SOLUTIONS OF THE SYSTEM OF OPERATOR EQUATIONS

\[ BXA = B = AXB \] VIA *-ORDER

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Abstract. In this paper, we establish some necessary and sufficient conditions for the existence of solutions to the system of operator equations \[ BXA = B = AXB \] in the setting of bounded linear operators on a Hilbert space, where the unknown operator \( X \) is called the inverse of \( A \) along \( B \). After that, under some mild conditions we prove that an operator \( X \) is a solution of \( BXA = B = AXB \) if and only if \( B^* \leq AXA \), where the \(*\)-order \( C^* \leq D \) means \( CC^* = DC^* \), \( C^*C = C^*D \). Moreover we present the general solution of the equation above. Finally, we present some characterizations of \( C \leq D \) via other operator equations.

1. Introduction and preliminaries

Throughout the paper, \( \mathcal{H} \) and \( \mathcal{K} \) are complex Hilbert spaces. We denote the space of all bounded linear operators from \( \mathcal{H} \) into \( \mathcal{K} \) by \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \), and write \( \mathcal{B}(\mathcal{H}) \) when \( \mathcal{H} = \mathcal{K} \). Recall that an operator \( A \in \mathcal{B}(\mathcal{H}) \) is positive if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \) and then we write \( A \geq 0 \). We shall write \( A > 0 \) if \( A \) is positive and invertible. An operator \( A \in \mathcal{B}(\mathcal{H}) \) is a generalized projection if \( A^2 = A^* \). Let \( \mathcal{S}(\mathcal{H}), \mathcal{Q}(\mathcal{H}), \mathcal{OP}(\mathcal{H}), \mathcal{GP}(\mathcal{H}) \) be the set of all self-adjoint operators on \( \mathcal{H} \), the set of all idempotents, the set of orthogonal projections and the set of all generalized projections on \( \mathcal{H} \), respectively. For \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), let \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \) be the range and the null space of \( A \), respectively. The projection corresponding to a closed subspace \( M \) of \( \mathcal{H} \) is denoted by \( P_M \). The symbol \( A^- \) stands for an arbitrary generalized inner inverse of \( A \), that is, an operator \( A^- \) satisfying \( AA^- A = A \). The Moore–Penrose inverse of a closed range operator \( A \) is the unique operator \( A^\dagger \in \mathcal{B}(\mathcal{H}) \) satisfying the following equations

\[
AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.
\]

Then, \( A^*AA^\dagger = A^* = A^\dagger AA^* \) and we have the following properties

\[
\mathcal{R}(A^\dagger) = \mathcal{R}(A^*), \quad \mathcal{N}(A^\dagger) = \mathcal{N}(A^*), \quad P_{\mathcal{R}(A)} = AA^\dagger \text{ and } P_{\mathcal{N}(A)} = A^\dagger A.
\]

(1.1)

For \( A, B \in \mathcal{S}(\mathcal{H}) \), \( A \leq B \) means \( B - A \geq 0 \). The order \( \leq \) is said to be the Löwner order on \( \mathcal{S}(\mathcal{H}) \). If there exists \( C \in \mathcal{S}(\mathcal{H}) \) such that \( AC = 0 \) and \( A + C = B \), then we write \( A \preceq B \). The order \( \preceq \) is said to be the logic order on
\( \mathcal{J}(\mathcal{H}) \). For \( A, B \in \mathcal{B}(\mathcal{H}) \), let \( A^{*} \) mean

\[
AA^{*} = BA^{*}, \quad A^{*}A = A^{*}B. \tag{1.2}
\]

It is known that, for \( A, B \in \mathcal{J}(\mathcal{H}) \), \( A \preceq B \) if and only if \( A^{*} \preceq B \); see [6]. We denote by \( A \wedge B \) the infimum (or the greatest lower bound) of \( A \) and \( B \) over the \( *- \) order and \( A \vee B \) the supremum (or the least upper bound) of \( A \) and \( B \) over the \( *- \) order, if they exist; cf. [12].

It is known that if \( A \in \mathcal{B}(\mathcal{H},\mathcal{H}) \) has closed range, then by considering

\[ \mathcal{H} = \mathcal{R}(A^{*}) \oplus \mathcal{N}(A) \] and \( \mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A^{*}) \)

we can write

\[
A = \begin{bmatrix} A_{1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^{*}) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^{*}) \end{bmatrix}, \tag{1.3}
\]

where \( A_{1} : \mathcal{R}(A^{*}) \rightarrow \mathcal{R}(A) \) is invertible; see [8, Lemma 2.1]. Therefore, the Moore–Penrose generalized inverse of \( A \) can be represented as

\[
A^{†} = \begin{bmatrix} A_{1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^{*}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^{*}) \\ \mathcal{N}(A) \end{bmatrix}. \tag{1.4}
\]

Many results have been obtained on the solvability of equations for matrices and operators on Hilbert spaces and Hilbert \( C^{*} \)-modules. In 1976, Mitra [11] considered the matrix equations \( AX = B, AXB = C \) and the system of linear equations \( AX = C, XB = D \). He got the necessary and sufficient conditions for existence and expressions of general Hermitian solutions. In 1966, the celebrated Douglas Lemma was established in [9]. It gives some conditions for the existence of a solution to the equation \( AX = B \) for operators on a Hilbert space. Using the generalized inverses of operators, in 2007, Dajić and Koliha [4] got the existence of the common Hermitian and positive solutions to the system \( AX = C, XB = D \) for operators acting on a Hilbert space. In 2008, Xu [17] extended these results to the adjointable operators. Several general operator equations and systems in some general settings such as Hilbert \( C^{*} \)-modules have been studied by some mathematicians; see, e.g., [7, 10, 13, 16].

The matrix equation \( AXB = C \) is consistent if and only if \( AA^{-}CB^{-}B = C \) for some \( A^{-}, B^{-} \), and the general solution is \( X = A^{-}CB^{-} + Y - A^{-}AYBB^{-} \), where \( Y \) is an arbitrary matrix; see [11]. In 2010, Gonzalez [1] got some necessary and sufficient conditions for existence of a solution to the equation \( AXB = C \) for operators on a Hilbert space.

Let \( A, B \) or \( C \) have closed range. Then, the operator equation \( AXB = C \) is solvable if and only if \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(C^{*}) \subseteq \mathcal{R}(B^{*}) \); see [1, Theorem 3.1]. Therefore, if \( A \) or \( C \) has closed range, then the equation \( AXC = C \) is solvable if and only if \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \), and \( CXA = C \) is solvable if and only if \( \mathcal{R}(C^{*}) \subseteq \mathcal{R}(A^{*}) \). Deng [5] investigated the equation \( CAX = C = XAC \), which is essentially different from ours. In this paper, we first characterize the existence of solutions of the system of operator equations \( BXA = B = AXB \) by means
of *— order. After that, we generalize the solutions to the system of operator equations $BXA = B = AXB$ in a new fashion.

2. THE EXISTENCE OF SOLUTIONS OF THE SYSTEM $BXA = B = AXB$

We start our work with the celebrated Douglas lemma.

**Lemma 2.1 (Douglas Lemma).** [9] Let $A, C \in \mathbb{B}(\mathcal{H})$. Then, the following statements are equivalent:

(a) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$.

(b) There exists $X \in \mathbb{B}(\mathcal{H})$ such that $AX = C$.

(c) There exists a positive number $\lambda$ such that $CC^* \leq \lambda^2 AA^*$.

If one of these conditions holds, then there exists a unique solution $\tilde{X} \in \mathbb{B}(\mathcal{H})$ of the equation $AX = C$ such that $\mathcal{R}(\tilde{X}) \subseteq \mathcal{R}(A^*)$ and $\mathcal{N}(\tilde{X}) = \mathcal{N}(C)$.

**Lemma 2.2.** Let $A, B \in \mathbb{B}(\mathcal{H})$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, then $B = B_1 \oplus 0$, where $B_1 \in \mathbb{B}(\mathcal{R}(A^*), \mathcal{R}(A))$.

**Proof.** Let $A, B$ be operators from the decomposition $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ into the decomposition $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then, by Lemma 2.1, there exists $C \in \mathbb{B}(\mathcal{H})$ such that $B = AC$ and $\mathcal{N}(C) = \mathcal{N}(B)$. Since $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, so $\mathcal{R}(C^*) \subseteq \mathcal{R}(C^*) = \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) = \mathcal{N}(P_{\mathcal{R}(A)})$. Hence, $P_{\mathcal{R}(A)}C^* = 0$ and so $CP_{\mathcal{R}(A)} = 0$. It follows from $\mathcal{N}(C) = \mathcal{N}(B)$ that $BP_{\mathcal{R}(A)} = 0$.

If $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, then a similar reasoning shows that $P_{\mathcal{R}(A^*)}B = 0$. Therefore, $P_{\mathcal{R}(A)}BP_{\mathcal{R}(A)} = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A)} = 0$. Hence, $B = B_1 \oplus 0$, where $B_1 = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A)}$.

**Theorem 2.3.** Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathcal{I}(\mathcal{H})$. If $A$ has closed range, then the following statements are equivalent:

1. The system of operator equations $BXA = B = AXB$ is solvable;

2. $AA^\dagger BA^\dagger A = B$;

3. $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$.

**Proof.** $(1) \implies (2) :$ Using (1.1) and $B = BXA$, we get that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A)$. Hence, by Lemma 2.1, there exists $C^* \in \mathbb{B}(\mathcal{H})$ such that $B = A^\dagger AC^*$. Hence, $B = CA^\dagger A$. Applying (1.1) and $AXB = B$, we derive that $\mathcal{R}(B) \subseteq \mathcal{R}(A) = \mathcal{R}(AA^\dagger)$. Thus, by Lemma 2.1, there exists $\tilde{C} \in \mathbb{B}(\mathcal{H})$ such that $B = A^\dagger \tilde{C}$. It follows that $AA^\dagger BA^\dagger A = AA^\dagger (AA^\dagger \tilde{C})A^\dagger A = AA^\dagger \tilde{C}A^\dagger A = BA^\dagger A = (CA^\dagger A)A^\dagger A = CA^\dagger A = B$.  

$$
AA^\dagger BA^\dagger A = AA^\dagger (AA^\dagger \tilde{C})A^\dagger A = AA^\dagger \tilde{C}A^\dagger A = BA^\dagger A = (CA^\dagger A)A^\dagger A = CA^\dagger A = B.
$$
(2) → (3): Let $AA^*BA^*A = B$. Then, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. It follows from $B = B^* = (AA^*BA^*A)^* = A^*ABAA^*$ and (1.1) that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$. 

(3) → (1): Let $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$. Upon applying Lemma 2.2, $B = B_1 \bigoplus 0$, where $B_1 = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A^*)}$. Since $A$ has closed rang, so by using (1.3) and (1.4) we have

\[
A = \begin{bmatrix}
A_1 & 0 \\
0 & 0 
\end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix}
A_1^{-1} & 0 \\
0 & 0 
\end{bmatrix}.
\]

Hence, $AA^*B = B$ and $BA^*A = B$. Thus $X = A^*$ is a solution of the system $BXA = B = AXB$. \hfill \Box

**Proposition 2.4.** Let $A, B, X \in \mathbb{B}(\mathcal{H})$. Then,

\[
\mathcal{R}(A) \subseteq \mathcal{R}(B), \quad \mathcal{N}(B) \subseteq \mathcal{N}(A) \quad \text{and} \quad BXA = B = AXB
\]

if and only if

\[
\mathcal{N}(B) = \mathcal{N}(A), \quad \mathcal{R}(B) = \mathcal{R}(A) \quad \text{and} \quad AXA = A.
\]

**Proof.** ($\Rightarrow$): Suppose that $\mathcal{R}(A) \subseteq \mathcal{R}(B), \mathcal{N}(B) \subseteq \mathcal{N}(A)$ and $BXA = B = AXB$. It follows from $BXA = B$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ that $\mathcal{N}(A) \subseteq \mathcal{N}(B) \subseteq \mathcal{N}(A)$. Hence, $\mathcal{N}(A) = \mathcal{N}(B)$. It follows from $AXB = B$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ that $\mathcal{R}(A) \subseteq \mathcal{R}(B) \subseteq \mathcal{R}(A)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(B)$. Moreover, $(I - AX)B = 0$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ Hence, we derive that $(I - AX)A = 0$. So, $AXA = A$. 

($\Leftarrow$): Suppose that $\mathcal{N}(B) = \mathcal{N}(A), \mathcal{R}(B) = \mathcal{R}(A)$ and $AXA = A$. Hence, 

\[
(I - AX)A = 0 \implies \mathcal{R}(A) \subseteq \mathcal{N}(I - AX) \implies \mathcal{R}(B) \subseteq \mathcal{N}(I - AX) \implies B = AXB,
\]

\[
A(I -XA) = 0 \implies \mathcal{R}(I -XA) \subseteq \mathcal{N}(A) \implies \mathcal{R}(I -XA) \subseteq \mathcal{N}(B) \implies B = BXA.
\]

\hfill \Box

### 3. System of operator equations $BXA = B = AXB$ via $*$-order

We know that $(\mathbb{B}(\mathcal{H}), \leq)$ is a partially ordered set; see [2]. Let $G_1, G_2 \in \mathbb{B}(\mathcal{H})$ be invertible and $G_1 \leq A, G_2 \leq A$. Then, $G_1G_1^* = AG_1^*$ and $G_2G_2^* = AG_2^*$. Hence, we obtain $G_1 = G_2 = A$. This fact leads us to consider the characterizations of $A \leq B$. Now we state the necessary and sufficient conditions in which the common $*$- lower or $*$- upper bounds of $A$ and $B$ exist.

We need the following essential lemmas.

**Lemma 3.1.** [18, Lemma 2.1] Let $A, B \in \mathbb{B}(\mathcal{H})$ and $\overline{\mathcal{H}}$ denote the closure of a space $\mathcal{H}$.

(a) $AA^* = BA^* \iff A = BP_{\mathcal{R}(A)} \iff A = BQ$ for some $Q \in \mathcal{G}(\mathcal{H})$;

(b) $A^*A = A^*B \iff A = P_{\mathcal{R}(A)}B \iff A = PB$ for some $P \in \mathcal{G}(\mathcal{H})$;

(c) $A \leq B \iff B = A + P_{\mathcal{N}(A^*)}BP_{\mathcal{N}(A)}$;
(d) \( A^* \leq B \iff A = P_{\mathcal{R}(A)}B = BP_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A)}BP_{\mathcal{R}(A^*)}; \)

(e) \( A^* \leq B \iff A = A_1 \bigoplus 0, B = A_1 \bigoplus B_1; \)

where \( A_1 \in \mathcal{B}(\mathcal{R}(A^*), \mathcal{R}(A)), B_1 \in \mathcal{B}(\mathcal{N}(A), \mathcal{N}(A^*)) \) and \( A \bigoplus B \) means the block matrix \[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}.
\]

The following Lemma is a version of Lemma 2.1 when the operator \( A \) has closed range.

**Lemma 3.2.** [4, Theorem 3.1]. Let \( A \in \mathcal{B}(\mathcal{H}) \) have closed range. Then, the equation \( AX = C \) has a solution \( X \in \mathcal{B}(\mathcal{H}) \) if and only if \( AA^*C = C \), and this if and only if \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \). In this case, the general solution is \( X = A^*C + (I - A^*A)T, \) where \( T \in \mathcal{B}(\mathcal{H}) \) is arbitrary.

**Proposition 3.3.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then,

(a) If \( A \) has closed range and \( B^* \leq A \), then \( X = A^* \) is a solution of the system \( BXA = B = AXB \).

(b) If \( B \) has closed range and \( B^* \leq A \), then \( X = B^* \) is a solution of the system \( BXA = B = AXB \).

**Proof.** (a) Let \( A \) be a closed range operator and \( B \leq A \). It follows from Lemma 3.1(d) that \( B = AP_{\mathcal{R}(B^*)} \) and \( B = P_{\mathcal{R}(B^*)}A \). Hence, \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \). It follows from \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) and Lemma 3.2 that \( AA^*B = B \). It follows from \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \) and Lemma 3.2 that \( BA^*A = ((A^*A)^*B^*)^* = (A^*A)^*B^* \). Hence, \( X = A^* \) is a solution of the system of operator equations \( BXA = B = AXB \).

(b) Let \( B \) be a closed range operator and \( B \leq A \). It follows from Lemma 3.1 that \( B = AP_{\mathcal{R}(B^*)} \) and \( B = P_{\mathcal{R}(B^*)}A \). Applying (1.1), we conclude that \( AB^*B = B \) and \( BB^*A = A \). Hence, \( X = B^* \) is a solution of the system \( BXA = B = AXB \). \( \square \)

**Proposition 3.4.** Let \( A, B, X \in \mathcal{B}(\mathcal{H}) \).

If \( A \leq B \) and \( BXA = B = AXB \), then \( \mathcal{N}(B) = \mathcal{N}(A), \mathcal{R}(B) = \mathcal{R}(A) \) and \( AXA = A \).

**Proof.** Let \( A \leq B \) and \( BXA = B = AXB \). Applying Lemma 3.1(d) we have \( A = P_{\mathcal{R}(A)}B = BP_{\mathcal{R}(A)} \). Hence, \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \) and \( \mathcal{N}(B) \subseteq \mathcal{N}(A) \). Using Proposition 2.4,

\[ \mathcal{N}(B) = \mathcal{N}(A), \mathcal{R}(B) = \mathcal{R}(A) \text{ and } AXA = A. \]

\( \square \)

**Remark 3.5.** Note that the converse of Proposition 3.4 is not true, in general. Set \( A^*, A, A \) instead of \( A, B, X \). If \( A \in \mathcal{B}(\mathcal{H}) \) has closed range, then, by (1.1),
we have \( \mathcal{R}(A^*) = \mathcal{R}(A^t), \mathcal{N}(A^*) = \mathcal{N}(A^t) \) and \( A^t A A^t = A^t \) but not \( A^t \leq A^* \).

Indeed, if \( A^t \leq A^* \), then by utilizing Lemma 3.1(d), we have \( A^t = P_{\mathcal{R}(A^t)} A^* \). It follows from \( \mathcal{R}(A^t) = \mathcal{R}(A^*) \) that \( A^t = P_{\mathcal{R}(A^t)} A^* = A^* \).

**Theorem 3.6.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) and \( B \leq A \). Then, the following statements are equivalent:

(a) There exists a solution \( X \in \mathbb{B}(\mathcal{H}) \) of the system \( BXA = B = AXB \);

(b) \( B \leq AXA \)

*Proof.* \((a) \implies (b): \) Let \( X \in \mathbb{B}(\mathcal{H}) \) is a solution of the system \( BXA = B = AXB \).

Hence, \( B - BXA = 0 \) and \( B - AXB = 0 \). It follows from the assumption \( B \leq A \) and Lemma 3.1(d) that \( B = P_{\mathcal{R}(B)} A \) and \( B = AP_{\mathcal{R}(B^*)} \). Hence,

\[
P_{\mathcal{R}(B)}(B - AXA) = B - P_{\mathcal{R}(B)} AXA = B - BXA = 0
\]

and

\[
(B - AXA) P_{\mathcal{R}(B^*)} = B - AXAP_{\mathcal{R}(B^*)} = B - AXB = 0.
\]

Therefore, \( B \leq AXA \).

\((b) \implies (a): \) Suppose that \( B \leq AXA \). Applying Lemma 3.1(d), we infer that \( P_{\mathcal{R}(B)}(B - AXA) = 0 \) and \( (B - AXA) P_{\mathcal{R}(B^*)} = 0 \). It follows from the assumption \( B \leq A \) and Lemma 3.1(d) that \( B = P_{\mathcal{R}(B)} A \) and \( B = AP_{\mathcal{R}(B^*)} \), whence

\[
B - BXA = B - P_{\mathcal{R}(B)} AXA = P_{\mathcal{R}(B)}(B - AXA) = 0
\]

and

\[
B - AXB = B - AXAP_{\mathcal{R}(B^*)} = (B - AXA) P_{\mathcal{R}(B^*)} = 0.
\]

Therefore, \( X \) is a solution of the system \( BXA = B = AXB \). \( \square \)

Let \( A, B \in \mathbb{B}(\mathcal{H}) \) have closed ranges. It follows from Proposition 3.3 that \( A^t \) and \( B^t \) are solutions of the system \( BXA = B = AXB \). Therefore, we are interested in the study of the following system of operator equations:

\[
BXA = B = AXB; \tag{3.1}
\]

\[
BAX = B = XAB. \tag{3.2}
\]

Let \( A, B \in \mathbb{B}(\mathcal{H}) \). An operator \( C \in \mathbb{B}(\mathcal{H}) \) is said to be an inverse of \( A \) along \( B \) if it fulfills one of the equations (3.1) or (3.2). If \( A \in \mathbb{B}(\mathcal{H}) \) is invertible, then \( X = A^{-1} \) is a solution of the system \( AXA = I = AX \). Hence, \( A^{-1} \) is an inverse of \( A \) along \( I \), where \( I \) is the identity of \( \mathbb{B}(\mathcal{H}) \).

Let \( A \in \mathbb{B}(\mathcal{H}) \) have closed range. Using (1.1), we have \( AA^t A = A = AA^t A \).

Hence, \( A^t \) satisfies Eq. (3.1). Therefore, \( A^t \) is the inverse of \( A \) along \( A \).
It follows from (1.1) that $A^*AA^\dag = A^* = A^\dag AA^*$. Hence, $A^\dag$ satisfies Eq. (3.2). Therefore, $A$ is the inverse of $A$ along $A^*$.

**Lemma 3.7.** [11, Theorem 2.1] Let $C \in \mathbb{B}(\mathcal{H})$ and $A, B \in \mathbb{B}(\mathcal{H})$ have closed ranges. Then, the equation $AXB = C$ has a solution $X \in \mathbb{B}(\mathcal{H})$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A), \mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, and this if and only if $AA^\dag CB^\dag B = C$. In this case, $X = A^\dag CB^\dag + U - A^\dag AUBB^\dag$, where $U \in \mathbb{B}(\mathcal{H})$ is arbitrary.

In the next result we provide a general solution of the system $BXA = B = AXB$.

**Theorem 3.8.** Let $A, B \in \mathbb{B}(\mathcal{H})$ have closed ranges and $B \leq A$. Then, the general solution of the system of operator equations $BXA = B = AXB$ is

$$X = A^\dag BB^\dag + A^\dag \left[ (I - AA^\dag) + (A - B)S \right] (A - B)^\dag + T - A^\dag AT(A - B)^\dag(A - B)$$

$$-A^\dag B(I - AA^\dag)(A - B)^\dag BB^\dag - A^\dag (A - B)S(A - B)^\dag BB^\dag$$

$$-A^\dag ATBB^\dag + A^\dag AT(A - B)^\dag(A - B)BB^\dag.$$

where $S, T \in \mathbb{B}(\mathcal{H})$.

**Proof.** Let $A, B$ have closed ranges. It follows from the assumption $B \leq A$ and Lemma 3.1(d) that $B = AP_{\mathcal{R}(B^*)}$. Hence, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Using Lemma 3.2, we have $AA^\dag B = B$. It follows from $AA^\dag BB^\dag B = B$ and Lemma 3.7 that the equation $AXB = B$ is solvable. In this case, the general solution is

$$X = A^\dag BB^\dag + W - A^\dag AWBB^\dag,$$

where $W \in \mathbb{B}(\mathcal{H})$ is arbitrary. If $X$ satisfies the equation $BXA = B$, then

$$B(A^\dag BB^\dag + W - A^\dag AWBB^\dag)A = B.$$ 

It follows from the assumption $B \leq A$ and Lemma 3.1(d) that $B = P_{\mathcal{R}(B)}A$. Applying (1.1), $BB^\dag A = B$. Hence,

$$BA^\dag B + BWA - BA^\dag AWB = B.$$ 

Therefore, $B(A^\dag B + WA - A^\dag AWB) = B$. So, $A^\dag B + WA - A^\dag AWB$ is a solution of the equation $BX = B$. Utilizing Lemma 3.2 again, we have

$$A^\dag B + WA - A^\dag AWB = B^\dag B + (I - B^\dag B)S,$$

where $S \in \mathbb{B}(\mathcal{H})$ is arbitrary. Multiply the left hand side of Eq. (3.4) by $A$, to get

$$AA^\dag B + AWB - AA^\dag AWB = AB^\dag B + A(I - B^\dag B)S$$

It follows from the assumption $B \leq A$ and Lemma 3.1(d) that $B = AP_{\mathcal{R}(B^*)}$. Applying (1.1), $AB^\dag B = B$. We derive that

$$AA^\dag B + AWB - AWB = B + (A - B)S.$$ 

Now, we get $AW(A - B) = B(I - AA^\dag) + (A - B)S$. So, $W$ is a solution of the equation $AX(A - B) = B(I - AA^\dag) + (A - B)S$. Using Lemma 3.2, we get that

$$W = A^\dag \left[ B(I - AA^\dag) + (A - B)S \right] (A - B)^\dag T - A^\dag AT(A - B)^\dag(A - B),$$

where $T \in \mathbb{B}(\mathcal{H})$. Therefore, $T = U - A^\dag AUBB^\dag$, where $U \in \mathbb{B}(\mathcal{H})$ is arbitrary.
where \( T \in \mathbb{B}(\mathcal{H}) \) is arbitrary. By putting \( W \) in Eq. (3.3), we reach
\[
X = A^\dagger BB^\dagger + A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B) \\
- A^\dagger A(A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger \\
+ T - A^\dagger AT(A - B)^\dagger (A - B) BB^\dagger
\]
\[
= A^\dagger BB^\dagger + A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B) \\
- A^\dagger AA^\dagger B(I - AA^\dagger)(A - B)^\dagger BB^\dagger - A^\dagger AA^\dagger (A - B)S(A - B)^\dagger BB^\dagger \\
- A^\dagger ATBB^\dagger + A^\dagger AT(A - B)^\dagger (A - B) BB^\dagger
\]
\[
(\text{by } (1.1))
\]
\[
= A^\dagger BB^\dagger + A^\dagger [B(I - AA^\dagger) + (A - B)S] (A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger (A - B) \\
- A^\dagger B(I - AA^\dagger)(A - B)^\dagger BB^\dagger - A^\dagger (A - B)S(A - B)^\dagger BB^\dagger \\
- A^\dagger ATBB^\dagger + A^\dagger AT(A - B)^\dagger (A - B) BB^\dagger
\]

\[
\square
\]

**Theorem 3.9.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) where \( A \) has closed range. If the system \( BXA = B =AXB \) is solvable, then the system \( XB = A^\dagger B, BX = BA^\dagger \) is solvable. Conversely, If \( B \leq A \) and the system \( XB = A^\dagger B, BX = BA^\dagger \) is solvable, then the system \(BXA = B =AXB \) is solvable.

**Proof.** (\( \implies \)): Let \( X \) be a solution of the system \(BXA = B =AXB \). It follows from \( B = A\tilde{X}B \) that \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \). Using Lemma 3.2, \( AA^\dagger B = B \). It follows from (1.1) that
\[
P_{\mathcal{R}(A^\dagger)}XAA^\dagger B = (A^\dagger A)\tilde{X}(AA^\dagger)B = (A^\dagger A)\tilde{X}(AA^\dagger)B = A^\dagger (A\tilde{X}B) = A^\dagger B.
\]
So, \( P_{\mathcal{R}(A^\dagger)}XAA^\dagger \) is a solution of the equation \(XB = A^\dagger B \). Since \( B^* = (B\tilde{X}A)^* = A^*\tilde{X}B^* \), we have \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \). Applying Lemma 2.1, there exists \( Y \in \mathbb{B}(\mathcal{H}) \) such that \( B = YA \). Hence,
\[
BP_{\mathcal{R}(A^\dagger)}XAA^\dagger \quad = \quad B(A^\dagger A)\tilde{X}(AA^\dagger) = Y(AA^\dagger A)\tilde{X}(AA^\dagger) \\
\quad = \quad (YA\tilde{X}A)A^\dagger = (B\tilde{X}A)A^\dagger = BA^\dagger.
\]
Therefore, \( P_{\mathcal{R}(A^\dagger)}XAA^\dagger \) is a solution of the equation \( B = BA^\dagger \). Thus \( P_{\mathcal{R}(A^\dagger)}XAA^\dagger \) is a solution of the system \( XB = A^\dagger B, BX = BA^\dagger \).

(\( \impliedby \)) Suppose that \( \tilde{X} \) is a solution of the system \(XB = A^\dagger B, BX = BA^\dagger \). It follows from the assumption \( B \leq A \) that \( B = AP_{\mathcal{R}(B^*)} \) and \( B = P_{\mathcal{R}(B^*)}A \). Hence, \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \). It follows from \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) to Lemma 3.2 that \( AA^\dagger B = B \). Hence, \( A\tilde{X}B = A(A^\dagger B) = AA^\dagger B = B \). It follows from \( \mathcal{R}(B^*) \subseteq \mathcal{R}(A^*) \) and Lemma 2.1 that there exists \( Z \in \mathbb{B}(\mathcal{H}) \) such that \( B = ZA \). Hence,
\[
B\tilde{X}A = (BA^\dagger)A = BA^\dagger A = ZAA^\dagger A = ZA = B.
\]
Therefore, \( \tilde{X} \) is a solution of the system \(BXA = B =AXB \).  \( \square \)
Lemma 3.10. [4, Theorem 4.2] Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ and $A, B, M = B^*(I - A^1A)$ have closed ranges. Then, the system $AX = C$, $XB = D$ have a hermitian solution $X \in \mathcal{B}(\mathcal{H})$ if and only if

$$AA^\dagger C = C, \quad DB^\dagger B = D, \quad AD = CB$$

and $AC^*$ and $B^*D$ are hermitian. In this case, the general hermitian solution is

$$X = A^\dagger C + (I - A^\dagger A)M^\dagger s(T) + (I - A^\dagger A)(I - M^\dagger M) [A^\dagger C + (I - A^\dagger A)M^\dagger s(T)]^* + (I - A^\dagger A)(I - M^\dagger M)W(I - M^\dagger M)^*(I - A^\dagger A)^*,$$

where $W \in \mathcal{B}(\mathcal{H})$ is hermitian and $s(T) = D^* - B^*A^\dagger C$ is the so-called Schur complement of the block matrix $T = \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix}$.

Theorem 3.11. Suppose that $A, B \in \mathcal{B}(\mathcal{H})$ have closed ranges. If $B \leq A$ and $B^*A^\dagger B, BA^\dagger B^*$ are hermitian, then the system $BXA = B = AXB$ has a hermitian solution.

Proof. Replace $A, B, C, D$ in Lemma 3.10 by $B, BA^\dagger, A^\dagger B$ to get

$$AA^\dagger C = BB^\dagger(BA^\dagger) = BA^\dagger = C, \quad DB^\dagger B = (A^\dagger B)B^\dagger B = A^\dagger B = D$$

and

$$AD = B(A^\dagger B) = (BA^\dagger)B = CB, \quad AC^* = B(BA^\dagger)^* = BA^\dagger B^*, \quad B^*D = B^*A^\dagger B.$$ 

Using Lemma 3.10, the system $XB = A^\dagger B, BX = BA^\dagger$ has a hermitian solution, say, $\tilde{X}$. It follows from the assumption $B \leq A$ that $B = AP_{\mathcal{R}(B^*)}$ and $B = P_{\mathcal{R}(B^*)}A$. Hence, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$. It follows from $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and Lemma 3.2 that $AA^\dagger B = B$. Hence, $A\tilde{X}B = A(A^\dagger B) = AA^\dagger B = B$.

It follows from $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$ and Lemma 2.1 that there exists $Z \in \mathcal{B}(\mathcal{H})$ such that $B = ZA$. Hence,

$$B\tilde{X}A = (BA^\dagger)A = BA^\dagger A = ZAA^\dagger A = ZA = B.$$ 

Therefore, $\tilde{X}$ is a hermitian solution of the system $BXA = B = AXB$. □

4. $*$-ORDER VIA OTHER OPERATOR EQUATIONS

Generally speaking, the inequality $PB \leq B$ does not hold for any $P \in \mathcal{P}(\mathcal{H})$ even if $\mathcal{R}(P) \subseteq \overline{\mathcal{R}(B)}$. In [2, Lemma 2.6], some conditions are mentioned which give a one-sided description of the relation $A \leq B$ regarding (1.2).

The next result is known.

Proposition 4.1. [2, Proposition 2.6] Let $B \in \mathcal{B}(\mathcal{H})$. 

(a) If \( P \in \mathcal{O}(\mathcal{H}) \) and \( \mathcal{R}(P) \subseteq \overline{\mathcal{R}(B)} \), then \( PB \leq B \) if and only if \( PBB^* = BB^*P \).

(b) If \( Q \in \mathcal{O}(\mathcal{H}) \) and \( \mathcal{R}(Q) \subseteq \overline{\mathcal{R}(B^*)} \), then \( BQ \leq B \) if and only if \( QB^*B = B^*BQ \).

In the following, we state a generalization of Proposition 4.1.

**Proposition 4.2.** Let \( B \in \mathbb{B}(\mathcal{H}) \). If there exist \( P, Q \in \mathcal{O}(\mathcal{H}) \) such that \( \mathcal{R}(P) \subseteq \overline{\mathcal{R}(B)} \) and \( \mathcal{R}(Q) \subseteq \overline{\mathcal{R}(B^*)} \), then \( PBQ \leq B \) if and only if \( PBB^* = BB^*P \) and \( QB^*B = B^*BQ \).

**Proof.** \((\Rightarrow):\) Let \( PBQ \leq B \). Applying (1.2), we get that
\[
PBB^* = (PBQ)B^* = B(PBQ)^* = BB^*P
\]
and
\[
B^*PBQ = B^*(PBQ) = (PBQ)^*B = QB^*PB.
\]

\((\Leftarrow):\) Let \( PBQB^* = BQB^*P \) and \( QB^*PB = B^*PBQ \). Applying (1.2), we obtain that
\[
(PBQ)(PBQ)^* = PBQB^*P = (BQB^*P)P = BQB^*P = B(PBQ)^*
\]
and
\[
(PBQ)^*(PBQ) = QB^*PBQ = Q(QB^*PB) = QB^*PB = (PBQ)^*B.
\]
\(\Box\)

The next known theorem gives a characterization of the order \( \leq \).

**Theorem 4.3.** [6, Theorem 2.3] Let \( A \in \mathbb{B}(\mathcal{H}) \) and \( C \in \mathcal{Q}(\mathcal{H}) \). Then, \( C^* \leq A \) if and only if there exists \( X \in \mathbb{B}(\mathcal{H}) \) such that \( A = C + (I - C^*)X(I - C^*) \).

In the following, we establish an analogue of Theorem 4.3 for generalized projections on a Hilbert space. Recall that an operator \( A \in \mathbb{B}(\mathcal{H}) \) is a generalized projection if \( A^2 = A^* \).

**Lemma 4.4.** [14, Theorem A.2] Let \( A \in \mathbb{B}(\mathcal{H}) \) be a generalized projection. Then, \( A \) is a closed range operator and \( A^3 \) is an orthogonal projection on \( \mathcal{R}(A) \). Moreover, \( \mathcal{H} \) has decomposition
\[
\mathcal{H} = \mathcal{R}(A) \bigoplus \mathcal{N}(A)
\]
and \( A \) has the following matrix representation
\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},
\]
where the restriction \( A_1 = A|_{\mathcal{R}(A)} \) is unitary on \( \mathcal{R}(A) \).

**Theorem 4.5.** Let \( A \in \mathbb{B}(\mathcal{H}) \) and \( B \in \mathcal{Q}(\mathcal{H}) \). Then, \( B^* \leq A \) if and only if there exists \( X \in \mathbb{B}(\mathcal{H}) \) such that \( A = B + (I - BB^*)X(I - B^*B) \).
Proof. $(\implies)$: Let $B \in \mathcal{P}(\mathcal{H})$ and $B \leq A$. Employing Lemma 4.4, we infer that $B$ has closed range and $B^3 = P_{\mathcal{H}(B)}$. It follows from (1.1) that
\[ \mathcal{R}(B^*) = \mathcal{R}(B^*B) = \mathcal{R}(B^3) = \mathcal{R}(BB^*) = \mathcal{R}(B). \]
Hence, $P_{\mathcal{H}(B)} = P_{\mathcal{H}(B^*)} = BB^* = B^*B$. Therefore, $P_{\mathcal{H}(B)} = P_{\mathcal{H}(B^*)} = I - BB^* = I - B^*B$. Applying Lemma 3.1(c), we get $A = B + P_{\mathcal{H}(B^*)}AP_{\mathcal{H}(B)}$. Hence, $A = B + (I - BB^*)A(I - B^*B)$.

$(\impliedby)$: Let $X \in \mathcal{B}(\mathcal{H})$ be a solution of the equation $A = B + (I - BB^*)X(I - B^*B)$. Since $B$ is a generalized projection, so $B^*BB^* = B^*$. Hence,
\[ B^*A = B^*B + B^*(I - BB^*)X(I - B^*B) = B^*B \]
and
\[ AB^* = BB^* + (I - BB^*)X(I - B^*B)B^* = BB^*. \]
Therefore, $B \leq A$ by (1.2). \hfill \Box

In the next result, we show that if $A$ is a generalized projection and $B \leq A \land A^*$, then $AA^*$ can be written as the sum of two idempotents.

**Theorem 4.6.** Let $A \in \mathcal{P}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. If $B \leq A \land A^*$, then $B$ is an idempotent and there exist an idempotent $X$ such that $AA^* = B + X$ and $B^*X = XB^* = 0$.

**Proof.** Let $B \leq A \land A^*$. It follows from the assumption $A^2 = A^*$ and Lemma 3.1(d) that
\[ B^2 = (P_{\mathcal{H}(B)}A^*)(A^*P_{\mathcal{H}(B)}^*) = P_{\mathcal{H}(B)}A^2P_{\mathcal{H}(B)}^* = P_{\mathcal{H}(B)}AP_{\mathcal{H}(B)}^* = BP_{\mathcal{H}(B)}^* = B. \]
Using Lemma 3.1, we get that
\[ AB = A(AP_{\mathcal{H}(B)}^*) = A^2P_{\mathcal{H}(B)}^* = A^*P_{\mathcal{H}(B)}^* = B, \]
\[ BA = (P_{\mathcal{H}(B)}A)A = P_{\mathcal{H}(B)}A^2 = P_{\mathcal{H}(B)}A^* = B, \]
\[ A^*B = A^*(A^*P_{\mathcal{H}(B)}^*) = A^2P_{\mathcal{H}(B)}^* = AP_{\mathcal{H}(B)}^* = B \]
and
\[ BA^* = (P_{\mathcal{H}(B)}A^*)A^* = P_{\mathcal{H}(B)}A^*A^2 = P_{\mathcal{H}(B)}A = B. \]
Let $X = AA^* - B$. It follows from the assumption $B \leq A \land A^*$ that
\[ X^2 = (AA^* - B)^2 = (AA^*)^2 + B^2 - AA^*B - BA^* \]
\[ = AA^* + B - AB - BA^* \]
\[ = AA^* + B - B = AA^* - B = X. \]
Hence, $X$ is an idempotent. Applying (1.2), we have
\[ B^*X = B^*(AA^* - B) = B^*AA^* - B^*B = B^*A^*A - B^*B = B^*A - B^*B = 0 \]
and
\[ XB^* = (AA^* - B)B^* = AA^*B^* - BB^* = AB^* - BB^* = 0. \]
Lemma 4.7. Let $A \in \mathcal{Q}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then, $B \preceq A$ if and only if $B$ is an idempotent and there exists an idempotent $X$ such that $A = B + X$ and $B^*X = XB^* = 0$.

Proof. ($\Longrightarrow$): Let $B \preceq A$. It follows from the assumption $A^2 = A$ and Lemma 3.1(d) that

$$B^2 = (P_{\mathcal{Q}(\mathcal{H})}A)(AP_{\mathcal{Q}(\mathcal{H})}) = P_{\mathcal{Q}(\mathcal{H})}A^2P_{\mathcal{Q}(\mathcal{H})} = (P_{\mathcal{Q}(\mathcal{H})}A)P_{\mathcal{Q}(\mathcal{H})} = BP_{\mathcal{Q}(\mathcal{H})} = B.$$ 

Utilizing Lemma 3.1(d), we obtain that

$$AB = A(AP_{\mathcal{Q}(\mathcal{H})}) = A^2P_{\mathcal{Q}(\mathcal{H})} = AP_{\mathcal{Q}(\mathcal{H})} = B$$

and

$$BA = (P_{\mathcal{Q}(\mathcal{H})}A)A = P_{\mathcal{Q}(\mathcal{H})}A^2 = P_{\mathcal{Q}(\mathcal{H})}A = B.$$ 

Hence, $X = A - B$ is an idempotent and $B^*X = B^*(A - B) = 0$ and $XB^* = (A - B)B^* = 0$.

($\Longleftarrow$): Let $A = B + X$ and $B^*X = XB^* = 0$ for some idempotent $X$. Then, $B^*(A - B) = B^*X = 0$ and $(A - B)B^* = XB^* = 0$. Therefore, $B \preceq A$ by (1.2). $\square$

Corollary 4.8. Let $A \in \mathcal{Q}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then, $B \preceq AA^*$ if and only if $B$ is an idempotent and there exists an idempotent $X$ such that $AA^* = B + X$ and $B^*X = XB^* = 0$.

Proof. Let $A \in \mathcal{Q}(\mathcal{H})$. Then, $(AA^*)^2 = AA^*AA^* = AA^*$. Hence, $AA^*$ is an idempotent. Now apply Lemma 4.7. $\square$

We end our work with the following result.

Proposition 4.9. Let $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{Q}(\mathcal{H})$. Then, $B \in \mathcal{B}(\mathcal{H})$ is common $\preceq$ lower bound of $A$ and $CC^*$ if and only if $B$ is an idempotent and there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that

$$A = B + (I - B^*)X(I - B^*), \text{ and } CC^* = B + Y,$$

where $B^*Y = YB^* = 0$.

Proof. ($\Longrightarrow$): If $B$ be a common $\preceq$ lower bound of $A$ and $CC^*$, then $B \preceq A$ and $B \preceq CC^*$. It follows from the assumption $B \preceq CC^*$ and Lemma 4.7 that $B$ is an idempotent and there exists an idempotent $Y \in \mathcal{B}(\mathcal{H})$ such that $CC^* = B + R$, where $B^*R = RB^* = 0$. Since $B$ is an idempotent and $B \preceq A$, by Theorem 4.3, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $A = B + (I - B^*)S(I - B^*)$.

($\Longleftarrow$): If there exists an idempotent $Y$ such that $CC^* = B + Y$ with $B^*Y = 0$ and $YB^* = 0$, then $B \preceq CC^*$. The assumption $A = B + (I - B^*)S(I - B^*)$ and the fact that $B$ is an idempotent yield $B^*(A - B) = 0$ and $(A - B)B^* = 0$. Hence, $B \preceq A$ and $B$ is a common $\preceq$ lower bound of $A$ and $CC^*$. $\square$
REFERENCES

[1] M.L. Arias and M.C. Gonzalez. Positive solutions to operator equations $AXB = C$. *Linear Algebra and its Applications*, 433:1194–1202, 2010.

[2] J. Antezana, C. Cano, I. Mosconi and D. Stojanoff. A note on the star order in Hilbert spaces. *Linear and Multilinear Algebra*, 58:1037–1051, 2010.

[3] D. Cvetković-Ilić. Re-visited solutions of the matrix equation $AXB = C$. *Journal of the Australian Mathematical Society*, 84:63–72, 2008.

[4] A. Dajić and J. J. Koliha. Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators. *Journal of Mathematical Analysis and Applications*, 333:567–576, 2007.

[5] C. Deng. On the solutions of operator equation $CAX = C = XAC$. *Journal of Mathematical Analysis and Applications*, 398:664–670, 2013.

[6] C. Deng and A.Yu. Some relations of projection and star order in Hilbert space. *Linear Algebra and its Applications*, 474:158–168, 2015.

[7] F.O. Farid, M.S. Moslehian, Wang, Qing-Wen, Wu and Zh.Ch. Wu. On the Hermitian solutions to a system of adjointable operator equations. *Linear Algebra and its Applications*, 437:1854–1891, 2012.

[8] D.S. Djordjević. Characterizations of normal, hyponormal and EP operators. *Journal of Mathematical Analysis and Applications*, 329:1181–1190, 2007.

[9] R.G. Douglas. On majorization, factorization and range inclusion of operators in Hilbert space. *Proceeding of the American Mathematical Society*, 17:413–416, 1966.

[10] Z.-H. He and Q.-W. Wang. The general solutions to some systems of matrix equations. *Linear and Multilinear Algebra*, 63:2017–2032, 2015.

[11] C.G. Khatri and S.K. Mitra. Hermitian and nonnegative definite solutions of linear matrix equations. *SIAM Journal on Applied Mathematics*, 31:579–585, 1976.

[12] L. Long and S. Gudder. On the supremum and infimum of bounded quantum observables. *Journal of Mathematical physics*, 52:122101, 2011.

[13] Z. Mousavi, F. Mirzapour and M.S. Moslehian. Positive definite solutions of certain non-linear matrix equations. *Operators and Matrices*, 10:113–126, 2016.

[14] S. Radosavljević and D.S. Djordjević. On pairs of generalized and hypergeneralized projections on a Hilbert space. *Functional Analysis, Approximation and Computation*, 5:67–75, 2013.

[15] Z. Sebestyén. Restrictions of positive operators. *Acta Scientiarum Mathematicarum (Szeged)*, 46:299–301, 1983.

[16] Q.-W. Wang and C.-Z. Dong. Positive solutions to a system of adjointable operator equations over Hilbert $C^*$-modules. *Linear Algebra and its Applications*, 433:1481–1489, 2010.

[17] Q. Xu. Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$. *Linear Algebra and its Applications*, 429:1–11, 2008.

[18] X.M. Xu, H.K. Du, X.C. Fang and Y. Li. The supremum of linear operators for the $*$-order. *Linear Algebra and its Applications*, 433:2198–2207, 2010.

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