Computing the maximum matching width is NP-hard

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Abstract

The maximum matching width is a graph width parameter that is defined on a branch-decomposition over the vertex set of a graph. In this short paper, we prove that the problem of computing the maximum matching width is NP-hard.

1 Introduction

Tree-width and branch-width are two prominent graph width parameters extensively studied in both structural graph theory and theoretical computer science due to wide applications. For instance, the seminar work by Courcelle [2] proved that many NP-hard problems in graphs can be solved in polynomial time when the graph has bounded tree-width or branch-width, proving the usefulness the width parameters.

Recently, a new graph width parameter compatible with tree-width or branch-width was introduced, called the maximum matching width (mm-width for short) [8]. It is shown in [8] that if mmw(G), tw(G), and bw(G) are the mm-width, tree-width, and branch-width of a graph G, respectively, then, for every graph G,

\[ \text{mmw}(G) \leq \max(\text{bw}(G), 1) \leq \text{tw}(G) + 1 \leq 3 \text{mmw}(G). \]

Because of the similarity between the parameters one might anticipate that the new parameter can be replaced by the existing width parameters, questioning the necessity of studying the new parameter. On the other hand, it has been observed that the new parameter can lead to better results than others when employed to design algorithms for popular problems [5]. In particular, it is recently shown that the algorithm for the Minimum Dominating Set problem based on mm-width can enjoy faster time-complexity than one based on tree-width [4].

In this short paper, we investigate the hardness of computing the mm-width. While it is widely known that the problem of computing the tree-width or branch-width in general graphs are both NP-hard [1, 7], it has not been explored whether or not computing mm-width is NP-hard. We prove that computing the mm-width is also NP-hard by leveraging recent results in [4].

1.1 Notations

All graphs are simple, undirected, and finite. For a graph G, let V(G) and E(G) denote the vertex set and edge set of the graph, respectively. For a subset of vertices S \subseteq V(G), N_G(S) is the subset of vertices that are adjacent to at least one vertex in S. A tree is called nontrivial if it

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has at least one edge. We say that a tree is ternary if all vertices have degree 1 or 3. A function \( f : 2^X \to \mathbb{Z} \) is symmetric if \( f(A) = f(X \setminus A) \) for all \( A \subseteq X \), and a function \( f \) is submodular if \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) for all \( A, B \subseteq X \).

# Preliminaries

## 2.1 Maximum matching width

We provide a formal definition of the maximum matching width. A branch-decomposition over a finite set \( X \) is a pair \((T, \mathcal{L})\) of a ternary tree \( T \) together with a bijection \( \mathcal{L} \) from the leaves of \( T \) to \( X \). When we remove an edge \( ab \) of \( T \), the tree is divided into two connected components, inducing the partition \( X_1 \cup X_2 \) of \( X \) due to bijection \( \mathcal{L} \). Here, we call \( X_1 \cup X_2 \) the partition induced by \( ab \).

For an edge \( e \) of \( T \), and a function \( f \), which is symmetric and submodular, the \( f \)-value of \( e \) is \( f(A) \) where \((A, B)\) is the partition induced by \( e \). The \( f \)-width of a branch-decomposition \((T, \mathcal{L})\) is the maximum of \( f \)-values over \( E(T) \). The \( f \)-width of \( X \) is the minimum value of the \( f \)-width over all possible branch-decompositions over \( X \).

Let \( \text{mm}_G : 2^V(G) \to \mathbb{Z} \) be a function, where \( \text{mm}_G(A) \) is defined as the size of a maximum matching in \( G \) between \( A \) and its complement. Note that the function \( \text{mm}_G \) is symmetric and submodular \cite{6}. The maximum matching width of \( G \), denoted by \( \text{mmw}(G) \), is the \( \text{mm}_G \)-width of \( V(G) \).

Jeong, Telle, and Sæther \cite{4} gave a new characterization of graphs whose mm-width are at most \( k \), formally stated as the following theorem.

**Theorem 2.1** (Jeong, Telle, and Sæther \cite[Theorem 3.8]{4}). A nontrivial graph \( G \) has \( \text{mmw}(G) \leq k \) if and only if there exist a ternary tree \( T \) and nontrivial subtrees \( T_u \) of \( T \) for all vertices \( u \in V(G) \) such that

1. if \( uv \in E(G) \) then the subtrees \( T_u \) and \( T_v \) have at least one node of \( T \) in common, and
2. for every edge of \( T \) there are at most \( k \) subtrees using this edge.

A tree-representation of a graph \( G \) is a pair \((T, \{T_x\}_{x \in V(G)})\) where \( T \) is a ternary tree and a collection \( \{T_x\}_{x \in V(G)} \) of nontrivial subtrees of \( T \) satisfying the property (1). Theorem 2.1 states that a graph \( G \) has a tree-representation in which every edge of \( T \) is contained in at most \( k \) subtrees if and only if \( \text{mmw}(G) \leq k \).

## 2.2 Helly property of subtrees

A set system \( \mathcal{F} \) is said to satisfy the Helly property if the following holds for every subcollection \( \mathcal{G} \subseteq \mathcal{F} \):

\[
\text{if } A \cap B \neq \emptyset \text{ for all } A, B \in \mathcal{G}, \text{ then } \bigcap_{A \in \mathcal{G}} A \neq \emptyset.
\]

It is well known that a collection of the node sets of subtrees of a tree satisfies the Helly property:

**Proposition 2.2.** Let \( C \) be a clique of a graph \( G \) with at least two vertices. For every tree representation \((T, \{T_x\}_{x \in V(G)})\), \( \bigcap_{x \in C} V(T_x) \neq \emptyset \).
## 3 Computing maximum matching width is NP-hard

A graph $G$ is called a split graph if $V(G)$ can be partitioned into an independent set $I$ of $G$ and a clique $C$ of $G$, in which case we write $C = C(G)$ and $I = I(G)$. Inspired by [5], we prove, in particular, that computing the mmw of split graphs is NP-hard.

The following lemma characterizes the conditions satisfied by a certain type of split graphs with mmw equal to $k$:

**Lemma 3.1.** Let $G$ be a split graph with $|C(G)| = 3k$, $k \geq 1$. Then $\text{mmw}(G) = k$ if and only if $C(G)$ can be partitioned into three subsets $C_1$, $C_2$, and $C_3$ with $|C_1| = |C_2| = |C_3| = k$ such that, for each vertex $w \in I(G)$, $N_C(w)$ is contained exactly one of $C_1$, $C_2$, and $C_3$.

**Proof.** ($\Rightarrow$) Let $C = C(G)$. Assume that $(T, \{T_x\}_{x \in V(G)})$ is a tree-representation of $G$ in which every edge of $T$ is contained in at most $k$ subtrees, whose existence is ensured by Lemma [2.1]. By Proposition [2.2], there exists a vertex $v_0 \in \bigcap_{x \in C} V(T_x)$ in $T$. Then $v_0$ cannot be a leaf; otherwise, the unique edge incident with $v_0$ would be contained in $3k$ subtrees $T_x$ for all $x \in C$. Hence, the degree of $v_0$ is $3$, and let $e_1, e_2, e_3$ be the three incident edges in $T$. Let $s_i$ be the number of subtrees containing $e_i$ for every $i = 1, 2, 3$. Then, $s_i \leq k$ for every $i = 1, 2, 3$. For each $x \in C$, $T_x$ contains at least one edge among $e_1, e_2, e_3$ because it contains $v_0$. Hence, $s_1 + s_2 + s_3 \geq 3k$ and it follows that $s_i = k$ for each $i = 1, 2, 3$. This also implies that each tree $T_x$ with $x \in C$ contains exactly one edge among $e_1, e_2, e_3$. Therefore, by defining $C_1 := \{x \in C : e_i \in E(T_x)\}$, one can partition $C$ into $C_1 \cup C_2 \cup C_3$ with $|C_1| = |C_2| = |C_3| = k$.

Note that the graph obtained from $T$ by deleting $v_0$ consists of three disconnected components. Denote by $T^{(i)}$ the component that is incident with $e_i$ for each $i = 1, 2, 3$. For each $w \in I(G)$, $T_w$ contains none of $e_1, e_2, e_3$ because the edges are fully occupied by subtrees corresponding to vertices in $C$. Thus, $T_w$ should be entirely contained in $T^{(j)}$ for some $j \in \{1, 2, 3\}$. This implies that, for each $x \in C$, if $V(T_w) \cap V(T_x) \neq \emptyset$, then $x \in C_j$. In other words, $N_G(w) \subseteq C_j$.

($\Leftarrow$) Assume that $(T, \{T_x\}_{x \in V(G)})$ admits a partition $C_1 \cup C_2 \cup C_3$ satisfying the property. Let $C_j = \{c^{(j)}_1, c^{(j)}_2, \ldots, c^{(j)}_k\}$ for every $j = 1, 2, 3$. Moreover, assume that $I(G)$ is partitioned into $I_1 \cup I_2 \cup I_3$ so that for each $j = 1, 2, 3$, $N_G(I_j) \subseteq C_j$. Let $I_j = \{i^{(j)}_1, \ldots, i^{(j)}_\ell\}$ for every $j = 1, 2, 3$. For every tree-representation $(T, \{T_x\}_{x \in V(G)})$, Proposition [2.2] ensures that there exists a vertex $v_0 \in \bigcap_{x \in C} V(T_x)$ in $T$. As $T$ is ternary, $v_0$ is incident with at most three edges, and from the pigeonhole principle, at least one edge should be contained in at least $k$ subtrees. Thus, $\text{mmw}(G) \geq k$.

Now, we show that $\text{mmw}(G) \leq k$. It is enough to construct a tree-representation $(T, \{T_x\}_{x \in V(G)})$ of $G$ where every edge in $T$ is contained in at most $k$ subtrees by Lemma [2.1]. First, introduce $|V(G)| + (|V(G)| - 3) + 1$ many vertices $\{\beta_x\}_{x \in V(G)}, \{\alpha_x\}_{x \in (V(G) \setminus \{c^{(1)}_k, c^{(2)}_k, c^{(3)}_k\})}$ and $\alpha_0$. We construct a ternary tree $T$ as follows:

(a) Build three paths $\alpha^{(j)}_{i_1} \alpha^{(j)}_{i_2} \cdots \alpha^{(j)}_{i_j} \alpha^{(j)}_1 \alpha^{(j)}_2 \cdots \alpha^{(j)}_{k-1}$ for every $j = 1, 2, 3$.

(b) Join the three paths by adding three edges $\{\alpha^{(j)}_{i_j}, \alpha_0\}$ for every $j = 1, 2, 3$.

(c) For $x \in (V(G) \setminus \{c^{(1)}_k, c^{(2)}_k, c^{(3)}_k\})$, add an edge between $\beta_x$ and $\alpha_x$. In addition, attach $\beta^{(j)}_{i_j}$ to $\alpha^{(j)}_{i_j}$ for every $j = 1, 2, 3$. 

3
Figure 1: An example of the construction of a tree-representation when $k = 2$, $\ell_j = 1$ for every $j = 1, 2, 3$. For each $w \in I(G)$, $T_w$ is the subtree consisting of a single edge $\{\alpha_w, \beta_w\}$. For each $c \in C(G)$, $T_c$ is the unique path from $\beta_c$ to $\alpha_0$.

In this way, we obtain a ternary tree whose leaves are $\beta_x$’s. See Figure 1 for an example. Now we define the collection $\{T_x\}_{x \in I(G) \cup C(G)}$ of subtrees as follows:

- For each $w \in I(G)$, $T_w$ is the subtree consisting of a single edge $\{\alpha_w, \beta_w\}$.
- For each $c \in C(G)$, $T_c$ is the unique path from $\beta_c$ to $\alpha_0$.

Then, it is straightforward to check that this construction yields a desired tree-representation.

We now introduce a decision problem, which we call PARTITION-3: Given a multi-set $S$ (a set in which multiple elements are allowed) of positive integers, the task is to decide whether $S$ can be partitioned into three multi-subsets $S_1 \cup S_2 \cup S_3$ such that $\sum_{s \in S_1} s = \sum_{s \in S_2} s = \sum_{s \in S_3} s$.

For instance, when $S = \{3, 1, 2, 1\}$, the answer is NO as the sum of elements is not even divisible by 3; when $S = \{4, 4, 7\}$, the answer is NO as every subset containing the element 7 will exceed a third of the total sum; when $S = \{1, 1, 1, 3, 2, 1\}$, the answer is YES as $S_1 = \{1, 1, 1\}$, $S_2 = \{3\}$, $S_3 = \{2, 1\}$ gives a desired partition.

**Lemma 3.2.** PARTITION-3 is NP-hard.

**Proof.** We construct a polynomial reduction from PARTITION (problem [SP12] in [3]): The instance of PARTITION is the same as PARTITION-3, and the task is to decide whether a multiset can be partitioned into two multi-subsets of equal sums rather than three. The reduction is constructed as follows: for a given instance $S = \{s_1, \ldots, s_m\}$ of PARTITION, construct an instance of PARTITION as $S' = \{s_1, s_2, \ldots, s_m, \frac{1}{2} \sum_{i=1}^m s_i\}$. The correctness of this reduction is straightforward. Because PARTITION is known to be NP-hard [3], PARTITION-3 is NP-hard.

Now, we finish the proof.

**Theorem 3.3.** Computing the maximum matching width for graphs is NP-hard. In particular, computing the maximum matching width for split graphs is NP-hard.
Proof. The reduction is from $PARTITION-3$. For a given instance $S = \{s_1, \ldots, s_m\}$, we construct a split graph $G$ as follows:

1. Consider a complete graph on $\sum_{i=1}^{m} s_i$ vertices $V$. Partition $V$ into $m$ many subsets $V_1 \cup V_2 \cup \cdots \cup V_m$ so that $|V_i| = s_i$ for each $i = 1, 2, \ldots, m$.

2. Introduce $m$ more vertices $I = \{i_1, \ldots, i_m\}$, and connect $i_j$ to all vertices in $V_j$ for each $j = 1, 2, \ldots, m$.

By Lemma 3.1, such constructed split graph has $mm$-width equal to $\frac{1}{3} \sum_{i=1}^{m} s_i$ if and only if $V$ can be partitioned into three equal-sized partition $C_1 \cup C_2 \cup C_3$ such that for each $w \in I$, $N_G(w)$ is completely contained in one of three partitions. By letting $I_\ell := \{s_w : N_G(w) \subseteq C_\ell\}$ for every $\ell = 1, 2, 3$, one can see that $\sum_{w \in I_\ell} s_w = \frac{1}{3} \sum_{i=1}^{m} s_i$.

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