LAPLACE’S EQUATION ON N-DIMENSIONAL SINGULAR MANIFOLDS

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Abstract. We show that the Sobolev embedding is compact on punctured manifolds with conical singularities. On the other hand, we find the Sobolev inequality does not hold on punctured manifolds with Poincaré like metric, on which one has Poincaré inequality. Applying the results to the Laplace’s equation on the singular manifolds, we obtain the existence of the solution in both cases. In conical singularity case, we prove further that the solution can be extended to the singular points and it is Hölder continuous. However, the solution can not be continuously extended to the singular points in Poincaré like metric case. Moreover, on singular manifolds with conical singularities, we obtain the existence and regularity result of nontrivial nonnegative solutions for the semilinear elliptic equation with subcritical exponent.

1. Introduction

We study the existence and regularity of the solution for the Laplace’s equation

\[ \Delta u = f, \quad (P) \]

where \( f \in L^2 \) in Riemannian manifolds with conical singularities and with Poincaré like metrics, which will be defined as follows.

Similarly as the definition of conical Riemann surfaces in [4, 6, 24], a n-dimensional \( (n \geq 2) \) Riemannian manifold \( M \) with conical singularities is a compact manifold with smooth metric everywhere except at finitely many points \( p_1, p_2, \ldots, p_k \) on \( M \); locally, near the singular points \( p_i \), the manifold is diffeomorphic to a Euclidean cone of total angle \( \theta_i > 0 \), the metric can be written as

\[ ds^2 = |x|^{2\beta_i} ds_0^2, \quad \text{where } \beta_i = \left( \frac{\theta_i}{2\pi} \right) - 1 > -1, \quad (1.1) \]

in a local coordinate centered at \( p_i \) and \( ds_0^2 \) is the Euclidean metric. Then we extend this metric to \( M \) and denote it by \( g \).

Let \( \alpha := \min_i \{ \beta_i \} \). It follows from (1.1) that

\[ \alpha > -1. \quad (1.2) \]

Definition 1.1. A metric on \( M \) is called a conical metric if it is quasiisometric to the metric \( g \) we constructed above.

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We denote the n-dimensional singular manifold by a triple \((M, \beta, g)\) where \(\beta = \sum_i^k \beta_i p_i\) is a real divisor, i.e., a function: \(M \to \mathbb{R}\) with discrete support at \(p_1, \ldots, p_k\) and is equal to \(\beta_i\) at \(p_i\). It is called a manifold of dimension \(n\) with conical singularities of angles \(\theta_1, \ldots, \theta_k\), with \(\theta_i = 2\pi(\beta_i + 1)\). Obviously, the manifold is noncomplete with a finite volume.

We assume \(M^0\) is an open Riemannian manifold of dimension \(n\) \((n \geq 2)\) which comes from a compact Riemannian manifold by removing some finite points, that is, \(M^0 = M \setminus \{p_1, p_2, \ldots, p_k\}\), where \(M\) is compact. At each puncture \(p_i\) \((i = 1, 2, \ldots, k)\), we choose a neighborhood \(U_i\) of \(p_i\) such that \(U_i \cap U_j = \emptyset\) \((i \neq j)\) and \(\varphi_i : U_i \to B_1(0)\) is a diffeomorphism and \(\varphi_i(p_i) = 0\) where \(B_1(0)\) is the unit ball. We introduce the Poincaré metric in \(U_i^* = U_i \setminus \{p_i\}\), in the local coordinates \(ds^2 = \frac{ds^2}{|x|^{2(1-\log|x|)}^2}\) where \(x \in B_1^*(0) = B_1(0) \setminus \{0\}\). Then we extend this metric to \(M^0\) and denote it by \(w_0\). Clearly \(w_0\) is a complete metric with a finite volume.

We generalize the definition of Poincaré like metric in a Riemann surface of [19] to an n-dimensional Riemannian manifold.

**Definition 1.2.** A complete metric on \(M^0\) is called a Poincaré like metric if it is quasiisometric to the metric \(w_0\) we constructed above.

Conical singular Riemann surfaces have been extensively studied by many authors (for example, [2, 3, 7, 12, 13, 18, 22, 23]). Troyanov [24] studied the prescribing curvature problem on the surfaces with conical singularities. In order to study the problem on these surfaces, he proved the Sobolev inequality, a compactness result and the following Trudinger inequality:

\[
\int_M e^{bu^2} dV \leq c_b,
\]

for all \(u \in H^1(M)\) satisfying \(\int_M |\nabla u|^2 dV \leq 1\), \(\int_M u dV = 0\) and for all \(b < b_0\), where \(M\) is a compact Riemann surface with conical singularities of divisor \(\beta = \sum_i^k \beta_i p_i\), \(c_b\) is a constant related to \(b\) and \(b_0 = 4\pi \min_i \{1, 1 + \beta_i\}\). Then Chen [4] further proved a Trudinger inequality with the best constant on such surfaces. For recent development of this topic, we refer the reader to [21, 26, 27] and the references therein.

In this paper, on n-dimensional manifolds with conical singularities we establish a Sobolev type embedding

\[
H^1(M) \hookrightarrow L^p(M)
\]

for any \(1 \leq p \leq \frac{2n}{n-2}\) and show that the embedding is compact for any \(1 \leq p < \frac{2n}{n-2}\). Similar Sobolev type embeddings can be founded in [14, 15]. It is known from [14] that if we consider the embedding above for the function whose trace is equal to 0 on the boundary of an open bounded domain \(\Omega \subset \mathbb{R}^n\), we can get more generalized weighted embedding results than the results obtained here.

On a compact Riemann surface with a finite set of punctures, Li and Wang [19] introduced a complete Poincaré like metric near the punctures and obtain a smooth solution of \((P)\) with the singular metric in the Riemann surfaces excepting at finite purtures. Dey [11] studied the prescribed negative Gaussian curvature problem on a
punctured Riemann surface with a complete metric. We are interested in the basic analysis properties of the singular manifolds with Poincaré like metric, on which we find that the Sobolev’s inequality does not hold (even $n = 2$) by constructing a counter example. However, on the singular manifolds we can prove that the Poincaré inequality holds.

Using the results above, we study the Laplace’s equation $(P)$. We say that $u$ is a weak solution of $(P)$ if $u \in H^1(M)$ and

$$
\int_M \nabla u \nabla \phi \, dV = \int_M f \phi \, dV, \quad \forall \phi \in H^1(M). 
$$

In both cases, we prove the existence of the solution for the Laplace’s equation $(P)$, and the regularities of the solution for $(P)$ in $M \setminus \{p_1, p_2, \ldots, p_k\}$. In conical singularities case, we proved it by the compact embedding given above and the standard variational method. In Poincaré like metric case, we show it by the Poincaré inequality and the spectrum theory. It is interesting that we find the solution in the first case can be extended to the singularities and it is Hölder continuous at every singularity $p_i$ ($i = 1, 2, \ldots, k$) which can be proved by the Moser iteration and conversely the solution in the second case is unbounded near the punctures even if $f$ is smooth on $M^0$ which can be shown by constructing an example.

In this paper, we also consider the semilinear elliptic equation with subcritical exponent

$$
- \Delta u + hu = u^p, \quad 1 < p < \frac{n+2}{n-2} 
$$

where $h \geq m > 0$ with $h \in L^q(M)$ for some $q > \frac{n}{2}$ on singular manifolds with conical singularities. We obtain the existence of nontrivial nonnegative solutions of the equation (1.5) and show that the solutions are Hölder continuous at every singularity $p_i$ ($i = 1, 2, \ldots, k$).

The main results of this paper are the following proposition and theorems.

**Proposition 1.1.** The embedding on singular manifold $(M, \beta, g)$ with conical singularities

$$H^1(M) \hookrightarrow L^p(M)
$$

is compact for all $p \in \left[1, \frac{2n}{n-2}\right)$.

**Theorem 1.2.** Let $u \in H^1(M)$ be a solution of equation $(P)$ with conical metric and $f \in L^q(M)$ for some $q > \frac{n}{2}$. Then $u \in C^s(M)$ for some $s \in (0, 1)$ depending only on $n, q, \alpha$. Moreover, for any $B_r \subset M$

$$
|u(x) - u(y)| \leq C \left(\frac{|x - y|}{r}\right)^s \left\{ \frac{1}{r^{n(\alpha+1)}} \int_{B_r} u^2 \right\}^{\frac{1}{2}} + r^{(2-\frac{n}{q})(\alpha + 1)} ||f||_{L^q(B_r)}
$$

for any $x, y \in B_{\frac{r}{2}}$, where $C$ is a positive constant depending only on $n, q, \alpha$.

**Theorem 1.3.** Let $u \in H^1(M)$ be a solution of equation (1.5) with conical metric where $h \geq m > 0$ and $h \in L^q(M)$ for some $q > \frac{n}{2}$. Then $u \in C^s(M)$ for some
s ∈ (0, 1) depending only on n, q, α. Moreover, for any \( B_r \subset M \)
\[
|u(x) - u(y)| \leq C \left( \frac{|x - y|}{r} \right)^s \left( \frac{1}{r^{n(\alpha + 1)}} \int_{B_r} u^2 \right)^{\frac{1}{2}}
\]
for any \( x, y \in B_{\frac{r}{2}} \), where \( C \) is a positive constant depending only on \( n, q, \alpha \).

**Theorem 1.4.** The equation \( (P) \) with Poincaré like metric has a smooth solution \( u \in W^{1, 2}(M^0) \) for \( f \in C^\infty(M^0) \cap L^2(M^0) \) if and only if \( \int_{M^0} f \, dV = 0 \).

The paper is organized as follows. In section 2, we establish the Sobloev type embedding and prove that the embedding is compact on conical manifolds. In section 3, we obtain the existence of the solution for the equation \( (P) \) with the conical metric and give the proof of Theorem 1.2. In section 4, we obtain the existence of nontrivial nonnegative solutions of the equation (1.5) and give the proof of Theorem 1.3. In the final section, we give the proof of Theorem 1.4 and show that the solution of \( (P) \) can not be continuously extended to the singular points in Poincaré like metric case.

In this paper, without other special notices, we denote \( C \) to be a positive constant which may change from one line to another line.

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2. **The compactness of Sobolev’s embedding on a manifold with conical singularities**

In this section, we prove Sobolev’s inequality and a compactness result on the singular manifold \((M, \beta, g)\) with conical singularities. Hereafter we denote such \( n \)-dimensional singular manifold \((M, \beta, g)\) with conical singularities simply by \( M \).

Associated to the conical metric \( g \), one can define gradient \( \nabla \) and \( \Delta \) operator in the usual way. One can also define the Hilbert space \( H^1(M) \) with norm \( \| \nabla u \|_2 + \| u \|_2 \), where \( \| u \|_p = (\int_M |u|^p \, dv)^{1/p} \) is the \( L^p \)-norm.

**Proposition 2.1.** There exists a constant \( C \) such that for all \( u \in H^1(M) \) and all \( p \in \left[ 1, \frac{2n}{n-2} \right] \), we have
\[
\| u \|_{L^p(M)} \leq C \| u \|_{H^1(M)},
\]
where \( C \) depends on \( n, \alpha \).

**Proof.** Let \( \{ \Omega_l \} \) be a finite covering of \( M \), \( l = 1, 2, \ldots, N \), such that \( \Omega_i \) \( (i = 1, 2, \ldots, k) \) contains the singularity \( p_i \) respectively and \( \Omega_j \) \( (j = k + 1, k + 2, \ldots, N) \) does not contain any singularity \( p_i \) \( (i = 1, 2, \ldots, k) \). Assume that \( (\Omega_l, \varphi_l) \) are the corresponding charts. Consider \( \{ \alpha_l \} \) a \( C^\infty \) partition of unity subordinate to the covering \( \{ \Omega_l \} \). We need prove there exists a constant \( C_l \) such that every \( C^\infty \) function \( u \) on \( M \) satisfies
\[
\| \alpha_l u \|_{L^p(\Omega_l)} \leq C_l \| \alpha_l u \|_{H^1(\Omega_l)}.
\]
It suffices to show that (2.2) holds when \( i = 1, 2, \ldots, k \). Note that on each \( \Omega_j \) \( (j = k + 1, k + 2, \ldots, N) \) there is a smooth metric.
In fact, since \( |\nabla (\alpha l u)| \leq |\nabla u| + |u||\nabla \alpha| \), it is easily seen that

\[
\|u\|_{L^p(M)} \leq \sum_{l=1}^{N} \|\alpha l u\|_{L^p(\Omega_i)}
\]

\[
\leq \sum_{l=1}^{N} C_l \|\alpha l u\|_{H^1(\Omega_i)}
\]

\[
\leq \sum_{l=1}^{N} C_l (\|u\|_{L^2} + \sup_{1 \leq l \leq N} |\nabla \alpha l|||u||_{L^2} + ||\nabla u||_{L^2})
\]

\[
\leq \sup_{1 \leq l \leq N} C_l N \left( (1 + \sup_{1 \leq l \leq N} |\nabla \alpha l|||u||_{L^2} + ||\nabla u||_{L^2} \right)
\]

\[
\leq C \|u\|_{H^1(M)}
\]

where \( C \) is dependent of \( M, \alpha \). Therefore (2.1) holds for all \( u \in H^1(M) \) by density.

Let \( U_i = \varphi_i(\Omega_i) \) be an bounded domain in \( \mathbb{R}^n \). By (1.8) in [14] (also see [5, 8, 20]), we have

\[
\left( \int_{U_i} |x|^\theta |\alpha_i u|^p dx \right)^{1/p} \leq C \left( \int_{U_i} |x|^\theta |\nabla_0 (\alpha_i u)|^2 dx \right)^{1/2}, \quad \forall 1 \leq p \leq \frac{2n}{n-2},
\]

(2.3)

where \( \nabla_0 \) is the gradient with the Euclidean metric and \( C \) depends on \( U_i, \theta \) and \( l \), on the condition that

\[
n + \theta > 2, \quad \tau := l - \theta > -2, \quad \theta \geq \frac{n-2}{2} \tau.
\]

(2.4)

We choose

\[
\theta = (n-2)\alpha, \quad l = n\alpha,
\]

(2.5)

then it is clear that (2.4) is equivalent to

\[
\alpha > -1.
\]

We therefore have the weighted Sobolev’s inequality, \( \forall 1 \leq p \leq \frac{2n}{n-2} \),

\[
\left( \int_{U_i} |x|^{n\alpha} |\alpha_i u|^p dx \right)^{1/p} \leq C_i \left( \int_{U_i} |x|^{(n-2)\alpha} |\nabla_0 (\alpha_i u)|^2 dx \right)^{1/2},
\]

(2.6)

where \( C_i \) depends on \( \Omega_i \) and \( \beta_i \), and \( \nabla_0 \) is the gradient with respect to \( ds_0^2 \), if \( \alpha > -1 \). Obviously, we see

\[
\left( \int_{\Omega_i} |\alpha_i u|^p \ dV \right)^{1/p} = \left( \int_{U_i} |\alpha_i u|^p |x|^{n\alpha} \ dx \right)^{1/p}
\]

\[
\leq C_i \left( \int_{U_i} |x|^{(n-2)\alpha} |\nabla_0 (\alpha_i u)|^2 \ dx \right)^{1/2}
\]

\[
= C_i \left( \int_{\Omega_i} |\nabla (\alpha_i u)|^2 \ dV \right)^{1/2}
\]

\[
= C_i \|\alpha_i u\|_{H^1(\Omega_i)}, \quad \forall i = 1, 2, \ldots, k.
\]

(2.7)

Thus we obtain inequality (2.2).
The above Proposition 2.1 implies, that for any \( p \in [1, \frac{2n}{n-2}] \) the imbedding \( H^1(M) \hookrightarrow L^p(M) \) is continuous. Moreover, we will show that the embedding is compact in the following.

**Proof of Proposition 1.1.** Let \( (\Omega_l, \varphi_l) \ (l = 1, 2, \ldots, N) \) be a finite atlas of \( M \) similar as those in the proof of Proposition 2.1, each \( \Omega_l \) being homeomorphic to a ball \( B \) of \( \mathbb{R}^n \) \((n \geq 2)\). Consider a \( C^\infty \) partition of unity \( \{\alpha_l\} \) subordinate to the covering \( \{\Omega_l\} \).

Suppose that \( \{u_m\}_{m=1}^\infty \) is a bounded sequence in \( H^1(M) \). Then \( \{\alpha_l u_m\}_{m=1}^\infty \) is also a bounded sequence in \( H^1(\Omega_l) \). If \( l \neq 1, 2, \ldots, k \), since the metric on \( \Omega_l \) is smooth, then \( \{\alpha_l u_m\}_{m=1}^\infty \) is precompact in \( L^p(\Omega_l) \) by the standard Kondrakov’s Theorem for compact Riemannian manifolds.

It suffices to show that \( \{\alpha_l u_m\}_{m=1}^\infty \) is precompact in \( L^p(\Omega_l) \) when \( l = 1, 2, \ldots, k \). Consider the functions defined on \( B, l = 1, 2, \ldots, k \) being given:

\[
h_m(x) = (\alpha_l u_m) \circ \varphi_l^{-1}(x).
\]

By Proposition 1.1 in [14], we know that if \( \alpha > -1 \) the embedding

\[
H^1_0(n-2\alpha)(B) \hookrightarrow L^p_{\alpha}(B)
\]

is compact for any \( p \in [1, \frac{2n}{n-2}] \). Recall that, in [14] \( H^1_0(n-2\alpha)(B) \) is the completion of \( C^\infty_0(B) \) under the norm induced by the inner product

\[
(u, v) = \int_B |x|^{(n-2)\alpha} \nabla u \cdot \nabla v \, dx,
\]

and \( L^p_{\alpha}(B) \) \((p \geq 1)\) is the space of functions \( \varphi \) such that

\[
|x|^n \varphi \in L^p(B)
\]

with the norm

\[
||\varphi||_{n, p} = \left( \int_B |x|^n |\varphi|^p \, dx \right)^{\frac{1}{p}}.
\]

It is easily seen that \( \{h_m(x)\}_{m=1}^\infty \) is bounded in \( H^1_0(n-2\alpha)(B) \). Hence \( \{h_m(x)\}_{m=1}^\infty \) is precompact in \( L^p_{\alpha}(B) \). Repeating this argument successively for \( l = 1, 2, \ldots, N \), we may extract a subsequence \( \{\tilde{u}_m\} \) of the sequence \( \{u_m\} \) by a standard argument, such that \( \{\alpha_l \tilde{u}_m \circ \varphi_l^{-1}\}_{m=1}^\infty \) converges in \( L^p_{\alpha}(B) \) for each \( l \). Assume \( \{\alpha_l \tilde{u}_m \circ \varphi_l^{-1}\} \) converges to \( g_l \) as \( m \to \infty \), i.e.,

\[
\int_{\Omega_l} |\alpha_l \tilde{u}_m - g_l \circ \varphi_l|^p \, dV \to 0 \text{ as } m \to \infty \text{ for all } l,
\]

and \( g_l \circ \varphi_l \in L^p(\Omega_l) \).

We extend functions \( g_l \circ \varphi_l \) to be zero in \( M \setminus \Omega_l \) and set \( \tilde{u} = \sum_{l=1}^{N} g_l \circ \varphi_l \). We easily see from (2.9) that \( \tilde{u} \in L^p(M) \) and

\[
\left( \int_M |\tilde{u}_m - \tilde{u}|^p \, dV \right)^{\frac{1}{p}} = \left( \int_M \left| \sum_{l=1}^{N} (\alpha_l \tilde{u}_m - g_l \varphi_l)^p \right| \, dV \right)^{\frac{1}{p}} \leq \sum_{l=1}^{N} \left( \int_{\Omega_l} |\alpha_l \tilde{u}_m - g_l \circ \varphi_l|^p \, dV \right)^{\frac{1}{p}}.
\]
This also implies that $\bar{u}_m \to \bar{u}$ in $L^p(M)$ as $m \to \infty$ and the proof of this proposition is complete. 

As a direct consequence, we have the Poincaré inequality similar as the Proposition 5 in [24].

**Corollary 2.2.** Let $\psi \in L^2(M)$ be a function such that $\int_M \psi \, dV \neq 0$, then there exists a constant $C$ such that $\|u\|_{L^2(M)} \leq C \|\nabla u\|_{L^2(M)}$, for all $u \in H^1(M)$ with $\int_M u \psi \, dV = 0$.

3. Existence and Regularity of solutions for Laplace's equation on conical manifolds

In this section, we study the equation

$$\Delta u = f$$

on $M$ where $\Delta$ is the Laplace-Beltrami operator induced by the conical metric and $f$ belongs to $L^2(M)$. The existence theorem and regularity result of solutions for the equation (3.1) are obtained. Applying the compact embedding theorem proved in last section, the proof of the existence is almost standard.

**Theorem 3.1.** There exists a weak solution $u \in H^1(M)$ of (3.1) if and only if $\int_M f \, dV = 0$. The solution $u$ is unique up to a constant.

**Proof.** (i) The proof of necessity is obvious.

(ii) Existence of $u$. If $f \equiv 0$, the solutions of (3.1) are $u \equiv$ constant. Hence suppose $f \not\equiv 0$. Consider the functional $I(u) = \int_M |\nabla u|^2 \, dV$. Define $\mu = \inf I(u)$ for all $u \in \mathcal{B}$, with $\mathcal{B} = \{u \in H^1(M) : \int_M u \, dV = 0, \int_M uf \, dV = 1\}$.

It is clear that $0 \leq \mu \leq I(f\|f\|_2^2)$. Let $\{u_i\}_{i=1}^\infty$ be a minimizing sequence in $\mathcal{B}$. Thus the set $\{\|\nabla u_i\|\}_{i=1}^\infty$ is bounded in $L^2(M)$. It follows by Poincaré inequality that $\{u_i\}_{i=1}^\infty$ is bounded in $H^1(M)$.

By the compactness of the embedding with conical metric in Proposition 1.1, there exists a subsequence $\{u_k\}$ of $\{u_i\}$ and $u_0 \in H^1(M)$ such that $\|u_k - u_0\|_{L^2} \to 0$ and such that $I(u_0) \leq \mu$.

Hence $u_0 \in \mathcal{B}$ and $I(u_0) = \mu$. Since $u_0$ minimizes the variational problem, there exists two constants $\beta$ and $\gamma$ such that for all $\phi \in H^1$:

$$\int_M \nabla u_0 \nabla \phi \, dV = \beta \int_M f \phi \, dV + \gamma \int_M \phi \, dV.$$

Picking $\phi = 1$ yields $\gamma = 0$. Choosing $\phi = u_0$ implies $\beta = \mu$. Since $\int_M u_0 f \, dV = 1$, $u_0$ is not constant and $\mu = I(u_0) > 0$. Set $\bar{u} = u_0/\mu$. Then $\bar{u}$ satisfies equation (3.1) weakly in $H^1(M)$.

Furthermore, we not only obtain removable singularity for solutions to the equation (3.1), but also prove the Hölder continuity of the solution. Firstly, we prove local boundedness of the solutions.

**Theorem 3.2.** Suppose $u \in H^1(M)$ is a solution of equation (3.1) and $f \in L^q(M)$ for some $q > n/2$. Then we have for any ball $B_{2r} \subset M$ and $p > 1$,

$$\sup_{B_r} |u| \leq C \left( r^{-\frac{n(\alpha+1)}{p}} \|u\|_{L^p(B_{2r})} + r^{(2-\frac{n}{q})(\alpha+1)} \|f\|_{L^q(B_{2r})} \right)$$

(3.2)
where \( C = C(n, \alpha, q, p) \).

**Proof.** Similarly as before, we have to consider the boundedness of the solution near the conical singularity. Let \( r = 1 \). For some \( k > 0 \) and \( m > 0 \), set \( \bar{u} = u^+ + k \) and

\[
\bar{u}_m = \begin{cases} 
\bar{u} & \text{if } u < m, \\
k + m & \text{if } u \geq m.
\end{cases}
\]  

(3.3)

Then we have \( D\bar{u}_m = 0 \) in \( \{ u < 0 \} \) and \( \{ u > m \} \), and \( \bar{u}_m \leq \bar{u} \). Consider the test function

\[
\varphi = \eta^2 (\bar{u}_m^\beta \bar{u} - k^{\beta + 1}) \in H^1_0(B_2),
\]

for some \( \beta \geq 0 \) and some function \( \eta \in C^1_0(B_2) \). We only integrate in the set \( \{ u > 0 \} \) note that \( \varphi = 0 \) and \( D\varphi = 0 \) in \( \{ u \leq 0 \} \). Note also that \( k \leq \bar{u} \) and \( \bar{u}_m^\beta \bar{u} - k^{\beta + 1} \leq \bar{u}_m^\beta \bar{u} \) for \( k > 0 \). We have

\[
\int_{\{ u > 0 \}} \nabla u \nabla \varphi \\
\geq \int_{\{ 0 < u < m \}} \beta \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int_{\{ u > 0 \}} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - \int_{\{ u > 0 \}} |\nabla \bar{u}| |\nabla \eta| \bar{u}_m^\beta \bar{u} \eta \\
\geq \int_{\{ 0 < u < m \}} \beta \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{1}{2} \int_{\{ u > 0 \}} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - 2 \int_{\{ u > 0 \}} |\nabla \eta|^2 \bar{u}_m^\beta \bar{u}^2.
\]

Hence, we obtain

\[
\int_{\{ 0 < u < m \}} \beta \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int_{\{ u > 0 \}} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \\
\leq C \left( \int_{\{ u > 0 \}} |\nabla \eta|^2 \bar{u}_m^\beta \bar{u}_m^\beta + \int_{\{ u > 0 \}} |f| k^{-1} \bar{u}_m^\beta \bar{u} \right) \\
\leq C \left( \int_{\{ u > 0 \}} |\nabla \eta|^2 \bar{u}_m^\beta \bar{u}_m^\beta + \int_{\{ u > 0 \}} |f| k^{-1} \eta^2 \bar{u}_m^\beta \bar{u}^2 \right).
\]

Choose \( k = ||f||_{L^q} \) if \( f \) is not identically zero. Otherwise choose an arbitrary \( k > 0 \) and eventually let \( k \to 0^+ \). Set \( w = \bar{u}_m^\beta \bar{u} \). Note

\[
|Dw|^2 \leq (1 + \beta) \{ \beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2 \}.
\]

Therefore, we have

\[
\int_{B_{c_r}} |Dw|^2 \eta^2 \leq C \left( (1 + \beta) \int_M w^2 |D\eta|^2 + (1 + \beta) \int_M |f| k^{-1} w \eta^2 \right) \\
\leq C \left( (1 + \beta) \int_M w^2 |D\eta|^2 + (1 + \beta) ||w||_{L^q}^{2q} \right).
\]

By the Sobolev’s inequality and the interpolation with

\[
2 \leq \frac{2q}{q - 1} \leq \frac{2n}{n - 2} \text{ if } q > \frac{n}{2},
\]
we have
\[
||\eta w||_{L^{2n/2}} \leq \varepsilon ||\eta w||_{L^{2n/2}} + C(n, q)\varepsilon^{-\frac{n}{2(n-q)}} ||\eta w||_{L^2} \\
\leq \varepsilon ||D(\eta w)||_{L^2} + \varepsilon ||\eta w||_{L^2} + C(n, q)\varepsilon^{-\frac{n}{2(n-q)}} ||\eta w||_{L^2},
\]
for any \(\varepsilon > 0\). We choose a small constant \(\varepsilon\) so that
\[
\int |D(w\eta)|^2 \leq C(1 + \beta) \int w^2|D\eta|^2 + \int \eta^2 w^2 + C(1 + \beta)\frac{2n}{2n - n} \int \eta^2 w^2.
\]
The sobolev inequality implies
\[
\left( \int |\eta w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C(1 + \beta) \int w^2|D\eta|^2 + C \int \eta^2 w^2 + C(1 + \beta)\frac{2n}{2n - n} \int \eta^2 w^2
\]
\[
\leq C(1 + \beta)\frac{2n}{2n - n} \int (|D\eta|^2 + \eta^2) w^2.
\]
For any \(1 \leq r < R \leq 2\), consider an \(\eta \in C^\infty_0(B_R)\) with the property
\[
\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{2}{R - r}.
\]
Setting \(\chi = \frac{n}{n-2}\) for \(n > 2\) and \(\chi = 2\) for \(n = 2\), we obtain
\[
\left( \int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(1 + \beta)\frac{2n}{2n - n}}{(R - r)^2} \int_{B_R} w^2.
\]
Recalling the definition of \(w\), we have
\[
\left( \int_{B_r} \bar{u}^{2\chi} \bar{u}_m^\beta \right)^{\frac{1}{\chi}} \leq C \frac{(1 + \beta)\frac{2n}{2n - n}}{(R - r)^2} \int_{B_R} \bar{u}^2 \bar{u}_m^\beta.
\]
Set \(\gamma = \beta + 2\). Then we obtain
\[
\left( \int_{B_r} \bar{u}^{\gamma\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(\gamma - 1)\frac{2n}{2n - n}}{(R - r)^2} \int_{B_R} \bar{u}^\gamma.
\]
By letting \(m \to \infty\) we conclude that
\[
||\bar{u}||_{L^{\gamma\chi}(B_r)} \leq \left( C \frac{(\gamma - 1)\frac{2n}{2n - n}}{(R - r)^2} \right)^{\frac{1}{\chi}} ||\bar{u}||_{L^\gamma(B_R)},
\]
where \(C\) is a positive constant depending only \(n, q, \alpha\) and independent of \(\gamma\). Now, taking \(p > 1\), we set \(\gamma = \gamma_i = \chi^i p\) and \(r_i = 1 + 2^{-i}, i = 0, 1, \ldots\), so that for \(i = 1, 2, \ldots\),
\[
||\bar{u}||_{L^{\gamma_i}(B_{r_i})} \leq \left( C(n, p, q, \alpha) \right)^{\frac{1}{\chi^i}} ||\bar{u}||_{L^{\gamma_{i-1}}(B_{r_{i-1}})}.
\]
Hence, by an iteration we obtain
\[
||\bar{u}||_{L^{\gamma_i}(B_{r_i})} \leq \left( C(n, p, q, \alpha) \right)^{\sum \frac{1}{\chi^i}} ||\bar{u}||_{L^p(B_2)}.
\]
Letting \(i \to \infty\), we get
\[
\sup_{B_1} \bar{u} \leq C ||\bar{u}||_{L^p(B_2)},
\]
or
\[ \sup_{B_1} u^+ \leq C(||u^+||_{L^p(B_2)} + k). \]

Since \( u \) must be a supersolution of equation (3.1), we similarly have
\[ \sup_{B_1} (u^-) \leq C(||u^-||_{L^p(B_2)} + k), \]
where \( u^- = \max\{-u, 0\} \). Therefore, we obtain
\[ \sup_{B_1} |u| \leq C(||u||_{L^p(B_2)} + k). \]

Now, the inequality (3.2) can be proved by a dilation argument. \( \square \)

**Remark 3.3.** More generally, suppose \( f \in L^q(M) \) for some \( q > \frac{n}{2} \) and
\[ ||f||_{L^q} \leq \Lambda, \]
for some positive constants \( \Lambda \). Assume that \( u \in H^1(M) \) is a subsolution in the following sense
\[ \int_M \nabla u \nabla \varphi \leq \int_M fu \varphi + \int_M g \varphi \text{ for any } \varphi \in H^1(M) \text{ with } \varphi \geq 0 \text{ in } M. \]
Then if \( g \in L^q(M) \), the inequality
\[ \sup_{B_r} u^+ \leq C \left( r^{-\frac{n(\alpha+1)}{p}} ||u||_{L^p(B_{2r})} + r^{(2-\frac{n}{q})(\alpha+1)} ||g||_{L^q(B_{2r})} \right) \]
holds for any \( p > 0 \) where \( C \) is a positive constant depending only on \( n, \alpha, p, q \).

Then the next result is referred to as the weak Harnack inequality. To this aim, we need an important lemma.

Because the conical metric space satisfies doubling condition, i.e.
\[ \frac{\text{vol}(B_{2R})}{\text{vol}(B_R)} \leq C_{n, \alpha}. \]
Indeed, we can assume that \( B_R \) is a ball of radius \( R \) centered at \( p_i \) \((i = 1, 2, \ldots, k)\). Then
\[ \text{vol}(B_R) = \int_{B_R} |x|^{\alpha} dx = \int_0^R \int_{S^{n-1}} r^{\alpha+n-1} d\sigma dr = \frac{W_n}{n(\alpha+1)} R^{n(\alpha+1)}. \]
(3.4)

It was proved that (see [25]) the John-Nirenberg lemma in [17] generalizes easily to accommodate doubling measures by the Calderon-Zygmund decomposition.

**Lemma 3.4.** Let \( u_{x,r} = \frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} u \ dV \). Suppose \( u \in L^1(M) \) satisfies
\[ \frac{1}{\text{vol}(B_r(x))} \int_{B_r} |u - u_{x,r}| \ dV \leq M_0, \text{ for any } B_r(x) \subset M, \]
where \( M_0 \) is a positive constant. Then there holds for any \( B_r(x) \subset M \)

\[
\frac{1}{\text{vol}(B_r(x))} \int_{B_r(x)} e^{p_0 |u-u_{x,r}|} \, dV \leq C , \tag{3.5}
\]

for some positive \( p_0 \) and \( C \) depending only on \( n \).

**Theorem 3.5.** Suppose \( u \in H^1(M) \) is a nonnegative solution of equation (3.1) and \( f \in L^q(M) \) for some \( q > \frac{n}{2} \). Then we have for any ball \( B_{4r} \subset M \) and \( 0 < p < \frac{n}{n-2} \),

\[
r^{-n/2} \|u\|_{L^p(B_{2r})} \leq C \left( \inf_{B_r} u(y) + r^{\frac{n}{2} - 1} \|f(y)\|_{L^q(B_{2r})} \right), \tag{3.6}
\]

where \( C = C(n, \alpha, q, p) \).

**Proof.** We only prove for \( r = 1 \).

**Step 1.** We prove that the result holds for some \( p_0 > 0 \).

Set \( \bar{u} = u + k > 0 \), for some \( k > 0 \) to be determined and \( v = \bar{u}^{-1} \). For any \( \varphi \in H^1_0(B_4) \) with \( \varphi \geq 0 \) in \( B_4 \), consider \( \bar{u}^{-2} \varphi \) as the test function in the definition of weak solution. We have

\[
\int_{B_4} \nabla v \nabla \varphi + \tilde{f} v \varphi \leq 0 ,
\]

where \( \tilde{f} = \frac{f}{\bar{u}} \). Then \( v \) is a nonnegative subsolution of some homogeneous equation. Choosing \( k = \|f\|_{L^q} \), we have \( \|\tilde{f}\|_{L^q} \leq 1 \). Remark 3.3 implies that for any \( p > 0 \)

\[
\sup_{B_1} \bar{u}^{-p} \leq C \int_{B_2} \bar{u}^{-p} ,
\]

or,

\[
\inf_{B_1} \bar{u} \geq C \left( \int_{B_2} \bar{u}^{-p} \right)^{-\frac{1}{p}}
\]

\[
= C \left( \int_{B_2} \bar{u}^{-p} \int_{B_2} \bar{u}^p \right)^{-\frac{1}{p}} \left( \int_{B_2} \bar{u}^p \right)^{\frac{1}{p}} ,
\]

where \( C \) is a positive constant depending on \( n, \alpha, p, q \).

The key point next is to show that there exists a \( p_0 > 0 \) such that

\[
\int_{B_2} \bar{u}^{-p_0} \int_{B_2} \bar{u}^p_0 \leq C ,
\]

where \( C \) is a positive constant depending on \( n, \alpha, q \). We will show for any \( B_R \subset B_4 \)

\[
\int_{B_R} e^{p_0 |w|} \leq C ,
\]

where \( w = \log \bar{u} - \beta \) with \( \beta = |B_R|^{-1} \int_{B_R} \log \bar{u} \).

Consider \( \bar{u}^{-1} \varphi \) as the test function with \( \varphi \in L^\infty(B_4) \cap H^1_0(B_4) \) and \( \varphi \geq 0 \). By a direct calculation we have

\[
\int_{B_4} |D \varphi|^2 = \int_{B_4} D \varphi D \varphi - \int_{B_4} \tilde{f} \varphi . \tag{3.7}
\]
Replace \( \varphi \) by \( \varphi^2 \) in (3.7). Then the Cauchy inequality implies
\[
\int_{B_4} |Dw|^2 \varphi^2 \leq C \left\{ \int_{B_4} |D\varphi|^2 + \int_{B_4} |f|^2 \varphi^2 \right\},
\]
where
\[
\int_{B_4} |f|^2 \varphi^2 \leq ||f||_{L^\infty} ||\varphi||_{L^{2n}}^2 \leq C(n, q) ||\varphi||_{L^{2n}}^2 \leq C(n, q, \alpha) ||D\varphi||_{L^2}^2.
\]
Therefore, we have
\[
\int_{B_4} |Dw|^2 \varphi^2 \leq C(n, q, \alpha) \int_{B_4} |D\varphi|^2 dV.
\]
For any \( B_{2R}(y) \subset B_4 \), choose \( \psi \in C_0^\infty (B_2(y)) \) with
\[
\psi(\tilde{x}) = \begin{cases} 
1 & \tilde{x} \in B_1(y), \\
0 & \tilde{x} \in M \setminus B_2(y),
\end{cases}
\]
and \( |\nabla \psi(\tilde{x})| \leq C. \) (3.8)
By scaling, we consider the cut-off function
\[
\varphi(x) = \psi \left( \frac{x_1}{R}, \ldots, \frac{x_n}{R} \right)
\]
with
\[
\text{supp } \varphi \subset B_{2R}(y), \varphi \equiv 1 \text{ in } B_R(y), \ |D\varphi| \leq \frac{C}{R^{1+\alpha}}.
\]
Indeed,
\[
|\nabla \varphi(x)|^2 = |x|^{-2\alpha} \left| \nabla_0 \varphi(x) \right|^2
= |R|^{-2\alpha} |\tilde{x}|^{-2\alpha} R^{-2} \left| \nabla_0 \psi(\tilde{x}) \right|^2
= R^{-2(1+\alpha)} \left| \nabla \psi \right|^2.
\]
Hence, we obtain
\[
\int_{B_R(y)} |Dw|^2 \leq C R^{-2(\alpha+1)} \text{vol}(B_R(y)).
\]
By the Poincaré inequality
\[
\int_{B_R(y)} |w - w_{y,R}|^2 dV \leq C R^{2(\alpha+1)} \int_{B_R(y)} |Dw|^2 dV,
\]
we have
\[
\frac{1}{\text{vol}(B_R(y))} \int_{B_R(y)} |w - w_{y,R}| dV \leq \frac{1}{\text{vol}(B_R(y))} \left( \int_{B_R(y)} |w - w_{y,R}|^2 dV \right)^{\frac{1}{2}}
\leq \frac{1}{\text{vol}(B_R(y))} \left( C R^{2(\alpha+1)} \int_{B_R(y)} |Dw|^2 dV \right)^{\frac{1}{2}}
\leq C.
\] (3.9)
Then Lemma 3.4 implies

\[ \int_{B_r} e^{p_0 |w|} \leq C. \]

**Step 2.** The result holds for any positive \( p < n/(n - 2) \).

We need to prove for any \( 1 \leq r_1 < r_2 \leq 3 \) and \( 0 < p_2 < p_1 < n/(n - 2) \),

\[
\left( \int_{B_{r_1}} \bar{u}^{p_1} \right)^{\frac{1}{p_1}} \leq C \left( \int_{B_{r_2}} \bar{u}^{p_2} \right)^{\frac{1}{p_2}},
\]

(3.10)

where \( C \) is a positive constant depending on \( n, q, \alpha, r_1, r_2, p_1 \) and \( p_2 \).

Take \( \varphi = \bar{u}^{-\beta} \eta^2 \) for \( \beta \in (0, 1) \) as the test function and we have

\[
\int_{B_4} |D\bar{u}|^2 \bar{u}^{-\beta - 1} \eta^2 \leq C \left( \frac{1}{\beta^2} \int_{B_4} |D\eta|^2 \bar{u}^{1-\beta} + \frac{1}{\beta} \int_{B_4} |f|^2 \eta^2 \bar{u}^{1-\beta} \right).
\]

Set \( \gamma = 1 - \beta \in (0, 1) \) and \( w = \bar{u}^\gamma \). Then, we obtain

\[
\int_{B_4} |D(w\eta)|^2 \leq \frac{C}{(1-\gamma)^m} \int_{B_4} w^2 (|D\eta|^2 + \eta^2),
\]

for some positive \( m > 0 \). Proposition 2.1 and an appropriate choice of cut-off function imply, with \( \chi = n/(n - 2) \), for any \( 1 \leq r < R \leq 3 \)

\[
\left( \int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq \frac{C}{(1-\gamma)^m} \cdot \frac{1}{(R-r)^2} \int_{B_r} w^2,
\]

or

\[
\left( \int_{B_r} \bar{u}^{\gamma \chi} \right)^{\frac{1}{\chi}} \leq \left( \frac{C}{(1-\gamma)^m} \cdot \frac{1}{(R-r)^2} \right)^{\frac{1}{\gamma}} \left( \int_{B_r} \bar{u}^\gamma \right)^{\frac{1}{\gamma}}.
\]

This holds for any \( \gamma \in (0, 1) \). Now (3.10) follows after finitely many iterations. \( \Box \)

**Remark 3.6.** Suppose \( f \in L^q(M) \) for some \( q > \frac{n}{2} \) and

\[ \|f\|_{L^q} \leq \Lambda, \]

for some positive constants \( \Lambda \). Assume that \( u \in H^1(M) \) is a nonnegative supersolution in the following sense

\[ \int_M \nabla u \nabla \varphi \geq \int_M fu \varphi + \int_M g \varphi \text{ for any } \varphi \in H^1(M) \text{ with } \varphi \geq 0 \text{ in } M. \]

Then if \( g \in L^p(M) \), we have for any ball \( B_{4r} \subset M \) and \( 0 < p < \frac{n}{n-2} \),

\[
r^{-\frac{n(\alpha+1)}{p}} \|u\|_{L^p(B_{2r})} \leq C \left( \inf_{B_r} u(y) + r^{(2-\frac{n}{q})(\alpha+1)} \|g(y)\|_{L^q(B_{2r})} \right),
\]

(3.11)

where \( C = C(n, \alpha, q, p) \).

Now the Moser’s Harnack inequality is an easy consequence of above results.

**Corollary 3.7.** Let \( u \in H^1(M) \) be a nonnegative solution of equation (3.1) and \( f \in L^q(M) \) for some \( q > n/2 \). Then we have for any ball \( B_{4r} \subset M \),

\[ \sup_{B_r} u \leq C \left( \inf_{B_r} u + r^{(2-\frac{n}{q})(\alpha+1)} \|f\|_{L^q(B_{2r})} \right), \]

where \( C \) is a positive constant depending on \( n, \alpha, q \).
Remark 3.8. If \( u \in H^1(M) \) is a nonnegative solution of equation
\[
- \Delta u = f(x)u + g(x) \text{ in } M, \tag{3.12}
\]
and \( f, g \in L^q(M) \) for some \( q > n/2 \), then we have for any ball \( B_{4r} \subset M \),
\[
\sup_{B_r} u \leq C \left( \inf_{B_r} u + r^{(2-\frac{n}{q})(\alpha+1)} ||g||_{L^q(B_{2r})} \right),
\]
where \( C \) is a positive constant depending on \( n, \alpha, q \).

The Hölder continuity follows easily from Corollary 3.7.

Proof of Theorem 1.2. The proof is standard. We prove the estimate for \( r = 1 \).

Set for \( r \in (0, 1) \)
\[
M(r) = \sup_{B_r} u \quad \text{and} \quad m(r) = \inf_{B_r} u.
\]
Then \( M(r) < \infty \) and \( m(r) > -\infty \). It suffices to show
\[
w(r) \equiv M(r) - m(r) \leq Cr^\delta \left\{ ||u||_{L^2(B_1)} + ||f||_{L^q(B_1)} \right\} \quad \text{for any } r < \frac{1}{2},
\]
Set \( \delta = (2 - \frac{n}{q})(\alpha + 1) > 0 \). Apply Corollary 3.7 to \( M(r) - u \geq 0 \) in \( B_r \) to get
\[
\sup_{B_r} (M(r) - u) \leq C \left\{ \inf_{B_r} (M(r) - u) + r^\delta ||f||_{L^q(B_r)} \right\},
\]
i.e.,
\[
M(r) - m(r) \leq C \left\{ (M(r) - M(\frac{r}{2})) + r^\delta ||f||_{L^q(B_r)} \right\}. \tag{3.13}
\]
Similarly, apply Corollary 3.7 to \( u - m(r) \geq 0 \) in \( B_r \) to get
\[
M(\frac{r}{2}) - m(r) \leq C \left\{ (m(\frac{r}{2}) - m(r)) + r^\delta ||f||_{L^q(B_r)} \right\}. \tag{3.14}
\]
Then by adding (3.13) and (3.14) together, we obtain
\[
w(\frac{r}{2}) \leq \frac{C - \frac{1}{C + 1}}{C + 1} w(r) + Cr^\delta ||f||_{L^q(B_r)}.
\]
Obviously, we can show the desired result by Lemma 8.23 in [16]. \( \square \)

The Hölder continuity also follows easily from Remark 3.8.

Corollary 3.9. Let \( u \in H^1(M) \) be a solution of equation (3.12) with conical metric
and \( f, g \in L^q(M) \) for some \( q > \frac{n}{2} \). Then \( u \in C^s(M) \) for some \( s \in (0, 1) \) depending only on \( n, q, \alpha \). Moreover, for any \( B_r \subset M \)
\[
|u(x) - u(y)| \leq C \left( \frac{|x - y|}{r} \right)^s \left\{ \left( \frac{1}{r^{n(\alpha+1)}} \int_{B_r} u^2 \right)^{\frac{1}{2}} + r^{(2-\frac{n}{q})(\alpha+1)} ||g||_{L^q(B_r)} \right\}
\]
for any \( x, y \in B_{\frac{r}{2}} \), where \( C \) is a positive constant depending only on \( n, q, \alpha \).
4. Existence and Regularity of solutions for semilinear elliptic equations with subcritical exponent on conical manifolds

In this section, we consider the semilinear elliptic equation

$$-\Delta u + hu = u^p, \quad 1 < p < \frac{n+2}{n-2}$$

(4.1)

on $M$ where $\Delta$ is the Laplace-Beltrami operator induced by the conical metric, $h \geq m > 0$ and $h \in L^q(M)$ for some $q > \frac{n}{2}$. We allow the coefficient $h$ to be unbounded from above. The existence theorem and regularity result of nontrivial nonnegative solutions for the equation (4.1) are obtained.

Using the embeddings established in Proposition 1.1, we obtain existence of nontrivial nonnegative solutions to the semilinear elliptic equation (4.1) via variational methods.

**Theorem 4.1.** Suppose that $h \in L^q(M)$ for some $q > \frac{n}{2}$. Then there exists a weak solution $u \in H^1(M)$ of (4.1).

**Proof.** Set $A = \{u \in H^1(M) : ||u||^2_{L^{p+1}} = 1\}$. Consider the minimizing problem

$$J = \inf_{u \in A} \int_M |\nabla u|^2 + hu^2 \, dV.$$

On the one hand, we easily see from the assumption and the embedding theorem in Proposition 2.1 that

$$\int_M hu^2 \, dV = \left( \int_M h^q \, dV \right)^{\frac{1}{q}} \left( \int_M u^{\frac{2q}{q-1}} \, dV \right)^{\frac{q-1}{q}} \leq C \left( \int_M |\nabla u|^2 + |u^2| \, dV \right)^{\frac{q-1}{q}} \leq C.$$

Note that

$$\frac{2q}{q-1} < \frac{2n}{n-2}, \quad \text{if } q > \frac{n}{2}.$$

This implies $J < \infty$. On the other hand, we have

$$\int_M |\nabla u|^2 + hu^2 \, dV \geq \min\{1, m\} \int_M |\nabla u|^2 + u^2 \, dV \geq C \min\{1, m\} ||u||^2_{L^{p+1}}.$$

So it implies that $J > 0$. Using compactness of the embedding given in Proposition 1.1 and the standard variational method, we know that $J$ attains at some $u_0 \in A$. Moreover, $u_0 \not\equiv 0$ in $M$. Note that

$$\int_M |\nabla u|^2 + hu^2 \, dV \geq \int_M |\nabla u|^2 + hu^2 \, dV.$$

And $u_0 \not\equiv 0$ in $M$. We easily see that $u_0$ satisfies the equation

$$-\Delta u + hu = \lambda u^p$$
with
\[
\lambda = \frac{\int_M |\nabla u_0|^2 + hu_0^2 \, dV}{(\int_M |u_0|^{p+1} \, dV)^{\frac{2}{p+1}}}
\]
Then \(u = \lambda^{\frac{1}{p+1}} u_0\) is a nontrivial nonnegative solution of (4.1). The regularity of elliptic equations implies that \(u \in C^\infty(M \setminus \{p_1, p_2, \ldots, p_k\})\) when \(h \in C^\infty(M \setminus \{p_1, p_2, \ldots, p_k\})\). The strong maximum principle also implies that \(u > 0\) in \(M \setminus \{p_1, p_2, \ldots, p_k\}\). This completes the proof of Theorem 4.1. □

**Remark 4.2.** The proof of Theorem 4.1 implies that when \(p = 1\), the eigenvalue problem
\[
-\Delta u + hu = \lambda u \quad \text{in} \quad M
\]
admits a positive eigenvalue \(0 < \lambda_1 < \infty\), the corresponding eigenfunction \(\phi_1 \in C^\infty(M \setminus \{p_1, p_2, \ldots, p_k\})\) and \(\phi_1 > 0\) in \(M \setminus \{p_1, p_2, \ldots, p_k\}\), if \(h \in C^\infty(M \setminus \{p_1, p_2, \ldots, p_k\})\).

We will show that the solution \(u\) of (4.1) is bounded and Hölder continuous at every singularity \(p_i\) \((i = 1, 2, \ldots, k)\) in the following.

**Proof of Theorem 1.3.** We apply Corollary 3.9 to show the Hölder continuity of the solution at every singularity \(p_i\) \((i = 1, 2, \ldots, k)\) for the equation (4.1). In our case, we choose \(f(x) = u^{p-1} - h\) and \(g(x) = 0\) in the equation (3.12). Indeed, it follows from \(u \in H^1(M)\) and the embedding theorem in Proposition 2.1 that \(||u^{p-1}||_{L^q} < \infty\) provided \(q < \frac{2n}{(n-2)(p-1)}\). Note that
\[
\frac{2n}{(n-2)(p-1)} > \frac{n}{2} \quad \text{for} \quad 1 < p < \frac{n+2}{n-2}.
\]
Therefore, we can choose
\[
\frac{n}{2} < q < \frac{2n}{(n-2)(p-1)}
\]
such that \(||u^{p-1}||_{L^q} < \infty\). This completes the proof of Theorem 1.3. □

5. **Laplace’s equation on manifolds with Poincaré like metrics**

In this section we mainly prove the existence of the solution for \(\Delta u = f\) on punctured Riemannian manifolds with Poincaré like metrics.

Suppose \(W^{k,p}(M^0)\) is the Sobolev space on \(M^0\) with the Poincaré like metric. We denote the Laplacian of \(M^0\) by \(\Delta\) which is the closed extension of Laplacian acting on smooth functions with compact support. Moreover, the definition domain of \(\Delta\) called \(\mathcal{D}(\Delta)\) consists of those \(g \in L^2(M^0)\) such that \(\Delta g \in L^2(M^0)\) in the sense of distribution. Then \(\Delta\) is a self-adjoint operator and \(\mathcal{D}(\Delta) \subset W^{1,2}(M^0)\) because \(||\nabla g||_2^2 = (g, \Delta g)\). Since \(M^0\) has a finite volume, constants are eigenfunctions of \(\Delta\).

In the following theorem, we prove the Poincaré inequality for the complete Riemannian manifold \(M^0\).

**Theorem 5.1.** For any \(f \in W^{1,2}(M^0)\) with \(\int_{M^0} f \, dV = 0\), we have
\[
\int_{M^0} f^2 \, dV \leq \lambda_1^{-1} \int_{M^0} |\nabla f|^2 \, dV.
\]
Proof. Suppose \( \lambda_1(U_i^*) \) is the first Dirichlet eigenvalue of \( U_i^* \). We claim that \( \lambda_1(U_i^*) \geq C \). Clearly it suffices to show this statement using the metric \( w_0 \).

We set \( h(r) = r^2(1 - \log r)^2 \) where \( r = |x| \). Then the metric of \( U_i^* \) can be expressed by \( ds^2 = \frac{dx^2}{h(r)} \). The volume element \( dV = \frac{1}{h(r)^{n/2}} dx \), where \( x = (x_1, x_2, \ldots, x_n) \). If we use the spherical coordinates we have \( ds^2 = \frac{1}{h(r)}(dr^2 + r^2 d\sigma^2) \) and \( dV = \frac{r^{n-1}}{(h(r))^{n/2}} dr d\sigma \).

For any \( g \in C_0^\infty(U_i^*) \) we show that

\[
\int_{U_i^*} |g| \, dV \leq \frac{1}{n-1} \int_{U_i^*} |\nabla g| \, dV. \tag{5.1}
\]

Since

\[
|\nabla g|^2 = h(r) \left( \frac{\partial g}{\partial r} \right)^2 + \frac{h(r)}{r^2} \left( \frac{\partial g}{\partial \theta} \right)^2 \geq h(r) \left( \frac{\partial g}{\partial r} \right)^2,
\]

we have

\[
|\nabla g| \geq \sqrt{h(r)} \left| \frac{\partial g}{\partial r} \right| \geq -\sqrt{h(r)} \frac{\partial |g|}{\partial r}.
\]

Hence we obtain

\[
\int_0^1 |\nabla g| \frac{r^{n-1}}{(h(r))^{n/2}} \, dr \geq -\int_0^1 \sqrt{h(r)} \frac{\partial |g|}{\partial r} \frac{r^{n-1}}{(h(r))^{n/2}} \, dr.
\]

Because of the support assumption of \( g \), applying the integration by parts with no boundary terms yields

\[
\int_0^1 |\nabla g| \frac{r^{n-1}}{(h(r))^{n/2}} \, dr \geq -\int_0^1 \frac{\partial |g|}{\partial r} \frac{r^{n-1}}{(h(r))^{n/2}} \, dr
= \int_0^1 |g| \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(h(r))^{n/2}} \right) \frac{r^{n-1}}{(h(r))^{n/2}} \, dr. \tag{5.2}
\]

A straightforward computation gives

\[
\frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(h(r))^{n/2}} \right) = n - 1. \tag{5.3}
\]

Obviously (5.1) follows from (5.2) and (5.3).

Applying the same argument to \(|g|^2\) in place of \(|g|\) we obtain

\[
\int_{U_i^*} |g|^2 \, dV \leq \frac{2}{n-1} \int_{U_i^*} |\nabla g| \, dV.
\]

By Hölder’s inequality we have

\[
\int_{U_i^*} |g|^2 \, dV \leq \frac{4}{(n-1)^2} \int_{U_i^*} |\nabla g|^2 \, dV,
\]

which means \( \lambda_1(U_i^*) \geq \frac{(n-1)^2}{4} \). We clearly have \( \lambda_1(U_{i=1}^n U_i^* \geq \frac{(n-1)^2}{4} \).
By Donnelly’s decomposition principle in [9] (Lemma 5.1), which means that the essential spectrum of $\Delta$ does not depend on the changes of the metric in a compact domain of $M^0$, we have $\sigma_{ess}(\Delta) \subset [\frac{(n-1)^2}{4}, \infty)$. This implies
\[
\inf \sigma(\Delta) \setminus \{0\} > 0.
\]

Now we consider the existence theorem for the equation
\[\Delta u = f\] (5.4)
on $M^0$ where $\Delta$ is the Laplace-Beltrami operator induced by the Poincaré like metric and $f \in L^2(M^0)$.

**Proof of Theorem 1.4.** The necessity part follows easily from $\int_{M^0} \Delta u \, dV = 0$. For any $R > 0$, we choose a cut-off function $\psi$ satisfying
\[\text{supp } \psi \subset B_{2R}(p), \psi \equiv 1 \text{ in } B_R(p), |D\psi| \leq \frac{C}{R}.
\]
Hence, we have
\[
|\int_{B_{2R}} \Delta u \psi \, dV| \leq \int_{B_{2R}} |\nabla u||\nabla \psi| \, dV \\
\leq \frac{C}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \, dV \\
\leq \frac{C}{R(1 - \log R)^{\frac{n-1}{2}}} \|\nabla u\|_{L^2} \rightarrow 0 \text{ as } R \rightarrow \infty.
\]
So $\int_{B_R} \Delta u \, dV \rightarrow 0$, as $R \rightarrow \infty$.

To prove the sufficiency part, we consider the closed subset $L^2_0(M^0) = \{u \in L^2(M^0) : \int_{M^0} u \, dV = 0\}$ of $L^2(M^0)$. Then $\Delta$ is a closed operator from $D(\Delta) \cap L^2_0(M^0)$ to $L^2_0(M^0)$. Theorem 5.1 implies that 0 does not belong to the spectrum of $\Delta$ in $L^2_0(M^0)$. So $(\Delta)^{-1}$ exists and the equation has a solution in the distribution sense. The regularity result for elliptic equations and $f \in C^\infty(M^0)$ show that the solution is smooth.

Is there the corresponding Sobolev’s inequality on a punctured manifold with Poincaré like metric, similar as Proposition 2.1? The answer is negative. For any $p > 2$, we can construct a counter example $u \in W^{1,2}(M^0)$ and $u \notin L^p(M^0)$.

Let $r = |x|$ ($0 < r \leq 1$) and $p > 2$, set
\[
u(r) = \frac{1}{\alpha - 1}(1 - \log r)^{1-\alpha}, \text{ where } \frac{3-n}{2} < \alpha \leq 1 - \frac{n-1}{p}.
\]
We can check $u \in W^{1,2}(U_i^*)$ easily. Note that
\[
\int_{U_i^*} |\nabla u|^2 \, dV = \int_{B_i^*(0)} |\nabla_0 u|^2 \frac{1}{|x|^{n-2}(1 - \log |x|)^{n-2}} \, dx
\]
\[
= W_{n-1} \int_0^1 |\nabla_0 u|^2 r(1 - \log r)^{2-n} \, dr
\]
\[
= W_{n-1} \int_0^1 r^{-1}(1 - \log r)^{2-n} \alpha \, dr
\]
\[
= \frac{W_{n-1}}{n + 2\alpha - 3}, \text{ if } \alpha > \frac{3-n}{2}.
\]
Similarly, we have that
\[
\int_{U_i^*} |u|^2 \, dV < \infty, \text{ if } \alpha > \frac{3-n}{2}.
\]
But,
\[
\int_{U_i^*} |u|^p \, dV = W_{n-1} \int_0^1 r^{-1}(1 - \log r)^{(1-\alpha)p-n} \, dr
\]
\[
= \infty, \text{ if } \alpha \leq 1 - \frac{n-1}{p}.
\]
Hence, we can see that the Sobolev’s inequality with the Poincaré like metric does not hold.

Further, the solution of $\Delta u = f$ with the Poincaré like metric can not be continuous at each puncture $p_i (i = 1, 2, \ldots, k)$. In fact, the boundedness of the solution at each puncture $p_i (i = 1, 2, \ldots, k)$ does not hold. For example, we set
\[
u = \log(1 - \log r), \quad 0 < r \leq 1. \quad (5.5)
\]
By a direct computation, we have
\[
\Delta u = r^2(1 - \log r)^2 \Delta_0 u + (n - 2)r^3(1 - \log r)^3 \frac{\partial}{\partial r} \left( \frac{1}{r(1 - \log r)} \right) \frac{\partial u}{\partial r}
\]
\[
= (2 - n)(1 - \log r) - 1 - (n - 2) \log r
\]
\[
= 1 - n,
\]
where $\Delta_0$ is the Laplace-Beltrami operator with the Euclidean metric. Obviously, we can also show that $u \in W^{1,2}(U_i^*)$. Indeed,
\[
\int_{U_i^*} |\nabla u|^2 \, dV = \int_{B_i^*(0)} |\nabla_0 u|^2 r^{2-n}(1 - \log r)^{2-n} \, dV_0
\]
\[
= W_{n-1} \int_0^1 \frac{1}{r(1 - \log r)^n} \, dr
\]
\[
= \frac{W_{n-1}}{n - 1} < \infty,
\]
and

\[
\int_{U_i^*} u^2 \, dV = \int_{B_i^*(0)} u^2 \frac{1}{r^n(1-\log r)^n} \, dV_0 \\
= W_{n-1} \int_0^1 \left( \log(1-\log r) \right)^2 \frac{1}{r(1-\log r)^n} \, dr \\
= \frac{2}{(n-1)^3} W_{n-1} < \infty,
\]

where \(dV_0\) is the volume element with the Euclidean metric. We choose a cut-off function \(\psi \in C^\infty_0(U_i)\) and \(\psi \equiv 1\) in a neighborhood of \(p_i\), and set \(w = \psi u\). Then we have \(\Delta w \in C^\infty(M^0)\). But \(w\) is not bounded near each puncture \(p_i\) \((i = 1, 2, \ldots, k)\).

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