Snap-fits are versatile mechanical designs in industrial products that enable repeated assembling and disassembling of two solid parts. This important property is attributed to a fine balance between geometry, friction, and bending elasticity. In the present study, we combine theory, simulation, and experiment to reveal the fundamental physical principles of snap-fit functions in the simplest possible setup that consists of a rigid cylinder and a thin elastic shell. We construct a phase diagram using geometric parameters and identify four distinct mechanical phases. We develop analytical predictions based on the linear elasticity theory combined with the static friction law and observed excellent agreement between simulations and experiments. The study reveals how an operational asymmetry of snap-fits (i.e., easy to assemble but difficult to disassemble) emerges from an exquisite combination of geometry, elasticity, and friction and suggests optimization of the tunable functionalities in a range of mechanical designs.

Assembling and disassembling two solid components is a fundamental process for functional structures in nature and manmade systems. Examples encompass different length scales ranging from ligand-receptor interactions in biochemistry \[1\] and plastic shell covers in industrial products \[2\] to the docking of free-flying space vehicles \[3\]. In manufacturing industries, snap-fits are typically used to join two plastic parts without glueing, and this constitutes a simple design with necessary resilience to allow for assembly and disassembly numerous times \[4\]. Today, snap-fits are found everywhere including the cap of a marker pen, plastic zipper bag, and toys such as Lego blocks. A “click” sound is a familiar characteristic of a snap-fit in our daily experiences. In most snap-fit designs, assembling requires a relatively small effort while disassembling is more difficult. The mechanical asymmetry is a central property of snap-fits for industrial use and is a property that emerges from an interplay between flexibility, frictional interactions, and geometric structure of snap-fit parts. However, despite its familiarity and prevalent use in daily life, fundamental aspects of snap-fit mechanics are highly unexplored at least from the viewpoint of physics.

In the study, we propose a model experimental system that can illustrate snap-fit behaviors and investigate its physical properties by combining force measurements, numerical simulations, and theoretical analysis. We consider a semi-cylindrical shell of radius \(R_s\) and thickness \(t\) [Fig. 1(a)], which is pushed onto a surface of a rigid cylinder with radius \(R_c\) [Fig. 1(b) and (c)]. The shell either clutches the cylinder via snap-fit or buckles on the cylindrical surface based on their geometries. We integrate experimental and numerical data to construct a phase diagram in terms of the geometric parameters and rationalize the observed phase boundaries with an analysis based on the linear elasticity theory combined with a dry friction law. We demonstrate how an excellent combination of geometry, elasticity, and friction leads to emergent asymmetry between assembling and disassembling forces. The proposed model is minimal albeit versatile and potentially scalable and can be used as a building block in the design of artificial non-reciprocal mechanical meta-materials \[5,7\].

The problem is essentially two-dimensional because a shell deforms uniformly along the cylindrical axis of length corresponding to 20 mm. Hence, we focus on the shapes of a shell’s cross-section with relevant geometric parameters corresponding to the radius ratio \(\alpha = R_c/R_s\) and opening angle, \(\Phi\) [Fig. 1(a)]. A shell is prepared by adding a permanent intrinsic curvature to an initially flat sheet of polystyrene \((t = 0.2, 0.3, \text{and } 0.4 \text{ mm})\) via thermoforming with hot water. Image analysis of the cross-sectional shape in the resulting semi-cylinder confirms a sufficiently uniform radius of curvature that ranges from \(R_c = 25.2 - 29.5 \text{ mm}\) with various angles in the range of \(\Phi = 1.8 - 3.0 \text{ rad}\). The bending moduli of the shells

**FIG. 1:** (a) Definition of geometric parameters of our shell, i.e., radius \(R_s\) and opening angle \(\Phi\) in its natural configuration. (b-c) Sequence of snapshots of a thin plastic shell undergoing Type I snap-fit, (b) and Type II snap-fit (c) process. (d) Schematic photographic view of our experimental system for force measurements during assembling and disassembling processes.
$B = 6.7 \times 10^{-5} - 4.3 \times 10^{-4}$ N·m² for shell thickness $t = 0.2 - 0.4$ mm are independently measured. The shell is sufficiently stiff such that effects of gravity are negligible. We cover the surface of an acrylic cylinder with radius $R_c = 30,35$, and 40 mm with a thin oriented polypropylene (OPP) sheet of thickness 10 μm to ensure uniform frictional interactions with a shell. The stepping motor controls the vertical position of the top of the shell through a force gage while the rigid cylinder is fixed on a bottom substrate [Fig. 1(d)]. In the assembling process, the shell is moved downwards with a speed of 5 mm/s until the shell’s top touches the cylindrical surface. After a 6-s interval, the shell is then moved upwards with the same speed of 5 mm/s (i.e., the disassembling process). A "push-back" force $F$ exerted by the shell is measured with a force gage that is attached to the shell’s top. A snap instability in the assembling process implies that a force curve crosses $F = 0$ from positive to negative. In the $F > 0$ region, the shell repels from the cylinder without loading, while the shell spontaneously clutches the cylindrical surface without any further loading in the $F < 0$ region. The measured force $F$ is displayed in units of $B/R_c^2$.

To complement the experimental results, we also performed numerical simulations using a discrete analog of the continuum elastica model [8]. The frictional interaction between the shell and cylindrical surface is modeled based on Amonton–Coulomb’s law that states that the contact point remains stationary if the tangential force is below the critical value $\mu P$ where $P$ denotes the normal reaction and $\mu$ denotes the coefficient of static friction [9]. (Full details of the numerical method are given in Supplemental Material [10].) We compare the experimental force curves with those obtained from the simulations in Fig. 2(b)–(d) and an excellent agreement is realized. From this, the value of the static friction coefficient in the experiments is typically determined as $\mu = 0.21$.

Experimental and numerical investigations are summarized in Fig. 2(a). We identify four different phases that are broadly divided into two distinct domains, i.e., the snap-on domain and misfit domain. The snap-on domain consists of the two phases that are termed as (i) Type I snap-fit and (ii) Type II snap-fit phases. The misfit domain consists of the remaining two phases that are termed as (iii) Type I misfit and (iv) Type II misfit. In two Type I phases [(i) and (iii)], a shell is only moderately deflected. The shell is pushed onto the surface, and its groove opens and the force rapidly increases [Fig. 2(b) and (c)]. Past the force maximum, it decreases with increases in the vertical displacement $u$. For a sufficiently deep shell of $\Phi > \Phi_{sf}$ (explained below) and of a relatively small size difference $\alpha$, the force crosses $F = 0$ to snap. See Supplemental Material for Video S1 [10]. For $\Phi < \Phi_{sf}$ or for $\alpha > 1.5$, the top of the shell touches the cylindrical surface before $F$ corresponds to zero. If the loading is removed, the shell either pushes back or retains its instant shape based on $\mu$. In Type I region, the force discontinuously assumes a substantial negative value when the shell is pulled out from the cylinder. This is because the normal reaction increases discontinuously when its vertical component changes sign as the loading switches from pushing to pulling. The discontinuous force jump in Fig. 2(b)–(d) manifests the critical role of static friction on the hysteretic non-reciprocal responses. In simulations, it is confirmed that a force response is completely reversible in the absence of friction, $\mu = 0$. In contrast to a common cantilever snap-fit design [4] [11], purely geometry-based asymmetry is absent. A key issue in the present study is that the asymmetry is due to coupling between geometry and friction.

Conversely, the Type II phases [(ii) and (iv)] involve high deflections. It is noted that the maximum assembling force in (ii) is approximately ten times that in (i). In (ii) and (iv), a shell is strongly squeezed to assume an M-shape configuration with its two ends being rolled up. At a critical compression, the two ends suddenly jump outwards such that the shell is outstretched to create a surprisingly loud snapping sound. See Supplemental Material for Video S2 [10]. The shell eventually “snaps-on” to the cylindrical surface in (ii) whereas the misfit is observed in (iv) because $\alpha$ is excessively high in which the shell bounces back (or maintains its instant shape) if the loading is removed. It is noted that in (ii), the assembling force highly exceeds the disassembling force, which is completely opposite to that expected in industrial designs. However, the behavior is potentially interesting in terms of efficient energy-absorbing devices [12]. Specifically, the diagram is overall insensitive to the modulus $B$, thickness $t$, and friction coefficient ($0.2 < \mu < 0.5$), thereby indicating that it is highly geometrical.

To understand the qualitatively distinct behaviors, we now develop an analytical argument. We first define a coordinate system as shown in Fig. 1(d). An important insight is obtained from the simulations. For most parameter sets ($\alpha, \Phi, \mu$), a shell touches a cylinder only at its two edges, thereby indicating that the contact always corresponds to point-like (or, line-like in the three dimensional view). This might be counter-intuitive because an evidently areal contact is typically observed between the shell and cylinder surface around snap-on configurations in experiments. However, discrete contact is a direct consequence of the moment-free boundary conditions at shell edges combined with the mismatch of two natural curvatures, i.e., $\alpha \neq 1$ [13]. Given that external forces are only applied at three discrete points on the shell, the overall vertical force balance requires that the sum of the forces must vanish irrespective of the shape of a shell, $0 = -P + 2P\cos \varphi + 2Q\sin \varphi$, where $P$ and $Q$ denote the normal and tangential components of the force exerted from the surface and we assume the symmetric deformations of the shell. In a quasi-static process, the critical condition $Q = \mu P$ can hold, and this leads from the above force balance to the following expression

$$\frac{F}{F_\parallel} = \frac{2(1 + \mu \tan \varphi)}{\tan \varphi - \mu}.
(1)$$
where $F_\parallel = P \sin \varphi - Q \cos \varphi$ denotes the horizontal component of the force [14]. A similar formula is valid for the disassembling process with the replacement given by $\mu \to -\mu$ in Eq. (1). Several important conclusions are obtained from Eq. (1). First, the snap-fit point, $F = 0$, is given by $1 + \mu \tan \varphi^* = 0$. For snap-fit bifurcation to occur, the shell’s (half) contour length $R_c \Phi$ must exceed the arclength along the cylindrical surface, $R_c \varphi^*$, and thus we predict the following expression

$$\Phi > \Phi_{sf} \approx \alpha \left( \pi - \tan^{-1} \frac{1}{\mu} \right).$$

As shown in Fig. 2(a), Eq. (2) perfectly explains the phase boundary between (i) and (iii). For $\mu \to 0$, we obtain $\Phi_{sf} \to (\pi/2) \alpha$. A shell snaps when it crosses the "equator" of a cylinder, and this confirms our intuition.

Equation (1) indicates that $F_\parallel$ vanishes at $\varphi = \tan^{-1} \mu$. For $\varphi > \varphi_c$, we find $F_\parallel > 0$, and the surface tends to open the groove. Conversely, for $\varphi < \varphi_c$, we observed $F_\parallel < 0$; the reaction force from the surface acts to close the shell’s groove. Let $\varphi_0$ be an initial value of $\varphi$ at which a shell touches the cylinder. If $\varphi_0 < \varphi_c$, the shell’s edges are pinned when the shell touches the surface since $F_\parallel < 0$. The pinned configuration is increasingly stabilized when the shell is compressed further and finally leads to a high amplitude Type II snap. Thus, the condition, $R_c \sin \Phi = R_c \sin \varphi_0 \lesssim R_c \sin \varphi_c$, yields the sufficient condition for Type II snap-fit, i.e.,

$$\Phi > \Phi_{I-II} \approx \pi - \sin^{-1} \left( \frac{\alpha \mu}{\sqrt{1 + \mu^2}} \right).$$

In Fig. 2(a), we plot Eq. (3) and observe that $\Phi > \Phi_{I-II}$ yields a sufficient condition for Type II phase. Furthermore, although it is less predictive when compared to Eq. (2), it also captures a rough qualitative trend in Type I and II boundary lines.

In an ideal snap-fit design, two solid parts are reasonably easy to assemble, while disassembling the same is more difficult albeit not excessively. This type of a medium asymmetry can be read in force curves in Fig. 2(b) and (d) as the criterion in which the maximum force in magnitude in the disassembling process, $|F_D|$, is higher although not excessively higher than that in the assembling process, $F_A$. (It is noted that $F_D < 0$ for a snap-on configuration.)

In Fig. 3(a) and (b), $F_A$ and $|F_D|$ obtained from the experiment and simulation are plotted as a function of $\Phi$ for the frictionless ($\mu = 0$) and frictional ($\mu = 0.21$) cases. In Fig. 3(c), the data for $\mu = 0.21$ are replotted in a form as $\Phi$ vs. $|F_D|/F_A$, a metric defined as "locking ratio" in Ref. [15]. Overall, a desirable condition $|F_D|/F_A > 1$ is achieved for $2 < \Phi < 2.6$, and this highly overlaps with Type-I snap-fit regime. The trend is valid for other typical values of $\mu$, thereby suggesting
that a relative magnitude of $|F_D|$ and $F_A$ can only be tuned with shell geometry, $\Phi$.

We now rationalize the above findings with an analysis based on the theory of elastica with natural curvatures. We are interested in Type I phases, and thus we employ a small deflection approximation. An important observation is that a dominant shell deformation in the regime corresponds to its groove opening due to the horizontal component of the normal reaction. Conversely, the deformation of the shell's cross-sectional shape due to the vertical compressional force is significantly less than that due to the horizontal force. We note that the vertical force $F$ is related to the horizontal force $F_\parallel$ via Eq. (1), and thus we can approximately predict $F$ by knowing $F_\parallel$ for a given configuration of the shell without any vertical compression. Given the simplification, we significantly reduce the mathematical complexity in the following analysis.

We consider a small deformation of a naturally curved elastica subjected to outgoing horizontal forces $F_\parallel$ applied at the two edges. By expanding relevant equations in terms of $F_\parallel R_2^2/B$ up to the first order, and imposing the inextensibility constraint, we obtain a linear relation $F_\parallel R_2^2/B = K(\Phi)\Delta/R_s$, where $\Delta$ denotes the horizontal displacement and

$$K(\Phi) = \left[\frac{1}{2} \Phi - \cos \Phi \left(\frac{3}{2} \sin \Phi - \Phi \cos \Phi\right)\right]^{-1}$$

denotes the $\Phi$-dependent effective spring constant. (For details, see our Supplemental Material.) We combine this with Eq. (1) to obtain an analytic expression for $F$ in terms of $\Delta$, where $\varphi$ is related to $\Delta$ and $\Phi$ via $\alpha \sin \varphi = \sin \Phi + \Delta/R_s$. For $\mu = 0$, it assumes a particularly compact form given by $FR_2^2/B = 2K(\Phi) \cot \varphi(\alpha \sin \varphi - \sin \Phi)$. We maximize this with respect to $\varphi$ and obtain the following expression

$$\frac{F_A R_2^2}{B} = 2\alpha K(\Phi) \left[1 - \left(\frac{\sin \Phi}{\alpha}\right)^{2/3}\right]^{3/2},$$

which is in excellent agreement with the simulation data [Fig. 3(a)].

A frictional case ($\mu > 0$) is analyzed similarly although it is mathematically more involved. Given $\mu \ll 1$ for typical cases, we employ a perturbative approach and explore an explicit expression for $F_\parallel$ to the first order of $\mu$. The result is plotted as a dashed line in Fig. 3(b) and is in good agreement with simulations and experiments. Overall, the results validate the assumption that the opening of the lead-in part of the shell dominates its elastic response.

Similarly, the disassembling force $|F_D|$ is evaluated for a shell that is slightly deformed from its snap-on configuration. We assume that the shell has its uniform curvature close to $1/R_s$ with the exception of the edge regions where the shell curvature is $\sim 1/R_s$. Hence, we use Eq. (1) with $\mu \to -\mu$ and $\varphi \simeq \Phi/\alpha$. The resulting formula is then combined with the derived linear relation, as stated above, for $F_\parallel$ to yield

$$\frac{F_D R_2^2}{B} = \frac{2(1 - \mu \tan(\Phi/\alpha))}{\mu + \tan(\Phi/\alpha)} \left(1 - \alpha^{-1}\right)\Phi \sin \Phi - \Phi \cos \Phi. \quad (6)$$

Equation (6) is observed to agree well with our simulation and experiments as shown in Fig. 3(a) (for $\mu = 0$) and (b) (for $\mu = 0.21$). The agreement is less satisfactory for high $\Phi$ where the assumption that the shell curvature is $\sim 1/R_s$, i.e., $\varphi \simeq \Phi/\alpha$, becomes systematically inaccurate for increase in $\Phi$.

Equation (6) suggests that $F_D$ diverges as $\Phi \to \alpha(\pi - \tan^{-1} \mu)$. The "locking" occurs because the external pulling force increases the normal reaction to increase the tangential component of the friction force and further increase the external force necessary for the disassembling. The friction-mediated self-stiffening mechanism exists for structures with curved geometry and is more prominent for a deeper shell.

Interestingly, the snap-fit behavior favorable to industrial uses, i.e., Type I snap-fit, is only achieved for a limited range of the geometric designs. To expand a design space for the snap-fits, the formalism should be generalized to account for other mechanical aspects such as stretch of a shell and deformations of a cylinder, which are likely to be important for thicker shells and soft constituent materials such as elastomers. In real experiments, air-flow induced effects, such as a negative pressure by the suction of air during snapping, can alter frictional interactions. Therefore, the estimated value, $\mu = 0.21$, can be interpreted as an effective value that accounts for the aforementioned types of physical effects. Hence, the validity of Amonton’s law and possibilities of other friction laws should be examined based on systems under considerations.
To the best of the authors’ knowledge, this is the first detailed study of snap-fit mechanics in the context of the physics of thin structures [16]. The results of the study reveal a quantitative design space for snap-fits and illustrate how an exquisite combination of geometry, elasticity, and friction leads to emergent mechanical asymmetry between assembling and disassembling processes. The proposed model is potentially scalable, and the clarified route for such non-reciprocal force responses can inspire a new class of energy absorbing metamechancial materials [5,7]. The study is also potentially insightful to envision a future study suitable for sustainable materials that can ultimately contribute to reduce plastic waste.

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Appendix A: Linearized theory

Consider a small deformation of a naturally curved elastica of a constant arclength $L = 2R_s \Phi$ subjected to outgoing horizontal forces $F_\parallel$ applied at the two edges. We assume a planar, symmetric deformation on the $x$-$y$ plane, with the horizontal displacement $\Delta$ as shown in Fig. 4(a). We parameterize a centerline of the cross-section of the shell, i.e., our elastica, by its arc length parameter $s$, starting at the left end, with the unit tangent vector $\textbf{t}(s) = \cos \theta(s) \hat{e}_x + \sin \theta(s) \hat{e}_y$ and the unit normal $\textbf{n}(s) = -\sin \theta(s) \hat{e}_x + \cos \theta(s) \hat{e}_y$. See Fig. 4(b).

In the absence of any external body forces and moments, the mechanical equilibrium is described by the Kirchhoff rod equations given by

$$\textbf{F}'(s) = 0, \quad M'(s) + \hat{t}(s) \times F(s) = 0,$$  \hspace{0.5cm} (A1)

where $\textbf{F}(s)$ and $M(s)$ are the internal force and moments in the elastica and $\cdot'$ denotes a derivative with respect to $s$. We assume the linear constitutive relation (i.e., the Hooke’s law) $M(s) = B(\kappa(s) - \kappa_0) \hat{e}_z$, where $B$ is the bending modulus and $\kappa(s) = \textbf{t}'(s) \cdot \textbf{n}(s)$ is the curvature. Note that $\kappa_0 = -1/R_s (< 0)$ is the natural curvature of our elastica. Solving the force balance equation to obtain $\textbf{F}(s) = F_\parallel \hat{e}_x$ and plugging this back to the equation for $M$ in Eq. (A1), we obtain the elastica equation given by

$$B \theta''(s) - F_\parallel \sin \theta(s) = 0,$$  \hspace{0.5cm} (A2)

together with the relevant boundary conditions $\theta(0) = \theta_0$ and $\theta'(0) = \kappa_0 = -1/R_s$. Note that we will determine $\theta_0$ in the following calculations so that the inextensibility condition for elastica is satisfied.

To develop a linear response theory, we expand the first integral form of Eq. (A2) in powers of a small parameter

$$\epsilon = \frac{F_\parallel R_s^2}{B} \ll 1$$  \hspace{0.5cm} (A3)

as

$$\frac{ds}{R_s} = -\frac{d\theta}{\sqrt{1 - 2\epsilon(\cos \theta(s) - \cos \theta_0)}} \hspace{0.5cm} (A4)$$

$$= -d\theta \left[ 1 + \epsilon(\cos \theta(s) - \cos \theta_0) \right] + O(\epsilon^2).$$  \hspace{0.5cm} (A5)

Note that we have chosen the negative sign in front of the square root bracket on the r.h.s of Eq. (A4) since $\theta'(s) < 0$ in our coordinate system. Plugging Eq. (A5) into the inextensibility condition of the elastica, $\int_0^{L/2} ds = R_s \Phi$, we find

$$\theta_0 = \Phi - (\sin \Phi - \Phi \cos \Phi) \epsilon + O(\epsilon^2).$$  \hspace{0.5cm} (A6)

In the horizontal direction, we have a configurational relation given by

$$x(L/2) - x(0) = \int_0^{L/2} \cos \theta(s) ds = R_s \sin \Phi + \Delta.$$  \hspace{0.5cm} (A7)

Plugging Eq. (A5) into Eq. (A7), we have

$$x(L/2) - x(0) = -R_s \int_0^{\theta_0} [1 + \epsilon(\cos \theta(s) - \cos \theta_0)] \cos \theta(s) d\theta + O(\epsilon^2).$$  \hspace{0.5cm} (A8)

FIG. 4: (a) Schematics of a naturally curved planar elastic line (elastica) subjected to a pair of outward horizontal forces applied at both ends with corresponding displacements $\Delta$. (b) Definition of the coordinate system and the various geometrical and physical parameters. (c) Comparison between our numerical simulation results and the analytic predictions given in Eqs. (A10) and (A11), showing an excellent agreement for $\Phi = 2.5$. 

- \hspace{0.5cm} - \hspace{0.5cm} - \hspace{0.5cm} - \hspace{0.5cm} -
Performing this integral and substituting Eq. (A6) into the resulting expression, and retaining the first order in $\epsilon$, we can obtain $\Delta$ as a function of $\epsilon = F_\parallel R_s^2/B$ from Eq. (A7), which is expressed as

$$
\frac{F_\parallel R_s^2}{B} = K(\Phi) \frac{\Delta}{R_s},
$$

(A10)

where the rescaled spring constant is given by

$$
K(\Phi) \equiv \left[ \frac{1}{2} \Phi - \cos \Phi \left( \frac{3}{2} \sin \Phi - \cos \Phi \right) \right]^{-1}.
$$

(A11)

This is Eq. (4) given in the main text. As shown in Fig. 1 (c), we confirm that the prediction based on Eqs. (A10) and (A11) is in excellent agreement with the corresponding numerical simulation results.

**Appendix B: Assembling force in the frictional case**

For the frictional case $\mu > 0$, we use Eq. (1) in the main text to find an approximate expression of $F$ from $F_\parallel$ given in Eq. (A10) under the assumption explained in the main text. This leads to

$$
\frac{F R_s^2}{B} = 2\alpha K(\Phi) \frac{1 + \mu \tan \varphi}{\tan \varphi - \mu} \left( \sin \varphi - \frac{\sin \Phi}{\alpha} \right).
$$

(B1)

In principle, we can obtain $F_\parallel$ by maximizing $F$ given in Eq. (B1) with respect to $\varphi$, which, however, turns out to be involved to perform. Instead, we take advantage of the fact that $\mu \ll 1$ for typical materials, and seek for the maximal value of $F$ perturbatively with respect to $\mu$. By introducing a variable $q \equiv \sin \varphi$, we see $0 < q \leq 1$ because $0 < \varphi \leq \pi/2$, we can rewrite Eq. (B1) as

$$
\frac{F R_s^2}{B} = 2\alpha K(\Phi) \frac{1 + \mu q/\sqrt{1-q^2}}{\sqrt{1-q^2} - \mu} \left( q - q_0^3 \right).
$$

(B2)

Note that $q_0 \equiv \{(\sin \Phi)/\alpha\}^{1/3}$ is the solution that maximizes Eq. (B2) for $\mu = 0$. The extremal condition, $dF/dq = 0$, then leads to the equation

$$
q^3 - q_0^3 - \mu(2q^2 - 1)\sqrt{1-q^2} + O(\mu^2) = 0.
$$

(B3)

We explore a solution of Eq. (B3) in a form assumed as $q = q_0 + c_1 \mu + O(\mu^2)$, and substitute it into Eq. (B3), we obtain $c_1 = (2q_0^2 - 1)\sqrt{1-q_0^2}/(3q_0^3)$. The assembling force $F_\parallel$ is determined by substituting $q = q_0 + c_1 \mu$ back into Eq. (B1), which is plotted in Fig. 3 (b) in the main text as the dashed line.

**Appendix C: Simulations**

In this section, we explain the detailed procedure of our numerical simulation that is the discrete analog of the continuum elastica model. An elastic curve is modeled as the chain of straight segments of length $b_0$, and at each node $i$, where $i = 1, 2, ..., N$, the position vector $r_i$ with equal mass $m$ is assigned. In the discrete model, a unit tangent vector is $\hat{u}_i \equiv (r_{i+1} - r_i)/b_i$, where $b_i \equiv |r_{i+1} - r_i|$ is the instantaneous bond length. To take into account the bending elasticity, we define a bending angle between any two consecutive bonds as $\hat{u}_{i+1} \cdot \hat{u}_i = \cos \beta_i$ for $i = 1, 2, ..., N - 1$. Note that $\beta_i = \beta_0 = 2\pi/(N-1)$ for a stress-free configuration of our naturally curved elastic chain. The total elastic energy of our system consists of two parts, i.e., $E_{tot} = E_{str} + E_{bend}$, where

$$
E_{str} = \sum_i \frac{k}{2}(b_i - b_0)^2, \quad E_{bend} = \sum_i \frac{k_b}{2}(\beta_i - \beta_0)^2.
$$

(C1)

are the stretching and bending energies, respectively. The stretching energy $E_{str}$ ensures the connectivity of the bonds. The bending forces acting on $i$-th node, $F_i^b$, can be obtained by considering the variation $\delta E_{bend}$ with respect to $\delta r_i$, which is given by

$$
F_i^b = k_b H_{i-1}(\beta_{i-1} - \beta_{i-2}) - k_b H_i(\beta_i - \beta_{i-1}),
$$

(C2)

where

$$
H_i \equiv \hat{e}_z \times \hat{u}_i.
$$

(C3)

Note that the resulting expressions, Eq. (C2), has been found by resolving $\hat{u}_{i \pm 1}$ with respect to $\hat{u}_i$ and $\hat{e}_z \times \hat{u}_i$, and are free from any spurious divergence at $\beta = 0$. In order to prevent any overlap between the elastica and a rigid cylinder of radius $R_c$ whose center is fixed at the origin, we add forces from the cylindrical surface $F_{\parallel sub}$ which are given by

$$
F_{\parallel sub}^{s} = -k_{x_{\parallel sub}} (|r_i| - R_c) \frac{r_i}{|r_i|} \Theta(R_c - |r_i|),
$$

(C4)

where $\Theta(x)$ is the step function such that $\Theta(x) = 1$ for $x > 0$ otherwise $\Theta(x) = 0$. The time evolution of each node is described by the Newton’s equation of motion given by

$$
m \frac{d^2 r_i}{dt^2} = F_i^{s} + F_i^{b} + F_i^{\parallel sub} - \gamma \frac{dr_i}{dt},
$$

(C5)

where an appropriate damping term that is linearly proportional to the velocity with the damping coefficient $\gamma$ has been included so that the system can reach its mechanical equilibrium under given conditions.

We discretize Eq. (C5) with the time step $\Delta t$, choose the units of length, energy, and time as the bond length $b_0$, $k_b$, and $\tau = b_0(m/k_b)^{1/2}$, and rescale all the variables in Eq. (C5) appropriately. The stretching modulus $k$ is taken sufficiently large, i.e., $k/k_b = 50$, in order to restrict the bond length variations negligibly small. For a sufficient numerical accuracy, we choose a rescaled time step $\Delta t/\tau = 0.05$, and the rescaled damping coefficient
just above a rigid cylinder of radius $R_c$. The position of the center node of the elastic chain is controlled externally; it is allowed to move (either upwards or downwards) only vertically at a constant speed that is confirmed to be sufficiently slow so that the process is well regarded as quasi-static, with all the rest of the nodes evolve according to the discretized version of Eqs. (C5).

**Appendix D: Diagrams for low and high $\mu$**

In Fig. 5 (a) and (b), we show the phase diagrams obtained from our numerical simulations for small ($\mu = 0.1$, (a)) and large frictional coefficients ($\mu = 0.5$, (b)). For small frictional interactions (typically $\mu \lesssim 0.1$), we observe a phase that is not seen for larger $\mu$ cases, which we call "slip" phase, around $\Phi \gtrsim 2.8$. See Fig. 5 (a). In the slip phase, a shell does not open its groove, but slips on the cylindrical surface that involves the left-right symmetry breaking.

For large frictional interactions, $\mu = 0.5$, we again observe a phase that is not seen for $\mu = 0.21$ or any $\mu$ smaller than that, which we call "buckling" phase. See Fig. 5 (b). In this phase, a shell ends up with a characteristic M-shape configuration when it is maximally pushed onto the surface of the cylinder. We do not observe Type II phases in $\mu = 0.5$; they are dominated by the buckling phase.

With regard to Type I phases, the overall pattern of the diagram is robust, i.e., largely independent of the static friction coefficients $\mu$. However, the diagrams are qualitatively different for a sufficiently deep shells (i.e., at high $\Phi$ regime), and for different frictional coefficients $\mu$. This is probably because the assembling process often involves large shell deflections for high $\Phi$, which may lead to a variety of shell behaviors depending on $\mu$.

For low $\Phi$ (typically less than 2.5 rad), our analytical prediction given in Eq. (2) in the main text still describes the phase boundaries between Type I snap-fit and Type I misfit very well, both for $\mu = 0.1$ and 0.5, as seen in Fig. 5 (a) and (b), respectively. Moreover, the approximate analytical predictions for $F_A$ and $|F_D|$ also show reasonably well agreements with the corresponding simulation data for $\mu = 0.5$. See Fig. 5 (c).

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