ONE-COMPONENT INNER FUNCTIONS

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Abstract. We explicitly unveil several classes of inner functions \( u \) in \( H^\infty \) with the property that there is \( \eta \in ]0, 1[ \) such that the level set \( \Omega_u(\eta) := \{ z \in \mathbb{D} : |u(z)| < \eta \} \) is connected. These so-called one-component inner functions play an important role in operator theory.

Dedicated to the memory of Vadim Tolokonnikov

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Introduction

Definition 0.1. An inner function \( u \) in \( H^\infty \) is said to be a one-component inner function if there is \( \eta \in ]0, 1[ \) such that the level set (also called sublevel set or filled level set) \( \Omega_u(\eta) := \{ z \in \mathbb{D} : |u(z)| < \eta \} \) is connected.

One-component inner functions, the collection of which we denote by \( \mathcal{I}_c \), were first studied by B. Cohn [10] in connection with embedding theorems and Carleson-measures. It was shown in [10, p. 355] for instance that arclength on \( \{ z \in \mathbb{D} : |u(z)| = \varepsilon \} \) is such a measure whenever

\[ \Omega_u(\eta) = \{ z \in \mathbb{D} : |u(z)| < \eta \} \]

is connected and \( \eta < \varepsilon < 1 \).

A thorough study of the class \( \mathcal{I}_c \) was given by A.B. Aleksandrov [1] who showed the interesting result that \( u \in \mathcal{I}_c \) if and only if there is a constant \( C = C(u) \) such that for all \( a \in \mathbb{D} \)

\[ \sup_{z \in \mathbb{D}} \left| \frac{1 - \overline{u(a)}u(z)}{1 - \overline{a}z} \right| \leq C \left( \frac{1 - |u(a)|^2}{1 - |a|^2} \right). \]

Many operator-theoretic applications are given in [1, 2, 7, 3]. In our paper here we are interested in explicit examples, which are somewhat lacking in literature. For example, if \( S \) is the atomic inner function, which is given by

\[ S(z) = \exp \left( -\frac{1 + z}{1 - z} \right), \]
then all level sets \( \Omega_S(\eta), \, 0 < \eta < 1 \) are connected, because these sets coincide with the disks

\[
D_\eta := \left\{ z \in \mathbb{D} : \left| z - \frac{L}{L + 1} \right| < \frac{1}{L + 1} \right\}, \quad L := \log \frac{1}{\eta},
\]

which are tangential to the unit circle at \( p = 1 \).

The scheme of our note here is as follows: in section 1 we prove a general result on level sets which will be the key for our approach to the problem of unveiling classes of one-component inner functions. Then in section 2 we first present with elementary geometric/function theoretic methods several examples and then we use Aleksandrov’s criterion to achieve this goal. For instance, we prove that \( BS, B \circ S \) and \( S \circ B \) are in \( \mathcal{I}_c \) whenever \( B \) is a finite Blaschke product. Considered are also interpolating Blaschke products. It will further be shown that, under the supremum norm, \( \mathcal{I}_c \) is an open subset of the set of all inner functions and multiplicatively closed. In the final section we give counterexamples.

1. Level sets

We first begin with a topological property of the class of general level sets. Although statement (1) is “well-known” (the earliest appearance seems to be in [26, Theorem VIII, 31]), we could nowhere locate a proof. The argument that the result is a simple and direct consequence of the maximum principle is, in our viewpoint, not tenable.

**Lemma 1.1.** Given a non-constant inner function \( u \) in \( H^\infty \) and \( \eta \in \]0, 1[\], let \( \Omega := \Omega_\eta(\eta) = \{ z \in \mathbb{D} : |u(z)| < \eta \} \) be a level set. Suppose that \( \Omega_0 \) is a component (=maximal connected subset) of \( \Omega \). Then

1. \( \Omega_0 \) is a simply connected domain; that is, \( \mathbb{C} \setminus \Omega_0 \) has no bounded components

2. \( \inf_{\Omega_0} |u| = 0 \).

**Proof.** We show that (1) holds for every holomorphic function \( f \) in \( \mathbb{D} \); that is if \( \Omega_0 \) is a component of the level set \( \Omega_f(\eta), \, \eta > 0 \), then it is a simply connected domain. Note that each component \( \Omega_0 \) of the open set \( \Omega_f(\eta) \) is an open subset of \( \mathbb{D} \). We may assume that \( \eta \) is chosen so that \( \{ z \in \mathbb{D} : |f(z)| = \eta \} \neq \emptyset \).

Suppose, to the contrary, that \( D \) is a bounded component of \( \mathbb{C} \setminus \Omega_0 \). Note that \( D \) is closed in \( \mathbb{C} \). Then, necessarily, \( D \) is contained in \( \mathbb{D} \), because the unique unbounded complementary component of \( \Omega_0 \) contains \( \{ z \in \mathbb{C} : |z| \geq 1 \} \). Hence \( D \) is a compact subset of \( \mathbb{D} \). Let \( G := \Omega_0^* \) be the simply-connected hull of \( \Omega_0 \); that

\footnote{A shorter proof can be given by using the advanced definition that a domain \( G \) in \( \mathbb{C} \) is simply connected if every curve in \( G \) is contractible in \( G \), or equivalently, if every Jordan curve \( J \) in \( G \) the interior of \( J \) belongs to \( G \). That depends though on the Jordan curve theorem.}

\footnote{This proof, as well as two different ones, including the one mentioned in footnote 1, stem from the forthcoming book manuscript [22] of the second author together with R. Rupp.
is the union of $\Omega_0$ with all bounded complementary components of $\Omega_0$. Note that $G$ is open because it coincides with the complement of the unique unbounded complementary component of $\Omega_0$. Then, by definition of the simply connected hull, $D \subseteq G$. Now if $H$ is any bounded complementary component of $\Omega_0$ then (as it was the case for $D$) $H$ is a compact subset of $\mathbb{D}$ and so $\partial H \subseteq \mathbb{D}$. Moreover,

\[(1.1) \quad \partial H \subseteq \partial \Omega_0.\]

In fact, given $z_0 \in \partial H$, let $U$ be a disk centered at $z_0$. Then $U \cap \Omega_0 \neq \emptyset$, since otherwise $U \cup H$ would be a connected set strictly bigger than $H$ and contained in the complement of $\Omega_0$; a contradiction to the maximality of $H$. Since $z_0 \in \partial H \subseteq H \subseteq \mathbb{C} \setminus \Omega_0$, we conclude that $z_0 \in \partial \Omega_0$.

Now $\partial H \subseteq \partial \Omega_0$ and $\Omega_0 \subseteq \Omega_f(\eta)$ imply that $|f| \leq \eta$ on $\partial H$, and so, by the maximum principle, $|f| \leq \eta$ on $H$. Consequently, $|f| \leq \eta$ on $G$. By the local maximum principle, $|f| < \eta$ on $G$. Since $\partial D \subseteq D \subseteq G$,

\[(1.2) \quad |f| < \eta \text{ on } \partial D.\]

On the other hand,

\[(1.3) \quad \partial D \subseteq \partial \Omega_0 \cap \mathbb{D} \subseteq \{z \in \mathbb{D} : |f(z)| = \eta\}.\]

Note that the second inclusion follows from the fact that if $|f(z_0)| < \eta$ for $z_0 \in \partial \Omega_0 \cap \mathbb{D}$, then $\Omega_0$ would no longer be a maximal connected subset of $\Omega_f(\eta)$. Hence $|f| = \eta$ on $\partial D$. This is a contradiction to (1.2). Thus $\Omega_0$ is a simply connected domain.

(2) If $\overline{\Omega}_0 \subseteq \mathbb{D}$, then, due to $\partial \Omega_0 \subseteq \{z \in \mathbb{D} : |u(z)| = \eta\}$, we obtain from the minimum principle that $u$ must have a zero in $\Omega_0$. Now let $E := \overline{\Omega}_0 \cap \partial \mathbb{D} \neq \emptyset$. In view of achieving a contradiction, suppose that $u$ is bounded away from zero in $\Omega_0$. Then $1/|u|$ is subharmonic and bounded in $\Omega_0$ and

\[
\limsup_{\xi \to z; x \in \partial \Omega_0 \setminus E} |u(\xi)|^{-1} = \eta^{-1}.
\]

Since $u$ is an inner function, $E$ has linear measure zero (by [5, Theorem 4.2]).

The maximum principle for subharmonic functions with few exceptional points (here on the set $E$; see [6] or [12]), now implies that $|u|^{-1} \leq \eta^{-1}$ on $\Omega_0$. But $|u| < \eta$ on $\Omega$ is a contradiction. We conclude that $\inf_{\Omega_0} |u| = 0$.

\[\square\]

**Lemma 1.2.** [10] Let $u$ be an inner function. Then the connectedness of $\Omega_u(\eta)$ implies the one of $\Omega_u(\eta')$ for every $\eta' > \eta$.

**Proof.** Because $\Omega_u(\eta)$ is connected and $\Omega_u(\eta) \subseteq \Omega_u(\eta')$, $\Omega_u(\eta)$ is contained in a unique component $U_1(\eta')$ of $\Omega_u(\eta')$. Suppose that $U_0(\eta')$ is a second component of $\Omega_u(\eta')$. Then $|u| \geq \eta$ on $U_0(\eta')$, because $U_0(\eta')$ is disjoint with $U_1(\eta')$ and
hence with \( \Omega_u(\eta) \). By Lemma 1.1 though, \( \inf_{\Omega_u(\eta')} |u| = 0 \); a contradiction. Thus \( \Omega_u(\eta') \) is connected. \( \square \)

2. Explicit examples of one-component inner functions

Let

\[
\rho(z, w) = \frac{|z - w|}{|1 - \bar{z}w|}
\]

be the pseudohyperbolic distance of \( z \) to \( w \) in \( \mathbb{D} \) and

\[
D_\rho(z_0, r) = \{ z \in \mathbb{D} : \rho(z, z_0) < r \}
\]

the associated disks, \( 0 < r < 1 \). Here is a first class of examples of functions in \( \mathcal{I}_c \). Although the next Proposition must be known (in view of A.B. Aleksandrov’s criterion [1]), see 2.12 below), we include a simple geometric proof for the reader’s convenience.

**Proposition 2.1.** Let \( B \) be a finite Blaschke product. Then \( B \in \mathcal{I}_c \).

**Proof.** Denote by \( z_1, \ldots, z_N \) the zeros of \( B \), multiplicities included. Let \( \eta \in ]0, 1[ \) be chosen so close to 1 that \( G := \bigcup_{n=1}^N D_\rho(z_n, \eta) \) is connected (for example by choosing \( \eta \) so that \( z_j \in D_\rho(z_1, \eta) \) for all \( j \)). Now

\[
G \subseteq \{ z \in \mathbb{D} : |B(z)| < \eta \} = \Omega_B(\eta),
\]

because \( z \in G \) implies that for some \( n \),

\[
|B(z)| = \rho(B(z), B(z_n)) \leq \rho(z, z_n) < \sigma.
\]

Since \( G \) is connected, there is a unique component \( \Omega_1 \) of \( \Omega \) containing \( G \). In particular, \( Z(B) \subseteq G \subseteq \Omega_1 \). If, in view of achieving a contradiction, we suppose that \( \Omega := \Omega_B(\eta) \) is not connected, there is a component \( \Omega_0 \) of \( \Omega \) which is disjoint with \( \Omega_1 \), and so with \( G \). In particular,

\[
(2.1) \quad \rho(z, Z(B)) \geq \sigma \quad \text{for every} \quad z \in \Omega_0.
\]

Since \( \overline{\Omega_0} \subseteq \overline{\Omega_B(\eta)} \subseteq \mathbb{D} \), and \( |B| = \eta \) on \( \partial \mathbb{D} \), we deduce from the minimum principle that \( \Omega_0 \) contains a zero of \( B \); a contradiction. \( \square \)

We now generalize this result to a class of interpolating Blaschke products. Recall that a Blaschke product \( b \) with zero set/sequence \( \{ z_n : n \in \mathbb{N} \} \) is said to be an interpolating Blaschke product if \( \delta(b) := \inf(1 - |z_n|^2)|b'(z_n)| > 0 \). If \( b \) is an interpolating Blaschke product then, for small \( \varepsilon \), the pseudohyperbolic disks

\[
D_\rho(z_n, r) = \{ z \in \mathbb{D} : \rho(z, z_n) < \varepsilon \}
\]

are pairwise disjoint. Moreover, by Hoffman’s Lemma (see below and also [19]), for any \( \eta \in ]0, 1[ \), \( b \) is bounded away from zero on \( \{ z \in \mathbb{D} : \rho(z, Z(b)) \geq \eta \} \).
Theorem 2.2 (Hoffman’s Lemma). ([18] p. 86, 106 and [13] p. 404, 310). Let \( \delta, \eta \) and \( \epsilon \) be real numbers, called Hoffman constants, satisfying \( 0 < \delta < 1 \), \( 0 < \eta < \frac{1 - \sqrt{1 - \delta^2}}{\delta} \), (that is, \( 0 < \eta < \rho(\delta, \eta) \)) and

\[
0 < \epsilon < \eta \frac{\delta - \eta}{1 - \delta \eta}.
\]

If \( B \) is any interpolating Blaschke product with zeros \( \{ z_n : n \in \mathbb{N} \} \) such that

\[
\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)|B'(z_n)| \geq \delta,
\]

then

1) the pseudohyperbolic disks \( D_\rho(a, \eta) \) for \( a \in Z(B) \) are pairwise disjoint.

2) The following inclusions hold:

\[
\{ z \in \mathbb{D} : |B(z)| < \epsilon \} \subseteq \{ z \in \mathbb{D} : \rho(z, Z(B)) < \eta \} \subseteq \{ z \in \mathbb{D} : |B(z)| < \eta \}.
\]

A large class of interpolating Blaschke products which are one-component inner functions now is given in the following result.

Theorem 2.3. Let \( b \) be an interpolating Blaschke product with zero set \( \{ z_n : n \in \mathbb{N} \} \). Suppose that for some \( \sigma \in (0, 1) \) the set

\[
G := \bigcup_n D_\rho(z_n, \sigma)
\]

is connected. Then \( b \) is a one-component inner function. This holds in particular, if \( \rho(z_n, z_{n+1}) < \sigma < 1 \) for all \( n \); for example if \( z_n = 1 - 2^{-n} \).

Proof. As in the proof of Proposition 2.1

\[
G \subseteq \{ z \in \mathbb{D} : |b(z)| < \sigma \} =: \Omega.
\]

Since \( G \) is assumed to be connected, there is a unique component \( \Omega_1 \) of \( \Omega \) containing \( G \). In particular, \( Z(b) \subseteq G \subseteq \Omega_1 \). Now, if we suppose that \( \Omega \) is not connected, then there is a component \( \Omega_0 \) of \( \Omega \) which is disjoint with \( \Omega_1 \), and so with \( G \). In particular,

\[
(2.2) \quad \rho(z, Z(b)) \geq \sigma \quad \text{for every} \quad z \in \Omega_0.
\]

Let \( \delta := \delta(b) \),

\[
0 < \eta < \min\{(1 - \sqrt{1 - \delta^2})/\delta, \sigma\},
\]

\[
0 < \epsilon < \eta \frac{\delta - \eta}{1 - \delta \eta}.
\]

By Lemma 1.1, \( \inf_{\Omega_0} |b| = 0 \). Thus, there is \( z_0 \in \Omega_0 \) be so that \( |b(z_0)| < \epsilon \). We deduce from Hoffman’s Lemma 2.2 that \( \rho(z_0, Z(b)) < \eta < \sigma \). This is a contradiction to (2.2). We conclude that \( \Omega \) must be connected. It is clear that
the condition $\rho(z_n, z_{n+1}) < \sigma$ for every $n$ implies that $\bigcup_n D_\rho(z_n, \sigma)$ is connected. For the rest, just note that

$$\rho(1 - 2^{-n}, 1 - 2^{-n-1}) = \frac{2 - 2^{-n} - 2^{-n-1}}{2^{-n} + 2^{-n-1} + 2^{-n}2^{-n-1}} = \frac{1}{3 + 2^{-n}}.$$ 



\[ \square \]

**Corollary 2.4.** Let $B$ be a Blaschke product with increasing real zeros $x_n$ satisfying

$$0 < \eta_1 \leq \rho(x_n, x_{n+1}) \leq \eta_2 < 1.$$ 

Then $b \in \mathcal{I}_c$.

**Proof.** Just use Theorem 2.3 and the fact that by the Vinogradov-Hayman-Newman theorem, $B$ is interpolating if and only if

$$\sup_n \frac{1 - x_{n+1}}{1 - x_n} \leq s < 1$$

or equivalently

$$\inf_n \rho(x_n, x_{n+1}) \geq r > 0.$$ 

\[ \square \]

Using a result of Kam-Fook Tse [25], telling us that a sequence $(z_n)$ of points contained in a Stolz angle (or cone) \( \{ z \in \mathbb{D} : |1 - z| < C(1 - |z|) \} \) is interpolating if and only if it is separated (meaning that $\inf_{n \neq m} \rho(z_n, z_m) > 0$), we obtain:

**Corollary 2.5.** Let $B$ be a Blaschke product whose zeros $(z_n)$ are contained in a Stolz angle and are separated. Suppose that $\rho(z_n, z_{n+1}) \leq \eta < 1$. Then $B \in \mathcal{I}_c$.

Similarly, using a result by M. Weiss [27, Theorem 3.6] and its refinement in [4, Theorem B], we also obtain the following assertion for sequences that may be tangential at 1 (see also Wortman [28]).

**Corollary 2.6.** Let $B$ be a Blaschke product whose zeros $z_n = r_n e^{i\theta_n}$ satisfy:

- $r_n < r_{n+1}$, $\theta_{n+1} < \theta_n$,
- $r_n \nearrow 1$, $\theta_n \searrow 0$,

(2.3) $0 < \eta_1 \leq \rho(z_n, z_{n+1}) \leq \eta_2 < 1$.

Then $B$ is an interpolating Blaschke product contained in $\mathcal{I}_c$. This holds in particular if the zeros are located on a convex curve in $\mathbb{D}$ with endpoint 1 and satisfying (2.3).

Other classes of this type can be deduced from [14]. Here are two explicit examples of interpolating Blaschke products in $\mathcal{I}_c$ whose zeros are given by iteration of the zero of a hyperbolic, respectively parabolic automorphism of $\mathbb{D}$. These functions appear, for instance, in the context of isometries on the Hardy space $H^p$ (see [8]).
Example 2.7. Let \( \varphi(z) = \frac{z - 1/2}{1 - (1/2)z} \). Then \( \varphi \) is an hyperbolic automorphism with fixed points \( \pm 1 \). If \( \varphi_j := \varphi \circ \cdots \circ \varphi \) \( j \)-times, then \( \varphi_j \in \text{Aut}(\mathbb{D}) \) and vanishes exactly at the point

\[
x_j := \frac{3^j - 1}{3^j + 1} = 1 - \frac{2}{3^j + 1}.
\]

This can readily be seen by using that \( x_{j+1} = \varphi^{-1}(x_j) \) and

\[
\varphi_{j+1}(z) = (\varphi_j \circ \varphi)(z) = \frac{z - \frac{\frac{1}{2} + x_j}{1 + \frac{1}{2} x_j}}{1 - z \frac{\frac{1}{2} + x_j}{1 + \frac{1}{2} x_j}}.
\]

Since

\[
\rho(x_j, x_{j+1}) = \frac{3^{j+1} - 3^j}{3^{j+1} + 3^j} = \frac{1}{2},
\]

we deduce from Corollary 2.4 that the Blaschke product

\[
B(z) := \prod_{j=1}^{\infty} \frac{x_j - z}{1 - x_j z}
\]

associated with these zeros is in \( \mathcal{J}_c \).

Let \( \sigma \in \text{Aut}(\mathbb{D}) \) and \( \tau = \sigma \circ \varphi \circ \sigma^{-1} \). Then \( \tau \) is also a hyperbolic automorphism fixing the points \( \sigma(\pm 1) \), and where \( \xi := \sigma(1) \) is the Denjoy-Wolff point with \( \tau'(\xi) < 1 \). The zeros of the \( n \)-th iterate \( \tau_n \) of \( \tau \) are given by

\[
z_n = \tau^{-1}_n(0) = (\sigma \circ \varphi^{-1}_n \circ \sigma^{-1})(0).
\]

By the grand iteration theorem [23, p.78], since 1 is an attracting fixpoint with \( (\varphi^{-1})'(1) = 1/3 < 1 \), the sequence \( (\varphi^{-1}_n(\sigma^{-1}(0))) \) converges nontangentially to 1. Hence the points \( z_n \) are located in a cone with cusp at \( \xi \). Moreover, if \( n > k \) and \( a = \sigma^{-1}(0) \),

\[
\rho(z_n, z_k) = \rho((\varphi^{-1}_n \circ \sigma^{-1})(0), (\varphi^{-1}_k \circ \sigma^{-1})(0)) = \rho((\varphi^{-1}_n(a), a)
\]

Thus,

\[
\rho(z_n, z_{n+1}) = \rho(\varphi(a), a) \quad \text{for all } n \quad \text{and } \inf_{n \neq k} \rho(z_n, z_k) > 0.
\]

Now \( (z_n) \) is a Blaschke sequence \(^3\) ([23, Ex. 6, p. 85]); in fact, use d’Alembert’s quotient criterion and observe that by the Denjoy-Wolff theorem,

\[
\frac{1 - |z_{n+1}|}{1 - |z_n|} = \frac{1 - |\tau^{-1}(z_n)|}{1 - |z_n|} \to (\tau^{-1})'(\xi) < 1.
\]

\(^3\) This also follows form the inequalities \(1 - |\sigma(\xi_n)|^2 = \frac{(1-|a|^2)(1-|\xi_n|^2)}{|1-|a||^2} \leq \frac{1+|a|}{1-|a|} (1-|\xi_n|^2) \) and \(1 - |\psi_n(a)|^2 \leq \frac{1+|a|}{1-|a|} (1-|w_n|^2)\), whenever \( (w_n) \) is a Blaschke sequence and \( \psi_n(w_n) = \sigma(a) = 0 \).
Hence, by Corollary 2.5, \((z_n)\) is an interpolating sequence (see also [11, p.80]) and the associated Blaschke product \(b = \prod_{n=1}^{\infty} e^{i\theta_n} \tau_n\) belongs to \(\mathcal{I}_c\) (here \(\theta_n\) is chosen so that the \(n\)-th Blaschke factor is positive at the origin).

**Figure 1.** The parabolic automorphism

- Let \(\psi(z) = i \frac{z - \frac{1+i}{2}}{1 - \frac{1-i}{2} z}\). Then \(\psi\) is a parabolic automorphism with attracting fixed point 1. The automorphism \(\psi\) is conjugated to the translation \(w \mapsto w + 2\) on the upper half-plane (see figure 1) via the map \(M(z) = i(1+z)/(1-z)\) and \(\psi_n = M^{-1} \circ T_n \circ M\). The zeros of the \(n\)-th iterate \(\psi_n\) of \(\psi\) are given by

\[z_n = \frac{n}{n-i};\]

just use that \(z_n = (M^{-1} \circ T_n^{-1} \circ M)(0)\). These zeros satisfy \(|z_n - \frac{1}{2}| = \frac{1}{2}\) and of course also the Blaschke condition \(\sum_{n=1}^{\infty} 1 - |z_n|^2 < \infty\). Moreover,

\[\rho(z_n, z_{n+1}) = \frac{1}{\sqrt{2}}.\]

Thus, by Corollary 2.6, the Blaschke product associated with these zeros is interpolating and belongs to \(\mathcal{I}_c\).

**Proposition 2.8.** Let \(B\) be a finite Blaschke product or an interpolating Blaschke product with real zeros clustering at \(p = 1\). Then \(f := BS \in \mathcal{I}_c\).

**Proof.** i) Let \(B\) be a finite Blaschke product. Chose \(\eta \in ]0,1[\) so close to 1 that the disk \(D_\eta\) in (0.1), which coincides with the level set \(\Omega_S(\eta)\), contains all zeros of \(B\). Now \(D_\eta = \Omega_S(\eta) \subseteq \Omega_f(\eta)\). Now \(\Omega_f(\eta)\) must be connected, since otherwise there
would be a component $\Omega_0$ of $\Omega_f(\eta)$ disjoint from the component $\Omega_1$ containing $D_{\eta}$. But $f$ is bounded away from zero outside $D_{\eta}$; hence $f = BS$ is bounded away from zero on $\Omega_0$. This is a contradiction to Lemma 1.1 (2).

ii) If $B$ is an interpolating Blaschke product with zeros $(z_n)$, then, by Hoffman’s Lemma 2.2, $B$ is bounded away from zero outside $R := \bigcup D_\rho(z_n, \varepsilon)$ for every $\varepsilon \in ]0, 1[$. Now, if the zeros of $B$ are real, and bigger than $-\sigma$ for some $\sigma \in ]0, 1[$, this set $R$ is contained in a cone with cusp at 1 and aperture-angle strictly less than $\pi$ (see for instance [21]). Hence $R$ is contained in $D_{\eta}$ for all $\eta$ close to 1. Thus, as above, we can deduce that $\Omega_{BS}(\eta)$ is connected. 

The previous result shows, in particular, that certain non one-component inner functions (for example a thin Blaschke product with positive zeros, see Corollary 3.1), can be multiplied by a one-component inner function into $I_c$. In particular, $uv \in I_c$ does not imply that $u$ and $v$ belong to $I_c$. The reciprocal, though, is true: that is $I_c$ itself is stable under multiplication, as we are going to show below.

**Proposition 2.9.** Let $u, v$ be two inner functions in $I_c$. Then $uv \in I_c$.

**Proof.** Let $\Omega_u(\eta)$ and $\Omega_v(\eta')$ be two connected level sets. Due to monotonicity (Lemma 1.2), the fact that $\bigcup_{\lambda \in [\lambda_0, 1]} \Omega_f(\lambda) = D$, we may assume that $\sigma$ satisfies

$$\max\{\eta, \eta'\} \leq \sigma < 1$$

and is so close to 1 that $0 \in \Omega_u(\sigma) \cap \Omega_v(\sigma) \neq \emptyset$. Hence $U := \Omega_u(\sigma) \cup \Omega_v(\sigma)$ is connected. Now

$$\Omega_u(\sigma) \cup \Omega_v(\sigma) \subseteq \Omega_{uv}(\sigma).$$

If we suppose that $\Omega_{uv}(\sigma)$ is not connected, then there is a component $\Omega_0$ of $\Omega_{uv}(\sigma)$ which is disjoint from $U$. In particular, $u$ and $v$ are bounded away from zero on $\Omega_0$. This contradicts Lemma 1.1 (2). Hence $\Omega_{uv}(\sigma)$ is connected and so $uv \in I_c$. 

**Theorem 2.10.** The set of one-component inner functions is open inside the set of all inner functions (with respect to the uniform norm topology).

**Proof.** Let $u \in I_c$. Then, by Lemma 1.2, $\Omega_u(\eta)$ is connected for all $\eta \in [\eta_0, 1[$. Choose $0 < \varepsilon < \min\{\eta, 1 - \eta\}$ and let $\Theta$ be an inner function with $||u - \Theta|| < \varepsilon$. We claim that $\Theta \in I_c$, too. To this end we note that

$$\Omega_{\Theta}(\eta - \varepsilon) \subseteq \Omega_u(\eta) \subseteq \Omega_{\Theta}(\eta + \varepsilon),$$

where $0 < \eta - \varepsilon < \eta + \varepsilon < 1$. As usual, if we suppose that $\Omega_{\Theta}(\eta + \varepsilon)$ is not connected, then there is a component $\Omega_0$ of $\Omega_{\Theta}(\eta + \varepsilon)$ which is disjoint from the connected set $\Omega_u(\eta)$, hence disjoint with $\Omega_{\Theta}(\eta - \varepsilon)$. In other words, $|\Theta| \geq \eta - \varepsilon > 0$ or $\Omega_0$. This contradicts Lemma 1.1 (2). Hence $\Omega_{\Theta}(\eta + \varepsilon)$ is connected and so $\Theta \in I_c$. 

□
Next we look at right-compositions of $S$ with finite Blaschke products. We first deal with the case where $B(z) = z^2$.

**Example 2.11.** The function $S(z^2)$ is a one-component inner function.

**Proof.** Let $\Omega_S(\eta)$ be the $\eta$-level set of $S$. Then, as we have already seen, this is a disk tangent to the unit circle at the point 1. We may choose $0 < \eta < 1$ so close to 1 that 0 belongs to $\Omega_S(\eta)$. Let $U = \Omega_S(\eta) \setminus [\infty, 0]$. Then $U$ is a simply connected slitted disk on which exists a holomorphic square root $q$ of $z$. The image of $U$ under $q$ is a simply connected domain $V$ in the semi-disk \( \{ z : |z| < 1, \text{Re} \ z > 0 \} \). Let $V^*$ be its reflection along the imaginary axis. Then $E := V^* \cup V$ is mapped by $z^2$ onto the closed disk $\Omega_S(\eta)$. Then $E \setminus \partial E$ coincides with $\Omega_S(z^2)(\eta)$.

\[ \text{Figure 2. The level sets of } S(z^2) \]

Using Aleksandrov’s criterion (see below), we can extend this by replacing $z^2$ with any finite Blaschke product. Recall that the spectrum $\rho(\Theta)$ of an inner function $\Theta$ is the set of all boundary points $\zeta$ for which $\Theta$ does not admit a holomorphic extension; or equivalently, for which $\text{Cl}(\Theta, \zeta) = \mathbb{D}$, where

\[ \text{Cl}(\Theta, \zeta) = \{ w \in \mathbb{C} : \exists (z_n) \in \mathbb{D}^\mathbb{N}, \lim_{n \to \infty} z_n = \zeta \text{ and } \lim \Theta(z_n) = w \} \]

is the cluster set of $\Theta$ at $\zeta$ (see [13, p. 80]).

**Theorem 2.12** (Aleksandrov). [1, Theorem 1.11 and Remark 2, p. 2915] Let $\Theta$ be an inner function. The following assertions are equivalent:

1. $\Theta \in \mathcal{I}_c$.
2. There is a constant $C > 0$ such that for every $\zeta \in \mathbb{T} \setminus \rho(\Theta)$ we have
   
   i) $|\Theta''(\zeta)| \leq C |\Theta'(\zeta)|^2$,

   and

   ii) $\lim_{r \to 1} |\Theta(r\zeta)| < 1$ for all $\zeta \in \rho(\Theta)$. 


Note that, due to this theorem, $\Theta \in \mathcal{I}_c$ necessarily implies that $\rho(\Theta)$ has measure zero.

**Proposition 2.13.** Let $B$ be a finite Blaschke product. Then $S \circ B \in \mathcal{I}_c$.

**Proof.** Let us note first that $\rho(S \circ B) = B^{-1}(\{1\})$. Since the derivative of $B$ on the boundary never vanishes (due to (2.4))

\[
\frac{B'(z)}{B(z)} = \sum_{n=1}^{N} \frac{1 - |a_n|^2}{|a_n - z|^2}, \quad |z| = 1, \quad B(a_n) = 0,
\]

$B$ is schlicht in a neighborhood of 1. The angle conservation law now implies that for $\zeta \in B^{-1}(1)$ the curve $r \mapsto B(r\zeta)$ stays in a Stolz angle at 1 in the image space of $B$. Hence $\liminf_{r \to 1} S(B(r\zeta)) = 0$ for $\zeta \in \rho(S \circ B)$. Now let us calculate the derivatives:

\[
S'(z) = -S(z) \frac{2}{(1-z)^2},
\]

\[
S''(z) = S(z) \left[ \frac{4}{(1-z)^4} - \frac{4}{(1-z)^3} \right],
\]

\[
(S \circ B)' = (S' \circ B)B',
\]

\[
(S \circ B)'' = (S'' \circ B)B'^2 + (S' \circ B)B''
\]

(2.5)

\[
A := \frac{(S \circ B)''}{|(S \circ B)'|^2} = \frac{S'' \circ B}{(S' \circ B)^2} + \frac{(S' \circ B) B''}{(S' \circ B)^2 B'^2} = \frac{S'' \circ B}{(S' \circ B)^2} + \frac{1}{S' \circ B B'^2}.
\]

Hence, for $\zeta \in \mathbb{T} \setminus \rho(S \circ B)$, $|B(\zeta)| = 1$, but $\xi := B(\zeta) \neq 1$, and so, by (2.4),

\[
|A(\zeta)| \leq \sup_{\xi \neq 1} \frac{|S''(\xi)|}{|S'(\xi)|^2} + 2\sup_{\xi \neq 1} \frac{|1 - \xi|^2}{|S(\xi)|} C
\]

\[
\leq C' \sup_{\xi \neq 1} \frac{|1 - \xi|^4}{|1 - \xi|^4} + 8C < \infty.
\]

\[\Box\]

**Corollary 2.14.** Let $S_\mu$ be a singular inner function with finite spectrum $\rho(S_\mu)$. Then $S_\mu \in \mathcal{I}_c$.

**Proof.** Since $S$ is the universal covering map of $\mathbb{D} \setminus \{0\}$, each singular inner function $S_\mu$ writes as $S_\mu = S \circ v$ for some inner function $v$. Since $\rho(S_\mu)$ is finite, $v$ necessarily is a finite Blaschke product. (This can also be seen from [15, Proof of Theorem 2.2]). The assertion now follows from Proposition 2.13. \[\Box\]
Note that this result also follows in an elementary way from Proposition 2.9 and the fact that every such $S_\mu$ is a finite product of powers of the atomic inner function $S$. We now consider left-compositions with finite Blaschke products.

**Proposition 2.15.** Let $\Theta$ be a one-component inner function. Then each Frostman shift $(a - \Theta)/(1 - \bar{a}\Theta) \in \mathcal{J}_c$, too. Here $a \in \mathbb{D}$.

*Proof.* Let $\tau(z) = (a - z)/(1 - \bar{a}z)$. Then $\rho(\tau \circ \Theta) = \rho(\Theta)$. As above,

$$\lim_{r \to 1} \inf \left| \tau \circ \Theta(r\zeta) \right| < 1$$

for every $\zeta \in \rho(\tau \circ \Theta)$. Now

$$\tau(z) = \frac{1}{a} + \frac{|a|^2 - 1}{\bar{a}} \frac{1}{1 - \bar{a}z},$$

from which we easily deduce the first and second derivatives. By using the formulas 2.5, we obtain

$$A := \left| \frac{(\tau \circ \Theta)''}{((\tau \circ \Theta)')^2} \right| \leq C \frac{|1 - \bar{a}\Theta|^4}{|1 - \bar{a}\Theta|^3} + C' \frac{|1 - \bar{a}\Theta|^2}{|\Theta'|^2}.$$

Hence, the assumption $\Theta \in \mathcal{J}_c$ now yields (via Aleksandrov’s criterion 2.12) that $\sup_{\zeta \in \rho(\tau \circ \Theta)} A(\zeta) < \infty$. Thus $\tau \circ \Theta \in \mathcal{J}_c$.

□

**Corollary 2.16.** Given $a \in \mathbb{D} \setminus \{0\}$, the interpolating Blaschke products $(S - a)/(1 - \bar{a}S)$ belong to $\mathcal{J}_c$.

This also follows from Corollary 2.6 by noticing that the $a$-points of $S$ are located on a disk tangent at 1 and that the pseudohyperbolic distance between two consecutive ones is constant (see [20]). There it is also shown that the Frostman shift $(S - a)/(1 - \bar{a}S)$ is an interpolating Blaschke product.

**Corollary 2.17.** Let $B$ be a finite Blaschke product and $\Theta \in \mathcal{J}_c$. Then $B \circ \Theta \in \mathcal{J}_c$.

*Proof.* This is a combination of Propositions 2.15 and 2.9. □

### 3. Inner Functions Not Belonging to $\mathcal{J}_c$

Here we present a class of Blaschke products that are not one-component inner functions. Recall that a Blaschke product $b$ with zero-sequence $(z_n)$ is thin if

$$\lim_{n} \prod_{k \neq n} \rho(z_k, z_n) = \lim_{n \to 1} (1 - |z_n|^2)|b'(z_n)| = 1.$$

It was shown by Tolokonnikov [24, Theorem 2.3] that $b$ is thin if and only if

$$\lim_{|z| \to 1} (|b(z)|^2 + (1 - |z|^2)|b'(z)|) = 1.$$
Corollary 3.1. Thin Blaschke products are never one-component inner functions.

Proof. Let $\varepsilon \in ]0,1[$ be arbitrary close to 1. Choose $\eta > 0$ and $\delta > 0$ so close to 1 so that

$$\varepsilon < \eta^2 \text{ and } \eta < (1 - \sqrt{1 - \delta^2})/\delta.$$  

By deleting finitely many zeros, say $z_1, \ldots, z_N$ of $b$, we obtain a tail $b_N$ such that $(1 - |z_n|^2)|b_N'(z_n)| \geq \delta$ for every $n > N$. Hence, by Theorem 2.2,  

(3.1) \[ \{ z \in \mathbb{D} : |b_N(z)| < \varepsilon \} \subseteq \{ z \in \mathbb{D} : \rho(z, Z(b_N)) < \eta \} \]

and the disks $D(z_n, \eta)$ are pairwise disjoint. This implies that the level set $\{ z \in \mathbb{D} : |b_N(z)| < \varepsilon \}$ is not connected. Now choose $r$ so close to 1 that 

$$p(z) := \prod_{n=1}^{N} \rho(z, z_n) \geq \varepsilon$$

for every $z$ with $r \leq |z| < 1$. We show that the level set $\{|b| < \varepsilon^2\}$ is not connected. In fact, for some $r \leq |z| < 1$ we have $|b(z)| < \varepsilon^2$, then

$$|b_N(z)| = \frac{|b(z)|}{|p(z)|} < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$ 

Hence

$$\{ z : r < |z| < 1, |b(z)| < \varepsilon^2 \} \subseteq \{ |b_N(z)| < \varepsilon \} \subseteq \bigcup_{n>N} D(z_n, \eta).$$

Since the disks $D_p(z_n, \eta)$ are pairwise disjoint if $n > N$, we are done. 

Corollary 3.2. No finite product $B$ of thin interpolating Blaschke products belongs to $\mathcal{I}_c$.

Proof. Let $\varepsilon \in ]0,1[$ be arbitrary close to 1. By Corollary 3.1, if $b_j$, ($j = 1, 2$), are two thin Blaschke products with zero-sequence $(z_n^{(j)})_n$, 

$$\Omega_{b_j}(\varepsilon) \subseteq \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta)$$

for suitable $\eta$, the disks $D_\rho(z_n^{(j)}, \eta)$, being pairwise disjoint for $n$ large. Since $\lim_{n} \rho(z_n^{(j)}, z_{n+1}^{(j)}) = 1$, we see that a disk $D_\rho(z_n^{(1)}, \eta)$ can meet at most one disk $D_\rho(z_n^{(2)}, \eta)$ for $n$ large. Hence 

$$\Omega_{b_1b_2}(\varepsilon^2) \subseteq \bigcup_{j=1}^{2} \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta),$$

where the set on the right hand side obviously is disconnected. The general case works via induction. 

$\square$
Remark. The conditions
\[(3.2) \quad \eta^* := \sup_{n \in \mathbb{N}} \rho(z_n, Z(b) \setminus \{z_n\}) < 1,\]
or equivalently
\[(3.3) \quad D(z_n, \eta) \cap \bigcup_{m \neq n} D(z_m, \eta) \neq \emptyset \text{ for some } \eta \in ]0,1[,\]
are not sufficient to guarantee that the interpolating Blaschke product $b$ is a one-component inner function.

Proof. Take $z_{2n} = 1 - n^{-n}$ and $z_{2n+1} = 1 - (n^{-n} + n^{-n})$. Then $(z_{2n})$ and $(z_{2n+1})$ are (thin) interpolating sequences by [16, Corollary 2.4]. Using with $a = n^{-n}$ and $b = 2a$ the identity
$$\rho(1-a, 1-b) = \frac{|a-b|}{a+b-ab},$$
we conclude that
$$\rho(z_{2n}, z_{2n+1}) = \frac{n^{-n}}{1 - z_{2n}z_{2n+1}} \to 1/3,$$
and so the union $(z_n)$ is an interpolating sequence satisfying (3.3). By Corollary 3.2, the Blaschke product formed with the zero-sequence $(z_n)$ is not in $\mathcal{J}_c$.

Using the following theorem in [5], we can exclude a much larger class of Blaschke products from being one-component inner functions:

**Theorem 3.3 (Berman).** Let $u$ be an inner function. Then, for every $\varepsilon \in ]0,1[$, all the components of the level sets $\{z \in \mathbb{C} : |u(z)| < \varepsilon\}$ have compact closures in $\mathbb{D}$ if and only if $u$ is a Blaschke product and
$$\limsup_{r \to 1} |u(r\xi)| = 1 \text{ for every } \xi \in \mathbb{T}.$$

In particular this condition is satisfied by finite products of thin Blaschke products (see [17, Proposition 2.2]) as well as by the class of uniform Frostman Blaschke products
$$\sup_{\xi \in \mathbb{T}} \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|\xi - z_n|} < \infty.$$ 
Note that this Frostman condition implies that the associated Blaschke product has radial limits of modulus one everywhere [9, p. 33]. As a byproduct of Theorem 2.3 we therefore obtain

**Corollary 3.4.** If $b$ is a uniform Frostman Blaschke product with zeros $(z_n)$ clustering at a single point, then $\limsup_{r \to 1} \rho(z_n, z_{n+1}) = 1$.

**Questions 3.5.** To conclude, we would like to ask two questions and present three problems:
(1) Can every inner function \( u \) whose boundary spectrum \( \rho(u) \) has measure zero, be multiplied by a one-component inner function into \( \mathcal{I}_c \)?

(2) Let \( S_\mu \) be a singular inner function with countable spectrum. Give a characterization of those measures \( \mu \) such that \( S_\mu \in \mathcal{I}_c \). Do the same for singular continuous measures.

(3) In terms of the zeros, give a characterization of those interpolating Blaschke products that belong to \( \mathcal{I}_c \).

(4) Does the Blaschke product \( B \) with zeros \( z_n = 1 - n^{-2} \) belong to \( \mathcal{I}_c \)?

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