CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS VIA POSITIVITY PROPERTIES OF THE TANGENT BUNDLE

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Abstract. Let $X$ be a smooth complex projective variety. A recent conjecture of S. Kovács states that if the $p^{th}$-exterior power of the tangent bundle $T_X$ contains the $p^{th}$-exterior power of an ample vector bundle, then $X$ is either a projective space or a smooth quadric hypersurface. This conjecture is appealing since it is a common generalization of Mori’s, Wahl’s, Andreatta-Wisniewski’s, Kobayashi-Ochiai’s and Araujo-Druel-Kovács’s characterizations of these spaces. In this paper I give a proof affirming this conjecture for varieties with Picard number 1.

1. Introduction

Let $X$ be a smooth complex projective variety of dimension $n$. In a seminal paper [Mor79], S. Mori proved that the only such varieties having ample tangent bundle $T_X$ are projective spaces. This result finally settled Hartshorne’s conjecture [Har70], the algebraic analog of Frankel’s conjecture [Fra61] in complex differential geometry. (Another proof of Frankel’s conjecture was given around the same time by Y. Siu and S. Yau in [SY80] using harmonic maps.) Since then, the ideas of [Mor79] have been expanded significantly, and there are many results in the literature using positivity properties of $T_X$ to characterize projective spaces and quadric hypersurfaces. In this paper I will prove another characterization in this direction:

Theorem 1.1. Let $X$ be a smooth complex projective variety of dimension $n$ with Picard number 1. Assume that there exists an ample vector bundle $\mathcal{E}$ of rank $r$ on $X$ and a positive integer $p \leq r$ such that $\bigwedge^p \mathcal{E} \subseteq \bigwedge^p T_X$. Then either $X \simeq \mathbb{P}^n$, or $p = n$ and $X \simeq Q_p \subset \mathbb{P}^{p+1}$, where $Q_p$ denotes a smooth quadric hypersurface in $\mathbb{P}^{p+1}$.

Theorem 1.1 gives an affirmative answer for varieties with Picard number 1 of the following more general conjecture of S. Kovács:

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Conjecture 1.2 (Kovács). Let $X$ be a smooth complex projective variety of dimension $n$. If there exists an ample vector bundle $E$ of rank $r$ on $X$ and a positive integer $p \leq r$ such that $\wedge^p E \subseteq \wedge^p T_X$, then either $X \simeq \mathbb{P}^n$, $p = n$ and $X \simeq Q_p \subset \mathbb{P}^{p+1}$, where $Q_p$ denotes a smooth quadric hypersurface in $\mathbb{P}^{p+1}$.

Motivation for this conjecture comes from the desire to unify existing characterization results of this type into a single statement. Mori’s proof of the Hartshorne conjecture in 1979 was the first major result, and its method of studying rational curves of minimal degree has been a catalyst for much that has followed.

Theorem 1.3. [Mor79] Let $X$ be a smooth complex projective variety of dimension $n$, and assume that the tangent sheaf $T_X$ is ample. Then $X \simeq \mathbb{P}^n$.

In 1983, J. Wahl proved a related statement using algebraic methods:

Theorem 1.4. [Wah83] Let $X$ be a smooth complex projective variety of dimension $n$, and assume that the tangent sheaf $T_X$ contains an ample line bundle $\mathcal{L}$. Then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ or $(X, \mathcal{L}) \simeq (\mathbb{P}^1, O_{\mathbb{P}^1}(2))$.

Note that S. Druel gave a geometric proof of this theorem in [Dru04]. In 1998, F. Campana and T. Peternell generalized Wahl’s theorem to bundles of rank $r = n, n-1, \text{ and } n-2$ [CP98]. Finally, in 2001, M. Andreatta and J. Wiśniewski proved the analogous statement for vector bundles of arbitrary rank:

Theorem 1.5. [AW01] Let $X$ be a smooth complex projective variety of dimension $n$, and assume that the tangent sheaf $T_X$ contains an ample vector bundle $E$ of rank $r$. Then either $(X, E) \simeq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ or $r = n$ and $(X, E) \simeq (\mathbb{P}^n, T_{\mathbb{P}^n})$.

It is worth noting that in 2006 C. Araujo developed a different approach to Theorem 1.3 using the variety of minimal rational tangents [Ara06]. In 1973, S. Kobayashi and T. Ochiai proved the following theorem characterizing both projective spaces and quadric hypersurfaces:

Theorem 1.6. [KO73] Let $X$ be an $n$-dimensional compact complex manifold with ample line bundle $\mathcal{L}$. If $c_1(X) \geq (n + 1)c_1(\mathcal{L})$ then $X \simeq \mathbb{P}^n$. If $c_1(X) = nc_1(\mathcal{L})$ then $X \simeq Q_n$, where $Q_n \subset \mathbb{P}^{n+1}$ is a hyperquadric.

Most recently, the following conjecture of A. Beauville [Bea00] was verified by Araujo, Druel, and Kovács:
Theorem 1.7. [ADK08] Let $X$ be a smooth complex projective variety of dimension $n$, and let $\mathcal{L}$ be an ample line bundle on $X$. If $H^0(X, \wedge^p T_X \otimes \mathcal{L}^{-p}) \neq 0$ for some positive integer $p$, then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $p = n$ and $(X, \mathcal{L}) \simeq (Q_p, \mathcal{O}_{Q_p}(1))$, where $Q_p$ denotes a smooth quadric hypersurface in $\mathbb{P}^{p+1}$.

Theorems 1.3-1.7 are comparable in their direction but incongruous in the sense that no one of them implies all the others. Conjecture 1.2 is appealing since it simultaneously implies all of them: Mori’s theorem is covered by the case $p = 1$, $E = T_X$, Wahl’s theorem by $p = 1$, $r = 1$, and the result of Andreatta-Wiśniewski by taking $p = 1$. The main theorem of [ADK08] is covered by setting $E = \mathcal{L} \otimes r$ where $r = p$, and [KO73] by setting $E = \mathcal{L}^\oplus n$ and $E = \mathcal{L}^\oplus n - 1 \oplus \mathcal{L}^\otimes 2$.

Remark 1.8. Notice that 1.2 also generalizes 1.7 to the case where $\wedge^p T_X$ contains a product of $p$ distinct ample line bundles: $(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_p) \subseteq \wedge^p T_X$.

It is easy to check that Conjecture 1.2 holds in some simple cases, for example, when the dimension of $X$ is small: If $\text{dim } X = 1$, the only choice for the integer $p$ is $p = 1$. In this case, Conjecture 1.2 follows from Theorem 1.4 (and also Theorem 1.5.) When $\text{dim } X = 2$, Conjecture 1.2 follows easily from the following theorem:

Theorem 1.9. Let $X$ be a smooth complex projective variety of dimension 2, and assume that $-K_X = A + F$ where $F$ is an effective divisor and $A$ is an ample divisor such that $A \cdot C \geq 2$ for every smooth rational curve $C \subseteq X$, $C \simeq \mathbb{P}^1$. Then either $X \simeq \mathbb{P}^2$ or $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. First notice that $X$ has negative Kodaira dimension since $-K_X \cdot C > 0$ for every general curve $C \subseteq X$. Let $X \to X_{\min}$ be a minimal model obtained by blowing down sufficiently many $(-1)$-curves. Since $\kappa(X) < 0$, $X_{\min}$ is isomorphic to either $\mathbb{P}^2$ or a ruled surface over a curve $B$. Before addressing each case, I prove the following claim that will be used in the rest of the proof:

Claim 1.9.1. Let $X$, $F$, and $A$ be as in the statement of Theorem 1.9 above. If $C \subseteq X$ is a curve such that $C \simeq \mathbb{P}^1$ and $C^2 < 0$, then $F \cdot C < 0$ and hence $C \subseteq F$.

Proof. The following computation implies the claim:

$$F \cdot C = (-K_X - A) \cdot C = (-K_X \cdot C) - (A \cdot C) \leq (2 + C^2) - 2 < 0$$

Here the first inequality follows from adjunction and the initial assumption on the ample divisor $A$. \qed
Continuing with the proof of Theorem 1.9, assume that $X \not\simeq \mathbb{P}^2$. It follows that $X$ admits a morphism to a ruled surface $Y \to B$: If $X_{\text{min}} \simeq \mathbb{P}^2$ then $Y$ is the blow-up of $\mathbb{P}^2$ at a single point. Otherwise $Y \simeq X_{\text{min}}$. The ruling $Y \to B$ induces a morphism $\pi : X \to B$. I will show by contradiction that the fibers of $\pi$ are irreducible, hence $X$ itself is ruled: Suppose that $G$ is a reducible fiber of $\pi$. Then $G$ may be written as a sum $G = \sum G_i$ where $G_i \simeq \mathbb{P}^1$ and $G_i^2 < 0$. By 1.9.1 each $G_i$ (and hence $G$) is contained in the effective divisor $F$. Also, as $G$ is a fiber, $G \cdot G_i = 0$. It follows from 1.9.1 that $(F - G) \cdot G_i < 0$ for each $G_i$, therefore $G$ must be contained in $F - G$, i.e., $F$ contains $2G$. Repeating this computation, one may show that $nG \subseteq F$ for any positive integer $n$, but this is a contradiction since $F$ is a fixed effective divisor. Therefore the fibers of $\pi$ are irreducible as claimed, and $\pi : X \to B$ itself must be a ruling of $X$.

Using the notation of [Har77, V.2.8], there exists a distinguished locally free sheaf $E'$ of rank 2 and degree $-e$ such that $X \simeq \mathbb{P}(E')$. Furthermore, in this case there is a section $\sigma : B \to X$ with image $C_0$ such that $L(C_0) \simeq O_{\mathbb{P}(E')}(1)$. Continuing with the notation of [Har77, V.2], let $f$ be a fiber of $\pi$. In particular, recall that $C_0 \cdot f = 1$ and $f^2 = 0$. By the assumption on $A$ and the fact that $f$ is nef, one has: $-K_X \cdot f = A \cdot f + F \cdot f \geq 2$. On the other hand, by [Har77, V.2.11], $-K_X \cdot f = 2$. Therefore $A \cdot f = 2$ and $F \cdot f = 0$, and the latter inequality implies that $F = mf$ is nef. It follows that $-K_X$ is ample, (it is the sum of an ample and a nef divisor), and therefore $X$ is a Del Pezzo surface. This means that $X$ is both ruled and rational, hence it is a Hirzebruch surface, i.e., $E'$ is decomposable. By [Har77, 2.12], it follows that $e \geq 0$. On the other hand, since $C_0 \not\subseteq F$, 1.9.1 implies that $C_0^2 \geq 0$. But $e = -C_0^2$ by [Har77, V.2.9], therefore $e = C_0^2 = 0$. The only Hirzebruch surface with $e = 0$ is $\mathbb{P}^1 \times \mathbb{P}^1$, and this completes the proof of Theorem 1.9. 

**Corollary 1.10.** Conjecture 1.2 holds when $\dim X = 2$.

**Proof.** If $\dim X = 2$, there are two choices for the integer $p$. If $p = 1$, Conjecture 1.2 follows from Theorem 1.4 so we may assume that $p = 2$. Over a field of characteristic zero, the wedge product of an ample vector bundle is again ample [Har66, 5.3], so the condition $\wedge^2 E \subseteq \wedge^2 T_X$ implies that $\omega_X^{-1}$ contains an ample line bundle. In particular, one may write $-K_X = A + F$ where $A = c_1(\wedge^2 E)$ is the corresponding ample divisor and $F$ is an effective divisor. Notice that $A \cdot C \geq 2$ for every smooth rational curve $C \subseteq X$, $C \simeq \mathbb{P}^1$: Since $E$ is ample, the degree of $E|_C = A|_C$ is bounded below by the rank of $E$. Now Theorem 1.7 shows that Conjecture 1.2 holds when $\dim X = 2$. 

□
In this paper I will show that Conjecture 1.2 holds for all varieties with Picard number 1. The paper is organized as follows: Section 2 is devoted to gathering necessary definitions and results about minimal covering families of rational curves. Section 3 will cover some auxiliary results needed for the main proof. The proof of Theorem 1.1 is covered in Section 4.

**Notation:** I will follow the notation of [Kol96] in the discussion of rational curves. By a vector bundle I mean a locally free sheaf; a line bundle is an invertible sheaf. I will denote by \( \mathbb{P}(V) \) the natural projectivization of a vector space \( V \). A point \( x \in X \) is general if it is contained in a dense open subset of \( U \subseteq X \) where \( U \) is a fixed open subset determined by the context. Throughout the paper I will be working over the field of complex numbers.

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**Note:** Upon completion of this paper, I learned of a somewhat related result by Matthieu Paris [Par10].

2. **Rational Curves of Minimal Degree on Uniruled Varieties**

The proof of the main theorem relies on studying rational curves of minimal degree on \( X \). Starting with [Mor79], many tools have been developed for analyzing families of rational curves on uniruled varieties; for the reader’s convenience I summarize the most important developments here.

Let \( X \) be a smooth complex projective variety. If \( X \) is uniruled, one can find an irreducible component \( H \subset \text{RatCurves}^{n}(X) \) such that the natural map \( \text{Univ}_{H} \to X \) is dominant. Such a component is called a dominating family of rational curves on \( X \). The component \( H \) is called unsplit if it is proper, and is called minimal if the subfamily of curves parameterized by \( H \) passing through a general point \( x \in X \) is proper. A uniruled variety always admits a minimal dominating family of curves [Kol96, IV.2.4].

If \( C \subset X \) is a rational curve on \( X \) and \( f : \mathbb{P}^{1} \to C \subseteq X \) is its normalization, the corresponding point in \( \text{RatCurves}^{n}(X) \) is denoted by \([f]\). If \( H \) is a minimal dominating family, then the splitting type of
for any general \([f] \in H\) is:
\[
f^*T_X \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus d} \oplus O_{\mathbb{P}^1}^{\oplus (n-d-1)}
\]
where \(d := \deg(f^*T_X) - 2 \geq 0\) [Kol96, IV.2.9, IV.2.10]. The “positive part” of \(f^*T_X\) is the subbundle defined by:
\[(f^*T_X)^+ := \text{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes O_{\mathbb{P}^1}(1) \to f^*T_X] \hookrightarrow f^*T_X.
\]

If \(H\) is a fixed minimal dominating family of rational curves on \(X\), one can define an equivalence relation on the points of \(X\) via \(H\): Two points \(x_1, x_2 \in X\) are \(H\)-equivalent if they can be connected by a chain of rational curves parameterized by \(H\). By [Kol96, IV.4.16], there exists a proper surjective morphism \(\pi^0 : X^0 \to Y^0\) from a dense open subset \(X^0 \subseteq X\) onto a normal variety \(Y^0\) whose fibers are \(H\)-equivalence classes. The morphism \(\pi^0\) is often called the \(H\)-rationally connected quotient of \(X\). If \(Y^0\) is a point, then \(X\) is called \(H\)-rationally connected.

An important fact used later is that when the Picard number of \(X\) is 1, the \(H\)-rationally connected quotient is trivial:

**Proposition 2.1.** Let \(X\) be a smooth complex projective variety, \(H\) a minimal dominating family of rational curves on \(X\), and \(\pi^0 : X^0 \to Y^0\) the corresponding \(H\)-rationally connected quotient. If \(\rho(X) = 1\), then \(Y^0\) is a point.

**Proof.** Suppose that \(Y^0\) is positive dimensional. Let \(D_{Y^0}\) be an ample effective divisor on \(Y^0\), \(D_{X^0}\) its pullback on \(X^0\) and \(D_X\) the closure of \(D_{X^0}\) in \(X\). Since \(\rho(X) = 1\), every effective divisor is ample, and it follows that every rational curve parameterized by \(H\) has positive intersection with \(D_X\). Let \(C\) be a rational curve parameterized by \(H\) and contained in \(X^0\). By definition, \(\pi^0\) contracts \(C\) and hence \(D_{X^0} \cdot C = 0\), a contradiction. Therefore \(Y^0\) must be a point. \(\square\)

**Remark 2.2.** The converse of Proposition 2.1 is also true by [Kol96, IV.3.13.3] if one assumes additionally that \(H\) is unsplit, but this will not be needed here.

**Remark 2.3.** The equivalence relation above can be extended to a collection of families of rational curves \(H_1, H_2, \ldots, H_k\): Two points \(x_1, x_2 \in X\) are \((H_1, H_2, \ldots, H_k)\)-equivalent if they can be connected by a chain of rational curves parameterized by \(H_1, H_2, \ldots, H_k\). This induces a morphism on a dense open subset of \(X\) with \((H_1, H_2, \ldots, H_k)\)-rationally connected fibers, called the \((H_1, H_2, \ldots, H_k)\)-rationally connected quotient of \(X\).

It is worth noting that a minimal dominating family \(H\) may not always restrict to a minimal dominating family on the fibers of the
H-rationally connected quotient. To be precise, if \( X_y \) is a fiber of an H-rationally connected quotient of \( X \) and \( \iota \) is the natural map

\[
\iota : \text{RatCurves}^n(X_y) \hookrightarrow \text{RatCurves}^n(X)
\]

it is not always the case that \( \iota^{-1}(H) \subseteq \text{RatCurves}^n(X_y) \) is irreducible:

**Example 2.4.** Let \( Y \subseteq \mathbb{P}^9 \) be the open subset parameterizing smooth quadric surfaces in \( \mathbb{P}^3 \), \( X \) the corresponding open subset of the universal hypersurface in \( \mathbb{P}^3 \times \mathbb{P}^9 \), \( \pi_1 : X \to \mathbb{P}^3 \) and \( \pi_2 : X \to Y \subseteq \mathbb{P}^9 \) the restrictions of the usual projection morphisms. Let \( C \) be a rational curve on \( X \) corresponding to a line on a smooth quadric in \( \mathbb{P}^3 \). (In other words, \( C \) has the property of being contracted by \( \pi_2 \) and having image equal to a line under \( \pi_1 \).) Let \( H \subseteq \text{RatCurves}^n(X) \) be the irreducible component containing the point parameterizing \( C \).

I claim that \( H \) is in fact a dominating family on \( X \): First notice that \( H \) parameterizes all the rational curves in \( X \) that correspond to a line on a smooth quadric in \( \mathbb{P}^3 \). Indeed, if \( C' \) is any other rational curve with these properties, there exists a smooth deformation of \( C \) to \( C' \) in \( X \): The images of \( C \) and \( C' \) in \( \mathbb{P}^3 \) are lines, say \( L \) and \( L' \), and in \( \mathbb{P}^3 \) there exists a smooth deformation of \( L \) to \( L' \) by a family of lines \( \{ L_t \} \) parameterized by \( \mathbb{P}^1 \). One can extend this to a family of smooth quadrics \( \{ Q_t \} \) parameterized over the same base such that \( L_t \subset Q_t \) for each \( t \in \mathbb{P}^1 \). (For example, let \( Q \) be the image of \( \mathbb{P}^1 \times \mathbb{P}^1 \) under the Segre embedding, and let \( L \) be a distinguished line on \( Q \). There exists a one-parameter family of automorphisms \( \{ \alpha_t \} \) of \( \mathbb{P}^3 \) such that \( \alpha_t(L) = L_t \) for each \( t \in \mathbb{P}^1 \), (just choose an appropriate non-trivial morphism \( \mathbb{P}^1 \to \text{Aut}(\mathbb{P}^3) \)), and now the family \( \{ Q_t := \alpha_t(Q) \mid t \in \mathbb{P}^1 \} \) has the desired properties.) Since \( X \) is covered by the rational curves corresponding to the lines on the smooth quadrics of \( \mathbb{P}^3 \), \( H \) is a dominating family on \( X \).

Next notice that the H-rationally connected quotient is just \( \pi_2 : X \to Y \): On one hand, by construction, every rational curve parameterized by \( H \) is contained in a fiber of \( \pi_2 \). On the other hand, the fibers of \( \pi_2 \) are just the smooth quadrics in \( \mathbb{P}^3 \) and each is rationally connected by the lines it contains.

Finally, observe that the restriction of \( H \) to any fiber cannot be a minimal dominating family: There are two minimal dominating families on any \( \mathbb{P}^1 \times \mathbb{P}^1 \), (namely the two families of lines), and the restriction of \( H \) to any fiber will contain both of them.

**Remark 2.5.** The above example also shows that one cannot assume in general that the fibers of the H-rationally connected quotient have
Picard number 1, even when $H$ is unsplit. A necessary condition on $H$ for the fibers to have Picard number 1 is given by [ADK08, 2.3].

Next, recall the definition of the variety of minimal rational tangents: If $x \in X$ is a general point of $X$, let $H_x$ denote the normalization of the subscheme of $H$ parameterizing curves passing through $x \in X$. For general $x \in X$, $H_x$ is a smooth projective variety of dimension $d := \deg(f^*T_X) - 2$ [Ko96, II.1.7, II.2.16]. There exists a map $\tau_x : H_x \to \mathbb{P}(T_x X)$ called the tangent map defined by sending a curve that is smooth at $x \in X$ to its corresponding tangent direction at $x$. The closure of the image of $\tau_x$ in $\mathbb{P}(T_x X)$ is called the variety of minimal rational tangents at $x$ and is denoted $C_x \subseteq \mathbb{P}(T_x X)$. The tangent map is actually the normalization morphism of $C_x$, a fact proved by S. Kebekus [Keb02] and J. Hwang and N. Mok [HM04]:

**Theorem 2.6.**

(2.6.1) [Keb02] The tangent map $\tau_x : H_x \to C_x$ is a finite morphism.

(2.6.2) [HM04] The tangent map $\tau_x : H_x \to C_x$ is birational, hence it is the normalization.

The variety $C_x$ has a natural embedding into $\mathbb{P}(T_x X)$, and this embedding yields important geometric information about $X$. For example, Araujo shows that when $C_x$ is a linear subspace of $\mathbb{P}(T_x X)$, the $H$-rationally connected quotient of $X$ is a projective space bundle:

**Theorem 2.7.** [Ara06, 1.1] Assume that $C_x$ is a $d$-dimensional linear subspace of $\mathbb{P}(T_x X)$ for a general point $x \in X$. Then there is a dense open subset $X^o$ of $X$ and $\mathbb{P}^{d+1}$-bundle $\varphi^o : X^o \to T^o$ such that any curve from $H$ meeting $X^o$ is a line on a fiber of $\varphi^o$.

Lastly, note that the tangent space of $C_x$ at a point $\tau_x([f])$ is related to the splitting type of $f^*T_X$ in an important way. In particular, the tangent space of $C_x$ at the point $\tau_x([f])$ is cut out by the positive directions of $f^*T_X$ at $x \in X$:

**Lemma 2.8.** [Hwa01, 2.3] Let $[f] \in H$ be a general member, and let $T_x X^+_f \subseteq T_x X$ be the $(d+1)$-dimensional subspace corresponding to the positive factors of the splitting $f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)}$. Then $\mathbb{P}(T_x X^+_f)$ is the projectivized tangent space of $C_x$ at the point $\tau_x([f])$.

3. Preliminary Results

Before proving the main theorem, I prove a few auxiliary results. In particular, I will show that with the assumptions made in the statement
of Theorem 1.1, $X$ admits a nice cover of rational curves, and one can determine the splitting type of the ample vector bundle $E$ when restricted to these rational curves.

**Lemma 3.1.** Let $X$ be a smooth complex projective variety, $E$ an ample vector bundle of rank $r$ on $X$, and assume that $\wedge^p E \subseteq \wedge^p T_X$ for some positive integer $p \leq r$. Then $X$ is uniruled.

**Proof.** Uniruledness of $X$ follows almost immediately from a theorem of Miyaoka, that says that if $\Omega_X$ is not generically semipositive, then $X$ is uniruled [Miy87, 8.6]. Since generic semipositivity of $\Omega_X$ implies generic semipositivity of $\wedge^p \Omega_X$, it is enough to check that $\wedge^p \Omega_X$ is not generically semipositive: Let $C$ be a general complete intersection curve on $X$. Then $(\wedge^p E)|_C$ has positive degree since $\wedge^p E$ is ample. The dual of the inclusion $(\wedge^p E)|_C \hookrightarrow (\wedge^p T_X)|_C$ yields the desired result. \qed

Now let $H \subset \text{RatCurves}^n(X)$ be a minimal dominating family of rational curves on $X$ guaranteed by Lemma 3.1. The next lemma determines the splitting type of $f^* E$ for $[f] \in H$.

**Lemma 3.2.** Let $X$ be a smooth complex projective variety, $E$ an ample vector bundle of rank $r$ on $X$, and $p \leq r$ a positive integer such that $\wedge^p E \subseteq \wedge^p T_X$. Let $H$ be a minimal dominating family of rational curves on $X$. Then either $f^* E \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ for every $[f] \in H$, or $f^* E \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ for every $[f] \in H$.

**Proof.** First let $[f] \in H$ be a general member of $H$. Since $E$ is ample and $[f]$ parameterizes a rational curve, $f^* E$ splits as a direct sum of positive degree line bundles:

$$f^* E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(\alpha_i), \quad \alpha_i \geq 1.$$ 

It follows that $f^*(\wedge^p E)$ splits as a sum of line bundles of degree at least $p$:

$$f^*(\wedge^p E) \cong \bigoplus_{j=1}^p \mathcal{O}_{\mathbb{P}^1}(\beta_j), \quad \beta_j = \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_p} \geq p.$$ 

By assumption,

$$f^*(\wedge^p E) \subseteq f^*(\wedge^p T_X) \cong \wedge^p (\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}(n-d-1))$$

$$\cong \mathcal{O}_{\mathbb{P}^1}(p+1)^{\oplus q_1} \oplus \mathcal{O}_{\mathbb{P}^1}(p)^{\oplus q_2} \oplus \ldots$$

and the highest degree line bundle occurring on the right is $\mathcal{O}_{\mathbb{P}^1}(p+1)$. Therefore $p \leq \beta_j \leq p+1$ for each $1 \leq j \leq \binom{r}{p}$, but this leaves
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only two possibilities for $f^*E$: Either $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ or $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$.

Lastly, observe that $E$ must split the same way on every rational curve parameterized by $H$: Since $H$ is an irreducible component of $\text{RatCurves}^n(X)$, the intersection number of a fixed line bundle on $X$ and any curve $C$ parameterized by $H$ is independent of $C$. In particular, the degree of $\det(E)$ remains constant on all the rational curves parameterized by $H$, and it follows that $\deg(f^*E) = r$ for every $[f] \in H$ or $\deg(f^*E) = r+1$ for every $[f] \in H$. That $E$ splits in one of the above two ways on every (i.e., not just general) $[f] \in H$ is forced by the fact that $f^*E$ is ample and its rank and degree differ by at most 1. □

**Corollary 3.3.** Let $X$ and $E$ be as above. Unless $r = 1$ and $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(2)$, $X$ admits an unsplit minimal dominating family of rational curves.

**Proof.** Let $H$ be a minimal dominating family of rational curves on $X$, $[f] \in H$ a general member. By Lemma 3.2

$$r = \text{rank}(f^*E) \leq \text{deg}(f^*E) = r \text{ or } r+1$$

When $r > 1$ or when $r = 1$ and $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, the above inequality shows that it is impossible for the curve parameterized by $[f]$ to split as a sum of two or more rational curves $C_1, C_2, \ldots, C_k$: On the one hand $r = \text{rank}(f^*E) \leq \text{deg}(E|_{C_i})$ for each $C_i$ by ampleness of $E$. On the other hand, the sum of the degrees of the $E|_{C_i}$ must equal $r$ or $r+1$. Therefore $H$ is unsplit. □

**4. Proof of Theorem 1.1**

Let $X$ be a smooth complex projective variety of dimension $n$, $E$ an ample vector bundle of rank $r$ on $X$, and $p \leq r$ a positive integer such that $\wedge^p E \subseteq \wedge^p T_X$. By Theorem 1.4 one may assume that $r > 1$. Let $H$ be an unsplit minimal dominating family of rational curves on $X$ guaranteed by Corollary 3.3. Lemma 3.2 shows that there are two possible ways for the vector bundle $E$ to split on the curves parameterized by $H$; I address each case separately:

CASE I: First assume that $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ for every $[f] \in H$. The following result of Andreatta-Wisniewski deals with this situation:

**Theorem 4.1.** [AW01 1.2] Let $X$ be a smooth complex projective variety such that $p(X) = 1$, $E$ a vector bundle of rank $r$ on $X$, and $H$ an unsplit minimal dominating family of rational curves on $X$. If there exists an integer $a$ such that $f^*E \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for every $[f] \in H$, then
there is a uniquely defined line bundle $\mathcal{L}$ on $X$ such that $\deg(f^*\mathcal{L}) = a$ and $\mathcal{E} \simeq \mathcal{L}^{\oplus r}$.

**Remark 4.2.** I was unable to follow all of the argument made in [AW01, 1.2], therefore an alternative proof is provided below. The method of lifting rational curves to $\mathbb{P}(\mathcal{E})$ remains the same as the proof given in [AW01]; modifications were made to reflect the fact that a general fiber of a rationally connected quotient may not have Picard number 1. (See 2.4-2.5 for more.) In fact, Theorem 4.5 is a generalization of the original statement. Since then, M. Andreatta has explained to me a nice fix for the apparent gap in the original proof of [AW01, 1.2].

**Theorem 4.3.** Let $X$ be a smooth complex projective variety of dimension $n$, $\mathcal{E}$ a vector bundle of rank $r$ on $X$, and $H_1, H_2, \ldots, H_k$ a collection of families of rational curves on $X$ such that $X$ is $(H_1, H_2, \ldots, H_k)$-rationally connected. If there exists an integer $a \in \mathbb{Z}$ such that $f^*\mathcal{E} \simeq \mathcal{O}_P(a)^{\oplus r}$ for every $[f] \in H_1, H_2, \ldots, H_k$, then there exists a finite surjective morphism $q : Y \to X$ from a variety $Y$ such that:

1. There is a collection of families $V_1, V_2, \ldots, V_l \subseteq \text{RatCurves}^n(Y)$ and a proper surjective morphism $q_r : \bigcup_{i=1}^l V_i \to \bigcup_{j=1}^k H_k$ where $q_r([f]) = [f]$ is given by $q \circ \hat{f} = f$. The variety $Y$ is $(V_1, V_2, \ldots, V_l)$-rationally connected.

2. There is a (uniquely defined) line bundle $\mathcal{L}$ on $Y$ such that $\deg(f^*\mathcal{L}) = a$ and $q^*\mathcal{E} \simeq \mathcal{L}^{\oplus r}$.

**Proof.** The argument applies induction with respect to $r$. Let $p : \mathbb{P}(\mathcal{E}) \to X$ be the projectivization of $\mathcal{E}$ with relative tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. For any $[f] \in H_1, H_2, \ldots, H_k$ and $y \in p^{-1}(f(0))$ there is a unique lift $\hat{f} : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ with the property that $p \circ \hat{f} = f$ and $\deg(\hat{f}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = a$, $\hat{f}(0) = y$: Since $\mathbb{P}(f^*\mathcal{E}) = \mathbb{P}^1 \times \mathbb{P}^{r-1}$, the morphism $\hat{f}$ is obtained by composing $\mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ with the morphism $\mathbb{P}^1 \to \mathbb{P}^1 \times \{y\} \subset \mathbb{P}^1 \times \mathbb{P}^{r-1}$. Thus, for a generic $f$, $\hat{f}^*T_{\mathbb{P}(\mathcal{E})} = f^*T_X \oplus \mathcal{O}(r-1)$.

For each $1 \leq i \leq k$, one may choose an irreducible component $\hat{H}_i \subset \text{RatCurves}^n(\mathbb{P}(\mathcal{E}))$ parameterizing these lifts such that $\hat{H}_i$ dominates $H_i$. In fact, there exists a natural morphism $p_* : \text{RatCurves}^n(\mathbb{P}(\mathcal{E})) \to \text{RatCurves}^n(X)$ defined by $p_*(\hat{f}) = p \circ \hat{f}$.

**Claim 4.3.3.** For each $1 \leq i \leq k$, the morphism $p_* : \hat{H}_i \to H_i$ is proper and thus surjective.
Proof. The proof uses the valuative criterion of properness \cite{Har77, II.4.7}. Let $B$ be the spectrum of a discrete valuation ring (or a germ of a smooth curve in the analytic context) with a closed point $\delta$ and a general point $B^0$. Then for any family of morphisms $F_B : B \times \mathbb{P}^1 \rightarrow X$ coming from $B \rightarrow H_i$ one has $\mathbb{P}(F_B^*\mathcal{E}) = B \times \mathbb{P}^1 \times \mathbb{P}^{r-1}$. Now take $\widehat{F}_{B^0} : B^0 \times \mathbb{P}^1 \rightarrow \mathbb{P}(\mathcal{E})$, coming from a lift $B^0 \rightarrow \widehat{H}_i$ of $B \rightarrow H_i$. By construction $\widehat{F}_{B^0}$ is the composition of $\mathbb{P}(F_B^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ with the product $id \times \psi_0 : B^0 \times \mathbb{P}^1 \rightarrow (B^0 \times \mathbb{P}^1) \times \mathbb{P}^{r-1}$, for some constant morphism $\psi_0 : B^0 \rightarrow \mathbb{P}^{r-1}$. The morphism $\psi_0$ extends trivially to $\psi : B \rightarrow \mathbb{P}^{r-1}$, thus $\widehat{F}_{B^0}$ extends to $\widehat{F}_B$ which is the composition of $\mathbb{P}(F_B^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ with the product $id \times \psi$, hence $p_\ast$ is proper. \hfill \Box

Continuing the proof of Theorem 4.3 consider the $(\widehat{H}_1, \widehat{H}_2, \ldots, \widehat{H}_k)$-rationally connected quotient of $\mathbb{P}(\mathcal{E})$, and let $Y \subset \mathbb{P}(\mathcal{E})$ be a general fiber. Notice that $\widehat{H}_1, \widehat{H}_2, \ldots, \widehat{H}_k$ restricts to a collection of families $\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \ldots, \widehat{H}_{Y_m} \subseteq \text{RatCurves}_a(Y)$, and $Y$ is $(\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \ldots, \widehat{H}_{Y_m})$-rationally connected by construction. Also note that $Y$ is projective and smooth.

Since $X$ is $(H_1, H_2, \ldots, H_k)$-rationally connected and $p_\ast : \widehat{H}_i \rightarrow H_i$ is surjective for each $1 \leq i \leq k$, the restriction map $p_Y : Y \rightarrow X$ is surjective.

Claim 4.3.4. The morphism $p_Y$ has no positive dimensional fiber, hence it is a finite morphism.

Proof. By \cite[II.4.4]{Kol96}, the morphism $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ induces a surjective map
\begin{equation}
A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q} \xrightarrow{p_\ast} A_1(X)_\mathbb{Q} \rightarrow 0.
\end{equation}
Let $d$ be the dimension of $A_1(X)_\mathbb{Q}$. Then the dimension of $A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q}$ is $d + 1$ \cite[II.4.5]{Kol96}, \cite[Ex. II.7.9]{Har77}, and the kernel of $p_\ast$ is the one dimensional space of 1-cycles in $A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q}$ that are contained in the fibers of $p$. Since the fibers of $p$ are projective spaces, these 1-cycles are each rationally equivalent to a line in a fiber of $p$. Therefore, since $X$ is rationally connected, they must be rationally equivalent in $A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q}$. If by contradiction there exists a proper curve $C \subset Y$ contracted by $p_Y$, one may take $C$ as a generator for the kernel of $p_\ast$.

Now by \cite[IV.3.13.3]{Kol96}, $A_1(X)_\mathbb{Q}$ is generated by the classes of curves parameterized by $H_1, H_2, \ldots, H_k$, and $A_1(Y)_\mathbb{Q}$ is generated by the classes of curves parameterized by $\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \ldots, \widehat{H}_{Y_m}$. Therefore one may choose lifts of $d$ curves from $H_1, H_2, \ldots, H_k$, say $\widehat{C}_1, \widehat{C}_2, \ldots, \widehat{C}_d$, such that $\widehat{C}_i \subset Y$ for $1 \leq i \leq d$, and $A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q}$ is generated by
\[ \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_d \text{ and } C. \] But \( C \subset Y \) by assumption, so \( C \) is a \( \mathbb{Q} \)-linear combination of \( \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_d. \) This implies that \( A_1(\mathbb{P}(\mathcal{E}))_\mathbb{Q} \) can be generated by \( d \) elements, a contradiction. Therefore \( p_Y \) does not contract any proper curve in \( Y \), hence it is a finite morphism as desired. \qed

Now consider the pullback \( \tilde{p} : \mathbb{P}(p_Y^* \mathcal{E}) \to Y \) with the induced morphism \( \tilde{p}_Y : \mathbb{P}(p_Y^* \mathcal{E}) \to \mathbb{P}(\mathcal{E}) \) such that \( p \circ \tilde{p}_Y = p_Y \circ \tilde{p} \). By the universal property of the fiber product the projective bundle \( \tilde{p} \) admits a section \( s : Y \to \mathbb{P}(p_Y^* \mathcal{E}) \) such that \( \tilde{p}_Y \circ s \) is the embedding of \( Y \) into \( \mathbb{P}(\mathcal{E}) \). This induces a sequence of bundles over \( Y \):

\[
(4.3.6) \quad 0 \to \mathcal{E}' \to p_Y^* \mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Y \to 0
\]

where \( \mathcal{E}' \) is a bundle of rank \( r - 1 \) on \( Y \). In order to apply the inductive hypothesis to \( \mathcal{E}' \), it suffices to show that \( \mathcal{E}' \) splits in the desired way on the curves parameterized by \( \hat{H}_{Y_1}, \hat{H}_{Y_2}, \ldots, \hat{H}_{Y_m} \): First notice that \( \deg(f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Y) = a \) for any curve \( [f] \in \hat{H}_{Y_1}, \hat{H}_{Y_2}, \ldots, \hat{H}_{Y_m} \).

(This follows from the fact that \( \deg(\hat{f}^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = a \) for every \( [\hat{f}] \in \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_k \) as stated at the beginning of the proof.) Therefore, by restricting (4.3.6) to any \( [\hat{f}] \in \hat{H}_{Y_1}, \hat{H}_{Y_2}, \ldots, \hat{H}_{Y_m} \), one has:

\[
0 \to \hat{f}^* \mathcal{E}' \to \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r} \to \mathcal{O}_{\mathbb{P}^1}(a) \to 0
\]

Twisting this sequence by \( \mathcal{O}_{\mathbb{P}^1}(-a - 1) \) yields:

\[
0 \to \hat{f}^* \mathcal{E}'(-a - 1) \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0
\]

Now, one may write \( \hat{f}^* \mathcal{E}'(-a - 1) \simeq \oplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(\beta_i) \) where \( \Sigma_{i=1}^{r-1} \beta_i = -(r-1) \). The inclusion \( \hat{f}^* \mathcal{E}'(-a - 1) \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \) implies that \( \hat{f}^* \mathcal{E}'(-a - 1) \) has no global sections, hence \( \beta_i < 0 \) for \( 1 \leq i \leq r - 1 \). But since \( \Sigma_{i=1}^{r-1} \beta_i = -(r-1) \), \( \beta_i = -1 \) for all \( 1 \leq i \leq r - 1 \). It follows that \( \hat{f}^* \mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r-1} \) for every \( [\hat{f}] \in \hat{H}_{Y_1}, \hat{H}_{Y_2}, \ldots, \hat{H}_{Y_m} \).

Now let \( q' : Y' \to Y' \) be the finite surjective morphism obtained from induction, and \( V_1, V_2, \ldots, V_l \) the corresponding collection of families of rational curves in \( \text{RatCurves}^n(Y') \) satisfying the conditions in (4.3.4). Pulling back the exact sequence (4.3.6) to \( Y' \) one obtains:

\[
(4.3.7) \quad 0 \to \mathcal{L}^{\oplus r-1} \to \left(q' \circ p_Y\right)^* \mathcal{E} \to \mathcal{L}' \to 0
\]

where \( \mathcal{L} \) is the uniquely defined line bundle coming from induction, and \( \mathcal{L}' = q'^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Y \) for simplicity. (Note that (4.3.7) is exact since the sheaves in (4.3.6) are each locally free.) I claim that \( \mathcal{L} \simeq \mathcal{L}' \) as line bundles on \( Y' \): First notice that \( \mathcal{L} \) and \( \mathcal{L}' \) agree on all of the rational curves parameterized by \( V_1, V_2, \ldots, V_l \). Indeed, for any \( [f'] \in \)
\[ V_1, V_2, \ldots, V_l: \]
\[ f^* \mathcal{L}^{\oplus r-1} = (f' \circ q')^* \mathcal{E} = \hat{f}^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1(a)}^{\oplus r-1} \]
and
\[ f^* \mathcal{L}' = (f' \circ q')^* \mathcal{O}_{\mathbb{P}^1(1)}|_Y = \hat{f}^* \mathcal{O}_{\mathbb{P}^1(1)}|_Y \simeq \mathcal{O}_{\mathbb{P}^1(a)} \]
where \([\hat{f}] \in \hat{H}_{Y_1}, \hat{H}_{Y_2}, \ldots, \hat{H}_{Y_m}\) is the image of \([f']\) under the map \(q'_*\) given in \(H\). Now, \(N_1(Y')\) is generated by the classes of curves coming from \(V_1, V_2, \ldots, V_l\) [Kol96, IV.3.13.3] and there exists a nondegenerate bilinear pairing
\[ N_1(Y') \times N^1(Y') \rightarrow \mathbb{Q} \]
given by the intersection number of curves and divisors. Since the pairing is nondegenerate, it follows that \(\mathcal{L}^{-1} \otimes \mathcal{L}'\) is numerically equivalent to \(\mathcal{O}_{Y'}\), and therefore \(\mathcal{L}^{-1} \otimes \mathcal{L}'\) is torsion [Laz04, 1.1.20]. Let \(\text{Spec}(\mathcal{A}) \rightarrow Y'\) be the unramified cyclic cover of \(Y'\) induced by the \(\mathcal{O}_{Y'}\)-algebra \(\mathcal{A} = \mathcal{O}_Y \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}') \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}')^\oplus 2 \oplus \cdots \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}')^\oplus m-1\), where \(m\) is the smallest positive integer such that \((\mathcal{L}^{-1} \otimes \mathcal{L}')^\oplus m = \mathcal{O}_{Y'}\).

By the inductive assumption, \(Y'\) is rationally connected, hence simply connected, therefore \(\text{Spec}(\mathcal{A}) \rightarrow Y'\) must be trivial. Therefore \(m = 1\) and \((\mathcal{L}^{-1} \otimes \mathcal{L}') \simeq \mathcal{O}_{Y'}\), as desired.

Lastly, since \(Y'\) is rationally connected, [Kol96, IV.3.8] implies that \(0 = H^0(Y', \Omega^1_{Y'}) \simeq H^1(Y', \mathcal{O}_{Y'})\), and therefore the sequence \(4.3.7\) splits. In other words, \((q' \circ p_Y)^* \mathcal{E} \simeq \mathcal{L}^{\oplus r}\) on \(Y'\), and this completes the proof of Theorem 4.3. \(\square\)

**Lemma 4.4.** [AW01, 1.2.2] Let \(X\) be a smooth complex projective Fano variety with \(p : \mathbb{P}(\mathcal{E}) \rightarrow X\) a projectivization of a rank \(r\) bundle. Suppose that \(\Psi : Y \rightarrow X\) is a finite morphism. If \(\mathbb{P}(\Psi^*(\mathcal{E})) \simeq Y \times \mathbb{P}^{r-1}\) then \(\mathbb{P}(\mathcal{E}) \simeq X \times \mathbb{P}^{r-1}\). \(\square\)

**Theorem 4.5.** Let \(X\) be a smooth complex projective Fano variety, \(\mathcal{E}\) a vector bundle of rank \(r\) on \(X\), and \(H_1, H_2, \ldots, H_k \subseteq \text{RatCurves}^n(X)\) a collection of rational curves such that \(X\) is \((H_1, H_2, \ldots, H_k)\)-rationally connected. If there exists an integer \(a \in \mathbb{Z}\) such that \(f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1(a)}^{\oplus r}\) for every \([f] \in H_1, H_2, \ldots, H_k\), then there is a uniquely defined line bundle \(\mathcal{L}\) on \(X\) such that \(\deg(f^* \mathcal{L}) = a\) and \(\mathcal{E} \simeq \mathcal{L}^{\oplus r}\).

**Proof.** This is immediate from Theorem 4.3 and Lemma 4.4. \(\square\)

**Remark 4.6.** When \(X\) is both uniruled and \(\rho(X) = 1\), \(X\) must be Fano. Therefore Theorem 4.4 follows from Theorem 4.5.

Continuing with the proof of Theorem 4.1, use Theorem 4.3 to define a new vector bundle \(\mathcal{F} := \mathcal{L}^{\oplus p}\) on \(X\), and note that in our case \(\mathcal{L}\) is
ample. Recall that \( p \leq r \) by assumption, therefore \( F \subseteq E \). It follows that \( L_p = \det(F) \subseteq \wedge^p E \subseteq \wedge^p T_X \). By [ADK08, 6.3] (= Theorem 1.7), either \( X \simeq \mathbb{P}^n \) or \( X \simeq Q_p \subseteq \mathbb{P}^{p+1} \).

CASE II: Now assume that \( \rho(X) = 1 \) and consider the case that \( f^*E \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus r-1} \) for every \( [f] \in H \). I will show that there exists a vector bundle injection \( f^*E \hookrightarrow f^*T_X \), and use this fact to study the geometry of the variety of minimal rational tangents \( C_x \) at general points \( x \in X \).

**Lemma 4.7.** Let \( X \) be a smooth complex projective variety of dimension \( n \), and let \( E \) be an ample vector bundle of rank \( r \) on \( X \) such that \( \wedge^p E \subseteq \wedge^p T_X \) for some positive integer \( p \leq r \). Let \( H \) be a minimal dominating family of rational curves on \( X \), and let \( [f] \in H \) be a general member. If \( f^*E \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus r-1} \), then there exists a vector bundle injection \( f^*E \hookrightarrow f^*T_X \).

**Proof.** Since \([f] \in H \) is a general member, the splitting type of \( T_X \) on the curve parameterized by \([f]\) is \( f^*T_X \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus d} \oplus O_{\mathbb{P}^1}^{\oplus (n-d-1)} \). When \( f^*E \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus r-1} \), a simple counting argument shows that \( r-1 \leq d \): If \( f^*E \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus r-1} \) then \( f^*(\wedge^p E) \) splits as a sum of line bundles of which exactly \( \binom{r-1}{p-1} \) have degree \( p+1 \). A similar computation shows that the direct sum decomposition of \( f^*(\wedge^p T_X) \) includes exactly \( \binom{d}{p-1} \) line bundles of degree \( p+1 \). Since \( p+1 \) is the largest degree of any line bundle occurring in the decomposition of \( f^*(\wedge^p T_X) \) and since I assume \( f^*(\wedge^p E) \subseteq f^*(\wedge^p T_X) \), it follows that \( r-1 \leq d \). But then \( f^*E \hookrightarrow f^*T_X \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)^{\oplus d} \) as desired. \( \square \)

Now let \( x \in X \) be a general point, \( H_x \) the normalization of the subscheme of \( H \) parameterizing curves passing through \( x \in X \), and \( \tau_x : H_x \longrightarrow C_x \subseteq \mathbb{P}(T_X) \) the tangent map defined in Section 2. Let \( H^i_x \), \( 1 \leq i \leq k \), be the irreducible components of \( H_x \), and define \( C^i_x := \text{im}(\tau_x(H^i_x)) \). Fix an irreducible component \( H^i_x \) and let \([f] \in H^i_x \) be a general member \( f : \mathbb{P}^1 \longrightarrow X \) such that \( f(o) = x \) for a point \( o \in \mathbb{P}^1 \). The fiber \((f^*T_X)_o \) of \( f^*T_X \) over the point \( o \) is naturally isomorphic to \( T_x X \), and under this isomorphism the positive part \((f^*T_X)^+_o \subset (f^*T_X)_o \) cuts out a \((d+1)\)-dimensional linear subspace \( T_x X^+_o \subseteq T_x X \). By Lemma 4.7, \((f^*E)_o \hookrightarrow (f^*T_X)^+_o \), and this induces the inclusion \( E_x \subseteq T_x X^+_o \). By [Hwa01, 2.3] (= Lemma 2.8), it follows that \( \mathbb{P}(E_x) \subseteq \overline{T_{\tau_x([f])}C^i_x} \subseteq \mathbb{P}(T_x X) \). Now the argument in [Ara06, 4.1, 4.2, 4.3] implies that \( C^i_x \) is a linear subspace of \( \mathbb{P}(T_x X) \); I include an outline of the main steps here for the convenience of the reader. By Lemma 4.8...
below, the inclusion $\mathbb{P}(\mathcal{E}_x) \subseteq \overline{T_{\tau_x} f|f|^{\mathcal{E}_x}}$ forces $\mathcal{C}_x$ to have the structure of a cone in $\mathbb{P}(T_x X)$ with $\mathbb{P}(\mathcal{E}_x)$ contained in its vertex. Now the result follows from Lemma 4.9 and the fact that $H_x$ is smooth [Kol96, II.1.7, II.2.16] and $\tau_x : H_x \to \mathcal{C}_x$ is the normalization morphism ([Keb02], [HM04] = Theorem 2.6).

**Lemma 4.8.** [Ara06] 4.2] Let $Z$ be an irreducible closed subvariety of $\mathbb{P}^m$. Assume that there is a dense open subset $U$ of the smooth locus of $Z$ and a point $z_0 \in \mathbb{P}^m$ such that $z_0 \in \bigcap_{z \in U} T_z Z$. Then $Z$ is a cone in $\mathbb{P}^m$ and $z_0$ lies in the vertex of this cone.

**Lemma 4.9.** [Ara06] 4.3] If $Z$ is an irreducible cone in $\mathbb{P}^m$ and the normalization of $Z$ is smooth, then $Z$ is a linear subspace of $\mathbb{P}^m$.

From here one can conclude that the irreducible components of $\mathcal{C}_x$ are all linear subspaces of $\mathbb{P}(T_x X)$. The following proposition of J.M. Hwang shows that in this case $\mathcal{C}_x$ is actually irreducible, thus itself a linear subspace of $\mathbb{P}(T_x X)$:

**Proposition 4.10.** [Hwa01] 2.2] Let $X$ be a smooth complex projective variety, $H$ a minimal dominating family of rational curves on $X$, and $\mathcal{C}_x \in \mathbb{P}(T_x X)$ the corresponding variety of minimal rational tangents at $x \in X$. Assume that for a general $x \in X$, $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_x X)$. Then the intersection of any two irreducible components of $\mathcal{C}_x$ is empty.

Now since $H_x$ is the normalization of $\mathcal{C}_x$ and it is dimension $d := \deg(f^* T_X) - 2$ for a general point $x \in X$, $\mathcal{C}_x$ is in fact a linear subspace of $\mathbb{P}(T_x X)$ of dimension $d$ for every general point $x \in X$. Therefore one can apply the main theorem of [Ara06] (= Theorem 2.7) to conclude that the $H$-rationally connected quotient $\pi^* : X^\circ \to Y^\circ$ admits the structure of a projective space bundle. But since the Picard number of $X$ is 1, $Y^\circ$ is a point by Proposition 2.1. Therefore $X \simeq \mathbb{P}^n$ as desired, and this proves Theorem 1.1. □

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