Concurrence classes for an arbitrary multi-qubit state based on positive operator valued measure

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In this paper, we propose concurrence classes for an arbitrary multi-qubit state based on orthogonal complement of a positive operator valued measure, or POVM in short, on quantum phase. In particular, we construct concurrence for an arbitrary two-qubit state and concurrence classes for the three- and four-qubit states. And finally, we construct $W^m$ and $GHZ^m$ class concurrences for multi-qubit states. The unique structure of our POVM enables us to distinguish different concurrence classes for multi-qubit states.

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I. INTRODUCTION

Entanglement is an interesting feature of quantum theory which in recent years attract many researcher to quantify, classify, and to investigate its useful properties. Entanglement has already some applications such as quantum teleportation and quantum key distribution, and it surely will arrive new applications for this fascinating quantum phenomenon. For instance, multipartite entanglement has a capacity to offer new unimaginable applications in emerging fields of quantum information and quantum computation. One of widely used measures of entanglement for a pair of qubits is the concurrence that gives an analytic formula for the entanglement of formation $\frac{1}{2}$. In recent years, there have been made some proposals to generalize this measure into a general bipartite state, e.g., Uhlmann [3] has generalized the concept of concurrence by considering arbitrary conjugation, than Audenaert et al. [4] generalized this formula in spirit of Uhlmann’s work, by defining a concurrence vector for pure state. Moreover, Gerjuoy [5] and Albeverio and Fei [6] gave an explicit expression in terms of coefficient of a general pure bipartite state. Therefore, it could be interesting to try to generalize this measure from bipartite to multipartite system, see Ref. [7, 8, 9, 10]. An application of concurrence for a physically realizable state such as BCS state can be found in Ref. [11]. Quantifying entanglement of multipartite states has been discussed in [12, 13, 14, 15]. In [16, 17, 18, 19, 20, 21, 22, 23], we have proposed a degree of entanglement for a general pure multipartite state, based on the POVM on quantum phase. In this paper, we will define concurrence for an arbitrary two-qubit state based on orthogonal complement of our POVM. From our POVM we will construct an operator that can be seen as a tiled operation acting on the density operator. Moreover, we will define concurrences for different classes of arbitrary three- and four-qubit states. And finally, we will generalize our result into an arbitrary multi-qubit state. The structure of our POVM enables us to detect and to define different concurrence classes for multi-qubit states. The definition of concurrence is based on an analogy with bipartite state. For multi-qubit states, the $W^m$ class concurrences are invariant under stochastic local quantum operation and classical communication(SLOCC) [24]. Furthermore, all homogeneous positive functions of pure states that are invariant under determinant-one SLOCC operations are entanglement monotones [25]. However, invariance under SLOCC for the $W^m$ class concurrence for general multipartite states need deeper investigation. It is worth mentioning that Uhlmann [3] has shown that entanglement monotones for concurrence are related to antilinear operators. However, the $GHZ^m$ class concurrences for multipartite states need optimization over all local unitary operations. Classification of multipartite states has been discussed in [24, 25, 26, 27, 28, 29]. For example, F. Verstraete et al. [26] have considered a single copy of a pure four-partite state of qubits and investigated its behavior under SLOCC, which gave a classification of all different classes of pure states of four qubits. They have also shown that there exist nine families of states corresponding to nine different ways of entangling four qubits. A. Osterloh and J. Siewert [27] have constructed entanglement measures for pure states of multipartite qubit systems. The key element of their approach is an antilinear operator that they called comb. For qubits, the combs are invariant under the action of the special linear group. They have also discussed inequivalent types of genuine four-qubit entanglement, and found three types of entanglement for these states. This result coincides with our classification, where in section VI we construct three types of concurrence classes for four-qubit states. A. Miyake [28, 29] has also discussed classification of multipartite states in entanglement classes based on the hyper-determinant. He shown that two states belong to the same class if they are interconvertible under SLOCC. Moreover, the only paper that addressed the classification of higher-dimensional multipartite states is the paper by A. Miyake and F. Ver-
strategically \[28\], where they have classified multipartite entangled states in the \(2 \times 2 \times n\) quantum systems for \((n \geq 4)\). They have shown that there exist nine essentially different classes of states, and they give rise to a five-graded partially ordered structure, including GHZ class and W class of 3 qubits. F. Mintert et al. \[29\] have proposed generalizations of concurrence for multi-partite quantum systems that can distinguish distinct quantum correlations. However, their construction is not similar to our concurrence classes, since we can distinguish these classes based on joint phases of the orthogonal complement of our POVM by construction. Finally, A. M. Wang \[10\] has proposed two classes of the generalized concurrence vectors of the multipartite systems consisting of qubits. Our classification is similar to Wang’s classification of multipartite state. However, the advantage of our method is that our POVM can distinguish these concurrence classes without prior information about inequivalence of these classes under local quantum operation and classical communication (LOCC). Let us denote a general multipartite quantum system with \(m\) subsystems by \(Q = Q_m(N_1, N_2, \ldots, N_m) = Q_1 \otimes Q_2 \otimes \ldots \otimes Q_m\), consisting of a state \(|\Psi\rangle = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, \ldots, k_m} |k_1, \ldots, k_m\rangle\) and \(|\Psi^*\rangle = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, \ldots, k_m}^* |k_1, \ldots, k_m\rangle\), the complex conjugate of \(|\Psi\rangle\). Let \(\rho_Q = \sum_{n=1}^{N} p_n |\Psi_n\rangle \langle \Psi_n|\), for all \(0 \leq p_n \leq 1\) and \(\sum_{n=1}^{N} p_n = 1\), denote a density operator acting on the Hilbert space \(H_Q = H_{Q_1} \otimes H_{Q_2} \otimes \cdots \otimes H_{Q_m}\), where the dimension of the \(j\)th Hilbert space is given by \(N_j = \dim(H_{Q_j})\). We are going to use this notation throughout this paper, i.e., we denote a mixed pair of qubits by \(Q_2(2,2)\). The density operator \(\rho_Q\) is said to be fully separable, which we will denote by \(\rho_Q^{sep}\), with respect to the Hilbert space decomposition, if it can be written as \(\rho_Q^{sep} = \sum_{n=1}^{N} p_n \bigotimes_{j=1}^{m} \rho_{Q_j}^{n}\), where \(\rho_{Q_j}^{n}\) is the density operator on Hilbert space \(H_{Q_j}\). If \(\rho_Q^{n}\) represents a pure state, then the quantum system is fully separable if \(\rho_Q^{n}\) can be written as \(\rho_Q^{sep} = \bigotimes_{j=1}^{m} \rho_{Q_j}\), where \(\rho_{Q_j}\) is a density operator on \(H_{Q_j}\). If a state is not separable, then it is called an entangled state. Some of the generic entangled states are called Bell states and EPR states.

II. GENERAL DEFINITION OF POVM ON QUANTUM PHASE

In this section we will define a general POVM on quantum phase. This POVM is a set of linear operators \(\Delta(\varphi_1,2, \ldots, \varphi_1,N, \varphi_2,3, \ldots, \varphi_{N-1},N)\) furnishing the probabilities that the measurement of a state \(\rho\) on the Hilbert space \(H\) is given by

\[
p(\varphi_1,2, \ldots, \varphi_1,N, \varphi_2,3, \ldots, \varphi_{N-1},N) = \text{Tr}(\rho \Delta(\varphi_1,2, \ldots, \varphi_1,N, \varphi_2,3, \ldots, \varphi_{N-1},N)),
\]

where \((\varphi_1,2, \ldots, \varphi_1,N, \varphi_2,3, \ldots, \varphi_{N-1},N)\) are the outcomes of the measurement of the quantum phase, which is discrete and binary. This POVM satisfies the following properties, \(\Delta(\varphi_1,2, \ldots, \varphi_1,N, \varphi_2,3, \ldots, \varphi_{N-1},N)\) is self-adjoint, is positive, and is normalized, i.e.,

\[
\begin{align*}
\int_{2\pi} \cdots \int_{2\pi} d\varphi_1,2 \cdots d\varphi_1,N d\varphi_2,3 \\
\cdots d\varphi_{N-1},N \Delta(\varphi_1,2, \ldots, \varphi_{N-1},N) = 1,
\end{align*}
\]

where the integral extends over any \(2\pi\) intervals of the form \((\varphi_k - \varphi_k + 2\pi)\) and \(\varphi_k\) are the reference phases for all \(k = 1, 2, \ldots, N\). A general and symmetric POVM in a single \(N_j\)-dimensional Hilbert space \(H_{Q_j}\) is given by

\[
\Delta(\varphi_1,2, \ldots, \varphi_1,N_j, \varphi_2,3, \ldots, \varphi_{N_j-1},N_j) = \sum_{i_j \leq k_j=1}^{N_i} e^{i \varphi_k - i \varphi_j} |k_j\rangle \langle l_j|,
\]

where \(|k_j\rangle\) and \(|l_j\rangle\) are the basis vectors in \(H_{Q_j}\) and quantum phases satisfies the following relation \(\varphi_k - \varphi_j = (1 - \delta_{k_j,l_j}) \varphi_j\). The POVM is a function of the \(N_j(N_j - 1)/2\) phases \((\varphi_1,2, \ldots, \varphi_1,N_j, \varphi_2,3, \ldots, \varphi_{N_j-1},N_j)\). It is now possible to form a POVM of a multipartite system by simply forming the tensor product

\[
\Delta_Q(\varphi_1,1, \ldots, \varphi_1,m, \varphi_2,1, \ldots, \varphi_2,m, \ldots, \varphi_{N_m},1, \ldots, \varphi_{N_m},m) = \bigotimes_{j=1}^{m} \Delta_{Q_j}(\varphi_1,j,1, \ldots, \varphi_1,j,m) \\
\bigotimes_{j=1}^{m} \Delta_{Q_j}(\varphi_2,j,1, \ldots, \varphi_2,j,m) \\
\vdots \\
\bigotimes_{j=1}^{m} \Delta_{Q_j}(\varphi_{N_j},j,1, \ldots, \varphi_{N_j},j,m),
\]

where, e.g., \(\varphi_1,j,1, \ldots, \varphi_1,j,m\) is the set of POVMs relative phase associated with subsystems \(Q_j\), for all \(k_j, l_j = 1, 2, \ldots, N_j\), where we need only to consider when \(l_j > k_j\). This POVM will play a central role in constructing concurrence classes for multi-qubit states.

III. ENTANGLEMENT OF FORMATION AND CONCURRENCE

In this section we will review entanglement of formation and concurrence for a pair of qubits and a general bipartite state. For a mixed quantum system \(Q_2(N_1,N_2)\) the entanglement of formation is defined by

\[
\mathcal{E}_F(Q_2(N_1,N_2)) = \inf_n \sum_{n} p_n \mathcal{E}_F(\rho_{Q(n)}),
\]

where \(0 \leq p_n \leq 1\) is a probability distribution and the infimum is taken over all pure state decomposition of \(\rho_{Q_2}\). The entanglement of formation for a mixed quantum system \(Q_2(2,2)\) \[2\] can be written in term of the Shannon entropy and concurrence as follows

\[
\mathcal{E}_F(Q_2(2,2)) = H \left( \frac{1}{2} \left( 1 + (1 - C(Q_2(2,2)))^2 \right) \right),
\]

where \(C(Q_2(2,2))\) is called concurrence and is defined by

\[
C(Q_2(2,2)) = \max(0, \lambda_1 - \sum_{n>1} \lambda_n),
\]
where, $\lambda_n$, $n = 1, \ldots, 4$ are square roots of the eigenvalues of $\rho_2Q_2$ in descending order, where $\rho_2Q_2$ is given by

$$
\rho_2Q_2 = (\sigma_2 \otimes \sigma_2) \rho_2^Q (\sigma_2 \otimes \sigma_2),
$$

and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli matrix. Moreover, the concurrence of a pure two-qubit, bipartite state is defined as $C(\Psi) = ||\Psi\rangle \langle \Psi||$, where the tilde represents the "spin-flip" operation $|\Psi\rangle = \sigma_2 \otimes \sigma_2 |\Psi^*\rangle$. In the following section we will use the concept of orthogonal complement of our POVM to detect and to define concurrence for an arbitrary two-qubit state and concurrence classes for arbitrary three-, four-, and multi-qubit states.

IV. CONCURRENCE FOR AN ARBITRARY TWO-QUBIT STATE

In this section we will construct concurrence for an arbitrary two-qubit state based on orthogonal complement of our POVM. For two-qubit state $Q_2(2,2)$ the POVM is explicitly given by

$$
\Delta_\varnothing (\varnothing_{1,2}, \varnothing_{2,1,2}) = \Delta_{Q_1}(\varnothing_{1,2}) \otimes \Delta_{Q_2}(\varnothing_{2,1,2})
$$

(8)

$$
= \begin{pmatrix} 1 & e^{i\varnothing_{1,2}} & e^{i\varnothing_{1,2}} & 1 \\
1 & e^{-i\varnothing_{1,2}} & e^{-i\varnothing_{1,2}} & 1 
\end{pmatrix}.
$$

In this POVM, the only terms that has information about joint properties of both subsystems are phase sum $e^{\pm (\varnothing_{1,2} + \varnothing_{2,1,2})}$ and phase difference $e^{\pm (\varnothing_{1,2} - \varnothing_{2,1,2})}$. Now, from this observation we can assume that the phase sum gives a negative contribution that is $-1$ and phase difference gives a positive contribution that is $+1$ to a measurement. Then, we can mathematically achieve this construction by defining an operator $\Delta_{Q_j}(\varnothing_{j,1,2}) = I_2 - \Delta_{Q_j}(\varnothing_{j,1,2})$, where $I_2$ is a 2-by-2 identity matrix, for each subsystem $j$. Indeed by construction this operator is orthogonal complement of our POVM. Then, we define an operator that detects entanglement as follows

$$
\Delta^{EPR}_{Q_1} = \Delta_{Q_1} (\tilde{\varnothing}^{Q_1}_{1,2}) \otimes \Delta_{Q_2} (\tilde{\varnothing}^{Q_2}_{2,1,2})
$$

(9)

$$
= \sigma_y \otimes \sigma_y,
$$

where by choosing $\varnothing^{Q_1}_{1,2}, \varnothing^{Q_2}_{2,1,2} = \frac{\pi}{2}$ for all $k_j < l_j$, $j = 1, 2$, we get an operator which coincides with Pauli spin-flip operator $\sigma_y$ for a single-qubit. Now, in analogy with Wootters's formula for concurrence of a quantum system $Q_2(2,2)$ with the density operator $\rho_2$, we can define $\rho_2^{EPR}$ as

$$
\rho_2^{EPR} = \Delta^{EPR}_{Q_1} \rho_2 \Delta^{EPR}_{Q_2}
$$

(10)

and the concurrence is given by $C_{\Theta}(Q_2(2,2)) = \max(0, \lambda_1^{EPR} - \sum_{n>1} \lambda_n^{EPR})$, where $\lambda_n^{EPR}$, $n = 1, \ldots, 4$ are square roots of the eigenvalues of $\rho_2^{EPR}$ in descending order and $\rho_2^Q$ is the complex conjugation of $\rho_2$. Now, we would like to extend this result to a three-qubit state.

V. CONCURRENCE FOR AN ARBITRARY THREE-QUBIT STATE

The procedure of defining concurrence for an arbitrary three-qubit state is more complicated than for a pair of qubits since in the three-qubit case we have to deal with two different classes of three partite state, namely $W^3$ and $GHZ^3$ classes. For $W^3$ class, we have three types of entanglement: entanglement between subsystems one and two $Q_1Q_2$, one and three $Q_1Q_3$, and two and three $Q_2Q_3$. So there should be three operators $\Delta^{W^3}_{Q_1,2}$, $\Delta^{W^3}_{Q_1,3}$ and $\Delta^{W^3}_{Q_2,3}$ corresponding to entanglement between these subsystems, e.g., we have

$$
\Delta^{W^3}_{Q_1,2} = \Delta_{Q_1} (\varnothing^{Q_1}_{1,2}) \otimes \Delta_{Q_2} (\varnothing^{Q_2}_{2,1}) \otimes \mathcal{I}_2,
$$

(11)

$$
\Delta^{W^3}_{Q_1,3} = \Delta_{Q_1} (\varnothing^{Q_1}_{1,2}) \otimes \mathcal{I}_2 \otimes \Delta_{Q_3} (\varnothing^{Q_3}_{3,2}),
$$

(12)

$$
\Delta^{W^3}_{Q_2,3} = \mathcal{I}_2 \otimes \Delta_{Q_2} (\varnothing^{Q_2}_{2,1}) \otimes \Delta_{Q_3} (\varnothing^{Q_3}_{3,2}).
$$

(13)

Now, for a pure quantum system $Q_3(2,2,2)$ we define concurrence of $W^3$ class by

$$
C(Q_3^{W^3}(2,2,2)) = \frac{1}{2} \sum_{1 \leq r < r_2} \langle \Theta | \Delta^{W^3}_{Q_3}(\varnothing^{Q_3}_{3,2}) | \Theta \rangle^{1/2},
$$

(14)

where $\lambda_3^{W^3}$ is a normalization constant and for a quantum system $Q_3(2,2,2)$ with the density operator $\rho_2$, let

$$
\rho_3^{W^3} = \Delta^{W^3}_{Q_1,2} \rho_2 \Delta^{W^3}_{Q_2,3}.
$$

(15)

Then concurrence of a three-qubit mixed state of $W^3$ class could be defined by

$$
C(Q_3^{W^3}(2,2,2)) = \max(0, \lambda_1^{W^3}(r_1, r_2) - \sum_{n=1}^{N_3} \lambda_n^{W^3}(r_1, r_2)),
$$

(16)

where $\lambda_n^{W^3}(r_1, r_2)$ for all $1 \leq r_1 < r_2 \leq 3$ are square roots of the eigenvalues of $\rho_3^{W^3}$ in descending order. The second class of three-qubit state that we would like to consider is $GHZ^3$ class. For $GHZ^3$ class we have again three types of entanglement that give contribution to degree of entanglement, but there is a difference in construction of operators compare to $W^3$ class. The operators $\Delta^{GHZ^3}_{Q_1,2}$, $\Delta^{GHZ^3}_{Q_1,3}$ and $\Delta^{GHZ^3}_{Q_2,3}$ that can detect entanglement between these subsystems, are given by

$$
\Delta^{GHZ^3}_{Q_1,2} = \Delta_{Q_1} (\varnothing^{Q_1}_{1,2}) \otimes \Delta_{Q_2} (\varnothing^{Q_2}_{2,1}) \otimes \Delta_{Q_3} (\varnothing^{Q_3}_{3,2}.
$$

(16)
\[ \Delta_{Q_{1,3}}^{GHZ^3} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{3}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{3}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{3}), \]  
(17)
\[ \Delta_{Q_{2,3}}^{GHZ^3} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{3}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{3}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{3}), \]  
(18)
where \( \varphi_{Q_{j}}^{3} = \pi \) for all \( j \).

Now, for a pure quantum system \( Q_{0}^{3}(2,2,2) \) we define concurrence of \( GHZ^3 \) class by
\[ C(Q_{0}^{GHZ^3}(2,2,2)) = \sqrt{\lambda^3_{GHZ^3}(r_{1}, r_{2}) - \sum_{n>1} \lambda^3_{GHZ^3}(r_{1}, r_{2})}, \]  
(19)
where \( \lambda^3_{GHZ^3}(r_{1}, r_{2}) \) for all \( 1 \leq r_{1} < r_{2} \leq 3 \) are square roots of the eigenvalues of \( \rho_{Q}^3\Delta_{Q_{1,2,3}}^{GHZ^3} \) in descending order.

For three-qubit state the operators \( \Delta_{Q_{1,2,3}}^{W^{3}} \) and \( \Delta_{Q_{1,2,3}}^{GHZ^3} \) satisfies \( (\Delta_{Q_{1,2,3}}^{W^{3}})^{2} = 1 \) and \( (\Delta_{Q_{1,2,3}}^{GHZ^3})^{2} = 1 \). Now, for a state \( |\Psi_{\text{Q}}\rangle = \alpha_{1,2,2}|1,2,2\rangle + \alpha_{2,1,2}|2,1,2\rangle + \alpha_{2,2,1}|2,2,1\rangle \), the \( W^{3} \) class concurrence gives
\[ C(Q_{0}^{W^{3}}(2,2,2)) = (4\lambda^3_{W^{3}}(1,2,2)\alpha^{2}_{1,2,2}\alpha^{2}_{2,1,2} + \alpha^{2}_{1,2,2}\alpha^{2}_{2,2,1})^{1/2}. \]  
When \( \alpha_{1,2,2} = \alpha_{2,1,2} = \alpha_{2,2,1} = \frac{1}{\sqrt{3}} \), we get
\[ C(Q_{0}^{W^{3}}(2,2,2)) = (4\lambda^3_{W^{3}}(2,2,2))^{1/2} = C(Q_{0}^{GHZ^3}(2,2,2)) = 0. \]

Thus, for \( N_{W}^{4} = \frac{1}{2} \) we have \( C(Q_{0}^{W^{3}}(2,2,2)) = 1 \).

Moreover, let \( |\Psi_{\text{Q}}^{GHZ^3}\rangle = \alpha_{1,1,1}|1,1,1\rangle + \alpha_{2,2,2}|2,2,2\rangle \) and
\[ \rho_{Q}^{GHZ^3} = q|\Psi_{\text{Q}}^{W^{3}}\rangle\langle\Psi_{\text{Q}}^{W^{3}}| + (1-q)|\Psi_{\text{Q}}^{GHZ^3}\rangle\langle\Psi_{\text{Q}}^{GHZ^3}|. \]  
Then the \( GHZ^3 \) class concurrence gives
\[ C(Q_{0}^{GHZ^3}(2,2,2)) = \max(0, \lambda^3_{GHZ^3}(r_{1}, r_{2}) - \sum_{n>1} \lambda^3_{GHZ^3}(r_{1}, r_{2})) = \max(0, 2q - 1), \]  
where \( \lambda^3_{GHZ^3}(1,2) = q, \lambda^3_{GHZ^3}(2,1) = 1 - q, \) and \( 0 < q \leq 1 \).

As we have seen there are \( W^{3} \) and \( GHZ^3 \) classes concurrences for three-qubit state. However, we are not sure how we should deal with these two different classes, but there are at least two possibilities: the first possibility is to deal with them separately, and the second one is to define an overall expression for concurrence of three-qubit state by adding these two concurrences.

**VI. CONCURRENCE CLASSES FOR AN ARBITRARY FOUR-QUTIT-STATE**

In this section we will construct three different concurrences for four-qubit systems based on quantum phases of our POVM, namely the \( W^{4}, GHZ^{4} \), and \( GHZ^{3} \) class concurrences. Let us begin by constructing operators for \( W^{4} \) class of four-qubit states. For \( W^{4} \) class we have six different types of entanglement: entanglement between subsystem one and two \( Q_{1}Q_{2} \) one and three \( Q_{1}Q_{3} \), and two and three \( Q_{2}Q_{3} \), etc.. So, there are six operators \( \Delta_{Q_{1,2}}^{W^{4}}, \Delta_{Q_{1,4}}^{W^{4}}, \Delta_{Q_{2,3}}^{W^{4}}, \Delta_{Q_{2,4}}^{W^{4}}, \Delta_{Q_{3,4}}^{W^{4}}, \Delta_{Q_{4,1}}^{W^{4}} \) corresponding to entanglement between these subsystems, i.e., we have
\[ \Delta_{Q_{1,2}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \]  
(21)
\[ \Delta_{Q_{1,3}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \]  
(22)
\[ \Delta_{Q_{1,4}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{4}) \otimes \Delta_{Q_{4}}(\varphi_{Q_{4}}^{4}) \]  
(23)
\[ \Delta_{Q_{1,4}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{4}) \otimes \Delta_{Q_{4}}(\varphi_{Q_{4}}^{4}) \]  
(24)
\[ \Delta_{Q_{1,4}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{4}) \otimes \Delta_{Q_{4}}(\varphi_{Q_{4}}^{4}) \]  
(25)
\[ \Delta_{Q_{1,4}}^{W^{4}} = \Delta_{Q_{1}}(\varphi_{Q_{1}}^{4}) \otimes \Delta_{Q_{2}}(\varphi_{Q_{2}}^{4}) \otimes \Delta_{Q_{3}}(\varphi_{Q_{3}}^{4}) \otimes \Delta_{Q_{4}}(\varphi_{Q_{4}}^{4}) \]  
(26)
Now, for a pure quantum system \( Q_{0}^{4}(2,\ldots,2) \) we define concurrence of \( W^{4} \) class by
\[ C(Q_{0}^{W^{4}}(2,\ldots,2)) = \left( \lambda_{W}^{4} \sum_{1=1,r_{2}}^{4} |\langle \Psi_{\text{Q}}^{W^{4}}(r_{1}, r_{2}) \rangle|^{2} \right)^{1/2}. \]  
(27)
where \( \lambda_{W}^{4} \) is a normalization constant. Now, for a quantum system \( Q_{0}^{W^{4}}(2,\ldots,2) \) let \( \rho_{Q}^{W^{4}} = \Delta_{Q_{1,2}}^{W^{4}} \rho_{Q}^{3} \Delta_{Q_{1,2}}^{W^{4}} \).

Then concurrence of four-qubit mixed state of \( W^{4} \) class can be defined by
\[ C(Q_{0}^{W^{4}}(2,\ldots,2)) = \max(0, \lambda_{W}^{4}(r_{1}, r_{2})) - \sum_{n>1} \lambda_{W}^{4}(r_{1}, r_{2}), \]  
(28)
where \( \lambda_{W}^{4}(r_{1}, r_{2}) \) for all \( 1 \leq r_{1} < r_{2} \leq 4 \) are square roots of the eigenvalues of \( \rho_{Q}^{3}\rho_{Q}^{W^{4}} \) in descending order.

The operators \( \Delta_{Q_{1,2}}^{W^{4}} \) for \( W^{4} \) class satisfies \( (\Delta_{Q_{1,2}}^{W^{4}})^{2} = 1 \). Now, for a state \( |\Psi_{\text{Q}}^{W^{4}}\rangle = \alpha_{1,1,1,1}|1,1,1,1\rangle + \alpha_{1,1,1,2}|1,1,1,2\rangle + \alpha_{1,1,2,1}|1,2,1,1\rangle + \alpha_{2,1,1,1}|2,1,1,1\rangle \), the \( W^{4} \) class concurrence gives
\[ C(Q_{0}^{W^{4}}(2,\ldots,2)) = (4\lambda_{W}^{4}|\alpha_{1,1,2,1}|^{2} + |\alpha_{1,1,1,2}|^{2} + |\alpha_{1,1,2,1}|^{2} + |\alpha_{1,2,1,1}|^{2})^{1/2}. \]  
(29)
and for \( \alpha_{1,1,1,2} = \alpha_{1,2,1,1} = \alpha_{1,2,1,1} = \frac{1}{\sqrt{4}} \), we get \( C(Q_4^A(2, \ldots, 2)) = \left( \frac{3}{2} N_4^A \right)^{1/2} \), \( C(Q_4^{GHZ}(2, \ldots, 2)) = 0 \). The second class of four-qubit state that we would like to consider is \( GHZ^4 \) class. For \( GHZ^4 \), we have again six different types of entanglement and there are six operators defined as follows

\[
\Delta_{Q_{1,2}}^{GHZ} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes \Delta_{Q_2}(\varphi_{Q_2}^x), \tag{29}
\]

\[
\Delta_{Q_{1,3}}^{GHZ} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes \Delta_{Q_3}(\varphi_{Q_3}^x), \tag{30}
\]

\[
\Delta_{Q_{1,4}}^{GHZ} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes \Delta_{Q_4}(\varphi_{Q_4}^x), \tag{31}
\]

\[
\Delta_{Q_{2,3}}^{GHZ} = \Delta_{Q_2}(\varphi_{Q_2}^x) \otimes \Delta_{Q_3}(\varphi_{Q_3}^x), \tag{32}
\]

\[
\Delta_{Q_{2,4}}^{GHZ} = \Delta_{Q_2}(\varphi_{Q_2}^x) \otimes \Delta_{Q_4}(\varphi_{Q_4}^x), \tag{33}
\]

\[
\Delta_{Q_{3,4}}^{GHZ} = \Delta_{Q_3}(\varphi_{Q_3}^x) \otimes \Delta_{Q_4}(\varphi_{Q_4}^x). \tag{34}
\]

Now, for a pure four-qubit state \( Q_0^A(2, \ldots, 2) \) we define concurrence of \( GHZ^4 \) class by

\[
C(Q_4^{GHZ}(2, \ldots, 2)) = \left( \frac{1}{4} \sum_{1 \leq r_1 < r_2 \leq 4} \left| \langle \Psi \mid \Delta_{Q_{r_1,r_2}}^{GHZ} \Psi^* \rangle \right|^2 \right)^{1/2}, \tag{35}
\]

where \( N_4^{GHZ} \) is a normalization constant and for a quantum system \( Q_2^{GHZ}(2, \ldots, 2) \) with \( \rho^{GHZ} = \Delta_{Q_{r_1,r_2}}^{GHZ} \otimes I_{Q_{r_1,r_2}}^{GHZ} \), we define concurrence of four-qubit mixed state of \( GHZ^4 \) class by

\[
C(Q_4^{GHZ}(2, \ldots, 2)) = \max(0, \lambda_1^{GHZ^4}(r_1, r_2)) - \sum_{n>1} \lambda_n^{GHZ^4}(r_1, r_2)), \tag{36}
\]

where \( \lambda_n^{GHZ^4}(r_1, r_2) \) for all \( 1 \leq r_1 < r_2 \leq 4 \) are square roots of the eigenvalues of \( \rho_Q \rho^{GHZ^4} \) in descending order. And again we have \( (\Delta_{Q_{r_1,r_2}}^{GHZ^4})^2 = 1 \). Thus, we have detected and defined three different concurrences for four-qubit state based on our POVM construction.

### VII. CONCURRENCE CLASSES FOR AN ARBITRARY MULTI-QUBIT STATE

At this point, we can realize that, in principle, we could in a straightforward manner extend our construction into a multi-qubit state \( Q_m(2, \ldots, 2) \). In order to simplify our presentation, we will use \( k_m = l_1, l_2, \ldots; m, l_m \) as an abstract multi-index notation, where \( k_j = 1, l_j = 2 \) for all \( j \). The unique structure of our POVM enables us to distinguish different classes of multipartite states, which are inequivalent under LOCC operations. In the \( m \)-partite case, the off-diagonal elements of the matrix corresponding to

\[
\Delta_{Q_{r_1},r_2}(\varphi_{Q_1},k_1,l_1, \ldots, \varphi_{Q_m},k_m,l_m) = \Delta_{Q_1}(\varphi_{Q_1,k_1,l_1}) \otimes \cdots \otimes \Delta_{Q_m}(\varphi_{Q_m,k_m,l_m}), \tag{44}
\]

For \( GHZ^3 \), we have four different types of entanglement. So there are four operators defined as below

\[
\Delta_{Q_{1,2,3}}^{GHZ^3} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes \Delta_{Q_2}(\varphi_{Q_2}^x) \otimes \Delta_{Q_3}(\varphi_{Q_3}^x) \otimes I_{Q_4}^{GHZ^3}, \tag{37}
\]

\[
\Delta_{Q_{1,2,4}}^{GHZ^3} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes \Delta_{Q_2}(\varphi_{Q_2}^x) \otimes I_{Q_3}^{GHZ^3} \otimes \Delta_{Q_4}(\varphi_{Q_4}^x), \tag{38}
\]

\[
\Delta_{Q_{1,3,4}}^{GHZ^3} = \Delta_{Q_1}(\varphi_{Q_1}^x) \otimes I_{Q_2}^{GHZ^3} \otimes \Delta_{Q_3}(\varphi_{Q_3}^x) \otimes \Delta_{Q_4}(\varphi_{Q_4}^x), \tag{39}
\]

\[
\Delta_{Q_{2,3,4}}^{GHZ^3} = I_{Q_1}^{GHZ^3} \otimes \Delta_{Q_2}(\varphi_{Q_2}^x) \otimes \Delta_{Q_3}(\varphi_{Q_3}^x) \otimes \Delta_{Q_4}(\varphi_{Q_4}^x). \tag{40}
\]

Then, for a pure four-qubit state \( Q_0^A(2, \ldots, 2) \) we define concurrence for a \( GHZ^3 \) class by

\[
C(Q_4^{GHZ^3}(2, \ldots, 2)) = \left( \frac{1}{4} \sum_{1 \leq r_1 < r_2 < r_3} \left| \langle \Psi \mid \Delta_{Q_{r_1,r_2,r_3}}^{GHZ^3} \Psi^* \rangle \right|^2 \right)^{1/2},
\]

where \( N_3^{GHZ^3} \) is a normalization constant and for a quantum system \( Q_2(2, \ldots, 2) \) with density operator \( \rho_Q \), let

\[
\rho_Q^{GHZ^3} = \Delta_{Q_{r_1,r_2,r_3}}^{GHZ^3} \otimes I_{Q_{r_1,r_2,r_3}}^{GHZ^3}. \tag{42}
\]

Then concurrence for a four-qubit \( GHZ^3 \) class is defined by

\[
C(Q_4^{GHZ^3}(2, \ldots, 2)) = \max(0, \lambda_1^{GHZ^3}(r_1, r_2, r_3)) - \sum_{n>1} \lambda_n^{GHZ^3}(r_1, r_2, r_3), \tag{43}
\]

where \( \lambda_n^{GHZ^3}(r_1, r_2, r_3) \) for all \( 1 \leq r_1 < r_2 < r_3 \leq 4 \) are square roots of the eigenvalues of \( \rho_Q \rho_Q^{GHZ^3} \) in descending order. And again we have \( (\Delta_{Q_{r_1,r_2,r_3}}^{GHZ^3})^2 = 1 \). Thus, we have defined and defined three different concurrences for four-qubit state based on our POVM construction.
have phases that are sum or differences of phases originating from two and \(m\) subsystems. That is, in the later case the phases of \(\Delta Q_k(\varphi_{Q_1};k_1,l_1,\ldots,\varphi_{Q_m};k_m,l_m)\) take the form \((\varphi_{Q_1};k_1,l_1 \pm \varphi_{Q_2};k_2,l_2 \pm \ldots \pm \varphi_{Q_m};k_m,l_m)\) and identification of these joint phases makes our classification possible. Thus, we can define linear operators for the \(EPR\) class based on our POVM which are sum and difference of phases of two subsystems, i.e., \((\varphi_{Q_1};k_1,l_1,\ldots,\varphi_{Q_m};k_m,l_m)\).

That is, for the \(EPR\) class we have

\[
\bar{\Delta}_{EPR_{r_1,r_2}}^{Q_{r_1,r_2}(2,1)} = I_{2_1} \otimes \ldots \otimes \bar{\Delta}_{Q_{r_1}}(\varphi_{Q_{r_1}};k_1,l_1) \\
\ldots \otimes \bar{\Delta}_{Q_{r_2}}(\varphi_{Q_{r_2}};k_2,l_2) \\
\otimes \cdots \otimes I_{2_m}.
\]  

(45)

Let \(C(m,k) = \binom{m}{k}\) denotes the binomial coefficient. Then there is \(C(m,2)\) linear operators for the \(EPR\) class and the set of these operators gives the \(W^{m}\) class concurrence.

For the \(GHZ^{m}\) class, we define the linear operators based on our POVM which are sum and difference of phases of \(m\)-subsystems, i.e., \((\varphi_{Q_{r_1}};k_1,l_1,\varphi_{Q_{r_2}};k_2,l_2,\ldots,\varphi_{Q_m};k_m,l_m)\). That is, for the \(GHZ^{m}\) class we have

\[
\bar{\Delta}_{GHZ_{r_1,r_2}}^{Q_{r_1,r_2}(2,1)} = \bar{\Delta}_{Q_{r_1}}(\varphi_{Q_{r_1}};k_1,l_1) \otimes \bar{\Delta}_{Q_{r_2}}(\varphi_{Q_{r_2}};k_2,l_2) \\
\bar{\Delta}_{Q_{r_3}}(\varphi_{Q_{r_3}};k_3,l_3) \otimes \cdots \otimes \\
\bar{\Delta}_{Q_{m-1}}(\varphi_{Q_{m-1}};k_{m-1},l_{m-1}) \otimes I_{2_m}.
\]  

(46)

where by choosing \(\varphi_{Q_{r_j}};k_j,l_j = \pi\) for all \(k_j < l_j, j = 1,2,\ldots,m\), we get an operator which has the structure of Pauli operator \(\sigma_x\) embedded in a higher-dimensional Hilbert space and coincides with \(\sigma_x\) for a single-qubit. There are \(C(m,2)\) linear operators for the \(GHZ^{m}\) class and the set of these operators gives the \(GHZ^{m}\) class concurrence.

Moreover, we define the linear operators for the \(GHZ^{m-1}\) class of \(m\)-partite states based on our POVM which are sum and difference of phases of \(m-1\)-subsystems, i.e., \((\varphi_{Q_{r_1}};k_1,l_1,\varphi_{Q_{r_2}};k_2,l_2,\ldots,\varphi_{Q_{m-1}};k_{m-1},l_{m-1})\) \(\pm \varphi_{Q_{m-1}};k_{m-1},l_{m-1})\). That is, for the \(GHZ^{m-1}\) class we have

\[
\bar{\Delta}_{GHZ_{r_1,r_2}}^{Q_{r_1,r_2}(2,1)} = \bar{\Delta}_{Q_{r_1}}(\varphi_{Q_{r_1}};k_1,l_1) \otimes \\
\bar{\Delta}_{Q_{r_2}}(\varphi_{Q_{r_2}};k_2,l_2) \otimes \\
\bar{\Delta}_{Q_{r_3}}(\varphi_{Q_{r_3}};k_3,l_3) \otimes \cdots \otimes \\
\bar{\Delta}_{Q_{m-1}}(\varphi_{Q_{m-1}};k_{m-1},l_{m-1}) \otimes I_{2_m}.
\]  

(47)

where \(1 \leq r_1 < r_2 < \cdots < r_{m-1} < m\). There is \(C(m,m-1)\) such operators for the \(GHZ^{m-1}\) class. Now, for pure quantum system \(Q_{r_1}^{2,1}\) we define the \(EPR\) class concurrence as

\[
\mathcal{C}(Q_m^{EPR_{r_1,r_2}(2,1)}) = (4\mathcal{N}_m^{EPR_{r_1,r_2}})^{1/2},
\]

and the \(W^{m}\) class concurrence as

\[
\mathcal{C}(Q_m^{W_{r_1,r_2}(2,1)}) = \left(\sum_{r_2 \geq r_1=1}^{m} C^2(Q_m^{EPR_{r_1,r_2}(2,1)})\right)^{1/2},
\]

(48)

(49)

where \(\mathcal{N}_m^{EPR_{r_1,r_2}}\) are normalization constants. Moreover, the \(GHZ^{m}\) class concurrence for general pure quantum system \(Q_m^{2,1}\) with

\[
\mathcal{C}(Q_m^{GHZ^{m}_{r_1,r_2}(2,1)}) = \sum_{r_2 \geq r_1=1}^{m} \left|\langle \Psi | \tilde{\Delta}_{Q_{r_1,r_2}}^{GHZ_{r_1,r_2}(2,1)} \Psi^\dagger \rangle^2\right|,
\]

(50)

where \(\mathcal{N}_m^{GHZ}\) is a normalization constant. Now, let us address the monotonicity of these concurrence classes of multipartite states. For \(m\)-qubit states, the \(W^{m}\) class concurrences are entanglement monotones. Let \(A_j \in SL(2,C)\), for \(j = 1,2,\ldots,m\), and \(\mathcal{A} = A_1 \otimes A_2 \otimes \cdots \otimes A_m\), then \(\Delta_{Q_{r_1,r_2}}^{W_{r_1,r_2}(2,1)} A^T = \Delta_{Q_{r_1,r_2}}^{W_{r_1,r_2}}(2,1)\) for all \(1 \leq r_1 < r_2 < m\). Thus, the \(W^{m}\) class concurrences for multi-qubit states are invariant under SLOCC, and hence are entanglement monotones. Again, for general multipartite states we cannot give any proof on invariance of \(W^{m}\) class concurrence under SLOCC and this question needs further investigation. Moreover, for multi-partite states, the \(GHZ^{m}\) class concurrences are not entanglement monotone except under additional conditions. Since \(\Delta_{Q_{r_1,r_2}}^{GHZ_{r_1,r_2}(2,1)} \neq \Delta_{Q_{r_1,r_2}}^{GHZ_{r_1,r_2}(2,1)}\) for all \(1 \leq r_1 < r_2 < m\). The reason is that \(A_j \Delta_{Q_1} (\varphi_{Q_{1,1,2}};A_j) \neq \Delta_{Q_1} (\varphi_{Q_{1,1,2}};A_j)\). Thus, the \(GHZ^{m}\) class concurrence for three-qubit states are not invariant under SLOCC, and hence are not entanglement monotones. However, by construction the \(GHZ^{m}\) class concurrences are invariant under all permutations. Moreover, we have \((\Delta_{Q_{r_1,r_2}}^{GHZ_{r_1,r_2}(2,1,2)})^2 = 1\) and \((\Delta_{Q_1}(\varphi_{Q_{1,1,2}};1)^2 = 1\). Furthermore, we need to be very careful when we are using the \(GHZ^{m}\) class concurrences. This class can be zero even for an entangled multipartite state. Since we have more than two joint phases in our POVM for \(GHZ^{m}\) class concurrence. Thus, for the \(GHZ^{m}\) class concurrences we
need to perform an optimization over local unitary operations. For example, let $U = U_1 \otimes U_2 \otimes \cdots \otimes U_m$, where $U_j \in U(2, C)$. Then we maximize the $GHZ^m$ class concurrences for a given pure $m$-partite state over all local unitary operations $U$.

Finally, e.g., for the $W^m$ class for a general quantum system $Q_m(2, \ldots, 2)$ with density operator $\rho_Q$, we define

$$\rho_{Q_{\Delta}}^{W^m} = E_{Q_{\rho_{\Delta}}^{W^m}} U_{Q_{\rho_{\Delta}}^{W^m}}$$

and then the $W^m$ class concurrence is defined by

$$C(Q_{\Delta})^{W^m}(2, \ldots, 2) = \max(0, \lambda^W_{1, \rho}(r_1, r_2) - \sum_{n > 1} \lambda^W_{n, \rho}(r_1, r_2)),$$

where $\lambda^W_{n, \rho}(r_1, r_2)$ for all $1 \leq r_1 < r_2 \leq m$ are the square roots of the eigenvalues of $\rho_Q^{W^m}$ in descending order. The $GHZ^m$ class concurrences for a quantum system $Q_m(N2, \ldots, 2)$ can be defined in similar way. The definition of concurrence classes for multi-partite mixed states is only a well motivated suggestion and is a generalization of Wootters and Uhlmann definitions. Moreover, our operators $\Delta^W_{\rho_{\Delta}}$ satisfies $\Delta^W_{\rho_{\Delta}} = 1$. As an example of multi-qubit state let us consider a state $|W^m\rangle = \frac{1}{\sqrt{m}}(|1, 1, \ldots, 1, 2 \rangle + \ldots + |2, 1, \ldots, 1, 1\rangle)$. For this state the $W^m$ class concurrence is

$$C(Q_{\Delta})^{W^m}(2, \ldots, 2) = \frac{2(m - 1)}{m^2} \Lambda^W_{m}$$

This value coincides with the one given by Dür [20]. Finally, for some partially separable states the $C(Q_{\Delta})^{W^m}(2, \ldots, 2)$ class and $C(Q_{\Delta})^{GHZ^m}(2, \ldots, 2)$ class concurrences do not exactly quantify entanglement in general. Example of such states can be e.g., constructed for three-qubit states. Thus, we may need to define an overall concurrence by adding these concurrence classes.

VIII. CONCLUSION

In this paper we have expressed concurrence for an arbitrary two-qubit state, based on our POVM, which coincides with Wootters original formula. Moreover, we have generalized this result into arbitrary three- and four-qubit states. For three-qubit states, we have found two different concurrence classes and for four-qubit states, we have constructed three concurrence classes. And finally, we have generalized our result into arbitrary multi-qubit state and we have explicitly constructed $W^m$ and $GHZ^m$ class concurrences. We have investigate the monotonicity of the $W^m$ class and the $GHZ^m$ class concurrences for multi-qubit states. The $W^m$ class concurrence for multi-qubit states are entanglement monotones. However, $GHZ^m$ class concurrences need optimization over all local unitary operation. Our construction suggested the existence of different classes of multipartite entanglement which are in equivalent under LOCC. At least, we known that there is two different classes of entanglement for multi-qubit states which our methods could distinguish very well. But we can also define an overall expression for concurrence with a suitable normalization coefficient. However, we think that this work is a timely contribution to the relatively large effort presently being undertaken to quantify and classify multipartite entanglement.

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