EXISTENCE OF POSITIVE SOLUTIONS FOR A PARAMETER FRACTIONAL $p$-LAPLACIAN PROBLEM WITH SEMIPOSITONE NONLINEARITY

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Abstract. In this paper we prove the existence of at least one positive solution for the nonlocal semipositone problem

$$\begin{cases} (-\Delta)^s_p(u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N - \Omega, \end{cases}$$

whenever $\lambda > 0$ is a sufficiently small parameter. Here $\Omega \subseteq \mathbb{R}^N$ a bounded domain with $C^{1,1}$ boundary, $2 \leq p < N$, $s \in (0,1)$ and $f$ superlinear and subcritical. We prove that if $\lambda > 0$ is chosen sufficiently small the associated Energy Functional to the problem has a mountain pass structure and, therefore, it has a critical point $u_\lambda$, which is a weak solution. After that we manage to prove that this solution is positive by using new regularity results up to the boundary and a Hopf’s Lemma.

1. Introduction

We are interested in the study of the existence of positive solutions to the problem

$$\begin{cases} (-\Delta)^s_p(u) = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N - \Omega, \end{cases}$$

where $N > 2$ is an integer, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $s \in (0,1)$, $1 < p$ and $sp < N$ and $\lambda > 0$. Besides $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $(-\Delta)^s_p$ is the $s$-fractional $p$-Laplacian operator defined as

$$(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} \, dy.$$

Let us denote by $p^*_s := \frac{Np}{N-sp}$ the fractional critical Sobolev exponent. For any Lebesgue measurable set $U \subseteq \mathbb{R}^N$, $|U|$ will stand for the Lebesgue measure of $U$. In this work we will assume that there exist $p-1 < q < \min\{s\frac{np^*_s}{2}, p^*_s - 1\}$, $A, B > 0$ such that

$$A(s^q - 1) \leq f(s) \leq B(s^q + 1) \quad \text{for } s > 0,$$

$$f(s) = 0 \quad \text{for } s \leq -1 \quad (2)$$

2020 Mathematics Subject Classification. 35A15, 35R11, 35B51, 35R09.

Key words and phrases. mountain pass theorem, semipositone problem, positive solutions, fractional $p$-Laplacian, comparison principles.
Let us define
\[ F(t) := \int_0^t f(s) ds. \]
Therefore, there exist \( A_1, C_1, B_1 > 0 \) such that
\[ F(u) \leq B_1(|u|^{q+1} + 1) \quad \text{for all } u \in \mathbb{R} \]
and
\[ A_1 (u^{q+1} - C_1) \leq F(u) \quad \text{for all } u \geq 0. \]
Let us also assume that \( f \) satisfies an Ambrosetti-Rabinowitz type condition. More specifically, we will assume that there exist \( \theta > p \) and \( M \in \mathbb{R} \) such that for all \( s \in \mathbb{R} \),
\[ sf(s) \geq \theta F(s) + M. \]

Remark 1. The existence of at least one solution to our problem can be stated under the assumption \( q \in (p - 1, p^* - 1) \). The restriction \( p - 1 < q < \min\{\frac{N}{p^*}, p^* - 1\} \) is necessary to prove the positiveness of this.

The aim of this paper is to prove the following result.

Theorem 1 (Main Theorem). Let us assume that \( \Omega \) is a bounded domain with \( C^{1,1} \) boundary. Then there is \( \lambda_0 > 0 \) such that for all \( \lambda \in (0, \lambda_0) \) problem \( \mathfrak{b} \) has at least one positive weak solution \( u_\lambda \in C^\alpha(\bar{\Omega}), \) for some \( \alpha \in (0, 1) \).

This result extends the one in \( \mathfrak{b} \) where the authors considered the problem for the \( p \)-Laplacian operator, \( (2 \leq p < N) \). The difficulties to prove the positiveness of the solutions for Dirichlet problems with semipositone type nonlinearities are well documented, see for example \( \mathfrak{b}, \mathfrak{b} \) and references therein. Such issues persist in the nonlocal case. To the best of our knowledge this is the first result on the existence of positive solutions for a semipositone nonlinearity with the fractional \( p \)-Laplacian. In \( \mathfrak{b} \), the authors studied the problem \( \mathfrak{b} \) with \( p = 2, f(u) = u^q - 1, \) (semipositone) but \( 0 < q < 1 \). Indeed, they proved the existence of at least one positive solution if \( \lambda > 0 \) is sufficiently large. In \( \mathfrak{b} \), the authors proved the existence of positive solutions of a problem of semipositone type for the \( \Phi \)-Laplacian through Orlicz-Sobolev spaces.

Throughout this paper, \( C \) will denote positive constant, not the same at each occurrence.

2. Fractional frame

Definition 1. Let \( s \in (0, 1) \) and \( 1 \leq p < \infty \) and let
\[ W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\} \]
be the fractional Sobolev space endowed with the norm
\[ \|u\|_{s,p} = \left( \|u\|_p + [u]_{s,p}^p \right)^{1/p}, \]
where
\[ [u]_{s,p}^p := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy, \]
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is the Gagliardo seminorm and for every $1 \leq q \leq \infty$, $\| \cdot \|_q$ is the norm in $L^q(\Omega)$.

With this norm, $W^{s,p}(\mathbb{R}^N)$ is a Banach space. We shall work in the closed subspace

$$W^{s,p}_0(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N - \Omega \}$$

which can be equivalently renormed by setting $\| u \| = [u]_{s,p}$. The equivalence of these norms is a consequence of the Sobolev embedding theorem (see [8]).

Let us set for all $s \in \mathbb{R}$

$$\Phi_p(s) = |s|^{p-2} s.$$

A weak solution to the problem (1) is a function $u \in W^{s,p}_0(\Omega)$ such that for all $\varphi \in W^{s,p}_0(\Omega)$

$$\int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy = \lambda \int_\Omega f(u) \varphi \, dx.$$

We shall give to this problem a variational approach. Then, for each $\lambda > 0$ let us define the functional $E_\lambda : W^{s,p}_0(\Omega) \to \mathbb{R}$ as

$$E_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \lambda \int_\Omega f(u) \varphi \, dx. \quad (6)$$

Observe that $E_\lambda(u) := \frac{1}{p} \| u \|_p^p - \lambda \int_\Omega f(u) \varphi \, dx$. It is well known that $E_\lambda \in C^1$ and its derivative is given by

$$\langle E_\lambda'(u), \varphi \rangle = \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy - \lambda \int_\Omega f(u) \varphi \, dx. \quad (7)$$

Therefore, the critical points of $E_\lambda$ turns out to be the weak solutions of problem (1).

3. PRELIMINARY RESULTS

In this section we shall establish some lemmas that guarantee that $E_\lambda$ has a critical point, $u_\lambda$, whenever $\lambda > 0$ is sufficiently small. After that, we present some lemmas concerning the regularity of $u_\lambda$. Finally we prove our main result. The positive number

$$r := \frac{1}{q + 1 - p},$$

will be use repeatedly throughout this paper. Let $\varphi \in W^{s,p}_0(\Omega)$ be a positive function with $\| \varphi \| = 1$ and let

$$c := \left( \frac{2}{pA_1 \| \varphi \|^{q+1}_{q+1}} \right)^r > 0.$$

Finally, let us define $d_\Omega(x) := \text{dist}(x, \Omega^c)$, for all $x \in \mathbb{R}^N$.

**Lemma 1.** There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$ then $E_\lambda(c\lambda^{-r}\varphi) \leq 0$. 
Proof. Let $l = c\lambda^{-r}$. From the growth behaviour of $F$ (see (1)) and the fact that $\|\varphi\| = 1$ we have

$$E_\lambda(l\varphi) = \frac{1}{p} \|l\varphi\|^p - \lambda \int_\Omega F(l\varphi) dx$$

$$\leq \frac{\|\varphi\|^p}{p} - \lambda A_1 l^{q+1} \int_\Omega \varphi^{q+1} dx + \lambda A_1 C_1 |\Omega|$$

$$\leq \frac{\|\varphi\|^p}{p} - \lambda A_1 l^{q+1} \|\varphi\|^{q+1}_{q+1} + \lambda A_1 C_1 |\Omega|.$$  

(8)

Thus, if $0 < \lambda < \left(\frac{c^p}{2pA_1 C_1 |\Omega|}\right)^{1/(1+rp)} =: \lambda_1$, then

$$E_\lambda(l\varphi) \leq -\frac{c^p}{2p} \lambda^{-rp} \leq 0.$$  

(9)

\[\square\]

**Lemma 2.** There exist $\tau > 0$, $c_1 > 0$ and $0 < \lambda_2 < 1$ such that if $\|u\| = \tau \lambda^{-r}$ then $E_\lambda(u) \geq c_1 (\tau \lambda^{-r})^p$ for all $\lambda \in (0, \lambda_2)$.

Proof. Let $u \in W^{s,p}_0(\Omega)$ with $\|u\| = \lambda^{-r} \tau$, by the Sobolev embedding theorem, there exists $K_1 > 0$ such that for all $v \in W^{s,p}_0(\Omega)$, $\|v\|_{q+1} \leq K_1 \|v\|$, define $\tau = \min\{(2pK_1^{q+1}B_1)^{-r}, c\}$ then,

$$E_\lambda(u) = \frac{1}{p} \|u\|^p - \lambda \int_\Omega F(u) dx$$

$$\geq \frac{1}{p} (\lambda^{-r} \tau)^p - \lambda B_1 \|u\|_{q+1}^{q+1} - \lambda B_1 |\Omega|$$

$$\geq \frac{1}{p} (\lambda^{-r} \tau)^p - \lambda B_1 (K_1 \|v\|)^{q+1} - \lambda B_1 |\Omega|$$

$$= \frac{1}{p} (\lambda^{-r} \tau)^p - \lambda B_1 K_1^{q+1} (\lambda^{-r} \tau)^{q+1} - \lambda B_1 |\Omega|$$

$$\geq \lambda^{-rp} \left(\frac{\tau^p}{2p} - \lambda^{1+rp} |\Omega| B_1 \right)$$

$$\geq \lambda^{-rp} \frac{\tau^p}{4p}$$

taking $c_1 = \frac{1}{4p}$ and $\lambda_2 := \tau^{p/(1+rp)} (4pB_1 |\Omega|)^{-1/(1+rp)}$ we obtain the result.  

\[\square\]

**Lemma 3.** Let $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. Then, there exists a constant $c_2 > 0$ such that for all $\lambda \in (0, \lambda_3)$ the functional $E_\lambda$ has a critical point $u_\lambda$ which satisfies

$$c_1 \lambda^{-rp} \leq E_\lambda(u_\lambda) \leq c_2 \lambda^{-rp},$$

where $c_1 > 0$ is the constant given in Lemma 2.

Proof. First of all, we will prove that $E_\lambda$ satisfies the Palais-Smale condition. Let us assume that $\{u_n\}$ is a sequence in $W^{s,p}_0(\Omega)$ such that $\{E_\lambda(u_n)\}$ is bounded and $E'_\lambda(u_n) \rightarrow 0$, as $n \rightarrow \infty$. Hence, there exists $\nu > 0$ such that for all $n > \nu$
On the other hand, taking into account the Hölder inequality, we see that
\[ |⟨E'_λ(u_n), u_n⟩| \leq ∥u_n∥. \]
Moreover, from (7) we have
\[ -||u_n||^p - ||u_n|| \leq -λ \int_Ω f(u_n)u_n dx, \quad \text{for all } n > ν. \tag{10} \]
Let $K > 0$ such that for all $n$, $|E'_λ(u_n)| \leq K$. From the Ambrosetti-Rabinowitz condition (equation (5)) we see that
\[ \frac{1}{p}||u_n||^p - \frac{λ}{θ} \int_Ω f(u_n)u_n dx + \frac{λ}{θ} M|Ω| \leq \frac{1}{p}||u_n||^p - λ \int_Ω F(u_n) dx \leq K. \tag{11} \]
Using (10) and (11) we obtain
\[ \left( \frac{1}{p} - 1 \right) ||u_n||^p - \frac{λ}{θ} ||u_n|| \leq K - \frac{λ}{θ} M|Ω|, \]
which proves that \{u_n\} is bounded in $W^{s,p}_0(Ω)$. Therefore, up to a sub-sequence, \{u_n\} converges weakly to the function $u ∈ W^{s,p}_0(Ω)$. Since $p < q + 1 < p^*_s$, then $u_n → u$ (strongly) in $L^{q+1}(Ω)$. Applying the Hölder inequality this implies that
\[ \lim_{n → ∞} λ \int_Ω f(u_n)(u_n - u) dx = 0. \]
Then, since $\lim_{n → ∞} E'_λ(u_n) = 0$, we have
\[ \lim_{n → ∞} \int_{Ω \mathbb{R}^N} \frac{Φ_p(u_n(x) - u_n(y))(u_n(x) - u_n(y))}{|x - y|^{N + sp}} = 0. \tag{12} \]
Using again that $u$ is the weak limit of $u_n$ we have
\[ \lim_{n → ∞} \int_{Ω \mathbb{R}^N} \frac{Φ_p(u(x) - u(y))(u_n(x) - u_n(y))}{|x - y|^{N + sp}} = 0. \tag{13} \]
On the other hand, taking into account the Hölder inequality, we see that
\[ \int_Ω \frac{Φ_p(u_n(x) - u_n(y)) - Φ_p(u(x) - u(y))}{|x - y|^{N + sp}} \left( (u_n - u)(x) - (u_n - u)(y) \right) dx dy \]
\[ = \int_Ω \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} - \frac{Φ_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x - y|^{N + sp}} \]
\[ - \frac{Φ_p(u(x) - u(y))(u_n(x) - u_n(y))}{|x - y|^{N + sp}} + \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \] dx dy
\[ \geq ||u_n||^p - ||u_n||^{p-1} ||u|| - ||u_n|| ||u||^{p-1} + ||u||^p \]
\[ = ([[u_n]]^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \geq 0. \]
From (12), (13) we obtain
\[ \lim_{n → ∞} (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) = 0, \]
which implies
\[ \lim_{n → ∞} ||u_n|| = ||u||. \]
Since $u_n \to u$, then $u_n \to u$ strongly in $W^{s,p}_0(\Omega)$. This proves that $E_\lambda$ satisfies the Palais-Smale condition.

Let us observe that, from (8), for all $0 \leq l \leq c\lambda^{-r}$

$$E_\lambda(l\phi) \leq \frac{lp}{p} + \lambda A_1 C_1 |\Omega| \leq \frac{c^p}{p} \lambda^{-rp} + A_1 C_1 |\Omega| \lambda^{-rp} = c_2 \lambda^{-rp}.$$ 

where $c_2 := \frac{c_p}{p} + A_1 C_1 |\Omega|$. Therefore

$$\max_{0 \leq l \leq c\lambda^{-r}} E_\lambda(l\phi) \leq c_2 \lambda^{-rp}. \quad (14)$$

From Lemmas 1 and 2 and the Mountain Pass Theorem for each $\lambda \in (0, \lambda_3)$ there exist $u_\lambda \in W^{s,p}_0(\Omega)$ such that $E'_\lambda(u_\lambda) = 0$. Furthermore, this critical point is characterized by

$$E_\lambda(u_\lambda) = \min_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} E(\gamma(t)). \quad (15)$$

where $\Gamma$ is the set of continuous functions $\gamma : [0, 1] \to W^{s,p}_0(\Omega)$ with $\gamma(0) = 0$, $\gamma(1) = c\lambda^{-r} \varphi$. Moreover, from (14), (15) and Lemma 2 we see that

$$c_1 \tau^p \lambda^{-rp} \leq E_\lambda(u_\lambda) \leq c_2 \lambda^{-rp}.$$ 

Note that $c_1$, $c_2$ are independent of $\lambda$. \quad \square

Remark 2. There exists a constant $C > 0$ such that for all $0 < \lambda < \lambda_3$

$$\|u_\lambda\| \leq C\lambda^{-r}. \quad (16)$$

In fact, since $u_\lambda$ is a critical point of $E_\lambda$, then

$$\|u_\lambda\|^p = \lambda \int_\Omega f(u_\lambda) u_\lambda dx.$$ 

From the Ambrosetti-Rabinowitz condition and Lemma 3 we see that

$$\left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_\lambda\|^p \leq \frac{1}{p} \|u_\lambda\|^p - \frac{\lambda}{\theta} \int_\Omega f(u_\lambda) u_\lambda dx + \frac{\lambda}{\theta} M|\Omega|

\leq \frac{1}{p} \|u_\lambda\|^p - \lambda \int_\Omega F(u_\lambda) dx 

= E_\lambda(u_\lambda) 

\leq c_2 \lambda^{-rp}. \quad (17)$$
But taking into account the Remark 2, we have
\[ \|g\|_t \leq C\lambda\|u_\lambda\|_{tq}^q \leq C\lambda\|u_\lambda\|_q^q \leq C\lambda^{1-rq}. \]
Therefore, from (17) and \(-r = (1 - rq)/(p - 1)\), we see that
\[ \|u_\lambda\|_\infty \leq C\lambda^{-r}. \] (18)
Since \(u_\lambda \in L^\infty(\Omega)\) then \(g \in L^\infty(\Omega)\). From Theorem 1.1. in [10], we see that there exists \(\alpha \in (0, s]\) and \(C > 0\), depending only on \(N, p, s\) and \(\Omega\), such that the solution \(u_\lambda\) satisfies \(u_\lambda/d_\Omega^s \in C^{\alpha}(\Omega)\) and
\[ \|u_\lambda/d_\Omega^s\|_{C^{\alpha}(\Omega)} \leq C\|\lambda f(u_\lambda)\|_{L^t(\Omega)}^{1-\frac{1}{t}} \leq \lambda^{-r}, \]
where the last inequality was obtained taking into account (18), the growing condition of \(f\) and that \(1 - rq = -r(p - 1)\).

**Lemma 5.** Let \(u_\lambda\) be a weak solution of (1). Then there exists a constant \(C\) such that for all \(0 < \lambda < \lambda_3\)
\[ C\lambda^{-r} \leq \|u_\lambda\|_\infty. \]

**Proof.** From Lemma 3 there exists \(c_1\) such that \(c_1 \lambda^{-rp} \leq E_\lambda(u_\lambda)\). Moreover, since \(\min F > -\infty\) then
\[ \lambda \int_\Omega f(u_\lambda)u_\lambda dx = \|u_\lambda\|^p \]
\[ = pE_\lambda(u_\lambda) + p\lambda \int_\Omega F(u_\lambda)dx \]
\[ \geq pc_1 \lambda^{-rp} + p|\Omega|\lambda \min F \]
\[ \geq C_1 \lambda^{-rp}, \] (19)
for some \(C_1 > 0\). On the other hand, observe from (2) that there exists \(B_2 > 0\) such that for all \(s \in \mathbb{R}\), \(f(s)s \leq B_2(|s|^{q+1} + |s|)\). Thus
\[ \lambda \int_\Omega f(u_\lambda)u_\lambda dx \leq B_2 \lambda \int_\Omega (|u_\lambda|^{q+1} + |u_\lambda|)dx \]
\[ \leq B_2 \lambda \int_\Omega (|u_\lambda|^{q+1} + \|u_\lambda\|_\infty)dx \]
\[ \leq B\lambda\|u_\lambda\|_\infty^{q+1}, \] (20)
for some \(B > 0\). From (19) and (20) we obtain the result. \(\square\)

Finally we prove the Main Theorem.

**Proof of the Main Theorem.** Arguing by contradiction, let \(\{\lambda_j\}\) a sequence of positive numbers such that \(\lambda_j \to 0\), as \(j \to \infty\) and such that \(|\{x \in \Omega : u_{\lambda_j}(x) \leq 0\}| > 0\). Let \(w_j := \frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}\). Then
\[ (-\Delta)_p w_j = \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-r}. \]
By Lemma 5 and Theorem 1.1 of [10], there exists $\alpha \in (0, s]$ such that
\[
\left\| \frac{w_j}{d_\Omega^s} \right\|_{C^\alpha(\mathcal{O})} \leq \|\lambda_j f(u_{\lambda_j})\|_{L_{\infty}}^{1-p} \|\frac{1}{\bar{w}_j}\|^{p-1} \leq C,
\]
where $C$ does not depend on $\lambda_j$. Let us choose any $0 < \beta < \alpha$. Since $C^\alpha(\mathcal{O}) \subset \subset C^0(\mathcal{O})$ (see Theorem 5.14, [9]) then, up to a sub-sequence, $\lim_{j \to \infty} \frac{w_j}{d_\Omega^s} = \frac{w}{d_\Omega}$ in $C^0(\mathcal{O})$. Now, we will use comparison principle to prove that $w(x) \geq 0$. Let $v_0 \in W_0^{s,p}(\Omega)$ be the solution of
\[
\begin{cases}
(\Delta)^s u = 0 & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N - \Omega.
\end{cases}
\]
Let $K_j = \frac{\lambda_j}{\|u_{\lambda_j}\|_{L_{\infty}}^{s,p}} \min_{t \in \mathbb{R}} f(t)$. Observe that $K_j < 0$. Then, the solution $v_j \in W_0^{s,p}(\Omega)$ of
\[
\begin{cases}
(\Delta)^s u = K_j & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N - \Omega,
\end{cases}
\]
is given by $v_j = -(K_j)^{1/(p-1)} v_0$. Since $\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_{L_{\infty}}^{1-p} \geq K_j$. By the comparison principle stated in [12] (Proposition 2.10) $w_j \geq v_j$. Since $v_j \to 0$, as $j \to \infty$, then $w(x) \geq 0$.

Let us observe that since $\{\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_{L_{\infty}}^{1-p}\} j$ is bounded by a constant independent of $\lambda_j$, then there exists $t > 1$ such that $\{\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_{L_{\infty}}^{1-p}\} j$ is bounded in $L^t(\Omega)$. Thus, we may assume that it converges weakly in $L^t(\Omega)$. Let $z := \lim_{j \to 0} \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_{L_{\infty}}^{1-p}$, its weak limit. Since $f$ is bounded from below and $\lim_{j \to \infty} \lambda_j\|u_{\lambda_j}\|_{L_{\infty}}^{1-p} = 0$, then $z \geq 0$. We claim that $(-\Delta)^s_p w = z$. In fact, from remark 2 and Lemma 5 the sequence of functions
\[
\psi_j(x, y) := \frac{|w_j(x) - w_j(y)|}{|x - y|^{p+s}},
\]
is bounded in $L^p(\mathbb{R}^{2N})$. Therefore, following the same procedure made in Lemma 3 to prove the strong convergence of $\{u_n\}$ (see Lemma 7 in the appendix), we conclude that it converges to
\[
\psi(x, y) := \frac{|w(x) - w(y)|}{|x - y|^{p+s}},
\]
in $L^p(\mathbb{R}^{2N})$. Then there exists $h \in L^p(\mathbb{R}^{2N})$ such that $|\psi_j(x, y)| \leq h(x, y)$, a.e. $(x, y)$. Hence, from the Young’s inequality, for all $\varphi \in W_0^{s,p}(\Omega)$ we have
\[
\frac{|w_j(x) - w_j(y)|^{p-1} |\varphi(x) - \varphi(y)|}{|x - y|^{N+sp}} = \frac{|w_j(x) - w_j(y)|^{p-1} |\varphi(x) - \varphi(y)|}{|x - y|^{N+sp}} \leq \frac{1}{p'} \frac{|w_j(x) - w_j(y)|^{(p-1)p'}}{|x - y|^{N+sp'}} + \frac{1}{p} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N+sp}} \leq \frac{1}{p'} (h(x, y))^p + \frac{1}{p} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{N+sp}}.
\]
where $p'$ stands for the conjugate Hölder exponent of $p$. Since the last function belongs to $L^1(\mathbb{R}^{2N})$, by the Lebesgue Dominated Convergence Theorem we have

$$
\int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy = \lim_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{|w_j(x) - w_j(y)|^{p-2}(w_j(x) - w_j(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy
$$

(21)

Observe that we also proved that $w_j \to w$ in $W^{s,p}_0(\Omega)$, and thus $w \in W^{s,p}_0(\Omega)$. This proves the claim. Thus $w$ is a supersolution of the $(-\Delta)_p^s(w) = 0$ in $\Omega$. Since $\Omega$ has $C^{1,1}$ boundary then it satisfies the interior ball condition (see Theorem 1.0.9 in [2]). Therefore, by Theorems 1.4 and 1.5 of [6] we have $w > 0$ in $\Omega$ and for all $x_0 \in \partial \Omega$,

$$
\liminf_{x \to x_0} \frac{w(x)}{d_{B_R(x)}} > 0,
$$

where $B_R \subseteq \Omega$ and $x_0 \in \partial B_R$. From Lemma 6 (see appendix), there exists $j$ sufficiently large such that $w_j > 0$ in $\Omega$. Absurd. □

4. Appendix

In this section we shall prove some technical results. The first one is based on the Hopf’s Lemma established in [4]. The second, follows the same lines in part of the proof of Lemma 3.

**Lemma 6.** Let us assume that $\Omega \subseteq \mathbb{R}^N$ is bounded domain with $C^{1,1}$ boundary and $\frac{w_j}{d_{\Omega}^{\beta}} \to \frac{w}{d_{\Omega}^{\beta}}$ in $C^\beta(\overline{\Omega})$ with $w(x) = w_j(x) = 0$, for all $j$ and all $x \in \partial \Omega$. Let us assume that $w > 0$ in $\Omega$ and for all $x_0 \in \partial \Omega$,

$$
m := \liminf_{x \to x_0} \frac{w(x)}{d_{B_R(x)}^s} > 0.
$$

Then there exists $j$ such that $w_j(x) > 0$ for all $x \in \Omega$.

**Proof.** First of all, let us emphasize that, since $\frac{w}{d_{\Omega}^{\beta}} \in C^\beta(\overline{\Omega})$, then for all $x_0 \in \partial \Omega$, $\frac{w(x_0)}{d_{\Omega}^{\beta}(x_0)}$ is well defined in terms of limits. Now, let $B_R \subseteq \Omega$ be an interior ball such that $x_0 \in \partial B_R$ and let be $\varepsilon_0 > 0$ such that for all $x \in B_R \cap B(x_0, \varepsilon_0)$,

$$
\frac{w(x)}{d_{B_R(x)}^s} > \frac{m}{2}.
$$

Let us pick up a sequence $\{x_n\}$ in $B_R \cap B(x_0, \varepsilon_0)$ in the segment joining $x_0$ and the center of $B_R$ and such that $x_n \to x_0$. So that for all $n$, $x_n - x_0$ is orthogonal
to \( \partial B_R \) and \( \partial \Omega \) and \( d_{B_R}(x_n) = d_\Omega(x_n) \). Therefore

\[
\frac{w(x)}{d_\Omega(x)} = \lim_{n \to \infty} \frac{w(x_n)}{d_\Omega(x_n)} = \lim_{n \to \infty} \frac{w(x_n)}{d_{B_R}(x_n)} \geq \frac{m}{2} > 0.
\]

And, obviously, \( \frac{w(x)}{d_\Omega(x)} > 0 \) for all \( x \in \Omega \). Thus \( \frac{w}{d_\Omega} \) is positive in the compact \( \Omega \). Let

\[
\varepsilon := \min_{\Omega} \frac{w}{d_\Omega} > 0.
\]

Let \( \Omega_1 \) be a nonempty open set such that \( \Omega_1 \subseteq \Omega \). We claim that there exists \( j \) such that for all \( x \in \Omega_1 \), \( w_j(x) > 0 \). Indeed, there exists \( j \) sufficiently large such that

\[
\left\| \frac{w}{d_\Omega} - \frac{w_j}{d_\Omega} \right\|_{C^\beta(\overline{\Omega})} < \frac{\varepsilon}{2}.
\]

In particular for all \( x \in \Omega_1 \)

\[
\frac{\varepsilon}{2} \leq \frac{w_j(x)}{d_\Omega(x)} - \frac{w(x)}{d_\Omega(x)}.
\]

Then, for all \( x \in \Omega_1 \)

\[
\frac{\varepsilon}{2} \leq \frac{w(x)}{d_\Omega(x)} - \frac{\varepsilon}{2} < \frac{w_j(x)}{d_\Omega(x)}.
\]

Which proves the claim. Finally, we will prove that for all \( x \in \Omega - \overline{\Omega_1} \), \( w_j(x) > 0 \). Let us argue by contradiction. If there exists \( x_0 \in \Omega - \overline{\Omega_1} \) such that \( w_j(x_0) \leq 0 \), then, by the intermediate Value Theorem, there is \( z_0 \in \Omega - \overline{\Omega_1} \) such that \( w_j(z_0) = 0 \). Thus, from (23) and the definition of \( \varepsilon_1 \), we have

\[
\varepsilon \leq \left\| \frac{w(z_0)}{d_\Omega(z_0)} - \frac{w_j(z_0)}{d_\Omega(z_0)} \right\|_{C^\beta(\overline{\Omega})} < \frac{\varepsilon}{2}.
\]

Absurd. \( \Box \)

**Lemma 7.** Let \( \{w_j\} \) be a bounded sequence in \( W^{s,p}_0(\Omega) \), such that

\[
\begin{cases}
(\Delta)^s p(w_j) = \lambda_j g(w_j) & \text{in } \Omega \\
 w_j(x) = 0 & \text{in } \mathbb{R}^N - \Omega,
\end{cases}
\]

with \( \{\lambda_j g(w_j)\} \) bounded in \( L^\infty(\Omega) \). Then \( w_j \) converges strongly in \( W^{s,p}_0(\Omega) \).

**Proof.** Since \( \{w_j\} \) is bounded in \( W^{s,p}_0(\Omega) \), then, up to a subsequence, \( \{w_j\} \) converges weakly to the function \( v \in W^{s,p}_0(\Omega) \). Since \( p < q + 1 < p^*_s \), then \( w_j \to v \) (strongly) in \( L^{q+1}(\Omega) \). As \( \{\lambda_j g(w_j)\} \) bounded in \( L^\infty(\Omega) \), applying the Hölder inequality this implies that

\[
\lim_{j \to \infty} \lambda_j \int_\Omega g(w_j)(w_j - v)dx = 0.
\]
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Then, since $J_{\lambda_j}'(w_j) = 0$ (where $J_\lambda$ is the associated Energy Functional to this problem), we have

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} \frac{\Phi_p(w_j(x) - w_j(y))((w_j - v)(x) - (w_j - v)(y))}{|x - y|^{N+sp}} = 0. \quad (24)$$

Using again that $v$ is the weak limit of $w_j$ we have

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} \frac{\Phi_p(v(x) - v(y))((w_j - v)(x) - (w_j - v)(y))}{|x - y|^{N+sp}} = 0. \quad (25)$$

Thus, from the same argument that we use in the proof of Lemma 3 we obtain

$$\int_{\Omega} \frac{\Phi_p(w_j(x) - w_j(y)) - \Phi_p(v(x) - v(y))}{|x - y|^{N+sp}}((w_j - v)(x) - (w_j - v)(y))dx\,dy \geq (\|w_j\|^{p-1} - \|v\|^{p-1})(\|w_j\| - \|v\|) \geq 0.$$

From (24), (25) we obtain

$$\lim_{j \to \infty} (\|w_j\|^{p-1} - \|v\|^{p-1})(\|w_j\| - \|v\|) = 0,$$

which implies

$$\lim_{j \to \infty} \|w_j\| = \|v\|.$$

Since $w_j \to v$, then $w_j \to v$ strongly in $W_0^{s,p}(\Omega)$. \hfill \Box

ACKNOWLEDGMENT

E. Lopera was partially supported by Facultad de Ciencias, Universidad Nacional de Colombia, sede Manizales, Hermes codes 55156 and 51894, and 100,000 Strong in the Americas, Innovation Fund.

C. López was partially supported by Facultad de Ciencias, Universidad Nacional de Colombia, sede Manizales, Hermes code 55156 and 100,000 Strong in the Americas, Innovation Fund.

R. E. Vidal was partially supported by CONICET, FONCyT and SECyT-UNC.

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