LOGARITHMIC SUBMAJORISATION AND ORDER-PRESERVING LINEAR ISOMETRIES

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Abstract. Let $E$ and $F$ be symmetrically $\Delta$-normed (in particular, quasi-normed) operator spaces affiliated with semifinite von Neumann algebras $M_1$ and $M_2$, respectively. We establish the disjointness preserving property for an order-preserving isometry $T : E \rightarrow F$ and show further that it guarantees the existence of a Jordan $*$-monomorphism from $M_1$ into $M_2$, provided that $E$ has order continuous $\Delta$-norm. As an application, we obtain a general form of these isometries, which extends and complements a number of existing results such as [7, Theorem 1], [60, Corollary 1], [65, Theorem 2] and [12, Theorem 3.1]. In particular, we fully resolve the case when $F$ is the predual of $M_2$ and other untreated cases in [72].

1. Introduction

Let $M$ be a von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. The $*$-algebra $S(M, \tau)$ of all $\tau$-measurable operators affiliated with $M$ is fundamentally important in noncommutative integration theory and/or in (semifinite version of) noncommutative geometry because it contains all $M$-bimodules of interest in these fields. Noncommutative $L_p$-spaces, or, more generally, noncommutative symmetric spaces, associated with $M$ are solid subspaces in $S(M, \tau)$ [50, 73], which are equipped with unitarily invariant (quasi)-norms (or even $\Delta$-norms). We view such bimodules as the noncommutative counterpart of the rearrangement invariant function spaces (see e.g. [4, 39]) which are important examples of partially ordered topological vector spaces [40, 54]. Indeed, the real subspace $S_h(M, \tau)$ (respectively, $E_h(M, \tau)$) of $S(M, \tau)$ (respectively, a symmetrically $\Delta$-normed space $E(M, \tau)$) consisting of all self-adjoint elements in $S(M, \tau)$ (respectively, $E(M, \tau)$) is a partially ordered vector space. Here, the partial ordering is an extension of the natural ordering in $M_h$, the real subspace of $M$ consisting of all self-adjoint operators. The prime intention of this paper is to demonstrate that any order-preserving (or positive) linear isometry from one such

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bimodule into another is generated by a Jordan \(*\)-monomorphism. In particular, let \( \mathcal{E} = E(M_1, \tau_1) \) and \( \mathcal{F} = F(M_2, \tau_2) \) be symmetrically \( \Delta \)-normed operator spaces associated with semifinite von Neumann algebras \((M_1, \tau_1)\) and \((M_2, \tau_2)\), respectively. In this paper, we show that if there exists an order-preserving linear isometry \( T: \mathcal{E} \to \mathcal{F} \) (i.e., \( T(x) \geq 0, \forall 0 \leq x \in \mathcal{E} \)), then \( M_1 \) and a weakly closed \( \ast \)-subalgebra of \( M_2 \) are Jordan \( \ast \)-isomorphic.

We shall omit the adjective “linear” as we do not consider non-linear isometries in this paper. The description of isometries from one noncommutative space into/onto another has been widely studied since the 1950s [42]. In particular, Kadison [42] showed that a surjective isometry between two von Neumann algebras can be written as a Jordan \( \ast \)-isomorphism multiplied by a unitary operator, which should be considered as a noncommutative version of the Banach-Stone Theorem [5]. After the non-commutative \( L_p \)-spaces were introduced in the 1950s [62], the description of \( L_p \)-isometries was investigated by Broise [9], Russo [60], Arazy [2] and Tam [74]. Finally, the complete description (for the semifinite case) was obtained in 1981 by Yeadon [76], i.e., every isometry \( T: L_p(M_1, \tau_1) \xrightarrow{\text{into}} L_p(M_2, \tau_2), 1 \leq p \neq 2 < \infty \), is generated by a Jordan \( \ast \)-isomorphism from \( M_1 \) onto a weakly closed \( \ast \)-subalgebra of \( M_2 \) (see [9] for order-preserving isometries on noncommutative \( L_2 \)-spaces). In the present paper, we concentrate on the following general question.

**Question 1.1.** Let \( \mathcal{E} \) and \( \mathcal{F} \) be symmetrically \( \Delta \)-normed operator spaces, respectively associated with semifinite von Neumann algebras \((M_1, \tau_1)\) and \((M_2, \tau_2)\). What is the general form of the order-preserving isometries \( T \) from \( \mathcal{E} \) into \( \mathcal{F} \)? i.e., is every order-preserving isometry \( T: \mathcal{E} \to \mathcal{F} \) generated by a Jordan \( \ast \)-homomorphism from \( M_1 \) into \( M_2 \)?
isometries $T : \mathcal{E} \rightarrow \mathcal{F}$ is obtained only in the special case when $\mathcal{E} = L_p(M_1, \tau_1)$ and $\mathcal{F} = L_p(M_2, \tau_2)$, $p > 0$, $p \neq 2$ (see e.g. [11, 63, 64, 76]; see also [65, 68] for results for symmetric spaces affiliated with specific semifinite algebras). One of the initial motivations of the present paper is to resolve the problem left in [72, Section 5] and to present new approaches which allow for study of order-preserving isometries of quasi-normed spaces and $\Delta$-normed spaces. Our results are new even in the classical (commutative) setting (see e.g. [3, 45]), as we are able to treat injective isometries between symmetrically quasi-normed, and even $\Delta$-normed, spaces, which appear to be non-amenable to any previously used techniques mostly developed for Banach spaces setting.

To further elaborate this point, recall that every symmetrically normed operator space is a subspace of $L_1(M, \tau) + M$ (see e.g. [21, 49, 50]). The $L_{\log}$-space $L_{\log}(M, \tau)$ plays a similar role for $\Delta$-normed/quasi-normed spaces as $L_1(M, \tau)$ does in the normed case, which was recently introduced and studied in [26]. That is, the majority of symmetrically $\Delta$-normed spaces used in analysis are subspaces of $L_{\log}(M, \tau) + M$. In the present paper, we consider $\Delta$-normed operator spaces $\mathcal{F}$ which are subspaces of $L_{\log}(M_2, \tau_2) + M_2$.

The main method used for the description of isometries is to establish and employ the “disjointness preserving” property, which underlies all investigations in the general (noncommutative) case. This idea lurks in the background of Yeadon’s description [76] (see also [63, 64, 74]) of isometries of noncommutative $L^p$-spaces ($1 \leq p \neq 2 < \infty$), whose proof relies on the study when we have the equality in the Clarkson’s inequality. However, Abramovich [1, Remark 2, p.78] emphasised that order-preserving isometries from a (Banach) symmetric space $\mathcal{E}$ into another (Banach) symmetric space $\mathcal{F}$ may not necessarily enjoy the “disjointness preserving” property, even in the commutative setting. Still, in the commutative setting, the “disjointness preserving” property of order-preserving isometries can be guaranteed by the so-called strict monotonicity of the norm $\| \cdot \|_F$. That is, by the assumption that $0 \leq z_1 < z_2 \in F$, we have $\|z_1\|_F < \|z_2\|_F$. It is natural to consider the following question in order to answer Question 1.1.

**Question 1.2.** Let the conditions of Question 1.1 hold. Does $T$ preserve disjointness? That is, does the equality $T(x)T(y) = 0$ hold whenever $xy = 0$, $0 \leq x, y \in \mathcal{E}$?

The so-called strictly $K$-monotone norms (see [22] or Section 2) form a proper noncommutative counterpart to the notion of strictly monotone norms, which were introduced in [11, 23, 67, 71] as an important component in the characterisation of Kadec-Klee type properties. Using the “the triangle inequality for the Hardy-Littlewood preorder” introduced in [12, 70] and analysing when this inequality
turns into equality, it is shown in [72] that every order-preserving isometry into $F$ possesses the “disjointness preserving” property whenever $F \subset (L_1 + L_\infty)(M_2, \tau_2)$ is a symmetric space with strictly $K$-monotone norm $\| \cdot \|_F$. The drawback of this approach is the fact that many important symmetric norms fail to be strictly $K$-monotone. In particular, as mentioned before, even the usual $L_1$-norm is not strictly $K$-monotone. To rectify this drawback and cover maximally wide class of (quasi-normed and $\Delta$-normed) symmetric spaces, we introduce the notion of strictly log-monotone (SLM) $\Delta$-norms (see Section 2), which should be considered as a far-reaching generalisation of the strict $K$-monotonicity. The class of SLM $\Delta$-norms embraces an extensive class of symmetric $\Delta$-norms. For example, the usual $L_p$-norms ($0 < p < \infty$), Lorentz quasi-norms and the log-integrable-$F$-norm $\| \cdot \|_{\log}$, are all examples of SLM $\Delta$-norms (for which we refer to Section 2 and Section 6).

Using techniques developed from detailed study of logarithmic submajorisation, we show that every order-preserving isometry $T : E \rightarrow F$ possesses the “disjointness preserving” property if $\| \cdot \|_F$ is an SLM $\Delta$-norm. Surprisingly, this result appears to be new even for symmetrically $\Delta$-normed function spaces. With “disjointness preserving” property at hand, we provide a general description of all order-preserving injective isometries $T$ as above, showing that every such isometry is generated by a Jordan $*$-isomorphism from $M_1$ onto a weakly closed $*$-subalgebra of $M_2$. This result extends and strengthens a number of existing results in the literature. On the one hand, we extend [72, Proposition 6] (see also [48,60]) to the case of arbitrary semifinite von Neumann algebras. On the other hand, we establish that the main results of [9,12,48,60,72] continue to hold in a much wider setting than in those papers. In particular, we resolve the $L_1$-case which was not amenable to the techniques based on strict $K$-monotonicity used in [72]. When $M_2$ is a semifinite factor, we obtain a semifinite version of [60, Corollary 1], showing that every order-preserving isometry $T : E \rightarrow F$ coincides with a $*$-isomorphism or a $*$-anti-isomorphism multiplied by a positive constant on $E \cap M_1$. In particular, when $M_1$ and $M_2$ are finite factors, $T|_{M_1}$ is indeed a $*$-isomorphism or a $*$-anti-isomorphism from $M_1$ onto $M_2$ multiplied a positive constant, which recovers and substantially extends [60, Corollary 1].

2. Preliminaries

In this section, we recall main notions of the theory of noncommutative integration, introduce some properties of generalised singular value functions and define noncommutative symmetrically $\Delta$-normed spaces. In what follows, $\mathcal{H}$ is a Hilbert space and $B(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$, and $1$ is the identity operator on $\mathcal{H}$. Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$. For details on von Neumann algebra theory, the reader is referred to e.g. [17]...
or [73]. General facts concerning measurable operators may be found in [53], [62] (see also the forthcoming book [22]). For convenience of the reader, some of the basic definitions are recalled.

2.1. $\tau$-measurable operators and generalised singular values. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$, where the domain $\mathcal{D}(x)$ of $x$ is a linear subspace of $\mathcal{H}$, is said to be affiliated with $\mathcal{M}$ if $yx \subseteq xy$ for all $y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is termed measurable with respect to $\mathcal{M}$ if $x$ is closed, densely defined, affiliated with $\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in the logic of all projections of $\mathcal{M}$, $\mathcal{P}(\mathcal{M})$, such that $p_n \uparrow 1$, $p_n(\mathcal{H}) \subseteq \mathcal{D}(x)$ and $1 - p_n$ is a finite projection (with respect to $\mathcal{M}$) for all $n$. It should be noted that the condition $p_n(\mathcal{H}) \subseteq \mathcal{D}(x)$ implies that $xp_n \in \mathcal{M}$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products (denoted simply by $x + y$ and $xy$ for all $x, y \in S(\mathcal{M})$).

Let $x$ be a self-adjoint operator affiliated with $\mathcal{M}$. We denote its spectral measure by $\{e^x\}$. It is well known that if $x$ is a closed operator affiliated with $\mathcal{M}$ with the polar decomposition $x = u|x|$, then $u \in \mathcal{M}$ and $e \in \mathcal{M}$ for all projections $e \in \{e^{|x|}\}$. Moreover, $x \in S(\mathcal{M})$ if and only if $x$ is closed, densely defined, affiliated with $\mathcal{M}$ and $e^{\pi}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when $\mathcal{M}$ is a von Neumann algebra of type $\text{III}$ or a type $\text{I}$ factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type $\text{II}$ von Neumann algebras, this is no longer true. From now on, let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$.

For any closed and densely defined linear operator $x : \mathcal{D}(x) \to \mathcal{H}$, the null projection $n(x) = n(|x|)$ is the projection onto its kernel $\text{Ker}(x)$, the range projection $r(x)$ is the projection onto the closure of its range $\text{Ran}(x)$ and the support projection $s(x)$ of $x$ is defined by $s(x) = 1 - n(x)$.

An operator $x \in S(\mathcal{M})$ is called $\tau$-measurable if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathcal{P}(\mathcal{M})$ such that $p_n \uparrow 1$, $p_n(\mathcal{H}) \subseteq \mathcal{D}(x)$ and $\tau(1 - p_n) < \infty$ for all $n$. The collection of all $\tau$-measurable operators is a unital $*$-subalgebra of $S(\mathcal{M})$, denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator $x$ belongs to $S(\mathcal{M}, \tau)$ if and only if $x \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(e^{\pi}(\lambda, \infty)) < \infty$. Alternatively, an unbounded operator $x$ affiliated with $\mathcal{M}$ is $\tau$-measurable (see [28]) if and only if

$$\tau\left(e^{\pi}(n, \infty)\right) \to 0, \quad n \to \infty.$$ 

For any $x = x^* \in S(\mathcal{M}, \tau)$, we set $x_+ = xe^{\pi}(0, \infty)$ and $x_- = xe^{\pi}(-\infty, 0)$.

**Definition 2.1.** Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semi-finite trace $\tau$ and let $x \in S(\mathcal{M}, \tau)$. The generalised singular value
function \( \mu(x) : t \to \mu(t; x), t > 0, \) of the operator \( x \) is defined by setting

\[
\mu(t; x) = \inf \{ \|xp\| : p = p^* \in \mathcal{M} \text{ is a projection, } \tau(1 - p) \leq t \}.
\]

An equivalent definition in terms of the distribution function of the operator \( x \) is the following. For every self-adjoint operator \( x \in S(\mathcal{M}, \tau) \), setting

\[
d_x(t) = \tau(e^x(t, \infty)), \quad t > 0,
\]

we have (see e.g. \([28]\) and \([50]\))

\[
\mu(t; x) = \inf \{ s \geq 0 : d_x(s) \leq t \}.
\]

Note that \( d_x(\cdot) \) is a right-continuous function (see e.g. \([28]\) and \([22]\)).

Consider the algebra \( \mathcal{M} = L^\infty(0, \infty) \) of all Lebesgue measurable essentially bounded functions on \((0, \infty)\). Algebra \( \mathcal{M} \) can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space \( \mathcal{H} = L^2(0, \infty) \), with the trace given by integration with respect to Lebesgue measure \( m \). It is easy to see that the algebra of all \( \tau \)-measurable operators affiliated with \( \mathcal{M} \) can be identified with the subalgebra \( S(0, \infty) \) of the algebra of Lebesgue measurable functions which consists of all functions \( f \) such that \( m(\{|f| > s\}) \) is finite for some \( s > 0 \). It should also be pointed out that the generalised singular value function \( \mu(f) \) is precisely the decreasing rearrangement \( \mu(f) \) of the function \( |f| \) (see e.g. \([49]\)) defined by

\[
\mu(t; f) = \inf \{ s \geq 0 : m(\{|f| \geq s\}) \leq t \}.
\]

For convenience of the reader, we also recall the definition of the measure topology \( t_\tau \) on the algebra \( S(\mathcal{M}, \tau) \). For every \( \varepsilon, \delta > 0 \), we define the set

\[
V(\varepsilon, \delta) = \{ x \in S(\mathcal{M}, \tau) : \exists p \in \mathcal{P}(\mathcal{M}) \text{ such that } \| x(1 - p) \|_\infty \leq \varepsilon, \tau(p) \leq \delta \}.
\]

The topology generated by the sets \( V(\varepsilon, \delta), \varepsilon, \delta > 0 \), is called the measure topology \( t_\tau \) on \( S(\mathcal{M}, \tau) \) \([22,28,55]\). It is well known that the algebra \( S(\mathcal{M}, \tau) \) equipped with the measure topology is a complete metrizable topological algebra \([55]\). We note that a sequence \( \{x_n\}_{n=1}^\infty \subset S(\mathcal{M}, \tau) \) converges to zero with respect to measure topology \( t_\tau \) if and only if \( \tau(\mathcal{E}[|x_n|, \infty]) \to 0 \) as \( n \to \infty \) for all \( \varepsilon > 0 \) \([22]\).

The space \( S_0(\mathcal{M}, \tau) \) of \( \tau \)-compact operators is the space associated to the algebra of functions from \( S(0, \infty) \) vanishing at infinity, that is,

\[
S_0(\mathcal{M}, \tau) = \{ x \in S(\mathcal{M}, \tau) : \mu(\infty; x) = 0 \}.
\]

The two-sided ideal \( \mathcal{F}(\tau) \) in \( \mathcal{M} \) consisting of all elements of \( \tau \)-finite range is defined by

\[
\mathcal{F}(\tau) = \{ x \in \mathcal{M} : \tau(r(x)) < \infty \} = \{ x \in \mathcal{M} : \tau(s(x)) < \infty \}.
\]

Clearly, \( S_0(\mathcal{M}, \tau) \) is the closure of \( \mathcal{F}(\tau) \) with respect to the measure topology \([21]\).
A further important vector space topology on $S(M, \tau)$ is the local measure topology \[7\] \cite{22}. A neighbourhood base for this topology is given by the sets $V(\varepsilon, \delta; p), \varepsilon, \delta > 0, p \in P(M) \cap F(\tau)$, where

$$V(\varepsilon, \delta; p) = \{x \in S(M, \tau) : pxp \in V(\varepsilon, \delta)\}.$$ 

It is clear that the local measure topology is weaker than the measure topology \cite{21} \cite{22}. If $\{x_{\alpha}\} \subset S(M, \tau)$ is a net and if $x_{\alpha} \to x \in S(M, \tau)$ in local measure topology, then $x_{\alpha}y \to xy$ and $yx_{\alpha} \to yx$ in the local measure topology for all $y \in S(M, \tau)$ \cite{21} \cite{22}.

2.2. Symmetrically $\Delta$-normed spaces of $\tau$-measurable operators. For convenience of the reader, we recall the definition of $\Delta$-norms. Let $\Omega$ be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from $\Omega$ to $\mathbb{R}$ is a $\Delta$-norm, if for all $x, y \in \Omega$ the following properties hold:

\begin{align*}
(1) & \quad \|x\| \geq 0, \quad \|x\| = 0 \iff x = 0; \\
(2) & \quad \|\alpha x\| \leq \|x\|, \quad \forall |\alpha| \leq 1; \\
(3) & \quad \lim_{\alpha \to 0} \|\alpha x\| = 0; \\
(4) & \quad \|x + y\| \leq C_{\Omega} (\|x\| + \|y\|)
\end{align*}

for a constant $C_{\Omega} \geq 1$ independent of $x, y$. The couple $(\Omega, \|\cdot\|)$ is called a $\Delta$-normed space. We note that the definition of a $\Delta$-norm given above is the same with that given in \cite{43}. It is well-known that every $\Delta$-normed space $(\Omega, \|\cdot\|)$ is metrizable and conversely every metrizable space can be equipped with a $\Delta$-norm \cite{43}. Note that properties (2) and (4) of a $\Delta$-norm imply that for any $\alpha \in \mathbb{C}$, there exists a constant $M$ such that $\|\alpha x\| \leq M ||x||$, $x \in \Omega$, in particular, if $\|x_n\| \to 0$, $\{x_n\}_{n=1}^{\infty} \subset \Omega$, then $\|\alpha x_n\| \to 0$. In particular, when $C_{\Omega} = 1$, $\Omega$ is called an $F$-normed space \cite{43}.

Let $E(0, \infty)$ be a space of real-valued Lebesgue measurable functions on $(0, \infty)$ (with identification $m$-a.e.), equipped with a $\Delta$-norm $\|\cdot\|_E$. The space $E(0, \infty)$ is said to be absolutely solid if $x \in E(0, \infty)$ and $|y| \leq |x|, \ y \in S(0, \infty)$ implies that $y \in E(0, \infty)$ and $\|y\|_E \leq \|x\|_E$. An absolutely solid space $E(0, \infty) \subseteq S(0, \infty)$ is said to be symmetric if for every $x \in E(0, \infty)$ and every $y \in S(0, \infty)$, the assumption $\mu(y) = \mu(x)$ implies that $y \in E(0, \infty)$ and $\|y\|_E = \|x\|_E$ (see e.g. \cite{14} \cite{19}).

We now come to the definition of the main object of this paper.

**Definition 2.2.** Let $M$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Let $E$ be a linear subset in $S(M, \tau)$ equipped with a $\Delta$-norm $\|\cdot\|_E$. We say that $E$ is a symmetrically $\Delta$-normed space if for $x \in E$, $y \in S(M, \tau)$ and $\mu(y) \leq \mu(x)$ imply that $y \in E$ and $\|y\|_E \leq \|x\|_E$. 

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Let $E(\mathcal{M}, \tau)$ be a symmetrically $\Delta$-normed space. Since $\mu(axb) \leq \mu(\|a\|_\infty \|b\|_\infty x)$, $a, b \in \mathcal{M}$, $x \in E(\mathcal{M}, \tau)$, it follows that every symmetrically $\Delta$-normed space is an $\mathcal{M}$-bimodule. It is well-known that any symmetrically normed space $E(\mathcal{M}, \tau)$ is a complete symmetrically $\Delta$-normed space (see e.g. [21] and [22]). However, one should note that a symmetrically $\Delta$-normed space when $E(\mathcal{M}, \tau)$ does not necessarily satisfy $\|axb\|_E \leq \|a\|_\infty \|b\|_\infty \|x\|_E$, $a, b \in \mathcal{M}$, $x \in E(\mathcal{M}, \tau)$. For every $x \in E(\mathcal{M}, \tau)$ and \{y$_n \in \mathcal{M}$\} with $\|y_n\|_\infty \to 0$, $\mu(xy_n), \mu(y_n x) \leq \|y_n\|_\infty \mu(x) = \mu(\|y_n\|_\infty x)$ implies that $xy_n, y_n x \in E(\mathcal{M}, \tau)$ and

$$\|xy_n\|_E, \|y_n x\|_E \leq \|\|y_n\|_\infty x\|_E \xrightarrow{n \to \infty} 0.$$  

Definition [22] together with [50] Lemma 2.3.12 and Corollary 2.3.17 implies that

$$\|x\|_E = \|x^*\|_E = \|x\|_E, \ x \in E.$$  

There exists a strong connection between symmetric function spaces and operator spaces exposed in [44] (see also [63, 69]). The operator space $E(\mathcal{M}, \tau)$ defined by

$$E(\mathcal{M}, \tau) := \{x \in S(\mathcal{M}, \tau) : \mu(x) \in E(0, \infty)\}, \|x\|_{E(\mathcal{M}, \tau)} := \|\mu(x)\|_E$$

is a complete symmetrically $\Delta$-normed space whenever $(E(0, \infty), \|\cdot\|_E)$ is a complete symmetrically $\Delta$-normed function space on $(0, \infty)$ [37] (see also [44, 69]).

For a given symmetrically $\Delta$-normed space $E(\mathcal{M}, \tau)$, we denote $\mathcal{P}(E) := E(\mathcal{M}, \tau) \cap \mathcal{P}(\mathcal{M})$. If $p, q \in \mathcal{P}(E)$, then $p \vee q \in \mathcal{P}(E)$ (see e.g. [22] Chapter IV, Lemma 1.4 or [21] Lemma 4). The carrier projection $c_E \in \mathcal{M}$ of an $\mathcal{M}$-bimodule $E$ is defined by setting

$$c_E := \vee\{p : p \in \mathcal{P}(E)\}.$$ 

The following proposition is an extension of [22] Chapter IV, Lemma 4.4.

**Proposition 2.3.** If the carrier projection $c_E$ of a symmetrically $\Delta$-normed space $E(\mathcal{M}, \tau)$ is equal to 1, then

$$\{p \in \mathcal{P}(\mathcal{M}) : \tau(p) < \infty\} \subset \mathcal{P}(E)$$

and hence, $\mathcal{F}(\tau) \subset E(\mathcal{M}, \tau)$.

**Proof.** By [21] Lemma 4 (iii), the set $\mathcal{P}(E)$ is upwards directed and the normality of trace $\tau$ implies that

$$\sup\{\tau(p) : P \in \mathcal{P}(E)\} = \tau(1).$$

Suppose first that $\tau(1) = \infty$. If $q \in \mathcal{P}(\mathcal{M})$ satisfies $\tau(q) < \infty$, then (7) implies that $\tau(q) \leq \tau(p)$ for some $p \in \mathcal{P}(E)$ and hence, $q \in \mathcal{P}(E)$. This proves the assertion in the case that $\tau(1) = \infty$.

Assume that $\tau(1) < \infty$. It suffices to show that $1 \in E(\mathcal{M}, \tau)$. It follows from (7) that there exists $p \in \mathcal{P}(E)$ such that $\tau(p) \geq \frac{1}{2} \tau(1)$. Since $\tau(p^\perp) \leq \frac{1}{2} \tau(1) \leq \tau(p)$, it follows that also $p^\perp \in \mathcal{P}(E)$ and hence, $1 \in \mathcal{P}(E)$.
If \( x \in \mathcal{F}(\tau) \), then the support projection \( p = s(x) \) of \( x \) satisfies \( \tau(p) < \infty \) and \( |x| \leq \|x\|_p \). That is, \( \mu(x) = \mu(|x|) \leq \mu(\|x\|_p) \). Since \( p \in \mathcal{P}(E) \) and \( E(\mathcal{M}, \tau) \) is a linear space, it follows Definition 2.2 that \( x \in E(\mathcal{M}, \tau) \).

It is often assumed that the carrier projection \( c_E \) is equal to 1. Indeed, for any symmetrically \( \Delta \)-normed function space \( E(0, \infty) \) on the interval \( (0, \infty) \), the carrier projection of the corresponding operator space \( E(\mathcal{M}, \tau) \) is always 1 (see e.g. [25, see also 39]). In the present paper, we always assume that the carrier projection of a symmetrically \( \Delta \)-normed space is equal to 1.

2.3. Submajorisation. If \( x, y \in S(\mathcal{M}, \tau) \), then \( x \) is said to be submajorised by \( y \), denoted by \( x \prec \prec y \) (Hardy-Littlewood-Polya submajorisation), if

\[
\int_0^t \mu(s; x) \, ds \leq \int_0^t \mu(s; y) \, ds
\]

for all \( t \geq 0 \) (see e.g. [21, 22, 50]).

The algebra

\[
\mathcal{L}_{\log}(\mathcal{M}, \tau) := \{ x \in S(\mathcal{M}, \tau) : \|x\|_{\log} := \int_0^\infty \log(1 + \mu(t; x)) \, dt < \infty \}
\]

of log-integrable operators introduced in [20] is a complete symmetrically \( F \)-normed space. Denote \( \log_+ t := \max\{\log t, 0\} \). For \( x, y \in S(\mathcal{M}, \tau) \) with \( \log_+ \mu(x), \log_+ \mu(y) \in L_1(0, \infty) + L_\infty(0, \infty) \), \( x \) is said to be logarithmically submajorised by \( y \) [20, 30], denoted by \( x \prec \prec_{\log} y \), if

\[
\int_0^t \log(\mu(s; x)) \, ds \leq \int_0^t \log(\mu(s; y)) \, ds, \quad t \geq 0.
\]

In particular, we have \( \mu(xy) \prec \prec_{\log} \mu(x)\mu(y) \) (see [36] Theorem 1.18 or [20]).

For the sake of convenience, we denote \( \mathcal{M}^\Delta := (\mathcal{L}_{\log}(\mathcal{M}, \tau) + \mathcal{M}) \cap S(\mathcal{M}, \tau) \). In particular, for \( x \in S(\mathcal{M}, \tau) \), \( x \in \mathcal{M}^\Delta \) if and only if \( \log_+ \mu(x) \in L_1(0, \infty) + L_\infty(0, \infty) \) and \( \mu(\infty; x) = 0 \).

A (Banach) symmetric norm \( \| \cdot \|_E \) on \( E(\mathcal{M}, \tau) \) is called strictly \( K \)-monotone if and only if \( \|x\|_E < \|y\|_E \) whenever \( x, y \in E(\mathcal{M}, \tau) \), \( x \prec \prec y \) and \( \mu(x) \neq \mu(y) \). It is natural to introduce the following notion when considering symmetrically \( \Delta \)-normed (or quasi-normed) operator space. A symmetric \( \Delta \)-norm on \( E(\mathcal{M}, \tau) \subset (\mathcal{L}_{\log} + \mathcal{L}_\infty)(\mathcal{M}, \tau) \) is called a strictly \( \log \)-monotone (SLM) \( \Delta \)-norm if \( \|x\|_E < \|y\|_E \) whenever \( x, y \in E(\mathcal{M}, \tau) \) satisfies \( \mu(x) \prec \prec_{\log} \mu(y) \) and \( \mu(x) \neq \mu(y) \). \( \| \cdot \|_E \) is called \( \log \)-monotone if \( \|X\|_E \leq \|y\|_E \) whenever \( \mu(x) \prec \prec_{\log} \mu(y) \). Indeed, the usual \( L_p \)-norm \( \| \cdot \|_p \), \( 0 < p < \infty \), are SLM \( \Delta \)-norms. In the last section, we show that noncommutative Lorentz spaces associated with \( \mathcal{M} \) are SLM quasi-normed. It is clear that \( E(\mathcal{M}, \tau) \) has SLM \( \Delta \)-norm whenever \( \| \cdot \|_E \) is an SLM \( \Delta \)-norm on \( E(0, \infty) \).

We denote the decreasing rearrangement \( f^* \) of a measurable function \( f \) by

\[
f^*(t) = \inf\{s \geq 0 : m(\{f \geq s\}) \leq t\}.
\]
The following result is well-known, which is essentially an inequality of Hardy, Littlewood and Polya (see [57] Theorem 12.15 or [51] Chapter 1, Theorem D.2 for results imply the following, or [75] Lemma and [51] Chapter II, Lemma 3.4 for the sequence version).

Proposition 2.4. Assume that \( f = f^* \) and \( g = g^* \) are measurable function integrable on \((0, s), s > 0\). If \( \int_0^b f(t)dt \leq \int_0^b g(t)dt \) for every \( 0 < b \leq s \), then for every continuous convex function \( \varphi \) on \( \mathbb{R} \), we have \( \int_0^b \varphi(f(t))dt \leq \int_0^b \varphi(g(t)) dt \) for every \( 0 < b \leq s \).

The following corollary is an easy consequence of Proposition 2.4.

Corollary 2.5. Let \( x, y \in M^\Delta \). If \( \mu(x) \prec \prec \log \mu(y) \), then \( \mu(x)^p \prec \prec \mu(y)^p \), \( 0 < p < \infty \).

Proposition 2.6. \( \| \cdot \|_\log \) is an SLM symmetric \( F \)-norm on \( \mathcal{L}_{\log}(\mathcal{M}, \tau) \).

Proof. Since \( \| \cdot \|_\log \) is a symmetric \( F \)-norm on \( \mathcal{L}_{\log}(\mathcal{M}, \tau) \), it suffices to prove the SLM property. Assume that \( x, y \in \mathcal{L}_{\log}(\mathcal{M}, \tau) \) with \( \mu(x) \prec \prec \log \mu(y) \). Without loss of generality, we may assume that \( \mu(t; x) > 0 \) for every \( t > 0 \) (therefore, \( \mu(t; y) > 0 \), \( t > 0 \)). Note that \( \log(\mu(x)\chi_{(0,t)}) \), \( \log(\mu(y)\chi_{(0,t)}) \) are integrable functions on \((0, t)\).

Since \( \mu(x) \prec \prec \log \mu(y) \), it follows that

\[
\mu(x)^\frac{1}{t}\chi_{(0,t)} \prec \prec \log \mu(y)^\frac{1}{t}\chi_{(0,t)},
\]

and therefore,

\[
\mu(x)\chi_{(0,t)} \prec \prec \log \mu(y)^\frac{1}{t}\mu(x)^\frac{1}{t}\chi_{(0,t)} \prec \prec \log \mu(y)^\frac{1}{t}\chi_{(0,t)}.
\]

Taking a continuous convex function \( \varphi(t) := \log(1 + e^t), t \in \mathbb{R} \), by Proposition 2.4 we obtain that

\[
(\mu(x) + 1)\chi_{(0,t)} \prec \prec \log \left( \mu(y)^\frac{1}{t}\mu(x)^\frac{1}{t} + 1 \right)\chi_{(0,t)} \prec \prec \log (\mu(y) + 1)\chi_{(0,t)}
\]

for every \( t > 0 \), which implies that \( \| \cdot \|_\log \) is log-monotone. In addition, if \( \| \mu \|_\log = \| \mu \|_\log \), then (8) implies that

\[
\begin{align*}
2 \int_0^\infty \log(\mu(t; y)^\frac{1}{t}\mu(t; x)^\frac{1}{t} + 1)dt \\
= \int_0^\infty \log(\mu(t; x) + 1) + \log(\mu(t; y) + 1)dt.
\end{align*}
\]

Since \( \mu(t; y)^\frac{1}{t}\mu(t; x)^\frac{1}{t} + 1 \leq (\mu(t; x) + 1)(\mu(t; y) + 1) \) and the equality holds true only when \( \mu(t; x) = \mu(t; y) \), it follows from (9) that \( \mu(x) = \mu(y) \), a.e.. By the right-continuity of \( \mu(x) \) and \( \mu(y) \), we obtain that \( \mu(x) = \mu(y) \). Therefore, \( \| \cdot \|_\log \) is an SLM \( F \)-norm on \( \mathcal{L}_{\log}(\mathcal{M}, \tau) \). □
2.4. Order continuous ∆-norms. In this subsection, we introduce the notion of order continuous ∆-norms. For the introduction of order continuous norms, we refer to [19, 21, 22].

If $E(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau)$ is a symmetrically ∆-normed operator space, then the ∆-norm $\| \cdot \|_E$ is called order continuous if $\|x_{\alpha}\|_E \to_\alpha 0$ whenever $\{x_{\alpha}\}$ is a downwards directed net in $E(\mathcal{M}, \tau)^+$ satisfying $x_{\alpha} \downarrow 0$. See e.g. [21, Proposition 2], we obtain the following result immediately.

Lemma 2.4. Let $\mathcal{M}$ be such that $\text{dim} \mathcal{M} = 1$ and $\mathcal{M}$ be an increasing net in $\mathcal{M}$ and $x \in E_h(\mathcal{M}, \tau)$ with $\|x_{\alpha} - x\|_E \to 0$, then $x_{\alpha} \uparrow x$.

Proposition 2.8. If $E(\mathcal{M}, \tau) \subset S(\mathcal{M}, \tau)$ is a symmetrically ∆-normed space, then the following statements are equivalent:

(i) $E(\mathcal{M}, \tau)$ has order continuous ∆-norm;

(ii) $\|x_{\alpha}\|_n \downarrow 0$ for every decreasing sequence $\{x_{\alpha}\}_n \subset E(\mathcal{M}, \tau)^+$ satisfying $x_{\alpha} \downarrow 0$.

Proof. Since it is clear that (ii) follows from (i), it suffices to show that statement (ii) implies that $\| \cdot \|_E$ is order continuous.

Suppose that $\{x_{\alpha}\}$ is a decreasing net in $E(\mathcal{M}, \tau)^+$ satisfying $x_{\alpha} \downarrow 0$. It should be observed that this implies that $\{x_{\alpha}\}$ is a Cauchy net for the ∆-norm. Indeed, if $\{x_{\alpha}\}$ is not Cauchy, then there exists an $\varepsilon > 0$ and an increasing subsequence $\{x_{\alpha_\lambda}\}_{\lambda=1}^\infty$ such that $\|x_{\alpha_\lambda} - x_{\alpha_{\lambda+1}}\|_E \geq \varepsilon$ for all $\lambda$. By [21, Proposition 2 (ii)], we obtain that there exists $y \in S(\mathcal{M}, \tau)^+$ such that $x_{\alpha_\lambda} \downarrow y$. By assumption (ii), this implies that $\|x_{\alpha_\lambda} - y\|_E \to 0$. Hence, we obtain that

$$
\varepsilon \leq \|x_{\alpha_\lambda} - x_{\alpha_{\lambda+1}}\|_E \leq C_E(\|x_{\alpha_\lambda} - y\|_E + \|x_{\alpha_{\lambda+1}} - y\|_E) \to_\lambda 0,
$$

which is a contradiction. This implies that there exists a decreasing subsequence $\{x_{\alpha_\lambda}\}$ such that

$$
\|x_{\alpha_\lambda} - x_{\alpha}\|_E \leq 1/\lambda
$$

for every $\lambda \geq \alpha$. Let $x \in S(\mathcal{M}, \tau)^+$ be such that $x_{\alpha} \downarrow x$ (see [21, Proposition 2]). Since $x_{\alpha_\lambda} \geq x$, it follows that $x \in E(\mathcal{M}, \tau)^+$. It follows from (ii) that $\|x - x_{\alpha}\|_E \to 0$ as $n \to \infty$ and hence, by (10), we have $\|x - x_{\alpha}\|_E \to 0$. Appealing to Lemma [21, Proposition 2], we obtain that $x_{\alpha} \downarrow x$. Hence, $x = 0$ and so, $\|x_{\alpha}\|_E \to_\alpha 0$.

Assume that $E(\mathcal{M}, \tau)$ is a symmetrically ∆-normed space. The subset $E(\mathcal{M}, \tau)^{oc} \subset E(\mathcal{M}, \tau)$ is defined by setting

$$
E(\mathcal{M}, \tau)^{oc} = \{x \in E : |x| \geq x_\downarrow_\alpha 0 \Rightarrow \|x_{\alpha}\|_E \downarrow 0\}.
$$
Remark 2.9. $E(M, \tau)^{oc}$ is a subspace of the $\| \cdot \|_E$-closure of $F(\tau)$ in $E(M, \tau)$. Indeed, if $0 \leq x \in E(M, \tau)^{oc}$, then there exists an upwards directed net $\{x_\alpha\}$ in $F(\tau)^+$ such that $0 \leq x_\alpha \uparrow x$ (see e.g. [21] Corollary 8 (vi) or [22] Chapter IV, Corollary 1.9), that is, $x \geq x - x_\alpha \downarrow_\alpha 0$. Hence, $\|x - x_\alpha\|_E \downarrow_\alpha 0$.

Since $S_0(M, \tau)$ is closed in $S(M, \tau)$ with respect to the measure topology (see [21] Section 2.4) and the embedding of $E(M, \tau)$ into $S(M, \tau)$ is continuous with respect to the measure topology (see [24] Lemma 2.4), it follows from $F(\tau) \subset S_0(M, \tau)$ that $E(M, \tau)^{oc} \subset S_0(M, \tau)$.

Proposition 2.10. Let $E(0, \infty) \subset S(0, \infty)$ be a symmetrically $\Delta$-normed function space. If $x \in E(M, \tau)$ and $\mu(x) \in E(0, \infty)^{oc}$, then $x \in E(M, \tau)^{oc}$. In particular, if $E(0, \infty)$ has order continuous $\Delta$-norm $\| \cdot \|_E$, then $\| \cdot \|_E$ is an order continuous $\Delta$-norm on $E(M, \tau)$.

Proof. If $\mu(x) \in E(0, \infty)^{oc}$, then $x \in S_0(M, \tau)$ (see Remark 2.9), that is, $\lim_{t \to \infty} \mu(t;x) = 0$. Suppose that $\{x_\alpha\}$ is a net in $E(M, \tau)$ such that $|x| \geq x_\alpha \downarrow_\alpha 0$. It follows from [19] Lemma 3.5 (see also [22] Chapter III, Lemma 2.14) that $\mu(x) \geq \mu(x_\alpha) \downarrow_\alpha 0$ in $E(0, \infty)$. Since $\mu(x) \in E(0, \infty)^{oc}$, this implies that $\|\mu(x_\alpha)\|_E \downarrow_\alpha 0$, that is, $\|x_\alpha\|_E \downarrow_\alpha 0$. \Box

We obtain the following corollary immediately.

Corollary 2.11. $\| \cdot \|_{log}$ is an order continuous $F$-norm on $\mathcal{L}_{log}(M, \tau)$.

Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be two semifinite von Neumann algebras. The general form of order-preserving isometries $T : \mathcal{L}_{log}(M_1, \tau_1) \overset{int}{{\rightarrow}} \mathcal{L}_{log}(M_2, \tau_2)$ is obtained in Corollary 4.9. In particular, there is a Jordan $*$-homomorphism from $M_1$ into $M_2$.

2.5. Jordan $*$-isomorphism. Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be two semifinite von Neumann algebras. A complex-linear map $J : M_1 \overset{int}{{\rightarrow}} M_2$ is called Jordan $*$-homomorphism if $J(x^*) = J(x)^*$ and $J(x^2) = J(x)^2$, $x \in M_1$ (equivalently, $J(xy + yx) = J(x)J(y) + J(y)J(x)$, $x, y \in M_1$). The following definitions vary slightly in different literature. In the present paper, we stick to the following definitions. We call $J$ a Jordan $*$-monomorphism if it is injective. If $J$ is a bijective Jordan $*$-homomorphism, then it is called a Jordan $*$-isomorphism (see [3] Definition 3.2.1]). A Jordan $*$-homomorphism is called normal if it is completely additive (equivalently, ultraweakly continuous). Alternatively, we adopt the following equivalent definition: $J(x_\alpha) \uparrow J(x)$ whenever $x_\alpha \uparrow x \in M_1^+$ (see e.g. [17] Chapter I.4.3). We note that there are some literature in which injective Jordan $*$-homomorphisms are called Jordan $*$-isomorphisms, and bijective Jordan $*$-homomorphisms are called surjective (or onto) Jordan $*$-isomorphisms (see e.g. [63, 76]).
For details on Jordan $\ast$-homomorphism, the reader is referred to [8] or [17] (see also [12] and [66]). For the sake of convenience, we collect some properties of Jordan $\ast$-homomorphism/isomorphism. The next result is very simple and well-known (see e.g. [72, p. 12]) and we omit its proof.

**Proposition 2.12.** Assume that $J : M_1 \to M_2$ is a complex-linear positive (i.e., $J(a) \geq 0$, $a \in M_1^+$) or self-adjoint (i.e., $J(a) = J(a)^\ast$, $a = a^\ast \in M_1$) mapping. If $J$ satisfies that $J(x^2) = J(x)^2$ for every $x \in M_1^+$, then $J$ is a Jordan $\ast$-homomorphism.

The following result is fundamental in the study of Jordan $\ast$-homomorphism (see [66, Theorem 3.3], see also [42] or [59, Appendix]).

**Lemma 2.13.** If $J$ is a Jordan $\ast$-homomorphism from $M_1$ into $M_2$, then there exists a projection $z$ in the center of the ultra-weak closure of $J(M_1)$ such that $J(\cdot)z$ is a $\ast$-homomorphism and $J(\cdot)(1_{M_2} - z)$ is a $\ast$-anti-homomorphism on $M_1$.

**Proposition 2.14.** If $J$ is a Jordan $\ast$-homomorphism from $M_1$ into $M_2$, then for any commuting $x, y \in M_1$, we have

$$J(xy) = J(x)J(y) = J(y)J(x).$$

**Proof.** Let $z$ be a projection in the center of the ultra-weak closure of $J(M_1)$ such that $J(\cdot)z$ is a $\ast$-homomorphism and $J(\cdot)(1_{M_2} - z)$ is a $\ast$-anti-homomorphism on $M_1$ (see Lemma 2.13). It now follows that

$$J(xy) = J(xy)z + J(xy)(1_{M_2} - z) = J(xy)z + J(yx)(1_{M_2} - z) = J(x)J(y)z + J(x)J(y)(1_{M_2} - z) = J(x)J(y),$$

which completes the proof. □

If $J$ is a Jordan $\ast$-homomorphism, then for any self-adjoint $a \in M_1$, $J(a)$ is self-adjoint. It is well-known (see e.g. [66] or [8, Page 211]) that every Jordan $\ast$-homomorphism is positive, i.e., if $a \geq 0$, then

$$J(a) \geq 0.$$ (12)

The following proposition provides a criterion for verifying that a Jordan $\ast$-homomorphism is injective.

**Proposition 2.15.** If $J : M_1 \to M_2$ is a Jordan $\ast$-homomorphism, then $J$ is injective if and only if $J(p) > 0$ for every $\tau_1$-finite $0 \neq p \in \mathcal{P}(M_1)$.

**Proof.** It suffices to prove the “if” part.

For every $x > 0$, there exists a $\tau_1$-finite projection $p \in M_1$ such that $x \geq \lambda p$ for some $\lambda > 0$. Therefore, $J(x) = J(x - \lambda p) + J(\lambda p) \geq J(\lambda p) > 0$. 


Assume that \( x \in \mathcal{M}_1 \) with \( J(x) = 0 \). Since \( J \) is a Jordan \( * \)-homomorphism, it follows that \( J(\text{Re}(x)) \) and \( J(\text{Im}(x)) \) are self-adjoint. Therefore, by \( J(\text{Re}(x)) + iJ(\text{Im}(x)) = J(x) = 0 \), we obtain that \( J(\text{Re}(x)) = J(\text{Im}(x)) = 0 \).

Let \( a := \text{Re}(x) \) or \( \text{Im}(x) \). In particular, \( a \) is self-adjoint with \( J(a) = 0 \). Then, \( J(a_+) = J(a_-) \geq 0 \). By \( \{1\} \), we have \( J(a_+)J(a_-) = 0 \), which implies that \( J(a_+) = J(a_-) = 0 \), i.e., \( a_+ = a_- = 0 \). Hence, \( a = 0 \). That is, \( x = 0 \). \( \square \)

**Remark 2.16.** Assume that \( J : \mathcal{M}_1 \to \mathcal{M}_2 \) is a normal Jordan \( * \)-homomorphism. Clearly, \( J(\mathcal{M}_1) \subset J(1_{\mathcal{M}_1})\mathcal{M}_2 J(1_{\mathcal{M}_1}) \) (see Proposition 2.14). By [17, Part I, Chapter 4.3, Corollary 2], \( J(\mathcal{M}_1) \) is a weakly closed \( * \)-subalgebra of \( \mathcal{M}_2 \). In particular, if \( J \) is an injective, then \( J \) is a normal Jordan \( * \)-isomorphism from \( \mathcal{M}_1 \) onto \( J(\mathcal{M}_1) \).

For the sake of convenience, we denote by \( \mathcal{P}_{fin}(\mathcal{M}_1) \) the subset of \( \mathcal{P}(\mathcal{M}_1) \) whose elements have finite traces. Lemma 2.18 below is drawn from Yeadon’s proof [76], which contains a beautiful trick of constructing a Jordan \( * \)-homomorphism from Jordan \( * \)-homomorphisms on reduced von Neumann algebras \( e\mathcal{M}_1 e \), \( e \in \mathcal{P}_{fin}(\mathcal{M}_1) \).

Before proceeding to the proof, let us recall the following fact concerning on strong operator convergence.

**Proposition 2.17.** Let \( \{x_i\} \) be a uniformly bounded net in a von Neumann algebra \( \mathcal{M} \) and \( \{p_i\} \) be a increasing net of projections increasing to 1. If \( x_i = p_i x_j p_i \) for every \( j \geq i \), then so \( \lim_i x_i \) exists (denoted by \( x \)). In particular, \( x_i = p_i x p_i \).

**Proof.** By the compactness of the unit ball of a von Neumann algebra in the weak operator topology (see e.g. [14, Chapter IX, Proposition 5.5]), there exists a wo-converging subnet \( \{x_{i_k}\} \) of \( \{x_i\} \). Let \( x := \text{wo} - \lim_{i_k} x_{i_k} \). In particular, we have \( x_{i_k} = \text{wo} - \lim_{i_k \geq i_k} p_{i_k} x_{i_k} p_{i_k} = p_{i_k} x p_{i_k} \). By the assumption, for every \( i \leq i_k \), we have \( x_{i_k} = p_i x_{i_k} p_i = p_i p_{i_k} x p_{i_k} p_i = p_i x p_i \). Since \( \{x_{i_k}\} \) is a subnet of \( \{x_i\} \), it follows that \( x_i = p_i x p_i \) for every \( i \). Clearly, \( p_i x p_i \to x \) in strong operator topology. \( \square \)

**Lemma 2.18.** Let \( \{J_e : e\mathcal{M}_1 e \to \mathcal{M}_2 \}_{e \in \mathcal{P}_{fin}(\mathcal{M}_1)} \) be a family of normal Jordan \( * \)-homomorphisms. If for every \( e \leq f \in \mathcal{P}_{fin}(\mathcal{M}_1) \), we have \( J_f = J_e \) on \( e\mathcal{M}_1 e \), then there exists a normal Jordan \( * \)-homomorphism \( J : \mathcal{M}_1 \to \mathcal{M}_2 \) agreeing with \( J_e \) on \( e\mathcal{M}_1 e \) for every \( e \in \mathcal{P}_{fin}(\mathcal{M}_1) \). Moreover, if \( J_e \) is injective for every \( e \in \mathcal{P}_{fin}(\mathcal{M}_1) \), then \( J \) is a normal Jordan \( * \)-isomorphism from \( \mathcal{M}_1 \) onto \( J(\mathcal{M}_1) \) and \( J(\mathcal{M}_1) \) is a weakly closed \( * \)-subalgebra in \( \mathcal{M}_2 \).

**Proof.** Note that \( J_f(f) - J_e(e) = J_f(f - e) \geq 0 \), \( e \leq f \in \mathcal{P}_{fin}(\mathcal{M}_1) \). We define

\[
J(1_{\mathcal{M}_1}) := \sup \{J_f(f) : f \in \mathcal{P}_{fin}(\mathcal{M}_1) \}.
\]
Since $J_f$ is a Jordan $*$-homomorphism on $fM_1f$, it follows from Lemma 1 (or Lemma 2) that for every $x \in M_1$ and $e \in \mathcal{P}_{fin}(M_1)$, we have
\begin{equation}
J_f(e)J_f(xf)J_f(e) = J_f(exe).
\end{equation}
Note that we have \(\|J_f(eye)\| \leq \|J_f(e\|y\|_{\infty} e)\|_{\infty} \leq \|y\|_{\infty}, 0 \leq y \in M_1^+\).

Applying Proposition 2.17 to the reduced algebra $J(1_{M_1}, M_1)J(1_{M_1})$, for every $x \in M_1$, we get that \(\{J_e(exe)\}_{e \in \mathcal{P}_{fin}(M_1)}\) is a uniformly bounded net converging in the strong operator topology, where the net is indexed by the upwards-directed set of projections $e \in \mathcal{P}_{fin}(M_1)$. So, we can extend $J$ to the whole $M_1$ by defining
\begin{equation}
J(x) := \lim_{e \in \mathcal{P}_{fin}(M_1)} J_e(exe).
\end{equation}
Moreover, by (13), we obtain that
\begin{equation}
J(e)J(x)J(e) = J(exe).
\end{equation}
for every $x \in M_1$ and projection $e \in \mathcal{P}_{fin}(M_1)$. By the definition, $J$ is complex-linear and $J(x)$ is self-adjoint for every self-adjoint $x \in M_1$. It remains to prove that $J$ is a normal Jordan $*$-homomorphism.

Given a net $\{x_\alpha\}$ with $x_\alpha \uparrow x \in M_1^+$ and any $e \in \mathcal{P}_{fin}(M_1)$, by Proposition 1 (vi) and the normality of $J_e$ on $eM_1e$, $e \in \mathcal{P}_{fin}(M_1)$, we obtain that
\begin{equation}
J(e)J(x)J(e) = \sup_{\alpha} J(ex_\alpha e) \leq \sup_{\alpha} J(e)J(x_\alpha)J(e) = J(e)\sup_{\alpha} J(x_\alpha)J(e).
\end{equation}
Now, taking the (so)-limit, we obtain that
\begin{equation}
J(1_{M_1})J(x)J(1_{M_1}) = J(1_{M_1})\sup_{\alpha} J(x_\alpha)J(1_{M_1}).
\end{equation}
By the construction of $J$ (see and (15)), one see that $s(J(x)), s(J(x_\alpha)) \leq \sup\{J(f) : f \in \mathcal{P}_{fin}(M_1)\} = J(1_{M_1})$, which implies that $J(x) = \sup J(x_\alpha)$. That is, $J$ is normal.

For any $x \in M_1^+$, by the construction of $J$ (see (14) and (15)), it is easy to see that $so - \lim_{e \in \mathcal{P}_{fin}(M_1)} J(f) = J(1_{M_1}) \geq s(J(x))$. Hence, for any $x \in M_1^+$, the normality of $J$ and the (so)-continuity of multiplication on the unit ball of a von Neumann algebra, we have
\begin{align*}
J(x^2) = so - \lim_{e \in \mathcal{P}_{fin}(M_1)} J(exe) = so - \lim_{e \in \mathcal{P}_{fin}(M_1)} J(e)J(exe)J(e) \\
\leq so - \lim_{e \in \mathcal{P}_{fin}(M_1)} J_e(exe) = so - \lim_{e \in \mathcal{P}_{fin}(M_1)} (J_e(exe))^2 = J(x)^2.
\end{align*}
By Proposition 2.12, $J$ is a Jordan $*$-homomorphism.

Assume now that $J_e$ is injective for every $e \in \mathcal{P}_{fin}(M_1)$. By Proposition 2.15, $J$ is a Jordan $*$-monomorphism. Moreover, since $J$ is normal, it follows from Remark 2.10 that $J(M_1)$ is a weakly closed $*$-subalgebra in $M_2$. \qed
If $J$ is a normal Jordan $*$-monomorphism, then there exists a projection $p \in Z(M_1)$ (the center of $M_1$) such that $J$ is a $*$-monomorphism on $pM_1p$ and a $*$-anti-monomorphism on $(1_{M_1} - p)M_1(1_{M_1} - p)$. The following property is an easy consequence of this fact.

**Proposition 2.19.** Assume that $J : M_1 \to M_2$ is a normal Jordan $*$-monomorphism. There exists a projection $p \in Z(M_1)$ such that for every $x \in M_1$, we have

$$|J(x)| = J(p|x| + (1_{M_1} - p)|x^*|).$$

**Proof.** Let $p \in Z(M_1)$ be a projection such that $J$ is a $*$-monomorphism on $pM_1p$ and a $*$-anti-monomorphism on $(1_{M_1} - p)M_1(1_{M_1} - p)$ (see e.g. [11, Theorem 3.3], see also [70, p. 45]). By (11), $J(p) \perp J(1_{M_1} - p)$. It then follows that for every $x \in M_1$, we have

$$|J(x)|^2 = J(x)^*J(x) = (J(px)^* + J((1_{M_1} - p)x)^*)(J(px) + J((1_{M_1} - p)x))$$

which together with (12) implies the validity of (10). \hfill \Box

Jordan $*$-homomorphisms have strong connections with projection ortho-morphisms, that is, mappings $\varphi : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$ satisfying that $\varphi(p) \perp \varphi(q)$ and $\varphi(p \perp q) = \varphi(p) \perp \varphi(q)$ for all mutually orthogonal $p, q \in \mathcal{P}(M_1)$ (see e.g. [24]). Whenever $M_1$ has no type $I_2$ direct summands, every the projection ortho-morphism from $\mathcal{P}(M_1)$ into $\mathcal{P}(M_2)$ can be extended to a Jordan $*$-homomorphism from $M_1$ into $M_2$ (see [33, Theorem 8.1.1], see also [24]). In general, one can not expect that every ortho-morphism can be extended to a Jordan $*$-homomorphism [24, P.83].

**Lemma 2.20.** Let $\varphi : M_1 \to M_2$ be a linear mapping which is continuous in the uniform norm topology. If the reduction of $\varphi$ on $\mathcal{P}(M_1)$ is an ortho-homomorphism from $\mathcal{P}(M_1)$ into $\mathcal{P}(M_2)$, then $\varphi$ is a Jordan $*$-homomorphism from $M_1$ into $M_2$.

**Proof.** Let $0 \leq x \in M_1$. Without loss of generality, we may assume that $\|x\|_\infty = 1$. Define $x_n := \sum_{k=1}^{n} \frac{k-1}{n} e^x(\frac{k-1}{n}, \frac{k}{n})$. In particular, $\|x - x_n\|_\infty \leq \frac{1}{n}$.

By linearity, we have $\varphi(x_n^2) = \varphi(\sum_{k=1}^{n} \frac{k-1}{n})^2 e^x(\frac{k-1}{n}, \frac{k}{n})) = \sum_{k=1}^{n} \frac{(k-1)^2}{n} e^x(\frac{k-1}{n}, \frac{k}{n})$.

Since $\varphi|_{\mathcal{P}(M_1)}$ is an ortho-homomorphism from $\mathcal{P}(M_1)$ into $\mathcal{P}(M_2)$, it follows that

$$(\varphi(x_n))^2 = \left(\sum_{k=1}^{n} \frac{k-1}{n} e^x(\frac{k-1}{n}, \frac{k}{n})\right)^2 = \sum_{k=1}^{n} \frac{(k-1)^2}{n} e^x(\frac{k-1}{n}, \frac{k}{n}) = \varphi(x_n^2).$$
Since \( \|x_n - x\|_\infty \to_n 0 \), it follows from the \( \|\cdot\|_\infty \)-continuity of \( \varphi \) that \( \varphi(x^2) = (\varphi(x))^2 \). Moreover, since \( \varphi(x_n) \geq 0 \), it follows from the \( \|\cdot\|_\infty \)-continuity of \( \varphi \) that \( \varphi(x) \geq 0 \). By Proposition 2.12, \( J \) is a Jordan \( * \)-homomorphism. \( \square \)

3. Isometries and Jordan isomorphisms

Assume that \( E(\mathcal{M}_1, \tau_1) \) and \( F(\mathcal{M}_2, \tau_2) \) are symmetrically \( \Delta \)-normed operator spaces affiliated with semifinite von Neumann algebras \( (\mathcal{M}_1, \tau_1) \) and \( (\mathcal{M}_2, \tau_2) \), respectively. The main purpose of this section is to describe the general form of order-preserving isometries from \( E(\mathcal{M}_1, \tau_1) \) into \( F(\mathcal{M}_2, \tau_2) \). The case of semifinite von Neumann algebras is more complicated than that of finite von Neumann algebras because of the possible absence of the identity in the symmetric space \( E(\mathcal{M}_1, \tau_1) \). The following is quoted from [72, Section 5.2]: “the fact that the unit element 1 in \( \mathcal{M} \) does not belong to any symmetric space \( E(\mathcal{M}, \tau) \) with order continuous norm when \( \tau(1) = \infty \) presents additional technical obstacles”. In this section, we completely resolve the case when \( E(\mathcal{M}, \tau) \) has order continuous \( \Delta \)-norm, which is left unanswered in [72].

It is known that for two von Neumann algebras \( \mathcal{M}_1, \mathcal{M}_2 \) and \( p > 0, p \neq 2, \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are Jordan \( * \)-isomorphic if and only if \( L_p(\mathcal{M}_1, \tau_1) \) are isometrically isomorphic [63,64]. In this section, we prove that if there exists an order-preserving isometry from \( E(\mathcal{M}_1, \tau_1) \) into \( F(\mathcal{M}_2, \tau_2) \), then there exists a Jordan \( * \)-homomorphism from \( \mathcal{M}_1 \) into \( \mathcal{M}_2 \).

For the sake of convenience, we denote \( (\mathcal{M}_1)_{fin} := \{ x \in \mathcal{M}_1 \mid \tau_1(s(x)) < \infty \} \) and \( (\mathcal{M}_2)_{fin} := \{ x \in \mathcal{M}_2 \mid \tau_2(s(x)) < \infty \} \). We always assume that the carrier projections of the symmetrically \( \Delta \)-normed spaces considered in this section are 1.

By Proposition 2.3, it is easy to see that \( \mathcal{P}(E) = \mathcal{P}_{fin}(\mathcal{M}_1) := (\mathcal{M}_1)_{fin} \cap \mathcal{P}(\mathcal{M}_1) \) whenever \( E(\mathcal{M}_1, \tau_1) \subset S_0(\mathcal{M}_1, \tau_1) \) and \( c_E = 1_{\mathcal{M}_1} \).

If \( \tau_1(1_{\mathcal{M}_1}) < \infty \), the isometries from one symmetrically normed space into another are widely studied. Since \( 1_{\mathcal{M}_1} \in E(\mathcal{M}_1, \tau_1) \), one can obtain explicit form of the isometry \( T : E(\mathcal{M}_1, \tau_1) \to F(\mathcal{M}_2, \tau_2) \) directly (see e.g. [12,72,76]). For convenience of the reader, we present a proof for \( \Delta \)-normed case. We also rectify an oversight in the proof of [72, Proposition 6], where \( T(1_{\mathcal{M}_1})^{-1} \) was asserted to be a \( \tau_2 \)-measurable operator, which requires additional arguments when \( (\mathcal{M}_2, \tau_2) \) is infinite while is correct for a finite von Neumann algebra \( (\mathcal{M}_2, \tau_2) \). The latter matter though only affects the form of the Jordan \( * \)-homomorphisms and not result claimed in that proposition.

**Theorem 3.1.** Let \( (\mathcal{M}_1, \tau_1) \) be a finite von Neumann algebra having a faithful normal finite trace \( \tau_1 \) and \( (\mathcal{M}_2, \tau_2) \) be an arbitrary semifinite von Neumann algebra. Assume that \( E(\mathcal{M}_1, \tau_1) \) and \( F(\mathcal{M}_2, \tau_2) \) are symmetrically \( \Delta \)-normed operator spaces. If there exists an order-preserving isometry \( T : E(\mathcal{M}_1, \tau_1) \to F(\mathcal{M}_2, \tau_2) \)
which is disjointness preserving, then there exists a Jordan \(*\)-monomorphism $J : \mathcal{M}_1 \to \mathcal{M}_2$. Furthermore, $T(1_{\mathcal{M}_1})$ commutes with $T(x)$, $x \in \mathcal{M}_1$, and

$$T(1_{\mathcal{M}_1})J(x) = T(x), \ x \in \mathcal{M}_1.$$  \hspace{1cm} (17)

Moreover, if $\|\cdot\|_E$ is order continuous, then $J$ is a normal Jordan \(*\)-isomorphism onto a weakly closed \(*\)-subalgebra of $\mathcal{M}_2$.

Proof. Since $\tau_1(1_{\mathcal{M}_1}) < \infty$, it follows from Proposition 2.3 that $1_{\mathcal{M}_1} \in E(\mathcal{M}_1, \tau_1)$.

Let $e \in P(\mathcal{M}_1)$. The disjointness preserving property of $T$ implies that $[T(e), T(1_{\mathcal{M}_1})] = [T(e), T(1_{\mathcal{M}_1}) + T(e), T(1_{\mathcal{M}_1} - e)] = 0$. In particular, $T(1_{\mathcal{M}_1})$ commutes with $T(e)$, $e \in P(\mathcal{M}_1)$. In other words, $T(e)$ commutes with every spectral projection of $T(1_{\mathcal{M}_1})$.

Let $r := s(T(1_{\mathcal{M}_1}))$. Since $T$ is an order-preserving isometry, it follows that

$$0 \leq T(x) \leq \|x\|_\infty T(1_{\mathcal{M}_1}), \ 0 \leq x \in \mathcal{M}_1.$$  \hspace{1cm} (18)

Therefore,

$$0 \leq (1_{\mathcal{M}_1} - r)T(x)(1_{\mathcal{M}_1} - r) \leq \|X\|_\infty (1_{\mathcal{M}_1} - r)T(1_{\mathcal{M}_1})(1_{\mathcal{M}_1} - r) = 0.$$  

In other words, $T(x)(1_{\mathcal{M}_1} - r) = 0$, which implies that $T(x)$ is affiliated with the reduced von Neumann algebra $\mathcal{M}_2^r$ of $\mathcal{M}_2$. Without loss of generality, we assume that $r = s(T(1_{\mathcal{M}_1})) = 1_{\mathcal{M}_2}$.

Fix $0 \leq x \in \mathcal{M}_1$, $\|x\|_\infty \leq 1$. Set $x_n := \sum_{k=1}^n \frac{k-1}{n} e^{i(k-1)/n}, \ n \geq 1$. In particular, $\|x_n - x\|_\infty \to 0$. Hence, $\|x_n - x\|_E \leq \|\|x_n - x\|_\infty 1_{\mathcal{M}_1}\|_E \to 0$. Since $T$ is an isometry, it follows that $\|T(x_n) - T(x)\|_E \to 0$ and therefore, by [37, Lemma 2.4], $T(x_n) \to T(x)$ in measure topology.

Since $T(1_{\mathcal{M}_1})$ commutes with $T(e)$, $e \in P(\mathcal{M}_1)$, it immediately follows from the definition of $x_n$ that $T(1_{\mathcal{M}_1})$ commutes with $T(x_n)$. By the preceding paragraph, $[T(x_n), T(1_{\mathcal{M}_1})] \to [T(x), T(1_{\mathcal{M}_1})]$ in measure topology (see e.g. [21, 28]). Hence, $T(1_{\mathcal{M}_1})$ commutes with $T(x)$ for all $x \in \mathcal{M}_1$.

Let $e_n := E^T(1_{\mathcal{M}_1})(\frac{1}{n}, \infty)$, $n > 0$. In particular, $e_n \to s(T(1_{\mathcal{M}_1})) = 1_{\mathcal{M}_2}$ as $n \to \infty$. Now, we have $[e_n T(x)e_n, e_n T(1_{\mathcal{M}_1}) e_n] = 0$, $n > 0$, $x \in \mathcal{M}_1$. For every $n$, we define $J_n$ by setting

$$J_n(x) := (e_n T(1_{\mathcal{M}_1}) e_n)^{-1} e_n T(x) e_n, \ x \in \mathcal{M}_1.$$  \hspace{1cm} (19)

Here, $(e_n T(1_{\mathcal{M}_1}) e_n)^{-1}$ is taken from the algebra $e_n \mathcal{M}_2 e_n$. Note that $(e_n T(1_{\mathcal{M}_1}) e_n)^{-1}$ commutes with $T(x)$, $x \in \mathcal{M}_1$. By [18], we obtain that

$$0 \leq e_n T(x) e_n \leq \|x\|_\infty e_n T(1_{\mathcal{M}_1}) e_n.$$  

Hence,

$$0 \leq (e_n T(1_{\mathcal{M}_1}) e_n)^{-1/2} e_n T(x) e_n (e_n T(1_{\mathcal{M}_1}) e_n)^{-1/2} \leq \|x\|_\infty.$$  \hspace{1cm} (20)
That is, \( J_n(x) \leq \|x\|_\infty \). Moreover, since \( T(1_{\mathcal{M}_1}) \) commutes with \( T(x) \), and \( e_k, k \geq 1 \), is a spectral projection of \( T(1_{\mathcal{M}_1}) \), it follows that for every \( m \geq n \),

\[
J_m(x)e_n = (e_mT(1_{\mathcal{M}_1})e_m)^{-1}e_mT(x)e_m e_n
= (e_nT(1_{\mathcal{M}_1})e_n)^{-1}e_nT(x)e_n = J_n(x).
\]  

(21)

Hence, \( \{J_n(x)\}_n \) converges in the strong operator topology. Define

\[
J(x) := (\text{so-} \lim_n J_n(x)) \in \mathcal{M}_2.
\]

Clearly, \( J \) is a complex-linear mapping. Since \( T(1_{\mathcal{M}_1}) \) commutes with \( J_n(x) \), it follows that \( T(1_{\mathcal{M}_1}) \) commutes with \( J(x), x \in \mathcal{M}_1 \). Since every \( \|J_n(x)\|_\infty \leq \|x\|_\infty, x \in \mathcal{M}_1 \), for every \( n > 0 \) and \( J(x) \) is the (so)-limit of \( \{J_n(x)\} \), it follows from the Kaplansky density theorem (see e.g., [23, Theorem II 4.8]) that \( J \) is a bounded mapping with \( \|J(x)\|_\infty \leq \|x\|_\infty, x \in \mathcal{M}_1 \).

For every \( 0 \neq e \in \mathcal{P}(\mathcal{M}_1) \), since \( T(1_{\mathcal{M}_1} - e)T(e) = 0 \), it follows that \( T(1_{\mathcal{M}_1})s(T(e)) = (T(1_{\mathcal{M}_1} - e) + T(e))s(T(e)) = T(e) \). Hence, we have

\[
J_n(e) = (e_nT(1_{\mathcal{M}_1})e_n)^{-1}e_nT(e)e_n = (e_nT(1_{\mathcal{M}_1})e_n)^{-1}e_nT(1_{\mathcal{M}_1})e_n s(T(e))
= e_ns(T(e)) \neq 0.
\]

(22)

Taking \( n \to \infty \), we obtain that \( J(e) = s(T(e)) > 0 \). By the disjointness preserving property of \( T \), it is easy to see that \( J \) is an ortho-homomorphism from \( \mathcal{P}(\mathcal{M}_1) \) to \( \mathcal{P}(\mathcal{M}_2) \). By Lemma [23.24] \( J \) is a Jordan **-homomorphism. Moreover, Proposition [23.18] implies that \( J \) is a Jordan **-monomorphism.

By (21), we have \( J(x)e_n = J_n(x), x \in \mathcal{M}_1, n > 0 \). Hence, we have

\[
T(1_{\mathcal{M}_1})J(x)e_n = e_nT(1_{\mathcal{M}_1})e_nJ(x)e_n = e_nT(1_{\mathcal{M}_1})e_nJ_n(x) \quad (\text{so-} \lim_n T(x)e_n, x \in \mathcal{M}_1).
\]

Since \( e_n \to 1_{\mathcal{M}_2} \) in the local measure topology (see e.g., [21, Proposition 2.(y)]), it follows that \( T(1_{\mathcal{M}_1})J(x) = T(x) \) (see page 17), which proves the validity of (17).

Now, assume that \( E(\mathcal{M}_1, \tau_1) \) has order continuous \( \Delta \)-norm and \( \{x_\alpha\} \) is an increasing net such that \( x_\alpha \uparrow \alpha \in \mathcal{M}_1^+ \). Note that

\[
\|T(1_{\mathcal{M}_1})(J(x) - J(x_\alpha))\|_F = \|T(x) - T(x_\alpha)\|_F = \|x - x_\alpha\|_E \to 0.
\]

(23)

Since \( T(1_{\mathcal{M}_1}) \) commutes with \( J(x) - J(x_\alpha) \), it follows that

\[
\|T(1_{\mathcal{M}_1})^2(J(x) - J(x_\alpha))T(1_{\mathcal{M}_1})^2\|_F = \|T(1_{\mathcal{M}_1})(J(x) - J(x_\alpha))\|_F \to 0.
\]

Since Jordan **-homomorphism preserves order (see [12]), Lemma [27.7] together with (23) that \( T(1_{\mathcal{M}_1})^{1/2}J(x_\alpha)T(1_{\mathcal{M}_1})^{1/2} \uparrow T(1_{\mathcal{M}_1})^{1/2}J(x)T(1_{\mathcal{M}_1})^{1/2} \). By [21, Proposition 1], we have

\[
T(1_{\mathcal{M}_1})^{1/2} \sup_\alpha J(x_\alpha)T(1_{\mathcal{M}_1})^{1/2} = \sup_\alpha T(1_{\mathcal{M}_1})^{1/2}J(x_\alpha)T(1_{\mathcal{M}_1})^{1/2} = T(1_{\mathcal{M}_1})^{1/2}J(x)T(1_{\mathcal{M}_1})^{1/2}
\]
Proof. Let \( e \in \mathcal{P} \) be such that \( J \) is weakly closed. By Remark 3.4, the proof.

Remark 3.4. Yeadon’s proof \([70]\) (see also \([41, \text{Theorem 3.1}]\)) provides an alternative construction of the Jordan \(*\)-isomorphism which coincides with that given in Theorem \([71]\). By the “disjointness preserving” property of the \( L_p\)-isometry, \( 1 \leq p < \infty, p \neq 2 \), one can construct an projection ortho-morphism \( \phi \) on \( \mathcal{P}_{\text{fin}}(\mathcal{M}_1) \) by defining \( \phi(e) := s(T(e)), e \in \mathcal{P}_{\text{fin}}(\mathcal{M}_1) \). This ortho-morphism can be extended to a Jordan \(*\)-isomorphism.

Now, we can settle Lemma 2.18 to prove the semifinite case.

Theorem 3.3. Assume that \( E(\mathcal{M}_1, \tau_1) \) and \( F(\mathcal{M}_2, \tau_2) \) are symmetrically \( \Delta \)-normed operator spaces and \( E(\mathcal{M}_1, \tau_1) \) has order continuous \( \Delta \)-norm. If there exists an order-preserving isometry \( T : E(\mathcal{M}_1, \tau_1) \xrightarrow{\text{int}} F(\mathcal{M}_2, \tau_2) \) which is disjointness preserving, then there exists a normal Jordan \(*\)-isomorphism \( J \) from \( \mathcal{M}_1 \) onto a weakly closed \(*\)-subalgebra of \( \mathcal{M}_2 \) such that \( T(e)J(x) = T(x) \) for every \( x \in e \mathcal{M}_1 e, e \in \mathcal{P}_{\text{fin}}(\mathcal{M}_1) \).

Proof. If \( \tau_1(1_{\mathcal{M}_1}) < \infty \), then the assertion follows from Theorem 3.1.

If \( \tau_1(1_{\mathcal{M}_1}) = \infty \), then we define a normal Jordan \(*\)-homomorphism \( J_e \) on the reduced algebra \( e \mathcal{M}_1 e \), \( e \in \mathcal{P}_{\text{fin}}(\mathcal{M}_1) \), as in Theorem 3.1. If \( e \leq f \in \mathcal{P}_{\text{fin}}(\mathcal{M}_1) \), then \( J_f(e) = s(T(e)) \). By the normality of \( J_e \) and \( J_f \), we have

\[
J_f(x) = J_e(x), \quad 0 \leq x \in e \mathcal{M}_1 e,
\]

which implies that \( J_f \) coincides with \( J_e \) on \( e \mathcal{M}_1 e \). By Lemma 2.18 we complete the proof.

Remark 3.4. For every \( 0 \leq x \in E(\mathcal{M}_1, \tau_1) \), we have \( s(T(x)) = J(s(x)) \). Indeed, let \( e_n = e^x(\frac{1}{n}, n), n > 0 \). Since \( E(\mathcal{M}_1, \tau_1) \subseteq S_0(\mathcal{M}, \tau) \) (see Remark 2.18), it follows that \( \tau_1(e_n) < \infty \). We have

\[
T(x e_n) = T(e_n)J(x e_n) \rightarrow T(e_n)J(e_n),
\]

which implies that \( s(T(x e_n)) \leq J(e_n) \leq J(s(x)) \). Note that \( T(x e_n) \uparrow_n T(x) \) (by the order continuity of \( \| \cdot \|_E \) and Lemma 2.7). This implies, in particular that, \( s(T(x)) \leq J(s(x)) \).

On the other hand, \( T(x) \geq \lambda T(e_n) \) for some \( \lambda > 0 \), which implies that \( s(T(x)) \geq s(T(e_n)) \rightarrow J(e_n) \). Hence, taking the (so)-limit of \( J(e_n) \) as \( n \rightarrow \infty \), by the normality of \( J \), we obtain that \( s(T(x)) = J(s(x)) \).
The techniques used in Yeadon’s proof of [76, Theorem 2] rely on the fine properties of $L_p$-norms. However, for general symmetrically $\Delta$-normed spaces, we do not have explicit descriptions of the $\Delta$-norms, which was the main obstacle we encountered. In Theorem 3.5 below, we use an approach different from that used in [76] to describe disjointness preserving order-preserving isometries from general symmetrically $\Delta$-normed space into another, which unifies and extends the results in [9, 12, 72].

The main result of this section is the following.

**Theorem 3.5.** Assume that $E(\mathcal{M}_1, \tau_1), F(\mathcal{M}_2, \tau_2)$ are symmetrically $\Delta$-normed operator spaces and $E(\mathcal{M}_1, \tau_1)$ has order continuous $\Delta$-norm. If there exists an order-preserving isometry $T : E(\mathcal{M}_1, \tau_1) \rightarrow F(\mathcal{M}_2, \tau_2)$ which is disjointness preserving, then there exist a (possibly unbounded) positive self-adjoint $B$ affiliated with $\mathcal{M}_2$ and a normal Jordan $*$-isomorphism $J$ from $\mathcal{M}_1$ onto a weakly closed $*$-subalgebra of $\mathcal{M}_2$ such that

$$T(x) = BJ(x), \ x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1. \quad (25)$$

The existence of Jordan $*$-isomorphism follows from Theorem 3.3 above. To define the operator $B$ properly and prove that this Jordan $*$-isomorphism $J$ satisfies \(25\), we need some preparations. The proposition below defines $B$ as the strong resolvent limit of the net $\{T(e)\}_{e \in P_{fin}(\mathcal{M}_1)}$. We refer for the definition of strong resolvent convergence to [58, p. 284].

**Proposition 3.6.** There exists a limit $B$ of $\{T(e)\}_{e \in P_{fin}(\mathcal{M}_1)}$ in the strong resolvent sense. In particular, $B$ is affiliated with $\mathcal{M}_2$.

**Proof.** For each $e \in P_{fin}(\mathcal{M}_1)$, we have the following spectral resolution

$$T(e) = \int_0^\infty \lambda dP_e(\lambda).$$

In particular, $J(e) = s(T(e)) = 1_{\mathcal{M}_2} - P_e(0)$. For every projection $f \leq e$, by Theorem 3.1, we have $T(f) = T(e)J(f) = J(f)T(e)$ and hence, for $\lambda \geq 0$, by the spectral theorem, we have

$$1_{\mathcal{M}_2} - P_f(\lambda) = J(f)(1_{\mathcal{M}_2} - P_e(\lambda)) = (1_{\mathcal{M}_2} - P_e(\lambda))J(f). \quad (26)$$

It follows from \(26\) that $\{P_e(\lambda)\}_{E \in P_{fin}(\mathcal{M}_1)}$ is a decreasing net indexed by the upwards-directed set of projections in $P_{fin}(\mathcal{M}_1)$. Now, by Vigier’s theorem (see [50, Theorem 2.1.1]), we can define

$$P_\lambda := \lim_{e \in P_{fin}(\mathcal{M}_1)} P_e(\lambda), \ \lambda \in \mathbb{R}. \quad (27)$$

In particular, for every $\lambda < 0$, we have $P_\lambda = \lim_{e \in P_{fin}(\mathcal{M}_1)} P_e(\lambda) = 0$. 
To show that the limit $B = \lim_{E \in \mathcal{P}_{fin}(\mathcal{M}_1)} T(E)$ exists in strong resolvent sense, i.e.,

$$B := \int_{0}^{\infty} \lambda dP_{\lambda}$$

is well-defined, it suffices to show that $\{P_{\lambda}\}_{\lambda \in \mathbb{R}}$ is a spectral family.

It follows immediately from the definition that $P_{\lambda} P_{\mu} = P_{\mu} P_{\lambda} = P_{\lambda}$, $\lambda \leq \mu$.

On one hand, $P_{\lambda} = \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda) = \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} \inf_{\varepsilon > 0} P_{\varepsilon}(\lambda + \varepsilon) \geq \inf_{\varepsilon > 0} P_{\lambda + \varepsilon}$. On the other hand, $P_{\lambda} = \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda) \leq \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda + \varepsilon)$ for every $\varepsilon > 0$, that is, $P_{\lambda} \leq P_{\lambda + \varepsilon}$. It follows that $\lambda \mapsto P_{\lambda}$ is (so)-right-continuous.

Since $P_{\lambda} \in \mathsf{sp}(\mathcal{M}_2)$ so $\lim_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda)$, it follows from (20) that for every $f \in \mathcal{P}_{fin}(\mathcal{M}_2)$ that

$$\lambda > 0, \quad \inf_{\lambda > 0} P_{\lambda} = \inf_{\lambda > 0} \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda) = 0.$$

Taking $\lambda \rightarrow +\infty$, we obtain from (28) that $0 = \lim_{\lambda \rightarrow +\infty} (\mathbf{1}_{\mathcal{M}_2} - P_{\lambda}) J(f)$. Hence, (lim$_{\lambda \rightarrow +\infty} (\mathbf{1}_{\mathcal{M}_2} - P_{\lambda}) J(\mathcal{M}_1) = 0$. Since $\mathbf{1}_{\mathcal{M}_2} - P_{\lambda} \leq J(f) \leq J(\mathcal{M}_1)$ for every $\lambda > 0$, it follows that $\mathbf{1}_{\mathcal{M}_2} - P_{\lambda} \leq J(\mathcal{M}_1)$ for every $\lambda > 0$. Therefore, $\lim_{\lambda \rightarrow +\infty} (\mathbf{1}_{\mathcal{M}_2} - P_{\lambda}) = 0$, i.e., $\lim_{\lambda \rightarrow +\infty} P_{\lambda} = \mathbf{1}_{\mathcal{M}_2}$.

Note that $\lim_{\lambda \rightarrow +\infty} P_{\lambda} = \inf_{\lambda > 0} P_{\lambda} = \inf_{\lambda > 0} \inf_{\varepsilon \in \mathcal{P}_{fin}(\mathcal{M}_1)} P_{\varepsilon}(\lambda) = 0$.

Therefore, $\{P_{\lambda}\}$ is a spectral family and therefore, $B$ is a well-defined self-adjoint operator (see e.g. (31)). Since $\{P_{\lambda}\} \subset \mathcal{M}_2$, it follows that $B$ is affiliated with $\mathcal{M}_2$ (see e.g. (22) Proposition II 1.4)).

We should prove that $T(x) = BJ(x)$ for every $x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$ (see (25)). We first prove the following proposition.

**Proposition 3.7.** Let $B = \int_{0}^{\infty} \lambda dP_{\lambda}$ be defined as in Proposition 3.6. Then,

1. $P_{\lambda}$ commutes with $J(x)$ for every $x \in \mathcal{M}_1$.
2. for every $0 \leq x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$ and spectral projection $e \in (\mathcal{M}_1)_{fin}$ of $x$, we have

$$T(xe) = BJ(x)J(e).$$

**Proof.** (1). Recall that for every $f \in \mathcal{P}_{fin}(\mathcal{M}_1)$, we have $s(T(f)) = J(f)$ (see e.g. Remark 3.3). Hence,

$$\mathbf{1}_{\mathcal{M}_2} - P_{\lambda} = \mathbf{1}_{\mathcal{M}_2} - P_{\lambda} = \mathbf{1}_{\mathcal{M}_2} - P_{\lambda} = J(f)(\mathbf{1}_{\mathcal{M}_2} - P_{\lambda}) = J(f)(\mathbf{1}_{\mathcal{M}_2} - P_{\lambda})$$

for every $f \in \mathcal{P}_{fin}(\mathcal{M}_1)$, which implies that

$$P_{\lambda} J(f) = P_{\lambda} J(f) = J(f) P_{\lambda}.$$
(2). Since \( e^{T(f)}(\lambda, \infty) = 1_{M_2} - P_f(\lambda) \), \( (1_{M_2} - P_\lambda)J(f) = e^B(\lambda, \infty)J(f) \), it follows that

\[
T(f) = BJ(f), \quad \forall f \in \mathcal{P}_{fin}(M_1).
\]

The assertion follows from Proposition 2.14. Indeed, for every \( 0 \leq x \in E(M_1, \tau_1) \cap M_1 \) and spectral projection \( e \in \mathcal{P}_{fin}(M_1) \) of \( x \), we have

\[
T(xe) = T(e)x = BJ(e)x = BJ(xe) = BJ(x)J(x).
\]

Since \( J(x), x \in M_1 \), commutes with \( P_\lambda, \lambda \in \mathbb{R} \), it follows that \( BJ(x) \) is a self-adjoint operator [10, 61]. Moreover, since \( B \) is positive, it follows \( BJ(x) \) is also positive. Recall that \( B \) is affiliated with \( M_2 \) (see Proposition 3.6). It is easy to see that \( BJ(x) \) is affiliated with \( M_2 \) (see e.g. [21, Proposition II 1.4]). To avoid dealing with the domain of unbounded operators, we first show that \( BJ(x) \) is \( \tau_2 \)-measurable.

**Proposition 3.8.** \( BJ(x) = T(x) \) for every \( 0 \leq x \in M_1 \cap E(M_1, \tau_1) \).

**Proof.** Note that every \( x \in M_1 \cap E(M_1, \tau_1) \) is \( \tau \)-compact (see Remark 2.9). Let \( e_n = e^n[\frac{1}{n}, n), n > 1 \). In particular, \( e_n \) is \( \tau_1 \)-finite and \( \sup_n e_n = s(x) \). Since \( BJ(x) \) is positive self-adjoint, it follows that \( BJ(x) \) has a spectral resolution

\[
BJ(x) = \int_0^\infty \lambda dQ(\lambda).
\]

Since \( P_\lambda, J(x) \) and \( J(e_n) \) commute with each other (see Propositions 3.7 and 2.14), which implies that \( J(e_n) \) strongly commutes with \( BJ(x) \) (see e.g. [16, Theorem 1]). Defining \( Q_n(\lambda) \) by \( 1_{M_2} - Q_n(\lambda) = (1_{M_2} - Q(\lambda))J(e_n) \), we have

\[
T(xe_n) = BJ(x)J(e_n) = \int_0^\infty \lambda dQ_n(\lambda).
\]

By Remark 3.4, we know that \( T(xe_n) = T(xe_n)J(e_n) + T(x(1 - e_n))J(e_n) = T(x)J(e_n) \). Let \( T(x) := \int_0^\infty \lambda dQ_T(\lambda) \) be the spectral resolution.

In particular, \( 1_{M_2} - Q_n(\lambda) = (1_{M_2} - Q_T(\lambda))J(e_n) \). Hence, \( (1_{M_2} - Q_T(\lambda))J(e_n) = (1_{M_2} - Q_T(\lambda))J(e_n) \). Taking the (so)-limit of \( J(e_n) \) and using the normality of \( J \), we have

\[
(1_{M_2} - Q_T(\lambda))J(s(x)) = (1_{M_2} - Q(\lambda))J(s(x)).
\]

By Proposition 2.14, we have \( J(s(x)) \geq s(J(x)) \geq s(BJ(x)) \geq 1_{M_2} - Q(\lambda) \). On the other hand, Remark 3.4 implies that \( J(s(x)) \geq 1_{M_2} - Q_T(\lambda) \). We conclude that \( 1_{M_2} - Q_T(\lambda) = 1_{M_2} - Q(\lambda), \lambda > 0, \text{ i.e., } T(x) = BJ(x), 0 \leq x \in M_1 \cap E(M_1, \tau_1) \).
**Proof of Theorem 3.5** Now, we consider the general case when \( x \in \mathcal{M}_1 \cap E(\mathcal{M}_1, \tau_1) \) is not necessarily positive. For any \( x \in \mathcal{M}_1 \cap E(\mathcal{M}_1, \tau_1) \), let \( J(x)^* = u|J(x)^*| \) be polar decomposition. By (15), we have that

\[
BJ(x) = BJ(J(x)^*|u^* = BJ|J(x)^*|u^* = BJ(p|x^*| + (1_\mathcal{M}_1 - p)|x|)u^*,
\]

where \( p \) is a central projection in \( \mathcal{M}_1 \) defined as in (16). Since \( p|x^*| + (1_\mathcal{M}_1 - p)|x| \in \mathcal{M}_1 \cap E(\mathcal{M}_1, \tau_1) \), it follows from Proposition 3.3 that \( BJ(p|x^*| + (1_\mathcal{M}_1 - p)|x|) \in S(\mathcal{M}_2, \tau_2) \). Hence, we obtain that \( BJ(x) \in S(\mathcal{M}_2, \tau_2) \).

For every \( x \in \mathcal{M}_1 \cap E(\mathcal{M}_1, \tau_1) \), let \( x_{1+}, x_{1-}, x_{2+}, x_{2-} \in \mathcal{M}_1 \) be positive operators such that \( x = (x_{1+} - x_{1-}) + i(x_{2+} - x_{2-}) \). Since \( P_\lambda, \lambda > 0 \), commutes with \( J(x) \) (see Proposition 3.7), we obtain that

\[
T(x)P_\lambda = T((x_{1+} - x_{1-}) + i(x_{2+} - x_{2-}))P_\lambda = T(x_{1+})P_\lambda + T(x_{1-})P_\lambda + T(ix_{2+})P_\lambda + T(ix_{2-})P_\lambda = BJ(x_{1+})P_\lambda - BJ(x_{1-})P_\lambda + iBJ(x_{2+})P_\lambda - iBJ(x_{2-})P_\lambda = BP_\lambda J(x_{1+}) - BP_\lambda J(x_{1-}) + iBP_\lambda J(x_{2+}) - iBP_\lambda J(x_{2-}) = BP_\lambda J(x) = BJ(x)P_\lambda.
\]

Hence, we obtain that

\[
(T(x) - BJ(x))P_\lambda = 0.
\]

Since \( T(x), BJ(x) \in S(\mathcal{M}_2, \tau_2) \) and \( P_\lambda \uparrow 1_\mathcal{M}_2 \) as \( \lambda \to \infty \), it follows that \( T(x) = BJ(x) \) (see [21]. Proposition 2 and Section 2.5)).

4. **Order-preserving isometries into \( \Delta \)-normed spaces**

In this section, we extend [72]. Proposition 6] in two directions. Firstly, we can consider general semifinite von Neumann algebras instead of finite von Neumann algebras, resolving the case left in [72]. Secondly, we extend significantly the class of symmetrically \( \Delta \)-normed spaces (even in the normed case) to which the theorem is applicable. In particular, we can consider the usual \( L_1 \)-norm, which is not strictly \( K \)-monotone. Our approach is based on detailed study of logarithmic submajorisation.

Recall that for a finite von Neumann algebra \( \mathcal{N} \) with a faithful normal finite trace \( \tau \), the Fuglede-Kadison determinant was introduced in [30]. In [32] (see also [25][27]), Haagerup and Schultz defined the Fuglede-Kadison determinant \( det_\mathcal{N}(x) \geq 0 \) of \( x \in S(\mathcal{N}, \tau) \) such that \( \log_+ \mu(x) \in L_1(0, \infty) \) by the integral:

\[
\log det_\mathcal{N}(x) = \int_0^{\tau(1_\mathcal{N})} \log \mu(t; x) dt.
\]
Lemma 4.1. Assume that $\mathcal{N}$ is a finite von Neumann algebra with a faithful normal finite trace $\tau$. If $0 \leq a \in \mathcal{L}_{\log}(\mathcal{N}, \tau)$ and $b \in \mathcal{L}_{\log}(\mathcal{N}, \tau)$ is self-adjoint with $-a \leq b \leq a$, then $\det_{\mathcal{N}}(b) \leq \det_{\mathcal{N}}(a)$.

Proof. For every $\varepsilon > 0$, we define $a_\varepsilon := a + \varepsilon 1$. In particular, $a_\varepsilon$ is invertible and $a_\varepsilon^{-1} = \int_{\mathcal{N}} d\lambda a_\varepsilon$. In particular, $a_\varepsilon^{-1}$ is bounded. By [22, Proposition 2.5] (see also [20]), we obtain that $a_\varepsilon^{-1/2}ba_\varepsilon^{-1/2} \in \mathcal{L}_{\log}(\mathcal{N}, \tau)$. Moreover,

$$-1 = -a_\varepsilon^{-1/2}a_\varepsilon^{-1/2} \leq a_\varepsilon^{-1/2}ba_\varepsilon^{-1/2} \leq a_\varepsilon^{-1/2}a_\varepsilon^{-1/2} = 1.$$ 

Hence, $\mu(a_\varepsilon^{-1/2}ba_\varepsilon^{-1/2}) \leq 1$. That is, $\det_{\mathcal{N}}(a_\varepsilon^{-1/2}ba_\varepsilon^{-1/2}) \leq 1$. By [32, Proposition 2.5], we have

$$\frac{\det_{\mathcal{N}}(b)}{\det_{\mathcal{N}}(a_\varepsilon)} = \frac{\det_{\mathcal{N}}(b)}{\det_{\mathcal{N}}(a_\varepsilon^{1/2})\det_{\mathcal{N}}(a_\varepsilon^{1/2})} = \det_{\mathcal{N}}(a_\varepsilon^{-1/2}ba_\varepsilon^{-1/2}) \leq 1$$

Since $\varepsilon$ is arbitrary, it follows that $\det_{\mathcal{N}} b \leq \det_{\mathcal{N}} a$. \hfill \Box

This is an extension of the result in [70] (see also [20]).

Lemma 4.2. Assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a semifinite faithful normal trace $\tau$. Let $a, b \in \mathcal{M}^\Delta$. If $a \geq 0$ and $b$ is self-adjoint with $-a \leq b \leq a$, then $b \prec \prec_{\log} a$.

Proof. Without loss of generality, we may assume that $\mathcal{M}$ is atomless (see e.g. [50, Lemma 2.3.18]). For every $t > 0$, we can choose a $p \in \mathcal{P}(\mathcal{M})$ such that $\tau(p) = t$ and $\mu(s; b) = \mu(s; pbp)$, $s \in (0, t)$ (see e.g. [13, Page 953] or [55]). Note that $-pap \leq pbp \leq pap$. By Lemma 4.1, we obtain that

$$\int_0^t \log \mu(s; pbp)ds = \log \det_{p\mathcal{M}p}(pbp) \leq \log \det_{p\mathcal{M}p}(pap) = \int_0^t \log \mu(s; pap)ds.$$

Hence, we obtain that

$$\int_0^t \log \mu(s; b)ds = \int_0^t \log \mu(s; pbp)ds \leq \int_0^t \log \mu(s; pap)ds \leq \int_0^t \log \mu(s; a)ds.$$

Since $t$ is arbitrary, it follows that $\mu(b) \prec \prec_{\log} \mu(a)$. \hfill \Box

Our next lemma is folklore. We provide a short proof for the sake of convenience.

Lemma 4.3. Let $\mathcal{M}$ be von Neumann algebra with a faithful normal finite trace $\tau$ and let $0 \leq b \leq a \in S(\mathcal{M}, \tau)$. If $\mu(b) = \mu(a)$, then $a = b$.

Proof. Note that $1 \leq 1 + b \leq 1 + a$. Taking inverses, we obtain

$$1 \geq (1 + b)^{-1} \geq (1 + a)^{-1}.$$

Subtracting $1$, we obtain

$$0 \leq \frac{b}{1 + b} \leq \frac{a}{1 + a}.$$
Since the mapping \( t \to \frac{t}{1+t} \) is increasing, it follows from [50] Corollary 2.3.17 that
\[
\mu\left(\frac{b}{1+b}\right) = \frac{\mu(b)}{1+\mu(b)} = \frac{\mu(a)}{1+\mu(a)} = \mu\left(\frac{a}{1+a}\right).
\]
Since \( \tau \) is finite, it follows that
\[
\tau\left(\frac{b}{1+b}\right) = \tau\left(\frac{a}{1+a}\right) < \infty.
\] (34)

Letting \( x := \frac{a}{1+a} - \frac{b}{1+b} \geq 0 \), we have \( x \geq 0 \) and \( \tau(x) \leq 0 \). The faithfulness of \( \tau \) implies that \( x = 0 \). That is, \( \frac{b}{1+b} = \frac{a}{1+a} \). Subtracting 1, we obtain
\[
(1+b)^{-1} = (1+a)^{-1},
\]
which implies that \( a = b \). \( \square \)

The following lemma was known before for the special case \( z \in (L_1 + L_\infty)(\mathcal{M}) \) (see [13] Lemma 4.4 for a similar result).

**Lemma 4.4.** Let \((\mathcal{M}, \tau)\) be a semifinite von Neumann algebra and let \( 0 \leq z \in S(\mathcal{M}, \tau) \). Let \( r := e^z(\lambda, \infty), \lambda > 0 \), and let \( p \in \mathcal{P}(\mathcal{M}) \) be such that \( t := \tau(p) = \tau(r) < \infty \). If \( \mu(pzp) = \mu(z) \) on \((0, t)\), then \( p = r \).

**Proof.** Let \( z_1 = \max\{z, \lambda\} \). For any \( s \in (0, t) \), we have
\[
\mu(s; pz_1p) \geq \mu(s; pzp) = \mu(s; z), \quad \mu(s; pz_1p) \leq \mu(s; z_1) = \mu(s; z),
\]
where the last equality follows immediately from the definition of singular value functions (see also [22] Chapter III, Proposition 2.10]). Therefore,
\[
\mu(s; pz_1p) = \mu(s; z_1), \quad s \in (0, t). \tag{35}
\]

Setting \( z_2 := (z - \lambda)_+ \), by [50] Corollary 2.3.16] and the definition of \( z_1 \), we have
\[
\mu(pz_1p) = \mu(pzp + \lambda p) \leq \mu(pzp) + \lambda.
\]
On the other hand, by [50] Corollary 2.3.17] and definitions of \( z_1 \) and \( z_2 \), we have
\[
\lambda + \mu(z_2) = \mu(z_1) \tag{36}
\]
on \((0, t)\). Hence, we obtain that
\[
\lambda + \mu(z_2) \leq \lambda + \mu(pzp)
\]
on \((0, t)\). Since \( \mu(pzp) \leq \mu(z_2) \), it follows that
\[
\mu(pzp) = \mu(z_2)
\]
on \((0, t)\). Since both functions vanish outside of \((0, t)\) (see e.g. [22] Chapter III, Proposition 2.10]), it follows that
\[
\mu(pzp) = \mu(z_2)
\]
(36)
on $(0, \infty)$. Note that $z_2 = rz_2 r$. Let $pr = u |pr|$ be the polar decomposition. Hence, we obtain that

$$\mu(pz_2 p) = \mu(pr \cdot z_2 \cdot rp) = \mu(u |pr| \cdot z_2 \cdot |pr| u^*) = \mu(|pr| \cdot z_2 \cdot |pr|).$$

For all $0 \leq a, b \in S(M, \tau)$, we have (see e.g. \cite[Lemma 2.3.12 and Corollary 2.3.17]{58})

$$\mu(ab^2 a) = \mu(|ba|^2) = \mu^2(|ba|) = \mu^2(ba) = \mu^2((ba)^*)
= \mu^2(ab) = \mu^2(|ab|) = \mu(|ab|^2) = \mu(ba^2 b).$$

Hence,

$$\mu(pz_2 p) = \mu(z_2^\frac{1}{2} |pr|^2 z_2^\frac{1}{2}).$$

Consider the ($\tau$-finite) reduced von Neumann algebra $rMr$. Recall that $z_2 = rz_2 r$. We have

$$x := z_2, \quad y := z_2^\frac{1}{2} |pr|^2 z_2^\frac{1}{2} \in rMr.$$

Clearly, $0 \leq y \leq x$ and $\mu(x) = \mu(pz_2 p) = \mu(y)$. By Lemma \cite[4.3]{43} we obtain that $y = x$. Therefore, we have

$$z_2^\frac{1}{2} (1 - |pr|^2)z_2^\frac{1}{2} = 0.$$

Since $s(z_2) = r$, it follows that

$$r(1 - |pr|^2)r = 0 \implies r(1 - rpr)r = 0 \implies r = rpr.$$

Note that

$$p = rpr + (1 - r)p(1 - r) + rp(1 - r) + (1 - r)pr.$$

Since $p$ and $r$ are $\tau$-finite, it follows that

$$\tau(rp(1 - r)) = \tau((1 - r) \cdot rp) = 0, \quad \tau((1 - r)pr) = \tau(r \cdot (1 - r)p) = 0.$$

Hence,

$$\tau(p) = \tau(rpr) + \tau((1 - r)p(1 - r)).$$

By assumption that $\tau(p) = \tau(r)$ and by $r = rpr$, we have

$$\tau(p) = \tau(r) = \tau(rpr).$$

Thus,

$$\tau((1 - r)p(1 - r)) = 0.$$

Since $\tau$ is faithful, it follows that

$$(1 - r)p(1 - r) = 0 \implies p(1 - r) = 0 \implies p = pr \implies p \leq r.$$

Since $\tau(p) = \tau(r)$, it follows that $p = r$. \hfill \Box

The following result is well-known in the setting of $\mathcal{F}(\tau)$. We extend it to the case of the algebra $S_0(M, \tau)$ of $\tau$-compact operators.

**Proposition 4.5.** Let $x, y \in S_0(M, \tau)$. If $xy = -yx$, then $xy = 0$. 

Proof. Letting \( p_n := e^{\frac{x}{n}} [\frac{1}{n}, n] \), we have
\[
\begin{align*}
p_n x p_n y p_n = p_n x y p_n &= -p_n y x p_n = -p_n y p_n x p_n.
\end{align*}
\]
For the sake of convenience, we define \( x_n := p_n x p_n \geq 0 \) and \( y_n := p_n y p_n \geq 0 \). Hence, we have \( x_n y_n = -y_n x_n \). Clearly, \( x_n \in L_1(\mathcal{M}, \tau) \cap \mathcal{M} \). Let \( q_{m,n} := e^{y_n} [\frac{1}{m}, m] \). We have
\[
q_{m,n} x n q_{m,n} y n q_{m,n} = q_{m,n} x_n y_n q_{m,n} = -q_{m,n} y_n x_n q_{m,n}
\]
\[
= -q_{m,n} y_n q_{m,n} x_n q_{m,n}, \quad \forall m,n \geq 1.
\]
Denote \( z_{m,n}^1 := q_{m,n} x_n q_{m,n} \) and \( z_{m,n}^2 := q_{m,n} y_n q_{m,n} \). Note that \( 0 \leq z_{m,n}^1, z_{m,n}^2 \in L_1(\mathcal{M}, \tau) \cap \mathcal{M} \) and \( z_{m,n}^1 = -z_{m,n}^2 \). Hence,
\[
\tau(z_{m,n}^1 z_{m,n}^{-1}) = \tau(z_{m,n}^2 z_{m,n}^{-1}) = \tau(-z_{m,n}^{-1} z_{m,n}^1),
\]
i.e., \( \tau((z_{m,n}^1)^{1/2} z_{m,n}^2 (z_{m,n}^1)^{1/2}) = \tau(z_{m,n}^2 z_{m,n}^1) = 0 \). The faithfulness of \( \tau \) implies that \( (z_{m,n}^1)^{1/2} z_{m,n}^2 (z_{m,n}^1)^{1/2} = 0 \). Hence, we obtain that \( (z_{m,n}^1)^{1/2} (z_{m,n}^2)^{1/2} = 0 \), and therefore,
\[
q_{m,n} x_n y_n q_{m,n} = z_{m,n}^1 z_{m,n}^2 = 0.
\]
Passing \( m \to \infty \), we obtain that \( 0 = q_{m,n} x_n y_n q_{m,n} \to_m s(y_n) x_n y_n \) in the measure topology (see e.g. [21, Proposition 2] or [22, Chapter II, Proposition 6.4]), i.e., \( s(y_n) x_n y_n = 0 \). Since \( y_n x_n y_n = y_n s(y_n) x_n y_n = 0 \), it follows that \( x_n^{1/2} y_n = 0 \) and therefore, \( x_n y_n = x_n^{1/2} x_n^{1/2} y_n = 0 \). That is, \( p_n x y p_n \to 0 \) for every \( n \). Taking \( n \to \infty \), we obtain that \( 0 = p_n x y p_n \to_n x y s(x) \) in measure topology, which implies that \( x y x = x y s(x) x = 0 \). Hence, \( x y \frac{1}{2} = 0 \), and therefore, \( x y = 0 \). This completes the proof.

The following lemma is an extension of [72, Theorem 2].

Lemma 4.6. Let \( 0 \leq x, y \in \mathcal{M}^\Delta := (\mathcal{L}_{\log}(\mathcal{M}, \tau) + \mathcal{L}_\infty(\mathcal{M}, \tau)) \cap S_0(\mathcal{M}, \tau) \). If \( \mu(x - y) = \mu(x + y) \), then \( x y = 0 \).

Proof. Let
\[
p_\lambda = e^{\frac{|x - y|}{\lambda}}(\lambda, \infty), \quad r_\lambda = e^{\frac{x+y}{\lambda}}(\lambda, \infty), \quad \lambda > 0.
\]
Since \( \mu(x - y) = \mu(x + y) \), it follows that \( t_\lambda := \tau(r_\lambda) = \tau(p_\lambda) < \infty \) (see e.g. [22, Chapter 3.2]).

By definition, \( p_\lambda \) commutes with \( x - y \). Thus,
\[
p_\lambda |x - y| p_\lambda = |p_\lambda (x - y)p_\lambda|.
\]
On \((0, t_\lambda)\), we have the coincidence of the following functions (see e.g. [22, Chapter III, Proposition 2.10])
\[
\mu(x + y) = \mu(x - y) = \mu(p_\lambda x - y p_\lambda) = \mu(p_\lambda (x - y)p_\lambda).
\]
For positive operators $a, b \in \mathcal{M}^\Delta$, Lemma 4.2 implies that $a - b \prec \prec_{\log} a + b$. Therefore,

\[ \mu(x + y)\chi_{(0,t_\lambda)} \leq \mu(p_\lambda(x - y)p_\lambda) \prec \prec_{\log} \mu(p_\lambda(x + y)p_\lambda) \leq \mu(x + y)\chi_{(0,t_\lambda)}. \]

Thus,

\[ \mu(p_\lambda(x + y)p_\lambda) = \mu(x + y) \]

on $(0, t_\lambda)$. By Lemma 4.3, we have $p_\lambda = r_\lambda$.

Since $p_\lambda = r_\lambda$ for all $\lambda > 0$, it follows from the Spectral Theorem that $|x - y| = x + y$. Squaring both parts of the preceding equality, we arrive at $xy = -yx$. Now, we can apply Proposition 4.5 to conclude that $xy = 0$. \qed

The following example shows that one cannot expect for a similar result of Lemma 4.6 without the assumption of $\tau$-compactness.

**Example 4.7.** Let $\mathcal{M}$ be a semifinite infinite von Neumann algebra with a semifinite faithful normal trace $\tau$. Consider algebra $\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}$ equipped with trace $\tau \oplus \tau \oplus \tau$. Let $a := 1 \oplus 0 \oplus \frac{1}{2}$ and $b := 0 \oplus 1 \oplus \frac{1}{2}$. It is clear that $\mu(a - b) = \mu(a + b)$, but $ab \neq 0$.

In what follows, we always assume that $E(\mathcal{M}_1, \tau_1)$ and $F(\mathcal{M}_2, \tau_2)$ are symmetrically $\Delta$-normed spaces affiliated with semifinite von Neumann algebras $(\mathcal{M}_1, \tau_1)$ and $(\mathcal{M}_2, \tau_2)$, respectively.

**Proposition 4.8.** Let $T : E(\mathcal{M}_1, \tau_1) \xrightarrow{\text{into}} F(\mathcal{M}_2, \tau_2)$ be an order-preserving isometry, where $F(\mathcal{M}_2, \tau_2)$ has SLM symmetric $\Delta$-norm. If $0 \leq x, y \in E(\mathcal{M}_1, \tau_1)$ such that $T(x), T(y) \in \mathcal{M}_2^\Delta$ and $xy = 0$, then $T(x)T(y) = 0$.

**Proof.** Since $xy = 0$ implies that $|x - y| = |x + y|$, it follows that $\|x - y\|_E = \|x + y\|_E$, and therefore, $\|T(x) - T(y)\|_F = \|T(x) + T(y)\|_F$. It follows from Lemma 4.2 that $T(x) - T(y) \prec \prec_{\log} T(x) + T(y)$. By the Definition of SLM $\Delta$-norms, we obtain that $\mu(T(x) - T(y)) = \mu(T(x) + T(y))$. It follows from Lemma 4.6 that $T(x)T(y) = 0$. \qed

Every strongly symmetric space of $\tau$-compact operators, whose central carrier projection is the identity, is a subspace of $L_0(\mathcal{M}, \tau) := (L_1(\mathcal{M}, \tau) + \mathcal{M}) \cap S_0(\mathcal{M}, \tau)$ and therefore a subspace of $\mathcal{M}^\Delta$. For every symmetric space $E(0, \infty)$ of functions vanishing at infinity, the corresponding operator space $E(\mathcal{M}, \tau)$ is a subspace of $L_0(\mathcal{M}, \tau)$ (see e.g. [19, 22, 49, 50]). Note that if a symmetric norm $\| \cdot \|_F$ is strictly monotone with respect to the submajorisation, then it is strictly monotone with respect to the logarithmic submajorisation (see Proposition 2.3 or [20]). However, the inverse is not the case. For example, the $L_1$-norm is an SLM norm which fails to be strictly $K$-monotone. As an application of Theorem 3.5 and Proposition 4.8, we obtain the following result, which significantly extends [72, Proposition 6].
Corollary 4.9. Assume that $E(M_1, \tau_1)$ has order continuous symmetric $\Delta$-norm and assume that $F(M_2, \tau_2) \subset M_2^{\Delta}$ has SLM symmetric $\Delta$-norm. For every order-preserving isometry $T : E(M_1, \tau_1) \xrightarrow{\text{into}} F(M_2, \tau_2)$, there exists a positive operator $B$ and a Jordan $*$-isomorphism $J$ from $M_1$ onto a weakly closed $*$-subalgebra of $M_2$ such that $T(x) = BJ(x)$, $\forall x \in M_1 \cap E(M_1, \tau_1)$.

In the special case when $M_1$ and $M_2$ are finite, we obtain the following corollary of Theorem 3.1, which recovers and extends [48, Theorem 1] and [60, Theorem 2].

Corollary 4.10. Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be two finite von Neumann algebras with $\tau_1(1_{M_1}), \tau_2(1_{M_2}) < \infty$. Assume that $E(M_1, \tau_1)$ is a symmetrically $\Delta$-normed space and assume that $F(M_2, \tau_2) \subset M_2^{\Delta}$ has SLM symmetric $\Delta$-norm. If an order-preserving isometry $T : E(M_1, \tau_1) \xrightarrow{\text{into}} F(M_2, \tau_2)$ satisfies that $T(1_{M_1}) = 1_{M_2}$, then $T|_{M_1}$ is a Jordan $*$-homomorphism from $M_1$ into $M_2$. Moreover, if $\| \cdot \|_F$ is order continuous, then $T|_{M_1}$ is a Jordan $*$-isomorphism from $M_1$ onto a weakly closed $*$-subalgebra of $M_2$.

5. ORDER-PRESERVING ISOMETRIES ONTO $\Delta$-NORMED SPACES

Order-preserving isometries from a noncommutative $L_2$-space onto another was studied in [9, Theorem 1]. The description of order-preserving isometries onto a fully symmetric space affiliated with a finite von Neumann algebra is given in [12, Theorem 3.1]. In this section, we consider the general form of surjective isometries, which significantly extends [9, Theorem 1] and complements [12, Theorem 3.1].

Recall that $M^\Delta := (L_{\log}(M, \tau) + M) \cap S_0(M, \tau)$. Assume that $(M_1, \tau_1)$ and $(M_2, \tau_2)$ are semifinite von Neumann algebras. The following lemma has been obtained by Ju.A. Abramovich for order-preserving isometries of arbitrary normed lattices [1] (see also [12, Lemma 3.2]). We extend this result to surjective isometries on symmetrically $\Delta$-normed spaces.

Lemma 5.1. Assume that $E(M_1, \tau_1)$ and $F(M_2, \tau_2)$ are symmetrically $\Delta$-normed spaces. In addition, we assume that $\| \cdot \|_F$ is (not necessarily strictly) log-monotone. Let $T : E(M_1, \tau_1) \xrightarrow{\text{onto}} F(M_2, \tau_2)$ be an order-preserving isometry. If $T(x) \geq 0$, then $x \geq 0$.

Proof. For every self-adjoint $a$, $T(a) = T(a_+ - a_-) = T(a_+) - T(a_-)$, which implies that $T(a)$ is self-adjoint. Since $T(\text{Re}(x)) + iT(\text{Im}(x)) = T(x) > 0$, it follows that $\text{Im}(x) = 0$, that is $x = x^*$.

Let $x_+$ and $x_-$ be the positive part and negative part of $x$, respectively. If $x_+ = 0$, then $0 \geq T(-x_-) = T(x) \geq 0$ implies that $x = 0$. Hence, it suffices to consider the case when $x_+ \neq 0$ and prove that $x_- = 0$. 


Let $b_1 := T(x_+)$ and $b_2 := T(x_-)$. In particular, since $T$ is an order-preserving isometry, it follows that $b_1, b_2 \geq 0$. Moreover, $b := b_1 - b_2 = T(x) \geq 0$ and $b_1 + b_2 = T(x_+ + x_-) = T(|x|)$.

Note that

\begin{equation}
\|a(b_1 + b_2)\|_F = \|T(a|x|)\|_F = \|a|x|\|_E = \|ax\|_E
\end{equation}

for every $a \in \mathbb{C}$. We assert that

\begin{equation}
\|ax_+ + akx_-\|_E = \|ab_1 + akb_2\|_F \leq \|ax\|_E
\end{equation}

for all $k = 1, 2, \cdots$ and $a \in \mathbb{C}$. It follows from (41) that (42) holds for $k = 1$. Assume that it holds for $k = n$. Noting that $b, b_2 \geq 0$, we obtain that

\[-(b + nb_2) \leq b - nb_2 \leq b + nb_2.

By Lemma 4.2, the logarithmic monotonicity of $\| \cdot \|_F$ guarantees that

\begin{equation}
\|ab - \alpha nb_2\|_F \leq \|ab + \alpha nb_2\|_F
\end{equation}

for every $a \in \mathbb{C}$. Using the inequality $0 \leq b + nb_2 = b_1 + (n - 1)b_2 \leq b_1 + nb_2$ and the assumption of induction, we get

\begin{equation}
\|ab_1 - \alpha(n + 1)b_2\|_F = \|ab - \alpha \cdot nb_2\|_F \leq \|ab + \alpha \cdot nb_2\|_F
\end{equation}

for every $a \in \mathbb{C}$. Hence, using that $x_+x_- = x_-x_+ = 0$, we obtain that

\begin{align*}
\|ab_1 + \alpha(n + 1)b_2\|_F &= \|T(ax_+ + \alpha(n + 1)x_-)\|_F = \|ax_+ + \alpha(n + 1)x_-\|_E \\
&= \|ax_+ - \alpha(n + 1)x_-\|_E = \|ax_+ - \alpha(n + 1)x_-\|_E \\
&= \|ab_1 - \alpha(n + 1)b_2\|_F \leq \|ax\|_E.
\end{align*}

Thus, we obtain validity of (42) for all $k \geq 1$. Therefore, since $\alpha$ is arbitrary, it follows that

\begin{equation}
\|x_+\|_E \leq \left\| \frac{1}{n}x_+ + x_- \right\|_E \leq \left\| \frac{1}{n}x \right\|_E \to 0,
\end{equation}

which implies that $x_+ = 0$. That is, $x \geq 0$. \qed

Corollary 5.3 below is the main result of this section. In contrast with the results in 54, Corollary 5.3 covers the case of noncommutative $L_2$-spaces (see Section 4). It is very common to assume that symmetrically (quasi-)normed spaces $E(M_1, \tau_1)$ and $F(M_2, \tau_2)$ have order-continuous norms or the Fatou property (see e.g., 45 and 28 Section 5.2). When considering order-preserving isometries from $E(M_1, \tau_1)$ onto $F(M_2, \tau_2)$, $\| \cdot \|_E$ is automatically order continuous if $\| \cdot \|_F$ is order continuous. The following proposition shows that the order continuity of $\Delta$-norms is preserved under order-preserving isometries.
Proposition 5.2. Assume that $E(M_1, \tau_1)$ and $F(M_2, \tau_2)$ are symmetrically $\Delta$-normed spaces. In addition, we assume that $\| \cdot \|_F$ is (not necessarily strictly) log-monotone. If there exists an order-preserving isometry $T : E(M_1, \tau_1) \rightarrow F(M_2, \tau_2)$ and $\| \cdot \|_F$ is order continuous, then $\| \cdot \|_E$ is also order continuous.

Proof. Assume that $\{x_n \in E(M_1, \tau_1)\}^+$ is a sequence such that $x_n \downarrow 0$. Since $T(x_n) \downarrow 0$, it follows that $T(x_n) \downarrow y \geq 0$ for some $y \in F(M_2, \tau_2)$ (see e.g. [21 Proposition 2]). Since $T$ is a surjective, it follows from Lemma 5.1 that $x_n - T^{-1}y \geq 0$ for every $n$, that is, $0 \leq T^{-1}y \leq \inf x_n = 0$, which implies that $y = 0$. Hence, $\inf_n \|x_n\|_E = \inf_n \|T(x_n)\|_F = 0$.

Proposition 5.2 guarantees that we can use Theorem 5.3 to obtain the following corollary, which extends a number of existing results (see e.g. [9 Theorem 1] and [60 Theorem 2]).

Corollary 5.3. Assume that $E(M_1, \tau_1) \subset S(M_1, \tau_1)$ and $F(M_2, \tau_2) \subset M_2^\Delta$ are symmetrically $\Delta$-normed spaces, and $F(M_2, \tau_2)$ has SLM order continuous $\Delta$-norm $\|\cdot\|_F$. If there exists an order-preserving isometry $T : E(M_1, \tau_1) \rightarrow F(M_2, \tau_2)$, then there is a Jordan $*$-isomorphism $J : M_1 \rightarrow M_2$ and a positive self-adjoint operator $B$ affiliated with $Z(M_2)$ such that $T(x) = BJ(x)$ for every $x \in E(M_1, \tau_2) \cap M_1$.

Proof. Let the Jordan $*$-monomorphism $J$ be defined as in Theorem 5.3. It suffices to prove that $J(M_1) = M_2$.

Recall that $s(T(x)) = J(s(x))$ for every $0 \leq x \in E(M_1, \tau_1)$ (see Remark 3.4) and $(M_2)_{fin} \subset F(M_2, \tau_2)$ (see Proposition 2.3). By Lemma 5.1 for every $f \in P_{fin}(M_2)$, there exists $0 \leq x \in E(M_1, \tau_1)$ such that $T(x) = f$. Hence, $f = s(f) = s(T(x)) = J(s(x))$. This implies that $J(M_1)$ contains all $\tau_2$-finite projections in $M_2$. Since $J(M_1)$ is a weakly closed $*$-subalgebra of $M_2$ (see Remark 2.10) and the latter is semifinite von Neumann algebra, it follows that $J(M_1) = M_2$.

Recall that the spectral projections of $B$ commute with every $J(x), x \in M_1$ (see Proposition 5.4). Since $J$ is a surjective, it follows that $B$ is affiliated with $Z(M_2)$ (see e.g. [22 Chapter II, Proposition 1.4]).

The following corollary is an extension of [60 Theorem 2].

Corollary 5.4. Suppose that the assumption of Corollary 5.3 are met and, in addition, $\tau_1(1_{M_1}), \tau_2(1_{M_2}) < \infty$. If $T(1_{M_1}) = 1_{M_2}$, then $T$ is a Jordan $*$-isomorphism $M_1$ onto $M_2$.

Corollary 5.5. Let $(M_1, \tau_1)$ and $(M_2, \tau_2)$ be two semifinite von Neumann algebras. If there exists an order-preserving surjective isometry $T : E(M_1, \tau_1) \rightarrow F(M_2, \tau_2)$ for some symmetrically $\Delta$-normed spaces $E(M_1, \tau_1)$ and $F(M_2, \tau_2)$ ($\| \cdot \|_F$ is an order continuous SLM $\Delta$-norm), then $M_1$ and $M_2$ are Jordan $*$-isomorphic.
When $1 \leq p < \infty$ and $(M, \tau)$ is a finite factor, it is shown in \cite{60} Corollary 1 that every order-preserving $L_p$-isometry from $T : M \to M$ is indeed a $\ast$-isomorphism or $\ast$-anti-isomorphism (see also \cite{48} Theorem 1). The following corollary is a semifinite version of \cite{60, Corollary 1} with significant extension.

**Corollary 5.6.** Suppose that the assumption of Corollary 5.5 are met and, in addition, $M_2$ is a factor. Then, there is a constant $\alpha > 0$ and a $\ast$-isomorphism or a $\ast$-anti-isomorphism $J : M_1 \to M_2$ such that $T(x) = \alpha J(x)$ for every $x \in E(M_1, \tau_2) \cap M_1$ and $T(1_{M_1}) = \alpha 1_{M_2}$.

6. ORDER-PRESERVING ISOMETRIES INTO LORENTZ SPACES

It is known (see e.g. \cite{21, 49, 50}) that every symmetrically normed operator space is a subspace of $L_1(M, \tau) + M$. Indeed, in the $\Delta$-normed setting, the $L_{\log}(M, \tau) + M$ plays a similar role as $L_1(M, \tau) + M$ does in the normed case. In this section, we show that the class of SLM $\Delta$-normed space embraces a wide class of symmetrically $\Delta$-normed spaces used in analysis.

Let $p \in (0, \infty)$ and let $w$ be a weight (that is, a non-negative measurable function on $(0, \infty)$ that is not identically zero). The Lorentz space $\Lambda^w_p(0, \infty)$ is defined by

$$\left\{ f \in S(0, \infty) \mid \|f\|_{\Lambda^w_p} := \left( \int_0^\infty \mu(t; f)^p w(t) dt \right)^{1/p} < \infty \right\}.$$  

For a given weight $w$, we define $W(t) := \int_0^t w(s) ds$. We always assume that $W(t) > 0$ for every $t \in (0, \infty)$. It is shown in \cite{15} that $\Lambda^w_p(0, \infty)$ is a linear space if and only if $W$ satisfies the $\Delta_2$-condition, i.e., $W(2t) \leq CW(t)$ for some $C > 1$ and all $t > 0$. Moreover, $\|\| \cdot \|_{\Lambda^w_p}$ is a complete quasi-norm \cite{10, 46}. It is known that $\|\| \cdot \|_{\Lambda^w_p}$ is order continuous if and only if $W(\infty) = \infty$ (see e.g. \cite{46}). We define

$$\Lambda^w_p(M, \tau) := \{ x \in S(M, \tau) : \mu(x) \in \Lambda^w_p(0, \infty) \}.$$  

In particular, $\Lambda^w_p(M, \tau)$ is a quasi-Banach space equipped with quasi-norm $\|X\|_{\Lambda^w_p} = \|\mu(X)\|_{\Lambda^w_p}$, $X \in \Lambda^w_p(M, \tau)$. \cite{37, 69}.

Assume that $w$ is a strictly positive decreasing function on $(0, \infty)$ such that $W(\infty) = \infty$. Then, $W$ satisfies the $\Delta_2$-condition. Moreover, Proposition \cite{2, 10} implies that $\Lambda^w_p(M, \tau)$ has order continuous (quasi-)norm and therefore, $\Lambda^w_p(M, \tau) \subset M^\Delta \subset S_0(M, \tau)$. In this section, we show that all $\Lambda^w_p(M, \tau)$ has SLM quasi-norms.

We note that if there is an isometry $T$ from a $\Delta$-normed symmetric space $E(M_1, \tau_1)$ into $\Lambda^w_p(M_1, \tau)$, then $E(M_1, \tau_1)$ must be quasi-normed. Indeed, for every $X \in E(M_1, \tau_1)$ and $\lambda \in \mathbb{C}$, we have

$$\|\lambda x\|_E = \|T(\lambda x)\|_{\Lambda^w_p} = |\lambda| \|T(x)\|_{\Lambda^w_p} = |\lambda| \|x\|_E.$$
which implies that $E(M_1, \tau_1)$ is quasi-normed. Moreover, if this isometry is surjective, then $E(M_1, \tau_1)$ is a quasi-Banach space.

**Remark 6.1.** Let $x \in S(M, \tau)$. Assume that $w$ is a strictly positive decreasing function on $(0, \infty)$. It is easy to see that $\mu(x)^p w$ and $\mu(\mu(x)^p w)$ are equimeasurable (see e.g. [50, Chapter III, Section 1]). Since $\mu(x)^p w$ is a decreasing function and $\mu(\mu(x)^p w)$ is right-continuous, it is easy to see that $\mu(x)^p w = \mu(\mu(x)^p w)$ a.e..

The following result is an easy consequence of Corollary 2.5:

**Corollary 6.2.** Assume that $w$ is a strictly positive decreasing function on $(0, \infty)$. Let $a, b \in \Delta_w^p(M, \tau)$, $p \in (0, \infty)$. If $b \prec \prec \log a$, then $w(t)\mu(t; b)^p \prec \prec w(t)\mu(t; a)^p$. In particular, $\|b\|_{\Lambda_w^p} \leq \|a\|_{\Lambda_w^p}$.

**Proof.** It follows from $\mu(b) \prec \prec \log \mu(a)$ that

$$
\int_0^t \log(w(t))^{1/p} \mu(t; b)) dt = \int_0^t \left( \log w(t)^{1/p} + \log \mu(t; b) \right) dt 
\leq \int_0^t \left( \log w(t)^{1/p} + \log \mu(t; a) \right) dt = \int_0^t \log \left( w(t)^{1/p} \mu(t; a) \right) dt.
$$

Thus, Corollary 2.5 together with Remark 6.1 implies that $w(t)\mu(t; b)^p \prec \prec w(t)\mu(t; a)^p$. 

Recall the definition of strictly $K$-monotone norms defined in Section 2. The following lemma is an easy consequence of the strict $K$-monotonicity of $L_2$-norm, showing that $L_p(M, \tau)$ is SLM quasi-normed for every $p \in (0, \infty)$.

**Lemma 6.3.** Let $a, b \in L_p(M, \tau)$, $0 < p < \infty$, be such that $b \prec \prec \log a$. If $\|a\|_p = \|b\|_p$, then $\mu(b) = \mu(a)$.

**Proof.** Assume that $\mu(b) \neq \mu(a)$, i.e., $\mu(b)^{p/2} \neq \mu(a)^{p/2}$. Corollary 2.5 implies that $\mu(b)^{p/2} \prec \prec \mu(a)^{p/2}$. Since $\mu(a)^{p/2}, \mu(b)^{p/2} \in L_2(0, \infty)$ and the $L_2$-norm $\cdot \| \cdot \|_2$ is strictly $K$-monotone (see e.g. [72, Section 5] or [11]), it follows that

$$
\|b\|_p = \|\mu(b)\|_p^p = \|\mu(b)^{p/2}\|_2^2 < \|\mu(a)^{p/2}\|_2^2 = \|\mu(a)\|_p^p = \|a\|_p^p,
$$

which is a contradiction, that is, $\mu(b) = \mu(a)$.

The following result is an easy consequence of Lemma 6.3 showing that every $\cdot \| \cdot \|_{\Lambda_w^p}$ is an SLM quasi-norm.

**Theorem 6.4.** Assume that $0 < p < \infty$ and $w$ is a strictly positive decreasing function on $(0, \infty)$. Let $a, b \in \Delta_w^p(M, \tau)$ be such that $b \prec \prec \log a$. If $\|a\|_{\Lambda_w^p} = \|b\|_{\Lambda_w^p}$, then $\mu(b) = \mu(a)$. In particular, $\cdot \| \cdot \|_{\Lambda_w^p}$ is an SLM quasi-norm.
Proof. Since $\mu(a), \mu(b) \in \Lambda^p_w((0, \infty))$, it follows that $\mu(a)^p w, \mu(b)^p w \in L_1(0, \infty)$. It follows from Remark 6.1 that

$$
\int_0^s \log \mu(t; \mu(b)^p w) dt = \int_0^s \log \mu(t; b)^p w(t) dt = \int_0^s p \log \mu(t; b) dt + \int_0^s \log w(t) dt
$$

and, by the assumption, we have

$$
\|\mu(b)^p w\|_1 = \int_0^\infty \mu(t; \mu(b)^p w) dt = \int_0^\infty \mu(t; \mu(a)^p w) dt = \|\mu(a)^p w\|_1.
$$

By Lemma 4.3, we have $\mu(\mu(b)^p w) = \mu(\mu(a)^p w)$, which implies that $\mu(b)^p w = \mu(a)^p w$ a.e. (see Remark 6.1). Since $w$ is a strictly positive function on $(0, \infty)$, it follows from the right-continuity of $\mu(a)$ and $\mu(b)$ that $\mu(a) = \mu(b)$, which together with Corollary 6.2 implies that $\|\cdot\|_{\Lambda^p_w}$ is an SLM $\Delta$-norm.

Recall that $\Lambda^p_w(\mathcal{M}, \tau) \subset \mathcal{M}^\Delta$ and $\|\cdot\|_{\Lambda^p_w}$ is an order continuous $\Delta$-norm whenever $w$ is a strictly positive decreasing function on $(0, \infty)$ such that $W(\infty) = \infty$. Moreover, Theorem 6.4 guarantees that all $\Lambda^p_w(\mathcal{M}, \tau)$ have SLM quasi-norms. Appealing to Corollary 4.9 and Corollary 5.3, we obtain immediately the general form of order-preserving isometries into/onto Lorentz spaces, respectively, which complements the results in [12, Section 5].

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