Squeezed-light source for the superresolving microscopy

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We propose a source of multimode squeezed light that can be used for the superresolving microscopy beyond the standard quantum limit. This source is an optical parametric amplifier with a properly chosen diaphragm on its output and a Fourier lens. We demonstrate that such an arrangement produces squeezed prolate spheroidal waves which are the eigen modes of the optical imaging scheme used in microscopy. The degree of squeezing and the number of spatial modes in illuminating light, necessary for the effective object field reconstruction, are evaluated.

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Spatial behavior of nonclassical light is presently attracting an increasing interest both in theory and experiment\textsuperscript{1,2}. Quantum effects in optical imaging and other transverse spatial phenomena are studied within a framework of a European project "Quantum Imaging"\textsuperscript{3}. One of the problems recently addressed in this context is about the ultimate quantum limit of the optical resolution\textsuperscript{4}. A classical criterion of resolution was formulated at the end of the last century by Abbe and Rayleigh. It states that optical resolution is limited by diffraction on the system pupil. However, though it might be difficult to accept, the diffraction limit is not a fundamental limit like, for example the Heisenberg uncertainty relation. Recent publications have theoretically demonstrated that in principle this limit can be beaten for both writing and reading of optical information.

Sub-diffraction-limited optical recording of information, coined as "quantum lithography", uses a specially designed nonclassical light, N-photon entangled state, together with a N-photon absorber which allows to reach theoretically a resolution of $\lambda/N$\textsuperscript{5,6}. To go beyond the diffraction limit in read-out of optical information like in imaging, is in principle possible having some a priori information about the object using the so-called "superresolution" techniques. As shown recently in Ref.\textsuperscript{7,8} the ultimate limit in such superresolution is set not by diffraction but by the quantum fluctuations of light illuminating the object of finite size and by the vacuum fluctuations outside the object. These quantum fluctuations of the illuminating light together with the vacuum fluctuations outside the object set up the \textit{standard quantum limit of superresolution} which can be much smaller than the diffraction limit. Moreover, one can go beyond this standard quantum limit using a specially designed nonclassical light for illumination of the object. In recent experiment\textsuperscript{7,8} it was demonstrated that using multimode squeezed light allows to measure a displacement of 2.9 Å of a laser beam with wavelength $\lambda = 1064$ nm.

In Ref.\textsuperscript{7} Kolobov and Fabre have suggested a scheme that allows to improve the superresolution beyond the standard quantum limit in reconstruction of an optical object using a multimode squeezed light. This scheme was formulated in terms of the so-called prolate spheroidal waves which are the eigenfunctions of the optical imaging scheme. To achieve the superresolution with multimode squeezed light one has to prepare these prolate waves in squeezed state. The question remained on how to produce such squeezed prolate waves.

In this letter we provide the answer to this question. Precisely, we demonstrate that an optical parametric amplifier (OPA) with a properly chosen diaphragm on its output and a Fourier lens, produces squeezed prolate spheroidal waves used in the superresolving microscopy. We investigate the quantum statistics of the squeezed prolate spheroidal waves in our scheme in dependence of the physical parameters of the OPA and of the optical configuration. We formulate simple estimates on the number of the "object elements" to be reconstructed in connection with the number of degrees of freedom in the nonclassical illuminating light.

The scheme of optical imaging with multimode squeezed light is shown in Fig. 1. For simplicity we consider a one-dimensional case. The part of the scheme to the right of the object plane performs diffraction-limited imaging of an object of finite size $X$ located in the object plane\textsuperscript{9}. The first lens $L_1$ performs the Fourier transform of the object field into the pupil plane with a pupil of size $d$. After the second Fourier transform by the lens $L_2$ an image is created in the image plane. This image is a diffraction-limited copy of the object due to the finite size of the pupil.

The part to the left of the object plane is an illumination scheme. It consists of a traveling-wave OPA placed in the source plane and a Fourier lens $L$. It is well-known from the literature that a traveling-wave OPA with plane-wave pump and nonlinear crystal with large transverse area creates multimode squeezed vacuum on its output\textsuperscript{10}. A new feature of our scheme is a diaphragm of size $d_s$ on the output of the OPA which serves for selection of the transverse modes in squeezed state. As we demonstrate below, when the size of this diaphragm matches the size of the pupil, $d_s \geq d$, this setup squeezes exactly the prolate spheroidal waves which are the eigen modes of the imaging scheme.
FIG. 1: Schematic of the optical imaging with squeezed light.

This result can be easily understood qualitatively. Indeed, when all three lenses in the scheme have the same focal distance \( f \), as in Fig. 1, the lenses \( L \) and \( L_1 \) create a geometrical image of the diaphragm \( d_s \) in the pupil plane. Therefore, it is intuitively clear that one has to match the diaphragm size \( d_s \) and the pupil size \( d \) to select the modes of the source which will pass through the imaging scheme.

Let us introduce the dimensionless coordinates in the object and the image plane as \( s = \frac{2x}{X} \), and the dimensionless coordinates in the source and the pupil plane as \( \xi = \frac{2y}{d} \) (see Fig. 1). The dimensionless photon annihilation operators in the source, object, pupil, and image planes are denoted respectively as \( \hat{c}(\xi) \), \( \hat{a}(s) \), \( \hat{f}(\xi) \), and \( \hat{e}(s) \). These operators obey the standard commutation relations,

\[
[\hat{c}(\xi), \hat{c}^\dagger(\xi')] = \delta(\xi - \xi'), \quad [\hat{a}(s), \hat{a}^\dagger(s')] = \delta(s - s'),
\]

and similar for \( \hat{f}(\xi) \), and \( \hat{e}(s) \). Naturally, the annihilation and creation operators at different planes commute with each other. These operators are normalized so that \( \langle \hat{c}^\dagger(\xi) \hat{c}(\xi) \rangle \), for example, gives the mean photon number per unit dimensionless length in the source plane. Similar normalization is valid for the other planes. The Fourier transform \( (T\hat{c}) (s) \) physically performed by the lens \( L \) in Fig. 1 reads as follows,

\[
\hat{a}(s) = (T\hat{c})(s) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} d\xi e^{-ics} \hat{c}(\xi),
\]

where \( c = \frac{\pi dX}{2\lambda f} \) is the space-bandwidth product of the imaging system.

The quantum theory of the imaging scheme in Fig. 1 was formulated in Ref. 4 in terms of the prolate spheroidal functions \( \psi_k(s) \) (for some examples see Ref. 10). These are the eigen functions of the imaging operator of the scheme, orthonormal on the interval \(-\infty < s < \infty\). The photon annihilation operator \( \hat{a}(s) \) in the object plane can be written as decomposition over \( \psi_k(s) \),

\[
\hat{a}(s) = \sum_{k=0}^{\infty} \hat{A}_k \psi_k(s) + \hat{N}(s),
\]

where the operator-valued coefficients \( \hat{A}_k \) are evaluated as

\[
\hat{A}_k = \int_{-\infty}^{\infty} ds \hat{a}(s) \psi_k(s).
\]

The operators \( \hat{A}_k \) and \( \hat{A}^\dagger_k \) satisfy the standard commutation relations of the photon annihilation and creation operators for discrete modes,

\[
[\hat{A}_k, \hat{A}^\dagger_{k'}] = \delta_{k,k'}, \quad [\hat{A}_k, \hat{A}_{k'}] = 0.
\]

The Fourier transform \( \tilde{\psi}_k(\xi) \) of the prolate spheroidal functions \( \psi_k(s) \), performed by the lens \( L_1 \), is zero outside the interval \( |\xi| \leq 1 \). Consequently, the set of functions \( \{\psi_k(s)\} \) is not complete in the Hilbert space \( L^2(-\infty, \infty) \), and to
satisfy the commutation relations (1) one has to add an additional term \( \hat{N}(s) \). This term has zero Fourier spectrum in the interval \( |\xi| \leq 1 \) and does not contribute to the coefficients \( \hat{A}_k \),

\[
\int_{-\infty}^{\infty} ds \psi_k(s)\hat{N}(s) = 0. \tag{6}
\]

The field \( \hat{N}(s) \) can be decomposed over the complementary set of prolate functions \( \{\theta_k(s)\} \), orthogonal to \( \{\psi_k(s)\} \).

Physically speaking, the first term in Eq. (5) is the object field component that propagates in our scheme through the pupil to the image plane. The second object field component \( \hat{N}(s) \) is absorbed outside the pupil and is not observed. Therefore, in what follows we shall omit \( \hat{N}(s) \) in the object field (4).

To obtain the canonical transformation of the photon annihilation and creation operators in the imaging scheme on Fig. 1, one has to split the coordinates \( s \) and \( \xi \) into two regions, the "core", \(|s| \leq 1 \) and \(|\xi| \leq 1 \), corresponding to the central regions of the object (image) and the source (pupil), and the "wings", \(|s| > 1 \) and \(|\xi| > 1 \), outside these areas. The orthonormal bases in these regions of the object (image) plane are given by

\[
\varphi_k(s) = \begin{cases} 
\frac{1}{\lambda_k^{1/2}} \psi_k(s) & |s| \leq 1, \\
0 & |s| > 1,
\end{cases} \quad \chi_k(s) = \begin{cases} 
\frac{1}{\sqrt{\lambda_k}} \psi_k(s) & |s| \leq 1, \\
\frac{1}{\sqrt{1-\lambda_k}} \psi_k(s) & |s| > 1,
\end{cases}
\tag{7}
\]

where \( \lambda_k \) are the eigenvalues of the corresponding prolate spheroidal functions \( \psi_k(s) \), depending on the space-bandwidth product \( c \). We should note that the functions \( \varphi_k(s) \) and \( \chi_k(s) \) are complete in the Hilbert space \( L^2(-1,1) \).

It follows from (1) that the functions \( \psi_k(s) \) and \( \theta_k(s) \) can be written in the form

\[
\psi_k(s) = \sqrt{\lambda_k} \varphi_k(s) + \sqrt{1-\lambda_k} \chi_k(s), \quad \theta_k(s) = \sqrt{1-\lambda_k} \varphi_k(s) - \sqrt{\lambda_k} \chi_k(s). \tag{8}
\]

Similar relations take place in the source (pupil) plane.

In terms of two sets \( \{\varphi_k(s)\} \) and \( \{\chi_k(s)\} \) we can write the annihilation operators in the source and the object planes as

\[
\hat{c}(\xi) = \int_{-\infty}^{\infty} d\xi \hat{c}(\xi) \varphi_k(\xi), \quad \hat{d}(s) = \int_{-\infty}^{\infty} d\xi \hat{d}(\xi) \chi_k(\xi). \tag{9}
\]

and

\[
\hat{a}(s) = \sum_{k=0}^{\infty} \hat{a}_k \varphi_k(s) + \sum_{k=0}^{\infty} \hat{b}_k \chi_k(s). \tag{10}
\]

Here \( \hat{c}_k \) and \( \hat{a}_k \) are the annihilation operators of the prolate modes \( \varphi_k \) in the core region of the source and the object planes, while \( \hat{d}_k \) and \( \hat{b}_k \) are the annihilation operators of the prolate modes \( \chi_k \) in the wings regions. The operators \( \hat{c}_k \) and \( \hat{d}_k \) are expressed through the field operator \( \hat{c}(\xi) \) by

\[
\hat{c}_k = \int_{-\infty}^{\infty} d\xi \hat{c}(\xi) \varphi_k(\xi), \quad \hat{d}_k = \int_{-\infty}^{\infty} d\xi \hat{c}(\xi) \chi_k(\xi). \tag{11}
\]

Similar relations hold for \( \hat{a}_k, \hat{b}_k \) and \( \hat{a}(s) \).

In our analysis we shall use the following property of prolate spheroidal functions (10),

\[
\int_{-1}^{1} d\xi \psi_k(\xi)e^{-iq\xi} = (-i)^k \sqrt{\frac{2\pi\lambda_k}{c}} \psi_k(q/c). \tag{12}
\]

Using (7), (8), the field transform (4) between the source and the object plane and this equation, one can find the following propagation relations for the core and wings of the light wave emitted by the source:

\[
(T\varphi_k)(s) = (-i)^k \left[ \sqrt{\lambda_k} \varphi_k(s) + \sqrt{1-\lambda_k} \chi_k(s) \right] = (-i)^k \psi_k(s),
\]

\[
(T\chi_k)(s) = (-i)^k \left[ \sqrt{1-\lambda_k} \varphi_k(s) - \sqrt{\lambda_k} \chi_k(s) \right] = (-i)^k \theta_k(s). \tag{13}
\]
Taking into account (9) and (10) we obtain,
\[
\hat{a}(s) = (T\hat{c}) (s) = \sum_{k=0}^{\infty} (-i)^k \left[ \hat{c}_k \psi_k(s) + \hat{d}_k \theta_k(s) \right],
\]
and
\[
\hat{A}_k = (-i)^k \hat{c}_k.
\]
As expected, there is no contribution into \(\hat{A}_k\) from the second sum in Eq. (10) containing operators \(\hat{d}_k\) and describing the illumination coming from the wings area of the source. If the source diaphragm is larger or matches the size of the pupil, \(d_s \geq d\), it has no effect on the operator amplitudes \(\hat{c}_k\) and \(\hat{A}_k\), see (11), and we shall neglect the diaphragm in the calculation of \(\hat{A}_k\).

To obtain explicitly the coefficients \(\hat{A}_k\) in case of illumination of the scheme by a traveling-wave OPA we shall use simplified description of an OPA with a plane-wave undepleted pump (see Ref. [1]). In this approximation one can find analytically the spatial Fourier amplitudes \(\hat{c}(q)\),
\[
\hat{c}(q) = \int_{-\infty}^{\infty} d\xi e^{-i\xi q} \hat{c}(\xi),
\]
of the field at the output of the crystal as a linear transformation of the corresponding input Fourier amplitudes \(\hat{c}_{in}(q)\) and \(\hat{c}_{in}^\dagger(q)\),
\[
\hat{c}(q) = U(q)\hat{c}_{in}(q) + V(q)\hat{c}_{in}^\dagger(-q).
\]
Here the operators \(\hat{c}_{in}(q)\) at the input of the OPA are in the vacuum state. The complex coefficients \(U(q)\) and \(V(q)\) depend on the nonlinear susceptibility of the crystal, its length, and the matching conditions in the OPA. These coefficients have the property,
\[
|U(q)|^2 - |V(q)|^2 = 1,
\]
that guarantees the preservation of the commutation relations (11). Using the standard parameters of multimode squeezing (see Ref. [1]) we obtain:
\[
U(q) \pm V^*(-q) = e^{-i\varphi(q)} \left[ e^{\pm r(q)} \cos \theta(q) + ie^{\mp r(q)} \sin \theta(q) \right].
\]
The phase of the amplified (stretched) quadrature amplitude of the OPA output field \(\hat{c}(q)\) is \(\theta(q)\). The phase factor \(\exp(-i\varphi(q))\) specifies the phase of the quadrature amplitudes of the input field \(\hat{c}_{in}(q)\) which are squeezed or stretched. For the vacuum input field of the OPA the last phase is irrelevant.

The operator amplitudes \(\hat{A}_k\) are found explicitly with the use of Eqs. (15) and (11). The quantities \(\hat{c}_{in}(\xi)\) and \(\varphi_k(\xi)\) are expressed through their Fourier transforms (17) and (18). After some calculation we obtain,
\[
\hat{A}_k = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{\infty} dq \psi_k(q/c) \left[ U(q)\hat{c}_{in}(q) + V(q)\hat{c}_{in}^\dagger(-q) \right].
\]
We shall introduce the real quadrature components of the field amplitudes \(\hat{A}_k\) in the object plane as
\[
\hat{A}_k = \hat{A}_{1k} + i\hat{A}_{2k}.
\]
For the variances of these quadrature components we obtain
\[
\langle (\Delta \hat{A}_{1k})^2 \rangle = \frac{1}{4c} \int_{-\infty}^{\infty} dq \psi_k^2(q/c) \left[ e^{\pm 2r(q)} \cos^2 \theta(q) + e^{\mp 2r(q)} \sin^2 \theta(q) \right],
\]
\[
\langle (\Delta \hat{A}_{2k})^2 \rangle = \frac{1}{4c} \int_{-\infty}^{\infty} dq \psi_k^2(q/c) \left[ e^{\mp 2r(q)} \cos^2 \theta(q) + e^{\pm 2r(q)} \sin^2 \theta(q) \right],
\]
Here the upper and the lower sign correspond respectively to the even and the odd prolate spheroidal functions \(\psi_k\).
It follows from this result that the prolate spheroidal waves in the object illumination can be prepared in squeezed state. By proper choice of the squeezing phase $\theta(q)$ at low spatial frequencies $q$ one can minimize quantum fluctuations in one of the quadrature amplitudes $A_{\sigma k}$, $\sigma = 1, 2$ (namely, in the one detected in the image plane of our scheme). Taking the degree and the phase of squeezing as constant, $r(q) = r$, $\theta(q) = \theta$, we can estimate the variance of the squeezed quadrature amplitude:

$$\langle (\Delta A_{\sigma k})^2 \rangle \sim e^{-2r/4}. \quad (24)$$

In reality the spatial-frequency band of multimode squeezing is limited by the phase-matching condition in the OPA. One has to take into account the spatial-frequency dispersion of squeezing, that is, the frequency dependence of the squeezing phase $\theta(q)$ due to diffraction in free space and inside the OPA. Both phenomena deteriorate squeezing of prolate spheroidal waves.

As shown in [1, 11, 12], the effect of diffraction on squeezing can be almost perfectly compensated by means of adjustment of the lens array. If this is done, it follows from (22), (23), that the degree of squeezing of prolate spheroidal waves depends on the overlap in the object plane of two areas: (i) the area illuminated by the squeezed plane waves $\tilde{c}(q)$, which are focused by lens $L$ to the points $s = q/c$, and (ii) the area of the energy localization $\sim \psi_k^2(s)$ of the prolate spheroidal waves.

In analogy to other phenomena with spatially-multimode squeezed light, in the even and odd components of the object field the different quadrature amplitudes are squeezed; for, say, $\theta = 0$, the squeezed quadratures are $A_{1k}$ and $A_{2k}$ for the odd and the even prolate waves respectively. In measurement this implies, e.g., the spatially resolved homodyne detection with numerical evaluation of the odd or even squeezed amplitudes (depending on the local oscillator phase).

Finally, we can formulate the conditions on multimode squeezing in the object illumination in terms of the number $N$ of independent degrees of freedom in the light field, propagating through the diaphragm of the OPA (we assume here the optimum size $d_s = d$ of the diaphragm). The properties of the OPA emission can be characterized by the coherence length $l_c$ of the output field $\hat{c}(\xi)$. The spatial-frequency range $q_c$ of effective squeezing is related to the coherence length by the estimate

$$|q_c| \leq \frac{\pi d}{2l_c}. \quad (25)$$

The minimum requirement on the OPA is that the effectively squeezed waves $\tilde{c}(q_c)$ illuminate, after passing the lens $L$, the object region $|s| \leq 1$. That is, the waves

$$|q_c|/c \geq 1, \quad (26)$$

should be squeezed. This gives the following estimate for the coherence length,

$$l_c \leq \frac{\pi d}{2c}, \quad (27)$$

and for the number $N$ of independent degrees of freedom in the illuminating light, emitted from the region $d$:

$$N \sim d/l_c \geq \frac{dX}{\lambda f} = S. \quad (28)$$

Here $S$ is the Shannon number of our optical scheme.

As seen from [3], the wave profiles $\psi_k(s)$ can be expanded in terms of two bases, $\{\varphi_k(s)\}$ and $\{\chi_k(s)\}$, representing the core and the wings of the illuminating field in the object plane, where $\lambda_k < 1$ are the eigenvalues of the imaging transformation. As known from the theory of prolate spheroidal functions [8, 10], these eigenvalues are close to 1 only for $k \leq S$, where $S$ is the Shannon number. For higher values of the index $k$ the field energy in prolate spheroidal waves $\psi_k(s)$ is concentrated in the wings, that is, outside the object area $|s| \leq 1$. Hence, the condition $N \sim S$ provides an effective squeezing only of the prolate spheroidal waves with $k \leq S$. In order to minimize quantum noise of the higher prolate spheroidal waves (with $k > S$), it is necessary to use OPA with large number of effectively squeezed spatial modes of radiation,

$$N \gg S, \quad (29)$$

and to illuminate by non-classical light a spot in the object plane with the size much larger than the object itself.
When the condition $N \gg S$ is met, one can illuminate the object (within the area $|s| \leq 1$) by a bright classical wave with a properly chosen phase. Since for $\lambda_k \ll 1$ our imaging scheme is analogous to a highly non-symmetrical beamsplitter [4], the squeezed illumination concentrated mainly in the wings area of the object plane allows for the low-noise measurement of the relevant amplitudes of the image field.

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