ORTHOGONAL POLYNOMIALS
IN SEVERAL VARIABLES. I

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ABSTRACT. In this paper we introduce and discuss some classes of orthogonal polynomials in several non-commuting variables. The emphasis is on a non-commutative version of the orthogonal polynomials on the real line. We introduce recurrence equations for these polynomials, Christoffel-Darboux formulas, and Jacobi type matrices.

1. INTRODUCTION

Orthogonal polynomials in several variables are known for long time, see for instance [8], but their theory is less developed than in the one variable case. The commutative case is also studied more intensively (see, for instance, [2], [18]), while the studies for the non-commutative case appear to be quite sparse (see [11]).

Our goal is to introduce and study some classes of orthogonal polynomials in several non-commuting variables. In this paper we focus on polynomials that are viewed as analogues of the orthogonal polynomials on the real line. The main topics are: recurrence equations, Szegö kernels, and Jacobi matrices.

The paper is organized as follows. In Section 2 we introduce the main definitions and several examples. Especially, we briefly review a non-commutative version of the Szegö theory of orthogonal polynomials on the unit circle that was sketched in [7]. Section 3 deals with recurrence equations. In Section 4 we introduce a non-commutative Szegö type kernel which is viewed as a sort of reproducing kernel for the Siegel upper half-space. Section 5 deals with Jacobi type matrices associated to the recurrence equations introduced in Section 3.

2. ORTHOGONAL POLYNOMIALS

In this section we introduce the main definitions and briefly discuss several examples. Let $\mathbb{F}^+_M$ be the unital free semigroup on $M$ generators $g_1, \ldots, g_M$ with lexicographic order $\prec$. The empty word is the identity element and the length of the word $\sigma$ is denoted by $|\sigma|$. The length of the empty word is 0.

Let $\mathcal{O}_N^{0}$ be the algebra of polynomials in $2N$ non-commuting indeterminates $Y_1, \ldots, Y_N$, $Y_{N+1}, \ldots, Y_{2N}$ with complex coefficients. Each element $Q \in \mathcal{O}_N^{0}$ can be uniquely written in the form $Q = \sum_{\sigma \in \mathbb{F}_{2N}^+} c_{\sigma} Y_{\sigma}$, with only finitely many $c_{\sigma} \neq 0$ and $Y_{\sigma} = Y_{i_1} \cdots Y_{i_k}$ for $\sigma = i_1 \cdots i_k \in \mathbb{F}_{2N}^+$. An involution $\mathcal{I}$ can be introduced on $\mathcal{O}_N^{0}$ as follows:

$$\mathcal{I}(Y_k) = Y_{N+k}, \quad k = 1, \ldots, N,$$
\[ \mathcal{I}(Y_l) = Y_{l-N}, \quad l = N + 1, \ldots, 2N; \]
on monomials,
\[ \mathcal{I}(Y_{i_1} \ldots Y_{i_k}) = \mathcal{I}(Y_{i_k}) \ldots \mathcal{I}(Y_{i_1}), \]
and finally, if \( Q = \sum_{\sigma \in \mathbb{F}_2^+} c_\sigma Y_\sigma \), then \( \mathcal{I}(Q) = \sum_{\sigma \in \mathbb{F}_2^+} c_\sigma \mathcal{I}(Y_\sigma) \). Thus, \( \mathcal{O}_N^0 \) is a unital, associative, *-algebra over \( \mathbb{C} \).

Let \( \mathcal{P}_N^0 \) denote the algebra of polynomials in \( N \) non-commuting indeterminates \( Y_1, \ldots, Y_N \) with complex coefficients. Then \( \mathcal{P}_N^0 \) is a subalgebra of \( \mathcal{O}_N^0 \). We say that \( \mathcal{A} \subset \mathcal{O}_N^0 \) is \( \mathcal{I} \)-symmetric if \( P \in \mathcal{A} \) implies \( c \mathcal{I}(P) \in \mathcal{A} \) for some \( c \in \mathbb{C} - \{0\} \). We construct an associative algebra \( \mathcal{O}_N^0(\mathcal{A}) \) as the quotient of \( \mathcal{O}_N^0 \) by the two-sided ideal \( \mathcal{E}(\mathcal{A}) \) generated by \( \mathcal{A} \). We notice that \( \mathcal{O}_N^0(\emptyset) = \mathcal{O}_N^0 \). We let \( \pi = \pi_\mathcal{A} : \mathcal{O}_N^0 \to \mathcal{O}_N^0(\mathcal{A}) \) be the quotient map and since \( \mathcal{A} \) is \( \mathcal{I} \)-symmetric,
\[
(2.1) \quad \mathcal{I}_\mathcal{A}(\pi(P)) = \pi(\mathcal{I}(P))
\]
gives an involution on \( \mathcal{O}_N^0(\mathcal{A}) \). We will be interested in linear functionals \( \phi \) on \( \mathcal{O}_N^0(\mathcal{A}) \) with the property that \( \phi(\mathcal{I}_\mathcal{A}(\pi(P))\pi(P)) \geq 0 \) for all \( P \in \mathcal{P}_N^0 \). Without loss of generality we will assume that \( \phi \) is unital, \( \phi(\pi(1)) = 1 \). Such a functional will be called a positive functional on \( \mathcal{O}_N^0(\mathcal{A}) \). The proof of the following result is straightforward and can be omitted.

**Lemma 1.** Let \( \phi \) be a positive functional on \( \mathcal{O}_N^0(\mathcal{A}) \). Then

1) \( \phi(\mathcal{I}_\mathcal{A}(\pi(P))) = \overline{\phi(\pi(P))} \) for \( P \in \mathcal{P}_N^0 \).

2) \( |\phi(\mathcal{I}_\mathcal{A}(\pi(P_1))\pi(P_2))|^2 \leq \phi(\mathcal{I}_\mathcal{A}(\pi(P_1))\pi(P_1))\phi(\mathcal{I}_\mathcal{A}(\pi(P_2))\pi(P_2)) \) for \( P_1, P_2 \in \mathcal{P}_N^0 \).

We now consider the GNS construction associated to \( \phi \). Thus, we define on \( \pi(\mathcal{P}_N^0) \),
\[
(2.2) \quad \langle \pi(P_1), \pi(P_2) \rangle_\phi = \phi(\mathcal{I}_\mathcal{A}(\pi(P_2)\pi(P_1))),
\]
and factor out the subspace \( \mathcal{N}_\phi = \{ \pi(P) \mid P \in \mathcal{P}_N^0, \langle \pi(P), \pi(P) \rangle_\phi = 0 \} \). Completing this quotient with respect to the norm induced by \( (2.2) \) we obtain a Hilbert space \( \mathcal{H}_\phi \).

From now on we will assume that \( \phi \) is strictly positive, that is, \( \phi(\mathcal{I}_\mathcal{A}(\pi(P))\pi(P)) > 0 \) for all \( P \in \mathcal{P}_N^0 - \mathcal{E}(\mathcal{A}) \), so that \( \mathcal{N}_\phi = \{0\} \) and \( \pi(\mathcal{P}_N^0) \) can be viewed as a subspace of \( \mathcal{H}_\phi \). Let \( \mathcal{F} = \{ F_\alpha \}_{\alpha \in G} \) be the set of the distinct elements \( \pi(Y_\sigma), \sigma \in \mathbb{F}_N^+ \). The index set \( G \) is chosen as follows: the equality on \( \{ \pi(Y_\sigma) \mid \sigma \in \mathbb{F}_N^+ \} \) gives an equivalence relation and choose from each equivalence class the element \( \pi(Y_\sigma) \) with the least \( \sigma \) with respect to the lexicographic order. Then \( G \subset \mathbb{F}_N^+ \) and \( \emptyset \in G \). Let \( G_n = \{ \alpha \in G \mid |\alpha| = n \} \), then \( G_0 = \{ \emptyset \} \) and \( \{G_n\}_{n \geq 0} \) is a partition of \( G \).

Since \( \phi \) is strictly positive it follows that \( \mathcal{F} \) is a linearly independent family in \( \mathcal{H}_\phi \) and the Gram-Schmidt procedure gives a family \( \{ \varphi_\alpha \}_{\alpha \in G} \) of elements in \( \pi(\mathcal{P}_N^0) \subset \mathcal{O}_N^0(\mathcal{A}) \) such that
\[
(2.3) \quad \varphi_\alpha = \sum_{\beta \leq \alpha} a_{\alpha, \beta} F_\beta, \quad a_{\alpha, \alpha} > 0;
\]
The elements $\varphi_\alpha$, $\alpha \in G$, will be called the orthogonal polynomials associated to $\phi$. Typically, the theory of orthogonal polynomials deals with the study of algebraic and asymptotic properties of the orthogonal polynomials associated to strictly positive functionals on $O_N^0(A)$. An explicit formula for the orthogonal polynomials can be obtained in the same manner as in the classical (one variable) case. Define

\begin{equation}
\begin{aligned}
    s_{\alpha, \beta} = \phi(I_A(F_{\alpha})F_{\beta}) = \langle F_{\beta}, F_{\alpha} \rangle_\phi, \quad \alpha, \beta \in G,
\end{aligned}
\end{equation}

and

\begin{equation}
D_\alpha = \det [s_{\alpha', \beta'}]_{\alpha', \beta' \leq \alpha} > 0, \quad \alpha \in G.
\end{equation}

We notice that $\phi$ is a positive functional on $O_N^0(A)$ if and only if $K_\phi(\alpha, \beta) = s_{\alpha, \beta}$, $\alpha, \beta \in G$, is a positive definite kernel on $G$. From now on $\tau - 1$ denotes the predecessor of $\tau$ with respect to the lexicographic order $\prec$ on $\mathbb{F}_N$, while $\sigma + 1$ denotes the successor of $\sigma$. The determinant involved in the next result is defined by the same formula as in the scalar case, even though its entries are elements of $\pi(P_N^0)$.

**Theorem 2.** Let $\{\varphi_\alpha\}_{\alpha \in G}$ be the orthogonal polynomials associated to the strictly positive, unital functional $\phi$ on $O_N^0(A)$. Then $\varphi_\varnothing = 1$ and for $\varnothing \prec \alpha$,

\begin{equation}
\varphi_\alpha = \frac{1}{\sqrt{D_{\alpha-1}D_\alpha}} \det \begin{bmatrix} [s_{\alpha', \beta'}]_{\alpha' \prec \alpha; \beta' \leq \alpha} \\ F_\varnothing & \ldots & F_\alpha \end{bmatrix}.
\end{equation}

**Proof.** The proof is similar to the classical one. Thus, we deduce from the orthogonality condition (2.4) that $\langle \varphi_\alpha, F_{\beta'} \rangle_\phi = 0$ for $\varnothing \leq \beta' < \alpha$, which implies that $\sum_{\beta \leq \alpha} a_{\alpha, \beta} s_{\beta', \beta} = 0$ for $\varnothing \leq \beta' < \alpha$. Since the coefficients of the linear system

\begin{equation}
\begin{cases}
    \sum_{\beta \leq \alpha} a_{\alpha, \beta} s_{\beta', \beta} = 0, \quad \varnothing \leq \beta' < \alpha, \\
    \sum_{\beta \leq \alpha} a_{\alpha, \beta} F_{\beta} = \varphi_\alpha,
\end{cases}
\end{equation}

with unknowns $a_{\alpha, \beta}$ are complex numbers except for those of the last equation which are in $\pi(P_N^0)$, Cramer’s rule still holds in a form that gives

$$
\varphi_\alpha = \frac{a_{\alpha, \alpha}}{D_{\alpha-1}} \det \begin{bmatrix} [s_{\alpha', \beta'}]_{\alpha' \prec \alpha; \beta' \leq \alpha} \\ F_\varnothing & \ldots & F_\alpha \end{bmatrix}.
$$

Next we notice that

$$
\langle \det \begin{bmatrix} [s_{\alpha', \beta'}]_{\alpha' \prec \alpha; \beta' \leq \alpha} \\ F_\varnothing & \ldots & F_\alpha \end{bmatrix}, F_\alpha \rangle_\phi = D_\alpha,
$$

and since $F_\alpha = \frac{1}{a_{\alpha, \alpha}} \varphi_\alpha + \sum_{\beta < \alpha} c_{\beta} F_{\beta}$ for some complex coefficients $c_{\beta}$, $\beta < \alpha$, we deduce

$$
D_\alpha = \langle \frac{D_{\alpha-1}}{a_{\alpha, \alpha}} \varphi_\alpha, \frac{1}{a_{\alpha, \alpha}} \varphi_\alpha + \sum_{\beta < \alpha} c_{\beta} F_{\beta} \rangle_\phi = \frac{D_{\alpha-1}}{a_{\alpha, \alpha}^2},
$$

where $\langle \cdot, \cdot \rangle_\phi$ denotes the sesquilinear form of $\phi$.
so that
\[
\frac{1}{a^2_{\alpha,\alpha}} = \frac{D_\alpha}{D_{\alpha-1}},
\]
which gives (2.7).

Usually, the representation (2.7) is not very useful for the actual computation of the orthogonal polynomials. Instead, recurrence equations are obtained for each particular case of interest. We consider several such examples.

2.1. Orthogonal polynomials in one variable. Assume \( N = 1 \). Then a linear functional \( \phi \) on \( O_1^0 \) is positive if and only if \( K_\phi(n, m) = \phi(\mathcal{I}(Y_1^n)Y_1^m), \) \( n, m \in \mathbb{N} \), is a positive definite kernel on \( \mathbb{N} \). One simple example can be obtained by taking \( A_T = \{1 - \mathcal{I}(Y_1)Y_1\} \). In this case, \( \phi \) is a positive functional on \( O_1^0(A_T) \) if and only if the kernel \( K_\phi \) is positive definite and satisfy the Toeplitz condition, that is
\[
K_\phi(n + k, m + k) = K_\phi(n, m), \quad m, n, k \in \mathbb{N}.
\]

This shows that the orthogonal polynomials on \( O_1^0(A_T) \) are the orthogonal polynomials on the unit circle (\[17\]). The index set is \( \mathbb{N} \) in this case, and define \( \gamma_n = -a_{n,n} \varphi_n(0) \) for \( n \geq 1 \). Then \( |\gamma_n| < 1 \) and set \( d_n = (1 - |\gamma_n|^2)^{1/2} \). The orthogonal polynomials on the unit circle satisfy the following basic recurrence equations (see \[17\]):

\[
\varphi_{n+1}(z) = \frac{1}{d_{n+1}}(z\varphi_n(z) - \gamma_{n+1}\varphi_{n}^2(z)), \tag{2.8}
\]

where \( \varphi_0(z) = 1 \) and for \( n > 0 \),

\[
\varphi_{n+1}^\pm(z) = \frac{1}{d_{n+1}}(-\gamma_{n+1}z\varphi_n(z) + \varphi_{n}^\pm(z)). \tag{2.9}
\]

Another example is given by \( A_R = \{Y_1 - \mathcal{I}(Y_1)\} \). The index set is still \( \mathbb{N} \), but this time, \( \phi \) is a positive functional on \( O_1^0(A_R) \) if and only if the kernel \( K_\phi \) is positive definite and has the Hankel property, that is
\[
K_\phi(n + m, n + k) = K_\phi(n + k, m), \quad m, n, k \in \mathbb{N}.
\]

This shows that the orthogonal polynomials on \( O_1^0(A_R) \) are the orthogonal polynomials on the real line. In this case one obtains a basic three-terms recurrence equation (see \[17\]):

\[
x\varphi_n(x) = b_{n+1}\varphi_{n+1}(x) + a_n\varphi_n(x) + b_n\varphi_{n-1}(x), \tag{2.10}
\]

with initial conditions \( \varphi_0(x) = 1 \) and \( \varphi_{-1}(x) = 0 \).
2.2. Examples in several variables. In this case, there is a large number of interesting examples. Here we mention just one. More examples will be considered in the next sections. Let

\[ \mathcal{A}_1 = \{1 - I(Y_k)Y_k \mid k = 1, \ldots, N\} \cup \{I(Y_k)Y_l \mid k, l = 1, \ldots, N, k \neq l\}. \]

Then \( \phi \) is a positive functional on \( O^0_N(\mathcal{A}_1) \) if and only if \( K_\phi \) is a positive definite kernel that obeys the rules:

\[
(2.11) \quad K_\phi(\tau \sigma, \tau' \sigma') = K_\phi(\sigma, \sigma'), \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+.
\]

\[
(2.12) \quad K_\phi(\sigma, \tau) = 0 \quad \text{if there is no} \quad \alpha \in \mathbb{F}_N^+ \text{such that} \quad \sigma = \alpha \tau \quad \text{or} \quad \tau = \alpha \sigma.
\]

Such type of kernels appeared in the study of some classes of stochastic processes indexed by nodes on a tree (see, for instance, [3]). In this case the index set is \( \mathbb{F}_N^+ \). We also need to introduce

\[
(2.13) \quad D_{1,\sigma} = \det [K_{\sigma',\tau}^{\phi} 0_{\leq \sigma',\tau' \leq \sigma}] > 0.
\]

Define \( \gamma_\sigma = -\sqrt{D_\sigma / D_{1,\sigma}} a_{\sigma, \emptyset} \) and notice that \( |\gamma_\sigma| < 1 \). Then we can define \( d_\sigma = (1 - |\gamma_\sigma|^2)^{1/2} \) and the orthogonal polynomials associated to \( O^0_N(\mathcal{A}_1) \) obey the recurrence equations (see [4]):

\[
(2.14) \quad \varphi_{k\sigma} = \frac{1}{d_{k\sigma}} (Y_k \varphi_\sigma - \gamma_{k\sigma} \varphi_{k-1}^\sharp), \quad k = 1, \ldots, N, \quad \sigma \in \mathbb{F}_N^+,
\]

where \( \varphi_\emptyset^\sharp = 1 \) and for \( k \in \{1, \ldots, N\}, \quad \sigma \in \mathbb{F}_N^+,
\]

\[
(2.15) \quad \varphi_{k\sigma}^\sharp = \frac{1}{d_{k\sigma}} (-\gamma_{k\sigma} Y_k \varphi_\sigma + \varphi_{k-1}^\sharp).
\]

3. Recurrence equations

In this section we consider some algebraic properties of the orthogonal polynomials of \( O^0_N(\mathcal{A}_2) \), where \( \mathcal{A}_2 = \{Y_k - I(Y_k) \mid k = 1, \ldots, N\}. \) If we take \( \mathcal{A}'_2 = \mathcal{A}_2 \cup \{Y_k Y_l - Y_l Y_k \mid k, l = 1, \ldots, N\} \), then \( \pi_{\mathcal{A}'_2}(\mathcal{P}_N^0) \) is isomorphic to the symmetric algebra of \( \mathbb{C}^N \) and the orthogonal polynomials correspond to the orthogonal polynomials of several commuting variables, see [18]. Since \( \pi_{\mathcal{A}'_2}(\mathcal{P}_N^0) \) is a quotient of \( \pi_{\mathcal{A}_2}(\mathcal{P}_N^0) = \mathcal{P}_N^0 \), we expect that results for \( \mathcal{P}_N^0 \) would give corresponding results for \( \pi_{\mathcal{A}'_2}(\mathcal{P}_N^0) \) by standard (symmetrization) techniques, see [14]. Going in the opposite direction, we expect to deduce generalizations of results on \( \pi_{\mathcal{A}'_2}(\mathcal{P}_N^0) \) to similar results for \( \mathcal{P}_N^0 \). We illustrate this remark by obtaining a three-term recurrence relation for the orthogonal polynomials in \( O^0_N(\mathcal{A}_2) \).

Let \( \phi \) be a strictly positive functional on \( O^0_N(\mathcal{A}_2) \). Since \( \pi_{\mathcal{A}_2}(\mathcal{P}_N^0) = \mathcal{P}_N^0 \), the index set \( G \) is \( \mathbb{F}_N^+ \) and \( G_n \) is the set of words in \( \mathbb{F}_N^+ \) of length \( n \). Let \( \{\varphi_\sigma\}_{\sigma \in \mathbb{F}_N^+} \) be the orthogonal
polynomials associated to \( \phi \). The matrix-vector notation in [12] is easily adapted to \( P_N^0 \), by setting \( \Phi_n = [\varphi_\sigma]_{\sigma=0}^0, n \geq 0 \). We can show that (2.10) extends to \( O_N^0(A_2) \).

**Theorem 3.** The orthogonal polynomials on \( O_N^0(A_2) \) obey the following recurrence relations: for \( k = 1, \ldots, N \),

\[
Y_k \Phi_0 = \Phi_1 B_{0,k} + \Phi_0 A_{0,k},
\]

and for \( k = 1, \ldots, N \) and \( n \geq 1 \),

\[
Y_k \Phi_n = \Phi_{n+1} B_{n,k} + \Phi_n A_{n,k} + \Phi_{n-1} B_{n-1,k}^*.
\]

**Proof.** The proof is most like in the classical, one-dimensional case. Thus, we can write

\[
Y_k \varphi_\sigma = \sum_{\tau \leq k\sigma} c_{\tau \sigma}^{k \sigma} \varphi_\tau, \quad k \in \{1, \ldots, N\}, \quad \sigma \in F_N^+,
\]

where the coefficients \( c_{\tau \sigma}^{k \sigma} \) are calculated by the formula \( c_{\tau \sigma}^{k \sigma} = \langle Y_k \varphi_\sigma, \varphi_\tau \rangle_\phi \). We notice that for any \( P, Q \in P_N^0 \),

\[
\langle Y_k P, Q \rangle_\phi = \phi(I(Q)Y_k P) = \phi(I(Q)I(Y_k)P) = \langle P, Y_k Q \rangle_\phi.
\]

[More generally, we have \( s_{\alpha \sigma, \tau} = s_{\sigma, I(\alpha) \tau} \) for \( \alpha, \sigma, \tau \in F_N^+ \), where \( I \) denotes the involution on \( F_N^+ \) given by \( I(i_1 \ldots i_k) = i_k \ldots i_1 \). In particular, for \( |\tau| \leq |\sigma| - 2 \),

\[
c_{\tau \sigma}^{k \sigma} = \langle \varphi_\sigma, Y_k \varphi_\tau \rangle_\phi = 0,
\]

while for the remaining values of \( \tau \leq k\sigma \),

\[
c_{\tau \sigma}^{k \sigma} = \langle \varphi_\sigma, Y_k \varphi_\tau \rangle_\phi = \langle Y_k \varphi_\tau, \varphi_\sigma \rangle_\phi = c_{\sigma \tau}^{k \sigma}.
\]

We deduce that for \( k = 1, \ldots, N \),

\[
Y_k \Phi_0 = \Phi_1 B_{0,k} + \Phi_0 A_{0,k},
\]

while for \( k = 1, \ldots, N \) and \( n \geq 1 \),

\[
Y_k \Phi_n = \Phi_{n+1} B_{n,k} + \Phi_n A_{n,k} + \Phi_{n-1} B_{n-1,k}^*.
\]

Let \( X = [x_{ij}] \) be a given matrix. We will use the following notation: first, \( I(X) = [I(x_{ji})] \) and then \( \phi(X) = [\phi(x_{ij})] \). We deduce from (3.3), (3.4), and (2.4) that

\[
B_{n,k}^* = \phi(I(\Phi_n)Y_k \Phi_{n+1}),
\]

\[
A_{n,k}^* = \phi(I(\Phi_n)Y_k \Phi_n),
\]

and

\[
C_{n,k}^* = \phi(I(\Phi_n)Y_k \Phi_{n-1}) = \phi(I(\Phi_{n-1})Y_k \Phi_n)^* = B_{n-1,k}.
\]
We notice that (3.3) implies that $A_{n,k}^* = A_{n,k}$ and if we define $B_n = \begin{bmatrix} B_{n,1} & \cdots & B_{n,N} \end{bmatrix}$, $n \geq 0$, then $B_n$ is an $N^{n+1} \times N^{n+1}$ upper triangular invertible matrix. Under these conditions we can prove a converse of Theorem 3. This appears as a Favard type result and gives a construction of strictly positive functionals on $O^0_N(A_2)$.

**Theorem 4.** Let $\varphi_\sigma = \sum_{\tau \preceq \sigma} a_{\sigma,\tau} Y_\tau$, $\sigma \in F^+_N$, be elements in $P^0_N$ such that $\varphi_\emptyset = 1$ and $a_{\sigma,\sigma} > 0$. Assume that there exist families $\{A_{n,k} \mid n \geq 0, k = 1, \ldots, N\}$, $\{B_{n,k} \mid n \geq 0, k = 1, \ldots, N\}$ of matrices such that $A_{n,k}^* = A_{n,k}$ for all $n \geq 0$ and $k = 1, \ldots, N$, $B_n = \begin{bmatrix} B_{n,1} & \cdots & B_{n,N} \end{bmatrix}$ is an upper triangular invertible matrix for each $n \geq 0$, for $k = 1, \ldots, N$.

(3.10) \[ Y_k \varphi_\emptyset = [\varphi_\sigma]_{|\sigma| = 1} B_{0,k} + \varphi_\emptyset A_{0,k}, \]

and for $k = 1, \ldots, N$, $n \geq 1$,

(3.11) \[ Y_k [\varphi_\sigma]_{|\sigma| = n} = [\varphi_\sigma]^T_{|\sigma| = n+1} B_{n,k} + [\varphi_\sigma]^T_{|\sigma| = n} A_{n,k} + [\varphi_\sigma]^T_{|\sigma| = n-1} B_{n-1,k}. \]

Then there exists a strictly positive functional $\phi$ on $O^0_N(A_2)$ such that $\{\varphi_\sigma \}_{\sigma \in F^+_N}$ is the family of orthogonal polynomials associated to $\phi$.

**Proof.** Since $B_n$, $n \geq 0$, are invertible matrices, it follows that $\{\varphi_\sigma \}_{\sigma \in F^+_N}$ is a linearly independent family in $P^0_N$ and formula (2.4) suggests to define:

(3.12) \[ \phi(1) = 1 \quad \text{and} \quad \phi(\varphi_\sigma) = 0 \quad \text{for} \quad \sigma \in F^+_N - \{\emptyset\}. \]

These relations uniquely determine a linear functional on $O^0_N(A_2) = P^0_N$. We will use again the matrix-vector notation, $\Phi_n = [\varphi_\sigma]_{|\sigma| = n}$, $n \geq 0$. Also, set $\Phi_{-1} = 0$ and $B_{-1,k} = 0$. Then (3.10) can be included in (3.11) for $n = 0$.

We prove that

(3.13) \[ \phi(\mathcal{I}(\Phi_n)\Phi_m) = \begin{cases} 0, & n \neq m; \\ I, & n = m, \end{cases} \]

where 0 and $I$ denote the zero, respectively the identity matrix of a suitable dimension. This will imply that $\phi$ is a strictly positive functional on $P^0_N$ and that the orthogonal polynomials associated to $\phi$ are precisely $\varphi_\sigma$, $\sigma \in F^+_N$.

We prove (3.13) by induction on $m$. For $m = 0$,

\[ \phi(\mathcal{I}(\Phi_n)\Phi_0) = \Phi(\mathcal{I}(\Phi_0)\Phi_n)^* \]

\[ = [\phi(\varphi_\sigma)]^*_{|\sigma| = n}, \]
so that, by (3.12), \( \phi(I(\Phi_0)\Phi_0) = 1 \) and \( \phi(I(\Phi_n)\Phi_0) = 0 \) for \( n > 0 \). Assume (3.13) holds for \( n \geq 0 \) and \( k \leq m \). Then \( \phi(I(\Phi_n)\Phi_{m+1}) = 0 \) for \( n \leq m \) by the induction hypothesis. We deduce from (3.11) that

\[
[Y_1\Phi_t \ldots Y_N\Phi_t] = \Phi_{t+1}B_t + \Phi_tA_t + \Phi_{t-1}C_t,
\]
where \( A_t = [A_{t,1} \ldots A_{t,N}] \) and \( C_t = [B_{t-1,1} \ldots B_{t-1,N}] \), hence

\[
\Phi_{t+1} = [Y_1\Phi_t \ldots Y_N\Phi_t]B_t^{-1} - \Phi_tA_tB_t^{-1} - \Phi_{t-1}C_tB_t^{-1}
\]

where \( B_t^{-1} = [D_{t,1} \ldots D_{t,N}]^T \), \( T \) denoting the matrix transpose, \( E_t = A_tB_t^{-1} \), and \( F_t = C_tB_t^{-1} \). Using the induction hypothesis, the previous formula for \( l = m \), and (3.11), we deduce

\[
\phi(I(\Phi_{m+1})\Phi_{m+1}) = \sum_{j=1}^N D_{m,j}^* \phi(I(\Phi_{m}Y_j\Phi_{m+1}) - E_{m}^* \phi(I(\Phi_{m+1} \Phi_{m+1}) - F_m^* \phi(I(\Phi_{m-1})\Phi_{m+1})
\]

\[
= \sum_{j=1}^N D_{m,j}^* \phi(I(Y_j\Phi_{m})\Phi_{m+1})
\]

\[
= \sum_{j=1}^N D_{m,j}^*B_{m,j}^* \phi(I(\Phi_{m+1})\Phi_{m+1})
\]

\[
+ \sum_{j=1}^N D_{m,j}^*B_{m,j}^* \phi(I(\Phi_{m})\Phi_{m+1})
\]

\[
+ \sum_{j=1}^N D_{m,j}^*A_{m-1,j}^* \phi(I(\Phi_{m-1})\Phi_{m+1})
\]

\[
= \sum_{j=1}^N D_{m,j}^*B_{m,j}^* = I.
\]

Similar calculations show that \( \phi(I(\Phi_l)\Phi_{m+1}) = 0 \) for \( l > m + 1 \) and the proof is concluded.

4. Christoffel-Darboux formula

In the classical theory orthogonal polynomials are evaluated at points in some suitable domains. For instance, for orthogonal polynomials on the unit circle the domain is the open unit disk and the polynomials are closely related to the theory of analytic functions on that domain. A key to this connection can be considered to be the Szegö kernel \( K_S(z, w) = \frac{1}{1-z\overline{w}} \). A similar relation is established between orthogonal polynomials on the real line and function theory on the upper half plane. In this section we suggest an extension of these connections to several non-commuting variables. The analogue of the unit disk was already dealt with in [4], [5], and we begin by briefly reviewing that construction.
Let $\mathcal{E}$ be an infinite-dimensional Hilbert space and by $\mathcal{L}(\mathcal{E})$ we denote the set of bounded linear operators on $\mathcal{E}$. The $N$-dimensional unit ball of $\mathcal{E}$ is defined by

$$\mathcal{B}_N(\mathcal{E}) = \{ Z = (Z_1 \ldots Z_N) \mid (Z | Z) < I_E \},$$

where $I_E$ is the identity on $\mathcal{E}$ and for $Z = (Z_1 \ldots Z_N)$ and $Z' = (Z'_1 \ldots Z'_N)$ in $\mathcal{L}(\mathcal{E})^N$,

$$ (Z | Z') = \sum_{k=1}^N Z_k(Z'_k)^*.$$

A family of Hilbert spaces is associated to the Hilbert space $\mathcal{E}$ as follows: $\mathcal{E}_0 = \mathcal{E}$ and for $k \geq 1$,

$$\mathcal{E}_k = \underbrace{\mathcal{E}_{k-1} \oplus \ldots \oplus \mathcal{E}_{k-1}}_{N \text{ terms}} = \mathcal{E}^{\oplus N}_{k-1}.$$  

For $\mathcal{E} = \mathbb{C}$ we deduce $\mathbb{C}_k = (\mathbb{C}^N)^{\otimes k}$, the $k$-fold tensor product of $\mathbb{C}^N$ with itself, therefore $\oplus_{k \geq 0} \mathbb{C}_k$ is the Fock space $\mathcal{F}(\mathbb{C}^N)$ associated to $\mathbb{C}^N$. We also deduce that $\oplus_{k \geq 0} \mathcal{E}_k$ is isomorphic to $\mathcal{F}(\mathbb{C}^N) \otimes \mathcal{E}$.

For $Z \in \mathcal{B}_N(\mathcal{E})$ we define $E(Z) = [Z_\sigma]^\sigma_{\sigma=0}$ and notice that $E(Z)$ is a bounded linear operator from $\oplus_{k \geq 0} \mathcal{E}_k$ into $\mathcal{E}$. The Szegő kernel for $\mathcal{B}_N(\mathcal{E})$ is defined by the formula

$$K_B(Z, Z') = E(Z)E(Z')^*, \quad Z, Z' \in \mathcal{B}_N(\mathcal{E}).$$

The following result describes some of the basic properties of $K_B$.

**Lemma 5.** (a) $K_B$ is a positive definite kernel on $\mathcal{B}_N(\mathcal{E})$.

(b) The set $\{ E(Z)^* \mathcal{E} \mid Z \in \mathcal{B}_N(\mathcal{E}) \}$ is total in $\oplus_{k \geq 0} \mathcal{E}_k$.

(c) For any $T \in \mathcal{L}(\mathcal{E})$ and $Z, Z' \in \mathcal{B}_N(\mathcal{E})$,

$$E(Z)(T - \sum_{k=1}^N Z_kT(Z'_k)^{\oplus \infty})E(Z')^* = T,$$

where $(T - \sum_{k=1}^N Z_kT(Z'_k)^{\oplus \infty})$ is the diagonal operator in $\mathcal{L}(\oplus_{k \geq 0} \mathcal{E}_k)$ with diagonal entry $T - \sum_{k=1}^N Z_kT(Z'_k)^*.$

**Proof.** The proof of this result can be found in [4]. Thus, (a) and (c) are quite straightforward. The most interesting is (b). Its proof depends on the assumption the $\mathcal{E}$ is infinite dimensional. In fact, for $\mathcal{E} = \mathbb{C}$, the result is not true, since the set $\{ E(Z)^* \mathbb{C} \mid Z \in \mathcal{B}_N(\mathbb{C}) \}$ is total in the symmetric Fock space, see [1]. For this reason and sake of completeness, we sketch the proof of (b) here. Let $f = \{ f_\sigma \}_{\sigma \in \mathbb{F}}$ be an element of $\oplus_{k \geq 0} \mathcal{E}_k$ orthogonal to the linear span of $\{ E(Z)^* \mathcal{E} \mid Z \in \mathcal{B}_N(\mathcal{E}) \}$. Taking $Z = 0$, we deduce that $f_{\emptyset} = 0$. Next, we claim that for each $\sigma \in \mathbb{F}$ there exist $Z_l = (Z_1^l, \ldots, Z_N^l) \in \mathcal{B}_N(\mathcal{E}), \quad l = 1, \ldots, 2|\sigma|$, such that

$$\text{range} \left[ \begin{array}{ccc} Z_{\sigma}^{*1} & \ldots & Z_{\sigma}^{*2|\sigma|} \end{array} \right] = \mathcal{E},$$

where
and
\[ Z^l_\tau = 0 \quad \text{for all} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|, \quad l = 1, \ldots, 2|\sigma|. \]

Once this claim is proved, a simple inductive argument gives \( f = 0 \), so \( \{E(Z)^*E \mid Z \in \mathcal{B}_N(\mathcal{E})\} \) is total in \( \bigoplus_{k \geq 0} \mathcal{E}_k \). In order to prove the claim we need the following construction.

Let \( \{e_{ij}^n\}_{i,j=1}^n \) be the matrix units of the algebra \( M_n \) of \( n \times n \) matrices. Each \( e_{ij}^n \) is an \( n \times n \) matrix consisting of 1 in the \( (i,j) \)th entry and zeros elsewhere. For a Hilbert space \( \mathcal{E}_1 \) we define \( E_{ij}^n = e_{ij}^n \otimes I_{\mathcal{E}_1} \) and we notice that
\[ (4.2) \quad E_{ij}^n E_{kl}^n = \delta_{jk} E_{il}^n, \quad E_{ji}^{*n} = E_{ij}^n. \]

Let \( \sigma = i_1 \ldots i_k \) so that \( \mathcal{E} = \mathcal{E}_1^{\otimes 2|\sigma|} \) for some Hilbert space \( \mathcal{E}_1 \) (here we essentially use the assumption that \( \mathcal{E} \) is of infinite dimension). Also, for \( s = 1, \ldots, N \), we define
\[ J_s = \{ l \in \{1, \ldots, k\} \mid i_{k+1-l} = s \} \]
and
\[ Z_{sp}^s = \frac{1}{\sqrt{2}} \sum_{r \in J_s} E_{r+p-1,r+p}^{2|\sigma|}, \quad s = 1, \ldots, N, \quad p = 1, \ldots, |\sigma|. \]

We can show that for each \( p \in \{1, \ldots, |\sigma|\}, \)
\[ (4.3) \quad Z_{sp}^s = \frac{1}{\sqrt{2^k}} E_{p,k+p}^{2|\sigma|}, \]
\[ (4.4) \quad Z_{\tau}^p = 0 \quad \text{for} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|. \]

Using \((4.2)\), we deduce
\[ \sum_{s=1}^N Z_{sp}^s Z_{sp}^{*s} = \frac{1}{2} \sum_{s=1}^N \sum_{r \in J_s} E_{r+p,r+p-1}^{2|\sigma|} E_{r+p-1,r+p}^{2|\sigma|} \]
\[ = \frac{1}{2} \sum_{s=1}^N \sum_{r \in J_s} E_{r+p,r+p}^{2|\sigma|} \]
\[ = \frac{1}{2} \sum_{r=1}^k E_{r+p,r+p}^{2|\sigma|} < I, \]

hence \( Z_{sp} \in \mathcal{B}_N(\mathcal{E}) \) for each \( p = 1, \ldots, |\sigma| \). For each word \( \tau = j_1 \ldots j_k \in \mathcal{F}_N^+ - \{\emptyset\} \) we deduce by induction that
\[ (4.5) \quad Z_{j_k \ldots j_1}^{sp} \ldots Z_{j_1}^{sp} = \frac{1}{\sqrt{2^k}} \sum_{r \in A_r} E_{r+p-1,r+p+k-1}^{2|\sigma|}, \]
where \( A_r = \bigcap_{p=0}^{k-1} (J_{j_{k-p}} - p) \subset \{1, \ldots, N\} \) and \( J_{j_{k-p}} - p = \{l-p \mid l \in J_{j_{k-p}}\} \).

We show that \( A_\sigma = \{1\} \) and \( A_\tau = \emptyset \) for \( \tau \neq \sigma \). Let \( q \in A_\tau \). Therefore, for any \( p \in \{0, \ldots, k-1\} \) we must have \( q + p \in J_{j_{k-p}} \) or \( i_{k+1-q-p} = j_{k-p} \). For \( p = k-1 \) we deduce \( j_1 = i_{2-q} \) and since \( 2-q \geq 1 \), it follows that \( q \leq 1 \). Also \( q \geq 1 \), therefore the only element that can be in \( A_\tau \) is \( q = 1 \), in which case we must have \( \tau = \sigma \). Since \( l \in J_{j_{k+1-l}} \) for each \( l = 1, \ldots, k-1 \), hence \( A_\sigma = \{1\} \) and \( A_\tau = \emptyset \) for \( \tau \neq \sigma \). Formula \((4.5)\) implies
In a similar manner we can construct a family \( Z^p, p = |\sigma| + 1, \ldots, 2|\sigma| \), such that
\[
Z_{s}^* = \frac{1}{\sqrt{2^k}} E_{p+k,p},
\]
and
\[
Z_{r}^p = 0 \quad \text{for} \quad r \neq \sigma, \quad |r| \geq |\sigma|.
\]
Thus, for \( s = 1, \ldots, N \), we define
\[
K_s = \{ l \in \{1, \ldots, k\} \mid i_k = s \}
\]
and
\[
Z_{s}^* = \frac{1}{\sqrt{2^k}} \sum_{r \in K_s} E_{p+r-k,r+p-k-1}, \quad s = 1, \ldots, N, \quad p = |\sigma| + 1, \ldots, 2|\sigma|.
\]
Now,
\[
\begin{bmatrix}
Z_{s}^{*1} & \ldots & Z_{s}^{*2|\sigma|}
\end{bmatrix} = \frac{1}{\sqrt{2^k}} \begin{bmatrix}
E_{1,k+1}^{2|\sigma|} & \ldots & E_{k,2k}^{2|\sigma|} & E_{k+1,1}^{2|\sigma|} & \ldots & E_{2k,k}^{2|\sigma|}
\end{bmatrix},
\]
whose range is \( E \). This concludes the proof.

Define the vector space
\[
\mathcal{R} = \{ r_f : \mathcal{B}_N(E) \to E \mid r_f(Z) = E(Z)f, \ f \in \oplus_{k \geq 0} \mathcal{E}_k \},
\]
and consider the map \( U : \oplus_{k \geq 0} \mathcal{E}_k \to \mathcal{R} \) defined by \( Uf = r_f \). This map is linear and bijective (by Lemma 5 (b)), so we can define on \( \mathcal{R} \) an inner product by the formula
\[
\langle r_f, r_h \rangle_{\mathcal{R}} = (f, h)_{\oplus_{k \geq 0} \mathcal{E}_k}.
\]
\( \mathcal{R} \) becomes a Hilbert space and \( U \) is a unitary operator from \( \oplus_{k \geq 0} \mathcal{E}_k \) onto \( \mathcal{R} \). The space \( \mathcal{R} \) has the reproducing property
\[
\langle r_f(Z), e \rangle_{E} = \langle r_f, r_{E(Z)e} \rangle_{\mathcal{R}}, \quad f \in \oplus_{k \geq 0} \mathcal{E}_k, \ e \in \mathcal{E}.
\]
We now let \( P = \sum_{\sigma \in \mathcal{F}_N^+} c_\sigma Y_\sigma \in \mathcal{P}_N^0 \) take values on \( \mathcal{B}_N(E) \) by the the formula:
\[
P(Z) = \sum_{\sigma \in \mathcal{F}_N^+} c_\sigma Z_\sigma, \quad Z \in \mathcal{B}_N(E).
\]
Note that each \( f_e = [c_\sigma]_{\sigma \in \mathcal{F}_N^+} \oplus e, \ e \in \mathcal{E} \), belongs to \( \oplus_{k \geq 0} \mathcal{E}_k \), so that we deduce from (4.6) that
\[
\langle P(Z)e, e' \rangle_{E} = \langle r_{f_e}, r_{E(Z)e'} \rangle_{\mathcal{R}}, \quad e, e' \in \mathcal{E}.
\]
All these considerations suggest to introduce the following set as a convenient domain for the evaluation of the orthogonal polynomials on \( \mathcal{O}_N^0(\mathcal{A}_2) \). Define
\[
\mathcal{G}_N(E) = \{ (W_1 \ldots W_N) \in \mathcal{L}(E)^N \mid W_1 W_1^* + \ldots + W_{N-1} W_{N-1}^* < \frac{1}{2^k} (W_N - W_N^*) \},
\]
the Siegel upper half-space of \( E \) (see [1], [3]). There is a linear fractional map (Cayley transform) connecting \( \mathcal{G}_N(E) \) with \( \mathcal{B}_N(E) \), defined by the formula:
\[
\mathcal{C}(Z) = ((I + Z_N)^{-1} Z_1, \ldots, (I + Z_N)^{-1} Z_{N-1}, i(I + Z_N)^{-1} (I - Z_N)),
\]

(4.3)
for \( Z = (Z_1 \ldots Z_N) \in \mathcal{B}_N(\mathcal{E}) \). Note that since \((Z_1 \ldots Z_N) \in \mathcal{B}_N(\mathcal{E})\), each \(Z_k, 1 \leq k \leq N\), is a strict contraction (\(\|Z_k\| < 1\)), hence \(\mathcal{C}(Z)\) is well-defined.

**Proposition 6.** \(\mathcal{C}\) is a bijection from \(\mathcal{B}_N(\mathcal{E})\) onto \(\mathcal{G}_N(\mathcal{E})\).

**Proof.** For \(Z \in \mathcal{B}_N(\mathcal{E})\) we define \(W_k = (I + Z_N)^{-1}Z_k\) for \(1 \leq k \leq N - 1\), and \(W_N = i(I + Z_N)^{-1}(I - Z_N)\). Then

\[
\frac{1}{2i}(W_N - W_N^*) = (I + Z_N)^{-1}(I - Z_NZ_N^*)(I + Z_N^*)^{-1}
\]

\[
> (I + Z_N)^{-1}(Z_1Z_1^* + \ldots + Z_{N-1}Z_{N-1}^*)(I + Z_N^*)^{-1}
\]

\[
= W_1W_1^* + \ldots + W_{N-1}W_{N-1}^*,
\]

so that \(\mathcal{C}(Z) \in \mathcal{G}_N(\mathcal{E})\).

If \(W = (W_1 \ldots W_N) \in \mathcal{G}_N(\mathcal{E})\), then the imaginary part of \(W_N\) is strictly positive, therefore \(i + W_N\) is invertible and \(\mathcal{C}\) is one-to-one. Also, one easily verifies that for \(W \in \mathcal{G}_N(\mathcal{E})\),

\[
\mathcal{C}^{-1}(W) = (2i(i + W_N)^{-1}W_1, \ldots, 2i(i + W_N)^{-1}W_{N-1}, (i + W_N)^{-1}(i - W_N)).
\]

We now introduce the Szegő kernel of \(\mathcal{G}_N(\mathcal{E})\) by the formula:

\[
K_G(W, W') = F(W)F(W')^*, \quad W, W' \in \mathcal{G}_N(\mathcal{E}),
\]

where

\[
F(W) = 2E(\mathcal{C}^{-1}(W))((i + W_N)^{-1})^{\oplus \infty}.
\]

**Lemma 7.** (a) \(K_G\) is a positive definite kernel on \(\mathcal{G}_N(\mathcal{E})\).

(b) The set \(\{F(W)^* \mathcal{E} \mid W \in \mathcal{G}_N(\mathcal{E})\}\) is total in \(\oplus_{k \geq 0}\mathcal{E}_k\).

(c) For any \(T \in \mathcal{L}(\mathcal{E})\) and \(W, W' \in \mathcal{B}_N(\mathcal{E})\),

\[
F(W)(\frac{1}{2i}(W_NT - T(W')_N^*)) - \sum_{k=1}^{N-1} W_kT(W'_k)^{\oplus \infty}F(W')^* = T.
\]

**Proof.** (a) and (b) follow directly from Lemma 5. For (c) we notice that if \(W = \mathcal{C}(Z)\) and \(W' = \mathcal{C}(Z')\) for \(Z, Z' \in \mathcal{B}_N(\mathcal{E})\), then

\[
T - \sum_{k=1}^{N} Z_kT(Z'_k)^* = 4(i + W_N)^{-1}(\frac{1}{2i}(W_NT - T(W')_N^*)) - \sum_{k=1}^{N-1} W_kT(W'_k)^*(-i + (W')^*_N)^{-1},
\]

and together with Lemma 5(c), this concludes the proof.

We can define

\[
\mathcal{S} = \{s_f : \mathcal{G}_N(\mathcal{E}) \rightarrow \mathcal{E} \mid s_f(W) = F(W)f, \ f \in \oplus_{k \geq 0}\mathcal{E}_k\},
\]

and with the inner product

\[
\langle s_f, s_h \rangle_{\mathcal{S}} = \langle f, h \rangle_{\oplus_{k \geq 0}\mathcal{E}_k},
\]
\( \mathcal{S} \) becomes Hilbert space. The connection between the spaces \( \mathcal{R} \) and \( \mathcal{S} \) is consistent with the connection between the Hardy spaces \( H^2 \) and \( H^2(\Im z > 0) \) on the unit disk and, respectively, the upper half plane, to which \( \mathcal{R} \) and \( \mathcal{S} \) reduce for \( N = 1 \) and \( \mathcal{E} = \mathbb{C} \).

We now let \( P = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma \in \mathcal{P}_N^0 \) take values on \( \mathcal{G}_N(\mathcal{E}) \) by the formula:

\[
P(W) = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma W_\sigma, \quad W \in \mathcal{G}_N(\mathcal{E}).
\]

Let \( \phi \) be a strictly positive functional on \( \mathcal{O}_N^0 \) and let \( \{ \varphi_\sigma \}_{\sigma \in \mathbb{F}_N^+} \) be the associated orthogonal polynomials. Then

\[
K_n(W, W') = \sum_{|\sigma| \leq n} \varphi_\sigma(W) \varphi_\sigma(W')^*,
\]

is a positive definite kernel on \( \mathcal{G}_N(\mathcal{E}) \), called the Christoffel-Darboux kernel, and the Christoffel-Darboux formula is supposed to provide a connection between \( K_n \) and \( K_G \).

One of the reasons for the interest in \( K_n \) in the classical, one variable case, is that \( K_n \) is a reproducing kernel for the set of polynomials of degree at most \( n \) with respect to the inner product induced by \( \phi \). A similar result can be obtained in the non-commutative case. Thus, we have to consider \( \mathcal{P}_N^0(\mathcal{E}) \), the set of elements of the form

\[
P = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma,
\]

with only finitely many \( c_\sigma \in \mathcal{L}(\mathcal{E}) \) different from the zero operator on \( \mathcal{E} \). On \( \mathcal{P}_N^0(\mathcal{E}) \) we introduce the product

\[
\sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma \sum_{\tau \in \mathbb{F}_N^+} d_\tau Y_\tau = \sum_{\sigma, \tau \in \mathbb{F}_N^+} c_\sigma d_\tau Y_{\sigma \tau}
\]

and extend \( \varphi \) to \( \mathcal{P}_N^0(\mathcal{E}) \) by setting \( \varphi(\sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma) = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma \varphi(Y_\sigma) \). Also, extend \( \mathcal{I} \) by \( \mathcal{I}(\sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma) = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma \mathcal{I}(Y_\sigma) \) and define

\[
\langle P, Q \rangle_\phi = \phi(\mathcal{I}(Q)P)
\]

for \( P, Q \in \mathcal{P}_N^0(\mathcal{E}) \). Let \( P = \sum_{|\sigma| \leq n} a_\sigma Y_\sigma \in \mathcal{P}_N^0 \), then \( P = \sum_{|\sigma| \leq n} a_\sigma \varphi_\sigma \), where \( a_\sigma = \langle P, \varphi_\sigma \rangle_\phi \), so that \( K_n \) has the following reproducing property:

\[
P(W') = \sum_{|\sigma| \leq n} a_\sigma \varphi_\sigma(W')
\]

\[
= \sum_{|\sigma| \leq n} \langle P, \varphi_\sigma \rangle_\phi \varphi_\sigma(W')
\]

\[
= \langle P, \sum_{|\sigma| \leq n} \varphi_\sigma \varphi_\sigma(W')^* \rangle_\phi
\]

\[
= \langle P, K_n(W') \rangle_\phi,
\]

where \( K_n(W', W) = K_n(W, W') \) and \( W, W' \in \mathcal{G}_N(\mathcal{E}) \).

We can obtain now a Christoffel-Darboux type formula. It is not as simple and relevant as in the commutative case.
Theorem 8. The Christoffel-Darboux kernel and the Szegö kernel are related by the formula:

\[ K_n(W, W') = F(W) \left( \frac{1}{2} \left( \Phi_{n+1}(W)B_{n,N} \Phi_n(W')^* - \Phi_n(W)B_{n,N}^* \Phi_{n+1}(W')^* \right) \right) F(W')^* - \sum_{k=1}^{N-1} F(W)W_k K_n(W, W') (W')^*_k F(W')^*. \]

Proof. Using the recurrence equations for the orthogonal polynomials we deduce

\[ W_N K_n(W, W') = K_n(W, W') (W'_N)^* \]

\[ = \sum_{k=0}^{n} \Phi_{k+1}(W)B_{k,N} \Phi_k(W')^* + \sum_{k=0}^{n} \Phi_{k-1}(W)B_{k-1,N}^* \Phi_k(W')^* - \sum_{k=0}^{n} \Phi_k(W)B_{k-1,N}^* \Phi_{k-1}(W')^* \]

\[ = \Phi_{n+1}(W)B_{n,N} \Phi_n(W')^* - \Phi_n(W)B_{n,N}^* \Phi_{n+1}(W')^*. \]

The proof is concluded by an application of Lemma 7 (c).

5. Jacobi matrices

Let \( \phi \) be a strictly positive functional on \( \mathcal{O}_N^0(A_2) \). In addition to the Hilbert space \( \mathcal{H}_\phi \), the GNS construction produces a representation of \( \mathcal{O}_N^0(A_2) \) by operators on \( \mathcal{H}_\phi \). In this section we analyse this representation in some details by showing the connection with certain matrices of Jacobi type. Let \( \{ \varphi_\sigma \}_{\sigma \in \mathbb{F}_+^N} \) be the family of orthogonal polynomials associated to \( \phi \). Then we define for \( P \in \mathcal{O}_N^0(A_2) (= \mathcal{P}_N^0) \),

\[ \Psi_{\phi}(P) \varphi_\sigma = P \varphi_\sigma. \]

Formula (3.3) shows that each \( \Psi_{\phi}(P) \) is a symmetric operator on \( \mathcal{H}_\phi \) with dense domain \( \mathcal{D} \), the linear space generated by the polynomials \( \varphi_\sigma, \sigma \in \mathbb{F}_+^N \). Also, for \( P, Q \in \mathcal{P}_N^0 \),

\[ \Psi_{\phi}(PQ) = \Psi_{\phi}(P) \Psi_{\phi}(Q), \]

and \( \Psi_{\phi}(P) \mathcal{D} \subset \mathcal{D} \) for any \( P \in \mathcal{P}_N^0 \), hence \( \Psi_{\phi} \) is an unbounded representation of \( \mathcal{O}_N^0(A_2) \). Also, \( \phi(P) = \langle \Psi_{\phi}(P)1, 1 \phi \rangle \) for \( P \in \mathcal{P}_N^0 \). Of special interest are the operators \( \Psi_k = \Psi_{\phi}(Y_k), k = 1, \ldots, N, \) since \( \Psi_{\phi}(\sum_{\sigma \in \mathbb{F}_+^N} c_\sigma Y_\sigma) = \sum_{\sigma \in \mathbb{F}_+^N} c_\sigma \Psi_{\phi,\sigma} \), where \( \Psi_{\phi,\sigma} = \Psi_{i_1} \ldots \Psi_{i_k} \) for \( \sigma = i_1 \ldots i_k \). Since each \( \Psi_k \) commutes with the complex conjugation, it follows from von Neumann’s theorem (Theorem X.7 in [16]) that each \( \Psi_k \) admits self-adjoint extensions.

Let \( \{ e_1, \ldots, e_N \} \) be the standard basis of \( \mathbb{C}^N \), then \( \{ e_{i_1 \ldots i_k} = e_{i_1} \otimes \ldots \otimes e_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq N \} \) is an orthonormal basis of the Fock space \( \mathcal{F}(\mathbb{C}^N) \). Let \( W \) be the unitary operator from \( \mathcal{F}(\mathbb{C}^N) \) onto \( \mathcal{H}_\phi \) such that \( W(e_\sigma) = \varphi_\sigma, \sigma \in \mathbb{F}_+^N \). We see that \( W^{-1} \mathcal{D} \) is the linear space \( \mathcal{D}_0 \) generated by \( e_\sigma, \sigma \in \mathbb{F}_+^N \), so that we can define

\[ J_k = W^{-1} \Psi_{\phi,k} W, \quad k = 1, \ldots, N, \]
on $\mathcal{D}_0$. Each $J_k$ is a symmetric operator on $\mathcal{D}_0$ and by Theorem 3, the matrix of $J_k$ with respect to the orthonormal basis $\{e_{\sigma}\}_{\sigma \in \mathbb{F}_N^+}$ is

$$J_k = \begin{bmatrix}
A_{0,k} & B_{0,k}^* & 0 & \cdots \\
B_{0,k} & A_{1,k} & B_{1,k}^* & \\
0 & B_{1,k} & A_{2,k} & \\
& & & \ddots \\
& & & & \ddots
\end{bmatrix}.$$ 

We call $(J_1 \ldots J_N)$ a Jacobi $N$-family on $\mathcal{D}_0$. We can state now the main result of this section about the modeling of Jacobi $N$-families.

**Theorem 9.** Let $(J_1 \ldots J_N)$ a Jacobi $N$-family on $\mathcal{D}_0$ such that $B_n = [B_{n,1} \ldots B_{n,N}]$ is an upper triangular invertible matrix. Then there exists a unique strictly positive functional $\phi$ on $\mathcal{O}_N^0(\mathcal{A}_2)$ such that the map

$$W(e_{\sigma}) = \varphi_{\sigma}, \quad \sigma \in \mathbb{F}_N^+,$$

extends to a unitary operator and

$$J_k = W^{-1}\Psi_{\phi,k}W, \quad k = 1, \ldots, N.$$ 

**Proof.** Since $B_n, \ n \geq 0$, are invertible matrices, we can uniquely determine the elements $\varphi_{\sigma}, \ \sigma \in \mathbb{F}_N^+$, in $\mathcal{P}_N^0$ such that for $k = 1, \ldots, N$,

$$Y_k \varphi_{\emptyset} = [\varphi_{\sigma}]_{|\sigma|=1}B_{0,k} + \varphi_{\emptyset}A_{0,k},$$

and for $k = 1, \ldots, N, \ n \geq 1$,

$$Y_k [\varphi_{\sigma}]_{|\sigma|=n} = [\varphi_{\sigma}]_{|\sigma|=n+1}B_{n,k} + [\varphi_{\sigma}]_{|\sigma|=n}A_{n,k} + [\varphi_{\sigma}]_{|\sigma|=n-1}B_{n-1,k}^*.$$ 

Theorem 4 gives a unique strictly positive functional $\phi$ on $\mathcal{O}_N^0(\mathcal{A}_2)$ such that $\{\varphi_{\sigma}\}_{\sigma \in \mathbb{F}_N^+}$ is the family of orthogonal polynomials associated to $\phi$. The GNS construction for this $\phi$ will produce the required $W$ and $\Psi_{\phi,k}$, as explained above.

We conclude with an application concerning the moments of positive functionals. Let $\phi$ be a positive functional on $\mathcal{O}_N^0(\mathcal{A}_2)$. The numbers

$$s_{\sigma} = \phi(Y_{\sigma}), \quad \sigma \in \mathbb{F}_N^+,$$

are called the moments of $\phi$. A Hamburger type problem would be to determine conditions on the family $\{s_{\sigma}\}_{\sigma \in \mathbb{F}_N^+}$ so that there exists a positive functional on $\mathcal{O}_N^0(\mathcal{A}_2)$ satisfying (5.2). A solution to this problem can be obtained as follows.

**Theorem 10.** The numbers $s_{\sigma}, \ \sigma \in \mathbb{F}_N^+$, are the moments of a positive functional on $\mathcal{O}_N^0(\mathcal{A}_2)$ if and only if $K(\sigma, \tau) = s_{\tau(\sigma)\tau}, \ \sigma, \tau \in \mathbb{F}_N^+$, is a positive definite kernel on $\mathbb{F}_N^+$.  

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Proof. The classical approach extends to this setting. Thus, we notice first that

\[ K(\alpha \sigma, \tau) = s_{I(\alpha \sigma)\tau} = s_{I(\sigma)I(\alpha)\tau} = K(\sigma I(\alpha), \tau), \]

for \( \alpha, \sigma, \tau \in \mathbb{F}_N^+ \). Define the sesqui-linear form on \( \mathcal{P}_N^0 \) by the formula

\[ \langle \sum_{\sigma \in \mathcal{F}_N^+} c_\sigma Y_\sigma, \sum_{\tau \in \mathbb{F}_N^+} d_\tau Y_\tau \rangle = \sum_{\sigma, \tau \in \mathcal{F}_N^+} s_{I(\sigma)\tau} c_\tau d_\sigma. \]

Let \( \mathcal{N} = \{ P \in \mathcal{P}_N^0 \mid \langle P, P \rangle = 0 \} \) and let \( \mathcal{H} \) be the Hilbert space obtained by completing \( \mathcal{P}_N^0 / \mathcal{N} \) in the inner product \( \langle \cdot, \cdot \rangle \). Define the maps \( \Psi_k : \mathcal{P}_N^0 \to \mathcal{P}_N^0, \) \( k = 1, \ldots, N \), by

\[ \Psi_k \left( \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma \right) = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_{k\sigma}. \]

From (5.3) we deduce that each \( \Psi_k, k = 1, \ldots, N, \) is a symmetric operator and \( \Psi_k \mathcal{N} \subset \mathcal{N} \). Thus, each \( \Psi_k \) lifts to a symmetric operator, still denoted \( \Psi_k \), with domain \( \mathcal{P}_N^0 / \mathcal{N} \) in \( \mathcal{H} \). Also \( \Psi_k(\mathcal{P}_N^0 / \mathcal{N}) \subset \mathcal{P}_N^0 / \mathcal{N} \), so that \( \Psi_\sigma = \Psi_{i_1} \cdots \Psi_{i_k} \) is defined for any \( \sigma = i_1 \ldots i_k \in \mathbb{F}_N^+ \) and we can take

\[ \phi \left( \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma Y_\sigma \right) = \left\langle \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma \Psi_\sigma \hat{1}, \hat{1} \right\rangle, \]

where \( \hat{1} \) denotes the class of \( 1 \in \mathcal{P}_N^0 \) in \( \mathcal{P}_N^0 / \mathcal{N} \). One can easily check that \( \phi \) is a solution of the Hamburger problem with data \( \{ s_\sigma \}_{\sigma \in \mathbb{F}_N^+} \). \( \square \)

A more specialized version of this result was proved in [10].

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