Arithmetic complexity revisited

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Abstract

The arithmetic complexity counts the number of algebraically independent entries in the periodic continued fraction \( \theta = [b_1, \ldots, b_N, a_1, \ldots, a_k] \). If \( \mathcal{A}_\theta \) is a noncommutative torus corresponding to the rational elliptic curve \( \mathcal{E}(K) \), then the rank of \( \mathcal{E}(K) \) is given by a simple formula \( r(\mathcal{E}(K)) = c(\mathcal{A}_\theta) - 1 \), where \( c(\mathcal{A}_\theta) \) is the arithmetic complexity of \( \theta \). We prove that \( c(\mathcal{A}_\theta) \) is equal to the dimension of the Brock–Elkies–Jordan variety of \( \theta \) introduced in Brock et al. (Acta Arith 197: 379–420, 2021). Following Zagier and Lemmermeyer, we evaluate the Shafarevich-Tate group of \( \mathcal{E}(K) \).

Keywords

Elliptic curve · Noncommutative torus · Brock–Elkies–Jordan variety

Mathematics Subject Classification

Primary 11G05; Secondary 46L85

1 Introduction

Noncommutative geometry studies an interplay between spatial forms and algebras with non-commutative multiplication. To illustrate the idea, consider the noncommutative torus \( \mathcal{A}_\theta \), i.e. the \( C^* \)-algebra generated by the unitary operators \( U \) and \( V \) satisfying the relation \( VU = e^{2\pi i \theta} UV \), where \( \theta \) is a real constant [Rieffel 1990] [9]. The \( \mathcal{A}_\theta \) is a non-commutative analog of the coordinate ring of complex elliptic curve \( \mathcal{E}(\mathbb{C}) \) [Manin 2004] [5]. Namely, there exists a covariant functor \( F \) mapping isomorphic elliptic curves \( \mathcal{E}(\mathbb{C}) \) to the Morita equivalent noncommutative tori \( \mathcal{A}_\theta \) [6, Sect. 1.3]. A restriction of \( F \) to the number fields \( K \subset \mathbb{C} \) corresponds to the irrational roots of quadratic polynomials with integer coefficients. Such roots unfold into...
a continued fraction of the period \((a_1, \ldots, a_k)\) which we write in the form:

\[
\theta = [b_1, \ldots, b_N, a_1, \ldots, a_k].
\] (1.1)

Let \(\mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}}\) be the Mordell-Weil group of the \(K\)-points of \(\mathcal{E}(C)\). To recast the rank \(r \geq 0\) of \(\mathcal{E}(K)\) in terms of the continued fraction (1.1), an arithmetic complexity \(c(\mathcal{A}_\theta)\) was used [6, Sect. 6.2.1]. Roughly speaking, the \(c(\mathcal{A}_\theta)\) counts the independent entries \(a_i\) and \(b_j\) in (1.1). Namely, the \(c(\mathcal{A}_\theta)\) is equal to the Krull dimension of a projective variety \(\mathcal{V}_E\) associated to the continued fraction (1.1). The equation of \(\mathcal{V}_E\) for \(\theta = \{\sqrt{D} | D > 0\ \text{square-free integer}\}\) was first studied by [Euler 1765] [2], hence the notation. The rank of elliptic curve \(E(K)\) and the arithmetic complexity of \(\mathcal{A}_\theta = F(\mathcal{E}(K))\) are linked by a simple formula:

\[
r = c(F(\mathcal{E}(K))) - 1.
\] (1.2)

An affine variety \(V_{N,k}(C)\) associated to (1.1) has been studied by [Brock, Elkies & Jordan 2019] [1]. In this remarkable paper, the authors prove that the \(V_{N,k}(C)\) fibers over the Fermat-Pell conic \(\mathcal{D}\) [Brock, Elkies & Jordan 2019] [1, Theorem 3.5]. The \(\mathbb{Z}\)-points of \(V_{N,k}(C)\) correspond to the continued fractions (1.1) and the reader is referred to Sect. 2.1 for other details. We call the \(V_{N,k}(C)\) a Brock-Elkies-Jordan variety.

The aim of our note is to relate geometry of the variety \(V_{N,k}(C)\) with the rank \(r\), torsion \(\mathcal{E}_{\text{tors}}\) and the arithmetic complexity \(c(F(\mathcal{E}(K)))\). The maximal connected component of the Brock-Elkies-Jordan variety containing a point \((b_1, \ldots, b_N; a_1, \ldots, a_k)\) will be denoted by \(V_{0,N,k}^0\). (We refer the reader to Sect. 4.2 for quick examples of the \(V_{1,2}^0\).) Let \(\mathbb{A}^r\) be the \(r\)-dimensional affine space. Our main results can be formulated as follows.

**Theorem 1.1** \(V_{0,N,k}^0\) is a fiber bundle over the Fermat-Pell conic \(\mathcal{D}\) with the fibers \(\mathbb{A}^r\) and the structure group \(\mathcal{E}_{\text{tors}}\). In particular, the arithmetic complexity is given by the formula

\[
c(F(\mathcal{E}(K))) = \dim_{\mathbb{C}} V_{N,k}^0.
\] (1.3)

**Remark 1.2** Theorem 1.1 can be used to evaluate ranks of the elliptic curves via formula (1.2). As an example, we calculate such ranks in Sect. 4 using explicit formulas for the continued fractions (1.1) with \(N + k \leq 3\) developed by [Brock, Elkies & Jordan 2019] [1, Sects. 5-8].

Recall that the Shafarevich-Tate group \(\text{III}(\mathcal{E}(K))\) measures the failure of the Hasse principle for the elliptic curve \(\mathcal{E}(K)\). Denote by \(\text{Cl}(\Lambda)\) the (narrow) class group of the order \(\Lambda := \mathbb{Z} + \mathbb{Z}\theta\) in the ring of integers of the real quadratic field \(k = \mathbb{Q}(\theta)\). If \(\Lambda\) is the maximal order, then \(\text{Cl}(\Lambda)\) is the class group of the field \(k\). If \(\mathcal{D}\) is a conic, then \(\text{III}(\mathcal{D}) \cong \text{Cl}(\Lambda) \oplus \text{Cl}(\Lambda)\) [Zagier 1991] [10, Sect. 4.2] and [Lemmermeyer 2003] [4, Theorem 11]. Applying this formula to the fiber bundle \(V_{N,k}^0\) over the Fermat-Pell conic \(\mathcal{D}\), one gets the following result.
Corollary 1.3 \( \mathcal{III}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda) \).

The article is organized as follows. Preliminary facts can be found in Sect. 2. The proofs of Theorem 1.1 and Corollary 1.3 are given in Sect. 3. Two examples illustrating Theorem 1.1 are considered in Sect. 4.

2 Preliminaries

We briefly review the Brock–Elkies–Jordan varieties and arithmetic complexity of the noncommutative tori. For a detailed account we refer the reader to [Brock, Elkies & Jordan 2019] [1] and [6, Sect. 6.2.1], respectively.

2.1 Brock–Elkies–Jordan variety

By an infinite continued fraction one understands an expression of the form:

\[ [c_1, c_2, c_3, \ldots] := c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}, \tag{2.1} \]

where \( c_1 \) is an integer and \( c_2, c_3, \ldots \) are positive integers. The continued fraction \( (2.1) \) converges to an irrational number and each irrational number has a unique representation by \( (2.1) \). The expression \( (2.1) \) is called \( k \)-periodic, if \( c_{i+k} = c_i \) for all \( i \geq N \) and a minimal index \( k \geq 1 \). We shall denote the \( k \)-periodic continued fraction by

\[ [b_1, \ldots, b_N, a_1, \ldots, a_k], \tag{2.2} \]

where \( (a_1, \ldots, a_k) \) is the minimal period of \( (2.1) \). The continued fraction \( (2.2) \) converges to the irrational root of a quadratic polynomial

\[ Ax^2 + Bx + C \in \mathbb{Z}[x]. \tag{2.3} \]

Conversely, the irrational root of any quadratic polynomial \( (2.3) \) has a representation by the continued fraction \( (2.2) \). Notice that the following two continued fractions define the same irrational number:

\[ [b_1, \ldots, b_N, \overline{a_1}, \ldots, \overline{a_k}] = [b_1, \ldots, b_N, a_1, \ldots, a_k, \overline{a_1}, \ldots, \overline{a_k}], \tag{2.4} \]

But it is well known, that two infinite continued fraction with at most finite number of distinct entries must be related by the linear fractional transformation given by a matrix \( \mathcal{E} \in GL_2(\mathbb{Z}) \). Therefore Eq. \( (2.4) \) can be written in the form

\[ x = \frac{E_{11}x + E_{12}}{E_{21}x + E_{22}}, \tag{2.5} \]

where \( \mathcal{E} = (E_{ij}) \in GL_2(\mathbb{Z}) \) and \( x = [b_1, \ldots, b_N, \overline{a_1}, \ldots, \overline{a_k}] \).
Remark 2.1 It is easy to see, that \( x \) in (2.5) is the root of quadratic polynomial (2.3) with \( A = E_{21}, B = E_{22} - E_{11} \) and \( C = -E_{12} \).

Definition 2.2 ([1, Definition 3.1]) The Brock–Elkies–Jordan variety \( V_{N,k}(\mathbb{C}) \subset \mathbb{A}^{N+k} \) is an affine variety over \( \mathbb{Z} \) defined by the three equations:

\[
\begin{cases}
A[E_{22} - E_{11}](y_1, \ldots, y_N, x_1, \ldots, x_k) = B E_{21}(y_1, \ldots, y_N, x_1, \ldots, x_k) \\
-AE_{12}(y_1, \ldots, y_N, x_1, \ldots, x_k) = C E_{21}(y_1, \ldots, y_N, x_1, \ldots, x_k) \\
-BE_{12}(y_1, \ldots, y_N, x_1, \ldots, x_k) = C[E_{22} - E_{11}](y_1, \ldots, y_N, x_1, \ldots, x_k),
\end{cases}
\]

where \( A, B, C \) are constants and \( E_{ij} \in \mathbb{Z}[y_1, \ldots, y_N, x_1, \ldots, x_k] \), see Remark 2.1.

Remark 2.3 Our notation \( V_{N,k}(\mathbb{C}) \) corresponds to the variety \( V(B)_{N,k} \), where \( B \) is the multi-set of roots [Brock–Elkies–Jordan 2021] [1, Definition 3.1]. Such a change is justified, since we focus on the continued fractions with the integer entries.

It is verified directly from Remark 2.1 and the equality \( E_{11}E_{22} - E_{12}E_{21} = (-1)^k \), that

\[ CE_{21}^2 - BE_{21}E_{22} + AE_{22}^2 = (-1)^k A. \] (2.6)

Definition 2.4 By the Fermat-Pell conic \( \mathcal{Q} \) one understands the plane curve:

\[ Cx^2 - Bxy + Ay^2 = (-1)^k A. \] (2.7)

Theorem 2.5 (Brock, Elkies & Jordan [1]) The affine variety \( V_{N,k}(\mathbb{C}) \) fibers over the Fermat-Pell conic \( \mathcal{Q} \), i.e. there exists a map \( \pi : V_{N,k}(\mathbb{C}) \to \mathcal{Q} \), such that

\[ \pi(y_1, \ldots, y_N, x_1, \ldots, x_k) = (E_{21}, E_{22}). \] (2.8)

2.2 Arithmetic complexity

The noncommutative torus \( \mathcal{A}_\theta \) is said to have real multiplication, if \( \theta \) is the irrational root of a quadratic polynomial (2.3). To distinguish this case from the other values of \( \theta \), we shall write \( \mathcal{A}_{RM} \). It is clear, that we can write \( \theta \) as:

\[ \theta_d = \frac{a + b\sqrt{d}}{c}, \] (2.9)

where \( a, b, c \in \mathbb{Z} \) and \( d > 0 \) is a square-free integer. Consider a family of the irrational numbers \( \{\theta_x \mid x > 0\} \) of the form:

\[ \theta_x = \frac{a + b\sqrt{x}}{c}, \quad a, b, c = \text{Const} \quad \text{and} \quad x \in U_d, \] (2.10)
where $U_d$ is a set of the square-free integers containing $d$. A system of the polynomial equations in the ring $\mathbb{Z}[y_1, \ldots, y_N, x_1, \ldots, x_k]$ is called Euler’s, if each $\theta_x \mid x \in U_d$ can be written as

$$\theta_x = [b_1(x), \ldots, b_N(x), a_1(x), \ldots, a_k(x)], \quad (2.11)$$

where $a_i(x), b_j(x) \in \mathbb{Z}[x]$. It is not hard to see, that (2.11) is equivalent to the equations $A = E_{21}, B = E_{22} - E_{11}, C = -E_{12}$ in the ring $\mathbb{Z}[y_1, \ldots, y_N, x_1, \ldots, x_k]$, since $\theta_x$ must satisfy condition (2.4) and Eq. (2.5), see Remark 2.1. Thus the Euler equations define an algebraic set in the affine space $\mathbb{A}^{N+k}$. The Euler variety $\mathcal{V}_E$ is defined as the projective closure of an irreducible affine variety containing the point $x = d$ of this set.

**Remark 2.6** An immediate example of the Euler equations is given by formula (4.3). Such equations are equivalent to the Brock–Elkies–Jordan equations in Definition 2.2, see item (i) in the proof of Lemma 3.1.

**Remark 2.7** The case $a = 0$ and $c = 1$ of (2.10) was first studied by [Euler 1765] [2]; hence the name.

**Definition 2.8** By an arithmetic complexity $c(\mathcal{A}_{RM})$ of the noncommutative torus $\mathcal{A}_{RM}$ one understands the Krull dimension of the Euler variety $\mathcal{V}_E$.

**Remark 2.9** The $c(\mathcal{A}_{RM})$ counts the number of the algebraically independent entries in the continued fraction $\theta_d = [b_1, \ldots, b_N, a_1, \ldots, a_k]$.

**Theorem 2.10** ([6, Theorem 6.2.1]) The rank $r$ of the elliptic curve $\mathcal{E}(K)$ is given by the formula:

$$r = c(\mathcal{A}_{RM}) - 1, \quad (2.12)$$

where $\mathcal{A}_{RM} = F(\mathcal{E}(K))$.

### 3 Proofs

#### 3.1 Proof of Theorem 1.1

We shall split the proof in a series of lemmas.

**Lemma 3.1** The connected component $V_{N,k}^0$ of the Brock–Elkies–Jordan variety $V_{N,k}(\mathbb{C})$ is a fiber bundle over the Fermat-Pell conic $\mathcal{O}$:

$$\pi : V_{N,k}^0 \to \mathcal{O}, \quad (3.1)$$

such that each fiber $\{\pi^{-1}(q) \mid q \in \mathcal{O}\}$ is an $r$-dimensional affine space $\mathbb{A}^r$.
Proof Roughly speaking, the idea is to take the projective closure of (3.1) and apply [6, Lemma 6.2.2]. Such a lemma says the Euler variety $\mathcal{V}_E$ is a fiber bundle over the projective line $\mathbb{C}P^1$ with the fiber an abelian variety $A_E$.

(i) Let us show that

$$\text{Proj } V^0_{N,k} \cong \mathcal{V}_E,$$  \hspace{1cm} (3.2)

where $\text{Proj}$ is the projective closure of the affine variety $V^0_{N,k}$. Indeed, recall that the $V^0_{N,k}$ is defined by the coefficients $A$, $B$ and $C$ of the quadratic polynomial (2.3). Using these coefficients, one can write (2.9) in the form:

$$\theta_d = -B + b\sqrt{d}/2A,$$

where $b^2d = B^2 - 4AC$. \hspace{1cm} (3.3)

Thus the $V^0_{N,k}$ defines the Euler equations for the family $\theta_x$ and the Euler variety $\mathcal{V}_E$, see Sect. 2.2. Moreover, the equations in Definition 2.2 must coincide with the Euler equations for the family (2.11). We conclude from (3.3) that the $V^0_{N,k}$ is an irreducible affine variety containing the point $x = d$. Thus, $\text{Proj } V^0_{N,k} \cong \mathcal{V}_E$ by the definition of the Euler variety $\mathcal{V}_E$ in Sect. 2.2. Item (i) is proved.

(ii) To prove that $\{\pi^{-1}(q) \cong \mathbb{A}^r \mid q \in \mathcal{D}\}$, let $(\mathcal{V}_E, \mathbb{C}P^1, \pi', A_E)$ be the fiber bundle constructed in [6, Lemma 6.2.2]. Consider a morphism of the following fiber bundles:

$$(V^0_{N,k}, \mathcal{D}, \pi, X) \to (\mathcal{V}_E, \mathbb{C}P^1, \pi', A_E),$$  \hspace{1cm} (3.4)

where $X$ is a fiber $\pi^{-1}(q)$ over $q \in \mathcal{D}$. Recall from item (i), that we have an isomorphism $\text{Proj } V^0_{N,k} \cong \mathcal{V}_E$. Since the genus of the Fermat-Pell conic $\mathcal{D}$ is equal to zero, we conclude that $\text{Proj } \mathcal{D} \cong \mathbb{C}P^1$. The fiber map $\pi'$ is the projective extension of the map $\pi$.

To calculate the fiber $X$ in (3.4), recall that the $A_E$ is an abelian variety of dimension $r$ [6, Lemma 2.4]. In particular, the $A_E$ is a compact algebraic group (group variety). Thus one gets a group morphism

$$h : X \to A_E,$$  \hspace{1cm} (3.5)

where $X$ is an affine group variety. Since the fundamental group $\pi_1(X)$ is trivial, we conclude that $h$ is a covering map with $\ker(h) \cong \mathbb{Z}^r$, where $\mathbb{Z}^r$ is a discrete subgroup of $X$. Thus $X \cong \mathbb{A}^r$ is the $r$-dimensional affine space. Item (ii) is proved.

To finish the proof of Lemma 3.1, we apply [6, Lemma 6.2.2]. A pullback of the $(\mathcal{V}_E, \mathbb{C}P^1, \pi', A_E)$ along the morphism (3.4) defines a fiber bundle $(V^0_{N,k}, \mathcal{D}, \pi, \mathbb{A}^r)$. Lemma 3.1 is proved.

Lemma 3.2 The structure group of the fiber bundle $(V^0_{N,k}, \mathcal{D}, \pi, \mathbb{A}^r)$ is isomorphic to the abelian group $\mathbb{E}_{\text{tors}}$. □
Proof Let $\mathcal{Q}$ be the Fermat-Pell conic defined by the Eq. (2.7). Since $\theta_d$ in formula (3.3) is a real number, the coefficients $A, B, C$ must satisfy the inequality $B^2 - 4AC > 0$. Thus Eq. (2.7) defines a hyperbola. The completion of $\mathcal{Q}$ by the point at infinity is homeomorphic to the unit circle $S^1 = \mathcal{Q} \cup \{\infty\}$. Denote by $(\hat{V}_{N,k}, S^1, \pi, \mathbb{A}^r)$ the corresponding one-point completion of $(V^0_{N,k}, \mathcal{Q}, \pi, \mathbb{A}^r)$ by a fiber $\mathbb{A}^r$ at the infinity.

Consider the universal cover $\mathbf{R}$ of $S^1$ given by the formula
\[ t \mapsto e^{2\pi it}, \quad t \in \mathbf{R}. \] (3.6)

We denote by $(\mathbb{A}^r \times \mathbf{R}, \mathbf{R}, \pi, \mathbb{A}^r)$ the pullback of the fiber bundle $(\hat{V}_{N,k}^0, S^1, \pi, \mathbb{A}^r)$ along the map (3.6). Consider the integer points $\mathbf{Z}^r$ of the affine space $\mathbb{A}^r$. It follows from [6, Lemma 6.2.2], that $\mathcal{E}(K)$ is embedded into $\mathbb{A}^r \times \mathbf{R}$ according to the formula:
\[ \mathcal{E}(K) \cong \mathbf{Z}^r \oplus \mathcal{E}_{\text{tors}} \hookrightarrow \mathbb{A}^r \times \mathbf{R}/m\mathbf{Z}, \] (3.7)

where $m = |\mathcal{E}_{\text{tors}}|$. Using formula (3.7) and the map
\[ t \mapsto e^{2\pi imt}, \quad t \in \mathbf{R}, \] (3.8)

we conclude that the fiber bundle $(\hat{V}_{N,k}^0, S^1, \pi, \mathbb{A}^r)$ has the structure group
\[ \mathbf{Z}/m\mathbf{Z} \cong \mathcal{E}_{\text{tors}}. \] (3.9)

The same is true of the fiber bundle $(V^0_{N,k}, \mathcal{Q}, \pi, \mathbb{A}^r)$, provided the point $\infty$ of conic $\mathcal{Q}$ is endowed with the fiber $\mathbb{A}^r$. Lemma 3.2 is proved. □

Lemma 3.3 $c(\mathcal{A}_\theta) = \dim C V^0_{N,k}$.

Proof Consider the fiber bundle $(V^0_{N,k}, \mathcal{Q}, \pi, \mathbb{A}^r)$. The dimension of a fiber bundle is the sum of dimensions of the base space and the fibers, i.e.
\[ \dim C V^0_{N,k} = \dim C \mathbb{A}^r + \dim C \mathcal{Q} = r + 1. \] (3.10)

Comparing (3.10) with (2.12), we conclude that $c(\mathcal{A}_\theta) = \dim C V^0_{N,k}$. Lemma 3.3 is proved. □

Theorem 1.1 follows from Lemmas 3.1–3.3.

3.2 Proof of corollary 1.3

We shall split the proof in two lemmas.

Lemma 3.4 $\mathcal{H}(\mathcal{Q}) \cong C_l (\Lambda) \oplus C_l (\Lambda)$. □

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Consider the order \( \Lambda = \mathbb{Z} + \mathbb{Z} \theta \) in the ring of integers in the real quadratic field \( k = \mathbb{Q}(\sqrt{D}) \). Since \( \theta \) is the root of quadratic Eq. (2.3), we conclude that \( A = 1, B = 0 \) and \( C = -D \). Using (2.7) one can write the Fermat-Pell conic \( \mathcal{Q} \) in the form:

\[
y^2 - Dx^2 = (-1)^k.
\] (3.11)

By the analogy between the Birch–Swinnerton–Dyer conjecture for elliptic curves and the Dirichlet class number formula observed in [Zagier 1991] [10, beginning of Sect. 4.2] and proved for conics in [Lemmermeyer, around 2003] [4, Theorem 11], the Shafarevich-Tate group \( \text{III}(\mathcal{Q}) \) of the conic (2.7) is given by the formula:

\[
\text{III}(\mathcal{Q}) \cong Cl(\mathcal{Q}) \oplus Cl(\mathcal{Q}).
\] (3.12)

where \( Cl(\mathcal{Q}) \) is the class group of \( \mathcal{Q} \). But \( Cl(\mathcal{Q}) \) is equal to the narrow class group of the order \( \Lambda \). Thus \( \text{III}(\mathcal{Q}) \cong Cl(\Lambda) \oplus Cl(\Lambda) \). Lemma 3.4 is proved. □

**Lemma 3.5** \( \text{III}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda) \).

**Proof** Using formula (3.7), one can identify the \( K \)-rational points of the elliptic curve \( \mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}} \) with the integer points of the fiber bundle \((V^0_{N,k}, \mathcal{Q}, \pi, \mathbb{A}^r)\). Since \( \mathbb{A}^r \) is a rational variety, the group \( \text{III}(\mathbb{A}^r) \) is trivial. We conclude therefore, that the failure of the Hasse principle for \( \mathcal{E}(K) \) occurs only in the base space \( \mathcal{Q} \). On the other hand, Lemma 3.4 says that \( \text{III}(\mathcal{Q}) \cong Cl(\Lambda) \oplus Cl(\Lambda) \). Thus \( \text{III}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda) \). Lemma 3.5 is proved. □

Corollary 1.3 follows from Lemma 3.5.

**Remark 3.6** The formula \( \text{III}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda) \) can be proved in purely algebraic terms [7, Corollary 1.3]. However, the approach based on the Lemmermeyer’s Lemma 3.4 and geometry of the fiber bundle \((V^0_{N,k}, \mathcal{Q}, \pi, \mathbb{A}^r)\) seems to be more elegant.

**4 Examples**

We shall consider two examples illustrating Theorem 1.1. They correspond to the variety \( V_{1,2}(\mathbb{C}) \) [Brock, Elkies & Jordan 2019] [1, Sect. 8].

**4.1 Brock–Elkies–Jordan variety \( V_{1,2}(\mathbb{C}) \)**

Consider the variety \( V_{N,k}(\mathbb{C}) \) with \( N = 1 \) and \( k = 2 \). According to [Brock, Elkies & Jordan 2019] [1, Sect. 8], in this case the variety \( V_{1,2}(\mathbb{C}) \) is defined by the system of equations:

\[
\begin{align*}
A(x_1x_2 - 2y_1x_1) &= Bx_1 \\
A(y_1^2x_1 - y_1x_1x_2 - x_2) &= Cx_1 \\
B(y_1^2x_1 - y_1x_1x_2 - x_2) &= C(x_1x_2 - 2y_1x_1).
\end{align*}
\] (4.1)
Theorem 4.1 ([Brock, Elkies & Jordan 2019] [1, Sect. 8]) The Brock–Elkies–Jordan variety $V_{1,2}(\mathbb{C})$ has:

(i) one component $[y_1, 0, 0]$, if $A = 0 \neq B$ or $[y_1, 0, x_2]$, if $A = B = 0$;
(ii) two components $[y_1, 0, 0]$ and

$$\left[ y_1, -\frac{2Ay_1 + B}{Ay_1^2 + By_1 + C}, \frac{2Ay_1 + B}{A} \right], \text{ if } A \neq 0; \quad (4.2)$$

(iii) three components $[y_1, 0, 0]$, $(4.2)$ and $[-\frac{B}{2A}, x_1, 0]$, if $B^2 = 4AC$ and $A \neq 0$.

4.2 Arithmetic complexity $c(F(\mathcal{E}(K)))$

Consider the following two families of elliptic curves $\mathcal{E}(K)$.

Example 4.2 ([6, Sect. 1.4]) Let $b \geq 1$ be an integer. Denote by $\mathcal{E}_{CM}$ an elliptic curve with complex multiplication by the integers of the imaginary quadratic field $\mathbb{Q}(i\sqrt{b^2 + 2})$. Then $F(\mathcal{E}_{CM}) = \mathcal{A}_{\sqrt{b^2 + 2}}$ [6, Theorem 1.4.1]. It is easy to see, that

$$\sqrt{b^2 + 2} = [b, b, 2b]. \quad (4.3)$$

Therefore $N = 1$ and $k = 2$, i.e. the $V_{1,2}(\mathbb{C})$ is the Brock–Elkies–Jordan variety corresponding to the continued fraction $(4.3)$. Recall that a classification of the connected components of the variety $V_{1,2}(\mathbb{C})$ is provided by Theorem 4.1. It is clear, that the connected components $[y_1, 0, 0]$, $[y_1, 0, x_2]$ or $[-\frac{B}{2A}, x_1, 0]$ in Theorem 4.1 must be excluded. Thus the component $V_{1,2}^0$ containing continued fraction $(4.3)$ is given by the formulas $(4.2)$. To calculate dimension of the component $V_{1,2}^0$, notice that the substitution

$$\begin{cases} y_1 = uv \\ A = 1 \\ B = 0 \\ C = -v(u^2v + 1), \end{cases} \quad (4.4)$$

brings component $(4.2)$ to the form:

$$[uv, 2u, 2uv] = \sqrt{v(u^2v + 1)}. \quad (4.5)$$

The restriction $2u = b$, $v = 2$ shows that our fraction $(4.3)$ is contained in the component $V_{1,2}^0$ described by $(4.5)$. Moreover, the $V_{1,2}^0$ is the maximal component with such a property. Since the $V_{1,2}^0$ is parametrized by two complex variables $u$ and $v$, we conclude that

$$\dim_{\mathbb{C}} V_{1,2}^0 = 2 = c(F(\mathcal{E}_{CM})). \quad (4.6)$$
**Example 4.3** ([6, Sect. 6.2.4.2]) Let \( b \geq 3 \) be an integer. Denote by \( \mathcal{E}(\mathbb{Q}) \) an elliptic curve defined by the affine equation

\[
y^2 = x(x - 1)\left(x - \frac{b - 2}{b + 2}\right). \tag{4.7}
\]

It is known ([6, Sect. 6.2.4.2]) that \( F(\mathcal{E}(\mathbb{Q})) = \mathcal{V}_{\frac{1}{2}}(b + \sqrt{b^2 - 4}) \). The reader can verify that

\[
\frac{b + \sqrt{b^2 - 4}}{2} = [b - 1, 1, b - 2]. \tag{4.8}
\]

Thus the \( V_{1,2}(C) \) is the Brock–Elkies–Jordan variety corresponding to the continued fraction (4.8). Clearly, the connected components \([y_1, 0, 0], [y_1, 0, x_2] \) or \([\frac{-B}{2A}, x_1, 0] \) in Theorem 4.1 are excluded. Therefore the connected component \( V^0_{1,2} \) containing periodic continued fraction (4.8) is given by formula (4.2). Using the substitution

\[
\begin{align*}
y_1 &= u - 1 \\
A &= 1 \\
B &= -u \\
C &= 1,
\end{align*} \tag{4.9}
\]

one can write (4.2) in the form:

\[
[u - 1, 1, u - 2] = \frac{u + \sqrt{u^2 - 4}}{2}. \tag{4.10}
\]

The continued fractions (4.8) and (4.10) are identical after the substitution \( u = b \). Thus (4.8) parametrizes the component \( V^0_{1,2} \). Moreover, the \( V^0_{1,2} \) is the maximal component with such a property. Since the \( V^0_{1,2} \) depends on one complex variable \( u \), we conclude that

\[
\dim_C V^0_{1,2} = 1 = c(F(\mathcal{E}(\mathbb{Q}))). \tag{4.11}
\]

**Remark 4.4** Formula (4.11) is valid only for the odd values of \( b \geq 3 \) in (4.8). Indeed, the substitution \( b = 2k \) brings the LHS of (4.8) to the form \( k + \sqrt{k^2 - 1} \). The latter is a special case of the RHS of the family (4.5) with \( u = k \) and \( v = -1 \). Thus the even values of \( b \geq 3 \) must be excluded.

### 4.3 Ranks of elliptic curves \( \mathcal{E}(K) \)

The ranks of elliptic curves \( \mathcal{E}_{CM} \) and \( \mathcal{E}(\mathbb{Q}) \) can be evaluated using formulas (1.2), (4.6) and (4.11).

**Example 4.5** Let the \( \mathcal{E}_{CM} \) be as in Example 4.2. The formulas (1.2) and (4.6) say that

\[
r(\mathcal{E}_{CM}) = c(F(\mathcal{E}_{CM})) - 1 = 2 - 1 = 1. \tag{4.12}
\]
Remark 4.6 The arithmetic of elliptic curves $E_{CM}$ has been studied by [Gross 1980] [3]. In particular, it was proved that $r(E_{CM}) = 1$, whenever $E_{CM}$ (satisfying some additional assumptions [6, Sect. 1.4.2]) is an elliptic curve with complex multiplication by the integers of the field $\mathbb{Q}(\sqrt{-p})$, where $p \equiv 3 \mod 8$ is a prime number [Gross 1980] [3, Theorem 22.4.2]. The reader can verify, that letting $b = 2k + 1$ in formula (4.3) implies $b \equiv 3 \mod 8$. Therefore formula (4.12) is a generalization of the result of [Gross 1980] [3, Theorem 22.4.2]. We refer the reader to [6, Table 1.1] for more examples.

Example 4.7 Let the $E(\mathbb{Q})$ be as in Example 4.3. One gets using (1.2), (4.11) and Remark 4.4, that

$$r(E(\mathbb{Q})) = c(F(E(\mathbb{Q}))) - 1 = 1 - 1 = 0.$$ \hspace{1cm} (4.13)

Thus the rank of of the generic fiber $E(\mathbb{Q})$ of the family (4.7) must be equal to zero, see also [8]. In other words, for the infinitely many odd integers $b \geq 3$ the rank of the elliptic curve $E(\mathbb{Q})$ is zero.

Remark 4.8 The reader is advised against testing the validity of formula (4.13) using a vast database of the analytic values of $r(E(\mathbb{Q}))$. While such calculations give correct results for the generic fibers of (4.7), they are misleading on the special fibers and the values of $b$ lying outside an admissible set of values.

Declarations

Ethical statement The author refrain from misrepresenting research results which could damage the trust in the journal, the professionalism of scientific authorship, and ultimately the entire scientific endeavour.

References

1. Brock, B.W., N.D. Elkies, and B.W. Jordan. 2021. Periodic continued fractions over $S$-integers in number fields and Skolem’s $p$-adic method. Acta Arith. 197: 379–420.
2. Euler, L. 1765. De usu novi algorithmi in Problemate Pelliano solvendo, Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae. 11.
3. Gross, B.H. 1980. Arithmetic on Elliptic Curves with Complex Multiplication, Lecture Notes Math. 776, Springer.
4. Lemmermeyer, F. Conics—a poor man’s elliptic curves, http://www.fen.bilkent.edu.tr/~franz/publ/pell-sum.pdf
5. Manin, Yu.I. 2004. Real multiplication and noncommutative geometry, in “The legacy of Niels Hendrik Abel”, 685–727. Berlin: Springer.
6. Nikolaev, I.V. 2022. Noncommutative Geometry, De Gruyter Studies in Math. 66, Second Edition, Berlin.
7. Nikolaev, I.V. Quantum dynamics of elliptic curves, arXiv:1808.03493
8. Nikolaev, I.V. Noncommutative geometry of elliptic surfaces, arXiv:2106.11392
9. Rieffel, M.A. 1990. Non-commutative tori—a case study of non-commutative differentiable manifolds. Contemp. Math. 105: 191–211.
10. Zagier, D. 1991. The Birch-Swinnerton-Dyer conjecture from a naive point of view, Arithmetic algebraic geometry (Texel, 1989), 377-389, Progr. Math. 89, Birkhäuser Boston, Boston, MA.
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