ATTAINABILITY OF THE FRACTIONAL HARDY CONSTANT
WITH NONLOCAL MIXED BOUNDARY
CONDITIONS: APPLICATIONS

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To Rafa de la Llave in his 60th birthday, with our best wishes.

Abstract. The first goal of this paper is to study necessary and sufficient
conditions to obtain the attainability of the fractional Hardy inequality
\[
Λ_N \equiv Λ_N(Ω) := \inf_{\phi \in E^s(Ω,D), \phi \neq 0} \frac{a_{d,s}}{2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|φ(x) - φ(y)|^2}{|x - y|^{d+2s}} \, dx \, dy \right) \int_{Ω} \frac{φ^2}{|x|^{2s}} \, dx,
\]
where Ω is a bounded domain of \( \mathbb{R}^d \), 0 < s < 1, D \( \subset \mathbb{R}^d \setminus Ω \) a nonempty open set, \( N = (\mathbb{R}^d \setminus Ω) \setminus D \) and

\[
E^s(Ω, D) = \left\{ u \in H^s(\mathbb{R}^d) : u = 0 \text{ in } D \right\}.
\]

The second aim of the paper is to study the mixed Dirichlet-Neumann boundary
problem associated to the minimization problem and related properties; pre-
cisely, to study semilinear elliptic problem for the fractional Laplacian, that is,

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\[ P_{\lambda} = \begin{cases} (-\Delta)^{s}u = \lambda \frac{u}{|x|^{2s}} + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathcal{B}_{s}u := u \chi_{\partial D} + \mathcal{N}_{s} u = 0 & \text{in } \mathbb{R}^{d} \setminus \Omega, \end{cases} \]

with \( N \) and \( D \) open sets in \( \mathbb{R}^d \) such that \( N \cap D = \emptyset \) and \( N \cup D = \mathbb{R}^d \setminus \Omega \), \( d > 2s \), \( \lambda > 0 \), and \( 1 < p \leq 2^*_s - 1 \), \( 2^*_s = \frac{2d}{d - 2s} \). We emphasize that the nonlinear term can be critical.

The operators \((-\Delta)^{s}\), fractional Laplacian, and \(\mathcal{N}_{s}\), nonlocal Neumann condition, are defined below in (7) and (8) respectively.

1. Introduction. The problems studied in this paper are motivated by some recent results that we summarize below.

In first place we consider the classical Hardy inequality proved in [23] (see also [9, 21, 31, 33]).

**Theorem.** (Fractional Hardy inequality). Assume that \( s \in (0, 1) \) is such that \( 2s < d \), then for all \( u \in C_{0}^{\infty}(\mathbb{R}^d) \), the following inequality holds,

\[ \frac{a_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, dx \, dy = \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}|^2 \, d\xi \geq \Lambda \int_{\mathbb{R}^d} |x|^{-2s} u^2 \, dx, \tag{1} \]

where \( \hat{u} \) is the Fourier transform of \( u \) and

\[ \Lambda = 2^{2s} \frac{\Gamma^2 \left( \frac{d+2s}{2} \right)}{\Gamma^2 \left( \frac{d-2s}{4} \right)}. \tag{2} \]

Moreover the constant \( \Lambda \) is optimal and is not attained.

Note that the constant \( a_{d,s} \) is defined by the first identity in (1) and is given by

\[ a_{d,s} = 2^{2s-1} \pi^{-\frac{d}{2}} \frac{\Gamma(d+2s)}{\Gamma(-s)}, \tag{3} \]

(see for instance, [12, 15, 21, 27]). The optimal constant defined in (2) coincides for every bounded domain \( \Omega \) containing the pole of the Hardy potential. More precisely, if \( 0 \in \Omega \), then for all \( u \in C_{0}^{\infty}(\Omega) \) we have

\[ \frac{a_{d,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, dx \, dy \geq \Lambda \int_{\Omega} \frac{u^2}{|x|^{2s}} \, dx, \tag{4} \]

where

\[ D_{\Omega} := (\mathbb{R}^d \times \mathbb{R}^d) \setminus ((\mathbb{R}^d \setminus \Omega) \times (\mathbb{R}^d \setminus \Omega)). \tag{5} \]

The optimality of \( \Lambda \) here follows by a scaling argument.

The other starting points for the problems considered in this work are some results obtained in the articles [17], [25] and [16].

In [17] the authors consider a natural Neumann condition in the sense that Gauss and Green integration by parts formulas hold for such condition. More precisely, if \( \Omega \) is a bounded open set in \( \mathbb{R}^d \) with suitable regularity, then the Neumann problem for the fractional Laplacian takes the form,

\[ \begin{cases} (-\Delta)^{s}u = f & \text{in } \Omega, \\ \mathcal{N}_{s}u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \tag{6} \]

where \((-\Delta)^{s}\) is the fractional Laplacian operator defined by

\[ (-\Delta)^{s}u(x) = a_{d,s} \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} \, dy, \tag{7} \]
$a_{d,s} > 0$ being the normalization constant defined in (3) and

$$N_s u(x) = a_{d,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy, \quad x \in \mathbb{R}^d \setminus \Omega.$$ (8)

See e.g. [24], [26], [15] and the references therein for more properties of this operator.

Notice that as a consequence of the analysis of the sequence of eigenvalues with Neumann condition done in [17], we reach that the best constant for the Hardy inequality with Neumann condition is 0 and it is attained by any constant function.

With this meaning for the Neumann condition, the authors in [25] studied the behavior of the eigenvalues for mixed Dirichlet-Neumann problems in terms of the boundary conditions. In particular, they proved a necessary and sufficient condition for the convergence of the first eigenvalue of mixed problems to 0, the principal eigenvalue for Neumann problem.

These previous results are the inspiration for our main goal in this paper: to study the attainability of the fractional Hardy constant with mixed boundary condition.

More precisely, let $\Omega \subset \mathbb{R}^d$ be a regular bounded domain containing the origin and consider $N$ and $D$ to be two open sets of $\mathbb{R}^d \setminus \Omega$ such that

$$N \cap D = \emptyset \quad \text{and} \quad N \cup D = \mathbb{R}^d \setminus \Omega.$$ A such pair $(D, N)$ will be called a Dirichlet-Neumann configuration, D-N configuration to be short.

Notice that in the whole paper and for simplicity of typing, we set

$$d\nu = \frac{dx \, dy}{|x - y|^{d+2s}}.$$ (9)

We define

$$\Lambda_N \equiv \Lambda_N(\Omega) = \inf_{\{\phi \in E^s(\Omega, D), \phi \neq 0\}} \frac{a_{d,s}}{2} \int_{\partial D} (\phi(x) - \phi(y))^2 d\nu \int_{\Omega} \frac{\phi^2}{|x|^{2s}} \, dx$$ (10)

where

$$E^s(\Omega, D) = \{u \in H^s(\mathbb{R}^d) : u = 0 \text{ in } D\}.$$ The above minimizing problem is strongly related to the next eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \Lambda_N \frac{u}{|x|^{2s}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B_s u = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$ (11)

The mixed boundary condition $B_s$ for the D-N configuration given by $D$ and $N$ open sets in $\mathbb{R}^d \setminus \Omega$ such that $N \cap D = \emptyset$ and $\overline{N} \cup \overline{D} = \mathbb{R}^d \setminus \Omega$, is defined by

$$B_s u = u \chi_D + N_s u \chi_N,$$ (12)

and $N_s$ is defined in (8). As customary, in (12), we denoted by $\chi_A$ the characteristic function of a set $A$.

In the local case $s = 1$, we can mention the works [3] and [4] where the authors have found some conditions of monotonicity that ensure the attainability or not of the Hardy constant. As a consequence they analyze a doubly-critical problem related to a mixed Sobolev constant. In [4] the authors deal with the same type of problem associated to elliptic operators in divergence form associated to the Caffarelli-Kohn-Nirenberg inequalities.
We will extend the previous results to the fractional Laplacian framework without any condition of monotonicity and then we get a stronger results than in the local case.

It is worthy to point-out that a kind of nonlocal mixed boundary conditions for the spectral fractional Laplacian was defined recently by several authors, see for example [13], where the authors used the Caffarelli-Silvestre extension to define a suitable nonlocal Neumann boundary condition. We refer also to [22] for other type of Neumann condition.

The main result in this work related to the attainability of the constant $\Lambda_N$ defined in (10), is the following characterization, that is deeply related to the results in [25].

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain with $0 \in \Omega$. Let $N, D$ be two open sets of $\mathbb{R}^d \setminus \Omega$ such that $N \cap D = \emptyset$ and $\overline{N} \cup \overline{D} = \mathbb{R}^d \setminus \Omega$, then $\Lambda_N < \Lambda$ if and only if $\Lambda_N$ is attained.

Notice that in the local case one of the main tools to analyze the compactness of the minimizing sequence is to use a suitable Concentration-Compactness argument that allows to avoid any concentration in $\Omega$ or at the boundary of $\Omega$. In the nonlocal setting we will consider an alternative approach. Other point that gives the difference between the local and the nonlocal case is the fact that in the nonlocal case, the set $N$ can be unbounded and little is known about the regularity of the solution in this set.

The second goal in this paper is the analysis of some semilinear problems, even with critical growth. There is a large literature about semilinear perturbations of the fractional Laplacian with Dirichlet boundary conditions. Notice that the subcritical concave-convex problem with mixed boundary conditions is studied in [2]. However the doubly critical problem has been only considered in the case of the whole Euclidean space, see [16] for details.

Consider the Sobolev constant, $S_N$, defined in Proposition 4, and the critical constant

$$S_\lambda = \inf_{\{u \in C^0_0(\mathbb{R}^d), u \not\equiv 0, \|u\|_{L^{2^*_s}(\mathbb{R}^d)} = 1\}} \frac{a_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^2 \, dv - \lambda \int_{\mathbb{R}^d} u^2(x) \frac{|x|^{2s}}{d} \, dx,$$

studied in [16], where it is analyzed the problem in the whole space $\mathbb{R}^d$.

The relevant result with respect to the solvability of the doubly critical problem with mixed boundary condition is the following Theorem which gives the condition of existence in terms of $S_N$ and $S_\lambda$. More precisely we state the result.

**Theorem 1.2.** Let $(D, N)$ a D-N configuration and assume that $\lambda \in (0, \Lambda_N)$. Define

$$T_{\lambda,N} = \inf_{\{u \in E^s(\Omega, D), u \not\equiv 0, \|u\|_{L^{2^*_s}(\Omega)} = 1\}} \frac{a_{d,s}}{2} \int_D \int_{\mathbb{R}^d} |u(x) - u(y)|^2 \, dv - \lambda \int_{\Omega} u^2(x) \frac{|x|^{2s}}{d} \, dx.$$

Then if $T_{\lambda,N} < \min\{S_\lambda, S_N\}$, the problem

$$\begin{cases}
(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^{2^*_s - 1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
B_s u = 0 & \text{in } \mathbb{R}^d \setminus \Omega,
\end{cases}
$$

has a nontrivial solution.
See also Theorem 4.4 in the last section of the paper.

The paper is organized as follows. In Section 2, we introduce some analytical tools needed to study the problem \((P_\lambda)\), such as the natural fractional Sobolev space associated to problem \((P_\lambda)\), some classical functional inequalities, and the adaptation of a Picone inequality type obtained in [26].

The Hardy constant for mixed problems is treated in Section 3, where we prove Theorem 1.1. The proof is more involved than in the local case and we prove some previous sharp estimates that we need to obtain the main result. In Subsection 3.1 we give sufficient condition to guarantee the attainability of \(\Lambda_N\); the non-attainability is analyzed in Subsection 3.2. In both cases we give explicit examples where these the attainability and the non-attainability are realized.

In the last section, Section 4, among others results, we prove Theorem 1.2, that is, we study the solvability of the doubly-critical problem.

2. Preliminaries and functional setting. We introduce in this section the natural functional framework for our problem and we give some properties and some embedding results needed when we deal with problem \((P_\lambda)\).

According to the definition of the fractional Laplacian, see [15], and the integration by parts formula, see [17], it is natural to introduce the following spaces. We denote by \(H^s(\mathbb{R}^d)\) the classical fractional Sobolev space,

\[
H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{d}{2} + s}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\},
\]

endowed with the norm

\[
\|u\|^2_{H^s(\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \frac{a_{d,s}}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |u(x) - u(y)|^2 \, d\nu,
\]

where \(d\nu\) is defined in (9).

It is clear that \(H^s(\mathbb{R}^d)\) is a Hilbert space.

We recall now the classical Sobolev inequality that is proved for instance in [15]. See also [28] for an elegant geometrical proof.

**Proposition 1.** Let \(s \in (0, 1)\) with \(d > 2s\). There exists a positive constant \(S = S(d, s)\) such that, for any function \(u \in H^s(\mathbb{R}^d)\), we have

\[
S\|u\|^2_{L^{2^*_s}(\mathbb{R}^d)} \leq \frac{a_{d,s}}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |u(x) - u(y)|^2 \, d\nu,
\]

where \(2^*_s = \frac{2d}{d - 2s}\).

Beside to the Hardy inequalities (1) and (4), in the case of bounded domain \(\Omega\), we have the next regional version of the Hardy inequality whose proof can be found in [1].

**Proposition 2.** Let \(\Omega\) be a bounded regular domain such that \(0 \in \Omega\), then there exists a constant \(C = C(\Omega, s, d) > 0\) such that for all \(u \in C_0^\infty(\Omega)\), we have

\[
C \int_{\Omega} \frac{|u(x)|^2}{|x|^s} \, dx \leq \int_{\Omega} \int_{\Omega} |u(x) - u(y)|^2 \, d\nu.
\]

Since we are considering a problem with mixed boundary condition we need to specify the space where the solutions belong.
Definition 2.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ and $D \subset \mathbb{R}^d \setminus \Omega$ an open set. For $0 < s < 1$, we define the space

$$E^s(\Omega, D) = \{ u \in H^s(\mathbb{R}^d) : u = 0 \text{ in } D \}$$

which is a Hilbert space endowed with the norm induced by $H^s(\mathbb{R}^d)$.

For $u \in E^s(\Omega, D)$, we set

$$\|u\|_2^2 = \frac{a_{d,s}}{2} \int \int_{D_\Omega} |u(x) - u(y)|^2 \, dv,$$

where $D_\Omega$ is defined in (5) and the measure $dv$ is defined in (9).

The properties of this norm are described by the following result. We refer to [17], [8] and [25] for the proof and other properties of this space.

Proposition 3. The norm $\| \cdot \|$ in $E^s(\Omega, D)$ is equivalent to the one induced by $H^s(\mathbb{R}^d)$, and then $(E^s(\Omega, D), (\cdot, \cdot))$ is a Hilbert space with scalar product given by

$$\langle u, v \rangle = \frac{a_{d,s}}{2} \int \int_{D_\Omega} (u(x) - u(y))(v(x) - v(y)) \, dv,$$

moreover there exists a positive constant $C(\Omega)$ such that the next Poincaré inequality holds:

$$C(\Omega) \int_{\Omega} u^2(x) \, dx \leq \int \int_{D_\Omega} (u(x) - u(y))^2 \, dv \text{ for all } u \in E^s(\Omega, D). \quad (18)$$

As a consequence of the definition of $E^s(\Omega, D)$ and using the extension result proved in [17], we get the next Sobolev inequality in the space $E^s(\Omega, D)$.

Proposition 4. Let $(D, N)$ be a D-N configuration. Suppose that $s \in (0, 1)$ and $d > 2s$. There exists a positive constant $S_N > 0$ such that, for all $u \in E^s(\Omega, D)$, we have

$$S_N \|u\|_{L^2(\Omega)}^2 \leq \|u\|^2.$$

The following result justifies the choice of the nonlocal Neumann boundary condition.

Proposition 5. Let $(D, N)$ be a D-N configuration and $s \in (0, 1)$ then for all $u, v \in E^s(\Omega, D)$ we have,

$$\int_{\Omega} v(-\Delta)^s u \, dx = \frac{a_{d,s}}{2} \int \int_{D_\Omega} (u(x) - u(y))(v(x) - v(y)) \, dv - \int_N vN_s u \, dx. \quad (19)$$

The proof of this result follows by the application of the integration by parts formula given in Lemma 3.3 of [17].

We give the definition of weak solution for the elliptic problem with mixed boundary condition.

Definition 2.2. Consider the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{B}_s(u) = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \quad (20)$$

where $f \in (E^s(\Omega, D))'$, the dual space of $E^s(\Omega, D)$. We say that $u$ is a weak solution to problem (20) if

$$\frac{a_{d,s}}{2} \int \int_{D_\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} \, dx \, dy = \langle f, v \rangle.$$
for all \( v \in \mathbb{E}^s(\Omega, D) \).

It is clear that the existence of \( u \) follows using the classical Lax-Milgram Theorem, see [25] for more details.

In order to obtain some a priori estimates, we will use the next Picone type inequality that is an extension of the corresponding inequality in \( H_0^1(\Omega) \) obtained in [26]. For the reader convenience we give the proof.

**Theorem 2.3.** Let \((D, N)\) be a D-N configuration. Consider \( u, v \in \mathbb{E}^s(\Omega, D)\) with \( u \geq 0 \) in \( \mathbb{R}^d \) and \( u > 0 \) in \( \Omega \cup N \). Assume that \((-\Delta)^s u \geq 0\) is a bounded Radon measure in \( \Omega \), then

\[
\int_N \frac{|v|^2}{u} \mathcal{N}_s u \, dx + \int_\Omega \frac{|v|^2}{u} (-\Delta)^s u \, dx \leq \frac{a_d s}{2} \int \int_{\mathcal{D}_\Omega} (v(x) - v(y))^2 \, d\nu. \tag{21}
\]

In particular, if we have equality in (21), then there exists a constant \( C \) such that \( v = Cu \) in \( \mathbb{R}^d \).

**Proof.** Notice that for \( u, v \) as in the hypotheses of the Theorem we have the following simple identity

\[
(v(x) - v(y))^2 - \left( \frac{v^2(x)}{u(x)} - \frac{v^2(y)}{u(y)} \right) (u(x) - u(y)) = \left( v(x) \left( \frac{u(y)}{u(x)} \right)^{\frac{1}{s}} - v(y) \left( \frac{u(x)}{u(y)} \right)^{\frac{1}{s}} \right)^2.
\]

Integrating the previous identity with respect to \( d\nu \), defined in (9), we conclude.

Finally, if we have the equality in (21), then we conclude that

\[
\int \int_{\mathcal{D}_\Omega} \left( v(x) \left( \frac{u(y)}{u(x)} \right)^{\frac{1}{s}} - v(y) \left( \frac{u(x)}{u(y)} \right)^{\frac{1}{s}} \right)^2 \, d\nu = 0
\]

Thus \( v(x) \left( \frac{u(y)}{u(x)} \right)^{\frac{1}{s}} = v(y) \left( \frac{u(x)}{u(y)} \right)^{\frac{1}{s}} \) for almost all \((x, y) \in \mathcal{D}_\Omega\). In particular, if \( v(y_0) \neq 0 \) for some \( y_0 \in \Omega \cup N \), then \( \frac{v(x)}{v(y_0)} = \frac{u(x)}{u(y_0)} \). Thus \( v(x) = \frac{v(y_0)}{u(y_0)} u(x) \) and the result follows. \( \square \)

3. **Analysis of the mixed Hardy optimal constant.** Consider \( \Omega \subset \mathbb{R}^d \) a bounded domain and \( D, N \subset \mathbb{R}^d \setminus \Omega \) a D-N configuration. In this section we will analyze the condition for the attainability of the mixed Hardy constant defined by

\[
\Lambda_N \equiv \Lambda_N(\Omega) = \inf_{\phi \in \mathbb{E}^s(\Omega, D), \phi \neq 0} \frac{a_d s}{2} \int \int_{\mathcal{D}_\Omega} |\phi(x) - \phi(y)|^2 \, d\nu \int_{\Omega} |x|^{2s} \, dx. \tag{22}
\]

We start by proving the following result.

**Theorem 3.1.** Assume that \( \Omega \subset \mathbb{R}^d \) is a smooth bounded domain with \( \emptyset \in \Omega \). Let \( N, D \) be two nonempty open sets of \( \mathbb{R}^d \setminus \Omega \) such that \( N \cap D = \emptyset \) and \( \overline{N} \cup \overline{D} = \mathbb{R}^d \setminus \Omega \), then

\[
0 < \Lambda_N \leq \Lambda,
\]

where \( \Lambda \) is defined in (2).
Proof. We begin by proving the positivity of \( \Lambda_N \). Let \( u \in \mathbb{E}^s(\Omega,D) \) and fix \( \delta > 0 \) such that \( B_{2\delta}(0) \subset \Omega \). Consider \( \varphi \in C_0^\infty(\Omega) \) be such that \( 0 \leq \varphi \leq 1, \varphi \equiv 1 \) in \( B_\delta(0) \) and \( \varphi = 0 \) in \( \Omega \setminus B_{2\delta}(0) \). In what follows we denote by \( C \) or \( C(\Omega) \) any positive constant that depends on \( \Omega, d, s \), that is independent of \( u \) and that can change from line to line.

It is clear that \( u = \varphi u + (1 - \varphi)u \), thus
\[
\int_\Omega \frac{u^2}{|x|^{2s}} \, dx = \int_\Omega \frac{(u \varphi)^2}{|x|^{2s}} \, dx + \int_\Omega \frac{u^2(1 - \varphi)^2}{|x|^{2s}} \, dx + 2 \int_\Omega \frac{u^2 \varphi(1 - \varphi)}{|x|^{2s}} \, dx. \tag{23}
\]

Since \( 1 - \varphi = 0 \) in \( B_\delta(0) \), then using the Poincaré inequality in (17), we conclude that
\[
\int_\Omega \frac{u^2(1 - \varphi)^2}{|x|^{2s}} \, dx + 2 \int_\Omega \frac{u^2 \varphi(1 - \varphi)}{|x|^{2s}} \, dx \leq C(\Omega) \int_\mathcal{D}_0 (u(x) - u(y))^2 \, d\nu. \tag{24}
\]

We deal now with the term \( \int_\Omega \frac{(u \varphi)^2}{|x|^{2s}} \, dx \). Since \( u \varphi \in H_0^s(\Omega) \), then by the Hardy inequality (4), we obtain that
\[
\int_\Omega \frac{(u \varphi)^2}{|x|^{2s}} \, dx \leq C(\Omega) \int_\Omega \int_\Omega \left( (u \varphi)(x) - (u \varphi)(y) \right)^2 \, d\nu. \tag{25}
\]

The immediate algebraic identity
\[
\left( (u(x) \varphi(x) - u(y) \varphi(y)) \right)^2 = \left( u(x) - u(y) \right)^2 \varphi^2(x) + u^2(y) \left( \varphi(x) - \varphi(y) \right)^2
+ 2u(y) \varphi(x) \left( u(x) - u(y) \right) \left( \varphi(x) - \varphi(y) \right),
\]
implies that
\[
\int_\Omega \int_\Omega \left( (u(x) \varphi(x) - u(y) \varphi(y)) \right)^2 \, d\nu =
\int_\Omega \int_\Omega ((u(x) - u(y))^2 \varphi^2(x) \, d\nu + \int_\Omega \int_\Omega u^2(y) \left( \varphi(x) - \varphi(y) \right)^2 \, d\nu
+ 2 \int_\Omega \int_\Omega \left( u(y) \varphi(x)(u(x) - u(y)) \left( \varphi(x) - \varphi(y) \right) \, d\nu
= J_1 + J_2 + 2J_3.
\]

In first place, it is clear that
\[
J_1 \leq C(\Omega) \int_\mathcal{D}_0 (u(x) - u(y))^2 \, d\nu.
\]

Respect to \( J_2 \), since \( \Omega \) is a bounded domain, it holds that
\[
J_2 \leq C(\Omega) \int_\Omega \int_\Omega \frac{u^2(y)}{|x - y|^{d+2s-2}} \, dxdy
\leq C(\Omega) \int_\Omega u^2(y) \, dy \int_{|\xi| \leq R} \frac{1}{|\xi|^{d+2s-2}} \, d\xi.
\]
Since \( \int_{|\xi| \leq c} \frac{1}{|\xi|^{d+2s-2}} \, d\xi < \infty \), using the Poincaré inequality in Proposition 3, we get

\[
J_2 \leq C(\Omega) \int_\Omega \int_\Omega u^2(y) \, dy \leq C(\Omega) \int \int_{\mathcal{D}_\Omega} (u(x) - u(y))^2 \, d\nu.
\]

By using Young inequality, we reach that

\[
|J_3| \leq C_1 J_1 + C_2 J_2 \leq C(\Omega) \int \int_{\mathcal{D}_\Omega} (u(x) - u(y))^2 \, d\nu.
\]

Therefore, combining the above inequalities, we conclude that

\[
\int_\Omega \int_\Omega (u(x)\varphi(x) - u(y)\varphi(y))^2 \, d\nu \leq C(\Omega) \int \int_{\mathcal{D}_\Omega} (u(x) - u(y))^2 \, d\nu.
\]

Going back to (23), and by using the estimates (24), (25), we conclude that

\[
\int \int_{\mathcal{D}_\Omega} |w_n(x) - w_n(y)|^2 \, d\nu \rightarrow \Lambda_N.
\]

Hence \( \Lambda_N > 0 \) and then the first affirmation follows.

Finally, since \( H_0^s(\Omega) \subset E^s(\Omega, D) \), by definition it follows that \( \Lambda_N \leq \Lambda \).

The main result in this section is Theorem 1.1. We split the proof into two parts contained in the following two subsections.

3.1. **If \( \Lambda_N < \Lambda \), then \( \Lambda_N \) is attained.**

**Proposition 6.** In the hypotheses of Theorem 1.1, assume that \( \Lambda_N < \Lambda \). Then, \( \Lambda_N \) is attained.

**Proof.** Let \( \{u_n\}_n \subset E^s(\Omega, D) \) be a minimizing sequence for \( \Lambda_N \) defined in (22) with

\[
\int_\Omega \frac{u_n^2}{|x|^{2s}} \, dx = 1,
\]

then \( \{u_n\}_n \) is bounded in \( E^s(\Omega, D) \), and

\[
\frac{\alpha_{d,s}}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |u_n(x) - u_n(y)|^2 \, d\nu \rightarrow \Lambda_N.
\]

Without loss of generality we can assume that \( u_n \geq 0 \) for all \( n \). Hence we get the existence of \( \bar{u} \in E^s(\Omega, D) \) such that \( u_n \rightharpoonup \bar{u} \) weakly in \( E^s(\Omega, D) \), and up to a subsequence, \( u_n \rightarrow \bar{u} \) strongly in \( L^\sigma(\Omega) \) for all \( \sigma < 2^*_s \) and \( u_n \rightarrow \bar{u} \) a.e. in \( \Omega \).

We claim that \( \bar{u} \neq 0 \). We argue by contradiction. Assume that \( \bar{u} = 0 \) and let \( R > 0 \) be such that \( B_{4R}(0) \subset \Omega \). We consider \( \varphi \in C_0^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in \( B_R(0) \) and \( \varphi = 0 \) in \( \mathbb{R}^d \setminus B_{2R}(0) \). Define \( w_n = \varphi u_n \), then \( w_n \in H_0^s(\Omega) \) and

\[
\Lambda \leq \frac{\alpha_{d,s}}{2} \int \int_{\mathcal{D}_\Omega} |w_n(x) - w_n(y)|^2 \, d\nu \int \frac{w_n^2}{|x|^{2s}} \, dx.
\]
Notice that
\[
\int_{\Omega} \frac{u_n^2 \varphi^2}{|x|^{2s}} \, dx = \int_{\Omega} \frac{u_n^2 \varphi^2}{|x|^{2s}} \, dx + \int_{\Omega} \frac{u_n^2 (\varphi^2 - 1)}{|x|^{2s}} \, dx
\]
\[
= \int_{\Omega} \frac{u_n^2}{|x|^{2s}} \, dx + \int_{\Omega \setminus B_{R}(0)} \frac{u_n^2 (\varphi^2 - 1)}{|x|^{2s}} \, dx + \int_{\Omega \setminus B_{R}(0)} \frac{u_n^2 (\varphi^2 - 1)}{|x|^{2s}} \, dx
\]
\[
= 1 + \int_{\Omega \setminus B_{R}(0)} \frac{u_n^2 (\varphi^2 - 1)}{|x|^{2s}} \, dx
\]
\[
= 1 + o(1) \rightarrow 1 \text{ as } R \rightarrow \infty.
\]
We have
\[
\int \int_{\mathcal{D}_n} |w_n(x) - w_n(y)|^2 \, d\nu = \int \int_{\mathcal{D}_n} |u_n(x)\varphi(x) - u_n(y)\varphi(y)|^2 \, d\nu.
\]
Since
\[
(u_n(x)\varphi(x) - u_n(y)\varphi(y))^2 = ((u_n(x) - u_n(y))\varphi(x) + u_n(y)(\varphi(x) - \varphi(y)))^2
\]
\[
= (u_n(x) - u_n(y))^2 \varphi^2(x) + u_n^2(y)(\varphi(x) - \varphi(y))^2 + 2u_n(y)\varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)),
\]
it holds that
\[
\int \int_{\mathcal{D}_n} |w_n(x) - w_n(y)|^2 \, d\nu = \int \int_{\mathcal{D}_n} |u_n(x)\varphi(x) - u_n(y)\varphi(y)|^2 \, d\nu
\]
\[
= \int \int_{\mathcal{D}_n} (u_n(x) - u_n(y))^2 \varphi^2(x) \, d\nu + \int \int_{\mathcal{D}_n} u_n^2(y)(\varphi(x) - \varphi(y))^2 \, d\nu
\]
\[
+ 2\int \int_{\mathcal{D}_n} u_n(y)\varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu
\]
\[
= I_1(n) + I_2(n) + 2I_3(n).
\]
Let us begin by estimating the term $I_2(n)$. Recall that $d\nu = \frac{dx \, dy}{|x - y|^{d+2s}}$, then we have
\[
I_2(n) = \int \int_{\mathcal{D}_n} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu
\]
\[
= \int_{\Omega} \int_{\Omega} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu + \int_{\Omega} \int_{\Omega^c} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu
\]
\[
+ \int_{\Omega^c} \int_{\Omega} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu
\]
\[
= I_2^1(n) + I_2^2(n) + I_2^3(n).
\]
Taking into account that for all $(x, y) \in \Omega \times \Omega$, $(\varphi(x) - \varphi(y))^2 \leq C(\Omega)|x - y|^2$, we reach that
\[
I_2^1(n) \leq C(\Omega) \int_{\Omega} \int \frac{u_n^2(y)}{|x - y|^{d+2s-2}} \, dx \, dy \leq C(\Omega) \int_{\Omega} u_n^2(y) \, dy \int_{\xi \leq \varepsilon} \frac{1}{|\xi|^{d+2s-2}} \, d\xi.
\]
Since
\[
\int_{|\xi| \leq \varepsilon} \frac{1}{|\xi|^{d+2s-2}} \, d\xi < \infty,
\]
then $I_2^1(n) = o(1)$. 

Next we proceed to estimate $I_2^3(n)$. We have

$$I_2^3(n) = \int_{\Omega} \int_{\Omega} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu = \int_{\Omega} \int_{\Omega} \varphi^2(y) u_n^2(y) \, d\nu$$

$$= \int_{\Omega} \varphi^2(y) u_n^2(y) \int_{\Omega^c} \frac{1}{|x-y|^{d+2s}} \, dx \, dy$$

$$= \int_{|y| \leq 2R} \varphi^2(y) u_n^2(y) \int_{|x| \geq 4R} \frac{1}{|x|^{d+2s}} \, dx \, dy = o(1).$$

Finally, we consider the term $I_3(n)$.

$$I_3(n) = \int \int_{D_n} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$= \int \int_{\Omega} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$+ \int \int_{\Omega^c} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$+ \int \int_{\Omega^c} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$= I_3^1(n) + I_3^2(n) + I_3^3(n).$$

We start by the first term. Using Hölder inequality, we get

$$I_3^1(n) = \int \int_{\Omega} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$\leq \left( \int \int_{\Omega} (u_n(x) - u_n(y))^2 \varphi^2(x) \, d\nu \right)^{\frac{1}{2}} \left( \int \int_{\Omega} (\varphi(y) - \varphi(x))^2 u_n^2(y) \, d\nu \right)^{\frac{1}{2}}$$

$$\leq C I_1^2(n) = o(1).$$

Now, since $\varphi(x) = 0$ if $x \in \Omega^c$, then $I_3^2(n) = 0$.

Combining the above estimates, we conclude that

$$\int \int_{D_n} |w_n(x) - w_n(y)|^2 \, d\nu = \int \int_{D_n} (u_n(x) - u_n(y))^2 \varphi^2(x) \, d\nu + I_2^3(n) + 2I_3^3(n) + o(1).$$

(27)

Notice that

$$I_2^3(n) + 2I_3^3(n) = \int \int_{\Omega} (\varphi(x) - \varphi(y))^2 u_n^2(y) \, d\nu$$

$$+ 2 \int \int_{\Omega} u_n(y) \varphi(x)(u_n(x) - u_n(y))(\varphi(x) - \varphi(y)) \, d\nu$$

$$= \int \int_{\Omega} \varphi^2(x) u_n^2(y) \, d\nu + 2 \int \int_{\Omega} u_n(y) \varphi^2(x)(u_n(x) - u_n(y)) \, d\nu$$

$$= \int \int_{\Omega} \varphi^2(x) u_n^2(y) \, d\nu + 2 \int \int_{\Omega} u_n(y) u_n(x) \varphi^2(x) \, d\nu$$

$$- 2 \int \int_{\Omega} u_n^2(y) \varphi^2(x) \, d\nu$$

$$= - \int \int_{\Omega} \varphi^2(x) u_n^2(y) \, d\nu + 2 \int \int_{\Omega} u_n(y) u_n(x) \varphi^2(x) \, d\nu.$$
Using now Young inequality, we get that
\begin{align*}
I^2_3(n) + 2I^3_3(n) &\leq -\int_N \int_{\Omega} \varphi^2(x)u^2_n(y) \, d\nu + \varepsilon \int_N \int_{\Omega} u^2_n(y)\varphi^2(x) \, d\nu \\
&\quad + C\varepsilon \int_N \int_{\Omega} u^2_n(x)\varphi^2(x) \, d\nu
\end{align*}
Choosing \( \varepsilon \) small enough, we obtain that
\begin{align*}
I^2_3(n) + 2I^3_3(n) &\leq \varepsilon \int_{\Omega} \int_{\Omega} |\varphi^2(x)u^2_n(y)| \, d\nu \\
&\quad + C\varepsilon \int_{\Omega} \int_{\Omega} u^2_n(x)|\varphi^2(x)| \, d\nu.
\end{align*}
Therefore, from (27), it holds that
\begin{align*}
\int \int_{\Omega} |w_n(x) - w_n(y)|^2 \, d\nu &\leq \int \int_{\Omega} (u_n(x) - u_n(y))^2 \varphi^2(x) \, d\nu + o(1). \quad (28)
\end{align*}
Going back to (26), we conclude that
\begin{align*}
\Lambda_N &< \Lambda_N \quad \text{in } \Omega, \\
\bar{u} &\geq 0 \quad \text{in } \Omega,
\end{align*}
which is a contradiction with the hypothesis \( \Lambda_N < \Lambda \). Hence \( \bar{u} \neq 0 \) and then the claim follows.
To show that \( \Lambda_N \) is achieved we will use the Ekeland variational principle, see [18]. Then up to a subsequence, it holds that
\begin{align*}
\frac{d_{d,s}}{2} \int \int_{\Omega} |w_n(x) - w_n(y)|^2 \, d\nu &\leq \int \int_{\Omega} (u_n(x) - u_n(y))^2 |\varphi^2(x)| \, d\nu \\
&\quad + o(1)
\end{align*}
which is a contradiction with the hypothesis \( \Lambda_N < \Lambda \). Hence \( \bar{u} \neq 0 \) and then the claim follows.
To show that \( \Lambda_N \) is achieved we will use the Ekeland variational principle, see [18]. Then up to a subsequence, it holds that
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s u_n = \Lambda_N \frac{u_n}{|x|^{2s}} + o(1) \quad \text{in } \Omega, \\
B_s u_n = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega,
\end{array} \right.
\end{align*}
Let \( \varphi \in E^s(\Omega, D) \), by duality argument we obtain that
\begin{align*}
\int \Omega (-\Delta)^s u_n \varphi &\to \int \Omega (-\Delta)^s \bar{u} \varphi \quad \text{and} \\
\int \Omega \frac{u_n \varphi}{|x|^{2s}} &\to \int \Omega \frac{\bar{u} \varphi}{|x|^{2s}} \quad \text{as } n \to \infty.
\end{align*}
Thus \( \bar{u} \) solves the problem
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s \bar{u} = \Lambda_N \frac{\bar{u}}{|x|^{2s}} \quad \text{in } \Omega, \\
\bar{u} &\in E^s(\Omega, D), \\
\bar{u} &\geq 0 \quad \text{in } \Omega, \\
B_s \bar{u} = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega.
\end{array} \right.
\end{align*}
Choosing \( \bar{u} \) as a test function in (30), we obtain

\[
\Lambda_N = \frac{a_{d,s}}{2} \int_{D_\Omega} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x|^2} \, d\nu \int_{\Omega} \frac{\bar{u}^2}{|x|^{2s}} \, dx,
\]

and the result follows.

3.1.1. Properties of the spectral value \( \Lambda_N \) if \( \Lambda_N < \Lambda \). In this subsection we treat the case \( \Lambda_N < \Lambda \), thus \( \Lambda_N \) is achieved and we prove that it behaves like a principal eigenvalue of the mixed elliptic problem with the Hardy weight.

To start, we begin by giving some configurations of \((D, N)\) for which the constant \( \Lambda_N \) is reached. By the previous results it suffices to prove that \( \Lambda_N < \Lambda \). This last inequality is a straightforward consequence of some results contained in [25]. For the reader convenience we explain below some details.

We say that \( \Omega \) is an admissible domain if it is a \( C^{1,1} \) and it satisfies the exterior sphere condition. Now, let consider sequences of sets \( \{D_k\}_{k \in \mathbb{N}}, \{N_k\}_{k \in \mathbb{N}} \) such that \( N_k \cap D_k = \emptyset \) and \( N_k \cup D_k = \mathbb{R}^d \setminus \Omega \). Following closely the same argument as in [25], we obtain the next result.

**Theorem 3.2.** Let \( \Omega \) be an admissible domain and \( 0 < s < \frac{1}{2} \). Suppose that for all \( R > 0 \) we have \( \lim_{k \to \infty} |D_k \cap B_R| = 0 \), then \( \lim_{k \to \infty} \Lambda_{N_k} = 0 \) and as a consequence there exists \( k_0 \in \mathbb{N} \) such that \( \Lambda_{N_k} \) is attained for all \( k \geq k_0 \).

**Proof.** For \( \rho > 0 \) fixed, we define

\[
\lambda_{\rho,k} = \inf_{\{u \in \mathcal{E}_{s}(\Omega, D), u \neq 0\}} \frac{a_{d,s}}{2} \int_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x|^2} \, d\nu \int_{\Omega} \frac{u^2(x)}{|x|^{2s} + \rho} \, dx,
\]

it is clear that \( \Lambda_{N_k} \leq \lambda_{\rho,k} \) for all \( \rho > 0 \). From [25] we know that \( \lim_{k \to \infty} \lambda_{\rho,k} = 0 \), thus we conclude.

In the case \( \frac{1}{2} \leq s < 1 \), we have the following result.

**Theorem 3.3.** Assume that \( s \in (\frac{1}{2}, 1) \) and that the hypotheses of Theorem 3.2 hold. Assume in addition that for some \( \delta > 0 \), we have \( \text{dist}(D_k, \Omega) > \delta, \forall k \geq k_0 \), then \( \lim_{k \to \infty} \Lambda_{N_k} = 0 \).

Let us denote by \( \bar{u} \in \mathcal{E}_{s}(\Omega, D) \), the function that realizes the minimum in (22) with \( \int_{\Omega} \frac{\bar{u}^2}{|x|^{2s}} \, dx = 1 \), then \( \bar{u} \) solves the eigenvalue problem

\[
\begin{align*}
(-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} & \text{in } \Omega, \\
u &\in \mathcal{E}_{s}(\Omega, D), \\
u &> 0 & \text{in } \Omega, \\
\mathcal{B}_s u &= 0 & \text{in } \mathbb{R}^d \setminus \Omega.
\end{align*}
\]

with \( \lambda = \Lambda_N \).

In the next result we show the relevant spectral properties of \( \Lambda_N \) when it is reached.
Theorem 3.4. Assume that \( \Lambda_N < \Lambda \), then \( \Lambda_N \) is an isolated and simple eigenvalue, that is:

1. If \( v \) is an other solution to problem (31) with \( \lambda = \Lambda_N \), then \( v = C \hat{u} \), \( C \in \mathbb{R} \).
2. There exists \( \varepsilon > 0 \) such that for all \( \lambda \in (\Lambda_N, \Lambda_N + \varepsilon) \), problem (31) has only the trivial solution.

Proof. Let us begin by proving the first point. Suppose that \( v \) is another solution to problem (31) with \( \lambda = \Lambda_N \). If \( v \geq 0 \), then using the Picone alternative in Theorem 2.3, we conclude that \( v = C \hat{u} \) for some \( C \geq 0 \).

Now, suppose that \( v \) change sign. Since \( v \) is a weak solution in the sense of Definition 2.2 to the problem (31), in particular, \( v \in \mathcal{E}(\Omega, D) \) and then \( v_\pm \in \mathcal{E}(\Omega, D) \). Thus, by using \( v_+ \) (respectively \( v_- \)) as a test function in the equation of \( v \), we obtain that

\[
\Lambda_N \int_\Omega \frac{v(x)v_\pm(x)}{|x|^{2s}} \, dx = \frac{a_{d,s}}{2} \int_{D_0} \int_{\mathcal{D}_0} (v(x) - v(y))(v_\pm(x) - v_\pm(y)) \, d\nu.
\]

Notice that

\[
(v_+(x) - v_+(y))(v_-(x) - v_-(y)) \leq 0 \quad \text{a.e. in } \mathbb{R}^d,
\]

then

\[
(v(x) - v(y))(v_+(x) - v_+(y)) \leq (v_\pm(x) - v_\pm(y))^2 \quad \text{a.e. in } \mathbb{R}^d,
\]

and

\[
(v(x) - v(y))(v_-(x) - v_-(y)) \geq -(v_-(x) - v_-(y))^2 \quad \text{a.e. in } \mathbb{R}^d.
\]

Therefore, in any case, we conclude that

\[
\Lambda_N \int_\Omega \frac{v_\pm^2(x)}{|x|^{2s}} \, dx \leq \frac{a_{d,s}}{2} \int_{D_0} \int_{\mathcal{D}_0} (v_\pm(x) - v_\pm(y))^2 \, d\nu.
\]

Hence \( v \) realizes the minimum in (22) and then there are nonnegative solutions to problem (31) with \( \lambda = \Lambda_N \). Hence there exists \( C_\pm \geq 0 \) such that \( v_\pm = C_\pm \hat{u} \), thus \( v = v_+ - v_- = (C_+ - C_-)\hat{u} \).

We next prove (2), that is, the eigenvalue \( \Lambda_N \) is isolated. Assume the existence of a sequence \( \{ (\lambda_n, u_n) \} \subset (\Lambda_N, \infty) \times \mathcal{E}(\Omega, D) \) such that \( \lambda_n \downarrow \Lambda_N \) and \( u_n \) solves the problem (31) with \( \lambda = \lambda_n \). Without loss of generality we can assume that

\[
\int_\Omega \frac{u_n^2(x)}{|x|^{2s}} \, dx = 1.
\]

Since \( \lambda_n \downarrow \Lambda_N \) and \( \Lambda_N < \Lambda \), then there exists \( \Lambda_0 \in (\Lambda_N, \Lambda) \) such that, without loss of generality, we have \( \Lambda_N < \lambda_n < \Lambda_0 < \Lambda \) for all \( n \). Hence

\[
\frac{a_{d,s}}{2} \int_{D_0} \int_{\mathcal{D}_0} (u_n(x) - u_n(y))^2 \, d\nu = \lambda_n \leq \Lambda.
\]

That is, \( \{ u_n \} \) is bounded in \( \mathcal{E}(\Omega, D) \) and then there exists \( \hat{u} \in \mathcal{E}(\Omega, D) \) such that \( u_n \rightharpoonup \hat{u} \) weakly in \( \mathcal{E}(\Omega, D) \), \( u_n \rightarrow \hat{u} \) strongly in \( L^\sigma(\Omega) \) for all \( \sigma < 2s \) and a.e. in \( \Omega \cup D \). It is not difficult to show that \( \hat{u} \) solves problem (31) with \( \lambda = \Lambda_N \). Therefore, using the previous step there exists a constant \( \hat{C} \) such that \( \hat{u} = \hat{C} u \).

Now, taking \( \hat{u} \) as a test function in the equation of \( u_n \) it holds that

\[
\int_\Omega \frac{u_n \hat{u}}{|x|^{2s}} \, dx = 0 \quad \text{for all } n,
\]

hence

\[
\int_\Omega \frac{\hat{u} \hat{u}}{|x|^{2s}} \, dx = 0.
\]
Since \( \hat{u} = \hat{C} \tilde{u} \), it holds that \( \hat{C} = 0 \) and then \( \hat{u} = 0 \).

Going back to the equation of \( u_n \) and by Kato inequality it holds that

\[
(-\Delta)^s |u_n| \leq \lambda_n \frac{|u_n|}{|x|^{2s}} \quad \text{in} \quad \Omega.
\]

Since \( \lambda_n \leq \Lambda_0 \), then by Lemma 3.1 in [6], we conclude that

\[
|u_n(x)| \leq C|x|^{-\alpha_0} \quad \text{in} \quad B_{\eta}(0) \subset \subset \Omega, \quad \text{for all} \quad n,
\]

where \( \alpha_0 > 0 \) and \( \Lambda_0 \) are related with the following identity

\[
\Lambda_0 = \Lambda_0(\alpha_0) = \frac{22s \Gamma\left(\frac{d+2s+2\alpha_0}{4}\right) \Gamma\left(\frac{d+2s-2\alpha_0}{4}\right)}{\Gamma\left(\frac{d-2s+2\alpha_0}{4}\right) \Gamma\left(\frac{d-2s-2\alpha_0}{4}\right)}
\]

and, since \( \Lambda_0 < \Lambda \), then \( \alpha_0 < \frac{d-2s}{2} \), thus

\[
\frac{|u_n|^2}{|x|^{2s}} \leq C|x|^{-2\alpha_0-2s} \in L^1(B_{\eta}(0)).
\]

By the Dominated Convergence Theorem we reach that

\[
1 = \int_{\Omega} \frac{u_n^2}{|x|^{2s}} \, dx \to \int_{\Omega} \frac{\hat{u}^2}{|x|^{2s}} \, dx \quad \text{as} \quad n \to \infty,
\]

a contradiction with the fact that \( \hat{u} = 0 \), hence we conclude.

\[\square\]

3.2. If \( \Lambda_N = \Lambda \), then \( \Lambda_N \) is not attained. We prove the following result that complete the proof of Theorem 1.1.

**Proposition 7.** In the hypotheses of Theorem 1.1 assume that \( \Lambda_N = \Lambda \), then \( \Lambda_N \) is not attained.

**Proof.** We argue by contradiction. Assume that \( \Lambda_N \) is attained in \( \mathbb{E}^s(\Omega, D) \), then there exists \( u \in \mathbb{E}^s(\Omega, D) \) such that

\[
\begin{cases}
(-\Delta)^s u = \Lambda_N \frac{u}{|x|^{2s}} & \text{in} \ \Omega, \\
u \in \mathbb{E}^s(\Omega, D), & \text{in} \ \Omega, \\
u > 0 & \text{in} \ \Omega, \\
B_s u = 0 & \text{in} \ \mathbb{R}^d \setminus \Omega.
\end{cases}
\]

Let \( B_r(0) \subset \subset \Omega \) and define \( v \) to be the unique solution of the problem

\[
\begin{cases}
(-\Delta)^s v = 0 & \text{in} \ \Omega, \\
v = v_0 & \text{in} \ \mathbb{R}^d \setminus B_r(0),
\end{cases}
\]

where

\[
v_0(x) = \begin{cases}
u(x) & \text{if} \ x \in \Omega \setminus B_r(0), \\
0 & \text{if} \ x \in \mathbb{R}^d \setminus \Omega,
\end{cases}
\]

it is clear that \( u \geq v \). Setting \( w = u - v \) then \( w \geq 0 \) in \( \mathbb{R}^d \), \( w \in H^s(\Omega) \) and it solves

\[
\begin{cases}
(-\Delta)^s w = \Lambda \frac{u}{|x|^{2s}} = \Lambda \frac{w}{|x|^{2s}} + \Lambda \frac{v}{|x|^{2s}} & \text{in} \ \Omega, \\
w \geq 0 & \text{in} \ \mathbb{R}^d \setminus \Omega.
\end{cases}
\]

From Lemma 3.9 in [5], we obtain that we know that \( w(x) \geq C_1 |x|^{-\frac{d+2s}{2}} \) if \( x \in B_{r_0}(0) \subset B_r(0) \), hence

\[
\infty = C_1 \int_{B_{r_0}(0)} |x|^{-d} \, dx = C_1 \int_{B_{r_0}(0)} |x|^{-2^\ast \frac{d-2s}{2}} \, dx \leq \int_{B_{r_0}(0)} w^{2^\ast} \, dx
\]

which is a contradiction with the fact that \( w \in H^s(\Omega) \). Hence the result follows. \[\square\]
3.2.1. Examples for which we find $\Lambda_N = \Lambda$. In this subsection we give some geometrical condition to ensure that $\Lambda_N = \Lambda$. We have the next result.

**Theorem 3.5.** Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain such that $0 \in \Omega$. Let $w(x) = |x|^{-\frac{4}{d-2}}$ and suppose that $N_\omega w(x) \geq 0$ for all $x \in N$, then $\Lambda_N = \Lambda$. Moreover, the problem

$$
\begin{align*}
(-\Delta)^s u &= \frac{\Lambda_N}{|x|^{2s}} \quad \text{in } \Omega, \\
u &\in \mathbb{E}^s(\Omega, D), \\
u > 0 \quad \text{in } \Omega, \\
B_s u &= 0 \quad \text{in } \mathbb{R}^d \setminus \Omega,
\end{align*}
$$

(36)

has no solution.

**Proof.** We argue by contradiction. Assume that $N_\omega w(x) \geq 0$ for all $x \in N$ and that $\Lambda_N < \Lambda$. Then by Theorem 1.1, we get the existence of $u_1 \in \mathbb{E}^s(\Omega, D)$ a positive solution to problem (36). Setting $v_1(x) = \frac{u_1(x)}{w(x)}$, then

$$
|v_1(x) - v_1(y)|^2 w(x)w(y) = \\
|u_1(x) - u_1(y)|^2 + u_1^2(x)\left(\frac{w(y)}{w(x)} - 1\right) + u_1^2(y)\left(\frac{w(x)}{w(y)} - 1\right).
$$

Thus

$$
\frac{a_{d,s}}{2} \int_{D_\Omega} |v_1(x) - v_1(y)|^2 w(x)w(y) d\nu = \frac{a_{d,s}}{2} \int_{D_\Omega} |u_1(x) - u_1(y)|^2 d\nu \\
+ \frac{a_{d,s}}{2} \int_{D_\Omega} u_1^2(x)\left(\frac{w(y) - w(x)}{w(x)}\right) d\nu + \frac{a_{d,s}}{2} \int_{D_\Omega} u_1^2(y)\left(\frac{w(x) - w(y)}{w(y)}\right) d\nu.
$$

According to the symmetry of last two terms of the above identity, we obtain that

$$
\frac{a_{d,s}}{2} \int_{D_\Omega} |v_1(x) - v_1(y)|^2 w(x)w(y) d\nu = \frac{a_{d,s}}{2} \int_{D_\Omega} |u_1(x) - u_1(y)|^2 d\nu \\
- \frac{a_{d,s}}{2} \int_{D_\Omega} \left(\frac{u_1^2(x)}{w(x)} - \frac{u_1^2(y)}{w(y)}\right) (w(y) - w(x)) d\nu.
$$

Notice that

$$
\frac{a_{d,s}}{2} \int_{D_\Omega} \left(\frac{u_1^2(x)}{w(x)} - \frac{u_1^2(y)}{w(y)}\right) (w(y) - w(x)) d\nu = \\
\int_{\Omega} \frac{u_1^2(x)}{w(x)} (-\Delta)^s w dx + \int_{N} \frac{u_1^2(x)}{w(x)} N_\omega w(x) dx.
$$

Therefore,

$$
\frac{a_{d,s}}{2} \int_{D_\Omega} \left(\frac{u_1^2(x)}{w(x)} - \frac{u_1^2(y)}{w(y)}\right) (w(x) - w(y)) d\nu = \\
\Lambda \int_{\Omega} \frac{u_1^2(x)}{|x|^{2s}} dx + \int_{N} \frac{u_1^2(x)}{w(x)} N_\omega w(x) dx.
$$

Hence, it holds that

$$
\frac{a_{d,s}}{2} \int_{D_\Omega} |v_1(x) - v_1(y)|^2 w(x)w(y) d\nu + \int_{N} \frac{u_1^2(x)}{w(x)} N_\omega w(x) dx \\
= \frac{a_{d,s}}{2} \int_{D_\Omega} |u_1(x) - u_1(y)|^2 d\nu - \Lambda \int_{\Omega} \frac{u_1^2(x)}{|x|^{2s}} dx.
$$
Since
\[
\frac{a_{d,s}}{2} \int_{\Omega_1} |u_1(x) - u_1(y)|^2 \, d\nu = \Lambda_N \int_{\Omega} \frac{u_1^2(x)}{|x|^{2s}} \, dx,
\]
we deduce that
\[
\frac{a_{d,s}}{2} \int_{\Omega_1} |v_1(x) - v_1(y)|^2 \, w(x)w(y) \, d\nu + \int_{\Omega} \frac{u_1^2(x)}{w(x)}N_s w(x) \, dx = 0.
\]
Thus, if \(N_s(w(x)) \geq 0\) for all \(x \in N\), it follows that \(v_1 = 0\) and we get a contradiction. Hence we conclude.

Here we give an explicit bounded domain where the above situation holds. Define the set \(\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3\) where
\[
\Omega_1 = B_\varepsilon(0), \quad \Omega_2 = \{x \in \mathbb{R}^d, \varepsilon \leq x_1 \leq A\text{ and } |(x_2, x_3, \ldots, x_d)| < \varepsilon\},
\]
\[
\Omega_3 = \{x \in \mathbb{R}^d, \ A < |x| < \beta\}.
\]

Now, we consider
\[
D = \left\{x \in \mathbb{R}^d \setminus \Omega, |x| < \eta\right\}
\cup \left\{x \in \mathbb{R}^d, \eta \leq x_1 \leq A\text{ and } |(x_2, x_3, \ldots, x_d)| < m\right\}
\cup \left\{x \in \mathbb{R}^d, |x| > \beta\right\},
\]
and
\[
N = \{x \in \mathbb{R}^d \setminus \{\Omega \cup D\}, \eta < |x| < A\}.
\]

It is clear that \(\Omega\) is a bounded domain of \(\mathbb{R}^d\), \(N\) and \(D\) are two open sets of \(\mathbb{R}^d \setminus \bar{\Omega}\) with \(N \cap D = \emptyset\) and \(N \cup D = \mathbb{R}^d \setminus \Omega\). See Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Example 1}
\end{figure}
To prove that $A_N = \Lambda$, we will show the existence of $\varepsilon_0$ such that if $\varepsilon \leq \varepsilon_0$, then $\mathcal{N}_s w(x) \geq 0$ for all $x \in N$. Recall that $w(x) = |x|^{-\frac{d-2s}{2}}$, for simplicity of typing we set $\theta_0 = \frac{d-2s}{2}$, then

$$\mathcal{N}_s w(x) = \int_{\Omega} \frac{(w(x) - w(y))}{|x-y|^{d+2s}} \, dy$$

$$= \int_{\Omega_1} \frac{(w(x) - w(y))}{|x-y|^{d+2s}} \, dy + \int_{\Omega_2} \frac{(w(x) - w(y))}{|x-y|^{d+2s}} \, dy$$

$$+ \int_{\Omega_3} \frac{(w(x) - w(y))}{|x-y|^{d+2s}} \, dy$$

$$= J_1 + J_2 + J_3.$$

Since $x \in N$, then $\eta \leq |x| \leq A$.

Let us begin by estimating $J_1$. Setting $y = |y|y'$ and $x = |x|x'$, it holds that

$$J_1 = \int_{\Omega_1} \frac{(w(x) - w(y))}{|x-y|^{d+2s}} \, dy = \int_{B(0,\varepsilon)} \frac{|x|^\theta_0 - |y|^\theta_0}{|x-y|^{d+2s}} \, dy$$

$$= \int_0^\varepsilon \frac{(|x|^{-\theta_0} - \rho^{-\theta_0})\rho^{d-1}}{|x|\rho^{d+2s}} \, d\rho$$

where $\rho = |y|$. Let $\sigma = \frac{\rho}{|x|}$, then following closely the radial computation as in [19], it follows that

$$J_1 = \frac{1}{|x|^{2s+\theta_0}} \int_{|y|=1} \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^\pi \frac{\sin^{d-2}(\theta)}{(1 - 2\sigma \cos(\theta) + \sigma^2)^{\frac{d+2s}{2}}} \, d\theta.$$

Choosing $\varepsilon << \eta$, there results that $\frac{\varepsilon}{|x|} \leq \frac{\varepsilon}{\eta} << 1$, hence

$$|J_1| = \frac{1}{|x|^{\theta_0+2s}} \int_{|y|=1} \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^\pi \frac{\sin^{d-2}(\theta)}{(1 - 2\sigma \cos(\theta) + \sigma^2)^{\frac{d+2s}{2}}} \, d\theta.$$

We deal now with $J_2$. Without loss of generality we will assume that $\varepsilon < \min\{\frac{\eta}{4}, \frac{m}{3}\}$ and fix $\sigma = \min\{\frac{\eta}{3}, \frac{m}{3}\}$. It is clear that for all $x \in N$ and for all $y \in \Omega_2$, we have $|x-y| \geq \sigma$. Thus

$$|J_2| \leq \int_{\Omega_2 \cap |x-y| \geq \sigma} \frac{|w(x) - w(y)|}{|x-y|^{d+2s}} \, dy \leq \frac{C}{\sigma^{d+2s}} \int_{\Omega_2} |y^{-\theta_0} - |y|^{-\theta_0}| \, dy.$$

Since $|y|^{-\theta_0} \in L^1_{\text{loc}}(\mathbb{R}^d)$, then by the Dominated Convergence Theorem it holds $|J_2| = o(\varepsilon)$.

We deal now with $J_3$. Following closely the computation of $J_1$, we reach that

$$J_3 = \frac{1}{|x|^{\theta_0+2s}} \int_{\Omega_3} \frac{(\sigma^{\theta_0} - 1)\sigma^{-\theta_0-1}K(\sigma)d\sigma}{|x-y|^{d+2s}} \, dy \geq \frac{1}{A^{\theta_0+2s}} \int_{\Omega_2} (\sigma^{\theta_0} - 1)\sigma^{-\theta_0-1}K(\sigma)d\sigma.$$

Choosing $\beta >= 2$ and combining the above estimates, we conclude that

$$\mathcal{N}_s(w(x)) \geq \frac{1}{A^{\theta_0+2s}} \int_{\Omega_2} (\sigma^{\theta_0} - 1)\sigma^{-\theta_0-1}K(\sigma)d\sigma - o(\varepsilon).$$
Hence we conclude.

We have also the example described by the Figure 2, where the constant \( \Lambda_N = \Lambda \) and then it is not attained. We leave the details to the reader.

4. **Semilinear mixed problem involving the Hardy potential.** In this section we assume that \( \Lambda_N < \Lambda \), that is, \( \Lambda_N \) is the principal eigenvalue for the corresponding mixed problem. We will consider the following nonlinear problem

\[
\begin{cases}
(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
B_s u = 0 & \text{in } \mathbb{R}^d \setminus \Omega,
\end{cases}
\]  

(37)

where \( 1 < p < 2^*_s - 1 \) and \( \lambda < \Lambda_N \).

4.1. **Subcritical problems,** \( 1 < p < 2^*_s - 1 \). The next result is a direct consequence of Theorem 3.4 and the classical Rabinowitz bifurcation Theorem, see [29].

**Theorem 4.1.** Assume that \( 1 < p < 2^*_s - 1 \) and \( \lambda < \Lambda_N < \Lambda \), then the problem (37) has an unbounded branch \( \Sigma \) of positive solutions bifurcating from \((0, \Lambda_N)\).

More interesting is the following problem. Assume now that \( \lambda \in (\Lambda_N, \Lambda) \) and define

\[
I_{\lambda,p} = \inf_{\{\phi \in \mathcal{H}(\Omega, D), \phi \neq 0\}} \frac{a_{d,s}}{2} \int \int_{\partial \Omega} |\phi(x) - \phi(y)|^2 d\nu - \lambda \int_{\Omega} \frac{\phi^2}{|x|^{2s}} dx - \left( \int_{\Omega} |\phi|^{p+1} dx \right)^{\frac{2}{p+1}},
\]  

(38)

where \( p \in (1, 2^*_s - 1) \). It is clear that \( I_{\lambda,p} < 0 \), however we have the next result.

**Theorem 4.2.** Assume that \( \lambda \in (\Lambda_N, \Lambda) \) and \( 1 < p < 2^*_s - 1 \), then \( I_{\lambda,p} < 0 \), is finite and it is achieved. As a consequence the problem

\[
\begin{cases}
(-\Delta)^s u + u^p = \lambda \frac{u}{|x|^{2s}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
B_s u = 0 & \text{in } \mathbb{R}^d \setminus \Omega,
\end{cases}
\]  

(39)
has a positive solution.

**Proof.** We divide the proof into two steps.

**Step 1.** $|I_{\lambda,p}| < \infty$.

Let $u \in \mathcal{E}^*(\Omega, D)$ be such that $\int_{\Omega} |u|^{p+1} \, dx = 1$ and consider $\psi \in C^\infty_0(B_\rho(0))$ to be a cut-off function such that $0 \leq \psi \leq 1$ and $\psi = 1$ in $B_{\rho/4}(0)$ for $\rho > 0$ small enough, then $u = \psi u + (1 - \psi)u = u_1 + u_2$.

Since $p + 1 > 2$, then

$$\int_\Omega \frac{u^2}{|x|^{2s}} \, dx \leq \int_\Omega \frac{u^2}{|x|^{2s}} \, dx + C(\Omega). \quad (40)$$

On the other hand we have

$$(u(x) - u(y))^2 = (u_1(x) - u_1(y))^2 + (u_2(x) - u_2(y))^2 + 2(u_1(x) - u_1(y))(u_2(x) - u_2(y)).$$

Then

$$\int \int_{D_\rho} (u(x) - u(y))^2 \, d\nu = \int \int_{D_\rho} (u_1(x) - u_1(y))^2 \, d\nu + \int \int_{D_\rho} (u_2(x) - u_2(y))^2 \, d\nu + 2 \int \int_{D_\rho} (u_1(x) - u_1(y))(u_2(x) - u_2(y)) \, d\nu. \quad (41)$$

We estimate the last integral. By a direct computation it holds that

$$\int \int_{D_\rho} (u_1(x) - u_1(y))(u_2(x) - u_2(y)) \, d\nu$$

$$= \int \int_{\Omega} (u_1(x) - u_1(y))(u_2(x) - u_2(y)) \, d\nu + 2 \int \int_{\Omega} (u_1(x) - u_1(y))(u_2(x) - u_2(y)) \, d\nu$$

$$= K_1 + K_2.$$

By the elementary identity,

$$\left( u_1(x) - u_1(y) \right) \left( u_2(x) - u_2(y) \right) =$$

$$\psi(x) \left( 1 - \psi(x) \right) \left( u(x) - u(y) \right)^2$$

$$+ \left( 1 - 2 \psi(x) \right) u(y) \left( u(x) - u(y) \right) \left( \psi(x) - \psi(y) \right)$$

$$- u^2(y) \left( \psi(x) - \psi(y) \right)^2,$$

and using Young inequality, we obtain that

$$K_1 \geq -\varepsilon \int \int_{\Omega} (u(x) - u(y))^2 \, d\nu - C(\varepsilon) \int \int_{\Omega} u^2(y)(\psi(x) - \psi(y))^2 \, d\nu.$$
Notice that
\[
\int_{\Omega} \int_{\Omega} u^2(y)(\psi(x) - \psi(y))^2 d\nu \leq C \int_{\Omega} \int_{\Omega} \frac{u^2(y)}{|x-y|^{d+\sigma-2}} dxdy
\]
\[
\leq C \int_{\Omega} u^2(y) dy \leq C(\Omega),
\]
where we have used the fact that \( \sup_{\{x \in \Omega\}} \int_{\Omega} \frac{1}{|x-y|^{d+\sigma-2}} dx \leq C(\Omega) \). Thus
\[K_1 \geq -C(\Omega, \varepsilon) - \varepsilon \int_{\Omega} \int_{\Omega} (u(x) - u(y))^2 d\nu. \tag{42}\]

Let analyze now \( K_2 \). It is clear that
\[
K_2 = \int_{\Omega^c} \int_{B_{2\rho}(0)} \psi(x)u(x) \left(1 - \psi(x)\right)u(x) - u(y)\right) d\nu
\]
\[
= \int_{\Omega^c} \int_{B_{2\rho}(0)} \psi(x)(1 - \psi(x))u^2(x) d\nu - \int_{\Omega^c} \int_{B_{2\rho}(0)} \psi(x)u(x)u(y) d\nu
\]
Since
\[
\sup_{\{x \in B_{2\rho}(0)\}} \int_{\Omega^c} \frac{dy}{|x-y|^{d+\sigma-2}} \leq C(\Omega, B_{2\rho}(0)), \tag{43}\]
then
\[
\int_{\Omega^c} \int_{B_{2\rho}(0)} \psi(x)(1 - \psi(x))u^2(x) d\nu \leq C \int_{B_{2\rho}(0)} u^2(x) dx \leq C(\Omega, B_{2\rho}(0)).
\]
Now, using Young inequality and the estimate (43), we reach that
\[
\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} \psi(x)u(x)u(y) d\nu
\]
\[
\leq \varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} u^2(y) d\nu + C(\varepsilon) \int_{\Omega^c} \int_{B_{2\rho}(0)} \psi^2(x)u^2(x) d\nu
\]
\[
\leq \varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} u^2(y) d\nu + C(\Omega, B_{2\rho}(0), \varepsilon).
\]
Since \( u^2(y) \leq 2(u(x) - u(y))^2 + 2u^2(x) \), it follows that
\[
\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} u^2(y) d\nu
\]
\[
\leq 2\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} (u(x) - u(y))^2 d\nu + 2\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} u^2(x) d\nu
\]
\[
\leq 2\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} (u(x) - u(y))^2 d\nu + C(\Omega, B_{2\rho}(0), \varepsilon).
\]
Thus
\[K_2 \geq -2\varepsilon \int_{\Omega^c} \int_{B_{2\rho}(0)} (u(x) - u(y))^2 d\nu - C(\Omega, B_{2\rho}(0), \varepsilon). \tag{44}\]
Therefore combining estimates (40), (41), (42) and (44), we conclude that
\[
\frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u^2}{|x|^{2s}} \, dx \\
\geq \left( \frac{a_{d,s}}{2} \int_{D_{\Omega}} (u_1(x) - u_1(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u_1^2}{|x|^{2s}} \, dx \right) \\
+ \frac{a_{d,s}}{2} \int_{D_{\Omega}} (u_2(x) - u_2(y))^2 \, d\nu \\
- 3\varepsilon \frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - C(\Omega, B_{2p}(0), \varepsilon).
\]

Since \( u_1 \in H_0^s(\Omega) \) and \( \lambda < \Lambda \), then
\[
\frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u^2}{|x|^{2s}} \, dx \\
\geq (\Lambda - \lambda) \frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu.
\]

Thus
\[
\frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u^2}{|x|^{2s}} \, dx \\
\geq C(\lambda, \Lambda) \left( \int_{D_{\Omega}} (u_2(x) - u_2(y))^2 \, d\nu + \int_{D_{\Omega}} (u_1(x) - u_1(y))^2 \, d\nu \right) \\
- 3\varepsilon \frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - C(\Omega, B_{2p}(0), \varepsilon).
\]

Choosing \( \varepsilon \) small, we reach that
\[
\frac{a_{d,s}}{2} \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u^2}{|x|^{2s}} \, dx \\
\geq C(\lambda, \Lambda, \varepsilon) \int_{D_{\Omega}} (u(x) - u(y))^2 \, d\nu - C(\Omega, B_{2p}(0), \varepsilon). \tag{45}
\]

Thus \( |I_{\lambda,p}| < \infty \).

**Step 2.** \( I_{\lambda,p} \) is attained. Define
\[
I_{\lambda,n} = \inf_{\{\phi \in \mathcal{E}^s(\Omega, D), \phi \neq 0\}} \frac{a_{d,s}}{2} \int_{D_{\Omega}} |\phi(x) - \phi(y)|^2 \, d\nu - \lambda \int_{\Omega} \frac{\phi^2}{|x|^{2s} + \frac{1}{n}} \, dx \\
\left( \int_{\Omega} |\phi|^{p+1} \, dx \right)^{\frac{p}{p+1}},
\]

it is clear that \( I_{\lambda,n} \downarrow I_{\lambda,p} \) as \( n \to \infty \). Hence \( I_{\lambda,n} < 0 \) for \( n \geq n_0 \).

Since \( p + 1 < 2^*_s \), then using a variational argument we get that \( I_{\lambda,n} \) is achieved.

Hence there exists \( u_n \in \mathcal{E}^s(\Omega, D) \) that satisfies
\[
(P_n) \equiv \begin{cases} 
(-\Delta)^s u_n - \lambda \frac{u_n}{|x|^{2s} + \frac{1}{n}} = I_{\lambda,n} u_n^p & \text{in } \Omega, \\
u_n \geq 0 & \text{in } \Omega, \\
B_s u_n = 0 & \text{in } \mathbb{R}^d \setminus \Omega,
\end{cases}
\]

with \( \|u_n\|_{L^{p+1}(\Omega)} = 1 \).
We claim that \( \{u_n\}_n \) is bounded in the space \( \mathbb{E}^s(\Omega, D) \). Since
\[
\frac{a_{d,s}}{2} \int_{D_\Omega} (u_n(x) - u_n(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u_n^2}{|x|^{2s} + \frac{1}{n}} \, dx \geq \frac{a_{d,s}}{2} \int_{D_\Omega} (u_n(x) - u_n(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u_n^2}{|x|^{2s} + \frac{1}{n}} \, dx,
\]
then by (45), it follows that
\[
\frac{a_{d,s}}{2} \int_{D_\Omega} (u_n(x) - u_n(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{u_n^2}{|x|^{2s} + \frac{1}{n}} \, dx \geq C(\lambda, \Lambda, \varepsilon) \int_{D_\Omega} (u_n(x) - u_n(y))^2 \, d\nu - C(\Omega, B_{2p}(0), \varepsilon).
\]
Thus \( \int \int_{D_\Omega} (u_n(x) - u_n(y))^2 \, d\nu \leq C \) for all \( n \) and the claim follows.

Therefore, there exists \( u_0 \in \mathbb{E}^s(\Omega, D) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( \mathbb{E}^s(\Omega, D) \) and strongly in \( L^{p+1}(\Omega) \). Hence \( \|u_0\|_{L^{p+1}(\Omega)} = 1 \) and then \( u_0 \neq 0 \). Notice that by the weak convergence we obtain that \( u_0 \) is a weak solution to (39).

We claim now that \( \frac{u_n}{|x|^{2s}} \to \frac{u_0}{|x|^{2s}} \) strongly in \( L^1(\Omega) \). Define \( w_n = u_n - u_0 \), it is clear that \( w_n \rightharpoonup 0 \) weakly in \( \mathbb{E}^s(\Omega, D) \) and \( w_n \to 0 \) strongly in \( L^{p+1}(\Omega) \).

As in the previous step, we have
\[
\int \int_{D_\Omega} (w_n(x) - w_n(y))^2 \, d\nu \geq \int \int_{D_\Omega} \left( (\psi w_n)(x) - (\psi w_n)(y) \right)^2 \, d\nu + o(1)
\]
and
\[
\int_{\Omega} \frac{w_n^2}{|x|^{2s} + \frac{1}{n}} \, dx = \int_{\Omega} \frac{(\psi w_n)^2}{|x|^{2s} + \frac{1}{n}} \, dx + o(1).
\]

Since \( w_n \in H^s_0(\Omega) \) and \( u_0 \) and \( u_n \) are solution of the problems (39) and (P_n) respectively, it holds that
\[
o(1) \geq \frac{a_{d,s}}{2} \int_{D_\Omega} (w_n(x) - w_n(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{w_n^2}{|x|^{2s}} \, dx
\]
\[
\geq \frac{a_{d,s}}{2} \int_{D_\Omega} ((\psi w_n)(x) - (\psi w_n)(y))^2 \, d\nu - \lambda \int_{\Omega} \frac{(\psi w_n)^2}{|x|^{2s} + \frac{1}{n}} \, dx + o(1)
\]
\[
\geq (\Lambda - \lambda) \int_{\Omega} \frac{w_n^2}{|x|^{2s}} \, dx + o(1).
\]

Hence \( \int_{\Omega} \frac{w_n^2}{|x|^{2s}} \, dx = o(1) \) and the claim follows.

Combining the above estimates we reach that \( u_n \rightharpoonup u_0 \) strongly in \( \mathbb{E}^s(\Omega, D) \) and thus \( u_0 \) realize \( I_{\lambda,p} \). Hence up to a positive constant, \( cu_0 \) solves problem (39), then we conclude.

4.2. **Doubly-critical problem.** In this subsection we discuss conditions for the existence and the non existence to the following double critical problem
\[
\begin{aligned}
\begin{cases}
(-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^{2^*_s - 1} & \text{in } \Omega, \\
u &> 0 & \text{in } \Omega, \\
B_s u &= 0 & \text{in } \mathbb{R}^d \setminus \Omega,
\end{cases}
\end{aligned}
\]
(46)
where $\lambda \in (0, \Lambda_N)$ according to the D-N configuration. If $\Omega = \mathbb{R}^d$, problem (46) is related to the next constant

$$S_\lambda = \inf_{u \in C_0^\infty(\mathbb{R}^d), u \neq 0, \|u\|_{L_2^*(\mathbb{R}^d)} = 1} \frac{a_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^2 \, d\nu - \lambda \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^{2s}} \, dx. \quad (47)$$

The problem in the whole Euclidian space $\mathbb{R}^d$ has been studied in [16]. From the result of [16] we know that the constant $S_\lambda$ is independent of $\Omega$ containing the pole of the Hardy potential.

In the same way we consider for a D-N configuration the constant $T_{\lambda,N}$ defined by

$$T_{\lambda,N} = \inf_{u \in E^s(\Omega, D), \|u\| \neq 0, \|u\|_{W_2^s(\Omega)} = 1} \frac{a_{d,s}}{2} \int_{D_\Omega} \int_{D_\Omega} |u(x) - u(y)|^2 \, d\nu - \lambda \int_{\Omega} \frac{u^2(x)}{|x|^{2s}} \, dx. \quad (48)$$

It is clear that if $T_{\lambda,N}$ is achieved, then problem (46) has a nontrivial solution.

We have the next existence result.

**Theorem 4.3.** Let $(D, N)$ a D-N configuration and assume that $\lambda \in (0, \Lambda_N)$.

Suppose that $T_{\lambda,N} < \min\{S_\lambda, S_N\}$, then $T_{\lambda,N}$ is achieved and, as a consequence, problem (46) has a nontrivial solution.

**Proof.** Recall that $S_N$ is the Sobolev constant defined in Proposition 4. Since $\lambda < \Lambda_N$, then $T_{\lambda,N} \geq (1 - \frac{\lambda}{\Lambda_N})S_N > 0$.

Let $\{u_n\}_n \subset E^s(\Omega, D)$ be a minimizing sequence for $T_{\lambda,N}$ with $\int_{\Omega} |u_n|^{2^*_s} \, dx = 1$, then $\{u_n\}_n$ is bounded in $E^s(\Omega, D)$, and

$$\frac{a_{d,s}}{2} \int_{D_\Omega} |u_n(x) - u_n(y)|^2 \, d\nu - \lambda \int_{\Omega} \frac{u_n^2(x)}{|x|^{2s}} \, dx \to T_{\lambda,N}.$$ 

Without loss of generality we can choose $u_n \geq 0$ in $\mathbb{R}^d$. Hence there exists $\tilde{u} \in E^s(\Omega, D)$ such that $u_n \rightharpoonup \tilde{u}$ weakly in $E^s(\Omega, D)$, and up to a subsequence, $u_n \rightharpoonup \tilde{u}$ strongly in $L^{2^*_s}(\Omega)$ for all $\sigma < 2^*_s$ and $u_n \to \tilde{u}$ a.e in $\Omega$.

Using the Ekeland variational principle it holds that

$$\begin{cases} (-\Delta)^s u_n - \lambda \frac{u_n}{|x|^{2s}} = T_{\lambda,N} u_n^{2^*_s - 1} + o(1) & \text{in } \Omega, \\ B_{\|u_n\|} = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases} \quad (49)$$

It is clear that if $\tilde{u} \neq 0$, then $\tilde{u}$ solves the problem (46).

Assume by contradiction that $\tilde{u} = 0$. Let $\psi \in C_0^\infty(B_\rho(0))$ be a cut-off function such that $0 \leq \psi \leq 1$ and $\psi = 1$ in $B_{\rho/4}(0)$ for $\rho > 0$ small enough.

We claim that

$$\int_{\Omega} (-\Delta)^s u_n \psi^2 \, dx = \frac{a_{d,s}}{2} \int_{D_\Omega} \left( (\psi u_n)(x) - (\psi u_n)(y) \right)^2 \, d\nu + o(1). \quad (50)$$

Notice that

$$\int_{\Omega} (-\Delta)^s u_n \psi^2 \, dx = \frac{a_{d,s}}{2} \int_{D_\Omega} (u_n(x) - u_n(y))(\psi^2 u_n(x) - (\psi^2 u_n)(y)) \, d\nu + o(1).$$
Since
\[(u_n(x) - u_n(y))(\psi u_n(x) - (\psi u_n)(y)) - ((\psi u_n)(x) - (\psi u_n)(y))^2 = -u_n(x)u_n(y)(\psi(x) - \psi(y))^2,\]
we reach that
\[
\int_\Omega (-\Delta)^s u_n \psi^2 \, dx = \frac{a_d s}{2} \int_\Omega \int_{\Omega^c} (\psi u_n(x) - (\psi u_n)(y))^2 \, dy - \int_\Omega \int_{\Omega^c} u_n(x)u_n(y)(\psi(x) - \psi(y))^2 \, dy.
\]
Notice that
\[
\int_\Omega \int_{\Omega^c} u_n(x)u_n(y)(\psi(x) - \psi(y))^2 \, dy = \int_\Omega \int_{\Omega^c} u_n(x)u_n(y)(\psi(x) - \psi(y))^2 \, dy + 2 \int_{\Omega^c} \int_\Omega u_n(x)u_n(y)(\psi(x) - \psi(y))^2 \, dy = K_{1n} + 2K_{2n}.
\]
Let us begin by estimating $K_{1n}$. It is clear that
\[
K_{1n} = C(\Omega) \int_\Omega \int_{\Omega^c} \frac{u_n(x)u_n(y)}{|x - y|^{d+2s-2}} \, dx \, dy
\leq 2C(\Omega) \int_\Omega \int_{\Omega^c} \frac{u_n^2(x)}{|x - y|^{d+2s-2}} \, dx \, dy + 2C(\Omega) \int_\Omega \int_{\Omega^c} \frac{u_n^2(y)}{|x - y|^{d+2s-2}} \, dx \, dy.
\]
Since
\[
sup_{x \in \Omega} \int_{\Omega^c} \frac{dy}{|x - y|^{d+2s-2}} \leq C(\Omega) \quad \text{and} \quad sup_{y \in \Omega} \int_{\Omega^c} \frac{dx}{|x - y|^{d+2s-2}} \leq C(\Omega),
\]
we reach that
\[
K_{1n} \leq 4C(\Omega) \int_\Omega u_n^2(x) \, dx = o(1).
\]
Now, we deal with $K_{2n}$. We have
\[
K_{2n} = \int_{\Omega^c} \int_{B_r(0)} u_n(x)u_n(y)\psi^2(x) \, dy
\leq \left( \int_{\Omega^c} \int_{B_r(0)} u_n^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega^c} \int_{B_r(0)} u_n^2(y) \, dy \right)^{\frac{1}{2}}.
\]
It is clear that \(\int_{\Omega^c} \int_{B_r(0)} u_n^2(x) \, dx = o(1)\), now using the fact that \(\{u_n\}_n\) is bounded in \(E^s(\Omega, D)\) and \(u^2(y) \leq 2(u(x) - u(y))^2 + 2u^2(x)\), we get
\[
\int_{\Omega^c} \int_{B_r(0)} u_n^2(y) \, dy \leq 2 \int_{\Omega^c} \int_{B_r(0)} (u_n(x) - u_n(y))^2 \, dy \, dx + 2 \int_{\Omega^c} \int_{B_r(0)} u_n^2(x) \, dy \, dx
\leq C.
\]
Hence \(K_{2n} = o(1)\).

Combining the above estimate and going back to (51), we reach (50) and the claim follows.

Therefore using \(u_n\psi^2\) as a test function in (49), we conclude that
\[
\frac{a_d s}{2} \int_\Omega \int_{\Omega^c} (\psi u_n(x) - (\psi u_n)(y))^2 \, dy - \lambda \int_\Omega \frac{(|\psi u_n|)^2}{|x|^{2s}} \, dx
\]

Recall that 

\[ u \in \Lambda < \lambda \] 

since 

\[ \frac{1}{2} \int \Omega (\psi u_n)^2 \, dx \]

\[ \leq T_{\lambda,N} \left( \int \Omega (\psi u_n)^2 \, dx \right)^{\frac{2}{\gamma}} + o(1). \]

We set \( u_{1n} = u_n \psi \), then \( u_{1n} \in H^1_0(\Omega) \). If for a subsequence of \( \{u_n\}_n \), we have

\[ \int \Omega u_{1n}^2 \, dx \geq C, \]

then we conclude that

\[ S_{\lambda} \leq \frac{a_{d,s}}{2} \int_{\Omega} \left( u_{1n}(x) - u_{1n}(y) \right)^2 \, d\nu - \lambda \int_{\Omega} \frac{u_{1n}^2}{|x|^{2s}} \, dx \]

\[ \leq T_{\lambda,N} + o(1). \]

Thus \( S_{\lambda} \leq T_{\lambda,N} \) which is a contradiction with the hypothesis in the statement of the Theorem.

Hence \( \int \Omega u_{1n}^2 \, dx \rightarrow 0 \) as \( n \rightarrow \infty \). Thus \( \int \Omega u_{1n}^2 (1 - \psi)^2 \, dx \rightarrow 1 \) as \( n \rightarrow \infty \).

We set \( g = 1 - \psi \), by the computation above, we reach that

\[ \int \Omega (-\Delta)^s u_n u_n g^2 \, dx = \frac{a_{d,s}}{2} \int_{\Omega} \int_{\Omega} \left( (gu_n)(x) - (gu_n)(y) \right)^2 \, d\nu + o(1), \quad (52) \]

and

\[ \int \Omega \frac{(gu_n)^2}{|x|^{2s}} \, dx = o(1). \]

Thus using \( u_n g^2 \) as a test function in \((49)\) we conclude that

\[ \frac{a_{d,s}}{2} \int \Omega \left( (gu_n)(x) - (gu_n)(y) \right)^2 \, d\nu = T_{\lambda,N} \int \Omega u_{1n}^2 g^2 \, dx + o(1) \]

\[ \leq T_{\lambda,N} \left( \int \Omega (gu_n)^2 \, dx \right)^{\frac{2}{\gamma}} + o(1). \]

Setting \( u_{2n} = u_n g \), it follows that

\[ S_N \leq \frac{a_{d,s}}{2} \int_{\Omega} \left( u_{2n}(x) - u_{2n}(y) \right)^2 \, d\nu \]

\[ \leq \frac{a_{d,s}}{2} \int_{\Omega} \left( u_{2n}(x) - u_{2n}(y) \right)^2 \, d\nu \]

\[ \leq T_{\lambda,N} + o(1). \]

Thus \( S_N \leq T_{\lambda,N} \) which again is a contradiction with the hypothesis.

Hence, as a conclusion we obtain that \( u_0 \neq 0 \). It is clear that, up to a constant, \( u_0 \) solves problem \((46)\).

To finish we have just to show that \( u_0 \) realize \( T_{\lambda,N} \). Let

\[ Q_{\lambda,N}(u) = \frac{a_{d,s}}{2} \int \Omega \left| u(x) - u(y) \right|^2 \, d\nu - \lambda \int \Omega \frac{u^2}{|x|^{2s}} \, dx, \]

since \( \lambda < \Lambda_N \), then \( Q_{\lambda,N} \) define an equivalent norm to the norm of the space \( E^s(\Omega, D) \) hence, to conclude, we have to show that \( Q_{\lambda,N}(u_n - u_0) \rightarrow 0 \) as \( n \rightarrow \infty \).

Recall that \( u_n \rightharpoonup u_0 \) weakly in \( E^s(\Omega, D) \), then

\[ Q_{\lambda,N}(u_n) = Q_{\lambda,N}(u_0) + Q_{\lambda,N}(u_n - u_0) + o(1). \quad (53) \]
In the same way, using Brezis-Lieb Lemma (see [10]), we get
\[ \|u_n\|_{L^{2^*}_\Delta(\Omega)} = \|u_0\|_{L^{2^*}_\Delta(\Omega)} + \|u_n - u_0\|_{L^{2^*}_\Delta(\Omega)} + o(1). \]

Since \(Q_{\lambda,N}(u_n - u_0) \geq T_{\lambda,N}\|u_n - u_0\|_{L^{2^*}_\Delta(\Omega)}\), then
\[
\frac{Q_{\lambda,N}(u_0)}{\|u_0\|_{L^{2^*}_\Delta(\Omega)}^{\frac{2^*}{2}}} = \frac{Q_{\lambda,N}(u_n) - Q_{\lambda,N}(u_0) + o(1)}{\left(\|u_n\|_{L^{2^*}_\Delta(\Omega)} - \|u_n - u_0\|_{L^{2^*}_\Delta(\Omega)} + o(1)\right)^{\frac{2^*}{2}}} \leq T_{\lambda,N} \frac{Q_{\lambda,N}(u_n) - Q_{\lambda,N}(u_0) + o(1)}{\left(Q_{\lambda,N}(u_n) - Q_{\lambda,N}(u_0) + o(1)\right)^{\frac{2^*}{2}}}.
\] (54)

If \(\limsup_{n \to \infty} Q_{\lambda,N}(u_n - u_0) \neq 0\) then, by (53), it holds
\[ \limsup_{n \to \infty} Q_{\lambda,N}(u_n - u_0) = T_{\lambda,N} - Q_{\lambda,N}(u_0). \]

Then going back to (54), it follows that
\[ T_{\lambda,N} \leq \frac{Q_{\lambda,N}(u_0)}{\|u_0\|_{L^{2^*}_\Delta(\Omega)}^{\frac{2^*}{2}}} < T_{\lambda,N} \]
which leads to a contradiction. Hence \(\limsup_{n \to \infty} Q_{\lambda,N}(u_n - u_0) = 0\) and then \(u_0\) realize \(T_{\lambda,N}\).

To complete the paper we give an example where the constant \(T_{\lambda,N}\) is realized and then problem (46) has a positive solution.

**Theorem 4.4.** Consider a D-N configuration such that \(\Lambda_N < \Lambda\), then there exists \(0 < \tilde{\lambda} < \Lambda_N\) such that for all \(\lambda \in (\tilde{\lambda}, \Lambda_N)\), the constant \(T_{\lambda,N}\) is achieved.

**Proof.** Since \(\Lambda_N < \Lambda\), by Theorem 1.1, it follows that \(\Lambda_N\) is achieved. Hence there exists \(\tilde{u} \in \mathbb{E}^s(\Omega, D)\) such that \(\tilde{u}\) is a positive solution to the eigenvalue problem (30). Without loss of generality we can assume that \(\|\tilde{u}\|_{L^{2^*}_\Delta(\Omega)} = 1\), thus, using the definition of \(T_{\lambda,N}\), we conclude that
\[
T_{\lambda,N} \leq \frac{\alpha d^s}{2} \int_{D_\Omega} |\tilde{u}(x) - \tilde{u}(y)|^2 \, dv - \lambda \int_\Omega \tilde{u}^2(x) \frac{dx}{|x|^{2s}} = \left(\Lambda_N - \lambda\right) \int_\Omega \frac{\tilde{u}^2(x)}{|x|^{2s}} \, dx.
\]

Since \(\lambda < \Lambda_N < \Lambda\), by using the definition of \(S_\lambda\), it follows that \(S_\lambda > S_{\Lambda_N} \geq 0\). It is clear that
\[
\left(\Lambda_N - \lambda\right) \int_\Omega \frac{\tilde{u}^2(x)}{|x|^{2s}} \, dx \to 0 \quad \text{as} \quad \lambda \to \Lambda_N.
\]

Therefore, there exists \(\bar{\lambda} \equiv \tilde{\lambda}(S_{\Lambda_N}, S_N, \tilde{u}) < \Lambda_N\) such that if \(\lambda \in (\bar{\lambda}, \Lambda_N)\), hence
\[ T_{\lambda,N} \leq (\Lambda_N - \bar{\lambda}) \int_\Omega \frac{\tilde{u}^2(x)}{|x|^{2s}} \, dx < \min\{S_{\Lambda_N}, S_N\} < \min\{S_\lambda, S_N\}. \]

Then by Theorem 4.3, \(T_{\lambda,N}\) is achieved and the result follows.

**Remark 1.** The result in Theorem 4.4 goes in the spirit of the classical existence result by H. Brezis and L. Nirenberg in the seminal paper for the Dirichlet problem of the Laplacian with critical exponent [11]; this result was extended in [30] to the nonlocal case. More general conditions for the existence in the doubly critical problem with mixed boundary conditions seem to be unknown.
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REFERENCES

[1] B. Abdellaoui and R. Bentifour, Caffarelli-Kohn-Nirenberg Type Inequalities of Fractional Order and Applications, *J. Funct. Anal.*, 272 (2017), 3998–4029.

[2] B. Abdellaoui, A. Dieb and E. Valdinoci, A nonlocal concave-convex problem with nonlocal mixed boundary data, *Comm. on Pure Appl. Analysis*, 17 (2018), 1103–1120.

[3] B. Abdellaoui, E. Colorado and I. Peral, Some remarks on elliptic equations with singular potential and mixed boundary conditions, *Advanced Nonlinear Studies*, 4 (2004), 503–533.

[4] B. Abdellaoui, E. Colorado and I. Peral, Effect on the boundary conditions in the behaviour of the optimal constant of some Caffarelli-Kohn-Nirenberg inequalities. Application to some doubly critical nonlinear elliptic problems, *Advances in Differential Equations*, 11 (2006), 667–720.

[5] B. Abdellaoui, M. Medina, I. Peral and A. Primo, A note on the effect of the Hardy potential in some Calderon-Zygmund properties for the fractional Laplacian, *J. Differential Equations*, 260 (2016), 8160–8206.

[6] B. Abdellaoui, M. Medina, I. Peral and A. Primo, Optimal results for the fractional heat equation involving the Hardy potential, *Nonlinear Anal.*, 140 (2016), 166–207.

[7] B. Abdellaoui, I. Peral and A. Primo, A remark on the fractional Hardy inequality with a remainder term, *C. R. Acad. Sci. Paris, Ser. I*, 352 (2014), 299–303.

[8] B. Barrios and M. Medina, Strong maximum principles for fractional elliptic and parabolic problems with mixed boundary conditions. *arXiv:1607.01505*.

[9] W. Beckner, Pitt’s inequality and the uncertainty principle, *Proceedings of the American Mathematical Society*, 123 (1995), 1897–1905.

[10] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, 88 (1983), 486–490.

[11] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, 36 (1983), 437–477.

[12] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, 20. Springer; Unione Matematica Italiana, Bologna, 2016.

[13] X. Cabré and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. Math.*, 224 (2010), 2052–2093.

[14] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, 47 (1974), 324–353.

[15] F. Ferrari and I. Verbitsky, Radial fractional Laplace operators and Hessian inequalities, *J. Differential Equations*, 253 (2012), 244–272.

[16] G. Grubb, Local and nonlocal boundary conditions for mu-transmission and fractional order elliptic pseudodifferential operators, *Anal. PDE*, 7 (2014), 1649–1682.

[17] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Commun. Math. Phys.*, 53 (1977), 285–294.
[24] N. S. Landkof, Foundations of Modern Potential Theory, Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180, Springer-Verlag, New York-Heidelberg, 1972.

[25] T. Leonori, M. Medina, I. Peral, A. Primo and F. Soria, Principal eigenvalue of mixed problem for the fractional Laplacian: Moving the boundary conditions, J. Differential Equations, 265 (2018), no. 2, 593-619.

[26] T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, Discrete Contin. Dyn. Syst., 35 (2015), 6031–6068.

[27] G. Molica Bisci, V. Radulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, 2016.

[28] A. C. Ponce, Elliptic PDE’s, Measures and Capacities, Tracts in Mathematics 23, European Mathematical Society (EMS), Zurich, 2016.

[29] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Analysis, 7 (1971), 487–513.

[30] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, Trans. Amer. Math. Soc., 367 (2015), 67–102.

[31] E. M. Stein and G. Weiss, Fractional integrals on n-dimensional Euclidean space, J. Math. Mech., 7 (1958), 503–514.

[32] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations, 1 (1996), 241–264.

[33] D. Yafaev, Sharp constants in the Hardy-Rellich inequalities, J. Functional Analysis, 168 (1999), 121–144.

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