High-dimensional data segmentation in regression settings permitting heavy tails and temporal dependence

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May 15, 2023

Abstract

We propose a data segmentation methodology for the high-dimensional linear regression problem where regression parameters are allowed to undergo multiple changes. The proposed methodology, MOSEG, proceeds in two stages: first, the data scanned for multiple change points using a moving window-based procedure, which is followed by a location refinement stage. MOSEG enjoys computational efficiency thanks to the adoption of a coarse grid in the first stage, and achieves theoretical consistency in estimating both the total number and the locations of the change points, under general conditions permitting serial dependence and non-Gaussianity. We also propose MOSEG.MS, a multiscale extension of MOSEG which, while comparable to MOSEG in terms of computational complexity, achieves theoretical consistency for a broader parameter space where large parameter shifts over short intervals and small changes over long stretches of stationarity are simultaneously allowed. We demonstrate good performance of the proposed methods in comparative simulation studies and in an application to predicting the equity premium. R software implementations are available from https://github.com/Dom-Owens-UoB/moseg.

1 Introduction

Regression modelling in high dimensions has received great attention with the development of data collection and storage technologies, and numerous applications are found in natural and social sciences, economics, finance and genomics, to name a few. There is a mature literature on high-dimensional linear regression modelling under the sparsity assumption, see Bühlmann and van de Geer (2011) and Tibshirani (2011) for an overview. When observations are collected over time in highly nonstationary environments, it is natural to allow for shifts in the regression parameters. Permitting the parameters to vary over time in a piecewise constant
manner, data segmentation, a.k.a. multiple change point detection, provides a conceptually simple framework for handling nonstationarity in the data.

In this paper, we consider the problem of multiple change point detection under the following model: We observe $(Y_t, x_t)$, $t = 1, \ldots, n$, with $x_t = (X_{1t}, \ldots, X_{pt})^\top \in \mathbb{R}^p$ where

$$Y_t = \begin{cases} x_t^\top \beta_0 + \varepsilon_t & \text{for } \theta_0 < t \leq \theta_1, \\ x_t^\top \beta_1 + \varepsilon_t & \text{for } \theta_1 < t \leq \theta_2, \\ \vdots & \\ x_t^\top \beta_q + \varepsilon_t & \text{for } \theta_q < t \leq n = \theta_{q+1}. \end{cases}$$ (1)

Here, $\{\varepsilon_t\}_{t=1}^n$ denotes a sequence of errors satisfying $E(\varepsilon_t) = 0$ and $\text{Var}(\varepsilon_t) = \sigma_\varepsilon^2 \in (0, \infty)$ for all $t$, which may be serially correlated. At each change point $\theta_j$, the vector of parameters undergoes a change such that $\delta_j = \beta_j - \beta_{j-1} \neq \beta_j$ for all $j = 1, \ldots, q$. Then, our aim is to estimate the set of change points $\Theta = \{\theta_j, 1 \leq j \leq q\}$ by estimating both the total number $q$ and the locations $\theta_j$ of the change points.

The data segmentation problem under (1) is considered by Bai and Perron (1998), Qu and Perron (2007), Zhao et al. (2022) and Kirch and Reckrühm (2022), among others, when the dimension $p$ is fixed. In high-dimensional settings, when there exists at most one change point ($q = 1$), Lee et al. (2016) and Kaul et al. (2019b) consider the problem of detecting and locating the change point, respectively. For the general case with unknown $q$, several data segmentation methods exist which adopt dynamic programming (Leonardi and Bühlmann, 2016; Rinaldo et al., 2021; Xu et al., 2022), fused Lasso (Wang et al., 2022; Bai and Safikhani, 2022) or wild binary segmentation (Wang et al., 2021) algorithms for the detection of multiple change points, and Bayesian approaches also exist (Datta et al., 2019). A related yet distinct problem of testing for the presence of a single change point under the regression model has been considered in Wang and Zhao (2022) and Liu et al. (2022), and Gao and Wang (2022) consider the case where $\beta_j - \beta_{j-1}$ is sparse without requiring the sparsity of $\beta_j$, $j = 0, \ldots, q$.

Against the above literature background, we list the contributions made in this paper by proposing computationally and statistically efficient data segmentation methods.

(i) **Computational efficiency.** For the data segmentation problem under (1), often the computational bottleneck is the local estimation of the regression parameters via penalised $M$-estimation such as Lasso. We propose MOSEG, a moving window-based two-stage methodology, and its multiscale extension, which are both highly efficient computationally. In the first stage, MOSEG scans the data for multiple change points using a moving window of length $G$ on a coarse grid of size $O(nG^{-1})$, which is followed by a simple location refinement step minimising the local residual sum of squares. The adoption of a coarse grid in the first stage contributes greatly to the reduction of Lasso estimation steps while losing little detection power. Figure 1 demonstrates the compu-
Figure 1: Execution time in seconds of MOSEG and MOSEG.MS and competing methodologies on simulated datasets (y-axis is in log scale for ease of comparison). Left: $p$ varies while $n = 450$ is fixed. Right: $n$ varies while $p = 100$ is fixed. For each setting, 100 realisations are generated and the average execution time is reported. See Section 4.2 for full details.

(ii) **Multiscale change point detection.** We propose a multiscale extension of the single-bandwidth methodology MOSEG. Referred to as MOSEG.MS, it is fully adaptive to the difficult scenarios with multiscale change points, where large frequent parameter shifts and small changes over long stretches of stationarity are simultaneously present, while still enjoying computational competitiveness. To the best of our knowledge, MOSEG.MS is the only data segmentation methodology under the model (1) for which the detection and localisation consistency is derived explicitly for the broad parameter space that permits multiscale change points. Also, while there exist several data segmentation methods that propose to apply moving window-based procedures with multiple bandwidths, MOSEG.MS is the first extension in high dimensions with a guaranteed rate of localisation.

(iii) **Theoretical consistency in general settings.** We show the consistency of MOSEG and MOSEG.MS in estimating the total number and the locations of multiple change points. Under Gaussianity, their separation and localisation rates nearly match the min-max lower bounds up to a logarithmic factor. Moreover, in our theoretical investigation, we permit temporal dependence as well as tail behaviour heavier than sub-Gaussianity. This, compared to the existing literature where independence and (sub-)Gaussianity assumptions are commonly made, shows that the proposed methods work well in situations that are more realistic for empirical applications.

The rest of the paper is organised as follows. Section 2 introduces MOSEG, the single-bandwidth methodology, and establishes its theoretical consistency. Then in Section 3, we...
propose its multiscale extension, MOSEG.MS, and show that it achieves theoretical consistency in a broader parameter space. Numerical experiments in Section 4 demonstrate the competitiveness of the proposed methods in comparison with the existing data segmentation algorithms and Section 5 provides a real data application to equity premium data. In the Appendix, we provide a comprehensive comparison between the existing methods and MOSEG and MOSEG.MS both on their theoretical and computational properties, and present all the proofs and additional numerical results. The R software implementing MOSEG and MOSEG.MS is available from https://github.com/Dom-Owens-UoB/moseg.

**Notation.** For a random variable $X$, we write $\|X\|_\nu = \left(\mathbb{E}(|X|^\nu)\right)^{1/\nu}$ for $\nu > 0$. For $a = (a_1, \ldots, a_p)^\top \in \mathbb{R}^p$, we write $\text{supp}(a) = \{i, 1 \leq i \leq p : a_i \neq 0\}$, $|a|_0 = \sum_{i=1}^p I_{\{a_i \neq 0\}}$, $|a|_1 = \sum_{i=1}^p |a_i|$, $|a|_2 = (\sum_{i=1}^p a_i^2)^{1/2}$ and $|a|_\infty = \max_{1 \leq i \leq p} |a_i|$. For a square matrix $A$, let $\Lambda_{\text{max}}(A)$ and $\Lambda_{\text{min}}(A)$ denote its maximum and minimum eigenvalues, respectively. For a set $A$, we denote its cardinality by $|A|$. For sequences of positive numbers $\{a_n\}$ and $\{b_n\}$, we write $a_n \preceq b_n$ if there exists some constant $C > 0$ such that $a_n / b_n \leq C$ as $n \to \infty$. Finally, we write $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$.

# 2 Single-bandwidth methodology

We introduce MOSEG, a single-bandwidth two-stage methodology for data segmentation in regression settings. We first describe its two stages in Section 2.1, establish its theoretical consistency in Section 2.2 and verify meta-assumptions made for the theoretical analysis in Section 2.3 for a class of linear processes with serial dependence and heavier tails than that permitted under sub-Gaussianity.

## 2.1 MOSEG

### 2.1.1 Stage 1: Moving window procedure on a coarse grid

Single-bandwidth moving window procedures have successfully been adopted for univariate (Preuss et al., 2015; Yau and Zhao, 2016; Eichinger and Kirch, 2018), multivariate (Kirch and Reckrühm, 2022) and high-dimensional (Cho et al., 2023) time series segmentation. Often in a moving window-based data segmentation procedure, the key challenge is to carefully design a detector statistic which, when adopted for scanning the data for changes, has good detection power against the type of changes which is of interest to detect.

For a given bandwidth $G \in \mathbb{N}$ satisfying $G \leq n/2$, our proposed detector statistic is

$$ T_k(G) = \sqrt{G} \left| \hat{\beta}_{k,k+G} - \hat{\beta}_{k-G,k} \right|_2, \quad G \leq k \leq n-G. \quad (2) $$
Here, $\hat{\beta}_{s,e}$ denotes an estimator of the vector of parameters obtained from $(Y_t, x_t)$, $s + 1 \leq t \leq e$, for any $0 \leq s < e \leq n$. The statistic $T_k(G)$ contrasts the local parameter estimators from two adjacent data sections over $\{k - G + 1, \ldots, k\}$ and $\{k + 1, \ldots, k + G\}$. Then, $T_k(G)$ is expected to form local maxima near the change points where the local parameter estimators differ the most, and thus it is well-suited for detecting and locating the change points under the model (1).

We propose to obtain the local estimator $\hat{\beta}_{s,e}$ via Lasso, as

$$\hat{\beta}_{s,e}(\lambda) = \arg \min_{\beta \in \mathbb{R}^p} \sum_{t=s+1}^{e} (Y_t - x_t^\top \beta)^2 + \lambda \sqrt{e-s} |\beta|_1$$

for some tuning parameter $\lambda > 0$. In what follows, we suppress the dependence of this estimator on $\lambda$ when there is no confusion. The estimand of $\hat{\beta}_{k-G,k}$ is

$$\beta^\ast_{k-G,k} = \frac{1}{G} \sum_{j=L(k-G+1)}^{L(k)} \{(\theta_j + 1 \wedge k) - ((k - G) \vee \theta_j)\} \beta_j,$$

where $L(t) = \{j, 0 \leq j \leq q : \theta_j + 1 \leq t\}$ denotes the index of a change point $\theta_j$ that is the closest to $t$ while lying strictly to its left. In short, $\beta^\ast_{k-G,k}$ is a weighted sum of $\beta_j$ with the weights corresponding to the proportion of the intervals $\{k - G + 1, \ldots, k\}$ overlapping with $\{\theta_j + 1, \ldots, \theta_{j+1}\}$.

Scanning the detector statistic $T_k(G)$ over all $k \in \{G, \ldots, n - G\}$ requires the computation of the Lasso estimator $O(n)$ times. This is far fewer than $O(n^2)$ times required by dynamic programming algorithms for $\ell_0$-penalised cost minimisation [Rinaldo et al. 2021; Xu et al. 2022], but it may still pose a computational bottleneck when the data sequence is very long or its dimensionality ultra high. Instead, we propose to evaluate $T_k(G)$ on a coarser grid only for generating pre-estimators of the change points. Let $\mathcal{T}$ denote the grid over which we evaluate $T_k(G)$, which is given by

$$\mathcal{T} = \mathcal{T}(r, G) = \left\{G + \lfloor rG \rfloor m, 0 \leq m \leq \left\lfloor \frac{n - 2G}{rG} \right\rfloor \right\}$$

with some constant $r \in (G^{-1}, 1)$ that controls the coarseness of the grid. When $r = G^{-1}$, we have the finest grid $\mathcal{T} = \{G, \ldots, n - G\}$ and the grid becomes coarser with increasing $r$.

Motivated by Eichinger and Kirch (2018), who considered the problem of detecting multiple shifts in the mean of univariate time series using a moving window procedure, we propose to accept all significant local maximisers of $T_k(G)$ over $k \in \mathcal{T}$ as the pre-estimators of the change points. That is, for some threshold $D > 0$ and a tuning parameter $\eta \in (0, 1]$, we accept all
\( \tilde{\theta} \in T \) that simultaneously satisfy

\[
T_{\tilde{\theta}}(G) > D \quad \text{and} \quad \tilde{\theta} = \arg \max_{\theta \in T : |\theta - \tilde{\theta}| \leq \eta G} T_{\theta}(G). \tag{6}
\]

That is, at such \( \tilde{\theta} \), the detector \( T_{\tilde{\theta}}(G) \) exceeds the threshold and attains a local maximum over the interval of length \( \eta G \). We denote the set collecting all pre-estimators fulfilling (6), by \( \tilde{\Theta} = \{ \tilde{\theta}_j : 1 \leq j \leq \hat{q} : \tilde{\theta}_1 < \ldots < \tilde{\theta}_{\hat{q}} \} \) with \( \hat{q} = |\tilde{\Theta}| \) as the estimator of the number of change points. This grid-based approach substantially reduces the computational complexity by requiring the Lasso estimators to be computed only \( O(n/\lfloor r G \rfloor) \) times. Even so, it is sufficient for detecting the presence of all \( q \) change points, provided that \( r \) is chosen not too large (see Theorem 1 (i) below). We remark that the idea of utilising only a sub-sample of the data for detecting the presence of change points, has been proposed for univariate mean change point detection in [Lu et al. (2017)]. The next section describes the location refinement step applied to the pre-estimators of change point locations.

2.1.2 Stage 2: Location refinement

Once the set of pre-estimators \( \tilde{\Theta} \) is generated by the first-stage moving window procedure on a coarse grid, we further refine the location estimators. It involves the local evaluation and minimisation of the following objective function

\[
Q(k; a, b, \tilde{\gamma}^L, \tilde{\gamma}^R) = \sum_{t=a+1}^{k} (Y_t - x_t^\top \tilde{\gamma}^L)^2 + \sum_{t=k+1}^{b} (Y_t - x_t^\top \tilde{\gamma}^R)^2 \quad \text{for} \quad k = a + 1, \ldots, b, \tag{7}
\]

for suitably chosen \( a, b, \tilde{\gamma}^L \) and \( \tilde{\gamma}^R \). A similar idea has been considered for location refinement in the change point literature, see e.g. [Kaul et al. (2019b) and Xu et al. (2022)].

For each \( j = 1, \ldots, \hat{q} \), let \( \tilde{\theta}_j^L = \tilde{\theta}_j - \lfloor G/2 \rfloor \) and \( \tilde{\theta}_j^R = \tilde{\theta}_j + \lfloor G/2 \rfloor \), and consider the following local parameter estimators

\[
\hat{\beta}_j^L = \hat{\beta}_{0\vee(\tilde{\theta}_j^L - G), \tilde{\theta}_j^L} \quad \text{and} \quad \hat{\beta}_j^R = \hat{\beta}_{0\wedge(\tilde{\theta}_j^R + G), \tilde{\theta}_j^R}, \tag{8}
\]

which serve as the estimators of \( \beta_{j-1} \) and \( \beta_j \), respectively. Then in Stage 2, we propose to obtain a refined location estimator of \( \theta_j \) from its pre-estimator \( \tilde{\theta}_j \), as

\[
\hat{\theta}_j = \arg \min_{\theta_j - G + 1 \leq k \leq \theta_j + G} Q\left(k; \tilde{\theta}_j - G, \tilde{\theta}_j + G, \hat{\beta}_j^L, \hat{\beta}_j^R\right), \tag{9}
\]

for all \( j = 1, \ldots, \hat{q} \). Referring to the methodology combining the two stages as MOSEG, we provide its algorithmic description in Algorithm 1 of Appendix D.

6
2.2 Consistency of MOSEG

To establish the consistency of MOSEG, we make the following assumptions on \((x_t, \varepsilon_t), 1 \leq t \leq n\). Assumption 1 is commonly made in the literature on high-dimensional regression and change point problems thereof.

Assumption 1. We assume that \(E(x_t) = 0, E(\varepsilon_t) = 0\) and \(\text{Var}(\varepsilon_t) = \sigma^2\) for all \(t = 1, \ldots, n\), and that \(\text{Cov}(x_t) = \Sigma_x\) has its eigenvalues bounded, i.e. there exist \(0 \leq \omega \leq \bar{\omega} < \infty\) such that

\[
\omega \leq \Lambda_{\text{min}}(\Sigma_x) \leq \Lambda_{\text{max}}(\Sigma_x) \leq \bar{\omega}.
\]

Assumptions 2 and 3 below extend the deviation bound and restricted strong convexity (RSC) conditions required for high-dimensional \(M\)-estimation (van de Geer and Bühlmann [2009] Loh and Wainwright [2012] Negahban et al. [2012]), to change point settings. They are met e.g. by a class of linear processes accommodating serial dependence and non-Gaussian tail behaviour, as verified later in Section 2.3.

We explicitly state these meta-assumptions to highlight that the consistency of MOSEG derived in this section is not limited to such processes only.

Assumption 2 (Deviation bound). There exist fixed constants \(C_0, C_{\text{DEV}} > 0\) and some \(\rho_{n,p} \to \infty\) as \(n, p \to \infty\), such that \(P(\mathcal{D}^{(1)} \cap \mathcal{D}^{(2)}) \to 1\), where

\[
\mathcal{D}^{(1)} = \left\{ \max_{0 \leq s < e \leq n, e - s \geq C_0 \rho_{n,p}^2} \left| \frac{1}{\sqrt{e - s}} \sum_{t=s+1}^{e} \varepsilon_t x_t \right|_{\infty} \leq C_{\text{DEV}} \rho_{n,p} \right\},
\]

\[
\mathcal{D}^{(2)} = \left\{ \max_{0 \leq s < e \leq n, e - s \geq C_0 \rho_{n,p}^2} \left| \frac{1}{\sqrt{e - s}} \sum_{t=s+1}^{e} (y_t - x_t^T \beta_{s,e}) x_t \right|_{\infty} \leq C_{\text{DEV}} \rho_{n,p} \right\}.
\]

Assumption 3 (Restricted strong convexity). There exist fixed constants \(C_{\text{RSC}} > 0\) and \(\tau \in [0, 1)\) such that \(P(\mathcal{R}^{(1)} \cap \mathcal{R}^{(2)}) \to 1\), where

\[
\mathcal{R}^{(1)} = \left\{ \sum_{t=s+1}^{e} a^T x_t x_t^T a \geq (e - s) \omega |a|_2^2 - C_{\text{RSC}} \log(p)(e - s)^{\tau} |a|_1^2 \text{ for all } 0 \leq s < e \leq n \text{ satisfying } e - s \geq C_0 \rho_{n,p}^2 \text{ and } a \in \mathbb{R}^p \right\},
\]

\[
\mathcal{R}^{(2)} = \left\{ \sum_{t=s+1}^{e} a^T x_t x_t^T a \leq (e - s) \bar{\omega} |a|_2^2 + C_{\text{RSC}} \log(p)(e - s)^{\tau} |a|_1^2 \text{ for all } 0 \leq s < e \leq n \text{ satisfying } e - s \geq C_0 \rho_{n,p}^2 \text{ and } a \in \mathbb{R}^p \right\}.
\]

For each \(j = 0, \ldots, q\), we denote by \(S_j = \text{supp}(\beta_j)\) the support of \(\beta_j\), and by \(s = \max_{0 \leq j \leq q} |S_j|\) the maximum segment-wise sparsity of the regression parameters. We make the following assumptions on the size of change \(\delta_j = |\beta_j - \beta_{j-1}|_2\) and the spacing between the neighbouring change points through imposing conditions on \(G\).
Assumption 4. There exists some constant $C_0 > 0$ such that $\max_{1 \leq j \leq q} \delta_j \leq C_0$.

Assumption 5. The bandwidth $G$ fulfils the following conditions with $\tau$, $\rho_{n,p}$ and $\omega$ introduced in Assumptions 1, 2 and 3

(a) $2G \leq \min_{1 \leq j \leq q+1} (\theta_j - \theta_{j-1})$.

(b) There exists a fixed constant $C_1 > 0$ such that

$$
\min_{1 \leq j \leq q} \delta_j^2 G \geq C_1 \max \left\{ \omega^{-2} s^{-2}_{p,n}, \left( \omega^{-1} s \log(p) \right)^{1/(1-\tau)} \right\}.
$$

Assumption 4 is a technical condition under which we focus on the more challenging regime where the size of change is allowed to tend to zero; an analogous condition found in Lee et al. (2016), Kaul et al. (2019b), Wang et al. (2021) and Xu et al. (2022). In particular, it rules out the case where $\text{Var}(Y_t)$ diverges for some $t$, since $\text{Var}(Y_t) \geq s \sum_{j=0}^q |\beta_j| \mathbb{1}_{\theta_j+1 \leq t \leq \theta_j} + \sigma^2$. Assumption 5(a) relates the choice of bandwidth $G$ to the minimum spacing between the change points. Together, (a) and (b) specify the separation rate imposing a lower bound on $\Delta^{(1)} = \min_{1 \leq j \leq q} \delta_j^2 \cdot \min_{0 \leq j \leq q} (\theta_{j+1} - \theta_j)$, for all the $q$ change points to be detectable by MOSEG. Later in Section 3, we propose a multiscale extension of MOSEG which achieves consistency under a more relaxed condition than Assumption 5.

Theorem 1. Suppose that Assumptions 1, 2, 3, 4 and 5 hold. Let the tuning parameters satisfy $\lambda \geq 4C_{\text{DEV}} \rho_{n,p}$, $r \in [1/G, 1/4]$, $\eta \in (4r, 1]$ and

$$
\frac{48 \sqrt{s} \lambda}{\omega} < D < \frac{\eta}{4 \sqrt{2}} \min_{1 \leq j \leq q} \delta_j \sqrt{G}.
$$

Then on $\mathcal{D}^{(1)} \cap \mathcal{D}^{(2)} \cap \mathcal{R}^{(1)} \cap \mathcal{R}^{(2)}$, the following holds.

(i) Stage 1 of MOSEG returns $\hat{\Theta} = \{ \hat{\theta}_j, 1 \leq j \leq \hat{q} : \hat{\theta}_1 < \ldots < \hat{\theta}_{\hat{q}} \}$ which satisfies

$$
\hat{q} = q \quad \text{and} \quad |\hat{\theta}_j - \theta_j| \leq \frac{48 \sqrt{2s} G \lambda}{\omega \delta_j} + |rG| < \left[ \frac{G}{2} \right] \quad \text{for each} \quad j = 1, \ldots, q.
$$

(ii) There exists a large enough constant $c_0 > 0$ such that Stage 2 of MOSEG returns $\hat{\Theta} = \{ \hat{\theta}_j, 1 \leq j \leq \hat{q} : \hat{\theta}_1 < \ldots < \hat{\theta}_{\hat{q}} \}$ which satisfies

$$
\max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0 \max \left( \rho_{n,p}^2, (s \log(p))^{1/(1-\tau)} \right).
$$

Theorem 1(i) establishes that Stage 1 of MOSEG correctly estimates the number of change points as well as identifying their locations by the pre-estimators with some accuracy. There
is a trade-off between computational efficiency and theoretical consistency with respect to the choice of $r$. On one hand, increasing $r$ leads to a coarser grid $T$ with its cardinality $|T| = O(n/(rG))$, and thus reduces the computational cost. On the other, the pre-estimators lie in the grid such that the best approximation to each change point $\theta_j$ can be as far from $\theta_j$ as $|rG|/2$, which is reflected on the localisation property of the pre-estimators. Theorem 1(ii) derives the rate of estimation for the second-stage estimators $\hat{\theta}_j$ which shows that the location estimation is more challenging when the size of change $\delta_j$ is small. Finally, we always have $\max_{1 \leq j \leq q} \delta_j^{-2} \max(s_\rho^2_{n,p}, (s \log(p))^{1/(1-\gamma)}) \lesssim G \lesssim \min_{1 \leq j \leq q+1}(\theta_j - \theta_{j-1})$ under Assumption 4.

2.3 Verification of Assumptions 2 and 3

Assumptions 2 and 3 generalise the deviation bound and the RSC condition which are often found in the high-dimensional M-estimation literature, to accommodate change points, serial dependence and heavy-tailedness. Condition 1 gives instances of $\{(x_t, \varepsilon_t)\}_{t=1}^n$ that fulfil Assumptions 2 and 3 and specify the corresponding $\rho_{n,p}$ and $\tau$.

**Condition 1.** Suppose that for i.i.d. random vectors $\xi_t = (\xi_{t1}, \ldots, \xi_{tp+1}, t \in \mathbb{Z}$, with $E(\xi_t) = 0$ and $Cov(\xi_t) = I$, we have

$$\begin{bmatrix} x_t \\ \varepsilon_t \end{bmatrix} = \sum_{\ell = 0}^{\infty} D_t \xi_{t-\ell} \quad \text{with} \quad D_t = [D_{t,ik}, 1 \leq i, k \leq p+1] \in \mathbb{R}^{(p+1) \times (p+1)}$$

subject to $E(x_t \varepsilon_t) = 0$. Further, there exist constants $\Xi > 0$ and $\zeta > 2$ such that

$$|D_{t,ik}| \leq C_{ik}(1 + \ell)^{-\zeta} \quad \text{with} \quad \max\left\{ \max_{1 \leq k \leq p+1} \sum_{i=1}^{p+1} C_{ik}, \max_{1 \leq i \leq p+1} \sum_{k=1}^{p+1} C_{ik} \right\} \leq \Xi$$

for all $\ell \geq 0$. Finally, we impose either of the two conditions on $\xi_t$.

(a) There exist some constants $C_\xi > 0$ and $\gamma \in (0, 2]$ such that $(E(|\xi_t|^\nu))^{1/\nu} = \|\xi_t\|_\nu \leq C_\xi \nu^{\gamma}$ for all $\nu \geq 1$. In other words, $\|\xi_t\|_\nu := \sup_{\nu \geq 1} \nu^{-1/\gamma} ||\xi_t||_\nu \leq C_\xi$.

(b) $\xi_t \sim \text{iid } \mathcal{N}(0, 1)$.

**Proposition 2.** Suppose that Assumptions 1 and 4 and Condition 1 hold. Then, there exist some constants $c_1, c_2 > 0$ such that $P(\mathcal{D}^{(1)} \cap \mathcal{D}^{(2)} \cap \mathcal{R}^{(1)} \cap \mathcal{R}^{(2)}) \geq 1 - c_1(p \vee n)^{-c_2}$, with $\omega = \Lambda_{\min}(\Sigma_x)/2$, $\bar{\omega} = 3\Lambda_{\max}(\Sigma_x)/2$, and $\tau$ and $\rho_{n,p}$ chosen as below.

(i) Under Condition 1(a) we set $\tau = (4\gamma + 2)/(4\gamma + 3)$ and $\rho_{n,p} = \log^{2\gamma+3/2}(p \vee n)$.

(ii) Under Condition 1(b) we set $\tau = 0$ and $\rho_{n,p} = \sqrt{\log(p \vee n)}$.

Under Condition 1, $\{(x_t, \varepsilon_t)\}_{t=1}^n$ is a linear process with algebraically decaying serial dependence. Also, Condition 1(a) permits heavier tail behaviour than that allowed under sub-Gaussianity or sub-exponential distributions when $\gamma > 1/2$ and $\gamma > 1$, respectively.
Remark 1. The consistency of Lasso-type estimator (when $q = 0$) under serial dependence and non-Gaussianity, has been investigated under functional dependence or mixing conditions ([Wu and Wu 2016; Adamek et al. 2020; Han and Tsay 2020; Wong et al. 2020]). In the change point literature, [Wang and Zhao 2022] propose a change point test and investigate its properties under $\beta$-mixing, and [Xu et al. 2022] analyse the $\ell_0$-penalised least squares estimation approach when the functional dependence of $\{x_t\}_{t=1}^n$ and $\{\varepsilon_t\}_{t=1}^n$ decays exponentially. Relaxing the Gaussianity, it is typically required that $x_t$ is a sub-Weibull random vector, i.e. $\sup_{a \in B_2(1)} \|a^\top x_t\|_{\psi_\gamma} < \infty$ for some $\gamma > 0$ (where $B_d(r) = \{a : |a|_d \leq r\}$) and similarly, $\|\varepsilon_t\|_{\psi_\gamma} < \infty$. Under these assumptions, the common approach is to verify the deviation bound and RSC conditions analogous to those made in Assumptions 2–3, with which the consistency of the Lasso estimator is derived (locally in the case of the change point detection problem). Instead, we explicitly state the meta-assumptions and give Condition 1 as one scenario under which these assumptions are met. The proposed MOSEG achieves consistency in multiple change point detection as shown in Theorem 1 whenever Assumptions 2–3 are met, which can be verified using the arguments adopted in the aforementioned literature.

Corollary 3 follows immediately from Theorem 1 and Proposition 2.

**Corollary 3.** Suppose that Assumptions 1, 4 and 5 and Condition 1 hold, and $\lambda, r, \eta$ and $D$ are chosen as in Theorem 1. Then, there exist constants $c_i > 0$, $i = 0, 1, 2$, such that $\hat{\Theta} = \{\hat{\theta}_j, 1 \leq j \leq \hat{q} : \hat{\theta}_1 < \ldots < \hat{\theta}_{\hat{q}}\}$ returned by MOSEG satisfies the following.

(i) Under Condition 1 (a) we have

$$P\left(\hat{q} = q \text{ and } \max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0(s \log(p \vee n))^{4\gamma + 3}\right) \geq 1 - c_1(p \vee n)^{-c_2}.$$

(ii) Under Condition 1 (b) we have

$$P\left(\hat{q} = q \text{ and } \max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0 s \log(p \vee n)\right) \geq 1 - c_1(p \vee n)^{-c_2}.$$

Corollary 3(ii) shows that under Gaussianity, the rate of localisation attained by MOSEG matches the minimax lower bound up to $\log(p \vee n)$, see Lemma 4 of Rinaldo et al. (2021). At the same time, Assumption 5(b) translates to $\Delta^{(1)} \gtrsim s \log(p \vee n)$ in this setting, nearly matching the minimax lower bound on the separation rate derived in Lemma 3 of Rinaldo et al. (2021) up to the logarithmic term.

3 Multiscale methodology

The single-bandwidth methodology proposed in Section 2 enjoys theoretical consistency as well as computational efficiency, but faces the difficulty arising from identifying a bandwidth
that satisfies Assumption (a)–(b) simultaneously. In this section, we propose MOSEG.MS, a multiscale extension of MOSEG, and show that it achieves consistency in a parameter space broader than that allowed by Assumption 3 and thus alleviates the difficulty associated with the choice of a single bandwidth.

3.1 MOSEG.MS: Multiscale extension of MOSEG

Similarly to MOSEG, MOSEG.MS consists of moving window-based data scanning and location refinement but it takes a set of bandwidths as an input. The key innovation lies in that for each change point, MOSEG.MS learns the bandwidth best-suited for its detection and localisation from the given set of bandwidths. While there exist multiscale extensions of moving sum procedures, they are mostly developed for univariate time series segmentation (Messer et al., 2014; Cho and Kirch, 2021b) and to the best of our knowledge, this is a first attempt at rigorously studying such an extension in a high-dimensional setting. Below we describe MOSEG.MS step-by-step.

Step 1: Pre-estimator generation. Given a set of bandwidths \( G = \{G_h, 1 \leq h \leq H : G_1 < \ldots < G_H\} \), we generate the coarse grid associated with each \( G_h \) and the parameter \( r \) by \( T_h = T(r, G_h) \), see (5). As in Stage 1 of MOSEG, the sets of pre-estimators \( \hat{\Theta}(G_h) \) are generated for \( h = 1, \ldots, H \), and we denote by \( \hat{\Theta}(G) = \bigcup_{h=1}^{H} \hat{\Theta}(G_h) \) the pooled set of all such pre-estimators. By (5), at each \( \hat{\theta} \in \hat{\Theta}(G_h) \), we have \( T_{\hat{\theta}}(G_h) > D \) and \( \hat{\theta} = \arg \max_{k \in I_{\tilde{\eta}}(\hat{\theta}) \cap T_h} T_k(G_h) \), where \( I_{\tilde{\eta}}(\hat{\theta}) = \{\hat{\theta} - \lfloor \eta G_h \rfloor + 1, \ldots, \hat{\theta} + \lfloor \eta G_h \rfloor\} \) denotes the detection interval associated with \( \hat{\theta} \). For simplicity, we write \( I_1(\hat{\theta}) = I(\hat{\theta}) \). Below, we sometimes write \( T(\hat{\theta}) \in \hat{\Theta}(G) \) to highlight that the pre-estimator is obtained with the bandwidth \( G \), and denote by \( G(\hat{\theta}) \) the bandwidth involved in the detection of a pre-estimator \( \hat{\theta} \). If some \( \hat{\theta} \) is detected with more than one bandwidths, we distinguish between them.

Step 2: Anchor estimator identification. Next, we identify anchor change point estimators \( \hat{\theta}^A(G) \in \hat{\Theta}(G) \) detected at some \( G \in \mathcal{G} \) which satisfy

\[
\bigcup_{h: G_h < G} \bigcup_{k \in \hat{\Theta}(G_h)} \left\{ I(k) \cap I(\hat{\theta}^A(G)) \right\} = \emptyset. \tag{14}
\]

That is, each anchor change point estimator does not have its detection interval overlap with the detection interval of any pre-estimator that is detected with a finer bandwidth. Denote the set of all such anchor change point estimators by \( \hat{\Theta}^A = \{\hat{\theta}^A_j, 1 \leq j \leq \hat{q} : \hat{\theta}^A_1 < \ldots < \hat{\theta}^A_{\hat{q}}\} \), with \( \hat{q} = |\hat{\Theta}^A| \) as an estimator of the number of change points \( q \).

Step 3: Pre-estimator clustering. We find subsets of the pre-estimators in \( \hat{\Theta}(G) \) denoted by \( \mathcal{C}_j, j = 1, \ldots, \hat{q} \), as described below. Initialised as \( \mathcal{C}_j = \emptyset \), for each \( j \), we add to \( \mathcal{C}_j \) the \( j \)th
anchor estimator $\hat{\theta}_j^A$ as well as all $\tilde{\theta} \in \tilde{\Theta}(G)$ which simultaneously fulfil
\[
I(\hat{\theta}) \cap I(\hat{\theta}_j^A) \neq \emptyset, \quad \text{and}
\{\hat{\theta} - G(\hat{\theta}) - \lfloor G(\hat{\theta})/2 \rfloor + 1, \ldots, \hat{\theta} + G(\hat{\theta}) + \lfloor G(\hat{\theta})/2 \rfloor \} \cap I(\hat{\theta}_j^A) = \emptyset \quad \text{for all } j' \neq j. \tag{15}
\]

**Step 4: Location refinement.** For each $C_j$, $j = 1, \ldots, q$, we denote the smallest and the largest bandwidths associated with the detection of the pre-estimators in $C_j$, by $G_j^m$ and $G_j^M$, respectively, and the corresponding pre-estimators by $\hat{\theta}_j^m$ and $\hat{\theta}_j^M$ (when $|C_j| = 1$, we have $\hat{\theta}_j^m = \hat{\theta}_j^M = \hat{\theta}_j^A$ and $G_j^m = G_j^M$). Setting $G_j^* = \lfloor 3G_j^m / 4 + G_j^M / 4 \rfloor$, we identify the local minimiser of the objective function defined in (7), as
\[
\hat{\theta}_j = \arg \min_{\delta_j^m - G_j^*} \delta_j^m + 1 \leq \delta_j^m + G_j^* Q \left( k; \delta_j^m - G_j^*, \delta_j^m + G_j^*, \beta_j^y, \beta_j^r \right), \tag{16}
\]
with $\beta_j^y = \beta(\delta_j^m - G_j^*) \bigcap [0, \delta_j^m - G_j^m)$ and $\beta_j^r = \beta(\delta_j^m + G_j^* \bigcap [0, \delta_j^m + G_j^m]) \bigcap n$.

Repeatedly performing (16) for $j = 1, \ldots, q$, we obtain $\tilde{\Theta} = \{\tilde{\theta}_j, 1 \leq j \leq \tilde{q}\}$.

An algorithmic description of MOSEG.MS is given in Algorithm 2 of Appendix D. The identification of anchor change point estimators bears some resemblance with the bottom-up merging proposed in Messer et al. [2014], but the anchor estimators do not come with a guaranteed rate of localisation. Instead, we cluster the pre-estimators and learn the bandwidth $G_j^*$ well-suited for localising each $\tilde{\theta}_j$ in a data-driven way, with which we obtain a refined estimator.

**Remark 2** (Bandwidth generation). Cho and Kirch [2021b] propose to use $G$ generated as a sequence of Fibonacci numbers, for a multiscale extension of the moving sum procedure proposed in Eichinger and Kirch [2018] in the context of univariate mean change point detection. For some finest bandwidth $G_0 = G_1$, we iteratively produce $G_h$, $h \geq 2$, as $G_h = G_{h-1} + G_{h-2}$. Equivalently, we set $G_h = F_h G_0$ where $F_h = F_{h-1} + F_{h-2}$ with $F_0 = F_1 = 1$. This is repeated until for some $H$, it holds that $G_H < [n/2]$ while $G_{H+1} \geq [n/2]$. By induction, $F_h = O(((1+\sqrt{2})/2)^h)$ such that the thus-generated bandwidth set $G$ satisfies $|G| = O(\log(n))$.

### 3.2 Consistency of MOSEG.MS

We make the following assumption on the size of change $\delta_j$ and the spacing between the neighbouring change points.

**Assumption 5.** Let $G$ denote the set of bandwidths generated as in Remark 2 with $G_1 \geq C_0 \max \{\rho_{n,p}^2, (\omega^{-1}s \log(p))^{1/(1-\tau)}\}$. Then, for each change point $\theta_j$, $j = 1, \ldots, q$, there exists a bandwidth $G_{(j)} \in G$ such that
\[
(a) \quad 4G_{(j)} \leq \min(\theta_j + 1 - \theta_j, \theta_j - \theta_{j-1}), \quad \text{and}
\]
\[
(b) \quad \delta_j^2 G_{(j)} \geq 4C_1 \max \left\{\omega^{-2}\rho_{n,p}^2, (\omega^{-1}s \log(p))^{1/(1-\tau)}\right\} \quad \text{with } C_1 \text{ from Assumption 5}
\]
If there are multiple such bandwidths, let \( G_{(j)} \) denote the smallest one.

Assumption 5′ relaxes Assumption 5 by requiring that for each \( \theta_j \), there exists one bandwidth \( G_{(j)} \in \mathcal{G} \) fulfilling the requirements imposed on a single bandwidth in the latter for all \( j = 1, \ldots, q \). Assumption 5′ effectively places a condition on

\[
\Delta^{(2)} = \min_{1 \leq j \leq q} \delta_j^2 \cdot \min(\theta_{j+1} - \theta_j, \theta_j - \theta_{j-1})
\]  

(17)

for MOSEG.MS to detect all \( q \) changes. Compared to \( \Delta^{(1)} \) defined in (10), we always have \( \Delta^{(1)} \leq \Delta^{(2)} \) and, if frequent large changes and small changes over long stretches of stationarity are simultaneously present, the former can be considerably smaller than the latter, see also the discussion in Cho and Kirch (2021a). To the best of our knowledge, Theorem 4 below provides a first result obtained under the larger parameter space defined with \( \Delta^{(2)} \), in establishing the consistency of a data segmentation methodology for the problem in (1). We refer to Appendix A for further discussions and comprehensive comparison between MOSEG, MOSEG.MS and competing methodologies.

**Theorem 4.** Suppose that Assumptions 1, 2, 3, 4 and 5′ hold. Let the tuning parameters satisfy \( \lambda \geq 4C_{\text{DEV}} n, r \in \left[ G^{-1}, 1/4 \right] \), \( \eta \in (4r, 1] \) and

\[
\frac{48\sqrt{\delta} \lambda}{\omega} < D < \frac{\eta}{4\sqrt{2}} \min_{1 \leq j \leq q} \delta_j \sqrt{G_{(j)}},
\]  

(18)

Then, there exists a constant \( c_0 > 0 \) such that on \( \mathcal{D}^{(1)} \cap \mathcal{D}^{(2)} \cap \mathcal{R}^{(1)} \cap \mathcal{R}^{(2)} \), MOSEG.MS returns \( \hat{\Theta} = \{ \hat{\theta}_j, 1 \leq j \leq \hat{q} : \hat{\theta}_1 < \ldots < \hat{\theta}_{\hat{q}} \} \) which satisfies

\[
\hat{q} = q \quad \text{and} \quad \max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0 \max \left( s^2 n, (s \log(p))^{1+\gamma} \right).
\]

**Corollary 5.** Suppose that Assumptions 1, 4 and 5′ and Condition 1 hold, and \( \lambda, r \) and \( D \) are chosen as in Theorem 4. Then, there exist constants \( c_i > 0, i = 0, 1, 2 \), such that \( \hat{\Theta} = \{ \theta_j, 1 \leq j \leq \hat{q} : \hat{\theta}_1 < \ldots < \hat{\theta}_{\hat{q}} \} \) returned by MOSEG.MS satisfies the following.

(i) Under Condition 1(a) we have

\[
P \left( \hat{q} = q \quad \text{and} \quad \max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0 (s \log(p \lor n))^{4+3\gamma} \right) \geq 1 - c_1 (p \lor n)^{-c_2}.
\]

(ii) Under Condition 1(b) we have

\[
P \left( \hat{q} = q \quad \text{and} \quad \max_{1 \leq j \leq q} \delta_j^2 |\hat{\theta}_j - \theta_j| \leq c_0 s \log(p \lor n) \right) \geq 1 - c_1 (p \lor n)^{-c_2}.
\]
4 Numerical experiments

4.1 Choice of tuning parameters

We discuss the selection of tuning parameters involved in MOSEG and MOSEG.MS, namely the set of bandwidths $G$, the grid $T(r, G)$ in [3], $\eta \in (0, 1]$ involved in the pre-estimation of the change points (see [6]), the penalty parameter $\lambda$ and the threshold $D$.

Selection of $G$. The set of bandwidths $G$ is determined once the finest bandwidth $G_1$ is chosen. To gain insights about the minimum sample size requirement for the Lasso estimator, we performed numerical experiments by simulating datasets under (1) with $q = 0$ and varying $(n, p, s, G)$, 100 realisation for each setting, and recorded the relative $\ell_2$-error $\max_{0 \leq k \leq n-G} |\beta_0|_2^{-1} |\tilde{\beta}_{k,k+G} - \beta_0|_2$ for each realisation. Then, regressing the 90%-percentile of the estimation errors over 100 realisations onto $\log(G)$, $\log\log(p)$ and $\log\log(n)$ ($R^2 = 0.8945$), we obtain a simple rule to determine the finest bandwidth as $G_1 = G_1(n, p) = [c_i^* \exp(c_i^* \log\log(n) + c_i^* \log\log(p))]$ with pre-specified $c_i^* > 0$, $i = 0, 1, 2$. Adopting the Fibonacci rule in Remark 2 often gives a sequence of bandwidths that grows too quickly when the sample size $n$ is small. Therefore, with the finest bandwidth $G_1$ chosen as above, we recommend generating bandwidths as $G_h = [(h+2)G_1/3]$ for $h \geq 2$. Throughout the simulation studies and real data applications, we set $H = 3$.

Selection of $D$ and $\lambda$. Theorems 1 and 4 provide ranges of values for $\lambda$ and $D$ for theoretical consistency, but they involve unknown parameters as is typically the case in the change point literature. For their simultaneous selection, we adopt a cross validation (CV) method motivated by [Zou et al., 2020]. Let $\Lambda$ denote the grid of possible values for $\lambda$, which is chosen as an exponentially increasing sequence from $10^{-3} \lambda_{\text{max}}$ up to $\lambda_{\text{max}}$ with $\lambda_{\text{max}} = \max_{0 \leq k \leq n-G} |\sum_{t=k+1}^{k+G} x_t Y_t|_\infty / \sqrt{G}$ the smallest value with which we obtain $\tilde{\beta}_{k,G} = 0$ for all $0 \leq k \leq n-G$. For given $G \in G$ and $\lambda \in \Lambda$, we generate $\hat{\Theta}(G, \lambda) = \{\hat{\theta}_j(G, \lambda), 1 \leq j \leq \tilde{q}_0(G, \lambda)\}$, the set of pre-estimators with $D = 0$, i.e. we take all local maximisers of the MOSUM statistics according to [6]; due to the detection rule, we always have $\tilde{q}_0(G, \lambda) \leq n/(2\eta G)$.

Sorting the elements of $\hat{\Theta}(G, \lambda)$ in the decreasing order of the associated MOSUM detector values, we generate a sequence of nested change point models

$$\emptyset = \hat{\Theta}_{[0]}(G, \lambda) \subset \hat{\Theta}_{[1]}(G, \lambda) \subset \ldots \subset \hat{\Theta}_{[\tilde{q}_0(G, \lambda)]}(G, \lambda) = \hat{\Theta}(G, \lambda).$$

Then, using the odd-indexed observations $(Y_t, x_t)$, $t \in J_1 = \{2t + 1, t = 0, \ldots, \lfloor (n-1)/2 \rfloor\}$, we produce local estimators of the regression parameters and the even-indexed observations $(Y_t, x_t)$, $t \in J_0 = \{1, \ldots, n\} \setminus J_1$, is used for validation. Specifically, we evaluate $\text{CV}(G, \lambda, m) =$
RSS\(_0(\tilde{\Theta}_{[m]}(G, \lambda), \lambda)\), where for any \(\mathcal{L} = \{\ell_j, 1 \leq \ell_j \leq L : 0 = \ell_0 < \ell_1 < \ldots < \ell_L < \ell_{L+1} = n\},\)

\[
\text{RSS}_0(\mathcal{L}, \lambda) = \sum_{j=0}^{L} \sum_{t \in J_0 \cap \{\ell_j+1, \ldots, \ell_{j+1}\}} \left( Y_t - x_{t}^{\top} \tilde{\beta}_j^{(1)}(\mathcal{L}, \lambda) \right)^2 .
\]

Here, \(\tilde{\beta}_j^{(1)}(\mathcal{L}, \lambda)\) denotes the Lasso estimator obtained using \((Y_t, x_t), t \in J_1 \cap \{\ell_j, \ldots, \ell_{j+1}\}\) with the penalty parameter \(\lambda\). Then for each \(G_h \in \mathcal{G}\), we find

\[
(\lambda^*, m^*) = \arg\min_{(\lambda, m): \lambda \in \Lambda, \ 0 \leq m \leq \tilde{q}_0(G_h, \lambda)} \text{CV}(G_h, \lambda, m)
\]

and obtain the set of pre-estimators \(\tilde{\Theta}(G_h) = \tilde{\Theta}_{[m^*]}(G_h, \lambda^*)\) using \(\lambda^*\) and \(m^*\). This amounts to selecting the bandwidth-dependent threshold \(D\) at a value just below the \(m^*\)th largest MOSUM detector value. Such \(\tilde{\Theta}(G_h)\), \(G_h \in \mathcal{G}\), serve as an input to Steps 2–4 of MOSEG.MS.

In all numerical experiments reported in this paper, we set \(|\Lambda| = 5|\).

### Selection of other tuning parameters.
For change point estimation, we recommend to use \(\eta = 0.5\) in (6) based on extensive simulations, which show that the performance of MOSEG and MOSEG.MS is not too sensitive to its choice. As noted in Section 4.2, MOSEG.MS is highly competitive computationally against the existing methods even without adopting a coarse grid. Therefore, we report the results obtained with \(r = G^{-1}\) (i.e. \(T = \{G, \ldots, n - G\}\) in (5) ) in the main text and provide the results obtained with \(r = 1/10\) in Appendix B.1, where we observe that adopting a coarse grid does not undermine the performance of MOSEG.

### 4.2 Computational complexity and run time

Let Lasso\((p)\) denote the cost of solving a Lasso problem with \(p\) variables. For the coordinate descent algorithm (Friedman et al., 2010), each complete iteration of the coordinate descent has the cost \(O(p^2)\). Then, the combined computational cost of Stages 1 and 2 of MOSEG is \(O(n(rG)^{-1}\text{Lasso}(p))\), and the memory cost is \(O(np)\). Similarly, with the set of bandwidths generated as described in Remark 2, the complexity of the multiscale extension MOSEG.MS is \(O(n(rG_1)^{-1}\text{Lasso}(p))\) with \(G_1\) denoting the finest scale, which follows from that \(\sum_{h=1}^{H} n/(rG_h) \leq n/(rG_1) \sum_{h=1}^{\infty} F_h^{-1} = O(n(rG_1)^{-1})\) (see Remark 2 for the notations). The CV outlined in Section 4.1, we generate pre-estimators and evaluate the CV objective function on a sequence of nested models for each \(\lambda \in \Lambda\), which brings the computational complexity of the complete MOSEG.MS methodology to \(O(|\Lambda|n(rG_1)^{-1}\text{Lasso}(p))\).

We investigate the run time of change point detection methodologies for the problem in (1).

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\(^1\)All numerical work reported in this paper was carried out using the computational facilities of the Advanced Computing Research Centre at the University of Bristol.
in Section 4.1 and the finest grid (i.e. \( T = \{ G, \ldots, n - G \} \)), and we include the CV procedure in run time. For comparison, we consider VPWBS \cite{Wang2021}, DPDU \cite{Xu2022} and ARBSEG \cite{Kaul2019a} applied with the recommended tuning parameters. In particular, VPWBS and DPDU adopt a grid of size 3 for the Lasso tuning parameter while we use the set \( \Lambda \) with \(|\Lambda| = 5\). We generate the data as described in the model (M3) in Section 4.3 below, with \( \delta = 1.6 \) and varying \((n, p)\). Figure 1 reports the average execution time (in seconds) over 100 realisations for each setting, for the five methods in consideration. In the left panel, we fix \( n = 450 \) while varying \( p \in \{80, 100, \ldots, 220\} \) and in the right, we fix \( p = 100 \) while varying \( n \in \{240, 300, \ldots, 660\} \). Both MOSEG and MOSEG.MS take only a fraction of time taken by the competing methodologies in their computation even without the use of the coarse grid, and their run time does not vary much with increasing \( n \) or \( p \) in the ranges considered. As expected, MOSEG is faster than MOSEG.MS but the difference in execution time is much smaller than that between MOSEG.MS and other competitors.

4.3 Simulation settings

We apply MOSEG.MS to datasets simulated with varying \((n, p, s)\) and change point configurations. In each setting, we generate \( x_t \) as i.i.d. Gaussian random vectors with mean \( 0 \) and the covariance matrix \( \Sigma_x \) which are specified below, and \( \varepsilon_t \sim_{\text{iid}} \mathcal{N}(0, \sigma^2) \); unless specified otherwise, we use \( \sigma = 1 \). We report the results from non-Gaussian and serially dependent data in Appendix B.2 where overall, the results are not sensitive to tail behaviour or temporal dependence. While we consider \( p \geq 100 \) in the main text, we report the results when \( p = 1000 \) in Appendix B.3 which, together with Section 4.2, demonstrate the scalability of MOSEG.MS. The models (M1)-(M3) below are taken from \cite{Wang2021}; in (M2) we adapt their model by randomly generating the set \( S \) on each realisation while in (M3), we consider a broader range of values for \( \delta \). In what follows, we assume that for given \( S \subset \{1, \ldots, p\} \) with \(|S| = s\), the parameter vector \( \beta_0 = (\beta_{0,1}, \ldots, \beta_{0,p})^\top \in \mathbb{R}^p \) has \( \beta_{0,i} \neq 0 \) for \( i \in S \) and \( \beta_{0,i} = 0 \) otherwise, i.e. \( S \) is the support of \( \beta_0 \). For each setting, we generate 100 realisations.

(M1) Setting \( p = 100, q = 3 \) and \( \Sigma_x = I \), we vary \( n \in \{480, 560, 640, 720, 800\} \) and the change points are located at \( \theta_j = jn/4, j = 1, 2, 3 \). Fixing \( S = \{1, \ldots, s\} \) with \( s = 4 \), we set \( \beta_{0,i} = 0.4 \cdot (-1)^{i-1} \) for \( i \in S \) and \( \beta_j = (-1)^j \cdot \beta_0 \).

(M2) We set \( n = 300, p = 100 \) and \( q = 2 \), and \( \Sigma_x = \left[0.6|s_i-s_{i'}|\right]_{i,i'=1}^{p,q} \). The change points are located at \( \theta_j = jn/3, j = 1, 2 \), and we vary \( s \in \{10, 20, 30\} \). For each realisation, we randomly draw \( S \subset \{1, \ldots, p\} \) of size \( s \), and set \( \beta_{0,i} = 1/\sqrt{4s} \) for \( i \in S \), \( \beta_j = (-1)^j \cdot \beta_0 \).

(M3) We have \( n = 300, p = 100, q = 2, s = 10 \) and \( \Sigma_x = \left[0.6|s_i-s_{i'}|\right]_{i,i'=1}^{p,q} \). The change points are located at \( \theta_j = jn/3 \) and fixing \( S = \{1, \ldots, s\} \), we set \( \beta_{0,i} = \delta \cdot (-1)^{i-1} \) for \( i \in S \) with varying \( \delta \in \{0.2, 0.4, 0.8, 1.6\}/\sqrt{5} \), and \( \beta_j = (-1)^j \cdot \beta_0 \).
(M4) We set $n = 840$, $p = 50$, $q = 5$, $s = 10$ and $\Sigma_x = I$. The change points are located at $\theta_1 = 60$, $\theta_2 = 120$, $\theta_3 = 240$, $\theta_4 = 360$ and $\theta_5 = 600$ and fixing $S = \{1, \ldots, s\}$, we set $\beta_{0,i} = \delta \cdot (-1)^{i-1}$ for $i \in S$ with varying $\delta \in \{0.2, 0.4, 0.8, 1.6\}/\sqrt{5}$, and $\beta_1 = -\beta_2 = -2\beta_0$, $\beta_3 = -\beta_4 = -\sqrt{2}\beta_0$ and $\beta_5 = -\beta_6 = -\beta_0$.

(M5) The data is generated as in (M3) except for that $q = 0$, $\Sigma_x = [10 \cdot 0.6^{|i-i'|}p_{i,i'}=1]$ and $\sigma_\varepsilon = 10$, and we use $\delta \in \{1, 1.2, 1.4, 1.6\}$.

In setting (M4) the change points are multiscale in the sense that the size of change and spacing between the change points vary, but $\delta^2 \cdot \min(\theta_{j+1} - \theta_j, \theta_j - \theta_{j-1})$ is kept constant for $j = 1, 3, 5$ and $j = 2, 4$, respectively. This results in $\Delta^{(1)}$ in (10) being much smaller than $\Delta^{(2)}$ in (17). (M5) is designed to test the performance of data segmentation methods when $q = 0$, where we scale the data to examine the sensitivity of the tuning parameter choices discussed in Section 4.1.

4.4 Simulation results

We apply MOSEG.MS with the tuning parameters selected as described in Section 4.1. For the purpose of illustration only, we also apply MOSEG with the bandwidth chosen with the knowledge of the minimum spacing between the change points; for (M1)–(M3) where change points are evenly spaced, we set $G = 3/4 \cdot \min_{0 \leq j \leq q}(\theta_{j+1} - \theta_j)$. For (M4) with multiscale change points, there does not exist a single bandwidth that works well in detecting all change points so we simply set $G = 125$. For (M5) with $q = 0$, we set $G = G_1$ selected as described in Section 4.1. For comparison, we apply the VPWBS method proposed by Wang et al. (2021) with the default tuning parameters recommended by the authors, which shows better performance than the methods proposed in Leonardi and Bühlmann (2016) and Lee et al. (2016). We also considered the methods proposed by Kaul et al. (2019a) and Xu et al. (2022) but generally they performed poorly for the simulation models considered in this paper and we omit the results from these methods.

In Tables 1–4, we report the distribution of the bias in change point number estimation ($\hat{q} - q$) for each method over the 100 realisations generated under each setting. Additionally, we report the scaled Hausdorff distance between the sets of estimated ($\hat{\Theta}$) and true ($\Theta$) change points, i.e.

$$d_H(\hat{\Theta}, \Theta) = \frac{1}{n} \max \left\{ \max_{\hat{\theta} \in \hat{\Theta}} \min_{\theta \in \Theta} |\hat{\theta} - \theta|, \max_{\theta \in \Theta} \min_{\hat{\theta} \in \hat{\Theta}} |\hat{\theta} - \theta| \right\},$$

averaged over 100 realisations; by convention, we set $D(\emptyset, \Theta) = 1$. In Table 5 (considering the case $q = 0$), we report the proportion of realisations where any false positive is returned.

Generally, as expected, we observe better performance from all methods with increasing sample size in (M1) or increasing change size with $\delta$ in (M3)–(M4) while varying the sparsity level $s$. 
brings in less clear change in the performance. In the presence of homogeneous change points under \([M1], [M3]\) MOSEG performs as well as MOSEG.MS in terms of correctly estimating the number of change points, but it suffers from the lack of adaptivity in the presence of multiscale change points under \([M4]\) where both large frequent shifts and small changes over long intervals are present. Here, we observe the benefit of using multiple bandwidths by MOSEG.MS as \(\delta\) grows, where it achieves better accuracy in detection and localisation against MOSEG and VPWBS. Comparing the performance of MOSEG.MS and VPWBS, we note that the former generally attains better detection power while the latter exhibits better localisation properties under \([M2]\) and \([M3]\) (when \(\delta\) is large). We remark that the Hausdorff distance tends to favour the cases when the change points are over-detected, than when they are under-detected. Under \([M5]\) where no changes are present, our methods are shown to control the number of false positives well. Here, we do not include VPWBS in Table 5 as it tends to detect false positives in most cases.

Table 1: \([M1]\) Performance of MOSEG, MOSEG.MS and VPWBS over 100 realisations. The best performer in each setting is denoted in bold.

| n    | Method   | -3 | -2 | -1 | 0 | 1  | 2  | \(\geq 3\) | \(d_H\) |
|------|----------|----|----|----|---|----|----|------------|--------|
| 480  | MOSEG    | 3  | 3  | 6  | 81| 7  | 0  | 0          | 0.0852 |
|      | MOSEG.MS | 1  | 6  | 7  | 84| 2  | 0  | 0          | 0.0710 |
|      | VPWBS    | 1  | 3  | 14 | 58| 16 | 5  | 3          | 0.0795 |
| 560  | MOSEG    | 3  | 3  | 5  | 72| 17 | 1  | 0          | 0.0742 |
|      | MOSEG.MS | 0  | 1  | 5  | 93| 1  | 0  | 0          | 0.0299 |
|      | VPWBS    | 1  | 0  | 10 | 73| 5  | 8  | 3          | 0.0579 |
| 640  | MOSEG    | 1  | 3  | 5  | 64| 23 | 4  | 0          | 0.0652 |
|      | MOSEG.MS | 0  | 1  | 2  | 91| 6  | 0  | 0          | 0.0203 |
|      | VPWBS    | 0  | 1  | 3  | 89| 3  | 2  | 2          | 0.0291 |
| 720  | MOSEG    | 1  | 3  | 1  | 76| 18 | 1  | 0          | 0.0433 |
|      | MOSEG.MS | 0  | 0  | 0  | 97| 3  | 0  | 0          | 0.0104 |
|      | VPWBS    | 0  | 0  | 1  | 92| 3  | 3  | 1          | 0.0190 |
| 800  | MOSEG    | 2  | 3  | 7  | 61| 25 | 2  | 0          | 0.0753 |
|      | MOSEG.MS | 0  | 0  | 0  | 100| 0  | 0  | 0          | 0.0073 |
|      | VPWBS    | 0  | 0  | 2  | 92| 3  | 2  | 1          | 0.0202 |

5 Real data application

There exists an extensive literature on the prediction of the equity premium, which is defined as the difference between the compounded return on the S&P 500 index and the three month Treasury bill rate. Using 14 macroeconomic and financial variables (see Table E.1 for full descriptions), Welch and Goyal (2008) demonstrate the difficulty of this prediction problem, in part due to the time-varying nature of the data. Koo et al. (2020) note that the majority of the variables are highly persistent with strong, positive autocorrelations, and develop an \(\ell_1\)-
Table 2: Performance of MOSEG, MOSEG.MS and VPWBS over 100 realisations. The best performer in each setting is denoted in bold.

| s  | Method  | $\hat{q} - q$ | $d_H$ |
|----|--------|--------------|-----|
| 10 | MOSEG  | 27 28 35 10 0 0 | 0.4204 |
|    | MOSEG.MS | 11 13 | 44 13 0 0 | 0.3117 |
|    | VPWBS | 45 17 11 9 15 3 | **0.2465** |
| 20 | MOSEG  | 14 31 | 50 5 0 0 | 0.3016 |
|    | MOSEG.MS | 8 32 | 48 12 0 0 | 0.2726 |
|    | VPWBS | 44 13 18 13 9 3 | **0.2302** |
| 30 | MOSEG  | 14 25 | 50 11 0 0 | 0.2765 |
|    | MOSEG.MS | 11 30 | 41 18 0 0 | 0.2848 |
|    | VPWBS | 24 20 33 9 9 5 | **0.1843** |

Table 3: Performance of MOSEG, MOSEG.MS and VPWBS over 100 realisations. The best performer in each setting is denoted in bold.

| $\sqrt{105}$ | Method  | $\hat{q} - q$ | $d_H$ |
|--------------|--------|--------------|-----|
| 0.2 | MOSEG | 12 19 63 6 0 0 | 0.2367 |
|    | MOSEG.MS | 4 16 | 64 15 1 0 | **0.1609** |
|    | VPWBS | 77 9 5 6 1 2 | 0.3025 |
| 0.4 | MOSEG | 7 9 | 81 3 0 0 | **0.1401** |
|    | MOSEG.MS | 3 21 | 71 5 0 0 | 0.1488 |
|    | VPWBS | 53 20 12 8 4 3 | 0.2681 |
| 0.8 | MOSEG | 6 10 | 82 2 0 0 | 0.1242 |
|    | MOSEG.MS | 2 14 | 77 6 1 0 | 0.1099 |
|    | VPWBS | 53 20 12 8 4 3 | **0.1061** |
| 1.6 | MOSEG | 3 5 | 91 1 0 0 | 0.0732 |
|    | MOSEG.MS | 0 10 | 88 2 0 0 | 0.0737 |
|    | VPWBS | 1 1 84 10 4 0 | **0.0404** |

penalised regression method that identifies co-integration relationships among the variables. Accordingly, we transform the data by taking the first difference of any variable labelled as being persistent by Koo et al. (2020), and scale each covariate series to have unit standard deviation. With the thus-transformed variables, we propose to model the monthly equity premium observed from 1927 to 2005 as $Y_t$, with the 14 variables at lags 1, 2, 3 and 12 as regressors $x_t$ via piecewise stationary linear regression; in total, we have $n = 936$ and $p = 57$ including the intercept.

We apply MOSEG.MS with $G = \{72, 96, 120\}$ in line with the choice described in Section 4.1 but we select $G_h$ to be multiples of 12 for interpretability as the observation frequency is monthly. MOSEG.MS returns $\hat{q} = 7$ change point estimators reported in Table 5 and takes 45 seconds in total (including CV). When applied to the same dataset, DPDU takes 25 minutes and VPWBS takes 15 minutes, and neither detects any change point. In Figure 2 we plot the local parameter estimates obtained from each of the seven estimated segments. We can relate the change detected in 1954 to the findings reported in Rapach et al. (2010), where they
Table 4: Performance of MOSEG, MOSEG.MS and VPWBS over 100 realisations. The best performer in each setting is denoted in bold.

| $\sqrt{10}\delta$ | Method | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $\geq 3$ | $d_H$ |
|-------------------|--------|------|------|------|-----|-----|-----|---------|-------|
| 0.2               | MOSEG | 45   | 6    | 4    | 5   | 21  | 9   | 10      | 0.4518|
|                   | MOSEG.MS | 17   | 9    | 19   | 12  | 15  | 8   | 20      | **0.2263**|
|                   | VPWBS  | 95   | 1    | 2    | 1   | 1   | 0   | 0       | 0.4073|
| 0.4               | MOSEG | 44   | 7    | 10   | 10  | 8   | 5   | 16      | 0.4347|
|                   | MOSEG.MS | 15   | 12   | 16   | **17** | 7   | 16  | 17      | **0.2015**|
|                   | VPWBS  | 73   | 2    | 7    | 6   | 7   | 5   | 0       | 0.3247|
| 0.8               | MOSEG | 4    | 23   | 31   | 27  | 9   | 5   | 1       | 0.1978|
|                   | MOSEG.MS | 0    | 3    | 34   | **38** | 15  | 9   | 1       | **0.0834**|
|                   | VPWBS  | 13   | 40   | 29   | 11  | 4   | 2   | 1       | 0.1165|
| 1.6               | MOSEG | 0    | 7    | 45   | 43  | 3   | 2   | 0       | 0.0970|
|                   | MOSEG.MS | 0    | 1    | 32   | **59** | 7   | 1   | 0       | **0.0387**|
|                   | VPWBS  | 3    | 35   | 38   | 19  | 1   | 3   | 1       | 0.0900|

Table 5: Proportions of detecting false positives when $q = 0$ for MOSEG and MOSEG.MS over 100 realisations.

| $\delta$ | Method | 1 | 1.2 | 1.4 | 1.6 |
|----------|--------|---|-----|-----|-----|
| 1.2      | MOSEG | 0.05 | 0.01 | 0.01 | 0.02 |
|          | MOSEG.MS | 0.04 | 0.01 | 0.01 | 0.02 |

attribute the instability in the pairwise relationships between the equity premium and each of the 14 variables to the Treasury-Federal Reserve Accord and the transition from the wartime economy. Dividend price ratio ($d/p$, at lag two) is active throughout the observation period which agrees with the observations made in [Welch and Goyal (2008)](Welch and Goyal (2008)). They also remark that the recession from 1973 to 1975 due to the Oil Shock drives the good predictive performance of many models proposed for equity premium forecasting, and most perform poorly over the 30 year period (1975–2005) following the Oil Shock. The two last segments defined by the change point estimators reported in Table 6 are closely located with these important periods, which supports the validity of the segmentation returned by MOSEG.MS. We note that regardless of the choice of bandwidths, both of the two estimators in 1974 and 1975 defining the two periods are detected separately.

Table 6: Equity premium data: Change point estimators detected by MOSEG.MS.

| Estimator | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\hat{\theta}_5$ | $\hat{\theta}_6$ | $\hat{\theta}_7$ |
|-----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| Date      | Oct 1935         | Apr 1943         | Aug 1951         | Nov 1954         | Nov 1958         | May 1974         | Aug 1975         |
6 Conclusions

In this paper, we propose MOSEG, a high-dimensional data segmentation methodology for detecting multiple changes in the parameters under a linear regression model. It proceeds in two steps, first scanning the data for large changes in local parameter estimators over a moving window, followed by a computational efficient location refinement step. We further propose its multiscale extension, MOSEG.MS, which alleviates the necessity to select a single bandwidth. Both numerically and theoretically, we demonstrate the efficiency of the proposed methodologies. Computationally, they are highly competitive thanks to the careful design of the algorithms that limit the required number of Lasso estimators. Theoretically, we show the consistency of MOSEG and MOSEG.MS in a general setting permitting serial dependence and heavy tails and establish their (near-)minimax optimality under Gaussianity. In particular, the consistency of MOSEG.MS is derived for a parameter space that simultaneously permits large changes over short intervals and small changes over long stretches of stationarity, which is much broader than that typically adopted in the literature. Comparative simulation studies and findings from the application of MOSEG.MS to equity premium data support its efficacy.

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A Literature review and comparison with the existing methods

Table A.1 provides an overview of the theoretical properties of MOSEG and MOSEG.MS in comparison with the methods proposed in [Wang et al. (2021), Kaul et al. (2019a) and Xu et al. (2022)] for the change point problem in [1] under Gaussianity, as well as their computational complexity. For a given methodology, let $\hat{\Delta}$ denote the set of estimated change points. When the magnitude of change $\Delta$, measured by either

$$\Delta^{(1)} = \min_{1 \leq j \leq q} \delta_j^2 \cdot \min_{0 \leq q_j \leq q} (\theta_{j+1} - \theta_j) \quad \text{or} \quad \Delta^{(2)} = \min_{1 \leq j \leq q} \delta_j^2 \min_{0 \leq q_j \leq q} (\theta_j - \theta_{j-1}, \theta_{j+1} - \theta_j),$$

diverges faster than the separation rate $s_{n,p}$ associated with the method, all $q$ changes are detected by $\hat{\Delta}$ with asymptotic power one and their locations are consistently estimated with the rate $\ell_{n,p}$, such that $\min_{1 \leq j \leq q} \min_{k \in \hat{\Delta}} w_j |k - \theta_j| = O_P(\ell_{n,p})$. Here, $w_j$ refers to the relative difficulty in locating $\theta_j$ which is related to the jump size $\delta_j$.

Table A.1: Comparison of data segmentation methods developed for the model (1) in their theoretical properties under Gaussianity and computational complexity (for given tuning parameters). Here, $s = \max_{0 \leq j \leq q} |S_j|$ and $S = |\bigcup_{j=0}^{q} S_j|$. See the text for the definitions of $s_{n,p}$, $\ell_{n,p}$, $\Delta$ and $w_j$.

| Method                  | Separation $s_{n,p}$ Complexity | Localisation $\ell_{n,p}$ Complexity | Computational Complexity |
|-------------------------|---------------------------------|--------------------------------------|--------------------------|
| MOSEG                   | $s \log(p \vee n)$ $\Delta^{(1)}$ | $s \log(p \vee n)$ $\delta_j$ | $O(\frac{n}{\tau_{\ell_0}} \cdot \text{Lasso}(p))$ |
| MOSEG.MS                | $s \log(p \vee n)$ $\Delta^{(2)}$ | $s \log(p \vee n)$ $\delta_j$ | $O(\frac{n}{\tau_{\ell_0}} \cdot \text{Lasso}(p))$ |
| Wang et al. (2021)      | $s \log(p \vee n)$ $\Delta^{(1)}$ | $s \log(p)$ $\delta_j$ | $O(n \log^2(n) \cdot \text{GroupLasso}(p))$ |
| Kaul et al. (2019a)     | $s \log(p \vee n)$ $\Delta^{(1)}$ | $s \log(p)$ $\delta_j$ | $O(\tilde{q} \cdot \text{Lasso}(p) + \text{SA}(\tilde{q}))$ |
| Xu et al. (2022)        | $s \log(p \vee n)$ $\Delta^{(1)}$ | $s \log(p \vee n)$ $\delta_j$ | $O(n^2p + n^2 \cdot \text{Lasso}(p))$ |

[Wang et al. (2021)] propose a method which learns the projection that is well-suited to reveal a change over each local segment and combines it with the wild binary segmentation algorithm [Fryzlewicz (2014)] for multiple change point detection. [Kaul et al. (2019a)] propose to minimise an $\ell_0$-penalised cost function given a set of candidate estimators of size $\tilde{q}$. Their theoretical analysis implicitly assumes that $\min_j (\theta_{j+1} - \theta_j)$ scales linearly in $n$, and the simulated annealing adopted for minimising the penalised cost, denoted by SA($\tilde{q}$) in Table A.1, has complexity ranging from $O(\tilde{q}^4)$ on average to being exponential in the worst case. [Xu et al. (2022)] investigate the dynamic programming algorithm of [Rinaldo et al. (2021)] for minimising an $\ell_0$-penalised cost function in a more general setting. In Table A.1 we report the separation and localisation rates derived in [Xu et al. (2021)] for the pre-estimators from the dynamic programming algorithm; in their proposal, the pre-estimators are further refined and their exact minimax optimality is established under a stronger condition on the size of changes, namely that $\Delta^{(1)}/(s^2 \log^3(pn)) \to \infty$.

We also mention [Zhang et al. (2015)] where the data segmentation problem is treated as a
high-dimensional regression problem with a group Lasso penalty, which only provides that the
estimation bias is of $o_P(n)$. Leonardi and Bühlmann (2016) consider both dynamic program-
ing and binary segmentation algorithms are considered for change point estimation, and we
refer to Rinaldo et al. (2021) for a detailed discussion on their results.

From Table A.1 we conclude that MOSEG.MS is highly competitive both computationally
and statistically. In specifying the properties of Kaul et al. (2019a), the global sparsity $S = |igcup_{j=0}^q S_j|$ can be much greater than the segment-wise sparsity $s$, particularly when the number
of change points $q$ is large. We investigate the theoretical properties of MOSEG.MS in the
broadest parameter space possible which is formulated with $\Delta^{(2)}$ instead of $\Delta^{(1)}$ as in all the
other papers; recall that from the discussion following (17) comparing $\Delta^{(l)}$, $l = 1, 2$, we always
have $\Delta^{(1)} \leq \Delta^{(2)}$ and the former can be much smaller than the latter when large shifts over
short intervals and small changes over long stretches of stationarity are simultaneously present
in the signal.

Besides, the theoretical properties of MOSEG.MS reported in Table A.1 do not require in-
dependence unlike other works (with the exception of Xu et al. (2022)), and extend beyond
i.i.d. sub-Gaussianity. In the presence of serial dependence and sub-Weibull tails (through
having $\gamma > 1$ as in Condition 1(a), Xu et al. (2022) require that $\Delta^{(1)} \gtrsim (s \log(np))^{4\gamma + 2\gamma' - 1}$
for the detection of all change points, where a smaller value of $\gamma' \in (0, \infty)$ imposes a faster
decay of the serial dependence. This is comparable to the detection boundary of MOSEG,
$\Delta^{(1)} \gtrsim (s \log(np))^{4\gamma + 3}$ which is implied by Assumption 5 (a) under Condition 1 (a). We
remark that Condition 1 assumes algebraically decaying serial dependence whereas $\gamma'$ of Xu
et al. (2022) governs the rate of exponentially decaying serial dependence. The localisation
rate in Corollary 3 (i) is also comparable to that attained by the preliminary estimators of
Xu et al. (2022) produced by a dynamic programming algorithm; as noted above, under a
stronger condition on $\Delta^{(1)}$, they derive a further refined rate.
B Additional simulations

B.1 Choice of the grid

We compare the change point estimators obtained in Stages 1 and 2 of MOSEG when the finest grid \((r = G^{-1})\) and a coarse grid \((r = 1/10)\) \(T(r, G)\) are used in Stage 1, see \(\text{[5]}\). For this, we set \(n = 300, p = 100, s = 2, q = 1\), and randomly generate \(x_t \sim_{\text{iid}} N_p(0, I)\) and \(\varepsilon_t \sim_{\text{iid}} N(0, 1)\). The change point \(\theta_1\) is randomly sampled from \(\{51, \ldots, 250\}\) and varying \(\delta \in \{0.1, 0.2, 0.4, 0.8\}\), we generate \(\beta_0 = (\beta_{0,1}, \ldots, \beta_{0,p})^\top\) with \(\beta_{0,i} = \delta \cdot (-1)^{i-1}\) for \(i \in \{1, \ldots, s\}\) and have \(\beta_1 = -\beta_0\). Using \(G = 50\), we select the maximiser of the MOSUM statistic as the initial estimator \(\hat{\theta}_1\), which then is refined as in \(\text{[9]}\). Table B.1 reports the average and the standard error of estimation errors over 100 realisations when different grids are used. We observe that regardless of the coarseness of the grid, the refinement step performed in Stage 2 improves the localisation performance. At the same time, there is little loss in estimation accuracy attributable to the increasing coarseness of the grid regardless of \(\delta\).

Table B.1: Comparison of \(d_H\) for Stage 1 and Stage 2 estimators from MOSEG when different grids are used. The average and the standard error of estimation errors over 100 realisations are reported.

| \(\delta\) | \(r = G^{-1}\) | \(r = 1/10\) |
|---|---|---|
| \(\delta\) | Stage 1 | Stage 2 | Stage 1 | Stage 2 |
| 0.1 | 0.2197 | 0.1597 | 0.2141 | 0.1655 | 0.2238 | 0.1447 | 0.2153 | 0.1484 |
| 0.2 | 0.1795 | 0.1622 | 0.1472 | 0.1712 | 0.1876 | 0.1767 | 0.1696 | 0.1770 |
| 0.4 | 0.0102 | 0.0133 | 0.0045 | 0.0073 | 0.0135 | 0.0212 | 0.0053 | 0.0101 |
| 0.8 | 0.0053 | 0.0051 | 0.0010 | 0.0018 | 0.0072 | 0.0080 | 0.0010 | 0.0017 |

B.2 Heavy-tailedness and temporal dependence

We examine the performance of MOSEG.MS and VPWBS \(\text{[Wang et al., 2021]}\) in the presence of heavy-tailed noise and temporal dependence. For this, we generate datasets with \(n = 300, p = 100, s = 10, q = 1\) and the two change points are located at \(\theta_j = jn/3, j = 1, 2\). We use \(\beta_0\) obtained as in Appendix \(\text{[B.1]}\) with \(\delta \in \{0.2, 0.4, 0.8, 1.6\}\) and set \(\beta_j = (-1)^{j} \cdot \beta_0\). MOSEG.MS is applied with the recommended bandwidth set and the CV-based model selection discussed in Section \(\text{[4.1]}\). We consider the following three settings for the generation of \(x_t\) and \(\varepsilon_t\).

(E1) \(x_t \sim_{\text{iid}} N_p(0, I)\) and \(\varepsilon_t \sim_{\text{iid}} N(0, 1)\) for all \(t\).

(E2) \(X_{it} \sim_{\text{iid}} \sqrt{3/5} \cdot t_5\) for all \(i\) and \(t\) and \(\varepsilon_t \sim_{\text{iid}} \sqrt{3/5} \cdot t_5\) for all \(t\).

(E3) \(\{x_t, \varepsilon_t\}_{t=1}^n\) is generated as in \(\text{[12]}\) where \(D_1\) is a diagonal matrix with 0.3 on its diagonals, \(D_{\ell} = \mathbf{0}\) for \(\ell \geq 2\) and \(\xi_t \sim_{\text{iid}} N_{p+1}(0, \sqrt{1 - 0.3^2} I)\) for all \(t\).
Under (E2) (E3) the data is permitted to be heavy-tailed and serially correlated, respectively; (E1) serves as a benchmark. Table B.2 reports the average and standard error of the Hausdorff distance in (19) and \( \hat{q} - q \) over 100 realisations. It shows that generally, neither method is sensitive to heavy-tailedness or temporal dependence. VPWBS shows good localisation performance, while MOSEG.MS tends to achieve better detection accuracy when the size of change is small.

Table B.2: Performance of MOSEG.MS and VPWBS under (E1)–(E3) over 100 realisations. The best performer in each setting is denoted in bold.

| \( \delta \) | Setting | Method | \(-2\) | \(-1\) | \(0\) | \(1\) | \(2\) | \(\geq 3\) | \(d_H\) |
|---|---|---|---|---|---|---|---|---|---|
| 0.2 | (E1) | MOSEG.MS | 10 | 35 | \textbf{46} | 9 | 0 | 0 | 0.2905 |
| | | VPWBS | 56 | 10 | 9 | 14 | 9 | 2 | \textbf{0.2583} |
| | (E2) | MOSEG.MS | 6 | 53 | \textbf{34} | 6 | 1 | 0 | 0.2779 |
| | | VPWBS | 60 | 10 | 7 | 13 | 10 | 0 | \textbf{0.2663} |
| | (E3) | MOSEG.MS | 8 | 30 | \textbf{44} | 17 | 1 | 0 | 0.2644 |
| | | VPWBS | 10 | 15 | 18 | 13 | 28 | 16 | \textbf{0.1671} |
| 0.4 | (E1) | MOSEG.MS | 0 | 9 | 86 | 4 | 1 | 0 | 0.0766 |
| | | VPWBS | 1 | 3 | \textbf{87} | 7 | 2 | 0 | \textbf{0.0361} |
| | (E2) | MOSEG.MS | 1 | 11 | \textbf{83} | 5 | 0 | 0 | 0.0830 |
| | | VPWBS | 1 | 7 | \textbf{83} | 4 | 5 | 0 | \textbf{0.0567} |
| | (E3) | MOSEG.MS | 1 | 9 | \textbf{81} | 9 | 0 | 0 | 0.0678 |
| | | VPWBS | 0 | 1 | 80 | 14 | 4 | 1 | \textbf{0.0397} |
| 0.8 | (E1) | MOSEG.MS | 0 | 0 | \textbf{99} | 1 | 0 | 0 | 0.0119 |
| | | VPWBS | 0 | 0 | 97 | 3 | 0 | 0 | \textbf{0.0095} |
| | (E2) | MOSEG.MS | 0 | 0 | \textbf{98} | 2 | 0 | 0 | \textbf{0.0100} |
| | | VPWBS | 0 | 0 | \textbf{98} | 2 | 0 | 0 | 0.0104 |
| | (E3) | MOSEG.MS | 0 | 1 | 96 | 3 | 0 | 0 | 0.0153 |
| | | VPWBS | 0 | 0 | \textbf{98} | 2 | 0 | 0 | \textbf{0.0103} |
| 1.6 | (E1) | MOSEG.MS | 0 | 0 | 97 | 3 | 0 | 0 | 0.0097 |
| | | VPWBS | 0 | 0 | \textbf{100} | 0 | 0 | 0 | \textbf{0.0037} |
| | (E2) | MOSEG.MS | 0 | 0 | \textbf{100} | 0 | 0 | 0 | 0.0036 |
| | | VPWBS | 0 | 0 | \textbf{100} | 0 | 0 | 0 | \textbf{0.0033} |
| | (E3) | MOSEG.MS | 0 | 1 | 96 | 3 | 0 | 0 | 0.0076 |
| | | VPWBS | 0 | 0 | \textbf{99} | 1 | 0 | 0 | \textbf{0.0045} |

B.3 When \( p = 1000 \)

We additionally examine the case where \( p = 1000 \), adopting the simulation setting (E1) from Appendix B.2. We exclude VPWBS Wang et al. (2021) which, as shown in Section 4.2, tends to take considerably longer time to run compared to MOSEG.MS. Table B.3 shows that, in comparison to the the results under (E1) in Table B.2 obtained when \( p = 100 \), the greater sample size is required to detect smaller changes. Also, the localisation performance worsens as \( p \) increases. Nonetheless, MOSEG.MS demonstrates itself to be scalable as the dimensionality increases when the size of change is sufficiently large, which is in line with the theoretical requirements.
Table B.3: Performance of MOSEG.MS under [(E1)] when $p = 1000$ over 100 realisations.

| $\delta$ | $-2$ | $-1$ | 0 | 1 | 2 | $\geq 3$ | $d_H$   |
|----------|------|------|---|---|---|--------|--------|
| 0.2      | 6    | 47   | 29| 18| 0  | 0      | 0.3051 |
| 0.4      | 10   | 34   | 44| 11| 1  | 0      | 0.2972 |
| 0.8      | 1    | 22   | 65| 11| 1  | 0      | 0.1391 |
| 1.6      | 4    | 3    | 92| 1 | 0  | 0      | 0.0673 |
C Proofs

In what follows, for any vector $a \in \mathbb{R}^p$ and a set $\mathcal{A} \subset \{1, \ldots, p\}$, we denote by $a(\mathcal{A}) = (a_i, i \in \mathcal{A})^\top$ the sub-vector of $a$ supported on $\mathcal{A}$. We write the population counterpart of $T_k(G)$ with $\beta_{s,e}^*$ defined in (4) as

$$T_k^*(G) = \sqrt{\frac{G}{2}} |\beta_{k,k+G}^* - \beta_{k-G,k}^*|_2.$$  

Further, we write $S_{s,e} = \text{supp}(\beta_{s,e})$.

C.1 Proof of Theorem 1

C.1.1 Supporting lemmas

**Lemma C.1.** We have

$$T_k^*(G) = \begin{cases} \frac{1}{\sqrt{2G}} (G - |k - \theta_j|) \delta_j & \text{if } \{k - G + 1, \ldots, k + G\} \cap \Theta = \{\theta_j\}, \\ 0 & \text{if } \{k - G + 1, \ldots, k + G\} \cap \Theta = \emptyset \end{cases}$$

**Lemma C.2.** Define $\Delta_{s,e} = \hat{\beta}_{s,e} - \beta_{s,e}^*$. With $\lambda \geq 4C_{DEV} \rho_n p$, we have $P(B) \geq 1 - P(\mathcal{R}^{(1)} \cap D^{(2)})$ where

$$B = \left\{ |\Delta_{s,e}|_2 \leq \frac{12 \sqrt{2e} \lambda}{\omega \sqrt{e - s}} \text{ and } |\Delta_{s,e}(S_{s,e}^c)|_1 \leq 3 |\Delta_{s,e}(S_{s,e})|_1 \text{ for all } 0 \leq s < e \leq n \right\}.$$  

with $|\{s + 1, \ldots, e\} \cap \Theta| \leq 1$ and $e - s \geq C_0 \max \left[ (\omega^{-1}s \log(p))^{\frac{1}{\tau}}, \rho_n^2 \right]$.  

**Proof.** For given $0 \leq s < e \leq n$, we have

$$\sum_{t=s+1}^e \left( Y_t - x_t^\top \hat{\beta}_{s,e} \right)^2 + \lambda \sqrt{e - s} |\hat{\beta}_{s,e}|_1 \leq \sum_{t=s+1}^e \left( Y_t - x_t^\top \beta_{s,e}^* \right)^2 + \lambda \sqrt{e - s} |\beta_{s,e}^*|_1,$$

from which it follows that

$$\lambda \sqrt{e - s} \left( |\beta_{s,e}^*|_1 - |\hat{\beta}_{s,e}|_1 \right) \geq \sum_{t=s+1}^e \left[ (x_t^\top \hat{\beta}_{s,e})^2 - (x_t^\top \beta_{s,e}^*)^2 - 2Y_t x_t^\top (\hat{\beta}_{s,e} - \beta_{s,e}^*) \right]$$

$$= \sum_{t=s+1}^e \left[ \Delta_{s,e}^\top x_t x_t^\top \Delta_{s,e} - 2(Y_t - x_t^\top \beta_{s,e}^*) x_t^\top \Delta_{s,e} \right].$$

Then, noting that $\beta_{s,e}^*(S_{s,e}^c) = 0$,

$$\frac{1}{\sqrt{e - s}} \sum_{t=s+1}^e \left[ \Delta_{s,e}^\top x_t x_t^\top \Delta_{s,e} - 2(Y_t - x_t^\top \beta_{s,e}^*) x_t^\top \Delta_{s,e} \right] + \lambda |\hat{\beta}_{s,e}(S_{s,e}^c)|_1$$

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where the last inequality follows from \((C.1)\) that on \(\mathcal{D}^{(2)}\),

\[
\frac{1}{\sqrt{e-s}} \sum_{t=s+1}^{e} \Delta_{s,e}^T x_t x_t^T \Delta_{s,e} - \frac{\lambda}{2} |\Delta_{s,e}|_1 + \lambda |\Delta_{s,e}(S_{s,e})|_1 \leq \lambda |\Delta_{s,e}(S_{s,e})|_1 ,
\]

\[
\therefore 0 \leq \frac{1}{\sqrt{e-s}} \sum_{t=s+1}^{e} \Delta_{s,e}^T x_t x_t^T \Delta_{s,e} \leq \frac{\lambda}{2} \left( 3 |\Delta_{s,e}(S_{s,e})|_1 - |\Delta_{s,e}(S_{s,e})^c|_1 \right),
\]

such that

\[
|\Delta_{s,e}(S_{s,e})^c|_1 \leq 3 |\Delta_{s,e}(S_{s,e})|_1 . \tag{C.2}
\]

This in particular leads to

\[
|\Delta_{s,e}|_1 \leq 4 |\Delta_{s,e}(S_{s,e})|_1 \leq 4 \sqrt{2} \lambda |\Delta_{s,e}|_2
\]

from the definition of \(s\). Then on \(\mathcal{R}^{(1)}\), we have

\[
6 \sqrt{2} \lambda |\Delta_{s,e}|_2 \geq \frac{1}{\sqrt{e-s}} \sum_{t=s+1}^{e} \Delta_{s,e}^T x_t x_t^T \Delta_{s,e}
\]

\[
\geq \omega \sqrt{e-s} |\Delta_{s,e}|^2 \frac{32 C_{RSC} s \log(p) (e-s)^\tau}{\sqrt{e-s}} |\Delta_{s,e}|^2 \geq \frac{\omega}{2} \sqrt{e-s} |\Delta_{s,e}|^2 ,
\]

where the last inequality follows for \((e-s)^{1-\tau} \geq 64 C_{RSC} \omega^{-1} s \log(p)\). In summary,

\[
|\Delta_{s,e}|_2 \leq \frac{12 \sqrt{2} \lambda}{\omega \sqrt{e-s}} . \tag{C.3}
\]

Combining \((C.2)\) and \((C.3)\), the proof is complete. \(\square\)

**C.1.2 Proof of Theorem 1**

Let \(T_j = \{\theta_j - |\eta G| + 1, \ldots, \theta_j + |\eta G|\} \cap T\) for \(1 \leq j \leq q\). Under Assumptions 4 and 5, we have \(G \geq C_5^{-2} C_1 \max \{ \omega^{-2} \rho_{n,p}, (\omega^{-1} s \log(p))^{1/(1-\tau)} \}\) such that the lower bound on \((e-s)^{1-\tau}\) made in \(\mathcal{B}\) (see Lemma \(C.2\)) is met by all \(s = k\) and \(e = k + G, k = 0, \ldots, n - G\). By Lemma \(C.2\)

\[
\max_{G \leq k \leq n - G} |T_k(G) - T_k^*(G)| \leq
\]

\[
\max_{G \leq k \leq n - G} \sqrt{\frac{G}{2}} \left( |\hat{\beta}_{k-G,k} - \beta_{k-G,k}^*|_2 + |\hat{\beta}_{k,k+G} - \beta_{k,k+G}^*|_2 \right) \leq \frac{24 \sqrt{5} \lambda}{\omega} . \tag{C.4}
\]

First, consider some \(k\) for which \(\{k - G + 2, \ldots, k + G - 1\} \cap \Theta = \emptyset\). Then, we have \(T_k^*(G) = 0\)
from Lemma C.1 such that by (C.4),
\[
\max_{k: \min_{1 \leq j \leq q} |k - \theta_j| \geq G} T_k(G) \leq \max_{G \leq \ell \leq n - G} |T_\ell(G) - T_\ell(G)| \leq \frac{24\sqrt{5}\lambda}{\omega} \leq D. \tag{C.5}
\]
This ensures that any \( \tilde{\theta} \in \tilde{\Theta} \) satisfies \( \min_{1 \leq j \leq q} |\tilde{\theta} - \theta_j| < G \). Next, let \( \theta^l_j \) and \( \theta^r_j \) denote two points within \( T_j \) which are the closest to \( \theta_j \) from the left and and the right of \( \theta_j \), respectively, with \( \theta^l_j = \theta^r_j \) when \( r = 1/G \). Then by construction of \( T \),
\[
\max(k_j - \theta^l_j, \theta^r_j - \theta_j) \leq \lfloor rG \rfloor \quad \text{and} \quad \min(\theta_j - \theta^l_j, \theta^r_j - \theta_j) \leq \frac{rG}{2}, \tag{C.6}
\]
such that from Lemma C.1
\[
\max \left( T_{\theta^l_j}(G), T_{\theta^r_j}(G) \right) \geq \frac{\delta_j(G - \lfloor rG \rfloor/2)}{\sqrt{2G}} \geq \sqrt{\frac{G}{2}} \delta_j (1 - r/2).
\]
From this and (C.4), at \( \tilde{\theta}_j = \arg \max_{k \in T_j} T_k(G) \), we have
\[
T_{\tilde{\theta}_j}(G) \geq \max \left( T_{\theta^l_j}(G), T_{\theta^r_j}(G) \right) \geq \sqrt{\frac{G}{2}} \delta_j \left( 1 - \frac{r}{2} \right) - \frac{24\sqrt{5}\lambda}{\omega} > \frac{1 - r/2}{2} \sqrt{\frac{G}{2}} \delta_j > D,
\]
where the second last inequality follows from Assumption 5{(b)} and the last one from (11). When \( \eta = 1 \), this and (C.5) indicates that such \( \tilde{\theta}_j \) satisfies (6). When \( \eta < 1 \), note that
\[
\max \left( T_{\theta^l_j}(G), T_{\theta^r_j}(G) \right) - \max \left\{ T_k(G) : |k - \theta_j| > (1 - \eta)G, k \in T \right\} \geq \sqrt{\frac{G}{2}} \delta_j \left( \eta - \frac{3r}{2} \right) - \frac{48\sqrt{5}\lambda}{\omega} \geq \frac{5\eta}{8\sqrt{2}} \min_{1 \leq j \leq q} \delta_j \sqrt{G} - \frac{48\sqrt{5}\lambda}{\omega} > 0
\]
from (11). These arguments ensure that we detect at least one change point in \( T_j \) at \( t = \tilde{\theta}_j \) for each \( j = 1, \ldots, q \). For such \( \tilde{\theta}_j \), suppose that \( \theta_j = \arg \min_{k \in \{\theta^l_j, \theta^r_j\}} |\tilde{\theta}_j - k| \). Then,
\[
\frac{\delta_j}{\sqrt{2G}} (G - |\tilde{\theta}_j - \theta_j|) + \frac{24\sqrt{5}\lambda}{\omega} \geq T_{\tilde{\theta}_j}(G) \geq T_{\theta^l_j}(G) \geq \frac{\delta_j}{\sqrt{2G}} (G - |\theta^l_j - \theta_j|) - \frac{24\sqrt{5}\lambda}{\omega}
\]
and re-arranging, we obtain
\[
\frac{\delta_j}{\sqrt{2G}} (|\tilde{\theta}_j - \theta_j| - |\theta^l_j - \theta_j|) \leq \frac{48\sqrt{2}\sqrt{\lambda}}{\omega \delta_j}, \text{ such that } |\tilde{\theta}_j - \theta_j| \leq \frac{48\sqrt{2}\sqrt{\lambda}}{\omega \delta_j} + \lfloor rG \rfloor < \left\lfloor \frac{G}{2} \right\rfloor,
\]
for large enough \( C \) in Assumption 5{(b)}
Finally, let \( L_T(t) \) denote the largest time point \( k' \in T \) that satisfies \( k' \leq t \), and define \( R_T(t) \)
analogously. Then, we establish that

$$T_{\mathcal{L}}(\theta - \frac{\eta G}{2}, G) > \max \left\{ T_k(G) : \frac{\eta G}{2} (m + 1) \leq \theta - k \leq \frac{\eta G}{2} (m + 2), k \in \mathcal{T} \right\}, \quad (C.7)$$

$$T_{\mathcal{R}}(\theta + \frac{\eta G}{2}, G) > \max \left\{ T_k(G) : \frac{\eta G}{2} (m + 1) \leq \theta - k \leq \frac{\eta G}{2} (m + 2), k \in \mathcal{T} \right\}, \quad (C.8)$$

for $m = 0, \ldots, \lceil 2/\eta \rceil - 2$. The inequality in (C.7) follows from noting that

$$T_{\mathcal{L}}(\theta - \frac{\eta G}{2}, G) - \max \left\{ T_k(G) : \frac{\eta G}{2} (m + 1) \leq \theta - k \leq \frac{\eta G}{2} (m + 2), k \in \mathcal{T} \right\} \geq \sqrt{\frac{G}{2}} \delta_j \left( \frac{\eta}{2r} \right) - \frac{48\sqrt{2} \lambda \omega}{\sqrt{G}} > 0$$

under (11), and the inequality in (C.8) follows analogously. This ensures that $\tilde{\theta}_j$ by its construction is the unique local maximiser of $T_k(G)$ within the interval $\{ \theta_j - G + 1, \ldots, \theta_j + G \} \cap \mathcal{T}$ satisfying (6) for each $j = 1, \ldots, q$, which completes the proof.

C.1.3 Proof of Theorem 1 (ii)

Recalling (7), we write

$$Q_j(k) = \sum_{t=\theta_j - G + 1}^{k} (Y_t - \tilde{x}_t^\top \hat{\beta}_j^L)^2 + \sum_{t=k+1}^{\theta_j + G} (Y_t - \tilde{x}_t^\top \hat{\beta}_j^R)^2.$$

Theorem 1 (i) establishes that for each $j = 1, \ldots, q$, we have $\tilde{\theta}_j \in \hat{\Theta}$ that satisfies $|\tilde{\theta}_j - \theta_j| < G/2$, and $\hat{\Theta}$ contains no other estimator. Then under Assumption 5 (a), we have the following statements satisfied for all $j$.

(i) Defining $\mathcal{I}(\tilde{\theta}_j) = \{ \tilde{\theta}_j - G + 1, \ldots, \tilde{\theta}_j + G \}$, it fulfils $\mathcal{I}(\tilde{\theta}_j) \cap \Theta = \{ \theta_j \}$.

(ii) $\{ \tilde{\theta}_j^L - G + 1, \ldots, \tilde{\theta}_j^L \} \subset \{ \theta_{j-1} + 1, \ldots, \theta_j \}$ and $\{ \tilde{\theta}_j^R + 1, \ldots, \tilde{\theta}_j^R + G \} \subset \{ \theta_j + 1, \ldots, \theta_{j+1} \}$, such that denoting by $\Delta_j^L = \hat{\beta}_j^L - \beta_{j-1}$ and $\Delta_j^R = \hat{\beta}_j^R - \beta_j$, we have

$$\max \left( |\Delta_j^L|_2, |\Delta_j^R|_2 \right) \leq \frac{12\sqrt{25} \lambda}{\omega \sqrt{G}},$$

$$|\Delta_j^L(S_{j-1})|_1 \leq 3 |\Delta_j^L(S_j)|_1 \quad \text{and} \quad |\Delta_j^R(S_j)|_1 \leq 3 |\Delta_j^R(S_j)|_1 \quad (C.9)$$

in $B$, see Lemma C.2.
Then we show that for all \( k \in \mathcal{I}(\tilde{\theta}_j) \) satisfying \( \delta_j^2 |k - \theta_j| > v_{n,p} \), with

\[
v_{n,p} = \max \left( \sigma^2 n p, (\sigma \log(p))^{\frac{1}{1+\tau}} \right) \cdot \max \left\{ C^2 \max \left[ \frac{9 C_{\text{RSC}}}{2 \omega}, \frac{32 C_{\text{RSC}}}{\omega \log(p)} \right], \left( \frac{96 C_{\text{DEV}}}{\omega} \right)^2 \right\},\tag{C.10}
\]

we have \( Q_j(k) - Q_j(\theta_j) > 0 \), which completes the proof.

First, suppose that \( k \geq \theta_j + 1 \). Then,

\[
Q_j(k) - Q_j(\theta_j) = \sum_{t=\theta_j+1}^{k} \left[ (Y_t - x_t^T \hat{\beta}_L^k)^2 - (Y_t - x_t^T \hat{\beta}_R^k)^2 \right] - \sum_{t=\theta_j+1}^{k} (\hat{\beta}_L^k - \beta_j)^T x_t x_t^T (\hat{\beta}_L^k - \beta_j) - \sum_{t=\theta_j+1}^{k} (\hat{\beta}_R^k - \beta_j)^T x_t x_t^T (\hat{\beta}_R^k - \beta_j) + 2 \sum_{t=\theta_j+1}^{k} (\beta_j - \beta_{j-1})^T x_t x_t^T (\beta_j - \beta_{j-1}) + (\hat{\beta}_R^k - \beta_j) - (\hat{\beta}_L^k - \beta_{j-1}) \right] = I_1 + I_2 + I_3.
\]

From the definition of \( s \) and the Cauchy-Schwarz inequality,

\[
|\beta_j - \beta_{j-1}|_1 \leq \sqrt{2s} |\beta_j - \beta_{j-1}|_2 \tag{C.11}
\]

and from (C.9), we have

\[
|\Delta_j^1|_1 \leq 4 |\Delta_j^1(S_j)|_1 \leq 4 \sqrt{2s} |\Delta_j^r|_2 \text{ and analogously, } |\Delta_j^1|_1 \leq 4 \sqrt{2s} |\Delta_j^r|_2. \tag{C.12}
\]

From (C.11)–(C.12), we derive

\[
|\hat{\beta}_L^k - \beta_j|_2 \leq \delta_j \left( 1 + \frac{12 \sqrt{2s} \lambda}{\omega \delta_j \sqrt{G}} \right) \leq \frac{3 \delta_j}{2} \text{ and similarly, } |\hat{\beta}_R^k - \beta_j|_2 \geq \frac{\delta_j}{2},
\]

\[
|\hat{\beta}_L^k - \beta_j|_1 \leq \sqrt{5} \delta_j \left( 1 + \frac{96 \sqrt{5} \lambda}{\omega \delta_j \sqrt{G}} \right) \leq \frac{3 \sqrt{5} \delta_j}{2},
\]

for a large enough \( C_1 \) in Assumption 5(b). Then on \( \mathcal{R}^{(1)} \), we have

\[
I_1 \geq |k - \theta_j| \omega \delta_j^2 \left( 1 - \frac{9 C_{\text{RSC}} \log(p)}{4 |k - \theta_j|^{1-\tau} \omega} \right) \geq \frac{\omega}{8} \delta_j^2 |k - \theta_j| \tag{C.13}
\]

from that \( |k - \theta_j| > \delta_j^{-2} v_{n,p} \geq C_{\delta}^{-2} v_{n,p} \) (from Assumption 4) and (C.10). As for \( I_2 \), from
Lemma [C.2] (C.10) and [C.12] we have on $R^{(2)}$,

$$|I_2| \leq |\mathbf{\Delta}_{j_1}^{n_{k_1}}(k - \theta_j)\bar{\omega} + 32C_{RSC5}\log(p)|k - \theta_j|\bar{\omega}| \leq 2|k - \theta_j|\bar{\omega}|\mathbf{\Delta}_{j_1}^{n_{k_1}}| \leq \frac{576\bar{\omega}\bar{\delta}^2|k - \theta_j|\lambda^2}{\omega^2G}.$$  

(C.14)

Turning our attention to $I_3$, from (C.11), (C.12),

$$\left| (\beta_j - \beta_{j-1}) + (\hat{\beta}_j - \beta_j) - (\hat{\beta}_{j-1} - \beta_{j-1}) \right| \leq |\beta_j - \beta_{j-1}| + |\hat{\beta}_j - \beta_j| + |\hat{\beta}_{j-1} - \beta_{j-1}| \leq \sqrt{5}\delta_j \left( 1 + \frac{192\sqrt{5}\lambda}{\omega\delta_j\sqrt{G}} \right) \leq 2\sqrt{5}\delta_j,$$

where the last inequality follows from Assumption 3(b). Then on $D^{(1)},$

$$\frac{1}{2} |I_3| \leq \sum_{t=\ell,j+1}^k \varepsilon_t |x_t^l|_\infty \left| (\beta_j - \beta_{j-1}) + (\hat{\beta}_j - \beta_j) - (\hat{\beta}_{j-1} - \beta_{j-1}) \right|_1 \leq 2C_{DEV}\delta_j\sqrt{s|k - \theta_j|\rho_{n,p}}.$$

(C.15)

Then from (C.13), (C.14) and (C.15), we derive

$$\frac{|I_2|}{I_1} = \frac{4608\bar{\omega}\bar{\delta}^2\lambda^2}{\omega^2\delta^2G} \leq \frac{1}{3} \quad \text{and} \quad \frac{|I_3|}{I_1} = \frac{32C_{DEV}\sqrt{5}\rho_{n,p}}{\omega\delta_j\sqrt{k - \theta_j}} \leq \frac{1}{3}$$

under Assumption 3(b) for all $k \in I_j$ satisfying $\delta_j^2|k - \theta_j| > v_{n,p}$ from (C.10). Analogous arguments apply when $k \leq \theta_j$, and the above arguments are deterministic on $M$. In summary, we have

$$\min_{1 \leq j \leq q} \min_{k \in I_j} \min_{\delta_j^2|k - \theta_j| > v_{n,p}} (Q_j(k) - Q_j(\theta_j)) > \frac{\omega}{24}v_{n,p} > 0,$$

which concludes the proof.

### C.2 Proof of Proposition 2

#### C.2.1 Supporting lemmas

Define $\mathbb{K}(b) = \mathbb{B}_0(b) \cap \mathbb{B}_2(1)$ with some $b \geq 1$, where $\mathbb{B}_q(r) = \{a: |a|_q \leq r\}$ with the dimension of $a$ determined within the context. Let $e_i$ denote a vector that contains zeros except for its $i$th component set to be one. We denote the time-varying vector of parameters under [1] by $

\beta(t) = \sum_{j=1}^{q+1} \beta_j I_{[\theta_{j-1} + 1 \leq t \leq \theta_j]}.$

Denote by $Z_t = (x_t^l, e_t)^\top \in \mathbb{R}^{p+1}$ which admits $Z_t = \sum_{\ell=0}^\infty D_{\ell} \xi_{t-\ell}$ under [12]. For some $a, b \in \mathbb{B}_2(1)$, define $U_t(a) = a^\top Z_t$ and $W_t(a, b) = a^\top Z_t b$. Let $\xi_t^\ell$ denote an independent
copy of $\xi_t$, and define $Z_{t,0} = \sum_{\ell=0}^{\infty} D_t \xi_{t-\ell}$. We denote the functional dependence measure and the dependence-adjusted norm for $U_t(a)$ as defined in Zhang and Wu (2017), by

$$
\delta_{t,\nu}(a) = \left\| a^\top Z_t - a^\top Z_{t,0} \right\|_\nu
$$

and

$$
\left\| U(a) \right\|_\nu = \sum_{t=0}^{\infty} \delta_{t,\nu}(a),
$$

respectively. Analogously, we define

$$
\delta_{t,\nu}(a, b) = \left\| a^\top Z_t Z_t^\top b - a^\top Z_{t,0} Z_{t,0}^\top b \right\|_\nu
$$

and

$$
\left\| W(a, b) \right\|_\nu = \sum_{t=0}^{\infty} \delta_{t,\nu}(a, b)
$$

for $W_t(a, b)$. Finally, for some $\kappa \geq 0$, we denote the dependence adjusted sub-exponential norm of $W_t(a, b)$ by $\| W_t(a, b) \|_{\psi_{\kappa}} = \sup_{\nu \geq 2} \nu^{-\kappa} \| W_t(a, b) \|_{\nu}$. In what follows, we denote by $C_\Pi$ with $\Pi \subset \{\gamma, \nu, \Xi, \zeta\}$ a constant that depends on the parameters included in $\Pi$ which may vary from one occasion to another.

**Lemma C.3.** Suppose that Condition 1 holds.

(i) Under Condition 1(a), we have

$$
\sup_{a, b \in B_2(1)} \| W_t(a, b) \|_{\psi_{\kappa}} \leq C_{\gamma, \Xi, \kappa} \| \xi_t \|_2^2 < \infty
$$

with $C_{\gamma, \Xi, \kappa} = \max(1/(\nu - 1), \sqrt{\nu - 1})$, where the inequality follows from Lemma 2 of Chen et al. (2021) (Burkholder’s inequality) and Minkowski inequality, and the second from Condition 1 and from that $\| D_t \|_2 \leq \sqrt{\| D_t \|_1 \| D_t \|_\infty}$ (with $\| \cdot \|_a$ denoting the induced matrix norms).

(ii) Under Condition 1(b), we have $\sup_{a \in B_2(1)} \| U(a) \|_2 \leq C_{\Xi, \kappa}$.

**Proof.** In what follows, we denote by $\mu_{\nu} = \| \xi_t \|_\nu$. For given $\nu > 1$, we have

$$
\sup_{a \in B_2(1)} \delta_{t,\nu}(a) = \left\| a^\top D_t (\xi_0 - \xi_0) \right\|_\nu \leq C_{\nu} \mu_{\nu} \sqrt{2 \sup_{a \in B_2(1)} \| a^\top D_t \|_2^2} \leq C_{\nu} \mu_{\nu} \Xi (1 + t)^{-\zeta}
$$

with $C_{\nu} = \max(1/(\nu - 1), \sqrt{\nu - 1})$, which proves (ii). Note that by Hölder and Minkowski’s inequalities,

$$
\delta_{t,\nu}(a, b) \leq \sum_{\ell=0}^{\infty} \left\| a^\top D_t \xi_{t-\ell} \right\|_{2\nu} \left\| b^\top D_t (\xi_0 - \xi_0) \right\|_{2\nu}
$$

$$
+ \sum_{\ell=0, \ell \neq t}^{\infty} \left\| b^\top D_t \xi_{t-\ell} + b^\top D_t \xi_0 \right\|_{2\nu} \left\| a^\top D_t (\xi_0 - \xi_0) \right\|_{2\nu}.
$$

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For given $\nu > 2$, similarly as in (C.16), we can show that
\[
\sup_{a \in \mathbb{B}_2(1)} \left\| \sum_{\ell=0}^{\infty} a^\top D_{t-\ell} \xi_{t-\ell} \right\|_{2_\nu} \leq \sup_{a \in \mathbb{B}_2(1)} \left\| a^\top D_{t-\ell} \xi_{t-\ell} \right\|_{2_\nu} \leq C_{2_\nu} \mu_{2_\nu} \sum_{\ell=0}^{\infty} a^\top D_{t-\ell} \xi_{t-\ell} \leq C_{2_\nu} \mu_{2_\nu} \sum_{\ell=0}^{\infty} (1 + \ell)^{-\varsigma} \leq C_{\nu, \xi, \nu} \nu^{\nu + 1/2} \tag{C.17}
\]
under Condition 1(a). Then, (C.16)–(C.17) lead to
\[
\sup_{a, b \in \mathbb{B}_2(1)} \delta_{t, \nu}(a, b) \leq C_{\nu, \xi, \nu} C_{\xi}^2 \nu^{2\gamma + 1}(1 + t)^{2\gamma + 1}, \quad \text{and}
\]
\[
\sup_{a, b \in \mathbb{B}_2(1)} \left\| W(a, b) \right\|_{\nu} \leq C_{\nu, \xi, \nu} C_{\xi}^2 \nu^{2\gamma + 1}(1 + t)^{2\gamma + 1} \leq C_{\nu, \xi, \nu} C_{\xi}^2 \nu^{2\gamma + 1},
\]
such that we have $\sup_{a, b \in \mathbb{B}_2(1)} \left\| W(a, b) \right\|_{\nu} \leq C_{\nu, \xi, \nu} C_{\xi}^2$ with $\kappa = 2\gamma + 1$, which proves (i).

\[\square\]

**Lemma C.4.** Under Condition 1(a), there exist fixed constants $C', C'' > 0$ such that for all $0 \leq s < e \leq n$ and $z > 0$, we have
\[
\sup_{a, b \in \mathbb{B}_2(1)} \mathbb{P} \left( \frac{1}{\sqrt{e - s}} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b - \mathbb{E} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b \right) \right) \geq z \right) \leq C' \exp \left( -C'' z^2 \nu^{2\gamma + 1} \right).
\]

**Proof.** By Lemma C.3(ii) and Lemma C.4 of Zhang and Wu (2017), there exist constants $C', C'' > 0$ that depend on $\gamma, \Xi, \varsigma$ and $C_\xi$, such that for all $z > 0$,
\[
\sup_{a, b \in \mathbb{B}_2(1)} \mathbb{P} \left( \frac{1}{\sqrt{e - s}} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b - \mathbb{E} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b \right) \right) \geq z \right) \leq C' \exp \left( -C'' z^2 \nu^{2\gamma + 1} \right).
\]

\[\square\]

**Lemma C.5.** Under Condition 1(b), there exists a fixed constants $C''' > 0$ such that for all $0 \leq s < e \leq n$ and $0 < z < C_{\nu, \xi, \nu}^2 \nu \sqrt{e - s}$, we have
\[
\sup_{a, b \in \mathbb{B}_2(1)} \mathbb{P} \left( \frac{1}{\sqrt{e - s}} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b - \mathbb{E} \left( \sum_{t=s+1}^{e} a^\top Z_t Z_t^\top b \right) \right) \geq z \right) \leq 6 \exp(-C''' z^2).
\]

**Proof.** By Lemma C.3(iii) and Theorem 6.6 of Zhang and Wu (2021), there exists an absolute
constant $C > 0$ such that for all $0 < z < C^2 \Xi^2 \sqrt{e - s}$,

\[
\sup_{a \in B_2(1)} P \left( \frac{1}{\sqrt{e - s}} \left| \sum_{t=s+1}^e a^\top Z_t Z_t^\top a - E \left( \sum_{t=s+1}^e a^\top Z_t Z_t^\top a \right) \right| \geq z \right) \leq 2 \exp \left( -C \min \left( \frac{z^2}{C^2 \Xi^4}, \frac{z \sqrt{e - s}}{C \Xi} \right) \right).
\]

Then noting that

\[
\sup_{a, b \in B_2(1)} P \left( \frac{2}{\sqrt{e - s}} \left| \sum_{t=s+1}^e a^\top Z_t Z_t^\top b - E \left( \sum_{t=s+1}^e a^\top Z_t Z_t^\top b \right) \right| \geq \frac{z}{3} \right) \leq 6 \exp \left( -C \delta \frac{z^2}{9C^2 \Xi^4} \right),
\]

we can find $C''$ that depends on $\Xi$ and $\varsigma$.

\[\square\]

### C.2.2 Proof of Proposition 2 (i)

Recalling $C'$ from Lemma C.4, we set $c_1 = 3C'$.

**Verification of Assumption 2**

By assumption, we have $E(x_t \epsilon_t) = 0$. Then setting $a = e_i$, $i = 1, \ldots, p$, $b = e_{p+1}$ and $z = C_{\text{DEV}} \log^{2+3/2}(p \lor n)$ in Lemma C.4,

\[
P(\mathcal{D}(1)) \geq 1 - C' \delta \exp \left( -C'' C_{\text{DEV}}^2 \log (p \lor n) \right).
\]

Next, by construction,

\[
\sum_{t=s+1}^e (\beta(t) - \beta_{s,e}^*) = 0 \quad \text{and} \quad \max_{0 \leq s < s' \leq n} \max_{s \lor t \leq e} |\beta(t) - \beta_{s,e}^*|_2 \leq C_\delta
\]

under Assumption 4 and

\[
E \left[ \sum_{t=s+1}^e x_t x_t^\top (\beta(t) - \beta_{s,e}^*) \right] = \Sigma_x \sum_{t=s+1}^e (\beta(t) - \beta_{s,e}^*) = 0
\]

under Assumption 1. Then setting $a = e_i$, $i = 1, \ldots, p$, $b = \beta(t) - \beta_{s,e}^*$ for given $s, e$ and
where the last inequality follows with $t \in \{s + 1, \ldots, e\}$ and $z = C_{\text{DEV}} C_{\delta} \log^{2\gamma+3/2}(p \lor n)$ in Lemma C.4

$$P(D^{(2)}) \geq 1 - C'' p v^{3} \exp \left(-C''(C_{\text{DEV}} C_{\delta})^{2 \gamma+3} \log(p \lor n)\right), \quad (C.21)$$

from (C.19) and (C.20). Combining (C.18) and (C.21), we can find large enough $C_{\text{DEV}}$ that depends only on $C''$, $\gamma$, $C_{\delta}$ and $c_{2}$ such that $P(D^{(1)} \cap D^{(2)}) \geq 1 - 2c_{1}(p \lor n)^{-c_{2}/3}$.

**Verification of Assumption 3**

Let $b_{s,e}$ denote an integer that depends on $(e - s)$ for some $0 \leq s < e \leq n$, and define

$$R = \left\{ \sup_{a \in \mathbb{R}^{(2b_{s,e})}} \frac{1}{e - s} \sum_{t=s+1}^{e} a^{\top} (x_{t}x_{t}^{\top} - \Sigma_x) a \geq \frac{\Lambda_{\text{min}}(\Sigma_x)}{54} \middle| \text{for all } 0 \leq s < e \leq n \right\},$$

with $e - s \geq C_{0} \log^{4\gamma+3}(p \lor n)$ and $\{|s + 1, \ldots, e\} \cap \Theta = 1$.

By Lemma C.4 and Lemma F.2 of Basu and Michailidis (2015), we have

$$P(R^{C}) \leq \sum_{0 \leq s < e \leq n \atop e - s \geq C_{0} \log^{4\gamma+3}(p \lor n) \atop |\{s + 1, \ldots, e\} \cap \Theta| \leq 1} C' \exp \left[-C'' \left(\frac{\sqrt{e - s} \Lambda_{\text{min}}(\Sigma_x)}{54}\right)^{2^{-\gamma+3}} + 2b_{s,e} \log(p)\right]$$

$$\leq C' n^{2} \exp \left[-\frac{C''}{2} \left(\frac{C_{0}^{1/2} \Lambda_{\text{min}}(\Sigma_x)}{54}\right)^{2^{-\gamma+3}} \log(p \lor n)\right],$$

where the last inequality follows with

$$b_{s,e} = \left\lfloor \frac{C''}{4 \log(p)} \left(\frac{\sqrt{e - s} \Lambda_{\text{min}}(\Sigma_x)}{54}\right)^{2^{-\gamma+3}} \right\rfloor,$$

which satisfies $b_{s,e} \geq 1$ for large enough $C_{0}$. Further, we can find $C_{0}$ that depends only on $C''$, $\Lambda_{\text{min}}(\Sigma_x)$, $\gamma$ and $c_{2}$ which leads to $P(R) \geq 1 - c_{1}(p \lor n)^{-c_{2}/3}$. Then, by Lemma 12 of Loh and Wainwright (2012), on $R$, we have

$$\sum_{t=s+1}^{e} a^{\top} x_{t}x_{t}^{\top} a \geq \Lambda_{\text{min}}(\Sigma_x)(e - s)|a|_{2}^{2}$$

$$\geq \Lambda_{\text{min}}(\Sigma_x)(e - s) \left(\frac{|a|_{2}^{2} + 4 \log(p) \left(\frac{54}{\sqrt{e - s} \Lambda_{\text{min}}(\Sigma_x)}\right)^{2^{-\gamma+3}} |a|_{1}^{2}}{C''}\right)$$

$$\geq \omega (e - s)|a|_{2}^{2} - C_{\text{RSC}} \log(p)(e - s)^{2\gamma+2}|a|_{1}^{2}$$

for all $a \in \mathbb{R}^{p}$, with $\omega = \Lambda_{\text{min}}(\Sigma_x)/2$ and $C_{\text{RSC}}$ depending only on $C''$, $\gamma$ and $\Lambda_{\text{min}}(\Sigma_x)$.
Analogously we have on $\mathcal{R}$,

$$
\sum_{t=s+1}^{c} a^\top x_t x_t^\top a \leq \bar{\omega}(e - s)|a|^2 + C_{RSC} \log(p)(e - s)^{\frac{4 + \bar{\omega}}{2\tau + 3}}|a|^2
$$

for all $a \in \mathbb{R}^p$, with $\bar{\omega} = 3\Lambda_{\text{max}}(\Sigma_x)/2$.

Combining the arguments above, we have $P(D^{(1)} \cap D^{(2)} \cap \mathcal{R}^{(1)} \cap \mathcal{R}^{(2)}) \geq 1 - c_1(p \vee n)^{\beta}$, with $\tau = (4\gamma + 2)/(4\gamma + 3)$ and $\rho_{n,p} = \log^{2\gamma + 3/2}(p \vee n)$.

C.2.3 Proof of Proposition 2(ii)

We set $c_1 = 18$.

Verification of Assumption 2

By assumption, we have $E(x_t x_t^\top) = 0$. Then setting $a = e_i$, $i = 1, \ldots, p$, $b = e_{p+1}$ and $z = C_{\text{DEV}} \log(p \vee n)$ in Lemma C.5,

$$
P(D^{(1)}) \geq 1 - 6pn^2 \exp \left(-C'' C_{\text{DEV}}^2 \log(p \vee n)\right),
$$

(C.22)

provided that $C_0 > C_{2,\epsilon}^2 C_{\text{DEV}}^2$. Also, setting $a = e_i$, $i = 1, \ldots, p$, $b = \beta(t) - \beta_{s,e}^*$ for given $s, e$ and $t \in \{s + 1, \ldots, e\}$ and $z = C_{\text{DEV}} C_{\delta} \log(p \vee n)$ in Lemma C.5,

$$
P(D^{(2)}) \geq 1 - 6pn^3 \exp \left(-C'' C_{\text{DEV}}^2 C_{\delta}^2 \log(p \vee n)\right),
$$

(C.23)

from (C.19) and (C.20). Combining (C.22) and (C.23), we can find large enough $C_{\text{DEV}}$ that depends only on $C''$, $C_{\delta}$ and $c_2$ such that $P(D^{(1)} \cap D^{(2)}) \geq 1 - 2c_1(p \vee n)^{-\beta}/3$.

Verification of Assumption 3

Let $b_{s,e}$ denote an integer that depends on $(e - s)$ for some $0 \leq s < e \leq n$, and define

$$
\mathcal{R} = \left\{ \sup_{a \in \mathbb{R}^{(2b_{s,e})}} \frac{1}{e - s} \sum_{t=s+1}^{c} a^\top (x_t x_t^\top - \Sigma_x) a \geq \frac{\Lambda_{\text{min}}(\Sigma_x)}{54} \right| \text{for all } 0 \leq s < e \leq n,
$$

with $e - s \geq C_0 \log(p \vee n)$ and $|\{s + 1, \ldots, e\} \cap \Theta| \leq 1$.

Then by Lemma C.5 and Lemma F.2 of Basu and Michailidis (2015), we have

$$
P(\mathcal{R}^c) \leq \sum_{0 \leq s < e \leq n} 6 \exp \left[-C'''(e - s) \left(\frac{\Lambda_{\text{min}}(\Sigma_x)}{54}\right)^2 + 2b_{s,e} \log(p)\right]
$$

$$
\leq 6n^2 \exp \left[-C''' C_0 \left(\frac{\Lambda_{\text{min}}(\Sigma_x)}{54}\right)^2 \log(p \vee n)\right],
$$

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where the last inequality follows with
\[
b_{s,e} = \left[ \frac{C''(e-s)}{4 \log(p)} \left( \frac{\Lambda_{\min}(\Sigma_x)}{54} \right)^2 \right],
\]
which satisfies \(b_{s,e} \geq 1\) for large enough \(C_0\). Further, we can find \(C_0\) that depends only on \(C''\), \(\Lambda_{\min}(\Sigma_x)\) and \(c_2\) which leads to \(P(\mathcal{R}) \geq 1 - c_1(p \vee n)^{-c_2}/3\). Then, by Lemma 12 of Loh and Wainwright (2012), on \(\mathcal{R}\), we have
\[
\sum_{t=s+1}^{e} \mathbf{a}^\top \mathbf{x}_t \mathbf{x}_t^\top \mathbf{a} \geq \Lambda_{\min}(\Sigma_x)(e-s) \left| \mathbf{a} \right|_2^2 - \frac{\Lambda_{\min}(\Sigma_x)}{2} (e-s) \left( \frac{4 \log(p)}{C''(e-s)} \left( \frac{54 \Lambda_{\min}(\Sigma_x)}{(e-s) C''} \right)^2 \right) \left| \mathbf{a} \right|_2^2
\]
\[
\geq \omega(e-s) \left| \mathbf{a} \right|_2^2 - C_{\text{rsc}} \log(p) \left| \mathbf{a} \right|_1^2
\]
for all \(\mathbf{a} \in \mathbb{R}^p\), with \(\omega = \Lambda_{\min}(\Sigma_x)/2\) and \(C_{\text{rsc}}\) depending only on \(C''\) and \(\Lambda_{\min}(\Sigma_x)\). Analogously we have on \(\mathcal{R}\),
\[
\sum_{t=s+1}^{e} \mathbf{a}^\top \mathbf{x}_t \mathbf{x}_t^\top \mathbf{a} \leq \bar{\omega}(e-s) \left| \mathbf{a} \right|_2^2 + C_{\text{rsc}} \log(p) \left| \mathbf{a} \right|_1^2
\]
for all \(\mathbf{a} \in \mathbb{R}^p\), with \(\bar{\omega} = 3\Lambda_{\max}(\Sigma_x)/2\).

Combining the arguments above, we have \(P(D(1) \cap D(2) \cap \mathcal{R}(1) \cap \mathcal{R}(2)) \geq 1 - c_1(p \vee n)^{-c_2}\), with \(\tau = 0\) and \(\rho_{n,p} = \sqrt{\log(p \vee n)}\).

### C.3 Proof of Theorem 4

In what follows, we operate on \(\mathcal{M} = D(1) \cap D(2) \cap \mathcal{R}(1) \cap \mathcal{R}(2) \cap \mathcal{B}\). Under Assumption 5, we have all \(G \in \mathcal{G}\) satisfy \(G \geq C_0 \max\{\rho_{n,p}^2, (\omega^{-1}s \log(p))^{1/(1-\tau)}\}\) such that the lower bound on \((e-s)\) made in \(\mathcal{B}\) (see Lemma C.2) is met by all \(s = k\) and \(e = k + G\), \(k = 0, \ldots, n-G\).

For some \(k\) and \(G \in \mathcal{G}\), we write \(\mathcal{I}(k,G) = \{k-G+1, \ldots, k+G\}\). Recall that for each pre-estimator \(\tilde{\theta} \in \Theta(G)\), we denote by \(\mathcal{I}(\tilde{\theta}) = \mathcal{I}(\tilde{\theta},G)\) its detection interval. By the same arguments adopted in (C.4) and Lemmas C.1 and C.2 we have
\[
\max_{G \in \mathcal{G}} \max_{G \leq k \leq n-G, \mathcal{I}(k,G) \cap \Theta \subseteq \Theta} |T_k(G) - T_k^*(G)| \leq \frac{24 \sqrt{5} \lambda}{\omega} \quad \text{and} \quad T_k^*(G) = 0 \text{ if } \mathcal{I}(k,G) \cap \Theta = \emptyset. \quad (C.24)
\]

Then, we make the following observations.

(i) From (C.24) and the requirement on \(D\) in (18), we have \(\mathcal{I}(\tilde{\theta}) \cap \Theta \neq \emptyset\) for all \(\tilde{\theta} \in \Theta(G)\), i.e. each pre-estimator in \(\Theta(G)\) has (at least) one change point in its detection interval.
(ii) Under Assumption \([5]\) for each \(\theta_j, j = 1, \ldots, q\), there exists one pre-estimator \(\tilde{\theta} \in \tilde{\Theta}(G(j))\) such that \(I(\tilde{\theta}) \cap \Theta = \{\theta_j\}\) and \(|\tilde{\theta} - \theta_j| < [G(j)/2]\), by the arguments used in the proof of Theorem 1(iv).

Thanks to (ii), there exists an anchor estimator \(\tilde{\theta}^A \in \tilde{\Theta}^A\) for each \(\theta_j\), in the sense that \(\theta_j \in I(\tilde{\theta}^A)\) and further, this anchor estimator \(\tilde{\theta}^A\) is detected with some bandwidth \(G \leq G(j)\).

At the same time, there is at most a single anchor estimator \(\tilde{\theta}^A\) fulfilling \(\theta_j \in I(\tilde{\theta}^A)\) by its construction in (14), and (i) ensures that all anchor estimators contain one change point in its detection interval. Therefore, we have \(\hat{q} = |\tilde{\Theta}^A| = q\) and we may write \(\tilde{\Theta}^A = \{\tilde{\theta}^A, 1 \leq j \leq q : \tilde{\theta}^A < \ldots < \tilde{\theta}^A_q\}\).

Next, by (ii) there exists some \(\tilde{\theta} \in \tilde{\Theta}(G(j))\) fulfilling (15) for each \(j = 1, \ldots, q\). To see this, note that if \(\theta \in \tilde{\Theta}(G(j))\) detects \(\theta_j\) in the sense that \(\theta_j \in I(\theta)\),

\[
\left\{\tilde{\theta} - G(j) - \left[\frac{G(j)}{2}\right] + 1, \ldots, \tilde{\theta} + G(j) + \left[\frac{G(j)}{2}\right]\right\} \subset \left\{\theta_j - 2G(j) + 1, \theta_j + 2G(j)\right\},
\]

where \(I(\tilde{\theta}^A_{j-1}) \subset \{\theta_j-1 - 2G(j-1) + 1, \ldots, \theta_j-1 + 2G(j-1)\}\) and \(I(\tilde{\theta}^A_{j+1}) \subset \{\theta_j+1 - 2G(j+1) + 1, \ldots, \theta_j+1 + 2G(j+1)\}\),

and the sets on RHS do not overlap under Assumption (a). This in turn implies that we have \(|C_j| \geq 1\). Also for \(\tilde{\theta}^M \in C_j\), we have that its detection bandwidth \(G(j)^M\) satisfies

\[
\frac{3}{2} G(j)^M \leq \min(\theta_j+1 - \theta_j, \theta_j - \theta_j-1) \quad \text{and} \quad G(j)^M \geq G(j)
\]

by the construction of \(C_j\). Also, the bandwidths generated as in Remark 2 satisfy

\[
G_{\ell-1} + \frac{1}{2} G_{\ell-1} \leq G_{\ell-1} + G_{\ell-2} = G_{\ell} \leq 2G_{\ell-1}, \quad \text{such that} \quad \frac{1}{2} G_{\ell} \leq G_{\ell-1} \leq \frac{2}{3} G_{\ell} \quad \text{for} \ \ell \geq 2,
\]

and therefore

\[
\frac{1}{4} G(j) \leq G_j^* \quad \text{and} \quad G_j^* \leq \left(\frac{3}{4} \cdot \frac{2}{3} + \frac{1}{4}\right) G_j^M \leq \frac{1}{2} \min(\theta_j+1 - \theta_j, \theta_j - \theta_j-1). \quad (C.25)
\]

Further, by that \(\tilde{\theta}^m_j - \theta_j < G_j^m\) (see (i)) and

\[
2G_j^m + G_j^* = \frac{11}{4} G_j^m + \frac{1}{4} G_j^M \leq \frac{11}{4} G(j) + \frac{1}{4} G_j^M \leq \frac{41}{48} \min(\theta_j+1 - \theta_j, \theta_j - \theta_j-1),
\]

we have

\[
\{\tilde{\theta}^m_j - G_j^m - G_j^* + 1, \ldots, \tilde{\theta}^m_j - G_j^m\} \cap \{\tilde{\theta}^m_j + G_j^m + 1, \ldots, \tilde{\theta}^m_j + G_j^m + G_j^*\} \cap \Theta = \emptyset. \quad (C.26)
\]
From (C.25) and Assumption 5(b) we have
\[ \delta^2_j G^*_j \geq C_1 \max \left\{ \omega^{-2}\omega_{n,p}, (\omega^{-1}\omega \log(p))^{1/(1-\tau)} \right\}, \]
and from (C.26) and Lemma C.2, we have \( \Delta^L_j = \hat{\beta}^L_j - \beta_{j-1} \) and \( \Delta^R_j = \hat{\beta}^R_j - \beta_j \) satisfy
\[
\max \left( 12\sqrt{2s} \sqrt{G^*_j} \right) \leq \frac{24\sqrt{2s} \lambda}{\omega \sqrt{G^*_j}}, \]
\[ |\Delta^L_j(S^c_{j-1})|_1 \leq 3 |\Delta^L_j(S_{j-1})|_1 \] and \[ |\Delta^R_j(S^c_j)|_1 \leq 3 |\Delta^R_j(S_j)|_1, \]
such that the arguments analogous to those employed in the proof of Theorem 1(ii) are applicable to establish the localisation rate of \( \hat{\theta}_j \), which completes the proof.
D Algorithms

Algorithm 1: MOSEG: Single-bandwidth two-stage data segmentation methodology under a regression model.

**input**: Bandwidth $G$, grid resolution $r$, penalty $\lambda$, threshold $D$, $\eta \in (0, 1]$

**initialise**: $\tilde{\Theta} = \emptyset$, $\hat{\Theta} = \emptyset$

// Stage 1
Compute $T_k(G)$ in (2) for all $k \in T = T(r, G)$
Add all $\tilde{\theta}$ satisfying $T_{\tilde{\theta}}(G) > D$ and $\tilde{\theta} = \arg\min_{k \in \{\tilde{g} \in [\eta G] + 1, \ldots, \tilde{g} + |G| \cap T} T_k(G)$ to $\tilde{\Theta}$, and set $\tilde{\Theta} = \{\tilde{\theta}_j, 1 \leq j \leq \hat{q}\}$

// Stage 2
for $j = 1, \ldots, \hat{q}$ do
   Identify $\hat{\theta}_j = \arg\min_{\tilde{\theta}_j \leq \tilde{g} \leq \tilde{g} + G} Q(k; \tilde{\theta}_j - G, \tilde{\theta}_j + G, \hat{\beta}_L^j, \hat{\beta}_R^j)$ with $\hat{\beta}_L^j$ and $\hat{\beta}_R^j$ computed as in (8), and add it to $\hat{\Theta}$
end

return $\hat{\Theta}$

Algorithm 2: MOSEG.MS: Multiscale extension of MOSEG.

**input**: A set of bandwidths $G$, grid resolution $r$, penalty $\lambda$, threshold $D$, $\eta \in (0, 1]$

**initialise**: $\tilde{\Theta}^A = \emptyset$, $\hat{\Theta} = \emptyset$, $C_j = \emptyset$ for all $j$

// Pre-estimator generation
for $h = 1, \ldots, H$ do
   Initialise $\tilde{\Theta}(G_h) = \emptyset$
   Compute $T_k(G_h)$ in (2) for all $k \in T_h = T(r, G_h)$
   Add all $\tilde{\theta}$ satisfying $T_{\tilde{\theta}}(G_h) > D$ and $\tilde{\theta} = \arg\min_{k \in \{\tilde{g} \in [\eta G] + 1, \ldots, \tilde{g} + |G| \cap T_h} T_k(G_h)$ to $\tilde{\Theta}(G_h)$
end

// Anchor change point estimator identification
Identify all $\tilde{\theta}(G) \in \bigcup_{h=1}^H \tilde{\Theta}(G_h)$ satisfying (14), and add all such estimators to $\tilde{\Theta}^A$, which is denoted by $\tilde{\Theta}^A = \{\tilde{\theta}_1^A, 1 \leq j \leq \hat{q}: \tilde{\theta}_1^A < \ldots < \tilde{\theta}_\hat{q}^A\}$

for $j = 1, \ldots, \hat{q}$ do
   // Pre-estimator clustering
   Identify all $\tilde{\theta} \in \bigcup_{h=1}^H \tilde{\Theta}(G_h)$ satisfying (15) and add it to $C_j$
   // Location refinement
   Add $\hat{\theta}_j$ obtained as in (16) to $\hat{\Theta}$
end

return $\hat{\Theta}$
## E Further information on the real datasets

Table E.1 lists the covariates included in the dataset analysed in Section 5.

**Table E.1: Covariates contained in the equity premium dataset analysed in Section 5 (cf. Koo et al. (2020), Table 3)**

| Name | Description |
|------|-------------|
| d/p  | Dividend price ratio: difference between the log of dividends and the log of prices |
| d/y  | Dividend yield: difference between the log of dividends and the log of lagged prices |
| e/p  | Earnings price ratio: difference between the log of earnings and the log of prices |
| d/e  | Dividend payout ratio: difference between the log of dividends and the log of earnings |
| b/m  | Book-to-market ratio: ratio of book value to market value for the Dow Jones Industrial Average |
| nts  | Net equity expansion: ratio of 12-month moving sums of net issues by NYSE listed stocks over the total end-of-year market capitalization of NYSE stocks |
| tbl  | Treasury bill rates: 3-month Treasury bill rates |
| lty  | Long-term yield: long-term government bond yield |
| tms  | Term spread: difference between the long term bond yield and the Treasury bill rate |
| dfr  | Default return spread: difference between the returns of long-term corporate and government bonds |
| svar | Log of stock variance obtained as the sum of squared daily returns on S&P500 index |
| infl | Inflation: CPI inflation for all urban consumers |
| ltr  | Long-term return: return of long term government bonds |