Characterization of affine links in the projective space

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1 Introduction

A link in the real projective space $\mathbb{RP}^3$ is a smooth closed 1-dimensional submanifold of $\mathbb{RP}^3$. As usual, if a link is connected, then it is called a knot. A link $L \subset \mathbb{RP}^3$ is said to be affine if it is isotopic to a link, which does not intersect some plane $\mathbb{RP}^2 \subset \mathbb{RP}^3$.

The main theorem of this paper provides necessary and sufficient condition for a link in $\mathbb{RP}^3$ to be affine:

**Main Theorem.** A link $L \subset \mathbb{RP}^3$ is affine if and only if $\pi_1(\mathbb{RP}^3 \setminus L)$ contains a non-trivial element of order 2.

1.1 Known results

The problem of determining whether a link is affine was considered in literature and there are results in this direction, mostly about necessary conditions:

**Homology condition.** Each connected component of a link $L \subset \mathbb{RP}^3$ realizes a homology class, an element of $H_1(\mathbb{RP}^3; \mathbb{Z}/2) = \mathbb{Z}/2$. All components of an affine link $K \subset \mathbb{RP}^3$ realize $0 \in H_1(\mathbb{RP}^3; \mathbb{Z}/2)$. The converse is not true: there exist knots homological to zero in $\mathbb{RP}^3$, which are not affine. A few examples are shown in Figure 1.

**Self-linking number.** If a knot $K \subset \mathbb{RP}^3$ realizes $0 \in H_1(\mathbb{RP}^3; \mathbb{Z}/2)$, then a self-linking number $sl(K) \in \mathbb{Z}$ is defined as the linking number modulo 2 of the connected components of the preimage $\tilde{K} \subset S^3$ of $K$ under the covering $S^3 \to \mathbb{RP}^3$, see [3], §7.

If $K$ is affine, then $sl(K) = 0$, see [3], §7. For the knot $2_1$ shown in Figure 1, this invariant equals 2, and this is why $2_1$ is not affine.

**Exponents of monomials in the bracket polynomial.** If a knot $K \subset \mathbb{RP}^3$ is affine, then the exponents of all monomials of its bracket
polynomial $V_K$ defined by Drobotukhina in [3] are congruent to each other modulo 4.

The self-linking and exponent conditions are independent: $sl_{5_2} = 0$, while $V_{5_2} = A^4 + A^2 - 1 - 2A^{-2} + A^{-4} + 2A^{-6} - 2A^{-10} + A^{-14}$, on the other hand, $V_{5_9} = A^{-8} + A^{-12} - A^{-20}$, while $sl_{5_9} = 3$. See Figure 1 and Drobotukhina’s table [4].

1.2 Comparison to Main Theorem

The main theorem of this note provides necessary and sufficient condition for a link in $\mathbb{R}P^3$ to be affine. It is similar to the famous results of the classical knot theory, like the Dehn-Papakyriacopoulos characterization of the unknot as the only knot $K \subset \mathbb{R}^3$ with $\pi_1(\mathbb{R}^3 \setminus K)$ isomorphic to $\mathbb{Z}$.

However conditions formulated in terms of fundamental groups are not easy to check. Therefore the known necessary conditions mentioned above may happen to be more useful for checking if a specific knot is affine.

1.3 Reformulation of Main Theorem

and its conjectural generalization

The property of a projective link of being affine admits the following obvious reformulation:

* A link $L \subset \mathbb{R}P^3$ is affine if and only if there exists an embedding $i : D^3 \to \mathbb{R}P^3$ such that $L \subset i(D^3)$.

This property of links in $\mathbb{R}P^3$ admits the following generalization to links in an arbitrary 3-manifold. A link $L$ in a 3-manifold $M$ is called **localizable** if there exists an embedding $i : D^3 \to M$ of the ball $D^3$ such that $L \subset i(D^3)$.

It would be interesting to find a characterization of localizable links. The following conjecture would provide a generalization of Main Theorem to closed 3-manifolds with trivial $\pi_2$.  

![Figure 1:](image)
Conjecture. Let $M$ be a closed 3-manifold with $\pi_2(M) = [\mathbb{F}]$. Then a link $L \subset M$ is localizable if and only if $\pi_1(M \setminus L)$ contains a subgroup $G$ which is mapped isomorphically onto $\pi_1(M)$ by the inclusion homomorphism $\pi_1(M \setminus L) \to \pi_1(M)$.

2 Proofs

2.1 Proof of Main Theorem. Necessity.

Assume that $L$ is an affine link. Since the group $\pi_1(\mathbb{R}P^3 \setminus L)$ is invariant under isotopy, it suffices to prove that $\pi_1(\mathbb{R}P^3 \setminus L)$ contains an element of order 2 if $L$ does not intersect a plane $\mathbb{R}P^2 \subset \mathbb{R}P^3$.

If $L \cap \mathbb{R}P^2 = \emptyset$, then $L$ is contained in the affine part $\mathbb{R}P^3 \setminus \mathbb{R}P^2 = \mathbb{R}^3$ of $\mathbb{R}P^3$, and $\mathbb{R}P^3 \setminus L$ is a union of $\mathbb{R}P^3 \setminus (L \cup \mathbb{R}P^2)$ and a regular neighborhood of $\mathbb{R}P^2$. The van Kampen Theorem applied to this presentation of $\mathbb{R}P^3 \setminus L$ implies that $\pi_1(\mathbb{R}P^3 \setminus L)$ is the free product $\mathbb{Z}/2 * \pi_1(\mathbb{R}^3 \setminus L)$. Hence it contains a non-trivial element of order 2.

2.2 A lemma about a map of the projective plane

The proof of sufficiency is prefaced with the following simple homotopy-theoretic lemma:

Lemma 1. Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a map inducing isomorphism on fundamental group. Then the covering map $\tilde{f} : S^2 \to S^2$ is not null homotopic.

Proof. Consider the diagram

$$
\begin{array}{c}
0 \to H_2(\mathbb{R}P^2; \mathbb{Z}/2) \to H_2(S^2; \mathbb{Z}/2) \to H_2(\mathbb{R}P^2; \mathbb{Z}/2) \to H_1(\mathbb{R}P^2; \mathbb{Z}/2) \\
\downarrow f_* \downarrow \tilde{f}_* \downarrow f_* \\
0 \to H_2(\mathbb{R}P^2; \mathbb{Z}/2) \to H_2(S^2; \mathbb{Z}/2) \to H_2(\mathbb{R}P^2; \mathbb{Z}/2) \to H_1(\mathbb{R}P^2; \mathbb{Z}/2)
\end{array}
$$

in which the rows are segments of the Smith sequence (see, e.g., [2]) for the antipodal involution on $s : S^2 \to S^2 : x \mapsto -x$. The diagram is commutative, because $\tilde{f}$ commutes with $s$. Notice that all the groups in this diagram are isomorphic to $\mathbb{Z}/2$. Exactness of the Smith sequences implies that the middle

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As follows from the Papakyriakopoulos sphere theorem, this condition can be reformulated as non-existence of a 2-sphere $\Sigma$ embedded into the orientation covering space of $M$ in such a way that $\Sigma$ does not bound a 3-ball there.
horizontal arrows in both rows are trivial, and the horizontal arrows next to them are isomorphism. By assumption, the rightmost vertical arrow is an isomorphism. Therefore the next vertical arrow is an isomorphism. This isomorphism coincides with the homomorphism represented by leftmost vertical arrow. Hence, the next arrow \( \tilde{f}^* : H_2(S^2; \mathbb{Z}/2) \to H_2(S^2; \mathbb{Z}/2) \) is an isomorphism. Thus \( \tilde{f} \) is not null homotopic.

2.3 Proof of Main Theorem: Sufficiency

Assume that \( \pi_1(\mathbb{R}P^3 \setminus L) \) contains a non-trivial element \( \lambda \) of order two. Realize \( \lambda \) by a smoothly embedded loop \( l : S^1 \to \mathbb{R}P^3 \setminus L \).

2.3.1 From a loop of order 2 to a singular projective plane in \( \mathbb{R}P^3 \setminus L \)

Since \( \lambda^2 = 1 \), there exists a continuous map \( D^2 \to \mathbb{R}P^3 \setminus L \) such that its restriction to the boundary circle \( \partial D^2 \) is the square of \( l \). Together with \( l \), this map gives a continuous map of the projective plane \( P = D^2/\sim \), for \( x \in \partial D^2 \) to \( \mathbb{R}P^3 \setminus L \). Denote this map by \( g \). So, \( g : P \to \mathbb{R}P^3 \setminus L \) is a generic differentiable map which induces a monomorphism \( g_* \) of \( \pi_1(P) = \mathbb{Z}/2 \) to \( \pi_1(\mathbb{R}P^3 \setminus L) \).

2.3.2 Singular sphere over the singular projective plane

Let \( p : S^3 \to \mathbb{R}P^3 \) be the canonical two-fold covering. Consider its restriction \( S^3 \setminus p^{-1}(L) \to \mathbb{R}P^3 \setminus L \). Observe that \( \lambda \) does not belong to the group of this covering, because otherwise \( \pi_1(S^3 \setminus p^{-1}(L)) \) would contain a non-trivial element of order two, which is impossible - a link group does not have any non-trivial element of finite order.

Therefore the covering of the projective plane induced from \( p \) via \( g \) is a non-trivial two-fold covering. Its total space is a 2-sphere. Denote it by \( S \). The map \( \tilde{g} \) which covers \( g \) maps \( S \) to \( S^3 \setminus p^{-1}(L) \).

2.3.3 Non-contractibility in \( S^3 \setminus p^{-1}(L) \) of the singular sphere

Let us choose a point \( x \in L \). Its complement \( \mathbb{R}P^3 \setminus \{x\} \) is homotopy equivalent to \( \mathbb{R}P^2 \). Indeed, the projection from \( x \) to any projective plane, which does not contain \( x \), is a deformation retraction \( \mathbb{R}P^3 \setminus \{x\} \to \mathbb{R}P^2 \).

The composition of \( g : P \to \mathbb{R}P^3 \setminus L \) with the inclusion \( \mathbb{R}P^3 \setminus L \to \mathbb{R}P^3 \setminus \{x\} \) induces an isomorphism \( \pi_1(P) \to \pi_1(\mathbb{R}P^3 \setminus \{x\}) \). Both spaces,
$P$ and $\mathbb{R}P^3 \setminus \{x\}$, have homotopy type of $\mathbb{R}P^2$. Lemma \[1\] implies that $\tilde{g}: S \to S^3 \setminus p^{-1}(x)$ is not null-homotopic.

### 2.3.4 Existence of a non-singular sphere $\Sigma_0 \subset S^3$ splitting $p^{-1}L$

Denote by $\sigma$ the antipodal involution $S^3 \to S^3: x \mapsto -x$.

**Lemma 2.** There exists a non-singular polyhedral submanifold $\Sigma_0$ of $S^3$ homeomorphic to $S^2$ such that $\Sigma_0 \cap p^{-1}(L) = \emptyset$ and the two points of $p^{-1}(x)$ belong to different connected components of $S^3 \setminus \Sigma_0$.

**Proof.** First, let us apply Whitehead’s modification \[6\] of the Papakyriakopoulos Sphere Theorem \[5\].

Recall the statement of this theorem (Theorem (1.1) of \[4\]): For any connected, orientable triangulated 3-manifold $M$ and subgroup $\Lambda \subset \pi_2(M)$ which is invariant under the action of $\pi_1(M)$, if $\Lambda \neq \pi_2(M)$, then $M$ contains a non-singular polyhedral 2-sphere which is essential mod $\Lambda$.

We will apply this theorem to $M = S^3 \setminus p^{-1}(L)$, $\Lambda = \text{Ker}(\text{in}_*: \pi_2(S^3 \setminus p^{-1}(L)) \to \pi_2(S^3 \setminus p^{-1}(x)))$.

We know that $\pi_2(S^3 \setminus \{x\}) = \pi_2(S^2) = \mathbb{Z}$ and that the homotopy class of $\tilde{g}$ is non-trivial in $\pi_2(S^3 \setminus p^{-1}(x))$. Therefore the homotopy class of $\tilde{g}$ does not belong to $\Lambda$, and hence $\Lambda \neq \pi_2(S^3 \setminus p^{-1}(L))$. Thus, all the assumptions of the Whitehead theorem are fulfilled.

Let us denote by $\Sigma_0$ a non-singular polyhedral 2-sphere whose existence is stated by the Whitehead theorem. By the Alexander Theorem \[1\], $\Sigma_0$ bounds in $S^3$ two domains homeomorphic to ball. Since $\Sigma_0$ is not null homotopic in $S^3 \setminus p^{-1}(x)$, each of these domains contains a point of $p^{-1}(x)$.

\[\square\]

### 2.3.5 Improving the splitting sphere

**Lemma 3.** There exists a smooth submanifold $\Sigma$ of $S^3$ homeomorphic to $S^2$ such that $\Sigma \cap p^{-1}(L) = \emptyset$, the two points of $p^{-1}(x)$ belong to different connected components of $S^3 \setminus \Sigma$, and either $\Sigma = \sigma(\Sigma)$ or $\Sigma \cap \sigma(\Sigma) = \emptyset$.

**Proof.** Any polyhedral compact surface can be approximated by a smooth 2-submanifold. Let $\Sigma_1$ be a smooth submanifold of $S^3$ approximating $\Sigma_0$ in $S^3 \setminus p^{-1}(L)$.

The antipodal involution $\sigma: S^3 \to S^3$ is an automorphism of the covering $p: S^3 \to \mathbb{R}P^3$, therefore $p^{-1}(L)$ is invariant under $\sigma$ and $\sigma(\Sigma_1) \subset S^3 \setminus p^{-1}(L)$. 

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Let us assume that $\Sigma_1$ and $\sigma(\Sigma_1)$ are transversal – this can be achieved by an arbitrarily small isotopy of $\Sigma_1$. Then the intersection $\Sigma_1 \cap \sigma(\Sigma_1)$ consists of disjoint circles.

Take a connected component $C$ of $\Sigma_1 \cap \sigma(\Sigma_1)$ which is innermost in $\sigma(\Sigma_1)$ (i.e., which bounds in $\sigma(\Sigma_1)$ a disc $D$ containing no other components of $\Sigma_1 \cap \sigma(\Sigma_1)$).

First, assume that $C \neq \sigma(C)$. In this case, make surgery on $\Sigma_1$ along $D$: remove a regular neighborhood $N$ of $C$ from $\Sigma_1$ and attach to $\partial N$ two discs parallel to $D$. This surgery does not change the homology class with coefficients in $\mathbb{Z}/2$ realized by $\Sigma_1$ in $S^3 \setminus p^{-1}(x)$.

Denote by $\Sigma_2$ the result of this surgery on $\Sigma_1$. This is a disjoint union of two spheres. The sum of the homology classes realized by them is the same non-trivial element of $H_2(S^2 \setminus p^{-1}(x); \mathbb{Z}/2) = \mathbb{Z}/2$ which was realized by $\Sigma_1$. Therefore, one of the summands is non-trivial. The corresponding component of $\Sigma_2$ separates the two points of $p^{-1}(x)$. Denote this component by $\Sigma_3$. Since $C \neq \sigma(C)$, the number of connected components of $\Sigma_3 \cap \sigma(\Sigma_3)$ is less than the number of connected components of $\Sigma_1 \cap \sigma(\Sigma_1)$, all other properties of $\Sigma_1$ are inherited by $\Sigma_3$, and we are ready to continue with the next connected component of $\Sigma_3 \cap \sigma(\Sigma_3)$ which bounds in $\sigma(\Sigma_3)$ a disc containing no other components of $\Sigma_3 \cap \sigma(\Sigma_3)$).

Second, consider the case $C = \sigma(C)$. Then the disc $D \subset \sigma(\Sigma_1)$ together with its image $\sigma(D) \subset \Sigma_1$ form an embedded sphere, which is invariant under $\sigma$ and does not meet the rest of $\Sigma \cup \sigma(\Sigma_1)$ besides along $C$, that is $(D \cup \sigma(D)) \cap ((\Sigma_1 \setminus \sigma(D) \cup (\Sigma_1 \setminus D)) = \emptyset$.

If $D \cup \sigma(D)$ separates points of $p^{-1}(x)$, we are done: we can smoothen the corner of $D \cup \sigma(D)$ along $C$ keeping it invariant under $\sigma$ and take the result for $\Sigma$.

If $D \cup \sigma(D)$ does not separate points of $p^{-1}(x)$, then $D \cup (\Sigma_1 \setminus \sigma(D))$ separates points of $p^{-1}(x)$ (as well as its image under $\sigma$, that is $\sigma(D) \cup (\sigma(\Sigma_1) \setminus D)$). Indeed, the homology classes realized by $D \cup \sigma(D)$ and $D \cup (\Sigma_1 \setminus \sigma(D))$ in $S^3 \setminus p^{-1}(x)$ differ from each other by the homology class of $\sigma(D) \cup (\Sigma_1 \setminus \sigma(D)) = \Sigma_1$ which is known to be nontrivial. So, if the class of $D \cup \sigma(D)$ is trivial, then the class of $\sigma(D) \cup (\sigma(\Sigma_1) \setminus D)$ is not. Then smoothing of a corner along $C$ turns $D \cup (\Sigma_1 \setminus \sigma(D))$ into a new sphere $\Sigma_2$ such that $\Sigma_2 \cap \sigma(\Sigma_2)$ has less connected components than $\Sigma_1 \cap \sigma(\Sigma_1)$.

By repeating this construction, we will eventually build up a sphere $\Sigma \subset S^3 \setminus p^{-1}(L)$ with the required properties. \qed
2.3.6 Completion of the proof

Let us return to the proof of Main Theorem. If the sphere $\Sigma$ provided by Lemma 3 is invariant under $\sigma$, then $\Sigma$ divides $S^3$ into two balls which are mapped by $\sigma$ homeomorphically to each other. Let $B$ be one of them. The part of $p^{-1}(L)$ contained in $B$ can be moved by an isotopy fixed on a neighborhood of the boundary of $B$ inside an arbitrarily small metric ball in $S^3$. Using $\sigma$, extend this isotopy to a $\sigma$-equivariant isotopy of the whole $S^3$. The equivariant isotopy defines an isotopy of $\mathbb{RP}^3$ which moves $L$ to a link contained in a small metric ball. This proves that $L$ is an affine link.

Consider now the case in which the sphere $\Sigma$ provided by Lemma 3 is not $\sigma$-invariant, but rather is disjoint from its image $\sigma(\Sigma)$. Then spheres $\Sigma$ and $\sigma(\Sigma)$ divide $S^3$ into three domains: two of them are balls bounded by $\Sigma$ and $\sigma(\Sigma)$, respectively. Let us denote by $B$ the ball bounded by $\Sigma$, then its image $\sigma(B)$ is bounded by $\sigma(\Sigma)$. Denote the third domain by $E$. It is invariant under $\sigma$.

If one of the points from $p^{-1}(x)$ belonged to $E$, then the other one also would belong to $E$, and then the sphere $\Sigma$ would be contractible in $S^3 \setminus p^{-1}(x)$. Therefore $B \cap p^{-1}(L) \neq \emptyset$. Denote $B \cap p^{-1}(L)$ by $K$. This is a sublink of $p^{-1}(L)$. It can be moved by an isotopy fixed on a neighborhood of the boundary of $B$ inside an arbitrarily small metric ball in $S^3$. Then this isotopy can be extended to $\sigma$-equivariant isotopy of $S^3$ fixed on $E$. This equivariant isotopy defines an isotopy of $\mathbb{RP}^3$ which moves $p(K)$ to a link contained in a small metric ball.

Thus our link $L$ is presented as a disjoint sum of an affine link $p(K)$ and the rest $L \setminus p(K)$ of $L$. If $L \setminus p(K) = \emptyset$, then we are done. If not, then $\pi_1(S^3 \setminus L)$ is presented as a free product of $\pi_1(B \setminus K)$ and $\pi_1(\mathbb{RP}^3 \setminus (L \setminus p(K)))$. By the assumption, the group $\pi_1(S^3 \setminus L)$ has a non-trivial element of order 2. The first factor, $\pi_1(B \setminus K)$ cannot contain such an element, because this is a group of a classical link. Hence, the second factor, $\pi_1(\mathbb{RP}^3 \setminus (L \setminus p(K)))$, contains it, and we can apply the constructions and arguments above to the link $L \setminus p(K)$. This link contains less components than the original one, therefore, after several iterations, we will come to the situation in which $p^{-1}(L) \cap B = \emptyset$.

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