Abstract

Let $G$ be a finite group of odd order, $F$ a finite field of odd characteristic $p$ and $B$ a finite–dimensional symplectic $FG$-module. We show that $B$ is $FG$-hyperbolic, i.e., it contains a self–perpendicular $FG$-submodule, if it is $FN$-hyperbolic for every cyclic subgroup $N$ of $G$.

1 Introduction

Let $F$ be a finite field of odd characteristic $p$, $G$ a finite group and $B$ a finite–dimensional $FG$-module. If $B$ carries a non-singular alternating bilinear form $<\cdot, \cdot>$ (i.e., a symplectic form) that is invariant by $G$, then we call $B$ a symplectic $FG$-module. Following the notation in [3], for any $FG$-submodule $S$ of $B$, we write $S^\perp$ for the perpendicular subspace of $S$, i.e., $S^\perp := \{ t \in B | <S,t> = 0 \}$. We say that $S$ is isotropic if $S \leq S^\perp$, and $B$ is anisotropic if it contains no non–trivial isotropic $FG$-submodules. Furthermore, we say that $B$ is hyperbolic if it contains some self–perpendicular $FG$-submodule $S$, i.e., $S$ is an $FG$-submodule satisfying $S = S^\perp$.

Symplectic modules play an essential role in studying monomial characters. (An irreducible character $\chi$ of a finite group $G$ is monomial if it is induced from a linear character of a subgroup of $G$.) One of the most representative links between symplectic modules and monomial characters can be found in [3]. (For other examples one could look at [2, 7, 8, 9, 10, 11, 12, and 13].) There E. C. Dade proved the following theorem (Theorem 3.2 in [3]):

**Theorem 1.1 (Dade).** Suppose that $F$ is a finite field of odd characteristic $p$, that $G$ is a finite $p$-solvable group, that $H$ is a subgroup of $p$-power index in $G$, and that $B$ is a symplectic $FG$-module whose restriction $B_H$ to a symplectic $FH$-module is hyperbolic. Then $B$ is hyperbolic.

Using the above theorem, E. C. Dade was able to prove (Theorem 0 in [3]) that, given a $p$-solvable odd group $G$, an irreducible monomial character $\chi$ of $G$, and a subnormal subgroup $N$ of $G$, every irreducible constituent of the restricted character $\chi_N$ is monomial, provided that $\chi(1)$ is a power of $p$.

In this paper we prove

**Theorem A.** Suppose that $F$ is a finite field of odd characteristic $p$, that $G$ is a finite group of odd order, and that $B$ is a symplectic $FG$-module whose restriction $B_N$ to a symplectic $FN$-module is hyperbolic for every cyclic subgroup $N$ of $G$. Then $B$ is hyperbolic.

All groups considered here are of finite order, and all modules have finite dimension over $F$.

**Acknowledgments** I am indebted to Professor E. C. Dade for many helpful ideas and suggestions. Also, I would like to thank Professor M. Isaacs for useful conversations that helped me improve this paper.
2 Symplectic modules

We first give some elementary results about symplectic modules.

Assume that \( B \) is a symplectic \( FG \)-module, while \( S \) is an isotropic \( FG \)-submodule of \( B \). Then the factor \( FG \)-module \( \overline{S} = S^\perp/S \) is again a symplectic \( FG \)-module with the symplectic form defined as (see 1.4 in [3]),

\[
< s_1 + S, s_2 + S > = < s_1, s_2 >, \text{ for all } s_1, s_2 \in S^\perp.
\]

Furthermore, if \( S \) is an isotropic \( FG \)-submodule of \( B \), then its \( F \)-dimension \( \dim_F S \) is at most \((1/2) \dim_F B\) (see 19.3 in [1]).

We say that an isotropic \( FG \)-submodule \( S \) of \( B \) is maximal isotropic if it is not properly contained in any larger isotropic \( FG \)-submodule of \( B \). Clearly any self–perpendicular \( FG \)-submodule \( S \) of \( B \) is maximal isotropic. The converse is also correct under the extra assumption that \( B \) is \( G \)-hyperbolic (see Lemma 3.1 in [3]). Another way to get a self–perpendicular module from a maximal isotropic one is to control its dimension, as the following lemma shows.

**Lemma 2.1.** Assume that \( B \) is a symplectic \( FG \)-module, and that \( S \) is a maximal isotropic \( FG \)-submodule of \( B \). If \( \dim_F S = (1/2) \dim_F B \) then \( S \) is self–perpendicular and \( B \) is \( G \)-hyperbolic.

**Proof.** Let \( \hat{S} \) denote the dual of \( S \). Then \( B/S^\perp \cong \hat{S} \). But \( \dim_F \hat{S} = \dim_F S = (1/2) \dim_F B \). Hence \( \dim_F S^\perp = (1/2) \dim_F B \). Since \( S \leq S^\perp \) we conclude that \( S = S^\perp \). Thus the lemma holds. \( \square \)

The following is Proposition 2.1 in [3].

**Proposition 2.2.** Let \( G \) be a finite group and \( B \) be an anisotropic symplectic \( FG \)-module. Then \( B \) is an orthogonal direct sum:

\[
B = U_1 \perp U_2 \perp \ldots \perp U_k,
\]

where \( k \geq 0 \) and each \( U_i \) is a simple \( FG \)-submodule of \( B \) that is also symplectic.

**Remark 1.** If \( G \) has odd order then according to Proposition (1.10) and Corollary 2.10 in [3] all the \( U_i \) that appear in (2) are distinct.

**Lemma 2.3.** Let \( \mathcal{U} \) be an \( FG \)-module that affords a symplectic \( G \)-invariant form \( < \cdot, \cdot > \). Then \( \mathcal{U} \) is self–dual.

**Proof.** We write \( \hat{\mathcal{U}} \) for the dual \( FG \)-module of \( \mathcal{U} \). For every \( x \in \mathcal{U} \) the map \( \alpha_x : \mathcal{U} \to \mathcal{F} \) defined as:

\[
\alpha_x(u) = < u, x > \text{ for all } u \in \mathcal{U}
\]

is an element of \( \text{Hom}_F(\mathcal{U}, \mathcal{F}) \cong \hat{\mathcal{U}} \). Since \( < \cdot, \cdot > \) is \( G \)-invariant the map \( \alpha : x \to \alpha_x \) is an \( FG \)-homomorphism from \( \mathcal{U} \) to \( \hat{\mathcal{U}} \). Furthermore the kernel of \( \alpha \) is trivial, as \( \mathcal{U} \) is symplectic. Hence \( \mathcal{U} \cong \hat{\mathcal{U}} \). \( \square \)

**Corollary 2.4.** Let \( B \) be an anisotropic symplectic \( FG \)-module. Then each of the simple \( FG \)-modules \( U_i \) that appears in (2) is self–dual.

**Proof.** It follows easily from Proposition 2.2 and Lemma 2.3. \( \square \)

**Proposition 2.5.** Assume that \( \mathcal{U} \) is a simple symplectic \( FG \)-module. Let \( N \) be a normal subgroup of \( G \) such that \( |G : N| \) is odd. Then any simple \( FN \)-submodule of \( \mathcal{U} \) is self–dual. Hence any \( FN \)-submodule of \( \mathcal{U} \) is self–dual.
Proof. As $N$ is a normal subgroup of $G$, Clifford’s theorem implies that

$$U_N \cong e(V_1 \oplus \ldots \oplus V_n)$$

where $V = V_1$ is a simple $F^N$-submodule of $U$ and $V_1, \ldots, V_n$ are the distinct $G$-conjugates of $V$. So $n|G:N|$ and therefore $n$ is odd.

According to Lemma 2.3 the module $U$ is self-dual. Hence the dual, $\hat{V}_i$ of any $V_i$ should appear in (3). Therefore we can form pairs among the $V_i$, consisting of a simple $F^N$-module $V_k$ and its dual for $k \in \{1, \ldots, n\}$, where we take as the second part of the pair the module itself if it is self-dual. Since $G$ acts transitively on the $V_i$ for $i = 1, \ldots, n$, either all the $V_i$ are self-dual or none of them is. In the latter case we get that any of the above pairs consists of two distinct modules. This implies that $2|n$. As $n$ is odd, this case can never occur. Hence any one of the $V_i$ is self-dual and the proposition is proved.

Proposition 2.6. Assume that the symplectic $FG$-module $B$ is hyperbolic. Assume further that $B$ is a semi-simple $FG$-module. Then every self-dual simple $FG$-submodule of $B$ appears with even multiplicity in any decomposition of $B$ as a direct sum of simple $FG$-submodules.

Proof. Because $B$ is hyperbolic it contains a self-perpendicular $FG$-submodule $S$. For every $FG$-submodule $V$ of $B$ we have $B/V \cong \hat{V}$. So

$$B/S \cong \hat{S}$$

Now the proposition follows from (4) and the fact that $B$ is semi-simple.

Corollary 2.7. Let $B$ be an anisotropic symplectic $FG$-module. Let $N$ be a normal subgroup of $G$ such that $|G:N|$ is odd. Assume further that $B_N$ is a hyperbolic $FN$-module. Then any simple $FN$-submodule of $B_N$ appears with even multiplicity in any decomposition of $B_N$ as a direct sum of simple $FN$-submodules.

Proof. This is a straightforward application of Propositions 2.2, 2.5 and 2.6.

We close this section with a well known fact that we prove here for completeness.

Lemma 2.8. Assume that $U$ is a self-dual absolutely irreducible $FG$-module, where $G$ has odd order and $F$ is a finite field whose characteristic does not divide $|G|$. Then $U$ is trivial.

Proof. Let $\chi$ denote the $F$-absolutely irreducible character that $U$ affords, while $\phi$ denotes a Brauer character that $U$ affords. Then $\phi$ is defined for every element of $G$, since the characteristic of $F$ is coprime to $|G|$. Because $U$ is self-dual, the character $\phi$ is real valued. Let $\nu_2(\phi) = |G|^{-1} \sum_{g \in G} \phi(g^2)$ be the Frobenius–Schur indicator (see Chapter 4 in [4]) of $\phi$. Then Theorem 4.5 in [4] implies that $\nu_2(\phi) \neq 0$, since $\phi$ is real valued. But

$$\nu_2(\phi) = |G|^{-1} \sum_{g \in G} \phi(g^2) = |G|^{-1} \sum_{g \in G} \phi(g),$$

because $G$ has odd order. Hence $\nu_2(\phi)$ is the inner product $\nu_2(\phi) = [\phi, 1_G]$, where $1_G$ is the trivial character of $G$. We conclude that $[\phi, 1_G] \neq 0$. Hence $\phi = 1_G$. Therefore $\chi = 1_G$, and the lemma follows.

3
3 Proof of Theorem A

We can now prove our main result. The proof will follow from a series of lemmas, based on the hypothesis that $F, B, G$ form a minimal counter-example. All the groups considered in this section have odd order. We also fix the odd prime $p$ that is the characteristic of $F$, and we assume that

**Inductive Hypothesis.** $F, B, G$ have been chosen among all triplets satisfying the hypothesis but not the conclusion of Theorem A so as to minimize first the order $|G|$ of $G$ and then the $F$–dimension $\dim_F B$ of $B$.

**Remark 2.** For any proper subgroup $H$ of $G$ the minimality of $|G|$ easily implies that the restriction $B_H$ is a hyperbolic $FH$-module.

**Lemma 3.1.** $B$ is non–zero and anisotropic.

*Proof.* If $B$ were zero it would be hyperbolic contradicting the Inductive Hypothesis. So $B$ is non–zero. If $B$ is not anisotropic then it contains a non–zero isotropic $FG$–module $U$. Let $N$ be an arbitrary cyclic subgroup of $G$. Then the isotropic $FN$-submodule $U_N$ of $BN$ is contained in some maximal isotropic $FN$–submodule $V$ of $BN$. Since $BN$ is hyperbolic this maximal isotropic submodule is self–perpendicular, i.e., $V = V^\perp$. Hence

$$U \subseteq V = V^\perp \subseteq U^\perp.$$ 

Therefore the factor module $\bar{V} = V/U$ is a self–perpendicular $FN$–submodule of the symplectic $FG$–module $\bar{U} = U^\perp/U$. Hence $F, G, \bar{U}$ satisfy the hypothesis of the Main Theorem. As $\dim(\bar{U}) < \dim(B)$, the minimality of $\dim(B)$ implies that $\bar{U}$ is a hyperbolic $FG$–module. So there is a self–perpendicular $FG$–submodule $\bar{J}$ in $\bar{U}$. From the definition of the symplectic form on $\bar{U}$ (see (1)) it follows that the inverse image $\bar{J}$ of $\bar{J}$ in $U^\perp$ is a self–perpendicular $FG$–submodule of $B$ containing $U$. Therefore $B$ is hyperbolic, contradicting the Inductive Hypothesis. So the lemma holds. $\square$

**Lemma 3.2.** $p$ doesn’t divide the order $|G|$ of $G$.

*Proof.* Suppose that $p$ divides $|G|$. Because $G$ is solvable, it contains a Hall $p'$-subgroup $H$. If $G$ is a $p$-group we take $H = 1$. Since $p$ divides $|G|$, the subgroup $H$ is strictly smaller than $G$. Then according to Remark 2 the $FH$-module $B_H$ is hyperbolic. It follows (see Theorem 3.2 of [3]) that $B$ is a hyperbolic $FG$-module, contradicting the Inductive Hypothesis. Hence $(p,|G|) = 1$. $\square$

**Lemma 3.3.** $B$ is an orthogonal direct sum

$$B = U_1 \perp \ldots \perp U_k$$

where $k \geq 1$ and $\{U_i\}_{i=1, \ldots, k}$ are distinct, simple $FG$–submodules of $B$, that are also symplectic. Furthermore each $U_i$ is quasi–primitive (i.e., its restriction to every normal subgroup of $G$ is homogeneous).

*Proof.* The first statement follows from Lemma 3.1 Proposition 2.2 and Remark 1. For the rest of the proof we fix $U = U_i$ for some $i = 1, \ldots, k$. We also fix a normal subgroup $K$ of $G$. If the restriction of $U$ to $K$ is not homogeneous then Clifford’s Theorem implies that

$$U_K \cong e(V^{\sigma_1} \oplus \ldots \oplus V^{\sigma_r})$$

where $e$ is the number of inequivalent irreducible representations of $K$. But this contradicts the minimality of $|G|$.
where \( e \) is some positive integer, \( \mathcal{V} = \mathcal{V}^{\sigma_1} \) is a simple \( \mathcal{F}K \)-submodule of \( \mathcal{U} \) and \( \mathcal{V}^{\sigma_1}, \ldots, \mathcal{V}^{\sigma_r} \) are the distinct conjugates of \( \mathcal{V} \) in \( G \), with \( \sigma_1, \ldots, \sigma_r \) coset representatives of the stabilizer, \( G_V \), of \( \mathcal{V} \) in \( G \).

Let \( \mathcal{W} = \mathcal{U}(\mathcal{V}) \) be the \( \mathcal{V} \)-primary component of \( \mathcal{U}_K \). Then Clifford’s Theorem implies that \( \mathcal{W} \) is the unique simple \( \mathcal{FG}_V \)-submodule of \( \mathcal{U} \) that lies above \( \mathcal{V} \) and induces \( \mathcal{U} \), i.e., that satisfies \( \mathcal{W}^G \cong \mathcal{U} \) and \( \mathcal{W}_K \cong e\mathcal{V} \). Furthermore the dual \( \hat{\mathcal{W}} \) of \( \mathcal{W} \) induces in \( G \) the dual \( \hat{\mathcal{U}} \) of \( \mathcal{U} \) since \( \hat{\mathcal{W}}^G \cong \hat{\mathcal{W}}^G \) and \( \hat{\mathcal{W}}_K \cong e\mathcal{V} \). Hence \( \hat{\mathcal{W}}^G \cong \hat{\mathcal{U}}^G \), because \( \hat{\mathcal{U}}^G \) is self–dual (see Lemma 2.3). On the other hand, the restriction of \( \hat{\mathcal{W}} \) to \( K \) is isomorphic to \( e\mathcal{V} \) since \( \mathcal{V} \) is self–dual by Proposition 2.4. Hence the unicity of \( \mathcal{W} \) implies that \( \hat{\mathcal{W}} \) is self–dual.

According to Proposition 2.3 the self–dual \( \mathcal{FG}_V \)-module \( \mathcal{W} \) appears with even multiplicity as a direct summand of \( \mathcal{B}_{\mathcal{G}_V} \), because \( \mathcal{B}_{\mathcal{G}_V} \) is hyperbolic (\( \mathcal{G}_V < G \)). This, along with the fact that \( \mathcal{W} \) appears with multiplicity one in \( \mathcal{U}_{\mathcal{G}_V} \), implies that there is some \( j \in \{1, \ldots, k\} \) with \( j \neq i \) such that the \( \mathcal{V} \)-primary component \( \mathcal{U}_j(\mathcal{V}) \) of \( \mathcal{U}_j \) is isomorphic to \( \mathcal{W} \). So

\[
\mathcal{W} = \mathcal{U}(\mathcal{V}) \cong \mathcal{U}_j(\mathcal{V}).
\]

We conclude that

\[
\mathcal{U}_i = \mathcal{U} \cong \mathcal{W}^G \cong \mathcal{U}_j(\mathcal{V})^G \cong \mathcal{U}_j,
\]

as \( \mathcal{FG} \)-modules. This contradicts the fact that \( \{\mathcal{U}_i\}_{i=1}^k \) are all distinct, by the first statement of the lemma. Hence the lemma is proved.

From now on and until the end of the paper, we write \( \mathcal{E} \) for a finite algebraic field extension of \( \mathcal{F} \), that is a splitting field of \( G \) and all its subgroups.

**Lemma 3.4.** Assume that \( \mathcal{U}_i \), for \( i = 1, \ldots, k \), is a direct summand of \( \mathcal{B} \) appearing in \( \mathcal{E} \). Let \( N \leq G \). Then \( \mathcal{U}_i|_N \cong e_i\mathcal{V}_i \), where \( \mathcal{V}_i \) is an irreducible \( \mathcal{FN} \)-submodule of \( \mathcal{U}_i \) and \( e_i \) is an integer. If \( \mathcal{V}_i \) is non-trivial then \( e_i \) is odd.

**Proof.** We fix \( \mathcal{U} = \mathcal{U}_i \), for some \( i = 1, \ldots, k \). We also fix a normal subgroup \( N \) of \( G \). According to Lemma 2.3 the \( \mathcal{FG} \)-module \( \mathcal{U} \) is quasi–primitive. Hence there exists an irreducible \( \mathcal{FN} \)-submodule \( \mathcal{V} \) of \( \mathcal{U} \), and an integer \( e \) such that \( \mathcal{U}_N \cong e\mathcal{V} \). Thus, it remains to show that \( e \) is odd in the case that \( \mathcal{V} \) is non-trivial. So we assume that \( \mathcal{V} \), and thus \( \mathcal{U} \), is non-trivial.

We observe that if \( \mathcal{U} \) and \( \mathcal{V} \) were absolutely irreducible modules then it would be immediate that \( e \) is odd (even if \( \mathcal{V} \) was trivial), because for absolutely irreducible modules the integer \( e \) divides the order of \( G \) (see Corollary 11.29 in [6]). So we assume that \( \mathcal{F} \) is not a splitting field of \( G \), and we work with the algebraic field extension \( \mathcal{E} \) of \( \mathcal{F} \). We define \( \mathcal{U}^\mathcal{E} \) to be the extended \( \mathcal{EG} \)-module

\[
\mathcal{U}^\mathcal{E} = \mathcal{U} \otimes_{\mathcal{F}} \mathcal{E}.
\]

According to Theorem 9.21 in [6], there exist absolutely irreducible \( \mathcal{EG} \)-modules \( \mathcal{U}_i^\mathcal{E} \), for \( i = 1, \ldots, n \), such that

\[
\mathcal{U}^\mathcal{E} \cong \bigoplus_{i=1}^n \mathcal{U}_i^\mathcal{E}.
\]

Furthermore the \( \mathcal{U}_i^\mathcal{E} \), for all \( i = 1, \ldots, n \), constitute a Galois conjugacy class over \( \mathcal{F} \), and thus they are all distinct. In particular, if \( \xi_{\mathcal{U}} \) is the subfield of \( \mathcal{E} \) that is generated by all the values of the irreducible \( \mathcal{E} \)-character afforded by \( \mathcal{U}_i^\mathcal{E} \) (the same field for all \( i = 1, \ldots, n \)), then \( n = [\mathcal{E}_{\mathcal{U}} : \mathcal{F}] = \dim_{\mathcal{F}}(\xi_{\mathcal{U}}) \). (Note that \( \xi_{\mathcal{U}} \) is the unique subfield of \( \mathcal{E} \) isomorphic to the center of the endomorphism algebra \( \text{End}_{\mathcal{FG}}(\mathcal{U}_i) \)). Clearly \( n \cdot \dim_{\mathcal{E}} \mathcal{U}_i^\mathcal{E} = \dim_{\mathcal{F}} \mathcal{U}_i \). Hence \( n \) is even, because \( \dim_{\mathcal{F}} \mathcal{U}_i \) is even (as \( \mathcal{U} \) is symplectic) and \( \dim_{\mathcal{E}} \mathcal{U}_i^\mathcal{E} \) is odd (as \( G \) is odd and \( \mathcal{U}_i^\mathcal{E} \) is an absolutely irreducible \( \mathcal{EG} \)-module). In addition, each \( \mathcal{EG} \)-module \( \mathcal{U}_i^\mathcal{E} \), for \( i = 1, \ldots, n \), when consider as an
\[ \mathcal{F}_G \text{-module}, \text{is isomorphic to a direct sum of } [E : \mathcal{E}_U] \text{ copies of } \mathcal{U} \text{ (see Theorem 1.16 in Chapter 1 of [5]). Hence if we denote by } \mathcal{U}_\mathcal{F} \text{ the } \mathcal{E}_G \text{-module } \mathcal{U} \text{ regarded as an } \mathcal{F}_G \text{-module, we get} \]

\[ \mathcal{U}_\mathcal{F} \cong [E : \mathcal{E}_U] \mathcal{U}, \tag{6} \]

for all \( i = 1, \ldots, n. \)

We also write \( \mathcal{V}^\mathcal{E} \) for the extended \( \mathcal{E}_N \text{-module } \mathcal{V}^\mathcal{E} = \mathcal{V} \otimes_{\mathcal{F}} \mathcal{E}. \) Then according to Theorem 9.21 in [6] there exist absolutely irreducible \( \mathcal{E}_N \text{-modules } \mathcal{V}^j \text{ for } j = 1, \ldots, d, \text{ such that} \]

\[ \mathcal{V}^\mathcal{E} \cong \bigoplus_{j=1}^{d} \mathcal{V}^j. \tag{7} \]

In addition, the absolutely irreducible modules \( \mathcal{V}^j \), for all \( j = 1, \ldots, d \), form a Galois conjugacy class, and thus they are all distinct. Furthermore, \( d = [E : \mathcal{F}] = \dim_{\mathcal{F}} \mathcal{E}_V, \) where \( \mathcal{E}_V \) is the subfield of \( \mathcal{E} \) generated by all the values of the irreducible \( \mathcal{E} \text{-character afforded by } \mathcal{V} \) (the same field for all \( j = 1, \ldots, d \)). The field \( \mathcal{E}_V \) is the unique subfield of \( \mathcal{E} \) isomorphic to the center of the endomorphism algebra \( \text{End}_{\mathcal{F}N}(\mathcal{V}) \). Note that, according to Proposition 2.5, the \( \mathcal{F}_N \text{-submodule } \mathcal{V} \text{ of } \mathcal{U} \text{ is self–dual}. \) Hence \( \mathcal{V}^\mathcal{E} \) is also a self–dual \( \mathcal{E}_N \text{-module}. \) Because \( \mathcal{V} \text{ is non-trivial, } \mathcal{V}^j \text{ is also non-trivial, for all } j = 1, \ldots, d. \) Therefore the absolutely irreducible \( \mathcal{E}_N \text{-module } \mathcal{V}^j \text{ can’t be self–dual, because } N \text{ has odd order and } \mathcal{V}^j \text{ is non–trivial (see Lemma 2.8), for all such } j. \)

The fact that none of the \( \mathcal{V}^j \) is self–dual, for all \( j = 1, \ldots, d \), while they all appear in (7) in dual pairs, implies that \( d \) is even. Even more, if \( \mathcal{V}^j \) denotes the module \( \mathcal{V}^j \) regarded as an \( \mathcal{F}_N \text{-module}, \) then Theorem 1.16 of Chapter 1 in [5] implies that

\[ \mathcal{V}^j \cong [E : \mathcal{E}_V] \mathcal{V}^j, \tag{8} \]

for all \( j = 1, \ldots, d. \)

Without loss we may assume that \( \mathcal{V}^1, \ldots, \mathcal{V}^c \) are exactly those among the \( \mathcal{V}^j \), for \( j = 1, \ldots, d, \) that lie under \( \mathcal{U}^1 \), for some \( c = 1, \ldots, d. \) Thus Clifford’s theorem implies that

\[ \mathcal{U}^1 = e' (\mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^c), \tag{9} \]

where \( \mathcal{V}^1, \ldots, \mathcal{V}^c \) are the distinct \( G \text{-conjugates of } \mathcal{V}^1, \) and \( e', c \) are integers that divide \( |G| \) and thus are odd. (Note that here we are dealing with absolutely irreducible modules so \( e' \) does divide \( |G| \).) If we regard the modules of (8) as modules over the field \( \mathcal{F} \) then we can clearly have

\[ \mathcal{U}^1|_{\mathcal{F}N} \cong e' (\mathcal{V}^1|_{\mathcal{F}N} \oplus \cdots \oplus \mathcal{V}^c|_{\mathcal{F}N}). \]

This, along with (8) and (6), implies

\[ [E : \mathcal{E}_U]|_{\mathcal{F}N} \cong e' [E : \mathcal{E}_V] \mathcal{V}. \]

Since \( \mathcal{U}N \cong e \mathcal{V}, \) we have

\[ [E : \mathcal{E}_U]e = e' [E : \mathcal{E}_V]. \tag{10} \]

If \( \mathcal{D} \) is the subfield of \( \mathcal{E} \) generated by \( \mathcal{E}_V \) and \( \mathcal{E}_U \), then dividing both sides of (10) by \( [E : \mathcal{D}] \) we obtain

\[ e [D : \mathcal{E}_U] = e' [D : \mathcal{E}_V]. \tag{11} \]

Assume that \( e \) is even. Then (11) implies that \( [D : \mathcal{E}_V] \) is even, as \( e' \) and \( c \) are known to be odd. Let \( \Gamma \) be the Galois group \( \Gamma = \text{Gal}(\mathcal{D}/\mathcal{F}) \) of \( \mathcal{D} \) over \( \mathcal{F}. \) Because \( \Gamma \) is cyclic, it contains a unique involution \( \iota. \) Let \( \mathcal{E}_V^\mathcal{U} \) and \( \mathcal{E}_V^\mathcal{U} \) be the subgroups of \( \Gamma \) consisting of those elements of \( \Gamma \) that fix pointwise \( \mathcal{E}_V \) and \( \mathcal{E}_U, \) respectively. Then Galois theory implies that \( \mathcal{E}_V^\mathcal{U} = [\mathcal{E}_V^\mathcal{U} : 1] = [\mathcal{D} : \mathcal{E}_V] \) is even. We conclude that the unique involution \( \iota \) of \( \Gamma \) is an element of \( \mathcal{E}_V^\mathcal{U}. \) Therefore, \( \iota \) fixes the field \( \mathcal{E}_V \) pointwise. So \( \iota \) fixes, to within isomorphisms, each of the \( \mathcal{E}_N \text{-modules } \mathcal{V}^j. \) Because
Lemma 3.5. The group $\mathcal{G}$ is not abelian.

Proof. Assume that $\mathcal{G}$ is abelian. Then any cyclic subgroup $N = \langle \sigma \rangle$ of $\mathcal{G}$ is normal, for every $\sigma \in \mathcal{G}$. Because $B_N$ is hyperbolic, Lemmas 3.1 and 3.4 along with Corollary 2.7 imply that
\[ B_N \cong 2 \cdot \Delta(N), \]
where $\Delta(N)$ is a semi–simple $\mathcal{F}N$-submodule of $B$. Using the splitting field $\mathcal{E}$ of $\mathcal{G}$, we write $\mathcal{B}^\mathcal{E}$ for the extended $\mathcal{E}G$-module $\mathcal{B}^\mathcal{E} = B \otimes_\mathcal{F} \mathcal{E}$. Then
\[ \mathcal{B}^\mathcal{E}_N \cong 2 \cdot \Delta^\mathcal{E}(N), \tag{12} \]
where $\Delta^\mathcal{E}(N)$ is the extended $\mathcal{E}N$-module $\Delta(N) \otimes_\mathcal{F} \mathcal{E}$. Let $\phi$ be a Brauer character that the $\mathcal{E}G$-module $\mathcal{B}^\mathcal{E}$ affords. Because $(p, |\mathcal{G}|) = 1$, $\phi$ is defined for every element of $\mathcal{G}$. So $\phi$ coincides with a complex character of $\mathcal{G}$. In view of (12), for every cyclic subgroup $N = \langle \sigma \rangle$ of $\mathcal{G}$, the restriction $\phi_N$ of $\phi$ to $N$ equals $2 \cdot \delta(N)$, where $\delta(N)$ is a complex character of $N$. Hence, for every element $\sigma \in \mathcal{G}$, the integer 2 divides $\phi(\sigma)$ in the ring $\mathbb{Z}[\omega]$, where $\omega$ is a $|\mathcal{G}|$-primitive root of unity. We conclude that 2 also divides $\sum_{\sigma \in \mathcal{G}} \phi(\sigma) \cdot \lambda(\sigma^{-1})$, for any irreducible (linear) complex character $\lambda$ of $\mathcal{G}$. That is, 2 divides $|\mathcal{G}| \langle \phi, \lambda \rangle$, for any $\lambda \in \text{Irr}(\mathcal{G})$. The fact that $\mathcal{G}$ has odd order, implies that 2 divides $\langle \phi, \lambda \rangle$ in $\mathbb{Z}[\omega]$, for any $\lambda \in \text{Irr}(\mathcal{G})$. Because $\phi = \sum_{\lambda \in \text{Irr}(\mathcal{G})} \langle \phi, \lambda \rangle \cdot \lambda$, we get
\[ \phi = 2 \cdot \chi, \tag{13} \]
where $\chi$ is a complex character of $\mathcal{G}$.

On the other hand, Lemma 3.3 implies that $\mathcal{B} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$, where the $\mathcal{U}_i$ are distinct simple $\mathcal{F}G$-modules, for $i = 1, \ldots, k$. Hence the extended $\mathcal{E}G$-module $\mathcal{B}^\mathcal{E}$ will also equal the direct sum of the distinct $\mathcal{E}G$-modules $\mathcal{U}_i^\mathcal{E}$, $\mathcal{U}_k^\mathcal{E}$. By Theorem 9.21 in [B], for each $i = 1, \ldots, k$, there exist absolutely irreducible $\mathcal{E}G$–modules $\mathcal{U}_i^\mathcal{E}$, for $j = 1, \ldots, n_i$ such that
\[ \mathcal{U}_i^\mathcal{E} \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i^3. \]
Furthermore, the $\mathcal{U}_i^3$, for $j = 1, \ldots, n_i$, constitute a Galois conjugacy class over $\mathcal{F}$, and thus they are all distinct. In addition, the above absolutely irreducible $\mathcal{E}G$-modules $\mathcal{U}_i^3$, for all $i = 1, \ldots, k$ and all $j = 1, \ldots, n_i$, are distinct. Indeed, for all $i = 1, \ldots, k$, the corresponding simple $\mathcal{F}G$-modules $\mathcal{U}_i$ are distinct. We conclude that
\[ \mathcal{B}^\mathcal{E} \cong \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n_i} \mathcal{U}_i^3. \]
where $\mathcal{U}_i^j$ are all distinct absolutely irreducible $\mathcal{E}G$-modules. So the character $\phi$ that $\mathcal{B}^\mathcal{E}$ affords equals

$$\phi = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \chi_i^j,$$

where, for all $i = 1, \ldots, k$ and all $j = 1, \ldots, n_i$, the character $\chi_i^j$ is a Brauer character that $\mathcal{U}_i^j$ affords. So all these characters are distinct. This contradicts \([13]\). Hence the group $G$ is not abelian, and the lemma is proved. □

**Lemma 3.6.** $G$ acts faithfully on $\mathcal{B}$.

**Proof.** Suppose not. Let $K$ denote the kernel of the action of $G$ on $\mathcal{B}$ and $\bar{G} = G/K$. Thus $\vert \bar{G} \vert \leq \vert G \vert$ (as $K \neq 1$).

If $\bar{G}$ is not itself cyclic, then any cyclic subgroup $\bar{N}$ of $\bar{G}$ is the image $\bar{N} = N/G$ of some proper subgroup $N$ of $G$. Since $\mathcal{B}$ is $\mathcal{F}N$-hyperbolic, it is clearly $\mathcal{F}\bar{N}$-hyperbolic. Hence the triplet $\mathcal{F}, \mathcal{B}, G$ satisfies the hypothesis of the Main Theorem. The minimality of $\vert G \vert$ implies that $\mathcal{B}$ is a hyperbolic $\mathcal{F}G$-module, and therefore a hyperbolic $\mathcal{F}G$-module, because any $\mathcal{F}G$-submodule of $\mathcal{B}$ is also an $\mathcal{F}G$-submodule of $\mathcal{B}$. This contradicts the Inductive Hypothesis.

If $G$ is cyclic, then $\bar{G} = < \bar{\sigma} >$, where $\bar{\sigma}$ is the image in $\bar{G}$ of some $\sigma \in G$. Let $M = < \sigma >$. Then $M$ is a proper subgroup of $G$, because $G$ is not cyclic. In addition, the image of $M$ in $\bar{G}$ is $\bar{G}$. So $G = MK$ with $M \leq G$. Then Remark $2$ implies that $\mathcal{B}$ is $\mathcal{F}M$–hyperbolic and thus $\mathcal{F}G$–hyperbolic. This last contradiction implies the lemma. □

**Lemma 3.7.** Suppose $M$ is a minimal normal subgroup of $G$. Then $M$ is cyclic and central.

**Proof.** According to Lemmas $3.3$ and $3.4$ for each $i = 1, \ldots, k$ there is a simple $\mathcal{F}M$–submodule $\mathcal{V}_i$ of $\mathcal{U}_i$ and an odd integer $e_i$, such that $\mathcal{U}_i \mid M \cong e_i \mathcal{V}_i$. As $G$ acts faithfully on $\mathcal{B}$, there is some $i \in \{1, \ldots, k\}$ such that $\mathcal{V}_i \neq 1$ is non-trivial. Let $K_M(\mathcal{V}_i)$ be the kernel of the action of $M$ on $\mathcal{V}_i$. The fact that $\mathcal{V}_i$ is $G$-invariant implies that $K_M(\mathcal{V}_i)$ is a normal subgroup of $G$ contained in $M$. Hence $K_M(\mathcal{V}_i) = 1$. Therefore $M$ admits a faithful simple representation. In addition, $M$ is a $q$-elementary abelian group, for some prime $q$ that divides $\vert G \vert$, because $G$ is solvable. We conclude that $M \cong \mathbb{Z}_q$ is a cyclic group of order $q$.

It remains to show that $M$ is central. If $\mathcal{F}$ is a splitting field of $M$ (that is, it contains a primitive $q$-root of $1$), then the fact that there exists a faithful, simple and thus one-dimensional, $G$-invariant $\mathcal{F}M$-module $\mathcal{V}_i$ implies that $M$ is central in $G$. If $\mathcal{F}$ is not a splitting field of $M$, we work with the extension field $\mathcal{E}$ of $\mathcal{F}$. The extended module $\mathcal{B}^\mathcal{E} = \mathcal{B} \otimes_\mathcal{F} \mathcal{E}$ equals the direct sum of the extended $\mathcal{E}G$-modules $\mathcal{U}_i^\mathcal{E}$, $\mathcal{U}_j^\mathcal{E}$, because $\mathcal{B}$ is the direct sum of $\mathcal{U}_1, \ldots, \mathcal{U}_k$. As we have already seen, for each $i = 1, \ldots, k$, there exist absolutely irreducible $\mathcal{E}G$-modules $\mathcal{U}_i^j$, for $j = 1, \ldots, n_i$, that constitute a Galois conjugacy class over $\mathcal{F}$ and satisfy

$$\mathcal{U}_i^j \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i^j.$$  

(14)

Since $\mathcal{U}_i \mid M \cong e_i \mathcal{V}_i$ we have $\mathcal{U}_i^\mathcal{E} \mid M \cong e_i \mathcal{V}_i^\mathcal{E}$. In addition,

$$\mathcal{V}_i^\mathcal{E} \cong \bigoplus_{r=1}^{s_i} \mathcal{V}_r^\mathcal{E},$$

8
where the $V_r$, for $r = 1, \ldots, s_1$, are absolutely irreducible $EM$-modules, and thus of dimension one, that form a Galois conjugacy class over $F$. Therefore,
\[
U_i^E|_M \cong m_i^{n_i} \bigoplus_{j=1}^{n_i} U_j^E|_M \cong m_i^{s_i} \bigoplus_{r=1}^{s_i} V_r^E,
\]
for all $i = 1, \ldots, k$.

As we have already seen, there exists $i \in \{1, \ldots, k\}$ such that $V_i$ is a faithful $FM$-module. Without loss, we may assume that $i = 1$. Then it is clear that the $V_r^1$ are faithful $EM$-modules, for all $r \in \{1, \ldots, s_1\}$. If $V_r^1$ is $G$-invariant, for some $r \in \{1, \ldots, s_1\}$ (and thus for all such $r$) we are done.

Thus we may assume that the stabilizer $G_V$ of $V = V_1^1$ in $G$ is strictly smaller than $G$. Then $G_V = C_G(M)$, because $V$ is $EM$-faithful. Let $C := G_V = C_G(M)$. Note that $C$ is a normal subgroup of $G$, since $M \lhd G$. Furthermore, $C$ is also the stabilizer of $V_r^1$, for all $r = 1, \ldots, s_1$. According to Lemma 3.4, for all $i = 1, \ldots, k$, we have $U_i|_C = m_i \cdot Y_i$, where $Y_i$ is a simple $FC$-module, and $m_i$ some positive integer. For the extended $EC$-modules $Y_i^E$ we have
\[
U_i^E|_C \cong m_iY_i^E \cong m_i \bigoplus_{l=1}^{t_i} Y_i^l,
\]
where the $Y_i^l$, for $l = 1, 2, \ldots, t_i$, are absolutely irreducible $EC$-modules that constitute a Galois conjugacy class over $F$. Hence
\[
U_i^E|_C \cong m_i \bigoplus_{l=1}^{t_i} Y_i^l,
\]
for all $i = 1, \ldots, k$. We remark here that, because $U_i$ is quasi–primitive, all the group conjugates of $Y_i^1$ are among its Galois conjugates, for every $i = 1, \ldots, k$.

In the case $i = 1$, equations (15) and (16) imply
\[
U_1^E|_M \cong m_1 \bigoplus_{l=1}^{t_1} Y_1^l|_M \cong e_1 \bigoplus_{r=1}^{s_1} V_r^E.
\]

Without loss we may assume that $U_1^E$ lies above $Y_1^1$, and that $Y_1^1$ lies above $V_1^1 = V$. Clearly $Y_1^1$ is non–trivial as it restricts to a multiple of the non–trivial $FM$-module $V_1$. Hence Lemma 3.3 implies that $m_1$ is an odd integer. Because $C$ is the stabilizer of $V$ in $G$, Clifford’s theory implies that $Y_1^1$ is the unique simple $EC$-module that lies above $V_1^1$ and induces irreducibly to $U_1^1$ in $G$. Note that $Y_1^1$ appears with odd multiplicity $m_1$ as a summand of $U_1^E|_C$, because the $EC$-modules $Y_i^l$ are distinct for distinct values of $l$, as they form a Galois conjugacy class over $F$. Furthermore, if $Y_1^1$ lies under some $U_i^l$, for $i \neq 1$, then it induces $U_i^l$. So $U_i^l \cong U_1^l$. Hence the sum of the Galois conjugates of $U_1^1$ is isomorphic to the sum of the Galois conjugates of $U_1$. Therefore
\[
U_1^E \cong \bigoplus_{j=1}^{n_1} U_j^1 \cong \bigoplus_{j=1}^{n_1} U_j^1 \cong U_1^E.
\]
The above contradicts the fact that $U_1$ and $U_i$ are non-isomorphic simple $FG$-modules (see Lemma 3.3). We conclude that $Y_1^1$ appears with odd multiplicity $m_1$ in the decomposition of
\[
B^E|_C \cong \bigoplus_{i=1}^{k} U_i^E|_C \cong \bigoplus_{i=1}^{k} m_i \bigoplus_{l=1}^{t_i} Y_i^l.
\]
On the other hand, in view of Corollary 2.4 every simple $\mathcal{F}C$-submodule of $\mathcal{B}$ appears with even multiplicity in any decomposition of $\mathcal{B}C$, as $C$ is a normal subgroup of $G$ and $\mathcal{B}C$ is hyperbolic as an $\mathcal{F}C$-module, by Remark 2. Hence every absolutely irreducible $\mathcal{E}C$-submodule of $\mathcal{B}^C$ should also appear with even multiplicity in any decomposition of $\mathcal{B}^C|C$. This contradicts the conclusion of the preceding paragraph. So we must have $G_V = C = G$. Hence the lemma is proved. \hfill $\square$

Clearly Lemma 3.7 implies

**Corollary 3.8.** Suppose that $M$ is a minimal normal subgroup of $G$ and $\mathcal{E}$ a splitting field of $G$ and all its subgroups. Then every $\mathcal{E}M$-module is $G$-invariant.

We can now show

**Lemma 3.9.** Suppose that $M$ is a minimal normal subgroup of $G$. Then the restriction $\mathcal{B}_M$ is homogeneous. Furthermore $\mathcal{B}_M \cong e\mathcal{V}$, where $\mathcal{V}$ is a simple faithful $G$-invariant $\mathcal{F}M$-submodule of $\mathcal{B}_M$ and $e$ is a positive integer.

**Proof.** As in the previous lemma we write $U_i|_M = e_iV_i$, where $i = 1, \ldots, k$, and $V_i$ is a simple $G$-invariant $\mathcal{F}M$-submodule of $U_i$. If $\mathcal{F}M$ is not homogeneous, then there are at least two non-isomorphic simple $\mathcal{F}M$-submodules of $\mathcal{B}_M$, say $\mathcal{V}$ and $\mathcal{W}$. We may suppose that $\mathcal{V}$ is non-trivial. Assume that $V_i \cong \mathcal{V}$ as $\mathcal{F}M$–modules, for all $i = 1, \ldots, l$ and some $l$ such that $1 \leq l \leq k$, while $V_i \not\cong \mathcal{V}$ for $i = l + 1, \ldots, k$. Let $\mathcal{U}$ be the orthogonal direct sum

$$\mathcal{U} = U_1 \perp \ldots \perp U_l,$$

of the corresponding $\mathcal{FG}$–submodules of $\mathcal{B}$. We also write

$$\mathcal{R} = U_{l+1} \perp \ldots \perp U_k,$$

for the orthogonal direct sum of the remaining simple $\mathcal{FG}$-submodules of $\mathcal{B}$. Clearly

$$\mathcal{B} = \mathcal{U} \perp \mathcal{R},$$

while $\mathcal{U}_M$ and $\mathcal{R}_M$ have no simple $\mathcal{F}M$-submodules in common.

We will show

**Claim 1.** $\mathcal{U}$ is $\mathcal{FN}$-hyperbolic for every cyclic subgroup $N$ of $G$.

We first prove Claim 1 in the case that the product $NM$ is a proper subgroup of $G$. In this case Remark 2 implies that $\mathcal{B}_{NM}$ is hyperbolic. Hence there exists a self-orthogonal $\mathcal{FNM}$-submodule $\mathcal{S} > 0$ of $\mathcal{B}$. Then $\mathcal{S}$ is a maximal isotropic $\mathcal{FNM}$-submodule of $\mathcal{B}_{NM}$. Furthermore, $\mathcal{B}_{NM} = \mathcal{U}_{NM} \perp \mathcal{R}_{NM}$, where $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$ have no simple $\mathcal{FNM}$-submodule in common (otherwise $\mathcal{U}_M$ and $\mathcal{R}_M$ would have some common simple $\mathcal{F}M$-submodule). Hence

$$\mathcal{S} = (\mathcal{S} \cap \mathcal{U}_{NM}) \perp (\mathcal{S} \cap \mathcal{R}_{NM}).$$

Because $\mathcal{S}$ is isotropic, both $\mathcal{S} \cap \mathcal{U}_{NM}$ and $\mathcal{S} \cap \mathcal{R}_{NM}$ are also isotropic. Hence their $\mathcal{F}$-dimensions are at most $1/2$ the dimensions of $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$, respectively. But $\mathcal{S}$ is self-orthogonal and thus its $\mathcal{F}$-dimension is exactly $(1/2)\dim(\mathcal{B}_{NM})$. We conclude that the $\mathcal{F}$-dimensions of $\mathcal{S} \cap \mathcal{U}_{NM}$ and $\mathcal{S} \cap \mathcal{R}_{NM}$ are exactly $1/2$ the dimensions of $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$, respectively. Therefore $\mathcal{S} \cap \mathcal{U}_{NM}$ is a maximal isotropic $\mathcal{FNM}$-submodule of $\mathcal{U}_{NM}$ of dimension $1/2$ the dimension of $\mathcal{U}_{NM}$. So $\mathcal{S} \cap \mathcal{U}_{NM}$ is self-orthogonal, by Lemma 2.4. Thus $\mathcal{U}_{NM}$ is hyperbolic as an $\mathcal{FNM}$-module. Hence it is also hyperbolic as an $\mathcal{FN}$-module. So Claim 1 holds when $NM < G$. 

10
Assume now that $N$ is a cyclic subgroup of $G$ such that $NM = G$. Because $M$ is minimal, Lemma 3.7 implies that $M \cong \mathbb{Z}_q$ is central. Hence $G = MN$ is an abelian group. This contradicts Lemma 3.5. Therefore $NM < G$, for every cyclic subgroup $N$ of $G$. Thus Claim 11 holds.

Since $\mathcal{U} < \mathcal{B}$, the Inductive Hypothesis, along with Claim 11 implies that $\mathcal{U}$ is $FG$-hyperbolic. Hence $\mathcal{U}$ contains a self–perpendicular $FG$-submodule $\mathcal{T}$. Let $\mathcal{T}^\perp$ be the submodule of $\mathcal{B}$ that is perpendicular to $\mathcal{T}$. Then $\mathcal{R}$ as well as $\mathcal{T}$ are subsets of $\mathcal{T}^\perp$. We conclude that $\mathcal{T}$ is an isotropic $FG$-submodule of $\mathcal{B}$. Hence $\mathcal{B}$ is not anisotropic. This last contradiction implies that $\mathcal{U} = \mathcal{B}$, and completes the proof of Lemma 3.10. \hfill $\square$

**Lemma 3.10.** Every abelian normal subgroup of $G$ is cyclic.

*Proof.* Let $A$ be an abelian normal subgroup of $G$. By Lemma 3.11 there is a simple $FA$–submodule $\mathcal{R}_1$ of $\mathcal{U}_1$ and an integer $e_1$ such that

$$\mathcal{U}_1|_A \cong e_1 \mathcal{R}_1.$$ 

It follows from Lemma 3.9 that $\mathcal{R}_1$ is non-trivial, since its restriction to any minimal normal subgroup of $G$ is non-trivial. Let $K_1$ denote the corresponding centralizer of $\mathcal{R}_1$ in $A$. Then $K_1$ equals the centralizer $C_A(\mathcal{U}_1)$ of $\mathcal{U}_1$ in $A$, and therefore is a normal subgroup of $G$. If $K_1$ is not trivial then it contains a minimal normal subgroup $M$ of $G$. In view of Lemma 3.9 the restriction $\mathcal{U}_1|_M$, cannot be trivial, contradicting the definition of $K_1$. Hence $K_1$ is trivial. Thus $A$ is cyclic and the lemma is proved. \hfill $\square$

Let $F = F(G)$ be the Fitting subgroup of $G$. Assume further that $\{q_i\}_{i=1}^r$ are the distinct primes dividing $|F|$, and that $T_i$ is the $q_i$-Sylow subgroup of $F$, for each $i = 1, \ldots, r$. Then $F = T_1 \times T_2 \times \cdots \times T_r$. Every characteristic abelian subgroup of $F$ is cyclic, according to Lemma 3.10. Hence (see Theorem 4.9 in [4]) either $T_i$ is cyclic or $T_i$ is the central product $T_i = E_i \odot Z(T_i)$ of the extra special $q_i$-group $E_i = \Omega(T_i)$ of exponent $q_i$ and the cyclic group $Z(T_i)$. We complete the proof of Theorem A exploring the two possible types of $T_i$.

Assume first that $T_i$ is a cyclic group, for all $i = 1, \ldots, r$. In this case $F = T_1 \times \cdots \times T_r$ is also a cyclic group. Let $C/F$ be a chief factor of $G$. So $\bar{C} = C/F$ is an elementary abelian $q$-group, for some prime $q$, because $G$ is solvable. Then $\bar{C}$ acts coprimely on $T_i$ for all $i$ such that $q$ does not divide $|T_i|$. But $T_i$ is cyclic, and the minimal subgroup of $T_i$ is central in $G$. Hence $C_{T_i}(\bar{C}) \neq 1$. We conclude that $T_i = [T_i, \bar{C}] \times C_{T_i}(\bar{C}) = C_{T_i}(\bar{C})$. So any $q$-Sylow subgroup $C_q$ of $C$ centralizes the $q'$-Hall subgroup $R$ of $F$ that is also a $q'$-Hall subgroup of $C$. We conclude that $C = C_q \times R$. But $R$ is nilpotent as a subgroup of $F$. So $C$ is a nilpotent normal subgroup of $G$ bigger than the Fitting subgroup $F$ of $G$. Therefore $G = F$ is a cyclic group, contradicting the Inductive Hypothesis. Hence there exists a $T_i$ of $F = F(G)$ that is not cyclic.

Let $T = T_i$ be a non-cyclic $q$-Sylow subgroup of $F$, where $q = q_i$ for some $i = 1, \ldots, r$. Then $T = E \odot Z(T)$, where $E = \Omega(T)$ is an extra special $q$-group of exponent $q$ and $Z(T)$ is the center of $T$. Of course $E$ is a normal subgroup of $G$, since it is a characteristic subgroup of $F$. Furthermore, $Z(E)$ is a central subgroup of $G$ because it is a minimal (it has order $q$) normal subgroup of $G$. According to Lemma 3.9 there exists a faithful $G$-invariant $FZ(E)$-module $\mathcal{V}$ so that the restriction $\mathcal{B}_{Z(E)}$ of $\mathcal{B}$ to $Z(E)$ is a multiple of $\mathcal{V}$.

Using the extension field $E$ of $F$, we write $\mathcal{V}^E$ for the extended $EZ(E)$-module $\mathcal{V} \otimes_{F} E$. Then

$$\mathcal{V}^E \cong \bigoplus_{j=1}^s \mathcal{V}^j.$$
where \( \mathcal{V}^j \) is an absolutely irreducible \( E \mathcal{Z}(E) \)-module, for all \( j \) with \( j = 1, \ldots, s \). Furthermore, the \( \mathcal{V}^j \) constitute a Galois conjugacy class over \( \mathcal{F} \), and thus they are all distinct. As we have already seen (see Corollary 3.8 and Lemma 3.9), the module \( \mathcal{V}^j \) is a non-trivial \( G \)-invariant \( E \mathcal{Z}(E) \)-module. Because \( E \) is extra special, there exists a unique, up to isomorphism, absolutely irreducible \( E \mathcal{E}(E) \)-module \( \mathcal{W}^j \) lying above \( \mathcal{V}^j \), for every \( j = 1, \ldots, s \). Note that for all such \( j \) the \( E \mathcal{E}(E) \)-module \( \mathcal{W}^j \) is \( G \)-invariant because \( \mathcal{V}^j \) is \( G \)-invariant. According to Theorem 9.1 in \cite{6} (used for modules) there exists a canonical conjugacy class of subgroups \( H \leq G \) such that \( H \mathcal{E} = G \) and \( H \cap E = Z(E) \). Furthermore, for this conjugacy class there exists a one–to–one correspondence between the isomorphism classes of absolutely irreducible \( E \mathcal{G} \)-modules lying above \( \mathcal{W}^j \) and those classes of absolutely irreducible \( E \mathcal{H} \)-modules lying above \( \mathcal{V}^j \). In addition, the fact that \( G \) has odd order implies that if \( \Xi \) and \( \Psi \) are representatives of the above two isomorphism classes, then they correspond if \( \Xi_H \cong \Psi \oplus 2 \cdot H \Delta \), where \( \Delta \) is a completely reducible \( H \mathcal{E}_H \)-submodule of \( \Xi_H \).

Let \( \mathcal{U} = \mathcal{U}_i \) be one of the simple \( F \mathcal{G} \)-submodules of \( \mathcal{B} \) appearing in \( \mathcal{F} \). Then \( \mathcal{U}^i \cong \bigoplus_{j=1}^{n_i} \mathcal{U}^j \), where the \( \mathcal{U}^j \) are absolutely irreducible \( E \mathcal{G} \)-modules that form a Galois conjugacy class. As earlier, we write \( \mathcal{E}_H \) for the extension field of \( \mathcal{F} \) generated by all the values of the absolutely irreducible character that \( \mathcal{U}^j \) affords. Let \( \Gamma = \text{Gal}(\mathcal{E}_H/\mathcal{F}) \) be the Galois group of that extension. Then (see Theorem 9.21 in \cite{6}),

\[
\mathcal{U}^i \cong \bigoplus_{j=1}^{n_i} \mathcal{U}^j \cong \bigoplus_{\tau \in \Gamma} (\mathcal{U}^1)^{\tau},
\]

Clearly \( \mathcal{U}^{1} \) lies above \( \mathcal{W}^{j} \), for some \( j = 1, \ldots, s \), since \( \mathcal{U} = \mathcal{U}_i \) lies above \( \mathcal{V} \). Let \( \Psi \) be a representative of the isomorphism class of absolutely irreducible \( E \mathcal{H} \)-modules that corresponds to \( \mathcal{U}^{1} \) and lies above \( \mathcal{V}^{j} \). Then

\[
\mathcal{U}^{1}_H \cong \Psi \oplus 2 \cdot \Delta,
\]

for some completely reducible \( E \mathcal{H} \)-module \( \Delta \). Let \( \mathcal{E}_\Psi \) be the subfield of \( \mathcal{E} \) generated by \( \mathcal{F} \) and all the values of the absolutely irreducible character that \( \Psi \) affords. Then \( \mathcal{E}_\Psi \) is a Galois extension of \( \mathcal{F} \). Furthermore,

\[
\mathcal{E}_\Psi = \mathcal{E}_\mathcal{U}.
\]

Indeed, for any element \( \sigma \) in the Galois group \( \text{Gal}(\mathcal{E}/\mathcal{F}) \) of \( \mathcal{E} \) above \( \mathcal{F} \) we get

\[
(\mathcal{U}^{1})^{\sigma} \cong \Psi^{\sigma} \oplus 2 \cdot \Delta^{\sigma}.
\]

Hence \( (\mathcal{U}^{1})^{\sigma} \) corresponds to \( \Psi^{\sigma} \), as \( \Psi^{\sigma} \) is the only absolutely irreducible \( E \mathcal{H} \)-module that appears with odd multiplicity in \( (\mathcal{U}^{1})^{\sigma} \). Therefore, \( (\mathcal{U}^{1})^{\sigma} \neq \mathcal{U}^{1} \) iff \( \Psi^{\sigma} \neq \Psi \). This is enough to guarantee that \( (\mathcal{U}^{1})^{\sigma} \neq \mathcal{U}^{1} \) holds. We conclude that the sum \( \bigoplus_{\tau \in \Gamma} \Psi^{\tau} \) is the extension to \( \mathcal{E} \) of an irreducible \( \mathcal{F} \mathcal{H} \)-module, i.e., there exists an irreducible \( \mathcal{F} \mathcal{H} \)-module \( \mathcal{P} \) such that

\[
\mathcal{P}^{\mathcal{E}} \cong \bigoplus_{\tau \in \Gamma} \Psi^{\tau},
\]

where \( \mathcal{P}^{\mathcal{E}} \) is the extended \( E \mathcal{H} \)-module \( \mathcal{P} \otimes_{\mathcal{F}} \mathcal{E} \). Furthermore, \( \{8\} \) and \( \{10\} \) imply that \( \mathcal{P} \) appears with odd multiplicity as a summand of \( \mathcal{U}^{1}_H = \mathcal{U}^{1} \).

Next we observe that if \( \mathcal{P} \) appears as a summand of \( \mathcal{U}^{1}_i \), for some \( i = 2, \ldots, k \), then it appears with even multiplicity. The reason is that \( \mathcal{U}^{1}_i \neq \mathcal{U}^{1}_i \) for all such \( i \). As in \( \{11\} \) we choose a Galois conjugacy class \( \{\mathcal{U}^{1}_i\}_{i=1}^{n_i} \) of absolutely irreducible \( E \mathcal{G} \)-modules such that \( \mathcal{U}^{1}_i \cong \bigoplus_{j=1}^{n_i} \mathcal{U}^{1}_j \). Then \( \mathcal{U}^{1}_i \neq \mathcal{U} = \mathcal{U}^{1}_1 \) implies that \( \mathcal{U}^{1}_i \neq \mathcal{U}^{1} \), for all \( i = 2, \ldots, k \) and all \( j = 1, \ldots, n_i \). So the \( E \mathcal{H} \)-module \( \Psi \) can’t correspond to \( \mathcal{U}^{1}_i \), for any such \( i, j \). Therefore if \( \Psi \) appears as a summand of the restriction \( \mathcal{U}^{1}_i \mid H \) of \( \mathcal{U}^{1}_i \) to \( H \), then it appears only with even multiplicity. Hence the same holds.
for Π, i.e., Π appears only with even multiplicity as a summand of \( U_i \mid_H \), whenever \( i = 2, \ldots, k \). We conclude that Π appears with odd multiplicity as a summand of \( B_H = U_1 \mid_H \oplus \cdots \oplus U_k \mid_H \).

We complete the proof of Theorem A with one more contradiction, that follows the fact that Π is a self–dual \( F H \)-module. That we get a contradiction if Π is self–dual is easy to see, because according to Proposition 2.4 Π should appear with even multiplicity as a summand of the hyperbolic \( F H \)-module \( B_H \). Thus it suffices to show that Π is self–dual.

The fact that \( U = U_1 \) is self–dual implies that \( U^\tau \) is also self–dual. Hence the dual \( \hat{U}^1 \) of \( U^1 \) is a Galois conjugate \( (U^1)^\tau \) to \( U^1 \), for some \( \tau \in \Gamma \). Furthermore, (19) implies that

\[
\hat{U}^1 \mid_H \cong \mathfrak{P} \oplus 2 \cdot \Delta.
\]

Thus the dual \( \hat{U}^1 \) corresponds to the dual \( \mathfrak{P} \) of \( \mathfrak{P} \). Therefore the dual \( \mathfrak{P} \) of \( \mathfrak{P} \) is a Galois conjugate of \( \mathfrak{P} \). Hence \( \Pi^\times \cong \oplus_{\tau \in \Gamma} \mathfrak{P}^\tau \) is a self–dual \( EH \)-module. So Π is also self–dual.

This completes the proof of Theorem A.

References

[1] M. Aschbacher, “Finite Group Theory,” Cambridge Studies in Advanced Mathematics, 10.
[2] T. R. Berger, Representation Theory and Solvable Groups: Length Type Problems, The Santa Cruz Conference on Finite Groups, Proc. Sympos. Pure Math. 37 (1980), 431–441.
[3] E. C. Dade, Monomial Characters and Normal Subgroups, Math.Z. 178, 401–420 (1981).
[4] D. Gorenstein, “Finite Groups,” New York: Harper and Row 1968.
[5] B. Huppert, N. Blackburn, “Finite Groups II,” Berlin–Heidelberg–New York : Springer 1982.
[6] I. M. Isaacs, “Character Theory of Finite Groups,” Academic Press, New York, 1976.
[7] I. M. Isaacs, Characters of Solvable and Symplectic Groups, Amer. J. Math. 95 (1973), 594–635.
[8] I. M. Isaacs, Abelian Normal Subgroups of \( M \)-Groups, Math. Z. 182 (1983), 205–221.
[9] I. M. Isaacs, Primitive Characters, Normal Subgroups and \( M \)-Groups, Math. Z. 177 (1981), 267–284.
[10] M. L. Lewis, Characters of Maximal Subgroups of \( M \)-Groups, J. Algebra 183 (1996) 864–897.
[11] A. E. Parks, Nilpotent by Supersolvable \( M \)-Groups, Canad. J. Math. 37 (1985), 934–962.
[12] R. W. van der Waall, On Clifford’s Theorem and Ramification Indices for Symplectic Modules over a Finite Field, Proc. Edinburgh Math. Soc. 30 (1987), 153–167.
[13] R. W. van der Waall and N.S. Hekster, Irreducible Constituents of Induced Monomial Characters, J. Algebra 105 (1987), 255-267.