Precipitation regions of high energy electrons, injected by a point source moving along an elliptic near-Earth orbit on the Earth surface

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Abstract. The paper is devoted to the study of precipitation regions of high-energy electrons injected by a point source moving in near-Earth space along an elliptical orbit. The precipitation regions are constructed on the basis of numerical integration of an ordinary differential equations describing the motion of electrons in a dipole magnetic field. The influence of some parameters of the source orbit on the structure of precipitation regions is considered. They are eccentricity, focal parameter, incline angle to the selected axis in the plane of the geographic equator and the energy of injected electrons. The difference between the obtained regions and the precipitation regions formed for the motion of a point source along circular near-Earth orbit is discussed.

1. Introduction
The problems of external injection were widely discussed ones at the initial stage of the studying of charged particles dynamics in a geomagnetic field. The source of particles was assumed to be at an infinitely large distance from the center of the field [1, 2, 3]. In recent years the problems of dynamics of charged high-energy particles injected into near-Earth space (NES) at a finite distance from the Earth in the geomagnetic field have become of great importance. This fact is directly connected with the problem of radiation pollution of near space by charged particles with high energies from manmade origin [4, 5]. The sources of these particles can be secondary high-energy particles generated in the material of active and passive orbital objects by particles of primary cosmic radiation. High-energy space accelerators are also potential internal sources of high-energy particles [6, 7].

Some theoretical methods can be used to study the dynamics of high-energy charged particles in a geomagnetic field, in particular, the theory of allowed zones by C. Stormer [5, 8]. However, their application is very limited due to the complexity of the system of differential equations; it is possible to obtain information about the property of their solutions only for very small number of cases. Therefore, the main approach to study the motion of these particles is the numerical solution of the problem for multiple dynamics by the Monte Carlo method of statistical tests. Previously, precipitation regions were constructed for a stationary point source and various models of the geomagnetic field [5, 9] and also for a point source which moves along a low circular.
near-earth orbit with a constant modulus of velocity [10]. In this paper, the precipitation regions are calculated for an injector moving in the NES along a near-Earth elliptical orbit. The orbital motion of the injector was simulated on the basis of Kepler’s second law which establishes the constancy of the injector sector velocity. For calculations in this paper we used the previously developed procedure for determining the stationary configuration of the precipitation regions which is established at times longer than the injector rotation period. The effect of the injector orbit precession around the geographic axis of the Earth is not taken into account, since the characteristic time for the establishment of a stationary precipitation regions structure in our case is much less than the period of the injector orbit precession.

2. Statement of the problem

We will assume that a source of high-energy electrons moves in the NES along a given elliptical orbit. The motion of the source is subject to Kepler’s second law, which states that in equal time intervals the ray connecting the center of the Earth (that is one of the ellipse foci) and the current position of the source goes equal areas. Mathematically, this means that

$$r^2 \dot{\varphi} = \text{const},$$

where $r$ is the distance from the injector to the center of the Earth, $\varphi$ is the angle of the polar coordinate system.

We will also assume that the source injects electrons of constant energy during its motion. Let us consider the problem of determination of the stationary configuration of the precipitation region, the stationary configuration is understood to be one that is established for the time after the beginning of injection which is longer than the period of injector’s orbital revolution. We will neglect the difference between the Earth’s magnetic field and the dipole field, which is correct enough at small distances from the Earth for electrons of the energies under consideration.

We will consider the spherical coordinate system $(r, \theta, \varphi)$ with a radial distance measured from the center of the Earth $r$, geographic longitude $\varphi$ and complement to geographic latitude $\theta$. The system of differential equations of the motion for an electron with charge $e$ and relativistic mass $m$ has the form

$$\begin{align*}
\frac{dr}{dt} &= v_r, \\
\frac{d\theta}{dt} &= \frac{v_\theta}{r}, \\
\frac{d\varphi}{dt} &= \frac{v_\varphi}{r \sin \theta}, \\
\frac{dv_r}{dt} &= \frac{e \omega_r}{mc} + \frac{v_\theta^2 + v_\varphi^2}{r}, \\
\frac{dv_\theta}{dt} &= \frac{e \omega_\theta}{mc} - \frac{v_r v_\theta}{r} + \frac{v_\varphi v_\theta c \theta}{r}, \\
\frac{dv_\varphi}{dt} &= \frac{e \omega_\varphi}{mc} - \frac{v_r v_\varphi}{r} + \frac{v_\theta v_\varphi c \theta}{r},
\end{align*}$$

(2)

where $v$ is the velocity of the particle, which is an integral of motion, $\omega_r = v_\theta B_\varphi - v_\varphi B_\theta$, $\omega_\theta = B_r v_\varphi - v_r B_\varphi$, $\omega_\varphi = v_r B_\theta - v_\theta B_r$; $B_r$, $B_\varphi$, $B_\theta$ are the components of magnetic induction.

Using dimensionless variables

$$\tilde{v}_r = v_r/c, \quad \tilde{v}_\theta = v_\theta/c, \quad \tilde{v}_\varphi = v_\varphi/c, \quad \tilde{r} = r/R_E, \quad \tau = ct/R_E,$$

where $c$ is the speed of light, we can write the system (2) in the form
\[
\begin{align*}
\frac{d\hat{r}}{dt} &= \hat{v}_r, \\
\frac{d\theta}{dt} &= \frac{\hat{v}_\theta}{r}, \\
\frac{d\phi}{dt} &= \frac{\hat{v}_\phi}{r \sin \theta}, \\
\frac{d\hat{v}_r}{dt} &= K\hat{\omega}_r + \frac{\hat{v}_\theta \hat{v}_\phi}{r} - \frac{\hat{v}_\phi \hat{v}_\theta}{r} + \frac{\hat{v}_\phi^2 \cos \theta}{r}, \\
\frac{d\hat{v}_\theta}{dt} &= K\hat{\omega}_\theta - \frac{\hat{v}_r \hat{v}_\phi}{r} + \frac{\hat{v}_\phi \hat{v}_r \cos \theta}{r}, \\
\frac{d\hat{v}_\phi}{dt} &= K\hat{\omega}_\phi - \frac{\hat{v}_r \hat{v}_\theta}{r} + \frac{\hat{v}_\theta \hat{v}_r \cos \theta}{r}, \\
\end{align*}
\]

(3)

where

\[\hat{\omega}_r = \hat{v}_\theta \hat{B}_\phi - \hat{v}_\phi \hat{B}_\theta, \quad \hat{\omega}_\theta = \hat{B}_r \hat{v}_\phi - \hat{v}_\phi \hat{B}_r, \quad \hat{\omega}_\phi = \hat{v}_r \hat{B}_\theta - \hat{v}_\theta \hat{B}_r; \quad \hat{B}_r = \frac{B_r}{B_0}, \quad \hat{B}_\theta = \frac{B_\theta}{B_0}, \quad \hat{B}_\phi = \frac{B_\phi}{B_0},\]

\(B_0\) is the value of the induction at the magnetic equator, \(K = \text{sign} (e) C_{st}^2 / R_E^2\) is the dimensionless coefficient, \(C_{st} = \sqrt{eM/mv} = 1.11 \cdot 10^6 / E_k \) is the Stormer's length unit, \(M\) – the Earth’s magnetic moment, \(E_k\) is the kinetic energy of the electron in GeV.

Assuming further that injection of electrons is uniform on any time interval we will determine the initial positions of electrons by simulating of the longitude \(\phi\) in accordance with law (1) and then will substitute it into the equation of the ellipse. The direction of injection of electrons \(j\) will be defined by the angle \(i\) between \(j\) and the direction of the local vertical and the angle \(q\) between the projection of \(j\) onto the plane of the local horizon and the local azimuthal direction. \(\cos i\) and \(q\) are assumed to be random variables uniformly distributed in the intervals \([-1; 1]\) and \([0; 2\pi]\) respectively. The trajectory will be calculated while all the following conditions are satisfied:

(i) the electron trajectory does not cross the sphere of radius \(R_E\);
(ii) the total length of the calculated path segment does not exceed the set limit value of \(400R_E\);
(iii) the electron does not move away at a geocentric distance exceeding the critical value \(15R_E\), at which point further motion is accompanied by an unlimited monotonic increase in its radial coordinate.

If the electron trajectory intersect the Earth’s surface, the corresponding point of the precipitation region with geomagnetic coordinates \(\phi\) and \(\psi\) has been appeared.

The spatial position of the orbit is determined by the angle of orbit inclination \(\lambda_{\text{max}}\) to the magnetic equator plane, and the longitude of the ascending node \(\varphi_{\text{up}}\).

It is well known, in polar coordinates an ellipse can be given by the equation

\[r = \frac{p}{1 + e \cos \varphi},\]

(4)

where \(p\) is the focal parameter of the ellipse and \(e\) is its eccentricity that lies inside the interval \((0, 1)\). For definiteness, we take \(p = 1.8R_E\) and \(e = 0.5\). We will further omit the factor \(R_E\) in the value of \(p\), giving the focal parameter in the radii of the Earth \(R_E\). These values of the parameters will correspond to the orbital apogee altitude \(H = 16565\ \text{km}\) and the perigee altitude \(h = 1274\ \text{km}\). The precipitation region for the selected \(p\) and \(e\) values is shown in Fig. 1.
Figure 1. Precipitation region for the elliptical orbit in the plane of the geographic equator with $p = 1.8$ and $e = 0.5$

As we can see in Fig. 1, the precipitation region is latitude-limited and is simply connected region in this case. The minimum latitude is $-52.8^\circ$, the maximum one is $55.8^\circ$. The different density of filling the region is associated with a strong curvature of the particle trajectories near the Earth’s surface. Numerical analysis shows that the points of the precipitation region fill the entire region inside the outer boundaries outlined in Fig. 1 if the number of source revolutions increases or its intensity increases at a given number of source revolutions.

Considering the case $p = 1.8, e = 0.5$ as the main one, we investigate the influence of the following parameters on the precipitation region structure:

(i) eccentricity of the orbit $e$,
(ii) focal parameter $p$,
(iii) the incline angle of the orbit to the selected axis in the plane of the geographic equator,
(iv) kinetic energy of injected electrons $T_e$.

3. Dependence on the eccentricity of the orbit $e$
To determine the dependence on the eccentricity of the ellipse $e$, the precipitation regions were calculated for the values $e = 0.772$ ($h = 101$ km), and $e = 0.7$ ($h = 375$ km). The corresponding source orbits are shown in Fig. 2. The results of calculating the precipitation region for $e = 0.7$, as well as the the precipitation region boundaries for $e = 0.5, e = 0.7$ and $e = 0.772$ are shown in Fig. 3–4.

4. Dependence on the focal parameter of the ellipse $p$
We will now change the focal parameter of the ellipse $p$, assuming that the eccentricity $e$ is unchanged. It has been proved that there is a critical value $p_0$ that lies inside the interval $[2.1, 2.2]$ for $e = 0.5$ such that for $p > p_0$ the points of the precipitation region become separated by the axis $\psi = 0$ and, thus, the precipitation region becomes two-component (see Fig. 5). Construction of the boundaries at different $p$ shows that the boundaries expand with increasing $p$. It can be noted that the inner ones expand more quickly and the outer ones do it more
Figure 2. Elliptical orbits with different eccentricities $e$: $e = 0.5$, $e = 0.7$, $e = 0.772$

slowly, and this expansion is observed for the entire interval of longitudes. This circumstance is especially remarkable in connection with the fact that an increase in $p$ is accompanied by an increase in the average distance of the orbit to the center of the Earth. It reduces the probability of the trajectory crossing the Earth’s surface.
Figure 4. Dependence of upper (a) and lower (b) boundaries of precipitation regions on the longitude $\phi$ for some values of the eccentricity of the source orbit. The selected values are $e = 0.5$, $e = 0.7$, $e = 0.772$

Figure 5. Precipitation region for $e = 0.5$ and $p = 2.2$

Figure 6. Upper outer boundaries for $e = 0.5$ and different parameters $p$: $p = 1.8$, $p = 2.2$, $p = 2.5$, $p = 3$

5. Dependence on the incline angle
Fig. 7–8 show the precipitation regions and its upper boundaries for an elliptical orbit rotated around the semi-major ellipse axis $x$ by $45^\circ$. It can be seen from the figures that, when the rotation angle increases, a sinus-like component appears in the structure of the precipitation region, as well as in the case of a circular orbit [10]. This component becomes noticeable at incline angles that are large enough. At once the number of points decreases in the region of low latitudes. In other words, there is the predominant penetration of high-energy electrons from strongly inclined orbits into the circumpolar regions.

6. Dependence on the kinetic energy
Let us now turn to the dependence of the precipitation regions on the kinetic energy of injected electrons. For the main orbit with $e = 0.5$, $p = 1.8$, the precipitation region is shown in Fig. 1. The precipitation region for electron with $T_e = 15 \text{ GeV}$ is shown in Fig. 9, and the precipitation region boundaries for energies 7, 15 and 30 GeV are shown in Fig. 10. It can be seen from Fig. 9–10 that, when energy increases, the interval of precipitation latitudes increases. The
influence of a magnetic field on a particle weakens with an increase in its energy, the trajectories of high-energy particles are slightly curved near the dipole.

7. Conclusions
The precipitation regions formed during the movement of a point source in the near-Earth space along an elliptical orbit have been studied. It was found that the interval of latitudes for precipitation’s points increases, if the eccentricity of the ellipse increases. An increase of the focal parameter for a fixed eccentricity leads to a shift of the regions boundaries towards high latitudes, and the displacement of the internal boundaries is faster than the external ones. The existance of non-zero incline angle to the plane of the geographic equator is accompanied by the appearance of a new structure in the precipitation region. This structure becomes especially noticeable at large incline angles. The precipitation region is more sensitive to orbital rotations around the semi-minor axis of the ellipse than around the major one. When the ellipse is rotated at a sufficiently large angle, the precipitation region is divided into two connected components.
An increase of the energy of injected electrons expands the region of precipitation latitudes, making the shape of the precipitation region weakly dependent on energy.

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