Cheap control problem for micro drone quadcopter

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Abstract. The paper describes a broad class of problems with a cheap control. In contrast to existing works in this field, a simple solution of the problem of synthesis of optimal control with an integral quadratic performance index is proposed. As an example, we consider the micro drone quadcopter model.

1. Introduction
The problem with a cheap control is usually set for linear systems of the form
\[ \dot{x} = A(t)x + B(t)u, \quad x(0) = x_0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^r, \]  
with the quadratic performance index
\[ J = \frac{1}{2} x^T(t_f)F_1 x(t_f) + \frac{1}{2} \int_0^{t_f} (x^T(t)Qx + \mu^2 \dot{u}^T(t)Ru)dt, \]  
where \( \mu \) is a small positive parameter.

The problem (1), (2) was considered in [1], where it was shown that the solution of this problem can be obtained by use of asymptotic expansions in fractional powers of the small parameter \( \varepsilon = \mu^{-\lambda} \), where \( \lambda \) can be found from
\[ B_j^T Q B_j = 0, \quad j = 0, L - 2 \]
\[ B_{L-1}^T Q B_{L-1} > 0, \]  
with
\[ B_0 = B, B_j = AB_{j-1} - \dot{B}_{j-1}, j \geq 1. \]

In this case, we must consider the problem in a much larger dimension, namely, \( n + Lr \) instead of \( n \). The method of integral manifolds [2, 3] for the analysis of such problems was applied in [4]. In this paper it is shown that for a natural class of problems one can do without increasing the dimensionality and, moreover, without solving differential equations.

2. Construction of control law
We consider the problem of constructing a control law for a second-order vector differential equation
\[ \ddot{x} + G(t)\dot{x} + N(t)x = B(t)u, \]  
with the quadratic performance index
\[ J = \frac{1}{2} x^T(t_f)F_1 x(t_f) + \frac{1}{2} \mu x^T(t_f)F_2 x(t_f) \]
\[ + \frac{1}{2} \int_0^T \left[ x^T(t) Q_1(t)x(t) + \mu x^T(t) Q_2(t)x(t) + \mu^2 u^T(t) R(t)u(t) \right] dt. \] 

Note that below we use the capital bold and regular letters for different matrices.

Introducing the small parameter \( \varepsilon = \varepsilon_2 \) we can rewrite the problem (4), (5) in the form (1), (2), where the corresponding matrices take the form

\[
A = \begin{pmatrix} 0 & I \\ -N & -G \end{pmatrix}, \quad A^T = \begin{pmatrix} I & -N^T \\ -G^T \end{pmatrix}, \\
Q = \begin{pmatrix} Q_1 & 0 \\ 0 & \varepsilon^2 Q_2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \\
S = BR^{-1}B^T.
\]

The optimal control is

\[
u = -\varepsilon^{-4}R^{-1}(0 B^T)P \left( \begin{array}{c} X \\ \chi \end{array} \right),
\]

and the matrix

\[
P = \begin{pmatrix} \varepsilon P_1 & \varepsilon^2 P_2 \\ \varepsilon^2 P_2^T & \varepsilon^3 P_3 \end{pmatrix}
\]

satisfies the matrix differential Riccati equation

\[
P(t_f) = \left( \begin{array}{cc} F_1 & 0 \\ 0 & \varepsilon^2 F_2 \end{array} \right)
\]

with the boundary condition

\[
P(t_f) = \left( \begin{array}{cc} F_1 & 0 \\ 0 & \varepsilon^2 F_2 \end{array} \right)
\]

The corresponding equations are

\[
\varepsilon P_1 - \varepsilon^2 (P_2 N + N^T P_2^T) - P_2 S P_2^T + Q_1 = 0, \\
\varepsilon P_2 + P_1 - \varepsilon P_2 G - \varepsilon^2 N^T P_3 - P_2 S P_3 = 0, \\
\varepsilon P_2 + P_2 + P_2^T - \varepsilon (P_2 G + G^T P_3) + Q_2 - P_3 S P_3 = 0,
\]

with the boundary condition

\[
\varepsilon P_1(t_f) = F_1, \quad P_2(t_f) = 0, \quad \varepsilon P_3(t_f) = F_2.
\]

The Riccati equation in the equivalent form

\[
e^4(P + A^T P + PA + Q) - PSP = 0
\]

is singularly perturbed, since when the small parameter is equal to zero the ability to specify an arbitrary initial or boundary conditions is lost. Such systems play an important role as mathematical models of numerous nonlinear phenomena in different fields (see e.g. [2−8]). Moreover, we have so called critical case, since the limiting equation

\[
PSP = 0
\]

has multiple zero solution [4, 5]. Critical cases appear in many problems [9−18].

Setting the small parameter equal to zero in (7)-(9), we obtain the equations

\[
-P_2 S P_2^T + Q_1 = 0, \\
P_1 - P_2 S P_3 = 0, \\
P_2 + P_2^T + Q_2 - P_3 S P_3 = 0.
\]

Suppose that these equations have the solution

\[
P_1 = M_1(t), \quad P_2 = M_2(t), \quad P_3 = M_3(t),
\]

such that all eigenvalues \( \lambda_i(\varepsilon) \) of the matrix

\[
D = \begin{pmatrix} 0 & I \\ -\varepsilon^{-2} S M_2^T - N & -\varepsilon^{-1} S M_3 - G \end{pmatrix}
\]

have the negative real parts

\[
-\frac{\nu(t, \varepsilon)}{\varepsilon}, \quad \nu(t, 0) > \nu > 0.
\]

Then we can neglect the boundary conditions, and as a solution of the system of matrix equations (7) - (9) take the regular part of the solution of this system, which can be regarded as a zero-dimensional integral manifold of slow motions. In the stationary case, the role of this solution is
played by a positive definite solution of the corresponding matrix algebraic Riccati equation. In the nonautonomous case matrices $P_1, P_2, P_3$ can be found as asymptotic expansions [19]

$$
P_1 = M_1 + \varepsilon P_{11} + \varepsilon^2 P_{12} + \cdots,
$$

$$
P_2 = M_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots,
$$

$$
P_3 = M_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots,
$$

from (7)-(9). Substituting these expansions into (7)-(9) we obtain

$$
\varepsilon (\dot{M}_1 + \varepsilon P_{11} + \varepsilon^2 P_{12} + \cdots) - \varepsilon^2 \left( M_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots \right) N + N^T (M_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots) N + N^T (M_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots) + Q_1 = 0,
$$

$$
\varepsilon (\dot{M}_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots) N + N^T (M_2 + \varepsilon P_{21} + \varepsilon^2 P_{22} + \cdots) + \varepsilon (M_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) N + N^T (M_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) = 0,
$$

$$
\varepsilon (M_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) N + N^T (M_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) + \varepsilon (\dot{M}_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) N + N^T (\dot{M}_3 + \varepsilon P_{31} + \varepsilon^2 P_{32} + \cdots) = 0.
$$

Equating the similar powers of $\varepsilon$ we obtain the relationships for $P_{11}, P_{21}, P_{31}$:

$$
\varepsilon \dot{M}_1 - P_{11} S M_2 - M_2 S P_{21} = 0,
$$

$$
\varepsilon \dot{M}_2 + P_{11} - M_2 G - M_2 S P_{31} - P_{21} S M_3 = 0,
$$

$$
\varepsilon \dot{M}_3 + P_{31} - M_3 G - M_3 S P_{31} - P_{31} S M_3 - M_3 S P_{31} = 0,
$$

then for $P_{12}, P_{22}, P_{32}$:

$$
P_{11} - M_2 N - N^T M_2 - P_{31} S M_2 - M_2 S P_{22} = 0,
$$

$$
P_{21} - P_{21} G - N^T M_3 - P_{21} S P_{21} - P_{22} S M_3 - M_2 S P_{22} = 0,
$$

$$
P_{31} + P_{22} + P_{22} G - G^T P_{31} - P_{22} S P_{31} - P_{32} S M_3 - M_3 S P_{32} = 0,
$$

and so on.

It should be noted that in the case of the time invariant problem (7)-(8) are the algebraic equations. Moreover, if

$$
N = 0, \quad G = 0,
$$

then relationships

$$
P_1 = M_1, \quad P_2 = M_2, \quad P_3 = M_3
$$

give the exact solution of (7)-(8).

The obtained mathematical results are applied by the authors for solving the problem of optimal control of a micro drone quadcopter (see, for example, [20, 21]).

### 3. Control of micro drone quadcopter

Consider the optimal control problem (4)-(5) with the following matrices

$$
G = N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\alpha & -\alpha & \alpha & \alpha \\ \beta & -\beta & 0 & 0 \\ 0 & 0 & \beta & -\beta \end{pmatrix},
$$

$$
Q_1 = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_5 & 0 \\ 0 & 0 & q_3 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

i.e.,

$$
A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & \alpha & \alpha \\ \beta & -\beta & 0 & 0 \\ 0 & 0 & \beta & -\beta \end{pmatrix}.
$$
\[ Q = \begin{pmatrix} q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu q_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu q_7 \end{pmatrix}. \]

To verify (3) we calculate
\[ B^T Q B = \begin{pmatrix} \mu q_5 \beta^2 & -\mu q_5 \beta^2 & 0 & 0 \\ -\mu q_5 \beta^2 & \mu q_5 \beta^2 & 0 & 0 \\ 0 & 0 & \mu q_6 \beta^2 & -\mu q_6 \beta^2 \\ 0 & 0 & -\mu q_6 \beta^2 & \mu q_6 \beta^2 \end{pmatrix}. \]

This matrix is not as identically zero and it is not positively definite matrix if \( \mu \neq 0 \). But this matrix is identically zero for \( \mu = 0 \). Considering for simplicity the case when \( \alpha, \beta \) are the constants, we have
\[ B_1 = AB = \begin{pmatrix} -\alpha & -\alpha & \alpha & \alpha \\ \beta & -\beta & 0 & 0 \\ 0 & 0 & \beta & -\beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
and, finally,
\[ B_1^T Q B_1 = \begin{pmatrix} \alpha^2 q_1 + \beta^2 q_2 & \alpha^2 q_1 - \beta^2 q_2 & -\alpha^2 q_1 & -\alpha^2 q_1 \\ \alpha^2 q_1 - \beta^2 q_2 & \alpha^2 q_1 + \beta^2 q_2 & -\alpha^2 q_1 & -\alpha^2 q_1 \\ -\alpha^2 q_1 & -\alpha^2 q_1 & \alpha^2 q_1 + \beta^2 q_3 & \alpha^2 q_1 - \beta^2 q_3 \\ -\alpha^2 q_1 & -\alpha^2 q_1 & \alpha^2 q_1 - \beta^2 q_3 & \alpha^2 q_1 + \beta^2 q_3 \end{pmatrix}. \]

This matrix is not identically zero but it is not positive definite since the sum of all rows is equal to zero. This means that the condition (3) is not valid and we cannot apply the usual approach [1].

To continue the analysis, we return to the results of the previous section.

It is easy to check that
\[ S = \begin{pmatrix} 4\alpha^2 & 0 & 0 \\ 0 & 2\beta^2 & 0 \\ 0 & 0 & 2\beta^2 \end{pmatrix}, \]
and, therefore,
\[ S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4\alpha^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta^2 & 0 \end{pmatrix}. \]

The easy algebra shows that in the case under consideration the equation (10) has the solution
\[ P_2 = M_2 = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \]
where
\[ b_1 = \frac{\sqrt{q_1}}{2\alpha}, \quad b_2 = \frac{\sqrt{q_2}}{\beta\sqrt{2}}, \quad b_3 = \frac{\sqrt{q_3}}{\beta\sqrt{2}}. \]

Thus, we obtain for the equation (12)
\[ M_2 + M_2^T + Q_2 = \begin{pmatrix} 2b_1 & 0 & 0 \\ 0 & q_5 + 2b_2 & 0 \\ 0 & 0 & q_6 + 2b_3 \end{pmatrix}. \]

The solution of (12) takes the form
\[ P_3 = M_3 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \]

where
\[ c_1 = \frac{\sqrt{2b_1}}{2\alpha}, \quad c_2 = \frac{\sqrt{q_5 + 2b_2}}{\beta\sqrt{2}}, \quad c_3 = \frac{\sqrt{q_6 + 2b_3}}{\beta\sqrt{2}}. \]

The relationship (11) gives the expression for \( P_1 \):
\[ P_1 = M_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \]

where
\[ a_1 = 4\alpha^2 b_1 c_1, \quad a_2 = 2\beta^2 b_2 c_2, \quad a_3 = 2\beta^2 b_3 c_3. \]

It should be noted that the matrix
\[ M = \begin{pmatrix} a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 & b_3 & 0 \\ b_1 & 0 & 0 & c_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & c_2 & 0 \\ 0 & 0 & b_3 & 0 & 0 & c_3 \end{pmatrix}, \]

is positively definite if and only if
\[ a_i c_i - b_i^2 > 0, \quad i = 1,2,3. \]

The matrix \( D \) (see (13)) takes the form
\[ D = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -e^{-2}4\alpha^2 b_1 & 0 & 0 & -e^{-4}\alpha^2 c_1 & 0 & 0 \\ 0 & -e^{-2}\beta^2 b_2 & 0 & 0 & -e^{-4}\beta^2 c_2 & 0 \\ 0 & 0 & -e^{-2}\beta^2 b_3 & 0 & 0 & -e^{-4}\beta^2 c_3 \end{pmatrix}. \]

It is easy to see that the matrix \( D \) has the following eigenvalues
\[ \lambda_{1,2} = -e^{-4}\alpha^2 c_1 \pm e^{-1}2\alpha i\sqrt{b_1}, \]
\[ \lambda_{3,4} = -e^{-1}\beta^2 c_2 \pm e^{-1}\beta i\sqrt{2b_2}, \]
\[ \lambda_{5,6} = -e^{-1}\beta^2 c_3 \pm e^{-1}i\beta \sqrt{2b_3}. \]

The real parts of all eigenvalues are negative. This means that we can use the control law (6) of form
\[ u = -e^{-4}R^{-1}(0 \quad B^T)M(x, \dot{x}) = -e^{-2}B^TM_2x - e^{-1}B^TM_3\dot{x} \]

with
\[ B^TM_2 = \begin{pmatrix} -a_1 & \beta b_2 & 0 \\ -a_1 & -\beta b_2 & 0 \\ a_1 & 0 & \beta b_3 \\ a_1 & 0 & -\beta b_3 \end{pmatrix}, \]

and
\[ B^TM_3 = \begin{pmatrix} -a_1 & \beta c_2 & 0 \\ -a_1 & -\beta c_2 & 0 \\ a_1 & 0 & \beta c_3 \\ a_1 & 0 & -\beta c_3 \end{pmatrix}. \]

As the result we obtain the optimal control law in the form
\[ u_1 = -e^{-2}(-a_1 x_1 + \beta b_2 x_2) - e^{-1}(-a_1 \dot{x}_1 + \beta c_2 x_2), \]
\[ u_2 = -e^{-2}(-a_1 x_1 - \beta b_2 x_2) - e^{-1}(-a_1 \dot{x}_1 - \beta c_2 x_2), \]
\[ u_3 = -e^{-2}(a_1 x_1 + \beta b_3 x_3) - e^{-1}(a_1 \dot{x}_1 + \beta c_3 x_3), \]
\[ u_4 = -e^{-2}(a_1 x_1 - \beta b_3 x_3) - e^{-1}(a_1 \dot{x}_1 - \beta c_3 x_3). \]
A somewhat unexpectedly the controlled system is splitted into three independent equations
\[ \ddot{x}_1 + \varepsilon^{-1}4\alpha^2c_1x_1 + \varepsilon^{-2}4\alpha^2b_1x_1 = 0, \]
\[ \ddot{x}_2 + \varepsilon^{-1}2\beta^2c_2x_2 + \varepsilon^{-2}2\beta^2b_2x_2 = 0 \]
\[ \ddot{x}_3 + \varepsilon^{-1}2\beta^2c_3x_3 + \varepsilon^{-2}2\beta^2b_3x_3 = 0 \]

Consider now concrete numerical values of parameters which are specific for micro drone quadcopter models of 3 DOF Hover type. In this case we have the optimal control problem
\[ \ddot{x} = Bu, \ x \in \mathbb{R}^3, \ u \in \mathbb{R}^4 \] (20)
with the performance index
\[ J = \frac{1}{2} \int_{t_0}^{t_f} \left[ 500x_1^2 + 350x_2^2 + 350x_3^2 + 20\dot{x}_2^2 + 20\dot{x}_3^2 + 0.01(u_1^2 + u_2^2 + u_3^2 + u_4^2) \right] dt. \]

It is possible to multiply the expression in the square brackets by 0.0025 and obtain the equivalent optimal control problem for (20) with the performance index
\[ J = \frac{1}{2} \int_{t_0}^{t_f} \left[ 1.25x_1^2 + 0.875x_2^2 + 0.875x_3^2 + \mu(10\dot{x}_2^2 + 10\dot{x}_3^2) + \mu^2(u_1^2 + u_2^2 + u_3^2 + u_4^2) \right] dt. \]

Here the role of \( \mu \) is played by \( 1/200 = 0.005 \). Taking into account the corresponding numerical values \( \alpha = 0.03, \beta = 0.42, \ q_1 = 1.25, \ q_2 = q_3 = 0.875, \ q_5 = q_6 = 10. \) we obtain
\[ b_1 = \frac{\sqrt{q_1}}{2\alpha} \approx 18.6339, \ b_2 = b_3 = \frac{\sqrt{q_2}}{\beta\sqrt{2}} \approx 1.5749, \]
\[ c_1 = \frac{\sqrt{2b_1}}{2\alpha} \approx 101.7456, \ c_2 = c_3 = \frac{\sqrt{q_5 + 2b_2}}{\beta\sqrt{2}} \approx 6.1051, \]
and, therefore,
\[ B^TM_2 = \begin{pmatrix} -0.5599 & 0.6615 & 0 \\ -0.5599 & -0.6615 & 0 \\ 0.5599 & 0 & -0.6615 \\ -3.0524 & 2.5641 & 0 \\ 3.0524 & 0 & 2.5641 \\ 3.0524 & 0 & -2.5641 \end{pmatrix}, \]
\[ B^TM_3 = \begin{pmatrix} -0.5599 & 0.6615 & 0 \\ -0.5599 & -0.6615 & 0 \\ 0.5599 & 0 & -0.6615 \\ -3.0524 & 2.5641 & 0 \\ 3.0524 & 0 & 2.5641 \\ 3.0524 & 0 & -2.5641 \end{pmatrix}. \]

As the result we obtain the optimal control law of form
\[ u_1 = 111.80x_1 - 132.28x_2 + 43.16\dot{x}_1 - 36.26\dot{x}_2, \]
\[ u_2 = 111.80x_1 + 132.28x_2 + 43.16\dot{x}_1 + 36.26\dot{x}_2, \]
\[ u_3 = -111.80x_1 - 132.28x_3 - 43.16\dot{x}_1 - 36.26\dot{x}_3, \]
\[ u_4 = -111.80x_1 + 132.28x_3 - 43.16\dot{x}_1 + 36.26\dot{x}_3. \]

The corresponding differential equations are
\[ \ddot{x}_1 + 5.18\dot{x}_1 + 13.42x_1 = 0, \]
\[ \ddot{x}_2 + 30.46\dot{x}_2 + 111.12x_2 = 0, \]
\[ \ddot{x}_3 + 30.46\dot{x}_3 + 111.12x_3 = 0. \]

It is clear that the solutions of these equations decay sufficiently rapidly and it is not necessary to give any numerical calculations.

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