continuous distributed average tracking controller algorithm:
\[
u_i(t) = K_1 (p_i(t) - r_i(t)) + K_2 \tilde{x}_i(t) + \mu \phi_i h_{\varepsilon} [K_2 \tilde{x}_i(t)] \\
+ \alpha \dot{\theta}_i \varepsilon (\sum_{j \in N_i} K_1 (p_i(t) - p_j(t))),
\]  
(40)
and an continuous distributed average tracking filter algorithm
\[
u_i(t) = s_i(t) + r_i(t), \quad s_i(t) = A s_i(t) + B K_1 (p_i(t) - r_i(t)) \\
+ \alpha \dot{\theta}_i \varepsilon (\sum_{j \in N_i} K_1 (p_i(t) - p_j(t))).
\]  
(41)

Submitting (40) into (1), one obtains the closed-loop systems in matrix form like:
\[
\dot{\tilde{x}}(t) = (I \otimes (A + BK_2)) \tilde{x}(t) + (I \otimes B)(F(x, t) - F(r, t)) \\
+ (\mu \Phi \otimes B) h_{\varepsilon}((I \otimes K_2) \tilde{x}(t)).
\]  
(42)

It follows from (2) and (41) that
\[
\dot{\xi}(t) = (I \otimes (A + BK_1)) \xi(t) - (M \otimes BK_1) r(t) \\
+ (M \otimes B) F(r, t) + (\alpha \Theta \otimes B) H_{\varepsilon}((L \otimes K_1) \xi(t)).
\]  
(43)

**Theorem 3** Under Assumptions 1-4, the adaptive DAT algorithms (40) and (41) solve the DAT problem of the multi-agent system (1) with the reference dynamical system (2) if the parameters are given by Algorithm 1.

**Proof:** First, consider the Lyapunov candidate (33). As proved in Theorem 1, the derivation of (33) along (43) is given by
\[
\dot{V}_1 \leq \xi^T [L \otimes (A P_1 + P_1 A) - 2L \otimes P_1 B^T P_1] \xi \\
+ 2\sum_{i=1}^{N} (\gamma + \|BPT_1\|\|r_i\|) \left\| \sum_{j \in N_i} [BPT_1 (\xi_i(t) - \xi_j(t))] \right\| \\
- 2\sum_{i=1}^{N} \alpha \dot{\theta}_i \left( \sum_{j \in N_i} K_1 (\xi_i(t) - \xi_j(t)) \right)
\]  
\[ 
\]
\[
    h_\varepsilon \left( \sum_{j \in N_i} K_1(\xi_j(t) - \xi_j(t)) \right).
\]

Since \( \alpha > \gamma + \|B^TP_1\| \) and \( \beta > 0 \), one has
\[
    \dot{V}_1(t) \leq t^T[L \otimes (A^TP_1 + P_1A) - 2(L \otimes P_1BB^TP_1)]\xi
    + 2\sum_{i=1}^{N} \alpha \partial_i \left[ \left| \left| \sum_{j \in N_i} [B^TP_1(\xi_j(t) - \xi_j(t))] \right| \right| \right]
    - \left( \sum_{j \in N_i} K_1(\xi_j(t) - \xi_j(t)) \right)
    h_\varepsilon \left( \sum_{j \in N_i} K_1(\xi_j(t) - \xi_j(t)) \right)
    \leq -\eta \dot{V}_1 + 2\sum_{i=1}^{N} \alpha \partial_i \varepsilon e^{-ct}. \tag{45}
\]

In light of the well-known Comparison Lemma, one gets that
\[
    V_1(t) \leq e^{-\eta(t)} V_1(0) + 2\sum_{i=1}^{N} \alpha \partial_i \int_0^t \varepsilon e^{-\eta(t-\tau) - ct} d\tau, \tag{46}
\]
where \( \overline{t}_i \) is the upper bound of \( \partial_i \). According to \( \lim_{t \to \infty} \int_0^t \varepsilon e^{-\eta(t-\tau) - ct} d\tau = 0 \), one has \( V_1(t) \) exponentially converges to the origin as \( t \to \infty \). Therefore, \( \lim_{t \to \infty} \|p_i - \sum_{k=1}^{N} p_k\| = 0 \). Second, similar to Theorem 1, one has
\[
    \zeta(t) = (A + BK_1)\zeta(t) + \varpi(t, \varepsilon), \tag{47}
\]
where \( \varpi(t, \varepsilon) = \sum_{i=1}^{N} \alpha \partial_i h_\varepsilon \left( \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) \right) \). Since \( \lim_{t \to \infty} \sum_{j \in N_i} K_1(p_i(t) - p_j(t)) = 0 \). One has \( \lim_{t \to \infty} \varpi(t, \varepsilon) = 0 \). It follows that \( \lim_{t \to \infty} \zeta(t) = 0 \). Thus, \( \lim_{t \to \infty} \|p_i - \sum_{k=1}^{N} r_k\| = 0 \). Third, consider derivative of \( V_2 \) along (42), one gets
\[
    \dot{V}_2 \leq \tilde{\alpha}_T (I \otimes (A^TP_2 + P_2A - 2P_2BB^TP_1)) \tilde{x}
    + 2\sum_{i=1}^{N} \|B^TP_2\tilde{x}_i(t)\| \gamma \|x_i - r_i\|
    - 2\mu \sum_{i=1}^{N} \phi_i(B^TP_2\tilde{x}_i(t))^T h_\varepsilon(B^TP_2\tilde{x}_i)
    \leq -\eta_2 \dot{V}_2 + 2\sum_{i=1}^{N} \mu \phi_i \varepsilon e^{-ct}. \tag{48}
\]
Thus, \( \lim_{t \to \infty} V_2(t) = 0 \), which implies \( \lim_{t \to \infty} \|x_i(t) - p_i(t)\| = 0 \). Thus, \( \lim_{t \to \infty} \|x_i - \sum_{k=1}^{N} r_k\| = 0 \). This completes the proof.

4 Conclusions

In this paper, we have studied the distributed average tracking problem of multiple time-varying signals generated by nonlinear dynamical systems. In the distributed fashion, a pair of discontinuous algorithms with static and adaptive coupling strengths have been developed. Then, in light of the boundary layer concept, a continuous algorithm is designed. Besides, sufficient conditions for the existence of distributed algorithms are given. The future topic will be focused on the distributed average tracking problem for the case with only the relative output information of neighboring agents.

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