Sub-Planck spots of Schrödinger cats and quantum decoherence

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Heisenberg’s principle\(^1\) states that the product of uncertainties of position and momentum should be no less than Planck’s constant \(\hbar\). This is usually taken to imply that phase space structures associated with sub-Planck (\(\ll \hbar\)) scales do not exist, or, at the very least, that they do not matter. I show that this deeply ingrained prejudice is false: Non-local “Schrödinger cat” states of quantum systems confined to phase space volume characterized by ‘the classical action’ \(A \gg \hbar\) develop spotty structure on scales corresponding to sub-Planck \(a = \hbar ^2/\Lambda \ll \hbar\). Such structures arise especially quickly in quantum versions of classically chaotic systems (such as gases, modelled by chaotic scattering of molecules), that are driven into nonlocal Schrödinger cat-like superpositions by the quantum manifestations of the exponential sensitivity to perturbations\(^2\). Most importantly, these sub-Planck scales are physically significant: \(a\) determines sensitivity of a quantum system (or of a quantum environment) to perturbations. Therefore sub-Planck \(a\) controls the effectiveness of decoherence and einselection caused by the environment\(^3\)–\(^8\). It may also be relevant in setting limits on sensitivity of Schrödinger cats used as detectors.

One of the characteristic features of classical chaos is the evolution of the small scale structure in phase space probability distributions: As a consequence of the exponential sensitivity to initial conditions, an initially regular “patch” in phase space with a characteristic size \(\Delta\) will, after a time \(t\), develop structure on the scales:

\[
z \simeq \Delta \exp(-\Lambda t)
\]

where \(\Lambda\) is, in effect, the Lyapunov exponent\(^9\)\(^10\). This is a consequence of the exponential stretching and of the conservation of phase space volume in reversible evolutions.

What happens with small scale structure in quantum versions of classically chaotic systems? We shall investigate this question using Wigner function \(W(x, p)\) – the closest quantum analogue of the classical phase space distribution\(^11\):

\[
W(x, p) = \frac{1}{2\pi \hbar} \int \exp(ipy/\hbar)\rho(x - y/2, x + y/2) dy .
\]

Above, \(\rho\) is the density operator. When the quantum state is a pure \(\vert \psi >\), the integrand becomes \(\exp(ipy/\hbar)\psi^\ast(x + y/2)\psi(x - y/2)\).

In a quantum system the smallest spatial structures in the wave function \(\psi(x)\) will be set by the highest phase space frequencies available. Given a finite energy, \(\psi(x)\), and, hence, \(W(x, p)\) cannot reach scales as small as the exponential squeezing of Eq. (1) would eventually imply. Thus, in a quantum system, there must be a scale below which structure should not appear. That some limit must exist was apparent already some time ago: Berry and Balazs (who, as we shall see below in connection with Eq. (13)) have an enviable record of correctly anticipating various aspects of quantum chaos) have conjectured\(^12\)\(^13\) that structure saturates on scales given by Planck constant. If this were the case, \(W\) should be smooth on scales small compared to \(2\pi\hbar\).

I shall show that copious structure appears in the Wigner function \(W\) on much smaller sub-Planck scales associated with the action of the order of:

\[
a \simeq \hbar \ast \hbar/\Lambda ,
\]

and explain its origin. Most importantly, I shall demonstrate that \(a\) has physical consequences: It determines sensitivity of the system (or of the environment) to decoherence. Above \(A\) is the classical action of the system, given approximately by the product of the range of effective support of its state in position \(L\) and momentum \(P\):

\[
A \simeq L \ast P.
\]

The values of \(L\) and \(P\) are in turn set by the available energy \(E\), and by the form of the potential \(V(x)\) (i.e., \(P \leq \sqrt{2mE}, E - V(L) \geq 0\), etc.) that determine the effective support of the probability distribution in phase space. We shall eventually see that many of the calculations can be carried out using the state vector of the system in the appropriate representation. Nevertheless, intuitive understanding of the significance of the sub-Planck scale \(a\) is easiest to attain starting with a more comprehensive view of the phase space image of quantum states afforded by the Wigner representation.
It is evident from Fig. 1 that – at least in chaotic systems – the structure on sub-Planck scales appears and saturates quickly. Let us now illustrate its origin: The smallest scales in phase space arise from interference. Consider, for instance, a familiar “Schrödinger cat” coherent superposition of two minimum uncertainty Gaussians:

$$W(x, p) = \frac{G(x + x_0, p) + G(x - x_0, p)}{2} + (\pi h)^{-1} \exp(-\frac{p^2 \xi^2}{h^2} - \frac{x^2}{\xi^2}) \cos(p \frac{2x_0}{h}) ,$$  \hspace{1cm} (5)

where Wigner functions of the two Gaussians ‘east’ and ‘west’ of the center are;

$$G(x \pm x_0, p - p_0) = (\pi h)^{-1} \exp(- (x \pm x_0)^2/\xi^2 - (p - p_0)^2 \xi^2/h^2) .$$  \hspace{1cm} (6)

The last oscillatory term $W_{WE}$ in Eq. (5) is a symptom of interference. Its ripples have a frequency proportional to the separation $L = 2x_0$ between the two peaks. When we define the frequency of the ripple pattern in momentum $f_p$ through $\cos(Lp/h) = \cos(f_p p)$, then $f_p = L/h$. Ridges and valleys of such interference pattern are always parallel to the line of sight between the two Gaussian peaks. Thus, standing on top of one Gaussian peak, one could still see the other peak through the valleys (and between the ridges) of the interference term, even though its envelope is a factor of two higher than either Gaussian.

Coherent states form an overcomplete set. We can therefore express an arbitrary pure state as a superposition of coherent states. The smallest interference structures in such an expansion arise from pairs of coherent states separated by the whole range available to the system in phase space. Therefore, we expect smallest scales with:

$$f_x = P/h, \text{ or } \delta_x = h/P, \text{ or } f_p = L/h, \text{ or } \delta_p = h/L .$$  \hspace{1cm} (7, 8)

As an example, consider compass state, a Schrödinger cat-like superposition of four minimum uncertainty Gaussians, one pair located north and south of the common center, the other east and west (see Fig. 2). Wigner distribution is quadratic in the wave function. Therefore, $W$ of any superposition can be reconstructed from the contributions corresponding to, at most, pairs of states. The structure of Wigner distribution of a superposition of a pair of Gaussians, (Eq. (5) that is, *nota bene*, reflected in the patterns on the sides of the square in Fig. 2), can be then used to infer $W$ of the compass state:

\begin{align*}
W_{NWSE} &= (G_N + G_W + G_S + G_E)/4 \hspace{1cm} (9a) \\
&+ (W_{NW} + W_{WS} + W_{SE} + W_{EN})/2 \hspace{1cm} (9b) \\
&+ (W_{NS} + W_{EW})/2 \hspace{1cm} (9c)
\end{align*}

in an obvious “geographic” notation. The last line of Eq. (9):

$$W_{NS} + W_{EW} = (\pi h)^{-1} \exp(-\frac{p^2 \xi^2}{h^2} - \frac{x^2}{\xi^2})(\cos(p L/h) + \cos(x P/h)) .$$  \hspace{1cm} (10)

is of greatest interest. Above, we have assumed that shapes of all the Gaussians are identical, so that the exponential envelope in Eq. (9c) is common to both terms.

The resulting interference term (the center of Fig. 2) has a “checkerboard pattern”. The size of the single ‘tile’ can be obtained from zeros of the oscillatory factor of Eq. (10):

$$\cos(pL/h) + \cos(xP/h) = 2 \cos \frac{Px + Lp}{2h} \cos \frac{Px - Lp}{2h} .$$  \hspace{1cm} (11)

Individual squares, four per tile, are defined by zeros that occur when: $x = \pm \pi h/2L$, $p = \pm \pi h/2P$. The fundamental periodic tile has an area of:

$$a = \frac{2\pi h}{L} \times \frac{2\pi h}{P} = (2\pi h)^2/A ,$$  \hspace{1cm} (12)

that, with the identification of $A = L \ast P$, yields action associated with the smallest scales present in quantum phase space.

The above calculation shows by construction that a quantum state spread over a phase space of volume $A = L \ast P$ can accommodate Wigner distribution structures as small as $a$. Such states arise naturally: Evolution will force almost any system (with a notable exception of a harmonic oscillator) into a Schrödinger cat state – a coherent non-local
superposition that, after a time it takes the wave function to spread over the phase space volume $A$ – inevitably develops interference pattern on the scale given by Eqs. (7-10).

It is not necessary to invoke chaos: After sufficient time, even an integrable non-linear system may spread coherently throughout the available phase space, and, consequently, saturate small scale structure. It is just that when the evolution is chaotic, such small scales will be attained faster, on a time scale given by 14:

$$t_h = \Lambda^{-1} \ln \Delta p \chi / \hbar .$$ (13a)

Above, $\Delta p$ is the characteristic spread of the initial smooth probability distribution. $\chi$ characterizes the scale on which potential is significantly nonlinear. It is typically given by $\chi \approx \sqrt{V'/V''}$. Similar estimates obtain from the formula:

$$t_r = \Lambda^{-1} \ln A / \hbar$$ (13b)

deduct some time ago 12, 15.

There is one more suggestive way to express saturation scale $a$: The number of distinct (orthogonal) states that can fit within phase space of volume $A$ is $N = A/(2\pi \hbar)$. The structure we are discussing appears therefore on the scale $a \sim \hbar / N$. Here $N$ is, in effect, the dimension of the available Hilbert space.

In accord with Heisenberg’s principle, quantum system cannot be localized to a sub-Planck volume in phase space. Hence, one might be tempted to dismiss sub-Planck scales as unphysical even if they appear in the Wigner distribution. In particular, if a state cannot be confined, by measurements, to a volume less than $\sim \hbar$, then one might expect that it will not to be noticeably perturbed by displacements much smaller than $\sqrt{\hbar}$. We now show that this expectation is false, and that $a \sim \hbar$ plays a decisive role in determining sensitivity of quantum systems (or of quantum environments) to perturbations: Phase space displacements $\delta \sim \sqrt{a} \ll \sqrt{\hbar}$ shift the state with a dominant structure on scale $a$ enough to make it orthogonal – i.e., distinguishable – from the unshifted original. Sensitivity to perturbations in turn sets the limit on the efficiency of decoherence.

There are two complementary aspects to this connection between $a$ and decoherence: When the system with a sub-Planck scale in $W$ is coupled to the environment, decoherence can be thought of as monitoring, by the environment, of some of its observables 3-8. Its effect – suppression of quantum coherence – can be traced to the Heisenberg’s principle: The observable complementary to the one monitored by the environment become less determined, in effect smearing Wigner distribution along the corresponding phase space direction 8. When this smearing obliterator interference structures on scale $a$, coherence on the large scales corresponding to $A \sim \hbar^2 / a$ shall be suppressed, and the Schrodinger cat state would have lost its quantum nonlocality 2, 14, 17.

A complementary aspect of the same story will be our focus in the remainder of this paper. It involves the situation when the state of the environment – the cause of decoherence – is a “Schrödinger cat” spread over the phase space region $A$. Decoherence is caused by ‘monitoring’ of the system by the environment. Environment entangles with the system, acting as an apparatus 3-8. The sensitivity of the environment to perturbations is therefore of essence. We shall test sensitivity of such Schrödinger cat environment by allowing it to interact with a Schrödinger cat system, assumed to be initially in a superposition of two perfect pointer states $\{|+\rangle, |-\rangle\}$. Pointer states, taken one at a time, perturb the state of the environment but do not entangle with it. However, each pointer state perturbs the environment differently 3. Therefore, a system prepared in a general superposition state will leave a pointer state dependent imprint on the environment, and, hence, entangle with it. We shall show that the displacement $\delta \sim \sqrt{a}$ sets the size of the smallest perturbations distinguished by the environment, which in turn controls its ability to decohere the system. Thus, when $|+\rangle$ and $|-\rangle$ shift the state of the environment differently, the off-diagonal terms of the density matrix of the system will be suppressed by a factor $|<\varepsilon_+|\varepsilon_->|$, where:

$$|<\varepsilon_+|\varepsilon_->|^2 = 2\pi \hbar \int W_+ W_- dx dp$$ (14)

Above, kets $|\varepsilon_{\pm}\rangle$ and their Wigner distributions ($W_{\pm}$) represent states that evolve from the original state of the environment $|\varepsilon\rangle$ due to the interaction (induced, say, by the system-environment interaction Hamiltonian such as $\sim (|+\rangle<+|-\rangle<-|-\rangle)<\varepsilon_{\pm}\rangle$ with the states $|\pm\rangle$ of the system, respectively.

Let us now demonstrate how the behavior of the magnitude of the overlap:

$$|<\varepsilon_+|\varepsilon_->| = |\int e^* (x) e^{i\delta_p x / \hbar} \varepsilon(x + \delta_x) dx|,$$ (15)

is controlled by $a$. Above, $\delta = (\delta_x, \delta_p)$ is the net displacement between $|\varepsilon_+\rangle$ and $|\varepsilon_-\rangle$ corresponding to $W_+ = W(x + \delta_x^+, p + \delta_p^+)$ and $W_- = W(x + \delta_x^-, p + \delta_p^-)$, respectively. The displacement $\delta$ is the difference between the shifts.
caused by $|+>$ and $|->$, $\delta = \delta^+ - \delta^- \neq 0$. Heuristic argument for the size of displacement that causes orthogonality $<\varepsilon_+|\varepsilon_-> \approx 0$, and, hence, decoherence, is easiest to follow when phrased in terms of Wigner functions: Suppose $W_+$ and $W_-$ in Eq. (14) have a small scale structure with patches of alternating sign, as seen in Figs. 1-3. Integral of their product can reach a maximum value of unity only when $W_+$ and $W_-$ are not shifted with respect to each other. For shifts small compared to the typical size of the patches the integrand will still be almost everywhere positive, but with the increase of the magnitude of the shift the integral will decrease, as $W_+(x,p)W_-(x,p)$ is no longer positive definite. As the shift exceeds the size of the patches in $W(x,p)$, the integrand will oscillate around zero, and the integral of Eq. (14) will be small compared to unity. If the interference pattern in the Wigner distribution is periodic (as is the case in Fig. 2) the oscillation will be also periodic with a period related to the size of the fundamental 'tile'. When, however, patches are random (as for the typical case illustrated in Fig. 3), the overlap – having decayed after a displacement $|\delta| \sim \sqrt{\pi} - \delta$ shall not significantly recur. A more formal version of this heuristic argument is put forward and backed up by numerical simulations elsewhere (Karkuszewski et al, in preparation). A simple back-of-the-envelope calculation valid for a generalization of compass states (that is, when the state of the environment can be approximated by a "sparse" collection of identically shaped minimum uncertainty Gaussians) is given in the caption of Fig. 3.

This intuitive picture, already supported the example of Fig. 2, can be further confirmed by a general yet straightforward calculation based on Eq. (15). To simplify notation we consider the case when $\delta = (0, \delta_p)$, i.e., when the net shift is aligned with one of the axes. (This assumption can be made with no loss of generality, as the axes in phasespace can be rotated to align one of them with an arbitrary $\delta$.) In that case:

$$<\varepsilon_+|\varepsilon_-> = \int |\varepsilon(x)|^2 e^{i\frac{\delta_p x}{\hbar}} dx.$$  \hspace{1cm} (16)

This is a simple, general, and compelling result: Suppression the off-diagonal terms in the density matrix of the system is given by a Fourier transform of the probability distribution in the environment along the direction perpendicular to the net relative shift induced by its coupling with the system. This leads back to sub-Planck scales: For small displacements the exponent in the integrand can be expanded. This yields:

$$|<\varepsilon_+|\varepsilon_->|^2 \approx 1 - \delta_p^2(<x^2> - <x^2>)/\hbar^2.$$  \hspace{1cm} (17a)

The spatial extent of the wave function is naturally defined as $L = \sqrt{<x^2> - <x^2>}$, where the averages of the observable $x$ are over the state of the environment. The estimate of the shift $\delta_p$ leading to orthogonality is then:

$$\delta_p \simeq \hbar/\sqrt{<x^2> - <x^2>},$$  \hspace{1cm} (17b)

in accord with the estimate of the dimensions of the sub-Planck structures, Eq. (8), and in agreement with our thesis about their role. I show in Methods that the same simple formula holds when the environment is initially in a mixed state.

Obvious generalizations of Eqs. (16) and (17) are valid for arbitrary displacements, and for the case of many states of the system. Our treatment can be also extended to interactions between the system and the environment that cannot be represented as simple shifts, although calculations become more complicated. The role played by the spread $L$ of the state of the environment in the behavior of the overlap (and, hence, in decoherence) is a direct consequence of the properties of the Fourier transform. Detailed analysis of other consequences of Eq. (16) is beyond the scope of this paper. Using elementary properties of the Fourier transform readers can nevertheless immediately confirm that the overlap decreases to near zero around $\delta_p$, Eqs. (8) and (17), and that – for typical initial states of the environment – it remains much less than unity. “Revivals” of the overlap we have seen in the case of the compass state (Fig. 2) are now easily understood and dismissed as an exception. Equation (16) shows that they can happen only when the initial probability distribution of the environment in the direction perpendicular to displacement is localized to a few peaks, so that $|\varepsilon(x)|^2$ – the spectrum of the displacement-dependent overlap – is essentially discrete.

To conclude, a physical consequence – orthogonality – can be induced by displacements much smaller than these corresponding to the Planck scale $\sqrt{\hbar}$. This is surprising, as sub-Planck scales are often regarded as unphysical. We have shown that they have dramatic physical implications: Displacements given by the square root of $a$, Eq. (3); $\delta \approx \hbar/\sqrt{A}$, suffice to induce orthogonality. They are a factor $\sim \sqrt{N} = \sqrt{A/2\pi\hbar}$ smaller than $\delta \sim \sqrt{\hbar}$ dictated by the Planck constant and needed to move a typical minimum uncertainty Gaussian in a random direction enough to noticeably reduce its overlap with its (old) self.

Nonlinear, and, especially, chaotic dynamics leads to states that are spread over much of the available phase space.\textsuperscript{12,13} Hence, one would expect that environments with unstable dynamics will be much more efficient decoherers, as they are constantly evolving into delocalized states. Yet, standard models of decoherence\textsuperscript{18–20} employ harmonic oscillators. They make up for this inefficiency by using many (infinity) of them, so that each becomes slightly displaced by the interaction with the system.
On a more mundane level, our results allow one to anticipate structure of the mesh required to simulate evolution of a quantum system in phase space. They are related to the quantization of discrete chaotic maps on a torus, which turn out to require a mesh similar to this given by Eqs. (7) and (8). This is in contrast to simulations of classical systems, that are doomed by Eq. (1), which implies resolution exponentially increasing with time. Most importantly, a controls sensitivity of quantum states to perturbations, with obvious implications for decoherence we have outlined above. Moreover, sensitivity to perturbations will set limits on ‘hypersensitivity’ of quantum chaotic systems, and helps one understand enhanced capacity of quantum chaotic systems for entanglement.

It is tempting to imagine that the sensitivity of quantum systems in highly delocalized states may be not just a cause of accelerated decoherence (and, hence, an impediment to truly quantum applications) but that in certain settings it may be beneficial. This is not be as far-fetched as it may seem at first: After all, a detector in a compass-like state of Fig. 2 would be sensitive to perturbations \( \sim h/\sqrt{A} \) that are minute compared with the ‘standard quantum limit’.

Methods

Decoherence happens to a quantum system \( \mathcal{S} \) as a consequence of a measurement-like interaction with the environment \( \mathcal{E} \), which entangles their states:

\[
|s> |\varepsilon> \rightarrow (|\alpha> + \beta|> )|\varepsilon> \rightarrow |\alpha> + \beta|> \varepsilon> = |\Phi_{\mathcal{SE}}> .
\]

Above, we have assumed a system with a two-dimensional Hilbert space spanned by the \{\(|+> ,|-> \)\} orthonormal basis. The two conditional states of the environment \(|\varepsilon> \rightarrow U_{\pm} |\varepsilon> \) evolve under the unitary transformations \( U_+ (U_-) \) induced by the system in the state \(|+> (|-> \)\) respectively. When the effect of the interaction is simply a displacement in phase space, then \( U_{\pm} = D_{\pm} = \exp \{i (\delta^x_{\pm} p + \delta^y_{\pm} x)/\hbar \} \), where \( D \) is the displacement operator, and \( \delta^x_{\pm} = (\delta^x_\alpha, \delta^x_\beta) \) are the resulting shifts. Such displacement could be induced by a Hamiltonian of interaction \( H_{\mathcal{SE}} = g(|+< +|-|->< -|)\frac{\hbar}{2}\partial_x \), where \( g \) is a coupling constant. More general conditional evolutions of \( \mathcal{E} \) can be of course considered.

Following entanglement, the state of the system alone is described by the reduced density matrix obtained from \( |\Phi_{\mathcal{SE}}> \) by a trace over the environment:

\[
\rho_{\mathcal{S}} = Tr_{\mathcal{E}} |\Phi_{\mathcal{SE}}> < \Phi_{\mathcal{SE}}| = \alpha^2 + |z\alpha\beta| + |z^*\alpha\beta| + |\beta|^2 - |z^*\alpha\beta| + |\beta|^2
\]

Disappearance of the off-diagonal terms signifies perfect decoherence. In the \{\(|+> ,|-> \)\} basis this is guaranteed when the overlap \( z \equiv |\varepsilon_+|\varepsilon_-> = Tr|\varepsilon_-><\varepsilon_+| \) disappears. It is therefore natural to measure effectiveness of decoherence by the magnitude of the overlap of the overlap of the two conditional states of the environment, that in turn determines the degree of suppression of the off-diagonal terms in \( \rho_{\mathcal{S}} \).

This two paragraph “crash course” is no substitute for a more complete discussion of decoherence. We have swept under the rug a number of issues. Foremost among them is einselection – the emergence, in course of decoherence, of the preferred set of pointer states that habitually appear on the diagonal of \( \rho_{\mathcal{S}} \) essentially independently of the initial states of either \( \mathcal{E} \) or \( \mathcal{S} \). Stability of these pointer states (rather than the diagonality of the density matrix in some basis) is the key to the role played by decoherence in transition from quantum to classical.

Another important subject we have avoided in the paper is the likely situation when the state of the environment is represented by a mixture \( \rho_{\mathcal{E}} \). Detailed discussion of this case (treated extensively before, although not from the point of view of the sub-Planck structures) in the present context is beyond the scope of this letter, but the basic conclusion is easy to state: The estimated magnitude of the smallest displacement leading to orthogonality is still given by Eq. (17). That is, it is still related to the smallest scales compatible with the classical action \( A \) associated with \( \rho_{\mathcal{E}} \). Off-diagonal terms of \( \rho_{\mathcal{S}} \) are suppressed by \( z = TrU_- \rho_{\mathcal{E}} U_+ \), which is the relevant generalization of Eq. (16). This expression can be expanded for simple displacements \( U_+ = D_+ \) in the limit of small shifts to recover Eq. (17), with the only difference arising from the fact that now the mixture \( \rho_{\mathcal{E}} \) must be used to obtain the averages (e. g. \( <a^2> = Trx^2 \rho_{\mathcal{E}}, \) etc.).

Note that throughout the paper we have set our discussion in one spatial dimension. Generalization to \( d \) dimensions is as straightforward conceptually as it is notationally cumbersome. The structure saturates in volumes of \( a^d \), etc.
This has little effect on decoherence, as it depends on displacements that yield orthogonality, and these are still \( \delta \sim \sqrt{a} \).

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FIG. 1. Snapshots of the classical probability density and of the quantum Wigner distribution in phase space of an evolving chaotic system with the Hamiltonian: 

$$H = \frac{p^2}{2m} - \kappa \cos(x - l \sin t) + \frac{ax^2}{2}.$$ 

For parameters $m = 1$, $\kappa = 0.36$, $l = 3$, and $a = 0.01$ this system exhibits chaos with Lyapunov exponent $\Lambda \approx 0.2$. Initial probability density was given by the same Gaussian in both the quantum (a-c; $\hbar = 0.16$) and the classical (d) cases. Structure on sub-Planck scales appears and saturates quickly. Note the contrast between the smallest dimensions of the probability density in the classical case (d) and of the Wigner distribution at the corresponding time (a). Exponential shrinking of the smallest scales makes it impossible to simulate accurately (e.g., reversibly) classical evolution much beyond the time shown in panel (d) above. By contrast, structures in quantum Wigner distribution saturate on scales $a \simeq \hbar^2/A$ soon after $t \sim 20$ (which is of the order of the estimated $t_\hbar$, Eq. (13a)). The scale of structure saturation of a state can be inferred from the volume of the domain containing it: Smallest structures have dimensions $\delta_x = \hbar/P$ ($\delta_x = \hbar/L$), where $P$ and $L$ defines the extent of the envelope of the effective support of the state ($P^2 \simeq <p^2> - <p>^2$, $L^2 \simeq <x^2> - <x>^2$). Action associated with the smallest structures will be then $a \simeq \hbar^2/LP \simeq \hbar^2/A$ in one spatial (two phase space) dimensions. Generalization to the case of many dimensions is straightforward. One way to understand structure saturation is through the menu of wavevectors available in the system that has momenta restricted to range $P \sim \sqrt{E}$, where $E$ is the total energy. Complementary argument ($V(L) \sim E$, where $V(x)$ is the potential) can be made for the smallest scale of “corrugation” of the Wigner distribution in $p$. 


FIG. 2. The compass state, Eq. (11), is a Schrödinger cat-like superposition of four minimum uncertainty states $|N\rangle, |W\rangle, |S\rangle, |E\rangle$ placed ‘north’, … ‘east’, of the common center. The form of the central interference pattern in $W_{NWSE}$ can be inferred from the familiar structure of the superposition of two Gaussians, Eq. (5), also apparent on the sides of the square above: The interference terms corresponding to the north-south and east-west pairs superpose, creating a checkerboard pattern. The area of individual tiles is set by the dimensions of the NWSE cross and corresponds to the classical action $A$, of phase space area of the effective support of its envelope, Eq. (12). Appearance of this interference pattern explains the origin of the structure saturation seen in Figs. 1 and 3: A system that can be effectively confined to phase space volume $A$ cannot develop structure on scales smaller than $a$. Sensitivity of the compass state to perturbations is controlled by $a$. This is readily seen in the sparse limit, that is when when $L \gg \xi$, $P \gg \hbar/\xi$. For simplicity, consider a shift $\delta = \delta_x \hat{x} + \delta_p \hat{p}$ small compared to the sizes of the Gaussians, $\delta_x \ll \xi$, $\delta_p \ll \hbar/\xi$. The square of the overlap of the original and displaced states $|\langle N_{NWSE} | N_{NWSE} \rangle|^2 = 2\pi \hbar \int W_{NWSE}(x,p)W_{NWSE}(x+\delta_x,p+\delta_p) dx dp$ is approximately $|\langle N | N \rangle + \ldots + \langle E | E \rangle|^2 = (\cos \delta_x P/2\hbar + \cos \delta_p L/2\hbar)^2/4$. To get this simple result we have ignored both the additive corrections (such as $\langle N | E \rangle$) that are small in the sparse limit, and a multiplicative correction $\sim |\langle E | E \rangle|^2$ that is very close to unity in the limit of small displacements $|\delta| \ll \sqrt{\hbar}$. Striking kinship of this form of the overlap with the interference term, Eqs. (10) and (11), is no accident: The magnitude of the shifts that produce orthogonality are $\delta_x = 2\pi \hbar/P$, $\delta_p = 2\pi \hbar/L$. Thus, displacing the states by the size of the basic ‘tile’ in the central interference pattern defined by Eqs. (11) and (12) suffices to cause decoherence, if the environment starts in the NWSE state. larger shifts.
FIG. 3. Snapshots of area $2\pi\hbar$ extracted from Fig. 1a-c. Smallest scales saturate on sub-Planck $a$, so that $a$ as can be estimated from Fig. 1 – the individual patches in the figure above appear on the scale consistent with Fig. 1, 2, and Eq. (3). Such small phase space substructure is physically significant: Displacement of the order of $\sqrt{a}$ – size of a typical patch – suffices to make the perturbed state approximately orthogonal to its old (unperturbed) self. Thus, $a$ sets the limit on the sensitivity of the state to perturbations, and is therefore relevant for decoherence. Note that the overlap of the original and displaced states will behave differently when the interference pattern is irregular (as it is here) rather than essentially periodic (as was the case in the compass state of Fig. 2). This can be understood by expressing the state as a superposition $|\psi> = \sum_k \alpha_k |k>$ of identical minimum uncertainty Gaussians $|k> = G(x - x_k, p - p_k)$, each centered on $x_k, p_k$ (see Eq. (6)). In the sparse limit (i.e., when the Gaussians in the superposition of $|\psi>$ do not overlap significantly) the overlap of the original and displaced states is approximately: $<\psi|\psi'> \simeq |\sum_k |\alpha_k|^2 e^{i(\delta p x_k + \delta x p_k)/\hbar}| = |\sum_k w_k e^{i\delta_k}|$, where $w_k = |\alpha_k|^2$, $\sum_k w_k = 1$ are the weights – probabilities – of finding the system in different states $|k>$. It is obvious that, when many sparsely distributed Gaussians participate in $|\psi>$, so that $w_k \sim 1/N$, size of the overlap is given by the distance covered by an eventually random walk in a complex plane where individual steps have magnitude $w_k$ and directions determined by phases $\phi_k$. Hence, as $\phi_k$ become random, the overlap will rapidly decrease from unity to approximately $1/\sqrt{N}$, and will likely remain small. (Such sums are a standard way of recovering a Gaussian probability distribution, and have already been studied in the context of decoherence some time ago.)