A note on geometric algebras and control problems with SO(3)-symmetries

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1 | INTRODUCTION

Geometric control theory uses geometric methods to control various mechanical systems.\textsuperscript{1,2} We use the methods of sub-Riemannian geometry and Hamiltonian concept.\textsuperscript{3,4} As a reasonable starting point, we consider mechanisms moving in the plane, typically wheeled mechanisms like cars (with or without trailers) or robotic snakes.\textsuperscript{5,6} The movement of a planar mechanisms is always invariant with respect to the action of the Euclidean group SE(2). As the prototypes of planar mechanisms, we choose those consisting of the body in the shape of a triangle and three legs connected to the vertices of the body by joints of various types and combinations, see the Figure 1. Although such mechanisms have almost the same shape, the configuration spaces may differ. In particular, possible motions of the mechanism induce a specific filtration in the configuration space. We present two examples that carry the filtration (3, 6) and (4, 7), respectively.\textsuperscript{6,7}

To control the mechanisms locally, we consider the nilpotent approximations of the original control systems.\textsuperscript{8} Although the configuration spaces and their approximations have the same filtration, the approximations form Carnot groups that are generally endowed with more symmetries.\textsuperscript{9} One gets the symmetries generated by the right-invariant vector fields, and there may be additional symmetries acting nontrivially on the distribution. Our Carnot groups of filtrations (3, 6) and (4, 7) carry subgroups of the symmetries isomorphic to SO(3).\textsuperscript{7,10} This observation leads to the idea of the local control in geometric algebra approach.

We reformulate the control problems in the concept of geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_4$.\textsuperscript{11–13} We use the natural SO(3)-invariant operations in geometric algebras to reduce the set of geodesics to a simpler set of curves in the geometric algebra.\textsuperscript{14} Namely, each geodesic is a linear combination of orthogonal vectors, and SO(3) acts on the geodesics by means of...
the action on the appropriate orthonormal system of vectors. So it is sufficient to study geodesics for one fixed orthonormal basis, that is, we can study just geodesics in the moduli space over the action of SO(3).

We present the local control algorithm for finding geodesics passing through the origin and an arbitrary point in its neighborhood. The algorithm is based on the use of rotors in order to relate two orthogonal bases. We provide an efficient method to such comparison using geometric algebras. We illustrate our algorithm on two specific examples.

2 | NILPOTENT CONTROL PROBLEMS

We focus on two control problems such that their symmetry groups contain SO(3) as their subgroups. The first system has the growth vector (3, 6), and the other one has the growth vector (4, 7).7,9

2.1 | Control problems on Carnot groups of step 2

By nilpotent control problems, we mean the invariant control problems on Carnot groups and we consider the Carnot groups $G$ of step 2 with the filtration $(m, n)$,3,15,16. If we denote the local coordinates by $(x, z) \in \mathbb{R}^m \oplus \mathbb{R}^{n-m}$, we can model the corresponding Lie algebra $\mathfrak{g}$ of vector fields

$$X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^{n-m} c_{ij}^l x_j \partial_{z_l}, \ j = 1, \ldots, m,$$

$$X_{m+j} = \partial_{z_j}, \ j = 1, \ldots, m-n,$$

where $c_{ij}^k$ are the structure constants of the Lie algebra $\mathfrak{g}$ and the symbol $\partial$ stands for partial derivative. We discuss the related optimal control problem

$$\dot{q}(t) = u_1 X_1 + \ldots + u_m X_m$$

for $t > 0$ and $q$ in $G$ and the control $u = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m$ with the boundary condition $q(0) = q_1$, $q(T) = q_2$ for fixed points $q_1, q_2 \in G$, where we minimize the cost functional $\int_0^T (u_1^2 + \ldots + u_m^2) dt$. The solutions $q(t)$ then correspond to the sub-Riemannian geodesics, that is, admissible curves parametrized by a constant speed whose sufficiently small arcs are the length minimizers.

We use the Hamiltonian approach to this control problem.3 There are no strict abnormal extremals for the step 2 Carnot groups, so we focus on the normal geodesics and address them just as geodesics. The left-invariant vector fields $X_i$, $i = 1, \ldots, m$ form a basis of $TG$ and determine the left-invariant coordinates on $G$. We define the corresponding left-invariant coordinates $h_i, i = 1, \ldots, m$ and $w_i, i = 1, \ldots, n-m$ on the fibers of $T^*G$ by $h_i(\lambda) = \lambda(X_i)$ and $w_i(\lambda) = \lambda(X_{m+i})$, for arbitrary 1-forms $\lambda$ on $G$. Thus, we use $(x_i, w_i)$ as the global coordinates on $T^*G$.

The geodesics are exactly the projections of normal Pontryagin extremals, that is, the integral curves of the left-invariant normal Hamiltonian

$$H = \frac{1}{2}(h_1^2 + h_2^2 + \ldots + h_m^2),$$

FIGURE 1  Generalized trident snakes
on \( G \). Assume that \( \lambda(t) = (x_i(t), z_i(t), h(t), w_i(t)) \) in \( T^*G \) is a normal extremal. Then the controls \( u_j \) to system (2) satisfy \( u_j(t) = h_j(\lambda(t)) \) and the base system takes the form

\[
\begin{align*}
\dot{x}_i &= h_i, \quad i = 1, \ldots, m \\
\dot{z}_j &= -\frac{1}{2}\sum_{i=1}^{m} c_{ik} h_i x_k, \quad j = 1, \ldots, n - m
\end{align*}
\]

for \( q = (x, z) \). Using \( u_j(t) = h_j(\lambda(t)) \) and the equation \( \dot{\lambda}(t) = \dot{H}(\lambda(t)) \) for the normal extremals, we write the fiber system as

\[
\begin{align*}
\dot{h}_i &= -\sum_{m-n}^{m-m} \sum_{j=1}^{m} c_{ij} h_j w_i, \quad i = 1, \ldots, m, \\
\dot{w}_j &= 0, \quad j = 1, \ldots, n - m,
\end{align*}
\]

where \( c_{ij} \) are the structure constants of the Lie algebra \( \mathfrak{g} \) for the basis \( X_i \). The solutions \( w_i, i = 1, \ldots, n - m \) are constants that we denote by

\[
w_1 = K_1, \ldots, w_{n-m} = K_{n-m}.
\]

If \( K_1 = \ldots = K_{n-m} = 0 \), then \( h(t) = h(0) \) is a constant and the geodesic \( (x(t), z(t)) \) is a line in \( G \) such that \( z(t) = 0 \).

If at least one of \( K_i \) is nonzero, the first part of the fiber system (5) forms a homogeneous system of ODEs \( \dot{h} = -\Omega h \) with constant coefficients for \( h = (h_1, \ldots, h_m)^T \) and the system matrix \( \Omega \). Its solution is given by \( h(t) = e^{-\Omega t} h(0) \), where \( h(0) \) is the initial value of the vector \( h \) at the origin.

### 2.2 Left-invariant control problem with the growth vector (3, 6)

Let us consider three vector fields on \( \mathbb{R}^6 \) with the local coordinates \( (x_1, x_2, x_3, z_1, z_2, z_3) \) in the form

\[
\begin{align*}
X_1 &= \partial_{x_1} + \frac{x_1}{2} \partial_{z_1} - \frac{x_2}{2} \partial_{z_2}, \\
X_2 &= \partial_{x_2} + \frac{x_1}{2} \partial_{z_1} - \frac{x_2}{2} \partial_{z_2}, \\
X_3 &= \partial_{x_3} + \frac{x_2}{2} \partial_{z_1} - \frac{x_1}{2} \partial_{z_2}.
\end{align*}
\]

The only nontrivial Lie brackets are

\[
X_4 = [X_1, X_2] = \partial_{z_1}, \quad X_5 = [X_1, X_3] = -\partial_{z_2}, \quad X_6 = [X_2, X_3] = \partial_{z_1}.
\]

These six vector fields determine a step 2 nilpotent Lie algebra \( \mathfrak{m} \) with the multiplication table given by Table 1.

There is a Carnot group \( M \) such that the fields \( X_i, i = 1 \ldots, 6 \) are left-invariant for the corresponding group structure. When identified with \( \mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3 \), the group structure on \( M \) reads as

\[
(x, z) \cdot (x', z') = (x + x', z + z' + \frac{1}{2} x \times x')
\]

**TABLE 1** Lie algebra \( \mathfrak{m} \)

| \( \mathfrak{m} \) | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) |
|---|---|---|---|---|---|---|
| \( X_1 \) | 0 | \( X_4 \) | \( X_5 \) | 0 | 0 | 0 |
| \( X_2 \) | \( -X_4 \) | 0 | \( X_6 \) | 0 | 0 | 0 |
| \( X_3 \) | \( -X_5 \) | \( -X_4 \) | 0 | 0 | 0 | 0 |
| \( X_4 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( X_5 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( X_6 \) | 0 | 0 | 0 | 0 | 0 | 0 |
for \( x = (x_1, x_2, x_3) \) and \( z = (z_1, z_2, z_3) \), where \( \times \) stands for the vector product on \( \mathbb{R}^3 \). In particular, \( \mathcal{M} = \langle X_1, X_2, X_3 \rangle \) forms a three-dimensional left-invariant distribution on \( \mathcal{M} \). We define the left-invariant sub-Riemannian metric \( g_M \) on \( \mathcal{M} \) by declaring \( X_1, X_2, X_3 \) orthonormal.

The geodesics of the control problem are the solutions to control systems (4),(5), with \( (m, n) = (3, 6) \), and one can read the structure constants in Table 1. Hence, the fiber system is given by \( w_1 = K_1, w_2 = K_2, w_3 = K_3 \) for the constants \( K_1, K_2, K_3 \) and \( \dot{h} = -\Omega h \) for \( h = (h_1, h_2, h_3)^T \) and

\[
\Omega = \begin{pmatrix}
0 & K_1 & K_2 \\
-K_1 & 0 & K_3 \\
-K_2 & -K_3 & 0
\end{pmatrix}.
\]

(10)

Its solution is given by the exponential \( h(t) = e^{-\Omega h}(0) \), where \( h(0) \) is the initial value of the vector \( h \) at the origin. We write an explicit formula for the general solution in terms of eigenvectors of (10). If at least one of the constants \( K_i \) is nonzero, the kernel of \( \Omega \), that is, zero-eigenspace, is one-dimensional, generated by the vector \( (K_2, K_3, K_1)^T \). Its orthogonal complement corresponds to the sum of eigenspaces to the eigenvalues \( \pm iK \), where \( K := \sqrt{K_1^2 + K_2^2 + K_3^2} \) and is generated by the vectors \( (-K_1K_3, -K_1K_2, K_2^2 + K_3^2) \pm i(K_2, -K_3, 0) \). Thus, solution to the fiber system can be written as follows:

\[
h(t) = (C_1 \cos(Kt) - C_2 \sin(Kt))v_1 + (C_1 \sin(Kt) + C_2 \cos(Kt))v_2 + C_3 v_3,
\]

(11)

where \( v_1, v_2, v_3 \) is the eigenspace-adapted real orthonormal basis

\[
v_1 = \frac{1}{K \sqrt{K_2^2 + K_3^2}} \begin{pmatrix} -K_1K_3 \\ K_1K_2 \\ K_2^2 + K_3^2 \end{pmatrix},
v_2 = \frac{1}{\sqrt{K_2^2 + K_3^2}} \begin{pmatrix} -K_2 \\ -K_3 \\ 0 \end{pmatrix},
v_3 = \frac{1}{K} \begin{pmatrix} K_3 \\ -K_2 \\ K_1 \end{pmatrix}
\]

and \( C_1, C_2, C_3 \) are the constants that satisfy the level set condition \( H = 1/2 \), that is, \( \|h(t)\| = 1 \) that reads \( C_1^2 + C_2^2 + C_3^2 = 1 \). Let us note that the choice \( C_1 = C_2 = 0 \) leads to the constant solutions that are irrelevant as the control functions. Thus, we assume that at least one of the constants \( C_1, C_2 \) is nonzero.

Let us emphasize that the base system (4) can be written in terms of a vector product as follows:

\[
\dot{x} = h,
\]

(12)

\[
\dot{z} = \frac{1}{2} x \times h
\]

for vectors \( x = (x_1, x_2, x_3)^T \) and \( z = (z_1, z_2, z_3)^T \). One obtains the general solution by substituting (11) for \( h \) and by consequent direct integration. We are interested in the solutions passing through the origin, that is, we impose the initial condition

\[
x_i(0) = 0, z_i(0) = 0, \quad i = 1, 2, 3.
\]

(13)

However, it may be difficult to find the integration constants giving the geodesics through a fixed target point.

### 2.3 Left-invariant control problem with the growth vector (4, 7)

Let us consider four vector fields on \( \mathbb{R}^7 \) with the local coordinates \( (x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3) \) in the form

\[
Y_0 = \frac{\partial}{\partial x} - \frac{\ell_1}{2} \frac{\partial}{\partial y_1} - \frac{\ell_2}{2} \frac{\partial}{\partial y_2} - \frac{\ell_3}{2} \frac{\partial}{\partial y_3},
\]

\[
Y_1 = \frac{\partial}{\partial \ell_1} + \frac{x}{2} \frac{\partial}{\partial y_1},
\]

\[
Y_2 = \frac{\partial}{\partial \ell_2} + \frac{x}{2} \frac{\partial}{\partial y_2},
\]

\[
Y_3 = \frac{\partial}{\partial \ell_3} + \frac{x}{2} \frac{\partial}{\partial y_3}.
\]

(14)

The only nontrivial Lie brackets are as follows:

\[
Y_4 = [Y_0, Y_1] = \partial_{y_1},
\]

\[
Y_5 = [Y_0, Y_2] = \partial_{y_2},
\]

\[
Y_6 = [Y_0, Y_3] = \partial_{y_3}.
\]

(15)
These seven fields determine a step 2 nilpotent Lie algebra \( \mathfrak{n} \) with the multiplication table given by Table 2.

There is a Carnot group \( N \) such that the fields \( Y_i, i = 1, \ldots, 7 \) are left-invariant for the corresponding group structure. The group structure on \( N \), when identified with \( \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \), yields

\[
(x, \ell', y) \cdot (x', \ell'', y') = (x + x', \ell + \ell', y + y' + \frac{1}{2} \ell \times \ell')
\]

for \( \ell' = (\ell_1, \ell_2, \ell_3) \) and \( y = (y_1, y_2, y_3) \). In particular, \( \mathcal{N} = \langle Y_0, Y_1, Y_2, Y_3 \rangle \) forms a four-dimensional left-invariant distribution on \( N \). Moreover, there is a natural decomposition

\[
\mathcal{N} = \langle Y_0 \rangle \oplus \langle Y_1, Y_2, Y_3 \rangle
\]

into a one-dimensional distribution and a three-dimensional involutive distribution, both left-invariant. We define the left-invariant sub-Riemannian metric \( g_N \) on \( \mathcal{N} \) by declaring \( Y_0, Y_1, Y_2, Y_3 \) orthonormal.

The geodesics of the control problem are solutions to the control systems (4), (5), with \((m, n) = (4, 7)\), and we read the structure constants in Table 2. Hence, the first part of the fiber system (5) is given by \( w_1 = K_1, w_2 = K_2, w_3 = K_3 \), where \( K_1, K_2, K_3 \) are constants. The second part of the fiber system takes the form \( \dot{h} = -\Omega h \), where \( h := (h_0, h_1, h_2, h_3)^T \) and

\[
\Omega = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & 0 & 0 \\ -K_2 & 0 & 0 & 0 \\ -K_3 & 0 & 0 & 0 \end{pmatrix}.
\]

Its solution is given by \( h(t) = e^{-\Omega t}h(0) \), where \( h(0) \) is the initial value of the vector \( h \) at the origin, and we write its explicit form in terms of the eigenvectors of (18). If \( K_1 = K_2 = K_3 = 0 \), then \( h(t) = h(0) \) is a constant and the geodesic \((x(t), \ell(t), y(t)) \) is a line in \( N \) such that \( y_1 = 0 \). If at least one of the constants \( K \) is nonzero, the kernel of \( \Omega \), that is, zero-eigenspace, is two-dimensional and is generated by the vectors \((0, -K_3, 0, K_1)^T \) and \((0, -K_2, K_1, 0)^T \). Its orthogonal complement corresponds to the sum of the eigenspaces to the eigenvalues \( \pm iK \), where \( K := \sqrt{K_1^2 + K_2^2 + K_3^2} \) and is generated by the eigenvectors \((0, K_1, K_2, K_3)^T \pm i(K, 0, 0, 0)^T \). Thus, the solution to the vertical system for nonzero \( K \) takes the form

\[
\begin{align*}
\dot{h}_0 &= K(C_2 \cos(Kt) - C_1 \sin(Kt)) \\
\dot{h} &= K(C_2 \sin(Kt) + C_1 \cos(Kt))r_1 + Cr_2
\end{align*}
\]

where \( \tilde{h} = (h_1, h_2, h_3)^T \) and \( r_1, r_2 \) are eigenspace-adapted real orthonormal vectors

\[
r_1 = \frac{1}{K} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, \quad r_2 = \frac{1}{C} \begin{pmatrix} -K_3 \\ 0 \\ K_1 \end{pmatrix} + C \begin{pmatrix} -K_2 \\ K_1 \\ 0 \end{pmatrix}
\]

with the constants \( C_1, C_2, C_3, C_4 \) and the normalization factor \( C = \sqrt{(C_3K_3 + C_4K_2)^2 + K_1^2(C_2^2 + C_3^2)} \). The level set condition \( ||h(t)|| = 1 \) reads \( C_1^2 + C_2^2 + C_3^2 = 1 \). Let us note that the choice \( C_1 = C_2 = 0 \) leads to the constant solutions that are irrelevant as the control functions. Thus, we assume that at least one of the constants \( C_1, C_2 \) is nonzero.

| Table 2 | Lie algebra n |
|---------|---------------|
| n       | Y_0 | Y_1 | Y_2 | Y_3 | Y_4 | Y_5 | Y_6 |
| Y_0     | 0   | Y_4 | Y_5 | Y_6 | 0   | 0   | 0   |
| Y_1     | -Y_4| 0   | 0   | 0   | 0   | 0   | 0   |
| Y_2     | -Y_5| 0   | 0   | 0   | 0   | 0   | 0   |
| Y_3     | -Y_6| 0   | 0   | 0   | 0   | 0   | 0   |
| Y_4     | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| Y_5     | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| Y_6     | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
The base system (4) takes the explicit form of

\[
\begin{align*}
\dot{x} &= h_0, \\
\dot{\ell} &= \bar{h}, \\
\dot{y} &= \frac{1}{2}(x\bar{h} - h_0\ell).
\end{align*}
\]

We are interested in the solutions passing through the origin; that is, we impose the initial condition

\[
x(0) = 0, \ell_i(0) = 0, y_i(0) = 0, \quad i = 1, 2, 3.
\]

By substitution of (19), system (20) can be directly integrated. Again, it may be difficult to find the geodesics through a fixed target point. In Section 4, we show how the symmetries of the system and the geometric algebra approach are used for finding a geodesic towards a given point.

### 2.4 Symmetries of the control systems

Symmetries of the control system in question coincide with the symmetries of the corresponding left–invariant sub–Riemannian structure \((M, \mathcal{M}, g_M)\) and \((N, \mathcal{N}, g_N)\), respectively. These are precisely the automorphisms on groups preserving the distributions and sub–Riemannian metrics. The group \(SO(3)\) acts on \(\mathbb{R}^3\) and preserves the vector product which implies the following statement.

**Proposition 1.** For each \(R \in SO(3)\), the map

\[
(x, z) \mapsto (Rx, Rz)
\]

maps the geodesics of the system from Section 2.2 starting at the origin to the geodesics starting at the origin. For each \(R \in SO(3)\), the map

\[
(x, \ell', y) \mapsto (x, R\ell', Ry)
\]

maps the geodesics of the system from Section 2.3 starting at the origin to the geodesics starting at the origin.

**Proof.** Follows from the invariance of (12) and (20) with respect to the action of \(R \in SO(3)\). \[\square\]

### 3 Geometric Algebra

The construction of the universal real geometric algebra is well-known.\(^{11-13,17}\) We provide only a brief description in a special case \(\mathbb{G}_m\) that we use later. In general, geometric algebras are based on symmetric bilinear forms of arbitrary signature. Here, we deal with the real vector space \(\mathbb{R}^m\) endowed with a positive definite symmetric bilinear form \(B\) only.

#### 3.1 Geometric product

Let us consider a positive definite symmetric bilinear form \(B\) on \(\mathbb{R}^m\) and the associated orthonormal basis \((e_1, \ldots, e_m)\), that is,

\[
B(e_i, e_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \quad \text{where } 1 \leq i, j \leq m.
\]

The Grassmann algebra \(\Lambda(\mathbb{R}^m)\) is an associative algebra with the anti-symmetric outer product \(\wedge\) defined by the rule

\[
e_i \wedge e_j + e_j \wedge e_i = 0 \quad \text{for } 1 \leq i, j \leq m.
\]
The Grassmann blade of grade \( r \) is \( e_A = e_{i_1} \wedge \ldots \wedge e_{i_r} \), where the multi-index \( A \) is a set of indices ordered in the natural way \( 1 \leq i_1 \leq \ldots \leq i_r \leq m \), and we put \( e_\emptyset = 1 \). Blades of grades \( 0 \leq r \leq m \) form the basis of the Grassmann algebra \( \Lambda(\mathbb{R}^m) \) and for the outer product we have

\[
e_j \wedge e_A = \begin{cases} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_r} & \text{if } j \notin A, \\ 0 & \text{if } j \in A \end{cases}
\]

and \( 1 \wedge e_A = e_A \). For the vectors from \( \mathbb{R}^m \), the inner product \( e_i \cdot e_j = B(e_i, e_j) \) and the outer product \( e_i \wedge e_j \) lead to the so-called geometric product

\[
e_i e_j = e_i \cdot e_j + e_i \wedge e_j, \quad 1 \leq i, j \leq m.
\]

The definitions of inner and geometric products then extend to blades of the grade \( r \) as follows. For the inner product we put \( 1 \cdot e_A = 0 \) and

\[
e_j \cdot e_A = e_j \cdot (e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{k=1}^{r} (-1)^k B(e_j, e_k) e_{A \setminus \{i_k\}},
\]

where \( e_{A \setminus \{i_k\}} \) is the blade of grade \( r - 1 \) created by deleting \( e_k \) from \( e_A \). This product is also called the left contraction in literature. For the geometric product, we define

\[
e_j e_A = e_j \cdot e_A + e_j \wedge e_A.
\]

These definitions extend linearly to the whole vector space \( \Lambda(\mathbb{R}^m) \). Thus, we get an associative algebra over this vector space, the so-called real geometric algebra, denoted by \( \mathbb{G}_m \). Note that this algebra is naturally graded; the grade zero and grade one elements are identified with \( \mathbb{R} \) and \( \mathbb{R}^m \), respectively. Finally, we can define the norm of a blade as the magnitude of the blade \( |e_A| = \sqrt{e_A \cdot e_A} \). Note that \( e_A \cdot e_A \) where \( e_A \neq 0 \) is always positive in \( \mathbb{G}_m \).

### 3.2 Objects

The vectors in \( \mathbb{R}^m \) with the coordinates \((x_1, \ldots, x_m)\) are given by \( x = x_1 e_1 + \cdots + x_m e_m \), and the square with respect to the geometric product \( x^2 = x_1^2 + \cdots + x_m^2 \in \mathbb{R} \) coincides with the square of the Euclidean norm of \( x \). A vector \( x \) represents a one-dimensional subspace (line) \( p \) in \( \mathbb{R}^m \) given by the scalar multiples of \( x \) which in \( \mathbb{G}_m \) is expressed by the formula \( u \in p \iff u \wedge x = 0 \). In the same way, a plane \( \pi \) generated by two vectors \( x \) and \( y \) is represented by \( x \wedge y \) in the sense \( u \in \pi \iff u \wedge x \wedge y = 0 \). In general, any \( r \)-dimensional subspace \( V_r \subseteq \mathbb{R}^m \) is represented by a blade \( A_r \) of grade \( r \) such that

\[
V_r = NO(A_r) = \{ x \in \mathbb{R}^m : x \wedge A_r = 0 \}.
\]

Such a representation is called the outer product null space (OPNS) representation in the literature. In particular, the whole space \( \mathbb{R}^m \) is represented by a blade of maximal grade, so-called pseudoscalar. Similarly, one defines the inner product null space (IPNS) representation \( A^*_{m-r} \) of \( V_r \) as a blade of grade \( m - r \) such that \( x \in V_r \iff x \cdot A^*_{m-r} = 0 \). The OPNS and IPNS representations are mutually dual with respect to the duality on \( \mathbb{G}_m \) defined by the multiplication by pseudoscalar, namely,

\[
A^* = AI,
\]

where \( A \) is a blade and \( I \) is the pseudoscalar. Indeed, one can show that \( (x \wedge A)I = x \cdot (AI) \) for each vector \( x \in \mathbb{R}^m \), in particular

\[
x \wedge A = 0 \iff x \cdot A^* = 0.
\]

**Remark 1.** OPNS representations of objects in \( \mathbb{G}_3 \) are summarized in Table 3. For example, the plane generated by the vectors \( u, v \) has the OPNS representation \( u \wedge v \). Its IPNS representation \((u \wedge v)^*\) is a vector perpendicular to the plane. More specifically, for the pseudoscalar \( I = e_1 \wedge e_2 \wedge e_3 = e_1 e_2 e_3 \), we receive the usual vector product in geometric algebra form as

\[
u \times v = -(u \wedge v)I.
\]
3.3 | Transformations

Let us fix a vector \( n ∈ \Lambda^1 \mathbb{R}^m ⊂ \mathbb{G}_m \) such that \( n \cdot n = n^2 = 1 \) and \( x ∈ \Lambda^1 \mathbb{R}^m ⊂ \mathbb{G}_m \) arbitrary. The negative conjugation \(-nxn\) defines the reflection with respect to the hyperplane orthogonal to \( n \), because

\[
- nx^\perp n = -(n ∧ x^\perp) n = (x^\perp ∧ n) n = x^\perp n n = x^\perp,
\]
\[
- nx^\parallel n = -x^\parallel,
\]

where \( x = x^\perp + x^\parallel \) is the orthogonal decomposition of \( x \) with respect to \( n \). Conjugation preserves the grades of blades and is an outermorphism \( n(u_1 ∧ ⋯ ∧ u_l)n = (nu_1) ∧ ⋯ ∧ (nu_l)n \) for any vectors \( u_1, \ldots, u_l \); thus, minus the conjugation is the (anti)outermorphism depending on the dimension \( m \). Since each rotation is a composition of two reflections, a rotation in \( \mathbb{G}_m \) is represented by the conjugation with respect to the geometric product of two vectors. To find a rotor between vectors \( x \) and \( y \), we have a nice formula at hand.

**Lemma 1.** Let \( x \) and \( y \) be the unit vectors in \( \mathbb{G}_m \), that is, \( x, y ∈ \Lambda^1 \mathbb{G}_m \), then the formula

\[
R_{xy} = 1 + \hat{y} x,
\]

where the hat symbol stands for the normalization \( \hat{u} = u/\sqrt{u \cdot u} \), defines the rotation in the plane \( x ∧ y \) which maps vector \( x \) to \( y \) and acts trivially on \((x ∧ y)^*\).

**Proof.** Multiplication of two vectors \( x, y ∈ \Lambda^1 \mathbb{R}^m ⊂ \mathbb{G}_m \), such that \( x^2 = y^2 = 1 \), defines a multivector

\[
yx = \cos(\theta) + \sin(\theta) y ∧ x,
\]

where \( \theta \) is the angle between \( x \) and \( y \), Lemma 4.212. The conjugation by such multivector \( yx \) represents the rotation in the plane \( x ∧ y \) with respect to angle \( 2\theta \) in the positive way. Using standard trigonometric formulas, we can see by straightforward calculation that

\[
R_{xy} = 1 + \hat{y} x = \frac{1 + \cos(\theta) + \hat{y} x \sin(\theta)}{\sqrt{(1 + \cos(\theta))^2 + \sin^2(\theta)}} = \frac{1 + \cos(\theta) + \hat{y} x \sin(\theta)}{\sqrt{2 + 2 \cos(\theta)}}
\]
\[
= \sqrt{\frac{1 + \cos(\theta)}{2}} + \hat{y} x \sqrt{\frac{1 - \cos^2(\theta)}{2(1 + \cos(\theta))}} = \sqrt{\frac{1 + \cos(\theta)}{2}} + \hat{y} x \sqrt{\frac{1 - \cos(\theta)}{2}}
\]
\[
= \cos\left(\frac{\theta}{2}\right) + (y ∧ x) \sin\left(\frac{\theta}{2}\right).
\]

So \( R_{xy} \) is the rotation in the plane \( \hat{x} ∧ y \) in the positive way about the angle between \( x \) and \( y \) so the vector \( x \) goes to the vector \( y \).

Finally, \((x ∧ y)^*\) is orthogonal to \( x \) and \( y \) and the straightforward computation \( xy(x ∧ y)^* yx = xyyx(x ∧ y)^* = (x ∧ y)^* \) proves the rest of the statement.

**Remark 2.** One can see that \( yxy \) is an axial symmetry with respect to \( y \), and thus, \( x + yxy = 2 \cos(\theta)y \). We can compute the square of the norm

\[
(1 + yx)(1 + xy) = 1 + xy + yx + 1 = 2 + 2 \cos(\theta)
\]
and with the help of the geometric product, we compute

\[(1 + xy)x(1 + xy) = (x + y)(1 + xy) = (x + 2y + yxy) = (2 + 2\cos(\theta))y\]

so the conjugation by \((26)\) maps \(x\) to \(y\).

### 3.4 | Rotor construction

Let \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) be a pair of bases of \(\mathbb{R}^m\) such that

1. \(x_i \cdot x_j = y_i \cdot y_j\) for all \(i, j = 1, \ldots, m\), that is, all the scalar products are equal, and
2. \(x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m\), that is, the pseudoscalars are equal.

Let us remind that a complete flag \([V]\) in an increasing sequence of subspaces of the vector space \(\mathbb{R}^m\)

\[
\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_m = \mathbb{R}^m,
\]

such that \(\dim(V_i) = i\). We use the complete flags to find the explicit rotation \(R\) such that \(Rx_i = y_i\) for all \(i = 1, \ldots, m\). Our method can be summarized as follows:

- We consider the complete flags \([V]\) and \([W]\) by setting \(V_i = \langle x_1, \ldots, x_i \rangle = NO(x_1 \wedge \cdots \wedge x_i)\) and \(W_i = \langle y_1, \ldots, y_i \rangle = NO(y_1 \wedge \cdots \wedge y_i)\), respectively.
- We map the complete flag \([V]\) to the complete flag \([W]\) inductively in \(m\) steps. In the \(j^{th}\) step, we assume \(V_i = W_i\) for \(i < j\) and we find the rotation \(R_i\) such that \(R_iV_i = W_i\) for \(i > j - 1\).

Before we formulate the construction in detail, we need several technical lemmas.

**Lemma 2.** Let \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) be a pair of bases such that

1. \(x_i \cdot x_j = y_i \cdot y_j\) for all \(i, j = 1, \ldots, m\), and
2. \(x_1 \wedge \cdots \wedge x_i = y_1 \wedge \cdots \wedge y_i\) for all \(i = 1, \ldots, m\).

If \([V]\) and \([W]\) are the corresponding complete flags, respectively, and if \(V_i = W_i\) for all \(i = 1, \ldots, m\), then \(x_i = y_i\) for all \(i = 1, \ldots, m\).

**Proof.** The equality \(x_1 = y_1\) holds trivially from the assumptions. Then \(x_2 \cdot x_2 = y_2 \cdot y_2\) reads that \(x_2, y_2\) are of the same length, \(x_2 \cdot x_3 = y_2 \cdot y_1\) reads that the angles between \(x_1, x_2\) and \(y_1, y_2\) are identical and \(x_1 \wedge x_2 = y_1 \wedge y_2\) reads that they have the same orientation. Then \(V_2 = W_2\) and \(x_1 = y_1\) imply \(x_2 = y_2\) and so on for all the basis vectors. \(\square\)

**Lemma 3.** Let \((x_1, \ldots, x_i, z)\) and \((y_1, \ldots, y_i, z)\) be a pair of sets of independent vectors such that \(x_1 \wedge \cdots \wedge x_i \wedge z = y_1 \wedge \cdots \wedge y_i \wedge z\). If \(NO(x_1 \wedge \cdots \wedge x_i) = NO(x_1 \wedge \cdots \wedge x_i)\) then \(x_1 \wedge \cdots \wedge x_i = y_1 \wedge \cdots \wedge y_i\).

**Proof.** The independence of the sets of vectors implies \(x_1 \wedge \cdots \wedge x_i \wedge z \neq 0\), \(y_1 \wedge \cdots \wedge y_i \wedge z \neq 0\). If \(NO(x_1 \wedge \cdots \wedge x_i) = NO(y_1 \wedge \cdots \wedge y_i)\), then \(x_1 \wedge \cdots \wedge x_i = \beta x_1 \wedge \cdots \wedge x_i\) for \(\beta \in \mathbb{R}\).

Then

\[
x_1 \wedge \cdots \wedge x_i \wedge z = y_1 \wedge \cdots \wedge y_i \wedge z, \\
(x_1 \wedge \cdots \wedge x_i \wedge y_i \wedge z) \wedge z = 0, \\
(1 - \beta)x_1 \wedge \cdots \wedge x_i \wedge z = 0.
\]

Finally \(\beta = 1\) and \(x_1 \wedge \cdots \wedge x_i = y_1 \wedge \cdots \wedge y_i\). \(\square\)
Lemma 4. Consider two complete flags \( \{ V \} \) and \( \{ W \} \) in \( \mathbb{R}^m \) and \( i \leq m \) such that \( V_j = W_j \) for \( j > i \). The rotor \( R_i \) between the hyperplanes \( V_i \oplus V_{i+1}^\perp \) and \( W_i \oplus W_{i+1}^\perp \) constructed by the formula (26) maps \( V_i \) to \( W_i \).

Proof. The property \( V_i \subseteq V_{i+1} \) implies that \( V_{i+1}^\perp \subseteq V_i^\perp \) and thus \( V_i \oplus V_{i+1}^\perp \) is a hyperplane equipped with an orthogonal decomposition. Recall that \( V_{i+1} = W_{i+1} \), and thus, \( V_{i+1}^\perp = W_{i+1}^\perp \). Any rotation preserves an orthogonal decomposition, and thus, \( R_i \) acts as the identity on \( V_{i+1}^\perp = W_{i+1}^\perp \), so it maps \( V_i \) to \( W_i \). \( \square \)

We use all these lemmas to provide a constructive proof of the following theorem.

Theorem 3.5. Let \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) be a pair of bases of \( \mathbb{R}^m \) such that

1. \( x_i \cdot x_j = y_i \cdot y_j \) for all \( i, j = 1, \ldots, m \), and
2. \( x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m \).

Then we can construct a rotor \( R \) such that \( R x_i = y_i \) for all \( i = 1, \ldots, m \).

Proof. Let \( \{ V \} \) and \( \{ W \} \) be a pair of the corresponding complete flags \( V_i = NO(x_1 \wedge \cdots \wedge x_i) \) and \( W_i = NO(y_1 \wedge \cdots \wedge y_i) \). We construct a rotor \( R = R_1 \cdots R_m \) mapping the complete flag \( \{ V \} \) to the complete flag \( \{ W \} \) so that \( V_i = W_i \) for all \( i = 1, \ldots, m \). The result on bases \( x_i, y_i \) then follows by Lemmas 2 and 3.

We define \( R_m \) as the identity and proceed inductively. It follows from Lemma 1 that there is a rotation \( R_{m-1} \) between the hyperplanes \( V_{m-1} \oplus V_{m+1}^\perp \cong V_{m-1} \) and \( W_{m-1} \oplus W_{m+1}^\perp \cong W_{m-1} \) which maps the complete flag \( \{ V \} \) to the complete flag \( \{ RV_i \} \) in such a way that \( W_{m-1} = RV_{m-1}^\perp \), where \( R = R_{m-1} \). As the induction step, we consider the rotor \( R = R_j \cdots R_m \) such that \( RV_i = W_i \) for all indices \( j \geq i \). According to Lemma 4, the rotation \( R_{j-1} \) between the hyperplanes \( (RV_{j-1}^\perp) \oplus (RV_j)^\perp \) and \( W_{j-1} \oplus W_j^\perp \) maps the complete flag \( \{ RV_i \} \) to the complete flag \( \{ R_j RV_j^\perp \} \) in such a way that \( W_i = R_j RV_j^\perp R_i \) for all \( i \geq j - 1 \).

After \( m \) steps, the rotor \( R = R_1 \cdots R_m \) maps the complete flag \( \{ V \} \) to the complete flag \( \{ W \} \) in such a way that \( V_i = W_i \) for all \( i = 1, \ldots, m \) and so \( R x_i = y_i \) for all \( j = 1, \ldots, m \) because of Lemma 2. \( \square \)

The explicit construction in the proof of Theorem 3.5 gives us the following algorithm.

Calculate the rotor \( R = R_1 \cdots R_m \)

Require: \( x_i \cdot x_j = y_i \cdot y_j \) and \( x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m \)

Ensure: \( y_i = R x_i \)

\[
\text{for} \ m > i > 0 \ \text{do} \\
V_i \leftarrow x_1 \wedge \cdots \wedge x_i \\
W_i \leftarrow y_1 \wedge \cdots \wedge y_i \\
\text{end for} \\
R \leftarrow \text{Id} \\
\text{for} \ m > i > 0 \ \text{do} \\
V_i \leftarrow RV_i \bar{R} \\
H_V \leftarrow V_i \wedge W_i^\perp \\
H_W \leftarrow W_i \wedge W_i^\perp \\
R_i \leftarrow 1 + \frac{H_V \cdot H_W}{\|H_V\|^2} \\
R \leftarrow RR \\
\text{end for}
\]

4 | NILPOTENT CONTROL PROBLEMS IN GA APPROACH

We use the symmetries of \( SO(3) \) to define an equivalence relation on the set of geodesics passing through the origin; see Proposition 1. We find a convenient representative of any equivalence class and describe the moduli space in the language of GA.
4.1 | Geodesics of \((3, 6)\)

Since the vector product \(x \times h\) coincides with minus the dual of wedge product \(x \wedge h\) according to \((25)\), the horizontal system \((12)\) can be written in the form

\[
\begin{align*}
\dot{x} &= h, \\
\dot{z} &= -\frac{1}{2} x \wedge h,
\end{align*}
\]

where \(x \in \wedge^1 \mathbb{R}^3\) represents a line and \(z \in \wedge^2 \mathbb{R}^3\) represents a plane in \(\mathbb{R}^3\). In this way, we see the geodesics as curves in the geometric algebra \(G_3\).

**Proposition 2.** Each arc-length parameterized sub-Riemannian geodesic satisfying the initial condition \(x_i(0) = 0, z_i(0) = 0, i = 1, 2, 3\) is equivalent to a curve in \(M \cong \wedge^1 \mathbb{R}^3 \oplus \wedge^2 \mathbb{R}^3 \subset G_3\) and up to the action of a suitable \(R \in SO(3)\), it takes the form

\[
q(t) = x(t) + z(t) = \frac{D}{K} (1 - \cos(Kt)) e_1 + \frac{D}{K} \sin(Kt) e_2 + C_3 t e_3 - \frac{D^2}{2K^2} (Kt - \sin(Kt)) e_1 \wedge e_2 - \frac{C_1 D}{2K^2} (Kt - Kt \cos(Kt)) e_3 \wedge e_1 + \frac{C_1 D}{2K^2} (2 - Kt \sin(Kt) - 2 \cos(Kt)) e_2 \wedge e_3,
\]

where \(K > 0\) and \(D, C_3\) satisfy the level set equation \(D^2 + C_3^2 = 1\).

**Proof.** The solution to the vertical system \((11)\) can be rewritten as

\[
h(t) = D \sin(Kt) \bar{v}_1 + D \cos(Kt) \bar{v}_2 + C_3 \bar{v}_3,
\]

where we denote \(D = \sqrt{C_1^2 + C_2^2}\) and the orthonormal vectors \(\bar{v}_1, \bar{v}_2\) are obtained by the rotation of orthonormal vectors \(v_1, v_2\) as

\[
\bar{v}_1 = \frac{1}{\sqrt{C_1^2 + C_2^2}} (-C_1 v_1 + C_2 v_2), \quad \bar{v}_2 = \frac{1}{\sqrt{C_1^2 + C_2^2}} (C_2 v_1 + C_1 v_2).
\]

Thus, the vectors \(\bar{v}_1, \bar{v}_2, \bar{v}_3\) are orthonormal with respect to the Euclidean metric on \(\mathbb{R}^3\). So, there is an orthogonal matrix \(R \in SO(3)\) that aligns vectors \(\bar{v}_1, \bar{v}_2, \bar{v}_3\) with the standard basis of \(\mathbb{R}^3\). Thus, we get

\[
\bar{v}_1 = Re_1, \quad \bar{v}_2 = Re_2, \quad \bar{v}_3 = Re_3,
\]

where \(e_1, e_2,\) and \(e_3\) are the elements of the standard Euclidean basis of \(\mathbb{R}^3\). According to \((22)\), the rotor \(R\) defines a representative of a geodesic class \((R^3 x(t), R^3 z(t))\) which is a solution to \((27)\) for \(h(t) = D \sin(Kt) e_1 + D \cos(Kt) e_2 + C_3 e_3\).

The solution \((28)\) then follows by a direct integration when the initial condition is applied. Equation for the level set follows from the definition of \(D\).

The action of \(SO(3)\) on \(M \cong \mathbb{R}^6\) given by Equation \((22)\) defines a moduli space \(M/\text{SO}(3)\). We see \(M\) as a subset of \(G_3\) and the group \(\text{SO}(3)\) is represented by rotors instead of matrices, which act on \(M\) by conjugation. The action preserves the vector and bivector parts, inner product, norm, and dualization with respect to \(*\). We can see the elements of \(M\) as the pairs consisting of lines and planes. The natural invariants are the norms of lines’ directional vectors, norms of the planes’ normal vectors and angles between these pairs of vectors. Square norm of the normal vector of the plane \(z^* \cdot z^*\) is \(-z \cdot z\). Scalar product between the directional vector of the line \(x\) and the normal vector of the plane \(z\) can be rewritten as \((x \wedge z)^*\) because \((x \cdot z^*)^* = x \wedge z\) and \(x \cdot z^* = (x \wedge z)^*\). Altogether, we consider three invariants

- the square norm of the vector \(x\), that is, \(x \cdot x\),
- the square norm of the bivector \(z\), that is, \(z \cdot z\),
- the element \((x \wedge z)^*\),

where \(\cdot\) coincides with the inner product on \(G_3\). In particular, these invariants form a coordinate system on the moduli space \(M/\text{SO}(3)\).
Proposition 3. Each geodesic starting at the origin defines a curve in the moduli space \( M / SO(3) \), which is determined by the invariants in the following way

\[
\begin{align*}
    x \cdot x &= -\frac{2D^2}{K^2}(\cos(Kt) - 1) + C_1^2 t^2, \\
    z \cdot z &= -\frac{D^2}{4K^4}((4C_2^2 K^2 - 4C_1^2 - D^2) \cos(Kt)^2 + 2KC_1^2(2t(K - 1) \sin(Kt) + t^2 K - 4) \cos(Kt) \\
    & \quad - 2K(4C_1^2 + D^2) \sin(Kt) + t^2(2C_1^2 + D^2)K^2 + D^2 + 8C_2^2), \\
    (x \wedge z)^\ast &= \frac{D^2C_1}{2K^3}((-2K + 2) \cos(Kt)^2 + (2K + 2) \cos(Kt) + K^2 t^2 + Kt \sin(Kt) - 4).
\end{align*}
\]

Proof. Follows directly from (28). \( \square \)

4.2 Geodesics of (4, 7)

The base system (20) can be seen as a system in geometric algebra \( \mathbb{G}_4 \)

\[
\begin{align*}
    \dot{x} + \dot{\ell} &= h_0 + \tilde{h}, \\
    \dot{y} &= -x \wedge \tilde{h} - \ell \wedge h_0,
\end{align*}
\]

where we assume that \( x \) and \( h_0 \) are collinear with \( e_1 \) and \( \ell, \tilde{h} \) in the subspace generated by \( e_2, e_3, e_4 \). The form of the second equation implies that \( y \) is given by minus the wedge product of \( e_1 \) and a vector from this subspace. Hence, the solution \( y(t) \) can be viewed as a curve of planes in \( \mathbb{G}_4 \).

Proposition 4. Each arc-length parameterized sub-Riemannian geodesic satisfying the initial condition \( x(0) = 0, \ell_i(0) = 0, y_i(0) = 0, i = 1, 2, 3 \) is equivalent to a curve in \( N \cong \wedge^1 \mathbb{R}^4 \oplus \wedge^2 \mathbb{R}^4 \subset \mathbb{G}_4 \) and up to the action of suitable \( R \in SO(3) \), it takes the form

\[
\begin{align*}
    q(t) &= x(t) + \ell(t) + y(t) = (C_1 \cos(Kt) + C_2 \sin(Kt) - C_1)e_1 + (C_1 \sin(Kt) - C_2 \cos(Kt) + C_2)e_2 + Cte_3 \\
    & \quad + \frac{1}{2}(C_1^2 + C_2^2)(tK - \sin(Kt))e_1 \wedge e_2 + \frac{C}{2K}((2C_1 - C_2K)\sin(Kt) - (C_1K + 2C_2)\cos(Kt) + 2C_2 - tC_1K)e_1 \wedge e_3,
\end{align*}
\]

where \( K > 0 \) and the constants \( C_1, C_2, C \) satisfy the level condition \( K^2(C_1^2 + C_2^2) + C^2 = 1 \).

Proof. According to the vertical system (19), the vector \( \tilde{h}(t) \) lies in the subspace generated by the vectors \( r_1, r_2 \) for any \( t \). Since the vectors \( r_1 \) and \( r_2 \) are orthonormal, there is an orthogonal matrix \( R \in SO(3) \) that aligns these vectors with the second and third vectors of the standard basis of \( \mathbb{R}^3 \), that is,

\[
    r_1 = Re_2, \quad r_2 = Re_3.
\]

Due to the symmetry of this system, see (23) this rotor defines a representative of the geodesic class \( (x(t), R^T \ell(t), R^T y(t)) \) which is the solution to the horizontal system (20) for

\[
\begin{align*}
    h_0 &= K(C_2 \cos(Kt) - C_1 \sin(Kt)), \\
    \tilde{h}(t) &= K(C_2 \sin(Kt) + C_1 \cos(Kt))e_1 + Ce_2,
\end{align*}
\]

or, equivalently, a curve in \( \mathbb{R}^4 \oplus \wedge^2 \mathbb{R}^4 \in \mathbb{G}_4 \) given by the solution of (31). By direct integration of this equation and by imposing the initial conditions, we get the formula (32) for the solution. \( \square \)

The action of \( SO(3) \) on \( N \cong \mathbb{R}^7 \) given by (23) defines a moduli space \( N / SO(3) \). We see \( N \) as a subset of \( \mathbb{G}_4 \), and the group \( SO(3) \) is represented by rotors instead of matrices, which act on \( N \) by the conjugation. The action preserves the vector and bivector part, the split \( x + \ell \), inner product, norm, and dualization with respect to \( \ast \). The orbits of this action are
determined by natural invariants. For the same reason as in the case of \((3, 6)\) and due to the invariant split, we have three invariants as follows:

- the value of the coordinate \(x\),
- the square of the norm of the vector \(\ell\), that is, \(\ell \cdot \ell\),
- the square of the norm of the bivector \(y\), that is, \(y \cdot y\).

We need one more invariant for the dimensional reasons, but the element \((\ell \wedge y)^*\) is not scalar but vector. On the other hand, \((\ell \cdot y)\) is a multiple of the vector \(e_1\), so the value of \((\ell \cdot y)e_1\) is a scalar. As the last invariant, we consider

- the value of \((\ell \cdot y)e_1\).

These form the coordinate system on the moduli space \(N/\text{SO}(3)\).

**Proposition 5.** Each geodesic starting at the origin defines a curve in the moduli space \(N/\text{SO}(3)\), which is determined by the invariants in the following way

\[
\begin{align*}
  x &= C_1(\cos Kt - 1) + C_2 \sin Kt, \\
  \ell \cdot \ell &= [(C_1 \sin Kt + C_2(1 - \cos Kt))^2 + (Ct)^2], \\
  (\ell \cdot y)e_1 &= \frac{1}{2} \left( (C_1^2 + C_2^2)(C_1 \sin Kt + C_2(1 - \cos Kt))(Kt - \cos Kt) + \frac{C_1^2}{K}(C_1(2 \sin Kt - Kt \cos Kt - Kt) \\
  &\quad + C_2(2 - 2 \cos Kt - Kt \sin Kt)) \right), \\
  y \cdot y &= \frac{1}{4} \left( (C_1^2 + C_2^2)^2(Kt - \cos Kt)^2 + \frac{C_1^2}{K^2}(C_1(2 \sin Kt - Kt \cos Kt - Kt) + C_2(2 - 2 \cos Kt - Kt \sin Kt))^2 \right).
\end{align*}
\]

**Proof.** Follows directly from (32).

5 | EXAMPLES

In the sequel, we present two examples of controls based on the symmetries in geometric algebra approach. We have the following scheme based on Algorithm.

1. For the target point \(q_t\), compute the invariants of the chosen particular control system (2).
2. Solve the system of non-linear Equations (30) or (33) in the moduli space.
3. Find the family of curves (28) or (32) going from the origin to the same point \(q_o\) that belongs to the same \(\text{SO}(3)\) orbit of \(q_t\).
4. Find \(R \in \text{SO}(3)\), such that \(R(q_o) = q_t\).
5. Apply \(R\) on the set of curves (28) or (32) to get a family of curves going from the origin to the target point \(q_t\).

The explicit calculations were acquired using a CAS system Maple similarly to the paper.\(^{18}\)

5.1 | Example in (3, 6)

Our goal is to find the geodesic going from the origin to the target point

\[q_t = (x, z) = 2e_1 - e_2 + 3e_3 + e_1 \wedge e_2 - 2e_1 \wedge e_3 - 2e_2 \wedge e_3\]

using the invariants (30) in the target point. We have

\[
x \cdot x = 14, \quad z \cdot z = -9, \quad (x \wedge z)^* = 3,
\]

and together with the level set condition, we get the system with the invariants at \(q_t\). We solve the system numerically in Maple and present the solution with rounding up to four decimal digits.
Using constant (34), we get the geodesic in the moduli space from the origin to the point $q_0$ in the form

$$q = (x, z) = (0.6965(1 - \cos(0.9886t)))e_1 + 0.6965 \sin(0.9886t)e_2 + 0.7252te_3$$

$$- (0.2425(0.9886t - \sin(0.9886t)))e_1 \wedge e_2 + (0.2555(0.9886t - 2 \sin(0.9886t) + 0.9886t \cos(0.9886t)))e_1 \wedge e_3$$

$$+ (0.2555(2 - 0.9886t \sin(0.9886t) - 2 \cos(0.9886t)))e_2 \wedge e_3$$

and at the time $t = 5.0236$, we reach the point

$$q_0 = 0.5216e_1 - 0.6741e_2 + 3.643e_3 - 1.439e_1 \wedge e_2 + 2.082e_1 \wedge e_3 + 1.611e_2 \wedge e_3.$$

We are looking for the rotor which maps the multivector $q_0$ on the multivector $q_t$. We consider the complete flags

$$\{0\} \subset NO(x_t) \subset NO(x_t \wedge z^*_t) \subset NO(z_t \wedge z^*_t) \cong \mathbb{R}^3,$$

$$\{0\} \subset NO(x_0) \subset NO(x_0 \wedge z^*_0) \subset NO(z_0 \wedge z^*_0) \cong \mathbb{R}^3.$$

We set $R_m = R_3 = \text{id}$ and map the plane $x_0 \wedge z^*_0$ to the plane $x_t \wedge z^*_t$ by the rotor $R_{m-1} = R_2$ according to formula (26). Explicitly,

$$R_2 := 0.1334 + 0.7083e_1 \wedge e_2 - 0.5483e_1 \wedge e_3 - 0.4242e_2 \wedge e_3$$

and we can map the multivector $q_0$ on the multivector $q_t = (x_t, z_t) = R_2q_0R_2$ in such a way that $x_t$ and $z_t$ lie in the plane $x_0 \wedge z^*_0$. Explicitly,

$$q_t = (x_t, z_t) = -2.8510e_1 + 2.3208e_2 - 0.6956e_3 - 2.641e_1 \wedge e_2 + 1.2523e_1 \wedge e_3 + 0.6767e_2 \wedge e_3.$$

Finally, we map the plane $x_t \wedge (x_t \wedge z_t)^*$ to the plane $x_t \wedge (x_t \wedge z_t)^*$ by rotor

$$R_{m-2} = R_1 = 0.3727 + 0.1716e_1 \wedge e_2 + 0.6863e_1 \wedge e_3 - 0.6005e_2 \wedge e_3.$$

Altogether, we found the rotor $R = R_1R_2R_3$ and, when applied on (28), we got a geodesic going from the origin to the point $q_t$ in the form

$$q = (x, z) = (-0.5302 + 0.4673t + 0.5302 \cos(0.9886t) - 0.05136 \sin(0.9886t))e_1$$

$$- (0.008(22.124 + 36.347t - 22.124 \cos(0.9886t) + 76.628 \sin(0.9886t)))e_2$$

$$+ (0.4156 \cos(0.9886t) + 0.4723t + 0.4156 - 0.3267 \sin(0.9886t))e_3$$

$$- (0.2(-1.5244 + 0.7536t \sin(0.9886t) + 0.4086 \sin(0.9886t))$$

$$+ 1.5244 \cos(0.9886t) - 0.5923t \cos(0.9886t) + 0.1884t))e_1 \wedge e_2$$

$$+ (-0.3184t + 0.1299 - 0.2223t \cos(0.9886t) - 0.1299 \cos(0.9886t)$$

$$- 0.0641t \sin(0.9886t) + 0.547 \sin(0.9886t))e_1 \wedge e_3$$

$$+ (-0.389 + 0.1923t \sin(0.9886t) - 0.1358t + 0.0186t \cos(0.9886t)$$

$$+ 0.389 \cos(0.9886t) + 0.1186 \sin(0.9886t))e_2 \wedge e_3.$$
5.2 Example in (4, 7)

Our goal is to find the geodesic going from the origin to the target point

\[ q_t = (x_t, \ell_t, y_t) = e_1 + 2e_2 + e_3 + 3e_4 - e_1 \wedge e_2 + 2e_1 \wedge e_3 + 2e_1 \wedge e_4 \]

using the invariants (33) at the target point. We compute

\[ x = 1, \quad \ell \cdot \ell = 14, \quad y \cdot y = -9, \quad (\ell \cdot y)e_1 = -6, \]

and together with the level set condition, we get the system with the invariants at \( q_t \). We solve the system numerically in Maple, and we present the solution with the constants rounded up to four decimal digits as follows:

\[
\begin{align*}
C &= 0.6126, \\
C_1 &= -0.7816, \\
C_2 &= -0.5324, \\
K &= 0.8358, \\
t &= 6.0748.
\end{align*}
\]

Using constant (37), we get a geodesic in the moduli space from the origin to the point \( q_o \) in the form

\[
q = (x, \ell, y) = (-0.7816 \cos(0.8358t) - 0.5324 \sin(0.8358t) + 0.7816)e_1 \\
+ (-0.7816 \sin(0.8358t) + 0.5324 \cos(0.8358t) - 0.5324)e_2 \\
+ 0.6126e_3 + 0.4471(0.8358t - \sin(0.8358t))e_1 \wedge e_2 \\
+ 0.3665((0.4449t - 1.563) \sin(0.8358t) + (0.6533t + 1.065) \cos(0.8358t) - 1.065 + 0.6533t)e_1 \wedge e_3
\]

and at the time \( t = 6.0748 \), we reach the point

\[ q_o = (x_o, \ell_o, y_o) = e_1 + 0.3878e_2 + 3.722e_3 + 2.688e_1 \wedge e_2 + 1.332e_1 \wedge e_3. \]

We are looking for the rotor which maps the multivector \( q_o \) on the multivector \( q_t \). We shall consider the complete flags starting with the line \( NO(\ell) \) and ending with the space \( NO(\ell \wedge y) \). To find the middle one, we can use the projection of the line \( NO(\ell) \) onto the plane \( NO(\ell \wedge y) \). Thus, we get

\[
\{0\} \subset NO(\ell_1) \subset NO(\ell_1 \wedge (\ell_1 \wedge y_1)^*) \subset NO(\ell_1 \wedge y_1) \subset NO(y_1 \wedge y) \cong \mathbb{R}^4,
\]

\[
\{0\} \subset NO(\ell_o) \subset NO(\ell_o \wedge (\ell_o \wedge y_o)^*) = NO(\ell_o \wedge (\ell_o \wedge y_o)) \subset NO(y_o \wedge y) \subset NO(y_o \wedge y) \cong \mathbb{R}^4.
\]

First, we map the hyperplane \( \ell_o \wedge y_o \) to the hyperplane \( \ell_1 \wedge y_1 \) by the rotor \( R_3 \) according to formula (26). We obtain

\[ R_3 := 0.4863 - 0.4335e_2 \wedge e_4 - 0.7587e_3 \wedge e_4. \]
The next step is to map the hyperplane $\text{NO}(\ell_0 \wedge (\ell_0 \cdot y_0) \wedge (\ell_0 \wedge y_0)^*)$ to the hyperplane $\text{NO}(\ell_1 \wedge (\ell_1 \cdot y_1) \wedge (\ell_1 \wedge y_1)^*)$ by the rotor $R_2$ according to formula (26). Explicitly,

$$R_2 := 0.0387 + 0.9993e_2 \wedge e_3. \quad (40)$$

Finally, we map the hyperplane $\text{NO}(\ell_0 \wedge (\ell_0 \wedge (\ell_0 \wedge y_0^*)^*)^*)$ to the hyperplane $\text{NO}(\ell_1 \wedge (\ell_1 \wedge (\ell_1 \wedge y_1^*)^*)^*)$ by the rotor $R_1$ according to the formula (26). It turns out that $R_1 = 1$. Altogether, $R = R_1R_2R_3$ and, when applied on (5.2), we get a geodesic going from the origin to the point $q_t$ as

$$q = (x, \ell, y) = (-0.7816 \cos(0.8358t) - 0.5324 \sin(0.8358t) + 0.7816)e_1$$
$$+ (0.5261 \sin(0.8358t) - 0.3584 \cos(0.8358t) + 0.3946t + 0.3584)e_2$$
$$+ (-0.4749 \sin(0.8358t) + 0.3235 \cos(0.8358t) + 0.1235t - 0.3235)e_3$$
$$+ ((0.2513 + 0.1542t) \cos(0.8358t) + (-0.06803 + 0.1050t) \sin(0.8358t) - 0.2514 - 0.09731t)e_1 \wedge e_2$$
$$+ ((0.07868 + 0.04827t) \cos(0.8358t) + (-0.3871 + 0.03287t) \sin(0.8358t) - 0.07869 + 0.2753t)e_1 \wedge e_3$$
$$+ ((0.2880 + 0.1767t) \cos(0.8358t) + (0.1203t - 0.6112) \sin(0.8358t) + 0.3342t - 0.2880)e_1 \wedge e_4.$$ 

In Figure 3 we present trajectories $x, (\ell_2, \ell_3, \ell_4)$ and $(y_1, y_2, y_3)$, respectively.

6 | CONCLUSION

We presented the use of geometric algebra for the control systems invariant with respect to the orthogonal transformations. The main contribution of GA lies in a construction of the rotor between two bases of a vector space based only on algebraic computations in a chosen GA. This allows us to use the geometric objects effectively, and analogously to quaternions, the implementations are faster than the usual computations with matrices. We assessed an algorithm and illustrated its use on two particular examples with filtration (3, 6) corresponding to a trident snake robot control and (4, 7) corresponding to the control of a trident snake with flexible leg. All calculations were acquired using Maple packages Clifford\textsuperscript{19} and DifferentialGeometry.\textsuperscript{20}

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.
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