Parametric estimation. Finite sample theory

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Abstract

The paper aims at reconsidering the famous Le Cam LAN theory. The main features of the approach which make it different from the classical one are:

(1) the study is non-asymptotic, that is, the sample size is fixed and does not tend to infinity; (2) the parametric assumption is possibly misspecified and the underlying data distribution can lie beyond the given parametric family.

These two features enable to bridge the gap between parametric and nonparametric theory and to build a unified framework for statistical estimation. The main results include a large deviation bound for the (quasi) maximum likelihood and a local quadratic bracketing result for the log-likelihood process. The latter yields a number of important corollaries for statistical inference: concentration, confidence and risk bounds, expansion of the maximum likelihood estimate, etc. All these corollaries are stated in a non-classical way admitting a model misspecification and finite samples. However, the classical asymptotic results including the efficiency bounds can be easily derived as corollaries of the obtained non-asymptotic statements. At the same time, the new bracketing device works well in the situations with large or growing parameter dimension in which the classical parametric theory fails. The general results are illustrated for the i.i.d. set-up as well as for generalized linear modeling and median estimation. The results apply for any dimension of the parameter space and provide a quantitative lower bound on the sample size yielding the root-n accuracy.

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1 Introduction

One of the most popular approaches in statistics is based on the parametric assumption (PA) that the distribution $\mathcal{P}$ of the observed data $Y$ belongs to a given parametric family $(\mathcal{P}_\theta, \theta \in \Theta \subseteq \mathbb{R}^p)$, where $p$ stands for the number of parameters. This assumption allows to reduce the problem of statistical inference about $\mathcal{P}$ to recovering the parameter $\theta$. The theory of parameter estimation and inference is nicely developed in a quite general set-up. There is a vast literature on this issue. We only mention the book by Ibragimov and Khas’minskij (1981), which provides a comprehensive study of asymptotic properties of maximum likelihood and Bayesian estimators. The theory is essentially based on two major assumptions: (1) the underlying data distribution follows the PA; (2) the sample size or the amount of available information is large relative to the number of parameters.

In many practical applications, both assumptions can be very restrictive and limit the scope of applicability for the whole approach. Indeed, the PA is usually only an approximation of real data distribution and in most statistical problems it is too restrictive to assume that the PA is exactly fulfilled. Many modern statistical problems deal with very complex high dimensional data where a huge number of parameters are involved. In such situations, the applicability of large sample asymptotics is questionable. These two issues partially explain why the parametric and nonparametric theory are almost isolated from each other. Relaxing these restrictive assumptions can be viewed as an important challenge of the modern statistical theory. The present paper attempts at developing a unified approach which does not require the restrictive parametric assumptions but still enjoys the main benefits of the parametric theory. The main steps of the approach are similar to the classical local asymptotic normality (LAN) theory; see e.g. Chapters 1–3 in the monograph Ibragimov and Khas’minskij (1981): first one localizes the problem to a neighborhood of the target parameter. Then one uses a local quadratic expansion of the log-likelihood to solve the corresponding estimation problem. There is, however, one feature of the proposed approach which makes it essentially different from classical scheme. Namely, the use of the bracketing device instead of classical Taylor expansion allows to consider much larger local neighborhoods than in the LAN theory. More specifically, the classical LAN theory effectively requires a strict localization to a root-$n$ vicinity of the true point. At this point, the LAN theory fails in extending to the nonparametric situation. Our approach works for any local vicinity of the true point.
This opens the door to building a unified theory including most of classical parametric and nonparametric results.

Let $Y$ stand for the available data. Everywhere below we assume that the observed data $Y$ follow the distribution $\mathcal{P}$ on a metric space $\mathcal{Y}$. We do not specify any particular structure of $Y$. In particular, no assumption like independence or weak dependence of individual observations is imposed. The basic parametric assumption is that $\mathcal{P}$ can be approximated by a parametric distribution $\mathcal{P}_\theta$ from a given parametric family $(\mathcal{P}_\theta, \theta \in \Theta \subseteq \mathbb{R}^p)$. Our approach allows that the PA can be misspecified, that is, in general, $\mathcal{P} \not\in (\mathcal{P}_\theta)$.

Let $L(Y, \theta)$ be the log-likelihood for the considered parametric model: $L(Y, \theta) = \log \frac{d\mathcal{P}_\theta(Y)}{d\mu_0(Y)}$, where $\mu_0$ is any dominating measure for the family $(\mathcal{P}_\theta)$. We focus on the properties of the process $L(Y, \theta)$ as a function of the parameter $\theta$. Therefore, we suppress the argument $Y$ there and write $L(\theta)$ instead of $L(Y, \theta)$. One has to keep in mind that $L(\theta)$ is random and depends on the observed data $Y$. By $L(\theta, \theta^*) \overset{\text{def}}{=} L(\theta) - L(\theta^*)$ we denote the log-likelihood ratio. The classical likelihood principle suggests to estimate $\theta$ by maximizing the corresponding log-likelihood function $L(\theta)$:

$$\tilde{\theta} = \arg\max_{\theta \in \Theta} L(\theta).$$ \hspace{1cm} (1.1)

Our ultimate goal is to study the properties of the quasi maximum likelihood estimator (MLE) $\tilde{\theta}$. It turns out that such properties can be naturally described in terms of the maximum of the process $L(\theta)$ rather than the point of maximum $\tilde{\theta}$. To avoid technical burdens it is assumed that the maximum is attained leading to the identity $\max_{\theta} L(\theta) = L(\tilde{\theta})$. However, the point of maximum needs not to be unique. If there are many such points we take $\tilde{\theta}$ as any of them. Basically, the notation $\tilde{\theta}$ is used for the identity $L(\tilde{\theta}) = \sup_{\theta \in \Theta} L(\theta)$.

If $\mathcal{P} \not\in (\mathcal{P}_\theta)$, then the (quasi) MLE $\tilde{\theta}$ from (1.1) is still meaningful and it appears to be an estimator of the value $\theta^*$ defined by maximizing the expected value of $L(\theta)$:

$$\theta^* = \arg\max_{\theta \in \Theta} \mathbb{E}L(\theta)$$ \hspace{1cm} (1.2)

which is the true value in the parametric situation and can be viewed as the parameter of the best parametric fit in the general case.

The results below show that the main properties of the quasi MLE $\tilde{\theta}$ like concentration or coverage probability can be described in terms of the excess which is the difference between the maximum of the process $L(\theta)$ and its value at the “true” point $\theta^*$:

$$L(\tilde{\theta}, \theta^*) \overset{\text{def}}{=} L(\tilde{\theta}) - L(\theta^*) = \max_{\theta \in \Theta} L(\theta) - L(\theta^*),$$
The established results can be split into two big groups. A large deviation bound states some concentration properties of the estimator $\widehat{\theta}$. For specific local sets $\Theta_0(r)$ with elliptic shape, the deviation probability $P(\widehat{\theta} \notin \Theta_0(r))$ is exponentially small in $r$. This concentration bound allows to restrict the parameter space to a properly selected vicinity $\Theta_0(r)$. Our main results concern the local properties of the process $L(\theta)$ within $\Theta_0(r)$ including a bracketing bound and its corollaries.

The paper is organized as follows. Section 2 presents the list of conditions which are systematically used in the text. The conditions only concern the properties of the quasi log-likelihood process $L(\theta)$.

Section 3 appears to be central in the whole approach and it focuses on local properties of the process $L(\theta)$ within $\Theta_0(r)$. The idea is to sandwich the underlying (quasi) log-likelihood process $L(\theta)$ for $\theta \in \Theta_0(r)$ between two quadratic (in parameter) expressions. Then the maximum of $L(\theta)$ over $\Theta_0(r)$ will be sandwiched as well by the maxima of the lower and upper processes. The quadratic structure of these processes help to compute these maxima explicitly yielding the bounds for the value of the original problem. This approximation result is used to derive a number of corollaries including the concentration and coverage probability, expansion of the estimator $\widehat{\theta}$, polynomial risk bounds, etc. In contrary to the classical theory, all the results are non-asymptotic and do not involve any small values of the form $o(1)$, all the terms are specified explicitly. Also the results are stated under possible model misspecification.

Section 4 accomplishes the local results with the concentration property which bounds the probability that $\widehat{\theta}$ deviates from the local set $\Theta_0(r)$. In the modern statistical literature there is a number of studies considering maximum likelihood or more generally minimum contrast estimators in a general i.i.d. situation, when the parameter set $\Theta$ is a subset of some functional space. We mention the papers Van de Geer (1993), Birgé and Massart (1993), Birgé and Massart (1998), Birgé (2006) and references therein. The established results are based on deep probabilistic facts from empirical process theory; see e.g. Talagrand (1996, 2001, 2005), van der Vaart and Wellner (1996), Boucheron et al. (2003). The general result presented in Section B follows the generic chaining idea due to Talagrand (2005); cf. Bednorz (2006). However, we do not assume any specific structure of the model. In particular, we do not assume independent observations and thus, cannot apply the most developed concentration bounds from the empirical process theory.

Section 5 illustrates the applicability of the general results to the classical case of an i.i.d. sample. The previously established general results apply under rather mild conditions. Basically we assume some smoothness of the log-likelihood process and some minimal number of observations per parameter: the sample size should be at least of
order of the dimensionality $p$ of the parameter space. We also consider the examples of generalized linear modeling and of median regression.

It is important to mention that the non-asymptotic character of our study yields an almost complete change of the mathematical tools: the notions of convergence and tightness become meaningless, the arguments based on compactness of the parameter space do not apply, etc. Instead we utilize the tools of the empirical process theory based on the ideas of concentration of measures and nonasymptotic entropy bounds. Section A in the Appendix presents an exponential bound for a general quadratic form which is very essential for getting the sharp risk bounds for the quasi MLE. This bound is an important step in the concentration results for the quasi MLE. Section B explains how generic chaining and majorizing measure device by Talagrand (2005) refined in Bednorz (2006) can be used for obtaining a general exponential bound for the log-likelihood process.

The proposed approach seems to be very promising and can be useful in many further research directions. Here we briefly outline some of them. The results of Sections 3 and 4 indicate the important role of the dimensionality of the parameter space. Namely, in the regular case, the entropy of the parametric family and hence, all the established bounds are proportional to the number of parameters to be estimated. This makes the results almost uninformative when the dimensionality becomes large relative to the sample size. The nonparametric theory operates with parameter sets of infinite dimension under some entropy restriction. An important advantage of the developed technique is that it is extendable to the case of the penalized likelihood. A proper choice of penalization allows to reduce the entropy bounds and to get sharp results even in the case of a large dimensional parameter space in terms of the so called effective dimension; Spokoiny (2012).

This paper focuses on the maximum likelihood estimation. However, the bracketing device appears to be very helpful and efficient for analysis of the posterior measure in the Bayes approach as well. The subsequent paper Spokoiny (2012) shows how the prominent Bernstein – von Mises result for quasi posteriors can be obtained using the new methodology. It also explains an interesting relation between the penalized maximum likelihood estimation and the Bayes method with Gaussian priors.

The proposed methodology has been successfully applied to the problem of bandwidth selection in local quantile regression, Spokoiny et al. (2012). An extension of the theory to the case of a semiparametric estimation is another important issue. The modern semiparametric theory is quite involved and requires special tools like the hardest parametric subspace. The new approach can be directly applied to the situation when the target of analysis is a projection (mapping) of the parameter vector onto some subspace; Andresen and Spokoiny (2012). Under some restriction on the dimension or on the entropy of the nuisance parameter, the bracketing device continues to work. The established results
include the semiparametric efficiency of the MLE and the Wilks theorem.

2 Conditions

Below we collect the list of conditions which are systematically used in the text. It seems to be an advantage of the whole approach that all the results are stated in a unified way under the same conditions. Once checked, one obtains automatically all the established results. We do not try to formulate the conditions and the results in the most general form. In some cases we sacrifice generality in favor of readability and ease of presentation. It is important to stress that all the conditions only concern the properties of the quasi likelihood process $L(\theta)$. Even if the process $L(\cdot)$ is not a sufficient statistic, the whole analysis is entirely based on its geometric structure and probabilistic properties. The conditions are not restrictive and can be effectively checked in many particular situations. Some examples are given in Section 5 for i.i.d setup, generalized linear models, and for median regression.

The imposed conditions can be classified into the following groups by their meaning:

- smoothness conditions on $L(\theta)$ allowing the second order Taylor expansion;
- exponential moment conditions;
- identifiability and regularity conditions;

We also distinguish between local and global conditions. The global conditions concern the global behavior of the process $L(\theta)$ while the local conditions focus on its behavior in the vicinity of the central point $\theta^*$. Below we suppose that degree of locality is described by a number $r$. The local zone corresponds to $r \leq r_0$ for a fixed $r_0$. The global conditions concern $r > 0$.

2.1 Local conditions

Local conditions describe the properties of $L(\theta)$ in a vicinity of the central point $\theta^*$ from (1.2).

To bound local fluctuations of the process $L(\theta)$, we introduce an exponential moment condition on the stochastic component $\zeta(\theta)$:

$$\zeta(\theta) \overset{\text{def}}{=} L(\theta) - EL(\theta).$$

Below we suppose that the random function $\zeta(\theta)$ is differentiable in $\theta$ and its gradient $\nabla \zeta(\theta) = \partial \zeta(\theta)/\partial \theta \in \mathbb{R}^p$ has some exponential moments. Our first condition describes the property of the gradient $\nabla \zeta(\theta^*)$ at the central point $\theta^*$. 
There exist a positive symmetric matrix $V_0^2$, and constants $g > 0, \nu_0 \geq 1$ such that $\text{Var}\left\{ \nabla \zeta(\theta^*) \right\} \leq V_0^2$ and for all $|\lambda| \leq g$

$$\sup_{\gamma \in \mathbb{R}^p} \log \mathbb{E} \exp\left\{ \lambda \frac{\gamma^T \nabla \zeta(\theta^*)}{\|V_0 \gamma\|} \right\} \leq \nu_0^2 \lambda^2 / 2.$$ 

In typical situation, the matrix $V_0^2$ can be defined as the covariance matrix of the gradient vector $\nabla \zeta(\theta^*)$: $V_0^2 = \text{Var}(\nabla \zeta(\theta^*)) = \text{Var}(\nabla L(\theta^*))$. If $L(\theta)$ is the log-likelihood for a correctly specified model, then $\theta^*$ is the true parameter value and $V_0^2$ coincides with the corresponding Fisher information matrix. The matrix $V_0$ shown in this condition determines the local geometry in the vicinity of $\theta^*$. In particular, define the local elliptic neighborhoods of $\theta^*$ as

$$\Theta_0(\mathbf{r}) \overset{\text{def}}{=} \{ \theta \in \Theta : \|V_0(\theta - \theta^*)\| \leq \mathbf{r} \}. \quad (2.1)$$

The further conditions are restricted to such defined neighborhoods $\Theta_0(\mathbf{r})$.

**ED1** For each $\mathbf{r} \leq \mathbf{r}_0$, there exist a constant $\omega(\mathbf{r}) \leq 1/2$ such that it holds for all $\theta \in \Theta_0(\mathbf{r})$

$$\sup_{\gamma \in \mathbb{R}^p} \log \mathbb{E} \exp\left\{ \lambda \frac{\gamma^T \{ \nabla \zeta(\theta) - \nabla \zeta(\theta^*) \}}{\omega(\mathbf{r}) \|V_0 \gamma\|} \right\} \leq \nu_0^2 \lambda^2 / 2, \quad |\lambda| \leq g.$$ 

The main bracketing result also requires second order smoothness of the expected log-likelihood $\mathbb{E} L(\theta)$. By definition, $L(\theta^*, \theta^*) \equiv 0$ and $\nabla \mathbb{E} L(\theta^*) = 0$ because $\theta^*$ is the extreme point of $\mathbb{E} L(\theta)$. Therefore, $-\mathbb{E} L(\theta, \theta^*)$ can be approximated by a quadratic function of $\theta - \theta^*$ in the neighborhood of $\theta^*$. The local identifiability condition quantifies this quadratic approximation from above and from below on the set $\Theta_0(\mathbf{r})$ from (2.1).

**L0** There are a symmetric strictly positive-definite matrix $D_0^2$ and for each $\mathbf{r} \leq \mathbf{r}_0$ and a constant $\delta(\mathbf{r}) \leq 1/2$, such that it holds on the set $\Theta_0(\mathbf{r}) = \{ \theta : \|V_0(\theta - \theta^*)\| \leq \mathbf{r} \}$

$$\left| \frac{-2 \mathbb{E} L(\theta, \theta^*)}{\|D_0(\theta - \theta^*)\|^2} - 1 \right| \leq \delta(\mathbf{r}).$$

Usually $D_0^2$ is defined as the negative Hessian of $\mathbb{E} L(\theta^*)$: $D_0^2 = -\nabla^2 \mathbb{E} L(\theta^*)$. If $L(\theta, \theta^*)$ is the log-likelihood ratio and $\mathbb{P} = \mathbb{P}_{\theta^*}$ then $-\mathbb{E} L(\theta, \theta^*) = \mathbb{E}_{\theta^*} \log(d\mathbb{P}_{\theta^*} / d\mathbb{P}_{\theta}) = \mathcal{K}(\mathbb{P}_{\theta^*}, \mathbb{P}_{\theta})$, the Kullback-Leibler divergence between $\mathbb{P}_{\theta^*}$ and $\mathbb{P}_{\theta}$. Then condition (L0) with $D_0 = V_0$ follows from the usual regularity conditions on the family $(\mathbb{P}_{\theta})$; cf. Ibragimov and Khas’minskij (1981). If the log-likelihood process $L(\theta)$ is sufficiently
smooth in θ, e.g. three times stochastically differentiable, then the quantities ω(r) and δ(r) can be taken proportional to the value ω(r) defined as

\[ \omega(r) \overset{\text{def}}{=} \max_{\theta \in \Theta_0(r)} \| \theta - \theta^* \|. \]

In the important special case of an i.i.d. model one can take \( \omega(r) = \omega^* r / \sqrt{n} \) and \( \delta(r) = \delta^* r / \sqrt{n} \) for some constants \( \omega^*, \delta^* \); see Section 5.1.

The identifiability condition relates the matrices \( D_0^2 \) and \( V_0^2 \).

(I) There is a constant \( a > 0 \) such that \( a^2 D_0^2 \geq V_0^2 \).

2.2 Global conditions

The global conditions have to be fulfilled for all \( \theta \) lying beyond \( \Theta_0(r_0) \). We only impose one condition on the smoothness of the stochastic component of the process \( L(\theta) \) in terms of its gradient, and one identifiability condition in terms of the expectation \( \mathbb{E}L(\theta, \theta^*) \).

The first condition is similar to the local condition \( (ED_0) \) and it requires some exponential moment of the gradient \( \nabla \zeta(\theta) \) for all \( \theta \in \Theta \). However, the constant \( g \) may be dependent of the radius \( r = \| V_0(\theta - \theta^*) \| \).

(\( E_r \)) For any \( r \), there exists a value \( g(r) > 0 \) such that for all \( \lambda \leq g(r) \)

\[ \sup_{\theta \in \Theta_0(r)} \sup_{\gamma \in \mathbb{R}^p} \log \mathbb{E} \exp \left\{ \lambda \gamma^\top \nabla \zeta(\theta) \| V_0 \gamma \| \right\} \leq \nu_0^2 \lambda^2 / 2. \]

The global identification property means that the deterministic component \( \mathbb{E}L(\theta, \theta^*) \) of the log-likelihood is competitive with its variance \( \text{Var} L(\theta, \theta^*) \).

(\( L_r \)) There is a function \( b(r) \) such that \( r b(r) \) monotonously increases in \( r \) and for each \( r \geq r_0 \)

\[ \inf_{\theta : \| V_0(\theta - \theta^*) \| = r} | \mathbb{E}L(\theta, \theta^*) | \geq b(r) r^2. \]

3 Local inference

The Local Asymptotic Normality (LAN) condition since introduced in Le Cam (1960) became one of the central notions in the statistical theory. It postulates a kind of local approximation of the log-likelihood of the original model by the log-likelihood of a Gaussian shift experiment. The LAN property being once checked yields a number of important corollaries for statistical inference. In words, if you can solve a statistical problem
for the Gaussian shift model, the result can be translated under the LAN condition to the original setup. We refer to Ibragimov and Khas’minskij (1981) for a nice presentation of the LAN theory including asymptotic efficiency of MLE and Bayes estimators. The LAN property was extended to mixed LAN or Local Asymptotic Quadraticity (LAQ); see e.g. Le Cam and Yang (2000). All these notions are very much asymptotic and very much local. The LAN theory also requires that $L(\theta)$ is the correctly specified log-likelihood. The strict localization does not allow for considering a growing or infinite parameter dimension and limits applications of the LAN theory to nonparametric estimation.

Our approach tries to avoid asymptotic constructions and attempts to include a possible model misspecification and a large dimension of the parameter space. The presentation below shows that such an extension of the LAN theory can be made essentially by no price: all the major asymptotic results like Fisher and Cramér-Rao information bounds, as well as the Wilks phenomenon can be derived as corollaries of the obtained non-asymptotic statements simply by letting the sample size to infinity. At the same time, it applies to a high dimensional parameter space.

The LAN property states that the considered process $L(\theta)$ can be approximated by a quadratic in $\theta$ expression in a vicinity of the central point $\theta^*$. This property is usually checked using the second order Taylor expansion. The main problem arising here is that the error of the approximation grows too fast with the local size of the neighborhood. Section 3.1 presents the non-asymptotic version of the LAN property in which the local quadratic approximation of $L(\theta)$ is replaced by bounding this process from above and from below by two different quadratic in $\theta$ processes. More precisely, we apply the bracketing idea: the difference $L(\theta, \theta^*) = L(\theta) - L(\theta^*)$ is put between two quadratic processes $L_\varepsilon(\theta, \theta^*)$ and $L_\varepsilon(\theta, \theta^*)$:

$$L_\varepsilon(\theta, \theta^*) - \diamond_\varepsilon \leq L(\theta, \theta^*) \leq L_\varepsilon(\theta, \theta^*) + \diamond_\varepsilon, \quad \theta \in \Theta_0(r),$$

where $\varepsilon$ is a numerical parameter, $\varepsilon = -\varepsilon$, and $\diamond_\varepsilon$ and $\diamond_\varepsilon$ are stochastic errors which only depend on the selected vicinity $\Theta_0(r)$. The upper process $L_\varepsilon(\theta, \theta^*)$ and the lower process $L_\varepsilon(\theta, \theta^*)$ can deviate substantially from each other, however, the errors $\diamond_\varepsilon, \diamond_\varepsilon$ remain small even if the value $r$ describing the size of the local neighborhood $\Theta_0(r)$ is large.

The sandwiching result (3.1) naturally leads to two important notions: the value of the problem and the spread. It turns out that most of the statements like confidence and concentration probability rely upon the maximum of $L(\theta, \theta^*)$ over $\theta$ which we call the excess. Its expectation will be referred to as the value of the problem. Due to (3.1) the excess can be bounded from above and from below using the similar quantities $\max_\theta L_\varepsilon(\theta, \theta^*)$ and $\max_\theta L_\varepsilon(\theta, \theta^*)$ which can be called the lower and upper excess.
while their expectations are the values of the lower and upper problems. Note that
\[
\max_\theta \{ \mathbb{L}_e(\theta, \theta^*) - \mathbb{L}_e(\theta, \theta^*) \} = \infty.
\]
However, this is not crucial. What really matters is the difference between the upper and the lower excess. The spread \( \Delta_e \) can be defined as the width of the interval bounding the excess due to (3.1), that is, as the sum of the approximation errors and of the difference between the upper and the lower excess:

\[
\Delta_e \overset{\text{def}}{=} \diamond_e + \diamond_{\xi} + \{ \max_\theta \mathbb{L}_e(\theta, \theta^*) - \max_\theta \mathbb{L}_e(\theta, \theta^*) \}.
\]

The range of applicability of this approach can be described by the following mnemonic rule: “The value of the upper problem is larger in order than the spread.” The further sections explain in details the meaning and content of this rule. Section 3.1 presents the key bound (3.1) and derives it from the general results on empirical processes. Section 3.2 presents some straightforward corollaries of the bound (3.1) including the coverage and concentration probabilities, expansion of the MLE and the risk bounds. It also indicates how the classical results on asymptotic efficiency of the MLE follow from the obtained non-asymptotic bounds.

### 3.1 Local quadratic bracketing

This section presents the key result about local quadratic approximation of the quasi log-likelihood process given by Theorem 3.1 below.

Let the radius \( r \) of the local neighborhood \( \Theta_0(r) \) be fixed in a way that the deviation probability \( P(\hat{\theta} \not\in \Theta_0(r)) \) is sufficiently small. Precise results about the choice of \( r \) which ensures this property are postponed until Section 4. In this neighborhood \( \Theta_0(r) \) we aim at building some quadratic lower and upper bounds for the process \( L(\theta) \). The first step is the usual decomposition of this process into deterministic and stochastic components:

\[
L(\theta) = \mathbb{E}L(\theta) + \zeta(\theta),
\]

where \( \zeta(\theta) = L(\theta) - \mathbb{E}L(\theta) \). Condition \((\mathcal{L}_0)\) allows to approximate the smooth deterministic function \( \mathbb{E}L(\theta) - \mathbb{E}L(\theta^*) \) around the point of maximum \( \theta^* \) by the quadratic form \( -\|D_0(\theta - \theta^*)\|^2/2 \). The smoothness properties of the stochastic component \( \zeta(\theta) \) given by conditions \((\mathcal{E}D_0)\) and \((\mathcal{E}D_1)\) leads to linear approximation \( \zeta(\theta) - \zeta(\theta^*) \approx (\theta - \theta^*)^\top \nabla \zeta(\theta^*) \). Putting these two approximations together yields the following approximation of the process \( L(\theta) \) on \( \Theta_0(r) \):

\[
L(\theta, \theta^*) \approx \mathbb{L}(\theta, \theta^*) \overset{\text{def}}{=} (\theta - \theta^*)^\top \nabla \zeta(\theta^*) - \|D_0(\theta - \theta^*)\|^2/2. \quad (3.2)
\]

This expansion is used in most of statistical calculus. However, it does not suit our purposes because the error of approximation grows quadratically with the radius \( r \) and
starts to dominate at some critical value of \( r \). We slightly modify the construction by introducing two different approximating processes. They only differ in the deterministic quadratic term which is either shrunk or stretched relative to the term \( \|D_0(\theta - \theta^*)\|^2/2 \) in \( L(\theta, \theta^*) \).

Let \( \delta, \varrho \) be nonnegative constants. Introduce for a vector \( \epsilon = (\delta, \varrho) \) the following notation:

\[
L_\epsilon(\theta, \theta^*) \overset{\text{def}}{=} (\theta - \theta^*)^\top \nabla L(\theta^*) - \|D_\epsilon(\theta - \theta^*)\|^2/2 = \xi_\epsilon^\top D_\epsilon(\theta - \theta^*) - \|D_\epsilon(\theta - \theta^*)\|^2/2, \tag{3.3}
\]

where

\[
D_\epsilon^2 = D_0^2(1 - \delta) - \varrho V_0^2, \quad \xi_\epsilon \overset{\text{def}}{=} D_\epsilon^{-1} \nabla L(\theta^*).
\]

Here we implicitly assume that with the proposed choice of the constants \( \delta \) and \( \varrho \), the matrix \( D_\epsilon^2 \) is non-negative: \( D_\epsilon^2 \geq 0 \). The representation (3.3) indicates that the process \( L_\epsilon(\theta, \theta^*) \) has the geometric structure of log-likelihood of a linear Gaussian model. We do not require that the vector \( \xi_\epsilon \) is Gaussian and hence, it is not the Gaussian log-likelihood. However, the geometric structure of this process appears to be more important than its distributional properties.

One can see that if \( \delta, \varrho \) are positive, the quadratic drift component of the process \( L_\epsilon(\theta, \theta^*) \) is shrunk relative to \( L(\theta, \theta^*) \) in (3.2) for \( \epsilon \) positive and it is stretched if \( \delta, \varrho \) are negative. Now, given \( r \), define \( \delta = \delta(r), \ \varrho = 3\nu_0 \omega(r) \) with the value \( \delta(r) \) from condition \( (L_0) \) and \( \omega(r) \) from condition \( (ED_1) \). Finally set \( \epsilon = -\epsilon \), so that

\[
D_\epsilon^2 = D_0^2(1 + \delta) + \varrho V_0^2.
\]

**Theorem 3.1.** Assume \( (ED_1) \) and \( (L_0) \). Let for some \( r \), the values \( \varrho \geq 3\nu_0 \omega(r) \) and \( \delta \geq \delta(r) \) be such that \( D_0^2(1 - \delta) - \varrho V_0^2 \geq 0 \). Then

\[
L_\epsilon(\theta, \theta^*) - \hat{\ell}_\epsilon(r) \leq L(\theta, \theta^*) \leq L_\epsilon(\theta, \theta^*) + \hat{\ell}_\epsilon(r), \quad \theta \in \Theta_0(r), \tag{3.4}
\]

with \( L_\epsilon(\theta, \theta^*), L_\epsilon(\theta, \theta^*) \) defined by (3.3). The error terms \( \hat{\ell}_\epsilon(r) \) and \( \hat{\ell}_\epsilon(r) \) satisfy the bound (3.11) from Proposition 3.7.

**Remark 3.1.** This bracketing bound (3.4) describes some properties of the log-likelihood process and the estimator \( \tilde{\theta} \) is not shown there. However, it directly implies most of our inference results. We therefore formulate it as a separate statement. Section 3.3 below presents some exponential bounds on the error terms \( \hat{\ell}_\epsilon(r) \) and \( \hat{\ell}_\epsilon(r) \). The main message is that under rather broad conditions, these errors are small and have only minor impact on the inference for the quasi MLE \( \tilde{\theta} \).
3.2 Local inference

This section presents a list of corollaries from the basic approximation bounds of Theorem 3.1. The idea is to replace the original problem by a similar one for the approximating upper and lower models. It is important to stress once again that all the corollaries only rely on the bracketing result (3.1) and the geometric structure of the processes \( L_\epsilon \) and \( L_\xi \). Define the spread \( \Delta_\epsilon(r) \) by

\[
\Delta_\epsilon(r) \overset{\text{def}}{=} \Diamond_\epsilon(r) + \Diamond_\xi(r) + \left( \|\xi_\epsilon\|^2 - \|\xi_\xi\|^2 \right) / 2.
\]

Here \( \xi_\epsilon = D_\epsilon^{-1} \nabla L(\theta^*) \) and \( \xi_\xi = D_\xi^{-1} \nabla L(\theta^*) \). The quantity \( \Delta_\epsilon(r) \) appears to be the price induced by our bracketing device. Section 3.3 below presents some probabilistic bounds on the spread showing that it is small relative to the other terms. All our corollaries below are stated under conditions of Theorem 3.1 and implicitly assume that the spread can be nearly ignored.

3.2.1 Local coverage probability

Our first result describes the probability of covering \( \theta^* \) by the random set

\[
\mathcal{E}(\mathfrak{z}) = \{ \theta : 2L(\tilde{\theta}, \theta) \leq \mathfrak{z} \}.
\]

**Corollary 3.2.** For any \( \mathfrak{z} > 0 \)

\[
\mathbb{P}\{ \mathcal{E}(\mathfrak{z}) \not\ni \theta^*, \tilde{\theta} \in \Theta_0(r) \} \leq \mathbb{P}\{ \|\xi_\|\|^2 \geq \mathfrak{z} - \Diamond_\epsilon(r) \}.
\]

**Proof.** The bound (3.7) follows from the upper bound of Theorem 3.1 and the statement (3.12) of Lemma 3.8 below.

Below; see (3.14), we also present an exponential bound which helps to answer a very important question about a proper choice of the critical value \( \mathfrak{z} \) ensuring a prescribed covering probability.

3.2.2 Local expansion, Wilks theorem, and local concentration

Now we show how the bound (3.4) can be used for obtaining a local expansion of the quasi MLE \( \tilde{\theta} \). All our results will be conditioned to the random set \( C_\epsilon(r) \) defined as

\[
C_\epsilon(r) \overset{\text{def}}{=} \{ \tilde{\theta} \in \Theta_0(r), \|V_0D_\xi^{-1}\xi_\xi\| \leq r \}.
\]

Below in Section 3.3 we present some upper bounds on the value \( r \) ensuring a dominating probability of this random set.

The first result can be viewed as a finite sample version of the famous Wilks theorem.
Corollary 3.3. On the random set $C_{\epsilon}(r)$ from (3.8), it holds
\[ \|\xi_{\epsilon}\|^2/2 - \diamondsuit_{\epsilon}(r) \leq L(\tilde{\theta}, \theta^*) \leq \|\xi_{\epsilon}\|^2/2 + \diamondsuit_{\epsilon}(r). \] (3.9)

The next result is an extension of another prominent asymptotic result, namely, the Fisher expansion of the MLE.

Corollary 3.4. On the random set $C_{\epsilon}(r)$ from (3.8), it holds
\[ \|D_{\epsilon}(\tilde{\theta} - \theta^*) - \xi_{\epsilon}\|^2 \leq 2\Delta_{\epsilon}(r). \] (3.10)

The proof of Corollaries 3.3 and 3.4 relies on the solution of the upper and lower problems and it is given below at the end of this section.

Now we describe concentration properties of $\tilde{\theta}$ assuming that $\tilde{\theta}$ is restricted to $\Theta_0(r)$. More precisely, we bound the probability that $\|D_{\epsilon}(\tilde{\theta} - \theta^*)\| > z$ for a given $z > 0$.

Corollary 3.5. For any $z > 0$, it holds
\[ \mathbb{P}\{\|D_{\epsilon}(\tilde{\theta} - \theta^*)\| > z, C_{\epsilon}(r)\} \leq \mathbb{P}\{\|\xi_{\epsilon}\| > z - \sqrt{2\Delta_{\epsilon}(r)}\} \]

An interesting and important question is for which $z$ in (3.6) the coverage probability of the event $\{z(3) \supset \Theta^\star\}$ or for which $z$, the concentration probability of the event $\{\|D_{\epsilon}(\tilde{\theta} - \theta^*)\| \leq z\}$ becomes close to one. It will be addressed in Section 3.3.

3.2.3 A local risk bound

Below we also bound the moments of the excess $L(\tilde{\theta}, \theta^*)$ and of the normalized loss $D_{\epsilon}(\tilde{\theta} - \theta^*)$ when $\tilde{\theta}$ is restricted to $\Theta_0(r)$. The result follows directly from Corollaries 3.3 and 3.4.

Corollary 3.6. For $u > 0$
\[ \mathbb{E}\{L^u(\tilde{\theta}, \theta^*) \mathbb{I}(\tilde{\theta} \in \Theta_0(r))\} \leq \mathbb{E}\{\|\xi_{\epsilon}\|^2/2 + \diamondsuit_{\epsilon}(r)\}^u\].

Moreover, it holds
\[ \mathbb{E}\{|D_{\epsilon}(\tilde{\theta} - \theta^*)|^u \mathbb{I}(C_{\epsilon}(r))\} \leq \mathbb{E}\{\|\xi_{\epsilon}\| + \sqrt{2\Delta_{\epsilon}(r)}\}^u\].

3.2.4 Comparing with the asymptotic theory

This section briefly discusses the relation between the established non-asymptotic bounds and the classical asymptotic results in parametric estimation. This comparison is not
straightforward because the asymptotic theory involves the sample size or noise level as the asymptotic parameter, while our setup is very general and works even for a “single” observation. Here we simply treat $\epsilon = (\delta, \varrho)$ as a small parameter. This is well justified by the i.i.d. case with $n$ observations, where it holds $\delta = \delta(x) \asymp \sqrt{x/n}$ and similarly for $\varrho$; see Section 5 for more details. The bounds below in Section 3.3 show that the spread $\Delta_\epsilon(x)$ from (3.5) is small and can be ignored in the asymptotic calculations. The results of Corollary 3.2 through 3.6 represent the desired bounds in terms of deviation bounds for the quadratic form $\|\xi_\epsilon\|^2$.

For better understanding the essence of the presented results, consider first the “true” parametric model with the correctly specified log-likelihood $L(\theta)$. Then $D_0^2 = V_0^2$ is the total Fisher information matrix. In the i.i.d. case it becomes $nf_0$ where $f_0$ is the usual Fisher information matrix of the considered parametric family at $\theta^*$. In particular, $\text{Var}\{\nabla L(\theta^*)\} = nf_0$. So, if $D_\epsilon$ is close to $D_0$, then $\xi_\epsilon$ can be treated as the normalized score. Under usual assumptions, $\xi_\epsilon \overset{\text{def}}{=} D_0^{-1}\nabla L(\theta^*)$ is asymptotically standard normal $p$-vector. The same applies to $\xi_\epsilon$. Now one can observe that Corollary 3.2 through 3.6 directly imply most of classical asymptotic statements. In particular, Corollary 3.3 shows that the twice excess $2L(\tilde{\theta}, \theta^*)$ is nearly $\|\xi_\epsilon\|^2$ and thus nearly $\chi_p^2$ (Wilks Theorem). Corollary 3.4 yields the expansion $D_\epsilon(\tilde{\theta} - \theta^*) \approx \xi_\epsilon$ (the Fisher expansion) and hence, $D_\epsilon(\tilde{\theta} - \theta^*)$ is asymptotically standard normal. Asymptotic variance of $D_\epsilon(\tilde{\theta} - \theta^*)$ is nearly one, so $\tilde{\theta}$ achieves the Cramér-Rao efficiency bound in the asymptotic set-up.

### 3.3 Spread

This section presents some bounds on the value

$$\Delta_\epsilon(x) \overset{\text{def}}{=} \diamond_\epsilon(x) + \diamond_\xi(x) + (\|\xi_\epsilon\|^2 - \|\xi_\xi\|^2)/2.$$ 

This quantity is random but it can be easily evaluated under the conditions made. We present two different results: one bounds the errors $\diamond_\epsilon(x), \diamond_\xi(x)$, while the other presents a deviation bound on quadratic forms like $\|\xi_\epsilon\|^2$. The results are stated under conditions $(ED_0)$ and $(ED_1)$ in a non-asymptotic way, so the formulation is quite technical. An informal discussion at the end of this section explains the typical behavior of the spread. The first result accomplishes the bracketing bound (3.4).

**Proposition 3.7.** Assume $(ED_1)$. The error term $\diamond_\epsilon(x)$ in (3.4) fulfills

$$\mathbb{P}\{q^{-1}\diamond_\epsilon(x) \geq \delta_0(x, Q)\} \leq \exp(-x) \quad (3.11)$$
with $\zeta_0(x, Q)$ given for $g_0 = g\nu_0 \geq 3$ by

$$
\zeta_0(x, Q) \overset{\text{def}}{=} \begin{cases}
(1 + \sqrt{x + Q})^2 & \text{if } 1 + \sqrt{x + Q} \leq g_0, \\
1 + \{2g_0^{-1}(x + Q) + g_0\}^2 & \text{otherwise},
\end{cases}
$$

where $Q = c_1p$ with $c_1 = 2$ for $p \geq 2$ and $c_1 = 2.7$ for $p = 1$. Similarly for $\zeta_\epsilon(x)$.

**Remark 3.2.** The bound (3.11) essentially depends on the value $g$ from condition $(ED_1)$. The result requires that $g\nu_0 \geq 3$. However, this constant can usually be taken of order $n^{1/2}$; see Section 5 for examples. If $g^2$ is larger in order than $p + x$, then $\zeta_0(x, Q) \approx c_1 p + x$.

**Proof.** Consider for fixed $x$ and $\epsilon = (\delta, \theta)$ the quantity

$$
\diamondsuit_\epsilon(x) \overset{\text{def}}{=} \sup_{\theta \in \Theta_0(x)} \{ L(\theta, \theta^*) - IE L(\theta, \theta^*) - (\theta - \theta^*)^\top \nabla L(\theta^*) - \frac{\theta}{2} \| V_0(\theta - \theta^*) \|^2 \}.
$$

As $\delta \geq \delta(x)$, it holds $-IE L(\theta, \theta^*) \geq (1 - \delta)D_0^2$ and $L(\theta, \theta^*) - L_\epsilon(\theta, \theta^*) \leq \diamondsuit_\epsilon(x)$. Moreover, in view of $\nabla IE L(\theta^*) = 0$, the definition of $\diamondsuit_\epsilon(x)$ can be rewritten in the form

$$
\diamondsuit_\epsilon(x) \overset{\text{def}}{=} \sup_{\theta \in \Theta_0(x)} \{ \zeta(\theta, \theta^*) - (\theta - \theta^*)^\top \nabla \zeta(\theta^*) - \frac{\theta}{2} \| V_0(\theta - \theta^*) \|^2 \}.
$$

Now the claim of the theorem can be easily reduced to an exponential bound for the quantity $\diamondsuit_\epsilon(x)$. We apply Theorem B.12 to the process

$$
\mathcal{U}(\theta, \theta^*) = \frac{1}{\omega(x)} \{ \zeta(\theta, \theta^*) - (\theta - \theta^*)^\top \nabla \zeta(\theta^*) \}, \quad \theta \in \Theta_0(x),
$$

and $H_0 = V_0$. Condition (3E) follows from $(ED_1)$ with the same $\nu_0$ and $g$ in view of $\nabla \mathcal{U}(\theta, \theta^*) = \{ \nabla \zeta(\theta) - \nabla \zeta(\theta^*) \}/\omega(x)$. So, the conditions of Theorem B.12 are fulfilled yielding (3.11) in view of $\theta \geq 3\nu_0 \omega(x)$. \hfill \Box

Due to the main bracketing result, the local excess $\sup_{\theta \in \Theta_0(x)} \mathbb{L}_\epsilon(\theta, \theta^*)$ can be put between similar quantities for the upper and lower approximating processes up to the error term $\diamondsuit_\epsilon^*(x)$. The random quantity $\sup_{\theta} \mathbb{L}_\epsilon(\theta, \theta^*)$ can be called the upper excess while $\sup_{\theta} \mathbb{L}_\epsilon^*(\theta, \theta^*)$ is the lower excess. The quadratic (in $\theta$) structure of the functions $\mathbb{L}_\epsilon(\theta, \theta^*)$ and $\mathbb{L}_\epsilon^*(\theta, \theta^*)$ enables us to explicitly solve the problem of maximizing the corresponding function w.r.t. $\theta$. Next result describes the upper and lower excesses $\sup_{\theta} \mathbb{L}_\epsilon(\theta, \theta^*)$.

**Lemma 3.8.** It holds

$$
\sup_{\theta \in \mathbb{R}^p} \mathbb{L}_\epsilon(\theta, \theta^*) = \| \xi_\epsilon \|^2 / 2.
$$

(3.12)
On the random set \( \{ \| V D_\xi^{-1} \xi_\xi \| \leq r \} \), it also holds

\[
\sup_{\theta \in \Theta_0(r)} L_\epsilon(\theta, \theta) = \| \xi_\xi \|^2 / 2.
\]

**Proof.** The unconstrained maximum of the quadratic form \( L_\epsilon(\theta, \theta^*) \) w.r.t. \( \theta \) is attained at \( \tilde{\theta}_\epsilon = D_\epsilon^{-1} \xi_\epsilon = D_\epsilon^{-2} \nabla L(\theta^*) \) yielding the expression (3.12). The lower excess is computed similarly.

Our next step is in bounding the difference \( \| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 \). It can be decomposed as

\[
\| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 = \| \xi_\epsilon \|^2 - \| \xi_\epsilon \|^2 + \| \xi_\epsilon \|^2 - \| \xi_\xi \|^2
\]

with \( \xi = D_0^{-1} \nabla L(\theta^*) \). If the values \( \delta, \vartheta \) are small then the difference \( \| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 \) is automatically smaller than \( \| \xi_\epsilon \|^2 \).

**Lemma 3.9.** Suppose \((\mathcal{I})\) and let \( \tau_\epsilon \defeq \delta + \vartheta \alpha^2 < 1 \). Then

\[
D_\epsilon^2 \geq (1 - \tau_\epsilon) D_0^2, \quad D_\xi^2 \leq (1 + \tau_\epsilon) D_0^2,
\]

\[
\| I_p - D_\epsilon D_\xi^{-2} D_\epsilon \| \leq \alpha_\epsilon \defeq \frac{2\tau_\epsilon}{1 - \tau_\epsilon}.
\]

Moreover,

\[
\| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 \leq \frac{\tau_\epsilon}{1 - \tau_\epsilon} \| \xi_\epsilon \|^2, \quad \| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 \leq \frac{\tau_\epsilon}{1 + \tau_\epsilon} \| \xi_\epsilon \|^2,
\]

\[
\| \xi_\epsilon \|^2 - \| \xi_\xi \|^2 \leq \alpha_\epsilon \| \xi_\epsilon \|^2.
\]

Our final step is in showing that under \((\mathcal{E}D_0)\), the norm \( \| \xi_\| \) behaves essentially as a norm of a Gaussian vector with the same covariance matrix. Define for \( \mathcal{B} \defeq D_0^{-1} V_2 D_0^{-1} \)

\[
p_0 \defeq \text{tr}(\mathcal{B}), \quad v^2 \defeq 2 \text{tr}(\mathcal{B}^2), \quad \lambda_0 \defeq \| \mathcal{B} \|_\infty = \lambda_{\max}(\mathcal{B}).
\]

Under the identifiability condition \((\mathcal{I})\), one can bound

\[
\mathcal{B}^2 \leq \alpha^2 I_p, \quad p_0 \leq \alpha^2 p, \quad v^2 \leq 2 \alpha^4 p, \quad \lambda_0 \leq \alpha^2.
\]

Similarly to the previous result, we assume that the constant \( g \) from condition \((\mathcal{E}D_0)\) is sufficiently large, namely \( g^2 \geq 2 p_0 \). Define \( \mu_c = 2/3 \) and

\[
y^2 \defeq g^2 / \mu_c^2 - p_0 / \mu_c,
\]

\[
g_c \defeq \mu_c y_c = \sqrt{g^2 - \mu_c p_0},
\]

\[
2 n_c \defeq \mu_c y_c^2 + \log \det(I_p - \mu_c \mathcal{B}^2 / \lambda_0).
\]

It is easy to see that \( y^2_c \geq 3 g^2 / 2 \) and \( g_c \geq \sqrt{2/3} g \).
Theorem 3.10. Let \((ED_0)\) hold with \(\nu_0 = 1\) and \(g^2 \geq 2p_0\). Then \(\mathbb{E}\|\xi\|^2 \leq p_0\), and for each \(x \leq x_c\)

\[
\mathbb{P}\left(\|\xi\|^2/\lambda_0 \geq \mathfrak{z}(x, B)\right) \leq 2e^{-x} + 8.4e^{-x_c},
\]

where \(\mathfrak{z}(x, B)\) is defined by

\[
\mathfrak{z}(x, B) = \begin{cases} 
p_0 + 2\nu x^{1/2}, & x \leq \nu/18, 
p_0 + 6x & \nu/18 < x \leq x_c,
\end{cases}
\]

Moreover, for \(x > x_c\), it holds with \(\mathfrak{z}(x, B) = |yc + 2(x - x_c)/g_c|^2\)

\[
\mathbb{P}\left(\|\xi\|^2/\lambda_0 \geq \mathfrak{z}(x, B)\right) \leq 8.4e^{-x}.
\]

Proof. It follows from condition \((ED_0)\) that

\[
\mathbb{E}\|\xi\|^2 = \mathbb{E} \text{tr} \xi^\top = \text{tr} D_0^{-1}\left[\mathbb{E}\nabla L(\theta^*)\nabla L(\theta^*)^\top\right] D_0^{-1} = \text{tr} \left[D_0^{-2} \text{Var}\{\nabla L(\theta^*)\}\right]
\]

and \((ED_0)\) implies \(\gamma^\top \text{Var}\{\nabla L(\theta^*)\} \gamma \leq \gamma^\top V_0^2 \gamma\) and thus, \(\mathbb{E}\|\xi\|^2 \leq p_0\). The deviation bound (3.14) is proved in Corollary A.12. \(\square\)

Remark 3.3. This small remark concerns the term \(8.4e^{-x_c}\) in the probability bound (3.14). As already mentioned, this bound implicitly assumes that the constant \(g\) is large (usually \(g \asymp n^{1/2}\)). Then \(x_c \asymp g^2 \asymp n\) is large as well. So, \(e^{-x_c}\) is very small and asymptotically negligible. Below we often ignore this term. For \(x \leq x_c\), we can use \(\mathfrak{z}(x, B) = p_0 + 6x\).

Remark 3.4. The exponential bound of Theorem 3.10 helps to describe the critical value of \(\mathfrak{z}\) ensuring a prescribed deviation probability \(\mathbb{P}\left(\|\xi\|^2 \geq \mathfrak{z}\right)\). Namely, this probability starts to gradually decrease when \(\mathfrak{z}\) grows over \(\lambda_0 p_0\). In particular, this helps to answer a very important question about a proper choice of the critical value \(\mathfrak{z}\) providing the prescribed covering probability, or of the value \(z\) ensuring the dominating concentration probability \(\mathbb{P}\left(\|D\tilde{\theta} - \theta^*\|^2 \leq z\right)\).

The definition of the set \(C_\epsilon(x)\) from (3.8) involves the event \(\{\|V_0 D^{-1}_\xi \xi\_\xi > r\}\). Under \((\mathcal{I})\), it is included in the set \(\{\|\xi\| > (1 + \alpha_\epsilon)^{-1}a^{-1}r\}\), see (3.13), and its probability is of order \(e^{-x}\) for \(r^2 \geq C(x + p)\) with a fixed \(C > 0\).

By Theorem 3.7, one can use \(\max\{\Diamond_\epsilon(x), \Diamond_\xi(x)\} \leq \mathfrak{z}_0(x, Q)\) on a set of probability at least \(1 - 2e^{-x}\). Further, \(\|\xi\|^2/\lambda_0 \leq \mathfrak{z}(x, B)\) with a probability of order \(1 - 2e^{-x}\);
see (3.14). Putting together the obtained bounds yields for the spread $\Delta_\epsilon(x)$ with a probability about $1 - 4e^{-x}$

$$\Delta_\epsilon(x) \leq 2p \hat{y}_0(x, Q) + \alpha_\epsilon \lambda_0 \hat{y}(x, B).$$

The results obtained in Section 3.2 are sharp and meaningful if the spread $\Delta_\epsilon(x)$ is smaller in order than the value $E\left\|\xi\right\|^2$. Theorem 3.10 states that $\left\|\xi\right\|^2$ does not significantly deviate over its expected value $p_0 \overset{\text{def}}{=} E\left\|\xi\right\|^2$ which is our leading term. We know that $\hat{y}_0(x, Q) \approx p + x = c_1 p + x$ if $x$ is not too large. Also $\hat{y}(x, B) \leq p_0 + 6x$, where $p_0$ is of order $p$ due to (I). Summarizing the above discussion yields that the local results apply if the regularity condition (I) holds and the values $\varrho$ and $\alpha_\epsilon$, or equivalently, $\omega(x), \delta(x)$ are small. In Section 5 we show for the i.i.d. example that $\omega(x) \approx \sqrt{r^2/n}$ and similarly for $\delta(x)$.

### 3.4 Proof of Corollaries 3.3 and 3.4

The bound (3.4) together with Lemma 3.8 yield on $C_\epsilon(x)$

$$L(\tilde{\theta}, \theta^*) = \sup_{\theta \in \Theta_0(\epsilon)} L(\theta, \theta^*) \geq \sup_{\theta \in \Theta_0(\epsilon)} L_{\epsilon}(\theta, \theta^*) - \diamond_{\epsilon}(x) = \left\|\xi_{\epsilon}\right\|^2/2 - \diamond_{\epsilon}(x). \quad (3.15)$$

Similarly

$$L(\tilde{\theta}, \theta^*) \leq \sup_{\theta \in \Theta_0(\epsilon)} L_{\epsilon}(\theta, \theta^*) + \diamond_{\epsilon}(x) \leq \left\|\xi_{\epsilon}\right\|^2/2 + \diamond_{\epsilon}(x)$$

yielding (3.9). For getting (3.10), we again apply the inequality $L(\theta, \theta^*) \leq L_{\epsilon}(\theta, \theta^*) + \diamond_{\epsilon}(x)$ from Theorem 3.1 for $\theta$ equal to $\tilde{\theta}$. With $\xi_{\epsilon} = D_{\epsilon}^{-1} \nabla L(\theta^*)$ and $u_{\epsilon} \overset{\text{def}}{=} D_{\epsilon}(\tilde{\theta} - \theta^*)$, this gives

$$L(\tilde{\theta}, \theta^*) - \xi_{\epsilon}^T u_{\epsilon} + \left\|u_{\epsilon}\right\|^2/2 \leq \diamond_{\epsilon}(x).$$

Therefore, by (3.15)

$$\left\|\xi_{\epsilon}\right\|^2/2 - \diamond_{\epsilon}(x) - \xi_{\epsilon}^T u_{\epsilon} + \left\|u_{\epsilon}\right\|^2/2 \leq \diamond_{\epsilon}(x)$$

or, equivalently

$$\left\|\xi_{\epsilon}\right\|^2/2 - \xi_{\epsilon}^T u_{\epsilon} + \left\|u_{\epsilon}\right\|^2/2 \leq \diamond_{\epsilon}(x) + \diamond_{\epsilon}(x) + (\left\|\xi_{\epsilon}\right\|^2 - \left\|\xi_{\epsilon}\right\|^2)/2$$

and the definition of $\Delta_\epsilon(x)$ implies $\left\|u_{\epsilon} - \xi_{\epsilon}\right\|^2 \leq 2\Delta_\epsilon(x)$.
4 Upper function approach and concentration of the qMLE

A very important step in the analysis of the qMLE $\tilde{\theta}$ is localization. This property means that $\tilde{\theta}$ concentrates in a small vicinity of the central point $\theta^*$. This section states such a concentration bound under the global conditions of Section 2. Given $r_0$, the deviation bound describes the probability $\mathbb{P}(\tilde{\theta} \notin \Theta_0(r_0))$ that $\tilde{\theta}$ does not belong to the local vicinity $\Theta_0(r_0)$ of $\Theta$. The question of interest is to check a possibility of selecting $r_0$ in a way that the local bracketing result and the deviation bound apply simultaneously; see the discussion at the end of the section.

Below we suppose that a sufficiently large constant $x$ is fixed to specify the accepted level be of order $e^{-x}$ for this deviation probability. All the constructions below depend upon this constant. We do not indicate it explicitly for ease of notation.

The key step in this large deviation bound is made in terms of an upper function for the process $L(\theta, \theta^*) \overset{\text{def}}{=} L(\theta) - L(\theta^*)$. Namely, $u(\theta)$ is a deterministic upper function if it holds with a high probability:

$$\sup_{\theta \in \Theta} \left\{ L(\theta, \theta^*) + u(\theta) \right\} \leq 0$$

(4.1)

Such bounds are usually called for in the analysis of the posterior measure in the Bayes approach. Below we present sufficient conditions ensuring (4.1). Now we explain how the concentration bounds can be derived from (4.1). Let $u(\theta)$ be an upper function. It can be used for describing the concentration sets for $\tilde{\theta}$.

**Lemma 4.1.** Let $u(\theta)$ be an upper function in the sense

$$\mathbb{P}\left(\sup_{\theta \in \Theta} \left\{ L(\theta, \theta^*) + u(\theta) \right\} \geq 0 \right) \leq e^{-x}$$

(4.2)

for $x > 0$. Given a subset $\Theta_0 \subset \Theta$ with $\theta^* \in \Theta_0$, the condition $u(\theta) \geq 0$ for $\theta \notin \Theta_0$ ensures

$$\mathbb{P}(\tilde{\theta} \notin \Theta_0) \leq e^{-x}.$$

*Proof.* If $\Theta^c$ is a subset of $\Theta$ not containing $\theta^*$, then the event $\tilde{\theta} \in \Theta^c$ is only possible if $\sup_{\theta \in \Theta^c} L(\theta, \theta^*) \geq 0$, because $L(\theta^*, \theta^*) \equiv 0$. This yields the result. \(\square\)

A possible way of checking the condition (4.2) is based on a lower quadratic bound for the negative expectation $-\mathbb{E}L(\theta, \theta^*) \geq b(r)\|V_0(\theta - \theta^*)\|^2/2$ in the sense of condition $(\mathcal{L}r)$ from Section 2.2. We present two different results. The first one assumes that the values $b(r)$ can be fixed universally for all $r \geq r_0$. 

**Theorem 4.2.** Suppose \((E r)\) and \((Lr)\) with \(b(r) \equiv b\). Let, for \(r \geq r_0\),

\[
1 + \sqrt{x + Q} \leq 3\nu_0^2 g(r)/b, \tag{4.3}
\]

\[
6\nu_0 \sqrt{x + Q} \leq rb, \tag{4.4}
\]

with \(x + Q \geq 2.5\) and \(Q = c_1 p\). Then

\[
\mathbb{P}(\tilde{\theta} \not\in \Theta_0(r_0)) \leq e^{-x}. \tag{4.5}
\]

**Proof.** The result follows from Theorem B.8 with \(\mu = \frac{b}{\nu_0}, t(\mu) \equiv 0, \ U(\theta) = L(\theta) - E L(\theta) \) and \(M(\theta, \theta^*) = -E L(\theta, \theta^*) \geq \frac{b}{2}\|V_0(\theta - \theta^*)\|^2\).

**Remark 4.1.** The bound (4.5) requires only two conditions. Condition (4.3) means that the value \(g(r)\) from condition \((E r)\) fulfills \(g^2(r) \geq C(x + p)\), that is, we need a qualified rate in the exponential moment conditions. This is similar to requiring finite polynomial moments for the score function. Condition (4.4) requires that \(r\) exceeds some fixed value, namely, \(r^2 \geq C(x + p)\). This bound is helpful for fixing the value \(r_0\) providing a sensible deviation probability bound.

If \(b(r)\) decreases with \(r\), the result is a bit more involved. The key requirement is that \(b(r)\) decreases not too fast, so that the product \(rb(r)\) grows to infinity with \(r\). The idea is to include the complement of the central set \(\Theta_0\) in \(\Theta\) in the union of the growing sets \(\Theta_0(r_k)\) with \(b(r_k) \geq b(r_0)2^{-k}\), and then apply Theorem 4.2 for each \(\Theta_0(r_k)\).

**Theorem 4.3.** Suppose \((E r)\) and \((Lr)\). Let \(r_k\) be such that \(b(r_k) \geq b(r_0)2^{-k}\) for \(k \geq 1\). If the conditions

\[
1 + \sqrt{x + Q + ck} \leq 3\nu_0^2 g(r_k)/b(r_k),
\]

\[
6\nu_0 \sqrt{x + Q + ck} \leq r_k b(r_k),
\]

are fulfilled for \(c = \log(2)\), then it holds

\[
\mathbb{P}(\tilde{\theta} \not\in \Theta_0(r_0)) \leq e^{-x}.
\]

**Proof.** The result (4.5) is applied to each set \(\Theta_0(r_k)\) and \(x_k = x + ck\). This yields

\[
\mathbb{P}(\tilde{\theta} \not\in \Theta_0(r_0)) \leq \sum_{k \geq 1} \mathbb{P}(\tilde{\theta} \not\in \Theta_0(r_k)) \leq \sum_{k \geq 1} e^{-x+ck} = e^{-x}
\]

as required.
Remark 4.2. Here we briefly discuss the very important question: how one can fix the value $r_0$ ensuring the bracketing result in the local set $\Theta_0(r_0)$ and a small probability of the related set $C_\epsilon(r)$ from (3.8)? The event $\{\|V_0 D_{\xi^{-1}}^T \xi\| > \epsilon\}$ requires $r \geq C(x + p)$. Further we inspect the deviation bound for the complement $\Theta \setminus \Theta_0(r_0)$. For simplicity, assume $(\mathcal{L}x)$ with $b(r) \equiv b$. Then the most important condition (4.4) of Theorem 4.2 requires that $r^2 \geq Cb^{-2}(x + p)$. In words, the squared radius $r^2$ should be of order $p$. The other condition (4.3) of Theorem 4.2 is technical and only requires that $g(r)$ is sufficiently large while the local results only require that $\delta(r)$ and $\varrho(r)$ are small for such $r$. In the asymptotic setup one can typically bring these conditions together. Section 5 provides further discussion for the i.i.d. setup.

5 Examples

The model with independent identically distributed (i.i.d.) observations is one of the most popular setups in statistical literature and in statistical applications. The essential and the most developed part of the statistical theory is designed for the i.i.d. modeling. Especially, the classical asymptotic parametric theory is almost complete including asymptotic root-n normality and efficiency of the MLE and Bayes estimators under rather mild assumptions; see e.g. Chapter 2 and 3 in Ibragimov and Khas’minskij (1981). So, the i.i.d. model can naturally serve as a benchmark for any extension of the statistical theory: being applied to the i.i.d. setup, the new approach should lead to essentially the same conclusions as in the classical theory. Similar reasons apply to the regression model and its extensions. Below we try demonstrate that the proposed non-asymptotic viewpoint is able to reproduce the existing brilliant and well established results of the classical parametric theory. Surprisingly, the majority of classical efficiency results can be easily derived from the obtained general non-asymptotic bounds.

The next question is whether there is any added value or benefits of the new approach being restricted to the i.i.d. situation relative to the classical one. Two important issues have been already mentioned: the new approach applies to the situation with finite samples and survives under model misspecification. One more important question is whether the obtained results remain applicable and informative if the dimension of the parameter space is high – this is one of the main challenge in the modern statistics. We show that the dimensionality $p$ naturally appears in the risk bounds and the results apply as long as the sample size exceeds in order this value $p$. All these questions are addressed in Section 5.1 for the i.i.d. setup, Section 5.2 focuses on generalized linear modeling, while Section 5.3 discusses linear median regression.
5.1 Quasi MLE in an i.i.d. model

The basic i.i.d. parametric model means that the observations \( Y = (Y_1, \ldots, Y_n) \) are independent identically distributed from a distribution \( P \) which belongs to a given parametric family \( (P_\theta, \theta \in \Theta) \) on the observation space \( Y_1 \). Each \( \theta \in \Theta \) clearly yields the product data distribution \( P_\theta = P_\theta \otimes P_\theta \) on the product space \( Y = Y_1 \). This section illustrates how the obtained general results can be applied to this type of modeling under possible model misspecification. Different types of misspecification can be considered. Each of the assumptions, namely, data independence, identical distribution, parametric form of the marginal distribution can be violated. To be specific, we assume the observations \( Y_i \) independent and identically distributed. However, we admit that the distribution of each \( Y_i \) does not necessarily belong to the parametric family \( (P_\theta) \). The case of non-identically distributed observations can be done similarly at cost of more complicated notation.

In what follows the parametric family \( (P_\theta) \) is supposed to be dominated by a measure \( \mu_0 \), and each density \( p(y, \theta) = dP_\theta/d\mu_0(y) \) is two times continuously differentiable in \( \theta \) for all \( y \). Denote \( \ell(y, \theta) = \log p(y, \theta) \). The parametric assumption \( Y_i \sim P_{\theta^*} \in (P_\theta) \) leads to the log-likelihood

\[
L(\theta) = \sum \ell(Y_i, \theta),
\]

where the summation is taken over \( i = 1, \ldots, n \). The quasi MLE \( \tilde{\theta} \) maximizes this sum over \( \theta \in \Theta \):

\[
\tilde{\theta} \text{ def } \arg \max_{\theta \in \Theta} L(\theta) = \arg \max_{\theta \in \Theta} \sum \ell(Y_i, \theta).
\]

The target of estimation \( \theta^* \) maximizes the expectation of \( L(\theta) \):

\[
\theta^* \text{ def } \arg \max_{\theta \in \Theta} \mathbb{E}L(\theta) = \arg \max_{\theta \in \Theta} \mathbb{E}\ell(Y_1, \theta).
\]

Let \( \zeta_i(\theta) \defeq \ell(Y_i, \theta) - \mathbb{E}\ell(Y_i, \theta) \). Then \( \zeta(\theta) = \sum \zeta_i(\theta) \). The equation \( \nabla \mathbb{E}L(\theta^*) = 0 \) implies

\[
\nabla \zeta(\theta^*) = \sum \nabla \zeta_i(\theta^*) = \sum \nabla \ell_i(\theta^*). \quad (5.1)
\]

I.i.d. structure of the \( Y_i \)'s allows to rewrite the local conditions \((EX)\), \((ED_0)\), \((ED_1)\), and \((L_0)\), and \((I)\) in terms of the marginal distribution.

\textbf{(ed0)} \hspace{1cm} \text{There exists a positively definite symmetric matrix } \nu_0, \text{ such that for all } |\lambda| \leq g_1

\[
\sup_{\gamma \in \mathbb{R}^p} \log \mathbb{E} \exp \left\{ \lambda^\top \nabla \zeta(\theta^*) \right\} \leq \nu_0^2 \lambda^2 / 2.
\]
A natural candidate on $V_0^2$ is given by the variance of the gradient $\nabla \ell(Y_1, \theta^*)$, that is, $V_0^2 = \text{Var}\{\nabla \ell(Y_1, \theta^*)\} = \text{Var}\{\nabla \ell_i(\theta^*)\}$.

Next consider the local sets

$$\Theta_{\text{loc}}(u) = \{\theta : \|v_0(\theta - \theta^*)\| \leq u\}.$$ 

In view of $V_0^2 = nV_0^2$, it holds $\Theta_0(r) = \Theta_{\text{loc}}(u)$ with $r^2 = nu_0^2$.

Below we distinguish between local conditions for $u \leq u_0$ and the global conditions for all $u > 0$, where $u_0$ is some fixed value.

The local smoothness conditions $(ED_1)$ and $(L_0)$ require to specify the functions $\delta(x)$ and $g(x)$ for $x \leq x_0$ where $x_0^2 = nu_0^2$. If the log-likelihood function $\ell(y, \theta)$ is sufficiently smooth in $\theta$, these functions can be selected proportional to $u = r/n^{1/2}$.

$$(ED_1) \quad \text{There are constants } \omega^* > 0 \text{ and } g_1 > 0 \text{ such that for each } u \leq u_0 \text{ and } |\lambda| \leq g_1$$

$$\sup_{\gamma \in \mathbb{R}^p} \sup_{\theta \in \Theta_{\text{loc}}(u)} \log \mathbb{E} \exp \left\{ \lambda \gamma^T \frac{\nabla \ell(Y_1, \theta) - \nabla \ell(Y_1, \theta^*)}{\omega^* u \|v_0\|} \right\} \leq \nu_0^2 \lambda^2 / 2.$$ 

Further we restate the local identifiability condition $(L_0)$ in terms of the expected value $k(\theta, \theta^*) \equiv -\mathbb{E}\{\ell(Y_1, \theta) - \ell(Y_1, \theta^*)\}$ for each $i$. We suppose that $k(\theta, \theta^*)$ is two times differentiable w.r.t. $\theta$. The definition of $\theta^*$ implies $\nabla \mathbb{E}\ell(Y_i, \theta^*) = 0$. Define also the matrix $f_0 = -\nabla^2 \mathbb{E}\ell(Y_i, \theta^*)$. In the parametric case $P = P_{\theta^*}$, $k(\theta, \theta^*)$ is the Kullback-Leibler divergence between $P_{\theta^*}$ and $P_\theta$ while the matrices $v_0^2 = f_0$ are equal to each other and coincide with the Fisher information matrix of the family $(P_\theta)$ at $\theta^*$.

$L_0$ \quad There is a constant $\delta^*$ such that it holds for each $u \leq u_0$ 

$$\sup_{\theta \in \Theta_{\text{loc}}(u)} \left| \frac{2k(\theta, \theta^*)}{(\theta - \theta^*)^T f_0 (\theta - \theta^*)} - 1 \right| \leq \delta^* u.$$ 

(u) \quad There is a constant $a > 0$ such that $a^2 f_0^2 \geq v_0^2$.

$$(EU) \quad \text{For each } u > 0, \text{ there exists } g_1(u) > 0, \text{ such that for all } |\lambda| \leq g_1(u)$$

$$\sup_{\gamma \in \mathbb{R}^p} \sup_{\theta \in \Theta_{\text{loc}}(u)} \log \mathbb{E} \exp \left\{ \lambda \gamma^T \nabla \ell(Y_1, \theta) \right\} \leq \nu_0^2 \lambda^2 / 2.$$ 

$L(u)$ \quad For each $u > 0$, there exists $b(u) > 0$ such that 

$$\sup_{\theta \in \Theta : \|v_0(\theta - \theta^*)\| = u} \frac{k(\theta, \theta^*)}{\|v_0(\theta - \theta^*)\|^2} \geq b(u),$$
Lemma 5.1. Let \( Y_1, \ldots, Y_n \) be i.i.d. Then (\( eu \)), \( (ed_0) \), \( (ed_1) \), (i), and \( (\ell_0) \) imply (\( Ex \)), \( (ED_0) \), \( (ED_1) \), (I), and \( (L_0) \) with \( V_0^2 = n\nu_0^2 \), \( D_0^2 = nf_0 \), \( \omega(x) = \omega^* x / n^{1/2} \), \( \delta(x) = \delta^* x / n^{1/2} \), and \( g = g_1 \sqrt{n} \).

Proof. The identities \( V_0^2 = n\nu_0^2 \), \( D_0^2 = nf_0 \) follow from the i.i.d. structure of the observations \( Y_i \). We briefly comment on condition (\( Ex \)). The use once again the i.i.d. structure yields by (5.1) in view of

\[
\log \mathbb{E} \exp \left\{ \lambda \gamma^\top \nabla \zeta(\theta) \right\} = n \mathbb{E} \exp \left\{ \frac{\lambda}{n^{1/2}} \frac{\gamma^\top \nabla \zeta_1(\theta)}{\|\nu_0 \gamma\|} \right\} \leq \nu_0^2 \lambda^2 / 2
\]

as long as \( \lambda \leq n^{1/2} g_1(u) \leq g(r) \). Similarly for (\( ED_0 \)) and (\( ED_1 \)). \qed

Remark 5.1. This remark discusses how the presented conditions relate to what is usually assumed in statistical literature. One general remarks concern the choice of the parametric family \( (P_\theta) \). The point of the classical theory is that the true measure is in this family, so the conditions should be as weak as possible. The viewpoint of this paper is slightly different: whatever family \( (P_\theta) \) is taken, the true measure is never included, any model is only an approximation of reality. From the other side, the choice of the parametric model \( (P_\theta) \) is always done by a statistician. Sometimes some special stylized features of the model force to include an irregularity in this family. Otherwise any smoothness condition on the density \( \ell(y, \theta) \) can be secured by a proper choice of the family \( (P_\theta) \).

The presented list also includes the exponential moment conditions \( (ed_0) \) and \( (ed_1) \) on the gradient \( \nabla \ell(Y_1, \theta) \). We need exponential moments for establishing some nonasymptotic risk bounds, the classical concentration bounds require even stronger conditions that the considered random variables are bounded.

The identifiability condition (\( \ell u \)) is very easy to check in the usual asymptotic setup. Indeed, if the parameter set \( \Theta \) is compact, the Kullback-Leibler divergence \( k(\theta, \theta^*) \) is continuous and positive for all \( \theta \neq \theta^* \), then (\( \ell u \)) is fulfilled automatically with a universal constant \( b \). If \( \Theta \) is not compact, the condition is still fulfilled but the function \( b(u) \) may depend on \( u \).

Below we specify the general results of Section 3 and 4 to the i.i.d. setup.

5.1.1 A large deviation bound

This section presents some sufficient conditions ensuring a small deviation probability for the event \( \{ \tilde{\theta} \notin \Theta_{\text{loc}}(u_0) \} \) for a fixed \( u_0 \). Below \( Q = c_1 I \). We only discuss the case \( b(u) \equiv b \). The general case only requires more complicated notations. The next result follows from Theorem 4.2 with the obvious changes.
**Theorem 5.2.** Suppose \((e_u)\) and \((f_u)\) with \(b(u) \equiv b\). If, for \(u_0 > 0\),

\[
n^{1/2}u_0 b \geq 6\nu_0\sqrt{x + Q},
\]

\[
1 + \sqrt{x + Q} \leq 3b^{-1}g_1(u_0)n^{1/2},
\]

then

\[
P(\hat{\theta} \not\in \Theta_{loc}(u_0)) \leq e^{-x}.
\]

**Remark 5.2.** The presented result helps to qualify two important values \(u_0\) and \(n\) providing a sensible deviation probability bound. For simplicity suppose that \(g_1(u) \equiv g_1 > 0\). Then the condition (5.2) can be written as \(nu_0^2 \gg x + Q\). In other words, the result of the theorem claims a large deviation bound for the vicinity \(\Theta_{loc}(u_0)\) with \(u_0^2\) of order \(p/n\). In classical asymptotic statistics this result is usually referred to as root-\(n\) consistency. Our approach yields this result in a very strong form and for finite samples.

### 5.1.2 Local inference

Now we restate the general local bounds of Section 3 for the i.i.d. case. First we describe the approximating linear models. The matrices \(v_0^2\) and \(f_0\) from conditions \((ed_0)\), \((ed_1)\), and \((\ell_0)\) determine their drift and variance components. Define

\[
f_{\epsilon} \overset{\text{def}}{=} f_0(1 - \delta) - \varphi v_0^2.
\]

If \(\tau_{\epsilon} \overset{\text{def}}{=} \delta + a^2\varphi < 1\), then

\[
f_{\epsilon} \geq (1 - \tau_{\epsilon})f_0 > 0.
\]

Further, \(D_{\epsilon}^2 = nf_{\epsilon}\) and

\[
\xi_{\epsilon} \overset{\text{def}}{=} D_{\epsilon}^{-1}\nabla\zeta(\theta^*) = (nf_{\epsilon})^{-1/2}\nabla\ell(Y_i, \theta^*).
\]

The upper bracketing process reads as

\[
L_{\epsilon}(\theta, \theta^*) = (\theta - \theta^*)^\top D_{\epsilon}\xi_{\epsilon} - \|D_{\epsilon}(\theta - \theta^*)\|^2/2.
\]

This expression can be viewed as log-likelihood for the linear model \(\xi_{\epsilon} = D_{\epsilon}\theta + \varepsilon\) for a standard normal error \(\varepsilon\). The (quasi) MLE \(\tilde{\theta}_{\epsilon}\) for this model is of the form \(\tilde{\theta}_{\epsilon} = D_{\epsilon}^{-1}\xi_{\epsilon}\).

**Theorem 5.3.** Suppose \((ed_0)\). Given \(u_0\), assume \((ed_1)\), \((\ell_0)\), and \((\iota)\) on \(\Theta_{loc}(u_0)\), and let \(\varrho = 3\nu_0 \omega^*u_0\), \(\delta = \delta^*u_0\), and \(\tau_{\epsilon} \overset{\text{def}}{=} \delta + a^2\varrho < 1\). Then the results of Theorem 3.1...
and all its corollaries apply to the case of i.i.d. modeling with $x^2 = n u_0^2$. In particular, on the random set $C_\epsilon(r_0) = \{ \tilde{\theta} \in \Theta_{\loc}(u_0), \| \xi_\epsilon \| \leq r_0 \}$, it holds

$$
\| \xi_\epsilon \|^2 / 2 - \Delta_\epsilon(r_0) \leq L(\tilde{\theta}, \theta^*) \leq \| \xi_\epsilon \|^2 / 2 + \Delta_\epsilon(r_0),
$$

$$
\| \sqrt{n f_\epsilon(\tilde{\theta} - \theta^*) - \xi_\epsilon} \|^2 \leq 2 \Delta_\epsilon(r_0).
$$

The random quantities $\Delta_\epsilon(r_0)$ and $\Delta_\epsilon(r_0)$ follow the probability bounds of Theorem 3.7 and 3.10.

Now we briefly discuss the implications of Theorem 5.2 and 5.3 to the classical asymptotic setup with $n \to \infty$. We fix $u_0^2 = Cp/n$ for a constant $C$ ensuring the deviation bound of Theorem 5.2. Then $\delta$ is of order $u_0$ and the same for $\rho$. For a sufficiently large $n$, both quantities are small and thus, the spread $\Delta_\epsilon(r_0)$ is small as well; see Section 3.3.

Further, under $(ed_0)$ condition, the normalized score

$$
\xi \overset{\text{def}}{=} (nf_0)^{-1/2} \sum \nabla \ell(Y_i, \theta^*)
$$

is zero mean asymptotically normal by the central limit theorem. Moreover, if $f_0 = v_0^2$, then $\xi$ is asymptotically standard normal. The same holds for $\xi_\epsilon$. This immediately yields all classical asymptotic results like Wilks theorem or the Fisher expansion for MLE in the i.i.d. setup as well as the asymptotic efficiency of the MLE. Moreover, our results bounds yield the asymptotic result for the case when the parameter dimension $p = p_n$ grows linearly with $n$. Below $u_n = o_n(p_n)$ means that $u_n/p_n \to 0$ with $n$.

**Theorem 5.4.** Let $Y_1, \ldots, Y_n$ be i.i.d. $\mathcal{F}_{\theta^*}$ and let $(ed_0), (ed_1), (\ell_0), (i), (eu)$, and $(\ell u)$ with $b(u) \equiv b$ hold. If $n > Cp_n$ for a fixed constant $C$ depending on constants in the above conditions only, then

$$
\| \sqrt{n f_0(\tilde{\theta} - \theta^*) - \xi} \|^2 = o_n(p_n), \quad 2L(\tilde{\theta}, \theta^*) - \| \xi \|^2 = o_n(p_n).
$$

This result particularly yields that $\sqrt{n f_0}(\tilde{\theta} - \theta^*)$ is nearly standard normal and $2L(\tilde{\theta}, \theta^*)$ is nearly $\chi^2_p$.

### 5.2 Generalized linear modeling

Now we consider a generalized linear modeling (GLM) which is often used for describing some categorical data. Let $\mathcal{P} = (P_w, w \in \gamma)$ be an exponential family with a canonical parametrization; see e.g. McCullagh and Nelder (1989). The corresponding log-density can be represented as $\ell(y, w) = y w - d(w)$ for a convex function $d(w)$. The popular examples are given by the binomial (binary response, logistic) model with
\[ d(w) = \log(e^w + 1), \] the Poisson model with \( d(w) = e^w \), the exponential model with \( d(w) = -\log(w) \). Note that linear Gaussian regression is a special case with \( d(w) = w^2/2 \).

A GLM specification means that every observation \( Y_i \) has a distribution from the family \( \mathcal{P} \) with the parameter \( w_i \) which linearly depends on the regressor \( \Psi_i \in \mathbb{R}^p \):

\[ Y_i \sim \mathcal{P}_{\Psi_i \top \theta^*}. \quad (5.3) \]

The corresponding log-density of a GLM reads as

\[ L(\theta) = \sum \{ Y_i \Psi_i \top \theta - d(\Psi_i \top \theta) \}. \]

Under \( \mathbb{P}_{\theta^*} \) each observation \( Y_i \) follows (5.3), in particular, \( \mathbb{E}Y_i = d'(\Psi_i \top \theta^*) \). However, similarly to the previous sections, it is accepted that the parametric model (5.3) is misspecified. Response misspecification means that the vector \( f \overset{\text{def}}{=} \mathbb{E}Y \) cannot be represented in the form \( d'(\Psi \top \theta) \) whatever \( \theta \) is. The other sort of misspecification concerns the data distribution. The model (5.3) assumes that the \( Y_i \)'s are independent and the marginal distribution belongs to the given parametric family \( \mathcal{P} \). In what follows, we only assume independent data having certain exponential moments. The target of estimation \( \theta^* \) is defined by

\[ \theta^* \overset{\text{def}}{=} \arg\max_\theta \mathbb{E}L(\theta). \]

The quasi MLE \( \tilde{\theta} \) is defined by maximization of \( L(\theta) \):

\[ \tilde{\theta} = \arg\max_\theta L(\theta) = \arg\max_\theta \sum \{ Y_i \Psi_i \top \theta - d(\Psi_i \top \theta) \}. \]

Convexity of \( d(\cdot) \) implies that \( L(\theta) \) is a concave function of \( \theta \), so that the optimization problem has a unique solution and can be effectively solved. However, a closed form solution is only available for the constant regression or for the linear Gaussian regression. The corresponding target \( \theta^* \) is the maximizer of the expected log-likelihood:

\[ \theta^* = \arg\max_\theta \mathbb{E}L(\theta) = \arg\max_\theta \sum \{ f_i \Psi_i \top \theta - d(\Psi_i \top \theta) \} \]

with \( f_i = \mathbb{E}Y_i \). The function \( \mathbb{E}L(\theta) \) is concave as well and the vector \( \theta^* \) is also well defined.

Define the individual errors (residuals) \( \varepsilon_i = Y_i - \mathbb{E}Y_i \). Below we assume that these errors fulfill some exponential moment conditions.
(e₁) There exist some constants ν₀ and g₁ > 0, and for every i a constant sᵢ such that \( \mathbb{E}(\varepsilon_i/sᵢ)^2 \leq 1 \) and

\[
\log \mathbb{E} \exp(b \varepsilon_i/sᵢ) \leq \nu₀ b^2/2, \quad |b| \leq g₁. \tag{5.4}
\]

A natural candidate for sᵢ is σᵢ where \( \sigmaᵢ^2 = \mathbb{E}\varepsilonᵢ^2 \) is the variance of \( \varepsilonᵢ \); see Lemma B.17. Under (5.4), introduce a \( p \times p \) matrix \( V₀ \) defined by

\[
V₀^2 \overset{\text{def}}{=} \sum sᵢ^2 \Psiᵢ \Psiᵢᵀ. \tag{5.5}
\]

Condition (e₁) effectively means that each error term \( \varepsilonᵢ = Yᵢ - \mathbb{E}Yᵢ \) has some bounded exponential moments: for \( |b| \leq g₁ \), it holds \( f(b) \overset{\text{def}}{=} \log \mathbb{E} \exp(b \varepsilon_i/sᵢ) < \infty \). This implies the quadratic upper bound for the function \( f(b) \) for \( |b| \leq g₁ \); see Lemma B.17. In words, condition (e₁) requires light (exponentially decreasing) tail for the marginal distribution of each \( \varepsilonᵢ \).

Define also

\[
N^{-1/2} \overset{\text{def}}{=} \max \sup \frac{sᵢ|\Psiᵢᵀγ|}{\|V₀γ\|}. \tag{5.6}
\]

**Lemma 5.5.** Assume (e₁) and let \( V₀ \) be defined by (5.5) and \( N \) by (5.6). Then conditions (ED₀) and (Ex) follow from (e₁) with the matrix \( V₀ \) due to (5.5) and \( g = g₁ N^{1/2} \). Moreover, the stochastic component \( \zeta(\theta) \) is linear in \( \theta \) and the condition (ED₁) is fulfilled with \( \omega(\theta) \equiv 0 \).

**Proof.** The gradient of the stochastic component \( \zeta(\theta) \) of \( L(\theta) \) does not depend on \( \theta \), namely, \( \nabla \zeta(\theta) = \sum \Psiᵢ\varepsilonᵢ \) with \( \varepsilonᵢ = Yᵢ - \mathbb{E}Yᵢ \). Now, for any unit vector \( γ ∈ \mathbb{R}^p \) and \( λ ≤ g₁ \), independence of the \( \varepsilonᵢ \)’s implies that

\[
\log \mathbb{E} \exp\left( \frac{λ}{\|V₀γ\|} \sum \Psiᵢ\varepsilonᵢ \right) = \sum \log \mathbb{E} \exp\left( \frac{λsᵢ\Psiᵢᵀγ}{\|V₀γ\|} \varepsilonᵢ/sᵢ \right).
\]

By definition \( sᵢ|\Psiᵢᵀγ|/\|V₀γ\| ≤ N^{-1/2} \) and therefore, \( λsᵢ|\Psiᵢᵀγ|/\|V₀γ\| ≤ g₁ \). Hence, (5.4) implies

\[
\log \mathbb{E} \exp\left( \frac{λ}{\|V₀γ\|} \sum \Psiᵢ\varepsilonᵢ \right) ≤ \frac{\nu₀ λ^2}{2\|V₀γ\|^2} \sum sᵢ^2|\Psiᵢᵀγ|^2 = \frac{\nu₀ λ^2}{2}, \tag{5.7}
\]

and (ED₀) follows.

It only remains to bound the quality of quadratic approximation for the mean of the process \( L(\theta, \theta^*) \) in a vicinity of \( \theta^* \). An interesting feature of the GLM is that the effect of model misspecification disappears in the expectation of \( L(\theta, \theta^*) \).
Lemma 5.6. It holds

$$-IEL(\theta, \theta^*) = \sum \{ d(\Psi_i^\top \theta) - d(\Psi_i^\top \theta^*) - d'(\Psi_i^\top \theta^*)\Psi_i^\top (\theta - \theta^*) \}$$

$$= K(\mathcal{P}_{\theta^*}, \mathcal{P}_{\theta}), \quad (5.8)$$

where $K(\mathcal{P}_{\theta^*}, \mathcal{P}_{\theta})$ is the Kullback-Leibler divergence between measures $\mathcal{P}_{\theta^*}$ and $\mathcal{P}_{\theta}$. Moreover,

$$-IEL(\theta, \theta^*) = \|D(\theta^*)(\theta - \theta^*)\|^2/2,$$  

where $\theta^* \in [\theta, \theta]$ and

$$D^2(\theta^*) = \sum d''(\Psi_i^\top \theta^*)\Psi_i\Psi_i^\top.$$  

Proof. The definition implies

$$IEL(\theta, \theta^*) = \sum \{ f_i\Psi_i^\top (\theta - \theta^*) - d(\Psi_i^\top \theta) + d(\Psi_i^\top \theta^*) \}.$$  

As $\theta^*$ is the extreme point of $IEL(\theta)$, it holds $\nabla IEL(\theta^*) = \sum [f_i - d'(\Psi_i^\top \theta^*)]\Psi_i = 0$ and (5.8) follows. The Taylor expansion of the second order around $\theta^*$ yields the expansion (5.9). \hfill $\square$

Define now the matrix $D_0$ by

$$D_0^2 \overset{\text{def}}{=} D^2(\theta^*) = \sum d''(\Psi_i^\top \theta^*)\Psi_i\Psi_i^\top.$$  

Let also $V_0$ be defined by (5.5). Note that the matrices $D_0$ and $V_0$ coincide if the model $Y_i \sim \mathcal{P}_{\Psi_i^\top \theta^*}$ is correctly specified and $\tilde{s}_i^2 = d''(\Psi_i^\top \theta^*)$. The matrix $V_0$ describes a local elliptic neighborhood of the central point $\theta^*$ in the form $\Theta_0(r) = \{ \theta : \|V_0(\theta - \theta^*)\| \leq r \}$. If the matrix function $D^2(\theta)$ is continuous in this vicinity $\Theta_0(r)$ then the value $\delta(r)$ measuring the approximation quality of $-IEL(\theta, \theta^*)$ by the quadratic function $\|D_0(\theta - \theta^*)\|^2/2$ is small and the identifiability condition ($\mathcal{L}_0$) is fulfilled on $\Theta_0(r)$.

Lemma 5.7. Suppose that

$$\|I_p - D_0^{-1}D^2(\theta)D_0^{-1}\|_{\infty} \leq \delta(r), \quad \theta \in \Theta_0(r). \quad (5.10)$$

Then ($\mathcal{L}_0$) holds with this $\delta(r)$. Moreover, as the quantities $\omega(r), \delta_\epsilon(r), \delta_\xi(r)$ vanish, one can take $\rho = 0$ leading to the following representation for $D_\epsilon$ and $\xi_\epsilon$:

$$D_\epsilon^2 = (1 - \delta)D_0^2, \quad \xi_\epsilon = (1 + \delta)^{1/2}\xi$$

$$D_\xi^2 = (1 + \delta)D_0^2, \quad \xi_\xi = (1 - \delta)^{1/2}\xi$$
with
\[ \xi \overset{\text{def}}{=} D_0^{-1} \nabla \zeta = D_0^{-1} \sum \Psi_i (Y_i - EY_i). \]

Linearity of the stochastic component \( \zeta(\theta) \) in the considered GLM implies the important fact that the quantities \( \diamondsuit \epsilon_r, \diamondsuit \xi_r \) in the general bracketing bound (3.4) vanish for any \( r \). Therefore, in the GLM case, the deficiency can be defined as the difference between upper and lower excess and it can be easily evaluated:
\[ \Delta(r) = \|\xi_\epsilon\|^2 / 2 - \|\xi_\xi\|^2 / 2 = \delta \|\xi\|^2. \]

Our result assumes some concentration properties of the squared norm \( \|\xi\|^2 \) of the vector \( \xi \). These properties can be established by general results of Section A under the regularity condition: for some \( a \)
\[ V_0 \leq a D_0. \quad (5.11) \]

Now we are prepared to state the local results for the GLM estimation.

**Theorem 5.8.** Let \((e_1)\) hold. Then for \( \delta \geq \delta(r) \) any \( z > 0 \) and \( \delta > 0 \), it holds
\[ IP\left( \|D_0(\tilde{\theta} - \theta^*)\| > z, \|V_0(\tilde{\theta} - \theta^*)\| \leq r \right) \leq IP\left\{ \|\xi\|^2 > (1 - \delta)z^2 \right\} \]
\[ IP\left( L(\tilde{\theta}, \theta^*) > 3, \|V_0(\tilde{\theta} - \theta^*)\| \leq r \right) \leq IP\left\{ \|\xi\|^2 / 2 > (1 - \delta)3 \right\}. \]

Moreover, on the set \( C_\epsilon(r) = \{ \|V_0(\tilde{\theta} - \theta^*)\| \leq r, \|\xi_\epsilon\| \leq r \} \), it holds
\[ \|D_0(\tilde{\theta} - \theta^*) - \xi\|^2 \leq \frac{2\delta}{1 - \delta^2} \|\xi\|^2. \quad (5.12) \]

If the function \( d(w) \) is quadratic then the approximation error \( \delta \) vanishes as well and the expansion (5.12) becomes equality which is also fulfilled globally, a localization step in not required. However, if \( d(w) \) is not quadratic, the result applies only locally and it has to be accomplished with a large deviation bound. The GLM structure is helpful in the large deviation zone as well. Indeed, the gradient \( \nabla \zeta(\theta) \) does not depend on \( \theta \) and hence, the most delicate condition \((E)r\) is fulfilled automatically with \( g = n N^{1/2} \) for all local sets \( \Theta_0(r) \). Further, the identifiability condition \((L)r\) easily follows from Lemma 5.6: it suffices to bound from below the matrix \( D(\theta) \) for \( \theta \in \Theta_0(r) \):
\[ D(\theta) \geq b(r) V_0, \quad \theta \in \Theta_0(r). \]

An interesting question, similarly to the i.i.d. case, is the minimal radius \( r_0 \) of the local vicinity \( \Theta_0(r_0) \) ensuring the desirable concentration property. Suppose for the
moment that the constants \( b(r) \) are all the same for different \( r \): \( b(r) \equiv b \). Under the regularity condition (5.11), a sufficient lower bound for \( r_0 \) can be based on Corollary 4.3. The required condition can be restated as

\[
1 + \sqrt{x + Q} \leq 3\nu_0^2 g/b, \quad 6\nu_0\sqrt{x + Q} \leq rb.
\]

It remains to note that \( Q = c_1 p \) and \( g = g_1 N^{1/2} \). So, the required conditions are fulfilled for \( r^2 \geq r_0^2 = C(x + p) \), where \( C \) only depends on \( \nu_0, b, \) and \( g \).

### 5.3 Linear median estimation

This section illustrates how the proposed approach applies to robust estimation in linear models. The target of analysis is the linear dependence of the observed data \( Y = (Y_1, \ldots, Y_n) \) on the set of features \( \Psi_i \in \mathbb{R}^p \):

\[
Y_i = \Psi_i^\top \theta + \varepsilon_i,
\]

where \( \varepsilon_i \) means the \( i \)th individual error. As usual, the true data distribution can deviate from the linear model. In addition, we admit contaminated data which naturally leads to the idea of robust estimation. This section offers a qMLE view on the robust estimation problem. Our parametric family assumes the linear dependence (5.13) with i.i.d. errors \( \varepsilon_i \) which follow the double exponential (Laplace) distribution with the density \((1/2)e^{-|y|}\). Then the corresponding log-likelihood reads as

\[
L(\theta) = -\frac{1}{2} \sum |Y_i - \Psi_i^\top \theta|
\]

and \( \tilde{\theta} \) \( \overset{\text{def}}{=} \arg\max_\theta L(\theta) \) is called the least absolute deviation (LAD) estimate. In the context of linear regression, it is also called the linear median estimate. The target of estimation \( \theta^* \) is defined as usually by the equation \( \theta^* = \arg\max_\theta \mathbb{E} L(\theta) \).

It is useful to define the residuals \( \tilde{\varepsilon}_i = Y_i - \Psi_i^\top \theta^* \) and their distributions

\[
P_i(A) = \mathbb{P}(\tilde{\varepsilon}_i \in A) = \mathbb{P}(Y_i - \Psi_i^\top \theta^* \in A)
\]

for any Borel set \( A \) on the real line. If \( Y_i = \Psi_i^\top \theta^* + \varepsilon_i \) is the true model then \( P_i \) coincides with the distribution of each \( \varepsilon_i \). Below we suppose that each \( P_i \) has a positive density \( f_i(y) \).

Note that the difference \( L(\theta) - L(\theta^*) \) is bounded by \( \frac{1}{2} \sum |\Psi_i^\top (\theta - \theta^*)| \). Next we check conditions \((ED_0)\) and \((ED_1)\). Denote \( \xi_i(\theta) = \mathbb{I}(Y_i - \Psi_i^\top \theta \leq 0) - q_i(\theta) \) for \( q_i(\theta) = \mathbb{P}(Y_i - \Psi_i^\top \theta \leq 0) \). This is a centered Bernoulli random variable, and it is easy to check that

\[
\nabla \zeta(\theta) = -\sum \xi_i(\theta) \Psi_i.
\]
This expression differs from the similar ones from the linear and generalized linear regression because the stochastic terms \( \xi_i \) now depend on \( \theta \). First we check the global condition \((E\mathbf{r})\). Fix any \( g_1 < 1 \). Then it holds for a Bernoulli r.v. \( Z \) with \( P(Z = 1) = q \), \( \xi = Z - q \), and \( |\lambda| \leq g_1 \)

\[
\log \mathbb{E} \exp(\lambda \xi) = \log \left[ q \exp(\lambda(1 - q)) + (1 - q) \exp(-\lambda q) \right] \\
\leq \nu_0^2 q(1 - q) \lambda^2 / 2, 
\]

where \( \nu_0 \geq 1 \) depends on \( g_1 \) only. Let now a vector \( \gamma \in \mathbb{R}^p \) and \( \rho > 0 \) be such that \( \rho |\psi_i^\top \gamma| \leq g_1 \) for all \( i = 1, \ldots, n \). Then

\[
\log \mathbb{E} \exp(\rho \gamma^\top \nabla \zeta(\theta)) \leq \nu_0^2 \rho^2 \sum q_i(\theta) \{1 - q_i(\theta)\} |\psi_i^\top \gamma|^2 \\
\leq \nu_0^2 \rho^2 \|V(\theta)\gamma\|^2, 
\]

where

\[
V^2(\theta) = \sum q_i(\theta) \{1 - q_i(\theta)\} \psi_i \psi_i^\top. 
\]

Denote also

\[
V_0^2 = \frac{1}{4} \sum \psi_i \psi_i^\top. 
\]

Clearly \( V(\theta) \leq V_0 \) for all \( \theta \) and condition \((E\mathbf{r})\) is fulfilled with the matrix \( V_0 \) and \( g(\mathbf{r}) \equiv g = g_1 N^{1/2} \) for \( N \) defined by

\[
N^{-1/2} \equiv \max_{i} \sup_{\gamma \in \mathbb{R}^p} \frac{\psi_i^\top \gamma}{2\|V_0 \gamma\|}; 
\]

cf. (5.7).

Let some \( r_0 > 0 \) be fixed. We will specify this choice later. Now we check the local conditions within the elliptic vicinity \( \Theta_0(\mathbf{r}_0) = \{\theta : \|V_0(\theta - \theta^*)\| \leq r_0\} \) of the central point \( \theta^* \) for \( V_0 \) from (5.17). Then condition \((ED_0)\) with the matrix \( V_0 \) and \( g(\mathbf{r}) \equiv g = g_1 N^{1/2} \) is fulfilled on \( \Theta_0(\mathbf{r}_0) \) due to (5.16). Next, in view of (5.18), it holds

\[
|\psi_i^\top \gamma| \leq 2N^{-1/2}\|V_0 \gamma\| \text{ for any vector } \gamma \in \mathbb{R}^p. 
\]

By (5.14)

\[
\nabla \zeta(\theta) - \nabla \zeta(\theta^*) = \sum \psi_i \{\xi_i(\theta) - \xi_i(\theta^*)\}. 
\]

If \( \psi_i^\top \theta \geq \psi_i^\top \theta^* \), then

\[
\xi_i(\theta) - \xi_i(\theta^*) = \mathbb{I}(\psi_i^\top \theta^* \leq Y_i < \psi_i^\top \theta) - \mathbb{I}(\psi_i^\top \theta^* \leq Y_i < \psi_i^\top \theta). 
\]
Similarly for $Ψ_i^T \theta < Ψ_i^T \theta^*$

$$ξ_i(\theta) - ξ_i(\theta^*) = -\mathbb{I}(Ψ_i^T \theta \leq Y_i < Ψ_i^T \theta^*) + F(Ψ_i^T \theta \leq Y_i < Ψ_i^T \theta^*).$$

Define $q_i(\theta, \theta^*) \overset{\text{def}}{=} |q_i(\theta) - q_i(\theta^*)|$. Now (5.15) yields similarly to (5.16)

$$\log \mathbb{E} \exp \{ργ^T \{∇ζ(θ) - ∇ζ(θ^*)\} \} \leq \frac{\nu_0^2 ρ^2}{2} \sum q_i(\theta, \theta^*) |Ψ_i^T γ|^2$$

$$\leq 2ν_0^2 ρ^2 \max_{i \leq n} q_i(\theta, \theta^*) \|V_0 γ\|^2 \leq (r)(\nu_0^2 ρ^2 \|V_0 γ\|^2) / 2,$$

with

$$\omega(r) \overset{\text{def}}{=} 4 \max_{i \leq n} \sup_{θ \in Θ_0(r)} q_i(\theta, \theta^*).$$

If each density function $p_i$ is uniformly bounded by a constant $C$ then

$$|q_i(\theta) - q_i(\theta^*)| \leq C|Ψ_i^T (\theta - \theta^*)| \leq CN^{-1/2}\|V_0 (\theta - \theta^*)\| \leq CN^{-1/2}r.$$ 

Next we check the local identifiability condition. We use the following technical lemma.

**Lemma 5.9.** It holds for any $θ$

$$-\frac{\partial^2}{\partial^2 \theta} \mathbb{E} L(θ) = D^2(θ) \overset{\text{def}}{=} \sum p_i(Ψ_i^T (θ - \theta^*))Ψ_iΨ_i^T,$$ 

(5.19)

where $f_i(·)$ is the density of $\tilde{ξ}_i = Y_i - Ψ_i^T \theta^*$. Moreover, there is $θ^0 \in [θ, θ^*]$ such that

$$-\mathbb{E} L(θ, \theta^*) = \frac{1}{2} \sum |Ψ_i^T (θ - \theta^*)|^2 f_i(Ψ_i^T (θ^0 - \theta^*))$$

$$= (θ - \theta^*)^T D^2(θ^0)(θ - \theta^*) / 2.$$ 

(5.20)

**Proof.** Obviously

$$\frac{\partial \mathbb{E} L(θ)}{\partial \theta} = \sum \{F(Y_i \leq Ψ_i^T \theta) - 1/2\} \Psi_i.$$

The identity (5.19) is obtained by one more differentiation. By definition, $θ^*$ is the extreme point of $\mathbb{E} L(θ)$. The equality $∇ \mathbb{E} L(θ^*) = 0$ yields

$$\sum \{F(Y_i \leq Ψ_i^T \theta^*) - 1/2\} \Psi_i = 0.$$

Now (5.20) follows by the Taylor expansion of the second order at $θ^*$.

Define

$$D_0^2 \overset{\text{def}}{=} \sum |Ψ_i^T (θ - \theta^*)|^2 f_i(0).$$ 

(5.21)
Due to this lemma, condition \((L_0)\) is fulfilled in \(\Theta_0(\mathbf{r})\) with this choice \(D_0\) for \(\delta(\mathbf{r})\) from (5.10); see Lemma 5.7. Moreover, if \(f_i(0) \geq a^2/4\) for \(a > 0\), then the identifiability condition \((I)\) is also satisfied. Now all the local conditions are fulfilled yielding the general bracketing bound of Theorem 3.1 and all its corollaries.

It only remains to accomplish them by a large deviation bound, that is, to specify the local vicinity \(\Theta_0(\mathbf{r}_0)\) providing the prescribed deviation bound. A sufficient condition for the concentration property is that the expectation \(\mathbb{E}L(\theta, \theta^*)\) grows in absolute value with the distance \(\|V_0(\theta - \theta^*)\|\). We use the representation (5.19). Suppose that for some fixed \(\delta < 1/2\) and \(\rho > 0\)

\[
|f_i(u)/f_i(0) - 1| \leq \delta, \quad |u| \leq \rho. \tag{5.22}
\]

For any \(\theta\) with \(\|V_0(\theta - \theta^*)\| = \mathbf{r} \geq \mathbf{r}_0\), and for any \(i = 1, \ldots, n\), it holds

\[
|\Psi_i^\top(\theta - \theta^*)| \leq N^{-1/2}\|V_0(\theta - \theta^*)\| = N^{-1/2}\mathbf{r}.
\]

Therefore, for \(\mathbf{r} \leq \rho N^{1/2}\) and any \(\theta \in \Theta_0(\mathbf{r})\) with \(\|V_0(\theta - \theta^*)\| = \mathbf{r}\), it holds \(f_i(\Psi_i^\top(\theta^0 - \theta^*)) \geq (1 - \delta)f_i(0)\). Now Lemma 5.9 implies

\[-\mathbb{E}L(\theta, \theta^*) \geq \frac{1 - \delta}{2}\|D_0(\theta - \theta^*)\|^2 \geq \frac{1 - \delta}{2a^2}\|V_0(\theta - \theta^*)\|^2 = \frac{1 - \delta}{2a^2}\mathbf{r}^2.\]

By Lemma 5.9 the function \(-\mathbb{E}L(\theta, \theta^*)\) is convex. This easily yields

\[-\mathbb{E}L(\theta, \theta^*) \geq \frac{1 - \delta}{2a^2}\rho N^{1/2}\mathbf{r}\]

for all \(\mathbf{r} \geq \rho N^{1/2}\). Thus,

\[
\mathbf{r}_b(\mathbf{r}) \geq \begin{cases} 
(1 - \delta)(2a^2)^{-1}\mathbf{r} & \text{if } \mathbf{r} \leq \rho N^{1/2}, \\
(1 - \delta)(2a^2)^{-1}\rho N^{1/2} & \text{if } \mathbf{r} > \rho N^{1/2}.
\end{cases}
\]

So, the global identifiability condition \((L_1)\) is fulfilled if \(\mathbf{r}_0^2 \geq C_1a^2(\mathbf{x} + \mathbb{Q})\) and if \(\rho^2N \geq C_2a^2(\mathbf{x} + \mathbb{Q})\) for some fixed constants \(C_1\) and \(C_2\).

Putting all together yields the following result.

**Theorem 5.10.** Let \(Y_i\) be independent, \(\theta^* = \arg\max_{\theta} \mathbb{E}L(\theta)\), \(D_0^2\) be given by (5.21), and \(V_0^2\) by (5.17). Let also the densities \(f_i(\cdot)\) of \(Y_i - \Psi_i^\top\theta^*\) be uniformly bounded by a constant \(C\), fulfill (5.22) for some \(\rho > 0\) and \(\delta > 0\), and \(f_i(0) \geq a^2/4\) for all \(i\). Finally, let \(N \geq C_2\rho^{-2}a^2(\mathbf{x} + \mathbf{p})\) for some fixed \(\mathbf{x} > 0\) and \(C_2\). Then on the random set of probability at least \(1 - e^{-x}\), one obtains for \(\xi := D_0^{-1}\nabla L(\theta^*)\) the bounds

\[
\|\sqrt{D_0}(\tilde{\theta} - \theta^*) - \xi\|^2 = o(p), \quad 2L(\tilde{\theta}, \theta^*) - \|\xi\|^2 = o(p).
\]
A Deviation probability for quadratic forms

The approximation results of the previous sections rely on the probability of the form $\mathbb{P}(\|\xi\| > y)$ for a given random vector $\xi \in \mathbb{R}^p$. The only condition imposed on this vector is that

$$\log \mathbb{E} \exp (\gamma^\top \xi) \leq \nu_0^2 \|\gamma\|^2 / 2, \quad \gamma \in \mathbb{R}^p, \|\gamma\| \leq g.$$ 

To simplify the presentation we rewrite this condition as

$$\log \mathbb{E} \exp (\gamma^\top \xi) \leq \|\gamma\|^2 / 2, \quad \gamma \in \mathbb{R}^p, \|\gamma\| \leq g. \quad (A.1)$$

The general case can be reduced to $\nu_0 = 1$ by rescaling $\xi$ and $g$:

$$\log \mathbb{E} \exp (\gamma^\top \xi / \nu_0) \leq \|\gamma\|^2 / 2, \quad \gamma \in \mathbb{R}^p, \|\gamma\| \leq \nu_0 g,$$

that is, $\nu_0^{-1} \xi$ fulfills (A.1) with a slightly increased $g$. In typical situations like in Section 5, the value $g$ is large (of order root-$n$) while the value $\nu_0$ is close to one.

A.1 Gaussian case

Our benchmark will be a deviation bound for $\|\xi\|^2$ for a standard Gaussian vector $\xi$. The ultimate goal is to show that under (A.1) the norm of the vector $\xi$ exhibits behavior expected for a Gaussian vector, at least in the region of moderate deviations. For the reason of comparison, we begin by stating the result for a Gaussian vector $\xi$.

Theorem A.1. Let $\xi$ be a standard normal vector in $\mathbb{R}^p$. Then for any $u > 0$, it holds

$$\mathbb{P}(\|\xi\|^2 > p + u) \leq \exp \left\{ -(p/2) \phi(u/p) \right\}$$

with

$$\phi(t) \overset{\text{def}}{=} t - \log(1 + t).$$

Let $\phi^{-1}(\cdot)$ stand for the inverse of $\phi(\cdot)$. For any $x$,

$$\mathbb{P}(\|\xi\|^2 > p + \phi^{-1}(2x/p)) \leq \exp(-x).$$

This particularly yields with $\varkappa = 6.6$

$$\mathbb{P}(\|\xi\|^2 > p + \sqrt{\varkappa} \exp \vee (\varkappa x)) \leq \exp(-x).$$
Proof. The proof utilizes the following well known fact: for $\mu < 1$

$$\log \mathbb{E} \exp(\mu \|\xi\|^2 / 2) = -0.5 \rho \log(1 - \mu).$$

It can be obtained by straightforward calculus. Now consider any $u > 0$. By the exponential Chebyshev inequality

$$\mathbb{P}(\|\xi\|^2 > p + u) \leq \exp\{-\mu(p + u)/2\} \mathbb{E} \exp(\mu \|\xi\|^2 / 2) = \exp\{-\mu(p + u)/2 - (p/2) \log(1 - \mu)\}. \quad (A.2)$$

It is easy to see that the value $\mu = u/(u + p)$ maximizes $\mu(p + u) + p \log(1 - \mu)$ w.r.t. $\mu$ yielding

$$\mu(p + u) - p \log(1 - \mu) = u - p \log(1 + u/p).$$

Further we use that $x - \log(1 + x) \geq a_0 x^2$ for $x \leq 1$ and $x - \log(1 + x) \geq a_0 x$ for $x > 1$ with $a_0 = 1 - \log(2) \geq 0.3$. This implies with $x = u/p$ for $u = \sqrt{2xp}$ or $u = \kappa x$ and $\kappa = 2/a_0 < 6.6$ that

$$\mathbb{P}(\|\xi\|^2 \geq p + \sqrt{2xp} \lor (\kappa x)) \leq \exp(-x)$$

as required. \qed

The message of this result is that the squared norm of the Gaussian vector $\xi$ concentrates around the value $p$ and the deviation over the level $p + \sqrt{xp}$ are exponentially small in $x$.

A similar bound can be obtained for a norm of the vector $\mathbb{B} \xi$ where $\mathbb{B}$ is some given matrix. For notational simplicity we assume that $\mathbb{B}$ is symmetric. Otherwise one should replace it with $(\mathbb{B}^\top \mathbb{B})^{1/2}$.

**Theorem A.2.** Let $\xi$ be standard normal in $\mathbb{R}^p$. Then for every $x > 0$ and any symmetric matrix $\mathbb{B}$, it holds with $p_0 = \text{tr}(\mathbb{B}^2)$, $v^2 = 2 \text{tr}(\mathbb{B}^4)$, and $a^* = \|\mathbb{B}^2\|_\infty$

$$\mathbb{P}(\|\mathbb{B} \xi\|^2 > p_0 + (2v^2x^{1/2}) \lor (6a^*x)) \leq \exp(-x).$$

**Proof.** The matrix $\mathbb{B}^2$ can be represented as $U^\top \text{diag}(a_1, \ldots, a_p)U$ for an orthogonal matrix $U$. The vector $\tilde{\xi} = U \xi$ is also standard normal and $\|\mathbb{B} \xi\|^2 = \tilde{\xi}^\top U \mathbb{B}^2 U^\top \tilde{\xi}$. This means that one can reduce the situation to the case of a diagonal matrix $\mathbb{B}^2 = \text{diag}(a_1, \ldots, a_p)$. We can also assume without loss of generality that $a_1 \geq a_2 \geq \ldots \geq a_p$.

The expressions for the quantities $p_0$ and $v^2$ simplifies to

$$p_0 = \text{tr}(\mathbb{B}^2) = a_1 + \ldots + a_p,$$
$$v^2 = 2 \text{tr}(\mathbb{B}^4) = 2(a_1^2 + \ldots + a_p^2).$$
Moreover, rescaling the matrix $B^2$ by $a_1$ reduces the situation to the case with $a_1 = 1$.

**Lemma A.3.** It holds

$$\mathbb{E}\|B\xi\|_2^2 = \text{tr}(B^2), \quad \text{Var}(\|B\xi\|_2^2) = 2 \text{tr}(B^4).$$

Moreover, for $\mu < 1$

$$\mathbb{E}\exp\{\mu\|B\xi\|_2^2/2\} = \det(1 - \mu B^2)^{-1/2} = \prod_{i=1}^p (1 - \mu a_i)^{-1/2}. \tag{A.3}$$

*Proof.* If $B^2$ is diagonal, then $\|B\xi\|_2^2 = \sum_i a_i \xi_i^2$ and the summands $a_i \xi_i^2$ are independent. It remains to note that $\mathbb{E}(a_i \xi_i^2) = a_i$, $\text{Var}(a_i \xi_i^2) = 2a_i^2$, and for $\mu a_i < 1$,

$$\mathbb{E}\exp\{\mu a_i \xi_i^2/2\} = (1 - \mu a_i)^{-1/2}$$

yielding (A.3). \hfill \square

Given $u$, fix $\mu < 1$. The exponential Markov inequality yields

$$\mathbb{P}(\|B\xi\|_2^2 > p_0 + u) \leq \exp\left\{ -\frac{\mu(p_0 + u)}{2} \right\} \mathbb{E}\exp\left( \frac{\mu\|B\xi\|_2^2}{2} \right)$$

$$\leq \exp\left\{ -\frac{\mu u}{2} - \frac{1}{2} \sum_{i=1}^p [\mu a_i + \log(1 - \mu a_i)] \right\}. $$

We start with the case when $x^{1/2} \leq v/3$. Then $u = 2x^{1/2}v$ fulfills $u \leq 2v^2/3$. Define $\mu = u/v^2 \leq 2/3$ and use that $t + \log(1 - t) \geq -t^2$ for $t \leq 2/3$. This implies

$$\mathbb{P}(\|B\xi\|_2^2 > p_0 + u)$$

$$\leq \exp\left\{ -\frac{\mu u}{2} + \frac{1}{2} \sum_{i=1}^p \mu^2 a_i^2 \right\} = \exp(-u^2/(4v^2)) = e^{-x}. \tag{A.4}$$

Next, let $x^{1/2} > v/3$. Set $\mu = 2/3$. It holds similarly to the above

$$\sum_{i=1}^p [\mu a_i + \log(1 - \mu a_i)] \geq -\sum_{i=1}^p \mu^2 a_i^2 \geq -2v^2/9 \geq -2x.$$ 

Now, for $u = 6x$ and $\mu u/2 = 2x$, (A.4) implies

$$\mathbb{P}(\|B\xi\|_2^2 > p_0 + u) \leq \exp\left\{ -(2x - x) \right\} = \exp(-x)$$

as required. \hfill \square

Below we establish similar bounds for a non-Gaussian vector $\xi$ obeying (A.1).
A.2 A bound for the $\ell_2$-norm

This section presents a general exponential bound for the probability $\mathbb{P}(\|\xi\| > y)$ under (A.1). Given $g$ and $p$, define the values $w_0 = gp^{-1/2}$ and $w_c$ by the equation

$$w_c(1 + w_c) = w_0 = gp^{-1/2}. \tag{A.5}$$

It is easy to see that $w_0/\sqrt{2} \leq w_c \leq w_0$. Further define

$$\mu_c \triangleq w_c^2/(1 + w_c^2),$$

$$y_c \triangleq \sqrt{(1 + w_c^2)p},$$

$$x_c \triangleq 0.5p[w_c^2 - \log(1 + w_c^2)]. \tag{A.6}$$

Note that for $g^2 \geq p$, the quantities $y_c$ and $x_c$ can be evaluated as $y_c^2 \geq w_c^2$ and $x_c \geq pw_c^2/2 \geq g^2/4$.

**Theorem A.4.** Let $\xi \in \mathbb{R}^p$ fulfill (A.1). Then it holds for each $x \leq x_c$

$$\mathbb{P}(\|\xi\|^2 > p + \sqrt{2}\exp \vee (zx), \|\xi\| \leq y_c) \leq 2\exp(-x),$$

where $x = 6.6$. Moreover, for $y \geq y_c$, it holds with $g_c = g - \sqrt{\mu_c p} = gw_c/(1 + w_c)$

$$\mathbb{P}(\|\xi\| > y) \leq 8.4\exp\{-g_c y/2 - (p/2)\log(1 - g_c/y)\} \leq 8.4\exp\{-x_c - g_c(y - y_c)/2\}.$$ 

**Proof.** The main step of the proof is the following exponential bound.

**Lemma A.5.** Suppose (A.1). For any $\mu < 1$ with $g^2 > p\mu$, it holds

$$\mathbb{E} \exp\left(\frac{\mu\|\xi\|^2}{2}\right) \mathbb{I}(\|\xi\| \leq g/\mu - \sqrt{p/\mu}) \leq 2(1 - \mu)^{-p/2}. \tag{A.7}$$

**Proof.** Let $\varepsilon$ be a standard normal vector in $\mathbb{R}^p$ and $u \in \mathbb{R}^p$. The bound $\mathbb{P}(\|\varepsilon\|^2 > p) \leq 1/2$ implies for any vector $u$ and any $r$ with $r \geq \|u\| + p^{1/2}$ that $\mathbb{P}(\|u + \varepsilon\| \leq r) \geq 1/2$. Let us fix some $\xi$ with $\|\xi\| \leq g/\mu - \sqrt{p/\mu}$ and denote by $\mathbb{P}_\xi$ the conditional probability given $\xi$. It holds with $c_p = (2\pi)^{-p/2}$

$$c_p \int \exp\left(\gamma^T\xi - \frac{\|\gamma\|^2}{2\mu}\right) \mathbb{I}(\|\gamma\| \leq g)d\gamma$$

$$= c_p \exp(\mu\|\xi\|^2/2) \int \exp\left(-\frac{1}{2}\|\mu^{-1/2}\gamma - \mu^{1/2}\xi\|^2\right) \mathbb{I}(\mu^{-1/2}\|\gamma\| \leq \mu^{-1/2}g)d\gamma$$

$$= \mu^{p/2} \exp(\mu\|\xi\|^2/2) \mathbb{P}_\xi(\|\varepsilon + \mu^{1/2}\xi\| \leq \mu^{-1/2}g)$$

$$\geq 0.5\mu^{p/2} \exp(\mu\|\xi\|^2/2),$$
because $\|\mu^{1/2}\xi\| + p^{1/2} \leq \mu^{-1/2}g$. This implies in view of $p < g^2/\mu$ that

$$\exp(\mu\|\xi\|^2/2) \mathbb{I}(\|\xi\|^2 \leq g/\mu - \sqrt{p/\mu}) \leq 2\mu^{-p/2}c_p \int \exp\left(\gamma^\top \xi - \frac{\|\gamma\|^2}{2\mu}\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma.$$  

Further, by (A.1)

$$c_p \mathbb{E} \int \exp\left(\gamma^\top \xi - \frac{1}{2\mu}\|\gamma\|^2\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma \leq c_p \int \exp\left(-\frac{\mu - 1}{2}\|\gamma\|^2\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma \leq c_p \int \exp\left(-\frac{\mu - 1}{2}\|\gamma\|^2\right) d\gamma \leq \frac{1}{\mu - 1} \exp^{-p/2}$$

and (A.7) follows.

Due to this result, the scaled squared norm $\mu\|\xi\|^2/2$ after a proper truncation possesses the same exponential moments as in the Gaussian case. A straightforward implication is the probability bound $P(\|\xi\|^2 > p + u)$ for moderate values $u$. Namely, given $u > 0$, define $\mu = u/(u + p)$. This value optimizes the inequality (A.2) in the Gaussian case. Now we can apply a similar bound under the constraints $\|\xi\| \leq g/\mu - \sqrt{p/\mu}$. Therefore, the bound is only meaningful if $\sqrt{u + p} \leq g/\mu - \sqrt{p/\mu}$ with $\mu = u/(u + p)$, or, with $w = \sqrt{u/p} \leq \sqrt{c}$; see (A.5).

The largest value $u$ for which this constraint is still valid, is given by $p + u = y_c^2$. Hence, (A.7) yields for $p + u \leq y_c^2$

$$P(\|\xi\|^2 > p + u, \|\xi\| \leq y_c) \leq \exp\left\{-\frac{\mu(p + u)}{2}\right\} \mathbb{E} \exp\left(\frac{\|\xi\|^2}{2}\right) \mathbb{I}(\|\xi\| \leq g/\mu - \sqrt{p/\mu}) \leq 2 \exp\left\{-0.5 \left[\mu(p + u) + p \log(1 - \mu)\right]\right\} \leq 2 \exp\left\{-0.5 \left[u - p \log(1 + u/p)\right]\right\}.$$  

Similarly to the Gaussian case, this implies with $\kappa = 6.6$ that

$$P(\|\xi\| \geq p + \sqrt{\kappa xp} \vee (\kappa x), \|\xi\| \leq y_c) \leq 2 \exp(-x).$$

The Gaussian case means that (A.1) holds with $g = \infty$ yielding $y_c = \infty$. In the non-Gaussian case with a finite $g$, we have to accompany the moderate deviation bound with a large deviation bound $P(\|\xi\| > y)$ for $y \geq y_c$. This is done by combining the bound (A.7) with the standard slicing arguments.
Lemma A.6. Let $\mu_0 \leq \frac{g^2}{p}$. Define $y_0 = \frac{g}{\mu_0} - \sqrt{\frac{p}{\mu_0}}$ and $g_0 = \mu_0 y_0 = g - \sqrt{\mu_0 p}$. It holds for $y \geq y_0$

\[
\mathbb{P}(\|\xi\| > y) \leq 8.4(1 - \frac{g_0}{y})^{-p/2} \exp(-g_0 y/2) \tag{A.8}
\]

\[
\leq 8.4 \exp\left\{-x_0 -(g_0(y-y_0))/2\right\}. \tag{A.9}
\]

with $x_0$ defined by

\[
2x_0 = \mu_0 y_0^2 + p \log(1 - \mu_0) = \frac{g^2}{\mu_0} - p + p \log(1 - \mu_0).
\]

Proof. Consider the growing sequence $y_k$ with $y_1 = y$ and $g_0 y_{k+1} = g_0 y + k$. Define also $\mu_k = g_0 / y_k$. In particular, $\mu_k \leq \mu_1 = g_0 / y$. Obviously

\[
\mathbb{P}(\|\xi\| > y) = \sum_{k=1}^{\infty} \mathbb{P}(\|\xi\| > y_k, \|\xi\| \leq y_{k+1}).
\]

Now we try to evaluate every slicing probability in this expression. We use that

\[
\mu_{k+1} y_k^2 = \frac{(g_0 y + k - 1)^2}{g_0 y + k} \geq g_0 y + k - 2,
\]

and also $g/\mu_k - \sqrt{p/\mu_k} \geq y_k$ because $g - g_0 = \sqrt{\mu_0 p} > \sqrt{\mu_k p}$ and

\[
g/\mu_k - \sqrt{p/\mu_k} - y_k = \mu_k^{-1}(g - \sqrt{\mu_k p} - g_0) \geq 0.
\]

Hence by (A.7)

\[
\mathbb{P}(\|\xi\| > y) \leq \sum_{k=1}^{\infty} \mathbb{P}(\|\xi\| > y_k, \|\xi\| \leq y_{k+1})
\]

\[
\leq \sum_{k=1}^{\infty} \exp\left(-\frac{\mu_{k+1} y_k^2}{2}\right) \mathbb{E} \exp\left(\frac{\mu_{k+1} \|\xi\|^2}{2}\right) \mathbb{I}(\|\xi\| \leq y_{k+1})
\]

\[
\leq \sum_{k=1}^{\infty} 2(1 - \mu_{k+1})^{-p/2} \exp\left(-\frac{\mu_{k+1} y_k^2}{2}\right)
\]

\[
\leq 2(1 - \mu_1)^{-p/2} \sum_{k=1}^{\infty} \exp\left(-\frac{g_0 y + k - 2}{2}\right)
\]

\[
= 2e^{1/2}(1 - e^{-1/2})^{-1}(1 - \mu_1)^{-p/2} \exp(-g_0 y/2)
\]

\[
\leq 8.4(1 - \mu_1)^{-p/2} \exp(-g_0 y/2)
\]

and the first assertion follows. For $y = y_0$, it holds

\[
g_0 y_0 + p \log(1 - \mu_0) = \mu_0 y_0^2 + p \log(1 - \mu_0) = 2x_0
\]
and (A.8) implies $P(\|\xi\| > y_0) \leq 8.4 \exp(-x_0)$. Now observe that the function $f(y) = g_0 y/2 + (p/2) \log (1 - g_0/y)$ fulfills $f(y_0) = x_0$ and $f'(y) \geq g_0/2$ yielding $f(y) \geq x_0 + g_0(y - y_0)/2$. This implies (A.9).

The statements of the theorem are obtained by applying the lemmas with $\mu_0 = \mu_c = w^2_c/(1 + w^2_c)$. This also implies $y_0 = y_c$, $x_0 = x_c$, and $g_0 = g_c = g - \sqrt{\mu_c p}$; cf. (A.6).

The statements of Theorem A.8 can be simplified under the assumption $g^2 \geq p$.

**Corollary A.7.** Let $\xi$ fulfill (A.1) and $g^2 \geq p$. Then it holds for $x \leq x_c$

$$P(\|\xi\|^2 \geq \delta(x, p)) \leq 2e^{-x} + 8.4e^{-x_c}, \quad (A.10)$$

$$\delta(x, p) \overset{\text{def}}{=} \begin{cases} p + \sqrt{x xp}, & x \leq p/\kappa, \\ p/\kappa & p/\kappa < x \leq x_c, \end{cases} \quad (A.11)$$

with $\kappa = 6.6$. For $x > x_c$

$$P(\|\xi\|^2 \geq \delta_c(x, p)) \leq 8.4e^{-x}, \quad \delta_c(x, p) \overset{\text{def}}{=} |y_c + 2(x - x_c)/g_c|^2. \quad (A.11)$$

This result implicitly assumes that $p \leq \kappa x_c$ which is fulfilled if $w^2_0 = g^2/p \geq 1$:

$$\kappa x_c = 0.5\kappa [w^2_0 - \log(1 + w^2_0)]p \geq 3.3 [1 - \log(2)]p > p.$$  

In the zone $x \leq p/\kappa$ we obtain sub-Gaussian behavior of the tail of $\|\xi\|^2 - p$, in the zone $p/\kappa < x \leq x_c$ it becomes sub-exponential. Note that the sub-exponential zone is empty if $g^2 < p$.

For $x \leq x_c$, the function $\delta(x, p)$ mimics the quantile behavior of the chi-squared distribution $\chi^2_p$ with $p$ degrees of freedom. Moreover, increase of the value $g$ yields a growth of the sub-Gaussian zone. In particular, for $g = \infty$, a general quadratic form $\|\xi\|^2$ has under (A.1) the same tail behavior as in the Gaussian case.

Finally, in the large deviation zone $x > x_c$ the deviation probability decays as $e^{-c x^{1/2}}$ for some fixed $c$. However, if the constant $g$ in the condition (A.1) is sufficiently large relative to $p$, then $x_c$ is large as well and the large deviation zone $x > x_c$ can be ignored at a small price of $8.4e^{-x_c}$ and one can focus on the deviation bound described by (A.10) and (A.11).

**A.3 A bound for a quadratic form**

Now we extend the result to more general bound for $\|IB\xi\|^2 = \xi^\top I B I^2 \xi$ with a given matrix $IB$ and a vector $\xi$ obeying the condition (A.1). Similarly to the Gaussian case
we assume that $\mathcal{B}$ is symmetric. Define important characteristics of $\mathcal{B}$

$$p_0 = \text{tr}(\mathcal{B}^2), \quad v^2 = 2\text{tr}(\mathcal{B}^4), \quad \lambda^* \overset{\text{def}}{=} \|\mathcal{B}\|_\infty \overset{\text{def}}{=} \lambda_{\max}(\mathcal{B}^2).$$

For simplicity of formulation we suppose that $\lambda^* = 1$, otherwise one has to replace $p_0$ and $v^2$ with $p_0/\lambda^*$ and $v^2/\lambda^*$.

Let $g$ be shown in (A.1). Define similarly to the $\ell_2$-case $w_c$ by the equation

$$\frac{w_c(1 + w_c)}{(1 + w_c)^{1/2}} = g p_0^{-1/2}. \quad \text{(A.1)}$$

Define also $\mu_c = w_c^2/(1 + w_c^2) \wedge 2/3$. Note that $w_c^2 \geq 2$ implies $\mu_c = 2/3$. Further define

$$y_c^2 = (1 + w_c^2)p_0, \quad 2x_c = \mu_c y_c^2 + \log\det\{I_p - \mu_c\mathcal{B}^2\}. \quad \text{(A.12)}$$

Similarly to the case with $\mathcal{B} = I_p$, under the condition $g^2 \geq p_0$, one can bound $y_c^2 \geq g^2/2$ and $x_c \geq g^2/4$.

**Theorem A.8.** Let a random vector $\xi$ in $\mathbb{R}^p$ fulfill (A.1). Then for each $x < x_c$

$$\Pr\left(\|\mathcal{B}\xi\|^2 > p_0 + (2vx)^2 \vee (6x), \|\mathcal{B}\xi\| \leq y_c\right) \leq 2\exp(-x).$$

Moreover, for $y \geq y_c$, with $g_c = g - \sqrt{\mu_c p_0} = gw_c/(1 + w_c)$, it holds

$$\Pr\left(\|\mathcal{B}\xi\| > y\right) \leq 8.4\exp(-x_c - g_c(y - y_c)/2).$$

**Proof.** The main steps of the proof are similar to the proof of Theorem A.4.

**Lemma A.9.** Suppose (A.1). For any $\mu < 1$ with $g^2/\mu \geq p_0$, it holds

$$\mathbb{E}\exp\left(\mu\|\mathcal{B}\xi\|^2/2\right) \mathbb{I}\left(\|\mathcal{B}\xi\| \leq g/\mu - \sqrt{p_0/\mu}\right) \leq 2\det(I_p - \mu\mathcal{B})^{-1/2}. \quad \text{(A.13)}$$

**Proof.** With $c_p(\mathcal{B}) = (2\pi)^{-p/2} \det(\mathcal{B}^{-1})$

$$c_p(\mathcal{B}) \int \exp\left(-\frac{\gamma^\top\mathcal{B}^{-1}\gamma}{2}\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma$$

$$= c_p(\mathcal{B}) \exp\left(-\frac{\mu\|\mathcal{B}\xi\|^2}{2}\right) \int \exp\left(-\frac{1}{2}\|\mu^{1/2}\mathcal{B}\xi - \mu^{-1/2}\mathcal{B}^{-1}\gamma\|^2\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma$$

$$= \mu^{p/2} \exp\left(-\frac{\mu\|\mathcal{B}\xi\|^2}{2}\right) \Pr_{\xi}(\|\mu^{1/2}\mathcal{B}\varepsilon + \mathcal{B}\xi\| \leq g/\mu),$$

where $\varepsilon$ denotes a standard normal vector in $\mathbb{R}^p$ and $\Pr_{\xi}$ means the conditional probability given $\xi$. Moreover, for any $u \in \mathbb{R}^p$ and $x \geq p_0^{1/2} + \|u\|$, it holds in view of $\Pr(\|\mathcal{B}\xi\|^2 > p_0) \leq 1/2$

$$\Pr(\|\mathcal{B}\varepsilon - u\| \leq x) \geq \Pr(\|\mathcal{B}\varepsilon\| \leq \sqrt{p_0}) \geq 1/2.$$
This implies
\[
\exp\left(\mu \|B\xi\|^2 / 2\right) \mathbb{I}(\|B^2\xi\| \leq g / \mu - \sqrt{p_0 / \mu}) \\
\leq 2\mu^{-p/2}c_p(B) \int \exp\left(\mathcal{G}^\top \xi - \frac{1}{2\mu} \|B^{-1}\gamma\|^2\right) \mathbb{I}(\|\gamma\| \leq g) d\gamma.
\]

Further, by (A.1)
\[
c_p(B) \mathbb{E} \int \exp\left(\frac{\|\gamma\|^2}{2} - \frac{1}{2\mu} \|B^{-1}\gamma\|^2\right) d\gamma \\
\leq c_p(B) \int \exp\left(\frac{\|\gamma\|^2}{2} - \frac{1}{2\mu} \|B^{-1}\gamma\|^2\right) d\gamma \\
\leq \det(B^{-1}) \det(\mu^{-1}B^{-2} - I_p)^{-1/2} = \mu^{p/2} \det(I_p - \mu B^2)^{-1/2}
\]
and (A.13) follows.

Now we evaluate the probability \( P(\|B\xi\| > y) \) for moderate values of \( y \).

**Lemma A.10.** Let \( \mu_0 < 1 / (g^2 / p_0) \). With \( y_0 = g / \mu_0 - \sqrt{p_0 / \mu_0} \), it holds for any \( u > 0 \)
\[
P(\|B\xi\|^2 > p_0 + u, \|B^2\xi\| \leq y_0) \\
\leq 2 \exp\left\{-0.5\mu_0(p_0 + u) - 0.5 \log \det(I_p - \mu B^2)\right\}. \tag{A.14}
\]

In particular, if \( B^2 \) is diagonal, that is, \( B^2 = \text{diag}(a_1, \ldots, a_p) \), then
\[
P(\|B\xi\|^2 > p_0 + u, \|B^2\xi\| \leq y_0) \\
\leq 2 \exp\left\{-\frac{\mu_0 u}{2} - \frac{1}{2} \sum_{i=1}^{p} [\mu_0 a_i + \log(1 - \mu_0 a_i)]\right\}. \tag{A.15}
\]

**Proof.** The exponential Chebyshev inequality and (A.13) imply
\[
P(\|B\xi\|^2 > p_0 + u, \|B^2\xi\| \leq y_0) \\
\leq \exp\left\{-\frac{\mu_0(p_0 + u)}{2}\right\} \mathbb{E} \exp\left(\frac{\mu_0 \|B^2\xi\|^2}{2}\right) \mathbb{I}(\|B^2\xi\| \leq g / \mu_0 - \sqrt{p_0 / \mu_0}) \\
\leq 2 \exp\left\{-0.5\mu_0(p_0 + u) - 0.5 \log \det(I_p - \mu B^2)\right\}.
\]

Moreover, the standard change-of-basis arguments allow us to reduce the problem to the case of a diagonal matrix \( B^2 = \text{diag}(a_1, \ldots, a_p) \) where \( 1 = a_1 \geq a_2 \geq \ldots \geq a_p > 0 \). Note that \( p_0 = a_1 + \ldots + a_p \). Then the claim (A.14) can be written in the form (A.15).

Now we evaluate a large deviation probability that \( \|B\xi\| > y \) for a large \( y \). Note that the condition \( \|B^2\|_\infty \leq 1 \) implies \( \|B^2\xi\| \leq \|B\xi\| \). So, the bound (A.14) continues to hold when \( \|B^2\xi\| \leq y_0 \) is replaced by \( \|B\xi\| \leq y_0 \).
Lemma A.11. Let \( \mu_0 < 1 \) and \( \mu_0 p_0 < g^2 \). Define \( g_0 = g - \sqrt{\mu_0 p_0} \). For any \( y \geq y_0 \equiv g_0 / \mu_0 \), it holds

\[
\text{IP}(\| \mathcal{B} \xi \| > y) \leq 8.4 \text{det} \{ I_p - (g_0/y) \mathcal{B}^2 \}^{-1/2} \exp \left( -\frac{g_0 y}{2} \right).
\]

where \( x_0 \) is defined by

\[
x_0 = \frac{g_0 y_0 + \log \text{det} \{ I_p - (g_0/y) \mathcal{B}^2 \}}{2}.
\]

Proof. The slicing arguments of Lemma A.6 apply here in the same manner. One has to replace \( \| \xi \| \) by \( \| \mathcal{B} \xi \| \) and \( (1 - \mu_1)^{-p/2} \) by \( \text{det} \{ I_p - (g_0/y) \mathcal{B}^2 \}^{-1/2} \). We omit the details. In particular, with \( y = y_0 = g_0 / \mu_0 \), this yields

\[
\text{IP}(\| \mathcal{B} \xi \| > y) \leq 8.4 \exp(-x_0).
\]

Moreover, for the function \( f(y) = g_0 y + \log \text{det} \{ I_p - (g_0/y) \mathcal{B}^2 \} \), it holds \( f'(y) \geq g_0 \) and hence, \( f(y) \geq f(y_0) + g_0(y - y_0) \) for \( y > y_0 \). This implies (A.16). \( \square \)

One important feature of the results of Lemma A.10 and Lemma A.11 is that the value \( \mu_0 < 1 \wedge (g^2 / p_0) \) can be selected arbitrarily. In particular, for \( y \geq y_c \), Lemma A.11 with \( \mu_0 = \mu_c \) yields the large deviation probability \( \text{IP}(\| \mathcal{B} \xi \| > y) \). For bounding the probability \( \text{IP}(\| \mathcal{B} \xi \|^2 > p_0 + u, \| \mathcal{B} \xi \| \leq y_c) \), we use the inequality \( \log(1 - t) \geq -t - t^2 \) for \( t \leq 2/3 \). It implies for \( \mu \leq 2/3 \) that

\[
-\log \text{IP}(\| \mathcal{B} \xi \|^2 > p_0 + u, \| \mathcal{B} \xi \| \leq y_c)
\geq \mu(p_0 + u) + \sum_{i=1}^{p} \log(1 - \mu a_i)
\geq \mu(p_0 + u) - \sum_{i=1}^{p} (\mu a_i + \mu^2 a_i^2) \geq \mu u - \mu^2 v^2 / 2.
\]

(A.17)

Now we distinguish between \( \mu_c = 2/3 \) and \( \mu_c < 2/3 \) starting with \( \mu_c = 2/3 \). The bound (A.17) with \( \mu = 2/3 \) and with \( u = (2v x^{1/2}) \lor (6x) \) yields

\[
\text{IP}(\| \mathcal{B} \xi \|^2 > p_0 + u, \| \mathcal{B} \xi \| \leq y_c) \leq 2 \exp(-x);
\]

see the proof of Theorem A.2 for the Gaussian case.

Now consider \( \mu_c < 2/3 \). For \( x^{1/2} \leq \mu_c v / 2 \), use \( u = 2v x^{1/2} \) and \( \mu_0 = u / v^2 \). It holds \( \mu_0 = u / v^2 \leq \mu_c \) and \( u^2 / (4v^2) = x \) yielding the desired bound by (A.17). For \( x^{1/2} > \mu_c v / 2 \), we select again \( \mu_0 = \mu_c \). It holds with \( u = 4\mu_c^{-1} x \) that \( \mu_c u / 2 - \mu_c^2 v^2 / 4 \geq 2x - x = x \). This completes the proof. \( \square \)
Now we describe the value $\mathcal{I}(x, \mathbb{B})$ ensuring a small value for the large deviation probability $\mathbb{P}(\|\mathbb{B}\xi\|^2 \geq \mathcal{I}(x, \mathbb{B}))$. For ease of formulation, we suppose that $g^2 \geq 2p_0$ yielding $\mu_c^{-1} \leq 3/2$. The other case can be easily adjusted.

**Corollary A.12.** Let $\xi$ fulfill (A.1) with $g^2 \geq 2p_0$. Then it holds for $x \leq x_c$ with $x_c$ from (A.12):

$$
\mathbb{P}(\|\mathbb{B}\xi\|^2 \geq \mathcal{I}(x, \mathbb{B})) \leq 2e^{-x} + 8.4e^{-x_c},
$$

$$
\mathcal{I}(x, \mathbb{B}) \overset{\text{def}}{=} \begin{cases} 
p_0 + 2v1/2, & x \leq v/18, \\
p_0 + 6x & v/18 < x \leq x_c.
\end{cases} \quad (A.18)
$$

For $x > x_c$

$$
\mathbb{P}(\|\mathbb{B}\xi\|^2 \geq \mathcal{I}(x, \mathbb{B})) \leq 8.4e^{-x}, \quad \mathcal{I}(x, \mathbb{B}) \overset{\text{def}}{=} |y_c + 2(x - x_c)/g_c|^2.
$$

**A.4 Rescaling and regularity condition**

The result of Theorem A.8 can be extended to a more general situation when the condition (A.1) is fulfilled for a vector $\zeta$ rescaled by a matrix $V_0$. More precisely, let the random $p$-vector $\zeta$ fulfills for some $p \times p$ matrix $V_0$ the condition

$$
\sup_{\gamma \in \mathbb{R}^p} \log \mathbb{E} \exp\left(\lambda^T \frac{\gamma}{\|V_0\gamma\|}\right) \leq \nu_0^2 \lambda^2/2, \quad |\lambda| \leq g, \quad (A.19)
$$

with some constants $g > 0$, $\nu_0 \geq 1$. Again, a simple change of variables reduces the case of an arbitrary $\nu_0 \geq 1$ to $\nu_0 = 1$. Our aim is to bound the squared norm $\|D_0^{-1}\zeta\|^2$ of a vector $D_0^{-1}\zeta$ for another $p \times p$ positive symmetric matrix $D_0^2$. Note that condition (A.19) implies (A.1) for the rescaled vector $\xi = V_0^{-1}\zeta$. This leads to bounding the quadratic form $\|D_0^{-1}V_0\xi\|^2 = \|\mathbb{B}\xi\|^2$ with $\mathbb{B}^2 = D_0^{-1}V_0^2D_0^{-1}$. It obviously holds

$$
p_0 = \text{tr}(\mathbb{B}^2) = \text{tr}(D_0^{-2}V_0^2).
$$

Now we can apply the result of Corollary A.12.

**Corollary A.13.** Let $\zeta$ fulfill (A.19) with some $V_0$ and $g$. Given $D_0$, define $\mathbb{B}^2 = D_0^{-1}V_0^2D_0^{-1}$, and let $g^2 \geq 2p_0$. Then it holds for $x \leq x_c$ with $x_c$ from (A.12):

$$
\mathbb{P}(\|D_0^{-1}\zeta\|^2 \geq \mathcal{I}(x, \mathbb{B})) \leq 2e^{-x} + 8.4e^{-x_c},
$$

with $\mathcal{I}(x, \mathbb{B})$ from (A.18). For $x > x_c$

$$
\mathbb{P}(\|D_0^{-1}\zeta\|^2 \geq \mathcal{I}(x, \mathbb{B})) \leq 8.4e^{-x}, \quad \mathcal{I}(x, \mathbb{B}) \overset{\text{def}}{=} |y_c + 2(x - x_c)/g_c|^2.
$$
Finally we briefly discuss the regular case with $D_0 \geq aV_0$ for some $a > 0$. This implies $\|H\|_\infty \leq a^{-1}$ and

$$v^2 = 2 \text{tr}(H^4) \leq 2a^{-2}p_0.$$

### A.5 A chi-squared bound with norm-constraints

This section extends the results to the case when the bound (A.1) requires some other conditions than the $\ell_2$-norm of the vector $\gamma$. Namely, we suppose that

$$\log I E \exp(\gamma^T \xi) \leq \|\gamma\|^2/2, \quad \gamma \in \mathbb{R}^p, \|\gamma\|_\circ \leq g_\circ,$$

where $\|\cdot\|_\circ$ is a norm which differs from the usual Euclidean norm. Our driving example is given by the sup-norm case with $\|\gamma\|_\circ \equiv \|\gamma\|_\infty$. We are interested to check whether the previous results of Section A.2 still apply. The answer depends on how massive the set $A(r) = \{\gamma : \|\gamma\|_\circ \leq r\}$ is in terms of the standard Gaussian measure on $\mathbb{R}^p$. Recall that the quadratic norm $\|\varepsilon\|^2$ of a standard Gaussian vector $\varepsilon$ in $\mathbb{R}^p$ concentrates around $p$ at least for $p$ large. We need a similar concentration property for the norm $\|\cdot\|_\circ$. More precisely, we assume for a fixed $r_*$ that

$$I P(\|\varepsilon\|_\circ \leq r_*) \geq 1/2, \quad \varepsilon \sim N(0, I_p).$$

This implies for any value $u_0 > 0$ and all $u \in \mathbb{R}^p$ with $\|u\|_\circ \leq u_0$ that

$$I P(\|\varepsilon - u\|_\circ \leq r_* + u_0) \geq 1/2, \quad \varepsilon \sim N(0, I_p).$$

For each $\tilde{z} > p$, consider

$$\mu(\tilde{z}) = (\tilde{z} - p) / \tilde{z}.$$ 

Given $u_0$, denote by $\tilde{z}_0 = \tilde{z}_0(u_0)$ the root of the equation

$$\frac{g_0}{\mu(\tilde{z}_0)} - \frac{r_*}{\mu^{1/2}(\tilde{z}_0)} = u_0.$$ 

One can easily see that this value exists and unique if $u_0 \geq g_0 - r_*$ and it can be defined as the largest $\tilde{z}$ for which $\frac{g_0}{\mu(\tilde{z})} - \frac{r_*}{\mu^{1/2}(\tilde{z})} \geq u_0$. Let $\mu_0 = \mu(\tilde{z}_0)$ be the corresponding $\mu$-value. Define also $x_0$ by

$$2x_0 = \mu_0 \tilde{z}_0 + p \log(1 - \mu_0).$$

If $u_0 < g_0 - r_*$, then set $\tilde{z}_0 = \infty$, $x_0 = \infty$. 
Theorem A.14. Let a random vector $\xi$ in $\mathbb{R}^p$ fulfill (A.20). Suppose (A.21) and let, given $u_0$, the value $\gamma_0$ be defined by (A.22). Then it holds for any $u > 0$

$$
\mathbb{P}(\|\xi\|^2 > p + u, \|\xi\|_0 \leq u_0) \leq 2 \exp\{-(p/2)\phi(u)\}.
$$

(A.23)

yielding for $x \leq x_0$

$$
\mathbb{P}(\|\xi\|^2 > p + \sqrt{x} \exp (\sqrt{x}), \|\xi\|_0 \leq u_0) \leq 2 \exp(-x),
$$

(A.24)

where $x = 6.6$. Moreover, for $\gamma \geq \gamma_0$, it holds

$$
\mathbb{P}(\|\xi\|^2 > \gamma, \|\xi\|_0 \leq u_0) \leq 2 \exp\{-\mu_0\gamma/2 - (p/2)\log(1 - \mu_0)\}
$$

$$
= 2 \exp\{-x_0 - \gamma_0(\gamma - \gamma_0)/2\}.
$$

Proof. The arguments behind the result are the same as in the one-norm case of Theorem A.4. We only outline the main steps.

Lemma A.15. Suppose (A.20) and (A.21). For any $\mu < 1$ with $\gamma_0 > \mu^{1/2}r_*$, it holds

$$
\mathbb{E}\exp(\mu\|\xi\|^2/2) \mathbb{I}(\|\xi\|_0 \leq \gamma_0/\mu - r_*/\mu^{1/2}) \leq 2(1 - \mu)^{-p/2}.
$$

(A.25)

Proof. Let $\xi$ be a standard normal vector in $\mathbb{R}^p$ and $u \in \mathbb{R}^p$. Let us fix some $\xi$ with $\mu^{1/2}\|\xi\|_0 \leq \mu^{-1/2}\gamma_0 - r_*$ and denote by $\mathbb{P}_\xi$ the conditional probability given $\xi$. It holds by (A.21) with $c_p = (2\pi)^{-p/2}$

$$
c_p \int \exp\left(\gamma^\top\xi - \frac{1}{2\mu}\|\gamma\|^2\right) \mathbb{I}(\|\gamma\|_0 \leq \gamma_0) d\gamma
$$

$$
= c_p \exp(\mu\|\xi\|^2/2) \int \exp\left(-\frac{1}{2}\|\mu^{1/2}\xi - \mu^{-1/2}\gamma\|^2\right) \mathbb{I}(\|\mu^{-1/2}\gamma\|_0 \leq \mu^{-1/2}\gamma_0) d\gamma
$$

$$
= \mu^{p/2} \exp(\mu\|\xi\|^2/2) \mathbb{P}_\xi(\|\xi - \mu^{1/2}\xi\|_0 \leq \mu^{-1/2}\gamma_0)
$$

$$
\geq 0.5\mu^{p/2} \exp(\mu\|\xi\|^2/2).
$$

This implies

$$
\exp\left(\frac{\mu\|\xi\|^2}{2}\right) \mathbb{I}(\|\xi\|_0 \leq \gamma_0/\mu - r_*/\mu^{1/2})
$$

$$
\leq 2\mu^{-p/2}c_p \int \exp\left(\gamma^\top\xi - \frac{1}{2\mu}\|\gamma\|^2\right) \mathbb{I}(\|\gamma\|_0 \leq \gamma_0) d\gamma.
$$

Further, by (A.20)

$$
c_p \mathbb{E} \int \exp\left(\gamma^\top\xi - \frac{1}{2\mu}\|\gamma\|^2\right) \mathbb{I}(\|\gamma\|_0 \leq \gamma_0) d\gamma
$$

$$
\leq c_p \int \exp\left(-\frac{\mu^{-1} - 1}{2}\|\gamma\|^2\right) d\gamma \leq (\mu^{-1} - 1)^{-p/2}
$$
and (A.25) follows. □

As in the Gaussian case, (A.25) implies for $\mathbf{z} > p$ with $\mu = \mu(\mathbf{z}) = (\mathbf{z} - p)/\mathbf{z}$ the bounds (A.23) and (A.24). Note that the value $\mu(\mathbf{z})$ clearly grows with $\mathbf{z}$ from zero to one, while $g_0/\mu(\mathbf{z}) - r_*/\mu^{1/2}(\mathbf{z})$ is strictly decreasing. The value $\mathbf{z}_0$ is defined exactly as the point where $g_0/\mu(\mathbf{z}) - r_*/\mu^{1/2}(\mathbf{z})$ crosses $u_\circ$, so that $g_0/\mu(\mathbf{z}) - r_*/\mu^{1/2}(\mathbf{z}) \geq u_\circ$ for $\mathbf{z} \leq \mathbf{z}_0$.

For $\mathbf{z} > \mathbf{z}_0$, the choice $\mu = \mu(\mathbf{y})$ conflicts with $g_0/\mu(\mathbf{z}) - r_*/\mu^{1/2}(\mathbf{z}) \geq u_\circ$. So, we apply $\mu = \mu_\circ$ yielding by the Markov inequality

$$\mathbb{P}(\|\mathbf{z}\|^2 > \mathbf{z}_0, \|\mathbf{z}\| \leq u_\circ) \leq 2\exp\left\{-\mu_\circ\mathbf{z}_0/2 - (p/2) \log(1 - \mu_\circ)\right\},$$

and the assertion follows. □

It is easy to check that the result continues to hold for the norm of $\Pi \mathbf{z}$ for a given sub-projector $\Pi$ in $\mathbb{R}^p$ satisfying $\Pi = \Pi^\top$, $\Pi^2 \leq \Pi$. As above, denote $p_0 \overset{\text{def}}{=} \text{tr}(\Pi^2)$, $v^2 \overset{\text{def}}{=} 2\text{tr}(\Pi^4)$. Let $r_*$ be fixed to ensure

$$\mathbb{P}(\|\Pi \mathbf{z}\| \leq r_*) \geq 1/2, \quad \varepsilon \sim \mathcal{N}(0, I_\circ).$$

The next result is stated for $g_0 \geq r_* + u_\circ$, which simplifies the formulation.

**Theorem A.16.** Let a random vector $\mathbf{z}$ in $\mathbb{R}^p$ fulfill (A.20) and $\Pi$ follows $\Pi = \Pi^\top$, $\Pi^2 \leq \Pi$. Let some $u_\circ$ be fixed. Then for any $\mu_\circ \leq 2/3$ with $g_0\mu_\circ^{-1} - r_*\mu_\circ^{-1/2} \geq u_\circ$,

$$\mathbb{E}\exp\left\{-\frac{\mu_\circ}{2}(\|\Pi \mathbf{z}\|^2 - p_0)\right\} \mathbb{I}(\|\Pi^2 \mathbf{z}\| \leq u_\circ) \leq 2\exp(\mu_\circ^2 v^2/4), \quad (A.26)$$

where $v^2 = 2\text{tr}(\Pi^4)$. Moreover, if $g_0 \geq r_* + u_\circ$, then for any $\mathbf{z} \geq 0$

$$\mathbb{P}(\|\Pi \mathbf{z}\|^2 > \mathbf{z}_0, \|\Pi^2 \mathbf{z}\| \leq u_\circ)$$

$$\leq \mathbb{P}(\|\Pi \mathbf{z}\|^2 > p_0 + (2v\lambda^{1/2}) \vee (6x), \|\Pi^2 \mathbf{z}\| \leq u_\circ) \leq 2\exp(-x).$$

**Proof.** Arguments from the proof of Lemmas A.9 and A.15 yield in view of $g_0\mu_\circ^{-1} - r_*\mu_\circ^{-1/2} \geq u_\circ$

$$\mathbb{E}\exp\{\mu_\circ\|\Pi \mathbf{z}\|^2/2\} \mathbb{I}(\|\Pi^2 \mathbf{z}\| \leq u_\circ)$$

$$\leq \mathbb{E}\exp(\mu_\circ\|\Pi \mathbf{z}\|^2/2) \mathbb{I}(\|\Pi^2 \mathbf{z}\| \leq g_0/\mu_\circ - p_0/\mu_\circ^{1/2})$$

$$\leq 2\det(I_\circ - \mu_\circ \Pi^2)^{-1/2}.$$

Now the inequality $\log(1 - t) \geq -t - t^2$ for $t \leq 2/3$ implies

$$-\log \det(I_\circ - \mu_\circ \Pi^2) \leq \mu_\circ p_0 + \mu_\circ^2 v^2/2.$$
A.6 A bound for the $\ell_2$-norm under Bernstein conditions

For comparison, we specify the results to the case considered recently in Y. Baraud (2010). Let $\zeta$ be a random vector in $\mathbb{R}^n$ whose components $\zeta_i$ are independent and satisfy the Bernstein type conditions: for all $|\lambda| < c^{-1}$

$$\log \mathbb{E} e^{\lambda \zeta_i} \leq \frac{\lambda^2 \sigma^2}{1 - c|\lambda|}, \quad (A.27)$$

Denote $\xi = \zeta/(2\sigma)$ and consider $\|\gamma\|_o = \|\gamma\|_\infty$. Fix $g_o = \sigma/c$. If $\|\gamma\|_o \leq g_o$, then $1 - c\gamma_i/(2\sigma) \geq 1/2$ and

$$\log \mathbb{E} \exp (\gamma^T \xi) \leq \sum_i \log \mathbb{E} \exp \left(\frac{\gamma_i \zeta_i}{2\sigma}\right) \leq \sum_i \frac{|\gamma_i/(2\sigma)|^2 \sigma^2}{1 - c\gamma_i/(2\sigma)} \leq \|\gamma\|^2/2.$$ 

Let also $S$ be some linear subspace of $\mathbb{R}^n$ with dimension $p_0$ and $\Pi_S$ denote the projector on $S$. For applying the result of Theorem A.14, the value $r_*$ has to be fixed.

We use that the infinity norm $\|\varepsilon\|_\infty$ concentrates around $\sqrt{2 \log p}$.

**Lemma A.17.** It holds for a standard normal vector $\varepsilon \in \mathbb{R}^p$ with $r_* = \sqrt{2 \log p}$

$$\mathbb{P}(\|\varepsilon\|_o \leq r_*) \geq 1/2.$$ 

**Proof.** By definition

$$\mathbb{P}(\|\varepsilon\|_o > r_*) \leq \mathbb{P}(\|\varepsilon\|_\infty > \sqrt{2 \log p}) \leq p \mathbb{P}(|\varepsilon_1| > \sqrt{2 \log p}) \leq 1/2$$

as required. \qed

Now the general bound of Theorem A.14 is applied to bounding the norm of $\|\Pi_S \xi\|$.

For simplicity of formulation we assume that $g_o \geq u_o + r_*$. **Theorem A.18.** Let $S$ be some linear subspace of $\mathbb{R}^n$ with dimension $p_0$. Let $g_o \geq u_o + r_*$. If the coordinates $\zeta_i$ of $\zeta$ are independent and satisfy (A.27), then for all $x$,

$$\mathbb{P}\left(\frac{4\sigma^2}{\|\Pi_S \xi\|^2} > p_0 + \sqrt{2x p_0} \lor (\infty), \|\Pi_S \xi\|_\infty \leq 2\sigma u_o\right) \leq 2 \exp(-x),$$

The bound of Baraud (2010) reads

$$\mathbb{P}\left(\|\Pi_S \xi\|_2 > (3\sigma \lor \sqrt{6cu}) \sqrt{x + 3p_0}, \|\Pi_S \xi\|_\infty \leq 2\sigma u_o\right) \leq e^{-x}.$$ 

As expected, in the region $x \leq x_c$ of Gaussian approximation, the bound of Baraud is not sharp and actually quite rough.
B Some results for empirical processes

This chapter presents some general results of the theory of empirical processes. We assume some exponential moment conditions on the increments of the process which allows to apply the well developed chaining arguments in Orlicz spaces; see e.g. van der Vaart and Wellner (1996), Chapter 2.2. We, however, follow the more recent approach inspired by the notions of generic chaining and majorizing measures due to M. Talagrand; see e.g. Talagrand (1996, 2001, 2005). The results are close to that of Bednorz (2006). We state the results in a slightly different form and present an independent and self-contained proof.

The first result states a bound for local fluctuations of the process \( U(\upsilon) \) given on a metric space \( \Upsilon \). Then this result will be used for bounding the maximum of the negatively drifted process \( U(\upsilon) - U(\upsilon_0) - \rho d^2(\upsilon, \upsilon_0) \) over a vicinity \( \Upsilon_0(r) \) of the central point \( \upsilon_0 \). The behavior of \( U(\upsilon) \) outside of the local central set \( \Upsilon_0(r) \) is described using the upper function method. Namely, we construct a multiscale deterministic function \( u(\mu, \upsilon) \) ensuring that with probability at least \( 1 - e^{-x} \) it holds \( \mu U(\upsilon) + u(\mu, \upsilon) \leq z(x) \) for all \( \upsilon \not\in \Upsilon_0(r) \) and \( \mu \in \mathbb{M} \), where \( z(x) \) grows linearly in \( x \).

B.1 A bound for local fluctuations

An important step in the whole construction is an exponential bound on the maximum of a random process \( U(\upsilon) \) under the exponential moment conditions on its increments. Let \( d(\upsilon, \upsilon') \) be a semi-distance on \( \Upsilon \). We suppose the following condition to hold:

\[
(\mathcal{E}d) \quad \text{There exist } g > 0, \ r_0 > 0, \ \nu_0 \geq 1, \text{ such that for any } \lambda \leq g \text{ and } \upsilon, \upsilon' \in \Upsilon \text{ with } d(\upsilon, \upsilon') \leq r_0 \ 
\begin{align*}
\log \mathbb{E} \exp \left\{ \lambda \frac{U(\upsilon) - U(\upsilon')}{{d}(\upsilon, \upsilon')} \right\} & \leq \nu_0^2 \lambda^2 / 2.
\end{align*}
(\text{B.1})
\]

Formulation of the result involves a sigma-finite measure \( \pi \) on the space \( \Upsilon \) which is often called the majorizing measure and used in the generic chaining device; see Talagrand (2005). A typical example of choosing \( \pi \) is the Lebesgue measure on \( \mathbb{R}^p \). Let \( \Upsilon^0 \) be a subset of \( \Upsilon \), a sequence \( r_k \) be fixed with \( r_0 = \text{diam}(\Upsilon^0) \) and \( r_k = r_02^{-k} \). Let also \( \mathcal{B}_k(\upsilon) \overset{\text{def}}{=} \{ \upsilon' \in \Upsilon^0 : d(\upsilon, \upsilon') \leq r_k \} \) be the \( d \)-ball centered at \( \upsilon \) of radius \( r_k \) and \( \pi_k(\upsilon) \) denote its \( \pi \)-measure:

\[
\pi_k(\upsilon) \overset{\text{def}}{=} \int_{\mathcal{B}_k(\upsilon)} \pi(d\upsilon') = \int_{\Upsilon^0} \mathbb{1}(d(\upsilon, \upsilon') \leq r_k) \pi(d\upsilon').
\]
Denote also
\[ M_k \overset{\text{def}}{=} \max_{u \in \mathcal{Y}} \frac{\pi(\mathcal{Y})}{\pi_k(u)} \quad k \geq 1. \]  
(B.2)

Finally set \( c_1 = 1/3 \), \( c_k = 2^{-k+2}/3 \) for \( k \geq 2 \), and define the value \( Q(\mathcal{Y}) \) by
\[ Q(\mathcal{Y}) \overset{\text{def}}{=} \sum_{k=1}^{\infty} c_k \log(2M_k) = \frac{1}{3} \log(2M_1) + \frac{4}{3} \sum_{k=2}^{\infty} 2^{-k} \log(2M_k). \]

**Theorem B.1.** Let \( \mathcal{U} \) be a separable process following to (Ed). If \( \mathcal{Y} \) is a \( d \)-ball in \( \mathcal{Y} \) with the center \( u^0 \) and the radius \( r_0 \), i.e. \( d(u, u^0) \leq r_0 \) for all \( u \in \mathcal{Y} \), then for \( \lambda \leq g_0 = \nu_0 g \)
\[ \log \mathbb{E} \exp \left\{ \frac{\lambda}{3r_0} \sup_{u \in \mathcal{Y}} |\mathcal{U}(u) - \mathcal{U}(u^0)| \right\} \leq \lambda^2/2 + Q(\mathcal{Y}). \]  
(B.3)

**Proof.** A simple change \( \mathcal{U}(\cdot) \) with \( \nu_0^{-1} \mathcal{U}(\cdot) \) and \( g \) with \( g_0 = \nu_0 g \) allows to reduce the result to the case with \( \nu_0 = 1 \) which we assume below. Consider for \( k \geq 1 \) the smoothing operator \( S_k \) defined as
\[ S_k f(u^0) = \frac{1}{\pi_k(u^0)} \int_{\mathcal{B}_k(u^0)} f(u) \pi(du). \]

Further, define
\[ S_0 \mathcal{U}(u) \equiv \mathcal{U}(u^0) \]
so that \( S_0 \mathcal{U} \) is a constant function and the same holds for \( S_k S_{k-1} \ldots S_0 \mathcal{U} \) with any \( k \geq 1 \). If \( f(\cdot) \leq g(\cdot) \) for two non-negative functions \( f \) and \( g \), then \( S_k f(\cdot) \leq S_k g(\cdot) \). Separability of the process \( \mathcal{U} \) implies that \( \lim_k S_k \mathcal{U}(u) = \mathcal{U}(u) \). We conclude that for each \( u \in \mathcal{Y} \)
\[ |\mathcal{U}(u) - \mathcal{U}(u^0)| = \lim_{k \to \infty} |S_k \mathcal{U}(u) - S_k \ldots S_0 \mathcal{U}(u)| \]
\[ \leq \lim_{k \to \infty} \sum_{i=1}^{k} |S_k \ldots S_i(I - S_{i-1}) \mathcal{U}(u)| \leq \sum_{i=1}^{\infty} \xi_i^*. \]

Here \( \xi_k^* \overset{\text{def}}{=} \sup_{u \in \mathcal{Y}} \xi_k(u) \) for \( k \geq 1 \) with
\[ \xi_1(u) \equiv |S_1 \mathcal{U}(u) - \mathcal{U}(u^0)|, \quad \xi_k(u) \overset{\text{def}}{=} |S_k(I - S_{k-1}) \mathcal{U}(u)|, \quad k \geq 2 \]

For a fixed point \( u^* \), it holds
\[ \xi_k(u^*) \leq \frac{1}{\pi_k(u^*)} \int_{\mathcal{B}_k(u^*)} \frac{1}{\pi_k-1(u^*)} \int_{\mathcal{B}_{k-1}(u^*)} |\mathcal{U}(u) - \mathcal{U}(u')| \pi(du') \pi(du). \]
For each $v' \in B_{k-1}(v)$, it holds $d(v, v') \leq r_{k-1} = 2r_k$ and

$$|U(v) - U(v')| \leq r_{k-1} \frac{|U(v) - U(v')|}{d(v, v')}.$$ 

This implies for each $v^* \in T^o$ and $k \geq 2$ by the Jensen inequality and (B.2)

$$\exp\left\{ \frac{\lambda}{r_{k-1}} \xi_k(v^*) \right\} \leq M_k \int_{T^o} \left( \int_{B_{k-1}(v)} \frac{\lambda |U(v) - U(v')|}{d(v, v')} \frac{\pi(dv')}{\pi_k(v^*)} \right) \frac{\pi(dv)}{\pi(T^o)} \leq M_k \int_{T^o} \left( \int_{B_{k-1}(v)} \frac{\pi(dv')}{\pi_k(v^*)} \right) \frac{\pi(dv)}{\pi(T^o)} = 2M_k \exp(\lambda^2/2).$$

As the right hand-side does not depend on $v^*$, this yields for $\xi_k^* \equiv \sup_{v \in T^o} \xi_k(v)$ by condition (Ed) in view of $e^{|x|} \leq e^x + e^{-x}$

$$\mathbb{E} \exp\left\{ \frac{\lambda}{r_{k-1}} \xi_k^* \right\} \leq \mathbb{E} \exp\left\{ \frac{\lambda}{r_0} |\sum_{1} U(v) - U(v^*)| \right\} \leq 2 \exp(\lambda^2/2) \quad (B.4)$$

and thus

$$\mathbb{E} \exp\left\{ \frac{\lambda}{r_0} |\sum_{1} U(v) - U(v^*)| \right\} \leq \frac{1}{\pi_1(v)} \int_{B_1(v)} \mathbb{E} \exp\left\{ \frac{\lambda}{r_0} |\sum_{1} U(v) - U(v^*)| \right\} \pi(dv') \leq \frac{M_1}{\pi(T^o)} \int_{T^o} \mathbb{E} \exp\left\{ \frac{\lambda}{r_0} |\sum_{1} U(v) - U(v^*)| \right\} \pi(dv').$$

This implies by (B.4) for $\xi_1^* \equiv \sup_{v \in T^o} |\sum_{1} U(v) - U(v^*)|$

$$\mathbb{E} \exp\left\{ \frac{\lambda}{r_0} \xi_1^* \right\} \leq 2M_1 \exp(\lambda^2/2).$$

Denote $c_1 = 1/3$ and $c_k = r_{k-1}/(3r_0) = 2^{-k+2}/3$ for $k \geq 2$. Then $\sum_{k=1}^{\infty} c_k = 1$ and it holds by the Hölder inequality; see Lemma B.16 below:

$$\log \mathbb{E} \exp\left( \frac{\lambda}{3r_0} \sum_{k=1}^{\infty} \xi_k^* \right) \leq c_1 \log \mathbb{E} \exp\left( \frac{\lambda}{r_0} \xi_1^* \right) + \sum_{k=2}^{\infty} c_k \log \mathbb{E} \exp\left( \frac{\lambda}{r_{k-1}} \xi_k^* \right) \leq \lambda^2/2 + c_1 \log(2M_1) + \sum_{k=2}^{\infty} c_k \log(2M_k) < \lambda^2/2 + \mathcal{Q}(T^o).$$
This implies the result.

The exponential bound of Theorem B.1 can be used for obtaining a probability bound on the maximum of the increments \( \U(v) - \U(v') \) over \( \U^\circ \).

**Corollary B.2.** Suppose \((\Ed)\). If \( \U^\circ \) is a central set with the center \( v^\circ \) and the radius \( r_0 \), then it holds for any \( x > 0 \)

\[
\mathbb{P}\left( \frac{1}{3\nu_0 r_0} \sup_{v \in \U^\circ} \U(v, v_0) > \tilde{z}(x, Q) \right) \leq \exp(-x), \tag{B.5}
\]

where with \( g_0 = \nu_0 g \) and \( Q = Q(\U^\circ) \)

\[
\tilde{z}(x, Q) \overset{\text{def}}{=} \begin{cases} \sqrt{2(x + Q)} & \text{if } \sqrt{2(x + Q)} \leq g_0, \\ g_0^{-1} (x + Q) + g_0/2 & \text{otherwise.} \end{cases} \tag{B.6}
\]

**Proof.** By the Chebyshev inequality, it holds for the r.v. \( \xi = \sup_{v \in \U^\circ} \U(v, v_0)/(3\nu_0 r_0) \) for any \( \lambda \leq g_0 \) by (B.3)

\[
\log \mathbb{P}(\xi > \tilde{z}) \leq -\lambda \tilde{z} + \log \mathbb{E} \exp\{\lambda \xi\} \leq -\lambda \tilde{z} + \lambda^2/2 + Q.
\]

Now, given \( x > 0 \), we choose \( \lambda = \sqrt{2(x + Q)} \) if this value is not larger than \( g_0 \), and \( \lambda = g_0 \) otherwise. It is straightforward to check that \( \lambda \tilde{z} - \lambda^2/2 - Q \geq x \) in both cases, and the choice of \( \tilde{z} \) by (B.6) yields the bound (B.5). \( \square \)

**B.2 Application to a two-norms case**

As an application of the local bound from Theorem B.1 we consider the result from Baraud (2009), Theorem 3. For convenience of comparison we utilize the notation from that paper. Let \( T \) be a subset of a linear space \( S \) of dimension \( D \), endowed with two norms denoted by \( d(s, t) \) and \( \delta(s, t) \) for \( s, t \in T \). Let also \( (X_t)_{t \in T} \) be a random process on \( T \). The basic assumption of Baraud (2009) is a kind of a Bernstein bound: for some fixed \( c > 0 \)

\[
\log \mathbb{E} \exp\{\lambda(X_t - X_s)\} \leq \frac{\lambda^2 d(s, t)^2/2}{1 - \lambda \delta(s, t)} \quad \text{if } \lambda \delta(s, t) < 1. \tag{B.7}
\]

The aim is to bound the maximum of the process \( X_t \) over a bounded subset \( T_{v, b} \) defined for \( v, b > 0 \) and a specific point \( t_0 \) as

\[
T_{v, b} \overset{\text{def}}{=} \{ t : d(t, t_0) \leq v, \ c\delta(t, t_0) \leq b \}.
\]

Let \( Q = c_1 D \) with \( c_1 = 2 \) for \( D \geq 2 \) and \( c_1 = 2.7 \) for \( D = 1 \).
**Theorem B.3.** Suppose that \((X_t)_{t \in S}\) fulfills (B.7), where \(S\) is a \(D\)-dimensional linear space. For any \(\rho < 1\), it holds
\[
\log \mathbb{P} \left( \frac{\sqrt{1 - \rho}}{3v} \sup_{t \in T_{v,b}} (X_t - X_{t_0}) > \tilde{z}(x, Q) \right) \leq -x
\] (B.8)
where \(\tilde{z}(x, Q)\) from (B.6) with \(g_0 = \rho(1 - \rho)^{-1/2}b^{-1}v\).

**Proof.** Define the new semi-distance \(d^*(s, t)\) by
\[
d^*(s, t) \overset{\text{def}}{=} \max \{ d(s, t), b^{-1}vc\delta(s, t) \}.
\]
The set \(T_{v,b}\) can be represented as
\[
T_{v,b} = \{ t : d^*(t, t_0) \leq v \}
\]
Moreover, Lemma B.10 applied for the semi-distance \(d^*(t, s)\) yields \(Q(T_{v,b}) \leq c_1 D\), where \(c_1 = 2\) for \(D \geq 2\), and \(c_1 = 2.4\) for \(D = 1\).

Fix some \(\rho < 1\) and define \(g = \rho b^{-1}v\). Then for \(|\lambda| \leq g\), it holds
\[
\frac{\lambda}{d^*(s, t)}c\delta(s, t) \leq \frac{\lambda}{b^{-1}vc\delta(s, t)}c\delta(s, t) \leq \rho
\]
and by (B.7), it follows with \(v_0^2 = (1 - \rho)^{-1}\)
\[
\log \mathbb{E} \exp \left\{ \frac{X_t - X_s}{d^*(s, t)} \right\} \leq \log \mathbb{E} \exp \left\{ \frac{X_t - X_s}{d(s, t)} \right\} \leq \frac{\lambda^2/2}{1 - \rho} \leq \frac{v_0^2\lambda^2}{2}.
\]
So, condition \((Ed)\) is fulfilled. Now the result follows from Corollary B.2.

If \(v\) is large relative to \(b\), then \(g = \rho b^{-1}v\) is large as well. With moderate values of \(x\), this allows for applying the bound (B.8) with \(\tilde{z}(x, Q) = \sqrt{2(x + Q)}\). In other words, the value \(\tilde{z} \approx \tilde{z}(x, Q)\) ensures that the maximum of \(X_t - X_{t_0}\) over \(t \in T_{v,b}\) deviates over \(3v_3\) with the exponentially small probability \(e^{-x}\).

### B.3 A local central bound

Due to the result of Theorem B.1, the bound for the maximum of \(\mathcal{U}(v, v_0)\) over \(v \in \mathcal{B}_r(v_0)\) grows quadratically in \(r\). So, its applications to situations with \(r^2 \gg Q(Y^0)\) are limited. The next result shows that introducing a negative quadratic drift helps to state a uniform in \(r\) local probability bound. Namely, the bound for the process \(\mathcal{U}(v, v_0) - \rho d^2(v, v_0)/2\) with some positive \(\rho\) over a ball \(\mathcal{B}_r(v_0)\) around the point \(v_0\) only depends on the drift coefficient \(\rho\) but not on \(r\). Here the generic chaining arguments are accomplished with the slicing technique. The idea is for a given \(r^* > 1\) to split the ball \(\mathcal{B}_{r^*}(v_0)\) into the slices \(\mathcal{B}_{r+1}(v_0) \setminus \mathcal{B}_{r}(v_0)\) and to apply Theorem B.1 to each slice separately with a proper choice of the parameter \(\lambda\).
**Theorem B.4.** Let $r^*$ be such that $(Ed)$ holds on $B_{r^*}(v_0)$. Let also $Q(\mathcal{R}^0) \leq Q$ for $\mathcal{R}^0 = \mathcal{B}_r(v_0)$ with $r \leq r^*$. If $\rho > 0$ and $\bar{z}$ are fixed to ensure $\sqrt{2\rho^3} \leq g_0 = \nu_0g$ and $\rho(\bar{z} - 1) \geq 2$, then it holds

$$
\log \mathbb{P} \left( \sup_{v \in B_{r^*}(v_0)} \left\{ \frac{1}{3\nu_0} U(v, v_0) - \frac{\rho}{2} d^2(v, v_0) \right\} > \bar{z} \right) \\
\leq - \rho(\bar{z} - 1) + \log(4\bar{z}) + Q. \tag{B.9}
$$

Moreover, if $\sqrt{2\rho^3} > g_0$, then

$$
\log \mathbb{P} \left( \sup_{v \in B_{r^*}(v_0)} \left\{ \frac{1}{3\nu_0} U(v, v_0) - \frac{\rho}{2} d^2(v, v_0) \right\} > \bar{z} \right) \\
\leq - g_0\sqrt{\rho(\bar{z} - 1)} + g_0^2/2 + \log(4\bar{z}) + Q. \tag{B.10}
$$

**Remark B.1.** Formally the bound applies even with $r^* = \infty$ provided that $(Ed)$ is fulfilled on the whole set $\mathcal{R}^0$.

**Proof.** Denote

$$u(r) \overset{\text{def}}{=} \frac{1}{3\nu_0r} \sup_{v \in B_r(v_0)} \{U(v) - U(v_0)\}.
$$

Then we have to bound the probability

$$\mathbb{P} \left( \sup_{r \leq r^*} \{ru(r) - \rho r^2/2\} > \bar{z} \right).
$$

For each $r \leq r^*$ and $\lambda \leq g_0$, it follows from (B.3) that

$$
\log \mathbb{E} \exp \{ \lambda u(r) \} \leq \lambda^2/2 + Q.
$$

The choice $\lambda = \sqrt{2\rho^3}$ is admissible in view of $\sqrt{2\rho^3} \leq g_0$. This implies by the exponential Chebyshev inequality

$$
\log \mathbb{P} (ru(r) - \rho r^2/2 \geq \bar{z}) \leq -\lambda(\bar{z}/r + \rho r/2) + \lambda^2/2 + Q
$$

$$
= -\rho\bar{z}(x + x^{-1} - 1) + Q, \tag{B.11}
$$

where $x = \sqrt{\rho/(2\bar{z})}r$. We now apply the slicing arguments w.r.t. $t = \rho r^2/2 = \bar{z}x^2$. By definition, $ru(r)$ increases in $r$. We use that for any growing function $f(\cdot)$ and any $t \geq 0$, it holds

$$f(t) - t \leq \int_0^{t+1} \{f(s) - s + 1\} ds$$
Therefore, for any \( t > 0 \), it holds by (B.11) in view of \( \frac{dt}{z} = 2z x \, dx \)

\[
\begin{align*}
\mathbb{P} \left( \sup_{x \leq t^*} \{ x u(x) - \rho x^2 / 2 \} > \delta \right) & \leq \int_0^{t^*+1} \mathbb{P} \left\{ x u(x) - t \geq \delta - 1 \right\} dt \\
& \leq 2 \int_0^{t^*+1} \exp \left\{ \rho t (x + x^{-1} - 1) + Q \right\} x \, dx \\
& \leq 2z e^{-b + Q} \int_0^{\infty} \exp \left\{ -2(x + x^{-1} - 2) \right\} x \, dx
\end{align*}
\]

with \( b = \rho(\delta - 1) \) and \( t^* = \rho r^2 / 2 \). This implies for \( b \geq 2 \)

\[
\begin{align*}
\mathbb{P} \left( \sup_{x \leq t^*} \{ x u(x) - \rho x^2 / 2 \} > \delta \right) & \leq 2z e^{-b + Q} \int_0^{\infty} \exp \left\{ -2(x + x^{-1} - 2) \right\} x \, dx \\
& \leq 4z \exp \left\{ -\rho(\delta - 1) + Q \right\}
\end{align*}
\]

and (B.9) follows.

If \( \sqrt{2 \rho z} > g_0 \), then select \( \lambda = g_0 \). For \( x \leq t^* \)

\[
\begin{align*}
\log \mathbb{P} \{ x u(x) - \rho x^2 / 2 \geq \delta \} &= \log \mathbb{P} \{ x > \delta / r + \rho x / 2 \} \\
& \leq -\lambda(\delta / r + \rho x / 2) + \lambda^2 / 2 + Q \\
& \leq -\lambda \sqrt{\rho z}(x + x^{-1} - 2) / 2 - \lambda \sqrt{\rho z} + \lambda^2 / 2 + Q,
\end{align*}
\]

where \( x = \sqrt{\rho / \delta} \). This allows to bound in the same way as above

\[
\begin{align*}
\mathbb{P} \left( \sup_{x \leq t^*} \{ x u(x) - \rho x^2 / 2 \} > \delta \right) & \leq 4z \exp \left\{ -\lambda \sqrt{\rho(\delta - 1)} + \lambda^2 / 2 + Q \right\}
\end{align*}
\]

yielding (B.10).

This result can be used for describing the concentration bound for the maximum of \((3v_0)^{-1}u(v, v_0) - \rho d^2(v, v_0) / 2\). Namely, it suffices to find \( \delta \) ensuring the prescribed deviation probability. We state the result for a special case with \( \rho = 1 \) and \( g_0 \geq 3 \) which simplifies the notation.

**Corollary B.5.** Under the conditions of Theorem B.4, for any \( x \geq 0 \) with \( x + Q \geq 4 \)

\[
\begin{align*}
\mathbb{P} \left( \sup_{v \in B_v(v_0)} \left\{ \frac{1}{3v_0} u(v, v_0) - \frac{1}{2} d^2(v, v_0) \right\} > \delta_0(x, Q) \right) & \leq \exp(-x),
\end{align*}
\]

where with \( g_0 = v_0 g \geq 2 \)

\[
\delta_0(x, Q) \overset{\text{def}}{=} \begin{cases} 
(1 + \sqrt{x + Q})^2 & \text{if } 1 + \sqrt{x + Q} \leq g_0, \\
1 + (2g_0^{-1}(x + Q) + g_0)^2 & \text{otherwise.}
\end{cases}
\]

(B.12)
Proof. First consider the case \( 1 + \sqrt{x + Q} \leq g_0 \). In view of (B.9), it suffices to check that \( \mathfrak{z} = (1 + \sqrt{x + Q})^2 \) ensures

\[ \mathfrak{z} - 1 - \log(4\mathfrak{z}) - Q \geq x. \]

This follows from the inequality

\[ (1 + y)^2 - 1 - 2 \log(2 + 2y) \geq y^2 \]

with \( y = \sqrt{x + Q} \geq 2 \).

If \( 1 + \sqrt{x + Q} > g_0 \), define \( \mathfrak{z} = 1 + y^2 \) with \( y = 2g_0^{-1}(x + Q) + g_0 \). Then

\[ g_0 \sqrt{\mathfrak{z}} - 1 - \log(4\mathfrak{z}) - g_0^2/2 - Q - x = g_0 y/2 - \log\{4(1 + y^2)\} \geq 0 \]

because \( g_0 \geq 3 \) and \( 3y/2 - \log(1 + y^2) \geq \log(4) \) for \( y \geq 2 \).

If \( g \gg \sqrt{Q} \) and \( x \) is not too big then \( \mathfrak{z}_0(x, Q) \) is of order \( x + Q \). So, the main message of this result is that with a high probability the maximum of \( (3\nu_0)^{-1}\mathcal{U}(v, v_0) - d^2(v, v_0)/2 \) does not significantly exceed the level \( Q \).

### B.4 A multiscale upper function and hitting probability

The result of the previous section can be explained as a local upper function for the process \( \mathcal{U}(\cdot) \). Indeed, in a vicinity \( B_r(v_0) \) of the central point \( v_0 \), it holds \( (3\nu_0)^{-1}\mathcal{U}(v, v_0) \leq d^2(v, v_0)/2 + \mathfrak{z} \) with a probability exponentially small in \( \mathfrak{z} \). This section aims at extending this local result to the whole set \( \mathcal{Y} \) using multiscaling arguments. For simplifying the notations assume that \( \mathcal{U}(v_0) \equiv 0 \). Then \( \mathcal{U}(v, v_0) = \mathcal{U}(v) \). We say that \( u(\mu, v) \) is a multiscale upper function for \( \mu \mathcal{U}(\cdot) \) on a subset \( \mathcal{Y}^0 \) of \( \mathcal{Y} \) if

\[ \mathbb{P}\left( \sup_{\mu \in \mathcal{M}} \sup_{v \in \mathcal{Y}^0} \{\mu \mathcal{U}(v) - u(\mu, v)\} \geq \mathfrak{z}(x) \right) \leq e^{-x}, \quad \text{(B.13)} \]

for some fixed function \( \mathfrak{z}(x) \). An upper function can be used for describing the concentration sets of the point of maximum \( \bar{v} = \arg\max_{v \in \mathcal{Y}^0} \mathcal{U}(v) \); see Theorem B.8 below.

The desired global bound requires an extension of the local exponential moment condition \((E_d)\). Below we suppose that the pseudo-metric \( d(v, v') \) is given on the whole set \( \mathcal{Y} \). For each \( r \) this metric defines the ball \( \mathcal{Y}_0(r) \) by the constraint \( d(v, v_0) \leq r \). Below the condition \((E_d)\) is assumed to be fulfilled for any \( r \), however the constant \( g \) may be dependent of the radius \( r \).

\((E_r)\) For any \( r \), there exists \( g(r) > 0 \) such that (B.1) holds for all \( v, v' \in \mathcal{Y}_0(r) \) and all \( \lambda \leq g(r) \).
Let now a finite or separable set $M$ where $\mu$ is a set. One possible choice of the set $M$ is to take a geometric sequence $\mu_k = \mu_0 2^{-k}$ with any fixed $\mu_0$ and define $t(\mu_k) = k = \log_2(\mu_k/\mu_0)$ for $k \geq 0$.

Putting together the bounds (B.14) for different $\mu \in M$ yields the following result.

**Theorem B.6.** Suppose $(\mathcal{E}r)$ and (B.15). Then for any $x \geq 2$, there exists a random set $A(x)$ of a total probability at least $1 - 2e^{-x}$, such that it holds on $A(x)$ for any $r > 0$

$$\sup_{v \in B_r(r_0)} \sup_{\mu \in M(r,x)} \left\{ \frac{\mu}{3\nu_0} \mathcal{U}(v) - \frac{1}{2}\mu^2 r^2 - \left\{ 1 + \sqrt{x + Q + t(\mu)} \right\}^2 \right\} < 0,$$

where

$$M(r, x) \overset{\text{def}}{=} \{ \mu \in M : 1 + \sqrt{x + Q + t(\mu)} \leq \nu_0 g(r)/\mu \}.$$

**Proof.** For each $\mu \in M(r, x)$, Corollary B.5 implies

$$\mathbb{P} \left( \sup_{v \in B_r(r_0)} \frac{\mu}{3\nu_0} \mathcal{U}(v) - \frac{1}{2}\mu^2 r^2 \geq \left\{ 1 + \sqrt{x + Q + t(\mu)} \right\}^2 \right) \leq e^{-x - t(\mu)}.$$

The desired assertion is obtained by summing over $\mu \in M$ due to (B.15).

Moreover, the inequality $x + Q \geq 2.5$ yields

$$\left\{ 1 + \sqrt{x + Q + t(\mu)} \right\}^2 \leq 2\left\{ x + Q + t(\mu) \right\}.$$

This allows to take in (B.13) $u(\mu, v) = 3\nu_0 \left\{ \mu^2 r^2/2 + 2t(\mu) \right\}$ and $\zeta(x) = 2(x + Q)$.

**Corollary B.7.** Suppose $(\mathcal{E}r)$ and (B.15). Then for any $x$ with $x + Q \geq 2.5$, there exists a random set $\Omega(x)$ of a total probability at least $1 - 2e^{-x}$, such that it holds on $\Omega(x)$ for any $r > 0$

$$\sup_{v \in B_r(r_0)} \sup_{\mu \in M(r,x)} \left\{ \frac{\mu}{3\nu_0} \mathcal{U}(v) - \frac{1}{2}\mu^2 r^2 - 2t(\mu) \right\} < 2(x + Q).$$
Now we briefly discuss the hitting problem. Let $M(\nu)$ be a deterministic boundary function. We aim at bounding the probability that a process $U(\nu)$ hits this boundary on the set $\Upsilon$. This precisely means the probability that $\sup_{\nu \in \Upsilon} \{U(\nu) - M(\nu)\} \geq 0$. An important observation here is that multiplication by any positive factor $\mu$ does not change the relation. This allows to apply the multiscale result from Theorem B.6. For any fixed $x$ and any $\nu \in B(r(\nu_0))$, define

$$M^*(\nu) \overset{\text{def}}{=} \sup_{\mu \in M(r,x)} \left\{ \frac{1}{3\nu_0} \mu M(\nu) - \frac{1}{2} \mu^2 r^2 - 2t(\mu) \right\}.$$ 

**Theorem B.8.** Suppose $(E_d)$, (B.15), and $x + Q \geq 2.5$. Let, given $x$, it hold $M^*(\nu) \geq 2(x + Q)$, $\nu \in \Upsilon$. (B.16)

Then

$$P\left( \sup_{\nu \in \Upsilon} \{U(\nu) - M(\nu)\} \geq 0 \right) \leq 2e^{-x}.$$ 

Maximizing the expression $(3\nu_0)^{-1} \mu M(\nu) - \mu^2 r^2/2$ suggests the choice $\mu = M(\nu)/(3\nu_0 r^2)$ yielding $M^*(\nu) \geq M^2(\nu)/(6\nu_0^2 r^2) - 2t(\mu)$. In particular, the condition (B.16) requires that $M(\nu)$ grows with $r$ a bit faster than a linear function.

**B.5 Finite-dimensional smooth case**

Here we discuss the special case when $\Upsilon$ is an open subset in $\mathbb{R}^p$, the stochastic process $U(\nu)$ is absolutely continuous and its gradient $\nabla U(\nu) \overset{\text{def}}{=} dU(\nu)/d\nu$ has bounded exponential moments.

$(\mathcal{E}D)$ There exist $g > 0$, $\nu_0 \geq 1$, and for each $\nu \in \Upsilon$, a symmetric non-negative matrix $H(\nu)$ such that for any $\lambda \leq g$ and any unit vector $\gamma \in \mathbb{R}^p$, it holds

$$\log E \exp \left\{ \frac{\lambda \gamma^\top \nabla U(\nu)}{\|H(\nu)\gamma\|} \right\} \leq \nu_0^2 \lambda^2 / 2.$$ 

A natural candidate for $H^2(\nu)$ is the covariance matrix $\text{Var}(\nabla U(\nu))$ provided that this matrix is well posed. Then the constant $\nu_0$ can be taken close to one by reducing the value $g$; see Lemma B.17 below.

In what follows we fix a subset $\Upsilon^0$ of $\Upsilon$ and establish a bound for the maximum of the process $U(\nu, \nu^0) = U(\nu) - U(\nu^0)$ on $\Upsilon^0$ for a fixed point $\nu^0$. We will assume existence of a dominating matrix $H^* = H^*(\Upsilon^0)$ such that $H(\nu) \leq H^*$ for all $\nu \in \Upsilon^0$. We also assume that $\pi$ is the Lebesgue measure on $\Upsilon$. First we show that the differentiability condition $(\mathcal{E}D)$ implies $(\mathcal{E}d)$. 

Lemma B.9. Assume that (ED) holds with some \( g \) and \( H(v) \leq H^* \) for \( v \in \mathcal{Y}^o \). Consider any \( v, v^o \in \mathcal{Y}^o \). Then it holds for \( |\lambda| \leq g \)

\[
\log \mathbb{E} \exp \left\{ \lambda \frac{\mathcal{U}(v, v^o)}{\|H^*(v - v^o)\|} \right\} \leq \frac{\nu_0^2 \lambda^2}{2}.
\]

Proof. Denote \( \delta = \|v - v^o\| \), \( \gamma = (v - v^o)/\delta \). Then

\[
\mathcal{U}(v, v^o) = \delta \gamma^\top \int_0^1 \nabla \mathcal{U}(v^o + t\delta \gamma) dt
\]

and \( \|H^*(v - v^o)\| = \delta \|H^* \gamma\| \). Now the Hölder inequality and (ED) yield

\[
\mathbb{E} \exp \left\{ \lambda \frac{\mathcal{U}(v, v^o)}{\|H^*(v - v^o)\|} - \frac{\nu_0^2 \lambda^2}{2} \right\} = \mathbb{E} \exp \left\{ \int_0^1 \left[ \lambda \frac{\gamma^\top \nabla \mathcal{U}(v^o + t\delta \gamma)}{\|H^* \gamma\|} - \frac{\nu_0^2 \lambda^2}{2} \right] dt \right\} \leq \int_0^1 \mathbb{E} \exp \left\{ \lambda \frac{\gamma^\top \nabla \mathcal{U}(v^o + t\delta \gamma)}{\|H^* \gamma\|} - \frac{\nu_0^2 \lambda^2}{2} \right\} dt \leq 1
\]
as required. \( \square \)

The result of Lemma B.9 enables us to define \( d(v, v') = \|H^*(v - v^o)\| \) so that the corresponding ball coincides with the ellipsoid \( B(x, v^o) \). Now we bound the value \( \mathcal{Q}(\mathcal{Y}^o) \) for \( \mathcal{Y}^o = B(x_0, v^o) \).

Lemma B.10. Let \( \mathcal{Y}^o = B(x_0, v^o) \). Under the conditions of Lemma B.9, it holds

\[
\mathcal{Q}(\mathcal{Y}^o) \leq c_1 p, \text{ where } c_1 = 2 \text{ for } p \geq 2, \text{ and } c_1 = 2.7 \text{ for } p = 1.
\]

Proof. The set \( \mathcal{Y}^o \) coincides with the ellipsoid \( B(x_0, v^o) \) while the \( d \)-ball \( B_k(v) \) coincides with the ellipsoid \( B(x_k, v) \) for each \( k \geq 2 \). By change of variables, the study can be reduced to the case with \( v^o = 0 \), \( H^* \equiv I_p \), \( x_0 = 1 \), so that \( B(x, v) \) is the usual Euclidean ball in \( \mathbb{R}^p \) of radius \( r \). It is obvious that the measure of the overlap of two balls \( B(1, 0) \) and \( B(2^{-k+1}, v) \) for \( \|v\| \leq 1 \) is minimized when \( \|v\| = 1 \), and this value is the same for all such \( v \).

Now we use the following observation. Fix \( v^\sharp \) with \( \|v^\sharp\| = 1 \). Let \( r \leq 1 \), \( v^b = (1 - r^2/2)v^\sharp \) and \( x^b = x - r^2/2 \). If \( v \in B(x^b, v^b) \), then \( v \in B(x, v^\sharp) \) because

\[
\|v^\sharp - v\| \leq \|v^\sharp - v^b\| + \|v - v^b\| \leq r^2/2 + r - r^2/2 \leq r.
\]

Moreover, for each \( v \in B(x^b, v^b) \), it holds with \( u = v - v^b \)

\[
\|u\|^2 = \|v^\sharp\|^2 + \|u\|^2 + 2u^\top v^\sharp \leq (1 - r^2/2)^2 + \|x^b\|^2 + 2u^\top v^b \leq 1 + 2u^\top v^b.
\]
This means that either \( \nu = \nu^b + \nu^s \) or \( \nu^b - \nu^s \) belongs to the ball \( B(\tau_0, \nu^o) \) and thus, \( \pi(B(1,0) \cap B(\tau, \nu)) \geq \pi(B(\nu^b, \nu^o))/2 \). We conclude that

\[
\frac{\pi(B(1,0))}{\pi(B(1,0) \cap B(\tau, \nu^2))} \leq \frac{2\pi(B(1,0))}{\pi(B(\nu^b, \nu^o))} = 2(\tau - \tau^2/2)^{-p}.
\]

This implies for \( k \geq 1 \) and \( \tau_k = 2^{-k+1} \) that \( 2M_{k+1} \leq 2^{2+kp}(1-2^{-k-1})^{-p} \). The quantity \( Q(T^o) \) can now be evaluated as

\[
Q(T^o) \leq \frac{1}{3} \log(2^{2+p}) + \frac{2}{3} \sum_{k=1}^{\infty} 2^{-k} \log(2^{2+kp}) - \frac{2p}{3} \sum_{k=1}^{\infty} 2^{-k} \log(1 - 2^{-k-1})
\]

\[
= \frac{\log 2}{3} \left[ 2 + p + 2 \sum_{k=1}^{\infty} (2 + kp)2^{-k} \right] - \frac{2p}{3} \sum_{k=1}^{\infty} 2^{-k} \log(1 - 2^{-k-1}) \leq c_1 p,
\]

where \( c_1 = 2 \) for \( p \geq 2 \), and \( c_1 = 2.7 \) for \( p = 1 \), and the result follows.

Now we specify the local bounds of Theorem B.1 and the central result of Corollary B.5 to the smooth case.

**Theorem B.11.** Suppose (Ed). For any \( \lambda \leq \nu_0 g \), \( \tau_0 > 0 \), and \( \nu^o \in \mathcal{T} \)

\[
\log \mathbb{E} \exp \left\{ \frac{\lambda}{3\nu_0 \tau_0} \sup_{\nu \in B(\tau_0, \nu^o)} [\mathcal{U}(\nu) - \mathcal{U}(\nu^o)] \right\} \leq \frac{\lambda^2}{2} + Q,
\]

where \( Q = c_1 p \).

We consider the local sets of the elliptic form \( \mathcal{T}_o(\tau) = \{ \nu : \|H_0(\nu - \nu_0)\| \leq \tau \} \), where \( H_0 \) dominates \( H(\nu) \) on this set: \( H(\nu) \leq H_0 \).

**Theorem B.12.** Let (Ed) hold with some \( g \) and a matrix \( H(\nu) \). Suppose that \( H(\nu) \leq H_0 \) for all \( \nu \in \mathcal{T}_o(\tau) \). Then

\[
\mathbb{P} \left( \sup_{\nu \in \mathcal{T}_o(\tau)} \left\{ \frac{1}{3\nu_0} \|\mathcal{U}(\nu, \nu_0) - \frac{1}{2}\|H_0(\nu - \nu_0)\|^2 \right\} \geq \delta_0(x, p) \right) \leq \exp(-x), \quad (B.17)
\]

where \( \delta_0(x, p) \) coincides with \( \delta_0(x, Q) \) from (B.12) for \( Q = c_1 p \).

**Remark B.2.** An important feature of the established result is that the bound in the right hand-side of (B.17) does not depend on the value \( \tau \) describing the radius of the local vicinity around the central point \( \nu_0 \). In the ideal case one would apply this result with \( \tau = \infty \) provided that the conditions \( H(\nu) \leq H_0 \) is fulfilled uniformly over \( \mathcal{T} \).

**Proof.** Lemma B.10 implies (Ed) with \( d(\nu, \nu_0) = \|H_0(\nu - \nu_0)\|^2/2 \). Now the result follows from Corollary B.5.

The global result of Theorem B.6 applies without changes to the smooth case.
B.6 Roughness constraints for dimension reduction

The local bounds of Theorems B.1 and B.4 can be extended in several directions. Here we briefly discuss one extension related to the use of a smoothness condition on the parameter \( \nu \). Let \( t(\nu) \) be a non-negative penalty function on \( \Upsilon \). A particular example of such penalty function is the roughness penalty \( t(\nu) = \| G \nu \|^2 \) for a given matrix \( R^p \). Let \( t_0 \geq 1 \) be fixed. Redefine the sets \( \mathcal{B}_r(\nu^o) \) by the constraint \( t(\nu) \leq t_0 \): \[
\mathcal{B}_r(\nu^o) = \{ \nu \in \Upsilon : d(\nu, \nu^o) \leq r; t(\nu) \leq t_0 \},
\]
and consider \( \Upsilon^o = \mathcal{B}_{t_0}(\nu^o) \) for a fixed central point \( \nu^o \) and the radius \( r_o \). One can easily check that the results of Theorems B.1 and B.4 and their corollaries extend to this situation without any change. The only difference is in the definition of the value \( Q(\Upsilon^o) \) and \( Q \). Each value \( Q(\Upsilon^o) \) is defined via the quantities \( \pi_k(\nu) = \pi(\mathcal{B}_r(\nu)) \) which obviously change when each ball \( \mathcal{B}_r(\nu) \) is redefined. Examples below show that the use of the penalization can substantially reduce the value \( Q \).

Now we specify the results to the case of a smooth process \( U \) given on a local ball \( \Upsilon^o = \mathcal{B}_{t_0}(\nu^o) \) defined by the condition \( \{ \| H_0(\nu - \nu^o) \| \leq r_o \} \) and a smoothness constraint \( \| G \nu \|^2 \leq t_0 = r_o^2 \). The local set \( \mathcal{B}_r(\nu) \) are of the form:
\[
\mathcal{B}_r(\nu) \overset{\text{def}}{=} \{ \nu' : \| H_0(\nu - \nu') \| \leq r, \| G \nu' \| \leq r_o \}. \tag{B.18}
\]
The effective dimension \( p_e = p_e(S) \) can be defined as the dimension of the subspace on which \( H_0 \geq \mathcal{G} \). The formal definition uses the spectral decomposition of the matrix \( S = H_0^{-1} \mathcal{G} H_0^{-1} \). Let \( g_1 \leq g_2 \leq \ldots \leq g_p \) be the eigenvalue of \( S \). Define \( p_e(S) \) as the largest index \( j \) for which \( g_j < 1 \):
\[
p_e(S) \overset{\text{def}}{=} \max \{ j : g_j < 1 \}. \tag{B.19}
\]
In the non-penalized case, the entropy term \( Q \) is proportional to the dimension \( p \). The roughness penalty enables to reduce \( p \) to the effective dimension \( p_e(S) \) which can be much smaller than \( p \) depending on the relation between matrices \( H_0 \) and \( \mathcal{G} \). More precisely, if the eigenvalues \( g_j \) of \( S \) grow sufficiently fast, the entropy calculus effectively reduces to the coordinates with \( g_j \leq 1 \).

**Lemma B.13.** Let \( g_1 = 0 \). For each \( r_o \geq 1 \), it holds
\[
Q(\Upsilon_o(r_o)) \leq c_1 p_s
\]
with \( p_s = p_s(S) \) defined by
\[
p_s(S) \overset{\text{def}}{=} p_e(S) + \sum_{j=1}^p g_j^{-1} \log_+ (g_j). \tag{B.20}
\]

In the non-penalized case, the entropy term \( Q \) is proportional to the dimension \( p \). The roughness penalty enables to reduce \( p \) to the effective dimension \( p_e(S) \) which can be much smaller than \( p \) depending on the relation between matrices \( H_0 \) and \( \mathcal{G} \). More precisely, if the eigenvalues \( g_j \) of \( S \) grow sufficiently fast, the entropy calculus effectively reduces to the coordinates with \( g_j \leq 1 \).
If the sum $\sum_{j\geq 1} g_j^{-1} \log_+(g_j)$ is bounded by a fixed constant, then the value $p_s$ is close to the effective dimension $p_e(S)$ from (B.19).

Proof. We follow the proof of Lemma B.10. By a change of variables one can reduce the problem to the case when $H_0$ is the identity matrix and $r_e \equiv 1$. Moreover, one can easily see that $v^o = 0$ is the hardest case. The case of $v^o \neq 0$ can be considered similarly. By a further change the matrix $S = G^2$ can be represented in diagonal form: $S = \text{diag}\{g_1^2, \ldots, g_p^2\}$. Let $v^z$ be any point with $\|v^z\| \leq 1$ and $\|Gv^z\| \leq 1$, and $r \leq 1$. Simple arguments show that the measure of the set $\mathcal{B}_r(v^z)$ over all such vectors $v^z$ is minimized at $v^z = (1, 0, \ldots, 0)^\top$. Define $r^o = r - r^2/2$, $v^b = (1 - r^2/2)v^z$. Fix any $v$ such that $u = v - v^b$ fulfills $\|u\| \leq r^o$, $\|G\| \leq r$, and $\|u\| < 0$. As $Gv^o = 0$, it holds

$$\|G\| = \|G\|u\| \leq r \leq 1,$$

$$\|v - u^o\| \leq \|v - v^b\| + \|v^b - u^o\| \leq r^o + r^2/2 = r,$$

$$\|v\|^2 = \|v^b + u\|^2 = \|v^b\|^2 + \|u\|^2 + 2u^\top v^b \leq (1 - r^2/2)^2 + |r^o|^2 < 1.$$

This yields that $\pi(\mathcal{B}_1(0) \cap \mathcal{B}_r(v^z)) \geq \pi(\mathcal{B}_r(0))/2$. Moreover, let the index $p_e(r)$ be defined as the largest $j$ with $g_j < r$. Consider any $v \in \mathcal{B}_1(0)$ and construct another point $v(r)$ by multiplying with $r$ every element $v_j$ for $j \leq p_e(r)$. The construction ensures that $v(r) \in \mathcal{B}_r(0)$. This implies

$$\frac{\pi(\mathcal{B}_1(0))}{\pi(\mathcal{B}_1(0) \cap \mathcal{B}_r(v^z))} \leq \frac{2\pi(\mathcal{B}_1(0))}{\pi(\mathcal{B}_r(0))} \leq 2|r^o|^{-p_e(r)}.$$

Application of this bound for $k \geq 1$, $r_{k+1} = 2^{-k}$, and $p_k = p_e(r_{k+1})$ yields that $2M_{k+1} \leq 2^{2+kp_k}(1 - 2^{-k-1})^{-p_k}$. The quantity $Q(T^\circ)$ can now be evaluated as

$$Q(T^\circ) \leq \frac{1}{3} \log(2^{2+p_e}) + \frac{2}{3} \sum_{k=1}^\infty 2^{-k} \log(2^{2+kp_k}) - \frac{2}{3} \sum_{k=1}^\infty 2^{-k} p_k \log(1 - 2^{-k-1}).$$

Further, for each $g > 1$, it holds with $k(g) = \max\{k : g < 2^k\}$

$$s(g) \overset{\text{def}}{=} \sum_{k=1}^\infty 2^{-k+1} k \mathbb{I}(2^{-k} g < 1) \leq 2k(g)2^{-k(g)} \leq 2k(g)/g \leq 2g^{-1} \log_2(2g).$$

Thus,

$$\sum_{k=1}^\infty 2^{-k} p_k \leq \sum_{k=1}^\infty 2^{-k} \sum_{j=1}^p \mathbb{I}(2^{-k} g_j < 1) \leq 2 \sum_{j=1}^p s(g_j).$$

This easily implies the result (B.20); cf. the proof of Lemma B.10. \qed
The first result adjusts Theorem B.11 to the penalized case. The maximum of the process $U$ is taken over a ball $B_{\tau}(\upsilon)$ from (B.18) which is smaller than the similar ball in the non-penalized case. This explains the gain in the entropy term $Q$.

**Theorem B.14.** Let $(ED)$ hold with some $g$ and a matrix $H(\upsilon) \leq H_0$ for all $\upsilon \in \Upsilon$. Then for any $\lambda \leq \nu_0 g, \tau_0 \geq 1$, and $\upsilon^o \in \Upsilon$

$$\log \mathbb{E} \exp \left\{ \frac{\lambda}{3\nu_0 \tau_0} \sup_{\upsilon \in B_{\tau_0}(\upsilon^o)} |U(\upsilon) - U(\upsilon^o)| \right\} \leq \lambda^2 / 2 + Q,$$

where $B_{\tau_0}(\upsilon^o)$ is given by (B.18), $Q = e_1 p_s$, and $p_s$ is the effective dimension from (B.21).

**Proof.** The result follows from Corollary B.5. It is only required to evaluate the local entropy $Q(\Upsilon_0(\tau_0))$. This is done in Lemma B.13.

The magnitude of the process $U$ over $B_{\tau_0}(\upsilon^o)$ is of order $\tau_0$ and it grows with $\tau_0$. The use of the negative drift allows to establish a unified result.

**Theorem B.15.** Let $\tau_0 \geq 1$ be fixed and let $(ED)$ hold with some $g$ and a matrix $H(\upsilon) \leq H_0$ for all $\upsilon \in B_{\tau_0}(\upsilon_0)$. Then

$$\mathbb{P} \left( \sup_{\upsilon \in B_{\tau_0}(\upsilon_0)} \left\{ \frac{1}{3\nu_0} U(\upsilon, \upsilon_0) - \frac{1}{2} \|H_0(\upsilon - \upsilon_0)\|^2 \right\} \geq \beta(x, Q) \right) \leq \exp(-x),$$

where $\beta(x, Q)$ is given by (B.12) with $Q = e_1 p_s$.

The result of Theorem B.6 for the non-penalized case applies without big changes to the penalized case.

**B.7 Auxiliary facts**

**Lemma B.16.** For any r.v.'s $\xi_k$ and $\lambda_k \geq 0$ such that $\Lambda = \sum_k \lambda_k \leq 1$

$$\log \mathbb{E} \exp \left( \sum_k \lambda_k \xi_k \right) \leq \sum_k \lambda_k \log \mathbb{E} e^{\xi_k}.$$

**Proof.** Convexity of $e^x$ and concavity of $x^\Lambda$ imply

$$\mathbb{E} \exp \left\{ \frac{\Lambda}{A} \sum_k \lambda_k \left( \xi_k - \log \mathbb{E} e^{\xi_k} \right) \right\} \leq \mathbb{E}^{\Lambda} \exp \left\{ \frac{1}{A} \sum_k \lambda_k \left( \xi_k - \log \mathbb{E} e^{\xi_k} \right) \right\} \leq \left\{ \frac{1}{A} \sum_k \lambda_k \mathbb{E} \exp (\xi_k - \log \mathbb{E} e^{\xi_k}) \right\}^{A} = 1.$$ 

$\square$
Lemma B.17. Let a r.v. $\xi$ fulfill $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$ and $\mathbb{E}\exp(\lambda_1|\xi|) = \kappa < \infty$ for some $\lambda_1 > 0$. Then for any $\varrho < 1$ there is a constant $C_1$ depending on $\kappa$, $\lambda_1$ and $\varrho$ only such that for $\lambda < \varrho \lambda_1$

$$\log \mathbb{E} e^{\lambda \xi} \leq C_1 \lambda^2 / 2.$$ 

Moreover, there is a constant $\lambda_2 > 0$ such that for all $\lambda \leq \lambda_2$

$$\log \mathbb{E} e^{\lambda \xi} \geq \varrho \lambda^2 / 2.$$ 

**Proof.** Define $h(x) = (\lambda - \lambda_1)x + m \log(x)$ for $m \geq 0$ and $\lambda < \lambda_1$. It is easy to see by a simple algebra that

$$\max_{x \geq 0} h(x) = -m + m \log \frac{m}{\lambda_1 - \lambda}.$$ 

Therefore for any $x \geq 0$

$$\lambda x + m \log(x) \leq \lambda_1 x + \log \left( \frac{m}{e(\lambda_1 - \lambda)} \right)^m.$$ 

This implies for all $\lambda < \lambda_1$

$$\mathbb{E}|\xi|^m \exp(\lambda|\xi|) \leq \left( \frac{m}{e(\lambda_1 - \lambda)} \right)^m \mathbb{E} \exp(\lambda_1|\xi|).$$

Suppose now that for some $\lambda_1 > 0$, it holds $\mathbb{E} \exp(\lambda_1|\xi|) = \kappa(\lambda_1) < \infty$. Then the function $h_0(\lambda) = \mathbb{E} \exp(\lambda \xi)$ fulfills $h_0(0) = 1$, $h_0'(0) = \mathbb{E} \xi = 0$, $h_0''(0) = 1$ and for $\lambda < \lambda_1$,

$$h''_0(\lambda) = \mathbb{E} \xi^2 e^{\lambda \xi} \leq \mathbb{E} \xi^2 e^{\lambda_1|\xi|} \leq \frac{1}{(\lambda_1 - \lambda)^2} \mathbb{E} \exp(\lambda_1|\xi|).$$

This implies by the Taylor expansion for $\lambda < \varrho \lambda_1$ that

$$h_0(\lambda) \leq 1 + C_1 \lambda^2 / 2$$

with $C_1 = \kappa(\lambda_1)/\{\lambda_1^2(1 - \varrho)^2\}$, and hence, $\log h_0(\lambda) \leq C_1 \lambda^2 / 2$. \qed

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