Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

Chao-Guang Huang$^{1,2}$, Hai-Qing Zhang$^{1,3}$ and Han-Ying Guo$^{2,4}$

1 Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
2 Kavli Institute for Theoretical Physics China at Chinese Academy of Sciences, Beijing 100080, People’s Republic of China
3 Graduate University of the Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
4 Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
E-mail: huangcg@ihep.ac.cn, hqzhang@ihep.ac.cn and hyguo@ihep.ac.cn

Received 26 June 2008
Accepted 3 September 2008
Published 8 October 2008

Abstract. The torsion is shown to be vitally important in the explanation of the evolution of the universe in a large class of gravitational theories containing quadratic terms of curvature and torsion. The cosmological solutions with homogeneous and isotropic torsion in a model of the de Sitter gauge theory of gravity are presented, which may explain the observation data for type Ia supernovae when parameters are suitably chosen.

Keywords: classical tests of cosmology, gravity
1. Introduction

The observations of SN Ia [1] show that our universe is in a stage of accelerating expansion. In order to explain the accelerating expansion of the universe, many dark energy models are constructed phenomenologically. These models mainly fall into two categories. One is those models which introduce phenomenologically some fields (or matter) with particular properties to effectively describe the behaviours of the dark energy and the other is those models which modify the gravitational Lagrangian.

Since our universe is possibly asymptotic to a de Sitter (dS) spacetime, the physics with respect to dS spacetime has been paid increasing attention recently. It is found that in addition to Einstein’s special relativity, there are two other kinds of special relativities: the dS/AdS invariant special relativities (dSSR/AdSSR) on the dS/AdS spacetimes, respectively [2]–[5]. Remember that the Poincaré gauge theory of gravity (PGT) [6] can be inspired from Einstein’s special relativity and the localization of Poincaré symmetry. From the viewpoint of the three kinds of special relativities, there should be three kinds of gauge theories of gravity based on the localization of the relevant special relativity with full symmetry: the PGT, the de Sitter gauge theory of gravity (dS gravity) and the anti-de Sitter gauge theory of gravity. The localization of the corresponding special relativity with full symmetry requires that the full symmetry—$ISO(1, 3)$ for Einstein’s special relativity, $SO(1, 4)$ for dSSR and $SO(2, 3)$ for AdSSR—and the Minkowski, dS and AdS space, respectively, both be localized.

As far as the dS gravity is concerned, some efforts to construct an alternative gravitational theory with local dS symmetry have been made. A model of dS gravity can be formulated in the way that the Lorentz connection and tetrad are combined to form...
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

It has been shown that on a kind of totally umbilical submanifold and under the dS–Lorentz gauge, the dS connection of the model can be realized. The gravitational action takes the form of a Yang–Mills gauge theory. After variation of the action with respect to the Lorentz tetrad and connection, one may obtain the Einstein-like equations and Yang-like equations, respectively. The Yang-like equations are a generalization of the gravitational field equations in the theory of gravity proposed by Yang [19] and others previously [20]. These equations consist of a set of highly non-linear equations, which are, in general, very difficult to solve. Fortunately, it can be shown that all vacuum solutions of Einstein field equations with a positive cosmological constant are the vacuum solutions of the set of equations without torsion [9,21]. In particular, Schwarzschild–dS and Kerr–dS metrics are two solutions of the model. Therefore, the model may pass all solar-system-scale observation and experimental tests for general relativity (GR). In addition, there exist torsion-free and spin current-free big bang solutions in the model [22]. Unfortunately, the equation of state (EoS) for the perfect fluid in the solutions has to take the form of radiation plus an effective cosmological constant, namely, \( p = \rho/3 + c \). Obviously, such cosmological solutions cannot explain the evolution of the universe. The stringent requirement comes from the non-trivial Yang-like equations in addition to Einstein-like equations.

One of the purposes of the present paper is to show that when an isotropic and homogeneous torsion is taken into account, an arbitrary form of EoS for matter may be used to supplement the Einstein-like equations and Yang-like equations, and thus various cosmological solutions may be constructed which may be used to explain the evolution of the universe. For example, we may find a series of ‘dust-dominated’ cosmological solutions. In the cosmological solutions the contributions of the higher order curvature term and torsion term automatically serve as the dark energy and dark radiation. Another purpose of the paper is to investigate what kinds of ingredients of gravitational dark energy in the model may fit the SN Ia observation data.

The paper is arranged as follows. We first very briefly review the model of the dS gravity in section 2. In the third section, we show that the torsion-free and spin current-free solutions in a large class of gravitational theories containing quadratic terms of curvature and torsion cannot explain the evolution of the universe and that torsion is necessary for the purpose, at least in the model of the dS gravity. In section 4, we present some numerical solutions for the homogeneous and isotropic universe with torsion. Also in this section, we try to fix the ingredients of gravitational dark energy by comparing with the SN Ia observation data. We give some concluding remarks in section 5. In appendix A, a totally umbilical submanifold approach to the dS connection is given and appendix B gives the detailed derivation of equation (38).

2. On the model of the de Sitter gravity

In the dS–Lorentz frame, the dS-connection coefficients can be explicitly written as [7]–[18]

\[
\tilde{B}^{ab\mu} = \begin{pmatrix}
B^{ab\mu} & R^{-1}e^a_{\mu} \\
-R^{-1}e^b_{\mu} & 0
\end{pmatrix} \in \mathfrak{so}(1, 4),
\]

The same connection with different gravitational dynamics has also been studied (see, e.g., [10]–[18]).
where \( \tilde{B}_{AB}^{\mu} = \eta^{BC} \tilde{B}_{C\mu}^{A} \) and \( \tilde{B}_{AB}^{4} = \eta^{BC} \tilde{B}_{C4}^{A} \), \( \eta^{AB} = \text{diag}(1, -1, -1, -1, -1) \). Appendix A gives a sketch of the construction of the connection via an umbilical submanifold. Its curvature reads

\[
\not{\mathcal{F}}_{\mu\nu} = (\not{\mathcal{F}}_{\mu\nu})^{AB} = \left( \begin{array}{ccc} F_{ab}^{\mu\nu} + R^{-2}e_{ab}^{\mu\nu} & R^{-1}T_{a}^{\mu} & 0 \\ -R^{-1}T_{b}^{\mu} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \in \mathfrak{so}(1, 4),
\]

where \( e_{b\mu} = e_{a}^{\mu}e_{b\mu} - e_{a}^{\mu}e_{b\mu}, e_{a\mu} = \eta_{ab}e_{\mu}^{b} \), \( F_{ab}^{\mu\nu} \) and \( T_{a}^{\mu} \) are the curvature (A.10) and torsion (A.9) of the Lorentz connection.

The total action of the model with a source is taken as

\[
S_{T} = S_{GYM} + S_{M},
\]

where \( S_{M} \) is the action of the source with minimum coupling to the gravitational fields, and \( S_{GYM} \) the gauge-like action in the Lorentz gauge of the model as follows [5], [7]–[9]:

\[
S_{GYM} = \frac{\hbar}{4g^{2}} \int_{M} d^{4}x \, e \text{Tr}_{dS}(\not{\mathcal{F}}_{\mu\nu})^{AB},
\]

\[
= -\int_{M} d^{4}x \, \left[ \frac{\hbar}{4g^{2}} F_{ab}^{\mu\nu} F_{ab}^{\mu\nu} - \chi(F - 2\Lambda) - \frac{\chi}{2} T_{a}^{\mu} T_{a}^{\mu} \right].
\]

Here, \( e = \det(e_{\mu}^{a}) \), a dimensionless constant \( g \) should be introduced as usual in the gauge theory to describe the self-interaction of the gauge field, \( \chi \) is a dimensionless coupling constant related to \( g \) and \( R \), and \( F = -\frac{1}{2} F_{ab}^{\mu\nu} e_{ab}^{\mu\nu} \) is the scalar curvature of the Cartan connection, the same as the action in the Einstein–Cartan theory. In order the comparison with the Einstein–Cartan theory to make sense, we should take \( R = (3/\Lambda)^{1/2} \), \( \chi = 1/(16\pi G) \) and \( \hbar g^{-2} = 3\chi\Lambda^{-1} \). Thus, \( g \sim 10^{-61} \).

The field equations can be given via the variational principle with respect to \( e_{a}^{\mu} \), \( B_{a\mu}^{b} \),

\[
T_{a}^{\mu\nu} - F_{a}^{\mu\nu} + \frac{1}{2} F_{a}^{\mu\nu} - \Lambda e_{a}^{\mu} = 8\pi G(T_{Ma}^{\mu} + T_{Ga}^{\mu}),
\]

\[
F_{ab}^{\mu\nu} = R^{-2}(16\pi GS_{ab}^{\mu} + S_{ab}^{\mu}).
\]

In equations (5) and (6), \( || \) represents the covariant derivative compatible with the Christoffel symbol \( \{ \mu_{\nu}^{a} \} \) and Lorentz connection \( B_{a\mu}^{b} \). Besides, \( F_{a}^{\mu\nu} = -F_{ab}^{\mu\nu} e_{b\nu}^{a} \), \( F_{a}^{\mu\nu} = F_{b}^{\mu\nu} e_{b\nu}^{a} \), \( F = F_{a}^{\mu\nu} e_{a}^{\mu} \),

\[
T_{Ma}^{\mu} := -\frac{1}{e} \delta S_{M}^{a},
\]

\[
T_{Ga}^{\mu} := \hbar g^{-2} T_{Fa}^{\mu} + 2\chi T_{Ta}^{\mu}
\]

are the tetrad forms of the stress–energy tensors for matter and gravity, respectively, where

\[
T_{Fa}^{\mu} := -\frac{1}{4e} \frac{\delta}{\delta e_{a}^{\mu}} \int d^{4}x \, e \text{Tr}(F_{\mu\nu} F_{\nu\kappa}) = e_{a}^{\kappa} \text{Tr}(F_{a}^{\mu\lambda} F_{\kappa\lambda}) - \frac{1}{4} e_{a}^{\mu} \text{Tr}(F_{a}^{\lambda\sigma} F_{\lambda\sigma})
\]
is the tetrad form of the stress–energy tensor for curvature and

\[ T_{Ta} := -\frac{1}{4e} \delta \epsilon^a_{\mu} \int d^4x \, eT^b_{\nu\kappa}T_{b}^{\nu\kappa} + T_{a}^{\mu\nu} = e_a^\kappa T_b^{\mu\lambda}T_{\kappa\lambda} - \frac{1}{4} \epsilon_a^\mu T_b^{\lambda\sigma}T_{\lambda\sigma} \]  

(10)

is the tetrad form of the stress–energy tensor for torsion.

\[ S_{Mab}^\mu = \frac{1}{2\sqrt{-g}} \frac{\delta S_M}{\delta B_{ab}^\mu} \]  

(11)

and \( S_{Gab}^\mu \) are spin currents for matter and gravity, respectively. In particular, the spin current for gravity can be divided into two parts:

\[ S_{Gab}^\mu = S_{Fab}^\mu + 2S_{Tab}^\mu, \]  

(12)

where

\[ S_{Fab}^\mu := \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta B_{ab}^\mu} \int d^4x \, \sqrt{-g} F = -e_{ab}^{\mu\nu} Y_{ab}^{\mu\nu} = Y_{ab}^{\mu\nu} + Y^{\nu\lambda}e_{ab}^{\mu\lambda}, \]  

(13)

\[ S_{Tab}^\mu := \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta B_{ab}^\mu} \frac{1}{4} \int d^4x \, \sqrt{-g} T_{\nu\kappa}^cT_{\kappa\lambda}^\nu c = T_{[a}^{\mu\lambda}e_{b]}^{\mu\lambda} \]  

(14)

are the spin current for curvature \( F \) and torsion \( T \), respectively. Here,

\[ Y_{ab}^{\mu\nu} = \frac{1}{2}(T_{\mu\nu}^a + T_{\mu\nu}^b + T_{\mu\nu}^c) \]  

(15)

is the contortion.

The field equations (5) and (6) have the explicit imprint of gauge field theory: the equation of motion for a gauge field has a form where the divergence of the curvature (or field strength) is proportional to the corresponding current. In equation (5), the tetrad is treated as a ‘gauge potential’ and the torsion is the ‘field strength’ of the tetrad. The Einstein-like terms and stress–energy tensors can be regarded as the current. Equation (6) reads that the divergence of the curvature of the Lorentz connection is proportional to the spin current [7].

It has been shown that all solutions, including the dS, Schwarzschild–dS, and Kerr–dS spacetimes, of the vacuum Einstein equation with a positive cosmological constant also solve equations (5) and (6) for the vacuum and torsion-free case [9, 21]. It has been proved that the vacuum solutions of GR and the vacuum solutions of the gravitational theory with the Lagrangian \( R + \alpha R_{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \) are equivalent [23]. Therefore, the Birkhoff theorem is valid for that theory. For the vacuum and torsion-free case, the model of the dS gravity is equivalent to the theory with \( R - 2\Lambda + \alpha R_{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \) and \( \alpha = 4\pi G\hbar/g^2 \) when the Einstein–Palatini variation principle is used. The proof of Debney et al can be readily generalized to the gravitational theory with a cosmological constant. We shall discuss the equivalence in detail elsewhere. In addition, the problem of matching an exterior solution with an interior solution in the theory has been studied in [24]. So, this simple model may pass the observational tests on the solar system scale.
3. Cosmological solutions in the model of the de Sitter gravity

To deal with the cosmological solutions, we suppose, as usual, that the universe is homogeneous and isotropic and thus is described by the Friedmann–Robertson–Walker (FRW) metric

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] . \]  

(16)

The matter without spin current in the universe takes the perfect fluid form

\[ T_{\mu \nu} = (\rho + p)U_{\mu}U_{\nu} - pg_{\mu \nu} , \]  

(17)

where \( U_{\mu} = \{1, 0, 0, 0\} \) is the 4-velocity of comoving observers, \( \rho(t) \) and \( p(t) \) are the energy density and pressure, respectively. The requirement that the stress–energy tensor (17) satisfy the conservation law \([25]\)

\[ T_{\mu}^\parallel_{\nu} - T_{\mu}^\parallel T^\parallel_{\mu \nu} = 0 \]  

(18)

in the FRW metric demands that

\[ T_{\nu}^\parallel_{\mu} = T_{\mu}^\parallel_{\nu} = 0 , \]  

(19)

as in GR, where ‘;’ represents the covariant derivative compatible with Christoffel symbols.

### 3.1. Torsion-free cosmological model

When the spacetime is torsion-free and there is no spin current in it, equations (5) and (6) reduce to \([22]\)

\[ R_{\mu \nu} - \frac{1}{2}R g_{\mu \nu} + \Lambda g_{\mu \nu} = -8\pi G(T_{\mu \nu} + \hbar g^{-2}T^R_{\mu \nu}) , \]  

(20)

\[ R_{\mu \nu}^{\lambda \sigma} ;_{\lambda \sigma} = 0 , \]  

(21)

where

\[ T^R_{\mu \nu} = R^{\kappa \rho} \gamma_{\mu \kappa \rho \nu \lambda} - \frac{1}{4}g_{\mu \nu} R^{\kappa \rho \lambda \sigma} R_{\kappa \rho \lambda \sigma} = \frac{R}{3} \left( R_{\mu \nu} - \frac{1}{4}R g_{\mu \nu} \right) . \]  

(22)

The first equation is an Einstein-like equation and the second one is the Yang equation \([19]\). The validity of the second equality in equation (22) holds because the FRW metric is conformally flat. Obviously, the trace of the Einstein-like equation is the same as that of the Einstein equation:

\[ R = 8\pi GT + 4\Lambda . \]  

(23)

From the Bianchi identity, the Yang equation becomes

\[ R^{\lambda}_{\mu \nu} = R^{\lambda}_{\nu \mu} . \]  

(24)

Then, we can easily get the conservation law of the stress tensor, i.e.

\[ T_{\mu}^\parallel_{\nu} = 0 \Rightarrow (\rho - 3p) = 0 \Rightarrow p = \frac{1}{3}\rho + c . \]  

(25)

The same result has been obtained in \([22]\). It is obvious that such a cosmological model cannot explain the evolution of the universe.
In fact, the conclusion can be generalized to the theory that the gravitational Lagrangian contains more terms such as

\[\epsilon^\lambda_{\alpha_\beta} T^a_{\mu\lambda} T^b_{\mu\sigma}, \quad \epsilon^\sigma_{\alpha_\beta} \epsilon^{\mu\nu}_{\lambda\sigma} T^b_{\mu\lambda}, \quad \epsilon^{ab}_{\lambda\sigma} \epsilon^{cd}_{\mu\nu} F_{ab}^{\mu\nu} F_{cd}^{\lambda\sigma}, \]

\[\epsilon^{\mu\nu}_{\alpha_\beta} e a^\gamma F_{ab}^{\mu\nu} F_{ac}^{\nu\sigma}, \quad F_a^{\mu\nu} F_a^{\alpha_\beta}, \quad \epsilon^a_{\mu_\alpha} e b^{\mu_\nu} F_{a}^{\mu_\nu} F_{b}^{\nu}.\]

Obviously, the addition of the first two terms will not change the conclusion because they make no contribution to the torsion-free and spin current-free field equations. In the above derivation, only the field equations (23) and (24) are used. Equation (23) will not change because the traces of the stress–energy tensors for these gravitational terms all vanish. The middle two terms make the same contribution to (21), and thus to (24), as the term \(F_{ab}^{\mu\nu} F_{\mu\nu}^{ab}\) does. Therefore, they only alter the unimportant coefficients.

The last two terms will contribute \((R_{\mu\nu}^{\alpha_\beta})^{\cdot\cdot}\) terms to (21); thus the form of (24) will be retained. Hence, the torsion-free, spin current-free, homogeneous, isotropic cosmological solutions with perfect fluids in a general \(F - 2\Lambda + F^2\) theory including the torsion-squared term cannot explain the evolution of our universe except for the case where the coupling constants of these terms are suitably arranged so that equation (21) vanishes.

### 3.2. Field equations for the universe with isotropic and homogeneous torsion

The reason that the EoS has to take almost a radiation form is that the 24 components of the torsion are set to zero in the above model while the component equations obtained from the variation with respect to the connection (or equivalently to the contortion) do not disappear simultaneously. Thus, the constraint on the EoS appears. To remove the constraint, we should study the cosmological solutions in the model of the dS gravity with isotropic and homogeneous torsion. For simplicity, we will assume that the spin currents are zero in the universe.

It can be shown that the isotropic and homogeneous torsion takes the form

\[T^0 = 0,\]
\[T^1 = T_+ b^0 \land b^1 + T_- b^2 \land b^3,\]
\[T^2 = T_+ b^0 \land b^2 - T_- b^1 \land b^3,\]
\[T^3 = T_+ b^0 \land b^3 + T_- b^1 \land b^2,\]

where \(T_\pm\) is a function of time \(t\) and \(\cdot\) and \(\cdot\cdot\) represent even and odd parity, respectively. Also, \(T_+\) is the trace part of the torsion, \(\frac{1}{3} T^a_{0a}\), while \(T_-\) is the traceless part of the torsion. Again, the matter in the universe is assumed to take the perfect fluid form of equation (17).

The reduced Einstein-like equations are\(^6\)

\[\frac{\dot{a}^2}{a^2} - \left(\dot{T}_+ + 2 \frac{\dot{a}}{a} T_+ - 2 \frac{\dot{a}}{a} a \right)^2 + \frac{1}{4} \left(\dot{T}_+ + 2 \frac{\dot{a}}{a} T_+ \right)^2 + 1 = T_+^4 - 2 T_+ T_- + \frac{3}{2} T_+^2 T_-^2 + \frac{1}{16} T_-^4 + \left(\frac{5 a^2}{a^2} + \frac{k}{R^2} - \frac{3}{R^2}\right) T_+^2 - \frac{1}{2} \left(\frac{5 a^2}{a^2} + \frac{k}{R^2} - \frac{3}{R^2}\right) T_-^2\]

\(^6\) The resulting equations are consistent with those in [26] for the case where the coupling constants \(f_0, f_1\) and \(a_1\) are taken to be \(\chi, -b/(4g^2)\) and \(2\chi\), respectively, and other coupling constants are zero. The cosmological constant here is not zero due to the requirement of dS connection.

---

JCAP10(2008)010
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

\[ + \frac{2\dot{a}}{a} \left( \frac{\ddot{a}}{a} - \frac{\dddot{a}^2}{a^2} - \frac{2k}{a^2} + \frac{3}{R^2} \right) T_+ - \frac{\dot{a}}{a} \left( 4T_+^2 - 3T_+^2 \right) T_+ \]
\[ + \frac{\ddot{a}}{a^2} \left( \frac{\dddot{a}^2}{a^2} + \frac{1}{2} \left( 2 - \frac{2k}{a^2} - \frac{2}{R^2} \right) \right) + \frac{k^2}{a^4} - \frac{2k}{R^2 a^2} + \frac{2}{R^4} = \frac{2}{3} R^{-2} \left( 8\pi G T r^0 r^0 b_0 \right), \tag{27} \]
\[ \frac{\dddot{a}}{a^2} + \left( \dddot{T}_+ + 2\ddot{T}_+ + \frac{6}{R^2} \right) \ddot{T}_+ - \frac{1}{4} \left( \dddot{T}_+ + 2\ddot{T}_+ \right) \ddot{T}_+ - T_+^4 + \frac{3}{2} T_+^2 T_+^2 - \frac{1}{16} T_+^4 \]
\[ + \frac{\dot{a}}{a} \left( 4T_+^2 - 3T_+^2 \right) T_+ - \left( \frac{5\dddot{a}^2}{a^2} + 2 \frac{k}{a^2} + \frac{3}{R^2} \right) T_+ + \frac{1}{2} \left( 5\dddot{a}^2 + \frac{k}{a^2} + \frac{3}{R^2} \right) T_+^2 \]
\[ - 2\frac{\ddot{a}}{a} \left( \frac{\ddot{a}}{a} - \frac{\dddot{a}^2}{a^2} - \frac{2k}{a^2} - \frac{6}{R^2} \right) T_+ - \frac{4}{R^4} a^2 - \frac{\ddot{a}}{a^2} \left( \frac{\dddot{a}^2}{a^2} + \frac{2k}{a^2} + \frac{2}{R^2} \right) \]
\[ - \frac{k^2}{a^4} - \frac{2k}{R^2 a^2} + \frac{6}{R^4} = -2R^{-2} \left( 8\pi G T r^0 r^0 b_0 \right). \tag{28} \]

They are obtained from the 0–0 and 1–1 component equations, respectively. Other component equations are not independent. The reduced Yang-like equations are

\[ \dddot{T}_+ + 3\frac{\ddot{a}}{a} T_+ + \left( \frac{1}{2} T_+^2 - 6T_+^2 + 12\frac{\dddot{a}}{a} T_+ + \frac{\ddot{a}}{a} - 5\frac{\dddot{a}^2}{a^2} - \frac{2k}{a^2} + \frac{6}{R^2} \right) T_+ = 0, \tag{29} \]
\[ \dddot{T}_+ + 3\frac{\ddot{a}}{a} T_+ - \left( 2T_+^2 - \frac{3}{2} T_+^2 - 6\frac{\dddot{a}}{a} T_+ - \frac{\ddot{a}}{a} + 5\frac{\dddot{a}^2}{a^2} + 2 \frac{k}{a^2} - \frac{3}{R^2} \right) T_+ - \frac{3}{2} \frac{\ddot{a}}{a} T_+^2 \]
\[ - \frac{\ddot{a}}{a} - \frac{\ddot{a}^2}{a^2} - 2\frac{\dddot{a}^2}{a^3} + 2 \frac{\dddot{a} k}{a^2} = 0. \tag{30} \]

They are obtained from the \((r, \theta, \phi)\) and \((t, r, r)\) components, respectively. Other component equations differ from the two equations by overall constants.

Now, we have four independent gravitational equations for five independent variables: scale factor \(a\), torsion components \(T_+\) and \(T_-\), energy density \(\rho = T_+^r c_0^r b_0^r\), and pressure \(p = T_+^r c_0^r b_0^r\). They, with the EoS of the fluid, constitute the complete system of equations for the five variables. Thus, the constraint on the EOS has been removed and it is possible that the cosmological solutions with homogeneous and isotropic torsion may explain the evolution of the universe.

For even parity of the torsion, namely \(T_- = 0\), the independent component equations of the Einstein-like and Yang-like equations further reduce to

\[ -\frac{\dddot{a}^2}{a^2} - \left( \dddot{T}_+ + 2\ddot{T}_+ - \frac{6}{a} \right) \ddot{T}_+ + T_+^4 - \frac{4}{a} T_+^2 + \left( \frac{5\dddot{a}^2}{a^2} + 2k \frac{k}{a^2} - \frac{3}{R^2} \right) T_+^2 \]
\[ + \left( \dddot{T}_+ + 2\ddot{T}_+ \right) \ddot{T}_+ + \frac{1}{2} \left( 2 - \frac{2k}{a^2} - \frac{2}{R^2} \right) \ddot{T}_+^2 + \frac{\dddot{a}^2}{a^2} \left( \frac{\dddot{a}^2}{a^2} + 2k \frac{k}{a^2} - \frac{2}{R^2} \right) \]
\[ + \frac{k^2}{a^4} - \frac{2k}{R^2 a^2} + \frac{2}{R^4} = -\frac{2}{3} R^{-2} \left( 8\pi G T_+^r c_0^r b_0^r \right), \tag{31} \]
\[ \dddot{T}_+ + \left( \dddot{T}_+ + 2\ddot{T}_+ - \frac{6}{a} \right) \ddot{T}_+ - T_+^4 + 4\frac{\dddot{a}}{a} T_+^2 - \left( \frac{5\dddot{a}^2}{a^2} + 2k \frac{k}{a^2} + \frac{3}{R^2} \right) T_+^2 \]
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

\[
-2\frac{\ddot{a}}{a} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} - \frac{6}{R^2} \right) T_+ - \frac{4}{R^2 a} \left( \ddot{a}^2 + \frac{k}{a} + \frac{2}{R^2} \right) + \frac{k}{a^2} - \frac{2k}{R^2 a^2} \right) + \frac{6}{R^4} = -2R^{-2}(8\pi GT^\mu_\nu \epsilon^\mu_b b^3), \tag{32}
\]

\[
\ddot{T}_+ + 3\frac{\ddot{a}}{a} \dot{T}_+ - \left( 2T^2 - \frac{6}{a} T_+ - \frac{\dot{a}^2}{a} + 5\frac{\ddot{a}^2}{a^2} + \frac{2k}{a^2} - \frac{3}{R^2} \right) T_+ + \frac{\ddot{a}}{a} - \frac{\ddot{a}^2}{a^2} + \frac{2\ddot{a}^3}{a^3} + \frac{2\ddot{a} k}{a a^2} = 0. \tag{33}
\]

The three equations with the EoS of the fluid can determine the four variables \( T_+, a, \rho \) and \( p \). On the other hand, for odd parity of the torsion, namely \( T_+ = 0 \), there will still be four Einstein-like equations and Yang-like equations. The four equations with the EoS of the fluid are the overdetermined set of equations for the variables \( T_-, a, \rho \) and \( p \). Therefore, the cosmological model with odd parity of torsion in the model of the dS gravity cannot explain the evolution of the universe either.

For simplicity, we will focus on even parity, in which case equations (31) and (32) give rise to

\[
\frac{\ddot{a}}{a} = -H^2 - \frac{k}{a^2} + \frac{4}{3} \pi G (\rho - 3p) + \frac{2}{R^2} + \frac{3}{2} \left( \ddot{T}_+ + 3HT_+ - T_+^2 \right), \tag{34}
\]

where \( H = \dot{a}/a \) is the Hubble parameter. With the help of equation (34), equations (33) and (31) can be rewritten as

\[
\ddot{T}_+ = -3 \left( H + \frac{3}{2} T_+ \right) \dot{T}_+ + \left[ 13 \frac{T_+ - 3H}{2} T_+ + 6H^2 - \frac{8}{R^2} \right.
+ \left. \frac{3k}{a^2} - \frac{28}{3} \pi G (\rho - 3p) \right] T_+ - \frac{8}{3} \pi G (\rho - 3p), \tag{35}
\]

\[
\left[ \frac{4}{3} \pi G (\rho - 3p) \right]^2 + \frac{4}{3} \pi G (\rho - 3p) \left[ \ddot{T}_+ + 7HT_+ - 3T_+^2 - 2 \left( \frac{H^2 + \frac{k}{a^2} - \frac{2}{R^2}}{2} \right) \right] - \frac{16}{3R^2} \pi G \rho
+ \left( \frac{2}{R^2} - H^2 - \frac{k}{a^2} \right) \dot{T}_+ + \left( \frac{8}{R^2} - 3H^2 - \frac{3k}{a^2} \right) HT_+
+ \left( \frac{37}{4} H^2 + \frac{k}{a^2} - \frac{3}{R^2} \right) T_+^2 + \frac{7}{2} HT_+ T_+ - \frac{3}{2} T_+^2 \dot{T}_+ - \frac{13}{2} HT^3 + \frac{1}{4} T^2 + \frac{5}{4} T^4
- \frac{2}{R^2} \left( H^2 + \frac{k}{a^2} - \frac{1}{R^2} \right) = 0. \tag{36}
\]

In principle, one can solve equations (34)–(36) and the EoS to obtain the cosmological solution for \( a(t), \rho(t), p(t) \) and \( T_+(t) \). In fact, equation (36) can be replaced by

\[
\frac{\mathrm{d}(\rho a^3)}{\mathrm{d}a} = -3\rho a^2, \tag{37}
\]

obtained from the covariant conservation law, equation (19). In the following, we shall find some numerical solutions for the some reasonable initial parameters.
4. The evolution of the universe

In the present stage of the universe the dominant matter has the EoS $p = 0$. For convenience, to compare our model with the $\Lambda$CDM model in GR, we rewrite equation (31) for $p = 0$ as

$$1 = \Omega_m + \Omega_\Lambda + \Omega_k + \Omega_{D_r} + \Omega_{D_1},$$

where

$$\Omega_m = \frac{8\pi G \rho}{3H^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \quad \Omega_k = \frac{-k}{a^2 H^2},$$

$$\Omega_{D_r} = \frac{D_r}{H^2}, \quad \Omega_{D_1} = \frac{D_1}{H^2},$$

with

$$D_r := \frac{1}{3} \left( \frac{1}{2} R^2 T_{\mu \nu}^\mu + T_{\mu \nu} T^\mu_{\nu} \right) = \frac{1}{2} \left[ R^2 (B^2 - A^2) + T_{\mu \nu} T^\mu_{\nu} \right],$$

$$D_1 := -\frac{1}{3} T_{\mu \nu} T^\mu_{\nu} + 2HT_+ - T_+^2 = 3HT_+ - 2T_+^2,$$

where

$$A = T_+ + HT_+ - \frac{\dot{a}}{a}, \quad B = 2HT_+ - T_+^2 - H^2 - \frac{k}{a^2}.$$ (42)

$D_r/(8\pi G)$ is the energy density of dark radiation contributed from both curvature and torsion and $D_1/(8\pi G)$ is the dark energy of the first part from the torsion and its interaction with the curvature. They are new contributions in comparison with the $\Lambda$CDM model in GR, in which $1 = \Omega_m + \Omega_\Lambda + \Omega_k$, and play the role of the dynamical dark energy.

By virtue of equation (38), equation (34) for $p = 0$ can be rewritten as

$$q = \frac{1}{2} \Omega_m - \Omega_\Lambda + \Omega_{D_r} + \frac{1}{2} (\Omega_{D_1} - \Omega_{D_2}),$$

where $\Omega_{D_2} = D_2/H^2$ and

$$D_2 = -T_{\mu \nu} \nabla^\mu + 4HT_+ + 2T_+ - T_+^2 = 3T_+ + 6HT_+ - T_+^2.$$ (44)

$D_2/(8\pi G)$ is the dark energy of the second part from the torsion.

The dark radiation $\Omega_{D_r}$ and dark energy $\Omega_{D_1}$ and $\Omega_{D_2}$, as well as $\Omega_\Lambda$, are not directly observable at the present time. However, the present values of $\Omega_{D_1} + \Omega_{D_r}$ and $\Omega_{D_2} - \Omega_{D_r}$ can be determined from equations (38) and (43) if the present values of $q, \Omega_m, \Omega_\Lambda$ and $\Omega_k$ are known. This gives us a chance of estimating ‘the density of torsion’ in the universe.

The behaviour of the scale factor can be obtained by numerically integrating the above equations backward from today. The initial conditions for numerical calculation may be chosen on the basis of the following facts. The kinematical analysis of the data from the SN Ia observations shows that the present deceleration parameter should be about $q_0 \approx -0.7$ to $-0.81$ [27]–[29]. (A subscript 0 denotes the present value as usual.) The density of matter (including baryonic and dark matter) on the scale of galaxy clusters is estimated to be between $\Omega_m \approx 0.2$–0.3 [30], which is consistent with the cosmological estimate from the observation data of WMAP [31, 32], SDSS [33], etc in the framework of GR. The space of the universe is very flat, so we may suppose that $|\Omega_{k0}| \leq 0.02$.

---

7 Detailed discussion is given in appendix B.
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

Figure 1. Plots of the evolution of scale factor subject to different parameters. The horizontal axis is time in units of $H_0^{-1}$, while the vertical axis is the ratio of the scale factor to its present value. The upper left plot is for the flat universe. The lower left plot is for the slightly curved, open universe. The upper right plot is for the slightly curved, closed universe. The lower right plot is for the more curved, closed universe to show the role of $\Omega_{k0}$. When $q_0$ and $\Omega_{m0}$ are fixed, there are still two degrees of freedom among $\Omega_{k0}$, $\Omega_{\Lambda 0}$, $\Omega_{D,0} + \Omega_{D,0}$ and $\Omega_{D,0} - \Omega_{D,0}$. We choose $\Omega_{k0}$ and $\Omega_{\Lambda 0}$ as independent ones, fix $\Omega_{k0}$ first, and then plot curves for different values of $\Omega_{\Lambda 0}$. In the figure, the horizontal axis is time in units of $H_0^{-1}$ and the vertical axis is $a/a_0$. From the figure, we can find that in all cases considered, the larger the cosmological constant, the younger the universe. Obviously, some models have been ruled out because they cannot explain the ages of the oldest globular clusters [34], which are between 10 and 13 Gyr. But there are still wide parameter ranges (roughly speaking, $\Omega_{\Lambda 0} < 0.35$) for the models which might be used to explain the evolution of the universe. For example, when the space of the universe is a little bit curved so that $\Omega_{k0} = -0.02$, which has been indicated from the analysis of WMAP [31] and SDSS [33], the model with $\Omega_{\Lambda 0} = 0.345$ behaves as $a \to 0$ as $t \to 0$. In this case, $\Omega_{D,0} + \Omega_{D,0} = 0.435$, $\Omega_{D,0} - \Omega_{D,0} = 1.605$, $\frac{1}{2}(\Omega_{D,0} - \Omega_{D,0}) + \Omega_{D,0} = -0.585$.
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

Figure 2. Plots of the deceleration parameter versus redshift $z$. The transit from the decelerating expansion to the accelerating expansion occurs at $z < 1$ for all models plotted.

due to equations (38) and (43). This means that on the large scale, the effect of torsion cannot be ignored. The ratio of the energy density of the torsion to the critical energy density is even greater than those for the cosmological constant and matter.

Figure 2 plots the behaviour of the deceleration parameter versus the redshift $z$. We can see that there is a transit from the decelerating expansion to the accelerating expansion in the model and that the transit happens at around $z = 0.9$, which is qualitatively consistent with the analysis of the SN Ia observations [27].

5. Concluding remarks

The astronomical observations show that the universe is probably asymptotically dS. This suggests that there is a need to analyse the observation data on the basis of a theory with local dS symmetry.

Although the recent WMAP5 data combined with other observations set tightened constraints on $\Omega_k$ based on GR [32], the central value is still negative.
We have shown that the torsion is vitally important in the explanation of the evolution of the universe, not only for the model of the dS gravity first proposed in 1970s, but also for a large class of gravitational theories containing quadratic terms of curvature and torsion. In a wide parameter range in the model of the dS gravity, the spin current-free cosmological solutions with homogeneous and isotropic torsion may explain the SN Ia observations and supply a natural transit from decelerating expansion to accelerating expansion. The transit occurs at around $z = 0.9$, which is qualitatively consistent with the analysis of the SN Ia observations. The reason that the redshift of the transit is systematically greater than those from the previous analysis is that the relation between $q$ and $z$ is obviously not a linear one in our model, while the previous analysis is based on the assumption that $q = q_0 + q_1 z^{27}$. If we make a linear fitting for the $q-z$ curve and then translate the line in a parallel direction so that it goes through $q_0$ at $z = 0$, then we shall get a smaller redshift for the transit.

In the cosmological solutions with torsion that we considered, the effects of torsion and its interaction with the curvature cannot be ignored on the large scale, which is even greater than that of the matter density or cosmological constant. Even though it is very difficult to directly measure the energy density of the torsion by local experiments because its order of magnitude is the same as that of cosmological constant, it is worthwhile to study the method of detecting torsion and place an upper limit for the torsion. Moreover, in the model of the dS gravity the torsion may not only be generated by the spin current of matter, but also appear as an input to the geometry of the spacetime itself. And in our cosmological model the spin current of matter is set to zero. So, this is quite different from the models, such as that of [35], in which the torsion is assumed to be coming from the spin current.

Needless to say, to check whether the model of the dS gravity can really explain the gravitational phenomena in the universe, much work is needed. For example, a long enough radiation-dominated era should be matched with the dust-dominated era in the earlier stage of the evolution. A mechanism of primary perturbation should be explored. In particular, we should perturb the FRW metric and compare the anisotropic spectrum with WMAP data.

Acknowledgments

We thank Z Xu, Y-Z Zhang, X-N Wu, Y Tian, B Zhou and H-T Wu for useful discussions. HQZ would especially like to thank J-R Sun, Y-J Zhang, T-T Qiu and Y-F Cai for helpful discussions. This work was supported by NNSFC under Grant Nos 10775140, 90503002 and Knowledge Innovation Funds of CAS (KJCX3-SYW-S03).

Appendix A. The totally umbilical submanifold and de Sitter connection

It is well known that a 1+3-dimensional dS space can be embedded in a 1+4-dimensional Minkowski space as a hyperboloid, $H^{1,3} \subset M^{1,4}$. With the help of this embedding picture, there is one way to localize the dS space. Namely, at each point on a four-dimensional submanifold in a five-dimensional manifold there is a tangent space with this picture; patch them together—and so on. In surface theory [36], a surface is totally umbilical if the normal curvatures at each point are constants. A dS-hyperboloid $H^{1,3}$ is
The transformations which map $T$ to the unit vector in $\mathbb{R}^4$ satisfy the following conditions in addition to (A.2):

$$\langle \partial_\mu, \partial_\nu \rangle = g_{\mu \nu}, \quad \langle e_a, e_b \rangle = \eta_{ab} = \text{diag}(1, -1, -1, -1).$$  \hfill (A.2)

Let the radius of local dS space $H_p^{1,3}$ at all points on the totally umbilical submanifold $\mathcal{H}^{1,3}$ be $R$. The tangent space $T_{p}^{1,4}$ of the $(1 + 4)$-dimensional Riemann–Cartan manifold $\mathcal{M}^{1,4}$ at point $p$ can be written as $T_{p}^{1,3} \times \mathbb{R}_p$, where $T_{p}^{1,3}$ is the tangent space of $(1 + 3)$-dimensional spacetime at $p$, while $\mathbb{R}_p$ at $p$ is perpendicular to $T_{p}^{1,3}$ in $\mathcal{M}^{1,4}$. On the cotangent space $T_{p}^{1,3*}$ at the point $p \in \mathcal{H}^{1,3}$ there is a Lorentz frame 1-form such that

$$\mathbf{b}^b = e_\mu \, dx^\mu, \quad \mathbf{b}^b(\partial_\mu) = e_\mu, \quad e_\mu e_\nu = \delta_\mu^\nu, \quad e_\mu e_\nu = \delta_\nu^\mu, \hfill (A.1)$$

where $\partial_\mu$ and $dx^\mu$ are the coordinate bases of $T_{p}^{1,3}$ and $T_{p}^{1,3*}$, respectively. Their inner products define the metrics:

$$\langle \partial_\mu, \partial_\nu \rangle = g_{\mu \nu}, \quad \langle e_a, e_b \rangle = \eta_{ab} = \text{diag}(1, -1, -1, -1).$$  \hfill (A.2)

Let the unit vector in $\mathbb{R}_p$ and unit 1-form in $\mathbb{R}_p^*$ be $n$ and $\nu$, respectively. Then, both $\{e_a, n; b^b, \nu\}$ and $\{\partial_\mu, n; dx^\lambda, \nu\}$ span $T_{p}^{1,4} = T_{p}^{1,3} \times \mathbb{R}_p$ and $T_{p}^{1,4*} = T_{p}^{1,3*} \times \mathbb{R}_p^*$. They satisfy the following conditions in addition to (A.2):

$$n(\nu) = 1, \quad b^b(n) = dx^\lambda(n) = 0, \quad \nu(e_a) = \nu(\partial_\mu) = 0; \hfill (A.3)$$

$$\langle n(n) = -1, \quad \langle e_a, n \rangle = \langle \partial_\mu, n \rangle = 0. \hfill (A.4)$$

The dS–Lorentz base $\{\hat{E}_A\}$ and the dual $\{\hat{\Theta}^B\}$ $(A, B = 0, \ldots, 4)$ can be defined as

$$\{\hat{E}_A\} = \{e_a, n\}, \quad \{\hat{\Theta}^B\} = \{b^b, \nu\}. \hfill (A.5)$$

Equations (A.1)–(A.4) can be expressed as

$$\hat{\Theta}^B(\hat{E}_A) = \delta^B_A, \quad \langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB} = \text{diag}(1, -1, -1, -1, -1). \hfill (A.6)$$

The transformations which map $T_{p}^{1,4}$ to itself and preserve the inner products are

$$\hat{E}_A \rightarrow E_A = (S^t)_A^B \hat{E}_B, \quad \hat{\Theta}^A \rightarrow \Theta = ((S)^{-1})_B^A \hat{\Theta}^B, \quad (S^t)^C_A \eta_{CD} S^D_B = \eta_{AB}. \hfill (A.7)$$

where $S = (S_B^A) \in SO(1, 4)$; the superscript $t$ denotes the transpose.

In the tangent bundle $T\mathcal{H}^{1,3}$, there is a Lorentz covariant derivative à la Cartan:

$$\nabla_{e_a} e_b = \theta^c_{ab}(e_a)e_c; \quad \theta^a_{b}(\partial_\mu) = B^a_{b\mu} \, dx^\mu, \quad \theta^a_{b}(\partial_\mu) = B^a_{b\mu}. \hfill (A.8)$$

$B^a_{c\mu} \in \mathfrak{so}(1, 3)$ are connection coefficients of the Lorentz connection 1-form $\theta^a_{c}$. The torsion and curvature can be defined as

$$\Omega^a = dB^a + \theta^a_{b \mu} \wedge b^b = \frac{1}{2} T^a_{\mu\nu} \, dx^\mu \wedge dx^\nu, \hfill (A.9)$$

$$T^a_{\mu\nu} = \partial_\mu \epsilon^a_{\nu} - \partial_\nu \epsilon^a_{\mu} + B^a_{c\mu} \epsilon^c_{\nu} - B^a_{c\nu} \epsilon^c_{\mu}, \hfill (A.9)$$

$$\Omega^a = dB^a + \theta^a_{c \mu} \wedge \theta^c_{b \nu} = \frac{1}{2} F^a_{b\mu\nu} \, dx^\mu \wedge dx^\nu, \hfill (A.10)$$

$$F^a_{b\mu\nu} = \partial_\mu B^a_{b\nu} - \partial_\nu B^a_{b\mu} + B^a_{c\mu} B^c_{b\nu} - B^a_{c\nu} B^c_{b\mu} .$$
They satisfy the corresponding Bianchi identities. Similarly, the dS covariant derivative can be introduced:
\[ \nabla_{E_A} E_B = \Theta^C_B (E_A) E_C. \] (A.11)
\[ \Theta^A_C \in \mathfrak{so}(1,4) \] is the dS-connection 1-form. In the local coordinate chart \( \{ x^\mu \} \) on \( \mathcal{H}^{1,3} \),
\[ \nabla_{\partial_\mu} E_B = \Theta^C_B (\partial_\mu) E_C = B^C_B \partial_\mu E_C, \] (A.12)
\[ \nabla_n E_B = \Theta^C_B (n) E_C = B^C_B n E_C, \] (A.13)
\( B^A_{C\mu} \) and \( B^A_{C4} \) denote the dS-connection coefficients on \( \mathcal{H}^{1,3} \).

For the dS–Lorentz base, equation (A.12) reads
\[ \nabla_{\partial_\mu} e_b = \tilde{\Theta}_b^c (\partial_\mu) e_c + \tilde{\Theta}_4^c (\partial_\mu) n, \quad \nabla_{\partial_\mu} n = \tilde{\Theta}_4^c (\partial_\mu) e_a, \] (A.14)
where \( \tilde{\Theta} \) denotes the dS connection \( \Theta \) in the dS–Lorentz gauge. On the other hand, taking the de Sitter covariant derivative of (A.3) with the properties of \( b_a \) and \( \theta^a_b \) directly, one gets
\[ \nabla_{\partial_\mu} e_a = \theta^b_a (\partial_\mu) e_b - b_{ab} b^b (\partial_\mu) n, \quad \nabla_{\partial_\mu} n = b^a_b (\partial_\mu) e_a, \] (A.15)
where \( b_{ab} \) defines the second fundamental form of the hypersurface
\[ \Pi = -b_{ab} b^a b^b. \] (A.16)
These are the generalizations of the Gauss formula and Weingarten formula in surface theory, respectively (see, e.g., [36]). Since \( \mathcal{H}^{1,3} \) is a non-degenerate totally umbilical submanifold, its first fundamental form is proportional to its second fundamental form [37]. The proportional constant is taken as \( -R \). Then,
\[ \eta_{ab} = R b_{ab}, \] (A.17)
and the equations in (A.15) become
\[ \nabla_{\partial_\mu} e_a = \theta^b_a (\partial_\mu) e_b - R^{-1} \eta_{ab} b^b (\partial_\mu) n, \quad \nabla_{\partial_\mu} n = R^{-1} b^a_b (\partial_\mu) e_a. \] (A.18)
By comparing equation (A.18) with equation (A.14), the dS-connection coefficients can be explicitly obtained in the dS–Lorentz frame, as equation (1).

**Appendix B. The derivation of** \( 1 = \Omega_m + \Omega_\Lambda + \Omega_k + \Omega_D + \Omega_D^t \)**

Let
\[ F_\nu = F_{a\nu}^a, \quad T_\alpha^\mu = T_a^\mu b_a^\alpha, \quad T_X^\mu = T_X^a b_a^\mu, \quad X = F, T, M. \] (B.1)

A direct calculation shows that for the FRW metric (16) and \( T_- = 0 \),
\[ -F_t^t + \frac{1}{2} F_t^\nu F_\nu = 3 \left( \frac{\dot{a}^2 + k}{a^2} - 2 H T_+ + T_+^2 \right), \] (B.2)
\[ T_F^t t = 3 (B_2 - A^2), \] (B.3)
\[ T_T^t t = \frac{3}{2} T_+^2, \] (B.4)
\[ T_t^t (v_\nu = -3 H T_+ + 3 T_+^2, \] (B.5)
where $H = \dot{a}/a$ is the Hubble parameter and
\[
A = \dot{T}_+ + HT_+ - \frac{\dot{a}}{a}, \quad B = 2HT_+ - T_+^2 - H^2 - \frac{k}{a^2}.
\] (B.6)

The $t-t$ component of equation (5) (i.e. equation (31))
\[
T_t^\nu{}_{||\nu} - F_t^\nu{}_{||\nu} - \dot{\delta}_t^\nu F - \delta_t^\nu \Lambda = 8\pi G (h_\nu^2 T_{Ft}^\nu + 2\chi T_{Tr}^\nu) + 8\pi GT_{Mt}^\nu
\]
becomes
\[
\dot{a}^2 + k = \frac{8}{3} \pi G \rho + \frac{\Lambda}{3} + \frac{8}{3} \pi G (h_\nu^2 T_{Ft}^\nu + 2\chi T_{Tr}^\nu) - \frac{1}{3} T_{t}^\nu{}_{||\nu} + 2HT_+ - T_+^2.
\] (B.7)

Let
\[
\rho_c = \frac{3H^2}{8\pi G}, \quad \rho_k = -\frac{3k}{8\pi Ga^2}, \quad \rho_\Lambda = \frac{\Lambda}{8\pi G},
\]
\[
\rho_{D_r} = \frac{3D_r}{8\pi G}, \quad \rho_{D_1} = \frac{3D_1}{8\pi G},
\] (B.8)

where
\[
D_r := \frac{8}{3} \pi G (h_\nu^2 T_{Ft}^\nu + 2\chi T_{Tr}^\nu) = \frac{1}{3} [\frac{1}{2} R^2 T_{Ft}^\nu + T_{Tr}^\nu] = \frac{1}{2} [R^2 (B^2 - A^2) + T_+^2],
\] (B.9)
\[
D_1 := -\frac{1}{2} T_{t}^\nu{}_{||\nu} + 2HT_+ - T_+^2 = 3HT_+ - 2T_+^2.
\] (B.10)

We immediately get equation (38):
\[
1 = \Omega_m + \Omega_\Lambda + \Omega_k + \Omega_{D_r} + \Omega_{D_1},
\]
with $\Omega_m = \rho/\rho_c$ and $\Omega_x = \rho_x/\rho_c$, $x = \Lambda, k, D_r, D_1$.

References

[1] Riess A G et al., 1998 Astron. J. 116 1009 [SPIRES] [arXiv:astro-ph/9805201]
Perlmutter S et al., 1999 Astrophys. J. 517 565 [SPIRES] [arXiv:astro-ph/9812133]
Riess A G et al., 2000 Astrophys. J. 536 62 [SPIRES] [arXiv:astro-ph/0001384]
Riess A G et al., 2001 Astrophys. J. 560 49 [SPIRES] [arXiv:astro-ph/0104455]

[2] Look K-H (Q-K Lu), Why the Minkowski metric must be used?, 1970 unpublished (in Chinese)
Look K-H, Tsou C-L (Z-L Zou) and Kuo H-Y (H-Y Guo), 1974 Acta Phys. Sin. 23 225 (in Chinese)
Look K-H, Tsou C-L (Z-L Zou) and Kuo H-Y (H-Y Guo), 1980 Nature (Shanghai, Suppl.)—Modern Physics 1 97 (in Chinese)
Guo H-Y, 1977 Kexue Tongbao (Chin. Sci. Bull.) 22 487 (in Chinese)

[3] Guo H-Y, 1982 Proc. 2nd Marcel Grossmann Meet. on GR ed R Ruffini (Amsterdam: North-Holland) p 801
Guo H-Y, 1989 Nucl. Phys. B (Proc. Suppl.) 6 381
Huang C-G and Guo H-Y, Gravitation and astrophysics—an occasion of the 90th year of general relativity, 2007 Proc. 7th Asia–Pacific Int. Conf. ed J M Nester, C-M Chen and J-P Hsu (Singapore: World Scientific) p 260

[4] Guo H-Y, Huang C-G, Xu Z and Zhou B, 2004 Mod. Phys. Lett. A 19 1701 [SPIRES] (in Chinese)
Guo H-Y, Huang C-G, Xu Z and Zhou B, 2004 Phys. Lett. A 331 1 [SPIRES] (in Chinese)
Guo H-Y, Huang C-G, Xu Z and Zhou B, 2005 Chin. Phys. Lett. 22 2477 (in Chinese)
Guo H-Y, Huang C-G, Tian Y, Xu Z and Zhou B, 2005 Acta Phys. Sin. 54 2494 (in Chinese)
Guo H-Y, Huang C-G and Zhou B, 2005 Europhys. Lett. 73 1045 [SPIRES] (in Chinese)
Guo H-Y, 2007 Phys. Lett. B 653 88 [SPIRES] (in Chinese)

[5] Guo H-Y, 2008 Sci. China A 51 588 [arXiv:0707.3855]

[6] Kibble T W B, 1961 J. Math. Phys. 2 212 [SPIRES]
Hehl F W, von der Heyde P, Kerlick G D and Nester J M, 1976 Rev. Mod. Phys. 48 393 and references therein [SPIRES]
Cosmological solutions with torsion in a model of the de Sitter gauge theory of gravity

[7] Wu Y-S, Li G-D and Guo H-Y, 1974 Kexue Tongbao (Chin. Sci. Bull.) 19 509 (in Chinese)
An Y, Chen S, Zou Z-L and Guo H-Y, 1974 Kexue Tongbao (Chin. Sci. Bull.) 19 379 (in Chinese)
Guo H-Y, 1976 Kexue Tongbao (Chin. Sci. Bull.) 21 31 (in Chinese)
Zou Z L et al, 1979 Sci. Sin. XXII 628 (in Chinese)
Yan M-L, Zhao B-H and Guo H-Y, 1979 Kexue Tongbao (Chin. Sci. Bull.) 24 587 (in Chinese)
Yan M-L, Zhao B-H and Guo H-Y, 1984 Acta Phys. Sin. 33 1377 (in Chinese)
Yan M-L, Zhao B-H and Guo H-Y, 1984 Acta Phys. Sin. 33 1386 (in Chinese)
[8] Townsend P K, 1977 Phys. Rev. D 15 2795 [SPIRES]
Tseytlin A A, 1982 Phys. Rev. D 26 3327 [SPIRES]
[9] Guo H-Y, Huang C-G, Tian Y, Wu H-T and Zhou B, 2007 Class. Quantum Grav. 24 4009 [SPIRES]
Guo H-Y, Huang C-G, Tian Y and Zhou B, 2007 Front. Phys. China 2 358 [arXiv:hep-th/0607016]
[10] MacDowell S W and Mansouri F, 1977 Phys. Rev. Lett. 38 739 [SPIRES]
MacDowell S W and Mansouri F, 1977 Phys. Rev. Lett. 38 1376 (erratum)
[11] Stelle K S and West P C, 1980 Phys. Rev. D 21 1466 [SPIRES]
[12] Wilczek F, 1998 Phys. Rev. Lett. 80 4851 [SPIRES]
[13] Freidel L and Starodubtsev A, Quantum gravity in terms of topological observables, 2005
arXiv:hep-th/0501191
[14] Armenta J and Nieto J A, 2005 J. Math. Phys. 46 112503 [SPIRES]
[15] Leclerc M, 2006 Ann. Phys., NY 321 708 [SPIRES]
[16] Wise D K, MacDowell–Mansouri Gravity and Cartan Geometry, 2006 arXiv:gr-qc/0611154
[17] Mahato P, 2002 Mod. Phys. Lett. A 17 1991 [SPIRES] [arXiv:gr-qc/0604042]
Mahato P, 2004 Phys. Rev. D 70 124024 [SPIRES] [arXiv:gr-qc/0603100]
Mahato P, 2005 Int. J. Theor. Phys. 44 79 [SPIRES] [arXiv:gr-qc/0603109]
Mahato P, 2007 Int. J. Mod. Phys. A 22 835 [SPIRES] [arXiv:gr-qc/0603134]
[18] Tresguerres R, 2008 Int. J. Geom. Methods Mod. Phys. 5 171
[19] Yang C N, 1974 Phys. Rev. Lett. 33 445 [SPIRES]
[20] Stephenson G, 1958 Nuovo Cim. 9 263 [SPIRES]
Kilmister C W and Newman D J, 1961 Proc. Camb. Phil. Soc. 57 851
[21] Huang C-G, Tian Y, Wu X and Guo H-Y, 2008 Front. Phys. China 3 191 [arXiv:0801.0905]
[22] Huang P and Guo H-Y, 1974 Kexue Tongbao (Chin. Sci. Bull.) 19 512 (in Chinese)
Huang P, 1976 Kexue Tongbao (Chin. Sci. Bull.) 21 69 (in Chinese)
Han J-C, 1981 Acta Astrophys. Sin. 1 131 (in Chinese)
Guilfoyle B S and Nolan B C, 1998 Gen. Rel. Grav. 30 473 [SPIRES]
[23] Debnay G, Fairchild E E Jr and Siklos S T C, 1978 Gen. Rel. Grav. 9 879 [SPIRES]
[24] Zou Z-L et al, 1976 Acta Astron. Sin. 17 147 (in Chinese)
Chen S et al, 1976 Sci. Sin. 19 35 (in Chinese)
[25] Zhang Y-Z, 1992 Commun. Theor. Phys. 18 337 [SPIRES]
[26] Minkevich A V, Garkun A S and Kudin V I, 2007 Class. Quantum Grav. 24 5835 [SPIRES]
[27] Riess A G et al, 2004 Astrophys. J. 607 665 [SPIRES]
[28] John M V, 2004 Astrophys. J. 614 1 [SPIRES]
[29] Rapetti D et al, 2007 Mon. Not. R. Astron. Soc. 375 1510 [arXiv:astro-ph/0605683]
[30] Bahcall N and Fan X, 1998 Astrophys. J. 504 1 [SPIRES]
Kashlinsky A, 1998 Phys. Rep. 307 67 [SPIRES]
Carlb erg R G et al, 1999 Astrophys. J. 516 552 [SPIRES]
[31] Bennett C L et al, 2003 Astrophys. J. Suppl. 148 1 [arXiv:astro-ph/0302207]
Spergel D N et al, 2003 Astrophys. J. Suppl. 148 175 [arXiv:astro-ph/0302209]
Spergel D N et al, 2007 Astrophys. J. Suppl. 170 377 [arXiv:astro-ph/0603449]
[32] Hinshaw G et al, Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations: data
processing, sky maps, and basic results, 2008 arXiv:0803.0732
[33] Tegmark M, 2004 Phys. Rev. D 69 103501 [SPIRES] [arXiv:astro-ph/0310723]
[34] Reid I N, 1997 Astron. J. 114 161 [SPIRES] [arXiv:astro-ph/9704078]
Chaboyer B, Demarque P, Kernan P J and Krauss L M, 1998 Astrophys. J. 494 96 [SPIRES]
[arXiv:astro-ph/9706128v3]
[35] Mielke E W and Romero E S, 2006 Phys. Rev. D 73 043521 [SPIRES]
[36] Spivak M, 1999 A Comprehensive Introduction to Differential Geometry, 3rd edn (Boston, MA: Publish
or Perish)
[37] Perlick V, 2005 Nonlinear Anal. 63 e511