Long-time magnetic and cross helicities evolution in the free decaying MHD turbulence

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Abstract. We re-examine the free decay of MHD turbulence at large Reynolds numbers in the framework of shell models. We study the statistical properties based on representative sample of realisations (128 realisations for each type of initial conditions) over the period of $10^5$ large-scale turnover times. The performed simulations show that the free-decaying non-helical MHD turbulence can demonstrate two different scenarios of evolution in spite of similar initial conditions. Within the first scenario, the cross-helicity accumulation is so fast that the energy cascade vanishes before significant magnetic energy dissipates. Then the system approaches the state of maximal cross-helicity. Within the second scenario, the cascade process continues to remain active until time $10^4$ in units of large-scale turnover time. Then the magnetic field becomes vastly helical due to magnetic helicity conservation. Thus the magnetic energy does not dissipate with kinetic energy.

1. Introduction

Free decaying magnetohydrodynamic (MHD) turbulence provokes interest for two main reasons. First, it raises the possibility of application to the physics of the interstellar medium and cosmology in the context of evolution of the primordial magnetic field and its contribution to the present configuration of the magnetic field in the Universe. Second, the MHD turbulence differs from the conventional turbulence of incompressible fluids by an extended set of conservation laws, which form the basis of diverse scenarios of free evolution of turbulent motion. This is a strong fundamental motivation for studying this problem. Three ideal quadratic invariants are known in 3D incompressible magnetohydrodynamics: the total energy $E = E^u + E^b$, the cross-helicity $H^c = \langle v \cdot b \rangle$ and the magnetic helicity $H^b = \langle a \cdot b \rangle$, where $E^u = \langle |v|^2 / 2 \rangle$, $E^b = \langle |b|^2 / 2 \rangle$, $v$ is the velocity field, $b$ is the magnetic field, $a$ is the vector potential ($b = \nabla \times a$). It is generally believed that for fully developed isotropic MHD turbulence there are no reasons for essential correlation between the pulsations of velocity and magnetic field (which means a noticeable level of cross-helicity). However, the discovery of highly correlated pulsation of velocity and magnetic field in the solar wind (Belcher & Davis, 1971) awoke interest in cross-helicity. The influence of cross-helicity on the forced MHD turbulence was studied by Grappin et al. (1983) in the context of the Alfvenic turbulence. The role of cross-helicity in stationary forced isotropic (not Alfvenic) MHD turbulence has been studied by Mizeva et al. (2009). It was shown that the injection of cross-helicity suppresses the spectral energy transfer and leads to energy accumulation in the turbulent flow. Then the spectrum becomes steeper and the intermittency decreases. Turning back to decaying turbulence, let us note that cosmological applications require consideration of...
very-long-term evolution of MHD turbulence — the age of the Universe calculated in the unit of turnover time for the largest galactic turbulent scale \( (\tau \approx 10^7 \text{ years}) \) gives \( T \approx 10^4 \), which is about 100 times longer than the best DNS range. The required time series can be considered in the framework of shell models of turbulence. For the first time, the long-time evolution of free-decaying MHD turbulence has been considered by Antonov et al. (2001). A coherent state with high alignment between the magnetic and velocity fields, and essential reduction of the dissipation rate, were obtained for most realisations. At the same time, there were a few realisations that displayed different behaviour, characterised by a low level of cross-helicity.

In this talk we present detailed analysis of the role of magnetic helicity and cross-helicity in the evolution scenario. We use a massive computer cluster which allows us to consider the statistical properties based on the representative sample of realisations (128 realisations for each type of initial conditions).

2. Shell model for helical MHD turbulence

Shell models describe the dynamics of fully-developed MHD turbulence through a set of complex variables \( U_n, B_n \), which characterise the amplitudes of velocity and magnetic field pulsations in the shell of wave number \( k_n < |k| < k_{n+1} \), where \( k_n = \lambda^n \) (\( \lambda \) is the shell width in a logarithmic scale). The total energy in terms of shell model is obviously the sum of kinetic and magnetic energy of individual shells \( E = \sum (|U_n|^2 + |B_n|^2)/2 \). The cross-helicity is defined in a similar way as \( H^c = \sum (U_n B_n^* + B_n U_n^*)/2 \). The definition of magnetic helicity is not so evident. The most popular MHD shell models (Brandenburg et al., 1996; Frick & Sokoloff, 1998; Basu et al., 1998) introduce the magnetic helicity as \( H^b = \sum (-1)^n |B_n|^2/k_n \), which associates the magnetic energy of a given shell with a positive or negative magnetic helicity. Then the non-helical state can be obtained only through the balance of magnetic energy in the even and odd shells. Helical shell model for hydrodynamical turbulence had been suggested by Benzi et al. (1996) which introduces twice more variables. It was generalized later for the MHD case by Lessinnes et al. (2009). Following the another idea used for the hydrodynamical helicity by Melander (1997), we define the magnetic helicity as \( H^b = \sum k_n^{-1} (|B_n|^2 - B_n^2)/2 \). This definition allows us to get helicity of any sign in any shell. Recent studies of the helicity cascades demonstrate similarity of result which was obtained using both approaches (Stepanov et al., 2009; Lessinnes et al., 2011). So we use the model introduced by Mizyeva et al. (2009):

\[
\begin{align*}
    d_t U_n &= i k_n (\Lambda_n(U,U) - \Lambda_n(B,B)) - \frac{k_n^2 U_n}{\text{Re}}, \quad (1) \\
    d_t B_n &= i k_n (\Lambda_n(U,B) - \Lambda_n(B,U)) - \frac{k_n^2 B_n}{\text{Rm}}, \quad (2)
\end{align*}
\]

where the nonlinear terms are written as

\[
\begin{align*}
    \Lambda_n(X,Y) &= \lambda^2 (X_{n+1} Y_{n+1} + X_n^* Y_{n+1}^*) - X_{n-1} Y_n \\
    -X_n Y_{n+1}^* + i \lambda(2X_n^* Y_{n+1}^* + X_{n+1}^* Y_{n+1} - X_n Y_{n+1}^*) \\
    +X_{n-1} Y_{n-1} + X_n^* Y_{n-1}^* - \lambda^2 (X_{n+1} Y_n + X_n Y_{n+1}) \\
    +i \lambda(2X_n^* Y_{n+1}^* + X_{n-1}^* Y_{n+1} - X_n Y_{n-1}^*),
\end{align*}
\]

a star means complex conjugation, and superscripts \( r, i \) are real and imaginary parts. \( \text{Re} \) and \( \text{Rm} \) are the kinetic and magnetic Reynolds numbers. In the limit \( \text{Re}, \text{Rm} \to \infty \) these eqs. (1) and (2) conserve the total energy, the cross-helicity, and the magnetic helicity. Time is measured in dimensionless units equal to the eddy turnover time on the shell \( n = 0 \) (\( k_0 = 1 \)), which corresponds to the largest scale of the system. For the case of hydrodynamics, L’vov et al. (1998) suggested the parameter \( \lambda \) equal to the golden number \( (1 + \sqrt{5})/2 \) for optimal spectral resolution which is also successfully applicable for MHD turbulence (Stepanov & Plunian, 2006).
3. Numerical results

Equations (1) and (2) were integrated for $0 \leq n \leq 40$ up to the time $t = 10^5$, when any turbulent transfer has been finished and only pure exponential energy decay remains in the largest scales. In all simulations, $\lambda = 1.618$ and $Re = Rm = 10^5$. First we consider the decay of fully-developed turbulence with vanishing helicities $H^c$ and $H^b$. Initial values of shell variables are $U_0 = -\sqrt{2} + \delta, B_0 = \sqrt{2} + \delta$, where $\delta$ is a random complex additive with real and imaginary parts in a range $[-10^{-4}: 10^{-4}]$, and $U_n = B_n = 0$ for other shells ($n > 0$). It corresponds to the initial state with $E^u \approx 1$, $E^b \approx 1$ and a small quantity of helicities $|H^c| \leq 10^{-4}, |H^b| \leq 10^{-4}$.

We checked that the distribution of initial helicity values are symmetric in the respect to zero. Figure 1(a) shows the evolution of the total energy in the entire totality of realisations. We see that similar initial conditions lead to the different scenarios in the evolution of the system. At the early stage of development (on the periods from several units to several ten) the bundle of trajectories remains sufficiently dense and is limited by the power laws $E(t) \sim t^{-1}$ and $E(t) \sim t^{-1/2}$, shown in the figure by dashed lines. Note that these are two power laws suggested for the decay of non-helical and helical MHD turbulence (in sense of magnetic helicity) (Campanelli, 2004). At the next stage ($50 < t < 1000$), the bundle of trajectories is bounded from below by a power law as before (but more steep, like $E(t) \sim t^{-2}$), while from above separate trajectories leave the bundle practically horizontal, which indicates the vanishing cascade of energy and transition to the exponential dissipation of the energy.

The variety of scenarios of the evolution confirms fig. 1(b), in which we show the evolution of the normalised cross-helicity $C$. The quantity $C = H^c/E$ characterizes the part of energy, concentrated in the correlated pulsations of velocity and magnetic field. The limit $C \to \pm 1$ corresponds to a completely correlated state ($U_n = \pm B_n$), in which the nonlinear energy transfer is blocked. Figure 1(b) shows that the main part of the trajectories reaches this state in the range of time $100 < t < 1000$, but there are some trajectories (about 10–15%) for which the evolution of $C$ stops at some arbitrary level.

Figure 2(a) shows that the cross-helicity $H^c$ can be generated, in contrast to energy, which can decay only. The active cross-helicity production is mostly observed until $t \approx 10^2$. In some realisations the cross-helicity reaches a level $|H^c| \approx 0.2$. The source of the cross-helicity in the free-decaying evolution can be the dissipation term only. In fig. 2(b) we present the evolution of the spectrum of cross-helicity for one realisation (namely, we took the realisation that corresponds to the lowest trajectory in fig. 2(a)). One can see that most intensive production
of the $H^c$ happened at the dissipation scale. Then nonlinear terms transfer $H^c$ to the largest scale like an inverse cascade. The same high sensitivity of $C$ to initial conditions had been noted by Dar et al. (1998).

The cross-helicity, which can be produced in the smallest scales by dissipation, is transported towards large scales through the inertial range. Then the spectral energy flux is considerably reduced depending on the level of accumulated cross-helicity in a given realisation. This leads to the blocking of the turbulent energy cascade, which occurred in different realisations at substantially different moments in time and with substantially different values of the remaining total energy (see fig. 1a). However, the growth of the normalised cross-helicity does not result in the completely correlated state $C = \pm 1$ for all realisations (see fig. 1b). Some realisations continue to stay at values $-1 < C < 1$. These realisations develop a highly helical magnetic field. The magnetic helicity (which is weak at the initial state) does not cascade to small scales and practically does not dissipate. This has been demonstrated through various studies starting from Frisch et al. (1975). If the energy transfer (and dissipation) is not blocked by the cross-helicity, only the magnetic field with maximal helicity survives at the late stage of the evolution.

The tendencies of correlation evolution are illustrated fig. 3, in which the distribution of realisations is shown on the $(C^b = H^b/(k_0 E^b), C = H^c/E)$ plane at different moments in time. All realisations start from the origin of the graph (at $t = 0$). At the first stage (up to $t \approx 100$), points scatter along the vertical line, showing the rapid increase of $|C|$. Several realisations ($\approx 10\%$) deviate from the axis $C^b = 0$. At the time $t = 10^3$, this set of points forms a cloud in the center of the plane, while the rest are concentrated on the lines $C = \pm 1$. At the late stage ($t = 10^4$), practically all points are on the lines $C = \pm 1$ or $C^b = \pm 1$. The first case ($C = \pm 1$) means that magnetic and velocity fields are completely correlated, while the second case ($C^b = \pm 1$) means that only the helical magnetic field remains at the largest scale. One can see that both energies scatter randomly from 0.01 to 0.1 at early stage ($t = 100$). At time $t = 10^3$, kinetic and magnetic fields approach the equipartition state ($U_n = B_n$) and the points mainly lie close to the line $E^b = E^u$. At late stage ($t = 10^4$) some points move away from this line. These points correspond to realisations that follow the second scenario, in which the kinetic energy continues to decay while the magnetic one, being helical, does not change ($C^b = \pm 1$).

The performed simulations show that the free-decaying non-helical ($H^b \approx 0, H^c \approx 0$) MHD turbulence can demonstrate fundamentally different ways of evolution in spite of similar initial conditions. We distinguish two scenarios of evolution. Within the first scenario, the cross-helicity accumulation is so fast that the energy cascade vanishes before significant magnetic energy dissipates. Then the system comes to a state with $C = \pm 1$ and the value of $C^b$ depends
on the rest of the magnetic energy. Within the second scenario, the cascade process remains active until the late time $t \sim 10^4$, when the magnetic field becomes vastly helical and later magnetic energy does not dissipate with kinetic energy. Then the system comes to the final state with $C^b = \pm 1$ and an arbitrary value of $C$.

We stress the point that the dynamics of helicities has a signifiable influence on the evolution of free-decaying MHD turbulence. Our understanding is based on the idea of the inverse cascade of magnetic helicity (Frisch et al., 1975) which has been confirmed by numerous simulation (Brandenburg et al., 1996; Christensson et al., 2001; Brandenburg, 2001; Mininni et al., 2005; Alexakis et al., 2006). Alexakis et al. (2006) reported that smaller-in-amplitude direct cascade is observed from the largest scale to small scales. This may be a result of insufficient inertial range resolution, in which forcing and dissipation scales are not well separated. In shell models, the Reynolds number is always quite large ($10^5$ or more) and the direct cascade of magnetic helicity was never observed. From the other perspective, a shortcoming of shell models is ignoring the nonlocal (in scales) interactions, due to which weak direct cascade of magnetic helicity may occur. This shortcoming can be overcome in the framework of shell models – a detailed study of the shell-to-shell interactions (Debliquy et al., 2005) gives a base on which to build a nonlocal MHD shell model (Plunian & Stepanov, 2007).

Finally, we note that the relationship $|H_x^b(k)| \leq k^{-1}E^b(k)$ indicates that the character of evolution can be changed substantially if the magnetic helicity can move to scales larger than the scale in which the energy is concentrated at $t = 0$. Figure 4 shows influence of the magnetic helicity on the free decay MHD turbulence. The inverse cascade of magnetic helicity to $k < k_0$ leads to the fact that an increasingly smaller part of magnetic energy is blocked at the largest scales, being excluded from the direct energy cascade to small scales (Frick & Stepanov, 2010). In this case the modulus of $C^b$ is limited by $k_0/k_{\text{min}}$ and the probability of the second scenario (with highly helical magnetic field) vanishes with the decrease of $k_{\text{min}}$.

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Figure 4. Energy spectra for different time $t = 1$ (dashed), 10 (solid), 100 (dot dashed), $10^3$ (dotted) without (a) and with (b) magnetic helicity in an initial field.

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